Strong Convergence towards self-similarity for one-dimensional dissipative Maxwell models

G. Furioli *, A. Pulvirenti †, E. Terraneo ‡ and G. Toscani §

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Abstract

We prove the propagation of regularity, uniformly in time, for the scaled solutions of the one-dimensional dissipative Maxwell models introduced in [3]. This result together with the weak convergence towards the stationary state proven in [24] implies the strong convergence in Sobolev norms and in the $L^1$ norm towards it depending on the regularity of the initial data. As a consequence, the original non scaled solutions are also proved to be convergent in $L^1$ towards the corresponding self-similar homogenous cooling state. The proof is based on the (uniform in time) control of the tails of the Fourier transform of the solution, and it holds for a large range of values of the mixing parameters. In particular, in the case of the one-dimensional inelastic Boltzmann equation, the result does not depend of the degree of inelasticity. This generalizes a recent result of Carlen, Carrillo and Carvalho [11], in which, for weak inelasticity, propagation of regularity for the scaled inelastic Boltzmann equation was found by means of a precise control of the growth of the Fisher information.

1 Introduction

In 2003 Ben-Avraham and coworkers [3] introduced a one-dimensional model of the Boltzmann equation, in which binary collision processes are given by arbitrary linear collision rules

$$v^* = pv + qw, \quad w^* = qv + pw; \quad p \geq q > 0. \quad (1)$$

The positive constants $p$ and $q$ represent the mixing parameters, namely the portion of the pre-collisional velocities $(v, w)$ which generate the post-collisional ones $(v^*, w^*)$. Under the hypothesis of constant collision frequency, this mechanism of collision leads to the integro-differential equation of Boltzmann type,

$$\partial_t f(v, t) = \int_{\mathbb{R}} \left( \frac{1}{J} f(v_*, t)f(w_*, t) - f(v, t)f(w, t) \right) \, dw \quad (2)$$

where now $(v_*, w_*)$ are the pre-collisional velocities that generate the couple $(v, w)$ after the interaction and $J = p^2 - q^2$ is the Jacobian of the transformation of $(v, w)$ into $(v^*, w^*)$. As observed in [3], while in the long-time limit velocity distributions are generically self-similar, there is a wide spectrum of possible behaviors. The velocity distributions are characterized by algebraic or stretched exponential tails and the corresponding exponents depend sensitively on the collision parameters. Interestingly, when there is energy or momentum conservation, the behavior is universal.

* University of Bergamo, viale Marconi 5, 24044 Dalmine, Italy. giulia.furioli@unibg.it
† Department of Mathematics, University of Pavia, via Ferrata 1, 27100 Pavia, Italy. ada.pulvirenti@unipv.it
‡ Department of Mathematics, University of Milano, via Saldini 50, 20133 Milano, Italy. Elide.Terraneo@mat.unimi.it
§ Department of Mathematics, University of Pavia, via Ferrata 1, 27100 Pavia, Italy. giuseppe.toscani@unipv.it
Since the integrals $\int_{\mathbb{R}} x^n f(v, t) \, dv$, $n \geq 0$, obey a closed hierarchy of equations \([4]\), moments can be evaluated recursively, starting from mass conservation. In particular, choosing as initial density a normalized probability density $f_0$ satisfying

$$f_0 \geq 0, \quad \int_{\mathbb{R}} f_0(v) \, dv = 1, \quad \int_{\mathbb{R}} v f_0(v) \, dv = 0, \quad \int_{\mathbb{R}} v^2 f_0(v) \, dv = 1,$$ \hspace{1cm} (3)

it follows that both mass and momentum are preserved in time, while the second moment varies according to the law

$$E(t) = \int_{\mathbb{R}} v^2 f(v, t) \, dv = \exp \left\{ (p^2 + q^2 - 1) t \right\}.$$ \hspace{1cm} (4)

Special cases include the elastic model ($p^2 + q^2 = 1$), which is the analogous of the well-known Kac model \([18, 23]\), the inelastic collisions ($p + q = 1$) \([11]\), the granules model ($p + q < 1$) \([26]\), the inelastic Lorenz gas ($q = 0, p < 1$) \([24]\), and in addition energy producing models ($p > 1$) \([30]\).

By \([1]\) it follows that the second moment of the solution is not conserved, unless the collision parameters satisfy

$$p^2 + q^2 = 1.$$

If this is not the case, the energy can grow to infinity or decrease to zero, depending on the sign of $p^2 + q^2 - 1$. In both cases, however, stationary solutions of finite energy do not exist, and the large–time behavior of the system can at best be described by self-similarity properties. The standard way to look for self-similarity is to scale the solution according to the rule

$$g(v, t) = \sqrt{E(t)} f \left( \frac{v}{\sqrt{E(t)}}, t \right).$$

This scaling implies that $\int_{\mathbb{R}} v^2 g(v, t) \, dv = 1$ for all $t \geq 0$.

The large-time behavior of the density $f(v, t)$ has been studied in \([24]\), by resorting to the Fourier transform version of the Boltzmann equation \([2]\), which reads

$$\partial_t \mathcal{F}(\xi, t) = \mathcal{F}(p\xi, t) \mathcal{F}(q\xi, t) - \mathcal{F}(\xi, t).$$ \hspace{1cm} (5)

which converts for the scaled solution $g(t)$ into the following

$$\partial_t \mathcal{G}(\xi, t) + \frac{1}{2} \left( p^2 + q^2 - 1 \right) \xi \partial_\xi \mathcal{G}(\xi, t) = \mathcal{G}(p\xi, t) \mathcal{G}(p\xi, t) - \mathcal{G}(\xi, t).$$ \hspace{1cm} (6)

It is worth recognizing from equation \([5]\) an equivalent formulation of equation \([2]\) making use of the convolution operator

$$\partial_t f(v, t) = f_p * f_q(v, t) - f(v, t)$$

where we used the shorthand

$$f_p(v) = \frac{1}{p} f \left( \frac{v}{p} \right).$$

The key argument for studying equations \([3]\) and \([4]\) is the use of a metric for probability densities with finite and equal moments of order $[\alpha]$, where, as usual, $[\alpha]$ denotes the entire part of the real number $\alpha$:

$$d_\alpha (f, g) = \sup_{\xi \in \mathbb{R}} \frac{\left| \mathcal{F}(\xi) - \mathcal{G}(\xi) \right|}{|\xi|^\alpha}. \hspace{1cm} (7)$$

The metric \([7]\) has been introduced in \([16]\) to investigate the trend to equilibrium of the solutions to the Boltzmann equation for Maxwell molecules. Further applications of $d_\alpha$ can be found in \([12, 25, 17, 24]\).

The study of the time evolution of the $d_\alpha$-metric, with $\alpha = 2 + \delta$, for some suitable $0 < \delta < 1$, enlightened the range of the mixing parameters for which one can expect that the scaled function $g(t)$ converges (weakly) towards a steady profile $g_\infty$ at an exponential rate. More precisely, let us define, for fixed $p$ and $q$ the function

$$S_{p,q}(\delta) = p^{2+\delta} + q^{2+\delta} - 1 - \frac{2+\delta}{2} \left( p^2 + q^2 - 1 \right).$$ \hspace{1cm} (8)
Then, the sign of $S_{p,q}$ determines the asymptotic behavior of the distance $d_\alpha(g_1(t), g_2(t))$ between two scaled solutions $g_1(t), g_2(t)$ issued from two initial data $f_1,0, f_2,0$. In particular, it has been proved in [24] that if there exists $\delta \in (0,1)$ such that $S_{p,q}(\delta) < 0$ for $0 < \delta < \delta$, we can conclude that $d_{2+\delta}(g_1(t), g_2(t))$ converges exponentially to zero if initially finite according to the bound

$$d_{2+\delta}(g_1(t), g_2(t)) \leq \exp \{-|S_{p,q}(\delta)| t\} d_{2+\delta}(f_1,0, f_2,0), \quad \delta \in (0,\delta).$$

It has been also proved in [24] that in this case a unique steady state $g_\infty$ exists for equation (6) and for any initial data $f_0$ with $(2+\delta)$ finite moments, we have for $0 < \delta < \delta$:

$$d_{2+\delta}(g(t), g_\infty) \leq \exp \{-|S_{p,q}(\delta)| t\} d_{2+\delta}(f_0, g_\infty) \to 0, \quad t \to +\infty. \quad (9)$$

On the original non scaled solution $f(t)$, the limit behavior corresponds to a self-similar state $f_\infty(v,t) = \frac{1}{\sqrt{E(t)}} g_\infty \left(\frac{v}{\sqrt{E(t)}}\right)$. Note that, by construction, $S_{p,q}(0) = 0$, and thus \(\min_{\delta \in (0,1)} \{S_{p,q}\} \leq 0\). A numerical evaluation of the region where the minimum of the function $S_{p,q}$ is negative for $p, q \in [0, 2]$ is reported in [24]. This region includes the relevant cases of both inelastic ($p+q = 1$) and elastic ($p^2+q^2 = 1$) collisions, as well as the case, among others, in which $p = q = 1$.

Despite the fact that the large-time behavior of the solution to the Boltzmann–like equation (2) can be described in terms of the $d_\ast$-metric, which is equivalent to the weak* convergence of measures [13], the strong convergence of the scaled density $g(t)$ towards $g_\infty$ has never been proved before.

A similar problem occurs in dissipative kinetic theory, where the weak convergence of the (scaled) solution to the inelastic Boltzmann equation for Maxwell molecules towards the homogeneous cooling state with polynomial tails is known to hold [13] in the $d_\ast$-metric framework, but the strong convergence is still unknown in full generality. A recent paper by Carlen, Carrillo and Carvalho [11] shows that in some cases one can prove that the strong convergence holds. Their result, however, requires a small inelasticity regime, which in our setting of the mixing parameters means $p+q = 1$, and at the same time $1 - p^2 - q^2 << 1$.

In this case, in fact, one can resort to methods close to the elastic situation, in which the (controlled) growth of the Fisher information, coupled with the exponential decay of the $d_\ast$-metric allows to prove the uniform propagation of Sobolev regularity, and from this, by interpolation, the strong convergence.

Propagation of Sobolev regularity for both Kac equation and the elastic Boltzmann equation for Maxwellian molecules, together with the precise exponential rate of the strong convergence to the Maxwellian equilibrium $M$ has been proved in [12]. The advantage of working with the classical elastic Boltzmann equation relies on the fact that one can resort to the $H$-theorem. A careful reading of [12], however, allows to conclude that the proof of uniform propagation of regularity makes use of the following condition for the distance between the solution $f(t)$ and the Maxwellian $M$:

$$\sup_{\xi \in R} |\hat{f}(\xi, t) - \hat{M}(\xi)| \to 0, \quad t \to +\infty. \quad (10)$$

While the convergence of $H(f)(t)$ to $H(M)$ implies the $L^1$-convergence and therefore [10], the same condition continues to hold provided the Fourier transform of the solution to the (scaled) Boltzmann equation $g(t)$ and of the stationary state $g_\infty$ satisfy [13] together with a suitable (uniform) decay at infinity in the $\xi$ variable, of the type

$$(1 + \kappa|\xi|)\mu |\hat{g}(\xi, t)| \leq K, \quad \xi \in \mathbb{R} \quad (11)$$

for some positive constants $\kappa, K$ and $\mu$. In fact, in this case, for any given $R > 0$,

$$|\hat{g}(\xi, t) - \hat{g}_\infty(\xi)| \leq d_\ast(g(t), g_\infty) R^\alpha + \frac{2K}{(\kappa R)^{\mu}}.$$
which implies, optimizing over $R$,

\[ |\hat{g}(\xi, t) - \hat{g}_\infty(\xi)| \leq C(\alpha, \mu, \kappa, K)d_c(g(t), g_\infty)^{\mu/(\alpha + \mu)}, \quad \xi \in \mathbb{R}, \; t \geq 0. \]

Therefore, in presence of condition (11), the decay to zero of the $d_\alpha$-metric implies the decay to zero of $\sup_{\xi \in \mathbb{R}} |\hat{g}(\xi, t) - \hat{g}_\infty(\xi)|$ for $t \to +\infty$. We will remark in Section 3 that condition (11) on the initial density $f_0$ holds provided the square root of $f_0$ has some regularity. For example, the Fisher information of $f_0$, which controls the $H^1$-norm of the square root, controls $|\xi||f_0(\xi)|$ (cfr. the proof in [20]).

Condition (11) is difficult to prove directly from the equation satisfied by the scaled density $\hat{g}(t)$, due to the presence of the drift term in equation (6). To simplify the proof of the various bounds we will introduce a semi-implicit discretization of equation (6), that is, for a small time interval $\Delta t$, we will consider the solution to

\[ \frac{\hat{g}(\xi, t + \Delta t) - \hat{g}(\xi, t)}{\Delta t} + \frac{1}{2}(p^2 + q^2 - 1) \xi \partial_t \hat{g}(\xi, t + \Delta t) = \hat{g}(\rho_\xi, t)\hat{g}(\xi, t) - \hat{g}(\xi, t). \]  

In particular, we shall prove that the steady state has the Gevrey regularity of class $\lambda$, namely

\[ e^{\mu|\xi|^\lambda} |\hat{g}_\infty(\xi)| \leq 1, \quad |\xi| > \rho, \]

where the constants $\lambda, \mu$ and $\rho$ are related both to $p, q$ and to the number of moments which are bounded initially. Our main results are summarized into the following statements. In what follows we will denote $g_0$ the initial data of a scaled solution $g(t)$, even if of course $g_0 = f_0$, the initial data of the original, non scaled solution $f(t)$.

**Theorem 1**

Assume $0 < q \leq p$ satisfying $p^2 + q^2 < 1$ and such that there is $\hat{\delta} \in (0, 1)$ for which $\mathcal{S}_{\rho, \eta}(\hat{\delta}) < 0$, for $0 < \delta < \hat{\delta}$. Let $g(t)$ be the weak solution of the equation (7), corresponding to the initial density $g_0$ satisfying the normalization conditions (6), and

\[ \int_{\mathbb{R}} |v|^{2 + \hat{\delta}} g_0(v) \, dv < +\infty. \]

If in addition

\[ |\hat{g}_0(\xi)| \leq \frac{1}{(1 + \beta|\xi|)^r}, \quad |\xi| > R, \]

for some $R > 0, \nu > 0$ and $\beta > 0$, then there exist $\rho > 0, k > 0, \beta' > 0, \nu' > 0$ such that $g(t)$ satisfies

\[ |\hat{g}(\xi, t)| \leq \begin{cases} 
\frac{1}{1 + k|\xi|^2}, & |\xi| \leq \rho, \quad t \geq 0 \\
\frac{1}{(1 + \beta'|\xi|)^{\nu'}}, & |\xi| > \rho, \quad t \geq 0.
\end{cases} \]
Theorem 1 is proven in Section 4. The second result is concerned with the regularity of the steady state \( g_\infty \) and is proven in Section 5.

**Theorem 2**

Assume \( 0 < q \leq p \) satisfying \( p^2 + q^2 < 1 \) and such that there exists \( \tilde{\delta} \in (0, 1) \) for which \( S_{p,q}(\delta) < 0 \) for \( 0 < \delta < \tilde{\delta} \) so that a non-trivial steady state \( g_\infty \) to the Boltzmann equation (4) exists. Let us denote \( \lambda \in (0, 2) \) the exponent such that \( p^\lambda + q^\lambda = 1 \). Then \( g_\infty \) is a smooth function and belongs to the \( \lambda \)-state \( g \).

The previous results are enough to prove the convergence in strong norms towards the steady state \( g_\infty \) in Section 4. The second result is concerned with the regularity of the propagation of the Gevrey regularity for solutions \( g(t) \) issued from Gevrey initial data.

**Theorem 3**

Assume \( 0 < q \leq p \) satisfying \( p^2 + q^2 < 1 \) and such that there exists \( \tilde{\delta} \in (0, 1) \) for which \( S_{p,q}(\delta) < 0 \) for \( 0 < \delta < \tilde{\delta} \) and let us denote \( \lambda \in (0, 2) \) the exponent such that \( p^\lambda + q^\lambda = 1 \). Let \( g(t) \) be the weak solution of the equation (4), corresponding to the initial density \( g_0 \) satisfying

\[
\int_{\mathbb{R}} |v|^{2+\lambda} g_0(v) \, dv < +\infty.
\]

If in addition \( |\hat{g}_0(\xi)| \leq e^{-|\beta| |\xi|^\nu}, \quad |\xi| > R, \)

for some \( R > 0, \nu > 0 \) and \( \beta > 0 \), then there exist \( \rho > 0 \) and \( \kappa > 0 \) such that \( g(t) \) satisfies

\[
|\hat{g}(\xi, t)| \leq \begin{cases} 
    e^{-\kappa \xi^2}, & |\xi| \leq \rho, \quad t \geq 0 \\
    e^{-\kappa |\xi|^{\min(\nu, \lambda)}}, & |\xi| > \rho, \quad t \geq 0.
\end{cases}
\]

The previous results are enough to prove the convergence in strong norms towards the steady state \( g_\infty \). In consequence of both Theorems 1 and 2 we can show in fact the uniform in time propagation of regularity in Sobolev spaces of high degree

\[
\|g\|_{H^\eta(R)}^2 = \int_{\mathbb{R}} |\xi|^{2\eta} |\hat{g}(\xi)|^2 \, d\xi
\]

with \( \eta > 0 \). It is enough to apply the technique developed in [12] for the Boltzmann equation for Maxwell molecules for showing that whenever the equation propagates a tiny degree of regularity, as in Theorem 1 this implies that the equation propagates regularity of any degree.

Then, using the regularity in high Sobolev spaces, we can pass from the weak convergence in \( d_\alpha \)-metric obtained in [24] into convergence in all Sobolev norms, and strong \( L^1 \) convergence at an explicit exponential rate for a certain class of initial data. This is the objective of Section 6 and the main result is summarized as follows.

**Theorem 4**

Assume \( 0 < q \leq p \) satisfying \( p^2 + q^2 < 1 \) and such that there exists \( \tilde{\delta} \in (0, 1) \) for which \( S_{p,q}(\delta) < 0 \) for \( 0 < \delta < \tilde{\delta} \) and let \( g_\infty \) be the unique stationary solution of (6). Let the initial density \( g_0 \) satisfy the normalization conditions (3), and

\[
\int_{\mathbb{R}} |v|^{2+\lambda} g_0(v) \, dv < +\infty.
\]

If in addition \( g_0 \in H^\eta(R) \) for some \( \eta > 0, \sqrt{\nu} \in H^{\nu}(R) \) for some \( \nu > 0 \), then the solution \( g(t) \) of (6) converges strongly in \( L^1 \) with an exponential rate towards the stationary solution \( g_\infty \), i.e., there exist positive constants \( C \) and \( \gamma \) explicitly computable such that

\[
\|g(t) - g_\infty\|_{L^1(R)} \leq Ce^{-\gamma t}, \quad t \geq 0.
\]
Thanks to the scaling invariance of the $L^1$ norm, Theorem 4 allows to deduce also the strong convergence of the original non scaled solution $f(t)$ to the self-similar state $f_\infty(v,t) = \frac{1}{\sqrt{E(t)}} \frac{\sqrt{E(t)} g_\infty(|\xi|)}{v \sqrt{E(t)}}$:

$$\|f(t) - f_\infty(t)\|_{L^1(\mathbb{R})} \leq Ce^{-\gamma t}, \quad t \geq 0.$$  

Finally, in Section 7 we will discuss in details the relevant case in which $p + q = 1$, which corresponds to the one-dimensional inelastic Boltzmann equation for Maxwell molecules 1. In this case, in fact, it is known that an explicit stationary solution to equation (6) exists, 

$$\hat{g}_\infty(|\xi|) = (1 + |\xi|) e^{-|\xi|},$$  

or, in the physical space 

$$g_\infty(v) = \frac{2}{\pi(1 + v^2)^{2/3}}.$$  

This stationary solution is independent of the values of $p$ and $q$, and within the set of functions $f$ which satisfy the normalization conditions (3), is the minimum of the convex functional 

$$H(f) = -\int_\mathbb{R} \sqrt{f(v)} dv.$$  

This suggests the idea that $H$ is an entropy functional for the scaled equation (6), but the proof of this conjecture would require to satisfy an inequality which is the analogous of the Shannon entropy power inequality [6, 27], we are not able to prove.

It is interesting to remark that the ideas of the present paper, which are valid for the $p+q = 1$ case, can be fruitfully used for the three-dimensional dissipative Boltzmann equation for Maxwell molecules, extending the validity of the recent analysis of [11] to a general coefficient of restitution. However, the proof of the analogous of Theorem 4 to the three-dimensional case requires heavy computations, we will publish separately elsewhere.

2 Preliminary results

Let us consider the one-dimensional kinetic models of Maxwell-Boltzmann type:

$$\begin{cases}
\partial_t f(v,t) = \int_\mathbb{R} \left( \frac{1}{2} f(v_*,t) f(w_*,t) - f(v,t) f(w,t) \right) dw \\
f(v,0) = f_0(v)
\end{cases} \tag{17}$$

where $f(v,t) : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ denotes the distribution of particles with velocity $v \in \mathbb{R}$ at the time $t \geq 0$ and $(v_*, w_*)$ are the pre-collisional velocities that generate post-collisional $(v, w)$:

$$\begin{aligned}
v &= pv_* + qw_* \\
w &= qv_* + pw_ *
\end{aligned}$$

and $0 < q \leq p$.

Moreover let us suppose $f_0$ satisfying the normalization conditions (3). In order to avoid the presence of the Jacobian we write equation (17) in weak form, namely

$$\frac{d}{dt} \int_\mathbb{R} \Phi(v) f(v,t) dv = \int_{\mathbb{R}^2} f(v,t) f(w,t) (\Phi(v^*) - \Phi(v)) dw dv,$$

and

$$\lim_{t \rightarrow 0} \int_\mathbb{R} \Phi(v) f(v,t) dv = \int_\mathbb{R} \Phi(v) f_0(v) dv$$

for any $\Phi$ bounded and continuous on $\mathbb{R}$. At least formally by choosing $\Phi(v) = 1$ and $\Phi(v) = v$ one shows that, under conditions (3) both the mass and momentum are preserved. By choosing $\Phi(v) = v^2$ we obtain that the energy of the solution

$$E(t) = \int_\mathbb{R} v^2 f(v,t) dv$$
satisfies the equality
\[ E(t) = e^{(p^2 + q^2 - 1)t} E(0). \]
Therefore, unless \( p^2 + q^2 = 1 \), the energy is not preserved. In the dissipative case \( p + q = 1 \), one has additionally that the moment is always preserved, while the energy is decreasing.

Following Bobylev [7], the weak form of equation (17) is equivalent to the equation in the Fourier variables:
\[
\begin{align*}
\partial_t \hat{f}(\xi, t) &= \hat{f}(p\xi, t)\hat{f}(q\xi, t) - \hat{f}(\xi, t), \quad t > 0 \\
\hat{f}(\xi, 0) &= \hat{f}_0(\xi).
\end{align*}
\]

The existence and uniqueness of a solution for any initial data \( f_0 \) satisfying (5) can be established in the same way as for the elastic Kac equation.

**Theorem 5 (Theorem of existence and uniqueness [24, 25])**

We consider \( f_0 \) satisfying the normalization conditions [7] and the following Cauchy problem:

\[
\begin{align*}
\partial_t \hat{f}(\xi, t) &= \hat{f}(p\xi, t)\hat{f}(q\xi, t) - \hat{f}(\xi, t), \quad t > 0 \\
\hat{f}(\xi, 0) &= f_0(\xi).
\end{align*}
\]

Then, there exists a unique nonnegative solution \( f \in C^1 \left([0, +\infty), L^1(\mathbb{R})\right) \) to equation (15) satisfying for all \( t > 0 \):
\[
\int_\mathbb{R} f(v, t) \, dv = 1, \quad \int_\mathbb{R} f(v, t) \, v \, dv = 0.
\]

In order to investigate some properties of the solution is useful to introduce the following rescaling
\[
\hat{g}(\xi, t) = \hat{f}(\frac{\xi}{\sqrt{E(t)}}, t).
\]

The function \( \hat{g}(t) \) preserves the energy and it satisfies the equation:

\[
\begin{align*}
\partial_t \hat{g}(\xi, t) + \frac{1}{2} \left(p^2 + q^2 - 1\right) \xi \partial_\xi \hat{g}(\xi, t) &= \hat{g}(p\xi, t)\hat{g}(q\xi, t) - \hat{g}(\xi, t), \quad t > 0 \\
\hat{g}(\xi, 0) &= \hat{f}_0(\xi) := \hat{g}_0(\xi).
\end{align*}
\]

In the relevant case \( p + q = 1 \), equation (19) admits an explicit stationary state [1]
\[
\hat{g}_\infty(\xi) = (1 + |\xi|) e^{-|\xi|}.
\]

Pareschi and Toscani proved in [24] that in a quite large range of values of the mixing parameters, a unique stationary state exists and the unique weak solution \( g(t) \) converges to \( g_\infty \). In all the cases \( p + q \neq 1 \) (excluding \( p^2 + q^2 = 1 \)) the stationary state is not explicit, even if some properties can be extracted from the analysis of the evolution equation. The convergence takes place in a Fourier distance introduced in [16] to investigate the trend to equilibrium of the solution to the Boltzmann equation for Maxwellian molecules. We recall the definition of this distance. Let \( 0 \leq \delta < 1 \) and let
\[
M_{2+\delta} = \left\{ f \geq 0 : \int_\mathbb{R} f(v) \, dv = 1, \int_\mathbb{R} v \, f(v) \, dv = 0, \int_\mathbb{R} v^2 \, f(v) \, dv = 1, \int_\mathbb{R} |v|^{2+\delta} \, f(v) \, dv < \infty, \right\}.
\]

We introduce on \( M_{2+\delta} \) the distance:
\[
d_{2+\delta}(f, g) = \sup_{\xi \in \mathbb{R}} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|}{|\xi|^{2+\delta}}.
\]

Let us define, for \( \delta \geq 0 \),
\[
S_{p,q}(\delta) = p^{2+\delta} + q^{2+\delta} - 1 = \frac{2+\delta}{2} \left(p^2 + q^2 - 1\right).
\]

The result of Pareschi and Toscani is as follows.
Theorem 6 (Pareschi–Toscani [24])
Assume $0 < q \leq p$ and such that there exists $\bar{\delta} \in (0, 1)$ for which $\mathcal{S}_{p,q}(\delta) < 0$ for $0 < \delta < \bar{\delta}$.
Let $g(t)$ be the weak solution of equation (19), corresponding to the initial density $g_0$ satisfying the normalization conditions (3) and
\[ \int_{\mathbb{R}} |v|^{2+\delta} g_0(v) \, dv < +\infty. \]
Then $g(t)$ satisfies for $0 < \delta \leq \bar{\delta}$ and for $c_\delta > 0$:
\[ \int_{\mathbb{R}} |v|^{2+\delta} g(v, t) \, dv \leq c_\delta, \quad t \geq 0. \]
Moreover, there exists a unique stationary state $g_\infty$ to equation (19) which satisfies for $0 < \delta < \bar{\delta}$:
\[ \int_{\mathbb{R}} |v|^{2+\delta} g_\infty(v) \, dv < +\infty, \]
g(t) converges exponentially fast in Fourier metric towards $g_\infty$ and the following bound holds for $0 < \delta < \bar{\delta}$:
\[ d_{2+\delta}(g(t), g_\infty) \leq e^{-|\mathcal{S}_{p,q}(\delta)| t} d_{2+\delta}(g_0, g_\infty). \]

3 An iteration process

The goal of this section is to build up a sequence of functions $\{g^N(\xi, t)\}$ which approximates uniformly the solution $\hat{g}(\xi, t)$. In order to do this, for any fixed $T > 0$ we consider firstly a semi-implicit discretization in time of equation (19) by partitioning the interval $[0, T]$ into $N$ subintervals and we define thus the approximate solution at any time $t = j \frac{T}{N}$ for $j = 0, \ldots, N$.
Secondly, we define $g^N(\xi, t)$ on the whole interval $[0, T]$ by interpolation and lastly we show the convergence of the approximation to the solution.

In order to lighten the reading, we will postpone almost all the proofs of this technical section at the end of the paper in Appendix 8.

The approximate equation

In this paragraph we define the approximate solution at any time $t = j \frac{T}{N}$ for $j = 0, \ldots, N$ by an iteration process and study some of its properties.

Let $T > 0$ and $\Delta t = \frac{T}{N}$ for $N \in \mathbb{N}$, $N > T$. Let $\hat{\phi}^N_j(\xi)$, $j = 0, \ldots, N$ be the sequence:
\[ \left\{ \begin{array}{l}
\hat{\phi}^N_0(\xi) = \hat{g}_0(\xi) \\
\hat{\phi}^N_j(\xi) - \hat{\phi}^N_{j+1}(\xi) = \frac{1}{\Delta t} \xi \frac{d}{d\xi} \hat{\phi}^N_{j+1}(\xi) + \hat{\phi}^N_j(p\xi) \hat{\phi}^N_j(q\xi) - \hat{\phi}^N_j(\xi), \quad j = 0, \ldots, N - 1
\end{array} \right. \tag{20} \]

where $\frac{1}{\tau} = \frac{1}{r} - \frac{p^2 + q^2}{2}.$

Proposition 7

Assume $0 < q \leq p$. If $g_0$ verifies the normalization conditions (3), then there exists a unique sequence of bounded function $\hat{\phi}^N_j$ for $j = 1, \ldots, N$ satisfying (20).

Proof: Let us begin by proving that $\hat{\phi}^N_1$ is well defined. In a similar way as in [8] we multiply equation (20) by $(-\frac{d}{d\xi}) \text{sgn} \xi |\xi|^{r-1}$ and obtain
\[ \frac{d}{d\xi} \left( \hat{\phi}^N_1(\xi)|\xi|^{r-1} \right) = \left( -\frac{r}{\Delta t} \right) \text{sgn} \xi |\xi|^{r-1} \left( \Delta t \hat{\phi}^N_0(p\xi) \hat{\phi}^N_0(q\xi) + (1 - \Delta t) \hat{\phi}^N_0(\xi) \right). \]

We assume now $p^2 + q^2 < 1$. For any positive $\xi$ we integrate on $[\xi, +\infty)$ and since $\hat{\phi}^N_0(\xi)$ is bounded we get:
\[ \hat{\phi}^N_1(\xi)|\xi|^{r-1} = \frac{r}{\Delta t} \int_{\xi}^{+\infty} \left( \Delta t \hat{\phi}^N_0(p\xi) \hat{\phi}^N_0(q\xi) + (1 - \Delta t) \hat{\phi}^N_0(\xi) \right) s^{-r-1} \, ds. \]
Finally by the change of variables $\tau = s/\xi$ we are led to:

$$
\hat{\phi}_j^N(\xi) = \frac{r}{\Delta t} \int_1^{+\infty} \left( \Delta t \, \hat{\phi}_0^N(p\tau \xi) \hat{\phi}_0^N(q\tau \xi) + (1 - \Delta t) \, \hat{\phi}_j^N(\tau \xi) \right) \frac{d\tau}{\tau^{2\xi+1}}. 
$$

(21)

For any negative $\xi$ we integrate on $(-\infty, \xi]$ and in a similar way we obtain that equality (21) holds for any $\xi \neq 0$. Moreover since $g_0$ satisfies the conditions (3) then $\hat{\phi}_0^N = \hat{g}_0$ belongs to $C^1(\mathbb{R})$ and

$$
\hat{g}_0(0) = 1 \quad \left| \frac{d\hat{g}_0(\xi)}{d\xi} \right| \leq 1 \quad \frac{d\hat{g}_0}{d\xi}(0) = 0.
$$

(22)

Therefore the function $\hat{\phi}_j^N$ can be defined by continuity in $\xi = 0$ and it is the unique, bounded and $C^1(\mathbb{R})$ solution of (20). By an iteration argument the same conclusion holds for any $\hat{\phi}_j^N$ obtaining for $j = 0, \ldots, N - 1$

$$
\hat{\phi}_{j+1}^N(\xi) = \frac{r}{\Delta t} \int_1^{+\infty} \left( \Delta t \, \hat{\phi}_0^N(p\tau \xi) \hat{\phi}_0^N(q\tau \xi) + (1 - \Delta t) \, \hat{\phi}_j^N(\tau \xi) \right) \frac{d\tau}{\tau^{2\xi+1}}.
$$

For $p^2 + q^2 > 1$, we repeat the same argument by integrating on $[0, \xi]$ and $[\xi, 0]$ and we get in the end

$$
\hat{\phi}_{j+1}^N(\xi) = -\frac{r}{\Delta t} \left[ \int_0^{\xi} \left( \Delta t \, \hat{\phi}_0^N(p\tau \xi) \hat{\phi}_0^N(q\tau \xi) + (1 - \Delta t) \, \hat{\phi}_j^N(\tau \xi) \right) \frac{d\tau}{\tau^{2\xi+1}} \right].
$$

\[\square\]

Applying Fubini’s theorem we can remark that for $p^2 + q^2 < 1$ any $\hat{\phi}_{j+1}^N(\xi)$ is the Fourier transform of $\varphi_{j+1}^N(v)$ where for $j = 0, \ldots, N - 1$

$$
\begin{cases}
\varphi_0^N(v) = g_0(v) \\
\varphi_{j+1}^N(v) = \frac{r}{\Delta t} \int_1^{+\infty} \left( \Delta t \, \frac{1}{\tau} \left( \varphi_{j,p}^N * \varphi_{j,q}^N \right) \left( \frac{v}{\tau} \right) + (1 - \Delta t) \, \frac{1}{\tau} \varphi_j^N \left( \frac{v}{\tau} \right) \right) \frac{d\tau}{\tau^{2\xi+1}}.
\end{cases}
$$

(23)

with $\varphi_{j,p}^N(v) = \frac{1}{p} \varphi_j^N \left( \frac{v}{p} \right)$ and similarly for $\varphi_{j,q}^N$. Analogously, for $p^2 + q^2 > 1$, $\varphi_{j+1}^N(\xi)$ is the Fourier transform of $\varphi_{j+1}^N(v)$ where for $j = 0, \ldots, N - 1$

$$
\begin{cases}
\varphi_0^N(v) = g_0(v) \\
\varphi_{j+1}^N(v) = -\frac{r}{\Delta t} \int_0^{\xi} \left( \Delta t \, \frac{1}{\tau} \left( \varphi_{j,p}^N * \varphi_{j,q}^N \right) \left( \frac{v}{\tau} \right) + (1 - \Delta t) \, \frac{1}{\tau} \varphi_j^N \left( \frac{v}{\tau} \right) \right) \frac{d\tau}{\tau^{2\xi+1}}.
\end{cases}
$$

(24)

In what follows, the function $S_{p,q}(\delta)$ is defined as in [3].

Proposition 8

Assume $0 < q \leq p$ such that there exists $\tilde{\delta} \in (0, 1)$ for which $S_{p,q}(\delta) < 0$ for $0 < \delta < \tilde{\delta}$. Let $\varphi_{j}^N$, for $j = 0, \ldots, N$ defined as in (20) or (21), with $g_0$ satisfying the normalization conditions [3] and \( \int_\mathbb{R} |v|^{2+\delta} g_0(v) \, dv < +\infty \). Then, for $0 < \delta < \tilde{\delta}$ there exists $C_\delta > 0$ such that for $N$ large enough (depending on $\delta, T, p$ and $q$), for $j = 0, \ldots, N$ we get

$$
\varphi_j^N(v) \geq 0, \quad \int_\mathbb{R} \varphi_j^N(v) \, dv = 1, \quad \int_\mathbb{R} v \varphi_j^N(v) \, dv = 0, \quad \int_\mathbb{R} v^2 \varphi_j^N(v) \, dv = 1, \quad \int_\mathbb{R} |v|^{2+\delta} \varphi_j^N(v) \, dv \leq C_\delta.
$$

(25)

Remark 9

The equalities in (22) imply that there exists $C > 0$ such that

$$
\left| \varphi_j^N(\xi) \right| \leq 1, \quad \left| \frac{d\varphi_j^N(\xi)}{d\xi} \right| \leq C \quad \text{and} \quad \left| \frac{d^2\varphi_j^N(\xi)}{d\xi^2} \right| \leq 1
$$

(26)

for any $\xi \in \mathbb{R}$, for $N$ large enough and for $j = 0, \ldots, N$. The first two inequalities have already been proved in the proof of Proposition 4.
Definition of the sequence \( \{g^N(\xi, t)\}_N \)

We are now in position to define the sequence \( \{g^N(\xi, t)\}_N \). We define \( g^N(\xi, j \frac{t}{T}) = \check{\phi}_j^N(\xi) \) for \( j = 0, \ldots, N \). We extend the definition on the whole interval by interpolation. More precisely, let us define:

\[
g^N(\xi, t) = \begin{cases} 
\hat{g}_0(\xi) & t = 0 \\
\alpha(t) \check{\phi}_N^N(\xi) + (1 - \alpha(t)) \check{\phi}_{N-1}^N(\xi) & 0 < t \leq T 
\end{cases}
\]

where for \( 0 < t \leq T \) we have \((K_N - 1) \frac{T}{N} < t \leq K_N \frac{T}{N} \) for \( K_N \in \{1, \ldots, N\} \) and more precisely there is a function \( 0 \leq \alpha(t) < 1 \) such that \( t = \alpha(t)(K_N - 1) \frac{T}{N} + (1 - \alpha(t))K_N \frac{T}{N} \). Any \( g^N(\xi, t) \) is continuous on \( \mathbb{R} \times [0, T] \) and for any \( t \in [0, T] \) it belongs to \( \mathcal{C}^2(\mathbb{R}) \).

The result of convergence is therefore as follows.

**Proposition 10**

There is a subsequence \( \{g^N(\xi, t)\}_N \) of \( \{g^N(\xi, t)\}_N \) which converges uniformly on any compact set of \( \mathbb{R} \times [0, T] \) to the solution \( \hat{g}(\xi, t) \).

## 4 Propagation of regularity

In this section we prove Theorem \([3]\). Thanks to the uniform convergence of a subsequence of the approximate solutions \( g^N(\xi, t) \) to the solution \( \hat{g}(\xi, t) \) and to the definition of \( g^N(\xi, t) \), it is enough to prove the bounds \([10]\) for any \( \check{\phi}_j^N(\xi) \) uniformly for \( N \in \mathbb{N} \) and \( j = 0, \ldots, N \). The control of low frequencies is a direct consequence of the properties \([3]\) of the initial data and of the convergence to zero of the distance \( d_{2+\delta}(g(t), g_\infty) \) and it is proven in the following lemma.

**Lemma 11**

Assume \( 0 < q \leq p \) satisfying \( p^2 + q^2 < 1 \) and such that there is \( \tilde{\delta} \in (0, 1) \) for which \( S_{p,q}(\delta) < 0 \) for \( 0 < \delta < \tilde{\delta} \). Let \( g(t) \) be the weak solution of the equation \([2]\), corresponding to the initial density \( g_0 \) satisfying the normalization conditions \([3]\), and

\[
\int_\mathbb{R} |v|^{2+\tilde{\delta}} \hat{g}_0(v) \, dv < +\infty.
\]

Let \( \check{\phi}_j^N \) the approximation defined in \([20]\). For any \( 0 < k < \frac{1}{2} \) there exists \( \rho > 0 \) such that for any fixed \( T > 0 \) and any \( N \in \mathbb{N} \) large enough we get

\[
|\check{\phi}_j^N(\xi)| \leq \frac{1}{1 + k\xi^2}, \quad |\xi| \leq \rho, \quad j = 0, \ldots, N.
\]

**Proof:** Let us begin by proving \([27]\). We remark first that since for all \( t \geq 0 \)

\[
g(v, t) \geq 0, \quad \int_\mathbb{R} g(v, t) \, dv = 1, \quad \int_\mathbb{R} v g(v, t) \, dv = 0, \quad \int_\mathbb{R} v^2 g(v, t) \, dv = 1
\]

we can deduce at once

\[
\hat{g}(\xi, t) = 1 - \frac{\xi^2}{2} + o(\xi^2), \quad \xi \to 0
\]

and so for \( t \geq 0 \) and \( 0 < k < \frac{1}{2} \) there exists \( \rho = \rho(t) > 0 \) such that

\[
|\check{\phi}_j^N(\xi)| \leq \frac{1}{1 + k\xi^2}, \quad |\xi| \leq \rho(t)
\]
but this is not enough because we need an estimate independent of $t$. We have therefore to proceed differently. By Theorem [3] for any fixed $0 < \delta < \delta$ we have

$$d_{2+\delta}(g(t), g_\infty) = \sup_{\xi \in \mathbb{R}} \left| \frac{\hat{g}(\xi, t) - \hat{g}_\infty(\xi)}{|\xi|^{2+\delta}} \right| \leq e^{-|S_{2+\delta}(\delta)|t} d_{2+\delta}(g_0, g_\infty), \quad t \geq 0.$$ 

Moreover, since $g_\infty \geq 0$, $\int_0^1 v \, g_\infty(v) \, dv = 1$, $\int_0^1 v \, g_\infty(v) \, dv = 0$, $\int_0^1 v^2 \, g_\infty(v) \, dv = 0$ we have

$$\hat{g}_\infty(\xi) = 1 - \frac{\xi^2}{2} + o(\xi^2), \quad \xi \to 0,$$

so we get uniformly for $t \geq 0$ and $\xi \neq 0$:

$$|\hat{g}(\xi, t)| \leq |\hat{g}_\infty(\xi)| + \frac{|\hat{g}(\xi, t) - \hat{g}_\infty(\xi)|}{|\xi|^{2+\delta}} \leq 1 - \frac{\xi^2}{2} + o(\xi^2) + C e^{-|S_{2+\delta}(\delta)|t} |\xi|^{2+\delta}, \quad \xi \to 0.$$ 

This shows that for any $0 < k < \frac{1}{2}$ there exists $\rho > 0$ such that for all $t \geq 0$

$$|\hat{g}(\xi, t)| \leq 1 - k\xi^2, \quad |\xi| \leq \rho$$

or, which is equivalent, for any $0 < k < \frac{1}{2}$ there exists $\rho > 0$ such that for all $t \geq 0$

$$|\hat{g}(\xi, t)| \leq \frac{1}{1 + k\xi^2}, \quad |\xi| \leq \rho.$$

In order to prove (28), we would like to exploit again the $d_{2+\delta}$ distance. For $N \in \mathbb{N}$ and $j = 1, \ldots, N - 1$ let us estimate first $d_{2+\delta}(\varphi_j^{N+1}, \hat{g}_\infty)$. We recall that

$$0 = \frac{1}{r} \xi \frac{d}{d\xi} \hat{g}_\infty(\xi) + \hat{g}_\infty(p\xi) \hat{g}_\infty(q\xi) - \hat{g}_\infty(\xi)$$

(29)

with $\frac{1}{r} = \frac{1 - \nu^2 - \nu^2}{2}$. By considering the two equations (20) and (29) we have

$$\varphi_j^{N+1}(\xi) - \hat{g}_\infty(\xi) - (\varphi_j^N(\xi) - \hat{g}_\infty(\xi)) = \frac{1}{\Delta t} \left( \frac{d}{d\xi} \hat{\varphi}_j^{N+1}(\xi) - \frac{d}{d\xi} \hat{g}_\infty(\xi) \right)$$

$$+ \varphi_j^N(p\xi) \varphi_j^N(q\xi) - \hat{g}_\infty(p\xi) \hat{g}_\infty(q\xi) - (\varphi_j^N(\xi) - \hat{g}_\infty(\xi)).$$

In the same way as in Proposition [4] we obtain the integral form:

$$\varphi_j^{N+1}(\xi) - \hat{g}_\infty(\xi) = \frac{r}{\Delta t} \int_1^{t+\Delta t} \Delta t \left( \varphi_j^N(p\tau\xi) \varphi_j^N(q\tau\xi) - \hat{g}_\infty(p\tau\xi) \hat{g}_\infty(q\tau\xi) \right)$$

$$+ (1 - \Delta t) \left( \varphi_j^N(\tau\xi) - \hat{g}_\infty(\tau\xi) \right) \frac{d\tau}{\tau + \tau^2}.$$

Therefore for $\xi \neq 0$ we get:

$$\frac{|\varphi_j^{N+1}(\xi) - \hat{g}_\infty(\xi)|}{|\xi|^{2+\delta}} \leq \frac{r}{\Delta t} \int_1^{t+\Delta t} \left( \Delta t \frac{|\varphi_j^N(p\tau\xi) \varphi_j^N(q\tau\xi) - \hat{g}_\infty(p\tau\xi) \hat{g}_\infty(q\tau\xi)|}{|\xi|^{2+\delta}} + (1 - \Delta t) \frac{|\varphi_j^N(\tau\xi) - \hat{g}_\infty(\tau\xi)|}{|\xi|^{2+\delta}} \right) \frac{d\tau}{\tau + \tau^2}.$$

Now for $\tau \neq 0$

$$\frac{|\varphi_j^N(p\tau\xi) \varphi_j^N(q\tau\xi) - \hat{g}_\infty(p\tau\xi) \hat{g}_\infty(q\tau\xi)|}{|\xi|^{2+\delta}} \leq d_{2+\delta}(\varphi_j^N, g_\infty)(p^{2+\delta} + q^{2+\delta}) \tau^{2+\delta}$$

and so

$$\frac{|\varphi_j^{N+1}(\xi) - \hat{g}_\infty(\xi)|}{|\xi|^{2+\delta}} \leq \frac{r}{\Delta t} d_{2+\delta}(\varphi_j^N, g_\infty) \left( \int_1^{t+\Delta t} \left( \Delta t (p^{2+\delta} + q^{2+\delta}) + 1 - \Delta t \right) \tau^{2+\delta} \frac{d\tau}{\tau + \tau^2} \right)$$

$$\leq d_{2+\delta}(\varphi_j^N, g_\infty) \frac{r}{\Delta t} \int_1^{t+\Delta t} \left( \Delta t (p^{2+\delta} + q^{2+\delta} - 1) + 1 \right) \frac{d\tau}{\tau + \tau^2 - 1 - \delta}.$$
For \( N > \frac{T(2+\delta)}{\epsilon} \) we get
\[
d_{2+\delta}(\varphi^N_j, g_\infty) \leq d_{2+\delta}(\varphi^N_j, g_\infty) \frac{\sqrt{\Delta t}}{\sqrt{T}-(2+\delta)} \left( \Delta t(2^{2+\delta} + q^{2+\delta} - 1) + 1 \right)
\]
and remembering that
\[
S_{p,q}(\delta) = (p^{2+\delta} + q^{2+\delta} - 1) + \frac{2+\delta}{2}(1 - p^2 - q^2)
\]
we get
\[
\frac{\sqrt{\Delta t}}{\sqrt{T}-(2+\delta)} \left( \Delta t(p^{2+\delta} + q^{2+\delta} - 1) + 1 \right) = \frac{\Delta t(p^{2+\delta} + q^{2+\delta} - 1) + 1}{1 - \Delta t \frac{\Delta t}{2+\delta} (1 - p^2 - q^2)} = \frac{1 + S_{p,q}(\delta)}{1 - \frac{(\Delta t)^2}{2+\delta} (1 - p^2 - q^2)}.
\]
Since \( S_{p,q}(\delta) < 0 \) and
\[
1 - \frac{(\Delta t)^2}{2+\delta} (1 - p^2 - q^2) = 1 + o(\Delta t), \quad \Delta t \to 0,
\]
we get, for \( N \in \mathbb{N} \) large enough and \( j = 1, \ldots, N - 1 \):
\[
d_{2+\delta}(\varphi^N_j, g_\infty) \leq \left( 1 - \frac{|S_{p,q}(\delta)|}{2} \Delta t \right) d_{2+\delta}(\varphi^N_j, g_\infty) \leq d_{2+\delta}(\varphi^N_j, g_\infty).
\]
Recursively, we get
\[
d_{2+\delta}(\varphi^N_j, g_\infty) \leq d_{2+\delta}(g_0, g_\infty)
\]
where \( d_{2+\delta}(g_0, g_\infty) \) depends only on the quantities in \( \mathbb{C} \). Therefore, for \( j = 1, \ldots, N \) and \( \xi \neq 0 \) we get
\[
|\tilde{\varphi}^N_j(\xi)| \leq |\tilde{g}_\infty(\xi)| + |\xi|^{2+\delta} d_{2+\delta}(\varphi^N_j, g_\infty) \leq |\tilde{g}_\infty(\xi)| + |\xi|^{2+\delta} C
\]
for \( C = d_{2+\delta}(g_0, g_\infty) \) independent of \( N \) and \( j \). Since
\[
\tilde{g}_\infty(\xi) = 1 - \frac{\xi^2}{2} + o(\xi^2), \quad \xi \to 0
\]
we get that for any \( 0 < k < \frac{1}{2} \) there exists \( \rho > 0 \) such that for \( j = 0, \ldots, N \)
\[
|\tilde{\varphi}^N_j(\xi)| \leq \frac{1}{1 + k \xi^2}, \quad |\xi| \leq \rho
\]
which achieves the proof. \( \square \)

We are now in position to prove Theorem 1.

**Theorem 1**

Assume \( 0 < q \leq p \) satisfying \( p^2 + q^2 < 1 \) and such that there is \( \delta \in (0,1) \) for which \( S_{p,q}(\delta) < 0 \), for \( 0 < \delta < \delta \). Let \( g(t) \) be the weak solution of the equation \( \mathcal{D} \), corresponding to the initial density \( g_0 \) satisfying the normalization conditions \( \mathcal{G} \), and
\[
\int_{\mathbb{R}} |v|^{2+\delta} g_0(v) \, dv < +\infty.
\]
If in addition
\[
|\tilde{g}_0(\xi)| \leq \frac{1}{(1 + \beta|\xi|)^{\nu}}, \quad |\xi| > R,
\]
for some \( R > 0, \nu > 0 \) and \( \beta > 0 \), then there exist \( \rho > 0, k > 0, \beta' > 0, \nu' > 0 \) such that \( g(t) \) satisfies
\[
|\tilde{g}(\xi, t)| \leq \begin{cases} \frac{1}{1 + k \xi^2}, & |\xi| \leq \rho, \quad t \geq 0 \\ \frac{1}{(1 + \beta'|\xi|)^{\nu'}}, & |\xi| > \rho, \quad t \geq 0. \end{cases}
\]
Proof: The bound on the low frequencies $|\xi| \leq \rho$ has been established in Lemma \[14\]. Moreover, as a consequence of Proposition 2.4 in \[15\], we can suppose that condition (15) holds for any $|\xi| > \rho$ with a possibly smaller exponent $\nu'$. We will prove that for any $N \in \mathbb{N}$ and $j = 0, \ldots, N$ we get

$$|\phi_j^N(\xi)| \leq \frac{1}{(1 + \beta' |\xi|)^{\nu'}} \quad |\xi| > \rho$$

for positive $\nu'$ and $\beta'$ small enough. By induction we have only to check the bound on

$$\hat{\phi}_j^N(\xi) = \frac{r}{\Delta t} \int_1^{+\infty} [\Delta t \; \hat{g}_0(p\tau \xi) \hat{g}_0(q\tau \xi) + (1 - \Delta t) \; \hat{g}_0(\tau \xi)] \frac{1}{\tau^{|\alpha|+1}} d\tau$$

Let $|\xi| > \rho$ and $\tau > 1$. We are faced with three different situations.

Case I: If $|q\tau \xi| > \rho$, then

$$|\hat{g}_0(p\tau \xi) \hat{g}_0(q\tau \xi)| \leq \frac{1}{(1 + \beta |\xi|)^{\nu'}} \frac{1}{(1 + \beta |\xi|)^{\nu'}}$$

$$\leq \frac{1}{(1 + \beta |\xi|)^{\nu'}} \frac{1}{(1 + \beta |\xi|)^{\nu'}}$$

for $\beta' \leq \beta(p + q)$.

Case II: If $|p\tau \xi| > \rho$ and $|q\tau \xi| \leq \rho$, then

$$|\hat{g}_0(p\tau \xi) \hat{g}_0(q\tau \xi)| \leq \frac{1}{(1 + \beta |\xi|)^{\nu'}} \frac{1}{(1 + \beta |\xi|)^{\nu'}}$$

$$\leq \frac{1}{(1 + \beta |\xi|)^{\nu'}} \frac{1}{(1 + \beta |\xi|)^{\nu'}}$$

By choosing $\nu' > 0$ small enough we can show that

$$\frac{1}{(1 + \beta |\xi|)^{\nu'}} \frac{1}{(1 + \beta |\xi|)^{\nu'}} \leq \frac{1}{(1 + \beta |\xi|)^{\nu'}}.$$

Indeed, since $\frac{1 + \beta |\xi|}{1 + \beta |\xi|} \leq \frac{1}{\rho}$ for any $x \geq 0$ we obtain

$$\left(1 + \beta |\xi| \right)^\nu' \frac{1}{1 + kq^2 |\xi|^2} \leq \left(\frac{1}{\rho}\right)^\nu' \frac{1}{1 + kq^2 |\xi|^2}$$

$$\leq \left(\frac{1}{\rho}\right)^\nu' \frac{1}{1 + kq^2 \rho^2}.$$

Finally the last term is smaller than 1 for $\nu' \leq \log_\rho (1 + kq^2 \rho^2)$.

Case III: If $|p\tau \xi| \leq \rho$, then

$$|\hat{g}_0(p\tau \xi) \hat{g}_0(q\tau \xi)| \leq \frac{1}{(1 + k\rho^2 |\xi|^2)^{\nu'}} \frac{1}{(1 + k\rho^2 |\xi|^2)^{\nu'}}$$

$$\leq \frac{1}{(1 + k\rho^2 |\xi|^2)^{\nu'}} \frac{1}{(1 + k\rho^2 |\xi|^2)^{\nu'}}$$

and we have to show that

$$\frac{1}{(1 + k\rho^2 |\xi|^2)(1 + kq^2 |\xi|^2)} \leq \frac{1}{(1 + \beta |\xi|)^{\nu'}}.$$

Since

$$\frac{1}{(1 + k\rho^2 |\xi|^2)(1 + kq^2 |\xi|^2)} \leq \frac{1}{(1 + k(p^2 + q^2) |\xi|^2)}$$

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and \( p^2 + q^2 = C > 0 \), in order to establish the desired estimate we have only to prove that for \( \nu' \) small enough we have
\[
\frac{1}{1 + kC|\xi|^2} \leq \frac{1}{(1 + \beta|\xi|)^\nu'}
\]
or
\[
\nu' \log(1 + \beta|\xi|) \leq \log(1 + kC|\xi|^2)
\]
for any \(|\xi| > \rho\). This is true since the function
\[
F(\xi) = \frac{\log(1 + kC|\xi|^2)}{\log(1 + \beta|\xi|)}
\]
is bounded below by a positive constant on \(|\xi| > \rho\) and so by choosing \( \nu' > 0 \) smaller than this constant we get the desired estimate.

\[\square\]

**Remark 12**

The bounds in (16) are also equivalent to
\[
|\hat{g}(\xi, t)| \leq \frac{C}{(1 + \kappa|\xi|)\rho}, \quad \xi \in \mathbb{R}, \quad t \geq 0
\]
for some positive constants \( \kappa, \mu \) and \( C \) suitably chosen.

**Remark 13**

It wouldn’t have been possible to prove also the behavior of the solution \( g(t) \) on the low frequencies by induction and this is why we had to exploit the \( d_{2+\delta} \) convergence and the behavior of the steady state as in Lemma 11.

## 5 Smoothness of the steady state

In this section we prove the following Theorem 2.

**Theorem 2**

Assume \( 0 < q < p \) satisfying \( p^2 + q^2 < 1 \) and such that there is \( \delta \in (0, 1) \) for which \( S_{p,q}(\delta) < 0 \) for \( 0 < \delta < \delta \), so that a non-trivial steady state \( g_\infty \) to the Boltzmann equation (6) exists. Let us denote \( \lambda \in (0, 2) \) the exponent such that \( p^\lambda + q^\lambda = 1 \). Then \( g_\infty \) is a smooth function and belongs to the \( \lambda \)-th Gevrey class \( G^\lambda(\mathbb{R}) \), i.e.
\[
|\hat{g}_\infty(\xi)| \leq \exp \left( -\mu|\xi|^{\lambda} \right), \quad |\xi| > \rho
\]
with suitable positive numbers \( \rho \) and \( \mu \).

**Proof:** We are going to prove that \( g_\infty \in G^\lambda(\mathbb{R}) \), following a scheme already used in [22].

As in the proof of Lemma 11 we know the behavior of \( \hat{g}_\infty \) close to zero. In particular, by the same reasoning, there exist \( k \in (0, \frac{\delta}{2}) \) and \( \rho > 0 \) such that
\[
|\hat{g}_\infty(\xi)| \leq e^{-k\xi^2}, \quad |\xi| \leq \rho.
\]
As a consequence of Proposition 2.4 in [15] and without any loss in generality, we can also suppose \( \rho > 1 \). Let us remember that \( \hat{g}_\infty \) satisfies the equation
\[
\frac{1}{2} (p^2 + q^2 - 1) \xi \frac{d \hat{g}_\infty(\xi)}{d\xi} = \hat{g}_\infty(p\xi)\hat{g}_\infty(q\xi) - \hat{g}_\infty(\xi)
\]
and so, denoting again \( r = \frac{2}{1 - p^2 - q^2} \) and following [8], we get
\[
\hat{g}_\infty(\xi) = r \int_1^\infty \frac{\hat{g}_\infty(p\tau\xi)\hat{g}_\infty(q\tau\xi)}{\tau^{r+1}} d\tau, \quad \xi \in \mathbb{R}.
\]
Let us consider the space

\[ X_\rho = \{ \psi \in L^\infty(\mathbb{R}), \ |\psi(\xi)| \leq 1, \ \psi(\xi) = \hat{g}_\infty(\xi) \text{ for } |\xi| \leq \rho \} \]

dowed with the metric \( d_{2+\delta} \) defined in \( \mathbb{R}^n \), where \( \delta \) satisfies the assumptions of Theorem 2. The space \( X_\rho \) is a Fréchet space. Let us consider then the function \( R \) defined by

\[
R(\psi)(\xi) = \begin{cases} 
\hat{g}_\infty(\xi) & |\xi| \leq \rho \\
\int_1^\infty \frac{\psi(p\tau\xi)\psi(q\tau\xi)}{\tau^{r+1}} \, d\tau, & |\xi| > \rho.
\end{cases}
\]  

(30)

We are going to prove that \( R \) is a contraction on \( X_\rho \) and so \( \hat{g}_\infty \) is its unique fixed point. It is straightforward that \( R : X_\rho \to X_\rho \). As for the contractiveness, for \( \psi \) and \( \varphi \in X_\rho \) and \( |\xi| > \rho \) we have

\[
\frac{|R(\psi)(\xi) - R(\varphi)(\xi)|}{|\xi|^{2+\delta}} \leq \left( r \int_1^\infty \frac{\tau^{2+\delta}(p^{2+\delta} + q^{2+\delta})}{\tau^{r+1}} \, d\tau \right) d_{2+\delta}(\psi, \varphi)
\]

\[
\leq \left( r \frac{(p^{2+\delta} + q^{2+\delta})}{r - 2 - \delta} \right) d_{2+\delta}(\psi, \varphi).
\]

We remark that the assumption

\[
S_{p,q}(\delta) = p^{2+\delta} + q^{2+\delta} - 1 + \frac{2+\delta}{2}(1 - p^2 - q^2) < 0
\]

implies

\[
\frac{2+\delta}{2}(1 - p^2 - q^2) < 1 - (p^{2+\delta} + q^{2+\delta}) < 1
\]

and so,

\[
\frac{\delta}{2}(1 - p^2 - q^2) < p^2 + q^2
\]

which is precisely

\[
r - 1 - \delta > 1.
\]

So we get

\[
d_{2+\delta}(R(\psi), R(\varphi)) \leq \frac{r(p^{2+\delta} + q^{2+\delta})}{r - 2 - \delta} d_{2+\delta}(\psi, \varphi).
\]

Remembering that \( r = \frac{2}{1 - p^2 - q^2} \), we get

\[
\frac{r(p^{2+\delta} + q^{2+\delta})}{r - 2 - \delta} = \frac{2}{1 - p^2 - q^2} \left( p^{2+\delta} + q^{2+\delta} \right) \frac{1}{1 - \frac{q}{p^2 + q^2} - 2 - \delta} < 1 \iff S_{p,q}(\delta) < 0
\]

and this allows to conclude. Choosing \( \psi_0 \in X_\rho \) for example as

\[
\psi_0(\xi) = \begin{cases} 
\hat{g}_\infty(\xi) & |\xi| \leq \rho \\
0 & |\xi| > \rho
\end{cases}
\]

and defining by induction for \( n \geq 0 \)

\[
\psi_{n+1}(\xi) = \begin{cases} 
\hat{g}_\infty(\xi) & |\xi| \leq \rho \\
\int_1^\infty \frac{\psi_n(p\tau\xi)\psi_n(q\tau\xi)}{\tau^{r+1}} \, d\tau, & |\xi| > \rho,
\end{cases}
\]

we get automatically \( d_{2+\delta}(\psi_n, \hat{g}_\infty) \to 0 \) for \( n \to +\infty \), which implies convergence pointwise. We show now that there exists \( \mu > 0 \) such that for all \( n \in \mathbb{N} \)

\[
|\psi_n(\xi)| \leq e^{-\mu|\xi|^\lambda}, \quad |\xi| > \rho
\]

and this uniform estimate passes therefore to the limit and allows to conclude. The only thing to control is that for \( \tau > 1 \) and \( |\xi| > \rho \) we have

\[
|\psi_0(p\tau\xi)\psi_0(q\tau\xi)| \leq e^{-\mu|\xi|^\lambda}.
\]

We distinguish three cases:
Case I: If $|q| > p$, since $p^\lambda + q^\lambda = 1$ and $p > 1$ we get
\[
e^{\mu|\xi|^\lambda} \cdot |\psi_0(p\tau \xi)| \cdot |\psi_0(q\tau \xi)| \leq e^{\mu|\xi|^\lambda(1 - r^\lambda(p^\lambda + q^\lambda))} \leq e^{\mu|\xi|^\lambda(1 - r^\lambda)} \leq 1.
\]

Case II: If $p r |\xi| \leq p$, then $|\xi| > 1$ implies $\xi^2 \geq |\xi|^\lambda$. Denoting $p^2 + q^2 = C$, we conclude
\[
e^{\mu|\xi|^\lambda} \cdot |\psi_0(p\tau \xi)| \cdot |\psi_0(q\tau \xi)| \leq e^{\mu|\xi|^\lambda - k^2 q^2 (p^2 + q^2)} \leq e^{\mu|\xi|^\lambda - k^2 |\xi|^2} \leq e^{\mu|\xi|^\lambda} \leq 1,
\]
provided that $\mu \leq k C$.

Case III: Now assume that $q r |\xi| \leq p$ while $p r |\xi| > p$. Using the condition $p^\lambda + q^\lambda = 1$ once again, one finds
\[
e^{\mu|\xi|^\lambda} \cdot |\psi_0(p\tau \xi)| \cdot |\psi_0(q\tau \xi)| \leq e^{\mu|\xi|^\lambda - k^2 q^2 (p^2 + q^2)} \leq e^{\mu|\xi|^\lambda - k^2 |\xi|^2} \leq e^{\mu|\xi|^\lambda} \leq 1
\]
provided that $\mu \leq \frac{k^2 q^2}{q^2}$.

\[
\square
\]

Remark 14
By the proof of Theorem 3, the condition $g_{\infty} \in G^\lambda(\mathbb{R})$ seems to be sharp, since in the estimate of Case I, it wouldn’t have been possible to replace $\lambda$ by $\sigma > \lambda$. Moreover, in the case $p + q = 1$ the explicit stationary state is $\hat{g}_{\infty}(\xi) = (1 + |\xi|) e^{-|\xi|}$ and this proves sharpness at least in this special case. It is worth noticing that Bobylev and Cercignani in [8] (Theorem 5.3) proved that for $p + q > 1$ and $p^2 + q^2 < 1$ the stationary state $g_{\infty}$ satisfies the bounds
\[
e^{-\frac{\xi^2}{2}} \leq |\hat{g}_{\infty}(\xi)| \leq (1 + |\xi|) e^{-\frac{\xi^2}{2}}.
\]
For the values of $p$ and $q$ for which in addition there is $\tilde{\delta} \in (0, 1)$ such that $S_{p,q}(\delta) < 0$ for $0 < \delta < \tilde{\delta}$, our result improves the upper bound and gives also a new result for some $p$ and $q$ in the range $p + q < 1$.

By a careful reading of both proofs of Theorem 1 and Theorem 2, we can deduce that not only polynomial tails of the Fourier transform of the initial data $g_0$ are uniformly propagated by the solution $g(t)$, but also exponential ones, as long as the exponent does not exceed the exponent $\lambda$ characterizing the mixing parameters $p$ and $q$. More precisely, the result is as follows.

Theorem 3
Assume $0 < q \leq p$ satisfying $p^2 + q^2 < 1$ and such that there is $\tilde{\delta} \in (0, 1)$ for which $S_{p,q}(\delta) < 0$ for $0 < \delta < \tilde{\delta}$. Let us denote $\lambda \in (0, 2)$ the exponent such that $p^\lambda + q^\lambda = 1$. Let $g(t)$ be the weak solution of the equation 12, corresponding to the initial density $g_0$ satisfying the normalization conditions 13, and
\[
\int_R |v|^{2+\lambda} g_0(v) \, dv < +\infty.
\]
If in addition
\[
|\hat{g_0}(\xi)| \leq e^{-\beta|\xi|^{\nu}}, \quad |\xi| > R,
\]
for some $R > 0$, $\nu > 0$ and $\beta > 0$, then there exist $\rho > 0$ and $\kappa > 0$, such that $g(t)$ satisfies
\[
|\hat{g}(\xi,t)| \leq \begin{cases} e^{-\kappa t^2}, & |\xi| \leq \rho, \quad t \geq 0, \\ e^{-\gamma |\xi|^{\min(\nu,\lambda)}}, & |\xi| > \rho, \quad t \geq 0. \end{cases}
\]
The proof follows the same lines as the proof of Theorem\ref{thm:convergence} replacing the polynomial decreasing by the exponential one. The key argument for the low frequencies is that the normalization assumptions \eqref{eq:norm} on the steady state imply

$$\dot{g}_\infty(\xi) = 1 - \frac{\xi^2}{2} + o(\xi^2), \quad \xi \to 0,$$

which means

$$|\dot{g}_\infty(\xi)| \leq \frac{1}{1 + k\xi^2}, \quad |\xi| \leq \rho$$

or equivalently

$$|\dot{g}_\infty(\xi)| \leq e^{-k\xi^2}, \quad |\xi| \leq \rho$$

for any $k \in (0, \frac{1}{\rho}]$ and $\rho$ depending on $k$. The whole proof follows then without any particular difficulty, exploiting for the high frequencies the estimates performed in the proof of the Gevrey regularity of the steady state.

\section{Strong convergence}

In this section, we are going to prove Theorem\ref{thm:strong} on the strong $L^1$ convergence of the scaled solution $g(t)$ to the stationary state $g_\infty$.

\begin{thm}
Assume $0 < q \leq p$ satisfying $p^2 + q^2 < 1$ and such that there exists $\tilde{\delta} \in (0, 1)$ for which $S_{p,q}(\delta) < 0$ for $0 < \delta < \tilde{\delta}$ and let $g_\infty$ be the unique stationary solution of \eqref{eq:G}. Let the initial density $g_0$ satisfy the normalization conditions \eqref{eq:norm}, and

$$\int_\mathbb{R} |v|^{2+\delta} g_0(v) \, dv < +\infty.$$

If in addition $g_0 \in H^\eta(\mathbb{R})$ for some $\eta > 0$, $\sqrt{g_0} \in H^\nu(\mathbb{R})$ for some $\nu > 0$, then the solution $g(t)$ of \eqref{eq:G} converges strongly in $L^1$ with an exponential rate towards the stationary solution $g_\infty$, i.e., there exist positive constants $C$ and $\gamma$ explicitly computable such that

$$\|g(t) - g_\infty\|_{L^1(\mathbb{R})} \leq C e^{-\gamma t}, \quad t \geq 0.$$

Let us begin by the following lemma.

\begin{lem}
Let the initial density $g_0$ satisfy the normalization conditions \eqref{eq:norm}, and

$$\int_\mathbb{R} |v|^{2+\delta} g_0(v) \, dv < +\infty.$$

If in addition $\sqrt{g_0} \in H^\nu(\mathbb{R})$ for some $\nu > 0$, then $g_0$ satisfies

$$|\dot{g}_0(\xi)| \leq \frac{C}{(1 + \beta|\xi|)^\nu}, \quad \xi \in \mathbb{R}$$

for positive constants $C$ and $\beta$ and the solution $g(t)$ of \eqref{eq:G} satisfies

$$\sup_{\xi \in \mathbb{R}} |\dot{g}(\xi, t) - \dot{g}_\infty(\xi)| \leq C_1 e^{-C_2 t}; \quad t \geq 0 \tag{31}$$

for positive constants $C_1$ and $C_2$.

\begin{prf}
Since $g_0 = \sqrt{g_0} \sqrt{g_0}$, then $\dot{g}_0 = \dot{\sqrt{g_0}} \sqrt{g_0}$. So, for $\xi \in \mathbb{R}$ we get

$$|\xi|^\nu |\dot{g}_0(\xi)| \leq \int_\mathbb{R} |\xi|^\nu \left| \sqrt{g_0}(\xi - \tau) \sqrt{g_0}(\tau) \right| d\tau \leq K \int_\mathbb{R} \left( |\xi - \tau|^\nu + |\tau|^\nu \right) \left| \sqrt{g_0}(\xi - \tau) \sqrt{g_0}(\tau) \right| d\tau \leq 2K \|\sqrt{g_0}\|_{H^\nu}.$$

\end{prf}
Since moreover \(|\hat{g}_0(\xi)| \leq 1\), we can find positive \(C\) and \(\beta\) such that
\[
|\hat{g}_0(\xi)| \leq \frac{C}{(1 + \beta|\xi|)^{\gamma}}, \quad \xi \in \mathbb{R}.
\]

Thanks to Theorem 1 and to Remark 12 we get
\[
|\hat{g}(\xi, t)| \leq \frac{\tilde{C}}{(1 + \kappa|\xi|)^{\mu}}, \quad \xi \in \mathbb{R}, \quad t \geq 0
\]
for suitable \(\tilde{C}\), \(\kappa\) and \(\mu\). The steady state \(g_\infty\) belongs to a Gevrey class, so it satisfies an analogous estimate, with suitable constants which we can suppose to be the same. Let now \(R > 0\) to be chosen in a moment. We get, for \(\xi \in \mathbb{R}:
\[
|\hat{g}(\xi, t) - \hat{g}_\infty(\xi)| \leq d_{2+\delta}(g(t), g_\infty)R^{2+\delta} + \frac{2\tilde{C}}{(\kappa R)^{\mu}}.
\]
which implies, optimizing over \(R\),
\[
|\hat{g}(\xi, t) - \hat{g}_\infty(\xi)| \leq C_1d_{2+\delta}(g(t), g_\infty)^{\mu/(2+\delta+\mu)} = C_1e^{-C_2t}, \quad \xi \in \mathbb{R}, \quad t \geq 0.
\]
for \(C_1\) and \(C_2\) positive constants.

**Proof of Theorem 4** The result consists in converting the weak convergence in the Fourier distance of the solution \(g(t)\) to the stationary state \(g_\infty\) (Theorem 3) into a \(L^1\) convergence by interpolating this weak distance \(d_{2+\delta}\) with the uniformly boundedness in time of suitable moment and Sobolev norm of the solution itself. The only missing ingredient at this point is the boundedness of the Sobolev norm and we will go through the proof of it in a moment. Let us recall first how we can interpolate these results, following the scheme introduced in 4.2 in [12]: for \(\delta \in (0, \tilde{\delta})\), there exists a positive constant \(\gamma\) such that
\[
\|h\|_{L^1} \leq C\|\nu^{2+\delta}h\|_{L^1}^{\frac{\tilde{\delta}}{\tilde{\delta}+2\delta+\gamma}}\|h\|_{L^2}^{\frac{2\delta+\gamma}{\tilde{\delta}+2\delta+\gamma}}
\]
and for any \(s \geq 0\) there exist positive constants \(M, N, \beta\) and \(\gamma\) such that
\[
\|h\|_{H^s} \leq C\left(\sup_{\xi \in \mathbb{R}} \frac{|\hat{h}(\xi)|}{\xi^{2+\delta}}\right)^{\beta} (\|h\|_{H^M} + \|h\|_{H^N})^{\gamma}.
\]
(32)

So, letting \(h = g(t) - g_\infty\), and \(s = 0\) we get
\[
\|g(t) - g_\infty\|_{L^1} \leq C\left(\|\nu^{2+\delta}g(t)\|_{L^1} + \|\nu^{2+\delta}g_\infty\|_{L^1}\right)^{\tilde{\beta}} d_{2+\delta}(g(t), g_\infty)^{\tilde{\gamma}} (\|g(t)\|_{H^M} + \|g(t)\|_{H^N} + \|g_\infty\|_{H^M} + \|g_\infty\|_{H^N})^{\gamma}
\]
for suitable exponents \(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\). Concerning the stationary state \(g_\infty\), it have been proved in Theorem 6 that \(\|\nu^{2+\delta}g_\infty\|_{L^1} < \infty\) for \(\delta \in (0, \tilde{\delta})\) and thanks to the Gevrey regularity, we have \(g_\infty \in H^s(\mathbb{R})\) for all \(s \geq 0\). As for the scaled solution \(g(t)\), the uniform boundedness of the \((2+\delta)\)-th moment have been also proved in Theorem 4 so we will get the \(L^1\) exponential convergence as soon as we prove the uniform boundedness in time of \(g(t)\) in a suitable Sobolev space \(H^{\max(M,N)}(\mathbb{R})\). Of course, we need to assume \(g_0 \in H^{\max(M,N)}(\mathbb{R})\). As a byproduct of the uniform boundedness of Sobolev norms, we will also get from (32) the convergence of \(g(t)\) to \(g_\infty\) in Sobolev spaces.

Let us recall how to prove the uniform boundedness of \(g(t)\) in a generic homogenous Sobolev space \(H^\eta\) for \(\eta \geq 0\) under the assumption \(g_0 \in \hat{H}^\eta\). First of all, let us remark that
$g(t) \in H^\alpha$ for all $t > 0$, without any uniformity in time. Indeed, coming back to the original non scaled solution $f(t)$ we get
\[
\frac{d}{dt} \|f(t)\|^2_{H^\alpha} = \frac{d}{dt} \int_R |\xi|^{2\eta} \hat{f}(\xi, t) \hat{f}(\xi, t) \, d\xi
\]
\[
= 2 \int_R |\xi|^{2\eta} \text{Re} \left( \hat{f}(\xi, t) \partial_\xi \hat{f}(\xi, t) \right) \, d\xi
\]
\[
= 2 \int_R |\xi|^{2\eta} \text{Re} \left( \hat{f}(\xi, t) \left( \hat{f}(p_\xi, t) \hat{f}(q_\xi, t) - \hat{f}(\xi, t) \right) \right) \, d\xi
\]
\[
= -2 \int_R |\xi|^{2\eta} |\hat{f}(\xi, t)|^2 \, d\xi + 2 \int_R |\xi|^{2\eta} \text{Re} \left( \hat{f}(\xi, t) \hat{f}(p_\xi, t) \hat{f}(q_\xi, t) \right) \, d\xi
\]
\[
\leq -1 \int_R |\xi|^{2\eta} |\hat{f}(\xi, t)|^2 \, d\xi + \int_R |\xi|^{2\eta} |\hat{f}(p_\xi, t)\hat{f}(q_\xi, t)|^2 \, d\xi
\]
\[
\leq -1 \int_R |\xi|^{2\eta} |\hat{f}(\xi, t)|^2 \, d\xi + \frac{1}{2} \left( \frac{1}{q^{2\eta+1}} + \frac{1}{p^{2\eta+1}} \right) \int_R |\xi|^{2\eta} |\hat{f}(\xi, t)|^2 \, d\xi.
\]
Since $\hat{g}(e^{\frac{t^2}{2} + \frac{q^2}{2} + \frac{1}{2}} \xi, t) = \hat{f}(\xi, t)$ we get
\[
\frac{d}{dt} \left( \frac{1-p^2-q^2}{2} (2\eta+1) t \|g(t)\|^2_{H^\alpha} \right) \leq \left( -1 + \frac{1}{2} \left( \frac{1}{q^{2\eta+1}} + \frac{1}{p^{2\eta+1}} \right) \right) \frac{1-p^2-q^2}{2} (2\eta+1) t \|g(t)\|^2_{H^\alpha}
\]
and so
\[
\frac{d}{dt} \|g(t)\|^2_{H^\alpha} \leq \left[ -1 - \frac{1-p^2-q^2}{2} (2\eta+1) + \frac{1}{2} \left( \frac{1}{q^{2\eta+1}} + \frac{1}{p^{2\eta+1}} \right) \right] \|g(t)\|^2_{H^\alpha} = C_{p,q,\eta} \|g(t)\|^2_{H^\alpha},
\]
which leads to
\[
\|g(t)\|^2_{H^\alpha} \leq \|g_0\|^2_{H^\alpha} e^{C_{p,q,\eta} t}.
\]
Since $p^2 + q^2 < 1$, it is not difficult to be convinced that $C_{p,q,\eta} > 0$ as soon as for example $q$ is small enough.

Let us make estimate (33) more accurate. The goal is to get for example the following differential inequality: for two positive constants $H$ and $K$ and $t_0 > 0$:
\[
\frac{d}{dt} \|g(t)\|^2_{H^\alpha} \leq -H \|g(t)\|^2_{H^\alpha} + K, \quad t \geq t_0
\]
(34)
so that
\[
\|g(t)\|^2_{H^\alpha} \leq C \max(\|g(t_0)\|^2_{H^\alpha}, 1), \quad t \geq t_0.
\]
Let us come back to inequality
\[
\frac{d}{dt} \|f(t)\|^2_{H^\alpha} \leq -1 \int_R |\xi|^{2\eta} |\hat{f}(\xi, t)|^2 \, d\xi + \int_R |\xi|^{2\eta} |\hat{f}(p_\xi, t)|^2 |\hat{f}(q_\xi, t)|^2 \, d\xi
\]
which reads, on the scaled solution $g(t)$,
\[
\frac{d}{dt} \left( \frac{1-p^2-q^2}{2} (2\eta+1) t \|g(t)\|^2_{H^\alpha} \right)
\]
\[
\leq -e^{-\frac{1-p^2-q^2}{2} (2\eta+1) t} \int_R |\xi|^{2\eta} |\hat{g}(\xi, t)|^2 \, d\xi + e^{-\frac{1-p^2-q^2}{2} (2\eta+1) t} \int_R |\xi|^{2\eta} |\hat{g}(p_\xi, t)|^2 |\hat{g}(q_\xi, t)|^2 \, d\xi
\]
and so
\[
\frac{d}{dt} \|g(t)\|^2_{H^\alpha} \leq \left( -1 - \frac{1-p^2-q^2}{2} (2\eta+1) \right) \|g(t)\|^2_{H^\alpha} + \int_R |\xi|^{2\eta} |\hat{g}(p_\xi, t)|^2 |\hat{g}(q_\xi, t)|^2 \, d\xi.
\]
Since $p^2 + q^2 < 1$, it would be enough to obtain for example the following inequality
\[
\int_R |\xi|^{2\eta} |\hat{g}(p_\xi, t)|^2 |\hat{g}(q_\xi, t)|^2 \, d\xi \leq \frac{1}{2} \|g(t)\|^2_{H^\alpha} + K, \quad t \geq t_0
\]
(35)
where \( K > 0 \) is independent of \( t \). Let us prove inequality \((34)\). We split the integral in \((33)\) into two parts

\[
\int_{\mathbb{R}} |\xi|^{2\eta} \hat{g}(p\xi, t)|^2 \hat{g}(q\xi, t)|^2 \, d\xi = \int_{|\xi| \leq R} + \int_{|\xi| > R} = A + B
\]

where \( R \) will be chosen later. Let us estimate first the term in \( A \) (we will denote \( \varepsilon \) a constant which is allowed to vary from one line to another, depending at most on \( p \) and \( q \)). Since \(|\hat{g}(\xi, t)| \leq 1\) for \( \xi \in \mathbb{R} \) and \( t \geq 0 \) we simply get

\[
\int_{|\xi| \leq R} |\xi|^{2\eta} |\hat{g}(p\xi, t)|^2 |\hat{g}(q\xi, t)|^2 \, d\xi \leq \int_{|\xi| \leq R} |\xi|^{2\eta} \, d\xi = \frac{2}{2\eta + 1} R^{2\eta + 1}, \quad t \geq 0.
\]

Let us come to the term in \( B \), where we are going to exploit Lemma \([15]\). We remark that \( \hat{g}_\infty(\xi) \to 0 \) for \( \xi \to +\infty \) and so, by Lemma \([15]\) for any \( \varepsilon > 0 \) there exist \( R > 0 \) and \( t_0 \) depending on \( \varepsilon \) and \( p \) such that

\[
|\hat{g}(p\xi, t)| \leq |\hat{g}(p\xi, t) - \hat{g}_\infty(p\xi)| + |\hat{g}_\infty(p\xi)| \leq 2\varepsilon, \quad |\xi| > R, \quad t \geq t_0.
\]

We can deduce for \( t \geq t_0 \):

\[
\int_{|\xi| > R} |\xi|^{2\eta} |\hat{g}(p\xi, t)|^2 |\hat{g}(q\xi, t)|^2 \, d\xi \leq (2\varepsilon)^2 \int_{|\xi| > R} |\xi|^{2\eta} |\hat{g}(q\xi, t)|^2 \, d\xi \leq \frac{\varepsilon}{q^{2\eta + 1}} \int_{\mathbb{R}} |\xi|^{2\eta} |\hat{g}(\xi, t)|^2 \, d\xi = \varepsilon \|g(t)\|_{H^{\eta}}^2.
\]

We have obtained

\[
\int_{\mathbb{R}} |\xi|^{2\eta} |\hat{g}(p\xi, t)|^2 |\hat{g}(q\xi, t)|^2 \, d\xi \leq \varepsilon \|g(t)\|_{H^{\eta}}^2 + \frac{2}{2\eta + 1} R^{2\eta + 1}, \quad t \geq t_0.
\]

Letting \( \varepsilon \) be fixed such that \( \varepsilon \leq \frac{1}{2} \), we get the desired estimate.

\[\square\]

### 7 Lyapunov functionals and open questions

The case \( p + q = 1 \) separates in a natural way from the others. It corresponds to a one-dimensional dissipative Boltzmann equation in which the momentum is preserved in a microscopic collision of type \([1]\). Equation \((2)\) with \( p + q = 1 \) as been intensively studied in a series of papers \([1, 8, 24]\). Among other properties, this model possesses an explicit self similar solution, which has been first discovered in \([1]\). In fact, condition \( p + q = 1 \) implies

\[
\frac{1}{2} (p^2 + q^2 - 1) = -pq.
\]

Hence the Fourier transformed version of the scaled equation \((6)\) can be written as

\[
\hat{g}(p\xi, t)\hat{g}(q\xi, t) - \hat{g}(\xi, t) = \partial_\xi \hat{g}(\xi, t) - pq\xi \partial_\xi \hat{g}(\xi, t).
\]

The choice

\[
\hat{g}_\infty(\xi) = (1 + |\xi|) e^{-|\xi|}
\]

leads to

\[
\hat{g}_\infty(p\xi)\hat{g}_\infty(q\xi) - \hat{g}_\infty(\xi) = pq|\xi|^2 e^{-|\xi|} = -pq\xi \partial_\xi \hat{g}_\infty(\xi)
\]

and so \( \hat{g}_\infty \) solves \((36)\) as a stationary solution for any choice of the parameters \( p \) and \( q \) such that \( p + q = 1 \). It can be easily verified that in physical variables the steady solution reads

\[
g_\infty(v) = \frac{2}{\pi(1 + v^2)}.
\]
Note that this function satisfies the normalization conditions \textcolor{red}{[3]}. Under these constraints, however, it can be shown \textcolor{red}{[28] that } $g_\infty$ is the (unique) minimizer of the convex functional

$$H(f) = -\int_\mathbb{R} \sqrt{f(v)} \, dv.$$ \hfill (38)

It is a natural question to investigate whether the functional $H$ is a Lyapunov functional for the scaled equation for $g(v,t)$, which can be formally written as

$$\partial_t g(v,t) = g_p * g_q(v,t) - g(v,t) + \frac{1}{2} (p^2 + q^2 - 1) \partial_v (vg(v,t)). \hfill (39)$$

The results of both Section \textcolor{red}{[4]} and Section \textcolor{red}{[8]} lead to conclude that, under suitable regularity assumptions on the initial value, one can study the time derivative of the functional $H(g)(t)$ along solutions to equation \textcolor{red}{(39)}, obtaining

$$\frac{d}{dt} H(g)(t) = \frac{1}{2} \left\{ \int_\mathbb{R} g_p * g_q(v,t) \, dv - \frac{1 + p^2 + q^2}{2} \int_\mathbb{R} \sqrt{g(v,t)} \, dv \right\}. \hfill (40)$$

Hence, the functional $H$ is a Lyapunov functional for equation \textcolor{red}{(39)} provided the inequality

$$\frac{1 + p^2 + q^2}{2} \int_\mathbb{R} \sqrt{g(v)} \, dv \leq \int_\mathbb{R} g_p * g_q(v) \sqrt{g(v)} \, dv, \quad p + q = 1, \hfill (41)$$

is verified for all functions satisfying constraints \textcolor{red}{[3]}. We remark that inequality \textcolor{red}{(41)} is saturated by choosing $g = g_\infty$, with $g_\infty$ defined as in \textcolor{red}{[52]}. To our knowledge, this inequality has never been investigated before, but it can be conjectured that it holds true, even if we are not able to prove it.

A different way to attach the problem is to resort to the Fourier version of equation \textcolor{red}{(39)}. This idea has been fruitfully employed in \textcolor{red}{[9]} to recover Lyapunov functionals for the Boltzmann equation for Maxwell molecules. Let us consider the approximate solution \textcolor{red}{(13)} which is a convex combination of the probability densities $\hat{g}(\xi,t)$ and $\hat{g}(p\xi,t)\hat{g}(q\xi,t)$. For any convex functional $\tilde{H}$ acting on $\hat{g}$ we obtain

$$\tilde{H}(\hat{g}(\xi,t+\Delta t)) \leq \Delta t \int_1^{t+\infty} \left( \Delta t \tilde{H}(\hat{g}(p\xi,t)\hat{g}(q\xi,t)) + (1 - \Delta t) \tilde{H}(\hat{g}(\xi,t)) \right) \frac{dr}{r^{2r+1}}. \hfill (42)$$

If $\tilde{H}(\hat{g}) = H(g)$ is defined by \textcolor{red}{(38)}, we have

$$\tilde{H}(\hat{g}(\xi)) = \sqrt{\Delta t} \tilde{H}(\hat{g}(\xi)),$$

and this implies that inequality \textcolor{red}{(41)} becomes

$$\tilde{H}(\hat{g}(\xi,t+\Delta t)) \leq \Delta t \int \left( \Delta t \tilde{H}(\hat{g}(p\xi,t)\hat{g}(q\xi,t)) + (1 - \Delta t) \tilde{H}(\hat{g}(\xi,t)) \right).$$

Let us suppose that there exists $A(p,q)$ such that for all functions $g$ satisfying conditions \textcolor{red}{[3]} we get

$$\tilde{H}(\hat{g}(p\xi)\hat{g}(q\xi)) \leq A(p,q) \tilde{H}(\hat{g}(\xi)). \hfill (43)$$

Then, since

$$\frac{1}{r} = \frac{1 - p^2 - q^2}{2},$$

the condition

$$A(p,q) = \frac{3 + p^2 + q^2}{4}$$

would imply

$$\Delta t A(p,q) + (1 - \Delta t) = 1.$$ 

In this case, inequality \textcolor{red}{(41)} would imply

$$\tilde{H}(\hat{g}(\xi,t+\Delta t)) \leq \tilde{H}(\hat{g}(\xi,t)).$$
and $H$ would be a Lyapunov functional. Note that inequality $\text{(12)}$ corresponds to a reverse Young inequality first derived by Leindler [19]: for $0 < \alpha, \beta, \rho \leq 1$ and $f, g$ non-negative
\[
\|f * g\|_\rho \geq \|f\|_\alpha \|g\|_\beta, \quad 1/\alpha + 1/\beta = 1 + 1/\rho. \tag{44}
\]
In our case, $\rho = \alpha = 1/2$, $\beta = 1$ together with the second condition in [3] implies
\[
\|f_p * f_q\|_{1/2} \geq p\|f\|_{1/2},
\]
namely inequality $\text{44}$ with $A(p, q) = p$. Unlike, the direct application of inequality $\text{(44)}$ is not enough to obtain $\text{(12)}$. It remains an open question to prove that, under constraints [3] it holds the Young-type reverse inequality
\[
\|f_p * f_q\|_{1/2} \geq A(p, q)\|f\|_{1/2},
\]
where $p + q = 1$ and $A(p, q)$ is given by $\text{46}$.

8 Appendix

Proof of Proposition [8]. We will consider only the dissipative case $p^2 + q^2 < 1$, since the other case adapts straightforwardly. It is easy to get for all $N \in \mathbb{N}$ and $j = 0, \ldots, N - 1$:
\[
\varphi_j^N(v) \geq 0, \quad \int_R \varphi_j^N(v) \, dv = 1 \quad \int_R v \varphi_j^N(v) \, dv = 0. \tag{45}
\]
Let us consider therefore the evolution of the two other moments of the sequence $\varphi_j^N$. First of all, it is worth noticing that for any function $h \in L^1(\mathbb{R})$, $\alpha > 0$ and $\tau \neq 0$ we have
\[
\int_R |v|^\alpha \frac{1}{|\tau|} |h(\frac{v}{\tau})| \, dv = |\tau|^\alpha \|v|^\alpha h\|_{L^1}. \tag{46}
\]
Let us compute now the second moment of $\varphi_j^{N+1}$:
\[
\int_R v^2 \varphi_j^{N+1}(v) \, dv = \frac{r}{\Delta t} \int_1^{\infty} \left[ \Delta t \left( \int_R v^2 \frac{1}{\tau} \varphi_j^{N} \left( \frac{v}{\tau} \right) \, dv \right) + (1 - \Delta t) \left( \int_R v^2 \frac{1}{\tau} \varphi_j^{N} \left( \frac{v}{\tau} \right) \, dv \right) \right] \frac{d\tau}{\tau^{N+1}}
\]
\[
= \frac{r}{\Delta t} \left( \Delta t \|v^2 \varphi_j^{N} \varphi_j^{N} \|_{L^1} + (1 - \Delta t) \|v^2 \varphi_j^{N} \|_{L^1} \right) \int_1^{\infty} \frac{r^2 d\tau}{\tau^{N+1}}.
\]
For $N > \frac{2r}{\Delta t}$ we get
\[
\frac{r}{\Delta t} \int_1^{\infty} \frac{d\tau}{\tau^{N+1}} = 1 + \frac{2r}{\Delta t} = \frac{1}{1 + \Delta t(p^2 + q^2 - 1)}
\]
and so we are left with $\|v^2 \varphi_j^{N} \varphi_j^{N} \|_{L^1}$. We have
\[
\left( \int_{\mathbb{R}^2} v^2 \left( \frac{u - w}{p} \right) \frac{1}{q} \varphi_j^N \left( \frac{u}{q} \right) \, dw \right) = \int_{\mathbb{R}^2} \left( (v-w)^2 + 2(v-w)w \right) \frac{1}{p} \varphi_j^N \left( \frac{v-w}{p} \right) \frac{1}{q} \varphi_j^N \left( \frac{w}{q} \right) \, dw dv.
\]
Thanks to $\text{45}$, we get
\[
\|v^2 \varphi_j^{N} \varphi_j^{N} \|_{L^1} = (p^2 + q^2) \|v^2 \varphi_j^{N} \|_{L^1}
\]
and we end up with
\[
\|v^2 \varphi_j^{N+1} \|_{L^1} = \frac{\Delta t(p^2 + q^2 - 1) + 1}{1 + \Delta t(p^2 + q^2 - 1)} \|v^2 \varphi_j^{N} \|_{L^1} = \|v^2 \varphi_j^{N} \|_{L^1}.
\]
and so by a recursive procedure

\[ \int_{\mathbb{R}} v^2 \varphi_{j+1}^N(v) \, dv = 1, \quad j = 0, \ldots, N - 1. \]  

(47)

As for \( \int_{\mathbb{R}} |v|^{2+\delta} \varphi_{j+1}^N(v) \, dv \), we proceed in the same way:

\[
\int_{\mathbb{R}} |v|^{2+\delta} \varphi_{j+1}^N(v) \, dv \\
= \frac{r}{\Delta t} \int_1^{+\infty} \left[ \Delta t \left( \int_{\mathbb{R}} |v|^{2+\delta} \frac{1}{\tau} \left( \varphi_{j,p}^N * \varphi_{j,q}^N \right) \left( \frac{v}{\tau} \right) \, dv \right) + \right. \\
+ (1 - \Delta t) \left( \int_{\mathbb{R}} |v|^{2+\delta} \frac{1}{\tau} \varphi_j^N \left( \frac{v}{\tau} \right) \, dv \right) \right] \frac{d\tau}{\tau^{\frac{3}{2} + 1}} \\
= \frac{r}{\Delta t} \left( \Delta t \| |v|^{2+\delta} \varphi_{j,p}^N * \varphi_{j,q}^N \|_{L^1} + (1 - \Delta t) \| |v|^{2+\delta} \varphi_j^N \|_{L^1} \right) \int_1^{+\infty} \frac{\tau^{\frac{3}{2} + 1} \, d\tau}{\tau^{\frac{3}{2} + 1}}.
\]

Now, for \( N > \frac{(2+\delta)T}{2} \) we get

\[
\frac{r}{\Delta t} \int_1^{+\infty} \frac{d\tau}{\tau^{\frac{3}{2} + 1 - \delta}} = \frac{1}{1 - \frac{3\delta}{2}(\delta + 2)} = \frac{1}{1 - \Delta t \frac{3\delta}{2}(1 - p^2 - q^2)}.
\]

Let us estimate \( \| |v|^{2+\delta} \varphi_{j,p}^N * \varphi_{j,q}^N \|_{L^1} \). We will denote

\[
d_j^N = \int_{\mathbb{R}} |v|^{2+\delta} \varphi_j^N(v) \, dv.
\]

Since \( 0 < \delta < 1 \), we can write

\[
|v|^{2+\delta} = v^2 |v|^\delta = (v - w + w)^2 |v - w + w| \leq ((v - w)^2 + w^2 + 2(v - w)w) \left( |v - w|^\delta + |w|^\delta \right) \\
= |v - w|^{2+\delta} + w^2 |v - w|^\delta + 2(v - w)|v - w|w + (v - w)^2 |w|^\delta + |w|^{2+\delta} + 2(v - w)|w|^\delta w
\]

so, thanks to \( \mathbb{H} \) and \( \mathbb{L} \), we get

\[
d_j^N \leq \frac{1}{1 - \Delta t \frac{3\delta}{2}(1 - p^2 - q^2)} \left( d_j^N + \Delta t (p^{2+\delta} + q^{2+\delta} - 1)d_j^N + \Delta t (p^2 q^2 + q^2 p^2) \| |v|^{2+\delta} \varphi_j^N \|_{L^1} \right).
\]

By Hölder inequality and the conservation of the mass, we obtain

\[
\| |v|^{\delta} \varphi_j^N \|_{L^1} \leq \| |v|^{2} \varphi_j^N \|_{L^2} \| |v|^{2+\delta} \varphi_j^N \|_{L^2}^{-\frac{\delta}{2}} = 1
\]

and so we get the recursive estimate

\[
d_{j+1}^N \leq \frac{1}{1 - \Delta t \frac{3\delta}{2}(1 - p^2 - q^2)} \left( d_j^N + \Delta t (p^{2+\delta} + q^{2+\delta} - 1)d_j^N + \Delta t (p^2 q^2 + q^2 p^2) \right).
\]

Remembering that \( \Delta t = \frac{r}{\tau} \), we would like to neglect the low order terms. We have

\[
\frac{1}{1 - \Delta t \frac{3\delta}{2}(1 - p^2 - q^2)} \left( d_j^N + \Delta t (p^{2+\delta} + q^{2+\delta} - 1)d_j^N + \Delta t (p^2 q^2 + q^2 p^2) \right) \\
= \frac{1 + \Delta t \frac{3\delta}{2}(1 - p^2 - q^2)}{1 - (\Delta t)^2 \left( \frac{3\delta}{2} \right) (1 - p^2 - q^2)} \left( d_j^N + \Delta t (p^{2+\delta} + q^{2+\delta} - 1)d_j^N + \Delta t (p^2 q^2 + q^2 p^2) \right) \\
= \frac{d_j^N + \Delta t (p^{2+\delta} + q^{2+\delta} - 1 d_j^N + \Delta t (p^2 q^2 + q^2 p^2) + (\Delta t)^2 K d_j^N + (\Delta t)^2 H}{1 - (\Delta t)^2 \left( \frac{3\delta}{2} \right) (1 - p^2 - q^2)^2},
\]

where \( H, K \) are positive constants, depending only on \( p, q \) and \( \delta \). Denoting

\[
S_{p,q}(\delta) = p^{2+\delta} + q^{2+\delta} - 1 + \frac{2 + \delta}{2} (1 - p^2 - q^2), \quad B_{p,q}(\delta) = p^2 q^2 + q^2 p^2,
\]

we got

\[
d_{j+1}^N \leq \frac{d_j^N + \Delta t S_{p,q}(\delta) d_j^N + \Delta t B_{p,q}(\delta) + (\Delta t)^2 H d_j^N + (\Delta t)^2 K}{1 - (\Delta t)^2 \left( \frac{3\delta}{2} \right) (1 - p^2 - q^2)^2}
\]

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where, by assumption, we have \( S_{p,q}(\delta) < 0 \). Moreover,
\[
1 - (\Delta t)^2 \left( \frac{p}{2} (1 - p^2 - q^2) \right) = 1 + o(\Delta t), \quad \Delta t \to 0,
\]
so we get, for \( N \in \mathbb{N} \) large enough, \( j = 0, \ldots, N - 1 \):
\[
d_{j+1}^N \leq d_j^N - \frac{1}{2} \Delta t |S_{p,q}(\delta)| d_j^N + 2 \Delta t B_{p,q}(\delta).
\]
This recursive relation implies
\[
\int_{\mathbb{R}} |v|^{2+\delta} \varphi_j^N(v) \, dv \leq C_\delta, \quad j = 0, \ldots, N - 1,
\]
for any \( N \in \mathbb{N} \) large enough.

Proof of Proposition 10. We divide it into several steps.

I STEP: Existence of the limit \( g^*(\xi, t) \) of a subsequence. Thanks to inequalities \( \text{(26)} \) and to the definition of \( g^N \) we have therefore
\[
\sup_{[0,T] \times \mathbb{R}} \left| g^N(\xi, t) \right| \leq 1, \quad \sup_{[0,T] \times \mathbb{R}} \left| \partial_t g^N(\xi, t) \right| \leq C, \quad \sup_{[0,T] \times \mathbb{R}} \left| \partial^2_{\xi \xi} g^N(\xi, t) \right| \leq 1, \quad (48)
\]
where \( C \) is the same constant as in \( \text{(26)} \) and \( N \) is large enough. Moreover since \( g^N(\xi, t) \) satisfies
\[
\partial^- g^N(\xi, t) = \frac{1}{r} \xi \left( \frac{d}{d\xi} \mathcal{N}_N(\xi) + \mathcal{N}_N'(1)(p\xi) \mathcal{N}_N(\xi) - \mathcal{N}_N(\xi, \delta) \right)
\]
then for any compact set \( K \subset \mathbb{R} \) there exists a constant \( C > 0 \) such that
\[
\sup_{[0,T] \times K} \left| \partial_t g^N(\xi, t) \right| \leq C, \quad N \geq \frac{2T}{r}. \quad (50)
\]
For any compact \( K \subset \mathbb{R} \) the function \( g^N(\xi, t) \) belongs to \( C([0,T] \times K) \) and thanks to properties \( \text{(18)} \) and \( \text{(50)} \) the sequence is equibounded and equicontinuous. Therefore by Ascoli-Arzelà theorem and by taking the diagonal, there exists a subsequence \( \{ g^{N_j}(\xi, t) \} \) which converges uniformly on \( [0,T] \times K \) for any compact \( K \subset \mathbb{R} \). Let us call \( g^*(\xi, t) \) the limit function.

Since \( g^{N_j}(\xi, 0) = \hat{g}_0(\xi) \) then \( g^*(\xi, 0) = \hat{g}_0(\xi) \). Moreover for any \( t \in [0,T] \) the function \( g^*(\xi, t) \in C^1(\mathbb{R}) \) indeed by Ascoli-Arzelà theorem and the diagonal argument applied now to the sequence \( \{ \partial_t g^{N_j}(\xi, t) \} \), for any \( t \in [0,T] \) we get a subsequence that converges uniformly to \( \partial_t g^*(\xi, t) \) on any compact set \( K \subset \mathbb{R} \). Since the limit function is \( \partial_t g^*(\xi, t) \), the convergence holds for the original sequence \( \{ \partial_t g^{N_j}(\xi, t) \} \) and it is not necessary to pass to a subsequence.

In order to get a uniform convergence in both frequency and time, we remark that by \( \text{(15)} \) we have that \( \sup_{[0,T] \times \mathbb{R}} \left| \partial^2_{\xi \xi} g^N(\xi, t) \right| \leq 1 \) and thanks to \( \text{(10)} \) and \( \text{(26)} \) we control \( \partial_t \partial_{\xi} g^N(\xi, t) \) therefore \( \partial_t g^N(\xi, t) \) is Lipschitz continuous on \( [0,T] \times K \) and uniformly bounded. Again by Ascoli-Arzelà we prove that \( \{ \partial_t g^{N_j}(\xi, t) \} \) converges uniformly to \( \partial_t g^*(\xi, t) \) on \( [0,T] \times K \) for any compact set \( K \subset \mathbb{R} \). From the uniform convergence of \( g^{N_j}(\xi, t) \) to \( g^*(\xi, t) \) and of \( \partial_t g^{N_j}(\xi, t) \) to \( \partial_t g^*(\xi, t) \) we get that both \( g^*(\xi, t) \) and \( \partial_t g^*(\xi, t) \) belong to \( C([0,T] \times K) \).

II STEP: \( g^*(\xi, t) \) is a solution of equation \( \text{(20)} \). By a direct computation we obtain
\[
\partial^- g^N(\xi, t) = \frac{1}{r} \xi \partial_t g^N(\xi, t) + g^N(p\xi, t)g^N(q\xi, t) - g^N(\xi, t) + R_N(\xi, t) \quad (51)
\]
where
\[
R_N(\xi, t) = -\frac{1}{r} \xi \left( \frac{d}{d\xi} \mathcal{N}_N(\xi) - \frac{d}{d\xi} \mathcal{N}_{K_N}(\xi) \right) + (1 - \alpha) \left[ \alpha \left( \mathcal{N}_N(p\xi) - \mathcal{N}_{K_N}(p\xi) \right) \left( \mathcal{N}_N(q\xi) - \mathcal{N}_{K_N}(q\xi) \right) \left( \mathcal{N}_N(p\xi) - \mathcal{N}_{K_N}(p\xi) \right) + \mathcal{N}_{K_N}(p\xi) \left( \mathcal{N}_N(q\xi) - \mathcal{N}_{K_N}(q\xi) \right) + \mathcal{N}_N(p\xi) \left( \mathcal{N}_N(q\xi) - \mathcal{N}_{K_N}(q\xi) \right) \right].
\]
III STEP: we show that $g(\xi,t)$ of (53) is continuous in both variables therefore the same is true for $L(\xi,t)$ the limit function, so that
\[ L(\xi,t) = \frac{1}{\sqrt{\pi}} \xi \partial_\xi g^*(\xi,t) + g^*(p\xi,t)g^*(q\xi,t) - g^*(\xi,t) \]  
(53) and $\{ \partial_\xi g^{N_i}(\xi,t) \}$ converges to $L(\xi,t)$ for any $t \in [0,T]$ uniformly on any compact set $K \subset \mathbb{R}$. Thanks to property (50) and to Lebesgue’s dominated convergence theorem we remark that the whole right hand side of equation (51) converges; let us call it $\hat{L}(\xi,t)$. Denoting $\hat{L}(\xi,t)$ uniformly on $[0,T] \times K$ and so in $D’([0,T] \times K’)$ we have that $\{ \partial_\xi g^{N_i}(\xi,t) \}$ converges in distributions to $\partial_\xi g^*(\xi,t)$. For the uniqueness of the limit and the fact that in distributions $\partial_\xi g^{N_i}(\xi,t) = \partial_\xi g^{N_i}(\xi,t)$, we obtain $\partial_\xi g^*(\xi,t) = L(\xi,t)$ in the sense of distributions. Finally, since $g^*(\xi,t)$ and $\partial_\xi g^*(\xi,t)$ belong to $C([0,T] \times K)$, for any compact $K$ the right hand side of (58) is continuous in both variables therefore the same is true for $\partial_\xi g^*(\xi,t)$. This implies that $g^*(\xi,t)$ belongs to $C’([0,T] \times \mathbb{R})$.

By Gronwall Lemma we have
\[ \| \partial_\xi h + h \| \leq 2 \| h(t) \|_{\infty} \]
and since $\hat{L}(\xi,0) = F^*(\xi,0)$ this implies the desired equality $g^*(\xi,t) = \hat{g}(\xi,t)$.

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