A generalization of simple Harnack curves

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Abstract

Simple Harnack curves, introduced in [Mik00], are smooth real algebraic curves in maximal position in toric surfaces. In the present paper, we suggest a natural generalization of simple Harnack curves by relaxing the smoothness assumption. After mentioning some of their properties, we address the question of their construction. We define tropical Harnack curves and show that their approximation using Mikhalkin’s machinery produces many new example of simple Harnack curves. We determine the topological classification of simple Harnack curves with a hyperbolic node in the fashion of [Mik00], and show that the space of such curves can be parametrized by the space of their tropical avatars, in the spirit of [KO06].

Introduction

Borrowing the words of [KOS06], simple Harnack curves are real algebraic curves sitting in toric surfaces “in the best possible way”. The original topological definition of [Mik00] can be rephrased in term of maximality with respect to a finite collection of Harnack-Smith inequalities, see [Mik01]. This maximality manifests in several interesting and different ways, as shown in [MR01], [PR11] or [MO07]. In particular, the reformulation of [MR01] has both advantages of being short and explicit: for a simple Harnack curve $C$ in

\[\text{...}\]

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a toric surface $T_\Delta$, the amoeba map $A : C \to \mathbb{R}^2$ is at most 2-to-1. The latter property characterizes simple Harnack curves if one allows them to have singularities. Indeed, it can be shown that any real oval of a simple Harnack curve can be contracted to a solitary double point (see [KO06]), and that the 2-to-1 property of the amoeba map $A$ is still satisfied for these singular simple Harnack curves (see [MR01]).

In the present paper, we introduce a generalization of the notion of simple Harnack curve, unifying the two classes of curves mentioned above. This generalization consists in relaxing the smoothness assumption in the characterization given in [PR11]: for a curve $C$ in a toric surface $T_\Delta$, its logarithmic Gauss map $\gamma : C \dashrightarrow \mathbb{C}P^1$ is a rational map defined only on the smooth locus of $C$. Its pullback to the normalization $\tilde{C}$ of $C$ extends to an algebraic map $\tilde{\gamma} : \tilde{C} \to \mathbb{C}P^1$.

**Definition.** An irreducible real algebraic curve $C \subset T_\Delta$ is a (generalized) simple Harnack curve if and only if its logarithmic Gauss map $\tilde{\gamma}$ is totally real, that is

$$\tilde{\gamma}^{-1}(\mathbb{R}P^1) = \mathbb{R}\tilde{C}.$$ 

We then show that the characterization of [MO07] in terms of covering of the argument torus extends as well, implying that the area of the coamoeba of a simple Harnack curve is determined by its Euler characteristic.

In a second time, we address the question of construction of such curves. The existence of simple Harnack curves as introduced by Mikhalkin was proven using Viro’s patchworking method. In the present case, the construction of singular curves requires an enhanced version of patchworking, see [Shu12]. Alternatively, we choose here to invoke Mikhalkin’s approximation theorem for phase-tropical curves. We introduce the notion of tropical Harnack curves and show the following.

**Theorem.** The approximation of a tropical Harnack curve of degree $\Delta$ is a simple Harnack curve in $T_\Delta$.

As one can expect, the type of the singularities of the approximating curve is predicted by the singularities of the tropical Harnack curve. In particular, the latter theorem implies the existence of simple Harnack curves with a single hyperbolic node. We then classify all possible topological type of simple
Harnack curves with a single hyperbolic node obtained by tropical approximation. By topological type of a curve $C \subset T_\Delta$, we mean the topological triad

$$\left( \mathbb{R}T_\Delta, \mathbb{R}C, \bigcup_s \mathbb{R}D_s \right)$$

where the $D_s$’s are the toric divisors at infinity of the toric surface $T_\Delta$.

**Proposition.** For a fixed degree $\Delta$, the deformation classes of tropical Harnack curves with a single hyperbolic node are indexed by the smooth corners of $\Delta$. The topological type of any algebraic approximation within a deformation class indexed by a corner $\nu$ is constant, and denoted $\text{Top}(\Delta, \nu)$.

Here, smooth corners are corners of $\Delta$ corresponding to smooth points in the associated toric surface $T_\Delta$.

In the last part, we eventually undertake the topological classification of all simple Harnack curves with a single hyperbolic node, similarly to [Mik00]. We obtain a complete classification that shows that the algebraic picture is totally described by the tropical one.

**Theorem.** For any simple Harnack curve $C$ in $T_\Delta$, there exists a smooth corner $\nu$ of $\Delta$ such that the topological type of $C$ is given by $\text{Top}(\Delta, \nu)$.

This theorem is a manifestation of a deeper connection between simple Harnack curves and their tropical avatars. This connection has first been observed in [KO06]. Recall that the spine of an algebraic curve in $(\mathbb{C}^*)^2$ is a “canonical” tropical curve sitting inside its amoeba, see [PR04]. Here, we prove the following.

**Theorem.** The consideration of the spine induces a local diffeomorphism between the spaces of algebraic and tropical Harnack curves with a single hyperbolic node.

Up to now, simple Harnack curves are the only classes of curves that are in diffeomorphic correspondence with tropical curves, via the spine map.
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1 Prerequisites

1.1 Logarithmic geometry of planar curves

In this text, \( \mathbb{C}^o \subset (\mathbb{C}^*)^2 \) will denote an algebraic curve in the complex 2-torus. Such curve can be defined as the zero set of a Laurent polynomial

\[ f \in \mathbb{C}[z^{\pm 1}, w^{\pm 1}]. \]

The coordinates of \( (\mathbb{C}^*)^2 \) induce a canonical isomorphism \( z^\alpha w^\beta \mapsto (\alpha, \beta) \) between its space of characters and \( \mathbb{Z}^2 \). The Newton polygon \( \text{New}(f) \) is defined as the convex hull in \( \mathbb{R}^2 = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R} \) of the monomials \( z^\alpha w^\beta \) appearing in \( f \). Two Laurent polynomials \( g \) and \( f \) define the same zero set in \( (\mathbb{C}^*)^2 \) if and only if there exists \( a \in \mathbb{C}^* \) and \( (\alpha, \beta) \in \mathbb{Z}^2 \) such that

\[ f(z, w) = az^\alpha w^\beta g(z, w). \]
Then, New$(f)$ is the translation of New$(g)$ by $(\alpha, \beta)$. As a convention, one will always consider polynomials $f \in \mathbb{C}[z, w]$ such that New$(f)$ touches both the $z$- and $w$- axes. According to this convention, any curve $C^o \subset (\mathbb{C}^*)^2$ is defined by a polynomial $f$, unique up to a multiplicative constant $a \in \mathbb{C}^*$. The Newton polygon New$(f)$ is uniquely determined by $C^o$ and will be denoted $\Delta$.

The Newton polygon $\Delta$ of $C^o$ induces a toric compactification $(\mathbb{C}^*)^2 \subset T_{\Delta}$, whenever the interior of $\Delta$ is 2-dimensional. It can be constructed as the closure of the image of the following map

$$(\mathbb{C}^*)^2 \rightarrow \mathbb{CP}^m,$$

$$(z, w) \mapsto [z^{\alpha_0} w^{\beta_0} : ... : z^{\alpha_m} w^{\beta_m}]$$

where $\Delta \cap \mathbb{Z}^2 := \{ (\alpha_0, \beta_0), ..., (\alpha_m, \beta_m) \}$. The action of $(\mathbb{C}^*)^2$ onto itself extends to $T_{\Delta}$. $(\mathbb{C}^*)^2$ is an open dense orbit in $T_{\Delta}$. In addition, there is a $\mathbb{C}^*$-orbit for each side of $\Delta$, and an orbit reduced to a single point for each vertex of $\Delta$. Denote

$$S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}.$$ 

We will also describe $S^1$ as $\mathbb{R}/2\pi\mathbb{Z}$ without any difference. As a subgroup of $(\mathbb{C}^*)^2$, $(S^1)^2$ acts on $T_{\Delta}$. The moment map $\mu : T_{\Delta} \rightarrow \Delta$ is the quotient map of $T_{\Delta}$ by the latter action. For any side $s$ of $\Delta$, the toric divisor at infinity associated to $s$ is defined by $D_s := \mu^{-1}(s)$. It is isomorphic to $\mathbb{CP}^1$ and compactify one of the $\mathbb{C}^*$-orbit mentioned above. For any vertex $v$ of $\Delta$, $\mu^{-1}(v)$ is one of the orbits reduced to a single point. We will refer to it as the vertices of $T_{\Delta}$.

We will denote by $C \subset T_{\Delta}$ the closure of $C^o$. The curve $C$ intersects every divisor $D_s$ at $|s \cap \mathbb{Z}^2| - 1$ many points, counted with multiplicities. A curve $C \subset T_{\Delta}$ constructed as the closure of a curve $C^o \subset (\mathbb{C}^*)^2$ of Newton polygon $\Delta$ never contains any vertex of $T_{\Delta}$. We will always restrict to this case while considering curves $C \subset T_{\Delta}$. We define the points at infinity of $C$ by $C_\infty := C \setminus C^o$. We will also denote

$$b := |\partial \Delta \cap \mathbb{Z}^2| \quad \text{and} \quad g := |\text{Int} \Delta \cap \mathbb{Z}^2|.$$ 

The integer $g$ is the arithmetic genus of $C$ and $b$ is the intersection multiplicity of $C$ with the union of all the divisors at infinity, see for instance [Kho78].
and [Kus76]. In particular, if $\mathcal{C}$ intersects each of them transversally, then $|\mathcal{C}_\infty| = b$.

The moment map $\mu$ extends the amoeba map $A : (\mathbb{C}^*)^2 \rightarrow \mathbb{R}^2$

\[(z, w) \mapsto (\log |z|, \log |w|)\]

after a barycentric change of coordinates $\mathbb{R}^2 \rightarrow \text{Int} \, \Delta$.

**Definition 1.1.** Let $\mathcal{C}^o \subset (\mathbb{C}^*)^2$ be an algebraic curve. The amoeba of $\mathcal{C}^o$ is the subset $A(\mathcal{C}^o) \subset \mathbb{R}^2$. With a slight abuse, we will as well refer to the latter subset as the amoeba of $\mathcal{C}$ and denote it $A(\mathcal{C})$.

**Proposition 1.2** (see [FPT00]). For an algebraic curve $\mathcal{C}^o \subset (\mathbb{C}^*)^2$, its amoeba $A(\mathcal{C}^o)$ is a closed subset of $\mathbb{R}^2$. Moreover, every connected component of $\mathbb{R}^2 \setminus A(\mathcal{C}^o)$ is convex.

Denote by $\text{arg}(z) \in S^1$ the argument of the complex number $z \in \mathbb{C}^*$, the argument torus $T := (S^1)^2$ and

$$\text{Arg} : (\mathbb{C}^*)^2 \rightarrow T$$

\[(z, w) \mapsto (\text{arg}(z), \text{arg}(w))\]

**Definition 1.3.** Let $\mathcal{C}^o \subset (\mathbb{C}^*)^2$ be an algebraic curve. The coamoeba $\text{Arg}(\mathcal{C}^o)$ of $\mathcal{C}^o$ is the subset $\text{Arg}(\mathcal{C}^o) \subset \mathbb{R}^2$. With a slight abuse, we will as well refer to the latter subset as the coamoeba of $\mathcal{C}$ and denote it $\text{Arg}(\mathcal{C})$.

Amoebas and coamoebas are related by the fact that they are respectively real and imaginary part of algebraic curves in logarithmic coordinates. Interplays between them can be often described in term of the logarithmic Gauss map. For a smooth curve $\mathcal{C} \subset T_\Delta$, the logarithmic Gauss $\gamma : \mathcal{C} \rightarrow \mathbb{CP}^1$ is given on $\mathcal{C}^o$ by

$$\gamma(z, w) = [z \cdot \partial_z f(z, w) : w \cdot \partial_w f(z, w)]$$

where $f$ is a polynomial equation for $\mathcal{C}^o$. Geometrically, this map is locally the composition of the coordinate wise complex logarithm with the classical Gauss map which associates to every point of a smooth hypersurface its tangent hyperplane. Even though the complex logarithm is multivalued, this
map is well defined. In the case of singular curves $C$, $\gamma$ might not be defined globally. It is always well defined on the smooth part of $C$. If $\pi : \tilde{C} \to C$ is the normalization of $C$, $\gamma$ defines then a rational map on $\tilde{C}$. By the removable singularity theorem, it extends to an algebraic map $\tilde{\gamma} : \tilde{C} \to \mathbb{CP}^1$.

**Definition 1.4.** Let $C \subset T_\Delta$ be a curve, possibly singular, and $\pi : \tilde{C} \to C$ its normalization. The logarithmic Gauss map of $\tilde{C}$ is the map $\tilde{\gamma} : \tilde{C} \to \mathbb{CP}^1$ defined above.

For a singular curve $C \subset T_\Delta$, we will always denote its normalization by $\pi : \tilde{C} \to C$. We will also denote $\tilde{C}^\circ := \pi^{-1}(C^\circ)$ and $\tilde{C}_\infty := \tilde{C} \setminus \tilde{C}^\circ$. Remark that whenever $C$ has no singularity outside $(\mathbb{C}^*)^2$, $C_\infty$ and $\tilde{C}_\infty$ are in bijection.

Let us give an alternative and useful description of $\gamma$ (and then $\tilde{\gamma}$). Looking at $z$ and $w$ as 2 meromorphic functions on $\tilde{C}$, consider the 2 meromorphic differentials $d\log(z)$ and $d\log(w)$ on $C$. The quotient of two such differentials defines a meromorphic function on $\tilde{C}$. One has the following

**Proposition 1.5.** The Logarithmic Gauss map $\tilde{\gamma} : \tilde{C} \to \mathbb{CP}^1$ is given by

$$p \mapsto \left[ -d\log(w(p)) : d\log(z(p)) \right].$$

Moreover, the degree of $\tilde{\gamma}$ is $-\chi(\tilde{C}^\circ)$.

**Proof** If $p$ is a local coordinate on $\tilde{C}^\circ$, then

$$\left[ -d\log(w(p)) : d\log(z(p)) \right] = \left[ -\frac{d}{dp} \log(w(p)) : \frac{d}{dp} \log(z(p)) \right].$$

As $\tilde{C}^\circ$ is immersed in $(\mathbb{C}^*)^2$, $(\frac{d}{dp} \log(z(p)), \frac{d}{dp} \log(w(p)))$ is a non zero tangent vector for the coordinatewise logarithm of $C^\circ$ at the point corresponding to $p$. Hence, the tangent plane at $p$ is given by the equation

$$-d\log(w(p)) \cdot u + d\log(z(p)) \cdot v = 0,$$

which proves the first part of the statement.

The degree of $\tilde{\gamma}$ can be computed as the number of zeroes of $d\log(w) \cdot u + d\log(z) \cdot v$ for a generic non zero vector $(u, v)$. By genericity, one can assume that $\tilde{C}_\infty$ is exactly the set of poles of this differential and all of them are simple. By Riemann-Roch, such differential has degree $g(\tilde{C}) - 2$. Hence, it has $g(\tilde{C}) + |\tilde{C}_\infty| - 2$ zeroes.
Lemma 1.6. Let $s$ be a side of $\Delta$ and $(a, b) \in \mathbb{Z}^2$ a primitive integer vector supporting $s$. For any curve $C \subset T_\Delta$, and any point $p \in \mathcal{D}_s \cap C$, one has
\[
\gamma(p) = [a : b].
\]

**Proof** Suppose first that neither $a$ nor $b$ is zero. By the implicit function theorem, for any local coordinate $t$ of $C$ centred at $p$, there exists 2 holomorphic functions $h_z(t)$ and $h_w(t)$ having a simple zero at the origin and a positive integer $m$ such that
\[
z(t) = (h_z(t))^{-bm} \quad \text{and} \quad w(t) = (h_w(t))^{am}.
\]
The number $m$ is exactly the intersection multiplicity $\mathcal{D}_s \cap C$ at $p$. By 1.5,
\[
\gamma(p) = \lim_{t \to 0} \left[ -d \log(w(t)) : d \log(z(t)) \right]
\]
\[
= \lim_{t \to 0} \left[ -am \cdot \frac{h'_w(t)}{h_w(t)} : -bm \cdot \frac{h'_z(t)}{h_z(t)} \right]
\]
\[
= [a : b].
\]
If $a$ (resp. $b$) is zero, $h_w$ (resp. $h_z$) is a non vanishing holomorphic function. The same computation leads to the result.

\[\square\]

The map $\mathcal{A} : \tilde{\mathcal{C}}^o \to \mathbb{R}^2$ is a map between smooth surfaces. Following [Mik00], denote by $\tilde{F}^o \subset \tilde{\mathcal{C}}^o$ the critical locus of $\mathcal{A}$, that is the set of points where $\mathcal{A}$ is not submersive. Denote by $\tilde{F}$ its closure in $\tilde{\mathcal{C}}$.

Lemma 1.7.
\[
\tilde{F} = \tilde{\gamma}^{-1}(\mathbb{R} \mathbb{P}^1).
\]
Moreover, $\tilde{F}$ is also the closure of the critical locus of the map $\text{Arg} : \tilde{\mathcal{C}}^o \to T$.

**Proof** Let $\tilde{p} \in \tilde{\mathcal{C}}^o$ and denote $p := (p_1, p_2) := \pi(\tilde{p})$. The point $\tilde{p}$ is in $\tilde{F}^o$ if $T_{\tilde{p}} \mathcal{C}^o$ contains a vector $v$ tangent to the torus $|z| = |p_1|$, $|w| = |p_2|$ (if $p$ is a
singular point of $C^\circ$, consider the tangent line in $T_p C^\circ$ corresponding to $\tilde{p}$.

Equivalently, $z$ has purely imaginary logarithmic coordinates. This holds if and only if $\tilde{\gamma}(\tilde{p}) \in \mathbb{RP}^1$.

Similarly, $\tilde{p}$ is a critical point for the map $\text{Arg}$ if $T_p C^\circ$ contains a vector $v$ tangent to $\text{arg}(z) = \text{arg}(p_1)$, $\text{arg}(w) = \text{arg}(p_2)$, i.e., $z$ has real logarithmic coordinates. Once again, it holds if and only if $\tilde{\gamma}(\tilde{p}) \in \mathbb{RP}^1$.

\[\square\]

**Remark.** Any point of $\tilde{C}$ mapped to the boundary of $A(C)$ belongs to $\tilde{F}$. By the above lemma, the real part $\mathbb{R}\tilde{C}$ of any curve $\tilde{C}$ defined over $\mathbb{R}$ is always a subset of $\tilde{F}$.

**Corollary 1.8.** One has the following

\begin{itemize}
  \item $\tilde{C}_\infty \subset \tilde{F}$.
  \item $\tilde{F} \subset \tilde{C}$ is smooth if and only if $\tilde{\gamma}$ has no branching point on $\mathbb{RP}^1$. In this case, $\tilde{F}$ is a disjoint union of smoothly embedded circle in $\tilde{C}$.
  \item If $\tilde{F}$ is smooth, $\tilde{O}$ is a connected component of $\tilde{F}$ and $A_{|\tilde{O}}$ is non constant on $\tilde{O}$, then the logarithmic Gauss map of the parametrized curve $A_{|\tilde{O}}$ is given by the restriction of $\tilde{\gamma}$. In particular, it is monotonic and $A_{|\tilde{O}}$ has no inflection point.
  \item Under the same assumptions, the latter statement holds for $\text{Arg}_{|\tilde{O}}$.
\end{itemize}

**Proof** The first point is a direct consequence of lemmas \[1.6\] and \[1.7\]. For the second point, if $\tilde{\gamma}$ has no branching point on $\mathbb{RP}^1$, $\tilde{\gamma}_{|\tilde{F}}$ is a local diffeomorphism, and $\tilde{F}$ is a topological covering of $\mathbb{RP}^1$. It implies that $\tilde{F}$ is a disjoint union of smoothly embedded circle in $\tilde{C}$. If $\tilde{\gamma}$ has a branching point $q \in \mathbb{RP}^1$, then there exists $p \in \tilde{\gamma}^{-1}(q)$ such that $\tilde{F}$ near $p$ is diffeomorphic to the preimage of $\mathbb{R} \subset \mathbb{C}$ by $z \mapsto z^n$ for some $n \geq 2$. Hence, $\tilde{F}$ is not smooth at $p$. The last two points fall from the geometric interpretation of the logarithmic Gauss map. First, notice that $A$ and $\text{Arg}$ are analytic on $\tilde{O}$. Hence, they are either constant or locally injective. In the latter case, they admit a logarithmic Gauss map. Now, consider the tangent bundle of $\mathbb{C}^2$ restricted to the coordinate wise complex logarithm $\text{Log}(\tilde{C}^\circ)$. Considering real and imaginary parts gives a splitting $\mathbb{R}^2 \oplus i\mathbb{R}^2$ of the latter bundle. We
have seen in the previous lemma that the tangent bundle of $Log(\hat{C}^\circ)$ splits in a direct sum of two line bundles while restricted to $Log(\hat{F}^\circ)$. One of them is contained in the $\mathbb{R}^2$ factor of the previous splitting, and the other one is contained in the $i\mathbb{R}^2$ factor. Denote them by $\mathcal{L}_{\text{Re}}$ and $\mathcal{L}_{\text{Im}}$ respectively. Note that the maps $\mathcal{A}$ and $\mathcal{Arg}$ are just linear projections on $\mathbb{R}^2$ and $i\mathbb{R}^2$ in these logarithmic coordinates. Therefore, $\mathcal{A}$ (resp. $\mathcal{Arg}$) maps $\mathcal{L}_{\text{Re}}$ (resp. $\mathcal{L}_{\text{Im}}$) to the tangent line bundle of $\mathcal{A}(\hat{O})$ (resp. $\mathcal{Arg}(\hat{O})$). It follows that the Gauss maps of $\mathcal{A}|_{\hat{O}}$ and $\mathcal{Arg}|_{\hat{O}}$ are both given by $\hat{\gamma}$. By assumption, $\hat{\gamma}$ has no critical point, that is $\mathcal{A}|_{\hat{O}}$ and $\mathcal{Arg}|_{\hat{O}}$ have no inflection point.

Now let us recall that to any holomorphic function $f$ on $(\mathbb{C}^*)^2$, one can associate its Ronkin function

$$N_f(x, y) := \frac{1}{(2i\pi)^2} \int_{A^{-1}(x, y)} \frac{\log |f(z, w)|}{zw} dz \wedge dw$$

defined on $\mathbb{R}^2$. The function $N_f$ allows to describe the geometry of the amoeba $A(\{f = 0\})$. It is a convex function that is affine linear on the connected components of the complement of $A(\{f = 0\})$, see [PR04]. The gradient $\text{grad} N_f$ is then a constant function on such component. One defines the order of such component to be the value $\text{grad} N_f$ on it.

In the case where $f$ is a polynomial, $\text{grad} N_f$ takes values inside of $\text{New}(f)$. Moreover, its image is dense there.

**Proposition 1.9** (see [FPT00]). Let $C \subset T_\Delta$ be an algebraic curve. The order map defines an injection from the set of connected components of $\mathbb{R}^2 \setminus A(C)$ to $\Delta \cap \mathbb{Z}^2$. Compact components are sent in the interior of $\Delta$ and non-compact components are sent on its boundary. Moreover, the order map is surjective on the vertices of $\Delta$.

Consideration of the Hessian of $N_f$ gives rise to a so-called Monge-Ampère measure supported on $A(C)$. Comparison of this measure with the standard Lebesgue measure gives the following result.

**Proposition 1.10** (see [PR04]). Let $C \subset T_\Delta$ be an algebraic curve. Then

$$\text{Area}(A(C)) \leq \pi^2 \text{Area}(\Delta)$$

where the area is computed with respect to the standard Lebesgue measure.
There are other “areas” one can compute about amoebas and coamoebas. Let $x_1, x_2$ be the coordinates of $\mathbb{R}^2$ and $y_1, y_2$ be the coordinates on $T$. Define first

$$\text{Area}_{A,s}(C) = \int_{\tilde{C}\setminus \tilde{F}} A^*(dx_1 \wedge dx_2)$$

and

$$\text{Area}_{\text{Arg},s}(C) = \int_{\tilde{C}\setminus \tilde{F}} \text{Arg}^*(dy_1 \wedge dy_2).$$

It consists in computing areas of amoebas and coamoebas with local signed multiplicities, taking into account the number of preimages and the local changes of orientation of the maps $A$ and $\text{Arg}$.

Define as well

$$\text{Area}_{A,m}(C) = \int_{\tilde{C}\setminus \tilde{F}} |A^*(dx_1 \wedge dx_2)|$$

and

$$\text{Area}_{\text{Arg},m}(C) = \int_{\tilde{C}\setminus \tilde{F}} |\text{Arg}^*(dy_1 \wedge dy_2)|.$$

Here, one compute areas with multiplicities just by counting the number of preimages over any point of the target spaces. The following observation is due to Mikhalkin:

**Lemma 1.11.** For any algebraic curve $C \subset T_\Delta$, the 2-forms $A^*(dx_1 \wedge dx_2)$ and $\text{Arg}^*(dy_1 \wedge dy_2)$ are equal on $\tilde{C} \setminus \tilde{F}$. It implies that

$$\text{Area}_{A,s}(C) = \text{Area}_{\text{Arg},s}(C) = 0$$

and

$$\text{Area}_{A,m}(C) = \text{Area}_{\text{Arg},m}(C).$$

**Proof** Consider locally the coordinate wise complex logarithm on $\tilde{C}$. Its image is a holomorphic curve in $\mathbb{C}^2$. It implies that the restriction of the complex 2-form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$ to $\text{Log}(\tilde{C})$ is zero. Write $z_j = x_j + iy_j$ for $j = 1, 2$, then $\text{Re}(dz_1 \wedge dz_2) = dx_1 \wedge dx_2 - dy_1 \wedge dy_2$ is also zero on $\text{Log}(\tilde{C})$, meaning that the 2-forms $dx_1 \wedge dx_2$ and $dy_1 \wedge dy_2$ are equal on $\text{Log}(\tilde{C})$. As we already said, the projection on the $x$-plane (resp. $y$-plane) is nothing but $A$ (resp. $\text{Arg}$). It implies the first part of the statement.

The equalities $\text{Area}_{A,s} = \text{Area}_{\text{Arg},s}$ and $\text{Area}_{A,m} = \text{Area}_{\text{Arg},m}$ are direct
consequences. In the first case, the moment map \( \mu \) extends \( A \) and has a compact source space. Then it has a well defined degree \( d \) which is zero as \( \mu \) is not surjective. This degree is precisely the number of preimages of \( A \) counted with signs. Hence \( \text{Area}_{A,s}(C) = 0. \)

\[ \square \]

Denote by \( A_f \) the set of connected components of the complement of \( A(\{ f = 0 \}) \). For an element \( \alpha \in A_f \) denote by \( N^\alpha_f \) the affine linear function on \( \mathbb{R}^2 \) extending \( (N_f)_\alpha \). Then, the spine \( S_f \) is defined as the corner locus of the piecewise affine linear and convex function

\[ S_f := \max_{\alpha \in A_f} N^\alpha_f. \]

\( S_f \) is a piecewise linear graph in the plane. Equipped with some natural collection of weights, the spine turns out to be a tropical curve, see next subsection.

**Theorem 1.12** (see [PR04]). Let \( C \subset T_\Delta \) be an algebraic curve defined by a polynomial \( f \). Then, \( A(C) \) deformation retracts on \( S_f \).

We end up this section with a short description of the maps \( A \) and \( Arg \) near the points of \( C_\infty \).

By convexity and finiteness of the area of \( A(C) \), one deduces that a neighbourhood of any point of \( C_\infty \) is mapped by \( A \) onto a thin tentacle going off to infinity along a certain asymptotic direction. If this point belongs to \( D_s \), lemma 1.6 implicitly states that this direction is orthogonal to the corresponding side \( s \) of \( \Delta \).

In the case of \( Arg \), define \( \hat{C}_{\text{Arg}} \) to be the real oriented blow-up of \( \hat{C} \) at every point of \( \hat{C}_\infty \). Denote by \( S_p \subset \hat{C}_{\text{Arg}} \) the fiber of the blow-up over \( p \in \hat{C}_\infty \).

**Lemma 1.13.** The map \( Arg \) extends to \( \hat{C}_{\text{Arg}} \). Moreover, if \( s \) is a side of \( \Delta \), \( (a,b) \) a primitive integer vector supporting \( s \), and \( p \) belongs to \( D_s \), then \( Arg : S_p \to T \) is an \( m \)-covering over a geodesic of slope \( (-b,a) \), where \( m \) is the intersection multiplicity of \( \hat{C}_\infty \cap D_s \) at \( p \).

**Proof** As in the proof of lemma 1.6, use a local coordinate \( t \) and consider the expressions

\[ z(t) = (h_z(t))^{-bm} \quad \text{and} \quad w(t) = (h_w(t))^{am}. \]
For $t = re^{i\theta}$, one has
\[ z(t) = r^{-bm} \left( z_0 e^{-ibm \theta} + o(1) \right) \] and \[ w(t) = r^{am} \left( w_0 e^{iam \theta} + o(1) \right). \]

It follows that for any $e^{i\theta} \in S^1$
\[ \lim_{r \to 0} \arg(h_z(re^{i\theta})) = \arg(z_0) - bm \cdot \theta \]
and
\[ \lim_{r \to 0} \arg(h_w(re^{i\theta})) = \arg(w_0) + am \cdot \theta. \]

This gives the result when $a$ and $b$ are non zero. Otherwise, replace $z(t)$ or $w(t)$ by a non vanishing function and repeat the same computation, see 1.6.

\[ \square \]

1.2 Simple Harnack curves

Simple Harnack curves as considered here have been introduced in [Mik00]. Recall that a smooth real algebraic curve $C$ of genus $g$ is said to be an M-curve if $\mathbb{R}C \subset C$ has the maximal number of connected components, that is $g + 1$. Now, for a subset $I$ of the set of sides of $\Delta$, denote
\[ C_{\infty}^I := C \cap \bigcap_{s \in I} D_s \] and \[ \mathbb{R}C_{\infty}^I := \mathbb{R}C \cap \bigcap_{s \in I} \mathbb{R}D_s. \]

Here is the original definition of [Mik00].

**Definition 1.14.** A smooth real algebraic curve $C \subset T_\Delta$ is a simple Harnack curves if

* $C$ is an M-curve,

* there exists a connected component $\alpha \subset \mathbb{R}C$ and pairwise disjoint connected arcs $\alpha_\sigma \subset \alpha$ for every side $\sigma$ of $\Delta$ such that $C_{\infty}^\sigma \subset \alpha_\sigma$ and such that the cyclical ordering of the $\alpha_\sigma$’s on $\alpha$ corresponds to the cyclical ordering of the $\sigma$’s on $\partial \Delta$.

In [Mik00], Mikhalkin showed that the embedding of simple Harnack curves in toric surface depends only on the toric surface itself. Namely, one has
Theorem 1.15. For any simple Harnack curve in $\mathcal{T}_\Delta$, the topological triad

$$
\left( \mathbb{R} \mathcal{T}_\Delta, \mathbb{R} \mathcal{C}, \bigcup_s \mathbb{R} \mathcal{D}_s \right)
$$

is unique and depends only on $\Delta$. Here, the union runs over all the sides $s$ of $\Delta$.

Such theorem can be proved after a careful study of the amoeba of such a curve, showing that the latter is covered in a 2-to-1 fashion away from its boundary. It leads to the main theorem of [MR01]. In the latter, the authors showed that simple Harnack curves can only degenerate to singular curves with solitary double points, after contractions of real ovals. These curves are referred to as singular simple Harnack curves. They obtained the following.

Theorem 1.16. A real algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$ is a (possibly singular) simple Harnack curves if and only if one of the following conditions is satisfied

- $\mathcal{A}: \mathcal{C} \to \mathbb{R}^2$ is at most 2-to-1.
- $\text{Area}(\mathcal{A}(\mathcal{C})) = \pi^2 \text{Area}(\Delta)$.

There are several others equivalent definitions for simple Harnack curves. The following is of particular interest for us, as it will be the starting point of the generalization suggested in this text.
Figure 2: The amoeba map on a simple Harnack quartic.

**Theorem 1.17** (see [Mik00] and [PR11]). A smooth real algebraic curve $C \subset T_\Delta$ is a simple Harnack curves if and only if its logarithmic Gauss map $\gamma : C \to \mathbb{CP}^1$ is totally real, that is

$$\gamma^{-1}(\mathbb{RP}^1) = \mathbb{RC}.$$  

The equivalent definition given in [MO07] will be generalized in theorem 1. The equivalent definition given in [Mik01] will also be discussed at the end of the section 3.1. The last point we want to mention is the surprising occurrence of simple Harnack curves in the context of planar bi-periodic dimers configurations as treated in [KOS06]. Going further, the authors of [KO06] gave an explicit description of the space of projective simple Harnack curves of fixed degree in terms of the area of the holes of their amoeba. This description can be rephrased in the following way.

**Theorem 1.18.** The map

$$\mathcal{S} : \{\text{Hanack curves of degree } d\} \to \{\text{tropical curves of degree } d\}$$

that associates its spine to a simple Harnack curve is a global diffeomorphism.

**Proof** This is a reformulation of theorem 6 in [KO06], using the fact that the map from the intercept coordinates of the spine to the area of the holes is a change of coordinates, see propositions 6 and 7 and section 4.5 in [KO06].
Conceptually, this description of simple Harnack curve is of higher importance for us. Indeed, it establishes a very strong connection between simple Harnack curves and tropical curves. This connection will be extended in theorem 4.

1.3 Phase-tropical curves

1.3.1 Tropical curves

Let us recall briefly some classical notions about tropical curves in the plane. All definitions, statements and their proofs can be found in [Mik05] and [IMS09]. Consider the set $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$ and the two tropical operations

\[ "x + y" = \max\{x, y\} \quad \text{and} \quad "xy" = x + y. \]

Tropical operations will always be distinguished from usual ones by quotation marks. Equipped with this two operations, $\mathbb{T}$ is the tropical semifield. A tropical Laurent polynomial in two variables $x$ and $y$ is a function

\[ f(x, y) = " \sum_{(\alpha, \beta) \in A} c(\alpha, \beta)x^{\alpha}y^{\beta}" \]

where $A \subset \mathbb{Z}^2$ is a finite set. Such a function is piecewise affine linear and convex. As for classical polynomials, the Newton polygon $\text{New}(f)$ of $f$ is the convex hull of $A$ in $\mathbb{R}^2 = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R}$. The tropical zero set $V(f)$ of a tropical Laurent polynomial $f$ is defined as the subset of $\mathbb{R}^2$ where $f$ is not smooth. Equivalently,

\[ V(f) = \{ (x, y) \in \mathbb{R}^2 \mid \exists (\alpha, \beta) \neq (a, b) \in A \text{ s.t. } "c(\alpha, \beta)x^{\alpha}y^{\beta}" = "c(a, b)x^{a}y^{b}" \}. \]

As a first observation, a tropical zero set is a graph embedded in $\mathbb{R}^2$ with straight edges and leaves (unbounded edges) with rational slopes. If $g$ is another tropical Laurent polynomial given by

\[ g(x, y) := "c(\alpha, \beta)x^{\alpha}y^{\beta} \cdot f(x, y)", \]

then

\[ V(f) = V(g), \]

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but the converse fails to be true. Without loss of generality, we can and we do restrict once again to Newton polygons $\Delta$ contained in the positive quadrant and touching the two coordinate axes.

For a tropical polynomial $f$ of Newton polygon $\Delta$, consider its extended Newton polygon

$$\tilde{\Delta} := \text{ConvexHull} \left\{ (\alpha, \beta, t) \in \mathbb{R}^3 \mid (\alpha, \beta) \in A, \ t \geq c_{(\alpha, \beta)} \right\}.$$

The projecting on the first two coordinates maps down all closed bounded faces of $\tilde{\Delta}$ homeomorphically onto $\Delta$. It induces a subdivision $\text{Subdiv}_f$ of $\Delta$.

Using the Legendre transform, one can observe the following duality

**Proposition 1.19.** Let $f$ be a tropical polynomial in two variables. The subdivision of $\mathbb{R}^2$ by $V(f)$ is dual to the subdivision $\text{Subdiv}_f$ of $\Delta$ in the following sense

* 2-cells of $\mathbb{R}^2 \setminus V(f)$ are in bijection with vertices of $\text{Subdiv}_f$, and 2-cells of $\text{Subdiv}_f$ are in bijection with vertices of $V(f)$,

* leaves-edges of $V(f)$ are in bijection with edges of $\text{Subdiv}_f$, and their directions are orthogonal to each other,

* incidence relations are reversed.

Moreover, unbounded 2-cells of $\mathbb{R}^2 \setminus V(f)$ are dual to boundary points of $\Delta$ and leaves of $V(f)$ are dual to edges on the boundary of $\Delta$.

**Definition 1.20.** Let $f$ be a tropical polynomial in two variables. For any leaf-edge $\varepsilon$ of $V(f)$, the weight $w(\varepsilon)$ of $\varepsilon$ is the integer length of its dual edge $\varepsilon^\vee$ in $\text{Subdiv}_f$,

$$|\varepsilon^\vee \cap \mathbb{Z}^2| - 1.$$

**Definition 1.21.** A tropical curve $C \subset \mathbb{R}^2$ is a tropical zero set equipped with the weights defined in 1.20. If $\Delta$ is its Newton polygon, denote by $\text{Subdiv}_C$ the subdivision of $\Delta$ dual to $C$.

**Remark.** The convex piecewise affine linear function $S_f$ defining the spine of the curve $\{f = 0\} \subset (\mathbb{C}^*)^2$ is a tropical polynomial. Equipped with the corresponding collection of weights, the spine of an algebraic curve in $(\mathbb{C}^*)^2$ is a tropical curve.

It is a classical fact that tropical curves satisfy the so-called balancing condition given by the following
Proposition 1.22. Let $C \subset \mathbb{R}^2$ be a tropical curve. For any vertex $v$ of $C$, let $v_1, \ldots, v_k$ be the collection of outgoing primitive vectors supporting the $k$ leaves-edges adjacent to $v$ and $w_1, \ldots, w_k$ their respective weights, then

$$\sum_{j=1}^k w_j \cdot v_j = 0.$$ 

Among tropical curves, some of them will be of particular interest for us. They have both advantages of being very simple and generic.

Definition 1.23. A tropical curve $C \subset \mathbb{R}^2$ is simple if its dual subdivision $\text{Subdiv}_C$ contains solely triangles and parallelogram.

In other words, a simple tropical curve has only 3-valent vertices and 4-valent vertices given locally as the union of two segments. Simple tropical curves can be parametrized uniquely by an abstract 3-valent tropical curve, see [Mik05]. Rather than defining tropical morphisms on abstract tropical curve, we give the following definition instead.

Definition 1.24. The normalization $\tilde{C}$ of a simple tropical curve $C \subset \mathbb{R}^2$ is the 3-valent graph obtained as the proper transform of $C$ by the real blow-up of $\mathbb{R}^2$ at all its 4-valent vertices. Denote the blow-up by $\pi : \tilde{C} \to C$.

Definition 1.25. A simple tropical curve $C \subset \mathbb{R}^2$ is irreducible if its normalization $\tilde{C}$ is connected.

Definition 1.26. From now on, we define the vertices (resp. edges, resp. leaves) of a simple tropical curve $C \subset \mathbb{R}^2$ to be the image of the vertices (resp. edges, resp. leaves) of its normalization $\tilde{C}$ by the map $\pi$. They form the set $V(C)$ (resp. $E(C)$, resp. $L(C)$) and we denote $\text{LE}(C) := L(C) \cup E(C)$. As a convention, edges and leaves are always open. The points of $C$ having two preimages in $\tilde{C}$ are called the nodes of $C$ and form the set $N(C)$.

Definition 1.27. Let $C \subset \mathbb{R}^2$ be a simple tropical curve and $n \in N(C)$. The multiplicity of the node $n$ is the natural number

$$m(n) := 2 \cdot \text{Area}(n^\vee)$$

where $n^\vee$ is the 2-cell dual to $n$ in $\text{Subdiv}_C$. A node $n$ is hyperbolic if $m(n) = 2$. 

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We end up this subsection by recalling what is the stable intersection multiplicity of two tropical curves in the plane $C_1$ and $C_2$. There is a dense open subset $\mathcal{O} \subset \mathbb{R}^2$, such that for any $\mathbf{v} \in \mathcal{O}$, $C_1$ intersects $C_2 + \mathbf{v}$ transversally away from the vertices of both curves. Let $p$ be an intersection point, $v_1$ (resp. $v_2$) a primitive integer vector supporting the leaf-edge of $C_1$ (resp. $C_2 + \mathbf{v}$) containing $p$. Denote also $w_1$ (resp. $w_2$) the corresponding weight. Define the local intersection multiplicity

$$m_p(C_1, C_2 + \mathbf{v}) := w_1 w_2 \cdot |\det(v_1, v_2)|,$$

and the intersection multiplicity by

$$m(C_1, C_2 + \mathbf{v}) := \sum_{p \in C_1 \cap (C_2 + \mathbf{v})} m_p(C_1, C_2 + \mathbf{v}).$$

It is an easy consequence of the balancing condition that $m(C_1, C_2 + \mathbf{v})$ does not depend on $\mathbf{v}$.

**Definition 1.28.** The stable intersection multiplicity of $C_1$ and $C_2$ is given by $m(C_1, C_2 + \mathbf{v})$ for any $\mathbf{v} \in \mathcal{O}$.

The stable intersection considered as a divisor on tropical curves can be defined by using small perturbations. For more details, see [RST05].

### 1.3.2 Phase-tropical curves

Consider the diffeomorphism

$$H_t : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$$

$$(z, w) \mapsto \left(\left|z\right|^\frac{1}{\log(t)} z, \left|w\right|^\frac{1}{\log(t)} w\right).$$

This corresponds to a change of the holomorphic structure of $(\mathbb{C}^*)^2$, see section 6 in [Mik05]. Note that

$$A \circ H_t = \frac{1}{\log(t)} A.$$

Denote $L := \{(z, w) \in (\mathbb{C}^*)^2 \mid z + w + 1 = 0\}$. The sequence of topological surfaces $\{H_t(L)\}_{t>1}$ converges in Hausdorff distance to a topological surface $L$, when $t$ goes to $\infty$, see for instance [Mik05]. Topologically, $L$ is a sphere.
minus three points. It is obtained as the gluing of three holomorphic annuli to the coamoeba of $L$, as pictured in figure 3. It implies that $L$ comes naturally equipped with a piecewise conformal structure. Its amoeba $A(L)$ is the classical tropical line with vertex at the origin. Denote it by $\Lambda$.

Recall that a toric morphism $A : (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^2$ is a map of the form

$$(z, w) \mapsto (b_1 z^{a_{11}} w^{a_{12}}, b_2 z^{a_{21}} w^{a_{22}})$$

where $(b_1, b_2) \in (\mathbb{C}^*)^2$ and $(a_{ij}) \in M_{2 \times 2}(\mathbb{Z})$. It descends to an affine linear transformation on $\mathbb{R}^2$ (resp. on $T$) by composition with the projection $A$ (resp. $\text{Arg}$) that we still denote by $A$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{The fibration $A : L \rightarrow \Lambda$.}
\end{figure}

**Definition 1.29.** A general phase-tropical line $\Gamma \subset (\mathbb{C}^*)^2$ is the image of the phase-tropical line $L \subset (\mathbb{C}^*)^2$ by any toric morphism $A : (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^2$.

For a simple tropical curve $C \subset \mathbb{R}^2$ and any vertex $v \in V(C)$, denote by $T_v \subset C$ the tripod obtained as the union of $v$ and the three leaves-edges of $C$ adjacent to $v$.

**Definition 1.30.** A simple phase-tropical curve $V \subset (\mathbb{C}^*)^2$ is a topological surface such that :

- its amoeba $A(V)$ is a simple tropical curve $C \subset \mathbb{R}^2$,
- for any $v \in V(C)$, there exists a general phase-tropical line $\Gamma_v \subset (\mathbb{C}^*)^2$ such that $(A_{|v})^{-1}(T_v) = A^{-1}(T_v) \cap \Gamma_v$,
- for any $e \in E(C)$, $v_1$ and $v_2$ its two adjacent vertices in $C$, $\Gamma_{v_1}$ and $\Gamma_{v_2}$ coincide on $A^{-1}(e)$. 

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Remark. Phase-tropical curves have been introduced in [Mik05], with a slightly different terminology. They can be obtained as degeneration of sequence of algebraic curves in $(\mathbb{C}^*)^2$. Simple phase-tropical curve in the plane can also be parametrized by their abstract counterpart via phase-tropical immersions. Abstract phase-tropical curve already appeared in [Mik06] and are extensively studied in the unpublished work [Mik]. For more details, one refers to [Lan15].

Similarly to Riemann surfaces, simple phase-tropical curves can be described in terms of Fenchel-Nielsen coordinates, see [Bus10] for instance. This description is already underlined in [Mik06]. More details can be found in [Lan15].

Recall that the coamoeba $\text{Arg}(\mathcal{L})$ is the union of two open triangles delimited by three geodesics plus their three common vertices, see figure 4. The argument map is orientation preserving on the up-leftward triangle and orientation reversing on the other. As a convention, we fix a framing on each of the boundary geodesics of $\text{Arg}(\mathcal{L})$ according to the positive orientation on the latter triangle, see figure 4. By construction, these framings are kept globally unchanged by any of the 6 toric automorphisms of $\mathcal{L}$. With these framings, any boundary geodesic is canonically isomorphic as an abelian group to $S^1 \subset \mathbb{C}$.

Recall that for any general phase-tropical line $\Gamma := A(L)$, one has that $A(\Gamma) = A(\Lambda)$ and that the fiber in $\Gamma$ over $A(\Gamma)$ is $\text{Arg}(\Gamma) = A(\text{Arg}(\Lambda)) = A(\text{Arg}(\mathcal{L}))$. For any general phase-tropical line $\Gamma := A(L)$, one carries the framings of $\text{Arg}(\Lambda)$ to framings on $\text{Arg}(\Gamma) = A(\text{Arg}(\Lambda))$ by the map $A$. By construction, it does not depend on the choice of $A$.

![Figure 4: The coamoeba of $\mathcal{L}$, and the framings of its 3 boundary geodesics.](image-url)
Now, let us consider a simple phase-tropical curve $V$, with $C := \mathcal{A}(V)$. For any $e \in E(C)$ and $v_1, v_2 \in V(C)$ its two adjacent vertices, the holomorphic annulus $(\mathcal{A}_v)^{-1}(e)$ is glued to two boundary geodesics $\gamma_1$ and $\gamma_2$ of the two coamoebas $(\mathcal{A}_v)^{-1}(v_1)$ and $(\mathcal{A}_v)^{-1}(v_2)$. Let $\tau_1, \tau_2$ be the canonical group isomorphisms $\tau_i : S^1 \to \gamma_i$. As $(\mathcal{A}_v)^{-1}(e)$ maps to a geodesic in the argument torus, $\gamma_1$ and $\gamma_2$ are set theoretically identical. Then, the map $\tau_2^{-1} \circ \tau_1$ is an orientation reversing isometry of the form

$$\tau_2^{-1} \circ \tau_1 : S^1 \to S^1, \quad z \mapsto -e^{i\theta}z.$$ 

As this isometry is self inverse, the datum $e^{i\theta}$ does not depend on the order we have chosen for $\gamma_1$ and $\gamma_2$.

**Definition 1.31.** Let $V$ be a simple phase-tropical curve, with $C := \mathcal{A}(V)$. For any $e \in E(C)$, the element $e^{i\theta} \in S^1$ constructed above is called the twist parameter of the edge $e$.

**Definition 1.32.** A simple real-tropical curve $V \subset (\mathbb{C}^*)^2$ is a simple phase-tropical curve that is invariant under complex conjugation. We denote by $RV \subset (\mathbb{R}^*)^2$ its real point set and by $TV$ the image of $RV$ under the diffeomorphism

$$\mathcal{A}_s : (\mathbb{R}^*)^2 \to \mathbb{R}^2 \times (\mathbb{Z}_2)^2, \quad (x, y) \mapsto \left((\frac{x}{|x|} \ln |x|, \frac{y}{|y|} \ln |y|), (\frac{x}{|x|}, \frac{y}{|y|})\right).$$

**Proposition 1.33.** Let $V \subset (\mathbb{C}^*)^2$ be a simple phase-tropical curve with $C := \mathcal{A}(V)$. Then $V$ a is real if and only if one of the following equivalent conditions holds:

i) For every $v \in V(C)$, the general phase-tropical line $\Lambda_v$ above $T_v$ is invariant under complex conjugation.

ii) For every $v \in V(C)$, the closed coamoeba $(\mathcal{A}_v)^{-1}(v)$ is invariant under complex conjugation, that is the antipodal map on $T$.

iii) For every $v \in V(C)$, the 3 special points of the closed coamoeba $(\mathcal{A}_v)^{-1}(v)$ are in $\{(0,0), (\pi,0), (0,\pi), (\pi,\pi)\}$.
The proof is straightforward.

**Remark.** Note that by composing the map \( A_s \) with

\[
\text{Abs} : \mathbb{R}^2 \times (\mathbb{Z}_2)^2 \rightarrow \mathbb{R}^2 \\
((x, y), (\varepsilon, \delta)) \mapsto (\varepsilon x, \delta y),
\]

one recovers the map \( A \). Note moreover that the real locus of any holomorphic annulus \( (A_{|V})^{-1}(e) \) above \( e \in LE(C) \) has exactly two connected components, whenever it is defined over \( \mathbb{R} \).

As corollaries of the previous observations, we have

**Proposition 1.34.** Let \( V \subset (\mathbb{C}^*)^2 \) be a simple real tropical curve and denote by \( C := A(V) \) its amoeba. Then \( TV \) is a piecewise linear curve and \( \text{Abs} : TV \rightarrow C \) is 2-to-1.

**Proposition 1.35.** The twist parameters of a real-tropical curve \( V \subset (\mathbb{C}^*)^2 \) are always contained in \( \{-1, 1\} \subset S^1 \).

## 2 Definition and construction

### 2.1 Definition and first properties

From now on, the notion of simple Harnack curve will always refer to the following definition.

**Definition 2.1.** An irreducible real algebraic curve \( C \subset T_\Delta \) is a simple Harnack curve if it is irreducible and

\[ \tilde{F} = \mathbb{R}C. \]

By the lemma [1.7], one has the following reformulation.

**Definition 2.2.** An irreducible real algebraic curve \( C \subset T_\Delta \) is a simple Harnack curve if and only if its logarithmic Gauss map \( \tilde{\gamma} \) is totally real.
This generalizes the notion of simple Harnack curve originally given by Mikhalkin, via the theorem 1.17. The only difference here is that a simple Harnack curve is not required to be smooth. We will see later that it allows many other cases to appear.

**Remark.** The latter equivalent definition of simple Harnack curve implies that $\mathbb{R}\tilde{C} \subset \tilde{C}$ is of type 1, that is $\mathbb{R}\tilde{C} \setminus \tilde{C}$ has exactly two connected components.

In order to get the first general properties of such curves, let us reproduce a construction due to [MO07]: for a curve $C \subset T$ consider the map

$$Alga : C^0 \rightarrow T$$

$$(z, w) \mapsto (2 \arg(z), 2 \arg(w)).$$

Define $\tilde{C}_{Bl}$ to be the real blow-up of $\tilde{C}$ at every point of $\tilde{C}_\infty$. For any point $p \in \tilde{C}_\infty$ denote by $\mathbb{P}_p$ the fiber of $\tilde{C}_{Bl} \rightarrow \tilde{C}$ over $p$. By construction, one has the factorisation $\tilde{C}_{Arg} \rightarrow \tilde{C}_{Bl} \rightarrow \tilde{C}$ inducing a double covering $S_p \rightarrow \mathbb{P}_p$ for any $p \in \tilde{C}_\infty$.

**Lemma 2.3.** The map $Alga$ naturally extends to

$$Alga : \tilde{C}_{Bl} \rightarrow T.$$ 

**Proof** By lemma 1.13, $Alga$ extend to $\tilde{C}_{Arg}$. For any $p$ and any point in $\mathbb{P}_p$, its two preimages in $S_p$ are mapped to the same value by $Alga$. Hence, $Alga : \tilde{C}_{Arg} \rightarrow T$ factorizes through $\tilde{C}_{Bl}$.

Define the subset $\tilde{C}_0 \subset \tilde{C}_{Bl}$ to be $Alga^{-1}(\{0_T\})$. Note that

$$Alga^{-1}(\{0_T\}) = (\mathbb{R}^*)^2.$$ 

Thus, it implies that $\tilde{C}_0$ is the union of $\mathbb{R}\tilde{C}_{Bl}$ plus some isolated points, whenever $C$ is defined over $\mathbb{R}$. In this case, the isolated points of $\tilde{C}_0$ come either as the trace of solitary double points of $\mathbb{R}C$ or from non transverse intersection with a toric divisor at infinity. Indeed, by lemma [1.13], if $C$ intersects a divisor $D_s$ with multiplicity $m$ at a point $p \in C_\infty$, there are exactly $m$ points in the exceptional divisor $\mathbb{P}_p$ belonging to $\tilde{C}_0$, and exactly one of these belongs to $\mathbb{R}\tilde{C}_{Bl}$. 

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Lemma 2.4. An irreducible real algebraic curve $C \subset T_\Delta$ is a simple Harnack curve if and only if

$$\text{Alga} : \tilde{C}_{\text{Bl}} \setminus \tilde{C}_0 \to T \setminus \{0_T\}$$

is an unbranched covering.

Proof By lemma 1.7 and the remark above, the latter statement is an equivalent reformulation of definition 2.1.

\[\square\]

Define at last $\hat{T}$ to be the real blow-up of $T$ at $0_T$, and $\hat{C}$ to be the real blow-up of $\tilde{C}_{\text{Bl}}$ at $\tilde{C}_0$. As blowing-up at a smooth submanifold of codimension 1 doesn’t change the surface, blowing-up is effective only at isolated points of $\tilde{C}_0$.

Theorem 1. An irreducible real algebraic curve $C \subset T_\Delta$ is a simple Harnack curve if and only if the map $\text{Alga} : \tilde{C}_{\text{Bl}} \setminus \tilde{C}_0 \to T \setminus \{0_T\}$ extends to a covering

$$\text{Alga} : \hat{C} \to \hat{T}.$$ 

Proof Let $\mathcal{C}$ be a simple Harnack curve. By definition, the map $\text{Alga} : \tilde{C}_{\text{Bl}} \to T$ is regular at the isolated points of $\tilde{C}_0$. Hence, the map $\text{Alga} : \tilde{C}_{\text{Bl}} \setminus \tilde{C}_0 \to T \setminus \{0_T\}$ extends to $\hat{T}$ in a tautological way at these isolated points. At a point of $\mathbb{R}\tilde{C}_{\text{Bl}}$, blowing-up consists of considering the normal direction to $\mathbb{R}\tilde{C}_{\text{Bl}}$ in the tangent space. This actually specifies the line bundle $L_{\text{Im}}$ introduced in the proof of lemma 1.8. The projectivized tangent map of $\text{Alga}$ realizes a covering of the exceptional divisor of $\hat{T}$ by $L_{\text{Im}}$, giving the extension $\text{Alga} : \hat{C} \to \hat{T}$.

Conversely, if $C$ is such that $\text{Alga} : \hat{C} \to \hat{T}$, lemma 2.4 implies that $C$ is a simple Harnack curve.

\[\square\]

Corollary 2.5. If $C \subset T_\Delta$ is a simple Harnack curve, then

$$\text{Area}_{\text{Arg},m}(C) = \pi^2(-\chi(\hat{C})).$$

If moreover $C$ has no real solitary double points and if it intersecting transversally every toric divisors at infinity, then

$$\text{Area}_{\text{Arg},m}(C) = \pi^2(-\chi(\hat{C}^o)).$$
Proof It is clear by definition that $\text{Area}_{\text{Arg},m}(C) = 4 \text{Area}_{\text{Alga},m}(C)$, where $\text{Area}_{\text{Alga},m}(C)$ is defined similarly to $\text{Area}_{\text{Arg},m}(C)$. By theorem 1,

$$\text{Area}_{\text{Alga},m}(C) = \text{Area}(T) \cdot \deg \text{Alga} = 4\pi^2 \cdot \left( -\chi(\hat{C}) \right).$$

The first part of the statement follows. For the second one, the assumptions are such that $\hat{C}_0$ has no solitary double point. It implies that $\hat{C} = \hat{C}_\text{BI}$, but $\chi(\hat{C}_\text{BI}) = \chi(\hat{C}^\circ)$. □

### 2.2 Tropical Harnack curves

**Definition 2.6.** Let $C \subset \mathbb{R}^2$ be a simple tropical curve with normalization $\hat{C}$. For every oriented loop $\hat{\lambda} \subset \hat{C}$, and $\lambda$ the corresponding oriented loop in $C$, denote by $\Gamma_\lambda \subset E(C) \cap \lambda$ the subset of oriented edges that forms, together with its previous and following edges in $\lambda$, a non convex piecewise linear curve in $\mathbb{R}^2$.

**Definition 2.7.** An irreducible simple tropical curve $C \subset \mathbb{R}^2$ with normalization $\hat{C}$ is a tropical Harnack curve if for every oriented loop $\hat{\lambda} \subset \hat{C}$, one has

$$\sum_{e \in \Gamma_\lambda} w(e) \cdot v_e = 0 \mod 2$$

where $v_e$ is the primitive integer vector supporting the oriented edge $\bar{e}$, and $w(e)$ is the weight of $e$.

Note how the latter definition is similar to the condition of “twist-admissibility” of definition 3.3 in [BIMS15].

The above definition is practical in the sense that it gives an effective method to determine whether a simple tropical curve is Harnack or not. We now give another equivalent characterization for tropical Harnack curve of a more conceptual nature. We will see in the next section how it justifies the terminology.

**Definition 2.8.** An edge (resp. a leaf) of $T V$ is defined to be one of the two connected components mapping onto an edge (resp. a leaf) of $C$. They form a set denoted $E(TV)$ (resp. $L(TV)$). Their union is denoted $LE(TV)$.

Let $V \subset (\mathbb{C}^*)^2$ be a real-tropical curve. An inflection pattern of $TV$ is a collection of three consecutive elements of $LE(TV)$ that is not convex.
Figure 5: Tropical cubic, quadric and quintic. Both the cubic and the quintic are Harnack whereas the quartic does not satisfy (I) in definition 2.7.

**Proposition 2.9.** An irreducible simple tropical curve $C \subset \mathbb{R}^2$ is a tropical Harnack curve if and only if there exists a simple real-tropical curve $V \subset (\mathbb{C}^*)^2$ such that

1. $\mathcal{A}(V) = C$,
2. $TV$ has no inflection pattern.

In such case, $V$ is unique up to the four sign changes of the coordinates. We refer to such $V$ as a phase-tropical Harnack curve.

Let us now recall how one can recover the curve $TV$ of a phase-tropical Harnack curve from its underlying tropical Harnack curve $C := \mathcal{A}(V)$. The fiber of $\mathcal{A}_{|V}$ over $e \in LE(C)$ is a holomorphic annulus given by an equation of the form

$$z^{-b}w^a = c$$

where $(a, b) \in \mathbb{Z}^2$ is a primitive integer vector supporting $e$ and $c \in \mathbb{R}^*$. This annulus realizes the class $(a, b) \in H_1((\mathbb{C}^*)^2)$ and maps to a geodesic of slope $(a, b)$ in $T$ under $\text{Arg}$. This geodesic contains exactly two points of the real subtorus $\{(0, 0), (\pi, 0), (0, \pi), (\pi, \pi)\} \subset T$ that we identify with $(\mathbb{Z}_2)^2$. One of these points is obtained from the other by adding $(a, b) \mod 2$ in $(\mathbb{Z}_2)^2$. At the level of $TV$, it means that the two preimages of $e$ are obtained one from each other by the sign change $(a, b) \mod 2$.

Now according to proposition 2.9, each connected component of $TV$ is convex. Hence, one can recover $TV$ from $C$ by drawing the ribbon graph of its normalization (meaning that we neglect the 4-valent vertices) and by unfolding it in $(\mathbb{R}^*)^2 \simeq \mathbb{R}^2 \times (\mathbb{Z}_2)^2$, the unique rule being the sign change.
relating two preimages of any element of $LE(C)$. See figure 6 for an example.

We end up this section with the proof of proposition 2.9.

**Definition 2.10.** The edge of a simple real-tropical curve is twisted (or has a twist) if its twist parameter is $-1$. It is not twisted (or has no twist) otherwise.

Note that we made a slight abuse of language by speaking about edge of a simple phase-tropical curve rather than edge of its underlying simple tropical curve.

**Lemma 2.11.** Let $V \subset (\mathbb{C}^*)^2$ be simple real-tropical curve. Then the inflection patterns of $TV$ are in 2-to-1 correspondence with the twisted edges of $V$.

**Proof** Let $C := \mathcal{A}(V)$. The middle element of an inflection pattern has to
be an edge. Consider \( e \in E(C) \) and \( v_1, v_2 \in V(C) \) its two adjacent vertices. Let \( \gamma_1 \) be the geodesic corresponding to \( e \) in \( (A_{|v})^{-1}(v_1) \) and \( \gamma_2 \) the one corresponding to \( e \) in \( (A_{|v})^{-1}(v_2) \). Denote by \( e_1, e_2 \in LE(C) \) the 2 others elements adjacent to \( v_1 \) and \( \varepsilon_1, \varepsilon_2 \in LE(C) \) the two others elements adjacent to \( v_2 \), such that \( e_1, e_2, e \) and \( \varepsilon_1, \varepsilon_2, e \) are cyclically ordered. The origin of \( \gamma_1 \) connects an edge-leaf of \( TV \) above \( e_2 \) to an edge of \( TV \) above \( e \) and the origin of \( \gamma_2 \) connects an edge-leaf of \( TV \) above \( \varepsilon_2 \) to an edge of \( TV \) above \( e \). Similarly, the \(-1\) point of \( \gamma_1 \) connects an edge-leaf of \( TV \) above \( e_1 \) to an edge of \( TV \) above \( e \) and the \(-1\) point of \( \gamma_2 \) connects an edge-leaf of \( TV \) above \( \varepsilon_1 \) to an edge of \( TV \) above \( e \).

Recall that by definition of the twist parameter, the origin of \( \gamma_1 \) is connected to the origin of \( \gamma_2 \) if and only if there is a twist on \( e \). It follows that: either there is a twist on \( e \) and one has two inflection patterns in \( TV \), one mapping down to \( e_2, e, \varepsilon_2 \) and one mapping down to \( e_1, e, \varepsilon_1 \); or there is no twist and one has two patterns in \( TV \), one mapping down to \( e_2, e, \varepsilon_1 \) and one mapping down to \( e_1, e, \varepsilon_2 \), and they are not inflected.

\[\square\]

![Twist vs. No Twist](image)

**Figure 7:** The proof of lemma 2.11. Twisted vs. non twisted edges.

We also refers to section 3 in [BIMS15], for a description of twisted edges in combinatorial patchworking.

**Proof of proposition 2.9** Suppose there is a simple phase-tropical curve \( V \subset (\mathbb{C}^*)^2 \) such that \( \mathcal{A}(V) = C \) and \( TV \) has no inflection pattern. By the previous lemma, it is equivalent to the fact that \( V \) has no twisted edges.
For any vertex $v \in \lambda$, there is a distinguished point among the three special points of the coamoeba $\left(A_{|v}\right)^{-1}(v)$, namely the intersection point of the two geodesics corresponding to the two edges in $\lambda$ adjacent to $v$. Let us look at the position of this distinguished point in the argument torus $T$ while going around $\lambda$. Going from a vertex $v$ to the next one via an edge $e$, the point is moved according to the following rule: if $e$ is not in $\Gamma_{\lambda}$, then this point is fixed; if $e$ is in $\Gamma_{\lambda}$, this point is moved by $\pi \cdot w(e) \cdot v_T$ in $T$. After a full cycle, the distinguished point has to come back to its initial place. This is clearly equivalent to the condition stated in definition 2.7 on the loop $\lambda$. Hence $C$ is a tropical Harnack curve. Conversely, if $C$ satisfies the condition of definition 2.7 for any cycle, pick an initial vertex $v_0$ on $C$. There is exactly one possible general phase-tropical line $\Gamma_{v_0}$ such that $\left(A_{|v}\right)^{-1}(T_{v_0}) = \Gamma_{v_0} \cap A^{-1}(T_{v_0})$, up to the four changes of signs of the coordinates. The twists determine the gluing of the general phase-tropical lines above adjacent vertices along the common edge. Hence, once $\Gamma_{v_0}$ is fixed, the adjacent general phase-tropical lines are also fixed. The first part of the proof shows that the condition of definition 2.7 is necessary and sufficient for this construction to close up along every cycle $\lambda$. The proposition is proved.

2.3 Construction by tropical approximation

After Viro introduced his patchworking techniques in the late 70’s, see [Vir80], it has been extensively used to construct real algebraic hypersurfaces with prescribed topology. In the case of curves, Mikhalkin’s approximation theorem even extends the possibilities given by Viro’s method. In particular, the point of view of parametrized objects suits better for constructing singular curves. Here is a particular case of Mikhalkin’s theorem.

**Theorem 2.12 (Mikhalkin).** Let $V \subset (\mathbb{C}^*)^2$ be a simple real-tropical curve of Newton polygon $\Delta$ such that its normalization $\tilde{V}$ has genus $g$ and $n$ punctures. Then, there exists a family of Riemann surfaces $\{S_t\}_{t \gg 1} \subset \mathcal{M}_{g,n}$ together with a family of immersions $\iota_t : S_t \to (\mathbb{C}^*)^2$ such that

* $\iota(S_t)$ is a real algebraic curve of Newton polygon $\Delta$,

* $\iota(S_t)$ converges in Hausdorff distance to $V$. 

$\square$
Proof See [Lan15]. □

Definition 2.13. Let $C$ be a tropical Harnack curve of Newton polygon $\Delta$. Define $\text{Top}(C)$ to be the topological triad

$$(\mathbb{R}T_\Delta, \mathbb{R}V, \bigcup_s \mathbb{R}D_s)$$

up to homeomorphism. Here, $s$ runs over all the sides of $\Delta$ and $V$ is one of the four simple phase-tropical curves of proposition 2.9 sitting above $C$.

Thanks to Mikhalkin’s approximation theorem, one can prove the following.

Theorem 2. Let $C$ be a tropical Harnack curve of Newton polygon $\Delta$, then there exists a simple Harnack curve $\mathcal{C} \subset \mathcal{T}_\Delta$ such that

$$(\mathbb{R}T_\Delta, \mathbb{R}\mathcal{C}, \bigcup_s \mathbb{R}D_s) = \text{Top}(C).$$

Before giving the proof, let us briefly illustrate why the latter theorem provides new instances of simple Harnack curves. In the previous literature, simple Harnack curves have been considered with only real solitary double points as possible singularities, see [MR01] and [KO06] for example. By approximating the simple tropical curves drawn in figure 5 one can construct simple Harnack curve with hyperbolic nodes, as shown in figure 6. In particular, such curves has not been considered before.

As another new instance, One can construct simple Harnack curves with complex conjugated double points. In the figure 8 we illustrate the construction of a curve of bi-degree $(4, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Such curve has arithmetic genus 3. The curve pictured here has one hyperbolic node on its real part and the two edges coloured in blue are responsible for two complex conjugated double points.

We end up this section with the proof of theorem 2

Proposition 2.14. Let $C \subset \mathbb{R}^2$ be a tropical Harnack curve, and $V$ be one of the four phase-tropical Harnack curves sitting above $C$. Then the total logarithmic curvature of $\mathbb{R}V$ is equal to $-\chi(\tilde{V})$. 31
The notion of total logarithmic curvature needs to be clarified. Before doing so, let us note that the latter proposition is the tropical counterpart of the characterization 2.2 of simple Harnack curves, up to the proposition 1.5. Anticipating the proof of theorem 2, one foresees that approximation of phase-tropical Harnack curve are simple Harnack curves.

Now, we come to the definition of the total logarithmic curvature of $R_V$. More details can be found in [BLR13]. Consider the compactification of $R_V$ in $T_\Delta$, and normalize it. The result is a disjoint union of topological circle. Up to the change of coordinates $A_s$, it induces a compactification-normalization of $T_V$. Choose $\varepsilon > 0$ sufficiently small such that the $\varepsilon$-neighbourhood of the vertices of $T_V$ are pairwise disjoint, and smooth the immersion of the topological circles such that

* $T_V$ gets deformed only inside the chosen $\varepsilon$-neighbourhoods,
* the deformation has no inflection inside the chosen $\varepsilon$-neighbourhoods.

Then, each of the smoothed immersions have a well defined logarithmic Gauss map. In particular, it has a well defined degree, up to the choice of an orientation. For each of these circles $O$, define its total curvature $\kappa_O$ to be the absolute value of the degree of its Gauss map. Each of these numbers is clearly independent of the deformation and the choice of an orientation.

**Definition 2.15.** The total curvature of $R_V$ is the defined by

$$\kappa := \sum \kappa_O,$$

where the sum runs over all the topological circles $O$ of the compactification-normalization of $R_V$ defined above.
Proof of proposition 2.14. On one hand, \(-\chi(\tilde{V})\) is equal to the number of vertices of \(\tilde{C}\). To see this, cut \(\tilde{C}\) at the middle of any of its edges. As \(\tilde{C}\) has only 3-valent vertices, it splits into a collection of tripods, one for each vertex of \(\tilde{C}\). The part of \(\tilde{V}\) sitting above any of these tripods is a topological pair-of-pants, having Euler characteristic -1. Additivity of Euler characteristic implies the above claim.

On the other hand, one can compute the total logarithmic curvature of \(R V\) by computing its local contributions. By the very definition, these contributions are concentrated at the vertices of \(TV\). It corresponds to \(1/\pi\) times the measure of the solid angle between the two normals at the vertices of \(TV\) (see figure 4 in [BLR13]), counted with signs depending on how \(TV\) changes inflection between to consecutive vertices. As \(TV\) has no inflection pattern, this local contributions should all be counted positively. Now, for each vertex in \(C\), there are three vertices in \(TV\). One easily sees that the local contributions at these three vertices add up to 1, see figure 5 in [BLR13]. Hence the total logarithmic curvature of \(R V\) is also equal to the number of vertices of \(C\), that is the number of vertices of \(\tilde{C}\) by convention. The result follows.

Proof of theorem 2. For \(t\) large enough, the immersed curve \(\iota(S_t)\) in theorem 2.12 realizes the topological type \(Top(C)\). Denoting by \(C\) such immersed curve, it remains to show that, necessarily, \(C\) is a simple Harnack curve. By the very definition of 2.15, one has that the degree of \(\tilde{\gamma}\) is equal to the total logarithmic curvature of \(R V\), for any phase-tropical Harnack curve \(V\) sitting above \(C\). This is equal to \(-\chi(\tilde{V})\) by proposition 2.14. But \(-\chi(\tilde{V}) = -\chi(\tilde{C})\), by construction of \(C\), see theorem 2.12. By proposition 1.5, this is exactly the degree of \(\tilde{\gamma}\). Hence, the logarithmic Gauss map \(\tilde{\gamma}\) is totally real. By proposition 2.2, it implies that \(C\) is a simple Harnack curve.

2.4 Tropical Harnack curves with a single hyperbolic node

The purpose of this section is to show that the topological types for tropical
Harnack curves with a single hyperbolic node are indexed by the pairs $(\Delta, \nu)$ where $\Delta$ is the degree of the curve and $\nu$ is a smooth vertex of $\Delta$. Recall that a vertex is said to be smooth if the toric surface $T_\Delta$ is smooth at the corresponding vertex.

**Proposition 2.16.** Let $C \subset \mathbb{R}^2$ be a tropical Harnack curve of Newton polygon $\Delta$ with a single hyperbolic node $n$. Then the parallelogram $n^\vee$ dual to $n$ in $\text{Subdiv}_C$ has exactly three of its vertices on the boundary of $\Delta$. These three vertices are distributed on two sides of $\Delta$ that are adjacent to a smooth vertex $\nu$ of $\Delta$.

**Proof** Suppose $n^\vee$ has at least two vertices $v_1$ and $v_2$ in the interior of $\Delta$. Consider the polygonal domain $P$ of $\Delta$ obtained by taking the union of $n^\vee$ together with all the minimal triangle of $\text{Subdiv}_C$ having $v_1$ or $v_2$ as a vertex. Consider the subset of $C$ dual to $P$. Then, its normalization in $\bar{C}$ a single loop $\bar{\lambda}$. There are two cases: either $v_1$ and $v_2$ are consecutive or opposite in $n^\vee$. In the first case, $\Gamma_{\lambda}$ is a singleton. In the second, $\Gamma_{\lambda}$ is exactly composed of the two leaves-edges forming the node $n$. In both case, the condition of definition 2.7 is not fulfilled. Hence, we get a contradiction.

If $n^\vee$ has its four vertices on the boundary $\Delta$, then $C$ is reducible. This is a contradiction.

Then, $n^\vee$ exactly three of its vertices on the boundary of $\Delta$. Either the middle vertex is a vertex of $\Delta$, and this is then a smooth one, or one of the two sides of $n^\vee$ specified by the three vertices is on a side of $\Delta$ and a minimal triangle is attached to the other side. The vertex of this triangle not contained in $n^\vee$ is the smooth vertex of $\Delta$ we are looking for. The statement is proven.

Two projectively equivalent tropical polynomial of degree $\Delta$ give the same tropical curves, but the converse is not true in general. It only holds on the subspace of polynomials for which every tropical monomial dominate the others for at least one point. This is easily seen to be a closed polyhedral domain in $\mathbb{RP}^{k-1}$, where $k$ is the number of integer point in $\Delta$. Tropical curves of degree $\Delta$ will always be seen as point in this polyhedral domain.

**Definition 2.17.** Let $C \subset \mathbb{R}^2$ be a tropical Harnack curve with a single hyperbolic node $n$. We say that the node of $C$ is next to $\nu$ if $\nu$ is the smooth
The vertex of the Newton polygon of $C$ given in the previous proposition.

Denote by $\mathcal{T}_H_{\Delta,\nu}$ the interior of the closure of the space of tropical Harnack curve with a single hyperbolic node next to $\nu$ inside the space of tropical curves of degree $\Delta$.

**Remark.** Curves of $\mathcal{T}_H_{\Delta,\nu}$ are tropical curves with maximal number of compact holes and a 4-valent vertex as prescribed by proposition 2.16. The other vertices can be of any valency (greater than 3).

**Proposition 2.18.** The topological type $\text{Top}(C)$ of any simple Harnack curve $C \in \mathcal{T}_H_{\Delta,\nu}$ is unique and will be denoted by $\text{Top}(\Delta,\nu)$.

**Proof of proposition 2.18** Up to toric transformation of $(\mathbb{C}^*)^2$, one can assume that $\nu = (0,0)$ and that its two adjacent sides are supported on the $x$- and the $y$-axes. Consider the subdivision $\text{Subdiv}_C$ dual to $\Delta$. By proposition 2.16, the unique parallelogram of $\text{Subdiv}_C$ can only be of one of the following type

The connected components of the space of tropical Harnack curves inside $\mathcal{T}_H_{\Delta,\nu}$ are characterized by the combinatorial type of $\text{Subdiv}_C$. For a fixed position of the unique parallelogram of $\text{Subdiv}_C$, one can pass from one combinatorial type to another by allowing extra quadrilaterals. Doing so, one see that the unfolding procedure giving $\mathcal{T}V$ from $C$ (see end of section 2.2) gives the same topological type.

In order to change the parallelogram of $\text{Subdiv}_C$ according to an arrow of the above picture, one need first to move to a particular combinatorial type. Indeed, $\text{Subdiv}_C$ has to contains the minimal triangle such that its union with the parallelogram is the trapezium obtained as the union of the two parallelograms at each sides of the concerned arrow, see the picture below.
Once again, we see that passing from one parallelogram to another by such deformation does not change $\text{Top}(C)$.

\[ \square \]

\section{Simple Harnack curves with a single hyperbolic node}

\subsection{Statements of the main theorems}

In this section we undertake the classification of the topological triads

\[ (\mathbb{R}^\mathcal{T}_\Delta, \mathbb{R}^\mathcal{C}, \bigcup_s \mathbb{R}^\mathcal{D}_s) \]

for simple Harnack curves $\mathcal{C}$ with a single hyperbolic node in any toric surface $\mathcal{T}_\Delta$. We obtain the following.

\textbf{Theorem 3.} \textit{Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be a simple Harnack curve with a single hyperbolic node. Assume moreover that $\mathcal{C}$ intersects transversally every toric divisor at infinity. Then, there is a smooth vertex $\nu$ of $\Delta$ such that}

\[ (\mathbb{R}^\mathcal{T}_\Delta, \mathbb{R}^\mathcal{C}, \bigcup_s \mathbb{R}^\mathcal{D}_s) = \text{Top}(\Delta, \nu). \]

In the latter theorem, one had to specify the intersection profil at infinity, as the topological classification depends on it. Transversality is a genericity assumption, and the general case can be deduced easily from the generic one.
Definition 3.1. Let $C \subset \mathcal{T}_\Delta$ be a simple Harnack curve with a single hyperbolic node. The node of $C$ is said to be next to $\nu$ if $\nu$ is the smooth vertex of its Newton polygon given in the previous theorem. Denote by $\mathcal{H}_{\Delta,\nu}$ the space of simple Harnack curve with Newton polygon $\Delta$ and with a single hyperbolic node next to $\nu$.

The space of simple Harnack curves $\mathcal{H}_{\Delta,\nu}$ can be seen as a subset of the projective space of polynomials supported inside $\Delta$. Hence, $\mathcal{H}_{\Delta,\nu} \subset \mathbb{RP}^{k-1}$ where $k$ is the number of integer points of $\Delta$.

The following extends theorem 1.18 underlined in [KO06].

Theorem 4. The spine of a curve in $\mathcal{H}_{\Delta,\nu}$ is a tropical curve in $\mathcal{T}_{\mathcal{H}_{\Delta,\nu}}$. The induced map

$$\mathcal{S} : \mathcal{H}_{\Delta,\nu} \to \mathcal{T}_{\mathcal{H}_{\Delta,\nu}},$$

is a diffeomorphism on each connected component of the source.

Corollary 3.2. The only possible singularities of a simple Harnack curve in $\mathcal{H}_{\Delta,\nu}$ are either real isolated double points or a unique cuspidal point.

Proof Let $C$ be a curve in $\mathcal{H}_{\Delta,\nu}$. Choose any path $\rho$ in $\mathcal{H}_{\Delta,\nu}$ ending at $C$ and corresponding to curves with fixed boundary points. The path $\mathcal{S} \circ \rho$ in $\mathcal{T}_{\mathcal{H}_{\Delta,\nu}}$ corresponds to a contraction of one or several compact holes of the tropical curves. As the latter holes correspond either to real ovals or to the singular loop of the curve upstairs, their contraction produces either real isolated double points or a cuspidal point on $C$.

In the present case, simple Harnack curves can still be characterized in a topological manner, similar to the original definition of [Mik00] and definition 10 in [Mik01]. For the sake of concision, we state the followings as claims. The proofs can be recovered from the arguments developed further below.

Claim 1. An irreducible real algebraic curve $C \subset \mathcal{T}_\Delta$ with an hyperbolic node as single singularity is a simple Harnack curves if and only if

* $\tilde{C}$ is an M-curve,
there exists a connected component $\alpha \subset \mathbb{R}C$ and connected arcs $\alpha_\sigma \subset \alpha$ for every side $\sigma$ of $\Delta$ such that $C^\infty_\sigma \subset \alpha_\sigma$ and such that the $\alpha_\sigma$’s are pairwise disjoint except for two arcs $\alpha_{\sigma_1}$ and $\alpha_{\sigma_2}$: $\alpha_{\sigma_1}$ contains exactly one point of $C^\infty_{\sigma_2}$ and $\alpha_{\sigma_2}$ contains exactly one point of $C^\infty_{\sigma_1}$. Moreover, the cyclical ordering of the $\alpha_\sigma$’s on $\alpha$ corresponds to the cyclical ordering of the $\sigma$’s on $\partial \Delta$.

Recall that for a topological space $X$ and $Y$ the fixed point set of a continuous involution on $X$, one has the inequality

$$b^*(Y; \mathbb{Z}_2) \leq b^*(X; \mathbb{Z}_2)$$

where $b^*(.; \mathbb{Z}_2) = \dim H^*(.; \mathbb{Z}_2)$ denotes the total $\mathbb{Z}_2$-Betti number of $X$.

**Claim 2.** An irreducible real algebraic curve $C \subset T_\Delta$ with an hyperbolic node as single singularity is a simple Harnack curves if and only if

$$\sum_I b^*(C \setminus \mathcal{C}_I^c, \mathcal{C}_I^c; \mathbb{Z}_2) = \sum_I b^*(C \setminus C^c_\infty, C^c_\infty; \mathbb{Z}_2) - 4,$$

where each sum runs over the subsets $I$ of the set of sides of $\Delta$ such that $I$ is either a singleton or a pair of adjacent sides.

### 3.2 Some technical conventions

In the rest of the paper, $C \subset T_\Delta$ will be a simple Harnack curve with a single hyperbolic node $p$. We denote by $\varphi$ the connected component of $\mathbb{R}C^c$ containing $p$, and by $\tilde{\varphi}$ its normalization in $\mathbb{R}\tilde{C}^c$. Up to toric transformation of $(\mathbb{C}^*)^2$, one can and do assume that $\Delta$ has an horizontal side supported on the $x$-axis. We will denote

$$b := |\partial \Delta \cap \mathbb{Z}^2| \quad \text{and} \quad g := |Int(\Delta) \cap \mathbb{Z}^2|$$

and refer to $m$ as the number of sides of $\Delta$.

We will always give $\partial \Delta$ the counter-clockwise orientation. It induces a cyclical order on the set of sides of $\Delta$. We formalize it by a bijection

$$\mathbb{Z}/m\mathbb{Z} \rightarrow \text{the set of sides of } \Delta$$

$$j \mapsto s_j$$
such that $s_1$ is the horizontal side of $\Delta$ supported on the $x$-axis. For each side $s_j$, $j \in \mathbb{Z}/m\mathbb{Z}$, there is a unique primitive integer vector $v_j$, $j \in \mathbb{Z}/m\mathbb{Z}$, supporting $s_j$ and coherent with the orientation of $\partial \Delta$.

By corollary 1.8 we can and we do orient each connected component $\vartheta$ of $\mathbb{R}\tilde{C}$ such that $A(\mathbb{R}\tilde{C})$ is a locally concave parametrized curve.

For any u-oval $\vartheta$, this orientation induces a cyclical order on the set (with possible repetitions) of the toric divisors $D_s$ as they are encountered by $\vartheta$. We formalize it by a map

$$\mathbb{Z}/m\vartheta \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow \text{the set of sides of } \Delta$$

$$j \mapsto \vartheta(j) \mapsto s_{\vartheta(j)}$$

where $m_{\vartheta}$ is the number of points of $\vartheta_{\infty} := \tilde{C}_{\infty} \cap \vartheta$. Denote also $\vartheta^\circ := \vartheta \setminus \vartheta_{\infty}$.

Note that the map on the left is not necessarily injective, and is defined up to translation. For our purpose, we do not need to specify this map any further.

For two vectors $u$ and $v$ in the plane, we denote by $\angle(u, v)$ the measure of the oriented angle from $u$ to $v$ with values in $[0; 2\pi]$.

**Definition 3.3.** The index of a u-oval $\vartheta$ of $\mathbb{R}\tilde{C}$ is defined by

$$\text{ind}(\vartheta) = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}/m\vartheta \mathbb{Z}} \angle(v_{\vartheta(j)}, v_{\vartheta(j+1)})$$

### 3.3 Proof of the main theorems

The proofs of theorems 3 and 4 go by the following results.

#### 3.3.1 Topological maximality

Contrary to Mikhalkin’s original definition, no topological constraint explicitly appears in definition 2.1, except that simple Harnack curves are always of type 1. In particular, it is a priori not guaranteed that simple Harnack curves are $M$-curves. In the present case, one has the following.

**Definition 3.4.** Let $C \subset T_\Delta$ be a simple Harnack curve. A connected component of $\mathbb{R}\tilde{C}$ is called a u-oval if it intersects $\tilde{C}_{\infty}$, and a b-oval otherwise.

**Proposition 3.5.** Let $C \subset T_\Delta$ be a simple Harnack curve with a single hyperbolic node. Then, the normalization $\mathbb{R}\tilde{C} \subset \tilde{C}$ is an $M$-curve. Moreover, $\mathbb{R}\tilde{C}$ is the union of $(g - 1)$ b-ovals and a single u-oval, where $(g - 1)$ is the genus of $\tilde{C}$. 39
The proof goes by an estimation of the contribution of each connected component of the real part to the logarithmic Gauss map, case by case. We postpone it to the appendix.

3.3.2 Lifted coamoebas

The latter proposition implies that the normalization of such simple Harnack curve admits the following decomposition

\[ \tilde{C} := \tilde{C}^- \cup \mathbb{R}\tilde{C} \cup \tilde{C}^+ \]

where \( \tilde{C}^- \) and \( \tilde{C}^+ \) are exchanged by the complex conjugation \( \sigma \) on \( \tilde{C} \). As \( Arg \circ \sigma = -id \circ Arg \), the map \( Arg \) is orientation preserving on one “half” of \( \tilde{C} \) and orientation reversing on the other. By convention, fix \( \tilde{C}^+ \) to be the one where \( Arg \) is orientation preserving.

Let us also introduce the following notation: if \( \nu \) is a smooth vertex of a Newton polygon \( \Delta \), denote by \( \Delta_\nu \subset \Delta \) the polygonal domain obtained by removing the parallelogram of area 1 in the corresponding corner of \( \Delta \).

**Proposition 3.6.** Let \( \mathcal{C} \subset T_\Delta \) be a simple Harnack curve with a single hyperbolic node next to \( \nu \). Then, the restriction to \( \tilde{C}^+ \) of the argument map \( Arg \) lifts to the universal covering \( \mathbb{R}^2 \) of \( T \). Moreover, its lift \( Arg_0 \) is a diffeomorphism and

\[ Arg_0(\tilde{C}^+) = \tau(\Delta_\nu) \]

where \( \tau \) is the composition of a rotation by \(-\pi/2\) and a homothety by \( \pi \).

The proof goes as follows: first one sees that the boundary is a piecewise linear curve obtained by concatenation of the “side” of \( \Delta \) and, regarding this constraint, one shows then that the only possibility for this curve to enclose a domain of area predicted by corollary 2.5 is the one described above. We postpone it to the appendix.

**Remark.** The idea of lifting coamoebas already appeared in [NP10]. Here, the latter proposition replaces the 2-to-1 property of the amoeba map used for the topological classification of smooth simple Harnack curves in [Mik00]. Despite it could have been used there, it cannot be applied in general (consider for instance the quintic of figure 5).
3.3.3 Spines

We are ready to prove theorems 3 and 4. The theorem 3 falls as a corollary of the following.

**Proposition 3.7.** The tropical curve $C := \mathcal{S}(C)$ belongs to $\mathbb{T}H_{\Delta,\nu}$. Moreover, $\mathbb{R}C$ deforms isotopically on $\mathbb{R}C$.

From proposition 3.6, one deduces first the intersection profile of the unique $u$-oval of $\mathbb{R}\tilde{C}$ with the toric divisors at infinity from the boundary of $\text{Arg}_0(\tilde{C}^+)$. One also deduces that $\mathcal{A}(C)$ has $g$ holes. Finally, one shows that $\mathcal{S}(C)$ has an hyperbolic node next to $\nu$, giving the latter proposition. The proof is given in the appendix.

In order to end the proof of theorem 4, one will reproduce some of the arguments given in section 2.2.4 and 4 of [KO06].

The space of curves in $\mathcal{H}_{\Delta,\nu}$ with fixed boundary points is a $(g-1)$-dimensional. Indeed, the coefficients on the boundary of the Newton polygon are determined by the boundary conditions, and the singular point imposes a 1-dimensional condition on the $g$ inner monomials. The tropical counterpart is the space of curves of genus $g$ and fixed boundary points inside $\mathbb{T}H_{\Delta,\nu}$. The latter space is also $(g-1)$-dimensional. The intercepts of each tropical monomial dominating inside a hole can be taken as coordinates, except for the hole next to the tropical node, the corresponding intercept being fixed by the boundary conditions.

**Proof of theorem**

First, note that the coordinates of the boundary points are in correspondence via the logarithm. One only needs to prove the result for fixed boundary points on both sides. For self-contentedness, let us repeat the arguments of [KO06]. take $\alpha_1, \ldots, \alpha_{g-1}$ to be the $(g-1)$ smooth compact ovals of $\mathbb{R}C$. For each of these ovals, pick a point $(x_j, y_j)$ in the complement of $\mathcal{A}(C)$ enclosed by $\mathcal{A}(\alpha_j)$, and define

$$\beta_j := \{(z, w) \in (\mathbb{C}^*)^2 \mid |z| = e^{x_j}, \ |w| \leq e^{y_j}\}.$$

The $\alpha_j$'s and $\beta_j$'s can be taken as $a$- and $b$-cycles for $\tilde{C}$.

Recall that $\tilde{C}$ is given by a polynomial $f$ of Newton polygon $\Delta$ that contains the origin and has a vertical and an horizontal side adjacent to it. The origin has been chosen to be the vertex next to which $C$ has an hyperbolic node as
shown in the above picture, see definition 3.1. Then, the tangent space to $\mathcal{C}$ with given points at infinity and node is given by the polynomials vanishing at the points at infinity and the node of $\mathcal{C}$, modulo the polynomial $f$ itself. Such polynomials have the form $zw \cdot q(z, w)$ where the Newton polygon of $q$ is obtained from $\Delta$ by removing all its boundary points. Hence, the Newton polygon of $q$ has $g$ integer points. As $q$ has to vanish at the node of $\mathcal{C}$, the space of such $q$’s is $(g - 1)$ dimensional.

On the other hand, one can check that the holomorphic differentials on $\tilde{\mathcal{C}}$ have the form

$$\omega = \frac{q(z, w)}{\frac{\partial}{\partial w} f(z, w)} \, dz.$$ 

The intercept of the tropical monomial giving the $j$-th hole of $\mathcal{C}$ can be computed by

$$N_f(x, y) - (ax + by)$$

where $(x, y)$ lies inside the $j$-th hole, and $(a, b)$ is the order of this hole. Repeating the computation of [KO06], one has

$$\frac{d}{dt} N_{f + tw} q(x, y)\big|_{t=0} = \frac{1}{(2\pi i)^2} \int_{|z|=e^x, |w|=e^y} \frac{q(z, w)}{f(z, w)} \, dz \, dw$$

$$= \frac{1}{2\pi i} \int_{|z|=e^x} \sum_{f(z, w_r) = 0, |w_r| \leq e^y} \frac{q(z, w_r)}{\frac{\partial}{\partial w} f(z, w)} \, dz = \frac{1}{2\pi i} \int_{\beta_j} \frac{q(z, w)}{\frac{\partial}{\partial w} f(z, w)} \, dz,$$

see [KO06] and references therein. One deduces that the Jacobian of the map that associates to a curve of $\mathcal{H}_{\Delta, \nu}$ the intercepts of its Ronkin function is precisely the period matrix of the curve, in particular it is invertible. It implies that the map $\mathcal{S}$ is a local diffeomorphism.

Now suppose that $\mathcal{S}$ is not surjective and consider $C \in \partial \mathcal{T}m \mathcal{S}$. Pick a sequence $\{\mathcal{C}_t\}_{t \in \mathbb{N}} \subset \mathcal{H}_{\Delta, \nu}$ converging to a curve $C \in \mathbb{RP}^{k-1}$ and such that $\mathcal{S}(\mathcal{C}_t) = C$ (such sequence definitely exists). As $C$ belongs to $\mathcal{T}\mathcal{H}_{\Delta, \nu}$, and since the property of being simple Harnack is closed, $C$ is a simple Harnack curve, with an hyperbolic node and genus $g - 1$, hence a point of $\mathcal{H}_{\Delta, \nu}$. As $\mathcal{S}$ is a local diffeomorphism, it implies that $C$ belongs to $\text{int}(\text{Im} \mathcal{S})$, which leads to a contradiction. Hence, $\mathcal{S}$ is a global diffeomorphism on every connected component.
4 Appendix

4.1 Proof of proposition 3.5

Lemma 4.1. The only possible cases are the following

(α) ϕ is a self-intersecting arc in \((\mathbb{R}^*)^2\) joining the toric divisor \(D_{s_j}\) to the toric divisor \(D_{s_k}\), with \(\pi < \angle(v_j, v_k) < 2\pi\),

(β) ϕ is a self-intersecting arc in \((\mathbb{R}^*)^2\) joining the toric divisor \(D_{s_j}\) to the toric divisor \(D_{s_k}\), with \(0 \leq \angle(v_j, v_k) < \pi\),

(γ) ϕ is an immersed closed curve of rotational index 2 self-intersecting at \(p\),

(δ) ϕ is the union of 2 arcs intersecting transversally at \(p\).

Remark. The distinction we made between case (α) and (β) is not of topological nature \textit{a priori}, but will be motivated later.

Proof Suppose the normalization \(\tilde{\phi}\) of \(\phi\) is connected. Then it is either an open segment or a topological circle. The first possibility is split between (α) and (β). For the second possibility, \(\phi\) is an immersed circle self-intersecting at \(p\). One of the 2 possible smoothings of \(\phi\) gives 2 disjoint circles. If these circle are nested in the plane, then it corresponds to (γ). If they are not, then \(\phi\) is isotopic to the figure “∞” and has inflection. This contradicts corollary 1.8. Suppose now that the normalization \(\tilde{\phi}\) of \(\phi\) is not connected. Then it is the union of 2 open segments, but this is case (δ).

The classification of lemma 4.1 is relevant while computing the contribution to \(\tilde{\gamma}\) of the different ovals of \(\mathbb{R}\tilde{C}\). The goal here is to shorten this list using the fact that the sum of all this contributions is constrained by the total reality of \(\tilde{\gamma}\).

Remark. Note that all the cases of the list in lemma 4.1 can appear for simple Harnack curve. The case (α) was already illustrated by the cubic of figure 5. We will see in the sequel that the 3 other cases cannot appear solely. Their manifestation forces the curve to have some other singularities, see figure 10.
Lemma 4.2. For a u-oval $\vartheta \subset \mathbb{R}\tilde{C}$ such that $\mathcal{A}$ is an embedding on each connected component of $\vartheta^\circ$, one has

$$\deg_{\vartheta^\circ} \gamma|_{\vartheta} = |\vartheta_\infty| - 2 \cdot \text{ind}(\vartheta).$$

For a b-oval $\vartheta \subset \mathbb{R}\tilde{C}$ for which $\mathcal{A}$ is an embedding, one has

$$\deg_{\vartheta^\circ} \gamma|_{\vartheta} = 2.$$

Proof To prove the first formula, one has to compute the contribution on every connected component of $\vartheta^\circ$. As $\mathcal{A}$ is an embedding, lemma 1.6 implies first that $0 \leq \angle(v_{\vartheta(j)}, v_{\vartheta(j+1)}) \leq \pi$ for each $j \in \mathbb{Z}/m_{\vartheta}\mathbb{Z}$ and that the contribution between the $j$-th and $(j+1)$-th point of $\vartheta_\infty$ is given by

$$\frac{1}{\pi} \angle(-v_{\vartheta(j+1)}, v_{\vartheta(j)}) = \frac{1}{\pi} \left(\pi - \angle(v_{\vartheta(j)}, v_{\vartheta(j+1)})\right) \geq 0,$$

Figure 9: $\mathcal{A}(\varphi)$ in the 4 cases of lemma 4.1

Figure 10: The case $(\beta)$ on the left, and $(\gamma)$ on the right. The case $(\delta)$ appears in both.
according to our orientation convention. Summing over all $j$'s gives the desired formula. The second formula is the projective reformulation of the fact that a simple closed curve in the plane has rotational index 1.

\[\square\]

**Lemma 4.3.** In the cases $(\alpha)$, and $(\beta)$ of proposition 4.1, let $\vartheta$ be the unique $u$-oval in $\mathbb{R}\tilde{C}$ containing $\tilde{\varphi}$. Then, one has

\[\deg_{\gamma}|_{\vartheta} = |\vartheta_{\infty}| + 2 - 2 \cdot \text{ind}(\vartheta).\]

In the case $(\gamma)$ of proposition 4.1 let $\vartheta$ be the unique $b$-oval in $\mathbb{R}\tilde{C}$ containing the node $p$. Then, one has

\[\deg_{\gamma}|_{\vartheta} = 4.\]

**Proof** In the cases $(\alpha)$, and $(\beta)$ of proposition 4.1, the proof goes as the one of the first formula of the previous lemma, except that the contribution of the arc $\tilde{\varphi}$ is given for some $j$ by

\[\frac{1}{\pi}(\pi - \angle(v_{\vartheta(j)}, v_{\vartheta(j+1)})) + 2.\]

For the case $(\gamma)$, $\mathcal{A}(\vartheta)$ has rotational index 2 in the plane, that is 4 projectively.

\[\square\]

**Proof of proposition 3.5** By lemma 1.5 one has that

\[\deg \tilde{\gamma} = 2(g - 1) - 2 + b = 2g + b - 4.\]

Consider first the cases $(\alpha)$ and $(\beta)$ of proposition 4.1, and denote by $\vartheta$ the $u$-oval of $\mathbb{R}\tilde{C}$ containing $\tilde{\varphi}$. Let us compute

\[
\deg \tilde{\gamma} = \sum_{\vartheta \text{ oval of } \mathbb{R}\tilde{C}} \deg \tilde{\gamma}|_{\vartheta} = \sum_{\vartheta \text{ u-oval}} \deg \tilde{\gamma}|_{\vartheta} + \sum_{\vartheta \text{ b-oval}} \deg \tilde{\gamma}|_{\vartheta}
\]

\[
= \left(\deg \tilde{\gamma}|_{\vartheta} + \sum_{\vartheta \text{ u-oval}, \vartheta \neq \tilde{\varphi}} \deg \tilde{\gamma}|_{\vartheta}\right) + \sum_{\vartheta \text{ b-oval}} \deg \tilde{\gamma}|_{\vartheta}
\]

\[
= \left(|\vartheta_{\infty}| + 2 - 2 \cdot \text{ind}(\vartheta) + \sum_{\vartheta \text{ u-oval}, \vartheta \neq \tilde{\varphi}} (|\vartheta_{\infty}| - 2 \cdot \text{ind}(\vartheta))\right)
\]

\[+ 2 \# \{\vartheta \text{ b-oval}\}\]
by lemmas 4.2 and 4.3

\[
= \left( b + 2 - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} \text{ind}(\mathcal{O}) \right) + 2 \left( b_0(\mathbb{R}\bar{C}) - \# \{ \mathcal{O} \text{ u-oval} \} \right) \\
= 2b_0(\mathbb{R}\bar{C}) + b + 2 - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} (\text{ind}(\mathcal{O}) + 1).
\]

It follows that

\[
2b_0(\mathbb{R}\bar{C}) - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} (\text{ind}(\mathcal{O}) + 1) = 2g - 6. \tag{2}
\]

Moreover, \(\mathbb{R}\bar{C}\) is of type 1 inside of \(\bar{C}\) which is of genus \((g - 1)\), hence \(b_0(\mathbb{R}\bar{C})\) is constrained by

\[
b_0(\mathbb{R}\bar{C}) \leq g \text{ and } b_0(\mathbb{R}\bar{C}) \equiv g \bmod 2.
\]

If \(b_0(\mathbb{R}\bar{C}) = g - 2l\), then

\[
2g - 6 = 2b_0(\mathbb{R}\bar{C}) - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} (\text{ind}(\mathcal{O}) + 1) \\
\leq 2g - 4l - 4 \# \{ \mathcal{O} \text{ u-oval} \} \\
\leq 2g - 4(l + 1).
\]

It implies that \(l = 0\), \(b_0(\mathbb{R}\bar{C}) = g\), and

\[
\sum_{\mathcal{O} \text{ u-oval}} (\text{ind}(\mathcal{O}) + 1) = 3.
\]

Hence \(\vartheta\) is the unique u-oval and \(\text{ind}(\vartheta) = 2\). We proved the result for cases \((\alpha)\) and \((\beta)\).

Consider now the case \((\gamma)\) of proposition 4.1, and denote once again by \(\vartheta\) the u-oval of \(\mathbb{R}\bar{C}\) containing \(\bar{\gamma}\). We repeat the same computation

\[
deg \bar{\gamma} = \sum_{\mathcal{O} \text{ u-oval}} \deg \bar{\gamma}_\mathcal{O} + \sum_{\mathcal{O} \text{ b-oval}} \deg \bar{\gamma}_\mathcal{O} \\
= \sum_{\mathcal{O} \text{ u-oval}} (|\mathcal{O}| - 2 \cdot \text{ind}(\mathcal{O})) + \left( \sum_{\mathcal{O} \neq \vartheta \text{ u-oval}} 2 + 4 \right)
\]

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as \( \vartheta \) contributes to 4 according to \ref{4.3}

\[
= \left( b - 2 \cdot \sum_{\vartheta \text{ u-oval}} \text{ind}(\vartheta) \right) + \left( 2 \# \{ \vartheta \text{ b-oval} \} + 2 \right)
\]

\[
= 2 b_0(\mathbb{R}\hat{C}) + b + 2 - 2 \cdot \sum_{\vartheta \text{ u-oval}} \left( \text{ind}(\vartheta) + 1 \right).
\]

Once again, we end up with equation \ref{2}. The same arguments as above imply that \( b_0(\mathbb{R}\hat{C}) = g \), and that there is a unique u-oval \( \vartheta \) with \( \text{ind}(\vartheta) = 2 \). We proved the result for case \((\gamma)\).

Consider finally the case \((\delta)\) of proposition \ref{4.1}. Note that every oval of \( \mathbb{R}\hat{C} \) satisfies the assumptions of \ref{4.2}. We compute as before

\[
\text{deg} \hat{\gamma} = \sum_{\vartheta \text{ u-oval}} \text{deg} \hat{\gamma}_{|\vartheta} + \sum_{\vartheta \text{ b-oval}} \text{deg} \hat{\gamma}_{|\vartheta}
\]

\[
= \left( b - 2 \cdot \sum_{\vartheta \text{ u-oval}} \text{ind}(\vartheta) \right) + \left( 2 \# \{ \vartheta \text{ u-oval} \} \right)
\]

\[
= 2 b_0(\mathbb{R}\hat{C}) + b - 2 \cdot \sum_{\vartheta \text{ u-oval}} \left( \text{ind}(\vartheta) + 1 \right).
\]

It follows that

\[
2 b_0(\mathbb{R}\hat{C}) - 2 \cdot \sum_{\vartheta \neq \vartheta} \left( \text{ind}(\vartheta) + 1 \right) = 2g - 4,
\]

implying in turn that \( b_0(\mathbb{R}\hat{C}) = g \), that there is a unique u-oval \( \vartheta \) and that \( \text{ind}(\vartheta) = 1 \). Equivalently, the cyclical order on the \( b \) boundary points of \( \mathcal{C} \) induced by \( \vartheta \) and the one induced by the boundary \( \partial \Delta \) of the moment polygon \( \Delta \) are the same. In such case, the image by \( \mathcal{A} \) of any two connected components of \( \vartheta^p \) intersect at 0 or 2 points, but \( \mathcal{C} \) has exactly one singular point. This is a contradiction.

\[
\square
\]

Note that, along the latter proof, we obtained the following

**Lemma 4.4.** For a simple Harnack curve with only one hyperbolic node, the case \((\delta)\) of lemma \ref{4.1} cannot occur.
4.2 Proof of proposition 3.6

Lemma 4.5. The restriction to \( \tilde{C}^+ \) of the argument map \( \text{Arg} \) lifts to the universal covering \( \mathbb{R}^2 \) of \( T \). Moreover, its lift \( \text{Arg}_0 \) is a local diffeomorphism.

Proof This lemma is a corollary of proposition 3.5. Indeed, the latter implies that \( \tilde{C}_+ \) is homeomorphic to an open disc with exactly \( (g - 1) \) holes. Compactifying \( \tilde{C}_+ \) by attaching back \( \mathbb{R}\tilde{C} \), one sees that the fundamental group of \( \tilde{C}_+ \) is generated by the \( (g - 1) \) b-ovals of \( \mathbb{R}\tilde{C} \). The argument map contracts each of these ovals to one of the four points \{\((0,0), (0,\pi), (\pi,0), (\pi,\pi)\)\} in \( T \). In other words, the map

\[
\text{Arg} : \pi_1(\tilde{C}_+) \to \pi_1(T)
\]

is trivial. This is the necessary and sufficient condition for \( \text{Arg} \) to lift to a map

\[
\text{Arg}_0 : \tilde{C}^+ \to \mathbb{R}^2.
\]

By the definition of simple Harnack curve and lemma 1.7, \( \text{Arg} \) is a local diffeomorphism. So is \( \text{Arg}_0 \).

Now, define the topological disc \( D \) as follows: consider first the closure of \( \tilde{C}^+ \) in \( \tilde{C}_{\text{Arg}} \), see lemma 1.13. It is a closed disc with \( (g - 1) \) open holes bounded by the b-ovals of \( \mathbb{R}\tilde{C} \). Now contract to a point every connected component of \( \mathbb{R}\tilde{C}_{\text{Arg}} \) that is in the closure of \( \tilde{C}_+ \). The result is clearly a topological disc that we denote \( D \).

Lemma 4.6. \( \text{Arg}_0 \) extends to a differentiable map \( \text{Arg}_0 : D \to \mathbb{R}^2 \). Denote by \( \vartheta \) the unique u-oval of \( \tilde{C} \). Then, \( \text{Arg}_0 \) maps the boundary of \( D \) to the piecewise linear curve with vertices in \( \pi\mathbb{Z}^2 \) obtained by the concatenation of the vectors \( \tau(v_{\vartheta(j)}) \) according to the cyclical ordering \( j \in \mathbb{Z}/m_\vartheta\mathbb{Z} \), \( \tau \) being defined in proposition 3.6.

Remark. The points of \( \text{Arg}_0(D) \cap \pi\mathbb{Z}^2 \) are exactly the points of \( D \) coming from the contracted connected components of \( \mathbb{R}\tilde{C}_{\text{Arg}} \).

Proof The fact that \( \text{Arg}_0 \) extends has been proven in lemma 1.13. Differentiability at the boundary is implicitly given in the proof of lemma 1.13. To
see differentiability at the points obtained by contraction of the b-ovals, use the same arguments as in the proof of theorem 1.

The second part of the statement falls from lemma 1.13 and the way we defined $\tilde{C}^+$. 

□

Now, we are ready to prove the proposition 3.6. We have to solve a combinatorial isoperimetrical inequality problem. On one side, the area enclosed by the piecewise linear curve $Arg_0(\partial D)$ is given by theorem 1. On the other, the geometry and in particular the length of the piecewise linear curve $Arg_0(\partial D)$ is constraint by the previous lemma.

**Proof of proposition 3.6** By theorem 1 one has that

\[
\text{Area}(Arg_0(D)) = \text{Area}(Arg_0(\tilde{C}^+)) = \frac{1}{2}\text{Area}(Arg(C^o)) = -\frac{\pi^2}{2}\chi(\tilde{C}^o).
\]

In the present case, Khovanskii’s formula [Kho78] gives

\[-\chi(\tilde{C}^o) = (2g + b - 2) - 2.\]

Pick’s formula gives in turn

\[
\text{Area}(Arg_0(D)) = \pi^2((g + b/2 - 1) - 1) = \pi^2(\text{Area}(\Delta) - 1) \tag{3}
\]

Consider the case $(\alpha)$ of lemma 4.1 By the previous lemma, $Arg_0(\partial D)$ is a piecewise linear curve with vertices in $\pi\mathbb{Z}^2$. As we have seen in the proof of lemma 4.2, it is locally convex everywhere except at the vertex coming from $\phi \subset C_{Arg}$, where by assumption the angle interior to $Arg_0(D)$ is strictly between $\pi$ and $2\pi$. Let $j \in \mathbb{Z}/m\mathbb{Z}$ be such that this non convex angle is formed by $\tau(v_{\theta(j)})$ and $\tau(v_{\theta(j+1)})$. If one permutes these 2 vectors, the area of the domain of $\mathbb{R}^2$ enclosed by the piecewise linear curve increases at least by $\pi^2$, and exactly by $\pi^2$ if and only if $v_{\theta(j)}$ and $v_{\theta(j+1)}$ span a parallelogram of area 1. Suppose the new polygonal domain that we just obtained is still not convex. Then, one can repeat the previous construction, and increase the area until we end up with a convex domain. This convex domain can be nothing but $\Delta$ (up to translation). This contradicts (3). Hence, the result of the permutation were already convex and $v_{\theta(j)}$ and $v_{\theta(j+1)}$ have to span a
parallelogram of area 1, by (3). In other words, there exists a smooth vertex \( \nu \) of \( \Delta \) such that \( \text{Arg}_0(D) = \tau(\Delta_\nu) \).

In cases (\( \beta \)) and (\( \gamma \)) of lemma 4.1, one has that \( \text{Arg}_0(\partial D) \) is convex. In such case, its rotational index is exactly computed by \( \text{ind}(\vartheta) \), where \( \vartheta \) is the unique u-oval of \( \tilde{C} \). It has been shown in the proof of proposition 3.5 that this rotational index is 2. Then, there exists two distinct points \( p_1 \) and \( p_2 \) on \( \partial D \) mapped to the same point in \( \mathbb{R}^2 \). They cut \( \partial D \) into two arcs \( \gamma_1 \) and \( \gamma_2 \). Denote by \( \Delta_1 \) and \( \Delta_2 \) the polygonal domains enclosed by \( \gamma_1 \) and \( \gamma_2 \) respectively. Then

\[
\text{Area}(\text{Arg}_0(D)) = \text{Area}(\Delta_1) + \text{Area}(\Delta_2).
\]  

(4)

Two cases can occur: either \( p_1 \) and \( p_2 \) are mapped to a point of \( \pi \mathbb{Z}^2 \), or not.

In the first case, \( \tau^{-1}(\Delta_1) \) and \( \tau^{-1}(\Delta_2) \) are two polygonal domains in \( \mathbb{R}^2 \) with integer vertices. Reorder their edges in convex position in order to get two convex polygon \( \Box_1 \) and \( \Box_2 \). Then, one has the Minkowski sum

\[
\Delta = \Box_1 + \Box_2.
\]  

(5)

By (3), (4), one has

\[
\text{Area}(\Delta) - 1 \leq \text{Area}(\Box_1) + \text{Area}(\Box_1),
\]

and (5) provide the opposite inequality. It means that the mixed volume \( \text{Vol}(\Box_1, \Box_2) = 1 \). By (Kus76), this is the intersection number of two generic curves of respective Newton polygon \( \Box_1 \) and \( \Box_2 \). By (5), the union of two such curves has Newton polygon \( \Delta \). Such reducible curves form a component of the space of nodal curves in \( T_\Delta \). By Horn parametrization, this space is irreducible. Hence, every nodal curve in \( T_\Delta \) is reducible, in particular, \( C \) is. This is a contradiction.

In the second case, \( p_1 \) and \( p_2 \) are not mapped on a point of \( \pi \mathbb{Z}^2 \). Denote by \( v_1 \) the closest vertex of \( \text{Arg}_0(\partial D) \) before the self-intersection point \( \text{Arg}_0(p_1) \), and \( v_2 \) the closest vertex of \( \text{Arg}_0(\partial D) \) after it. Now cut the first edge of \( \text{Arg}_0(\partial D) \) after \( v_1 \) and past it before \( v_2 \) as shown below.

This construction has the following properties:

(\( \ast \)) the area enclosed by the resulting curve is strictly greater than \( \text{Area}(\text{Arg}_0(D)) \),
the angle at the vertex next the self-intersection point strictly decreases, if it is still convex.

The latter property implies that, repeating this process, one has to end up either in the case where the self-intersection point is moved to a point of $\pi\mathbb{Z}^2$, then the previous treatment leads to a contradiction; or in the case of a piecewise linear curve of rotational index 1 that is convex except at one vertex. This amounts to the treatment of case $(\alpha)$. The first property implies that

\[ \text{Area}(\text{Arg}_0(D)) < \pi^2 (\text{Area}(\Delta) - 1). \]

This is in contraction with (3). By lemma 4.4, the case $(\delta)$ cannot occur. The theorem is proved.

\[ \square \]

Note that during the proof, we obtained the following

**Corollary 4.7.** Only the case $(\alpha)$ of lemma 4.1 can occur.

### 4.3 Proof of proposition 3.7

Up to a toric transformation, one can assume that $\Delta$ contains the three points $(0,0)$, $(0,1)$ and $(1,0)$ and that $\nu = (0,0)$. It induces the local compactification of $(\mathbb{C}^*)^2$ by $\mathbb{C}^2$ in $\mathcal{T}_\Delta$, where the two toric divisors adjacent to $\nu$ are the two coordinate axis. By corollary 4.7, the two asymptotes of $\mathcal{A}(\varphi)$ are horizontal leftward and vertical downward. As before, let $p = (p_1, p_2) \in (\mathbb{C}^*)^2$ be the node of $\mathcal{C}$, and choose $\varepsilon_1, \varepsilon_2 > 0$ such that the point $(\log |p_1| + \varepsilon_1, \log |p_2| + \varepsilon_2)$ belongs to the compact connected component
of the complement of $A(\mathbb{R}C)$ delimited $A(\varphi)$. Define the following sets

\[ R := \{(x, y) \in \mathbb{R}^2 \mid x \leq \log|p_1| + \varepsilon_1, \ y \leq \log|p_2| + \varepsilon_2\}, \]
\[ H := \{(x, y) \in \mathbb{R}^2 \mid x \leq \log|p_1| + \varepsilon_1, \ y = \log|p_2| + \varepsilon_2\}, \]
\[ V := \{(x, y) \in \mathbb{R}^2 \mid x = \log|p_1| + \varepsilon_1, \ y \leq \log|p_2| + \varepsilon_2\}. \]

**Lemma 4.8.** $\partial(A(\mathcal{C}) \cap R^c) = A(\mathbb{R}C) \cap R^c$. $A|_{\mathcal{C}\setminus A^{-1}(R)}$ is at most 2-to-1.

For any connected component $C$ of $A^{-1}(R)$ in the normalization $\tilde{C}$, $A|_C$ is at most 2-to-1.

**Proof** From the way $R$ was chosen, $A$ is an embedding on each connected component of $\mathbb{R}C \cap A^{-1}(R^c)$. For any such component, its image by $A$ split $R^c$ into two halves. The proof is then the very same as the one of lemma 8 in [Mik00].

\[ \square \]

**Lemma 4.9.** One has the following decomposition $\mathcal{C} \cap A^{-1}(R) = \mathcal{C}_H \cup \mathcal{C}_V$ where $\mathcal{C}_H$ and $\mathcal{C}_V$ are two connected Riemann surfaces with boundary such that $A(\partial \mathcal{C}_H) \subset H$ and $A(\partial \mathcal{C}_V) \subset V$. Moreover, $A$ is at most 2-to-1 when restricted to $\mathcal{C}_H$ or $\mathcal{C}_V$.

**Proof** Draw a curve in the interior of $A(\mathcal{C})\cap R^c$ that starts on $A(\varphi)$ and ends on the boundary of a non compact component of the complement of $A(\mathcal{C})^c$. By the above 2-to-1 property, its lift in $\tilde{C}$ is a topological circle $\rho$ invariant under the complex conjugation that cuts the unique $u$-oval of $\tilde{C}$ in two connected components. Indeed, the single $u$-oval of $\mathbb{R}\tilde{C}$ intersects $A^{-1}(R)$ in exactly two connected components $\alpha_H$ and $\alpha_V$ intersecting respectively only $A^{-1}(H)$ and only $A^{-1}(V)$. It follows that $\rho$ cuts $\tilde{C}$ into two connected components intersecting either $A^{-1}(H)$ or $A^{-1}(V)$. Denote the intersection with $A^{-1}(R)$ of these components $\mathcal{C}_H$ and $\mathcal{C}_V$ respectively. $\mathcal{C}_H$ has a connected component containing $\alpha_H$. The amoeba of any other other connected component of $\mathcal{C}_H$ has to have a non compact complement component in $R$, bounded by an arc joining $H$ to $V$, contradicting the fact the $\rho$ splits $\tilde{C}$. The same argument applies to $\mathcal{C}_V$ and we conclude that both $\mathcal{C}_H$ and $\mathcal{C}_V$ are connected.

The proof of the 2-to-1 property goes the same way as before.
Lemma 4.10. There exists two functions $g$ and $h$ holomorphic on $\mathcal{A}^{-1}(R)$ such that $C_V$ (resp. $C_H$) is the zero set of $g$ (resp. $h$) and $g \cdot h = f$ where $f$ is a polynomial defining $C$.

Proof The closure of $\mathcal{A}^{-1}(R)$ in $T_\Delta$ is a polydisc $D$ centred at the origin, in the local compactification $\mathbb{C}^2$ of $(\mathbb{C}^*)^2$. Let us consider an irreducible component $C$ of $C_V$ or $C_H$. By the classical implicit function theorem, there exists an open covering $D \subset \bigcup_j U_j$ and a collection of function $g_j$ holomorphic on $U_j$ such that the zero set of $g_j$ is exactly $U_j \cap C$. Hence the quotient $g_j/g_k$ is a nowhere vanishing holomorphic function on the overlap $U_j \cap U_k$. We are looking for a global function $g$ holomorphic on $D$ such that its zero set is exactly $C$ or equivalently such that $g/g_j$ is a nowhere vanishing holomorphic function on $U_j$. This amounts to solve the second Cousin problem, in the holomorphic case. As $D$ is a Stein manifold, such that $H^2(D, \mathbb{Z}) = 0$, the result follows from theorem 7.4.4 in [H"{o}90].

The local factorization of $f$ on $\mathcal{A}^{-1}(R)$ induces a splitting of its associated Ronkin function on $R$, that is

$$N_f(x, y) := \frac{1}{(2i\pi)^2} \int_{\mathcal{A}^{-1}(x,y)} \frac{\log |f(z, w)|}{zw} dw$$

$$= \frac{1}{(2i\pi)^2} \int_{\mathcal{A}^{-1}(x,y)} \frac{\log |g(z, w)| + \log |h(z, w)|}{zw} dw$$

$$=: N_g(x, y) + N_h(x, y).$$

To each of the latter Ronkin functions, one can associate their respective spines $\mathcal{S}_g$ and $\mathcal{S}_h$, as in [PR04]. These are two tropical curves in the domain $R$ such that the amoeba $\mathcal{A}(C_V)$ (resp. $\mathcal{A}(C_H)$) deformation retracts on $\mathcal{S}_g$ (resp. $\mathcal{S}_h$), see theorem 1.17.

Recall that $A_f$ is the set of connected components of the complement of $\mathcal{A}(C)$. Similarly, denote $A_h$ (resp. $A_g$) the set of connected components of the complement of $\mathcal{A}(C_H)$ (resp. $\mathcal{A}(C_V)$) in $R$. The same properties hold for
\( N_h \) and \( N_g \) and \( S_g \) and \( S_h \) are defined the exact same way as \( S_f \). If one denotes by \( A_f^R \) the elements of \( A_f \) intersecting \( R \), convexity implies that

\[
(S_f)_{\mid R} = \max_{\alpha \in A_f^R} N^\alpha_f
\]

\[
= \max_{\alpha \in A_h^R} N^\alpha_h + \max_{\alpha \in A_g^R} N^\alpha_g
\]

\[
\leq \max_{\alpha \in A_h^R} N^\alpha_h + \max_{\alpha \in A_g^R} N^\alpha_g
\]

\[
= S_h + S_h,
\]

where the elements of \( A_f^R \) are seen as subsets of the elements of \( A_h \) and \( A_g \) in the second equality. There is equality if and only the maps \( A_f^R \to A_h \) and \( A_f^R \to A_g \) given by the inclusion are both surjective, or equivalently no connected component of the complement of \( A(C_H) \) is included in \( A(C_V) \) and vice versa. Note that in such case

\[
\mathcal{I}_f \cap R = \mathcal{I}_h \cup \mathcal{I}_g.
\]

In order to prove proposition \[3.7\] one will need few more lemmas.

**Lemma 4.11.** The stable intersection of \( \mathcal{I}_g \) and \( \mathcal{I}_h \) in \( R \) is 1.

**Proof** The idea is to deform smoothly \( C_H \) and \( C_V \) close to their respective spine in such a way that the intersection number of the deformations is kept constant equal to 1, and such that this intersection number corresponds to the stable intersection of \( \mathcal{I}_g \) and \( \mathcal{I}_h \).

Let us first deform \( C_H \). Consider a smooth foliation of \( A(C_H) \) modelled on the foliation \( \mathcal{F} \), as pictured in figure 6 of [Mik04]. The example given here is depicted in the neighbourhood of a 3-valent vertex of the spine but can easily be carried out for any higher valency. Now, let us deform \( A(C_H) \) in time \( t \) by applying on each leaf an homothety of ratio \( 1/t \) centered at the the spine. The result is a smooth deformation retraction of \( A(C_H) \) to \( \text{resp. } \mathcal{I}_h \), for \( t >> 1 \). Now, one can deform smoothly \( C_H \) in a smooth surface \( C_{H,t} \) lying above the deformation at time \( t \) of \( A(C_H) \), by keeping constant the argument of the points in a moving fiber. Now, perform a similar deformation \( C_{V,t} \) of \( C_V \). As \( A(C_{V,t}) \cap H = A(C_{H,t}) \cap V = \emptyset \) and \( \partial A(C_{V,t}) \subset V \), \( \partial A(C_{H,t}) \subset H \) for
any \( t >> 1 \), the homological intersection of \( \mathcal{C}_{H,t} \) and \( \mathcal{C}_{V,t} \) is constant in \( t \), that is equal to 1. At the level of spines, one can use a translation as small as desired in order to assume that the set-theoretical intersection of \( \mathcal{S}_g \) and \( \mathcal{S}_h \) is a finite collection of points, none of which is a vertex of any of the two spines, and that no intersection point went out of \( R \). Then, translating \( \mathcal{C}_{H,t} \) and \( \mathcal{C}_{V,t} \) accordingly, one clearly has that both intersection numbers under consideration are not affected by these translations.

Hence, for \( t \) large enough, \( \mathcal{A}(\mathcal{C}_{H,t}) \) does not contain any vertex of \( \mathcal{S}_g \) and \( \mathcal{A}(\mathcal{C}_{V,t}) \) does not contain any vertex of \( \mathcal{S}_h \). It implies that for any intersection point of \( \mathcal{S}_g \) and \( \mathcal{S}_h \), there exists a small neighbourhood \( U \) such that the quadruple \((\mathcal{A}(\mathcal{C}_{V,t}) \cap U, \mathcal{S}_g \cap U, \mathcal{A}(\mathcal{C}_{H,t}) \cap U, \mathcal{S}_h \cap U)\) is diffeomorphic to

\[
[[-2, 2] \times [-1, 1], -2, 2] \times \{0\}, [-1, 1] \times [-2, 2], \{0\} \times [-2, 2])
\]

in \([-2, 2]^2\). By the 2-to-1 property, see lemma 4.8, the preimages of the two stripes in \( U \) are two cylinders in \( \mathcal{C}_{V,t} \) and \( \mathcal{C}_{H,t} \). Locally, each of these cylinders separates two connected component of the complement. By the proof of lemma 11 in [Mik00], the homology class of each cylinder in \( H_1(T, \mathbb{Z}) \) is given by the difference of the orders of its two adjacent components of the complement. By the definition of the spine, the difference of these orders corresponds to the primitive integer vector supporting the corresponding edge of the spine, times its multiplicity. On one side, the homological intersection of the two cylinders is given by the intersection of their homology class in \( H_1(T, \mathbb{Z}) \), that is the corresponding lattice index. By the definition of stable intersection and the latter observations, this is also the local stable intersection of \( \mathcal{S}_g \) and \( \mathcal{S}_h \) in \( U \). The result follows.

\[\square\]

**Lemma 4.12.** \( \mathcal{S}_g \) and \( \mathcal{S}_h \) are trees. Their Newton polygons are either segments of integer length 1 or triangles without inner integer points.

**Proof** \( \mathcal{S}_h \) has exactly one vertical leaf going downward, and \( \mathcal{S}_g \) has exactly one horizontal leaf going leftward. Indeed, they have at least one by assumption, and cannot have more otherwise the stable intersection of \( \mathcal{S}_h \) and \( \mathcal{S}_g \) would be greater than 1, contradicting the previous lemma. If \( \Delta_h \) and \( \Delta_h \) are the respective Newton polygons, the latter implies that \( \Delta_h \) is bounded from below by an horizontal side \( s_H \) of length 1 and \( \Delta_g \) is bounded from
the left by a vertical side $s_V$ of length 1. For $\Delta_g$, the side attached at the
top of $s_V$, if there is, is strictly slanted toward the right, otherwise the cor-
responding leaf of $\mathcal{S}_g$ would intersect $H$. The side attached at the bottom
of $s_V$, if there is, is horizontal or strictly slanted toward the left, otherwise
the corresponding leaf of $\mathcal{S}_g$ would intersect the vertical leaf of $\mathcal{S}_h$, up to
translation, contradicting the previous lemma. The only possibility is that
$\Delta_g$ is either a binomial or a right angled triangle with integer height 1. The
same arguments apply for $\Delta_h$.

□

Proof of proposition 3.7 By the previous lemma, every connected compo-
nent of the complement of $\mathcal{A}(C_H)$ and $\mathcal{A}(C_V)$ are unbounded in $R$. It implies
that no connected components of the complement of $\mathcal{A}(C_H)$ is hidden by
$\mathcal{A}(C_V)$ and vice versa. By the previous remarks, it implies that
$$\mathcal{S}_f \cap R = \mathcal{S}_h \cup \mathcal{S}_g.$$ It implies also that $\mathcal{A}(\mathcal{C})$ has exactly $g$ visible holes, which is the maximal
possible. Indeed, the previous lemma implies that none of the $(g-1)$ b-ovals
of $\mathbb{R}\tilde{C}$ intersect $\mathcal{A}^{-1}(R)$, and $\mathcal{A}$ is at most 2-to-1 on this space, by lemma 4.8.
Hence, their image by $\mathcal{A}$ bounds a compact component of the complement.
The same holds for the singular loop of $\mathcal{A}(\varphi)$. Hence, the complement of $\mathcal{S}_f$
has also $g$ compact connected components. By lemma 4.11, $\mathcal{S}_f$ possess an
hyperbolic node. The first part of the proposition is proved.
The second part of the statement can be deduced from the existence of a
defformation retraction of $\mathcal{A}(\mathcal{C})$ onto $\mathcal{S}_f$ such that $\mathcal{A}(C_H)$ retracts on $\mathcal{S}_h$
and $\mathcal{A}(C_V)$ retracts on $\mathcal{S}_g$. The deformations of $\mathcal{A}(C_H)$ and $\mathcal{A}(C_V)$ have
been constructed in the proof of lemma 4.11. Deformations of $\mathcal{A}(\mathcal{C}) \cap R^c$
onto $\mathcal{S}_f \cap R^c$ exist by classical theory (see theorem 1 in [PR04]). From
this, one can easily construct a global deformation of $\mathcal{A}(\mathcal{C})$ with the required
properties.

□

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