THE IMPACT OF QUANTUM INTERFERENCE BETWEEN DIFFERENT J-LEVELS ON SCATTERING POLARIZATION IN SPECTRAL LINES

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ABSTRACT

The spectral line polarization produced by optically pumped atoms contains a wealth of information on the thermal and magnetic structure of a variety of astrophysical plasmas, including that of the solar atmosphere. A correct decoding of such information from the observed Stokes profiles requires a clear understanding of the effects that radiatively induced quantum interference (or coherence) between pairs of magnetic sublevels produces on these observables, in the absence of and in the presence of magnetic fields of arbitrary strength. Here we present a detailed theoretical investigation of the role of coherence between pairs of sublevels pertaining to different fine-structure J-levels, clarifying when it can be neglected for facilitating the modeling of the linear polarization produced by scattering processes in spectral lines. To this end, we apply the quantum theory of spectral line polarization and calculate the linear polarization patterns of the radiation scattered at 90° by a slab of stellar atmospheric plasma, both taking into account and neglecting the above-mentioned quantum interference. Particular attention is given to the 2S − 2P, 3S − 3P, and 3P − 3S multiplets. We point out the observational signatures of this kind of interference and analyze its sensitivity to the energy separation between the interfering levels, to the amount of emissivity in the background continuum radiation, to lower-level polarization, and to the presence of a magnetic field. Some interesting applications to the following spectral lines are also presented: Ca ii H and K, Mg ii h and k, Na i D1 and D2, the Ba ii 4554 Å and 4934 Å resonance lines, the Cr i triplet at 5207 Å, the O i triplet at 7773 Å, the Mg i b-lines, and the Hα and Lyα lines of H1.

Key words: polarization – scattering – stars: atmospheres – Sun: atmosphere – Sun: surface magnetism

1. INTRODUCTION

Over the last few years, we have witnessed renewed interest in the spectral line polarization produced by the presence of population imbalances and quantum interference (or coherence) among the magnetic sublevels of atomic energy levels (e.g., Casini & Landi Degl’Innocenti 2007; Trujillo Bueno 2009; Stenflo 2009; Manso Sainz 2011). This so-called atomic level Jℓ at the center of the H line (a sign reversal between the two lines, and zero polarization – scattering – stars: atmospheres – Sun: atmosphere – Sun: surface magnetism interference between the two upper J-levels of these Ca ii resonance lines.

A rigorous theoretical framework for describing the spectral line polarization produced by radiatively induced population imbalances and quantum coherence, in the presence of arbitrary magnetic fields, is the density-matrix theory described in the monograph “Polarization in Spectral Lines” (Landi Degl’Innocenti & Landolfi 2004, hereafter LL04). This theory is based on the hypothesis that the pumping radiation field has no spectral structure over frequency intervals larger than the frequency separation between the interfering levels (flat-spectrum approximation). Because of this approximation, the theory is very suitable for treating spectral lines that can be described under the hypothesis of complete frequency redistribution, while it cannot account for the effects of partial redistribution in frequency (PRD). Despite this limitation, this theory represents the most robust quantum approach to the physics of polarization developed so far. In particular, it accounts for the role of quantum interference in a very general, self-consistent way (e.g., the review by Belluzzi 2011). As a matter of fact, although the Ca ii H and K lines are 35 Å apart (the flat-spectrum approximation thus appearing rather restrictive), the theory explains very well the sign reversal observed in the blue wing of the H line in terms of coherence between the two upper levels of these lines (see LL04). Theoretical approaches aimed at including PRD effects in the presence of J-state interference can be found in Landi Degl’Innocenti et al. (1997) and Smitha et al. (2011).

It is known that interference between pairs of magnetic sublevels pertaining to different J-levels is smaller the larger the energy separation is between them. When this separation is large, its effects are negligible in the core of the lines,
becoming more important in the far wings (see Stenflo 1980; LL04). On the other hand, far from the line center the line emissivity is very low, and the presence of the continuum generally masks the effects of such interference. Because of these basic arguments, interference between different $J$-levels is usually neglected when investigating the scattering polarization properties of many spectral lines with a considerable simplification of the problem. However, this kind of qualitative consideration should not be applied as a rigorous rule, as testified by the Ca II H and K lines and by several other signals of the linearly polarized solar limb spectrum (or second solar spectrum) which show the signatures of $J$-state interference (see Gandorfer 2000, 2002, 2005). As far as the solar Ca II H and K lines are concerned, it is now clear that the reason why quantum interference between such significantly separated lines produces observable effects is because these strong lines have very extended wings, so that the continuum is not immediately reached when moving away from line center, and because the large optical depth of the solar atmosphere compensates for the low line emissivity at the wavelengths where such effects appear (cf. Stenflo 1980; LL04).

In this paper, we present a systematic theoretical investigation of the role of quantum interference between different $J$-levels. We aim to clarify its observable effects in various interesting multiplets and to provide a series of general criteria for establishing under which circumstances its effects are expected to be observable, and when, on the contrary, the modeling of spectropolarimetric observations can be safely carried out ignoring its contribution. The investigation is carried out within the framework of the above-mentioned quantum theory of polarization. After briefly introducing the density-matrix formalism (Section 2) and presenting the scattering polarization model that is applied for our investigation (Section 3), we focus our attention on the $^2S-^2P$ multiplet, the simplest one where interference between different $J$-levels occurs (Section 4). We show that although the large energy separation generally present between different $J$-levels makes this kind of interference usually very small, as far as fractional polarization signals (i.e., ratios, such as $Q/I$) are considered, its effects are not necessarily negligible. Indeed, when only line processes are considered (no continuum), the fractional polarization patterns obtained by taking into account and neglecting $J$-state interference co-occur within a small spectral interval around the center of the lines, but are found to be very different at all other wavelengths, even when the $J$-levels are very separated from each other (see Section 4.2). Particular attention is given to the analysis of the effects of the Doppler broadening of the lines, a physical aspect that is found to play an important role for establishing where and when $J$-state interference can be neglected.

As previously mentioned, what makes the effects of interference between different $J$-levels vanish in the observed polarization patterns is the continuum, which starts dominating over the line emissivity moving from the line center to the wings. In Section 4.3, we show that the amount of continuum needed to mask the signatures of this kind of interference is larger the smaller the separation is between the $J$-levels. The effects of $J$-state interference are thus expected to be observable either when the interfering lines have very extended wings, so that the continuum is not immediately reached between them (as in the case of the Ca II H and K lines), or when the separation between the $J$-levels is sufficiently small but, as we will see, still large when compared to the Doppler width of the corresponding spectral lines.

In the first part of the paper, we analyze the scattering polarization profiles of hypothetical multiplets characterized by different values of the energy separation between the interfering $J$-levels, of the Doppler width, and of the continuum intensity. The results of this analysis, and their consequences for practical applications, are then investigated in detail on various multiplets of particular interest for the diagnostics of the magnetism of the solar atmosphere (Sections 4.5, 5, 6, and 7). The effects due to the presence of magnetic fields of various intensities and configurations are also investigated (Section 8).

2. STATISTICAL EQUILIBRIUM EQUATIONS IN THE DENSITY-MATRIX FORMALISM

A convenient way to describe the populations of the various magnetic sublevels of an atomic system, as well as the quantum interference between pairs of them, is through the matrix elements of the density operator. The density operator is a very useful theoretical tool for describing any physical system that is in a statistical mixture of states (e.g., Fano 1957). If $p_1, p_2, \ldots, p_i, \ldots$ are the probabilities of a given physical system of being in the dynamical states represented by the vectors $|1\rangle$, $|2\rangle, \ldots, |i\rangle, \ldots$, respectively, the corresponding density operator is defined by

$$\rho = \sum_i p_i |i\rangle \langle i|.$$  \hfill (1)

Like any other quantum operator, the density operator also is completely specified once its matrix elements, evaluated on a given basis of the Hilbert space associated with the physical system, are known. The matrix elements of the density operator contain all the accessible information about the system.

On the basis of the atomic energy eigenvectors $|\langle m|\rangle$, the matrix elements of the density operator are given by

$$\langle m|\rho|m'\rangle \equiv \rho_{mm'}.$$  \hfill (2)

the diagonal elements represent the populations of the various energy levels, while the off-diagonal elements represent the quantum interference between pairs of them. On this basis, the statistical equilibrium equations (SEEs) for the atomic density matrix are given by (see Equation (6.62) of LL04)

$$\frac{d}{dt} \rho_{mm'} = -2\pi i w_{mn} \rho_{mm'} + \sum_{nn'} \rho_{mn} T_A(m, m', n, n')$$

$$+ \sum_{pp'} \rho_{pp'} T_E(m, m', p, p') + \sum_{pp'} \rho_{pp'} T_S(m, m', p, p')$$

$$- \sum_m \{ R_A(m, m', m) + \rho_{mm'} R_A(m', m, m) \}$$

$$- \sum_{mm'} \{ R_E(m', m, m') + \rho_{mm'} R_E(m, m', m') \}$$

$$- \sum_{mm'} \{ R_S(m', m, m') + \rho_{mm'} R_S(m, m', m') \}.$$  \hfill (3)

These equations describe the transfer ($T$ rates) and relaxation ($R$ rates) of populations and coherence due to absorption (index $A$), spontaneous emission (index $E$), and stimulated emission (index $S$) processes. They also describe the effects due to the presence of a magnetic field. A complete derivation of these equations, as well as the explicit expression of the various rates, can be found in LL04.
Here we focus our attention on the first term on the right-hand side of Equations (3). This term, proportional to the Bohr frequency \( v_{\text{man}} = (E_\alpha - E_\beta)/\hbar \), with \( E_i \) being the energy of level \( |i\rangle \) and \( \hbar \) being the Planck constant, is zero both for populations (i.e., for the diagonal elements \( \rho_{\alpha\alpha} \)) and for coherence between degenerate levels, while it produces a relaxation of coherence between pairs of non-degenerate levels. The coherence \( \rho_{\alpha\beta} \) is thus smaller, the larger the energy separation is between the levels \( |m\rangle \) and \( |m'\rangle \). Modifying the frequency separation \( v_{\text{man}} \) among the various magnetic sublevels, a magnetic field modulates the corresponding coherence \( \rho_{\alpha\beta} \), and consequently, the polarization of the emitted radiation. This is the basic physical mechanism at the origin of the Hanle effect.

Finally, it is important to recall that Equations (3) are valid under the so-called flat-spectrum approximation. This approximation requires the pumping radiation field to be specrtrally flat over frequency intervals larger than the Bohr frequency relating pairs of levels between which quantum interference is considered, and larger than the inverse lifetime of the same levels.

For an atomic system devoid of hyperfine structure (HFS), the matrix elements of the density operator are often defined on the basis of the eigenvectors of the angular momentum \( \{ |\alpha J M \rangle \} \), with \( J \) and \( M \) being the quantum numbers associated with the total angular momentum and its projection along the quantization axis, and \( \alpha \) being a set of inner quantum numbers associated with the atomic Hamiltonian.\(^5\) On this basis, the matrix elements of the density operator are

\[
\langle \alpha J M | \rho | \alpha' J' M' \rangle \equiv \rho(\alpha J M, \alpha' J' M').
\]

The diagonal elements represent the populations of the various magnetic sublevels, while the off-diagonal elements represent the quantum interference between pairs of them.

An atomic model accounting for coherence between pairs of magnetic sublevels pertaining either to the same \( J \)-level or to different \( J \)-levels within the same term is generally referred to as a “multi-term atom” (see Section 7.5 of LL04). In this case, the flat-spectrum approximation requires the radiation field incident on the atom to be flat over frequency intervals larger than the frequency separation among the \( J \)-levels belonging to each term. This implies that the incident radiation field must be flat across the frequency interval covered by the transitions of a given multiplet.

When coherence between different \( J \)-levels is neglected, the corresponding atomic model is generally referred to as a “multi-level atom” (see Section 7.1 of LL04). In this case, the flat-spectrum approximation is less restrictive since it requires the radiation field to be constant across frequency intervals larger than the Zeeman splitting among the various magnetic sublevels pertaining to each \( J \)-level, and larger than their inverse lifetimes. In a multi-level atom, the incident radiation field can thus vary across the frequency interval spanned by the transitions of a given multiplet.

The SEEs and the expressions of the radiative transfer coefficients for a multi-term and a multi-level atom can be found in Chapter 7 of LL04.

\(^4\) We recall that in the presence of intense magnetic fields, in the Paschen-Back effect regime, this is not the basis of the energy eigenvectors.

\(^5\) For an atomic system described by the \( L-S \) coupling scheme, \( \alpha \) could represent, for example, the set of quantum numbers \( (\beta, L, S) \) which describe the electronic configuration, the total orbital angular momentum, and the total electronic spin.

3. THE SCATTERING POLARIZATION MODEL

In order to analyze the effects of interference between different \( J \)-levels, we consider an atomic system composed of two terms (each being characterized by the total orbital angular momentum \( L \), by the total electronic spin \( S \), and by given fine-structure (FS) splittings of the various \( J \)-levels), and we compare the polarization patterns of the radiation emitted across the transitions of the corresponding multiplet as obtained by both taking into account and neglecting such coherence (i.e., as calculated within the framework of a two-term atomic model, which accounts for \( J \)-state interference, and within the framework of the corresponding multi-level atomic model, which neglects it). Although in the multi-level atom case the pumping radiation field can be different at the frequencies of the various transitions of the multiplet, in order to analyze the net effect of the coherence between different \( J \)-levels, we consider the same incident field, flat across the whole frequency interval of the multiplet, both in the case of the two-term atom and in the case of the corresponding multi-level atom.

Interference between different \( J \)-levels has a double role: on one hand it enters the SEEs and therefore, in principle, its presence may modify populations and interference within the same \( J \)-level with respect to the case in which it is neglected; on the other hand it contributes (exactly as populations and interference within the same \( J \)-level do) to the radiative transfer coefficients.

We focus our attention on the radiation scattered at 90° by a plane-parallel slab of plasma illuminated by the solar continuum radiation field. In this simple scenario, the fractional polarization of the scattered radiation can be calculated through the approximate formula (see Trujillo Bueno 2003)

\[
\frac{X}{T} \approx \frac{\epsilon_X}{\epsilon_f + \epsilon_g} - \frac{\eta_X}{\eta_f + \eta_g}, \quad \text{with} \quad X = Q, U, V,
\]

where \( \epsilon_i \) and \( \eta_i \) \((i = I, Q, U, V)\) are the line emission and absorption coefficients, respectively, in the four Stokes parameters, while \( \epsilon_f \) and \( \eta_f \) are the continuum intensity emission and absorption coefficients, respectively. The incident continuum radiation is assumed to have axial symmetry around the normal to the slab (the local vertical), to be unpolarized, and to be flat over the frequency intervals covered by the multiplets that will be considered in this investigation. The reference direction for positive \( Q \) is assumed perpendicular to the scattering plane. The Doppler width used in the calculation of the line emission and absorption coefficients is derived from the temperature and microturbulent velocity of a solar model atmosphere at the height where the line-center optical depth is unity for an observation at \( \mu = 0.1 \). The semi-empirical atmospheric model FAL-C of Fontenla et al. (1993) has been used.

The first step is to write down and solve the SEEs for the given incident continuum radiation field. We describe the incident radiation through the radiation field tensor \( J_0(\nu) \) (see Equation (5.157) of LL04 for its definition), taking the quantization axis along the local vertical. Due to the cylindrical symmetry of the incident continuum radiation around this direction, only two components of the radiation field tensor are non-zero: \( J_0^2 \) and \( J_2^2 \). The former describes the average intensity of the radiation field over all the directions of propagation, while the latter gives a measure of the anisotropy of the radiation field. Under such circumstances, it is customary to describe the radiation field through two equivalent dimensionless quantities, the mean number of photons per mode \( \langle \bar{n} \rangle \) and the anisotropy factor...
discussed. the calculation of the Doppler width of each line, as previously
transition under investigation and at the temperature chosen for
where \( B \) the upper levels
We investigate the effects of the quantum interference between
Figure 1. (a) \( p_Q \) profiles obtained taking into account (solid line) and neglecting (dashed line) interference between the upper \( J \)-levels, plotted as a function of the reduced wavelength \( w = (\lambda - \lambda_0)/\Delta \lambda_D \). The results refer to a hypothetical \( ^2S - ^2P \) multiplet with a wavelength separation between the two FS components equal to 500 times the Doppler width of the lines (the same Doppler width has been assumed for the two lines). The vertical dotted lines indicate the position of the two FS components. The reference wavelength \( (\lambda_0) \) is the one corresponding to the energy difference between the centers of gravity of the two terms. All the calculations have been performed assuming the \( L - S \) coupling scheme to hold. Panel (b): detail of the “plateau” (see the text) shown by the \( p_Q \) profiles around transition 2. Panel (c): detail of the “plateau” shown by the \( p_Q \) profiles around transition 1.

\( (w) \), defined by (LL04)

\[
\tilde{n} = \frac{c^2}{2h\nu^3} J^0_0, \quad \nu = \sqrt{2J^0_0/J^0_0}.
\]  

In the absence of magnetic fields and of other mechanisms able to break the cylindrical symmetry of the problem, if the quantization axis is taken along the symmetry axis (as we are assuming here), the only interference that can be excited is that between pairs of magnetic sublevels pertaining to different \( J \)-levels and characterized by the same value of the magnetic quantum number \( M \).

Once the SEEs are solved, and the density-matrix elements are known, we calculate the line emission and absorption coefficients appearing in Equation (5), according to the expressions given in LL04 (Chapter 7). The continuum emission coefficient \( \varepsilon^\ell_0 \) will be a free parameter of the problem, used to analyze the effect of the continuum in masking the observational signatures of the \( J \)-state interference under investigation. The value of the continuum absorption coefficient is calculated here through the equation

\[
\eta^\ell_0 = \frac{\varepsilon^\ell_0}{B(T)},
\]

where \( B(T) \) is the Planck function at the wavelength of the transition under investigation and at the temperature chosen for the calculation of the Doppler width of each line, as previously discussed.

4. THE \(^2S - ^2P\) MULTIPLET

We start our investigation considering the \(^2S - ^2P\) multiplet, the simplest one where interference between different \( J \)-levels occurs, and one of the most interesting given the large number of strong spectral lines belonging to this multiplet that are observed on the Sun (Ca \( \text{ii} \) H and K, Mg \( \text{ii} \) h and k, Na \( \text{I} \) D\(_1\) and D\(_2\), Ly\( \alpha \), etc.).

This multiplet consists of the following two transitions: \( J_\ell = 1/2 \rightarrow J_u = 1/2 \) (transition of \( \text{H}\) or \( \text{D}_1\) type, in the following referred to as transition 1) and \( J_\ell = 1/2 \rightarrow J_u = 3/2 \) (transition of \( \text{K}\) or \( \text{D}_2\) type, in the following referred to as transition 2). We investigate the effects of the quantum interference between the upper levels \( J_a = 1/2 \) and \( J_u = 3/2 \). Since the common lower level of the transitions of this multiplet, having \( J_\ell = 1/2 \), cannot be polarized by the unpolarized incident radiation field, the SEEs considerably simplify with respect to the general case. Moreover, \( \eta^\ell_0 = 0 \) so that the second term on the right-hand side of Equation (5), which describes the effects of dichroism, does not contribute to the emergent polarization. If stimulation effects are neglected (which is a good approximation in the solar atmosphere), and no magnetic fields are considered, it is possible to find an analytical solution of the SEEs, and rather simple analytical expressions for the ratio \( p_Q = \varepsilon^\ell_0/\varepsilon^\ell_1 \), both in the case in which the interference between the two upper \( J \)-levels is taken into account, and in the case in which it is neglected (see Equations (A15) and (A16) in the Appendix).

The \( p_Q \) profiles obtained by taking into account (solid line) and neglecting (dashed line) interference between the upper \( J \)-levels, assuming \( w = 0.1 \) and a wavelength separation between the two components \( \Delta \lambda = 500\Delta \lambda_D \), with \( \Delta \lambda_D \) the Doppler width of the two lines, are plotted in panel (a) of Figure 1. As seen in the figure, the two profiles coincide in the core of the two lines, while they are very different at all the other wavelengths (cf. Stenflo 1980; LL04). The most remarkable difference is the sign reversal between the two lines shown by the \( p_Q \) profile calculated by taking into account the quantum interference between the two upper \( J \)-levels. This particular signature of \( J \)-state interference has been clearly observed between the H and K lines of Ca \( \text{ii} \) (see Stenflo 1980). The negligible contribution of interference between different \( J \)-levels in the center of the single lines of an arbitrary multiplet has already been discussed in Section 10.17 of LL04, where an analytical expression for the value of \( p_Q \) in the core of the lines is derived (cf. Equation (A14)).

As discussed in LL04, under the hypotheses previously introduced, the asymptotic value of \( p_Q \) for an arbitrary two-term atom is equal to the (constant) \( p_Q \) value of a two-level atom with \( J_a = L_u \) and \( J_\ell = L_\ell \). The analytical expression of \( p_Q \) for a two-level atom can be found in the Appendix (Equation (A13)).

4.1. Dependence on \( \tilde{n} \) and \( w \)

As shown by Equations (A15) and (A16), the quantity \( p_Q \), as calculated by both taking into account and neglecting interference between different \( J \)-levels, depends on the value of \( w \) but not on the value of \( \tilde{n} \).\(^6\) Concerning the dependence on the anisotropy factor, it should be observed that for small values

\(^6\) Note that this result is correct provided that \( \tilde{n} \ll 1 \), as implicit in Equations (A15) and (A16) which are derived neglecting stimulation effects.
of $w$ the second term in the denominator of Equations (A15) and (A16) can be neglected with respect to the first one, so that $w$ represents just a scaling factor of the whole profile. This property can be clearly appreciated from Figure 2, where the absolute value of the ratio between the values of $p_Q$ (calculated taking into account interference) at the line center of transition 2 and at the wavelength position of the (negative) minimum between transitions 2 and 1, plotted as a function of the anisotropy factor (the interval between 0 and 1 has been considered). The inset graphic shows the same plot with the abscissa in logarithmic scale.

4.2. Dependence on the Wavelength Separation between the Two Components and the Effect of a Finite Doppler Width

As shown in panels (b) and (c) of Figure 1, at the wavelength positions of the two transitions, the $p_Q$ profiles obtained by taking into account and neglecting $J$-state interference coincide and are constant over a wavelength interval of about five Doppler widths. The mathematical and physical origin of these “plateaus” is discussed in the Appendix, where it is shown that their boundaries are determined by the wavelength positions at which the behavior of the corresponding Voigt profiles changes from Gaussian to Lorentzian.

Since the extension of the plateaus does not show significant variations with the wavelength separation between the two components of the multiplet, their presence only marginally affects the $p_Q$ profile when the separation between the two components is much larger than the Doppler width of the two lines (see panel (a) of Figures 1 and 3), while it starts modifying the overall shape of the $p_Q$ profile when the distance between the two components is of the same order of magnitude as the Doppler width of the lines (see panel (b) of Figure 3).

When the wavelength separation between the two components is equal to or smaller than the extension of the two plateaus, these start merging, so that between the two lines all the signatures of the interference between the upper $J$-levels (such as the negativity previously discussed) disappear (see panels (c) and (d) of Figure 3). Since the width of each plateau is about five Doppler widths, it is important to observe that their merging, and therefore the disappearance of the effects of $J$-state interference, starts when the intensity profiles of the two lines are still separated from each other (see panel (c) of Figure 3).

The effects of $J$-state interference become appreciable in the line core when the separation between the two components is of the same order of magnitude as the width of the two interfering $J$-levels. Indeed, in the limiting case in which the separation between the two levels reaches zero, the effects (and in particular the depolarizing effect) of the FS have to disappear, and the polarization pattern of a two-level atom with $J_u = L_u$ and $J_l = L_l$ must be recovered (because of the principle of spectroscopic stability). This effect is shown in panels (e) and (f) of Figure 3, where a frequency separation between the two components of the same order of magnitude as the Einstein coefficient for spontaneous emission has been considered. It must be observed that the FS splitting of the various $J$-levels is always much larger (also in the case of hydrogen) than the natural width of the levels. For example, in the case of the Ly$\alpha$ line, the separation between the two FS components is of the order of $10^{10}$ s$^{-1}$, while the Einstein coefficient for spontaneous emission is equal to $6.265 \times 10^8$ s$^{-1}$. However, we have to remember that collisions (neglected here) may play an important role in these phenomena (both for their broadening and depolarizing effects).

Another interesting result to be pointed out is that when the separation between the two components is smaller than the Doppler width of the two lines, so that the two plateaus completely merge, the FS depolarizing effect takes place on a wavelength interval of about five Doppler widths (the width of a single plateau), independently of the actual separation between the two components.

4.3. The Role of the Continuum

We now analyze how the fractional polarization profiles previously obtained by taking into account only line processes are modified when the contribution of an unpolarized continuum is added according to Equation (5). As shown by the various panels of Figure 4, the main effect of the continuum is to make the fractional polarization vanish as one moves from the core of the lines toward the wings. Although this overall effect makes the patterns obtained by taking into account and neglecting $J$-state interference coincide, if the continuum is not too strong, clear signatures of the interference between different $J$-levels can still be noticed.

Let us consider first the case in which the wavelength separation between the two transitions is much larger than the Doppler width (Figure 4, panels (a) and (b)). As far as the profile calculated by taking into account $J$-state interference is concerned, we observe that if the continuum is not sufficiently strong, the maximum value of the polarization is not found at the center of transition 2, but at slightly shorter wavelengths. This circumstance can be seen in the observations of the Ca II H and K system (see Stenflo et al. 1980; Gandorfer 2002). The amount of continuum needed in order to produce the maximum signal falling at the center of transition 2 is larger the smaller the wavelength.

\footnote{Note that the intensity profiles are in emission since we are considering the radiation scattered at 90° by an optically thin slab.}
Figure 3. Each panel shows the $I/Q$ profiles obtained by taking into account (solid line) and neglecting (dashed line) interference between the upper $J$-levels for decreasing values of the separation $\Delta \lambda$ between the two components of the multiplet. The profiles are plotted as a function of the reduced wavelength $u$. The reference wavelength is the one corresponding to the energy difference between the centers of gravity of the two terms. The vertical dotted lines indicate the wavelength position of the two components (in panels (e) and (f) they cannot be distinguished). In panel (c) also the intensity profiles of the two lines (normalized to the maximum value of $I/\mu$ over the whole multiplet) are plotted as a function of the reduced wavelength (dash-dotted line), according to the scale shown on the right ordinate axis.

separation is between the two components. The presence of the continuum reduces the negativity of the $Q/I$ pattern between the two transitions; it moves the negative minimum toward the wavelength position of transition 1, but it does not affect the wavelength position of the sign reversals. Moreover, it produces a positive peak in the red wing of transition 1, so that an antisymmetrical profile across this component appears. This latter effect, due to interference between the different $J$-levels of the upper term, can be clearly observed across the Na\textsc{i} $D_1$ line (see Stenflo & Keller 1997). As the value of the continuum is increased, the amplitude of this antisymmetrical pattern is decreased, and the overall profile becomes more similar to the one obtained by neglecting $J$-state interference. The amount of continuum needed for this signature to be cancelled out is larger the smaller the wavelength separation is between the two components (this explains why this structure can be observed across the $D_1$ line of Na\textsc{i}, but not across the corresponding Ba\textsc{ii} line at 4934 Å).

We consider now the opposite case in which the various components are very close to each other with respect to the Doppler width of the single lines (Figure 4, panels (c) and (d)). In this case, as previously discussed, interference between different $J$-levels does not produce any observable effect across a spectral interval of about five Doppler widths centered at the wavelength of the two transitions, irrespective of the continuum value. Just outside this interval, on the other hand, the $Q/I$ profile obtained by taking into account $J$-state interference shows, if the continuum is not too strong, two peaks. These $Q/I$ peaks disappear as the value of the continuum is increased. It should be emphasized that the physical origin of this two-peak structure, which is obtained only when $J$-state interference is taken into account, lies in the way the continuum is included in the slab model we are considering here. This does not mean that it is an artifact, but it is clear that its reliability is intimately related to the suitability of the model that we are using. We finally observe that for the same value of the continuum, the $Q/I$ profiles obtained by taking into account $J$-state interference always show more extended wings than the corresponding profiles obtained by neglecting it.

4.4. The Effect of Radiative Transfer

All the results shown in this investigation are based on the slab model described in Section 3, which allows us to study...
the polarization properties of the scattered radiation through the approximate analytical expression of Equation (5). This allows us to analyze in great detail the atomic aspects of the problem without introducing the complications due to radiative transfer processes in stratified model atmospheres.\(^8\)

The analysis of the effects of radiative transfer on the polarization signatures produced by interference between different \(J\)-levels is beyond the scope of this paper. Nevertheless, we can make some brief, qualitative considerations on this topic. The main consequence of radiative transfer is that, depending on its frequency across the spectral line profile, the emergent radiation comes from different atmospheric heights, characterized by different values of the anisotropy factor of the local radiation field. The fractional polarization patterns previously obtained using a single value of the anisotropy factor will thus be modified by radiative transfer effects through a modulation caused by the different values of \(w\). Such modifications will take place mainly between the center and the wings of the single spectral lines. Very likely, when radiative transfer effects in solar atmospheric models are considered, the amplitude and shape of the \(Q/I\) profile around the line center will not be exactly identical to those shown in the plots of Figure 4. For example, the flat plateaus previously discussed might not be obtained or might be different. On the other hand, the results of the analysis of the spectral intervals where the effects of \(J\)-state interference are negligible, and where the FS depolarization takes place, should remain valid, as well as the qualitative shapes of the signatures of interference, as far as these appear in the far wings of the lines, where assuming a constant value of \(w\) is quite a good approximation.

Finally, we emphasize that, despite its simplicity, the slab model of Section 3 has already been applied with success for interpreting several peculiarities observed in scattering polarization signals (e.g., Belluzzi et al. 2009), thus showing that it represents a suitable approximation, at least for the modeling and understanding of the physical mechanisms producing polarization in the solar atmosphere.

4.5. Application to Particular \(^2S -^2P\) Multiplets

We consider now various \(^2S -^2P\) multiplets of particular interest. If not explicitly stated, we keep assuming \(w = 0.1\) and \(\bar{n} = 10^{-3}\).

Figure 5 shows the fractional linear polarization patterns obtained across the \(D_1\) and \(D_2\) lines of Ba\(^{II}\),\(^9\) the H and K lines of Ca\(^{II}\), the h and k lines of Mg\(^{II}\), and the \(D_1\) and \(D_2\) lines of Na\(^{I}\), taking into account the interference between the two upper \(J\)-levels. Although some of these elements (Ba, Mg, and Na) have isotopes with non-zero nuclear spin, HFS has been neglected in this investigation. For all these ions, the wavelength separation between the two lines of the multiplet is much larger than their Doppler width, so that, depending on the continuum emissivity, patterns similar to those shown in panels (a) and (b) of Figure 4 are obtained.

Comparing the profiles of the various ions plotted in Figure 5, it can be clearly observed how the amount of continuum needed for masking the signatures of \(J\)-state interference is larger the smaller the separation is between the two lines. For example, in the case of barium, a continuum \(\varepsilon_c/\varepsilon_{\ell}(\text{max}) = 10^{-5}\) is already sufficient to completely cancel out the sign reversal between the two lines and the ensuing antisymmetric pattern across the \(D_1\)

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\(^8\) Note, however, that the second term on the right-hand side of Equation (5) describes the possibility of dichroism (i.e., selective absorption of polarization components), which is absent in optically thin media.

\(^9\) It should be noted that although these barium lines, being produced by a singly ionized atomic species, should be considered, from a spectroscopic point of view, of H and K type, they are often indicated as \(D_1\)- and \(D_2\)-type lines (e.g., Stenflo & Keller 1997; Belluzzi et al. 2007) due to the similarities of the \(Q/I\) scattering polarization signals that they produce at the limb with the corresponding signals produced by the \(D_1\) and \(D_2\) lines of Na\(^{I}\).
The fact that these two lines are very close to each other (their wavelength separation of the magnesium (and sodium) lines, if the same continuum is considered. From Figure 5, it can also be observed that for the wavelength separation of the magnesium (and sodium) lines, the amount of continuum needed in order to obtain a given ratio between the amplitude of the negative minimum (in absolute value) and the amplitude of the polarization peak of the k (or D2) line is more than 10 times stronger than in the case of the calcium lines. In agreement with these considerations, the above-mentioned signatures of interference between different J-levels can be clearly observed in calcium and sodium, but not in barium.

Unfortunately, there are no high sensitivity observations available as far as the h and k lines of Mg II are concerned. The fact that these two lines are very close to each other (their separation is similar to that between the sodium D lines), with very extended wings (so that, as in the case of calcium, the continuum is not completely reached between them), suggests that these signatures should also be observable across these lines. On the other hand, at the wavelengths of these lines the continuum has a very high degree of linear polarization whose effect should be carefully taken into account. We note that no sign reversal was found in the pioneering observations of these Q/I signals performed by Henze & Stenflo (1987) using the Ultraviolet Spectrometer and Polarimeter on board the Solar Maximum Mission satellite, which, however, was not designed for detecting weak scattering polarization signals.

As a last example of the 2S → 2P multiplet, we consider the H I Lyα line (see Figure 6). Since the separation between the two components is in this case much smaller than the Doppler width of the line, the profiles that are obtained for different values of the continuum are very similar to those shown in panels (c) and (d) of Figure 4. As previously pointed out, in this case J-state interference does not produce any effect in the core of the line, across a wavelength interval of about five Doppler widths. It can be noted that this is the wavelength interval over which the depolarizing effect of FS takes place. Signatures of interference between the upper J-levels, consisting of two peaks appearing in the wings of the line, can be produced if the continuum is sufficiently weak (of the order of $10^{-5}$ or smaller). However, we already pointed out that these features in the calculated Q/I profile might be due to...
The abbreviations p.l.l. and u.l.l. stand for polarized and unpolarized lower level, respectively. Calculated by neglecting the second term on the right-hand side of Equation (5). Panel (d): same as panel (c), but neglecting interference between different J-levels. Panel (c): Q/I profiles obtained by taking into account (dotted line) and neglecting (solid line) lower-level polarization. Interference between different J-levels is taken into account. The same continuum as in panel (b) is considered. The profile obtained by taking into account lower-level polarization has been calculated by neglecting the second term on the right-hand side of Equation (5). Panel (d): same as panel (c), but neglecting interference between different J-levels. The abbreviations p.l.l. and u.l.l. stand for polarized and unpolarized lower level, respectively.

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Figure 7. Panel (a): $p_Q$ profiles obtained by taking into account (solid line) and neglecting (dashed line) interference between the upper J-levels. The vertical dotted lines indicate the wavelength positions of the various lines. Lower-level polarization has been neglected. Panel (b): same as panel (a), but including the contribution of the continuum. Panel (c): Q/I profiles obtained by taking into account (dotted line) and neglecting (solid line) lower-level polarization. Interference between different J-levels is taken into account. The same continuum as in panel (b) is considered. The profile obtained by taking into account lower-level polarization has been calculated by neglecting the second term on the right-hand side of Equation (5). Panel (d): same as panel (c), but neglecting interference between different J-levels. The abbreviations p.l.l. and u.l.l. stand for polarized and unpolarized lower level, respectively.

Figure 7. Panel (a): $p_Q$ profiles obtained by taking into account (solid line) and neglecting (dashed line) interference between the upper J-levels. The vertical dotted lines indicate the wavelength positions of the various lines. Lower-level polarization has been neglected. Panel (b): same as panel (a), but including the contribution of the continuum. Panel (c): Q/I profiles obtained by taking into account (dotted line) and neglecting (solid line) lower-level polarization. Interference between different J-levels is taken into account. The same continuum as in panel (b) is considered. The profile obtained by taking into account lower-level polarization has been calculated by neglecting the second term on the right-hand side of Equation (5). Panel (d): same as panel (c), but neglecting interference between different J-levels. The abbreviations p.l.l. and u.l.l. stand for polarized and unpolarized lower level, respectively.

The analysis carried out in this section on particular multiplets, characterized by different values of the wavelength separation between the two components, and of the Doppler width, provides complete information on the role of J-state interference in $^2S-^2P$ doublets. The extension of these results to other interesting doublets of this kind should not present particular difficulties.

5. THE $^3S-^3P$ MULTIPLET

In this section, we investigate the effects of interference between different J-levels on the scattering polarization patterns produced across $^3S-^3P$ multiplets. The possibility for the (common) lower level of the transitions of this multiplet to carry atomic polarization ($J_\ell = 2$) allows us to investigate the effects of dichroism on the polarization signatures produced by J-state interference. The analysis will be carried out on the Cr I triplet at 5207 Å and on the O I triplet at 7773 Å.

5.1. The Cr I Triplet at 5207 Å

The Cr I inverted multiplet at 5207 Å is composed by the following transitions: $J_\ell = 2 \rightarrow J_a = 1$ (line 1 at 5205.50 Å), $J_\ell = 2 \rightarrow J_a = 2$ (line 2 at 5206.04 Å), and $J_\ell = 2 \rightarrow J_a = 3$ (line 3 at 5208.42 Å). The HFS of the only stable isotope of chromium with non-zero nuclear spin ($^{53}$Cr, abundance 9.5%) is neglected in the present investigation. As in the previous sections, we consider the radiation scattered at 90° by a slab of solar atmospheric plasma illuminated by the solar continuum radiation field ($ω = 0.1$, $n = 10^{-3}$).

As can be observed in panel (a) of Figure 7, the $p_Q$ profile obtained by taking into account interference between different J-levels shows in this multiplet two sign reversals, one between lines 1 and 2, and a larger one between lines 2 and 3. The presence of small plateaus can be observed in the core of the three lines; here, as in the case of the $^2S-^2P$ multiplet, interference between different J-levels does not produce any observable effect. Note that the profiles of panel (a) of Figure 7 have been obtained by neglecting lower-level polarization and stimulation effects; the value of $p_Q$ in the core of the lines is thus given by Equation (A14).

The effect of the continuum in masking the signatures of interference is similar to the case of the $^2S-^2P$ multiplet. As shown in panel (b) of Figure 7, the continuum makes the polarization vanish in the far blue wing of line 1 and in the far red wing of line 3; it reduces the amplitude of the negative patterns between the lines, and it produces an antisymmetrical profile across transition 1 ($J_a = 1$). A qualitatively similar polarization feature across this line as well as a small sign reversal (with respect to the continuum polarization level) between lines 2 and 3, can be observed in Gandorfer’s (2000) atlas of the second solar spectrum.

The lower level of this triplet, having $J_\ell = 2$, can carry atomic polarization. The first thing that has to be pointed out is the appreciable feedback that the presence of lower-level polarization has on the atomic polarization of the upper levels. As shown in panels (c) and (d) of Figure 7, the value of $Q/I$ (as

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10 We recall that a multiplet is said to be regular when the energy of the J-levels increases with the value of J, while it is said to be inverted in the opposite case.
calculated still neglecting the contribution of dichroism, given by the second term in the right-hand side of Equation (5) is modified by the presence of lower-level polarization in the core of the three lines (it is increased in lines 1 and 3, and decreased in line 2). As far as the profiles obtained by taking into account interference between the upper \( J \)-levels are concerned (see panel (c)), we observe that lower-level polarization slightly modifies the negative pattern between lines 2 and 3, as well as the antisymmetrical profile across line 1. Interference between different \( J \)-levels also does not produce any effect in the core of the lines when lower-level polarization is taken into account.

When lower-level polarization is taken into account, \( \eta^{Q}_{J} \) is non-zero, and also the second term on the right-hand side of Equation (5) (i.e., dichroism) brings a contribution to the polarization of the emergent radiation. The effects of dichroism on the scattering polarization pattern of this multiplet can be observed in Figure 8. As shown in panel (b) of this figure, dichroism modifies the value of \( Q/I \) in the core of the three lines. Particularly interesting is its effect on the polarization pattern of line 1. In the presence of dichroism, the polarization in the core of this line becomes negative, while a small positive bump due to interference between different \( J \)-levels (see panel (a)) is obtained in the blue wing. A profile with a sign reversal (qualitatively similar to the antisymmetric profile which is obtained across transition 1 when interference between different \( J \)-levels and lower-level polarization are taken into account, and when the continuum is not too strong (see the dotted profile in panel (c) of Figure 9) is very similar to the one observed by Keller & Sheeley (1999) outside the solar limb (see their Figure 5). On the other hand, the profiles plotted in panel (d) of Figure 9, obtained assuming a continuum sufficiently strong in order to cancel out all the observational signatures of \( J \)-state interference, are, as expected, strongly reduced (see panels (c) and (d) of Figure 9).

It should be noted that the antisymmetric profile which is obtained across transition 1 when interference between different \( J \)-levels and lower-level polarization are taken into account, and when the continuum is not too strong (see the dotted profile in panel (c) of Figure 9) is very similar to the one observed by Keller & Sheeley (1999) outside the solar limb (see their Figure 5). On the other hand, the profiles plotted in panel (d) of Figure 9, obtained assuming a continuum sufficiently strong in order to cancel out all the observational signatures of \( J \)-state interference, are, as expected, strongly reduced (see panels (c) and (d) of Figure 9).

Also the role of lower-level polarization is the same as in the case of chromium, both concerning its feedback on the atomic polarization of the upper levels (see panels (c) and (d) of Figure 9), as well as concerning the ensuing contribution of dichroism (see panels (a) and (b) of Figure 10, and note the negative \( Q/I \) signal present in the core of line 1). If the same, large value of \( \varepsilon^{Q}_{J} \) as in panel (d) of Figure 9 is considered, the signatures of \( J \)-state interference also practically disappear in the presence of dichroism (see panel (a) of Figure 10). In particular, we note that no positive bump is obtained in the red wing of line 1 (unlike the Cr\( i \) case shown in Figure 8 which, being an inverted multiplet, shows such a positive bump in the blue wing of transition 1). The profiles obtained by taking into account dichroism are similar to the ones calculated by Trujillo Bueno (2009) and show a very good agreement with the close to the limb, on-disk observation presented in Figure 4 of Trujillo Bueno (2009). Note that it is natural that off-limb and inside-limb observations may show different polarization features since the effects of dichroism are enhanced for on-disk observations.

Figure 8. Panel (a): \( Q/I \) profiles calculated in the presence of dichroism (i.e., including the second term on the right-hand side of Equation (5)) by taking into account (solid line) and neglecting (dashed line) interference between different \( J \)-levels. The value of \( \varepsilon^{Q}_{J}/\varepsilon^{Q}_{J}(\text{max}) \) is the same as in panel (b) of Figure 7, the value of \( \eta^{Q}_{J} \) has been calculated according to Equation (7). Panel (b): \( Q/I \) profiles calculated by taking into account (solid line) and neglecting (dotted line) dichroism. Interference between different \( J \)-levels is taken into account. The various parameters have the same values as in panel (a).
6. THE $^3P - ^3S$ MULTIPLET: THE ROLE OF LOWER-TERM INTERFERENCE IN THE Mg i $b$-LINES

The multiplets considered in the previous sections allowed us to analyze the effects of interference between different upper $J$-levels. In order to investigate the role of interference between different $J$-levels of the lower term, we now consider the $^3P - ^3S$ triplet of Mg i at 5178 Å. This triplet is composed of the following transitions: $J_l = 2 \rightarrow J_u = 1$ ($b_1$ line at 5183.60 Å), $J_l = 1 \rightarrow J_u = 1$ ($b_2$ line at 5172.68 Å), and $J_l = 0 \rightarrow J_u = 1$ ($b_4$ line at 5167.32 Å).

We first note that under the hypothesis that the incident field is flat across the whole multiplet, if lower-term polarization is neglected, the emission coefficient $\epsilon_L^0$ is identically zero both for the two-term atom and for the corresponding four-level atom.\footnote{As pointed out in LL04, this is due to the presence of the $b$-$j$ symbol \[
\left\{ \frac{1}{L_u - L_v} \frac{1}{K} \right\} \]
in Equations (A1) and (A7), which is zero for $L_u = 0$ and $K = 2$.}

We emphasize that this is a consequence of the flat-spectrum approximation required for the theory of the two-term atom to hold. If we describe this triplet within the framework of a multi-level atom approach, so that we are allowed to consider a pumping field which varies among the various transitions, the $\epsilon_L^0$ coefficient will in general be different from zero, also under the hypothesis of unpolarized lower levels (see Trujillo Bueno

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**Figure 9.** Panel (a): $pQ$ profiles obtained taking into account (solid line) and neglecting (dashed line) interference between the upper $J$-levels. The vertical dotted lines indicate the wavelength positions of the various lines. Lower-level polarization has been neglected. Panel (b): same as panel (a), but including the contribution of the continuum. Panel (c): $Q/I$ profiles obtained by taking into account (dotted line) and neglecting (solid line) lower-level polarization. Interference between different $J$-levels is taken into account. The profile obtained by taking into account lower-level polarization has been calculated neglecting the second term on the right-hand side of Equation (5). The value of the continuum (higher than in panel (b)) is indicated on the plot. Panel (d): same as panel (c), but for a higher value of the continuum. The abbreviations p.l.l. and u.l.l. stand for polarized and unpolarized lower level, respectively.

**Figure 10.** Panel (a): $Q/I$ profiles calculated in the presence of dichroism, by taking into account (solid line) and neglecting (dashed line) interference between different $J$-levels (the two profiles are practically indistinguishable). The value of $\epsilon_L^I/\epsilon_L^J$ (max) is the same as in panel (d) of Figure 9 ($10^{-3}$), the value of $\eta_L$ has been calculated according to Equation (7). Panel (b): $Q/I$ profiles calculated by taking into account (solid line) and neglecting (dotted line) dichroism. Interference between different $J$-levels is taken into account. The various parameters have the same values as in panel (a).
Nevertheless, Trujillo Bueno (1999, 2001) showed that the presence of a given amount of atomic polarization in the lower levels of this triplet is required in order to explain the observations presented in Stenflo et al. (2000), which show positive $Q/I$ signals of similar amplitude in all three lines.

Although lower-term polarization plays an important role in the Mg I $b$-lines, interference between different lower $J$-levels is found to produce negligible effects on the $\varepsilon'_Q/\varepsilon'_I$ pattern of this multiplet (see panel (a) of Figure 11). This result implies that the atomic polarization of the upper level is practically unaffected by the presence of this kind of $J$-state interference in the SEEs (as already pointed out in Section 10.21 of LL04).

Interestingly, interference between different lower $J$-levels is also found to play a negligible role in the ratio $\eta'_Q/\eta'_I$ (see panel (c) of Figure 11). Detailed analytical calculations performed in Section 10.21 of LL04 on the simpler $2^2P - 2^2S$ multiplet show that interference between different $J$-levels of the lower term is smaller when the ratio $x_\ell$ between the FS splitting of the lower term and its natural width (given by $B(L_u \rightarrow L_\ell)\eta_\ell^0 = \bar{n}A(L_u \rightarrow L_\ell)(2L_u + 1)/(2L_\ell + 1)$) is larger. Indeed, such a negligible effect of this interference on the $\eta'_Q/\eta'_I$ profile is due to the low value of $\bar{n}$ that we are considering ($10^{-3}$). As shown in panel (d) of Figure 11, if larger values of $\bar{n}$ (of the order of 0.1 or larger) are considered, clear differences can be observed between the $\eta'_Q/\eta'_I$ profiles obtained by taking into account and neglecting interference between lower $J$-levels.

Such differences are larger in the wings of the lines, while they disappear in the cores, where the effects of this kind of $J$-state interference also remain negligible when large values of $\bar{n}$ are considered. As discussed in LL04 for the $2^2P - 2^2S$ multiplet, when lower-term polarization is taken into account, not only the interference between different lower $J$-levels but also the interference between magnetic sublevels pertaining to the same $J$-level, as well as the populations of the various magnetic sublevels, are sensitive to the value of $x_\ell$. This explains the variation with $\bar{n}$ of the $\eta'_Q/\eta'_I$ profiles calculated by neglecting the interference between lower $J$-levels. It can be noted that the sensitivity of these latter profiles to the value of $\bar{n}$ is limited to the cores of the lines (the asymptotic values do not change significantly with $\bar{n}$), while a variation with $\bar{n}$ of the overall pattern (from the core to the far wings) takes place when interference between different $J$-levels of the lower term is taken into account.

The effect of interference between lower $J$-levels on the $\varepsilon'_Q/\varepsilon'_I$ profiles is also found to be negligible when large values of $\bar{n}$ are considered (see panel (b) of Figure 11). Indeed, these profiles are much more sensitive to the value of $\bar{n}$ in the core of the lines than in the wings.

As expected from the previous discussion, interference between lower $J$-levels does not produce any observable effect on the $Q/I$ profile calculated by including the contribution of the continuum, taking into account dichroism, and assuming $\bar{n} = 10^{-3}$ (see panel (a) of Figure 12). Note that within the modeling assumptions considered here (and in particular the assumption that the anisotropy of the pumping radiation field is the same for the three lines), a negative signal is obtained in the $b_4$ line (to understand why the observations of Stenflo et al. 2000 show positive signals in the three Mg I $b$-lines).

12 When this ratio is very large, interference between different lower $J$-levels vanishes, and the levels become completely uncorrelated (like in a multi-level atom). Vice versa, when this ratio goes to zero, the lower $J$-levels are degenerate, and for the principle of spectroscopic stability the lower level has to behave like a $J$-level with $J_l = L_l$.

13 Large $\bar{n}$ values are typical of masers.

Figure 11. Panel (a): $\varepsilon'_Q/\varepsilon'_I$ profiles calculated by taking into account (solid line) and neglecting (dashed line) interference between different lower $J$-levels, and assuming $\bar{n} = 10^{-3}$. Panel (b): same as panel (a) but for different values of $\bar{n}$ (indicated on the plot). Panel (c): same as panel (a) but for the ratio $-\eta'_Q/\eta'_I$. Panel (d): same as panel (c) but for different values of $\bar{n}$ (indicated on the plot). In panels (a)–(c) the profiles obtained by taking into account and neglecting interference cannot be distinguished. The vertical dotted lines indicate the wavelength positions of the various lines. Stimulated emission (not negligible when $\bar{n}$ assumes values of the order of 0.1 or larger) has been taken into account.
lines, see Trujillo Bueno 2009 and references therein). The $Q/I$ profiles obtained for larger values of $\bar{n}$ are shown in panel (b) of Figure 12. As can be observed, the $Q/I$ pattern calculated through Equation (5) is quite sensitive to the value of $\bar{n}$ in the wings of the lines, while the line-center polarization does not show any variation with $\bar{n}$. We conclude pointing out that for the values of the continuum here considered no differences can be observed between these profiles and the corresponding ones calculated neglecting such interference).

Figure 12. Panel (a): $Q/I$ profiles calculated according to Equation (5) by taking into account (solid line) and neglecting (dashed line) interference between different lower $J$-levels, assuming $\bar{n} = 10^{-3}$. The contribution of dichroism is included; the value of the continuum is indicated on the plot ($\nu_m$ is calculated according to Equation (7)). Note that the two profiles cannot be distinguished. Panel (b): $Q/I$ profiles calculated according to Equation (5) assuming different values of $\bar{n}$ (indicated on the plot). The value of the continuum is the same as in panel (a). The profiles are calculated by taking into account interference between lower $J$-levels (note that for the values of the continuum here considered no differences can be observed between these profiles and the corresponding ones calculated neglecting such interference).

7. THE Hα LINE

Here we investigate the role of interference between different $J$-levels on the scattering polarization profile of the Hα line. Since this line is composed by seven FS components, belonging to three different multiplets, a multi-term atomic model must be applied for the analysis of $J$-state interference in this line. One of the lower terms is the upper term of the strong Lyα line, while one of the upper terms is also the upper term of the Lyβ line. The inclusion of these lines in the atomic model is required for a correct analysis of the scattering polarization properties of Hα (e.g., Štĕpán & Trujillo Bueno 2011). The Grotrian diagrams of the multi-level and multi-term atomic models considered are shown in Figure 13. We keep assuming $\bar{n} = 10^{-3}$ and $w = 0.1$ for all the lines considered in the model (the impact of interference between different $J$-levels does not depend critically on the relative values of these quantities in the various lines).

Since the wavelength separation among the various components is much smaller than the Doppler width of the line, interference between different $J$-levels does not produce any observable signature in the core of the line, across the whole wavelength interval over which the FS depolarization takes place (see the left panel of Figure 14). In agreement with the previous results, the width of this wavelength interval is about five times the Doppler width of the line.

The polarization present in the $2p^2P$ lower term (the upper term of Lyα) has been taken into account. However, its effect on the atomic polarization of the upper levels/terms through the SEE is found to be negligible.

Also in the case of Hα, the $Q/I$ profile calculated by taking into account $J$-state interference shows, for the same value of the continuum, slightly more extended wings than the profile calculated by neglecting it (see the right panel of Figure 14). Finally, it is important to point out that the line-core asymmetry shown by the $Q/I$ profile in the right panel of Figure 14 is not present when radiative transfer effects are fully taken into account in given semi-empirical models of the solar atmosphere (see Štĕpán & Trujillo Bueno 2010), although an asymmetric $Q/I$ profile similar to that observed by Gandorfer (2000) can be produced in the presence of magnetic field gradients (see Štĕpán & Trujillo Bueno 2010, 2011).

8. THE EFFECTS OF A MAGNETIC FIELD

We investigate now how the fractional polarization patterns described in the previous sections are modified by the presence of a magnetic field. Before carrying out this analysis, we recall that in the multi-level atom approximation, any kind of correlation between different $J$-levels is neglected by definition. This implies that when this approach is applied in the presence of a magnetic field, it must always be assumed that the field is sufficiently weak for the Zeeman effect regime to hold (the splitting of the magnetic sublevels must be much smaller than the separation among the various $J$-levels). This limitation is not required as far as the multi-term atom approximation is considered. In this latter case, magnetic fields going from the Zeeman effect regime to the complete Paschen–Back effect regime can be considered. As we will see (and as it is discussed in detail in LL04), in the incomplete Paschen–Back effect regime, interference between different $J$-levels is at the origin of interesting phenomena (e.g., “level-crossing” and “anti-level-crossing” effects), which may leave their signatures in the observed polarization profiles.

8.1. The Impact of the Hanle Effect on the Polarization Pattern of the Mg ii h and k Lines

We start considering the $^2S - ^2P$ doublet of Mg ii at 2800 Å. The energy separation between the two $J$-levels of the upper term is sufficiently large for the Zeeman effect regime to hold, at least for the magnetic field intensities of the solar atmospheric plasma.

In the Zeeman effect regime, the magnetic sublevels pertaining to the same $J$-level split monotonically with the magnetic field strength. Because of the relaxation term proportional to $v_{nun'}$ appearing in the SEE (recall Equations (3) in Section 2, and the discussion therein), this splitting causes a decrease of
the interference between different magnetic sublevels pertaining to the same $J$-level,\(^{14}\) which in turn produces a decrease of the linear polarization degree of the emitted radiation with respect to the non-magnetic case. This is the Hanle effect for quantum interference within the same $J$-level. In the Mg II doublet under investigation, this effect can be clearly observed in the core of the $k$ line (see panel (a) of Figure 15).

In this regime, on the other hand, the variation of the energy separation between magnetic sublevels pertaining to different $J$-levels is extremely small, so that interference between different $J$-levels does not show significant variations with respect to the unmagnetized case. The Hanle effect has therefore the same impact on the linear polarization patterns calculated by taking into account and neglecting $J$-state interference, as shown in panel (b) of Figure 15. As it can be clearly observed in the same figure, the Hanle effect takes place only in the core of the lines, right in the wavelength interval where the contribution of interference between different $J$-levels is negligible.

Since the Hanle effect takes place only in the core of the lines, if the same continuum as in the unmagnetized case is considered, a two-peak $Q/I$ profile is obtained, the central dip being produced by the Hanle depolarization (see panel (c) of Figure 15). This structure is gradually lost as the continuum intensity is increased. On the other hand, if a continuum sufficiently strong to cancel out this two-peak structure is considered, the signatures of interference between different $J$-levels (such as the antisymmetric pattern across the $h$ line) also become strongly reduced (see panel (d) of Figure 15). As in Section 4.3, we emphasize that the physical origin of this two-peak structure lies in the way the continuum is included in the slab model.

In the presence of a magnetic field with a longitudinal component, an appreciable $\epsilon'_{ij}/\epsilon_{ij}^{\text{max}}$ profile is obtained in the $k$ line. This signal, due to the Hanle effect, appears in the core of the $k$ line, over the same wavelength interval where the Hanle depolarization of the $\epsilon'_{ij}/\epsilon_{ij}^{\text{max}}$ profile takes place. This is also the wavelength interval (of about five Doppler widths) where interference between different $J$-levels does not produce any appreciable effect on the $\epsilon'_{ij}/\epsilon_{ij}^{\text{max}}$ pattern. Indeed, the profiles obtained by taking into account and neglecting such interference cannot be distinguished (see panel (a) of Figure 16). We note that a very weak antisymmetrical $\epsilon'_{ij}/\epsilon_{ij}^{\text{max}}$ profile (of the order of $10^{-5}$, not visible in the figure) is obtained in the core of the $h$ line when $J$-state interference is taken into account. This signal, too weak to be observable in practice, and of purely academic importance, is probably due to the small variation of the interference between different $J$-levels produced by the magnetic field in the Zeeman effect regime. As previously pointed out, this kind of effect (i.e., Hanle effect for interference between different $J$-levels) is generally negligible for magnetic fields in this regime, since the variation of the energy separation between the interfering magnetic sublevels (pertaining to different $J$-levels) is very small. These effects become more important as the Paschen–Back effect regime is reached. In the presence of the same field, a $\epsilon'_{ij}/\epsilon_{ij}^{\text{max}}$ profile due to the Zeeman effect is also obtained (see panel (b) of Figure 16). There are no differences

\(^{14}\) We observe that in a medium with cylindrical symmetry, this interference is in general non-zero if the quantization axis is not taken along the symmetry axis (cf. the discussion in Section 3).
Figure 15. Effects of a horizontal magnetic field of 100 G perpendicular to the line of sight on the polarization pattern of the radiation scattered at 90° by an optically thin slab in the Mg II doublet at 2800 Å. Panel (a): $pQ$ profiles, calculated by taking into account interference between the upper $J$-levels, in the absence of (dotted line) and in the presence of (solid line) the above-mentioned magnetic field. Panel (b): $pQ$ profiles calculated in the presence of the same field, taking into account (solid line) and neglecting (dashed line) interference between the upper $J$-levels. Panel (c): same as panel (a) but introducing the contribution of the continuum according to Equation (5). Panel (d): same as panel (c), but in the presence of a stronger continuum (the profile corresponding to the unmagnetized case is not shown).

Figure 16. $\epsilon_\ell^I/\epsilon_\ell^I$ (panel (a)) and $\epsilon_\ell^V/\epsilon_\ell^I$ (panel (b)) profiles calculated by taking into account (solid line) and neglecting (dashed line) interference between different $J$-levels, in the presence of a longitudinal magnetic field of 50 G. The profiles obtained by taking into account and neglecting $J$-state interference cannot be distinguished.

between the $\epsilon_\ell^I/\epsilon_\ell^I$ profiles calculated by taking into account and neglecting $J$-state interference.

The Mg II h and k lines are an example of a doublet in which the separation between the two lines is much larger than their Doppler width. The results obtained in this section can be extended to any other $^2S - ^2P$ multiplet in which the two components are sufficiently separated from each other. The results can also be extended to different multiplets, provided that the various lines are well separated, that lower-term polarization is absent or negligible (in this multiplet it is zero by definition), and that the same hypotheses here considered can be made.

8.2. The Effect of a Magnetic Field on the Antisymmetric Interference Profile of the H/D1-type Lines

In this section, we investigate the effects of a magnetic field on the antisymmetric interference pattern characterizing the emergent $Q/I$ profile across the $1/2 - 1/2$ transition of the $^2S - ^2P$ multiplets. In the absence of HFS, no Hanle depolarization takes place in this line since its line-core polarization is already identical to zero in the unmagnetized case (see, for example, the previous analysis of the Mg II doublet at 2800 Å). Nevertheless, it is interesting to observe how the interference pattern is modified by the presence of a magnetic field strong enough for the transverse Zeeman effect to be appreciable. As shown in Figure 17, when the typical signatures of the transverse Zeeman effect superimpose on the interference pattern, a peculiar profile, qualitatively similar to the one observed by Stenflo et al. (2000) in the Na I D1 line, is obtained (see also the theoretical investigations by Trujillo Bueno et al. 2002 and Casini & Manso Sainz 2005, which took HFS into account).

The exact shape of the resulting profile depends on the relative amplitude and width of the antisymmetric interference pattern, and of the transverse Zeeman effect pattern (compare the three panels of Figure 17). The shape of the interference pattern is controlled by the continuum intensity (recall the discussion in Section 4.3), while the shape of the Zeeman pattern by the magnetic field strength. As expected, the intensity of the magnetic field required for the Zeeman pattern to be appreciable
The profiles have been calculated according to Equation (5), assuming the presence of magnetic fields of different intensities and orientations (indicated on the plot). The profiles have been calculated according to Equation (5), assuming for the continuum the value \( \varepsilon_c \) of 2 kG (i.e., in the incomplete Paschen–Back effect regime). The value of the continuum emissivity has been chosen in order to obtain the same amplitude of the antisymmetrical profile in the three lines.

Figure 17. Modification of the antisymmetrical patterns produced by interference between different \( J \)-levels in the emergent \( Q/I \) profiles of the \( h \) line of \( Mg \, \iota \) (left panel), of the \( H \) line of \( Ca \, \iota \) (center panel), and of the \( D_1 \) line of \( Na \, \iota \) (right panel), due to the transverse Zeeman effect produced by a horizontal magnetic field perpendicular to the line of sight of 100 G (dashed line) and 300 G (solid line). The \( Q/I \) profiles have been calculated neglecting the second term on the right-hand side of Equation (5), in order to show the observational signature of the transverse Zeeman effect in optically thin structures. The reference profiles corresponding to the unmagnetized case are shown by the dotted line.

The overall profile is higher the shorter the wavelength is (compare the three panels of Figure 17).

### 8.3. The \( \text{Ly}_\alpha \) Line Case

In order to investigate the effects of a magnetic field on a multiplet in which the separation among the various components is much smaller than their Doppler width, we now consider the \( \text{Ly}_\alpha \) line. As discussed in Section 4.5, interference between different \( J \)-levels does not produce any observable signature in the core of this line. Also in this case, the impact of the Hanle effect due to a magnetic field with strength in the Zeeman effect regime is the same on the polarization profiles obtained by taking into account and neglecting interference between different \( J \)-levels (see Figure 18).

In the \( \text{Ly}_\alpha \) line, on the other hand, the separation between the two upper \( J \)-levels is sufficiently small for the incomplete Paschen–Back effect regime to be reached for magnetic fields of the order of 1 kG. In this regime, the variation of the energy separation between pairs of magnetic sublevels pertaining to different \( J \)-levels is no longer negligible. In particular, when the separation between two magnetic sublevels is of the same order of magnitude as their natural width (i.e., when the two sublevels overlap), the first term on the right-hand side of Equations (3) produces a significant modification of the corresponding coherence (generally an increase in absolute value), which may produce observable effects in the polarization of the emergent radiation. It can be demonstrated that while pairs of magnetic sublevels with \( \Delta M \neq 0 \) can approach and cross each other (note that when this happens the relaxation term is exactly zero), pairs of magnetic sublevels with \( \Delta M = 0 \) can approach but never cross. The terminology of “level-crossing” and “anti-level-crossing” (see Bommier 1980) effects is often used to indicate the Hanle effect produced by these particular behaviors of the magnetic sublevels in the incomplete Paschen–Back effect regime. A detailed description of these phenomena can be found inLL04.

Here we want to focus attention on the following aspect: the possibility, when interference between different \( J \)-levels is considered, of also having the Hanle effect in the presence of a “vertical” magnetic field. As discussed in Section 3, in a medium with cylindrical symmetry around a given direction (e.g., the vertical), if the quantization axis is taken along this direction, it can be shown that no quantum interference can be induced between pairs of magnetic sublevels pertaining to the same \( J \)-level. Such interference, which is the only kind to be accounted for in a multi-level atom, also remains zero in the presence of a magnetic field directed along the symmetry axis of the incident radiation, since this does not break the symmetry of the problem. Since the magnetic field does not modify the populations (note that under the same hypotheses population imbalances can be induced among the various magnetic sublevels), it follows that, as known, for the multi-level atom model there is no Hanle effect in the presence of a vertical magnetic field.

This is no longer true when interference between different \( J \)-levels is considered. As previously shown, in a medium with cylindrical symmetry, interference between magnetic sublevels with the same value of \( M \), pertaining to different \( J \)-levels, is actually non-zero. This \( J \)-state interference is modified by a vertical magnetic field, in the incomplete Paschen–Back effect regime, with appreciable effects on the polarization of the scattered radiation. This particular example of the Hanle effect for interference between different \( J \)-levels can be clearly observed in Figure 18: in the presence of a vertical magnetic field of about 2 kG (thus in the incomplete Paschen–Back effect regime), the amplitude of the scattering polarization signal is increased with respect to the zero-field case. This enhancement of the polarization due to a vertical magnetic field was already pointed out by Trujillo Bueno et al. (2002) for the \( Na \, \iota \) \( D_2 \) line, and by Belluzzi et al. (2007) for the \( Ba \, \iota \) \( D_2 \) line. However, it
should be observed that in these latter cases the effect is due to interference between different HFS $J$-levels, and not between different $J$-levels as in the $\lambda\alpha_2$ line (the physical mechanism is exactly the same). It is clear that this effect is of more practical interest when HFS is present, since relatively weak magnetic fields are sufficient for the incomplete Paschen–Back effect regime of HFS to be reached.

8.4. Hanle Effect and Lower-level Polarization

We analyze here the effect of a magnetic field on the Mg I triplet at 5178 Å. As discussed in Section 6, the polarization of the radiation emitted in these lines is very sensitive to the atomic polarization of the lower levels. The so-called lower-level Hanle effect can thus be clearly observed in this triplet. If the lower level is metastable (as in this case), the lower-level Hanle effect is sensitive to rather weak magnetic fields. In these lines, assuming $\bar{n} = 10^{-5}$, it can already be appreciated for magnetic field intensities of the order of a few mG (see Figure 19), while for a magnetic field of 1 G it is already saturated.

Also in the presence of magnetic fields, interference between different lower $J$-levels does not produce any observable signature on the $Q/I$ profile of the emergent radiation, and it can be safely neglected.

An interesting property of the lower-level Hanle effect is that it does not vanish in the wings of the lines, but it depolarizes the whole pattern (see panel (a) of Figure 19). As a consequence, if the same continuum as in the unmagnetized case is considered, no double-peak structures are obtained as a result of Hanle depolarization (see panel (b) of Figure 19).

We recall that the property of the (upper-level) Hanle effect to vanish in the wings of the lines is strictly verified under the assumptions that the lower level is unpolarized, and that stimulated emission and elastic collisions are negligible (see Section 10.4 of LL04 for an analytical proof). It is clear that this property remains valid whenever stimulation effects are very weak (like in the solar atmosphere), and whenever the influence of lower-level polarization and collisions on the polarization properties of the line is negligible. An analysis of the influence of elastic collisions on the Hanle effect can be found in Section 10.6 of LL04. A detailed analysis of the physical conditions under which the “wing-Hanle-effect” can be observed will be presented in a forthcoming paper.

9. CONCLUSIONS

We have investigated the effects of quantum interference between different $J$-levels on the linear polarization pattern of the radiation scattered at 90° by a slab of stellar atmospheric plasma. The investigation has been carried out within the framework of the quantum theory of polarization presented in LL04, which is based on the flat-spectrum approximation.

We started focusing our attention on the $^2S - ^2P$ doublet. We analyzed the effects of the interference between the two upper $J$-levels as a function of the wavelength separation ($\Delta\lambda$) between the two FS components, and assuming a finite Doppler width ($\Delta\lambda_D$) for the two spectral lines, in the absence of the continuum. The most important results of this part of our study, carried out for the unmagnetized case, and neglecting the effects of collisions and stimulated emission, are the following.

1. The effects of interference between different $J$-levels are negligible in the core of the two lines, while they become important moving from the center to the wings (see panel (a) of Figure 1). If the separation $\Delta\lambda$ between the two lines is much larger than their Doppler width, the shape of the overall interference pattern (shown in panel (a) of Figure 1) does not depend on the particular value of $\Delta\lambda$.

2. In the core of the two lines, the fractional polarization profiles obtained by taking into account and neglecting $J$-state interference show flat “plateaus” of about $5\Delta\lambda_D$, across which they perfectly coincide (see panels (b) and (c) of Figure 1).

3. When $\Delta\lambda \ll 5\Delta\lambda_D$ (so that the two plateaus merge), all the signatures due to $J$-state interference disappear between the two lines (note that this merging starts when the intensity profiles are still well separated from each other; see panel (c) of Figure 3).

4. When $\Delta\lambda < \Delta\lambda_D$ (so that the two plateaus completely merge), the FS depolarization takes place on a wavelength interval of about $5\Delta\lambda_D$ (the width of a single plateau), irrespectively of the actual separation between the two components (see panels (d)–(f) of Figure 3).

5. Signatures of $J$-state interference become appreciable in the line core when the separation between the interfering $J$-levels is of the same order of magnitude as their natural width (see panels (e) and (f) of Figure 3).

Although the flat plateaus that appear in the core of the lines might not be obtained once radiative transfer effects in realistic solar model atmospheres are taken into account, the fact that across the corresponding wavelength intervals the effects of $J$-state interference are negligible is a physical result that is expected to also remain valid when full radiative transfer is properly considered. A detailed analysis of the physical and
needed in order to observe its signatures on the ratio $\eta_{Q}/\eta_{I}$ is taken into account, finding the following main results.

1. The amount of the continuum emissivity needed to mask the signatures of $J$-state interference is larger the smaller the wavelength separation is between the interfering lines (see Figures 4 and 5).

2. In the case of the $^2S - ^2P$ doublet, the effect of the continuum is to reduce the amplitude of the negative minimum between the two lines, to move it toward the 1/2 − 1/2 transition, and to produce an antisymmetrical pattern across this transition (see Figure 4). For a given value of the continuum emissivity, the amplitude of the above-mentioned antisymmetrical pattern is smaller the larger the separation is between the two components of the multiplet (see Figure 5).

The above-mentioned results are not all limited to the $^2S - ^2P$ doublet, but they can be generalized to any other multiplet.

We carried out an analysis of the signatures produced by interference between different $J$-levels on the following multiplets:

1. Ba ii $^2S - ^2P$ doublet (4554 Å and 4934 Å resonance lines),
2. Ca ii $^2S - ^2P$ doublet (H and K lines),
3. Mg ii $^2S - ^2P$ doublet (h and k lines),
4. Na i $^2S - ^2P$ doublet (D1 and D2 lines),
5. H i $^2S - ^2P$ doublet (Lyα),
6. Cr i $^3S - ^3P$ triplet at 5207 Å,
7. O i $^3S - ^3P$ triplet at 7773 Å,
8. Mg i $^3P - ^3S$ triplet (b1, b2, and b4 lines),
9. H i Hα (line composed of seven FS components belonging to three different multiplets).

The analysis of the Cr i and O i triplets allowed us to investigate in detail the combined effect on the emergent scattering polarization profiles of interference between different $J$-levels, lower-level polarization (see Figures 7 and 9), and dichroism (see Figures 8 and 10).

The analysis of the Mg i b-lines allowed us to investigate the role of interference between different $J$-levels of the lower term. We found that for the typical solar values of $\bar{n}$ (the mean number of photons per mode of the pumping radiation field), its effect on the scattering polarization profiles is completely negligible (see panels (a) and (c) of Figure 11, and panel (a) of Figure 12). Values of $\bar{n}$ of the order of 0.1 or larger are needed in order to observe its signatures on the ratio $\eta_{Q}/\eta_{I}$, its effect being in any case negligible in the core of the lines (see panel (d) of Figure 11). The atomic polarization of the upper level is practically unaffected by the presence of this kind of $J$-state interference in the SEEs, also for high values of $\bar{n}$ (see panel (b) of Figure 11). The signatures of interference between different lower $J$-levels (appreciable only when high values of $\bar{n}$ are considered) are in any case strongly masked by the presence of the continuum. Our calculations also showed an appreciable sensitivity of the $Q/I$ profile of the emergent radiation to the value of $\bar{n}$ (see panel (b) of Figure 12).

We finally investigated whether or not, and to which extent, the signatures due to interference between different $J$-levels are modified by the presence of a magnetic field. The results can be summarized as follows.

1. In the Zeeman effect regime, the influence of the magnetic field on the interference between different $J$-levels is extremely small. The Hanle effect thus leaves the same signatures on the fractional polarization patterns calculated by taking into account and neglecting interference between different $J$-levels. As shown by the profiles calculated for the Mg ii h and k lines, as far as the $\epsilon_{Q}/\epsilon_{I}$ pattern is concerned, the Hanle effect takes place in the core of the k line, right in the wavelength interval where the effect of $J$-state interference is negligible (see Figure 15). Also the $\epsilon_{Q}/\epsilon_{I}$ signal, produced by the Hanle effect in the same line in the presence of a longitudinal field, appears across this wavelength interval (see panel (a) of Figure 16).

2. In the incomplete Paschen–Back effect regime, interference between different $J$-levels is significantly modified by the magnetic field, thus producing observable effects on the emergent radiation. The physical mechanisms at the origin of these effects (e.g., level-crossings, anti-level-crossings, alignment-to-orientation conversion mechanism) are described in detail in LL04. In this paper, we focused our attention on the possibility of also having the Hanle effect in the presence of “vertical” fields when interference between different $J$-levels is taken into account (see Figure 18). We should remember that the Paschen–Back effect regime is more typical for HFS multiplets. Even for the Lyα line considered here, for which the separation between the two interfering $J$-levels is particularly small if compared to that of other FS multiplets, fields of about 2 kG are needed for producing this kind of effects.

In summary, when the energy separation between pairs of magnetic sublevels pertaining to different $J$-levels is much larger than their width (which is always the case in the absence of magnetic fields or in the presence of weak magnetic fields in the Zeeman effect regime), the contribution of the interference between such magnetic sublevels modifies the fractional linear polarization pattern in the wings of the lines (outside a $\Delta \lambda \approx 5\Delta \lambda_{D}$ wavelength interval around the line center). However, at these wavelengths the line contribution to the total emissivity and absorptivity is no longer dominating with respect to that of the continuum. The presence of the continuum strongly masks the effects of interference between different $J$-levels, which indeed are observable only in rather strong spectral lines (e.g., Ca ii H and K, Na i D1 and D2). Because of the crucial role of the continuum, it is not possible to establish a simple quantitative criterion for deciding whether or not interference between different $J$-levels is expected to produce observable effects. On the other hand, in the presence of a magnetic field sufficiently intense for the Paschen–Back effect regime to be reached (or, in other words, when the energy separation between pairs of magnetic sublevels pertaining to different $J$-levels is no longer negligible with respect to their width), interference between different $J$-levels may modify the amplitude of the fractional polarization pattern in the core of the lines through the mechanisms described in detail in Sections 10.18 and 10.20 of LL04.

We conclude pointing out that all the results presented in this paper on the effects of interference between different $J$-levels can be generalized to the case of interference between different HFS $F$-levels. The only remarkable difference for practical applications is the fact that the Paschen–Back effect regime for HFS is generally reached for magnetic fields much weaker than those needed for reaching the Paschen–Back effect regime for FS.

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APPENDIX

ANALYTICAL RESULTS

In this appendix, we first recall some equations and analytical results derived in LL04 that have been mentioned in the text. These equations will be the starting point for the following derivation of a series of analytical expressions that will allow us to gain more insight into some results presented in this paper on the $2S - 2P$ multiplets.

Let us consider a two-term atom, the upper and lower terms being characterized by the spin $S$ and by the total orbital angular momentum $L$. Each term is composed of $(L + S - |L - S| + 1)$ FS $J$-levels, each having $(2J + 1)$ magnetic sublevels $M$, for a total of $(2S + 1)$ $(2L + 1)$ sublevels. As shown in Section 10.16 of LL04, under the simplifying hypotheses of an unpolarized lower term, no stimulation effects, no collisions, no magnetic field, and in the flat-spectrum approximation it is possible to find an analytical solution of the SEEs for the multipole moments of the density matrix of the upper term (see Equation (10.126) of LL04). If this solution is substituted into the expression of the emission coefficient for a two-term atom (see Equation (7.47e) of LL04), one obtains (see Equation (10.129) of LL04)

$$e_i(v, \Omega) = \frac{h\nu_0}{4\pi} \psi(v_{uL} - v) = \frac{1}{\sqrt{\pi \Delta \nu_\beta}} L(u, a).$$  \hspace{1cm} (A4)

The Voigt function $H(u, a)$ and the associated dispersion profile $L(u, a)$ are functions of the reduced frequency $u = (v_{uL} - v)/\Delta \nu_\beta$, with $\Delta \nu_\beta$ the Doppler width in frequency units and $v_{uL}$ the Bohr frequency between levels $u$ and $\ell$, and of the damping constant $\Gamma_{u\ell}$. The broadening constant $\Gamma_{u\ell}$ is given by

$$\Gamma_{u\ell} = \frac{\gamma_u + \gamma_\ell}{4\pi}.$$  \hspace{1cm} (A5)

where $\gamma_u$ and $\gamma_\ell$ are the inverse lifetimes of the levels $u$ and $\ell$, respectively.

The contribution of interference between different $J$-levels to the emission coefficient is described by the terms with $J_u \neq J'_{\ell}$ in Equation (A1). If these terms are neglected, recalling the relation

$$B(J_\ell \rightarrow J_u) = (2J_\ell + 1)\left\{ \begin{array}{ccc} J_u & J_u & J_\ell \\ J_u & J_u & J_\ell \\ J_u & J_u & J_\ell \\ \end{array} \right\} B(L_{\ell u} \rightarrow L_u),$$

and introducing the quantity (see Equation (10.147) of LL04)

$$W_K(L_{\ell u} S, J_\ell J_u) = (-1)^{2L_u - J_\ell + K} 2(2J_u + 1)$$

where $L = 2, 3, \ldots$ and $K = 0, 1, \ldots, J_u$, one obtains the expression of the emission coefficient for a two-level atom (see Equation (7.47e) of LL04)

$$B(J_\ell \rightarrow J_u) = (2J_\ell + 1)\left\{ \begin{array}{ccc} J_u & J_u & J_\ell \\ J_u & J_u & J_\ell \\ J_u & J_u & J_\ell \\ \end{array} \right\} B(L_{\ell u} \rightarrow L_u),$$

and introducing the quantity (see Equation (10.147) of LL04)

$$W_K(L_{\ell u} S, J_\ell J_u) = (-1)^{2L_u - J_\ell + K} 2(2J_u + 1)$$

Equation (A1) reduces to

$$[e_i(v, \Omega)]_{\text{int.}} = \frac{h\nu_0}{4\pi} N_\ell \frac{1}{2S + 1} \frac{1}{2L_u + 1} \times \sum_{J_u J_\ell} (2J_\ell + 1) B(J_\ell \rightarrow J_u) \psi(v_{uL} - v)$$

where $N_\ell$ is the number density of atoms in the lower term, $A(L_u \rightarrow L_\ell)$ and $B(L_\ell \rightarrow L_u)$ are the Einstein coefficients for spontaneous emission and for absorption, respectively, between the two terms, $T^Q_{\ell u}(i, \Omega)$ is a geometrical tensor that depends on the direction of the emitted radiation $\Omega$, and on the reference direction for positive $Q$ (see Section 5.11 of LL04 and equations therein for its explicit expression), $v_{J_u J_\ell}$ is the Bohr frequency between levels $J_u$ and $J_\ell$, and $f^Q_{\ell u}(v_0)$ is the radiation field tensor introduced in Section 3, describing the incident (pumping) radiation field. We recall that because of the flat-spectrum approximation it is sufficient to calculate it at a single frequency $v_0$ within the frequency interval covered by the multiplet. The profile $\Phi(v_{uL} - v)$ is given by

$$\Phi(v_{uL} - v) = \phi(v_{uL} - v) + i\psi(v_{uL} - v),$$  \hspace{1cm} (A2)

where

$$\phi(v_{uL} - v) = \frac{1}{\sqrt{\pi \Delta \nu_\beta}} H(u, a).$$  \hspace{1cm} (A3)
atom is recovered (see Equation (10.16) of LL04):

\[ [\varepsilon_i(v, \Omega)]_{\text{two-lev}} = \frac{h v_0}{4 \pi} N_{\text{ele}} B(J_i \rightarrow J_u) \phi(v_{J_i - J_u} - v) \times \sum_{Q \Omega} W_K(J_i, J_u) (-1)^Q \mathcal{T}_Q^K(i, \Omega) J_{\ell}^{K}(v_0). \]  

(A10)

The quantity \( W_K(L_i L_u S, J_i J_u) \) is thus a sort of generalization of \( W_K(J_i, J_u) \) that takes into account (though under the flat-spectrum approximation) the effects due to the presence of the various components of a multiplet on the polarization properties of a given transition. Besides the trivial case of \( S = 0 \), it can be shown that the symbol \( W_K(L_i L_u S, J_i J_u) \) coincides with the symbol \( W_K(J_i, J_u) \) in all the \( S-P \) multiplets (all the multiplets with \( L_i = 0, L_u = 1 \), and any value of the spin).

We consider an unmagnetized plane-parallel atmosphere, and we take a Cartesian reference system with the \( z \)-axis (quantization axis) directed along the local vertical and focus our attention on the radiation scattered at 90° by an optically thin slab of solar plasma. Choosing the reference frame for positive \( Q \) parallel to the atmosphere, and recalling that, because of the symmetry of the problem, the only non-zero components of the radiation field tensor are \( J_J^0 \) and \( J_J^2 \), the geometrical tensors \( \mathcal{T}_Q^K(i, \Omega) \) that enter the previous expressions of the emission coefficients assume the values

\[ \mathcal{T}_0^0(I, \Omega) = 1, \quad \mathcal{T}_0^0(I, \Omega) = -\frac{1}{2\sqrt{2}}, \]  

(A11)

\[ \mathcal{T}_0^0(Q, \Omega) = 0, \quad \mathcal{T}_0^0(Q, \Omega) = \frac{3}{2\sqrt{2}}. \]  

(A12)

Substituting these values into Equation (A10), we can easily obtain the analytical expression of the fractional polarization pattern \( p_Q(v) = \varepsilon_Q(v)/\varepsilon_I(v) \) for the radiation scattered at 90° by a two-level atom:

\[ [p_Q(v)]_{\text{two-lev}} = \frac{3W_2(J_i, J_u)}{w - W_2(J_i, J_u)}. \]  

(A13)

where \( w \) is the anisotropy factor defined in Equation (6).

As discussed in LL04, in a two-term atom, when the separation among the various lines of the multiplet is much larger than their natural width, the emission coefficients in the neighborhood of a single line with \( J_u = \tilde{J}_u \) and \( J_i = \tilde{J}_i \) can be evaluated by restricting the summation over \( J_u, J_u' \), and \( J_i \) in Equation (A1) to the values \( J_u = \tilde{J}_u \) and \( J_i = \tilde{J}_i \) (i.e., neglecting, in particular, the terms corresponding to the interference between different \( J \)-levels). From Equation (A8), we can thus obtain the analytical expression of the fractional polarization of the radiation scattered by a two-term atom in the core of the various lines of the corresponding multiplet. For 90° scattering we obtain

\[ [p_Q(\text{core})]_{\text{two-term}} = \frac{3W_2(L_i L_u S, J_i J_u)}{w - W_2(L_i L_u S, J_i J_u)}. \]  

(A14)

We now focus our attention on the \( ^2S \rightarrow ^2P \) multiplet. Taking into account that for this multiplet \( W_K(L_i L_u S, J_i J_u) = W_K(J_i, J_u) \) and that \( W_2(1/2, 1/2) = 0 \) (transition 1), while \( W_2(1/2, 3/2) = 1/2 \) (transition 2), starting from Equations (A1) and (A8), it is possible to find rather compact analytical expressions for the fractional polarization pattern \( p_Q(v) \) of the radiation scattered at 90°, both neglecting or taking into account \( J \)-state interference. Using the more compact notation \( \phi_j \equiv \phi(v_{J_j} - v) \) and \( \psi_j \equiv \psi(v_{J_j} - v) \) for the profiles defined in Equation (A4), with \( v_0 \) being the frequency of transition \( j \), after some algebra (including the calculation of several 6-\( j \) symbols, and the use of Equation (A6)), one obtains the following expressions:

\[ [p_Q(v)]_{\text{no int}} = \frac{3/2\phi_2}{w (\phi_1 + \phi_2) - 1/2\phi_2}, \]  

(A15)

\[ [p_Q(v)]_{\text{int}} = \frac{3(2\phi_2 + \chi)}{w (\phi_1 + \phi_2) - 1/2(\phi_2 + \chi)}, \]  

(A16)

where

\[ \chi = \frac{1}{1 + \alpha^2} (\phi_1 + \phi_2) + \frac{\alpha}{1 + \alpha^2} (\psi_1 - \psi_2). \]  

(A17)

with \( \alpha = (2\pi \Delta \nu)/A(L_u \rightarrow L_i) \). The quantity \( \Delta \nu = \nu_{02} - \nu_{01} \) is the frequency separation between the two components of the multiplet. As it can be observed by comparing Equations (A15) and (A16), the contribution of interference between different \( J \)-levels is fully described by the quantity \( \chi \).

In Section 4.2, we observed that in the neighborhood of the single transitions, the \( p_Q(v) \) profiles calculated by taking into account and neglecting \( J \)-state interference coincide and are constant over a wavelength interval of about five Doppler widths. The analytical expressions of \( p_Q \) given by Equations (A15) and (A16) allow us to analyze in detail the origin of this behavior. As far as \( [p_Q(v)]_{\text{no int}} \) is concerned, Equation (A15) can be rewritten as

\[ [p_Q(v)]_{\text{no int}} = \frac{3/2}{w (1 + 2\phi_2)} - \frac{1/2}{w}. \]  

(A18)

As shown in panel (a) of Figure 20, the ratio \( \phi_1/\phi_2 \) is much smaller than unity for frequencies close to transition 2, while it is extremely large for frequencies close to transition 1. In the core of the two transitions, we thus obtain

\[ [p_Q(v \approx \nu_{02})]_{\text{no int}} = \frac{3/2}{w - 1/2}, \quad [p_Q(v \approx \nu_{01})]_{\text{no int}} = 0. \]  

(A19)

These expressions coincide with those valid for a two-level atom (see Equation (A13)).

If the two lines are very separated from each other, the boundaries of the intervals over which the ratio \( \phi_1/\phi_2 \) is very small or very large (and thus the fractional polarization \( [p_Q(v)]_{\text{no int}} \) is practically constant) are determined by the points where the shape of the Voigt profile changes from Gaussian to Lorentzian (see panel (a) of Figure 20). Exploiting a series of properties of the functions \( H(u, a) \) and \( L(u, a) \) discussed in Section 5.4 of LL04, it can be shown that an asymptotic expansion of the Voigt function in power series of \( a \) is given by

\[ H(u, a) \simeq e^{-u^2} + \frac{1}{\sqrt{\pi}} \frac{a}{u^2}. \]  

(A20)

The value \( u_c \), for which the behavior of the Voigt function changes from Gaussian to Lorentzian can thus be evaluated through the equality

\[ e^{-u_c^2} = \frac{1}{\sqrt{\pi}} \frac{a}{u_c^2}. \]  

(A21)
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Figure 20. Panel (a): ratio between the Voigt profile of transition 1 ($\phi_1$), and the Voigt profile of transition 2 ($\phi_2$), plotted as a function of the reduced wavelength $u = (\lambda - \lambda_0)/\Delta \lambda_D$, with $\Delta \lambda_D$ the Doppler width of the two lines, and $\lambda_0$ the wavelength corresponding to the energy difference between the centers of gravity of the two terms. The separation between the two components is $\Delta \lambda = 30 \Delta \lambda_D$. The damping parameter is $a = 10^{-3}$. Panel (b): plot of $u_c$ (see the text) as a function of $\log(a)$.

Figure 21. Panel (a): plot of the interference term $\chi/(1/2 \phi_1 + \phi_2)$ as a function of the reduced wavelength $u = (\lambda - \lambda_0)/\Delta \lambda_D$, with $\Delta \lambda_D$ the Doppler width of the two lines, and $\lambda_0$ the wavelength corresponding to the energy difference between the centers of gravity of the two terms. As in panel (a) of Figure 20, the separation between the two components is $\Delta \lambda = 30 \Delta \lambda_D$, while the damping parameter is $a = 10^{-3}$. Panel (b): ratio between the dispersion profile of line 2 ($\psi_2$) and the Voigt profile of line 2 ($\phi_2$). The inner panel shows in more detail the behavior of this ratio around the wavelength position of transition 2. Panel (c): same as panel (b), with the ratio $\psi_2/\phi_2$ multiplied by $K = \alpha/(1 + \alpha^2) = 6.67 \times 10^{-5}$. Panel (d): plot of the first derivative of the ratio $\psi_2/\phi_2$.

It can be verified that the solution of this transcendent equation is given by the following recursive expression (E. Landi Degl’Innocenti 2011, private communication):

$$u_c = \sqrt{\ln(C \ln(C \ln(C \ln(\cdots))))},$$  \hspace{1cm} (A22)

with $C = \sqrt{\pi}/a$. The value of $u_c$ as a function of $\log(a)$ is shown in panel (b) of Figure 20. We see that $u_c$ varies almost linearly with $\log(a)$, going from a value of 3.844 for $a = 10^{-5}$ to a value of 2.673 for $a = 10^{-2}$, in agreement with the extension (of about five Doppler widths) of the plateaus observed in the plots of Section 4.2.

The situation is similar as far as the $[p_Q(v)]_{\text{int}}$ profile is concerned. Noting that for small values of $w$ the second term in the denominator of Equation (A16) is much smaller than the first one, this equation can be simplified as

$$[p_Q(v)]_{\text{int}} = \frac{3w}{8} \left[ \frac{\phi_2}{\frac{1}{2}\phi_1 + \phi_2} + \frac{\chi}{\frac{1}{2}\phi_1 + \phi_2} \right].$$  \hspace{1cm} (A23)

The contribution of $J$-state interference is described by the term $\chi/(\frac{1}{2}\phi_1 + \phi_2)$ (see panel (a) of Figure 21), the remaining part of the expression being the same as in the case without interference. In order to analyze the behavior of this term around transition 2, recalling Equation (A17), we rewrite it as

$$\frac{\chi}{\frac{1}{2}\phi_1 + \phi_2} = \frac{1}{1 + \frac{1}{2}\phi_2} \left[ \frac{1}{1 + \alpha^2} \left( \frac{\phi_1}{\phi_2} + \frac{\alpha}{\phi_2} \right) - \frac{\psi_1}{\phi_2} \right].$$  \hspace{1cm} (A24)

Noting that when the two transitions are sufficiently separated from each other, as we are assuming here, $\phi_1$ and $\psi_1$ are
practically constant close to transition 2, it is clear that both the ratios $\phi_1/\phi_2$ (as observed in panel (a) of Figure 20) and $\psi_1/\psi_2$ are practically zero in the frequency interval over which $\phi_2$ has a Gaussian behavior. More complex is the behavior of the Voigt function $\phi_2$, this ratio is exactly zero for $u = 0$ but, as shown in panel (b) of Figure 21, it immediately increases (in absolute value) moving away from the line center, assuming values of transition 2. This ratio shows an abrupt increase at the frequencies where the behavior of the Voigt function $\phi_2$, when the multiplicative factor $K = \alpha/(1 + \alpha^2)$ (which is extremely small when the separation between the two components is much larger than the Einstein coefficient) is taken into account. Noting that the multiplicative factor $1/(1 + \alpha^2)$ makes the contribution of the first term in the square bracket of Equation (A24) negligible within the interval over which $\phi_2$ has a Gaussian behavior, it follows that $J$-state interference brings a negligible contribution on this spectral interval around transition 2.

With analogous considerations, it can be shown that $J$-state interference brings a negligible contribution around transition 1, within the interval over which the Voigt profile $\phi_1$ has a Gaussian behavior.

REFERENCES

Belluzzi, L., 2011, in ASP Conf. Ser. 437, Solar Polarization 6, ed. J. R. Kuhn, S. V. Berdyugina, D. M. Harrington, S. Keil, H. Lin, T. Rimmele, & J. Trujillo Bueno (San Francisco, CA: ASP), 29

Belluzzi, L., Landi Degl’Innocenti, E., & Trujillo Bueno, J. 2009, ApJ, 705, 218

Belluzzi, L., Trujillo Bueno, J., & Landi Degl’Innocenti, E. 2007, ApJ, 666, 588

Bommier, V. 1980, A&A, 87, 109

Casini, R., & Landi Degl’Innocenti, E. 2007, in Plasma Polarization Spectroscopy, ed. T. Fujimoto & A. Iwamae (Berlin: Springer), 249

Casini, R., & Manso Sainz, R. 2005, ApJ, 624, 1025

Fano, U. 1957, Rev. Mod. Phys., 29, 74

Fontenla, J. M., Avrett, E. H., & Loeser, R. 1993, ApJ, 406, 319

Gandorfer, A. 2000, The Second Solar Spectrum: A High Resolution Polarimetric Survey of Scattering Polarization at the Solar Limb in Graphical Representation, Vol. I: 4625 Å to 6995 Å (Zürich: vdf ETH)

Gandorfer, A. 2002, The Second Solar Spectrum: A High Resolution Polarimetric Survey of Scattering Polarization at the Solar Limb in Graphical Representation, Vol. II: 3910 Å to 4630 Å (Zürich: vdf ETH)

Gandorfer, A. 2005, The Second Solar Spectrum: A High Resolution Polarimetric Survey of Scattering Polarization at the Solar Limb in Graphical Representation, Vol. III: 3160 Å to 3920 Å (Zürich: vdf ETH)

Henze, W., & Stenflo, J. O. 1987, Sol. Phys., 111, 243

Keller, C. U., & Sheeley, N. R., Jr. 1999, in ASSL 243, Solar Polarization, ed. K. N. Nagendra & J. O. Stenflo (Boston, MA: Kluwer), 17

Landi Degl’Innocenti, E., & Landolfi, M. 2004, Polarization in Spectral Lines (Dordrecht: Kluwer) (LL04)

Landi Degl’Innocenti, E., Landi Degl’Innocenti, M., & Landolfi, M. 1997, in Science with THEMIS, ed. N. Mein & S. Sahal-Bréchot (Paris: Obs. Paris-Meudon), 59

Manso Sainz, R. 2011, in ASP Conf. Ser. 437, Solar Polarization 6, ed. J. R. Kuhn, S. V. Berdyugina, D. M. Harrington, S. Keil, H. Lin, T. Rimmele, & J. Trujillo Bueno (San Francisco, CA: ASP), 19

Sheeley, N. R., Jr., & Keller, C. U. 2003, ApJ, 594, 1085

Smitha, H. N., Sampoorna, M., Nagendra, K. N., & Stenflo, J. O. 2011, ApJ, 733, 4

Stenflo, J. O. 1980, A&A, 84, 68

Stenflo, J. O. 2009, in IAU Symp. 259, Cosmic Magnetic Fields: From Planets, to Stars and Galaxies, ed. K. G. Strassmeier, A. G. Kosovichev, & J. E. Beckman (Cambridge: Cambridge Univ. Press), 211

Stenflo, J. O., Baur, T. G., & Elmore, D. F. 1980, A&A, 84, 60

Stenflo, J. O., Gandorfer, A., & Keller, C. U. 2000, A&A, 355, 781

Stenflo, J. O., & Keller, C. U. 1997, A&A, 321, 924

Stépán, J., & Trujillo Bueno, J. 2010, ApJ, 711, L133

Stépán, J., & Trujillo Bueno, J. 2011, ApJ, 732, 80

Trujillo Bueno, J. 1999, in ASSL 243, Solar Polarization, ed. K. N. Nagendra & J. O. Stenflo (Boston, MA: Kluwer), 73

Trujillo Bueno, J. 2001, in ASP Conf. Ser. 236, Advanced Solar Polarimetry: Theory, Observations and Instrumentation, ed. M. Sigwarth (San Francisco, CA: ASP), 161

Trujillo Bueno, J. 2003, in ASP Conf. Ser. 307, Solar Polarization 3, ed. J. Trujillo Bueno & J. Sánchez Almeida (San Francisco, CA: ASP), 407

Trujillo Bueno, J. 2009, in ASP Conf. Ser. 405, Solar Polarization 5, ed. S. Berdyugina, K. N. Nagendra, & R. Ramelli (San Francisco, CA: ASP), 65

Trujillo Bueno, J., Casini, R., Landolfi, M., & Landi Degl’Innocenti, E. 2002, ApJ, 566, L53