Skyrme black holes in the isolated horizons formalism

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We study static, spherically symmetric, Skyrme black holes in the context of the assumption that they can be viewed as bound states between ordinary bare black holes and solitons. This assumption and results stemming from the isolated horizons formalism lead to several conjectures about the static black hole solutions. These conjectures are tested against the Skyrme black hole solutions. It is shown that, while there is in general good agreement with the conjectures, a crucial aspect seems to violate one of the conjectures.

I. INTRODUCTION

The isolated horizons formalism [1] has been used to formulate several conjectures regarding static black holes [2, 3]. In this article we study the static, spherically symmetric Einstein-Skyrme black hole solutions found previously [4, 5]. The Einstein-Skyrme solutions have a number of features generic to certain models involving non-abelian gauge fields [7] that complement features of models previously considered in the context of the isolated horizons conjectures [8, 9, 10, 11]. Like a large class of models admitting solitons in flat space, Skyrme black holes have been shown to be linearly stable [5, 13, 14] and thus important to the conjectures. Our metric convention is the usual commutation relations [τ − iA, τ c] and 

\[ \tau \cdot \tau = \tau_r, \quad \tau \cdot \hat{x} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \]

and τ a are the usual Cartesian Pauli matrices. Thus

\[ -i A \equiv -i A_\mu dx^\mu = \tau_a \chi' dr \]

\[ + [\cos \chi \sin \chi (\partial_\theta \tau_r) - i \sin^2 \chi (\tau_r \partial_\theta \tau_r)] d\theta \]

\[ + [\cos \chi \sin \chi (\partial_\phi \tau_r) - i \sin^2 \chi (\tau_r \partial_\phi \tau_r)] d\phi \]

The Pauli matrices τ r, τ φ = ∂θ τ r, τ φ = ∂θ τ r satisfy the usual commutation relations [τ r, τ θ] = 2iτ φ (and cyclical permutations) and τ r = τ r = τ φ = 1. The energy-momentum tensor for the Skyrme lagrangian density above is

\[ T_{ab} = -\frac{1}{2} f^2 \text{Tr}(A_a A_b - \frac{1}{2} g_{ab} A_c A^c) + \frac{1}{8g^2} \text{Tr}(F_{ac} F_{bd} g^{cd} - \frac{1}{4} g_{ab} F_{cd} F^{cd}) \]

Note that the trace of the energy-momentum tensor is in general non-zero. The fact that the Skyrme model has
its own length scale permits the existence of flat-space solitons. It has been conjectured that it is this length scale that is responsible for the upper bound on the size of the black hole within the soliton as has also been observed in other theories that admit flat-space solitons (see [10] and references therein). This is not the case for the Einstein-Yang-Mills black holes considered in [3], since Yang-Mills theory does not admit flat space solitons and the trace of its energy-momentum tensor is always zero. However, in the Einstein-Yang-Mills-Higgs model considered in [3], the trace of the energy-momentum tensor is generically non-zero and black holes exist of arbitrary size.

The baryon current is given by
\[ b^a = \frac{1}{24\pi^2} \epsilon^{abcd} \text{Tr}(U^a \nabla_b U U^a \nabla_d U). \] (5)

The topological charge on a given hypersurface \( \Sigma \) is then
\[ B = \int_{\Sigma} B^a n_a dS = \int_{\Sigma} q^3 x \sqrt{-g} B^0 \] (6)

where \( dS \) is the volume form for \( \Sigma \) and \( n_a \) is the unit normal to the hypersurface. Thus, the spherically symmetric hedgehog ansatz above, for the regular soliton, gives
\[ B = \frac{1}{\pi} \left[ \chi(r) - \frac{1}{2} \sin 2\chi(r) \right]_\infty^0. \] (7)

Consequently for a regular Skyrme soliton (Skyrmion) the baryon number will be an integer. In the black hole case it has been argued that the integration should only run from the horizon \( r_\Delta \) to \( \infty \) and hence the black hole solutions do not have integer baryon number [3]. In addition, for \( B = n > 1 \) the solutions will not be the lowest energy configurations and thus probably will not be stable to decay into \( n \) well isolated \( B = 1 \) skyrmions. In particular, the \( B = 2 \) lowest energy configuration is known to be axially symmetric, the black hole versions of which have been given in [12].

The ansatz for a spherically symmetric metric can be written as
\[ ds^2 = -A^2(r) \left( 1 - \frac{2Gm(r)}{r} \right) dt^2 + \left( 1 - \frac{2Gm(r)}{r} \right)^{-1} dr^2 + r^2 d\Omega^2. \] (8)

The \( G'_t \) and \( G'_r \) components of the Einstein equations for matter given by [3] and [14] are:
\[ \frac{2m'}{r^2} = 4\pi \left[ f^2 \left( \chi'^2 (1 - \frac{2Gm}{r}) + \frac{2}{r^2} \sin^2 \chi \right) \right. \]
\[ \left. + \frac{1}{g^2} \left( \frac{2}{r^2} \chi'^2 (1 - \frac{2Gm}{r}) \sin^2 \chi + \frac{\sin^4 \chi}{r^4} \right) \right] \] (9)

and
\[ \frac{4A'm - 2A'r}{r^2 A} + \frac{2m'}{r^2} = 4\pi \left[ f^2 \left( -\chi'^2 (1 - \frac{2Gm}{r}) + \frac{2}{r^2} \sin^2 \chi \right) \right. \]
\[ \left. + \frac{1}{g^2} \left( \frac{2}{r^2} \chi'^2 (1 - \frac{2Gm}{r}) \sin^2 \chi + \frac{\sin^4 \chi}{r^4} \right) \right] \] (10)

The \( G'_r \) equation can be more simply written as
\[ \frac{A' A}{r} = 4\pi G \left[ f^2(\chi'^2) + \frac{2}{g^2 r^2} (\chi'^2) \sin^2 \chi \right]. \] (11)

Note that \( A' \) is always positive if \( A \) is initially positive. These equations of motion should be supplemented by the boundary conditions for asymptotically flat black hole solutions:
\[ m(r_\Delta) = \frac{r_\Delta}{2G} \]
\[ A(r_\infty) = 1 \]
\[ \frac{2Gm(r)}{r} \bigg|_{r=\infty} = 0 \]
\[ \chi(r_\infty) = 0. \] (12)

The shooting parameter for the black hole to be used in the numerical integration is \( \chi(r_\Delta) \). As is shown in [3] this set of equations will have solutions for all horizon values \( r_\Delta \), up to some maximum \( r_{\Delta,\max} \) and this \( r_{\Delta,\max} \) will depend on the dimensionless coupling constant \( \alpha = 4\pi GF^2 \). A smaller \( \alpha \) will give larger maximum radii and in the limit \( \alpha \to 0 \) the Skyrme field totally decouples from gravity and we recover the Schwarzschild solution with no upper bound on the black hole radius. The maximum radius \( r_{\Delta,\max} \) will also depend on the topological baryon number \( B \). A larger baryon number black hole will have a lower maximum radius. For a given \( r_\Delta \), less than \( r_{\Delta,\max} \), there will be, in general, several classes of solutions, each class labelled by a unique baryon number and each class containing a stable branch and an unstable branch. In the limit of \( r_\Delta \to 0 \) the two branches will tend to the two separate solutions of the regular soliton case [3].

For the regular Skyrmion case, the boundary condition for \( \chi(r) \) in [12] has the result of making \( U(r) \) a mapping from \( S^3 \) into \( SU(2) \) characterized by the third homotopy group of \( SU(2) \), \( \pi_3(SU(2)) \). Thus they fall into topologically distinct equivalence classes labelled by an integer \( B \) given by [14]. Solutions with different integer values of \( B \) are therefore topologically distinct and cannot be deformed into one another. However, in the case of black hole solutions, the solution is only defined on a domain that excludes a central ball (the black hole) and thus all the mappings are topologically trivial. Thus there is no expectation of conservation of the, now non-integer, baryon number \( B \) and it is possible that the stable Skyrme black hole solution could decay into a
where \( \beta \) should be taken over the black hole solutions with horizon mass function (8) we have for the form of the metric given explicitly in terms of the Schwarzschild black hole via quantum tunnelling \[5\]. It is also possible that a Skyrme black hole with \( B > 1 \) could decay into several widely separated Skyrmions and a black hole or indeed a spherically symmetric Skyrme black hole with \( B > 1 \) could decay into a non-spherically symmetric black hole \[12\].

### III. ISOLATED HORIZONS CONJECTURES

#### A. Surface Gravity

First of all we need to define the surface gravity. This will in general depend on a normalization. For a static, spherically symmetric black hole we can use the fact that the horizon will be a Killing horizon and thus the surface gravity is just given by

\[
\kappa = \lim_{r \to r_H} \left( \frac{1}{2} \partial_t g_{tt} \right). \tag{13}
\]

For the form of the metric given explicitly in terms of the mass function \[5\] we have

\[
\kappa = \kappa_s A(1 - 2Gm'). \tag{14}
\]

In order to chose a normalization such that the time-translational Killing vector becomes unity at infinity the value of \( A \) (which is really just pure gauge since it does not affect the dynamics) should be set equal to one at infinity. This fixes the overall normalization for the surface gravity.

#### B. Horizon Mass and Conjectures

The conjectures for static black holes start from the physical idea that a hairy black hole can be viewed as a bound state of an ordinary hairless black hole and a soliton of the matter theory, in this case a Skyrmion \[2\,3\]. Using the fact that the value of the total Hamiltonian is constant on any connected component of static solutions, the Arnowitt-Deser-Misner (ADM) mass of a given black hole solution should be decomposable into a mass associated with the horizon and a mass associated with the soliton:

\[
M_{\text{ADM}} = M_{\text{sol}} + M_\Delta. \tag{15}
\]

The ADM mass is simply \( m(\infty) \). Taking the isolated horizon to be an internal boundary for the spacetime manifold, the horizon mass can be chosen to take the form

\[
M_\Delta = \frac{1}{2G} \int_{0}^{\hat{\Delta}} \beta(\hat{r}_\Delta) d\hat{r}_\Delta \tag{16}
\]

where \( \beta = 2r_\Delta \kappa = A(1 - 2Gm') \) and the integration should be taken over the black hole solutions with horizon radii up to \( r_\Delta \) (see \[2\] and references therein). Each branch of solutions can be labelled by an integer \( n \). Since for each value of \( B \) there is both a stable branch and an unstable branch, we can take \( n = 2B - 1 \) for the stable branches and \( n = 2B \) for the unstable branches. While \( B \) is non-integer for the black holes, this will still be ‘approximately’ true since the values do not deviate much from the integer values. Thus for a given horizon radius \( r_\Delta \) on a given branch \( n \)

\[
M_{\text{ADM}}^{(n)} = M_{\text{sol}}^{(n)} + M_\Delta^{(n)} + E_{\text{bind}} \tag{17}
\]

Writing the ADM mass as the sum of the two constituent parts and their binding energy,

\[
E_{\text{bind}} = M_{\text{sol}}^{(n)} - M_\Delta^{(n)}. \tag{19}
\]

This provides the motivation for the following conjectures \[18\]. Firstly, since the binding energy must be negative,

1. \( M_{\text{ADM}}^{(n)}(r_\Delta) < M_{\text{ADM}}^{(0)}(r_\Delta) \) for all \( n > 0 \) and all \( r_\Delta \)

The definition of the horizon mass then implies:

2. \( \kappa^{(n)}(r_\Delta) < \kappa^{(0)}(r_\Delta) \) for all \( n > 0 \) and all \( r_\Delta \).

Since the magnitude of the gravitational binding energy should increase (i.e. the binding energy should become more negative) as the mass of either of the two bound objects grows, when we consider fixing the radius \( r_\Delta \) of the black hole and increasing \( n \), we get:

3. For a fixed value of \( r_\Delta \) the horizon mass \( M_\Delta^{(n)} \) and the surface gravity \( \kappa^{(n)} \) are monotonically decreasing functions of \( n \).

When fixing the mass of the soliton and increasing the mass of the black hole by increasing \( r \), we get:

4. \( \beta^{(n)}(r_\Delta) < 1 \) for all \( n > 0 \) and all \( r_\Delta \).

Since \( M_{\text{ADM}} \) is monotonically increasing with \( r_\Delta \) and \( M_{\text{sol}} \) is fixed for fixed \( n \), by \[15\] :

5. \( M_{\Delta}^{(n)} \) is a monotonically increasing function of \( r_\Delta \), is positive for all values of \( n \) and vanishes at \( r_\Delta = 0 \).

As mentioned earlier one of the key features that distinguishes Einstein-Skyrme black holes from their Einstein-Yang-Mills counterparts is the existence of an upper radius for black hole solutions. This maximum radius forms a ‘crossing point’ at which two different branches of solutions meet, one of which is stable to linear perturbations and one of which is unstable. Using the expression for the ADM mass in terms of the soliton mass and horizon
mass and requiring the ADM mass to be uniquely defined even at the crossing point one obtains \[2\]

\[
M^\text{stab} - M^\text{unstab} = \frac{1}{2G} \int_0^{r_{\text{max}}^-} \beta^\text{stab}(r_\Delta) dr_\Delta
\]

\[
- \frac{1}{2G} \int_0^{r_{\text{max}}^+} \beta^\text{unstab}(r_\Delta) dr_\Delta
\]

\[
= \frac{1}{2G} \oint \beta(r_\Delta) dr_\Delta
\]

(20)

where the integral should be taken along the closed contour formed by the two branches of solutions that meet at the crossing point and the vertical axis between their endpoints at \(r_\Delta = 0\).

IV. RESULTS

![Graph showing \(\beta(x_\Delta)\) for two branches of the \(B \sim 1\) (blue line), \(B \sim 2\) (red line) and \(B \sim 3\) (green line) Skyrme Black Holes with \(\alpha = 4\pi G f^2 = 0.001\).](image)

FIG. 1: \(\beta(x_\Delta)\) for the two branches of the \(B \sim 1\) (blue line), \(B \sim 2\) (red line) and \(B \sim 3\) (green line) Skyrme Black Holes with \(\alpha = 4\pi G f^2 = 0.001\). The dotted lines are the unstable solutions.

The results of the numerical integration of the equations of motion for the Einstein-Skyrme system are displayed in the figures. From the figures it is easy to read off the behaviour of the various functions appearing in the conjectures.

1. \(M_\Delta^{(n)}(r_\Delta) < M_\Delta^{(0)}(r_\Delta)\) for all \(n > 0\) and all \(r_\Delta\).

Since the value of \(M_\Delta^{(n)}(r_\Delta)\) is just half the value of the area under the curve \(\beta(r_\Delta)\) (Fig. 1) and \(M_\Delta^{(0)}\) is just given by half the area under the curve \(\beta = 1\) we can see that this conjecture is true for all the values and solutions investigated. Note that the validity of this conjecture just follows from the validity of conjecture 2.

2. \(\kappa^{(n)}(r_\Delta) < \kappa^{(0)}(r_\Delta)\) for all \(n > 0\) and all \(r_\Delta\).

Using \(\beta(r_\Delta) = 2r_\Delta \kappa(r_\Delta)\) this can be seen to be true for all curves in Fig. 1. In addition since we have \(\kappa = \kappa_s A(1 - 2Gm')\) and \(A(r)\) is always less than one (for example see Fig. 3 and equation (11)) this conjecture simply follows from the assumption of static, spherical symmetry. It also implies conjecture 1 directly.

3. For a fixed value of \(r_\Delta\), the horizon mass \(M_\Delta^{(n)}\) and the surface gravity \(\kappa^{(n)}\) are monotonically decreasing functions of \(n\).

This will hold for the stable solution branches but will not hold for the unstable branches (see Fig. 1), where both \(M_\Delta^{(n)}\) and \(\kappa^{(n)}\) will be monotonically increasing. One possible solution to this would be to argue that the unstable branches should not be viewed as bound states of a Schwarzschild black hole and the corresponding soliton but instead should be viewed as bound states between the stable branch and the corresponding unstable soliton. The binding energy would then be

\[
E_{\text{bind}} = M^\text{unstab}_\Delta - M^\text{stab}_\Delta
\]

\[
= \frac{1}{2G} \int_0^{r_\Delta} (\beta^\text{unstab} - \beta^\text{stab}) dr_\Delta.
\]

(21)

As can be seen from Fig. 1 the binding energy would then decrease in magnitude as \(n\) increases for a fixed horizon size \(r_\Delta\). However, for fixed \(r_\Delta\) and increasing \(n\), while the mass of the unstable soliton would be increasing, the mass of the stable black hole solution would be...
decreasing. Since the mass of both objects in the bound system is changing the original conjecture loses its motivation. It can also be seen from Fig. 3 that for all cases considered the magnitude of the binding energy will increase as $r_\Delta$ increases. Note also that in this case, if one still wished to view the stable Skyrme black holes as bound systems between Schwarzschild black holes and solitons, then one would effectively be viewing the unstable Skyrme black holes as bound systems between Schwarzschild black holes and both the stable and unstable solitons.

4. $\beta^{(n)}(r_\Delta) < 1$ for all $n > 0$ and all $r_\Delta$.

In the Einstein-Skyrme model the case $B = 0$ is just the familiar Schwarzschild solution. Thus $\beta^{(0)} = 1$ and this is essentially the same as conjecture 2 and Fig. 3. Notice also that since there is an upper bound on the radius of the black hole $\beta^{(n)}(r_\Delta)$ does not tend to one asymptotically.

5. $M^{(n)}_\Delta$ is a monotonically increasing function of $r_\Delta$, is positive for all values of $n$ and vanishes at $r_\Delta = 0$.

This will be true if $\mu'$ is never greater than a half and this is certainly the case for all the solutions considered (see Fig. 4). For the case of the conjecture explicitly related to the crossing point of the stable and unstable branches

$$M^{unstab}_\text{sol} - M^{stab}_\text{sol} = \frac{1}{2G} \int f(r_\Delta)dr_\Delta$$

the following values can be computed using the numerical solutions.

| $B$ | $M^{stab}_\text{sol}$ | $M^{unstab}_\text{sol}$ | $\frac{1}{2G} \int f(r)dr_\Delta$ | Difference |
|-----|----------------------|-------------------------|-------------------------------|------------|
| $B \sim 1$ | 0.00578 | 0.2637 | 0.02078 | 0.00019 |
| $B \sim 2$ | 0.01706 | 0.3223 | 0.01532 | 0.00016 |
| $B \sim 3$ | 0.03363 | 0.4331 | 0.00979 | 0.00011 |

While there is a slight mismatch in the left-hand side and right-hand side it is most likely that this is simply due to the numerical accuracy of the numerical solutions. The convergence of the numerical integration on the right hand side of (20) was checked and was found to converge quickly to the values given in the table.

V. CONCLUSIONS

We have investigated the static black hole conjectures stemming from the isolated horizons formalism using numerically generated solutions to the Einstein-Skyrme model. We have found that the numerical results are in impressive agreement with the conjectures. The only mismatch is in the behaviour of $\beta(r_\Delta)$ for the linearly unstable branches. However, since conjecture three is the only conjecture that relies critically on the assumption of the hairy black holes being bound states between bare black holes and solitons, it would be useful to investigate whether this behaviour is repeated in other models such as the Einstein-Proca system. It is possible that this
FIG. 5: The function $\chi(x)$ against dimensionless length $x = fgr$ for the two branches of the $B \sim 1$ (blue line), $B \sim 2$ (red line) and $B \sim 3$ (green line) Skyrme Black Hole with $\alpha = 4\pi G f^2 = 0.001$ and $x_\Delta = 0.03$. The dotted lines are the unstable solutions.

conjecture will fail generically for models exhibiting the two-branch-merging behaviour seen in the Skyrme model and will need to be replaced in these cases with something else, such as the modification discussed here, whereby the unstable branches are viewed as bound states of the solitons and the stable black holes. Further investigations would however need to be carried out to test this idea. While we have restricted ourselves to spherically symmetric, asymptotically flat solutions, similar tests could be carried out on Skyrme black holes with negative cosmological constant provided that due care is taken that the Hamiltonian formalism is well defined in the non-flat asymptotic region. However, it is not clear how to extend the analysis to the axisymmetric $B \sim 2$ solutions of since these have a lower bound on the horizon radius and hence are not connected to their corresponding regular soliton solutions and it is not clear in this situation how to define the horizon mass. Since these axisymmetric solutions represent minimal energy solutions in the $B \sim 2$ sector there may be some interesting relation to the $BPS$ bound. In addition, it would be interesting to see whether anything could be said about the lowest energy configurations in the $B > 2$ sector since these are expected to have discrete tetrahedral and octahedral symmetries.

VI. ACKNOWLEDGMENTS

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[1] A. Ashtekar, C. Beetle, O. Dreyer, S. Fairhurst, B. Krishnan, J. Lewandowski and J. Wisniewski, Phys. Rev. Lett. 85 (2000) 3564 [arXiv:gr-qc/0006006].
[2] A. Ashtekar, A. Corichi and D. Sudarsky, Class. Quant. Grav. 18 (2001) 919 [arXiv:gr-qc/0011081].
[3] A. Corichi, U. Nucamendi and D. Sudarsky, Phys. Rev. D 62 (2000) 044046 [arXiv:gr-qc/0002078].
[4] H. Luckock, in “String theory, quantum cosmology and quantum gravity, integrable and conformal invariant theories: Proceedings of the Paris-Meudon Colloquium, 22-26 September 1986.” (World Scientific, Singapore, 1987)
[5] P. Bizon and T. Chmaj, Phys. Lett. B 297 (1992) 55.
[6] S. Droz, M. Heusler and N. Straumann, Phys. Lett. B 268 (1991) 371.
[7] M. S. Volkov and D. V. Gal’tsov, Phys. Rept. 319 (1999) 1 [arXiv:hep-th/9810070].
[8] N. Breton, Phys. Rev. D 67 (2003) 124004 [arXiv:hep-th/0301254].
[9] A. Corichi, U. Nucamendi and M. Salgado, Phys. Rev. D 73 (2006) 084002 [arXiv:gr-qc/0504126].
[10] B. Kleihaus and J. Kunz, Phys. Lett. B 494 (2000) 130 [arXiv:hep-th/0008034].
[11] R. Ibadov, B. Kleihaus, J. Kunz and M. Wirschins, Phys. Lett. B 627 (2005) 180 [arXiv:gr-qc/0507110].
[12] N. Sawado, N. Shiiki, K. i. Maeda and T. Torii, Gen. Rel. Grav. 36 (2004) 1361 [arXiv:gr-qc/0401020].
[13] M. Heusler, S. Droz and N. Straumann, Phys. Lett. B 285 (1992) 21.
[14] M. Heusler, N. Straumann and Z. H. Zhou, Helv. Phys. Acta 66 (1993) 614.
[15] R. M. Wald, General Relativity, (The University of Chicago Press, Chicago, 1984)
[16] B. Hartmann, B. Kleihaus and J. Kunz, Phys. Rev. D 65 (2002) 024027 [arXiv:hep-th/0108129].
[17] N. Shiiki and N. Sawado, Phys. Rev. D 71 (2005) 104031 [arXiv:gr-qc/0502107].
[18] While these conjectures were first proposed in the context of colored, that is to say Einstein-Yang-Mills, black holes it is stated in section 4. of that they should apply to more general black holes, such as Einstein-Skyrme, provided care is taken over the definition of the horizon mass at the crossover point.