Once again about the reaction $\phi(1020) \to \gamma\pi\pi$

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Abstract

Gauge invariance constraints and analytical properties of the amplitudes $e^+e^- \to \gamma\pi\pi$ and $\phi(1020) \to \gamma\pi\pi$ are discussed in line with the criticism of N.N. Achasov in hep-ph/0606003. I show that the formulae of our paper (Yad. Fiz. 68 1614 (2005) [PAN 68 1554 (2005)], hep-ph/0403123) have nothing common with those N.N. Achasov claims to be ours - his criticism is therefore misplaced.

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In the paper [1], we investigated the reactions $e^+e^- \to \gamma\pi\pi$, $\phi(1020) \to \gamma\pi\pi$ and came to the conclusion that available experimental data do not contradict the assumed $q\bar{q}$ structure of $f_0(980)$. This paper was recently criticized by N.N. Achasov in [2]. He wrote that ”the realization of gauge invariance as a consequent of cancellation between the $\phi(1020) \to \gamma f_0(980) \to \gamma\pi^0\pi^0$ resonance contribution and background one, suggested in Ref. [1], is misleading.”

In this short note, I would like to remind the logic of reasoning in Ref. [1] and, comparing it with that of N.N. Achasov [2], to argue in which points the logic of N.N. Achasov is incorrect.

1 The logic of paper [1]

Let us start with the general formula for the transition amplitude $e^+e^- \to \gamma\pi\pi$ assuming that the $e^+e^-$ system is in the $1^{--}(V)$ state, $\pi\pi$ system in the $I = 0, 0^{++}(S)$ state and the photon is real (eq. (20) in [1]):

$$A_{\mu\alpha}^{(V\to\gamma S)}(s_V, s_S, q^2 = 0) = \left( g_{\mu\alpha} - \frac{2q_{\mu}P_{V\alpha}}{s_V - s_S} \right) A_{V\to\gamma S}(s_V, s_S, 0).$$  \hspace{1cm} (1)

The indices $\mu$ and $\alpha$ refer to the initial vector state (total momentum $P_V$ and $P_V^2 = s_V$) and photon (momentum $q$ and $q^2 = 0$). We have $(P_V - q)^2 = s_S$ and $(P_Vq) = (s_V - s_S)/2$. The spin operator $[g_{\mu\alpha} - 2q_{\mu}P_{V\alpha}/(s_V - s_S)]$ is gauge invariant:

$$[g_{\mu\alpha} - 2q_{\mu}P_{V\alpha}/(s_V - s_S)]q_{\alpha} = 0, \quad P_{V\mu}[g_{\mu\alpha} - 2q_{\mu}P_{V\alpha}/(s_V - s_S)] = 0.$$
The requirement of analyticity (absence of the pole at $s_V = s_S$) leads to the condition (eq. (21) in [1]):

$$[A_{V\to S}(s_V, s_S, 0)]_{s_V\to s_S} \sim (s_V - s_S),$$

that is the threshold theorem for the transition amplitude $V \to \gamma S$.

It should be now emphasized that the form of the spin operator in eq. (1), $[g_{\mu\alpha} - 2q_\mu P_{V\alpha} / (s_V - s_S)]$, is not unique. Alternatively, one can write the spin factor as a metric tensor $g_{\mu\alpha}^{\perp\perp}$ which works in the space orthogonal to $P_{V\mu}$ and $q_\alpha$, i.e. $P_{V\mu} g_{\mu\alpha}^{\perp\perp} = 0$ and $g_{\mu\alpha}^{\perp\perp} q_\alpha = 0$. Ambiguities in choice of the spin operator for the process $V \to \gamma S$ are due to the fact that the difference

$$g_{\mu\alpha}^{\perp\perp} - [g_{\mu\alpha} - 2q_\mu P_{V\alpha} / (s_V - s_S)]$$

is the nilpotent operator, for the detail see [3, 4] and eqs. (22)-(25) in [1]. (Note that the use of the operator $g_{\mu\alpha}^{\perp\perp}$ was also criticized by N.N. Achasov [5] but he did not pay attention to the presence of nilpotent operators).

1.1 Transitions $e^+e^- \to \gamma\pi\pi$, $\phi \to \gamma f_0$ and $\phi \to \gamma\pi\pi$.

The amplitude of the transition $e^+e^- \to \gamma\pi\pi$ determines the amplitudes $\phi \to \gamma f_0$ and $\phi \to \gamma\pi\pi$ as corresponding residues of the pole terms.

1.1.1 The amplitude $e^+e^- \to \gamma\pi\pi$: the pole and background terms

The $A_{\phi\to \gamma f_0}$ amplitude is defined as the amplitude residue of the corresponding double-pole term in the amplitude $e^+e^- \to \gamma\pi\pi$. For $e^+e^- \to \gamma\pi\pi$, see Fig. 1, the amplitude with singled
out double-pole term reads (eq. (26) in [1]):

\[
A_{\mu\alpha}^{(e^-e^+\rightarrow\gamma\pi\pi)}(s_V, s_S, 0) = \left(g_{\mu\alpha} - \frac{2q_\mu P_{V\alpha}}{s_V - s_S}\right) \times \\
\times \left[ G_{e^+e^-\rightarrow\phi} \frac{A_{\phi\rightarrow\gamma f_0}(m_{\phi}^2, m_{f_0}^2, 0)}{(s_V - m_{\phi}^2)(s_S - m_{f_0}^2)} \ g_{f_0\rightarrow\pi\pi} + B(s_V, s_S, 0) \right]. \tag{3}
\]

Here, \(A(m_{\phi}^2, m_{f_0}^2, 0)\), up to the factors \(G_{e^+e^-\rightarrow\phi}\) and \(g_{f_0\rightarrow\pi\pi}\), is the residue in the amplitude poles \(s_V = m_{\phi}^2\) and \(s_S = m_{f_0}^2\), just this value supplies us with the transition amplitude for the reaction with bound states, \(\phi \rightarrow \gamma f_0\).

In eq. (3), we deal with the double-pole term only. The amplitude \(B(s_V, s_S, 0)\) may contain single-pole terms too. Writing down all pole terms, one has:

\[
A_{\mu\alpha}^{(e^+e^-\rightarrow\gamma\pi\pi)}(s_V, s_S, 0) = \left(g_{\mu\alpha} - \frac{2q_\mu P_{V\alpha}}{s_V - s_S}\right) \left[ G_{e^+e^-\rightarrow\phi} \frac{A_{\phi\rightarrow\gamma f_0}(m_{\phi}^2, m_{f_0}^2, 0)}{(s_V - m_{\phi}^2)(s_S - m_{f_0}^2)} \ g_{f_0\rightarrow\pi\pi} + B_0(s_V, s_S, 0) \right] \\
+ G_{e^+e^-\rightarrow\phi} \frac{B_0(m_{\phi}^2, s_S, 0)}{s_V - m_{\phi}^2} + \frac{B_{f_0}(s_V, m_{f_0}^2, 0)}{s_S - m_{f_0}^2} \ g_{f_0\rightarrow\pi\pi} + B_0(s_V, s_S, 0) \right]. \tag{4}
\]

The threshold theorem (2) for the amplitude (1) reads:

\[
\left[ G_{e^+e^-\rightarrow\phi} \frac{A_{\phi\rightarrow\gamma f_0}(m_{\phi}^2, m_{f_0}^2, 0)}{(s_V - m_{\phi}^2)(s_V - m_{f_0}^2)} \ g_{f_0\rightarrow\pi\pi} + \\
+ G_{e^+e^-\rightarrow\phi} \frac{B_0(m_{\phi}^2, s_V, 0)}{s_V - m_{\phi}^2} + \frac{B_{f_0}(s_V, m_{f_0}^2, 0)}{s_V - m_{f_0}^2} \ g_{f_0\rightarrow\pi\pi} + B_0(s_V, s_V, 0) \right] = 0. \tag{5}
\]

### 1.1.2 The \(\phi \rightarrow \gamma f_0\) amplitude

If we deal with stable composite particles, in other words, if \(\phi\) and \(f_0\) can be included into the set of fields \(\{in\}\) and \(\{out\}\), the transition amplitude \(\phi \rightarrow \gamma f_0\) can be written in the form similar to (1) (see eq. (27) in [1]):

\[
A_{\mu\alpha}^{(\phi\rightarrow\gamma f_0)}(m_{\phi}^2, m_{f_0}^2, 0) = \left(g_{\mu\alpha} - \frac{2q_\mu P_{\phi\alpha}}{m_{\phi}^2 - m_{f_0}^2}\right) A_{\phi\rightarrow\gamma f_0} \left(m_{\phi}^2, m_{f_0}^2, 0\right). \tag{6}
\]

Here, we have substituted \(P_V \rightarrow P_{\phi}\) where

\[
P_{\phi}^2 = m_{\phi}^2, \quad (P_{\phi} - q)^2 = m_{f_0}^2, \quad q^2 = 0.
\]

For \(A_{\phi\rightarrow\gamma f_0} \left(m_{\phi}^2, m_{f_0}^2, 0\right)\), the threshold theorem reads (eq. (28) in [1]):

\[
\left[ A_{\phi\rightarrow\gamma f_0}(m_{\phi}^2, m_{f_0}^2, 0) \right]_{m_{\phi}^2 \rightarrow m_{f_0}^2} \sim (m_{\phi}^2 - m_{f_0}^2), \tag{7}
\]

that means that the threshold theorem of eq. (7) reveals itself as a requirement of analyticity of the amplitude (1).
1.1.3 The $\phi \to \gamma \pi \pi$ amplitude

To describe the reaction $\phi \to \gamma \pi \pi$ considering $\phi$ as a stable particle, one needs to separate from (4) the pole factors: $\sim (s_V - m_\phi^2)^{-1}$. The amplitude $\phi \to \gamma \pi \pi$ is the residue in the pole $s_V = m_\phi^2$:

$$A_{\mu \alpha}^{(\phi \to \gamma \pi \pi)}(m_\phi^2, s_S, 0) = \left( g_{\mu \alpha} - \frac{2q_\mu P_{\phi \alpha}}{m_\phi^2 - s_S} \right) \left[ \frac{A_{\phi \to \gamma f_0}(m_\phi^2, m_{f_0}^2, 0)}{s_S - m_{f_0}^2} g_{f_0 \to \pi \pi} + B_{\phi}(m_\phi^2, s_S, 0) \right], \quad (8)$$

where

$$P_{\phi}^2 = m_\phi^2, \quad (P_{\phi} - q)^2 = s_S, \quad q^2 = 0.$$

Let us emphasize once more that $B_{\phi}(m_\phi^2, s_S, 0)$ does not contain pole terms and $m_\phi$ is fixed ($m_\phi = 1020$ MeV).

The analyticity condition reads:

$$\left[ \frac{A_{\phi \to \gamma f_0}(m_\phi^2, m_{f_0}^2, 0)}{s_S - m_{f_0}^2} g_{f_0 \to \pi \pi} + B_{\phi}(m_\phi^2, s_S, 0) \right]_{s_S \to m_\phi^2} \sim (s_S - m_\phi^2). \quad (9)$$

1.2 Example of idealistic description of $\phi(1020) \to \gamma \pi \pi$

To make clear our way of calculations, let us consider the idealistic case: the $f_0$ is a standard Breit-Wigner resonance, while $K\bar{K}$ channel is strongly suppressed in the region under consideration and may be neglected. The $\phi$ is considered as a stable particle (the width of $\phi(1020)$ is small).

In this case, the $\phi \to \gamma \pi \pi$ amplitude is given by eq. (9), with

$$m_{f_0}^2 = M_0^2 - i\Gamma M_0. \quad (10)$$

To be illustrative, let us rewrite (9) extracting the Breit-Wigner resonance pole explicitly:

$$A_{\mu \alpha}^{(\phi \to \gamma \pi \pi)}(m_\phi^2, s_S, 0) = \left( g_{\mu \alpha} - \frac{2q_\mu P_{\phi \alpha}}{m_\phi^2 - s_S} \right) \left[ \frac{A_{\phi \to \gamma f_0}(m_\phi^2, m_{f_0}^2, 0)}{s_S - M_0^2 + i\Gamma M_0} g_{f_0 \to \pi \pi} + B_{\phi}(m_\phi^2, s_S, 0) \right], \quad (11)$$

Recall that $B_{\phi}(m_\phi^2, s_S, 0)$ is the background contribution, i.e. it does not contain the pole term related to $f_0(980)$ and $m_\phi$ is fixed, $m_\phi = 1020$ MeV. In the fitting procedure, $B_{\phi}(m_\phi^2, s_S, 0)$ can be approximated by a smooth term or as a contribution of other poles (for example, such as $\sigma$). The fitting to the parameters of $B_{\phi}(m_\phi^2, s_S, 0)$ should be carried out under two constraints:

$$\left( \frac{A_{\phi \to \gamma f_0}(m_\phi^2, m_{f_0}^2, 0)}{s_S - M_0^2 + i\Gamma M_0} g_{f_0 \to \pi \pi} + B_{\phi}(m_\phi^2, s_S, 0) \right) = \left| \frac{A_{\phi \to \gamma f_0}(m_\phi^2, m_{f_0}^2, 0)}{s_S - M_0^2 + i\Gamma M_0} g_{f_0 \to \pi \pi} + B_{\phi}(m_\phi^2, s_S, 0) \exp \left( i\delta_0^0(s_S) \right) \right| \quad (12)$$
and

\[
\frac{A_{\phi \to f_0}(m_{\phi}^2, m_{f_0}^2, 0)}{m_{\phi}^2 - M_0^2 + i\Gamma M_0} \ g_{f_0 \to \pi \pi} + B_{\phi}(m_{\phi}^2, m_{f_0}^2, 0) = 0 .
\]

The factor \(\exp (i\delta_0(s_s))\), where \(\delta_0(s_s)\) is the \(\pi \pi\) scattering phase shift, appears in (12) because of the final state interactions of pions, while the condition (13) results from the threshold theorem (9).

1.3 Description of the reaction \(\phi(1020) \to \gamma \pi \pi\) in paper [1]

The vector meson \(\phi(1020)\) has rather small decay width, \(\Gamma_{\phi(1020)} \simeq 4.5\) MeV; from this point of view there is no doubt that treating \(\phi(1020)\) as a stable particle is reasonable. As to \(f_0(980)\), the picture is not so determinate. In the PDG compilation [6], the \(f_0(980)\) width is given in the interval \(40 \leq \Gamma_{f_0(980)} \leq 100\) MeV, and the width uncertainty is related not to the data unaccuracy (experimental data are rather good) but is due to a vague definition of the width.

![Figure 2: Complex-M plane and location of the poles corresponding to \(f_0(980)\); the cut related to the \(K\bar{K}\) threshold is shown as a broken solid line (here we denote \(M = \sqrt{s_s}\)).](image)

The definition of the \(f_0(980)\) width is aggravated by the \(K\bar{K}\) threshold singularity that leads to the existence of two, not one, poles. According to the \(K\)-matrix analyses [8, 9], there are two poles in the \((IJPC = 00^{++})\)-wave at \(s \sim 1.0\) GeV²,

\[
M_I \simeq 1.020 - i0.040 \text{ GeV} , \quad M_{II} \simeq 0.960 - i0.200 \text{ GeV} ,
\]

which are located on different complex-\(M\) sheets related to the \(K\bar{K}\)-threshold, see Fig. 2.

A significant trait of the \(K\)-matrix analysis is that it also gives us, along with the characteristics of real resonances, the positions of levels before the onset of the decay channels, i.e. it determines the bare states. In addition, the \(K\)-matrix analysis allows us to observe the transform of bare states into real resonances. In Fig. 3, one can see such a transform of the \(00^{++}\)-amplitude poles by switching off the decays \(f_0 \to \pi \pi, K\bar{K}, \eta \eta, \eta' \pi \pi\). One may see
that, after switching off the decay channels, the $f_0(980)$ turns into stable state, approximately 300 MeV lower:

$$f_0(980) \rightarrow f_0^{\text{bare}}(700 \pm 100).$$  \hspace{1cm} (15)
Figure 4: The $f_0$-levels in the potential well depending on the onset of the decay channels: bare states (a) and real resonances (b).

with the following replacement (see Fig. 5):

$$\left[ \frac{A_{\phi \rightarrow \gamma f_0}(m_\phi^2, m_{f_0}^2, 0)}{s_S - m_{f_0}^2} g_{f_0 \rightarrow \pi \pi} + B_{\phi}(m_\phi^2, s_S, 0) \right] \rightarrow \sum_a \left( \sum_n \frac{F_{\phi(1020) \rightarrow \gamma f_0}^{(\text{bare})}(n)}{M_n^2 - s_S} g_{\text{bare}}^a(n) \right) + b_a(s_S) \left( \frac{1}{1 - i\hat{\rho}(s_S)\hat{K}(s_S)} \right)_{a, \pi \pi},$$

where the $K$-matrix elements $K_{ab}(s_S)$ contain the poles corresponding to bare states:

$$K_{ab}(s) = \sum_n \frac{g_{\text{bare}}^a(n) g_{\text{bare}}^b(n)}{M_n^2 - s_S} + f_{ab}(s_S).$$

Here $M_n$ is the mass of bare state, $g_{\text{bare}}^a(n)$ is the coupling for the transition $f_0^{\text{bare}}(n) \rightarrow a$, where

$$a = \pi\pi, \ K\bar{K}, \ \eta\eta, \ \eta\eta', \ \pi\pi\pi\pi.$$  

The matrix element $(1 - i\hat{\rho}(s_S)\hat{K}(s_S))^{-1}$ takes into account of the rescattering of the formed mesons. Here $\hat{\rho}(s_S)$ is the diagonal matrix of phase spaces for hadronic states (for example, for the $\pi\pi$ system it reads: $\rho_{\pi\pi}(s_S) = \sqrt{(s_S - 4m_\pi^2)/s_S}$). The functions $b_a(s_S)$ and $f_{ab}(s)$ describe background contributions, they are smooth ones in the right-hand side half-plane, at $\text{Re} \ s_S > 0$.

The formulae (16) and (17) are presented in (eqs. (46),(47)), with the renotation $s_S \rightarrow s$.

To be scrupulous, let us present the amplitude $\phi(1020) \rightarrow \gamma\pi\pi$ explicitly:

$$A_{\mu\alpha}^{(\phi \rightarrow \gamma\pi\pi)}(m_\phi^2, m_\phi^2, 0) = \left( g_{\mu\alpha} - \frac{2q_\mu P_{\phi\alpha}}{m_\phi^2 - s_S} \right) \times$$
Figure 5: Diagram for the transition $\phi(1020) \rightarrow \gamma\pi\pi$ with final state interaction taken in terms of the K-matrix representation (the right-hand side block $hh \rightarrow \pi\pi$).

\[
\times \left[ \sum_a \left( \sum_n \frac{F_{\phi(1020)\rightarrow\gamma f_0}(n)}{M^2_n - m^2_{\phi}} g^\text{bare}_a(n) + b_a(s_S) \right) \left( \frac{1}{1 - i\tilde{\rho}(s_S)K(s_S)} \right)_{a,\pi\pi} \right]. \quad (19)
\]

Therefore, the threshold condition reads:

\[
\left[ \sum_a \left( \sum_n \frac{F_{\phi(1020)\rightarrow\gamma f_0}(n)}{M^2_n - m^2_{\phi}} g^\text{bare}_a(n) + b_a(s_S) \right) \left( \frac{1}{1 - i\tilde{\rho}(s_S)K(s_S)} \right)_{a,\pi\pi} \right]_{s_S \rightarrow m^2_{\phi}} \sim m^2_{\phi} - s_S. \quad (20)
\]

The fitting procedure of the reaction $\phi \rightarrow \gamma\pi\pi$ should be performed with the threshold constraint only:

\[
\left[ \sum_a \left( \sum_n \frac{F_{\phi(1020)\rightarrow\gamma f_0}(n)}{M^2_n - m^2_{\phi}} g^\text{bare}_a(n) + b_a(m^2_{\phi}) \right) \left( \frac{1}{1 - i\tilde{\rho}(m^2_{\phi})K(m^2_{\phi})} \right)_{a,\pi\pi} \right] = 0 \quad (21)
\]

because here the final state interaction is taken into account explicitly (in contrast to eq. (18) where one needs to account for the constraint (12)).

Formula (19) was used in [1] for the calculation of residues in the poles $s_S = M^2_I$ and $s_S = M^2_{II}$, see [14] and Fig. 2. The residues are $A_{\phi\rightarrow\gamma f_0}(m^2_{\phi}, s_S = M^2_I, 0)$ and $A_{\phi\rightarrow\gamma f_0}(m^2_{\phi}, s_S = M^2_{II}, 0)$ (see Section 5.3 in [1]) which characterize the production of the considered state; this calculation was in line with singling out the pole in [8]. (Discussion of the role of the pole residues can be found, for example, in [9, 10] and references therein). Then, we compare the amplitude $A_{\phi\rightarrow\gamma f_0}(m^2_{\phi}, s_S = M^2_I, 0)$ (for the pole which is the nearest one to the physical region)
with quark model predictions (Section 7) and conclude that experimental data on the reaction \( \phi(1020) \to \gamma f_0(980) \) do not contradict the suggestion about the dominance of the \( q\bar{q} \) component in \( f_0(980) \).

### 1.3.1 Illustrative examples of the K-matrix description of the decay \( \phi \to \gamma \pi\pi \)

Following [8], the K-matrix consideration of the decay \( \phi(1020) \to \gamma \pi\pi \) was performed in [1] with the use of five channels (18) and five resonance states in the \( 0^+0^+ \) wave.

To make the reader more acquainted with this method, consider illustrative examples similar to that given in Section 1.1. We present formulae for the two cases when in the \( 0^+0^+ \pi\pi \) channel we have (i) one resonance and (ii) two of them.

**i) One resonance in the \( 0^+0^+ \pi\pi \) channel.**

The decay amplitude \( \phi \to \gamma \pi\pi \) in the K-matrix representation is written as

\[
A_{\mu\alpha}^{(\phi \to \gamma \pi\pi)}(m_{\phi}^2, s_S, 0) = \left( g_{\mu\alpha} - \frac{2q_{\mu}P_{\phi\alpha}}{m_{\phi}^2 - s_S} \right) \times \\
\times \left( \frac{F_{\phi \to \gamma f_0}^{(bare)} g_{\pi\pi}^{(bare)}}{M_0^2 - s_S} + b(s_S) \right) \left( 1 - i\rho_{\pi\pi}(s_S) \frac{g_{\pi\pi}^{(bare)}^2}{M_0^2 - s_S} + f(s_S) \right)^{-1}.
\]  

The first factor in the right-hand side of (22) describes a direct production of \( \pi\pi \), while the second one is due to rescattering of pions with the K-matrix factor equal to

\[
K_{\pi\pi \to \pi\pi} = \frac{g_{\pi\pi}^{(bare)}^2}{M_0^2 - s_S} + f(s_S).
\]

Note that the form factor \( F_{\phi \to \gamma f_0}^{(bare)} \) and coupling \( g_{\pi\pi}^{(bare)} \) do not depend on \( s_S \), they are constant.

The functions \( b(s_S) \) and \( f(s_S) \) are smooth ones in the region under consideration.

The threshold theorem for (22) reads:

\[
\frac{F_{\phi \to \gamma f_0}^{(bare)} g_{\pi\pi}^{(bare)}}{M_0^2 - m_{\phi}^2} + b(m_{\phi}^2) \left( 1 - i\rho_{\pi\pi}(m_{\phi}^2) \frac{g_{\pi\pi}^{(bare)}^2}{M_0^2 - m_{\phi}^2} + f(m_{\phi}^2) \right)^{-1} = 0.
\]

One may rewrite (22) in the form similar to that of the Breit-Wigner resonance:

\[
A_{\mu\alpha}^{(\phi \to \gamma \pi\pi)}(m_{\phi}^2, s_S, 0) = \left( g_{\mu\alpha} - \frac{2q_{\mu}P_{\phi\alpha}}{m_{\phi}^2 - s_S} \right) \times \\
\times \frac{F_{\phi \to \gamma f_0}^{(bare)} g_{\pi\pi}^{(bare)} + b(s_S)(M_0^2 - s_S)}{M_0^2 - s_S - i\rho_{\pi\pi}(s_S) [g_{\pi\pi}^{(bare)}^2 + f(s_S)(M_0^2 - s_S)]}.
\]

The position of the \( f_0 \)-resonance is determined by zero of the denominator in (24):

\[
M_0^2 - s_S - i\rho_{\pi\pi}(s_S) [g_{\pi\pi}^{(bare)}^2 + f(s_S)(M_0^2 - s_S)] = 0.
\]
Let the resonance pole exist at
\[ s_S = m^2_{\text{Res}} - i\Gamma_{\text{Res}} m_{\text{Res}} = M^2_{\text{Res}}. \]  
(26)

With this definition, one can rewrite (24) in the form of eq. (8):
\[ A^{(\phi \rightarrow \gamma \pi \pi)}(m^2_{\phi}, s_S, 0) = \left( g_{\mu \alpha} - \frac{2q_{\mu} P_{\phi \alpha}}{m^2_{\phi} - s_S} \right) \left[ \frac{F^{(\text{bare})}_{\phi \rightarrow \gamma f_0^{\text{bare}} g^{(\text{bare})}_{\pi \pi}} + B(m^2_{\phi}, s_S, 0)}{M^2_{\text{Res}} - s_S} \right] \]
(27)

where
\[ B_{\phi}(m^2_{\phi}, s_S, 0) = \frac{F^{(\text{bare})}_{\phi \rightarrow \gamma f_0^{\text{bare}} g^{(\text{bare})}_{\pi \pi}} + b(s_S)(M^2_{\text{Res}} - s_S)}{M^2_{\text{Res}} - s_S - i\rho_{\pi \pi}(s_S)[(g^{(\text{bare})}_{\pi \pi})^2 + f(s_S)(M^2_{\text{Res}} - s_S)]} - \frac{F^{(\text{bare})}_{\phi \rightarrow \gamma f_0^{\text{bare}} g^{(\text{bare})}_{\pi \pi}} + b(M^2_{\text{Res}})}{M^2_{\text{Res}} - s_S}. \]
(28)

The function \( B_{\phi}(m^2_{\phi}, s_S, 0) \) determined by (28) does not contain the resonance pole. The threshold theorem (28) reads now as a cancellation of the pole and the smooth background term \( B_{\phi}(m^2_{\phi}, s_S, 0) \) at \( s_S = m^2_{\phi} \).

(ii) Two resonances in the channel 00^{++}\pi\pi.

One may try to describe the \( \pi\pi \) background by a broad resonance. In the case of two resonances the decay amplitude in the K-matrix representation for \( \phi \rightarrow \gamma \pi\pi \) gives us:
\[ A^{(\phi \rightarrow \gamma \pi \pi)}(m^2_{\phi}, s_S, 0) = \]
\[ = \left( g_{\mu \alpha} - \frac{2q_{\mu} P_{\phi \alpha}}{m^2_{\phi} - s_S} \right) \left( \frac{F^{(\text{bare})}_{\phi \rightarrow \gamma f_0^{\text{bare}}(1) g^{(\text{bare})}_{\pi \pi}(1)}}{M^2_{1} - s_S} + \frac{F^{(\text{bare})}_{\phi \rightarrow \gamma f_0^{\text{bare}}(2) g^{(\text{bare})}_{\pi \pi}(2)}}{M^2_{2} - s_S} + b(s_S) \right) \times \]
\[ \times \left( 1 - i\rho_{\pi \pi}(s_S)[(g^{(\text{bare})}_{\pi \pi})^2(1) + (g^{(\text{bare})}_{\pi \pi})^2(2) + f(s_S)] \right)^{-1}, \]
(29)

with the following threshold condition:
\[ \left( \frac{F^{(\text{bare})}_{\phi \rightarrow \gamma f_0^{\text{bare}}(1) g^{(\text{bare})}_{\pi \pi}(1)}}{M^2_{1} - m^2_{\phi}} + \frac{F^{(\text{bare})}_{\phi \rightarrow \gamma f_0^{\text{bare}}(2) g^{(\text{bare})}_{\pi \pi}(2)}}{M^2_{2} - m^2_{\phi}} + b(m^2_{\phi}) \right) \times \]
\[ \times \left( 1 - i\rho_{\pi \pi}(m^2_{\phi}) \left[ (g^{(\text{bare})}_{\pi \pi})^2(1) + (g^{(\text{bare})}_{\pi \pi})^2(2) + f(m^2_{\phi}) \right] \right)^{-1} = 0. \]
(30)

1.4 Approximate description of the \( \pi^0\pi^0 \) spectra in \( \phi \rightarrow \gamma \pi\pi \) with Flatté formula for \( f_0(980) \)

The \( \pi^0\pi^0 \) spectrum in the reaction \( \phi \rightarrow \gamma \pi^0\pi^0 \) is shown in Fig. 6.

The resonance \( f_0(980) \) has two dominant decay channels \( f_0(980) \rightarrow \pi\pi, K\bar{K} \), so the precise description of the \( \pi\pi \) spectrum needs the K-matrix technique. But the K-matrix description
Figure 6: The $\pi\pi$ spectrum of the reaction $\phi(1020) \to \gamma\pi^0\pi^0$ calculated with the Flatté formula (notation for the $\pi\pi$ invariant mass is redenoted here, $\sqrt{s_S} \to M_{\pi\pi}$).

requires more information, in particular, about the reaction $f_0(980) \to \gamma K\bar{K}$, that is not available now. Therefore, a reasonable compromise may be the use of the Breit–Wigner-type formula, where the $K\bar{K}$ threshold singularity is taken into account: it is the Flatté formula [11] or that suggested in [12] (where the transition length is taken into account).

In case of using the Flatté formula, the reaction $\phi(1020) \to \gamma\pi^0\pi^0$ is described by formulae (11) - (13) of Section 1.2, with a change of the Breit-Wigner factor:

$$
\frac{1}{s_S - M_0^2 + i\Gamma M_0} \rightarrow \frac{1}{s_S - M_0^2 + ig_{\pi\pi}\rho_{\pi\pi}(s_S) + ig_{\rho\rho_{\rho\rho}}K\bar{K}(s_S)},
$$

where $\rho_{K\bar{K}}(s_S) = \sqrt{(s_S - 4m_K^2)/s_S}$ is the $K\bar{K}$ phase space.

In [1], the amplitude $A_{\phi \to \gamma f_0}(m_{\phi}^2, m_{f_0}^2, 0)$ (see [11]) was determined supposing the $q\bar{q}$ structure for $f_0(980)$. Fitting to the $\pi\pi$ spectrum, see Fig. 6 [13], was performed under the con-
straints (12) and (13), and the background term in [1] was parametrized as follows:

\[ B_\phi(m_\phi^2, s_S, 0) = C(s_S) \left[ 1 + a(s_S - m_\phi^2) \right] \exp \left[ -\frac{m_\phi^2 - s_S}{\mu^2} \right]. \] (32)

The result is shown in Fig. 6.

Note that the condition (12) is not valid in the region

\[ 2m_K \leq \sqrt{s_S} \leq m_\phi, \]

but at such \( \sqrt{s_S} \) the contribution from \( \sigma(\phi \rightarrow \pi^0\pi^0) \) is negligently small as compared to error bars in Fig. 6.

2 The logic of paper [2]

The aim of paper [2] is to demonstrate that formulae used in our paper [1] are incorrect. To this aim, N.N. Achasov starts from our formula for \( \phi \rightarrow \gamma\pi\pi \):

\[ \sigma(\phi \rightarrow \pi\pi) \sim \left| \frac{1}{s_S - M_\phi^2 + ig_\pi^2\rho_{\pi\pi}(s_S) + ig_K^2\rho_{KK}(s_S)} + B_\phi(m_\phi^2, s_S, 0) \right|^2. \] (33)

Then, he generalizes it for the reaction \( e^+e^- \rightarrow \phi \rightarrow \gamma\pi^0\pi^0 \) in the following way (see eq. (9) of [2]):

\[ \sigma(e^+e^- \rightarrow \phi \rightarrow \gamma\pi^0\pi^0, \sqrt{s_V}) \sim \left| \frac{1}{s_V - M_\phi^2 + i\sqrt{s_V}\Gamma_\phi(\sqrt{s_V})} \right|^2 \left| \frac{g_\pi}{s_S - M_0^2 + ig_\pi^2\rho_{\pi\pi}(s_S) + ig_K^2\rho_{KK}(s_S)} + B_\phi(m_\phi^2, s_S, 0) \right|^2. \] (34)

Hereafter, to avoid possible confusion, we use the notation of our paper, replacing the notations of [2] as follows: \( E^2 \rightarrow s_V \), \( M_{\pi\pi}^2 \rightarrow s_S \) and \( B(M_{\pi\pi}^2) \rightarrow (-)B_\phi(m_\phi^2, s_S, 0) \).

The formula (34) is incorrect. Correct formula for the amplitude \( e^+e^- \rightarrow \gamma\pi^0\pi^0 \) is given by eq. (1) which is characterised by four terms, with corresponding substitution of the pole factors:

\[ (s_V - m_\phi^2)^{-1} \rightarrow \left( s_V - M_\phi^2 + iM_\phi\Gamma_\phi \right)^{-1}, \]

\[ (s_V - m_\phi^2)^{-1} \rightarrow \left( s_S - M_0^2 + ig_\pi^2\rho_{\pi\pi}(s_S) + ig_K^2\rho_{KK}(s_S) \right)^{-1}. \] (35)

The threshold theorem is given by (5), it does not lead to eq. (10) of Achasov’s paper [2].
3 Conclusion

In Section 1, I present the logic of calculation of the reactions $e^+e^- \rightarrow \gamma\pi\pi$, $\phi \rightarrow \gamma f_0$ and $\phi \rightarrow \gamma\pi\pi$ which was accepted in [1] (and in previous papers [14, 15]). Also, the main formulae for the considered reactions are given for the case of a description of resonances by the Breit-Wigner poles as well as in the K-matrix technique.

The formulae presented in Section 1 have nothing common with those N.N. Achasov claims to be ours. His criticism is therefore misplaced.

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