Steady states of asymmetric exclusion processes with inhomogeneous hopping

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We study the nonequilibrium steady states in totally asymmetric exclusion processes (TASEP) with open boundary conditions having spatially inhomogeneous hopping rates. Considering smoothly varying hopping rates, we show that the steady states are in general classified by the steady state currents in direct analogy with open TASEPs having uniform hopping rates. We calculate the steady state bulk density profiles, which are now spatially nonuniform. We also obtain the phase diagrams in the plane of the control parameters, which though have phase boundaries that are in general curved lines, have the same topology as their counterparts for conventional open TASEPs, independent of the form of the hopping rate functions. This reveals a type of universality, not encountered in critical phenomena. Surprisingly and in contrast to the phase transitions in an open TASEP with uniform hopping, our studies on the the phase transitions in the model indicate that all the three transitions are first order.

I. INTRODUCTION

Many natural systems are driven by some external fields or are made of self-propelled particles. In the long time limit, these systems evolve into stationary states which carry steady currents, which are hallmarks of nonequilibrium systems. Such states are characterized by a constant gain or loss of energy, which distinguishes them from systems in thermal equilibrium. Examples of such driven systems range from live cell biological systems like ribosomes moving along mRNA or motor molecules “walking” along molecular tracks known as microtubules to ions diffusing along narrow channels, or even vehicles traveling along roads. In order to elucidate the nature of such nonequilibrium steady states and in the absence of a general theoretical framework, it is useful to study purpose-built simple models. To this end, a variety of driven lattice gas models have been introduced and studied extensively.

In this work, we focus on driven one-dimensional (1D) models with open boundaries, where particles preferentially move in one direction. In particular, we work on the totally asymmetric simple exclusion process (TASEP), that has become one of the paradigms of nonequilibrium physics in low-dimensional systems (see, e.g., Ref. [2] for reviews). In this model identical particles hop unidirectionally and with a uniform rate along a 1D lattice. The hopping movement is subject to exclusion, i.e., when the target site is empty, since a given site can accommodate maximum one particle. Particles enter the system at one side at a specified rate α, and leave the system through the other end at a given rate β; α and β are the two control parameters of TASEP. It is known that the steady states of TASEPs with open boundaries are rather sensitive to the boundary conditions: by varying the boundary conditions, i.e., by varying α, β, the steady states of open TASEPs can be varied, resulting into boundary-induced nonequilibrium phase transitions. These are genuine nonequilibrium effects, since equilibrium systems are usually insensitive to precise boundary conditions.

In the original TASEP model, the hopping rate in the bulk is assumed to be a constant (of unit value), which is of course an idealisation. In real life examples it is generally expected to have nonuniformity along the bulk of the TASEP channel leading to nonuniform hopping rates. For instance, mRNA in cells are known to have pause sites, where the effective hopping rates are lower. This is a potentially important issue even in urban transport, where the speeds of vehicles (which is the analogue of the hopping rates here) depend sensitively on the bottlenecks along the roads. Such spatially varying hopping rates can either be smoothly varying along the TASEP lanes, or be random quenched disorders with given distribution. We focus here on the case with smoothly varying hopping rates, for which the generic nature of the steady states in TASEPs are still not known. There have been some studies on quenched heterogeneous TASEP; see, e.g., Refs. [6, 7] for previous studies on different aspects of heterogeneous TASEP. Recently, the steady states of TASEPs with periodic boundary conditions are affected by smoothly varying hopping rates are studied; see also Ref. [8] for a study on periodic TASEP with random quenched disordered hopping rates. A type of universality has been uncovered, showing the topological equivalence of the phase diagrams independent of the precise form of the space dependence of the hopping rates. Our studies here complement these results by considering the problem in an open TASEP with space-dependent hopping. Following Ref. [8], we set up the analytical mean-field theory (MFT) framework to calculate the steady state density profiles for generic smoothly varying hopping rates. We illustrate the theoretical predictions by...
II. MODEL

The model consists of a 1D lattice of size $L$. The particles enter through the left end at rate $\alpha$, hop unidirectionally from the left to the right, all subject to exclusion, i.e., a single site can accommodate maximum one particle at a time, and finally leave the system at a rate $\beta$. Labelling each site by an index $i$ that runs from 1 to $L$, the hopping rate at site $i$ is given by $q_i \leq 1$; see Fig. 1 for a schematic model diagram.

![Schematic model diagram](image)

FIG. 1: (Colour online) Schematic model diagram. Broken line represents the TASEP lattice. Particles enter and exit at rates $\alpha$ and $\beta$, respectively, and hop from left to right, subject to exclusion.

A microscopic configuration of the model is characterised by a distribution of identical particles on the lattice, i.e., by configurations $C = \{n_1, n_2, \ldots, n_L\}$, where each of the occupation numbers $n_i$ is equal to either zero (vacancy) or one (particle), as it should be in a model with exclusion. Physically, a hard core repulsion between the particles is imposed, resulting into prohibition of a double or higher occupancy of sites in the model. The full state space then consists of $2^L$ configurations. The following elementary processes fully define the microscopic dynamical update rules of this model:

(a) At any site $i = 1, \ldots, L - 1$ a particle can jump to site $i + 1$ if unoccupied with a rate $q_i \leq 1$.

(b) At the site $i = 1$ a particle can enter the lattice with rate $aq(1)$ only if is unoccupied; and

(c) At the site $i = N$ a particle can leave the lattice with rate $\beta q(L)$ when it is occupied.

In general, $q_i \neq q_j$ for $i \neq j$. Processes (a)-(c) formally define a TASEP with open boundary conditions. If all of $q_i = 1$ for all $i$ identically, then this model reduces to the conventional TASEP with open boundary conditions [2].

II. STEADY-STATE DENSITIES

We are interested to calculate the density profiles in the steady states. To this end, we set up MFT which can be solved analytically. We supplement the MFT results by extensive Monte-Carlo simulations (MCS).

A. Monte-Carlo simulations

We consider a lattice of $L$ sites, labelled by an index $i$ with $i \in [1, L]$. Let $n_i(t)$, which is either 0 or 1, be the occupation at site $i$ at time $t$. We perform MCS studies of the model subject to the update rules (a)-(c) described above in Sec. II by using a random sequential updating scheme. The particles enter the system through the left most site $(i = 1)$ at a fixed rate $\alpha$, subject to exclusion, i.e., if $n_1 = 0$. After hopping through the system from $i = 1$ to $L$, subject to exclusion, the particles exit the system from $i = L$ at a fixed rate $\beta$. Here, $\alpha$ and $\beta$ are the two simulation parameters, which are varied to produce different steady states. After reaching the steady states, the density profiles are calculated and temporal averages are performed. This produces time-averaged, space-dependent density profiles, given by $\langle n_i(t) \rangle$, which are parametrised by $\alpha$ and $\beta$; here $\langle \ldots \rangle$ implies temporal averages over steady states. The simulations have been performed with $L = 10000$ up to $10^7$ Monte-Carlo steps. Lastly, all the measurements are made in the steady states, which are reached by the system after spending certain transient times. In an open TASEP, a steady state is easily ascertained by observing the spatio-temporal constancy of the average density $\langle n_i(t) \rangle$ (ex-

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calculating the density profiles for a few representative examples of spatially varying hopping rates in Monte-Carlo simulation (MCS) studies. We show that the notion of universality illustrated in Ref. [3] can be extended to systems with open boundaries as well, establishing the robustness of the notion of universality of the phase diagrams here. We further study the phase transitions in the model. Both MFT and MCS studies show that all the phase transitions are first order or discontinuous in nature, a surprising and unexpected outcome from this work. This is in contrast to an open TASEP with uniform hopping. The rest of the article is organised as follows. In Sec. II we define and construct our model. Next, in Sec. IIIA we discuss the algorithm of the MCS study of the model to numerically calculate the steady state densities. Then in Sec. IIIB we set up the MFT, and solve it to obtain the steady state densities for smoothly varying hopping rates. In Sec. IIIC we present the phase diagrams of the model. Then in Sec. IV we discuss the phase transitions in the model. We summarise our results in Sec. V.

In general, $q_i \neq q_j$ for $i \neq j$. Processes (a)-(c) formally define a TASEP with open boundary conditions. If all of $q_i = 1$ for all $i$ identically, then this model reduces to the conventional TASEP with open boundary conditions [2]. We consider some specified choices of $q_i$ that depends explicitly on $i$, and study their effects on the nonequilibrium steady states of the model. Recall that the steady states of an open TASEP with $\alpha$ and $\beta$ as the entry and exit rates, and a uniform hopping rate are characterised by the mean bulk density $ρ_T$: For $\alpha < \beta$ and $\alpha < \beta/2$, one has $ρ_T = \alpha$ giving the low density (LD) phase, for $\beta < \alpha$ and $\beta < 1/2$, one has $ρ_T = 1 - \beta$ giving the high density (HD) phase, and for $\alpha, \beta > 1/2$, one has $ρ_T = 1/2$ giving the maximal current (MC) phase. This immediately gives the phase boundary in the $\alpha - \beta$ plane [2]. The principal aim of the present study is to find the phases and phase boundaries, and the principles behind obtaining them when the hopping rate is not constant, but spatially smoothly varying.
including the domain walls) in the bulk of the system. In
the present problem, such a way to confirm the steady
state fails due to the (expected) spatially varying steady
state density in the bulk. Instead, we use the constancy
of the current $J$ in the steady state, a condition that
holds both in the present study and also for a uniform
open TASEP. In our MCS studies, all our measurements
are done only after this condition is satisfied.

B. Mean-field theory

The dynamics of TASEP is formally given by rate
equations for every site which are not closed. In MFT
approximation, we neglect correlation effects and replace
the average of product of densities by the product of av-
gerage of densities [10]. While this is an approximation,
this has worked with high degree of accuracy in the orig-
inal TASEP problem and its many variants (see, e.g.,
Refs. [11–13] as representative examples); we use MFT
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\( \alpha, \beta \). However, \( J \) may not reach \( J_{\max} \) for any \( \alpha, \beta \); see below. In the limit of uniform hopping with \( q(x) = 1 \) everywhere, \( J_{\max} = 1/4 \), corresponding to the MC phase current in the conventional TASEP. We thus note that in the present model steady states with current \( J = J_{\max} \) for a given \( q(x) \) should generalise the standard MC phase.

We now systematically derive the conditions for the different phases. To do this, we must calculate \( J \) to specify the solutions \( \rho_+(x) \) and \( \rho_-(x) \) completely.

Recall that in the LD phase of conventional open TASEPs with uniform hopping, the steady state is described by the incoming current, which in the bulk is given by \( J_{\text{LD}}^I = \alpha(1 - \alpha) < J_{\text{HD}}^I = \beta(1 - \beta) \), the outgoing current. This corresponds to \( \rho = \alpha \) as the bulk density in the LD phase. In contrast, in the HD phase, \( J_{\text{HD}} < J_{\text{LD}} \), giving \( 1 - \beta \) as the HD phase bulk density; a superscript \( T \) refers to an open TASEP. The MC phase is associated with the current \( J_{\text{MC}}^T = 1/4 \) in the bulk. The LD-HD phase boundary is given by the condition \( J_{\text{LD}}^T = J_{\text{HD}}^T \), giving \( \alpha = \beta \); the LD-MC and HD-MC phase boundaries likewise are given by \( J_{\text{LD}}^T = J_{\text{MC}}^T \) and \( J_{\text{HD}}^T = J_{\text{MC}}^T \), giving, respectively, \( \alpha = 1/2 \) and \( \beta = 1/2 \) as the phase boundaries. We now generalise this picture by finding out the forms of \( J \) in the present problem.

\( b. \) LD phase:- We start by noting that the current in the bulk of the TASEP channel is \( J = q(x)\rho(x)(1 - \rho(x)) \), where \( \rho(x) \) is \( \rho_+(x) \) or \( \rho_-(x) \). Since \( \rho(0) = \alpha \), we obtain

\[
J_{\text{LD}} = q(0)\alpha(1 - \alpha). \tag{III.11}
\]

Since, the steady state bulk density in the LD phase is less than 1/2 everywhere, the density profile \( \rho_{\text{LD}}(x) \) in the LD phase is given by

\[
\rho_{\text{LD}}(x) = \frac{1}{2} \left[ 1 - \sqrt{1 - \frac{4q(0)}{q(x)}\alpha(1 - \alpha)} \right] < \frac{1}{2}. \tag{III.12}
\]

c. HD phase:- The logic we have developed above to obtain \( \rho_{\text{LD}}(x) \) can be used to obtain \( \rho_{\text{HD}}(x) \), the steady state density in the HD phase. Noting that \( \rho(1) = 1 - \beta \), we obtain the HD phase current

\[
J_{\text{HD}} = q(1)\beta(1 - \beta). \tag{III.15}
\]

Since, the steady state bulk density everywhere is more than 1/2, the density profile \( \rho_{\text{HD}}(x) \) in the HD phase is given by

\[
\rho_{\text{HD}}(x) = \frac{1}{2} \left[ 1 + \sqrt{1 - \frac{4q(1)}{q(x)}\beta(1 - \beta)} \right] > \frac{1}{2}. \tag{III.16}
\]

Therefore, we must have \( \beta < 1/2 \) as for an open TASEP with uniform hopping. In contrast to \( \rho_{\text{LD}}(x) \), given by (III.12) above, \( \rho_{\text{HD}}(x) \) in (III.16) depends on \( q(x) \) and the exit rate \( \beta \), but not on \( \alpha \), as expected in the HD phase. With \( q(x) = q(1) = \text{const.} \) everywhere, \( \rho_{\text{HD}}(x) = 1 - \beta \),

Then we must have \( \alpha < 1/2 \) as in an open TASEP with a uniform hopping rate. Equation (III.12) gives the steady state density in the LD phase for a given \( q(x) \) and depends on the entry rate \( \alpha \), but not on the exit rate \( \beta \), as expected in the LD phase. With \( q(x) = q(0) = \text{const.} \) everywhere, \( \rho_{\text{LD}}(x) = \alpha \), neglecting the other solution \( 1 - \alpha > 1/2 \) for \( \alpha < 1/2 \) for an open TASEP with a uniform hopping rate.

We have plotted \( \rho_{\text{LD}}(x) \) versus \( x \) in Fig. 2 for two different and simple choices of the hopping rate function \( q(x) \):

Choice I: \( q(x) = \frac{1}{1 + 2x}, \quad 0 \leq x \leq 1/2, \)

\[
= \frac{1}{3 - 2x}, \quad 1/2 \leq x \leq 1, \quad \text{(III.13)}
\]

Choice II: \( q(x) = \frac{1}{2} \left[ 2 - \frac{x^2}{0.49} \right], \quad 0 \leq x \leq 0.7, \)

\[
= \frac{1}{2} \left[ 2 - \frac{(x - 1.4)^2}{0.49} \right], \quad 0.7 \leq x \leq 1. \quad \text{(III.14)}
\]

Clearly, \( q(x) \) in Choice I is symmetric about \( x = 1/2 \), whereas \( q(x) \) in Choice II has no particular symmetry. Results on the steady state densities from MFT and MCS studies are plotted together in Fig. 2 which show good agreements between MFT and MCS results.

In an open TASEP with uniform hopping, \( \alpha < 1/2 \) and \( \alpha < \beta \) specify the LD phase, whereas \( \beta < \alpha \) and \( \beta < 1/2 \) specify the HD phase. What are the analogous conditions here? These conditions in the present case may be obtained by considering the steady state currents. We recall that the above conditions for the LD and HD phases in an open TASEP with uniform hopping can be recast in terms of the steady state currents as \( J_{\text{LD}} < 1/4 \) and \( J_{\text{LD}} < J_{\text{HD}} \) for the LD phase, and \( J_{\text{HD}} < 1/4 \) and \( J_{\text{HD}} < J_{\text{LD}} \) for the HD phase. These conditions may be generalised to the present case with non-uniform hopping. The LD phase now exists for

\[
J_{\text{LD}} = q(0)\alpha(1 - \alpha) < J_{\text{HD}} = q(1)\beta(1 - \beta), \quad J_{\text{LD}} < \frac{q_{\text{min}}}{4}, \quad \text{(III.17)}
\]
Similarly, for the HD phase to exist we must have
\[ J_{\text{HD}} < J_{\text{LD}}, \quad J_{\text{HD}} < \frac{q_{\text{min}}}{4}. \quad \text{(III.18)} \]

We have plotted \( \rho_{\text{HD}}(x) \) versus \( x \) in Fig. 3 for \( q(x) \) as defined in Choice I above. The HD phase density plots for \( q(x) \) as given in Choice II above, likewise, can be obtained from corresponding plots in the LD phase by using the particle-hole symmetry. Results from MFT and MCS studies are plotted together in Fig. 3 which again reveal good agreements between MFT and MCS results.

d. **MC phase:** The steady density in the MC phase, \( \rho_{\text{MC}}(x) \), is somewhat tricky to calculate. We already know that the steady state current in the MC phase
\[ J_{\text{MC}} = q_{\text{min}}/4, \quad \text{(III.19)} \]
which can be used either in \( \rho_+(x) \) or \( \rho_-(x) \), with these two solutions become identical at \( x = x_0 \), at which \( q(x) = q_{\text{min}} \). MCS studies reveal that a part of \( \rho_{\text{MC}}(x) \) is bigger than \( 1/2 \), whereas elsewhere it is smaller than \( 1/2 \). Thus in order to construct \( \rho_{\text{MC}}(x) \), we must use both \( \rho_+(x) \) and \( \rho_-(x) \), i.e., \( \rho_{\text{MC}}(x) \) is a combination of \( \rho_+(x) \) and \( \rho_-(x) \), with the two segments meeting at \( x_0 \). This further implies that if \( x_0 \), the location of \( q_{\text{min}} \) is not in the bulk, but at the extreme ends (i.e., \( x = 0, 1 \)), \( \rho_{\text{MC}}(x) \) will consist of only \( \rho_-(x) \) or \( \rho_+(x) \). Interestingly, this means in general the average density in the MC phase (averaged over the whole TASEP) can be more or less than \( 1/2 \)! This is clearly in contrast to TASEP with uniform hopping, where the average density in the MC phase is \( 1/2 \). We have plotted \( \rho_{\text{MC}}(x) \) versus \( x \) in Fig. 4.

Results from MFT and MCS studies are plotted together for several choices of \( q(x) \), which again show good agreements between MFT and MCS results.
FIG. 3: (Colour online) Plots of the steady state density $\rho(x)$ versus $x$ in the HD phase for the hopping rate function $q(x)$ as given in Choice I above for two different sets of values of $\alpha$ and $\beta$. These are connected to the corresponding $\rho_{LD}(x)$ via the particle-hole symmetry discussed above. In each plot, the green line represents the MFT prediction, the red points are from the corresponding MCS study; the blue line represents $q(x)$. Good agreement between the MFT and MCS predictions can be seen (see text).

C. Phase diagram

We now discuss the conditions to obtain the phase diagram and the phase boundaries in the $\alpha,\beta$-plane. First consider the boundary between the LD and HD phases. In the LD phase, the bulk current is given by $J_{LD}$ in (III.11), whereas the bulk current in the HD phase is given by $J_{HD}$ in (III.15). The two phases meet when $J_{LD} = J_{HD}$, which gives the phase boundary between the LD and HD phases that has the form

$$q(0)\alpha(1-\alpha) = q(1)\beta(1-\beta).$$  \hspace{1cm} (III.20)

This is a quadratic equation in $\beta$ in terms of $\alpha$ with two solutions $\beta_\pm$:

$$\beta_\pm = \frac{1}{2} \left[ 1 \pm \sqrt{1-4\mu \alpha (1-\alpha)} \right],$$  \hspace{1cm} (III.21)

where $\mu \equiv q(0)/q(1)$, which can be bigger or smaller than unity. Since $\beta < 1/2$ for the HD phase, $\beta = \beta_-$ gives the LD-HD phase boundary. Phase boundary (III.21) automatically reduces to $\alpha = \beta$, the well-known result for the phase boundary or the coexistence line between the LD and HD phases in an open TASEP with a uniform $q(x)$. In fact, even with nonuniform $q(x)$, $\alpha = \beta$ is the phase boundary, so long as $q(0) = q(1)$ is maintained,
FIG. 4: (Colour online) Plots of the steady state density $\rho(x)$ versus $x$ in the MC phase for different choices of the hopping rate functions and for different sets of values of $\alpha$ and $\beta$. (top) $q(x)$ as given in Choice I, (bottom) $q(x)$ as given in Choice II. In each plot, the green line represents the MFT prediction, the red points are from the corresponding MCS study; the blue line represents $q(x)$. Good agreement between the MFT and MCS predictions can be seen (see text).

The steady state density profile at the LD-HD coexistence line has a special structure. In an open TASEP with uniform hopping rate, it occurs on the line $\alpha = \beta < 1/2$ in the $\alpha - \beta$ plane, and is actually a delocalised domain wall (DDW), which is a domain wall or a density “shock” whose position is not fixed but fluctuates along the whole length of the TASEP length. Moreover, the position of the domain wall is equally likely to be anywhere in the TASEP. This means the long time average of the density profile, that essentially captures the envelop of the DDW, is an inclined straight line connecting $\rho_{LD} = \alpha$ at the entry end and $\rho_{HD} = 1 - \beta$ at the exit end. Since the MFT neglects all fluctuations, it cannot capture this DDW. We numerically investigate the analogue of a DDW in a uniform open TASEP in the present problem for the case $q(x)$ in Choice I above, for which the coexistence line is still $\alpha = \beta$ due to the symmetry of the function $q(x)$ chosen. We numerically calculate $\rho(x)$ at $\alpha = \beta = 0.07$ and present our result in Fig. 5. More specifically, we calculate $\rho(x)$ over short time windows $\sim 10^6$ MCS steps, and also over MCS steps of $10^7$ to obtain the long-time averaged envelop of the density profile. Due to the diffusive nature of the DDW fluctuations, good statistics for the DDW envelop requires averaging over $\sim L^2$ MCS steps. Due to this, we have restricted this particular study to $L = 1000$. We find that over short time windows, $\rho(x)$ has the form of a step function, reminiscent of a localised domain wall (LDW) in heterogeneous TASEPs in ring geometries [11, 12]. However, the position of the LDW is different at different times, implying a DDW. The long-time averaged envelop of $\rho(x)$ takes the form of an inclined line, as shown in Fig. 5. Due to the computational requirement of a very large time required to obtain the long-time averaged shape of the DDW, we cannot precisely determine its geometrical shape. Going beyond MFT by taking into account of fluctuations should allow us write down a Fokker-Planck
equation for the instantaneous position of the density shock \[17\]. Solving this equation one can in principle determine the mathematical form of the envelop, which is outside the scope of the present study.

Similar considerations allow us to obtain the LD-MC and HD-MC phase boundaries. For example, the LD-MC phase boundary is given by the condition \( J_{LD} = J_{MC} \), which gives

\[
\alpha = \frac{1}{2} \left[ 1 - \sqrt{1 - \frac{q_{\min}}{q(0)}} \right], \quad \text{(III.22)}
\]

since \( \alpha < 1/2 \) for the LD phase. Assuming \( q(i) = q_{\min} \) for some \( i \) in the bulk, the effect of a nonuniform \( q(x) \) is to shift the boundary line (III.22) towards the \( \beta \)-axis. Likewise, the HD-MC phase boundary is given by the condition \( J_{HD} = J_{MC} \), giving

\[
\beta = \frac{1}{2} \left[ 1 - \sqrt{1 - \frac{q_{\min}}{q(1)}} \right], \quad \text{(III.23)}
\]

since \( \beta < 1/2 \) for the HD phase. Again with \( q(i) = q_{\min} \) for some \( i \) in the bulk, the effect of a nonuniform \( q(x) \) is to shift the boundary line (III.23) towards the \( \alpha \)-axis. The three phase boundaries meet at \( \left( \frac{1}{2}, \frac{1}{2} \right) \). Since \( q(0), q(1) \geq q_{\min} \), the general effect of a nonuniform hopping rate appears to be to enlarge the MC phase region and shrink the LD and HD phase regions in the \( \alpha - \beta \)-plane. Furthermore, since \( q(0) \neq q(1) \) in general, the phase diagram could be asymmetric under interchange of \( \alpha \) and \( \beta \). Phase diagrams for \( q(x) \) in Choice I and \( q(x) \) in Choice II are shown in Fig. 6 (top) and Fig. 6 (bottom), respectively. Phase boundaries (III.21), (III.22) and (III.23) calculated from MFT, and the corresponding results from MCS studies are superposed. Good agreement between the two are found.

Let us now make some general observations on the phase diagrams in Fig. 6. Clearly, the phase diagrams in Fig. 6 are quantitatively different from the well-known phase diagram of an open TASEP with uniform hopping. First of all, the MC region of the phase diagrams is now distinctly bigger with space-dependent \( q(x) \) than in the corresponding phase diagram with a constant hopping.
rate. Secondly, between the two phase diagrams presented in Fig. 6 the one with \( q(x) \) as given in Choice I above with \( q(0) = q(1) \) [Fig. 6 (top)] remains unchanged under the interchange of \( \alpha \) and \( \beta \), same as for the phase diagram for an open TASEP with uniform hopping rate. In contrast, the phase diagram with \( q(x) \) as given in Choice II, such that \( q(0) \neq q(1) \) [Fig. 6 (bottom)] has no such symmetry under the interchange of \( \alpha \) and \( \beta \). These properties are consistent with our discussions above; see Eqs. (III.21), (III.22), and (III.23). Nonetheless, the phase diagrams above have the same topology as that for an open TASEP with uniform hopping: all of them have three phases, which meet at a common point. This establishes the universality of the phase diagrams for different choices of the hopping rate function, a key qualitative outcome from the present study, and complements the results of Ref. 8.

IV. PHASE TRANSITIONS

The original TASEP model with open boundaries and uniform hopping, the transition between the LD and HD phases are first order transitions, whereas those between the MC and LD or HD phases are second order transitions. The difference in the average bulk densities of the two phases serves as the order parameter in each of these transitions. We can generalise this in the present study within the MFT. To start with, we define the mean density

\[
\bar{\rho}_a \equiv \frac{1}{L} \int_0^1 \rho_a(x) \, dx, \tag{IV.1}
\]

for the phase \( a \), where \( a = \text{LD, HD or MC phase} \). Since if \( \rho_{\text{LD}}(x) < 1/2 \) in the bulk of the system, \( \bar{\rho}_{\text{LD}} < 1/2 \) necessarily. Similarly, if \( \bar{\rho}_{\text{MC}} > 1/2 \) necessarily. Interestingly, \( \bar{\rho}_{\text{MC}} \) need not be 1/2, in contrast to conventional open TASEPs with uniform hopping. In fact, in the present study, with \( q(x) \) in Choice I above, \( \bar{\rho}_{\text{MC}} = 1/2 \) due to the symmetry of \( q(x) \) and hence \( \rho_{\text{MC}} \) about \( x = 1/2 \). In contrast, for \( q(x) \) in Choice II above, \( \bar{\rho}_{\text{MC}} > 1/2 \). With this, considering the mean density as the order parameter, the transition between the LD and HD phases is a first order transition with

\[
\Delta_{\text{HD-LD}} \equiv \bar{\rho}_{\text{HD}} - \bar{\rho}_{\text{LD}} \tag{IV.2}
\]

showing a jump across the LD-HD phase boundary. This jump, given by the magnitude of \( \Delta_{\text{HD-LD}} \) is to be calculated on the phase boundary between the LD and HD phases, and clearly depends on \( \alpha \) (or equivalently \( \beta \)). The finite jump of \( \Delta_{\text{HD-LD}} \) tells us that the phase transition in question is a first order transition. To study the phase transitions between the MD and LD or HD phases, we similarly consider

\[
\begin{align*}
\Delta_{\text{LD-MC}} & \equiv \bar{\rho}_{\text{LD}} - \bar{\rho}_{\text{MC}}, \tag{IV.3} \\
\Delta_{\text{HD-MC}} & \equiv \bar{\rho}_{\text{HD}} - \bar{\rho}_{\text{MC}}. \tag{IV.4}
\end{align*}
\]

The transitions would be second order if the respective order parameters defined above would vanish at the corresponding phase boundaries. Else, if instead they show discontinuities, the transitions are first order in nature. In this Section, we focus on the phase transitions in the restricted case with symmetric \( q(x) \) with a single minimum (as above). The more general cases including asymmetric \( q(x) \) will be discussed elsewhere in future. We use both MFT and MCS to analyse the phase transitions. We first study the LD-HD transition. To that end, we consider Eq. (III.7) or Eq. (III.16) for \( \rho_{\text{HD}}(x) \), and correspondingly Eq. (III.8) or Eq. (III.12) for \( \rho_{\text{LD}}(x) \). Since both \( J_{\text{LD}} = J_{\text{HD}} > J_{\text{MC}} = q_{\text{min}}/4 \), we conclude that the discriminant in each of Eq. (III.7) or Eq. (III.16) and Eq. (III.8) or Eq. (III.12), giving \( \rho_{\text{HD}}(x) > 1/2 \) and \( \rho_{\text{LD}}(x) < 1/2 \) at all \( x \). This in turn implies \( \bar{\rho}_{\text{HD}} > 1/2 \) and \( \bar{\rho}_{\text{LD}} < 1/2 \) necessarily. This holds even at the transition point making \( \bar{\rho}_{\text{HD}} \neq \bar{\rho}_{\text{LD}} \), which in turn means \( \Delta_{\text{HD-LD}} \) is finite at the transition. Thus the LD-HD transition in the MFT is first order, as in an open TASEP with constant hopping rates. We now turn to the LD-MC transition, which is a second order transition in an open TASEP with constant hopping. To analyse this for symmetric \( q(x) \), we use the fact that \( \rho_{\text{MC}} = 1/2 \) (see above), and again consider Eq. (III.5) or Eq. (III.13) for \( \rho_{\text{MC}}(x) \). At the LD-MC transition \( \bar{\rho}_{\text{MC}} = 1/2 \) (see above), \( \rho_{\text{MC}} \) behaving like a symmetric \( q(x) \) is the mean LD phase density having one minimum. To verify these MFT predictions for our model numerically, we have studied the nature of the phase transitions numerically across the LD-HD, LD-MC and HD-MC phase boundaries. More specifically, we calculate:

(i) \( \bar{\rho} \) as a function of \( \alpha \) for a fixed \( \beta = 0.7 \), as \( \alpha \) approaches the LD-HD phase boundary; see Fig. 7(left) for a plot of the average density as a function of \( \alpha \). On one side of the transition, \( \bar{\rho} \) is the mean LD phase density \( \bar{\rho}_{\text{LD}} \) that rises with \( \alpha \); on the other side of it, \( \bar{\rho} \) is \( \bar{\rho}_{\text{HD}} \) that remains independent of \( \alpha \) as well as \( \beta \). Surprisingly,
this plot clearly shows a jump in $\rho$ across the transition, meaning a first order transition, in contrast to an open TASEP with uniform hopping.

(iii) $\overline{\rho}_{\text{HD}}$ as a function of $\beta$ for a fixed $\alpha = 0.7$, as $\beta$ approaches the HD-MC phase boundary; see Fig. 7 (right) for a plot of the average density as a function of $\beta$. On one side of the transition, $\overline{\rho}$ is the mean HD phase density $\overline{\rho}_{\text{HD}}$ that decreases as $\beta$ increases; on the other side of it, $\overline{\rho}$ is $\overline{\rho}_{\text{MC}}$ that remains independent of $\alpha$ as well as $\beta$. Surprisingly, this plot clearly shows a jump in $\overline{\rho}$ across the transition, again implying a first order transition, again in contrast to an open TASEP with uniform hopping.

Our MCS studies show that in all these cases there is a jump in the density at the respective phase boundary, implying a discontinuous or a first order transition, in agreement with the MFT prediction for the same. This is a truly novel result, that shows how quenched disorder can alter the order of phase transitions. To benchmark our numerical codes, in Fig. 8 in Appendix A we have shown the analogous plots for the LD-HD, LD-MC and HD-MD transitions for an open TASEP with uniform hopping. Unsurprisingly, the plots in Fig. 8 show a first order LD-HD transition and second order LD-MC and HD-MC transitions, both in the MCS and MFT studies. That quenched disorder can change the order of transitions in pure models is well-known. For instance, Ref. [18] shows that sufficiently strong quenched disorder can make the magnetic transition in ferromagnetic manganites first order. Similarly, quenched disorder can introduce a first order transition in the well-known Kuramoto model of oscillator synchronization [19]. The current study forms yet another such example, and possibly the first of its kind in TASEP-like driven models with open boundary conditions.

FIG. 7: Plots showing phase transitions with $q(x)$ in Choice I given in [III.13] from MFT (continuous lines) and MCS (points) studies: (left) $\overline{\rho}$ versus $\alpha$ for a fixed $\beta = 0.7$. The horizontal line in the top gives $\overline{\rho}_{\text{HD}} = 0.9$ that is independent of $\alpha$, and the inclined line near the origin gives $\rho_{\text{LD}}$ which grows with $\alpha$. (middle) $\overline{\rho}$ versus $\alpha$ for a fixed $\beta = 0.6$. The top horizontal line gives $\overline{\rho}_{\text{MC}} \approx 0.5$ that is independent of $\alpha$ and $\beta$, and the inclined line near the origin gives $\rho_{\text{LD}}$ which grows with $\alpha$. (right) $\overline{\rho}$ versus $\beta$ for a fixed $\alpha = 0.7$. The horizontal line at the bottom gives $\overline{\rho}_{\text{MC}} \approx 0.5$ that is independent of both $\alpha$ and $\beta$, and the inclined line at the top gives $\overline{\rho}_{\text{HD}}$ that is independent of $\alpha$. All the transitions appear to be discontinuous.

V. SUMMARY AND OUTLOOK

We have thus studied the totally asymmetric exclusion process with open boundaries having spatially smoothly varying hopping rates. Our study reveals the universal form of the phase diagrams for generic smooth hopping rates. Our results are sufficiently general and applies to any smoothly varying hopping rate functions. We construct the mean-field theory, and use that to outline a scheme to calculate steady state density profiles. Our method is complementary to those used in Ref. [8]. It directly gives the steady state densities in terms of the current $J$ almost immediately by using the spatial constancy of the latter in the steady states. The different phases are then analysed by varying the boundary conditions in a straightforward manner. Unsurprisingly, the bulk steady state densities are generically space varying, unlike those in the conventional TASEP with uniform hopping (except along the special line $\alpha = \beta < 1/2$). These match well with those obtained from the MCS studies, lending credence to our mean-field analysis. Because of the spatially varying densities, the conventional way to characterise the phases via the densities, i.e., $\rho_{\text{LD}} < 1/2$, $\rho_{\text{HD}} > 1/2$ and $\rho_{\text{MC}} = 1/2$ in the bulk of the TASEP no longer holds. Rather one needs to resort to the equivalent conditions to decide the phases, since the current $J$ is a constant. This together with the condition that $\rho_{\text{LD}}(x) < 1/2$ and $\rho_{\text{HD}}(x) > 1/2$ everywhere in the bulk, allows us to distinguish the LD and HD phases. Further, the maximum steady state current that the system can sustain is no longer $1/4$, but is $q_{\text{min}}/4$, where $q_{\text{min}}$ is the minimum hopping rate. Surprisingly, our theory shows that the average bulk density in the MC phase can be more or less than $1/2$, in direct contrast with conventional open TASEPs.
with uniform hopping. We show that the general effect of spatially varying hopping rates is to enlarge the MC region of the phase space, while shrinking the LD and HD regions. Furthermore, our work elucidates the universal phase diagram for various choices of the hopping rate function, highlighting the robustness of asymmetric exclusion process in an open system. Lastly but not the least, our MFT and MCS studies clearly show that both the LD-MC and HD-MC transitions in the model are first order for any symmetric hopping rate function $q(x)$ having one minimum. This forms a truly novel outcome from this study. How this generalises to other forms of $q(x)$ is an interesting question to be investigated in the future.

Our MFT scheme is sufficiently general. It applies for any $q(x)$ that is smoothly and slowly varying. It would be interesting to extend our scheme to situations where $q(x)$ is smooth and slowly varying in general, but can have a few finite discontinuities. This will be discussed in the future. It will also be important to study the effects of interactions in the systems [14]. In addition, there are in vivo situations, where $q(x)$ is rapidly fluctuating in space [20], which breaks down the assumption of slowly varying $q(x)$. How an equivalent analysis may be carried out for such a system, and to what degree the present results may be valid there are interesting questions to study. It would also be interesting to apply the boundary layer theory developed in Ref. [21] on our model, and determine the stationary densities and phases. We hope our work here will provide impetus to studies along these lines in future.

Our results may be verified in model experiments on the collective motion of driven particles with light-induced activity [22] passing through a narrow channel. Spatial modulations of the hopping rate can be created by applying patterned or spatially varying illumination.

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Appendix A: Phase transitions in an open TASEP with uniform hopping

We numerically study the LD-HD, LD-MC and HD-MC transitions. As expected, our studies show that the LD-HD transition is a first order transition, whereas the LD-MC and HD-MC transitions are second order in nature.

FIG. 8: Plots of the densities showing phase transitions from MFT and MCS studies in an open TASEP with a constant hopping rate: (a) LD-HD phase transition, and (b) HD-MC transition. Both MFT and MCS show that (a) is a first order, and (b) and (c) are second order transitions, as is well-known for open TASEPs with uniform hopping.

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