Kolmogorov equations on spaces of measures associated to nonlinear filtering processes

Mattia Martini
Università degli Studi di Milano

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1 Stochastic filtering
- Nonlinear filtering problem
- Nonlinear filtering equations

2 Kolmogorov equations associated to filtering equations
- Itô formula
- Backward equation associated to the Zakai equation
- Backward equation associated to the K.-S. equation
We want to introduce and study a class of backward Kolmogorov equations on

- $\mathcal{M}^+_2(\mathbb{R}^d)$, $\mathcal{P}_2(\mathbb{R}^d)$: positive and probability measures with finite second moment;
- $\langle \mu, \psi \rangle = \mu(\psi) = \int_{\mathbb{R}^d} \psi(x) \mu(\,dx)$;

SDEs for measure-valued processes arise naturally in the stochastic filtering framework.

- Many results when there is a density, using stochastic calculus on Hilbert spaces (e.g. Rozovsky [9], Pardoux [8]).
- New tools for calculus on spaces of (probability) measures (e.g. Ambrosio, Giglio & Savarè [1], P.-L. Lions [5], Carmona & Delarue [3]).
- Optimal control with partial observation (e.g. Gozzi & Święch [4] in the Hilbert setting, or recently Bandini, Cosso, Fuhrman & Pham [2] on $\mathcal{P}_2(\mathbb{R}^d)$).
We want to introduce and study a class of backward Kolmogorov equations on

- $\mathcal{M}_2^+(\mathbb{R}^d)$, $\mathcal{P}_2(\mathbb{R}^d)$: positive and probability measures with finite second moment;
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Signal process

\[ dX_t = b(X_t) \, ds + \sigma(X_t) \, dW_t, \quad X_0 \in L^2(\Omega, F_0), \quad t \in [0, T]. \]  

(1)
Signal process

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(1)

Observation process

For every \( t \in [0, T] \),

\[ dY_t = h(X_t) \, dt + dB_t, \quad Y_0 = 0, \]

\[ \mathcal{F}_t^Y = \sigma(Y_s, 0 \leq s \leq t) \lor \mathcal{N}, \]

where \( \mathcal{N} \) are \( \mathbb{P} \)-negligible sets.
### Stochastic filtering: The problem

#### Signal process

\[
dx_t = b(x_t) \, ds + \sigma(x_t) \, dW_t, \quad x_0 \in L^2(\Omega, \mathcal{F}_0), \quad t \in [0, T]. \tag{1}
\]

#### Observation process

For every \( t \in [0, T] \),

\[
dy_t = h(x_t) \, dt + dB_t, \quad y_0 = 0,
\]

\[
\mathcal{F}^y_t = \sigma(y_s, 0 \leq s \leq t) \vee \mathcal{N},
\]

where \( \mathcal{N} \) are \( \mathbb{P} \)-negligible sets.

#### Goal

- The signal \( X \) is not directly observed;
- The available information is given by \( Y \);
- We want to provide an approximation of \( X \) given the observation \( Y \).
• Given the information $\mathcal{F}_t^\mathcal{Y}$, the best estimate for $\varphi(X_t)$ is

$$\mathbb{E} \left[ \varphi(X_t) | \mathcal{F}_t^\mathcal{Y} \right];$$
• Given the information $\mathcal{F}_t^Y$, the best estimate for $\varphi(X_t)$ is

$$\mathbb{E} \left[ \varphi(X_t) | \mathcal{F}_t^Y \right];$$

• Let $\Pi_t$ be the regular conditional probability distribution of $X_t$ given $\mathcal{F}_t^Y$: for any $A \in \mathcal{B}(\mathbb{R}^d)$

$$\Pi_t(A, \omega) = \mathbb{P} \left( X_t \in A | \mathcal{F}_t^Y \right)(\omega), \quad \text{a.e. } \omega.$$
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$$\Pi_t(A, \omega) = \mathbb{P} \left( X_t \in A | \mathcal{F}_t^Y \right)(\omega), \; \text{a.e. } \omega.$$

• For every $\varphi \in C_b(\mathbb{R}^d)$ and $t \in [0, T]$,

$$\langle \Pi_t, \varphi \rangle = \mathbb{E} \left[ \varphi(X_t) | \mathcal{F}_t^Y \right], \; \text{a.s.}$$
• Given the information $\mathcal{F}_t^Y$, the best estimate for $\varphi(X_t)$ is

$$E \left[ \varphi(X_t) | \mathcal{F}_t^Y \right] ;$$

• Let $\Pi_t$ be the regular conditional probability distribution of $X_t$ given $\mathcal{F}_t^Y$: for any $A \in \mathcal{B}(\mathbb{R}^d)$

$$\Pi_t(A, \omega) = P \left( X_t \in A | \mathcal{F}_t^Y \right) (\omega), \quad \text{a.e.} \ \omega.$$

• For every $\varphi \in C_b(\mathbb{R}^d)$ and $t \in [0, T]$,

$$\langle \Pi_t, \varphi \rangle = E \left[ \varphi(X_t) | \mathcal{F}_t^Y \right], \quad \text{a.s.}$$

$\{\Pi_t = \text{Law}(X_t | \mathcal{F}_t^Y)\}_{t \in [0, T]}$ is a $\mathcal{P}(\mathbb{R}^d)$-valued process called filter.
Define $Q$ by 

$$
\frac{dQ}{dp}|_{\mathcal{F}_t} = M_t^{-1} = \exp \left\{ -\frac{1}{2} \int_0^t |h(X_s)|^2 \, ds - \int_0^t h(X_s) \, dB_s \right\}.
$$
Define $\mathbb{Q}$ by
$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = M_t^{-1} = \exp\left\{-\frac{1}{2} \int_0^t |h(X_s)|^2 \, ds - \int_0^t h(X_s) \, dB_s\right\}.$$ 

**Theorem (Kallianpur-Striebel formula)**

The filter $\Pi$ can be represented as
$$\langle \Pi_t, \varphi \rangle = \frac{\langle \rho_t, \varphi \rangle}{\langle \rho_t, 1 \rangle}, \quad t \in [0, T], \varphi \in C_b(\mathbb{R}^d),$$

where $\langle \rho_t, \varphi \rangle = \mathbb{E}^\mathbb{Q}[M_t \varphi(X_t)|\mathcal{F}_t]$. 

\{\rho_t\}_{t\in[0,T]} is a $\mathcal{M}^+(\mathbb{R}^d)$-valued process called **unnormalized filter**.
Stochastic filtering The unnormalized filter

Define $\mathcal{Q}$ by
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where
\[ \langle \rho_t, \varphi \rangle = \mathbb{E}^{\mathcal{Q}} \left[ M_t \varphi(X_t) | \mathcal{F}_t^Y \right]. \]

\{\rho_t\}_{t \in [0, T]} is a $\mathcal{M}^+(\mathbb{R}^d)$-valued process called **unnormalized filter**.

$Y$ is a brownian motion under $\mathcal{Q}$. 
Stochastic filtering

The unnormalized filter

Define \( Q \) by
\[
\frac{dQ}{dP} |_{\mathcal{F}_t} = M_t^{-1} = \exp \left\{ -\frac{1}{2} \int_0^t |h(X_s)|^2 \, ds - \int_0^t h(X_s) \, dB_s \right\}.
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Theorem (Kallianpur-Striebel formula)

The filter \( \Pi \) can be represented as
\[
\langle \Pi_t, \varphi \rangle = \frac{\langle \rho_t, \varphi \rangle}{\langle \rho_t, 1 \rangle}, \quad t \in [0, T], \varphi \in C_b(\mathbb{R}^d),
\]
where \( \langle \rho_t, \varphi \rangle = \mathbb{E}^Q [M_t \varphi(X_t) | \mathcal{F}_t^Y] \).

\( \{\rho_t\}_{t \in [0, T]} \) is a \( \mathcal{M}^+(\mathbb{R}^d) \)-valued process called unnormalized filter.

\( Y \) is a brownian motion under \( Q \). By Itô formula applied to \( M_t \varphi(X) \) we obtain

The Zakai equation (Z)

The unnormalized filter satisfies, for every test \( \varphi \),
\[
d\langle \rho_t, \varphi \rangle = \langle \rho_t, A \varphi \rangle \, dt + \langle \rho_t, h \varphi \rangle \, dY_t, \quad t \in (0, T],
\]
where \( A \) is the infinitesimal generator of \( X \).
Stochastic filtering Kushner-Stratonovitch equation

Let $A$ be the generator of $X$: $A \varphi = b^T (D_x \varphi) + \frac{1}{2} \text{tr}\{(D_x^2 \varphi)\sigma \sigma^T\}$.

The Zakai equation (Z)

The unnormalized filter satisfies, for every test $\varphi$,

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where $Y$ is a Brownian motion under $\mathbb{Q}$. 

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Let $A$ be the generator of $X$: $A\varphi = b^\top (D_x \varphi) + \frac{1}{2} \text{tr}\{ (D_x^2 \varphi) \sigma \sigma^\top \}$.

**The Zakai equation (Z)**

The unnormalized filter satisfies, for every test $\varphi$,

$$d\langle \rho_t, \varphi \rangle = \langle \rho_t, A \varphi \rangle \, dt + \langle \rho_t, h \varphi \rangle \, dY_t, \quad t \in (0, T],$$

where $Y$ is a Brownian motion under $Q$.

**Using the Kallianpur-Striebel formula**

**The Kushner-Stratonovich equation (KS)**

The filter satisfies, for every test $\varphi$,

$$d\langle \Pi_t, \varphi \rangle = \langle \Pi_t, A \varphi \rangle \, dt + (\langle \Pi_t, h \varphi \rangle - \langle \Pi_t, \varphi \rangle \langle \Pi_t, h \rangle) \, dl_t, \quad t \in (0, T],$$

where $\{l_t\}_{t \in [0, T]}$ is called **innovation process** and is a Brownian motion under $P$. 
Stochastic filtering

Example: Kalman-Bucy filter

Signal:

\[ dX_t = b_t X_t \, dt + \sigma_t \, dW_t, \quad a^{ij}_t = \sigma_t \sigma^T_t, \]

\[ A_t \varphi(x) = D_x \varphi(x)^T b_t x + \frac{1}{2} \sum_{i,j} a^{ij}_t \partial^2_{ij} \varphi(x). \]
Stochastic filtering  Example: Kalman-Bucy filter

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\]
\[
A_t \varphi(x) = D_x \varphi(x)^\top b_t x + \frac{1}{2} \sum_{i,j} a_{ij}^t \partial_{ij}^2 \varphi(x).
\]

Observation:
\[
dY_t = h_t X_t \, dt + dB_t, \quad Y_0 = 0.
\]

\((X, Y)\) is a gaussian process.
Stochastic filtering Example: Kalman-Bucy filter

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\[ dX_t = b_t X_t \, dt + \sigma_t \, dW_t, \quad a_{ij}^t = \sigma_t \sigma_t^\top, \]
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The filter \( \Pi \) solves
\[ d\langle \Pi_t, \varphi \rangle = \langle \Pi_t, A_s \varphi \rangle \, dt + \langle \Pi_t, \varphi h_t^\top \iota \rangle \, dl_t - \langle \Pi_t, \varphi \rangle \langle \Pi_t, h^\top \iota \rangle \, dl_t, \]
\[ \iota(x) = x. \]
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Signal:
\[ dX_t = b_t X_t \, dt + \sigma_t \, dW_t, \quad a_t^{ij} = \sigma_t \sigma_t^T, \]
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\[ \iota(x) = x. \] Moreover, for \( \omega \in \Omega \) fixed, \( \Pi_t(\omega) \) is gaussian with
- **Mean** \( \hat{X}_t \) that solves the SDE
  \[ d\hat{X}_t = b_t \hat{X}_t \, dt + \gamma_t h_t \, dl_t, \quad l_t = Y_t - \int_0^t h_s \hat{X}_s \, ds. \]
- **Deterministic variance** that solves the Riccati equation
  \[ \frac{d}{dt} \gamma_t = \gamma_t b_t^T + b_t \gamma_t + a_t - \gamma_t (h^T h) \gamma_t^T. \]
Let $\{\rho_t\}_{t \in [0, T]}$ be a solution to (Z), i.e. for every test $\varphi$
\[ d\langle \rho_t, \varphi \rangle = \langle \rho_t, A\varphi \rangle \, dt + \langle \rho_t, h\varphi \rangle \, dY_t, \quad t \in (0, T). \]
Itô formula for the Zakai equation

Let \( \{ \rho_t \}_{t \in [0,T]} \) be a solution to (Z), i.e. for every test \( \varphi \)

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\]

Hypotheses (H)

a. \( b, \sigma, h \) are Borel-measurable and bounded, \( b, \sigma \) are Lipschitz;

b. The matrix \( \sigma \sigma^\top (x) \) is positive definite for every \( x \in \mathbb{R}^d \).
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Proposition (M. [6])

Let \( u \) be in \( \mathcal{C}^2_L(\mathcal{M}_2^+(\mathbb{R}^d)) \) and let us assume (H). Then, for every \( t \in [0, T] \):
\[
du(\rho_t) = \langle \rho_t, A\delta u(\rho_t) \rangle \, dt + \langle \rho_t, h\delta u(\rho_t) \rangle \, dY_t + \frac{1}{2} \langle \rho_t \otimes \rho_t, h^\top h\delta^2 u(\rho_t) \rangle \, dt.
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\[
du(\rho_t) = \langle \rho_t, A \delta_{\mu} u(\rho_t) \rangle \, dt + \langle \rho_t, h \delta_{\mu} u(\rho_t) \rangle \, dY_t + \frac{1}{2} \langle \rho_t \otimes \rho_t, h^\top h \delta_{\mu}^2 u(\rho_t) \rangle \, dt.
\]

- \( \delta_{\mu} u \) is a notions of derivatives for \( u : \mathcal{M}^+(\mathbb{R}^d) \to \mathbb{R} \):
  \[
  \delta_{\mu} u : \mathcal{M}^+(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}, \quad \delta_{\mu}^2 u : \mathcal{M}^+(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R};
  \]
- Proof by cylindrical approximation: \( u(\mu) := g(\langle \mu, \psi_1 \rangle, \ldots, \langle \mu, \psi_n \rangle) \).

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The generator $\mathcal{L} : C^2_{L}(\mathcal{M}_2^+(\mathbb{R}^d)) \to C_b(\mathcal{M}_2^+(\mathbb{R}^d))$
The infinitesimal generator of the Zakai equation

The generator $\mathcal{L} : C^2_L(\mathcal{M}_2^+(\mathbb{R}^d)) \to C_b(\mathcal{M}_2^+(\mathbb{R}^d))$

$$(\mathcal{L}u)(\mu) = \langle \mu, A\delta_\mu u(\mu) \rangle + \frac{1}{2} \langle \mu \otimes \mu, h^\top h\delta_\mu^2 u(\mu) \rangle$$

$$= \int_{\mathbb{R}^d} (A\delta_\mu u)(\mu, x) \mu(\,dx) + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x)^\top h(y)\delta_\mu^2 u(\mu, x, y) \mu(\,dx) \mu(\,dy).$$
The infinitesimal generator of the Zakai equation

The generator \( \mathcal{L} : C^2_L(\mathcal{M}_2^+ (\mathbb{R}^d)) \to C_b(\mathcal{M}_2^+ (\mathbb{R}^d)) \)

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(\mathcal{L}u)(\mu) = \langle \mu, A\delta_\mu u(\mu) \rangle + \frac{1}{2} \langle \mu \otimes \mu, h^\top h\delta_\mu^2 u(\mu) \rangle \\
= \int_{\mathbb{R}^d} (A\delta_\mu u)(\mu, x)\mu(\,dx) + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x)^\top h(y)\delta_\mu^2 u(\mu, x, y)\mu(\,dx)\mu(\,dy).
\]

Remark

- Formally \( d\rho_t = A^* \rho_t \, dt + h^\top \rho_t \, dY_t \), so:

\[
du(\rho_t) = \langle A^* \rho_t, \delta_\mu u(\rho_t) \rangle \, dt + \langle h^\top \rho_t, \delta_\mu u(\rho_t) \rangle \, dY_t + \frac{1}{2} \langle h^\top \rho_t \otimes h\rho_t, \delta_\mu^2 u(\rho_t) \rangle \, dt.
\]
The infinitesimal generator of the Zakai equation

The generator $\mathcal{L} : C^2_{\mathcal{L}}(\mathcal{M}^+_2(\mathbb{R}^d)) \to C_b(\mathcal{M}^+_2(\mathbb{R}^d))$

$$(\mathcal{L}u)(\mu) = \langle \mu, A\delta_{\mu}u(\mu) \rangle + \frac{1}{2} \langle \mu \otimes \mu, h^\top h\delta^2_{\mu}u(\mu) \rangle$$

$$= \int_{\mathbb{R}^d} (A\delta_{\mu}u)(\mu, x)\mu(dx) + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x)^\top h(y)\delta^2_{\mu}u(\mu, x, y)\mu(dx)\mu(dy).$$

Remark

- Formally $d\rho_t = A^\ast \rho_t \, dt + h^\top \rho_t \, dY_t$, so:

$$du(\rho_t) = \langle A^\ast \rho_t, \delta_{\rho_t}(\rho_t) \rangle \, dt + \langle h^\top \rho_t, \delta_{\rho_t}(\rho_t) \rangle \, dY_t + \frac{1}{2} \langle h^\top \rho_t \otimes h\rho_t, \delta^2_{\rho_t}(\rho_t) \rangle \, dt.$$

- On $\mathbb{R}$, if $dX_t = bX_t \, dt + \sigma X_t \, dB_t$, then

$$du(X_t) = bX_t \, D_x u(X_t) \, dt + \sigma X_t \, D_x u(X_t) \, dB_t + \frac{1}{2} \sigma^2 X_t^2 \, D_x^2 u(X_t) \, dt.$$
The backward Kolmogorov equation

Existence and uniqueness

Let

\[(Lu)(\mu) = \langle \mu, A\delta_\mu u(\mu) \rangle + \frac{1}{2} \langle \mu \otimes \mu, h^\top h\delta_{\mu}^2 u(\mu) \rangle.\]  

(5)
Let

$$(Lu)(\mu) = \langle \mu, A\delta_{\mu} u(\mu) \rangle + \frac{1}{2} \langle \mu \otimes \mu, h^\top h \delta_{\mu}^2 u(\mu) \rangle.$$  \hfill (5)

Given $\Phi: \mathcal{M}_2^+(\mathbb{R}^d) \to \mathbb{R}$, the Backward Kolmogorov equation (BEZ) reads as

$$\begin{cases}
    \partial_s u(\mu, s) + Lu(\mu, s) = 0, \quad (\mu, s) \in \mathcal{M}_2^+(\mathbb{R}^d) \times [0, T], \\
    u(\mu, T) = \Phi(\mu), \quad \mu \in \mathcal{M}_2^+(\mathbb{R}^d).
\end{cases}$$
Let
\[(L u)(\mu) = \langle \mu, A \delta \mu u(\mu) \rangle + \frac{1}{2} \langle \mu \otimes \mu, h^\top h \delta^2 \mu u(\mu) \rangle.\] (5)

Given $\Phi : M_2^+ (\mathbb{R}^d) \to \mathbb{R}$, the Backward Kolmogorov equation (BEZ) reads as
\[
\begin{cases}
\partial_s u(\mu, s) + Lu(\mu, s) = 0, & (\mu, s) \in M_2^+ (\mathbb{R}^d) \times [0, T], \\
u(\mu, T) = \Phi(\mu), & \mu \in M_2^+ (\mathbb{R}^d).
\end{cases}
\]

Let $\{\rho_t^{s, \mu}\}_{t \in [s, T]}$ be a solution to (Z) starting at time $s$ from $\mu \in M_2^+ (\mathbb{R}^d)$. 

Theorem (M. [6]) Let $\Phi \in C^2 L (M_2^+ (\mathbb{R}^d))$. Let (H) holds and let us set
\[
u(\mu, s) := E[\Phi(\rho_s^{s, \mu})], \quad (\mu, s) \in M_2^+ (\mathbb{R}^d) \times [0, T].\] (6)
Then $u$ is the unique classical solution to (BEZ).
Let
\[(\mathcal{L}u)(\mu) = \langle \mu, A\delta_\mu u(\mu) \rangle + \frac{1}{2} \langle \mu \otimes \mu, h^\top h \delta^2_\mu u(\mu) \rangle.\] (5)

Given \(\Phi: \mathcal{M}^+_2(\mathbb{R}^d) \to \mathbb{R}\), the **Backward Kolmogorov equation** (BEZ) reads as
\[
\begin{aligned}
\partial_s u(\mu, s) + \mathcal{L} u(\mu, s) &= 0, \quad (\mu, s) \in \mathcal{M}^+_2(\mathbb{R}^d) \times [0, T], \\
u(\mu, T) &= \Phi(\mu), \quad \mu \in \mathcal{M}^+_2(\mathbb{R}^d).
\end{aligned}
\]

Let \(\{\rho_t^{s, \mu}\}_{t \in [s, T]}\) be a solution to (Z) starting at time \(s\) from \(\mu \in \mathcal{M}^+_2(\mathbb{R}^d)\).

**Theorem (M. [6])**
Let \(\Phi \in C^2_L(\mathcal{M}^+_2(\mathbb{R}^d))\). Let (H) holds and let us set
\[
u(\mu, s) := \mathbb{E}[\Phi(\rho_t^{s, \mu})], \quad (\mu, s) \in \mathcal{M}^+_2(\mathbb{R}^d) \times [0, T].\] (6)

Then \(u\) is the unique classical solution to (BEZ).
Uniqueness:

- By the Itô formula, every classical solution to (BEZ) has the form

\[ u(\mu, s) = \mathbb{E} \left[ \Phi(\rho_{s}^{s, \mu}) \right]. \]
Proof (key steps)

Uniqueness:

- By the Itô formula, every classical solution to (BEZ) has the form

\[ u(\mu, s) = \mathbb{E} \left[ \Phi(\rho_{T}^{s, \mu}) \right] . \]

Existence:

- Prove that \( \mu \mapsto u(\mu, s) := \mathbb{E} \left[ \Phi(\rho_{T}^{s, \mu}) \right] \) is in \( C_{L}^{2}(\mathcal{M}_{2}^{+}(\mathbb{R}^{d})) \):
Proof (key steps)

Uniqueness:

• By the Itô formula, every classical solution to (BEZ) has the form

\[ u(\mu, s) = \mathbb{E} \left[ \Phi(\rho^s_{\tau}, \mu) \right]. \]

Existence:

• Prove that \( \mu \mapsto u(\mu, s) := \mathbb{E} \left[ \Phi(\rho^s_{\tau}, \mu) \right] \) is in \( C^2_{\mathcal{L}}(\mathcal{M}_2^+(\mathbb{R}^d)) \):
  
  • given a suitable notion of derivative for functions from \( C^2_{\mathcal{L}}(\mathcal{M}_2^+(\mathbb{R}^d)) \) to \( C^2_{\mathcal{L}}(\mathcal{M}_2^+(\mathbb{R}^d)) \), we show that \( \mu \mapsto \rho^s_{\tau, \mu} \) is twice differentiable;
Proof (key steps)

Uniqueness:

- By the Itô formula, every classical solution to (BEZ) has the form
  \[ u(\mu, s) = \mathbb{E} [\Phi(\rho_{T}^{s,\mu})] . \]

Existence:

- Prove that \( \mu \mapsto u(\mu, s) := \mathbb{E} [\Phi(\rho_{T}^{s,\mu})] \) is in \( C_{L}^{2}(\mathcal{M}_{2}^{+}(\mathbb{R}^{d})) \):
  - given a suitable notion of derivative for functions from \( C_{L}^{2}(\mathcal{M}_{2}^{+}(\mathbb{R}^{d})) \) to \( C_{L}^{2}(\mathcal{M}_{2}^{+}(\mathbb{R}^{d})) \), we show that \( \mu \mapsto \rho_{T}^{s,\mu} \) is twice differentiable;
  - since \( \Phi \in C_{L}^{2}(\mathcal{M}_{2}^{+}(\mathbb{R}^{d})) \) and by the previous point, we conclude by a chain rule.
Proof (key steps)

Uniqueness:

• By the Itô formula, every classical solution to (BEZ) has the form

$$u(\mu, s) = \mathbb{E} \left[ \Phi(\rho^s_{T, \mu}) \right].$$

Existence:

• Prove that $\mu \mapsto u(\mu, s) := \mathbb{E} \left[ \Phi(\rho^s_{T, \mu}) \right]$ is in $C^2_L(\mathcal{M}_2^+(\mathbb{R}^d))$:
  
  - given a suitable notion of derivative for functions from $C^2_L(\mathcal{M}_2^+(\mathbb{R}^d))$ to $C^2_L(\mathcal{M}_2^+(\mathbb{R}^d))$, we show that $\mu \mapsto \rho^s_{T, \mu}$ is twice differentiable;
  - since $\Phi \in C^2_L(\mathcal{M}_2^+(\mathbb{R}^d))$ and by the previous point, we conclude by a chain rule.

• By Itô formula and Markov property

$$\lim_{h \to 0} \frac{1}{h} \left[ u(\mu, s + h) - u(\mu, s) \right] = -\lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \int_s^{s+h} \mathcal{L}u(\rho^s_{T, \mu}, s + h) \, d\tau \right] = -\mathcal{L}u(\mu, s).$$
The Kushner-Stratonovich equation case

The operator $L^{KS}: C^2_L(\mathcal{P}_2(\mathbb{R}^d)) \rightarrow C_b(\mathcal{P}_2(\mathbb{R}^d))$

$$L^{KS}u(\pi) = \langle \pi, A\delta_\mu u(\pi) \rangle + \frac{1}{2} \langle \pi \otimes \pi, (h - \pi(h))^\top(h - \pi(h))\delta_{\mu}^2 u(\pi) \rangle.$$
The Kushner-Stratonovich equation case

The operator $\mathcal{L}^\text{KS}: \mathcal{C}^2_L(\mathcal{P}_2(\mathbb{R}^d)) \to \mathcal{C}_b(\mathcal{P}_2(\mathbb{R}^d))$

\[ \mathcal{L}^\text{KS} u(\pi) = \langle \pi, A\delta_\mu u(\pi) \rangle + \frac{1}{2} \langle \pi \otimes \pi, (h - \pi(h))^\top (h - \pi(h))\delta_\mu u(\pi) \rangle. \]

Given $\Phi: \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, the Backward Kolmogorov equation (BEKS) reads as

\[
\begin{cases}
\partial_s u(\pi, s) + \mathcal{L}^\text{KS} u(\pi, s) = 0, & (\pi, s) \in \mathcal{P}_2(\mathbb{R}^d) \times [0, T], \\
u(\pi, T) = \Phi(\pi), & \pi \in \mathcal{P}_2(\mathbb{R}^d).
\end{cases}
\]
The Kushner-Stratonovich equation case

The operator $L^{KS}: C^2_L(\mathcal{P}_2(\mathbb{R}^d)) \to C_b(\mathcal{P}_2(\mathbb{R}^d))$

$$L^{KS}u(\pi) = \langle \pi, A\delta\mu u(\pi) \rangle + \frac{1}{2} \langle \pi \otimes \pi, (h - \pi(h))^\top (h - \pi(h))\delta^2\mu u(\pi) \rangle.$$

Given $\Phi: \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, the Backward Kolmogorov equation (BEKS) reads as

$$ \begin{cases} 
\partial_s u(\pi, s) + L^{KS}u(\pi, s) = 0, & (\pi, s) \in \mathcal{P}_2(\mathbb{R}^d) \times [0, T], \\
u(\pi, T) = \Phi(\pi), & \pi \in \mathcal{P}_2(\mathbb{R}^d). 
\end{cases} $$

Let $\{\Pi_t^{s,\pi}\}_{t \in [s, T]}$ be a solution to (KS) starting at time $s$ from $\pi \in \mathcal{P}_2(\mathbb{R}^d)$:

$$d\langle \Pi_t, \psi \rangle = \langle \Pi_t, A\psi \rangle \, dt + (\langle \Pi_t, h\psi \rangle - \langle \Pi_t, \psi \rangle \langle \Pi_t, h \rangle) \cdot dl_t, \quad t \in (0, T]. \quad (7)$$

**Theorem (M. [6])**

Let $\Phi \in C^2_L(\mathcal{P}_2(\mathbb{R}^d))$. Let $(H)$ holds and let us set

$$u(\pi, s) = E \left[ \Phi(\Pi_T^{s,\pi}) \right], \quad (\pi, s) \in \mathcal{P}_2(\mathbb{R}^d) \times [0, T].$$

Then $u$ is the unique classical solution to (BEKS).
The Kushner-Stratonovich equation case Viscosity approach

\[ K \subset \mathbb{R}^d \text{ compact, } \Phi \in C_b(\mathcal{P}_2(K)): \]

\[
\begin{cases}
  \partial_s u(\pi, s) + \mathcal{L}^{KS} u(\pi, s) = 0, \quad (\pi, s) \in \mathcal{P}_2(K) \times (0, T], \\
  u(\pi, T) = \Phi(\pi), \quad \pi \in \mathcal{P}_2(K).
\end{cases}
\]

Let \( \{\Pi_t^{s, \pi}\}_{t \in [s, T]} \) be a solution to (KS) confined in \( \mathcal{P}_2(K) \).

**Theorem (M. [7])**

Let \( \Phi \in C_b(\mathcal{P}_2(K)) \). Let (H) holds and let us set

\[
u(\pi, s) = \mathbb{E}\left[ \Phi(\Pi_t^{s, \pi}) \right], \quad (\pi, s) \in \mathcal{P}_2(K) \times (0, T].
\]

Then \( u \) is the unique viscosity solution to (BEKS).
Proof of the comparison principle (Key steps)

Let $u_1$ and $u_2$ be respectively a subsolution and a supersolution to (BEKS). Moreover, let $u(\pi, s) := \mathbb{E} \left[ \Phi(\Pi_s^\pi) \right]$. We want to show that $u_1 \leq u_2$.

- Show: $u_1 \leq u$ and $u \leq u_2$.
- Introduce a family of approximated problems:

$$\begin{cases}
\partial_s u(\pi, s) + \mathcal{L}^{KS} u(\pi, s) = 0, & (\pi, s) \in \mathcal{P}_2(K) \times (0, T], \\
u(\pi, T) = \Phi_n(\pi) \in C^2_L(\mathcal{P}_2(K)), & \pi \in \mathcal{P}_2(K).
\end{cases}$$

- $u^n(\pi, s) := \mathbb{E} \left[ \Phi_n(\Pi_s^\pi) \right]$ is a classical solution to the approximated problem which converges to $u$.
- Using the Borwein-Preiss variational principle with a suitable smooth gauge-type function, we introduce a suitable test function that allows us to conclude.
Thank you!
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**Spaces of measures Linear functional derivatives**

**Linear functional derivative**

\[ u: \mathcal{M}^+(\mathbb{R}^d) \to \mathbb{R} \text{ is in } C^1_b(\mathcal{M}^+(\mathbb{R}^d)) \text{ if it is continuous, bounded and if exists} \]

\[ \delta_\mu u: \mathcal{M}^+(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mu, x) \mapsto \delta_\mu u(\mu, x) \in \mathbb{R}, \]

bounded, continuous and such that for all \( \mu \) and \( \mu' \) in \( \mathcal{M}^+(\mathbb{R}^d) \), it holds:

\[
 u(\mu') - u(\mu) = \int_0^1 \int_{\mathbb{R}^d} \delta_\mu u \left( t \mu' + (1-t)\mu, x \right) [\mu' - \mu](dx) \, dt. \tag{8}
\]

Similarly we can define \( C^k_b(\mathcal{M}^+(\mathbb{R}^d)), k \in \mathbb{N} \).

**Example**

Let \( g \in C^2_b(\mathbb{R}) \) and let \( \psi \in C_b(\mathbb{R}^d) \). We define

\[ u: \mathcal{M}^+(\mathbb{R}^d) \ni \mu \mapsto g(\langle \mu, \psi \rangle) \in \mathbb{R}. \]

Then \( u \in C^2_b(\mathcal{M}^+(\mathbb{R}^d)) \) and it holds:

\[
 \delta_\mu u(\mu, x) = g'(\langle \mu, \psi \rangle) \psi(x), \quad \delta^2_\mu u(\mu, x, y) = g''(\langle \mu, \psi \rangle) \psi(x)\psi(y).\]
The space $C^2_L(M^+(\mathbb{R}^d))$

$u: M^+(\mathbb{R}^d) \rightarrow \mathbb{R}$ is in $C^2_L(M^+(\mathbb{R}^d))$ if:

a. $u$ is in $C^2_B(M^+(\mathbb{R}^d))$;

b. $\mathbb{R}^d \ni x \mapsto \delta_\mu u(\mu, x) \in \mathbb{R}$ is twice differentiable, with continuous and bounded derivatives on $M^+(\mathbb{R}^d) \times \mathbb{R}^d$.

We set

$$D_\mu u(\mu, x) := D_x \delta_\mu u(\mu, x) \in \mathbb{R}^d,$$

Remark

On $\mathcal{P}_2(\mathbb{R}^d)$, the derivative $D_\mu u$ coincides with the one introduced by P.-L. Lions through the lifting procedure in the context of mean field games ([5, 3]).
1. Prove the formula for functions of the form

\[ u: \mathcal{M}_2^+ (\mathbb{R}^d) \ni \mu \mapsto g (\langle \mu, \psi_1 \rangle, \ldots, \langle \mu, \psi_n \rangle), \]

exploiting classical Itô formula and the Zakai equation.

2. Prove the formula for functions of the form

\[ u(\mu) = \langle \frac{\mu^r}{\mu(\mathbb{R}^d)^r}, \varphi(\cdot, \ldots, \cdot, \mu(\mathbb{R}^d)) \rangle \]

by approximation, where \( \varphi: \mathbb{R}^{d \times r+1} \rightarrow \mathbb{R} \) is symmetrical in the first \( r \) arguments.

3. Prove the formula for functions in \( C^2_L (\mathcal{M}_2^+ (\mathbb{R}^d)) \) by approximation.
The backward Kolmogorov equation  Existence and uniqueness

Theorem (M. [6])

Let us set

\[ u(\mu, s) = \mathbb{E} \left[ \Phi(\rho_T^{s, \mu}) \right], \tag{9} \]

where \( \rho_T^{s, \mu} \) is the weak solution to the Zakai equation starting at time \( s \) from \( \mu \in \mathcal{M}_2^+ (\mathbb{R}^d) \), \( \Phi \in C^2_{L}(\mathcal{M}_2^+ (\mathbb{R}^d)) \) and let (H) hold. Then \( u \) is the unique classical solution to the backward Kolmogorov equation (BEZ).

Proof (uniqueness)

We show that if \( u \) is a classical solution to (BEZ), then \( u(\mu, s) = \mathbb{E} \left[ \Phi(\rho_T^{s, \mu}) \right] \).

- By the Itô formula

\[ u(\rho_T^{s, \mu}, T) - u(\rho_s^{s, \mu}, s) = \int_s^T \left\{ \partial_s u(\rho_T^{s, \mu}, \tau) + \mathcal{L} u(\rho_T^{s, \mu}, \tau) \right\} \, d\tau + \int_s^T \mathcal{G} u(\rho_T^{s, \mu}, \tau) \cdot dY_\tau. \]

- By taking the expectation and since \( u \) solves (BEZ)

\[ \mathbb{E} \left[ \Phi(\rho_T^{s, \mu}) \right] - u(\mu, s) = \mathbb{E} \left[ \int_s^T \mathcal{G} u(\rho_T^{s, \mu}, \tau) \cdot dY_\tau \right]. \]

- The rhs is zero since the integral is a martingale, thus \( u(\mu, s) = \mathbb{E} \left[ \Phi(\rho_T^{s, \mu}) \right] \).
The backward Kolmogorov equation  
Existence and uniqueness

Proof (existence)

Let $u(\mu, s) = \mathbb{E} \left[ \Phi(\rho^s_T, \mu) \right]$ be our candidate solution.

1. Prove that $\mu \mapsto u(\mu, s)$ is in $C^2_L(M^+_T(R^d))$:
   - given a suitable notion of derivative for functions from $C^2_L(M^+_T(R^d))$ to $C^2_L(M^+_T(R^d))$, we show that $\mu \mapsto \rho^s_T, \mu$ is twice differentiable;
   - since $\Phi \in C^2_L(M^+_T(R^d))$ and by the previous point, we conclude by a chain rule.

2. Prove the continuity of

   $[0, T] \ni s \mapsto Lu(\mu, s), \quad [s, T] \times [0, T] \ni (\tau, \sigma) \mapsto Lu(\rho^s_T, \mu, \tau) \in L^2(\Omega)$.

3. By the Itô formula and the Markov property

   $$\lim_{h \to 0} \frac{1}{h} \left[ u(\mu, s + h) - u(\mu, s) \right] = -\lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \int_s^{s+h} Lu(\rho^s_T, \mu, s + h) \, d\tau \right] = -Lu(\mu, s).$$