TWIST INTERVAL FOR TWIST MAPS

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Abstract. The twist interval of a twist map on the annulus \( A = T \times [0,1] \) has nonempty interior if \( f \) preserves the area, but could be degenerate for general twist maps. In this note, we show that if a twist map \( f \) is non-wandering, then the twist interval of \( f \) is non-degenerate. Moreover, if there are two disjoint invariant curves of \( f \), then their rotation numbers must be different (no matter if they are rational or irrational).

1. Introduction

Let \( f \) be an orientation-preserving homeomorphism on the closed annulus \( A = T \times [0,1] \), \((x,y) \mapsto (x_1,y_1)\). Suppose \( f \) preserves the two boundaries of \( A \): \( y_1 = 0 \) if \( y = 0 \), and \( y_1 = 1 \) if \( y = 1 \). The restriction of \( f \) to each boundary \( T \times \{i\} \), denoted by \( f_i \), is a circle homeomorphism, \( i = 0,1 \). Let \( \rho(f_i) \) be the rotation number of \( f_i \). More generally, one can define the rotation set \( I_f \) of \( f \) on the whole annulus \( A \), see Section 2 for more details. The rotation set \( I_f \) could be complicated for general annulus maps.

The map \( f \) on \( A \) is said to satisfy a (positive) twist condition if for each \( x \in T \), the map \( y \mapsto x_1(x,y) \) is strictly increasing. We will call such \( f \) a twist map. For example, the map \( f(x,y) = (x + y, y) \) satisfies the twist condition.

It is easy to see that for a twist map \( f \) on \( A \), the rotation set of \( f \) satisfies \( I_f \subset [\rho(f_0), \rho(f_1)] \). In the following, \([\rho(f_0), \rho(f_1)]\) will be called the twist interval of the twist map \( f \). Note that it is possible that \( \rho(f_0) = \rho(f_1) \) for a twist map, and hence the twist interval can degenerate to a single point, a phenomena caused by the mode-locking effect, see [3] for examples.

We need an extra condition to guarantee the non-degeneracy of twist intervals. Recall that a map \( f \) is called non-wandering if the non-wandering set of \( f \) equals the whole space.

Theorem 1.1. Let \( f \) be a non-wandering twist map on \( A \). Then \( \rho(f_0) < \rho(f_1) \).

As an application of the above theorem, we have the following result.

Corollary 1.2. Let \( f \) be a non-wandering twist map on \( A \). Then the rotation numbers of any two disjoint invariant curves of \( f \) are different.

A condition slightly ‘weaker’ than the twist condition is the so-called boundary twist condition. Recall that an orientation-preserving homeomorphism \( f \) on \( A \) is said to satisfy the boundary twist condition if \( \rho(f_0) < \rho(f_1) \). As Example 4.3 shows, some twist map does not necessarily satisfy the ‘weaker’ boundary twist condition. It follows from Theorem 1.1 that

Corollary 1.3. A non-wandering twist map on \( A \) always satisfies the boundary twist condition.

2. Preliminary

In this section we introduce some notations and results that will be used later.

2000 Mathematics Subject Classification. 37E40 37E45.

Key words and phrases. Twist map, twist interval, rotation numbers, rotation set, non-wandering, invariant curves.
2.1. **Non-wandering set.** Let \( f \) be a homeomorphism on a compact topological space \( X \). A point \( x \in X \) is called *wandering* if there is an open neighborhood \( U \) of \( x \) such that \( f^nU \cap U = \emptyset \) for each \( n \geq 1 \). Let \( \Omega(f) \) be the set of points that are not wandering, which is called the non-wandering set of \( f \). Then \( f \) is said to be *non-wandering* if \( \Omega(f) = X \). A point \( x \in X \) is said to be *recurrent* if \( f^n(x) \to x \) for some \( n \to \infty \). Note that if \( f : X \to X \) is non-wandering, then the set of recurrent points are dense in \( X \) and \( f^n : X \to X \) is non-wandering for any \( n \geq 1 \).

2.2. **Lifts to the universal cover.** Let \( A = \mathbb{T} \times [0, 1] \) be the closed annulus, \( \tilde{A} = \mathbb{R} \times [0, 1] \) be the universal cover of \( A \), \( \pi_1 \) be the projection from \( \tilde{A} \) to \( \mathbb{R} \). We will use the same notation for the projection from \( \tilde{A} \) to \( \mathbb{R} \). Let \( \tilde{f} : \tilde{A} \to \tilde{A} \) be an orientation-preserving homeomorphism on \( \tilde{A} \). Then one can lift the map \( f \) from \( A \) to its universal cover \( \tilde{A} \). The lift is unique up to an integer shift \( T_k : (x, y) = (x + k, y) \), where \( k \in \mathbb{Z} \). Let \( \tilde{f} \) be such a lift of \( f \) to \( \tilde{A} \). Let \( f_i \) be the projection of the restriction \( f \) on \( \mathbb{T} \times \{i\} \) to \( \mathbb{T} \). That is, \( f_i(x) = \pi_1(f(x, i)) \), \( i = 0, 1 \). In the same way we define the projection \( \tilde{f}_i \) of the restriction \( \tilde{f} \) on \( \mathbb{R} \times \{i\} \) to \( \mathbb{T} \), \( i = 0, 1 \).

2.3. **Rotation numbers of circle homeomorphisms.** Let \( g \) be an orientation-preserving homeomorphism on \( \mathbb{T} \), \( \tilde{g} \) be a lift of \( g \) from \( \mathbb{T} \) to \( \mathbb{R} \). Poincaré proved that the limit \( \lim_{n \to \infty} \frac{\tilde{g}^n(\tilde{x}) - \tilde{x}}{n} \) exists and is independent of the choices of \( \tilde{x} \in \mathbb{R} \). Denote the limit by \( \rho(\tilde{g}) \), which will be called the rotation number of \( g \). A different choice of the lift \( \tilde{g} \) of \( g \) results in an integer shift of the rotation number.

It follows from the definition of rotation numbers that if the lifts of two circle homeomorphism \( g_1 \) and \( g_2 \) satisfy \( \tilde{g}_1(\tilde{x}) \leq \tilde{g}_2(\tilde{x}) \) for each \( \tilde{x} \in \mathbb{R} \), then \( \rho(\tilde{g}_1) \leq \rho(\tilde{g}_2) \). However, a stronger condition \( \tilde{g}_1(\tilde{x}) < \tilde{g}_2(\tilde{x}) \) for each \( \tilde{x} \in \mathbb{R} \) does not necessarily lead to the stronger result that \( \rho(\tilde{g}_1) < \rho(\tilde{g}_2) \).

**Proposition 2.1.** Assume \( \tilde{g}_1(\tilde{x}) < \tilde{g}_2(\tilde{x}) \) for each \( \tilde{x} \in \mathbb{R} \). If \( \rho(\tilde{g}_1) \) is irrational, then \( \rho(\tilde{g}_1) < \rho(\tilde{g}_2) \).

See [3, Chapter 1] or [5, Proposition 11.1.9] for proofs of this result.

2.4. **Rotation sets of annulus maps.** Next we define the rotation set of a map \( f \) on the annulus \( A \). For a general point \( (x, y) \in A \), we lift it to some point \( (\tilde{x}, \tilde{y}) \in \tilde{A} \), and denote \( (\tilde{x}_n, \tilde{y}_n) = \tilde{f}^n(\tilde{x}, \tilde{y}) \). Then we define the lower and upper rotation numbers of \( (x, y) \) under \( f \) as \( \rho_*(x, y, f) := \limsup_{n \to \infty} \frac{\tilde{x}_n - \tilde{x}}{n} \), \( \rho_*(x, y, f) := \liminf_{n \to \infty} \frac{\tilde{x}_n - \tilde{x}}{n} \). The two limits coincide for \( \mu \)-a.e. \( x \in A \) for every \( f \)-invariant probability measure \( \mu \). Denote the common value by \( \rho(x, y, f) \). More generally, the rotation set of \( f \) on \( A \) is defined by

\[
I_f = \left\{ \rho \in \mathbb{R} : \frac{\pi_1(f^{n_i}(\tilde{x}_i, y_i)) - \tilde{x}_i}{n_i} \to \rho \text{ for some } (\tilde{x}_i, y_i) \in \tilde{A}, n_i \to \infty \right\}.
\]

(2.1)

Note that \( I_f \) is always closed. See [7] for a detailed discussion of rotation sets.

2.5. **Birkhoff’s theorem on invariant curves.** Let \( f \) be a twist map on \( A \). An *invariant curve* of \( f \) is an invariant circle in \( A \) that goes around the annulus (hence not null-homotopic in \( A \)).

**Proposition 2.2.** Let \( f \) be a twist map on \( A \). Then there exists a constant \( L(f) > 0 \) such that any invariant curve of \( f \) is the graph of some Lipschitz continuous function whose Lipschitz constant is bounded by \( L(f) \).

For a proof of Birkhoff’s theorem, see [1], or [5, Lemma 13.1.1].
3. Twist interval of twist maps

Let $A = \mathbb{T} \times [0, 1]$ be the annulus, $f : A \to A$ be an orientation-preserving homeomorphism that satisfies the twist condition. In the following we will simply say that $f$ is a twist map. Let $f_i$ be the projection of the restriction of $f$ on the boundary $\mathbb{T} \times \{i\}$, and $\rho_i := \rho(f_i)$ be the rotation number of $f_i$, $i = 0, 1$, via some lift $\tilde{f}$. A different choice of the lift $\tilde{f}$ results in a shift of $\rho_0$ and $\rho_1$ by the same integer. We make the following convention:

Convention. We always pick the lift $\tilde{f}$ of $f$ that satisfies $\rho(\tilde{f}_0) \in [0, 1)$.

3.1. The rotation set of twist maps. In the following we will call $[\rho_0, \rho_1]$ the twist interval of $f$. Let $I_f$ be the rotation set of $f$ on $A$. See Section 2 for the definition of these quantities.

Lemma 3.1. Let $f : A \to A$ be a twist map. Then the rotation set of $f$ satisfies $I_f \subset [\rho_0, \rho_1]$.

Proof. Let $\rho \in I_f$. Then according to (2.1), $\rho = \lim_{i \to \infty} \frac{\pi_1(\tilde{f}_i^n(x_i, y_i)) - x_i}{n_i} \in I_f$ for some $(x_i, y_i) \in \tilde{A}$ and $n_i \to \infty$. Let us fix the index $i$ for now.

Let $(\tilde{x}_{i,n}, y_{i,n}) = \tilde{f}_n(\tilde{x}_i, y_i)$ be the $n$-th iterate of $(\tilde{x}_i, y_i)$, and $(\tilde{x}_{i,n}', 0) = \tilde{f}_n(\tilde{x}_i, 0)$ be a comparison orbit. We claim that $\tilde{x}_{i,n} \geq \tilde{x}_{i,n}'$ for each $n \geq 1$.

Proof of the claim. By the twist condition, we have $\tilde{x}_{i,1} \geq \tilde{x}_{i,1}'$. Assume $\tilde{x}_{i,k} \geq \tilde{x}_{i,k}'$ for each $1 \leq k \leq n$. Then for $k = n + 1$, we have

$$\tilde{x}_{i,n+1} = \pi_1(\tilde{f}(\tilde{x}_{i,n}, y_{i,n})) \geq \pi_1(\tilde{f}(\tilde{x}_{i,n}, 0)) = \tilde{f}_0(\tilde{x}_{i,n}) \geq \tilde{f}_0(\tilde{x}_{i,n}') = \tilde{x}_{i,n+1}',$$

since $\tilde{f}_0$ preserves the order of the points. Therefore, $\tilde{x}_{i,n} \geq \tilde{x}_{i,n}'$ for each $n \geq 1$.

It follows from the above claim that $\frac{\pi_1(\tilde{f}_i^n(\tilde{x}_i, y_i)) - \tilde{x}_i}{n_i} \geq \frac{\pi_1(\tilde{f}_i^n(\tilde{x}_i, 0)) - \tilde{x}_i}{n_i}$ for any $n \geq 1$. Setting $n = n_i$ and then letting $i \to \infty$, we see that $\rho \geq \rho_0$. In the same way we have $\rho \leq \rho_1$. This holds for any $\rho \in I_f$. Therefore, $I_f \subset [\rho_0, \rho_1]$. 

Without some extra assumption of $f$, it is possible that the rotation set $I_f \subsetneq [\rho_0, \rho_1]$. See 4 for examples of twist maps with $I_f = \{\rho_0, \rho_1\}$.

By our twist condition, we know that $\tilde{f}_0(\tilde{x}) < \tilde{f}_1(\tilde{x})$ for any $\tilde{x} \in \mathbb{R}$. It follows from the definition of rotation numbers that $\rho_0 \leq \rho_1$. This inequality may not necessarily be a strict one. See Example 4.3, where a twist map has a degenerate twist interval.

3.2. Non-wandering twist maps. In this subsection we consider the case when $f$ is non-wandering. Recall that if $f$ is non-wandering, so is $f^n$ for each $n \geq 1$.

Theorem 3.2. Let $f : A \to A$ be a twist map. If $f$ is non-wandering, then $\rho_0 < \rho_1$.

Proof. We will assume $\rho_0 = \rho_1$ and derive a contradiction from it.

Case 1. $\rho_0$ is irrational. The twist condition implies $f_0(x) < f_1(x)$ for any $x \in \mathbb{T}$. Then Proposition 2.1 states $\rho_0 < \rho_1$, a contradiction.

Case 2. $\rho_0 = p/q$ is a rational number. We start with the special case $\rho_0 = 0$ and then extend our proof to the general case.

Case 2a. $\rho_0 = 0$. It means $f_i$ admits some fixed point for each $i = 0, 1$. By our choice of the lift $\tilde{f}$, we see that $\tilde{f}_i$ also admits some fixed point, $i = 0, 1$. Let $\tilde{x}_0 \in \mathbb{R}$ be a fixed point of $\tilde{f}_0$ and $\tilde{x}_1 \in (\tilde{x}_0, \tilde{x}_0 + 1]$ be the corresponding fixed point of $\tilde{f}_1$. Then $(\tilde{x}_i, i) \in \tilde{A}$, $i = 0, 1$ are two fixed
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Applying the twist condition again, we see that $\tilde{f}^n L_0$ for any $n \geq 1$. The orientation-preserving assumption of $f$ implies that $f^{n+1} L_0$ lies on the right hand side of $f^n L_0$, and the domain bounded by them is exactly $f^n (D_0)$. So $f^n (D_0)$, $n \geq 0$, are mutually disjoint, and lie on the right side of $L_0$. Therefore, $D_0$ is a wandering domain with respect to $\tilde{f}$ on $A$.

Let $D_0$ be the projection of $\tilde{D}_0$ on $A$. Since $f$ is non-wandering, $f^n (D_0) \cap D_0 \neq \emptyset$ for some $n \geq 1$. Lifting to $\tilde{A}$, we see that $\tilde{f}^n L_0$ has to reach to the right side to the shifted segment $L_0 + (1,0) = (\tilde{x}_0 + 1) \times [0,1]$. Then $\tilde{f}^n L_0$ crosses $L_0 + (1,0)$ at least twice since the two endpoints of $\tilde{f}^n L_0$ are kept on the left side of the the vertical segment $\{ \tilde{x}_1 \} \times [0,1]$, and $\tilde{x}_1 \leq \tilde{x}_0 + 1$. Projecting this structure from $\tilde{A}$ to $A$, we see that there exists a topological horseshoe $\Lambda \subset A$ that is invariant under $f^n$. Let $w \in \Lambda$ be a point fixed by $f^n$. Then the lift $\tilde{w} \in A$ of $w$ satisfies $\tilde{f}^n (\tilde{w}) = \tilde{w} + (1,0)$, and hence $\rho (\tilde{w}, \tilde{f}) = 1/n$. It contradicts the hypothesis that $I_f = \{ 0 \}$.

Case 2b. Now we deal with the general case that $\rho_0 = p/q$. Our argument is similar to Case 2a, with not one, but $q$ moving screens. Let $x_0$ be a periodic point $f_0$, $\tilde{x}_0$ be a lift of $x_0$. Then $\tilde{f}_0^n (\tilde{x}_0) = \tilde{x}_0 + np$ for any integer $n$. Let $\tilde{x}_1 \in (\tilde{x}_0, \tilde{x}_0 + 1)$ be the corresponding periodic point of $\tilde{f}_1$. We see that $\tilde{f}_1^n (\tilde{x}_0) \subset (\tilde{x}_0 + np, \tilde{x}_1 + np) \subset \mathbb{R}$ for any $n \geq 1$. On $\tilde{A}$, the lift $\tilde{f}$ satisfies $\tilde{f}^q (\tilde{x}_0,0) = (\tilde{x}_0 + p,0)$. Let $L_k = \{ \tilde{f}_k^n (\tilde{x}_0) \times [0,1] \}$ for each $k \geq 0$. Then $\tilde{f} (L_k)$ lies on the right side of $L_{k+1}$ by the twist condition. Let $\tilde{D}_k$ be the domain bounded by $\tilde{f} (L_k)$ and $L_{k+1}$, $0 \leq k \leq q-1$. Then the domain bounded by $\tilde{f}^q (L_0)$ and $L_q = (\tilde{x}_0 + p) \times [0,1]$ is $\tilde{U} := \tilde{D}_{q-1} \cup \tilde{f} (\tilde{D}_{q-2}) \cup \cdots \cup \tilde{f}^{q-1} (\tilde{D}_0)$.

Let $\tilde{g}(\tilde{x},y) = \tilde{f}^q (\tilde{x},y) - (p,0)$. Then the domain bounded by $L_0$ and $\tilde{g} (L_0)$ is exactly $\tilde{U} - (p,0)$. Therefore, $\tilde{U} - (p,q)$ is a wandering domain with respect to $\tilde{g}$. The projection $g$ of $\tilde{g}$ satisfies $g(x,y) = f^q (x,y)$, and hence is non-wandering. Using the same argument as in Case 2a, we see that there exists a $g$-periodic point $w \in A$ with $\rho (\tilde{w}, \tilde{g}) > 0$. On the other hand, $I_g = q \cdot I_f - p = \{ 0 \}$, which leads to a contradiction. This completes the proof. 

We consider a special case when $f$ preserves a fully supported measure. Let $\mu$ be a probability measure on $A$ that is fully supported. That is, $\mu (U) > 0$ for any nonempty open set $U \subset A$.

**Corollary 3.3.** If a twist map $f : A \to A$ preserves a fully supported measure, then $\rho_0 < \rho_1$.

**Remark 3.4.** Let $f$ be a non-wandering twist map. Then Theorem 3.2 shows $\rho_0 < \rho_1$. Combining with Franks’ generalized Poincaré–Birkhoff Theorem [4], we see that $I_f \supseteq \mathbb{Q} \cap [\rho_0, \rho_1]$. Therefore $I_f = [\rho_0, \rho_1]$ since $I_f$ is closed. Note that Franks proved much stronger results in [4] than what we need here.

The following is a direct corollary of Theorem 3.2

**Corollary 3.5.** Let $f$ be a non-wandering twist map on $A$. Then any two disjoint invariant curves of $f$ have different rotation numbers.

**Proof.** Let $C_1$ and $C_2$ be two invariant curves of $f$ that are disjoint. Proposition 2.2 states that each $C_i$ is the graph of some continuous (in fact Lipschitz) function $\phi_i : \mathbb{T} \to [0,1]$. Since $C_1$ and $C_2$ are disjoint, we assume $\phi_1 (x) < \phi_2 (x)$ for any $x \in \mathbb{T}$. Then the region $A'$ between $C_1$ and $C_2$ is a smaller annulus and the restriction $f|_{A'}$ is a twist map that is also non-wandering. Then we can apply Theorem 3.2 and conclude that the two rotation numbers of $f$ on $C_1$ and $C_2$ are different. □
Remark 3.6. Note that for any twist map, there is at most one (disjoint or not) invariant curve with rotation number $\rho$ if $\rho \in I_f$ is irrational. See [6] or [5] Theorem 13.2.9. The phase portrait of an elliptic billiards (see [2] Page 12) indicates that there can be more than one (non-disjoint) invariant curves of the same rational rotation number (even when the map preserves a smooth measure). If $f$ admits more than one invariant curves with the same rotation number $\rho$, then $\rho$ is rational, and all these curves intersect along some common Birkhoff periodic orbits of $f$.

Proof. It follows from [5] Theorem 13.2.9 that $\rho$ must be rational, say $p/q$. Let $\{C_{\alpha} : \alpha \in A\}$ be the collection of invariant curves with the rotation number $\rho$. It follows from Proposition 2.2 that for each $\alpha \in A$, $C_{\alpha} = \{(x, \phi_{\alpha}(x)) : x \in \mathbb{T}\}$ for some Lipschitz functions $\phi_{\alpha} : \mathbb{T} \to [0,1]$ with a uniform Lipschitz constant. Let $\psi_1(x) = \inf_{\alpha}\{\phi_{\alpha}(x)\}$ and $\psi_2(x) = \sup_{\alpha}\{\phi_{\alpha}(x)\}$. Then $\psi_1$ and $\psi_2$ are two Lipschitz functions, whose graphs $\gamma_1$ and $\gamma_2$ are invariant curves of $f$ with rotation number $\rho$.

It follows from Corollary 3.5 that the intersection $E := \gamma_1 \cap \gamma_2$ is nonempty, which is also closed and $f$-invariant. Let $X = \pi_1(E) = \{x \in \mathbb{T} : \psi_1(x) = \psi_2(x)\}$, and enumerate the component complement \mathbb{T}\setminus X, say $I_n$, $n \geq 1$.

Then for each $n \geq 1$, the two invariant curves $\gamma_1$ and $\gamma_2$ bound an open disk $D_n$ over $I_n$. Since both $\gamma_1$ and $\gamma_2$ are invariant, these disks are permuted by $f$. The non-wandering property of $f$ implies all of the disks are periodically permuted, and the corresponding points in the intersection $E$ must be periodic. These periodic points are of Birkhoff type $p/q$, since they lie on an invariant curve of $f$ of rotation number $p/q$. \hfill \Box

4. Some examples of twist maps

In this section we give some example to illustrate the different situations of twist maps. We start with a standard one, and then give some variations of it. These classes of examples show that some extra assumption is needed to get non-degenerate twist intervals and rotation sets with nonempty interior.

Example 4.1. Let $f(x,y) = (x + \phi(y), y)$, $(x,y) \in A$, where $\phi : [0,1] \to \mathbb{R}$ is a continuous and increasing function. Then $f$ is a twist map that preserves the Lebesgue measure on $A$, and the rotation set $I_f = [\phi(0), \phi(1)]$.

Our first class of variations of the standard twist map is

Example 4.2. Let $f(x,y) = (x + \phi(y), \psi(y))$, $(x,y) \in A$, where $\phi : [0,1] \to \mathbb{R}$ is increasing, and $\psi$ is a homeomorphism on $[0,1]$ that fixes the two endpoints. Let $\text{Fix}(\psi) = \{y \in [0,1] : \psi(y) = y\}$ be the set of points fixed by $\psi$. Then $f$ is a twist map whose nonwondering set is $\Omega(f) = \mathbb{T} \times \text{Fix}(\psi)$. The rotation set $I_f = \{\phi(y) : y \in \text{Fix}(\psi)\}$ can be any closed subset of $[\phi(0), \phi(1)]$.

To introduce the second variations, we briefly recall the Mode Locking phenomena in circle dynamics. See [3] Section 1.4 for more details. Let $g_0 : \mathbb{T} \to \mathbb{T}$ be a circle homeomorphism with a periodic point $x_0 \in \mathbb{T}$ of period $p/q$. Then $\rho(g_0) = p/q$. Assume the graph of the $q$-th iterate $g_q^0$ crosses the diagonal at $x_0$. Then for any circle map $g$ that is close to $g_0$, the graph of $g^q$ also crosses the diagonal, which implies that the rotation number $\rho(g)$ of $g$ is locked at $p/q$.

Example 4.3. Let $f_0$ be a circle homeomorphism with locked mode $p/q$. Consider the one-parameter family $\{f_t : t \in \mathbb{R}\}$ of maps, where $f_t : x \in \mathbb{T} \mapsto f_0(x) + t$. Then there exists $\epsilon_0 > 0$ such that $\rho(f_t) = p/q$ for any $t \in [0, \epsilon_0]$. Consider the map $f$ on $\mathbb{T} \times [0, \epsilon_0]$, define by $f(x,t) = (f_t(x), t)$. It is easy to see that $f$ satisfies the twist condition, $\rho_0 = \rho_1 = p/q$ and $I_f = \{p/q\}$.
One might wonder what one can say when $\Omega(f)$ has nonempty interior. To construct our next examples, we first recall the phase portrait of the billiard map inside an ellipse. See [2 §1.4] for more details.

![Phase portrait of the billiard dynamics inside an ellipse.](image)

Let $\Gamma$ be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b > 0$. Consider the billiards system inside $\Gamma$. Then the phase space of the billiard map $F$ is given by $M := (\mathbb{R}/|\Gamma|) \times [0, \pi]$, where $x \in \mathbb{R}/|\Gamma|$ is the arc-length parameter of $\Gamma$, and $\theta \in [0, \pi]$ measures the angle from the tangent vector $\Gamma'(x)$ to the velocity vector of the orbit right after the impact on $\Gamma$. Note that $\frac{dx}{dt} = \frac{\tau(x, x_1)}{\sin \theta}$, where $\tau(x, x_1)$ is the Euclidean distance from $\Gamma(x)$ to $\Gamma(x)$, see [2 §2.11]. There exists a constant $c = c(\Gamma) > 0$ such that $\frac{dx_1}{dt} \geq c$. So $F$ satisfies the twist condition, and $I_F = [0, 1]$.

**Example 4.4.** The phase space $M$ is divided into three parts. Let $M_1$ be the upper part, $M_2$ be the lower part, and $M_3$ be the center part. We make a smooth perturbation $G$ of $F$ on the interior of $M_1$ that pushes the invariant curves in $M_1$ upward, and on the interior of $M_2$ that pushes the invariant curves in $M_2$ downward, while keeps $F$ unchanged on $M_3$. Clearly $G$ satisfies the boundary twist condition and $\Omega(G) \supseteq M_3$. Moreover, $G$ still satisfies the twist condition (as long as the perturbation is $C^1$-small) while $I_G = \{0, 1/2, 1\}$.

For our last example, we insert the dynamics of the elliptic billiards on the eye-shape domain $M_3$ into twist map with degenerate twist interval. Let $f_0 : \mathbb{T} \to \mathbb{T}$ be a diffeomorphism with rotation number $\rho_0 = 1/2$, such that $f_0(x) = x + 1/2$ for $1/6 \leq x \leq 1/3$ and $f_0(x) = x - 1/2$ for $2/3 \leq x \leq 5/6$, and all other points are wandering. See Part (a) of Fig. 2 There exists $\epsilon_0 > 0$ such that $f_\epsilon(x) := f_0(x) + \epsilon$ has a unique periodic orbit of period 2 whenever $0 < |\epsilon| \leq \epsilon_0$. Consider the induced twist map $f$ on $A := \mathbb{T} \times [-\epsilon_0, \epsilon_0]$, $f(x, y) = (f_0(x) + y, y)$. Then $\frac{\partial f_\epsilon}{\partial y} = 1 > 0$ and $\rho(f-\epsilon_0) = \rho(f_\epsilon) = 1/2$. See Part (b) of Fig. 2 for the non-wandering set of $f$.

**Example 4.5.** Now we make a (piecewise) $C^1$ small perturbation $g$ of $f$ over the two cylinders, $[1/6, 1/3] \times [-\epsilon_0, \epsilon_0]$ and $I_2 = [2/3, 5/6] \times [-\epsilon_0, \epsilon_0]$. We first cut $A$ along the two flat segments $I_1 = [1/6, 1/3] \times \{0\}$ and $I_2 = [2/3, 5/6] \times \{0\}$, push the upper copy to the right, and the lower copy to the right and then paste the restriction of the dynamics of the elliptic billiards $F$ on $M_3$. This perturbation resembles the change of the billiard map when one deforms the billiard table from a unit disk to an elliptic domain. Note that $g$ is $C^1$-close to $f$ outside the two eyes, and equals to $F$ on the two eyes. So $g$ is a twist map. It is easy to see $\Omega(f)$ has nonempty interior, while $\rho_0 = \rho_1 = 1/2$.

**Acknowledgments**

The author is very grateful to Zhihong Xia for useful discussions.
Figure 2. Construction of a twist map by Cut-and-Paste: (a) graph of the function $f_0$; (b) the non-wandering set of the twist map $f$; (c) the perturbation $g$ of $f$. Blue color indicates the periodic orbit is attracting, while red color indicates repelling.

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