No-passing Rule in the Ground State Evolution of the Random-Field Ising Model

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We exactly prove the no-passing rule in the ground state evolution of the random-field Ising model (RFIM) with monotonically varying external field. In particular, we show that the application of the no-passing rule can speed up the calculation of the zero-temperature equilibrium \( M(H) \) curve dramatically.

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I. INTRODUCTION

The no-passing rule was first introduced by Middleton in the study of sliding charge-density waves (CDW’s) [1]. The CDW problem belongs to the more general class of driven elastic manifolds in random media. If one defines a simple one-dimensional order parameter within the model, then a natural partial ordering of the configurations can be defined. In the simple CDW model considered by Middleton, the CDW configuration \( \{ \varphi_i(t) \} \) describes the CDW distortions at \( N \) lattice sites indexed by \( i \), with \( \varphi_i(t) \) real phase variables and \( t \) the time. The equation of motion for an overdamped CDW is governed by the Langevin dynamics

\[
\dot{\varphi}_i = \Delta^2 \varphi_i - V'_i(\varphi_i) + f(t) \tag{1}
\]

Here, the \( \Delta^2 \varphi_i \) term represents the simple elastic interactions. \( V'_i(\varphi_i) \) is the pinning force at site \( i \) due to the \( 2\pi \) periodic pinning potential \( V(\varphi_i) \). And \( f(t) \) stands for the external driving force. Then one can define the natural partial ordering of two configurations:

\[
C^G = \{ \varphi_i^G \} \geq C^L = \{ \varphi_i^L \} \quad \text{if} \quad \varphi_i^G \geq \varphi_i^L \quad \text{for each} \quad \text{site} \quad i \quad \text{of the system.}
\]

The no-passing rule states that given a driving force \( f \) if initially \( C^G(0) \geq C^L(0) \), then \( C^G(t) \geq C^L(t) \) for all \( t > 0 \), i.e. the “greater” \( C^G \) is never passed by the “lesser” \( C^L \). As stressed by Middleton, this rule relies crucially on the elastic potential being convex. In other words, the elastic potential tends to decrease the separation of nearest-neighbor \( \varphi \)’s. More recently, Krauth et al found a similar no-passing rule in the study of driven elastic strings in disordered media [2, 3]. Obviously, this is the same general problem. Again, the rule is crucially dependent on the fact that the elastic potential is convex.

The no-passing rule can be used to prove many useful properties, such as the asymptotic uniqueness of the sliding state for CDW’s [1] and the intriguing memory effects [4]. In fact, just after its introduction by Middleton, the no-passing rule was used in the non-equilibrium zero-temperature random-field Ising model (RFIM) by Sethna et al to prove the return point memory effect [3]. The RFIM is obtained by adding a random field \( h_i \) at each site of the Ising model

\[
\mathcal{H} = - \sum_{<i,j>} J s_i s_j - \sum_i (H + h_i) s_i \tag{2}
\]

The distribution of \( h_i \) values is usually taken to be Gaussian, with standard deviation \( R \) and mean 0. \( J \) is the nearest-neighbor ferromagnetic coupling strength and \( H \) is the uniform external field. In this case, the natural partial ordering of two configurations can be defined similarly as in the CDW case. A difference is that \( h_i = \pm 1 \) while \( \varphi_i \) is real. The no-passing rule states: Let a system \( C^G(t) \) be evolved under the fields \( H^G(t) \) and similarly \( C^L(t) \) evolved under \( H^L(t) \). Suppose the fields \( H^G(t) \geq H^L(t) \) and the initial configurations satisfy \( C^G(0) \geq C^L(0) \), then \( C^G(t) \geq C^L(t) \) at all times later \( t > 0 \), i.e. the partial ordering will be preserved by the dynamics. With a local metastable single-spin-flip dynamics, i.e. a spin flips when its effective local field

\[
h_i^{\text{eff}} = J \sum_j s_j + h_i + H \tag{3}
\]

changes sign, the proof of the no-passing rule is straightforward [4]. Even with a two-spin-flip dynamics, it has been shown by Vives et al that the no-passing rule is still true at zero-temperature [5]. Note that for the magnetization process, the no-passing rule is equivalent to the fact that the flipped spins can never flip back as \( H \) is swept monotonically. Again, this rule is not unconditionally true. It relies crucially on the nearest-neighbor interaction being ferromagnetic \( (J > 0) \). Just like the convex elastic potential, the ferromagnetic interaction also tends to decrease the separation of nearest-neighbor degrees of freedom, i.e. it tends to align the spins.

Recently, in the study of the equilibrium zero-temperature RFIM, Vives et al conjectured that when the external field \( H \) is swept from \(-\infty \) to \( \infty \), flipped spins cannot flip back in the equilibrium \( M(H) \) curve [6]. In other words, the no-passing rule is valid even for the zero-temperature equilibrium dynamics, i.e. the evolution of the ground state (GS). Vives et al further conjectured that this rule can be used to speed up the calculation of the equilibrium \( M(H) \) curve since flipped spins at a lower field can be removed from the GS calculation for all higher fields. Unfortunately, this simple but powerful
rule has not been proven so far for the equilibrium RFIM. This is the main motivation of our work.

This paper is organized as follows. In Sec.II, we give a short introduction to the calculation of the equilibrium $M(H)$ curve of the zero-temperature RFIM. In Sec.III, we work out some basic steps for the proof of the equilibrium no-passing rule. In Sec.IV, we present the proof. In Sec.V, we show the direct application of this rule to the calculation of the equilibrium $M(H)$ curve. Finally, in Sec.VI we discuss its validity in other systems.

II. THE EQUILIBRIUM $M(H)$ CURVE

To calculate the equilibrium $M(H)$ curve of the zero-temperature RFIM, we first need to calculate the exact GS in the arbitrary applied external field $H$. This is the basic step of calculating the equilibrium $M(H)$ curve, i.e. the GS evolution for varying $H$. Fortunately, there is a well-known mapping of the RFIM GS problem to a min-cut/max-flow problem in combinatorial optimization. The mapping and the so-called push-relabel algorithm for the min-cut/max-flow problem has been well described in the literatures [7, 8]. For RFIM, the run time of the push-relabel algorithm scales as $O(N^{4/3})$ with $N$ the system size [9, 11].

The equilibrium $M(H)$ curve can be simulated with the method reported in Ref. [9, 11]. It is essentially based on the fact that the GS energy $E(S_i, H)$ is convex up in $H$, which allows for estimates of the fields $H$ where the magnetization jumps (called “avalanches” occur). This algorithm finds steps by narrowing down ranges where the magnetization jumps with an efficient linear interpolation scheme. An illustration of the algorithm is shown in Fig. 1. An example of the calculated equilibrium $M(H)$ curve is shown in Fig. 2. The details have been explained extensively in Ref. [11]. Here, we just introduce some important propositions.

In the $E-H$ diagram, for each state $\{s_i\}$, the energy $E$ is represented by a straight line with slope $-M \equiv -\sum_i s_i$ since

$$E(\{s_i\}, H) = E_0(\{s_i\}) - HM$$

with

$$E_0(\{s_i\}) = -\sum_{<i,j>} J s_i s_j - \sum_i h_i s_i$$

the energy axis intercept, i.e. the total energy of the configuration when $H = 0$. Consider a D-dimensional hypercubic lattice of size $N = L^D$. Let $h_{\text{max}}$ ($h_{\text{min}}$) be the maximum (minimum) values of $h_i$ for a certain realization of the random fields. (Usually, random fields are chosen from a Gaussian distribution with mean 0 and standard deviation $R$. $R$ is often called the disorder parameter.) Most of the following simple propositions have been proven in Ref. [11].

**Proposition 1** For $H < -h_{\text{max}}$ ($H > -h_{\text{min}}$), the ground state is $\{s_i = -1\}$ ($\{s_i = +1\}$).

**Proposition 2** Let the spin configuration $C_1$ ($C_2$) be the ground state for $H = H_1$ ($H = H_2$). They have magnetization $M_1$ and $M_2$, respectively. If $C_1 \neq C_2$ and $H_2 > H_1$, then $M_2 > M_1$.

This proposition gives an important insight into the structure of the energy axis intercept, i.e. the total energy of the ground state.

**Proposition 3** Let the spin configuration $C_1$ ($C_2$) be the ground state for $H = H_1$ ($H = H_2$), $C_1 \neq C_2$. If the crossing field between $C_1$ and $C_2$ is $H^*$, then $H^*$ is defined as the crossing field between $C_1$ and $C_2$. According to the definition, one has

$$H^*(C_1, C_2) = \frac{E_0(C_2) - E_0(C_1)}{M_2 - M_1}$$

For example, we can calculate the crossing field between the two simplest ground states: $C_1 = \{s_i = -1\}$ with $M_1 = -N$ and $C_2 = \{s_i = +1\}$ with $M_2 = N$. We have $H^* = -1/N \sum_i h_i = -\bar{h}$. For a zero-mean distribution of the random fields, we should have $H^* = 0$.

**Proposition 4** If the ground state is non-degenerate, then there can not be more than one avalanche connecting two nearest ground states in the $E-H$ diagram.

The proof of this interesting proposition is shown in the Appendix. This proposition says that if the GS is non-degenerate, when we increase the external field $H$ adiabatically slowly, we can trigger just one avalanche at a time.

III. PREPARATIONS OF THE PROOF

In this section, we will work out the total energy change of the spin configuration due to multiple spin flips and external field varying. The spin configuration is not necessarily the ground state.

First, let’s consider the simplest case of a single spin flip. Suppose only one spin ($s_i$) flips during the evolution of configuration $C$ at $H$ to configuration $C'$ at $H'$. 

- The proof of this interesting proposition is shown in the Appendix. This proposition says that if the GS is non-degenerate, when we increase the external field $H$ adiabatically slowly, we can trigger just one avalanche at a time.
FIG. 1: An illustration of the algorithm to calculate the equilibrium $M-H$ curve. Calculate the energies $E_1$ and $E_2$ of the two simplest states $C_1 = \{s_i = -1\}$ and $C_2 = \{s_i = +1\}$, respectively, as a function of $H$. According to Proposition. 1 $C_1$ (or $C_2$) would be the ground state for $H < -h_{max}$ (or $H > -h_{min}$). Calculate the crossing field $H^*(C_1, C_2)$ where $E_1 = E_2$. Check whether there is a GS at $H^*$ which is different from $C_1$ and $C_2$. If no, the algorithm ends. If yes, denote the GS as $C$. Calculate the crossing field $H^*(C, C_1)$ and $H^*(C, C_2)$. At the new crossing fields, check whether there is a GS which is different from the two intersected states. The algorithm will not end until all the crossing fields have been checked. An example of the calculated equilibrium $M-H$ curve is shown in Fig. 2.

It is easy to get the bond energy change $4J(n_i - D)$. And the total energy change due to the single spin-flip and the varying external field is given by

$$f_i(H, \Delta H) = f_i(H) - \Delta HM'$$

(7)

Here we have defined

$$f_i(H) \equiv f_i(H, 0) = 4J(n_i - D) - (h_i + H)\Delta s_i$$

(8)

which is the energy change due to spin $i$ flipping for the configuration $C$ just at the field $H$, i.e. $\Delta s_i = \pm 2$ with $\Delta H = 0$. It is easy to check that

$$f_{i, \pm}(H) = f_{i, \pm}(0) \pm 2H = \pm 2h_i^{eff}$$

(9)

with ‘$\pm$’ represents $s_i = \pm 1$ and $\Delta s_i = \mp 2$ accordingly.

Second, we consider two spin flips. Suppose two different spins ($s_i$ and $s_j$) flip during the evolution of configuration $C$ at $H$ to configuration $C'$ at $H'$. There are two subcases.

(1) $s_i$ and $s_j$ are not next to each other. The energy change is

$$f_{i,j}(H, \Delta H) = f_i(H) + f_j(H) - \Delta HM'$$

(10)

(2) $s_i$ and $s_j$ are next to each other. The energy change is

$$f_{(i,j)}(H, \Delta H) = f_i(H) + f_j(H) - 4J(s_i, s_j) - \Delta HM'$$

(11)

with $\Delta = H' - H$ and $\Delta M = M' - M$. Define $n_i$ (or $n_i'$) to be the number of the $i$-th spin’s nearest neighbors that point in the same direction as the spin in configuration $C$ (or $C'$). We call these spins the same-direction nearest neighbors (SDNN) of the $i$-th spin. Note that $n_i = 0, 1, 2, ..., Z$ with $Z = 2D$ the coordination number of the $D$-dimensional hypercubic lattice.

FIG. 2: The equilibrium $M-H$ curve (the ground state evolution) for the Gaussian RFIM with $D = 3$, $L = 32$ and $R = 2.837$. Here $R$ is the standard deviation of the Gaussian random field distribution. The inset shows a detail of the $M(H)$ curve near $H = 0$, where magnetization jumps are clearly seen.
Note that the term $-4J(s_i \cdot s_j)$ is just due to the fact that the energy of the $i - j$ bond will not change during the flip.

Finally, let’s consider the general case. See Fig. 3(a). There are many spin flips during the evolution of configuration $C$ at field $H$ to $C'$ at field $H'$. It is easy to check that the total energy change is given by

$$
\Delta E(H, \Delta H) = \left[ f_i(H) + f_j(H) + \cdots \right] - 4J \sum_{s_i \cdot s_j} - \Delta H(M + \Delta s_i + \Delta s_j + \cdots)
$$

(12)

On the RHS, the first term includes all the flipping spins. The last term is due to the varying external field. In particular, if all the flipping spins flip at the same $H$ and they are connected to each other and have the same spin value $-1$ (or $+1$) before the flip, then this collective spin flip is called an avalanche (or a reverse avalanche).

Denote the energy change due to an avalanche $A_\alpha$ as $f_{A_\alpha}(H, \Delta H)$, we have

$$
f_{A_\alpha}(H, \Delta H) = \left[ f_i(H) + f_j(H) + \cdots \right] - 4J N_b(A_\alpha) - \Delta H(M + 2S_\alpha)
$$

(13)

with $N_b(A_\alpha)$ defined as the number of interacting bonds in $A_\alpha$, $S_\alpha$ the size of the avalanche and $f_{A_\alpha}(H)$ the energy change due to the avalanche when $\Delta H = 0$. Similarly, for the reverse avalanche, we have

$$
f_{A_\beta}(H, \Delta H) = \left[ f_i(H) + f_j(H) + \cdots \right] - 4J N_b(A_\beta) - \Delta H(M - 2S_\beta)
$$

(14)

Due to Eq. 12 we have

$$f_{A_\alpha}(H) = f_{A_\alpha}(0) - 2S_\alpha H$$

(15)

$$f_{A_\beta}(H) = f_{A_\beta}(0) + 2S_\beta H$$

(16)

Now we can rewrite Eq. (12) in terms of $f_{A_\alpha}$ and $f_{A_\beta}$. The total energy change due to avalanches and reverse avalanches is given by

$$
\Delta E(H, \Delta H) = F_A(H) + F_{A'}(H) + 4J N_b(A, A') - \Delta H(M + 2S_A - 2S_{A'})
$$

(17)

with notations $F_A(H) \equiv \sum_{A_\alpha} f_{A_\alpha}(H)$, $F_{A'}(H) \equiv \sum_{A_\beta} f_{A_\beta}(H)$, $S_A \equiv \sum_{A_\alpha} S_\alpha$, $S_{A'} \equiv \sum_{A_\beta} S_\beta$. Here $N_b(A, A')$ denotes the number of interacting bonds between avalanches and reverse avalanches. For example, in Fig. 3(a), $N_b(A, A') = 3$.

IV. PROOF OF THE NO-PASSING RULE

Now we are ready for the proof of the no-passing rule. Let the spin configuration $C_1$ ($C_2$) be the ground state for $H = H_1$ ($H = H_2$). $H_2 > H_1$. Suppose $C_1$ and $C_2$ are connected with multiple avalanches: $A_1, A_2, \ldots, A_n$, with sizes $S_1, S_2, \ldots, S_n$ and reverse avalanches $A_1', A_2', \ldots, A_m'$, with sizes $S_1', S_2', \ldots, S_m'$ respectively. To compensate these reverse avalanches (so as to make sure $M$ is monotonically increasing, see Proposition. 2), we must have $S_A = \sum_{n=1}^n S_\alpha > S_{A'} = \sum_{m=1}^m S_\beta$. See Fig. 3(a).

The idea is that if $C_2$ is the GS at field $H_2$, then it should have lower energy than any other spin configuration at $H_2$. But we will prove this is NOT true. Just consider another spin configuration $\tilde{C}$. The only difference between $C_2$ and $\tilde{C}$ is that $\tilde{C}$ is evolved from $C_1$ without any reverse avalanches. See Fig. 3(b). We now try to prove that $E(\tilde{C}, H_2) < E(C_2, H_2)$, so $C_2$ can NOT be the GS at $H_2$. But this is equivalent to proving that $\Delta \tilde{E} < \Delta E$. Here,

$$
\Delta E \equiv E(C_2, H_2) - E(C_1, H_1) = F_{A'}(H_1) + F_A(H_1) + 4J N_b(A, A') - \Delta H(M - 2S_{A'} + 2S_A).
$$

(18)

On the other hand,

$$
\Delta \tilde{E} \equiv E(\tilde{C}, H_2) - E(C_1, H_1) = F_A(H_1) - \Delta H(M + 2S_A)
$$

(19)

Therefore,

$$
\Delta E - \Delta \tilde{E} = F_{A'}(H_1) + 4J N_b(A, A') + 2S_{A'} \Delta H > 0
$$

(20)

Here we have used the fact that $C_1$ is the ground state for $H = H_1$ such that any kinds of spin flip will increase the energy: $f_{A'}(H_1) > 0 \Rightarrow F_{A'}(H_1) > 0$. Also, for the ferromagnetic RFIM, $J > 0$. Since each term is positive, so the sum is positive, i.e. $\Delta E > \Delta \tilde{E}$ or $E > \tilde{E}$. Actually, for any state $C_2$ which evolved from $C_1$ with reverse avalanches, we can find a corresponding state $\tilde{C}$ which evolved from $C_1$ without any reverse avalanches that has lower energy than $C_2$ at field $H_2$. So reverse spin flips are impossible for ground state evolution when increasing external field. Generally, flipped spins can never flip back when we sweep the external field monotonically.

V. APPLICATION

The straightforward application of the no-passing rule is very useful to accelerate the calculation of the ground states when varying the external field. Suppose the GS $C_1$ at field $H_1$ has already been obtained, and we want to calculate the GS $C_2$ at field $H_2$ with $H_2 > H_1$. According to the no-passing rule, the UP spins in $C_1$ will
FIG. 3: States evolved from $C_1$, with only the change of the spin configuration, i.e. avalanches and reverse avalanches are explicitly shown. (a) State $C_2$: evolved from state $C_1$ with both avalanches and reverse avalanches. (Black dot) spins flip UP, forming avalanches ($A_1$, $A_2$, $A_3$ and $A_4$). (White dot) spins flip DOWN (reverse flip), forming reverse avalanches ($A_r1$, $A_r2$). Note that there are three interacting bonds between avalanche $A_2$ and reverse avalanche $A_r2$. (b) State $\tilde{C}$ evolved from state $C_1$ without reverse avalanches. We stay UP in $C_2$, i.e. those spins are frozen, so we needn’t consider them in the ground state analysis. We just need to consider the DOWN spins in $C_1$. The only cost is that we have to deal with the frozen UP spins as complicated fixed boundary conditions for the DOWN spins. At first sight, one may think that only when the density of the frozen spins is big enough can we make the GS calculation faster. But how big is enough? To optimize our calculation, we consider the running time difference ($\Delta t$) between the two methods: (I) without using the earlier solution $C_1$; (II) using the earlier solution $C_1$. For both methods, ground states are found by using the push-relabel algorithm. The numerical experiments are conducted on a desktop with 2.80 GHz CPU and 2GB Memory. And we tune the UP-spin density $n_{up}$ (Down-spin density $n_{down}$) by varying $H_1$. The result is shown in Fig. 4. It is found that for $H_2 > H_1$, as long as $n_{up} \gtrsim 0.07$ in GS $C_1$, method II will be faster than I. Symmetrically, for $H_2 < H_1$, as long as $n_{down} \gtrsim 0.07$ in GS $C_1$, method II will be faster than I. This suggests it is not necessary to have an extremely large portion of frozen spins to use the earlier solution. Freezing a tiny part of spins will accelerate the GS calculation already. Furthermore, for larger and larger density of the frozen spins, using the earlier solution will save more and more running time. (Keep in mind that for RFIM, the running time of the push-relabel algorithm scales as $O(N^{4/3})$.) Consequently, the calculation of the whole $M(H)$ curve will be sped up dramatically.

VI. DISCUSSIONS

Throughout our proof of the no-passing rule, we don’t assume that the ground state is unique. In other words, the no-passing rule is correct even when the ground state is degenerate. For example, this happens for the RFIM when the random fields are chosen from a bimodal distribution.

In the proof we explicitly use the fact that the nearest neighbor interaction should be ferromagnetic ($J > 0$). This means any antiferromagnetic interactions will destroy the no-passing rule. Thus, for other random magnet models, if $J_{ij}$ could be negative, such as the random-bond Ising model (RBIM) with negative $J_{ij}$ or the spin glasses, the rule will be violated.

Finally, we conjecture that for elastic manifolds in random media, there could be a similar equilibrium no-passing rule at zero temperature, provided that the elastic potential is convex and partial ordering of the configurations can be clearly defined.

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FIG. 4: Running time difference ($\Delta t$) between methods with and without using the earlier solution, i.e the GS $C_1$ at field $H_1$, to calculate the GS $C_2$ at field $H_2$. The time difference $\Delta t$ (given in seconds) is plotted against the UP-spin (or DOWN-spin) density of the GS $C_1$. $\Delta t < 0$ means using the earlier solution will save the running time. Calculations are done for 3D Gaussian RFIM (with disorder parameter $R = 2.27$) for different system sizes. (Top) $H_2 > H_1$. UP spins in $C_1$ at field $H_1$ will stay UP at field $H_2$. (Bottom) $H_2 < H_1$. DOWN spins in $C_1$ at field $H_1$ will stay DOWN at field $H_2$.

APPENDIX A: A SINGLE AVALANCHE CONNECTS TWO NEAREST GROUND STATES

Here, we show the proof of Proposition 4.

**Proof:** Suppose when the field is increased from $H_1$ to $H_2$, the GS $C_1$ evolves to the nearest GS $C_2$ with two avalanches ($A_1$ and $A_2$ with size $S_1 \geq 1$ and $S_2 \geq 1$, respectively).

The crossing field is given by

$$H^*(C_1, C_2) = \frac{E_0(C_2) - E_0(C_1)}{M_2 - M_1} = \frac{f_{A_1}(0) + f_{A_2}(0)}{2(S_1 + S_2)} \quad (A1)$$

The last line is due to Eq. [17]. We can choose a trial state $C$ which is evolved from $C_1$ with only avalanche $A_1$ occurring. We want the following relation to hold

$$E(C, H^*) < E(C_1, H^*) = E(C_2, H^*) \quad (A2)$$

which is equivalent to

$$E(C, H^*) - E(C_1, H^*) = f_{A_1}(H^*) = f_{A_1}(0) - 2S_1H^* < 0 \quad (A3)$$

Plugging Eq. [A1] in it, we just need to prove

$$S_2f_{A_1}(0) < S_1f_{A_2}(0). \quad (A4)$$

If we do have $S_2f_{A_1}(0) < S_1f_{A_2}(0)$, then we choose the trial state $C$, which has lower energy than $C_1$ and $C_2$ at the field $H^*$. If $S_2f_{A_1}(0) > S_1f_{A_2}(0)$, then we can choose another trial state $C'$, which is evolved from $C_1$ through only avalanche $A_2$ and has lower energy than both $C_1$ and $C_2$ at $H^*$. In both cases, we have shown that $C_2$ cannot be the nearest GS at $H_2$ for the GS $C_1$ at $H_1$, if $C_1$ evolves to $C_2$ with two avalanches. If $S_2f_{A_1}(0) = S_1f_{A_2}(0)$, it is easy to show that $E(C_1, H^*) = E(C, H^*) = E(C', H^*) = E(C_2, H^*)$, then there will be degenerate ground states at $H^*$, which is in contradiction to the hypothesis of Proposition 4.

Q.E.D

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