Abstract

Noncommutative geometry applied to the standard model of electroweak and strong interactions was shown to produce fuzzy relations among masses and gauge couplings. We refine these relations and show then that they are exhaustive.
1 Introduction

Connes’ geometric version of the standard model \([1]\) needs no further introduction \([2]\). For the physicist, its most interesting feature is the explanation of spontaneous symmetry breaking. Starting from the fermionic mass matrix and ‘noncommutative gauge couplings’, this explanation produces the bosonic mass matrices, spin 0 and 1, and the ordinary gauge couplings. Recall that the ordinary gauge couplings \(g_i, i \in \{2, 3\}\), parameterize the most general invariant scalar product on the Lie algebra \(\mathfrak{g}\), e.g.,

\[
(X, X') := \frac{2}{g_i^2} \text{tr}(X^*X'), \quad X, X' \in \text{su}(i).
\]

In noncommutative geometry, the Lie algebra \(\mathfrak{g}\) is contained in the involution algebra \(\mathcal{A}\), \(\mathfrak{g} = \{X \in \mathcal{A}, X^* = -X\}\) and the invariant scalar product is constructed from the fermion representation \(\rho\), which now is a representation of \(\mathcal{A}\) on a Hilbert space \(\mathcal{H}\),

\[
(a, a') := \text{tr}(z\rho(a)^*\rho(a')), \quad a, a' \in \mathcal{A}.
\]

The noncommutative gauge coupling \(z\) is a positive matrix on \(\mathcal{H}\) that commutes with \(\rho(\mathcal{A})\) and with the fermionic mass matrix. \(z\) unifies ordinary gauge couplings and boson masses. In the standard model, \(z\) contains six positive numbers \(x, y_1, y_2, y_3, \tilde{x}, \tilde{y}\) and the boson masses and gauge couplings as functions of these six numbers are \([3]\):

\[
m_W^2 = \frac{xq + y_1m_2^2 + y_2m_\mu^2 + y_3m_\tau^2}{3x + y_1 + y_2 + y_3},
\]

\[
m_H^2 = \frac{xr^2 + 3(y_1m_\epsilon^4 + y_2m_\mu^4 + y_3m_\tau^4)}{xq + y_1m_\epsilon^2 + y_2m_\mu^2 + y_3m_\tau^2} - (xq + y_1m_\epsilon^2 + y_2m_\mu^2 + y_3m_\tau^2)
\]

\[
\left(\frac{1}{3x + y_1 + y_2 + y_3} + \frac{1}{3x + (y_1 + y_2 + y_3)/2}\right),
\]

\[
g_1^{-2} = 3x + \frac{2}{3} \tilde{x} + \frac{1}{2} (y_1 + y_2 + y_3) + \frac{3}{2} \tilde{y},
\]

\[
g_2^{-2} = 3x + y_1 + y_2 + y_3,
\]

\[
g_3^{-2} = 4\tilde{x}.
\]

Here, we have denoted the mass of a particle \(p\) by \(m_p\) and put,

\[
q := m_1^2 + m_2^2 + m_3^2 + m_\mu^2 + m_\tau^2,
\]

\[
r^2 := 3(m_\epsilon^4 + m_\mu^4 + m_\tau^4 + m_\epsilon^4 + m_\mu^4 + m_\tau^4)
\]

\[
+ 2 \left[(m_\mu m_\tau |V_{ud}|)^2 + (m_\mu m_\tau |V_{us}|)^2 + (m_\mu m_\tau |V_{ub}|)^2
\]

\[
+ (m_\mu m_\tau |V_{cd}|)^2 + (m_\mu m_\tau |V_{cs}|)^2 + (m_\mu m_\tau |V_{cb}|)^2
\]

\[
+ (m_\tau m_\mu |V_{td}|)^2 + (m_\tau m_\mu |V_{ts}|)^2 + (m_\tau m_\mu |V_{tb}|)^2\right],
\]

and the \(V_\cdot\) are the Cabbibo-Kobayashi-Maskawa mixings.
In the ordinary formulation, the standard model has 18 positive input parameters: the three
gauge couplings, \(g_1, g_2, g_3\), the \(W\) and \(H\) masses, three lepton and six quark masses, and four
angles contained in the unitary Cabbibo-Kobayashi-Maskawa matrix \(V\).

In its geometric formulation, there are 19 positive input parameters, the 9 fermionic masses
and 4 mixing angles and 6 parameters from the noncommutative gauge coupling. In order to
derive the constraint equations for \(m_W, m_H, g_1, g_2, g_3\), one has to distinguish several cases
in terms of the 13 independent parameters of the fermionic mass matrix. The equations (1-5)
apply to the case where the Cabbibo-Kobayashi-Maskawa matrix is non-degenerate, i.e. not
block diagonal up to permutations of basis elements. In physical terms, this means that there
are no simultaneous mass and weak interaction eigenstates.

The following abbreviations will be useful:

\[
\begin{align*}
  e &:= m_e^2, \quad \mu := m_\mu^2, \quad \tau := m_\tau^2, \\
  t &:= m_t^2, \quad b := m_b^2, \quad c := m_c^2, \quad \ldots, \\
  W &:= m_W^2, \quad H := m_H^2.
\end{align*}
\]

Now, we are in a more symmetric situation of five equations for \(m_W, m_H, g_1, g_2, g_3\) as functions of five unknowns noncommutative gauge parameters \(y_1, y_2, y_3, x, y\) and five effective
parameters \(q, r, e, \mu, \tau\).

Our task is to describe the open subset of the five dimensional space of \((m_W, m_H, g_1, g_2, g_3)\)
that is the image under equations (1-5) of the five positive noncommutative gauge parameters.
Of course this image varies with the effective parameters. Again, we have to distinguish cases
in terms of the five effective parameters \(q, r, e, \mu, \tau\).

Here, we treat only one simple case given by the following hierarchies,

\[
\begin{align*}
  e < \mu < \tau < W, \\
  u + d < \min\{c, s\} < (1 + \epsilon)^{-1}\max\{c, s\}, \\
  c + s + \min\{c, s\} < \min\{t, b\} < (1 + \epsilon)^{-1}\max\{t, b\},
\end{align*}
\]

where \(\epsilon := 1 - \min\{|V_{tb}|^2, |V_{cs}|^2, |V_{ud}|^2\}\) measures the deviation of the Cabbibo-Kobayashi-
Maskawa matrix from the identity.

These hierarchies are simply used for getting the positivity of the following constant [4]

\[
C := \frac{r^2 - q^2}{3W^2} > 0.
\]

Actually, we write \(C\) under the form

\[
\frac{3}{2}CW^2 = [t^2 + b^2 + c^2 + s^2 + u^2 + d^2] \\
+ [tb (|V_{tb}|^2 - 1) + cs (|V_{cs}|^2 - 1) + ud (|V_{ud}|^2 - 1)] \\
+ [us |V_{us}|^2 + ub |V_{ub}|^2 + cd |V_{cd}|^2 + cb |V_{cb}|^2 + td |V_{td}|^2 + ts |V_{ts}|^2] \tag{8}
\]
In (8), a lower bound of the second term of the right-hand side is \(-\epsilon(tb + cs + ud)\), 0 for the third term and \(-[(t+b) \min\{t,b\} + (c+s) \min\{c,s\}\) for the last one. According to the definition of \(\epsilon\), \(3CW^2/2 > u^2 + d^2 - \epsilon ud > 0\).

2 Fuzzy relations for the masses and coupling constants

Since the previous hierarchies (6) are experimentally true (cf the appendix), this hypothesis is not restrictive and we have the following

Theorem. Assume (7): the heaviest lepton \(\tau\) is lighter than the W and there is a hierarchy between quarks and mixings. Then, the image, in the five dimensional space \((m_W, m_H, g_1, g_2, g_3)\), of the six strictly positive noncommutative gauge parameters \(x, y_1, y_2, y_3, \tilde{x}, \tilde{y}\), is characterized by the following five inequalities,

\[
\tau < \frac{W}{3}, \quad W < q/3, \quad (9)
\]

\[
H_{\text{min}}(W) < H(W) < H_{\text{max}}(W), \quad (10)
\]

\[
\sin^2\theta_w < \frac{2}{3} \left(1 + \frac{W - \tau}{q - 3\tau} + \left(\frac{g_2}{3g_3}\right)^2\right)^{-1}. \quad (11)
\]

The saturated bounds are given by

\[
H_{\text{max}}(W) := \frac{r^2 - 9e^2}{q - 3e} - \frac{(r^2 - 3qe)e}{q - 3e} \frac{1}{W} - \frac{3q + 3W - 12e}{q + 3W - 6e} W, \quad (12)
\]

\[
H_{\text{min}}(W) := \frac{r^2 - 9\tau^2}{q - 3\tau} - \frac{(r^2 - 3q\tau)\tau}{q - 3\tau} \frac{1}{W} - \frac{3q + 3W - 12\tau}{q + 3W - 6\tau} W. \quad (13)
\]

In particular,

\[
H_{\text{max}}(W) - H_{\text{min}}(W) = (\tau - e) (q - 3W) \left[\frac{r^2 - 3q(e + \tau) + 9e\tau}{(q - 3e)(q - 3\tau)} \frac{1}{W} + \frac{6W}{(q + 3W - 6e)(q + 3W - 6\tau)} \right]. \quad (14)
\]

Note that the 3 in (9) is the number of generations and that the intermediate lepton \(\mu\) does not appear in these formulæ.

This is the first time that we see a mass relation affected by a small conceptual uncertainty. We call it a fuzzy mass relation [4].

Proof: Inequalities (8) follow immediately from equation (11).

The proof of (10) is more involved. Since the equations (11,12) are homogeneous in the \(x, y_1, y_2, y_3\) variables, we will assume temporarily

\[3x = 1.\]
As in [4], we introduce two variables:

\[ X := 1 + \sum_{j=1}^{3} y_j, \]
\[ Y := \alpha_0^2 + \sum_{j=1}^{3} \alpha_j^2 y_j. \]

with the following abbreviations

\[ \alpha_0 := q/3W, \quad \alpha_1 := e/W, \quad \alpha_2 := \mu/W, \quad \alpha_3 := \tau/W. \]

The hierarchy (6) and (9) imply

\[ \alpha_1 < \alpha_2 < \alpha_3 < 1 < \alpha_0. \]

In terms of \( X \) and \( Y \), the mass relations (1-2) read:

\[ \frac{H}{W} + 1 = \frac{C}{X} + 3 \frac{Y}{X} - 2 \frac{X}{1+X}, \quad (15) \]
\[ X = \alpha_0 + \sum_{j=1}^{3} \alpha_j y_j. \quad (16) \]

It is convenient to define

\[ X_j := \frac{\alpha_0 - \alpha_j}{1 - \alpha_j}, \]
\[ Y_j := \beta_j X_j, \]
\[ \beta_j := \alpha_0 + \alpha_j - \alpha_0 \alpha_j, \quad j \in \{1, 2, 3\}. \]

Recall the following result of [4] and its proof for completeness:

**Lemma 1.** \( D := \{ y = (y_1, ..., y_3) / y_j > 0, \sum_{j=1}^{3} (1 - \alpha_j) y_j = \alpha_0 - 1 \} \) is a convex open set in \( \mathbb{R}^3 \). Moreover, on \( D \), the variables \( X \) and \( Y \) are independent and satisfy \( X \in [X_1, X_3], Y \in [Y_1, Y_3] \).

**Proof.** \( D \) is convex and bounded: Indeed, for \( j \in \{1, 2, 3\} \), we have

\[ (1 - \alpha_j) y_j < \sum_{j=1}^{3} (1 - \alpha_j) y_j = \alpha_0 - 1, \]

and \( 0 < y_j < (\alpha_0 - 1)/(1 - \alpha_j)^{-1} \). Let

\[ A_1 := (X_1 - 1, 0, 0), \quad A_2 := (0, X_2 - 1, 0) \quad \text{and} \quad A_3 := (0, 0, X_3 - 1). \]

Clearly, the \( A_j \) are in the closure of \( D \) and \( D \) is the interior of the convex envelope of the vectors \( A_j \): every \( y = (y_1, y_2, y_3) \in D \) can be written as

\[ y = \sum_{j=1}^{3} \lambda_j A_j \quad \text{with} \quad \lambda_j := \frac{1 - \alpha_j}{\alpha_0 - 1} y_j > 0 \quad \text{and} \quad \sum_{j=1}^{3} \lambda_j = 1. \]
because of the constraint (16). Therefore
\[
X = 1 + \sum_{j=1}^{3} y_j = \sum_{j=1}^{3} \lambda_j \left( 1 + \frac{\alpha_0 - 1}{1 - \alpha_j} \right) = \sum_{j=1}^{3} \lambda_j \frac{\alpha_0 - \alpha_j}{1 - \alpha_j},
\]
and as \((\alpha_0 - \alpha)/(1 - \alpha)\) is an increasing function of \(\alpha\),
\[
\frac{\alpha_0 - \alpha_1}{1 - \alpha_1} < X < \frac{\alpha_0 - \alpha_3}{1 - \alpha_3}.
\]
Similarly, we obtain the bounds on \(Y\),
\[
Y = \alpha_0^2 + \sum_{j=1}^{3} \alpha_j^2 y_j = \sum_{n=1}^{3} \lambda_n \left( \alpha_0^2 + (\alpha_0 - 1) \frac{\alpha_n^2}{1 - \alpha_n} \right)
\]
by noting that \(\alpha^2/(1 - \alpha)\) is increasing in \(\alpha\):
\[
\alpha_0^2 + (\alpha_0 - 1) \frac{\alpha_1^2}{1 - \alpha_1} < Y < \alpha_0^2 + (\alpha_0 - 1) \frac{\alpha_3^2}{1 - \alpha_3}.
\]
In particular, \(X \in [X_1, X_3]\) and \(Y \in [Y_1, Y_3]\).
The independence of \(X\) and \(Y\) follows from a non-vanishing functional determinant. Solving the constraint,
\[
y_3 = -\frac{1 - \alpha_0}{1 - \alpha_3} - \frac{1 - \alpha_1}{1 - \alpha_3} y_1 - \frac{1 - \alpha_2}{1 - \alpha_3} y_2,
\]
we eliminate \(y_3\):
\[
X = \frac{\alpha_0 - \alpha_3}{1 - \alpha_3} + \frac{\alpha_1 - \alpha_3}{1 - \alpha_3} y_1 + \frac{\alpha_2 - \alpha_3}{1 - \alpha_3} y_2,
\]
\[
Y = \left( \alpha_0^2 - \alpha_3^2 \frac{1 - \alpha_0}{1 - \alpha_3} \right) + \left( \alpha_1^2 - \alpha_3^2 \frac{1 - \alpha_1}{1 - \alpha_3} \right) y_1 + \left( \alpha_2^2 - \alpha_3^2 \frac{1 - \alpha_2}{1 - \alpha_3} \right) y_2,
\]
and compute the functional determinant
\[
\det \begin{pmatrix} \frac{\partial X}{\partial y_1} & \frac{\partial X}{\partial y_2} \\ \frac{\partial Y}{\partial y_1} & \frac{\partial Y}{\partial y_2} \end{pmatrix} = \frac{(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)}{1 - \alpha_3} \neq 0.
\]
ending the proof of the lemma.

The next lemma characterizes the domain \(D\) as function of the variables \(X\) and \(Y\).

**Lemma 2.** Let \(T\) be the map from \(\mathbb{R}^3\) to \(\mathbb{R}^2\) defined by \(T(y_1, y_2, y_3) := (X, Y)\). Then, the image \(T(D)\) is the interior of the triangle delimited by the points \(T(A_j) = (X_j, Y_j), j \in \{1, 2, 3\}\).

**Proof:** Since \(y_3\) is positive, (17) implies
\[
0 < y_2 < -\frac{1 - \alpha_1}{1 - \alpha_2} y_1 + \frac{\alpha_0 - 1}{1 - \alpha_2}.
\]
This upper bound being a line in the \((y_1, y_2)\) plane, the projection of \(D\) on this plane is contained in the triangle defined by the points \((X_1 - 1, 0), (0, X_2 - 1)\) and \((0,0)\). These points are nothing but the projection of \(A_1, A_2, A_3\) which are in the closure of \(D\). The projection on the plane preserves convexity and the previous Lemma yields the result because

\[
X_{T(A_j)} = 1 + X_j - 1 = X_j \\
Y_{T(A_j)} = \alpha_0^2 + \alpha_j^2(X_j - 1) = (\alpha_0 - \alpha_j)(\alpha_0 + \alpha_j) + \alpha_j^2X_j \\
= (1 - \alpha_j)X_j(\alpha_0 + \alpha_j) + \alpha_j^2X_j = (\alpha_0 + \alpha_j - \alpha_0\alpha_j)X_j = Y_j.
\]

Thanks to \([13]\), we need to control the function

\[f(X,Y) := \frac{C}{X} + 3 \frac{Y}{X} - 2 \frac{X}{1 + X} \]

\(C\) being positive by \([7]\), \(f\) is decreasing in \(X\) and increasing in \(Y\). So the minimum and maximum of \(f(X,Y)\) for \((X,Y) \in T(D)\) lie on the three segments \([T(A_j), T(A_k)], j \neq k\), which are the boundaries of \(T(D)\). The points of these segments have coordinates of the form \((X, a_{jk}X + b_{jk})\) where \(a_{jk}, b_{jk}\) are real numbers. The derivative of \(g_{jk}(X) := f(X, a_{jk}X + b_{jk})\) is \(g'_{jk}(X) = -(C + 3b_{jk})(1 + X^2)^{-1} - (1 + X)^{-2}\) and the functions \(g_{jk}\) will be decreasing if \(b_{jk}\) is positive which is the case as proved in the next lemma, because \(b_{jk} = (\alpha_0 - \alpha_j)(\alpha_0 - \alpha_k)\). This shows that

\[
\max\{f(X,Y) \mid (X,Y) \in D\} = \max\{f(X,Y) \mid (X,Y) \in [T(A_1), T(A_3)]\} = \max\{g_{13}(X) \mid X \in [X_1, X_3]\} = g_{13}(X_1) = f(X_1, Y_1), \\
\min\{f(X,Y) \mid (X,Y) \in D\} = g_{13}(X_3) = g_{23}(X_3) = f(X_3, Y_3).
\]

Now, by \([15]\),

\[
H_{\max} = W(3\beta_1 + \frac{C}{X_1} - \frac{2X_1}{1 + X_1} - 1) = q + 3e - \frac{qe}{W} + 3C \frac{W - eW}{q - 3eW} - \frac{3q + 3W - 12eW}{q + 3W - 6eW}
\]

yielding \([12]\). This proves \([12-14]\).

Note that

\[H_{\min} = W(3\beta_3 + \frac{C}{X_3} - \frac{2X_3}{1 + X_3} - 1)\]

is positive because \(-2X(1 + X)^{-1} - 1 > -3\) for any positive \(X\) and \(\beta_3 = \alpha_0 - \alpha_3(\alpha_0 - 1) > \alpha_0 - (\alpha_0 - 1) = 1\).

**Lemma 3.** The equation of the line passing through the points \(T(A_i)\) and \(T(A_j)\) in the \((X,Y)\) plane is \(Y = (\alpha_i + \alpha_j - \alpha_i\alpha_j)X + (\alpha_0 - \alpha_i)(\alpha_0 - \alpha_j)\).
Proof: This line is \[ Y = (Y_j - Y_i)(X_j - X_i)^{-1} X + (Y_i X_j - Y_j X_i)(X_j - X_i)^{-1} \] and
\[
\frac{Y_i X_j - Y_j X_i}{X_j - X_i} = (\beta_i - \beta_j) \frac{X_i X_j}{X_j - X_i} = (\alpha_j - \alpha_i)(\alpha_0 - \alpha_j) \frac{(\alpha_0 - \alpha_i)}{(\alpha_0 - \alpha_j)(1 - \alpha_j) - (\alpha_0 - \alpha_i)(1 - \alpha_i)} = (\alpha_0 - \alpha_i)(\alpha_0 - \alpha_j).
\]
Moreover, the slope is \((Y_j - Y_i)(X_j - X_i)^{-1} = (\alpha_i + \alpha_j - \alpha_i \alpha_j)\).

To include the coupling constants, equations (3-5), we remark that the W and Higgs masses are homogeneous in \(x, y_1, y_2, y_3\) and independent of \(\tilde{x}, \tilde{y}\). Consequently, the image under equations (1,2,4,5) is a cylinder with \(g_2 > 0, g_3 > 0\) with basis given by the inequalities (9,10) and shown in Figure 1. At this point, \(x\) is arbitrary positive as \(\tilde{x}\) and so are \(g_2\) and \(g_3\). To solve the last constraint (3), we write
\[
(y_1 + y_2 + y_3) \epsilon < y_1 \epsilon + y_3 \tau < (y_1 + y_2 + y_3) \tau,
\]
and from (14)
\[
\left(\frac{q}{3} - W\right)g_2^{-2} = (y_1 + y_2 + y_3) \frac{q}{3} - (y_1 \epsilon + y_3 \tau),
\]
we obtain two optimal inequalities
\[
\frac{q/3 - W}{q/3 - \epsilon} g_2^{-2} < y_1 + y_2 + y_3 < \frac{q/3 - W}{q/3 - \tau} g_2^{-2}.
\]
This solves the constraint (3) on \(g_1\):
\[
\frac{1}{2} g_2^{-2} \left(1 + \frac{W - \tau}{q/3 - \tau} \right) + \frac{1}{6} g_3^{-2} + \frac{3}{2} \tilde{y} < g_1^{-2} < \frac{1}{2} g_2^{-2} \left(1 + \frac{W - \epsilon}{q/3 - \epsilon} \right) + \frac{1}{6} g_3^{-2} + \frac{3}{2} \tilde{y}.
\]
Since \(\tilde{y}\) is an arbitrary positive number, we finally get
\[
\frac{1}{2} g_2^{-2} \left(1 + \frac{W - \tau}{q/3 - \tau} \right) + \frac{1}{6} g_3^{-2} < g_1^{-2}
\]
which is nothing else but (11) with \(\sin^2 \theta_w = g_2^{-2}(g_1^{-2} + g_2^{-2})^{-1}\) and the theorem is proved.

Problem: It would be interesting to get the Theorem without the hierarchy (6).

3 Physical consequences

The inequality (9) is
\[
m_W < \sqrt{\frac{q}{3}} = 104 \text{ GeV}.
\]
The three inequalities (10-11) deserve a few graphic representations. Figure 1 shows the allowed domain for the Higgs mass as a function of $m_W$ with $m_\tau$ as a parameter. The upper curve is $m_{H_{\text{max}}}$ which is independent of $m_\tau$. All parameters not explicitly mentioned in a figure or its caption are set to their experimental central values e.g. in Figure 1, $m_t = 180$ GeV. For the experimental values $m_W = 80$ GeV and $m_\tau = 1.8$ GeV, the allowed interval for the Higgs mass collapses in Figure 1. Indeed, this conceptual uncertainty, ‘fuzziness’, is

$$m_{H_{\text{max}}} - m_{H_{\text{min}}} = 34 \text{ MeV}.$$ 

The fuzziness is controlled by the $\tau$ mass:

$$\frac{m_{H_{\text{max}}} - m_{H_{\text{min}}}}{m_{H_{\text{max}}} + m_{H_{\text{min}}}} \sim \frac{m_\tau^2 - m_e^2}{m_t^2} \sim 10^{-4} \quad \text{at } m_W = 80 \text{ GeV}$$

and disappears at the upper bound $m_W = 104$ GeV since

$$H_{\text{min}}(\frac{q}{3}) = H_{\text{max}}(\frac{q}{3}) = \frac{q^2}{3q} - \frac{2q}{3} = (275 \text{ GeV})^2.$$ 

Note that this value of $m_H$ is independent of the lepton masses.

![Figure 1: $m_{H_{\text{max}}}$ and $m_{H_{\text{min}}}$ as function of $m_W$ for $m_\tau = 1.8, 30$ and 60 GeV](image)

In any case, the experimental uncertainties on the masses, completely drown the fuzziness. Since, today, the major experimental uncertainty is on the top mass, ±12 Gev, it is worth to represent the fuzziness as function of $m_t$ with $m_\tau$ as parameter. Figures 2 and 3 illustrate again the mentioned mass collapse. To incorporate inequality (11), we include in these figures $\sin^2 \theta_w$ and $g_3$. A second collapse in $\sin^2 \theta_w$ is plotted in Figure 4.

Neglecting the fuzziness with respect to experimental accuracy, the inequalities (9-11) reduce to:

$$m_e < m_W \lesssim m_t/\sqrt{3},$$
Note that $m_W > m_e$ does not use the hierarchy (6).

The last inequality has two physical consequences: if we know the $W$ and fermion masses, then, the weak angle is constrained by $\sin^2 \theta_w < 0.54$ (Figure 4). Recall the experimental values, $\sin^2 \theta_w = 0.23$, $g_3 = 1.2$. If we know the $W$ and fermion masses and the electroweak couplings, then, the strong coupling cannot be too weak: $g_3 > 0.17$ at the $Z$ mass. At this point, the following fact [5] is intriguing: If we know the fermion representation under electroweak interactions, then, the strong interactions must be vectorlike. If not, the noncommutative generalization of Poincaré duality breaks down.
One of the attractive features of noncommutative geometry in Yang-Mills theories is that gauge couplings and boson masses are correlated. Numerically, this unification is visible in the inequality \( \sin^2 \theta_w < \frac{1}{2} \frac{1}{1 + g_2^2/(12g_3^2)} \).

However in this case, gauge couplings and masses decouple and also Poincaré duality breaks down.

4 Conclusion

The fuzzy mass relation for the Higgs raises the question of stability under renormalization. We feel that this question can only be answered by taking seriously the revolution that noncommutative geometry operates on spacetime. Spacetime becomes fuzzy \( \mathbb{Q} \), just as phase space becomes fuzzy in quantum mechanics. Let us try to explain this feeling by an analogy with electrodynamics. Unifying electricity and magnetism, Maxwell obtained an expression for the speed of light in terms of the two static coupling constants \( \epsilon_0 \) and \( \mu_0 \). His relation was confirmed by already existing data and no-one really dared to ask, what could be the meaning of an equation between a quantity depending on the reference system and a constant. Later, Einstein answered the question with the help of Minkowskian geometry. This geometry was already inherent in Maxwell’s equations, but not accepted by the community of physicists. Noncommutative geometry tells us that the Higgs field with its spontaneous symmetry breaking is only a magnetic field and therefore the Higgs mass is fixed, fuzzily. This seems in contradiction with large renormalization flow. However the origin of this flow is a small distance divergence that ignores the new spacetime uncertainty. In this context, quantum field theory has to be redone \( \mathbb{F} \).
Meanwhile, we are looking forward to the LHC verdict concerning equation \([20]\),

\[ m_H = 288 \pm 22 \text{ GeV} \quad \text{if} \quad m_t = 180 \pm 12 \text{ GeV}. \]

## 5 Appendix

The present experimental constraints \([9]\) on the 18 parameters of the standard model are listed below. The gauge couplings are given at the Z mass and all masses are pole masses.

\[
\begin{align*}
    g_1 &= 0.3575 \pm 0.0001, \\
    g_2 &= 0.6507 \pm 0.0007, \\
    g_3 &= 1.207 \pm 0.026,
\end{align*}
\]

\[
\begin{align*}
    m_e &= 0.51099906 \pm 0.00000015 \text{ MeV}, \\
    m_\mu &= 0.105658389 \pm 0.000000034 \text{ GeV}, \\
    m_\tau &= 1.7771 \pm 0.0005 \text{ GeV},
\end{align*}
\]

\[
\begin{align*}
    m_u &= 5 \pm 3 \text{ MeV}, \\
    m_d &= 10 \pm 5 \text{ MeV}, \\
    m_c &= 1.3 \pm 0.3 \text{ GeV}, \\
    m_s &= 0.2 \pm 0.1 \text{ GeV}, \\
    m_t &= 180 \pm 12 \text{ GeV}, \\
    m_W &= 80.22 \pm 0.26 \text{ GeV}, \\
    m_H &> 58.4 \text{ GeV}.
\end{align*}
\]

The Cabbibo-Kobayashi-Maskawa matrix is a unitary matrix

\[
V := \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix}
\]

and the absolute values of its matrix elements are:

\[
\begin{bmatrix}
0.9753 \pm 0.0006 & 0.221 \pm 0.003 & 0.004 \pm 0.002 \\
0.221 \pm 0.003 & 0.9745 \pm 0.0007 & 0.040 \pm 0.008 \\
0.010 \pm 0.006 & 0.039 \pm 0.009 & 0.9991 \pm 0.0004
\end{bmatrix}
\]

For physical purposes, the Cabbibo-Kobayashi-Maskawa matrix can be parameterized by three angles, \(\theta_{12}, \theta_{23}, \theta_{13}\) and one \(CP\) violating phase \(\delta\):

\[
V = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},
\]

with \(c_{kl} := \cos \theta_{kl}, \ s_{kl} := \sin \theta_{kl}\).

## References

[1] A. Connes & J. Lott, *The metric aspect of noncommutative geometry*, in the proceedings of the 1991 Cargèse Summer Conference, eds.: J. Fröhlich et al., Plenum Press (1992)

A. Connes, *Noncommutative Geometry*, Academic Press (1994)

A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. 36 (1995) 6194
[2] D. Kastler, A detailed account of Alain Connes’ version of the standard model in non-commutative geometry, I and II, Rev. Math. Phys. 5 (1993) 477
D. Kastler, A detailed account of Alain Connes’ version of the standard model in non-commutative geometry, III, Rev. Math. Phys. 8 (1996) 103
D. Kastler & M. Mebkhout, Lectures on Non-Commutative Differential Geometry, World Scientific, to be published
J. C. Várilly & J. M. Gracia-Bondía, Connes’ noncommutative differential geometry and the standard model, J. Geom. Phys. 12 (1993) 223
J. Madore, An Introduction to Noncommutative Differential Geometry and its Physical Applications, Cambridge University Press (1995)
C. P. Martín, J. M. Gracia-Bondía & J. C. Várilly, The standard model as a non-commutative geometry: the low mass regime, UCR-FM-6-96

[3] D. Kastler & T. Schücker, The standard model à la Connes-Lott, CPT-94/P.3091, hep-th/9412185
D. Kastler & T. Schücker, A detailed account of Alain Connes’ version of the standard model in non-commutative geometry, IV, Rev. Math. Phys., to appear

[4] B. Iochum, D. Kastler & T. Schücker, Fuzzy mass relations in the standard model, CPT-95/P.3235, hep-th/9507150

[5] R. Asquith, Non-commutative geometry and the strong force, Phys. Lett. B 366 (1996) 220

[6] J. M. Gracia-Bondía, Connes’ interpretation of the Standard model and massive neutrinos, Phys. Lett. B 351 (1995) 510

[7] J. Madore, Quantum mechanics on a fuzzy sphere, Phys. Lett. 263B (1991) 245
J. Madore, The fuzzy sphere, Class. Quant. Grav. 9 (1992) 69
J. Madore, Fuzzy physics, Ann. Phys. 219 (1992) 187

[8] S. Doplicher, K. Fredenhagen & J. E. Roberts, Space-time quantization induced by classical gravity, Phys. Lett. B 331 (1994) 39
S. Doplicher, K. Fredenhagen & J. E. Roberts, The quantum structure of spacetime at the Planck scale and quantum fields, Comm. Math. Phys. 172 (1995) 187

[9] L. Montanet et al. Review of Particle Properties, Phys. Rev. D50 (1994) 1173 and 1995 update from http://pdg.lbl.gov/