Objective thermomechanics

Abstract: An irreversible thermodynamical theory of solids is presented where the kinematic quantities are defined in an automatically objective way. Namely, auxiliary elements like reference frame, reference time and reference configuration are avoided by formulating the motion of the continuum on spacetime directly. Solids are distinguished from fluids by possessing not only an instantaneous metric tensor but also a relaxed metric. The elastic state variable is defined through comparing these two metrics. Thermal expansion is conceived as temperature dependence of the relaxed metric and plasticity, an irreversible change in the relaxed metric, is described via a plastic change rate tensor. Thermomechanics is built around these – finite deformation – kinematic quantities by starting from mechanics and adding thermodynamical requirements gradually. The obtained theory is not restricted to isotropic media.

1 Introduction

In Euclid's geometry, one operates with points, lines, triangles, vectors, rotations, reflections, and there may seem no need for coordinates. Descartes' suggestion to use coordinates comes convenient for applicational calculations for complicated geometric objects. It is inconvenient, on the other side, for principles and understanding. Nevertheless, along the many successful applications, coordinates became the standard language of geometric description in physics and related areas.

Reference frames are analogous objects on spacetime. The Galilean principle of relativity – the equivalence of (inertial) reference frames – unfolds not only that motion is not absolute but also that space is not absolute. Namely, in terms of the Galilean transformation rule in customary notation, while time is absolute, \( t' = t = f(t) \), space is not: \( r' = r - Vt \neq g(r) \): space is inevitably intertwined with time.

When Newton created his dynamics, he assumed an absolute space because he knew no other way to formulate his action-at-a-distance type description of gravitation – neither mathematics nor physics was developed enough to provide him a more appropriate framework. Along the success of his dynamics, absolute space also became a standard part of mechanics.

Later, time proved relative as well, and the Galilean transformation rule has been superseded by the Lorentz transformation rule:

\[
\begin{align*}
t' &= (t - \frac{V}{c^2}r_\parallel) / \sqrt{1 - V^2/c^2}, \\
r'_{\parallel} &= (r_{\parallel} - Vt) / \sqrt{1 - V^2/c^2}, \\
r'_{\perp} &= r_{\perp}.
\end{align*}
\]

When, correspondingly, Einstein formulated his relativity theory, he spoke in terms of reference frames, and about their relationships. Along the success of his ideas, reference frames also became the de facto standard when treating spacetime.

Nevertheless, Weyl \(^1\) and later independently Matolcsi \(^2, 3\), have pointed out that a reference frame free description of spacetime is possible. According to this, in both the Galilean and the special relativistic case, spacetime is a four dimensional affine space, equipped by some further structure: In the Galilean case, an absolute time structure (foliation, "slicing") and a Euclidean structure on the equal-time subspaces ("slices"), while in the special relativistic case a Lorentz metric (pseudo-Euclidean form).

In the continuum thermodynamics (thermomechanics, etc.) of solids, various customary auxiliary elements are used, like a reference frame – including a coordinate system –, an initial/reference time \( t_0 \), a reference configuration (the distribution of material points at reference time in the space of the reference frame), and a constant temperature \( T_0 \) at reference time. Objectivity is the requirement that there must be a physical content behind our description, which content is independent of our description. The customary formulation of objectivity is telling that, when changing auxiliary elements, what transforms how.
This approach has turned out to lead to controversies, artifacts and mistakes. Moreover, the essence (of the phenomena) is hidden behind the formulation (via the auxiliary elements). The situation is analogous to electromagnetism where, for more easily solvable equations, it is customary to replace the field strength four-tensor by a four-potential. The four-potential is not unique, there is a freedom – the so-called gauge freedom – in its choice so a gauge fixing condition is required. This condition does not belong to the physical phenomenon but to this type of its description. While formally convenient, physically the four-potential is misleading, which is revealed, for example, in approximations like perturbative solutions, where the smallness of a perturbative correction turns out to depend heavily on the gauge fixing condition.

The term ‘auxiliary’ is, in the present context, related to the meanings ‘something nonessential, arbitrary, and misleading’. A direct, i.e., auxiliary element free, spacetime description of continuum thermodynamics of solids would have various advantages. First, objectivity would automatically be guaranteed. Next, essence would be more directly visible (what causes what, etc.). Quantities which one ordinarily considers as scalars, three-vectors or three-tensors could be realized as relative components of absolute spacetime four-quantities: four-scalars, four-vectors or four-tensors. Auxiliary elements would be applied only in concrete applications – calculating a given process of a given sample of given properties –, they should be introduced only to the necessary amount, and should be used with one eye continuously on what distortions those auxiliary elements may cause on our physical picture of the phenomenon.

However, if one uses no reference frame, no reference time and no reference configuration then one has no displacement field \( \mathbf{u} \), no deformation gradient \( \mathbf{F} \), no strain tensor \( \varepsilon \) nor any customary deformation measure: Is it then possible to formulate elasticity? Is it possible to formulate plasticity? If one has no initial temperature \( T_0 \) then how to formulate thermoelasticity?

The task of this paper is to show that, yes, everything necessary can be formulated. Partial answers have already been published in recent works: The spacetime-friendly elastic and plastic kinematic quantities for solid continua have been introduced in (4), thermal expansion was added in (5), and thermomechanics in the small-strain regime and for isotropic materials was presented and used for evaluating experimental data in (6; 7; 8). The present paper extends the treatment to large-deformation thermomechanics and to anisotropic materials.

The spacetime perspective is quite an abstract one, while the impression ‘this theory is so abstract that it cannot be used for concrete situations’ should be avoided. Therefore, thermomechanics is built here step by step starting from mechanics and in terms of experimentally readily accessible quantities. This attitude also helps in ensuring thermodynamical consistency. Namely, one could postulate thermodynamical potentials and the Gibbs relation but, historically, the Gibbs relation was born in the context of gases, and who knows a priori how to generalize it for large deformations of anisotropic solid bodies, with their tensorial – and therefore not necessarily commuting – quantities? During the present step-by-step approach, one can see what one can have and how one can ensure positive entropy production in the end.

This paper treats the case of Galilean spacetime model only. Formally, there seems no obstacle to generalize the obtained formulae for special and general relativistic background. Physically, however, many constitutive assumptions may break. For example, elasticity has been born to express a type of deviation
from rigid body behaviour, and assumes internal forces depending on the equal-time distances among material points, but special/general relativistically there are no rigid (accelerated) bodies and ‘equal time’ is a frame dependent notion. Other interactions among the different parts of the continuum also propagate with a finite speed so thermodynamical aspects are also nontrivial to realistically generalize.

In the Galilean spacetime model, many notions – spacetime points, world lines, four-velocity etc. – are analogous to those of special relativistic spacetime theory so a knowledge of the latter helps a lot to understand the former. Concerning reading background for the differences, Weyl’s book (1) does not work out the technical details for practical applicability, and Matolcsi’s English books (2,3) are not easy to find, but the Appendix of (4) provides a helpful introduction and summary.

2 Kinematic quantities for solid continua

As the first step, a continuum is modelled by a three dimensional smooth manifold, called hereafter the material manifold. (Boundaries are not treated in the present paper.) The tangent vectors of this manifolds are referred to as material vectors. These give rise to, along usual manifold theory, tensors (called material tensors) of various type. It is worth emphasizing that covectors (real valued linear forms on a tangent space) are not identified with vectors (of that tangent space).

The existence of any material point (i.e., point of the material manifold) in spacetime is described by a world line. This provides a system of world lines $r$: a given material point $P$ at a given time $t$ is at spacetime point $r(t, P)$. Differentiation with respect to the time variable will be denoted by overdot and in the material variable by $\nabla K$. Here and hereafter, Penrose’s abstract index notation is used, where indices do not refer to components with respect to a coordinate system but to vectorial/tensorial type (without the need for a coordinate system). Upper indices indicate vectors and lower ones covectors, and an index appearing twice, once in upper and once in lower position, indicates tensorial contraction. We need to treat various manifolds, and the conventions applied are best explained on the example of the world line gradient, $J^k_k := \nabla_K r^k$ which is, at time $t$, the differential map from the material manifold to the three dimensional flat Riemannian manifold of equal $t$ spacetime points, the metric of which is the Euclidean inner product $h_{ij}$ of spacelike spacetime vectors. Namely, capital letters like $K$ here refer to material indices, small ones with overhat like $\hat{k}$ to four-indices, i.e., indices for spacetime as a manifold, and small ones without overhat like $k$ here to spacelike spacetime vectors. Though no coordinate systems are assumed here, formulae $\text{look like}$ ones with coordinate systems introduced, and most calculational rules for coordinate indices are valid here, too. When it is important to indicate that a certain formula does not hold in the coordinate sense but only in the abstract sense, this will be denoted by braces like in \{ $J^k_K$ \}.

The derivative $\nu^k := \dot{r}^k$ is the four-velocity field, the gradient of which defines the velocity gradient $L^k_k := \nabla_L \nu^k = J^k_k$, \ One can observe here an example of the general differential geometrical role of the differential map $J^k_k$: It transports material vectors to spacelike spacetime ones and vice versa, material covectors to spacelike spacetime covectors, etc. – depending on context, its inverse or transpose or both may be needed.

Another important notion is the instantaneous metric $h_{KL} := J^k_k h_{kl} J^l_L$, which describes the instantaneous distance of any two material points. This metric makes the material manifold a flat Riemannian manifold for any instant. The instantaneous metric is heavily process dependent, motion dependent, and one finds for its time derivative

\[
\dot{h}_{KL} = J^k_k h_{kl} J^l_L + J^k_k h_{kl} J^l_L = L^k_K (J^{-1})^M_K h_{ML} + h_{KM} (J^{-1})^M_L L^l_M = L^M_K h_{ML} + h_{KM} L^M_L. \tag{2}
\]

In the spirit of the general methodology of Matolcsi (2), we intend to give a mathematical model to every physical notion (so typically any notion gets modelled by either a set or a map from a set to another). Our description so far has not distinguished fluids from solids. How to formulate the difference?

Our everyday experience says that solids are “solid”. As a zeroth approximation, solids could be modelled by a rigid body. Naturally, for elasticity we need to go beyond that level: Distances of an elastic body
are not time independent. Nevertheless, for any pair of material points, there seems to be a distinguished distance, encoded somehow in the body, which is the distance when the body is in an undisturbed and completely relaxed state. This intuitive picture can be modelled in the following way: In addition to the instantaneous metric, solids are equipped with a relaxed metric \( g_{KL} \), too. In a relaxed state, \( h_{KL} = g_{KL} \).

In general, \( h_{KL} \neq g_{KL} \), and the extent to which they differ is measured conveniently by the elastic shape tensor \( A^K_L := (g^{-1})^{KM} h_{ML} \), which is the unit tensor \( I^K_L \) in relaxed state and some other, but still nondegenerate, tensor otherwise (its determinant is never zero). Historically, we are accustomed to a deformation measure that is zero in relaxed state, and to which Hookean elastic stress is proportional, i.e., which generalizes Cauchy’s small-strain tensor — this explains why the Biot, Hencky, Almansi etc. tensors have been defined. Hence, the elastic deformedness tensor is also practical to introduce: \( \{D^K_L\} := \frac{1}{2} \ln \{A^K_L\} \). This logarithmic definition is a distinguished choice. Indeed, on one side, it can be shown that the spherical part of this tensor, \( (D^K_L)^S \), is zero for volume-preserving motions while its deviatoric part, \( (D^K_L)^D \), is zero for isotropic volumetric changes. Furthermore, it also gives rise to the property

\[
\frac{df}{dD^K_L} = 2 \frac{df}{dA^K_L} A^K_L = 2 A^K_L \frac{df}{dA^K_L}
\]

for any isotropic scalar function \( f \). Property \((3)\) can be proved by writing \( f \) as a function of the isotropic invariants of \( A^K_L \), and differentiating it as a composite function. For the isotropic invariants themselves property \((3)\) is easy to see. As a part of this, why \( A^K_L \) commutes with the derivative of \( f \) with respect to it follows from that the derivative of these isotropic invariants with respect to \( A^K_L \) are some powers of \( A^K_L \), multiplied by some scalar so commuting holds term by term.

As long as we remain in the range of elastic phenomena, the relaxed metric is considered constant. Accordingly, one can derive the following evolution equation for the spacetime version \( A^i_j = J^i_k A^K_L (J^{-1})^L_j \) from \((2)\): \( \dot{A}^i_j = L^i_k A^j_k + A^i_k (h^{-1})^{kl} L^i_l m h_{mj} \).

Also while staying within the range of elasticity, there seems no need to expect that \( g_{KL} \) is not flat. Correspondingly to flatness, a condition can be derived for the Ricci tensor of the relaxed metric, which is the kinematic compatibility condition for \( g_{KL} \). In the present, large deformation, description, the compatibility condition proves to be a rather complicated formula for \( A^i_j \). Its small-deformedness leading order is the well-known simple form of the compatibility condition (left + right curl of \( D^i_j \) is zero).

As a consequence of systematically distinguishing covectors from vectors, we can see that elastic shape and elastic deformedness are not simply symmetric tensors. The proper — and apparent — property is that they are \( h \)-symmetric and \( g \)-symmetric: Combinations like

\[
h_{IJ} A^I_K = h_{IJ} (g^{-1})^{JK} m h_{KL}, \quad A^I_J (g^{-1})^{JK} = (g^{-1})^{IL} m h_{LM} (g^{-1})^{MK}
\]

and the corresponding spacetime tensorial versions, are symmetric, following from the symmetricity of \( h_{IJ} \), \( h_{ij} \), \( g_{IJ} \) and \( g_{ij} \). Mixed tensors like \( A^i_j \) can never be symmetric or antisymmetric. On the other side, only mixed tensors can have a determinant, a trace and eigenvalues—eigenvectors, such as \( A^i_j a^j = \lambda a^i \).

Thermal expansion is a phenomenon which enforces us to go beyond elasticity. In the language of the relaxed metric, thermal expansion can be formulated by allowing \( g_{KL} = g_{KL}(T) \). Accordingly,

\[
\alpha^J_L := \frac{1}{2} \frac{dg_{KL}}{dT} (g^{-1})^{JK}
\]

the thermal expansion coefficient tensor, can be defined. For isotropic materials, \( \alpha^L_L = \alpha g^L_L \). On the other side, in anisotropic cases, the order of the matrix product \((5)\) is relevant — why this product order is preferred to \( \frac{1}{2} (g^{-1})^{IK} \frac{dg_{KL}}{dT} \) will become clear in Sect. \( 5 \). The relaxed metric now becoming process dependent, the following generalization of the above kinematic time evolution equation can be found:

\[
\dot{A}^i_j = L^i_k A^j_k + A^i_k (h^{-1})^{kl} L^i_l m h_{mj} - 2 A^i_k (h^{-1})^{kl} \alpha^m_l m h_{mj} \dot{T},
\]

where \( \alpha^m_l = (J^{-1})^K_i \alpha^K_L J^m_L \), according to the rules of transporting a material tensor to spacetime.
Within the range of elasticity, the relaxed metric could be flat, in other words, it could make the material manifold a Euclidean affine space. With thermal expansion, this no longer holds in general: The relaxed metric makes the material manifold a curved Riemannian space.

Plasticity [see, e.g., (3)] is another known source of change of the relaxed structure, but, contrary to thermal expansion, plastic changes are permanent. Related to the plastic deformation originated change of the relaxed metric, the plastic change rate tensor

$$Z^J_i := \frac{1}{2} \left( \frac{dg_{JK}}{dt} \right)_{\text{plastic}} (g^{-1})^{KJ}$$

(7)
can be defined. Altogether then, we obtain

$$\dot{A}^i_j = L^i_k A^k_j + A^i_k (h^{-1})^{kl} L^m_l h_{mj} - 2A^i_k (h^{-1})^{kl} \left( \alpha^m_l \dot{T} + Z^m_l \right) h_{mj}. \quad (8)$$

The small-deformedness regime is when the norm of the deformedness tensor is small, \( \| \{D^i_j \} \| \ll 1 \) and thus \( \{A^i_j\} = \exp \left( 2 \left\{ D^i_j \right\} \right) \approx \{I^{i} + 2D^{i}j\} \), and then \( \{S\} \) leads, in the leading order of \( D^i_j \), to

$$\left( L^{i} j \right)^{j}_{j} = D^{i} j + (h^{-1})^{ik} \left( \alpha^{l} k \dot{T} + Z^{l} k \right) h_{lj}, \quad (9)$$

with \( S \) standing for symmetric – more closely, \( h \)-symmetric – part, \( \left( L^{i} j \right)^{j}_{j} := \frac{1}{2} \left[ L^{i} j + (h^{-1})^{ik} L^{i} k h_{lj} \right] \).

Both sources of a changing relaxed metric, thermal expansion and plastic processes, may ruin the flatness property of \( g_{ij} \). As an example, for elasticity plus thermal expansion one can find that, in the small-deformedness regime, only temperature distributions with a space independent gradient result in a zero Ricci tensor [a classic result (10)] while for large deformations, any nonhomegeneous temperature distribution causes nonzero Ricci tensor (3) so the compatibility condition for the relaxed metric is violated.

One can observe that we have used the relaxed metric as some kind of reference quantity to define elastic shape and deformedness, but it fundamentally differs from comparison to a reference configuration: reference configuration is an auxiliary element – involving choosing a reference frame and a reference time – which is not part of the phenomenon to describe. In contradistinction, relaxed metric is a state quantity, in other words, a physical field. The existence of this additional field is what makes solids differ from fluids.

One could build thermomechanics with \( g_{ij} \) as one of the state variables, but there are a few reasons to use \( A^i_j \), instead. First, the relaxed metric is not readily accessible through measurements; second, already well-known elastic energy expressions are more straightforward to make spacetime compatible via \( A^i_j \), and numerical calculations are also expected to be more cumbersome in terms of the relaxed metric.

3 Comparison to the usual kinematic framework

Starting with the usual deformation gradient, its meaning can be freed from reference frame but not from reference time. One can derive, with time dependence emphasized,

$$F^i_j(t, t_0) = J^i_K(t) (J^{-1})^K_j(t_0) \quad \text{and, for constant } g_{KL}, \quad A^i_j(t) = F^i_k(t, t_0) A^k_l(t_0)(h^{-1})^{lm} F_{jm}^n(t, t_0) h_{nj}$$

(10)

[suppressing spatial (material point) dependence]. The first of these equations shows how, by identifying material points \( P \) with their spacetime position \( r(t_0, P) \) at an instant \( t_0 \) chosen as reference time, the material manifold can be identified with the Euclidean affine space of spacetime points at \( t_0 \): material vectors and tensors are mapped to spatial spacetime vectors and tensors via \( J^i_K(t_0) \), and later world line gradients \( J^i_K(t) \) can be accessed as \( J^i_K(t) = F^i_l(t, t_0) J^l_K(t_0) \). One danger is that thus one also automatically – and unnoticed – transports the Euclidean (i.e., flat Riemannian) structure of the Euclidean affine space of spacetime points at \( t_0 \) to the material manifold. The result is naturally the instantaneous metric \( h_{KL}(t_0) \), which is indeed a legitimate Riemann metric on the material manifold, but is not the only one that may be relevant for constitutive and other purposes. For example, comparison to \( h_{KL}(t_0) \)
implies that the continuum is considered undeformed at reference time, and is supposed to fulfil the
kinematic compatibility condition. This assumption of an initial relaxed state at every material point is
not necessarily satisfied in practical applications. When opening an underground tunnel, initial (in situ)
stress is not zero; a laboratory loading machine must apply some initial stress on the sample to keep it firmly
fixed; etc. Moreover, a generic plastic preceding history or any inhomogeneous temperature distribution
cause to violate the kinematic compatibility condition mentioned above, the relaxed metric is not flat so
initial instantaneous metric – a flat one by definition – cannot be equal to the relaxed one everywhere,
generating elastic stress [plasticity originated remanent (“frozen”) stress and unavoidable thermal stress].

In parallel, various formulae emerging in continuum theory can require a metric structure and, at
each such situation, one should be able to decide explicitly which metric, \( h_{KL} \) or \( g_{KL} \), is relevant at that
situation. Such a decision situation emerges for Fourier heat conduction, as we will find at (29).

Similarly to the assumption of an everywhere relaxed situation at \( t_0 \), considering a homogeneous
initial temperature distribution is also artificial in general. This has consequences for thermal expansion
and thermal stresses.

The distinction between ‘right’ and ‘left’ vectors (tensors, etc.) is that ‘right’ vectors are material
vectors, tangent vectors of the material manifold, while ‘left’ vectors are spacelike spacetime vectors. If,
through \( J^I_K(t_0) \), one brings ‘right’ vectors/tensors to the same vector space in which the ‘left’ ones live,
that may lead – and indeed leads here and there – to \( t_0 \) dependent, hence, objectively forbidden, formulae.

It is also to be remembered that the classification ‘right’ and ‘left’ leaves out non-spacelike spacetime
vectors, which are also legitimate objective quantities. In fact, similarly to how various physical areas have
been rewritten to be compatible with special and general relativistic spacetime, continuum theory must
also be possible to rewrite to be totally compatible with Galilean, special and general relativistic spacetime
each. Such efforts, parallel to the described here, can be found, among others, at (11; 12; 13).

The second formula of (10) gives, in the special case of zero deformedness (unit elastic shape tensor) at
\( t_0 \), \( A^i_j(t) = F^i_k(t,t_0)(h^{-1})^{km} F^m_n(t,t_0)h_{nj} \) which tells that elastic shape generalizes the Cauchy–Green
tensor \( (A^i_j \text{ the‘left’one, and } A^I_j \text{ turns out to generalize the ‘right’ one). Correspondingly, to obtain, for
example, an objectively safe elastic energy of the variable \( A^I_j \) [or of \( D^I_j \) like in \( \frac{1}{2} C^I_K L D^L_J D^J_K \) from
a usually used version written with the Cauchy–Green tensor, the Cauchy–Green tensor variable is to be
replace by the elastic shape tensor. More generally, functions and formulae written in terms of \( F^i_j(t,t_0) \) can
be made spacetime compatible if rewritable as functions of \( t_0 \) independent spacetime compatible quantities
seen above, via inserting at appropriate places the silent assumption behind, \( I^i_j = A^i_j(t_0). \) Otherwise
objectivity of their content is questionable. During doing the rewriting, covectors should be distinguished
from vectors, and wherever a metric needs to be inserted, a decision must be made whether to use the
instantaneous metric or the relaxed one.

Deformation and strain are change type quantities, somewhat like heat and work in thermodynamics,
in the sense that they express some change occurring between two instants, while elastic shape and elastic
deformedness are state describing quantities at one given instant. This conceptual difference explains why,
for these new quantities, new names are introduced.

Quantities like plastic deformation gradient may also fail to carry a spacetime compatible meaning as
they stand. It has to be a topic of detailed future investigations how customary formulae of plasticity can
be brought into coherence with the approach of spacetime friendly quantities.

4 From mechanics to thermodynamics: isotropic case

Let us start from elastic mechanics. Until Sect. [5] let us restrict ourselves to isotropic solids only. The
customary balances for mass and linear momentum are, fortunately, straightforward to rewrite as frame
free four-equations. Throughout the paper, only differential equations in the bulk will be considered so it
is just a side remark here that, concerning boundary conditions, the switch to the spacetime compatible
Quantities do not currently seem to require any modification. The mechanical equations
\[ \dot{\mathbf{\varepsilon}} = -\varrho \nabla \mathbf{v}, \quad \dot{\mathbf{\varepsilon}}^T = \nabla \mathbf{v} \left[ (h^{-1})^{ij} \mathbf{\sigma}^T \right], \quad \dot{\mathbf{A}}^i_j = L^i_k A^k_j + A^i_k (h^{-1})^{kl} L^m_l h_{mj} \]

(11)
together with a constitutively known \( \mathbf{\sigma}^i_j = \mathbf{\sigma}^i_j \left( \{ A^i_j \} \right) \), or, equivalently, \( \mathbf{\sigma}^i_j = \mathbf{\sigma}^i_j \left( \{ D^i_j \} \right) \), form a closed set of equations. As an initial value problem, the initial distribution of \( \mathbf{A}^i_j \) is arbitrary as long as \( g_{KL} = h_{KM} A^M_L \) is flat, and later evolution of \( \mathbf{A}^i_j \) happens according to the velocity field. In other words, the time evolution equation for \( \mathbf{A}^i_j \) is equivalent to that \( g_{KL} \) is time independent. The time evolution equation will become practical at later stages. For the purpose of \( \mathbf{\sigma}^i_j = \mathbf{\sigma}^i_j \left( \{ D^i_j \} \right) \), we may think, for example, of
\[ \mathbf{\sigma}^i_j = E^d (D^d)^i_j + E^s (D^s)^i_j \quad \left( E^d = 2G, \quad E^s = 3K \right). \]

(12)
Note that \( \mathbf{\sigma}^i_j \) is not directly symmetric but is to be \( h_{kl} \)-symmetric, like seen for \( \mathbf{A}^i_j \) and \( D^i_j \).

Elasticity assumes, further, the existence of a specific elastic energy \( e_{el} \left( \{ D^i_j \} \right) \), an isotropic scalar function of \( \{ D^i_j \} \), with the properties
\[ \mathbf{\sigma}^i_j = \frac{\partial e_{el}}{\partial D^i_j}, \quad \dot{e}_{el} = \mathbf{\sigma}^i_j \left( L^S \right)^i_j. \]

(13)
Here, the second formula can be proved from the first one and from (9) (its ‘left’ version) as follows:
\[ \dot{e}_{el} = \frac{\partial e_{el}}{\partial D^i_j} \dot{D}^i_j = \frac{\partial e_{el}}{\partial D^i_j} \left[ A^i_k (h^{-1})^{km} L^n_m h_{nj} \right] \]
\[ = \frac{\partial e_{el}}{\partial D^i_j} A^i_k (h^{-1})^{km} L^n_m h_{nj} = \mathbf{\sigma}^i_j \left( L^S \right)^i_j. \]

(14)
Apparently, this simple result relies on the logarithmic definition of elastic deformedness.

It is interesting to observe that, by (13), we have also obtained \( \mathbf{\sigma}^i_j \dot{D}^i_j = \mathbf{\sigma}^i_j \left( L^S \right)^i_j \), since
\[ \mathbf{\sigma}^i_j \left( L^S \right)^i_j = \dot{e}_{el} = \frac{\partial e_{el}}{\partial D^i_j} \dot{D}^i_j = \mathbf{\sigma}^i_j \dot{D}^i_j. \]

(15)
However, \( \mathbf{\sigma}^i_j \dot{D}^i_j = \mathbf{\sigma}^i_j \left( L^S \right)^i_j \) holds not because \( \left( L^S \right)^i_j \) itself equals \( \dot{D}^i_j \) (that would be true only in the small-deformedness regime) but because of some much less trivial reasons (including isotropy).

Thermal effects enforce elasticity to be generalized from various aspects. On one side, \( e_{el} = e_{el} \left( T, \{ D^i_j \} \right) \) in general (e.g., in (12), \( E^d = E^d (T), \ E^s = E^s (T) \)). Second, there may be thermal expansion: \( g_{KL} = g_{KL} (T) \). Third, in addition to mechanical power, a heat flux \( j_s^i \) may also be present. Then, having the first law of thermodynamics in mind, we expect \( \dot{e}_{el} = \mathbf{\sigma}^i_j \left( L^S \right)^i_j \) to be generalized to a balance
\[ \dot{e} = -\nabla_i (j_e)^i + \mathbf{\sigma}^i_j \left( L^S \right)^i_j. \]

(16)
But with what \( e \left( T, \{ D^i_j \} \right) \) and \( (j_e)^i \)? In parallel, on thermodynamical grounds, we anticipate the existence of a specific entropy \( s \left( T, \{ D^i_j \} \right) \) as well, with a balance
\[ \dot{s} = -\nabla_i (j_s)^i + \pi_s \]

(17)
where the source term \( \pi_s \) – entropy production – is expected to be only heat-related since elasticity and thermal expansion are experienced as reversible phenomena. Taking a look at nonequilibrium thermodynamics, we can assume \( (j_s)^i = \frac{1}{T} (j_e)^i \), but what \( s \left( T, \{ D^i_j \} \right) \) would fulfil our hope?

Let us start with a calculation generalizing (12), and utilizing (9):
\[ \dot{e}_{el} = \frac{\partial e_{el}}{\partial T} \dot{T} + \frac{\partial e_{el}}{\partial A^i_j} \dot{A}^i_j = \frac{\partial e_{el}}{\partial T} \dot{T} + \mathbf{\sigma}^i_j \left( L^S \right)^i_j - \mathbf{\sigma}^i_j (h^{-1})^{kl} \alpha^j_k m_{mj} \dot{T}. \]

(18)
Now let us subtract this from the balance (16), and aim at forming (17) multiplied by $T$:

$$
\varrho(e - e_{cl})' = -\nabla_i(Tj_s)^i - \varrho \frac{\partial e_{cl}}{\partial T} \hat{T} + \sigma_j^i (h^{-1})^{kl} \alpha_l^m h_{mj} \hat{T},
$$

(19)

$$
\varrho \left( T \frac{e - e_{cl}}{T} \right)' + \left( \frac{\partial e_{cl}}{\partial T} - \sigma_j^i (h^{-1})^{kl} \alpha_l^m h_{mj} \right) \hat{T} = -T \nabla_i(j_s)^i - (\nabla_i T) (j_s)^i,
$$

(20)

$$
\varrho T \left( \frac{e - e_{cl}}{T} \right)' + \left[ e - e_{cl} + T \left( \frac{\partial e_{cl}}{\partial T} - \frac{1}{\varrho} \sigma_j^i (h^{-1})^{kl} \alpha_l^m h_{mj} \right) \right] \frac{\hat{T}}{T} = -T \nabla_i(j_s)^i - \left( \nabla_i T \right) \frac{1}{T} (j_e)^i,
$$

(21)

$$
\varrho T \left( \frac{e - e_{cl}}{T} \right)' + \varrho \hat{e}_{th} \frac{\hat{T}}{T} = -T \nabla_i(j_s)^i + T \left( \nabla_i \frac{1}{T} \right) (j_e)^i,
$$

(22)

with

$$
e_{th} = e_{th}(T, \{D^i_j\}) = e - e_{cl} + T \left( \frac{\partial e_{cl}}{\partial T} - \frac{1}{\varrho} \sigma_j^i (h^{-1})^{kl} \alpha_l^m h_{mj} \right).
$$

(23)

This is of the form (17) multiplied by $T$, where the would-be entropy production is related to heat only (no elastic or thermal expansion contribution). The only snag is the second term on the lhs of (22); it also should be of the form $\varrho T(\text{something})'$. This can be satisfied if $e_{th} = e_{th}(T)$. Then, writing (22) as

$$
\varrho T \left[ \frac{e_{th}}{T} - \left( \frac{\partial e_{cl}}{\partial T} - \frac{1}{\varrho} \sigma_j^i (h^{-1})^{kl} \alpha_l^m h_{mj} \right) \right]' + \varrho \hat{e}_{th} \frac{\hat{T}}{T} = -T \nabla_i(j_s)^i + T \left( \nabla_i \frac{1}{T} \right) (j_e)^i,
$$

(24)

and rearranging this as

$$
\varrho T \left( \frac{1}{\varrho} \sigma_j^i (h^{-1})^{kl} \alpha_l^m h_{mj} - \frac{\partial e_{cl}}{\partial T} \right)' + \varrho T \left[ \frac{e_{th}}{T^2} \right]' = -T \nabla_i(j_s)^i + T \left( \nabla_i \frac{1}{T} \right) (j_e)^i,
$$

(25)

we have

$$
\left( \frac{e_{th}}{T} \right)' + \frac{e_{th}}{T^2} \hat{T} = \frac{d e_{th}}{dT} \frac{\hat{T}}{T} = s_{th}(T)' = \frac{ds_{th}}{dT} = \frac{1}{T^2} \frac{d e_{th}}{dT},
$$

(26)

Then we can read off

$$
e = e_{th}(T) + e_{cl} + T \left( \frac{1}{\varrho} \sigma_j^i (h^{-1})^{kl} \alpha_l^m h_{mj} - \frac{\partial e_{cl}}{\partial T} \right),
$$

(27)

$$
s = s_{th}(T) + \left( \frac{1}{\varrho} \sigma_j^i (h^{-1})^{kl} \alpha_l^m h_{mj} - \frac{\partial e_{cl}}{\partial T} \right),
$$

(28)

The term $e_{th}(T)$ is related to specific heat, $c_l D_{j}^i = 0$, while the last equation here is the entropy production which we can ensure to be positive definite via, e.g., Fourier heat conduction, for which both forms

$$
(j_e)^i = \lambda \left( g^{-1} \right)^i j \nabla_j 1 \frac{1}{T} \quad \text{or} \quad (j_e)^i = \lambda (h^{-1})^i j \nabla_j 1 \frac{1}{T}
$$

(29)

are allowed (with a non-negative heat conduction coefficient $\lambda$). The former, i.e., choosing the relaxed metric for connecting the covector gradient with the vector lhs, is more plausible physically but it must be a topic of further study what metric to choose here.

Note that, in (27) - (28), stress is purely elastic and may well be $\sigma_j^i = \sigma_j^i(\{D^k_{lj}\})$ (no $T$ dependence). Nevertheless, the terms coupling $T$ and $D^k_{lj}$ are sources of thermal stress and of Joule–Thomson effect. As a special limiting case, small-strain Duhamel–Neumann thermoelasticity can be obtained (10).

When incorporating plastic changes, the only modification is an additional entropy production term

$$
\varrho \sigma_j^i (h^{-1})^{kl} Z_l^m h_{mj}.
$$

Its positive definiteness can be ensured, for example, with the simple yet plausible

$$
Z_l^j = \Gamma h_i k (\dot{\sigma}^d)^k (h^{-1})^{ij} \quad \text{with} \quad \Gamma = \gamma H \left( (\sigma^d)^i_j (\sigma^d)^j_i - \frac{4}{3} \sigma^2 \right) H \left( (\sigma^d)^i_j (\dot{\sigma}^d)^j_i \right), \quad \gamma > 0,
$$

(30)

the first Heaviside function $H$ embodying von Mises type yield criterion. Note how irreversible thermodynamics switches off plastic change during unloading (second Heaviside function): Entropy production is not allowed to be negative.
5 Anisotropy

Anisotropy means distinguished material directions. In a linear elastic model, for example, it is not \( \sigma^i_j \) and \( D^i_k \) but their material version \( \sigma^i_j \) and \( D^i_k \) between which the linear coefficient tensor \( C^i_j L^k K \) is constant, in \( \sigma^i_j = C^i_j L^k K D^i_k \). Other constitutive properties are also expected to be connected to the material form rather than to the spacelike spacetime form. (In the meantime, balances primarily live on spacetime.)

Next, we must realize that no part of (3) may hold for no isotropy, including that the factors in the products two terms become simple, and (32) reduces to

\[
\dot{A}^I_J = -(g^{-1})^{IK} g_{KL} (g^{-1})^{LM} h_{MJ} + 2A^I_K (L^S)^K_J
\]

\[
= -2A^I_M (J^{-1})^M_i (h^{-1})^{ij} (\alpha_j^k \hat{T} + Z_j^k) h_{kl} J^I_J + 2A^I_K (L^S)^K_J.
\]

(31)

This is analogous to (19) so, from here, we can proceed the same way, and the result will also be (27)–(28) [the entropy production being extended by the same plasticity related term as in the isotropic case].

Therefore, after settling (33), nothing differs from the isotropic formulae. Note that, if we had used the opposite product order in (5) and (7), the calculation would not have led to such a nice outcome.

As a summary, the objective thermomechanics obtained here works with basic fields \( r^k, A^K_L, g \) and \( T \) as functions of \( (t, P) \), with derived fields \( v^k, J^k_K, L^k_i \) and \( h_{KL} \), with basic constitutive functions \( e_{th}(T), e_{cl}(T, \{D^j_j\}), \alpha_K^L(T), Z_K^L \) and \( (j_e)_K \) \{variables for the latter two can be written in various ways\} and derived constitutive functions \( \sigma^i_j(T, \{D^j_j\}), s(T, \{D^j_j\}) \) and \( (j_e)_K^L \).

Apparently, this formulation is not a thermodynamically elegant one, but is at least a thermodynamically consistent as well as objective one, and the special cases of a constant specific heat, thermal expansion coefficient, heat conduction coefficient and elasticity coefficients \( 2G, 3K \) – important for many engineering applications – can be explicitly expressed. Naturally, more elegant reformulations are worth exploring.

6 Discussion and outlook

The framework presented here guarantees objectivity via the frame free spacetime formulation, and describes elasticity, thermal expansion and plasticity in a thermodynamically consistent theory. Naturally,
there are various tasks for the future. Continuing comparison with the literature, especially on the plasticity side, is necessary, both for the kinematic quantities and on the constitutive content. Rheology/viscoelasticity is to be incorporated – here, the tensorial internal variable methodology realized so far for isotropic solids and small deformations\(^\text{[14]}\) seems to pose no problem against generalization, and the possibility of nonequilibrium thermodynamical coupling of this tensorial phenomenon to plasticity promises interesting predictions. This may still not be enough to describe all large-deformation complex rheological phenomena like considered in\(^\text{[15]}\), but the number of internal variables can be increased and other possibilities are also to be investigated.

Further phenomena like damage and failure\(^\text{[16; 17]}\) are to be added, too. To explore the spacetime aspects of GENERIC and other nonequilibrium thermodynamical frameworks is also an important mission.

The frame free approach can be advantageous not only because it helps avoiding artifacts and mistakes: It also catalyses and even enforces better physical understanding. For example, the notion of the relaxed metric was born motivated by the urgent need of distinguishing solids from fluids. When discussing other thermodinamical four-quantities like four-energy-momentum\(^\text{[13]}\) and related issues, further similar outcomes and discoveries can be expected.

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