Abstract

We explore the cycle types of a class of biased random derangements, described as a random game played by some children labeled \(1, \ldots, n\). Children join the game one by one, in a random order, and randomly form some circles of size at least 2, so that no child is left alone. The game gives rise to the cyclic decomposition of a random derangement, inducing an exchangeable random partition. The rate at which the circles are closed varies in time, and at each time \(t\), depends on the number of individuals who have not played until \(t\). A \(\{0, 1\}\)-valued Markov chain \(X_n\) records the cycle type of the corresponding random derangement in that any 1 represents a hand-grasping event that closes a circle. Using this, we study the cycle counts and sizes of the random derangements and their asymptotic behavior. We approximate the total variation distance between the reversed chain of \(X^n\) and its weak limit \(X^\infty\), as \(n \to \infty\). We establish conditional (and push-forward) relations between \(X^n\) and a generalization of the Feller coupling, given that no 11-pattern (1-cycle) appears in the latter. We extend these relations to \(X^\infty\) and apply them to investigate some asymptotic behaviors of \(X^n\).

Keywords: Generalized Feller Coupling, biased random permutations and derangements, conditioning, exchangeable random partitions, probabilistic combinatorics

MSC: 60C05, 60J10, 60F05, 05A05, 65C40
1 Introduction

This paper studies biased signed or unsigned random derangements as random permutations conditioned on having no fixed points. A simplified version of the random derangement models studied in this paper may be described as the initial configuration of the following playground game played by $n$ children at camp. The children are labeled $1, \ldots, n$ and their right hands and left hands by $+1, \ldots, +n$ and $-1, \ldots, -n$, respectively. We denote by $|i|$ a child with hand $i \in [-n] := \{\pm 1, \ldots, \pm n\}$. The goal is to form an ordered collection of circles of children holding hands, with some children looking in and others looking out of the circles. Of course no circle of size 1 is allowed, which means a child is not allowed to grasp their other hand. The process is completed in $n$ steps. Each step corresponds to a hand grasping event. First, a label $\sigma_n$ is chosen, uniformly at random, from $[|+n|] := \{+1, \ldots, +n\}$. Child $|\sigma_n|$ starts the game by initiating the first circle. This is represented by $(\sigma_n, \ldots, \sigma_n)$, where $\sigma_n \in [|+n|]$ (i.e. it is positive) indicates that child $|\sigma_n|$ looks in. While looking in, with their left hand child $|\sigma_n|$ grasps a hand $\sigma_{n-1}$ of another child chosen uniformly at random from $[\pm n] \setminus \{\pm |\sigma_n|\}$. We record this as an incomplete circle $(\sigma_n, \sigma_{n-1}, \ldots)$. Note that $\sigma_{n-1} \in [+n]$ or it is equivalently a right hand, if and only if child $|\sigma_{n-1}|$ looks in. In the second step, child $|\sigma_{n-1}|$, with their free hand grasps a free hand chosen uniformly at random from the set of all remaining $2(n-2) + 1$ free hands. As a result, child $|\sigma_{n-1}|$ chooses to grab $\sigma_n$ (the right hand of $|\sigma_n|$) with probability $1/(2(n-2) + 1)$. In this case, the first circle is completed and $\sigma_{n-2}$ is chosen uniformly at random from $[|+n|] \setminus \{\pm \sigma_n, \pm \sigma_{n-1}\}$. Child $|\sigma_{n-2}|$ starts the second circle. Again, we assume $|\sigma_{n-2}|$ looks in. This is recorded as $(\sigma_n, \sigma_{n-1}, \sigma_{n-2}, \ldots)$. In the case that child $|\sigma_{n-1}|$ chooses to not grab $\sigma_n$, they grab a hand $\sigma_{n-2}$ chosen uniformly at random from $[\pm n] \setminus \{\pm |\sigma_n|, \pm |\sigma_{n-1}|\}$. We write $(\sigma_n, \sigma_{n-1}, \sigma_{n-2}, \ldots)$ for the updated circle.

In the same manner, right after the grasping event which results in the completion of a circle, a new circle gets started by a child whose label is chosen uniformly at random from the set of all unused labels. We always assume the child who starts a circle looks in, so we pick a positive label for the first child of each circle. Note also that, in this way, there is exactly one incomplete circle before every hand grasping event. In each step, the last child in the incomplete circle chooses a hand uniformly at random from the set of all remaining free hands which do not make any circle of size 1, and grasps it with their free hand (left hand if they are the child who starts the circle). Assuming that $|\sigma_2|$, the last-but-one child who joins the game, always grasps a randomly chosen hand of the last child $|\sigma_1|$, so as not to leave the last child alone, the process ends up with an ordered collection of circles of size at least 2. We emphasize that the side at which a child looks is represented by signs $+/-$ such that $+i$ means the child $i \in [n] := \{1, \ldots, n\}$ looks in while $-i$ indicates the child $i$ looks out (cf. Figure 1).

Note that when there are $n - j$ children left who are not playing yet and there are at least two children in the current incomplete circle, with their free hand the last child in the incomplete circle grasps any of the other $2(n - j) + 1$ free hands (including the free hand of the first child in the incomplete circle) with the same probability, $0.5/(0.5 + n - j)$. We can generalize this model by changing the proportional weights of closing a circle versus grasping the right and left hands of children who are not playing yet. More precisely, we assume that the weight of the right hand of the first child in the circle is given by $\theta$, while the right and the left hands of each of the other $n - j$ children with both hands free are given by $\kappa$ and $1 - \kappa$, respectively. This means that the probability of closing the circle is $\theta/(\theta + n - j)$, and the probability that a new child joins the circle while they look in (or look out) is $\kappa(n - j)/(\theta + n - j)$ (or $(1 - \kappa)(n - j)/(\theta + n - j)$, respectively).
in two steps. To see this, let $p_i = q_i = (i - 1)/(i - 1 + \theta)$ for $i \geq 3$. Let $p = (p_i)_{i \geq 1}$ and $q = (q_i)_{i \geq 1}$, where $q_i = 1 - p_i$. In the first step, we can model the sizes of the circles in the order of their formation via a $\{\pm 0, 1\}$-valued Markov chain $X^{\sigma,p,\kappa} = (X^{\sigma,p,\kappa}_i)_{j=0}^\infty$. More precisely, set $X^{\sigma,p,\kappa}_n = 1$ and for $j \leq n$, at the $j^{th}$ step of the playground game, let $X^{\sigma,p,\kappa}_{n+1-j} = 1$ if the last child in the incomplete circle grasps the right hand of the first child in that circle and hence closes the circle, let $X^{\sigma,p,\kappa}_{n+1-j} = 0$ if he or she grasps the right hand of a child with two free hands, and let $X^{\sigma,p,\kappa}_{n+1-j} = -1$ if he or she grasps the left hand of a child with two free hands. Letting $Q^{\sigma,p,\kappa}_r(u, v) := \mathbb{P}(X^{\sigma,p,\kappa}_r = v | X^{\sigma,p,\kappa}_{r+1} = u)$ for $1 \leq r \leq n$ and $u, v \in \{\pm 0, 1\}$, this can be formalized as a $\{\pm 0, 1\}$-valued Markov chain with the transition probability matrices

$$
\tilde{P}^{\sigma,p,\kappa}_r := \begin{pmatrix}
Q^{\sigma,p,\kappa}_r(+0, +0) & Q^{\sigma,p,\kappa}_r(+0, -0) & Q^{\sigma,p,\kappa}_r(+0, 1) \\
Q^{\sigma,p,\kappa}_r(-0, +0) & Q^{\sigma,p,\kappa}_r(-0, -0) & Q^{\sigma,p,\kappa}_r(-0, 1) \\
Q^{\sigma,p,\kappa}_r(1, +0) & Q^{\sigma,p,\kappa}_r(1, -0) & Q^{\sigma,p,\kappa}_r(1, 1)
\end{pmatrix} = \begin{pmatrix}
\kappa p_r & (1 - \kappa)p_r & q_r \\
\kappa p_r & (1 - \kappa)p_r & q_r \\
\kappa & 1 - \kappa & 0
\end{pmatrix},
$$

for $2 \leq r \leq n - 1$ and

$$
\tilde{P}^{\sigma,p,\kappa}_n = \begin{pmatrix}
\kappa & 1 - \kappa & 0 \\
\kappa & 1 - \kappa & 0 \\
\kappa & 1 - \kappa & 0
\end{pmatrix}, \quad \tilde{P}^{\sigma,p,\kappa}_1 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}.
$$

Having defined $X^{\sigma,p,\kappa}$, in the second step, we use an auxiliary randomization to generate a random signed permutation representing the playground game process. More explicitly, choose $\tilde{\sigma}_n$ uniformly at random from $\{+n\}$. Suppose we have $\tilde{\sigma}_n, \ldots, \tilde{\sigma}_{i+1}$ for $i < n$. Then pick $\tilde{\sigma}_i$ uniformly at random from $\{+1, \ldots, +n\} \setminus \{\pm \tilde{\sigma}_{i+1}, \ldots, \pm \tilde{\sigma}_n\}$ if $X^{\sigma,p,\kappa}_{i+1} = 1$ or $X^{\sigma,p,\kappa}_{i+1} = +0$, and pick $\tilde{\sigma}_i$ uniformly at random from $\{-1, \ldots, -n\} \setminus \{\pm \tilde{\sigma}_{i+1}, \ldots, \pm \tilde{\sigma}_n\}$ if $X^{\sigma,p,\kappa}_{i+1} = 0$. Denote by $K_n = \sum_{j=1}^n X^{\sigma,p,\kappa}_j$ the number of circles in the $X^{\sigma,p,\kappa}$ process and by $A^{(n)}_j, j \leq K_n$, the size of the $j^{th}$ circle in the $X^{\sigma,p,\kappa}$ process, in order of formation of circles. Letting $\tau_j = n + 1 - \sum_{i=1}^j A^{(n)}_i$, we can represent the random playground game with parameters $\theta$ and $\kappa$ by

$$
(\tilde{\sigma}_n, \ldots, \tilde{\sigma}_{\tau_1})(\tilde{\sigma}_{\tau_1+1}, \ldots, \tilde{\sigma}_{\tau_2}) \cdots (\tilde{\sigma}_{\tau_{k-1}+1}, \ldots, \tilde{\sigma}_{\tau_k}), \quad k = K_n.
$$

Figure 1: An example of a playground game with 14 children forming two circles with some children looking in and some looking out of the circle. The associated signed permutation is $\sigma = (+1 - 4 + 6 + 2 + 8 - 13 + 9 + 10)(+3 + 12 + 14 + 11 + 7 - 5)$. Modified figure from [1].
Note that the circles are placed in order, and \((x_1, \ldots, x_r)\) means that \(|x_i|\) holds a hand of \(|x_{i+1}|\), for \(i = 1, \ldots, r-1\) and \(|x_r|\) holds the right hand of \(|x_1|\). Furthermore, child \(|x_i|\) looks in if and only if \(x_i > 0\). Forgetting the signs and the order among the circles in this representation, one obtains the cycle decomposition of a random permutation on \([n]\). It is clear from the definition that the random partition induced from the cycles of this random permutation is exchangeable. On the other hand, keeping the signs gives rise to a specific random signed permutation on \(\{\pm 1, \ldots, \pm n\}\), which will not be discussed further in this paper.

The case \(p_i = (i - 1)/(\theta + i - 1)\) for \(\theta > 0\) is very special and can be extended to any \(p_i \in (0, 1)\) for \(i \geq 3\). Henceforth, for \(p = (p_i)_{i=1}^\infty\) in the above construction, we assume that \(p_1 = 0\), \(p_2 = 1\), and \(p_i \in (0, 1)\), for \(i \geq 3\).

The random signed permutations discussed above may be interpreted in various ways. For instance, one can consider \(n\) strands, labeled \(1, \ldots, n\). Each strand \(i\) may represent a gene or a marker with two ends denoted by \(-i\) and \(+i\), where the signs indicate the genes’ polarity or orientation (also called strandedness in the biological literature). As a result of the process discussed above, one can obtain a random genome with some circular chromosomes. See [7] for further applications of signed permutations in Mathematical Genomics. As another example, tying randomly the ends of \(n\) cooked spaghetti strands such that the ends of any strand are not tied together (i.e. there is no circle of size 1), one can construct a random spaghetti loop as discussed in [12]. In this paper, we only follow the playground game perspective. Of particular interest is the probability distribution of cycle numbers and sizes, their asymptotic behavior, and their relationship to the so-called generalized Feller coupling conditional on having no cycle of size 1.

### 2 Description of results

This section briefly discusses the main results of this paper. In particular, in Section 2.1, we explain some results, mostly related to the number and type of the cycles in \(\bar{X}^{n,p,\kappa}\), for which the sign information is not needed, hence can be ignored. This can be done via a projected \(\{0,1\}\)-valued Markov chain which is simply obtained from \(\bar{X}^{n,p,\kappa}\) by dropping the signs from \(-0\) and \(+0\). In Section 2.2, we explain how more complex quantities for \(\bar{X}^{n,p,\kappa}\), specifically those for which the signs are involved, can be derived from the corresponding quantities for the projected chain. The outline of this paper will be given at the end of this section (see Section 2.3).

#### 2.1 When signs do not matter

As already mentioned, we can ignore and drop the signs in studying those properties of \(\bar{X}^{n,p,\kappa}\) for which signs are not involved. The major examples are the total number and the type of the cycles. More formally, by projecting \(\{\pm 0,1\}\) onto \(\{0,1\}\) such that \(\pm 0 \mapsto 0\) and \(1 \mapsto 1\), we derive a \(\{0,1\}\)-valued inhomogeneous Markov chain \(X^{n,p} := (X_{n+1} = 1, X_n, \ldots, X_1)\) from \(\bar{X}^{n,p,\kappa}\), whose transition matrices are given by

\[
P^n_p := \begin{pmatrix}
\mathbb{P}(X_{r+1}^n = 0 | X_r^n = 0) & \mathbb{P}(X_{r+1}^n = 1 | X_r^n = 0) \\
\mathbb{P}(X_{r+1}^n = 0 | X_r^n = 1) & \mathbb{P}(X_{r+1}^n = 1 | X_r^n = 1)
\end{pmatrix} = \begin{pmatrix}
p_r & q_r \\
1 & 0
\end{pmatrix},
\]

for \(r = n-1, \ldots, 2\), and

\[
P^n_1 = \begin{pmatrix}
1 & 0 \\
1 & 0
\end{pmatrix}, \quad P^n_1 = \begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix},
\]
where \( p_1 = 0, p_2 = 1, \) and \( p_i \in (0,1) \), for \( i \geq 3 \). Note that under the above projection, the law of \( X^{n,P} \) can be obtained as the push-forward measure of the law of \( \tilde{X}^{n,P,\kappa} \). For \( \theta > 0 \), we denote by \( \eta = \eta^{n,\theta} \) the special case of \( X^{n,P} \) with \( p_r = (r-1)/(r+1+\theta) \). We also denote by \( P^{\eta,P}_{r} = P^{\eta^{n,\theta}}_{r} \), \( r = 1, \cdots, n \), the transition probability matrices of \( \eta \). In what follows, we refer to \( X^{n,P} \) and \( \eta^{n,\theta} \) as the Random Derangement and Playground Game (PG) processes, respectively.

In Section 3, after some useful results such as the marginal distribution of \( X^n \) and \( \eta^n \) (Section 3.1), we find the expected number of cycles and \( j \)-cycles and their asymptotics for \( X^n \) and \( \eta^n \); see Theorems 1 and 2, Proposition 1, and Lemmas 3 and 4.

In Section 4.1, we represent the law of the derangement Markov chain \( X^{n,P} \) as the law of the cycle type of a biased random permutation conditioned on not having any fixed points. In fact, we will see that the cycle type of this biased random permutation may be generated as the spacings (each \( j \)-spacing represents a \( j \)-cycle) between consecutive 1s in a sequence of independent \( \{0,1\} \)-valued random variables called the Generalized Feller Coupling.

To better describe this, note that the dependency between the pairs of \( \eta^{n,\theta}_{r+1} \) and \( \eta^{n,\theta}_{r} \) (or more generally between \( X^{n,P}_{r+1} \) and \( X^{n,P}_{r} \)) is a result of the discrepancy between the rows of \( P^{\eta^{n,\theta}}_{P} \) (or \( P^{\eta^{n,\theta}}_{P} \)). While the second row in \( P^{\eta^{n,\theta}}_{P} \) ensures the occurrence of no consecutive 1s, the first row distributes the same chances to 0 and 1 as does the \( r^{th} \) component of the classic Feller coupling, introduced in [2]. So by replacing the lower row of \( P^{\eta^{n,\theta}}_{P} \) by \( \frac{\theta}{r+\theta} \) \( \frac{\theta}{r+\theta} \), one can obtain the Feller coupling with parameter \( \theta \) that is the sequence of independent Bernoulli random variables \( \tilde{\xi}^{\theta} := (\tilde{\xi}^\theta_i)_{i=1}^{\infty} \) with \( \mathbb{P}_\theta(\tilde{\xi}^\theta_i = 1) = 1 - \mathbb{P}_\theta(\tilde{\xi}^\theta_i = 0) = \frac{\theta}{1+\theta} \). As before, we drop \( \theta \) if there is no risk of confusion.

Let \( C^{\tilde{\xi}}_j(n) \) be the number of \( j \)-spacings between 1s in \( \tilde{\xi}_n, \cdots, \tilde{\xi}_1 \), i.e., the number of sub-patterns \( 10^{j-1}1 \) in it. The distribution of the cycle counts \( C^{\tilde{\xi}}(n) := (C^{\tilde{\xi}}_1(n), \cdots, C^{\tilde{\xi}}_n(n)) \) is given by the well-known Ewens Sampling Formula (ESF) [6]

\[
ESF_n(\theta)(c_1, \cdots, c_n) := \mathbb{P}_\theta(C^{\tilde{\xi}}_1(n) = c_1, \cdots, C^{\tilde{\xi}}_n(n) = c_n) = 1 \left( \sum_{j=1}^{n} j c_j = n \right) \frac{n!}{\theta(n)} \sum_{j=1}^{n} \begin{pmatrix} \theta \\ j \end{pmatrix} \frac{1}{c_j!}
\]

It is tempting to think that the law of \( \eta^{n,\theta} \) is obtained from the \( ESF_n(\theta) \) conditioned on \( C^{\tilde{\xi}}_1(n) = 0 \), but this is not true. Letting \( \lambda_1(\theta) = 0 \) and

\[
\lambda_n(\theta) := \mathbb{P}_\theta(C^{\tilde{\xi}}_1(n) = 0) = \frac{n!}{\Gamma(n+\theta)} \sum_{j=0}^{n} (-1)^j \frac{\theta^j \Gamma(n+\theta-j)}{j!(n-j)!}, n = 2, 3, \ldots
\]

the authors showed in [5] that for a Markov chain \( \tilde{\eta} = \tilde{\eta}^{n,\theta} = (\tilde{\eta}^{(n)}_{n+1} = 1, \cdots, \tilde{\eta}^{(n)}_{1}) \) with the transition probabilities

\[
P^{\tilde{\eta}}_{r} = \begin{pmatrix}
\mathbb{P}(\tilde{\eta}^{(n)}_{r} = 0 | \tilde{\eta}^{(n)}_{r+1} = 0) & \mathbb{P}(\tilde{\eta}^{(n)}_{r} = 1 | \tilde{\eta}^{(n)}_{r+1} = 0) \\
\mathbb{P}(\tilde{\eta}^{(n)}_{r} = 0 | \tilde{\eta}^{(n)}_{r+1} = 1) & \mathbb{P}(\tilde{\eta}^{(n)}_{r} = 1 | \tilde{\eta}^{(n)}_{r+1} = 1)
\end{pmatrix}
\]

defined by

\[
P^{\tilde{\eta}}_{r} = \begin{pmatrix}
(\theta + r - 1)\lambda_r(\theta) & \theta \lambda_{r-1}(\theta) \\
(\theta + r - 1)\lambda_r(\theta) + \theta \lambda_{r-1}(\theta) & (\theta + r - 1)\lambda_r(\theta) + \theta \lambda_{r-1}(\theta)
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0
\end{pmatrix}
\]

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for \(2 < r < n\), and
\[ P_n^\eta = P_2^\eta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_1^\eta = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \]
we have the conditional relation
\[ \mathcal{L}(C_2^\eta(n), \ldots, C_n^\eta(n)) = \mathcal{L}(C_2^\xi(n), \ldots, C_n^\xi(n) \mid C_1^\xi(n) = 0), \]
where \(C_j^\eta(n)\) is the number of \(j\)-spacings between consecutive 1s in \(\eta\).

A similar question can be asked about \(\eta\) and more generally \(X^\eta,p\): Is there a sequence of independent random variables \(Y = (Y_i)_{i=1}^\infty\) for which \(P_\theta(Y = 1) = 1 - P_\theta(Y = 0) = \theta/(i-1+\theta)\) for \(i \in \mathbb{N}\), and for any \(n \in \mathbb{N}\), the laws of \(X_n^\eta,p\) and \(Y\) satisfy
\[ \mathcal{L}(X_n^\eta,p, \ldots, X_1^\eta,p) = \mathcal{L}(Y_n, \ldots, Y_1 \mid C_1^Y(n) = 0), \]
where \(C_j^Y(n)\) is the number of \(j\)-spacings between 1s (i.e., 1 \(0^{i-1}\) patterns) in the sequence \(1, Y_n, \ldots, Y_1\)? Theorem 3 shows that this is not necessarily possible for a fixed \(\theta > 0\). Instead, it shows that for a vector \(\theta := (\theta_i)_{i \geq 1}\) with \(\theta_i > 0\) for \(i \in \mathbb{N}\), defining \(Y^\theta = (Y_i^\theta)_{i=1}^\infty\) with
\[ P_\theta(Y_1 = 1) = 1 - P_\theta(Y_1 = 0) = \frac{\theta_i}{i-1+\theta_i}, \]
and letting \(C_j^{Y^\theta}(n)\) count the number of \(j\)-spacings in \(1, Y_n^\theta, \ldots, Y_1^\theta\), we have
\[ \mathcal{L}(X_n^\eta,p, \ldots, X_1^\eta,p) = \mathcal{L}(Y_n^\theta, \ldots, Y_1^\theta \mid C_1^{Y^\theta}(n) = 0), \]
if and only if \(\theta_i = (i-1)q_i/(p_ip_{i-1})\) for \(i = 3, \ldots, n-1\). We call \(Y^\theta\) the generalized Feller coupling with vector parameter \(\theta\). The corresponding Feller coupling for \(\eta^{n,\theta}\) is denoted by the sequence of independent \(\{0,1\}\)-valued random variables \(\xi^\theta = (\xi_i^\theta)_{i=1}^\infty\) with
\[ P_\theta(\xi_1^\theta = 1) = 1 - P_\theta(\xi_1^\theta = 0) = \theta_i^*/(i-1+\theta_i^*) \]
where \(\theta_i^* = \theta/(1+\theta/(i-2))\), for \(i \geq 4\), and \(\theta_3^* = \theta\); cf. Corollary 1. Some basic properties of \(Y^\theta\) (e.g., its cycle count distribution) along with some of its applications for \(X_n^\eta,p\) via the conditional relation are given in Sections 4.1, 4.2 and 4.3.

Recall that, to avoid 1-cycles, we always assume \(X_n^\eta,p = 0\), for any \(n \in \mathbb{N}\), and this restriction is the reason that \(\mathcal{L}(X_n^\eta,p)\) and \(\mathcal{L}(X_n^\eta,p)\) cannot couple, for \(n > m \geq 2\). In other words, \(\mathcal{L}(X_n^\eta,p)\) cannot be obtained by projecting \(\mathcal{L}(X_n^\eta,p)\) on the first \(m\) components. Assuming \(\sum_{j=1}^\infty p_j = \infty\), however ensures \(\varphi_m(p) := \lim_{n \to \infty} \mathbb{P}(X_n^\eta,p = 1)\) exists and \(0 < \varphi_m(p) < 1\) (Lemma 5). Under this condition, we find an infinite Markov chain \(X^{\infty,p}\) arising as the weak limit of \(X_n^\eta,p\), as \(n \to \infty\). In particular, Theorem 5 and Corollary 3 estimate the total variation distance between \(X_n^\eta,p\) and \(X^{\infty,p}\), and provides certain conditions under which this distance converges to zero. In Section 4.5, we investigate conditions under which the conditional relation holds between \(X^{\infty,p}\) and \(Y^\theta\), namely \(\mathcal{L}(X^{\infty,p}) = \mathcal{L}(Y^\theta \mid \sum_{j=1}^\infty Y_j^\theta Y_{j+1}^\theta = 0)\), where the condition on the right means there is no 11 pattern (or equivalently, 1-cycle) in the infinite chain \(Y^\theta\).

As discussed above, the conditional relation (1) holds for \(\theta_i = (i-1)q_i/(p_ip_{i-1})\), not for \(q_i = \theta_i/(i-1+\theta_i)\). For the latter, rather than a conditional relation, we have a push-forward relation under a specific 11-erasing map. More precisely, let \(\chi_n\) be a map that erases 11 patterns, from
left to right, from any given \( \{0,1\} \)-valued sequence \( 1, y_2, y_1 = 1 \). In Section 5 we will see that for \( p_i = (i - 1)/(i + 1 + \theta_i) \), \( \mathcal{L}(X^n P) = \chi_n * \mathcal{L}(Y^n \theta, \cdots, Y^n \theta) \), where the r.h.s. represents the push-forward of \( \mathcal{L}(Y^n \theta, \cdots, Y^n \theta) \) under \( \chi_n \). We extend this push-forward relation to \( X^n P \) and apply it in Section 5.1 to prove a central limit theorem for the total number of cycles in \( X^n P \), see Theorem 8 and Corollary 4. Another application of this is given in Section 5.2, where we show if \( \theta_n \to \theta \) as \( n \to \infty \), the normalized cycle-lengths of \( X^n P \), in order of formation, converges weakly to \( GEM(\theta) \); see Theorem 9.

The main purpose of this paper is to build a theory to relate \( X^n P, X^n P \), and \( Y^n \theta \) together in a very general set up where \( \theta_i > 0 \) and \( p_i \in (0, 1) \), for \( i \geq 3 \). This is done using the conditional and push-forward relations which connects the random permutations and derangements. These relations are applied to study the asymptotic behavior of the cycle counts of \( X^n P \). Although, most of the results hold for a broad range of choices for \( \theta_i \) and \( p_i \), when it comes to computational examples, we focus on calculating different quantities either for \( \eta \) and \( \xi \) where \( \theta_i = \theta_i^* = \theta(1 + \theta/(i - 2)) \), \( i \geq 4 \), or for \( \tilde{\eta} \) and \( \tilde{\xi} \) where \( \theta_i = \theta, i \geq 3 \). It is worth mentioning that in addition to \( \xi \) and \( \tilde{\xi} \), another specific example of \( Y^n \theta \), defined in this paper, has been studied by some authors. More specifically, [11], [8] and [9] consider different versions of a sequence of independent Bernoulli random variables \( \tilde{Y}_1, \tilde{Y}_2, \cdots \) for which \( \mathbb{P}(\tilde{Y}_i = 1) = a/(b + i - 1) \), for \( 0 < a \leq b \) and \( i \in \mathbb{N} \). For \( i \geq 2 \), this is in fact equivalent to take \( \theta_i = a(i - 1)/(i - 1 + b - a) \) in our definition of \( Y^n \theta \). Of course \( \mathbb{P}(Y^n \theta = 1) \neq \mathbb{P}(\tilde{Y}_1 = 1) \) for \( a < b \), but this is not a significant difference. [10] considers an extended version in which \( \mathbb{P}(\tilde{Y}_i = 1) = a/(b + (i - 1)c) \), for \( c > 0 \), that can be translated to \( \theta_i = a(i - 1)/(b - a + (i - 1)c) \), for \( i \geq 2 \). As already mentioned, although these are specific cases of \( Y^n \theta \), we focus on a different set of problems in this paper.

### 2.2 When signs should be tracked

In this section we briefly discuss the connections between the signed and unsigned models for more complicated problems in which signs are involved. We see how certain quantities of interest for the signed process can be easily obtained from the corresponding quantities for the unsigned model.

To establish this, let \( \mathcal{X} = (\mathcal{X}_1, \cdots, \mathcal{X}_n) \) be a general sequence of \( n \), possibly correlated, \( \{\pm 0, 1\} \)-valued random variables, and for any \( k = 1, \cdots, n \), let \( \mathcal{C}_k \) count the number of \( k \)-circles (\( k \)-spacings) in \( \mathcal{X} \). Also, for \( i = 1, \cdots, k \), let \( \mathcal{C}_{ki}(n) \) be the number of \( k \)-circles with exactly \( i \) children looking in, i.e. the number of \( k \) spacings with exactly \( i - 1, +0 \) and \( k - i, -0 \) between the 1s.

Suppose the random vectors \( (\mathcal{C}_{1i}(n))_{i=1}^{1}, (\mathcal{C}_{2i}(n))_{i=1}^{1}, \cdots, (\mathcal{C}_{ni}(n))_{i=1}^{n} \) are conditionally independent given \( \mathcal{C}_1, \cdots, \mathcal{C}_n \), and suppose for each \( k \), \( \mathcal{C}_{k1}(n), \cdots, \mathcal{C}_{kk}(n) \) conditioned on \( \mathcal{C}_k \), is multinomially distributed with weights \( \omega_{ki} \), where \( \sum_{i=1}^{k} \omega_{ki} = 1 \), that is the probability that there are exactly \( i \) children looking in, in any given \( k \)-circle is \( \omega_{ki} \). For instance, assuming the first child in each circle always looks in while any other child looks in or out with probability \( \kappa \) and \( 1 - \kappa \) respectively, we get

\[
\omega_{ki} = \binom{k - 1}{i - 1} \kappa^{i - 1} (1 - \kappa)^{k - i}.
\]

Under the above assumptions, we can write

\[
\mathbb{P}(\mathcal{C}_{ki} = c_{ki}, k = 1, \cdots, n, i = 1, \cdots, k | \mathcal{C}_1 = c_1, \cdots, \mathcal{C}_n = c_n) = \mathbb{1}\{\sum_{k=1}^{n} k c_k = n\} \prod_{k=1}^{n} \binom{k c_k}{c_{k1}, \cdots, c_{k k}} \prod_{i=1}^{k} (\omega_{ki})^{c_{ki}},
\]
where $c_{ki} \in \mathbb{Z}_+$ and $c_k = \sum_{i=1}^{k} c_{ki}$. Note that there is at least one positive number in each circle, as the first child in each circle always looks in. From the definition, $e_k = \sum_{i=1}^{k} e_{ki}$. Then

$$
P(e_{ki} = c_{ki}, k = 1, \ldots, n, i = 1, \ldots, k) = P(e_1 = c_1, \ldots, e_n = c_n) \times \prod_{k=1}^{n} k c_k = n \prod_{k=1}^{n} (c_{k1}, \ldots, c_{kk}) \prod_{i=1}^{k} (\omega_{ki})^{c_{ki}}.
$$

In the special case that $\mathcal{X}$ is either $(Y^n_1, \ldots, Y^n_n)$ or $X^{n-p}$, the joint distribution of $e_k$ is given by (29) or (30), respectively. We can also find the distribution of $e_{ki}$ once we know the distribution of $e_k$. In fact, we can write

$$
P(e_{ki} = \ell) = \mathbb{1}\{k \leq n\} \sum_{m=\ell}^{n} \binom{m}{\ell} \omega_{ki}^\ell (1 - \omega_{ki})^{m-\ell} P(e_k = m).$$

As another example, let $e^*_j(n)$ be the number of circles in which exactly $j$ children look in. Since $e^*_j(n) = \sum_{k=j}^{n} e_{kj}(n)$, we have

$$
E[e^*_j(n)] = \sum_{k=j}^{n} \sum_{r=0}^{\lfloor \frac{n}{r} \rfloor} P(e_k = r) E(e_{kj} \mid e_k = r)
$$

$$
= \sum_{k=j}^{n} \omega_{kj} \sum_{r=0}^{\lfloor \frac{n}{r} \rfloor} r P(e_k = r) = \sum_{k=j}^{n} \omega_{kj} E(e_k)
$$

and $E(e_k)$ is given in (12) for $X^{n-p}$ process and in (13) for $\eta$.

In addition, as $e_{ki}$ and $e_{kj}$ are conditionally independent, given $e_k$ and $e_k'$, we have $E(Cov(e_{ki}, e_{kj}' \mid e_1, \ldots, e_n)) = -\omega_{ki} \omega_{kj} E(e_k) \mathbb{1}\{k = k'\}$. Hence

$$
Cov(e^*_i, e^*_j) = \sum_{k_i, k_j' \geq j} \{E(Cov(e_{ki}, e_{kj}' \mid e_k, e_k'))
$$

$$
+ Cov(E(e_{ki} \mid e_k), E(e_{kj}' \mid e_k'))\}
$$

$$
= \sum_{k_i, k_j' \geq j} \omega_{ki} \omega_{kj'} Cov(e_k, e_k') - \sum_{k_i, k_j' \geq j} \omega_{ki} \omega_{kj} E(e_k).
$$

Furthermore, let $\mathcal{A}_i$ and $\mathcal{A}^*_i$ be the size of the $i$-th circle and the number of children looking in, in the $i$-th circle. Denote by $\mathcal{N}_n$ the total number of circles. Then for $\sum_{i=1}^{k} a_i < n$,

$$
P(\mathcal{A}^*_1 = a_1, \ldots, \mathcal{A}^*_k = a_k, \mathcal{N}_n > k)
$$

$$
= \sum_{a_1 \leq r_1, \ldots, a_k \leq r_k, \sum a_i < n} P(\mathcal{A}_1 = r_1, \ldots, \mathcal{A}_k = r_k, \mathcal{N}_n > k) \prod_{l=1}^{k} P(\mathcal{A}^*_l = a_l \mid \mathcal{A}_l = r_l)
$$

$$
= \sum_{a_1 \leq r_1, \ldots, a_k \leq r_k, \sum a_i < n} P(\mathcal{A}_1 = r_1, \ldots, \mathcal{A}_k = r_k, \mathcal{N}_n > k) \prod_{l=1}^{k} \omega_{r_la_l}.
where $P(\mathcal{A}_1 = r_1, \ldots, \mathcal{A}_k = r_k, \mathcal{X}_n > k)$ is given in (46) for the $Y^\theta$ process.

Now, let $\Lambda_n$ be the total number of children looking in. In the case that the probability of a child looking in is $\kappa$, we have

$$P(\Lambda_n = r) = \sum_{k=1}^{n-r} \binom{n-k}{r-k} (1-\kappa)^{n-r-k} \kappa^r, n \geq k,$$

and $P(\mathcal{X}_n = k)$ is given in (26) and (27) for $Y^\theta$ and $X^{n-P}$ processes, respectively. Also,

$$E[\Lambda_n] = E[\mathcal{X}_n] + \sum_{i=1}^{n} P(\mathcal{X}_i = +0) = E[\mathcal{X}_n] + n\kappa - \kappa \sum_{i=1}^{n} P(\mathcal{X}_i = 1)$$

Therefore, knowing $E[\mathcal{X}_n]$, we can obtain $E[\Lambda_n]$. For $X^{n-P}$, $E[\mathcal{X}_n]$ is given in (4), and for $\eta$ in (10).

2.3 Outline

The remainder of the paper is organized as follows. Section 3 explores some basic properties of the Markov chain $X^{n-P}$ directly deduced from the transition probabilities. This includes the average number of cycles, the average number of $j$-cycles and their asymptotics. Section 4 provides necessary and sufficient conditions for the conditional law (1). The probability generating function of the total number of cycles and the joint distribution of the cycle counts are obtained using the conditional relation. The last part of Section 4 is devoted to finding the weak limit of $X^{n-P}$ and its conditional relation with $Y^\theta$, under certain conditions. Finally, Section 5 uses a specific coupling between $X^{n-P}$ and $Y^\theta$, for $p_i = (i-1)/(\theta_i + i - 1)$, to derive a central limit theorem for the total number of cycles, and also to deduce the asymptotic behavior of the joint distribution of the normalized cycle lengths of the $X^{n-P}$ process, in order of their formation.

3 Random derangements via Markov chains

In the following sections we study some properties of the Markov chain $X^{n-P} = (1, X_n, \ldots, X_2, 1)$ derived directly from its transition matrices. We also explore the particular case of the playground game process $\eta$ via its corresponding chain. As in the Feller coupling process, we denote by $C_j(n)$ the number of spacings of length $j$ between the 1s in $X^{n-P}$, that is equivalent to the number of cycles of size $j$. Also, we denote by $K_n$ the total number of cycles of $X^{n-P}$. We first compute the marginal distribution of $X_i$, and then use it to study the expected value of $K_n$ and its asymptotics.

3.1 Marginal distributions and transition probabilities

For $j \geq i \geq 1$ and $u, v \in \{0, 1\}$, denote $Q_{j,i}(u, v) = P(X_i^{j-n-P} = v)$. From the Markov property, for any $n \geq j \geq i \geq 1$, and any $u, v \in \{0, 1\}$ we have

$$P(X_i^{n-P} = v \mid X_j^{n-P} = u) = Q_{j,i}(u, v),$$

and as $P(X_{n+1}^{n-P} = 0) = P(X_{n}^{n-P} = 1) = 0$, we assume by convention that (2) holds for any $n+1 \geq j \geq i \geq 1$. On the other hand, it is clear from the definition of $X^{n-P}$ that $Q_{ji}(0, 1) = Q_{j+1,i}(1, 1),$
for \( j \geq i \). Hence, to determine the values of \( Q_{ji}(u, v) \) for any \( j > i \) and any \( u, v \in \{0, 1\} \), we need to find the values of \( Q_{j+1,i}(1, 1) = \mathbb{P}(X_j = 1) \), for any \( j \geq i \). The following lemma may be proved by induction on \( i \) and using the fact that

\[
\mathbb{P}(X_i^n = 1) = \sum_{u \in \{0, 1\}} \mathbb{P}(X_i^n = u) Q_{i+1,i}(u, 1) = q_i(1 - \mathbb{P}(X_i^n = 1)).
\]

**Lemma 1.** For \( 3 \leq i < n \),

\[
\mathbb{P}(X_i^n = 1) = \sum_{j=0}^{n-i-1} (-1)^j \prod_{l=i}^{j+i} q_l.
\]

Summarizing the above discussion, for \( j > i > 2 \) and \( u, v \in \{0, 1\} \), we get

\[
Q_{ji}(u, v) = \mathbb{P}(X_j = u, p_i = v)
\]

\[
= v \left( \sum_{r=0}^{j-u-1-i} (-1)^r \prod_{s=i}^{i+r} q_s \right) + (1 - v) \left( 1 - \sum_{r=0}^{j-u-1-i} (-1)^r \prod_{s=i}^{i+r} q_s \right)
\]

\[
= (1 - v) + (2v - 1) \sum_{r=0}^{j-u-1-i} (-1)^r \prod_{s=i}^{i+r} q_s,
\]

where, by convention, we assume \( \sum_{r=0}^{j-u-1-i} a_r = 0 \). In particular, for \( i_0 = n + 1 > i_1 > \cdots > i_k > 2 \), we have

\[
\mathbb{E}_{n,p}[X_{i_1}\ldots X_{i_k}] = \prod_{l=0}^{k-1} \sum_{r=0}^{i_{l+1}-i_l-2} (-1)^r \prod_{s=i_l+1}^{i_l+i_{l+1}} q_s.
\]

For \( a, b, z \in \mathbb{C} \), denote by

\[
M(a, b, z) = \sum_{j=0}^{\infty} \frac{a(j) z^j}{b(j) j!},
\]

the confluent hypergeometric function. For \( Re b > Re a > 0 \), the integral representation of \( M \) is given by

\[
M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a) \Gamma(b - a)} \int_0^1 e^{zu} u^{a-1} (1 - u)^{b-a-1} du.
\]

(3)

We have the following lemma.

**Lemma 2.** For any \( 3 \leq i < n \),

\[
\mathbb{P}_\theta(\eta_i = 1) = \sum_{j=1}^{n-i} (-1)^{j+1} \frac{\theta^j}{(\theta + i - 1)_{(j)}}.
\]

Furthermore, for any \( i \geq 3 \)

\[
\lim_{n \to \infty} \mathbb{P}_\theta(\eta_i = 1) = \frac{\theta}{\theta + i - 1} M(1, \theta + i, -\theta) = \theta \int_0^1 e^{-\theta u} (1 - u)^{\theta + i - 2} du.
\]
Proof. The first part is clear from Lemma 1 by letting $q_l = \theta / (\theta + l - 1)$. For the second part, write

$$\lim_{n \to \infty} P(\eta_i = 1) = -\sum_{j=1}^{\infty} \frac{(-\theta)^j}{(\theta + i - 1)(j)} = 1 - \sum_{j=0}^{\infty} \frac{1_{(j)(-\theta)^j}}{(\theta + i - 1)(j)} = 1 - M(1, \theta + i - 1, -\theta).$$

Using (3) and integrating by parts, the second part of the lemma follows. \qed

For $i_0 = n + 1 > i_1 > \cdots > i_k > 2$,

$$E_\theta[\eta_{i_1} \cdots \eta_{i_k}] = \prod_{r=1}^{k} \sum_{l=0}^{i_r - 1 - i_r - 2} (-1)^l \frac{\theta^{l+1}}{(\theta + i_r - 1)(l+1)}.$$

As a result, for $2 < i < j < n - 1$,

$$\text{Cov}(\eta_j, \eta_i) = (-1)^{i+j+1} \Gamma(\theta + i - 1) \Gamma(\theta + j - 1) \left( \frac{\theta^l}{\Gamma(\theta+l)} \right) \left( \sum_{l=j-1}^{n-1} (-1)^l \frac{\theta^l}{\Gamma(\theta+l)} \right).$$

### 3.2 Number of cycles

Let $H_n := \sum_{i=1}^{n} 1/i$ and let $\gamma := \lim_{n \to \infty} H_n - \log n \simeq 0.5772$ denote the Euler constant. As in the Feller Coupling case, the following lemma shows the linear relationship of the expected value of $K_n$ and $\log n$, under specific conditions.

**Lemma 3.** For $n > 3$,

$$E_p(K_n) = \sum_{i=1}^{n-1} \sum_{j=1}^{n-i-1} (-1)^j \prod_{l=i}^{j+i} q_l. \quad (4)$$

If, in addition, there exists a constant $\alpha \geq 0$ such that $nq_n \to \alpha$, as $n \to \infty$, and $|\psi_\alpha(p)| < \infty$ for

$$\psi_\alpha(p) = \psi(p) := \sum_{i=1}^{\infty} \left( q_i - \frac{\alpha}{i} \right), \quad (5)$$

then

$$\lim_{n \to \infty} E_p(K_n) - \alpha \log n = \alpha \gamma + \psi(p) + \sum_{j=1}^{\infty} (-1)^j \bar{a}_j, \quad (6)$$

where $\bar{a}_j = \sum_{i=1}^{j+i} \prod_{l=1}^{i} q_l < \infty$, for $j \in \mathbb{N}$. 


Proof. The first part is straightforward from Lemma 1. Now note that for any fixed \( j \geq 1 \),
\[
\lim_{i \to \infty} i_{(j+1)} \prod_{l=i}^{j+i} q_l = \alpha^j \geq 0.
\]
Therefore, from the limit comparison test, as \( \sum_{i=1}^{\infty} 1/i_{(j+1)} \) converges so does \( \bar{a}_j \). We have
\[
\sum_{i=1}^{n-2} \sum_{j=1}^{i} (-1)^{j+1} \prod_{l=i}^{j+i} q_l \leq \sum_{i=1}^{n-2} q_i q_{i+1} \leq \sum_{i=1}^{\infty} q_i q_{i+1} < \infty.
\]
Also one can easily see that the l.h.s. of (7) equals \( \sum \) and similarly, the l.h.s. of (7) is greater than or equal to \( j \) decreases as \( n \) increases, since \( q_i < 1 \) for \( i \geq 3 \). Hence, by interchanging the sums, for any \( m \geq 1 \),
\[
\lim_{n \to \infty} \sum_{i=1}^{n-2} \sum_{j=1}^{i} (-1)^{j+1} \prod_{l=i}^{j+i} q_l = \lim_{n \to \infty} \sum_{j=1}^{n-2} (-1)^j a_j(n) \leq \sum_{j=1}^{2m-1} (-1)^j \bar{a}_j + \bar{a}_{2m},
\]
and similarly, the l.h.s. of (7) is greater than or equal to
\[
\sum_{j=1}^{2m+1} (-1)^j \bar{a}_j \geq \sum_{j=1}^{2m-1} (-1)^j \bar{a}_j - \bar{a}_{2m+1}.
\]
Hence, to see that the l.h.s. of (7) equals \( \sum_{i=1}^{\infty} (-1)^j \bar{a}_j \), it suffices to show \( \bar{a}_{2m} \to 0 \), as \( m \to \infty \).
This is clear from
\[
\sum_{i=1}^{\infty} \prod_{l=i}^{i+j} q_l \leq \sum_{i=1}^{\infty} q_i q_{i+1} < \infty, \quad j = 2, 3, \ldots,
\]
and
\[
\lim_{j \to \infty} \bar{a}_j = \lim_{j \to \infty} \sum_{i=1}^{\infty} \prod_{l=i}^{i+j} q_l = \sum_{i=1}^{\infty} \lim_{j \to \infty} \prod_{l=i}^{i+j} q_l = 0.
\]
Now, write
\[
\lim_{n \to \infty} \mathbb{E}_p(K_n) - \alpha \log n = \lim_{n \to \infty} \left( \sum_{i=1}^{n-1} q_i - \alpha H_n \right) + \alpha \left( \lim_{n \to \infty} H_n - \log n \right)
- \lim_{n \to \infty} \sum_{i=1}^{n-2} \sum_{j=1}^{i} (-1)^{j+1} \prod_{l=i}^{j+i} q_l
= \psi(p) + \alpha \gamma + \sum_{j=1}^{\infty} (-1)^j \bar{a}_j.
\]
\[\square\]

Remark 1. Note that (7) and (8) give upper and lower bounds for the sum on the right of (6).

Denote by \( K^n_\eta \) the number of cycles of the playground game \( \eta^n \). The following result is an application of Lemma 3. Before stating the next theorem, recall that for any \( p, q \in \mathbb{N} \), the generalized hypergeometric function \( pF_q \) is defined by
\[
pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{j=0}^{\infty} \frac{(a_1)_{(j)} \cdots (a_p)_{(j)} z^j}{(b_1)_{(j)} \cdots (b_q)_{(j)} j!}.
\]
Note that $1F_1(a; b; z) = M(a, b, z)$ the confluent hypergeometric function. From the Euler’s integral transform
\[ p+1 F_{q+1}(a_1, \ldots, a_p, a; b_1, \ldots, b_p; b; z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 x^{a-1}(1-x)^{b-a-1} \ pF_q(a_1, \ldots, a_p; b_1, \ldots, b_p; zx) \, dx. \tag{9} \]

**Theorem 1.** For $n > 3$,
\[ \mathbb{E}(K_n^q) = 1 + \sum_{i=3}^{n-1} \frac{\theta}{\theta + i - 1} + \sum_{j=2}^{n-3} \sum_{i=3}^{n-j} \frac{(-1)^j \theta^j (\theta + i - 1)(j)}{(\theta + i - 1)(j)}, \tag{10} \]
and
\[ \lim_{n \to \infty} \mathbb{E}(K_n^q) - \theta \log n = 1 - \theta H_{\theta+1} + \theta \gamma - \theta^2 \int_0^1 \int_0^1 e^{-\theta xy}(1-x)^{\theta+1} \, dx \, dy, \tag{11} \]
where
\[ H_y = \int_0^1 \frac{1-x^y}{1-x} \, dx. \]

*Proof.* Letting $q_i = \theta/(\theta + i - 1)$, for $i \geq 3$, we have $\lim_{n \to \infty} nq_i = \theta$, and
\[ \psi_\theta(p) = \lim_{n \to \infty} \left( -\theta H_{n-1} + \sum_{i=3}^{n-1} \frac{\theta}{\theta + i - 1} \right) = 1 + \theta \lim_{n \to \infty} (-H_{n-1} + H_{n-2+\theta} - H_{\theta+1}) = 1 - \theta H_{\theta+1}, \]
where the last equality follows from $\lim_{n \to \infty}(H_{n-2+\theta} - H_{n-1}) = 0$. Also,
\[ a_j = \lim_{n \to \infty} a_j(n) = \frac{\theta^{j+1} (\theta + i - 1)(j+1)}{j(\theta + 2)(j)}, \]

Now we have
\[ \sum_{j=1}^{\infty} (-1)^j a_j = \frac{-\theta^2}{\theta + 2} \sum_{j=0}^{\infty} \frac{1}{2(j)(\theta+3)(j)} \frac{(-\theta)^j}{j!} = \frac{-\theta^2}{\theta + 2} \ 2F2(1,1;2,\theta+3;\theta), \]
where from (9), the l.h.s. reduces to
\[ \frac{-\theta^2 \Gamma(\theta + 3)}{(\theta + 2)\Gamma(1)\Gamma(\theta + 2)} \int_0^1 (1-x)^{\theta+1} M(1, 2, -\theta x) \, dx, \]
which, from (3), in turn reduces to the term with the double integral on the right of (11). Applying Lemma 3 completes the proof. \hfill \Box

Note that, as before, the double integral on the right of (11) is bounded by
\[ \sum_{j=2}^{2m+1} \frac{(-1)^j \theta^j}{(j-1)(\theta + 2)(j-1)}, \]
for any $m \in \mathbb{N}$. As an example, for $\theta = 1/2$ and $m = 3$, we obtain $\lim_{n \to \infty} \mathbb{E}(K_n^q) - \frac{1}{2} \log n \approx 0.555069$, with the error $\leq 1.23333 \times 10^{-7}$. 

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3.3 Cycles of size \( j \)

The following lemma gives the expected value of \( C_j(n) \) and its asymptotics.

**Lemma 4.** For \( j > 1 \),

\[
\mathbb{E}_p(C_j(n)) = q_{n-j+1}^{n-1} \prod_{l=n-j+2}^{n-1} p_l + \sum_{i=j+1}^{n-1} \sum_{k=0}^{n-i-1} (-1)^k q_{i-j} \prod_{l=i-j+1}^{n-i-1} p_l \prod_{l=i}^{k+i} q_l. \tag{12}
\]

Furthermore, if there exists a constant \( \alpha \geq 0 \) such that \( \lim_{n \to \infty} n q_n = \alpha \), then for any \( m \geq 1 \)

\[
\lim_{n \to \infty} \mathbb{E}_p(C_j(n)) = 2^{m-1} \sum_{k=0}^{2m-1} (-1)^k \bar{b}_k(j) + \varepsilon(m, p, j),
\]

with \( \varepsilon(m, p, j) \leq \bar{b}_{2m}(j) \), where

\[
\bar{b}_k(j) = \lim_{n \to \infty} b_k(n, j) = \lim_{n \to \infty} \sum_{i=j+1}^{n-k-1} q_{i-j} \prod_{l=i-j+1}^{n-i} p_l \prod_{l=i}^{k+i} q_l.
\]

**Proof.** Write

\[
\mathcal{R}_{j,i}(n) := \mathbb{P}_p \left( X_i^n X_{i-j}^{j-1} \prod_{l=1}^{j-1} (1 - X_{i-l}^n) = 1 \right) = \mathbb{P}(X_i^n = 1) q_{i-j} \prod_{l=i-j+1}^{n-i-1} p_l = \sum_{k=0}^{n-i-1} (-1)^k q_{i-j} \prod_{l=i-j+1}^{n-i-1} p_l \prod_{l=i}^{k+i} q_l,
\]

for \( j + 1 \leq i \leq n - 1 \), and

\[
\mathcal{R}_{j,n+1}(n) = q_{n-j+1}^{n-1} \prod_{l=n-j+2}^{n-1} p_l.
\]

The first part of the theorem follows from

\[
\mathbb{E}_p(C_j(n)) = \sum_{i=j+1}^{n+1} \mathcal{R}_{j,i}(n).
\]

For the second part, note that for any \( j \geq 2 \), \( \mathcal{R}_{j,n+1}(n) \to 0 \) as \( n \to \infty \). As \( q_1 = 1 \)

\[
\sum_{i=j+1}^{n} \sum_{k=0}^{n-i-1} (-1)^k q_{i-j} \prod_{l=i-j+1}^{n-i} p_l \prod_{l=i}^{k+i} q_l \leq \sum_{i=j+1}^{n} q_i q_{i-j} \prod_{l=i-j+1}^{n-i} p_l \prod_{l=i-j+1}^{n-i-1} q_l \leq \sum_{i=j+1}^{\infty} q_i q_{i-j} < \infty,
\]

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where the l.h.s. is positive. In fact, for any fixed \( j > 1 \), there exists \( N_j \) large enough that the l.h.s. of the last inequalities is an increasing function of \( n \), for \( n \geq N_j \). To see this, temporarily denote by \( h_j(n) \) the l.h.s. of the last inequality. Then

\[
h_j(n + 1) - h_j(n) = \sum_{i=j+1}^{n} (-1)^{n-i} q_{i-j} \prod_{l=i}^{n} q_l \prod_{l=i-j+1}^{i-2} p_l = q_n q_{n-j} \prod_{l=n-j+1}^{n-2} p_l + o(n^{-2}),
\]

for large enough \( n \). Thus, \( \lim_{n \to \infty} h_j(n) \) and therefore \( \lim_{n \to \infty} \mathbb{E}_p(C_j(n)) \) exist and are finite. On the other hand, as in the proof of Lemma 3, from the limit comparison test \( \bar{b}_k(j) = \lim_{n \to \infty} b_k(n, j) \) exists and is finite. Now, as \( b_k(n, j) \), for fixed \( n \), is a decreasing sequence as \( k \) increases, for any \( m \geq 1, \)

\[
\lim_{n \to \infty} \mathbb{E}_p(C_j(n)) = \lim_{n \to \infty} h_j(n) = \lim_{n \to \infty} \sum_{k=0}^{n-j-2} (-1)^k b_k(n, j)
\]

\[
= \sum_{k=0}^{2m-1} (-1)^k \bar{b}_k(j) + \varepsilon(m, p, j),
\]

where \( \varepsilon(m, p, j) \leq \bar{b}_{2m}(j) \). \( \square \)

Let \( C_j^\eta(n) \) be the number of the cycles of size \( j \) in the playground game \( \eta^\theta \). Applying Lemma 4 to the \( \eta \) process, we obtain the following theorem. By convention, we let \( \theta_{(-1)} = \theta_{(0)} = 1 \) and \( \sum_{i=m}^{n} a_i = 0 \), for \( m > n \).

**Theorem 2.** For \( 2 \leq j \leq n - 2 \),

\[
\mathbb{E}_\theta(C_j^\eta(n)) = \frac{\theta(n-j+1)(j-2)}{(\theta + n - j)(j-1)} + \sum_{k=1}^{n-j-1} \frac{(-1)^{k+1} \theta^k(j-2)!}{(\theta+2)(j-3)(\theta+j)(k)}
\]

\[
+ \sum_{i=j+3}^{n-1} \sum_{k=1}^{n-i} \frac{(-\theta)^{k+1}(i-j)(j-2)}{(\theta+i-j-1)(j-1)(\theta+i-1)(k)},
\]

while \( \mathbb{E}_\theta(C_n^\eta(n)) = (n-2)!/(\theta + 2)(n-3) \), and \( \mathbb{E}_\theta(C_{n-1}^\eta(n)) = 0 \). For \( j \geq 2 \), we have

\[
\lim_{n \to \infty} \mathbb{E}(C_j^\eta(n)) = \int_0^1 \int_0^1 \theta^2 e^{-\theta x} \theta^{1-1} (1-x)^{-2} (1-y)^{\theta+j-1} dxdy
\]

\[
+ 1 \{j \geq 3\} \frac{\theta^3(j-2)!}{\theta^{(j+1)}} \left( \theta + j - 1 - ((\theta + j - 1)^2 + j - 1) \int_0^1 e^{-\theta x} (1-x)^{\theta+j} dx \right)
\]

\[
- 1 \{j = 2\} \frac{\theta^2}{\theta + 1} \int_0^1 e^{-\theta x} (1-x)^{\theta+2} dx.
\]

**Proof.** Having \( q_i = \theta/(\theta + i - 1) \), for \( i \geq 3 \), (12) gives the first part of the theorem. Also, as \( nq_n \to \theta \), as \( n \to \infty \), from Lemma 4, \( \lim_{n \to \infty} \mathbb{E}_\eta(C_j^\eta(n)) \) exists and is finite. To find this limit, note
Thus, we get
\[
0 \leq \lim_{n \to \infty} \sum_{i=n}^{\infty} \frac{\mathbb{P}_\theta(\eta_i^n = 1)\theta(i - j)(j - 2)}{(\theta + i - j - 1)(j-1)} \leq \lim_{n \to \infty} \sum_{i=n}^{\infty} \mathbb{P}(\eta_i^n = 1)q_{i-j} \\
\leq \lim_{n \to \infty} \sum_{i=n}^{\infty} q_i q_{i-j} = 0.
\]

Thus, we get
\[
\lim_{n \to \infty} \mathbb{E}(C_j^n(n)) = \sum_{i=j+1}^{\infty} \frac{\theta(i - j)(j - 2)}{(\theta + i - j - 1)(j-1)} \lim_{n \to \infty} \mathbb{P}_\theta(\eta_i^n = 1) \\
- \frac{\theta(j - 1)!}{(\theta + 1)(j-1)} \lim_{n \to \infty} \mathbb{P}_\theta(\eta_{j+2}^n = 1) - \frac{\theta(j - 2)!}{\theta(j-1)} \lim_{n \to \infty} \mathbb{P}_\theta(\eta_{j+1}^n = 1) \\
+ \frac{(\theta + 1)\theta(j - 2)!}{\theta(j-1)} \lim_{n \to \infty} \mathbb{P}_\theta(\eta_{j+1}^n = 1),
\]
where the second term on the right of the last equation is subtracted from the sum as \(\mathbb{P}_\theta(\eta_{j+2}^n = \eta_2^n = 1, \eta_{j+1}^n = \cdots = \eta_3^n = 0) = 0\) while the two last terms appear because of the discrepancy between the first term of the sum (for \(i = j + 1\)) and the value of \(\mathbb{P}_\theta(\eta_{j+1}^n = \eta_1^n = 1, \eta_{j+1}^n = \cdots = \eta_2^n = 0) = (\theta + 1)\theta(j - 2)!\mathbb{P}(\eta_{j+1}^n = 1)/\theta(j-1)\). To calculate (15), we first note that
\[
\sum_{r=0}^{\infty} \frac{(r + 1)(j - 2)}{(\theta + r)(j-1)} (1 - y)^r = \frac{(j - 2)!}{\theta(j-1)} \sum_{r=0}^{\infty} \frac{(j - 1)(r)\theta(r)}{(\theta + j - 1)(r)r!}(1 - y)^r \\
= \frac{\Gamma(j - 1)\Gamma(\theta)}{\Gamma(\theta + j - 1)} _2 F_1(j - 1, \theta; \theta + j - 1; 1 - y) \\
= \int_0^1 x^{\theta-1}(1 - x)^{j-2}(1 - x(1 - y))^{-(j-1)}dx,
\]
where the hypergeometric function \(_2 F_1(a, b; c; z) = \sum_{r=0}^{\infty} a_r b_r z^r / (c_r r!)_r\), and the last equality is given by the Euler type integral representation for \(_2 F_1\), for \(\text{Re}(c) > \text{Re}(b) > 0\). Having this, from Lemma 2, after interchanging the sum and integral, the first term on the right of (15) reduces to
\[
\int_0^1 \theta^2 e^{-\theta y} (1 - y)^{\theta + j - 1} \sum_{r=0}^{\infty} \frac{(r + 1)(j - 2)}{(\theta + r)(j-1)} (1 - y)^r dy \\
= \int_0^1 \int_0^1 \frac{\theta^2 e^{-\theta y} x^{\theta-1}(1 - x)^{j-2}(1 - y)^{\theta + j - 1}}{(1 - x + xy)^{j-1}}dxdy.
\]
Again from Lemma 2, the second, third and the fourth terms on the right of (15) equal
\[
\frac{\theta^3(j - 2)!}{\theta(j)} \left( -(j - 1) \int_0^1 e^{-\theta u} (1 - u)^{\theta + j} du + (\theta + j - 1) \int_0^1 e^{-\theta u} (1 - u)^{\theta + j - 1} du \right)
\]
\[
= \frac{\theta^3(j - 2)!}{\theta(j)} \left\{ -(j - 1) \int_0^1 e^{-\theta u} (1 - u)^{\theta + j} du 
- \left( \frac{\theta + j - 1}{\theta + j} \right) \left( -1 + \theta \int_0^1 e^{-\theta u} (1 - u)^{\theta + j} du \right) \right\},
\]
which completes the proof of this part for \( j \geq 3 \), after simplification. Similarly, for \( j = 2 \), we can write
\[
\lim_{n \to \infty} \mathbb{E}(C_{j}^n(n)) = \sum_{i=j+1}^{\infty} \frac{\theta(i-j)(j-2)}{(\theta + i - j - 1)(j-1)} \lim_{n \to \infty} \mathbb{P}_\theta(\eta_i^n = 1)
- \frac{\theta}{\theta + 1} \lim_{n \to \infty} \mathbb{P}_\theta(\eta_4^n = 1)
\]
where the first term on the r.h.s. equals the first term of (15) and the last term on the r.h.s. equals
\[
- \frac{\theta^2}{\theta + 1} \int_0^1 e^{-\theta x} (1 - x)^{\theta + 2} dx.
\]

Although very useful for numerical evaluations, the double integral in (14) does not provide a simple expression for \( \lim_{n \to \infty} \mathbb{E}_\theta C_{j}^n(n) \). The following result gives an approximation for the limit.

**Proposition 1.** For any \( j \geq 2 \) and \( m \geq 1 \), we have
\[
\lim_{n \to \infty} \mathbb{E}(C_{j}^n(n)) = 1(j \geq 3) \frac{\theta(j - 2)!}{(\theta + 2)(j-3)} \int_0^1 e^{-\theta x} (1 - x)^{\theta + j - 1} dx 
+ 1(j = 2) \theta \int_0^1 e^{-\theta x} (1 - x)^{\theta + 1} dx + \sum_{k=1}^{2m} (-1)^{k+1} \tilde{b}_k(\theta, j) + \varepsilon(m, \theta, j), \quad (16)
\]
with \( \varepsilon(m, \theta, j) \leq \tilde{b}_{2m+1}(\theta, j) \), where
\[
\tilde{b}_k(\theta, j) = \frac{\theta^k(k - 1)! (k(j - 1) + \theta(k + j - 1))}{(\theta + 1)(k)(j-1)(k+1)} 
- \frac{\theta^k(j - 2)! ((k - 1 + (\theta + 1)j)(\theta + j) - k)}{(\theta + 1)(k+j)}.
\]

**Proof.** As \( n \to \infty \), the first term on the right of (13) converges to 0 and the second term converges to the first and the second terms on the right of (16), for \( j \geq 3 \) and \( j = 2 \), respectively. Let
\[
b_k(n, \theta, j) = \sum_{i=j+3}^{n-k} \frac{\theta^{k+1}(i-j)(j-2)}{(\theta + i - j - 1)(j-1)(\theta + i - 1)(k)}.
\]
and note that \( \lim_{n \to \infty} b_k(n, \theta, j) = \bar{b}_k(\theta, j) \). From Lemma 4, the limit of the double sum on the r.h.s. of (13) can be written

\[
\lim_{n \to \infty} \sum_{k=1}^{n-j-3} (-1)^{k+1} b_k(n, \theta, j) = \lim_{n \to \infty} \sum_{k=1}^{\infty} (-1)^{k+1} b_k(n, \theta, j)
\]

\[
= \sum_{k=1}^{2m} (-1)^{k+1} \bar{b}_k(\theta, j) + \varepsilon(m, \theta, j),
\]

where

\[
0 \leq \varepsilon(m, \theta, j) = \lim_{n \to \infty} \sum_{k=2m+1}^{\infty} (-1)^k b_k(n, \theta, j) \leq \lim_{n \to \infty} b_{2m+1}(n, \theta, j).
\]

As an example, Table 1 provides the approximation for the \( \mathbb{E}(C_j^\eta(n)) \) as \( n \) tends to infinity, for \( 2 \leq j \leq 7 \) and \( m = 2 \). The numerical results perfectly match with the exact values derived from (14). The values are compared to their counterpart for the classical Feller coupling \( \tilde{\xi} \), \( \lim_{n \to \infty} \mathbb{E}(C_j^{\tilde{\xi}}(n)) = \theta/j \).

Table 1: The approximation of the \( \mathbb{E}C_j^\eta(n) \) as \( n \) tends to infinity for \( \theta = 0.5 \), \( 2 \leq j \leq 7 \), by applying Proposition 1 for \( m = 2 \).

| \( j \) | \( \lim_{n \to \infty} \mathbb{E}C_j^\eta(n) \) approx. | Error | \( \theta/j \) |
|---|---|---|---|
| 2 | 0.255318 | 9.86668 \times 10^{-7} | 0.250 |
| 3 | 0.19468 | 4.38404 \times 10^{-7} | 0.167 |
| 4 | 0.137891 | 2.20947 \times 10^{-7} | 0.125 |
| 5 | 0.107192 | 1.21856 \times 10^{-7} | 0.100 |
| 6 | 0.0878281 | 7.19514 \times 10^{-8} | 0.083 |
| 7 | 0.0744583 | 4.48278 \times 10^{-8} | 0.072 |

Although the details are omitted, one can also easily obtain the variance

\[
\text{Var}(C_j(n)) = \sum_{i=j+1}^{n+1} \mathcal{R}_{j,i}^{(n)} (1 - \mathcal{R}_{j,i}^{(n)}) + 2 \sum_{l \leq i-j} \mathcal{R}_{j,i}^{(n)} \mathcal{R}_{j,l}^{(n)} (i-l-1) - 2 \sum_{l<i} \mathcal{R}_{j,i}^{(n)} \mathcal{R}_{j,l}^{(n)}.
\]

Some numerical examples of the variance for the \( \eta \) process are displayed in Table 2.

### 4 Conditioning on generalized Feller coupling

In this section we establish necessary and sufficient conditions for a conditional relation between the derangement Markov chains \( X^\eta \) and \( (Y_n, \cdots, Y_1) \), for \( n \in \mathbb{N} \). We explore some properties of
Table 2: The variance of $C_j^\eta(n)$ for $\theta = 0.5$, $3 \leq j \leq 7$.

| $j$ | $n = 20$ | $n = 50$ | $n = 100$ |
|-----|-----------|-----------|-----------|
| 3   | 0.185732  | 0.177823  | 0.175253  |
| 4   | 0.142278  | 0.133938  | 0.131308  |
| 5   | 0.116493  | 0.107688  | 0.104996  |
| 6   | 0.0996403 | 0.090335  | 0.087578  |
| 7   | 0.087877  | 0.078045  | 0.075221  |

the Generalized Feller Coupling (GFC) $Y^\theta = (Y^\theta)_i^{\infty}$, along with some of its applications for $X^{n,p}$ via the conditional relation. We also obtain the weak limit of $X^{n,p}$, as $n \to \infty$, as a $\{0,1\}$-valued infinite Markov chain, and provide a conditional relation between the limit and the infinite sequence $Y^\theta$. We also apply the theory developed in this section to the specific example of the playground process $\eta$.

4.1 A conditional relation

[5] constructs a Markov chain $\tilde{\eta}_n, \cdots, \tilde{\eta}_1$ generating the cycle counts of the random derangement sampled from ESF$_n(\theta)$ conditioned on not having fixed points. In other words,

$\mathcal{L}(C_2^\eta(n), \ldots, C_n^\eta(n)) = \mathcal{L}(C_2^\xi(n), \ldots, C_n^\xi(n) \mid C_1^\xi(n) = 0)$,

where as before, $\tilde{\xi}$ denotes the classical Feller coupling. Studying a Markov chain is not always easy if one directly uses its transition probabilities, while using the conditional relations such as (17) sometimes makes computations easier. Motivated by this, one can ask if there exists an infinite sequence of independent $\{0,1\}$-valued random variables $Y_1^\theta = 1, Y_2^\theta, \cdots$ for which the law of $(Y_i)_i^{\infty}$ conditional on having no 11 patterns in $Y_n, Y_{n-1}, \cdots, Y_1 = 1$, coincides with that of the Markov chain $X_n^\eta, X_{n-1}^\eta, \cdots, X_1^\eta$. More specifically, we define a generalized Feller coupling (GFC) as a sequence of independent Bernoulli $\{0,1\}$-valued random variables $Y^\theta = Y := (Y_i)_i^{\infty}$ such that

$\mathbb{P}_\theta(Y_i = 0) = \frac{i - 1}{i - 1 + \theta_i}$,

where $\theta = (\theta_i)_{i \in \mathbb{N}}$ is a sequence of strictly positive real numbers. In this paper, we always assume that $\theta_1 = 1$. For $j \in \mathbb{N}$, let

$\Delta_j := \{(a_j, \ldots, a_1) \in \{0,1\}^j : a_1 = 1, a_j + \sum_{i=1}^{j-1} a_ia_{i+1} = 0\}$,

and let $Y^{n,\theta} = Y^n := (Y_n, \cdots, Y_1)$. Theorem 3 gives the necessary and sufficient conditions under which

$\mathbb{P}_p(X_n^\eta = a_n, \ldots, X_1^\eta = a_1) = \mathbb{P}_\theta(Y_n = a_n, \ldots, Y_1 = a_1 | Y^n \in \Delta_n)$,

for $(a_n, ..., a_1) \in \Delta_n$. In other words, we investigate the necessary and sufficient conditions under which the cycle counts $(C_2(n), \ldots, C_n(n))$ of the permutation generated by $X^{n,p}$ have a distribution
determined by
\[ \mathcal{L}(C_2(n), \ldots, C_n(n)) = \mathcal{L}(\tilde{C}_2(n), \ldots, \tilde{C}_n(n) \mid \tilde{C}_1(n) = 0), \] (20)
where \((\tilde{C}_2(n), \ldots, \tilde{C}_n(n))\) is the cycle counts of \(Y_n, \ldots, Y_1 = 1\). Furthermore, having a Markov chain \(X^nP\), we can find a sequence \(\theta\) such that (19) holds, and vice versa, having a sequence of independent random variables \(Y^n\), we can find a sequence \(p\) such that (19) holds. To make this precise, let \(\gamma_i(\theta) = \mathbb{P}_\theta((Y_i, \ldots, Y_1) \in \Delta_i)\) and note that
\[
\gamma_i(\theta) = \frac{i - 1}{i - 1 + \theta_i} \left( \gamma_{i-1}(\theta) + \frac{\theta_{i-1}}{i - 2 + \theta_{i-1}} \gamma_{i-2}(\theta) \right),
\] (21)
for \(i \geq 3\), with initial conditions \(\gamma_1 = 0\) and \(\gamma_2 = \mathbb{P}(Y_2 = 0) = 1/(1 + \theta_2)\). Let
\[ \theta_{<n>} := \theta_1(\theta_2 + 1)(\theta_3 + 2) \cdots (\theta_n + n - 1), \]
and recall we assume \(\theta_1 = 1\) in this paper. The next proposition gives an exact formula for \(\gamma(\theta)\).

**Proposition 2.** For \(i \geq 2\)
\[
\gamma_i(\theta) = G_{i-1}(\theta) \frac{(i - 1)!}{\theta_{<i>>}},
\]
where \(G_0(\theta) = 0\) and \(G_1(\theta) = G_2(\theta) = 1\),
\[
G_{i}(\theta) := 1 + \sum_{k=1}^{\lfloor \frac{i-1}{2} \rfloor} \sum_{(i_1, \ldots, i_k)} \frac{\theta_{i_1} \cdots \theta_{i_k}}{(i_1 - 1) \cdots (i_k - 1)},
\]
for \(i \geq 3\), and the last sum is over all \((i_1, \ldots, i_k)\) such that \(2 < i_j \leq i\) and \(i_{j+1} > i_j + 1\).

**Proof.** Note that \(G_0(\theta) = \gamma_1(\theta) = 0\) and \(G_1(\theta)/\theta_{<2>>} = 1/(1 + \theta_2) = \gamma_2(\theta)\). For any \(i \geq 3\),
\[
\frac{i - 1}{i - 1 + \theta_i} \left( \frac{(i - 2)!}{\theta_{<i-1>>}} G_{i-2}(\theta) + \frac{\theta_{i-1}}{i - 2 + \theta_{i-1}} \frac{(i - 3)!}{\theta_{<i-2>>}} G_{i-3}(\theta) \right)
\]
\[= \frac{(i - 1)!}{\theta_{<i>>}} \left( G_{i-2}(\theta) + \frac{\theta_{i-1}}{i - 2} G_{i-3}(\theta) \right) = \frac{(i - 1)!}{\theta_{<i>>}} G_{i-1}(\theta), \]
where the last equality follows from the definition of \(G_i(\theta)\). Hence for any \(i\), \(G_{i-1}(\theta)(i - 1)!/\theta_{<i>>}\) satisfies (21), hence the result. \(\square\)

Now we are ready to state the main theorem of this section as follows.

**Theorem 3.** For any \(4 \leq n \in \mathbb{N}\), the following are equivalent.

(i) For any \((a_n, \ldots, a_1) \in \Delta_n\),
\[ \mathbb{P}_p(X_n^n = a_n, \ldots, X_1^n = a_1) = \mathbb{P}_\theta(Y_n = a_n, \ldots, Y_1 = a_1 \mid Y^n \in \Delta_n). \]

(ii) \(p_i = \frac{(i - 1 + \theta_i)\gamma_i(\theta)}{(i - 1 + \theta_i)\gamma_i(\theta) + \theta_i\gamma_{i-1}(\theta)} = \frac{G_{i-1}(\theta)}{G_i(\theta)}\), for \(i = 3, \ldots, n - 1\).

(iii) \(\theta_i = \frac{(i - 1)q_i}{p_ip_{i-1}}\), for \(i = 3, \ldots, n - 1\).
Proof. First note that, from (21) and Proposition 2, the last equality in (ii) holds for \(i \geq 3\). The same lines of argument as those in [5] proves (i) \(\Rightarrow\) (ii). More precisely, suppose (i) holds, then for \((r_n, \ldots, r_{i+2}, 0, 1) \in \Delta_{n-i+1}\)

\[
p_i = \mathbb{P}_p(X_i = 0 \mid X_{i+1} = 0) = \mathbb{P}_p(X_i = X_{i+1} = 0) / \mathbb{P}_p(X_{i+1} = 0)
\]

\[
= \frac{\gamma_{n-1}^{-1}(\theta) \mathbb{P}_\theta(Y_n = r_n, \ldots, Y_{i+2} = r_{i+2}, Y_{i+1} = 0) \gamma_i(\theta)}{\gamma_{n-1}^{-1}(\theta) \mathbb{P}_\theta(Y_n = r_n, \ldots, Y_{i+2} = r_{i+2}, Y_{i+1} = 0)(\gamma_i(\theta) + \mathbb{P}(Y_i = 1) \gamma_{i-1}(\theta))}
\]

\[
= \frac{(i - 1 + \theta_i) \gamma_i(\theta)}{(i - 1 + \theta_i) \gamma_i(\theta) + \theta_i \gamma_{i-1}(\theta)}.
\]

To show (iii) \(\Rightarrow\) (i), for \(n \in \mathbb{N}\), suppose that

\[
\mathbb{P}(Y_m = a_m, \ldots, Y_1 = a_1 | Y^m \in \Delta_m) = \mathbb{P}(X_m^n = a_m, \ldots, X^n_1 = a_1)
\]

\[
= \mathbb{P}(X_m^n = a_m, \ldots, X^n_1 = a_1 | X^n_{m+1} = 1)
\]

holds for any \((a_m, \ldots, a_1) \in \Delta_m\) and \(m < n\). Let \((a_n, \ldots, a_1) \in \Delta_n\) be such that \(\prod_{i=2}^{n}(1 - a_i) = 0\), which means there exists at least one index \(2 < i < n\) s.t. \(a_i = 1\). Let \(1 < j < n - 1\) be the largest index for which \(a_{j+1} = 1\). Then \((a_j, \ldots, a_1) \in \Delta_j\) and

\[
\mathbb{P}(X^n_j = a_j, \ldots, X^n_1 = a_1)
\]

\[
= \frac{\mathbb{P}(Y_j = 0, Y_{j-1} = a_{j-1}, \ldots, Y_1 = a_1) \gamma_j(\theta)}{\gamma_j(\theta)} \prod_{i=j+2}^{n-1} p_i \gamma_{i+1}(\theta)
\]

\[
= \mathbb{P}(Y_n = a_n, \ldots, Y_1 = a_1 | Y^n \in \Delta_n),
\]

where in the last-but-one line we used (21). In the case that \(a_i = 0\), for \(2 \leq i \leq n\), we have

\[
\mathbb{P}(X^n_i = a_n, \ldots, X^n_1 = a_1) = \prod_{i=2}^{n-1} p_i \frac{\mathbb{P}(Y_1 = 0) \gamma_i(\theta)}{\gamma_i(\theta)} \mathbb{P}(Y_1 = a_1 | Y^n \in \Delta_n),
\]

since \(\gamma_2 = \mathbb{P}(Y_2 = 0)\) and \(\mathbb{P}(Y_1 = 1) = 1\), hence (iii) \(\Rightarrow\) (i) for \(n \in \mathbb{N}\).

Note that (iii) is straightforward from (ii). More precisely, using (21) and (ii), for \(i = 3, \ldots, n-1\), we get

\[
p_i = \frac{i \gamma_i}{(i + \theta_i) \gamma_i+1}, \quad q_i = \frac{i \theta_i \gamma_{i-1}}{(i + \theta_i)(i - 1 + \theta_i) \gamma_{i+1}},
\]

where substituting these into \((i - 1)q_i/p_i p_{i-1}\) leads to the equation in (iii). To prove (iii) \(\Rightarrow\) (ii), we first show if (iii) holds, we have

\[
\gamma_i(\theta) = \frac{p_i}{1 + \theta_2 \prod_{j=2}^{i-1} p_j p_{j+1} + q_{j+1}},
\]
for $i = 3, \ldots, n - 1$. To establish this, denote, temporarily, by $\tilde{\gamma}_i(\theta)$ the r.h.s. of (23). We show the $\tilde{\gamma}_i(\theta)$ satisfies (21), hence $\tilde{\gamma}_i(\theta) = \gamma_i(\theta)$. It is easy to see $\tilde{\gamma}_i(\theta) = \gamma_i(\theta)$, for $i = 3, 4$, just from the definition, so the calculation is omitted here. Now for $i \geq 4$, substituting $\theta_i = (i - 1)q_i/(p_ip_{i-1})$ and $\theta_{i+1} = iq_{i+1}/(p_{i+1}p_i)$, and simplifying, we get

$$
\frac{i}{i + \theta_{i+1}}(\tilde{\gamma}_i(\theta) + \frac{\theta_i}{1 + \theta_i}(\tilde{\gamma}_{i-1}(\theta))) = \frac{p_{i+1}p_i}{p_{i+1}p_i + q_{i+1}}(\tilde{\gamma}_i(\theta) + \frac{q_i}{p_ip_{i-1} + q_i}(\tilde{\gamma}_{i-1}(\theta)))
$$

$$
= \frac{p_{i+1}p_i}{p_{i+1}p_i + q_{i+1}}\tilde{\gamma}_{i-1}(\theta) \left( \frac{p_i}{p_{i-1}p_i + q_i} + \frac{q_i}{p_ip_{i-1} + q_i} \right)
$$

$$
= \frac{p_{i+1}p_i}{(p_{i+1}p_i + q_{i+1})(p_ip_{i-1} + q_i)} \tilde{\gamma}_{i-1}(\theta) = \tilde{\gamma}_{i+1}(\theta),
$$

as claimed, hence $\tilde{\gamma}_i = \gamma_i$, for $i = 3, \cdots, n - 1$, therefore (23) holds if we assume (iii). Thus, applying (23) and $\theta_i = (i - 1)q_i/(p_ip_{i-1})$, the middle term in (ii) simplifies to $p_i$, hence (iii) $\Rightarrow$ (ii).

As discussed before, the last theorem gives a simple way to construct $Y^n,\theta$ from $X^n,p$, and vice versa. Note also that (23) provides an interesting way to compute the $\gamma_i(\theta)$, that does not involve usual inclusion-exclusion arguments for such quantities. We now apply Theorem 3 to establish the main conditioning result for the playground game $\eta$. Denote by

$$
B(\tilde{z}_1, \tilde{z}_2) = \frac{\Gamma(\tilde{z}_1)\Gamma(\tilde{z}_2)}{\Gamma(\tilde{z}_1 + \tilde{z}_2)} = \int_0^1 u^{\tilde{z}_1-1}(1 - u)^{\tilde{z}_2-1} du,
$$

the Beta function for complex variables $\tilde{z}_1, \tilde{z}_2$ with $Re(\tilde{z}_1), Re(\tilde{z}_2) > 0$.

**Corollary 1.** For given $0 < \theta_1, \theta_2 \leq 1$, there exists a unique sequence of independent Bernoulli \{0,1\}-valued random variables $\xi := (\xi_i)_{i=1}^\infty$, with

$$
P_\theta(\xi_i = 0) = \frac{i - 1}{i - 1 + \theta_i^*},
$$

where $\theta_i^* = \theta(\theta_i^2 + 1)$, for $i \geq 4$, and $\theta_3^* = \theta$, such that for any $4 \leq n \in \mathbb{N}$ and $(a_n, \ldots, a_1) \in \Delta_n$, letting $\xi^n = (\xi_n, \cdots, \xi_1)$, we have

$$
P_\theta(\eta^n = a_n, \ldots, \eta_1^n = a_1) = P_\theta(\xi_n = a_n, \ldots, \xi_1 = a_1 | \xi^n \in \Delta_n).
$$

In addition, for $3 \leq n \in \mathbb{N}$

$$
\delta_n(\theta) := P_\theta(\xi^n \in \Delta_n) = \frac{(n - 1)!((\theta^2 + \theta + 2)(\theta + 2)(n - 3))}{(1 + \theta_3^*)(\theta + 2) \prod_{k=1}^{n-2} (k(k + 1) + \theta(\theta + k))},
$$

while $\delta_1 = 0$ and $\delta_2 = P(\xi_2 = 0) = 1/(1 + \theta_2^*)$. Also,

$$
\lim_{n \to \infty} \delta_n(\theta) = \frac{\theta^2 + \theta + 2}{1 + \theta_2^*} B(z_1(\theta), z_2(\theta)),
$$

where $z_1(\theta) = \frac{3 + \theta - \sqrt{(1-\theta)(1+3\theta)}}{2}$, $z_2(\theta) = \frac{3 + \theta + \sqrt{(1-\theta)(1+3\theta)}}{2}$.
Proof. First note that from Theorem 3, for any \( n \geq 4 \), (24) holds if and only if \( \theta_i^* = (i-1)q_i/(p_ip_{i-1}) \), for \( 3 \leq i \leq n-1 \), which letting \( q_i = 1 - p_i = \theta/(\theta + i - 1) \) and \( p_2 = 1 \), simplifies to the values of \( \theta_i^* \) given in the statement of the corollary, and results in (25), if we substitute them into (23) for \( i \geq 3 \). Note that \( \delta_1(\theta) = 0, \delta_2(\theta) = 1/(1 + \theta_i^2) \) are obvious from the definition.

For the limit, as \( k(k + 1) + \theta(\theta + k) = (k - 1 + z_1(\theta))(k - 1 + z_2(\theta)) \) and \( z_1(\theta) + z_2(\theta) = \theta + 3 \), writing \( x_{(m)} = \Gamma(x + m)/\Gamma(x) \) concludes

\[
\delta_n(\theta) = \frac{1}{(1 + \theta_i^2)(z_1(\theta))_{(n-2)}(z_2(\theta))_{(n-2)}}
\]

\[
\frac{(\theta^2 + \theta + 2)B(z_1(\theta), z_2(\theta))}{1 + \theta_i^2} = \frac{\Gamma(n)\Gamma(n + \theta - 1)}{\Gamma(n + z_1(\theta) - 2)\Gamma(n + z_2(\theta) - 2)}
\]

where for large \( n \), the term

\[
\frac{\Gamma(n)\Gamma(n + \theta - 1)}{\Gamma(n + z_1(\theta) - 2)\Gamma(n + z_2(\theta) - 2)} \sim n^{\theta - 1 + 2 - z_2(\theta)} n^{\theta + 3 - z_1(\theta) - z_2(\theta)} = 1,
\]

as, once again, \( z_1(\theta) + z_2(\theta) = \theta + 3 \), completing the proof. \( \square \)

4.2 Number of cycles revisited

For the classical Feller coupling \( \tilde{\xi}_n^\theta, \ldots, \tilde{\xi}_1^\theta \), the probability generating function (pgf) of the number of cycles \( K_n^\tilde{\xi} \) is given by

\[
\mathbb{E}_\theta(s^{K_n^\tilde{\xi}}) = \frac{(\theta s)_{(n)}}{\theta_{(n)}}.
\]

The above pgf indeed has an interesting relation with the pgf of the number of cycles \( K_n^\tilde{\eta} \) of the \( \tilde{\eta} \) process [5], that is

\[
\mathbb{E}_\theta(s^{K_n^\tilde{\eta}}) = \frac{\lambda_n(\theta s)}{\lambda_n(\theta)} \mathbb{E}_\theta(s^{K_n^\tilde{\xi}}),
\]

where \( \lambda_n(\theta) = \mathbb{P}_\theta((\tilde{\xi}_n, \ldots, \tilde{\xi}_1) \in \Delta_n) \) represents the probability that a \( \theta \)-biased random permutation of size \( n \) is a derangement. A similar relation holds for the total number of cycles of \( X_n^\theta \) and \( Y_n^\theta \).

To see this, denote by \( K_n = K_n^\theta \) and \( \tilde{K_n} = \tilde{K}_n^\theta \) the total number of cycles of \( X_n^\theta \) and \( Y_n^\theta \). We have

\[
\mathbb{P}_\theta(\tilde{K}_n = k) = \sum_{1 = i_1 < \ldots < i_k \leq n} \frac{(n-1)!\theta_{i_1} \cdots \theta_{i_k}}{(i_2 - 1) \cdots (i_k - 1) \theta_{<n}}, \tag{26}
\]

where \( \mathbb{P}_\theta(\tilde{K}_n = 1) = \theta_1(n - 1)!/\theta_{<n} \). Note that \( \tilde{K}_n \) has the Poisson-binomial distribution. The
pgf of $\tilde{K}_n$ is given by

$$E_\theta(s^{\tilde{K}_n}) = \sum_{k=1}^{n} s^k \mathbb{P}_\theta(\tilde{K}_n = k)$$

$$= \sum_{k=1}^{n} s^k \sum_{1=i_1<...<i_k \leq n} \frac{(n-1)! \theta_{i_1} \cdots \theta_{i_k}}{(i_2-1) \cdots (i_k-1) \theta_{<n}}$$

$$= \frac{(s\theta)_{<n>}}{\theta_{<n>}} \sum_{k=1}^{n} \sum_{1=i_1<...<i_k \leq n} \frac{(n-1)! (s\theta_{i_1}) \cdots (s\theta_{i_k})}{(i_2-1) \cdots (i_k-1) (s\theta)_{<n>}}$$

$$= \frac{(s\theta)_{<n>}}{\theta_{<n>}}.$$

Similarly, for the number $K_n$ of cycles of $X^n$ we have

$$P_p(K_n = k) = \sum_{1=i_1<...<i_k \leq n} \frac{(n-1)! \theta_{i_1} \cdots \theta_{i_k}}{(i_2-1) \cdots (i_k-1) \gamma_n(\theta) \theta_{<n>}}.$$  (27)

Therefore, we obtain the following relation for the probability generating functions of $\tilde{K}_n$ and $K_n$.

**Theorem 4.** Suppose $\theta_i = (i-1)q_i/(p_ip_{i-1})$, for $i = 3, ..., n-1$. Then

$$E_p(s^{K_n}) = \frac{\gamma_n(s\theta)}{\gamma_n(\theta)} E_\theta(s^{\tilde{K}_n}).$$

**Proof.** As in the above discussion, we have

$$E_p(s^{K_n}) = \frac{\gamma_n(s\theta)(s\theta)_{<n>}}{\gamma_n(\theta) \theta_{<n>}} \sum_{k=1}^{n} \sum_{1=i_1<...<i_k \leq n} \frac{(n-1)! (s\theta_{i_1}) \cdots (s\theta_{i_k})}{(i_2-1) \cdots (i_k-1) \gamma_n(\theta) (s\theta)_{<n>}}$$

$$= \frac{\gamma_n(s\theta)(s\theta)_{<n>}}{\gamma_n(\theta) \theta_{<n>}} = \frac{\gamma_n(s\theta)}{\gamma_n(\theta)} E_\theta(s^{\tilde{K}_n}).$$

\[\square\]

### 4.3 Cycle counts

In this section, we find the distribution of the cycle counts for the GFC. In fact, this is the counterpart of the Ewens sampling formula when $\theta$ is replaced by $\theta = (\theta_i)_{i=1}^\infty$, hence more involved. For a vector $r \in \{0,1\}^n$ with $r_1 = 1$,

$$P_\theta(Y^n = r) = \frac{(n-1)!}{\theta_{<n>}} \prod_{1<i\leq n:r_i=1} \frac{\theta_i}{i-1}.$$  (28)

Letting $\prod_{i=1}^{0} x_i = 1$, by convention, this is equivalent to

$$P_\theta(Y^n = r) = \frac{(n-1)!}{\theta_{<n>}} \prod_{i=1}^{\|r|-1} \frac{\theta_{n+1-a_1-\cdots-a_i}}{n-a_1-\cdots-a_i},$$
where \( \|r\| \) is the number of cycles of \( r \) and \( a_i \) is the size of the \( i \)-th cycle (in the order of formation of the cycles). In other words, \( a_1 = n + 1 - \max \{ i \leq n; r_i = 1 \} \), \( a_2 = n + 1 - a_1 - \max \{ i \leq n - a_1; r_i = 1 \} \), and so on.

To compute \( \mathbb{P}(C_1 = c_1, \ldots, C_n = c_n) \) we sum \( \mathbb{P}_\theta(Y^n = r) \) over all possible \( r \in \{0, 1\}^n \), \( r_1 = 1 \), which have the cycle type \((c_1, \ldots, c_n)\). To this end, for any \( c = (c_1, \ldots, c_n) \in \mathbb{Z}_+^n \) satisfying \( \sum_{i=1}^n ic_i = n \), let \( \|c\| = \sum_{i=1}^n c_i \). We define \( \bar{c} = (\bar{c}_1, \ldots, \bar{c}_{\|c\|}) \) as follows. Let \( i_1, \ldots, i_k \) be all numbers in \( \{1, \ldots, n\} \) such that \( c_{i_j} \neq 1 \). For \( 0 < l \leq c_{i_l} \), let \( \bar{c}_l = i_l \). Similarly, for any \( m = 2, \ldots, k \), if \( \sum_{j=1}^{m-1} c_{i_j} < l \leq \sum_{j=1}^m c_{i_j} \), let \( \bar{c}_l = i_m \). For example, if \( c = (2, 0, 1, 0, 4) \), then \( \bar{c} = (1, 1, 3, 5, 5, 5) \).

Denote by \( S_l \) the permutation group of size \( l \in \mathbb{N} \).

**Proposition 3.** For any \( n \in \mathbb{N} \) and \( c = (c_1, \ldots, c_n) \in \mathbb{Z}_+^n \) with \( \sum_{i=1}^n ic_i = n \),

\[
\mathbb{P}_\theta(C_1(n) = c_1, \ldots, C_n(n) = c_n) = \frac{(n-1)!}{\theta_{\|c\|}} \theta_1 \prod_{i=1}^n \frac{1}{c_i!} \sum_{\sigma \in S_{\|c\|}} \prod_{i=1}^{\|c\|} \frac{\theta_{\epsilon(i, c, \sigma)}}{\epsilon(i, c, \sigma) - 1},
\]

where \( \epsilon(i, c, \sigma) = \epsilon_n(i, c, \sigma) := n + 1 - \sum_{j=1}^i \bar{c}_{\sigma(j)} \), for \( i = 1, \ldots, \|c\| \). Furthermore, if \( \theta_1 = (i - 1)q_i/p_i p_{i-1} \), \( i \geq 3 \), then for any \( 2 \leq n \in \mathbb{N} \) and any \( c = (c_2, \ldots, c_n) \) with \( \sum_{i=2}^n ic_i = n \), we have

\[
\mathbb{P}_\mathbf{p}(C_2(n) = c_2, \ldots, C_n(n) = c_n) = \frac{(n-1)!}{\gamma_n(\theta) \theta_{\|c\|}} \prod_{i=2}^n \frac{1}{c_i!} \sum_{\sigma \in S_{\|c\|}} \prod_{i=1}^{\|c\|} \frac{\theta_{\epsilon(i, c, \sigma)}}{\epsilon(i, c, \sigma) - 1} \prod_{i=2}^n \frac{q_{\epsilon(i, c, \sigma)}}{p_{\epsilon(i, c, \sigma)} p_{\epsilon(i, c, \sigma)} - 1}.
\]

**Proof.** The first part is straightforward from (28), by summing over all possible \( r \in \{0, 1\}^n \), with \( r_1 = 1 \). The first equality in (30) follows from (29) and the conditional relation given in Theorem 3, while the second equality comes from the relation of \( \mathbf{p} \) and \( \theta \), and (23).

Let by convention \( \epsilon(0, c, \sigma) = \infty \). Note that \( \epsilon(\|c\|, c, \sigma) = 1 \) and \( \epsilon(\|c\| - 1, c, \sigma) \geq 3 \), for any \( \sigma \) and \( c \). The next corollary follows immediately.

**Corollary 2.** For any \( n \geq 3 \) and \( c = (c_2, \ldots, c_n) \in \mathbb{Z}_+^{n-1} \), with \( \sum_{i=2}^n ic_i = n \),

\[
\mathbb{P}(C_2^n(n) = c_2, \ldots, C_n^n(n) = c_n) = \frac{(\theta + 1)(n - 2)! \theta^{\|c\|}}{\theta_{(n-1)}} \prod_{i=2}^n \frac{1}{c_i!} \sum_{\sigma \in S_{\|c\|}} \epsilon^*(c, \sigma) \prod_{i=1}^{\|c\| - 1} \frac{\theta + \epsilon(i, c, \sigma) - 2}{(\epsilon(i, c, \sigma) - 1)(\epsilon(i, c, \sigma) - 2)},
\]

where \( \epsilon(i, c, \sigma) \) is defined as in Proposition 3, and

\[
\epsilon^*(c, \sigma) = \mathbb{1} \{ \epsilon(\|c\| - 1, c, \sigma) > 3 \} + \frac{1}{\bar{\theta} + 1} \mathbb{1} \{ \epsilon(\|c\| - 1, c, \sigma) = 3 \}.
\]
4.4 The weak limit of $X^{n,p}$

As $X^n_m = 0$, for any $m > 2$, one cannot recover the outcomes or random derangements of $X^n$ from those of $X^m$, $n > m$. In other words, one cannot obtain the law of $X^m$ from the law of $X^n$ by projecting on the first $m$ components, i.e.

$$\tilde{\mu}_n \pi^{-1} \neq \tilde{\mu}_m, \ n > m \geq 2,$$

where $\tilde{\mu}_m$ is the law of $X^m$, $\pi_m(x) = (x_i)_{i=1}^m$ for $x \in \{0,1\}^N$ or $x \in \{0,1\}^n$, $n \geq m$, and finally $\tilde{\mu}_n \pi^{-1}$ is the image of $\tilde{\mu}_n$ under $\pi_m$. Now, from (2), as

$$Q_{j,i}(u,v) = P(X^n_i = v | X^n_j = u) = P(X^m_i = v | X^m_j = u),$$

for any $n > m \geq j > i \geq 1$, assuming $\lim_{n \to \infty} P(X^m_n = 1)$ exists, for any $m \in \mathbb{N}$, we can conclude that $(X^n_m, \ldots, X^n_1)$ has a weak limit, as $n \to \infty$. A natural framework to see this is through the reversed chain. From the time-reversal transformation,

$$P(X^n_{i+1} = v | X^n_i = u) = \frac{Q_{i+1,i}(v,u)P(X^n_{i+1} = v)}{P(X^n_i = u)}, \ u,v \in \{0,1\}.$$

We need the following lemma.

Lemma 5. Suppose

$$\sum_{j=1}^{\infty} p_j = \infty. \quad (31)$$

Then $\varphi_i = \varphi_i(p) := \lim_{n \to \infty} P(X^n_i = 1)$ exists, and $0 < \varphi_i < 1$ for $i \geq 3$.

Proof. We have $\sum_{j=1}^{\infty} p_j = \infty$ if and only if $\prod_{i=1}^{\infty} q_i = 0$ for any $j \in \mathbb{N}$. Now from Lemma 1, $\lim_{n \to \infty} P(X^n_i = 1)$ exists and $0 < q_i p_{i+1} < \varphi_i < q_i < 1$.

Note that we always have $\varphi_1 = 1 - \varphi_2 = 1$. In the rest of this section, we assume condition (31) holds, hence $\varphi_i \in (0,1)$ is well-defined for $i \geq 3$. We are now ready to formally define $X^{\infty,p} = (X^{\infty}_i)_{i \geq 1}$ by $X^{\infty}_1 = 1$ and

$$P_p(X^{\infty}_{i+1} = 1 | X^{\infty}_i = 0) = 1 - P_p(X^{\infty}_{i+1} = 0 | X^{\infty}_i = 0) = \lim_{n \to \infty} \frac{P_p(X^n_{i+1} = 1)}{P_p(X^n_i = 0)} = \frac{\varphi_{i+1}(p)}{1 - \varphi_i(p)}, \quad (32)$$

and

$$P_p(X^{\infty}_{i+1} = 1 | X^{\infty}_i = 1) = 1 - P_p(X^{\infty}_{i+1} = 0 | X^{\infty}_i = 1) = 0.$$  

Letting $X^{n,p}_i = 0$, for $i > n + 1$, we have as $n \to \infty$

$$(X^{n,p}_1, \ldots, X^{n,p}_n) \Rightarrow (X^{\infty,p}_1, X^{\infty,p}_2, \ldots).$$
To see this notice that, for any \( m \in \mathbb{N} \) and \( x = (x_i)_{i=1}^{\infty} \in \{0,1\}^\mathbb{N} \) with \( x_1 = 1 \) and \( x_2 = 0 \)

\[
\lim_{n \to \infty} \mathbb{P}_p(X_1^n = x_i; i = 1, \ldots, m) = \lim_{n \to \infty} \mathbb{P}_p(X_m^n = x_m) \prod_{i=1}^{m-1} Q_{i+1,i}(x_{i+1}, x_i)
\]

\[
= \prod_{i=1}^{m-1} Q_{i+1,i}(x_{i+1}, x_i) \frac{\lim_{n \to \infty} \mathbb{P}_p(X_{i+1}^n = x_{i+1})}{\lim_{n \to \infty} \mathbb{P}_p(X_i^n = x_i)}
\]

\[
= \mathbb{P}_p(X_i^\infty = x_i; i = 1, \ldots, m),
\]

where the last equality follows from the time-reversal transformation. Denote by \( \mu_n \) the law of \((X_1^n, \ldots, X_m^n)\) and by \( \mu \) the law of \( X_\infty^\infty \cdot \mathbb{P}_p \), and recall the definition of \( \Delta_n \) from (18). The next theorem gives the total variation distance between \( \mu_{\pi_n^{-1}} \) and \( \mu_n \).

**Theorem 5.** For any \( n \in \mathbb{N} \),

\[
d_{TV}(\mu_{\pi_n^{-1}}, \mu_n) = \varphi_n \mathbb{1}\{n > 1\}.
\]

**Proof.** For \( n = 1, 2 \), both sides of the above equality equal 0. For \( n > 2 \), write

\[
d_{TV}(\mu_{\pi_n^{-1}}, \mu_n)
\]

\[
= \frac{1}{2} \sum_{x \in \Delta_n} |\mathbb{P}_p(X^n_1 = x_i; i = 1, \cdots, n) - \mathbb{P}_p(X^n_\infty = x_i; i = 1, \cdots, n)|
\]

\[
+ \frac{1}{2} \sum_{x \in \Delta_{n-1}} \mathbb{P}_p(X^n_\infty = 1, X^n_\infty = x_i; i = 1, \cdots, n-1)
\]

\[
= \frac{1}{2} \sum_{x \in \Delta_n} \left| 1 - \prod_{i=1}^{n-1} \frac{\lim_{n \to \infty} \mathbb{P}_p(X_{i+1}^n = x_{i+1})}{\lim_{n \to \infty} \mathbb{P}_p(X_i^n = x_i)} \right| \prod_{i=1}^{n-1} Q_{i+1,i}(x_{i+1}, x_i)
\]

\[
+ \frac{1}{2} \sum_{x \in \Delta_{n-1}} \frac{Q_{n,n-1}(1,0) \varphi_n}{\lim_{n \to \infty} \mathbb{P}_p(X_{n-1}^n = x_{n-1})} \prod_{i=1}^{n-2} Q_{i+1,i}(x_{i+1}, x_i) \frac{\lim_{n \to \infty} \mathbb{P}_p(X_{i+1}^n = x_{i+1})}{\lim_{n \to \infty} \mathbb{P}_p(X_i^n = x_i)}
\]

\[
= \frac{1}{2} \sum_{x \in \Delta_n} (1 - \mathbb{P}_p(X_\infty^n = 0)) \prod_{i=1}^{n-1} Q_{i+1,i}(x_{i+1}, x_i)
\]

\[
+ \frac{1}{2} \sum_{x \in \Delta_{n-1}} Q_{n,n-1}(1,0) \varphi_n \prod_{i=1}^{n-2} Q_{i+1,i}(x_{i+1}, x_i) = \frac{1}{2} \varphi_n + \frac{1}{2} \varphi_n = \varphi_n.
\]

\( \square \)

We notice that (31) does not guarantee \( d_{TV}(\mu_{\pi_n^{-1}}, \mu_n) = \varphi_n \to 0 \), as \( n \to \infty \). For instance, if \( q_n = q \in (0,1) \) for \( n \in \mathbb{N} \), then \( \varphi \) and \( X_\infty^\infty \cdot \mathbb{P}_p \) are well-defined but \( \varphi_n = q/(1+q) > 0 \). Note that \( q_n - q_n q_{n+1} \leq \varphi_n \leq q_n \), and \( q_n (1 - \varphi_{n+1}) = \varphi_n \). Therefore, as \( n \to \infty \), \( q_n \to 0 \) if and only if \( \varphi_n \to 0 \), hence the following result.

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Corollary 3. The following are equivalent.

(i) $\sum_{n=1}^{\infty} p_n = \infty$ and as $n \to \infty$, $d_{TV}(\mu \pi_n^{-1}, \mu_n) \to 0$.

(ii) As $n \to \infty$, $q_n \to 0$.

As examples, $\lim_{n \to \infty} q_n = 0$, for both $\eta_n, \theta$, and hence $\eta^\infty, \tilde{\eta}^\infty$ are well-defined. Furthermore, $0 < d_{TV}(\mu \pi_n^{-1}, \mu_n) = \varphi_n \to 0$, as $n \to \infty$, for both processes. More exactly, from Lemma 2, for $\eta$ we have

$$\varphi_n^\eta = \lim_{m \to \infty} P_\theta(\eta_m = 1) = P_\theta(\eta^\infty = 1) = \theta \int_0^1 e^{-\theta u} (1 - u)^{\theta + n - 2} du. \quad (33)$$

For $\tilde{\eta}$, from the conditional relation between $\tilde{\eta}$ and $\tilde{\xi}$, given in [5] and this paper,

$$\varphi_n^\tilde{\eta} = \lim_{m \to \infty} P_\theta(\tilde{\eta}_m = 1) = P_\theta(\tilde{\eta}^\infty = 1) = \frac{\lambda_{n+1,\infty}(\theta) P_\theta(\tilde{\xi}_n = 1) \lambda_{n-1}(\theta)}{\lambda_\infty(\theta)},$$

where $\lambda_n(\theta)$, the probability that a $\theta$-biased random permutation of size $n$ is a derangement, given in Equation (7) in [5],

$$\lambda_{n+1,\infty} := P_\theta(\tilde{\xi}_{n+1} + \sum_{j=n+1}^{\infty} \tilde{\xi}_j \tilde{\xi}_{j+1} = 0) = M(\theta + 1, \theta + n, -\theta),$$

as given in Theorem 6 in [5], and

$$\lambda_\infty(\theta) := P_\theta(\sum_{j=1}^{\infty} \tilde{\xi}_j \tilde{\xi}_{j+1} = 0) = \lim_{n \to \infty} \lambda_n(\theta) = e^{-\theta}.$$

Therefore,

$$\varphi_n^\tilde{\eta} = \frac{\theta e^{\theta} \lambda_{n-1}(\theta)}{\theta + n - 1} M(\theta + 1, \theta + n, -\theta).$$

The transition matrices of the limit Markov chains $\eta^\infty$ and $\tilde{\eta}^\infty$ can be easily obtained from (32).

4.5 Conditional relation between $X^\infty$ and $Y^\theta$

Theorem 3 implies that, when $\theta_i = (i - 1)q_i/(p_i p_{i-1})$, for $i \geq 3$

$$P_p(X_i^n = 1) = 1 - P_p(X_i^n = 0) = \frac{\theta_i \gamma_{i+1,n}(\theta) \gamma_{i-1}(\theta)}{\gamma_n(\theta)},$$

where $\gamma_0(\theta) = 1$, $\gamma_{n,n}(\theta) = P_\theta(Y_n = 0)$, $\gamma_{n+1,n}(\theta) = 0$, and for $2 \leq j < n$

$$\gamma_{j,n}(\theta) = P_\theta \left( Y_j + Y_n + \sum_{k=j}^{n-1} Y_k Y_{k+1} = 0 \right).$$
This reduces the transition probabilities of a finite reversed chain to
\[
\mathbb{P}_p(X_{i+1}^n = 1 \mid X_i^n = 0) = 1 - \frac{\mathbb{P}_p(X_{i+1}^n = 0 \mid X_i^n = 0)}{\mathbb{P}_p(X_i^n = 0)} = \frac{\theta_{i+1} \gamma_{i+2,n}(\theta)}{\theta_{i+1} \gamma_{i+2,n}(\theta) + (i + \theta_{i+1}) \gamma_{i+1,n}(\theta)}.
\] (34)

We look for conditions under which we can extend the conditional relation for $X^\infty$ and $Y$. We first record some useful properties of $Y$. For $i \geq 2$, let
\[
\gamma_{i,\infty}(\theta) := \mathbb{P}(Y_i + \sum_{j=1}^{\infty} Y_j Y_{j+1} = 0),
\]
and $\gamma_{\infty}(\theta) := \gamma_{2,\infty}(\theta)$. Consider the following conditions
\[
\sum_{i=1}^{\infty} \frac{\theta_i \theta_{i+1}}{(i - 1 + \theta_i)(i + \theta_{i+1})} < \infty, \quad (35)
\]
\[
\theta_n/n \to 0, \quad n \to \infty, \quad (36)
\]
\[
\sum_{i=1}^{\infty} \left(\frac{\theta_i}{i - 1 + \theta_i}\right)^2 < \infty. \quad (37)
\]

Note that condition (36) is equivalent to $\theta_n/(n - 1 + \theta_n) \to 0$, as $n \to \infty$. Also, (37) implies (35) and (36), but not vice versa. For $j \geq 2$, let
\[
\tilde{C}_j(n) = Y_{n-j+1} \prod_{l=2}^{j} (1 - Y_{n-j+l}) + \sum_{i=1}^{n-j} Y_i Y_{i+j} \prod_{l=1}^{j-1} (1 - Y_{i+l}),
\]
and $\tilde{C}_1(n) = Y_n + \sum_{i=1}^{n-1} Y_i Y_{i+1}$. Similarly, for $j \geq 2$,
\[
\tilde{C}_1(\infty) = \sum_{i=1}^{\infty} Y_i Y_{i+1}, \quad \text{and} \quad \tilde{C}_j(\infty) = \sum_{i=1}^{\infty} Y_i Y_{i+j} \prod_{l=1}^{j-1} (1 - Y_{i+l}).
\]

From the definition $\gamma_{\infty}(\theta) = \mathbb{P}_\theta(\tilde{C}_1(\infty) = 0)$. We have the following result.

**Theorem 6.** (i) If (35) and (36) hold, then as $n \to \infty$, $\gamma_{n,\infty}(\theta) \to 1$ and $\gamma_{i,n}(\theta) \to \gamma_{i,\infty}(\theta)$, for $i \geq 2$. In particular, $\gamma_{n}(\theta) \to \gamma_{\infty}(\theta)$, as $n \to \infty$.

(ii) If (35) holds, then $\gamma_{\infty}(\theta) > 0$, $\gamma_{i,\infty}(\theta) > 0$, $i \in \mathbb{N}$.

(iii) If (37) holds, then for any $m \in \mathbb{N},$
\[
d_{TV}\left(\mathcal{L}(\tilde{C}_1(n), \ldots, \tilde{C}_m(n)), \mathcal{L}(\tilde{C}_1(\infty), \ldots, \tilde{C}_m(\infty))\right) \to 0, \quad (38)
\]
as $n \to \infty$. If in addition $\theta$ is a bounded sequence, then (38) holds for $m = m(n) = o(n)$, as $n \to \infty.$

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Proof. For (i), applying Borel-Cantelli lemma, we can write
\[
\lim_{n \to \infty} \mathbb{P}_\theta \left( \sum_{j=n}^{\infty} Y_j Y_{j+1} = 0 \right) = 1 - \mathbb{P}_\theta(Y_n = Y_{n+1} = 1 \text{ i.o.}) = 1; \tag{39}
\]
\[
\lim_{n \to \infty} \gamma_{n,\infty}(\theta) = \lim_{n \to \infty} \mathbb{P}_\theta(Y_n = 0) \mathbb{P}_\theta \left( \sum_{j=n+1}^{\infty} Y_j Y_{j+1} = 0 \right) = 1.
\]

Now, we have
\[
\gamma_{i,\infty}(\theta) = \lim_{n \to \infty} \sum_{\ell=1}^{2} \mathbb{P}_\theta(Y_i + \sum_{j=i}^{\infty} Y_j Y_{j+1} = 0, Y_n = \ell)
\]
\[
= \lim_{n \to \infty} \left( \frac{\gamma_{i,n}(\theta) \gamma_{\infty}(\theta)}{\mathbb{P}_\theta(Y_n = 0)} + \gamma_{i,n-1}(\theta) \mathbb{P}_\theta(Y_n = 1) \gamma_{n+1,\infty}(\theta) \right)
\]
\[
= \lim_{n \to \infty} \gamma_{i,n}(\theta),
\]
since \(\gamma_{n,\infty}(\theta), \mathbb{P}_\theta(Y_n = 0) \to 1\) and \(\mathbb{P}_\theta(Y_n = 1) \to 0\), as \(n \to \infty\). Noting \(\gamma_{2,n}(\theta) = \gamma_n(\theta)\) and \(\gamma_{2,\infty}(\theta) = \gamma_{\infty}(\theta)\), finishes (i).

For (ii), note that from (39), \(\mathbb{P}_\theta(\sum_{j=n}^{\infty} Y_j Y_{j+1} = 0)\) gets very close to 1 for large \(n\), and therefore
\[
\gamma_{i,\infty}(\theta) \geq \mathbb{P}_\theta \left( Y_n = 0, Y_i + \sum_{j=i}^{\infty} Y_j Y_{j+1} = 0 \right) = \gamma_{i,n}(\theta) \mathbb{P}_\theta \left( \sum_{j=n+1}^{\infty} Y_j Y_{j+1} = 0 \right) > 0.
\]

For (iii), note that
\[
d_{TV} \left( \mathcal{L}(\tilde{C}_1(n), \ldots, \tilde{C}_m(n)), \mathcal{L}(\tilde{C}_1(\infty), \ldots, \tilde{C}_m(\infty)) \right)
\]
\[
\leq \mathbb{P}_\theta \left( (\tilde{C}_1(n), \ldots, \tilde{C}_m(n)) \neq (\tilde{C}_1(\infty), \ldots, \tilde{C}_m(\infty)) \right)
\]
\[
\leq \mathbb{P}_\theta \left( \{Y_{n-m+1} = \cdots = Y_n = 0\} \mathbb{P}_\theta(Y_{n+1} = 0) + \sum_{i \geq n} \mathbb{P}(Y_i = 1) \mathbb{P}_\theta(\{Y_{i+1} = \cdots = Y_{i+m} = 0\}) \right)
\]
which is bounded by
\[
\leq \frac{n}{n + \theta_{n+1}} \sum_{j=1}^{m} \frac{\theta_{n-m+j}}{n - m + j - 1 + \theta_{n-m+j}} + \sum_{i \geq n} \frac{\theta_i \theta_{i+j}}{(i-1 + \theta_i)(i+j-1 + \theta_{i+j})}
\]
\[
\leq \frac{n}{n + \theta_{n+1}} \sum_{j=1}^{m} \frac{\theta_{n-m+j}}{n - m + j - 1 + \theta_{n-m+j}} + m \sum_{i \geq n} \left( \frac{\theta_i}{i-1 + \theta_i} \right)^2.
\]

For fixed \(m \in \mathbb{N}\), the r.h.s. converges to 0, as \(n \to \infty\), if (37) holds. Now if \(\theta_n \leq c\) for \(n \in \mathbb{N}\), the first and second terms on the right of the last inequality are bounded by \(c m(n)/(n - m(n))\) and \(m(n) c^2/(n - 1)\). Therefore, the r.h.s. converges to 0, if \(m(n) = o(n)\).  

\[\square\]

Remark 2. Note that \(\{Y_{2i-1} Y_{2i} = 1\}\) are independent for \(i \in \mathbb{N}\), and also \(\{Y_{2i} Y_{2i+1} = 1\}\) are independent for \(i \in \mathbb{N}\). Hence from Borel-Cantelli lemma, \(\tilde{C}_1(\infty) < \infty\ a.s.\) if and only if \(\gamma_{i,\infty}(\theta) > 0\) for \(i \in \mathbb{N}\), if and only if (35) holds.
Remark 3. We can indeed show that for every \( i \geq 2 \), \( \gamma_{i,n}(\theta) \to \gamma_{i,\infty}(\theta) > 0 \), as \( n \to \infty \), if and only if (35) and (36) hold.

Proof. The proof of sufficiency was given in Theorem 6, part (ii). For the necessity part, consider \( (p_i)_{i=3}^{\infty} \), \( p_1 = 0, p_2 = 1 \) s.t. \( \theta_i = (i-1)q_i/(p_ip_{i-1}) \), for \( i \geq 3 \). From (23) and (22), for \( n \geq 4 \), we can write

\[
\gamma_n(\theta) = \frac{P_\theta(Y_n = 0)}{1 + \theta_2} \prod_{j=2}^{n-2} \frac{p_j}{p_jp_{j+1} + q_{j+1}},
\]

where the limit of the product on the right of the last equation always exists, as \( p_j/(p_jp_{j+1} + q_{j+1}) \leq 1 \), for \( j \geq 2 \). Furthermore, this limit is strictly positive if and only if

\[
\sum_{j=3}^{\infty} \left(1 - \frac{p_j}{p_jp_{j+1} + q_{j+1}}\right) = \sum_{j=3}^{\infty} \frac{q_jq_{j+1}}{p_jp_{j+1} + q_{j+1}} < \infty.
\]

On the other hand, as \( p_{j-1}p_j + q_j \leq 1 \),

\[
\sum_{j=3}^{\infty} \frac{q_jq_{j+1}}{p_jp_{j+1} + q_{j+1}} \leq \sum_{j=3}^{\infty} \frac{q_jq_{j+1}}{(p_{j-1}p_j + q_j)(p_jp_{j+1} + q_{j+1})} = \sum_{j=3}^{\infty} \frac{\theta_j\theta_{j+1}}{(j-1 + \theta_i)(j + \theta_{j+1})},
\]

where the last equality follows by substituting \( \theta_i = (i-1)q_i/(p_ip_{i-1}) \), for \( i \geq 3 \). Thus (35) implies (41).

Now to complete the proof of the necessity part, note that from Theorem 6, part (ii) and Remark 2, \( \gamma_{i,\infty}(\theta) = 0 \) if (35) does not hold. On the other hand, if (35) holds but \( \lim_{n \to \infty} P_\theta(Y_n = 0) \) does not exist, the limit of the product on the r.h.s. of (40) is strictly positive, as (41) holds. Therefore, from (40), \( \lim_{n \to \infty} \gamma_n(\theta) \) does not exist. This shows that, if \( \gamma_n \to \gamma_\infty > 0 \), as \( n \to \infty \), then (35) and (36) both hold.

For the general case of \( \gamma_{i,n} \to \gamma_{i,\infty} > 0 \), as \( n \to \infty \), use the same argument for the shifted sequence \( \bar{Y}_j = Y_{j+i-2}, j \geq 2 \), with

\[
P_\theta(\bar{Y}_j = 0) = \frac{\bar{\theta}_j}{j - 1 + \bar{\theta}_j} = \frac{\theta_{j+i-2}}{\theta_{j+i-2} + j + i - 3},
\]

which is equivalent to take \( \bar{\theta}_j = (j-1)\theta_{j+i-2}/(j + i - 3) \), for \( j \geq 2 \). This finishes the proof of the remark.

The following theorem provides conditions under which \( X^{\infty,p} \) is well-defined, while its law coincides with the law of \( Y^\theta \) conditional on no fixed point.

**Theorem 7.** Suppose that

(i) \( \theta_n = (n-1)q_n/(p_n p_{n-1}) \), for \( n \geq 3 \);

(ii) as \( n \to \infty \), \( q_n \to 0 \) (or equivalently \( \theta_n/n \to 0 \));

(iii) \( \sum_{i=3}^{\infty} \frac{q_iq_{i+1}}{(q_i+p_ip_{i-1})(q_{i+1}+p_{i+1}p_i)} = \sum_{i=3}^{\infty} \frac{\theta_i\theta_{i+1}}{(i-1+\theta_i)(i+\theta_{i+1})} < \infty \).
Then
\[ \mathcal{L}(X^\infty P) = \mathcal{L}(Y^\theta | \tilde{C}_1(\infty) = 0). \] (42)

Moreover, under assumptions (i), (ii) and (iii),
\[ P_p(X_{i+1}^\infty = 1 | X_i^\infty = 0) = 1 - P_p(X_{i+1}^\infty = 0 | X_i^\infty = 0) = \frac{\theta_{i+1} \gamma_{i+2,\infty}(\theta)}{\theta_{i+1} \gamma_{i+2,\infty}(\theta) + (i + \theta_{i+1}) \gamma_{i+1,\infty}(\theta)}, \]
and
\[ P_p(X_{i+1}^\infty = 1 | X_i^\infty = 1) = 1 - P_p(X_{i+1}^\infty = 0 | X_i^\infty = 1) = 0. \]

Proof. We first notice that, assuming (i)
\[ \frac{(n-1)q_n}{n} \leq \frac{\theta_n}{n} = \frac{(n-1)q_n}{np_n p_{n-1}}, \]
hence \( q_n \to 0 \) if and only if \( \theta_n/n \to 0 \), as \( n \to \infty \). Also the equality in (iii) is a direct result of applying (i). Now, from Theorem 6, \( \gamma_{\infty}(\theta), \gamma_{i,\infty}(\theta) > 0, i \geq 2 \), and thus from Corollary 3 and Theorem 3,
\[ P_p(X_{i+1}^\infty = 1 | X_i^\infty = 0) = \frac{\varphi_{i+1}(p)}{1 - \varphi_1(p)} = \lim_{n \to \infty} \frac{P_p(X_{i+1}^n = 1)}{P_p(X_i^n = 0)} = \lim_{n \to \infty} \frac{\theta_{i+1} \gamma_{i+2,n}(\theta)}{\theta_{i+1} \gamma_{i+2,\infty}(\theta) + (i + \theta_{i+1}) \gamma_{i+1,\infty}(\theta)} \quad \text{(from (34))} \]
\[ = \frac{\theta_{i+1} \gamma_{i+2,\infty}(\theta)}{\theta_{i+1} \gamma_{i+2,\infty}(\theta) + (i + \theta_{i+1}) \gamma_{i+1,\infty}(\theta)} \quad \text{(from Theorem 6)} \]
\[ = \frac{\theta_{i+1} \gamma_{i+2,\infty}(\theta)}{\theta_{i+1} \gamma_{i+2,\infty}(\theta) + \gamma_{i+1,\infty}(\theta)} \cdot \frac{\gamma_i(\theta)/\gamma_{\infty}(\theta)}{\gamma_i(\theta)/\gamma_{\infty}(\theta)} \]
\[ = \frac{P_\theta(Y_{i+1} = 1, Y_i = 0 | \tilde{C}_1(\infty) = 0)}{P_\theta(Y_i = 0 | \tilde{C}_1(\infty) = 0)}, \]
for any \( i \geq 2 \), hence (42) holds. \( \square \)

Remark 4. Condition (ii) ensures that \( X^\infty P \) is well-defined and serves as the weak limit of \( X^n P \) as \( n \to \infty \). Assuming (iii), we get \( \gamma_{\infty}(\theta), \gamma_{i,\infty}(\theta) > 0 \), and the conditional relation (42). Note that (iii) in Theorem 7 holds for \( \eta \) and \( \tilde{\eta} \), so does the conditional relation (42) for both.

In the last section, the transition probabilities of \( \eta_{\infty,\theta} \) were given in terms of \( \varphi_n^\theta \). But it is still useful to find the exact value of \( \delta_{i,\infty} \).

Proposition 4. For any \( i \geq 4 \) and \( \theta > 0 \),
\[ \delta_{i,\infty}(\theta) := P_\theta(\xi_i + \sum_{j=i}^{\infty} \xi_j \xi_{j+1} = 0) = M(1, \theta + i - 1, -\theta) \frac{B(z_1(\theta) + i - 3, z_2(\theta) + i - 3)}{B(i - 2, \theta + i - 1)}, \]
where \( z_1(\theta), z_2(\theta) \) are defined as in Corollary 1.
\textit{Proof.} From Theorem 6, \( \lim_{n \to \infty} \delta_{i,n}(\theta) = \delta_{i,\infty}(\theta) > 0 \), and in particular, from Corollary 1, getting the limit of (25), we have

\[
0 < \delta_{\infty}(\theta) = \lim_{n \to \infty} \delta_{n}(\theta) = \frac{(\theta^2 + \theta + 2)\Gamma(z_1(\theta))\Gamma(z_2(\theta))}{(1 + \theta_2^2)\Gamma(z_1(\theta) + z_2(\theta))}.
\]  

(43)

Now, for \( i \geq 4 \),

\[
\varphi_{i-1} = \mathbb{P}_\theta(\eta_{i-1}^\infty = 1) = \frac{\delta_{i-2}(\theta)}{\theta \delta_{i-1}} \frac{\theta_{i-1}^\infty - \theta_{i-1} \delta_{i,\infty}(\theta)}{\delta_{\infty}(\theta)},
\]

recalling \( \theta_{i}^* = \theta(1 + \theta/(i - 2)) \), for \( i \geq 4 \), and \( \theta_{3}^* = \theta \). Hence \( \delta_{4,\infty} = (\theta + 2)\varphi_3\delta_{\infty}/(\theta\delta_2) \) and

\[
\delta_{i,\infty}(\theta) = \frac{\varphi_{i-1}((i - 2)(i - 3) + \theta(i - 3 + \theta))\delta_{\infty}(\theta)}{\theta(i - 3 + \theta)\delta_{i-2}(\theta)}, \quad i \geq 5.
\]

Applying (25), (33), (43), after simplification, this reduces to

\[
\frac{\theta_{(3)} B(z_1(\theta), z_2(\theta))(z_1(\theta))(z_2(\theta))(3)}{(i - 3)! \theta_{(i-2)}} \int_{0}^{1} e^{-\theta u}(1 - u)^{\theta + i - 3} du,
\]

which concludes the proposition, after further simplification. \( \square \)

\textbf{Remark 5.} Note that \( \delta_{2,\infty}(\theta) = \delta_{\infty}(\theta) \) given in (43), and \( \delta_{3,\infty}(\theta) \) can be obtained using the recursion

\[
\gamma_{i,\infty}(\theta) = \frac{i - 1}{i - 1 + \theta_i} \left( \gamma_{i+1,\infty}(\theta) + \frac{\theta_{i+1}}{i + \theta_{i+1}} \gamma_{i+2,\infty}(\theta) \right), \quad i \geq 3.
\]

5 A coupling between \( X \) and \( Y \): a push-forward relation

We have seen so far that, for \( \theta > 0 \), the conditional relation (1) does not hold between \( \eta_{n,\theta}^n \) and the classic Feller coupling \( (\tilde{\xi}_n^\theta, \cdots, \tilde{\xi}_1^\theta) \), given there is no 11 patterns observed in the latter. In this section, we will see that in fact the distributions of these two are related by a simple push-forward relation, in the sense that the law of \( \eta_{n,\theta}^n \) is the image of that of \( (\tilde{\xi}_n^\theta, \cdots, \tilde{\xi}_1^\theta) \) under a natural 11-erasing mapping. More generally, in this section, we establish a push-forward relation between \( X^{n,\theta} \) and \( Y^{n,\theta} \), for

\[
p_i = \frac{i - 1}{i - 1 + \theta_i}, \quad (44)
\]

or equivalently \( \theta_i = (i - 1)q_i/p_i \), for \( n \in \mathbb{N} \) and \( 3 \leq i \leq n - 1 \). Under (31), which ensures the existence of the Markov chain \( X^{\infty,\theta} \), we also provide a similar coupling relation between \( X^{\infty,\theta} \) and \( Y^{\theta} \), where once again \( p_i \) and \( \theta_i \) are related by (44), for \( i \geq 3 \). We use the coupling relations to prove a central limit theorem for \( K_n \). We also see that when \( \theta_n \to \theta > 0 \), as \( n \to \infty \), the asymptotic behavior of the joint distribution of the normalized cycle lengths of \( X^{n,\theta} \), in order of their formation, can be studied through that of \( Y^{n,\theta} \).
To make this precise, suppose (44) holds for $\theta_i$ and $p_i$, for $3 \leq i \in \mathbb{N}$. For any $n, p \in \mathbb{N}$, and $y = (y_1, y_2, \cdots) \in \{1\} \times \{0, 1\}^N$, let

$$\beta_i^n(y) := \mathbb{1}\{y_i = 1\} \cdot \max\{0 \leq j \leq n - i - 1 : \prod_{k=0}^{j} y_{i+k} = 1\};$$

$$\beta_i^\infty(y) := \mathbb{1}\{y_i = 1\} \cdot \sup\{0 \leq j : \prod_{k=0}^{j} y_{i+k} = 1\}.$$  

In fact, $\beta_i^n(y)$ and $\beta_i^\infty(y)$ count the number of consecutive 1’s right after $y_i = 1$, in $(y_{i+1}, \cdots, y_{n-1})$ and $(y_{i+1}, y_{i+2}, \cdots)$, respectively. Note that $\beta_i^\infty(y) = \infty$ for some $i \in \mathbb{N}$, if and only if there exists a unique $r \leq i$ s.t. $(1 - y_r) \prod_{j=1}^{\infty} y_{r+j} = 1$. We now define the mappings $\chi_n$ and $\chi_\infty$ that erase the 11 patterns in $Y^n$ and $Y$, starting from the end of the sequence. More explicitly, for $y \in \{1\} \times \{0, 1\}^N$, and $n \in \mathbb{N}$, let $\Phi^1_i(y) = \Phi^\infty_i(y) = 1, \Phi^2_i(y) = \Phi^\infty_i(y) = 0$, for $j \geq n$, and let

$$\Phi^n_i := \mathbb{1}\{y_1 = 1, \beta_i^n(y) \text{ even}\}, \quad 3 \leq i \leq n - 1;$$

$$\Phi^\infty_i := \mathbb{1}\{y_1 = 1, \beta_i^\infty \text{ finite and even}\}, \quad i \geq 3.$$  

It is clear from the definition that $\Phi^n$ and $\Phi^\infty$ indeed remove consecutive 1’s (i.e. 11 patterns) from $y$, starting from the end. To make use of these for defining our mappings, for any $n \in \mathbb{N}$, let

$$\Delta_n := \left\{ y \in \{1\} \times \{0, 1\}^N, \ y_1 = 1, \ y_2 + \sum_{j=2}^{n-1} y_j y_{j+1} + \sum_{j=n}^{\infty} y_j = 0 \right\},$$

$$\Delta_\infty := \left\{ y \in \{1\} \times \{0, 1\}^N, \ y_1 = 1, \ y_2 + \sum_{j=2}^{\infty} y_j y_{j+1} = 0 \right\},$$

and define $\chi_n : \{1\} \times \{0, 1\}^N \to \Delta_n$ and $\chi_\infty : \{1\} \times \{0, 1\}^N \to \Delta_\infty$ by $\chi_n(y) = (\Phi^i_n(y))_{i=1}^{\infty}$ and $\chi_\infty(y) = (\Phi^i_\infty(y))_{i=1}^{\infty}$. For example, for $y = (11111001111000 \cdots)$, we have $\chi_n(y) = (10101010101000 \cdots)$ and $\chi_\infty(y) = (1010100101000 \cdots)$.  

Let $F^n_\theta(n) = F^n_\theta(Y^n)$ and $F^\infty_\theta = F^\infty_\theta(Y^n)$ be the image of $Y^n$ under these mappings. From the definition,

$$P(F^n_\theta(n) = 0 \mid F^i_{\theta+1}(n) = 0) = P_{\theta}(Y_i = 0 \mid \Phi^i_{\theta+1}(Y) = 0) = P_{\theta}(Y_i = 0) = p_i,$$

$$P(F^n_\theta(n) = 0 \mid F^{i+1}_\theta(n) = 1) = P_{\theta}(Y_i \in \{0, 1\} \mid \Phi^i_{\theta+1}(Y) = 1) = 1,$$

which implies $L(F^n_\theta(n), F^{i+1}_\theta(n), \cdots) = L(X^n, X^{i+1}, \cdots)$, where $X^n = 0$, for $j \geq n$. Hence, for any $n \in \mathbb{N}$ and $\theta$ and $p$ related by (44),

$$L(X^n, X^{i+1}, \cdots) = \chi_n * L(Y^n),$$

where the r.h.s. is the push-forward of the law of $Y^n$ under $\chi_n$. Likewise, from the definition $L(F^n_\theta(\infty), F^{i+1}_\theta(\infty), \cdots) = \chi_\infty * L(Y^n)$. It follows from Borel-Cantelli lemma that $P(F^n_\theta(\infty) = \infty$ for some $i \in \mathbb{N}) = 0$ under (31). So for any $m \in \mathbb{N}$, $(F^n_\theta(n), \cdots, F^m_\theta(n)) \to (F^n_\theta(\infty), \cdots, F^m_\theta(\infty))$
a.s., as \( n \to \infty \), concluding \( \mathcal{L}(X_{1_p}^n, X_{2_p}^n, \cdots) \Rightarrow \chi_\infty \ast \mathcal{L}(Y^\theta) \). Therefore, we have the following representation of the limit chain

\[
\mathcal{L}(X^{\infty,p}) = \chi_\infty \ast \mathcal{L}(Y^\theta).
\]

We can also see that, assuming (44), for any \( i \geq 3 \), we readily have

\[
d_{TV}(\mathcal{L}(X^{n,p}), \mathcal{L}(Y^{n,\theta})) \leq \mathbb{P}(\{F_1(n), \cdots, F_n(n) \neq (Y_1, \cdots, Y_n)\})
\leq \mathbb{P}(Y^\theta = 1) + 1 - \gamma_n(\theta),
\]

which leads to

\[
d_{TV}(\mathcal{L}(X^{\infty,p}), \mathcal{L}(Y^\theta)) \leq 1 - \gamma_\infty(\theta),
\]

if we additionally assume (35) and (36). In the rest of this paper, we assume that (44) holds and \( X^{n,p} \) and \( Y^{n,\theta} \) are coupled as explained above, i.e. we assume \( X^{n,p} = (F_n(n), \cdots, F_1(n)) \).

5.1 Central limit theorem for the number of cycles

The next theorem provides a central limit theorem for \( X^{n,p} \).

**Theorem 8.** Let \( \tilde{q}_n = \sum_{i=1}^n q_i \) and \( \bar{q}_n = \sum_{i=1}^n q_i^2 \), and suppose \( (\tilde{q}_n)^2/\bar{q}_n \to 0 \) as \( n \to \infty \). Then as \( n \to \infty \),

\[
\frac{K^p_n - \tilde{q}_n}{\sqrt{\bar{q}_n}} \Rightarrow \mathcal{N}
\]

where \( \mathcal{N} \sim \text{Normal}(0,1) \).

**Proof.** To prove the central limit theorem for \( K^p_n \), we consider \( \tilde{K}^\theta_n \) coupled with \( K^p_n \), where \( p \) and \( \theta \) are related by (44). Form the way we coupled \( K_n \) and \( \tilde{K}_n \), we conclude \( \tilde{K}^\theta_n - K^p_n \leq (\tilde{C}_1(n) + 1)/2 \). From [4], Theorem 1, we get

\[
d_{TV}(\mathcal{L}(\tilde{K}_n), Po(\tilde{q}_n)) \leq \bar{q}_n(1 - e^{-\tilde{q}_n})/\tilde{q}_n.
\]

Hence, from the assumption \( \tilde{q}_n/\bar{q}_n \to 0 \), as \( n \to \infty \), so \( d_{TV}(\mathcal{L}(\tilde{K}_n), Po(\tilde{q}_n)) \to 0 \) as \( n \to \infty \). Therefore, as \( n \to \infty \)

\[
\frac{\tilde{K}_n - \tilde{q}_n}{\sqrt{\bar{q}_n}} \Rightarrow \mathcal{N}.
\]

It now suffices to prove \( (\tilde{C}_1(n) - Y_n)/\sqrt{\tilde{q}_n} \to 0 \), in probability, as \( n \to \infty \). This implies \( (\tilde{K}_n - K_n)/\sqrt{\bar{q}_n} \to 0 \), in probability, as \( n \to \infty \), and hence

\[
\frac{K_n - \tilde{q}_n}{\sqrt{\bar{q}_n}} = \frac{\tilde{K}_n - \tilde{q}_n}{\sqrt{\tilde{q}_n}} - \frac{K_n - K_n}{\sqrt{\bar{q}_n}} \Rightarrow \mathcal{N}.
\]

To complete the proof, write \( \tilde{C}_1(n) - Y_n = \sum_{i=1}^{n-1} Y_i Y_{i+1}, \)

\[
\mathbb{E}_\theta[(\tilde{C}_1(n) - Y_n)^2] = n \sum_{i=1}^{n-1} q_i q_{i+1} + 2 \sum_{i=1}^{n-2} q_i q_{i+1} q_{i+2} + 2 \sum_{i,j} q_i q_{i+1} q_j q_{j+1},
\]

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where the last term of the r.h.s. is over all $i,j$ such that $2 \leq i + 1 < j \leq n - 1$. Now as $q_i^2 + q_{i+1}^2 \geq 2q_iq_{i+1}$, for $i \in \mathbb{N}$, we get
\[
\sum_{i=1}^{n-2} q_i q_{i+1} q_{i+2} \leq \sum_{i=1}^{n-1} q_i q_{i+1} \leq \tilde{q}_n,
\]
\[
\sum_{i,j} q_i q_{i+1} q_{j+1} \leq \left( \sum_{i=1}^{n-1} q_i q_{i+1} \right)^2 \leq (\tilde{q}_n)^2.
\]

Hence, for any $\varepsilon > 0$, as $n \to \infty$,
\[
\mathbb{P} \left( \tilde{C}_1(n) - Y_n \geq \varepsilon \sqrt{q_n} \right) \leq \frac{\mathbb{E}_{\theta}[(\tilde{C}_1(n) - Y_n)^2]}{\varepsilon^2 q_n} \leq \frac{3\tilde{q}_n + (\tilde{q}_n)^2}{\varepsilon^2 q_n} \to 0.
\]

Recall the definition of $\psi(p)$ from (5). The following is an immediate application of the last theorem.

**Corollary 4.** Suppose $np_n \to \theta > 0$. Then $(K_n - \tilde{q}_n)/\sqrt{q_n} \Rightarrow \mathcal{N}$, as $n \to \infty$. Furthermore if $|\psi(p)| < \infty$, then as $n \to \infty$
\[
\frac{K_n - \theta \log n}{\sqrt{\theta \log n}} \Rightarrow \mathcal{N}.
\] (45)

In particular, (45) holds if there exists $c > 0$ such that $|\theta_n - \theta| \asymp n^{-c}$.

### 5.2 Ordered and longest cycles

Let $\tilde{A}_j(n)$ be the length of the $j^{th}$ cycle in the natural order of the cycles determined by the GFC, with $\tilde{A}_j(n) = 0$ for $j > K_n$. It is easy to see that, for $r, a_1, \ldots, a_r \in \mathbb{N}$, with $a_1 + \cdots + a_r = m < n$, we have
\[
\mathbb{P}(\tilde{A}_1(n) = a_1, \ldots, \tilde{A}_r(n) = a_r, \tilde{K}_n > r) = \frac{n! \theta_{<n-m}>}{(n-m)! \theta_{<n>}} \frac{\theta_{n+1-a_1} \theta_{n+1-a_1-a_2} \cdots \theta_{n+1-m}}{n(n-1) \cdots (n-a_1 - \cdots - a_{r-1})}.
\] (46)

Assuming $\theta_n \to \theta$ as $n \to \infty$, we have $\theta_{<n-m>} \sim \frac{\theta_{<n>}}{\theta_{<n>}}$, and therefore it is straightforward from the corresponding result for the Feller Coupling that
\[
n^{-r} \mathbb{P}(\tilde{A}_1(n) = [nx_1], \tilde{A}_2(n) = [nx_2], \ldots, \tilde{A}_r(n) = [nx_r], \tilde{K}_n > r) \to \theta^r (1 - x_1 - \cdots - x_r)^{\theta - 1}
\]
\[
(1 - x_1)(1 - x_1 - x_2) \cdots (1 - x_1 - x_2 - \cdots - x_{r-1})
\] (47)

for a fixed $r$, and $x_1, x_2, \ldots, x_r > 0$ satisfying $x_1 + \cdots + x_r < 1$, which for $\theta_n \to \theta$ as $n \to \infty$, implies
\[
n^{-1}(\tilde{A}_1(n), \tilde{A}_2(n), \ldots) \Rightarrow (A_1, A_2, \ldots) \text{ as } n \to \infty,
\] (48)

where $A_1, A_2, \ldots$ is GEM($\theta$), with density given in [3], equation (5.28).

Now, let $A_j(n)$ be the length of the $j^{th}$ cycle in the natural order of the cycles determined by $X^np$, with $A_j(n) = 0$ for $j > K_n$. As in the GFC case, we can deduce the asymptotic behavior of $n^{-1}(A_1(n), A_2(n), \ldots)$, as stated in the following theorem.
Theorem 9. Suppose $\theta_n \to \theta$ as $n \to \infty$. Then
\[ n^{-1}(A_1(n), A_2(n), \ldots) \Rightarrow (A_1, A_2, \ldots) \text{ as } n \to \infty. \]

Proof. For a fixed $r \in \mathbb{N}$ as $n \to \infty$, we show
\[ n^r \mathbb{P}(A_1(n) = m_1, A_2(n) = m_2, A_r(n) = m_r, K_n > r) \]
\[ \to \frac{\theta^r(1 - x_1 - \cdots - x_r)^{\theta - 1}}{(1 - x_1)(1 - x_1 - x_2) \cdots (1 - x_1 - x_2 - \cdots - x_{r-1})} \]
for $a_i = \lfloor nx_i \rfloor, i = 1, \ldots, r$ satisfying $x_1, x_2, \ldots, x_r > 0, x_1 + \cdots + x_r < 1$.

We notice that the l.h.s. differs from the corresponding probability under the generalized Feller Coupling (l.h.s. in (47)) by the reciprocal of
\[ \frac{n - 1}{(1 + \theta_2)(n - 1 + \theta_n)} \prod_{j=1}^{r-1} \frac{n - 1 - a_1 - \cdots - a_j}{n - 1 - a_1 - \cdots - a_j + \theta_{n-a_1-\cdots-a_j}} \to 1 \]
as $n \to \infty$. The result immediately follows from the corresponding result for the Feller Coupling. □

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