Weighted symmetrization results for a problem with variable Robin parameter

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Abstract
By means of a suitable weighted rearrangement, we obtain various apriori bounds for the solutions to a Robin problem. Among other things, we derive a family of Faber-Krahn type inequalities.

Keywords Weighted symmetrization · Robin problem · Comparison results

Mathematics Subject Classification 35J25 · 35B45

1 Introduction
The last two decades have seen a growing interest in the study of the Robin-Laplacian (see, e.g., [6, 9–11, 15]). Recently, in [5], it has been introduced a method that allows to obtain Talenti-type results for this type of operator. The results in [5] are quite surprising since, as well known, the techniques introduced by Talenti in [23] are tailored for problems whose solutions have level sets that do not intersect the domain where the problem is defined, a phenomenon that typically occurs when Robin boundary conditions are imposed.

Such an analysis is pushed even further in [2], where the Robin parameter is allowed to be a function bounded from above and below by two positive constants. Let us emphasize that the methods used in [5] and [2] are based on the classical Schwarz symmetrization, where, as well known, the standard isoperimetric inequality plays a crucial role. For related results see also [12, 21] and the reference therein.

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Here we consider the following problem

\[
\begin{align*}
\left\{ \begin{array}{ll}
-\Delta u &= f(x) |x|^l \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} + \beta |x|^{l/2} u &= 0 \quad \text{on } \partial \Omega 
\end{array} \right.
\end{align*}
\]

(1.1)

where, here and throughout the paper, \(\Omega\) is a bounded Lipschitz domain of \(\mathbb{R}^2\) containing the origin, \(\beta > 0\), \(l \in (-2, 0]\), \(\nu\) denotes the outer unit normal to \(\partial \Omega\) and \(f(x)\) is a non-negative function in \(L^2(\Omega, |x|^l \, dx)\), see Sects. 2 and 4 for definitions and some properties of this weighted space. A weak solution to problem (1.1) is a function \(u \in H^1(\Omega)\) such that

\[
\int_{\Omega} \nabla u \nabla \psi \, dx + \beta \int_{\partial \Omega} u \psi |x|^l/2 \, dH^1 = \int_{\Omega} f \psi |x|^l \, dx \quad \forall \psi \in H^1(\Omega).
\]

(1.2)

Our main results are based on a family of isoperimetric inequalities where two different weights (which are powers of the distance from the origin) appear in the perimeter and in the area element.

We will denote by \(\Omega^\sharp\) the disk centered at the origin of radius \(r^\sharp\), where \(r^\sharp\) is uniquely defined by the following identity

\[
|\Omega|_l := \int_{\Omega} |x|^l \, dx = \int_{\Omega^\sharp} |x|^l \, dx.
\]

Let us introduce the so-called symmetrized problem

\[
\begin{align*}
\left\{ \begin{array}{ll}
-\Delta v &= f^\sharp(x) |x|^l \quad \text{in } \Omega^\sharp \\
\frac{\partial v}{\partial \nu} + \beta (r^\sharp)^{l/2} v &= 0 \quad \text{on } \partial \Omega^\sharp,
\end{array} \right.
\end{align*}
\]

(1.3)

where \(f^\sharp(x)\) is the unique radial and radially decreasing function such that

\[
|\{x \in \Omega : f(x) > t\}|_l = |\{x \in \Omega : f^\sharp(x) > t\}|_l \quad \text{for any } t \geq 0.
\]

Our main results are contained in the following three theorems.

**Theorem 1.1** Let \(u\) and \(v = v^\sharp\) be the solutions to problems (1.1) and (1.3), respectively. If \(f(x) \in L^2(\Omega; |x|^l \, dx)\) and \(f(x) \geq 0\) a.e. in \(\Omega\) then

\[
\|v\|_{L^1(\Omega^\sharp; |x|^l \, dx)} = \int_{\Omega^\sharp} |v(x)| |x|^l \, dx \geq \int_{\Omega} |u(x)| |x|^l \, dx = \|u\|_{L^1(\Omega; |x|^l \, dx)}
\]

(1.4)

and

\[
\|v\|_{L^2(\Omega^\sharp; |x|^l \, dx)} = \int_{\Omega^\sharp} v^2(x) |x|^l \, dx \geq \int_{\Omega} u^2(x) |x|^l \, dx = \|u\|_{L^2(\Omega; |x|^l \, dx)}.
\]

(1.5)

Theorem above can be significantly improved when the datum \(f\) is constant. More precisely the following holds true

**Theorem 1.2** If \(f(x) = 1\) a.e. in \(\Omega\) then

\[
u^\sharp(x) \leq v(x) \quad x \in \Omega^\sharp.
\]

(1.6)
Let $\lambda_{1,l}(\Omega)$ and $\lambda_{1,l}(\Omega^2)$ be the first eigenvalues of the problems

$$
\begin{cases}
-\Delta u = \lambda(\Omega) |x|^l u & \text{in } \Omega \\
\frac{\partial u}{\partial v} + \beta |x|^{l/2} u = 0 & \text{on } \partial\Omega
\end{cases}
$$

and

$$
\begin{cases}
-\Delta v = \lambda(\Omega^2) |x|^l v & \text{in } \Omega^2 \\
\frac{\partial u}{\partial v} + \beta (r^2)^{l/2} v = 0 & \text{on } \partial\Omega^2,
\end{cases}
$$

respectively. Then the following Faber–Krahn type inequality holds true.

**Theorem 1.3** It holds that

$$
\lambda_{1,l}(\Omega) \geq \lambda_{1,l}(\Omega^2).
$$

The paper is organized as follows. In Sect. 2 we give the basic definitions and results about the rearrangement with respect the measure $|x|^l \, dx$. We also recall the weighted isoperimetric inequality on which such a rearrangement relies. Sections 3 and 4 contain the proofs of Theorems 1.1 and 1.2, respectively. The last section is devoted to the proof of Theorem 1.3. There we include further comments on the relation between the spaces $H^1(\Omega)$ and $L^2(\Omega; |x|^l \, dx)$.

**Remark 1.1** Since $0 \not\in \partial\Omega$, our results still hold true if one assumes that the Robin parameter is a function $\beta(x) \in L^\infty(\partial\Omega)$ such that for some positive constant $C$ it holds

$$
\beta(x) > C \quad \text{on } \partial\Omega.
$$

Under this assumption the proofs are conceptually equivalent, they just turn out to be more cumbersome.

Let us now make a few comments on the present note.

To the best of our knowledge, neither Talenti-type comparison results nor the Faber-Krahn-type principle have been discovered yet for problems of the kind (1.1).

The weighted isoperimetric inequality we need requires the assumptions made on $l$ and $n$ ($l \in (-2, 0]$ and $n = 2$).

Finally, it is a quite delicate issue trying to relax the hypothesis $\beta > 0$. For instance, if $\beta = 0$ the operator becomes the Neumann Laplacian. Therefore, by the compatibility condition, one can no longer assume that the function $f$ is positive. Nevertheless, in this case, only some weaker versions of Talenti’s Theorem have been obtained (see [4, 7, 22] and the references therein).

The case $\beta < 0$ is certainly the more challenging. Even if $l = 0$, i.e. if the Robin parameter is a constant, the issue about the Faber-Krahn inequality’s validity has not yet been fully clarified (see [8, 17, 18] and the references therein).

### 2 Preliminary results

Let $\Omega$ be a Lebesgue measurable subset of $\mathbb{R}^2$ and let $l \in (-2, 0]$. Define

$$
|\Omega|_l = \int_\Omega |x|^l \, dx
$$

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and
\[
P_{\frac{1}{2}}(\Omega) = \begin{cases} 
\int_{\partial\Omega} |x|^\frac{1}{2} \, d\mathcal{H}^1 & \text{if } \Omega \text{ is } 1 - \text{rectifiable} \\
+\infty & \text{otherwise.}
\end{cases}
\]

In the sequel by \(D_\rho\) we will denote the disk centered at the origin of radius \(\rho\).

The following result is a particular case of a two-parameter family of isoperimetric inequalities (see, e.g., [3, 13, 16] and the references therein).

**Theorem 2.1** It holds that
\[
P_{\frac{1}{2}}(\Omega) \geq P_{\frac{1}{2}}(\Omega^\sharp),
\]
where \(\Omega^\sharp = D_{r^\sharp}\) with \(r^\sharp > 0\):
\[
|\Omega|_l = |\Omega^\sharp|_l.
\]

**Remark 2.1** Note that the isoperimetric inequality above can be written equivalently as follows
\[
P_{\frac{1}{2}}^2(\Omega) \geq 2\pi (l + 2) |\Omega|_l.
\]  
(2.1)

In fact an elementary computation shows that
\[
P_{\frac{1}{2}}(\Omega^\sharp) = 2\pi (r^\sharp)^{\frac{l+2}{l}}
\]
and
\[
|\Omega|_l = |\Omega^\sharp|_l = \int_{\Omega^\sharp} |x|^l \, dx = \frac{2\pi}{l+2} (r^\sharp)^{l+2}.
\]

The previous identities imply
\[
r^\sharp = \left[\frac{l+2}{2\pi} |\Omega|_l\right]^\frac{1}{l+2}.
\]

Finally we deduce that
\[
P_{\frac{1}{2}}(\Omega) \geq P_{\frac{1}{2}}(\Omega^\sharp) = 2\pi \left[\frac{l+2}{2\pi} |\Omega|_l\right]^\frac{1}{2}
\]
and, hence, the claim.

Starting from this isoperimetric inequality one can consider the corresponding weighted rearrangement of a function. For further reading on this topic the reader can consult, for instance, [19, 20, 24] and the references therein.

Let \(u : \Omega \to \mathbb{R}\) be a measurable function.

**Definition 2.1** The distribution function \(\mu : t \in [0, \infty) \to [0, \infty)\) of \(u\) is defined as
\[
\mu(t) = |\{x \in \Omega : |u(x)| > t\}|_l.
\]

**Definition 2.2** The decreasing rearrangement \(u^\ast : s \in [0, |\Omega|_l] \to [0, \infty)\) of \(u\) is defined as
\[
u^\ast (s) = \inf \{t \geq 0 : \mu(t) < s\}.
\]
Definition 2.3 The weighted Schwarz symmetrization \( u^\#(x) : x \in \Omega^2 \to [0, \infty] \) of \( u \) is defined as
\[
u^\#(x) = u^* \left( |D|x| l \right) = u^* \left( \frac{2\pi}{l+2} |x|^{l+2} \right).
\]
Equivalently one can say that \( u^\# \) is the unique radial and radially non-increasing function such that
\[
|\{ x \in \Omega : |u(x)| > t \}|_l = |\{ x \in \Omega : u^\#(x) > t \}|_l \text{ for any } t \geq 0.
\]

Definition 2.4 If \( p \in [1, +\infty) \), we will denote by \( L^p(\Omega, |x|^l dx) \) the space of all Lebesgue measurable real-valued functions \( u \) such that
\[
\|u\|_{L^p(\Omega, |x|^l dx)} = \left( \int_\Omega |u|^p |x|^l dx \right)^{1/p} < +\infty.
\]

Note that since, by construction, \( u, u^* \) and \( u^\# \) are equimeasurable we have that
\[
\int_\Omega |u|^p |x|^l dx = \int_{\Omega^2} (u^*)^p |x|^l dx = \int_0^{|\Omega|} (u^*)^p ds \text{ for any } p \geq 1.
\]
We will need in the sequel the following well-known result (see, e.g., [14, 19, 20]).

Proposition 2.1 Let \( u \in L^1(\Omega, |x|^l dx) \) be a non-negative function and let \( E \subseteq \Omega \) be a measurable set. Then we have
\[
\int_E u(x) |x|^l dx \leq \int_0^{\|E\|} u^*(s) ds.
\]

We end this section by recalling the following version of Gronwall’s Lemma

Lemma 2.1 Let \( \xi(\tau) \) be a continuously differentiable function, satisfying, for some constant \( C \geq 0 \) the following differential inequality
\[
\tau \xi'(\tau) \leq \xi(\tau) + C \text{ for all } \tau \geq \tau_0 > 0.
\]
Then
\[
\xi(\tau) \leq \tau \frac{\xi(\tau_0) + C}{\tau_0} - C \text{ for all } \tau \geq \tau_0
\]
and
\[
\xi'(\tau) \leq \frac{\xi(\tau_0) + C}{\tau_0} \text{ for all } \tau \geq \tau_0.
\]

3 The case \( f(x) \in L^2(\Omega, |x|^l dx) \)

Let \( u \) and \( v \) the solutions to problems (1.1) and (1.3), respectively. In the sequel the following notation will be in force.

For \( t \geq 0 \) we denote
\[
U_t = \{ x \in \Omega : u(x) > t \}, \quad \partial U_t^{\text{int}} = \partial U_t \cap \Omega, \quad \partial U_t^{\text{ext}} = \partial U_t \cap \partial \Omega,
\]
\[
\begin{align*}
\partial U_t^{\text{int}} &= \partial U_t \cap \Omega, \\
\partial U_t^{\text{ext}} &= \partial U_t \cap \partial \Omega,
\end{align*}
\]
and
\[ \mu(t) = |U_t| \text{ and } P_u(t) = P_{1/2}(U_t). \] (3.2)

Analogously if \( t \geq 0 \) we denote
\[ V_t = \{ x \in \Omega^2 : v(x) > t \}, \quad \phi(t) = |V_t| \text{ and } P_v(t) = P_{1/2}(V_t). \] (3.3)

The proof of our main theorems requires several auxiliary results, that may have an interest of their own.

**Lemma 3.1** The following inequalities hold true
\[ 0 \leq u_m \leq v_m, \] (3.4)
where
\[ u_m := \inf_{\Omega} u, \quad v_m := \min_{\Omega^2} v. \] (3.5)

**Proof** Using \( u^- = \max \{0, -u\} \) as test function in (1.2), we obtain
\[ 0 \geq -\int_{\Omega} |\nabla u^-|^2 \, dx - \int_{\partial \Omega} (u^-)^2 |x|^l/2 \, dH^1 = \int_{\Omega} (u^-) |x|^l f(x) \, dx. \]
Since \( f(x) \geq 0 \), we deduce that \( u^- = 0 \) a.e. in \( \Omega \), and the first inequality in (3.5) is verified.

We observe that the function \( v(x) \) is radial and, therefore, \( v = v_m \) on \( \partial \Omega^# \). From equations (1.1) and (1.3) we easily deduce that
\[ v_m P_{1/2}(\Omega^2) = \int_{\partial \Omega^2} v(x) |x|^l/2 \, dH^1 = -\frac{1}{\beta} \int_{\partial \Omega^2} \frac{\partial v}{\partial \nu} \, dH^1 = \frac{1}{\beta} \int_{\Omega} f \, |x|^l \, dx. \]
\[ \geq u_m P_{1/2}(\Omega) \geq u_m P_{1/2}(\Omega^#), \]
where in last inequality we have used the weighted isoperimetric inequality. The claim is hence proven. \( \Box \)

**Lemma 3.2** It holds
\[ \int_0^t \tau \left( \int_{\partial U_t^\prime} |x|^l/2 \, dH^1 \right) \, d\tau \leq \int_0^{\tilde{\Omega}_l} \frac{f^*(s) \, ds}{2\beta}. \]

**Proof** Fubini’s Theorem gives
\[
\int_0^\tau \left( \int_{\partial \Omega_t^{\text{ext}}} \frac{|x|^l/2}{|u(x)|} d\mathcal{H}^1 \right) d\tau \leq \int_0^\infty \tau \left( \int_{\partial \Omega_t^{\text{ext}}} \frac{|x|^l/2}{|u(x)|} d\mathcal{H}^1 \right) d\tau
\]

\[
= \int_{\partial \Omega} \left( \int_0^\infty \tau d\tau \right) \frac{|x|^l/2}{|u(x)|} d\mathcal{H}^1
\]

\[
= \frac{1}{2} \int_{\partial \Omega} u(x) |x|^l/2 d\mathcal{H}^1 = \frac{1}{2\beta} \int_{\Omega} f(x) |x| d\mathcal{H}^1 = \frac{\int_{\Omega_t^l} f^*(s) ds}{2\beta}.
\]

Now in order to render the notations less heavy we introduce two constants that will appear often in the following

\[
C(l) = 2\pi \left( l + 2 \right) \quad \text{and} \quad C(\Omega) = \frac{1}{2\beta^2} \left( \int_{\Omega_t^l} f^*(s) ds \right)^2.
\]  

(3.6)

**Lemma 3.3** For almost every \( t > 0 \) it holds that

\[
C(l)\mu(t) \leq \left( -\mu'(t) + \frac{1}{\beta} \int_{\partial \Omega_t^{\text{ext}}} \frac{|x|^l/2}{|u(x)|} d\mathcal{H}^1 \right) \left( \int_0^{\mu(t)} f^*(s) ds \right)
\]

(3.7)

and

\[
C(l)\phi(t) = \left( -\phi'(t) + \frac{1}{\beta} \int_{\partial \Omega_t^{\text{ext}}} \frac{|x|^l/2}{v(x)} d\mathcal{H}^1 \right) \left( \int_0^{\phi(t)} f^*(s) ds \right).
\]

(3.8)

**Proof** Let \( t, h > 0 \). Using the following test functions in (1.1)

\[
\varphi_h(x) = \begin{cases} 
0 & \text{if } 0 < u \leq t \\
h & \text{if } u > t + h \\
u - t & \text{if } t < u \leq t + h 
\end{cases}
\]

we obtain

\[
\int_{U_t \setminus U_{t+h}} |\nabla u|^2 \, dx + \beta h \int_{\partial U_t^{\text{ext}}} u |x|^{l/2} d\mathcal{H}^1 + \beta \int_{\partial U_t^{\text{int}} \setminus \partial U_t^{\text{ext}}} u(u - t) |x|^{l/2} d\mathcal{H}^1
\]

\[
= \int_{U_t \setminus U_{t+h}} f(u - t) |x|^l \, dx + h \int_{U_{t+h}} f |x|^l \, dx.
\]

Dividing by \( h \) and then letting \( h \) go to 0 in the previous equality and, finally, using coarea formula, we obtain

\[
\int_{\partial \Omega_t} g(x) \, d\mathcal{H}^1 = \int_{\partial U_t^{\text{int}}} |\nabla u| \, d\mathcal{H}^1 + \beta \int_{\partial U_t^{\text{ext}}} |u| |x|^{l/2} \, d\mathcal{H}^1 = \int_{U_t} f |x|^l \, dx
\]

where

\[
g(x) = \begin{cases} 
|\nabla u| & \text{if } x \in \partial U_t^{\text{int}} \\
\beta u |x|^{l/2} & \text{if } x \in \partial U_t^{\text{ext}}.
\end{cases}
\]

\(\Box\)
On the other hand we have
\[
P^2_u(t) \leq \left( \int_{\mathcal{U}_t^1} g(x) \, d\mathcal{H}^1 \right) \left( \int_{\mathcal{U}_t^1} \frac{|x|^l}{g(x)} \, d\mathcal{H}^1 \right),
\]
\[
= \left( \int_{\mathcal{U}_t^1} g(x) \, d\mathcal{H}^1 \right) \left( \int_{\mathcal{U}_t^1} \frac{|x|^l}{\nabla u} \, d\mathcal{H}^1 + \frac{1}{\beta} \int_{\mathcal{U}_t^{1/m}} \frac{|x|^{l/2}}{u(x)} \, d\mathcal{H}^1 \right)
\]
\[
\leq \left( -\mu'(t) + \frac{1}{\beta} \int_{\mathcal{U}_t^{1/m}} \frac{|x|^{l/2}}{u(x)} \, d\mathcal{H}^1 \right) \left( \int_0^t f^*(s) \, ds \right).
\]

Using the isoperimetric inequality (2.1) we get the claim (3.7).

Note that the distribution function \( \phi \) of \( v \) fulfills equality (3.8), in place of inequality, since, as it is straightforward to check, \( v \) is a radial and radially decreasing function. \( \square \)

Now we are in position to prove our first main result.

**Proof of Theorem 1.1** Multiplying both sides of (3.7) by \( t \) and then integrating over \((0, \tau)\), with \( \tau \geq v_m \), we get
\[
C(l) \int_0^\tau t \mu(t) \, dt 
\]
\[
\leq \int_0^\tau \left( -t \mu'(t) \int_0^{\mu(t)} f^*(s) \, ds \right) \, dt + \int_0^\tau \left( \frac{t}{\beta} \int_{\mathcal{U}_t^{1/m}} \frac{|x|^{l/2}}{u(x)} \, d\mathcal{H}^1 \left( \int_0^{\mu(t)} f^*(s) \, ds \right) \right) \, dt
\]
\[
\leq \int_0^\tau \left( -t \mu'(t) \int_0^{\mu(t)} f^*(s) \, ds \right) \, dt + \frac{1}{\beta} \left( \int_0^{\Omega_l} f^*(s) \, ds \right) \int_0^\tau \left( \int_{\mathcal{U}_t^{1/m}} \frac{|x|^{l/2}}{u(x)} \, d\mathcal{H}^1 \right) \, dt.
\]

Lemma 3.2 yields
\[
C(l) \int_0^\tau t \mu(t) \, dt \leq \int_0^\tau -t \left( \int_0^{\mu(t)} f^*(s) \, ds \right) \mu'(t) \, dt + \frac{1}{2\beta^2} \left( \int_0^{\Omega_l} f^*(s) \, ds \right)^2,
\]
or equivalently
\[
C(l) \int_0^\tau t \mu(t) \, dt \leq \int_0^\tau -t \left[ \left( \int_0^{\mu(t)} f^*(\sigma) \, d\sigma \right) \mu'(t) \right] \, dt + C(\Omega), \tag{3.9}
\]
where \( C(l) \) and \( C(\Omega) \) are the constants defined in (3.6). An integration by parts of the left-hand side of (3.9) gives
\[
C(l) \int_0^\tau t \mu(t) \, dt = C(l) \left[ \tau \int_0^\tau \mu(t) \, dt - \int_0^\tau \left( \int_0^{\mu(t)} \mu(\sigma) \, d\sigma \right) \, dt \right]. \tag{3.10}
\]

Setting
\[
F(\omega) := \int_0^\omega \left( \int_0^{\eta} f^*(s) \, ds \right) \, d\eta,
\]
an integration by parts of the right-hand side of (3.9) gives
\[
\int_0^\tau -t \left( \int_0^{\mu(t)} f^*(s) \, ds \right) \, d\mu(t) = -\tau F(\mu(\tau)) + \int_0^\tau F(\mu(s)) \, ds. \tag{3.11}
\]
From (3.10) and (3.11) we deduce that
\[
\tau F(\mu(\tau)) + C(l) \tau \int_0^\tau \mu(t)dt \leq \int_0^\tau F(\mu(s))ds + C(l) \int_0^\tau \left( \int_0^t \mu(\sigma)d\sigma \right) dt + C(\Omega).
\]  
(3.12)

Defining
\[
\xi_1(\tau) := \int_0^\tau F(\mu(s))ds + C(l) \int_0^\tau \left( \int_0^t \mu(\sigma)d\sigma \right) dt
\]
we can rewrite inequality (3.12) as follows
\[
\tau \xi_1'(\tau) \leq \xi_1(\tau) + C(\Omega).
\]  
(3.13)

Gronwall Lemma (2.5), with \(\tau_0 = v_m\), gives
\[
F(\mu(\tau)) + C(l) \int_0^\tau \mu(t)dt \leq \tilde{C}, \ \tau \geq v_m.
\]  
(3.13)

where
\[
\tilde{C} = \frac{\xi_1(v_m) + C(\Omega)}{v_m} = \int_0^{v_m} F(\mu(s))ds + C(l) \int_0^{v_m} \left( \int_0^t \mu(\sigma)d\sigma \right) dt + C(\Omega)
\]

While for \(\phi(t)\), the distribution function of \(v\), we have the equality sign
\[
F(\phi(\tau)) + C(l) \int_0^\tau \phi(t)dt = \tilde{C}.
\]  
(3.14)

Inequalities (3.13) and (3.14) clearly imply that
\[
F(\phi(\tau)) + C(l) \int_0^\tau \phi(t)dt \geq F(\mu(\tau)) + C(l) \int_0^\tau \mu(t)dt, \ \tau \geq v_m.
\]

Since
\[
\lim_{\tau \to +\infty} F(\phi(\tau)) = \lim_{\tau \to +\infty} F(\mu(\tau)) = F(0) = 0
\]
we get
\[
\|v\|_{L^1(\Omega;|x|')} = \int_0^\infty \phi(t)dt \geq \int_0^\infty \mu(t)dt = \|u\|_{L^1(\Omega;|x|')},
\]
i.e. our first claim, inequality (1.4).

Now we want to establish the same comparison between the weighted \(L^2\) -norms of \(u\) and \(v\), i.e. (1.5). If \(\tau\) goes to \(+\infty\) in (3.9) we obtain
\[
C(l) \int_0^\infty t\mu(t)dt \leq \int_0^\infty F(\mu(\sigma))d\sigma + C(\Omega),
\]
while for \(\phi\) it holds
\[
C(l) \int_0^\infty t\phi(t)dt = \int_0^\infty F(\phi(\sigma))d\sigma + C(\Omega).
\]
Therefore we get the claim once we show that
\[
\int_0^\infty F(\mu(\sigma))d\sigma \leq \int_0^\infty F(\phi(\sigma))d\sigma.
\]  
(3.15)
Multiplying both sides of (3.7) by \( \frac{t F(\mu(t))}{\mu(t)} \) and then integrating over \((0, \tau)\) we get

\[
A := C(l) \int_0^\tau t F(\mu(t)) d\tau \leq \int_0^\tau \left( -\mu'(t) + \frac{1}{\beta} \int_{\partial U_{\text{ext}}} \frac{|x|^{l/2}}{u(x)} dH^1 \right) \frac{t F(\mu(t))}{\mu(t)} \left( \int_0^{\mu(t)} f^*(s) ds \right) dt =: B
\]

(3.16)

Again an integration by parts gives

\[
A = C(l) \left[ t \int_0^\tau F(\mu(t)) d\tau \right]_0^\tau - C(l) \int_0^\tau \left( \int_0^{\mu(t)} F(\mu(\sigma)) d\sigma \right) d\tau = C(l) \tau \int_0^\tau F(\mu(\sigma)) d\sigma - C(l) \int_0^\tau \left( \int_0^{\mu(t)} F(\mu(\sigma)) d\sigma \right) d\tau.
\]

(3.17)

Now set

\[
H(\rho) := \int_0^\rho \frac{F(\sigma)}{\sigma} \int_0^\sigma f^*(s) ds
\]

and

\[
B := \int_0^\tau \left( -\mu'(t) \right) \frac{t F(\mu(t))}{\mu(t)} \left( \int_0^{\mu(t)} f^*(s) ds \right) dt + \int_0^\tau \left( \frac{1}{\beta} \int_{\partial U_{\text{ext}}} \frac{|x|^{l/2}}{u(x)} dH^1 \right) \frac{t F(\mu(t))}{\mu(t)} \left( \int_0^{\mu(t)} f^*(s) ds \right) dt.
\]

Furthermore define

\[
B_1 := \int_0^\tau t \left[ \frac{F(\mu(t))}{\mu(t)} \left( \int_0^{\mu(t)} f^*(s) ds \right) \left( -\mu'(t) \right) \right] dt = - \int_0^\tau t \left[ \frac{d}{dt} H(\mu(t)) \right] dt
\]

and

\[
B_2 := \int_0^\tau \left( \frac{1}{\beta} \int_{\partial U_{\text{ext}}} \frac{|x|^{l/2}}{u(x)} dH^1 \right) \frac{t F(\mu(t))}{\mu(t)} \left( \int_0^{\mu(t)} f^*(s) ds \right) dt,
\]

so that

\[
B = B_1 + B_2.
\]

Integrating by parts in \(B_1\) we obtain

\[
B_1 = - \left[ t H(\mu(t)) \right]^\tau_0 + \int_0^\tau H(\mu(t)) dt = -\tau H(\mu(\tau)) + \int_0^\tau H(\mu(t)) dt.
\]

(3.18)

Since, as it is easy to verify, \( \frac{F(\rho)}{\rho} \) is a nondecreasing function, using Lemma 3.2 we derive
\begin{align*}
B_2 & \leq \frac{1}{\beta} \frac{F(|\Omega|)}{|\Omega|} \int_0^\tau \left( t \int_{\partial U^{ext}} \frac{|x'|^2}{u(x)} dH^1 \right) \left( \int_0^\mu(t) f^*(s) ds \right) dt \\
& \leq \frac{1}{2\beta^2} \frac{F(|\Omega|)}{|\Omega|} \left( \int_0^{|\Omega|} f^*(s) ds \right)^2 =: \hat{C}. \quad (3.19)
\end{align*}

Collecting (3.16), (3.17), (3.18) and (3.19) we infer that
\begin{align*}
\tau \left[ C(l) \int_0^\tau F(\mu(\sigma)) d\sigma + H(\mu(\tau)) \right] \\
& \leq C(l) \int_0^\tau \left( \int_0^s F(\mu(\sigma)) d\sigma \right) + \int_0^\tau H(\mu(t)) dt + \hat{C}. \quad (3.20)
\end{align*}

Define
$$
\xi_2(\tau) = C(l) \int_0^\tau \left( \int_0^s F(\mu(\sigma)) d\sigma \right) ds + \int_0^\tau H(\mu(t)) dt,
$$
then (3.20) can be rewritten as follows
$$
\tau \xi_2'(\tau) \leq \xi_2(\tau) + \hat{C}, \quad \tau \geq v_m.
$$

At this point Gronwall’s Lemma (2.5) ensures that
$$
\xi_2'(\tau) \leq \frac{\xi_2(v_m) + \hat{C}}{v_m}, \quad \tau \geq v_m.
$$

Therefore
\begin{align*}
\xi_2'(\tau) & = C(l) \int_0^\tau F(\mu(\sigma)) d\sigma + H(\mu(\tau)) \\
& \leq \frac{1}{v_m} \left[ C(l) \int_0^{v_m} \left( \int_0^s F(\mu(\sigma)) d\sigma \right) ds + \int_0^{v_m} H(\mu(t)) dt + \hat{C} \right], \quad \tau \geq v_m.
\end{align*} \quad (3.21)

Since \( F, H \) are nondecreasing functions and \( \mu(\sigma) \leq |\Omega| \) we have
\begin{align*}
C(l) \int_0^\tau F(\mu(\sigma)) d\sigma + H(\mu(\tau)) \\
& \leq \frac{1}{v_m} \left[ C(l) \int_0^{v_m} F(|\Omega|) d\sigma \right] ds + \int_0^{v_m} H(|\Omega|) dt + \hat{C} \right], \quad \tau \geq v_m.
\end{align*}

The last inequality can be equivalently written as follows
\begin{align*}
C(l) \int_0^\tau F(\mu(\sigma)) d\sigma + H(\mu(\tau)) \\
& \leq C(l) F(|\Omega|) \frac{v_m}{2} + H(|\Omega|) + \frac{\hat{C}}{v_m} =: G(|\Omega|), \quad \tau \geq v_m, \quad (3.22)
\end{align*}

where
\begin{align*}
G(t) := C(l) F(t) \frac{v_m}{2} + H(t) + \frac{\hat{C}}{v_m}.
\end{align*}
It is easy to verify that
\[ C(l) \int_0^\tau F(\phi(\sigma))d\sigma + H(\phi(\tau)) = G(|\Omega|). \] (3.23)

Therefore from (3.22) and (3.23) we get
\[ C(l) \int_0^\tau F(\mu(\sigma))d\sigma + H(\mu(\tau)) \leq C(l) \int_0^\tau F(\phi(\sigma))d\sigma + H(\phi(\tau)) \]

which implies
\[ \lim_{\tau \to \infty} \left[ C(l) \int_0^\tau F(\mu(\sigma))d\sigma + H(\mu(\tau)) \right] \leq \lim_{\tau \to \infty} \left[ C(l) \int_0^\tau F(\phi(\sigma))d\sigma + H(\phi(\tau)) \right]. \]

Since
\[ \lim_{\tau \to \infty} H(\mu(\tau)) = \lim_{\tau \to \infty} H(\phi(\tau)) = 0 \]

we conclude that
\[ \int_0^\infty F(\mu(\sigma))d\sigma \leq \int_0^\infty F(\phi(\sigma))d\sigma, \]

i.e. our claim (3.15). \(\square\)

4 The case \(f(x) = 1\)

As already pointed out when \(f\) is constant the previous result can be considerable sharpened. In this short Section we provide the proof of Theorem (1.2). Here we will use the same notation of Sect. 3.

Let \(u\) and \(v\) the solutions of the problems (1.1) and (1.3) with \(f(x) = 1\) a.e. in \(\Omega\), that is
\[ \begin{cases} -\Delta u = |x|^l & \text{in } \Omega \\ \frac{\partial u}{\partial v} + \beta |x|^{l/2} u = 0 & \text{on } \partial \Omega \end{cases} \] (4.1)

and
\[ \begin{cases} -\Delta v = |x|^l & \text{in } \Omega^2 \\ \frac{\partial v}{\partial v} + \beta (r^2)^{l/2} v = 0 & \text{on } \partial \Omega^2. \end{cases} \] (4.2)

**Proof of Theorem 1.2** Note, firstly, that when \(f = 1\) inequality (3.7) and equality (3.8) become
\[ C(l) \leq -\mu'(t) + \frac{1}{\beta} \int_{\partial U_{\text{ext}}} |x|^{l/2} u(x) dH^1 \] (4.3)

and
\[ C(l) = -\phi'(t) + \frac{1}{\beta} \int_{\partial U_{\text{ext}}} |x|^{l/2} v(x) dH^1, \]
respectively. Multiplying (4.3) by \( t \) and then integrating over \((0, \tau)\), with \( \tau \geq v_m \), we get

\[
2\pi (l + 2) \frac{\tau^2}{2} \leq \int_0^\tau (-\mu'(\sigma)\sigma) \, d\sigma + \frac{1}{\beta} \int_0^\tau t \left( \int_{\partial U_{\text{ext}}}^{\text{ext}} \frac{|x|^{l/2}}{u(x)} \, d\mathcal{H}^1 \right) \, dt.
\]

The last inequality together with Lemma 3.2 with \( f = 1 \) yield

\[
2\pi (l + 2) \tau \leq \int_0^\tau (-\mu'(\sigma)\sigma) \, d\sigma + \frac{1}{2\beta} |\Omega|_l, \quad \tau \geq v_m. \tag{4.4}
\]

Again for the solution \( v \) of the symmetrized problem we have the equality sign

\[
2\pi (l + 2) \tau = \int_0^\tau (-\phi'(\sigma)\sigma) \, d\sigma + \frac{1}{2\beta} |\Omega|_l, \quad \tau \geq v_m. \tag{4.5}
\]

From (4.4) and (4.5) we immediately deduce that

\[
\int_0^\tau (-\phi'(\sigma)\sigma) \, d\sigma \leq \int_0^\tau (-\mu'(\sigma)\sigma) \, d\sigma, \quad \tau \geq v_m.
\]

An integration by parts ensures that

\[
\mu(\tau) \leq \phi(\tau), \quad \tau \geq v_m. \tag{4.6}
\]

Our claim holds true, since, clearly, \( \phi(\tau) = |\Omega|_l \geq \mu(\tau) \) for all \( \tau \in [0, v_m] \). \( \square \)

### 5 A Faber-Krahn inequality

Consider the functional

\[
F : u \in H^1(\Omega) \to \int_\Omega \frac{|\nabla u|^2 \, dx + \beta \int_{\partial \Omega} u^2 |x|^{l/2} \, d\mathcal{H}^1}{\int_\Omega u^2 |x|^l \, dx}. \tag{5.1}
\]

Firstly note that \( F \) is well defined on \( H^1(\Omega) \). Indeed, as well-known (see, e.g. [1]) \( L^q(\partial \Omega) \) is compactly embedded in \( H^1(\Omega) \) for any \( q \geq 1 \). Therefore, since \( 0 \in \Omega \), we have that \( \exists C = C(\Omega, l) : \)

\[
\int_{\partial \Omega} u^2 |x|^{l/2} \, d\mathcal{H}^1 \leq C(\Omega, l) \int_{\partial \Omega} u^2 \, d\mathcal{H}^1 \leq C(\Omega, l) \int_\Omega (u^2 + |\nabla u|^2) \, dx \quad \forall u \in H^1(\Omega).
\]

Finally let us show that there exists a constant \( C = C(\Omega, l) \) such that

\[
\int_\Omega u^2 |x|^l \, dx = \| u \|_{L^2(\Omega; |x|^l)}^2 \leq C \int_\Omega (u^2 + |\nabla u|^2) \, dx \quad \forall u \in H^1(\Omega).
\]

Since the embedding of \( H^1(\Omega) \) in \( L^p(\Omega) \) is compact for any \( p \geq 1 \) we have

\[
\int_\Omega u^2 |x|^l \, dx \leq \left( \int_E u_{\text{ext}}^2 \, dx \right)^{\frac{1}{p}} \left( \int_E \frac{1}{|x|^{|\frac{l}{p}|}} \, dx \right)^{\frac{1}{q}}, \tag{5.2}
\]

\( \square \)
where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( p, q > 1 \). Now choose \( q = \tilde{q} \in \left( 1, \frac{2}{|l|} \right) \), so that \( |l| q < 2 \), and \( p = \tilde{p} = \frac{\tilde{q}}{\tilde{q} - 1} \) in (5.2) we obtain

\[
\int_{\Omega} u^2 |x|^l \, dx \leq \left( \int_{\Omega} u^{2\tilde{p}} \, dx \right)^{\frac{1}{\tilde{p}}} \left( \int_{\Omega} \frac{1}{|x|^{l\tilde{q}}} \, dx \right)^{\frac{1}{\tilde{q}}} \leq C \left( \int_{\Omega} 1 \, dx \right)^{\frac{2}{\tilde{q}}} \left( \int_{\partial\Omega} \frac{1}{|x|^{l\tilde{q}}} \, d\mathcal{H}^1 \right)^{\frac{1}{\tilde{q}}} \leq C |\Omega|^{\frac{2}{\tilde{q}}} \left( \int_{\partial\Omega} \frac{1}{|x|^{l\tilde{q}}} \, d\mathcal{H}^1 \right)^{\frac{1}{\tilde{q}}} \left( \int_{\Omega} |x|^l \, dx \right)^{\frac{1}{\tilde{q}}} \leq C \left( \int_{\partial\Omega} \frac{1}{|x|^{l\tilde{q}}} \, d\mathcal{H}^1 \right)^{\frac{1}{\tilde{q}}} |\Omega|^{\frac{2}{\tilde{q}}}.
\]

Incidentally note that the arguments above also show that \( H^1(\Omega) \) is compactly embedded in \( L^2(\Omega; |x|^l \, dx) \).

Now we can consider the problem

\[
\inf_{w \in H^1(\Omega) \setminus \{0\}} F(u) = \inf_{w \in H^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla w|^2 \, dx + \beta \int_{\partial\Omega} w^2 |x|^l \, d\mathcal{H}^1}{\int_{\Omega} w^2 |x|^l \, dx}.
\]

We claim that the infimum above is attained. To this aim let \( \{u_n\}_{n \in \mathbb{N}} \) be a minimizing sequence. Without loss of generality we may assume that

\[
\int_{\Omega} u_n^2 |x|^l \, dx = 1 \quad \forall n \in \mathbb{N}. \tag{5.3}
\]

Clearly such a sequence is bounded in \( H^1(\Omega) \). Therefore, up to a subsequence, we have that there exists \( u \in H^1(\Omega) \) such that

\[
\begin{align*}
  \begin{cases}
    u_n \to u \text{ a.e. in } \Omega \\
    u_n \to u \text{ in } L^p(\Omega) \quad \forall \, p \geq 1 \\
    u_n \to u \text{ weakly in } H^1(\Omega)
  \end{cases}
  \tag{5.4}
\end{align*}
\]

with

\[
\int_{\partial\Omega} u_n^2 |x|^{l/2} \, d\mathcal{H}^1 \to \int_{\partial\Omega} u^2 |x|^{l/2} \, d\mathcal{H}^1
\]

Hence, using (5.3), (5.4) and the weak lower semicontinuity of the \( L^2 \)—norm of the gradient we obtain our claim

\[
\liminf_{n \to \infty} F(u_n) = F(u) = \min_{w \in H^1(\Omega) \setminus \{0\}} F(w).
\]

Let \( \lambda_{1,l}(\Omega) \) denote the first eigenvalue of the problem

\[
\begin{cases}
  -\Delta u = \lambda(\Omega) \ |x|^l \ u \quad \text{in } \Omega \\
  \frac{\partial u}{\partial v} + \beta |x|^{l/2} \ u = 0 \quad \text{on } \partial\Omega.
\end{cases} \tag{5.5}
\]
By the consideration above $\lambda_{1,l}(\Omega)$ has the following variational characterization

$$\lambda_{1,l}(\Omega) = \min_{w \in H^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla w|^2 \, dx + \beta \int_{\partial \Omega} w^2 |x|^{l/2} \, d\mathcal{H}^1}{\int_{\Omega} w^2 |x|^l \, dx}.$$ 

Now we are in position to prove the weighted Faber-Krahn inequality (1.9), see also [20] and [5].

**Proof of Theorem 1.3** Let $u_1$ be an eigenfunction corresponding to $\lambda_{1,l}(\Omega)$, that is

$$\begin{cases}
-\Delta u_1 = \lambda_{1,l}(\Omega) |x|^l u_1 & \text{in } \Omega \\
\frac{\partial u_1}{\partial \nu} + \beta |x|^{l/2} u_1 = 0 & \text{on } \partial\Omega.
\end{cases} \tag{5.6}$$

Denoting by $z$ the solution to

$$\begin{cases}
-\Delta z = \lambda_{1,l}(\Omega) |x|^l u_1^* & \text{in } \Omega^z \\
\frac{\partial z}{\partial \nu} + \beta (r^z)^{l/2} z = 0 & \text{on } \partial\Omega^z.
\end{cases} \tag{5.7}$$

Inequality (1.5) gives

$$\int_{\Omega} u_1^2 |x|^l \, dx = \int_{\Omega^z} \left( u_1^* \right)^2 |x|^l \, dx \leq \int_{\Omega^z} z^2 |x|^l \, dx,$$

which, together with the Cauchy–Schwarz inequality, implies

$$\int_{\Omega^z} u_1^* z \, dx \leq \left( \int_{\Omega^z} \left( u_1^* \right)^2 |x|^l \, dx \right)^{1/2} \left( \int_{\Omega^z} z^2 |x|^l \, dx \right)^{1/2} \leq \int_{\Omega^z} z^2 |x|^l \, dx. \tag{5.8}$$

Multiplying equation (5.7) by $z$, integrating and taking into account of (5.8) we finally obtain

$$\lambda_{1,l}(\Omega) \geq \frac{\int_{\Omega^z} |\nabla z|^2 \, dx + \beta \int_{\partial\Omega^z} z^2 |x|^{l/2} \, d\mathcal{H}^1}{\int_{\Omega^z} u_1^* z \, |x|^l \, dx} \geq \frac{\int_{\Omega^z} |\nabla z|^2 \, dx + \beta \int_{\partial\Omega^z} z^2 |x|^{l/2} \, d\mathcal{H}^1}{\int_{\Omega^z} z^2 |x|^l \, dx} \geq \lambda_{1,l}(\Omega^z). \qedhere$$

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