SOME CORRELATORS OF $SU(3)_3$ WZW MODELS ON HIGHER-GENUS RIEMANN SURFACES

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Abstract

Using the conformal embedding on the torus, we can express some characters of $SU(3)_3$ in terms of $SO(8)_1$ characters. Then with the help of crossing symmetry, modular transformation and factorization properties of Green functions, we will calculate a class of correlators of $SU(3)_3$ on arbitrary Riemann surfaces. This method can apply to all $k > 1$ WZW models which can be conformally embedded in some $k = 1$ WZW models.

I. Introduction

Understanding of conformal field theory, with infinite dimensional conformal symmetry, and Wess-Zumino-Witten (WZW) models, with extra affine Kac-Moody symmetry$^{[1-5]}$, demands treating them on arbitrary world sheet. Therefore studying WZW models on higher-genus Riemann surfaces (HGRS’s) is one of the important branches of the string theory.

So far the results that have been achieved toward the solution of the above problem, can be summarized as follows (for brevity in the following we denote the Green function
and partition function by G.f. and p.f. respectively). Most of the calculations have been done for the level-one WZW models, since these are rather easier to handle. The equivalence of the level-one simply-laced group with compactified free bosons, was used to calculate the G.fs. of the primary field of these models on HGRS’s $^6$.$^8$. As an another approach, the behavior of the correlators under zero-homology pinching (ZHP) and non-zero homology pinching (NZHP) (or its factorization properties) can be used to calculate the G.fs. By using this method, the $n$-point functions of the primary fields, and also a class of the descendant fields, was calculated for the level-one simply-laced and also non-simply-laced ($SO(2N + 1)_1$) WZW models on HGRS’s $^9$.$^11$.

Similar calculations for the higher-level ($k > 1$) WZW models, are more important and of course more complicated. The only method that have been used in these cases is based on the Coulomb gas representation of these models. The complete structure of these representations (that is their BRST cohomology structure) was only found for $SU(2)_k$ $^{12,13}$, and therefore the calculation of G.fs. was only done for this case (only the partition function on genus-two was explicitly calculated $^{14}$).

As a result, searching for other methods of calculations of $k > 1$ WZW models is of great importance. In this paper, we will present another approach which can be applied to a large class of $k > 1$ models. This method is based on: conformal embedding, crossing symmetry, modular transformation and factorization properties of G.fs.

Conformal embedding allows one to express some characters of a class of $k > 1$ models ($H_k$) in terms of the characters of some specific level-one WZW models ($G_1$). By $H_k$ we mean a WZW model with group H and level k. It is of crucial importance to note that this equivalence of p.fs. of $H_k$ and $G_1$ is restricted to genus-one (torus). By increasing the genus, the equivalence is lost due to the fact that the difference between the number of diagrams, in the two theories, increases rapidly. Therefore we allow to use the conformal embedding only in the first step of our calculations (that is on the torus), and to find the higher-genus correlators we must seek another techniques (those mentioned above).

Here we will study a simple example of this kind, that is $SU(3)_3$ (which can be conformally embedded in $SO(8)_1$), but our method can be easily generalized to other cases which can be embedded in some level-one models. In this papers, we will compute a class of G.fs. on arbitrary Riemann surfaces. This correlators correspond to the diagrams in which $\Phi_8, \Phi_{10}$ and $\Phi_{10}$ are external legs, $\Phi_8$ is the loop field and $\Phi_1, \Phi_{10}$ and $\Phi_{10}$ are propagators (the indices denote the representations of $SU(3)$).
In sec. II we will briefly explain the conformal embedding and the branching rules of $SU(3)_3$ in $SO(8)_1$. Sec. III is devoted to the calculation of the fusion rules of $SU(3)_3$ and specifying the propagators and Sec.IV to the calculation of genus-two p.f.s. All the above mentioned G.f.s. will be calculated in sec.V. The necessary mathematical tools will be provided in the appendix.

II. Some $SU(3)_3$ Characters on the Torus

Consider a subalgebra $h$ of a finite dimensional Lie algebra $g$. One can associate to $h$, a Kac-Moody subalgebra $\hat{h}$ of an untwisted kac-Moody algebra $\hat{g}$ by identifying the derivations \cite{i.e. identifying the basic gradations}. Doing this, the central charges of the Kac-Moody algebras get identified with a relative factor $j$ given by the Dynkin index of the embedding of $h$ in $g$. This implies that every level $k$ highest weight $g$-module $L(\Lambda)$ is reducible into a sum of level $\bar{k} = jk$ highest weight $h$-module $L(\bar{\Lambda})$. In general this branching is not finite,i.e. the highest weight $g$-module $L(\Lambda)$ is not finitely reducible under $\hat{h}$. One can prove however that $L(\Lambda)$ is finitely reducible under $\hat{h}$ if and only if the corresponding Virasoro algebras have equal central charge \cite{16}

\[ c_g(k) = c_h(jk) \] (1)

In particular, it can be shown that, Eq.(1) can only be satisfied if $k=1$. In this case $h_j$ is conformally embedded in $g_1$. All of these embeddings has been classified \cite{17}.

Since $SU(N)$ is a subgroup of $SO(N^2 - 1)$ and the Dynkin index of this embedding is $N$, $SU(N)_N$ can be embedded (conformally) in $SO(N^2 - 1)_1$. As the first nontrivial example, we will consider the N=3 case.

As mentioned above, the characters of $SO(8)_1$ can be decomposed in terms of $SU(3)_3$ characters. But before discussing the branching rules, we must specify the primary fields of these two models. As is well known, the primary fields of $SU(N)_k$ are those representations for which the width of the corresponding Young tableaux are less than or equal to $k$. Therefore the primary fields of $SU(3)_3$ are:

\[ \Phi_1, \Phi_3, \Phi_5, \Phi_6, \Phi_8, \Phi_{10}, \Phi_{12}, \Phi_{15}, \Phi_{18} \] (2)

The indices show the dimension of representation. The conformal weights of primary fields are

\[ h_r = \frac{C_r/\psi^2}{2k/\psi^2 + g} \] (3)
where $C_r$ is the quadratic Casimir of representation, $\psi$ is the highest root, $k$ is the level and $g$ is the dual coaxter number. For representation $(m,n)$ of $SU(3), C_{mn}/\psi^2 = \frac{1}{3}(m^2 + n^2 + mn + 3m + 3n)$ and $g=3$. By choosing $\psi^2 = 2$, we find the following conformal weights for the primary fields of $SU(3)_3$

$$h_1 = 0, h_3 = h_5 = \frac{2}{9}, h_8 = \frac{1}{2}, h_6 = h_9 = \frac{5}{9}, h_{10} = h_{10} = 1, h_{15} = h_{15} = \frac{8}{9}. \quad (4)$$

The primary fields of $SO(8)_1$ can also be determined as $\Phi_0, \Phi_v, \Phi_s$ and $\Phi_{\overline{s}}$ where the indices stand for $0=$trivial, $v=$vector and $s(\overline{s})=$spinor (antispinor) representations of $SO(8)$. The conformal weights of these primary fields are: (see Eq.(3))

$$h_o = 0, h_v = h_s = h_{\overline{s}} = \frac{1}{2}. \quad (5)$$

Comparison of Eqs.(4) and (5), shows that if we expand the character $\chi_0$ in terms of the characters of $SU(3)_3$, only those characters enter which the corresponding primary fields have integer conformal weights. To see this let’s remind the definition of the character of a primary field of conformal dimension $h$.

$$\chi_h(q) = q^{-c/24}trq^Lq_o = q^{-c/24}(a_1q^h + a_2q^{h+1} + \cdots), \quad q = e^{2\pi i \tau} \quad (6)$$

Neglecting the factor $q^{-c/24}$ (as is the same for both models, i.e. $c_{SO(8)_1} = c_{SU(3)_3} = 4$), the powers of $q$ in $\chi_0(q)$ are all integers (as $h_0 = 0$) and therefore only the primary fields of integer conformal weight can take part in $\chi_0$’s decomposition. Direct calculation shows that $^{[18]}$

$$\chi_0 = \chi_1 + \chi_{10} + \chi_{10}, \quad (7)$$

and in the same way

$$\chi_s = \chi_{\overline{s}} = \chi_8. \quad (8)$$

Using these relations, it is possible to write the above characters of $SU(3)_3$ in terms of $SO(8)_1$ characters.

The characters of $SO(8)_1$ can be calculated in two ways, either by using the string functions of $SO(8)_1$, in the same way that the characters of $SO(2N + 1)_1$ was found in Ref.[11], or by using the free field representation of this theory $^{[19]}$. By either way the result is

$$\chi_0(\tau) = \frac{1}{2\eta^4(\tau)} \left\{ \Theta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0|\tau) + \Theta^4 \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (0|\tau) \right\}. \quad (9)$$
\[ \chi_v(\tau) = \frac{1}{2\eta^4(\tau)} \left\{ \Theta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0|\tau) - \Theta^4 \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (0|\tau) \right\} \]  

(9-b)

\[ \chi_s(\tau) = \chi_v(\tau) = \frac{1}{2\eta^4(\tau)} \Theta^4 \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (0|\tau). \]  

(9-c)

Since \[ \Theta^4 \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (0|\tau) = \Theta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0|\tau) - \Theta^4 \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (0|\tau) , \] we have:

\[ \chi_v = \chi_s = \chi_v = \frac{1}{2\eta^4(\tau)} \Theta^4 \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (0|\tau). \]  

(10)

On the other hand, the bosonic p.f. on the torus is

\[ Z_B(\tau) = (Im\tau)^{-1/2}|\eta(\tau)|^{-2}, \]  

(11)

therefore

\[ \hat{Z}_8(\tau) = \frac{(Im\tau)^2}{2} |\Theta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (0|\tau)|^8 \]  

(12)

\[ (\hat{Z}_1 + \hat{Z}_{10} + \hat{Z}_{10})(\tau) = \frac{(Im\tau)^2}{2} |\Theta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0|\tau) + \Theta^4 \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (0|\tau)|^2, \]  

(12-b)

where by \( \hat{Z} \) we mean

\[ \hat{Z} = \frac{Z}{Z_B^c} \]  

(13)

The reason of division of p.f. by \( Z_B^c \) is to write the p.f. in a metric independent way \[20\], and in fact only this ratio is well defined as the p.f. and must be generalized to HGRS’s.

Before concluding this section, it is worth noting that the decompositions similar to Eqs. (7) and (8) also occur in other cases, and the techniques which we will develop in this paper, can be extended to apply to these more general cases also. In the following we quote some of these branching rules \[18\]:

\[ ch_{10} = ch_{10} = \chi_{10} \]

\[ ch_5 = ch_5 = \chi_5, \]

(14)

where \( ch_i \in SU(5)_1 \) and \( \chi_i \in SO(5)_2, \)

\[ ch_{16} = ch_{16} = \chi_{16} \]

(15)
with \( c_h \in SO(10)_1 \) and \( \chi_i \in SO(5)_3 \), etc.

III. \( SU(3)_3 \) Fusion Rules

In this section, we will find \( SU(3)_3 \) fusion rules of the conjugate fields, by orthogonal polynomials technique, developed by Gepner \[21\]. Let us briefly review this technique.

Let \( \mu_i(i = 1, \cdots, N) \) be the fundamental weights of \( SU(N) \). Then any arbitrary representation with highest weight \( \mu \), can decompose as
\[
\mu = \sum_{i=1}^{N-1} \sum_{a_i} a_i \mu_i.
\]
In this way, we can denote any primary field of \( SU(N)_k \) by \( [a_1, \cdots, a_k] \) where \( 0 \leq a_1 \leq a_2 \cdots a_k \leq N - 1 \). Now if we denote the fully antisymmetric (fundamental) representations of \( SU(N) \) by
\[
\overline{c}_r = [0, 0, \cdots, 0, r], \quad r = 0, 1, \cdots, N - 1,
\]
whose Young tableaux is , then Gepner has shown that there exist a natural map from \( [a_1, \cdots, a_k] \) to the polynomials of \( N \) indeterminates \( \overline{c}_i \), such that one can express this representation as follows:
\[
[a_1, \cdots, a_k] = \text{det}_{1 \leq i, j \leq k} a_{i, i-j}
\]
The \text{det} stands for the determinant of the matrix \( A_{ij} = \overline{c}_{a_i, a_j} \), which is defined by the convention \( \overline{c}_0 = \overline{c}_N = 1 \) and \( \overline{c}_i = 0 \) for \( i > N \) or \( i < 0 \). It can also be proved that the fusion rule of \( \overline{c}_r \) with other representations can be read from the following equation \[21\]:
\[
\overline{c}_r[a_1, \cdots, a_k] = \sum_{a_i \leq b_i \leq a_i+1} \sum_{b_i=r+\sum a_i} \overline{c}_b[a_1, \cdots, a_k]
\]
In this equation \( b_i \)'s must be written mod\( N \), and must be ordered as \( 0 \leq b_1 \leq b_2 \cdots \leq b_k \).
In the RHS of Eq.(18) each representation must be considered only once.

Now let us apply this method to \( SU(3)_3 \). In this case, the variables of polynomials are two fundamental representations of \( SU(3)_3 \), that is \( 3 = [0, 0, 1] = x \) and \( \overline{3} = [0, 0, 2] = y \). Then Eq.(17) allows us to write for each representation of \( SU(3)_3 \) a corresponding polynomial
\[
1 = [0, 0, 0] = 0 \quad 3 = [0, 0, 1] = x \quad \overline{3} = [0, 0, 2] = y \quad 6 = [0, 1, 1] = x^2 - y \\
\overline{6} = [0, 2, 2] = y^2 - x \quad 8 = [0, 1, 2] = xy - 1 \quad 10 = [1, 1, 1] = x^3 - 2xy + 1 \\
10 = [2, 2, 2] = y^3 - 2xy + 1 \quad 15 = [1, 1, 2] = x^2y - x - y^2 \quad 15 = [1, 2, 2] = xy^2 - x^2 - y
\]
Now it is possible to find the fusion rules by using Eq.(18). For example:

\[ \Phi_6 \times \Phi_6 = (x^2 - y).[0, 2, 2] \]

but

\[ y.[0, 2, 2] = [0, 1, 2] + [2, 2, 2] \]
\[ x.[0, 2, 2] = [0, 0, 2] + [1, 2, 2] \]
\[ x^2.[0, 2, 2] = x.[0, 0, 2] + x.[1, 2, 2] = [0, 0, 0] + 2[0, 1, 2] + [2, 2, 2] \]

therefore

\[ \Phi_6 \times \Phi_6 = [0, 0, 0] + [0, 1, 2] = \Phi_1 + \Phi_8 \]

In this way we obtain the following fusion rules of the conjugate representations of SU(3)_3

\[ \Phi_3 \times \Phi_3 = \Phi_1 + \Phi_8 , \Phi_6 \times \Phi_6 = \Phi_1 + \Phi_8 , \Phi_{10} \times \Phi_{10} = \Phi_1 \]
\[ \Phi_8 \times \Phi_8 = \Phi_1 + 2\Phi_8 + \Phi_{10} + \Phi_{15} , \Phi_{15} \times \Phi_{15} = \Phi_1 + \Phi_8 \]  

(20)

These fusion rules are sufficient to know all the multiloop diagrams of SU(3)_3. The reason is the following: As is obvious from sewing procedure, we can construct any loop by sewing (fusing) two legs of a three-point vertex, only when their corresponding representations are conjugate. Therefore to know the allowed propagators of a theory, it is enough to only consider the fusion rules of the conjugate fields (Fig.1). Eq.(20) shows that in SU(3)_3 diagrams on HGRS’s, only the fields [\Phi_1], [\Phi_8], [\Phi_{10}] and [\Phi_{15}] are propagators. By [\Phi_i], we mean the family of \Phi_i, that is the primary field and all the descendant fields which can be built on \Phi_i.

**IV. Genus-two Partition Functions**

Everything is now ready for calculation of multiloop diagrams. As is clear from Eq.(20), in the genus-two p.f.s., whenever \Phi_8 is the field which circulate in the loops, only the fields \Phi_1, \Phi_8, \Phi_{10} or \Phi_{15} can propagate between the two loops (Fig.2). (From now on, whenever we do not specify the fields of the loops, we mean that they are \Phi_8).

For \Phi_i = \Phi_1 (Fig.3), the calculation of the p.f. is straightforward. As was extensively discussed in Refs. [9-11], the ZHP and NZHP behaviors of the G.f.s. are very important for the calculation of the correlators of a conformal field theory on HGRS’s. In particular,
in the cases which the identity operator is propagator, these limits determine the p.fs. uniquely. Therefore we can find the p.f. of Fig.3 as:

\[
\hat{Z}_{88}^1(\Omega) = (\det \text{Im} \ \Omega)^2 \left| \Theta \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} (01\Omega) \right|^8,
\]

where by the index \(i\) in \(\hat{Z}_i\), we mean that the propagator of the p.f. (in Fig.2) is \(\Phi_i\). In Eq.(21) \(\Omega\) is the period matrix of the Riemann surface and \(\Theta\) are the genus-two Riemann theta functions (Eq. A.5). It can be shown that Eq. (21) behaves correctly under ZHP and NZHP limits, that is, it leads in these limits to:

\[
\hat{Z}_{88}^1(\Omega) \xrightarrow{ZHP} \hat{Z}_8(\tau_1)\hat{Z}_8(\tau_2) + O(t)
\]

where \(t = \frac{1}{2}\), it is obvious that Eq.(21) behaves as predicted by Eqs. (A.2) and (A.3).

In this way we could find one of the p.fs. of Fig.(2). Now what about the others. The main reason that we could write Eq.(21), was its easy ZHP behaviour (Eq. A.2), and this came from the fact that in the corresponding diagram, the propagator is the scalar field \(\Phi_1\). But this is not true for other diagrams and we must look for other techniques to calculate them.

The method that we are going to apply it to the remaining diagrams, is based on the using the modular transformation properties and crossing symmetry of Green functions. So let's begin with the definition of the generators of the modular transformations. As is known, the Dehn twist about the cycle \(c\), \(D_c\), is equivalent to cutting the surface along the cycle \(c\), rotating one cut end by \(2\pi\), and then glueing the surface back together again. Now if we choose a homology basis \((a_i, b_i)\), then it can be shown that \([22]\) the generators of modular transformation are in a 1-1 correspondence with the set of Dehn twists about the cycles \(a_i, \ a_i^{-1}a_{i+1}\) and \(\prod_{i=1}^{q} a_i^{-1}b_ia_i^{-1}\).

Now consider the genus-two diagram of Fig.4. Cutting this surface along the \(a\) cycles, hence transforming it to a sphere with four puncture, and using the crossing symmetry (which interchanges the \(t\) and \(s\) channels) we arrive at Fig.5. The sum over \(\ell\) is restricted by fusion rule. Next consider the cycle \(a_1^{-1}a_2\) of Fig.4 (which is the same as the cycle \(c\) in Fig.5) and let's apply the Dehn twist \(D_c\) in both sides of Fig.5. There is a crucial difference
between these two twists. In the LHS, there is no specific field which propagates along the cycle $c$, and therefore $D_c$ is nontrivial. But in the RHS, there is such a field, $\Phi_\ell$, and therefore $D_c$ is trivial and only produce the phase factor $e^{2\pi i h_\ell}$. In the following, we will use the above mentioned points to find the unknown diagrams.

Consider the diagram of Fig.3. If we cut the loops, we find a four-point function which can be expanded as follows:

\begin{equation}
\hat{Z}_{88}^1(\Omega) = \sum_{\ell=1,8,10,15} c_\ell(\Omega) \hat{Z}_{88}^\ell(\Omega) \tag{24}
\end{equation}

The propagators in the RHS is determined by fusion rule. Note that at this stage, and before any twisting, it is clear that $c_1 = 1$ and $c_8 = c_{10} = 0$. If we again glue the external legs of Eq.(23), we find the following relation between the p.fs.

Now let's twisting both sides about the cycle $c$ ($D_c$). Any modular transformation is specified by four $g \times g$ matrices $A, B, C$ and $D$ which determine the transformation of the period matrix

\begin{equation}
\Omega \rightarrow \widetilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1} \tag{25}
\end{equation}

In the case of $D_{a_1^{-1}a_2}$ these matrices are (in $g = 2$):

\begin{equation*}
A = D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, C = 0.
\end{equation*}

By Using the above matrices, and Eq.(A.8), we find that $\hat{Z}_{88}^1(\Omega)$ is transformed as (by Eq.(21)):

\begin{equation}
\hat{Z}_{88}^1(\Omega) \xrightarrow{D_{a_1^{-1}a_2}} \frac{(det \Im \Omega)^2}{4} |\Theta \left( \begin{array}{cc} 1/2 & 1/2 \\ 1/2 & 1/2 \end{array} \right) (0|\Omega) |^8 \tag{26}
\end{equation}

The transormation of the RHS of Eq.(24) will only leads to an $e^{2\pi i h_\ell}$ phase factor, and therefore Eq.(24) leads to

\begin{equation*}
\frac{(det \Im \Omega)^2}{4} |\Theta \left( \begin{array}{cc} 1/2 & 1/2 \\ 1/2 & 1/2 \end{array} \right) (0|\Omega) |^8 = \sum_\ell c'_\ell e^{2\pi i h_\ell} \hat{Z}_{88}^\ell(\Omega)
\end{equation*}
\[ \hat{Z}_{88}^1 = c'_1 \hat{Z}_{88}^1 - c'_8 \hat{Z}_{88}^8 + c'_{10} \hat{Z}_{88}^{10} + c'_{10} \hat{Z}_{88}^{\overline{10}}. \] (27)

To determine the coefficients \( c'_\ell \), we study the ZHP behavior of both sides of Eq.(27). Note that both sides must behave identically. From Eq.(A.6) one can easily verify that in the ZHP limit, the pinching parameter \( t \) appears in the LHS of Eq.(27) with the power \( t^8 \). On the other hand, in the ZHP limit of a p.f. of a primary field with conformal weight \( h \), \( t \) appears with the power \( (2h + n) \), \( (n = 0, 1, 2, \cdots) \), where \( n \) denotes the level of its descendants (we have supposed the left-right symmetry for p.f. i.e. \( h = \overline{h} \)). Therefore one finds that Eq.(27) can be written in this limit as:

\[ t^8 \sim c'_o (t^o + t^2 + \cdots) + c'_8 (t^1 + t^3 + \cdots) + c'_{10} (t^2 + t^4 + \cdots) + c'_{10} (t^2 + t^4 + \cdots) \] (28)

where we have suppressed the coefficient of \( t^n \)'s for simplicity. Using the fact that the ZHP limit of \( \hat{Z}_{88}^1 \) appears as \( t^o + t^2 + \cdots \) (Eq.22), the above relation leads:

\[ c'_o = c'_8 = 0 \] (29)

To determine \( c'_{10} \) and \( c'_{10} \), we use the fact that the representations 10 and \( \overline{10} \) appears symmetricaly in Eqs.(4,12.b and 20) and therefore \( c'_{10} = c'_{10} \) and \( \hat{Z}_{88}^{10} = \hat{Z}_{88}^{\overline{10}} \). Without loss of generality we set \( c'_{10} = c'_{10} = 1/2 \) and finally:

\[ \hat{Z}_{88}^{10} = \hat{Z}_{88}^{\overline{10}} = \frac{(det \text{ Im } \Omega)^2}{4} \left\{|0\rangle \langle 0| \right\} (0|\Omega|^8) \] (30)

Let us convince ourselves, why in the ZHP limit, the leading terms of \( \hat{Z}_{88}^{10} \) and \( \hat{Z}_{88}^{\overline{10}} \) is \( t^8 \). This means, with the help of Eq.(A.1), that the first nonvanishing one-point function of \( < \Phi_{10} > \), On the torus, appears at level three. From the general properties of the Kac-Moody algebras, one can show the following condition for the one-point functions of the descendant fields\[^{[10]}\]:

\[ < 0|J_{-m_1}^{\beta_1} \cdots J_{-m_k}^{\beta_k} \Phi_{\Lambda}\rangle |0> = 0 \quad \text{unless} \quad \lambda + \sum_{i=1}^k \beta_i = 0, \] (31)

where \( \lambda \) is a specific weight of a representation \( \Lambda \). This relation can help us to interpret our results.

In our case \( \Lambda = 10 \). Suppose that \( \lambda \) is the highest weight of this representation, that is \( \lambda = 3\nu_1 \) (\( \nu_1 \) is the highest weight of representation 3, \( i.e., R_3 \)). Therefore the selection rule (31) implies that the G.f.s. is zero, unless \( \sum_{i=1}^k \beta_i = -3\nu_1 \). One of the solution of
this relation is $\beta_1 = \beta_2 = -\alpha_1$ and $\beta_2 = -\alpha_2$, where $\alpha_i$'s are the simple roots of $SU(3)$. Hence we encounter the descendant fields of the form $J_{-m_1} J_{-m_2} J_{-m_3} \Phi_{10}^{3\nu}$. These fields have conformal weights $1 + m_1 + m_2 + m_3$. As the minimum value of $m_i$'s are one, so the minimum level of the non-zero one-point function of the descendant fields of $\Phi_{10}$ is three. This is exactly the same result that we found in Eq.(18), that is the first non-zero one-point function of $\Phi_{10}$ (with minimum conformal weight) appear at level three.

There is yet a remaining question. In our analysis, we only consider a special set of solutions, that is $\lambda = 3\nu$, and a special decomposition of $\sum \beta_i$ (in terms of the roots). How we can use this, to explain the general result of Eq.(28), which does not depend on any specific weight and root of descendant fields. The answer is: There is some indications that the G.f. of descendant $J_{-m_1} \cdots J_{-m_k} \Phi_{\lambda}$ are equal, as long as $\lambda + \sum_{i=1}^k \beta_i$ and $\sum_{i=1}^k m_i$ are the same. So our result about the necessity of level three, is not restricted to the above case which we considered, and is correct for the general case.

Let us conclude this section with some comments about the information that are contained in the second part of Eq.(12), that is (12.b). Unfortunately we can not separate out the $Z_{1\overline{10}}$, $Z_{10\overline{10}}$ and $Z_{1\overline{10}}$ from this equation and therefore it only helps us to find a combination of p.f.s. On HGRS’s. For instance, consider the genus-two p.f. It is obvious from Eq.(20) that in the diagrams in which the fields $\Phi_1$, $\Phi_{10}$ and $\Phi_{\overline{10}}$ are their loop fields, only the field $\Phi_1$ can propagate between successive loops. This makes the calculation of these diagrams doable. Using a similar reasoning which led to Eq.(21), we can find the following sum of p.f.s.:

$$\sum_{i,j=1,10,10} \hat{Z}_{ij}^1(\Omega) = \frac{(\text{det} \text{ Im } \Omega)^2}{4}$$

$$\left\{ \Theta^4 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \Theta^4 \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix} + \Theta^4 \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix} + \Theta^4 \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix} \right\} (0|\Omega)$$

(32)

It can be easily checked that the above relation will produce the correct ZHP behavior, that is

$$\sum_{i,j=1,10,10} \hat{Z}_{ij}^1(\Omega) \xrightarrow{ZHP} \sum_{i=1,10,\overline{10}} \hat{Z}_i(\tau_1) \sum_{j=1,10,\overline{10}} \hat{Z}_j(\tau_2)$$

The generalization of this combination to HGRS’s, that is $\sum_{i_1,\ldots,i_g=1,10,\overline{10}} \hat{Z}_{i_1,\ldots,i_g}^{1\ldots g}(\Omega_g)$, can be found in a similar way.

V. Higher-Genus Correlators
By similar techniques which led to Eq.(30), one can compute the higher-genus p.fs.
We first consider the genus-three, in detail, and then give the results for arbitrary genus
p.fs. Hereafter, we will only consider the diagrams with \( \Phi_1, \Phi_{10} \) and \( \Phi_{10} \) as propagator
(and also \( \Phi_8 \) as the loops fields).

In genus three there are nine diagrams of this type (Fig.6). In five of them, there is,
at least, one \( \Phi_1 \) propagator and the resulting p.fs. can be written by considering its ZHP
behavior. For example

\[
\hat{Z}_{888}^{10,1} = \frac{\left(\det \text{Im} \Omega\right)^2}{2^3} \left| \Theta \left[ \begin{array}{ccc} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{array} \right] \right| \left(0|\Omega\right)^8 \tag{33}
\]

In the remaining four diagrams both of the propagators are 10 or \( \overline{10} \), and these must
be calculated in the same way which led to Eq.(30). But thanks to the equality of the
contributions of 10 and \( \overline{10} \), all these four diagrams reduces to \( \hat{Z}_{888}^{10,10} \) and only this p.f.
must be calculated.

Consider \( \hat{Z}_{888}^{10,1} \) and apply the previous procedure to one of its loops:

\[
\Theta \left[ \begin{array}{ccc} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{array} \right] \xrightarrow{D_{a_2^{-1}a_3}} \Theta \left[ \begin{array}{ccc} 1/2 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{array} \right], \tag{34}
\]

or:

\[
\Theta \left[ \begin{array}{ccc} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{array} \right] \xrightarrow{D_{a_2^{-1}a_3}} \Theta \left[ \begin{array}{ccc} 1/2 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{array} \right].
\]

Obviously before any Dehn twisting, we have \( c_1 = 1 \) and \( c_i = 0(i \neq 1) \). Now lets Dehn
twist about the cycle \( a_2^{-1}a_3 \). The LHS of (34) changes as

\[
\Theta \left[ \begin{array}{ccc} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{array} \right] \xrightarrow{D_{a_2^{-1}a_3}} \Theta \left[ \begin{array}{ccc} 1/2 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{array} \right], \tag{35}
\]
and the RHS changes (trivially)

\[ \sum c'_\ell e^{2\pi i h_\ell} \]

By considering the ZHP behavior of RHS of (33), and noting that the pinching parameter appear with the power \(4 + 4\), one can prove that \(c'_1 = c'_8 = 0\), and as \(c'_{10} = c'_{10}\), one finds

\[ \hat{Z}_{888}^{10,10}(\Omega) = \frac{(\det \text{Im} \Omega)^2}{2^3} \left[ \begin{array}{ccc} 1/2 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{array} \right] (0|\Omega)^8 \] (36)

This method can be similarly applied to the general higher-genus cases and the final result for the genus-\(g\) partition function of Fig.7 is:

\[ \hat{Z}^m(\Omega_g) = A_g \left| \Theta \left[ \begin{array}{c} \vec{\alpha} \\ \vec{\beta} \end{array} \right] (0|\Omega_g)^8 \right| \] (37)

where \(A_g = \frac{(\det \text{Im} \Omega_g)^2}{2^g}\) and \(\vec{\alpha}, \vec{\beta}\) and \(m\) are \(g, g\) and \(g + 1\) component objects

\[ \alpha_k = \frac{1}{2}, \quad k = 1, 2, \ldots, g \\
\beta_k = \frac{1}{2}(1 - \delta_{m_{k-1}, m_k}), \quad k = 1, 2, \ldots, g \\
\text{and} \quad m = (1, m_1, \ldots, m_{g-1}, 1). \] (38)

Note that \(m_i\) specifies the propagators (as shown in Fig.7) and takes the value 1, 10 and \(\overline{10}\). Note also that \(\delta_{10,\overline{10}}\) is to be set equal to one (due to the symmetry of these representations).

To find the \(n\)-point functions, it is necessary to study the NZHP limits of Eq.(37). Various factors of this equation behave under NZHP of \(g\)-th loop as following

\[ Z_B(\Omega_g)^{NZHP} \rightarrow Z_B(\Omega_{g-1})(\text{Im} \ln t)^{-1/2}|t|^{-1/12} \] (39)

\[ \det \text{Im} \Omega_g^{NZHP} \rightarrow (\det \text{Im} \Omega_{g-1})(\text{Im} \ln t) \] (39-b)

\[ Z^m(\Omega_g)^{NZHP} \rightarrow Z^c(\Omega_{g-1})|t|^{1/2+1/2-c/24-c/24} \left[ \frac{2\Theta \left[ \begin{array}{c} \vec{\alpha} \\ \vec{\beta} \end{array} \right] (\frac{1}{2} \int_{\mathbb{C}} \mathbf{u}|\Omega_{g-1})}{E^{1/4}(z, w)} \right]^8 \] (39-c)

Where in the last equation, (39-c), use has been made of the first two one and also Eq.(A.7). In the above relations \(c\) is centred charge, \(\vec{\alpha}\) and \(\vec{\beta}\) have \(g - 1\) components and
$E(z, w)$ is prime form on genus-$g$ Riemann surface $(\Sigma_g). \mathbf{u} = (u_1, \cdots, u_g)$ are differential one-forms on $\Sigma_g$. Eq.(39-c) shows that Eq.(37) behaves correctly under NZHP. This is obvious by Eq. (A.3) and note that $h_8 = 1/2$ (on cylinder $h \to h - c/24$). Therefore we will find the two-point function of $\Phi_8 - \Phi_8$ on $\Sigma_g$ as (Fig.8):

$$<\Phi_8(z)\Phi_8(w)> \mathbf{m} (\Omega_g) = 2^7 A_g \left| \Theta \left( \frac{\alpha}{\beta} \right) \left( \frac{1}{2} \int_w \mathbf{u} |\Omega_g \right) \right|^8 E^{1/4}(z, w)$$

(40)

$$<< \cdots >> = \frac{<\cdots>} {Z_h^B}$$

(41)

and $\mathbf{m} = (1, m_1, \cdots, m_g)$. On can repeat this NZHP technique for one the loops of Fig.8, and after obtaining the correct power for $t$, find the four-point function and again continue this procedure. In this way, one may find the $2n$-point function of $\Phi_8$ as following (Fig.9)

$$<\prod_{i=1}^n \Phi_8(z_i)\Phi_8(w_i)> \mathbf{m} (\Omega_g) = 2^{8-n} A_g \left| \xi_n \left[ \frac{\alpha}{\beta} \right] \left( \prod_{j=1}^n E^{1/4}(z_j, w_j) \right) \right|^8$$

(42)

where $\mathbf{m} = (1, m_1, \cdots, m_g)$ and

$$\xi_n \left[ \frac{\alpha}{\beta} \right] = \prod_{p=2}^n (1 + d_p) \prod_{i,j=1}^n R_{ij}^{\gamma_{ij}} \Theta \left( \frac{\alpha}{\beta} \right) \left( \sum_{i=1}^n \gamma_i \int_{w_i} u_i |\Omega_g \right).$$

(43)

The index $i$ in $\gamma_i, z_i$ and $w_i$ refers to the $i$th external leg (with arguments $z_i$ and $w_i$ in Fig.9), and all $\gamma_i$’s are 1/2. $d_p$ is an operator which acts on any arbitrary function of $\gamma_p$ as following

$$d_p f(\gamma_p) = \exp[\pi i (1 - \delta_{m_p,n-p,m_p+n-p+1})] f(-\gamma_p),$$

(44)

and $R_{ij}$ is defined as follows:

$$R_{ij} = \frac{E(z_i, z_j)E(w_i, w_j)}{E(z_i, w_j)E(w_i z_j)}$$

(45)

Finally we will describe the method of calculating the odd-point functions. with the help of Eqs.(A.6) and (11), the ZHP limit of Eq.(30) will leads to:

$$Z_{88}^{10} \overset{ZHHP}{\longrightarrow} 2^7 |\pi \eta^2(\tau_1) u_1(p_1)|^8 \ |t|^{4+4} \ 2^7 \ |\pi \eta^2(\tau_2) u_2(p_2)|^8$$

(46)
$u_1(u_2)$ is the holomorphic differential one-forms on the torus $T_1(T_2)$ (which is the result of pinching of $g = 2$ surface) and $p_1(p_2)$ is the point which is created on $T_1(T_2)$ by ZHP. By using Eqs.(46) and (A.1), One can read the following one-point function of $\Phi_{10}(z)$ on torus (Fig. 10):

$$<\Phi_{10}(z)>(\tau) = 2^7|\pi\eta^2(\tau)u(z)|^8$$

(47)

This is the one-point function of third-level descendant fields of $\Phi_{10}$ (as explained in the previous section).

Next consider the diagram of Fig.7 when $m_{g-1} = 10$, and pinch the cycle $a_g$. In this way we will find the correct power for $|t|$ (that is $|t|^{4+4}$) and the resulting relation becomes the product of the $<\Phi_{10}>(\tau)$ (Eq.(47)) and $<\Phi_{10}>(\Omega_{g-1})$. So we will find the $<\Phi_{10}>(\Omega_g)$ on HGRS’s as (Fig. 11)

$$<<\Phi_{10}(z)>>_{m}(\Omega_g) = Ag\left|\sum_{i=1}^{g} u_i(z)\partial_j \Theta \left[\frac{\bar{\alpha}}{\beta}\right] \Omega_g \right|^8$$

(48)

where $m = (1,m_1,\cdots,m_{g-1},10)$.

By the further ZHP and NZHP of the previous G.fs., one could find all the Green function in which $\Phi_8$ is their loop fields, $\Phi_1$, $\Phi_{10}$ and $\Phi_{10}$ are their propagators and $\Phi_8, \Phi_{10}$ or $\Phi_{10}$ are external legs.

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Appendix

In this section, we will summarize some properties of the Green functions and theta function on HGRS’s which have been extensively used in the paper. First we will review the factorization properties of correlators.

The factorization property of the genus-$g$ partition function was studied by Friedan and Shenker$^{[23]}$. On a surface $\Sigma_{g,k}$ (a surface which becomes, on removal of some node, two disconnected surfaces one of genus $k$ and one of genus $g-k$, each with one puncture) in the limit of $t \longrightarrow 0$ the partition function becomes:

$$Z_g \xrightarrow{\text{ZHP}} <\phi(x_1)>_k<\phi(x_2)>_{g-k} |t|^{h+\tilde{h}}$$

(A.1)
$\phi(x) >_g$ is the normalized one-point function on $\Sigma_g$, $\phi$ is the field that propagates at that node and $t$ is the pinching parameter. In the case that the propagator is $\phi_0$, it reduces to:

$$Z_g \xrightarrow{\text{ZHP}} Z_k Z_{g-k}$$  \hspace{1cm} (A.2)

Under NZHP the partition function becomes $[21]$:

$$Z_g \xrightarrow{\text{NZHP}} <\phi(x_1)\tilde{\phi}(x_2) >_{g-1} |t|^{h + \bar{h}}$$  \hspace{1cm} (A.3)

$\phi$ is the field that propagates along the pinched handle. When this field is $\phi_0$ Eq.(A.3) reduces to:

$$Z_g \xrightarrow{\text{NZHP}} Z_{g-1}$$  \hspace{1cm} (A.4)

Next we need to know the ZHP and NZHP behaviors of theta functions. The Riemann theta function is defined as:

$$\Theta \left[ \frac{\alpha}{\beta} \right] (z|\Omega) = \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \left[ \frac{1}{2} (n + \alpha)^t \Omega (n + \text{vec} \alpha) + (n + \alpha)^t (z + \beta) \right]$$  \hspace{1cm} (A.5)

where $\alpha, \beta$ and $z$ are $g$-components objects, $\alpha \text{pha}$ and $\beta$ belongs to $\{0, \frac{1}{2}\}^g$ and $z \in C^g$. Under ZHP, $\sum_g$ divide to $\Sigma_1$ and $\Sigma_2$ (with $g = g_1 + g_2$). Then $[24]$:

$$\Theta \left[ \frac{\alpha}{\beta} \right] (\int_w v|\Omega_g)^{\text{ZHP}} \Theta \left[ \frac{\alpha_1}{\beta_1} \right] (\int_w u|\Omega_1) \Theta \left[ \frac{\alpha_2}{\beta_2} \right] (0|\Omega_2)$$

$$+ t\omega_{z-w}(p_1) \left[ \sum_{i=1}^{g_1} u_i(p_1) \partial_i + \sum_{j=g_1+1}^{g_1+g_2} u_j(p_2) \partial_j \right] \Theta \left[ \frac{\alpha}{\beta} \right] (\int_w u|\Omega_1) \Theta \left[ \frac{\alpha_2}{\beta_2} \right] (0|\Omega_2) + O(t^2)$$  \hspace{1cm} (A.6)

In the above equation $v_i, u_i$ and $u_j$ are holomorphic differential one-forms on $\Sigma_g$, $\Sigma_1$ and $\Sigma_2$, respectively and $\left[ \frac{\alpha}{\beta} \right] = \left[ \frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2} \right]$. $p_1(p_2)$ is the point that is generated on $\Sigma_1(\Sigma_2)$ by ZHP, $z$ and $w$ are points on $\Sigma_1$. $\omega_{z-w}(p_1)$ is defined as:

$$\omega_{z-w}(x) = \partial_x \ln \frac{E(z, x)}{E(w, x)}$$

Under NZHP $\Sigma_{g+1}$ transforms to $\Sigma_g$ and Eq.(A.5) reduces to $[24]$:
\[ \Theta \left[ \frac{\alpha'}{\beta'} \right] \left( \int_{w_1}^{z_1} v | \Omega_{g+1} \right) \frac{N_{ZHP}}{\delta_{\alpha_{g+1},0}} \left( \Theta \left[ \frac{\alpha}{\beta} \right] \left( \int_{w_1}^{z_1} u | \Omega_g \right) + \frac{t^2}{E(z_2, w_2)} e^{2\pi i \beta g+1} \left[ R_{12} \Theta \left[ \frac{\alpha}{\beta} \right] \left( \int_{w_1}^{z_1} u + \frac{1}{2} \int_{w_2}^{z_2} u | \Omega_g \right) + R_{12}^{-1} \Theta \left[ \frac{\alpha}{\beta} \right] \left( \int_{w_1}^{z_1} u - \frac{1}{2} \int_{w_2}^{z_2} u | \Omega_g \right) \right] + \delta_{\alpha_{g+1},\frac{t^2}{E(z_2, w_2)} e^{2\pi i \beta g+1} \left( R_{12} \Theta \left[ \frac{\alpha}{\beta} \right] \left( \int_{w_1}^{z_1} u + \frac{1}{2} \int_{w_2}^{z_2} u | \Omega_g \right) + R_{12}^{-1} \Theta \left[ \frac{\alpha}{\beta} \right] \left( \int_{w_1}^{z_1} u - \frac{1}{2} \int_{w_2}^{z_2} u | \Omega_g \right) \right] \right) \right) \] (A.7)

In the above equation \( v_i \) and \( u_i \) are one-forms on \( \Sigma_{g+1} \) and \( \Sigma_g \), \( z_2 \) and \( w_2 \) are the two new points that are created on \( \Sigma_g \) (under NZHP), and \( \left[ \frac{\alpha'}{\beta'} \right] = \left[ \frac{\alpha}{\beta} \right] \left( \Omega_{g+1}, 0 \right) \). \( R_{12} \) is defined in Eq.(45).

The other relations that we need, is the modular transformation properties of the theta functions. One can show that under a general transformation of the period matrix, Eq.(25), \( \det \text{ Im} \Omega \) and theta function transformed as \([24]\):

\[ \det \text{ Im} \Omega \rightarrow \left| \det (C \Omega + D) \right|^{-2} \left( \det \text{ Im} \Omega \right) \]

\[ \Theta \left[ \frac{\alpha'}{\beta'} \right] \left( 0 | \Omega \right) \rightarrow e^{-i \pi \varphi} \det^{1/2} (C \Omega + D) \Theta \left[ \frac{\alpha'}{\beta'} \right] \left( 0 | \Omega \right) \] (A.8)

where \( \left[ \frac{\alpha'}{\beta'} \right] \) is specified as:

\[ \left( \begin{array}{c} \alpha' \\ \beta' \end{array} \right) = \left( \begin{array}{cc} D & -C \\ -B & A \end{array} \right) \left( \begin{array}{c} \alpha' \\ \beta' \end{array} \right) + \frac{1}{2} \left( \begin{array}{c} (CD)_d \\ (AB)_d \end{array} \right) \] (A.9)

In the above relation \((M)_d\) is meant a column matrix built from the diagonal elements of \( M \). In Eq.(A.8), \( \varphi \) is a phase which depends on \( \varphi = \varphi(A, B, C, D, \alpha, \beta) \).

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**Figure Captions**

Fig.1: The fusion rule $\Phi_i \times \Phi_i = \sum_j \Phi_j$ shows which fields appear as propagator of $\Phi_i$-loop ($\Phi_i$ stands for conjugate representation of $\Phi_i$).

Fig.2: Genus-two partition function of $\Phi_8$. In this diagram $\Phi_i$ is $\Phi_1, \Phi_8, \Phi_{10}$ or $\Phi_{\bar{10}}$.

Fig.3: One of the genus-two p.f.s. of $\Phi_8$.

Fig.4: The cycle $a_1^{-1}a_2$ in genus-two.

Fig.5: Expanding the LHS diagram in terms of the RHS diagrams (crossing symmetry).

Fig.6: Genus-three partition function of $\Phi_8$. $\Phi_i$ and $\Phi_j$ are $\Phi_1, \Phi_{10}$ or $\Phi_{\bar{10}}$

Fig.7: A typical diagram of $SU(3)_3$ partition function on HGRS’s. $m_i$ are 1, 10 or $\bar{10}$

Fig.8: Two-point $\Phi_8 - \Phi_8$ correlator on HGRS’s.

Fig.9: $2n$-point $\Pi_{i=1}^n (\Phi_8(z_i) - \Phi_8(w_i))$ correlator on HGRS’s

Fig.10: Correlator of $\langle \Phi_{10}(z) \rangle$ on the torus.

Fig.11: One-point function of $\Phi_{10}(z)$ on HGRS’s