Backward problems in time for fractional diffusion-wave equation

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Abstract
In this article, for a time-fractional diffusion-wave equation \( \partial_t^\alpha u(x, t) = -Au(x, t) \), \( 0 < t < T \) with fractional order \( \alpha \in (1, 2) \), we consider the backward problem in time: determine \( u(\cdot, t), 0 < t < T \) by \( u(\cdot, T) \) and \( \partial_t u(\cdot, T) \). We prove that there exists a countably infinite set \( \Lambda \subset (0, \infty) \) with a unique accumulation point 0 such that the backward problem is well-posed for \( T \notin \Lambda \).

Keywords: backward problem, fractional diffusion-wave equation, well-posedness

1. Introduction and main results
Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) with sufficiently smooth boundary \( \partial \Omega \). We consider a fractional differential equation:
\[
\partial_t^\alpha u(x, t) = -Au(x, t), \quad x \in \Omega, \quad 0 < t < T,
\] (1.1)
where \(-A\) is a uniformly elliptic operator. Henceforth for \( n - 1 < \alpha < n \) with \( n \in \mathbb{N} \), we define the Caputo derivative by
\[
\partial_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} g(s) ds.
\]
For \( \alpha \in (0, 1) \cup (1, 2) \), equation (1.1) is widely studied not only by mathematical interests but also for the modeling of various types of diffusion phenomena of substances such as...
contaminants in heterogeneous media. Among them, we particularly refer to the anomalous diffusion which cannot be modeled by a classical advection–diffusion equation which corresponds to \( \alpha = 1 \). More precisely, field data of diffusion of e.g., contaminants in soil often indicate long-tailed profiles, which cannot be interpreted by a classical advection–diffusion equation whose solution decays very fast, i.e., exponentially. In particular, for the case \( 1 < \alpha < 2 \), equation (1.1) is called a fractional diffusion-wave equation, and is considered related to a fractal dimension of the media (e.g., Metzler, Glöckle and Nonnenmacher [14]). We refer to Mainardi [13], Nigmatullin [15] as a few early works concerning other physical backgrounds.

There are tremendously many works on mathematical analysis and here we are strongly limited to some references. As for the well-posedness of the initial boundary value problem for (1.1), we refer to Kubica, Ryszewska and Yamamoto [7], Kubica and Yamamoto [8], Sakamoto and Yamamoto [17], Zacher [28], and for inverse problems and related topics the readers can consult the handbook chapters Li, Liu and Yamamoto [9], Li and Yamamoto [10], Liu, Li and Yamamoto [12].

A solution to equation (1.1) with \( \alpha \neq 1 \) shows behavior which is essentially different from the case of \( \alpha = 1 \) and can characterize the anomaly of the diffusion in the heterogeneous media. Among such characteristic properties, the backward stability in time is important and this is the main subject of this article. In the case of \( \alpha = 1 \), the classical diffusion equation possesses the strong smoothing property, so that we cannot solve the equation with final value condition, and cannot have good stability but with given a priori bound assumptions, one can prove only conditional stability of logarithmic type (e.g., Imanuvilov and Yamamoto [5], Isakov [6], section 9 in Yamamoto [26]).

In the case of \( 0 < \alpha < 1 \), Sakamoto and Yamamoto [17] established the well-posedness of the backward problem in time with data \( u(\cdot, T) \in H^2(\Omega) \cap H^1_0(\Omega) \) for finding \( u(\cdot, 0) \in L^2(\Omega) \). After [17], as for \( 0 < \alpha < 1 \), there have been many theoretical and numerical works on the backward problems, and here we can refer to Floridia, Li and Yamamoto [3], Liu and Yamamoto [11], Tuan, Huynh, Ngoc and Zhou [18], Tuan, Lung and Tatar [19], Tuan, Thach, O’Regan and Can [20], Wang, Wei and Zhou [21], Wang and Liu [22], Wei and Wang [23], Xiong, Wang and Li [25], Yang and Liu [27], and we do not intend comprehensive references.

However, to the best knowledge of the authors, except for Wei and Zhang [24], there are still no works on the backward problem as long as the case \( 1 < \alpha < 2 \) is concerned, although the case \( 1 < \alpha < 2 \) is used for the modeling, and we can expect that the backward well-posedness may indicate an intermediate character between the cases \( \alpha = 1 \) and \( \alpha = 2 \).

The purpose of this article is to sharpen the stability and the uniqueness, which improves the theoretical achievements of [24] for the backward problem for the case of \( 1 < \alpha < 2 \). Since the backward problems are related to several real world problems such as identification of initial density distribution of the current contamination by density data at the present time, numerical reconstruction methods are important. However the numerical approach is out of the scope of this article, and our stability and uniqueness results should be the theoretical foundation for future researches for reasonable numerical reconstruction methods.

For the formulation of the problem, we introduce an operator and function spaces. We assume that all functions under consideration are real-valued. Henceforth \( L^2(\Omega) \) and \( H^1(\Omega) \), \( H^2_0(\Omega), H^3_0(\Omega), \) etc denote the Lebesgue space and usual Sobolev spaces (e.g., Adams [1]), and by \( \| \cdot \|_X \) we denote the norm in the space \( X \). We set \( (a, b) = \int_{\Omega} a(x) b(x) \, dx \). Identifying the dual space \((L^2(\Omega))'\) with itself, we denote \( H^{-1}(\Omega) = (H^1_0(\Omega))' \) and \( H^{-2}(\Omega) = (H^2_0(\Omega))' \).
We set

$$-Av(x) = \sum_{i,j=1}^{d} \partial_i (a_{ij}(x) \partial_j v(x)) + c(x)v(x), \quad x \in \Omega$$

for \(v \in C^2(\Omega)\), where \(\partial_i = \frac{\partial}{\partial x_i}\) for \(1 \leq i \leq d\), \(a_{ij} = a_{ji} \in C^1(\overline{\Omega})\), \(c \in C(\overline{\Omega})\), \(c \leq 0\) on \(\overline{\Omega}\). Then we define an operator \(A\) in \(L^2(\Omega)\) by

$$Av = Av, \quad v \in \mathcal{D}(A) := H^2(\Omega) \cap H_0^1(\Omega).$$

In particular, \(v \in \mathcal{D}(A)\) means that \(v = 0\) on \(\partial \Omega\) in the sense of the trace (e.g., [1]). Then it is also known that the operator defined by \((1.2)\) has eigenvalues and we can number the set of all the eigenvalues:

$$0 < \mu_1 < \mu_2 < \ldots \rightarrow \infty.$$

Let \(\{\varphi_n\}_{1 \leq j \leq \ell_n}\) be an orthonormal basis of \(\text{Ker} (A - \mu_n)\). More precisely, \(\ell_n\) is the dimension of the eigenspace \(\text{Ker} (A - \mu_n)\) corresponding to the eigenvalue \(\mu_n\), \(n \in \mathbb{N}\), and \(A\varphi_n = \mu_n\varphi_n\) and \((\varphi_n, \varphi_m) = \delta_{nm}\delta_{ij}\) where we set \(\delta_{ij} = 1\) if \(i = j\) and \(\delta_{ij} = 0\) if \(i \neq j\). Then we see that \(\{\varphi_n; n \in \mathbb{N}, 1 \leq j \leq \ell_n\}\) is an orthonormal basis in \(L^2(\Omega)\), and

$$\|a\|_{L^2(\Omega)} = \left(\sum_{n=1}^{\infty} \sum_{j=1}^{\ell_n} |(a, \varphi_n)|^2\right)^{\frac{1}{2}}.$$

Throughout this article, we always assume

$$1 < \alpha < 2.$$

In terms of \(A\), we rewrite \((1.1)\) as

$$\begin{cases}
\partial_t^\alpha u(x, t) = -Au(x, t), \quad x \in \Omega, \quad t > 0, \\
u(x, 0) = a(x), \quad \partial_t u(x, 0) = b(x), \quad x \in \Omega, \\
u(\cdot, t) \in H_0^1(\Omega), \quad t > 0.
\end{cases}$$

By \(E_{\alpha, \beta}(z)\) we denote the Mittag–Leffler function:

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

with \(\alpha > 0\) and \(\beta \in \mathbb{C}, z \in \mathbb{C}\). It is known that \(E_{\alpha, \beta}(z)\) is an entire function in \(z \in \mathbb{C}\) (e.g., Gorenflo, Kilbas, Mainardi and Rogosin [4], Podlubny [16]).

Before stating the main results, we show the well-posedness and the regularity of the solution \(u_{a,b}\) to \((1.3)\) (e.g., [17]).

**Proposition 1.1.** Let \(a, b \in L^2(\Omega)\). Then there exists a unique solution \(u = u_{a,b}\) to \((1.3)\) such that

$$\begin{cases}
u_{a,b} \in C([0, T]; L^2(\Omega)) \cap C(0, T); H^2(\Omega) \cap H_0^1(\Omega), \\
\lim_{t \to 0} \|\nu(\cdot, t) - a\|_{L^2(\Omega)} = \lim_{t \to 0} \|\partial_t\nu(\cdot, t) - b\|_{H^2(\Omega)} = 0
\end{cases}$$
and

\[
\begin{align*}
    u(x,t) &= \sum_{n=1}^{\infty} \sum_{j=1}^{\ell_n} \{(a, \varphi_{nj})E_{\alpha,1}(-\mu_n t^\alpha) + (b, \varphi_{nj})\eta E_{\alpha,2}(-\mu_n t^\alpha)\} \varphi_{nj}(x) \\
    \partial_t u(x,t) &= \sum_{n=1}^{\infty} \sum_{j=1}^{\ell_n} \{-\mu_n t^{\alpha-1}(a, \varphi_{nj})E_{\alpha,\alpha}(-\mu_n t^\alpha) + (b, \varphi_{nj})E_{\alpha,1}(-\mu_n t^\alpha)\} \varphi_{nj}(x)
\end{align*}
\]

in \(C([0,T];L^2(\Omega))\).

Moreover, by (3.8) proved in section 3, we can see that

\[
u(\cdot, t), \partial_t u(\cdot, t) \in H^2(\Omega) \cap H^1_0(\Omega), \quad 0 < t \leq T.
\]

Now we formulate

**Backward problem:**

Let \(T > 0\) and \(a_T, b_T \in L^2(\Omega)\) be given. Then determine \(u = u(x,t)\) satisfying

\[
\begin{align*}
    \partial_t^\alpha u &= -Au, \quad x \in \Omega, \ t > 0, \\
    u(x, T) &= a_T(x), \quad \partial_t u(x, T) = b_T(x), \quad x \in \Omega, \\
    u(\cdot, t) &\in H^2(\Omega), \quad t > 0.
\end{align*}
\]

We set

\[
\psi(\eta) \equiv E_{\alpha,1}(-\eta^\alpha) + \eta E_{\alpha,2}(-\eta)E_{\alpha,\alpha}(-\eta), \quad \eta > 0.
\] (1.5)

By the definition of the Mittag–Leffler function, we have \(\psi(0) = 1\). The Mittag–Leffler functions \(E_{\alpha,\beta}(-\eta)\) can be represented explicitly only for special values \(\alpha, \beta\) (e.g., pp 17–18 in [16]). Such representations are extremely difficult for \(E_{\alpha,\alpha}(-\eta)\) with \(1 < \alpha < 2\) and \(\kappa = 1, 2, \alpha\), and accordingly also for \(\psi(\eta)\). Thus in order to study the function \(\psi(\eta)\), we have to rely on the asymptotic behavior of the Mittag–Leffler functions and we can prove

**Lemma 1.2.** The set \(\{\eta > 0; \psi(\eta) = 0\}\) is a non-empty and finite set.

We set

\[
\{\eta_1, \ldots, \eta_N\} = \{\eta > 0; \psi(\eta) = 0\}
\] (1.6)

with \(\eta_1 < \cdots < \eta_N\). We have no information of the number \(N\) of the zeros of \(\psi\), except that it exists. In lemma 4.1 in section 4, we will provide an upper bound of the largest zero \(\eta_N\).

Now we are ready to state our main result:

**Theorem 1.3.**

(a) We assume

\[
T \notin \bigcup_{n=1}^{\infty} \left\{ \left( \frac{\eta_1}{\mu_n} \right)^{\frac{1}{\alpha}}, \ldots, \left( \frac{\eta_N}{\mu_n} \right)^{\frac{1}{\alpha}} \right\}.
\] (1.7)
Then for any $a_T, b_T \in H^2(\Omega) \cap H^1_0(\Omega)$, there exist $a, b \in L^2(\Omega)$ such that the solution $u_{a,b}$ to (1.3) satisfies
\begin{align}
  u_{a,b}(\cdot, T) &= a_T, \\
  \partial_T u_{a,b}(\cdot, T) &= b_T. 
\end{align}
(1.8)

Moreover there exists a constant $C > 0$ such that
\begin{align}
  C^{-1}(\|a_T\|_{H^2(\Omega)} + \|b_T\|_{H^2(\Omega)}) \leq \|a\|_{L^2(\Omega)} + \|b\|_{L^2(\Omega)} \\
  \leq C(\|a_T\|_{H^2(\Omega)} + \|b_T\|_{H^2(\Omega)}) 
\end{align}
(1.9)
for all $a_T, b_T \in H^2(\Omega) \cap H^1_0(\Omega)$.

(b) We assume
\begin{align}
  T \in \bigcup_{n=1}^{\infty} \left\{ \left( \frac{\eta_1}{\mu_n} \right)^{\frac{1}{\ell}}, \ldots, \left( \frac{\eta_N}{\mu_n} \right)^{\frac{1}{\ell}} \right\}. 
\end{align}
(1.10)
Then there exists $(a, b) \neq (0, 0)$ in $\Omega$ such that $(u_{a,b}(\cdot, T), \partial_T u_{a,b}(\cdot, T)) \equiv (0, 0)$ in $\Omega$. Furthermore, we assume that $(u_{a,b}(\cdot, T), \partial_T u_{a,b}(\cdot, T)) \equiv (0, 0)$ in $\Omega$. Then for $n_0 \in \mathbb{N}$, we have
\begin{align}
  (a, \varphi_{n_0}) = (b, \varphi_{n_0}) = 0, \quad 1 \leq j \leq n_0 
\end{align}
if
\begin{align}
  T \notin \left\{ \left( \frac{\eta_1}{\mu_0} \right)^{\frac{1}{\ell}}, \ldots, \left( \frac{\eta_N}{\mu_0} \right)^{\frac{1}{\ell}} \right\}. 
\end{align}
Henceforth we set
\begin{align}
  \Lambda = \Lambda(\alpha, A) := \bigcup_{n=1}^{\infty} \left\{ \left( \frac{\eta_1}{\mu_n} \right)^{\frac{1}{\ell}}, \ldots, \left( \frac{\eta_N}{\mu_n} \right)^{\frac{1}{\ell}} \right\}. 
\end{align}
We note that $\Lambda$ is a countably infinite set. Theorem 1.3 (a) implies that the backward problem in time for $1 < \alpha < 2$, is well-posed if $T \notin \Lambda$. The part (b) means that we cannot determine the $\varphi_{n_0}$-components of initial values where $n_0 \in \mathbb{N}$ for which $T \in \left\{ \left( \frac{\eta_1}{\mu_0} \right)^{\frac{1}{\ell}}, \ldots, \left( \frac{\eta_N}{\mu_0} \right)^{\frac{1}{\ell}} \right\}$, that is, such exceptional values of the final time $T$ actually cause the non-uniqueness for the backward problem.

We sum up the property of $\Lambda$:

Lemma 1.4. The set $\Lambda$ is a countably infinite set with a unique accumulation point $0$, and
\begin{align}
  \Lambda \subset \left( 0, \left( \frac{\eta_N}{\mu_1} \right)^{\frac{1}{\ell}} \right]. 
\end{align}
Proof of lemma 1.4. Since $0 < \mu_1 < \mu_2 < \ldots$ and $\lim_{n \to \infty} \mu_n = \infty$, we can readily see that the set $\Lambda$ has an accumulation point $0$, and $\Lambda \subset \left( 0, \left( \frac{\eta_N}{\mu_1} \right)^{\frac{1}{\ell}} \right]$. Let $\lambda_0 > 0$ be an accumulation point. Then for $n \in \mathbb{N}$, we can choose $k(n) \in \{1, 2, \ldots, N\}$ and $m(n) \in \mathbb{N}$ such that
\begin{align}
  \lim_{n \to \infty} \frac{\eta_{k(n)}}{\mu_{m(n)}} = \lambda_0, \quad \frac{\eta_{k(n)}}{\mu_{m(n)}} \neq \lambda_0 \quad \text{for all } n \in \mathbb{N}. 
\end{align}
Then we can verify $\lim_{n \to \infty} m(n) = \infty$. Indeed, otherwise, $\{m(n); n \in \mathbb{N}\}$ is a finite set. Since $\{\eta_{kn}; n \in \mathbb{N}\}$ is also finite, the set $\{\frac{\eta_{kn}}{\mu_{kn}}\}_{k \in \mathbb{N}}$ is finite. Therefore $\lim_{n \to \infty} \frac{\eta_{kn}}{\mu_{kn}} = \lambda_0$ implies $\lambda_0 = \frac{\eta_{kn}}{\mu_{kn}}$ with some $n_1 \in \mathbb{N}$, which is impossible. Therefore $\lim_{n \to \infty} m(n) = \infty$. Hence $\lambda_0 = 0$. Thus we proved that 0 is a unique accumulation point of $\Lambda$. \hfill \square

Moreover,

**Corollary 1.5.** If 

$$T > \left( \frac{\eta_{\nu\eta}}{\mu_{\eta}} \right)^{\frac{1}{\nu}},$$

(1.11)

then for any $a_T, b_T \in H^2(\Omega) \cap H^1_0(\Omega)$, there exists a unique $(a, b) \in L^2(\Omega) \times L^2(\Omega)$ such that $u_{a,b}$ satisfies (1.8) and (1.9).

In corollary 4.2 in section 4, we provide a more concrete estimate of $T$ than (1.11).

The backward problem for $0 < \alpha < 2$ is rather different from the case $0 < \alpha < 1$, which is well-posed for any $T > 0$. We can sum up the results for the backward problems for $0 < \alpha \leq 2$:

**Backward problem in time.**

- $0 < \alpha < 1$: well-posed for any $T > 0$.
- $\alpha = 1$: severely ill-posed but we have the uniqueness and some conditional stability for any $T > 0$.
- $1 < \alpha < 2$: well-posed for $T > 0$ not belonging to a countably infinite set $\Lambda$. Even non-uniqueness occurs for such exceptional values of $T$.
- $\alpha = 2$: well-posed. Also we have the conservation quantity such as energy, which is impossible for $\alpha \neq 2$.

The well-posedness of the backward problems is sensitive according to $0 < \alpha < 1$, $\alpha = 1$, $1 < \alpha < 2$ and $\alpha = 2$, and in the case $1 < \alpha < 2$, a quite new aspect of the non-uniqueness happens by choices of $T$.

This article is composed of four sections. In section 2, we prove lemma 1.2, and section 3 is devoted to the proof of theorem 1.3. Section 4 gives concluding remarks.

### 2. Proof of lemma 1.2

We recall that $\psi(\eta)$ is defined by (1.5). By the analyticity of the Mittag–Leffler function, we see that $\psi(\eta)$ is analytic in $\eta > 0$ and continuous in $[0, \infty)$. Moreover by the asymptotics of the Mittag–Leffler functions (e.g., theorem 1.4 (pp 33–34) in [16]), we see that

$$
\begin{align*}
E_{\alpha,1}(-\eta) &= \frac{1}{\Gamma(1 - \alpha)} \frac{1}{\eta} + O \left( \frac{1}{\eta^\nu} \right), \quad E_{\alpha,2}(-\eta) = \frac{1}{\Gamma(2 - \alpha)} \frac{1}{\eta} + O \left( \frac{1}{\eta^\nu} \right), \\
E_{\alpha,0}(-\eta) &= -\frac{1}{\Gamma(-\alpha)} \frac{1}{\eta^2} + O \left( \frac{1}{\eta} \right),
\end{align*}
$$

(2.1)

Therefore

$$
\psi(\eta) = E_{\alpha,1}(-\eta)^2 + \eta E_{\alpha,2}(-\eta) E_{\alpha,0}(-\eta)
= \left( \frac{1}{\Gamma(1 - \alpha)} \frac{1}{\eta} + O \left( \frac{1}{\eta^\nu} \right) \right)^2 - \eta \left( \frac{1}{\Gamma(2 - \alpha)} \frac{1}{\eta} + O \left( \frac{1}{\eta^\nu} \right) \right)
$$
\[ \{0\} = (0) \quad \text{(3.1)} \]

Since \( \Gamma(1 - \alpha) = -\alpha \Gamma(-\alpha) \) and \( \Gamma(2 - \alpha) = (1 - \alpha) \Gamma(1 - \alpha) = (\alpha^2 - \alpha) \Gamma(-\alpha) \), we obtain

\[
\psi(\eta) = -\frac{1}{\alpha^2(\alpha - 1)\Gamma(-\alpha)} \frac{1}{\eta^2} + O\left(\frac{1}{\eta^3}\right) \quad \text{as} \quad \eta \to \infty. \tag{2.2}
\]

By \( \alpha \in (a, 0] \) or \( \alpha \in (0, b] \), there exists a constant \( M > 0 \) such that \( \psi(\eta) < 0 \) for \( \eta \geq M \). Since \( \psi(0) = 1 \), by the continuity of \( \psi \) in \([0, \infty)\) we can find a sufficiently small constant \( \varepsilon > 0 \) such that \( \psi(\eta) > 0 \) for \( 0 \leq \eta \leq \varepsilon \). Therefore the intermediate value theorem yields that there exists \( \eta_0 \in (\varepsilon, M] \) such that \( \psi(\eta_0) = 0 \). Moreover, since \( \psi \) is analytic in \([\varepsilon, M]\), the set \( \{\eta \in [\varepsilon, M]; \psi(\eta) = 0\} \) is a finite set. Otherwise \( \psi(\eta) = 0 \) for each \( \eta \in [\varepsilon, M] \), which implies \( \psi(0) = 0 \) by the continuity of \( \psi(\eta) \) at \( \eta = 0 \), which contradicts \( \psi(0) = 1 \). Thus the proof of lemma 1.2 is complete.

3. Proof of theorem 1.3

First, for convenience, we describe the norm in \( H^2(\Omega) \). By the classical \( a \) priori elliptic estimate and \( 0 < \mu_1 < \mu_2 < \ldots \), we have

\[
C^{-1} ||a||_{H^2(\Omega)} \leq ||Aa||_{L^2(\Omega)} \leq C ||a||_{H^2(\Omega)}
\]

for each \( a \in H^2(\Omega) \cap H^1_0(\Omega) \). Thus we henceforth identify

\[
||a||_{H^2(\Omega)} = ||Aa||_{L^2(\Omega)} = \left( \sum_{n=1}^{\infty} \mu_n^2 \sum_{j=1}^{l_n} |(a, \varphi_j)|^2 \right)^{\frac{1}{2}}.
\]

We set

\[
a_{nj} = (a, \varphi_{nj}), \quad b_{nj} = (b, \varphi_{nj}),
\]

and

\[
\begin{align*}
p_{nj} &:= a_{nj}E_{n,1}(-\mu_n T^n) + b_{nj}TE_{n,2}(-\mu_n T^n), \\
q_{nj} &:= -\mu_n T^n a_{nj}E_{n,2}(-\mu_n T^n) + b_{nj}E_{n,3}(-\mu_n T^n).
\end{align*}
\]

Since \( \{\varphi_j\}_{1 \leq j \leq \ell_n} \in \mathbb{V} \) is an orthonormal basis in \( L^2(\Omega) \), we see that

\[
\sum_{n=1}^{\infty} \sum_{j=1}^{l_n} |(g, \varphi_{nj})|^2 = ||g||_{L^2(\Omega)}^2, \quad g \in L^2(\Omega),
\]

\[
\sum_{n=1}^{\infty} \sum_{j=1}^{l_n} \mu_n^2 |(g, \varphi_{nj})|^2 = ||g||_{H^1(\Omega)}^2, \quad g \in \mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega).
\]
Hence, by (1.4), we have
\[
\|u(\cdot,T)\|^2_{H^1(\Omega)} + \|\partial_t u(\cdot,T)\|^2_{H^0(\Omega)}
\] (3.2)
\[
= \sum_{n=1}^{\infty} \sum_{j=1}^{\ell_n} \mu_n^2 \left( |a_{nj} E_{\alpha,1}(\mu_n T^n) + b_{nj} T E_{\alpha,2}(\mu_n T^n)|^2 + | - \mu_n T^{n-1} a_{nj} E_{\alpha,0}(\mu_n T^n) + b_{nj} E_{\alpha,1}(\mu_n T^n)|^2 \right)
\]
\[
= \sum_{n=1}^{\infty} \sum_{j=1}^{\ell_n} \mu_n^2 (|p_{nj}|^2 + |q_{nj}|^2).
\]

Now we proceed to

**Proof of theorem 1.3 (a).** We assume (1.7), and so \(\psi(\mu_n T^n) \neq 0\) for all \(n \in \mathbb{N}\). Then we can solve (3.1) with respect to \(a_{nj}\) and \(b_{nj}\):
\[
\begin{cases}
\frac{1}{\psi(\mu_n T^n)}(p_{nj} E_{\alpha,1}(\mu_n T^n) - q_{nj} T E_{\alpha,2}(\mu_n T^n)) = a_{nj}, \\
\frac{1}{\psi(\mu_n T^n)}(p_{nj} \mu_n T^{n-1} E_{\alpha,0}(\mu_n T^n) + q_{nj} E_{\alpha,1}(\mu_n T^n)) = b_{nj}.
\end{cases}
\] (3.3)

By (2.1) and (2.2), we can choose a large constant \(M_0 > 0\) such that
\[
|E_{\alpha,1}(\eta)| \leq \frac{2}{\eta} \left| \frac{1}{\Gamma(1-\alpha)} \right|, \quad |E_{\alpha,2}(\eta)| \leq \frac{2}{\eta} \left| \frac{1}{\Gamma(2-\alpha)} \right|,
\]
\[
|E_{\alpha,0}(\eta)| \leq \frac{2}{\eta^2} \left| \frac{1}{\Gamma(-\alpha)} \right|, \quad |\psi(\eta)| \geq \frac{1}{2\eta^2} \left| \frac{1}{\alpha^2(\alpha-1)\Gamma(-\alpha)} \right|, \quad \eta \geq M_0.
\]

Here we note that \(\Gamma(1-\alpha) < 0\) and \(\Gamma(2-\alpha), \Gamma(-\alpha) > 0\). Consequently we can fix \(N_0 \in \mathbb{N}\) such that
\[
\begin{cases}
|\psi(\mu_n T^n)| \geq \frac{1}{2T^{2\alpha}} \frac{\mu_n^2}{\alpha^2(\alpha-1)\Gamma(-\alpha)^2} = C_1, \\
|E_{\alpha,1}(\mu_n T^n)|, \quad |E_{\alpha,2}(\mu_n T^n)|, \quad |\mu_n E_{\alpha,0}(\mu_n T^n)| \leq C_1 \frac{\mu_n}{\mu_n}, \quad n \geq N_0.
\end{cases}
\] (3.4)

Here and henceforth \(C_k, k = 1, 2, \ldots, 5, 6\) denote generic constants which are independent of \(n\) and \(j\), but dependent on \(T, N_0, \alpha\).

Therefore (3.3) implies
\[
|a_{nj}| \leq C_2 \mu_n (|p_{nj}| + |q_{nj}|), \quad |b_{nj}| \leq C_2 \mu_n (|p_{nj}| + |q_{nj}|), \quad n \geq N_0, \quad 1 \leq j \leq \ell_j. \tag{3.5}
\]

On the other hand, by bounds of the the Mittag–Leffler functions (e.g., theorem 1.6 (p 35) in [16]), we see that
\[
|E_{\alpha,1}(\mu_n T^n)|, \quad |E_{\alpha,2}(\mu_n T^n)| \leq \frac{C_3}{1 + \mu_n} \leq C_4, \quad n \in \mathbb{N}. \tag{3.6}
\]

Moreover the estimate of \(|E_{\alpha,0}(\mu_n T^n)|\) in (3.4) implies
\[
|E_{\alpha,0}(\mu_n T^n)| \leq \frac{C_4}{1 + \mu_n^2}, \quad n \in \mathbb{N}. \tag{3.7}
\]
Since $\psi(\mu_i T^\alpha) \neq 0$ for each $n \in \mathbb{N}$, by (3.3) and (3.6), we have
\[
|a_n| \leq C_5 \max_{1 \leq n \leq N_0-1} \left| \frac{1}{\psi(\mu_n T^\alpha)} \right| (|p_n| + T|q_n|),
\]
\[
|b_n| \leq C_5 \max_{1 \leq n \leq N_0-1} \left| \frac{1}{\psi(\mu_n T^\alpha)} \right| (\mu_n T^{\alpha-1}|p_n| + |q_n|), \quad 1 \leq n \leq N_0 - 1, \ 1 \leq j \leq \ell_a,
\]
so that (3.5) holds for each $n \in \mathbb{N}$ and $1 \leq j \leq \ell_a$. Hence
\[
\sum_{n=1}^{\ell_a} \sum_{j=1}^{\ell_a} (|a_n|^2 + |b_n|^2) \leq C_5 \sum_{n=1}^{\ell_a} \sum_{j=1}^{\ell_a} \mu_n^2 (|p_n|^2 + |q_n|^2),
\]
and applying (3.2), we obtain
\[
\|a\|_{L_2(\Omega)}^2 + \|b\|_{L_2(\Omega)}^2 \leq C_5 (\|u(\cdot, T)\|_{L_2(\Omega)}^2 + \|\partial u(\cdot, T)\|_{L_2(\Omega)}^2).
\]
Next we prove the reverse inequality. Applying (3.6) and (3.7) in (3.1), we have
\[
\mu_n |p_n| \leq C_6 (|a_n| + |b_n|),
\]
\[
\mu_n |q_n| \leq C_6 \left( \frac{C_3}{1 + \mu_n} |a_n| + |b_n| \right) \leq C_6 (|a_n| + |b_n|)
\]
for all $n \in \mathbb{N}$ and $1 \leq j \leq \ell_a$. Hence, in view of (3.2), we see
\[
\|u(\cdot, T)\|_{L_2(\Omega)}^2 + \|\partial u(\cdot, T)\|_{L_2(\Omega)}^2
\]
\[
\leq C_6 \sum_{n=1}^{\ell_a} \sum_{j=1}^{\ell_a} (|a_n|^2 + |b_n|^2) = C_6 (\|a\|_{L_2(\Omega)}^2 + \|b\|_{L_2(\Omega)}^2),
\]
which completes the proof of theorem 1.3 (a). $\square$

**Proof of theorem 1.3 (b).** By (1.4) we see
\[
u(\cdot, T) = \sum_{n=1}^{\ell_a} \sum_{j=1}^{\ell_a} p_n \eta_{nj}, \quad \partial u(\cdot, T) = \sum_{n=1}^{\ell_a} \sum_{j=1}^{\ell_a} q_n \eta_{nj}.
\]
Therefore $u(\cdot, T) = \partial u(\cdot, T) = 0$ in $\Omega$ is equivalent to $p_n = q_n = 0$ for $n \in \mathbb{N}$ and $1 \leq j \leq \ell_a$. By (1.10), we can choose $m_0 \in \mathbb{N}$ and $k_0 \in \{1, \ldots, N\}$ such that $T = \left( \frac{m_0}{m_0} \right) \sqrt{\cdot}$, that is, $\eta_{k_0} = \mu_{m_0} T^{\alpha}$. Consequently $\psi(\mu_{m_0} T^{\alpha}) = 0$. Recalling the definition of $\psi(\mu_{m_0} T^{\alpha})$, we see that it is the determinant of the coefficient matrix of the linear system (3.1) with respect to $a_{nj}$ and $b_{nj}$. Hence there exist $(a_{m_0}, b_{m_0}) \neq (0, 0)$ satisfying
\[
\begin{cases}
\alpha_{m_0}^1 \varepsilon_{\alpha,1} (\mu_{m_0} T^{\alpha}) + b_{m_0}^1 T \varepsilon_{\alpha,2} (\mu_{m_0} T^{\alpha}) = 0, \\
-\mu_{m_0} T^{\alpha-1} \alpha_{m_0}^1 \varepsilon_{\alpha,1} (\mu_{m_0} T^{\alpha}) + b_{m_0}^1 \varepsilon_{\alpha,1} (\mu_{m_0} T^{\alpha}) = 0.
\end{cases}
\]
Setting $a = u(\cdot, 0) := \alpha_{m_0}^1 \varepsilon_{m_0,1}$ and $b = \partial u(\cdot, 0) := b_{m_0}^1 \varepsilon_{m_0,1}$, we see that either $a \neq 0$ in $\Omega$ or $b \neq 0$ in $\Omega$, and $u_{m_0}(\cdot, T) = \partial u_{m_0}(\cdot, T) = 0$ in $\Omega$. The former part of (b) is now proved. The latter part follows from (3.1). Indeed let $T \notin \left\{ \left( \frac{m_0}{m_0} \right)^{\frac{1}{2}} \right\}$, for some $m_0 \in \mathbb{N}$. Then
ψ(μ^Tν) ≠ 0. Therefore the determinant ψ(μ^Tν) of the coefficient matrix of (3.1) is not zero, and so a_{n,j} = b_{n,j} = 0, that is, (a, ϕ_{n,j}) = (b, ϕ_{n,j}) = 0 for 1 ≤ j ≤ ℓ_{n,j}. Thus the proof of theorem 1.3 is complete.

4. Concluding remarks

4.1. Estimation of η_N and T

We recall (1.6). First we give an upper bound for η_N. For simplicity, we set

\[ \alpha_1 := \frac{-1}{\Gamma(1 - \alpha)} = \frac{1}{\alpha \Gamma(-\alpha)}, \]
\[ \alpha_2 := \frac{1}{\Gamma(2 - \alpha)} = \frac{1}{(\alpha^2 - \alpha) \Gamma(-\alpha)}, \]
\[ \alpha_3 := \frac{1}{\Gamma(-\alpha)}. \]

Here we used \( \Gamma(1 - \alpha) = -\alpha \Gamma(-\alpha) \) and \( \Gamma(2 - \alpha) = (1 - \alpha) \Gamma(1 - \alpha) = (1 - \alpha)(-\alpha) \Gamma(-\alpha) \). By \( \Gamma(-\alpha) > 0 \), we see that \( \alpha_1, \alpha_2, \alpha_3 > 0 \).

For \( 1 < \alpha < 2 \), we choose \( \theta \) such that

\[ \frac{\pi \alpha}{2} < \theta < \pi. \] (4.1)

By \( \gamma \) we denote the contour in \( \mathbb{C} \) which is directed from \( e^{-\sqrt{\alpha} \theta} \) to \( e^{\sqrt{\alpha} \theta} \) and consists of

(a) \( \arg z = -\theta, |z| > 1 \)
(b) \( -\theta \leq \arg z \leq \theta, |z| = 1 \)
(c) \( \arg z = \theta, |z| \geq 1 \).

Moreover we set

\[ \nu_1 = \frac{1}{2\pi \alpha \sin \theta} \int_{\gamma} |\exp(\frac{\alpha}{2}|\zeta|)| |\zeta| d\zeta, \]
\[ \nu_2 = \frac{1}{2\pi \alpha \sin \theta} \int_{\gamma} |\exp(\frac{\alpha}{2}|\zeta|)| |\zeta|^{1 - \frac{1}{2}} d\zeta, \]
\[ \nu_3 = \frac{1}{2\pi \alpha \sin \theta} \int_{\gamma} |\exp(\frac{\alpha}{2}|\zeta|)| |\zeta|^{1 + \frac{1}{2}} d\zeta. \]

Since there exists a constant \( C_0 > 0 \) such that \( |\exp(\frac{\alpha}{2}|\zeta|)| \leq \exp \left( -C_0 |\zeta|^{\frac{1}{2}} \right) \) for \( \zeta \in \gamma \), we can directly verify that \( 0 < \nu_1, \nu_2, \nu_3 < \infty \).

Then we can prove

**Lemma 4.1.**

\[ \eta_N < \max \left\{ \frac{1}{|\cos \theta|^2}, \alpha^2 (\alpha - 1) \Gamma(-\alpha)^2 (\alpha^2 \nu_3 + \alpha_3 \nu_2 + 2 \alpha_1 \nu_1 + \nu_1^2 + \nu_2 \nu_3) \right\}. \]

**Proof.** First by formula (1.145) (p 34) in [16], we see that

\[ \begin{align*}
E_{\alpha,1}(-\eta) &= -\frac{\alpha_1}{\eta} + I_{\alpha,1}(\eta), & E_{\alpha,2}(-\eta) &= \frac{\alpha_2}{\eta} + I_{\alpha,2}(\eta), \\
E_{\alpha,\alpha}(-\eta) &= -\frac{\alpha_3}{\eta^2} + I_{\alpha,\alpha}(\eta), & \eta &\geq 1, \end{align*} \] (4.2)
where
\[
I_{\alpha,1}(\eta) = -\frac{1}{2\pi\alpha\sqrt{1-\eta^2}} \int_\gamma \exp(\frac{\eta}{\sqrt{(\zeta + \eta)^2 + 1}}) \frac{d\zeta}{\zeta + \eta},
\]
\[
I_{\alpha,\alpha}(\eta) = \frac{1}{2\pi\alpha\sqrt{1-\eta^2}} \int_\gamma \exp(\frac{\eta}{\sqrt{(\zeta + \eta)^2 + 1}}) \frac{d\zeta}{\zeta + \eta}, \quad \ell = 1, 2, \eta \geq 1. \tag{4.3}
\]

Next we will prove
\[
|I_{\alpha,1}(\eta)| \leq \frac{\nu_1}{\eta^\alpha}, \quad |I_{\alpha,2}(\eta)| \leq \frac{\nu_2}{\eta^\alpha}, \quad |I_{\alpha,\alpha}(\eta)| \leq \frac{\nu_3}{\eta^\alpha}, \quad \text{for } \eta \geq \frac{1}{\cos \theta}. \tag{4.4}
\]

Proof of (4.4). For \(\eta \geq \frac{1}{\cos \theta}\), we can directly verify that
\[
\min_{\zeta \in \gamma} |\zeta + \eta| = \min_{r \geq 1} |r e^{\sqrt{-1}\theta} - (-\eta)| = | - \eta | \sin(\pi - \theta) = \eta \sin \theta > 0.
\]

Indeed the intersection point of the perpendicular from \(-\eta\) with the half-line \(r e^{\sqrt{-1}\theta}; r \geq 1\) is outside of \(\{ z \in \mathbb{C}; |z| \leq 1 \}\) if \(|\eta| > \frac{1}{\cos \theta}\). Hence with fixed \(\eta\) satisfying \(\eta \geq \frac{1}{\cos \theta}\), the function \(|\zeta + \eta|\) in \(\zeta \in \gamma\), attains the minimum at such an intersection point \(\zeta\).

Therefore
\[
|I_{\alpha,1}(\eta)| \leq \frac{1}{2\pi\alpha \sin \theta} \frac{1}{\eta^\alpha} \int_\gamma |\exp(\frac{\eta}{\sqrt{(\zeta + \eta)^2 + 1}})| d\zeta = \frac{\nu_1}{\eta^\alpha},
\]
\[
|I_{\alpha,\alpha}(\eta)| \leq \frac{1}{2\pi\alpha \sin \theta} \frac{1}{\eta^\alpha} \int_\gamma |\exp(\frac{\eta}{\sqrt{(\zeta + \eta)^2 + 1}})| d\zeta = \frac{\nu_3}{\eta^\alpha}, \quad \eta \geq \frac{1}{\cos \theta}.
\]

Hence (4.4) is proved.

Now we will complete the proof of lemma 4.1. Applying (4.2) and (4.4) in (1.5), we obtain
\[
\psi(\eta) = E_{\alpha,1}(-\eta)^2 + \eta E_{\alpha,2}(-\eta)E_{\alpha,\alpha}(-\eta)
\]
\[
= \left( \frac{\alpha_1}{\eta} - I_{\alpha,1}(\eta) \right)^2 + \left( \frac{\alpha_2}{\eta} + I_{\alpha,2}(\eta) \right) \left( \frac{\alpha_3}{\eta} + \eta I_{\alpha,\alpha}(\eta) \right)
\]
\[
= \frac{1}{\eta^2} (\alpha_1^2 + \alpha_2 \alpha_3) + \left\{ -2 \frac{\alpha_1}{\eta} I_{\alpha,1}(\eta) + I_{\alpha,1}(\eta)^2 + \alpha_2 I_{\alpha,\alpha}(\eta) \right. \\
\left. - \frac{\alpha_3 I_{\alpha,2}(\eta)}{\eta} + \eta I_{\alpha,2}(\eta) I_{\alpha,\alpha}(\eta) \right\}
\]
\[
\leq \frac{1}{\eta^2} (\alpha_1^2 + \alpha_2 \alpha_3) + \frac{2(\alpha_1)}{\eta} |I_{\alpha,1}(\eta)| + |I_{\alpha,1}(\eta)|^2 + \alpha_2 |I_{\alpha,\alpha}(\eta)| \\
+ \frac{\alpha_3 |I_{\alpha,2}(\eta)|}{\eta} + \eta |I_{\alpha,2}(\eta)| |I_{\alpha,\alpha}(\eta)|
\]
\[
\leq \frac{1}{\alpha^2(\alpha - 1) \Gamma(-\alpha)} \frac{1}{\eta^2} + (\alpha_2 \nu_3 + \alpha_3 \nu_2) \frac{1}{\eta^\alpha} + (\nu_1^2 + \nu_2 \nu_3) \frac{1}{\eta^3}
\]
\[
\leq \frac{1}{\alpha^2(\alpha - 1) \Gamma(-\alpha)} \frac{1}{\eta^2} + (\alpha_2 \nu_3 + \alpha_3 \nu_2 + 2\alpha_1 \nu_1) \frac{1}{\eta^\alpha} + (\nu_1^2 + \nu_2 \nu_3) \frac{1}{\eta^3}
\]
for $\eta \geq \frac{1}{\cos \theta} \geq 1$. For the last inequality, we use $\frac{1}{\eta} \leq \frac{1}{\eta}$ for $\eta \geq 1$. Hence

$$
\psi(\eta) = \frac{-1}{\alpha^2(\alpha - 1)\Gamma(-\alpha)^2} \frac{1}{\eta^2} \left\{ 1 - \alpha^2(\alpha - 1)\Gamma(-\alpha)^2 \right\}
$$

$$
\times (\alpha_2\nu_3 + \alpha_3\nu_2 + 2\alpha_1\nu_1 + \nu_1^2 + \nu_2\nu_3) \frac{1}{\eta}
$$

for $\eta \geq \frac{1}{\cos \theta}$. Thus the proof of lemma 4.1 is complete.

Applying lemma 4.1 to corollary 1.5, we obtain a lower bound of $T$ which is described more concretely than (1.11) for guaranteeing the well-posedness of the backward problem.

**Corollary 4.2.** If

$$
T > \left( \frac{1}{\alpha_1} \max \left\{ \frac{1}{\cos \theta} | \alpha_2(\alpha - 1)\Gamma(-\alpha)^2(\alpha_2\nu_3 + \alpha_3\nu_2 + 2\alpha_1\nu_1 + \nu_1^2 + \nu_2\nu_3) \frac{1}{\eta} \right\} \right)^{\frac{1}{2}},
$$

then for any $a_T, b_T \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique $(a, b) \in L^2(\Omega) \times L^2(\Omega)$ such that $u_{a, b}$ satisfies (1.8) and (1.9).

### 4.2. More general equations

Our arguments are based on the eigenfunction expansions (1.4) and the asymptotics of the Mittag–Leffler functions. Thus it is not easy to generalize the results in the cases where $A$ is not a symmetric operator and the coefficients of $A$ depend also on $t$. These issues including the nonlinear equations should be future subjects.

#### 4.3. Backward fractional ordinary differential equations

Let $1 < \alpha < 2$ and $\lambda > 0$. We consider a backward fractional ordinary differential equation:

$$
\partial_t^\alpha v(t) = -\lambda v(t), \quad v(T) = a_T, \quad \partial_t v(T) = b_T, \quad 0 < t < T. \quad (4.5)
$$

By (1.6), we can prove that if $T \notin \left\{ \left( \frac{\eta_1}{\lambda} \right)^{\frac{1}{\alpha}}, \ldots, \left( \frac{\eta_N}{\lambda} \right)^{\frac{1}{\alpha}} \right\}$, then (4.5) possesses a unique solution for arbitrary $a_T, b_T \in \mathbb{R}$. Moreover if $T \in \left\{ \left( \frac{\eta_1}{\lambda} \right)^{\frac{1}{\alpha}}, \ldots, \left( \frac{\eta_N}{\lambda} \right)^{\frac{1}{\alpha}} \right\}$, then there exists a non-zero solution $v$ to (4.5) with $a_T = b_T = 0$, and there may be no solutions with some $a_T, b_T$. Thus for the case $1 < \alpha < 2$, the backward problem for a fractional ordinary differential equation is not always uniquely solvable for all $T > 0$. In general, even for nonlinear backward fractional ordinary differential equations, under suitable conditions, we can apply the contraction mapping theorem to prove the well-posedness for sufficiently small $T$. However, as theorem 1.3 (a) asserts, the backward problem for fractional partial differential equations with $1 < \alpha < 2$ may not be well-posed even for sufficiently small $T > 0$. We can refer to an example (p 374) in Diethelm and Ford [2] which indicates the non-uniqueness in the case of $b_T = 0$ with some value of $\lambda$.

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References

[1] Adams R A 1975 Sobolev Spaces (New York: Academic)
[2] Diethelm K and Ford N J 2018 A note on the well-posedness of terminal value problems for
fractional differential equations J. Integral Equ. Appl. 30 371–6
[3] Floridia G, Li Z and Yamamoto M 2020 Well-posedness for the backward problems in time for
general time-fractional diffusion equation Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. 31
593–610
[4] Gorenflo R, Kilbas A A, Mainardi F and Rogosin S V 2014 Mittag–Leffler Functions, Related Topics
and Applications (Berlin: Springer)
[5] Imanuvilov O Y and Yamamoto M 2014 Conditional stability in a backward parabolic system Appl.
Anal. 93 2174–98
[6] Isakov V 2006 Inverse Problems forPartial Differential Equations 2nd edn (Berlin: Springer)
[7] Kubica A, Ryszewska K and Yamamoto M 2020 Time-fractional Differential Equations: A Theore-
etical Introduction (Berlin: Springer)
[8] Kubica A and Yamamoto M 2018 Initial-boundary value problems for fractional diffusion equations
with time-dependent coefficients Fractional Calculus Appl. Anal. 21 276–311
[9] Li Z, Liu Y and Yamamoto M 2019 Inverse problems of determining parameters of the fractional
partial differential equations Handbook of Fractional Calculus with Applications vol 2 ed J A
Teneiro Machado, A N Kochubei and Y Luchko (Berlin: De Gruyter) pp 431–42
[10] Li Z and Yamamoto M 2019 Inverse problems of determining coefficients of the fractional partial
differential equations Handbook of Fractional Calculus with Applications vol 2 ed J A Teneiro
Machado, A N Kochubei and Y Luchko (Berlin: De Gruyter) pp 443–64
[11] Liu J J and Yamamoto M 2010 A backward problem for the time-fractional diffusion equation Appl.
Anal. 89 1769–88
[12] Liu Y, Li Z and Yamamoto M 2019 Inverse problems of determining sources of the fractional partial
partial differential equations Handbook of Fractional Calculus with Applications vol 2 ed J A Teneiro
Machado, A N Kochubei and Y Luchko (Berlin: De Gruyter) pp 411–29
[13] Mainardi F 1996 Fractional relaxation-oscillation and fractional diffusion-wave phenomena Chaos
Solutions Fractals 7 1461–77
[14] Metzler R, Glöckle W G and Nonnenmacher T F 1994 Fractional model equation for anomalous
diffusion Physica A 211 13–24
[15] Nigmatullin R R 1986 The realization of the generalized transfer equation in a medium with fractal
geometry Phys. Status. Solidi b 133 425–30
[16] Podlubny I 1999 Fractional Differential Equations (New York: Academic)
[17] Sakamoto K and Yamamoto M 2011 Initial value/boundary value problems for fractional diffusion-
wave equations and applications to some inverse problems J. Math. Anal. Appl. 382 426–47
[18] Tuan N H, Huyinh L N, Ngoc T B and Zhou Y 2019 On a backward problem for nonlinear fractional
diffusion equations Appl. Math. Lett. 92 76–84
[19] Tuan N H, Long L D and Tatar S 2018 Tikhonov regularization method for a backward problem for
theinhomogeneous time-fractional diffusion equation Appl. Anal. 97 842–63
[20] Tuan N H, Thach T N, O’Regan D and Can N H 2019 Backward problem for time fractional reaction-
diffusion equation with nonlinear source and discrete data (arXiv:1910.14204)
[21] Wang J-G, Wei T and Zhou Y-B 2013 Tikhonov regularization method for a backward problem for the time-fractional diffusion equation Appl. Math. Modelling 37 8518–32
[22] Wang L and Liu J J 2013 Total variation regularization for a backward time-fractional diffusion problem Inverse Problems 29 115013
[23] Wei T and Wang J-G 2014 A modified quasi-boundary value method for the backward time-fractional diffusion problem ESAIM: Math. Modelling Numer. Anal. 48 603–21
[24] Wei T and Zhang Y 2018 The backward problem for a time-fractional diffusion-wave equation in a bounded domain Comput. Math. Appl. 75 3632–48
[25] Xiong X T, Wang J X and Li M 2012 An optimal method for fractional heat conduction problem backward in time Appl. Anal. 91 823–40
[26] Yamamoto M 2009 Carleman estimates for parabolic equations and applications Inverse Problems 25 123013
[27] Yang M and Liu J J 2013 Solving a final value fractional diffusion problem by boundary condition regularization Appl. Numer. Math. 66 45–58
[28] Zacher R 2009 Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces Funkc. Ekvacioj 52 1–18