Remarks on global regularity of 2D generalized MHD equations

Baoquan Yuan* and Linna Bai
School of Mathematics and Information Science, Henan Polytechnic University, Henan, 454000, China. (bqyuan@hpu.edu.cn, blinna@163.com)

Abstract
In this paper, we investigate the global regularity of 2D generalized MHD equations, in which the dissipation term and magnetic diffusion term are $\nu(-\Delta)^\alpha u$ and $\eta(-\Delta)^\beta b$ respectively. Let $(u_0, b_0) \in H^s$ with $s \geq 2$, it is showed that the smooth solution $(u(x, t), b(x, t))$ is globally regular for the case $0 \leq \alpha \leq \frac{1}{2}, \alpha + \beta > \frac{3}{2}$.

AMS Subject Classification 2000: 35Q35, 35B65.

Key words: Generalized MHD equations, smooth solution, global regularity.

1 Introduction
In this paper, we consider the following 2D generalized magnetohydrodynamic (GMHD) equations

\[
\begin{aligned}
&u_t + \nu \Lambda^{2\alpha} u + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\
&b_t + \eta \Lambda^{2\beta} b + u \cdot \nabla b = b \cdot \nabla u, \\
&\nabla \cdot u = \nabla \cdot b = 0,
\end{aligned}
\]

(1.1)

where $\alpha \geq 0$, $\beta \geq 0$, $\nu \geq 0$ and $\eta \geq 0$ are real parameters, and $u$ is the velocity of the flow, $b$ is the magnetic field, $p$ is the scalar pressure, $\Lambda = (-\Delta)^{\frac{1}{2}}$ is defined in terms of Fourier transform by

\[
\widehat{\Lambda f}(\xi) = |\xi| \widehat{f}(\xi).
\]

If $\alpha = \beta = 1$, (1.1) is the viscous MHD equations, and the global wellposedness of classical solution is well-known [6]. If $\nu = \eta = 0$, (1.1) is the invisid magnetohydrodynamic equations.

We know that the 2D Euler equation is globally wellposed for smooth initial data. But for the 2D invisid MHD equations, the global wellposedness of classical solution is still a big open problem. So the GMHD equations has attracted much interest of many mathematicians and has motivated a large number of research papers concerning various generalizations and improvements [8, 9, 10, 11, 13, 14]. People pay attention to how the parameters $\nu, \eta, \alpha, \beta$ influence the global regularity of the GMHD equations. It is well-known that the d-dimensional GMHD equations [1, 1] with $\nu > 0$ and $\eta > 0$ has a unique global classical solution for every initial data.

*Corresponding Author: B. Yuan
\((u_0,b_0) \in H^s\) with \(s \geq \max\{2\alpha, 2\beta\}\) if \(\alpha \geq \frac{1}{2} + \frac{d}{4}\) and \(\beta \geq \frac{1}{2} + \frac{d}{4}\) \([9]\). An improved result by Wu \([11]\) was established by reducing the requirement for \(\alpha\) and \(\beta\) and the dissipation in \((1.1)\) by a logarithmic factor. It is showed that the system is globally regular as long as the following conditions \(\alpha \geq \frac{1}{2} + \frac{d}{4}, \beta > 0, \alpha + \beta \geq 1 + \frac{d}{4}\) are satisfied. As a special consequence, smooth solutions of the 2D GMHD equations with \(\alpha \geq 1, \beta > 0, \alpha + \beta \geq 2\) are global.

However, for the 2D incompressible MHD equations with partial dissipation, the global regularity of the classical solutions is still a difficult problem. In 2011, Cao and Wu \([2]\) showed an interesting result which considered the 2D MHD equations of the form

\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= -p + \nu_1 u_{xx} + \nu_2 u_{yy} + b \cdot \nabla b, \\
\partial_t b + u \cdot \nabla b &= \eta_1 b_{xx} + \eta_2 b_{yy} + b \cdot \nabla u, \\
\nabla \cdot u &= \nabla \cdot b = 0,
\end{align*}
\]

they stated that the classical solutions of the equations \((1.2)\) with either \(\nu_1 = 0, \nu_2 = \nu > 0, \eta_1 = \eta > 0\) and \(\eta_2 = 0\) or \(\nu_1 = \nu > 0, \nu_2 = 0, \eta_1 = 0\) and \(\eta_2 = \eta > 0\) are globally existed for all time. If \(\nu_1 = \nu_2 = 0\) and \(\eta_1 = \eta_2 > 0\), the MHD equations \((1.2)\) has a global \(H^1\) weak solution \([2, 5]\). But the existence of global classical solution is an open problem. When \(\eta_1 = \eta_2 = 0\) and \(\nu_1 = \nu_2 > 0\), it is also unknown for the existence of global classical solutions.

Recently, Tran, Yu and Zhai \([8]\) obtained the global regularity of 2D GMHD equations \((1.2)\) for the following three cases: (1) \(\alpha \geq \frac{1}{2}, \beta \geq 1; (2) 0 \leq \alpha < \frac{1}{2}, 2\alpha + \beta > 2; (3) \alpha \geq 2, \beta = 0\). Combining them with the result of \([11]\), we know that if \(\alpha + \beta \geq 2, (1.1)\) with \(\nu > 0\) and \(\eta > 0\) possesses a global smooth solution. Note that in this case, the end point \(\alpha = 0 (\nu = 0)\) and \(\beta = 2\) is not included and it cannot ensure the global regularity for the system \((1.1)\).

Motivated by Tran, Yu and Zhai \([8]\), we carried on a thorough investigation on whether the smooth solutions are global in the case \(\alpha = 0\) and \(\beta = 2\) for 2D GMHD equations. In fact, the system \((1.1)\) has a global classical solution for this case. What is more, we find that when \(\alpha = 0\), the condition \(\beta = \alpha + \beta \geq 2\) can be reduced to \(\beta > \frac{3}{2}\). When \(0 < \alpha \leq \frac{1}{2}\), we also conclude that the system is globally regular provided that \(\alpha\) and \(\beta\) satisfy the relation \(\alpha + \beta > \frac{3}{2}\).

To this end, we state our regularity criteria as follows.

**Theorem 1.1.** Consider the GMHD equations \((1.1)\) in 2D case. Assume \((u_0, b_0) \in H^s\) with \(s \geq 2\). Then the system is globally regular for \(\alpha\) and \(\beta\) satisfying \(0 \leq \alpha \leq \frac{1}{2}, \alpha + \beta > \frac{3}{2}\).

**Remark 1.1.** In the special case \(\alpha = \frac{1}{2}, \beta = 1\), reference \([8]\) showed that the equation \((1.1)\) is globally regular. However, the global regularity of \((1.1)\) with \(0 \leq \alpha \leq \frac{1}{2}, \alpha + \beta = \frac{3}{2}\) is still a difficult problem.

**Remark 1.2.** To simplify the presentation, we will set \(\nu = \eta = 1\). It is a standard exercise to adjust various constants to accommodate other values of \(\nu, \eta\), as long as both are positive.

## 2 Proof of the main result

In this section, we shall prove Theorem \((1.1)\). The key idea here is to apply the standard \(L^2\)-energy estimates to carry out the \(H^1, H^2\) and higher estimates.

### 2.1 \(L^2\) and \(H^1\)-energy estimates

We consider the 2D GMHD equations \((1.1)\) with \(\alpha \geq 0\) and \(\beta \geq 1\). It is easy to get the standard \(L^2\)-energy estimate. Multiplying the first two equations of \((1.1)\) by \(u\) and \(b\), respectively, integrating and adding the resulting equations together it follows that

\[
\|u\|_2^2 + \|b\|_2^2 + 2 \int_0^t \|\Lambda^\alpha u\|_2^2 ds + 2 \int_0^t \|\Lambda^\beta b\|_2^2 ds = \|u_0\|_2^2 + \|b_0\|_2^2, \hspace{1cm} (2.1)
\]
where we have used the incompressibility condition $\nabla \cdot u = \nabla \cdot b = 0$.

As $\beta \geq 1$, we can easily get

$$b \in L^2(0, T; H^\beta(\mathbb{R}^2)) \Rightarrow \nabla b \in L^2(0, T; L^2(\mathbb{R}^2)).$$

Let $\omega = \nabla \times u = -\partial_2 u_1 + \partial_1 u_2$ be the vorticity and $j = \nabla \times b = -\partial_2 b_1 + \partial_1 b_2$ be the current density. Applying $\nabla \times$ to the first two equations of (1.1) we obtain the governing equations.

$$\begin{cases}
\partial_t u + u \cdot \nabla \omega = b \cdot \nabla \omega - \Lambda^{2\alpha} \omega, \\
\partial_t j + u \cdot \nabla j = b \cdot \nabla \omega + T(\nabla u, \nabla b) - \Lambda^{2\beta} j.
\end{cases}
$$

(2.2)

Here

$$T(\nabla u, \nabla b) = 2\partial_t b_1 (\partial_1 u_2 + \partial_2 u_1) + 2\partial_2 b_2 (\partial_1 b_2 + \partial_2 b_1).$$

Multiplying the two equations of (2.2) by $\omega$ and $j$, respectively, integrating and applying the incompressibility condition we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (\omega^2 + j^2) \, dx + \int_{\mathbb{R}^2} (\Lambda^\alpha \omega)^2 \, dx + \int_{\mathbb{R}^2} (\Lambda^\beta j)^2 \, dx = \int_{\mathbb{R}^2} T(\nabla u, \nabla b) j \, dx.
$$

(2.3)

According to the Biot-Savart law, we have the representations

$$\frac{\partial u}{\partial x_k} = R_k(R \times \omega); k = 1, 2,$n

and

$$\frac{\partial b}{\partial x_k} = R_k(R \times j); k = 1, 2,$n

where $R = (R_1, R_2)$, $R_k = -\frac{\partial x_k}{(-\Delta)^{-\frac{1}{2}}}$ denotes Riesz transformation. For details about the Riesz transformation please refer to [7]. By the boundedness of Riesz operator $R$ in $L^p$ space ($1 < p < \infty$), we arrive at

$$\|\nabla u\|_{L^2} \leq C \|\omega\|_{L^2} \quad \text{and} \quad \|\nabla b\|_{L^4} \leq C \|j\|_{L^4}.$n

Using Hölder and Young’s inequalities one has

$$\int_{\mathbb{R}^2} T(\nabla u, \nabla b) j \, dx \leq C \|\nabla u\|_{L^2} \|\nabla b\|_{L^4} \|j\|_{L^4} \leq C \|\omega\|_{L^2} \|j\|_{L^2}^{2-\frac{1}{2}} \|\Lambda^\beta j\|_{L^2}^{\frac{1}{2}} \leq C(\varepsilon) \|\omega\|_{L^2}^{\frac{2}{3}} \|j\|_{L^2} + \varepsilon \|\Lambda^\beta j\|_{L^2}^2 \leq C(\varepsilon)(\|\omega\|_{L^2}^2 + 1) \|j\|_{L^2}^2 + \varepsilon \|\Lambda^\beta j\|_{L^2}^2,$n

where we have used the following Gagliardo-Nirenberg inequality

$$\|j\|_{L^4} \leq C \|j\|_{L^2}^{\frac{1}{2}} \|\Lambda^\beta j\|_{L^2}^{\frac{3}{2}}.$n

Inserting the above estimate into (2.3), and taking $\varepsilon$ small enough so that $\varepsilon < 1$ we have

$$\frac{d}{dt} \left( \|\omega\|_{L^2}^2 + \|j\|_{L^2}^2 \right) + \|\Lambda^\alpha \omega\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2 \leq C(\varepsilon) \left( \|\omega\|_{L^2}^2 + 1 \right) \|j\|_{L^2}^2.$n

Gronwall’s inequality [4, Appendix B.1] and $L^2$ energy estimate imply that

$$\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha \omega\|_{L^2}^2 \, ds + \int_0^t \|\Lambda^\beta j\|_{L^2}^2 \, ds \leq \left( \|\omega_0\|_{L^2}^2 + \|j_0\|_{L^2}^2 \right) \exp \left[ \int_0^t \|j\|_{L^2}^2 \, ds \right] < \infty.$$
2.2 Higher estimates for $\alpha = 0$

In this case we have $\beta > \frac{3}{2}$, and the GMHD equations now read

\[
\begin{align*}
\begin{cases}
u_t + u \cdot \nabla u &= -\nabla p + b \cdot \nabla b, \\
\dot{b}_t + u \cdot \nabla b &= b \cdot \nabla u - \Lambda^{2\beta} b, \\
\nabla \cdot u &= \nabla \cdot b = 0.
\end{cases}
\end{align*}
\]

(2.4)

First of all, we estimate $b_t$. Taking the inner product of the second equation of (2.4) with $\dot{b}_t$ and using Hölder and Young’s inequalities we obtain

\[
\begin{align*}
\|b_t\|^2_{L^2} + \frac{1}{2} \frac{d}{dt} \|\Lambda^\beta b\|^2_{L^2} &\leq \int \|u \cdot \nabla b \cdot b_t\| dx + \int \|b \cdot \nabla u \cdot b_t\| dx \\
&\leq \frac{1}{2} \|b_t\|^2_{L^2} + \frac{1}{2} \left( \|u\|^2_{L^4} \|\nabla b\|^2_{L^4} + \|\nabla u\|^2_{L^2} \|b\|^2_{L^\infty} \right).
\end{align*}
\]

Application of the following Gagliardo-Nirenberg inequalities

\[
\|f\|_{L^p} \leq C \|f\|^\frac{2}{p}_{L^2} \|\nabla f\|^\frac{2}{p}_{L^2},
\]
\[
\|f\|_{L^\infty} \leq C \|f\|^\frac{1}{2}_{L^2} \|\Lambda^\beta f\|^\frac{1}{2}_{L^2},
\]
yields that

\[
\begin{align*}
\|b_t\|^2_{L^2} + \frac{d}{dt} \|\Lambda^\beta b\|^2_{L^2} &\leq C \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla b\|_{L^2} \|\nabla j\|_{L^2} + C \|\nabla u\|^2_{L^2} \|\nabla b\|^2_{L^2} \|\Lambda^\beta b\|^\frac{1}{2}_{L^2} \\
&\leq C \|\nabla j\|_{L^2} + C \|\Lambda^\beta b\|^\frac{1}{2}_{L^2}.
\end{align*}
\]

By the results of the $L^2$-energy estimate and $H^1$ estimate, we deduce that

\[
\|\Lambda^\beta b\|^2_{L^2} + \int_0^t \|b_t\|^2_{L^2} ds \leq \|\Lambda^\beta b_0\|^2_{L^2} + C \int_0^t \|\Lambda^\beta b\|^\frac{1}{2}_{L^2} ds + C \int_0^t \|\nabla j\|_{L^2} ds < \infty. \tag{2.5}
\]

Now we go back to the equation $b_t + u \cdot \nabla b = b \cdot \nabla u - \Lambda^{2\beta} b$, and using the similar way with the estimate of $b_t$ we get

\[
\|\Lambda^{2\beta} b\|^2_{L^2} \leq \|b_t\|^2_{L^2} + \|u \cdot \nabla b\|^2_{L^2} + \|b \cdot \nabla u\|^2_{L^2} \leq \|b_t\|^2_{L^2} + C \|\nabla j\|_{L^2} + C \|\Lambda^\beta b\|^\frac{1}{2}_{L^2}.
\]

Recall that $j = \nabla \times b$, one can deduce, thanks to (2.5), that

\[
\begin{align*}
\int_0^t \|\nabla j\|^2_{H^{2\beta-2}} ds &\leq \int_0^t \|\Lambda^{2\beta} b\|^2_{L^2} ds \\
&\leq \int_0^t \|b_t\|^2_{L^2} ds + C \int_0^t \|\nabla j\|_{L^2} ds + C \int_0^t \|\Lambda^\beta b\|^\frac{1}{2}_{L^2} ds < \infty. \tag{2.6}
\end{align*}
\]

Since $\beta > \frac{3}{2}$, by Sobolev embedding theorem, it is easily to see

\[
\nabla j \in L^2(0, T; H^{2\beta-2}(\mathbb{R}^2)) \hookrightarrow L^2(0, T; L^\infty(\mathbb{R}^2)).
\]

Secondly, we estimate $\omega$. From the first equation of (2.4), we have the vorticity equation $\omega_t + u \cdot \nabla \omega = b \cdot \nabla j$. Multiplying both sides of it by $p|\omega|^{p-2} \omega$ and integrating both sides over $\mathbb{R}^2$, it follows, by Hölder inequality, that

\[
\frac{d}{dt} \|\omega\|^p_{L^p} + p \int \nabla \omega \cdot |\omega|^{p-2} \omega dx \leq p \|b \cdot \nabla j\|_{L^p} \|\omega\|^{p-1}_{L^{p-1}} \leq p \|\omega\|_{L^\infty} \|\nabla j\|_{L^p} \|\omega\|^{p-1}_{L^{p-1}}.
\]
Noting that \( p \int_{\mathbb{R}^2} u \cdot \nabla \omega \cdot |\omega|^{p-2} \omega \, dx = 0 \). Now let \( p \to \infty \), we infer that

\[ \| \omega \|_{L^\infty} \leq \| \omega_0 \|_{L^\infty} + \int_0^t \| b \|_{L^\infty} \| \nabla j \|_{L^\infty} \, ds < \infty. \]

This leads to

\[ \omega \in L^\infty(0, T; L^\infty(\mathbb{R}^2)). \]

Lastly, according to the classical BKMT-type blow up criterion \([1]\) which is the MHD system stays regular beyond \( T \) provided that \( \int_0^T (\| \omega \|_{L^\infty} + \| j \|_{L^\infty}) \, dt < \infty \), the proof of the case \( \alpha = 0 \) is thus completed.

### 2.3 Higher estimates for \( 0 < \alpha \leq \frac{1}{2}, \alpha + \beta > \frac{3}{2} \)

In this case, we can easily get \( \beta > 1 \). Firstly, we estimate \( \| \omega \|_{L^p} \). Multiplying both sides of the first equation of (2.2) by \( p|\omega|^{p-2} \omega \) and integrating both sides over \( \mathbb{R}^2 \), it follows that

\[ \frac{d}{dt} \| \omega \|_{L^p}^p + p \int_{\mathbb{R}^2} \Lambda^{2\alpha} \omega \cdot |\omega|^{p-2} \omega \, dx \leq p \| b \cdot \nabla j \|_{L^p} \| \omega \|_{L^p}^{p-1}. \]

For the dissipation term, we know by the property of Riesz potential that \( \int_{\mathbb{R}^2} \Lambda^{2\alpha} \omega \cdot |\omega|^{p-2} \omega \, dx \geq 0 \). For the details on it see \([3]\). Thus, we have

\[ \| \omega \|_{L^p} \leq \| \omega_0 \|_{L^p} + \int_0^t \| b \cdot \nabla j \|_{L^p} \, ds. \] (2.7)

By the Gagliardo-Nirenberg inequality, one has the following estimate

\[ \| b \cdot \nabla j \|_{L^p} \leq C \| b \|_{L^\infty} \| \nabla j \|_{L^p} \leq C \| b \|_{L^p} \frac{1}{\| b \|_{L^\infty}} \| \Lambda^\beta j \|_{L^\frac{\beta\alpha}{\beta-1}} \| j \|_{L^\frac{2\alpha}{\beta-1}} \| \Lambda^{2\beta-1} j \|_{L^\frac{2(p-1)}{p}} \], (2.8)

where \( p \) satisfies \( p > \frac{1}{\alpha} \). So, inserting (2.8) into (2.7) and applying with \( L^2 \) and \( H^1 \) estimates and (2.6), it can be derived that

\[ \| \omega \|_{L^p} \leq \| \omega_0 \|_{L^p} + C \int_0^t \| b \|_{L^p} \frac{1}{\| b \|_{L^\infty}} \| \Lambda^\beta j \|_{L^\frac{\beta\alpha}{\beta-1}} \| j \|_{L^\frac{2\alpha}{\beta-1}} \| \Lambda^{2\beta-1} j \|_{L^\frac{2(p-1)}{p}} \, ds \]

\[ \leq \| \omega_0 \|_{L^p} + C \int_0^t \| \Lambda^\beta j \|_{L^\frac{\beta\alpha}{\beta-1}} \| \Lambda^{2\beta-1} j \|_{L^\frac{2(p-1)}{p}} \, ds \]

\[ \leq \| \omega_0 \|_{L^p} + C \int_0^t (\| \Lambda^\beta j \|_{L^2}^2 + \| \Lambda^{2\beta-1} j \|_{L^2}^2) \, ds < \infty. \]

Note that as long as \( p > \frac{1}{\alpha} \), we have \( \frac{4(p-1)(1+\beta)}{(2\beta-1)(1+\beta)p} \leq 2 \).

Secondly, we derive the estimates of \( \| \omega \|_{H^1} \) and \( \| j \|_{H^1} \). We differentiate the equations (2.2) with respect to \( x_i \) over \( \mathbb{R}^2 \), then multiply the resulting equations by \( \partial_{x_i} \omega \) and \( \partial_{x_i} j \) for \( i = 1, 2 \), integrate with respect to \( x \) over \( \mathbb{R}^2 \) and sum them up. It follows that

\[
\frac{1}{2} \frac{d}{dt} (\| \nabla \omega \|_{L^2}^2 + \| \nabla j \|_{L^2}^2) + \| \Lambda^\alpha \nabla \omega \|_{L^2}^2 + \| \Lambda^\beta \nabla j \|_{L^2}^2 \\
\leq \int |\nabla u| |\nabla \omega|^2 \, dx + \int |\nabla b| |\nabla j| |\nabla \omega| \, dx + \int |\nabla u| |\nabla j|^2 \, dx \\
+ \int |\nabla b| |\nabla \omega| |\nabla j| \, dx + \int |\nabla^2 u| |\nabla b| |\nabla j| \, dx + \int |\nabla u| |\nabla^2 b| |\nabla j| \, dx \\
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \] (2.9)
It is easy to see that the estimates of $I_4$ and $I_5$ are the same as $I_2$ while $I_6$ is the same as $I_3$. Therefore, it suffices to estimate $I_1$, $I_2$, $I_3$.

 Hölder, Young and Gagliardo-Nirenberg inequalities together give

$$I_1 \leq \|\nabla u\|_{L^p} \|\nabla \omega\|_{L^2}^2 \leq C \|\omega\|_{L^p} \|\nabla \omega\|_{L^2}^{2(\alpha - 1)} \|\Lambda^\alpha \nabla \omega\|_{L^2} \leq C(\varepsilon) \|\nabla \omega\|_{L^2}^2 + \varepsilon \|\Lambda^\alpha \nabla \omega\|_{L^2},$$

where $p$ and $q$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and $p > \frac{1}{\alpha}$.

Arguing similarly as the estimate of $I_1$, thanks to the $L^2$ and $H^1$ estimates, one has

$$I_2 \leq \|\nabla b\|_{L^\infty} \|\nabla j\|_{L^2} \|\nabla \omega\|_{L^2} \leq C \|\nabla b\|_{L^2} |\frac{1}{2} - \frac{1}{p}| \|\Lambda^\beta \nabla b\|_{L^2} \leq C(\varepsilon) \|\nabla j\|_{L^2}^2 + \varepsilon \|\Lambda^\beta \nabla j\|_{L^2}^2,$$

where use has been made of the following Gagliardo-Nirenberg inequality

$$\|\nabla b\|_{L^\infty} \leq C \|\nabla b\|_{L^2}^{1 - \frac{4}{\beta}} \|\Lambda^\beta \nabla b\|_{L^2}^{\frac{4}{\beta}}.$$ 

The estimate of $I_3$ can also be obtained by Hölder, Young and Sobolev embedding inequalities

$$I_3 \leq \|\nabla u\|_{L^2} \|\nabla j\|_{L^2}^2 \leq C \|\nabla u\|_{L^2} \|\nabla j\|_{L^2}^{2(\alpha - 1)} \|\Lambda^\beta \nabla j\|_{L^2} |\frac{1}{2} - \frac{1}{p}| \leq C(\varepsilon) \|\nabla j\|_{L^2}^2 + \varepsilon \|\Lambda^\beta \nabla j\|_{L^2}^2.$$ 

Combining the above estimates into (2.9), and taking $\varepsilon$ small enough we get

$$\frac{1}{2} \frac{d}{dt}(\|\nabla \omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2) + \|\Lambda^\alpha \nabla \omega\|_{L^2}^2 + \|\Lambda^\beta \nabla j\|_{L^2}^2 \leq C(\varepsilon) \|\Lambda^\beta \nabla b\|_{L^2} \|\nabla \omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2).$$

Gronwall’s inequality and $H^1$ estimate imply that

$$\|\nabla \omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha \nabla \omega\|_{L^2}^2 ds + \int_0^t \|\Lambda^\beta \nabla j\|_{L^2}^2 ds \leq C(\|\nabla \omega_0\|_{L^2}^2 + \|\nabla j_0\|_{L^2}^2).$$

Thus, we arrive at

$$\omega \in L^\infty(0, T; H^1(\mathbb{R}^2)) \cap L^2(0, T; H^{\alpha+1}(\mathbb{R}^2)),
\quad j \in L^\infty(0, T; H^1(\mathbb{R}^2)) \cap L^2(0, T; H^{\beta+1}(\mathbb{R}^2)).$$

In the end, by the embedding relation $H^s(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ for $s > 1$, we can get $\omega \in L^2(0, T; L^\infty(\mathbb{R}^2)), j \in L^2(0, T; L^\infty(\mathbb{R}^2))$, and combining the BKM-type blow-up criterion [12], this completes the proof. Obviously, the fact $H^1(\mathbb{R}^2) \hookrightarrow \text{BMO}(\mathbb{R}^2)$ and the blow-up criterion [12] can also give the proof. □

Acknowledgements The research of B Yuan was partially supported by the National Natural Science Foundation of China (No. 11071057). Innovation Scientists and Technicians Troop Construction Projects of Henan Province (No. 104100510015).

References

[1] R. E. Caflisch, I. Klapper and G. Steele, Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD, Comm. Math. Phys., 184 (1997), 443-455.

[2] C. S. Cao and J. H. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, Adv. Math., 226 (2011), 1803-1822.
[3] A. Córdoba and D. Córdoba, A maximum principle applied to quasi-geostrophic equations, *Comm. Math. Phys.*, **249** (2004), 511-528.

[4] L. C. Evans, Partial Differential Equations, American Mathematical Society, Providence, Rhode Island, 1998.

[5] Z. Lei and Y. Zhou, BKM’s criterion and global weak solutions for magnetohydrodynamics with zero viscosity, *Discrete Contin. Dyn. Syst.*, **25** (2009), 575-583.

[6] M. Sermange and R. Temam, Some mathematical questions related to the MHD equations, *Comm. Pure Appl. Math.*, **36** (1983), 635-644.

[7] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

[8] C. V. Trann, X. W. Yu and Z. C. Zhai, On global regularity of 2D generalized magnetohydrodynamic equations, *J. Differential Equations*, **254** (2013), 4194-4216.

[9] J. H. Wu, Generalized MHD equations, *J. Differential Equations*, **195** (2003), 284-312.

[10] J. H. Wu, Regularity criteria for the generalized MHD equations, *Comm. Partial Differential equations*, **33** (2008), 285-306.

[11] J. H. Wu, Global regularity for a class of generalized magnetohydrodynamic equations, *J. Math. Fluid Mech.*, **13** (2011), 295-305.

[12] Z. F. Zhang and X. F. Liu, On the blow-up criterion of smooth solutions to the 3D ideal MHD equations, *Acta Math. Appl. Sin. Engl.Ser.*, **4** (2004), 695-700.

[13] Z. J. Zhang, Remarks on the regularity criteria for the generalized MHD equations, *J. Math. Anal.Appl.*, **375** (2011), 799-802.

[14] Y. Zhou, Regularity criteria for the generalized viscous MHD equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **24** (2007), 491-505.