The statistical dynamics of classic particles ensemble in gravitational field

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Abstract

The article is the translation of authors paper [1], printed earlier in inaccessible edition and devoted to the formulation of basic concepts of dynamic description of particles’ statistic ensemble in a gravitational field. Later on, the results of this article were used by numbers of authors in papers of relativistic kinetics.

1 Introduction

During last 10 years the independent direction in GRG - the general relativistic kinetic theory of matter, bases of which were laid in papers of N.A.Chernikov [2]-[7] and A.A.Vlasov [8], segregated and roughly develops. Methods of general relativistic kinetics are applied in cosmology [9]-[14], at researching of gravitational [15]-[25] and electromagnetic [26]-[32] waves propagation processes in relativistic plasma, in the theory of Universe gravitational instability [33]-[38], as well as in the theory of equilibrium gravitating plasma configurations [39]-[50]. There were publicized some papers, devoted to the quantum-field substantiation of classic general relativistic kinetic theory [51]-[53], as well as to the quantum-statistical description of particles’ ensemble in the gravitational field [54]-[56]. The kinetic consideration is necessary in the cases, when high temp of running processes in the system doesn’t allow thermodynamical equilibrium to establish. However, the strict kinetic model is necessary also for the substantiation of macroscopic hydrodynamics in General Theory of Relativity. Thus, for instance, the kinetic analysis of cosmological expansion process display [11],[13], that processes, leading to the deformation of equilibrium spectrum of heavy particles, arise in expanding plasma. According to Weinberg, [57], this fact can have an important meaning on the earlier stages of cosmological expansion.

An existing general relativistic theory of gases presents itself the phenomenological theory (although deeper label, than hydrodynamics), describing diluted gas, particles of which interact between themselves on external fields background via contact binary collisions. The dynamical substantiation of general relativistic kinetics becomes an urgent problem. In non-relativistic physics the similar problem was solved in papers of Bogolyubov, Borne, Grin, Kirkwood, and Evon (BBGKE-chain). Significant progress in special theory of relativity was reached by Yu.L.Klimontovich [58]-[60] and R.Balescu [61]-[63].

At attempt of constructing of statistical theory in GRG we face significant difficulties, related with nonlinearity of gravitational field. First, nonlinearity of of Einstein field equations doesn’t allow in strict sense to apply kinetic theory to the description of interacting particles system [64], [65]. In fact, averaging
Einstein equations by different states of particles, we’ll obtain

\[ \langle G_{ik} \rangle = \kappa \sum_a \langle T_{ik} \rangle_a, \]

where \( \langle \ldots \rangle \) means averaging, \( a \) - index of particle. Nonlinear structure of Einstein tensor \( G_{ik} \) does not allow us to write the equality

\[ \langle G_{ik}(g) \rangle = G_{ik}(\langle g \rangle), \]

in consequence of that, macroscopic Einstein tensor should be presented in the form of infinite series

\[ \langle G_{ik}(g) \rangle =
\begin{align*}
= G_{ik}(\langle g \rangle) + \sum_{n=2}^{\infty} \frac{\text{Correlations of local fluctuations of an order } n}{\text{of an order } n}.
\end{align*}
\]

Hence, the macroscopic momentum-energy tensor, in strict sense, doesn’t define the macroscopic metric of space-time. This means, that at constructing of macroscopic picture of Universe, fluctuations of metric tensor are related with correlated microscopic particles’ motions and will stand forward for some effective momentum-energy tensor. Physically, correlations of metric tensor local fluctuations can lead to the change of equation of matter state at high densities (according to Sakharov [66], the equation of state can change by radical way). Let’s note, that this moment will play an important role also at constructing of overall statistic quantum picture of the world.

At constructing of dynamical relativistic theory of interacting particles we face also with difficulty of retarding field interactions’ description. The finiteness of speed of waves’ propagation lead to the necessity of including to the statistical system an infinite number of dynamical field variables. In electrodynamics, mentioned above difficulty, can be overcome, in principle, due to the linearity of Maxwell field equations. In GRG the nonlinearity of field equations doesn’t allow to present a metric tensor in form of linear superposition of gravitational field independent states. Consequently, there is no obvious way of precise statistical description of interacting particles’ ensemble in GRG. The quantum theory of gravitational field faces with the same problem.

We suggest an avoiding line of problem solving instead of direct solution of it. The smallness of gravitational interaction constant allows to hope, that at not very high values of matter density, when gravitational radiiuses of particles don’t overlap, the macroscopic gravitational field in zero approximation is determined sufficiently well by macroscopic substance and fields’ distribution. In this case it is possible to develop the perturbation theory by order of correlation parameter smallness, and to find on that way the corrections to the macroscopic momentum-energy tensor. At the initial stages of theory constructing, we will neglect influence of thermal motion of field on particles’ correlations, \(^1\) that

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\(^1\) This approximation is justified by smallness of electromagnetic (\( \alpha = e^2/\hbar c = 1/137 \)) and gravitational (\( g_p = Gm_p^2/\hbar c \approx 6 \cdot 10^{-39} \)) interaction constants, at not very high values of particles energy.
corresponds to the exclusion of field degrees of freedom from the ensemble’s
distribution function. Such approximation accounts only retarding particles’
interaction and leads to the kinetic equations without accounting of radiation,
i.e. to the equations of Landau type \[67\] or to the equations of Belyaev-Budker
\[68\]. Mentioned program is particularized in Ref. \[65\].

This article, according to author’s opinion, should be considered not alter-
natively to completed theory, but rather as a designs of future statistical theory,
as an attempt to collect together the principles of such theory, and to develop,
as far as possible, it’s certain methods. It is hard to predict now, how the right
statistical theory will look like, however, as it seems to Author, one of the main
features of it will be a multitime character of statistical and field equations.
Observed averages at that should be determined as a synchronized in one or
another way of multitime functions’ contraction. This idea presents the key one
for given article.

All denotations, unconditioned in this text specially, are borrowed from ar-
ticles \[24],\[65],\[69].

2 The Statistical Description Of Medium In Gra-
vitational Field

In disposal of modern gravitationist there is a sufficient number of exact and
established surely equations of fields and of particles’ motion for basic types of
interactions. Let’s mentally try to write down a self-consistent system of these
equations. The microscopic sources of these fields, determinable by correspond-
ing “charges” and their detail distribution, will found in the right parts of field
equations. In consequence of pointlike character of particles the field sources are
singular. “Forces” of interactions, determinable by summary “tensions” of fields
in points of locations, will found in the right parts of motions equations. For
picture’s completeness, it is necessary to add to the written system of equations,
an Einstein equations, in the right-hand sides of which is situated summary mi-
croscopic momentum-energy tensor of particles and generated by them fields.
In the latter case, the necessity to write down the equations of motion, which
present themselves differential consequences of field equations, disappears. In
voluminous system of equations, which appeared before our mental view, all
information about classic particles interaction (including gravitational) is en-
closed. Let’s preset on the spacelike hypersurface \[2\] in moment of time \(\tau_0\) all
“potentials” and their first derivatives by time, and also coordinates and ve-
locities of all \(N\) particles. Then the solutions of this system of equations will
exactly describe an evolution of classical interacting particles’ ensemble.

\[2\]
Here it is necessary to clarify a matter. We’ll call “spacelike hypersurface” such hyper-
surface, the normal vector of which is timelike. Often it is done vise versa: the name, directly
coinciding with the name of type of normal to hypersurface vector, appropriates to hypersur-
face. In our opinion it is a violence upon language. According this scheme, the magnetic field
should be named as a pro-electric only because it’s orthogonality to electric field.
Let’s immerse the observer together with macroscopic apparatus into ensemble of particles. Let spacelike volume of apparatus is $\Delta V$. Since observer using this apparatus provides local measures, and the motion of each particle, falling into apparatus, is determined by the whole history of this particle’s interaction with others in ranges, uncontrollable by apparatus, then from observer’s point of view the concrete parameters of each registrable particle (microscopic parameters), and also macroscopic field parameters are random, unpredictable. However, if apparatus’ volume will be large enough so as to sufficiently great number of statistically indistinguishable (by given apparatus) particles will synchronically locate in it, or sufficiently great number of wave packages will keep within it, then the observer will fix certain average macroscopic parameters of particles and fields. For instance, by simple summation of momentums of statistically indistinguishable particles, lying in apparatus’ volume, the observer can measure the total momentum, transferring through apparatus by particles of given sort. Dividing this momentum by apparatus’ volume, one will come to the conception of momentum average density. This value will experience so small fluctuations (random deviations) in due course, as large will be the number of particles, synchronically lying in apparatus. Thus, an observer comes to the idea of measurement of physical values’ certain average macroscopic densities instead of measurement of these values essentially.

For description of values’ densities theorist introduces the conception of probability density as a function, dependent from the certain array of microscopic characteristics of particles and fields, using which he can calculate any macroscopic densities of these values. Which requirements should array of these values satisfy? Let’s mentally fulfill the whole space-time by observers, equipped by macroscopic apparatuses and clocks, synchronized by certain way. Mathematically such operation is equivalent to introduction in $\mathbb{V}^4$ of certain timelike vector field (field of observers), vector lines of which in each point coincide with the direction of observer’s proper time. The realization of such field of observers with synchronized clocks is the semi-geodesic coordinate system \cite{70}, relatively to which observers rest, - synchronic frame of reference \cite{71}. In time point $\tau_0$ in this frame of reference we will carry on the measurement of probability density on the whole hypersurface $\mathbb{V}$, orthogonal to the field of observers, -$\mathcal{R}_{\tau_0}(\mathbb{V})$. Let’s require the r-presentation $\mathcal{R}_{\tau_0}(\mathbb{V})$ to determine completely the further evolution of all measurable values’ densities. Probability density for it should be a function of certain array of values, completely determining the whole ensemble’s dynamics. We’ll denote mentioned above values through dynamic variables, and their full array - through full dynamic array. Dynamic variables for particles are c-o-dinates and velocities (or momentums) - assignment of these values at defined fields completely determines particle’s trajectory. Dynamic variables for fields are potentials and their time derivatives. Here the explanation is necessary. Field equations correspond themselves equations in partial derivatives in contrast to particles’ equations of motion, which represent themselves ordinary differential equations. The construction of total dynamic array of field variables is usually done in such way. Further the total array of eigenfunctions of vacuum field equations - $\psi_k(x)$, where $k$ - quantities, composed by certain rule
from quadruple of quantities $k_a$ (4-dimensional Fourier-representation, spherical photon waves etc) should be determined. In accordance with well-known theorem about eigenfunctions of Hermit operator, any function can be represented in form of expansion by $\psi_k(x) : \sum C_k \psi_k(x)$. General solution of field equations with source is not an exclusion. Let’s represent time derivatives of potential in form of $\sum \dot{C}_k \psi_k(x)$, where $\{\dot{C}_k\}$ - recent array of constants. These infinite arrays $\{C_k\}$ and $\{\dot{C}_k\}$ represent themselves exactly the dynamic array of field variables. The number of field degrees of freedom consequently is equal to $8 \cdot \infty$ (two arrays, $\{C_k\}$ and $\{\dot{C}_k\}$, each of them includes four arrays of quantities $k_a$).

Thus, probability density must be a function of type

$$D(\tau, \tilde{x}_1, \ldots, \tilde{x}_N, C_1, \dot{C}_1, \ldots),$$

where $\tilde{x}_a$ - totality of dynamic variables of $a-$ particle.

In given article we will neglect ensemble’s field degrees of freedom, i.e. will consider functions (2), averaged by field variables. Thereby we exclude free fields (photons) from the system and limit ourself by accounting of such fields, which are generated by moving charged particles.

In case of, for instance, electromagnetic interactions such approximation is fair under the assumption \[68 \]

$$< E > \ll mc^2 \sqrt{137L},$$

where $L$- Coulomb logarithm. Though the statistic theory with excluded degrees of freedom is not complete, really relativistic theory, it nevertheless seems justified and reasonable at first stages to develop the apparatus of such theory.

In consequence of simple logical construction of statistic theory with excluded field degrees of freedom, on the basis of this theory can be testified statistic methods and prepared, perhaps, more rational and simple formulas for complete statistic theory.

### 3 Sources Functions

On account of exclusive importance of $\delta$-Dirac function in further calculations we’ll pay our fixed attention on it’s features. We will term as an invariant symmetrical double-point $\delta$-Dirac function, determined on $n$- dimensional Riemannian manifold $R$, a function $D(x_1|x_2)$, possessing next features:

$$\int_{X_2} D(x_1|x_2)F(x_2)dX_2 = \begin{cases} F(x_1); & x_1 \in X_2, \\ 0; & x_2 \notin X_2, \end{cases}$$

$(X_2 \subset R); F(x) - random tensor field, dX = \sqrt{-g(x)}dx^1 \ldots dx^n - invariant differential of volume R)$.

$$D(x_1|x_2) = D(x_2|x_1);$$ \hspace{1cm} (5)

$$D(x'_1|x'_2) = D(x_1|x_2),$$ \hspace{1cm} (6)
if \( x' = \varphi(x^1, \ldots, x^n) \)- non-degenerate transformation of coordinates

\[
J = \det \left| \frac{\partial x'}{\partial x} \right| \neq 0. \tag{7}
\]

Alongside with invariant \( \delta \)- function it is possible to consider also the scalar density \( \Delta(x_1|x_2) \), that is what usually termed \( \delta \)- Dirac function,

\[
\Delta(x_1|x_2) = \frac{1}{\sqrt{-g(x_2)}} D(x_1|x_2) \tag{8}
\]

with the law of transformation:

\[
\Delta(x'_1|x'_2) = |J^{-1}(x_2)| \Delta(x_1|x_2). \tag{9}
\]

Let us consider an expression of type

\[
\int_R F(x)D[\psi(x)|0]dx = \int_R F(x)\Delta[\psi(x)|0]d^n x, \tag{10}
\]

where \( \psi^a(x) \)- univalent functions \( x \), where

\[
J(\psi) = \det \left| \frac{\partial x^b}{\partial \psi^a} \right| \neq 0.
\]

Last criterion allows us to choose \( \psi^a \) by way of new coordinates and adduce an integral to the form

\[
\int_{\psi} F[x(\psi)]\Delta[\psi|0]dx|J(\psi)|d^n \psi
\]

and, consequently [11], to obtain formula

\[
\int_R F(x)D[\psi(x)|0]dx = \sum_a |J[\psi(x_a)]| F(x_a), \tag{11}
\]

that can be written down in form of symbolic rule

\[
D[\psi(x)|0] = \sum_a |J[\psi(x_a)]| D(x|x_a), \tag{12}
\]

where \( x_a \)- roots of equation \( \psi(x) = 0; \)

\[
J[\psi(x_a)] = \det^{-1} \left| \frac{\partial \psi^a}{\partial x^b_a} \right|.
\]

Formula (12) is the generalization of well-known property of one-dimensional \( \delta \)-Dirac function:

\[
\delta[\varphi(x)] = \sum_a |\varphi'(x_a)|^{-1} \delta(x - x_a). \tag{13}
\]
If in certain frame of reference \( R \) is representable in form of simple product of three-dimensional anisotropic hypersurface \( V_k \) and normal to it in each point of coordinate lines’ \( x_k \) congruence, then \( \delta \)-Dirac function in this frame of references can be represented in form of product of invariant on hypersurface \( V_k \) three-dimensional \( \delta \)-function \( \delta(x_1 | x_2) \) and one-dimensional \( \delta \)-function

\[
\delta(\tilde{x}_1 | \tilde{x}_2) = \delta(x_1 | x_2)
\]

(14)

where \( \tilde{x} \)-coordinates on the \( V_k \). In this frame of references differential of volume \( R \) also represents in form of product \( dX = dV_k dx_k \), where \( dV_k = \sqrt{-g(x)} d^{n-1}\tilde{x} \)-differential of hypersurface area \( V_k \). Henceforward we will often carry out such an operation in synchronic frame, when normal vector \( k_i \) is timelike. Metric \( R \) in this frame has a form

\[
ds^2 = d\tau^2 + g_{\alpha\beta} dx^\alpha dx^\beta, \quad (\alpha, \beta = 1, 2, 3).
\]

(15)

In this case 3-dimensional hypersurface appears spacelike, differential of it’s area we will denote through \( dV \).

Let’s consider \( \delta \)-function derivatives. Extending integration in (4) on the whole space and differentiating this relation, we obtain

\[
\frac{\partial F(x_1)}{\partial x^i_1} = \int_{\mathcal{R}} F(x_2) \frac{\partial D(x_1 | x_2)}{\partial x^i_1} dX_2.
\]

From the other hand, by definition

\[
\frac{\partial F(x_1)}{\partial x^i_1} = \int_{\mathcal{R}} \frac{\partial F(x_2)}{\partial x^i_2} D(x_1 | x_2) dX_2.
\]

Comparing these expressions, we will obtain the symbolic rule of invariant \( \delta \)-Dirac function differentiation

\[
\frac{\partial}{\partial x^i_1} D(x_1 | x_2) = D(x_1 | x_2) \frac{\partial}{\partial x^i_2}.
\]

(16)

Replicating analogous calculations for tensor object \( F(x) \), we will obtain more common differentiation rule

\[
\nabla^i D(x_1 | x_2) = D(x_1 | x_2) \frac{\partial}{\partial x^i_2},
\]

(17)

where \( \nabla^i \)-operator of differentiation in point \( x_a \).

Geometric image of particle is timelike world line \( x^i = x^i(s_a) \equiv x^i_a \), along which the certain geometric object \( \omega_a(x_a) \equiv \omega_a \), characterizing it’s physical properties, prescribed. We will term this object a source. Only one tensor object \( 3 \)-velocity vector \( u^i_a = dx^i_a / ds_a \) and some scalar “charges” conform to the classic

\footnote{Quantum particle in quasi-classic representation can be conformed also with spinor, determined on the trajectory.}
point particle. When accounting only electromagnetic and gravitational interactions, charges are equal $-m_a \, \text{(mass)}, e_a \, \text{(electric charge)}$. Thus, classic particle’s source can have only a structure of type $\omega_a(s_a) = \{1, e_a, m_a\} \times u_{a1}^i \ldots u_{an}^i$. Let’s determine source’s density field

\[
\Omega_a(x) = \int_{\text{Along whole trajectory}} \omega_a(s_a)D(x \mid x_a)ds_a. \tag{18}
\]

In consequence of invariant $\delta$-function definition, integration in (18) extends tensor properties of object $\omega_a$ from the particle’s trajectory to the whole manifold $R$, presetting tensor field $\Omega_a(x)$.

Let’s introduce into consideration the following sources’ densities, having simple physical meaning:

\[
n^i(x) = \sum_a n^i_a(x) = \sum_a \int u_{a1}^i(s_a)D(x \mid x_a)ds_a \tag{19}
\]

- density vector of particles’ number (numerical vector),

\[
j^i(x) = \sum_a e_a cn_{ai}^i(x) =
\]

\[
= \sum_a e_a c \int u_{a1}^i(s_a)D(x \mid x_a)ds_a \tag{20}
\]

- current’s density vector,

\[
T^{ik}_p(x) = \sum_a T^{ik}_{a1}^a(x) =
\]

\[
= \sum_a m_a c^2 \int u_{a1}^i(s_a)u_{a1}^k(s_a)D(x \mid x_a)ds_a \tag{21}
\]

- tensor of particles’ energy-momentum without accounting of interacting fields (tensor of stripped particles’ energy-momentum).

Let’s calculate the covariant divergencies from these values. First we’ll consider an expression of type

\[
\nabla_i n^i_a(x) = \int_S u_{a1}^i(s_a) \frac{\partial D(x \mid x_a)}{\partial x^i} ds_a,
\]

Let’s represent an integral in this expression as a curvilinear integral of 2 type, taken along the whole particle’s trajectory:

\[
\nabla_i n^i_a(x) = \int_S \frac{\partial D(x \mid x_a)}{\partial x^i} dx_a.
\]
Also we will account the symbolic rule \[^{10}\] , according to which operator must act in our case to the unit. Thus, relation
\[
\nabla_i n^i_a(x) = \int_S u^i_a(s_a) \frac{\partial D(x \ | x_a)}{\partial x^i} ds_a = 0.
\]
always takes place. Mentioned above relation has a form of conservation law and establishes the evident fact of particle’s existence on it’s own trajectory. In consequence of \((22)\) conservation laws fulfil:
\[
\nabla_i n^i(x) = 0; \quad (23) \\
\nabla_{ij} n^i(x) = 0. \quad (24)
\]
For clarification of physical meaning of these laws we’ll obtain an explicit expression \((18)\) in synchronic frame of reference, in which, according to \((14)\), coordinates \(\tilde{x}\) are given on three-dimensional spacelike \(\mathcal{V}\). In consequence of timelikeness of velocity vector \(u^i_a\):
\[
d\tau_a \neq 0,
\]
therefore we can proceed from the integration by proper time \(s_a\) in \((18)\) to integration by coordinate \(\tau_a\). Thus, after integration we will receive
\[
\Omega_a(x) = \omega_a(\tau) \frac{D[\tilde{x} \ | \tilde{x}_a(\tau)]}{u^4_a(\tau)}, \quad (25)
\]
where \(\tilde{x}_a(\tau)\) means, that coordinates of all particles are taken in moment \(\tau\) of coordinate time. Relation \((25)\) can be written also in tensor form. Let’s fill up all \(R\) by observer’s unitary timelike vector field
\[
(k, k) = 1 \quad (26)
\]
and measure all events by clocks of observers, associated with field \(R\). Space of time, measured by clocks of this observer and required for infinitely small shift of particle along it’s trajectory, is
\[
d\tau_a = \frac{d\tau}{k \cdot u_a}.
\]
between integrals \(d\tau\) and \(ds_a\). At each \(\tau = \text{const}\) we will construct the spacelike hypersurface \(\mathcal{V}\), orthogonal to the field \(k_1\). The equation of this hypersurface as is well known
\[
k_i(x) dx^i = 0.
\]
As a result instead \((25)\) we have
\[
\Omega_a(x) = \omega_a(\tau) \frac{D[\tilde{x} \ | \tilde{x}_a(\tau)]}{(k, u_a)_{\tau_a=\tau}}. \quad (28)
\]
Let’s integrate now \((23)\) all along the hypersurface \(\mathcal{V}\), accounting \((25)\), - as a result we’ll obtain
\[
\int_V \frac{\partial}{\partial x^i} \left\{ \sum \sqrt{-g(\tilde{x})} \frac{u^i_a(\tau)}{u^4_a(\tau)} D[\tilde{x} \ | \tilde{x}_a(\tau)] \right\} d^3\tilde{x} = 0.
\]
Let’s unfold in details this expression, taking the differentiation by \( \tau = x^4 \) outside integration and introducing velocity 3-vector of particle on hypersurface \( V \):
\[
v_a^\alpha(\tau)/d\tau
\]
\[
\frac{\partial}{\partial \tau} \sum_a \int_V D(\tilde{x} | \tilde{x}_a(\tau))dV + 
\]
\[
\int_V \frac{\partial}{\partial \tilde{x}_a} \left\{ \sqrt{-g} \sum_a v_a^\alpha(\tau)D(\tilde{x} | \tilde{x}_a(\tau)) \right\} d^3 \tilde{x} = 0.
\]
Let’s apply the Gauss formula to the second integral, proceeding to the integration all along closed two-dimensional surface \( \Sigma \), limiting region \( V \):
\[
\frac{\partial}{\partial \tau} \sum_a \int_V D(\tilde{x} | \tilde{x}_a(\tau))dV = 
\]
\[
- \int \int_\Sigma \sum_a v_a^\alpha(\tau)D(\tilde{x} | \tilde{x}_a(\tau))d\Sigma_a.
\]
In consequence of \( \delta \)-function definition integral in the left side of (29) is equal to the number of particles, lying in three-dimensional range \( V \) in point of time \( \tau \)
\[
\sum_a \int_V D(\tilde{x} | \tilde{x}_a(\tau))dV = N(\tau).
\]
Integral in the right side of (29) is equal to the flux of number of particles, crossing closed two-dimensional surface \( \Sigma \), which limits three-dimensional range \( V \). Thus, (29) posits, that variation of particles’ number in range \( V \) is induced only by particles’ leaving from that range or arrival to it - relations (23), (24) are differential forms of particles’ and charge’s conservation laws correspondingly.

Let’s calculate, finally, the covariant divergence of stripped particles’ energy-momentum tensor. Using relations (17), (22), we’ll obtain
\[
\nabla_k T^i_k(x) = \sum_a m_a c^2 \int \frac{\partial}{\partial s_a} \int u_k \nabla_k u_a D|x | x_a(s_a)ds_a.
\]

### 4 Field equations

Field equations for system of particles, interacting via massive scalar (\( \Phi \)), vector (\( A_i \)) and gravitational fields, can be written down using sources’ densities in following form
\[
R^{ik} - \frac{1}{2} R g^{ik} = \kappa T^{ik} = \kappa (T^{ik}_p + T^{ik}_s + T^{ik}_v),
\]
\[ F^{ik} - \mu_v^2 A^i = -4\pi \sum e_a \int u^i_a D(x | x_a) ds_a, \quad (33) \]

\[ F^{ik} = 0, \quad (34) \]

\[ \Box \Phi + \mu_s^2 \Phi = -4\pi \sum q_a \int D(x | x_a) ds_a, \quad (35) \]

where

\[ T^{ik}_p = \sum m_a c^2 \left( 1 + \frac{q_a \Phi}{m_a c^2} \right) \int u^i_a u^k_a D(x | x_a) ds_a \quad (36) \]

- tensor of particles’ energy-momentum,

\[ T^{ik}_s = \frac{1}{4\pi} \left\{ \Phi^{.i} \Phi^{.k} + \frac{1}{2} g^{ik} (\mu_s^2 \Phi^2 - \Phi^{.j} \Phi^{.j}) \right\} \quad (37) \]

- tensor of massive scalar field’s energy-momentum\(^4\)

\[ T^{ik}_v = \frac{1}{4\pi} \left\{ F^{ij} F^{k.j} + \mu_v^2 v^A_i A^k \right\} \]

- tensor of massive vector field’s momentums, \( e_a, q_a \) - vector and scalar charges of particles, \( \mu_v = m_v c / \hbar, \mu_s = m_s c / \hbar; m_v, m_s \) - vector and scalar masses of bosons.

Field equations (32) - (35) include ordinary motion equations. In order to show it, let’s apply to both sides of (32) the \( \nabla_k \) operator and account Bianchi identity law:

\[ 0 = \sum a = 1 \left\{ m_a c^2 \left( 1 + \frac{q_a \Phi}{m_a c^2} \right) \nabla_k \int u^i_a u^k_a D(x | x_a) ds_a + + q_a \Phi_k \int u^i_a u^k_a D(x | x_a) ds_a \right\} + \frac{1}{4\pi} \left\{ F^{i.j} (F^{k.j} - \mu_v^2 A^k) + + A^i A^k_{,k} \mu_v^2 \right\} + \frac{1}{4\pi} \left\{ \Phi^{.i} (\Box \Phi - \mu_s^2 \Phi) \right\}. \quad (39) \]

Let’s affect via \( \nabla_i \) operator to both sides of equation (33). In consequence of skew-symmetry of tensor \( F^{ik} \) we will obtain

\[ \mu_v^2 A^k_{,k} = 4\pi \sum e_a \nabla_k \int u^k_a D(x | x_a) ds_a, \]
and subject to (22) we’ll have the Lorentz calibration at
\[ \mu_v \equiv 0. \]

Using this fact, as well as field equations (33) and (35) and relation (31), we’ll adduce (39) to the form

\[ 0 = \sum_{a=1}^{N} \left\{ m_a c^2 \left( 1 + \frac{q_a \Phi}{m_a c^2} \right) \int u_a^k \nabla_k u_a^i D(x \mid x_a) ds_a - \right. \]
\[ - \epsilon_a F^i_{\cdot k} \int u_a^k D(x \mid x_a) ds_a - q_a \nabla_k \Phi \left[ g^{ik} \int D(x \mid x_a) ds_a - \right. \]
\[ \left. - \int u_a^i u_a^k D(x \mid x_a) \right\}. \]

Let’s choose the proper time of particle \( b \) as a coordinate \( x^4 \) and integrate the last expression all along three-dimensional spacelike hypersurface \( V_b \) (\( dV_b = \sqrt{-g} d^3 \tilde{x}_b \)). In consequence of definition of invariant \( \delta \)-function (4) we will obtain motion equations:

\[ \frac{dp_a^i}{ds_a} = \]
\[ = \frac{1}{1 + \frac{q_a \Phi}{m_a c^2}} \left\{ \frac{\epsilon_a F^i_{\cdot k} u_a^k + \frac{q_a \Phi}{c} \nabla_k \Phi (g^{ik} - u_a^i u_a^k)}{1 + \frac{q_a \Phi}{m_a c^2}} \right\}, \quad (41) \]

where \( p_a^i = m_a c u_a^i \) - particle’s momentum. It is important to notice, that fields \( \Phi \) and \( A_i \) are calculated in point of particle’s location, i.e. \( A_i = A_i(x_a), \Phi = \Phi(x_a) \); in the same point also calculated operator \( \nabla_i \), shift of field coordinates to the point \( x_a \) occurs in consequence of integration of \( \delta \)-function all along the spacelike hypersurface.

Let’s now discuss the problem with initial conditions for system (32) - (35). We’ll incorporate, as we did it in previous section, the unit timelike field of geodesic observers \( k_i(x) \), and will fix events with the help of their clocks. Let’s construct the spacelike hypersurface \( V \), orthogonal to \( k_i \). We will define on this three-dimensional surface in point of time \( \tau_0 \) the components of potentials \( \Phi(\tilde{x}, \tau_0) \equiv \Phi_0(\tilde{x}), \ A_i(\tilde{x}, \tau_0) \equiv A^0_i(\tilde{x}), \ g_{\alpha\beta}(\tilde{x}, \tau_0) \equiv g^0_{\alpha\beta}(\tilde{x}) \) and their derivatives by time:

\[ (\partial_{\tau}, \Phi)(\tau=\tau_0) \equiv \dot{\Phi}_0(\tilde{x}), \ (\partial_{\tau} A_i)(\tau=\tau_0) \equiv \dot{A}^0_i(\tilde{x}), \]
\[ (\partial_{\tau} g_{\alpha\beta})(\tau=\tau_0) \equiv \ddot{g}^0_{\alpha\beta}(\tilde{x}), \]

as well as the coordinates and momentums of all particles: \( x_a(\tau_0) \equiv \tilde{x}_a^0, \ u_a(\tau_0) \equiv \tilde{u}_a^0 \). The solution of problem with initial conditions for system (32) - (35) has a form

\[ \psi(x) = \psi(\tau, \tilde{x} | \tilde{x}_1^0, \tilde{u}_1^0, \ldots, \tilde{x}_N^0, \Phi_0, \dot{\Phi}_0 \ldots), \quad (42) \]
where \( \psi(x) \) - certain field functions, depending from values \( g_{ik}, \tilde{x}_a, \Phi \) and from the parameters. Thus, field values \((g_{ik}, A_i, \Phi) \) can be considered as functionals of form

\[
\psi(x) = \psi(\tau, \tilde{x}_1, \tilde{u}_1, \ldots, \tilde{x}_N, \tau_0).
\]

This notation should be understood in the following way: field values are calculated in point \( \tilde{x} \) in instant of time \( \tau_0 \), if coordinates and velocities of particles on hypersurface \( V \) in this instant \( \tau_0 \) were defined by values \( \tilde{x}_a, \tilde{u}_a \).

## 5 Many-time formalism

Field equation’s character (32) - (35) includes one feature, besides mentioned in the introduction, which present problems of invariant construction of statistical model in GRG. This problem consists in the fact, that motion of each particle is described by means of invariant proper time \( s_a \), while this time in field equations (32) must be associated with certain coordinate \( \tau \) in point where field is calculated. (Let’s recall, that in (43) values \( \tilde{x}_a, \tilde{u}_a \) are taken in the same point of coordinate time in geodesic frame of references). This problem, however, can be bypassed, using instead of field observed values many-time field values \( \psi_N(x \mid x_1(s_1), \ldots, x_N(s_N), u_N(s_N)) \), associated with observed \( \psi \) with the help of the rule

\[
\psi(x \mid \tilde{x}_1, \tilde{u}_1, \ldots, \tilde{x}_N, \tilde{u}_N) =
\]

\[
= \mathcal{S}^k N(s_1, \ldots, s_N) \psi_N(x \mid x_1(s_1), \ldots, u_N(s_N)),
\]

where \( \mathcal{S}_N^k \) - synchronization operator, determined by relation:

\[
\mathcal{S}_N^k(s_1, \ldots, s_N) \psi_N(x \mid x_1(s_1), \ldots, u_N(s_N)) =
\]

\[
= \prod_{a=1}^N \delta(s_a - s_k) ds_a ds_k \psi_N(x \mid x_1(s_1), \ldots, u_N(s_N)).
\]

Thus,

\[
\psi(x, \tau \mid x_1(\tilde{\tau}), \ldots, u_N(\tilde{\tau})) =
\]

\[
= \int ds_k \psi_N(x \mid x_1(s_k), \ldots, u_N(s_N)).
\]

Integration in (45) - (46) is carried out along the whole trajectory of particles and observer. The association between observer’s proper time \( s_k \) and coordinate time \( \tau \) is realized by means of relations (47)

\[
ds_k = k_i dx^i.
\]

\(^5\)Vector \( k \) just chooses time direction.
From (45) it follows, that synchronization operator commutates with any operator, acting solely on coordinates $x$ of field’s “measuring” point:

$$[k_x, S^k_N] = 0.$$  \hfill (48)

Equations for many-time fields we’ll write down in form

$$\nabla R^{ik} - \frac{1}{2} R g_{ik} = \kappa T^{ik},$$  \hfill (49)

$$\nabla F^{ik} = -\mu_v^2 A^i = -4\pi \sum e_a u^i_a(s_a) D[x | x_a(s_a)],$$  \hfill (50)

$$\nabla F^{ik} = 0,$$  \hfill (51)

$$\Box \Phi + \mu^2 \Phi = -4\pi \sum q_a D[x | x_a(s_a)],$$  \hfill (52)

where many-time tensor of fields’ energy-momentum is constructed by rules \(37\), \(38\), but relatively many-time fields,

$$\nabla T^{ik} =$$

$$\sum m_a c^2 \left( 1 + \frac{q_a \Phi}{m_a c^2} \right) u^i_a(s_a) u^k_a(s_a) D[x | x_a(s_a)].$$  \hfill (53)

In consequence of (48) and (45) the application of operator $S^k_N$ to both sides of (49) - (52) brings us to initial field equations (32) - (35). From the definition of many-time field functions is clear, that they should have $\delta$ - type character.

To make the sense of all told transparent, we’ll consider the example of solution of field equations for massless tensor field $\psi^{i_1 \ldots i_n}(x)$ in flat space-time. Corresponding many-time equations field equations have form

$$\Box \psi^{i_1 \ldots i_n} = 4\pi \sum q_a u^{i_1}_a \ldots u^{i_n}_a D(x | x_a).$$  \hfill (54)

For solution of problem with initial conditions let’s affect on both sides of (54) by Fourier operator

$$\int_{-\infty}^{+\infty} dx^4 \int_{-\infty}^{+\infty} d^3x e^{i(k,x)},$$  \hfill (55)

as a result we’ll obtain the solution

$$\nabla \psi^{i_1 \ldots i_n}(k | s_1, \ldots, s_N) =$$

$$= -\frac{4\pi}{(k, k)} \sum q_a u^{i_1}_a \ldots u^{i_n}_a e^{i(k,x)}(s_a)$$  \hfill (56)
Index “+” of symbol ∑ means, that only \( x_a^4 \geq 0 \) are selected (in consequence of \((55)\)). Performing backward Fourier transformation, after standard calculations we’ll find

\[
\psi^{i_1...i_n}(x | s_1, \ldots, s_N) =
\]

\[
= 2 \sum_+ q_a u_a^{i_1} \ldots u_a^{i_n} \delta[(R_a, R_a)],
\]

(57)

where

\[
R_a^i = x_a^i(s_a) - x^i.
\]

(58)

Let’s calculate now the observed field \( \psi^{i_1...i_n}(x) \) using relations (56):

\[
\psi^{i_1...i_n}(x | \tilde{x}_1, \ldots, \tilde{x}_N) =
\]

\[
= 2 \sum_+ q_a \int ds_k u_a^{i_1}(s_k) \ldots u_a^{i_n}(s_k) \delta[(R_a, R_a)],
\]

(59)

where now \( x_a^i = x_a^i(s_k) \). For taking of this integral it is necessary to proceed from integration by variable \( s_k \) to integration by variable \( z_a = (R_a, R_a) \). Differentiating, we’ll find the relation

\[
\frac{dz_a}{ds_a} = 2(u_a, R_a).
\]

Thus, from (59) we’ll receive

\[
\psi^{i_1...i_n}(x | \tilde{x}_1, \ldots, \tilde{x}_N) =
\]

\[
= \sum_+ q_a u_a^{i_1} \ldots u_a^{i_n} | (u_a, R_a) \mid z_a = 0.
\]

(60)

Equation \( z_a = 0 \) has two roots

\[
t_a - t = \pm R(t_a)/c,
\]

(61)

where \( R(t_a) = |\vec{r}_a(t) - \vec{r}| \). These roots correspond to retarded and advanced solutions. Since we solve the problem with initial conditions, then condition \( t > 0 \) always fulfills, therefore from (61) the “retarded root” is selected

\[
t_a + R(t_a)/c = t, \quad t_a \geq 0.
\]

(62)

The relation (61) is necessary to consider as an equation for the determination of \( t_a(t, \vec{r}) \), which after that is essential to substitute in (60). At \( n = 1, N = 1 \) (60) describes well-known potentials of uniformly moving charges, at \( n = 2 \), averaging the distribution of “charges”, we’ll obtain the solution of linearized Einstein equations with source [67].

In order to display how index \( k \) of operator \( s_N^k \) can be used, we will consider \((59)\) in case of \( n = 1, N = 1 \). Then orientating vector \( k \) along 4 - velocity of
particle $u^i_a = \delta^i_4$, $ds_k = dx^4_k(s_k)$, at once we’ll receive from (59) the Coulomb’s law: $\psi^i = q\delta^i_4/r$.

We have mentioned already, that many-time field functions have $\delta$-type character. In flat space for massless fields these functions are equal to zero everywhere, except isotropic cone, connecting the coordinates of each particle and coordinates of field’s measuring point. In case of massive field, these functions are different from zero on certain hyperbolic surfaces, lying inside the light cone, slope ratio of these cones is equal to relation $v(x, t)/c$, where $v$ - speed of field propagation. It is obvious by the example of solution of many-time equations for massive field in flat space-time. To which facts the account of gravitational field leads? Firstly, in gravitational field isotropic surfaces are not strictly conic; secondly, as it has been shown in papers of Sibgatullin, Ibrahimov and others, wave packet in gravitational field spreads, that may lead to the occurrence of statistical tails of initially monochromatic wave. Strictly speaking, while gravitational field presents, only the high-frequency component of field moves along isotropic hypersurfaces. As a result of this, low-frequency components will arrive to the point of observation later, summing there with high-frequency components, radiated in later instants of time. This will lead to the occurrence of the continuous spectrum in many-time functions, i.e. to the spreading of $\delta$-function. The account of gravitational interaction should lead to the same effect. Actually, in light cones’ region of overlap appears gravitational interaction, implication of which will be the appearance in region of overlap of effective statistical part, which itself will be the source.

6 Relativistic Hamilton Dynamics Of Particle

Most naturally statistic dynamics is formulated in terms of canonical formalism. Relativistic motion equations of massive particle have form

$$\frac{dx^i}{ds} = \frac{\partial H}{\partial P^i}, \quad \frac{dP^i}{ds} = -\frac{\partial H}{\partial x^i},$$

where $P_i$ - generalized particle’s momentum, association of which with ordinary momentum is

$$P^i = mc\frac{dx^i}{ds} \equiv mcu^i,$$

is formed by equations (63), $H(x, P)$ - invariant Hamilton function. Let $\psi[s(x), P(s)] \equiv \psi(s)$ - is certain dynamic function, i.e. function of dynamic variables. Let’s calculate this function’s derivative along particle’s trajectory

$$\frac{d\psi}{ds} = \frac{\partial \psi}{\partial x^i} \frac{dx^i}{ds} + \frac{\partial \psi}{\partial P_i} \frac{dP_i}{ds}.$$

Thus, subject to (63) we will obtain

$$\frac{d\psi}{ds} = \{H, \psi\},$$

\[ \text{6 About relativistic canonical formalism see [14], [58], [65], [72], [73].} \]
where relativistic Poisson brackets are incorporated
\[
\{ H, \psi \} = \frac{\partial H}{\partial P_i} \frac{\partial \psi}{\partial x^i} - \frac{\partial H}{\partial x_i} \frac{\partial \psi}{\partial P_i} .
\] (66)

Poisson brackets possess following algebraic properties:
\[
\{ A, B \} = - \{ B, A \} ;
\] (67)
\[
\{ \alpha A + \beta B, C \} = \alpha \{ A, C \} + \beta \{ B, C \} ;
\] (68)
\[
\{ AB, C \} = A \{ B, C \} + \{ A, C \} B ,
\] (69)

where \( \alpha, \beta \) - numbers. Properties (67, 68) determine Lie algebra. We can introduce an operator \([A] \), acting on the random function in such way, that
\[
[A] B \equiv \{ A, B \} .
\] (69)

By means of this operator we can built the formal solution of the equation (65)
\[
\psi(s) = e^{s[H]} \psi(s_0) \equiv \hat{U}(s) \psi(s_0).
\] (70)

Since last relations formally do not differ from classical ones [74], we may assert, that transformations \( \hat{U}(s) \) form Lie group — the group of canonical transformations, relatively to which equations (63) are invariant. One more important property of Poisson bracket is its linearity as a differential operator, in consequence of that
\[
\{ A, \psi(B) \} = \frac{d\psi}{dB} \{ A, B \} .
\] (71)

In consequence of (67) and (71) \([H, \psi(H)] = 0\), therefore relativistic Hamilton function serves as an integral of motion equations (63). We will let this integral equal to particle’s rest momentum
\[
H(x, P) = mc.
\] (72)

Hamilton function of charged massive particle, lying in gravitational, vector and scalar fields, can be represented in form [14]
\[
H(x, P) = \sqrt{\left( P, P \right)} - \frac{q\Phi}{c}.
\] (73)

Using (63), (64) and (72) we’ll find the association between \( P_j \) and \( p^i \):
\[
p^i = (P_j - \frac{e}{c} A_j) \frac{g^{ij}}{1 + \frac{q\Phi}{mc^2}} \equiv \frac{P^i - \frac{e}{c} A^i}{1 + \frac{q\Phi}{mc^2}}.
\] (74)

Then in terms of variables \( x^i \), \( p^i \) Hamilton function (73) is equal to
\[
H(x, p) = \left\{ 1 + \frac{q\Phi}{mc^2} \right\} \sqrt{(p, p)} - \frac{q\Phi}{c} .
\]
from which subject to (72) we’ll obtain the normalization relation
\[(p, p) = m^2 c^2\] (75)
or \((u, u) = 1\). Using (75) motion equation (63) with Hamilton function (73) during conversion to non-canonical variables \(x^i, p^i\) transforms to the form (41).

7 Many-time Hamilton Formalism

As we have mentioned in section 4, field functions in motion equations (41) are taken in whereabouts of a particle. In future we will denote the coordinates of a-particle by means of \(x^a\), and the totality of its dynamic variables \(\{x_a, P_a\}\) by means of \(\xi_a\). Thus, according to the meaning of equations (32) - (35), as well as (41) in motion equations of a particle are included field values (see (43)):

\[
\psi(x_a | \tilde{\xi}_1, \ldots, \tilde{\xi}_N),
\]
where \(\xi_a = \xi_a(s_a)\). In accordance to meaning of motion equations (41) field values \(\psi(x_a)\) include also the contribution from proper fields, i.e.

\[
\psi(x_a | \tilde{\xi}_1, \ldots, \tilde{\xi}_a, \ldots, \tilde{\xi}_N).
\]
The account of proper fields, as is well known, leads to unremovable divergences, therefore we won’t cast out proper fields in motion equations of each particle, supposing that particle moves as a sampling one in summary field of others.

Then many-time Hamilton function of system of \(N\) particles can be represented in form

\[
H[\xi_1(s_1), \ldots, \xi_N(s_N)] = \sum H_a[\xi_a(s_a) | \tilde{\xi}_1, \ldots, \tilde{\xi}_N],
\]
where, for instance, in case of scalar interaction’s missing

\[
H_a[\xi_a(s_a) | \tilde{\xi}_1, \ldots, \tilde{\xi}_N] = \left\{ g_{jk}(x_a | \tilde{\xi}_1, \ldots, \tilde{\xi}_N) \left[ P^a_j - \frac{e_a}{c} A_j(x_a | \tilde{\xi}_1, \ldots, \tilde{\xi}_N) \right] \times \left[ P^a_k - \frac{e_a}{c} A_k(x_a | \tilde{\xi}_1, \ldots, \tilde{\xi}_N) \right] \right\}^{\frac{1}{2}}.
\]
Since Hamilton function is many-time, and these times aren’t connected by any relation, motion equations of particles’ ensemble can be written down in form

\[
\frac{dx^i_a}{ds_a} = \frac{\partial H}{\partial P^i_a} \left( \equiv \frac{\partial H_a}{\partial P^i_a} \right) \quad (a = 1, \ldots, N),
\]
\[
\frac{dP^a_i}{ds_a} = -\frac{\partial H}{\partial x^i_a} \left( \equiv -\frac{\partial H_a}{\partial x^i_a} \right); \quad (78)
\]

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Field functions’ differentiation by variables, placed on the right from vertical line, is not carried out, since it does not depend explicitly from the time \( s_a \). Exactly such motion equations during conversion to variables \( x^i_a, p^i_a \) give equations, coinciding with (41).

Let’s introduce many-time dynamic function

\[
\psi_N[\xi_1(s_1), \ldots, \xi_N(s_N)] \equiv \psi_N(s_1, \ldots, s_N)
\]  

(79)

and timelike field of geodesic observers \( k_i(x) \), by clocks of which \( \tau \) we will carry out the synchronization of this function

\[
\psi_N(\tau) = S_N(\tau)\psi_N(s_1, \ldots, s_N),
\]  

(80)

where operator of synchronization by particles’ ensemble is determined via relation

\[
S_N(\tau)\psi_N(s_1, \ldots, s_N) = 
\int \prod_{a=1}^N \delta[s_a - s^*_a(\tau)]\psi_N(s_1, \ldots, s_N) ds_a,
\]  

(81)

and \( s^*_a(\tau) \) is resulted via solution of motion equations (78) relatively to canonical parameter \( dx^i_a/ds_a^* = p^i_a(\tau) \). In synchronic frame of reference \( \delta \)-functions in (81) are transformed to the form

\[
\delta(x^4_a - \tau)\frac{d\tau}{ds^*_a} \equiv \delta(x^4_a - \tau)p^4_a(\tau)/m_a c.
\]

Introduced operator differs from operator \( S^k_N \) in consequence of the fact, that ensemble of particles is determined by \( N \) times, whereas fields of this ensemble - \( N + 1 \). The connection of these operators is following:

\[
S^k_N = \int S_N(\tau)d\tau.
\]  

(82)

Let’s calculate the derivative from \( \psi_N(\tau) \). Taking into account the rule of \( \delta \)-functions’ differentiation, as well as equations (78), we’ll find

\[
\frac{d\psi_N}{d\tau} = \int \prod_{a=1}^N \delta[s_a - s^*_a(\tau)]\psi_N(s_1, \ldots, s_N) \sum_{b=1}^N \frac{ds^*_b}{d\tau} \{H_b, \psi\}_b.
\]  

(83)

Let’s introduce for convenience the many-time Poisson bracket

\[
\{H, \psi\}_N = \sum_{a=1}^N \frac{ds_a}{d\tau_a} \left( \frac{\partial H}{\partial P^i_a} \frac{\partial \psi}{\partial x^i_a} - \frac{\partial H}{\partial x^i_a} \frac{\partial \psi}{\partial P^i_a} \right)
\]  

(84)

And rewrite (83) in form

\[
\frac{d\psi_N}{ds} = S_N(\tau) \{H, \psi\}_N.
\]  

(85)

The expression (85) should be understood by following way: first we calculate the many-time Poisson bracket, next employ the operator of ensemble synchronization to it.
8 The Relativistic Phase Space

At non-degenerate transformations of coordinates $x^i_a$ in Riemannian space

$$x'^i = \varphi^i(x_1, x_2, x_3, x_4) \equiv \varphi^i(x)|_{x=x_a},$$

$$J_a = \det \left| \frac{\partial \varphi^i}{\partial x^k} \right|_{x=x_a} \neq 0$$

components of generalized momentum $P_i$ (as well as $p_i$) are transformed like covariant vector components

$$P^a_i = P^a_k \frac{\partial x^k_a}{\partial x^i_a}.$$

Overhauling various values of particle’s coordinates and in every point—various values of momentum, we’ll arrive to the conception of phase space of single particle. This space is fiber, representing skew product upon base $X_a$ with fiber $P_a(x_a)$. Base $X_a$ is 4-dimensional Riemannian space, provided with metric $g_{ik}(x_a)$ and various fields: $\Phi(x_a)$, $A_i(x_a)$ etc. We will term the base the configuration space of particle. Fiber $P_a(x_a)$ is the tangent space to $X_A$ with point of contact $x_a$ and represents the bundle of all kinds of momentums, applied to the points $x_a$. We will term the fiber $P_a(x_a)$ a momentum space of particle. The topology of momentum space is the topology of infinite 4-dimensional parallelepiped, coordinates $P_i$ (or $p_i$) of which possess various values in open interval:

$$-\infty < P_i < +\infty.$$

It is evident, that (89) is invariant with respect to transformations (86) - (88). It is necessary to note, that singling out of conditions only with positive energy on mass surface (72) is not always possible (see concerning to it [73]).

Invariant with respect to transformation of coordinates (86) - (88) differentials of volumes of configuration and momentum spaces are

$$dX_a = \sqrt{-g_a} \, dx^1_a \, dx^2_a \, dx^3_a \, dx^4_a,$$

$$dP_a = \frac{1}{\sqrt{-g_a}} dP^a_1 \, dP^a_2 \, dP^a_3 \, dP^a_4.$$

In accordance to phase space’s definition an invariant differential of its volume is equal to product

$$d\Gamma_a = dx^1_a \, dx^2_a \, dx^3_a \, dx^4_a \, dP^a_1 \, dP^a_2 \, dP^a_3 \, dP^a_4.$$

This definition remains also at conversion to momentums $p^a_i$. If contravariant components $P^a_i$ (or $p^a_i$) are taken for coordinates of momentum space, then

$$dP_a = -\sqrt{-g_a} dP^a_1 \, dP^a_2 \, dP^a_3 \, dP^a_4.$$
Many-time phase space of whole particles’ ensemble is the direct product of single particles’ phase spaces

\[ \Gamma = \Gamma_1 \otimes \Gamma_2 \otimes \ldots \otimes \Gamma_N. \]  
(93)

According to this definition, the invariant differential of volume is equal to

\[ d\Gamma = \prod_{a=1}^{n} d\Gamma_a. \]  
(94)

Thus, phase space of ensemble has the dimensionality \( 8N \); ensemble is reflected by point in this space.

9 Differential And Integral Operations In Phase Space

Since phase space of ensemble is the direct product of single particles’ phase spaces, it is enough to determine these operations in phase space of one particle. By the same reason we will omit particle’s index in this section. Let \( f(x, p) \equiv f(\xi) \) - certain function of particle’s phase coordinates. Let’s consider an integral of type

\[ \int f(\xi) d\Gamma. \]  
(95)

Since invariant differential of phase space’s volume is equal to product of configuration \( (dX) \) and momentum \( (dP) \) spaces’ differentials of volumes, \( (95) \) is understood as a skew integration all along these spaces. It is necessary to emphasize especially that since \( P(x) \) is a fiber, operation

\[ \int dP \int f(x, P) \]

is not determined and is insensible. Therefore \( (95) \) is necessary to understood by following way

\[ \int f(\xi) d\Gamma = \int dX \int_{P(x)} dP f(x, P), \]  
(96)

where it is forbidden to change the order of integration.

Let \( f(\xi) \) is tensor

\[ f^{i_1 \ldots i_n}(x', p') = f^{i_1 \ldots i_n}(x, p) \frac{\partial x^{i_1}}{\partial x^{i_1}} \ldots \frac{\partial x^{i_n}}{\partial x^{i_n}} \]  
(97)
with respect to transformations (86)-(88). This tensor can have only structures of type
\[ p^{i_1\ldots i_n} f(x, p), \quad a_i^{i_1\ldots i_n}(x) f(x, p), \]
or structure of mixed type, where \( f(x, p) \) - scalar function in phase space, \( a_i^{i_1\ldots i_n}(x) \) - tensor in configuration. As it is easy to see, arbitrary scalar with respect to transformations (86)-(88) function \( f(\xi) \) can be only the function of variables \( a_i(\xi) \):
\[ f(\xi) = f(a_0, a_1, \ldots, a_s), \quad (98) \]
where \( a_s(\xi) \) - \( S \)-linear forms:
\[ a_s(\xi) = a_{i_1\ldots i_s}(x)p^{i_1\ldots i_s}, \quad (99) \]
and \( a_{i_1\ldots i_s}(x) \) — \( s \)-valent completely symmetrical tensors in configurational space; in particular \( a_0(\xi) = a_0(x) \) - scalar in \( X \).

Let’s introduce in consideration tensor field in configurational space
\[ f^{i_1\ldots i_n}(x) = \int_{\mathcal{P}(x)} p^{i_1\ldots i_n} f(x, p) dp, \quad (100) \]
which we will term the \( n \) moment about scalar \( f(x, p) \). Integration in (100) is carried out throughout infinite 4-dimensional parallelepiped.

Let’s consider the derivative from scalar \( f(x, p) \) by configurational space
\[ \frac{\partial f(x, p)}{\partial x^i} = \nabla_i f(x, p). \quad (101) \]
Let’s keep in mind, that \( f(x, p) \) has a structure (98), then
\[ \frac{\partial f(x, p)}{\partial x^i} = \sum_s \frac{\partial f}{\partial a_s} \nabla_i a_s(x, p). \quad (102) \]
Carrying out the covariant differentiation in (102) with account of \( \partial p_i/\partial x^k = 0 \), we’ll find
\[ \nabla_i a_s(x, p) = a_{i_1\ldots i_s} p^{i_1\ldots i_s} + a_{i_1\ldots i_s} p^{i_1\ldots i_s+1} \Gamma^{i_1}_{i_{i+1}} r^{i_{i+1} p^{i_2\ldots i_s}}. \quad (103) \]
Thus, values \( \nabla_i a_s(x, p) \), and (101) along with them, are not covector’s components in phase space. This means, that it is necessary to redefine the operation of covariant differentiation in phase space. From (103) it is obvious, that instead \( \nabla_i \) in phase space operator of covariant differentiation is [8]:
\[ \tilde{\nabla}_i = \nabla_i - \Gamma^j_{ik} p^k \frac{\partial}{\partial p^j}. \quad (104) \]
This operator is determined by such way, that
\[ \tilde{\nabla}_i p^k = 0. \quad (105) \]
We will term an operator \( \tilde{\nabla}_i \) the operator of covariant differentiation by Cartan \([74]\), or simply - Cartan derivative. Then from (103) we’ll find at once
\[
\tilde{\nabla}_i a_s(x,p) = p_{i1} \ldots p_{is} a_{i_1 \ldots i_s},
\]
i.e. according to (102) we’ll obtain the following symbolic rule of differentiation of functions \( [68] \):
\[
\tilde{\nabla}_i f(x,p) = \nabla_i [f(x),p],
\]
which means, that for calculation of Cartan derivative from function \( f(x,p) \) is enough to calculate the ordinary covariant derivative from it, temporarily supposing vector of momentum at that is covariant constant.

Using the operator of covariant differentiation by Cartan it is possible to
to attach to the canonical motion equations (63) more convenient form \([64]\):
\[
\frac{ds^i}{ds} = \frac{\partial H}{\partial P_i}, \quad \frac{dP_i}{ds} = -\tilde{\nabla}_i H.
\]
In that case Poisson brackets \([69]\) will be written down in form
\[
\frac{d\psi}{ds} = \{H,\psi\} = \tilde{\nabla}_i \psi \frac{\partial H}{\partial P_i} - \frac{\partial \psi}{\partial P_i} \tilde{\nabla}_i H.
\]
In particular, at conversion to ordinary momentums \( p^i \) and using of Hamiltonian’s explicit form \([73]\) we will obtain \([64], [38] (\Phi = 0)\):
\[
\frac{d\psi}{ds} = \left\{ p^i \tilde{\nabla}_i + \frac{c}{\xi} F_{ik} p^k \frac{\partial}{\partial p^i} \right\} \psi.
\]
In conclusion of this section we will consider an expression of type
\[
\int_{P(x)} \tilde{\nabla}_i \{ f(x,p)p^{i_1} \ldots p^{i_n} \} \, dP.
\]
Using rule \([104]\) and definition \([100]\), we will obtain at once an important relation \([68]\)
\[
\nabla_i f^{i_1 \ldots i_n}(x) = \int_{P'(x)} \tilde{\nabla}_i \{ f(x,p)p^{i_1} \ldots p^{i_n} \} \, dP.
\]

10 Many-time And Synchronized Distribution Functions

Let’s introduce an invariant many-time distribution function of \( N \) particles’
ensemble, \( D_{N_s}(s_1, \ldots, s_N) \), which we will define in such way:
\[
dW_{N_s} = \prod_{a=1}^{N} \delta(s_a - s'_a) D_{N_s}[\xi_1(s'_1), \ldots, \xi_N(s'_N)] d\Gamma =
\]
Why the multi-time probability is introduced by singular way? The fact is that at any specified array of time $s_1, \ldots, s_N$ the probability to find the ensemble on all $N$ 3-dimensional spacelike hypersurfaces at whole momentums’ array must be equal to one in consequence of that particles do not disappear anywhere and do not appear from anywhere. Integration of this one by $N$ times in infinite limits gives $\infty^N$ degrees and represents itself an absurd operation. Probability density by its own meaning is determined on 3-dimensional volume, but not on 4-dimensional one. In accordance to (112) normalization relation must fulfil

$$\int_{\Gamma} dW_N = \prod_{a=1}^{N} \delta(s_a - s_a^*) D_N_s[\xi_1(s_1^*), \ldots, \xi_N(s_N^*)] d\Gamma = 1,$$

where integration is carried out by whole phase space. Function $D_N_s$ is symmetrical by permutations of identical particles “a” “b”:

$$D_N_s(\xi_1, \ldots, \xi_a, \ldots, \xi_b, \ldots, \xi_N) = D_N_s(\xi_1, \ldots, \xi_b, \ldots, \xi_a, \ldots, \xi_N).$$

In consequence of function’s $D_N_s$ many-time character time coordinates of particles are not connected with anything, since probability (112) is calculated in different, non-correlated by any means times $s_1, \ldots, s_N$. It is necessary to synchronize the distribution function so that it has got its ordinary physical meaning. For synchronization we’ll introduce in $V_4$, as we did it earlier, a unitary timelike field of geodesic observers $k_i(x)$, supposing the whole this field sample, not influencing on the ensemble’s dynamics. Then at conversion to phase space’s formalism field of observers will reflect in each particle’s phase subspace, like any other field on $V_4$: $g_{ik}(x), A_i(x), \Phi(x)$ etc; $k_i(x) \rightarrow k_i(x_a)$. For synchronization it is necessary to set the time coordinate of particle $x^4_a$ in synchronic frame of reference equal to the proper (synchronic) time $\tau$ of observer, situated in the same space point $x^a_a$. Then probability, synchronized by field of observers $k_i(x)$ can be determined using relation

$$dW_s = S_N(\tau) dW_{N_s} = \prod_{a=1}^{N} \delta(s_a - s_a^*(\tau)) D_{N_s}(s_1, \ldots, s_N) d\Gamma.$$

Let’s introduce also $s$- particle synchronized distribution functions:

$$F_s(\xi_1, \ldots, \xi_s) \prod_{a=1}^{s} d\Gamma_s = S_{N-s}(\tau) dW_{N_s} =$$
\[ \prod_{a=1}^{s} \delta(s_a - s'_a) \int \prod_{b=s+1}^{N} \delta(s_b - s'_b) \times \]
\[ \times D_{N_s} [\xi_1(s'_1), \ldots, \xi_s(s'_s), \xi_{s+1}(s'_{s+1}) \ldots] d\Gamma. \tag{116} \]

These functions are many-time by first \( s \) particles. Integral \( (116) \), multiplied on corresponding differential of phase volume, is equal to probability of first \( s \) particles finding in range \( d\Gamma_1 \times \ldots \times d\Gamma_s \) with center in point \( \{\xi_1, \ldots, \xi_s\} \) regardless from the rest \( N-s \) particles’ position in phase space in point of time \( \tau \). If ensemble consists not from identical particles, it is necessary to consider besides functions \( (117) \) other arrays.

It is possible in integral \( (116) \) to carry out an integration by time variables. For that we’ll built three-dimensional spacelike hypersurfaces \( V_a \), orthogonal to field \( k_i(x_a) \) and represent differential of volume of configurational space \( dX_a \) in form
\[ dX_a = dV^a d\tau_a. \]

Accounting between \( \tau_a \) and \( s_a \) \( (27) \) we’ll write down:
\[ d\tau_a = (k, p)_a^{m_a s} ds_a. \tag{117} \]
Then
\[ \delta[s_a - s'_a(\tau)] dX_a = \delta[s_a - s'_a(\tau)] dV_a^{(k, p)_a^{m_a c}} \equiv \]
\[ \equiv \delta(\tau_a - \tau) (k, p)_a^{m_a s} dV^a d\tau_a, \tag{118} \]
i.e.
\[ \delta[s_a - s'_a(\tau)] ds_a = \delta(\tau_a - \tau) d\tau_a, \tag{119} \]
and after integration in \( (116) \) we’ll obtain
\[ F_s(\xi_1, \ldots, \xi_s) = \prod_{a=1}^{s} \delta(s_a - s'_a) \int \prod_{b=s+1}^{N} \delta(s_b - s'_b) \times \]
\[ \times D_{N_s} [\xi'_1, \ldots, \xi'_s, \xi_{s+1}, \ldots, \xi_N], \tag{120} \]
where
\[ d\Sigma^a_i = dV^a(k_1(x_a)). \tag{121} \]

Most simple meaning is possessed by 1-s and 2-s particle distribution functions, - \( F_1(\xi_a) d\Gamma_a \) - is probability to find “a” particle in differential of volume of phase space. In consequence of that
\[ \frac{1}{m_a c} \int_{V^a} d\Sigma^a_i \int_{P_a(x_a)} F_1(\xi_a) p^i dP = 1. \tag{122} \]
Let’s note also the fact, that normalization relation \((113)\) for synchronized distribution function has a form
\[
\int \prod_{a=1}^{N} (m_a c)^{-1} d\Sigma_a p_a^i dP_a \mathcal{D}_{N_s}(\xi_1, \ldots, \xi_N) = 1.
\] (123)

For multiplicative function
\[
\mathcal{D}_{N_s}(\xi_1, \ldots, \xi_N) = \prod_{a=1}^{N} F_1(\xi_a)
\] (124)
\((123)\) comes to \((122)\).

11 Ensemble-Averages From Dynamic Functions

Let \(\psi[x | \xi_1(s_1), \ldots, \xi_N(s_N)]\) - is many-time field function of coordinates of observation point \(x\) and dynamic variables \(\xi_1(s_1), \ldots, \xi_N(s_N)\). According to meaning of probability \((112)\), expected value of this function in condition \(\{s_1, \ldots, s_N\}\) is equal to
\[
\langle \langle \psi(x | s_1, \ldots, s_N) \rangle \rangle =
\int_{\Gamma} \psi(x | s_1, \ldots, s_N) dW_{N_s}(s_1, \ldots, s_N).
\] (125)

This record should be understood in following way: in the left side of \((125)\) is written down the macroscopic value of many-time field function \(\psi\), measured, when proper time of first particle is \(s_1\), second - \(s_2\), -etc. Such values we’ll call many-time ensemble-averages, and operation \(\langle \langle \ldots \rangle \rangle\) - the statistical average. According to the definition, operator of statistical average commutes with any operator \(K_x\), forcing only on field coordinates \(x\):
\[
[K_x, \langle \langle \ldots \rangle \rangle] = 0.
\] (126)

Forcing now on both sides of \((125)\) operator of synchronization by field \(k_i\), we’ll obtain ensemble - averages from observed field values
\[
\langle \langle \psi(x | \tilde{\xi}_1, \ldots, \tilde{\xi}_N) \rangle \rangle_N = S^k_N \langle \langle \psi(x | s_1, \ldots, s_N) \rangle \rangle_N =
\int_{\Gamma} S^k_N \psi[x | \xi_1(s_1), \ldots, \xi_N(s_N)] dW_{N_s}[\xi_1(s_1), \ldots, \xi_N(s_N)]
\]
\[
= \int_{\Gamma} \psi(x | \tilde{\xi}_1, \ldots, \tilde{\xi}_N) \times dW_{N_s}[\xi_1(\tau), \ldots, \xi_N(\tau)].
\] (127)
Hence we see, that operator of statistical average commutates with operator of synchronization $S_N^k$

$$[S_N^k, \ll \ldots \gg_N] = 0.$$  

(128)

Subject to results of previous section  can be written down in form

$$\ll \psi(x | \tilde{\xi}_1, \ldots, \tilde{\xi}_N) \gg_N =$$

$$\int \prod_{a=1}^N \frac{d\Sigma_a p^a dP_a}{m_a c} D_N(\tilde{\xi}_1, \ldots, \tilde{\xi}_N) \psi(x | \tilde{\xi}_1, \ldots, \tilde{\xi}_N).$$

(129)

In particular, from  and  follows, that ensemble-average from value, independent from dynamic variables, is equal to this value itself:

$$\ll \psi(x) \gg_N = \psi(x).$$

(130)

## 12 Liouville Equation

Now we have in our hands all essential apparatus to prosecute the main problem - an establishment of the equation for the distribution function. We’ll limit ourself with consideration of only elastic processes of interaction, at which particles do not disappear and do not born. Let all synchronized observers completely filling in point of time $s_k = \tau = \tau_0$ activate their clocks. The task of each observer is fixation of passage time through its point upon particles (if it will appear hear), and also the establishment of particle’s sort. These data should be recorded by them on cards and after time $\Delta \tau$, they should finish observations, assemble together and discuss the information about ensemble’s behavior. In language of phase space observers will reconstruct phase trajectory of ensemble $\xi_1(\tau), \ldots, \xi_N(\tau)$. It is obvious, that if observers had an opportunity to carry out exact measurements, then they would ascertain by theirs card-file, that in each point of time $s_k = \tau$ in whole space there were only $N$ particles and exactly of those sorts, that were in the beginning of the experiment. Moreover, connecting coordinates of particle’s location points in each point of time, they will make sure in fact, that particle had moved along continuous line, and had never disappeared from it. In other words, in each point of time was realized not only global conservation law of particles, but also the local one. In language of probability $dW_\tau$, this means, that along the ensemble’s phase trajectory it is conserved the synchronized probability:

$$\frac{\delta}{\delta \tau} \{ dW_N(\tilde{\xi}_1, \ldots, \tilde{\xi}_N) \} = 0$$

(131)

and the local law of particles’ number conservation and

$$\frac{d}{d\tau} \int dW_N(\tilde{\xi}_1, \ldots, \tilde{\xi}_N) = 0$$

(132)
- the global law of particles' number conservation. It is obvious, that (132) is the consequence of (131), and (131) is the consequence of simple fact of existing of all particles on their own phase trajectory.

Let’s engage now in mathematics. Modifying \( dW_\tau \) along trajectory, we’ll obtain in consequence of (115)

\[
\delta dW_\tau = \frac{\delta}{\delta \tau} \left\{ \prod_{a=1}^{N} \delta [s_a - s^*_a(\tau)] D_{N_\tau}(s_1, \ldots, s_N) d\Gamma \right\} \delta \tau.
\]

Taking into account, that from \( \tau \) depend only arguments of \( \delta \)-functions, and remembering simple rule of \( \delta \)-functions’ differentiation (16), we’ll obtain

\[
\delta dW_\tau = \prod_{a=1}^{N} \delta [s_a - s^*_a(\tau)] \delta \tau \sum_{a=1}^{N} \frac{ds^*_a}{d\tau} \frac{\delta}{\delta s_a} \times
\]

\[
\times \{D_{N_\tau},[\xi_1(s_1), \ldots, \xi_N(s_N)] d\Gamma =
\]

\[
= \prod_{a=1}^{N} \delta [s_a - s^*_a(\tau)] \delta \tau \left\{ \sum_{a=1}^{N} \frac{ds^*_a}{d\tau} \frac{\delta}{\delta s_a} \times
\]

\[
\times D_{N_\tau},[\xi_1(s_1), \ldots, \xi_N(s_N)] d\Gamma +
\]

\[
+D_{N_\tau},[\xi_1(s_1), \ldots, \xi_N(s_N)] \sum_{a=1}^{N} \frac{ds^*_a}{d\tau} \frac{\delta d\Gamma}{\delta s_a} \right\} \delta \tau =
\]

\((133)\)

In consequence of (78) we have for \( \delta D_{N_\tau}/\delta s_a \)

\[
\frac{\delta D_{N_\tau}}{\delta s_a} = \{H_a, D_{N_\tau}\}_a.
\]

(134)

Let’s calculate now the variation \( \delta d\Gamma/\delta s_a \). It is obvious, by virtue of many-time character \( d\Gamma \), that

\[
\frac{\delta d\Gamma}{\delta s_a} = \frac{\delta d\Gamma}{\delta s_b} \prod_{b=1}^{N} \Gamma_b,
\]

(135)

where \(^'\) means, that product is taken by all particles, except \( a \). Variation \( \delta d\Gamma_a \) can be calculated in the following way: let in point of time \( s_a \) dynamic variables are equal to \( x^i_a, P^i_a \). Carrying out the infinite small shift along the trajectory, we’ll obtain these magnitudes’ values in infinitely near point of time \( s'_a = s_a + \delta s_a \):

\[
x^i_a' = x^i_a + \frac{dx^i_a}{ds_a} \delta s_a + O^2(\delta s_a),
\]

\[
P^i_a' = P^i_a + \frac{dP^i_a}{ds_a} + O^2(\delta s_a).
\]
Using equation (78) here, we’ll obtain

\[ x'_a = x^i_a + \frac{\partial H_a}{\partial x^i_a} \delta s_a + O^2(\delta s_a), \quad (136) \]

\[ P'_a = P^a_i - \frac{\partial H_a}{\partial x^i_a} \delta s_a + O^2(\delta s_a). \]

Relations (136) can be considered as the transformation of coordinates in differential of volume \( d\Gamma_a \), i.e.

\[ d\Gamma'_a(s'_a) = J \left( \frac{\partial \xi'_a}{\partial \xi_a} \right) d\Gamma_a(s_a). \quad (137) \]

Carrying out the differentiation of (136), we’ll obtain elements of determinant

\[ \frac{\partial x^i_a}{\partial x^k_a} = \delta^i_k + \frac{\delta^2 H_a}{\partial x^k_a \partial x^i_a} \delta s_a + O(\delta s_a), \]

\[ \frac{\partial P^i_a}{\partial P^k_a} = \delta^i_k - \frac{\delta^2 H_a}{\partial P^k_a \partial x^i_a} \delta s_a + O(\delta s_a). \quad (138) \]

As to elements of determinant, which belong to the right upper and left lower blocks, of type \( \partial x^i_a / \partial P^k_a \), they aren’t interesting for us, since they make contribution in Jacobian of superior infinitesimal order, than \( o^1(\delta s_a) \). Let’s calculate the Jacobian with the help of (138), we’ll obtain

\[ J \left( \frac{\partial \xi'_a}{\partial \xi_a} \right) = 1 + O(\delta s_a), \quad (139) \]

in consequence of that we’ll obtain from (19)

\[ \frac{\delta d\Gamma}{\delta s_a} = 0, \quad (140) \]

i.e. ensemble’s phase space is conserved during motion along trajectories of each particle. In classics relations (140) are named Liouville theorem. Accounting now (131), (133), (134) and (140), we’ll obtain the common-relativistic Liouville equation

\[ \prod_{a=1}^{N} \lambda[s_a - s_a^*(\tau)] \sum_{b=1}^{N} \{H_b, D_{N_a}\}_b \frac{ds_b}{d\tau_b} = 0. \quad (141) \]

Let’s note, that Liouville equation allows us to calculate only synchronized averages by ensemble. However, we can go farther, following ours many-time ideology, and extend equation (141) outside the limits of phase trajectory.

\[ \sum_{b=1}^{N} \frac{ds_b}{d\tau_b} \{H_b, D_{N_a}\}_b = 0. \quad (142) \]
Such equation as a matter of fact is postulated in papers \[64\], \[65\]. We’ll note, that this equation’s many-time character will not be reflected on values, observed averages by ensemble, in consequence of synchronization operation’s application.

Let’s note, that in consequence of motion equations \[78\] along the trajectory of each of particles it’s own Hamilton function is conserved. Therefore on each real trajectory is fulfilled the normalization relation

\[ H_a(x, P) = m_a c, \quad (143) \]

which describes pairs of non-intersecting hypersurfaces (pseudospheres) in particles’ momentum spaces. Let \( \overline{\mathcal{D}}_{N_s}(\xi_1, \ldots, \xi_N) \)- solution of Liouville many-time equation \[142\], i.e.

\[ \{ H, \overline{\mathcal{D}}_{N_s} \} = 0. \]

In that case in consequence of Poisson bracket’s linearity the solution of Liouville equation will be also

\[ \mathcal{D}_{N_s}(\xi_1, \ldots, \xi_N) = \]

\[ = \overline{\mathcal{D}}_{N_s}(\xi_1, \ldots, \xi_N) \psi[H(\xi_1, \ldots, \xi_N)], \quad (144) \]

where \( \psi \) - random function. Any real particle with fixed rest mass can move only all along hypersurface \[143\] of momentums’ space, therefore ensemble’s distribution function should have form

\[ \mathcal{D}_{N_s}(\xi_1, \ldots, \xi_N) = \]

\[ = \overline{\mathcal{D}}_{N_s}(\xi_1, \ldots, \xi_N) \prod_{a=1}^{N} \delta(H_a - m_a c), \quad (145) \]

where \( \overline{\mathcal{D}}_{N_s}(\xi_1, \ldots, \xi_N) \)- non-singular on pseudospheres \[143\] solution of Liouville equation \[142\]. It is necessary to emphasize, that \( \delta(H_a - m_a c) \) commutates with Poisson bracket:

\[ [\delta(H_a - m_a c), \{ H_a, \mathcal{D}_{N_s} \}_a] = 0. \quad (146) \]

At integration of distribution function there occur integrals of type

\[ \int dP_a \delta(H_a - m_a c) F_a(x_a, P_a), \]

for calculation of which we’ll use the property of \( \delta \)- function \[13\]. An equation \[143\] has two roots, responding to conditions with positive and negative energies

\[ p_4 = \mathcal{E}_\pm(x, p)/c \]

\[ (in \ synchronic \ frame \ of \ reference \ roots \ differ \ only \ by \ sign). \]

Then

\[ \delta(H_a - m_a c) = \frac{m_a c}{|p_4^+|} \delta(p_4^+ - p_4) + \frac{m_a c}{|p_4^-|} \delta(p_4^+ - p_4^-), \]

\[ 30 \]
and we have
\[ \int dP_a \delta(H_a - m_a c) F_a(x_a, P_a) = \]
\[ = m_a c \sum_{\pm} \int dP_a^\mp \overline{F}_a^\pm(x_a, P_a), \quad (148) \]
\[ dP^\pm = \sqrt{-g} dp_1 dp_2 dp_3 \frac{p_4^2}{p_4^2} \quad (149) \]
- an invariant differential of volume of corresponding pseudosphere (see for example [5]), and \( \overline{F}_a^\pm(x_a, p_a) \) - functions on the same pseudosphere, corresponding to conditions of particles with positive and negative energy. In accordance to (148) we'll obtain for them
\[ \overline{F}_a(\xi_1, \ldots, \xi_s) = \int \prod_{a=s+1}^N d\Sigma_k^a p_k^a (dP_a^+ \sigma_a^+ + dP_a^- \sigma_a^-) \times \]
\[ \times D_{N_s}(\xi_1, \ldots, \xi_s, \tilde{\xi}_{s+1}, \ldots, \tilde{\xi}_N) = \]
\[ = \int \prod_{a=s+1}^N d\Sigma_k^a P_a^+ dP_a^+ \overline{D}_{N_s}(\xi_1, \ldots, \xi_s, \tilde{\xi}_{s+1}, \ldots, \tilde{\xi}_N) + \]
\[ + \int \prod_{a=s+2}^N d\Sigma_j^a P_a^+ d\Sigma_{s+1}^k P_{s+1}^- \times \]
\[ \times D_{N_s}(\xi_1, \ldots, \xi_s, \tilde{\xi}_{s+1}, \tilde{\xi}_{s+2}, \ldots, \tilde{\xi}_N) + \]
\[ \int \prod_{a=s+1}^N d\Sigma_k^a P_k^a dP_a^+ \overline{D}_{N_s}(\xi_1, \ldots, \xi_s, \tilde{\xi}_{s+1}, \ldots, \tilde{\xi}_N), \quad (150) \]
where \( \sigma_a^\pm \) - operator of sorting by pseudospheres
\[ \sigma_a^\pm \overline{D}_{N_s}(\xi_1, \ldots, \xi_s, \ldots, \xi_N) = \overline{D}_{N_s}(\xi_1, \ldots, \xi_a^\pm, \ldots, \xi_N). \]

Sum (150) contains \( 2^{N-s} \) members, obtained by every possible permutations of pluses and minuses in coordinates \( \xi_{s+1}, \ldots, \xi_N \).

13 The Integral Representation of Liouville Equation

Let us advert to Liouville equation (141). We will consider integrals of type
\[ \int d\Gamma_a \delta(s_a - s_a^*(\tau)) \frac{ds_a}{d\tau} \{ H_a, D_{N_s} \} \quad (151) \]
We will carry out several transformations with Poisson bracket:

\[ \{H_a, \mathcal{D}_N\}_a = \frac{\partial H_a}{\partial P_a} \frac{\partial \mathcal{D}_N}{\partial x_a} - \frac{\partial H_a}{\partial x_a} \frac{\partial \mathcal{D}_N}{\partial P_a} \equiv \]

\[ \equiv \mathcal{D}_N \frac{\partial^2 H_a}{\partial x_a \partial P_i} + \frac{\partial}{\partial x_a} \left( \mathcal{D}_N \frac{\partial H_a}{\partial dP_a} \right) - \mathcal{D}_N \frac{\partial^2 H_a}{\partial P_i \partial x_a} - \frac{\partial}{\partial P_i} \left( \mathcal{D}_N \frac{\partial H_a}{\partial x_a} \right) \equiv \]

\[ = \frac{\partial}{\partial x_a} \left( \mathcal{D}_N \frac{\partial H_a}{\partial P_i} \right) - \frac{\partial}{\partial P_i} \left( \mathcal{D}_N \frac{\partial H_a}{\partial x_a} \right). \quad (152) \]

It easily can be seen, that by means of (109) Poisson bracket can be presented in form

\[ \{H_a, \mathcal{D}_N\}_a = \hat{\nabla}_i \left( \mathcal{D}_N \frac{\partial H_a}{\partial P_i} \right) - \frac{\partial}{\partial P_i} \left( \mathcal{D}_N \hat{\nabla}_i H_a \right). \quad (153) \]

Let us note, that following relations are always fair

\[ \delta [s_a - s^*_a(\tau)] \frac{ds^*_a}{d\tau} = \delta (\tau - \tau_a), \quad (154) \]

\[ \delta [s_a - s^*_a(\tau)] ds_a = \delta (\tau - \tau_a) d\tau_a. \]

Therefore integral (151) takes form

\[ J_a = \int dV_a \int_{P_a} \mathcal{D}_N \hat{\nabla}_i \left( \mathcal{D}_N \frac{\partial H_a}{\partial P_i} \right) - \frac{\partial}{\partial P_i} \left( \mathcal{D}_N \hat{\nabla}_i H_a \right) \]

Let us now take into account the relation (111), according to which

\[ J_a = \int dV_a \hat{\nabla}_i \left( \mathcal{D}_N \frac{\partial H_a}{\partial P_i} \right) - \frac{\partial}{\partial P_i} \left( \mathcal{D}_N \hat{\nabla}_i H_a \right) \]

Carrying out 3 + 1 -partition in the first integral, we will obtain

\[ \frac{\partial}{\partial \tau} \int dV_a \int_{P_a} \mathcal{D}_N P_i(\alpha) + \int dV_a \hat{\nabla}_a \int_{P_a} dP_a \mathcal{D}_N P_i(\alpha). \]
We will transform the second item in the last expression using Gauss theorem, and the first item with account of relation \[ \partial H / \partial p^a = dx^a / ds_a \] we will adduce to the form:

\[
\frac{\partial}{\partial \tau} \int d\Gamma_a \delta(s_a - s_a^*(\tau)) D_N \equiv \frac{\partial}{\partial \tau} F_{N-1} \equiv \left( \frac{\partial}{\partial \tau_a} F_{N-1} \right)_{\tau=\tau_a}
\]

-this item turns to zero, since \( F_{N-1} \) does not depend from time coordinate of a particle. We will apply Gauss theorem also in the second integral in (155) and carry out an integration in it all along the mass hypersurface, as a result we will obtain

\[
J_a = \int \int_{S_a} ds_a^{\pm} \sum_{\pm} \int dP_a p_a^\alpha D_N \equiv \int \int_{V_a} dV_a^{\pm} \sum_{\pm} \int dP_a p_a^\alpha D_N \equiv \int \int_{P_a} dP_a^{\pm} p_a^\alpha D_N \equiv \int \int_{P_a} dP_a^{\pm} \hat{\nabla}_k H_a.
\]

(156)

Where external integration in the first member is carried out all along closed two-dimensional hypersurface \( S_a \), limiting the three-dimensional spacelike hypersurface \( \Sigma_a \) of a particle’s configurational space; internal integration in the second member is carried out all along two-dimensional hypersurfaces \( P_a \), limiting corresponding pseudospheres in the momentum space. On these surfaces

\[
p^2 = -g_{\alpha\beta} p^\alpha p^\beta \to +\infty, \quad p_4 \to \pm\infty.
\]

Generalizing accepted in classical statistics hypothesis [8], we may suppose, that distribution function of particles’ ensemble \( D_N \) will sufficiently fast tend to zero at any particle’s configurational coordinates approaching to the two-dimensional hypersurface \( S_a \), limiting spacelike hypersurface \( V_a \) of its configurational space \( X_a \), or at particle’s momentums approaching to the two-dimensional hypersurfaces \( P_a \), limiting pseudospheres of its momentum space \( P_a \).

At this supposition we will obtain finally

\[
\int d\Gamma_a \delta(s_a - s_a^*(\tau)) \{ H_a, D_N \}_a \frac{ds_a}{d\tau} = 0.
\]

(157)

Integrating Liouville equation (141) sequentially by phase coordinates of various particles, we will receive the chain of equations

\[
\sum_{s=1}^N \prod_{b=s+1}^N d\Gamma_b^k \{ H_a, D_N \}_a \frac{ds_a}{d\tau} = 0,
\]

(158)

\[(s = 1, \ldots, N - 1),
\]

\[
d\Gamma_b^k = \frac{d\sum_{i} m_b^i dP_b^i}{m_b^c} = \sum_{i} d\sum_{i} m_b^i dP_b^i \sigma_b^i \pm.
\]
where

\[ d\Sigma^k_i = dV^b i_k(x_b), \quad dP^\pm_b = \frac{1}{\sqrt{-g_b}} \frac{dp_1 dp_2 dp_3}{|p^4_{\pm}|}. \]  

Equations (158) are the integral representation of Liouville equation (141). It is necessary to add to them the equations, obtained by every possible permutations of identical particles.

14 Bogolubov’s Chain Of Zero Approximation

In consequence of non-linear character of gravitational interaction of particles from system (158) it is impossible to obtain the chain of equations of Bogolubov’s chain type, i.e. equations, connecting \( s \)-particle distribution functions with \( s + 1 \)-particle. It is caused by circumstance, that every particle’s Hamilton function includes coordinates of all the rest particles. Now let us consider the case, when gravitational interaction of particles can be neglected in comparison with other interactions. In this case gravitational field there is the background, independent from instant phase coordinates of particles (i.e. \( g_{ik} = g_{ik}(x) \)), and Bogolubov’s chain can be obtained for any interactions, satisfying superposition principle and conserving the integral of rest mass: \( H_a = m_a c \). From methodical considerations we will not go beyond the consideration of vector interactions.

Accounting the structure of charged particle’s Hamiltonian in vector and gravitational fields (77) we will obtain

\[
\{H_a, D_{N_s}\}_a = p^i_a \nabla_i D_{N_s} + \frac{e_a}{c} F^i_{k}(x_a) p^k_a \frac{\partial D_{N_s}}{\partial p^a},
\]  

where

\[ F_{ik} = F_{ik}(x_a | \xi_1, \ldots, \xi_N). \]

In consequence of superposition principle for vector field

\[ F_{ik}(x_a | \xi_1, \ldots, \xi_N) = \sum_{b=1}^{N} F_{ik}(x_a | \xi_b), \]

i.e. \( F_{ik}(x_a) \) - external vector field’s tensor, \( F_{ik}(x_a | \xi_b) \) - tensor of vector field, produced by particle “a” in point \( x_b \) and observed in point \( x_a \); mark at sum symbol in (161) means, that \( b \neq a \). Tensor \( \nabla \cdot F_{ik}(x_a | \xi_b) \), describing pairwise interaction can be represented by means of vector potential

\[ \nabla \cdot F_{ik} (x_a | \xi_b) = \nabla_i A_i k (x_a | \xi_b) - \nabla_k A_i (x_a | \xi_b), \]

which in consequence of (33) and (34) satisfies equations

\[
\left\{ \delta^i_{\xi_b} (\Delta_2 + \mu^2) - R^i_{\xi_b} (x_a) \right\} A (x_a | \xi_b) = 4\pi e_b u^i_b D(x_a | x_b),
\]
\[ \nabla_i A^i (x_a|\xi_b) = 0. \]  

(164)

Thus, according to (160) we have

\[
\{H_a, \mathcal{D}_{N_s}\}_a = \left\{ p_i^a \nabla_i + \frac{e_a}{c} F^{i,k}_a (x_a)p^b_k \frac{\partial}{\partial p^a_i} \right\} \mathcal{D}_{N_s} +
\]

\[ + \frac{e_a}{c} F^{i,k}_a \frac{\partial}{\partial p^a_i} \sum_{b=1}^{N} F^{i,k}(x_a|\bar{\xi}_b) \mathcal{D}_{N_s}. \]

(165)

Thus, in consequence of superposition principle the Poisson bracket (160) can be represented in form

\[
\{H_a, \mathcal{D}_{N_s}\} = \left\{ H^o_a, \mathcal{D}_{N_s}\right\}_a + \sum_{b=1}^{N} \{H_{ab}, \mathcal{D}_{N_s}\}_a, \]

(166)

where \( H^o_a \) - operator, dependent only from phase coordinates of a particle, \( H_{ab} \) - operator, dependent from couple of particles’ “a” “b” phase coordinates. Such representation is fair not only for vector interactions, but for any others, satisfying superposition principle and conserving the integral of rest mass 7.

Using (166) in (158), we’ll obtain

\[
\sum_{a=1}^{s} \frac{ds_a}{d\tau} \sum_{b=s+1}^{N} d\Gamma^k_b \left\{ H_a, \mathcal{D}_{N_s}\right\}_a =
\]

\[
= \sum_{a=1}^{s} \frac{ds_a}{d\tau} \sum_{b=s+1}^{N} d\Gamma^k_b \left\{ H^o_a, \mathcal{D}_{N_s}\right\}_a +
\]

\[ + \sum_{a=1}^{s} \frac{ds_a}{d\tau} \sum_{b=1}^{s} \sum_{c=s+1}^{N} d\Gamma^k_c \left\{ H_{ab}, \mathcal{D}_{N_s}\right\}_a +
\]

\[
\sum_{a=1}^{s} \frac{ds_a}{d\tau} \sum_{b=s+1}^{N} \sum_{c=s+1}^{N} d\Gamma^k_c \left\{ H_{ab}, \mathcal{D}_{N_s}\right\}_a = 0.
\]

Using now the definition of s-particle distribution functions (116), we will come to the common-relativistic Bogolubov’s chain (64)

\[
\sum_{a=1}^{s} \frac{ds_a}{d\tau} \left\{ H_a, \mathcal{P}_s(\xi_1, \ldots, \xi_s)\right\}_a +
\]

(167)

7For scalar interactions, for instance,

\[
\{H_{ab}, \mathcal{D}_{N_s}\}_a = -\frac{q_a}{c} \nabla_i \Phi(x_a|\bar{\xi}_b) \frac{\partial \mathcal{D}_{N_s}}{\partial p^a_i}.
\]
\[ + \sum_{a=1}^{s} \frac{dS_a}{d\tau} \sum_{b=1}^{s'} \{ H_{ab}, \mathbf{F}_s(\xi_1, \ldots, \xi_s) \}_a + \sum_{a=1}^{s} \frac{dS_a}{d\tau} \times \]

\[ \sum_{b=s+1}^{N} \int d\Gamma_b^k \delta(s_b - s'_b(\tau)) \{ H_{ab}, \mathbf{F}_{s+1}(\xi_1, \ldots, \xi_s, \xi_b) \}_a = 0. \]

Index \(b\) at \(s + 1\)-particle distribution function points to the fact, that not all \(F_{s+1}^b\) are equal between themselves in consequence of particles' nonidentity. In case of identical particles' ensemble we will obtain, carrying out summation in the last member

\[ \sum_{a=1}^{s} \frac{dS_a}{d\tau} \{ \hat{H}_a, \mathbf{F}_s \}_a + \sum_{a=1}^{s} \frac{dS_a}{d\tau} \sum_{b=1}^{s'} \{ H_{ab}, \mathbf{F}_s \} + \]

\[ +(N-s) \sum_{a=1}^{s} \frac{dS_a}{d\tau} \int d\Gamma_{s+1}^k \delta(s_{s+1} - s'_{s+1}(\tau)) \]

\[ \times \{ H_{a,s+1}, \mathbf{F}_{s+1}(\xi_1, \ldots, \xi_{s+1}) \} = 0. \]

In particular, for vector interaction we have from (167)

\[ \sum_{a=1}^{s} \frac{dS_a}{d\tau} \left\{ \hat{p}_a^i \hat{\nabla}_i + \frac{e_a}{c} F_{i,k} p^k_a \frac{\partial}{\partial p^i_a} \right\} \mathbf{F}_s + \sum_{a=1}^{s} \frac{dS_a}{d\tau} \frac{e_a}{c} p^k_a \frac{\partial}{\partial p^i_a} \]

\[ \times \sum_{b=s+1}^{N} \int d\Gamma_b^k \delta(s_b - s'_b(\tau)) F_{i,k}^l(x_a | \xi_b) \mathbf{F}_{s+1} = 0, \]

\((s = 1, 2, \ldots, N - 1).\)

These equations jointly with field equation (32) - (35) form the full system of equations, describing gas of interacting charged particles on the background of Riemannian space.

### 15 Conservation Laws In The Statistical Model

The consequence of Liouville equation (141) should be the macroscopic laws of conservation. Let us display this. We will affect by means of statistical average’s operator to the local macroscopic values \(j^i(x), n^i(x), T^{ik}_p\) (19) - (21). At calculation of averages there occur integrals of type

\[ \int d\Gamma_a \delta(s'_a - s_a) \int D(x|x'_a) \psi[\xi_a(s'_a)] ds'_a. \]
Let us find using (28) values of these integrals:

\[ \int \psi(x, p_a) dP_a. \]

Thus, accounting definition (120), we’ll obtain

\[ \ll n^i(x) \gg_N = \sum \frac{1}{m_a c} \int p_a^i F_1(x, p_a) dP_a, \quad (170) \]

\[ \ll j^i(x) \gg_N = \sum \frac{e_a}{m_a c} \int p_a^i F_1(x, p_a) dP_a, \quad (171) \]

\[ \ll T^{ik}_p (x) \gg_N = \sum \frac{1}{m_a} \int p_a^i p_a^k F_1(x, p_a) dP_a. \quad (172) \]

The values of these macroscopic magnitudes coincide with theirs phenomenological definitions \[76\].

For establishment of macroscopic conservation laws we will consider the first group of integral equations (158) for vector interaction

\[ \int \left\{ p_a^i \frac{\partial}{\partial p_a^i} D_N s + \frac{e_a}{c} F^i k p_a^k \frac{\partial D_N s}{\partial p_a^i} \right\} \times \]

\[ \times \frac{ds_a}{d\tau} \prod_{b=1}^N d \Gamma_b^k = 0. \quad (173) \]

Let’s integrate these equations all along the momentum space of a particle, in consequence of (111) we’ll obtain

\[ \nabla_i \int dP_a p_a^i \int \prod_{b=1}^N d \Gamma_b^k D_N s + \]

\[ + \frac{e_a}{c} \int p_a^k dP_a \frac{\partial}{\partial p_a^i} \int D_N s F^i k \prod_{b=1}^N d \Gamma_b^k = 0. \]

Integrating the second member by parts, accounting the skew-symmetry of \( F^{ik} \) and the definition of many-particle functions \[120\], we will find

\[ \nabla_i \int F_1(x_a, p_a) p^i dP_a = 0. \quad (174) \]

Under the integral \[173\] is situated the magnitude, which is equal, according to \[170\] to

\[ \ll n^i_a(x_a) \gg_N = \int F_1(x_a, p_a) p_a^i dP_a (m_a c)^{-1}. \quad (175) \]

Substituting coordinates \( x_a \) with \( x \), we will obtain

\[ \nabla_i \ll n^i_a(x) \gg_N = 0 \quad (176) \]
- conservation law of particle. In consequence of (176) there fulfil the macroscopic laws of conservation of full number of particles and charges

\[ \nabla_i \ll n_i(x) \gg_N = 0, \quad \nabla_i \ll j_i(x) \gg_N = 0. \quad (177) \]

Let’s now multiply (173) by \( p_i^k \) and repeat the whole made procedure; after that let’s sum the result by all particles’ sorts, then we’ll obtain

\[ a \nabla_i \ll T_{p_i}^{sk}(x) \gg_N = \]

\[ - \sum \frac{e_a}{c} \int p_a^k dP_a \int \prod_{b=1}^{N} dT_b^{sk} D_N, F_{i,k}(x, |\xi_1, \ldots, \xi_n) = 0. \quad (178) \]

From macroscopic equations we have the well-known relation

\[ \nabla_k T_{f_j}^{ik} = - j_{ik}. \quad (179) \]

Let’s average this relation by ensemble, accounting the fact, that averaging operator commutates with operator \( \nabla_k \):

\[ \nabla_k \ll T_{p_i}^{ik} \gg_N = \]

\[ = \sum \frac{e_a}{c} \int p_a^k dP_a \int \prod_{b=1}^{N} dT_b^{sk} D_N, F_{i,k}. \quad (180) \]

Here we have carried out an integration by \( \delta \)-functions of sources, summing (178) and (180), we have the macroscopic law of energy conservation:

\[ \nabla_k \ll T_{p_i}^{ik} + T_{f_j}^{ik} \gg_N = 0. \]

16 Vlasov Equations

Let’s consider an equation for the one-particle distribution function \( F_1(\xi) \), for that we’ll put in (169) \( s = 1 \). In consequence of definition of operation \( \Sigma' \) the second member in left part of (169) will disappear and we will obtain

\[ \left\{ \frac{a}{p_i^o} \nabla_i + \frac{e_a}{c} F_{i,k} p_a^k \frac{\partial}{\partial p_a^o} \right\} F_1(\xi_o) + \]

\[ + \frac{e_a}{c} p_a^k \frac{\partial}{\partial p_a^o} \sum_{b=2}^{N} dT_b^{sk} F_{i,k}(\xi_o, \xi_b) F_{i,k}(x, |\xi_b) = 0. \quad (181) \]

Let us suppose, that distribution function of particles’ ensemble is multiplicative, i.e.

\[ D_N(x, |\xi_1, \ldots, \xi_N) = \prod_{a=1}^{N} F_1(\xi_o), \quad (182) \]
then

\[ F_s(\xi_1, \ldots, \xi_s) = \prod_{a=1}^{N} F(\xi_a). \]  

(183)

Correlation between single particles’ motion at that is absent, i.e. interaction - particles interact with each other just via macroscopic smoothed field \( <F_{ik}>_a \).

According to the distribution function’s meaning, average value of this field in point is

\[ <F_{ik}(x_a)>_a = \int d\Gamma_b^k F_{ik}(x_a|\xi_1, \ldots, \xi_n) D_N(\xi_1, \ldots, \xi_n, \ldots, \xi_N). \]

(184)

Using a superposition principle, we’ll receive from here

\[ <F_{ik}(x_a)>_a = \sum_{b=1}^{N'} \int F(\xi_b) F_{ik}(x_a|\xi_b) d\Gamma_b^k, \]

(185)

Thus, supposing in (181) no correlations, we’ll obtain finally

\[ \left\{ p^i \nabla_i + \frac{e_a}{c} F_{i,k} p^k \frac{\partial}{\partial p^i} \right\} F_1(x, p) = 0, \]

(186)

where

\[ F_{ik} = \hat{F}_{ik}(x_a) + \sum_{b=1}^{N} \int F(\xi_b) F_{ik}(x_a|\xi_b) d\Gamma_b^k \]

(187)

- summary vector field. We have obtained so-called collisionless kinetic equation [65]. Let’s act now by average operator (129) on both sides of field equations (33) - (34). In consequence of field equations’ linearity and observer’s coordinates’ independence from particles’ phase coordinates, we will obtain

\[ -4\pi \Sigma e_a \int d\Gamma_a^k \int u^i_a D(x|x_a) dS_a = \]

\[ = \nabla_k \ll F^{ik}(x) \gg_N -\mu^2 \ll A^i(x) \gg_N, \]

(188)

\[ \nabla_k \ll \hat{F}^{ik}(x) \gg_N = 0. \]

(189)

Carrying out integration by all particles except \( a \) one in (188), in consequence of one-particle distribution function’s definition we will transform an expression in the right side of (188) to the form

\[ -4\pi \Sigma e_a \int d\Gamma_a^k F_1(\xi_a) \int u^i_a D(x|x_a) dS \equiv \]

39
\[ -4\pi \Sigma e_a \int _{\mathcal{V}_a} d\xi \sum _{k} \sum _{\pm} \int _{p_a} dP_a p_a^{k} F_1 (\xi_a) \int _{\xi_a} u_a^{k} D (x|\xi_a) dS_a. \]

Let’s use relation (28) for invariant \( \delta \)-Dirac function

\[ \int _{\mathcal{V}_a} u_a^{i} D (x|\xi_a) dS_a = u_a^{i} (\tau) \frac{D (\tilde{x} | \tilde{x}_a (\tau))}{(u,k)_{a}} \]

where \( \tau \)-proper time by clocks of observer \( k \), \( D (\tilde{x} | \tilde{x}_a (\tau)) \)-an invariant \( \delta \)-function on hypersurface \( V_a \); then we’ll receive

\[ \ll j^i \gg _N = \sum _{a=1}^{N} e_a \frac{m_a}{c} \int _{p_a} p_a^{i} F_1 (x,p_a) dP_a \equiv \]

\[ \ll \sum _{a=1}^{N} e_a c \ll n_a^i \gg _N, \]

where \( \ll n_a^i \gg _N \)-vector of particles’ number flux density:

\[ \ll n_a^i (x) \gg _N = \frac{1}{m_a c} \int _{p_a} p_a^{i} F_1 (x,p_a) dP_a \]

In full accordance with (175). Thus, (180) is transformed to the form

\[ \nabla _k \ll F^{ik} \gg _N - \mu_v^2 \ll A^i \gg _N = - \frac{4\pi}{c} \ll j^i \gg _N . \]

It is necessary to note, that these are the exact equations, independent from the supposition about distribution function’s multiplicativity. Representing in form of linear superposition by fields of single sources

\[ \forall F_{ik} (x|s_1, \ldots, s_N) = \sum _{a=1}^{N} \forall F_{ik} (x|s_a) \]

and averaging this expression

\[ \ll F_{ik} (x) \gg _N = \sum _{a=1}^{N} \int _{\xi_a} d\Gamma_a^{k} F_1 (\xi_a) F_{ik} (x|\xi_a), \]

and then comparing it with (183), we will obtain

\[ \ll F_{ik} (x) \gg _N = \ll F_{ik} (x) \gg _a + \int d\Gamma_a^{k} F_1 (\xi_a) F_{ik} (x|\xi_a). \]

Thus, average macroscopic field \( \ll F_{ik} \gg _N \) differs from average macroscopic \( \ll F_{ik} \gg _a \), acting on a particle, only on a value of one particle’s average field. If this difference can be neglected, at supposition \( \ll F_{ik} \gg _N \ll F_{ik} \gg _a \), we will come to the system of common-relativistic Vlasov equations (186), (190), (192). Let us note, that in consequence of equation (186) 4-current and flux of particles’ number automatically conserve.
17 Postscript

Following stated here program, we can obtain kinetic equations for one or another type of particles’ interactions on the gravitational field’s background. For that it is necessary to solve the field equations (33), (34) (or (35)) and substitute the solution into Bogolubov’s chain, in which it is essential to account particles’ pair correlations and neglect triple correlations. Such program, undoubtedly, will develop in further papers. Not having in view to analyze here the problem of kinetic equations’ constructing on the gravitational field’s background in detail, we’ll point, however, the most striking differences of such theory from corresponding theory in flat space-time. First of all, in flat space-time in intervals between collisions particle moves with constant velocity and does not radiate. In gravitational field in intervals between collisions particle moves along geodesic and, as a matter of fact, radiates. This radiation can lead to the extension of interaction’s effective radius. Since character wavelength of bremsstrahlung is \( \lambda \sim L \) (\( L \) - character scale of space curvature), stated effect can play a noticeable part just under the condition

\[ L^3 n \lesssim 1, \]  

(194)

where \( n \) - particles’ number density. Secondly, gravitational field redistributes moving particles’ fields (imports anisotropy in these distributions), that also can influence on final value of dispersion’s differential cut set. And, finally, thirdly, particle’s radiation, strictly speaking, does not spread along geodesic lines (geodesic are just the trajectories of short-wave quantum \( \lambda \ll L \)). Low-frequency radiation (\( \lambda \gtrsim L \)) can spread with smaller velocity and, moreover, in a number of cases produce the static tales, that besides can influence on interaction cut-set’s value. The range, where all these three effects become sufficient, as it is easy to see, is described by formula (194). Thus, gravitation can influence sufficiently upon processes’ kinetics of only sufficiently rarefied medium. The situation, however, changes, if the medium itself serves the source if gravitational field. In this case \( L \sim 1/\sqrt{\kappa E} \) [17] and (194) has a form

\[ \kappa E \gtrsim n^{2/3}. \]  

(195)

For ultrarelativistic medium with equation of state \( E = 3nT \) we will receive from here

\[ T \gtrsim \frac{1}{3\kappa n^{1/3}} \]

and, for example, in hot Universe \( \kappa \) is realized at times, lesser than Planck ones: \( t \lesssim t_{\text{pe}} = \sqrt{\hbar c} \sim 10^{-43}. \) Stated above theory is in essence the theory on the background of gravitational field. Let us clarify, whether it is impossible to adapt by some way this theory for the description of particles’ gravitational interactions. Here we right away run against insoluble obstacles, which are not limited by obstacles of construction of Einstein equation’s general solution [32]. The separation of such an important for us timelike field of geodesic observers, or timelike hypersurface, on which initial conditions are preset, turns into the
insensible operation. Actually, for the separation of such field it is necessary to know metric’s microscopic structure in every point of time (which?), but in order to know this structure, it is necessary to know this structure on timelike hypersurface (which?). The vicious circle has become closed. Indeed, we do not need random observer, but exactly the macroscopic one, which moves along geodesics of macroscopic gravitational field. Consequently, we need to know this macroscopic field. But we can not define it again, before we will not determine metric’s detail structure, i.e. while we will not solve completely the problem, in which on each point of calculation the macroscopic observer appears.

But nevertheless there is an overcome from, seemingly, stalemate, and as a matter of fact, it has been found in papers [64], [65] (although it has not been formulated there sufficiently distinctly). The gravitational field can not be turned on or off. Exactly this circumstance, which, seemingly, leads to the vicious circle and to the impossibility to measure anything until the macroscopic dynamics of ensemble’s geometry will not be known, is the overcome from the deadlock. For any type of non-gravitational interaction particles’ ensemble always has a “zero condition”, in which macroscopic fields are absent, that is the consequence of the existence of opposite signs “charges”. Therefore any macroscopic field of non-gravitational character can be turned off. Gravitational “charges” of all particles and fields have the same sign - and it is impossible to turn off the macroscopic gravitational field. It is impossible to represent particles’ ensemble without gravitational field. Even in the case of annihilation, concrete particle’s disappearance, its gravitational field does not disappear, since energy of given particle does not disappear, but solely grades into another forms of matter. Gravitational field serves more fundamental, more inert form of matter, than ones or another particles of ensemble. Thus, the conception of gravitational field’s statistic inertness forms: gravitational field of ensemble of sufficiently great number of particles has at its heart the macroscopic character and just small on average macroscopic constituent, which is defined by correlated motion of particles and fields. In mathematical form this conception looks:

\[ g_{ik}(x|\tilde{\xi}_1,\ldots,\tilde{\xi}_N) = g_{ik}(x) + \delta g_{ik}(x|\tilde{\xi}_1,\ldots,\tilde{\xi}_N), \]  

at that

\[ \ll \delta g_{ik} \gg_N = 0, \]  

and

\[ \ll \delta g_{ik}\delta g_{jk} \gg_N \ll 1, \]

where average is carried out by field of macroscopic observer. It is seen from this considerations, that (198) can be broken only at condition (195), i.e. at times, smaller than Planck one, where quantization of gravitational field is already necessary. At condition (198) the microscopic constituent of gravitational field can be considered as the small linear perturbation of metric tensor, i.e. as an ordinary field on the background and can be described within the limits of proposed above scheme (see [65]).
What is the location of observer in gravitational fluctuating world? According to the meaning of medium’s statistic description this observer must be the macroscopic one, i.e. observations should be carried out in scales lot more than lengths (or times) of healing of single particles’ local gravitational fields. GRG imposes its indelible impress upon the statistic picture - clocks and bars on microlabel do not coincide with macroscopic clocks and bars. Therefore particle’s motion in terms of macroscopic synchronized time will not be geodesic even at non-gravitational origin forces absence. So, for example, motion of particle with zero rest mass by clocks and scales of macroscopic observer will not be described by means of isotropic geodesic line - dodging in microscopic gravitational fields particle “dresses” by interaction and in terms of macroscopic observer must looks like particle with non-zero rest mass. This allows us to suggest following dependence of graviton’s energy from the momentum in medium:

$$\mathcal{E}^2 = c^2 p^2 + m_g^2 c^4,$$

where \(m_g\) - its effective macroscopic mass. The last relation greatly resembles the relation between gravitational wave’s frequency in medium and its wave vector, determined in [15]:

$$\omega^2 = k^2 c^2 + \omega_g^2.$$

Thus, the statistic picture of graviton’s motion in gravitating medium can give \(m_g \sim \omega_g\) and, thereby, determine physical correspondence between these two pictures.

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