Smooth solutions to the heat equation which are nowhere analytic in time

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Abstract

The existence of smooth but nowhere analytic functions is well-known (du Bois-Reymond, Math. Ann., 21(1):109-117, 1883). However, smooth solutions to the heat equation are usually analytic in the space variable. It is also well-known (Kowalevsky, Crelle, 80:1-32, 1875) that a solution to the heat equation may not be time-analytic at \( t = 0 \) even if the initial function is real analytic. Recently, it was shown in [6, 8, 26] that solutions to the heat equation in the whole space, or in the half space with zero boundary value, are analytic in time under essentially optimal conditions. In this paper, we show that time analyticity is not always true in domains with general boundary conditions or without suitable growth conditions. More precisely, we construct two bounded solutions to the heat equation in the half plane which are nowhere analytic in time. In addition, for any \( \delta > 0 \), we find a solution to the heat equation on the whole plane, with exponential growth of order \( 2 + \delta \), which is nowhere analytic in time.

1 Introduction

The study of the existence of nowhere-analytic smooth functions has a rich history (see e.g. [1]) since the pioneering works du Bois-Reymond [9], Lerch [18] and Cellerier [5]. Later, many other examples were found with different methods, see e.g. [2, 12, 20, 22]. For the heat equation, the space analyticity of the classical solution in a space-time domain is usually expected as a consequence of parabolic regularity. But the time analyticity is more delicate and is not true in general, see e.g. the well-known examples in Kowalevsky [16] and Tychonoff [21]. Under extra assumptions, however, many time-analyticity results for the heat equation, Navier-Stokes equations, and some other parabolic equations may still be justified, see e.g. [10, 11, 15, 19, 23].

Recently, in [8, 26], it was discovered that for any complete and noncompact Riemannian manifold \( M \) whose Ricci curvature is bounded from below, solutions to the heat equation on \( M \) with exponential growth of order 2 are analytic in time. In particular, as a corollary to Theorem 2.1 in [8], for any time interval \((a, b] \subseteq \mathbb{R}\), if \( u \) is a smooth solution to the heat

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equation \( \partial_t u - \partial_x^2 u = 0 \) on \( \mathbb{R} \times (a, b) \) that satisfies for two positive constants \( A_1 \) and \( A_2 \),

\[
|u(x, t)| e^{-A_2 x^2} \leq A_1, \quad \forall (x, t) \in \mathbb{R} \times (a, b),
\]

(1.1)

then \( u \) must be time analytic in \( t \in (a, b) \). The growth restraint (1.1) is sharp due to the Tychonoff’s non-uniqueness example with suitable modifications (e.g. see Remark 2.3 in [8] for more details). Later, similar phenomena were also found in other types of PDEs [7, 24] and in domains with boundary [6]. In particular, by denoting \( \mathbb{R}^+ = (0, \infty) \), Theorem 2.1 in [6] implies that for any time interval \( (a, b) \subseteq \mathbb{R} \), if \( v \) is a smooth solution to the heat equation on \( \mathbb{R}^+ \times (a, b) \) with the Dirichlet boundary condition \( v(0, t) = 0 \) and with the growth constraint \( |v(x, t)| e^{-A_2 x^2} \leq A_1 \) for any \( (x, t) \in \mathbb{R}^+ \times (a, b) \), then \( v \) is time analytic in \( t \in (a, b) \).

The study of analyticity of solutions to PDEs has both a long history, see e.g. the famous Cauchy-Kowalevsky theorem in [4, 16], and many applications, such as the time reversibility, the solvability of backward equations and the control theory. One particular application is about control problems involving heat type equations. For these problems, it is well known that the set of reachable states, though hard to describe exactly, is just a little larger than the set of those that can be reached by the free heat flow. But until the papers [8, 26], it’s even not clear how to characterize the latter in general (see, e.g. the comment on page 1 in [17]). For a precise characterization of the reachable states by the free heat flow, see Corollary 2.2 and Remark 2.5 in [8]. Later, an explicit formula was derived for the control function by representing solutions with power series in time thanks to the time analyticity, see Theorem 2.1 in [25].

In this paper, however, we discover that the time analyticity is hopeless for general boundary conditions or without suitable growth conditions. More precisely, we construct solutions to the heat equation on the half space-time plane \( \{ x \geq 0, \ t \in \mathbb{R} \} \) that satisfy the growth condition (1.1) but are nowhere analytic in time. As a byproduct, we also find a solution to the heat equation on the whole space which is nowhere analytic in time and almost satisfies the growth condition (1.1). This example will demonstrate the sharpness of the growth condition (1.1) even if the solution is only required to be analytic in time at a single point.

Denote the space-time domain \( \Omega_1 \) as

\[
\Omega_1 = \mathbb{R}^+ \times \mathbb{R}.
\]

(1.2)

We will construct two bounded solutions to the heat equation on \( \Omega_1 \) which are nowhere analytic in time. Our first example (1.4) can be regarded as an extension to the space-time case of du Bois-Reymond [9], which itself is based on the Weierstrass function: a continuous but nowhere differentiable trigonometric series on \( \mathbb{R} \). Our second example (1.5) takes advantage of the heat kernel \( \Phi \) on \( \mathbb{R} \), defined as in (1.3), and the method of the condensation of
singularities \[3,13\].

\[
\Phi(x, t) = \begin{cases} 
(4\pi t)^{-\frac{1}{2}} \exp \left( -\frac{x^2}{4t} \right) & \text{if } x \in \mathbb{R}, \ t > 0, \\
0 & \text{if } x \in \mathbb{R}, \ t \leq 0.
\end{cases}
\]  

(1.3)

Although the construction of \(u_2\) is direct via the method of condensation of singularities, we remark that the method of constructing \(u_1\) in (1.4) by the Weierstrass type functions may be more flexible to study other evolutionary PDEs such as the Schrödinger equation and the wave equation.

**Theorem 1.1.** Define two functions \(u_1, u_2 : \Omega_1 \to \mathbb{R}\) by

\[
\begin{align*}
  u_1(x, t) &= \sum_{k=1}^{\infty} e^{-2^k} e^{-2^k x} \sin \left( 2^{2k+1} t - 2^k x \right), \\
  u_2(x, t) &= \sum_{k=1}^{\infty} 2^{-k} \Phi(x + 1, t - r_k),
\end{align*}
\]

(1.4)

where \(\{r_k\}_{k=1}^{\infty}\) is an enumeration of all the rational numbers. Then for \(i = 1, 2\), \(u_i \in C^\infty(\Omega_1) \cap L^\infty(\Omega_1)\) and \(u_i\) satisfies the heat equation on \(\Omega_1\). However, for any fixed \(x_0 \in [0, \infty)\), the function \(u_i(x_0, \cdot)\) is nowhere analytic in \(t \in \mathbb{R}\).

The functions in Theorem 1.1 are only defined on \(\Omega_1\). If we want to construct smooth solutions to the heat equation on the whole plane \(\Omega_2 := \mathbb{R} \times \mathbb{R}\), then the solutions have to break the growth constraint (1.1). In addition, it is well-known that this growth constraint is sharp for everywhere time-analyticity. More precisely, for any \(\delta > 0\), there exists a solution to the heat equation on \(\Omega_2\) which grows slower than \(e^{A_2|x|^{2+\delta}}\) but is not time analytic at some point. Then it is interesting to investigate the following question: *Is the growth condition (1.1) sharp for somewhere time-analyticity?* The next result, which is inspired by (1.4), gives a positive answer to this question.

**Theorem 1.2.** Let \(\Omega_2 = \mathbb{R} \times \mathbb{R}\) and \(\epsilon \in (0, 1)\). Define \(w_\epsilon : \Omega_2 \to \mathbb{R}\) by

\[
w_\epsilon(x, t) = \sum_{k=1}^{\infty} e^{-2(1+\epsilon)k} e^{-2^k x} \sin \left( 2^{2k+1} t - 2^k x \right).
\]

(1.6)

Then \(w_\epsilon \in C^\infty(\Omega_2)\) and \(w_\epsilon\) satisfies the heat equation on \(\Omega_2\). However, for any fixed \(x_0 \in \mathbb{R}\), the function \(w_\epsilon(x_0, \cdot)\) is nowhere analytic in \(t \in \mathbb{R}\). Meanwhile, there exist positive constants \(A_1\) and \(A_2\), which only depend on \(\epsilon\), such that

\[
\sup_{x, t \in \mathbb{R}} |w_\epsilon(x, t)| \exp \left( - A_2|x|^{1+\frac{1}{\epsilon}} \right) \leq A_1.
\]

(1.7)

**Remark 1.3.** For any \(\epsilon \in (0, 1)\), the function \(w_\epsilon\) in the above theorem, when restricted to \(\Omega_1\) where \(x \geq 0\), is also a bounded nowhere time-analytic solution to the heat equation on \(\Omega_1\).
Furthermore, for any $\delta > 0$, by choosing $\varepsilon = \frac{1}{1+\delta}$, $w_\varepsilon$ is bounded by $A_1 e^{A_2|x|^2+\delta}$ but is nowhere time-analytic on $\Omega_2$.

2 Proofs of Theorems 1.1 and 1.2

2.1 Proof of Theorem 1.1

- We first study $u_1$. It is straightforward to check that $u_1 \in C^\infty(\Omega_1) \cap L^\infty(\Omega_1)$ and $u_1$ satisfies the heat equation on $\Omega_1$. Next, for any fixed $x_0 \geq 0$, we define $h(t) = u_1(x_0, t), \forall t \in \mathbb{R}$.

Then it reduces to prove that $h$ is not analytic at any point $t_0 \in \mathbb{R}$. By Cauchy-Hadamard theorem, it suffices to show

$$\limsup_{n \to \infty} \left( \frac{|h^{(n)}(t_0)|}{n!} \right)^{\frac{1}{n}} = \infty. \quad (2.1)$$

For any integer $m \geq 1$, the $(2m)^{th}$ and the $(2m + 1)^{th}$ derivatives of $h$ at $t_0$ can be written as

$$h^{(2m)}(t_0) = (-1)^m \sum_{k=1}^{\infty} e^{-2^k(1+x_0)/2} \sin(2^k t_0 - 2^k x_0),$$

$$h^{(2m+1)}(t_0) = (-1)^m \sum_{k=1}^{\infty} e^{-2^k(1+x_0)/2} \cos(2^k t_0 - 2^k x_0).$$

For any $N \in \mathbb{Z}^+$, there exists a unique $m_N \in \mathbb{Z}^+$ such that

$$2^N(1 + x_0) \leq 4m_N < 2^N(1 + x_0) + 4. \quad (2.2)$$

Define $F_N : \mathbb{Z}^+ \to \mathbb{R}^+$ as

$$F_N(k) = e^{-2^k(1+x_0)/2} 2^{2m_N(k+1)}. \quad (2.3)$$

Then

$$h^{(2m_N)}(t_0) = (-1)^{m_N} \sum_{k=1}^{\infty} F_N(k) \sin(2^{2k+1} t_0 - 2^k x_0),$$

$$h^{(2m_N+1)}(t_0) = (-1)^{m_N} \sum_{k=1}^{\infty} 2^{2k+1} F_N(k) \cos(2^{2k+1} t_0 - 2^k x_0).$$
By the triangle inequality,
\[
|h^{(2mN)}(t_0)| \geq F_N(N)|\sin(2^{2N+1}t_0 - 2^Nx_0)| - \sum_{k \neq N} F_N(k),
\]
\[
|h^{(2mN+1)}(t_0)| \geq 2^{2N+1}F_N(N)|\cos(2^{2N+1}t_0 - 2^Nx_0)| - \sum_{k \neq N} 2^{2k+1}F_N(k). 
\tag{2.4}
\]

Since $|\sin(\theta)| + |\cos(\theta)| \geq 1$ for any $\theta \in \mathbb{R}$, adding the two inequalities in (2.4) yields
\[
|h^{(2mN)}(t_0)| + |h^{(2mN+1)}(t_0)| \geq F_N(N) - 4\left(\sum_{k \neq N} 2^{2k}F_N(k)\right). \tag{2.5}
\]

By direct computation, it follows from (2.3) that for any $k \geq 1$,
\[
\frac{F_N(k+1)}{F_N(k)} = \frac{2^{4mN}}{e^{2N(1+x_0)}} = \exp\left[4mN \ln 2 - 2^k(1+x_0)\right]. \tag{2.6}
\]

For any fixed $N$, thanks to the choice (2.2) of $m_N$ and the fact that $\frac{1}{2} < \ln 2 < 1$, $F_N(N)$ is the largest term in the sequence $\{F_N(k)\}_{k \geq 1}$. Moreover, when $N$ is large enough, $F_N(N)$ is much larger than the other terms in the sequence $\{F_N(k)\}_{k \geq 1}$. Actually, it is not difficult to find a positive constant $N_0$, which only depends on $x_0$, such that
\[
\sum_{k \neq N} 2^{2k}F_N(k) \leq \frac{1}{100}F_N(N), \quad \forall N \geq N_0. \tag{2.7}
\]

Plugging (2.7) into (2.5) leads to
\[
|h^{(2mN)}(t_0)| + |h^{(2mN+1)}(t_0)| \geq \frac{1}{2}F_N(N), \quad \forall N \geq N_0. \tag{2.8}
\]

By (2.3),
\[
F_N(N) = e^{-2^N(1+x_0)}2^{2m_N(2N+1)} \geq e^{-2^N(1+x_0)(2^N)^{4m_N}}.
\]

Reorganizing (2.2) gives rise to
\[
\frac{4(m_N - 1)}{1 + x_0} < 2^N \leq \frac{4m_N}{1 + x_0}. \tag{2.9}
\]

Consequently,
\[
F_N(N) \geq e^{-4m_N}\left(\frac{4(m_N - 1)}{1 + x_0}\right)^{4m_N} \geq 2\left(\frac{m_N - 1}{1 + x_0}\right)^{4m_N}.
\]

Thus, for any $N \geq N_0$, it follows from (2.8) that
\[
|h^{(2mN)}(t_0)| + |h^{(2mN+1)}(t_0)| \geq \left(\frac{m_N - 1}{1 + x_0}\right)^{4m_N}. \tag{2.10}
\]
As a result,
\[
\frac{|h^{(2mN)}(t_0)| + |h^{(2mN+1)}(t_0)|}{(2mN + 1)!} \geq \left[ \frac{(mN - 1)^2}{(1 + x_0)^2(2mN + 1)} \right]^{2mN}.
\] (2.10)

Since \( m_N \to \infty \) as \( N \to \infty \), then (2.1) follows immediately from (2.10).

• Now we consider \( u_2 \). Although the expression (1.5) looks complicated, the conclusion follows directly from an elegant result in [22].

Lemma 2.1. ([22]) Let \( \varphi \) be a bounded \( C^\infty \) function which is analytic on \( \mathbb{R} \setminus \{0\} \) but not analytic at 0. Assume there are positive constants \( \delta_0, A \) and \( L \) such that for any \( |t| > A \),
\[
\sup_{n \geq 0} \frac{\left| \partial^n \varphi(t) \right|}{n!} \delta_0^n < L.
\] (2.11)

Let \( \{a_k\}_{k \geq 1} \) be a sequence of non-zero real numbers such that \( \sum_{k=1}^{\infty} |a_k| < \infty \). Let \( \{r_k\}_{k \geq 1} \) be an enumeration of all the rational numbers. Define a function \( f : \mathbb{R} \to \mathbb{R} \) by
\[
f(t) = \sum_{k=1}^{\infty} a_k \varphi(t - r_k).
\]

Then \( f \in C^\infty(\mathbb{R}) \) but \( f \) is nowhere analytic on \( \mathbb{R} \).

Next, we will apply Lemma 2.1 to prove the desired result for \( u_2 \). First, we recall that the heat kernel \( \Phi \) is defined as in (1.3). In addition, by noticing \( x + 1 \) is away from 0 for any \( x \geq 0 \), we know for any integer \( n \geq 1 \), there exists some constant \( M_n > 0 \) such that
\[
|\partial^n_x \Phi(x + 1, t)| + |\partial^n_t \Phi(x + 1, t)| \leq M_n, \quad \forall \ x \geq 0, \ t \in \mathbb{R}.
\]

As a result, \( u_2 \in C^\infty(\overline{\Omega}_1) \cap L^\infty(\Omega_1) \) and \( u_2 \) satisfies the heat equation on \( \overline{\Omega}_1 \) since
\[
(\partial_t - \partial_x^2)u_2 = (\partial_t - \partial_x^2) \left( \sum_{k=1}^{\infty} 2^{-k} \Phi(x + 1, t - r_k) \right)
= \sum_{k=1}^{\infty} 2^{-k}(\partial_t - \partial_x^2)\Phi(x + 1, t - r_k) = 0.
\]

Then for any fixed \( x_0 \geq 0 \), define
\[
\varphi(t) = \Phi(x_0 + 1, t), \quad \forall \ t \in \mathbb{R}.
\]

According to classical estimates on the heat kernel \( \Phi \) (see e.g. formula (3.3) in [14]),
there exists some constant $C > 0$ such that for any $n \in \mathbb{N}$,

$$|\partial_x^n \Phi(x, t)| \leq \frac{C^n}{t^{(n+1)/2}} e^{-\frac{x^2}{8t}}, \quad \forall x \in \mathbb{R}, \ t > 0.$$ 

Consequently,

$$|\partial^n_t \Phi(x, t)| = |\partial_t^n \Phi(x, t)| \leq \frac{2^n C^n}{t^{n+\frac{1}{2}}} e^{-\frac{x^2}{8t}}, \quad \forall x \in \mathbb{R}, \ t > 0.$$ 

In particular, there exists some constant $C_1 > 0$ such that for any $n \in \mathbb{N},$

$$|\partial^n_t \varphi(t)| = |\partial_t^n \Phi(x_0 + 1, t)| \leq \frac{C^n}{t^{n+\frac{1}{2}}} e^{-(x_0+1)^2/(8t)}, \quad \forall t > 0. \tag{2.12}$$

So by choosing $A = 1$ and $\delta_0 = \frac{1}{2C_1}$, it follows from (2.12) that for any $t > A$,

$$\frac{|\partial^n_t \varphi(t)|}{n!} \delta_0^n \leq \frac{C^n}{n!} \frac{1}{(2C_1)^n} = \frac{n^n}{n!} \frac{1}{2^n}.$$ 

Thanks to the Sterling formula, we conclude that there exists some constant $L > 0$ such that

$$\frac{|\partial^n_t \varphi(t)|}{n!} \delta_0^n < L, \quad \forall t > A. \tag{2.13}$$

Noticing that $\partial^n_t \varphi(t) = 0$ for any $t < 0$, so (2.13) is also valid for $|t| > A$. Therefore, (2.11) is justified for $\varphi$. Finally, in Lemma 2.1 by setting $a_k = \frac{1}{2^k}$, we conclude that the function

$$u_2(x_0, ·) = \sum_{k=1}^{\infty} \frac{1}{2^k} \Phi(x_0 + 1, · - r_k) = \sum_{k=1}^{\infty} a_k \varphi(· - r_k)$$

is nowhere analytic in $t \in \mathbb{R}$.

### 2.2 Proof of Theorem 1.2

Fix any $\epsilon \in (0, 1)$, it is readily seen that $w_\epsilon \in C^\infty(\overline{\Omega}_2)$ and $w_\epsilon$ satisfies the heat equation on $\overline{\Omega}_2$.

Next, for any fixed $x_0 \in \mathbb{R}$, we will prove that the function $w_\epsilon(x_0, ·)$ is nowhere analytic on $\mathbb{R}$. Define $h_\epsilon(t) = w_\epsilon(x_0, t)$ for $t \in \mathbb{R}$. Then it reduces to prove $h_\epsilon$ is not analytic at any point $t_0 \in \mathbb{R}$. By Cauchy-Hadamard theorem, it suffices to show

$$\limsup_{n \to \infty} \left( \frac{|h_\epsilon^{(n)}(t_0)|}{n!} \right)^{\frac{1}{n}} = \infty. \tag{2.14}$$

The proof of (2.14) is similar to that for the function $u_1$ in Theorem 1.1, so we will only sketch the process. For any large $N$ such that $2^\epsilon N \geq 2 + |x_0|$, there exists a unique $m_N \in \mathbb{Z}^+$ such that

$$(2^\epsilon N + x_0)2^N \leq 4m_N < (2^\epsilon N + x_0)2^N + 4. \tag{2.15}$$
Define $F_N : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ as

$$F_N(k) = e^{-2^{(1+\epsilon)k}} e^{-2^{k}x_0} 2^{2m_N}(2k+1). \quad (2.16)$$

Then similar to (2.5), we have

$$\left| h_\epsilon^{(2m_N)}(t_0) \right| + \left| h_\epsilon^{(2m_N+1)}(t_0) \right| \geq F_N(N) - 4 \left( \sum_{k \neq N} 2^{2k} F_N(k) \right).$$

Thanks to the choice (2.15) of $m_N$, it is not difficult to find a positive constant $N_0$, which only depends on $x_0$ and $\epsilon$, such that

$$\sum_{k \neq N} 2^{2k} F_N(k) \leq \frac{1}{100} F_N(N), \quad \forall N \geq N_0. \quad (2.17)$$

Hence,

$$\left| h_\epsilon^{(2m_N)}(t_0) \right| + \left| h_\epsilon^{(2m_N+1)}(t_0) \right| \geq \frac{1}{2} F_N(N), \quad \forall N \geq N_0. \quad (2.18)$$

Reorganizing (2.15) leads to

$$\frac{4(m_N - 1)}{2\epsilon N + x_0} < 2^N \leq \frac{4m_N}{2\epsilon N + x_0}. \quad (2.19)$$

Based on (2.18) and (2.16), for any $N \geq N_0$,

$$\left| h_\epsilon^{(2m_N)}(t_0) \right| + \left| h_\epsilon^{(2m_N+1)}(t_0) \right| \geq \frac{1}{2} e^{-2^{(1+\epsilon)N}} e^{-2^{N}x_0} 2^{2m_N}(2N+1)$$

$$\geq e^{-2^{N}(2\epsilon N + x_0)} (2^N)^{4m_N}.$$

Then it follows from (2.19) that

$$\left| h_\epsilon^{(2m_N)}(t_0) \right| + \left| h_\epsilon^{(2m_N+1)}(t_0) \right| \geq e^{-4m_N} \left( \frac{4(m_N - 1)}{2\epsilon N + x_0} \right)^{4m_N} \geq \left( \frac{m_N - 1}{2\epsilon N + x_0} \right)^{4m_N}.$$

As a result,

$$\left| h_\epsilon^{(2m_N)}(t_0) \right| + \left| h_\epsilon^{(2m_N+1)}(t_0) \right| \geq \left( \frac{m_N - 1}{(2\epsilon N + x_0)^2(2m_N + 1)} \right)^{2m_N}. \quad (2.20)$$

When $N \to \infty$, it follows from (2.15) that $m_N \to \infty$ and

$$\frac{m_N}{(2\epsilon N + x_0)^2} \sim 2^{(1-\epsilon)N} \to \infty.$$ 

Then (2.14) follows from (2.20).

Finally, we need to establish the growth constraint (1.7). Fix $0 < \epsilon < 1$. Then it suffices
to find constants $A_1$ and $A_2$, which only depend on $\epsilon$, such that for any $x \geq 0$,

$$
\sum_{k=1}^{\infty} \exp \left( -2^{(1+\epsilon)k} + 2^k x \right) \leq A_1 \exp \left( A_2 x^{1+\frac{1}{\epsilon}} \right) .
$$

Define $g_k(x) = \exp[2^k(x - 2^k\epsilon)]$ for any $k \geq 1$. Then it reduces to justify

$$
\sum_{k=1}^{\infty} g_k(x) \leq A_1 \exp \left( A_2 x^{1+\frac{1}{\epsilon}} \right) . \tag{2.21}
$$

If $0 \leq x \leq 100$, then it is readily seen that the above series in (2.21) is uniformly bounded by a constant $B_1$ which only depends on $\epsilon$, so it reduces to consider the case when $x > 100$. Define $K \in \mathbb{Z}^+$ to be the unique positive integer such that

$$
\frac{1}{\epsilon} \log_2 \left( \frac{x}{2^{1+\epsilon} - 1} \right) \leq K < 1 + \frac{1}{\epsilon} \log_2 \left( \frac{x}{2^{1+\epsilon} - 1} \right) , \tag{2.22}
$$

which can be rewritten as

$$
x < 2^{1+\epsilon} - 1 < 2^K x < 2^{1+\epsilon} - 1 . \tag{2.23}
$$

In addition, since $x > 100$ and $0 < \epsilon < 1$, $K \geq 4/\epsilon > 4$. By direct computation,

$$
g_{k+1}(x) / g_k(x) = \exp \left( 2^k [x - 2^k(2^{1+\epsilon} - 1)] \right) . \tag{2.24}
$$

**Case 1: Estimate of $\sum_{k=1}^{K} g_k(x)$.**

For any $1 \leq k \leq K - 1$, it follows from (2.24) and (2.23) that $g_{k+1}(x) \geq g_k(x)$. As a consequence,

$$
\sum_{k=1}^{K} g_k(x) \leq Kg_K(x) \leq K \exp(2^K x) . \tag{2.25}
$$

Since $x > 100$ and $K > 4$, then $K \leq \exp(2^K x)$. Moreover, we can see from (2.25) and (2.23) that

$$
\sum_{k=1}^{K} g_k(x) \leq \exp \left[ 2 \left( \frac{2^\epsilon x}{2^{1+\epsilon} - 1} \right)^{\frac{1}{\epsilon}} x \right] = \exp \left( B_2 x^{1+\frac{1}{\epsilon}} \right) , \tag{2.26}
$$

where $B_2$ is a constant which only depends on $\epsilon$.

**Case 2: Estimate of $\sum_{k=K}^{\infty} g_k(x)$.**

For any $k \geq K$, it follows from (2.24) and (2.23) that $g_{k+1}(x) \geq g_k(x)$. In particular, by choosing $k = K$ and recalling (2.26), we know

$$
g_{K+1} \leq g_K(x) \leq \exp \left( B_2 x^{1+\frac{1}{\epsilon}} \right) . \tag{2.27}
$$
From (2.23), we have \( x \leq (2^{1+\epsilon} - 1)2^K \). Plugging this inequality into (2.24) yields

\[
\frac{g_{k+1}(x)}{g_k(x)} \leq \exp \left[ - (2^{1+\epsilon} - 1)2^k(2^k - 2^\epsilon K) \right].
\]

So for any \( k \geq K + 1 \),

\[
\frac{g_{k+1}(x)}{g_k(x)} \leq \exp \left[ - (2^{1+\epsilon} - 1)2^k(2^{(K+1)} - 2^\epsilon K) \right] \leq \exp \left[ - 2^k(2^\epsilon - 1) \right].
\]

This implies that for any \( k \geq K + 1 \),

\[
g_{k+1}(x) = \left( \prod_{i=K+1}^{k} \frac{g_{i+1}(x)}{g_i(x)} \right) g_{K+1}(x)
\leq \exp \left[ - (2^\epsilon - 1)2^k(2^k - 2^\epsilon K - 1) \right] g_{K+1}(x)
\leq \exp \left[ - (2^\epsilon - 1)(2^k - 2^\epsilon - 1) \right] g_{K+1}(x)
\]

Thus, by setting \( j = k - K \) and adding \( j \) from 1 to \( \infty \),

\[
\sum_{k=K+2}^{\infty} g_k(x) \leq g_{K+1}(x) \sum_{j=1}^{\infty} \exp \left[ - (2^\epsilon - 1)(2^j - 1) \right] = B_3 g_{K+1}(x), \tag{2.28}
\]

where \( B_3 \) is a positive constant which only depends on \( \epsilon \).

Combining (2.26), (2.27) and (2.28) together leads to the desired estimate (2.21).

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