SEIDEL ELEMENTS AND POTENTIAL FUNCTIONS OF HOLOMORPHIC DISC COUNTING

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ABSTRACT. Let $M$ be a symplectic manifold equipped with a Hamiltonian circle action and let $L$ be an invariant Lagrangian submanifold of $M$. We study the problem of counting holomorphic disc sections of the trivial $M$-bundle over a disc with boundary in $L$ through degeneration. We obtain a conjectural relationship between the potential function of $L$ and the Seidel element associated to the circle action. When applied to a Lagrangian torus fibre of a semi-positive toric manifold, this degeneration argument reproduces a conjecture (now a theorem) of Chan-Lau-Leung-Tseng \cite{CLLT} relating certain correction terms appearing in the Seidel elements with the potential function.

1. INTRODUCTION

Let $M$ be a symplectic manifold with a Hamiltonian circle action. Seidel \cite{Seidel} constructed an invertible element of the quantum cohomology of $M$ by counting pseudo-holomorphic sections of the associated $M$-bundle $E$ over $S^2$:

$$E = (M \times S^3)/S^1$$

where $S^1$ acts by the diagonal action and $S^3 \to S^2$ is the Hopf fibration. Seidel elements have been used to detect essential loops in the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms. McDuff-Tolman \cite{McDuff-Tolman} used them to verify Batyrev’s presentation of quantum cohomology rings for toric varieties.

In the previous paper \cite{Gonzalez-Iritani}, we computed Seidel elements of semi-positive toric manifolds and found that they are closely related to Givental’s mirror transformation \cite{Givental}. Chan-Lau-Leung-Tseng \cite{CLLT} conjectured that certain correction terms appearing in our computation of Seidel elements determine the potential function of a Lagrangian torus fibre. The potential function here is given by counting holomorphic discs with boundary in a Lagrangian torus fibre and is thought of as a mirror of the toric variety. The conjecture was proved by themselves \cite{CLLT} in a recent preprint. In this paper, we propose an alternative approach which relates Seidel elements and potential functions via degeneration. Our method should apply to a general symplectic manifold $M$ with a Hamiltonian $S^1$-action and an invariant Lagrangian.

We assume that $M$ is a smooth projective variety, equipped with a $\mathbb{C}^\times$-action and an $S^1$-invariant Kähler form $\omega$. Let $L$ be an $S^1$-invariant Lagrangian submanifold of $M$. Let $M_1(\beta)$ denote the moduli space of genus-zero bordered stable holomorphic maps from $(\Sigma, \partial \Sigma)$ to $(M, L)$ with one boundary marking and representing $\beta \in H_2(M, L)$. By the fundamental work of Fukaya-Oh-Ohta-Ono \cite{FOOO}, $M_1(\beta)$ is compact and carries a Kuranishi structure with boundary and corner. Let $\beta$ be a class of Maslov index two. Under certain assumptions
(see §2,[1]), the virtual fundamental chain of $\mathcal{M}_1(\beta)$ is a cycle of dimension $\dim \mathbb{R} L$ and one can define the open Gromov-Witten invariant $n_\beta \in \mathbb{Q}$ by

$$\text{ev}_* [\mathcal{M}_1(\beta)]^{\text{vir}} = n_\beta [L]$$

where $\text{ev}: \mathcal{M}_1(\beta) \to L$ is the evaluation map. The potential function $W$ is

$$W = \sum_{\beta \in H_2(M,L) \mu(\beta) = 2} n_\beta z^\beta.$$ 

The idea of degeneration is that instead of counting discs in $(M, L)$, we consider the problem of counting disc sections of the trivial bundle $M \times \mathbb{D} \to \mathbb{D}$ with boundary in $L \times S^1$. Then we degenerate the target $M \times \mathbb{D}$ to the union $E \cup_M (M \times \mathbb{D})$. From this geometry we expect the following degeneration formula (see §3.3 for details):

$$\varphi_* \text{ev}_* [\mathcal{M}_1(\hat{\beta})]^{\text{vir}} = \sum_{(r(\hat{\beta}) - r(\beta)) \sigma + \hat{\alpha}} \text{ev}_* [\mathcal{M}_S(\sigma) \times_M \mathcal{M}_1^\text{rel}(\hat{\alpha})]^{\text{vir}}$$

if the both-hand sides carry virtual fundamental cycles, instead of chains. Here $\hat{\beta} \in H_2(M \times \mathbb{D}, L \times S^1)$ denotes a disc section class corresponding to $\beta \in H_2(M, L)$ and $\mathcal{M}_1(\hat{\beta})$ is the corresponding moduli space of disc sections. The summation in the right-hand side is taken over all possible decompositions $\sigma + \hat{\alpha}$ of the class $\hat{\beta}$ into a section class $\sigma$ of $E$ and another disc section class $\hat{\alpha}$ under the degeneration. Also $\mathcal{M}_S(\sigma)$ is a moduli space of holomorphic sections of $E$ in the class $\sigma$, which is relevant to the Seidel element. This formula relates disc counts of different boundary types; the boundary classes $\partial \alpha$ and $\partial \beta$ from the both-hand sides differ exactly by the $S^1$-action.

The degeneration formula predicts a relationship between the Seidel element of the $S^1$-action and the potential function $W$. We need the following conditions in order to extract meaningful information from the formula (1):

(i) $\mathcal{M}_1(\beta)$ is empty for all $\beta \in H_2(M, L)$ with $\mu(\beta) \leq 0$.

(ii) The maximal fixed component $F_{\text{max}} \subset M$ of the $\mathbb{C}^\times$-action (see §2.2) is of complex codimension one and the $\mathbb{C}^\times$-weight on the normal bundle is $-1$.

(iii) $c_1(M)$ is semi-positive.

(iv) $\text{ev}(\mathcal{M}_S(\sigma))$ is disjoint from $L$ for all $\sigma \in H_2^\text{sec}(E)$ such that $\langle c_1^\text{vert}(E), \sigma \rangle = -1$.

**Theorem 1.1** (Corollary §3.2). Assume that $M$ is simply-connected and $L$ is connected. Assume that the degeneration formula (1) holds (see Conjecture §3.7 for a precise formulation) and that the above conditions (i)–(iv) are satisfied. Then

$$z^{\alpha_0} = \langle \hat{S}^{(2)}, dW \rangle + \hat{S}^{(0)}$$

holds in a certain “open” Novikov ring $\Lambda^{\text{op}}$ (see §2.1), where

- $\alpha_0 \in H_2(M, L)$ is the maximal disc class defined by rotating a path connecting $L$ and $F_{\text{max}}$ by the $S^1$-action (see §1.2);
- $dW = \sum_{\mu(\beta) = 2} \beta \otimes n_\beta z^\beta$ is the logarithmic derivative of $W$;
- $\hat{S} = \hat{S}^{(0)} + \hat{S}^{(2)}$ is the Seidel element associated to the $S^1$-action and $\hat{S}^{(i)} \in H^i(M) \otimes \Lambda$ ($\Lambda$ is the “closed” Novikov ring in Remark §2.9);
- $\hat{S}^{(2)} \in H^2(M, L) \otimes \Lambda$ is a lift of $\hat{S}^{(2)}$ (see Definition §3.19).
In particular,

$$KS(\tilde{S}) = [z^{\alpha_0}]$$

holds in a certain Jacobi algebra of $W$, where $KS$ denotes the Kodaira-Spencer mapping (see the end of §3.3.3).

In the second half of the paper, we apply these to a semi-positive toric manifold $X$ and calculate the potential function of a Lagrangian torus fibre $L \subset X$. In toric case, the potential function can be regarded as a function on the moduli space $\mathcal{M}_{\text{opcl}}$ of Lagrangian torus fibres $L$ together with complexified Kähler classes $-\omega + iB$ and lifts $h \in H^2(X, L; U(1))$ of $\exp(iB)$ (see §4.2.1, $h$ defines a $U(1)$-local system on $L$ when $B = 0$). The potential function is of the form:

$$W = w_1 + \cdots + w_m$$

with $w_i = f_i(q)z_i$, where $f_i(q) \in \Lambda$ is the correction term defined by

$$f_i(q) = \sum_{d \in H_2(X; \mathbb{Z}) : c_1(X) \cdot d = 0} n_{\beta_i + d}q^d.$$

Each term $w_i$ corresponds to a prime toric divisor $D_i \subset X$ and arises from disc counting of fixed boundary type $b_i \in H_1(L)$. Applying the degeneration formula, we get:

**Theorem 1.2** (Theorem 4.13). Assume that the degeneration formula (1) (Conjecture 3.17) holds for $(X, L)$ equipped with the $\mathbb{C}^\times$-action $\rho_j$ rotating around the prime toric divisor $D_j$ (see §4.3). Let $\tilde{S}_j \in H^2(X) \otimes \Lambda$ be the Seidel element $\rho_j$ and let $\hat{S}_j \in H^2(X, L) \otimes \Lambda$ be its lift. Then we have

$$\langle \hat{S}_j, dw_k \rangle = \delta_{jk}z_j.$$

In particular we have $KS(\tilde{S}_j) = [z_j]$.

We observe in Theorem 4.14 that the degeneration formula reproduces the following conjecture (now a theorem) of Chan-Lau-Leung-Tseng [7, 8].

**Theorem 1.3** ([7, Conjecture 4.12], [8, Theorem 1.1]). Let $g_0^{(j)}(y)$, $j = 1, \ldots, m$ be explicit hypergeometric functions in variables $y_1, \ldots, y_r$ ($r = \dim H^2(X)$) given in equation (37). Then we have

$$f_j(q) = \exp \left( g_0^{(j)}(y) \right)$$

under an explicit change of variables (mirror transformation) of the form $\log q_i = \log y_i + g_i(y)$, $i = 1, \ldots, r$ with $g_i(y) \in \mathbb{Q}[y_1, \ldots, y_r]$ and $g_i(0) = 0$.

In [19], we introduced Batyrev elements $\tilde{D}_j$ as mirror analogues of the divisor classes $D_j$. They satisfy the relations of Batyrev’s quantum ring [4] for toric varieties. The hypergeometric functions $g_0^{(j)}(y)$ originally appeared in our computation [19] as the difference between the Seidel and the Batyrev elements:

$$\tilde{D}_j = \exp \left( g_0^{(j)}(y) \right) \tilde{S}_j.$$

Hence by Theorem 1.2, $\tilde{S}_j$ and $\tilde{D}_j$ correspond respectively to $[z_j]$ and $[w_j]$ under the Kodaira-Spencer mapping (see also [8, Theorem 1.5]).
Finally we discuss briefly the method of Chan-Lau-Leung-Tseng \cite{8}. Their approach is different from ours but is closely related to it. They observed that a holomorphic disc in $(X, L)$ whose boundary class is $b_j \in H_1(L)$ can be completed to a holomorphic sphere in the $M$-bundle $E'_j$ associated to the inverse $\mathbb{C}^\times$-action $\rho^{-1}_j$. Using this, they identified open Gromov-Witten invariants of $(X, L)$ with certain closed invariants of $E'_j$. The associated bundle $E'$ of the inverse action also appears in our story as the central fibre $E \cup M E'$ of the degeneration of the closed manifold $M \times \mathbb{P}^1$ (instead of $M \times \mathbb{D}$) in §3.1.

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2. Preliminaries

In this section, we review a potential function of a Lagrangian submanifold and a Seidel element associated to a Hamiltonian circle action.

2.1. Potential function of a Lagrangian submanifold. The potential of a Lagrangian submanifold arises as the 0-th operation $m_0$ of the corresponding $A_\infty$-algebra in Lagrangian Floer theory of Fukaya-Oh-Ohta-Ono \cite{12}. In this paper, we do not use the full generality of $A_\infty$-formalism developed in \cite{12}; instead we consider potential functions under certain restrictive assumptions.

Let $(M, \omega)$ be a closed symplectic manifold and $L$ be a Lagrangian submanifold. For simplicity, we restrict ourselves to the case where $M$ is a smooth projective variety. We assume that $L$ is oriented, relatively-spin and fix a relative spin structure \cite{12}, Definition 8.1.2] of $L$ so that the moduli space of bordered stable maps to $(M, L)$ has an oriented Kuranishi structure. Let $\mu : H_2(M, L) \rightarrow \mathbb{Z}$ denote the Maslov index. It takes values in $2\mathbb{Z}$ since $L$ is oriented.

Let $\mathcal{M}_1(\beta)$ denote the moduli space of stable holomorphic maps from a genus-zero bordered Riemann surface $(\Sigma, \partial \Sigma)$ to $(M, L)$ with one boundary marked point and in the class $\beta \in H_2(M, L)$. This was denoted by $\mathcal{M}_1^{\text{main}}(\beta)$ in \cite{12}. By \cite{12} Proposition 7.1.1] (see also \cite{15} Theorem 15.3)), $\mathcal{M}_1(\beta)$ is compact and equipped with an oriented Kuranishi structure (with boundary and corner) and has virtual dimension $n + \mu(\beta) - 2$, where $n = \dim_{\mathbb{R}} L$. Let $\text{ev} : \mathcal{M}_1(\beta) \rightarrow L$ denote the evaluation map. Define an open version of Novikov ring $\Lambda^{op}$ to be the space of all formal power series

$$\sum_{\beta \in H_2(M, L)} c_\beta z^\beta$$

with $c_\beta \in \mathbb{Q}$ such that

$$\# \left\{ \beta : c_\beta \neq 0, \int_\beta \omega < E \right\} < \infty$$

holds for all $E \in \mathbb{R}$.
Definition 2.1. Assume that $M_1(\beta)$ is empty for all $\beta \in H_2(M, L)$ with $\mu(\beta) \leq 0$. Then $M_1(\beta)$ with $\mu(\beta) = 2$ has no boundary and carries a virtual fundamental cycle of dimension $n = \dim_{\text{alg}} L$ [12, Lemma A.1.32]. We define open Gromov-Witten invariants $n_\beta \in \mathbb{Q}$ by

$$\text{ev}_* [M_1(\beta)]^\text{vir} = n_\beta [L]$$

for $\beta$ with $\mu(\beta) = 2$, where $[L] \in H_n(L)$ is the fundamental class of $L$. The potential function of $L$ is defined to be the formal sum:

$$W = \sum_{\beta \in H_2(M, L) : \mu(\beta) = 2} n_\beta z^\beta.$$

This is an element of $\Lambda^\text{op}$.

We can decompose $W$ according to boundary classes of discs.

Definition 2.2. Under the same assumption as in Definition 2.1, we write

$$W = \sum_{\gamma \in H_1(L)} W_\gamma$$

with $W_\gamma \in \Lambda^\text{op}$ given by

$$W_\gamma := \sum_{\beta \in H_2(M, L) : \mu(\beta) = 2, \partial \beta = \gamma} n_\beta z^\beta.$$

Remark 2.3. The potential function does depend on the choice of a complex structure on $M$ and this is a reason why we restricted to a smooth projective variety $M$. For example, the Hirzebruch surfaces $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $F_2$ together with their Lagrangian torus fibres are symplectomorphic to each other, but the potential functions are different. See Auroux [2] for a wall-crossing of disc counting.

2.2. Seidel elements. Seidel element is an invertible element of quantum cohomology associated to a loop in the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms of a symplectic manifold $(M, \omega)$. In this paper we restrict to the case where $M$ is a smooth projective variety equipped with an algebraic $\mathbb{C}^\times$-action. In this case, the associated $S^1$-action is Hamiltonian and yields a loop in $\text{Ham}(M, \omega)$. We refer the reader to [25, 22, 23] for the original definitions and to [24, 18] for applications in symplectic topology.

Let $M$ be a smooth projective variety, equipped with a $\mathbb{C}^\times$-action.

Definition 2.4. The associated bundle of the $\mathbb{C}^\times$-action on $M$ is the $M$-bundle over $\mathbb{P}^1$

$$E := M \times (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times \to \mathbb{P}^1,$$

where $\mathbb{C}^\times$ acts with the diagonal action $\lambda \cdot (x, (z_1, z_2)) = (\lambda x, (\lambda z_1, \lambda z_2))$.

Remark 2.5. In symplectic geometric terms, the associated bundle is in fact a clutching bundle obtained by gluing two trivial $M$-bundles over the unit disc, along the boundary, using the action. More precisely,

$$E = (M \times \mathbb{D}_0) \cup_g (M \times \mathbb{D}_\infty)$$

where $\mathbb{D}_0 = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\mathbb{D}_\infty = \{z \in \mathbb{C} : |z| \geq 1\} \cup \{\infty\}$ and the gluing map $g: M \times \partial \mathbb{D}_0 \to M \times \partial \mathbb{D}_\infty$ is given by

$$g(x, e^{i\theta}) = (e^{-i\theta} \cdot x, e^{i\theta}).$$
This construction can be generalized to a loop in the group of Hamiltonian diffeomorphisms and yields a Hamiltonian bundle \( E \to \mathbb{P}^1 \) in general. One can equip a symplectic form \( \omega_E \) on the total space \( E \) of the Hamiltonian bundle such that \( \omega_E \) restricts to the symplectic form \( \omega_M \) on each fibre \([25]\).

There exists a unique \( \mathbb{C}^\times \)-fixed component \( F_{\text{max}} \subset M^{\mathbb{C}^\times} \) such that the normal bundle of \( F_{\text{max}} \) has only negative \( \mathbb{C}^\times \)-weights. For a Hamiltonian function \( H \) generating the \( S^1 \)-action, \( F_{\text{max}} \) is the locus where \( H \) takes the maximum value. Each fixed point \( x \in M^{\mathbb{C}^\times} \) defines a section \( \sigma_x \) of \( E \). We denote by \( \sigma_0 \) the section associated to a fixed point in \( F_{\text{max}} \). We call it a maximal section. This defines a splitting\(^1\)

\[
H_2(E; \mathbb{Z}) \cong \mathbb{Z}[\sigma_0] \oplus H_2(M; \mathbb{Z}).
\]

Let \( \operatorname{NE}(M) \subset H_2(M, \mathbb{R}) \) denote the Mori cone, that is the cone generated by effective curves and set \( \operatorname{NE}(M)_Z := \{ d \in H_2(M; \mathbb{Z}) : d \in \operatorname{NE}(M) \} \). We introduce \( \operatorname{NE}(E) \) and \( \operatorname{NE}(E)_Z \) similarly.

**Lemma 2.6** ([19] Lemma 2.2). \( \operatorname{NE}(E)_Z = \mathbb{Z}_{\geq 0}[\sigma_0] + \operatorname{NE}(M)_Z \).

Let \( H^{\text{sec}}_2(E; \mathbb{Z}) \) denote the affine subspace of \( H_2(E; \mathbb{Z}) \) which consists of section classes, i.e. the classes that project to the positive generator of \( H_2(\mathbb{P}^1; \mathbb{Z}) \). We set \( \operatorname{NE}(E)_{Z}^{\text{sec}} := \operatorname{NE}(E)_Z \cap H^{\text{sec}}_2(E; \mathbb{Z}) \). The above lemma shows that

\[
\operatorname{NE}(E)_Z^{\text{sec}} = [\sigma_0] + \operatorname{NE}(M)_Z.
\]

For \( d \in \operatorname{NE}(X)_Z \) and \( \sigma \in \operatorname{NE}(E)_Z \), we denote by \( q^d \) and \( q^\sigma \) the corresponding elements in the group ring \( \mathbb{Q}[\operatorname{NE}(X)_Z] \) and \( \mathbb{Q}[\operatorname{NE}(E)_Z] \) respectively. We write:

\[
q^\sigma = q_0^k q^d \quad \text{when} \quad \sigma = k[\sigma_0] + d
\]

where \( q_0 = q^{\sigma_0} \) is the variable corresponding to the maximal section \( \sigma_0 \). For \( \sigma \in \operatorname{NE}(E)_Z^{\text{sec}} \), let \( \mathcal{M}_S(\sigma) \) denote the moduli space of stable maps from genus-zero closed nodal Riemann surfaces to \( E \) in the class \( \sigma \) with one marked point whose image lies in a fixed fibre \( M \subset E \). We can write

\[
\mathcal{M}_S(\sigma) = \mathcal{M}_1(\sigma) \times_E M
\]

using the usual moduli space \( \mathcal{M}_1(\sigma) \) of genus-zero one-pointed stable maps to \( E \) in the class \( \sigma \). Since \( \mathcal{M}_1(\sigma) \) has a Kuranishi structure (without boundary) of virtual real dimension \( 2n + 2 \langle c_1(E), \sigma \rangle - 2 \) (with \( n := \dim_{\mathbb{C}} M \)) and we may assume that the evaluation map \( \mathcal{M}_1(\sigma) \to E \) is weakly submersive, the fibre product \( \mathcal{M}_S(\sigma) \) is equipped with the induced Kuranishi structure of virtual dimension:

\[
\operatorname{vir. dim}_{\mathbb{R}} \mathcal{M}_S(\sigma) = 2n + 2 \langle c_1^{\text{vert}}(E), \sigma \rangle.
\]

Here \( c_1^{\text{vert}}(E) \) denotes the 1st Chern class of the vertical tangent bundle \( T_{\text{vert}}E \),

\[
T_{\text{vert}}E := \ker(d\pi: TE \to \pi^*T\mathbb{P}^1)
\]

with \( \pi: E \to \mathbb{P}^1 \) the natural projection. (Note that \( \langle c_1(E), \sigma \rangle = \langle c_1^{\text{vert}}(E), \sigma \rangle + 2 \).) Let \( \text{ev}: \mathcal{M}_S(\sigma) \to M \) be the evaluation map and let \( [\mathcal{M}_S(\sigma)]^{\text{vir}} \) be the virtual fundamental cycle of \( \mathcal{M}_S(\sigma) \).

\(^1\)The section \( \sigma_0 \) gives a splitting of the Serre spectral sequence. In general one has a non-canonical splitting \( H^*(E; \mathbb{Q}) \cong H^*(M; \mathbb{Q}) \otimes H^*(\mathbb{P}^1; \mathbb{Q}) \) for any Hamiltonian bundle \( E \to \mathbb{P}^1 \) \([23]\).
Definition 2.7. The Seidel element associated to the $\mathbb{C}^\times$-action on $M$ is the class

$$S := \sum_{\sigma \in \text{NE}(E)^{\text{sec}}} \text{PD} \left( \text{ev}_* [M_S(\sigma)]^\text{vir} \right) q^\sigma$$

in $H^*(M; \mathbb{Q}) \otimes \mathbb{Q}[\text{NE}(E)_Z]$. Here PD stands for the Poincaré duality isomorphism. By (3), we can factorize $S$ as $S = q_0 \tilde{S}$ with $\tilde{S}$ in the small quantum cohomology ring $QH(M) := H(X; \mathbb{Q}) \otimes \mathbb{Q}[\text{NE}(M)_Z]$ and $q_0 := q^{\sigma_0}$ as above. Then $\tilde{S}$ is an invertible element of $QH(M)[q^{-d}: d \in \text{NE}(M)_Z]$ with respect to the quantum product [25, 22, 23].

Remark 2.8. Using genus zero one-point Gromov-Witten invariants for $E$, we can write

$$S = \sum_{\sigma \in \text{NE}(E)^{\text{sec}}} \sum_i \langle \iota_* \phi_i \rangle^{E}_{0,1,\sigma} \phi_i^i q^\sigma$$

where $\{\phi_i\}$ is a basis of $H^*(M; \mathbb{Q})$, $\{\phi^i\}$ is the dual basis with respect to the Poincaré pairing and $\iota: M \to E$ is the inclusion of a fibre. (We followed the standard notation of Gromov-Witten invariants as in [10].)

Remark 2.9. For a general symplectic manifold $M$, we use the Novikov ring $\Lambda$

$$\Lambda := \left\{ \sum_{d \in H_2(M; \mathbb{Z})} c_d q^d : c_d \in \mathbb{Q}, \text{ if } \{d : c_d \neq 0, \langle \omega, d \rangle \leq E \} < \infty \text{ for all } E \in \mathbb{R} \right\}.$$ 

instead of $\mathbb{Q}[\text{NE}(M)_Z]$. The Seidel elements associated to loops in $\text{Ham}(M, \omega)$ define a group homomorphism [25, 22, 23]:

$$\pi_1(\text{Ham}(M, \omega)) \to QH(M)_{\Lambda} / \{ q^d : d \in H_2(M; \mathbb{Z}) \}$$

which is called the Seidel representation, where $QH(M)_{\Lambda} = H^*(M; \mathbb{Q}) \otimes \Lambda$ denotes the quantum cohomology ring over $\Lambda$.

3. Degeneration Formula

Let $M$ be a smooth projective variety equipped with a $\mathbb{C}^\times$-action. We take an $S^1$-invariant Kähler form $\omega$ on $M$. Let $L$ be a Lagrangian submanifold of $M$ which is preserved by $S^1 \subset \mathbb{C}^\times$, i.e. $\lambda L \subset L$ for $\lambda \in S^1$. Instead of counting holomorphic discs in $(M, L)$, we shall consider the problem of counting holomorphic disc sections of the bundle $M \times \mathbb{D} \to \mathbb{D}$ with boundary in $L \times S^1$. Then we degenerate the target $M \times \mathbb{D}$ into the union of the associated bundle $E$ and $M \times \mathbb{D}$. From this we expect a certain relationship between Seidel elements and disc counting invariants. We assume that $M$ is a smooth projective variety with a $\mathbb{C}^\times$-action for simplicity, but the degeneration formula in this section makes sense for a symplectic manifold with a Hamiltonian circle action (or a loop in the group of Hamiltonian diffeomorphisms) in general.
3.1. Degeneration of $M \times \mathbb{D}$. Let $\mathbb{D}$ denote the unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. A degeneration of the disc $\mathbb{D}$ into the union $\mathbb{D} \cup \mathbb{P}^1$ is given by the blowup $\text{Bl}_{(0,0)}(\mathbb{D} \times \mathbb{C})$ of $\mathbb{D} \times \mathbb{C}$ at the origin. The projection $\pi : \text{Bl}_{(0,0)}(\mathbb{D} \times \mathbb{C}) \to \mathbb{C}$ satisfies $\pi^{-1}(t) \cong \mathbb{D}$ for $t \neq 0$ and $\pi^{-1}(0) \cong \mathbb{D} \cup \mathbb{P}^1$. Explicitly:

\[
\text{Bl}_{(0,0)}(\mathbb{D} \times \mathbb{C}) = \left\{(z, t, [\alpha, \beta]) \in \mathbb{D} \times \mathbb{C} \times \mathbb{P}^1 : z\beta - t\alpha = 0\right\}.
\]

An $M$-bundle $\mathcal{E}$ over $\text{Bl}_{(0,0)}(\mathbb{D} \times \mathbb{C})$ is defined as follows.

\[
\mathcal{E} := \left\{(x, z, t, (\alpha, \beta)) \in M \times \mathbb{D} \times \mathbb{C} \times (\mathbb{C}^2 \setminus \{0\}) : z\beta - t\alpha = 0\right\} / \mathbb{C}^\times
\]

where $\mathbb{C}^\times$ acts as $(x, z, t, (\alpha, \beta)) \mapsto (\lambda x, z, t, (\lambda\alpha, \lambda\beta))$. We have a natural projection $\pi : \mathcal{E} \to \mathbb{C}$. One can see that

\[
\mathcal{E}_t = \pi^{-1}(t) = \begin{cases}
M \times \mathbb{D} & \text{if } t \neq 0; \\
E \cup_M (M \times \mathbb{D}) & \text{if } t = 0
\end{cases}
\]

where $E$ is the associated bundle (Definition 2.4) of the $\mathbb{C}^\times$-action on $M$. One can also construct $\mathcal{E}$ as a symplectic quotient:

\[
\mathcal{E} = \left\{(x, z, t, (\alpha, \beta)) : z\beta - t\alpha = 0, \ H(x) + |\alpha|^2 + |\beta|^2 = c\right\} / S^1
\]

where $H : M \to \mathbb{R}$ is the moment map of the $S^1$-action and $c > \max_{x \in M} H(x)$ is a real number. We can equip $\mathcal{E}$ with a symplectic structure. The boundary $\partial \mathcal{E}_t$ can be identified with $M \times S^1$ via the map:

\[
M \times S^1 \ni (x, z) \mapsto [x, z, t, (z, t)] \in \partial \mathcal{E}_t.
\]

Via this identification, $\mathcal{E}_t$ contains a Lagrangian submanifold $\widetilde{L}_t := L \times S^1$ in the boundary $M \times S^1 \cong \partial \mathcal{E}_t$.

We can also close $\mathcal{E}_t$ by attaching $M \times \mathbb{D}$ to the boundary for each $t$ and get a degenerating family $\mathcal{E}$ of closed manifolds. More explicitly, we define:

\[
\mathcal{E} = \left\{(x, z, w, t, (\alpha, \beta)) \in M \times (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C} \times (\mathbb{C}^2 \setminus \{0\}) : t\omega = z\beta\right\} / \mathbb{C}^\times \times \mathbb{C}^\times
\]

where $\mathbb{C}^\times \times \mathbb{C}^\times$ acts as

\[
(x, (z, w), t, (\alpha, \beta)) \mapsto (\lambda_1^{-1}\lambda_2 x, (\lambda_1 z, \lambda_1 w), t, (\lambda_2\alpha, \lambda_2\beta)).
\]

This is an $M$-bundle over

\[
\text{Bl}_{(0,0)}(\mathbb{P}^1 \times \mathbb{C}) = \left\{(z, w), t, (\alpha, \beta) \in \mathbb{P}^1 \times \mathbb{C} \times \mathbb{P}^1 : t\omega = z\beta\right\}.
\]

With respect to the projection $\pi : \mathcal{E} \to \mathbb{C}$ to the $t$-plane, we have

\[
\mathcal{E}_t = \pi^{-1}(t) = \begin{cases}
M \times \mathbb{P}^1 & \text{if } t \neq 0; \\
E \cup_M E' & \text{if } t = 0
\end{cases}
\]

where $E'$ is the associated bundle of the $\mathbb{C}^\times$-action on $M$ inverse to the original one. Note that $\mathcal{E}$ is contained in $\mathcal{E}$ as the locus $\{w = 1, |z| \leq 1\}$ and $\mathcal{E} = \mathcal{E} \cup M \times S^1 \times \mathbb{C} (M \times \mathbb{D}^2 \times \mathbb{C})$.

We can also equip $\mathcal{E}$ with a symplectic structure by describing it as a symplectic quotient in a similar manner.

A topological description is given as follows. We start from a trivial $M$-bundle $M \times \mathbb{P}^1$ over $\mathbb{P}^1$. We cut $\mathbb{P}^1$ into 3 pieces: $\mathbb{P}^1 = \mathbb{D}_0 \cup A \cup \mathbb{D}_{\infty}$, where $\mathbb{D}_0 = \{|z| \leq 1/2\}$, $A = \{1/2 \leq$
where the clutching functions

Remark 3.2. We shall consider stable holomorphic discs in $E \subset M \times A$ down to $M$, we get the singular central fibre $E \cup_M E'$. In fact, for $|t| < 1$, one can decompose $\mathcal{E}_t$ as

$$\mathcal{E}_t = \{[x, (tz, 1), t, (z, 1)] : |z| \leq 1\}$$

where we set $\partial A = \partial_0 A \cup \partial_\infty A$. Collapsing $M \times S^1 \subset M \times A$ lying in the boundary of $M \times \mathbb{D}_0 \cup_{g_1} M \times \mathbb{D}_\infty$. In the degeneration family, we have a family of Lagrangian submanifolds $L \times S^1$ lying in the boundary of $M \times \mathbb{D}_0 \cup_{g_1} M \times A$.

3.2. Relative homology classes of degenerating discs. We write $\mathcal{L} = \bigcup_{t \in \mathbb{C}} \hat{L}_t$. The total space $(\mathcal{E}, \mathcal{L})$ of the family has a deformation retraction to the central fibre $(\mathcal{E}_0, \hat{L}_0)$. This gives a retraction map for $t \neq 0$:

$$r : H_2(\mathcal{E}_t, \hat{L}_t) \to H_2(\mathcal{E}_0, \hat{L}_0).$$

Let $\pi : \mathcal{E} \to \text{Bl}_{(0,0)}(\mathbb{D} \times \mathbb{C})$ denote the natural projection. We have the following commutative diagram:

$$\begin{array}{ccc}
H_2(\mathcal{E}_t, \hat{L}_t) & \xrightarrow{r} & H_2(\mathcal{E}_0, \hat{L}_0) \\
\pi_* \downarrow & & \pi_* \\
H_2(\mathbb{D}, S^1) & \xrightarrow{r} & H_2(\mathbb{P}^1 \cup \mathbb{D}, S^1)
\end{array}$$

Under the natural identifications $H_2(\mathbb{D}, S^1; \mathbb{Z}) \cong \mathbb{Z}$ and $H_2(\mathbb{P}^1 \cup \mathbb{D}, S^1; \mathbb{Z}) \cong H_2(\mathbb{P}^1; \mathbb{Z}) \oplus H_2(\mathbb{D}, S^1; \mathbb{Z}) \cong \mathbb{Z}^2$, the bottom arrow is given by $n \mapsto (n, n)$. We are interested in section classes lying in the following groups:

$$H_2^\text{sec}(\mathcal{E}_t, \hat{L}_t) = \pi_*^{-1}(1), \text{ for } t \neq 0, \text{ and } H_2^\text{sec}(\mathcal{E}_0, \hat{L}_0) = \pi_*^{-1}(1, 1).$$
There is an induced retraction map $r : H_2^\text{sec} (\mathcal{E}_t, \hat{L}_t) \to H_2^\text{sec} (\mathcal{E}_0, \hat{L}_0)$ for $t \neq 0$.

**Lemma 3.3.** Assume that $M$ is simply connected and $L$ is connected. Then we have

$$
H_2^\text{sec} (\mathcal{E}_t, \hat{L}_t) \cong H_2 (M, L) \quad \text{for} \ t \neq 0
$$

(9)

$$
H_2^\text{sec} (\mathcal{E}_0, \hat{L}_0) \cong H_2^\text{sec} (E) \times_{H_2 (M)} H_2 (M, L)
$$

**Proof.** Recall that $(\mathcal{E}_t, \hat{L}_t) \cong (M \times \mathbb{D}, L \times S^1)$ for $t \neq 0$. We show the isomorphism:

$$(p_{1*}, p_{2*}) : H_2 (M \times \mathbb{D}, L \times S^1) \cong H_2 (M, L) \times H_2 (\mathbb{D}, S^1)$$

where $p_1, p_2$ are natural projections. Because we have sections $i_1, i_2 : (M, L) \to (M \times \mathbb{D}, L \times S^1)$ such that $p_1 \circ i_1 = \text{id}$, $p_2 \circ i_2 = \text{id}$, $p_2 \circ i_1 = \text{const}$ and $p_1 \circ i_2 = \text{const}$, the map $(p_{1*}, p_{2*})$ is surjective. To show that it is injective, we use the commutative diagram:

$$
\begin{array}{cccc}
0 & \longrightarrow & 0 & \longrightarrow & H_2 (\mathbb{D}, S^1) & \longrightarrow & H_1 (S^1) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_2 (L \times S^1) & \longrightarrow & H_2 (M \times \mathbb{D}) & \longrightarrow & H_2 (M \times \mathbb{D}, L \times S^1) & \longrightarrow & H_1 (L \times S^1) \\
\text{epi} & \downarrow & \cong & \downarrow & \text{p}_1 & \downarrow & \downarrow \\
H_2 (L) & \longrightarrow & H_2 (M) & \longrightarrow & H_2 (M, L) & \longrightarrow & H_1 (L).
\end{array}
$$

Here all the horizontal sequences are exact. The injectivity of $(p_{1*}, p_{2*})$ follows from the diagram chasing and $H_1 (L \times S^1) \cong H_1 (L) \oplus H_1 (S^1)$ (here we use the condition that $L$ is connected). Then $H_2^\text{sec} (\mathcal{E}_t, \hat{L}_t) \cong H_2 (M, L)$ for $t \neq 0$ follows.

The Mayer-Vietoris exact sequence for $\mathcal{E}_0 = E \cup_M (M \times \mathbb{D})$ gives

$$
H_2 (M) \longrightarrow H_2 (E) \oplus H_2 (M \times \mathbb{D}, L \times S^1) \longrightarrow H_2 (\mathcal{E}_0, \hat{L}_0) \longrightarrow 0
$$

Here we used $H_1 (M) = 0$. The formula for $H_2^\text{sec} (\mathcal{E}_0, \hat{L}_0)$ follows. \hfill $\square$

Henceforth we assume that $L$ is connected and $M$ is simply-connected.

**Remark 3.4.** The natural map $H_2 (\mathcal{E}_t, \hat{L}_t) \to H_2 (\mathcal{E}_0, \hat{L}_0)$ is injective because the composition:

$$
H_2 (M \times \mathbb{D}, L \times S^1) \to H_2 (M \times \mathbb{P}^1, L \times S^1) \overset{(p_{1*}, p_{2*})}{\longrightarrow} H_2 (M, L) \oplus H_2 (\mathbb{P}^1, S^1)
$$

is injective.

**Notation 3.5.** We denote by

$$
\hat{\beta} \in H_2^\text{sec} (\mathcal{E}_t, \hat{L}_t) \cong H_2^\text{sec} (M \times \mathbb{D}, L \times S^1) \quad (t \neq 0)
$$

$$
\sigma + \hat{\beta} \in H_2^\text{sec} (\mathcal{E}_0, \hat{L}_0)
$$

the homology classes corresponding to $\beta \in H_2 (M, L)$ and to $[\sigma, \beta] \in H_2^\text{sec} (E) \times_{H_2 (M)} H_2 (M, L)$ respectively, under the isomorphism (9) in Lemma 3.3.

Let $u : \mathbb{D} \to M$ be a disc such that $u(e^{i\theta}) = e^{i\theta} \cdot u(1)$, namely, $u$ is a disc contracting an $S^1$-orbit in $M$. This defines a section $\sigma (u)$ of the associated bundle $E \to \mathbb{P}^1$:

$$
\sigma (u)|_{\mathbb{D}_0} : \mathbb{D}_0 \to E|_{\mathbb{D}_0} \cong M \times \mathbb{D}_0, \quad z \mapsto (z, u(1))
$$

$$
\sigma (u)|_{\mathbb{D}_\infty} : \mathbb{D}_\infty \to E|_{\mathbb{D}_\infty} \cong M \times \mathbb{D}_\infty, \quad z \mapsto (z, u(z^{-1}))
$$
where \(D_0 = \{ z \in \mathbb{C} : |z| \leq 1 \} \) and \(D_\infty = \{ z \in \mathbb{C} : |z| \geq 1 \} \cup \{ \infty \} \); here we used the gluing construction of \(E \) in Remark 2.5.

Recall the maximal section class \(\sigma_0 \) of \(E \) in [2.2]. We introduce a similar maximal disc class \(\alpha_0 \in H_2(M, L)\) as follows. Take a path \(\gamma : [0, 1] \to M\) such that \(\gamma(0) \in F_{\max}\) and \(\gamma(1) \in L\), where \(F_{\max}\) is the maximal fixed component. We define \(\alpha_0\) to be the class represented by the disc \(D \ni re^{i\theta} \mapsto e^{-i\theta} \cdot \gamma(r) \in M\). The homotopy class here is independent of the choice of a path \(\gamma\) because \(M\) is simply-connected and \(L\) is connected. The boundary of \(\alpha_0\) is an inverse \(S^1\)-orbit on \(L\).

**Proposition 3.6.** The retraction map \(r : H_2^{sec}(\mathcal{E}_t, \widehat{L}_t) \to H_2^{sec}(\mathcal{E}_0, \widehat{L}_0)\) (for \(t \neq 0\)) of section classes is an isomorphism. It is given by (under Notation 3.5)

\[
r(\beta) = \sigma(u) - \hat{u} + \hat{\beta} = \sigma_0 + \hat{\alpha}_0 + \hat{\beta} \quad \text{for } \beta \in H_2(M, L)
\]

where \(u : D \to M\) is an arbitrary disc whose boundary is an \(S^1\)-orbit in \(L\) and \(\sigma_0, \alpha_0\) are the maximal classes. In particular we have the commutative diagram

\[
\begin{array}{ccc}
H_2^{sec}(\mathcal{E}_t, \widehat{L}_t) & \xrightarrow{\cong} & H_2^{sec}(\mathcal{E}_0, \widehat{L}_0) \\
\downarrow & & \downarrow \\
H_1(L) & \xrightarrow{\cong} & H_1(L)
\end{array}
\]

where the bottom map is the subtraction of the class \(\lambda = [\partial u]\) of an \(S^1\)-orbit on \(L\).

**Proof.** Consider a constant section \(s_{\text{triv}}(z) = (x, z)\) of \(M \times D \cong \mathcal{E}_t\) with \(x \in L\). By the topological description of the degeneration given in [3.1] we see that \(s_{\text{triv}}\) can degenerate to the union:

\[
\sigma(u) \cup \hat{u} : \mathbb{P}^1 \cup D \to E \cup_M (M \times D)
\]

where \(u : D \to M\) is a disc contracting the \(S^1\)-orbit \(e^{i\theta}x \) on \(L\) and \(\hat{u} : D \to M \times D\) is given by \(z \mapsto (u(z), z)\). This shows that \(r([s_{\text{triv}}]) = \sigma(u) - \hat{u}\). Since the retraction map is a homomorphism of \(H_2(M, L)\)-modules, we have \(r(\beta) = \sigma(u) - \hat{u} + \hat{\beta}\) in general. When \(u\) is a disc of the form: \(D \ni re^{i\theta} \mapsto e^{i\theta} \cdot \gamma(r) \in M\), where \(\gamma : [0, 1] \to M\) is a path such that \(\gamma(0) \in F_{\max}\) and \(\gamma(1) \in L\), \(\sigma(u)\) is homotopic to the maximal section \(\sigma_0\) and \([u] = -\alpha_0\). This shows the formula \(r(\hat{\beta}) = \sigma_0 + \hat{\alpha}_0 + \hat{\beta}\). It is easy to check that \(r\) is an isomorphism between section classes. \(\square\)

**Remark 3.7.** The latter statement is a consequence of the difference of trivializations of \(\partial \mathcal{E}_t\) (\(t \neq 0\)) and \(\partial \mathcal{E}_0\). Recall that we have a trivialization \(\partial \mathcal{E}_t \cong M \times S^1\) in [7] depending smoothly on \(t \in \mathbb{C}\). For \(t \neq 0\), this trivialization is induced from the isomorphism \(\mathcal{E}_t \cong M \times D\) in [6]; however for \(t = 0\), this trivialization differs by the \(S^1\)-action from the one induced by the isomorphism \(\mathcal{E}_0 \cong E \cup_M (M \times D)\) in [6].

**Lemma 3.8** (Maslov index and vertical Chern number). Let \(u : D \to M\) be a disc with boundary an \(S^1\)-orbit on \(L\), i.e. \(u(e^{i\theta}) = e^{i\theta} \cdot u(1)\) and \(u(1) \in L\). Then \(u\) defines a class in \(\pi_2(M, L)\) and we have \(\mu(u) = 2 \langle c_1^{vert}(E), [\sigma(u)] \rangle\).

**Proof.** We recall the definition of Maslov index of a disc \(u : (\mathbb{D}, S^1) \to (M, L)\). We set \(\gamma = u|_{\partial \mathbb{D}}\). Note that \(u^*TM|_{S^1}\) is a complexification of the subbundle \(\gamma^* TL\). Thus \(\det(u^*TM)|_{S^1}\)
is a complexification of the real line bundle $\det_{\mathbb{R}}(\gamma^*TL)$. On the other hand $\det_{\mathbb{R}}(\gamma^*TL)^{\otimes 2}$ has a canonical orientation. Take a positive (nowhere vanishing) section $s_0$ of $\det_{\mathbb{R}}(\gamma^*TL)^{\otimes 2}$. The Maslov index of $u$ is the signed count of zeros of a transverse section $s$ of $\det(u^*TM)^{\otimes 2}$ such that $s|_{\partial D} = s_0$.

When $u|_{\partial D}$ is an $S^1$-orbit of $L$, we can take $s_0$ above to be $S^1$-equivariant. A transverse section $s$ of $\det(u^*TM)^{\otimes 2}$ with $s|_{\partial D} = s_0$ defines a section $t \in \det(\sigma(u)^*T_{\text{vert}}E)^{\otimes 2}$ by

$$t|_{\partial D}(z) = s_0(1), \quad t|_{D_\infty}(z) = s(z^{-1}).$$

Then the numbers of zeros of $t$ and $s$ coincide. The lemma follows. \hfill \square

Proposition \ref{prop:degformula} and Lemma \ref{lem:degformula} show the following corollary:

**Corollary 3.9.** Let $r: H^\text{sec}_2(\mathcal{E}_t, \widehat{L}_t) \to H^\text{sec}_2(\mathcal{E}_0, \widehat{L}_0)$ be the retraction map for $t \neq 0$. Suppose that $r(\beta) = \sigma + \hat{\alpha}$ with $\alpha, \beta \in H_2(M, L)$. Then $\mu(\beta) = 2 \langle c_1^\text{vert}(E), \sigma \rangle + \mu(\alpha)$.

**Remark 3.10.** We have $\mu(\hat{\beta}) = \mu(\beta) + 2$ for $\beta \in H_2(M, L)$.

3.2.1. *Example.* We give an example of degenerating holomorphic discs. Consider a family of (constant) holomorphic disc sections $u_t: (\mathbb{D}, S^1) \to (\mathcal{E}_t, \widehat{L}_t)$ given by

$$u_t(z) = [x_0, z, t, (z, t)]$$

for some $x_0 \in L$. For a fixed non-zero $z \in \mathbb{D}$, we have

$$\varphi(z) := \lim_{t \to 0} u_t(z) = [x_0, z, 0, (z, 0)].$$

This can be completed to a holomorphic disc section $\varphi: \mathbb{D} \to M \times \mathbb{D} \subset \mathcal{E}_0$. Note that the limit

$$\lim_{z \to 0} \varphi(z) = [x_1, z, 0, (1, 0)] \quad \text{where} \quad x_1 := \lim_{z \to 0} z^{-1}x_0 \in M,$$

exists by the completeness of $M$. On the other hand, we can see a bubbling off holomorphic sphere at $z = 0$ by the usual rescaling:

$$\psi(z) := \lim_{t \to 0} u_t(tz) = \lim_{t \to 0} [t^{-1}x_0, tz, (z, 1)] = [x_1, 0, 0, (z, 1)].$$

This defines a holomorphic section $\psi: \mathbb{P}^1 \to E \subset \mathcal{E}_0$ associated to the $\mathbb{C}^$-fixed point $x_1 \in M$. Note that $\psi(\infty) = \varphi(0)$ and $\partial \varphi$ is an inverse $S^1$-orbit on $L$.

3.3. *Degeneration formula.* In what follows, we propose a conjectural degeneration formula and discuss its consequences. As before, $M$ denotes a smooth projective variety equipped with a $\mathbb{C}^*$-action and an $S^1$-invariant Kähler form $\omega$; $L$ is a Lagrangian submanifold which is preserved by the $S^1$-action. We assume that $M$ is simply-connected and $L$ is connected. Moreover we assume that $L$ is oriented and relatively spin and we fix a relative spin structure \cite[Definition 8.1.2]{12}.

Take $\beta \in H_2(M, L)$. We consider the moduli space $\mathcal{M}_1(\hat{\beta})$ of stable holomorphic maps from genus zero bordered Riemann surface $(\Sigma, \partial \Sigma)$ to $(\mathcal{E}_t, \widehat{L}_t) \cong (M \times \mathbb{P}^1, L \times S^1)$ with one boundary marked point and in the class $\hat{\beta} \in H^\text{sec}_2(\mathcal{E}_t, \widehat{L}_t)$ (where $t \neq 0$; see Notation \ref{not:degformula}). Such stable maps project onto the disc $(\mathbb{D}, S^1) \subset (\mathbb{P}^1, S^1)$ on the base and so are contained in $\mathcal{E}_t$ (see Remarks \ref{rem:degformula} and \ref{rem:degformula}). The virtual dimension of $\mathcal{M}_1(\hat{\beta})$ is $n + 1 + \mu(\hat{\beta}) - 2 = \text{the expression}$. \hfill \square
$n + 1 + \mu(\beta)$ with $n := \dim_{\mathbb{C}} M$. The corresponding moduli space at $t = 0$ should be described as the fibre product:

$$\bigcup_{r(\hat{\beta}) = \sigma + \hat{\alpha}} \mathcal{M}_S(\sigma) \times_M \mathcal{M}^{rel}_{1,1}(\hat{\alpha})$$

where $\mathcal{M}_S(\sigma)$ is the moduli space of holomorphic sections of $E$ appearing in §2.2 and $\mathcal{M}^{rel}_{1,1}(\hat{\alpha})$ is the moduli space of stable holomorphic maps from genus zero bordered Riemann surfaces to $(M \times \mathbb{P}^1, L \times S^1)$ in the class $\hat{\alpha} \in H^2_{sec}(M \times \mathbb{D}, L \times S^1)$ with one boundary marked point and one interior marked point such that the image of the interior marked point lies in $M \times \{0\}$. The superscript “rel” (which means “relative”) signifies the last condition. The fibre product above is taken with respect to the interior evaluation maps. One can write:

$$\mathcal{M}^{rel}_{1,1}(\hat{\alpha}) = \mathcal{M}_{1,1}(\hat{\alpha}) \times_{M \times \mathbb{P}^1} (M \times \{0\})$$

using the moduli space $\mathcal{M}_{1,1}(\hat{\alpha})$ of bordered stable maps to $(M \times \mathbb{P}^1, L \times S^1)$ of class $\hat{\alpha}$ with one boundary marking and one interior marking. Then a Kuranishi structure on $\mathcal{M}^{rel}_{1,1}(\hat{\alpha})$ is induced from the Kuranishi structure on $\mathcal{M}_{1,1}(\hat{\alpha})$ (as defined in [12 §7.1]) via this presentation. The virtual dimension is

$$\text{vir. dim } \mathcal{M}^{rel}_{1,1}(\hat{\alpha}) = n + 1 + \mu(\alpha).$$

We write $\text{ev}^{(i)}: \mathcal{M}^{rel}_{1,1}(\hat{\alpha}) \to M$ for the interior evaluation map and $\text{ev}^{(b)}: \mathcal{M}^{rel}_{1,1}(\hat{\alpha}) \to L \times S^1$ for the boundary evaluation map.

When the virtual fundamental chains on the moduli spaces $\mathcal{M}_{1}(\hat{\beta})$ and $\mathcal{M}_S(\sigma) \times_M \mathcal{M}^{rel}_{1,1}(\hat{\alpha})$ happen to be cycles, we expect the following degeneration formula:

$$\varphi_* \text{ev}^{(i)}_* [\mathcal{M}_{1}(\hat{\beta})]^{\text{vir}} = \sum_{r(\hat{\beta}) = \sigma + \hat{\alpha}} \text{ev}^{(b)}_* [\mathcal{M}_S(\sigma) \times_M \mathcal{M}^{rel}_{1,1}(\hat{\alpha})]^{\text{vir}}$$

in $H_*(L \times S^1)$. Here $\text{ev}$ on the both-hand sides denotes the evaluation map at the boundary markings taking values in $\hat{\mathcal{L}}_t \cong L \times S^1$ and $\varphi: L \times S^1 \to L \times S^1$ is the map $(x, e^{i\theta}) \mapsto (e^{-i\theta} \cdot x, e^{i\theta})$ which corresponds to the difference of boundary trivializations (see Remark 3.7). We will study below when the both-hand sides of (11) make sense as cycles; then will calculate them in terms of Seidel elements and open Gromov-Witten invariants.

### 3.3.1. The left-hand side of (11)

When $\beta = 0$, $\mathcal{M}_{1}(\hat{\beta})$ consists of constant disc sections and $\text{ev}: \mathcal{M}_{1}(\hat{\beta}) \to L \times S^1$ is a homeomorphism. All constant disc sections are Fredholm regular. When $\beta \neq 0$, we have a natural map

$$\mathcal{M}_{1}(\hat{\beta}) \to \mathcal{M}_{1}(\beta)$$

induced by the projection $\mathcal{E}_t \to M$, where $\mathcal{M}_{1}(\beta)$ is the moduli space of one-pointed bordered stable maps to $(M, L)$ in the class $\beta$. By taking the graph of a disc component, we can see that this map is surjective. Therefore, for $\beta \neq 0$, $\mathcal{M}_{1}(\hat{\beta})$ is non-empty if and only if $\mathcal{M}_{1}(\beta)$ is non-empty. Moreover, if $\mathcal{M}_{1}(\beta)$ is non-empty, $\mathcal{M}_{1}(\hat{\beta})$ has boundary since a bordered stable map of class $\hat{\beta}$ can be constructed as a union of a constant disc section and a disc of class $\beta$ (which is constant in the $\mathbb{D}$-direction). Therefore, we have
Lemma 3.11. The virtual cycle $\varphi_* [\mathcal{M}_1(\hat{\beta})]^{vir}$ is well-defined if $\mathcal{M}_1(\beta) = \emptyset$. We have

$$\varphi_* [\mathcal{M}_1(\hat{\beta})]^{vir} = \begin{cases} [L \times S^1] & \text{if } \beta = 0; \\ 0 & \text{if } \beta \neq 0 \text{ and } \mathcal{M}_1(\beta) = \emptyset. \end{cases}$$

3.3.2. The right-hand side of (11). Take $(\sigma, \alpha) \in H_2^{sec}(E) \times H_2(M, L)$ such that $\sigma + \hat{\alpha} = r(\hat{\beta})$. By Corollary 3.9 and Proposition 3.6, we have

$$(12) \quad \mu(\beta) = 2 \langle c_1^{vert}(E), \sigma \rangle + \mu(\alpha)$$

$$(13) \quad \partial \beta = \partial \alpha + \lambda$$

where $\lambda \in H_1(L)$ is the class of an $S^1$-orbit.

Suppose $\alpha = 0$. This can happen only when $\partial \beta = \lambda$ by (13). Since $\alpha = 0$, $\mathcal{M}_{1,1}^{rel}(\hat{\alpha})$ consists of constant disc sections and $ev^{(b)}: \mathcal{M}_{1,1}^{rel}(\hat{\alpha}) \cong L \times S^1$. The interior evaluation $ev^{(i)}: \mathcal{M}_{1,1}^{rel}(\alpha) \to M$ is given by the projection $L \times S^1 \to L \subset M$. Thus

$$(14) \quad ev_* [\mathcal{M}_S(\sigma) \times_M \mathcal{M}_{1,1}^{rel}(\hat{\alpha})]^{vir} = ev_* [\mathcal{M}_S(\sigma) \times_M (L \times S^1)]^{vir}$$

$$= (S_\sigma \cap [L]) \times [S^1]$$

where

$$(15) \quad S_\sigma := PD \left( ev_* [\mathcal{M}_S(\sigma)]^{vir} \right) \in H^{-\mu(\beta)}(M).$$

Here we used the virtual dimension formula (4) and (12).

Suppose $\alpha \neq 0$. By the same argument as in 3.3.1, $\mathcal{M}_{1,1}^{rel}(\hat{\alpha})$ is non-empty if and only if $\mathcal{M}_1(\alpha)$ is non-empty; also $\mathcal{M}_{1,1}^{rel}(\hat{\alpha})$ has boundary if $\mathcal{M}_1(\alpha)$ is non-empty. Assume that $\mathcal{M}_1(\alpha)$ has no boundary. This means that every stable map in $\mathcal{M}_1(\alpha)$ has only one disc component \(\hat{\alpha}\) (but possibly with sphere bubbles). Let us study the moduli space $\mathcal{M}_{1,1}^{rel}(\hat{\alpha})$ and its boundary. Since $\alpha \neq 0$, we have a map

$$f = (f_1, f_2): \mathcal{M}_{1,1}^{rel}(\hat{\alpha}) \to \mathcal{M}_1(\alpha) \times S^1.$$ \hfill (16)

The first factor $f_1$ is given by projecting bordered stable maps to $M$, forgetting the interior marking and collapsing unstable components; the second factor $f_2$ is the boundary evaluation $ev^{(b)}: \mathcal{M}_{1,1}^{rel}(\hat{\alpha}) \to L \times S^1$ followed by the projection $L \times S^1 \to S^1$. The map $f$ can be viewed as a tautological family of stable discs over $\mathcal{M}_1(\alpha) \times S^1$. In fact we have the following result.

Lemma 3.12. Let $u: (\Sigma, \partial \Sigma) \to (M, L)$ be a one-pointed bordered stable map of class $\alpha$ and $x \in \partial \Sigma$ be the boundary marking. Suppose that $\Sigma$ has only one disc component. Then the fibre $f^{-1}([u, \Sigma, x], z)$ at $[u, \Sigma, x], z) \in \mathcal{M}_1(\alpha) \times S^1$ can be identified with the oriented real blow-up $\hat{\Sigma}$ of $\Sigma$ at $x$ (see the proof below for the definition of $\hat{\Sigma}$) and the interior evaluation $ev^{(i)}$ on $f^{-1}([u, \Sigma, x], z)$ can be identified with the map $\hat{\Sigma} \to \Sigma \stackrel{u}{\to} M$.

Proof. The assumption that $\Sigma$ has only one disc component was made for simplicity’s sake (and this is the case we are interested in). In general, the fibre of $f$ can be identified with a smoothing of $\hat{\Sigma}$ at the boundary singularities. See [12, Lemma 7.1.45] for a similar statement.

\textsuperscript{2}See [12] §7.1.1 for the boundary description of the moduli spaces.
We identify a neighbourhood of $x \in \Sigma$ with the upper-half disc $D_+ = \{ w \in \mathbb{D} : \text{Im}(w) \geq 0 \}$ where $x$ corresponds to $0 \in D_+$. The oriented real blow-up $\hat{\Sigma}$ is defined by replacing this neighbourhood with $[0, \pi] \times [0, 1]$: 

$$\hat{\Sigma} = (\Sigma \setminus \{ x \}) \cup_{D_+ \setminus \{ 0 \}} ([0, \pi] \times [0, 1])$$

where $D_+ \setminus \{ 0 \}$ is identified with $[0, \pi] \times (0, 1]$ by the map $w \mapsto (\text{arg}(w), |w|)$. Note that $\hat{\Sigma}$ is a real analytic manifold (with boundary and corner) equipped with a natural projection $\hat{\Sigma} \to \Sigma$.

For a point $p \in \hat{\Sigma}$, we shall construct a bordered stable map in the fibre $\mathfrak{f}^{-1}((u, \Sigma, x), z)$. Suppose $p \in \hat{\Sigma} \setminus \partial \hat{\Sigma} \cong \Sigma \setminus \partial \Sigma$. Note that $\Sigma$ is a union of one disc component $\Sigma_0$ and trees of sphere bubbles. If $p$ is in a tree of spheres bubbles, let $q$ be the intersection point of the tree (on which $p$ lies) and the disc $\Sigma_0$. If $p$ is in the interior of $\Sigma_0$, set $q := p$. Take a unique holomorphic map $v : \Sigma_0 \to \mathbb{D}$ which sends $q$ to $0 \in \mathbb{D}$ and $x \in \partial \Sigma_0$ to $z \in S^1$. Extend $v$ to the whole $\Sigma$ so that it is constant on each sphere component. Then we obtain a bordered stable map $\hat{u} = (u, v) : \Sigma \to M \times \mathbb{D}$ of class $\alpha$ with $p$ a new interior marked point. (If $p$ is a node, we insert at the node a trivial sphere with an interior marking.)

Next consider the case $p \in \partial \hat{\Sigma}$. In this case, the corresponding bordered stable map is in the boundary of $\mathcal{M}_{1,1}^{\text{rel}}(\hat{\alpha})$. See Figure 1 below. If $p$ is not in the exceptional locus $[0, \pi]$ of $\hat{\Sigma} \to \Sigma$, we attach a disc $\mathbb{D}$ to $\Sigma$ by identifying $1 \in \partial \mathbb{D}$ with $p \in \partial \Sigma$ and define a map $\hat{u} : \mathbb{D} \cup_p \Sigma \to M \times \mathbb{D}$ by 

$$\hat{u}|_{\mathbb{D}}(w) = (u(p), zw), \quad \hat{u}|_{\Sigma}(y) = (u(y), z).$$

A new interior marking is taken to be $0 \in \mathbb{D}$. If $p$ corresponds to an interior point $\theta \in (0, \pi)$ of the exceptional locus $[0, \pi]$ of $\hat{\Sigma} \to \Sigma$, we attach a disc $\mathbb{D}$ to $\Sigma$ by identifying $1 \in \partial \mathbb{D}$ with $x \in \partial \Sigma$ and define a map $\hat{u} : \mathbb{D} \cup_1 \Sigma \to M \times \mathbb{D}$ by 

$$\hat{u}|_{\mathbb{D}}(w) = (u(x), e^{-2i\theta}z w), \quad \hat{u}|_{\Sigma}(y) = (u(y), e^{-2i\theta}z).$$

We put a new boundary marking at $e^{2i\theta} \in \mathbb{D}$ and a new interior marking at $0 \in \mathbb{D}$. If $p$ is a boundary point of the exceptional locus $[0, \pi]$, say, $0 \in [0, \pi]$, we consider the domain $\mathbb{D}^{(1)} \cup_{-1} \mathbb{D}^{(2)} \cup_{\Sigma} \Sigma$ (subscripts signify how to identify boundary points) with a boundary marking $i \in \mathbb{D}^{(2)}$ and an interior marking $0 \in \mathbb{D}^{(1)}$ and define a map $\hat{u} : \mathbb{D}^{(1)} \cup \mathbb{D}^{(2)} \cup \Sigma \to M \times \mathbb{D}$ by:

$$\hat{u}|_{\mathbb{D}^{(1)}}(w) = (u(x), zw), \quad \hat{u}|_{\mathbb{D}^{(2)}}(w) = (u(x), z), \quad \hat{u}|_{\Sigma}(y) = (u(y), z).$$

When $p$ corresponds to $\pi \in [0, \pi]$, we take $-i \in \mathbb{D}^{(2)}$ in place of $i \in \mathbb{D}^{(2)}$ as a boundary marking. One can see that the above construction defines a homeomorphism $\hat{\Sigma} \cong \mathfrak{f}^{-1}([u, \Sigma, x], z)$. The latter statement is obvious.

From the previous lemma and its proof, we have:

**Corollary 3.13.** Suppose $\partial \mathcal{M}_1(\alpha) = \emptyset$. The boundary $\partial \mathcal{M}_{1,1}^{\text{rel}}(\hat{\alpha})$ maps to $L$ under the interior evaluation map $\text{ev}^{(1)} : \mathcal{M}_{1,1}^{\text{rel}}(\hat{\alpha}) \to M$.

**Corollary 3.14.** Suppose $\partial \mathcal{M}_1(\alpha) = \emptyset$ and $\text{ev}((\mathcal{M}_S(\sigma)) \cap L = \emptyset$. Then the fibre product $\mathcal{M}_S(\sigma) \times_M \mathcal{M}_{1,1}^{\text{rel}}(\hat{\alpha})$ has no boundary. In particular, the virtual fundamental cycle $\text{ev}_*[\mathcal{M}_S(\sigma) \times_M \mathcal{M}_{1,1}^{\text{rel}}(\hat{\alpha})]^{\text{vir}}$ is well-defined (see \cite{12} Lemma A.1.32]).
We proceed to calculate the cycle $ev_*[\mathcal{M}_S(\sigma) \times_M \mathcal{M}_{1,1}^\text{rel}(\hat{\alpha})]^{\text{vir}}$ under the assumption of Corollary 3.14. By Corollary 3.13 taking a sufficiently small perturbation, we get a virtual fundamental chain

$$\mathcal{P}_\alpha := (ev^{(i)} \times ev^{(b)})_*[\mathcal{M}_{1,1}^\text{rel}(\hat{\alpha})]^{\text{vir}}$$

whose boundary lies in $\nu(L) \times L \times S^1$, where $\nu(L) \subset M$ is an arbitrarily small tubular neighbourhood of $L$. In other words, $\mathcal{P}_\alpha$ defines a relative homology class of the pair $(M \times L \times S^1, \nu(L) \times L \times S^1)$ whose dimension is $n + 1 + \mu(\alpha)$ (where $n = \dim_M M$). On the other hand, since $ev(\mathcal{M}_S(\sigma)) \cap L = \emptyset$, taking a sufficiently small perturbation again, we obtain a virtual cycle $ev_*[\mathcal{M}_S(\sigma)]^{\text{vir}}$ in $M \setminus \nu(L)$. By Poincaré-Lefschetz duality this defines a class

$$(17) \quad \hat{S}_\sigma := \text{PD} (ev_*[\mathcal{M}_S(\sigma)]^{\text{vir}}) \in H^{\mu(\alpha)-\mu(\beta)}(M, \nu(L)).$$

Here we used the virtual dimension formula (4) and (12). (Note that we put “hat” to distinguish $\hat{S}_\sigma$ from the element $S_\sigma \in H^*(M)$ appearing in (15).) The virtual cycle of the fibre product can be evaluated as the pairing of the two classes:

$$(18) \quad ev_*[\mathcal{M}_S(\sigma) \times_M \mathcal{M}_{1,1}^\text{rel}(\hat{\alpha})]^{\text{vir}} = \langle \hat{S}_\sigma, \mathcal{P}_\alpha \rangle$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between relative cohomology and homology (with Künneth decomposition):

$$H^{\mu(\alpha)-\mu(\beta)}(M, \nu(L)) \otimes H_{n+1+\mu(\alpha)}(M \times L \times S^1, \nu(L) \times L \times S^1) \to H^{\mu(\alpha)-\mu(\beta)}(M, \nu(L)) \otimes H_{n+1+\mu(\beta)}(L \times S^1) \to H_n(L \times S^1).$$

**Lemma 3.15.** Suppose $\partial \mathcal{M}_1(\alpha) = \emptyset$. Then the relative homology class $\mathcal{P}_\alpha$ is given by

$$(19) \quad \mathcal{P}_\alpha = \alpha \otimes (ev_*[\mathcal{M}_1(\alpha)]^{\text{vir}} \times [S^1])$$

in $H_*(M \times L \times S^1, \nu(L) \times L \times S^1) \cong H_*(M, L) \otimes H_*(L \times S^1)$. 

**Figure 1.** Three types of boundary points of $\mathcal{M}_{1,1}^\text{rel}(\hat{\alpha})$. The horizontal direction is $M$ and the vertical direction is $\mathbb{D}$. The boundary/interior markings are denoted by $b$ and $i$ respectively. The shaded disc is a component where the map is constant.
Proof. Consider the diagram:

\[
\begin{array}{c}
\mathcal{M}_{1,1}^{\text{rel}}(\hat{\alpha}) \\
\downarrow \ev^{(i)} \\
M
\end{array} \xrightarrow{f} \begin{array}{c}
\mathcal{M}_1(\alpha) \times S^1 \\
\downarrow \ev \times \id \\
L \times S^1
\end{array}
\]

where \( f \) is given in (16). The composition of the horizontal arrows is the boundary evaluation map \( \ev^{(b)} \). As we saw in Lemma 3.12, \((f, \ev^{(i)})\) can be viewed as a universal family of bordered stable maps of class \( \alpha \). Ignoring the issues on virtual chains, the lemma follows from this diagram. In the sequel, we compare the Kuranishi structures on \( \mathcal{M}_{1,1}^{\text{rel}}(\hat{\alpha}) \) and \( \mathcal{M}_1(\alpha) \) and see that the virtual chain of \( \mathcal{M}_{1,1}^{\text{rel}}(\hat{\alpha}) \) can be chosen to be a “fibre bundle” over a virtual chain of \( \mathcal{M}_1(\alpha) \times S^1 \) with fibre the corresponding stable discs.

First, we review the construction of a Kuranishi structure on \( \mathcal{M}_1(\alpha) \). We refer the reader to [12, §7.1], [15, Part 3, 4] for the details. Recall [12, Definition A1.1] that a Kuranishi neighbourhood of a point \( x_0 \in \mathcal{M}_1(\alpha) \) is a tuple \((V, E, \Gamma, \psi, s)\) where

- \( V \) is a finite dimensional manifold (possibly with boundary and corner);
- \( E \) is a finite dimensional real vector space;
- \( \Gamma \) is a finite group; it acts on \( V \) smoothly and effectively and on \( E \) linearly;
- \( s \) is a smooth \( \Gamma \)-equivariant map \( V \to E \);
- \( \psi \) is a homeomorphism between \( s^{-1}(0)/\Gamma \) and an open neighbourhood of \( x_0 \) in \( \mathcal{M}_1(\alpha) \).

These data are constructed as follows. Let \((u_0: \Sigma_0 \to M, x_0 \in \partial \Sigma_0)\) be a marked bordered stable map representing \( x_0 \in \mathcal{M}_1(\alpha) \). The finite group \( \Gamma \) is given by the set of holomorphic automorphisms \( \varphi: \Sigma_0 \to \Sigma_0 \) such that \( u_0 \circ \varphi = u_0 \) and \( \varphi(x_0) = x_0 \). Since \( \Sigma_0 \) has only one marking, it is unstable if we forget the map \( u_0 \). We add additional interior markings \( w_{0,1}, \ldots, w_{0,l} \) on \( \Sigma_0 \) so that \( \Sigma_0 \) becomes stable. We also require that the set \( \{w_{0,1}, \ldots, w_{0,l}\} \) is preserved by the \( \Gamma \)-action [15, Definition 17.5]. Since \( \Gamma \) permutes the additional markings, we can regard it as a subgroup of the symmetric group \( \mathcal{G}_l \). We also take real codimension 2 submanifolds \( Q_1, \ldots, Q_l \) of \( M \) such that \( \Sigma_0 \) intersects with \( Q_i \) transversely at \( w_{0,i} \) (so \( u_0 \) is necessarily an immersion at \( w_{0,i} \)); moreover we require that \( Q_i = Q_{\sigma(i)} \) for every permutation \( \sigma \in \Gamma \subset \mathcal{G}_l \). Let \( \mathcal{M}_{1,l} \) denote the moduli space of genus-zero stable bordered Riemann surfaces with one boundary and \( l \) interior markings and let \( x_0 \) be the point represented by \((\Sigma_0, x_0, \{w_{0,1}, \ldots, w_{0,l}\})\). The group \( \Gamma \) acts on \( \mathcal{M}_{1,l} \) by permutation of \( l \) interior markings and \( x_0 \) is fixed by \( \Gamma \). Let \( N \subset \mathcal{M}_{1,l} \) be a \( \Gamma \)-invariant small open neighbourhood of \( x_0 \). It is equipped with a tautological family \( \mathcal{R} \to N \) of stable bordered Riemann surfaces. Note that \( \Gamma \) also acts on \( \mathcal{R} \). We take a \( \Gamma \)-invariant compact subset \( K \subset \mathcal{R} \) such that the fibre \( K_0 = \Sigma_0 \cap K \) at \( x_0 \) is the complement (in \( \Sigma_0 \)) of small neighbourhoods of nodes of \( \Sigma_0 \) and that the family \( K \to N \) is \( C^\infty \)-trivial. We choose a \( \Gamma \)-equivariant \( C^\infty \)-trivialization \( K \cong K_0 \times N \) which preserves the markings. See [15] Definitions 16.2, 16.4, 16.6, 16.7. \( K \) is called the “core” and its complement is called the “neck region”. For a bordered Riemann surface \( \Sigma \) appearing as a fibre of \( \mathcal{R} \to N \), the core \( K = \Sigma \cap K \) is identified with \( K_0 \) by the given trivialization \( K \cong K_0 \times N \), and thus \( u_0 \) induces a map \( u_0: K \to M \). We consider an infinite

\footnote{Fukaya-Oh-Ohta-Ono [15] constructed \( N \) in two steps: first they considered a subset \( \mathcal{W} \subset \mathcal{M}_{1,l} \) consisting of deformations of \( x_0 \) having the same dual graph as \( x_0 \); then they introduced smoothing (or gluing) parameters \( T, \theta \) to construct a neighbourhood \( N \) of \( \mathcal{W} \).}
dimensional space $\mathcal{U}$ consisting of tuples $(u, \Sigma, x, \{w_1, \ldots, w_l\})$, where $(\Sigma, x, \{w_1, \ldots, w_l\})$ represents a point of $N$ and $u : (\Sigma, \partial \Sigma) \to (M, L)$ is a smooth map of degree $\alpha$ which is sufficiently “close” to $u_0$ in the sense that (see [15] Definitions 17.12, 18.10)

- $u$ is $C^{10}$-close to $u_0$ on the core $K = \Sigma \cap K$;
- $u$ is holomorphic on the neck region $\Sigma \setminus K$;
- the diameter of the image of each connected component of the neck region under $u$ is small.

The group $\Gamma$ acts on $\mathcal{U}$ by permutation of interior marked points. Next we choose an obstruction bundle $E$ over $\mathcal{U}$ as follows (see [15] Definitions 17.7, 17.15). We take a $\Gamma$-equivariant smooth family of finite dimensional subspaces $\mathcal{E}_a$

$$\mathcal{E}_a \subset C_c^\infty(\text{Int}(K), u_0^*TM \otimes \Lambda^{0,1})$$

parametrized by $a = (\Sigma, x, \{w_1, \ldots, w_l\}) \in N$, where $K = \Sigma \cap K$ is the core of $\Sigma$ and $\Lambda^{0,1}$ is the bundle of $(0,1)$-forms on $\Sigma$. Then we extend this family to the whole $\mathcal{U}$ via parallel transport, i.e. for each point $\mathbf{r} = (u, \Sigma, x, \{w_1, \ldots, w_l\}) \in \mathcal{U}$ over $a = (\Sigma, x, \{w_1, \ldots, w_l\}) \in N$, we define

$$\mathcal{E}_\mathbf{r} \subset C_c^\infty(\text{Int}(K), u^*TM \otimes \Lambda^{0,1})$$

as the parallel transport of $\mathcal{E}_a$ along geodesics joining $u(y)$ and $u_0(y)$. Here we use a connection on $TM$ such that $TL$ is preserved by parallel translation [15 §11]. By construction, the bundle $\mathcal{E} \to \mathcal{U}$ is $\Gamma$-equivariant. The Kuranishi neighbourhood $V \subset \mathcal{U}$ is now cut out by the equations:

$$\overline{\partial}u \equiv 0 \mod \mathcal{E}_\mathbf{r}$$

$$u(w_i) \in Q_i \quad i = 1, \ldots, l$$

for $\mathbf{r} = (u, \Sigma, x, \{w_1, \ldots, w_l\}) \in \mathcal{U}$. We need to choose $\mathcal{E}$ so that the equations (20) are transversal (see below). The $\Gamma$-action on $\mathcal{U}$ preserves $V$ and the obstruction bundle restricts to a $\Gamma$-equivariant vector bundle $E = \mathcal{E}|_V$ over $V$. The Cauchy-Riemann operator $\overline{\partial}$ induces a section $s$ of $E \to V$ and $s^{-1}(0)/\Gamma$ gives a neighbourhood of $\mathbf{r}_0 \in \mathcal{M}_1(\alpha)$.

The required transversality for (20) is stated as follows (see [15] Lemmata 18.16, 20.7]). For a smooth map $u : (\Sigma, \partial \Sigma) \to (M, L)$, let $L^2_{m,\delta}(\Sigma, \partial \Sigma; u^*TM, u^*TL)$ denote a certain weighted Sobolev space consisting of $L^2_{m,\delta}$-sections of $u^*TM$ which take values in $u^*TL$ along the boundary $\partial \Sigma$, see [15] Definitions 10.1, 19.8] ($m$ is sufficiently large and $\delta > 0$ is a parameter relevant to the weighted Sobolev norm). Let $L^2_{m,\delta}(\Sigma, u^*TM \otimes \Lambda^{0,1})$ denote a similar weighted Sobolev space of sections of $u^*TM \otimes \Lambda^{0,1}$ (see [15] Definition 19.9]). Let $D_{\mathbf{r}}\overline{\partial}$ denote the linearized operator of $\overline{\partial}$ at $\mathbf{r} = (u, \Sigma, x, \{w_1, \ldots, w_l\}) \in \mathcal{U}$:

$$D_{\mathbf{r}}\overline{\partial} : L^2_{m+1,\delta}(\Sigma, \partial \Sigma; u^*TM, u^*TL) \to L^2_{m,\delta}(\Sigma, u^*TM \otimes \Lambda^{0,1})$$

where the connection on $TM$ is used to define the derivative $D_{\mathbf{r}}\overline{\partial}$ (see [15] Remark 12.5]). We require that $\text{Im}(D_{\mathbf{r}}\overline{\partial})$ and $\mathcal{E}_\mathbf{r}$ span $L^2_{m,\delta}(\Sigma, u^*TM \otimes \Lambda^{0,1})$ for each $\mathbf{r} \in \mathcal{U}$. (This is called “Fredholm regularity”.) Let $\mathcal{M} \subset \mathcal{U}$ denote the subspace cut out only by the first equation of (20). Let $ev_{\text{ad}} : \mathcal{M} \to M^l$ be the evaluation map at the $l$ additional markings. We also require that $ev_{\text{ad}}$ is transversal to $\prod_{i=1}^l Q_i \subset M^l$. Then we have $V = ev_{\text{ad}}^{-1}(\prod_{i=1}^l Q_i)$.

Now we construct a Kuranishi neighbourhood of $\mathfrak{f}^{-1}(\mathbf{r}_0 \times S^1) \subset \mathcal{M}^w_{1,\delta}(\alpha)$ from the Kuranishi neighbourhood $(V, E, \Gamma, \psi, s)$ of $\mathbf{r}_0 \in \mathcal{M}_1(\alpha)$ above. Recall that $\mathfrak{f}^{-1}(\mathbf{r}_0 \times S^1) \cong \mathcal{S}_0 \times S^1$
by Lemma 3.12 where \( \tilde{\Sigma}_0 \) is the oriented real blow-up of \( \Sigma_0 \) at \( x_0 \). Here we perform oriented real blow-ups in families. The family \( \mathcal{R} \to N \) is equipped with a section \( x : N \to \mathcal{R} \) corresponding to the boundary marked point. Let \( \hat{\mathcal{R}} \) denote the oriented-real-blow-up along the section \( x \). The proof of Lemma 3.12 shows that a point \( p \in \hat{\mathcal{R}} \) parametrizes a marked stable bordered Riemann surface \( \hat{\Sigma} = (\hat{\Sigma}, x, p, \{w_1, \ldots, w_1\}) \) with a new interior marking \( p \) (see [12, Lemma 7.1.45]). More precisely, letting \( p \in \hat{\mathcal{R}} \) be on the blow-up of a fibre \( \Sigma \subset \mathcal{R} \): if \( p \) is neither a node nor a boundary point, \( \hat{\Sigma} = \Sigma \); if \( p \) is an interior node, \( \hat{\Sigma} \) is obtained from \( \Sigma \) by adding a sphere bubble at the node; if \( p \) is a boundary point, \( \hat{\Sigma} \) is obtained from \( \Sigma \) by adding at most two disc bubbles (see Figure 1). We allow \( p \) to coincide with one of \( w_i \)'s. Let \( \mathfrak{N} \to \hat{\mathcal{R}} \) denote the corresponding family of marked stable bordered Riemann surfaces. The group \( \Gamma \) acts on \( \mathfrak{N} \) by permutation of the markings \( w_1, \ldots, w_1 \). The core \( K \subset \mathcal{R} \) canonically induces a \( \Gamma \)-invariant core \( \mathfrak{K} \subset \mathfrak{N} \) equipped with a \( \Gamma \)-equivariant \( C^\infty \)-trivialization \( \mathfrak{K} \cong K_0 \times \hat{\mathcal{R}} \). Here \( \text{Int}(\mathfrak{K}) \) is disjoint from the components contracted under \( \tilde{\Sigma} \to \Sigma \). We consider the space \( \mathfrak{U} \) consisting of tuples \((\hat{u}, \tilde{\Sigma}, x, p, \{w_1, \ldots, w_1\}) \) where \((\tilde{\Sigma}, x, p, \{w_1, \ldots, w_1\}) \) is a marked bordered Riemann surface corresponding to a point of \( \hat{\mathcal{R}} \) (i.e. arises as a fibre of \( \mathfrak{N} \to \hat{\mathcal{R}} \)) and \( \hat{u} : (\tilde{\Sigma}, \partial \tilde{\Sigma}) \to (M \times \mathbb{P}^1, L \times S^1) \) is a smooth map of class \( \hat{\alpha} \) which satisfies the following conditions:

- \( \pi_M \circ \hat{u} \) is \( C^{10} \)-close to \( u_0 \) on the core \( \hat{K} = \tilde{\Sigma} \cap \mathfrak{K} \), where \( \pi_M : M \times \mathbb{P}^1 \to M \) is the projection (since \( \hat{K} \) is identified with \( K_0 \) via the given trivialization \( \mathfrak{K} \cong K_0 \times \hat{\mathcal{R}} \), \( u_0 \) defines a map \( \hat{u}_0 : \hat{K} \to M \));
- \( \hat{u} \) is holomorphic on the neck region \( \tilde{\Sigma} \setminus \hat{K} \);
- the diameter of the image of each connected component of the neck region under \( \pi_M \circ \hat{u} \) is small.

We use the obstruction bundle \( E \to \mathfrak{U} \) induced from \( E \to \mathcal{U} \) as follows. Take an element \( s = (\hat{u}, \tilde{\Sigma}, x, p, \{w_1, \ldots, w_1\}) \in \mathfrak{U} \) and let \( a = (\Sigma, x, \{w_1, \ldots, w_1\}) \in \mathcal{N} \) denote the marked Riemann surface given by forgetting \( p \) and collapsing unstable components of the source. Define the obstruction space at \( s \in \mathfrak{U} \)

\[
E_s \subset C^\infty_c(\text{Int}(\hat{K}), (\pi_M \circ \hat{u})^*TM \otimes \Lambda^{0,1}) \subset C^\infty_c(\text{Int}(\hat{K}), \hat{u}^*T(M \times \mathbb{P}^1) \otimes \Lambda^{0,1})
\]

(with \( \hat{K} = \tilde{\Sigma} \cap \mathfrak{K} \)) to be the parallel transport of \( E_a \subset C^\infty_c(\text{Int}(\hat{K}), u_0^*TM \otimes \Lambda^{0,1}) \) along geodesics joining \( u_0(y) \) and \( (\pi_M \circ \hat{u})(y) \). Let \( C \subset \tilde{\Sigma} \) be the contracted components of \( \tilde{\Sigma} \to \Sigma \). Because \( \pi_M \circ \hat{u} \mid \hat{K} \) is sufficiently close to \( u_0 \) and is holomorphic on \( C \), by choosing a smaller neck region from the beginning if necessary (see “extending the core” [15, Definition 17.21]), we may assume that \( \pi_M \circ \hat{u} \) is constant on \( C \) (since the symplectic area of the neck region has to be small). Hence \( \pi_M \circ \hat{u} \) induces a map \( u : (\Sigma, \partial \Sigma) \to (M, L) \) belonging to \( \mathcal{U} \). Therefore we have a projection \( \mathfrak{U} \to \mathcal{U} \) and \( E \) is identified with the pull-back of \( E \). The group \( \Gamma \) acts on \( \mathfrak{U} \) and \( E \).
$M_{1,1}^{\text{rel}}(\hat{\alpha})$ is cut out from $\mathcal{U}$ by the equations:

\begin{equation}
\begin{aligned}
\mathcal{U} & \equiv 0 \mod E_s \\
\hat{u}(p) & \in M \times \{0\} \\
\hat{u}(w_i) & \in Q_i \times \mathbb{P}^1 \quad i = 1, \ldots, l
\end{aligned}
\end{equation}

for $s = (\hat{u}, \hat{\Sigma}, x, p, \{w_1, \ldots, w_l\}) \in \mathcal{U}$. The second equation of (21) corresponds to the fibre product presentation (10) of $M_{1,1}^{\text{rel}}(\hat{\alpha})$. Let $\hat{M} \subset \mathcal{U}$ denote the subspace cut out by the first and the second equations of (21). Consider the map $\mathcal{U} \to U \times S^1$, where the first factor is the projection we discussed and the second factor is the evaluation map at the boundary marking $x$ followed by the projection $L \times S^1 \to S^1$. We claim that $\hat{M}$ is a tautological family of (blown-up) Riemann surfaces over $M \times S^1$ under the map $\hat{M} \subset \mathcal{U} \to U \times S^1$. (Recall that $M \subset \mathcal{U}$ is cut out by the first equation of (20).) More precisely, it is identified with the restriction to $M \times S^1$ of the family $\pi_{\mathbb{P}^1}^* \hat{R} \to U \times S^1$ where $\pi_{\mathbb{P}^1} : U \times S^1 \to N \times S^1$ is the natural projection. By the choice of $E$, each element $(\hat{u}, \hat{\Sigma}, x, p, \{w_1, \ldots, w_l\})$ of $\hat{M}$ is holomorphic in the $\mathbb{P}^1$-factor and its image $(u, \Sigma, x, \{w_1, \ldots, w_l\})$ in $\mathcal{U}$ belongs to $M$. By the same argument as in the proof of Lemma 5.12, it follows that $\hat{u}$ is uniquely reconstructed from $u : \Sigma \to M$, $p \in \hat{\Sigma}$ and $(\pi_{\mathbb{P}^1} \circ \hat{u})(x) \in S^1$. This proves the claim. Cutting down the moduli space $\hat{M}$ by the third equation of (21), we obtain $\hat{V}$ as a tautological family of (blown-up) Riemann surfaces over $V \times S^1$, with $V$ the Kuranishi neighbourhood of $\tau_0 \in \mathcal{M}_1(\alpha)$. The obstruction bundle $\hat{E} = E|_{\hat{\Sigma}}$ and its section $\hat{s} := \mathcal{D}$ are the pull-backs of $E \to V$ and $s = \mathcal{D}$ respectively. These data $(\hat{V}, \hat{E}, \hat{s})$ are $\Gamma$-equivariant and give a Kuranishi neighbourhood $(\hat{V}, \hat{E}, \Gamma, \hat{\psi}, \hat{s})$ of $\mathcal{U}^{-1}(\tau_0 \times S^1) \subset M_{1,1}^{\text{rel}}(\alpha)$.

We need to check the transversality of (21). To see the transversality of the $\partial$-equation, it suffices to show that, for a map $\hat{u}$ satisfying the first equation of (21), the degree-one holomorphic map $v := \pi_{\mathbb{P}^1} \circ \hat{u} : (\hat{\Sigma}, \partial \hat{\Sigma}) \to (\mathbb{P}^1, S^1)$ is Fredholm regular (since $\pi_{\mathbb{P}^1} \circ \hat{u}$ is already known to be Fredholm regular with respect to $E$). The Fredholm regularity here can be rephrased as the vanishing of sheaf cohomology (see [21] §3.4, [9] §6):

$$H^1(\mathcal{U}, (v^*T\mathbb{P}^1, v^*TS^1)) = 0$$

where $(v^*T\mathbb{P}^1, v^*TS^1)$ denotes the sheaf of holomorphic sections of $v^*T\mathbb{P}^1$ which take values in $v^*TS^1$ on $\partial \hat{\Sigma}$. We can prove this by using the standard normalization sequence and [9, Lemma 6.4]. Let $\mathcal{N} \subset \mathcal{U}$ denote the moduli space cut out only by the first equation of (21). The holomorphic automorphism group $\text{Aut}(\mathbb{D})$ acts on the target $(M \times \mathbb{P}^1, L \times S^1)$ and also on the moduli space $\mathcal{N}$. The transversality for the second equation of (21) follows from the fact that the $\text{Aut}(\mathbb{D})$-action on $\text{Int}(\mathbb{D})$ is transitive. The first and the second equations of (21) define the moduli space $\hat{M}$. The evaluation map $e_{\text{val}} : \hat{M} \to (M \times \mathbb{P}^1)^l$ at the markings $w_1, \ldots, w_l$ is transversal to $\prod_{i=1}^l (Q_i \times \mathbb{P}^1)$ by the transversality assumption for the second equation of (20). The transversality for (21) follows.

Finally we compare virtual cycles. A virtual cycle is defined by multi-valued perturbations (multisections) of $s$ on Kuranishi neighbourhoods which are compatible under co-ordinate changes (see [12] §A1.1, [15] Part 2). By the above construction of Kuranishi neighbourhoods, we can define a virtual cycle $[M_{1,1}^{\text{rel}}(\hat{\alpha})]^{\text{vir}}$ by pulling back multisections used to define
a virtual cycle $[\mathcal{M}_1(\alpha)]^{\text{vir}}$. Then $[\mathcal{M}_{1,1}^{\text{rel}}(\hat{\alpha})]^{\text{vir}}$ becomes a fibre bundle over $[\mathcal{M}_1(\alpha)]^{\text{vir}} \times S^1$ with fibre the corresponding stable bordered Riemann surfaces. Each fibre is of class $\alpha$ under the interior evaluation map. The lemma follows. □

Summarizing the discussion, we obtain (see (14), (18), (19)):

**Lemma 3.16.** The virtual cycle $\text{ev}_* [\mathcal{M}_S(\sigma) \times_M \mathcal{M}_{1,1}^{\text{rel}}(\hat{\alpha})]^{\text{vir}}$ is well-defined if one of the following holds:

(a) $\mathcal{M}_S(\sigma) = \emptyset$ or;
(b) $\mathcal{M}_1(\alpha) = \emptyset$ or;
(c) $\partial \mathcal{M}_1(\alpha) = \emptyset$ and $\text{ev}(\mathcal{M}_S(\sigma)) \cap L = \emptyset$.

When one of the above conditions holds, we have:

$$\text{ev}_* [\mathcal{M}_S(\sigma) \times_M \mathcal{M}_{1,1}^{\text{rel}}(\hat{\alpha})]^{\text{vir}} = \begin{cases} (S_\sigma \cap [L]) \times [S^1] & \text{if } \alpha = 0 \text{ (then (b) holds)}; \\ \langle \hat{S}_\sigma, \alpha \rangle \text{ev}_*[\mathcal{M}_1(\alpha)]^{\text{vir}} \times [S^1] & \text{if (c) holds}; \\ 0 & \text{if } \alpha \neq 0 \text{ and (a) or (b) holds}. \end{cases}$$

### 3.3.3. Conjecture and expected results

We now state our conjecture:

**Conjecture 3.17** (Degeneration Formula). Let $\beta \in H_2(M, L)$ be such that $\mathcal{M}_1(\beta) = \emptyset$. Assume that every pair $(\sigma, \alpha) \in H^2_{\text{sec}}(E) \times H_2(M, L)$ with $r(\hat{\beta}) = \sigma + \hat{\alpha}$ satisfies one of the three conditions (a), (b), (c) in Lemma 3.16. Then the degeneration formula (11)

$$\varphi_* \text{ev}_*[\mathcal{M}_1(\hat{\beta})]^{\text{vir}} = \sum_{r(\hat{\beta})=\sigma+\hat{\alpha}} \text{ev}_*[\mathcal{M}_S(\sigma) \times_M \mathcal{M}_{1,1}^{\text{rel}}(\hat{\alpha})]^{\text{vir}}$$

holds. This implies, by Lemmata 3.11 and 3.16 that

$$(22) \quad \delta_{\beta,0} [L] = \sum_{(\sigma, \alpha): r(\hat{\beta})=\sigma+\hat{\alpha}, \alpha \neq 0 \text{ satisfying (c) of Lemma 3.16}} \langle \hat{S}_\sigma, \alpha \rangle \text{ev}_*[\mathcal{M}_1(\alpha)]^{\text{vir}} + \delta_{\beta, \lambda} \sum_{\sigma: r(\hat{\beta})=\sigma+0} S_\sigma \cap [L]$$

holds in $H_{\text{rel}}(L; \mathbb{Q})$. Here $S_\sigma$, $\hat{S}_\sigma$ are defined in (15), (17) and $\lambda \in H_1(L)$ is the class of an $S^1$-orbit.

Note that the second term in the right-hand side of (22) arises from the case $\alpha = 0$ (recall the discussion around (14)).

In practice it is not easy to make all the assumptions here to be satisfied and to obtain a non-trivial result from (22). Notice that the both-hand sides of (22) are zero unless $\mu(\beta) \leq 0$ for dimensional reason. Also the term $\langle \hat{S}_\sigma, \alpha \rangle$ is zero unless $\hat{S}_\sigma \in H^2(M, L)$, i.e. $\langle c_1^{\text{vert}}(E), \sigma \rangle = -1$. Hence by (12), the first term of the right-hand side is the sum over classes $\alpha$ satisfying $\mu(\alpha) = \mu(\beta) + 2$. This motivates the following (rather restrictive) assumption:

**Assumption 3.18.** (i) $\mathcal{M}_1(\beta)$ is empty for all $\beta \in H_2(M, L)$ with $\mu(\beta) \leq 0$.

(ii) The maximal fixed component $F_{\max} \subset M$ of the $\mathbb{C}^\times$-action (see §2.2) is of complex codimension one and the $\mathbb{C}^\times$-weight on the normal bundle is $-1$.

(iii) $c_1(M)$ is semi-positive.

(iv) $\text{ev}(\mathcal{M}_S(\sigma))$ is disjoint from $L$ for all $\sigma \in H^2_{\text{sec}}(E)$ such that $\langle c_1^{\text{vert}}(E), \sigma \rangle = -1$. 

We assume Assumption 3.18 in the rest of this section. Recall from Definition 2.1 that open Gromov-Witten invariants $n_\alpha$ are defined when $\mu(\alpha) = 2$ by the assumption (i) and so the potential function $W$ of $L$ is also defined. The role of the assumptions (ii) and (iii) is as follows. The assumption (ii) implies that $\langle c^\text{vert}_1(E), \sigma_0 \rangle = -1$ for a maximal section $\sigma_0$. Note that by (3) $M_\Sigma(\sigma)$ is empty unless $\sigma = \sigma_0 + d$ for some $d \in \text{NE}(M)_\mathbb{Z}$. Therefore by (iii), $M_\Sigma(\sigma)$ is empty unless $\langle c^\text{vert}_1(E), \sigma \rangle \geq -1$. This implies that the Seidel element $S$ in Definition 2.7 is in $H^{\leq 2}(M; \mathbb{Q}) \otimes \mathbb{Q}[\text{NE}(E)_\mathbb{Z}]$.

**Definition 3.19.** Under Assumption 3.18 we can decompose the Seidel element as

$$S = q_0 \tilde{S} = q_0(\tilde{S}^{(0)} + \tilde{S}^{(2)})$$

with $\tilde{S}^{(i)} \in H^i(M; \mathbb{Q}) \otimes \mathbb{Q}[\text{NE}(M)_\mathbb{Z}]$ and $q_0 = q^{\sigma_0}$. Furthermore, we can define a lift $\tilde{S}^{(2)}$ of $S^{(2)}$ as follows:

$$\tilde{S}^{(2)} := \sum_{\sigma; \langle c^\text{vert}_1(E), \sigma \rangle = -1} \tilde{S}_\sigma q^{\sigma - \sigma_0}$$

where $\tilde{S}_\sigma \in H^2(M, L; \mathbb{Q})$ (see (17)) is well-defined by Assumption 3.18 (iv). The lift $\tilde{S}^{(2)}$ is an element of $H^2(M, L; \mathbb{Q}) \otimes \mathbb{Q}[\text{NE}(M)_\mathbb{Z}]$ which maps to $S^{(2)}$ under the natural map $H^2(M, L) \to H^2(M)$.

Under Assumption 3.18 the conditions in Conjecture 3.17 are satisfied for all $\beta$ with $\mu(\beta) = 0$. In fact, if $r(\beta) = \sigma + \hat{\alpha}$, then $\mu(\beta) = 2 \langle c^\text{vert}_1(E), \sigma \rangle = 0$ by (12), and thus

- if $\mu(\alpha) \leq 0$ and $\langle c^\text{vert}_1(E), \sigma \rangle \geq 0$, then $M_1(\alpha) = \emptyset$ by the assumption (i);
- if $\mu(\alpha) \geq 4$ and $\langle c^\text{vert}_1(E), \sigma \rangle \leq -2$, then $M_\Sigma(\sigma) = \emptyset$ by the assumptions (ii), (iii);
- if $\mu(\alpha) = 2$ and $\langle c^\text{vert}_1(E), \sigma \rangle = -1$, then $M_1(\alpha)$ has no boundary by the assumption (i) and $\text{ev}(M_\Sigma(\sigma)) \cap L = \emptyset$ by the assumption (iv).

Fix a class $\gamma \in H_1(L)$. We now apply the formula (22) for $\beta$ with $\mu(\beta) = 0$ and $\partial \beta = \gamma + \lambda$. In this case, (22) yields the following equality in $H_n(L; \mathbb{Q}) \cong \mathbb{Q}$:

$$\delta_{\beta,0} = \sum_{(\sigma, \alpha); r(\beta) = \sigma + \hat{\alpha}, \mu(\alpha) = 2; \langle c^\text{vert}_1(E), \sigma \rangle = -1} \langle \tilde{S}_\sigma, \alpha \rangle n_\alpha + \delta_{\partial \beta, \lambda} \sum_{\sigma; r(\beta) = \sigma + \hat{0}} S_\sigma$$

where $n_\alpha$ is the open Gromov-Witten invariant defined in Definition 2.1. Note that $S_\sigma$ in the second term of the right-hand side lies in $H^0(L; \mathbb{Q}) \cong \mathbb{Q}$. We consider a generating function in the “open” Novikov ring $\Lambda^{op}$ which was introduced before Definition 2.1. We have a (not necessarily injective) homomorphism from the “closed” Novikov ring $\Lambda$ (see Remark 2.9) to the “open” Novikov ring $\Lambda^{op}$

$$\Lambda \to \Lambda^{op}, \quad q^d \mapsto z^d.$$ 

Thus $\Lambda^{op}$ is a $\Lambda$-algebra. Note that $r(\beta) = \sigma + \hat{\alpha}$ means

$$z^{\alpha_0 + \beta} = q^{\sigma - \sigma_0} z^\alpha \in \Lambda^{op}$$

by Proposition 3.6 where $\sigma_0, \alpha_0$ are maximal section/disc classes. We multiply the both-hand sides of (23) by $z^{\alpha_0 + \beta} = q^{\sigma - \sigma_0} z^\alpha$ and sum over all $\beta$ with $\mu(\beta) = 2$ and $\partial \beta = \gamma + \lambda$. About the first term of the right-hand side, this summation boils down to the sum over all $(\sigma, \alpha)$ with $\langle c^\text{vert}_1(E), \sigma \rangle = -1, \mu(\alpha) = 2, \partial \alpha = \gamma$ (see (13)); about the second term of the right-hand
side (which occurs when and only when \( \gamma = 0 \)), this boils down to the sum over all \( \sigma \) with \( \langle c_1^\text{vert}(E), \sigma \rangle = 0 \). Therefore we have:

**Theorem 3.20.** Assume that the degeneration formula (Conjecture 3.17) and Assumption 3.18 hold for \((M, L)\). For any \( \gamma \in H_1(L) \), we have

\[
\delta_{\gamma + \lambda, 0} z_{\alpha_0} = \langle \tilde{S}^{(2)}, dW_\gamma \rangle + \delta_{\gamma, 0} \tilde{S}^{(0)}
\]

in \( \Lambda^{\text{op}} \), where \( \tilde{S}^{(0)} \) and \( \tilde{S}^{(2)} \) are in Definition 3.19, \( W_\gamma \) is in Definition 2.2 and \( dW_\gamma \) is its logarithmic derivative:

\[
dW_\gamma := \sum_{\alpha \in H_2(M, L): \mu(\alpha) = 2, \partial\alpha = \gamma} \alpha \otimes n_\alpha z^\alpha \in H_2(M, L) \otimes \Lambda^{\text{op}}.
\]

Recall that \( \alpha_0 \) is the maximal disc class introduced before Proposition 3.6 and \( \lambda \in H_1(L) \) is the class of an \( S^1 \)-orbit on \( L \).

Summing over all \( \gamma \in H_1(L) \) in (24), we obtain:

**Corollary 3.21.** Assume that the degeneration formula (Conjecture 3.17) and Assumption 3.18 hold for \((M, L)\). Then we have

\[
z_{\alpha_0} = \langle \hat{S}^{(2)}, dW \rangle + \tilde{S}^{(0)} \text{ in } \Lambda^{\text{op}}.
\]

Via the natural map \( H^1(L) \to H^2(M, L) \), an element of \( H^1(L) \) can be regarded as a vector field tangent to the fibre of the map \( \text{Spec } \Lambda^{\text{op}} \to \text{Spec } \Lambda \). We define the (relative) Jacobi algebra of the potential \( W \) as

\[
\text{Jac}(W) := \Lambda^{\text{op}} / \Lambda^{\text{op}} \langle H^1(L), dW \rangle
\]

where \( \Lambda^{\text{op}} \langle H^1(L), dW \rangle \) denotes the ideal of \( \Lambda^{\text{op}} \) generated by \( \langle \varphi, dW \rangle, \varphi \in \text{Im}(H^1(L) \to H^2(M, L)) \). As a class in the Jacobi algebra, the right-hand side of (25) depends only on the Seidel element \( \tilde{S} \) itself, not on the lift \( \tilde{S}^{(2)} \). We can interpret it as the derivative of the bulk-deformed potential \( W + t^0 \) with respect to \( \tilde{S} \), where \( t^0 \) is a co-ordinate on \( H^0(M) \). The derivative of \( W + t^0 \) defines the so-called Kodaira-Spencer mapping:

\[
\text{KS}: H^{\leq 2}(M) \otimes \Lambda \to \text{Jac}(W).
\]

Then the equation (25) implies

\[
\text{KS}(\tilde{S}) = [z_{\alpha_0}] \text{ in } \text{Jac}(W).
\]

**Remark 3.22.** Assumption 3.18 (i)–(iii) ensures that the conditions in Conjecture 3.17 hold for all \( \beta \) with \( \mu(\beta) \leq -2 \). Using the formula (22) for \( \beta \) with \( \mu(\beta) \leq -2 \) and \( \partial\beta = \lambda \), we find:

\[
\sum_{\sigma \in i_* H_2(L)} S_{\sigma + d} \cap [L] = 0 \text{ if } \langle c_1^\text{vert}(E), \sigma \rangle \leq -1.
\]

This supports the validity of Assumption 3.18 (iv).

**Remark 3.23.** A more intuitive explanation for the formula (25) is as follows. One can think of the moduli space \( \mathcal{M}_{1,1}(\beta) \) of stable holomorphic discs with boundaries in \( L \) and with one interior and one boundary marked points as giving a correspondence between \( M \) and the
free loop space $\mathcal{L}L = \text{Map}(S^1, L)$ of $L$. This correspondence should give rise to a map (bulk-boundary map) 
$$
C_\ast(M) \rightarrow C_\ast(\mathcal{L}L)
$$
of chain complexes. One can view this as an analogue of the Kodaira-Spencer map. One can speculate that this map is an intertwiner between the Seidel homomorphism $S: C_\ast(M) \rightarrow C_\ast(M)$ and the map $\mathcal{L}L \rightarrow \mathcal{L}L$ induced by the $S^1$-action.

4. Potential function of a semi-positive toric manifold

Using the degeneration formula (Conjecture 3.17), we compute the potential function of a Lagrangian torus fibre of a semi-positive toric manifold $X$. This confirms a conjecture (now a theorem [8]) of Chan-Lau-Leung-Tseng [7].

4.1. Toric manifolds. We fix notation on toric geometry. For more details we refer the reader to [1, 10, 11]. For this paper a toric manifold $X$ is a smooth projective toric variety, as constructed from the following data.

(a) An integral lattice $N \cong \mathbb{Z}^n$ and its dual $M = \text{Hom}(N, \mathbb{Z})$. We denote by $\langle \cdot, \cdot \rangle$ the natural pairing between $N$ and $M$.

(b) A fan $\Sigma$ in $N_\mathbb{R} := N \otimes \mathbb{R}$ consisting of a collection of strongly convex rational polyhedral cones $\sigma \subset N_\mathbb{R}$, which is closed under intersections and taking faces.

In order for $X$ to be smooth and projective, we need to assume that $\Sigma$ is complete, regular and admits a strongly convex piecewise-linear function. Let $\Sigma(1)$ denote the set of 1-cones (rays) in $\Sigma$, and we let $b_1, \ldots, b_m$ denote integral primitive generators of the 1-cones. The fan sequence of $X$ is the exact sequence

$$
0 \rightarrow \mathbb{L} \rightarrow \mathbb{Z}^m \rightarrow N \rightarrow 0,
$$
where the third arrow takes the canonical basis to the primitive generators $b_1, \ldots, b_m \in N$ and $\mathbb{L}$ is defined to be the kernel of the third arrow. The dual of the sequence (26) is the divisor sequence

$$
0 \rightarrow M \rightarrow \mathbb{Z}^m \rightarrow \mathbb{L}^\vee \rightarrow 0.
$$
The second arrow takes $v \in M$ into the tuple $\langle (b_i, v) \rangle_{i=1}^m$. The third arrow is denoted by $\kappa: \mathbb{Z}^m \rightarrow \mathbb{L}^\vee$.

The fan sequence tensored with $\mathbb{C}^\times$ gives the exact sequence of tori:

$$
1 \rightarrow G \rightarrow (\mathbb{C}^\times)^m \rightarrow T \rightarrow 1
$$
with $G := \mathbb{L} \otimes \mathbb{C}^\times$ and $T := N \otimes \mathbb{C}^\times$. Let the torus $G$ act on $\mathbb{C}^m$ by the second arrow $G \rightarrow (\mathbb{C}^\times)^m$. The combinatorics of the fan defines a stability condition of this action as follows. Let $Z(\Sigma)$ denote the union

$$
Z(\Sigma) := \bigcup_{I \in \mathcal{A}} \mathbb{C}^I, \quad \mathbb{C}^I = \{(x_1, \ldots, x_m) : x_i = 0 \text{ for } i \notin I\},
$$
where $\mathcal{A}$ is the collection of anti-cones, that is the subsets of indices that do not yield a cone in the fan

$$
\mathcal{A} := \left\{ I : \sum_{i \in I} \mathbb{R}_{\geq 0} b_i \notin \Sigma \right\}.
$$
The toric variety $X$ is defined as the quotient
$$X := U_{\Sigma} / G; \quad U_{\Sigma} := C^m \setminus Z(\Sigma).$$

The torus $T = (C^*)^m / G$ acts naturally on $X$. The toric manifold $X$ contains $T$ as an open free orbit; $X$ is a compactification of $T$ along the rays in $\Sigma(1)$.

Each character $\xi : G \to C^*$ defines a line bundle
$$L_\xi := \mathbb{C} \times_{\xi, G} U_{\Sigma} \to X.$$

The correspondence $\xi \mapsto L_\xi$ yields an identification of the Picard group with the character group of $G$. Thus, we have
$$L_\xi \lor = \text{Hom}(G, \mathbb{C}^*) \cong \text{Pic}(X) \cong H^2(X; \mathbb{Z}).$$

The $i$th toric divisor is given by
$$D_i := \{ [x_1, \ldots, x_m] : x_i = 0 \} \subset X$$

The Poincaré dual of $D_i$ is the image $\kappa(e_i) \in \mathbb{L}^\lor \cong H^2(X; \mathbb{Z})$ of the standard basis $e_i \in \mathbb{Z}^m$ under the map $\kappa$ in (27). By abuse of notation, $D_i$ sometimes also denotes the corresponding cohomology class $\kappa(e_i)$ of $X$. The first Chern class $c_1(X)$ of $X$ is given by $D_1 + \cdots + D_m$.

The Kähler cone $C_X$ of $X$, the cone consisting of Kähler classes, is given by
$$C_X := \bigcap_{I \in \mathcal{A}} \sum_{i \in I} \mathbb{R}_{>0} \kappa(e_i) \subset \mathbb{L}^\lor \otimes \mathbb{R} = H^2(X; \mathbb{R}).$$

The cone $C_X$ is nonempty if and only if $X$ is projective. Set $r := m - n$. We choose a nef integral basis $p_1, \ldots, p_r$ of $H^2(X; \mathbb{Z})$, that is an integral basis such that $p_a \in C_X$ for all $a = 1, \ldots, r$. Then we write the toric divisor classes as

$$D_j = \kappa(e_j) = \sum_{a=1}^r m_{aj} p_a,$$

for some integer matrix $(m_{aj})$. The Mori cone $\text{NE}(X) \subset H_2(X, \mathbb{R})$ is the dual of the cone $C_X$. We set $\text{NE}(X)_\mathbb{Z} := \text{NE}(X) \cap H_2(X; \mathbb{Z})$.

The toric manifold $X$ can be alternatively defined as a symplectic quotient. Let $G_R \cong (S^1)^r$ be the maximal compact subgroup in $G$. The $G_R$-action on $C^m$ is generated by the moment map
$$\phi : C^m \to g_R^\lor, \quad \phi(x_1, \ldots, x_m) = \kappa(|x_1|^2, \ldots, |x_m|^2)$$

where $\kappa : \mathbb{R}^m \to \mathbb{L}^\lor \otimes \mathbb{R}$ is the map in the divisor sequence (27) tensored with $\mathbb{R}$. For any Kähler class $\omega \in C_X$, we have a diffeomorphism ([11, 20])
$$\phi^{-1}(\omega) / G_R \cong X.$$
Let $T_{\mathbb{R}} \cong (S^1)^n$ be the maximal compact subgroup of $T$. The $T_{\mathbb{R}}$-action on the symplectic toric manifold $(X, \omega)$ admits a moment map:

\[
\Phi_\omega: X \rightarrow \kappa^{-1}(\omega),
\]
\[
\Phi_\omega([x_1, \ldots, x_m]) = ([x_1]^2, \ldots, [x_m]^2) \quad \text{with} \quad (x_1, \ldots, x_m) \in \phi^{-1}(\omega).
\]

where the affine subspace $\kappa^{-1}(\omega) \subset \mathbb{R}^m$ can be identified with $M_{\mathbb{R}} = M \otimes \mathbb{R} \cong \mathbb{T}^\vee$ up to translation. The image of the moment map $\Phi_\omega$ is the convex polytope:

\[
P(\omega) = \{(t_1, \ldots, t_m) \in \mathbb{R}^m : t_i \geq 0, \kappa(t_1, \ldots, t_m) = \omega\}
\]
\[
\cong \{v \in M_{\mathbb{R}} : \langle b_i, v \rangle \leq -c_i, \ i = 1, \ldots, m\}.
\]

In the second line, we took a lift $(c_1, \ldots, c_m)$ of $\omega$ (such that $\omega = \kappa(c_1, \ldots, c_m)$) to identify $\kappa^{-1}(\omega)$ with $M_{\mathbb{R}}$. This is called the momentum polytope. The facet $F_i \subset P$ of $P(\omega)$ normal to $b_i \in N$ corresponds to the toric divisor $D_i = \Phi_\omega^{-1}(F_i) \subset X$.

### 4.2. Potential function of a Lagrangian torus fibre

Cho-Oh [9] calculated potentials of Lagrangian torus fibres for Fano toric manifolds and matched them up with mirror Landau-Ginzburg potentials of Givental and Hori-Vafa. Fukaya-Oh-Ohta-Ono [14] studied potentials for general symplectic toric manifolds. Chan [5], Chan-Lau [6] and Chan-Lau-Leung-Tseng [7, 8] have studied the potential functions for semi-positive toric manifolds by establishing an equality between open and closed Gromov-Witten invariants.

Let $X$ be a toric manifold in the previous section. Every free $T_{\mathbb{R}}$-orbit in $X$ is a fibre of the moment map $\Phi_\omega: X \rightarrow P(\omega)$ of an interior point in $P(\omega)$, and vice versa. We call it a Lagrangian torus fibre of $X$. For a Lagrangian torus fibre $L$, we have a homotopy exact sequence:

\[
0 \longrightarrow \pi_2(X) \longrightarrow \pi_2(X, L) \xrightarrow{\partial} \pi_1(L) \longrightarrow 0.
\]

Let $\beta_i \in \pi_2(X, L)$ denote the class represented by the holomorphic disc $u_i: \mathbb{D} \rightarrow X$:

\[
u_i(z) = [c_1, \ldots, c_{i-1}, c_i z, c_{i+1}, \ldots, c_m], \quad |z| \leq 1
\]

where $[c_1, \ldots, c_m] \in X$ is a point on the Lagrangian $L$ (thus $c_i \neq 0$ for all $i$). The class $\beta_i$ intersects with toric divisors as

\[
\beta_i \cdot D_j = \delta_{ij}.
\]

The relative homotopy group $\pi_2(X, L)$ is an abelian group freely generated by the classes $\beta_1, \ldots, \beta_m$ and the toric divisors $D_1, \ldots, D_m$ define a dual basis of $H^2(X, L)$. Under the identification:

\[
\pi_2(X) \cong H_2(X; \mathbb{Z}) \cong \mathbb{L}, \quad \pi_2(X, L) \cong H_2(X, L; \mathbb{Z}) \cong \mathbb{Z}^m, \quad \pi_1(L) \cong N
\]

the exact sequence (30) above can be identified with the fan sequence (26), i.e. $\partial \beta_i = b_i$. The Maslov index

\[
\mu: \pi_2(X, L) \longrightarrow \mathbb{Z}
\]

is given by the intersection with $2(D_1 + \cdots + D_m) \in H^2(X, L)$ [9 Theorem 5.1].

We consider the potential function (Definition 2.1) of a Lagrangian torus fibre $L \subset X$. As before, let $M_1(\beta)$ denote the moduli space of bordered stable maps to $(X, L)$ in the class $\beta \in \pi_2(X, L)$ with one boundary marked point.
Proposition 4.1. Suppose that $c_1(X)$ is semi-positive. Then $\mathcal{M}_1(\beta)$ is empty for all $\beta$ with $\mu(\beta) \leq 0$. If $\mathcal{M}_1(\beta)$ is non-empty for $\beta$ with $\mu(\beta) = 2$, then $\beta = \beta_i + d$ for some $i$ and $d \in \text{NE}(X)_\mathbb{Z}$ such that $\langle c_1(X), d \rangle = 0$.

Proof. Let $\beta$ be a class of a bordered stable map to $(X, L)$. By the classification of holomorphic discs by Cho-Oh [9], we find that $\beta$ is of the form:

$$\beta = \sum_{i=1}^{m} k_i \beta_i + d$$

for some $k_i \geq 0$ and $d \in \text{NE}(X)_\mathbb{Z}$. Here $\sum_{i=1}^{m} k_i \beta_i$ is the degree of disc components and $d$ is the degree of sphere bubbles. Hence $\mu(\beta) = 2 \sum_{i=1}^{m} k_i + 2 \langle c_1(X), d \rangle \geq 0$. We claim that $(k_1, \ldots, k_m) = 0$ implies $\mu(\beta) \geq 4$. If $(k_1, \ldots, k_m) = 0$, a bordered stable map of class $\beta$ is the union of a constant disc and sphere bubbles. In this case, at least one non-trivial sphere component has to touch $L$. Let $d_1$ be the degree of a non-trivial sphere component touching $L$ and let $d_2$ be the degree of the remaining sphere bubbles. Then $d = d_1 + d_2$ with $d_1, d_2 \in \text{NE}(X)_\mathbb{Z}$. Since $D_i$ is disjoint from $L$, we have $\langle D_i, d_1 \rangle \geq 0$. Since $d_1 \neq 0$, we have $\sum_{i=1}^{m} \langle D_i, d_1 \rangle \geq 1$. Also it is impossible that $\sum_{i=1}^{m} \langle D_i, d_1 \rangle = 1$ since $d_1$ gives the relation $\sum_{i=1}^{m} \langle D_i, d_1 \rangle b_i = 0$ in $N$. Thus

$$\mu(\beta) = 2 \langle c_1(X), d \rangle \geq 2 \sum_{i=1}^{m} \langle D_i, d_1 \rangle \geq 4.$$

The claim follows. Consequently, $\mu(\beta) \leq 2$ implies $(k_1, \ldots, k_m) \neq 0$. The proposition follows easily. □

In particular, the potential function of a Lagrangian torus fibre (Definition 2.1) is well-defined for a semi-positive toric manifold.

Remark 4.2. Fukaya-Oh-Ohta-Ono [14] defined the potential function of a Lagrangian torus fibre even without semi-positivity assumption. They defined virtual cycles and open Gromov-Witten invariants $n_\beta \in \mathbb{Q}$ for all $\beta$ with $\mu(\beta) = 2$ using $\mathbb{T}_\mathbb{R}$-equivariant perturbations, see [14] Lemmata 11.2, 11.5, 11.6, 11.7. In general, since every effective stable disc class $\beta$ is of the form (32), the potential $W$ lies in the completed group ring:

$$\mathbb{Q}[[\mathbb{Z}_{\geq 0})^m + \text{NE}(X)_\mathbb{Z}] \subset \Lambda^{op}$$

where $\text{NE}(X)_\mathbb{Z}$ is regarded as a subset of $\mathbb{Z}^m$ via the second arrow in the fan sequence (26). Notice that $(\mathbb{R}_{\geq 0})^m + \text{NE}(X)$ is a strictly convex cone.

Example 4.3 ([9]). When $\beta = \beta_i$, the moduli space $\mathcal{M}_1(\beta_i)$ consists of holomorphic discs of the form (31) and $ev: \mathcal{M}_1(\beta_i) \cong L$; moreover all such discs are Fredholm regular [9, Theorem 6.1]. Therefore we have $n_{\beta_i} = 1$.

We write

$$z^\beta = z_1^{k_1}z_2^{k_2}\cdots z_m^{k_m} \in \mathbb{Q}[H_2(X, L; \mathbb{Z})]$$

for $\beta = k_1 \beta_1 + \cdots + k_m \beta_m$. Also we write

$$q^d = q_1^{(p_1, d)}q_2^{(p_2, d)}\cdots q_r^{(p_r, d)} \in \mathbb{Q}[H_2(X; \mathbb{Z})]$$
for \( d \in H_2(X; \mathbb{Z}) \), where \( p_1, \ldots, p_r \) is the nef integral basis of \( H^2(X; \mathbb{Z}) \cong \mathbb{L}^r \) we chose in \( \S 4.1 \). Note that we have a natural inclusion of the group rings:

\[
\mathbb{Q}[H_2(X; \mathbb{Z})] \hookrightarrow \mathbb{Q}[H_2(X, L; \mathbb{Z})].
\]

By this we identify \( q^d \) with \( z^d \), in co-ordinates:

\[
q^d = z^d = z_1^{(D_1,d)} z_2^{(D_2,d)} \cdots z_m^{(D_m,d)} \quad \text{or} \quad q_a = \prod_{i=1}^m z_i^{m_{ai}}
\]

where \((m_{ai})\) is the divisor matrix in \( (29) \). Using these notations and Proposition \( 4.1 \) we can write the potential function of \((X, L)\) in the following form when \( c_1(X) \) is semi-positive.

**Definition 4.4.** Let \( X \) be a semi-positive toric manifold. We present the potential function \( W \) of a Lagrangian torus fibre in the form:

\[
W = w_1 + \cdots + w_m
\]

where \( w_i = f_i(q)z_i \) and

\[
f_i(q) = \sum_{d \in \text{NE}(X)_2 : (c_1(X),d) = 0} n_{\beta_i+d} q^d.
\]

We call \( f_i(q) \) the *correction term*. This decomposition of \( W \) is parallel to Definition \( 2.2 \).

Note that we have \( f_i(q) = 1 + O(q) \) by Example \( 4.3 \). The correction term \( f_i(q) \) was denoted by \( 1 + \delta_i(q) \) in \( [7] \). When \( X \) is Fano, all the correction terms are 1 and

\[
W = z_1 + \cdots + z_m.
\]

This is the result of Cho-Oh [9]. By the *fan polytope*, we mean the convex hull of the ray vectors \( b_1, \ldots, b_m \in \mathbb{N} \). In the proof of \([7, \text{Corollary 4.12}]\), Chan-Lau-Leung-Tseng showed the following:

**Proposition 4.5** (Chan-Lau-Leung-Tseng [7]). Let \( f_i(q) \) be the correction terms of the potential of a semi-positive toric manifold \( X \). If the vector \( b_i \) is a vertex of the fan polytope of \( X \), then \( f_i(q) = 1 \).

4.2.1. *Open-closed moduli space.* We explain that the potential \( W \) of a Lagrangian torus fibre can be interpreted as a formal function on the *open-closed moduli space* introduced below.

The *closed moduli space* \( \mathcal{M}_{cl} \) of \( X \) is defined to be:

\[
\mathcal{M}_{cl} = \{ \exp(-\omega + iB) \in \mathbb{L}^r \otimes \mathbb{C}^\times : \omega, B \in \mathbb{L}^r \otimes \mathbb{R}, \omega \in C_X \}.
\]

This is also called the *complexified Kähler moduli space*. The nef basis \( p_1, \ldots, p_r \) of \( \mathbb{L}^r \cong H^2(X; \mathbb{Z}) \) in \( \S 4.1 \) defines \( \mathbb{C}^\times \)-valued co-ordinates \((q_1, \ldots, q_r)\) on \( \mathcal{M}_{cl} \subset \mathbb{L}^r \otimes \mathbb{C}^\times \).

The *open-closed moduli space* \( \mathcal{M}_{opcl} \) is defined to be the set of triples \((q, L, h)\) such that

- a closed moduli \( q = \exp(-\omega + iB) \in \mathcal{M}_{cl} \); 
- a Lagrangian torus fibre \( \tilde{L} = L_\eta = \Phi_\omega^{-1}(\eta) \) at \( \eta \in P(\omega)^0 \); 
- a class \( h \in H^2(X, L; U(1)) \) which maps to \( \exp(iB) \in H^2(X; U(1)) \).
When the $B$-field vanishes $B = 0$, the class $h$ defines a $U(1)$-local system on $L$ via the exact sequence:

$$0 \longrightarrow H^1(L; U(1)) \longrightarrow H^2(X, L; U(1)) \longrightarrow H^2(X; U(1)) \longrightarrow 0$$

Let $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m$ be the co-ordinates of $\eta$ and write $h = (h_1, \ldots, h_m)$ using the identification $H^2(X, L; U(1)) \cong (S^1)^m$; and set

$$z_i := \exp(-\eta_i)h_i.$$  

The parameter $z = (z_1, \ldots, z_m)$ here determines $\eta_i \in \mathbb{R}$, $h_i \in S^1$ by polar decomposition; then $\eta$ determines $\omega$ by the condition $\eta \in P(\omega)^0$ (as $\omega = \kappa(\eta)$) and $h$ determines $\exp(iB)$. Thus $z$ determines a point of $\mathcal{M}_{\text{opcl}}$. We have:

$$\mathcal{M}_{\text{opcl}} \cong \{ z = (z_1, \ldots, z_m) \in (\mathbb{C}^\times)^m : |z_i| < 1 \text{ for all } i, \kappa_{\mathbb{C}^\times}(z) \in \mathcal{M}_{\text{cl}} \}$$

where $\kappa_{\mathbb{C}^\times} : (\mathbb{C}^\times)^m \to \mathbb{L}^Y \otimes \mathbb{C}^\times$ is the third arrow of the divisor sequence (27) tensored with $\mathbb{C}^\times$. A point $z = (z_1, \ldots, z_m)$ of the right-hand side parametrizes

- a closed moduli $q = \exp(-\omega + iB) = \kappa_{\mathbb{C}^\times}(z)$;
- a Lagrangian torus fibre $L = L_\eta$ at $\eta = (-\log |z_1|, \ldots, -\log |z_m|) \in P(\omega)^0$;
- a class $h = (z_1/|z_1|, \ldots, z_m/|z_m|) \in H^2(X, L_\eta)$ which is a lift of $\exp(iB)$.

We regard $W$ as a formal function on $\mathcal{M}_{\text{opcl}}$ via these co-ordinates $(z_1, \ldots, z_m)$. The open-closed moduli is fibred over $\mathcal{M}_{\text{cl}}$:

$$\pi : \mathcal{M}_{\text{opcl}} \to \mathcal{M}_{\text{cl}}, \quad z \mapsto \kappa_{\mathbb{C}^\times}(z).$$

By pulling-back the co-ordinates $q_1, \ldots, q_r$ by $\pi$, we obtain the same relation between $z_i$ and $q_\alpha$ as in (33). The fibre $\mathcal{M}_{\text{opcl}, q} = \pi^{-1}(q)$ has the structure of an $(M_{\mathbb{R}}/M) \cong (S^1)^n$-bundle over $P(\omega)^0$ via the map:

$$\mathcal{M}_{\text{opcl}, q} \to P(\omega)^0, \quad (z_1, \ldots, z_m) \mapsto \eta = (-\log |z_1|, \ldots, -\log |z_m|).$$

This is a torus fibration dual to the moment map $\Phi_\omega : X \to P(\omega)$; we can view it as a mirror of $(X, q)$.

**Proposition 4.6.** Via the co-ordinates $(z_1, \ldots, z_m)$ on $\mathcal{M}_{\text{opcl}}$, the potential function of a Lagrangian torus fibre is identified with the following formal sum of functions on $\mathcal{M}_{\text{opcl}}$:

$$W(q, L, h) = \sum_{\beta \in \pi_2(X, L) : \mu(\beta) = 2} n_{\beta}h(\beta)e^{-\int_\beta \omega}$$

where $q = \exp(-\omega + iB)$.

**Proof.** For $\beta = \beta_1$, we have (see [9, Theorem 8.1])

$$h(\beta_1) = \frac{z_i}{|z_i|}, \quad \int_{\beta_1} \omega = \eta_i = -\log |z_i|$$

and thus $h(\beta_1)e^{-\int_{\beta_1} \omega} = z_i$ (cf. (34)). Therefore $h(\beta)e^{-\int_{\beta} \omega} = z^\beta$ for every $\beta$. \qed

**Remark 4.7.** When $B = 0$, the term $h(\beta)$ is the holonomy along the loop $\partial \beta \in \pi_1(L)$ of the $U(1)$-local system associated to $h$. This matches with the usual interpretation. In general, this term cannot be interpreted just as holonomy.
Fukaya-Oh-Ohta-Ono [13, Theorem 2.32] showed that the Jacobi algebra of the potential function restricted to the fibre $M_{opc,q} = \pi^{-1}(q)$ is isomorphic to the quantum cohomology ring of $(X, q)$ in a certain $q$-adic sense.

4.3. Seidel elements for toric varieties and Givental’s mirror transformation. We review our previous computation [19] relating Seidel elements for toric varieties to Givental’s mirror transformation [17]. Let $X$ be a toric manifold from §4.1 with $c_1(X)$ semi-positive.

4.3.1. Seidel elements associated to the $\mathbb{C}^\times$-actions fixing toric divisors. For each toric divisor $D_j$ of $X$, we can associate a $\mathbb{C}^\times$-action $\rho_j$ on $X$ rotating around $D_j$. It is given by:

$$\rho_j(\lambda): [x_1, \ldots, x_m] \mapsto [x_1, \ldots, \lambda^{-1} x_j, \ldots, x_m], \quad \lambda \in \mathbb{C}^\times.$$ 

The toric divisor $D_j = \{ x_j = 0 \}$ is the maximal fixed component of this action. Let $E_j$ denote the associated bundle of this $\mathbb{C}^\times$-action and let $S_j$ denote the corresponding Seidel element. We also write $S_j = q_0 \tilde{S}_j$ with $\tilde{S}_j \in QH^*(X)$ following Definition [2.7]. Using the Seidel representation (see Remark 2.9), McDuff-Tolman [24] showed the following multiplicative relations in $QH(X)[q^{-d} : d \in NE(X)]$:

$$\prod_{j=1}^m \tilde{S}_j^{(D_j,d)} = q^d \quad \text{for } d \in H_2(X; \mathbb{Z}).$$

4.3.2. Givental’s mirror theorem. Givental [17] introduced the two cohomology-valued functions

$$I(y, z) = e^{\sum_{i=1}^r p_i \log y_i / z} \sum_{d \in NE(X) \mathbb{Z}} \prod_{i=1}^m \left( \prod_{k=-\infty}^0 (D_i + k z) \prod_{k=\infty}^1 (D_i + k z) \right) y^d$$

$$J(q, z) = e^{\sum_{i=1}^r p_i \log q_i / z} \left( 1 + \sum_{j} \sum_{d \in NE(X) \mathbb{Z}} \left\langle \frac{\phi_j}{z - \psi} \right\rangle_{0,1,d}^X \phi^d q^d \right)$$

called the $I$-function and the $J$-function respectively. Here we used a nef basis $\{ p_1, \ldots, p_r \} \subset H^2(X)$ in §4.1 and write

$$q^d = q_1^{(p_1,d)} \ldots q_r^{(p_r,d)}, \quad y^d = y_1^{(p_1,d)} \ldots y_r^{(p_r,d)},$$

and $\{ \phi_1 \}$ and $\{ \phi^d \}$ are mutually dual basis of $H^*(X)$. The variables $y = (y_1, \ldots, y_r)$ are called mirror co-ordinates, i.e. co-ordinates of the complex moduli of the mirror Landau-Ginzburg model. Givental [17] showed the following mirror theorem:

**Theorem 4.8 (17).** We have $I(y, z) = J(q, z)$ under a change of coordinates of the form

$$\log q_i = \log y_i + g_i(y), \quad i = 1, \ldots, r, \quad g_i(y) \in \mathbb{Q}[y_1, \ldots, y_r] \text{ with } g_i(0) = 0.$$ 

The functions $g_i(y)$ here are uniquely determined by the asymptotics:

$$I(y, z) = e^{\sum_{i=1}^r p_i \log y_i / z} \left( 1 + \sum_{i=1}^r g_i(y) \frac{p_i}{z} + o(z^{-1}) \right).$$

The change of co-ordinates is called mirror transformation (or mirror map).
4.3.3. Batyrev elements and Seidel elements. In [19], we introduced Batyrev elements \( \tilde{D}_j \), \( j = 1, \ldots, m \). They are defined by
\[
\tilde{D}_j := \sum_{a=1}^{r} m_{aj} \bar{p}_a, \quad \bar{p}_a := \sum_{b=1}^{r} \frac{\partial \log q_b}{\partial \log y_a} p_b.
\]
Note that \( \tilde{D}_j \) is an element corresponding to the vector field \( \sum_{a=1}^{r} m_{aj} y_a \partial/\partial y_a \) whereas the genuine divisor class \( D_j \) corresponds to the vector field \( \sum_{a=1}^{r} m_{aj} q_a \partial/\partial q_a \) (see (29)). Batyrev elements are determined by, and determine the Jacobi matrix \( (\partial \log q_b/\partial \log y_a) \) of the mirror transformation. Using Givental’s mirror theorem, we find that the Batyrev elements satisfy the multiplicative relations (see [19, Proposition 3.8])
\[
\prod_{j=1}^{m} \tilde{D}_j^{\langle D_j, d \rangle} = y^d \quad d \in H_2(X; \mathbb{Z})
\]
in the quantum cohomology ring. These are very similar to the multiplicative relations (35) of Seidel elements, but note that co-ordinates \( q \) are replaced with mirror co-ordinates \( y \). Moreover, the Batyrev elements satisfy the following linear relations:
\[
\begin{align*}
\sum_{j=1}^{m} c_j \tilde{D}_j &= 0 \quad \text{whenever} \quad \sum_{j=1}^{m} c_j D_j = 0.
\end{align*}
\]
The linear relations are obvious from the definition. These multiplicative and linear relations show that \( \tilde{D}_j \) satisfy the relations of Batyrev’s quantum ring \([4]\). It turns out that the Seidel elements are multiples of the Batyrev elements.

**Theorem 4.9** ([19, Theorem 1.1]). Let \( g^{(j)}_0(y) \) be the following hypergeometric series in mirror co-ordinates:
\[
g^{(j)}_0(y_1, \ldots, y_r) = \sum_{\langle c_1(X), d \rangle = 0, \langle D_j, d \rangle < 0, \langle D_i, d \rangle \geq 0 \text{ for all } i \neq j} \frac{(-1)^{\langle D_j, d \rangle} (-\langle D_j, d \rangle - 1)!}{\prod_{i \neq j} \langle D_i, d \rangle!} y^d.
\]
Then under the mirror transformation we have
\[
\tilde{S}_j = \exp -g^{(j)}_0(y) \tilde{D}_j.
\]

Conversely, one can recover the Batyrev elements from the Seidel elements in the following way.

**Theorem 4.10** ([19, Theorem 1.2]). Given the Seidel elements \( \tilde{S}_1, \ldots, \tilde{S}_m \), the Batyrev elements \( \tilde{D}_j \in H^*(X) \otimes \mathbb{Q}[q_1, \ldots, q_r] \), \( j = 1, \ldots, m \) are uniquely characterized by the following conditions:
(a) \( \tilde{D}_j = H_j \tilde{S}_j \) for some \( H_j \in \mathbb{Q}[q_1, \ldots, q_r] \);
(b) \( \tilde{D}_j = \tilde{S}_j \) if \( b_j \) is a vertex of the fan polytope;
(c) \( \tilde{D}_j \) satisfy the linear relations (36).
In particular, the Seidel elements determine the mirror transformation \( q \mapsto y \) and the functions \( g^{(j)}_0(y) \).
4.4. Correction terms of potential functions and Seidel elements. Chan-Lau-Leung-Tseng [7] gave a conjecture relating the correction terms of the potential function and the Seidel elements for a semi-positive toric manifold.

Conjecture 4.11 ([7] Conjecture 5.2). For a semi-positive toric manifold, the correction term $f_j(q)$ of the potential function (Definition 4.4) coincides with $\exp(g_0^{(j)}(y))$ in Theorem 4.9 under mirror transformation.

Originally Chan-Lau-Leung-Tseng [7] proved this conjecture under the convergence assumption for $W$ using an isomorphism [13] of Jacobi ring and quantum cohomology. Recently they gave an alternative proof [8] which does not require the convergence assumption. They identified open Gromov-Witten invariants with certain closed Gromov-Witten invariants around the prime toric divisor.

4.5. Degeneration formula for toric manifolds.

Proposition 4.12. Assumption 3.18 holds for a pair $(X, L)$ equipped with the $\mathbb{C}^\times$-action $\rho_j$ around the prime toric divisor $D_j$ we considered in §4.3.

Proof. The statement (i) is shown in Proposition 4.1 and (ii), (iii) are obvious. To verify the statement (iv), it is enough to show that every stable map $u: C \to E_j$ representing a class $\sigma \in H^2_{\text{sec}}(E_j)$ with $\langle c_1^{\text{vert}}(E_j), \sigma \rangle = -1$ is contained in $\bigcup_{i=1}^m \hat{D}_i$, where

$$\hat{D}_i = D_i \times (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times$$

is a toric divisor of $E_j$. Let $C = \bigcup C_{\alpha}$ be an irreducible decomposition of $C$. If $u_*[C_{\alpha}]$ is a section class, we have $\langle c_1^{\text{vert}}(E_j), u_*[C_{\alpha}] \rangle \geq -1$ by (3) and the semi-positivity of $c_1(X)$. If $u_*[C_{\alpha}]$ is not a section class, $u(C_{\alpha})$ is contained in a fibre $X$ and we have $\langle c_1^{\text{vert}}(E_j), u_*[C_{\alpha}] \rangle = \langle c_1(X), u_*[C_{\alpha}] \rangle \geq 0$ again by the semi-positivity. Since $\langle c_1^{\text{vert}}(E_j), \sigma \rangle = -1$, we have

$$\langle c_1^{\text{vert}}(E_j), u_*[C_{\alpha}] \rangle = \begin{cases} -1 & \text{if } u(C_{\alpha}) \text{ is a section;} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $u(C) \not\subset \bigcup_{i=1}^m \hat{D}_i$. Then we can find a component $C_{\alpha}$ such that $u(C_{\alpha})$ is not a point and $u(C_{\alpha}) \not\subset \bigcup_{i=1}^m \hat{D}_i$. Then $\langle \hat{D}_i, u_*[C_{\alpha}] \rangle = 0$ for all $i$. Note that $\sum_{i=1}^m \hat{D}_i$ is the Poincaré dual of $c_1^{\text{vert}}(E_j)$. By the above calculation we see that $\langle \hat{D}_i, u_*[C_{\alpha}] \rangle = 0$ for all $i$ and $u(C_{\alpha})$ is contained in a certain fibre $X$. Then $\langle D_i, u_*[C_{\alpha}] \rangle = 0$ for all $i$. A homology class $d \in H_2(X)$ satisfying $\langle D_i, d \rangle = 0$ for all $i$ is zero. This is a contradiction.

Recall from Remark 4.2 that the potential function $W = W(z_1, \ldots, z_m)$ of a toric manifold $X$ is an element of

$$R := \mathbb{Q}[\text{NE}(X)_{\mathbb{Z}} + (\mathbb{Z}_{\geq 0})^m] \subset \Lambda^{\text{op}}.$$

We also set

$$K := \mathbb{Q}[\text{NE}(X)_{\mathbb{Z}}] \subset \Lambda.$$
Then $R$ is a $K$-algebra (cf. (33)). For $f \in R$, we write (following notation in Theorem 3.20):

$$df = \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_m} \right) \in \mathbb{Z}^m \otimes R \cong H_2(X, L) \otimes R.$$ 

In other words,

$$dz^\beta = \beta \otimes z^\beta$$

for $\beta \in H_2(X, L)$.

We apply Theorem 3.20 to the $\mathbb{C}^\times$-action $\rho_j$ rotating around $D_j$. Note that the $k$-th term $w_k$ of the potential $W$ in Definition 4.4 corresponds to the boundary class $b_k \in N \cong H_1(L)$ and $w_k = W_{b_k}$ in the notation of Definition 2.2. Since the Seidel element $\tilde{S}_j$ in §4.3 belongs to $H^2(X) \otimes K$, we have $\tilde{S}_j^{(0)} = 0$ and $\tilde{S}_j = \tilde{S}_j^{(2)}$. By Proposition 4.12, we can define the lift

$$\tilde{S}_j \in H^2(X, L) \otimes K$$

of $\tilde{S}_j = \tilde{S}_j^{(2)}$ as in Definition 3.19. The class $\lambda$ of an $S^1$-orbit on $L$ is $-b_j \in H_1(L)$ and the maximal disc class $\alpha_0$ is $\beta_j$. Hence we obtain:

**Theorem 4.13.** Assume that the degeneration formula (Conjecture 3.17) holds for $(X, L)$ equipped with the $\mathbb{C}^\times$-action $\rho_j$ around the toric divisor $D_j$ (see §4.3). Then we have

$$\langle \tilde{S}_j, dW \rangle = z_j.$$  

In particular, we have $\langle \tilde{S}_j, dW \rangle = z_j$.

### 4.5.1. Example

Consider the second Hirzebruch surface $\mathbb{F}_2 = \mathbb{P}(O_{\mathbb{P}^1}(-2) \oplus O_{\mathbb{P}^1})$, a compactification of $O_{\mathbb{P}^1}(-2)$. The divisor matrix (29) is:

$$(m_{ai}) = \begin{bmatrix} 0 & -2 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$ 

The column vectors give toric divisors classes $D_1, D_2, D_3, D_4$. Here $D_1$ is the $\infty$-section, $D_2$ is the zero-section ($-2$ curve) and $D_3, D_4$ are fibres. The potential function has been calculated by Auroux [3], Fukaya-Oh-Ohta-Ono [16] and Chan-Lau [6]:

$$W = z_1 + (1 + q_1)z_2 + z_3 + z_4.$$ 

Therefore we have

$$\begin{bmatrix} dw_1 \\ dw_2 \\ dw_3 \\ dw_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - q_1 \begin{bmatrix} z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} q_1 z_2 \\ q_1 z_2 \\ q_1 z_3 \\ q_1 z_4 \end{bmatrix}.$$ 

where we used $q_1 = z_2 z_3 z_4$ (see (33)) and $d(q_1 z_2) = [0, -q_1 z_2, q_1 z_2, q_1 z_2]$. Assuming the degeneration formula (38), we obtain

$$\begin{bmatrix} \tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{S}_4 \end{bmatrix} = [D_1, D_2, D_3, D_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ 

This is compatible with the calculations of $\tilde{S}_j$ by McDuff-Tolman [24] and González-Iritani [19].
4.6. **Kodaira-Spencer map.** Recall from Definition 4.4 that \( w_i = f_i(q)z_i \) for some \( f_i(q) \in K \). We have (using (33))

\[
\frac{z_i}{\partial w_j} = \left( \delta_{ij} + z_i \frac{\partial f_j(q)}{\partial z_i} \right) z_j = \left( \delta_{ij} + \sum_{a=1}^{r} m_{ai} q_a \frac{\partial f_j(q)}{\partial q_a} \right) z_j \in K z_j.
\]

Therefore we have an isomorphism of \( K \)-modules:

\[
\xi_s : H^2(X, L) \otimes K \cong \bigoplus_{j=1}^{m} K z_j, \quad D_i \mapsto \left( z_i \frac{\partial w_1}{\partial z_i}, \ldots, z_i \frac{\partial w_m}{\partial z_i} \right).
\]

The degeneration formula (38) says that \( \xi_s(\tilde{S}_i) = z_i \). For \( \varphi \in H^1(L) = M \), we have

\[
\xi_s(\delta \varphi) = \bigoplus_{j=1}^{m} \sum_{i=1}^{m} \langle \varphi, b_i \rangle z_i \frac{\partial w_j}{\partial z_i} = \bigoplus_{j=1}^{m} \sum_{i=1}^{m} \langle \varphi, b_i \rangle \left( z_i \delta_{ij} f_j(q) + z_i z_j \frac{\partial f_j(q)}{\partial z_i} \right) = \bigoplus_{j=1}^{m} \langle \varphi, b_j \rangle w_j \in \bigoplus_{j=1}^{m} K z_j,
\]

where \( \delta : H^1(L) \cong M \to H^2(X, L) \cong \mathbb{Z}^m \) is a coboundary map. Hence \( \xi_s \) induces an isomorphism

\[
\xi_s : H^2(X) \otimes K \cong \bigoplus_{j=1}^{m} K z_j / \left\langle \bigoplus_{j=1}^{m} \langle \varphi, b_j \rangle w_j : \varphi \in M \right\rangle_K.
\]

This satisfies \( \xi_s(\tilde{S}_i) = [z_i] \). Set \( B_j := f_j(q)\tilde{S}_j, j = 1, \ldots, m \). Then \( \xi_s(B_j) = f_j(q)[z_j] = [w_j], j = 1, \ldots, m \) satisfy the linear relations

\[
\sum_{j=1}^{m} \langle \varphi, b_j \rangle [w_j] = 0
\]

for all \( \varphi \in M \). Consequently,

- \( B_j = f_j(q)\tilde{S}_j \);
- \( f_j(q) = 1 \) if \( b_j \) is a vertex of the fan polytope (Proposition 4.5);
- \( B_j, j = 1, \ldots, m \) satisfy the linear relations (by the injectivity of \( \xi_s \)).

By the characterization of the Batyrev elements (see Theorem 4.10), we know that \( B_j = \tilde{D}_j \), i.e. \( f_j(q) = \exp(g_0^{(j)}(y)) \). This shows the conjecture of Chan-Leung-Tseng:

**Theorem 4.14.** Assume that the degeneration formula (Conjecture 3.17) holds for \( (X, L) \) equipped with the \( \mathbb{C}^\times \)-actions \( \rho_j, j = 1, \ldots, m \) in \( \S 4.3 \). Then Conjecture 4.11 holds.

**Remark 4.15.** Via the natural map \( \bigoplus_{j=1}^{m} K z_j \to R \), the map \( \xi_s \) induces the so-called Kodaira-Spencer map (cf. the discussion at the end of \( \S 3.3.3 \)):

\[
KS : H^2(X) \otimes K \to Jac(W)
\]

where the Jacobi algebra \( Jac(W) \) is defined to be

\[
Jac(W) := R/R(\langle H^1(L), dW \rangle).
\]
Then we have $KS(\tilde{S}_i) = [z_i]$ and $KS(\tilde{D}_i) = [w_i]$. In other words, the Seidel elements are the inverses of $[z_i]$ and the Batyrev elements are the inverses of $[w_i]$.

4.7. **Consistency check: computing equivariant Seidel elements.** Here we give a consistency check concerning Chan-Lau-Leung-Tseng Conjecture 4.11 and our degeneration formula (38). We calculate the lifts $\tilde{S}_j$ of Seidel elements assuming Conjecture 4.11 and (38) and see that the result is compatible with our previous calculation [19]. The lifts $\tilde{S}_j$ here should be viewed as the $\mathbb{T}$-equivariant Seidel elements since $H^2_\mathbb{T}(X) \cong H^2(X, L)$.

**Lemma 4.16.** Suppose that Conjecture 4.11 holds. Then $w_i = f_i(q)z_i$, $i = 1, \ldots, m$ satisfy the multiplicative relation

$$\prod_{j=1}^m w_j^{(D_j,d)} = y^d \quad \text{for all } d \in H_2(X; \mathbb{Z}).$$

In other words, $y_a = \prod_{j=1}^m w_j^{m_{aj}}$, $a = 1, \ldots, r$.

**Proof.** Recall that the Seidel and the Batyrev elements satisfy the multiplicative relations with respect to the quantum product (4.3):

$$\prod_{j=1}^m \tilde{D}_j^{(D_j,d)} = y^d, \quad \prod_{j=1}^m \tilde{S}_j^{(D_j,d)} = q^d.$$

Hence we have

$$\prod_{j=1}^m f_j(q)^{(D_j,d)} = y^d / q^d.$$

Therefore

$$\prod_{j=1}^m w_j^{(D_j,d)} = \prod_{j=1}^m (f_j(q)^{(D_j,d)}z_j^{(D_j,d)}) = (y^d / q^d) \cdot q^d = y^d.$$

\[\square\]

**Theorem 4.17.** Assume Conjecture 4.11 and the degeneration formula (38). The lifts $\tilde{S}_j$ of the Seidel elements are given by

$$\tilde{S}_j = e^{-g_0^{(j)}(w)} \left( D_j - \sum_{i=1}^m D_i \sum_{c_1(X) = 0, D_i < 0, D_k \cdot d \geq 0 \text{ for all } k \neq i} (-1)^{(D_j,d)} \langle D_j, d \rangle \frac{(D_i, d) - 1!}{\prod_{k \neq i} \langle D_k, d \rangle} y^d \right)$$

under the mirror transformation.

**Proof.** Note that $(dw_1, \ldots, dw_m)^T$ can be viewed as the Jacobi matrix between the two coordinate systems $(w_1, \ldots, w_m)$ and $(\log z_1, \ldots, \log z_m)$ on the open-closed moduli space. The degeneration formula (38) implies that $(z_1^{-1} \tilde{S}_1, \ldots, z_m^{-1} \tilde{S}_m)$ is the inverse Jacobi matrix, i.e.

$$z_j^{-1} \tilde{S}_j = \sum_{i=1}^m \frac{\partial \log z_i}{\partial w_j} D_i = w_j^{-1} \sum_{i=1}^m \frac{\partial \log z_i}{\partial \log w_j} D_i.$$
in $H^2(X, L)$. Assuming Conjecture 4.11, we have
\[ \log z_i = \log w_i - g_0^{(i)}(y). \]
Hence
\[
\hat{S}_j = \frac{z_j}{w_j} \sum_{i=1}^{m} \left( \delta_{ij} - w_j \frac{\partial g_0^{(i)}}{\partial w_j} \right) D_i
\]
\[
= \exp \left( -g_0^{(j)}(y) \right) \left( D_j - \sum_{i=1}^{m} \sum_{a=1}^{r} m_{aj} y_a \frac{\partial g_0^{(i)}}{\partial y_a} D_i \right)
\]
In the second line, we used Lemma 4.16. The conclusion follows.

Note that we did not use the lifts $\hat{S}_j$ of the Seidel elements (but used only the original Seidel elements $\tilde{S}_j$) in the proof of Theorem 4.14.

**Remark 4.18.** This result is compatible with the calculation of $\tilde{S}_j$ in our previous paper [19]. Note however that the formula in [19] Lemma 3.17] contains a mistake. It occurred from an erroneous cancellation between the factors $\langle D_j, d \rangle$ in the numerator and $\langle D_j, d \rangle!$ in the denominator.

**Remark 4.19.** It is not difficult to generalize the computation in [19] to the $\mathbb{T}$-equivariant setting and to check the above computation of $\hat{S}_j$ without using Conjecture 4.11 and the degeneration formula (38). Since Chan-Lau-Leung-Tseng’s conjecture 4.11 was proved by themselves [8], it follows that the degeneration formula (38) holds true in toric case.

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