A Local Variational Theory for the Schmidt Metric

Fredrik Ståhl

Department of Mathematics, University of Umeå, Umeå, Sweden
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Abstract

We study local variations of causal curves in a space-time with respect to b-length (or generalised affine parameter length). In a convex normal neighbourhood, causal curves of maximal metric length are geodesics. Using variational arguments, we show that causal curves of minimal b-length in sufficiently small globally hyperbolic sets are geodesics. As an application we obtain a generalisation of a theorem by B. G. Schmidt, showing that the cluster curve of a partially future imprisoned, future inextendible and future b-incomplete curve must be a null geodesic. We give examples which illustrate that the cluster curve does not have to be closed or incomplete. The theory of variations developed in this work provides a starting point for a Morse theory of b-length.

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*e-mail: fredriks@abel.math.umu.se
I. INTRODUCTION

The classical singularity theorems by Hawking and Penrose predict that in a physically reasonable model of space-time, singularities will inevitably occur in the form of incomplete, inextendible causal geodesics. Unfortunately the nature of these singularities are still not fully understood. From a physical viewpoint one would expect some sort of divergence of the curvature when approaching a singularity. More precisely, we might expect divergence of some scalar polynomial in the Riemann tensor and the metric along an incomplete curve ending at the singularity, or equivalently that the Riemann tensor components are unbounded in all frames along the curve.

There are however singular space-times where the scalar polynomials are bounded, i.e. there exists some frame where the Riemann tensor components are bounded, along an incomplete inextendible causal curve. In this paper we concentrate on space-times containing b-imcomplete causal curves partially imprisoned in a compact set. The existence of imprisoned curves is equivalent to the existence of a cluster point to some causal curve, and thus implies a non-Hausdorff behaviour of the b-completion.

A well-known example of totally imprisoned inextendible incomplete curves can be found in the Taub-NUT space-time. Instead of analysing the Taub-NUT space-time directly we will consider a two-dimensional space-time with similar properties given by Misner. We will refer to this space-time as the Misner space-time. The manifold is $S^1 \times \mathbb{R}$ with the metric

$$ds^2 = 2\,dt\,d\psi + t\,d\psi^2 \quad \text{for} \quad t \in \mathbb{R} \text{ and } \psi \in [0, 2\pi).$$

The vertical lines $t = u, \psi = \text{const.}$ are complete null geodesics. In addition there are families of null geodesics with affine parameter $u$ which follow infinite spirals as they approach $t = 0$, of the type $t = Cu, \psi = -\ln u^2$ where $C$ is a non-zero constant (Fig. 1). The affine parameter is bounded on these inextendible geodesics as they approach $t = 0$ so they are in fact incomplete. The closed null geodesic given by $t = 0, \psi = -\ln u^2$ is incomplete, and there are no curvature singularities.

If we endow a neighbourhood of $t = 0$ with a continuous frame field we find that the tangent vector to one of the spiralling incomplete curves are boosted by an unbounded amount relative to the frame field as the curve approaches $t = 0$. The reason that the Riemann components remain bounded along the curve is that the Riemann tensor is highly specialised in this case, and it is likely that the situation is unstable because of the back reaction of test particle travelling along the curve.

Indeed one can show that a b-boundary point $p$ corresponding to an inextendible incomplete causal curve $\lambda$ must be a p.p. singularity, i.e. the Riemann tensor components diverge in any parallelly propagated frame along the curve, if either i) space-time is globally hyperbolic with the Riemann tensor non-specialised at $p$ or ii) $\lambda$ has a cluster point where the Riemann tensor is non-specialised.

Schmidt has found the following result concerning totally imprisoned curves:

**Theorem 1** If a future inextendible, future b-incomplete causal curve is totally future imprisoned in a compact set $\mathcal{N}$, then there is a future inextendible null geodesic totally imprisoned in $\mathcal{N}$. 
Our aim is now to develop a local theory of variations for b-length, and to use this to prove a similar theorem for partially imprisoned curves. Note that partial imprisonment is a weaker concept than total imprisonment, since a totally imprisoned curve is also partially imprisoned in a small compact neighbourhood of one of its cluster points.

The plan of the paper is as follows. In section II we discuss the appropriate definitions and section III contains the variational arguments. The generalisation of Theorem II is provided in section IV and section V is devoted to discussions and examples.

II. DEFINITIONS

Similar to the terminology in Ref. [1], \((\mathcal{M}, g)\) is a space-time consisting of a connected four-dimensional Hausdorff \(C^\infty\) manifold \(\mathcal{M}\) and a Lorentz metric \(g\) of signature \((- + + +)\). Throughout the rest of this paper, “incomplete” will mean “future incomplete”, “incomplete” is “future incomplete” and so on. This is for convenience only, since all results hold equally well for past incomplete, past inextendible curves.

Let \(\lambda(t)\) be a causal curve in \(\mathcal{M}\) from \(p = \lambda(t_p)\) to \(q = \lambda(t_q)\) with tangent vector \(V\). The metric length of \(\lambda\) is given by
\[
L(\lambda) := \int_{t_p}^{t_q} \left( -g(V, V) \right)^{1/2} dt.
\]
(1)

The generalised affine parameter length, or bundle length of \(\lambda\) with respect to a given orthonormal basis \(E_k\) is
\[
l(\lambda, E_k) := \int_{t_p}^{t_q} \left( \sum_{k=0}^{3} (V^k)^2 \right)^{1/2} dt
\]
(2)

where \(V^k\) are the components of the tangent vector \(V\) in the basis \(E_k\). Note that the b-length coincides with affine parameter length for geodesics and that for causal curves, the b-length is always greater than the metric length, i.e.
\[
l(\lambda, E_k) \geq L(\lambda).
\]

Let \(\phi : \mathcal{A} = (-\varepsilon, \varepsilon) \times [t_p, t_q] \to \mathcal{M}\) be a one-parameter causal variation of \(\lambda\) in \(\mathcal{M}\), i.e. \(\phi\) satisfies the conditions

1. \(\phi(0, t) = \lambda(t)\)

2. \(\phi\) is \(C^3\) except at a finite number of points

3. \(\phi(u, t_p) = p\) and \(\phi(u, t_q) = q\) for all \(u \in (-\varepsilon, \varepsilon)\)

4. \(\phi(u_0, t)\) is a causal curve for each \(u_0 \in (-\varepsilon, \varepsilon)\).

Starting with an orthonormal basis \(E_k\) at \(p\) with \(E_0\) timelike, we parallelly propagate \(E_k\) along \(\phi(u_0, t)\) for each constant \(u_0\), obtaining functions \(\mathcal{A} \to T_{\phi(u,t)} \mathcal{M}\), \((u, t) \mapsto E_k(u, t)\) for \(k = 0, 1, 2, 3\). We denote the tangent vectors in \(T_{(u,t)} \mathcal{A}\) with respect to \(u\) and \(t\) by \(\frac{\partial}{\partial u}\) and \(\frac{\partial}{\partial t}\) and their corresponding vectors in \(T_{\phi(u,t)} \mathcal{M}\) by \(X := \phi_* \frac{\partial}{\partial u}\) and \(V := \phi_* \frac{\partial}{\partial t}\) respectively.
The map $\phi$ is not necessarily injective, which means that the vectors $V$, $X$ and $E_0$ do not in general constitute vector fields in $\mathcal{M}$. This could prove to be a problem when we try to study the variation of $l(\lambda, E_k)$. The problem can be avoided however if we define the covariant derivatives needed as

\[ (\frac{D}{\partial t} E_0)^a := \frac{d}{dt} E_0^a + \Gamma_{bc}^a E_0^b V^c \] (3a)

\[ (\frac{D}{\partial u} E_0)^a := \frac{d}{du} E_0^a + \Gamma_{bc}^a E_0^b X^c \] (3b)

\[ (\nabla_{E_0} X)^a := X^a_{,b} E_0^b + \Gamma_{bc}^a X^b E_0^c. \] (3c)

It is then possible to do the usual calculations, in particular

\[ \frac{D}{\partial t} \frac{D}{\partial u} E_0 = R(V, X) E_0 + \frac{D}{\partial u} \frac{D}{\partial t} E_0 + \nabla_{[V,X]} E_0 = R(V, X) E_0 \] (4)

since $[V,X] = 0$ and $\frac{D}{\partial u} E_0 = 0$.

For $V, W \in T_{\phi(u,t)} \mathcal{M}$ let

\[ h_{(u,t)}(V, W) := g(V, W) + 2g(V, E_0(u, t))g(W, E_0(u, t)). \] (5)

We may then rewrite (2) as

\[ l(\lambda, E_k) := \int_{t_p}^{t_q} \left[ h_{(u,t)}(V, V) \right]^\frac{1}{2} dt \bigg|_{u=0}. \] (6)

Again, $h_{(u,t)}$ might not be a tensor field since $E_0(u, t)$ might have different values at the same point in $\mathcal{M}$. From now on we will write $h$ for $h_{(u,t)}$ and $E_0$ for $E_0(u, t)$, the dependence of $u$ and $t$ being understood. We will refer to $h$ as the Schmidt metric.

Given two timelike separated points $p$ and $q$ in $\mathcal{M}$ we denote the space of continuous causal curves from $p$ to $q$ by $C(p,q)$, and the subset of $C(p,q)$ consisting of timelike $C^1$ curves by $C'(p,q)$. On these spaces we will use the $C^0$ topology where a neighbourhood of $\gamma \in C(p,q)$ is defined by all curves in $C(p,q)$ which lie in a neighbourhood of the image of $\gamma$ in $\mathcal{M}$. Note that this topology is non-Hausdorff if strong causality is violated, since it does not distinguish how many times a closed curve is traversed. For the discussion at hand this problem is avoided since we are only interested in the structure on some small globally hyperbolic neighbourhood, and such a set can always be found around any point. In this context, $C'(p,q)$ is dense in $C(p,q)$. (Remember that a globally hyperbolic set is a strongly causal set $\mathcal{U}$ where $C(p,q)$ is compact for all points $p, q \in \mathcal{U}$.)

### III. VARIATIONAL THEORY

In this section we develop a theory of variations for $b$-length by adapting the usual procedure for metric length. We will be concerned with the properties of curves in a small globally hyperbolic set, and we will also need to restrict the set further to put bounds on the additional terms introduced by the parallelly propagated basis used in the $b$-length definition. We start by computing the first variation of (3).
Lemma 1 The first variation of \( l(\lambda, E_k) \) is
\[
\frac{d}{du} l(\lambda, E_k) = - \int_{t_p}^{t_q} f^{-1} h(\frac{D}{\partial t} V, P_{V^\perp} h) dt + 2 \int_{t_p}^{t_q} f^{-1} g(V, E_0) g(V, \frac{D}{\partial u} E_0) dt
\] (7)
where \( f := [h(V, V)]^{\frac{1}{2}} \) and \( P_{V^\perp} \) is the projection with respect to \( h \) onto the space of tangent vectors orthogonal to \( V \).

Proof. First note that from (5) and \( \mathcal{L}_X V = 0 \) we have
\[
\frac{\partial}{\partial u} [h(V, V)] = 4 g(V, E_0) g(V, \frac{D}{\partial u} E_0) + 2 h(\frac{D}{\partial t} X, V). \] (8)
We use this to rewrite \( \frac{d}{du} l(\lambda, E_k) \) as
\[
\frac{d}{du} l(\lambda, E_k) = \int_{t_p}^{t_q} \frac{\partial}{\partial u} \left( [h(V, V)]^{\frac{1}{2}} \right) dt
\]
\[
= \int_{t_p}^{t_q} f^{-1} h(\frac{D}{\partial t} X, V) dt + 2 \int_{t_p}^{t_q} f^{-1} g(V, E_0) g(V, \frac{D}{\partial u} E_0) dt.
\]
The first term can now be reformulated as a sum involving a total \( t \) derivative which vanishes because of the fixed endpoints of the variation.
\[
\int_{t_p}^{t_q} f^{-1} h(\frac{D}{\partial t} X, V) dt = \int_{t_p}^{t_q} \frac{\partial}{\partial t} \left[ f^{-1} h(X, V) \right] dt - \int_{t_p}^{t_q} f^{-1} h(\frac{D}{\partial t} V, X - f^{-2} h(X, V) V) dt.
\]
Defining the projection with respect to the Schmidt metric \( h \) as \( P_{V^\perp}^h = X - f^{-2} h(X, V) V \) we get (6). □

The first term in Lemma 1 is similar to the expression occurring in the metric case (c.f. Ref. [3]). The additional second term corresponds to the variation of the basis vector \( E_0 \) when parallelly transporting \( E_0 \) along different curves. The change in \( E_0 \) is determined by the Riemann tensor and one would expect it to be small for short curves. Indeed, we have

Lemma 2
\[
g(V, E_0) g(V, \frac{D}{\partial u} E_0) \bigg|_t = K(t) \sup_{[t_p, t_q]} \left\{ h(X, X)^{\frac{1}{2}} \right\}
\] (9)
where \( K(t) = O(t - t_p) \).

Proof. We start by rewriting the second factor on the left side as
\[
g(V, \frac{D}{\partial u} E_0) = g_{\hat{a} \hat{b}} V^{\hat{a}} E_0^{\hat{b}, \hat{c}} X^{\hat{c}}
\]
where hatted indices denotes components with respect to the basis \( E_k \). Now let
\[
r := \sup_{[t_p, t_q]} \left\{ \| R^{\hat{c}}_{\hat{a} \hat{b}} A^{\hat{a}} B^{\hat{b}} \| : A, B \in \mathbb{R}^4, \| A \| = \| B \| = 1 \right\}
\]
\[
g := \sup_{[t_p, t_q]} \left\{ |g_{ab} A^a B^b| : A, B \in \mathbb{R}^4, \|A\| = \|B\| = 1 \right\}
\]

where \(\| \cdot \|\) is the usual Euclidean norm in \(\mathbb{R}^4\). Then

\[
E_0^b \tilde{\varepsilon} X^c \bigg|_t = \int_{t_p}^t (E_0^b \tilde{\varepsilon} X^c)_t V^d dt = \int_{t_p}^t R^b_{0cd} X^c V^d dt
\]

by (8), and

\[
\left\| \int_{t_p}^t R^b_{0cd} X^c V^d dt \right\| \leq r(t - t_p) \|V\|_\lambda \|X\|_\lambda
\]

where \(\| \cdot \|_\lambda := \sup_{[t_p, t_q]} \left\{ h(\cdot, \cdot)^{\frac{1}{2}} \right\}\). It follows immediately that

\[
\left| g(V, E_0)g(V, \frac{D}{\partial t} E_0) \right| \leq r g^2 \|V\|_\lambda^2 (t - t_p) \|X\|_\lambda.
\]

\[\square\]

In the metric case, timelike geodesics are curves of extremal metric length provided there are no conjugate points along the geodesic. In order to prove a similar result for the \(b\)-length, we need to construct a variation of any non-geodesic causal curve which results in a shorter curve. This is done in the following somewhat lengthy lemma.

**Lemma 3** Let \(\lambda\) be a non-geodesic causal curve from \(p\) to \(q\). Then there is a variation of \(\lambda\) giving a causal curve from \(p\) to \(q\) with smaller \(b\)-length than \(\lambda\).

**Proof.** We want to construct a causal variation of \(\lambda\) with variation vector \(X\) such that \(\frac{\partial}{\partial u}[h(V, V)] < 0\). We proceed by the following steps:

1. Choose \(X\) such that the second term in (8) is everywhere non-positive, and strictly negative somewhere, i.e.

\[
h(\frac{D}{\partial t} X, V) \leq 0
\]

with strict inequality on some part of \(\lambda\).

2. Check that the variation is causal.

3. Show that by a suitably small variation along \(X\) and by restricting the variation to a sufficiently small portion of \(\lambda\), the second term in (8) dominates the first, i.e.

\[
g(V, E_0)g(V, \frac{D}{\partial u} E_0) < \frac{1}{2} \left| h(\frac{D}{\partial t} X, V) \right|.
\]
Case 1. We start with the case when the tangent vector $V$ is continuous. Let $\lambda$ be parameterised by b-length. Then $h(V, V) = 1$ and

$$0 = \frac{\partial}{\partial t} [h(V, V)] = 2h(V, \frac{D}{\partial t} V)$$

so $\frac{D}{\partial t} V$ is orthogonal to $V$ with respect to $h$ whenever $\frac{D}{\partial t} V$ is non-zero, which has to be the case somewhere along $\lambda$ since $\lambda$ is non-geodesic.

Let the variation vector be

$$X := xV + y\frac{D}{\partial t} V$$

where $x$ and $y$ are functions vanishing outside some interval $[\alpha, \beta] \subset [t_p, t_q]$ such that $\frac{D}{\partial t} V \neq 0$ on $[\alpha, \beta]$.

Step 1. We restrict our attention to the interval $[\alpha, \beta]$. On this interval, (12) and (13) implies

$$h\left(\frac{D}{\partial t} X, V\right) = \left. \frac{\partial}{\partial t} [h(X, V)] - h(X, \frac{D}{\partial t} V) \right|_{\alpha} = \frac{dx}{dt} - ay$$

where $a := h\left(\frac{D}{\partial t} V, \frac{D}{\partial t} V\right)$. Choose $x$ as

$$x(t) := \int_{\alpha}^{t} (ay - 1) \, dt.$$  

Then

$$h\left(\frac{D}{\partial t} X, V\right) = -1$$

for any function $y$ on $[\alpha, \beta]$ with

$$y(\alpha) = y(\beta) = \int_{\alpha}^{\beta} (ay - 1) \, dt = 0.$$  

We may choose $y$ to be positive.

Step 2. If $\lambda$ has a timelike segment, a sufficiently small variation of that segment will give a timelike curve. If $\lambda$ is null, $g(V, V) = 0$ gives

$$1 = h(V, V) = 2g(V, E_0)^2$$

so $g(V, E_0)^2 = \frac{1}{2}$. Using $\mathcal{L}_X V = 0$ and the definition of $h$ (Eq. 5) we then get

$$\frac{\partial}{\partial u} [g(V, V)] = 2 \frac{\partial}{\partial t} [g(X, V)] - 2g(X, \frac{D}{\partial t} V)$$

$$= 2 \frac{\partial}{\partial t} [h(X, V)] - 2h(X, \frac{D}{\partial t} V) - 4 \left( \frac{\partial}{\partial t} [g(X, E_0)g(V, E_0)] - g(X, E_0)g\left(\frac{D}{\partial t} V, E_0\right) \right)$$
\[ = -2 - 4 \frac{\partial}{\partial t} [g(X, E_0)] g(V, E_0) \]

where we have used \( h(\frac{D}{dt} X, V) = -1 \) to obtain the last equality. Finally, (14) implies \( g(\frac{D}{dt} V, E_0) = 0 \) so

\[ \frac{\partial}{\partial u} [g(V, V)] = -2 - 4 \frac{dx}{dt} g(V, E_0)^2 = -2ay < 0 \]

since both \( a \) and \( y \) are positive on \([\alpha, \beta]\), and hence the curve will remain causal under a small variation along \( X \).

**Step 3.** We restrict \( y \) such that \( ay \) ≤ \( 2 \) and \([\alpha, \beta]\) such that \( \beta - \alpha \leq 1 \) and \( K < \frac{1}{6} \inf_{[\alpha, \beta]} \{1, \sqrt{a}\} \) in (3) applied to \( \lambda_{|[\alpha, \beta]} \). Then Lemma 2 gives

\[ g(V, E_0)g(V, \frac{D}{\partial u} E_0) \leq K \sup_{[\alpha, \beta]} \left\{ h(X, X)^{\frac{1}{2}} \right\} \]

\[ \leq K \sup_{[\alpha, \beta]} \left\{ \sqrt{|x| + \sqrt{ay}} \right\} \leq K \sup_{[\alpha, \beta]} \left\{ |ay - 1| (\beta - \alpha) + K \sup_{[\alpha, \beta]} \sqrt{ay} \right\} \]

\[ < \frac{1}{6} + \frac{1}{6} \sup_{[\alpha, \beta]} \{|ay\} \leq \frac{1}{2}. \]

Thus \( \frac{\partial}{\partial u} [h(V, V)] < 0 \) and the lemma is true for curves with a continuous tangent vector.

**Case 2.** Suppose now that \( \lambda \) is made up of a finite number of geodesic segments. Let \( \lambda \) be parameterised by b-length which is equivalent to affine parameterisations of the geodesic segments. It is sufficient to study the case when \( V \) is discontinuous at one point \( \lambda(t_0) \). Let \( W \) be the discontinuity at \( \lambda(t_0) \), i.e.

\[ W := V_+ - V_- \]

where

\[ V_+ := \lim_{t \to t_0^+} V \quad \text{and} \quad V_- := \lim_{t \to t_0^-} V_{\lambda(t)}. \]

Parallelly propagate \( W \) along \( \lambda \). Then

\[ \frac{\partial}{\partial t} [h(W, V)] = h(\frac{D}{dt} W, V) + h(W, \frac{D}{dt} V) = 0 \]

on each geodesic section of \( \lambda \). We know that \( h(W, V)|_{\lambda(t)} = h(W, V_-)|_{\lambda(t_0)} = h(V_+, V_-) - 1 \) if \( t \in [t_p, t_0) \) and

\[ h(W, V)|_{\lambda(t)} = h(W, V_+)|_{\lambda(t_0)} = 1 - h(V_+, V_-) \]
if \( t \in (t_0, t_q) \), i.e. \( h(W, V) \) is negative on \([t_p, t_0]\) and positive on \((t_0, t_q]\). Let the variation vector be \( X := xW \) where
\[
x(t) := \begin{cases} 
-h(W, V)^{-1}(\beta - t_0)(t - \alpha) & \text{when } t \in [\alpha, t_0) \\
h(W, V)^{-1}(\beta - t)(t_0 - \alpha) & \text{when } t \in (t_0, \beta] 
\end{cases}
\]
(15)
on some interval \([\alpha, \beta] \subset [t_p, t_q]\) to be chosen below, and zero otherwise.

**Step 1.** On the interval \([\alpha, \beta]\),
\[
h(D\frac{\partial}{\partial t} X, V) = \frac{d}{dt} [xh(W, V)] = \begin{cases} 
-(\beta - t_0) & \text{when } t \in [\alpha, t_0) \\
-(t_0 - \alpha) & \text{when } t \in (t_0, \beta] 
\end{cases}
\]
which is negative.

**Step 2.** If one of the geodesic segments is null, we must ensure that the varied curve remains causal. Since \( L_X V, \frac{\partial}{\partial t} V \) and \( \frac{\partial}{\partial t} W \) all vanish we have
\[
\frac{\partial}{\partial u} [g(V, V)] = \frac{d}{dt} [xg(W, V)] = g(W, V) \frac{dx}{dt}.
\]
Suppose that \( V_- \) is null. On \([\alpha, t_0]\),
\[
g(W, V) = g(V_+, V_-) - g(V_-, V_-) = g(V_+, V_-) \leq 0
\]
since \( V_+ \) and \( V_- \) are causal vectors. But on this interval
\[
\frac{dx}{dt} = -h(W, V)^{-1}(\beta - t_0) \geq 0
\]
so \( \frac{\partial}{\partial u} [g(V, V)] \leq 0. \) The case when \( V_+ \) is null is similar.

**Step 3.** We choose \([\alpha, \beta]\) such that \( \beta - \alpha \leq 1 \) and
\[
K < \frac{1}{2\sqrt{2}} (1 - h(V_+, V_-))^\frac{1}{2}
\]
in (9). Lemma 2 gives
\[
g(V, E_0) g(V, \frac{\partial}{\partial u} E_0) \leq K \sup_{[\alpha, \beta]} \left\{ h(X, X)^\frac{1}{2} \right\} = K \sup_{[\alpha, \beta]} \left\{ |x| (2 - 2h(V_+, V_-))^\frac{1}{2} \right\}.
\]
(16)
Now we can use the definition of \( x \) (Eq. 15) to get an estimate.
\[
|x| \leq |h(V, W)|^{-1}(\beta - t_0)(t_0 - \alpha)
\]
\[
\leq K \sup_{[\alpha, \beta]} \left\{ (1 - h(V_+, V_-))^{-1} (2 - 2h(V_+, V_-))^\frac{1}{2} \right\} (\beta - t_0)(t_0 - \alpha).
\]
Substituting this into (16) gives
\[
g(V, E_0) g(V, \frac{\partial}{\partial u} E_0) \leq \frac{1}{2} (\beta - t_0)(t_0 - \alpha)
\]
so again \( \frac{\partial}{\partial u} [h(V, V)] < 0. \)

We have now established that if a causal curve has minimal b-length, it must be a geodesic. It remains to prove the existence of causal curves with minimal b-length. First we need a result on the continuity properties of the b-length.
Lemma 4 Suppose that all the curves in $C'(p,q)$ are contained in a strongly causal region. Then the b-length $l$ is lower semi-continuous in the $C^0$-topology on $C'(p,q)$.

Proof. Let $\lambda \in C'(p,q)$ be a timelike curve from $p$ to $q$, parameterised by b-length $t$ such that the map $t \mapsto \lambda(t)$ is injective. This has to be possible since otherwise strong causality would be violated. Let $f$ be a function in a neighbourhood $U$ of $\lambda$ such that $f|_{\lambda} = t$ and the surfaces of constant $f$ are spacelike and orthogonal to the tangent $V$ of $\lambda$ with respect to $h$. Any curve $\mu \in C'(p,q) \cap U$ can be parameterised by $f$, and the tangent vector of $\mu$ may be expressed as

$$\frac{\partial}{\partial f}\bigg|_{\mu} = Z + W,$$

where $Z^a = g^{ab} f_b$ and $h(Z,W) = 0$. Then

$$h \left( \frac{\partial}{\partial f}\bigg|_{\mu}, \frac{\partial}{\partial f}\bigg|_{\mu} \right) = h(Z,Z) + h(W,W) \geq h(Z,Z).$$

But $Z|_{\lambda} = V$ and $\lambda$ is parameterised by b-length, so $h(Z|_{\lambda}, Z|_{\lambda}) = 1$. Given $\epsilon > 0$ we can choose a neighbourhood $U' \subset U$ of $\lambda$ such that $h(Z,Z) > 1 - \epsilon$ on $U'$. Then for all curves $\mu$ in $U'$,

$$l(\mu, E_k) \geq \int_{t_p}^{t_q} h(Z|_{\mu}, Z|_{\mu}) \frac{1}{2} dt > \sqrt{1 - \epsilon} \int_{t_p}^{t_q} dt = \sqrt{1 - \epsilon} l(\lambda, E_k).$$

We summarise the results of this section in a theorem.

Theorem 2 If $p$ and $q$ are causally separated and belong to a globally hyperbolic set, there exists a causal curve from $p$ to $q$ with minimal b-length. Moreover, any such curve is geodesic.

Proof. $l$ is lower semi-continuous by Lemma 4 and bounded below by 0 on the closure of $C'(p,q)$, which is compact since $p$ and $q$ belong to a globally hyperbolic set. Thus there is a curve $\gamma$ in the closure of $C'(p,q)$ with minimal b-length. $\gamma$ must be geodesic since if otherwise, $\gamma$ can be varied to give a shorter curve by Lemma 3.

IV. IMPRISONED CURVES

We can now use Theorem 2 to generalise Schmidt’s theorem (Theorem 1) to partially imprisoned curves.

Theorem 3 Every cluster curve $\gamma$ of a partially imprisoned incomplete inextendible causal curve $\lambda$ is a null geodesic.
Proof. The intuitive picture is that the tangent of \( \lambda \) must become more and more null as one follows \( \lambda \) to the future. We will prove this by contradiction; if \( \gamma \) is timelike somewhere, the b-length of \( \lambda \) must be infinite.

If \( \gamma \) is not a null geodesic, we can find a small portion \( \gamma' \) of \( \gamma \) contained in a globally hyperbolic, convex normal neighbourhood \( \mathcal{N} \) such that the endpoints \( p \) and \( q \) of \( \gamma' \) are timelike separated. We can also find portions \( \lambda_i \) of \( \lambda \) contained in \( \mathcal{N} \) such that \( \gamma' \) is the limit curve of the \( \lambda_i \).

Let \( \pi \) be the b-shortest causal curve between \( p \) and \( q \) and let \( \pi_i \) be the b-shortest causal curve between the endpoints of \( \lambda_i \). Then \( \pi \) and \( \pi_i \) are geodesics by Theorem 2 and \( l(\lambda_i) \geq l(\pi_i) \geq L(\pi_i) \). In a convex normal neighbourhood, geodesics are uniquely defined as curves with maximal metric length between two points, so \( L(\pi_i) \to L(\pi) \) as \( i \to \infty \).

Now \( l(\lambda) \geq \sum_i l(\lambda_i) \) and \( l(\lambda) \) is finite since \( \lambda \) is incomplete. But \( p \) and \( q \) are timelike separated so

\[
l(\lambda_i) \geq L(\pi_i) \to L(\pi) > 0 \quad \text{as} \quad i \to \infty
\]

which is a contradiction. \( \square \)

In the Misner space-time (c.f. section I and Fig. 1) the statement of Theorem 3 is not surprising since the closed null geodesic at \( t = 0 \) is a cluster curve of totally imprisoned null geodesics. In this particular case the cluster curve is inextendible and incomplete, but that is not always true as the following example shows.

Example. Consider the manifold \( S^1 \times \mathbb{R} \) with the metric

\[
ds^2 = 2dtd\psi + t^2d\psi^2.
\]

This space-time exhibits a similar behaviour as the Misner space-time. There are complete vertical null geodesics given by \( t = u, \psi = \text{const.} \) and incomplete, inextendible spiralling geodesics approaching the cluster curve at \( t = 0 \) of the type \( t = u, \psi = 2u^{-1} \). However, in this case the closed null geodesic is complete; it is given by \( t = 0, \psi = u \) (Fig. 2).

Another property of the cluster curve in the Misner space-time is that it is imprisoned in a compact set. An example where this is not the case can be found by simply removing a point on \( t = 0 \) in the Misner space-time (Fig. 3).

Note that closed timelike curves are partially imprisoned and that closed non-geodesic causal curves can be varied to give closed timelike curves. Thus Theorem 3 implies the following.

**Corollary 1** If a closed causal curve is non-geodesic or timelike, it must be complete.

This result is intuitively clear since a closed timelike curve \( \lambda \) must be metrically complete. When parallelly propagating an orthogonal basis along \( \lambda \) we get an infinite sequence of orthogonal bases along the image of \( \lambda \), each corresponding to one “circulation” of \( \lambda \). In each of these bases, \( h(V,V) \geq g(V,V) \) where \( V \) is the tangent of \( \lambda \), with equality if and only if \( V \) is orthogonal to the timelike basis vector \( E_0 \) with respect to \( g \). Then the b-length has to be greater than or equal to the metric length which is infinite, so \( \lambda \) must be complete.
V. CONCLUDING REMARKS

The variational arguments used in this article are of a very local nature, and we have focused on the extremal properties of geodesics in small neighbourhoods. The techniques are based on variations in a globally hyperbolic neighbourhood small enough to constrain the second term in the first variation of the b-length. The next logical step is to shift attention to the global connection between geodesic behaviour and the b-length, i.e. to construct a Morse theory for the Schmidt metric similar to the Morse theory for metric length. The relation of conjugate points to extrema of the b-length is of particular interest. This calls for some modifications of the techniques used in this article. For example, causality might not hold in \( \mathcal{M} \), which has the consequence that the topology on \( C(p, q) \) is no longer Hausdorff. A bigger problem is that in general, we expect that the curvature will have a severe impact on variations on a larger scale, since the parallel propagation of the basis used in the definition of b-length might give a large difference in b-length even for small variations. This is subject to further study.

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FIG. 1. Misner’s two-dimensional space-time. In region A causality is preserved but in region B there are closed timelike curves through every point. The horizon is generated by the closed incomplete null geodesic at $t = 0$.

FIG. 2. A two-dimensional space-time with a complete cluster curve at $t = 0$. Causality is preserved in both regions A and B, but violated by the closed complete null geodesic at $t = 0$. 