LARGE $q$ CONVERGENCE OF RANDOM CHARACTERISTIC POLYNOMIALS TO RANDOM PERMUTATIONS AND ITS APPLICATIONS

GILYOUNG CHEONG, HAYAN NAM, AND MYUNGJUN YU

ABSTRACT. We extend an observation due to Stong that the distribution of the number of degree $d$ irreducible factors of the characteristic polynomial of a random $n \times n$ matrix over a finite field $\mathbb{F}_q$ converges to the distribution of the number of length $d$ cycles of a random permutation in $S_n$, as $q \to \infty$, by having any finitely many choices of $d$, say $d_1, \ldots, d_r$. This generalized convergence will be used for the following two applications: the distribution of the cokernel of an $n \times n$ Haar-random $\mathbb{Z}_p$-matrix when $p \to \infty$ and a matrix version of Landau’s theorem that estimates the number of irreducible factors of a random characteristic polynomial for large $n$ when $q \to \infty$.

1. INTRODUCTION

The purpose of this paper is to give some concrete arithmetic applications of the following fact: for any finitely many distinct $d_1, \ldots, d_r \in \mathbb{Z}_{\geq 1}$, the distribution of numbers of length $d_1, \ldots, d_r$ cycles of a random $\sigma \in S_n$, as $q$ goes to infinity.

**Proposition 1.1.** Let $d_1, \ldots, d_r \in \mathbb{Z}_{\geq 1}$ be distinct and $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$ not necessarily distinct. We have

$$\lim_{q \to \infty} \text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)} \left( f_A(t) \text{ has } k_j \text{ degree } d_j \text{ irreducible factors for } 1 \leq j \leq r \right) \text{ counting with multiplicity } = \text{Prob}_{\sigma \in S_n} \left( \sigma \text{ has } k_j \text{ disjoint } d_j \text{-cycles for } 1 \leq j \leq r \right).$$

where $f_A(t)$ is the characteristic polynomial of $A \in \text{Mat}_n(\mathbb{F}_q)$.

**Remark 1.2.** After the previous version of this paper, we have been noted that Proposition 1.1 is known due to Hansen and Schumutz ([HS1993], Theorem 1.1). Their result is for $\text{GL}_n(\mathbb{F}_q)$ rather than $\text{Mat}_n(\mathbb{F}_q)$, but these two are easily interchangeable. Indeed, the last line of their proof (but not the statement), together with a well-known fact regarding some statistics of degree $n$ monic polynomials in $\mathbb{F}_q[t]$ for large $q$, implies Proposition 1.1. The proof of Hansen and Schumutz uses Stong’s work [Sto1988], on which our proof also depends, and they even compute some asymptotic errors with respect to $q$ and $r$. Since our work only considers the limit in $q$, our proof is simpler, so we decided to keep it in the paper.

In the following subsections, we discuss two applications of Proposition 1.1. The first one, Theorem 1.3 considers the distribution of the cokernel of a random $n \times n$ matrix over $\mathbb{Z}_p$, the ring of $p$-adic integers, with respect to the Haar (probability) measure on $\text{Mat}_n(\mathbb{Z}_p) = \mathbb{Z}_p^{n^2}$, for large $p$ and $n$. According to our best knowledge and the consultations with the experts we have talked to, Theorem 1.3 is new. The second one, Theorem 1.7 is a matrix version of Landau’s theorem in number theory that estimates the number of irreducible factors of a random characteristic polynomial for large $n$ when $q \to \infty$. Just as Proposition 1.1 can be deduced from the work of Hansen and Schumutz [HS1993], Theorem 1.7 can also be deduced from their work if we apply another result of S. Cohen in [Coh1963]. The interesting difference in our exposition

---

*Date: May 19, 2020.*

1The event on the right-hand side means that when we write $\sigma$ as a unique product of disjoint cycles, there are $k_j$ cycles of length $d_j$ in the product for $1 \leq j \leq r$. The special case $r = 1$ is indirectly mentioned in “Conclusion” section of [Sto1988].
is that we instead used Jordan’s result in [Jor1947] about the statistics of a random permutation in $S_n$ for large $n$ rather than the statistics of degree $n$ monic polynomials in $\mathbb{F}_q[t]$ for large $q$ and $n$ in order to deduce Theorem 1.6.

1.1. The distribution of cokernels of Haar $\mathbb{Z}_p$-random matrices with large $p$. Let $\mathbb{Z}_p$ be the ring of $p$-adic integers, namely the inverse limit of the system of projection maps $\cdots \twoheadrightarrow \mathbb{Z}/(p^3) \twoheadrightarrow \mathbb{Z}/(p^2) \twoheadrightarrow \mathbb{Z}/(p)$.

**Theorem 1.3.** Fix any distinct $d_1, \ldots, d_r \in \mathbb{Z}_{\geq 1}$. For any $d \in \mathbb{Z}_{\geq 1}$ denote by $\mathcal{M}(d, p)$ the set of all monic irreducible polynomials of degree $d$ in $\mathbb{F}_p[t]$. Then the limit

$$
\lim_{p \to \infty} \lim_{n \to \infty} \Pr_{A \in \text{Mat}_n(\mathbb{Z}_p)} \left( \text{coker}(P(A)) = 0 \right)
$$

exists, where for $P(A)$, we use any lift of $\bar{P}(t) \in \mathbb{F}_p[t]$ in $\mathbb{Z}_p[t]$ under the reduction modulo $p$ and the probability is according to the Haar measure on $\text{Mat}_n(\mathbb{Z}_p)$. Moreover, we have

$$
\lim_{n \to \infty} \lim_{p \to \infty} \Pr_{A \in \text{Mat}_n(\mathbb{Z}_p)} \left( \text{coker}(P(A)) = 0 \right) = e^{-1/d_1} \cdots e^{-1/d_r}.
$$

**Remark 1.4.** Taking the two limits in the particular order matters in Theorem 1.3. For instance, we are unsure whether we can change the order of these limits, which may lead to an interesting consequence regarding the Cohen-Lenstra measure introduced in [CL1983]. It is worth noting that our proof can be easily modified so that one may replace $\mathbb{Z}_p$ with $\mathbb{F}_q[t]$, reduce modulo $(t)$ in place of $p$, and let $q \to \infty$ instead of $p \to \infty$ in Theorem 1.3, although we will just focus on $\mathbb{Z}_p$. The reader does not need to have much expertise of the Haar measure on $\text{Mat}_n(\mathbb{Z}_p)$ to follow our proof. The only property of the Haar measure on $\text{Mat}_n(\mathbb{Z}_p)$ we will use is that each fiber under the mod $p$ projection map $\text{Mat}_n(\mathbb{Z}_p) \to \text{Mat}_n(\mathbb{F}_p)$ has the same measure, which is necessarily $1/p^{n^2}$.

We conjecture that the large $p$ limit in Theorem 1.3 has to do with independent Poisson random variables with means $1/d_1, \ldots, 1/d_r$ in more generality. However, we have little empirical evidence, so it would be interesting to see any more examples that either support or disprove this conjecture.

**Conjecture 1.5.** Fix any distinct $d_1, \ldots, d_r \in \mathbb{Z}_{\geq 1}$ and not necessarily distinct $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$. Then the limit

$$
\lim_{p \to \infty} \lim_{n \to \infty} \Pr_{A \in \text{Mat}_n(\mathbb{Z}_p)} \left( |\text{coker}(P_j(A))| = p^{d_j k_j} \right)
$$

must exist. Moreover, we must have

$$
\lim_{n \to \infty} \lim_{p \to \infty} \Pr_{A \in \text{Mat}_n(\mathbb{Z}_p)} \left( |\text{coker}(P_j(A))| = p^{d_j k_j} \right) = \frac{e^{-1/d_1}(1/d_1)^{k_1}}{k_1!} \cdots \frac{e^{-1/d_r}(1/d_r)^{k_r}}{k_r!}.
$$

1.2. Jordan-Landau theorem for matrices over finite fields. It is a folklore that the cycle decomposition of a random permutation is analogous to the prime decomposition of a random positive integer. For instance, Granville (Section 2.1 of [Gra] notes that given $k \in \mathbb{Z}_{\geq 1}$, a theorem of Jordan (p.161 of [Jor1947]) says

$$
\Pr_{\sigma \in S_n}(\sigma \text{ has } k \text{ disjoint cycles}) \sim \frac{(\log n)^{k-1}}{(k-1)!n}.
$$
for large $n \in \mathbb{Z}_{\geq 1}$, while a theorem of Landau (p.211 of [Lan1909] or Theorem 437 on p.491 of [HW2008]) says

$$\text{Prob}_{1 \leq N \leq x} \left( \begin{array}{c} N \text{ has } k \text{ distinct prime factors} \\ \text{counting with multiplicity} \end{array} \right) \sim \text{Prob}_{1 \leq N \leq x} \left( \begin{array}{c} N \text{ has } k \text{ prime factors} \\ \text{counting with multiplicity} \end{array} \right) \sim \frac{(\log \log x)^{k-1}}{(k-1)! \log x}$$

for large $x \in \mathbb{R}_{\geq 1}$. The first probability is given uniformly at random from the set $S_n$ of permutations on $n$ letters. The second probability is given by choosing the integer $N$ uniformly at random from the set $\{1,2,\ldots,|x|\}$. We wrote “$f(x) \sim g(x)$ for large $x$” to mean that $f(x)/g(x) \to 1$ as $x \to \infty$. This is a generalization of the Prime Number Theorem (i.e., the case $k = 1$).

By the Prime Number Theorem, for large $x$, we may expect one prime in every interval of length $\log x$. For large $n$, we expect one cycle in $n$ permutations (since there are $(n-1)!$ $n$-cycles while there are far less cycles of shorter lengths), so Jordan’s result is analogous to Landau’s result. We take this analogy further. In particular, summing the identity given in Proposition 1.1 over all tuples $(d_1,\ldots,d_r) \in (\mathbb{Z}_{\geq 1})^r$ and $(k_1,\ldots,k_r) \in (\mathbb{Z}_{\geq 0})^r$ such that

- $r \in \mathbb{Z}_{\geq 0}$,
- $n = \sum_{j=1}^{r} k_j d_j$, and
- $k = \sum_{j=1}^{r} k_j$,

we have

$$\lim_{q \to \infty} \text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)} \left( f_A(t) \text{ has } k \text{ irreducible factors} \right) = \lim_{q \to \infty} \text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)} \left( f_A(t) \text{ has } k \text{ disjoint cycles} \right) \sim \frac{(\log n)^{k-1}}{(k-1)! n}$$

which is asymptotically $(\log n)^{k-1}/((k-1)! n)$ for large $n$, as noted above. Using the facts that almost all matrices $A \in \text{Mat}_n(\mathbb{F}_q)$ have square-free characteristic polynomials and almost all monic polynomials of degree $n$ are square-free when $q \to \infty$ (see Section 3), we may obtain the following.

**Theorem 1.6** (Jordan-Landau theorem for $\mathbb{F}_q$-matrices). Given $k \in \mathbb{Z}_{\geq 1}$, for large $n \in \mathbb{Z}_{\geq 1}$, we have

$$\lim_{q \to \infty} \text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)} \left( f_A(t) \text{ has } k \text{ distinct irreducible factors} \right) = \lim_{q \to \infty} \text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)} \left( f_A(t) \text{ has } k \text{ irreducible factors} \right) \sim \frac{(\log n)^{k-1}}{(k-1)! n}$$

**Remark 1.7.** There is an analogous result of Theorem 1.6 where the sample space is the set $\mathcal{A}^n(\mathbb{F}_q)$ of all degree $n$ monic polynomials in $\mathbb{F}_q[t]$ due to S. Cohen [Coh1968]. It says given $k \in \mathbb{Z}_{\geq 1}$, for large $n \in \mathbb{Z}_{\geq 1}$, we have

$$\lim_{q \to \infty} \text{Prob}_{f \in \mathcal{A}^n(\mathbb{F}_q)} \left( f(t) \text{ has } k \text{ distinct irreducible factors} \right) \sim \lim_{q \to \infty} \text{Prob}_{f \in \mathcal{A}^n(\mathbb{F}_q)} \left( f(t) \text{ has } k \text{ irreducible factors} \right) \sim \frac{(\log n)^{k-1}}{(k-1)! n}$$

With these two results in mind, we note that the push-forward measure $\text{Mat}_n(\mathbb{F}_q) = \mathcal{A}^n(\mathbb{F}_q) \to \mathcal{A}^n(\mathbb{F}_q)$ given by taking the characteristic polynomial of a matrix does not change the probability of the event we compute in $\mathcal{A}^n(\mathbb{F}_q)$, when we take $q \to \infty$. This can be also taken as another example to be added in the list of similarly behaving combinatorial examples in [ABT1997], a survey paper by Arratia, Barbour, and Tavaré.
1.3. Fulman’s generalization and further directions. Recently, we have also realized that Fulman generalized Proposition 1.1 to certain linear algebraic groups $G$ over $\mathbb{F}_q$, where $G = \text{GL}_n$ is the desired special case for Proposition 1.1 ([Ful1999], Theorem 15), where $S_n$ occurs as the Weyl group of $G(\mathbb{F}_q) = \text{GL}_n(\mathbb{F}_q)$. Fulman’s argument needs to assume that the probability that a random element in $G(\mathbb{F}_q) \subset G(\mathbb{F}_q)$ is regular semisimple converges to 1, as $q \to \infty$, which we previously thought as a mere heuristics. There is a statement in Fulman’s paper ([Ful1999], Theorem 5 and Theorem 7) that proves a similar property related to this hypothesis for $G = \text{GL}_n$, but it takes $n \to \infty$ before we can let $q \to \infty$. However, for $g \in \text{GL}_n(\mathbb{F}_q)$ saying that $g$ is regular semisimple turns out to mean that the characteristic polynomial of $g$ is square-free in $\mathbb{F}_q[t]$, so using a similar argument as in Lemma 2.2 we see that $G = \text{GL}_n$ satisfies Fulman’s hypothesis because the number of degree $n$ monic square-free polynomials $f(t) \in \mathbb{F}_q[t]$ such that $f(0) \neq 0$ is precisely
\[
q^n - 2q^{n-1} + 2q^{n-2} - \cdots + (-1)^{n-1}2q + (-1)^n,
\]
which replaces Lemma 2.1 in the argument. We are not sure whether the above count is explicitly written in literature, but this is certainly well-known (e.g., more general phenomena are dealt in various works such as BE1997, Kim1994, and VW2015). More specifically, one can obtain this count by counting the $\mathbb{F}_q$-points of the unordered configuration space $\text{Conf}^n(\mathbb{A}^1 \setminus \{0\})$ of $n$ points on the affine line minus the origin over $\mathbb{F}_q$. The generating function for such count $|\text{Conf}^n(\mathbb{A}^1 \setminus \{0\})(\mathbb{F}_q)|$ is given by
\[
\sum_{n=0}^\infty |\text{Conf}^n(\mathbb{A}^1 \setminus \{0\})(\mathbb{F}_q)|t^n = \frac{Z_{\mathbb{A}^1 \setminus \{0\}}(t)}{Z_{\mathbb{A}^1 \setminus \{0\}}(t^2)} = \frac{1 - qt^2}{(1+t)(1-qt)},
\]
where $Z_X(t)$ means the zeta series of an algebraic variety over $\mathbb{F}_q$, and expanding the right-hand side, one may obtain the desired count. We are not sure whether this hypothesis may or may not be satisfied by other linear algebraic groups $G$ over $\mathbb{F}_q$. The reader must note that the reasons that this works for $G = \text{GL}_n$ are as follows:

1. we are aware of the size of each fiber of the map $\text{Mat}_n(\mathbb{F}_q) = \mathbb{A}^{n^2}(\mathbb{F}_q) \to \mathbb{A}^n(\mathbb{F}_q)$ given by taking the characteristic polynomials and

2. the preimage of the set of degree $n$ monic polynomials with nonzero constant terms under this map is precisely $\text{GL}_n(\mathbb{F}_q)$.

Nevertheless, it would be extremely interesting if one can identify which algebraic groups $G$, other than $\text{GL}_n$, over $\mathbb{F}_q$ produce the sets $G(\mathbb{F}_q)$ of $\mathbb{F}_q$-points that can replace $\text{Mat}_n(\mathbb{F}_q)$ with the Weyl groups of $G(\mathbb{F}_q)$, replacing $S_n$ in Proposition 1.1. If such algebraic groups are given in a sequence $G_n$ in $n \in \mathbb{Z}_{\geq 1}$, then we can hope that studying the corresponding sequence $W_n$ of Weyl groups asymptotically in $n$ may produce analogous statements for Theorem 4.3 and Theorem 4.6.

1.4. Organization for the rest. In Section 2 we will show that Proposition 1.1 implies Theorem 1.4. In Section 3 we will show that Proposition 1.1 implies Theorem 1.5. We will provide a crucial formula due to Stong in Section 4 but our proof is slightly different from the original one, as we use direct counting of Young diagrams. Stong’s formula will be used in proving Proposition 1.1 in Section 5. Finally, in Section 6 we will provide another proof of Lemma 3.2, an influential result of Shepp and Lloyd, which states that the number of length $d$ cycles of a random permutation in $S_n$ asymptotically follows the Poisson distribution with mean $1/d$ when $n$ is large, and the probability is given independently for finitely many different choices of $d$. It is interesting that our proof is entirely combinatorial, while the original proof was probabilistic. Since the result is quite popular in literature, our alternative proof of Lemma 3.2 may be known to experts, although we were unable to locate it.

1.5. Acknowledgment. We thank Yifeng Huang, Nathan Kaplan, and Jeff Lagarias for helpful conversations. G. Cheong was supported by NSF grant DMS-1162181 and the Korea Institute for Advanced Study for his visits to the institution regarding this research. M. Yu was supported by a KIAS Individual Grant (SP075201) via the Center for Mathematical Challenges at Korea Institute for Advanced Study. He was also
supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1C1C1A01007604).

2. Proposition 1.1 implies Theorem 1.6

We already know that Proposition 1.1 implies that

$$\lim_{q \to \infty} \text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)} \left( f_A(t) \text{ has } k \text{ irreducible factors counting with multiplicity} \right) \sim \frac{(\log n)^{k-1}}{(k-1)!n}$$

for large $n$ in the statement of Theorem 1.6. We need to justify the statement about distinct irreducible factors. We need two lemmas, well-known to the experts, to provide a complete exposition.

**Lemma 2.1.** Let $\mathcal{A}^n(\mathbb{F}_q)$ be the set of all monic polynomials of degree $n$ in $\mathbb{F}_q[t]$. For $n \geq 2$, we have

$$|\{ f \in \mathcal{A}^n(\mathbb{F}_q) : f(t) \text{ is square-free in } \mathbb{F}_q[t] \}| = q^n(1 - q^{-1}).$$

In particular, we have

$$\lim_{q \to \infty} \text{Prob}_{f \in \mathcal{A}^n(\mathbb{F}_q)}(f(t) \text{ is square-free in } \mathbb{F}_q[t]) = 1.$$

**Proof.** This is a well-known fact in literature. For a proof, see Lemma 3.4 of [CWZ2015]. A more combinatorial proof using partitions is also available (e.g., Proposition 5.9 in [VW2015] with $a = 2$).

**Lemma 2.2.** For any $n \in \mathbb{Z}_{\geq 1}$, we have

$$\lim_{q \to \infty} \text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)}(f_A(t) \text{ is square-free in } \mathbb{F}_q[t]) = 1.$$

**Proof.** Fix any monic square-free polynomial

$$f(t) = P_1(t) \cdots P_r(t),$$

where $P_i(t)$ are distinct monic irreducible polynomials in $\mathbb{F}_q[t]$. By Theorem 2 in [Rei1960], the number of $A \in \text{Mat}_n(\mathbb{F}_q)$ such that $f_A(t) = f(t)$ is

$$q^{n^2-n} \cdot \frac{(1 - q^{-1})(1 - q^{-2}) \cdots (1 - q^{-n})}{(1 - q^{-\deg(P_1)})(1 - q^{-\deg(P_2)}) \cdots (1 - q^{-\deg(P_r)})} \geq q^{n^2-n} \cdot \frac{(1 - q^{-1})(1 - q^{-2}) \cdots (1 - q^{-n})}{(1 - q^{-n})^r}.$$

For $n = 1$, the map $A \mapsto f_A(t)$ is bijective, and thus letting $q \to \infty$, we are done. For $n \geq 2$, by Lemma 2.1 we have

$$\text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_q)}(f_A(t) \text{ is square-free in } \mathbb{F}_q[t]) \geq (1 - q^{-1}) \cdot \frac{(1 - q^{-1})(1 - q^{-2}) \cdots (1 - q^{-n})}{(1 - q^{-n})^r},$$

so letting $q \to \infty$ gives the result.
\textit{Proof that Proposition 1.1 implies Theorem 1.6} Again, we only need to show the first identity in the statement. By Lemma 2.2 we have

\[
\lim_{q \to \infty} \text{Prob}_{\text{Mat}_n(F_q)} \left( f_A(t) \text{ has } k \text{ distinct irreducible factors} \right) = \lim_{q \to \infty} \text{Prob}_{\text{Mat}_n(F_q)} \left( f_A(t) \text{ is square-free in } \mathbb{F}_q[t] \text{ and has } k \text{ irreducible factors counting with multiplicity} \right) = \lim_{q \to \infty} \text{Prob}_{\text{Mat}_n(F_q)} \left( f_A(t) \text{ has } k \text{ irreducible factors counting with multiplicity} \right),
\]

because the number of monic non-square-free polynomials of degree \( n \) is negligible as \( q \to \infty \).

\[\Box\]

3. Proposition 1.1 implies Theorem 1.3

3.1. Useful lemmas. The following result, due to Shepp and Lloyd, will be crucial in applying Proposition 1.1 to obtain Theorem 1.3. In [SL1966], Shepp and Lloyd obtained the result by showing that the characteristic function of the distribution of the numbers of cycles of fixed lengths \( d_1, \ldots, d_r \) of a random permutation converges to the characteristic function of the distribution given by the independent Poisson random variables with the means \( 1/d_1, \ldots, 1/d_r \) and then applying Lévy’s theorem. We will provide our own combinatorial proof of this result in Section 6.

\textbf{Notation 3.1.} For any permutation \( \sigma \in S_n \), we denote by \( m_d(\sigma) \) the number of \( d \)-cycles in the cycle decomposition of \( \sigma \).

\textbf{Lemma 3.2} (cf. p.342 in [SL1966]). Fix \( r \in \mathbb{Z}_{\geq 0} \). Given distinct \( d_1, \ldots, d_r \in \mathbb{Z}_{\geq 1} \) and not necessarily distinct \( k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0} \), we have

\[
\lim_{n \to \infty} \text{Prob}_{\sigma \in S_n}(m_{d_j}(\sigma) = k_j \text{ for } 1 \leq j \leq r) = \frac{e^{-1/d_1} (1/d_1)^{k_1}}{k_1!} \cdots \frac{e^{-1/d_r} (1/d_r)^{k_r}}{k_r!},
\]

which means that the number of cycles of length \( d_1, \ldots, d_r \) of a random permutation of \( n \) letters are asymptotically given by independent Poisson random variables with means \( 1/d_1, \ldots, 1/d_r \) when \( n \) is large.

We need the following lemma to deduce Theorem 1.3 from Proposition 1.1. This lemma contains some ideas used for proving Theorem C of [CH2019], but the proof here is simpler, since we only need a special case of their argument.

\textbf{Lemma 3.3.} Let \( P_1, \ldots, P_r \in \mathbb{Z}_p[t] \) be any monic polynomials whose images in \( \mathbb{F}_p[t] \) under the mod \( p \) reduction are distinct irreducible polynomials. Then

\[
\text{Prob}_{\text{Mat}_n(Z_p)} \left( \text{coker}(P_j(A)) = 0 \text{ for } 1 \leq j \leq r \right) = \text{Prob}_{\text{Mat}_n(F_p)} \left( A[\bar{P}_j^{\infty}] = 0 \text{ for } 1 \leq j \leq r \right),
\]

where \( A[\bar{P}_j^{\infty}] \) means the \( \bar{P}_j \)-part of the \( \mathbb{F}_p[t] \)-module structure given by the matrix action \( A \subset \mathbb{F}_p^n \).

\textbf{Proof.} First, note that for any \( A \in \text{Mat}_n(F_p) \), we have \( \dim_{\mathbb{F}_p} \ker(P_j(A)) = \dim_{\mathbb{F}_p} \ker(\bar{P}_j(A)) \) and \( \ker(\bar{P}_j(A)) = A[\bar{P}_j^{\infty}]/\bar{P}_j(t)A[\bar{P}_j^{\infty}] \), as \( \mathbb{F}_p[t] \)-modules where the action of \( t \) is given by multiplying the matrix \( A \) on the left. Hence, the following are equivalent:

- \( A[\bar{P}_j^{\infty}] = 0 \);
- \( \ker(\bar{P}_j(A)) = 0 \);
- \( \ker(\bar{P}_j(A)) = 0 \),
so it is enough to show that

\[
\text{Prob}_{A \in \text{Mat}_n(\mathbb{Z}_p)}\left( \begin{array}{c}
\text{coker}(P_j(A)) = 0 \\
\text{for } 1 \leq j \leq r
\end{array} \right) = \text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_p)}\left( \begin{array}{c}
\text{coker}(\tilde{P}_j(A)) = 0 \\
\text{for } 1 \leq j \leq r
\end{array} \right).
\]

For any \( B \in \text{Mat}_n(\mathbb{Z}_p) \), the right exactness of \((-) \otimes_{\mathbb{Z}_p} \mathbb{F}_p\) implies that

\[
\text{coker}(B) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \cong \text{coker}(\tilde{B}),
\]

where \( \tilde{B} \in \text{Mat}_n(\mathbb{F}_p) \) is the reduction of \( B \) mod \( p \). Hence, by Nakayama’s lemma, we have \( \text{coker}(B) = 0 \) if and only if \( \text{coker}(\tilde{B}) = 0 \). Thus, if we consider the mod \( p \) reduction map \( \pi : \text{Mat}_n(\mathbb{Z}_p) \rightarrow \text{Mat}_n(\mathbb{F}_p) \), we have

\[
\bigcap_{i=1}^{\pi-1} \{ A \in \text{Mat}_n(\mathbb{F}_p) : \text{coker}(\tilde{P}_j(A)) = 0 \} = \bigcap_{i=1}^{\pi-1} \{ A \in \text{Mat}_n(\mathbb{Z}_p) : \text{coker}(P_j(A)) = 0 \}.
\]

Since each fiber under \( \pi \) has the same measure \( 1/p^n \), this finishes the proof. \( \square \)

3.2. **Proof of Theorem 1.3 given Proposition 1.1.** We now prove Theorem 1.3 assuming Proposition 1.1. Proposition 1.1 will be proven in Section 5.

Proof of Theorem 1.3 given Proposition 1.1. Let \( \mathcal{M}(q,d) \) be the set of degree \( d \) monic irreducible polynomials in \( \mathbb{F}_q[t] \) so that \( \mathcal{M}(q,d) = |\mathcal{M}(q,d)| \). By Lemma 5.3, we have

\[
\text{Prob}_{A \in \text{Mat}_n(\mathbb{Z}_p)}\left( \begin{array}{c}
\text{coker}(P(A)) = 0 \\
\text{for all } P \in \bigcup_{j=1}^{\pi} \mathcal{M}(p, d_j)
\end{array} \right) = \text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_p)}\left( \begin{array}{c}
A[P] = 0 \\
\text{for all } P \in \bigcup_{j=1}^{\pi} \mathcal{M}(p, d_j)
\end{array} \right) = \text{Prob}_{A \in \text{Mat}_n(\mathbb{F}_p)}\left( f_A(t) \text{ has no irreducible factors of degrees } d_1, \ldots, d_r \right).
\]

Hence, we may apply Proposition 1.1 and Lemma 5.2 to finish the proof. \( \square \)

4. Strong’s formula

4.1. **Set-up.** Let \( R \) be a PID, and say \( m \) is a maximal ideal of \( R \) such that \( R/m = \mathbb{F}_q \). Denote by \( \mathcal{P} \) the set of all partitions including the empty partition \( \emptyset \). Then any finite length (or equivalently, finite size) \( m^\infty \)-torsion \( R \)-module is isomorphic to

\[
H_{m,\lambda}^R := R/m^{\lambda_1} \oplus \cdots \oplus R/m^{\lambda_l},
\]

for a unique partition \( \lambda = (\lambda_1, \ldots, \lambda_l) \in \mathcal{P} \). We will always assume \( \lambda_1 \geq \cdots \geq \lambda_l > 0 \), and we write \( |\lambda| := \lambda_1 + \cdots + \lambda_l \). The case \( l = 0 \) will correspond to the empty partition \( \emptyset = \emptyset \). Note that

\[
|H_{m,\lambda}^R| = q^{\lambda_1 + \cdots + \lambda_l} = q^{|\lambda|}.
\]

We will write \( H_{m,\lambda} := H_{m,\lambda}^R \) if the meaning of \( R \) is evident from the context. Denote by \( m_d(\lambda) \) the number of parts of size \( d \) in \( \lambda \), and define

\[
n(\lambda) := 0 \cdot \lambda_1 + 1 \cdot \lambda_2 + 2 \cdot \lambda_3 + \cdots + (l - 1) \cdot \lambda_l.
\]

The number \( |\text{Aut}_R(H_{m,\lambda}^R)| \) of automorphisms of \( H_{m,\lambda}^R \) can be computed by noting the fact that

\[
|\text{Aut}_R(H_{m,\lambda}^R)| = |\text{Aut}_{R_m}(H_{mR_m,\lambda}^R)|
\]
Lemma 4.1 (1.6 on p.181 in [Mac1995]). Let \((R, m)\) be a DVR (discrete valuation ring) with the finite residue field \(R/m = \mathbb{F}_q\). Then we have

\[ |\text{Aut}_R(H_{m,\lambda})| = q^{|\lambda|+2n(\lambda)} \prod_{d=1}^{\infty} \prod_{i=1}^{m_d(\lambda)} (1 - q^{-i}). \]

To our best knowledge, the following result is due to Stong in [Sto1988], but we rephrase Stong’s result in a more general setting and provide a slightly different proof. We will essentially go through some key ideas in Lemma 6 in [Sto1988], which relies on the Fine-Herstein theorem (i.e., the number of \(n \times n\) nilpotent matrices over \(\mathbb{F}_q\) is \(q^{n(n-1)}\)), but unlike the reference, we will avoid the use of Möbius inversion along with exponentiation, logarithm, differentiation and integration of power series by counting partitions (i.e., Young diagrams) instead.

Lemma 4.2 (cf. Proposition 19 in [Sto1988]). Let \(R\) be any Dedekind domain with a maximal ideal \(m\) such that \(R/m = \mathbb{F}_q\). Then

\[ \sum_{M \in \text{Mod}_{R/m}} u^{\dim_{\mathbb{F}_q}(M)} \frac{|\text{Aut}_R(M)|}{|\text{Aut}_R(H_{m,\lambda})|} = \prod_{i=1}^{\infty} \frac{1}{1 - q^{-i}u^i}, \]

where \(u\) is a complex variable with \(|u| < q^{1/q}\). Equivalently, we have

\[ \sum_{\lambda \in \mathcal{P}} \frac{|\text{Aut}_R(H_{m,\lambda})|}{|\text{Aut}_R(H_{m,\lambda})|} = \prod_{i=1}^{\infty} \frac{1}{1 - q^{-iy}}, \]

where \(y\) is a complex variable with \(|y| < q\).

Proof. As explained in the beginning of this section, finite \(m^\infty\)-torsion \(R\)-modules are parametrized by partitions. Hence, taking \(y = u^q\), we see that the two given statements are equivalent. We will prove the second statement. Applying Lemma 4.1, this reduces our problem to the case where \(R = \mathbb{F}_q[t]\) and \(m = \langle t \rangle\) by replacing \(R_m\) with \(\mathbb{F}_q[t]_{\langle t \rangle}\) or \(\mathbb{F}_q[t]_{\langle t \rangle}\). This reduction lets us rewrite the left-hand side of the desired identity as

\[ \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \frac{y^n}{|\text{Stab}_{\text{GL}_n(\mathbb{F}_q)}(J_{0,\lambda})|}, \]

where \(J_{0,\lambda}\) is the Jordan canonical form given by the 0-Jordan blocks of whose sizes are equal to the parts of the partition \(\lambda\), and the action \(\text{GL}_n(\mathbb{F}_q) \supset \text{Mat}_n(\mathbb{F}_q)\) is given by the conjugation. By the orbit-stabilizer theorem, we have

\[ \frac{1}{|\text{Stab}_{\text{GL}_n(\mathbb{F}_q)}(J_{0,\lambda})|} = \frac{|\text{GL}_n(\mathbb{F}_q) : J_{0,\lambda}|}{|\text{GL}_n(\mathbb{F}_q)|}. \]

Recall that \(n \times n\) matrices all of whose eigenvalues are 0 are precisely nilpotent matrices. Thus, we have
\[
\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \frac{y^n}{|\text{Stab}_{\text{GL}_n(F_q)}(J_{0,\lambda})|} = \sum_{n=0}^{\infty} \frac{|\text{Nil}_n(F_q)|y^n}{|\text{GL}_n(F_q)|}
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{q^n(n-1)y^n}{(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})}
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{(q^{-1}y)^n}{(1 - q^{-1})(1 - q^{-2}) \cdots (1 - q^{-n})}
\]

\[
= \sum_{n=0}^{\infty} (q^{-1}y)^n \left( \sum_{j_1=1}^{\infty} q^{-j_1} \right) \left( \sum_{j_2=0}^{\infty} q^{-2j_2} \right) \cdots \left( \sum_{j_n=0}^{\infty} q^{-nj_n} \right)
\]

\[
= \prod_{n=0}^{\infty} (1 + q^{-i}(q^{-1}y) + q^{-2i}(q^{-1}y)^2 + \cdots)
\]

\[
= \prod_{i=0}^{\infty} \frac{1}{1 - q^{-(i+1)}y}
\]

\[
= \prod_{i=1}^{\infty} \frac{1}{1 - q^{-i}y},
\]

where for the second equality we used the elementary identity \(|\text{GL}_n(F_q)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})\), and Fine-Herstein theorem (e.g., Theorem 1 of [FH1958]), which gives the number \(|\text{Nil}_n(F_q)|\) of nilpotent matrices in \(\text{Mat}_n(F_q)\):

\[
|\text{Nil}_n(F_q)| = q^{n(n-1)}.
\]

For the sixth equality, we have used the fact that the coefficient of \(Y^n\) of the following product

\[
\prod_{i=0}^{\infty} (1 + X^iY + X^{2i}Y^2 + \cdots)
\]

is equal to

\[
\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} X^{j_1+2j_2+\cdots+nj_n},
\]

where \(X, Y\) are considered to be complex numbers varying within the open unit disk centered at 0 (i.e., \(|X|, |Y| < 1\)) so that we can take \(X = q^{-1}\) and \(Y = q^{-1}y\) with \(|y| < q\) in our proof for the sixth equality in the chain of equalities above. Indeed, when we expand the given product, we have

\[
\prod_{i=0}^{\infty} (1 + X^iY + X^{2i}Y^2 + \cdots) = \sum_{m_0, m_1, m_2, \cdots \geq 0} X^{0m_0 + 1m_1 + 2m_2 + \cdots} Y^{m_0 + m_1 + m_2 + \cdots}
\]

\[
= \sum_{m_0, m_1, m_2, \cdots \geq 0} X^{m_1 + 2m_2 + \cdots},
\]

so it is enough to show that
\[
\sum_{m_0, m_1, m_2, \ldots \geq 0 \atop m_0 + m_1 + m_2 + \cdots = n} X^{m_1 + 2m_2 + \cdots} = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} X^{j_1 + 2j_2 + \cdots + nj_n}.
\]

Note that we have a bijection

\[\{(m_0, m_1, m_2, \ldots) \in \mathbb{Z}_{\geq 0}^\infty : m_0 + m_1 + m_2 + \cdots = n\} \leftrightarrow \{(m_1, m_2, \ldots) \in \mathbb{Z}_{\geq 0}^\infty : m_1 + m_2 + \cdots \leq n\}\]

given by \((m_0, m_1, m_2, \ldots) \mapsto (m_1, m_2, \ldots)\). This reduces our problem to the following:

\[
\sum_{m_1, m_2, \ldots \geq 0 \atop m_1 + m_2 + \cdots \leq n} X^{m_1 + 2m_2 + \cdots} = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} X^{j_1 + 2j_2 + \cdots + nj_n}.
\]

If \(n = 0\), both sides are 1, so let \(n \geq 1\). Let \(A_n\) be the set of partitions whose number of parts counting with multiplicity (i.e., the row lengths of Young diagrams) is \(\leq n\) and \(B_n\) the set of partitions whose parts (i.e., the column lengths of Young diagrams) are \(\leq n\). Then we have a bijection \(A_n \leftrightarrow B_n\) given by taking conjugation, so in particular, we have

\[
\sum_{\lambda \in A_n} X^{\left|\lambda\right|} = \sum_{\lambda \in B_n} X^{\left|\lambda\right|}.
\]

Now, noting that

\[
\sum_{\lambda \in A_n} X^{\left|\lambda\right|} = \sum_{m_1, m_2, \ldots \geq 0 \atop m_1 + m_2 + \cdots \leq n} X^{m_1 + 2m_2 + \cdots}
\]

and

\[
\sum_{\lambda \in B_n} X^{\left|\lambda\right|} = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} X^{j_1 + 2j_2 + \cdots + nj_n},
\]

we finish the proof. \(\square\)

5. PROOF OF PROPOSITION 1.1

For our proof of Proposition 1.1, we need to analyze polynomials that encode some information about random matrices in \(\text{Mat}_n(F_q)\) and random permutations in \(S_n\). Such a polynomial is called a “cycle index”.

5.1. Cycle index. Given any permutation group \(G \leq S_n\), we define the cycle index of \(G\) as the following polynomial:

\[
Z(G, x) := \frac{1}{|G|} \sum_{g \in G} x_1^{m_1(g)} \cdots x_n^{m_n(g)},
\]

where we recall that \(m_d(g)\) means the number of \(d\)-cycles in the cycle decomposition of \(g\). Cycle indices were used to solve various enumeration problems, and to our best knowledge, these were invented independently by Redfield [Red1927] and Pólya [Pol1937]. (Pólya’s paper is in German, but there is an English translation [PRI1987] by Read). We will use the cycle index \(Z(S_n, x)\) of the full symmetric group \(S_n\). Note that for any \(\sigma, \tau \in S_n\), we note that \(\sigma\) and \(\tau\) are conjugate to each other in \(S_n\) if and only if \(x_1^{m_1(\sigma)} \cdots x_n^{m_n(\sigma)} = x_1^{m_1(\tau)} \cdots x_n^{m_n(\tau)}\), so \(Z(S_n, x)\) captures some information about the conjugate action \(S_n \subset S_n\).

We will also make use of the following polynomial:
\[
Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x) := \frac{1}{|\mathbb{F}_q^{\lambda}|} \sum_{A \in \text{Mat}_n(\mathbb{F}_q)} \prod_{P \in |A^\lambda_q|} x_{P, \mu_P(A)},
\]
where \( \mu_P(A) = (\mu_1, \ldots, \mu_i) \) is the partition defined by the \( P \)-part of the \( \mathbb{F}_q[t] \)-module structure defined by the matrix multiplication \( A \subset \mathbb{F}_q^n \):
\[
A[P^\lambda] \approx \mathbb{F}_q[t]/(P)^{\mu_1} \oplus \cdots \oplus \mathbb{F}_q[t]/(P)^{\mu_i}.
\]

We denoted by \( |A^\lambda_q| \) the set of all monic irreducible polynomials in \( \mathbb{F}_q[t] \). For \( P \in |A^\lambda_q| \) and a nonempty partition \( \lambda \), the notation \( x_{P, \lambda} \) means a formal variable associated to the pair \( (P, \lambda) \). For the empty partition, we define \( x_{P, \emptyset} := 1 \). We call \( Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x) \) the cycle index of the conjugate action \( \text{GL}_n(\mathbb{F}_q) \subset \text{Mat}_n(\mathbb{F}_q) \). The terminology makes sense because each monomial \( \prod_{P \in |A^\lambda_q|} x_{P, \mu_P(A)} \) is characterized by an orbit of such action. That is, two matrices \( A \) and \( B \) are in the same orbit if and only if \( \prod_{P \in |A^\lambda_q|} x_{P, \mu_P(A)} = \prod_{P \in |B^\lambda_q|} x_{P, \mu_P(B)} \). The notion of \( Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x) \) is introduced by Stong \[Sto1988\], generalizing a similar definition for the conjugate action \( \text{GL}_n(\mathbb{F}_q) \subset \text{GL}_n(\mathbb{F}_q) \) introduced by Kung \[Kun1981\]. We will use the following factorization results for the generating functions for the cycle indices.

**Lemma 5.1** (p.14 of \[PRI1987\]). In the ring \( \mathbb{Q}[x][u] \) of formal power series in \( u \) over the polynomial ring \( \mathbb{Q}[x] = \mathbb{Q}[x_1, x_2, \ldots] \), we have
\[
\sum_{n=0}^{\infty} Z(S_n, x)u^n = \prod_{d=1}^{\infty} e^{x_d u^d/d}.
\]

**Proof.** Given \( \lambda \vdash n \) (i.e., a partition of \( n \)), the number of permutations in \( S_n \) with cycle type \( \lambda \) is
\[
\frac{n!}{m_1(\lambda)! m_1(\lambda) \cdots m_n(\lambda)! m_n(\lambda)},
\]
so
\[
Z(S_n, x) = \sum_{\lambda \vdash n} \frac{x_1^{m_1(\lambda)} \cdots x_n^{m_n(\lambda)}}{m_1(\lambda)! m_1(\lambda) \cdots m_n(\lambda)! m_n(\lambda)}.
\]
Thus, we have
\[
\sum_{n=0}^{\infty} Z(S_n, x)u^n = \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \frac{x_1^{m_1(\lambda)} \cdots x_n^{m_n(\lambda)}}{m_1(\lambda)! m_1(\lambda) \cdots m_n(\lambda)! m_n(\lambda)}
\]
\[
= \sum_{\lambda \vdash n} \prod_{d=1}^{\infty} \frac{(x_d u^d/d)^{m_d(\lambda)}}{m_d(\lambda)!}
\]
\[
= \prod_{d=1}^{\infty} \sum_{m=0}^{\infty} \frac{(x_d u^d/d)^m}{m!}
\]
\[
= \prod_{d=1}^{\infty} e^{x_d u^d/d},
\]
as desired. \( \square \)
Lemma 5.2 (Lemma 1 (2) in [Sto1988]). In the ring $\mathbb{Q}[x][u]$ of formal power series in $u$ over the polynomial ring $\mathbb{Q}[x] = \mathbb{Q}[x_P, \lambda]_{P \in \mathbb{A}^1, \lambda \in P \setminus \{\emptyset\}}$, we have
\[
\sum_{n=0}^{\infty} Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x) u^n = \prod_{P \in \mathbb{A}^1} \sum_{\lambda \in P} \frac{\lambda^{|\text{deg}(P)|}}{|\text{Aut}_{\mathbb{F}_q}[t](H_{P, \lambda})|},
\]
where $H_{P, \lambda} := H_{(P), \lambda}$, following the notation defined in the beginning of Section 4.

Remark 5.3. Lemma 5.2 says that on the left-hand side, the coefficient of $u^n$ is given by
\[
\sum_{|\lambda| = |\text{deg}(P_1)| + \cdots + |\lambda| = |\text{deg}(P_n)| = n} \frac{x_{P_1, \lambda(1)} \cdots x_{P_n, \lambda(n)}}{|\text{Aut}_{\mathbb{F}_q}[t](H_{P_1, \lambda(1)})| \cdots |\text{Aut}_{\mathbb{F}_q}[t](H_{P_n, \lambda(n)})|},
\]
where the sum is over distinct $P_1, \ldots, P_n \in \mathbb{A}^1$ and not necessarily distinct $\lambda(1), \ldots, \lambda(n)$ with the stipulated condition. This is a finite sum, so it makes sense to evaluate any complex numbers in the variables $x_{P, \lambda}$ for $\lambda \neq \emptyset$ of the infinite product in the statement of Lemma 5.2 and get an identity in $\mathbb{C}[u]$. (Recall that $x_{P, \emptyset} = 1$.)

5.2. Connection between two cycle indices. We define $Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x)$ to be the polynomial given by specializing $x_{P, \lambda} = x_{\lambda}$ in the cycle index $Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x)$, which also makes sense for the empty partition because $x_{P, \emptyset} = 1 = x^0_{\text{deg}(P)}$. More explicitly, we have
\[
Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x) = \frac{1}{|\text{GL}_n(\mathbb{F}_q)|} \sum_{A \in \text{Mat}_n(\mathbb{F}_q)} \prod_{P \in \mathbb{A}^1} x_1^{|\mu_P(A)|} = \frac{1}{|\text{GL}_n(\mathbb{F}_q)|} \sum_{A \in \text{Mat}_n(\mathbb{F}_q)} x_1^{|A|} \cdots x_n^{|A|},
\]
where $m_d(A)$ is the number of degree $d$ monic irreducible polynomials, counting with multiplicity, in $\mathbb{F}_q[t]$ dividing the characteristic polynomial $f_A(t)$ of $A$. Note that only $x_1, \ldots, x_n$ will occur in $Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x)$ because the degree of $f_A(t)$ is $n$ for $A \in \text{Mat}_n(\mathbb{F}_q)$, so $f_A(t)$ is not divisible by any irreducible polynomial of degree $> n$. In the concluding remark of [Sto1988], Stong essentially observed that there is a close relationship between $Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x)$ and $Z(S_n, x)$ when $q$ is large. We rigorously formulate what he might have meant.

Lemma 5.4 (cf. “Conclusion” in [Sto1988]). We have
\[
\lim_{q \to \infty} Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x) = Z(S_n, x),
\]
meaning that the coefficients of $Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x)$ converge to the coefficients of $Z(S_n, x)$ as $q \to \infty$.

Proof. Let $x_d \in \mathbb{C}$ such that $|x_d| \leq 1$ for any $d \in \mathbb{Z}_{\geq 1}$. Applying Lemma 5.2 and then Lemma 4.2, we have
\[
\sum_{n=0}^{\infty} \frac{Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x) u^n}{n!} = \prod_{P \in \mathbb{A}^1} \frac{\lambda^{|\text{deg}(P)|}}{|\text{Aut}_{\mathbb{F}_q}[t](H_{P, \lambda})|} = \prod_{P \in \mathbb{A}^1} \prod_{i=1}^{\infty} \prod_{d=1}^{\infty} (1 - x_{\text{deg}(P)}(q^{-i} u)^{\text{deg}(P)})^{-1} = \prod_{d=1}^{\infty} \prod_{i=1}^{\infty} (1 - x_d(q^{-i} u)^{d})^{-M(q, d)},
\]
for \( |u| < 1 \), where \( M(q, d) \) is the number of monic irreducible polynomials in \( \mathbb{F}_q[t] \) with degree \( d \). Since \( |x_d| \leq 1 \) for all \( d \geq 1 \) so that we have

\[
|Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x)|^{1/n} \leq \left( \prod_{i=1}^n \frac{1}{1 - q^{-i}} \right)^{1/n} \leq \frac{1}{1 - q^{-1}},
\]

the radius of convergence of the power series \( \sum_{n=0}^{\infty} Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x)u^n \) in \( u \) is at least \( 1 - q^{-1} > 0 \). Since our statement only involves finitely many \( x_d \), we may set all but finitely many of them to be 0, say \( x_d = 0 \) for all \( d > m \) for fixed \( m \in \mathbb{Z}_{\geq 1} \). This lets us have

\[
\sum_{n=0}^{\infty} Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x)u^n = \prod_{d=1}^{m} \prod_{i=1}^{\infty} (1 - x_d(q^{-1}u)^d)^{-M(q,d)}.
\]

In what follows, we will use the fact that

\[
\lim_{q \to \infty} \frac{M(q, d)}{q^d/d} = 1,
\]

for any \( d \geq 1 \), which can be found as Theorem 2.2 in [Ros2002]. (Note that for \( M(q, 1) = q \), so we do not need to take the limit to see this for \( d = 1 \).) Now, for \( |u| < 1 - q^{-1} \), we have

\[
\lim_{q \to \infty} \sum_{n=0}^{\infty} Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x)u^n = \prod_{d=1}^{m} \lim_{q \to \infty} \prod_{i=1}^{\infty} (1 - x_d(q^{-1}u)^d)^{-M(q,d)}
\]

\[
= \prod_{d=1}^{m} \lim_{q \to \infty} \prod_{i=1}^{\infty} (1 - x_d(q^{-1}u)^d)^{-q^d/d}^{M(q,d)/q^d}
\]

\[
= \prod_{d=1}^{m} \prod_{i=1}^{\infty} \lim_{q \to \infty} (1 - x_d(q^{-1}u)^d)^{-q^d/d}
\]

\[
= \prod_{d=1}^{\infty} \left( \lim_{q \to \infty} (1 - x_du^d)^{-q^d/(x_du^d)} \right)^{(x_du^d)/d}
\]

\[
= \prod_{d=1}^{\infty} e^{-x_du^d/d}.
\]

Applying Lemma [5.1] to the last expression, we see that

\[
\lim_{q \to \infty} f_q(u) = f(u),
\]

where

\[
f_q(u) := \sum_{n=0}^{\infty} Z([\text{Mat}_n/\text{GL}_n](\mathbb{F}_q), x)u^n
\]

and

\[
f(u) := \sum_{n=0}^{\infty} Z(S_n, x)u^n
\]
are holomorphic functions in $u$ defined for $|u| < 1 - q^{-1}$. Take any $0 < \epsilon < 1 - q^{-1}$ and write $C_{\epsilon} := \{ u \in \mathbb{C} : |u| = \epsilon \}$. By the Cauchy integral formula, we have
\[
Z([\text{Mat}_n / GL_n](\mathbb{F}_q), x) = \frac{1}{2\pi i} \int_{z \in C_{\epsilon}} f_q(z) \frac{dz}{z^{n+1}}
\]
and
\[
Z(S_n, x) = \frac{1}{2\pi i} \int_{z \in C_{\epsilon}} f(z) \frac{dz}{z^{n+1}}.
\]
Thus, by the Dominated Convergence Theorem, we have
\[
\lim_{q \to \infty} Z([\text{Mat}_n / GL_n](\mathbb{F}_q), x) = \lim_{q \to \infty} \frac{1}{2\pi i} \int_{z \in C_{\epsilon}} f_q(z) \frac{dz}{z^{n+1}} = \frac{1}{2\pi i} \int_{z \in C_{\epsilon}} f(z) \frac{dz}{z^{n+1}} = Z(S_n, x).
\]
Since $x_1, \ldots, x_m$ are arbitrary with the restriction $|x_d| \leq 1$, this is enough to prove the desired statement about the convergence of the coefficients in $x_1, \ldots, x_m$ by the Cauchy integral formula for several variables (e.g., Theorem 2.1.3 of [Fie1982]) with the Dominate Convergence Theorem.

5.3. Proof of Proposition 1.1. Again, Lemma 5.4 says that the coefficients of the polynomial
\[
Z([\text{Mat}_n / GL_n](\mathbb{F}_q), x) = \frac{1}{|GL_n(\mathbb{F}_q)|} \sum_{A \in \text{Mat}_n(\mathbb{F}_q)} x_1^{m_1(A)} \cdots x_n^{m_n(A)} \in \mathbb{Q}[x_1, \ldots, x_n]
\]
converge to the coefficients of the polynomial
\[
Z(S_n, x) = \frac{1}{|S_n|} \sum_{\sigma \in S_n} x_1^{m_1(\sigma)} \cdots x_n^{m_n(\sigma)} \in \mathbb{Q}[x_1, \ldots, x_n]
\]
as $q \to \infty$, where the convergences are happening in $\mathbb{R}$ or $\mathbb{C}$, although the limits still lie in $\mathbb{Q}$. This implies that, setting $x_{d,0} := 1$ and letting $x_{d,m}$ be formal for any $d, m \in \mathbb{Z}_{\geq 1}$, the coefficients of
\[
\hat{Z}([\text{Mat}_n / GL_n](\mathbb{F}_q), x) := \frac{1}{|GL_n(\mathbb{F}_q)|} \sum_{A \in \text{Mat}_n(\mathbb{F}_q)} x_1^{m_1(A)} \cdots x_n^{m_n(A)} \in \mathbb{Q}
\]
converge to the coefficients of
\[
\hat{Z}(S_n, x) := \frac{1}{|S_n|} \sum_{\sigma \in S_n} x_1^{m_1(\sigma)} \cdots x_n^{m_n(\sigma)} \in \mathbb{Q}.
\]
To refer back to this fact, we record this as follows.

Lemma 5.5. As $q \to \infty$, the coefficients of the polynomial $\hat{Z}([\text{Mat}_n / GL_n](\mathbb{F}_q), x)$ converge to those of $\hat{Z}(S_n, x)$. In particular, for any evaluation of the variables $x_{d,m}$ in $\mathbb{C}$, we have
\[
\lim_{q \to \infty} \hat{Z}([\text{Mat}_n / GL_n](\mathbb{F}_q), x) = \hat{Z}(S_n, x).
\]

We are now ready to prove Proposition 1.1.
Proof of Proposition 1.1. Recall the statements of Proposition 1.1 since we will use the notations in them. We will also use the notations given in this section. If we evaluate

- \( x_{d_1,k_1} = \cdots = x_{d_r,k_r} = 1 \),
- \( x_{d_j,m} = 0 \) for all \( m \neq 0, k_j \) for \( 1 \leq j \leq r \), and
- \( x_{d,m} = 1 \) for all the other cases from above,

then we have

\[
\hat{Z}([\text{Mat}_n/\text{GL}_n](F_q), x) = \left\{ \frac{A \in \text{Mat}_n(F_q) : f_A(t) \text{ has } k_j \text{ irreducible factors of degree } d_j \text{ for } 1 \leq j \leq r}{|\text{GL}_n(F_q)|} \right\},
\]

where we count the factors with multiplicity, and

\[
\hat{Z}(S_n, x) = \left\{ \frac{\sigma \in S_n : \sigma \text{ has } k_j \text{ cycles of length } d_j \text{ for } 1 \leq j \leq r}{|S_n|} \right\}.
\]

Thus, applying Lemma 5.5 and noting that

\[
\lim_{q \to \infty} \frac{|\text{Mat}_n(F_q)|}{|\text{GL}_n(F_q)|} = \lim_{q \to \infty} (1 - q^{-1})^{-1} \cdots (1 - q^{-n})^{-1} = 1,
\]

we have

\[
\lim_{q \to \infty} \text{Prob}_{A \in \text{Mat}_n(F_q)} \left( A \in \text{Mat}_n(F_q) : f_A(t) \text{ has } k_j \text{ irreducible factors of degree } d_j \text{ for } 1 \leq j \leq r \right) = \text{Prob}_{\sigma \in S_n} \left( \sigma \in S_n : \sigma \text{ has } k_j \text{ cycles of length } d_j \text{ for } 1 \leq j \leq r \right),
\]

as desired. \( \square \)

6. Combinatorial proof of Lemma 3.2

In this section, we give a proof of Lemma 3.2 a result due to Shepp and Lloyd in [SL1966], originally proven by computing the characteristic functions of the distributions. Unlike their proof, we will directly compute the desired probability with a more combinatorial method. Most of our argument will be encoded in the following lemma.

Lemma 6.1. Take

- \( x_{d_j,m} = 0 \) with \( 1 \leq j \leq s \) and \( m \neq 0 \),
- \( x_{d_j,m} = 0 \) with \( s + 1 \leq j \leq r \) and \( m \neq 0, k_j \), and
- \( x_{d,m} = 1 \) for any other ones not on the above list.

Then

\[
\sum_{n=0}^{\infty} \hat{Z}(S_n, x) u^n = \left( \frac{e^{-u^{d_1}/d_1} \cdots e^{-u^{d_r}/d_r}}{1 - u} \right) \cdot \left( \prod_{j=s+1}^{r} \left( 1 + \frac{(u^{d_j}/d_j)^{k_j}}{k_j!} \right) \right) \in \mathbb{C}[u].
\]

Remark 6.2. Before proving Lemma 6.1, we first see how it implies Lemma 3.2. We will make use of the following useful observation: for any \( f(u) = c_0 + c_1 u + c_2 u^2 + \cdots \in \mathbb{C}[u] \), if \( f(1) = c_0 + c_1 + c_2 + \cdots \) exists, then the limit of the coefficient sequence of the power series

\[
\frac{f(u)}{1 - u} = b_0 + b_1 u + b_2 u^2 + \cdots
\]
is given by
\[
\lim_{n \to \infty} b_n = f(1),
\]
because \((1 - u)^{-1} = 1 + u + u^2 + \cdots\) so that \(b_n = c_0 + c_1 + \cdots + c_n\), which is the coefficient of \(u^n\) for the power series \(f(u)(1-u)^{-1}\).

**Proof of Lemma 6.1** Using Lemma 6.1 with \(s = r\), we have
\[
\sum_{n=0}^{\infty} \Prob_{\sigma \in S_n} \left( m_{d_j}(\sigma) = 0 \text{ for } 1 \leq j \leq r \right) u^n = \frac{e^{-u^{d_1}/d_1} \cdots e^{-u^{d_r}/d_r}}{1 - u}.
\]
Using Remark 6.2, this implies that
\[
\lim_{n \to \infty} \Prob_{\sigma \in S_n} \left( m_{d_j}(\sigma) = 0 \text{ for } 1 \leq j \leq r \right) = e^{-1/d_1} \cdots e^{-1/d_r},
\]
as claimed. For the general case, we work with the induction on the number of nonzero integers among \(k_1, \ldots, k_r\). The base case is when such number is 0, which is exactly what we have proved above. Suppose that the result is true when such number is 0, 1, \ldots, \(s - 1\), where \(s - 1 \geq 0\). Then we assume that there are \(s\) nonzero elements among \(k_1, \ldots, k_r\), so permuting them if necessary, say
\[
k_1 = \cdots = k_s = 0,
\]
while
\[
k_{s+1}, \ldots, k_r \neq 0.
\]
Applying Lemma 6.1 we get
\[
\sum_{n=0}^{\infty} \Prob_{\sigma \in S_n} \left( m_{d_j}(\sigma) = 0 \text{ for } 1 \leq j \leq s, \text{ and } d_{s+1}(\sigma) \in \{0, k_{s+1}\}, \ldots, d_r(\sigma) \in \{0, k_r\} \right) u^n = \frac{e^{-u^{d_1}/d_1} \cdots e^{-u^{d_r}/d_r}}{1 - u} \prod_{j=s+1}^{r} \left( 1 + \frac{(u^{d_j}/d_j)^{k_j}}{k_j!} \right),
\]
so
\[
\sum_{\epsilon_{s+1} \in \{0, k_{s+1}\}} \cdots \sum_{\epsilon_r \in \{0, k_r\}} \lim_{n \to \infty} \Prob_{\sigma \in S_n} \left( m_{d_1}(\sigma) = \cdots = m_{d_s}(\sigma) = 0, m_{d_{s+1}}(\sigma) = \epsilon_{s+1}, \ldots, m_{d_r}(\sigma) = \epsilon_r \right) = e^{-1/d_1} \cdots e^{-1/d_r} \prod_{j=s+1}^{r} \left( 1 + \frac{(1/d_j)^{k_j}}{k_j!} \right)
\]
Applying the induction hypothesis after expanding the right-hand side of the above identity implies that
\[
\Prob_{\sigma \in S_n} \left( m_{d_1}(\sigma) = \cdots = m_{d_s}(\sigma) = 0, m_{d_{s+1}}(\sigma) = k_{s+1}, \ldots, m_{d_r}(\sigma) = k_r \right) = e^{-1/d_1} \cdots e^{-1/d_r} \frac{(1/d_{s+1})^{k_{s+1}}}{k_{s+1}!} \cdots \frac{(1/d_r)^{k_r}}{k_r!},
\]
which finishes the proof. \(\square\)

Finally, we provide a proof of Lemma 6.1.
Proof of Lemma 6.1. For now, let us not evaluate any variables in $x = (x_{d,m})$. Recall that

$$\hat{Z}(S_n, x) = \frac{1}{|S_n|} \sum_{\sigma \in S_n} x_{1,m_1(\sigma)} \cdots x_{n,m_n(\sigma)},$$

where $x_{d,0} = 1$ and $x_{d,m}$ are defined to be formal variables for $d, m \geq 1$. We observe that

$$\hat{Z}(S_n, x) = \sum_{\lambda \vdash n} \frac{x_{1,m_1(\lambda)} \cdots x_{n,m_n(\lambda)}}{m_1(\lambda)! m_2(\lambda) \cdots m_n(\lambda)! n^m(\lambda)},$$

so

$$\sum_{n=0}^{\infty} \hat{Z}(S_n, x) u^n = \prod_{d=1}^{\infty} \sum_{m=0}^{\infty} \frac{x_{d,m} u^m}{m!} = \prod_{d=1}^{\infty} \sum_{m=0}^{\infty} \frac{x_{d,m} (u^d/d)^m}{m!}.$$

Note that taking all $x_{d,m} = 1$ in the identity, we get

$$\frac{1}{1-u} = 1 + u + u^2 + \cdots = \prod_{d=1}^{\infty} \sum_{m=0}^{\infty} \frac{(u^d/d)^m}{m!}.$$

Hence, with the given evaluation in the variables $x = (x_{d,m})$, we get

$$\sum_{n=0}^{\infty} \hat{Z}(S_n, x) u^n = \left( \prod_{d=1}^{\infty} \sum_{m=0}^{\infty} \frac{(u^d/d)^m}{m!} \right) \cdot \left( \prod_{j=1}^{r} \left( \sum_{m=0}^{\infty} \frac{(u^j/d_j)^m}{m!} \right)^{-1} \right) \cdot \left( \prod_{j=s+1}^{r} \left( 1 + \frac{(u^j/d_j)^{k_j}}{k_j!} \right) \right)$$

$$= \left( \frac{e^{-u^d/d} \cdots e^{-u^r/d}}{1-u} \right) \cdot \left( \prod_{j=s+1}^{r} \left( 1 + \frac{(u^j/d_j)^{k_j}}{k_j!} \right) \right).$$

Since the identities make sense as those of holomorphic functions in $u \in \mathbb{C}$ with $|u| < 1$, this finishes the proof. \qed
