FINITE ENTROPY FOR MULTIDIMENSIONAL CELLULAR AUTOMATA

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ABSTRACT. Let $X = S^G$ where $G$ is a countable group and $S$ is a finite set. A cellular automaton (CA) is an endomorphism $T : X \to X$ (continuous, commuting with the action of $G$). Shereshevsky [14] proved that for $G = \mathbb{Z}^d$ with $d > 1$ no CA can be forward expansive, raising the following conjecture: For $G = \mathbb{Z}^d$, $d > 1$ the topological entropy of any CA is either zero or infinite. Morris and Ward [11], proved this for linear CA’s, leaving the original conjecture open. We show that this conjecture is false, proving that for any $d$ there exist a $d$-dimensional CA with finite, nonzero topological entropy. We also discuss a measure-theoretic counterpart of this question for measure-preserving CA’s.

1. INTRODUCTION

Cellular automata form a class of dynamical systems which has been extensively studied since the 1940’s, going back to some work of von Neumann [15] and others. For some survey papers on different aspects of cellular automata see [2, 3, 9, 16]. The following is an intuitive description of a cellular automaton: Consider an infinite mesh of finite state machines, called cells interconnected locally with each other. These cells change their states synchronously depending on the states of some nearby cells, as determined by a local update rule. All the cells use the same update rule so that the system is homogeneous.

The entropy of a cellular automaton is, informally, the rate of information flow required to describe the long term evolution of any finite number of cells. If the mesh of cells is a one dimensional array (bi-infinite or one sided), a simple argument shows that the entropy must be finite. Intuitively, when the cells are arranged in a lattice of dimension 2 or more, it can be expected the entropy is either zero (the evolution of a finite number of cells is eventually determined by the history of these cells) or infinite (the rate of information to describe the evolution of a set $F$ of cells grows to infinity with the cardinality of $F$). This intuition, backed up by some supporting results gave raise to the following conjecture [11], attributed to Shereshevsky:

Conjecture 1.1. A $d$-dimensional cellular automaton, with $d > 1$, can not have finite, positive entropy.

The main result of this paper is a counter-example to this conjecture. For any $d \geq 2$ we describe a $\mathbb{Z}^d$ cellular automaton and give non-trivial bounds in its topological entropy. An important tool implied in this construction is a certain aperiodic set of tiles, associated to a substitution system. The cellular automaton we describe was introduced by Kari [8] for $d = 2$, to prove certain undecidability results on cellular automata.

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The paper is organized as follows: In section 2 we introduce notation and give brief definitions of cellular automata, subshifts and entropy. Section 3 concludes with previous results and background on the entropy problem for cellular automata. Section 4 consists of a construction of an automaton with non-trivial topological entropy, along with entropy estimations. Section 5 contains an explanation of a key ingredient for this construction - the so-called “Hilbert-Tiles”. Some tools developed in sections 3 and 4 are applied in section 5 to study measure-theoretic entropy of surjective cellular automata. The last section of this paper contains some further comments and open questions.

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2. Preliminaries and background

2.1. Subshifts and cellular automata. Let $S$ be some finite set, which we call the possible states of each cell. A configuration is an element $x \in S^G$, where $G$ is a countable set. For $n \in G$ we denote by $x_n$ the state of the cell at coordinate $n$ in the configuration $x$. If $F \subset G$, we denote by $x_F$ the restriction of $x$ to $F$ - the states of those cells located in $F$. We put the product topology on $S^G$, which is compact and metrizable. If $G$ is a semigroup, we can define an action $\sigma$ of the group $G$ on $X$ by translations: For $h, g \in G$ and $x \in S^G$, $(\sigma g x)_h = x_{hg}$. Each map $\sigma_g$ is a continuous transformation of $X$, and if $G$ is a group, $\{\sigma_g: g \in G\}$ are homeomorphisms. The classic cases are $G = \mathbb{Z}$ and $G = \mathbb{N}$, in which case the action $\sigma$ is generated by one transformation, called the shift map. In this case, $S^\mathbb{Z}$ (or $S^\mathbb{N}$) is called a full shift space (one sided or two sided, respectively). If $G = \mathbb{Z}^d$ where $d \geq 1$ then $S^\mathbb{Z}^d$ is a $d$-dimensional full shift. A subset $X \subset S^\mathbb{Z}^d$ is called a $\mathbb{Z}^d$-subshift. Any subshift can be described by a set of forbidden blocks $FB = \{b_1, \ldots, b_n, \ldots\}$ where $b_i \in S^{F_i}$ and $F_i \subset \mathbb{Z}^d$ are finite sets as follows:

$$X = \{x \in S^\mathbb{Z}^d : x_{F_i+m} \neq b_i \forall m \in \mathbb{Z}^d, i \in \mathbb{N}\}$$

A subshift $X$ is called a subshift of finite type (SFT) if there exist a finite set $FB$ of forbidden blocks for $X$. We say that a subshift $X \subset S^\mathbb{Z}^d$ is a factor of a subshift $Y \subset S^\mathbb{Z}^d$ (or $Y$ extends $X$) if there exist a surjective map $\pi : Y \to X$ which commutes with the shift action $\sigma$.

For a countable group $G$, a $G$-cellular automaton is a pair $(S^G, T)$ where $T : S^G \to S^G$ is a continuous transformation which commutes with the $G$-action $\sigma$ of translations. We abbreviate CA for cellular automaton. The transformation $T$ is always of the form $(Tx)_n = t(x_{F+n})$ where $F \subset G$ is a finite set, and $t : S^F \to S$ is called the local rule of the CA. Mostly, we restrict attention to the case $G = \mathbb{Z}^d$.

Often a wider definition of a $\mathbb{Z}^d$-CA is used, and any continuous, shift commuting transformation of an SFT is called a $\mathbb{Z}^d$-CA. In this paper we deal specifically with CA on a full-shift.

2.2. Topological and measure-theoretic entropy. Let $(X, T)$ be a CA with $X = S^G$, and $F$ a finite subset of $G$. We denote:

$$W(F, n, T) := \{((x)_F, (Tx)_F, \ldots, (T^{n-1}x)_F) : x \in X\}$$
$W(F, n, T)$ is the set of possible configurations for cells inside $F$ along $n$ iterations of $T$.

The topological entropy of $(X, T)$ is:

$$h_{\text{top}}(X, T) = \sup_{F \subset \mathbb{G}} \lim_{n \to \infty} \frac{1}{n} \log |W(F, n, T)|$$

where the supremum above is over all finite subsets $F$ of $\mathbb{G}$, and the limit above exists by subadditivity of the sequence $\log |W(F, n, T)|$.

For $w = (w_0, \ldots, w_{n-1}) \in W(F, n, T)$, let $[w] = \{x \in X : (T^ix)_F = w_i \, \forall \, 0 \leq i < n\}$. Let $\mathcal{P}(X, T)$ be the set of probability measures on the Borel $\sigma$-algebra $\mathcal{B}$ of $X$ which are invariant to $T$ ($\mu(T^{-1}B) = \mu(B)$ any Borel $B \subset X$). For $\mu \in \mathcal{P}(X, T)$, the measure-theoretic entropy of $(X, \mu, T)$ is:

$$h_\mu(X, B, T) = \sup_{F \subset \mathbb{G}} \lim_{n \to \infty} \frac{1}{n} H_n$$

with:

$$H_n = -\sum_{w \in W(F, n, T)} \mu([w]) \log(\mu([w]))$$

where the supremum is again over all finite subsets $F$ of $\mathbb{G}$, and the limit above exists by subadditivity of the sequence $H_n$. Both the quantities $h_{\text{top}}(X, T)$ and $h_\mu(X, B, T)$ indicate the "information flow" of the long term evolution of any finite number of cells by $T$. The conceptual difference is that the topological entropy gives the rate of information required to describe any evolution of finitely many cells, while as $h_\mu(X, B, T)$ gives the rate of information required to describe a "typical" evolution, where "typical" is with respect to the probability $\mu$.

An important result connecting the topological entropy and the measure-theoretic entropy, known as the variational principle states that:

$$h_{\text{top}}(X, T) = \sup_{\mu \in \mathcal{P}(X, T)} h_\mu(X, B, T)$$

Entropy of a transformation (topological or measure theoretic) is defined in a broader context. See [2] for definitions, proofs and a detailed discussion in the context of compact spaces.

2.3. Substitution systems and tilings. A $\mathbb{Z}^d$-tiling system consists of a finite set $S$ of "square tiles" and some adjacency rules $R \subset S^F$ with $F = \{0, \pm e_1, \ldots, \pm e_d\}$, which determine when a tile $s_1$ is allowed to be placed next to a tile $s_2$ (and in which directions). A configuration $x \in S^F$ for some $F \subset \mathbb{Z}^d$ is valid at $n \in F$ if the neighbors of the cell at $n$ obey the adjacency rules: $x_{F+n} \in R$. Evidently, the set of infinite valid configurations in $S^{\mathbb{Z}^d}$ are an SFT. We call this SFT the associated subshift of $(S, R)$ and denote it by $X_R$.

A $\mathbb{Z}^d$-substitution system consists of a finite set $S$ and a substitution rule $\rho$ which is a function $\rho : S \to S^{F_k}$ where $F_k = \{1, \ldots, k\}^d$. The function $\rho$ naturally extends to a map $\rho^n : S \to S^{F_{kn}}$ by applying $n$ iterations of $\rho$, and to a map $\rho^n : S^{2^{d^k}} \to S^{2^{d^k}}$. For $F \subset \mathbb{Z}^d$ and a configuration $x_F \in S^F$, we say that $x_F$ is admissible for the substitution system $(S, \rho)$, if it appears as a sub-configuration of $\rho^n(s)$ for some $s \in S$ and $n \in \mathbb{N}$. To a substitution system $(S, \rho)$ there is an associated subshift $X_\rho$, whose forbidden blocks are the non-admissible blocks for $(S, \rho)$.

We say that a substitution system $\rho$ has the unique derivation property if any for $x \in X_\rho$ there exist a unique $n \in F_k$ and $y \in X_\rho$ such that $x = \sigma_n \rho(y)$. 


Equivalently, this means that given a part of a finite configuration \( x \in S^{F_k^n} \) such that \( x = \rho^n(s) \) for some \( s \in S \) which has been shifted, the shift can be recovered uniquely modulo \( k\mathbb{Z}^d \). Mozes [12], based on some earlier work of Robinson [13], devised an algorithm for implementing a subshift associated with a substitution system via local constraints - a tiling system. Namely, Mozes proved the following theorem:

**Theorem 2.1.** (Theorem 4.5 of [12]) Let \( X_\rho \subset S^\mathbb{Z}^d \) be the \( \mathbb{Z}^d \)-subshift associated to a substitution rule \( s : S \to S^{F_k} \) with unique derivation. If \( d > 1 \), then there exists a tiling system \( (\tilde{S}, R) \) such that the associated subshift \( X_R \) extends \( X_\rho \).

To be precise, Mozes proved this theorem for \( d = 2 \), but the proof extends to any \( d > 1 \). In fact, Goodman-Strauss [6] proved a more general result, relaying on Mozes’ techniques, which implies theorem [2.1] for any \( d > 1 \). For \( d = 1 \), theorem 2.1 does not hold: there are classic examples of one-dimensional substitution systems which are not factors of an SFT.

### 2.4. The CA entropy problem

We turn to explain some motivation for conjecture [1]. A \( \mathbb{G} \)-CA \( (S^G, T) \) is **expansive** if there exist some finite \( F \subset \mathbb{G} \) such that for any \( x \neq y \in S^G \) there exist \( n \geq 0 \) such that \( (T^n x)_F \neq (T^n y)_F \). Expansiveness of a CA means that by observing the states of some finite number of cells under iterations of \( F \), we can eventually distinguish between any 2 configurations. It can be shown that an expansive CA always has finite, non-zero topological entropy. Shereshevsky [14] proved that a \( \mathbb{Z}^d \)-CA with \( d > 1 \) can not be expansive, which gave raise to conjecture [1] stated in the introduction. Motivated by this, Morris and Ward [11] proved a result on entropy of group automorphisms, which implies this conjecture for some subclass of cellular automata called **linear cellular automata**. This result has been refined [4], to show that linear CA are either “sensitive to initial conditions”, in which case the entropy is infinity, or equicontinuous, in which case the entropy is zero. Lakshtanov and Langvagen [10] prove that any multidimensional cellular automaton which admits a spaceship has infinite entropy. A spaceship for a CA \( (X, T) \) is configuration \( x \in X \) which differs from a \( \sigma \)-invariant configuration in only finitely many cells, and such that \( T^k x = \sigma_n x \) for some \( n \in \mathbb{Z}^d \setminus \{0\} \) and some integer \( k > 0 \), but \( x \) is not itself \( \sigma \)-invariant.

We point out that for any \( d \geq 1 \) there are trivial examples of continuous, shift commuting transformations \( T : X \to X \) with finite, positive entropy where \( X \subset S^\mathbb{Z}^d \) is an SFT:

Let

\[
X = \{ x \in S^\mathbb{Z}^d : x_n = x_{n+i}e_i , \forall 1 < i \leq d \}
\]

where \( \{e_1, \ldots, e_d\} \) are the standard generators of \( \mathbb{Z}^d \) and \( T(x) = \sigma_{e_1} x \) is the shift in the direction of \( e_1 \). In this case, the example is essentially 1-dimensional, and the topological entropy of \( T \) is \( \log |S| \).

### 3. Surjective \( \mathbb{Z}^d \) CA with finite nonzero entropy

In this section we present a construction of a surjective \( \mathbb{Z}^d \)-CA, associated with a set of directed tiles \( S \) and a finite group \( \Gamma \). This is a simple generalization of Kari’s CA [8]. We prove that if \( S \) has some special properties, the associated CA has non-trivial entropy. The existence of a set of directed tiles with the required properties is proved in section [11].
A set of directed tiles is a tiling system with a forward direction \( d(s) \in \{ \pm e_1, \ldots, \pm e_d \} \) associated to each tiles \( s \in S \). Given a configuration \( x \in S^{Z^d} \), a path defined by \( x \) is a sequence \( p_1, p_2, \ldots \) with \( p_n \in Z^d \) obtained by traversing the forward directions of \( x \): \( p_{n+1} = p_n + d(x_{p_n}) \). Given \( x \in S^{Z^d} \), a path \( \{ p_n \} \) is valid if \( x \) is valid at every \( p_n \). A set of directed tiles \( S \), such that no valid path in \( x \in S^{Z^d} \) forms a loop is called an acyclic set of tiles.

Fix a finite group \( \Gamma \). The group operation of \( \Gamma \) is written here in additive notation. Let \( S \) be a set of directed set of tiles. Recall that for \( s \in S \), \( d(s) \) denotes the forward direction of the tile \( s \).

Define \( T_S : X \to X \) with \( X = (S \times \Gamma)^{Z^d} \) as follows:

\[
T_{S,\Gamma}(x, y)_n := \begin{cases} (x_n, y_n + d(x_n)) & \text{if } x \text{ is valid at } n \\ (x_n, y_n) & \text{otherwise} \end{cases}
\]

In this cellular automaton cells “transmit-information” along valid paths. We prove that when the number of infinite valid paths is bounded, the topological entropy is finite.

For \( x \in X \), denote by \( \gamma(x)_n \) the \( \Gamma \)-part of state of the cell at \( n \) in \( x \), and by \( s(x)_n \) the \( S \)-part of the state.

**Lemma 3.1.** With \( S \) an acyclic set of tiles as above, and \( T_{S,\Gamma} \) defined according to equation \( (3.2) \), \( T_{S,\Gamma} \) is surjective.

**Proof.** It is sufficient to prove that the image of \( T_{S,\Gamma} \) is dense in \( X \), that is, to any finite \( F \subset Z^d \) and any \( y \in X \) there is some \( x \in X \) such that \( (T_{S,\Gamma}x)_F = y_F \). Let \( y \in X \) and \( F \subset Z^d \) a finite set. We describe \( x \in X \) as follows: Set \( s(x)_n = s(y)_n \) for any \( n \in Z^d \). For \( n \in Z^d \setminus F \), set \( \gamma(x)_n = 0 \) (the identity of the group \( \Gamma \)). It remains to define the \( \gamma(x)_n \) for \( n \in F \). We define \( \gamma(x)_n \) iteratively, according to the length of maximal valid path in \( x \) beginning at \( n \), which remains in \( F \). Since \( S \) is acyclic and \( F \) is finite, any such path must be finite. If \( x \) is not valid at \( n \), or the successor of \( n \) in \( x \) is not in \( F \), define \( \gamma(x)_n = \gamma(y)_n \). Suppose now that the length of the maximal valid path in \( x \) beginning at \( n \) is \( k > 1 \), and we have defined \( \gamma(x)_{n+d(x_{n+1})} \). So we define \( \gamma(x)_n = \gamma(y)_n - \gamma(x)_{n+d(x_n)} \). It can now be verified that indeed \((T_{S,\Gamma}x)_F = y_F \). □

Suppose \( S \) is an acyclic set of tiles, and \( w \in S^{Z^d} \). Let \( X_w = \{(x, y) \in S^{Z^d} \times \Gamma^{Z^d} : x = w \} \) and identify it with naturally as a subset of \( X \). The set \( X_w \) is a closed subset of \( X \), and is invariant under \( T_{S,\Gamma} \). Let us define a directed graph \( G_w = (V_w, E_w) \) with vertex set \( V_w = Z^d \) and the edges

\[
E_w = \{(n, n + d(w_n)) : n \in Z^d, w \text{ is valid at } n \}
\]

We say that \( K \subset Z^d \) is connected in \( G_w \) if for any two cells in \( K \) there is a directed path in \( G_w \) from one to the other (but not necessarily in both directions). A connected component of \( G_w \) is a connected set in \( G_w \) which is maximal with respect to inclusion. For any connected component \( K \) of \( G_w \), and any \( x \in X_w \), the CA \( T_{S,\Gamma} \) acts on \( x_K \) independently of the states outside \( K \).

Fix \( w \in S^{Z^d} \). Suppose \( G_w \) has \( c \) infinite components, denoted by \( K_1, \ldots, K_c \). Let \( K_0 = Z^d \setminus \bigcup_{c} K_c \) denote the union of cells which are not part of forward infinite valid path in \( w \). Let \( (X_i, T_i) \) denote the system corresponding to those cells in \( K_i \) with the action of \( T_{S,\Gamma} \). The systems \( (X_i, T_i) \) are factors of \( (X_w, T_{S,\Gamma}) \).
For \( w \in S^{Z_d} \), denote by \( I(w) \) the number of infinite connected components of \( G_w \), which is equal to the maximal number of pairwise disjoint, forward infinite valid paths in \( w \). For a directed set of tiles \( S \), let:

\[
I(S) = \sup_{w \in S^{Z_d}} I(w)
\]

Lemma 3.2. Let \((X_0, T_0)\) be the system corresponding to the cells which are not part of forward infinite valid paths as above, \( h_{top}(X_0, T_0) = 0 \).

Proof. Since any valid path in \( X_0 \) is forward finite, a simple induction on the length of the path shows that the states of the cells in a valid path of length at most \( k \) is \( 2^k \)-periodic. Thus, \((X_0, T_0)\) is isomorphic to an inverse limit of (finite) periodic systems, and thus has 0 topological entropy. \( \square \)

Lemma 3.3. Let \( X_w \) and \( T_{S, \Gamma} \) be as above. If \( w \in S^{Z_d} \) contains exactly \( c \) forward infinite disjoint valid paths, then \( h_{top}(X_w, T_{S, \Gamma}) = c \log |\Gamma| \).

Proof. With the above notations, since \( T \) acts independently on cells in different connected components, \((X_w, T_{S, \Gamma})\) is isomorphic to \( \prod_{i=0}^c (X_i, T_i) \), and so

\[
h_{top}(X_w, T_{S, \Gamma}) = \sum_{i=0}^c h_{top}(X_i, T_i)
\]

By lemma 3.2, the topological entropy of \((X_0, T_0)\) is 0. Each of the other systems \((X_i, T_i)\) has topological entropy \( \log |\Gamma| \) since it is isomorphic to an inverse limit of finite extensions of the \( N \)-CA on \( \Gamma^N \) defined by: \( (Tx)_n = x_n + x_{n+1} \).

In the next section we describe Kari’s directed set of tiles from \([8]\), called \( S_H \). The tile set \( S_H \) is acyclic and has \( 0 < I(S_H) < \infty \). In fact, we show that this construction can be carried out in any dimension \( d > 1 \). For this set of tiles \( S_H \) and some fixed group \( \Gamma \), denote \( T_H = T_{S_H, \Gamma} \). Assuming this, we have our main result:

Theorem 3.4. For any \( d \geq 1 \) there exist a surjective \( Z^d \)-CA with positive, finite topological entropy.

Proof. Let \( S_H \) be an acyclic set of directed tiles with \( 0 < I(S_H) < \infty \). By lemma 3.3, this means that \( \sup_{w \in S_H^{Z_d}} h_{top}(X_w, T_H) = I(S_H) \log |\Gamma| < \infty \). Since \( X = \bigcup_{w \in S_H^{Z_d}} X_w \), we have that \( h_{top}(X, T_H) = \sup_{w \in S_H^{Z_d}} h_{top}(X_w, T_H) \). In section 4 we will prove the existence of \( S_H \) as above, which will complete the proof.

Before describing the set of tiles required to complete the proof of theorem 3.4, we note that a slight modification of the above construction yields the following result about the possible values the topological entropy of multidimensional CA can obtain:

Proposition 3.5. For any \( d \geq 1 \), the set of entropies of surjective \( d \)-dimensional cellular automata is dense in \([0, \infty)\).

Proof. Since a product of surjective \( d \)-dimensional cellular automata also a surjective \( d \)-dimensional CA with topological entropy equal to the sum of the entropies, it is sufficient to prove that there exist surjective \( d \)-dimensional CA with arbitrarily small positive topological entropy.
FIGURE 4.1. The four basic paths through the squares of size $2 \times 2$ tiles.

Retaining the notations from the beginning of this section, let $\Gamma$ be some finite group, $S$ an acyclic set of tiles with $0 < I(S) < \infty$ and $\mathbb{Z}_m$ the cyclic group of order $m$. We define a CA

$$T_{S,\Gamma,m} : (S \times \Gamma \times \mathbb{Z}_m) \to (S \times \Gamma \times \mathbb{Z}_m)$$

by:

$$T_{S,\Gamma,m}(x, y, t)_n := \begin{cases} (x_n, y_n + y_{n+d(x_n)}, 1) & \text{if } x \text{ is valid at } n \text{ and } t = 0 \\ (x_n, y_n, t + 1) & \text{otherwise} \end{cases}$$

Evidently, $T_{S,\Gamma,m}$ is surjective (the proof is similar to lemma 3.1). Also note that $T_{S,\Gamma,m}^m$ is conjugate to $T_{S,\Gamma \times \text{id}}$, and so

$$h_{\text{top}}(T_{S,\Gamma,m}) = \frac{1}{m} h_{\text{top}}(T_{S,\Gamma})$$

Since $0 < h_{\text{top}}(T_{S,\Gamma}) < \infty$ and $m$ was arbitrary, the proof of this proposition is complete. \qed

4. KARI'S TILES AND HILBERT SPACE FILLING PATHS

In this section we describe Kari’s set of directed tiles $S_H$, which has the properties announced above. In order to prove our main result, we only need two properties from this set of tiles $S_H$: The first property is $0 < I(S_H)$, that is, there exist $x \in S_H^Z$ with a forward infinite valid path. The second property, which is harder to prove, is $I(S_H) < \infty$. This property of $S_H$ follows from a lemma proved by Kari (quoted here as lemma 4.2). In the first part of this section (subsection 4.1) we describe a substitution system which is associated with Kari’s set of tiles. Subsection 4.2 contains a technical description of Kari’s tiles, and it may be skipped by readers who are familiar with [8]. The last part of this section contains a proof that Kari’s tiles have a bounded number of forward-infinite valid paths.

4.1. Hilbert space filling paths. Before describing Kari’s tiles, let us describe a certain substitution system, which is closely related with these tiles. Consider the discrete version of Hilbert’s plane filling curve. This is a path in $\mathbb{Z}^2$ which, starting at $(0, 0)$ visits each point of the non-negative quarter of $\mathbb{Z}^2$ once. Following is an inductive definition of this path: Define four basic paths which visit each point in a $2^n \times 2^n$-square. We denote these paths by $P^a_n$, $P^b_n$, $P^c_n$, and $P^d_n$. $P^a_n$ is defined as $(0, 0) \to (0, 1) \to (1, 1) \to (1, 0)$. The other basic paths are obtained by rotations and reflections. These 4 basic paths are described in figure 4.1. The path $P^a_{n+1}$ is obtained by walking according to $P^b_n$, $P^a_n$, $P^c_n$, and $P^d_n$. Similarly, $P^b_{n+1}$, $P^c_{n+1}$ and $P^d_{n+1}$ are defined inductively as shown in figure 4.2.
A \( \mathbb{Z}^d \)-Hilbert path can also be defined in an analogous manner: the basic paths are obtained by traversing the vertices of a \( d \)-dimensional hypercube using Gray-code. In dimension \( d \) there are \( 2^d \) basic paths, corresponding to the \( d \) generators and their inverses (see Alber and Niedermeier \[1\] for more on \( \mathbb{Z}^d \)-Hilbert type paths). For simplicity, we first consider only the case \( d = 2 \).

We can associate a substitution system to the Hilbert path: there are 12 tiles in this system, each corresponding to a basic path with the inward and outward directions indicated. The substitution rules reflect the inductive step of the definition of this path. The tiles and one of the rules of this substitution system is described in figure 4.4. The other substitution rules follow in a symmetric manner, according the definition of the Hilbert path.

**Lemma 4.1.** The substitution system associated with the Hilbert tiles (described in figure 4.4) has unique derivation.
Proof. It can be directly verified that any tile in the interior of a valid configuration is a part of a unique path of length 4 which fills a $2 \times 2$ square. Now consider a $\mathbb{Z}^2$ configuration in the associated subshift. The lattice $\mathbb{Z}^2$ is partitioned into $2 \times 2$ squares each of which is tiled by a path of length 4. This partition uniquely determines the substitution rule applied to derive the $\mathbb{Z}^2$ configuration. \qed

By the result of Mozes (theorem 2.1), there exist a set of tiles which implements the Hilbert substitution system. We do not know however, if any tiling system $S$ which implements the Hilbert substitution system has the property that $0 < I(S) < \infty$. Kari’s tiles, which are a specific set of tiles which implement this substitution system, do have this property. We suspect that by applying the algorithm described by Mozes in [12] on the Hilbert substitution system, one also ends up with a set of tiles with the required properties.

4.2. Description of Kari’s tiles. We now describe Kari’s tiles, which we denote by $S_H$. This directed set of tiles has “direction” labels, which are defined in such a manner that any valid path follows part of the Hilbert path. The construction is similar to an aperiodic set of tiles constructed by Robinson [13], which is also at the heart of Mozes’ result [12].

Each tile has a basic label which is either a blank cross, a bold cross a blank arm, a bold arm or a mixed arm. The five types of basic labels are represented in figure 4.5. A tile with a basic label which is an arm can face in one of the four main directions (north,south,east or west). Each cross also has an orientation label in NE,NW,SE,SW referring to the four cornerwise directions. The tiles labeled with an arm contain one arrow which faces outwards, and two arrows facing inwards. We call the arrow facing outwards a principal arrow and the arrows facing inwards side...
Figure 4.6. The possible orientation labels of the two side arrows of a bold arm. $X \in \{E, W\}$ and $Y \in \{N, S\}$.

Figure 4.7. The possible orientation labels of the two side arrows of a mixed arm.

Figure 4.8. The labeling of mixed arms whose principle arrow has the label $a$.

*arrows.* Each arrow in an arm also contains an orientation label in NE,NW,SE,SW. The principle arrow of an arm can be labeled with any of the four orientations.

There are restrictions on the orientation labels of the side arrows, determined by the direction of the arm, and the type of the arm (see figure 4.6). For a bold arm in which the principle arrow is horizontal, there are two possibilities: either the upper side arrow is labeled with SE and the lower one with NE, or the upper arrow is labeled with SW and the lower one with NW. If the arm is vertical, then either the left arrow is NE and the right arrow is NW or the left arrow is SE and the right arrow is SW. The possible orientation labels for size arrows in bold arms are shown in figure 4.6. The orientation labels for size arrows in a blank arm are similar to a bold arm (figure 4.6). The possible orientation labels for arrows in mixed arms are shown in figure 4.7.

Each arrow head also has a *Hilbert-label* in $\{a, b, c, d\}$. The only restriction imposed on the Hilbert-labels concern crosses and mixed arms. In each cross all four arrow heads must have the same Hilbert-label. The Hilbert-labels of the side arrows on the mixed arms are restricted according to the direction and Hilbert-label of the principle arrow. The restrictions can be obtained from figure 4.2 as follows: If the Hilbert-label of the principle arrow is $x \in \{a, b, c, d\}$, and the principle arrow is facing right, then the upper side arrow has the label of the path in the upper right corner of the square of $P_{n+1}^x$, and the lower side arrow has the label of the path in the lower right corner of the square of $P_{n+1}^x$. For mixed arms facing the other directions, the restrictions are defined in a similar manner. For example, figure 4.8 shows the allowed labels of side arrows for mixed arms whose primary arrow is labeled $a$. 

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Each tile has four *corner-parity label* in \(\{0, 1\}\), each corresponding to a corner of the tile. In a tile which has a horizontal arm the top corner labels are both 0, and the bottom corner labels are both 1. In a tile which has a vertical arm the top corner labels are both 1 and the bottom corner labels are both 0. A tile with a cross has two possible labeling for the corner parities, with opposite corners having the same parity and adjacent corners having different parities. The possible corner parity labels are shown in Figure 4.9.

There are two more labels for each tile: a *horizontal-parity label* and a *vertical-parity label*, both of which are in \(\{0, 1\}\). The restrictions on these labels are as follows: For a blank cross, both these parity labels must be 0. For all other basic tiles (bold crosses and arms), at least one of these parity labels must be 1.

In addition each tile has a direction label, which is in \(\{N, S, E, W\}\). These direction labels determine the direction function \(d : S_H \rightarrow \{N, S, E, W\}\), which make \(S_H\) a directed set of tiles.

Let us describe the adjacency rules for these tiles, which define when a tiling is locally valid. For a configuration to be valid around a tile the following conditions must hold:

1. For any arrow head facing outwards, the arrow head must meet an arrow tail.
2. Any arrow head facing inwards must have an arrow head which meets it’s tail.
3. The orientation labels and Hilbert-labels of meeting arrow heads and tails agree.
4. Corner-parity labels of adjacent corners agree.
5. Both horizontal and vertical parity labels alternate.
6. The direction label is valid, as explained below.

We now explain the rules which determine when the direction label is valid (rule 6). the direction label of a blank cross is valid if and only if one of the following conditions are fulfilled:

1. The direction is N and either the NW neighbor is a bold cross with Hilbert-label \(b\) or a vertical arm whose right side arrow has Hilbert-label \(a\) or \(d\).
2. The direction is W and either the NW neighbor is a bold cross with Hilbert-label \(d\) or a horizontal arm whose lower side-arrow has Hilbert-label \(c\) or \(b\).
3. The direction is S and either the SE neighbor is a bold cross with Hilbert-label \(c\) or a vertical arm whose left side arrow has Hilbert-label \(a\) or \(d\).
4. The direction is W and either the SE neighbor is a bold cross with Hilbert-label \(a\) or a horizontal arm whose lower side-arrow has Hilbert-label \(c\) or \(b\).
4.3. Bounding the number of infinite valid paths. To complete the proof of theorem 3.4 for \( d = 2 \), it remains to show that \( 0 < I(S_H) < \infty \).

We now show that there exist an infinite valid path for \( S_H \). By induction, we define sequences of valid \( S_H \)-configurations of squares of size \((2^{n-1}+1) \times (2^{n-1}+1)\), and denote these by \( B_{XY}(n) \) with \( X \in \{N, S\} \) and \( Y \in \{E, W\} \). \( B_{XY}(0) \) is a blank cross with orientation-label \( XY \). \( B_{XY}(n+1) \) is obtained by surrounding bold cross with orientation label \( XY \) by 4 configurations \( B_{SE}(n), B_{SW}(n), B_{NW}(n), B_{NE}(n) \), with arms labeled correctly as in figure 4.10. The configuration \( B_{XY}(2) \) with Orientation-labels on the crosses is illustrated in figure 4.11. The reader can verify that it is possible to fill the parity labels and corner-parity labels of these configurations so they remain valid. This was proved by Kari [8]. We now explain how to add Hilbert-labels and directions to \( B_{XY}(n) \) so that the configuration remains valid. By induction on \( n \), we can show that there exists a unique way to add these labels so that the central cross has Hilbert-label \( x \in \{a, b, c, d\} \). The Hilbert-labels of each of the four surrounding squares \( B_{XY}(n-1) \) are determined by the inductive definition of the Hilbert path (as in figure 4.2). The labeling of the paths is determined by a Hilbert-path which traverses all blank-crosses within this square. These configurations define arbitrarily long valid paths, and so by compactness there exist an infinite valid path for Kari’s tiles (corresponding to a Hilbert path).

Our next goal is to prove that \( I(S_H) < \infty \). For this we quote the following technical lemma, about the structure of valid paths in the tiling system \( S_H \):
Lemma 4.2. (Kari [3], lemma 5) For each \( x \in S_H^2 \) if \( p_1, \ldots, p_N \) is a valid path in \( x \), and \( N \geq 2 \cdot 4^n \), then there are integers \( 1 \leq i < j \leq N \) with \( j - i = 4^n \) so that the path \( p_i, \ldots, p_j \) fills a \( 2^n \times 2^n \) square.

Kari’s proof of this lemma involves a delicate examination of the tiles \( S_H \). We comment that it is easier to prove the corresponding statement about paths which are admissible configurations for the substitution system \((S_H, \rho_H)\). However, applying theorem 2.1 to deduce the result for the tiling system is non-trivial, since this lemma refers to configurations of which only some part (the path) is assumed to be valid. Using lemma 4.2 we obtain the following:

Lemma 4.3. Suppose \( x \in S_H^2 \) and the cells in \( p_1, \ldots, p_N \) form a valid path in \( x \). There exist a square \( F \subset \mathbb{Z}^n \) centered at \( p_1 \) which contains all the cells of the path, such that the path fills up an \( \epsilon \)-fraction of the cells in \( F \). The constant \( \epsilon \) is independent of \( x \) and of the path.

Proof. Suppose \( p_1, \ldots, p_N \) is a valid path with \( 2 \cdot 4^n \leq N < 3 \cdot 4^n \) (if \( 3 \cdot 4^n \leq N \leq 4 \cdot 4^n \), we look at a prefix). By lemma 4.2 there must by some \( 1 \leq i_1 \leq 4^n \) such that the path \( p_{i_1}, \ldots, p_{i_1+4^n} \) fills a square of size \( 2^n \times 2^n \). If \( i_1 > 1 \), let \( n_1 = \lfloor \log_4(i_1) \rfloor \). Apply lemma 4.2 on the path \( p_{i_1-4^n}, \ldots, p_{i_1+4^n-1} \), and deduce that \( p_{i_1-4^n}, \ldots, p_{i_1} \) fills a square of size \( 2^{n_1} \times 2^{n_1} \). Let \( i_2 = i_1 - 4^n \), continue in this manner applying lemma 4.2 on path segments of sizes \( 2 \cdot 4^n \). Each time set \( n_j = \lfloor \log_4(i_{j}) \rfloor \), and \( i_{j+1} = i_j - 4^{n_j} \), until \( i_j = 1 \). Each path segment fills a square of size \( 2^{n_j} \times 2^{n_j} \). The sizes of the \( n_j \)'s are non-increasing, and there can be no more than \( 3 \) \( n_j \)'s of the same size. Repeat the same procedure on the suffixes of the path \( p_{i_1+4^n}, \ldots, p_N \), this time the \( i_j \)'s increase until \( i_j = N \). We obtain that \( \max_{1 \leq i,j \leq N} \| p_i - p_j \|_{\infty} \leq 6 \sum_{i=0}^{n} 2^i \), and so the entire path is contained inside a square of size \( 2^{n+3} \times 2^{n+3} \), centered at \( p_1 \), of which it fills at least \( 2 \cdot 4^n \) cells. This completes the proof of the lemma, with \( \epsilon = \frac{4^n}{4^{n+3}} = \frac{1}{64} \). \( \square \)
We now use lemma [4.3] to show that the number of forward infinite valid paths in any configuration on the Hilbert tiles is bounded:

**Lemma 4.4.** For each $x \in S_H^{2^2}$ there exist at most $M = \left\lceil \frac{1}{\epsilon} \right\rceil$ disjoint forward infinite valid paths in $x$, where $\epsilon$ is the constant from lemma [4.3].

**Proof.** Fix some sufficiently large $N$. Suppose there are more then $M = \left\lceil \frac{1}{\epsilon} \right\rceil$ forward infinite valid paths in $x$, and fixes cells $c_1, \ldots, c_{M+1}$ with $c_i$ a cell in the $i$'th path. For any sufficiently large $N$, by lemma [4.3] each of these paths fill fills up an $\epsilon$-fraction of the cells in a $2^N \times 2^N$-square centered around $c_i$. As $N$ tends to $\infty$, each of these paths also fill almost an $\epsilon$-fraction of the same square of size $2^N \times 2^N$. As the paths are disjoint, we reach a contradiction. □

It follows from lemma [4.4] that $I(S_H) \leq M$. On the other hand, we have seen that $I(S_H) > 0$. This concludes the proof if theorem [3.4]. We remark that it can actually be proved by a more detailed examination of this system that there can be at most 4 disjoint forward infinite valid paths in any $x \in S_H^{2^2}$. Also, there exist $x \in S_H^{2^2}$ with exactly 4 disjoint infinite valid paths in $x$. Using these observations one can exactly compute the topological entropy of $S_H$.

5. Measure-theoretic entropy of surjective CA

This section contains a discussion of surjective CA’s as measure preserving dynamical systems, and the measure-theoretic entropy of these systems with respect to a “natural” measure.

Denote the symmetric Bernoulli measure by $\mu$: the state of each cell is distributed uniformly and independently of the other cells. The following simple proposition was already noted in the earliest dynamical systems study of CA for the case $G = Z$ (see Hedlund’s fundamental paper [7]):

**Proposition 5.1.** If $G$ is an amenable group, any surjective $G$-CA is measure preserving with respect to the symmetric Bernoulli measure $\mu$.

**Proof.** First, note that the action $\sigma_g : S^G \rightarrow S^G$ of $G$ on $S^G$ by translations preserves $\mu$. The measure theoretic entropy of this $G$-action is equal to the topological entropy of this action, and $\mu$ is the unique $\sigma$-invariant measure with this property- $\mu$ is the unique probability measure of maximal entropy for $\sigma$. Now consider the set $S = \{ \nu \in \mathcal{P}(S^G, \mathcal{B}, \sigma) : \nu = \nu \circ T^{-1} \}$ of $\sigma$-invariant probability measures. This set is non-empty since $G$ is amenable. For any $\nu \in S$, $T : (S^G, \mathcal{B}, \nu, \sigma) \rightarrow (S^G, \mathcal{B}, \mu, \sigma)$ is a measure-theoretic factor map, and so $h_{\nu}(S^G, \mathcal{B}, \sigma) \geq h_{\mu}(S^G, \mathcal{B}, \sigma)$, but since $\mu$ is the unique measure of maximal entropy for $\sigma$, it follows that $\nu = \mu$. □

The closed support of $\mu$ is $S^G$ for any countable $G$, and so any continuous map which preserves $\mu$ must be surjective. It follows that for an amenable group $G$, a $G$-CA is surjective iff it is $\mu$-preserving.

It is interesting to note that the above characterization of surjective CA does not hold for general countable groups: Consider the free group on two generators, denoted by $F_2$. We describe a surjective $F_2$-CA which does not preserve the symmetric Bernoulli measure $\mu$ on $\{0,1\}^{F_2}$: Denote 2 generators of $F_2$ by $a, b$, and consider the CA $M : \{0,1\}^{F_2} \rightarrow \{0,1\}^{F_2}$ defined by the local rule determined by “majority vote” of $x_{wa}, x_{wb}$ and $x_{wa^{-1}}$. 

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**Proposition 5.2.** The cellular automaton $M : \{0,1\}^{F_2} \rightarrow \{0,1\}^{F_2}$ defined as above is surjective, but does not preserve $\mu$.

**Proof.** To see that $M$ does not preserve $\mu$, let

$$A = \{ x \in \{0,1\}^{F_2} : x_{a^{-1}} \neq x_a \}$$

Obviously, $\mu(A) = \frac{1}{2}$. The pre-image of $A$ under $M$ is all the points $x \in \{0,1\}^{F_2}$ such that the state of cell at the identity disagrees either with both the cells at $a^{-2}$ and $a^{-1}b$ or with both the cells at $a^2$ and at $ab$:

$$M^{-1}A = \{ x_1 \neq x_{a^{-2}} = x_{a^{-1}b} \} \Delta \{ x_1 \neq x_{a^2} = x_{ab} \}$$

It follows that $\mu(A) = \frac{1}{2}$ but $\mu(M^{-1}A) = \frac{1}{8}$, so $T$ does not preserve the measure $\mu$.

To see that $M$ is surjective, we prove that for any finite $F \subset F_2$ and any $y \in \{0,1\}^{F_2}$ there exist $x \in \{0,1\}^{F_2}$ such that $(Tx)_F = y_F$. Consider the finite graph $G$ with vertex set $F$ and edges $\{(w_1,w_2) \in F \times F : w_1w_2^{-1} \in \{a,b,a^{-1},b^{-1}\}\}$. Since $G$ has no cycles, its edges can be directed so that each connected component is a directed tree. By adding extra cells to $F$, we can assume that $G$ is a 3-regular directed tree. Now we can define $x$: If $w_1, w_2, w_3$ are the children of $w \in F$, define $x_{w_i} := y_w$ for $i = 1, \ldots, 3$. Define the other states of other cells in $x$ in an arbitrary way. It follows that indeed $(Tx)_F = y_F$. \qed

Since the symmetric Bernoulli measure is preserved by any surjective $\mathbb{Z}^d$-CA $T$, one can study the measure theoretic entropy of $T$, which is bounded by the topological entropy.

**Proposition 5.3.** The $\mathbb{Z}^2$-CA associated with Kari’s Hilbert-tiles which we denoted by $T_H$ in section 6 has measure theoretic entropy zero with respect to the symmetric Bernoulli measure.

**Proof.** Recall that for each $w \in S_H^{\mathbb{Z}^2}$, we denoted by $X_w$ all the points $x \in X$ with directions determined by $w$. Since the sets $X_w$ are all $T_H$-invariant, and form a partition of $X$, almost any ergodic component of $\mu$ with respect to $T_H$ is contained in some $X_w$. Let $F_1$ and $F_2$ be squares centered around the origin of dimensions $2^N \times 2^N$ and $4 \cdot 2^N \times 4 \cdot 2^N$ respectively. Suppose there is a valid path staring inside $F_1$ and leaving $F_2$. By lemma 4.2 such path must fill a square of size $2^N \times 2^N$ contained inside $F_2$. $F_2$ contains less then $16 \cdot 4^N$ squares of size $2^N \times 2^N$, and at least one of them must be valid. The $\mu$-probability for a square of size $2^N \times 2^N$ to be valid is exponentially small in $4^N$, and so the $\mu$-probability of the event that a valid path starting in $F_1$ leaves $F_2$ tends to 0 exponentially as $N \rightarrow \infty$. The Borel-Cantelli lemma implies that almost surely this does not happen for infinitely many $N$’s, and so with probability 1 there is no infinite valid path. By lemma 5.2 when $w$ has no infinite valid path, $\mu_w$ is supported on a set with zero topological entropy. We conclude that $\mu$-almost every ergodic component of $\mu$ has 0 measure-theoretic entropy. \qed

In a private communication, Ron Peled suggested that with an appropriate set of acyclic tiles, $T_{S, \Gamma}$ defined by equation 3.2 can be a (surjective) CA with finite non-zero measure theoretic entropy with respect to $\mu$. To see this, apply the $T_{S, \Gamma}$ construction with the acyclic tile set $S = \{\uparrow, \rightarrow\}$ with no adjacency restrictions. With probability 1 a configuration has 1 forward infinite path, and any pair of
forward paths eventually coincide. Using lemma 3.3 we deduce that \( \mu \)-almost any ergodic component of this CA has entropy \( \log |\Gamma| \). On the other hand, this CA has infinite topological entropy.

6. Concluding remarks and questions

Our investigation of the existence of a multi-dimensional CA with finite non-zero entropy was motivated by the misleading intuition explained in the introduction. We conclude with some related questions and remarks.

Since the CA \( T_H \) described in section 4 is not injective, a question which follows naturally is:

**Question 6.1.** Does there exist an automorphism of a \( \mathbb{Z}^d \) full shift (an injective \( \mathbb{Z}^d \)-CA) with positive, finite topological entropy for \( d > 1 \)?

The discussion of measure-theoretic entropy of surjective CA in section 5 raises the following question:

**Question 6.2.** Does there exist a 2-dimensional surjective cellular automaton with positive measure theoretic entropy (with respect to the symmetric Bernoulli measure), and finite topological entropy?

One can study dynamical properties such as entropy of cellular automata over any countable group. We remark that if \( G \) is a countable group which is not finitely generated, \( X = \Sigma^G \) and \( T = X \to X \) is an \( G \)-cellular automaton then \( h(T) \in \{0, \infty\} \). Here is an explanation of this fact: Since \( T \) is given by some local rule, there exist some finitely generated subgroup \( H < G \) such that each cell only interacts via \( T \) with other cells in the same \( H \)-coset. Since \( G \) is not countably generated, \( [G : H] = \infty \), and so \((X,T)\) is conjugate to the infinite product \( \prod_{g \in G/H}(S_H, T) \), and so must have infinite or zero entropy. We ask:

**Question 6.3.** Does there exist a finitely generated, countable group \( G \) such that any \( G \)-CA has either zero or infinite topological entropy?

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