INCREASING-DECREASING PATTERNS IN THE ITERATION OF AN ARITHMETIC FUNCTION

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Abstract. Let \( \Omega \) be a set of positive integers and let \( f : \Omega \to \Omega \) be an arithmetic function. Let \( V = (v_1)_{i=1}^{n} \) be a finite sequence of positive integers. An integer \( m \in \Omega \) has increasing-decreasing pattern \( V \) with respect to \( f \) if, for all odd integers \( i \in \{1, \ldots, n\} \),
\[
f^{v_1+\cdots+v_i-1}(m) < f^{v_1+\cdots+v_i-1+1}(m) < \cdots < f^{v_1+\cdots+v_i-1+v_i}(m)
\]
and, for all even integers \( i \in \{2, \ldots, n\} \),
\[
f^{v_1+\cdots+v_i-1}(m) > f^{v_1+\cdots+v_i-1+1}(m) > \cdots > f^{v_1+\cdots+v_i-1+v_i}(m).
\]
The arithmetic function \( f \) is wildly increasing-decreasing if, for every finite sequence \( V \) of positive integers, there exists an integer \( m \in \Omega \) such that \( m \) has increasing-decreasing pattern \( V \) with respect to \( f \). This paper gives a proof that the Syracuse function is wildly increasing-decreasing.

1. Iterations and patterns

An arithmetic function is a function whose domain is a subset \( \Omega \) of the set \( \mathbb{N} = \{1, 2, 3, \ldots \} \) of positive integers. Let \( f : \Omega \to \Omega \) be an arithmetic function and, for all \( j \in \mathbb{N} \), let \( f^j : \Omega \to \Omega \) be the \( j \)th iterate of \( f \). The function \( f^0 : \Omega \to \Omega \) is the identity function. The trajectory of \( m \in \Omega \) is the sequence of positive integers \( (f^j(m))_{j=0}^{\infty} \). The pair \( (\Omega, f) \) is a discrete dynamical system.

Let \( \Omega^{\text{fix}} = \{m \in \Omega : f(m) = m\} \) be the set of fixed points of \( f \) and let
\[
\Omega^{\text{per}} = \{m \in \Omega : f^k(m) = m \text{ for some positive integer } k\}
\]
be the set of periodic points of \( f \). The period of \( m \in \Omega^{\text{per}} \) is the smallest positive integer \( k \) such that \( f^k(m) = m \). The fixed points are the points of period one.

The trajectory \( (f^j(m))_{j=0}^{\infty} \) is eventually constant if \( f^k(m) \in \Omega^{\text{fix}} \) for some \( k \in \mathbb{N} \). The trajectory \( (f^j(m))_{j=0}^{\infty} \) is eventually periodic if \( f^k(m) \in \Omega^{\text{per}} \) for some \( k \in \mathbb{N} \). The trajectory is bounded if and only if \( f^k(m) \in \Omega^{\text{per}} \) for some \( k \in \mathbb{N} \). The trajectory is unbounded if and only if \( f^k(m) \in \Omega^{\text{per}} \) for some \( k \in \mathbb{N} \). We are interested in the pattern of changes (increases and decreases) in the trajectory \( (f^j(m))_{j=0}^{\infty} \).

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Here are some examples. Let $\Omega$ be the set of odd positive integers and let $\ell \in \Omega$. For all $m \in \Omega$, the integer $\ell m + 1$ is even. We define the arithmetic function $S_{\ell} : \Omega \to \Omega$ by

$$S_{\ell}(m) = \frac{\ell m + 1}{2^e}$$

where $2^e$ is the highest power of 2 that divides $\ell m + 1$. The function $S_{\ell}$ has a fixed point if and only if $\ell = 2^k - 1$ for some positive integer $k$, and in this case the unique fixed point is $m = 1$. The function $S_5$ has no fixed point, but it does have periodic points. For example, $S_5(1) = 3$ and $S_5(3) = 1$.

The Syracuse function (also called the Collatz function) is the arithmetic function

$$S(m) = S_3(m) = \frac{3m + 1}{2^e}$$

where $2^e$ is the highest power of 2 that divides $3m + 1$. The unique fixed point of this function is $m = 1$. There is a large literature on the Syracuse function (cf. Eliahou [3], Everett [7], Lagarias [11, 12], Tao [17], Terras [18]).

Let $\Omega = \mathbb{N}$. The sum of the proper divisors function is the arithmetic function $s : \mathbb{N} \to \mathbb{N}$ defined by

$$s(m) = \sum_{d \mid m, 1 \leq d < m} d.$$ 

The trajectory of the positive integer $m$, that is, $(s^i(m))_{i=1}^{\infty}$, is called an aliquot sequence. The fixed points of $s$ are the perfect numbers 6, 28, 496, $\ldots$. If $m$ is a point of period 2 and if $n = s(m)$, then $m \neq n$ and $s(n) = m$. The pair of integers $(m, n)$ is called an amicable pair. For example, $(220, 284)$ and $(1184, 1210)$ are amicable pairs. Integers whose trajectories are periodic under $s$ are called sociable numbers. There exist sociable numbers of period 4 (such as 1264460), but no sociable number of period 3 is known. The behavior of aliquot sequences is poorly understood (cf. Guy [8], P. Erdős, A. Granville, C. Pomerance, and C. Spiro [6], Pollack-Pomerance [15]).

Let $\Omega$ be a set of positive integers and let $f : \Omega \to \Omega$ be an arithmetic function. Let $V = (v_i)_{i=1}^{\infty}$ be a finite sequence of positive integers. We say that an integer $m$ in $\Omega$ has increasing-decreasing pattern $V$ with respect to $f$ if

$$m < f(m) < f^2(m) < \cdots < f^{v_1}(m)$$

$$f^{v_1}(m) > f^{v_1+1}(m) > \cdots > f^{v_1+v_2}(m)$$

$$f^{v_1+v_2}(m) < f^{v_1+v_2+1}(m) < \cdots < f^{v_1+v_2+v_3}(m)$$

and, in general, if $i$ is odd, then

$$f^{v_1+\cdots+v_{i-1}}(m) < f^{v_1+\cdots+v_{i-1}+1}(m) < \cdots < f^{v_1+\cdots+v_{i-1}+v_i}(m)$$

and if $i$ is even, then

$$f^{v_1+\cdots+v_{i-1}}(m) > f^{v_1+\cdots+v_{i-1}+1}(m) > \cdots > f^{v_1+\cdots+v_{i-1}+v_i}(m).$$

The arithmetic function $f$ is wildly increasing-decreasing if, for every finite sequence $V$ of positive integers, there exists an integer $m \in \Omega$ such that $m$ has increasing-decreasing pattern $V$ with respect to $f$.

More generally, let $F = (f_j)_{j=0}^{\infty}$ be a sequence of arithmetic functions $f_j : \Omega \to \Omega$. The $F$-trajectory of $m \in \Omega$ is the sequence $(f_j(m))_{j=1}^{\infty}$. Let $V = (v_i)_{i=1}^{\infty}$ be a
finite sequence of positive integers. We say that an integer \( m \) in \( \Omega \) has increasing-decreasing pattern \( V \) with respect to the sequence \( F \) if, for odd \( i \),
\[
f_{v_1+\cdots+v_{i-1}}(m) < f_{v_1+\cdots+v_{i-1}+1}(m) < \cdots < f_{v_1+\cdots+v_{i-1}+v_i}(m)
\]
and, for even \( i \),
\[
f_{v_1+\cdots+v_{i-1}}(m) > f_{v_1+\cdots+v_{i-1}+1}(m) > \cdots > f_{v_1+\cdots+v_{i-1}+v_i}(m).
\]
The sequence \( F \) is wildly increasing-decreasing if, for every finite sequence \( V \) of positive integers, there exists \( m \in \Omega \) such that \( m \) has increasing-decreasing pattern \( V \) with respect to the sequence \( F \).

Fix an arithmetic function \( f : \Omega \to \Omega \). Let \( (k_j)_{j=0}^\infty \) be a strictly increasing sequence of nonnegative integers. If \( f \) is wildly increasing-decreasing, then the sequence of arithmetic functions \( F = (f^{k_j})_{j=0}^\infty \) is also wildly increasing-decreasing.

It is not known if aliquot sequences are wildly increasing-decreasing. Lenstra [13] proved that, for every positive integer \( v_1 \), there are infinitely many integers \( m \) such that \( m < s(m) < s^2(m) < \cdots < s^{v_1}(m) \). Erdős [5] subsequently refined this result. Pomerance [16] proved that, for every positive integer \( v_2 \), there are infinitely many integers \( m \) such that \( m > s(m) > s^2(m) > \cdots > s^{v_2}(m) \). It is an open problem to determine if, for every pair of positive integers \( v_1, v_2 \) there are integers \( m \) such that
\[
m < s(m) < s^2(m) < \cdots < s^{v_1}(m)
\]
and
\[
s^{v_1}(m) > s^{v_1+1}(m) > s^{v_1+2}(m) > \cdots > s^{v_1+v_2}(m).
\]

There are two competing conjectures about the trajectories of the sum of the proper divisors function \( s(m) \). Catalan [11] and Dickson [2] conjectured that every aliquot sequence is bounded. Guy and Selfridge [9, 10] conjectured that infinitely many aliquot sequences go to infinity.

The goal of this paper is to prove that Syracuse function \( S(m) \) defined by (1) is wildly increasing-decreasing. The proof uses integer matrices and systems of linear diophantine equations.

2. INTEGER MATRICES

We consider vectors in \( \mathbb{R}^n \). A vector is positive if all of its coordinates are positive and negative if all of its coordinates are negative. A vector is integral if all of its coordinates are integers.

Let \( \mathbb{Z}^n \) denote the set of \( n \)-dimensional integral vectors. An integral vector is odd if its coordinates are odd integers and even if its coordinates are even integers. An integral vector is primitive if its coordinates are relatively prime (not necessarily pairwise relatively prime) positive integers.

**Theorem 1.** Let \( (a_i)_{i=1}^n \) and \( (b_j)_{j=2}^{n+1} \) be sequences of nonzero real numbers. Consider the \( n \times (n + 1) \) matrix
\[
M = \begin{pmatrix}
a_1 & -b_2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & a_2 & -b_3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & a_3 & -b_4 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & a_4 & -b_5 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & a_n & -b_{n+1}
\end{pmatrix}
\]
(a) The matrix $M$ has rank $n$ and the kernel of $M$ has dimension 1.

(b) Let $(a_i)_{i=1}^n$ and $(b_j)_{j=2}^{n+1}$ be sequences of positive real numbers. If $x \in \mathrm{ker}(M)$ and $x \neq 0$, then the vector $x$ is positive or negative.

(c) Let $(a_i)_{i=1}^n$ and $(b_j)_{j=2}^{n+1}$ be sequences of positive integers. There is a unique primitive vector $z$ that generates $\mathrm{ker}(M)$.

(d) Let $(a_i)_{i=1}^n$ be a sequence of odd positive integers and let $(b_j)_{j=2}^{n+1}$ be a sequence of even positive integers. Let $x$ be a nonzero vector of the matrix $M$.

(e) Let $(a_i)_{i=1}^n$ be a sequence of odd positive integers and let $(b_j)_{j=2}^{n+1}$ be a sequence of even positive integers. Let $h \in \mathbb{Z}^n$ be an odd vector. If $Mx = h$ for some integral vector $x \in \mathbb{Z}^{n+1}$, then there exists an odd positive vector $g \in \mathbb{Z}^{n+1}$ such that $Mg = h$.

Proof.

(a) The $n$ row vectors of the matrix $M$ are linearly independent and so $M$ has rank $n$.

(b) Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix} \in \mathrm{ker}(M)$. If $Mx = 0$, then for all $i \in \{1, \ldots, n\}$ we have

$$a_ix_i = b_{i+1}x_{i+1}.$$ 

If $x \neq 0$, then $x_{i_0} \neq 0$ for some $i_0 \in \{1, \ldots, n+1\}$. The positivity of the numbers $a_i$ and $b_j$ implies that if $x_{i_0} > 0$, then $x_i > 0$ for all $i \in \{1, \ldots, n+1\}$. Similarly, if $x_{i_0} < 0$, then $x_i < 0$ for all $i \in \{1, \ldots, n+1\}$. Therefore, $x$ is either a positive vector or a negative vector.

(c) Consider $M$ as a matrix with coordinates in the field $\mathbb{Q}$ of rational numbers. The kernel of $M$ is one-dimensional. Let $x \in \mathbb{Q}^{n+1}$ be a nonzero vector in the kernel of $M$. By (b), the coordinates of $x$ are either all positive rational numbers or all negative rational numbers. Multiplying, if necessary, by -1, we can assume that the coordinates are positive rational numbers. Multiplying by a common denominator of the denominators of the coordinates, we obtain a vector whose coordinates are positive integers. Dividing by the greatest common divisor of these integers, we obtain a primitive vector in the kernel of $M$.

Let $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \\ z_{n+1} \end{pmatrix}$ and $z' = \begin{pmatrix} z'_1 \\ \vdots \\ z'_n \\ z'_{n+1} \end{pmatrix}$ be primitive vectors in $\mathrm{ker}(M)$.

Because the kernel is one-dimensional, there is a positive rational number $p/q$ with $\gcd(p, q) = 1$ such that $(p/q)z = z'$. Multiplying by $q$ we obtain

$$\begin{pmatrix} pz_1 \\ \vdots \\ pz_n \\ pzn+1 \end{pmatrix} = pqz = \begin{pmatrix} qz'_1 \\ \vdots \\ qz'_n \\ qz'_{n+1} \end{pmatrix}.$$
Theorem 2. Let \((a_i)_{i=1}^n\) and \((b_j)_{j=2}^{n+1}\) be sequences of nonzero integers such that
\[
\gcd(a_i, b_j) = 1
\]
for all \(i \in \{1, \ldots, n\}\) and \(j \in \{2, \ldots, n+1\}\). Let \(M\) be the \(n \times (n+1)\) matrix defined in Theorem \(\text{[1]}\). The homomorphism \(M : \mathbb{Z}^{n+1} \to \mathbb{Z}^n\) is surjective.

and so
\[
pz_i = qz_i'
\]
for all \(i \in \{1, \ldots, n, n+1\}\). The divisibility condition \(\gcd(p, q) = 1\) implies that \(p\) divides \(z_i'\) and \(q\) divides \(z_i\) for all \(i \in \{1, \ldots, n, n+1\}\). Because the vectors \(z\) and \(z'\) are primitive, we have \(p = q = 1\) and so \(z = z'\). Thus, \(\text{kernel}(M)\) contains a unique primitive vector.

(d) Let \(z = \begin{pmatrix} z_1 \\ \vdots \\ z_{n+1} \end{pmatrix} \) be the unique primitive vector in \(\text{kernel}(M)\). For all \(i \in \{1, \ldots, n\}\) we have
\[
a_i z_i = b_{i+1} z_{i+1}.
\]
Because \(b_i\) is even and \(a_i\) is odd, it follows that \(z_i\) is even for all \(i \in \{1, \ldots, n\}\). We have \(\gcd(z_1, \ldots, z_n, z_{n+1}) = 1\) because the vector \(z\) is primitive, and so \(z_{n+1}\) must be odd.

(e) Let \(h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \) be an odd vector. If \(x = \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \) \(\in \mathbb{Z}^{n+1}\) and \(Mx = h\), then for all \(i \in \{1, \ldots, n\}\) we have
\[
a_i x_i - b_{i+1} x_{i+1} = h_i.
\]
Because the integer \(b_{i+1}\) is even and the integers \(a_i\) and \(h_i\) are odd, it follows that the integer \(x_i\) is odd for all \(i \in \{1, \ldots, n\}\). The integer \(x_{n+1}\) is not necessarily odd, nor are the integers \(x_1, \ldots, x_n, x_{n+1}\) necessarily positive.

Let \(z = \begin{pmatrix} z_1 \\ \vdots \\ z_{n+1} \end{pmatrix} \) be the unique primitive vector in \(\text{kernel}(M)\). By (d), the integers \(z_i\) are even for all \(i \in \{1, 2, \ldots, n\}\) and the integer \(z_{n+1}\) is odd. We have \(Mz = 0\) and so
\[
M(x + kz) = Mx = h
\]
for all integers \(k\), where
\[
x + kz = \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} + k \begin{pmatrix} z_1 \\ \vdots \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} x_1 + k z_1 \\ \vdots \\ x_{n+1} + k z_{n+1} \end{pmatrix}.
\]
The coordinates \(z_i\) are positive, and so the vector \(x + kz\) is positive for all sufficiently large \(k\).

For all \(i \in \{1, \ldots, n\}\) the integer \(x_i\) is odd and the integer \(z_i\) is even, and so \(x_i + k z_i\) is an odd integer. The integer \(z_{n+1}\) is odd, and so the integers \(x_{n+1} + k z_{n+1}\) and \(x_{n+1} + (k+1) z_{n+1}\) have opposite parity. It follows that for all sufficiently large \(k\), either \(g = x + kz\) or \(g = x + (k+1)z\) is an odd positive vector such that \(Mg = h\).

This completes the proof. \(\square\)
Proof. We give two proofs. The first proof uses induction on \( n \). Because \((a_1, b_2) = 1\), for every integer \( h \) there are integers \( x_1, x_2 \) such that \( a_1 x_1 + b_2 x_2 = h \). Equivalently, for the \( 2 \times 1 \) matrix \( M = \begin{pmatrix} a_1 & b_2 \end{pmatrix} \) and for \( h = (h_1) \), we have
\[
M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = h.
\]
This is the case \( n = 1 \).

For \( n \geq 2 \), let \( M \) be the \( n \times (n + 1) \) matrix defined in Theorem 1 and let \( h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \in \mathbb{Z}^n \). There exists an integral vector \( x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix} \in \mathbb{Z}^{n+1} \) such that
\[
Mx = h
\]
if and only if the diophantine system of \( n \) linear equations in \( n + 1 \) variables
\[
\begin{align*}
a_1 x_1 - b_2 x_2 &= h_1 \\
a_2 x_2 - b_3 x_3 &= h_2 \\
&\quad \vdots \\
a_{i-1} x_{i-1} - b_i x_i &= h_{i-1} \\
a_i x_i - b_{i+1} x_{i+1} &= h_i \\
&\quad \vdots \\
a_n x_n - b_{n+1} x_{n+1} &= h_n
\end{align*}
\]
has a solution in integers \( x_1, \ldots, x_n, x_{n+1} \).

For \( n = 2 \), we have two equations
\[
\begin{align*}
a_1 x_1 - b_2 x_2 &= h_1 \\
a_2 x_2 - b_3 x_3 &= h_2
\end{align*}
\]
The divisibility condition \( (\text{iv}) \) gives \( \gcd(a_1, b_2) = \gcd(a_2, b_3) = 1 \), and so both equations have solutions in integers. If \((c_1, d_2)\) is a particular solution of the first equation, then the general solution of the first equation is
\[
\begin{align*}
x_1 &= c_1 + b_2 y_1 \\
x_2 &= d_2 + a_1 y_1
\end{align*}
\]
for any integer \( y_1 \). If \((c_2, d_3)\) is a particular solution of the second equation, then the general solution of the second equation is
\[
\begin{align*}
x_2 &= c_2 + b_3 y_2 \\
x_3 &= d_3 + a_2 y_2
\end{align*}
\]
for any integer \( y_2 \). We have a simultaneous solution of the system of two equations if and only if there exist integers \( y_1 \) and \( y_2 \) such that
\[
x_2 = d_2 + a_1 y_1 = c_2 + b_3 y_2
\]
or, equivalently,
\[
a_1 y_1 - b_3 y_2 = c_2 - d_2.
\]
Because \( \gcd(a_1, b_3) = 1 \), this equation has a solution in integers. This proves the Theorem for \( n = 2 \).
Let $n \geq 3$ and assume that the Theorem is true for $n - 1$ equations in $n$ variables. Consider the diophantine system of $n$ equations in $n + 1$ variables such that, for all $i \in \{1, \ldots, n\}$, equation $(i)$ is
\[ a_i x_i - b_{i+1} x_{i+1} = h_i. \]
This equation has a solution in integers because $\gcd(a_i, b_{i+1}) = 1$. If $(c_i, d_{i+1})$ is a particular solution of the equation, then the general solution of equation $(i)$ is
\[ x_i = c_i + b_{i+1} y_i \]
\[ x_{i+1} = d_{i+1} + a_i y_i \]
for any integer $y_i$. Similarly, for all $i \in \{2, \ldots, n+1\}$, equation $(i - 1)$ is
\[ a_{i-1} x_{i-1} - b_i x_i = h_{i-1} \]
This equation has a solution in integers because $\gcd(a_{i-1}, b_i) = 1$. If $(c_{i-1}, d_i)$ is a particular solution of the equation, then the general solution of equation $(i - 1)$ is
\[ x_{i-1} = c_{i-1} + b_i y_{i-1} \]
\[ x_i = d_i + a_{i-1} y_{i-1} \]
for any integer $y_{i-1}$. Equations $(i - 1)$ and $(i)$ have a simultaneous solution in integers if and only if there exist integers $y_{i-1}$ and $y_i$ such that
\[ d_i + a_{i-1} y_{i-1} = c_i + b_{i+1} y_i \]
or, equivalently, if
\[ a_{i-1} y_{i-1} - b_{i+1} y_i = c_i - d_i - 1. \]
It follows that the original system of $n$ equations in $n + 1$ variables has a solution in integers if and only if the following system of $n - 1$ equations in $n$ variables has a solution in integers:
\[
\begin{align*}
a_1 y_1 - b_3 y_2 &= c_2 - d_1 \\
a_2 y_2 - b_4 y_3 &= c_3 - d_2 \\
& \vdots \\
a_{n-1} y_{n-1} - b_{n+1} y_n &= c_n - d_{n-1}.
\end{align*}
\]
The divisibility condition and the induction hypothesis imply that this system of equations does have an integral solution. This completes the first proof.

The second proof uses the Smith normal form of an integral matrix (Marcus and Minc [14 pp. 40–48]). The Smith normal form of an $m \times n$ matrix $M$ of rank $k$ is the unique diagonal matrix $\text{SNF}(M)$
\[
\begin{pmatrix}
s_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & s_2 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & s_3 & 0 & \cdots & 0 & 0 \\
\vdots & & & & \ddots & & \vdots \\
0 & 0 & 0 & 0 & \cdots & s_k & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & & & \ddots & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]
constructed from $M$ by integral elementary row and column operations and whose coordinates $s_1, \ldots, s_k$ are positive integers. The $k$th determinantal divisor of $A$ is the greatest common divisor of all of the $k \times k$ minors of $A$. If the $k$th determinantal divisor of $A$ is 1, then $s_i = 1$ for all $i = 1, \ldots, k$. If $A$ has rank $n$ and the $n$th determinantal divisor of $A$ is 1, then $s_i = 1$ for all $i = 1, \ldots, n$ and

$$SNF(M) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The homomorphism $M : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ is surjective if and only if the homomorphism $SNF(M) : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ is surjective if and only if rank$(M) = n$ and $s_i = 1$ for all $i = 1, \ldots, n$.

Let $M$ be the $n \times (n+1)$ matrix constructed in Theorem 1 from integer sequences $(a_i)_{i=1}^n$ and $(b_j)_{j=2}^{n+1}$ such that gcd$(a_i, b_j) = 1$ for all $i \in \{1, \ldots, n\}$ and $j \in \{2, \ldots, n+1\}$. The determinant of the $n \times n$ minor obtained by deleting the first column of $M$ is $(-1)^n \prod_{j=2}^{n+1} b_j$. The determinant of the $n \times n$ minor obtained by deleting the last column of $M$ is $\prod_{i=1}^n a_i$. The divisibility condition gcd$(a_i, b_j) = 1$ implies that these determinants are relatively prime integers and so the $n$th determinantal divisor of $M$ is 1. It follows that the matrix $M$ has Smith normal form (5) and the homomorphisms $SNF(M) : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ and $M : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ are surjective. This completes the second proof. □

**Theorem 3.** Let $(v_i)_{i=1}^{n+1}$ be a sequence of positive integers. Define the positive integral sequences $(a_i)_{i=1}^n$ and $(b_j)_{j=2}^{n+1}$ as follows:

$$a_i = 3^{v_i}$$

and

$$b_j = \begin{cases} 4^{v_j} & \text{if } j \text{ is even} \\ 2^{v_j} & \text{if } j \text{ is odd} \end{cases}$$

Let

$$c_{n+1} = \begin{cases} 4 & \text{if } n+1 \text{ is even} \\ 2 & \text{if } n+1 \text{ is odd} \end{cases}$$

The system of linear diophantine equations

$$3^{v_1}x_1 - 4^{v_2}x_2 = 1$$

$$3^{v_2}x_2 - 2^{v_3}x_3 = -1$$

$$3^{v_3}x_3 - 4^{v_4}x_4 = 1$$

$$\vdots$$

$$3^{v_n}x_n - c_{n+1} x_{n+1} = (-1)^{n+1}$$

has a solution in odd positive integers $w_1, w_2, \ldots, w_n, w_{n+1}$. 
Proof. Let
\[ M = \begin{pmatrix}
3^{v_1} & -4^{v_2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 3^{v_2} & -2^{v_3} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 3^{v_3} & -4^{v_4} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 3^{v_4} & -2^{v_5} & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\end{pmatrix}
\]
be the \( n \times (n + 1) \) matrix constructed in Theorem 1. Let \( h_i = (-1)^{i+1} \) for \( i \in \{1, \ldots, n\} \) and \( h = \begin{pmatrix}
h_1 \\
h_2 \\
h_3 \\
\vdots \\
h_n \\
\end{pmatrix} \in \mathbb{Z}^n \). By Theorem 2 there exists an integral vector \( x \in \mathbb{Z}^{n+1} \) such that \( Mx = h \). By Theorem 1(e), there exists an odd positive vector \( w = \begin{pmatrix}
w_1 \\
w_2 \\
w_3 \\
\vdots \\
w_n \\
w_{n+1} \\
\end{pmatrix} \in \mathbb{Z}^{n+1} \) such that \( Mw = h \). This completes the proof. \( \square \)

3. Iterations of the Syracuse function

Let \( \Omega \) be the set of odd positive integers. The Syracuse function is the arithmetic function \( S : \Omega \to \Omega \) defined by
\[ S(m) = \frac{3m + 1}{2^e} \]
where \( e \) is the largest integer such that \( 2^e \) divides \( 3m + 1 \). Note that \( S(m) = 1 \) if and only if \( m = (4^e - 1)/3 \) for some positive integer \( e \).

Lemma 1. Let \( m \) be a positive integer such that
\[ m \equiv -1 \pmod{4} \]
and let \( v \) and \( w \) be the unique positive integers with \( w \) odd such that
\[ m = 2^{v+1}w - 1. \]
Then
\[ m < S(m) < S^2(m) < \cdots < S^v(m) \]
and
\[ S^v(m) = 2 \cdot 3^v w - 1. \]
Moreover,
\[ S^{v+1}(m) = \frac{3^v w - 1}{2^e} \]
for some positive integer \( e \).

Proof. We use induction on \( j \) to prove that if
\[ 0 \leq j \leq v \]
then
\[ S^j(m) = 2^{v+1-j}3^j w - 1 = \left(\frac{3}{2}\right)^j (m + 1) - 1. \]
This is true for $j = 0$. For $j = 1$ we have

$$S(m) = \frac{3(2^{v+1}w - 1) + 1}{2} = \left(\frac{3}{2}\right) 2^{v+1}w - 1 = \left(\frac{3}{2}\right) (m + 1) - 1.$$  

If $j \leq v - 1$ and

$$S^j(m) = 2^{v+1-j}3^j w - 1$$
then

$$S^{j+1}(m) = 2^{v+1-(j+1)}3^{j+1}w - 1 = \left(\frac{3}{2}\right)^{j+1} (m + 1) - 1.$$  

This proves (8), and (8) implies (6).

Note that

$$S^v(m) = 2 \cdot 3^w - 1$$
implies

$$S^{v+1}(m) = S (2 \cdot 3^w - 1) = \frac{3^{v+1}w - 1}{2e}$$
for some positive integer $e$. This completes the proof. □

**Lemma 2.** Let $m$ be a positive integer such that

$$m \equiv 1 \pmod{8}$$
and let $v$ and $w$ be the unique positive integers with $v \geq 3$ and $w$ odd such that

$$m = 2^v w + 1.$$  

If

$$v_0 = \left\lfloor \frac{v - 1}{2} \right\rfloor$$
then

$$v = 2v_0 + 1 \quad \text{or} \quad v = 2v_0 + 2$$
and

(9) \quad m > S(m) > S^2(m) > \cdots > S^{v_0-1}(m) > S^{v_0}(m).$$

If $v = 2v_0 + 1$, then

(10) \quad S^{v_0}(m) = 2 \cdot 3^{v_0} w + 1

If $v = 2v_0 + 2$, then

$$S^{v_0}(m) = 4 \cdot 3^{v_0} w + 1$$

Proof. We have $v - 2j \geq 1$ if and only if

$$j \leq v_0 = \left\lfloor \frac{v - 1}{2} \right\rfloor$$
and $v \geq 3$ implies $v_0 \geq 1$. We use induction on $j$ to prove that if

$$0 \leq j \leq v_0$$
then

(11) \quad S^j(m) = 2^{v-2j}3^j w + 1 = \left(\frac{3}{4}\right)^j (m - 1) + 1.$$
This is true for \( j = 0 \). For \( j = 1 \) we have
\[
S(m) = S(2^v w + 1) = \frac{3(2^v w + 1) + 1}{4} = 2^{v-2} 3^w + 1 = \frac{3}{4}(2^v w) + 1 = \frac{3}{4}(m - 1) + 1.
\]

If \( 1 \leq j \leq v_0 - 1 \) and
\[
S^j(m) = 2^{v-2j} 3^j w + 1
\]
then
\[
v - 2(j + 1) \geq v - 2v_0 = v - 2 \left\lfloor \frac{v - 1}{2} \right\rfloor \geq 1
\]
and
\[
S^{j+1}(m) = S(2^{v-2j} 3^j w + 1) = \frac{2^{v-2j} 3^{j+1} w + 4}{4} = 2^{v-2(j+1)} 3^{j+1} w + 1 = \left(3 \cdot \frac{3}{4}ight)^{j+1} (2^v w) + 1
\]
\[= \left(3 \cdot \frac{3}{4}ight)^{j+1} (m - 1) + 1.
\]
This proves \((11)\), and \((11)\) implies \((9)\).

We have
\[
S^{v_0}(m) = 2^{v-2v_0} 3^{v_0} w + 1
\]
If \( v = 2v_0 + 1 \), then
\[
S^{v_0}(m) = 2 \cdot 3^{v_0} w + 1
\]
If \( v = 2v_0 + 2 \), then
\[
S^{v_0}(m) = 4 \cdot 3^{v_0} w + 1
\]
This completes the proof. \(\Box\)

**Theorem 4.** The Syracuse function \( S \) is wildly increasing-decreasing.

*Proof.* Let
\[
c_{n+1} = \begin{cases} 
4 & \text{if } n + 1 \text{ is even} \\
2 & \text{if } n + 1 \text{ is odd}.
\end{cases}
\]

Let \( V = (v_i)_{i=1}^{n} \) be a finite sequence of positive integers. By Lemma 1 if
\[
m = 2^{v_1+1} w_1 - 1
\]
for some odd positive integer \( w_1 \), then the odd positive integer \( m \) increases for \( v_1 \) iterations of the Syracuse function \( S \). From \((7)\) we have the odd positive integer
\[
S^{v_1}(m) = 2 \cdot 3^{v_1} w_1 - 1.
\]

By Lemma 2 if
\[
S^{v_1}(m) = 2 \cdot 4^{v_2} w_2 + 1
\]
for some odd positive integer \( w_2 \), then the odd positive integer \( S^{v_1}(m) \) decreases for \( v_2 \) iterations of the Syracuse function \( S \). The integers \( w_1 \) and \( w_2 \) are solutions of the diophantine equation
\[
2 \cdot 3^{v_1} w_1 - 1 = 2 \cdot 4^{v_2} w_2 + 1
\]
or, equivalently,
\[ 3^{v_1}w_1 - 4^{v_2}w_2 = 1. \]

From (10) we have the odd positive integer
\[ S^{v_1+v_2}(m) = 2 \cdot 3^{v_2}w_2 + 1. \]

By Lemma 1 if
\[ S^{v_1+v_2}(m) = 2^{v_3+1}w_3 - 1 \]
for some odd positive integer \( w_3 \), then the integer \( S^{v_1+v_2}(m) \) increases for \( v_3 \) iterations of the Syracuse function \( S \). The integers \( w_2 \) and \( w_3 \) are solutions of the diophantine equation
\[ 2 \cdot 3^{v_2}w_2 + 1 = 2^{v_3+1}w_3 - 1 \]
or, equivalently,
\[ 3^{v_2}w_2 - 2^{v_3}w_3 = -1. \]

Continuing inductively, we obtain an odd positive integer \( m \) that has increasing-decreasing pattern \( V \) with respect to the Syracuse function \( S \) if the system of linear diophantine equations
\[
\begin{align*}
3^{v_1}w_1 - 4^{v_2}w_2 &= 1 \\
3^{v_2}w_2 - 2^{v_3}w_3 &= -1 \\
3^{v_3}w_3 - 4^{v_4}w_4 &= 1 \\
&\vdots \\
3^{v_n}w_n - 4^{v_{n+1}}w_{n+1} &= (-1)^{n+1}
\end{align*}
\]
has a solution \( w_1, \ldots, w_{n+1} \) in odd positive integers. This is precisely what Theorem 2 provides. This completes the proof. \( \square \)

4. A SIMPLER PROOF

I thank an anonymous referee for the reference to Everett [7] and the following variant of Theorem 4.

**Theorem 5.** Let \( k \) be a positive integer and let \( \Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) be a \( k \)-tuple with \( \lambda_i \in \{ \text{increase, decrease} \} \) for all \( i \in \{1, 2, \ldots, k\} \). Let \( S(\Lambda) \) be the set of all odd positive integers \( m \) such that
\[ S^{i-1}(m) < S^i(m) \quad \text{if } \lambda_i = \text{increase} \]
and
\[ S^{i-1}(m) > S^i(m) \quad \text{if } \lambda_i = \text{decrease} \]
for all \( i \in \{1, 2, \ldots, k\} \). There are positive integers \( \ell = \ell(\Lambda) \) and \( r = r(\Lambda) \) with \( r \) odd such that, for all \( m \geq 1 \),
\[ m \equiv r \pmod{2^\ell} \quad \text{implies} \quad m \in S(\Lambda). \]

This result immediately implies that the Sylvester function is wildly increasing-decreasing.
Proof. The proof is by induction on $k$.

Let $k = 1$. If $\Lambda = \{\text{increase}\}$, then choose $\ell = 2$ and $r = 3$. By Lemma 1 the
congruence $m \equiv 3 \pmod{2^3}$ implies $m \in S(\Lambda)$. If $\Lambda = \{\text{decrease}\}$, then choose
$\ell = 3$ and $r = 1$. By Lemma 2 the congruence $m \equiv 1 \pmod{2^3}$ implies $m \in S(\Lambda)$. This proves the Theorem for $k = 1$.

Let $k \geq 1$ and assume that for every $k$-tuple $\Lambda$ there exist positive integers
$\ell = \ell(\Lambda)$ and $r = r(\Lambda)$ with $r$ odd that satisfy (12) for all $m \geq 1$. Let $\Lambda' = (\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_{k+1})$ be a $(k+1)$-tuple with $\lambda_i \in \{\text{increase}, \text{decrease}\}$ for all $i \in \{1, 2, \ldots, k, k+1\}$. Consider the $k$-tuple $\Lambda = (\lambda_2, \ldots, \lambda_k, \lambda_{k+1})$. By the induction hypothesis, there exist positive integers $\ell'$ and $r'$ with $r'$ odd such that, for all
$m' \geq 1$,

\[ m' \equiv r' \pmod{2^\ell'} \implies m' \in S(\Lambda'). \]

Suppose $\lambda_1 = \text{increase}$. The odd positive integer $m$ is in $S(\Lambda)$ if

\[ S(m) = \frac{3m + 1}{2} \equiv r' \pmod{2^\ell'} \]

or, equivalently, if

\[ 3m \equiv 2r' - 1 \pmod{2^\ell' + 1}. \]

Let $\ell = \ell' + 1$. Choose $t \in \mathbb{N}$ such that $3t \equiv 1 \pmod{2^\ell}$. Then $t$ and $r = (2r' - 1)t$
are odd and $\ell$ and $r$ satisfy (12).

Suppose $\lambda_1 = \text{decrease}$. The odd positive integer $m$ is in $S(\Lambda)$ if

\[ S(m) = \frac{3m + 1}{4} \equiv r' \pmod{2^\ell'} \]

or, equivalently, if

\[ 3m \equiv 4r' - 1 \pmod{2^\ell' + 2}. \]

Let $\ell = \ell' + 2$. Choose $t \in \mathbb{N}$ such that $3t \equiv 1 \pmod{2^\ell}$. Then $t$ and $r = (4r' - 1)t$
are odd and $\ell$ and $r$ satisfy (12). Thus, the Theorem holds for all $(k+1)$-tuples $\Lambda$. This completes the proof. \qed

5. Open problems

5.1. The infinite constant sequence $v_i = 1$. The Collatz conjecture states that
for every positive integer $m$ there is an integer $k_m$ such that $S^{k_m}(m) = 1$ and
so the Syracuse trajectory $(S^j(m))_{j=0}^\infty$ is eventually constant. This would imply
that there is no infinite sequence of positive integers $V = (v_j)_{j=1}^\infty$ and no positive integer $m$ for which the trajectory $(S^j(m))_{j=0}^\infty$ satisfies the increasing-decreasing
conditions (2) and (4) for all $i \in \mathbb{N}$. Is it possible to prove that some particular
infinite sequence $V = (v_j)_{j=1}^\infty$ is not the increasing-decreasing pattern of any positive integer $m$ under iterations of the Syracuse function? For example, can the constant
sequence $V = (v_j)_{j=1}^\infty$ with $v_i = 1$ for all $i$ be proven impossible? This is equivalent
to proving that if $M$ is the infinite matrix with $(i,j)$th coordinate

\[ M_{i,j} = \begin{cases} 3 & \text{if } j = i \\ -4 & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases} \]

for all $i, j \in \mathbb{N}$ and if $h = (h_i)_{i=1}^\infty$ is the infinite vector with $h_i = (-1)^{i+1}$ for all $i \in \mathbb{N}$, then there exists no infinite odd positive vector $w = (w_j)_{j=1}^\infty$ such that
$Mw = h$. 


5.2. **The finite constant sequence** $v_i = 1$. For every positive integer $n$ there exists an odd positive integer $m$ that has the increasing-decreasing pattern $V = (v_i)_{i=1}^n$ with $v_i = 1$ for all $i \in \{1, 2, \ldots, n\}$. Let $m_n$ be the least such integer. The sequence $(m_n)_{n=1}^\infty$ is increasing. Is it strictly increasing? Can $m_n$ be efficiently computed?

5.3. **Sufficient conditions for wildly increasing-decreasing.** The method of proof of Theorem 5 can be extended to prove that a large class of arithmetic functions is wildly increasing decreasing. Are there necessary and sufficient conditions for an arithmetic function $f$ to be wildly increasing-decreasing?

Are there sufficient conditions for a sequence $F = (f_j)_{j=0}^\infty$ of arithmetic functions to be wildly increasing-decreasing?

**References**

[1] E. Catalan, Propositions et questions diverses, Bull. Soc. Math. France 16 (1888), 128–129.
[2] L. E. Dickson, Theorems and tables on the sums of divisors of a number, Quart. J. Math. 44 (1913), 264-296.
[3] S. Eliahou, The 3x+1 problem: new lower bounds on nontrivial cycle lengths, Discrete Math. 118 (1993), 45–56.
[4] P. Erdős, Über die Zahlen der Form $\sigma(n) - n$ und $n - \varphi(n)$, Elem. Math 11 (1973), 83–86.
[5] P. Erdős, On asymptotic properties of aliquot sequences, Math. Computation 30 (1976), 641–645.
[6] P. Erdős, A. Granville, C. Pomerance, and C. Spiro, On the normal behavior of the iterates of some arithmetic functions, Analytic Number Theory (Allerton Park, IL, 1989), Prog. Math. 85, Birkhäuser Boston, Boston, MA, 1990, pp. 165–204.
[7] C. J. Everett, Iteration of the number-theoretic function $f(2n) = n, f(2n + 1) = 3n + 2$, Adv. Math. 25 (1977), 42–45.
[8] R. K. Guy, Unsolvable Problems in Number Theory, 3rd edition, Springer-Verlag, New York, 2004.
[9] R. K. Guy and J. L. Selfridge, J. L., Interim report on aliquot series, in: Proceedings of the Manitoba Conference on Numerical Mathematics (Univ. Manitoba, Winnipeg, Man., 1971), 557–580.
[10] R. K. Guy and J. L. Selfridge, J. L., What drives an aliquot sequence?, Math. Computation 29 (1975), 101–107.
[11] J. C. Lagarias, The 3x+1 problem and its generalizations, Amer. Math. Monthly 92 (1985), 3–23.
[12] J. C. Lagarias, The ultimate challenge: the 3x+1 problem, Amer. Math. Soc., Providence, RI, 2010.
[13] H. W. Lenstra, Jr., Problem 6064, Amer. Math. Monthly 82 (1975), 1016. Solution by the proposer, Amer. Math. Monthly 84 (1977), 580.
[14] M. Marcus and H. Minc, A Survey of Matrix Theory and Matrix Inequalities, Dover Publications, New York, 1992.
[15] P. Pollack and C. Pomerance, Some problems of Erdős on the sum-of-divisors function, Trans. Amer. Math. Soc. Ser. B 3 (2016), 1–26.
[16] C. Pomerance, Aliquot sequences, University of Calgary lecture, October 2, 2020, https://math.dartmouth.edu/~carlp/upintconf.pdf.
[17] T. Tao, Almost all orbits of the Collatz map attain almost bounded values, Forum Math. Pi 10 (2022), Paper No. e12, 56.
[18] R. Terras, A stopping time problem on the positive integers, Acta Arith. 30 (1976), 241–252.

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