A LOWER BOUND FOR $L_2$ LENGTH OF SECOND FUNDAMENTAL FORM ON MINIMAL HYPERSURFACES

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Abstract. We prove a weak version of the Perdomo Conjecture, namely, there is a positive constant $\delta(n) > 0$ depending only on $n$ such that on any closed embedded, non-totally geodesic, minimal hypersurface $M^n$ in $S^{n+1}$,

$$\int_M S \geq \delta(n) \text{Vol}(M^n),$$

where $S$ is the squared length of the second fundamental form of $M^n$. The Perdomo Conjecture asserts that $\delta(n) = n$ which is still open in general. As byproducts, we also obtain some integral inequalities and Simons-type pinching results on closed embedded (or immersed) minimal hypersurfaces, with the first positive eigenvalue $\lambda_1(M)$ of the Laplacian involved.

1. Introduction

Half a century ago, S. S. Chern [13] proposed the following famous conjecture.

Conjecture 1.1. Let $M^n$ be a closed minimal hypersurface of constant scalar curvature (CSC) $R_M$ immersed in the unit sphere $S^{n+1}$. Then the set of all possible values of $R_M$ is discrete.

S. T. Yau raised it again as the 105th problem in his Problem Section [64]. The refined version of the Chern Conjecture can be stated as follows [50].

Conjecture 1.2 (Chern Conjecture). Let $M^n$ be a closed immersed minimal CSC hypersurface of $S^{n+1}$. Then $M^n$ is isoparametric.

The classification of isoparametric hypersurfaces in unit spheres was initiated in late 1930s by Cartan and finally completed till the year 2020 by many mathematicians (cf. Cecil-Chi-Jenson [7], Immervoll [33], Chi [16, 17, 18], Dorfmeister-Neher [26] and Miyaoka [39, 40], etc.), please see the excellent book [8] and the elegant survey [15] for more details and references.
In 1968, J. Simons [53] gave the first pinching result which motivated the Chern Conjecture, since the discreteness of $R_M$ is equivalent to that of $S := |A|^2$ ($A$ is the shape operator) on minimal hypersurfaces by the Gauss equation $R_M = n(n-1) - S$.

**Theorem 1.3 (Simons inequality).** Let $M^n$ be a closed immersed minimal hypersurface of $S^{n+1}$ with squared length of the second fundamental form $S$. Then

$$\int_M S(S - n) \geq 0.$$ 

In particular, if $0 \leq S \leq n$, one has either $S \equiv 0$ or $S \equiv n$ on $M^n$.

The classification of $S \equiv n$ in Theorem 1.3 was characterized by Chern-do Carmo-Kobayashi [14] and Lawson [34] independently. Namely, the Clifford tori are the only closed minimal CSC hypersurfaces in $S^{n+1}$ with $S \equiv n$, i.e.,

$$M^n = S^k\left(\sqrt{\frac{k}{n}}\right) \times S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right), \quad 1 \leq k \leq n - 1.$$ 

In 1983, Peng and Terng [43, 44] made the first breakthrough towards Chern Conjecture 1.1, namely, for minimal CSC hypersurfaces in $S^{n+1}$, if $S > n$, then $S > n + \frac{1}{12n}$. Moreover, if $S > 3$ for $n = 3$, then $S \geq 6$. In 1993, Chang [9] finished the proof of Chern Conjecture 1.2 for $n = 3$. Yang-Cheng [63] and Suh-Yang [54] improved the second gap from $\frac{1}{12n}$ to $\frac{3}{7n}$. However, it is still an open problem for higher dimensional case whether $S \geq 2n$ if $S > n$. As for minimal isoparametric hypersurfaces in $S^{n+1}$ with $g$ distinct principal curvatures, it indeed satisfies $S = (g-1)n \geq 2n$ when we exclude the $g = 1$ case (the equators) and the $g = 2$ case (the Clifford tori).

Strongly supporting Chern Conjecture 1.2, a recent remarkable progress by Tang-Wei-Yan [55] and Tang-Yan [58] generalized the theorem of de Almeida and Brito [22] for $n = 3$ to arbitrary dimension $n$. Namely, a closed immersed hypersurface $M^n$ in $S^{n+1}$ having constant $1, 2, \cdots, (n - 1)$-th mean curvatures and nonnegative scalar curvature $R_M \geq 0$ is isoparametric. de Almeida-Brito-Scherfner-Weiss [23] showed that a closed immersed hypersurface $M^n$ in $S^{n+1}$ having constant Gauss-Kronecker curvature $K_M$ and $3$ distinct principal curvatures everywhere is isoparametric. For the four dimensional case, Deng-Gu-Wei [24] proved that if $M^4$ is a closed Willmore minimal CSC hypersurface in $S^5$, then it is isoparametric. In other words, in dimension four [24] dropped the nonnegativity assumption $R_M \geq 0$ of [58], under the new condition of being Willmore which is equivalent to that the third mean curvature vanishes other than being only a constant as in [58]. In fact, given some pinching restrictions other than identities to the third mean curvature and the Gauss-Kronecker curvature, one can also remove the nonnegativity assumption $R_M \geq 0$ of [58] in dimension four (cf. [36]).

More related results can be found in the surveys by Scherfner-Weiss [49], Scherfner-Weiss-Yau [50] and Ge-Tang [29]. Very recently, we [28] gave a characterization to the condition of Chern Conjecture 1.2, a Takahashi-type theorem, i.e., An immersed
hypothesis $M^n$ in $S^{n+1}$ is minimal and has constant scalar curvature if and only if $\Delta \nu = \lambda \nu$ for some constant $\lambda$, where $\nu$ is a unit normal vector field of $M^n$.

Without assuming constant scalar curvature, Peng and Terng \[43, 44\] obtained that there exists a positive constant $\delta(n)$ depending only on $n$, such that if $n \leq S \leq n + \delta(n)$, $n \leq 5$, then $S \equiv n$, i.e., $M^n$ is a minimal Clifford torus. Later, Cheng-Ishikawa \[10\] improved the previous pinching constant for $n \leq 5$, Wei-Xu \[60\] extended it to $n = 6, 7$, and Zhang \[66\] extended it to $n \leq 8$. Finally, Ding and Xin \[25\] proved the second gap for all dimensions, in particular, they showed $\delta(n) = \frac{n}{23}$ for $n \geq 6$. Xu-Xu \[62\] improved this pinching constant to $\delta(n) = \frac{n}{22}$ and Li-Xu-Xu \[37\] further improved it to $\delta(n) = \frac{n}{18}$. As a matter of fact in the pinching results above, the condition $S \geq n$ is indispensable owing to some counterexamples of Otsuki \[42\].

In this paper, without assuming constant scalar curvature, we are interested in whether there is a universal lower bound for the mean value of $S$ on non-totally geodesic minimal hypersurfaces. Inspired by the preceding paper \[28\], we answer this question affirmatively for embedded hypersurfaces.

**Theorem 1.4 (Main Theorem).** Let $M^n$ be a closed embedded, non-totally geodesic, minimal hypersurface in $S^{n+1}$. Then there is a positive constant $\delta(n) > 0$, depending only on $n$, such that

$$\int_M S \geq \delta(n) \text{Vol}(M^n).$$

In fact, Theorem 1.4 provides an evidence to the following Perdomo Conjecture.

**Conjecture 1.5 (Perdomo Conjecture \[46\]).** Let $M^n$ be a closed embedded, non-totally geodesic, minimal hypersurface in $S^{n+1}$, then

$$\int_M S \geq n \text{Vol}(M^n).$$

Moreover, the equality holds if and only if $S \equiv n$, i.e., $M^n$ is a minimal Clifford torus.

The equality assertion of Perdomo Conjecture 1.5 for surfaces is equivalent to the Lawson Conjecture \[35\], i.e., *The only embedded minimal torus in $S^3$ is the Clifford torus.* This is because by the Gauss equation and the Gauss-Bonnet theorem, for genus $g$ minimal surface $M^2 \subset S^3$, we have

$$\int_M S = 8\pi (g - 1) + 2 \text{Vol}(M^2).$$

Notice that the inequality also holds for genus $g \geq 2$ surfaces, while for the $g = 0$ case, it follows from Almgren \[3\] and Calabi \[6\] that any embedded minimal sphere in $S^3$ is totally geodesic (which is not true for higher dimensions by Hsiang \[31\]). The Lawson Conjecture has been proven by Brendle \[4, 5\].
For general dimension, Perdomo \cite{46} also conjectured that the only minimal immersed hypersurfaces in $S^{n+1}$ with $\text{Index}(M^n) = n + 3$ are the minimal Clifford tori. With an additional assumption on the symmetries of $M^n$, this conjecture was verified by Perdomo himself \cite{45}. In particular, if Conjecture 1.5 is true, then the conjecture above also holds for embedded hypersurfaces \cite{46}. In addition, Perdomo \cite{47} proved that the condition “$M^n$ is embedded” is needed in Conjecture 1.5. This is due to the fact that rotational minimal hypersurfaces (those with exactly two principal curvatures) satisfy the reverse inequality and it is known that none of these examples are embedded. For more relations between the Jacobi (or stability) operator and Perdomo’s conjectures we refer to \cite{1, 2, 45, 46}. Besides, assume there are $(n+2)$ great hyperspheres of $S^{n+1}$ perpendicular to each other, such that $M^n$ is symmetric with respect to them, then Conjecture 1.5 was verified by Wang and Wang \cite{59} very recently. There are some similar pinching results (cf. \cite{30, 38, 52, 61}).

In fact, Theorem 1.4 is a corollary of the general inequality below.

**Theorem 1.6.** Let $M^n$ be a closed embedded hypersurface in $S^{n+1}$ and $\lambda_1(M)$ be the first positive eigenvalue of the Laplacian. Then

$$\int_M S \geq \lambda_1(M) \frac{\text{Vol}^2(M^n) - \text{Vol}^2(S^n)}{\text{Vol}(M^n)},$$

where the equality holds if and only if $M^n$ is totally geodesic.

The following bound of the first eigenvalue (which can be compared with the Yang-Yau inequality $\lambda_1(M) \leq \frac{8\pi(1+g)}{\text{Vol}(M^n)}$ for genus $g$ (minimal) surfaces, see \cite{51}), and the Simons-type pinching result for general (not only minimal) closed hypersurfaces are immediate corollaries of Theorem 1.6.

**Corollary 1.7.** Let $M^n$ be a closed embedded hypersurface in $S^{n+1}$. If $\text{Vol}(M^n) > \text{Vol}(S^n)$, then

$$\lambda_1(M) < \frac{\text{Vol}(M^n) \int_M S}{\text{Vol}^2(M^n) - \text{Vol}^2(S^n)}.$$ 

**Corollary 1.8.** Let $M^n$ be a closed embedded hypersurface in $S^{n+1}$. If

$$\int_M S \leq \lambda_1(M) \frac{\text{Vol}^2(M^n) - \text{Vol}^2(S^n)}{\text{Vol}(M^n)},$$

then $M^n$ is totally geodesic.

Cheng-Li-Yau \cite{12} proved in 1984 that if $M^n$ is a closed immersed minimal hypersurface in $S^{n+1}$ and $M^n$ is non-totally geodesic, then there is a positive constant $c(n) > 0$, depending only on $n$, such that the area of $M^n$ satisfies

$$\text{Vol}(M^n) > (1 + c(n)) \text{Vol}(S^n).$$

Thus, Theorem 1.4 follows from Theorem 1.6 as $\lambda_1(M) \geq \frac{n^2}{2}$ by Choi and Wang \cite{21}.
Throughout this paper, we denote by $S$ the squared length of the second fundamental form and its maximum and minimum by

$$S_{\text{max}} = \sup_{p \in M^n} S(p), \quad S_{\text{min}} = \inf_{p \in M^n} S(p).$$

For immersed case we also obtain similar inequalities and pinching results as follows.

**Theorem 1.9.** Let $M^n$ be a closed immersed minimal hypersurface in $S^{n+1}$.

(i) If $S \not\equiv 0$, then

$$\int_M S \geq \left( \lambda_1(M) - \frac{2(n-1)}{\lambda_1(M)(2n-1)} (S_{\text{max}} - n) (S_{\text{max}} - S_{\text{min}}) \right) \Vol(M^n).$$

(ii) The following Simons-type inequality holds:

$$\int_M S^2 \geq \frac{n}{n-1} \int_M S \left( \frac{2n-1}{n} \lambda_1(M) - \frac{\int_M S}{\Vol(M^n)} \right).$$

If $S \not\equiv 0$, then

$$n \int_M S + (n-1)S_{\text{max}} \Vol(M^n) \geq (2n-1)\lambda_1(M)\Vol(M^n).$$

**Corollary 1.10.** Let $M^n$ be a closed immersed minimal hypersurface in $S^{n+1}$. If

(i) $\int_M S \leq \frac{(2n-1)\lambda_1^2(M)}{2n-1} \Vol(M^n);$  
(ii) $S_{\text{max}} - S_{\text{min}} < n;$

then $M^n$ is totally geodesic.

**Remark 1.11.** If $M^n$ is embedded, then by Choi and Wang [21] we can replace $\lambda_1(M)$ with $n/2$ in Theorems 1.9 and 1.10. These results can be improved further by replacing $\lambda_1(M)$ with $n$ if the following Yau Conjecture is true.

**Conjecture 1.12 (Yau Conjecture).**

(i) Let $M^n$ be a closed embedded minimal hypersurface of $S^{n+1}$, then $\lambda_1(M) = n$.

(ii) The area of one of the minimal Clifford tori gives the lowest value of area among all non-totally geodesic closed minimal hypersurfaces of $S^{n+1}$.

In particular, Tang and Yan [57] proved Yau Conjecture 1.12 (i) in the isoparametric case. There are some classical results on the first eigenvalue for minimal hypersurfaces in spheres (cf. [5, 19, 20, 21, 56], etc.). For (ii) of Yau Conjecture 1.12 (also called the Solomon-Yau Conjecture [27]), among minimal rotational hypersurfaces Perdomo and Wei [48] showed numerical evidences that it is true if $2 \leq n \leq 100$ and Cheng-Wei-Zeng [11] showed in all dimensions. Remarkably, in the asymptotic sense, Ilmanen-White [32] verified the Solomon-Yau Conjecture in the class of topologically nontrivial hypercones.
2. Preliminary lemmas and a Simons-type pinching result

In this section, we will give some necessary lemmas which also lead to a Simons-type pinching result (see Proposition 2.4).

Let $x : \mathbb{M}^n \hookrightarrow \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be a closed immersed hypersurface in the unit sphere $\mathbb{S}^{n+1}$ and $\nu(x)$ denote the unit normal vector field, $\nabla$ and $\overline{\nabla}$ be the Levi-Civita connections on $\mathbb{M}^n$ and $\mathbb{S}^{n+1}$, respectively. Following [28], for any unit vector $a \in \mathbb{S}^{n+1}$, we consider the height functions on $\mathbb{M}^n$, 

$$\varphi_a(x) = \langle x, a \rangle, \quad \psi_a(x) = \langle \nu, a \rangle.$$ 

Then we have the following basic properties.

**Proposition 2.1.** [28, 41] For all $a \in \mathbb{S}^{n+1}$, we have

$$\nabla \varphi_a = a^T, \quad \nabla \psi_a = -Aa^T,$$

$$\Delta \varphi_a = -n \varphi_a + nH \psi_a, \quad \Delta \psi_a = -n \langle \nabla H, a \rangle + nH \varphi_a - S \psi_a.$$

where $a^T \in \Gamma(TM)$ denotes the tangent component of $a$ along $\mathbb{M}^n$, $A$ is the shape operator with respect to $\nu$, i.e., $A(X) = -\overline{\nabla}_X \nu$, $S = \|A\|^2 = \text{tr} AA^t$, $H = \frac{1}{n} \text{tr} A$ is the mean curvature.

**Lemma 2.2.** (Choi and Wang [21]) Let $\mathbb{M}^n$ be a closed embedded minimal hypersurface in $\mathbb{S}^{n+1}$, then $\lambda_1(\mathbb{M}) \geq n/2$.

A careful argument (see [5, Theorem 5.1]) shows that the strict inequality holds, i.e., $\lambda_1(\mathbb{M}) > n/2$ in Lemma 2.2.

**Lemma 2.3.** Let $\mathbb{M}^n$ be a closed immersed minimal hypersurface in $\mathbb{S}^{n+1}$, then

$$\int_{\mathbb{M}} S (S - n) (S - S_{\min}) \geq \lambda_1(\mathbb{M}) \left( \int_{\mathbb{M}} S^2 - \frac{\left( \int_{\mathbb{M}} S \right)^2}{\text{Vol}(\mathbb{M}^n)} \right).$$

Moreover, if $\mathbb{M}^n$ is embedded, then

$$\int_{\mathbb{M}} S (S - n) (S - S_{\min}) \geq \frac{n}{4} \left( \int_{\mathbb{M}} S^2 - \frac{\left( \int_{\mathbb{M}} S \right)^2}{\text{Vol}(\mathbb{M}^n)} \right).$$

**Proof.** Without loss of generality, we suppose $S$ is not a constant on $\mathbb{M}^n$. Let $\mathcal{F}$ be the set of non-constant functions $f : \mathbb{M}^n \to \mathbb{R}$ with $\int_{\mathbb{M}} f = 0$ and $f \in H^1(\mathbb{M})$. Recall the Simons identity (cf. [25])

$$\frac{1}{2} \Delta S = |\nabla A|^2 + S(n - S).$$

(2.1)
Let \( u = S - \frac{\int_M S}{\text{Vol}(M)} \in \mathcal{F} \), one has

\[
\lambda_1(M) = \inf_{f \in \mathcal{F}} \frac{\int_M |\nabla f|^2}{\int_M f^2} \leq \frac{\int_M |\nabla u|^2}{\int_M u^2}
\]

\[
= \frac{\int_M |\nabla S|^2}{\int_M u^2} = \frac{-\int_M S \Delta S}{\int_M u^2}
\]

\[
= -2 \int_M S (|\nabla A|^2 + S(n - S)) \frac{1}{\int_M u^2} \leq -2 \int_M (S_{\text{min}} |\nabla A|^2 + S^2(n - S)) \frac{1}{\int_M u^2}
\]

\[
= 2 \int_M S (S - n)(S - S_{\text{min}}) \frac{1}{\int_M u^2},
\]

and

\[
\int_M u^2 = \int_M S^2 - \frac{(\int_M S)^2}{\text{Vol}(M^n)} \geq 0.
\]

This proves the first inequality and the second follows from Lemma 2.2.

From Lemma 2.3, we have the following pinching result immediately.

**Proposition 2.4.** Let \( M^n \) be a closed embedded minimal hypersurface in \( S^{n+1} \). If

(i) \( \int_M S \leq n \text{Vol}(M^n) \);

(ii) \( \int_M S^2 (S - n) \leq S_{\text{max}} \int_M S (S - n) \);

(iii) \( S_{\text{max}} - S_{\text{min}} < \frac{n}{4} \);

then \( S \equiv n \) or \( S \equiv 0 \), i.e., \( M^n \) is a minimal Clifford torus or an equator. Moreover, if Yau Conjecture 1.12 (i) is true, condition (iii) can be replaced by

\( S_{\text{max}} - S_{\text{min}} < \frac{n}{2} \).

**Proof.** By the Simons identity (2.1) and condition (i), we have

\[
\int_M S^2 - \frac{(\int_M S)^2}{\text{Vol}(M^n)} = n \int_M S - \frac{(\int_M S)^2}{\text{Vol}(M^n)} + \int_M |\nabla A|^2 \geq \int_M |\nabla A|^2.
\]

Then it follows from Lemma 2.3 condition (ii) and (2.1) that

\[
\int_M S (S - n) (S_{\text{max}} - S_{\text{min}} - \frac{n}{4}) = (S_{\text{max}} - S_{\text{min}} - \frac{n}{4}) \int_M |\nabla A|^2 \geq 0.
\]

Therefore by condition (iii), \( |\nabla A| \equiv 0 \) and \( S \equiv n \) or \( S \equiv 0 \), i.e., \( M^n \) is a minimal Clifford torus or an equator. The proof is similar if Yau Conjecture 1.12 (i) is true. □
Lemma 2.5. Let $M^n$ be a closed immersed, non-totally geodesic, minimal hypersurface in $S^{n+1}$, then for all $a \in S^{n+1}$, we have
\[
\int_M S \psi_a^2 \geq \lambda_1(M) \frac{\left( \int_M S \right)^2 \int_M \psi_a^2}{\text{Vol}(M^n) \int_M S^2}.
\]

Proof. For all $a \in S^{n+1}$, by Proposition 2.1, we have
\[
\Delta \psi_a = -S \psi_a, \quad \int_M S \psi_a = 0.
\]
Hence, for all constant $K \in \mathbb{R}$, one has
\[
\left( \int_M \psi_a \right)^2 = \left( \int_M (1 - KS) \psi_a \right)^2 \leq \int_M (1 - KS)^2 \int_M \psi_a^2.
\]
Set
\[
K = \frac{\int_M S}{\int_M S^2},
\]
then by similar argument as in Lemma 2.3, we deduce
\[
\int_M S \psi_a^2 = \int_M -\psi_a \Delta \psi_a = \int_M |\nabla \psi_a|^2 \geq \lambda_1(M) \left( \int_M \psi_a^2 - \frac{\left( \int_M \psi_a \right)^2}{\text{Vol}(M^n)} \right)
\]
\[
\geq \lambda_1(M) \frac{\int_M (2KS - K^2S^2)}{\text{Vol}(M^n)} \int_M \psi_a^2 = \lambda_1(M) \frac{\left( \int_M S \right)^2 \int_M \psi_a^2}{\text{Vol}(M^n) \int_M S^2}.
\]

Remark 2.6. If Yau Conjecture 1.12 (i) is true, i.e., $\lambda_1(M) = n$ for all embedded minimal hypersurface of $S^{n+1}$, we have
\[
\int_M S \psi_a^2 \geq n \frac{\left( \int_M S \right)^2 \int_M \psi_a^2}{\text{Vol}(M^n) \int_M S^2},
\]
for all $a \in S^{n+1}$. Hence, summing up this inequality over an orthonormal basis $\{a_j\}_{j=1}^{n+2}$ for $a = a_j$ and noting that $\sum_{j=1}^{n+2} \psi_{a_j}^2 = 1$, we obtain $\int_M S^2 \geq n \int_M S$. This gives another proof of the Simons inequality for embedded hypersurfaces.

Lemma 2.7. Let $M^n$ be a closed immersed minimal hypersurface in $S^{n+1}$, then for all $a \in S^{n+1}$, we have
\[
\int_M S \psi_a^2 \leq \frac{n - 1}{2n - 1} \int_M S.
\]

Proof. Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of $A$ with $\lambda_1^2 \geq \lambda_2^2 \geq \cdots \geq \lambda_n^2$. Then we have
\[
\sum_{i=1}^n \lambda_i = 0, \quad \sum_{i=1}^n \lambda_i^2 = ||A||^2 = S.
\]
Thus
\[ 0 = \left( \sum_{i=1}^{n} \lambda_i \right)^2 = \lambda_1^2 + 2 \lambda_1 \sum_{i=2}^{n} \lambda_i + \left( \sum_{i=2}^{n} \lambda_i \right)^2 = -\lambda_1^2 + \left( \sum_{i=2}^{n} \lambda_i \right)^2 \]
\[ \leq -\lambda_1^2 + (n-1) \sum_{i=2}^{n} \lambda_i = (n-1)S - n\lambda_1^2. \]

Hence
\[ \lambda_1^2 \leq \frac{n-1}{n} S, \]

where the equality holds if and only if \( \lambda_1 = (1-n)\lambda_2 \) and \( \lambda_2 = \lambda_3 = \cdots = \lambda_n \). It follows from Proposition 2.1 that
\[
\int_M S \psi_a^2 = \int_M -\psi_a \Delta \psi_a = \int_M |\nabla \psi_a|^2 = \int_M \langle Aa^T, Aa^T \rangle \\
\leq \int_M \lambda_1^2 \langle a^T, a^T \rangle = \int_M \lambda_1^2 (1 - \psi_a^2 - \varphi_a^2) \\
\leq \frac{n-1}{n} \int_M S (1 - \psi_a^2 - \varphi_a^2).
\]

Then
\[
\int_M S \psi_a^2 \leq \frac{n-1}{2n-1} \int_M S (1 - \varphi_a^2) \leq \frac{n-1}{2n-1} \int_M S. \quad \square
\]

**Lemma 2.8.** Let \( M^n \) be a closed immersed hypersurface in \( S^{n+1} \), then there exists an equator \( S^n \subset S^{n+1} \) such that for all \( a \in S^n \),
\[
\int_M \psi_a = 0.
\]

**Proof.** Observe that the function \( I(a) \) on \( a \in \mathbb{R}^{n+2} \) defined by \( I(a) = \int_M \psi_a \) is a linear function. Therefore the kernel of \( I(a) \) is a linear subspace of \( \mathbb{R}^{n+2} \) of dimension at least \( n+1 \), whose intersection with \( S^{n+1} \) is the required equator. \( \square \)

**Remark 2.9.** For a closed immersed (or embedded), non-totally geodesic, minimal hypersurface \( M^n \) in \( S^{n+1} \), we conjecture that \( \int_M \psi_a = 0 \) for all \( a \in S^{n+1} \). For example, if \( M^n \) is invariant under the antipodal map and the unit normal vector field along \( M^n \) is odd, i.e., \( \nu(-x) = -\nu(x) \), then the conjecture holds. This would improve the inequality of Theorem 1.14 into
\[ \int_M S \geq \lambda_1(M) \text{Vol}(M^n). \]

Furthermore, it would prove the inequality part of Perdomo Conjecture 1.3 if Yau Conjecture 1.12 (i) is true.
3. Proof of the theorems

**Proof of Theorem 3.6.** Let $M^n$ be a closed embedded hypersurface in $S^{n+1}$. Denote the components of $S^{n+1}\setminus M^n$ by $U_1$ and $U_2$, then we have $S^{n+1} = U_1 \cup M^n \cup U_2$. Obviously, we can extend the height function $\varphi_a(x) = \langle x, a \rangle$ on $M^n$ to $S^{n+1}$. It is easy to prove that on $x \in S^{n+1}$,

$$\nabla \varphi_a = a^T, \quad \Delta \varphi_a = -(n + 1) \varphi_a,$$

where $a^T = a - \varphi_a x \in \Gamma(TS^{n+1})$ denotes the tangent component of $a$ at $x \in S^{n+1}$. Then for all $a \in S^{n+1}$ and $i \in \{1, 2\}$, by the divergence theorem we have

$$\left| \int_M \psi_a \right| = \left| \int_{U_i} \text{div}(a^T) \right| = \left| \int_{U_i} \Delta \varphi_a \right| = (n + 1) \left| \int_{U_i} \varphi_a \right|.$$

By

$$\left| \int_{U_i} \varphi_a \right| \leq \left| \int_{U_i} |\varphi_a| \right|,$$

and

$$\int_{U_1} |\varphi_a| + \int_{U_2} |\varphi_a| = \int_{S^{n+1}} |\varphi_a| = 2 \text{Vol}(\mathbb{S}^{n+1}) = \frac{2}{n + 1} \text{Vol}(\mathbb{S}^n),$$

we can choose some $i_0 \in \{1, 2\}$ for every fixed $a \in S^{n+1}$, such that

$$(3.1) \quad \left| \int_M \psi_a \right| = (n + 1) \left| \int_{U_{i_0}} \varphi_a \right| \leq (n + 1) \int_{U_{i_0}} |\varphi_a| \leq \text{Vol}(\mathbb{S}^n).$$

By Lemma 2.8 we can choose an orthonormal basis $\{a_j\}_{j=1}^{n+2}$ of $\mathbb{R}^{n+2}$ such that $a_i$ ($i = 1, \cdots, n+1$) lie in the kernel of $I(a)$, i.e.,

$$(3.2) \quad I(a_i) = \int_M \psi_{a_i} = 0, \quad i = 1, \cdots, n+1.$$

Let $\{e_k\}_{k=1}^n$ be a local orthonormal frame of $M^n$ such that $Ae_k = \lambda_k e_k$. Then the tangent component $a^T$ along $M^n$ and $Aa^T$ can be expressed by

$$a^T = \sum_{k=1}^n \langle a, e_k \rangle e_k, \quad Aa^T = \sum_{k=1}^n \langle a, e_k \rangle \lambda_k e_k.$$

This implies

$$\sum_{j=1}^{n+2} |Ae_j|^2 = \sum_{j=1}^{n+2} \sum_{k=1}^n \langle a_j, e_k \rangle^2 \lambda_k^2 = \sum_{k=1}^n \lambda_k^2 = |A|^2 = S.$$
Then, by $\sum_{j=1}^{n+2} \psi_{a_j}^2 = 1$, (3.1) (3.2) and Proposition 2.1, we deduce

$$\int_M S = \int_M \sum_{j=1}^{n+2} |Aa_j|^2 = \sum_{j=1}^{n+2} \int_M |\nabla \psi_{a_j}|^2$$

$$\geq \lambda_1(M) \sum_{j=1}^{n+2} \left( \int_M \psi_{a_j}^2 - \left( \frac{\int_M \psi_{a_{n+2}}}{\text{Vol}(M^n)} \right)^2 \right)$$

$$= \lambda_1(M) \left( \text{Vol}(M^n) - \left( \frac{\int_M \psi_{a_{n+2}}}{\text{Vol}(M^n)} \right)^2 \right)$$

$$\geq \lambda_1(M) \frac{\text{Vol}^2(M^n) - \text{Vol}^2(S^n)}{\text{Vol}(M^n)}.$$

This proves the inequality of the theorem. If the equality holds, the equal signs of (3.1) also hold. Then there exists some $a = a_{n+2} \in S^{n+1}$ such that

$$\varphi_a \geq 0 \ (\text{or } \varphi_a \leq 0) \quad \text{on } U_{i_0},$$

and thus $U_{i_0}$ lies in the hemisphere $S^{n+1}_+ := \{ x \in S^{n+1} \mid \varphi_a(x) \geq 0 \}$, moreover,

$$\int_{U_{i_0}} \varphi_a = \frac{1}{n+1} \text{Vol}(S^n).$$

On the other hand, we have

$$\int_{S^{n+1}_+} \varphi_a = \frac{1}{n+1} \text{Vol}(S^n).$$

Therefore we have $U_{i_0} = S^{n+1}_+$ and thus $M^n = S^n$ is totally geodesic. \qed

**Proof of Theorem 1.9**: Case (i). As in the proof of Theorem 1.6, by Lemma 2.8, we can choose an orthonormal basis $\{a_j\}_{j=1}^{n+2}$ of $\mathbb{R}^{n+2}$ such that

$$\int_M \psi_{a_i} = 0, \quad i = 1, \ldots, n+1.$$

Hence by Proposition 2.1,

$$\int_M S \psi_{a_i}^2 = \int_M |\nabla \psi_{a_i}|^2 \geq \lambda_1(M) \int_M \psi_{a_i}^2, \quad i = 1, \ldots, n+1.$$

Due to Lemma 2.5 and (3.3), we have

$$\sum_{i=1}^{n+1} \int_M S \psi_{a_i}^2 + \frac{\text{Vol}(M^n) \int_M S^2}{(\int_M S)^2} \int_M S \psi_{a_{n+2}}^2 \geq \lambda_1(M) \sum_{i=1}^{n+2} \int_M \psi_{a_i}^2.$$

Since $\sum_{i=1}^{n+2} \psi_{a_i}^2 = 1$, we have

$$\int_M S + \frac{\text{Vol}(M^n) \int_M S^2}{(\int_M S)^2} \int_M S \psi_{a_{n+2}}^2 \geq \lambda_1(M) \text{Vol}(M^n).$$
By Lemma 2.3, Lemma 2.7, (2.2) and (3.4), we obtain
\[
\int_M S + \frac{\int_M S(S-n)(S-S_{\min})}{\int_M S} \frac{2(n-1)\text{Vol}(M^n)}{\lambda_1(M)(2n-1)} \geq \lambda_1(M)\text{Vol}(M^n).
\]
By Theorem 1.3 and \(\lambda_1(M) \leq n\) for minimal hypersurfaces, without loss of generality, we can suppose \(S_{\min} \leq n \leq S_{\max}\). It follows that
\[
(S-n)(S-S_{\min}) \leq (S_{\max}-n)(S_{\max}-S_{\min}).
\]
Thus we obtain the required inequality
\[
\int_M S \geq \left(\lambda_1(M) - \frac{2(n-1)}{\lambda_1(M)(2n-1)} (S_{\max}-n)(S_{\max}-S_{\min})\right) \text{Vol}(M^n).
\]

**Case (ii).** By Lemma 2.7, (2.2) and (3.4), we have
\[
\int_M S + \frac{n-1}{2n-1} \frac{\text{Vol}(M^n) \int_M S^2 - (\int_M S)^2}{\int_M S} \geq \lambda_1(M)\text{Vol}(M^n).
\]
Thus
\[
\int_M S^2 \geq \frac{n}{n-1} \int_M S \left(\frac{2n-1}{n} \lambda_1(M) - \frac{\int_M S}{\text{Vol}(M^n)}\right),
\]
which implies
\[
n \int_M S + (n-1)S_{\max}\text{Vol}(M^n) \geq (2n-1)\lambda_1(M)\text{Vol}(M^n),
\]
if \(S \neq 0\). \(\square\)

**Proof of Corollary 1.10.** If \(S \neq 0\), then by Theorem 1.9 we have
\[
\int_M \left(S + \frac{2(n-1)}{\lambda_1(M)(2n-1)} (S_{\max}-n)(S_{\max}-S_{\min})\right) \geq \lambda_1(M)\text{Vol}(M^n).
\]
Substituting condition (ii) \(S_{\max} < S_{\min} + n\) into the inequality above, we get
\[
\int_M S \left(1 + \frac{2(n-1)n}{\lambda_1(M)(2n-1)}\right) \geq \int_M \left(S + \frac{2(n-1)n}{\lambda_1(M)(2n-1)}S_{\min}\right) > \lambda_1(M)\text{Vol}(M^n).
\]
Therefore
\[
\int_M S > \frac{(2n-1)\lambda_1^2(M)}{(2n-1)\lambda_1(M) + 2n(n-1)} \text{Vol}(M^n),
\]
contradicting to condition (i). \(\square\)

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