ON THE ADJUNCTION FORMULA FOR 3-FOLDS IN CHARACTERISTIC $p > 5$

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ABSTRACT. In this article we prove a relative Kawamata-Viehweg vanishing-type theorem for PLT 3-folds in characteristic $p > 5$. We use this to prove the normality of minimal log canonical centers and the adjunction formula for codimension 2 subvarieties on $\mathbb{Q}$-factorial 3-folds in characteristic $p > 5$.

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1. Introduction

Let $(X, \Delta)$ be a log canonical pair and $W$ a minimal log canonical center, then (under mild technical assumptions) by Kawamata’s celebrated subadjunction theorem, it is known that $W$ is normal and we can write $(K_X + \Delta)|_W = K_W + \Delta_W$ where $(W, \Delta_W)$ is Kawamata log terminal [Kaw98] (see also [Kaw97b] and the references therein). The proof of this result is based on the Kawamata-Viehweg vanishing theorem and Hodge theory. These results are known to fail in characteristic $p > 0$ and therefore one may expect that Kawamata’s subadjunction also fails in this context. It should however be noted that related results have been obtained in the closely related context of $F$-singularities (see for example [Sch09], [HX15] and [Das15]) and that the minimal model program has been established for 3-folds in characteristic $p > 5$ (see [HX15] and [Bir13]). In particular [HX15] exploits the fact that PLT singularities in dimension 3 and characteristic $p > 5$ are closely related to the analogous notion of Purely $F$-regular singularities. In this paper, using the results from [HX15] and [Bir13], we show that in dimension 3 and characteristic $p > 5$ a relative version of the Kawamata-Viehweg vanishing theorem
holds and we use this to establish that (under some mild technical conditions) the analog of Kawamata’s subadjunction result holds.

**Theorem (Theorem 3.5).** Let \( f : (X, S + B \geq 0) \rightarrow Z \) be either a pl-divisorial contraction or a pl-flipping contraction. If the maximum dimension of the fibers of \( f \) is 1, then \( R^i f_* O_X(-S) = 0 \) for all \( i > 0 \).

This result allows us to prove the normality of the minimal LC centers for 3-folds.

**Theorem (Theorem 3.6, 4.9).** Let \((X, \Delta)\) be a \(\mathbb{Q}\)-factorial 3-fold log canonical pair such that \(X\) has Kawamata Log Terminal singularities. If \(W\) is a minimal log canonical center of \((X, \Delta)\), then \(W\) is normal. If moreover the coefficients of \(D\) belong to a DCC set \(I \subseteq [0, 1]\) and char \(k > \max\{5, \delta/2\}\), where \(\delta > 0\) is the minimum of the set \(D(I) \cap (0, 1]\) (where \(D(I)\) is defined in 4.1). Then the following hold:

1. There exists effective \(\mathbb{Q}\)-divisors \(\Delta_W\) and \(M_W\) on \(W\) such that \((K_X + \Delta)|_W \sim \mathbb{Q} K_W + \Delta_W + M_W\). Moreover, if \(\Delta = \Delta' + \Delta''\) with \(\Delta'\) (resp. \(\Delta''\)) the sum of all irreducible components which contain (resp. do not contain) \(W\), then \(M_W\) is determined only by the pair \((X, \Delta')\).

2. There exists an effective \(\mathbb{Q}\)-divisor \(M'_W\) such that \(M'_W \sim \mathbb{Q} M_W\) and the pair \((W, \Delta_W + M'_W)\) is KLT.

All of the results in this article hold in characteristic \(p > 5\) unless stated otherwise. We will use the standard terminologies and notations from [KM98]. We also use the abbreviations: LC for log canonical, KLT for Kawamata log terminal, PLT for purely log terminal, DLT for divisorially log terminal, NLC for non-log canonical centers, NKLT centers for non-Kawamata log terminal centers, and lct for log canonical thresholds. If \((X, \Delta)\) is LC, then the NKLT centers are also known as log canonical centers or LC centers.

## 2. Properties of Log Canonical Centers

In this section we establish some basic properties of LC centers.

**Lemma 2.1.** Let \(X\) be a \(\mathbb{Q}\)-factorial KLT 3-fold and \((X, \Delta \geq 0)\), a log canonical pair. Let \(W_1\) and \(W_2\) be two log canonical centers of \((X, \Delta)\). Then every irreducible component of \(W_1 \cap W_2\) is a log canonical center of \((X, \Delta)\).

**Proof.** There are three cases depending on the codimension of \(W_1\) and \(W_2\).
**Case I**: \( \text{codim}_X W_1 = \text{codim}_X W_2 = 1 \). In this case \( W_1 \) and \( W_2 \) are components of \( \Delta \). Let \( \Delta = W_1 + W_2 + \Delta \). Then by adjunction we have

\[
(K_X + W_1 + W_2 + \Delta)|_{W_1^n} = K_{W_1^n} + \text{Diff}_{W_1}(\Delta) + W_2|_{W_1^n},
\]

where \( W_1^n \to W_1 \) is the normalization. By localizing at the generic point of an irreducible component of \( W_1 \cap W_2 \) we reduce to a surface problem. Now, on a surface in characteristic \( p > 0 \), the relative Kawamata-Viehweg vanishing and Kollár’s connectedness theorem hold (see [Kol13, 10.13] and [Das15, 3.1]). Thus on a surface the intersection of two LC centers is a LC center and we are done by the usual argument (cf. [Kaw97a, Proposition 1.5]).

**Case II**: \( \text{codim}_X W_1 = 1 \) and \( \text{codim}_X W_2 = 2 \). Since \( X \) is \( \mathbb{Q} \)-factorial, \( (X, (1 - \epsilon)\Delta) \) is KLT for any \( 0 < \epsilon < 1 \). Thus by [Bir13, 7.7] there exists a \( \mathbb{Q} \)-factorial model \( f' : X' \to X \) of relative Picard number \( \rho(X'/X) = 1 \) such that \( \text{Ex}(f') \) is a unique exceptional divisor \( E' \) with center \( W_2 \), and

\[
K_{X'} + E' + W_1' + \Delta' = f'^*(K_X + \Delta), \tag{2.1}
\]

where \( \Delta' \geq 0 \), and \( W_1' \) is the strict transform of \( W_1 \) under \( f' \).

Since \( W_1' \) and \( E' \) are \( \mathbb{Q} \)-Cartier, they intersect along a curve (possibly reducible). Let \( C' \) be an irreducible component of \( W_1' \cap E' \). Then by Case I, \( C' \) is a LC center of \( (X', E' + W_1' + \Delta' \geq 0) \). Since every irreducible component of \( W_1 \cap W_2 \) is dominated by an irreducible component of \( W_1' \cap E' \), we are done by relation (2.1).

**Case III**: \( \text{codim}_X W_1 = \text{codim}_X W_2 = 2 \). Again, since \( X \) is \( \mathbb{Q} \)-factorial, \( (X, (1 - \epsilon)\Delta) \) is KLT for any \( 0 < \epsilon < 1 \). Thus by [Bir13, 7.7], there exists a \( \mathbb{Q} \)-factorial DLT model \( f' : X' \to X \) extracting two exceptional divisors (one at a time) \( E_1' \) and \( E_2' \) such that \( E_1' \cap E_2' \neq \emptyset \), \( f'(E_1') = W_1 \) and \( f'(E_2') = W_2 \), and

\[
K_{X'} + E_1' + E_2' + \Delta' = f'^*(K_X + \Delta). \tag{2.2}
\]

Since \( E_1' \) and \( E_2' \) are \( \mathbb{Q} \)-Cartier, they intersect along a curve (possibly reducible). Let \( C' \) be an irreducible component of \( E_1' \cap E_2' \). Then by Case I, \( C' \) is a LC center of \( (X', E_1' + E_2' + \Delta' \geq 0) \). Since every irreducible component of \( W_1 \cap W_2 \) is dominated by an irreducible component of \( E_1' \cap E_2' \), we are done by relation (2.2).

\[\square\]

The following proposition is a characteristic \( p > 5 \) version of Fujino’s adjunction theorem for DLT pairs (see [Cor07, 3.9.2] and [Kol13, 4.16]) on a \( \mathbb{Q} \)-factorial 3-fold.
Proposition 2.2 (DLT Adjunction). Let \((X, \Delta \geq 0)\) be a \(\mathbb{Q}\)-factorial DLT \(n\)-fold with \(n \leq 3\) such that \(\Delta = D_1 + D_2 + \cdots + D_r + B\) and \(\lfloor \Delta \rfloor = D_1 + D_2 + \cdots + D_r\). Assume that \(X\) has KLT singularities. Then following hold:

1. The \(s\)-codimensional log canonical centers of \((X, \Delta)\) are exactly the irreducible components of the various intersections \(D_{i_1} \cap \cdots \cap D_{i_s}\) for some \(\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, r\}\).
2. Every irreducible component of \(D_{i_1} \cap \cdots \cap D_{i_s}\) is normal and of pure codimension \(s\).
3. Let \(W\) be a log canonical center of \((X, \Delta)\), then there exists an effective \(\mathbb{Q}\)-divisor \(\Delta_W \geq 0\) on \(W\) such that \((K_X + \Delta)\vert_W \sim \mathbb{Q}K_W + \Delta_W\) and \((W, \Delta_W)\) is DLT.
4. If \(D_i \cap D_j = \emptyset\) for all \(i \neq j\), then \((X, \Delta)\) is in fact PLT.

Proof. The result is well known in dimension \(\leq 2\). (1) follows from the proof in [Kol13, Theorem 4.16].

Since \(X\) is \(\mathbb{Q}\)-factorial, \((X, D_i)\) is also PLT and then by adjunction \((D^n_i, \text{Diff}_{D^n_i})\) is KLT, where \(D^n_i \rightarrow D_i\) is the normalization. Since \(\text{Diff}_{D^n_i}\) has standard coefficients, by [Har98] and [HX15, 3.1], \((D^n_i, \text{Diff}_{D^n_i})\) is strongly \(F\)-regular in characteristic \(p > 5\). Then by [HX15, 4.1] and [Das15, 4.1, 5.4], \(D_i\) is normal. This proves that every irreducible component of \(\lfloor \Delta \rfloor\) is normal and hence (2) holds for \(s = 1\).

It is easy to see that \((D_i, \text{Diff}_{D_i}(\Delta - D_i))\) is DLT, and so \(D_i\) is a \(\mathbb{Q}\)-factorial surface by [FT12, 6.3]. (2) and (3) now follow from the result in dimension 2. (4) is immediate. \(\square\)

3. Vanishing Theorem and Minimal Log Canonical Centers

In this section we will prove some relative vanishing theorems and then use them to prove the normality of minimal log canonical centers and rationality of KLT singularities.

Definition 3.1. Let \(f : X \rightarrow Z\) be a projective birational morphism between normal quasi-projective varieties with relative Picard number \(\rho(X/Z) = 1\). Let \((X, S + B \geq 0)\) be a \(\mathbb{Q}\)-factorial PLT pair such \([S + B] = S\) is irreducible, and \(-S\) and \(-(K_X + S + B)\) are both \(f\)-ample.

1. If \(\dim \text{Ex}(f) = \dim X - 1\), then \(f : X \rightarrow Z\) is called a pl-divisorial contraction.
2. If \(\dim \text{Ex}(f) < \dim X - 1\), then \(f : X \rightarrow Z\) is called a pl-flipping contraction.
**Proposition 3.2.** Let \((X, S + B)\) be a \(\mathbb{Q}\)-factorial 3-fold PLT pair, where \(S\) is a prime Weil divisor. Assume that \((p^e - 1)(K_X + S + B)\) is an integral Weil divisor for some \(e > 0\). Then there exists an integer \(e_0 \gg 0\) such that the following sequence

\[
0 \rightarrow \mathcal{B}_{ne_0} \rightarrow F^*_{ne_0} \mathcal{O}_X((1 - p^{ne_0})(K_X + B) - p^{ne_0}S) \rightarrow \mathcal{O}_X(-S) \rightarrow 0
\]

is exact at all codimension 2 points of \(X\) contained in \(S\), for all \(n \geq 1\), where \(\phi_{ne_0}\) is defined by the trace map (see [Sch14] and [Pat14]) and \(\mathcal{B}_{ne_0}\) is the kernel of \(\phi_{ne_0}\).

**Proof.** By Proposition 2.2, \(S\) is normal. Since the question is local on \(X\), we may assume that \(X\) is affine. Then by [HX15, 2.13], we can choose an effective \(\mathbb{Q}\)-Cartier divisor \(G \geq 0\) not containing \(S\) and with sufficiently small coefficients such that \(K_X + S + B + G\) is \(\mathbb{Q}\)-Cartier with index not divisible by \(p\).

Localizing \(X\) at a codimension 2 point of \(X\) contained in \(S\), we may assume that \(X\) is an excellent surface. Then by adjunction we have \((K_X + S + B + G)|_S = K_S + B_S + G|_S\), where \(B_S\) is the Different. Since \((X, S + B)\) is PLT, \((S, B_S)\) is KLT by adjunction. Now, \((S, B_S)\) is strongly \(F\)-regular by [HX15, 2.2], since \(S\) is a smooth curve. Since the coefficients of \(G\) are sufficiently small, \((S, B_S + G|_S)\) is also strongly \(F\)-regular. Therefore we get the following surjection

\[
F^*_{e} \mathcal{O}_S((1 - p^{e})(K_S + B_S + G|_S)) \rightarrow \mathcal{O}_S,
\]

for all \(e \gg 0\) and sufficiently divisible.

We have the following commutative diagram

\[
\begin{array}{ccc}
F^*_{e} \mathcal{O}_X((1 - p^{e})(K_X + S + B + G)) & \rightarrow & F^*_{e} \mathcal{O}_S((1 - p^{e})(K_S + B_S + G|_S)) \\
\mathcal{O}_X & \rightarrow & \mathcal{O}_S
\end{array}
\]

To see the surjectivity of the top arrow note that since \(F^*_{e}\) is exact, it suffices to show that \((1 - p^{e})(K_X + S + B + G)|_S = (1 - p^{e})(K_S + B_S + G|_S)\), and since \((1 - p^{e})(K_X + S + B + G)\) and \((1 - p^{e})(K_S + B_S + G|_S)\) are Cartier for \(e \gg 0\), it suffices to show that this equality holds at codimension 1 points of \(S\), but this is clear since \((K_X + S + B + G)|_S = K_S + B_S + G|_S\). Since the ring \(\mathcal{O}_X\) is local, the surjectivity of the second vertical map (along with Nakayama’s Lemma) implies the surjectivity of the first vertical map, i.e.,

\[
F^{ne_0} \mathcal{O}_X((1 - p^{ne_0})(K_X + S + B + G)) \rightarrow \mathcal{O}_X
\]
is surjective for all \( n \geq 1 \), where \( e_0 \gg 0 \) is sufficiently divisible.

Since the map (3.3) factors through \( F^\neq_0 \mathcal{O}_X((1 - p^\neq_0)(K_X + B)) \), we get the following surjectivity

\[
F^\neq_0 \mathcal{O}_X((1 - p^\neq_0)(K_X + B)) \xrightarrow{\psi^\neq_0} \mathcal{O}_X.
\]

Let \( s \) be a pre-image of 1 under \( \psi^\neq_0 \), then we get the following splitting of \( \psi^\neq_0 \)

\[
\mathcal{O}_X \xrightarrow{s} F^\neq_0 \mathcal{O}_X((1 - p^\neq_0)(K_X + B)) \xrightarrow{\psi^\neq_0} \mathcal{O}_X.
\]

Twisting (3.5) by \( \mathcal{O}_X(-S) \) and taking reflexive hulls we get the following splitting

\[
\mathcal{O}_X(-S) \xrightarrow{s} F^\neq_0 \mathcal{O}_X((1 - p^\neq_0)(K_X + B) - p^\neq_0 S) \xrightarrow{\psi^\neq_0} \mathcal{O}_X(-S).
\]

In particular the morphism

\[
F^\neq_0 \mathcal{O}_X((1 - p^\neq_0)(K_X + B) - p^\neq_0 S) \xrightarrow{\psi^\neq_0} \mathcal{O}_X(-S)
\]

is surjective for all \( n \geq 1 \).

\[\square\]

**Remark 3.3.** In Proposition 3.2, if we further assume that the coefficients of \( B \) are in the standard set \( I = \{1 - \frac{1}{n} : n \geq 1\} \), then it follows that the sequence (3.1) is exact at all codimension 2 points of \( X \). Indeed, by localizing at a codimension 2 point \( P \in X \setminus S \), we may assume that \( (X, B) \) is an excellent surface. In this case the RHS of the sequence (3.1) takes the following form

\[
F^\neq_0 \mathcal{O}_X((1 - p^\neq_0)(K_X + B)) \xrightarrow{\phi^\neq_0} \mathcal{O}_X.
\]

Since \( (X, B) \) is a PLT surface and \( |B| = 0 \), \( (X, B) \) is KLT. Thus by [Har98] and [HX15, 3.1], \( (X, B) \) is strongly \( F \)-regular in char \( p > 5 \). Since the coefficients of \( G \) are sufficiently small, \( (X, B + G) \) is also strongly \( F \)-regular. Therefore we get the following surjectivity

\[
F^\neq_0 \mathcal{O}_X((1 - p^\neq_0)(K_X + B + G)) \xrightarrow{\phi^\neq_0} \mathcal{O}_X,
\]

for some \( e_0 \gg 0 \) and sufficiently divisible, and for all \( n \geq 1 \). Since the map in (3.9) factors through \( F^\neq_0 \mathcal{O}_X((1 - p^\neq_0)(K_X + B)) \), we get the following surjectivity

\[
F^\neq_0 \mathcal{O}_X((1 - p^\neq_0)(K_X + B)) \xrightarrow{\phi^\neq_0} \mathcal{O}_X.
\]
**Remark 3.4.** In Proposition 3.2, we can further show that the sequence (3.1) is exact at all codimension 3 points of \(X\) contained in \(S\), if the coefficients of \(B\) are in the standard set \(I = \{1 - \frac{1}{n} : n \geq 1\}\). Indeed, by localizing at a codimension 3 point of \(X\) contained in \(S\), we may assume that \(X\) is an excellent 3-fold. Then \((S, B_S)\) is a KLT surface pair. By [Har98] and [HX15, 3.1], \((S, B_S)\) is strongly \(F\)-regular in char \(p > 5\). The rest of the proof runs without any changes.

**Theorem 3.5.** Let \(f : (X, S + B \geq 0) \to Z\) be either a pl-divisorial contraction or a pl-flipping contraction. If the maximum dimension of the fibers of \(f\) is 1, then \(R^i f_* \mathcal{O}_X(-S) = 0\) for all \(i > 0\).

**Proof.** Since \(X\) is \(\mathbb{Q}\)-factorial, by perturbing the coefficients of \(B\) we may assume that \((p^e - 1)(K_X + S + B)\) is an integral Weil divisor for some \(e > 0\). Since \(f\) is birational and \(\text{Ex}(f) \subseteq \text{Supp}(S)\), it is enough to show that \(R^i f_* \mathcal{O}_X(-S) = 0\) in a neighborhood of \(f(S)\). Thus by restricting \((X, S + B)\) on a suitable neighborhood of \(S\) and by Proposition 3.2, we may assume that the following sequence is exact at all codimension 2 points of \(X\)

\[
\begin{align*}
0 & \longrightarrow B_e \longrightarrow F^e_* \mathcal{O}_X((1 - p^e)(K_X + B) - p^e S) \xrightarrow{\phi_e} \mathcal{O}_X(-S) \longrightarrow 0,
\end{align*}
\]

for all \(e \gg 0\) and sufficiently divisible.

The sequence (3.11) can be split into the following two exact sequences

\[
\begin{align*}
(3.12) & \quad 0 \longrightarrow B_e \longrightarrow F^e_* \mathcal{O}_X((1 - p^e)(K_X + B) - p^e S) \xrightarrow{\phi_e} \text{Im}(\phi_e) \longrightarrow 0 \\
& \quad \text{and} \\
(3.13) & \quad 0 \longrightarrow \text{Im}(\phi_e) \longrightarrow \mathcal{O}_X(-S) \longrightarrow Q_e \longrightarrow 0,
\end{align*}
\]

where \(Q_e\) is the corresponding quotient.

Pushing forward the exact sequence (3.12) by \(f_*\) we get

\[
R^i f_* (F^e_* \mathcal{O}_X((1 - p^e)(K_X + B) - p^e S)) \to R^i f_* \text{Im}(\phi_e) \to R^{i+1} f_* B_e.
\]

Now \(R^{i+1} f_* B_e = 0\) for all \(i > 0\), since the maximum dimension of the fiber of \(f\) is 1.

Let \(r\) be the index of \(K_X + S + B\) and \(H = -(K_X + S + B)\). By the division algorithm, there exist integers \(k \geq 0\) and \(0 \leq b < r\) such that \((p^e - 1) = r \cdot k + b\).

Then by Serre vanishing

\[
R^i f_* (F^e_* \mathcal{O}_X((1 - p^e)(K_X + B) - p^e S)) = F^e_* (R^i f_* \mathcal{O}_X(k \cdot r H - b(K_X + S + B) - S)) = 0,
\]

for all \(i > 0\).
for all \( e \gg 0 \) and sufficiently divisible, and \( i > 0 \), since \( H \) is \( f \)-ample.

Thus from (3.14) we get
\[
R^i f_* \text{Im}(\phi_e) = 0,
\]
for all \( i > 0 \).

Again, pushing forward the exact sequence (3.13) by \( f_* \) we get
\[
R^i f_* \text{Im}(\phi_e) \rightarrow R^i f_* O_X(-S) \rightarrow R^i f_* Q_e.
\]
\( R^i f_* Q_e = 0 \) for all \( i > 0 \), since \( Q_e \) is supported at finitely many points, by (3.11). Thus we have
\[
R^i f_* O_X(-S) = 0,
\]
for all \( i > 0 \).

\textbf{Theorem 3.6.} Let \((X, \Delta)\) be a \( \mathbb{Q} \)-factorial 3-fold log canonical pair such that \( X \) has KLT singularities. If \( W \) is a minimal log canonical center of \((X, \Delta)\), then \( W \) is normal.

\textbf{Proof.} Since \( X \) is \( \mathbb{Q} \)-factorial and KLT, \((X, (1-\epsilon)\Delta)\) is KLT for any \( 0 < \epsilon < 1 \), and all log canonical centers of \((X, \Delta)\) are contained in \( \Delta \). Then by Reid’s Tie Breaking trick (see [Cor07, 8.7.1]) we may assume that \( W \) is the unique log canonical center of \((X, \Delta)\) with a unique divisor over \( X \) of discrepancy \(-1\). There are two cases depending on the codimension of \( W \).

\textbf{Case I:} \( \text{codim}_X(W) = 1 \). Since \( X \) is \( \mathbb{Q} \)-factorial, \((X, W)\) is log canonical. By adjunction \((K_X + W)|_{W^n} = K_{W^n} + \text{Diff}_{W^n}\), where \( W^n \rightarrow W \) is the normalization and \((W^n, \text{Diff}_{W^n})\) is KLT. Thus by [Har98] and [HX15, 3.1], \((W^n, \text{Diff}_{W^n})\) is strongly \( F \)-regular in characteristic \( p > 5 \). Then \( W^n = W \), i.e., \( W \) is normal by [HX15, 4.1] or [Das15, 4.1].

\textbf{Case II:} \( \text{codim}_X(W) = 2 \). Let \( f : (Y, S + \Delta') \rightarrow (X, \Delta) \) be an extraction of the unique exceptional divisor \( S \) over \( X \) such that
\[
K_Y + S + \Delta' = f^*(K_X + \Delta).
\]
Note that \(-S\) is a \( f \)-ample. Since \((Y, S + \Delta')\) is PLT, \( S \) is normal by Proposition 2.2. Also, since \( Y \) is \( \mathbb{Q} \)-factorial, \((Y, S)\) is PLT.

Consider the following exact sequence
\[
0 \rightarrow O_Y(-S) \rightarrow O_Y \rightarrow O_S \rightarrow 0.
\]
Since $W$ is contained in the support of $\Delta$, $\Delta' \cap S \neq \emptyset$, and hence $-(K_Y + S)$ is $f$-ample. Thus $f : (Y, S) \to X$ is a pl-divisorial contraction. Then by Theorem 3.5, $R^1 f_* \mathcal{O}_Y(-S) = 0$, and we get the following exact sequence

$$(3.18) \quad 0 \longrightarrow f_* \mathcal{O}_Y(-S) \longrightarrow f_* \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_S \longrightarrow 0.$$ 

Since $f_* \mathcal{O}_Y(-S) = \mathcal{I}_W$ and $f_* \mathcal{O}_Y = \mathcal{O}_X$, we get

$$(3.19) \quad 0 \longrightarrow \mathcal{I}_W \longrightarrow \mathcal{O}_X \longrightarrow f_* \mathcal{O}_S \longrightarrow 0.$$ 

Now $\mathcal{O}_X \to f_* \mathcal{O}_S$ factors in the following way

$$(3.20) \quad \mathcal{O}_W \longrightarrow \mathcal{O}_X \longrightarrow f_* \mathcal{O}_S$$

where $\nu : W^n \to W$ is the normalization morphism.

Hence $\mathcal{O}_W = \nu_* \mathcal{O}_{W^n}$, i.e. $W$ is normal. \hfill \square

4. ADJUNCTION FORMULA

In this section we will prove an adjunction formula for 3-folds in characteristic $p > 5$. To start with we will need the following definitions and results.

**Definition 4.1** (DCC sets). We say that a set $I$ of real numbers satisfies the *descending chain condition* or DCC, if it does not contain any infinite strictly decreasing sequence. For example,

$$I = \left\{ \frac{r-1}{r} : r \in \mathbb{N} \right\}$$

satisfies the DCC.

Let $I \subseteq [0, 1]$. We define

$$I_+ := \{ j \in [0, 1] : j = \sum_{p=1}^{l} i_p \text{ for some } i_1, i_2, \ldots, i_l \in I \}$$

and

$$D(I) := \{ a \leq 1 : a = \frac{m-1+f}{m}, m \in \mathbb{N}, f \in I_+ \}.$$
Lemma 4.2. [MP04, 4.4] Let $I \subseteq [0, 1]$. Then
(1) $D(D(I)) = D(I) \cup \{1\}$.
(2) $I$ satisfies DCC if and only if $\bar{I}$ satisfies the DCC, where $\bar{I}$ is the closure of $I$.
(3) $I$ satisfies DCC if and only if $D(I)$ satisfies the DCC.

Lemma 4.3. [CGS14, Lemma 2.3][MP04, Lemma 4.3][HMX14, Lemma 4.1] Let $(X, \Delta \geq 0)$ be a log canonical pair such that the coefficients of $\Delta$ belong to a set $I \subseteq [0, 1]$. Let $S$ be a normal irreducible component of $[\Delta]$ and $\Theta \geq 0$ be the $\mathbb{Q}$-divisor on $S$ defined by adjunction:

$$(K_X + \Delta)|_S = K_S + \Theta.$$  

Then, the coefficients of $\Theta$ belong to $D(I)$.

Definition 4.4 (Divisorial part and Moduli part). Let $f : X \to Z$ be a surjective proper morphism between two normal varieties and $K_X + D \sim_{\mathbb{Q}} f^*L$, where $D$ is a boundary divisor on $X$ and $L$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $Z$. Let $(X, D)$ be LC near the generic fiber of $f$, i.e., $(f^{-1}U, D|_{f^{-1}U})$ is LC for some Zariski dense open subset $U \subseteq Z$. Then we define two divisors $D_{\text{div}}$ and $D_{\text{mod}}$ on $Z$ in the following way:

$$D_{\text{div}} = \sum (1 - c_Q)Q,$$

where $Q \subseteq Z$ are prime Weil divisors of $Z$,

$$c_Q = \sup \{c \in \mathbb{R} : (X, D + cf^*Q) \text{ is LC over the generic point } \eta_Q \text{ of } Q \}$$

and $D_{\text{mod}} = L - K_Z - D_{\text{div}}$, so that $K_X + D \sim_{\mathbb{Q}} f^*(K_Z + D_{\text{div}} + D_{\text{mod}})$.

Remark 4.5. Observe that $D_{\text{div}}$ is a fixed divisor on $Z$, called the Divisorial part and $D_{\text{mod}}$ is a $\mathbb{Q}$-linear equivalence class on $Z$, called the Moduli part. For other properties of $D_{\text{div}}$ and $D_{\text{mod}}$ see [PS09, Section 7] and [Amb99, Section 3].

Let $\overline{\mathcal{M}}_{0,n}$ be the moduli space of $n$-pointed stable curves of genus 0, $f_{0,n} : \overline{U}_{0,n} \to \overline{\mathcal{M}}_{0,n}$ the universal family, and $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n$, the sections of $f_{0,n}$ which correspond to the marked points (see [Kee92] and [Kmu83]). Let $d_j$ $(j = 1, 2, \cdots, n)$ be rational numbers such that $0 < d_j \leq 1$ for all $j$, $\sum_j d_j = 2$ and $D = \sum_j d_j \mathcal{P}_j$.

Lemma 4.6.  
(1) There exists a smooth projective variety $\mathcal{U}_{0,n}^*$, a $\mathbb{P}^1$-bundle $g_0 : \mathcal{U}_{0,n}^* \to \overline{\mathcal{M}}_{0,n}$, and a sequence of blowups with smooth centers

$$\overline{U}_{0,n} = \mathcal{U}^{(1)} \xrightarrow{\sigma_2} \mathcal{U}^{(2)} \xrightarrow{\sigma_3} \cdots \xrightarrow{\sigma_{n-2}} \mathcal{U}^{(n-2)} = \mathcal{U}_{0,n}^*.$$

(2) Let $\sigma : \overline{U}_{0,n} \to \mathcal{U}_{0,n}^*$ be the induced morphism, and $D^* = \sigma_* D$. Then $K_{\overline{U}_{0,n}} + D - \sigma^*(K_{\mathcal{U}_{0,n}^*} + D^*)$ is effective.
(3) There exists a semi-ample $\mathbb{Q}$-divisor $L$ on $\overline{M}_{0,n}$ such that
$$K_{\overline{M}_{0,n}} + D^* \sim_\mathbb{Q} g^*_0(M_{0,n} + L).$$

Proof. The proof in [Kaw97b, Theorem 2] works in positive characteristic without any change (see also [CTX13, 6.7], [PS09, 8.5] and [KMM94, Section 3]). □

**Lemma 4.7** (Stable Reduction Lemma). Let $B$ be a smooth curve and $f : X \to B$, a flat family of rational curves such that the general fiber is isomorphic to $\mathbb{P}^1$, and a unique singular fiber $X_0$ over $0 \in B$. Also assume that $f|_{X^*} : (X^* = X \setminus X_0; \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n) \to B^* = B - \{0\}$ is a flat family of $n$-pointed stable rational curves sitting in the following commutative diagram

$$
\begin{array}{ccc}
X^* &=& B^* \times \overline{M}_{0,n}
\end{array}
$$

Then there exists a unique flat family $\hat{f} : \hat{X} \to B$ of $n$-pointed stable rational curves satisfying the following commutative diagram

$$
\begin{array}{ccc}
X^* &=& B^* \times \overline{M}_{0,n}
\end{array}
$$

where the broken horizontal map is a birational map such that $f^{-1}B^* \cong \hat{f}^{-1}B^*$.

Proof. Since $\overline{M}_{0,n}$ is a proper scheme, by the valuative criterion of properness any morphism $B^* \to \overline{M}_{0,n}$ extends uniquely to a morphism $B \to \overline{M}_{0,n}$. Now since $\overline{M}_{0,n}$ has a universal family $\overline{U}_{0,n}$, the existence of $\hat{f} : \hat{X} \to B$ follows by taking the fiber product. □

**Theorem 4.8** (Canonical Bundle Formula). Let $f : X \to Z$ be a proper surjective morphism, where $X$ is a normal surface and $Z$ is a smooth curve over an algebraically closed field $k$ of char $k > 0$. Assume that $Q = \sum_i Q_i$ is a divisor on $Z$ such that $f$ is smooth over $Z - \text{Supp}(Q)$ with fibers isomorphic to $\mathbb{P}^1$. Let $D = \sum d_j P_j$ be a $\mathbb{Q}$-divisor on $X$, where $d_j = 0$ is allowed, which satisfies the following conditions:

1. $(X, D \geq 0)$ is KLT.
2. $D = D^h + D^v$, where $D^h = \sum f(D_j) = Z d_j D_j$ and $D^v = \sum f(D_j) \neq Z d_j D_j$.

An irreducible component of $D^h$ (resp. $D^v$) is called horizontal (resp. vertical) component.
\( \text{char } k = p > \frac{2}{\delta}, \) where \( \delta \) is the minimum non-zero coefficient of \( D_h \).

(4) \( K_X + D \sim_{\mathbb{Q}} f^*(K_Z + M) \) for some \( \mathbb{Q} \)-Cartier divisor \( M \) on \( Z \).

Then there exist an effective \( \mathbb{Q} \)-divisor \( D_{\text{div}} \geq 0 \) and a semi-ample \( \mathbb{Q} \)-divisor \( D_{\text{mod}} \geq 0 \) on \( Z \) (as defined in 4.4) such that

\[
K_X + D \sim_{\mathbb{Q}} f^*(K_Z + D_{\text{div}} + D_{\text{mod}}).
\]

Proof. The sketch of the proof of this formula is given in [CTX13, 6.7]. We include a complete proof following the idea of the proof of [PS09, Theorem 8.1].

First we reduce the problem to the case where all components of \( D_h \) are sections. Let \( D_{i_0} \) be a horizontal component of \( D \) and \( Z' \to D_{i_0} \) be the normalization of \( D_{i_0} \). Then \( \nu : Z' \to Z \) is a finite surjective morphism of smooth curves. Let \( X' \) be the normalization of the component of \( X \times_Z Z' \) dominating \( Z \).

\[
X \xrightarrow{\nu} X' \quad f \quad f' \quad Z \xleftarrow{\nu} Z'.
\]

Let \( k = \text{deg}(\nu : Z' \to Z) \) and \( l \) be a general fiber of \( f \). Then

\[
k = D_i \cdot l \leq \frac{1}{d_i}(D \cdot l) = \frac{1}{d_i}(-K_X \cdot l) = \frac{2}{d_i} \leq \frac{2}{\delta} < \text{char } k.
\]

Therefore \( \nu : Z' \to Z \) is a separable morphism.

Let \( D' \) be the log pullback of \( D \) under \( \nu' \), i.e.,

\[
K_{X'} + D' = \nu'^*(K_X + D).
\]

More precisely we have (by [Kol92, 20.2])

\[
D' = \sum_{i,j} d'_{ij} D'_{ij}, \quad \nu'(D'_{ij}) = D_i, \quad d'_{ij} = 1 - (1 - d_i) e_{ij},
\]

where \( e_{ij} \)'s are the ramification indices along the \( D_{ij} \)'s.

By construction \( X \) dominates \( Z \). Also, since \( \nu \) is etale over a dense open subset of \( Z \), say, \( \nu^{-1} U \to U \), and etale morphisms are stable under base change, \( (f' \circ \nu)^{-1} U \to f^{-1} U \) is etale. Thus the ramification locus \( \Lambda \) of \( \nu' \) does not contain any horizontal divisor of \( f' \), i.e., \( f'(\Lambda) \neq Z' \). Therefore \( D' \) is a boundary near the generic fiber of \( f' \), i.e., \( D'^h \) is effective. We observe that the coefficients of \( D'^h \) can be computed by intersecting with a general fiber of \( f' : X' \to Z' \), hence they are equal to the coefficients of \( D^h \subseteq X \). Thus the
condition \( p > \frac{2}{5} \) remains true for \( D' \) on \( X' \).

After finitely many such base changes let \( g : \tilde{X} \to \tilde{Z} \) be a family such that all of the horizontal components of \( D_{\tilde{X}} \) are sections of \( g \), where \( D_{\tilde{X}} \) is the log pullback of \( D \), i.e., \( K_{\tilde{X}} + D_{\tilde{X}} = \psi^*(K_X + D) \).

\[
\begin{array}{c}
X \xleftarrow{\psi} \tilde{X} \\
\downarrow f \quad \downarrow g \\
Z \xleftarrow{\psi_0} \tilde{Z}
\end{array}
\]

By Lemma 4.7, we get a family of \( n \)-pointed stable rational curves \( \bar{X} = \tilde{Z} \times_{\overline{\mathcal{M}}_{0,n}} \overline{U}_{0,n} \to \tilde{Z} \). Let \( X' \) be the common resolution of \( \bar{X} \) and \( \tilde{X} \). Let \( \tilde{X} = \tilde{Z} \times_{\overline{\mathcal{M}}_{0,n}} \overline{U}^*_{0,n} \). By the universal property of fiber products there exists a morphism \( \mu : X' \to \tilde{X} \). Since \( X' \), \( \bar{X} \) and \( \tilde{X} \) are all isomorphic \( \mathbb{P}^1 \)-bundles over a dense open subset \( U \subseteq \tilde{Z} \), \( \mu : X' \to \tilde{X} \) is birational.

\[
\begin{array}{c}
\begin{array}{c}
X' \quad X \\
\pi \quad \psi \\
\downarrow \lambda \quad \downarrow f
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\tilde{X} \quad \bar{U}_{0,n} \\
\phi \quad \psi_0 \\
\downarrow \phi_0 \quad \downarrow f_{0,n}
\end{array}
\end{array}
\end{array}
\]

Let \( D' \) and \( \hat{D} \) be \( \mathbb{Q} \)-divisors on \( X' \) and \( \tilde{X} \) respectively, defined by

\[
K_{X'} + D' = \pi^*(K_X + D).
\]

and

\[
K_{\tilde{X}} + \hat{D} = \mu_*(K_{X'} + D').
\]

Since \( K_{X'} + D' \) is a pullback from the base \( \tilde{Z} \) (by (4.7)), by the Negativity lemma we get

\[
K_{X'} + D' = \mu^*(K_{\tilde{X}} + \hat{D}).
\]

Since the definition of the divisorial part of the adjunction does not depend on the birational modification of the family (see [PS09, Remark 7.3(ii)] or [Amb99, Remark 3.1]), we will define it with respect to \( \tilde{f} : \tilde{X} \to \tilde{Z} \). First we will show that the \( \mathbb{Q} \)-divisor \( \hat{D}_{\text{mod}} \) on \( \tilde{Z} \) is semi-ample.
Since $\hat{\phi}$ is finite and $D^*$ is horizontal it follows that $\hat{\phi}^* D^*$ is horizontal too. Since $\hat{D}^h$ is also horizontal one sees that
\begin{equation}
\hat{D}^h = \hat{\phi}^* D^*.
\end{equation}
From the construction of $\sigma : \mathcal{U}_{g_0,n} \to \mathcal{U}_{\tilde{g}_0,n}$ we see that $(F, D^*|_F)$ is log canonical for any fiber $F$ of $g_0,n : \mathcal{U}_{g_0,n} \to \mathcal{M}_{g_0,n}$. Since the fibers of $\tilde{f} : \tilde{X} \to \tilde{Z}$ are isomorphic to the fibers of $g_0,n$, $(\tilde{F}, \hat{D}^h|_{\tilde{F}})$ is also log canonical, where $\tilde{F}$ is a fiber of $\tilde{f}$. Finally, since $\tilde{X}$ is a surface, by inversion of adjunction $(\tilde{X}, \tilde{D}^h)$ is log canonical near $\tilde{F}$. Thus, since the fibers of $\tilde{f}$ are reduced, the lct of $(\tilde{X}, \tilde{D}; \tilde{F})$ over the generic point of $\tilde{F}$ is $(1 - \text{coeff} \cdot \tilde{F})$. Hence we get
\begin{equation}
\hat{D}^v = \tilde{f}^* \hat{D}^\text{div}.
\end{equation}
By definition of $D_{\text{mod}}$ we have
\begin{equation}
K_{\tilde{X}} + \hat{D}^h \sim_{\mathbb{Q}} \hat{\phi}^*(K_{\tilde{Z}} + \hat{D}_{\text{mod}}).
\end{equation}
Then we have
\begin{equation}
K_{\tilde{X}} + \hat{D}^h - \hat{\phi}^*(K_{\tilde{Z}} + \phi_0^* L) = K_{\tilde{X}/\tilde{Z}} + \hat{D}^h - \hat{\phi}^* K_{\mathcal{U}_{g_0,n}/\mathcal{M}_{g_0,n}} - \phi^* D^* \sim_{\mathbb{Q}} 0,
\end{equation}
where the first equality follows from (4.12) and Lemma 4.6, and the second relation from (4.10) and [Liu02, Chapter 6, Theorem 4.9 (b) and Example 3.18].

Since $\tilde{f}$ has connected fibers, by (4.12) and (4.13) and the projection formula for locally free sheaves, we get
\begin{equation}
\hat{D}_{\text{mod}} \sim_{\mathbb{Q}} \phi_0^* L
\end{equation}
i.e., $\hat{D}_{\text{mod}}$ is semi-ample.

Now, since $\psi : \tilde{Z} \to Z$ is a composition of finite morphisms of degree strictly less than char $k$, by [Kol13, Corollary 2.43] and [Amb99, Theorem 3.2] (also see [CTX13, 6.6]) we get
\begin{equation}
K_{\tilde{Z}} + \hat{D}_{\text{div}} \sim_{\mathbb{Q}} \psi_0^*(K_Z + D_{\text{div}}).
\end{equation}
Therefore
\begin{equation}
\psi_0^* D_{\text{mod}} \sim_{\mathbb{Q}} \hat{D}_{\text{mod}}
\end{equation}
Since $Z$ and $\tilde{Z}$ are both smooth curves, $D_{\text{mod}}$ is semi-ample.

Theorem 4.9. Let $(X, D \geq 0)$ be a $\mathbb{Q}$-factorial 3-fold log canonical pair such that the coefficients of $D$ are contained in a DCC set $I \subset [0, 1]$. Let $W$ be a minimal log canonical center of $(X, D)$, and codimension of $W$ is 2. Also...
assume that $X$ has KLT singularities and char $k > \max\{5, \frac{3}{2}\}$, where $\delta$ is the non-zero minimum of the set $D(I)$ (defined in 4.1). Then the following hold:

1. $W$ is normal.
2. There exists effective $\mathbb{Q}$-divisors $D_W$ and $M_W$ on $W$ such that $(K_X + D)|_W \sim_{\mathbb{Q}} K_W + D_W + M_W$. Moreover, if $D = D' + D''$ with $D'$ (resp. $D''$) the sum of all irreducible components which contain (resp. do not contain) $W$, then $M_W$ is determined only by the pair $(X, D')$.
3. There exists an effective $\mathbb{Q}$-divisor $M'_W$ such that $M'_W \sim_{\mathbb{Q}} M_W$ and the pair $(W, D_W + M'_W)$ is KLT.

Proof. Normality of $W$ follows from Theorem 3.6.

Since $X$ is $\mathbb{Q}$-Cartier, $(K_X + D)|_W = (K_X + D'| + D'')|_W = (K_X + D')|_W + D''|_W$. Thus we may assume that all components of $D$ contain $W$. Since $W$ is a minimal log canonical center of $(X, D)$ and $\text{codim}_X W = 2$, it does not intersect any other LC center of codimension $\geq 2$, by Lemma 2.1. Thus by shrinking $X$ (removing closed subsets of codimension $\geq 2$ which do not intersect $W$) if necessary we may assume that $W$ is the unique log canonical center of codimension $\geq 2$ of $(X, D)$.

Let $f : (X', D') \rightarrow (X, D)$ be a $\mathbb{Q}$-factorial DLT model over $(X, D)$ such that

\begin{equation}
(4.16) \quad K_{X'} + D' = f^*(K_X + D).
\end{equation}

Such $f$ exists by [KK10, 3.1] and [Bir13].

Note that, since $X$ is $\mathbb{Q}$-factorial, the exceptional locus of $f$ supports an effective anti-ample divisor. In particular all positive dimensional fibers of $f$ are contained in the support of $[D']$.

Let $E$ be an exceptional divisor dominating $W$. Then $E$ is normal by Proposition 2.2. Write $D' = E + \sum d_i f_*^{-1} D_i$. By adjunction we have

\begin{equation}
(4.17) \quad K_E + D'_E = (K_{X'} + D')|_E = f^*((K_X + D)|_W)
\end{equation}

and $(E, D'_E)$ is DLT, by Proposition 2.2 and the coefficients of $D'_E$ are in the set $D(I)$ by Lemma 4.3.

By Theorem 4.8, there exist $\mathbb{Q}$-divisors $D_W \geq 0$ and $M_W \geq 0$ on $W$ such that

\begin{equation}
(4.18) \quad K_E + D'_E \sim_{\mathbb{Q}} f|_E^*(K_W + D_W + M_W).
\end{equation}
Since \( f|_E : E \to W \) has connected fibers, from (4.17), (4.18) and the projection formula for locally free sheaves, we get
\[
(4.19) \quad (K_X + D)|_W \sim_Q K_W + D_W + M_W.
\]
Lemma 4.10 given below shows that \( D_W \) is independent of the choice of the exceptional divisor \( f \) dominating \( W \).

From the definition of \( D_W \) we see that \( D_W \geq 0 \), since \( D'_E \geq 0 \). Also, since \( D_W \) is independent of the birational modifications (by [PS09, Remark 7.3(ii)]) and \( W \) is a minimal LC center, by taking a log resolution of \((X', D')\) and working on the strict transform of \( E \), we see that the coefficients of \( D_W \) are strictly less than 1. Thus \( \lfloor D_W \rfloor = 0 \).

Since \( M_W \) is semi-ample and \( W \) is a smooth curve, either \( M_W = 0 \) or \( M_W \) is ample. In the later case by Bertini’s theorem there exists an effective \( \mathbb{Q} \)-divisor \( M'_W \sim_Q M_W \) such that \( \lfloor M'_W \rfloor = 0 \) and \( \text{Supp}(M'_W) \cap \text{Supp}(D_W) = \emptyset \). Hence \((W, D_W + M'_W)\) is KLT.

\[\square\]

**Lemma 4.10.** With the same hypothesis as in Theorem 4.9, the divisor \( D_W = D_{\text{div}} \) on \( W \) is independent of the choice of the exceptional divisors dominating \( W \).

**Proof.** Let \( E_1 \) and \( E_2 \) be two exceptional divisors of \( f \) dominating \( W \) such that
\[
(4.20) \quad K_{X'} + E_1 + E_2 + \Delta' = f^*(K_X + D),
\]
where \( f : X' \to X \) is the DLT model as above and \( D' = E_1 + E_2 + \Delta' \).

Notice that if \( \eta_W \) is the generic point of \( W \), then \( f^*\eta_W \cap \text{NKLT}(X', E_1 + E_2 + \Delta) \) is connected (By localizing at \( \eta_W \), this follows from a surface computation involving relative Kawamata-Viehweg vanishing theorem). Therefore we may assume that \( E_1 \cap E_2 \neq \emptyset \).

By adjunction on \( E_1 \) we get
\[
(4.21) \quad K_{E_1} + C + \Delta'_{E_1} = f^*((K_X + D)|_W),
\]
where \( C \) is an irreducible component of \( E_1 \cap E_2 \).

Adjunction on \( C \) gives
\[
(4.22) \quad K_C + \Delta'_C = f^*((K_W + D)|_W).
\]
Let \( Q \) be a point on \( W \), and \( t = \text{lct}(E_1, C + \Delta'_{E_1}; f^*Q) \) and \( s = \text{lct}(C, \Delta'_C; f^*Q|_C) \). Since \( C \) is an irreducible component of \( E_1 \cap E_2 \) dominating \( W \), it is enough to
Thus the curves $G_{NKLT(\lambda)}$ are contained in $NLC(E_1+C+\Delta_{E_1}+\lambda C)$ of some appropriate divisors from the base to $\lambda C$. Let $\lambda > 0$ such that
\[
(H - \lambda C') \cdot F_\eta = 0.
\]
Then $(H - \lambda C')|_{F_\eta} \sim_{\mathbb{Q}} 0$. Thus by [Cor07, 8.3.4], $H \sim_{\mathbb{Q}} \lambda C' - \sum \lambda_i F_i$, where the $F_i$'s are irreducible components of some fibers of $f$. By adding pullback of some appropriate divisors from the base to $\lambda C' - \sum \lambda_i F_i$, we may assume that $\lambda_i > 0$ for all $i$ and $\lambda C' - \sum \lambda_i F_i$ is $f$-ample.

Assume that there exists a point $P \in f^{-1}Q$ but $P \notin C$ such that $(E_1, C + \Delta_{E_1} + (t + \epsilon)f*Q)$ is not LC at $P$, where $0 < \epsilon \ll 1$ such that $t + \epsilon < s$. Then by choosing $0 < \lambda, \lambda_i \ll 1$ we can assume that $(C + \Delta_{E_1} - \lambda C' + \sum \lambda_i F_i) \geq 0$, $(E_1, C + \Delta_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon)f*Q)$ still not LC at $P$, and
\[
(4.23) \quad -(K_{E_1} + C + \Delta_{E_1} - \lambda C' + \sum \lambda_i F_i) = -f*((K_X + D)|_W) + (\lambda C' - \sum \lambda_i F_i)
\]
is $f$-ample.

Then by [Bir13, 8.3], $NKLT(E_1, C + \Delta_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon)f*Q) \cap f^{-1}Q$ is connected. Let $R \in C \cap f^{-1}Q$. Then there exists a chain of curves $G_i$’s connecting $R$ and $P$, and contained in $NKLT(E_1, C + \Delta_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon)f*Q) \cap f^{-1}Q$.

Now $NKLT(E_1, C + \Delta_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon)f*Q) \subseteq NKLT(E_1, C + \Delta_{E_1} + \sum \lambda_i F_i + (t + \epsilon)f*Q)$. Since we are only concentrating on the NKL centers along $f^{-1}Q$, we may assume that $F_i$’s are all contained in $f^{-1}Q$. Then by choosing $0 < \lambda_i \ll 1$ for all $i$, such that $t + \epsilon' = t + \epsilon + \max \{\lambda_i\} < s$, we see that $NKLT(E_1, C + \Delta_{E_1} + \sum \lambda_i F_i + (t + \epsilon)f*Q) \subseteq NKLT(E_1, C + \Delta_{E_1} + (t + \epsilon')f*Q)$. Thus the curves $G_i$’s are contained in the $NKLT(E_1, C + \Delta_{E_1} + (t + \epsilon')f*Q)$. Hence $G_i$’s are contained in $NLC(E_1, C + \Delta_{E_1} + sf*Q)$. This implies that $(E_1, C + \Delta_{E_1} + sf*Q)$ is not LC at $R \in C$. Then by inversion of adjunction
we get a contradiction to the fact that \((C, \Delta'_C + sf^*Q|_C)\) is LC.

**Case II:** \(C\) intersects the general fiber with degree 2. In this case \(E_1 \cap E_2 = C\) and \(\Delta'_E_1 = \Delta'_E_2 = 0\). Since \(D \neq 0\) and every component of \(D\) contains \(W\), one of the \(E_i\)'s, say \(E_2 = f_1^{-1}D_i\), where \(D_i\) is an irreducible component of \(D\). Thus in this case the exceptional divisors of \(f\) do not intersect each other. Since \(X\) is \(\mathbb{Q}\)-factorial, the exceptional locus \(\text{Ex}(f)\) of \(f : X' \to X\) supports an effective anti-ample divisor and hence \(\text{Ex}(f) \cap f^{-1}(w)\) is connected for all \(w \in W\). Thus \(f\) has a unique exceptional divisor in this case and we are done.

\(\square\)

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