WEIGHTED COMPOSITION OPERATORS:
ISOMETRIES AND ASYMPTOTIC BEHAVIOUR

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Abstract. This paper studies the behaviour of iterates of weighted composition operators acting on spaces of analytic functions, with particular emphasis on the Hardy space $H^2$. Questions relating to uniform, strong and weak convergence are resolved in many cases. Connected to this is the question when a weighted composition operators is an isometry, and new results are given in the case of the Hardy and Bergman spaces.

1. Introduction and notation

Let $H^2$ denote the Hardy space of the disc, and for $w \in H^2$ and $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic, we consider the (a priori densely defined) weighted composition operator $T_{w,\varphi}$ on $H^2$ given by

$$(T_{w,\varphi} f)(z) = w(z)f(\varphi(z)) \quad (f \in H^2, \ z \in \mathbb{D}).$$

In general it is hard to determine when $T_{w,\varphi}$ is bounded but, by Littlewood’s subordination theorem, $w \in H^\infty$ is always a sufficient condition. The extreme case that $T_{w,\varphi}$ is bounded if and only if $w \in H^\infty$ occurs precisely when $\varphi$ is a finite Blaschke product \cite{7, 17}. At the other extreme, $w \in H^2$ is necessary and sufficient for boundedness if and only if $\|\varphi\|_\infty < 1$ \cite{11}.

Continuing some of the work of \cite{11}, which considered composition operators $C_\varphi$ defined by $(C_\varphi f)(z) = f(\varphi(z))$ for $z \in \mathbb{C}$, we shall be considering the following properties that powers of $T_{w,\varphi}$ may possess:

- uniform convergence, convergence in the operator norm;
- strong convergence, the norm convergence of $T_{w,\varphi}^n f$ for all $f \in H^2$.

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2 I. CHALENDAR AND J.R. PARTINGTON

- weak convergence, the weak convergence of $T^n_{w,\varphi}f$ for all $f \in H^2$
  (implying local uniform convergence, the convergence of $T^n_{w,\varphi}f$
  in the Fréchet space Hol($\mathbb{D}$) of holomorphic functions on $\mathbb{D}$).

Each of the properties above implies those below it.

Using the notation $\varphi_k$ for the $k$th iterate of $\varphi$, we have the following
easily-derived formula for $T^n_{w,\varphi}$ for $n = 2, 3, \ldots$:

$$(T^n_{w,\varphi}f)(z) = w(z)w(\varphi(z))\ldots w(\varphi_{n-1}(z))f(\varphi_n(z)).$$

Recall that every holomorphic mapping $\varphi : \mathbb{D} \to \mathbb{D}$ that is not an
automorphism with a fixed point in the disc possesses a Denjoy–Wolff
point, $DW(\varphi) \in \overline{\mathbb{D}}$ such that the iterates of $\varphi$ tend to $DW(\varphi)$ uniformly
on compact subsets of $\mathbb{D}$. (See, for example [8, Chap. 2].)

As regards $H^2$, some of the results of [1] may be summarized as
follows:

- Theorem 3.4. $(C^n_{\varphi})$ converges in operator norm if and only if
  the essential spectral radius satisfies $r_e(C_{\varphi}) < 1$.
- Theorem 4.3. If $DW(\varphi) \in \mathbb{T}$, then $\sup_n \|C^n_{\varphi}\| = \infty$ and so $(C^n_{\varphi})$
  cannot even converge weakly.
- Theorem 4.8, Theorem 4.10. If $DW(\varphi) = \alpha \in \mathbb{D}$ and $\varphi$ is inner,
  then $(C^n_{\varphi}f)$ converges weakly to $P_\alpha f := f(\alpha)1$ for all $f \in H^2$,
  where $1$ is the constant function; however, it does not converge
  strongly.
- Corollary 4.16. If $DW(\varphi) = \alpha \in \mathbb{D}$ and $\varphi$ is not inner, then
  $(C^n_{\varphi})$ converges in operator norm to $P_\alpha$.

For mappings $\varphi$ with a fixed point $a \in \mathbb{D}$, we may study the iterates
of $T_{w,\varphi}$, reducing to the case where the fixed point is 0 by using the
involution

$$(\psi_a : z \mapsto \frac{a - z}{1 - \bar{a}z}).$$

This provides an isomorphism $\phi_{\psi_a}$ of $H^2$ and satisfies

$$(C_{\psi_a}T_{w,\varphi}C_{\psi_a}f)(z) = w(\psi_a(z))f(\psi_a \circ \varphi \circ \psi_a(z)).$$
where now $\psi_a \circ \varphi \circ \psi_a$ has its fixed point at the origin.

In the next few sections we analyse convergence for various classes of $\varphi$. As will be seen, some of our results apply to other spaces of functions, although our main focus is on $H^2$.

We write $\sigma(T)$ and $\sigma_p(T)$ for the spectrum and point spectrum (eigenvalues), respectively, of an operator $T \in \mathcal{L}(X)$, the algebra of bounded operators on a Banach space $X$. Also $r_e(T)$ denotes the essential spectral radius of $T$.

2. Convergence criteria for weighted composition operators

There is a general convergence criterion for convergence of powers of an operator $T \in \mathcal{L}(X)$, where $X$ is a Banach space, which may be found in [2, A-III, Sec. 3.7]; it was applied to composition operators in [1], and we shall also make use of it in this section.

An operator $T \in \mathcal{L}(X)$ is power-bounded if $\sup_{n \geq 1} \|T^n\| < \infty$.

Proposition 2.1. Let $T \in \mathcal{L}(X)$ be power-bounded. Then the following assertions are equivalent:

(i) $P := \lim T^n$ exists in $\mathcal{L}(X)$ and is of finite rank.

(ii) (a) $r_e(T) < 1$;

(b) $\sigma_p(T) \cap \mathbb{T} \subseteq \{1\}$;

(c) if $1$ is in $\sigma(T)$ then it is a pole of the resolvent of order 1.

In that case, $P$ is the residue at 1.

In this section we take a Banach space $X$ of holomorphic functions on $\mathbb{D}$ containing the constant function 1, such that there is a continuous injection $X \to \text{Hol}(\mathbb{D})$, and hence point evaluations in $\mathbb{D}$ are continuous functionals on $X$. There are numerous examples, including the Hardy spaces $H^p$ and weighted Bergman spaces $A^p_\alpha$ for $p \geq 1$ and $\alpha > -1$. We also suppose that $T_{w,\varphi}$ is a power-bounded weighted composition operator on $X$, excluding the case when $\varphi$ is an elliptic automorphism.
Theorem 2.2. With \( X \) and \( T_{w,\varphi} \) as above, suppose also that \( \alpha := \text{DW}(\varphi) \in \mathbb{D} \). Then the sequence \( (T_n^{w,\varphi}) \) converges weakly as \( n \to \infty \) if and only if (i) \( |w(\alpha)| < 1 \) or \( w(\alpha) = 1 \), and (ii) \( \sup_n \|T_n^{w,\varphi}\| < \infty \).

Proof. (i) Since \( T_n^{w,\varphi}(1)(z) = w(z) \cdots w(\varphi(z)) \), it follows that

\[ T_n^{w,\varphi}(1)(\alpha) = w(\alpha)^n. \]

So we have \( |w(\alpha)| < 1 \) or \( w(\alpha) = 1 \), as well as the power-boundedness condition, if \( (T_n^{w,\varphi}) \) converges weakly.

(ii) Conversely, if \( |w(\alpha)| < 1 \) or \( w(\alpha) = 1 \), then we may first suppose without loss of generality that \( \alpha = 0 \), by applying the transformation given in (3). We also have by Schwarz’s lemma that under these circumstances \( |\varphi(z)| \leq \delta|z| \) on compact discs \( r\mathbb{D} \) for \( 0 < r < 1 \), for some fixed \( \delta \in (0,1) \), and so \( |\varphi_k(z)| \leq \delta^k|z| \) for \( k = 1, 2, \ldots \). Thus \( f(\varphi_n(z)) \to 0 \) for \( f \in X \).

If \( |w(0)| < 1 \), then clearly \( w(z)w(\varphi(z)) \cdots w(\varphi_{n-1}(z)) \to 0 \) uniformly on any compact \( K \subset \mathbb{D} \).

If \( w(0) = 1 \), then \( |w(\varphi_k(z)) - 1| \leq C\epsilon^k \) uniformly on \( K \) for some fixed \( C > 0 \) and \( \epsilon \in (0,1) \), depending on the modulus of \( w' \). This ensures the convergence of the infinite product \( w(z)w(\varphi(z)) \cdots w(\varphi_{n-1}(z)) \) to a function \( \tilde{w}(z) \). Since the \( T_n^{w,\varphi} \) are uniformly bounded in norm, this implies the weak convergence of \( (T_n^{w,\varphi}) \).

We may now apply Proposition 2.1 to obtain the following result.

Theorem 2.3. Under the hypotheses of Theorem 2.2, suppose that either \( |w(\alpha)| < 1 \) or \( w(\alpha) = 1 \), and that \( \sup_n \|T_n^{w,\varphi}\| < \infty \). Then \( (T_n^{w,\varphi}) \) converges uniformly if and only if \( r_e(T_{w,\varphi}) < 1 \).

Proof. Once more we may assume without loss of generality that \( \alpha = 0 \). Given that \( (T_n^{w,\varphi}) \) converges weakly, we also have the following condition (D):

\[ X = \text{fix}(T_{w,\varphi}) \oplus \text{Im}(I - T_{w,\varphi}) \]

where \( \text{fix}(T_{w,\varphi}) = \{ f \in X : f(z) = w(z)f(\varphi(z)) \} \) as in [1].
Now if $|w(0)| < 1$ and $T_{w,\varphi} f = \lambda f$, we have

$$\lambda^n f(z) = T^n_{w,\varphi} f(z) = w(z)w(\varphi(z)) \ldots w(\varphi_{n-1}(z)) f(\varphi_n(z)),$$

and therefore if $|\lambda| \geq 1$ and $|z| < 1$, we have

$$f(z) = \lim_{n \to \infty} \lambda^{-n} w(z)w(\varphi(z)) \ldots w(\varphi_{n-1}(z)) f(\varphi_n(z)) = 0.$$  

Thus $T_{w,\varphi}$ has no point spectrum on the circle.

Suppose now that $w(0) = 1$. If $f \in \text{fix}(T_{w,\varphi})$ we have

$$f(z) = \lim_{n \to \infty} \lambda^{-n} w(z)w(\varphi(z)) \ldots w(\varphi_{n-1}(z)) f(\varphi_n(z)),$$

and so $\text{fix}(T_{w,\varphi})$ is one-dimensional and spanned by

$$\tilde{w}(z) := \lim_{n \to \infty} w(z)w(\varphi(z)) \ldots w(\varphi_{n-1}(z)).$$

Next, if $T_{w,\varphi} f = \lambda f$ with $\lambda \in \mathbb{T} \setminus \{1\}$; then we have $w(z) f(\varphi(z)) = \lambda f(z)$, and taking $z = 0$ we have that $f(0) = \lambda f(0)$ so $f(0) = 0$.

We may then write $f(z) = zf_1(z)$ for some $f_1 \in \text{Hol} (\mathbb{D})$, and likewise $\varphi(z) = z\varphi_1(z)$, to obtain $w(z)z\varphi_1(z)f_1(\varphi(z)) = \lambda zf_1(z)$, so that $\varphi_1(0)f_1(0) = \lambda f_1(0)$. But $|\varphi_1(0)| < 1$, since $\varphi_1$ is not an automorphism of $\mathbb{D}$. So $f_1(0) = 0$ and we may write $f_1(z) = zf_2(z)$ for some $f_2 \in \text{Hol}(\mathbb{D})$.

We now have $w(z)\varphi(z)^2 f_2(\varphi(z)) = \lambda z^2 f_2(z)$, giving $\varphi_1(0)^2 f_2(0) = \lambda f_2(0)$ and now $f_2(0) = 0$.

Continuing in this way we conclude that $f$ is identically zero, and so $\sigma_p(T) \cap \mathbb{T} \subseteq \{1\}$ in both the cases $w(0) = 0$ and $w(0) = 1$; moreover, if $\text{re}(T_{w,\varphi}) < 1$ then $\text{Id} - T_{w,\varphi}$ is Fredholm, and so $\text{Im}(\text{Id} - T_{w,\varphi})$ is closed.

Moreover, since $\ker(\text{Id} - T_{w,\varphi})$ is either 0 or 1-dimensional, we see that 1 is a pole of the resolvent of order at most 1.

We now have all the conditions of Proposition 2.1, and may conclude that $\|T^n_{w,\varphi} - P\| \to 0$, where the projection $P$ is given by $P = 0$ if $|w(0)| < 1$ and $Pf(z) = \tilde{w}(z)f(0)$ if $w(0) = 1$.

Some applications of these results will be given in the next section.
3. Automorphisms

As mentioned above, in the case that \( \varphi \) is a finite Blaschke product, a necessary and sufficient condition for boundedness of \( T_{w,\varphi} \) on \( H^2 \) is that \( w \in H^\infty \). We consider some special cases of this situation (we return to looking at operators on more general spaces in Subsection 4.3).

We begin with the case that \( \varphi \) is an elliptic automorphism of finite order.

Suppose that \( \varphi(z) = \lambda z \), where \( \lambda \) is a primitive \( k \)th root of unity. In this situation an important role is played by the function \( v \in H^\infty \) defined by

\[
v(z) = w(z)w(\lambda z) \cdots w(\lambda^{k-1}z) \quad (z \in \mathbb{D}).
\]

We have that

\[
(T_{w,\varphi}^{mk+j} f)(z) = v(z)^m w(z)w(\varphi(z)) \cdots w(\varphi_{j-1}(z)) f(\lambda^j z).
\]

for \( m \geq 0 \) and \( 0 \leq j < m \).

**Theorem 3.1.** Suppose that \( \varphi(z) = \lambda z \), where \( \lambda \) is a primitive \( k \)th root of unity, and let \( v \) be as in (5).

(i) If \( \|v\|_\infty < 1 \), then \( \|T_{w,\varphi}^n\| \to 0 \) as \( n \to \infty \).

(ii) If \( \|v\|_\infty > 1 \), then \( T_{w,\varphi}^n \) does not converge weakly.

(iii) If \( \|v\|_\infty = 1 \) and \( v \) is non-constant, then

(a) if \( |v| < 1 \) a.e., it follows that \( T_{w,\varphi}^n \to 0 \) strongly, although not uniformly;

(b) if \( |v| = 1 \) on a set of positive measure, then \( \|v^n\|_2 \nrightarrow 0 \) and so there is weak convergence of \( T_{w,\varphi}^n \) to 0, but not strong convergence.

**Proof.** The proof is made simpler by observing that, apart from the factor \( v^m \), there are only finitely many possibilities for the other factor in \( T_{w,\varphi}^n f(z) \), and we may consider subsequences determined by \( n = mk + j \) for a fixed \( j \).

Then (i) is clear, and (ii) follows on noting that there is a \( z_0 \in \mathbb{D} \) with \( |v(z_0)| > 1 \) and examining the values at \( (T_{w,\varphi}^n f)(z_0) \).
For (iii) we note that in case (a) \( v^n \to 0 \) a.e. on \( T \), and the strong convergence follows, by dominated convergence, although \( \|v^n\|_\infty \not\to 0 \). Finally in case (b), weak convergence holds since \( |v(z)| < 1 \) for all points \( z \in \mathbb{D} \), but strong convergence to 0 is clearly not satisfied.

The results from Section 2 can be used to obtain results on the convergence of iterates of weighted composition operators, although there is little published work on the essential spectra of such operators, except in the case of automorphims \( \varphi \). Here we mention \([13, 14]\), and in particular, we have the following results from \([14]\), which are clearly relevant to our analysis, at least in the case that \( w \) lies in the disc algebra \( A(\mathbb{D}) \) and is bounded away from 0 on \( \mathbb{D} \).

- If \( \varphi \) is an elliptic automorphism of infinite order with fixed point \( \alpha \in \mathbb{D} \) and \( w \) is as above, then
  \[
  \sigma(T_{w,\varphi}) = \sigma_e(T_{w,\varphi}) = \{ \lambda \in \mathbb{C} : |\lambda| = |w(\alpha)| \}.
  \]
- The same holds if \( \varphi \) is a parabolic automorphism with fixed point \( \alpha \in \mathbb{T} \).
- If \( \varphi \) is a hyperbolic automorphism with attractive/repulsive fixed points \( \alpha, \beta \) respectively. Then
  \[
  \sigma(T_{w,\varphi}) = \sigma_e(T_{w,\varphi}) = \{ \lambda \in \mathbb{C} : |w(b)|/\varphi'(b)^{1/2} \leq |\lambda| \leq |w(a)|/\varphi'(a)^{1/2} \}.
  \]

Clearly, the essential spectra of weighted composition operators is a matter for further investigation, but we record here one easy corollary of the above results.

**Corollary 3.2.** Let \( \varphi \) be an elliptic automorphism of infinite order with fixed point \( \alpha \in \mathbb{D} \) and \( w \in A(\mathbb{D}) \) bounded away from 0 on \( \mathbb{D} \). Suppose that \( \sup_n \|T_{w,\varphi}^n\| < \infty \). Then \( (T_{w,\varphi}^n) \) converges uniformly if and only if \( |w(\alpha)| < 1 \).

**Proof.** This follows directly from Theorem 2.3 on applying the above result from \([14]\). \( \square \)
Remark 3.3. In general, suppose that $\varphi$ is a finite Blaschke product with $\varphi(0) = 0$ and that $w \in H^\infty$. Then for weak (or indeed uniform) convergence we require $w(0) \in \{0, 1\}$ (also $r_e(T_{w,\varphi}) < 1$ to ensure uniform convergence).

In this situation $C_{\varphi}$ is a contraction, so we have the power-boundedness condition $\sup_n \|T_{w,\varphi}^n\| < \infty$ whenever $\|w\|_\infty \leq 1$, and hence also weak convergence. Indeed, $\|T_{w,\varphi}^n\| \to 0$ whenever $\|w\|_\infty < 1$.

4. Particular classes of operator

4.1. Isometries. If $T$ is an isometry on a Hilbert space, not equal to the identity, then clearly the sequence $(T^n)$ cannot converge uniformly, or even strongly; however, if $T$ is completely non-unitary then $(T^n)$ will converge weakly to $0$.

The isometric weighted composition operators on $H^2$ were characterised in [16] as follows.

Proposition 4.1. $T_{w,\varphi}$ is an isometry on $H^2$ if and only if $\varphi$ is inner, $\|w\|_2 = 1$ and $\langle w, w\varphi^n \rangle = 0$ for all $n \geq 1$.

It is worth exploring this further, and in fact all inner functions $\varphi$ can occur as part of an isometry $T_{w,\varphi}$.

Proposition 4.2. For every inner function $\varphi$ there is a weight $w$ such that $T_{w,\varphi}$ is an isometry.

Proof. First, if $\varphi(0) = 0$, then $C_{\varphi}$ is an isometry, and any inner function $w$ provides an isometry $T_{w,\varphi}$.

Next, if $\varphi$ is inner and has a zero at $\beta \in \mathbb{D}$, then we claim that $w = \theta k_\beta/\|k_\beta\|$ provides an isometry $T_{w,\varphi}$ for any inner function $\theta$, where $k_\beta$ is the reproducing kernel at $\beta$.

For $\|w\|_2 = 1$, and for $n \geq 1$ we have

$$\langle w, w\varphi^n \rangle = \langle \theta k_\beta, \theta k_\beta \varphi^n \rangle/\|k_\beta\|^2 = 0,$$

since $\varphi(\beta) = 0$. 


Finally, if $\varphi$ is a singular inner function, let
\[ w = \theta(\varphi - \varphi(0)) / \|\varphi - \varphi(0)\|, \]
where again $\theta$ is inner. We have, for $n \geq 1$,
\[ \langle \theta(\varphi - \varphi(0)), \theta(\varphi - \varphi(0)) \varphi^n \rangle = \langle (\varphi - \varphi(0)), (\varphi - \varphi(0)) \varphi^n \rangle \]
\[ = \langle 1, (\varphi - \varphi(0)) \varphi^{n-1} \rangle - \varphi(0) \langle 1, (\varphi - \varphi(0)) \varphi^n \rangle = 0, \]
using the fact that 1 is the reproducing kernel $k_0$. Hence $T_{w,\varphi}$ is an isometry in this case as well. \[\square\]

Thus, unlike in the unweighted case $C_\varphi$, when the isometries require $\varphi$ to be inner and $\varphi(0) = 0$ (see, e.g. [19, 4]), the Denjoy–Wolff point does not play a role here. Some further ideas about shifts and unitary operators are given in [18].

Isometries on the Bergman space $A^2$ are harder to characterise. Some conditions are given by Zorboska [20], expressed mainly in terms of the measure $\mu$ defined on Borel subsets of $\mathbb{D}$ by
\[ \mu(A) = \int_{\varphi^{-1}(A)} w(z)^2 \, dm(z), \]
where now $m$ denotes normalized area measure. This has the property that
\[ \|T_{w,\varphi}f\|^2 = \int_{\mathbb{D}} |f|^2 \, d\mu(z) \]
for $f \in A^2$ (see [16]).

In the case of isometries, we have the following result, which can be seen as a reformulation of [20, Thm. 2.1 (ii)], with a different proof.

**Theorem 4.3.** The weighted composition operator $T_{w,\varphi}$ is an isometry on $A^2$ if and only if the measure $\mu$ defined in (6) coincides with Lebesgue measure $m$.

**Proof.** The operator $T_{w,\varphi}$ is an isometry if and only if
\[ \int_{\mathbb{D}} |f(z)|^2 \, dm(z) = \int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \]
for $f \in A^2$, taking
so clearly this happens if $\mu = m$. For the converse, we introduce a new algebra $\mathcal{A} \subset C(\overline{D})$, defined by

$$\mathcal{A} = \left\{ \sum_{k=1}^{n} c_k |p_k|^2 : n \in \mathbb{N}, c_k \in \mathbb{C}, p_k \in \mathbb{C}[z] \right\}.$$ 

This algebra is self-adjoint, separates points (if $z_1 \neq z_2$, simply choose a polynomial that vanishes at $z_1$ but not $z_2$), and contains the constant functions. Hence, by the Stone–Weierstrass theorem it is dense in $C(\overline{D})$. Using the isometry condition we see that $m$ and $\mu$ induce identical functionals on $\mathcal{A}$, hence on $C(\overline{D})$, and thus $\mu = m$. \qed

**Remark 4.4.** The above methods can be used to characterise isometries on the weighted Bergman spaces $A^2_\alpha$ for $\alpha > -1$, defined by replacing $m$ by the measure $(1 - |z|^2)^\alpha dm(z)$. We omit the details.

Zorboska notes that if $\varphi$ is a finite Blaschke product of degree $N$, then $T_{w,\varphi}$ is isometric on $A^2$ if we take $w(z) = \frac{1}{\sqrt{N}} \varphi'(z)$. In fact similar methods can be used to prove the following more general result.

**Proposition 4.5.** Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic and that there is an integer $N \in \mathbb{N}$ such that

$$m(\mathbb{D} \setminus \{ z \in \mathbb{D} : \# \{ \varphi^{-1}(\{ z \}) \} = N \}) = 0.$$ 

Then $w = c\varphi'/\sqrt{N}$ induces an isometric weighted composition operator $T_{w,\varphi}$ on $A^2$ for each constant $c$ with $|c| = 1$.

**Proof.** Let $\Omega$ denote $\{ z \in \mathbb{D} : \# \{ \varphi^{-1}(\{ z \}) \} = N \}$. Then

$$\frac{1}{N} \int_{\mathbb{D}} |f \circ \varphi(z)|^2 |\varphi'(z)|^2 dm(z) = \frac{1}{N} \int_{\Omega} |f \circ \varphi(z)|^2 |\varphi'(z)|^2 dm(z) = \frac{N}{N} \int_{\mathbb{D}} |f(s)|^2 dm(s) = \| f \|_{A^2}^2.$$ 

This allows us to construct some interesting new examples of isometries. For example, let $\psi$ denote the conformal mapping (Riemann mapping) from $\mathbb{D}$ onto the left semidisc $\mathbb{D}_L = \{ z \in \mathbb{D} : \Re z < 0 \}$ mapping real $z$ to real $z$. Then $\varphi := \psi^4$ is a double cover of the disc...
\( \mathbb{D} \) with the exception of the positive real axis, which is covered just once (and 0 is not covered at all). Thus \( T_{\varphi'/\sqrt{2}\varphi} \) is an isometry of \( A^2 \). Explicit formulae for \( \psi \) may be found in [5, p. 64].

However, it is not necessary to use \( w = c\varphi'/\sqrt{N} \) to obtain Bergman space isometries. There are many instances when a Blaschke product \( \varphi \) of degree \( N \) has the property that \( \varphi \circ \psi = \varphi \) for some elliptic automorphism \( \psi \) of order \( N \) (see, particularly, [3, Cor. 2.6] for details). In this case we have the following.

**Corollary 4.6.** Let \( \varphi \) be a Blaschke product of degree \( N \) such that \( \varphi \circ \psi = \varphi \), where \( \psi \) is an elliptic automorphism with fixed point \( p \in \mathbb{D} \) and order \( N \). Then the composition operator \( T_{w,\varphi} \) is an isometry on \( A^2 \) if
\[
T_{w,\varphi} = c \frac{1}{\sqrt{N}} \varphi'(\psi(z)) \quad (z \in \mathbb{D})
\]
with \( |c| = 1 \).

**Proof.** Let \( A \subset \mathbb{D} \) be a Borel set and let \( B \) be a Borel set mapped bijectively to \( A \) by \( \varphi \) to within sets of measure 0 (this can be obtained by dividing \( \mathbb{D} \) into \( N \) regions, each mapped almost everywhere onto \( \mathbb{D} \)). Then
\[
\int_{\varphi^{-1}(A)} |w(z)|^2 \ dm(z) = \int_B \sum_{k=1}^{N} |w(\psi_k(z))|^2 \ dm(z),
\]
\[
= N \int_B \sum_{k=1}^{N} |\varphi'(\psi_k(z))|^2 \ dm(z) = N \int_{\varphi^{-1}(A)} |\varphi'(z)|^2 \ dm(z)
\]
and since this equals \( m(A) \) we see that \( T_{w,\varphi} \) is an isometry by applying Theorem 4.3. \( \square \)

4.2. **Hardy spaces of simply-connected domains.** Iterates of (unweighted) composition operators \( C_\Phi \) on the right half-plane \( \mathbb{C}_+ \) have been studied in [15]. We recall a few basic facts about such operators:

- For \( \Phi : \mathbb{C}_+ \rightarrow \mathbb{C}_+ \) holomorphic, the composition operator \( C_\Phi \) on \( H^2(\mathbb{C}_+) \) is bounded if and only if \( \Phi(\infty) = \infty \) and its angular
derivative satisfies \( \Phi'(\infty) > 0 \). Then the norm, essential norm, and spectral radius of \( C_\Phi \) all equal \( \Phi'(\infty)^{1/2} \). (See [10].)

- Let \( M \) denote the self-inverse conformal mapping \( M(z) = (1 - z)/(1 + z) \) between \( \mathbb{D} \) and \( \mathbb{C}_+ \). Then \( C_\Phi \) on \( H^2(\mathbb{C}_+) \) is unitarily equivalent to the operator \( T_{w,\varphi} \) on \( H^2(\mathbb{D}) \), where \( \varphi = M \circ \Phi \circ M \) and \( w(z) = \frac{1 + \varphi(z)}{1 + z} \). (See [4].)

- For bounded \( T_{w,\varphi} \) as above, we have \( DW(\varphi) = -1 \) and \( \varphi'(-1) = \Phi'(\infty) \). (See [15].)

Some of the conclusions of [15] are the following:

**Theorem 4.7.** For \( \Phi : \mathbb{C}_+ \to \mathbb{C}_+ \) holomorphic, such that \( \Phi(\infty) = \infty \) and \( 0 < \Phi'(\infty) < \infty \), with \( \Phi'(\infty) \neq 1 \), the following are equivalent:

(i) \( C^n_\Phi \to 0 \) strongly;

(ii) \( DW(\Phi) = \infty \) and \( \Phi'(\infty) < 1 \);

(iii) \( \text{re}(C_\Phi) < 1 \);

(iv) \( C^n_\Phi \to 0 \) uniformly.

Now let \( \Omega \) be a bounded simply-connected domain with rectifiable (finite-length) boundary, and \( \beta : \mathbb{D} \to \Omega \) a conformal bijection. Then let \( E^2(\Omega) \) be the Hardy–Smirnoff space of all functions \( f : \Omega \to \mathbb{C} \) such that the function \( Jf : f \mapsto f(\beta(z))\beta'(z) \) lies in \( H^2(\mathbb{D}) \), endowed with the norm \( \|f\|_{E^2(\Omega)} = \|Jf\|_{H^2} \).

The space \( E^2(\Omega) \) coincides with Rudin’s space \( H^2(\Omega) \) of all analytic functions \( f \) such that \( |f|^2 \) has a harmonic majorant, if and only if there exist constants \( a, b > 0 \) such that \( a \leq |\beta'(z)| \leq b \) for all \( z \in \mathbb{D} \). We refer to the book of Duren [9] for further background on these spaces.

For \( \Phi : \Omega \to \Omega \) holomorphic, the composition operator \( C_\Phi \) on \( E^2(\Omega) \) is unitarily equivalent to the weighted composition \( T_{w,\varphi} \) on \( H^2 \), where \( \varphi = \beta^{-1} \circ \Phi \circ \beta \), and \( w(z) = (\beta'(z)/\beta'(\varphi(z)))^{1/2} \). (See, e.g. [16].)

**Theorem 4.8.** Suppose that \( DW(\varphi) \in \mathbb{T} \). Then \( (T^n_{w,\varphi}) \) is unbounded in norm, and thus cannot converge even weakly.

**Proof.** Let us write \( k_w \) for the reproducing kernel on \( E^2(\Omega) \), so that \( \langle f, k_w \rangle = f(w) \) for all \( f \in E^2(\Omega) \) and \( w \in \Omega \). Then we have the
standard formula $C^*_k w = k_{\Phi(w)}$, and hence $(C^*_k)^n w = k_{\Phi_n(w)}$ for all $w \in \Omega$.

Fix any $w \in \Omega$, and then the sequence $(\Phi_n(w)) = (\beta(\varphi_n(\beta^{-1}(w))))$ has at least one accumulation point on the boundary, since $\varphi_n(\beta^{-1}(w)) \to DW(\varphi)$.

Now for each point $a \in \partial \Omega$ there are functions in $E^2(\Omega)$ that are unbounded near $a$, such as a continuous branch of $(z - a)^{-1/4}$, and it follows that $(k_{\Phi_n(w)})$ is unbounded, and hence $T_{w,\varphi}$ is not power-bounded.

\[ \square \]

4.3. Operators on weighted spaces. In [6] the following theorem is given; it is based on the notion of an operator represented by an infinite lower-triangular matrix. For a sequence $d = (d_n)_{n=0}^{\infty}$ of positive real numbers, $H^2(d)$ denotes the space of holomorphic functions $f : z \mapsto \sum_{n=0}^{\infty} c_n z^n$ such that

\[ \|f\|_{H^2(d)}^2 := \sum_{n=0}^{\infty} |c_n|^2 d_n^2 < \infty. \]

Also, for $a \in \mathbb{D}$, $\psi_a$ is the automorphism defined by (1).

**Theorem 4.9.** Let $\varphi$ be a holomorphic self-mapping of the open unit disc $\mathbb{D}$, and let $h \in H^2$ be such that $T_{w,\varphi}$ is a bounded operator on $H^2$. Let $(d_n)$ be a decreasing sequence of positive reals such that $H^2(d)$ is automorphism-invariant. Then $T_{w,\varphi}$ is also a bounded operator on $H^2(d)$ with

\[ \|T_{w,\varphi}\|_{H^2(d)} \leq \|C_{\psi_{\varphi(0)}}\|_{H^2} \|C_{\psi_{\varphi(0)}}\|_{H^2} \|T_{w,\varphi}\|_{H^2}. \]

Easy examples of such $H^2(d)$ spaces are the weighted Bergman spaces $A^2_\alpha(\mathbb{D})$ for $\alpha > -1$ with $d_n = (n + 1)^{-\alpha - 1}$ for $n = 0, 1, 2, \ldots$.

As regards convergence of iterates, we may use this theorem to analyse the case that $DW(\varphi) \in \mathbb{D}$.

**Theorem 4.10.** Let $\varphi$ be a holomorphic self-mapping of the open unit disc $\mathbb{D}$ with $DW(\varphi) \in \mathbb{D}$, and let $h \in H^2$ be such that $T_{w,\varphi}$ is a bounded operator on $H^2$. Let $(d_n)$ be a decreasing sequence of positive reals such
that $H^2(d)$ is automorphism-invariant.

(i) If $\|T^n_{w,\varphi}\|_{H^2} \to 0$, then $\|T^n_{w,\varphi}\|_{H^2(d)} \to 0$.

(ii) If $T^n_{w,\varphi} \to 0$ strongly on $H^2$, then $T^n_{w,\varphi} \to 0$ strongly on $H^2(d)$.

Proof. (i) We have

$$\|T^n_{w,\varphi}\|_{H^2(d)} \leq \|C_{\psi,\varphi_n(0)}\|_{H^2} \|C_{\psi,\varphi_n(0)}\|_{H^2(d)} \|T^n_{w,\varphi}\|_{H^2}.$$ 

Now $\varphi_n(0) \to \text{DW}(\varphi)$, and it remains to show that this implies that the operators $C_{\psi,\varphi_n(0)}$ are uniformly bounded on $H^2(d)$ – clearly they are uniformly bounded on $H^2$. But this follows from [12, Cor. 2.5], where the bound

$$\|C_{\psi,\alpha}\|_{H^2(d)} \leq \left(\frac{1 + |\alpha|}{1 - |\alpha|}\right)^{\Lambda/2}$$

is given, with $\Lambda$ being a constant depending on $d$.

(ii) Given that the iterates $T^n_{w,\varphi}$ converge strongly to 0 on $H^2$ it follows from the fact that $(d_n)$ is bounded that $\|T^n_{w,\varphi}f\|_{H^2(d)} \to 0$ for all polynomials $f$, which form a dense subset of $H^2(d)$. But the sequence of operators $(T^n_{w,\varphi})$ is uniformly bounded on $H^2$, and hence also on $H^2(d)$ by arguments similar to those in (i), and so $\|T^n_{w,\varphi}f\|_{H^2(d)} \to 0$ for all $f \in H^2(d)$. \qed

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