EVENTUAL REGULARITY OF THE SOLUTIONS TO THE SUPERCRITICAL DISSIPATIVE QUASI-GEOSTROPHIC EQUATION

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Abstract. We show that a solution to the supercritical surface quasi-geostrophic equation that is smooth up to a certain time must remain smooth forever.

1 Introduction

The setting of this paper will be the $d$-dimensional torus, $\mathbb{T}^d$. We may equivalently think of the problem in the setting of $\mathbb{R}^d$ with periodic initial data. Throughout the paper we will consider only real-valued functions. We consider the Cauchy problem for the dissipative equation

$$\begin{cases}
\theta_t = (u \cdot \nabla) \theta - (-\Delta)^{\alpha/2} \theta, \\
\theta(x, 0) = \theta_0(x),
\end{cases}$$

where $u = R\theta$, $R$ is a certain divergence-free operator, and $(-\Delta)^{\alpha/2}$ is the fractional Laplacian. In the case of the surface quasi-geostrophic equation (SQG for brevity), $d = 2$ and $u = (-R_2\theta, R_1\theta)$, where the $R_j$’s are the Riesz transforms, which are defined on a suitably smooth class of functions by multiplication on the Fourier side. For $n \in \mathbb{Z}^d$, if

$$\hat{\theta}(n) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \theta(x) e^{-in \cdot x} dx$$

is the $n^{th}$ Fourier coefficient of $\theta$, then for $n \neq 0$

$$\langle R_j \hat{\theta} \rangle(n) = \frac{n_j}{|n|} \hat{\theta}(n) \quad \text{and} \quad \langle (-\Delta)^{\alpha/2} \hat{\theta} \rangle(n) = |n|^\alpha \hat{\theta}(n),$$

and $\langle R_j \hat{\theta} \rangle(0) = \langle (-\Delta)^{\alpha/2} \hat{\theta} \rangle(0) = 0$.

The parameter $\alpha$ ranges between 0 and 2. The case when $\alpha \in (1, 2]$ is referred to as the subcritical case. In the subcritical case, the global well-posedness has been established in the case of smooth initial data (see [CoCW] and the references therein). The critical case, $\alpha = 1$, has been the source of much study in recent years ([CoCW], [CV], [KiN], [KiNV]). The global existence problem for the supercritical

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case, $0 < \alpha < 1$, remains wide open. The best previously known result in this direction belongs to Silvestre [S], who showed that every weak solution of the dissipative SQG with slightly supercritical ($\alpha = 1 - \epsilon$, $\epsilon \leq \epsilon_0$) dissipative term becomes smooth after a certain time. We show that this eventual regularization property holds for arbitrarily small powers of the Laplacian. Our result can be stated as follows:

**Theorem 1** (Eventual regularization for the supercritical SQG). There is a time $T = T(\alpha, \|\theta_0\|_{\infty})$ such that if $\theta \in C^{\infty}(\mathbb{T}^2 \times [0, T])$ is a solution to

\[
\begin{align*}
\theta_t &= (R_{\perp} \theta \cdot \nabla)\theta - (-\Delta)^{\alpha/2} \theta, \\
\theta(x, 0) &= \theta_0(x),
\end{align*}
\]

then $\theta$ extends to a solution in $C^{\infty}(\mathbb{T}^2 \times [0, \infty))$.

We will discuss how to obtain a Silvestre-type result at the end of this paper. Theorem 1 is an immediate corollary of the following more general theorem that can be applied to other dissipative equations of this type:

**Theorem 2.** Suppose that $R$ is a divergence-free vector-valued operator that, for every $k \geq 0$ and every $1 < p < \infty$, satisfies

\[
\|Rf - Rg\|_{k,p} \leq C(k, p)\|f - g\|_{k,p},
\]

and, for every $\epsilon > 0$, satisfies

\[
\|\nabla(Rf)\|_{\infty} \leq C(\epsilon)\|\nabla f\|_{C^\epsilon}
\]

for some constants $C(k, p)$ and $C(\epsilon)$. There is a time $T = T(\alpha, \|\theta_0\|_{\infty})$ such that if $\theta \in C^{\infty}(\mathbb{T}^d \times [0, T])$ is a solution to the Cauchy problem

\[
\begin{align*}
\theta_t &= (R \theta \cdot \nabla)\theta - (-\Delta)^{\alpha/2} \theta, \\
\theta(x, 0) &= \theta_0(x),
\end{align*}
\]

then $\theta$ extends to a solution in $C^{\infty}(\mathbb{T}^d \times [0, \infty))$.

In the statement of Theorem 2 we have used the notation

\[
\|f\|_{k,p} = \|\nabla^k f\|_p.
\]

For mean zero functions, this norm can be shown to be equivalent to the standard Sobolev norm by the Poincaré inequality. Classical results about Riesz transforms imply $R_{\perp}$ satisfies the conditions in Theorem 1 (see [St]). Both theorems tell us that for any value of $\alpha$ in the supercritical range, if we have a solution that is smooth up to a certain time, then it remains smooth forever.

We will prove Theorem 2 by investigating how a certain class of test functions, which in some sense is dual to Hölder continuous functions, evolves under the dynamics of the equation. The use of test functions to prove global existence was introduced by Kiselev and Nazarov in [KiN]. The result of Kiselev and Nazarov was given as an alternative to the approach of Caffarelli and Vasseur in [CV], who proved the same result by using De Giorgi iteration. In the supercritical regime, Constantin and Wu [CoW] showed that a uniform bound on the $C^{1-\alpha+\delta}$ ($\delta > 0$) norm of a certain weak solution to the SQG on a time interval implies the solution is smooth on that interval. Recently, Kiselev [Ki] gave an alternative proof of the