Decoupling constant for $\alpha_s$ and the effective gluon-Higgs coupling to three loops in supersymmetric QCD

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Abstract

We compute the three-loop QCD corrections to the decoupling constant for $\alpha_s$ which relates the Minimal Supersymmetric Standard Model to Quantum Chromodynamics with five or six active flavours. The new results can be used to study the stability of $\alpha_s$ evaluated at a high scale from the knowledge of its value at $M_Z$. We furthermore derive a low-energy theorem which allows the calculation of the coefficient function of the effective Higgs boson-gluon operator from the decoupling constant. This constitutes the first independent check of the matching coefficient to three loops.

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1 Introduction

The decoupling of particles with masses much heavier than the considered energy scale has a long history [1]. It is tightly connected to the construction of an effective theory containing only the light active degrees of freedom in the dynamical part of the Lagrange density. Within the framework of QCD decoupling constants for the strong coupling \( \alpha_s \) are known at two- [2–4], three [4] and even four-loop order [5, 6]. Recently also the expression for the simultaneous decoupling of two heavy quarks has been computed at the three-loop level [7].

Decoupling relations are also important in the context of supersymmetry where the Standard Model constitutes the effective theory. Two-loop corrections for a degenerate supersymmetric mass spectrum are known from Ref. [8, 9] and the general result can be found in Ref. [10]. In this paper we compute the three-loop corrections for several different assumptions on the masses of the MSSM.

There is an interesting connection between the decoupling constants and the effective coupling of a CP neutral Higgs boson to gluons which is defined via the Lagrange density (the superscript 0 marks bare quantities)

\[
\mathcal{L}_{Y,\text{eff}} = -\frac{\phi^0}{v^0} G^0_1 \mathcal{O}^0_1 + \mathcal{L}^{(5)}_{\text{QCD}},
\]

with

\[
\mathcal{O}^0_1 = \frac{1}{4} G^0_{\mu \nu} G^{0,\mu \nu},
\]

where \( \phi \) is the Higgs field \( v \) is the vacuum expectation value and \( G_{\mu \nu} \) the field strength tensor in QCD. \( \mathcal{L}^{(5)}_{\text{QCD}} \) is the QCD Lagrange density with five active flavours. The first term in Eq. (1) describes the coupling of the Higgs boson to two, three and four gluons.

In Ref. [4] an all-order low-energy theorem (LET) has been derived which connects \( C_1 \) to the derivative of the decoupling constant for \( \alpha_s \) with respect to the top quark mass. As far as supersymmetry is concerned a next-to-leading order (NLO) version of the LET has been derived in Ref. [14] (see also Ref. [15]). In this way the NLO supersymmetric QCD (SQCD) corrections to \( C_1 \) obtained in Ref. [16] could be confirmed. We re-derive the LET, apply it at three loops and thus obtain the coefficient function \( C_1 \) which is needed for NNLO prediction of Higgs boson production and decay within the MSSM. With our calculation we confirm the result for \( C_1 \) obtained in Ref. [17, 18] by an explicit calculation of the vertex diagrams.

The outline of this paper is as follows: In the next Section we describe the calculation of the decoupling constant for \( \alpha_s \) to three loops and discuss the numerical influence in the computation of \( \alpha_s(M_{\text{GUT}}) \). Afterwards we derive in Section 3 an all-order low-energy-theorem which we use to compute \( C_1 \) to NNLO accuracy. We summarize and conclude

\footnote{Discussions about the LET applied at one and two loops can, e.g., be found in Refs. [11, 13].}
the paper in Section 4. In the Appendix we present a compact expression of the exact two-loop result for the decoupling coefficient.

2 Decoupling of heavy supersymmetric particles

In order to compute the decoupling effects of heavy particles from the running of $\alpha_s$ one can use the well-established formalism derived in Ref. [4]. It has been applied to supersymmetry in Refs. [8–10] where two-loop corrections have been computed.

The starting point is the relation between the strong coupling in the full theory, which is in our case the MSSM, respectively, SQCD, and the effective theory, QCD

$$\alpha_s^{(QCD)}(\mu) = \zeta_{\alpha_s}(\mu)\alpha_s^{(SQCD)}(\mu). \quad (3)$$

At that point some comments are in order:

- $\alpha_s^{(QCD)}(\mu)$ is defined in the five or six flavour theory, depending on whether the top quark is integrated out together with the supersymmetric particles or not.

- $\alpha_s^{(QCD)}(\mu)$ is defined in the $\overline{MS}$ scheme based on Dimensional Regularization (DREG). $\alpha_s^{(SQCD)}(\mu)$ is defined in the DR scheme since the supersymmetric theory is regularized using Dimensional Reduction (DRED). DRED is implemented with $\varepsilon$ scalars, where the details can be found in Refs. [18, 19].

- $\zeta_{\alpha_s}(\mu)$ as introduced in Eq. (3) has two tasks: (i) it has to decouple the heavy particles not present in the effective theory, and (ii) $\zeta_{\alpha_s}(\mu)$ has to ensure the change of regularization from DRED to DREG. In principle the two tasks can be performed in two steps as it has been proposed in Refs. [8–10]. However, it is more convenient to choose the same renormalization scale for the decoupling and the change of scheme. Calculations along these lines have also been performed in Ref. [17, 20].

- In principle each vertex containing $\alpha_s$ can be used in order to compute $\zeta_{\alpha_s}$. It is, however, convenient to use the gluon-ghost vertex in order to compute the decoupling constant via [4]

$$\zeta_{\alpha_s}^0 = \left( \frac{\tilde{\zeta}_0}{\bar{\zeta}_3 \sqrt{\bar{\zeta}_3}} \right)^2, \quad (4)$$

where the superscript “0” marks bare quantities. $\tilde{\zeta}_1$, $\tilde{\zeta}_3$ and $\zeta_3$ are the decoupling constants of the gluon-ghost vertex, ghost and gluon propagator, respectively. They are obtained from the hard part of the corresponding Green’s function (see Fig. 1 for sample Feynman diagrams up to three loops). The corresponding formulae can be found in Ref. [4] where a derivation has been performed in the framework of QCD.
Figure 1: Sample diagrams contributing to \( \zeta_3 \) (top row), \( \tilde{\zeta}_3 \) (middle row) and \( \tilde{\zeta}_1 \) (bottom row) up to three loops. The symbols \( t, \tilde{t}, q, \tilde{q}, g, \tilde{g}, c \) and \( \varepsilon \) denote top quarks, top squarks, light quarks and the corresponding squarks, gluons, gluinos, ghosts and \( \varepsilon \) scalars, respectively. \( \sigma_\varepsilon \) and \( \sigma \) are auxiliary particles used for the implementation of the four-\( \varepsilon \) and four-gluon vertices, respectively.

It can be taken over to SQCD without modifications. The renormalized decoupling constant is obtained from

\[
\zeta_{\alpha_s} = \frac{Z_{\alpha_s}}{Z_{\alpha'_s}} \zeta_{0,\alpha_s},
\]

where \( Z_{\alpha_s} \) and \( Z_{\alpha'_s} \) are the renormalization constants for \( \alpha_s \) in the full and effective theory, respectively.

- All occurring parameters are renormalized in the \( \overline{\text{DR}} \) scheme, except the \( \varepsilon \) scalar mass which is renormalized on-shell with the condition \( M_\varepsilon = 0 \). The corresponding counterterms can, e.g., be found in Ref. \[21\].

Assuming a strong hierarchy among the quarks one encounters in the case of QCD vacuum diagrams which contain only one mass scale. The occurring integrals can even be computed up to four-loop order \[5,6\]. Two scales appear if two quarks are integrated out simultaneously. This has been done in Ref. \[7\] to three-loop accuracy.
In the case of supersymmetry significantly more mass scales have to be considered. In our approach we have the gluino and top squark masses ($m_{\tilde{g}}, m_{\tilde{t}_1}, m_{\tilde{t}_2}$) and a generic squark mass $m_{\tilde{q}}$ which we take as the average of the up, down, strange, charm and bottom squarks. In addition there is the $\varepsilon$ scalar ($M_\varepsilon$) and the top quark ($m_t$) mass. The latter only appears if we match to five-flavour QCD since $m_t = 0$ is chosen for the matching to six-flavour QCD. Up to two loops $\zeta_\alpha$ can nevertheless be computed exactly [10] taking into account the dependence on all mass parameters. The analytical result can be found in the Appendix. At three-loop order, however, approximations have to be adopted in order to be able to compute the integrals. Motivated by scenarios which are currently discussed in the literature we have chosen

\begin{align}
(h1) & \quad m_{\tilde{g}} \approx m_{\tilde{t}_1} \approx m_{\tilde{t}_2} \approx m_{\tilde{g}} \gg m_t , \\
(h2) & \quad m_{\tilde{g}} \approx m_{\tilde{t}_2} \approx m_{\tilde{g}} \gg m_{\tilde{t}_1} \gg m_t , \\
(h3) & \quad m_{\tilde{g}} \approx m_{\tilde{t}_2} \approx m_{\tilde{g}} \gg m_{\tilde{t}_1} \approx m_t ,
\end{align}

where in the case of “$\gg$” an asymptotic expansion in the corresponding hierarchy is performed. In the case of “$\approx$” a naive Taylor expansion in the difference of the particle masses is sufficient. For all hierarchies we assume that $M_\varepsilon$ is not zero but much smaller than all other masses. In this way we ensure that the $\varepsilon$ scalar is integrated out and not present in the effective theory. Thus, in the latter dimensional regularization can be used. In what follows the heavy mass scales for each hierarchy are also denoted by $m_{SUSY}$ in case they are identified.

Whereas at one- and two-loop order only 12 and 362 Feynman diagrams have to be considered there are more than 20000 at three-loop order. It goes without saying that it is thus necessary to automate the calculation as much as possible. We rely on a chain of programs which work hand-in-hand in order to minimize the error-prone manual interaction: All Feynman diagrams are generated with QGRAF [22] and afterwards transformed to FORM [23] notation with the help of q2e. The rules of asymptotic expansion (see, e.g., Ref. [24]) are applied on a diagrammatic level using exp [25, 26] and finally we evaluate the resulting vacuum integrals which (after asymptotic expansion) only contain a single scale with the help of the package MATAD [27]. The automated setup allows us to perform the calculation for general gauge parameter $\xi$. Whereas $\zeta_1, \zeta_3$ and $\zeta_3$ individually depend on $\xi$ it drops out in the combination for $\zeta_\alpha$ which serves as a welcome check for our calculation. A further check is provided by the overlap of the numerical results of the three hierarchies defined in Eq. (6) as we will discuss below.

At three-loop order terms up to $O(1/m_{SUSY}^{10})$ have been computed for (h1) and (h3) and up to $O(1/m_{t_1}^6)$ and $O(1/m_{SUSY}^6)$ for (h2). For each mass difference at least four expansion terms (i.e. terms including $(m_i^2 - m_j^2)^3$) could be evaluated. It is either possible to expand in the linear or the quadratic mass difference. Formally both choices are equivalent, however, in practice it turns out that depending on the actual numerical values of the parameters one can be significantly better behaved than the other. Similarly there is a freedom to choose a mass parameter, $m_R$, around which the expansion is performed. $m_R$ should be of the order of the involved masses. Note that for (h1) and (h2) only one
reference mass $m_R$ is required whereas for (h3) one needs two as can be seen from Eq. (9). Again there may be significant numerical differences and thus we adopt the following choices when evaluating the three-loop corrections to the decoupling coefficient

\begin{align}
\text{(h1)} & \quad m_R = m_{\tilde{t}1}, m_R = m_{\tilde{t}2}, m_R = m_{\tilde{q}}, m_R = m_{\tilde{q}}, m_R = \frac{m_{\tilde{t}1} + m_{\tilde{t}2} + 10m_{\tilde{q}} + m_{\tilde{q}}}{13}, \\
\text{(h2)} & \quad m_R = m_{\tilde{t}2}, m_R = m_{\tilde{q}}, m_R = m_{\tilde{q}}, m_R = \frac{m_{\tilde{t}2} + 10m_{\tilde{q}} + m_{\tilde{q}}}{12}, \\
\text{(h3)} & \quad m_{R1} = m_{\tilde{t}2}, m_{R1} = m_{\tilde{q}}, m_{R1} = m_{\tilde{q}}, m_{R1} = \frac{m_{\tilde{t}2} + 10m_{\tilde{q}} + m_{\tilde{q}}}{12}, \\
& \quad m_{R2} = m_t, m_{R2} = m_{\tilde{t}1}, m_{R2} = \frac{m_t + m_{\tilde{t}1}}{2}. \tag{7}
\end{align}

In the following it is convenient to consider the perturbative expansion of $\zeta_{\alpha_s}$ which we define as

$$
\zeta_{\alpha_s}(\mu) = 1 + \frac{\alpha_s^{(\text{SQCD})}}{\pi} \zeta^{(1)}_{\alpha_s} + \left(\frac{\alpha_s^{(\text{SQCD})}}{\pi}\right)^2 \zeta^{(2)}_{\alpha_s} + \left(\frac{\alpha_s^{(\text{SQCD})}}{\pi}\right)^3 \zeta^{(3)}_{\alpha_s} + \ldots, \tag{8}
$$

where the $\mu$ dependence of $\alpha_s^{(\text{SQCD})}$ and $\zeta^{(i)}_{\alpha_s}$ is suppressed on the right-hand side.

The general results are quite lengthy and will not be presented in this paper. However, in order to get an impression of the results we present $\zeta_{\alpha_s}$ for the hierarchy (h1) with a degenerate supersymmetric mass spectrum which reads

$$
\begin{align*}
\zeta^{(1)}_{\alpha_s} & = -\frac{1}{4} - l_S - \frac{l_t}{6}, \\
\zeta^{(2)}_{\alpha_s} & = \frac{307}{288} + \left(\frac{77}{72} + \frac{7}{3} l_t\right) l_t + \frac{49}{36} l_t^2 + \frac{25}{36} l_x + \frac{l_x^2}{2} + x_{tS} \left(\frac{1}{432} + \frac{1}{9} l_t + \frac{13}{72} l_x\right) \\
& \quad + \frac{x_{tS}^2}{2\pi} \left(-\frac{1597}{21600} + \frac{61}{720} l_x\right) + \ldots, \\
\zeta^{(3)}_{\alpha_s} & = \frac{162443}{62208} - \frac{8509}{3456} \zeta^{(3)} + \left(-\frac{27013}{5184} + \frac{2581}{432} l_x - \frac{7 l_x^2}{2}\right) l_t + \left(\frac{6361}{1728} - \frac{49}{12} l_x\right) l_t^2 \\
& \quad - \frac{343 l_t^3}{216 l_x} - \frac{21583}{5184} l_x + \frac{641}{288} l_x^2 - \frac{l_x^3}{3} + x_{tS} \left(-\frac{90481643}{3888000} + \frac{47429}{2304} \zeta^{(3)}\right) \\
& \quad + \frac{12163}{21600} - \frac{122}{135} l_x \left(l_t - \frac{79}{216} l_t^2 + \frac{51353}{86400} l_x^{-1} - \frac{69}{128} l_x^2\right) + x_{tS}^2 \left(\frac{1542497350769}{64012032000} - \frac{104479}{181440} l_x^2\right) \\
& \quad - \frac{2330095}{110592} \zeta^{(3)} + \left(\frac{585083}{12700800} - \frac{26807}{60480} l_x\right) l_t - \frac{2}{27} l_t^2 + \frac{3208403}{3386880} l_x^{-1} - \frac{104479}{181440} l_x^2\right) \\
& \quad + \ldots, \tag{9}
\end{align*}
$$

where $x_{tS} = m_t^2/m_{\text{SUSY}}^2$, $l_t = \ln(\mu^2/m_t^2)$, $l_S = \ln(\mu^2/m_{\text{SUSY}}^2)$ and $l_x = \ln(x_{tS})$. The ellipses denote terms of order $x_{tS}^3$. The corresponding results where the matching is performed
to six-flavour QCD, i.e. where the top quark is not integrated out and thus treated as massless in the loop integrals, reads

\[ \zeta_{\alpha_s}^{(1)} = -\frac{1}{4} - l_S, \]

\[ \zeta_{\alpha_s}^{(2)} = \frac{77}{96} - \frac{7}{12} l_S + l_S^2, \]

\[ \zeta_{\alpha_s}^{(3)} = -\frac{11203}{4608} - \frac{1495}{576} l_S + \frac{541}{288} l_S^2 - \frac{1349}{9216} l_S^3. \] (10)

All analytical expressions corresponding to the hierarchies of Eq. (6) can be found in the file `decsusy31.m` obtained from Ref. [28].

Let us in the following test our approximation at two loops by comparing to the exact result. For this purpose we adopt the following values for the input parameters

\[
\begin{align*}
    m_t &= 150 \text{ GeV}, & A_t &= 100 \text{ GeV}, & M_{\tilde{Q}_3} &= 500 \text{ GeV}, & \mu_{\text{SUSY}} &= 100 \text{ GeV}, \\
    \tan \beta &= 10, & M_Z &= 91.2 \text{ GeV}, & \sin^2 \theta_W &= 0.2233,
\end{align*}
\] (11)

where \(A_t\) is the trilinear coupling, \(\mu_{\text{SUSY}}\) is the Higgs-Higgsino bilinear coupling from the super potential, \(\tan \beta\) is the ratio of the vacuum expectation values of the two Higgs doublets, \(M_Z\) is the \(Z\) boson mass, \(\theta_W\) the weak mixing angle and \(M_{\tilde{Q}_3}\), a soft SUSY breaking parameter for the squark doublet of the third family. Furthermore we set the renormalization scale to \(\mu = 500 \text{ GeV}\). These parameters can be used to compute \(m_{\tilde{t}_1}, m_{\tilde{t}_2}\) and \(\theta_t\) as a function of the singlet soft SUSY breaking parameter of the top squark, \(M_{\tilde{u}_3,R}\) (see, e.g., Ref. [29]) by diagonalizing the corresponding mass matrix. The result is shown in Fig. 2(a). Furthermore we choose for simplicity \(m_{\tilde{t}_2} = m_{\tilde{q}} = m_{\tilde{g}}\). This allows us to consider in Fig. 2(b) both the exact result for \(\zeta_{\alpha_s}^{(2)}\) (solid line) and the approximations (dashed lines) based on the hierarchies (h1), (h2) and (h3). The latter are obtained from the (naive) averages over the various representations, i.e., the different choices of \(m_R\) according to Eq. (7). One observes that in the whole range of \(M_{\tilde{u}_3,R}\) at least one of the hierarchies approximates the exact to a high degree, which provides the motivation to proceed in a similar way at three loops.

Since at three-loop order the exact result is not known a criterion is needed in order to select the best approximation among the various choices at hand. For this reason we define

\[
\delta_{\text{app}} = \left| \frac{\zeta_{\text{app}}^{(2)} - \zeta_{\text{exact}}^{(2)}}{\zeta_{\text{exact}}^{(2)}} \right| + \left| \frac{\zeta_{\text{app}}^{(3)c} - \zeta_{\text{app}}^{(3)}}{\zeta_{\text{app}}^{(3)}} \right|,
\] (12)

where “app” marks an approximation result and the superscript “c” indicates that the highest terms in the expansions are cut. For each set of input parameters we choose the representation which leads to the minimal value of \(\delta_{\text{app}}\). The first term on the right-hand side of Eq. (12) guarantees that the approximation works well at two-loop order whereas the second term assures the convergence of the expansion.
Figure 2: (a) $m_{\tilde{t}_1}$, $m_{\tilde{t}_2}$ and $\theta_t$ obtained from the diagonalization of the top squark mass matrix as a function of the soft SUSY breaking parameter $M_{\tilde{u}_3,R}$. (b) $\zeta_{\alpha_s}^{(2)}$ as a function of $M_{\tilde{u}_3,R}$ using the parameters of Eq. (11). The exact result is shown as solid black line.
The three-loop result $\zeta^{(3)}$ is shown in Fig. 3 as a function of $M_{\tilde{u}_3,R}$. The notation for the three hierarchies is as in Fig. 2. The thick lines are obtained using all available expansion terms whereas for the thin curves the highest order is set to zero. Thus the difference between the thick and the corresponding thin lines is a measure for the quality of the convergence.

One observes a similar behaviour as at two-loop order: For small values of $M_{\tilde{u}_3,R}$, which correspond to small values of $m_{\tilde{t}_1}$, both (h2) and (h3) provide good approximations. With increasing $M_{\tilde{u}_3,R}$ (h3) becomes worse whereas $\zeta^{(3)}_{app}$ and $\zeta^{(3)c}_{app}$ for (h2) are still practically on top of each other. For values $300 \text{ GeV} \lesssim M_{\tilde{u}_3,R} \lesssim 800 \text{ GeV}$ the top squark masses are relatively close to each other which is the region of validity for (h1). For higher values one observes again a strong hierarchy between $m_{\tilde{t}_1}$ and $m_{\tilde{t}_2}$ and thus (h2) takes over. It is interesting to note that for each value of $M_{\tilde{u}_3,R}$ there is at least one hierarchy with a small value of $\delta_{app}$ and thus an expected good approximation to the unknown exact result. Furthermore, the approximations show a significant overlap so that the whole range of $M_{\tilde{u}_3,R}$ is covered.
Figure 4: $\alpha_s^{(SQCD)}(M_{\text{GUT}})$ as a function of $\mu_{\text{dec}}$. Thick and thin lines correspond to the one- and two-step scenario, respectively. Thin lines are only shown for three- and four-loop running.

Let us in the following briefly discuss the numerical impact of the three-loop corrections computed in this paper. In Figs. 4 we show the strong coupling at the GUT scale, $\alpha_s^{(SQCD)}(M_{\text{GUT}})$ with $M_{\text{GUT}} = 2 \cdot 10^{16}$ GeV as a function of the decoupling scale $\mu_{\text{dec}}$ which is obtained by the following procedure. The starting point is $\alpha_s^{(5,\overline{MS})}(M_Z)$. In a first step we run in the SM from $\mu = M_Z$ to $\mu = \mu_{\text{dec}}$ where the decoupling of the top quark and the SUSY particles is performed simultaneously and $\alpha_s^{(5)}(\mu_{\text{dec}})$ is transformed to $\alpha_s^{(SQCD)}(\mu_{\text{dec}})$. The use of the SQCD $\beta$ function finally leads to $\alpha_s^{(SQCD)}(M_{\text{GUT}})$. The thick lines in Fig. 4 correspond to this procedure, i.e., we use the following chain in order to arrive at $\alpha_s^{(SQCD)}(M_{\text{GUT}})$

$$
\alpha_s^{(5,\overline{MS})}(M_Z) \Rightarrow \alpha_s^{(5,\overline{MS})}(\mu_{\text{dec}}) \Rightarrow \alpha_s^{(SQCD)}(\mu_{\text{dec}}) \Rightarrow \alpha_s^{(SQCD)}(M_{\text{GUT}}).
$$

For a degenerate supersymmetric mass spectrum the decoupling constant can be found in Eq. (9).

Alternatively, in order to obtain the thin lines we integrate out the top quark in a separate
step with \( \mu = M_t \) (\( M_t \) is the on-shell top quark mass) and transform afterwards \( \alpha_s^{(6),\overline{\text{MS}}} \) to \( \alpha_s^{(\text{SQCD})}(M_{\text{GUT}}) \) in analogy to Eq. (13). Thus we have

\[
\alpha_s^{(5),\overline{\text{MS}}}(M_Z) \xrightarrow{\text{run.}} \alpha_s^{(5),\overline{\text{MS}}}(M_t) \xrightarrow{\text{dec.}} \alpha_s^{(6),\overline{\text{MS}}}(M_t) \\
\alpha_s^{(6),\overline{\text{MS}}}(\mu_{\text{dec}}) \xrightarrow{\text{dec.}} \alpha_s^{(\text{SQCD})}(\mu_{\text{dec}}) \xrightarrow{\text{run.}} \alpha_s^{(\text{SQCD})}(M_{\text{GUT}}).
\]

The decoupling constant needed for the transition from \( \alpha_s^{(6),\overline{\text{MS}}} \) to \( \alpha_s^{(\text{SQCD})} \) in the limit of degenerate SUSY masses is given in Eq. (10).

In order to obtain the numerical results in Fig. 4 we have used the measured result for \( \alpha_s^{(5)}(M_Z) \) which reads [30]

\[
\alpha_s^{(5)}(M_Z) = 0.1184 \pm 0.0007. \tag{15}
\]

Furthermore we have adopted a mSUGRA scenario with

\[
m_0 = 700 \, \text{GeV}, \quad m_{1/2} = 600 \, \text{GeV}, \quad \tan \beta = 10, \quad A_0 = 0, \quad \mu_{\text{SUSY}} > 0. \tag{16}
\]

as input for softsusy [31] in order to compute the supersymmetric mass spectrum. Note that there is only a weak dependence of the general features of our numerical result on the particular spectrum. However, it is convenient to make use of a spectrum generator in order to obtain directly the DR values for the masses at the scale \( \mu_{\text{dec}} \). To our knowledge the running of the DR parameters is only implemented to two-loop accuracy which poses a slight inconsistency in our analysis. However, this is only an minor effect and does not influence the main conclusions. In order to get an impression about the numerical values for the physical masses we show the DR results for a typical scale \( \mu_{\text{dec}} = 1000 \, \text{GeV} \)

\[
m_t = 146.7 \, \text{GeV}, \quad m_{\tilde{t}_1} = 1022 \, \text{GeV}, \quad m_{\tilde{t}_2} = 1271 \, \text{GeV}, \\
m_{\tilde{g}} = 1348 \, \text{GeV}, \quad m_{\tilde{g}} = 1326 \, \text{GeV}, \quad \theta_t = 1.26. \tag{17}
\]

At three-loop order the best approximation is provided by the hierarchy (h1). In fact the quantity \( \delta_{\text{app}} \) in Eq. (12) takes the value \( \delta_{\text{app}} = 0.002 \).

For consistency N-loop running has to be accompanied with \( N-1 \)-loop decoupling relations. Thus, we can show curves for \( N = 1, 2, 3 \) and 4 which corresponds to the (thick) dotted, dash-dotted, dashed and solid line, respectively. Within QCD the beta function is known to four-loop accuracy [32,33], however, the supersymmetric analogue only to three loops [19,34,35] \(^2\) As a consequence for the four-loop curve in Fig. 4 we only use three-loop running above \( \mu_{\text{dec}} \).

\( \mu_{\text{dec}} \) is an unphysical scale not predicted by theory. Thus, on general grounds, the dependence on \( \mu_{\text{dec}} \) has to diminish if higher order corrections are included. This is clearly visible in Fig. 4 where the dotted, dash-dotted, dashed and solid lines correspond to one-, two-, three- and four-loop running, respectively. Around the central scale of approximately

\(^2\)The four-loop SQCD \( \beta \) function is not yet complete [36].
1000 GeV all loop orders lead to predictions which are quite close. However, a variation of $\mu_{\text{dec}}$ leads to a relatively strong variation of the two-loop result which gets stabilized at three-loops and which furthermore gets to a large extend $\mu_{\text{dec}}$ independent at four loops. Actually, varying $\mu_{\text{dec}}$ between 100 GeV and 10 000 GeV changes $\alpha_s^{(\text{QCD})}(M_{\text{GUT}})$ by only 0.07%.

It is interesting to compare the variation of the individual curves with respect to $\mu_{\text{dec}}$ with the experimental uncertainty induced from $\alpha_s^{(5),\text{MS}}(M_Z)$ which is indicated by the band around the four-loop curve. The two-loop prediction is inside the band for $300 \text{ GeV} \lesssim \mu_{\text{dec}} \lesssim 1800 \text{ GeV}$ whereas the three-loop curve leaves the band only for $\mu_{\text{dec}} \gtrsim 13 000 \text{ GeV}$. It is also interesting to mention that all higher order corrections are very small for $\mu_{\text{dec}} \approx 650 \text{ GeV}$.

Note that often $\mu_{\text{dec}} = M_Z$ is chosen for the matching between the SM and the MSSM. This choice leads to strong deviation at two-loops. At three-loop order the results are already quite stable which is further supported at four loops.

Let us finally remark on the step-by-step decoupling of the top quark and the supersymmetric particles. The corresponding three- and four-loop results are shown as thin lines in Fig. 4. One observes even flatter curves than for the one-step scenario, however, the difference is numerically small and well within the uncertainty band. In this context we want to stress the wide range of $\mu_{\text{dec}}$ which is considered in Fig. 4.

3 Low-energy theorem and Higgs-gluon coupling in supersymmetric QCD

In Ref. [4] the following formula valid for all orders in perturbation theory has been derived in the framework of QCD:

\begin{equation}
C_1 = D_h^{\text{QCD}} \ln \zeta_{\alpha_s} \tag{18}
\end{equation}

where

\begin{equation}
D_h^{\text{QCD}} = -m_h \frac{\partial}{\partial m_h} \tag{19}
\end{equation}

describes the derivative with respect to the heavy mass $m_h$. Thus the $N$-loop corrections to $\zeta_{\alpha_s}$ immediately leads to $N$-loop corrections to $C_1$. Since in Eq. (18) a logarithmic derivative is taken and furthermore the dependence of $\zeta_{\alpha_s}$ on $m_h$ only occurs via $\ln(\mu^2/m_h^2)$ even the $(N+1)$ corrections of $C_1$ can be computed once the renormalization scale dependence of $\zeta_{\alpha_s}$ at $(N+1)$-loop order is re-constructed with the help of the renormalization group equations.

\textsuperscript{3}Note the different normalization of the operator $O_1$ in Ref. [4].
The LET of Eq. (18) can easily be extended to the case where more than one heavy quark is present. This version has been used in Ref. [7] in order to derive $C_1$ for theories with several heavy quarks which couple in a Yukawa-like way to the Higgs boson.

The extension of Eq. (18) to NLO corrections in the framework of the MSSM has been considered in Ref. [14]. Because of the different setup of our calculation, which is mainly due to the $\varepsilon$ scalars, we cannot take over the derivation of Ref. [14]. However, following the same line of reasoning as in Ref. [4] we obtain a version of the LET which is appropriate for the decoupling constants computed in the previous chapter. For this purpose it is convenient to consider the bare decoupling constant (see Eq. (4)) expressed in terms of bare parameters. This leads to the LET in the form

$$C_1^0 = D_h^0 \ln \zeta_{\alpha_s}^0.$$  

$D_h^0$ contains derivatives with respect to bare parameters (indicated by the superscript “0”) and can be written as

$$D_h^0 = D_t^0 + D_q^0 + V_{t i}^0 \frac{\partial}{\partial m_i^0} + (\Lambda_{\varepsilon}^0)^2 \frac{\partial}{\partial (m_\varepsilon^0)^2}.$$  

$\Lambda_\varepsilon$ is the evanescent Higgs boson-$\varepsilon$ scalar coupling which is best defined through the corresponding part of the Lagrange density [17]

$$\mathcal{L}_{\varepsilon} = -\frac{1}{2} (M_{\varepsilon}^0)^2 \varepsilon_\sigma^0 \varepsilon_{\sigma}^0 - \phi_0^0 (\Lambda_{\varepsilon}^0)^2 \varepsilon_\sigma^0 \varepsilon_{\sigma}^0.$$  

For convenience we have also displayed the mass term for the $\varepsilon$ scalar.

The derivative operators in Eq. (21) are defined through

$$D_t \equiv V_{11}^t \frac{\partial}{\partial m_{11}^t} + V_{22}^t \frac{\partial}{\partial m_{22}^t} + \frac{V_{12}^t + V_{21}^t}{2(m_{11}^t - m_{22}^t)} \frac{\partial}{\partial \theta_t},$$  

$$D_q \equiv V_{11}^q \frac{\partial}{\partial m_{11}^q} + V_{22}^q \frac{\partial}{\partial m_{22}^q} + \frac{V_{12}^q + V_{21}^q}{2(m_{11}^q - m_{22}^q)} \frac{\partial}{\partial \theta_q},$$  

where the prefactors in the top quark sector are obtained from the relations

$$V_t = -m_t \frac{\cos \alpha}{\sin \beta},$$  

$$V_{LL}^t = -2m_t^2 \frac{\cos \alpha}{\sin \beta} + M_Z^2 \cos^2 \theta_W \left(1 - \frac{1}{3} \tan^2 \theta_W\right) \sin(\alpha + \beta),$$  

$$V_{RR}^t = -2m_t^2 \frac{\cos \alpha}{\sin \beta} + \frac{4}{3} M_Z^2 \sin^2 \theta_W \sin(\alpha + \beta),$$  

$$V_{LR}^t = V_{RL}^t = \frac{m_t}{\sin \beta} (-\mu_{\text{susy}} \sin \alpha - A_t \cos \alpha),$$  

$^4$In order to keep the notation simple we omit the superscript “0” in these expressions.
\[
\begin{pmatrix} V_{11}^\tilde{t} & V_{12}^\tilde{t} \\ V_{21}^\tilde{t} & V_{22}^\tilde{t} \end{pmatrix} = R(\theta_t)\dagger \begin{pmatrix} V_{11}^\tilde{t} & V_{12}^\tilde{t} \\ V_{21}^\tilde{t} & V_{22}^\tilde{t} \end{pmatrix} R(\theta_t),
\]

with
\[
R(\theta_t) = \begin{pmatrix} \cos \theta_t & -\sin \theta_t \\ \sin \theta_t & \cos \theta_t \end{pmatrix}.
\]

All light quark masses are set to zero. Their averaged contribution is denoted by \( q \) and thus we have
\[
V_{11}^{\tilde{q}} = \frac{1}{n_l} \left( \frac{n_l + n_t}{2} V_{11}^{\tilde{d}} + \frac{n_l - n_t}{2} V_{11}^{\tilde{u}} \right),
\]
\[
V_{22}^{\tilde{q}} = \frac{1}{n_l} \left( \frac{n_l + n_t}{2} V_{22}^{\tilde{d}} + \frac{n_l - n_t}{2} V_{22}^{\tilde{u}} \right),
\]
\[
V_{12}^{\tilde{q}} = V_{21}^{\tilde{q}} = 0,
\]
\[
V_{11}^{\tilde{u}} = M_Z^2 \cos^2 \theta_W \left( 1 - \frac{1}{3} \tan^2 \theta_W \right) \sin(\alpha + \beta),
\]
\[
V_{22}^{\tilde{u}} = \frac{4}{3} M_Z^2 \sin^2 \theta_W \sin(\alpha + \beta),
\]
\[
V_{11}^{\tilde{d}} = M_Z^2 \cos^2 \theta_W \left( 1 - \frac{1}{3} \tan^2 \theta_W \right) \sin(\alpha + \beta),
\]
\[
V_{22}^{\tilde{d}} = -\frac{2}{3} M_Z^2 \sin^2 \theta_W \sin(\alpha + \beta),
\]

where “\( u \)” and “\( d \)” denote generic up- and down-type squarks, respectively, and the labels \( n_l = 5 \) and \( n_t = 1 \) are kept arbitrary for convenience. \( V_{ij}^{\tilde{q}} \) with \( i, j = 1, 2 \) are obtained in analogy to \( V_{ij}^{\tilde{t}} \).

After applying \( D_0^h \) to \( \zeta_{\alpha_s}^0 \) of Section 2 we obtain the coefficient function \( C_1 \) expressed in terms of bare parameters. Thus, in a next step one has to perform the parameter renormalization. Furthermore, it is necessary to take into account the operator renormalization constant, often denoted by \( Z_{\alpha_s} \) [17], to obtain a finite result for the coefficient function which can then be compared to [17, 18].

An alternative version of the LET (compared to Eq. (20)) is obtained by exploiting the fact that \( Z_{\alpha_s} \) and \( Z_{\alpha_s'} \) are independent of the parameters occurring in \( D_0^h \). Thus we can write
\[
C_1 = D_0^h \ln \zeta_{\alpha_s},
\]

where it is still understood that \( C_1 \) and \( \zeta_{\alpha_s} \) are expressed in terms of unrenormalized parameters. After computing \( C_1 \) with the help of Eq. (23) the parameters have to be renormalized as before, however, the operator renormalization constant is not necessary anymore.

\[\text{[17, 18]}\]
A third version of the LET reads

\[ C_1 = D_h \ln \zeta_{\alpha_s} . \]  

(24)

In this equation all quantities are expressed in terms of \( \overline{\text{DR}} \) renormalized quantities and \( \alpha_s^{\text{SQCD}} \), except the evanescent couplings (\( M_\varepsilon \) and \( \Lambda_\varepsilon \)) which are renormalized to zero. It is very convenient to use Eq. (24) since it directly leads to a finite result for \( C_1 \). It is worth noting that the computation of \( C_1 \) from Eq. (24) avoids the introduction of the evanescent coupling \( \Lambda_\varepsilon \). This can be understood by considering the renormalized version of \( D_h \) in Eq. (21) where the last term vanishes due to the condition \( \Lambda_\varepsilon^2 = (\Lambda_0^0)^2 - \delta \Lambda_\varepsilon^2 = 0 \). Due to the derivatives in Eq. (21) the expansion depth available for \( \zeta_{\alpha_s} \) is reduced. Nevertheless we can compare the results to the findings of Ref. [17, 18] where \( C_1 \) has been computed from vertex diagrams. For all three hierarchies we found complete agreement for the first three terms in the mass difference, i.e. up to order \( (m_i^2 - m_j^2)^2 \). Furthermore, for (h1) [(h3)] terms up to \( 1/m_{\text{USY}}^6 \) [1/m_{\text{USY}}^4] could be compared successfully and for (h2) all terms including \( \mathcal{O}(1/m_{\text{USY}}^4) \) and \( \mathcal{O}(1/m_{\text{USY}}^4) \) agree. Thus the calculation of the decoupling constant together with the application of the LET provides an independent confirmation of the Higgs-gluon coupling at three-loop order.

The LET in Eq. (20) differs from the one presented in [14] by the term involving \( \Lambda_0^0 \) (see Eq. (21)). Up to NLO it is possible to avoid such a contribution [14, 16], at three-loop order, however, a renormalization of the Higgs boson-\( \varepsilon \) scalar coupling is mandatory (see Ref. [17, 18] for a detailed discussion) in case derivatives with respect to bare parameters are taken.

4 Conclusions

In this paper we have computed the three-loop SQCD corrections to the decoupling constant relating \( \alpha_s \) defined in full MSSM to the one defined in QCD. The occurring three-loop integrals have been evaluated by applying expansions in various hierarchies and thus results are obtained which are valid in a large part of the parameter space. The decoupling constant constitutes an important ingredient in the relation of \( \alpha_s(M_Z) \) and \( \alpha_s(M_{\text{GUT}}) \). We have shown that the inclusion three-loop terms to the decoupling constant in combination with four-loop corrections to the \( \beta \) function leads to results for \( \alpha_s(M_{\text{GUT}}) \) which are practically independent of the decoupling scale \( \mu_{\text{dec}} \), where the effective theory is matched to the full one, even when considering a variation of \( \mu_{\text{dec}} \) by more than two orders of magnitude.

A further interesting application of the decoupling constant is its relation to the effective Higgs-gluon coupling \( C_1 \) which is obtained by simple derivatives with respect to the involved parameters. This calculation constitutes an independent check of the results obtained in Ref. [17, 18] by an explicit calculation. In this paper we provide the corresponding LET which contains all features also present at higher orders in perturbation theory. It is
valid to all orders in perturbation theory. We have checked that the renormalized version (cf. Eq. (24)) works including three-loop SQCD corrections.

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Appendix: Exact one- and two-loop result for $\zeta_{\alpha_s}$

In this Section we present the results for $\zeta_{\alpha_s}$ up to two loops taking into account the exact dependence on the occurring masses. All parameters are renormalized in the $\overline{\text{DR}}$ scheme except $M_\epsilon$ which is renormalized on-shell.

In contrast to Eq. (8) the coefficients of $\alpha_s^{(5)}$ defined through

$$\zeta_{\alpha_s}(\mu) = 1 + \frac{\alpha_s^{(5)}}{\pi} \tilde{\zeta}_{\alpha_s}^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \tilde{\zeta}_{\alpha_s}^{(2)} + \ldots,$$

is presented. The results read

$$\tilde{\zeta}_{\alpha_s}^{(1)} = \left[- \frac{1}{4} \left( C_A \left[ \frac{1}{3} \left( \frac{1}{3} l_1 + \frac{2}{3} l_2 \right) \right] + T_F \left[ N_t \left( \frac{1}{3} l_1 + \frac{1}{3} l_2 + \frac{4}{3} l_t \right) + \frac{2 N_q}{3} l \tilde{q} \right] \right) + \epsilon \left[ T_F \left( N_t \left( \frac{1}{6} l_1^2 + \frac{1}{6} l_2^2 + \frac{2}{3} l_t^2 + \zeta_2 \right) + N_q \left( l_2^2 + \frac{1}{3} \zeta_2 \right) \right) \right] + C_A \left( \frac{1}{3} L_\epsilon + \frac{1}{3} \tilde{g}^2 + \frac{1}{3} \tilde{z}_2 \right) \right] \right],$$

$$\tilde{\zeta}_{\alpha_s}^{(2)} = \frac{1}{16} \left( C_A^2 \left[ \frac{7}{36} - \frac{2}{3} l \tilde{g} \right] + C_A T_F \left[ N_q \left( \frac{5}{9} + \frac{2 m_q^2}{3 \tilde{g}^2} l \tilde{g} - \frac{2 m_q^2}{3 \tilde{g}} l \tilde{g} \right) \right) + N_t \left( 1 + \frac{4 N_{i1}}{3 D_{t_1}} + \frac{4 m_g^2 m_{t_1}^2 m_{\bar{g}}^2 N_{i1}}{3 D_{t_1}} + \Phi(m_t, m_{\bar{g}}, m_\tilde{g}) \right) \right. - \frac{2 N_{3i1}}{3 D_{t_1}^2} l_t + \left. \left[ - \frac{8}{3} + \frac{16 m_g^2 m_{t_1}^2 N_{i1}}{3 D_{t_1}^2} - \frac{2 N_{21}}{3 D_{t_1}} \right] l_t + \left[ \frac{2 m_g^2 N_{19}}{3 D_{t_1}} - \frac{8 m_g^2 m_{t_1}^2 N_{i1}}{3 D_{t_1}^2} \right] l \tilde{g} \right].
where $C_F = 4/3$, $C_A = 3$, $T_F = 1/2$, $N_t = 1$, $N_q = 5$, $\zeta_n$ is the Riemann zeta function, $l_x = \ln(\mu^2/m_x^2)$, $L_t = \ln(\mu^2/M_t^2)$ and $M_t$ is the $\epsilon$ scalar mass. Furthermore we have

\[
\begin{align*}
D_{t_i} &= m_i^4 + \left( m_{t_i}^2 - m_t^2 \right)^2 - 2m_m^2 \left( m_{t_i}^2 + m_t^2 \right), \\
D_{q\bar{q}} &= m_{q\bar{q}}^2 - m_{q\bar{q}}^2, \\
N_{1i} &= m_{t_i}^2 - m_{t_i}^2, \\
N_{2i} &= m_t^2 + m_t^2 - 3m_t^2 - m_t^2 \left( m_{t_i}^2 + m_{t_i}^2 - 2m_t^2 \right) + m_t^2 \left( m_{t_i}^2 + m_{t_i}^2 \right), \\
N_{13i} &= m_\phi^2 - m_\phi^2 \left( m_{t_i}^2 + m_{t_i}^2 \right)^2 - m_\phi^2 \left( 3m_{t_i}^2 + m_\phi^2 \right) + m_\phi^2 \left( 3m_{t_i}^2 + m_\phi^2 \right), \\
N_{23i} &= m_\phi^2 - 2m_\phi^2 + 2m_\phi^2 m_t^2 + m_\phi^2 \left( m_{t_i}^2 - m_{t_i}^2 \right), \\
N_{33i} &= m_\phi^2 - m_\phi^2 \left( 3m_{t_i}^2 + 4m_\phi^2 \right) + m_\phi^2 \left( -3m_{t_i}^2 + m_\phi^2 \right) \left( -m_{t_i}^2 m_t + m_\phi^2 \right)^2.
\end{align*}
\]

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\[ + m_y^4 \left( 3m_i^4 + 13m_i^2 m_t^2 + 6m_t^4 \right) - m_y^2 \left( m_i^6 + 6m_i^4 m_t^2 + 5m_i^2 m_t^4 + 4m_t^6 \right), \]

\[ N_{4i} = \left( m_y^2 - 3m_i^2 \right) m_t \left( m_y^2 + 3m_i^2 - m_t^2 \right), \]

\[ N_{8i} = m_y^4 - 2m_y^2 m_t^2 + m_i^4 - m_t^4, \]

\[ N_{10i} = \left( m_y^4 + m_i^4 - m_i^2 m_t^2 - m_t^2 \right) \left( 2m_i^2 + m_t^2 \right), \]

\[ N_{21i} = m_y^4 + m_i^4 \left( m_i^2 + m_t^2 \right) + m_t^2 \left( 22m_i^2 + m_t^2 \right), \]

where following abbreviations have been introduced

\[ \lambda(x, y) = \sqrt{(1 - x - y)^2 - 4xy}, \]

\[ \text{Cl}_2(x) = \text{Im} \left[ \text{Li}_2(e^{ix}) \right], \]

\[ \Phi_1(x, y) = \lambda^{-1}(x, y) \left\{ 2 \ln \left[ \frac{1}{2}(1 + x - y - \lambda(x, y)) \right] \ln \left[ \frac{1}{2}(1 - x + y - \lambda(x, y)) \right] + \frac{1}{2} \pi^2 \right. - \ln x \ln y - 2 \text{Li}_2 \left[ \frac{1}{2}(1 + x - y - \lambda(x, y)) \right] - 2 \text{Li}_2 \left[ \frac{1}{2}(1 - x + y - \lambda(x, y)) \right] \} , \]

\[ \Phi_2(x, y) = \frac{2}{\sqrt{-\lambda^2(x, y)}} \left\{ \text{Cl}_2 \left( 2 \arccos \frac{-1 + x + y}{2\sqrt{xy}} \right) + \text{Cl}_2 \left( 2 \arccos \frac{1 + x - y}{2\sqrt{x}} \right) + \text{Cl}_2 \left( 2 \arccos \frac{1 - x + y}{2\sqrt{y}} \right) \} , \]
[83x641]Φ(m_1, m_2, m_3) = \begin{cases} 
 m_3^2 \lambda^2 \left( \frac{m_1^2}{m_1^2}, \frac{m_2^2}{m_2^2}, \frac{m_3^2}{m_3^2} \right) \Phi_2 \left( \frac{m_1^2}{m_1^2}, \frac{m_2^2}{m_2^2}, \frac{m_3^2}{m_3^2} \right) & \text{Re} \left[ \lambda^2 \left( \frac{m_1^2}{m_1^2}, \frac{m_2^2}{m_2^2}, \frac{m_3^2}{m_3^2} \right) \right] < 0 \\
 m_2^3 \lambda^2 \left( \frac{m_1^2}{m_1^2}, \frac{m_2^2}{m_2^2}, \frac{m_3^2}{m_3^2} \right) \Phi_1 \left( \frac{m_1^2}{m_1^2}, \frac{m_2^2}{m_2^2}, \frac{m_3^2}{m_3^2} \right) & m_1 + m_2 \leq m_3 \\
 m_1^2 \lambda^2 \left( \frac{m_1^2}{m_1^2}, \frac{m_2^2}{m_2^2}, \frac{m_3^2}{m_3^2} \right) \Phi_1 \left( \frac{m_1^2}{m_1^2}, \frac{m_2^2}{m_2^2}, \frac{m_3^2}{m_3^2} \right) & m_2 + m_3 \leq m_1 \\
 m_3^2 \lambda^2 \left( \frac{m_1^2}{m_1^2}, \frac{m_2^2}{m_2^2}, \frac{m_3^2}{m_3^2} \right) \Phi_1 \left( \frac{m_1^2}{m_1^2}, \frac{m_2^2}{m_2^2}, \frac{m_3^2}{m_3^2} \right) & m_1 + m_3 \leq m_2. 
\end{cases}

The one-loop result agrees with Ref. [8]. The two-loop result has also been considered in Ref. [10], however, no compact result has been presented. Furthermore, the decoupling has only been considered within DRED, i.e., the transition from DREG to DRED has been performed in a separate step.

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