EXAMPLES OF NONPOLYGONAL LIMIT SHAPES IN I.I.D. FIRST-PASSAGE PERCOLATION AND INFINITE COEXISTENCE IN SPATIAL GROWTH MODELS

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We construct an edge-weight distribution for i.i.d. first-passage percolation on $\mathbb{Z}^2$ whose limit shape is not a polygon and whose extreme points are arbitrarily dense in the boundary. Consequently, the associated Richardson-type growth model can support coexistence of a countably infinite number of distinct species, and the graph of infection has infinitely many ends.

1. Introduction. Throughout this note $\mu$ denotes a Borel probability measure on $[0, \infty)$ with finite mean and such that $\mu([0]) < p_c$, the critical probability for bond percolation in $\mathbb{Z}^2$, and $\mathcal{M}$ denotes the family of such measures. Let $E$ denote the set of nearest-neighbor edges of the lattice $\mathbb{Z}^2$, and let $\{\tau_e : e \in E\}$ be a family of i.i.d. random variables with marginal $\mu$ and joint distribution $P = \mu^E$. The passage time of a path $\gamma = (e_1, \ldots, e_n) \in E^n$ in the graph $(\mathbb{Z}^2, E)$ is $\tau(\gamma) = \sum_{i=1}^n \tau_{e_i}$, and for $x, y \in \mathbb{Z}^2$ the passage time from $x$ to $y$ is

$$\tau(x, y) = \min_{\gamma} \tau(\gamma),$$

where the minimum is over all paths $\gamma$ joining $x$ to $y$. A minimizing path is called a geodesic from $x$ to $y$.

The theory of first passage percolation (FPP) is concerned with the large-scale geometry of the metric space $(\mathbb{Z}^2, \tau)$. The following fundamental result concerns the asymptotic geometry of balls. Write $B(t) = \{x \in \mathbb{Z}^2 : \tau(0, x) \leq t\}$ for the ball of radius $t$ at the origin, and for $S \subseteq \mathbb{R}^2$ and $a \geq 0$, write $aS = \{ax : x \in S\}$.

THEOREM 1.1 (Cox and Durrett [1]). For every $\mu \in \mathcal{M}$ there exists a deterministic, compact,\textsuperscript{3} convex set $B_\mu$, with nonempty interior, such that for every

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\textsuperscript{3}This is the only place where we use the assumptions on $\mu$. If $\mu([0])$ exceeds the critical percolation probability, then $B_\mu = \mathbb{R}^2$ in an appropriate sense, and without finite mean we could have $B_\mu = \{0\}$. 

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\( \varepsilon > 0, \)
\[
\mathbb{P}\left( (1 - \varepsilon)B_\mu \subseteq \frac{1}{t} B(t) \subseteq (1 + \varepsilon)B_\mu \text{ for all large } t \right) = 1.
\]

Little is known about the geometry of \( B_\mu \), which is called the \textit{limit shape}. It is conjectured to be strictly convex when \( \mu \) is nonatomic, and nonpolygonal in all but the most degenerate cases, but, in fact, there are currently no known examples of \( \mu \) for which these properties are verified (see [8]). For a compact, convex set \( C \subseteq \mathbb{R}^2 \) write \( \text{ext}(C) \) for the set of extreme points and \( \text{sides}(C) = |\text{ext}(C)| \), so that \( C \) is a polygon if and only if \( \text{sides}(C) < \infty \). The best result to date, due to Marchand [13], is that under mild assumptions, \( \text{sides}(B_\mu) \geq 8 \). Building on results of Marchand, our purpose of this note is to give the first examples of distributions for which the limit shape is not a polygon. If \( A \) and \( B \) are subsets of \( \mathbb{R}^2 \) (with, say, the \( \ell^1 \)-metric), we say that \( A \) is \( \varepsilon \)-dense in \( B \) if for each \( x \in B \) there exists \( y \in A \) such that \( \|x - y\|_1 < \varepsilon \).

**Theorem 1.2.** For every \( \varepsilon > 0 \) there exists \( \mu \in \mathcal{M} \) (with atoms) such that \( B_\mu \) is not a polygon, that is, \( \text{sides}(B_\mu) = \infty \), and \( \text{ext}(B_\mu) \) is \( \varepsilon \)-dense in \( \partial B_\mu \). There exist nonatomic \( \mu \) such that \( \text{sides}(B_\mu) > 1/\varepsilon \) and \( \text{ext}(B_\mu) \) is \( \varepsilon \)-dense in \( \partial B_\mu \).

It is tempting to try to obtain a strictly convex limit shape by taking a limit of measures \( \mu_n \) such that \( B_{\mu_n} \) have progressively denser sets of extreme points, but unfortunately the limit one gets in our example is the unit ball of \( \ell^1 \).

We also obtain examples of measures \( \mu \) such that, at the points \( v \in \text{ext}(B_\mu) \) which lie on the boundary of the \( \ell^1 \)-unit ball, \( \partial B_\mu \) is infinitely differentiable. This should be compared with the work of Zhang [16], where such behavior was ruled out for certain \( \mu \). Last, as we explain in Section 3, we can produce measures \( \mu \) which are not purely atomic that have \( \text{sides}(B_\mu) = \infty \).

Theorem 1.2 has implications for the Richardson growth model, whose definition we recall next. Fix \( x_1, \ldots, x_k \in \mathbb{Z}^2 \) and imagine that at time 0 the site \( x_i \) is inhabited by a species of type \( i \). Each species spreads at unit speed, taking time \( \tau_e \) to cross an edge \( e \in E \). An uninhabited site is exclusively and permanently colonized by the first species that reaches it, that is, \( y \in \mathbb{Z}^2 \) is occupied at time \( t \) by the \( i \)th species if \( \tau(y, x_i) \leq t \) and \( \tau(y, x_j) < \tau(y, x_i) \) for all \( j \neq i \). When there are unique geodesics, that is, \( \mathbb{P}\text{-a.s.} \) no two paths have the same passage time, as is the case when \( \mu \) is continuous, each site is eventually occupied by a unique species. We shall also want to consider measures \( \mu \) with atoms. The definition of the model in this case is formally the same, but note that there may be sites which are never colonized, that is, those sites \( y \) for which \( \min_{1 \leq i \leq k} \tau(y, x_i) \) is achieved by multiple \( x_i \)'s.

Given initial sites \( x_1, \ldots, x_k \), consider the set colonized by the \( i \)th species,
\[
C_i = \{ y \in \mathbb{Z}^2 : y \text{ is eventually occupied by } i \}.
\]
One says that \( \mu \) admits coexistence of \( k \) species if for some choice of \( x_1, \ldots, x_k \),
\[
P(|C_i| = \infty \text{ for all } i = 1, \ldots, k) > 0.
\]
Coexistence of infinitely many species is defined similarly. Notice that if \( \mu \) has atoms and a site \( x \) is colonized by species \( i \), then the same will be true if we change the model by introducing an arbitrary tie-breaking rule governing the infections of sites which are reached simultaneously by more than one species. Thus, if coexistence of \( k \) species holds in our model, then the same is true under any tie-breaking rule.

In the past ten years, there have been many studies related to Richardson-type models, for instance, in questions related to the asymptotic shape of infected regions \([7, 15]\) and to coexistence \([6, 9–11]\). Pertaining to the latter, it is not known, even in simple examples, how many species can coexist. When \( \mu \) is the exponential distribution, Häggström and Pemantle \([9]\) proved coexistence of 2 species (see \([3]\) for a review of recent results on Richardson models, focused on exponential passage times). Shortly thereafter, Garet and Marchand \([6]\) and Hoffman \([10]\) independently extended these results to prove coexistence of 2 species for a broad class of translation-invariant measures on \((0, \infty)^E\), including some non-i.i.d. ones. Later, Hoffman \([11]\) demonstrated coexistence of 8 species for a similarly broad class of measures by establishing a relation with the number of sides of the limit shape in the associated FPP. Using the same relation we obtain the following theorem.

**Theorem 1.3.** There exists \( \mu \in \mathcal{M} \) (with atoms) which admits coexistence of infinitely many species. For each \( k \) there exist nonatomic \( \mu \in \mathcal{M} \) admitting coexistence of \( k \) species.

When \( \mu \) is nonatomic Theorem 1.3 follows from Theorem 1.2 and from Hoffman \([11]\), Theorem 1.4. For the atomic case we provide the necessary modifications of Hoffman’s arguments in Section 4.

Finally, the graph of infection \( \Gamma(0) \subseteq \mathbb{E} \) is the union over \( x \in \mathbb{Z}^d \) of the edges of geodesics from 0 to \( x \). This terminology is consistent with the Richardson model when there are unique passage times, in which case it is a tree, but note that in general the graph of infection may also contain sites which were not infected, that is, those where a tie condition exists, and in this way we may obtain loops. A graph has \( m \) ends if, after removing a finite set of vertices, the induced graph contains at least \( m \) infinite connected components, and, if there are \( m \) ends for every \( m \in \mathbb{N} \), we say there are infinitely many ends. Letting \( K(\Gamma(0)) \) be the number of ends in \( \Gamma(0) \), Newman \([14]\) has conjectured for a broad class of \( \mu \) that \( K(\Gamma(0)) = \infty \). Hoffman \([11]\) showed for continuous distributions that in general \( K(\Gamma(0)) \geq 4 \) almost surely.

**Theorem 1.4.** There exist \( \mu \in \mathcal{M} \) (with atoms) such that \( \mathbb{P}\text{-a.s., } K(\Gamma(0)) = \infty \). For each \( k \) there exist nonatomic \( \mu \in \mathcal{M} \) such that \( \mathbb{P}\text{-a.s., } K(\Gamma(0)) \geq k \).
When there are unique geodesics, Hoffman’s results imply that \( K(\Gamma(0)) \) is at least \( \text{sides}(B_\mu)/2 \) ([11], Theorem 1.4), which proves Theorem 1.4 for nonatomic \( \mu \). In the case that \( \mu \) has atoms, Theorem 1.4 follows directly from Theorem 1.2 (using the fact that the measure can be made to be not purely atomic) and the following result, which we prove in Section 5.

**Theorem 1.5.** If \( \mu \) is not purely atomic and \( B_\mu \) has at least \( s \in \mathbb{N} \) sides, then \( K(\Gamma(0)) \) is \( \mathbb{P} \)-a.s. at least

\[
(1.1) \quad k = 4 \left\lfloor \frac{s - 4}{12} \right\rfloor.
\]

See below Theorem 5.1 in Section 5 for an explanation of this bound.

2. **Background on the limit shape.** For any \( x \in \mathbb{Z}^2 \) let \( m_\mu(x) = \lim_{n \to \infty} \tau(0, nx)/n \). This limit exists by Theorem 1.1 and by a theorem of Cox and Kesten, the map \( \mu \mapsto m_\mu((1,0)) \) is continuous [2]. By [12], Remark 6.18, this continuity can be extended to other unit vectors \( x \) and is actually uniform over all of them. To describe this, endow \( \mathcal{M} \) with the topology of weak convergence and for convenience fix a compatible metric \( d(\cdot, \cdot) \) on \( \mathcal{M} \). Next, fix the \( \ell^1 \)-metric on \( \mathbb{R}^2 \), and write \( A(\varepsilon) \) for the \( \varepsilon \)-neighborhood of \( A \subseteq \mathbb{R} \). Let \( C \) denote the space of nonempty, closed, convex subsets of \( \mathbb{R}^2 \) endowed with the Hausdorff metric \( d_H \),

\[
d_H(A, B) = \inf \{ \varepsilon : A \subseteq B^{(\varepsilon)} \text{ and } B \subseteq A^{(\varepsilon)} \}.
\]

**Theorem 2.1 (Kesten).** The map \( \mu \mapsto B_\mu \) from \( \mathcal{M} \) to \( C \) is continuous.

We shall use the following elementary semicontinuity property of the map \( A \mapsto \text{ext}(A) \) for \( A \in C \).

**Lemma 2.1.** Let \( A \in C \) and \( x \in \text{ext} A \). For every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that, if \( A' \in C \) and \( d_H(A, A') < \delta \), then there exists \( x' \in \text{ext} A' \) with \( \|x - x'\|_1 < \varepsilon \).

**Proof.** Choose a linear functional \( T : \mathbb{R}^2 \to \mathbb{R} \) and \( \beta > 0 \) such that \( T(x) > \beta \) and the set \( B = \{ y \in A : T(y) \geq \beta \} \) has diameter less than \( \varepsilon/2 \). Note that \( x \in B \). Since \( T \) is continuous, for small enough \( \delta \), if \( d_H(A, A') < \delta \) then the set \( B' = \{ y \in A' : T(y) \geq \beta \} \) is nonempty and satisfies \( d_H(B, B') < \varepsilon/2 \). Since \( T \) is linear, its maximum on the convex set \( A' \) is attained at some extreme point \( x' \in \text{ext} A' \), and by definition \( x' \in B' \). Now if \( y \) is the closest point in \( B \) to \( x' \) then \( \|x' - y\|_1 < \varepsilon/2 \) and we also have \( \|x - y\|_1 < \varepsilon/2 \), so \( \|x - x'\|_1 < \varepsilon \). \( \square \)

Combining this lemma with Theorem 2.1, we have the following corollary.
COROLLARY 2.1. Let $\mu \in \mathcal{M}$. For every $x_1, \ldots, x_k \in \text{ext}(B_\mu)$ and $\varepsilon > 0$ there is a $\delta > 0$ such that, if $v \in \mathcal{M}$ and $d(v, \mu) < \delta$ then there are $y_1, \ldots, y_k \in \text{ext}(B_v)$ such that $\|x_i - y_i\| < \varepsilon$ for $i = 1, \ldots, k$.

Next we recall some results about limit shapes for a special class of measures. Given $0 < p < 1$, let $\mathcal{M}_p \subseteq \mathcal{M}$ denote the set of measures $\mu \in \mathcal{M}$ with an atom of mass $p$ located at $x = 1$, that is, $\mu(\{1\}) = p$, and no mass to the left of 1, that is, $\mu((-\infty, 1)) = 0$. Limit shapes for $\mu$ of this form were first studied by Durrett and Liggett [5]. Writing $\tilde{\rho}_c$ for the critical parameter of oriented percolation on $\mathbb{Z}^2$ (see Durrett [4] for background), it was shown that when $p > \tilde{\rho}_c$ and $\mu \in \mathcal{M}_p$, the limit shape $B_\mu$ contains a “flat edge,” or more precisely, $\partial B_\mu$ has sides which lie on the boundary of the $\ell^1$-unit ball. The nature of this edge was fully characterized by Marchand in [13]. For $p \geq \tilde{\rho}_c$, let $\alpha_p$ be the asymptotic speed of super-critical oriented percolation on $\mathbb{Z}^2$ with parameter $p$ (see [4]). Define points $w_p, w'_p \in \mathbb{R}^2$ by

$$w_p = (1/2 + \alpha_p/\sqrt{2}, 1/2 - \alpha_p/\sqrt{2}),$$

$$w'_p = (1/2 - \alpha_p/\sqrt{2}, 1/2 + \alpha_p/\sqrt{2}).$$

Let $[w_p, w'_p] \subseteq \mathbb{R}^2$ denote the line segment with endpoints $w_p$ and $w'_p$. It will be important to note that $\alpha_p$ is strictly increasing in $p > \tilde{\rho}_c$, so the same is true of $[w_p, w'_p]$.

THEOREM 2.2 (Marchand [13]). Let $\mu \in \mathcal{M}_p$. Then:

1. $B_\mu \subseteq \{x \in \mathbb{R}^2 : \|x\|_1 \leq 1\}$.
2. If $p < \tilde{\rho}_c$, then $B_\mu \subseteq \{x \in \mathbb{R}^2 : \|x\|_1 < 1\}$.
3. If $p > \tilde{\rho}_c$, then $\partial B_\mu \cap \{(x, y) \in \mathbb{R}^2 : x + y = 1\} = [w_p, w'_p]$.
4. If $p = \tilde{\rho}_c$, then $\partial B_\mu \cap \{(x, y) \in \mathbb{R}^2 : x + y = 1\} = \{(1/2, 1/2)\}$.

As noted by Marchand, this implies $\text{side}(B_\mu) \geq 8$ for $\mu \in \mathcal{M}_p$ and $\tilde{\rho}_c < p < 1$, since $w_p, w'_p$ and their reflections about the axes are extreme points.

3. Proof of Theorem 1.2. Our aim is to construct a $\mu \in \mathcal{M}$ with $\text{side}(B_\mu) = \infty$. Fix any $p_0 > \tilde{\rho}_c$, $\mu_0 \in \mathcal{M}_{p_0}$ and a real parameter $\eta_0 > 0$. We will inductively define a sequence $p_1 > p_2 > \cdots > \tilde{\rho}_c$, measures $\mu_1 \in \mathcal{M}_{p_1}, \mu_2 \in \mathcal{M}_{p_2}, \ldots$, and $\eta_1, \eta_2, \ldots > 0$ such that for every $n \geq 0$ and all $k \leq n$,

1. If $v \in \mathcal{M}$ and $d(v, \mu_k) < \eta_k$ then $\text{side}(B_v) \geq k$, and
2. $d(\mu_k, \mu_n) < \frac{1}{2} \eta_k$.

Note that, in particular, $\text{side}(B_{\mu_k}) \geq k$ by (1). Assuming $p_k, \mu_k$ and $\eta_k$ are defined for $k \leq n$, we define them for $n + 1$. Fix $p_{n+1} \in (\tilde{\rho}_c, p_n)$ and set
For $y > 1$ construct $\mu_{n+1}^y$ from $\mu_n$ by moving an amount $r$ of mass from the atom at 1 to $y$, that is,

$$\mu_{n+1}^y = \mu_n - r\delta_1 + r\delta_y.$$  

We claim that for small enough $y > 1$ and any sufficiently small choice of $\eta_{n+1} > 0$ (depending on the previous parameters), the measure $\mu_{n+1} = \mu_{n+1}^y$ has the desired properties. First, $\mu_{n+1}^y \to \mu_n$ weakly as $y \downarrow 1$, so, since $\mu_n$ satisfies (2), so does $\mu_{n+1}$ for all sufficiently small $y$.

Second, we claim that $\text{sides}(B\mu_{n+1}^y) \geq n+1$ for $y$ close enough to 1. Indeed, since $r > 0$ we have $w_{p_{n+1}} \neq w_{p_n}$. Using (1), choose $n$ extreme points $x_1, \ldots, x_n \in \text{ext}(B\mu_n)$ and let

$$a = \min\{||x_i - x_j||_1, ||x_i - w_{p_{n+1}}||_1 : i \neq j\}.$$  

Note that $a > 0$ by Marchand’s theorem. By Corollary 2.1, for $y$ close enough to 1, for each $i = 1, \ldots, n$ we can choose an extreme point $x_i$ of $B\mu_{n+1}^y$ with $||x_i - x_i||_1 < a/2$. By Marchand’s theorem, $B\mu_{n+1}^y$ also has an extreme point at $w_{p_{n+1}}$. By definition of $a$, these extreme points are distinct, giving $\text{sides}(B\mu_{n+1}^y) \geq n+1$.

Finally, by Corollary 2.1, $\mu_{n+1}^y$ satisfies (1) for any sufficiently small choice of $\eta_{n+1}$.

Let $\mu$ be a weak limit of $\mu_n$. Then $d(\mu, \mu_n) \leq \frac{1}{2} \eta_n$ for all $n$, so by (2), $\text{sides}(B\mu) = \infty$. The proof is complete.

One can modify the construction in a number of ways in order to control the resulting measure $\mu$. First, at each step, rather than creating a new atom at $y$, one can instead add, for example, Lebesgue measure on a small interval around $y$. In this way one can make the atom at 1 be the only atom of $\mu$.

Regarding the degree of denseness of the extreme points in the boundary, note that at each stage, if $y$ is small enough and $p_{n+1}$ is close enough to $p_n$, the new extreme point we introduce can be made arbitrarily close to $w_{p_n}$ (here we use that $\alpha_p$ is continuous in $p > \bar{p}_c$, from [4]), and in the limit we can ensure an extreme point close to it. Thus, if we begin from $\mu_0 = \delta_1$ and choose $p_n$ so that $\lim p_n = \bar{p}_c$, and using Marchand’s result that the flat edge in $B\mu_n$ then shrinks to a point (and symmetry of the limit shape about the axes), we can ensure $\varepsilon$-density of the extreme points of $B\mu$.

For the second part of Theorem 1.2, choose a sequence $\nu_n \in \mathcal{M}$ of continuous measures converging weakly to $\mu$. By Corollary 2.1, $\text{sides}(B\nu_n) \to \infty$ and if $\text{ext}(B\mu)$ is $\varepsilon$-dense in $\partial B\mu$, then the same holds for $B\nu_n$ for sufficiently large $n$.

Regarding the remark after Theorem 1.2, one may verify that if at each stage $y$ is chosen close enough to 1 and $p = \lim p_n$, then $w_p$ is a $C^\infty$-point of $\partial B\mu$.

4. Proof of Theorem 1.3. Let us recall Hoffman’s argument relating coexistence to the geometry of the limit shape for continuous $\mu$ ([11], Theorem 1.6).
Extend $\tau$ to $\mathbb{R}^2 \times \mathbb{R}^2$ by $\tau(x, y) = \tau(x', y')$ where $x'$ is the unique lattice point in $x + [-1/2, 1/2]^2$. Similarly, a geodesic between $x$, $y$ is a geodesic between $x'$, $y'$. For $S \subseteq \mathbb{R}^2$, the Busemann function $B_S : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$B_S(x, y) = \inf_{z \in S} \tau(x, z) - \inf_{w \in S} \tau(y, w).$$

For $v \in \mathbb{R}^2$, write $S + v = \{s + v : s \in S\}$. If $v \in \partial B_\mu$ is a point of differentiability and $w$ is a tangent vector at $v$, let $\pi_v$ denote the linear functional $av + bw \mapsto a$. Define the lower density of a set $A \subseteq \mathbb{N}$ by $d(A) = \liminf \frac{1}{k} |A \cap \{1, \ldots, N\}|$. The following is a rephrasing of [11], Lemma 4.6.

**Theorem 4.1** (Hoffman). Let $\mu \in M$ and let $v \in B_\mu$ be a point of differentiability of $\partial B_\mu$ with tangent line $L \subseteq \mathbb{R}^2$. Then for every $\varepsilon > 0$ there exists an $M = M(v, \varepsilon) > 0$ such that, if $x, y \in \mathbb{R}^2$ satisfy $\pi_v(x - y) > M$, then the set of $n$ such that

$$\mathbb{P}(B_{L + n v}(y, x) > (1 - \varepsilon)\pi_v(x - y)) > 1 - \varepsilon$$

has lower density at least $1 - \varepsilon$.

Hoffman’s proof of this result does not use unique passage times.

Theorem 4.1 is related to coexistence as follows. Suppose sides$(B_\mu) \geq k$. We can then find $k$ points of differentiability $v_1, \ldots, v_k \in \partial B_\mu$ with distinct tangent lines $L_i$, and in particular $\pi_{v_j}(v_i - v_j) > 0$ for all $j \neq i$. Fix $\varepsilon > 0$ and choose $R > 0$ large enough so that the points $x_i = R v_i$ satisfy $\pi_{v_j}(x_i - x_j) > M(v_i, \varepsilon/k^2)$. Using the elementary relation $d(\cap_{i=1}^n A_i) \geq 1 - \sum_{i=1}^n (1 - d(A_i))$, for each $i$ we see that the set of $n$ such that

$$\mathbb{P}(B_{L + n v_i}(x_j, x_i) > 0 \text{ for all } j \neq i) > 1 - \frac{\varepsilon}{k}$$

has lower density at least $1 - \varepsilon/k$. Hence, with positive probability (which can be made arbitrarily close to 1 by decreasing $\varepsilon$), for each $i$ there are infinitely many $n$ such that $B_{L_i + n v_i}(x_j, x_i) > 0$ for all $j \neq i$. For such an $n$, take $y_{i,n} \in L_i + n v_i$ to be the closest point (in the sense of passage times) to $x_i$; by definition $y_{i,n}$ is reached first by species $i$. The points $y_{i,n}$ are in $C_i$, so $|C_i| = \infty$ for $i = 1, \ldots, k$, that is, coexistence occurs. Note that this argument does not use unique passage times.

When sides$(B_\mu) = \infty$, one proves coexistence of infinitely many types using the above arguments. Choose a sequence $\{v_i\}_{i=1}^\infty \subseteq \partial B_\mu$ of points of differentiability of the boundary, ordered clockwise, say. Given $\varepsilon > 0$, define the points $x_i$ inductively by $x_{i+1} = x_i + R_i(v_{i+1} - v_i)$ for a sufficiently large $R_i > 0$ so as to ensure that for $i \neq j$, $\pi_{v_i}(x_i - x_j) > M(v_i, \varepsilon_{i,j})$, where $\sum_{i,j} \varepsilon_{i,j} < \varepsilon$. Using the argument above, we see that with probability $> 1 - \varepsilon$, any finite subcollection of the $x_i$’s coexist. We now need to show that this implies that all species coexist with probability $> 1 - \varepsilon$. Fix a configuration in which every finite set of species coexist,
and suppose that they do not all coexist. Let \( C_i \) denote the set of sites colonized by the \( i \)th species when all species compete simultaneously. Since we are assuming that coexistence does not occur, we have \( |C_{i_0}| < \infty \) for some \( i_0 \). Let \( \partial C_{i_0} \) denote the set of sites in \( \mathbb{Z}^2 \setminus C_{i_0} \) which are adjacent to \( C_{i_0} \), so that this is a finite set. Every \( u \in \partial C_{i_0} \) is colonized by some species \( j = j(u) \) at or before the time the species \( i_0 \) reaches \( u \). It follows that when the species \( i_0 \) competes against the finite set of species \( \{ j(u) : u \in \partial C_{i_0} \} \), it still only colonizes the set \( C_{i_0} \). This contradiction completes the proof.

5. Proof of Theorem 1.5. Recall that \( \Gamma(0) \) denotes the infection graph and that \( K(\Gamma(0)) \) is the number of ends of \( \Gamma(0) \). In the following discussion we fix a measure \( \mu \in \mathcal{M} \) and assume that \( \mu \) is not purely atomic. Consequently, there exists a Borel set \( Q \subset (0, \infty) \) such that \( \mu(Q) > 0 \) and \( \mu(\{q\}) = 0 \) for all \( q \in Q \). Note that \( B_\mu \) has nonempty interior, is symmetric with respect to reflections through the axes and is convex. Therefore, the origin is in its interior.

**Proposition 5.1.** Suppose that \( \mathbb{P}\text{-a.s. there exist } k \text{ infinite geodesics } \gamma_1, \ldots, \gamma_k \text{ starting at 0, edges } e_1, \ldots, e_k \text{ and a finite set } V \subseteq \mathbb{Z}^2, \text{ such that: (a) } e_i \text{ lies on } \gamma_i \text{ but not on } \gamma_j \text{ for } i \neq j, \text{ (b) the endpoints of } e_i \text{ are in } V, \text{ (c) } \tau_{e_i} \in Q \text{ and (d) each pair of geodesics is disjoint outside of } V. \text{ Then } \mathbb{P}\text{-a.s., } K(\Gamma(0)) \geq k. \)

**Proof.** Under our assumptions on \( \mu \), with probability 1 every pair of edges with passage times in \( Q \) has distinct passage times. Consequently, geodesics between \( x, y \in \mathbb{Z}^2 \) can differ only in edges \( e \) with \( \tau_e \notin Q \), and must share edges with \( \tau_e \in Q \). We assume we are in this probability-1 event.

We claim that no two of the given geodesics are connected in \( \Gamma(0) \setminus V \). Suppose, for instance, that \( \gamma_1, \gamma_2 \) were connected in \( \Gamma(0) \setminus V \) by a path \( \sigma \) which we may assume is simple (not self-intersecting) and with endpoints \( y_1 \in \gamma_1 \) and \( y_2 \in \gamma_2 \). Denote the sequence of vertices in \( \sigma \) by \( y_1 = v_1, v_2, \ldots, v_m = y_2 \). Write \( e = e_1 \) and let \( J \subseteq \{1, \ldots, m\} \) denote the set of \( j \) such that there exists a geodesic \( \sigma_j \) from 0 to \( v_j \) which contains \( e \). We claim that \( m \in J \). This leads to a contradiction because then \( \sigma_1 \) and \( \gamma_2 \) are both geodesics connecting 0 and \( \gamma_2 \), but only one of them, \( \sigma_m \), contains \( e \).

Clearly \( 1 \in J \). Suppose now that \( j \in J \) with corresponding geodesic \( \sigma_j \). Write \( f \) for the edge between \( v_j \) and \( v_{j+1} \), and note that \( f \neq e \) because the endpoints of \( e \) are in \( V \) while those of \( f \) are not. If \( \tau(f) = 0 \), then we can adjoin \( f \) to \( \sigma_j \) to form \( \sigma_{j+1} \). Suppose, therefore, that \( \tau(f) > 0 \), so that \( \tau(0, v_j) \neq \tau(0, v_{j+1}) \). If \( \tau(0, v_{j+1}) > \tau(0, v_j) \) then we adjoin \( f \) to \( \sigma_j \) and obtain a geodesic \( \sigma_{j+1} \) with the desired properties. If \( \tau(0, v_{j+1}) < \tau(0, v_j) \) and \( v_{j+1} \) lies on \( \sigma_j \), we remove \( f \) from \( \sigma_j \) to obtain \( \sigma_{j+1} \). On the other hand, if \( v_{j+1} \) does not lie on \( \sigma_j \), but \( \tau(0, v_{j+1}) < \tau(0, v_j) \), then there is a geodesic \( \sigma'_{j+1} \) from 0 to \( v_j \) whose last edge is \( f \). Because \( \sigma'_{j+1} \) must reach \( v_j \) in the same time as \( \sigma_j \) does, it must pass through each of
those edges of $\sigma_j$ which have passage times in $Q$, and in particular through $e$. We remove $f$ from $\sigma_{j+1}'$ to obtain $\sigma_{j+1}$. □

Our goal is to establish the hypotheses of the proposition for $k$ as in (1.1). It is enough to show that there exist random variables $m < M$ such that with probability one:

1. There exist $k$ infinite geodesics $\gamma_1, \ldots, \gamma_k$ which are disjoint outside of $mB_\mu$.
2. There are edges $e_i$ in $\gamma_i$, with endpoints in $MB_\mu \setminus mB_\mu$, such that $\tau_{e_i} \in Q$.

This suffices because we can then set $V = MB_\mu$ in the proposition. To show that such $m, M$ exist, it is enough to show that for every $\varepsilon > 0$ there exist deterministic integers $m < M$ such that each of the conditions above holds on an event of probability $> 1 - \varepsilon$.

Given $u, v, w \in \partial B_\mu$ which are points of differentiability of $\partial B_\mu$, let $C(u, v, w)$ denote the open arc in $\partial B_\mu$ from $u$ to $w$ containing $v$. We rely on the following result, whose proof does not require unique geodesics. It is a rephrasing of [11], Lemma 4.7.

**Theorem 5.1 (Hoffman).** Let $u, v, w \in \partial B_\mu$ be points of differentiability of $\partial B_\mu$, let $L$ be the tangent line at $v$ and write $C = C(u, v, w)$. Then for every $\varepsilon > 0$, there is an $M_0 = M_0(\varepsilon)$ such that for every $M > M_0$, the set of $n$ such that

$$\mathbb{P}(\gamma \cap M\partial B_\mu \subseteq MC \text{ for all geodesics } \gamma \text{ from } 0 \text{ to } L + n\nu) > 1 - \varepsilon$$

has lower density at least $1 - \varepsilon$.

Henceforth, fix $k$ as in (1.1) and $\varepsilon > 0$ and for $i = 1, \ldots, k$ choose points $u_i, v_i, w_i \in \partial B_\mu$ and lines $L_i$ as in the theorem, and such that the closed sets $C_i = C(u_i, v_i, w_i)$ are pairwise disjoint and do not intersect the boundary of the $\ell^1$ unit ball; write $C = \bigcup_{i=1}^k C_i$. Note that $k$ was picked so that such a choice is possible: $\frac{1}{4}(\text{sides}(B_\mu) - 4)$ is the number of distinct sides on each of the four curves in $\partial B_\mu$ which constitute the complement of the $\ell^1$ unit ball; dividing this number by 3 gives an upper bound on the number of triples we can choose in each of these curves. Taking integer part and multiplying by 4 gives $k$.

**Claim 5.1.** There exists $M_0$ and $\rho > 0$ such that with probability at least $1 - \varepsilon$, for all $M > M_0$, every $x \in MC$ and every geodesic $\gamma$ from $0$ to $x$, at least $\rho M$ edges of $\gamma$ have passage times in $Q$.

**Proof.** Define edge weights $\{\tau'_e : e \in E\}$ by the rule that if $\tau_e \in Q$ then $\tau'_e = \tau_e + 1$ and $\tau'_e = \tau_e$ otherwise. Let $\mu'$ denote the marginal distribution of $\tau'_e$.

Choose $0 < \eta < 1$ so that $(1 - \eta)C \cap B_{\mu'} = \emptyset$ (we can do so by a theorem of Marchand [13], Theorem 1.5, and the fact that $C$ is disjoint from the $\ell^1$ unit ball).
For a path \( \sigma \) let \( N_Q(\sigma) \) be the number of edges of \( \sigma \) with passage time in \( Q \). By Theorem 1.1, there is an event \( A \) with \( P(A) > 1 - \varepsilon \) and an \( M_0 \) such that for all \( M > M_0 \) and \( y \in M \partial B_\mu \), the \( \tau \)-geodesic \( \gamma \) from 0 to \( y \) satisfies \( (1 - \eta^2)M < \tau(\gamma) < (1 + \eta^2)M \), and similarly for \( y' \in M \partial B_\mu' \) and the \( \tau' \)-length of \( \tau' \)-geodesics from 0 to \( y' \). We claim that \( A \) is the desired event. Indeed, let \( M > M_0 \) and let \( \gamma \) be a \( \tau \)-geodesic from 0 to some \( x \in MC \). We have

\[
\tau'(y) = \tau(y) + N_Q(\gamma) \leq (1 + \eta^2)M + N_Q(\gamma).
\]

On the other hand, \( x = \frac{M}{s} y \) for some \( y \in \partial B_\mu' \) and \( s < 1 - \eta \), so

\[
\tau'(y) \geq (1 - \eta^2)\frac{M}{1 - \eta}.
\]

Combining these we find that \( N_Q(\gamma) \geq (\eta - \eta^2)M \). We take this to be \( \rho M \). \( \Box \)

Let \( \alpha > 0 \) denote the quantity

\[
\alpha = \frac{1}{2} \min \{ \pi_{vi}(x_i - x_j) : x_i \in C_i, x_j \in C_j, i \neq j \}.
\]

Choose finite sets \( D_i \subseteq C_i \cap \partial B_\mu \) with the property that for every \( i = 1, \ldots, k \),

\[
C_i \subseteq \bigcup_{x \in D_i} \left( x + \frac{\alpha}{10} B_\mu \right).
\]

This property can be satisfied by compactness of \( \bigcup_i C_i \) and the fact that \( B_\mu \) contains a neighborhood of the origin. Write \( D = \bigcup_{i=1}^k D_i \).

We can choose large integers \( m \) and \( M \gg m \) and a set \( I \subseteq \mathbb{N} \) of density \( > 1 - \varepsilon \) such that, for \( n \in I \), the following statements hold with probability \( > 1 - \varepsilon \).

(A) If \( i \neq j \), then \( B_{L_i + mvi}(x_j, x_i) \geq m\alpha \) for all \( x_i \in mD_i \) and \( x_j \in mD_j \).

(B) Every geodesic \( \gamma_{i,n} \) from 0 to \( L_i + nvi \) intersects \( m \partial B_\mu \) in \( mC_i \) and intersects \( M \partial B_\mu \) in \( MC_i \).

(C) \( |\tau(0, x) - m| < \frac{ma}{10} \) for all \( x \in mD \).

(D) If \( x \in mD \) and \( y \in x + \frac{ma}{10} B_\mu \), then \( \tau(y, x) < \frac{ma}{5} \).

(E) At least one edge on \( \gamma_{i,n} \cap (MB_\mu \setminus mB_\mu) \) has passage time in \( Q \).

Indeed, for \( m, M \) large enough the first two properties follow from Theorems 5.1 and 4.1, and the third and fourth from Theorem 1.1 [for (D) we apply Theorem 1.1 to each of the finitely many points in \( mD \) and intersect the events; note that the probabilities do not depend on the point in question, only on \( m \)]. Last, (E) follows from the previous claim, since when \( M \) is large the number of edges on any geodesic from 0 to \( m \partial B_\mu \) is smaller than \( \rho M \).

Call \( A_n \) the intersection of the above five events. Since \( P(A_n) > 1 - \varepsilon \) for all \( n \in I \), the event \( A \) that \( A_n \) occurs for infinitely many \( n \) has \( P(A) > 1 - \varepsilon \). We now consider only configurations in \( A \). For each \( n \) and \( i \), fix \( \gamma_{i,n} \) as in (B). We may choose a (random) infinite set \( I' \subseteq I \) such that for all \( n \in I' \), \( A_n \) occurs and \( J \subseteq I' \)
such that \( \lim_{n \in J} \gamma_{i,n} \to \gamma_i \) for some infinite geodesics \( \gamma_i \) originating at 0, that is, for every \( r > 0 \) we have \( \gamma_i \cap [-r, r]^2 = \gamma_{i,n} \cap [-r, r]^2 \) for all large enough \( n \in J \). Henceforth, we only consider such \( n \).

Let \( y_{i,n} \) be the first intersection point of \( \gamma_{i,n} \) with \( mC_i \), and choose \( x_{i,n} \in mD_i \) such that \( y_{i,n} \in x_{i,n} + \frac{ma}{10} B_\mu \). Then by (D) we have \( \tau(x_{i,n}, y_{i,n}) \leq \frac{ma}{5} \), so by (A),

\[
(5.2) \quad |B_{L_{1+nv_1}}(y_{j,n}, y_{i,n}) - B_{L_{1+nv_1}}(x_{j,n}, x_{i,n})| < \frac{2ma}{5} \quad \text{for } i \neq j.
\]

**Claim 5.2.** The \( \gamma_i \)'s are disjoint outside of \( mB_\mu \).

**Proof.** Suppose, for example, that \( \gamma_1, \gamma_2 \) intersect at some point \( z \) outside of \( mB_\mu \). Then for large enough \( n \in J \) the same is true of \( \gamma_{1,n} \) and \( \gamma_{2,n} \). Then

\[
\tau(0, y_{1,n}) + \tau(y_{1,n}, z) = \tau(0, y_{2,n}) + \tau(y_{2,n}, z).
\]

By (C) we have \( |\tau(0, y_{1,n}) - \tau(0, y_{2,n})| < \frac{2ma}{10} \), so

\[
|\tau(y_{1,n}, z) - \tau(y_{2,n}, z)| < \frac{2ma}{10}.
\]

Write \( \sigma_1 \) for the part of \( \gamma_{1,n} \) from \( y_{1,n} \) to \( L_1 + nv_1 \). Let \( \sigma_2 \) be path which starts at \( y_{2,n} \), follows \( \gamma_{2,n} \) until \( z \) and then follows \( \gamma_{1,n} \) until \( L_1 + nv_1 \). We find that \( |\tau(\sigma_1) - \tau(\sigma_2)| < \frac{2ma}{10} \). But \( \gamma_{1,n} \) is a shortest path from 0 to \( L_1 + nv_1 \), so \( \sigma_1 \) is a shortest path from \( y_{1,n} \) to \( L_1 + nv_1 \). Hence, \( B_{L_1+nv_1}(y_{2,n}, y_{1,n}) \leq \frac{2ma}{10} \). Combined with (5.2), this contradicts (A). \( \Box \)

Finally, combining the last claim with (E) establishes the two claims stated after Proposition 5.1. This completes the proof of Theorem 1.4.

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