The contact layer stiffness influence assessment on the stress-strain state of a multilayer beam

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Abstract. The article is devoted to the stress-strain state of multilayer beams’ study of the. The beam model is represented by the main and connecting layers. For the main layers, the classical Euler-Bernoulli hypotheses are used. The contact layer model is used to describe the connecting layers. As a result, a system of resolving equations for solving numerous problems is obtained. As an example, the analytical solution to the problem of bending a two-layer beam is given. The resulting solution describes the stress-strain state of the beam in the range from the ideal contact of the main layers to the complete absence of the contact. It is shown that the used model of the contact layer allows satisfying all the boundary conditions, including the fact that the tangent stresses at corner points are equal to zero. The influence of the contact layer’s stiffness on the model’s stress-strain state is analyzed.

1. The problem’s statement
Let us consider a multilayer beam consisting of \( m+1 \) main layers, and \( m \) contact layers connecting the main ones. A general view of this model is given below.

![Figure 1. General view of the multilayer beam’s model with contact layers](image)

The resolving equations’ system for this problem is represented by two groups. The first group includes the equations that describe the stress-strain state of the main layers under the action of external loads and forces arising in adjacent contact layers. The second group of equations describes the stress-strain state of the contact layers.
A solution to a similar problem is given in [1]. The main difference of this work is the use of the contact layer’s refined model, which takes into account the nonlinearity of the displacements’ distribution within the contact layer. This allows us to satisfy all the boundary conditions, including the fact that the tangent stresses at the corner points are equal to zero.

2. Resolution equations for the contact layer

When solving the problem, the hypothesis proposed in [2] and implemented in many works [3-5] by R. A. Turusov, according to which, the layers’ interaction is carried out using a contact layer in which the adhesive substance interacts with the substrate.

To obtain the resolving equations for the contact layer, we consider the plane problem of elasticity theory for an orthotropic body. The models taking into account elastic-plastic behavior can be found in [6].

We write down the generalized Hooke law for a plane problem.

$$\sigma_x = \frac{E_x}{1 - \mu_{xy}} (e_x + \varepsilon_{yx} \mu_{xy}); \quad \sigma_y = \frac{E_y}{1 - \mu_{xy}} (e_y + \varepsilon_{xy} \mu_{xy}). \quad (1.1)$$

We take into account the symmetry of the elastic properties.

$$\frac{\mu_{xy}}{E_y} = \frac{\mu_{yx}}{E_x} \Rightarrow E_y \mu_{yx} = E_x \mu_{xy}. \quad (1.2)$$

We represent the contact layer as an orthotropic medium [10, 11] with such properties, that it can be represented as a set of short elastic rods not connected with each other. Since the rods are not connected, $E_x \rightarrow 0$. From the equality (1.2) it follows $E_y \mu_{yx} = 0 \cdot \mu_{xy}$. As $E_y \neq 0$, then $\mu_{yx} = 0$.

As a result of the system (1.1) we get:

$$\sigma_x = 0; \quad \sigma_y = E_y \varepsilon_y. \quad (1.3)$$

The received expressions (1.3) make it possible to integrate the basic system of equations of the elasticity theory for the contact layer:

![Figure 2. Contact layer and adjacent base layers](image)

When integrating, we take into account the following boundary conditions:

$$\begin{align*}
v(x, -h/2) &= v_i (x); \quad v(x, h/2) = v_b (x); \\
u(x, -h/2) &= u_i (x); \quad u(x, h/2) = u_b (x).
\end{align*} \quad (1.4)$$

in which the indices $t$ and $b$ mean the upper and lower faces of the contact layer, respectively. In the general case, these quantities are equal to the main adjoining layers’ displacements at the contact boundary. As a result, we find the expressions for the movements in the contact layer:

$$\begin{align*}
v &= \frac{1}{E_y} \left( \frac{h^2}{8} \frac{y^2}{2} \frac{d^2r_{xy}}{dx^2} + \frac{h}{4} (v_b - v_i) + \frac{1}{2} (v_b + v_i) \right); \\
u &= \frac{y}{6} \left( \frac{y^2 - h^2}{4} \frac{d}{dx} \left( \frac{1}{E_y} \frac{d^2r_{xy}}{dx^2} \right) + \frac{1}{2} \left( \frac{h}{4} \frac{y^2}{h} \frac{dv_b}{dx} - \frac{dv_i}{dx} \right) + \frac{y}{h} (u_b - u_i) + \frac{1}{2} (u_b + u_i) \right).
\end{align*} \quad (1.5)$$

Using (1.5) it is possible to calculate the stress in the contact layer.
\[
\frac{\tau_{xy}}{G_{xy}} = \frac{h^2}{12} \frac{d}{dx} \left( \frac{1}{E_y} \frac{d^2 \tau_{xy}}{dx^2} \right) + \frac{1}{2} \left( \frac{dv_b}{dx} + \frac{dv_l}{dx} \right) + \frac{1}{h} (u_b - u_l); \quad \sigma_y = -y \frac{d \tau_{xy}}{dx} + \frac{E_y}{h} (v_b - v_l).
\] (1.6)

Thus expressions (1.5) and (1.6) fully determine the stress-strain state of the contact layer. Further, all the quantities related to the contact layer will be marked with \*.

3. Resolution equations for the main layers

Next, we dwell on the resolution equations’ derivation for the main layers of the beam using the classical Euler – Bernoulli hypotheses. We take into account that axial displacements \( u \) are nonzero, i.e. \( u(x, 0) = u_0(x) \). The \( k \) layer index is discarded.

![Figure 3. Boundary conditions for the midline of layer](image)

The main difference from the classical equations of beams lies in the boundary conditions used in their preparation.

\[
\tau_{xy} \left( x, \frac{h}{2} \right) = p_l; \quad \tau_{xy} \left( x, \frac{h}{2} \right) = p_b; \\
\sigma_y \left( x, \frac{h}{2} \right) = q_l; \quad \sigma_y \left( x, \frac{h}{2} \right) = q_b.
\] (1.7)

![Figure 3. Boundary conditions for the main layers](image)

Considering (1.7), we write the final expressions for the stresses and displacements in the main layers.

\[
u = u_0 - y \frac{dv}{dx}; \quad \sigma_x = E \left( \frac{du_0}{dx} - y \frac{d^2 v}{dx^2} \right); \quad \tau_{xy} = E \left( y^2 - \frac{h^2}{4} \right) \frac{d^2 y}{dx^2} - Ey \frac{d^2 u_0}{dx^2} + \frac{1}{2} (p_b + p_l); \\
\sigma_y = E \left( \frac{h^2 y}{2} \right) \frac{d^3 y}{dx^3} + E \left( \frac{y^2}{4} \right) \frac{d^3 y}{dx^3} - \frac{h^2}{2} \frac{d^4 y}{dx^4} + \frac{1}{2} \left( dp_b + dp_l \right) + \frac{1}{2} (q_l + q_b).
\] (1.8)

Below the resolution equations for the carrier layers are written.

\[
Eh \frac{d^2 u_0}{dx^2} + (p_b - p_l) = 0; \quad Eh \frac{d^3 v}{dx^3} \frac{1}{2} \frac{h}{h} \frac{dp_b + dp_l}{dx} - (q_b - q_l) = 0.
\] (1.9)

4. Complete system of resolving equations for a multilayer beam

The complete system of resolving equations for a multilayer beam consists of the first equation (1.6) and the equations (1.9). In the first equation (1.6) the displacements of adjacent main layers are calculated by the following formulas:

\[
v_{l,k} = v_{k-1}; \quad u_{r,k} = u_{k-1} \left( x, \frac{h_k}{2} \right) = u_{0,k-1} \frac{h_k}{2} - \frac{h_{k-1}}{2} \frac{dv_{k-1}}{dx}; \quad v_{b,k} = v_k; \quad u_{b,k} = u_k \left( x, \frac{h_k}{2} \right) = u_{0,k} + \frac{h_k}{2} \frac{dv_k}{dx}.
\] (1.10)

In the equation (1.9) \( p_l \) and \( q_l \) should be replaced with the sum of external forces and stresses acting in adjacent contact layers (1.6).

\[
p_{l,k} = \tau_{l,k} + s_{l,k}; \quad q_{l,k} = \sigma_{l,k} \left( x, \frac{h_k}{2} \right) + r_{l,k} = - \frac{h_k}{2} \frac{d \tau_{l,k}}{dx} + \frac{E_{l,k}}{h_k} (v_{k-1} - v_{k-1}) + \frac{E_{l,k}}{h_k} (v_{k+1} - v_{k+1}); \\
p_{b,k} = \tau_{b,k+1} + s_{b,k}; \quad q_{b,k} = \sigma_{b,k+1} \left( x, \frac{h_{k+1}}{2} \right) + n_{b,k} = \frac{h_{k+1}}{2} \frac{d \tau_{b,k+1}}{dx} + \frac{E_{b,k+1}}{h_{k+1}} (v_{k+1} - v_{k+1}) + \frac{E_{b,k+1}}{h_{k+1}} (v_{k+1} - v_{k+1}).
\] (1.11)

The following quantities are included in the expression (1.11): \( s_{b,l,k} \) — external tangential load; \( n_{b,l,k} \) — external normal load applied to the upper (lower) edge of the layer.

As a result, we obtain a complete resolving equations’ system of a multilayer beam with respect to the functions \( u_0, v, \tau \).
\[ E_k h_k \frac{d^2 u_{0,k}}{dx^2} + (\tau^*_{k+1} - \tau^*_k) = -(s_{b,k} - s_{r,k}), \quad u - \text{equation}; \]
\[ E_k h_k^3 \frac{d^3 v_k}{dx^4} - \frac{E_k h_k^3 v_k}{h_k} = -v_k \left( \frac{E_{k+1} h_{k+1}^3}{h_{k+1}^3} + \frac{E_k h_k^3}{h_k} \right) - \frac{E_{k+1} h_{k+1}^3 v_{k+1}}{h_{k+1}^3} \]
\[ - \frac{d \tau^*_k}{dx} \left( \frac{h_k}{2} \frac{d^2 \tau^*_k}{dx^2} + h_k \right) = \eta_{b,k} - \eta_{r,k} + \frac{h_k}{2} \left( \frac{ds_{b,k}}{dx} + \frac{ds_{r,k}}{dx} \right), \quad v - \text{equation}; \]
\[ \frac{1}{12E_k h_k^2} \frac{d^2 v_k}{dx^2} - \frac{\tau^*_k}{2} = -\frac{d^2 \tau^*_k}{dx^2} - \frac{\tau^*_k}{2} \]
\[ \frac{1}{2} \frac{d v_k}{dx} \frac{h_k^2}{2} + \frac{1}{2} \frac{d v_{k-1}}{dx} h_{k-1} + \frac{1}{2} \frac{d v_{k-1}}{dx} h_{k-1} + \frac{u_{0,k} - u_{0,k-1}}{h_k} = 0, \quad \tau - \text{equation}, \]

in which \( G^*_k = G^*_{xy,k}; \quad E^*_k = E^*_y \).

5. Statically equivalent internal forces

The internal forces in the bearing layers of the beam are determined from the stresses by the integration over the layer’s area. We write them down:

\[ N_k = b \int_{-h_k/2}^{h_k/2} \sigma_{x,k} dy; \quad M_k = b \int_{-h_k/2}^{h_k/2} y \sigma_{x,k} dy; \quad Q_k = b \int_{-h_k/2}^{h_k/2} \tau_{xy,k} dy. \]  \hspace{1cm} (1.13)

Considering (1.8), we will find:

\[ N_k = E_k b h_k \frac{d u_{0,k}}{dx}; \quad M_k = -E_k b h_k^3 \frac{d^2 v_k}{dx^2}; \quad Q_k = -E_k b h_k^3 \frac{d^3 v_k}{dx^4} + b h_k \left( \frac{\tau^*_k+1}{2} + \frac{\tau^*_k}{2} \right) \]

The total efforts in the beam are recorded below:

\[ N_{beam} = \sum_{k=0}^{k_{max}} \left( b \int_{-h_k/2}^{h_k/2} \sigma_{x,k} dy \right) = \sum_{k=0}^{k_{max}} N_k; \quad M_{beam} = \sum_{k=0}^{k_{max}} \left( b \int_{-h_k/2}^{h_k/2} y \sigma_{x,k} dy \right) = \sum_{k=0}^{k_{max}} M_k; \]
\[ Q_{beam} = \sum_{k=0}^{k_{max}} \left( b \int_{-h_k/2}^{h_k/2} \tau_{xy,k} dy \right) + \sum_{k=1}^{k_{max}} \left( b \int_{-h_k/2}^{h_k/2} \tau_{xy,k} dy \right) = \sum_{k=0}^{k_{max}} Q_k + \sum_{k=1}^{k_{max}} (b h_k \tau^*_k) \]  \hspace{1cm} (1.15)

6. Model of a two-layer beam. Analytical solution

As an example of calculation, we consider the double layer beam model, given below:

![Double Layer Beam Model](image)

To obtain a system of resolving equations for a two-layer beam, it is necessary to substitute the system into the first two equations (1.12) \( k = 0, 1 \), and in the third equation (for the contact layer) \( k = 1 \).

Moreover, we take into account that there are no external tangential loads, and normal loads \( r \) are applied only to the upper face of layer 0, i.e. \( r_{x,0} = -q \).
When considering the problem of the stress-strain state of a two-layer beam, it follows from the conditions of static equilibrium that $N_1 = -N_0$. Of (1.14) with a symmetrical load application relative to the beam center, we find the following relation: $u_{0,1} = -(E_0 h_0 u_{0,0})/(E_1 h_1)$.

After all the transformations, we obtain a system of resolving equations:

$$
\begin{align*}
D_0 \frac{d^4 u_{0,0}}{dx^4} + \eta G^* (v_0 - v_1) + B_0 \left( \frac{h_0}{2} + \frac{h^*}{2} \right) \frac{d^3 u_{0,0}}{dx^3} = q, \\
D_1 \frac{d^4 v_1}{dx^4} + \eta G^* (v_1 - v_0) + B_0 \left( \frac{h_0}{2} + \frac{h^*}{2} \right) \frac{d^3 u_{0,0}}{dx^3} = 0; \\
\frac{d^4 u_{0,0}}{dx^4} B_0 (h^* + h_0) - \frac{d^3 u_{0,0}}{dx^3} B_0 + B_1 \frac{d v_1}{dx} \frac{h^* + h_0}{2} - \frac{d v_0}{dx} \frac{h^* + h_0}{2} = 0, \\
\end{align*}
$$

wherein $D_0 = E_0 h_0^3/12$; $D_1 = E_1 h_1^3/12$; $B_0 = E_0 h_0$; $B_1 = E_0 h_1$; $E^* = \eta G^*$. The tangential stresses in the contact layer are associated with the displacements by the expression:

$$
\tau^* = -E_0 h_0 \frac{d^2 u_{0,0}}{dx^2}. 
$$

The following is a solution to the differential equations’ system (1.16):

$$
\begin{align*}
u_{0,0} &= C_0 \frac{x^2}{2} + C_{10} x + C_{11} + \frac{2}{B_0 \left( h_0 + 2h^* + h_1 \right)} \frac{q x^3}{6} - \frac{2(D_0 + D_1)}{B_0 \left( h_0 + 2h^* + h_1 \right)} \int F_{i}^{IV} dx, \\
v_1 &= v_0 + F_{i}^{IV} \frac{(D_0 - D_1) h^*}{2\eta G^*} \frac{d^3 u_{0,0}}{dx^3} B_0 h_0 (h_0 - h_1) \frac{q h^*}{G^*} + \frac{2(B_0 + B_1)}{B_1 \left( h_0 + 2h^* + h_1 \right)} \int \left[ u_{0,0} dx - F_{i}^{IV} \frac{(D_0 - D_1) h^*}{2\eta G^*} (h_0 + 2h^* + h_1) \right] + C_{12}, \\
F_{i}^{IV} &= \psi \frac{d}{d x} \sum_{i=1}^{n} \left[ (C_i + C_{i+4}) \cosh(x \sqrt{s_i}) + (C_i - C_{i+4}) \sinh(x \sqrt{s_i}) \right], \\
\end{align*}
$$

The roots $s_i$ are determined from the characteristic equation $\lambda_{12} y_1^4 - \lambda_{10} y_1^3 + \lambda_8 y_1^2 - \lambda_6 y_1 + \lambda_4 = 0$, in which:

$$
\begin{align*}
\lambda_{12} &= B_0 D_0 D_1 (h^*)^2 / (12 \eta G^*), \\
\lambda_{10} &= B_0 D_0 D_1 / G^*; \\
\lambda_8 &= 4 D_0 D_1 h^* \left[ \left( h_0 + h^* \right)^2 / h_0 \right] + B_0 \left[ \left( h_0 + h^* \right)^2 / h_0 \right] - B_1 \left[ \left( h_0 + h^* \right)^2 / h_0 \right] - B_0 \left( D_0 + D_1 \right) \eta h^*, \\
\lambda_6 &= B_0 \left( D_0 + D_1 \right) \eta h^* / h^*, \\
\lambda_4 &= \eta G^* \left( B_0 \left( D_0 + D_1 \right) + (D_1 + 4D_0) + B_0 \left( \frac{h_0 h_1 + h^* (h_0 + h_1 + h^*)}{h_0} \right) \right) \left( h^* \right)^2 \psi_v = q \eta G^* (B_1 + B_0) / h^* B_1. \\
\end{align*}
$$

The numerical examples of the calculation can be found in [7, 8, 11].

7. Calculation example

Let us consider a model of a double layer beam with the following geometric and physical and mechanical parameters:
We will call these parameters basic. The magnitude of the applied load and the beam width, for simplicity, we take equal to one.

The boundary conditions of the model are written below.

$$
\begin{align*}
    v_0 \left( \pm \frac{l}{2} \right) &= 0; \\
    M_0 \left( \pm \frac{l}{2} \right) &= 0; \\
    M_1 \left( \pm \frac{l}{2} \right) &= 0; \\
    Q_1 \left( \pm \frac{l}{2} \right) &= 0; \\
    N_0 \left( \pm \frac{l}{2} \right) &= 0; \\
    \tau^* \left( \pm \frac{l}{2} \right) &= 0.
\end{align*}
$$

(1.19)

**Figure 5.** Boundary conditions of the model

The border conditions (1.19) allow to define all 12 integration constants included in (1.18). Due to their bulkiness, they are not given here.

Figure 6 shows the distribution graph of displacements and internal forces (1.14) in the main layers. It can be seen from them that the boundary conditions (1.19) for the core layers are fully implemented. The main check of the obtained solution is the fulfillment of the static equilibrium conditions. To do this, the function graphs are plotted in Fig. 7. (1.15). As you can see, they fully correspond to the known diagrams of the hinged beam.

**Figure 6.** Relative displacements and internal forces in the main layers.

**Figure 7.** Total effort in the beam.

$$
\begin{align*}
    1 \quad &- v_0(x)/v_0(0); \\
    1' \quad &- v_1(x)/v_1(0); \\
    2 \quad &- M_0(x)/M_0(0); \\
    2' \quad &- M_1(x)/M_1(0); \\
    3 \quad &- Q_0(x)/Q_0(0.5l); \\
    3' \quad &- Q_1(x)/Q_1(0.36l); \\
    4 \quad &- u_0,0(x)/u_{0,0}(0.5l); \\
    4' \quad &- N_0(x)/N_0(0)
\end{align*}
$$

A distinctive feature of this model is the ability to strictly meet the boundary conditions. In particular, the tangential stresses are equal to zero at the corner points at the contact boundaries (Figure 8).
Obviously, the stiffness of the contact layer has a significant effect on the stress-strain state of the model. For example, for the basic model parameters used, with increasing contact stiffness, tensile stresses disappear in the layer 0, and compressive stresses disappear in the layer 1 (Figure 9, curves 1' and 2).

\[ \tau^* \left( \frac{l}{2} \right) = 0. \]

**Figure 8.** Tangent stresses in the contact layer. \( \tau^* \left( \frac{l}{2} \right) = 0. \)

**Figure 9.** The dependence of the contact layer stiffness normal stresses.

\[
\begin{align*}
1 - \sigma_{x,0}(0,-h_0/2) ;
1' - \sigma_{x,0}(0,h_0/2) ;
2 - \sigma_{x,1}(0,-h_1/2) ;
2' - \sigma_{x,1}(0,h_1/2).
\end{align*}
\]

**Figure 10.** Dependence of the contact layer stiffness displacements.

\[
\begin{align*}
1 - \nu(0) ;
2 - \text{perfect layer contact};
3 - \text{lack of layers’ contact.}
\end{align*}
\]

**Summary**

The system of resolving equations obtained in the work based on the contact layer model allows predicting the stress-strain state of multilayer beams.

A calculation example, demonstrating that the equations obtained describe a wide range of problems: from models with perfect contact to models with weak interaction of layers (Figure 10) is given in the work. In this case, all boundary conditions are satisfied and the conditions of the beam’s static equilibrium as a whole are satisfied. This opens up the possibilities for processing the experimental data, determining the true strength of adhesive interaction and the rigidity of the contact layer [10, 11].

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