The top-order energy of quasilinear wave equations in two space dimensions is uniformly bounded

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\section*{A B S T R A C T}

Alinhac solved a long-standing open problem in 2001 and established that quasilinear wave equations in two space dimensions with quadratic null nonlinearities admit global-in-time solutions, provided that the initial data are compactly supported and sufficiently small in Sobolev norm. In this work, Alinhac obtained an upper bound with polynomial growth in time for the top-order energy of the solutions. A natural question then arises whether the time-growth is a true phenomenon, despite the possible conservation of basic energy. In the present paper, we establish that the top-order energy of the solutions in Alinhac theorem remains globally bounded in time.

\section{1. Introduction}

1.1. Background

We consider quasilinear wave equations in two space dimensions with quadratic null nonlinearities. In 2001 Alinhac established a global-in-time existence theory \cite{Alinhac2001} when the initial data are compactly supported and sufficiently small in Sobolev norm, which was a long-standing open problem then. In this work, Alinhac established an upper bound with polynomial growth (in time) for the top-order energy of the solutions. A natural question is whether the time-growth of the top-order energy is a true phenomenon or not. In fact, Alinhac has explicitly raised this question in Alinhac \cite{Alinhac2001} (see the remark below Theorem 27 therein), which is the main motivation of our study. In addition, Alinhac conjectured \cite{Alinhac2001a, Alinhac2001b, Alinhac2001c} that the time-growth should be a true phenomenon and called it “blowup-at-infinity”, in the context of three-dimensional, quasilinear wave equations enjoying a weak null structure of Lindblad and Rodnianski \cite{Lindblad2001}. Interestingly and importantly, Alinhac’s conjecture was recently established by Deng and Pusateri \cite{Deng2022} for a class of three-dimensional nonlinear wave equations satisfying the weak null condition, and the authors did prove the high-order energy of the solutions cannot be uniformly bounded in time. Thus, it is interesting to consider the same question for two-dimensional quasilinear wave equations with null nonlinearities, which remained open until now. Yet, there has been substantial progress concerning this question in recent years.

In ref. \cite{Cai2022}, Cai, Lei, and Masmoudi showed the uniform boundedness of the top-order energy for a class of two-dimensional quasilinear wave equations with (hidden) strong null structure in the sense of Lei \cite{Lei2022}. More recently, Cai \cite{Cai2021} considered the system of two-dimensional incompressible elastodynamics, and again proved the uniform boundedness of the top-order energy. There are also some related results for three-dimensional wave equations. For instance, Wang \cite{Wang2022} proved that quasilinear wave equations with quadratic null nonlinearities in $\mathbb{R}^{+}\times\mathbb{R}^3$ admit compactly supported small global-in-time solutions with uniformly bounded energy. Lei and Wang in \cite{Lei2023} showed the uniform boundedness of the top-order energy for the three-dimensional incompressible isotropic elastodynamics. Using the hyperboloidal foliation
method, LeFloch and Ma [30] treated nonlinear systems coupling wave equations and Klein-Gordon equations and proved a uniform energy bound; see also [18,32]. The uniform energy bounds results of refs. [17,38,39] are also relevant to our result. Last but not least, we want to draw one’s attention to the very recent series of interesting works [11,12,35] on the two-dimensional quasilinear wave equations under null condition by Li et al.

1.2 Other models

In addition to wave equations, the study of the time growth behavior of the (higher-order) energy is also of central importance in the theory of the Schrödinger equations, two-dimensional incompressible Euler equations, as well as Hamiltonian systems. The time-growth of the top-order energy reflects the cascade of energy to its high frequency part in some cases. In ref. [6] (and [7], respectively), Bourgain constructed a wave equation (Schrödinger equation, resp.) whose solutions have unbounded high-order energy, and later in ref. [8] raised a conjecture concerning the time-growth of high-order Sobolev norms for Schrödinger equations. Later on, Colliander, Keel, Staffilani, Takaoka, and Tao [14] proved that the \( H^s \)-energy (with \( s > 1 \)) of the cubic defocusing nonlinear Schrödinger equation posed on the two-dimensional torus has the property of generating high-frequency modes within a certain bounded time interval. This result can be regarded as a partial answer to Bourgain conjecture in [8]. Recently, in the study of the two-dimensional incompressible Euler equations posed on a disk, Kiselev and Sverak [26] constructed initial data such that the vorticity gradient grows as a double exponential in time. (See also Lei and Shi [33] for a result on the torus and an exponential growth.) Various other interesting results related to this topic can be found in the aforementioned references as well as the references cited therein.

At this point, we would like to mention one important difference between the time-growth of the energy for wave equations and the energy cascade for Schrödinger equations, for instance. What is concerned in Alinhac’s blowup-at-infinity conjecture is the generalised energy with weighted commuting vector fields; however, the study of the time-growth behavior of the energy for Schrödinger equations involves \( H^s \)-energy (with \( s > 1 \)) without any weights. In this point of view, the time-growth of the generalised energy for wave equations might be caused by the weights in the commuting vector fields, see for instance the work ref. [15], while the time-growth of the \( H^s \)-energy (with \( s > 1 \)) for Schrödinger equations reflects the cascade of energy to its high frequency part (combined with the boundedness of the \( L^2 \)-energy).

1.3 Main result

In the present paper, we establish the top-order energy of the solutions to Eq. 1.1 remains globally bounded in time. This is achieved by applying the hyperboloidal foliation method and taking advantage of the null structure in order to uncover extra decay in the hyperbolic time \( t = \sqrt{t^2 - |x|^2} \). We emphasize that such a phenomenon substantially differs from that of Alinhac’s blowup-at-infinity conjecture [3–5,15].

We are interested in the quasilinear wave equation:

\[
-\Box u + P^{\alpha\beta} \partial_\alpha \psi_\beta \partial_\beta u = 0
\]

when compactly supported initial data are prescribed on a constant time slice \( t = t_0 = 2 \):

\[
(u_0, \partial_t u_0)|_{t=t_0} = (u_0, \partial_t u_0)
\]

the data \( u_0, u_1 \) having Sobolev regularity (see below). The wave operator is defined as \( \Box = -c^2 \partial_t^2 + \partial_\alpha \partial^\alpha \) with \( m = \text{diag}(1, 1, 1) \), while for simplicity in the presentation \( P^{\alpha\beta} \) are constants satisfying the standard null condition, i.e. \( P^{\alpha\beta} c_{\alpha\beta} = 0 \) for all \( c_{\alpha\beta} = \xi_{\alpha} \xi_\beta + \xi_{\beta} \xi_\alpha \) and, in addition, the symmetry condition \( P^{\alpha\beta} = P^{\beta\alpha} \). While we focus here on a model problem, our method should be applicable well beyond (1.1) and, for instance, apply to systems of coupled wave equations with both, quasilinear and semilinear null nonlinearities.

Latin letters \( a, b, \ldots \in \{1, 2\} \) are used for spatial indices and Greek letters \( \alpha, \beta, \ldots \in \{0, 1, 2\} \) represent spacetime indices, while Einstein summation convention is adopted unless otherwise specified. Without loss of generality, the initial data \( (u_0, u_1) \) are assumed to be supported in a ball, say \( \{r(x,t): t = 2, |x| \leq 1\} \), so that the solution is supported within the region \( \{r(x,t): t \geq 1, 2 \leq x \} \). We write \( A \leq B \) to indicate \( A \leq CB \) with \( C \) a generic constant.

Our main result is as follows, and provides us with a uniform control of the energy associated with any vector field in the following list, referred to as admissible vector fields:

\[
\Gamma \in \{ \partial_t, L_0 = x_0 \partial_t + x_\alpha \partial_\alpha, L_\omega = \partial_t + x^a \partial_a \}.
\]

Theorem 1.1. Let \( N \geq 4 \) be an integer. Consider the wave Eq. 1.1 together with initial data \( (u_0, u_1) \) on the time slice \( t = 2 \) supported in the ball \( \{ x : |x| \leq 1 \} \). For any \( \delta > 0 \) there exists \( \epsilon_0 > 0 \), such that for all \( \epsilon \leq \epsilon_0 \) and all initial data satisfying:

\[
\|u_0\|_{H^k} + \|u_1\|_{H^k} \leq \epsilon
\]

(1.4)

1.2. The Cauchy problem Eq. 1.1–1.2 admits a global-in-time solution \( u \), which decays according to:

\[
|u(x,t)| \leq e^{-\epsilon t^{1/2}/2}, \quad |u(x,t)| \leq e^{-\epsilon t/2}.
\]

In addition, the total energy of this solution is uniformly bounded, i.e. for any admissible field \( \Gamma \):

\[
\|\partial^\Gamma u\|_{L^2(\mathbb{R}^2)} \leq \epsilon, \quad |\Gamma| \leq N.
\]

(1.5)

The global existence part in this theorem has been established by Alinhac using the ghost weight energy method in ref. [1]. Here the improvement concerns the uniform boundedness of the top-order energy as stated in Eq. 1.6 (together with the decay rate for the solution itself in Eq. 1.5). Our strategy is first to show the energy for the restriction of the solution on a hyperboloidal foliation is uniformly bounded, and then to deduce that the energy on constant time \( \tau \) slices is also uniformly bounded. The analysis is based on the hyperboloidal foliation method developed in LeFloch and Ma [30] for coupled systems of wave and Klein-Gordon equations. As far as either (uncoupled) wave equations or Klein-Gordon equations are concerned this strategy was investigated first in pioneering work by Klainerman [28] and Hörmander [21].

In addition to showing the uniform boundedness of the top-order energy for the model problem Eq. 1.1, our method of proof also applies to systems of coupled wave equations with quasilinear and semilinear null nonlinearities.

At this point, we would like to make a comparison between the two-dimensional quasilinear wave Eq. 1.1 satisfying the null condition and the three-dimensional quasilinear wave equation:

\[
-\Box u + u = u u, \quad \Delta = \sum_{j=1,2} \partial_j \partial_j
\]

(1.7)

satisfying the weak null condition. The main reason why Alinhac conjectured the blowup-at-infinity phenomenon for the three-dimensional case Eq. 1.7 is that an exponential growth indeed occurs in the reduced Eq. 1.7, see for instance [3, Section 6]. As a contrast, the reduced equation for two-dimensional wave Eq. 1.1 is a linear one (see for instance [3, Section 6]) because of the null condition. However, due to the slower decay of waves compared to the three-dimensional case, it remains an interesting open question to verify whether the top-order energy has certain time-growth, as raised in Alinhac [1].

Shortly after our work was completed, Li in ref. [35] obtained a similar result using a novel alternative approach, in which the Lorentz boosts \( L_\omega (\omega = 1, 2) \) can be avoided. The analysis by Li in ref. [35] is significant, and is expected to have further applications, for instance to wave systems with multi-speeds.

1.4 Brief history on related topics

In seminal work by Klainerman [27] and Christodoulou [13], the wave equation with null nonlinearities and sufficiently small initial data was shown to admit global-in-time solutions in \( \mathbb{R}^{1+3} \). However, due to the slow decay of linear waves in \( \mathbb{R}^{1+4} \), nonlinear wave equations in
$\mathbb{R}^{1+2}$ are somewhat more difficult to handle. In the framework of Klainerman's vector field method, Alinhac found a class of "good derivatives" and proved a new kind of energy estimate, which is called the "ghost weight" energy estimate. Based on this idea, Alinhac [1] succeeded to prove that quasilinear wave equations with nonlinearities admit small global-in-time solutions in $\mathbb{R}^{1+2}$. In ref. [1], the top-order energy to the solution grows polynomially in time. Whether or not the time-growth on this two-dimensional problem is a true phenomenon as raised in ref. [1] remains an open question until now.

Following Alinhac's pioneering work [1] on two-dimensional quasilinear wave equations, several interesting advances were also made in recent years. In Hou and Yin [22] concerning the Eq. 1.1, the authors removed the compactness assumption on the initial data and, to this end, relied on a class of weighted $L^m-\dot{L}^\infty$ estimates. A similar result for two-dimensional, fully nonlinear, wave equations satisfying the null condition was obtained in Cai et al. [10], and this result was achieved by relying on an inherent strong null structure [32] enjoyed by the nonlinearities. Another interesting work is He et al. [20], in which a detailed description of the scattering properties of solutions was derived.

Next, we turn to the description of various progress related to Alinhac conjecture on obtaining time-growth properties or deriving uniform bounds for the top-order energy for the wave-type equations. On the one hand, in Deng and Puateri [15], the authors studied the large-time behavior of the solutions to a class of three-dimensional nonlinear wave equations satisfying the weak null condition, and proved that the decay of solutions at high-order is strictly lower than $r^{-1}$; this means that the high-order energy of the solutions cannot be uniformly bounded, and this establishes Alinhac's conjecture concerning the blowup-at-infinity for this problem. On the other hand, there exists many progress on establishing uniform bounds for the top-order energy of solutions for various classes of wave-type equations. In the recent work [10] on a class of two-dimensional quasilinear wave equations with hidden strong null structure, as well as in ref. [9] on two-dimensional incompressible elastodynamics, the authors obtained a uniform bound for the top-order energy. In their proof, the main ingredients include the use of the ghost weight method [1] and the inherent strong null structure [32]. In the work refs. [42] and [34], the authors derived a uniform bound for the top-order energy for three-dimensional quasilinear null wave equations and for the system of three-dimensional incompressible isotropic elastodynamics, respectively; their proof was based on a novel use of the null condition at the top-order energy level. As mentioned earlier when introducing the hyperboloidal foliation method, similar results were discovered for three-dimensional coupled wave-Klein-Gordon equations [18,30] and for evolving relativistic membranes [31]. Furthermore, the uniform energy bounds for the Klein-Gordon-Zakharov equations were also recently established in ref. [17]. The works refs. [2,19,24,25,29] are also relevant to our study.

1.5 Main challenge

Let us describe here the main challenge in deriving a uniform bound for the top-order energy of the solutions to (1.1), which is mainly due to the nature of the (slow!) decay of two-dimensional waves. Recall that the ghost energy estimate for the equation $\Box u = f$ reads:

$$\int_{\mathbb{R}^2} |\partial_t u|^2 + \sum_{\delta > 0} \int_{\mathbb{R}^2} |G_{\delta} \partial_x u|^2 \, dx \, dt \leq \int_{\mathbb{R}^2} |\partial_t u_0|^2 \, dx + \int_{\mathbb{R}^2} |\partial_x u|^2 \, dx dt,$$  

(1.8)

in which $\delta > 0$ and $G_{\delta} = (\chi_\delta / |x|) \delta_t + \delta_x$ are the so-called "good derivatives" associated with the ghost weight energy estimate. Returning to our model Eq. 1.1, we see that, when estimating the top-order energy of the solution $u$, we need to control the time-integral:

$$\int_{0}^{t_0} \|\partial_x u|^2 + \|\partial_x u|^2 \Box u\|_{L^1} \, dt,$$  

(1.9)

with $|I| = N$ and $\Gamma \in \{\delta_t, \partial_x, \partial_x, \partial_x\}$. (See (3.13) below for further details.) Then we find:

$$\int_{0}^{t_0} \|\partial_x u|^2 + \|\partial_x u|^2 \Box u\|_{L^1} \, dt \leq \sum_{\delta > 0} \int_{0}^{t_0} \|G_{\delta} \partial_x u\|^2 \Box u\|_{L^1} \, dt + \text{similar},$$

$$\int_{0}^{t_0} \|\partial_x u|^2 + \|\partial_x u|^2 \Box u\|_{L^1} \, dt \leq \left( \int_{0}^{t_0} \|\partial_x u|^2 + \|\partial_x u|^2 \Box u\|_{L^1} \right) \sup_{r \in [0,t_0]} \|\partial_x u(r)\|^2 \sim L^1,$$

(1.10)

where "similar" stands for integral terms that will be similarly controlled. It is easy to see that even if we assume that $u$ enjoys the properties enjoyed by linear waves, i.e.

$$\int_{0}^{t_0} \|\partial_x u|^2 + \|\partial_x u|^2 \Box u\|_{L^1} \, dt \leq \left( \int_{0}^{t_0} \|\partial_x u|^2 + \|\partial_x u|^2 \Box u\|_{L^1} \right) \sup_{r \in [0,t_0]} \|\partial_x u(r)\|^2 \sim L^1.$$

1.6. Our strategy of the proof

We now present our new key technique which helps to overcome the aforementioned challenge. In our approach we use hyperboloids $H_t = \{x, t) : x^2 = t^2 + |x|^2\}$ in order to foliate the spacetime, and we estimate the energy of the wave equation along these hyperboloids, as in refs. [21, 28, 30]; see Lemma 2.1 below. We find that in the energy estimate we integrate with respect to the hyperbolic time $s = \sqrt{t^2 - |x|^2}$ instead of $t$, and this allows us to take advantage of the $t - |x|$ decay, which is precisely provided by the ghost weight energy estimate [1]. A second key ingredient for our proof is the conformal–type energy estimate and its quasilinear version [23, 40, 43], leading to almost sharp bounds on the norms $\| \partial_x^s u \|_{L^1}$ (defined in (2.5), below) of the following terms $(I \leq N)$:

$$\int_{I} (s/t) L^r u, \quad (s/t)L^r \partial_x u, \quad (s/t)L^r \partial_x u, \quad (s/t)L^r \partial_x u.$$

In addition, we also rely on a version of the estimate for null forms, which is well-adapted to the hyperboloidal foliation setting. This reads as follows [30]:

$$\int_{I} |\partial_t \delta_t u| \Box u \|_{L^1} \leq \left( \int_{I} (s/t)^2 \Box u \Box u \right) \int_{I} \|L^r u\|_{L^1} + \int_{I} \|L^r u|_{L^1} \, dt + \int_{I} |\partial_t \Box u\|_{L^1},$$

(1.11)

We refer to Lemma 3.2 for further details. Based on this null form estimate, we roughly expect:

$$\|\partial_t \delta_t u| \Box u \|_{L^1} \leq s^{-2},$$

(1.12)

which, now, is an integrable function. For a better understanding that null condition can compensate the slow decay of two-dimensional waves, we provide here a comparison with three-dimensional wave equations with general quadratic nonlinearity. Namely, if $u$ denotes a (local) solution to:

$\Box u = \partial_{\alpha \beta} \partial_{\alpha \beta} u$, \quad in $\mathbb{R}^{1+3},$

then we find that the best decay we can get is:

$$\|\partial_t \delta_t u\|_{L^1} \leq \left( \int_{I} (s/t)^2 \Box u \Box u \right) \int_{I} \|L^r u\|_{L^1} \|L^r \Box u\|_{L^1} \leq s^{-1}$$

which is non-integrable. Clearly, the null condition forces the nonlinearities to decay fast, even if in lower dimension.

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1.7. Outline of this paper
In Section 2, we provide some basic properties of wave equations and we introduce the hyperboloidal foliation framework. Next, we derive a quasilinear version of conformal-type energy estimates on hyperboloids; see Section 3. Finally, Section 4 is devoted to the proof of Theorem 1.1.

2. Fundamental energy estimate

2.1. Notation

We work in the spacetime $\mathbb{R}^{1+2}$ with metric $m = \text{diag}(-1, 1, 1)$, and the indices are raised or lowered by the metric $m$. As usual, we use:

\[ \partial_a = \partial_{\omega_a} \]

to denote partial derivatives, with $z^0 = t$ (i.e. $x_0 = -t$). The rotating vector fields are denoted by:

\[ \Omega_{a\omega} = x_a \partial_\omega - x_\omega \partial_a, \quad a, b \in \{1, 2\}. \]

We represent the Lorentz boosts by:

\[ L_a = x_a \partial_t + t \partial_a, \quad a \in \{1, 2\} \]

and the scaling vector field is denoted by:

\[ L_0 = x^0 \partial_t. \]

We will use $\Gamma$ to denote a vector field in the set $V := \{\partial_0, \partial_1, L_1, L_2, \Omega_1, \Omega_2, L_0\}$.

On the other hand, we use $s \geq s_0 = 2$ to denote the hyperbolic time, and the hyperboloids are denoted by:

\[ H_s = \{(t,x) : t^2 = |x|^2 + s^2\}. \]

In order to fit well with the hyperboloidal foliation of the interior of the cone:

\[ \mathcal{K} = \{(t,x) : t \geq |x| + 1\}, \]

we first introduce (see LeFloch and Ma [30]) the so-called semi-hyperboloidal frame:

\[ \Gamma_0 = \partial_t, \quad \Gamma_{\omega} = \frac{L_a}{r}, \quad a = 1, 2. \]

The original partial derivatives can be expressed by the hyperboloidal frame, which reads:

\[ \partial_a = \Psi_a \Gamma_{\omega}, \quad \Psi_a = \begin{pmatrix} \frac{t}{s} & 0 & 0 \\ -\frac{x_1}{s} & 1 & 0 \\ -\frac{x_2}{s} & 0 & 1 \end{pmatrix} \]

with the transition matrix:

\[ (\Psi_a^\dagger) = \begin{pmatrix} \frac{t}{t/s} & 0 & 0 \\ -\frac{x_1}{x_2/s} & 1 & 0 \\ -\frac{x_2}{x_2/s} & 0 & 1 \end{pmatrix} \]

Next, we introduce (see LeFloch and Ma [30]) the so-called semi-hyperboloidal frame:

\[ \partial_0 = \partial_t, \quad \partial_a = \frac{L_a}{t}, \quad a = 1, 2. \]

The usual partial derivatives can also be expressed by the semi-hyperboloidal frame:

\[ \partial_t = \partial_0, \quad \partial_a = -\frac{x_a}{t} \partial_0 + \frac{L_a}{t}, \quad a = 1, 2. \]

and this will be used when estimating the null forms.

Worth to mention, for all points $(t,x) \in H_s \cap \mathcal{K}$ with $s \geq 2$ we have:

\[ |x| \leq t, \quad s \leq t \leq s^2, \quad t \leq t + |x| \leq 2t. \]

Given a sufficiently nice function $u$ with support $\mathcal{K}$, its weighted norms on hyperboloids $H_s (s \geq 2)$ are defined by:

\[ \|u\|^2_{L^2(H_s)} = \int_{H_s^+} |u(t,x)|^2 \, dt := \int_{H_s^+} \bar{u}(s,x) \bar{u}(s,x) \, dt, \quad \bar{u}(s,x) := u \left( \sqrt{s^2 + |x|^2} \right) \]

and:

\[ \|u\|^2_{L^2(H_s)} = \int_{H_s^+} |u(t,x)|^2 \, dt := \int_{H_s^+} \bar{u}(s,x) \bar{u}(s,x) \, dt, \quad \bar{u}(s,x) := u \left( \sqrt{s^2 + |x|^2} \right) \]

for all $1 \leq p < +\infty$, in which $H_s^+$ represents the projection of $H_s$ onto the slice $\{(t,x) : t = 2\}$, and the $L^\infty$ norm $\|\cdot\|_{L^\infty(H_s)}$ is defined in the natural way.

2.2. Energy estimate

Given a sufficiently nice function $u$ on $H_s$, we define its (natural) energy and conformal energy respectively by (following Huang and Ma [23], LeFloch and Ma [30], Ma [40])

\[ E(u, s) := \int_{H_s^+} \left( (\partial_t u)^2 + \sum_{a} (\partial_a u)^2 + 2x^a x^b \partial_a u \partial_b u \right) \, dx \]

\[ = \int_{H_s^+} \left( (x^a / t) \partial_a u)^2 + \sum_{a} (\partial_a u)^2 \right) \, dx \]

\[ = \int_{H_s^+} \left( \partial_a u)^2 + \sum_{a} (\partial_a (x^a / t) \partial_a u + (t^1 \partial_0 u)^2 \right) \, dx. \]

with $a, b \in \{1, 2\}$, the orthogonal vector field $\Gamma_0 := L_0 / t = \partial_t + (x^a / t) \partial_a$ and $\partial_0 := (x_a \partial_t + t \partial_a) / t$. Easily, we see that:

\[ \int_{H_s^+} \left( (x^a / t) \partial_a u)^2 + (t^1 / t^2) \partial_0 u \right) \, dx \leq E(u, s), \]

with $\Gamma \in \{\partial_0, L_0, \Omega_0\}$.

**Lemma 2.1** (Energy estimates on hyperboloids). Consider the wave equation

\[ -\square u = h, \quad (a, \partial_a u)(s_0, \cdot) = (a_0, u_0). \]

We have the following three kinds of energy estimates on hyperboloids.

- The standard energy estimates (see ref. [30]):

\[ E(u, s)^{1/2} \leq E(u, s_0)^{1/2} + \int_0^s \|u\|^2_{L^2(H_t)} \, dt. \]

- The conformal energy estimates (see refs. [23, 40, 43]):

\[ E_{con}(u, s)^{1/2} \leq E_{con}(u, s_0)^{1/2} + \int_0^s \|u\|^2_{L^2(H_t)} \, dt. \]

- The $L^2$ norm estimates (see ref. [40]):

\[ \|u\|^2_{L^2(H_t)} \leq \|u\|^2_{L^2(H_t)} + \int_0^s \|u\|^2_{L^2(H_t)} \, dt. \]

**Proof.** We revisit the proof for the conformal energy estimates only. First recall:

\[ \partial_t = \frac{t}{s} \partial_0, \quad \partial_a = -\frac{x_a}{s} \partial_0 + \frac{L_a}{t} \]

and the wave operator $-\square$ can thus be expressed in terms of $\partial_0$, which reads:

\[ -\square u = \partial_t \partial_0 u - \partial_0 \partial_t u \]

\[ = \frac{1}{s^2} \partial_t \partial_0 u - \frac{1}{s^2} \partial_0 \partial_t u + \frac{1}{s^2} \partial_0 \partial_0 u + \frac{1}{s^2} \partial^2_0 u \]

\[ = \frac{1}{s^2} \partial_0 \partial_0 u + \frac{1}{s^2} \partial^2_0 u \]

\[ = \partial_0 \partial_t u + \frac{1}{s^2} \partial_0 \partial_0 u + \frac{1}{s^2} \partial^2_0 u \]

\[ = s^{-1} \partial_0 \partial_t u + s^{-2} \partial^2_0 u \]

In succession, we have:

\[ s \partial_t \partial_0 u + 2s \partial_0 \partial_t u + u \]
and
\[ su_\partial \frac{\partial u}{\partial u} = s \bar{u}_\partial \frac{\partial u}{\partial u} \]
we arrive at:
\[ s(x_0, u) + 2s \bar{u}_\partial \frac{\partial u}{\partial u} - 2s \bar{u}_\partial \frac{\partial u}{\partial u} - 2s \bar{u}_\partial \frac{\partial u}{\partial u} \]
Integrating the above identity yields the desired energy estimates.

The following lemma from ref. [16] helps bound \( L^2 \)-type norm for \( L_0u \).

**Lemma 2.2.** Let \( u \) be supported in \( K \), then it holds that:
\[
\left\| (s/\bar{u}) L_0 u \right\|_{L^2(K)} + \left\| (s/\bar{u}) L_0 u \right\|_{L^2(K)} + \left\| (s/\bar{u}) \frac{\partial u}{\partial u} \right\|_{L^2(K)} \leq E_{\text{con}}, (s/\bar{u}) \frac{\partial u}{\partial u} \right\|_{L^2(K)}^{1/2}.
\] (2.10)

**Proof.** We revisit the proof for Eq. 2.10 in ref. [16].

First, the definition of the conformal energy easily deduces that:
\[
\left\| (s/\bar{u}) L_0 u \right\|_{L^2(K)} \leq E_{\text{con}}, (s/\bar{u}) \frac{\partial u}{\partial u} \right\|_{L^2(K)}^{1/2}.
\]

Then the relation \( \Omega_{12} = 1 (x_1 + x_2 L_2) \) implies:
\[
\left\| (s/\bar{u}) \frac{\partial u}{\partial u} \right\|_{L^2(K)} \leq \left\| (s/\bar{u}) \frac{\partial u}{\partial u} \right\|_{L^2(K)}^{1/2}
\]

Next, we consider the estimate for the scaling vector field \( L_0 \), and we write the scaling vector field in the hyperboloidal frame:
\[
L_0 = \delta_1 + x^2 \delta_2 = \delta_1 + x^2 \delta_2.
\]

Note that:
\[
Ku = (x_1, 2x^2 \bar{u}) u = L_0 u + x^2 \bar{u} u.
\]

By the triangle inequality, we find:
\[
\left\| \left( (s/\bar{u}) L_0 u \right) \right\|_{L^2(K)} \leq \left\| (s/\bar{u}) (K + 1) \right\|_{L^2(K)}^{1/2} + \left\| (s/\bar{u}) (x^2 \bar{u} u) \right\|_{L^2(K)}^{1/2} + \left\| (s/\bar{u}) (x^2 \bar{u} u) \right\|_{L^2(K)}^{1/2}
\]

Finally we recall that within the cone \( \mathcal{K} \) it holds that \( s < r, [x^0] < t \), and we thus obtain:
\[
\left\| \left( (s/\bar{u}) L_0 u \right) \right\|_{L^2(K)} \leq \left\| (s/\bar{u}) \frac{\partial u}{\partial u} \right\|_{L^2(K)}^{1/2}.
\]

The proof is now complete.

**3. Conformal energy for quasilinear wave operators**

For the quasilinear wave equation (later on, we will take \( a = \Gamma^x u \) with \( |I| = N \)
\[ g^{\delta \partial} \left( a_\partial \partial u \right) = h, \quad g^{\delta \delta} = -m^{\delta \delta} + p^{\delta \delta}, \]
we want to show the following conformal type energy estimates in analogues to (2.8), which have been proved in [23,40].

**Proposition 3.1.** Consider the quasilinear wave equation
\[ g^{\delta \partial} \left( a_\partial \partial u \right) = h, \quad g^{\delta \delta} = -m^{\delta \delta} + p^{\delta \delta}, \]
then we have:
\[
E_{\text{con}}(u, t) = E_{\text{con}}(u, t_0) + \int_{t_0}^{t} \int_{H_t^1} r Z h - r Z M_1 - M_4 \ d x \ d t.
\] (3.2)

In the above, we used the notations:
\[
E_{\text{con}}(u, s) = \int_{H_t^1} \frac{1}{2} Z^2 + M_2 \ d x.
\]

\[ Z = g^{\delta \delta} \left( a_\partial \partial u \right) + 2g^{\delta \delta} \left( a_\partial \partial u \right) + \frac{1}{2} g^{\delta \delta} \left( a_\partial \partial u \right) + g^{\delta \delta}, \]

\[ M_1 = \frac{1}{2} \left( g^{\delta \delta} \left( a_\partial \partial u \right) + \frac{1}{2} g^{\delta \delta} \left( a_\partial \partial u \right) + 2g^{\delta \delta} \left( a_\partial \partial u \right) \right), \]

\[ M_2 = \frac{1}{2} \left( g^{\delta \delta} \left( a_\partial \partial u \right) + \frac{1}{2} g^{\delta \delta} \left( a_\partial \partial u \right) - 2g^{\delta \delta} \left( a_\partial \partial u \right) \right), \]

\[ M_4 = \frac{1}{2} \left( g^{\delta \delta} \left( a_\partial \partial u \right) + \frac{1}{2} g^{\delta \delta} \left( a_\partial \partial u \right) \right), \]

\[ + \frac{1}{2} \left( g^{\delta \delta} \left( a_\partial \partial u \right) + \frac{1}{2} g^{\delta \delta} \left( a_\partial \partial u \right) \right) \]

\[
(3.3)
\]

**Proof.** Inserting the relation:
\[ a_\partial \partial u = \Psi^\partial \partial u \]
into the above equation, we have:
\[
g^{\delta \partial} \left( a_\partial \partial u \right) = \Psi^\partial \partial u \Psi^\partial \partial u \]

To succeed, we get:
\[
g^{\delta \partial} \left( a_\partial \partial u \right) = g^{\delta \partial} \left( a_\partial \partial u \right) - 2g^{\delta \partial} \left( a_\partial \partial u \right) \]

in which we used the relation \( \delta_\partial \Psi^\partial = 0 \).

Furthermore, we find:
\[
g^{\delta \partial} \left( a_\partial \partial u \right) = g^{\delta \partial} \left( a_\partial \partial u \right) - 2g^{\delta \partial} \left( a_\partial \partial u \right) \]

in which we used the relation \( g^{\delta \partial} = \Psi^\partial \Psi^\partial \).

Next, we denote:
\[
Z = g^{\delta \delta} \left( a_\partial \partial u \right) + 2g^{\delta \delta} \left( a_\partial \partial u \right) + \frac{1}{2} g^{\delta \delta} \left( a_\partial \partial u \right) - g^{\delta \delta} \left( a_\partial \partial u \right) \]

then we are led to:
\[
s Z g^{\delta \partial} \left( a_\partial \partial u \right) = \frac{1}{2} \left( Z \right) + s Z M_1 + s Z g^{\delta \partial} \left( a_\partial \partial u \right) \]

As for the last term, we split it into four parts:
\[
A_1 = s g^{\delta \delta} \left( a_\partial \partial u \right) \left( a_\partial \partial u \right) \]
\[
+ s Z g^{\delta \partial} \left( a_\partial \partial u \right) - s \delta_\partial \left( g^{\delta \delta} \left( a_\partial \partial u \right) \right) \]

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\[
- \frac{1}{2} s g^{\alpha \beta} \partial_{\gamma} \partial_{\delta} \left( \partial_{\gamma} \partial_{\delta} \right) = s^2 \partial_{\gamma} \partial_{\delta} \partial_{\gamma} \partial_{\delta} + \frac{1}{2} \partial_{\gamma} \partial_{\delta} \partial_{\gamma} \partial_{\delta}
\]

and

\[
A_2 = 2 s g^{\alpha \beta} \partial_{\gamma} \partial_{\delta} \partial_{\gamma} \partial_{\delta} = 2 s^2 g^{\alpha \beta} \partial_{\gamma} \partial_{\delta} \partial_{\gamma} \partial_{\delta} - s^2 g^{\alpha \beta} \partial_{\gamma} \partial_{\delta} \partial_{\gamma} \partial_{\delta}
\]

and

\[
A_1 = s g^{\alpha \beta} \partial_{\gamma} \partial_{\delta} \partial_{\gamma} \partial_{\delta} = s^2 g^{\alpha \beta} \partial_{\gamma} \partial_{\delta} \partial_{\gamma} \partial_{\delta} - s^2 g^{\alpha \beta} \partial_{\gamma} \partial_{\delta} \partial_{\gamma} \partial_{\delta}
\]

Then, we get:

\[
s Z g^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u = A_1 + A_2 + A_3 + A_4 = \partial_{\gamma} M_2 + M_3 + M_4
\]

and we find that the term \(M_3\) will contribute to the left hand side, \(M_4\) will be gone after the spatial integration, and \(M_2\) will be moved to the right hand side as a source term.

Now, we have:

\[
s Z g^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u = s Z h
\]

and we further get:

\[
\int_{\mathcal{H}^r} Z g^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u = \int_{\mathcal{H}^{r_0}} Z g^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u + \int_{r_0}^{r} \int_{\mathcal{H}^r} s Z h = \int_{\mathcal{H}^r} Z M_1 - M_4 d x d \tau.
\]

3.1 Null form estimates and commutator estimates

We need the following result to estimate the classical null forms, which can be found in ref. [30] for instance.

Lemma 3.2. Let \(P^{\alpha \beta}\) satisfy the null condition, then for all nice functions \(u, v, w\) supported in \(\{t, x\}: t \geq |x| + 1\), it holds:

\[
|P^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u| \leq \left( s f^{1/2} |\partial_{\gamma} \partial_{\delta} u| + \sum_{n} |\partial_{\gamma} \partial_{\delta} u|\right)
\]

\[
+ \sum_{n} |\partial_{\gamma} \partial_{\delta} u| t + |t| |\partial_{\gamma} \partial_{\delta} u|.
\]

Proof. The proof can be found in LeFloch and Ma [30], but here we also provide one elementary proof.

We express the term \(P^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u\) in the semi-hyperboloidal frame \((\partial_\gamma, \partial_\delta)\), with the relation

\[
\partial_\gamma = -\frac{s}{2} \partial_t + \partial_x,
\]

We find that:

\[
P^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u = P^{00} \partial_{\gamma} \partial_{\delta} u + P^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u + P^{\alpha 0} \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u.
\]

But

\[
P^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u = P^{00} \partial_{\gamma} \partial_{\delta} u + P^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u + P^{\alpha 0} \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u.
\]

and

\[
P^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u = P^{00} \partial_{\gamma} \partial_{\delta} u + P^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u + P^{\alpha 0} \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u.
\]

In which

\[
B_1 = \partial_{\gamma} \partial_{\delta} u + P^{00} \partial_{\gamma} \partial_{\delta} u + P^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u + P^{\alpha 0} \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u.
\]

and

\[
B_2 = \partial_{\gamma} \partial_{\delta} u + P^{00} \partial_{\gamma} \partial_{\delta} u + P^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u + P^{\alpha 0} \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u.
\]

Since \(P^{\alpha \beta}\) is null, we have (recall \(r = |x|\) is the spacial radius of a point):

\[
P^{00} \partial_{\gamma} \partial_{\delta} u + P^{\alpha \beta} \partial_{\gamma} \partial_{\delta} u + P^{\alpha 0} \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u + \partial_{\gamma} \partial_{\delta} u.
\]
+ p^{\text{blo}}(\frac{-\gamma}{7})\frac{\gamma}{7} + p^{\text{bloc}}(\frac{-\gamma}{7})\frac{\gamma}{7} \\
+ p^{\text{blo}}(\frac{-\gamma}{7})\frac{\gamma}{7} = 0.

This deduces that:

\begin{align*}
B_1 &= \partial_t v_0 \partial_t u + p^{\text{blo}}(1 - \frac{\gamma}{7}) + p^{\text{bloc}}(1 - \frac{\gamma}{7})\\
&+ p^{\text{blo}}(\frac{\gamma}{7}) + p^{\text{bloc}}(\frac{\gamma}{7})
\end{align*}

Then the facts:

\begin{align*}
\frac{\gamma}{7} &\leq 1, \quad |1 - \frac{\gamma}{7}| + \frac{\gamma}{7} \\
&\leq \frac{\gamma}{7} + \frac{\gamma}{7} \\
&\leq \frac{\gamma}{7} + \frac{\gamma}{7}
\end{align*}

yield that:

\[|B_1| \leq \frac{\gamma}{7} \left|\partial_t v_0 \partial_t u\right|.

As for \(B_2\), part, the facts:

\begin{align*}
\frac{\gamma}{7} &\leq 1, \quad \left|a \partial_t u\right| \leq |\partial u|
\end{align*}

imply that:

\[|B_2| \leq \sum a \left|\partial_t u\right| + \sum \left|\partial_t v_0 \partial_t u\right| + \frac{\gamma}{7} \left|\partial_t v_0 \partial_t u\right|.

In a similar (and easier) way, we can get the desired bound for

\[p^{\text{sto}} \partial_t v_0 \partial_t u , \partial_t u |

The following result tells us that acting some vector fields to the null forms still leads to null forms, and one refers to ref. [21] for more details.

Lemma 3.3. Let \(N(v, u) = p^{\text{sto}} \partial_t v_0 \partial_t u, \) then we have (\(\Gamma \in \{\partial_0, \partial_1, \partial_2, \Omega_\Sigma, L_0, L_1, L_2\})

\[\Gamma^k (p^{\text{sto}} \partial_t v_0 \partial_t u) = \sum_{|t| + |r| = |\Gamma|} N_{(t, r, \Omega_\L)}, \quad (3.5)

We will also need to frequently use the following estimates for commutators, which can be found in refs. [30,36,41].

Lemma 3.4. Let \(u \) be a sufficiently nice function supported in \(K = \{(t, x) : x \geq |x| + 1\}, \) then the following inequalities are valid (\(a, b \in \{1, 2\}, a, b \in \{1, 2\}\))

\begin{align*}
\partial_0 L_0 u &\leq L_0 \partial_0 u + \sum a \left|\partial_0 u\right|, \\
L_0 \partial_0 u &\leq L_0 \partial_0 u + \sum a \left|\partial_0 u\right|, \\
L_0 L_0 u &\leq L_0 L_0 u + \sum a \left|\partial_0 u\right|, \\
\partial_0 (\alpha s) f &\leq (\alpha s) \partial_0 u + s^{-1} u, \\
L_0 (\alpha s) f &\leq (\alpha s) L_0 u + (\alpha s) u, \\
L_0 L_0 (\alpha s) f &\leq (\alpha s) L_0 L_0 u + (\alpha s) L_0 u + \sum a \left|\partial_0 u\right|.
\end{align*}

3.2. Sobolev-type inequality

In order to obtain wave decay, we need the following Sobolev-type inequality, whose proof can be found in [30]. The rotation vector field \(\Omega_\Sigma\) is not needed in the inequality (3.7), which can be seen from the fact that it can be bounded by the Lorentz boosts within the domain of interest \(\{(t, x) : x \geq |x| + 1\}, \) which reads:

\[\Omega_{\Sigma} = r^{-1} (x_1 L_2 - x_2 L_1).

Lemma 3.5. Let \(u = u(x, t) \) be a sufficiently smooth function with support \(\{(t, x) : x \geq |x| + 1\}, \) then for all \(s \geq 2, \) one has:

\[\sup_{N_t} \|u(x, t)\| \leq \sum_{|\gamma| \leq 2} \|L_\gamma u\|_L^2 / (N_t), \quad (3.6)

and the symbol \(L^\gamma\) above represents the Lorentz boosts with \(J\) a multi-index.

Together with the commutator estimates in Lemma 3.4, the following inequality, which can be more conveniently applied, holds:

\[\sup_{N_t} \|u(x, t)\| \leq \sum_{|\gamma| \leq 2} \|L_\gamma u\|_L^2 / (N_t), \quad (3.7)

with \(s \geq 2.\)

3.3. Technical identities and inequalities

We prepare some calculations for later use, which will play an important role in the analysis.

Step I.

Lemma 3.6. Within the cone \(K = \{(t, x) : x \geq |x| + 1\}, \) we have:

\[\partial_0 \Psi_0 = -\frac{1}{r} \Psi_0 + \Delta u_{\gamma}, \quad \partial_0 \Psi_0 \leq \frac{1}{r}.

Proof. We note:

\[\partial_0 (\Psi_0 + \partial_0 \Psi_0) = \partial_0 \Psi_0 \leq \partial_0 \Psi_0, \quad \partial_0 \Psi_0 = \frac{1}{r}.

Step II.

Lemma 3.7. Let \(u \) be supported in the cone \(K, \) satisfying \(\partial u \leq r^{-1}, \) then it holds:

\[\partial_0 \partial_0 \Psi_0 \leq \frac{1}{r}, \quad (3.8)

In addition, we have:

\[\partial_0 \partial_0 \Psi_0 \leq \frac{1}{r}, \quad \partial_0 \partial_0 \Psi_0 \leq \frac{1}{r}.

Proof. We find:

\[\partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0 \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0.

We have:

\[\partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0 + \partial_0 \partial_0 \Psi_0 + \partial_0 \partial_0 \Psi_0 + \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0, \quad \partial_0 \partial_0 \Psi_0 = \partial_0 \partial_0 \Psi_0.

and since \(\partial_0 \partial_0 \Psi_0 \) is null, we get:
\[
p^{\rho \phi} \Psi_0 \Phi = P^{\rho \Phi} \Psi_0 \Phi = \frac{\partial}{\partial t} \left( \frac{\rho}{\rho} \right)^3 + \frac{3}{8} P^{\rho \Phi} \Phi + P^{\rho \phi} \Psi_0 + \frac{3}{8} P^{\rho \phi} \Psi_0 + \frac{3}{8} P^{\rho \phi} \Psi_0 + \frac{3}{8} P^{\rho \phi} \Psi_0.
\]

Recall that:
\[
\Psi_0 = \frac{x}{a}.
\]

then using the fact that \( P^{\rho \phi} \) is null, we get:
\[
\left| P^{\rho \phi} \Psi_0 \Phi \right| \leq \frac{1}{s^2}.
\]

If \( |\partial_t u| \leq s^{-1} - 1, |\partial_x u| \leq s^{-1} - 1 - 1 \), then we have:
\[
\left| P^{\rho \phi} \Psi_0 \Phi \right| \leq \frac{1}{s^2}.
\]

Furthermore, we act \( \tilde{\sigma} \) on Eq. (3.12) to get:
\[
\left| \partial_t \left( P^{\rho \phi} \Psi_0 \Phi \right) \right| \leq \frac{1}{s^2}.
\]

Step III.
Similar to the last lemma, we also have the following estimates.

Lemma 3.8. With \( \Phi = (1, \pm x_1/t, \pm x_2/t) \) one has:
\[
\left| P^{\rho \phi} \Phi \right| \leq \frac{1}{s^2}.
\]

Step IV.

Lemma 3.9. Consider the quasilinear wave equation:
\[
-\square u + P^{\rho \phi} \partial_t u \partial_x u \partial_y u = h,
\]

and assume \( |\partial_t u| \leq \frac{1}{100} s^{-1}, |\partial_x u| \leq \frac{1}{100} s^{-1} - 1 - 1 \), then we have:
\[
E(u, s) \leq E(u, s_0) + \int_{s_0}^s \int_{\Gamma} 2(\tau/\tau_f) P^{\rho \phi} \partial_\gamma u \partial_\gamma u + 2(\tau/\tau_f) \partial_t u \partial_x u d\tau d\tau.
\]

Proof. Multiplying the equation with \( \partial_t u \), we have:
\[
\begin{align*}
\frac{1}{2} \frac{d}{d\tau} \left( (\partial_t u)^2 \right) &+ \sum_{\gamma} \left( \partial_\gamma u \partial_\gamma u \right) - \frac{1}{2} P^{\rho \phi} \partial_\gamma \left( \partial_\gamma u \partial_\gamma u \right) \\
&= \frac{1}{2} P^{\rho \phi} \partial_\gamma u \partial_\gamma u \partial_\gamma u + \frac{1}{2} P^{\rho \phi} \partial_\gamma u \partial_\gamma u \partial_\gamma u + h \partial_t u.
\end{align*}
\]

Thus we have the energy estimates:
\[
\int_{H^2} \left| \partial_\gamma u \right|^2 + 2 P^{\rho \phi} \partial_\gamma u \partial_\gamma u \partial_\gamma u - P^{\rho \phi} \partial_\gamma u \partial_\gamma u \partial_\gamma u d\tau d\tau
\]
\[
- \int_{H^2} \left| \partial_\gamma u \right|^2 + 2 P^{\rho \phi} \partial_\gamma u \partial_\gamma u \partial_\gamma u - P^{\rho \phi} \partial_\gamma u \partial_\gamma u \partial_\gamma u d\tau d\tau
\]
\[
= \int_{H^2} 2(\tau/\tau_f) P^{\rho \phi} \partial_\gamma u \partial_\gamma u \partial_\gamma u d\tau d\tau
\]
\[
= \int_{H^2} (\tau/\tau_f) P^{\rho \phi} \partial_\gamma u \partial_\gamma u \partial_\gamma u d\tau d\tau + \int_{H^2} 2(\tau/\tau_f) \partial_\gamma u \partial_\gamma u \partial_\gamma u d\tau d\tau
\]

in which
\[
n = (1, -s_x/t).
\]

We observe that:
\[
n_x = \frac{s_x}{s}.
\]

Since \( P^{\rho \phi} \) is null, and \( |\partial_t u| \leq \frac{1}{100} s^{-1}, |\partial_x u| \leq \frac{1}{100} s^{-1} - 1 - 1 \), thus we have:
\[
\left| \int_{H^2} 2 P^{\rho \phi} \partial_\gamma u \partial_\gamma u \partial_\gamma u - P^{\rho \phi} \partial_\gamma u \partial_\gamma u \partial_\gamma u d\tau d\tau \right| \leq \frac{1}{2} \int_{H^2} \left| \partial_\gamma u \right|^2 d\tau
\]

which yields the desired estimates.

4. Proof of the boundedness of the energy

Bootstrap assumptions and direct consequences

We take the following bootstrap assumptions on \( s_0 = 2, s_1 \):
\[
E(T^\gamma u, s)^{1/2} \leq C_e, \quad |I| \leq N,
\]
\[
E_{\text{cor}}(T^\gamma u, s)^{1/2} \leq C_e s^{d_1}, \quad |I| \leq N - 1,
\]
\[
E_{\text{cor}}(T^\gamma u, s)^{1/2} \leq C_e s^{d_2}, \quad |I| \leq N
\]

with \( 0 < \delta \ll 1, C_1 \gg 1 \) some large constant to be determined later, and \( \epsilon \ll 1 \) the size of the initial data, and:
\[
s_1 := \sup \{ s > s_0 : (4.1) \text{ holds} \}.
\]

Lemma 4.1. Under the assumptions in Eq. (4.1), for all \( s \in \{ s_0, \ldots, s_1 \} \) it is true that:
\[
\left\| \left( s/\tau_f \right) T^\gamma u \right\|_{L^2_{\text{loc}}(H_s)} + s^{-d} \left\| \left( s/\tau_f \right) T^\gamma u \right\|_{L^2_{\text{loc}}(H_s)} \leq C_e, \quad |I| \leq N - 1,
\]
\[
\left\| \left( s/\tau_f \right) T^\gamma u \right\|_{L^2_{\text{loc}}(H_s)} + s^{-d} \left\| \left( s/\tau_f \right) T^\gamma u \right\|_{L^2_{\text{loc}}(H_s)} \leq C_e, \quad |I| \leq N.
\]

Proof. The proof follows from the definition of the natural energy \( E(u, s) \) and the conformal energy \( E_{\text{cor}}(u, s) \) as well as the estimates in Lemma 2.1 and Lemma 2.2.

The use of the Sobolev inequality in Lemma 3.5 (more precisely Eq. (3.8)) leads us to the following pointwise decay for the solution \( u \).

Lemma 4.2. Let the bootstrap assumptions in Eq. (4.1) hold, then for all \( s \in \{ s_0, \ldots, s_1 \} \) we have:
\[
\left\| \left( s/\tau_f \right) T^\gamma u \right\|_{L^2_{\text{loc}}(H_s)} \leq C_e s^{-1}, \quad |I| \leq N - 2,
\]
\[
\sum_{l_1 + l_2 + 2d+l = |I|} N_{\text{d}}(T^\gamma u, l_1, l_2) \leq C_e s^{-1}, \quad |I| \leq N - 3.
\]

Proof. We first act the vector field \( T^\gamma \) (with \( |I| \leq N - 1 \)) on the wave Eq. 1.1, and arrive at:
\[
-\square T^\gamma u = -T^\gamma N(u, u) = -\sum_{l_1 + l_2 + 2d+l = |I|} N_{\text{d}}(T^\gamma u, l_1, l_2).
\]

In order to refine the bound for \( E(T^\gamma u, s)^{1/2} \), we apply the energy estimates Eq. 2.7 to get:
\[
E(T^\gamma u, s)^{1/2} \leq E(T^\gamma u, s_0)^{1/2} + \sum_m \left\| T^\gamma N(u, u) \right\|_{L^2_{\text{loc}}(H_s)} dt.
\]

We find:
\[
\left\| T^\gamma N(u, u) \right\|_{L^2_{\text{loc}}(H_s)} \leq \sum_{l_1 + l_2 + 2d+l = |I|} \left\| N_{\text{d}}(T^\gamma u, l_1, l_2) \right\|_{L^2_{\text{loc}}(H_s)}
\]
\[
\leq \sum_{l_1 + l_2 + 2d+l = |I|} \left( (\tau_f)^2 \right)^2 \left\| \partial_\gamma u \right\|_{L^2_{\text{loc}}(H_s)}
\]
\[
+ \sum \left\| \partial_\gamma u \right\|_{L^2_{\text{loc}}(H_s)} + \left\| \tau_f \right\|_{L^2_{\text{loc}}(H_s)}
\]
\[
+ \sum \left\| \partial_\gamma u \right\|_{L^2_{\text{loc}}(H_s)} + \left\| \tau_f \right\|_{L^2_{\text{loc}}(H_s)}.
\]

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Now we estimate each of the terms. First, we have:

\[
\sum_{|l|,|j|+|k|\leq|I|} \left\| (\tau(t))^{2} dI^{j} w \partial_{a} dI^{k} u \right\|_{L^{2}(\mathcal{H})} \\
\leq \sum_{|l|,|j|+|k|\leq|I|-1} \left\| (\tau(t))^{2} (x-|x|)^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
+ \sum_{|l|,|j|+|k|\leq|I|-1} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
+ \sum_{|l|,|j|+|k|\leq|I|-1} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
\leq \sum_{|l|,|j|+|k|\leq|I|-1} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
+ \sum_{|l|,|j|+|k|\leq|I|-1} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
\leq (C_{e})^{3} r^{-2}\tag{4.6}
\]

in which we used the relation $|dw| \leq (t-|x|)^{-1} |\Gamma|u|$. Next, we proceed by estimating:

\[
\sum_{|l|,|j|+|k|\leq|I|-1} \left\| dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
\leq \sum_{|l|,|j|+|k|\leq|I|-1} \left\| dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
\leq \sum_{|l|,|j|+|k|\leq|I|-1} \left\| dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
+ \sum_{|l|,|j|+|k|\leq|I|-1} \left\| dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
\leq (C_{e})^{3} r^{-2}\tag{4.6}
\]

Finally, we easily get (without details):

\[
\sum_{|l|,|j|+|k|\leq|I|-1} \left\| dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
\leq (C_{e})^{3} r^{-2}\tag{4.6}
\]

To sum things up, we arrive at:

\[
\left\| \Gamma^{j} T N(w, u) \right\|_{L^{2}(\mathcal{H})} \leq (C_{e})^{3} r^{-2}\tag{4.6}
\]

Then the energy estimates give us:

\[
\left\| \Gamma^{j} T^{3/2} N(w, u) \right\|_{L^{2}(\mathcal{H})} \leq \left\| \Gamma^{j} T^{3/2} N(w, u) \right\|_{L^{2}(\mathcal{H})} \\
\leq \epsilon + (C_{e})^{3} f^{|s|} r^{-2} \tag{4.6} \quad |s| \leq N - 1.
\]

Similarly, the conformal energy estimates lead us to:

\[
E_{\text{conf}}(\Gamma^{j} T^{3/2} N(w, u)) \leq E_{\text{conf}}(\Gamma^{j} T^{3/2} N(w, u)) \\
\leq \epsilon + (C_{e})^{3} f^{|s|} r^{-2} \tag{4.6} \quad |s| \leq N - 1.
\]

Proposition 4.4. Under the assumptions in Eq. 4.1, we have the following improved estimates for all $s \in [s_{0}, s_{1})$

\[
E(\Gamma^{j} T^{3/2}) \leq \epsilon + (C_{e})^{3} r^{-2} \tag{4.7} \quad |s| \leq N.
\]

Proof. Applying the vector field $\Gamma^{j}$ (with $|j| = N$) to the wave Eq. 1.1, we have:

\[
\square^{j} T w = T^{3} N(w, u) = \sum_{|I|,|J|+|K|\leq|I|+|j|} N_{d}(\Gamma^{j} T^{3/2} N(w, u))
\]

which is:

\[
\square^{j} T w + P^{\alpha} \partial_{\alpha} w \partial_{\beta} \partial_{\gamma} \partial_{\delta} T^{3/2} N(w, u) = - \sum_{|I|,|J|+|K|\leq|I|+|j|} N_{d}(\Gamma^{j} T^{3/2} N(w, u)).
\]

Recall the energy estimates for quasilinear wave, and we have:

\[
E(\Gamma^{j} T^{3/2}) \leq E(\Gamma^{j} T^{3/2}) + \int_{s_{0}}^{s_{1}} \int_{\mathcal{H}} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} d\tau dt \\
+ \int_{s_{0}}^{s_{1}} \int_{\mathcal{H}} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} d\tau dt \\
+ \int_{s_{0}}^{s_{1}} \int_{\mathcal{H}} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} d\tau dt \\
\leq (C_{e})^{3} r^{-2}\tag{4.6}
\]

which leads us to:

\[
\int_{s_{0}}^{s_{1}} \int_{\mathcal{H}} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} d\tau dt \leq (C_{e})^{3} \int_{s_{0}}^{s_{1}} \epsilon \tag{4.6}
\]

Next, we find that:

\[
\sum_{|l|,|j|+|k|\leq|I|-1} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
\leq \sum_{|l|,|j|+|k|\leq|I|-1} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
+ \sum_{|l|,|j|+|k|\leq|I|-1} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
\leq \sum_{|l|,|j|+|k|\leq|I|-1} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
+ \sum_{|l|,|j|+|k|\leq|I|-1} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
\leq (C_{e})^{3} r^{-2}\tag{4.6}
\]

which implies that:

\[
\sum_{|l|,|j|+|k|\leq|I|-1} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} \\
\leq (C_{e})^{3} r^{-2}\tag{4.6}
\]

and we further have:

\[
\int_{s_{0}}^{s_{1}} \int_{\mathcal{H}} \left\| (\tau(t))^{2} dI^{j} u \right\|_{L^{2}(\mathcal{H})} d\tau dt \leq (C_{e})^{3} r^{-2}\tag{4.6}
\]

The proof is completed. \qed

The estimates left to be improved are $E_{\text{con}}(\Gamma^{j} T^{3/2})$ with $|j| = N$. We need the following result, which tells us that the conformal en-
ergy $E_{\text{con}}(\Gamma^I w, s)$ and $\tilde{E}_{\text{con}}(\Gamma^I w, s)$ in Proposition 3.1 can be somehow bounded by each other.

**Lemma 4.5.** With the assumptions in Eq. 4.1, we have the following estimates for all $s \in [s_0, s]$:

$$|E_{\text{con}}(\Gamma^I w, s) - \tilde{E}_{\text{con}}(\Gamma^I w, s)| \leq (C_1 c)^3 s^{2/3}, \quad |I| = N. \tag{4.9}$$

**Proof.** We take $u = \Gamma^I w$ with $|I| = N$ in Proposition 3.1, and we recall the formulas for $E_{\text{con}}(\Gamma^I w, s)$, $\tilde{E}_{\text{con}}(\Gamma^I w, s)$, which read:

$$E_{\text{con}}(\Gamma^I w, s) = \int_{\Omega} \sum_{\alpha \beta} \left(\frac{\partial}{\partial s} \Gamma^I w + 2x^\alpha \partial^\alpha \Gamma^I w + \Gamma^I w \partial^I \right)^2 \, dx,$$

$$\tilde{E}_{\text{con}}(\Gamma^I w, s) = \int_{\Omega} \frac{1}{2} \partial^s + M_t \, dx,$$

$$Z = \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w + 2t^{\alpha \beta} \psi^{\alpha \beta} \partial^\alpha \Gamma^I w + \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w - \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w,$$

$$M_s = \frac{1}{2} \psi_0^{\alpha \beta} \partial^\alpha \gamma \partial^\beta \Gamma^I w,$$

in which $\psi_0^{\alpha \beta} = -m^{\alpha \beta} + P^{\alpha \beta} \partial^\alpha \Gamma^I w$. We note that $E_{\text{con}}(\Gamma^I w, s)$ can also be expressed with:

$$E_{\text{con}}(\Gamma^I w, s) = \frac{1}{2} \tilde{Z}^2 + M_t \, dx,$$

$$\tilde{Z} = \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w + 2 \psi_0^{\alpha \beta} \partial^\alpha \partial^\beta \Gamma^I w + 2 \psi_0^{\alpha \beta} \partial^\alpha \partial^\beta \Gamma^I w + \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w + \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w$$

Thus we have:

$$E_{\text{con}}(\Gamma^I w, s) - \tilde{E}_{\text{con}}(\Gamma^I w, s) \leq \int_{\Omega} \left( \psi_0^2 \partial^s + M_t \right) \, dx. \tag{4.10}$$

First, we note:

$$Z - \tilde{Z} = \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w + 2 \psi_0^{\alpha \beta} \partial^\alpha \partial^\beta \Gamma^I w + \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w - \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w$$

Next, we estimate each of the terms. Recall that $|\tilde{\alpha}_0 w| \leq C_1 c t^{-1}$, $|P^{\alpha \beta} \partial^\alpha \Gamma^I w| \leq (t/s)$, and we find:

$$|P^{\alpha \beta} \partial^\alpha \Gamma^I w| \leq C_1 c t^{-1} |P^{\alpha \beta} \partial^\alpha \Gamma^I w| \leq (t/s).$$

Recall that $|\tilde{\alpha}_0 w| \leq C_1 c t^{-1} s^3$, and we find:

$$|P^{\alpha \beta} \partial^\alpha \partial^\beta \Gamma^I w| \leq C_1 c t^{-1} s^3.$$  

Recall that $|\tilde{\alpha}_0 w| \leq C_1 c t^{-1} s^3$, and we find:

$$|2 \psi_0^{\alpha \beta} \partial^\alpha \partial^\beta \Gamma^I w| \leq C_1 c t^{-1} s^3.$$  

In view of $|\tilde{\alpha}_0 w| \leq C_1 c t^{-1}$, $|\tilde{\alpha}_0 w| \leq C_1 c t^{-1} s^3$, $|P^{\alpha \beta} \partial^\alpha \partial^\beta \Gamma^I w| \leq (t/s)$, we find:

$$Z = \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w + 2 \psi_0^{\alpha \beta} \partial^\alpha \partial^\beta \Gamma^I w + \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w - \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w$$

Finally, the combination of Eq. 4.11 and Eq. 4.12 yields the desired estimates in Eq. 4.9.

**Proposition 4.6.** One has

$$E_{\text{con}}(\Gamma^I w, s)^{1/2} \leq c + (C_1 c)^{1/2} s^{2/3}, \quad |I| = N. \tag{4.13}$$

**Proof.** By the estimates in Lemma 4.5, it suffices to show:

$$E_{\text{con}}(\Gamma^I w, s) \leq c + (C_1 c)^{3/2}.$$  

Recall the conformal energy estimates for quasilinear wave equations in Proposition 3.1 with $u = \Gamma^I w$ with $|I| = N$, and we thus have:

$$E_{\text{con}}(\Gamma^I w, s) = \tilde{E}_{\text{con}}(\Gamma^I w, s) + \int_{\Omega} \tau \rho - \tau M_t \, dx. \tag{4.14}$$

in which:

$$h = \sum_{|I| \geq 2} \sum_{|I| = n} N_j (\Gamma^I w, \Gamma^I w) - \sum_{|I| \geq 2} N_j (\Gamma^I w, \Gamma^I w)$$

and

$$Z = \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w + 2 \psi_0^{\alpha \beta} \partial^\alpha \partial^\beta \Gamma^I w + \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w - \psi_0^{\alpha \beta} \partial^\alpha \Gamma^I w$$

Finally, the combination of Eq. 4.11 and Eq. 4.12 yields the desired estimates in Eq. 4.9.
we have:

\[ M_h = M_0 + \frac{1}{s} \left( \langle \frac{\partial (\langle \partial_s u \rangle \langle \partial_s \psi \rangle \psi) \rangle \rangle - 1 \right) - 1 \]

Recall that \( |\partial_s u| \leq C \epsilon^{-1}, \quad |\tilde{u}| \leq C \epsilon^{-1}, \quad |\tilde{u}| \quad (t - r)^{-1} \) [for \( t \leq C \epsilon^{-1} \), and we have:

\[ \| \partial_s (\langle \partial_s u \rangle \langle \partial_s \psi \rangle \psi) \|_{L^2(\Omega)} \leq C \epsilon^{-1}, \quad \| \tilde{u} \|_{L^2(\Omega)} \leq C \epsilon^{-1} \]

In view of \( |\partial_s u| \leq C \epsilon^{-1} \), we have:

\[ \| 2 \partial_s (\langle \partial_s u \rangle \langle \partial_s \psi \rangle \psi) \|_{L^2(\Omega)} \leq C \epsilon^{-1}, \quad \| \tilde{u} \|_{L^2(\Omega)} \leq C \epsilon^{-1} \]

Recalling that \( |\partial_s u| \leq C \epsilon^{-1}, \quad |\tilde{u}| \leq C \epsilon^{-1} \), we have:

\[ \| \partial_s (\langle \partial_s u \rangle \langle \partial_s \psi \rangle \psi) \|_{L^2(\Omega)} \leq C \epsilon^{-1}, \quad \| \tilde{u} \|_{L^2(\Omega)} \leq C \epsilon^{-1} \]

From \( |\partial_s u| \leq C \epsilon^{-1}, \quad |\tilde{u}| \leq C \epsilon^{-1} \), we deduce that:

\[ \| \partial_s (\langle \partial_s u \rangle \langle \partial_s \psi \rangle \psi) \|_{L^2(\Omega)} \leq C \epsilon^{-1}, \quad \| \tilde{u} \|_{L^2(\Omega)} \leq C \epsilon^{-1} \]

in which we used the observations:

\[ \partial_s \partial_s \psi_u = \frac{1}{2} \psi_u - \frac{1}{2} \psi_u - \frac{1}{2} \psi_u - \frac{1}{2} \psi_u \]

and

\[ |\partial_s u| = \frac{1}{2} \partial_s u + \frac{1}{2} \partial_s u - \frac{1}{2} \partial_s u \leq C \epsilon^{-1} \]

Gathering the estimates leads us to:

\[ \int_0^\infty \int_{\Omega} \| z \|_{L^2(\Omega)} \| M_h \|_{L^2(\Omega)} \| \psi \|_{L^2(\Omega)} \leq (C \epsilon)^3 \]

Estimates for \( M_0 \): Not that:

\[ \| \partial_s (\langle \partial_s u \rangle \langle \partial_s \psi \rangle \psi) \|_{L^2(\Omega)} \leq C \epsilon^{-1}, \quad \| \tilde{u} \|_{L^2(\Omega)} \leq C \epsilon^{-1} \]

with all terms.

\[ M_h = M_0 + \frac{1}{s} \left( \langle \frac{\partial (\langle \partial_s u \rangle \langle \partial_s \psi \rangle \psi) \rangle \rangle - 1 \right) - 1 \]

noting that:

\[ \| \partial_s (\langle \partial_s u \rangle \langle \partial_s \psi \rangle \psi) \|_{L^2(\Omega)} \leq C \epsilon^{-1}, \quad \| \tilde{u} \|_{L^2(\Omega)} \leq C \epsilon^{-1} \]

Since \( |\partial_s u| \leq C \epsilon^{-1} \), we have:

\[ \| \partial_s (\langle \partial_s u \rangle \langle \partial_s \psi \rangle \psi) \|_{L^2(\Omega)} \leq C \epsilon^{-1}, \quad \| \tilde{u} \|_{L^2(\Omega)} \leq C \epsilon^{-1} \]

From \( |\partial_s u| \leq C \epsilon^{-1}, \quad |\tilde{u}| \leq C \epsilon^{-1} \), we deduce that:

\[ \| \partial_s (\langle \partial_s u \rangle \langle \partial_s \psi \rangle \psi) \|_{L^2(\Omega)} \leq C \epsilon^{-1}, \quad \| \tilde{u} \|_{L^2(\Omega)} \leq C \epsilon^{-1} \]
Estimates for the term $H_1$. For the term $H_1$, recall that $\frac{\partial}{\partial t} u = \nabla \cdot \mathbf{u} + m_\delta \nabla \phi \phi_\delta$, and we have:

\[
\frac{1}{2} \left\| \partial_s \left( s^2 \rho_\tau \rho_\alpha \partial_s \phi_\delta \right) \right\|_{L^2(H)} = -1,
\]
and we get:

\[
\frac{1}{2} \left\| \partial_s \left( s^2 \rho_\tau \rho_\alpha \partial_s \phi_\delta \right) \right\|_{L^2(H)} \leq (C_1 \epsilon)^{\frac{1}{2}} s^\delta.
\]
and recall $\rho_\tau \rho_\alpha \phi_\delta \partial_s \phi_\delta (\rho_\tau \rho_\alpha \partial_s \phi_\delta) \leq C_1 \epsilon s^\delta$

\[
\frac{1}{2} \left\| \partial_s \left( s^2 \rho_\tau \rho_\alpha \partial_s \phi_\delta \right) \right\|_{L^2(H)} \leq (C_1 \epsilon)^{\frac{1}{2}} s^\delta.
\]
which lead to:

\[
\frac{1}{2} \left\| \partial_s \left( s^2 \rho_\tau \rho_\alpha \partial_s \phi_\delta \right) \right\|_{L^2(H)} \leq (C_1 \epsilon)^{\frac{1}{2}} s^\delta.
\]
Thus we obtain:

\[
H_1 \leq C_1 \epsilon s^\delta \sum_{\gamma} \left\| \bar{\phi} \Gamma^\gamma \right\|_{L^2(H)} \leq (C_1 \epsilon)^{\frac{1}{2}} s^\delta.
\]
which is an integrable quantity provided $\delta \ll 1$.

Estimates for the term $H_2$. For the term $H_2$, recall again that $\frac{\partial}{\partial t} u = \nabla \cdot \mathbf{u} + m_\delta \nabla \phi \phi_\delta$, and we have:

\[
\frac{1}{2} \left\| \partial_s \left( s^2 \rho_\tau \rho_\alpha \partial_s \phi_\delta \right) \right\|_{L^2(H)} \leq (C_1 \epsilon)^{\frac{1}{2}} s^\delta.
\]
We start with:

\[
\frac{1}{2} \left\| \partial_s \left( s^2 \rho_\tau \rho_\alpha \partial_s \phi_\delta \right) \right\|_{L^2(H)} \leq (C_1 \epsilon)^{\frac{1}{2}} s^\delta.
\]
On the one hand, we have, by recalling $|\bar{\phi} (\rho_\tau \rho_\alpha \phi_\delta) | \leq s^{-1}$

\[
\frac{1}{2} \left\| \partial_s \left( s^2 \rho_\tau \rho_\alpha \partial_s \phi_\delta \right) \right\|_{L^2(H)} \leq (C_1 \epsilon)^{\frac{1}{2}} s^\delta.
\]
Next, we estimate:

\[
\left\| (s/t) \bar{\phi} (\rho_\tau \rho_\alpha \partial_s \phi_\delta) \right\|_{L^2(H)} \leq \left\| \bar{\phi} \Gamma^\gamma \right\|_{L^2(H)} \leq (C_1 \epsilon)^{\frac{1}{2}} s^\delta.
\]
and:

\[
\left\| (s/t) \bar{\phi} (\rho_\tau \rho_\alpha \partial_s \phi_\delta) \right\|_{L^2(H)} \leq \left\| \bar{\phi} \Gamma^\gamma \right\|_{L^2(H)} \leq (C_1 \epsilon)^{\frac{1}{2}} s^\delta.
\]
Thus we have:

\[
H_1 \leq C_1 \epsilon s^\delta \sum_{\gamma} \left\| \bar{\phi} \Gamma^\gamma \right\|_{L^2(H)} \leq (C_1 \epsilon)^{\frac{1}{2}} s^\delta.
\]
which is integrable as long as $\delta \ll 1$.

Tendid but similar computations allow us to get:

\[
\int_{t_0}^{t} \int_{H} |\mathbf{M}| dt dx \leq (C_1 \epsilon)^{\frac{1}{2}} s^\delta.
\]

Gathering the above estimates, we finally arrive at the desired conclusion:

\[
E_{\text{en}} (\Gamma^\gamma t, w, s) \leq C_s (C_1 \epsilon)^{\frac{1}{2}} s^\delta, \quad |\Gamma| \leq N.
\]

□
Proof of Theorem 1.1. According to the improved estimates in Proposition 4.3 and Proposition 4.4, if we choose $C_1$ sufficiently large, and $\epsilon$ sufficiently small (such that $C_\epsilon \ll 1/2$), then we are led to
\[
\begin{align*}
E(T^4(w_\epsilon, x, t)+1/2) &\leq C_1 \epsilon, \quad |I| \leq N, \\
E_{2,\alpha}(T^4(w_\epsilon, x, t)+1/2) &\leq C_1 \epsilon^2, \quad |I| \leq N - 1, \\
E_{2,\alpha}(T^4(w_\epsilon, x, t)+1/2) &\leq C_1 \epsilon^2 \psi^2, \quad |I| \leq N.
\end{align*}
\]

This means that $s_1 > s_0$ cannot be of finite value, otherwise, we can extend the solution to a larger hyperbolic time $s_1 > s_1$, which is thanks to the improved estimates in Eq. 4.19, but this will contradict to the definition of $s_1$ in Eq. 4.2. Thus we conclude that $s_1 = +\infty$.

Then given any time $T \geq t_0 + 1$, we integrate Eq. 4.8 over the space-time region (see ref. [30]) to get:
\[
\int_{T_0} \left[ \int_{\mathbb{R}^3} \left| \frac{\partial}{\partial t} w(t) + \sum_{|\alpha|=1} \nabla w(t) \frac{\partial}{\partial x} \Gamma_{\alpha}(t) \right|^2 \right] \, dt \leq C_1 \epsilon \int_{\mathbb{R}^3} \left| \frac{\partial}{\partial t} w(t) \right| \, dt,
\]

we observe that:
\[
\int_{T_0} \left| \int_{\mathbb{R}^3} \left| \frac{\partial}{\partial t} w(t) \right| \, dt \right| \, dx \leq C_1 \epsilon \int_{\mathbb{R}^3} \left| \frac{\partial}{\partial t} w(t) \right| \, dx,
\]

and we deduce that $\| \partial T^4 w \|_{L^2(I)} \leq C_1 \epsilon$ for all $|I| \leq N$. □

Declaration of competing interest
The authors declare that they have no conflicts of interest in this work.

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References
[1] S. Alinhac, The null condition for quasilinear wave equations in two space dimensions I, Invent. Math. 145 (2001) 597-618.
[2] S. Alinhac, The null condition for quasilinear wave equations in two space dimensions II, Amer. J. Math. 123 (2001) 1071-1101.
[3] S. Alinhac, A minicourse on global existence and blowup of classical solutions to multidimensional quasilinear wave equations, 2002, Journées “Équations aux Dérivées Partielles” (Forges-les-Eaux, 2002), Exp. No. 1, 33 pp., Univ. of Nantes.
[4] S. Alinhac, An example of blowup at infinity for a quasilinear wave equation. autour de l’analyse microlocale, Astérisque 284 (2003) 1-91.
[5] S. Alinhac, Geometric analysis of hyperbolic differential equations: An introduction , London Mathematical Society Lecture Note Series, vol. 374, Cambridge University Press, New York, 2010.
[6] J. Bourgain, On the growth in time of higher Sobolev norms of smooth solutions of Hamiltonian PDE, Int. Math. Res. Not. 6 (1996) 277-304.
[7] J. Bourgain, Growth of linear schrödinger equations with quasi-periodic potential, Commun. Math. Phys. 204 (1999) 207-247.
[8] J. Bourgain, Problems in hamiltonian PDE’s, Geom. Funct. Anal. (2000) 32-56.
[9] Y. Cai, Uniform bound of the highest-order energy of the 2d incompressible elastodynamics, (2020). Preprint arXiv:2010.08718.
[10] Y. Cai, Z. Lei, N. Masmoudi, Global well-posedness for 2d nonlinear wave equations without compact support, J. Math. Pures Appl. 114 (2018) 211-234.
[11] X. Cheng, D. Li, J. Xu, Uniform boundedness of highest norm for 2D nonlinear wave equation, (2014). Preprint arXiv:2104.01019.
[12] X. Cheng, D. Li, J. Xu et al., Global well-posedness for 2D nonlinear wave equation without compact support, (2014). Preprint arXiv:1010.0193.
[13] D. Christodoulou, Global solutions of nonlinear hyperbolic equations for small initial data, Comm. Pure Appl. Math. 39 (1986) 267-282.
[14] J. Colliander, M. Keel, G. Staffilani, et al., Transfer of energy to high frequencies in the cubic defocusing nonlinear schrödinger equation , Invent. Math. 181 (2010) 39-113.
[15] Y. Deng, F. Pusateri, On the global behavior of weak null nonlinear wave equations, Commun. Pure Appl. Math. 73 (2020) 1035-1099.
[16] S. Dong, Stability of a wave and Klein-Gordon system with mixed coupling, (1912). Preprint arXiv:912.05578.
[17] S. Dong, Global solution to the kleinogrodzakharov equations with uniform energy bounds, SIAM J. Math. Anal. 54 (1) (2022) 595-615.
[18] S. Dong, P.G. LeFloch, Z. Wyatt, Global evolution of the u(1) big boson: Nonlinear stability and uniform energy bounds, Ann. Henri Poincaré 22 (3) (2021) 677-713.
[19] P. Godin, Lifespan of solutions of semilinear wave equations in two space dimensions, Commun. Partial Differ. Equ. 18 (1993) 993-916.
[20] D. He, J. Liu, K. Wang, Scattering for the quasilinear wave equations with null conditions in two dimensions, J. Differential Equations 269 (2020) 3067-3098.
[21] L. Hörmander, Lectures on nonlinear hyperbolic differential equations, Springer Verlag,Berlin, 1997.
[22] F. Hou, H. Yin, Global small data smooth solutions of 2D null-form wave equations with non-compactly supported initial data, J. Differential Equations 268 (2020) 490-512.
[23] H. Huang Y. Ma, A conformal-type energy inequality on hyperboloids and its application to quasi-linear wave equation in $\mathbb{R}^n$, Preprint arXiv:1711.00498.
[24] F. John, Blow-up solutions for quasi-linear wave equations in three space dimensions, Comm. Pure Appl. Math. 34 (1981) 291-251.
[25] S. Katayama, Global solutions and the asymptotic behavior for nonlinear wave equations with small initial data, (2017). MDJ Memoirs,vol.36,Mathematical Society of Japan,Tokyo.
[26] A. Kiselev, V.S. ak, Small scale creation for solutions of the incompressible two-dimensional Euler equation, Ann. of Math. (2) 180 (3) (2014) 1205-1220.
[27] S. Klainerman, The null condition and global existence to nonlinear wave equations, Lect. Appl. Math. 23 (1986) 293-326.
[28] S. Klainerman, Global existence of small amplitude solutions to nonlinear klein-Gordon equations in four spacetime dimensions, Comm. Pure Appl. Math. 38 (1985) 631-641.
[29] P.G. LeFloch, Y. Ma, The global nonlinear stability of minkowski space for self-gravitating massive fields, The wave-Klein-Gordon model, Commun. Math. Phys. 346 (2016) 603-665.
[30] P.G. LeFloch, Y. Ma, The hyperbolic foliation method for nonlinear wave equations, World Scientific Press,Singapore, 2014.
[31] P.G. LeFloch, C.H. Wei, Boundeness of the total energy of relativistic membranes evolving in a curved spacetime, J. Differential Equations 265 (2018) 325-331.
[32] Z. Lei, Global well-posedness of incompressible elastodynamics in two dimensions, Comm. Pure Appl. Math. 69 (2016) 2072-2106.
[33] Z. Lei, J. Shi, Infinite-time exponential growth of the euler equation on two-dimensional torus, Preprint arXiv:1608.07010.
[34] Z. Lei, F. Wang, Uniform bound of the highest energy for the three dimensional incompressible elastodynamics, Arch. Ration. Mech. Anal. 216 (2015) 593-622.
[35] D. Li, Uniform estimates for 2D quasilinear wave, Preprint arXiv:2106.06419.
[36] T. Li, Y. Zhou, Nonlinear wave equations, Vol.2, Translated from the Chinese by Yachun Li. Series in Contemporary Mathematics,2. Shanghai Science and Technical Publishers,Shanghai,Berlin, Berlin, 17th, XIV + 391
[37] H. Lindblad, I. Rodnianski, The global stability of minkowski spacetime in harmonic gauge, Ann. of Math. 171 (3) (2010) 1401-1477.
[38] S. Ma, Uniform energy bound and morawetz estimate for extreme components of spin fields in the exterior of a slowly rotating Kerr black hole II: Linearized gravity, Comm. Math. Phys. 377 (3) (2020) 2489-2551.
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[39] S. Ma, Uniform energy bound and morawetz estimate for extreme components of spin fields in the exterior of a slowly rotating Kerr black hole I: Maxwell field, Ann. Henri Poincaré 21 (3) (2020) 815–863.

[40] Y. Ma, Global solutions of nonlinear wave-Klein-Gordon system in two spatial dimensions: Weak coupling case, Preprint ArXiv:1907.03516.

[41] C.D. Sogge, Lectures on nonlinear wave equations, International Press, Boston, 2008.

[42] F. Wang, Uniform bound of sobolev norms of solutions to 3d nonlinear wave equations with null condition, J. Differ. Equ. 256 (2014) 4013–4032.

[43] W.W. Wong, Small data global existence and decay for two dimensional wave maps, Preprint arXiv:1712.07684, to appear in Annales Henri Lebesgue.

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