NOTES ON FUNDAMENTAL ALGEBRAIC SUPERGEOMETRY.

HILBERT AND PICARD SUPERSCHEMES

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Abstract. These notes aim at providing a complete and systematic account of some foundational aspects of algebraic supergeometry, namely, the extension to the geometry of superschemes of many classical notions, techniques and results that make up the general backbone of algebraic geometry, most of them originating from Grothendieck’s work. In particular, we extend to algebraic supergeometry such notions as projective and proper morphisms, finiteness of the cohomology, vector and projective bundles, cohomology base change, semi-continuity theorems, relative duality, Castelnuovo-Mumford regularity, flattening, Hilbert and Quot schemes, faithfully flat descent, quotient étale relations (notably, Picard schemes), among others. Some results may be found elsewhere, and, in particular, there is some overlap with [49]. However, many techniques and constructions are presented here for the first time, notably, a first development of Grothendieck relative duality for proper morphisms of superschemes, the construction of the Hilbert superscheme in a more general situation than the one already known (which in particular allows one to treat the case of sub-superschemes of supergrassmannians), and a rigorous construction of the Picard superscheme for a locally superprojective morphism of noetherian superschemes with geometrically integral fibres. Moreover, some of the proofs given here are new as well, even when restricted to ordinary schemes. In a final section we construct a period map from an open substack of the moduli of proper and smooth supercurves to the moduli stack of principally polarized abelian superschemes.

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1. Introduction

The introduction of a geometry that encompasses both even (bosonic) and odd (fermionic) coordinates was motivated by the supersymmetric field theories that were formulated in the 1970s and 1980s. After the first notions introduced by physicists (see e.g. [59]), more mathematically sound theories were proposed, where different kinds of “supermanifolds” were considered (see [5] and references therein). The Berezin-Leĭtes approach [6], also developed by Kostant [39] and Manin [44, 45, 46, 47] among others, closer in spirit to algebraic geometry, has become the standard approach to superschemes. As in classical algebraic geometry, superschemes are locally ringed spaces which locally look like spectra of rings, the only basic difference being that the rings are $\mathbb{Z}_2$-graded commutative.

Many objects have been introduced and studied in algebraic supergeometry, as, for instance, super Riemann surfaces, or supersymmetric (SUSY) curves [26, 4, 44], and their supermoduli spaces, which are supposed to be relevant to the perturbative approach to superstring theories. Supermoduli spaces were constructed by LeBrun and Rothstein [42] as superorbifolds (see also [14]). In [18] a supermoduli space for SUSY curves with Neveu-Schwarz (NS) punctures was constructed as an Artin algebraic superspace, that is, as a quotient of an étale equivalence relation of superschemes. The interest in this topic was revived by the works of Witten [65] and Donagi-Witten [19, 20]. A relatively recent paper is [13], which contains a construction of the supermoduli of SUSY curves as a Deligne-Mumford (DM) superstack, together with a new theory of stacks in algebraic supergeometry, or superstacks. A supermoduli space of SUSY curves with both Neveu-Schwarz and Ramond-Ramond punctures was constructed in [9], again as an Artin algebraic superspace, and in [49] as a DM superstack. The last paper also proves the existence of a superstack which compatifies the supermoduli of SUSY curves with any kind of punctures. Moreover, substantial work has been done about superperiods and the generalization of the Mumford formula to the supermoduli of SUSY curves [65, 16, 13, 23, 24] and about the relationship of supermoduli problems with relevant aspects of string theory [49]. All these developments have required an increasing amount of tools in algebraic supergeometry, which, in most cases, have been introduced and studied by the corresponding authors according to their contingent needs.

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1 This paper was published in 2019 but first appeared in 2012 as preprint arXiv:1209.2459.
Goal of the paper. There exist some accounts of different aspects of algebraic supergeometry and of one of its basic ingredients, namely, the algebra of the $\mathbb{Z}_2$-commutative rings. Some elements were given in [5]; moreover, one can mention Westra’s dissertation [64], Carmeli, Caston and Fioresi’s work (eg. [11]) and Cacciatori and Noja’s paper [10]. However, a systematic treatment of algebraic supergeometry in the spirit of Grothendieck’s work, as laid down in Grothendieck’s FGA [29] and in the EGA series, is still missing. That should include superprojective and proper morphisms of superschemes, finiteness of cohomology, supervector and projective superbundles, cohomology base change, semicontinuity theorems, relative duality, Castelnuovo-Mumford regularity, flattening, Hilbert and Quot superschemes, faithfully flat descent, quotients of étale relations, Picard superschemes, and more. The scope of these notes is to give a contribution to the building of such a systematic approach. The extension of some classical results to supergeometry is sometimes very easy, while sometimes requires quite a bit of work. In other cases it is better, or simply unavoidable, to develop the constructions from scratch, often using the original ideas but exploiting new techniques. In doing so, some generalizations of the classical statements or simplifications of their proofs are achieved. Some results about the algebra of graded commutative rings are necessary, as are suitable extensions to the super case of theorems like the existence of the Grothendieck-Mumford complex or the super Nakayama Lemma for half exact functors.

While we were preparing these notes the preprint [49] by Moosavian and Zhou appeared. It contains some results and constructions given here up to Section 4, so there is some overlap between the two works. Also, Jang in [36] constructed and studied the Hilbert superscheme of 0-dimensional subspaces of a $(1,1)$-dimensional supercurve. We address here many topics not covered in [49], and some proofs and approaches are different (they are different also in the classical case). Moreover, we prove the representability of the super Hilbert functor under more general assumptions than in [49]. This added generality is needed to prove the representability of the super Picard functor, which is done here for the first time.

It is important to note here that the assumption of projectivity, which is quite natural in the classical algebraic geometry, is rather restrictive in supergeometry. As is well known, many interesting superschemes cannot be embedded into superprojective spaces, for instance, supergrassmannians in general are not superprojective [58]. It seems however that all interesting superschemes (with projective bosonization) can be embedded into supergrassmannians. For example, we were informed by Vera Serganova that this is true for all proper homogeneous superspaces. In this paper we make a useful observation that in some sense a superscheme embeddable into a supergrassmannian is very close to being superprojective. Namely, we show that there always exists a smooth surjective morphism of purely odd relative dimension from a superprojective scheme to such a scheme (see Corollary 2.30). Because of this, projective superschemes are still of great significance, and in particular, we first prove the existence of the Hilbert superscheme for them and then deduce it for more general superschemes.

Structure of the paper. In Section 2 we introduce and study projective superschemes by means of the projective superspectrum $\text{Proj}$ of a bigraded algebra over a superring, that is, an algebra which has two compatible gradings, one over $\mathbb{Z}$, as in the classical case, and another over $\mathbb{Z}_2$. Projective superschemes have been considered in other papers, such as [10, 49], where they were called “superprojective superschemes.” In subsection 2.7 we compute the cohomology of the sheaves $\mathcal{O}(p)$ and prove the finiteness of the cohomology of coherent sheaves on projective superspaces, as well as the super extension of some classical facts about them, such as Serre’s Theorem 2.35. Some of these results were already proved in [10], however

\footnote{See also [22] as a kind of update to FGA.}
our method does not rely on computations for the classical case, but rather we extend from
scratch Grothendieck’s methods to algebraic supergeometry. We also introduce supervector
bundles and projective superbundles, or more generally, linear superspaces and projective
superschemes associated to coherent sheaves as the superspectrum and the projective super-
spectrum, respectively, of the (graded) symmetric algebra of the corresponding module.
Supergrassmannians are also introduced and the proof of their existence is referred to Manin’s
original treatment \[46\].

In Section 3 we extend to supergeometry the classical results on the relative cohomology of
coherent sheaves with respect to proper morphisms of locally noetherian superschemes. The
main statements are the cohomology base change Theorem 3.4, the semicontinuity Theorem
3.9 and Grauert’s Theorem 3.11. Our treatment is simpler than others available in the
literature (e.g. [34]) due to the use of the Nakayama Lemma for half exact functors, which
we extend to supergeometry in Subsection A.4. In particular, the use of the Formal Function
Theorem ([34] or [49, Thm. 7.4] in the super setting) is no more necessary. Section 3 also
includes a treatment of Grothendieck’s relative duality for superschemes. This differs in some
parts from the treatments of Grothendieck’s duality usually found in the literature. We also
include a proof that the relative dualizing complex of a smooth proper morphism is a shift of
the relative Berezinian sheaf.

Section 4 is devoted to the proof of the existence of the Hilbert superscheme, following an
analogous procedure as in the classical case (see e.g. the original exposition by Grothendieck
[29] or [22, 53]). So the strategy is first to apply the super version of Castelnuovo-Mumford
regularity (Subsection 4.3) to embed the super Hilbert functor into the functor of points of
a supergrassmannian, and then prove, using flattening (Subsection 4.4), that this embedding
is representable by immersions, so that the Hilbert superscheme is a sub-superscheme of a
supergrassmannian, which is proved to be proper by the valuative criterion for properness.
We first prove the existence of the Hilbert superscheme in the superprojective case (Theorem
4.3) and then we also give a more general version of this result (Theorem 4.4), which allows
one to treat also the case of sub-superschemes of a supergrassmannian, and will be needed
to prove the existence of the Picard superscheme. As we discuss in a simple example, the
ordinary scheme underlying a Hilbert superscheme has a richer structure than the usual
Hilbert scheme.

As an application, in Section 4.8 the superscheme of morphisms between two projective
superschemes is constructed. The existence of this superscheme is necessary to prove that
the supermoduli of SUSY curves is not only a category fibred in groupoids, but it is also
a Deligne-Mumford stack [49, 7.3] (this is also true in the presence of Neveu-Schwarz and
Ramond-Ramond punctures, and for the compactified supermoduli). This section is quite
similar to [49, 7.1.8]. In Section 4.9 we prove that a superscheme is embeddable into some
supergrassmannian if and only if it is a target of a smooth morphism of odd relative dimension
from a superprojective superscheme.

Section 5 offers the first construction of the (relative) Picard superscheme for a superpro-
jective morphism of noetherian superschemes that are cohomologically flat in dimension 0
and whose bosonic fibres are geometrically integral schemes. The total Picard superscheme
is the disjoint union of the Picard superschemes parameterizing the even and the odd line
bundles on the fibres, and it suffices to construct explicitly the even one. The proof mostly fol-
lows Grothendieck’s construction [29], or more precisely Kleiman’s version of it [37], however,
to overcome some difficulties arising from the fact that we are working with superschemes,
sometimes we need to take a different route. We also take advantage of a significant simplification, which can also be used when constructing the classical Picard scheme. Our procedure consists in proving the existence of the open subfunctor of the even super Picard sheaf that parameterizes (relatively) acyclic even line bundles generated by their global sections, and the proof that translating them one obtains a covering of the even Picard sheaf by representable open subfunctors, so that it is representable as well. In this way one avoids the theory of $(b)$-sheaves used in the classical case to prove that the Picard functors with fixed Hilbert polynomials are representable.

The final part of Section 5 discusses examples where the conditions for the existence of the Picard superscheme can be relaxed when the base superscheme is even affine as well as examples where the super Picard functor is not representable.

In Section 6, as an application, we construct a period map from an open substack of the moduli of proper and smooth supercurves to the moduli stack of principally polarized abelian superchemes.

Finally, we have gathered in the Appendix some auxiliary results, which are partly original.

We assume that the reader is familiar with the basic definitions and notions of algebraic supergeometry. However, for convenience in Subsection A.5 we recall some properties of morphisms of superschemes.

Just to mention a few references that have been important to us in connection with algebraic supergeometry, we would like to cite Kostant’s paper [39], Manin’s book [46], the early papers by Penkov and Skornyakov [57, 58], the above mentioned paper by Le Brun and Rothstein [42], the study of superprojective embeddings by LeBrun, Poon and Wells [41], Deligne’s letter [15] and the already cited works of Donagi and Witten [19, 20, 65]. We also used some basic results in superalgebra and supergeometry by Cacciatori and Noja [10], Carmeli, Caston and Fioresi [11], and Westra [64].

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2. Basic algebraic supergeometry

In this section we collect some basic definitions in supergeometry.

We adopt the following convention: for every $\mathbb{Z}_2$-graded module $M$, we write $M_+$ for its even part and $M_-$ for its odd part, so that $M = M_+ \oplus M_-$. 

2.1. Superrings. By a superring $\mathbb{A}$ we mean a $\mathbb{Z}_2$-graded supercommutative ring such that one of the following equivalent conditions is satisfied, where $J$ is the ideal generated by the odd elements:

(1) $J$ is finitely generated;

(2) $J^n = 0$ for some $n > 0$ and $J/J^2$ is a finitely generated module over $A = \mathbb{A}/J$.

Note that under these conditions the graded ring $Gr_J(\mathbb{A}) = \bigoplus_{i \geq 0} J^i/J^{i+1}$ is finitely generated as an $A$-module.

We shall say that $A = \mathbb{A}/J$ is the bosonic reduction of $\mathbb{A}$. We also say that $\mathbb{A}$ is split if there exists a finitely generated projective $A$-module $M$ such that $\mathbb{A} \simeq \bigwedge_A M$.

We define the odd dimension of $\mathbb{A}$ as the smallest number odd-dim($\mathbb{A}$) of generators of the ideal $J$, or, equivalently, the smallest number of generators of the $A$-module $J/J^2$. 
There is actually another odd dimension parameter one may consider, i.e., the smallest integer number $p$ such that $J^{p+1} = 0$. We write $\text{ord}(\mathbb{A})$ for this number. One easily sees that $\text{ord}(\mathbb{A}) = \text{odd-dim}(\mathbb{A})$ and the two numbers are coincide when $\mathbb{A}$ is split.

The notion of noetherian superring generalizes the usual one, i.e., every ascending chain of $\mathbb{Z}_2$-graded ideals stabilizes [64]. For a noetherian local superring $\mathbb{A}$ with maximal ideal $m$, the odd dimension $\text{odd-dim}(\mathbb{A})$ of $\mathbb{A}$ is the odd dimension of $m/m^2$ as a graded $\mathbb{A}/m$-vector space [64, 7.1.4].

2.2. Superschemes. We recall the notion of superscheme. All schemes and superschemes in these notes are locally noetherian and all superscheme morphisms are locally of finite type.

Definition 2.1. A locally ringed superspace is a pair $\mathscr{X} = (X, \mathcal{O}_X)$, where $X$ is a topological space, and $\mathcal{O}_X$ is a sheaf of $\mathbb{Z}_2$-graded commutative rings such that for every point $x \in X$ the stalk $\mathcal{O}_{\mathscr{X}, x}$ is a local superring.

Definition 2.2. A morphism of locally ringed superspaces is a pair $(f, f^2)$, where $f: X \to Y$ is a continuous map, and $f^2: \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a homogeneous morphism of graded commutative sheaves, such that for every point $x \in X$, the induced morphism of local superrings $\mathcal{O}_{\mathscr{X}, f(x)} \to \mathcal{O}_{\mathscr{Y}, x}$ is local.

Given a locally ringed superspace $\mathscr{X} = (X, \mathcal{O}_X)$, we can consider the homogeneous ideal $J = (\mathcal{O}_{\mathscr{X}, -})^2 \oplus \mathcal{O}_{\mathscr{X}, -}$ generated by the odd elements. Then $\mathcal{O}_X := \mathcal{O}_X/J$ is a purely even sheaf of rings. We say that the locally ringed space $X = (X, \mathcal{O}_X)$ is the ordinary locally ringed space underlying $\mathscr{X}$. Sometimes we also call it the bosonic reduction of $\mathscr{X}$ and denote as $\mathscr{X}_{\text{bos}}$.

There is a closed immersion of locally ringed superspaces

$$i: X \hookrightarrow \mathscr{X}$$

induced by the epimorphism $\mathcal{O}_{\mathscr{X}} \to \mathcal{O}_X$. $\mathscr{X}$ is said to be projected if there exists a morphism of locally ringed superspaces $\rho: \mathscr{X} \to X$ such that $\rho \circ i = \text{Id}$. As in the case of supersurfaces, we said that $\mathscr{X}$ is split when $\mathcal{O}_{\mathscr{X}} \cong \bigwedge_{\mathcal{O}_X} \mathcal{E}$ for a locally free sheaf $\mathcal{E}$ on $X$ which generates the ideal $J$, so that $\mathcal{E} \cong J/J^2$. Analogously, $\mathscr{X}$ is called locally split if it can be covered by split open locally ringed superspaces.

The group $\Gamma = \{\pm 1\}$ acts on $\mathcal{O}_{\mathscr{X}}$ by $f(-1) = f_+ - f_-$ and defines another locally ringed superspace $\mathscr{X}/\Gamma = (X, \mathcal{O}_{\mathscr{X}}, \Gamma)$, called the bosonic quotient of $\mathscr{X}$. A morphism $f: \mathscr{X} \to \mathcal{Y}$ of locally ringed superspaces induces a morphism $\rho: X \to Y$ of the underlying locally ringed spaces and a morphism $f/\Gamma: \mathscr{X}/\Gamma \to \mathcal{Y}/\Gamma$ between the bosonic quotients, so that

$$\mathscr{X} = (X, \mathcal{O}_X) \mapsto (X, \mathcal{O}_X), \quad \mathscr{X} \mapsto \mathscr{X}/\Gamma$$

are functors. In fact, it is easy to see that these functors are the right and left adjoint functors, respectively, to the natural inclusion from the category of (purely even) locally ringed spaces to that of locally ringed superspaces: for a locally ringed superspace $Y$, we have functorial isomorphisms

$$\text{Mor}(Y, \mathscr{X}) \cong \text{Mor}(Y, X), \quad \text{Mor}(\mathscr{X}, Y) \cong \text{Mor}(\mathscr{X}/\Gamma, Y).$$

The sheaves $\text{Gr}^j(\mathcal{O}_{\mathscr{X}}) = J^j/J^{j+1}$ are annihilated by $J$ so that they are $\mathcal{O}_X$-modules. Then we can consider the sheaf of $\mathcal{O}_X$-modules

$$\text{Gr}(\mathcal{O}_{\mathscr{X}}) = \bigoplus_{j \geq 0} \text{Gr}^j(\mathcal{O}_{\mathscr{X}}) = \bigoplus_{j \geq 0} J^j/J^{j+1}$$
which comes with a natural \( \mathbb{Z}_2 \) grading.

The **superspectrum** of a superring \( \mathcal{A} \) is the locally ringed superspace \( \text{Spec} \mathcal{A} = (X, \mathcal{O}) \), where \( X \) is the spectrum of the bosonic reduction \( A \) of \( \mathcal{A} \), and \( \mathcal{O} \) is a sheaf of \( \mathbb{Z}_2 \)-graded commutative rings defined as follows: any non-nilpotent element \( f \in \mathcal{A} \) defines in the usual way a basic open subset \( D(f) \subset X \), and one defines \( \mathcal{O}_f(D(f)) = \mathcal{A}_f \), the localization of \( \mathcal{A} \) at the multiplicative subsystem defined by \( f \). The locally ringed superspaces of this form are called affine superschemes.

**Definition 2.3.** A superscheme is a locally ringed superspace \( \mathcal{X} = (X, \mathcal{O}_X) \) which is locally isomorphic to the superspectrum of a superring.

A superscheme \( \mathcal{X} = (X, \mathcal{O}_X) \) is noetherian if \( X \) has a finite open cover \( \{U\} \) such that every restriction \( \mathcal{X}|_U \) is the superspectrum of a noetherian superring.

It is easy to see that

1. the bosonic reduction and the bosonic quotient of a superscheme are usual schemes;
2. \( \text{Gr}(\mathcal{O}_X) \) is coherent as an \( \mathcal{O}_X \)-module.

**Definition 2.4.** The odd dimension of a superscheme \( \mathcal{X} \) is the supremum of the odd dimensions of the local superrings \( \mathcal{O}_{\mathcal{X},x} \) for all the points \( x \in X \). The even dimension of \( \mathcal{X} \) is the dimension of the scheme \( X \). Both dimensions may be infinite. The dimension of \( \mathcal{X} \) is the pair \( \dim \mathcal{X} = (\text{even-dim } \mathcal{X}, \text{odd-dim } \mathcal{X}) \). We say that a morphism \( f : \mathcal{X} \to \mathcal{Y} \) of superschemes has relative dimension \((m,n)\) if the fibres \( \mathcal{X}_x \) are superschemes of dimension \((m,n)\).

For a locally split superscheme the odd dimension equals the rank of the locally free \( \mathcal{O}_X \)-module \( \mathcal{J}/\mathcal{J}^2 \).

A \( \mathbb{Z}_2 \)-graded sheaf \( \mathcal{M} \) of \( \mathcal{O}_X \)-modules on a superscheme \( \mathcal{X} = (X, \mathcal{O}_X) \) can be regarded as a sheaf of abelian groups on \( X \), and the cohomology groups of \( \mathcal{M} \) as an \( \mathcal{O}_X \)-modules are the same as its cohomology groups as an abelian sheaf, so that the usual cohomology vanishing \([28, 34]\) also holds for \( \mathbb{Z}_2 \)-graded sheaves of \( \mathcal{O}_X \)-modules.

**Proposition 2.5.** If \( \mathcal{X} = (X, \mathcal{O}_X) \) is a noetherian superscheme, and \( \mathcal{M} \) a \( \mathbb{Z}_2 \)-graded sheaf of \( \mathcal{O}_X \)-modules, then \( H^i(X, \mathcal{M}) = 0 \) for every \( i > \dim X \). \( \square \)

### 2.3. Projective superspaces and quasicoherent sheaves on them.

Let \( \mathbb{P}^m_Z \) be the projective \( m \)-space over \( \mathbb{Z} \), i.e., \( \mathbb{P}^m_Z = \text{Proj} \mathbb{Z}[x_0, \ldots, x_m] \to \text{Spec} \mathbb{Z} \).

**Definition 2.6.** The projective \((m,n)\)-superspace over \( \mathbb{Z} \) is the split superspace over \( \mathbb{Z} \)

\[
\mathbb{P}^{m,n}_Z = (\mathbb{P}^m_Z, \wedge_{\mathcal{O}_{\mathbb{P}^m_Z}}(\mathcal{O}_{\mathbb{P}^m_Z}(-1))^{\oplus n}.
\]

If \( \mathcal{J} \) is a superscheme, the projective \((m,n)\)-superspace over \( \mathcal{J} \) is

\[
\mathbb{P}^{m,n}_\mathcal{J} = \mathbb{P}^{m,n}_Z \times_{\text{Spec} \mathbb{Z}} \mathcal{J}.
\]

When \( \mathcal{J} = \text{Spec} \mathcal{A} \) is the superspectrum of a superring \( \mathcal{A} \), we also use the notation \( \mathbb{P}^{m,n}_\mathcal{A} \).

Its underlying scheme is the projective \( m \)-space \( \mathbb{P}^m_\mathcal{A} = \text{Proj} \mathcal{A}[x_0, \ldots, x_m] \), with \( \mathcal{A} = \mathcal{A}/\mathcal{J}_\mathcal{A} \), where \( \mathcal{J}_\mathcal{A} \) is the ideal generated by the odd elements.

One can define a bigraded \( \mathcal{A} \)-algebra

\[
\mathbb{B}(m,n) = \mathcal{A}[x_0, \ldots, x_m, \theta_1, \ldots, \theta_n]
\] (2.1)
where the \( x_i \) are even and the \( \theta_J \) are odd, and all variables are free of \( \mathbb{Z} \)-degree 1. It has both a \( \mathbb{Z} \)-grading and a \( \mathbb{Z}_2 \)-grading. We call it the free polynomial superalgebra. Let \( \mathbb{B} = \mathbb{B}(m, n)/H = \Lambda[\xi_0, \ldots, \xi_m, \eta_1, \ldots, \eta_n] \), where \( H \) is an ideal, homogeneous for both gradings. Then, mimicking the construction of the projective spectrum, we can define a projective superspectrum \( \mathcal{X} = \text{Proj}_A \mathbb{B} = (X, \mathcal{O}) \) by letting

- \( X = \text{Proj} A[\xi_0, \ldots, \xi_m] \) (it is a projective scheme over \( A \)).
- The structure sheaf \( \mathcal{O} \) is defined by \( \mathbb{Z} \)-homogeneous localization: on the open set \( U_i = X - (\xi_i)_{0} \),
  \[
  \mathcal{O}(U_i) = \left\{ \frac{P_q(\xi_j, \eta_j)}{\xi_i^q} \mid P_q \text{ is \( \mathbb{Z} \)-homogeneous of degree } q \right\}.
  \] (2.2)

The following properties are standard.

**Proposition 2.7.** \( \text{Proj}_A \mathbb{B} \) is projected. Moreover, if \( \mathbb{B} = \mathbb{B}(m, n) \), we recover the projective superspace \( \text{Proj}_A \mathbb{B}(m, n) \cong \mathbb{P}^{m, n}_{A} \), which is split. \( \square \)

The homogeneous localization can be used, as in the classical (nonsuper) case, to construct quasi-coherent sheaves.

**Definition 2.8.** Let \( M \) be a bigraded \( \mathbb{B} \)-module. The sheaf of \( \mathbb{Z} \)-homogeneous localizations of \( M \) is the sheaf \( M^h \) associated to the presheaf whose sections on \( U_i = X - (\xi_i)_{0} \) as above are given by

\[
M^h_{\xi_i} := \left\{ \frac{m_q}{\xi_i^q} \mid m_q \in M \text{ is } \mathbb{Z} \text{-homogeneous of degree } q \right\}.
\]

It is a quasi-coherent sheaf of \( \mathcal{O} \)-modules.

Given a bigraded module \( M \), we denote by \( M_q \) the homogeneous component of \( \mathbb{Z} \)-degree \( q \). For any integer \( r \) we define a new bigraded module \( M(r) \) by shifting the \( \mathbb{Z} \) grading by \( -r \), that is, \( M(r)_s = M_{s+r} \).

**Definition 2.9.** For any integer \( r \) the sheaf \( \mathcal{O}(r) \) on \( \mathcal{X} = \text{Proj}_A \mathbb{B} \) is defined as

\[
\mathcal{O}(r) := \mathbb{B}^h(r).
\]

It is a line bundle of rank \( (1, 0) \). For any quasi-coherent \( \mathbb{Z}_2 \)-graded sheaf \( \mathcal{M} \) on \( \mathcal{X} \), we define

\[
\mathcal{M}(r) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{O}(r).
\]

Note that the restriction to \( \mathcal{X} \hookrightarrow \mathbb{P} = \mathbb{P}^{m, n}_{A} \) of the sheaf \( \mathcal{O}(r) \) is the sheaf \( \mathcal{O}_{\mathbb{P}}(r) \).

All the results in the remainder of this subsection are proved as in the classical case.

**Proposition 2.10.**

1. If \( M = \mathbb{B}^{p, q} = \mathbb{B}^{p} \oplus \mathbb{B}^{q} \), then \( \mathbb{M}^h = \mathcal{O}^{p, q} \).

2. For any bigraded module and any integer \( r \), there is an isomorphism \( \mathbb{M}[r]^h \cong \mathbb{M}^h(r) \).

3. If \( M_r = 0 \) for \( r > 0 \), then \( \mathbb{M}^h = 0 \).

4. Any bihomogeneous morphism of \( \mathbb{A} \)-modules \( N \rightarrow M \) induces a morphism \( \mathbb{N}^h \rightarrow \mathbb{M}^h \) of \( \mathcal{O} \)-modules. Moreover, if \( 0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0 \) is an exact sequence of bihomogeneous morphisms, the sequence \( 0 \rightarrow \mathbb{N}^h \rightarrow \mathbb{M}^h \rightarrow \mathbb{P}^h \rightarrow 0 \) is also exact.

5. If a bihomogeneous morphism \( N \rightarrow M \) of \( \mathbb{A} \)-modules induces an isomorphism \( N_r \cong M_r \) for \( r > 0 \), then \( \mathbb{N}^h \rightarrow \mathbb{M}^h \) is an isomorphism of sheaves.

Assume now that \( \mathbb{A} \) is noetherian. Then one has:
(6) If $M$ is finitely generated, then $\tilde{\mathcal{M}}^h$ is coherent.

(7) If $M$ is finitely generated and $\tilde{\mathcal{M}}^h = 0$, then $M_r = 0$ for $r \gg 0$.

(8) If a bihomogeneous morphism $N \to M$ of $\mathbb{A}$-modules induces an isomorphism of sheaves $\tilde{N}^h \cong \tilde{M}^h$, then $M_r \cong N_r$ for $r \gg 0$. □

For every quasi-coherent sheaf $\mathcal{M}$ of $(\mathbb{Z}_2\text{-graded}) \mathcal{O}$-modules, one can define a bigraded $\mathbb{A}$-module by

$$\Gamma_s(\mathcal{M}) = \bigoplus_{r \geq 0} \Gamma(X, \mathcal{M}(r)).$$

**Proposition 2.11.**

1. There is an isomorphism

$$\tilde{\Gamma}_s(\mathcal{M})^h \cong \mathcal{M},$$

that is, every quasi-coherent sheaf on a projective superspectrum $\mathcal{X}$ can be obtained by $\mathbb{Z}$-homogeneous localization of a bigraded $\mathbb{A}$-module.

2. If $\Gamma(X, \mathcal{M}(r)) = 0$ for $r \gg 0$, then $\mathcal{M} = 0$.

3. Let $g: \mathcal{M} \to \mathcal{N}$ be a morphism of quasi-coherent sheaves on $\mathcal{X}$. If the induced morphism $\Gamma(X, \mathcal{M}(r)) \to \Gamma(X, \mathcal{N}(r))$ is an isomorphism for $r \gg 0$, then $g$ is an isomorphism, $g: \mathcal{M} \cong \mathcal{N}$. (Note that different bigraded $B$-modules can induce the same quasi-coherent sheaf.)

If $\mathbb{A}$ is noetherian, and $\mathcal{M}$ is a coherent sheaf on $\mathcal{X}$, one also has:

4. for $r \gg 0$, the sheaf $\mathcal{M}(r)$ is generated by its global sections, i.e., the natural morphism $\Gamma(X, \mathcal{M}(r)) \otimes_{\mathbb{A}} \mathcal{O} \to \mathcal{M}(r)$ is an epimorphism.

5. $\mathcal{M}$ is the cokernel of a morphism of locally free sheaves; more precisely, there is an exact sequence

$$\bigoplus_{1 \leq j \leq M} \mathcal{O}(-r_j) \oplus \bigoplus_{1 \leq j' \leq M'} \Pi \mathcal{O}(-r'_j) \to \bigoplus_{1 \leq i \leq N} \mathcal{O}(-r_i) \oplus \bigoplus_{1 \leq i' \leq N'} \Pi \mathcal{O}(-r'_i) \to \mathcal{M} \to 0.$$

2.4. **Supervisor bundles, projective superbundles and supergrassmannians.** Grothendieck’s classical definition of vector bundle associated with a locally free sheaf generalizes directly to superschemes.

Let $\mathcal{M}$ be a coherent sheaf on a superscheme $\mathcal{X}$. For every affine open sub-superscheme $\mathcal{U} = \text{Spec } \mathbb{A}_U$ of $\mathcal{X}$, one has the graded-commutative $\mathbb{A}_U$-algebra $\text{Sym } \mathbb{A}_U, \mathcal{M}_U$, where $\mathcal{M}_U = \Gamma(U, \mathcal{M})$, and an affine superscheme $\text{Spec } (\text{Sym } \mathbb{A}_U, \mathcal{M}_U)$ over $\mathbb{A}_U$. For a covering of $\mathcal{X}$ by affine open sub-superschemes, the symmetric algebras glue together to give a sheaf $\text{Sym } \mathcal{M}$ of graded-commutative $\mathcal{O}_\mathcal{X}$-algebras and the corresponding superspectra glue to give a relatively affine superscheme $\text{Spec } (\text{Sym } \mathfrak{M}_\mathcal{X}) \to \mathcal{X}$ over $\mathcal{X}$.

**Definition 2.12.** The linear superscheme associated to $\mathcal{M}$ is the $\mathcal{X}$-superscheme

$$\pi: \mathbb{V}(\mathcal{M}) = \text{Spec } (\text{Sym } \mathfrak{M}) \to \mathcal{X}.$$

The supervisor bundle associated to a locally free sheaf $\mathcal{M}$ is the linear superscheme

$$\tilde{\mathbb{V}}(\mathcal{M}) := \mathbb{V}(\mathcal{M}^*) \to \mathcal{X},$$

associated to its dual.

For every $\mathcal{X}$ superscheme $\phi: \mathcal{T} \to \mathcal{X}$ one has

$$\text{Sym } \mathfrak{M}_\mathcal{T}(\mathcal{M}_\mathcal{T}) \simeq \text{Sym } \mathcal{M}_{\mathcal{X}}\mathcal{X}_{\mathcal{T}}, \tag{2.3}$$
where we denote, as it is customary, the pull-back by $\phi$ by a subscript $\mathcal{T}$. One then has:

$$V(M) \times \mathcal{T} = V(M)_{\mathcal{T}} \simeq V(M_{\mathcal{T}}),$$

(2.4)

that is, the formation of the linear superscheme $V(M)$ is compatible with base change.

The functor of points of the supervector bundle associated to a locally free sheaf can be easily described. Note that

$$\tilde{V}(M)^*(\mathcal{T}) = \text{Hom}_{\mathcal{T}}(\mathcal{T}, V(M^*)) = \Gamma(\tilde{V}(M)_{\mathcal{T}}/\mathcal{T}),$$

where $\Gamma(\tilde{V}(M)_{\mathcal{T}}/\mathcal{T})$ is the set of sections $\sigma$ of the projection $\pi_\mathcal{T}: \tilde{V}(M)_{\mathcal{T}} \to \mathcal{T}$, so that there is a diagram

$$\begin{array}{ccc}
\tilde{V}(M)_{\mathcal{T}} & \xrightarrow{\phi} & \tilde{V}(M) \\
\downarrow{\sigma} & & \downarrow{\pi} \\
\mathcal{T} & \xrightarrow{\phi} & \mathcal{T}
\end{array}$$

One has:

**Proposition 2.13.** If $M$ is a locally free sheaf on $\mathcal{X}$, one has

$$\tilde{V}(M)^*(\mathcal{T}) = \Gamma(\tilde{V}(M)_{\mathcal{T}}/\mathcal{T}) \simeq \Gamma(\mathcal{T}, M_{\mathcal{T}}).$$

In particular, for every point $s \in S$, the fibre of $V(M^*)_s \simeq V(M^*) \times_{\mathcal{T}} \kappa(s)$ over $s$ is given by

$$\tilde{V}(M)_s \simeq M_s,$$

where $M_s = M \otimes_{\mathcal{T}} \kappa(s)$. Thus, the fibres of the supervector bundle associated with a locally free sheaf $M$ are the fibres of $M$.

**Proof.** It follows from

$$\text{Hom}_{\mathcal{T}}(\mathcal{T}, V(M^*)) \simeq \text{Hom}_{\mathcal{T}}(\mathcal{T}, V(M^*) \times_{\mathcal{T}} \kappa(s)) \simeq \text{Hom}_{\mathcal{T}}(\mathcal{T}, M_{\mathcal{T}}).$$

This is the motivation for the above definition of supervector bundle.

The functor of points of $V(M)$ is not so easily described when $M$ is not locally free. However we shall see in Proposition 3.18 a result in that direction. For every coherent sheaf $M$ on $\mathcal{X}$, there is a surjection $\text{Sym}_\mathcal{X}(M) \twoheadrightarrow \mathcal{O}_{\mathcal{X}} \to 0$ of algebras, whose kernel is the ideal generated by $M$.

**Definition 2.14.** The zero section of $\pi: V(M) \to \mathcal{X}$ is the closed immersion

$$\sigma_0: \mathcal{X} \hookrightarrow V(M)$$

of $\mathcal{X}$-superschemes induced by $\rho$.

One can also define projective superbundles or, more generally, projective superspaces associated to a coherent sheaf. For this, we simply consider the projective superspectrum

$$\pi: \text{Proj} (\text{Sym} (M)) \to \mathcal{X},$$

defined by gluing the projective superspectra of the bigraded $\mathbb{A}U$-algebras $\text{Sym}_{\mathbb{A}U} M_U$ of a cover of $\mathcal{X}$ by affine open sub-superschemes.
Definition 2.15. The projective superscheme associated to $\mathcal{M}$ is the $S$-superscheme
$$\pi : \mathbb{P}(\mathcal{M}) := \text{Proj} \left( \text{Sym} (\mathcal{M}) \right) \to \mathcal{S}.$$ 

The projective superbundle associated to a locally free $\mathcal{O}_{\mathcal{S}}$-module $\mathcal{E}$ is the projective super-scheme associated to $\mathcal{E}^*$,
$$\tilde{\mathbb{P}}(\mathcal{E}) := \mathbb{P}(\mathcal{E}^*) \to \mathcal{S}.$$ 

Proceeding as in Definition 2.8 we can define a natural "hyperplane" line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{M})}(1)$ on $\mathbb{P}(\mathcal{M})$. If
$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{P} \to 0$$ 
is an exact sequence of coherent sheaves on $\mathcal{S}$, there is an epimorphism of bigraded $\mathcal{O}_{\mathcal{S}}$-algebras
$$\text{Sym} \mathcal{O}_{\mathcal{S}}(\mathcal{M}) \to \text{Sym} \mathcal{O}_{\mathcal{S}}(\mathcal{P}) \to 0,$$ 
whose kernel is the bihomogeneous ideal $\mathcal{N} \cdot \text{Sym} \mathcal{O}_{\mathcal{S}}(\mathcal{M})$ generated by $\mathcal{N}$. This induces a closed immersion of $\mathcal{S}$-schemes
$$\mathbb{P}(\mathcal{P}) \hookrightarrow \mathbb{P}(\mathcal{M}),$$
which is an isomorphism with the closed sub-superscheme defined, on every open affine sub-superscheme $\mathcal{U}$ of $\mathcal{S}$, by the homogeneous localization of $\mathcal{N} \cdot \text{Sym} \mathcal{O}_{\mathcal{S}}(\mathcal{M})$ on the projective superschemes $\mathbb{P}(\mathcal{M})_{\mathcal{U}}$. One has
$$\mathcal{O}_{\mathbb{P}(\mathcal{P})}(1) \sim \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1)|_{\mathbb{P}(\mathcal{P})}.$$ 

The following results are straightforward.

Proposition 2.16.

1. If $\mathcal{M} = \mathcal{O}_{\mathcal{S}}^{m,n}$ is free, then $\mathbb{P}(\mathcal{M}) \simeq \mathbb{P}_{\mathcal{S}}^{m,n}$.
2. If $\mathcal{M}$ is quotient of a free sheaf, $\mathcal{O}_{\mathcal{S}}^{m,n} \to \mathcal{M} \to 0$, there is a closed immersion $\mathbb{P}(\mathcal{M}) \hookrightarrow \mathbb{P}_{\mathcal{S}}^{m,n}$ of superschemes over $\mathcal{S}$.
3. If $\mathcal{E}$ is locally free of rank $(m,n)$, the projective superbundle $\mathbb{P}(\mathcal{E}^*) \to \mathcal{S}$ is locally a projective superspace over the base, that is, there is a cover of $\mathcal{S}$ by affine open sub-superschemes $\mathcal{U} = \text{Spec} \, \mathcal{A}$ such that $\tilde{\mathbb{P}}(\mathcal{E})_{\mathcal{U}} \simeq \mathbb{P}_{\mathcal{A}}^{m,n}$ as $\mathcal{A}$-superschemes.
4. For every coherent sheaf $\mathcal{M}$ there is a cover of $\mathcal{S}$ by affine open sub-superschemes $\mathcal{U} = \text{Spec} \, \mathcal{A}$ such that there is a closed immersion $\mathbb{P}(\mathcal{M})_{\mathcal{U}} \hookrightarrow \mathbb{P}_{\mathcal{A}}^{m,n}$ of superschemes over $\mathcal{A}$. 

Let $\mathcal{E}$ be a locally free sheaf on $\mathcal{S}$ and let $\pi : \mathbb{P}(\mathcal{E}) \to \mathcal{S}$ be the corresponding projective superbundle. The even associated line bundle $\mathcal{O}(1)$ is relatively generated by its global sections $\pi^* \mathcal{O}(1) \simeq \mathcal{E}$, that is, there is an epimorphism
$$\pi^* \mathcal{E} \to \mathcal{O}(1) \to 0.$$ 

For every superscheme $\phi : \mathcal{I} \to \mathcal{S}$ over $\mathcal{S}$, every morphism $f : \mathcal{I} \to \mathbb{P}(\mathcal{E})$ of $\mathcal{S}$-superschemes gives rise to an even line bundle $\mathcal{L} = f^* \mathcal{O}(1)$, which is a quotient
$$\phi^* \mathcal{E} \simeq f^* \pi^* \mathcal{E} \xrightarrow{\cong} \mathcal{L} = f^* \mathcal{O}(1) \to 0,$$ 
(2.5)
of $\phi^* \mathcal{E}$. Conversely, if $\mathcal{L}$ is an even line bundle on $\mathcal{I}$ which a quotient of $\phi^* \mathcal{E}$ as in Equation (2.5), we have an epimorphism of bigraded $\mathcal{O}_{\mathcal{S}}$-algebras
$$\text{Sym} \mathcal{O}_{\mathcal{S}}(\phi^* \mathcal{E}) \xrightarrow{\cong} \text{Sym} \mathcal{O}_{\mathcal{S}}(\mathcal{L}) \to 0,$$
and a morphism $f: \mathcal{F} \to \mathbb{P}(\mathcal{E})$ of $\mathcal{F}$-superschemes

$$
\begin{array}{ccc}
\mathcal{F} & \simeq & \text{ProjSym}_{\mathcal{O}_\mathcal{F}}(\mathcal{L}) \\
\phi & \downarrow & \downarrow \pi \\
& \mathbb{P}(\phi^* \mathcal{E}) & \simeq \mathbb{P}(\mathcal{E}) \times_{\mathcal{F}} \mathcal{F} & \mathbb{P}(\mathcal{E})
\end{array}
$$

(2.6)

such that $\mathcal{L} \simeq f^* \mathcal{O}(1)$. Thus

**Proposition 2.17.** The functor $SGrass(\mathcal{E}, (1, 0))$ on $\mathcal{F}$-superschemes which associates to any morphism $\phi: \mathcal{F} \to \mathcal{F}$ the set of the quotients of $\phi^* \mathcal{E} \to \mathcal{L} \to 0$, where $\mathcal{L}$ is an even line bundle, is representable by the projective superbundle

$$
\mathbb{P}(\mathcal{E}^*) = \mathbb{P}(\mathcal{E}).
$$

The above Proposition can be reformulated to say that the projective superbundles are supergrassmannians of a particular kind. More generally, for every pair $(p, q)$ of integer numbers, we have:

**Definition 2.18.** Let $\mathcal{E}$ be a locally free sheaf on a superscheme $\mathcal{F}$. The supergrassmannian functor of locally free quotients of rank $(p, q)$ of $\mathcal{E}$ on $\mathcal{F}$ superschemes is the functor which associates to every superscheme $\phi: \mathcal{F} \to \mathcal{F}$ over $\mathcal{F}$ the family

$$
SGrass(\mathcal{E}, (p, q))(\mathcal{F}),
$$

of all the quotients $\phi^* \mathcal{E} \to \mathcal{E}_{p, q} \to 0$ of $\phi^* \mathcal{E}$ that are locally free of rank $(p, q)$.

In the classical case, the construction of the grassmannian scheme, that is, the proof of the representability of the grassmannian functor, was first done by Grothendieck using “big cells”, but the easiest way to do it is by means of the Plücker embedding into a projective space. In supergeometry, the proof of the existence of the supergrassmannian is due to Manin [46, §3] using the super-analogue of the classical “big cells”. His proof also shows that the supergrassmannian is of finite type. In this situation, the proof using an analogue of the Plücker embedding is not available, since it is known that the supergrassmannians may fail to be superprojective [58]. The precise statement is Theorem 1.3.10 of [46], which we recall here for convenience:

**Proposition 2.19.** Let $(p, q)$ be a pair of integer numbers and $\mathcal{E}$ a locally free sheaf of rank $(c + p, d + q)$ on a superscheme $\mathcal{F}$. The supergrassmannian functor $SGrass(\mathcal{E}, (p, q))$ is representable by an $\mathcal{F}$-superscheme $SGrass(\mathcal{E}, (p, q))$. Moreover, the natural morphism $SGrass(\mathcal{E}, (p, q)) \to \mathcal{F}$ is proper of relative dimension $(cp + dq, cq + dp)$.

**Proof.** We need only to prove the properness. This follows from Proposition A.13 since $SGrass(\mathcal{E}, (p, q))$ is of finite type over $\mathcal{F}$ and the underlying scheme to $SGrass(\mathcal{E}, (p, q))$ is the fibre product $\text{Grass}(\mathcal{E}_0, p) \times_S \text{Grass}(\mathcal{E}_1, q)$ of two ordinary grassmannians. □

**Remark 2.20.** Recall that a subbundle of a locally free sheaf $\mathcal{E}$ is a locally free subsheaf of $\mathcal{E}$ such that $\mathcal{E}/\mathcal{F}$ is locally free. We shall use freely this terminology through this paper. The supergrassmannian of quotients $SGrass((c, d), \mathcal{E})$ can also be seen as a supergrassmannian of subbundles of $\mathcal{E}$, that is, it represents the functor $SGrass((c, d), \mathcal{E})$ which associates to
every $\mathcal{S}$-superscheme $\mathcal{S}$ the family of all rank $(c,d)$ subbundles of $\mathcal{E}_\mathcal{S}$. At times it may be convenient to write

$$\mathsf{SGrass}((c,d),\mathcal{E}) \simeq \mathsf{SGrass}(\mathcal{E},(p,q))$$

if we want to describe supergrassmannians as superschemes that parametrize the rank $(c,d)$ subbundles of $\mathcal{E}$.

2.4.1. Flag superschemes. As in the ordinary case, one can construct flag superschemes. We equip $\mathbb{Z} \times \mathbb{Z}$ with the following order: for any two pairs of integer numbers we declare that

$$(h_0, h_1) < (h'_0, h'_1) \iff \begin{cases} h_0 < h'_0 \text{ and } h_1 \leq h'_1, \text{ or} \\ h_0 \leq h'_0 \text{ and } h_1 < h'_1. \end{cases} \quad (2.7)$$

Let $\mathcal{E}$ be a locally free sheaf of rank $(m,n)$ of a superscheme $\mathcal{S}$.

Definition 2.21. For every choice of an integer $i$ with $1 \leq i \leq m+n$ and of a collection of dimensions $(0,0) \leq (m_1, n_1) \leq \cdots \leq (m_i, n_i) \leq (m, n)$, the superflag functor of $\mathcal{E}$ associates to every $\mathcal{S}$-superscheme $\mathcal{S}$ the family

$$\mathsf{SFlag}((m_1, n_1), \ldots, (m_i, n_i), \mathcal{E})(\mathcal{S})$$

of all the flags

$$\mathcal{F}_1 \subset \ldots \mathcal{F}_i \subset \mathcal{E}$$

where $\mathcal{F}_j$ is a rank $(m_j, n_j)$ subbundle of $\mathcal{E}$.

Thus, there is an immersion of functors on the category of $\mathcal{S}$-superschemes

$$\mathsf{SFlag}((m_1, n_1), \ldots, (m_i, n_i), \mathcal{E}) \hookrightarrow \mathsf{SGrass}((m_1, n_1), \mathcal{E}) \times \cdots \times \mathsf{SGrass}((m_i, n_i), \mathcal{E}).$$

(cf.–Remark 2.20). As in the ordinary case, one proves that this is representable by closed immersions. Then one has:

Proposition 2.22. The superflag functor $\mathsf{SFlag}((m_1, n_1), \ldots, (m_i, n_i), \mathcal{E})$ is representable by a closed sub-superscheme $\mathsf{SFlag}((m_1, n_1), \ldots, (m_i, n_i), \mathcal{E})$ (the superflag superscheme) of a product of supergrassmannians of $\mathcal{E}$. \hfill $\square$

2.5. Superprojective morphisms. Let $\mathbb{B} = \mathbb{B}(m,n)/H = \mathbb{A}[\xi_0, \ldots, \xi_m, \eta_1, \ldots, \eta_n]$ as above, where $H$ is an ideal, homogeneous for both gradings. The projection $\mathbb{B}(m,n) \to \mathbb{B} = \mathbb{B}(m,n)/H$ induces a closed immersion of superschemes over $\mathbb{A}$

$$\delta: \text{Proj}_\mathbb{A}\mathbb{B} \hookrightarrow \mathbb{P}^{m,n}_\mathbb{A}$$

which identifies $\mathcal{X}$ with the closed sub-superscheme defined by the ideal sheaf $\mathcal{H}$ obtained by $\mathbb{Z}$-homogeneous localization of the graded ideal $H$. Conversely, if $\mathcal{X} \hookrightarrow \mathbb{P}^{m,n}_\mathbb{A}$ is a closed immersion of superschemes defined by a quasi-coherent ideal $\mathcal{H}$ of $\mathcal{O} = \mathcal{O}_{\mathbb{P}^{m,n}_\mathbb{A}}$, the $\mathbb{A}$-module

$$H = \Gamma_\mathbb{A}(\mathcal{H}) = \bigoplus_{r \geq 0} \Gamma(X, \mathcal{H}(r)),$$

is a homogeneous ideal for the two gradings. Since $\Gamma_\mathbb{A}(\mathcal{H}) \Rightarrow \mathcal{H}$ by Proposition 2.11, we see that $\mathcal{X} \simeq \text{Proj}_\mathbb{B}$.

This motivates the following definition.

Definition 2.23. A projective superscheme$^3$ over $\mathbb{A}$ is an $\mathbb{A}$-superscheme of the form

$$\mathcal{X} = \text{Proj}_\mathbb{A}\mathbb{B} \xrightarrow{f} \text{Spec} \mathbb{A},$$

$^3$Superprojective superscheme in [10, 49].
for a bigraded \( \mathbb{A} \)-algebra \( \mathbb{B} = \mathbb{B}(m,n) / H = \mathbb{A}[\xi_0, \ldots, \xi_m, \eta_1, \ldots, \eta_n] \), with \( H \) a homogeneous ideal of \( \mathbb{B}(m,n) \) for the two gradings. Equivalently, a superscheme \( \mathcal{X} \) over \( \mathbb{A} \) is projective if it is endowed with a closed immersion of \( \mathbb{A} \)-superschemes into a projective superspace \( \mathbb{P}_\mathbb{A}^n \) over \( \mathbb{A} \).

A morphism \( f: \mathcal{X} \to \mathcal{I} \) of superschemes is locally superprojective if there is a covering of the base \( \mathcal{I} \) by affine superschemes \( \mathcal{W} = \mathbb{Spec} \mathbb{A} \) such that each restriction \( f_\mathcal{W}: \mathcal{X}_\mathcal{W} \to \mathcal{W} \) is a projective superscheme \( \mathcal{X}_\mathcal{W} \simeq \mathbb{Proj} \mathbb{B} \) over \( \mathbb{A} \).

Note that, if \( f: \mathcal{X} \to \mathbb{Spec} \mathbb{A} \) is a locally superprojective morphism, \( \mathcal{X} \) may fail to be a projective superscheme over \( \mathbb{A} \).

A first example of a locally superprojective morphism is the projective superscheme \( \pi: \mathbb{P}(\mathcal{M}) \to \mathcal{I} \) (Definition 2.15) associated to a coherent sheaf \( \mathcal{M} \) on \( \mathcal{I} \), as Proposition 2.16 shows.

Then, following Grothendieck, we define:

**Definition 2.24.** A morphism \( f: \mathcal{X} \to \mathcal{I} \) of superschemes is superprojective if there is a coherent sheaf \( \mathcal{M} \) on \( \mathcal{I} \) such that \( f \) factors through a closed immersion \( \mathcal{X} \hookrightarrow \mathbb{P}(\mathcal{M}) \) and the natural projection \( \pi: \mathbb{P}(\mathcal{M}) \to \mathcal{I} \).

A morphism \( f: \mathcal{X} \to \mathcal{I} \) of superschemes is quasi-superprojective if it is the composition of an open immersion \( \mathcal{X} \hookrightarrow \tilde{\mathcal{X}} \) and a superprojective morphism \( \tilde{\mathcal{X}} \to \mathcal{I} \).

**Remark 2.25.** A closed immersion \( \mathcal{X} \hookrightarrow \mathbb{P}(\mathcal{M}) \) gives rise to a line bundle \( \mathcal{O}_{\mathcal{X}}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1)|_{\mathcal{X}} \). We then say that \( \mathcal{O}_{\mathcal{X}}(1) \) is a relatively very ample line bundle on \( f: \mathcal{X} \to \mathcal{I} \). Since there are different closed immersions in different projective superschemes \( \mathbb{P}(\mathcal{M}') \), a superprojective morphism has many different relatively ample line bundles. Whenever we say that \( f: \mathcal{X} \to \mathcal{I} \) is a superprojective morphism with a relatively very ample line bundle \( \mathcal{O}_{\mathcal{X}}(1) \), we mean that we have chosen a closed immersion \( \mathcal{X} \hookrightarrow \mathbb{P}(\mathcal{M}) \) and \( \mathcal{O}_{\mathcal{X}}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1)|_{\mathcal{X}} \). In the case when the base \( \mathcal{I} \) is locally Noetherian, for any superprojective \( (\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \) over \( \mathcal{I} \), locally over \( \mathcal{I} \) there exists an embedding of \( \mathcal{X} \) into a relative superprojective space inducing \( \mathcal{O}_{\mathcal{X}}(1) \). Indeed, it is enough to represent locally \( \mathcal{M} \) as a quotient of \( \mathcal{O}_{\mathcal{I}}^{(m,n)} \).

The classical theory of relatively ample and very ample line bundles and their relationship with embeddings in projective superschemes or superbundles can be extended to the super setting, but we shall not develop that theory in these notes.

**Lemma 2.26.** Let \( \mathcal{O}_{\mathcal{X}/\mathcal{I}}(1) \) be a relatively very ample line bundle on \( \mathcal{X}/\mathcal{I} \). Then for any \( r > 0 \) the line bundle \( \mathcal{O}_{\mathcal{X}/\mathcal{I}}(r) \) is also relatively very ample.

**Proof.** This corresponds to the natural embedding of \( \mathbb{P}(\mathcal{M}) \) into \( \mathbb{P}(\text{Sym}^r(\mathcal{M})) \) associated with the surjection \( \pi^*\text{Sym}^r(\mathcal{M}) \to \mathcal{O}_{\mathbb{P}(\mathcal{M})}(r) \).

**Definition 2.27.** A morphism \( f: \mathcal{X} \to \mathcal{I} \) is called strongly superprojective if \( f \) factors through a closed immersion \( \mathcal{X} \hookrightarrow \mathbb{P}(\mathcal{E}) \) for a projective superbundle \( \mathbb{P}(\mathcal{E}) \to \mathcal{I} \) associated with a locally free sheaf \( \mathcal{E} \) on \( \mathcal{I} \).

In the case when \( \mathcal{I} \) is affine (remember that we are assuming the noetherian condition) or superprojective over a graded-commutative ring (\( \mathbb{Z} \) or a field \( k \), for instance), then every coherent sheaf is the quotient of a free sheaf (Proposition 2.11), and thus, strong superprojectivity is equivalent to superprojectivity in that case. Also, as in the classical case, a flat superprojective morphism over a noetherian base is strongly superprojective (see Corollary 3.8).
Let us say that a line bundle (of rank \((1, 0)\)) \(L\) over \(X\) is strongly relatively ample over \(S\) if there exists \(n > 0\), a locally free sheaf \(E\) over \(S\) and a closed immersion \(\phi: X \rightarrow \mathbb{P}(E)\) over \(S\), such that \(L^n \simeq \phi^* O_X/S(1)\). One has the following useful criterion of strong superprojectivity of \(X \rightarrow S\) (extending a criterion for smooth families in [41]). Let \(X \rightarrow S\) denote the corresponding morphism between the bosonizations.

[24, Prop. A.2]

**Proposition 2.28.** Let \(f: X \rightarrow S\) be a flat morphism of noetherian superschemes. If a line bundle \(L\) on \(X\) is such that \(L|_X\) is strongly relatively ample over \(S\) then \(L\) is strongly relatively ample over \(S\).

### 2.6. Superprojectivity of some partial superflag varieties.

For further use we note that some superflag varieties, contrary to supergrassmannians, are projective. Let \(E\) be a locally free sheaf of rank \((m,n)\) on a superscheme \(S\). For \(a \leq m, b \leq n\), let us consider the relative partial flag variety \(F := S\text{Flag}((a,0),(a,b);E)\) (Proposition 2.22).

**Proposition 2.29.** \(F\) is strongly superprojective over \(S\).

**Proof.** It is easy to see that \(F_{\text{bos}} \simeq \text{Grass}(a;E_+ \times_S \text{Grass}(b,E_-),\) where \(E_+ = (E_S)_+\) and \(E_- = (E_S)_-\) are the components of the restrictions of \(E\) to the underlying scheme \(S\) and \(\text{Grass}(a;E_+\) and \(\text{Grass}(b,E_-)\) are their relative grassmannians. By Proposition 2.28, it is enough to prove that there exists an even line bundle on \(F\) whose restriction to the underlying scheme \(F_{\text{bos}}\) is strongly relatively ample.

Denote by \(\pi: F \rightarrow S\) the natural projection. Let \(E(a,0) \subset E(a,b) \subset \pi^* E\) be the tautological subbundles over \(F\) and let \(E_{+,a} \subset E_+, \ E_{-,b} \subset E_-\) be the tautological bundles on the respective bosonic grassmannians (see Remark 2.20). Then we have natural isomorphisms \(E(a,0)|_{F_{\text{bos}}} \simeq p_1^* E_{+,a} \) and \(\Pi(E(a,b)/E(a,0))|_{F_{\text{bos}}} \simeq p_2^* E_{-,b}\) where \(p_1\) and \(p_2\) are the projections of \(F_{\text{bos}}\) onto its factors. But it is well known that the line bundles \(L_+ := \det^{-1}(E_{+,a})\) and \(L_- = \det^{-1}(E_{-,b})\) are strongly relatively ample on \(\text{Grass}(a;E_+)\) and \(\text{Grass}(b,E_-)\), respectively. Hence, \(p_1^* L_+ \otimes p_2^* L_-\) is strongly relatively ample on \(F_{\text{bos}}\). It remains to observe that this extends to the line bundle \(\text{Ber}^{-1} E(a,0) \otimes \text{Ber}^{-1} \Pi(E(a,b))/E(a,0)\) on \(F\).

**Corollary 2.30.** Let \(\widetilde{X}\) be a closed sub-superscheme of a relative supergrassmannian over a base superscheme \(S\). Then there exists a smooth morphism \(\widetilde{X} \rightarrow X\) of relative dimension \((0,n)\) (in particular this morphism is finite) such that \(\widetilde{X}\) is strongly superprojective over \(S\).

(The notion of smooth superscheme morphism is given in Section A.6.)
**Proof.** We just have to prove that for the relative partial superflag variety \( F \) considered in Proposition 2.29 the natural projection

\[
F \rightarrow S\text{Grass}((a, b); E)
\]

is smooth of relative dimension \((0, n)\). But \( F \) can be identified with the relative supergrassmannian of rank \((a, 0)\) subbundles associated with the tautological subbundle \( W \) of rank \((a, b)\) on \( S\text{Grass}((a, b); E) \), so we see that \( F \) is smooth of relative dimension \((0, ab)\) over \( S\text{Grass}((a, b); E) \).

\[ \Box \]

### 2.7. Cohomology of even line bundles on the projective superspace.

Let \( A \) be a noetherian superring and \( B = A[x_0, \ldots, x_m, \theta_1, \ldots, \theta_n] \). In this Subsection we denote by \( X \) the projective \((m, n)\)-superspace \( \mathbb{P}^m_n = \text{Proj}_A B \) over \( A \), so that the underlying scheme \( X \) is the projective space \( \mathbb{P}^m_A \).

If we denote by \( B' \) the algebra

\[
B' = B(m - 1, n) = A[x_0, \ldots, x_{m-1}, \theta_1, \ldots, \theta_n]
\]

there is an exact sequence of bigraded \( B \)-modules

\[
0 \rightarrow B[-1] \rightarrow B \rightarrow B' \rightarrow 0
\]

which induces a closed immersion

\[
\delta : \mathcal{X}' := \text{Proj}_A B' \hookrightarrow \mathcal{X}
\]

of the projective \((m - 1, n)\)-superspace \( \mathcal{X}' \cong \mathbb{P}^{m-1,n}_A \) as the hyperplane defined by the homogeneous ideal \((x_m)\). Moreover, for every integer number \( r \), there is an exact sequence of bigraded \( B \)-modules

\[
0 \rightarrow B[r - 1] \rightarrow B[r] \rightarrow B'[r] \rightarrow 0
\]

which induces an exact sequence of sheaves

\[
0 \rightarrow \mathcal{O}(r - 1) \rightarrow \mathcal{O}(r) \rightarrow \mathcal{O}'(r) \rightarrow 0,
\]

where we have written \( \mathcal{O} = \mathcal{O}_\mathcal{X} \) and \( \mathcal{O}' = \delta_* \mathcal{O}_{\mathcal{X}'} \).

**Lemma 2.31.**

1. If \( m > 0 \), for every integer \( r \) the natural morphism \( \mathbb{B}_r \rightarrow H^0(X, \mathcal{O}(r)) = \Gamma(X, \mathcal{O}(r)) \) is an isomorphism. So, \( H^0(X, \mathcal{O}(r)) = 0 \) for \( r < 0 \).
2. If \( m = 0 \), then \( H^0(X, \mathcal{O}(r)) = x_0^r \cdot \bigwedge A E_\theta \) for any integer \( r \), where \( E_\theta \) is the free \( A \)-module generated by \((\theta_1/x_0, \ldots, \theta_n/x_0)\).
3. The sequence of global sections of Equation (2.8) is exact when either
   - \( m > 1 \) and \( r \geq 0 \), or
   - \( m = 1 \) and \( r \geq n \).

For \( m = 1 \) and \( 0 \leq r < n \), the image of \( \mathbb{B}_r = H^0(X, \mathcal{O}(r)) \rightarrow H^0(X, \mathcal{O}'(r)) \) is the free submodule \( \bigoplus_{p=0}^r \bigwedge A E_\theta \).

**Proof.** (1) is proved as in the classical case. For (2), recall that when \( m = 0 \), \( \mathcal{X} = \mathbb{P}^0_n \) is the affine superscheme associated with \( \bigwedge A E_\theta \). Then the line bundle \( \mathcal{O}(r) \) is free over \( A \) with basis \( x_0^r \cdot (\theta_1/x_0, \ldots, \theta_n/x_0) \) for any integer \( r \) (positive or negative).
(3) The case \( m > 1 \) and \( r \geq 0 \) follows directly from (1). When \( m = 1 \), \( \mathcal{X}' = \mathbb{P}^0_{\mathbb{A}} \) so that \( H^0(X, \mathcal{O}'(r)) \cong x_0^r \cdot \wedge H^0 \mathbb{A}E_\theta \). The image of \( \mathbb{B}_r = H^0(X, \mathcal{O}(r)) \) consists of the elements of the form

\[
P_r(x_0, \theta_1, \ldots, \theta_n) = \sum_{0 \leq p \leq r} \sum_{i_1 < \cdots < i_p} a_{i_1 \ldots i_p} x_0^{-p} \theta_{i_1} \cdots \theta_{i_p} = x_0^r \sum_{0 \leq p \leq r} \sum_{i_1 < \cdots < i_p} a_{i_1 \ldots i_p} \left( \frac{\theta_{i_1}}{x_0} \cdots \frac{\theta_{i_p}}{x_0} \right).
\]

Then, the image equals \( \bigoplus_{p=0}^r \wedge^p E_\theta \) when \( r < n \) and equals the total space \( H^0(X, \mathcal{O}'(r)) \) if \( r \geq n \).

\[
\square
\]

Let us introduce the numbers

\[
h_{(m,n)}(r)_0 = \sum_{\begin{array}{c} 0 \leq p \leq n, p \text{ even} \\ 0 \leq p + s = r \end{array}} \left( \begin{array}{c} m + s \\ m \end{array} \right) \left( \begin{array}{c} n \\ p \end{array} \right),
\]

\[
h_{(m,n)}(r)_1 = \sum_{\begin{array}{c} 0 < p \leq n, p \text{ odd} \\ 0 \leq p + s = r \end{array}} \left( \begin{array}{c} m + s \\ m \end{array} \right) \left( \begin{array}{c} n \\ p \end{array} \right), \quad \text{and}
\]

\[
h_{(m,n)}(r) = (h_{(m,n)}(r)_0, h_{(m,n)}(r)_1).
\]

for \( r \geq 0 \). Note for future use that, for every value of the dimension \( (m,n) \), \( h_{(m,n)}(r)_0 \) and \( h_{(m,n)}(r)_1 \) are polynomials in \( r \) with rational coefficients. When \( m > 0 \), these numbers give for each \( r \) the rank of the component of \( \mathbb{Z} \)-degree \( r \) of \( \mathbb{B} = \mathbb{B}(m,n) \):

\[
\text{rk}_{\mathbb{A}} \mathbb{B}(m,n)_r = h_{(m,n)}(r) = (h_{(m,n)}(r)_0, h_{(m,n)}(r)_1).
\]

In the case \( m = 0 \) we get the rank of the image of the morphism \( \mathbb{B}_r = H^0(X, \mathcal{O}(r)) \to H^0(X, \mathcal{O}'(r)) \) (Lemma 2.31):

\[
\text{rk}_{\mathbb{A}} \text{Im}(H^0(X, \mathcal{O}(r)) \to H^0(X, \mathcal{O}'(r))) = (h_{(0,n)}(r)_0, h_{(0,n)}(r)_1).
\]

Let \( \mathcal{Y} = \mathcal{X} - \mathcal{X}' \) be the complementary open sub-superscheme of the hyperplane \( x_m = 0 \). The injective morphism \( \mathcal{O} \to \mathcal{X}' \mathcal{O}(r) \) (Equation (2.8)) induces an isomorphism \( \mathcal{O}_{\mathcal{Y}} \cong \mathcal{O}(r)_{\mathcal{Y}} \). Then we have injective morphisms \( 0 \to \mathcal{O}(r) \to \iota_* \mathcal{O}_{\mathcal{Y}} \) where \( \iota : \mathcal{Y} \to \mathcal{X} \) is the open immersion of \( \mathcal{Y} \) into \( \mathcal{X} \), consisting of the composition of the product by \( x_m^{-r} \) and the restriction to \( \mathcal{Y} \). Moreover, one has a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}(r) & \xrightarrow{\iota_*} & \mathcal{O}_{\mathcal{Y}} \\
x_m & \searrow & \\
& \mathcal{O}(r-1) \\
\end{array}
\]

This gives rise to an injective morphism \( \lim_{r \to} \mathcal{O}(r) \to \iota_* \mathcal{O}_{\mathcal{Y}} \). As in the classical case one proves:

**Lemma 2.32.** The above morphism induces an isomorphism \( \lim_{r \to} \mathcal{O}(r) \cong \iota_* \mathcal{O}_{\mathcal{Y}} \).

We now get (see also [10]):

**Proposition 2.33** (Cohomology of the sheaves \( \mathcal{O}(r) \)). (1) If \( r \geq n - 1 \), the sheaf \( \mathcal{O}(r) \) is acyclic and \( H^0(X, \mathcal{O}(r)) \) is a free \( \mathbb{A} \)-module of rank \( h_{(m,n)}(r) \) (See Equation (2.9)).
(2) If $0 \leq r < n - 1$, then $H^i(X, \mathcal{O}(r)) = 0$ for $i \neq 0, m$, and these groups are free $\mathbb{A}$-modules.

(3) If $r > 0$, $H^i(X, \mathcal{O}(-r)) = 0$ for $i \neq m$ and $H^m(X, \mathcal{O}(-r))$ is a free $\mathbb{A}$-module.

Proof. (1) The statement about the rank was already proved. To prove the acyclicity, one proceeds by induction on the even dimension $m$, the case $m = 0$ being trivial. Consider the exact sequence

$$0 \to \mathcal{O}(r) \to \mathcal{O}(r + 1) \to \mathcal{O}'(r + 1) \to 0,$$

(Equation (2.8)). Since $r + 1 \geq n$, the sequence of global sections is exact by Lemma 2.31. Moreover, $\mathcal{O}'(r + 1)$ is acyclic by induction on $m$, and we have isomorphisms

$$H^i(X, \mathcal{O}(r)) \cong H^i(X, \mathcal{O}(r + 1)), \quad \text{for } i > 0.$$

Lemma 2.32 now gives

$$H^i(X, \mathcal{O}(r)) \cong H^i(X, \iota_* \mathcal{O}_Y) = 0, \quad \text{for } i > 0,$$

where the first equality is due to the fact that the cohomology commutes with direct limits, $X$ being a noetherian space, and the second follows because $\iota$ is an affine morphism and $Y$ is affine.

(2) We also proceed by induction on $m \geq 1$.

Let us consider first the case $m = 1$. We need only to prove that $H^1(X, \mathcal{O}(r))$ is a free $\mathbb{A}$-module for $0 \leq r \leq n - 2$. By Lemma 2.31 we have an exact sequence of $\mathbb{A}$-modules

$$0 \to \bigoplus_{r + 1 \leq p \leq n} \bigoplus_{\ell \leq p \leq n} E_\theta \to H^1(X, \mathcal{O}(r)) \to H^1(X, \mathcal{O}(r + 1)) \to 0.$$

Since the first module is free, we get

$$H^1(X, \mathcal{O}(r)) \cong \bigoplus_{r + 1 \leq p \leq n} \bigoplus_{\ell \leq p \leq n} E_\theta \oplus H^1(X, \mathcal{O}(r + 1)).$$

Moreover $H^1(X, \mathcal{O}(n - 1)) = 0$ by (1), and by descending induction we get

$$H^1(X, \mathcal{O}(r)) \cong \bigoplus_{r + 1 \leq q \leq n} (\bigoplus_{\ell \leq q \leq n} E_\theta)^q \cong \bigoplus_{r + 1 \leq q \leq n} (\bigoplus_{\ell \leq q \leq n} E_\theta)^{q-r-1}, \quad 0 \leq r \leq n - 2.$$ (2.13)

For $m \geq 2$, by Lemma 2.31 and induction on $m$, Equation (2.8) yields $H^i(X, \mathcal{O}(r)) = 0$ for $i \neq 0, m - 1, m$ and an exact sequence

$$0 \to H^{m-1}(X, \mathcal{O}(r)) \to H^{m-1}(X, \mathcal{O}(r + 1)) \to H^{m-1}(X, \mathcal{O}'(r + 1)) \to \cdots \to H^{m}(X, \mathcal{O}(r)) \to H^{m}(X, \mathcal{O}(r + 1)) \to 0.$$

Since $H^{m-1}(X, \mathcal{O}(n - 1)) = 0$ by (1), by descending induction we have $H^{m-1}(X, \mathcal{O}(r)) = 0$ for $0 \leq r \leq n - 2$, and an isomorphism

$$H^{m}(X, \mathcal{O}(r)) \cong H^{m-1}(X, \mathcal{O}'(r + 1)) \oplus H^{m}(X, \mathcal{O}(r + 1))$$

as $H^{m-1}(X, \mathcal{O}'(r + 1))$ is free by induction. Moreover,

$$H^{m}(X, \mathcal{O}(n - 1)) = 0 \quad \text{and} \quad H^{m-1}(X, \mathcal{O}'(n - 1)) = 0$$

by (1), so that (remember that $m \geq 2$):

$$H^{m}(X, \mathcal{O}(r)) \cong \begin{cases} 0 & r = n - 2 \\ H^{m-1}(X, \mathcal{O}'(r + 1)) & 0 \leq r < n - 2. \end{cases}$$ (2.14)
(3) Since $H^0(X, \mathcal{O}(-r)) = 0$ for $r > 0$ by Lemma 2.31, we need to prove that $H^i(X, \mathcal{O}(-r)) = 0$ for $i \neq 0, m$ and $r \geq 0$; the inclusion of the case $r = 0$, which is part of (1), makes the proof easier. We proceed by induction on $m$, the case $m = 0$ being obvious. For $m \geq 1$, do induction on $r \geq 0$. For $r \geq 1$ we consider the sequence

$$0 \to \mathcal{O}(-r) \to \mathcal{O}(-(r-1)) \to \mathcal{O}'(-(r-1)) \to 0,$$

By induction on $m$, $H^i(X, \mathcal{O}'(-(r-1))) = 0$ for $i \neq m-1$ and $H^{m-1}(X, \mathcal{O}'(-(r-1)))$ is a free $\mathbb{A}$-module.

By induction on $r$, $H^i(X, \mathcal{O}(-(r-1))) = 0$ for $i \neq m$ and $H^m(X, \mathcal{O}(-(r-1)))$ is a free $\mathbb{A}$-module. Thus, $H^i(X, \mathcal{O}(-r)) = 0$ for $i \neq m$ (the case $i = 0$ is part of Lemma 2.31), and there is an exact sequence of $\mathbb{A}$-modules

$$0 \to H^{m-1}(X, \mathcal{O}'(-(r-1))) \to H^m(X, \mathcal{O}(-r)) \to H^m(X, \mathcal{O}(-(r-1))) \to 0.$$

Since the first and the third modules are free, one has

$$H^m(X, \mathcal{O}(-r)) \simeq H^{m-1}(X, \mathcal{O}'(-(r-1))) \oplus H^m(X, \mathcal{O}(-(r-1)))$$

(2.15) and one finishes. □

Remark 2.34. The ranks of the free $\mathbb{A}$-modules $H^i(X, \mathcal{O}(r))$ can be computed from Equations (2.13), (2.14) and (2.15). △

2.8. Cohomology of coherent sheaves on projective superschemes. Let $f : \mathcal{X} \to \mathcal{S}$ a locally superprojective morphism (Definition 2.23) over a locally noetherian superscheme $\mathcal{S}$.

The following result is the super analogue of a classical theorem by Serre.

Theorem 2.35 (Serre’s Theorem). Let $\mathcal{M}$ be a coherent sheaf on $\mathcal{X}$.

1. The higher direct images $R^i f_* \mathcal{M}$ are coherent sheaves on $\mathcal{S}$.

Moreover, if $\mathcal{S}$ is noetherian and $f$ is superprojective:

2. there exists an integer $r_0$, depending only on $\mathcal{M}$, such that $R^i f_* \mathcal{M}(r) = 0$ for every $i > 0$ and $r \geq r_0$;

3. there exists an integer $r_0$, depending only on $\mathcal{M}$, such that $\mathcal{M}(r)$ is relatively generated by its global sections for every $r \geq r_0$, that is, the natural morphism $f^* f_* \mathcal{M}(r) \to \mathcal{M}(r)$ is surjective;

4. if $f_* \mathcal{M}(r)$ is locally free for $r \gg 0$, then $\mathcal{M}$ is flat over $\mathcal{S}$.

Proof. The first and fourth questions are local on the base, and for $\mathcal{S}$ noetherian, the second and the third too. We can then assume that $\mathcal{S} = \text{Spec} \mathbb{A}$ for a noetherian superring $\mathbb{A}$ and $\mathcal{X}$ is a projective superscheme over $\mathbb{A}$, so that there is a closed immersion $\delta : \mathcal{X} \hookrightarrow \mathbb{P}^{m,n}_\mathbb{A}$ into a projective superspace. Since $\delta_*$ preserves coherence and cohomology, we are reduced to the case $\mathcal{X} = \mathbb{P}^{m,n}_\mathbb{A}$.

1. By Proposition 2.11 there is an exact sequence

$$0 \to \mathcal{N} \to \bigoplus_{1 \leq i \leq \mathcal{N}} \mathcal{O}_\mathcal{X}(-(r_i)) \to \mathcal{M} \to 0$$

(2.16) and $\mathcal{N}$ is coherent. Since all cohomology groups $H^i$ vanish for $i > m$ by cohomology vanishing (Proposition 2.5), we have an exact sequence

$$\cdots \to H^m(X, \mathcal{N}) \to H^m(X, \bigoplus_{1 \leq i \leq \mathcal{N}} \mathcal{O}_\mathcal{X}(-(r_i))) \to H^m(X, \mathcal{M}) \to 0,$$
so that $H^m(X, \mathcal{M})$ is finitely generated for every coherent sheaf by Proposition 2.33. In particular, $H^m(X, \mathcal{N})$ is finitely generated, and from

$$H^{m-1}(X, \bigoplus_{1 \leq i \leq N} \mathcal{O}_X(-r_i)) \to H^{m-1}(X, \mathcal{M}) \to H^m(X, \mathcal{N})$$

one gets that $H^{m-1}(X, \mathcal{M})$ is finitely generated for every coherent sheaf, because $\mathbb{A}$ is noetherian. Continuing by descending induction one proves the claim.

(2) From Equation (2.16) one gets $H^m(X, \mathcal{M}(r)) = 0$ for every $r \geq r_i$ and all $i$. By descending induction as in (1) the result follows.

(3) This is (4) of Proposition 2.11.

(4) One has $\mathcal{M} \simeq \widetilde{M}^h$, where $\mathcal{M}$ is the bigraded $\mathbb{B}(m, n)$-module $\bigoplus_{r \geq r_0} f_* \mathcal{M}(r)$. If $f_* \mathcal{M}(r)$ is locally free for $r \geq r_0$, then $\mathcal{M}$ is flat over $\mathbb{A}$. It follows that for every $i = 0, \ldots, m$ also the $\mathbb{Z}$-homogeneous localization $M_h^i$ (Definition 2.8) is flat over $\mathbb{A}$, as it is a direct summand of the total localization $M^i$. But $M^i$ are the sections of $\mathcal{M}$ over the complementary open sub-superscheme of the homogeneous zeroes of $x_i$, which proves that $\mathcal{M}$ is flat over $\mathbb{A}$. □

Part (1) of this theorem immediately implies the finiteness of the cohomology of coherent sheaves.

**Corollary 2.36.** If $\mathcal{X}$ is a projective superscheme over a field $k$ and $\mathcal{M}$ is a coherent sheaf on it, then all groups $H^i(X, \mathcal{M})$ are finite-dimensional over $k$.

### 2.9. Filtrations associated to a sheaf.

To define the super Hilbert polynomial, and for other purposes, we shall need to consider some filtrations of quasi-coherent sheaves. Let $f: \mathcal{X} \to \mathcal{Y}$ be a morphism of superschemes and consider the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{j} & \mathcal{X}_S & \xrightarrow{i} & \mathcal{X} \\
\downarrow{f_{\text{bas}}} & & \downarrow{f_S} & & \downarrow{f} \\
S & \xrightarrow{i} & \mathcal{Y}
\end{array}
$$

(2.17)

For every quasi-coherent sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules we have two filtrations naturally associated with this diagram. The first is the **base filtration**: let $\mathcal{J}$ be the ideal sheaf of $S$ in $\mathcal{Y}$ and $r$ the order of $\mathcal{J}$, that is, the first integer such that $\mathcal{J}^{r+1} = 0$. The base filtration of $\mathcal{M}$ is

$$0 \subset \mathcal{J}^0 \mathcal{M} \subset \cdots \subset \mathcal{J} \mathcal{M} \subset \mathcal{M}. \quad (2.18)$$

The successive quotients $M^{(p)} = \mathcal{J}^p \mathcal{M}/\mathcal{J}^{p+1} \mathcal{M}$ are annihilated by $\mathcal{J}$, so that they are supported on $\mathcal{X}_S$, $M^{(p)} = i_*(i^* \mathcal{J}^p \mathcal{M})$, and one has

$$0 \to \mathcal{J}^{p+1} \mathcal{M} \to \mathcal{J}^p \mathcal{M} \to M^{(p)} \to 0. \quad (2.19)$$

Analogously, there is a **total filtration**. One considers the ideal $\mathcal{I}$ of $X$ as a closed super subscheme of $\mathcal{X}$ and the filtration

$$0 \subset \mathcal{I}^0 \mathcal{M} \subset \cdots \subset \mathcal{I} \mathcal{M} \subset \mathcal{M}, \quad (2.19)$$

where $n$ is the order of $\mathcal{I}$ as above, whose successive quotients $M^{(p)} = \mathcal{I}^p \mathcal{M}/\mathcal{I}^{p+1} \mathcal{M}$ are supported on $X$; one has $M^{(p)} \simeq \iota_*(\iota^* \mathcal{I}^p \mathcal{M})$, where $\iota = i \circ j$ is the immersion of $X$ into $\mathcal{X}$, and there is an exact sequence

$$0 \to \mathcal{I}^{p+1} \mathcal{M} \to \mathcal{I}^p \mathcal{M} \to M^{(p)} \to 0. \quad (2.20)$$

One has:
Proposition 2.37.

(1) If $\mathcal{M}$ is coherent all the quotients of the base filtration (Equation (2.18)) are coherent sheaves on $\mathcal{X}_S$. Moreover, all the quotients of the total filtration (Equation (2.19)) are coherent sheaves on $X$.

(2) If $\mathcal{M}$ is flat over $\mathcal{S}$ all the successive quotients of the total filtration are flat over $\mathcal{S}$. When $\mathcal{S} = S$ is an ordinary scheme, the converse is also true.

Proof. (1) is straightforward. For (2), if $\mathcal{M}$ is flat over $\mathcal{S}$, then $\mathcal{M}_S$ is flat over $S$, so that we are reduced to the case when $\mathcal{S} = S$ is an ordinary scheme. Now, the question is local on $X$ so we can assume $\mathcal{O}_X = Gr_I \mathcal{O}_X = \mathcal{O}_X \oplus I/I^2 \oplus \ldots$; hence there is a decomposition $\mathcal{M} = \bigoplus_{p \geq 0} \mathcal{M}^{(p)}$ of $\mathcal{M}$ as an $\mathcal{O}_X$-module. It follows that $\mathcal{M}$ is flat over $S$ if and only if all the sheaves $\mathcal{M}^{(p)}$ are flat over $S$. □

2.10. The super Hilbert polynomial. Let $\mathcal{X}$ be a projective superscheme over a field $k$ (Definition 2.23). We are going to define the super Hilbert polynomial of a coherent sheaf $\mathcal{M}$ on $X$. It would be possible to adapt the proof for ordinary projective schemes based on the Snapper polynomials, but we prefer to follow a simpler approach which relies on the existence of the Hilbert polynomial for a coherent sheaf on a projective scheme.

Let us consider the Euler characteristics of $\mathcal{M}$, defined as the pair of integer numbers
\[
\chi(X, \mathcal{M}) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{M}),
\]
which is well defined due to the finiteness of the cohomology (Corollary 2.36) and the cohomology vanishing (Proposition 2.5).

Lemma 2.38. The Euler characteristic of a coherent sheaf $\mathcal{M}$ is the sum of the Euler characteristics of the quotients of its total filtration (Equation (2.19)):
\[
\chi(X, \mathcal{M}) = \sum_{0 \leq p \leq n} \chi(X, \mathcal{M}^{(p)}) = \sum_{0 \leq p \leq n} \left( \chi(X, \mathcal{M}^{(p)}_0), \chi(X, \mathcal{M}^{(p)}_1) \right).
\]

Proof. The result is proved by repeatedly applying the additivity of the Euler characteristic and Equation (2.20). □

Since $\mathcal{M}^{(p)}_0$ and $\mathcal{M}^{(p)}_1$ are coherent sheaves on the ordinary projective scheme $X$, they have Hilbert polynomials, that is, there exists polynomials $H(\mathcal{M}^{(p)}_0, r), H(\mathcal{M}^{(p)}_1, r)$ with rational coefficients, such that
\[
H(\mathcal{M}^{(p)}_0, r) = \chi(X, \mathcal{M}^{(p)}_0(r)), \quad H(\mathcal{M}^{(p)}_1, r) = \chi(X, \mathcal{M}^{(p)}_1(r))
\]
for $r \gg 0$. So there are polynomials
\[
H(\mathcal{M}, r) := \sum_{0 \leq p \leq n} H(\mathcal{M}^{(p)}_0, r), \quad H(\mathcal{M}, r) := \sum_{0 \leq p \leq n} H(\mathcal{M}^{(p)}_1, r).
\]

Definition 2.39. The super Hilbert polynomial of a coherent sheaf $\mathcal{M}$ is the pair
\[
\mathbf{H}(\mathcal{M}, r) = (H(\mathcal{M}, r)_+, H(\mathcal{M}, r)_-)
\]
of polynomials with rational coefficients.

By Lemma 2.38 one has
\[
\mathbf{H}(\mathcal{M}, r) = \chi(X, \mathcal{M}(r)) \quad \text{for } r \gg 0.
\]
Proposition 2.40. Let \( f : \mathcal{X} \to \mathcal{I} \) be a superprojective morphism of locally noetherian superschemes with \( S \) connected, and \( \mathcal{O}_\mathcal{X}(1) \) a relatively very ample line bundle (Remark 2.25). For every coherent sheaf \( M \) on \( \mathcal{X} \), flat over \( \mathcal{I} \), the super Hilbert polynomials \( H(M_s, r) \) of the restrictions \( M_s = M \otimes \kappa(s) \) of \( M \) to the fibres \( f_s : \mathcal{X}_s \to \text{Spec} \kappa(s) \) of \( f \) are independent of the choice of the point \( s \in S \). In other words, the function

\[
 s \in S \mapsto H(M_s, r)
\]

is constant on \( S \).

Proof. The restriction \( M|_S \) of \( M \) to \( \mathcal{X}_S \) is flat over \( S \) and then its odd and even part are flat as well. We finish by the corresponding statement for the classical case. \( \square \)

We can then define:

Definition 2.41. Let \( f : \mathcal{X} \to \mathcal{I} \) be a superprojective morphism of locally noetherian superschemes, \( \mathcal{O}_\mathcal{X}(1) \) a relatively very ample line bundle (Remark 2.25), and \( P = (P_+, P_-) \) a pair of polynomials with rational coefficients. We say that a coherent sheaf \( M \) on \( \mathcal{X} \) has super Hilbert polynomial \( P \) on the “fibre” of a “point” \( s : \mathcal{T} \to \mathcal{I} \) if

1. \( M_s \) is flat over \( \mathcal{T} \).
2. \( M_s \) has super Hilbert polynomial \( P \) on every fibre of \( f_s : \mathcal{X}_s \to \mathcal{T} \), that is, \( H(M_t, r) = P(r) \) for every point \( t \in T \).

3. Cohomology of proper morphisms

3.1. Finiteness theorems and Grothendieck-Mumford complex. In the classical case the finiteness of the the cohomology for proper morphisms of locally noetherian superschemes is proved from Serre’s Theorem 2.35, using an argument based on “dévissage” and on Chow lemma. Although a super version of the Chow lemma is available,[49, 7.1.3], we prefer a different approach, using the filtrations of a sheaf defined in Subsection 2.9 and the classical finiteness result.

Proposition 3.1. Let \( f : \mathcal{X} \to \mathcal{I} \) be a proper morphism of locally noetherian superschemes. For every coherent sheaf \( M \) on \( \mathcal{X} \) and for every \( i \geq 0 \), the higher direct images \( R^i f_* M \) are coherent sheaves on \( \mathcal{I} \).

Proof. Consider the total filtration of \( M \). The exact sequence of higher direct images of Equation (2.20) gives

\[
0 \to f_*(\mathcal{I}^{p+1}M) \to f_*(\mathcal{I}^pM) \to f_*M^{(p)} \to R^1 f_*(\mathcal{I}^{p+1}M) \to R^1 f_*(\mathcal{I}^pM) \to \\
R^1 f_*M^{(p)} \to R^2 f_*(\mathcal{I}^{p+1}M) \to R^2 f_*(\mathcal{I}^pM) \to R^2 f_*M^{(p)} \to \ldots
\]

By Proposition 2.37 \( \iota^* \mathcal{I}^pM \) is coherent on \( X \). Due to

\[
R^i f_*M^{(p)} \cong i_* R^i f_{\text{bos}}(\iota^* \mathcal{I}^pM)
\]

these sheaves are coherent as a consequence of the corresponding property for classical schemes. By descending induction on \( p \), we can assume that the sheaves \( R^i f_*(\mathcal{I}^{p+1}M) \) are coherent. By local noetherianity the sheaves \( R^i f_*(\mathcal{I}^pM) \) are coherent as well. \( \square \)
The finiteness theorem allows us to generalize straightforwardly many classical results. We reproduce the relevant statements offering proofs only when they are different from the classical ones. Among the many references for those classical results we mention Grothendieck’s original treatment [30], Hartshorne [34] and Nitsure [53].

The first result we would like to report on is the existence of a complex of finitely generated modules that computes the cohomology of a coherent sheaf on a proper superscheme.

**Proposition 3.2** (Grothendieck-Mumford complex). Let \( X \) be a proper superscheme over a noetherian superring \( A \) and \( M \) a coherent sheaf on \( X \) flat over \( A \). There exists a finite complex 
\[
K^\bullet := 0 \to K_0 \xrightarrow{\partial_1} K_1 \xrightarrow{\partial_2} \ldots \xrightarrow{\partial_s} K_s \to 0,
\]
of finitely generated and locally free \( \mathbb{Z}_2 \)-graded \( A \)-modules, and a functorial \( A \)-linear isomorphism
\[
H^i(X, M \otimes_A N) \cong H^i(K^\bullet \otimes_A N),
\]
in the category of all \( (\mathbb{Z}_2\text{-graded}) \) \( A \)-modules \( N \).

**Proof.** The same proof as in the classical case gives that the modules of the complex are projective. By the super Nakayama lemma ([5, 11] or [64, 6.4.5]) this is equivalent to the local freeness. \( \square \)

Let \( A \) be a noetherian superring and \( f : X \to \mathcal{S} = \text{Spec} A \) a proper morphism of superschemes. For every coherent sheaf \( M \) of \( \mathcal{O}_X \)-modules, flat over \( \mathcal{S} \), and every index \( i \geq 0 \) we consider the linear half exact functor (Subsection A.4) \( T^i \) from the category of \( (\mathbb{Z}_2\text{-graded}) \) finitely generated \( A \)-modules to itself given by
\[
T^i(N) := \Gamma(S, R^if_*(M \otimes_A N)) = H^i(X, M \otimes_A N) = H^i(K^\bullet \otimes_A N),
\]
where \( K^\bullet \) is the Grothendieck-Mumford complex associated to \( M \) (Proposition 3.2). The functors \( T^i \) form a \( \delta \)-functor, that is, for every exact sequence \( 0 \to N' \to N \to N'' \to 0 \) of finitely generated \( A \)-modules, there is an exact sequence
\[
\ldots T^{i-1}(N'') \xrightarrow{\delta} T^i(N') \to T^i(N) \to T^i(N'') \xrightarrow{\delta} T^{i+1}(N') \to \ldots
\]

For every point \( s \in \mathcal{S} \), denote by \( T^i(s) \) the functor \( T^i \) for the morphism \( X \times_{\mathcal{S}} \text{Spec} A_s \to \text{Spec} \ A_A \) and the sheaf \( M \otimes_A A_s \) obtained by the localization flat base change \( A \to A_s \).

**Lemma 3.3.**

1. \( T^i \) is left exact if and only if \( W_i := \text{coker} \partial_{i-1} \) is a locally free \( A \)-module. Therefore:
   (a) If \( M \) is flat over \( \mathcal{S} \), \( T^0 \) is left exact.
   (b) If \( T^i(s_0) \) is left exact for a point \( s_0 \), \( T^i(s) \) is also left exact for all the points \( s \) in an open neighbourhood of \( s_0 \).
2. \( T^i \) is right exact if and only if \( T^{i+1} \) is left exact.
Proof. (1) If \( N' \to N \) is an injective morphism, one has a commutative diagram of exact rows and columns:

\[
\begin{array}{ccc}
0 & \to & T^i(N') \\
\downarrow & & \downarrow \\
0 & \to & W_i \otimes_{\mathbb{A}} N' \\
\downarrow \alpha & & \downarrow \beta \\
0 & \to & K_i+1 \otimes_{\mathbb{A}} N' \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \to & T^i(N) \\
\downarrow & & \downarrow \\
0 & \to & W_i \otimes_{\mathbb{A}} N \\
\downarrow \alpha & & \downarrow \beta \\
0 & \to & K_i+1 \otimes_{\mathbb{A}} N \\
\end{array}
\]

Now \( \text{coker}(K_{i-1} \otimes_{\mathbb{A}} N' \to K_i \otimes_{\mathbb{A}} N) = W_i \otimes_{\mathbb{A}} N \), so that \( \alpha \) is injective if and only if \( \beta \) is injective. Thus, \( T^i \) is left exact if and only if \( W_i \) is flat and then locally free.

Now (a) is a consequence of Proposition 3.2 as \( W_0 = K_0 \), and (b) follows from the fact that if a sheaf of modules is free at a point, then it is free in an open neighbourhood.

(2) follows from Equation (3.2). \( \square \)

3.2. Cohomology base change and semicontinuity. We are going to apply to the functors \( T^i \) the Nakayama Lemma A.9 for half exact functors and Proposition A.10. This provides a version of the cohomology base change that does not require flatness over \( \mathcal{U} \) of the relevant sheaf \( \mathcal{M} \).

**Theorem 3.4** (Cohomology base change). Let \( f : \mathcal{X} \to \mathcal{I} \) be a proper morphism of locally noetherian superschemes and \( \mathcal{M} \) a coherent sheaf on \( \mathcal{X} \) flat over \( \mathcal{I} \). If \( s \in S \) is a point of \( S \), one has:

1. If for some \( i \) the base change map \( \varphi_s^i : (R^i f_* \mathcal{M})_s \to H^i(X_s, \mathcal{M}_s) \) is surjective, then it is an isomorphism and the same happens for all points in an open neighbourhood of \( s \).

2. If (1) is true, there exists an open sub-superscheme \( \mathcal{U} \) of \( \mathcal{I} \) containing \( s \), such that for any quasi-coherent \( \mathcal{O}_\mathcal{U} \)-module \( \mathcal{N} \) the natural morphism

\[
(R^i f_{\mathcal{U}*} \mathcal{M}_{|\mathcal{U}}) \otimes_{\mathcal{O}_\mathcal{U}} \mathcal{N} \to R^i f_{U*} (\mathcal{M}_{|\mathcal{U}} \otimes f_\mathcal{U}^* \mathcal{N}),
\]

is an isomorphism.

3. If (1) is true for \( i > 0 \), then \( \varphi_s^{i-1} : (R^{i-1} f_* \mathcal{M})_s \to H^{i-1}(X_s, \mathcal{M}_s) \) is surjective if and only of \( R^i f_* \mathcal{M} \) is locally free on an open sub-superscheme \( \mathcal{U} \) for \( \mathcal{I} \) with \( s \in U \).

4. If \( H^i(X_s, \mathcal{M}_s) = 0 \) for some \( i > 0 \), the morphism \( \varphi_s^{i-1} : (R^{i-1} f_* \mathcal{M})_s \to H^{i-1}(X_s, \mathcal{M}_s) \) is an isomorphism.

Proof. The question is local, so that we can assume that \( \mathcal{I} \) is affine, \( \mathcal{I} = \text{Spec} \, \mathbb{A} \), and that the superring \( \mathbb{A} \) is noetherian. After the flat base change \( \mathbb{A} \to \mathbb{A}_s \) we can also assume that \( \mathbb{A} \) is local. Applying Proposition A.10 (Nakayama’s Lemma for half exact functors) to the functors \( T^i \) and Lemma 3.3 we obtain the first three statements.

(4) \( T^i = 0 \) by Lemma A.9, and then \( T^{i-1} \) is right exact by Lemma 3.3. The statement now follows from Proposition A.10. \( \square \)

**Corollary 3.5.** Let \( f : \mathcal{X} \to \mathcal{I} \) be a proper morphism of locally noetherian superschemes and \( \mathcal{M} \) a coherent sheaf on \( \mathcal{X} \) flat over \( \mathcal{I} \). If there is an index \( j \) such that \( R^i f_* \mathcal{M} = 0 \) for every \( i \geq j \), then \( H^i(X_s, \mathcal{M}_s) = 0 \) for every \( i \geq \dim X_s \) and every point \( s \in S \).

Proof. It follows from Theorem 3.4 by descending induction on \( j \), taking into account that \( H^i(X_s, \mathcal{M}_s) = 0 \) for every \( i \geq \dim X_s \) and every point \( s \in S \) by Proposition 2.5. \( \square \)
Corollary 3.6 (Boundedness of higher direct images). Let \( f: \mathcal{X} \to \mathcal{I} \) be a proper morphism of noetherian superschemes, \( \mathcal{M} \) a coherent sheaf on \( \mathcal{X} \) flat over \( \mathcal{I} \). Let \( r \) denote the maximum of the dimensions \( \dim X_s \) of the bosonic fibres, which is well defined as \( \mathcal{I} \) is noetherian. Then \( R^i f_* \mathcal{M} = 0 \) for \( i > r \).

Proof. For every point \( s \) one has \( (R^i f_* \mathcal{M})_s = 0 \) for \( i > r \) by cohomology boundedness (Proposition 2.5) and by (4) of Theorem 3.4. Then \( R^i f_* \mathcal{M} = 0 \) for \( i > r \) by super Nakayama ([5, 11] or [64, 6.4.5]).

One can also deduce the local freeness of the direct images of relatively acyclic coherent sheaves, assuming that the latter are flat over the base:

Proposition 3.7. Let \( f: \mathcal{X} \to \mathcal{I} \) be a proper morphism of locally noetherian superschemes and \( \mathcal{M} \) a coherent sheaf on \( \mathcal{X} \) flat over \( \mathcal{I} \).

1. If either \( R^i f_* \mathcal{M} = 0 \) for every \( i > 0 \) or \( H^i(X_s, \mathcal{M}_s) = 0 \) for every \( i > 0 \) and every point \( s \in S \), then \( f_* \mathcal{M} \) is locally free. Moreover, for every morphism \( \mathcal{I} \to \mathcal{I} \), the base change morphism \( (f_* \mathcal{M})_{\mathcal{S}} \to f_{\mathcal{S}}(\mathcal{M}_{\mathcal{S}}) \) is an isomorphism.

2. If \( f \) is superprojective and \( \mathcal{I} \) is noetherian then \( f_* \mathcal{M}(r) \) is locally free for \( r \gg 0 \).

Proof. (1) We can now assume that \( \mathcal{I} = \text{Spec} \mathbb{A} \) and \( \mathbb{A} \) noetherian. Reasoning as in the proof of Corollaries 3.5 and 3.6, one sees that the two conditions are equivalent and imply \( T^1 = 0 \). Then \( T^0 \) is right exact by Lemma 3.3 and thus \( T^0 = f_*(\mathcal{M}) \otimes_{\mathbb{A}} \) by Proposition A.10. Again by Lemma 3.3, \( T^0 \) is left exact because \( \mathcal{M} \) is flat over \( \mathcal{I} \), so that \( f_*(\mathcal{M}) \) is flat over \( \mathcal{A} \) and \( (f_*(\mathcal{M}))_s \cong H^0(X_s, \mathcal{M}_s) \) for every point \( s \in S \). Since \( \mathcal{M}_\mathcal{S} \) is flat over \( \mathcal{I} \), we also have an isomorphism \( (f_{\mathcal{S}}(\mathcal{M}))_s \cong H^0(X_s, \mathcal{M}_s) \) for every point \( s \in T \), and \( f_{\mathcal{S}}(\mathcal{M}) \) is locally free. Now \( (f_* \mathcal{M})_{\mathcal{S}} \to f_{\mathcal{S}}(\mathcal{M}_{\mathcal{S}}) \) is a morphism of locally free sheaves inducing isomorphisms \( ((f_* \mathcal{M})_{\mathcal{S}})_s \cong (f_{\mathcal{S}}(\mathcal{M}_{\mathcal{S}}))_s \) for every point; then, it is an isomorphism by Nakayama’s Lemma (Proposition A.10).

(2) It follows from (1) and Serre’s Theorem 2.35.

Corollary 3.8. Let \( f: \mathcal{X} \to \mathcal{I} \) be a flat superprojective morphism, where \( \mathcal{I} \) is noetherian. Then \( f \) is strongly superprojective (see Definition 2.27).

Proof. Let \( O_{\mathcal{X}/\mathcal{I}}(1) \) be a relatively very ample line bundle for \( \mathcal{X}/\mathcal{I} \). By Proposition 3.7(2) and Theorem 2.35, there exists \( r > 0 \) such that \( f_* O_{\mathcal{X}/\mathcal{I}}(r) \) is locally free and the natural map \( f^*(f_* O_{\mathcal{X}/\mathcal{I}}(r)) \to O_{\mathcal{X}/\mathcal{I}}(r) \) is surjective. Hence, for \( n \geq n_0 \), the morphism

\[
f_* O_{\mathcal{X}}(r) \otimes f_* O_{\mathcal{X}}(n) \to f_* (O_{\mathcal{X}}(r + n))
\]

is surjective. This implies that

\[
(f_* O_{\mathcal{X}/\mathcal{I}}(r)) \otimes f_* O_{\mathcal{X}/\mathcal{I}}(n) \to f_* (O_{\mathcal{X}/\mathcal{I}}(rm + n))
\]

is surjective for \( n \geq n_0 \) and \( m > 0 \). Hence,

\[
(f_* O_{\mathcal{X}/\mathcal{I}}(r)) \otimes f_* O_{\mathcal{X}/\mathcal{I}}(mn_0 r)
\]

is surjective for any \( m > 0 \). Thus, we get a surjection of sheaves of graded algebras

\[
\text{Sym}(f_* O_{\mathcal{X}/\mathcal{I}}(d)) \to \bigoplus_{m \geq 0} O_{\mathcal{X}/\mathcal{I}}(md),
\]

where \( d = n_0 r \). This induces a closed immersion of \( \mathcal{X} \) into the projectivization of the dual vector bundle to \( f_* O_{\mathcal{X}/\mathcal{I}}(d) \).
We can now state the super semicontinuity theorem, also given in [49, Theorem 7.3], whose proof is the same as in the classical case (see, for instance, [34, Thm. III.12.8] or [30, 7.7.5 and 7.6.9]). Recall that we have equipped \( \mathbb{Z} \times \mathbb{Z} \) with the order given by Equation (2.7).

**Theorem 3.9** (semicontinuity). Let \( f : \mathcal{X} \to \mathcal{Y} \) be a proper morphism of locally noetherian superschemes, and \( \mathcal{M} \) a coherent sheaf on \( \mathcal{X} \), flat over \( \mathcal{Y} \). One has:

1. for every integer \( i \) the function
   \[
   S \to \mathbb{Z} \times \mathbb{Z}, \quad s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{M}_s);
   \]
   is upper semicontinuous on \( S \), where \( \mathbb{Z} \times \mathbb{Z} \) is equipped with the order given above.

2. the function \( s \mapsto \sum_{i \geq 0} \dim_{\kappa(s)} H^i(X_s, \mathcal{M}_s) \) is locally constant on \( S \).

**Proof.** The question is local, so that we can assume \( \mathcal{Y} = \text{Spec} \ A \) with \( A \) noetherian. For every point \( s \in S \) and every index \( i \), we have exact sequences
   \[
   0 \to T^i(\kappa(s)) \to W_i \otimes_A \kappa(s) \to K_{i+1} \otimes_A \kappa(s)
   \]
   and then an exact sequence
   \[
   0 \to T^i(\kappa(s)) \to W_i \otimes_A \kappa(s) \to K_{i+1} \otimes_A \kappa(s) \to W_{i+1} \otimes_A \kappa(s) \to 0.
   \]
   Since \( \dim K_{i+1} \otimes_A \kappa(s) \) is constant because \( K_{i+1} \) is free, and, for every \( (p, q) \), the set of points \( s \) such that \( \dim(W_i \otimes W_{i+1}) \otimes_A \kappa(s) > (p, q) \) is closed by super Nakayama ([5, 11] or [64, 6.4.5]), one proves (1). To prove (2) it is enough to take alternate sums in Equation (3.5).

**Lemma 3.10.** Let \( \mathcal{Y} \) be a noetherian superscheme with \( S \) reduced, and \( \mathcal{N} \) a coherent sheaf on \( \mathcal{Y} \) such that \( \dim \mathcal{N} \otimes_{\mathcal{O}_\mathcal{Y}} \kappa(s) = (p, q) \) for some integers \( p, q \) and for every point \( s \in S \). Then \( \mathcal{N} \) is locally free of rank \( (p, q) \).

**Proof.** If \((n_1, \ldots, n_p, \eta_1, \ldots, \eta_q)\) is a family of \( p \) even and \( q \) odd sections of \( \mathcal{N} \) on an open neighbourhood \( U \) of a point \( s \) such that the classes \((\bar{n}_1, \ldots, \bar{n}_p, \bar{\eta}_1, \ldots, \bar{\eta}_q)\) in \( \mathcal{N} \otimes_{\mathcal{O}_\mathcal{Y}} \kappa(s) \) are a basis, then by super Nakayama ([5, 11] or [64, 6.4.5]), there is an open sub-superscheme \( \mathcal{V} \) with \( s \in V \subseteq U \) such that \( \mathcal{N}_V \) is generated by \((n_1, \ldots, n_p, \eta_1, \ldots, \eta_q)\). Then, there is an exact sequence
   \[
   \mathcal{O}_{\mathcal{V}}^{p,q} \to \mathcal{N}_V \to 0.
   \]
   Since \( \dim \mathcal{N} \otimes_{\mathcal{O}_\mathcal{Y}} \kappa(s') = (p, q) \) for every point \( s' \), by shrinking \( \mathcal{V} \) if necessary, we have that \( \ker \phi \subseteq \mathcal{p}_{s'} \cdot \mathcal{O}_{s'}^{p,q} \) for every point point \( s' \in V \). Then \( \ker \phi = 0 \) because \( V \) is reduced.

**Theorem 3.11** (Grauert). Under the same hypotheses of Theorem 3.9, and assuming that \( S \) is reduced, the function \( s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{M}_s) \) is locally constant for some \( i \geq 0 \) if and only if \( R^if_*\mathcal{M} \) is locally free and \( (R^if_*\mathcal{M})_s \simeq H^i(X_s, \mathcal{M}_s) \) for every point \( s \in S \). If these conditions are satisfied, the base change morphism \( (R^{i-1}f_*\mathcal{M})_s \to H^{i-1}(X_s, \mathcal{M}_s) \) is an isomorphism as well.

**Proof.** We can assume that \( \mathcal{Y} = \text{Spec} \ A \) with \( A \) noetherian. Suppose that the function \( s \mapsto \dim_{\kappa(s)} H^i(X_s, \mathcal{M}_s) \) is locally constant. Taking dimensions in Equation (3.5) for every point \( s \in S \), and applying Lemma 3.10, we have that \( W_i \oplus W_{i+1} \) is locally free, and then all summands are flat. By Lemma 3.3, \( T^i \) and \( T^{i+1} \) are left exact, so that \( T^i \) is also right exact.
3.3. Cohomological flatness in dimension 0. In this section we give the definition of cohomological flatness in dimension 0 for morphisms of superschemes, which is necessary in the proof of the existence of the Picard superscheme, as we shall see in Section 5.

Definition 3.12. Let $\mathbb{A}$ be a superring. Recall that $J \subset A$ denotes the ideal generated by odd elements. We say that $\mathbb{A}$ is integral if $A = A/J$ is an integral ring (i.e., a domain) and every element of $\mathbb{A} \setminus J$ is not a zero divisor in $\mathbb{A}$.

For example, the superring $k[x, \theta]/(x\theta)$ is not integral, while $k[x, \theta_1, \theta_2]/(\theta_1\theta_2)$ is (where $\theta$ and $\theta_i$ are odd variables). Note that if $A = \mathbb{A}/J$ is integral and all the $A$-modules $J^i/J^{i+1}$ are torsion free, then $\mathbb{A}$ is integral.

Definition 3.13. A superscheme $\mathcal{X} = (X, \mathcal{O}_X)$ is called integral if $\mathcal{O}_\mathcal{X}(U)$ is integral for every open $U$. A superscheme over a field $k$ is called geometrically integral if for every field extension $k \hookrightarrow \bar{k}$, where $\bar{k}$ is algebraically closed, the base change superscheme $\mathcal{X} = \mathcal{X} \times_{\text{Spec } k} \text{Spec } \bar{k}$ is an integral superscheme.

Clearly for a integral superscheme $\mathcal{X}$ its bosonization $X$ is an integral scheme. Given a superscheme $\mathcal{X}$ with integral bosonization $X$, let $i_\eta : \eta \to X$ denote the embedding of the general point. It is easy to see that $\mathcal{X}$ is integral if and only if in addition the natural morphism of sheaves

$$\mathcal{O}_\mathcal{X} \to i_{\eta, *}^*i_{\eta}^*\mathcal{O}_X$$

is injective.

The following easy fact, which follows immediately from the definition, will be needed later on.

Proposition 3.14. Every morphism $\mathcal{L} \to \mathcal{N}$ of line bundles on an integral superscheme $\mathcal{X} = (X, \mathcal{O}_X)$ whose boson restriction $\mathcal{L}|_X \to \mathcal{N}|_X$ to $X$ is nonzero, is injective. \hfill \Box

Definition 3.15. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of superschemes is cohomologically flat in dimension 0 if the natural morphism

$$\mathcal{O}_\mathcal{Y} \to f_{\mathcal{Y}}^*\mathcal{O}_\mathcal{X}$$

is an isomorphism for every base change $\mathcal{Y} \to \mathcal{J}$. A superscheme $\mathcal{X}$ over a superring $\mathbb{A}$ is is cohomologically flat in dimension 0 if so is the natural morphism $\mathcal{X} \to \text{Spec } \mathbb{A}$.

Example 3.16.

(1) Let $f : X \to S$ be a proper flat morphism of locally noetherian ordinary schemes. If the geometric fibres, that is, fibres over spectra of algebraically closed fields, are irreducible and reduced, then $f$ is cohomologically flat in dimension 0 (see [30, Proposition 7.8.6 and Corollary 7.8.8]).

(2) If $f : \mathcal{X} = \mathcal{P}(\mathcal{E}) \to \mathcal{J}$ is the superprojective bundle associated to a locally free sheaf $\mathcal{E}$ of rank $(m,n)$ on $\mathcal{J}$ (Definition 2.15) with $m \geq 1$, then $f$ is cohomologically flat in dimension 0. Since $\mathcal{X}_\mathcal{Y} \simeq \mathcal{P}(\mathcal{E}_\mathcal{Y})$, one has only to prove that $\mathcal{O}_\mathcal{Y} \cong f_{\mathcal{Y}}^*\mathcal{O}_\mathcal{X}$, and this follows from Lemma 2.31 because $\mathcal{J}$ can be covered by affine superschemes $\mathcal{Y} = \text{Spec } \mathbb{A}$ such that $\mathcal{X}_\mathcal{Y} \simeq \mathbb{P}^m_n(\mathbb{A})$ (Proposition 2.16).
(3) A split superscheme $\mathcal{X} = (X, O_{\mathcal{X}}(E))$ over a field $k$ is cohomologically flat in dimension 0, if and only if $\Gamma(X, O_X) = k$ and $\Gamma(X, O_{\mathcal{X}}(E)) = 0$ for every $p > 0$.

We are going to extend (1) of Example 3.16 of superschemes. The proof is similar to the one for ordinary schemes given in [37].

Given a superscheme $\mathcal{X} = (X, O_{\mathcal{X}})$ over a field $k$ and a field extension $k \hookrightarrow \bar{k}$, we denote by $\mathcal{X}^{\bar{k}} = (X = X \times_{\text{Spec} k} \text{Spec} \bar{k}, O_{\mathcal{X}}^{\bar{k}} = O_{\mathcal{X}} \otimes_k \bar{k})$ the base change of $\mathcal{X}$ under $\text{Spec} k \to \text{Spec} \bar{k}$.

**Proposition 3.17.** Let $f: \mathcal{X} \to \mathcal{S}$ be a proper and flat morphism of locally noetherian superschemes whose fibres are geometrically integral. Then $f$ is cohomologically flat in dimension 0 if and only if the geometric fibres of $f$ satisfy

$$H^0(\mathcal{X}_s, O_{\mathcal{X}_s}) \simeq \kappa(s). \quad (3.6)$$

Proof. It is clear that if $f$ is cohomologically flat in dimension 0 then its geometric fibres satisfy (3.6). For the converse, since our conditions are stable under base change, it is enough to prove that the morphism $O_{\mathcal{X}} \to f_* O_{\mathcal{X}}$ is an isomorphism.

Let us consider the left exact functor $T^0(N) = f_* (f^* N)$ defined on the category of finitely-generated $k$-modules and associated to $\mathcal{M} = O_{\mathcal{X}}$ as in Equation (3.1).

By the flat base change, the condition (3.6) implies that for every point $s \in \mathcal{S}$ one has

$$T^0(\kappa(s)) = H^0(\mathcal{X}_s, O_{\mathcal{X}_s}) \simeq \kappa(s).$$

Hence, the composition of $\kappa(s) \to f_* O_{\mathcal{X}} \otimes \kappa(s) = T^0(\mathcal{A}) \otimes \kappa(s)$ with $T^0(\mathcal{A}) \otimes \kappa(s) \to T^0(\kappa(s))$ is an isomorphism, so that the second morphism is surjective. Then by Proposition A.10 one has that $T^0(\mathcal{A}) \otimes \kappa(s) \cong T^0(\kappa(s))$, $T^0 = T^0(\mathcal{A}) \otimes -$ and that $T^0$ is right exact. Since $T^0$ is also left exact, we get that $T^0(\mathcal{A}) = f_* O_{\mathcal{X}}$ is free of rank $(1, 0)$. Since the morphism $O_{\mathcal{X}} \to f_* O_{\mathcal{X}}$ induces an isomorphism of fibres at every point, it is an isomorphism.

3.4. The superscheme of homomorphisms between coherent sheaves. The following result on the representability of morphisms between coherent sheaves will be quite useful in the construction of the Hilbert superscheme. The proof is the same as in the classical case (see, for instance, [53, Thm. 3.5]).

Let $f: \mathcal{X} \to \mathcal{S}$ be a superprojective morphisms of superschemes, and let $\mathcal{M}, \mathcal{N}$ be coherent sheaves on $\mathcal{X}$. Let us consider the functor $\text{Hom}_{\mathcal{X}}(\mathcal{M}, \mathcal{N})$ defined on the category on $\mathcal{S}$-superschemes $\mathcal{T} \to \mathcal{S}$ by letting

$$\text{Hom}_{\mathcal{X}}(\mathcal{M}, \mathcal{N})(\mathcal{T}) = \text{Hom}_{\mathcal{S}}(\mathcal{M}_\mathcal{T}, \mathcal{N}_\mathcal{T}).$$

**Proposition 3.18.** [30, 7.7.8] If $\mathcal{N}$ is flat over $\mathcal{S}$, the above functor is representable by a linear superscheme $\mathcal{V}(\mathcal{Q})$ associated to a coherent sheaf $\mathcal{Q}$ on $\mathcal{S}$, that is, there is a “universal” morphism $\Phi: \mathcal{M}_{\mathcal{V}(\mathcal{Q})} \to \mathcal{N}_{\mathcal{V}(\mathcal{Q})}$ and a functorial isomorphism

$$\text{Hom}_{\mathcal{S}}(\mathcal{T}, \mathcal{V}(\mathcal{Q})) \cong \text{Hom}_{\mathcal{X}}(\mathcal{M}, \mathcal{N})(\mathcal{T})$$

$$\gamma \mapsto \gamma^* \Phi. \quad \square$$

**Corollary 3.19.** The zero section $\mathcal{V}_0(\mathcal{Q})$ of $\mathcal{V}(\mathcal{Q})$ (Definition 2.14) is the locus where the universal morphism $\Phi$ vanishes, that is, for every $\mathcal{S}$-superscheme $\mathcal{T}$ and every morphism $\phi: \mathcal{M}_\mathcal{T} \to \mathcal{N}_\mathcal{T}$, corresponding to a morphism $\gamma: \mathcal{T} \to \mathcal{V}(\mathcal{Q})$ of $\mathcal{S}$-superschemes ($\phi = \gamma^* \Phi$), the closed sub-superscheme

$$\gamma^{-1}(V_0(\mathcal{Q})) \hookrightarrow \mathcal{T},$$

\[\square\]
has the following universal property: a morphism \( \psi : \mathcal{X} \to \mathcal{Y} \) satisfies that \( \psi^* \phi = 0 \) if and only if it factors through \( \gamma^{-1}(\mathcal{V}_0(Q)) \).

3.5. **Relative Grothendieck duality.** Relative Grothendieck duality can be developed also for morphisms of superschemes. In the mid eighties a simple version of it, namely, Serre duality for projective smooth superschemes, was proved in [55] in its simplest version, Penkov later extended duality to complete smooth complex superschemes [57]. A remarkable result in these papers is the fact that the Berezinian sheaf is the the dualizing sheaf for smooth complex complete superschemes. Relative Grothendieck duality for proper morphisms was stated in the super analytic case in [62, 1.5.1]; however no proof was provided there in addition to a reference to the ordinary analytic case.

So a description of relative duality for superschemes seems to be missing in the literature, and the aim of this Subsection is to fill that gap. We state the main facts, stressing which proofs are straightforward translations of the corresponding ones for schemes, and what is new in the super setting. Albeit most likely this is not the simplest approach, we use Brown’s representability theorem as in Neeman [52] (see also the references therein for a more complete perspective of the topic).

For every superscheme \( \mathcal{X} \) we denote by \( D(\mathcal{X}) \) the unbounded derived category of the category of quasi-coherent \( \mathcal{O}_\mathcal{X} \)-modules. If \( \mathfrak{Mod}(\mathcal{X}) \) is the category of all \( \mathcal{O}_\mathcal{X} \)-modules, there is a natural functor \( D(\mathcal{X}) \to D(\mathfrak{Mod}(\mathcal{X})) \). If \( \mathcal{X} \) is quasi-compact and separated, this induces an equivalence of categories between \( D(\mathcal{X}) \) and the full subcategory \( D_{qc}(\mathfrak{Mod}(\mathcal{X})) \) of complexes with quasi-coherent cohomology. This is proved by translating the formal proof given in [8, Cor. 5.5] to our setting.

Again as in the ordinary case, one can prove, following Spaltenstein [61], that for every quasi-compact morphism \( f : \mathcal{X} \to \mathcal{Y} \) of superschemes the direct and inverse images

\[
\mathbf{R}f_* : D(\mathcal{X}) \to D(\mathcal{Y}), \quad \mathbf{L}f^* : D(\mathcal{Y}) \to D(\mathcal{X})
\]

are defined for unbounded complexes of quasi-coherent sheaves, and the latter functor is a right adjoint to the former. Moreover, if \( f \) is quasi-compact and quasi-separated, there are also derived functors

\[
\mathbf{R}f_* : D(\mathcal{X}) \to D(\mathcal{Y}), \quad \mathbf{L}f^* : D(\mathcal{Y}) \to D(\mathcal{X}).
\]

Following [43, 3.9.2] one can prove that the two derived direct images agree, that is, there is a commutative diagram

\[
\begin{array}{ccc}
D(\mathcal{X}) & \xrightarrow{\mathbf{R}f_*} & D(\mathcal{Y}) \\
\downarrow & & \downarrow \\
D(\mathfrak{Mod}(\mathcal{X})) & \xrightarrow{\mathbf{R}f_*} & D(\mathfrak{Mod}(\mathcal{Y})).
\end{array}
\]

From this one sees that also the functors \( \mathbf{R}f_* : D(\mathcal{X}) \to D(\mathcal{Y}) \) and \( \mathbf{L}f^* : D(\mathcal{Y}) \to D(\mathcal{X}) \) are adjoint to each other.

To simplify the exposition, in this Subsection we assume that *all superschemes and all morphisms between them are quasi-compact and separated.* We note that there is a projection formula

\[
\mathbf{R}f_* (\mathcal{M}^\bullet \otimes^L \mathbf{L}f^* \mathcal{N}^\bullet) \cong \mathbf{R}f_* \mathcal{M}^\bullet \otimes^L \mathcal{N}^\bullet,
\]

which is proved as in [52, Prop. 5.3], and also that

\[
\mathbf{R}\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{M}^\bullet, \mathcal{K}^\bullet \otimes^L \mathcal{G}^\bullet) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\mathcal{M}^\bullet, \mathcal{K}^\bullet) \otimes^L \mathcal{G}^\bullet
\]
for $M^\bullet$, $K^\bullet$ and $G^\bullet$ in $D(\mathcal{X})$, whenever either $M^\bullet$ or $G^\bullet$ has finite homological dimension. A simple proof that extends directly to our setting can be found in \cite[1.2]{35}.

Another interesting formula concerns cohomology flat base change in the derived category. It says that if

$$
\begin{array}{ccc}
X & \phi & Y \\
\downarrow f & & \downarrow f \\
\mathcal{X} & \phi & \mathcal{Y}
\end{array}
$$

is a cartesian diagram of morphisms of superschemes and $\phi$ is flat, there is an isomorphism

$$
\phi^* Rf_* \cong Rf_{\mathcal{X}*}\phi^*_{\mathcal{X}}
$$

of functors from $D(\mathcal{X})$ to $D(\mathcal{Y})$. This can be proved as in \cite[3.9.5]{43}.

Let $f: \mathcal{X} \to \mathcal{Y}$ be a morphism of superschemes. We want to apply Brown’s representability theorem \cite[Thm. 3.1]{52} to $Rf_*: D(\mathcal{X}) \to D(\mathcal{Y})$. To this end, one needs two preliminary results. The first is the following, which is proved as \cite[Proposition 2.5]{52}:

**Lemma 3.20.** The derived category $D(\mathcal{X})$ is a compactly generated triangulated category, that is:

1. it contains all small coproducts,\footnote{This is why we need to work with the unbounded derived category.}
2. there exists a small set $P$ of objects in $D(\mathcal{X})$ such that if $\text{Hom}(P^\bullet, M^\bullet) = 0$ for every $P^\bullet \in P$ then $M^\bullet = 0$ in $D(\mathcal{X})$.

The second result, whose proof follows again \cite[Lemma 1.4]{52}, is:

**Lemma 3.21.** If $f: \mathcal{X} \to \mathcal{Y}$ is a morphism of superschemes, the functor $Rf_*: D(\mathcal{X}) \to D(\mathcal{Y})$ preserves (small) coproducts.

One can then apply Brown’s representability to obtain an extension to the super setting of \cite[Example 4.2 and 6]{52}:

**Proposition 3.22** (Relative Grothendieck duality). If $f: \mathcal{X} \to \mathcal{Y}$ is a morphism of superschemes, the functor $Rf_*: D(\mathcal{X}) \to D(\mathcal{Y})$ has a right adjoint $f^!: D(\mathcal{Y}) \to D(\mathcal{X})$. So there is a functorial isomorphism

$$
\text{Hom}_{D(\mathcal{X})}(Rf_* M^\bullet, N^\bullet) \cong \text{Hom}_{D(\mathcal{Y})}(M^\bullet, f^! N^\bullet)
$$

for $M^\bullet$ in $D(\mathcal{X})$ and $N^\bullet$ in $D(\mathcal{Y})$. Moreover, if $f: \mathcal{X} \to \mathcal{Y}$ is proper, there is a sheaf version of the duality isomorphism, namely, there is a functorial isomorphism

$$
R\text{Hom}_{\mathcal{Y}}(Rf_* M^\bullet, N^\bullet) \cong Rf_* R\text{Hom}_{\mathcal{X}}(M^\bullet, f^! N^\bullet).
$$

**Definition 3.23.** The object $D_f^\bullet := f^! O_\mathcal{X}$ in $D(\mathcal{X})$ is called the dualizing complex of $f$.

The dualizing complex determines in many cases the functor $f^!$. The following particular case of \cite[Thm. 5.4]{52} will suffice to our purposes.
Proposition 3.24. Let $f : \mathcal{X} \to \mathcal{I}$ be a proper morphism of superschemes. If $\mathcal{N}^\bullet$ is of finite homological dimension in $D(\mathcal{I})$ there is an isomorphism

$$f^! \mathcal{N}^\bullet \cong Lf^* \mathcal{N}^\bullet \otimes L f^! \mathcal{O}_{\mathcal{I}}.$$ 

Proof. By Grothendieck duality and Equations 3.8 and 3.7 one has

$$Rf_* R\text{Hom}_{\mathcal{O}_{\mathcal{X}}} (\mathcal{M}^\bullet, Lf^* \mathcal{N}^\bullet \otimes L f^! \mathcal{O}_{\mathcal{I}}) \cong Rf_* (R\text{Hom}_{\mathcal{O}_{\mathcal{X}}} (\mathcal{M}^\bullet, f^! \mathcal{O}_{\mathcal{I}})) \otimes L f^* \mathcal{N}^\bullet$$

$$\cong R\text{Hom}_{\mathcal{O}_{\mathcal{X}}} (Rf_* \mathcal{M}^\bullet, \mathcal{O}_{\mathcal{I}}) \otimes L \mathcal{N}^\bullet$$

for every $\mathcal{M}^\bullet$ in $D(\mathcal{X})$. Then one concludes. \hfill \Box

3.5.1. Local properties. Relative Grothendieck duality is local both on the base and on the source. The locality on the base, that is to say, the the base change property under open immersions, was proved for schemes by Neeman [52, Lemma 6.1] as a preliminary lemma for the sheaf version of relative Grothendieck duality. His proof passes directly to the super setting, so that one has:

Proposition 3.25. Let $f : \mathcal{X} \to \mathcal{I}$ be a proper morphism of superschemes and let $j : \mathcal{U} \hookrightarrow \mathcal{I}$ be an open subsuperscheme. Consider the cartesian diagram

$$\xymatrix{ \mathcal{X}_\mathcal{U} \ar[d]^{f_\mathcal{U}} \ar[r]^{i} & \mathcal{X} \ar[d]^{f} \\ \mathcal{U} \ar[r]^{j} & \mathcal{I}. }$$

There is an isomorphism

$$i^* f_\mathcal{U}^! \cong f_\mathcal{X}^! j^*$$

of functors $D(\mathcal{I}) \to D(\mathcal{X}_\mathcal{U})$. \hfill \Box

Locality on the source is also known as independence of the compactification. The statement is:

Proposition 3.26. Let $f_1 : \mathcal{X}_1 \to \mathcal{I}$, $f_2 : \mathcal{X}_2 \to \mathcal{I}$ be proper morphism of superschemes, and let $j_1 : \mathcal{Y} \hookrightarrow \mathcal{X}_1$, $j_2 : \mathcal{Y} \hookrightarrow \mathcal{X}_2$ be open immersions such that the diagram

$$\xymatrix{ \mathcal{Y} \ar[r]^{j_1} & \mathcal{X}_1 \ar[d]^{f_1} \ar[r]^{f_2} & \mathcal{X}_2 \ar[d]^{f_2} \ar[l]_{j_2} \\ \mathcal{Y} \ar[r]_{j_1} & \mathcal{X}_1 \ar[d]^{f_1} \ar[r]^{f_2} & \mathcal{X}_2 \ar[d]^{f_2} \ar[l]_{j_2} }$$

commutes. There is a functorial isomorphism

$$j_1^* f_1^! \cong j_2^* f_2^!.$$ 

Proof. If $\bar{\mathcal{X}}^\prime$ is the superscheme-theoretic closure of the immersion $\mathcal{Y} \hookrightarrow \mathcal{X}_1 \times_\mathcal{I} \mathcal{X}_2$ induced by $(j_1, j_2)$, we have a commutative diagram

$$\xymatrix{ \mathcal{Y} \ar[r]^{j_1} \ar[d]_{j_2} & \mathcal{X}_1 \ar[r]^{f_1} \ar[d]^g & \mathcal{X}_2 \ar[d]_{f_2} \ar[l]_{j_2} \\ \mathcal{Y} \ar[r]_{j_1} \ar[d]_{j_2} & \bar{\mathcal{X}}^\prime \ar[r]^{f_1} \ar[d]^g & \mathcal{X}_2 \ar[d]_{f_2} \ar[l]_{j_2} }$$
where \( \pi_1, \pi_2 \) and \( g \) are proper. One is then reduced to proving that \( j_1^* f_1^! \cong j^* g_1^! \) and \( j_2^* f_2^! \cong j^* g_1^! \). It follows that we can assume that there is a proper morphism \( g: X_1 \to X_2 \) such that \( f_1 = f_2 \circ g \) and \( j_2 = g \circ j_1 \). Let us consider the cartesian diagram

\[
\begin{array}{ccc}
X_1 \times_{X_2} V & \xrightarrow{g} & X_1 \\
\downarrow s & & \downarrow g \\
V & \xrightarrow{j_1 \circ g} & X_1 \\
\end{array}
\]

where \( s = (j_1, \text{Id}) \) and \( V \hookrightarrow X_1 \times_{X_2} V \). Notice that \( s \) is a closed immersion because it is a section of the proper morphism \( g \), and since it is also an open immersion, it has to be an isomorphism with a connected component of \( X_1 \times_{X_2} V \). Then \( s^! = s^* \) and then applying Proposition 3.25 one has

\[
j_2^* f_2^! \cong j_1^* \circ g \circ j_2^* f_1^! \cong j_2^* f_1^!
\]

It follows that \( j_2^* f_2^! \cong j_1^* f_1^! \).

3.5.2. Flat base change for duality. Relative Grothendieck duality is sometimes compatible with base change, in particular it is compatible with flat base changes. We shall give a simple proof in the particular case of flat morphisms that locally of finite type, since the proofs available for general flat morphisms of schemes (see e.g. [43, Thm. 4.4.1]) are not easily translated to superschemes.

Let us consider a cartesian diagram of morphisms of superschemes

\[
\begin{array}{ccc}
\mathcal{X}^0 & \xrightarrow{\phi} & \mathcal{X} \\
\downarrow f & & \downarrow f \\
\mathcal{T} & \xrightarrow{\phi_*} & \mathcal{T} \\
\end{array}
\]

where \( \phi \) is flat. Taking right adjoints in Equation (3.9) one gets a functorial isomorphism

\[
R \phi_* f_!^* \cong f^! R \phi_* .
\]

Composing the inverse morphism with the adjunction morphism \( \text{Id} \to R \phi_* \circ \phi^* \) one has a morphism \( f_!^* \to R \phi_* f_!^* \phi^* \) and then, by adjunction of \( R \phi_* \) and \( \phi^* \), a base change morphism

\[
\phi_! f_!^* \to f_!^* \phi^* . \tag{3.10}
\]

Proposition 3.27. If \( f \) is locally of finite type, the flat base change morphism 3.10 yields a functorial isomorphism

\[
\phi_! f_!^* N^* \cong f_!^* \phi^* N^*
\]

for every object \( N^* \) in \( D(\mathcal{X}) \) of finite homological dimension. In particular, one has an isomorphism

\[
\phi_!^* D_!^* \cong D_!^* .
\]

Proof. By Proposition 3.25 we can assume that \( \mathcal{T} = \text{Spec} \mathcal{A} \) is affine. Then \( \mathcal{X} \) can be covered by affine open sub-superschemes of finite type over \( \mathcal{A} \), that can be embedded as open sub-superschemes of a superprojective superscheme over \( \mathcal{A} \). By the Proposition 3.26 on the independence of the compactification, and again by Proposition 3.25, we can assume that \( \mathcal{T} \) is affine too and that \( f \) is superprojective.

Take a relatively very ample line bundle \( \mathcal{O}_X(1) \). If we write \( g = \phi \circ f_\mathcal{T} = f \circ \phi_{\mathcal{T}} \), by applying cohomology base change (Equation (3.9)), relative Grothendieck duality for \( f \) and
$f_\mathcal{X}$ (Proposition 3.22), and Proposition 3.24, one gets for every pair $r, s$ of integer numbers and every object $\mathcal{N}^\bullet$ in $D(\mathcal{I})$ of finite cohomological dimension, functorial isomorphisms:

$$
\text{Hom}_{D(\mathcal{I})}(\mathcal{O}_{\mathcal{X}}(r)[s], \phi_2^r f_1^r \mathcal{N}^\bullet) \simeq \text{Hom}_{D(\mathcal{I})}(\phi_2^r \mathcal{O}_{\mathcal{X}}(r)[s], \phi_2^r f_1^r \mathcal{N}^\bullet)
$$

$$
\simeq \text{Hom}_{D(\mathcal{I})}(\mathcal{O}_{\mathcal{X}}(r)[s], \mathcal{R}\phi_2^r \phi_2^r f_1^r \mathcal{N}^\bullet)
$$

$$
\simeq \text{Hom}_{D(\mathcal{I})}(\mathcal{O}_{\mathcal{X}}(r)[s], \mathcal{L}f_1^r \phi_2^r \mathcal{O}_{\mathcal{X}_\mathcal{Y}} \otimes^L \mathcal{N}^\bullet)
$$

$$
\simeq \text{Hom}_{D(\mathcal{I})}(\mathcal{O}_{\mathcal{X}}(r)[s], f_1^r (\mathcal{R}\phi_2^r \mathcal{O}_{\mathcal{X}_\mathcal{Y}} \otimes^L \mathcal{N}^\bullet))
$$

$$
\simeq \text{Hom}_{D(\mathcal{I})}(\mathcal{R}f_2, \mathcal{O}_{\mathcal{X}}(r)[s], \phi_2^r \mathcal{N}^\bullet) \simeq \text{Hom}_{D(\mathcal{I})}(\mathcal{O}_{\mathcal{X}}(r)[s], f_1^r \phi_2^r \mathcal{N}^\bullet).
$$

Then $\text{Hom}_{D(\mathcal{I})}(\mathcal{O}_{\mathcal{X}}(r)[s], \mathcal{C}^\bullet) = 0$ for every $r, s$, where $\mathcal{C}^\bullet$ is the cone of $\phi_2^r f_1^r \mathcal{N}^\bullet \to f_2^r \phi_2^r \mathcal{N}^\bullet$. Proceeding as in [52, Example 1.10], one sees that the family of objects $\mathcal{O}_{\mathcal{X}}(r)[s]$ is a compact generating set of $D(\mathcal{I})$, so that one gets $\mathcal{C}^\bullet = 0$ in $D(\mathcal{I})$ and then $\phi_2^r f_1^r \mathcal{N}^\bullet \to f_2^r \phi_2^r \mathcal{N}^\bullet$ is an isomorphism.

From the base change property one deduces the compatibility of the dualizing complex with products.

**Corollary 3.28.** Let $f_1: \mathcal{X}_1 \to \mathcal{I}$, $f_2: \mathcal{X}_2 \to \mathcal{I}$ proper, flat and locally of finite type morphisms of superschemes. Assume that $\mathcal{D}^\bullet_{f_2}$ is of finite homological dimension. Then the dualizing complex of the product $f_1 \times f_2: \mathcal{X}_1 \times_{\mathcal{I}} \mathcal{X}_2 \to \mathcal{I}$ is the (cartesian) product of the dualizing complexes of the factors, that is, there is an isomorphism

$$
\mathcal{D}^\bullet_{f_1 \times f_2} \simeq \mathcal{D}^\bullet_{f_1} \otimes^L_{\mathcal{O}_{\mathcal{I}}} \mathcal{D}^\bullet_{f_2} := p_1^* \mathcal{D}^\bullet_{f_1} \otimes^L \mathcal{D}^\bullet_{f_2},
$$

where $p_i$ are the projections of $\mathcal{X}_1 \times_{\mathcal{I}} \mathcal{X}_2$ onto its factors.

**Proof.** Consider the cartesian diagram

$$
\begin{array}{ccc}
\mathcal{X}_1 & \xrightarrow{p_1} & \mathcal{X}_1 \\
\downarrow{p_2} & & \downarrow{f_1} \\
\mathcal{X}_2 & \xleftarrow{f_2} & \mathcal{I}
\end{array}
$$

Propositions 3.24 and 3.27 give

$$
\mathcal{D}^\bullet_{f_1 \times f_2} = (f_1 \times f_2)^* \mathcal{O}_{\mathcal{I}} \simeq p_1^* f_2^* \mathcal{O}_{\mathcal{X}_2} \simeq p_2^* f_1^* \mathcal{O}_{\mathcal{X}_2} \simeq p_2^* f_1^* \mathcal{O}_{\mathcal{X}_2} \otimes^L p_2^* f_1^* \mathcal{O}_{\mathcal{X}_2} = p_1^* \mathcal{D}^\bullet_{f_1} \otimes^L p_2^* \mathcal{D}^\bullet_{f_2}.
$$

3.5.3. **Duality for affine morphisms.** When $f: \mathcal{X} \to \mathcal{I}$ is an affine morphism of superschemes, the derived functor $\mathcal{R}f_*$ is isomorphic to $f_*$. One then has:

**Proposition 3.29.** Let $f: \mathcal{X} \to \mathcal{I}$ be a proper affine morphism of (separated) superschemes. For every object $\mathcal{N}^\bullet$ in $D(\mathcal{I})$ one has a functorial isomorphism

$$
f_* f^! \mathcal{N}^\bullet \simeq \mathcal{R}\text{Hom}_{\mathcal{O}_{\mathcal{I}}}(f_* \mathcal{O}_{\mathcal{X}}, \mathcal{N}^\bullet).
$$

□
Since any quasi-coherent sheaf on $\mathcal{X}$ is determined by its direct image, we can write

$$f^!\mathcal{N}^* \simeq \mathcal{R}\text{Hom}_{\mathcal{O}_\mathcal{X}}(\mathcal{O}_\mathcal{X}, \mathcal{N}^*),$$

where the second member is endowed with its natural structure of $\mathcal{O}_\mathcal{X}$-module.

An interesting case is when $f$ is a closed immersion $f: \mathcal{X} \hookrightarrow \mathcal{Y}$. Then for every quasi-coherent sheaf $\mathcal{N}$ on $\mathcal{X}$ Equation (3.11) gives the cohomology sheaves of $f^!\mathcal{N}$:

$$\mathcal{H}^i(f^!\mathcal{N}) \simeq \begin{cases} \mathcal{Ext}^i_{\mathcal{O}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y}, \mathcal{N}) & i \geq 0 \\ 0 & i < 0 \end{cases}$$

(3.12)

When the closed immersion $f: \mathcal{X} \hookrightarrow \mathcal{Y}$ is locally regular, that is, when its ideal $\mathcal{I}$ is locally generated by a regular sequence $(a_1, \ldots, a_p, \eta_1, \ldots, \eta_q)$ with the $a_i$ even and the $\eta_j$ odd [55], the sheaves $\mathcal{H}^{-i}(f^!\mathcal{N})$ can be computed in terms of the normal sheaf $\mathcal{N}_f := (\mathcal{I}/\mathcal{I}^2)^*$, as we shall see in the following paragraphs.

If $\mathbb{A}$ is a superring, $(a_1, \ldots, a_p, \eta_1, \ldots, \eta_q)$ ($|a_i| = 0, |\eta_j| = 1$), the associated Koszul complex is defined as the graded symmetric algebra $\mathcal{K}(a, \eta) = \text{Sym}_\mathbb{A} L^\Pi$, where $L$ is a free module of rank $(p, q)$ with homogeneous basis $(x_1, \ldots, x_p, \theta_1, \ldots, \theta_q)$ ($|x_i| = 0, |\theta_j| = 1$) equipped with the differential $d = \sum_i a_i \frac{\partial}{\partial x_i} + \sum_j \eta_j \frac{\partial}{\partial \theta_j}$.

**Proposition 3.30.** [55] Let $I$ be an ideal generated by a regular sequence $(a, \eta) = (a_1, \ldots, a_p, \eta_1, \ldots, \eta_q)$ and $\mathbb{A} = \mathbb{A}/I$. One has:

1. $H_i(K(a, \eta)) = 0$ for $i \neq 0$ and $H_0(K(a, \eta)) = \mathbb{A}$, that is, the Koszul complex is a free resolution of $\mathbb{A}$ as an $\mathbb{A}$-module.
2. $H^i(K(a, \eta)^*) = 0$ if $i \neq p$ and $H^p(K(a, \eta)^*) = \text{Ber}(I/I^2)^*$, so that there is an isomorphism

   $$\text{Ext}_\mathbb{A}^i(\mathbb{A}, \mathbb{A}) \xrightarrow{\gamma(a, \eta)} \begin{cases} 0 & i \neq p \\ \text{Ber}(I/I^2)^* & i = p \end{cases}$$

(3)

Moreover, if $(a', \eta') = (a'_1, \ldots, a'_p, \eta'_1, \ldots, \eta'_q)$ is another regular sequence generating $I$, then $\gamma(a', \eta') = \text{Ber}(A)\gamma(a, \eta)$, where $A$ is the matrix relating the bases induced by the two regular sequences.

□

**Proposition 3.31.** Let $f: \mathcal{X} \hookrightarrow \mathcal{Y}$ be a closed immersion of codimension $(p, q)$ given by an ideal $\mathcal{I}$, and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. If $\mathcal{U}$ is the open subsuperscheme of $\mathcal{Y}$ where $f_!: \mathcal{X}_U \hookrightarrow \mathcal{U}$ is locally regular, there is an isomorphism

$$ (f^!\mathcal{L})|_{\mathcal{X}_U} \simeq (f_!\mathcal{L}_U) \cong \text{Ber}N_{f!\mathcal{U}} \otimes f^*\mathcal{L}_U[-p], $$

in the derived category $D(\mathcal{X}_U)$, where $\mathcal{N}_f = (\mathcal{I}/\mathcal{I}^2)^*$ is the normal sheaf to $f$.

**Proof.** By Proposition 3.25 one can assume that $f$ is locally regular. The result follows from Proposition 3.30 as the local isomorphisms $\text{Ext}_\mathbb{A}^i(\mathbb{A}, \mathbb{A}) \cong \text{Ber}(I/I^2)^*$ glue because of property (3).

□

3.5.4. Duality for smooth morphisms. We can now compute the dualizing complex of a proper smooth morphism. The first step is the following result on the dualizing complex of projective superspace.
Lemma 3.32. Let \( \mathcal{X} = \mathbb{P}^{m,n}_A \to \mathcal{I} = \text{Spec } A \) be the projective superspace over a supersing \( A \). The relative dualizing complex is of the form

\[
\mathcal{D}_\pi^* \simeq \mathcal{L}[m],
\]

where \( \mathcal{L} \) is a line bundle.

Proof. One has \( \mathcal{X} = \text{Proj } A[x_0, \ldots, x_m, \theta_1, \ldots, \theta_n] \), and \( \mathcal{X} \) is covered by the affine open sub-superschemes \( j_i: \mathcal{U}_i \hookrightarrow \mathcal{X} \), complementary to the closed sub-superschemes defined by the ideal \( \mathfrak{p}_i \) generated by homogeneous localization with respect to \( x_i \). Then,

\[
j_* j^* \mathcal{D}_\pi^* \cong \varprojlim_r \mathcal{R} \text{Hom}_{\mathcal{O}_\mathcal{X}}(p_i^*, \mathcal{D}_\pi^*) \cong \varprojlim_r \mathcal{R} \text{Hom}_A(\mathcal{R} f_* p_i^*, \mathcal{L})
\]

by duality. Since \( p_i \) is isomorphic with \( \mathcal{O}_{\mathcal{X}}(-1) \), one has that \( R^i f_* p_i^* = 0 \) for \( i \neq m \) and that \( R^m f_* p_i^* \simeq H^m(\mathcal{X}, \mathcal{O}_\mathcal{X}(-r)) \) is a free \((\mathbb{Z}_2, \text{graded})\) \( A \)-module for \( r \gg 0 \) (Proposition 3.26). It follows that

\[
j_* j^* \mathcal{D}_\pi^* \cong \varprojlim_r \mathcal{R} \text{Hom}_A(R^m f_* p_i^*[-m], \mathcal{L}) \simeq (\varprojlim_r H^m(\mathcal{X}, \mathcal{O}_\mathcal{X}(-r))^r)[m].
\]

This proves that \( \mathcal{D}_\pi^* \simeq \mathcal{L}[m] \) for a quasi-coherent sheaf \( \mathcal{L} \) on \( \mathcal{X} \) such that

\[
\Gamma(U_i, \mathcal{L}) \cong \varprojlim_r H^m(\mathcal{X}, \mathcal{O}_\mathcal{X}(-r))^r \quad \text{for every } i.
\]

Let us prove that \( \mathcal{L} \) is a line bundle. In the case of the ordinary projective space \( X = \mathbb{P}^m_A \) one has

\[
\varprojlim_r H^m(X, \mathcal{O}_X(-r))^r \cong A[x_0/x_1, \ldots, x_i/x_1, \ldots, x_m/x_1].
\]

In our setting, from the expression \( \mathcal{O}_X \simeq \mathcal{O}_X \oplus \bigoplus_{1 \leq j_1 < \cdots < j_p \leq n} \mathcal{O}_X(-p) \theta_{i_{j_1}} \cdots \theta_{i_{j_p}} \) we obtain isomorphisms (that do not glue on \( U_i \cap U_j \))

\[
\varprojlim_r H^m(\mathcal{X}, \mathcal{O}_\mathcal{X}(-r))^r \cong \mathcal{O}_{\mathcal{X}}[x_0/x_1, \ldots, x_i/x_1, \ldots, x_m/x_1, \theta_1, \ldots, \theta_n] \simeq \Gamma(U_i, \mathcal{O}_\mathcal{X}),
\]

and then \( \mathcal{L} \) is a line bundle by Equation (3.13). \( \square \)

As in the classical case one proves that:

Lemma 3.33. If \( f: \mathcal{X} \to \mathcal{I} \) is smooth and \( j: \mathcal{V} \hookrightarrow \mathcal{X} \) is a closed immersion such that the composition \( g = f \circ j: \mathcal{V} \to \mathcal{I} \) is smooth, then \( j \) is locally regular. \( \square \)

We now recover the computation of the dualizing sheaf of a smooth morphism \([55, 57]\).

Lemma 3.34. Let \( f: \mathcal{X} \to \mathcal{I} \) be a proper smooth morphism of superschemes of relative even dimension \( m \). The dualizing complex of \( f \) has a unique cohomology sheaf in degree \( -m \), that is, there is an isomorphism

\[
\mathcal{D}_f^* \cong \omega_f[m]
\]

in the derived category \( D(\mathcal{X}) \), where \( m \) is the relative even dimension of \( f \). Moreover, \( \omega_f \) is a line bundle.

Proof. This is a local question so that we can assume that \( \mathcal{I} \) is affine, \( \mathcal{I} = \text{Spec } A \) (Proposition 3.25). Then we can cover \( \mathcal{X} \) by affine superschemes \( \mathcal{V} \) smooth of finite type over \( A \). We have a closed immersion \( j: \mathcal{V} \hookrightarrow \mathcal{U} = \text{Spec } A[x_1, \ldots, x_p, \theta_1, \ldots, \theta_q] \), where \( (x_1, \theta_j) \) are free variables with \( |x_1| = 0, |\theta_j| = 1 \). Moreover \( j \) is locally regular by Lemma 3.33. If we denote by \( j: \mathcal{V} \hookrightarrow \mathbb{P}^p_A \) the induced closed immersion, by Proposition 3.26 we have to prove that \( \mathcal{D}_f^*|_{\mathcal{V} \to \mathcal{I}} \cong \omega[m] \) for a line bundle \( \omega \) on \( \mathcal{V} \), where \( \pi: \mathbb{P}^p_A \to \mathcal{I} \) is the projection. But \( \mathcal{D}_\pi^* \cong \mathcal{L}[p] \) for a line bundle \( \mathcal{L} \) on \( \mathbb{P}^p_A \) by Lemma 3.32 and one finishes by Proposition 3.31. \( \square \)
Proposition 3.35. In the situation of Lemma 3.34, there is an isomorphism
\[ \omega_f \simeq \text{Ber}(\mathcal{X}/\mathcal{I}) = \text{Ber}(\Omega_{\mathcal{X}/\mathcal{I}}). \]

Proof. If \( \delta: \mathcal{X} \hookrightarrow \mathcal{X} \times_{\mathcal{I}} \mathcal{X} \) is the diagonal immersion, so that \( f = p_1 \circ (f \times f) \), one has
\[ D^*_f = f^! O_S \simeq \delta^! D^*_{f \times f}. \]

By Lemma 3.34 \( D^*_f \simeq \omega_f[m] \). where \( \omega_f \) is a line bundle, which is of finite homological dimension. Then we can apply Corollary 3.28 to get \( D^*_{f \times f} \simeq D^*_f \otimes_{D^*_f} D^*_f \simeq \omega_f \otimes_{O_{\mathcal{I}} \omega_f}[2m]. \)

Thus
\[ \omega_f[m] \simeq D^*_f \simeq \delta^!(\omega_f \otimes_{O_{\mathcal{I}} \omega_f} \omega_f)[-2m] \simeq \delta^*(\omega_f \otimes_{O_{\mathcal{I}} \omega_f} \omega_f) \otimes D^*_f[2m] \simeq \omega_f \otimes \omega_f \otimes D^*_f[2m]. \]

by Proposition 3.24. Since \( f \) is smooth, \( \delta \) is locally a regular by Lemma 3.33; then Proposition 3.31 yields \( D^*_f \simeq \text{Ber}(\Delta_f/\Delta^2_f)[-m] = \text{Ber}(\Omega_{\mathcal{X}/\mathcal{I}})^*[n-m] \) and the result follows. \( \square \)

Corollary 3.36. Let \( \pi: \mathcal{X} = \mathbb{P}^{m,n}_S \to \mathcal{I} \) be a projective superspace over a locally noetherian superscheme \( \mathcal{I} \). Then
\[ \omega_\pi \simeq O_{\mathcal{I}}(n-m-1). \]

Proof. One has \( \text{Ber}(\mathbb{P}^{m,n}_S/\mathcal{I}) \simeq O_{\mathcal{I}}(n-m-1) \) [46]. \( \square \)

4. The Hilbert superscheme

4.1. The super Hilbert functor. All superschemes we consider are locally noetherian.

Let \( f: \mathcal{X} \to \mathcal{I} \) be a morphism of superschemes. For every superscheme \( \phi: \mathcal{I} \to \mathcal{I} \) over \( \mathcal{I} \) consider the cartesian diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\phi} & \mathcal{X} \\
\downarrow f \mathcal{I} & & \downarrow f \mathcal{I} \\
\mathcal{I} & \xrightarrow{\phi} & \mathcal{I} \\
\end{array}
\]

Definition 4.1. The relative super Hilbert functor is the functor on the category of superschemes over \( \mathcal{I} \) that to every superscheme morphism \( \mathcal{I} \to \mathcal{I} \) associates the family \( \text{SHilb}_{\mathcal{X}/\mathcal{I}}(\mathcal{I}) \) of all the closed sub-superschemes

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{X} \\
\downarrow g \mathcal{I} & & \downarrow f \mathcal{I} \\
\mathcal{I} & & \\
\end{array}
\]

that are proper and flat over \( \mathcal{I} \).

The super Hilbert functor is a sheaf for the Zariski topology of superschemes.

Assume that \( f: \mathcal{X} \to \mathcal{I} \) is superprojective morphism with a relative very ample line bundle \( O_{\mathcal{X}}(1) \) (Remark 2.25). We can decompose the super Hilbert functor into subfunctors parametrizing closed sub-superschemes of the fibres with fixed super Hilbert polynomial (Definition 2.41):

Definition 4.2. Let \( P = (P_+, P_-) \) be a pair of polynomials with rational coefficients. The super Hilbert functor \( \text{SHilb}^P_{\mathcal{X}/\mathcal{I}} \) is the subsheaf of \( \text{SHilb}_{\mathcal{X}/\mathcal{I}} \) given by the relative closed sub-superschemes of \( f_\mathcal{I}: \mathcal{X}_{\mathcal{I}} \to \mathcal{I} \) whose super Hilbert polynomial is \( P \).
When $f: \mathcal{X} \to \mathcal{I}$ is quasi-superprojective, so that $f = \bar{f} \circ j$, where $j: \mathcal{X} \to \bar{\mathcal{X}}$ is an open immersion and $\bar{f}$ is superprojective, as the objects of $\mathcal{SH}(\mathcal{X}/\mathcal{I})$ are also objects of $\mathcal{SH}(\bar{\mathcal{X}}/\mathcal{I})$ one can decompose $\mathcal{SH}(\mathcal{X}/\mathcal{I})$ as the union of the subfunctors $\mathcal{SH}(\mathcal{X}/\mathcal{I})$ given by the relative closed sub-superschemes of $f_{\mathcal{I}}: \mathcal{X} \to \mathcal{I}$, proper over $\mathcal{I}$, whose super Hilbert polynomial is $P$.

4.2. Statement of the existence theorems. Our aim is to prove the following representability theorems. The first deals with the superprojective case, while the second establishes the result under more general conditions. This generality will be needed to prove the representability of the super Picard functor.

**Theorem 4.3** (Existence, superprojective case). Let $f: \mathcal{X} \to \mathcal{I}$ be a superprojective or quasi-superprojective morphism with a relative very ample line bundle $O_{\mathcal{X}}(1)$.

1. For any super Hilbert polynomial $P$, the super Hilbert functor $\mathcal{SH}(\mathcal{X}/\mathcal{I})$ is representable by an $\mathcal{I}$-superscheme $\mathcal{SH}(\mathcal{X}/\mathcal{I}) \to \mathcal{I}$. Moreover, if $f$ is superprojective (resp., quasi-superprojective), $\mathcal{SH}(\mathcal{X}/\mathcal{I})$ is proper over $\mathcal{I}$ (resp., an open sub-superscheme of a proper superscheme over $\mathcal{I}$). Then, $\mathcal{SH}(\mathcal{X}/\mathcal{I}) \to \mathcal{I}$ is of finite type and separated.

2. As a consequence, the super Hilbert functor $\mathcal{SH}(\mathcal{X}/\mathcal{I})$ is representable by the disjoint union of $\mathcal{SH}(\mathcal{X}/\mathcal{I})$ of the $\mathcal{I}$-superschemes $\mathcal{SH}(\mathcal{X}/\mathcal{I})$ corresponding to the various super Hilbert polynomials.

**Theorem 4.4** (Existence, general case). Let $\mathcal{I}$ be noetherian and let $f: \mathcal{X} \to \mathcal{I}$ be a separated morphism of superschemes. Assume that there exists a faithfully flat morphism $\bar{\mathcal{X}} \to \mathcal{X}$ such that $\bar{\mathcal{X}} \to \mathcal{I}$ is superprojective (resp. quasi-superprojective). Then the super Hilbert functor of $\mathcal{X}/\mathcal{I}$ is representable by a closed sub-superscheme (resp. open sub-superscheme) $\mathcal{SH}(\mathcal{X}/\mathcal{I})$ of the super-Hilbert scheme $\mathcal{SH}(\bar{\mathcal{X}}/\mathcal{I})$ of $\bar{\mathcal{X}}/\mathcal{I}$. Moreover, $\mathcal{SH}(\mathcal{X}/\mathcal{I})$ is locally of finite type and separated over $\mathcal{I}$.

**Remark 4.5.** Notice that, due to Corollary 2.30, Theorem 4.4 can be applied taking for $\mathcal{X}$ any sub-superscheme of a supergrassmannian over $\mathcal{I}$.

**Proof.** We prove Theorem 4.4 using Theorem 4.3, which will be in turn proved later on. We give a proof when $\bar{\mathcal{X}} \to \mathcal{I}$ is superprojective, as the other case follows easily from it. Set $\mathcal{U} = \bar{\mathcal{X}} \times_{\mathcal{I}} \mathcal{X}$. Note that $\mathcal{U}$ can be identified with the superschematic preimage of the relative diagonal under the morphism

$$\bar{\mathcal{X}} \times_{\mathcal{I}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{I}} \mathcal{X}.$$ 

Since $\mathcal{X}$ is separated over $\mathcal{I}$, we deduce that $\mathcal{U}$ is a closed sub-superscheme of $\bar{\mathcal{X}} \times_{\mathcal{I}} \mathcal{X}$, and then it is superprojective over $\mathcal{I}$. Thus by Theorem 4.3 we have a Hilbert superscheme $\mathcal{SH}(\mathcal{U}/\mathcal{I})$ which is separated over $\mathcal{I}$.

Note that, as $\bar{\mathcal{X}}$ is superprojective over $\mathcal{I}$, the morphism $\bar{\mathcal{X}} \to \mathcal{X}$ is quasi-compact. By the effective descent for sub-superschemes (Proposition A.30), to give a sub-superscheme of $\mathcal{X} \times_{\mathcal{I}} \mathcal{X}$ (where $\mathcal{I}$ is a superscheme over $\mathcal{I}$) is equivalent to giving a sub-superscheme $\mathcal{U}$ of $\bar{\mathcal{X}} \times_{\mathcal{I}} \mathcal{Y}$ such that $p_{1}^{-1}(\mathcal{Y}) = p_{2}^{-1}(\mathcal{Y})$ in $\mathcal{U}$, where $p_{1}$, $p_{2}$ are the projections of $\mathcal{U}$ onto its factors.

Thus, the superHilbert functor of $\mathcal{X}/\mathcal{I}$ is represented by the closed sub-superscheme of $\mathcal{SH}(\mathcal{X}/\mathcal{I})$ given as the superschematic preimage of the relative diagonal under the
morphism
\[ \mathcal{SHilb}(X/\mathcal{I}) \xrightarrow{(p_1^{-1}, p_2^{-1})} \mathcal{SHilb}(Y/\mathcal{I}) \times_{\mathcal{I}} \mathcal{SHilb}(Y/\mathcal{I}). \]

We now prove Theorem 4.3. This will be done in several steps and reducing to simpler situations.

**Lemma 4.6.** Let \( f: X \to S \) be a superprojective or quasi-superprojective morphism.

1. If there is a covering of \( S \) by open sub-superschemes \( U \) such that the functors \( \mathcal{SHilb}^P_{X/\mathcal{I}} \) are representable, then \( \mathcal{SHilb}^P_{X/\mathcal{I}} \) is representable as well. Moreover \( \mathcal{SHilb}^P(X/\mathcal{I}) \to \mathcal{U} \) is proper if and only if all the local morphisms \( \mathcal{SHilb}^P(X_U/\mathcal{I}) \to \mathcal{U} \) are proper.

2. If \( j: Y \hookrightarrow X \) is a closed (open) immersion of \( S \)-schemes, then the functor morphism \( j^*: \mathcal{SHilb}^P_{Y/\mathcal{I}} \to \mathcal{SHilb}^P_{X/\mathcal{I}} \) is representable by closed (open) immersions.

**Proof.** (1) It follows from the sheaf condition for the relative super Hilbert functor.

(2) One has to show that given a superscheme morphism \( \phi: T \to S \) and a closed (open) sub-superscheme \( \delta: Z \hookrightarrow T \) flat and proper over \( T \) with Hilbert polynomial \( P \), there exists a closed (resp. open) sub-superscheme \( T' \hookrightarrow T \) with the following universal property: for any \( T \)-superscheme \( \psi: U \to T \), \( \psi \) factors through \( T' \hookrightarrow T \) if and only if \( Z_U \) is the image of \( j^*: \mathcal{SHilb}^P_{Y/\mathcal{I}}(U) \to \mathcal{SHilb}^P_{X/\mathcal{I}}(U) \).

In the case of a closed immersion, this is equivalent to saying that the pull-back epimorphism \( O_{X_U} \xrightarrow{\psi^*(\delta^*)} O_{X_U} \to 0 \) factors through \( \psi^*(j^*): \)

\[
\begin{array}{ccc}
0 & \xrightarrow{\psi^*(\delta^*)} & O_{X_T} \\
\xrightarrow{\ker \psi^*(j^*)} & & \xrightarrow{\psi^*(j^*)} O_{X_U} \\
& & \xrightarrow{0}
\end{array}
\]

Then \( T' \) is the zero locus of the composition morphism \( \ker \psi^*(j^*) \to O_{X_T} \to O_{X_T} \), which exists by Corollary 3.19 because \( O_{X_T} \) is flat over \( T \).

When \( j: Y \hookrightarrow X \) is an open immersion, if it's enough to take \( T' \) as the complementary of sub-superscheme \( f(X_T - j(Y_T)) \), which is closed because \( f \) is proper. \( \square \)

One then has:

**Proposition 4.7.** It is enough to prove Theorem 4.3 when \( \mathcal{I} = \text{Spec} \mathbb{A} \) is affine and noetherian, \( X = \mathbb{P}^{n,m}_\mathbb{A} \) is a projective superspace over \( \mathbb{A} \), and \( f \) is the natural projection \( \mathbb{P}^{n,m}_\mathbb{A} \to X \). \( \square \)

**Strategy 4.8.** Once we are reduced to the case of the projective superspace over a superring, the proof of the existence of the Hilbert superscheme is similar to the strategy for the classical case:

1. Using cohomology base change (Subsection 3.2) and Castelnuovo-Mumford regularity (Subsection 4.3), one proves that, for any pair \( P = P(r) = (P_+(r), P_-(r)) \) of polynomials with rational coefficients, and every closed sub-superscheme \( \mathcal{Z} \) with \( \mathbb{P}^{m,n}_\mathbb{A} \) of
super Hilbert polynomial $P$, there exists an integer $q$, depending only on the coefficients of $P$, such that $f_* \mathcal{O}_\mathscr{X}(q)$ is a locally free quotient of rank $P$ of the locally free sheaf $f_* \mathcal{O}(q)$ on $\text{Spec } \mathbb{A}$ (where $\mathcal{O} = \mathcal{O}_{\mathbb{P}^m,n}$). This gives an immersion of the super Hilbert functor into a supergrassmannian.

(2) Using generic flatness and the flattening (Subsection 4.4), one proves that this morphism of functors is representable by immersions. This proves that the Hilbert superscheme exists as a sub-superscheme of that supergrassmannian.

The same results are also contained in [49, 7.1.8, 7.19] in the more general situation of Quot superschemes.

Remark 4.9. The existence of the Hilbert superscheme implies the existence of the superscheme of morphisms between two projective superschemes, see Subsection 4.8. △

In the following Subsections we complete the proof of Theorem 4.3.

4.3. $m$-regularity for projective superschemes. In the classical case, the Castelnuovo-Mumford $m$-regularity is used to embed the Hilbert functor into a grassmannian functor. This procedure can be adapted to our situation (see [49, 7.1.6]). Let $k$ be a field and $\mathscr{X} = \mathbb{P}^{m,n}_k$ the projective superspace over $k$. We write $\mathcal{O}(r) = \mathcal{O}_\mathscr{X}(r)$ for every integer $r$.

Definition 4.10. A coherent sheaf $\mathcal{M}$ on $\mathscr{X}$ is $r$-regular if one has

$$H^i(X, \mathcal{M}(r-i)) = 0 \text{ for } i > 0.$$  

We now state a super Castelnuovo Theorem, whose proof in view of Remark A.2 is the same as in the classical case [50, 53].

Proposition 4.11. If $\mathcal{M}$ is a coherent $r$-regular sheaf on $\mathscr{X}$, then for every $r' \geq r$ one has:

1. $\mathcal{M}$ is $r'$-regular;
2. the natural morphism $H^0(X, \mathcal{M}(r')) \otimes_k H^0(X, \mathcal{O}(1)) \rightarrow H^0(X, \mathcal{M}(r'+1))$ is surjective;
3. $\mathcal{M}(r')$ is generated by its global sections.

□

For the construction of the super Hilbert scheme we shall need the following result on $r$-regularity, which corresponds to Mumford’s theorem [50, Thm. 2.3].

Theorem 4.12. For every pair $(m, n)$ of non-negative integers, there exist universal polynomials $F^m_{+n}, F^m_{0n}$ with integer coefficients having the following property:

for any field $k$ and every coherent sheaf $p$ of ideals on $\mathscr{X} = \mathbb{P}^{m,n}_k$ with super Hilbert polynomial

$$H(X, p)(r) = \left( \sum_{0 \leq p \leq m} a_{p,0} \binom{r}{p}, \sum_{0 \leq p \leq m} a_{p,1} \binom{r}{p} \right),$$

the sheaf $p$ is $q$-regular, with $q = \max(F^m_{+n}(a_{00}, \ldots, a_{m0}), F^m_{0n}(a_{01}, \ldots, a_{m1}))$.

We state a preliminary Lemma. Let $\mathscr{X}' \simeq \mathbb{P}^{m-1,n}_k \hookrightarrow \mathscr{X}$ be a super-hypersurface which does not meet any of the points corresponding to the associated primes of $\mathcal{M}$, as in Remark A.2, so that there is an exact sequence

$$0 \rightarrow \mathcal{M}(-1) \rightarrow \mathcal{M} \rightarrow \mathcal{M}' := \mathcal{M}|_{\mathscr{X}'} \rightarrow 0.$$  

Lemma 4.13. If $\mathcal{M}'$ is $r$-regular, then:
\(1\) if \(i \geq 2\), then \(H^i(X, \mathcal{M}(p)) = 0\) for \(p \geq r - i\);

\(2\) \(H^i(X, \mathcal{M}(p)) = 0\) for \(p \geq r - 1 + q\), with \(q = \max(h_1(r-1)_0, h_1(r-1)_1)\), where \((h_1(r-1)_0, h_1(r-1)_1) = \dim_k H^1(X, \mathcal{M}(r-1))\).

Thus, \(\mathcal{M}\) is \((r + q)\)-regular.

**Proof.** (1) For \(i \geq 2\), \(p \geq r - (i - 1)\), the exact sequence of cohomology together with (1) of Proposition 4.11, give isomorphisms

\[H^i(X, \mathcal{M}(p - 1)) \cong H^i(X, \mathcal{M}(p)) \cong \ldots \cong 0\.

(2) If \(s \geq r - 1\), one has an exact sequence

\[0 \to H^0(X, \mathcal{M}(p - 1)) \to H^0(X, \mathcal{M}(p)) \xrightarrow{\alpha_p} H^0(X, \mathcal{M}'(p)) \to H^1(X, \mathcal{M}(p - 1)) \to H^1(X, \mathcal{M}(p)) \to 0.

If \(\alpha_p\) is surjective, from

\[\begin{array}{c}
H^0(X, \mathcal{M}(p)) \otimes_k H^0(X, \mathcal{O}(1)) \\
\downarrow \alpha_p \otimes 1 \\
H^0(X, \mathcal{M}'(p)) \otimes_k H^0(X, \mathcal{O}(1)) \\
\downarrow \alpha_{p+1} \\
H^0(X, \mathcal{M}(p + 1)) \\
\downarrow \\
0
\end{array}
\]

whose second column is exact by Proposition 4.11, it follows that \(\alpha_{p+1}\) is surjective as well. Thus, \(H^1(X, \mathcal{M}(p - 1)) \cong H^1(X, \mathcal{M}(p)) \cong \ldots \cong 0\).

If \(\alpha_p\) is not surjective, then \(\dim_k H^1(X, \mathcal{M}(p)) < \dim_k H^1(X, \mathcal{M}(p - 1))\), (with respect to the order given by Equation (2.7). Then, \(\dim H^1(X, \mathcal{M}(p))\) decreases until it is zero. \(\square\)

**Proof of Theorem 4.12.** Since cohomology is compatible with the flat base change \(\text{Spec} K \to \text{Spec} k\) induced by a field extension, we can assume that \(k\) is infinite. We proceed by induction on \(m\), the case \(m = 0\) being trivial. Since \(k\) is infinite, there exists a hypersurface \(\mathbb{P}_k^{n-1,n} = \mathcal{X}' \to \mathcal{X}\) which does not meet the points corresponding to the associated primes of \(\mathcal{O}/p\) (Remark A.2). Then \(\mathcal{T}or^2_{\mathcal{O}'}(\mathcal{O}', \mathcal{O}/p) = 0\) (Equation (A.1)) so that \(p' := p \otimes_{\mathcal{O}} \mathcal{O}' = p_{\mathcal{X}'}\) is a coherent ideal on \(\mathcal{X}'\), and one has

\[0 \to p(r - 1) \to p(r) \to p'(r) \to 0,
\]
for every integer \(r\).

Since

\[
\chi(X, p'(r)) = \chi(X, p(r)) - \chi(X, p(r - 1))
\]

\[= \left( \sum_{0 \leq p \leq m} a_{p,0} \left( \frac{r}{p} - \frac{r-1}{p} \right) \right) \sum_{0 \leq p \leq m} a_{p,1} \left( \frac{r}{p} - \frac{r-1}{p} \right), \]

by induction on \(m\), \(p'\) is \(q'\)-regular with \(q' = \max(Q_0^{m,n}(a_{1,0}, \ldots, a_{m,0}), Q_1^{m,n}(a_{1,1}, \ldots, a_{m,1}))\) for certain universal polynomials \(Q_0^{m,n}, Q_1^{m,n}\). By Lemma 4.13, \(p\) is \((q' + q)\)-regular, where

\[q = \max(\dim_k H^1(X, p(r - 1)), \dim_k H^1(X, p(r - 1))_1),
\]
and \( H^i(X, p(r - 1)) = 0 \) for \( i \geq 2 \). Moreover, by Proposition 2.33

\[
\dim_k H^1(X, p(r - 1)) = \dim_k H^0(X, p(r - 1)) - \chi(X, p(r - 1)) \\
\leq \dim_k H^0(X, \mathcal{O}(r - 1)) - \mathcal{P}(r - 1) \\
= h_{(m,n)}(r) - \mathcal{P}(r - 1) \\
= \left( H^0_{m,n}(r, a_{1,0}, \ldots, a_{m,0}), H^{m,n}_1(r, a_{1,1}, \ldots, a_{m,1}) \right)
\]

for certain universal polynomials \( H^0_{m,n}, H^{m,n}_1 \). Then \( p \) is \((q + \bar{q})\)-regular, where

\[
\bar{q} = \max(q + H^0_{m,n}(r, a_{1,0}, \ldots, a_{m,0}), q + H^{m,n}_1(r, a_{1,1}, \ldots, a_{m,1})).
\]

\[ \square \]

4.4. Generic flatness and flattening. We start this Subsection with a result about base change without any flatness conditions.

Let \( \mathcal{I} \) be a superscheme and \( \mathcal{M} \) a coherent sheaf on \( \mathbb{P}^{m,n}_\mathcal{I} \). For every \( \mathcal{I} \)-superscheme \( \phi : \mathcal{I} \to \mathcal{I} \) we have the base change cartesian diagram

\[
\begin{array}{ccc}
\mathbb{P}^{m,n}_\mathcal{I} & \xrightarrow{\phi} & \mathbb{P}^{m,n}_\mathcal{I} \\
\downarrow f_{\mathcal{I}} & & \downarrow f \\
\mathcal{I} & \xrightarrow{\phi} & \mathcal{I}
\end{array}
\]

**Proposition 4.14.** If \( \mathcal{I} \) and \( \mathcal{I} \) are noetherian, there exists an integer \( r_0 \) (which may depend on \( \phi \) and on \( \mathcal{M} \)) such that the base change morphism

\[
\phi^*_f \mathcal{M}(r) \to f_{\mathcal{I}}^* \mathcal{M}_\mathcal{I}(r)
\]

is an isomorphism for every \( r \geq r_0 \).

**Proof.** Since \( \mathcal{I} \) and \( \mathcal{I} \) are noetherian, we can assume that they are affine, \( \mathcal{I} = \text{Spec} \mathbb{A}, \mathcal{I} = \text{Spec} \mathbb{A}' \). The sheaf \( \mathcal{M} \) is the sheaf associated by \( \mathbb{Z} \)-homogeneous localization to the bigraded \( \mathbb{A} \)-module \( \Gamma_\ast(\mathcal{M}) = \bigoplus_{r \geq 0} f_* \mathcal{M}(r) \) (Proposition 2.11). Then \( \mathcal{M}_\mathcal{I} \) is the sheaf associated by \( \mathbb{Z} \)-homogeneous localization to the bigraded \( \mathbb{A}' \)-module \( \Gamma_\ast(\mathcal{M}_\mathcal{I}) \simeq \Gamma_\ast(\mathcal{M}) \otimes_\mathbb{A} \mathbb{A}' \). On the other hand, \( \mathcal{M}_\mathcal{I} \) is also the sheaf associated by \( \mathbb{Z} \)-homogeneous localization to \( \Gamma_\ast(\mathcal{M}_\mathcal{I}) = \bigoplus_{r \geq 0} f_{\mathcal{I}} \mathcal{M}_\mathcal{I}(r) \). Thus the morphism of bigraded \( \mathbb{A}' \)-modules \( \Gamma_\ast(\mathcal{M}_\mathcal{I}) \to \Gamma_\ast(\mathcal{M}_{\mathcal{I}}) \), given by the base change morphisms, induces an isomorphism between the sheaves associated by \( \mathbb{Z} \)-homogeneous localization. One concludes by Proposition 2.10. \[ \square \]

We now prove a generic flatness result for the simple case we need. The proof adapts to our setting that for the classical case given in [7, Thm. 3.7.3], and is similar to the one in [49, 7.1.7].

**Proposition 4.15** (Generic flatness). Let \( S \) be a noetherian integral scheme and \( f : \mathcal{X} \to S \) a superprojective morphism. For every coherent sheaf \( \mathcal{M} \) on \( \mathcal{X} \) there is a nonempty affine open subscheme \( U \) of \( S \) such that \( \mathcal{M}_U \) is flat over \( U \).

**Proof.** One can assume that \( S = \text{Spec} \mathbb{A} \) is affine and that \( \mathcal{X} = \mathbb{P}^{m,n}_\mathbb{A} \) is a projective superspace over \( \mathbb{A} \). We now proceed by induction on \( m \). If \( m = 0 \), then \( \mathcal{X} = \text{Spec} \mathbb{A}[\theta_1, \ldots, \theta_n] \) and \( f : \mathcal{X} \to S \) is finite. It follows that \( \mathcal{M} \) is finitely generated over \( \mathbb{A} \) and we conclude by [64, Thm. 6.4.4].

Take now \( m > 0 \). If \( S \) has only a finite number of points, we can assume it reduces to its generic point. In this case \( \mathbb{A} \) is a field and flatness is automatic. If \( S \) has an infinite
number of points, there is a hyperplane \( \mathcal{H} \cong \mathbb{P}^{m-1,n} \) which does not meet any point of \( \mathcal{H} \) corresponding to the (finitely many) associated primes of \( M \). Then, for every integer \( r \), one has an exact sequence

\[
0 \to M(r) \to M(r+1) \to M'(r+1) \to 0,
\]

where \( M' = M|_{\mathcal{H}} \), as in Remark A.2. By induction on \( m \), \( M'_V \) is flat over \( V \) for an affine open subscheme \( V \) of \( S \). After restriction to that open, we can assume that \( M' \) is flat over \( S \). Then, by Theorem 2.35 and Proposition 3.7, there exists \( r_0 \) such that for every \( r \geq r_0 \) one has \( R^if_*M(r) = 0 \), \( R^if_*M'(r+1) = 0 \) for \( i > 0 \), and \( f_*M'(r+1) \) is locally free. Moreover, by the case \( m = 0, n = 0 \), there is a nonempty affine open subscheme \( U \) of \( S \) such that \( R^if_*M(r_0)_U \) is flat, and then locally free. From the exact sequence

\[
0 \to f_*M(r_0) \to f_*M(r_0+1) \to f_*M'(r_0+1) \to 0,
\]

one obtains that \( f_*M(r_0+1) \) is locally free on \( U \). By ascending induction one sees that \( f_*M(r) \) is locally free on \( U \) for every \( r \geq r_0 \). By Theorem 2.35 \( M_U \) is flat over \( U \).

The result on flattening will need a preliminary result. In its proof we shall use the following notation: for every affine superscheme \( \mathcal{I} = \text{Spec} A \), \( \mathcal{I}_rA \) will stand for the affine superscheme \( \text{Spec}(A \otimes_A A_{red}) \). It is a closed sub-superscheme of \( \mathcal{I} \) whose ordinary underlying scheme is the reduced scheme \( S_{red} = \text{Spec} A_{red} \).

**Lemma 4.16.** Let \( \mathcal{I} \) be a noetherian superscheme, \( f : \mathcal{I} \to \mathcal{I} \) a superprojective morphism of superschemes, \( O_{\mathcal{I}}(1) \) a relatively very ample line bundle, and \( M \) a coherent sheaf on \( \mathcal{I} \).

1. Only finitely many pairs of polynomials occur as super Hilbert polynomials \( H(X_s,M) \) of the fibres of \( M \).
2. There exists an integer \( r_0 \) such that for every \( \mathcal{I} \)-superscheme \( \mathcal{I} \to \mathcal{I} \) and every \( r \geq r_0 \) one has
   a. \( R^if_*M(\mathcal{I})(r) = 0 \) for \( i > 0 \);
   b. the base change morphism
   \[
   ((f_*M(\mathcal{I})(r))_s \to H^0(X_s,M_s(r))
   \]
   is an isomorphism for every point \( s \in T \).

**Proof.** We can assume that \( \mathcal{I} \) is affine, \( \mathcal{I} = \text{Spec} A \), and that \( \mathcal{I} \) is the superprojective space \( \mathcal{I} = \mathbb{P}^{m,n}_A \) over \( A \). By Serre’s Theorem 2.35 there exists \( r_0 \) such that \( R^iI_s(M(r)) = 0 \) for \( i > 0 \) and \( r \geq r_0 \). Then \( H^0(X_s,M_s(r)) = 0 \) for \( i > 0 \) and \( r \geq r_0 \) by Corollary 3.5. If \( \mathcal{I} \to \mathcal{I} \) is a superscheme morphism, the same formula holds for every point \( s \in T \), and then \( R^iI_s(M(\mathcal{I})(r)) = 0 \) for \( i > 0 \) and \( r \geq r_0 \) by (1) of the cohomology base change Theorem 3.4. This proves (2a).

Notice now that to prove (1) we can assume that \( \mathcal{I} = S \) is an ordinary scheme. Moreover, as \( ((f_*M(\mathcal{I}))_s(\mathcal{I}))_s \cong ((f_*M(\mathcal{I}))_T)_s \) to prove (2b) we can also assume that \( \mathcal{I} = T \) is an ordinary scheme. Let \( S' \) be one of the finitely many irreducible components of \( S \). By Proposition 4.15 on generic flatness there is a nonempty open sub-superscheme \( U' \) of \( S' \) such that \( M_U \) is flat over \( U \). We can assume that \( U \) is affine of the form \( U = \text{Spec} A'_r \) where \( A' \) is the ring of \( S' \) and \( f \in A' \). Then, \( Z = \text{Spec} A'/f \to A' \) is the complementary closed sub-superscheme of \( U \). We now take \( Z \) as a new base superscheme and repeat the procedure. By noetherian induction this recursive process ends up with a finite number of subschemes \( V_\alpha \to S \) such that \( M_{V_\alpha} \) is flat over \( V_\alpha \), for every \( \alpha \) and the induced morphism \( \coprod_\alpha V_\alpha \to S \) is topologically surjective. Since \( M_{V_\alpha} \) is flat over \( V_\alpha \), the super Hilbert polynomials are constant on the fibres, so that
there are only a finite number of super Hilbert polynomials $\mathbf{H}(X_s, M_s)$ of the fibres of $\mathcal{M}$ over $\mathcal{S}$. This proves (1).

By Proposition 4.14 for every $\alpha$ there exists an integer $r_\alpha$ such that
\begin{equation}
(f_* M(r))_\alpha \cong f_{V_\alpha} M_{V_\alpha}(r) \quad (4.1)
\end{equation}
for every $r \geq r_\alpha$. Write $T_\alpha = V_\alpha \times_{S T} T \rightarrow T$. Since all higher direct images $R^i f_{V_\alpha} M_{V_\alpha}(r)$ vanish by (2a) and $M_{V_\alpha}$ is flat over $V_\alpha$, Proposition 3.7 implies that
\begin{equation}
(f_{V_\alpha} M_{V_\alpha}(r))_{T_\alpha} \cong f_{T_\alpha} M_{T_\alpha}(r),
\end{equation}
\begin{equation}
(f_{T_\alpha} M_{T_\alpha}(r))_s \cong H^0(X_s, M_s(r)) \quad (4.2)
\end{equation}
for every $s \in T_\alpha$ and $r \geq r_\alpha$. Combining Equations (4.1) and (4.2) we obtain an isomorphism
\begin{equation}
((f_* M(r))_{T_\alpha})_s \cong H^0(X_s, M_s(r))
\end{equation}
for every $r \geq r_\alpha$ and every point $s \in T_\alpha$. Corollary 3.5 and the cohomology base change Theorem 3.4 imply that for every point $s \in T_\alpha$ and every $r \geq r_\alpha$ the base change morphism is an isomorphism
\begin{equation}
(f_{T_\alpha} M_{T_\alpha}(r))_s \cong H^0(X_s, M_s(r))
\end{equation}
for $r \geq r_\alpha$ and every $s \in T_\alpha$. It follows that
\begin{equation}
((f_* M(r))_T)_s \cong H^0(X_s, M_s(r))
\end{equation}
for every $r \geq r_0$ and $s \in T$.

Let $\mathcal{S}$ be a noetherian superscheme and $\mathcal{M}$ a coherent sheaf on $\mathcal{S}$. For any pair of integer numbers $(p, q)$ the geometric locus of the points $\mathcal{S} \rightarrow \mathcal{S}$ where $\mathcal{M}$ is locally free of rank $(p, q)$ is a sub-superscheme of $\mathcal{S}$. In other words, there exists a sub-superscheme $\mathcal{Y}_{p,q}(\mathcal{M}) \rightarrow \mathcal{S}$ such that a morphism $\mathcal{S} \rightarrow \mathcal{S}$ of superschemes factors through $\mathcal{Y}_{p,q}(\mathcal{M}) \rightarrow \mathcal{S}$ if and only if $\mathcal{M}_S$ is locally free of rank $(p, q)$.

Proof. The question is local, so that it is enough to prove that for every point $s \in S$ there is an open sub-superscheme $\mathcal{U}$ with $s \in U$ where the Lemma is true. If $\mathcal{M}$ has rank $(p, q)$ at $s \in S$ by the super Nakayama Lemma ([5, 11],[64, 6.4.5]) there is an open sub-superscheme $\mathcal{U}$ with $s \in U$ and an exact sequence
\begin{equation}
\mathcal{O}_{\mathcal{U}}^{p,s} \rightarrow \mathcal{O}_{\mathcal{U}}^{q,s} \rightarrow \mathcal{M}_\mathcal{U} \rightarrow 0.
\end{equation}
Then $\mathcal{M}$ has rank smaller or equal to $(p, q)$, with respect to the order given by Equation (2.7), at every point of $U$. Taking $\mathcal{Y}_{p,q}(\mathcal{M})$ as the closed sub-superscheme of $\mathcal{U}$ defined by the ideal generated by the entries of the graded matrix of $\Psi$, one finishes. 

The following result is part of what in the classical case is the existence of a “flattening stratification” (see e.g. [53]).

Proposition 4.18. Let $\mathcal{S}$ be a noetherian superscheme, $f : \mathcal{X} \rightarrow \mathcal{S}$ a superprojective morphism of superschemes, $\mathcal{O}_\mathcal{X}(1)$ a relatively very ample line bundle, and $\mathcal{M}$ a coherent sheaf on $\mathcal{X}$. For each pair of polynomials with rational coefficients $P(r) = (P_+, r, P_−(r))$, the geometric locus of the points $\mathcal{S} \rightarrow \mathcal{S}$ where $\mathcal{M}$ has super Hilbert polynomial $P$ (Definition 4.2) is a sub-superscheme $\mathcal{S}_P$ of $\mathcal{S}$. More precisely, there is a sub-superscheme $\mathcal{S}_P \rightarrow \mathcal{S}$...
with the following universal property: given a morphism \( \phi: \mathcal{I} \to \mathcal{J} \) of superschemes, \( \mathcal{M}_\mathcal{J} \) is flat over \( \mathcal{J} \) with super Hilbert polynomial \( \mathbf{P}(r) \) if and only if \( \phi \) factors through \( \mathcal{J}_\mathcal{P} \hookrightarrow \mathcal{J} \).

**Proof.** The question is local on the base, so we can assume that \( \mathcal{I} = \text{Spec} \mathbb{A} \) is affine. Moreover, we can assume that \( \mathcal{J} = \mathbb{P}^n_{\mathbb{A}} \) is a projective superspace over \( \mathbb{A} \).

By Lemma 4.16 there exists \( r_0 \) such that for every \( \mathcal{J} \)-superscheme \( \mathcal{J} \to \mathcal{I} \) one has \( R^i \mathcal{F}_* \mathcal{M}_\mathcal{J}(r) = 0 \) for \( i > 0 \) and \( ((\mathcal{F}_* \mathcal{M}(r))_\mathcal{J})_\mathcal{J} \cong H^0(X, \mathcal{M}_\mathcal{J}(r)) \) for every \( r \geq r_0 \). If \( \mathcal{M}_\mathcal{J} \) is flat over \( \mathcal{J} \) with super Hilbert polynomial \( \mathbf{P} \), then \( (\mathcal{F}_* \mathcal{M}(r))_{\mathcal{J}} \cong \mathcal{F}_* \mathcal{M}_\mathcal{J}(r) \) is locally free of rank \( \mathbf{P}(r) \) (Proposition 4.17), and then, by Lemma 4.17, \( \mathcal{J} \to \mathcal{I} \) factors through the sub-superscheme \( \mathcal{J}_\mathcal{P}(r)(f_\mathcal{I}, \mathcal{M}(r)) \) associated to the sheaf \( f_\mathcal{I} \mathcal{M}(r) \), where \( \mathbf{P}(r) = (P_+(r), P_-(r)) \) for every \( r \geq r_0 \). Thus \( \mathcal{J} \to \mathcal{I} \) factors through the intersection sub-superscheme \( \bigcap_{r \geq r_0} \mathcal{J}_\mathcal{P}(r)(f_\mathcal{I}, \mathcal{M}(r)) \) of \( \mathcal{J} \) if one proves that the intersection makes sense as a superscheme.

Conversely, if \( \mathcal{J} \) factors through the sub-superscheme \( \mathcal{J}_\mathcal{P}(r)(f_\mathcal{I}, \mathcal{M}(r)) \) for every \( r \geq r_1 \) for some value \( r_1 \geq r_0 \), then \( f_\mathcal{J} \mathcal{M}_\mathcal{J}(r) \) is locally free of rank \( \mathbf{P}(r) \) for \( r \geq r_1 \) and then \( \mathcal{M}_\mathcal{J} \) is flat over \( \mathcal{J} \) by Theorem 2.35.

To complete the proof one has to show that for some value \( r_1 \geq r_0 \), the sub-superscheme \( \bigcap_{r \geq r_1} \mathcal{J}_\mathcal{P}(r)(f_\mathcal{I}, \mathcal{M}(r)) \) of \( \mathcal{J} \) exists. This follows from the following statement:

- (\( \bullet \)) For every integer \( N \geq r_0 \) the sub-superschemes \( \mathcal{V}_N = \bigcap_{r_0 \leq r \leq N} \mathcal{J}_\mathcal{P}(r)(f_\mathcal{I}, \mathcal{M}(r)) \) of \( \mathcal{J} \) form a stationary chain as \( N \) grows.

To prove (\( \bullet \)), increase first \( r_0 \) so that Proposition 4.14 is true for the immersion \( S \hookrightarrow \mathcal{J} \). Then, if \( f_\mathcal{I} \mathcal{M}(r) \) is locally free of rank \( \mathbf{P}(r) \), the same happens after the base change \( S \to \mathcal{J} \), so that we can assume that the base is the ordinary scheme \( S \). Consider then the sub-superschemes \( V_\alpha \hookrightarrow S \) such that \( \mathcal{M}_{V_\alpha} \) is flat over \( V_\alpha \) for every \( \alpha \) constructed in the proof of Lemma 4.16. Since the induced morphism \( \prod_\alpha V_\alpha \to S \) is surjective it is enough to prove (\( \bullet \)) after the base change \( \prod_\alpha V_\alpha \to S \), that is, we can assume that \( \mathcal{M} \) is flat over \( S \). Now, the super Hilbert polynomials of \( \mathcal{M} \) on the fibres of \( \mathcal{J} \to S \) are all equal to a pair \( H = (H_+, H_-) \) of polynomials.

If \( \mathbf{P} \neq \mathbf{H} \), one has \( \mathbf{P}(r) \neq \mathbf{H}(r) \) for \( r \gg r_0 \) and then \( \mathcal{V}_N = \emptyset \) for \( N \) big enough.

If \( \mathbf{P} = \mathbf{H} \), then \( \mathcal{V}_N = \mathcal{V}_{r_0} \) for every \( N \geq r_0 \) and one finishes. \( \square \)

4.5. **Embedding into supergrassmannians.** Let \( f: \mathcal{X} = \mathbb{P}^{m,n}_{\mathcal{X}} \to \mathcal{J} \) be the projective superspace over \( \mathcal{J} \), with \( \mathcal{J} \) noetherian, \( \mathcal{O}_{\mathcal{J}}(1) \) a relatively very ample line bundle, and let \( \mathbf{P} = \mathbf{P}(r) = (P_0(r), P_1(r)) \) be a pair of polynomials with rational coefficients. From Mumford’s Theorem 4.12 on regularity and by Proposition 3.7 one obtains:

**Proposition 4.19.** There exists an integer \( r_0 \), depending only on the coefficients of the polynomials of \( \mathbf{P} \), such that for every closed sub-superscheme \( \mathcal{Z} \hookrightarrow \mathcal{X}_\mathcal{J} \) in \( \mathbb{S} \text{Hilb}_{\mathcal{J}/\mathcal{J}}(\mathcal{J}) \), and every integer \( r \geq r_0 \), the following conditions hold:

1. \( R^i f_\mathcal{X}_*(\mathcal{O}_\mathcal{J}(r)) = 0 \) for \( i > 0 \) and \( f_\mathcal{X}_*(\mathcal{O}_\mathcal{J}(r)) \) is a locally free sheaf on \( \mathcal{J} \) of rank \( \mathbf{P}(r) \).

2. If \( p_\mathcal{X} \) is the ideal of \( \mathcal{Z} \) in \( \mathcal{X}_\mathcal{J} \), then \( R^i f_\mathcal{X}_*(p_\mathcal{X}(r)) = 0 \) for \( i > 0 \), so that
   (a) there is an exact sequence
   \[
   0 \to f_\mathcal{X}_*(p_\mathcal{X}(r)) \to f_\mathcal{X}_*(\mathcal{O}_{\mathcal{X}_\mathcal{J}}(r)) \to f_\mathcal{X}_*(\mathcal{O}_{\mathcal{X}}(r)) \to 0,
   \]
   of locally free sheaves of ranks \( h_{(m,n)}(r) - \mathbf{P}(r) \), \( h_{(m,n)}(r) \) and \( \mathbf{P}(r) \), respectively (Proposition 2.33);
Corollary 4.20. The above functor morphism is injective, that is, one has a functorial immersion

\[ \mathcal{SHilb}_{\mathcal{J}/\mathcal{I}} \rightarrow \mathcal{SG}^* \]

into the (functor of the points of the) supergrassmannian \( \pi: \mathcal{SG} := \mathcal{SG}^*(\mathcal{E}, \mathcal{P}(p)) \rightarrow \mathcal{I} \).

Proof. Let us see that the quotient \( q: \mathcal{O}_{\mathcal{J}/\mathcal{I}} \rightarrow \mathcal{O}_{\mathcal{I}} \rightarrow 0 \) can be recovered from

\[ f_{\mathcal{J}}(q(p)): f_{\mathcal{J}}^* \mathcal{E} \rightarrow f_{\mathcal{J}}^* f_{\mathcal{I}}^* (\mathcal{O}_{\mathcal{I}}(p)) \rightarrow 0. \]

On \( \mathcal{SG} \) the sheaf \( \mathcal{E}_{\mathcal{SG}} \simeq \pi^* \mathcal{E} \simeq f_{\mathcal{SG}}^* (\mathcal{O}_{\mathcal{SG}}(p)) \) has a universal locally free quotient

\[ 0 \rightarrow \mathcal{K} \xrightarrow{\mu} \mathcal{E}_{\mathcal{SG}} \xrightarrow{\rho} \mathcal{Q} \rightarrow 0 \]

so that, if \( v: \mathcal{J} \rightarrow \mathcal{SG} \) is the morphism corresponding to the surjection \( f_{\mathcal{J}}^*(q(p)) \), one has \( f_{\mathcal{J}}^*(q(p)) = v^*(p) \). This gives an exact sequence

\[ 0 \rightarrow v^* \mathcal{K} \simeq f_{\mathcal{J}}^* (\mathcal{O}_{\mathcal{J}}(p)) \xrightarrow{v^*(\mu)} f_{\mathcal{J}}^* (\mathcal{O}_{\mathcal{J}/\mathcal{I}}(p)) \xrightarrow{f_{\mathcal{J}}(q(p)) = v^*(p)} v^* \mathcal{Q} \simeq f_{\mathcal{J}}^* (\mathcal{O}_{\mathcal{I}}(p)) \rightarrow 0. \]

If \( g: f_{\mathcal{J}}^* (\mathcal{O}_{\mathcal{J}}(p)) \rightarrow \mathcal{O}_{\mathcal{J}/\mathcal{I}}(p) \) is the composition of

\[ f_{\mathcal{J}}^* (v^*(\mu)): f_{\mathcal{J}}^* f_{\mathcal{J}}^* (\mathcal{O}_{\mathcal{J}}(p)) \rightarrow f_{\mathcal{J}}^* f_{\mathcal{J}}^* (\mathcal{O}_{\mathcal{J}/\mathcal{I}}(p)) \]

and \( f_{\mathcal{J}}^* f_{\mathcal{J}}^* (\mathcal{O}_{\mathcal{I}}(p)) \rightarrow \mathcal{O}_{\mathcal{J}/\mathcal{I}}(p) \), the diagram of (2)-(b) in Proposition 4.19 gives an exact sequence

\[ f_{\mathcal{J}}^* f_{\mathcal{J}}^* (\mathcal{O}_{\mathcal{J}}(p)) \xrightarrow{g} \mathcal{O}_{\mathcal{J}/\mathcal{I}}(p) \xrightarrow{q(p)} \mathcal{O}_{\mathcal{I}}(p) \rightarrow 0. \]

Thus, \( q(p) \) can be recovered from \( v^*(\mu) \), and so from \( f_{\mathcal{J}}^*(q(p)) \), as the cokernel of \( g \). Twisting by \(-p\) we recover \( q: \mathcal{O}_{\mathcal{J}/\mathcal{I}} \rightarrow \mathcal{O}_{\mathcal{I}} \) as well. \( \square \)
4.6. Existence of the Hilbert superscheme. Now to prove that \( \mathcal{SH}ilb^{P}_{X/\mathcal{I}} \) is representable it is enough to show that \( \mathcal{SH}ilb^{P}_{X/\mathcal{I}} \hookrightarrow \mathcal{S}G \) is representable by immersions, that is, there exists a sub-superscheme \( \mathcal{W} \) of \( \mathcal{S}G \) whose points \( g: \mathcal{I} \rightarrow \mathcal{W} \) are precisely the points \( g: \mathcal{I} \rightarrow \mathcal{S}G \) belonging to the image of \( \mathcal{SH}ilb^{P}_{X/\mathcal{I}}(\mathcal{I}) \).

We use the notation of the universal Equation (4.5) of the supergrassmannian \( \mathcal{S}G \). Let 
\[
g: f_{\mathcal{S}G}^{*}\mathcal{K} \rightarrow O_{\mathcal{S}G}(s)
\]
be the composition
\[
f_{\mathcal{S}G}^{*}\mathcal{K} \xrightarrow{f_{\mathcal{S}G}^{*}(\mu)} f_{\mathcal{S}G}^{*}\mathcal{E} \cong f_{\mathcal{S}G}^{*}(O_{\mathcal{S}G}(s)) \xrightarrow{g} O_{\mathcal{S}G}(s) .
\]
and \( \psi: O_{\mathcal{S}G}(s) \rightarrow \mathcal{N} \) its cokernel. By generic flatness and the flattening (Proposition 4.18) for the sheaf \( \mathcal{N} \) and the super Hilbert polynomial \( \mathcal{P}(r) = \mathcal{P}(r + s) \), there exists a sub-superscheme \( \mathcal{H} \hookrightarrow \mathcal{S}G \) such that a morphism \( \mathcal{I} \rightarrow \mathcal{S}G \) of \( \mathcal{I} \)-superschemes factors through \( \mathcal{H} \) if and only if \( \mathcal{N} \) is flat over \( \mathcal{I} \) with super Hilbert polynomial \( \mathcal{P}(r) \) (so that \( \mathcal{N}(-s) \) defines a \( \mathcal{I} \)-valued point of \( \mathcal{SH}ilb^{P}_{X/\mathcal{I}} \)).

The superscheme \( \mathcal{H} \) clearly represents the super Hilbert functor. One then has the following result, which completes the proof of the existence Theorem 4.3 due to Proposition 4.7.

**Proposition 4.21.** Let \( \mathcal{K} = \mathbb{P}^{m,n}_{\mathcal{I}/\mathcal{S}G} \rightarrow \mathcal{I} \) be the projective superspace over \( \mathcal{I} \), with \( \mathcal{I} \) noetherian and let \( \mathcal{P} = (P_{0}(r), P_{1}(r)) \) be a pair of polynomials with rational coefficients. The super Hilbert functor \( \mathcal{SH}ilb^{P}_{X/\mathcal{I}} \) is representable by a sub-superscheme
\[
\mathcal{SH}ilb^{P}(\mathcal{K}/\mathcal{I}) \hookrightarrow \mathcal{S}G
\]
of a supergrassmannian \( \mathcal{S}G = \mathcal{S}Grass(\mathcal{E}, \mathcal{P}(s)) \) associated to a locally free sheaf \( \mathcal{E} \) over \( \mathcal{I} \). Moreover, \( \mathcal{SH}ilb^{P}(\mathcal{K}/\mathcal{I}) \rightarrow \mathcal{I} \) is a proper morphism of superschemes.

**Proof.** By the above discussion we have only to prove that \( \mathcal{SH}ilb^{P}(\mathcal{K}/\mathcal{I}) \rightarrow \mathcal{I} \) is proper. This follows straightforwardly from the valuative criterion for properness for a morphism of superschemes (see Corollary A.14). \( \square \)

**Remark 4.22.** (Relation with the Hilbert scheme \( \mathcal{H}ilb(X/S) \).) If \( \mathcal{K} : \mathcal{X} \rightarrow \mathcal{I} \) is a superprojective superscheme morphism, and \( f_{bos}: X \rightarrow S \) is the underlying projective morphism of ordinary schemes, one can ask about the relation between the underlying ordinary scheme \( \mathcal{S}H := \mathcal{SH}ilb(\mathcal{X}/\mathcal{I})_{bos} \) of the Hilbert superscheme and the ordinary Hilbert scheme \( H := \mathcal{H}ilb(X/S) \), and the relation between the super Hilbert polynomial in the first case and the usual Hilbert polynomial in the second. The scheme \( \mathcal{S}H \) represents the restriction of the super Hilbert functor to the category of schemes. Since every morphism \( T \rightarrow \mathcal{I} \) from an ordinary scheme \( T \) factors through \( S \), we see that \( \mathcal{S}H \) is as well the underlying scheme to the Hilbert superscheme \( \mathcal{SH}ilb(\mathcal{X}_{S}/S) \).

Now, every closed subscheme of \( X \rightarrow S \) flat over \( S \) is also a closed sub-superscheme of \( \mathcal{X} \rightarrow S \), so that there is an immersion of schemes \( \omega: H \rightarrow \mathcal{S}H \) which maps \( \mathcal{K}^{P} \) to \( \mathcal{S}H^{(P,b)} \). Moreover, \( \omega \) is an open immersion, as for every scheme morphism \( T \rightarrow S \) and every closed sub-superscheme \( \mathcal{K} \rightarrow \mathcal{X} \) flat over \( T \), the geometric locus \( U \subseteq T \) where \( \mathcal{K} \) is an ordinary closed subscheme of \( X_{U} \) is the open subset complementary to the image by the proper morphism \( f \) of the support of the ideal \( \mathcal{J}_{\mathcal{K}} \) defining \( Z \) in \( \mathcal{X} \).

However, \( \mathcal{S}H \) is usually bigger than \( H \) as there are non-purely bosonic families of closed sub-superschemes of \( \mathcal{X} \rightarrow S \) that only have even parameters. This is entirely analogous to what happens with the supergrassmannian of a supervector bundle \( \mathcal{E} = \mathcal{E}_{0} \oplus \mathcal{E}_{1} \), whose
underlying ordinary scheme is strictly bigger than the grassmannian of $E_0$ (cf. Proposition 2.19).

Actually one can say something more: if $\mathcal{X} \hookrightarrow \mathcal{X}_{S}$ is a closed superscheme flat over $S$, the bosonic reduction $Z$ is a closed subscheme of $X$ and it is flat over $S$ by Proposition 2.37, so that we have a scheme morphism $\rho: SH \to H$ such that $\rho \circ \varpi = \text{Id}$. $\triangle$

4.7. Hilbert superschemes of 0-cycles. Let $f: \mathcal{X} \to \mathcal{I}$ be a superprojective morphism of superschemes. We call Hilbert superscheme of 0-cycles of $\mathcal{X}$ a superscheme which represents the super Hilbert functor associated with a constant super Hilbert polynomial $P = (p, q)$, where $p, q$ are nonnegative integers. It turns out that the ordinary scheme underlying this Hilbert superscheme has a richer structure than the ordinary Hilbert scheme of 0-cycles.

Consider for instance a smooth superprojective superscheme $\mathcal{X} \to S$ over an ordinary scheme $S$, of relative dimension $(m, 1)$. Then $\mathcal{X}$ is split, and moreover its structure sheaf $\mathcal{O}_{\mathcal{X}}$ has the form $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{X} \oplus \mathcal{L}$, where $X \to S$ is the underlying ordinary scheme, and $\mathcal{L}$ is a line bundle on $X$. Now let $\mathcal{X} \subseteq \mathcal{X}$ be a closed sub-superscheme of $\mathcal{X}$ with super Hilbert polynomial $(p, q)$, flat over $S$. Let $\mathcal{I}_{\mathcal{X}}$ be the ideal sheaf of $\mathcal{X}$; it has the structure

$$\mathcal{I}_{\mathcal{X}} = \mathcal{I}_0 \oplus \mathcal{I}_1 \cdot \mathcal{L},$$

where $\mathcal{I}_0, \mathcal{I}_1$ are ideals of $\mathcal{O}_X$, which correspond to 0-cycles $Z_0$ and $Z_1$ of $X$ of length $p$ and $q$, respectively. Actually for $\mathcal{I}_{\mathcal{X}}$ to be an ideal of $\mathcal{O}_X$ one also needs that $\mathcal{I}_0 \subseteq \mathcal{I}_1$, i.e., $Z_1 \subseteq Z_0$. Therefore, when $p \geq q$ the ordinary scheme underlying the Hilbert superscheme $\text{SHilb}^{(p, q)}(\mathcal{X}/S)$ is a (2-step) nested Hilbert scheme of 0-cycles of $X$, parameterizing pairs of 0-cycles $(Z_0, Z_1)$ of $X$, with $Z_1 \subseteq Z_0$. When $p < q$ the Hilbert superscheme $\text{SHilb}^{(p, q)}(\mathcal{X}/S)$ is empty. For a formal definition of the (2-step) nested Hilbert scheme of 0-cycles see e.g. [27].

Nested Hilbert schemes have been recently studied intensively, for instance in connection with enumerative invariants and moduli spaces of quiver representations. One may envision that Hilbert superschemes of 0-cycles can find applications to study these problems.

4.8. Superschemes of morphisms of superschemes. Our next aim is to prove that, under quite reasonable hypotheses, the space of morphisms between two superschemes can be given the structure of a superscheme. We start with a preliminary result.

Lemma 4.23. Let $\mathcal{I}$ be a noetherian superscheme and let $f: \mathcal{X} \to \mathcal{I}$, $g: \mathcal{Z} \to \mathcal{I}$ be flat morphisms of superschemes, locally of finite type, and let $\phi: \mathcal{Z} \to \mathcal{X}$ be a morphism of $\mathcal{I}$-superschemes. Let $z \in Z$ be a point of $Z$ and write $x = \phi(z)$, $s = g(z) = f(x)$. If the restriction $\phi_*: \mathcal{Z}_s \to \mathcal{X}_z$ to the fibre is flat at $z$, then $\phi$ is flat at $z$ as well.

Proof. Directly from Corollary A.8. $\square$

Lemma 4.24. Let $\mathcal{I}$ be a noetherian superscheme, let $f: \mathcal{X} \to \mathcal{I}$, $g: \mathcal{Z} \to \mathcal{I}$ be proper and flat morphisms of $\mathcal{I}$-superschemes, and let $\phi: \mathcal{Z} \to \mathcal{X}$ be a superprojective morphism. Then, the loci of the points where $\phi$ is flat or an isomorphism are open. In other words, there exists open subsuperschemes $\mathcal{U} \hookrightarrow \mathcal{V}$ of $\mathcal{I}$ with the following universal property: For every $\mathcal{I}$-superscheme $h: \mathcal{J} \to \mathcal{I}$, the base change morphism $\phi_{\mathcal{J}}: \mathcal{Z}_{\mathcal{J}} \to \mathcal{X}_{\mathcal{J}}$ is flat (respect. an isomorphism) if and only if $\phi$ factors through $\mathcal{V} \hookrightarrow \mathcal{I}$ (respect. $\mathcal{U} \hookrightarrow \mathcal{I}$).

Proof. By Proposition 4.15 on generic flatness there is an open sub-superscheme $\mathcal{X}'$ of $\mathcal{X}$ which is the locus of the points where $\phi$ is flat. $\mathcal{X}'$ can be covered by open sub-superschemes $\mathcal{X}_i$ such that $\phi: \mathcal{Z}_i = \phi^{-1}(\mathcal{X}_i) \to \mathcal{X}_i$ is flat and superprojective. Then there exist integers $r_i$ such that $R^i\phi_*\mathcal{O}_{\mathcal{X}_i}(r) = 0$, $\mathcal{O}_{\mathcal{Z}_i}(r)$ is relatively generated by its global sections, and
\( \phi_* \mathcal{O}_{X_i}(r) \) is locally free for \( r \geq r_i \) (Theorem 2.35 and Proposition 3.7). Since \( X \) is noetherian, we can choose an \( r \) that satisfies \( R^i\phi_* \mathcal{O}_{X}(r) = 0 \), \( \phi^r \phi_* \mathcal{O}_{X}(r) \to \mathcal{O}_{Y}(r) \), and \( \phi_* \mathcal{O}_{X}(r) \) is locally free. Let \( Y \) be the open sub-superscheme of \( X \) where \( L = \phi_* \mathcal{O}_{X}(r) \) has rank \( (1,0) \) and \( \mathcal{L} = \phi^{-1}(Y) \). By Proposition 2.17 \( \mathcal{O}_{X_i}(r) \) induces a morphism
\[
\mathcal{L} \to \mathcal{F}(\mathcal{L}) \simeq Y
\]
of superschemes over \( Y \) which is an isomorphism. Now, \( U = S - f(X - V) \) is an open subset of \( S \) because \( f \) is proper. If we give \( U \) the induced structure \( \mathcal{U} \) of sub-superscheme of \( \mathcal{T} \), one sees that \( \mathcal{U} \) satisfies the required universal property.

Let \( X \to \mathcal{T} \), \( Y \to \mathcal{T} \) be superschemes over \( \mathcal{T} \).

**Definition 4.25.** The functor of morphisms of \( \mathcal{T} \)-superschemes from \( X \) to \( Y \) is the functor on the category of \( \mathcal{T} \)-superschemes that associates to every \( \mathcal{T} \)-superscheme \( \mathcal{T} \to \mathcal{T} \) the family
\[
\text{Hom}_{\mathcal{T}}(X, Y)(\mathcal{T}) = \text{Hom}(X, Y),
\]
of all the morphisms \( X \to Y \) of \( \mathcal{T} \)-superschemes.

**Proposition 4.26.** Let \( \mathcal{T} \) be a noetherian superscheme and let \( X \to \mathcal{T} \), \( Y \to \mathcal{T} \) be super-projective morphisms. If \( X \to \mathcal{T} \) is flat, the functor of morphisms \( \text{Hom}_{\mathcal{T}}(X, Y) \) is representable by an open sub-superscheme \( \text{Hom}_{\mathcal{T}}(X, Y) \) of the Hilbert superscheme \( \text{SHilb}(X \times \mathcal{T} Y / \mathcal{T}) \).

**Proof.** If \( f: X \to Y \) is a morphism of \( \mathcal{T} \)-superschemes, the graph morphism
\[
(1, f): X \to Y \times \mathcal{T} Y \simeq (X \times \mathcal{T} Y),
\]
is a closed immersion as \( f \) is separated. Then, its graph \( \Gamma(f) = \text{Im}(1 \times f) \) is a closed sub-superscheme of \( (X \times \mathcal{T} Y)_{\mathcal{T}} \). Moreover, the first projection \( \Gamma(f) \to X_{\mathcal{T}} \) is an isomorphism as it is the inverse of \( (1, f): X_{\mathcal{T}} \to \Gamma(f) \). Then \( \Gamma(f) \) is flat over \( \mathcal{T} \), that is, it defines a \( \mathcal{T} \)-valued point of the super Hilbert functor \( \text{SHilb}(X \times \mathcal{T} Y / \mathcal{T}) \). This defines a morphism of functors
\[
\gamma: \text{Hom}_{\mathcal{T}}(X, Y) \to \text{SHilb}(X \times \mathcal{T} Y / \mathcal{T}),
\]
given, for every \( \mathcal{T} \)-superscheme \( \mathcal{T} \), by
\[
\gamma(\mathcal{T}): \text{Hom}_{\mathcal{T}}(X, Y)(\mathcal{T}) \to \text{SHilb}(X \times \mathcal{T} Y / \mathcal{T})(\mathcal{T})
\]
\( f \mapsto \Gamma(f) \).

One needs only to prove that \( \gamma \) is representable by open immersions. Take an \( \mathcal{T} \)-superscheme \( \mathcal{T} \) and an element \( X \in \text{SHilb}(X \times \mathcal{T} Y / \mathcal{T})(\mathcal{T}) \), that is, a closed sub-superscheme \( X \to X_{\mathcal{T}} \) flat over \( \mathcal{T} \). Since \( X_{\mathcal{T}} \to \mathcal{T} \) is locally superprojective, the same is true for the projection \( (X \times \mathcal{T} Y)_{\mathcal{T}} \to X_{\mathcal{T}} \) and then the restriction of that projection to \( X \) is a locally superprojective morphism \( X \to X_{\mathcal{T}} \) as well. By Lemma 4.24 there is an open sub-superscheme \( Y \) of \( \mathcal{T} \) with the following universal property: a morphism of superschemes \( \mathcal{R} \to \mathcal{T} \) factors through \( Y \to \mathcal{T} \) if and only if the base change map \( Y \to X_{\mathcal{T}} \) is an isomorphism, that is, if \( Y \) is the graph of a well-determined morphism \( X_{\mathcal{R}} \to X_{\mathcal{T}} \) of \( \mathcal{R} \)-superschemes. This finishes the proof.

**Corollary 4.27.** Let \( \mathcal{T} \) be a noetherian superscheme and let \( X \to \mathcal{T} \), \( Y \to \mathcal{T} \) be super-projective and flat morphisms. Then, the functor of isomorphisms of \( \mathcal{T} \)-superschemes from
\( \mathcal{X} \) to \( \mathcal{Y} \) is representable. That is, there exists an \( \mathcal{I} \)-superscheme \( \text{Isom}_{\mathcal{I}}(\mathcal{X}, \mathcal{Y}) \) such that for every \( \mathcal{I} \)-superscheme \( \mathcal{T} \) there is a functorial isomorphism

\[
\text{Hom}_{\mathcal{I}}(\mathcal{T}, \text{Isom}_{\mathcal{I}}(\mathcal{X}, \mathcal{Y})) \cong \text{Isom}_{\mathcal{I}}(\mathcal{T}, \mathcal{Y}) .
\]

\[ \square \]

4.9. Superschemes embeddable into supergrassmannians. Using super Hilbert superschemes we get the following characterizations of superschemes embeddable into supergrassmannians.

Proposition 4.28. Let \( \mathcal{X} \to \mathcal{I} \) be a proper morphism of Noetherian superschemes. Then the following conditions are equivalent:

1. there exists a closed immersion of \( \mathcal{X} \) into the relative supergrassmannian of some supervector bundle over \( \mathcal{I} \);
2. there exists a smooth surjective morphism \( \overline{\mathcal{X}} \to \mathcal{X} \) of relative dimension \((0, n)\) such that \( \overline{\mathcal{X}} \) is strongly superprojective over \( \mathcal{I} \);
3. there exists a faithfully flat morphism \( g : \overline{\mathcal{X}} \to \mathcal{X} \) such that \( \overline{\mathcal{X}} \) is strongly superprojective over \( \mathcal{I} \).

Proof. (1) \( \implies \) (2). This follows immediately from Corollary 2.30.

(2) \( \implies \) (3). We just have to observe that a smooth surjective morphism is faithfully flat.

(3) \( \implies \) (1). The idea is to show that \( \mathcal{X} \) embeds into the Hilbert superscheme component \( \text{SFhilb}^P(\overline{\mathcal{X}}/\mathcal{I}) \) for some \( P \). Recall that by Theorem 4.4, the Hilbert superscheme \( \text{SFhilb}(\mathcal{X}/\mathcal{I}) \) exists and we have a closed immersion

\[
\text{SFhilb}(\mathcal{X}/\mathcal{I}) \xrightarrow{g^{-1}} \text{SFhilb}(\overline{\mathcal{X}}/\mathcal{I}) .
\]

Since \( \mathcal{X} \) is separated over \( \mathcal{I} \), the diagonal \( \delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{I}} \overline{\mathcal{X}} \) is a closed immersion, so viewing it as a flat family of subschemes in \( \mathcal{X} \) parametrized by \( \mathcal{X} \), we get a morphism

\[
\mathcal{X} \to \text{SFhilb}(\mathcal{X}/\mathcal{I}) \to \text{SFhilb}(\overline{\mathcal{X}}/\mathcal{I}) .
\]

Assume that \( \mathcal{X} \) is connected. Then the above morphism factors through \( \text{SFhilb}^P(\overline{\mathcal{X}}/\mathcal{I}) \) for some super Hilbert polynomial \( P \). We claim that the obtained morphism

\[
\mathcal{X} \to \text{SFhilb}(\mathcal{X}/\mathcal{I}) \times_{\text{SFhilb}(\mathcal{X}/\mathcal{I})} \text{SFhilb}^P(\overline{\mathcal{X}}/\mathcal{I})
\]

is an isomorphism. Indeed, let \( \mathcal{I} \) be an \( \mathcal{I} \)-superscheme and suppose we have a flat family of sub-superschemes \( \mathcal{Z} \subseteq \mathcal{X} \) such that \( g^{-1}(\mathcal{Z}) \subseteq \overline{\mathcal{X}} \) has super Hilbert polynomial \( P \). It is enough to show that \( \mathcal{Z} \) is a graph of a morphism \( \mathcal{I} \to \mathcal{X} \) of \( \mathcal{I} \)-superschemes. In other words, we need to show that the projection \( \mathcal{Z} \to \mathcal{I} \) is an isomorphism. Since \( \mathcal{Z} \) is flat over \( \mathcal{I} \), it suffices to show that for every geometric point \( t : \text{Spec}(K) \to \mathcal{I} \), the fibre \( \mathcal{Z}_t \) is isomorphic to \( \text{Spec}(K) \). We know that \( g^{-1}(\mathcal{Z}_t) \) is a closed subscheme in \( P_s \), where \( s : \text{Spec}(K) \to \mathcal{I} \) is the corresponding \( K \)-point of \( \mathcal{I} \), with the same Hilbert polynomial as \( g^{-1}(x) \), where \( x \) is a \( K \)-point of \( \mathcal{X} \). This implies that \( \mathcal{Z}_t \) is nonempty, so picking a \( K \)-point \( z \) of \( \mathcal{Z}_t \), we obtain that the sub-superschemes \( g^{-1}(\mathcal{Z}_t) \) and \( g^{-1}(z) \) of \( \mathcal{Z}_s \) have the same super Hilbert polynomial. Since one sub-superscheme contains the other, this is possible only if \( \mathcal{Z}_t \) is the reduced point, as claimed.

Thus, we obtained a closed immersion of \( \mathcal{X} \) into \( \text{SFhilb}^P(\overline{\mathcal{X}}/\mathcal{I}) \), and hence, into a relative supergrassmannian.
In the case when \( \mathcal{X} \) is not connected, the above argument gives an immersion of each connected component of \( \mathcal{X} \) into a relative supergrassmannian. It remains to observe that the disjoint union of two supergrassmannians \( S\text{Grass}((c_i, d_i), \mathcal{X}_i), i = 1, 2 \), can be embedded into the supergrassmannian

\[
S\text{Grass}((c_1 + c_2, d_1 + d_2), \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \bigoplus_{i=1}^{c_1+c_2} \mathcal{O}_{c_i} \oplus \bigoplus_{j=1}^{d_1+d_2} \mathcal{O}_{f_j}).
\]

Namely, a subbundle \( \mathcal{F}_1 \subset \mathcal{E}_1 \) of rank \((c_1, d_1)\) (resp., \( \mathcal{F}_2 \subset \mathcal{E}_2 \) of rank \((c_2, d_2)\)) is sent to

\[
\mathcal{F}_1 \oplus 0 \oplus \bigoplus_{i=c_1+1}^{c_1+c_2} \mathcal{O}_{c_i} \oplus \bigoplus_{j=d_1+1}^{d_1+d_2} \mathcal{O}_{f_j} \text{ (resp., } 0 \oplus \mathcal{F}_2 \oplus \bigoplus_{i=1}^{c_1} \mathcal{O}_{c_i} \oplus \bigoplus_{j=1}^{d_1} \mathcal{O}_{f_j}).
\]

\[\square\]

5. The Picard superscheme

All the superschemes considered in this Section are locally noetherian.

5.1. The super Picard functors. If \( \mathcal{X} \) is a superscheme, we can associate to it two Picard groups: the group \( \text{Pic}^+(\mathcal{X}) \) of isomorphism classes of all even line bundles on \( \mathcal{X} \), and the group \( \text{Pic}(\mathcal{X}) = \text{Pic}^+(\mathcal{X}) \coprod \text{Pic}^-(\mathcal{X}) \) of all even and odd line bundles on \( \mathcal{X} \), that is, all locally free sheaves of \( \mathcal{O}_\mathcal{X} \)-modules of rank either \((1, 0)\) or \((0, 1)\). Moreover, \( \text{Pic}^+(\mathcal{X}) \) is a subgroup of \( \text{Pic}(\mathcal{X}) \).

Notice that if \( \mathcal{O}\Pi_\mathcal{X} = \mathcal{O}_\mathcal{X,1} \oplus \mathcal{O}_\mathcal{X,0} \) is the sheaf obtained by changing the parity of the components of the structure ring, the multiplication by \( \mathcal{O}\Pi_\mathcal{X} \) gives a one-to-one correspondence between \( \text{Pic}^+(\mathcal{X}) \) and \( \text{Pic}^-(\mathcal{X}) \).

**Lemma 5.1.** One has a natural isomorphism

\[
\text{Pic}(\mathcal{X}) \simeq H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X},+}^*) \simeq \text{Pic}(\mathcal{X}/\Gamma),
\]

where \( \mathcal{X}/\Gamma \) is the bosonic quotient of \( \mathcal{X} \).

**Proof.** The proof of the first isomorphism is the same as in the classical case: we just observe that transition functions of a line bundle are even invertible functions. The isomorphism with the usual Picard group \( \text{Pic}(\mathcal{X}/\Gamma) \) follows from this. \[\square\]

Let \( f: \mathcal{X} \rightarrow \mathcal{Y} \) be a morphism of superschemes. Then the inverse image of line bundles maps \( \text{Pic}(\mathcal{Y}) \) to \( \text{Pic}(\mathcal{X}) \) preserving the parity.

**Definition 5.2.** The relative even Picard group and the relative Picard group of \( f \) are, respectively, the quotient groups

\[
\text{Pic}^+(\mathcal{X}/\mathcal{Y}) := \text{Pic}^+(\mathcal{X})/f^*\text{Pic}^+(\mathcal{Y}), \quad \text{Pic}(\mathcal{X}/\mathcal{Y}) := \text{Pic}(\mathcal{X})/f^*\text{Pic}(\mathcal{Y}).
\]

We denote by \([L]\) the class in \( \text{Pic}(\mathcal{X}/\mathcal{Y}) \) of a line bundle \( L \) on \( \mathcal{X} \). If \( L' \in \text{Pic}^-(\mathcal{X}) \) and \( L \in \text{Pic}^+(\mathcal{X}) \) have the same class \([L'] = [L]\) in the relative Picard group \( \text{Pic}(\mathcal{X}/\mathcal{Y}) \), then \( L' \simeq L \otimes f^*\mathcal{N} \) for an odd line bundle on \( \mathcal{Y} \). Then,

\[
L' \simeq L \otimes f^*(\mathcal{O}_{\mathcal{Y}}^\Pi \otimes \mathcal{N}^\Pi) \simeq L^\Pi \otimes f^*\mathcal{N}^\Pi,
\]

and now \( f^*(\mathcal{O}_{\mathcal{Y}}^\Pi \otimes \mathcal{N}^\Pi) \) is odd and \( \mathcal{N}^\Pi \in \text{Pic}^+(\mathcal{Y}) \). This proves that

\[
\text{Pic}(\mathcal{X}/\mathcal{Y}) := \text{Pic}(\mathcal{X})/f^*\text{Pic}(\mathcal{Y}) \simeq \text{Pic}(\mathcal{X})/f^*\text{Pic}^+(\mathcal{Y}),
\]
so that $\text{Pic}_+(\mathcal{X}/\mathcal{I})$ is a subgroup of $\text{Pic}(\mathcal{X}/\mathcal{I})$. Moreover, if we write $\text{Pic}_-(\mathcal{X}/\mathcal{I}) := \text{Pic}_-(\mathcal{X})/f^*\text{Pic}_+(\mathcal{X})$, then

$$\text{Pic}(\mathcal{X}/\mathcal{I}) = \text{Pic}_+(\mathcal{X}/\mathcal{I}) \bigoplus \text{Pic}_-(\mathcal{X}/\mathcal{I}) \cong \text{Pic}_+(\mathcal{X}/\mathcal{I}) \bigoplus \text{Pic}_+(\mathcal{X}/\mathcal{I}) \otimes O_\mathcal{I} \text{Pic}^+(\mathcal{X}/\mathcal{I}).$$

(5.1)

**Definition 5.3.** The super Picard presheaf of $f$ is the group functor on the category of (locally noetherian) superschemes over $\mathcal{I}$ that associates to every superscheme morphism $T \to \mathcal{I}$ the relative Picard group

$$\mathcal{SPic}_{\mathcal{X}/\mathcal{I}}(T) = \text{Pic}(\mathcal{X} \times \mathcal{I}/T).$$

Analogously, the even super Picard presheaf of $f$ is the subgroup functor of $\mathcal{SPic}_{\mathcal{X}/\mathcal{I}}$ given by

$$\mathcal{SPic}^+_{\mathcal{X}/\mathcal{I}}(T) = \text{Pic}^+(\mathcal{X} \times \mathcal{I}/T).$$

The super Picard functor, or super Picard sheaf of $f$, is the sheaf

$$\mathcal{SPic}_{\mathcal{X}/\mathcal{I}}(\text{ét})$$

associated with $\mathcal{SPic}_{\mathcal{X}/\mathcal{I}}$ for the étale topology of superschemes. Analogously, the even super Picard functor, or even super Picard sheaf of $f$, is the sheaf

$$\mathcal{SPic}^+_{\mathcal{X}/\mathcal{I}}(\text{ét})$$

associated with $\mathcal{SPic}^+_{\mathcal{X}/\mathcal{I}}$ for the étale topology of superschemes.

This Section is devoted to proving a representability theorem for the super Picard functor, that is, the existence, under suitable hypotheses, of the Picard superscheme associated to $f$.

**Theorem 5.4.** Let $\mathcal{I}$ be a locally noetherian superscheme and $f: \mathcal{X} \to \mathcal{I}$ a locally superprojective flat morphism which has geometrically integral fibres (Definition 3.13) and is cohomologically flat in dimension 0 (Definition 3.15).

1. The even super Picard functor is representable by an $\mathcal{I}$-supergroup

$$\mathcal{SPic}^+_{\mathcal{X}/\mathcal{I}} \to \mathcal{I}$$

which is locally of finite type over $\mathcal{I}$.

2. As a consequence, the total super Picard functor is representable by the $\mathcal{I}$-supergroup

$$\mathcal{SPic}_{\mathcal{X}/\mathcal{I}} \to \mathcal{I}$$

given by

$$\mathcal{SPic}_{\mathcal{X}/\mathcal{I}} \cong \mathcal{SPic}^+_{\mathcal{X}/\mathcal{I}} \times \mathbb{Z}_2$$

where the subgroup $\mathbb{Z}_2$ is generated by the class of $O_{\mathcal{X}}$.

**Proof.** Claim (2) easily follows from Claim (1), so that we only need to prove the latter. Also note that we can assume that $\mathcal{I}$ is noetherian as the representability of the even super Picard sheaf is local on $\mathcal{I}$ in the Zariski topology of superschemes.

The strategy for a proof consists of the study of the super Abel morphism (or Abel contraction). This is the map from positive superdivisors to even line bundles that associates to a positive superdivisor the dual of the corresponding ideal sheaf. We shall see that this map is a projective superbundle on an open part of the even Picard sheaf, so that that open part is actually representable. Since the super Picard functor is a group functor which can be covered by translates of that open subfunctor, this will imply that it is representable as well.

The proof will be actually given in the next Subsections, and will be completed by the end of Subsection 5.6. \qed
We start by describing the relation between the super Picard presheaf and its associated étale sheaf.

**Lemma 5.5.** Let \( f: \mathcal{X} \to \mathcal{I} \) be a morphism of superschemes. Assume that \( f \) is cohomologically flat in dimension 0 (Definition 3.15). For every base change \( \mathcal{T} \to \mathcal{I} \) the morphisms
\[
\phi^*_f: \text{Pic}(\mathcal{T}) \to \text{Pic}(\mathcal{X}_{\mathcal{T}}), \quad \phi^*_f: \text{Pic}_+(\mathcal{T}) \to \text{Pic}_+(\mathcal{X}_{\mathcal{T}}),
\]
are injective.

**Proof.** We prove the statement for even line bundles, as the odd case is similar. Let \( \mathcal{N} \) be an even line bundle on \( \mathcal{T} \) such that \( \phi^*_f \mathcal{N} \cong \mathcal{O}_{\mathcal{X}_{\mathcal{T}}} \). Then \( \phi^*_f \mathcal{O}_{\mathcal{X}_{\mathcal{T}}} \cong \phi^*_f \phi^*_f \mathcal{N} \cong \mathcal{N} \otimes \phi^*_f \mathcal{O}_{\mathcal{X}_{\mathcal{T}}} \). When \( f \) is cohomologically flat in dimension 0, we have \( \mathcal{N} \cong \mathcal{O}_{\mathcal{X}_{\mathcal{T}}} \), which finishes the proof in this case.

**Proposition 5.6.** Let \( f: \mathcal{X} \to \mathcal{I} \) be proper and flat.

1. If \( f \) is cohomologically flat in dimension 0 the natural morphisms of group functors
\[
S\text{Pic }\mathcal{X}/\mathcal{I} \to S\text{Pic }\mathcal{X}/\mathcal{I} (\text{ét}) , \quad S\text{Pic}_+\mathcal{X}/\mathcal{I} \to S\text{Pic}_+\mathcal{X}/\mathcal{I} (\text{ét})
\]
are injective;
2. if in addition \( f: \mathcal{X} \to \mathcal{I} \) has a section, then
\[
S\text{Pic }\mathcal{X}/\mathcal{I} \cong S\text{Pic }\mathcal{X}/\mathcal{I} (\text{ét}) \quad \text{and} \quad S\text{Pic}_+\mathcal{X}/\mathcal{I} \cong S\text{Pic}_+\mathcal{X}/\mathcal{I} (\text{ét}).
\]

**Proof.** We prove the statements for the total Picard functors, as the proof for the even Picard ones is the same.

1. Let \( \mathcal{T} \to \mathcal{I} \) be an \( \mathcal{I} \)-superscheme and \( \mathcal{L} \) be a class in \( S\text{Pic }\mathcal{X}/\mathcal{I} (\mathcal{T}) \) whose image in \( S\text{Pic }\mathcal{X}/\mathcal{I} (\text{ét})(\mathcal{T}) \) is trivial. This means that there exist an étale covering \( \phi: \mathcal{V} \to \mathcal{T} \) and an isomorphism \( \mathcal{L}_{\mathcal{V}} \cong \phi^*_f \mathcal{N} \) for some even line bundle \( \mathcal{N} \) on \( \mathcal{V} \). Let us write \( \mathcal{R} = \mathcal{V} \times_{\mathcal{I}} \mathcal{V} \) and \( (p_1, p_2): \mathcal{R} \to \mathcal{V} \) for the projections. Consider the cartesian diagram
\[
\begin{array}{ccc}
\mathcal{X}_{\mathcal{R}} & \cong & \mathcal{X}_{\mathcal{V}} \times_{\mathcal{I}} \mathcal{X}_{\mathcal{V}} \\
\downarrow f_{\mathcal{R}} & & \downarrow f_{\mathcal{V}} \\
\mathcal{R} & \cong & \mathcal{V} \\
\uparrow \phi_{\mathcal{R}} & & \uparrow \phi_{\mathcal{V}} \\
\mathcal{X} & \cong & \mathcal{I}
\end{array}
\]
Since \( \mathcal{L}_{\mathcal{V}} = \phi^*_f \mathcal{L}_{\mathcal{V}} \), there is an isomorphism \( q^*_1 \phi^*_f \mathcal{N} \cong q^*_1 \mathcal{L}_{\mathcal{V}} \cong q^*_2 \mathcal{L}_{\mathcal{V}} = q^*_2 \phi^*_f \mathcal{N} \), that is, an isomorphism \( \phi^*_f p^*_1 \mathcal{N} \cong f^*_f p^*_1 \mathcal{N} \).

By Lemma 5.5 there is an isomorphism \( p^*_1 \mathcal{N} \cong p^*_2 \mathcal{N} \), that is, a descent data for \( \mathcal{N} \). By faithfully flat descent (Proposition A.29), there is an even line bundle \( \mathcal{N}' \) on \( \mathcal{T} \) such that \( \mathcal{N} \cong \phi^* \mathcal{N}' \). Moreover, the isomorphism \( \phi: \mathcal{L}_{\mathcal{V}} \cong \phi^*_f \mathcal{N} \) satisfies \( q^*_1 \alpha = q^*_2 \alpha \), so that, by faithfully flat descent for morphisms (Proposition A.28), \( \alpha \) descends to an isomorphism \( \mathcal{L} \cong \phi^*_f \mathcal{N}' \). Then \( \mathcal{L} \) is trivial in \( S\text{Pic }\mathcal{X}/\mathcal{I} (\text{ét}) \).

2. Let \( \sigma: \mathcal{I} \to \mathcal{X} \) be a section of \( f: \mathcal{X} \to \mathcal{I} \). If \( \mathcal{T} \to \mathcal{I} \) is a superscheme over \( \mathcal{I} \), a section \( \xi \) of the super Picard sheaf over \( \mathcal{T} \) is given by an étale covering \( \phi: \mathcal{V} \to \mathcal{T} \) and a class \( \xi = [\mathcal{L}] \) of an even line bundle \( \mathcal{L} \) on \( \mathcal{X}_{\mathcal{V}} \) in the relative super Picard group. Let us consider the diagram
\[
\begin{array}{ccc}
\mathcal{X}_{\mathcal{R}} & \cong & \mathcal{X}_{\mathcal{V}} \times_{\mathcal{I}} \mathcal{X}_{\mathcal{V}} \\
\downarrow f_{\mathcal{R}} & & \downarrow f_{\mathcal{V}} \\
\mathcal{R} & \cong & \mathcal{V} \\
\uparrow \sigma_{\mathcal{R}} & & \uparrow \sigma_{\mathcal{V}} \\
\mathcal{X} & \cong & \mathcal{I}
\end{array}
\]
where $\sigma_\mathcal{X}$, $\sigma_\mathcal{Y}$ and $\sigma_\mathcal{Z}$ are the sections of $f_\mathcal{X}$, $f_\mathcal{Y}$ and $f_\mathcal{Z}$ induced by $\sigma$. Since $q_1^*\phi_\mathcal{X}^*(\xi) = q_2^*\phi_\mathcal{Y}^*(\xi)$ in $\text{SPic } \mathcal{X}/\mathcal{Y}(\text{ét})(\mathcal{R})$, and the map
\[
\text{SPic } \mathcal{X}/\mathcal{Y}(\mathcal{R}) \to \text{SPic } \mathcal{X}/\mathcal{Y}(\text{ét})(\mathcal{R})
\]
is injective by (1), one has $q_1^*L \cong q_1^*L \otimes f_\mathcal{X}^*N$ for an even line bundle $N$ on $\mathcal{R}$. Take $\tilde{L} = L \otimes f_\mathcal{Y}^*\sigma_\mathcal{X}^*-1$. Then $[\tilde{L}] = [L]$ in the relative super Picard group, and $q_1^*\tilde{L} \cong q_2^*L$. By faithfully flat descent (Proposition A.29), there is an even line bundle $L'$ on $\mathcal{X}_\mathcal{Y}$ such that $\tilde{L} \cong \phi_\mathcal{Y}^*L'$, so that $\xi = [L']$. Then the immersion $\text{SPic } \mathcal{X}/\mathcal{Y}(\mathcal{R}) \hookrightarrow \text{SPic } \mathcal{X}/\mathcal{Y}(\text{ét})(\mathcal{R})$ is also surjective. \hfill $\square$

Let us also make some simple general observations about the Picard functor.

**Lemma 5.7.**

1. For a base change $\mathcal{X}' \to \mathcal{X}$, the restriction of the Picard functor $\text{SPic } \mathcal{X}/\mathcal{Y}(\text{ét})$ to the category of $\mathcal{X}'$-superschemes is naturally isomorphic to $\text{SPic } \mathcal{X'}/\mathcal{Y}(\text{ét})$, where $\mathcal{X}'/\mathcal{Y} = \mathcal{X} \times_\mathcal{Y} \mathcal{X}'$. In particular, if $\text{SPic } \mathcal{X}/\mathcal{Y}(\text{ét})$ is representable by $\text{SPic } (\mathcal{X}/\mathcal{Y})$ then $\text{SPic } \mathcal{X'}/\mathcal{Y}(\text{ét})$ is representable by
\[
\text{SPic } (\mathcal{X'}/\mathcal{Y}) \cong \text{SPic } (\mathcal{X}/\mathcal{Y}) \times_\mathcal{Y} \mathcal{X}'.
\]

2. Assume that $\mathcal{X} = S$ is a usual scheme. Then the restriction of $\text{SPic } +\mathcal{X}/S(\text{ét})$ to the category of usual schemes over $S$ is naturally isomorphic to the (even) Picard functor of the bosonic quotient $\mathcal{X}/\Gamma$ over $S$. If the Picard functor of $\mathcal{X}/S$ is representable by a superscheme then the Picard functor of $(\mathcal{X}/\Gamma)/S$ is also representable and we have
\[
\text{Pic } ((\mathcal{X}/\Gamma)/S) \cong \text{SPic } +\mathcal{X}/S_{\text{bos}}.
\]

**Proof.** Part (1) follows directly from the definitions. To prove (2) we use Lemma 5.1. Namely, for any $S$-scheme $T$ we have
\[
\text{Pic } +\mathcal{X}_T/T \cong \text{Pic } ((\mathcal{X}_T/\Gamma)/T) \cong \text{Pic } ((\mathcal{X}/\Gamma)_T/T).
\]

It follows the étale sheafifications of these functors are also isomorphism. Assuming that the Picard functor of $\mathcal{X}/S$ is representable, we obtain for any $S$-scheme $T$,
\[
\text{Mor } (T, \text{SPic } +\mathcal{X}/S_{\text{bos}}) \cong \text{Mor } (T, \text{SPic } +\mathcal{X}/S) \cong \text{Pic } ((\mathcal{X}/\Gamma)/S(\text{ét})),
\]

which shows that the Picard functor of $(\mathcal{X}/\Gamma)/S$ is represented by $\text{SPic } +\mathcal{X}/S_{\text{bos}}$. \hfill $\square$

**Remark 5.8.** Lemma 5.7(2) explains why we impose the superprojectivity assumption in order to prove the representability of the super Picard functor. Even in the case when $S$ is even, the representability of super Picard functor of $\mathcal{X}$ over $S$ implies the representability (by a scheme) of the usual Picard functor of the bosonic quotient $\mathcal{X}/\Gamma$ over $S$. But even for superschemes embeddable into a relative supergrassmannian, $\mathcal{X}/\Gamma$ is not necessarily projective over $S$, so its Picard functor may only be representable by an algebraic space. \hfill $\triangle$

### 5.2. Quotients of flat equivalence relations of superschemes.

**Definition 5.9.** Let $f: \mathcal{X} \to \mathcal{I}$ be a morphism of superschemes. An equivalence relation of superschemes $\mathcal{I} \to \mathcal{X}$ is a closed sub-superscheme $\mathcal{R} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ that induces an equivalence relation for the functor of points, that is, for every $\mathcal{I}$-superscheme $\mathcal{I}$, the inclusion $\mathcal{R}^*(\mathcal{I}) \hookrightarrow \mathcal{X}^*(\mathcal{I}) \times_{\mathcal{Y}^*(\mathcal{I})} \mathcal{X}^*(\mathcal{I})$ is an equivalence relation of sets.
Here, as customary, we write $\mathcal{V}^*(\mathcal{I}) = \text{Hom}_{\mathcal{I}}(\mathcal{I}, \mathcal{V})$ for every $\mathcal{I}$-superscheme $\mathcal{V}$.

Following [38, I.5.1] we consider the notion of effective relation of superschemes, and the associated quotient superscheme.

**Definition 5.10.** An equivalence relation of superschemes is effective if

1. there exists the cokernel $q: \mathcal{X} \to \mathcal{Z}$ of $(p_1, p_2): \mathcal{R} \rightrightarrows \mathcal{X}$, that is, $q: \mathcal{X} \to \mathcal{Z}$ is a categorical quotient;

2. $\mathcal{R} \simeq \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$.

We then say that $\mathcal{Z}$ is the quotient superscheme of $\mathcal{R}$.

**Example 5.11.** Let $p: \mathcal{W} \to \mathcal{H}$ be a morphism of $\mathcal{I}$-superschemes, $\mathcal{X} \to \mathcal{W}$ a sub-superscheme and $\mathcal{R} = \mathcal{X} \times_{\mathcal{W}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Then $\mathcal{R}$ is an equivalence relation of superschemes, and it is effective if and only if it descends to a sub-superscheme $\mathcal{Z} \to \mathcal{H}$, that is, if and only if there exists a sub-superscheme $\mathcal{Z} \hookrightarrow \mathcal{H}$ such that $\mathcal{X} = p^{-1}(\mathcal{Z})$.

Proposition A.30 can be stated as follows:

**Proposition 5.12** (Grothendieck’s effective descent for sub-superschemes). With the notation of Example 5.11, assume that $p: \mathcal{W} \to \mathcal{H}$ is quasi-compact and faithfully flat. Then $\mathcal{R}$ is effective if and only if $\mathcal{W} \times_{\mathcal{Z}} \mathcal{X} \simeq \mathcal{X} \times_{\mathcal{Z}} \mathcal{W}$ as sub-superschemes of $\mathcal{W} \times_{\mathcal{Z}} \mathcal{W}$. □

**Proposition 5.13.** Let $f: \mathcal{X} \to \mathcal{I}$ be a morphism of superschemes such that the Hilbert superscheme $\mathcal{H} = \text{Hilb}(\mathcal{X}/\mathcal{I})$ exists. An equivalence relation $\mathcal{R} \rightrightarrows \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ such that the second projection $p_2: \mathcal{R} \to \mathcal{X}$ is flat and proper is effective. The quotient morphism $q: \mathcal{X} \to \mathcal{Z}$ is flat and proper, and $\mathcal{Z}$ is a closed sub-superscheme of the Hilbert superscheme $\mathcal{H}$, so that $\mathcal{X}$ is of finite type and separated over $\mathcal{I}$ when $\mathcal{H}$ is so.

**Proof.** By hypothesis $\mathcal{R}$ is an $\mathcal{X}$-valued point of the Hilbert superscheme $\mathcal{H}$, that is, it defines a morphism of superschemes $g: \mathcal{X} \to \mathcal{H}$ such that $\mathcal{R} \rightrightarrows \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is the pull-back by $1 \times g: \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{H}$ of the universal closed sub-superscheme $\mathcal{W} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{H}$.

Notice that for every $\mathcal{I}$-superscheme $\mathcal{T}$ two $\mathcal{I}$-valued points $x_1, x_2: \mathcal{T} \to \mathcal{X}$ are equivalent (i.e. $(x_1, x_2) \in \mathcal{I}^*(\mathcal{T})$) if and only if $g(x_1) = g(x_2)$. Then $\mathcal{R} \simeq \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Moreover, as $x_1, x_2$ are equivalent if and only if $(x_1, g(x_2)) \in \mathcal{W}^*(\mathcal{T})$, the graph $\Gamma_g: \mathcal{X} \rightrightarrows \mathcal{X} \to \mathcal{H}$ of $g$ gives a closed immersion $\Gamma_g: \mathcal{X} \rightrightarrows \mathcal{H}$. We conclude by applying Proposition 5.12 to the natural morphism $\mathcal{W} \to \mathcal{H}$, which is faithfully flat, and to the closed sub-superscheme $\mathcal{R} \rightrightarrows \Gamma_g(\mathcal{X}) \rightrightarrows \mathcal{W}$. The required condition that $\mathcal{W} \times_{\mathcal{Z}} \Gamma_g(\mathcal{X})$ and $\Gamma_g(\mathcal{X}) \times_{\mathcal{Z}} \mathcal{X}$ coincide in $\mathcal{W} \times_{\mathcal{Z}} \mathcal{W}$ is readily verified.

The quotient morphism $q : \mathcal{X} \to \mathcal{Z}$ is flat and proper as it is obtained from $\mathcal{W} \to \mathcal{H}$ by base change. Moreover, $\mathcal{Z}$ is a closed sub-superscheme of the Hilbert superscheme $\mathcal{H}$ and we conclude. □

The following lemma corresponds to [37, Exercise 9.4.9] and has an analogous proof.

**Lemma 5.14.** Let $q : S\mathcal{F} \to \mathcal{A} \mathcal{G}$ be a morphism of étale sheaves on $\mathcal{I}$-superschemes. Then $q$ is an epimorphism of sheaves if and only if it is the cokernel (coequalizer) of the two morphisms $(p_1, p_2): S\mathcal{F} \times_{\mathcal{G}} S\mathcal{F} \rightrightarrows S\mathcal{F}$. □

The following result will be necessary to get a proof of the representability of the super Picard sheaf, and is a strengthened version for superschemes of [37, Lemma 9.9].
Proposition 5.15. Let $f : \mathcal{X} \to \mathcal{I}$ be a morphism of superschemes such that the Hilbert superscheme $\text{Hilb}(\mathcal{X}/\mathcal{I})$ exists.

1. Let $\mathcal{R} \hookrightarrow \mathcal{X} \times_\mathcal{I} \mathcal{X}$ be an equivalence relation on $f : \mathcal{X} \to \mathcal{I}$ such that the second projection $p_2 : \mathcal{R} \to \mathcal{X}$ is smooth and proper, so that it is effective by Proposition 5.13. Then, the quotient morphism $q : \mathcal{X} \to \mathcal{R}$ is smooth and proper as well and $\mathcal{R}$ is a closed sub-superscheme of $\text{Hilb}(\mathcal{X}/\mathcal{I})$. Moreover $\mathcal{R}^\bullet \hookrightarrow \mathcal{X}^\bullet \times_\mathcal{I} \mathcal{X}^\bullet$ is an effective equivalence relation of étale sheaves on $\mathcal{I}$-superschemes and $q : \mathcal{X}^\bullet \to \mathcal{R}^\bullet$ is the corresponding quotient morphism.

2. Let $\mathcal{S} \mathcal{G}$ be an étale sheaf on $\mathcal{I}$-superschemes and $\psi : \mathcal{X}^\bullet \to \mathcal{S} \mathcal{G}$ an epimorphism of étale sheaves such that
   (a) $\mathcal{X}^\bullet \times_\mathcal{S} \mathcal{G} \mathcal{X}^\bullet$ is representable by a sub-superscheme $\mathcal{R} \hookrightarrow \mathcal{X} \times_\mathcal{I} \mathcal{X}$.
   (b) $\mathcal{R}$ is an equivalence relation of superschemes that satisfies the conditions of (1);

Then, $\mathcal{S} \mathcal{G}$ is representable by $\mathcal{R}$ and $\psi = q$.

Proof. (1). Since the horizontal $q$ in the cartesian diagram

$$
\begin{array}{ccc}
\mathcal{R} & \xrightarrow{p_1} & \mathcal{X} \\
\downarrow{p_2} & & \downarrow{q} \\
\mathcal{X} & \xrightarrow{q} & \mathcal{R}
\end{array}
$$

is flat and surjective, and so faithfully flat, and $p_2$ is smooth, the vertical $q$ is smooth. Since $\mathcal{R}^\bullet \simeq \mathcal{X}^\bullet \times \mathcal{X}^\bullet$ by Lemma 5.14 we have only to see that $q : \mathcal{X}^\bullet \to \mathcal{R}^\bullet$ is a surjective morphisms of étale sheaves. Let $\mathcal{I} \to \mathcal{I}$ be a $\mathcal{I}$-superscheme and $\xi : \mathcal{I} \to \mathcal{I}$ an element of $\mathcal{X}^\bullet(\mathcal{I})$. Then, $q \mathcal{I}_\mathcal{I} : \mathcal{X}_\mathcal{I} \to \mathcal{X}_\mathcal{I}$ is smooth, and thus there exists an étale covering $\mathcal{V} \to \mathcal{I}$ such that $\mathcal{X}_\mathcal{V} \to \mathcal{I}$ has a section $\sigma : \mathcal{V} \to \mathcal{X}_\mathcal{V}$. The image of $\sigma$ maps to the element of $\mathcal{X}^\bullet(\mathcal{V})$ induced by $\xi$, and we finish.

(2). By (1) $q : \mathcal{X}^\bullet \to \mathcal{R}^\bullet$ is the cokernel of the two morphisms $\mathcal{R}^\bullet \xrightarrow{\pi} \mathcal{X}^\bullet$, and by Lemma 5.14 $\psi : \mathcal{X}^\bullet \to \mathcal{S} \mathcal{G}$ is also the cokernel of the same arrows. Since the cokernel is unique, one has $\mathcal{S} \mathcal{G} \simeq \mathcal{X}^\bullet$ and $\psi \simeq q$. \hfill \Box

Remark 5.16. If $\mathcal{R} \hookrightarrow \mathcal{X} \times_\mathcal{I} \mathcal{X}$ is an effective étale equivalence relation of superschemes on $\mathcal{X} \to \mathcal{I}$ (by this we mean that $p_1$ and $p_2$ are étale and proper), the quotient morphism $q : \mathcal{X} \to \mathcal{R}$ is an étale morphism. This can be seen as in Proposition 5.15 (see also [38, I.5.6]). Again by Proposition 5.15, the induced equivalence categorical relation $\mathcal{R}^\bullet \hookrightarrow \mathcal{X}^\bullet \times \mathcal{X}^\bullet$ on the category of étale sheaves on $\mathcal{I}$-superschemes is effective and $q : \mathcal{X}^\bullet \to \mathcal{R}^\bullet$ is a categorical quotient. The converse is not true, and this is what motivates the definition of Artin algebraic superspace [9, Def. 2.19]. \hfill \triangle

5.3. Relative positive divisors. The Abel morphism. Let $f : \mathcal{X} \to \mathcal{I}$ be a morphism of superschemes.

Definition 5.17. A relative positive superdivisor in $\mathcal{X}/\mathcal{I}$ (or in $f$) is a closed sub-superscheme $j : \mathcal{L} \to \mathcal{X}$ flat over $\mathcal{I}$ (i.e. such that $f \circ i : \mathcal{L} \to \mathcal{I}$ is flat), whose ideal sheaf $\mathcal{L}_\mathcal{X}$ in $\mathcal{X}$ is an even line bundle.

In this situation, we write $\mathcal{O}_\mathcal{X}(-\mathcal{L}) := \mathcal{L}_\mathcal{X}, \quad \mathcal{O}_\mathcal{X}(\mathcal{L}) := \mathcal{O}_\mathcal{X}(-\mathcal{L})^{-1}$.
Definition 5.18. The functor of relative positive superdivisors is the functor $\text{Div}_{\mathcal{X}/\mathcal{I}}$ on $\mathcal{I}$-superschemes that associates to an $\mathcal{I}$-superscheme $\mathcal{I} \to \mathcal{X}$ the family $\text{Div}_{\mathcal{X}/\mathcal{I}}(\mathcal{I})$ of the relative positive superdivisors in $f_{\mathcal{I}}: \mathcal{X}_{\mathcal{I}} \to \mathcal{I}$.

Since every relative positive divisor is a closed sub-superscheme which is flat over the base, there is a natural functor immersion into the super Hilbert functor:

$$\text{Div}_{\mathcal{X}/\mathcal{I}} \hookrightarrow S\text{Hilb}_{\mathcal{X}/\mathcal{I}}.$$

Proposition 5.19. If $f$ is proper, the above immersion is representable by open immersions. Then, if the Hilbert superscheme $S\text{Hilb}(\mathcal{X}/\mathcal{I})$ exists (see Theorem 4.4), $\text{Div}_{\mathcal{X}/\mathcal{I}}$ is representable by a superscheme $S\text{Div}(\mathcal{X}/\mathcal{I}) \to \mathcal{I}$ which is an open sub-superscheme of $S\text{Hilb}(\mathcal{X}/\mathcal{I})$, and then it is locally of finite type and separated over $\mathcal{I}$.

Proof. One has to prove that, for every $\mathcal{I}$-superscheme $\mathcal{I} \to \mathcal{I}$ and every closed sub-superscheme $\mathcal{Z} \hookrightarrow \mathcal{X}_{\mathcal{I}}$ flat over $\mathcal{I}$, there is an open sub-superscheme $\mathcal{U} \hookrightarrow \mathcal{I}$ with the following universal property. A morphism of superschemes $\mathcal{V} \to \mathcal{I}$ factors through $\mathcal{U} \hookrightarrow \mathcal{I}$ if and only if $\mathcal{Z} \hookrightarrow \mathcal{X}_{\mathcal{I}}$ is a positive relative superdivisor. This is equivalent to impose that the ideal sheaf $\mathcal{I}_{\mathcal{Z}_{\mathcal{I}}}$ is locally free $\mathcal{O}_{\mathcal{Z}_{\mathcal{I}}}$-module. Since $\mathcal{I}_{\mathcal{Z}_{\mathcal{I}}}$ is coherent, the locus of the points where it is free is an open subset $W$ of $\mathcal{X}_{\mathcal{I}}$. Then $f_{\mathcal{I}}(\mathcal{X}_{\mathcal{I}} - W)$ is closed in $\mathcal{I}$ as $f$ is proper. One has then to take $U = T - f_{\mathcal{I}}(\mathcal{X}_{\mathcal{I}} - W)$ and endow it with the natural structure $\mathcal{U}$ of an open sub-superscheme of $\mathcal{I}$. The second part follows from the existence Theorem 4.3. □

Definition 5.20. The Abel morphism is the morphism of functors

$$\text{Ab}: \text{Div}_{\mathcal{X}/\mathcal{I}} \to S\text{Pic}_{\mathcal{X}/\mathcal{I}},$$

where, for an $\mathcal{I}$-superscheme $\mathcal{I}$, the morphism $\text{Div}_{\mathcal{X}/\mathcal{I}}(\mathcal{I}) \to S\text{Pic}_{\mathcal{X}/\mathcal{I}}(\mathcal{I})$ is defined by

$$\mathcal{Z} \mapsto [\mathcal{O}_{\mathcal{Z}_{\mathcal{I}}}((\mathcal{Z})]]$$

for every relative positive superdivisor $\mathcal{Z} \hookrightarrow \mathcal{X}_{\mathcal{I}}$ in $\mathcal{X}_{\mathcal{I}}/\mathcal{I}$.

It defines an equivalence relation in the functor $\text{Div}_{\mathcal{X}/\mathcal{I}}$, the relative linear equivalence, by saying that two relative positive divisors are relatively linearly equivalent when they have the same image by the Abel morphism, that is, when the duals of the corresponding ideal sheaves are isomorphic up to the pull-back of a line bundle on the base. The relative linear equivalence is then given by the fibre product functor

$$\text{Div}_{\mathcal{X}/\mathcal{I}} \times_{S\text{Pic}_{\mathcal{X}/\mathcal{I}}} S\text{Pic}_{\mathcal{X}/\mathcal{I}} \to \text{Div}_{\mathcal{X}/\mathcal{I}} \times_{\mathcal{I}} \text{Div}_{\mathcal{X}/\mathcal{I}}.$$

On our way to prove Theorem 5.4, we take a flat superprojective morphism $f: \mathcal{X} \to \mathcal{I}$ where $\mathcal{I}$ is assumed to be noetherian and connected. Let us denote by $O_\mathcal{X}(1)$ the fixed ample line bundle for $f: \mathcal{X} \to \mathcal{I}$. Since $f$ is flat, every even line bundle on $\mathcal{X}$ has constant super Hilbert polynomial on the various fibres. This means that the even super Picard presheaf decomposes as the disjoint union of the various even super Picard presheaves $S\text{Pic}_{\mathcal{X}/\mathcal{I}}^+ \mathcal{P}$ of the classes of even line bundles $\mathcal{L}$ such that the dual $\mathcal{L}^{-1}$ has super Hilbert polynomial $\mathcal{P}$, that is,

$$\mathcal{P}(n) = \chi(\mathcal{X}, \mathcal{L}_{s}^{-1}(n))$$

for every $s \in S$. The same happens for the corresponding étale sheaves and for the superschemes they represent. Let $H$ be the super Hilbert étale of $O_\mathcal{X}$, and $Q(n) = H(n) - \mathcal{P}(n)$. Then the pre-image of $S\text{Pic}_{\mathcal{X}/\mathcal{I}}^+ \mathcal{Q}$ by the Abel morphism is the functor $\text{Div}_{\mathcal{X}/\mathcal{I}}$
of relative superdivisors of super Hilbert polynomial $Q$ and $\mathcal{D}iv_{\mathcal{X}/S}$ is the disjoint union of the various $\mathcal{D}iv_{\mathcal{X}/S}^Q$.

The Abel morphism decomposes as the union of the various

$$\text{Ab}: \mathcal{D}iv_{\mathcal{X}/S}^Q \rightarrow \mathcal{S}P\text{ic}_{+a,\mathcal{X}/S},$$

and we have relative linear equivalences

$$\mathcal{D}iv_{\mathcal{X}/S}^Q \times \mathcal{S}P\text{ic}_{+a,\mathcal{X}/S} \cong \mathcal{D}iv_{\mathcal{X}/S}^Q \times_{\mathcal{X}/S} \mathcal{D}iv_{\mathcal{X}/S}^Q.$$

Proceeding as in Proposition 5.19, we have

**Proposition 5.21.** $\mathcal{D}iv_{\mathcal{X}/S}^Q$ is representable by a superscheme $\mathcal{S}d\mathcal{D}iv_{\mathcal{X}/S}^Q \rightarrow \mathcal{X}$ which is an open sub-superscheme of the super Hilbert scheme $\mathcal{S}h\mathcal{D}iv_{\mathcal{X}/S}^Q$, and then it is a sub-superscheme of a supergrassmannian over $\mathcal{X}$. In particular, it is of finite type and separated over $\mathcal{X}$.

**Remark 5.22.** The existence of the superscheme of positive divisors $\mathcal{S}d\mathcal{D}iv_{\mathcal{X}/S}^Q$, in the particular case of a smooth supercurve $f: \mathcal{X} \rightarrow S$ (i.e., the relative dimension is $(1, 1)$) over an ordinary scheme $S$, is known from the 90s [17]. Since $\mathcal{O}_{\mathcal{X}} \simeq \mathcal{O}_S \otimes \mathcal{L}$ for a line bundle $\mathcal{L}$ on $X$, one can define the dual supercurve as the superscheme with the same bosonic reduction. In particular, it is of finite type and separated over $\mathcal{X}$.

The strategy for the proof of Theorem 5.4 is the following.

**Strategy 5.23.**

1. There is an open subfunctor of $\mathcal{D}iv_{\mathcal{X}/S}$, given by the divisors such that the dual of their ideal sheaf is acyclic and generated by global sections, which is representable by a superscheme $\mathcal{D}a_a$.

2. The relative linear equivalence for $\mathcal{D}a_a$ is given by a sub-superscheme $\mathcal{R}$ of $\mathcal{D}a_a \times \mathcal{X} \mathcal{D}a_a$, that is, $\mathcal{R}$ defines an equivalence relation of superschemes. Analogously, the relative linear equivalence for $\mathcal{D}a_a^Q$ is given by a sub-superscheme $\mathcal{R}^Q$ of $\mathcal{D}a_a^Q \times \mathcal{X} \mathcal{D}a_a^Q$.

3. Moreover, when $f$ is flat and superprojective with geometrically integral fibres, and is cohomologically flat in dimension 0, then for every super polynomial $Q$, $\mathcal{R}^Q$ is proper and flat over $\mathcal{D}a_a^Q$ w.r.t. the second projection, so that the quotient superscheme exists by Proposition 5.13. Notice that Proposition 5.13 can be applied here because by Proposition 5.21 $\mathcal{D}a_a^Q$ is a sub-superscheme of a supergrassmannian, so that Corollary 2.30 and Theorem 4.4 imply that the Hilbert superscheme of $\mathcal{D}a_a^Q/\mathcal{X}$ exists.

4. This quotient superscheme represents the open subfunctor $\mathcal{S}P\text{ic}_{+a,\mathcal{X}/S(\mathfrak{d}t)}$ of the even super Picard sheaf associated to the classes of even line bundles that satisfy the following properties: they are relatively acyclic, are generated by their global sections, and their duals have super Hilbert polynomial $P$. The disjoint union of the various quotient superschemes for all the super Hilbert polynomial $P$ represents the open subfunctor $\mathcal{S}P\text{ic}_{+a,\mathcal{X}/S(\mathfrak{d}t)}$ of the even super Picard sheaf associated to the classes of even line bundles that relatively acyclic and are generated by their global sections.
Finally, one proves that the even super Picard sheaf and its open subsheaf $\text{Spic}_{\mathcal{X}/\mathcal{Y}}$ satisfy the conditions of Proposition A.31. This implies that the even super Picard functor is representable, thus finishing the proof of Theorem 5.4.

5.4. Representability of acyclic divisors. In this subsection we prove (1) of Strategy 5.23, assuming that $f: \mathcal{X} \to \mathcal{Y}$ is proper and flat.

Let $\text{Spic}_{\mathcal{X}/\mathcal{Y}}$ be the subfunctor of $\text{Spic}_{\mathcal{X}/\mathcal{Y}}$ given by the even line bundles that are relatively acyclic and are generated by their global sections, and let $\pi: \mathcal{F} \to \mathcal{Y}$ be the functor of the relative positive supervisors such that the duals of their ideal sheaf are relatively acyclic and generated by their global sections, and let

$$\text{Div}_{\mathcal{X}/\mathcal{Y}} = \text{Ab}^{-1}(\text{Spic}_{\mathcal{X}/\mathcal{Y}})$$

be the functor of the relative positive supervisors such that the duals of their ideal sheaf are relatively acyclic and generated by their global sections.

**Proposition 5.24.** The functor morphisms

$$\text{Spic}_{\mathcal{X}/\mathcal{Y}} \hookrightarrow \text{Spic}_{\mathcal{X}/\mathcal{Y}}, \quad \text{Div}_{\mathcal{X}/\mathcal{Y}} \hookrightarrow \text{Div}_{\mathcal{X}/\mathcal{Y}}$$

are representable by open immersions. So, if $f: \mathcal{X} \to \mathcal{Y}$ is superprojective, $\text{Div}_{\mathcal{X}/\mathcal{Y}}$ is representable by an open sub-superscheme $\text{SDiv} (a, \mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ of the superscheme $\text{SDiv} (\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ of relative positive supervisors of $\mathcal{X}/\mathcal{Y}$. Analogously, $\text{Div}_{\mathcal{X}/\mathcal{Y}}$ is representable by an open sub-superscheme $\text{SDiv} (a, \mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ of $\text{SDiv} (\mathcal{X}/\mathcal{Y})$.

**Proof.** We prove the statement for the Picard functors, since the other is similar. Let $T$ be a superscheme over $\mathcal{Y}$ and $\Phi: T \to \text{Spic}_{\mathcal{X}/\mathcal{Y}}$ a morphism of $\mathcal{Y}$-superschemes. Then $\Phi$ is given by a class $[L]$ in the relative Picard group of $\mathcal{X}/\mathcal{Y}$ of even line bundles on $\mathcal{X}/\mathcal{Y}$. By the cohomology base change Theorem 3.4, there is an open sub-superscheme $\mathcal{U}$ of $\mathcal{Y}$ such that a morphism $T' \to T$ of $\mathcal{Y}$-superschemes factors through $\mathcal{U}$ if and only if $L_{T'}$ is relatively acyclic and generated by global sections with respect to $f: \mathcal{X}_{T'} \to \mathcal{Y}$. Let $\mathcal{U} \hookrightarrow \mathcal{X}_{T'}$ be the support of the cokernel of $f_{T'}^*L_{T'}$. Since $f_T$ is proper, $f_T(Y)$ is closed in $T$. If we endow $V = U - f_T(Y)$ with the structure of an open sub-superscheme of $\mathcal{U}$, we obtain an open sub-superscheme $V$ of $\mathcal{T}$ with the following universal property: a morphism $T' \to \mathcal{T}$ of $\mathcal{Y}$-superschemes factors through $V$ if and only if $L_{T'}$ is relatively acyclic with respect to $f_{T'}: \mathcal{X}_{T'} \to \mathcal{T}$ and it is relatively generated by its global sections.

The final part now follows from Proposition 5.19. The statement about $\text{Div}_{\mathcal{X}/\mathcal{Y}}$ is proved in the same way using Proposition 5.21. \qed

To prove a similar statement for the Picard sheaves we need a preliminary lemma.

**Lemma 5.25.** Let $G \to F$ be a morphism of prestacks on superschemes, and let $G_{(\text{ét})} \to F_{(\text{ét})}$ be the induced morphisms between the associated étale sheaves. If $F \to F_{(\text{ét})}$ is injective and $G \to F$ is representable by open immersions, then $G_{(\text{ét})} \to F_{(\text{ét})}$ is representable by open immersions as well.

**Proof.** Let $T \to \mathcal{T}$ be an $\mathcal{Y}$-superscheme and $\lambda: T \to \mathcal{F}_{(\text{ét})}$ a morphism of functors on the category of $\mathcal{Y}$-superschemes. There exists an open covering $\pi: T' \to T$ such that $\lambda \circ \pi: T' \to \mathcal{F}_{(\text{ét})}$ factors through a morphism $\lambda': T' \to \mathcal{F}$ and the immersion $i: \mathcal{F} \hookrightarrow \mathcal{F}_{(\text{ét})}$. Then $G_{(\text{ét})} \times \mathcal{F}_{(\text{ét})} \lambda \circ \pi T'$ is representable by an open sub-superscheme $V' \hookrightarrow T'$. (In the notation of the fibre product we stress the second projection to avoid confusion.) In particular, $V \simeq G \times \mathcal{F}_{(\text{ét})} \lambda \circ \pi T'$ is a sheaf, so that it coincides with its associated sheaf
\( S_C(\tilde{\mathcal{E}}) \times S_F(\tilde{\mathcal{E}}) : \mathcal{C} \to \mathcal{F}' \). Let us consider the projections \( p_1, p_2 \) of \( \mathcal{C} \times \mathcal{F} \to \mathcal{F}' \). Then

\[
p^{-1}_1(\mathcal{F}) = \mathcal{F} \times \mathcal{F}' \cong (S_C(\tilde{\mathcal{E}}) \times S_F(\tilde{\mathcal{E}})) \times \mathcal{F}'
\]

\[
\cong S_C(\tilde{\mathcal{E}}) \times S_F(\tilde{\mathcal{E}}) \cdot \lambda \circ p_1 \cdot (\mathcal{C} \times \mathcal{F})
\]

\[
\cong S_C(\tilde{\mathcal{E}}) \times S_F(\tilde{\mathcal{E}}) \cdot \lambda \circ p_2 \cdot (\mathcal{C} \times \mathcal{F}) \cong \mathcal{C} \times \mathcal{F} \cong p^{-1}_2(\mathcal{F})
\]

where we have used that \( \lambda \circ p_1 = \lambda \circ p_2 = \lambda \circ \pi \circ p_2 \) and \( i : S \hookrightarrow S_F(\tilde{\mathcal{E}}) \) is injective.

By Grothendieck effective descent for superschemes (Proposition 5.12) there exists an open sub-superscheme \( U \hookrightarrow \mathcal{F} \) such that \( \mathcal{V} = \pi^{-1}(U) \). One now sees that \( U \cong S_G(\tilde{\mathcal{E}}) \times S_F(\tilde{\mathcal{E}}) \mathcal{F} \).

**Proposition 5.26.** If \( f \) is cohomologically flat in dimension 0 the sheaf morphisms

\[
SPic_{+a, \mathcal{F}/\mathcal{T}(\tilde{\mathcal{E}})} \hookrightarrow SPic_{+a_1, \mathcal{F}/\mathcal{T}(\tilde{\mathcal{E}})}, \quad SPic_{+a, \mathcal{F}/\mathcal{T}(\tilde{\mathcal{E}})} \hookrightarrow SPic_{+a_1, \mathcal{F}/\mathcal{T}(\tilde{\mathcal{E}})}
\]

are representable by open immersions.

**Proof.** This follows from Lemma 5.25 and Proposition 5.24 taking also Proposition 5.6 into account. \( \Box \)

5.5. **Structure of the Abel morphism.** We start by defining the complete linear series \( |L| \) associated to an even line bundle \( L \) as the fibre of the Abel morphism (Definition 5.20) over the point of the Picard superscheme corresponding to \( L \). More precisely:

**Definition 5.27.** Let \( L \) be an even line bundle on \( \mathcal{F} \). The complete linear series of \( L \) is the subfunctor

\[
|L| = \text{Ab}^{-1}(L) \hookrightarrow \text{Div}_{\mathcal{F}/\mathcal{F}}
\]

which associates to every superscheme \( \mathcal{F} \to \mathcal{F} \) the set \( |L|(\mathcal{F}) := \text{Ab}^{-1}(L)(\mathcal{F}) \) of the relative positive superdivisors \( \mathcal{F} \hookrightarrow \mathcal{F} \) such that

\[
[L] = [\mathcal{O}_{\mathcal{F}}(\mathcal{F})]
\]

in \( SPic_{+a, \mathcal{F}/\mathcal{F}}(\mathcal{F}) \). This is equivalent to claiming that

\[
\mathcal{O}_{\mathcal{F}}(\mathcal{F}) \cong L \otimes f^\ast \mathcal{N}
\]

for an even line bundle \( \mathcal{N} \) on \( \mathcal{F} \).

In general, the structure of the complete linear series of \( L \) can be determined as in the classical case [2]. However, we only need to consider the simple case of acyclic line bundles generated by their global sections.

Let \( f : \mathcal{F} \to \mathcal{F} \) a flat proper morphism of superschemes with \( \mathcal{F} \) noetherian and let \( L \) be an even line bundle on \( \mathcal{F} \) which is \( f \)-acyclic, that is, one has \( R^if_\ast L = 0 \) for every \( i > 0 \). Then \( f_\ast L \) is locally free by Proposition 3.7, and the formation of \( f_\ast L \) commutes with arbitrary base change, that is, for every morphism \( \mathcal{F} \to \mathcal{F} \) of superschemes one has

\[
(f_\ast L)(\mathcal{C}) \cong f_\mathcal{C} \ast L_\mathcal{C}
\]

Assume moreover that \( L \) is relatively generated by its global sections, i.e., the morphism \( f^\ast f_\ast L \to L \) is surjective. Then \( f_\ast L \) is nonzero.

If we consider the locally free sheaf \( Q = (f_\ast L)^{-1} \), Equation (5.2) implies that the formation of \( Q \) is compatible with base change, that is, one has

\[
Q_\mathcal{C} \cong (f_\mathcal{C} \ast L_\mathcal{C})^{-1}
\]
for every morphism $\mathcal{I} \to \mathcal{J}$ of superschemes, and there is a functorial isomorphism

$$\text{Hom}_{\mathcal{O}_X}(Q, N) \cong f_{\mathcal{J}}^*(L_{\mathcal{J}}) \otimes N \cong f_{\mathcal{J}}^*(L_{\mathcal{J}} \otimes f_{\mathcal{J}}^*N)$$

(5.4)

for every quasi-coherent sheaf $N$ on $\mathcal{J}$. In particular, for every base change $\mathcal{I} \to \mathcal{J}$ there is an isomorphism

$$\gamma: \text{Hom}_{\mathcal{J}}(Q, N) \cong H^0(\mathcal{I}, L_{\mathcal{J}} \otimes f_{\mathcal{J}}^*N).$$

(5.5)

**Proposition 5.28.** Let $\mathcal{I}$ be a noetherian superscheme and let $f: \mathcal{X} \to \mathcal{I}$ be a flat proper morphism of superschemes which is cohomologically flat in dimension 0 and has geometrically integral fibres. Let $\mathcal{L}$ be an even line bundle on $\mathcal{X}$ which is $f$-acyclic and relatively generated by its global sections. Then the complete linear series of $\mathcal{L}$ is represented by the projective superbundle $\overline{\mathcal{P}}(f, \mathcal{L}) = \mathbb{P}(Q)$.

**Proof.** As we have seen, $f_!\mathcal{L}$, and then $Q$, are locally free and nonzero. Given a superscheme $\mathcal{I} \to \mathcal{J}$ over $\mathcal{J}$, the relative positive divisors $\mathcal{Z}$ in the linear series $|\mathcal{L}|(\mathcal{I})$ are identified with the exact sequences

$$0 \to \mathcal{L}_{\mathcal{I}}^{-1} \otimes f_{\mathcal{J}}^*N^{-1} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{I}} \to 0,$$

where $\rho$ is the natural projection and $\mathcal{N}$ is an even line bundle on $\mathcal{I}$. The sheaf $\mathcal{N}$ is uniquely determined by $\rho$ because if $\mathcal{L}_{\mathcal{I}}^{-1} \otimes f_{\mathcal{J}}^*N^{-1} \cong \mathcal{L}_{\mathcal{I}}^{-1} \otimes f_{\mathcal{J}}^*\mathcal{N}'^{-1}$, then $f_{\mathcal{J}}^*\mathcal{N} \cong f_{\mathcal{J}}^*\mathcal{N}'$ so that $\mathcal{N} \cong \mathcal{N}'$ by Lemma 5.5. The morphism $\sigma$ can be seen as a section of $\mathcal{L}_{\mathcal{J}} \otimes f_{\mathcal{J}}^*\mathcal{N}$, and by Equation (5.5) it defines a morphism $\gamma(\sigma): Q_{\mathcal{J}} \to \mathcal{N}$ such that

$$\gamma(\sigma) = \gamma_{\mathcal{J}}(\sigma_{\mathcal{J}})$$

(5.6)

for every morphism $\mathcal{J}' \to \mathcal{J}$. The condition that $\mathcal{O}_{\mathcal{J}}$ is flat over $\mathcal{I}$ is tantamount to the fact that for every $t \in T$ the restriction $\sigma_t$ of $\sigma$ to the fibre of $f_{\mathcal{J}}: \mathcal{X}_{\mathcal{J}} \to \mathcal{I}$ over $t$ is still injective. Since the fibres of $f$ are geometrically integral, Proposition 3.14 implies that the injectivity of $\sigma_t$ for every $t \in T$ is equivalent to its nonvanishing. Thus by Equation (5.6)

$$\gamma(\sigma): Q \otimes \kappa(t) \to \mathcal{N} \otimes \kappa(t)$$

is nonzero for every $t \in T$, and then it is surjective as $\mathcal{N}$ is an even line bundle. By the super Nakayama Lemma ([5, 11] or [64, 6.4.5]), $\gamma(\sigma): Q_{\mathcal{J}} \to \mathcal{N}$ is surjective, that is, it yields a $\mathcal{J}$-valued point of the supergrassmanian $\text{SGrass}(Q,(1,0))$.

Moreover, if $\sigma'$ is another section of $\mathcal{L}_{\mathcal{J}} \otimes f_{\mathcal{J}}^*\mathcal{N}$ inducing the same positive relative divisor, then $\sigma' = \phi \circ \sigma$ where $\phi$ is an invertible section of $\mathcal{O}_{\mathcal{X}_{\mathcal{J}}}$, and $f$ is cohomologically flat in dimension 0, so that $\mathcal{O}_{\mathcal{J}} \cong f_{\mathcal{J}}^*\mathcal{O}_{\mathcal{X}_{\mathcal{J}}}$, $\phi$ is an invertible section of $\mathcal{O}_{\mathcal{J}}$ and $\gamma(\sigma') = \phi \circ \gamma(\sigma)$. Then, the two surjections $\gamma(\sigma): Q_{\mathcal{J}} \to \mathcal{N}$ and $\gamma(\sigma'): Q_{\mathcal{J}} \to \mathcal{N}$ define the same $\mathcal{J}$-valued point of $\text{SGrass}(Q,(1,0))$. It follows that there is a functor morphism $|\mathcal{L}| \to \text{SGrass}(Q,(1,0))$, which by the above discussion is an isomorphism. This concludes the proof as $\text{SGrass}(Q,(1,0)) \cong \mathbb{P}(Q)$ by Proposition 2.17.

**Remark 5.29.** The requirements on $\mathcal{L}$ in Proposition 5.28 are actually superfluous. There always exists a sheaf $Q$ fulfilling Equations 5.3 and 5.4 (see [30, 7.7.3] for the corresponding classical statement), and one can follow the same proof of Proposition 5.28 to prove that $|\mathcal{L}| \cong \mathbb{P}(Q)$ (see [37, Theorem 9.3.13]). However, the latter superscheme is not a superprojective bundle when $Q$ is not locally free.

**Remark 5.30.** The hypothesis of cohomological flatness in dimension 0 cannot be removed from Proposition 5.28. Indeed, we can produce examples of a morphism and a line bundle $\mathcal{L}$ satisfying all the hypotheses of Proposition 5.28 except for the cohomological flatness and such that $|\mathcal{L}|$ is not representable by $\overline{\mathcal{P}}(f, \mathcal{L}) = \mathbb{P}(Q)$.
If \( f : \mathbb{P}^0_\mathcal{F} = \mathbb{A}^0_\mathcal{F} \to \mathcal{F} \) is the affine superline over a superscheme \( \mathcal{F} \), for every \( \mathcal{F} \)-superscheme \( \mathcal{F} \) there are no nonzero relative effective divisors in \( f : \mathbb{A}^0_\mathcal{F} \to \mathcal{F} \) (as there are no codimension \((1,0)\) subschemes). This means that for even line bundle \( \mathcal{L} \) on \( \mathbb{A}^0_\mathcal{F} \) the complete linear system \( |\mathcal{L}| \) is

\[
|\mathcal{L}|(\mathcal{F}) = \begin{cases} 
\emptyset & \text{if } \mathcal{L}_\mathcal{F} \text{ is nontrivial} \\
\{0\} & \text{if } \mathcal{L}_\mathcal{F} \text{ is trivial}
\end{cases}
\]

Now take \( \mathcal{L} \) to be the trivial line bundle on \( \mathbb{A}^0_\mathcal{F} \), where \( k \) is a field. Then the corresponding linear system is represented by the point, but the corresponding projective superspace \( \mathbb{P}((f_*\mathcal{L})^{-1}) \) is \( \mathbb{P}^1_k \), which has more than one \( \mathcal{F} \)-point whenever the \( k \)-superscheme \( \mathcal{F} \) has nonzero odd functions. \( \triangle \)

### 5.6. Construction of the Picard superscheme

In this Subsection we prove (4) and (5) of Strategy 5.23, thus finishing the proof of the existence Theorem 5.4. Remember that we are assuming that \( \mathcal{F} \) is noetherian and connected and that \( f : \mathcal{X} \to \mathcal{F} \) is flat and projective with geometrically integral fibres and is cohomologically flat in dimension 0.

For simplicity let us write \( \mathcal{D} = \text{SDiv}(\mathcal{X}/\mathcal{F}) \to \mathcal{F} \) and \( O = O_{\mathcal{X} \times \mathcal{F}}. \) If \( \mathcal{D} = \mathcal{X} \times \mathcal{F} \) is the relative universal divisor over \( \mathcal{D} \), the open sub-superscheme \( \text{SDiv}(a, \mathcal{X}/\mathcal{F}) \to \mathcal{F} \) (Proposition 5.26) is the locus \( \mathcal{D}_a \) of the points of \( \mathcal{D} \) where \( O(\mathcal{D}) \) is relatively acyclic and generated by its global sections over \( \mathcal{D} \).

In the rest of this paragraph we will confuse superschemes with their functors of points. Now, the Abel morphism

\[ \text{Ab} : \mathcal{D}_a \to \mathcal{SPic}^{+a, \mathcal{X}/\mathcal{F}} \]

is defined by the class of \( [O(\mathcal{D})] \) in the relative Picard group of \( \mathcal{X} \times \mathcal{F} \mathcal{D}_a/\mathcal{D}_a \). Then in the cartesian diagram of morphisms of functors

\[
\begin{array}{ccc}
\mathcal{D}_a \times \mathcal{SPic}^{+a, \mathcal{X}/\mathcal{F}} & \xrightarrow{p_1} & \mathcal{D}_a \\
\downarrow^{p_2} & & \downarrow^{\text{Ab}} \\
\mathcal{D}_a & \xrightarrow{\text{Ab}} & \mathcal{SPic}^{+a, \mathcal{X}/\mathcal{F}}
\end{array}
\]

the projection \( p_2 \) identifies \( \mathcal{D}_a \times \mathcal{SPic}^{+a, \mathcal{X}/\mathcal{F}} \mathcal{D}_a \) with the fibre functor of the (vertical) Abel morphism over the universal line bundle \( O(\mathcal{D}) \):

\[ \mathcal{D}_a := \mathcal{D}_a \times \mathcal{SPic}^{+a, \mathcal{X}/\mathcal{F}} \mathcal{D}_a \cong [O(\mathcal{D})] = \text{Ab}^{-1}([O(\mathcal{D})]) \to \mathcal{D}_a \times \mathcal{D}_a, \]

so that the relative linear equivalence on \( \mathcal{D}_a \) is an equivalence relation of superschemes, thus proving (2) of Strategy 5.23.

If we fix the super Hilbert polynomials as in Subsection 5.3 we have similar formulas:

\[
\begin{array}{ccc}
\mathcal{D}_a^Q \times \mathcal{SPic}^{+a, \mathcal{X}/\mathcal{F}} & \xrightarrow{p_1^Q} & \mathcal{D}_a^Q \\
\downarrow^{p_2^Q} & & \downarrow^{\text{Ab}} \\
\mathcal{D}_a^Q & \xrightarrow{\text{Ab}} & \mathcal{SPic}^{P+a, \mathcal{X}/\mathcal{F}}
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{D}_a^Q := \mathcal{D}_a^Q \times \mathcal{SPic}^{+a, \mathcal{X}/\mathcal{F}} & \xrightarrow{\text{Ab}} & \mathcal{D}_a^Q \\
\downarrow^{\text{Ab}} & & \downarrow^{\text{Ab}} \\
\mathcal{D}_a^Q & \xrightarrow{\text{Ab}} & \mathcal{SPic}^{P+a, \mathcal{X}/\mathcal{F}},
\end{array}
\]

where \( O(\mathcal{D}^P) \) is over the universal line bundle whose dual has super Hilbert polynomial \( P \).

Since \( \mathcal{F} \) is noetherian, \( \mathcal{D}_a^Q \) is noetherian as well by Proposition 5.21. By Proposition 5.28 \( p_2 \).
is a projective superbundle $p_2: |O(\mathcal{L})| \simeq \overline{P}(f_{\mathcal{L_a}^*:O(\mathcal{L}))} \to \mathcal{D}_a$, so that it is proper and flat. Analogously, $p_2^Q$ is also a projective superbundle $p_2^Q: |O(\mathcal{L}^P)| \simeq \overline{P}(f_{\mathcal{L_a}^*:O(\mathcal{L}^P)}) \to \mathcal{D}_a^Q$.

One has following result, which proves (3) of Strategy 5.23.

**Proposition 5.31.** The relative linear equivalence $\mathcal{D}_a^Q := \mathcal{D}_a^Q \times_{\mathcal{SPic}^{X_a, X}/X} \mathcal{D}_a^P$ on $\mathcal{D}_a^Q \to \mathcal{I}$ is effective, so that the quotient $\mathcal{I}$-superscheme exists. The relative linear equivalence $\mathcal{D}_a := \mathcal{D}_a \times_{\mathcal{SPic}^{X_a, X}/X} \mathcal{D}_a$ on $\mathcal{D}_a \to \mathcal{I}$ is also effective, so that the quotient $\mathcal{I}$-superscheme exists; we denote those quotients by $q: \mathcal{D}_a^Q \to \mathcal{D}_a^Q/\sim$ and $q: \mathcal{D}_a \to \mathcal{D}_a/\sim$. Moreover, $\mathcal{D}_a^Q/\sim$ is of finite type and separated over $\mathcal{I}$ for every $Q$, and then $\mathcal{D}_a/\sim$ is of finite type and separated over $\mathcal{I}$ as well.

**Proof.** Since $\mathcal{D}_a$ is the disjoint union of the various $\mathcal{D}_a^Q$, it is enough to prove the first statement. By Proposition 5.21 $\mathcal{D}_a^Q$ is a sub-superscheme of a supergrassmannian, so that Corollary 2.30 and Theorem 4.4 imply that the Hilbert superscheme of $\mathcal{D}_a^Q/\mathcal{I}$ exists. Then, Proposition 5.13 implies that the relative linear equivalence $\mathcal{D}_a^Q$ is effective and that the quotient $\mathcal{D}_a^Q/\sim$ is of finite type and separated over $\mathcal{I}$.

We can now prove that the quotient superscheme $\mathcal{D}_a/\sim$ represents the Picard sheaf $\mathcal{SPic}^{+a, X}/\mathcal{I}(\text{et})$ and that $\mathcal{D}_a^Q/\sim$ represents the Picard sheaf $\mathcal{SPic}^{P, X}/\mathcal{I}(\text{et})$. For simplicity we introduce the notation
\[ \mathcal{SP}_a := \mathcal{SPic}^{+a, X}/\mathcal{I}(\text{et}). \]

Note that if we compose the Abel morphism with the immersion $\mathcal{SPic}^{+a, X}/\mathcal{I} \hookrightarrow \mathcal{SP}_a$ (Proposition 5.6) we have another Abel morphism
\[ \text{Ab}_{\text{et}}: \mathcal{D}_a \to \mathcal{SP}_a. \]

Moreover one has an isomorphism
\[ \mathcal{R}_a = \mathcal{D}_a \times_{\mathcal{SPic}^{+a, X}/\mathcal{I}} \mathcal{D}_a \simeq \mathcal{D}_a \times_{\mathcal{SP}_a} \mathcal{D}_a. \]

**Proposition 5.32.** Under the hypotheses of Theorem 5.4 there are isomorphisms of étale sheaves
\[ \mathcal{D}_a^Q/\sim \cong \mathcal{SPic}^{P, X}/\mathcal{I}(\text{ét}), \quad \mathcal{D}_a/\sim \cong \mathcal{SPic}^{+a, X}/\mathcal{I}(\text{ét}) \]
for the étale topology on $\mathcal{I}$-superschemes. Thus, the Picard sheaf $\mathcal{SPic}^{P, X}/\mathcal{I}(\text{ét})$ is representable by the superscheme $\mathcal{SPic}^{P, X}(X/\mathcal{I}) := \mathcal{D}_a^Q/\sim$, and the Picard sheaf $\mathcal{SPic}^{+a, X}/\mathcal{I}(\text{ét})$ is representable by the superscheme $\mathcal{SPic}^{+a, X}(X/\mathcal{I}) := \mathcal{D}_a/\sim$. Both Picard superschemes are of finite type and separated over $\mathcal{I}$.

**Proof.** It is enough to prove the first case. Since $\mathcal{R}_a^Q = \mathcal{D}_a^Q \times_{\mathcal{SP}_a} \mathcal{D}_a^Q$ and the projection $p_2^Q: \mathcal{R}_a^Q \to \mathcal{D}_a^Q$ is smooth and proper, if we prove that $\text{Ab}: \mathcal{R}_a^Q \to \mathcal{SPic}^{P, X}/\mathcal{I}(\text{ét})$ is an epimorphism of étale sheaves, the result will follow from Proposition 5.15. Let $\mathcal{F} \to \mathcal{I}$ be a superscheme over $\mathcal{I}$. A $\mathcal{I}$-valued point $\xi: \mathcal{I} \to \mathcal{SPic}^{P, X}/\mathcal{I}(\text{ét})$ is given by an étale covering $\phi: \mathcal{F} \to \mathcal{I}$ together with the class $\xi' = [\mathcal{L}] \in \mathcal{SPic}^{P, X}/\mathcal{I}(\mathcal{I}')$ of an even line bundle $\mathcal{L}$ on $\mathcal{X} \to \mathcal{F}$ that relatively acyclic and is generated by its global sections. One has to prove that there exist an étale covering $\psi: \mathcal{V} \to \mathcal{I}'$ and a morphism $\sigma: \mathcal{V} \to \mathcal{D}_a^Q$ of $\mathcal{I}$-superschemes such that $\text{Ab} \circ \sigma = \xi' \circ \psi$. 

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By Proposition 5.28 and Proposition 5.6 the fibre of the Abel morphism over $\xi'$ is a projective bundle $p: \tilde{\mathbb{P}}(f_{\mathcal{I}^*}\mathcal{L}) \to \mathcal{I}'$:

$$
\begin{array}{ccc}
\tilde{\mathbb{P}}(f_{\mathcal{I}^*}\mathcal{L}) & \xrightarrow{\xi'_Q} & \mathbb{P}^Q \\
p & & Ab \\
\mathcal{I}' & \xrightarrow{\xi'} & Pic^P_{\mathcal{I}^*}.
\end{array}
$$

Since $p$ is smooth, there exist an étale covering $\psi: \mathcal{V} \to \mathcal{I}'$ and a morphism $\varpi: \mathcal{V} \to \tilde{\mathbb{P}}(f_{\mathcal{I}^*}\mathcal{L})$ satisfying $\psi = p \circ \varpi$. Now one can take for $\sigma$ the composition $\xi'_Q \circ \varpi: \mathcal{V} \to \mathbb{P}^Q$. □

To finish the proof of Theorem 5.4 we check that the even Picard sheaf $\mathcal{S}\mathcal{P}ic_{+\mathcal{I}^*/\mathcal{I}(\acute{e}t)}$ and the open subsheaf $\mathcal{S}\mathcal{P}ic_{+\mathcal{A}^t/\mathcal{I}(\acute{e}t)}$ satisfy condition (1) of Proposition A.31. This is the content of the following statements. In the proof we use the following notation: we write $\mathcal{O} = \mathcal{O}_{\mathcal{I}^*}$ and $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$, where $\mathcal{O}(1)$ is the even ample line bundle which makes $f$ a superprojective morphism.

**Lemma 5.33.** Let us consider the set of the $\mathcal{I}$-valued points of the Picard sheaf defined by the classes $[\mathcal{O}(-n)]$ ($n > 0$). One has:

$$
\bigcup_{n \in \mathbb{N}} \mathcal{S}\mathcal{P}ic_{+\mathcal{A}^t/\mathcal{I}(\acute{e}t)} \cdot [\mathcal{O}(-n)] = \mathcal{S}\mathcal{P}ic_{+\mathcal{I}^*/\mathcal{I}(\acute{e}t)},
$$

that is, the even super Picard sheaf is the union of the translated of the even Picard sheaf $\mathcal{S}\mathcal{P}ic_{+\mathcal{A}^t/\mathcal{I}(\acute{e}t)}$ by the classes $[\mathcal{O}(-n)]$.

**Proof.** Let us write $\mathcal{S}\mathcal{P} = \mathcal{S}\mathcal{P}ic_{+\mathcal{I}^*/\mathcal{I}(\acute{e}t)}$ and $\mathcal{S}\mathcal{P}_n = \mathcal{S}\mathcal{P}ic_{+\mathcal{A}^t/\mathcal{I}(\acute{e}t)}$ for simplicity. One has to prove that for every $\mathcal{I}$-superscheme $\mathcal{I} \to \mathcal{I}$ and every section $\lambda \in \mathcal{S}\mathcal{P}(\mathcal{I})$, that is, for every functor morphism $\lambda: \mathcal{I} \to \mathcal{S}\mathcal{P}$, the open sub-superschemes $\mathcal{I}_n(\lambda) = \mathcal{S}\mathcal{P}_n \cdot [\mathcal{O}(-n)] \times _{\mathcal{S}\mathcal{P}, \lambda} \mathcal{I} \to \mathcal{I}$ for the various $n \in \mathbb{Z}$ yield a covering of $\mathcal{I}$. Let $\pi: \mathcal{I}' \to \mathcal{I}$ be an étale covering such that $\pi^*\lambda = \lambda \circ \pi: \mathcal{I}' \to \mathcal{S}\mathcal{P}$ is defined by the class $[\mathcal{L}]$ of an even line bundle $\mathcal{L}$ on $\mathcal{I}$. Then, it is enough to prove that the open sub-superschemes $\mathcal{I}'_n(\lambda) = \mathcal{S}\mathcal{P}_n \cdot [\mathcal{O}(-n)] \times _{\mathcal{S}\mathcal{P}, \pi^*\lambda} \mathcal{I}' \to \mathcal{I}'$ for the various $n \in \mathbb{Z}$ yield a covering of $\mathcal{I}'$. By Proposition 5.24 $\mathcal{I}'_n(\lambda)$ is the open sub-superscheme of the points $t \in \mathcal{I}'$ such that the restriction of the sheaf $\mathcal{L}_n(t)$ to the fibre $\mathcal{X}_t$ of $\mathcal{X}^\mathcal{I}, \to \mathcal{I}'$ over $t$ is acyclic and generated by its global sections. It follows then from Serre's Theorem 2.35 that for every point $t \in \mathcal{I}'$ there is $n$ such that $t \in \mathcal{I}'_n(\lambda)$. □

Then one has:

**Proposition 5.34 (End of the proof of Theorem 5.4).** Under the hypotheses of Theorem 5.4, the even super Picard sheaf $\mathcal{S}\mathcal{P}ic_{+\mathcal{I}^*/\mathcal{I}(\acute{e}t)}$ is representable by a superscheme $\mathcal{S}\mathcal{P}ic_{+\mathcal{I}^*/\mathcal{I}}$ which can be covered by open sub-superschemes of the form $\mathcal{S}\mathcal{P}ic_{+\mathcal{A}^t/\mathcal{I}} \cdot [\mathcal{O}(-n)]$. In particular, the Picard superscheme $\mathcal{S}\mathcal{P}ic_{+\mathcal{I}^*/\mathcal{I}}$ is locally of finite type over $\mathcal{I}$.

**Proof.** The representability follows from Propositions A.31 and 5.32 together with Lemma 5.33. Moreover, the superschemes $\mathcal{S}\mathcal{P}ic_{+\mathcal{A}^t/\mathcal{I}} \cdot [\mathcal{O}(-n)]$ are of finite type over $\mathcal{I}$ by Proposition 5.32 and then $\mathcal{S}\mathcal{P}ic_{+\mathcal{I}^*/\mathcal{I}}$ is locally of finite type over $\mathcal{I}$. □

5.7. Representability of the Picard functor over an even affine base.
5.7.1. General criterion. The conditions that according to Theorem 5.4 ensure the existence of the Picard superscheme of a morphism \( f : \mathcal{X} \to \mathcal{Y} \) and, in particular the requirement of projectivity and of cohomological flatness in dimension 0 may look quite restrictive. Using the following technical criterion for representability of the Picard functor for proper superschemes over an even affine base, we will show that in some cases these conditions can be relaxed.

Let us fix a Noetherian even ring \( R \).

**Proposition 5.35.** Let \( \mathcal{X} \) be a proper superscheme over \( S = \text{Spec}(R) \). Assume that 
\[(\mathcal{O}_{\mathcal{X},-})^{N+1} = 0 \quad \text{and} \quad N! \text{ is invertible in } R \text{ for some } N.\]
Then \( \mathcal{P}ic_{+\mathcal{X}/S(\ell)} \) is representable if and only if

- the usual Picard functor of the bosonic quotient \( \mathcal{X}/\Gamma \) over \( S \) is representable by an \( R \)-scheme;
- the functor \( M \mapsto H^1(\mathcal{X}/\Gamma, \mathcal{O}_{\mathcal{X},-} \otimes_R M) \) on the category of \( R \)-modules is left exact.

Furthermore, if this is the case then \( \mathcal{S}Pic_{+\mathcal{X}/S} \) splits as follows:
\[\mathcal{S}Pic_{+\mathcal{X}/S} \simeq \text{Pic}(\mathcal{X}/\Gamma) \times_S \text{Spec}(\bigwedge_R F),\]
for some finitely generated \( R \)-module \( F \) (where we think of elements of \( F \) as odd).

**Proof.** Let us set for brevity \( X_0 = \mathcal{X}/\Gamma \). For an \( R \)-superalgebra \( A \) the set of \( A \)-points \( \mathcal{S}Pic_{+\mathcal{X}/S}(A) \) can be identified with
\[H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X},+}^*) / \text{Pic}((\mathcal{X}/\Gamma)/S).\]
Furthermore, we have
\[\mathcal{O}_{\mathcal{X},+} = \mathcal{O}_{\mathcal{X},+} \otimes_R \mathcal{A}_+ + \mathcal{O}_{\mathcal{X},-} \otimes_R \mathcal{A}_-. \tag{5.7}\]
Hence, \((\mathcal{O}_{\mathcal{X},+} \otimes_R \mathcal{A}_+)^*\) is naturally a subgroup in \( \mathcal{O}_{\mathcal{X},+}^* \). On the other hand, we claim that there is a well defined homomorphism of sheaves of groups,
\[\exp : \mathcal{O}_{\mathcal{X},-} \otimes_R \mathcal{A}_- \to \mathcal{O}_{\mathcal{X},+}^*.\]
Indeed, this follows from the assumption that \((\mathcal{O}_{\mathcal{X},-})^{N+1} = 0 \) and \( N! \) is invertible in \( R \).

Next, we claim that the homomorphism
\[
(\mathcal{O}_{\mathcal{X},+} \otimes_R \mathcal{A}_+)^* \times \mathcal{O}_{\mathcal{X},-} \otimes_R \mathcal{A}_- \to \mathcal{O}_{\mathcal{X},+}^* : (x_0, x_1) \mapsto x_0 \exp(x_1)
\]
is an isomorphism. Indeed, the components of \( x_0 \exp(x_1) \) with respect to the decomposition (5.7) are given by
\[f_0 = x_0 \cosh(x_1) = x_0(1 + \frac{x_1^2}{2!} + \ldots), \quad f_1 = x_0 \sinh(x_1) = x_0(x_1 + \frac{x_1^3}{3!} + \ldots).\]
Hence, \( f_0^{-1} f_1 = \tanh(x_1) \), and we can recover \((x_0, x_1)\) from \((f_0, f_1)\) by
\[(f_0, f_1) \mapsto \left( \frac{f_0}{\cosh(\tanh^{-1}(f_0^{-1} f_1))}, \tanh^{-1}(f_0^{-1} f_1) \right),\]
where \( \tanh^{-1}(t) = t + \frac{3}{5} t^3 + \frac{8}{15} t^5 + \ldots \) (note that the compatibility with the group law corresponds to the addition theorem for \( \tanh \)).

The above isomorphism of sheaves of groups induces an isomorphism of functors
\[\mathcal{S}Pic_{+\mathcal{X}/S}(A) \simeq \mathcal{P}ic_{X_0/S}(\mathcal{A}_+) \times H^1(X_0, \mathcal{O}_{\mathcal{X},-} \otimes_R \mathcal{A}_-).\]
We claim that this implies the isomorphism for the sheafified functors
\[\mathcal{S}Pic_{+\mathcal{X}/S(\ell)}(A) \simeq \mathcal{P}ic_{X_0/S(\ell)}(\mathcal{A}_+) \times H^1(X_0, \mathcal{O}_{\mathcal{X},-} \otimes_R \mathcal{A}_-).\]
Namely, the fact that the second factor does not change follows by taking odd components in the identification
\[ H^1(X_0, \mathcal{O}_{X,-} \otimes_R \mathbb{A}) = \ker(H^1(X_0, \mathcal{O}_{X,-} \otimes_R \mathbb{B}) \to H^1(X_0, \mathcal{O}_{X,-} \otimes_R (\mathbb{B} \otimes_{A} \mathbb{B}))) \]
for any faithfully flat extension \( A \to \mathbb{B} \). Indeed, this follows from the faithfully flat descent for modules using the isomorphism
\[ H^1(X_0, \mathcal{O}_{X,-} \otimes_R \mathbb{B}) \simeq H^1(X_0, \mathcal{O}_{X,-} \otimes_R \mathbb{B}) \otimes_A \mathbb{B} \]
(and a similar isomorphism for \( \mathbb{B} \otimes_{A} \mathbb{B} \)) which follows from the flat base change.

Assume first that \( \mathcal{SPic}_{+X/S(\acute{\text{e}}t)} \) is representable. Then by Lemma 5.7(ii), \( \mathcal{Pic}_{X_0/S} \) is also representable. Now let us consider \( R \)-superalgebras of the form
\[ A = R \oplus M_-, \]
where \( M_- \) is any \( R \)-module (the multiplication on \( A \) is such that \((M_-)^2 = 0\)). Note that for such superalgebras we have
\[ \mathcal{SPic}_{+X/S(\acute{\text{e}}t)}(R \oplus M_-) \simeq \mathcal{Pic}_{X_0/S(\acute{\text{e}}t)}(R) \times T^1(M_-), \]
where
\[ T^1(M_-) := H^1(X_0, \mathcal{O}_{X,-} \otimes_R M_-). \]
Now let us consider any fibred product diagram of \( R \)-modules
\[
\begin{array}{ccc}
M_- & \times_{P_-} N_- & \to N_- \\
\downarrow & & \downarrow \\
M_- & \to P_- \\
\end{array}
\]
with \( M_- \to P_- \) surjective. We can form the corresponding diagram of \( R \)-superalgebras (by adding \( R \) as the even part). Then the corresponding spectra form a fibred coproduct diagram in the category of \( R \)-superschemes (this is checked similarly to the even case considered in [60]). Hence, taking \( N_- = 0 \) and applying the isomorphisms
\[ \mathcal{Mor}(\text{Spec}(R \oplus M_-), \mathcal{SPic}(X/S)) \simeq \mathcal{Pic}_{X_0/S(\acute{\text{e}}t)}(R) \times T^1(M_-) \]
we get the fibred product diagram of sets
\[
\begin{array}{ccc}
\mathcal{Pic}_{X_0/S(\acute{\text{e}}t)}(R) \times T^1(\ker(M_- \to P_-)) & \to & \mathcal{Pic}_{X_0/S(\acute{\text{e}}t)}(R) \times \{0\} \\
\downarrow & & \downarrow \\
\mathcal{Pic}_{X_0/S(\acute{\text{e}}t)}(R) \times T^1(M_-) & \to & \mathcal{Pic}_{X_0/S(\acute{\text{e}}t)}(R) \times T^1(P_-) \\
\end{array}
\]
which implies that \( T^1 \) is left exact.

Conversely, assume that \( \mathcal{Pic}_{X_0/S} \) is representable and \( T^1 \) is left exact. Then there exists a finitely generated \( R \)-module \( F \) such that there is a functorial isomorphism
\[ T^1(M_-) \simeq \text{Hom}_R(F, M_-). \]
This means that an element of \( \mathcal{SPic}_{+X/S(\acute{\text{e}}t)}(A) \) is given by a pair: a morphism \( \text{Spec}(A) \to \text{Pic}(X_0/S) \) (which automatically factors through \( \text{Spec}(A_+) \)) and a homomorphism of \( R \)-modules \( F \to A_- \). Giving such a homomorphism is equivalent to giving a homomorphism of \( R \)-superalgebras \( \bigwedge_R F \to \mathbb{A} \), so we deduce that \( \mathcal{SPic}_{+X/S(\acute{\text{e}}t)} \) is represented by
\[ \text{Pic}(X_0/S) \times_S \text{Spec}(\bigwedge_R F). \]
\[ \square \]
Remark 5.36. We claim that if the morphism $\mathcal{X} \to \text{Spec}(R)$ is flat and cohomologically flat in dimension 0 then the functor $M \mapsto H^1(\mathcal{X}/\Gamma, \mathcal{O}_{\mathcal{X},-} \otimes_R M)$ is left exact. Indeed, by Proposition 3.2, the functor $M \mapsto T^i(M) = H^i(\mathcal{X}/\Gamma, \mathcal{O}_{\mathcal{X},-} \otimes_R M)$ can be calculated as

$$T^i(M) = H^i(K^\bullet \otimes_R M),$$

where $K^\bullet$ is a complex of finitely generated projective $R$-modules, concentrated in degrees $[0,n]$. The condition of cohomological flatness in dimension 0 applied to $R$-superalgebras of the form $R \oplus M$ implies that $T^0(M) = 0$ for all $R$-modules $M$. This implies that the differential $\partial_0 : K^0 \to K^1$ is injective with projective quotient $C^1 = \text{coker}(\partial_0)$. It follows that

$$T^1(M) = \ker(C^1 \otimes_R M \to K^2 \otimes_R M),$$

so this functor is left exact. △

Corollary 5.37. Under the assumptions of Proposition 5.35, assume in addition that $\mathcal{X}$ is a flat over $R$ and the relative dimension of $\mathcal{X}/\Gamma$ over $R$ is $\leq 1$. Then $\mathcal{Pic}_{+\mathcal{X}/S(\text{ét})}$ is representable if and only if

- the usual Picard functor of the bosonic quotient $\mathcal{X}/\Gamma$ over $S$ is representable by an $S$-scheme;
- the $R$-module $H^1(\mathcal{X}/\Gamma, \mathcal{O}_{\mathcal{X},-})$ is projective.

Proof. We just observe that using the notation above, the functor $T^1$ associated with $\mathcal{O}_{\mathcal{X},-}$ (which is flat over $R$) is right exact due to the assumption on the relative dimension. Hence, it is left exact if and only if it is exact if and only if the $R$-module $H^1(\mathcal{X}/\Gamma, \mathcal{O}_{\mathcal{X},-})$ is projective (see [34, Cor. 12.6]). □

Corollary 5.38. Let $f : \mathcal{X} \to S$ be a flat family of $(1,1)$-curves over $S = \text{Spec}(R)$, given by $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X \oplus \mathcal{L}$, where $f_{\text{bos}} : X \to S$ is a family of smooth projective curves, and $\mathcal{L}$ is a line bundle over $X$. Then $\mathcal{Pic}_{+\mathcal{X}/S(\text{ét})}$ is representable if and only if $R^1f_{\text{bos}*}(\mathcal{L})$ is locally free.

For example, if $\mathcal{L} = \omega_{X/S}$ in the last corollary then $\mathcal{Pic}_{+\mathcal{X}/S(\text{ét})}$ is representable since $R^1f_{\text{bos}*}(\omega_{X/S})$ is locally free but $f : \mathcal{X} \to S$ is not cohomologically flat in dimension 0 if genus is $\geq 1$.

5.7.2. Generic representability and representability over a field.

Theorem 5.39. Let $S = \text{Spec}(R)$ where $R$ is an (even) integral domain. Assume that $\mathcal{X}$ is a proper superscheme over $S$ and that $(\mathcal{O}_{\mathcal{X},-})^{N+1} = 0$ and $N!$ is invertible in $R$, for some $N$. Then there exists a nonempty open subset $V \subset S$ such that $\mathcal{SPic}_{\mathcal{X}/V(\text{ét})}$ is representable by

$$\mathcal{Pic}((\mathcal{X}/\Gamma)_V/V) \times \mathbb{A}^{0,n}$$

for some $n$, where $\mathcal{Pic}((\mathcal{X}/\Gamma)_V/V)$ is a disjoint union of quasi-projective schemes over $V$.

Proof. We would like to apply Proposition 5.35 to $\mathcal{X}_V$ over $V$ for some open affine subset $V \subset S$. First, we apply the classical result [22, Thm. 9.4.18.2] that implies that there exists $V$ such that the Picard functor of $(\mathcal{X}/\Gamma)_V$ over $V$ is representable by a disjoint union of quasi-projective schemes.

Next we claim that after replacing $R$ with its nonzero localization, we can achieve that the functor $M \mapsto H^1(\mathcal{X}/\Gamma, \mathcal{O}_{\mathcal{X},-} \otimes_R M)$ on the category of $R$-modules is left exact. Indeed, let $K^0 \to \ldots \to K^n$ be the complex of finitely generated projective $R$-modules such that

$$T^i(M) = H^i(\mathcal{X}/\Gamma, \mathcal{O}_{\mathcal{X},-} \otimes_R M) \simeq H^i(K^\bullet \otimes_R M)$$
(it exists by Proposition 3.2, since $\mathcal{O}_\mathcal{X}$ is flat). Furthermore, $T^1$ is left exact if and only if coker$(\partial_0 : K^0 \to K^1)$ is projective (Lemma 3.3). But the latter condition is satisfied after replacing $R$ with its localization.

Applying Proposition 5.35, we get the representability of the Picard functor and an isomorphism

$$\text{SPic}_+(\mathcal{X}/R) \simeq \text{Pic}((\mathcal{X}/\Gamma)/R) \times_{\text{Spec}R} \text{Spec}(\bigwedge_R F),$$

for some finitely generated $R$-module $F$. Localizing $R$ further we can achieve that $F$ is a free $R$-module.

**Corollary 5.40.** Let $\mathcal{X}$ be a proper superscheme over a field $k$, such that for some $N$, one has $(\mathcal{O}_{\mathcal{X},-})^{N+1} = 0$ and $N!$ does not divide the characteristic of $k$. Then $\text{SPic}_+ \mathcal{X}/k(\acute{\text{e}}t)$ is representable by

$$\text{Pic}((\mathcal{X}/\Gamma)/k) \times_{\text{Spec}(k)} \mathbb{A}^{0,n},$$

where $n = \dim_k H^1(\mathcal{X}/\Gamma/k, \mathcal{O}_{\mathcal{X},-})$.

**Example 5.41.** Using Corollary 5.40 one can deduce that for the $\Pi$-projective space $\mathbb{P}^n_\Pi$ over a field $k$ of characteristic zero (see e.g., [54]) one has $\text{SPic}(\mathbb{P}^n_\Pi/k) \simeq \mathbb{A}^{0,1}$. Indeed, the corresponding reduced space is the usual projective space $\mathbb{P}^n$ and it is well known that only the trivial line bundle on $\mathbb{P}^n$ extends to a line bundle on $\mathbb{P}^n_\Pi$. Our assertion follows from the fact that this remains true over any base (even) ring.

5.7.3. **Example of non-representability of the super Picard functor.** Extending Corollary 5.37, we are going to give an example of a projective morphism of superschemes of relative dimension $(1,1)$ for which not only super Picard functor is not representable, but even the corresponding deformation functor is not pro-representable.

Let $X$ be a family of smooth projective curves over Spec$(R)$, where $R = k[t]$, and let $\mathcal{L}$ be a line bundle over $X$. Let us consider the split $(1,1)$-dimensional smooth superscheme $\mathcal{X}$ over Spec$(R)$, with the usual underlying scheme $X$, given by $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X \oplus \mathcal{L}$.

Let $K = R[t^{-1}] = k[t]$. Let $\mathcal{L}_0 = \mathcal{L}_{|X_0}$, where $X_0$ is the fibre of $X$ over Spec$k \subset$ Spec$R$ and let $\mathcal{L}_K = \mathcal{L}_{|X_K}$, where $X_K = X \times_{\text{Spec}(R)} \text{Spec}(K)$.

**Proposition 5.42.** Assume that $H^1(X_0, \mathcal{L}_0) \neq 0$ while $H^1(X_K, \mathcal{L}_K) = 0$. Then the relative even Picard functor of $\mathcal{X}/\text{Spec}(R)$ is not representable. Moreover, the corresponding functor on Artin super $R$-algebras (deforming the trivial line bundle over $\mathcal{X}_0$) is not pro-representable.

Here is an example when the above situation can realize. Consider a fixed smooth projective curve $X_0$ of genus $g \geq 1$ over $k$, and let $X = X_0 \times \text{Spec}(R)$. Let $\mathcal{L}_0$ be a line bundle of degree $g-1$ with $H^1(X_0, \mathcal{L}_0) \neq 0$, i.e., the corresponding point sits on the theta divisor in $\text{Pic}^{g-1}(X_0)$. Now consider any curve in $\text{Pic}^{g-1}(X_0)$ passing through this point and not contained in the theta-divisor. Then take the induced family of line bundles over $k[t]$.

Furthermore, by allowing to vary the curve as well, we can make sure that $\mathcal{L}$ is a theta-characteristic on $X$, so $\mathcal{X}$ would be a family of SUSY curves.

**Proof of Proposition.** Let us consider the functor $\mathbb{A} \mapsto \text{SPic}_+ (\mathcal{X}_\mathbb{A}/\mathbb{A})$ on local Artin $R$-superalgebras with the residue field $k$. Restricting this functor to local Artin $R$-algebras with $\mathbb{A}_+ = k$ and arguing as in the proof of Proposition 5.35, we see that if the deformation functor were pro-representable then the functor

$$M_- \mapsto H^1(X, \mathcal{L} \otimes_R M_-)$$
on the category of finite-dimensional torsion $R$-modules would be left exact. Note that by the base change,

$$H^1(X, \mathcal{L} \otimes_R M_-) \simeq H^1(X, \mathcal{L}) \otimes_R M_-.$$ 

But $H^1(X, \mathcal{L})$ is a finitely generated $R$-module with $H^1(X, \mathcal{L}) \otimes_R k = H^1(X, \mathcal{L}_0)$ and $H^1(X, \mathcal{L}) \otimes_R k = 0$ (still by the base change). Hence, $H^1(X, \mathcal{L})$ is a nonzero torsion $R$-module, so it is a direct sum of $R$-modules of the form $R/(t^n)$. Thus, it is enough to observe that the functor

$$M_- \mapsto R/(t^n) \otimes_R M_-$$ 

is not left exact on finite-dimensional torsion $R$-modules. Indeed, for the exact sequence

$$0 \to R/(t^n) \xrightarrow{t^n} R/(t^{n+1}) \to R/(t^n) \to 0$$

its tensor product with $R/(t^n)$ fails to be left exact. \hfill \Box

6. The super period map

Our goal in this section is to construct a morphism from an open substack of the moduli of proper and smooth supercurves to the moduli stack of principally polarized abelian supercurves.

**Definition 6.1.** Let $\mathcal{I}$ be a superscheme. An abelian superscheme over $\mathcal{I}$ is a group superscheme $\pi: \mathcal{X} \to \mathcal{I}$ that is proper and smooth and has connected geometric fibres.

The notion of abelian superscheme is stable under base change, that is, if $\pi: \mathcal{X} \to \mathcal{I}$ is an abelian superscheme over $\mathcal{I}$, for every $\mathcal{I}$-superscheme $\mathcal{J} \to \mathcal{I}$ the induced morphism $\pi_{\mathcal{J}}: \mathcal{X}_{\mathcal{J}} \to \mathcal{J}$ is an abelian superscheme over $\mathcal{J}$.

6.1. Smoothness of the Picard scheme. To begin with, we need a standard criterion of smoothness of the Picard superscheme. Let $f: \mathcal{X} \to \mathcal{I}$ be a flat proper morphism of superschemes, with $\mathcal{I}$ Noetherian, for which the even Picard superscheme $\mathbb{S}\text{Pic}^+(\mathcal{X}/\mathcal{I})$ exists and represents the super Picard sheaf in the étale topology.

**Proposition 6.2.** Assume that for some point $s \in S$ one has $H^2(X_s, \mathcal{O}_{\mathcal{X}_s}) = 0$. Then $\mathbb{S}\text{Pic}^+(\mathcal{X}/\mathcal{I})$ is smooth over an open neighbourhood of $s$.

**Proof.** This is a standard argument similar to the classical case. By semicontinuity and base change replacing $\mathcal{I}$ by an affine neighbourhood of $s$ we get that $H^2(X_s, \mathcal{O}_{\mathcal{X}_s}) = 0$ for every $s \in S$, and for every affine $\mathcal{I}$-superscheme $\mathcal{J} = \text{Spec} \,(\mathbb{B})$ one has $H^2(X_{\mathcal{J}}, \mathcal{O}_{\mathcal{X}_{\mathcal{J}}}) = 0$.

Now let $\text{Spec} \,(\mathbb{A})$ be an affine superscheme over $\mathcal{I}$ with a zero-square ideal $\mathcal{N} \subset \mathbb{A}$, and let $\mathbb{B} = \mathbb{A}/\mathcal{N}$. We can view $\mathcal{X}_\mathbb{A} = \mathcal{X} \times_{\mathbb{S}} \text{Spec} \,(\mathbb{A})$ as an infinitesimal thickening of $\mathcal{X}_{\mathcal{I}} = \mathcal{X}_{\mathbb{B}}$. The exact sequence

$$0 \to (f^*_\mathbb{B}\mathcal{N})_+ \to (\mathcal{O}_{\mathcal{X}_{\mathbb{A}}})^* \to (\mathcal{O}_{\mathcal{X}_{\mathbb{B}}})^* \to 0$$

leads to a long exact sequence

$$\ldots \to \text{Pic}(\mathcal{X}_\mathbb{A}) \to \text{Pic}(\mathcal{X}_\mathbb{B}) \to H^2(X_B, f^*_\mathbb{B}\mathcal{N})_+ \to \ldots$$

Thus, the obstruction to lifting a line bundle on $\mathcal{X}_\mathbb{B}$ to a line bundle on $\mathcal{X}_\mathbb{A}$ lies in the even part of $H^2(X_T, f^*_\mathbb{B}\mathcal{N})$. On the other hand,

$$H^2(X_T, f^*_\mathbb{B}\mathcal{N}) \simeq H^0(T, R^2 f^*_\mathbb{B}(f^*_\mathbb{N})) \simeq H^0(T, \mathcal{N} \otimes R^2 f^*_\mathbb{B}(\mathcal{O}_{\mathcal{X}_{\mathbb{B}}})) = 0$$
as \( R^2 f_\ast (\mathcal{O}_{X_S}) = 0 \). It follows that \( \mathbb{SPic}^+ (\mathcal{X}/\mathcal{S}/\mathcal{B}) \) is formally smooth (see Definition A.18). Since it is locally of finite type by Theorem 5.4, the conclusion follows (cf. Proposition A.20).

\[ \blacksquare \]

6.2. Abelian superschemes associated with Picard superschemes. In this section we construct abelian superschemes which are naturally associated with Picard superschemes of relative supercurves.

6.2.1. From the Picard superscheme to an abelian superscheme. Let \( f : \mathcal{X} \to \mathcal{S} \) be a proper and smooth morphism of relative dimension \((1, 1)\). Proceeding as in [49, Lemma 7.24], one can see that \( f \) is locally superprojective (see also [24] or [12]). Assume that \( H^s(X_s, \mathcal{O}_{X_s}) = 0 \) for each \( s \) in \( \mathcal{S} \). Then \( f \) is cohomologically flat in dimension 0 by Proposition 3.17, and Theorem 5.4 ensures the existence of the even Picard superscheme \( \mathbb{SPic}^+ (\mathcal{X}/\mathcal{S}) \) of \( f \).

**Theorem 6.3.** In this situation there exists an open subgroup scheme \( \mathbb{SPic}^0 (\mathcal{X}/\mathcal{S}) \hookrightarrow \mathbb{SPic}^+ (\mathcal{X}/\mathcal{S}) \) such that for every point \( s \in \mathcal{S} \), one has

\[
\mathbb{SPic}^0 (\mathcal{X}/\mathcal{S}) \times_{\mathcal{S}} \{ s \} = \mathbb{Pic}^0 (X_s),
\]

where \( \mathbb{Pic}^0 (X_s) \) is the connected component of zero of the Picard scheme \( \mathbb{Pic} (X_s) \). Furthermore, \( \mathbb{SPic}^0 (\mathcal{X}/\mathcal{S}) \) is smooth and proper of even dimension over \( \mathcal{S} \). Hence, there exists an abelian scheme \( A \) over the bosonic quotient \( \mathcal{S}/\Gamma \) such that \( \mathbb{SPic}^0 (\mathcal{X}/\mathcal{S}) \simeq \mathcal{X} \times_{\mathcal{S}/\Gamma} A \).

**Proof.** First, we observe that by Proposition 6.2 the superscheme \( \mathbb{SPic}^+ (\mathcal{X}/\mathcal{S}) \) is smooth over \( \mathcal{S} \). Furthermore, for every \( s \in \mathcal{S} \) the tangent space to \( \mathbb{SPic}^+ (\mathcal{X}/\mathcal{S}) \) at any point is isomorphic to \( H^1(X_s, \mathcal{O}_{X_s}) \). By assumption, \( H^1(X_s, \mathcal{O}_{X_s}) = 0 \). Hence, we derive that \( \mathbb{SPic}^+ (\mathcal{X}/\mathcal{S}) \) is a disjoint union of schemes which are smooth of even relative dimension over \( \mathcal{S} \).

Next, let us consider the case when \( \mathcal{S} = S \) is purely even. By Corollary A.27, in this case \( \mathbb{SPic}^+ (\mathcal{X}/S) \) is also a purely even superscheme (since it is a disjoint union of smooth superschemes of even dimension over \( S \)).

Let \( C = \mathcal{X}_{bos} = \mathcal{X}/\Gamma \) be the underlying family of usual curves over \( S \). From Lemma 5.7(ii) we get an isomorphism

\[
\mathbb{Pic} (C/S) \simeq \mathbb{SPic}^+ (\mathcal{X}/S)_{bos} \simeq \mathbb{SPic}^+ (\mathcal{X}/S).
\]

Thus, in the case of purely even \( \mathcal{S} = S \), using the isomorphism (6.1) we can define \( \mathbb{SPic}^0 (\mathcal{X}/S) \) as the subgroup scheme of \( \mathbb{SPic}^+ (\mathcal{X}/S) \) corresponding to \( \mathbb{Pic}^0 (C/S) \), which is an open subgroup scheme of \( \mathbb{Pic} (C/S) \). Hence, by the classical result on the Picard scheme of a family of curves (see [29, no 236, Thm. 2.1]), the scheme \( \mathbb{SPic}^0 (\mathcal{X}/S) \) is proper over \( S \).

Now let us return to the case of a general base \( \mathcal{S} \). Let us set \( S = \mathcal{X}_{bos} \) and let \( \mathcal{X}_S \to S \) be the base change of our family. By Lemma 5.7(i), one has

\[
\mathbb{SPic}^+ (\mathcal{X}/\mathcal{S}) \times_{\mathcal{S}} S \simeq \mathbb{SPic}^+ (\mathcal{X}/S). 
\]

In particular, the underlying topological space of \( \mathbb{SPic}^+ (\mathcal{X}/\mathcal{S}) \) is the same as that of \( \mathbb{SPic}^+ (\mathcal{X}/S) \), so we define \( \mathbb{SPic}^0 (\mathcal{X}/\mathcal{S}) \) as the open sub-superscheme of \( \mathbb{SPic}^+ (\mathcal{X}/\mathcal{S}) \) corresponding to the open subset \( \mathbb{SPic}^0 (\mathcal{X}/S) \). Note that the base change of the morphism

\[
\alpha : \mathbb{SPic}^0 (\mathcal{X}/\mathcal{S}) \times_{\mathcal{S}} \mathbb{SPic}^0 (\mathcal{X}/\mathcal{S}) \to \mathbb{SPic}^+ (\mathcal{X}/\mathcal{S})
\]

\[(a, b) \mapsto a \cdot b^{-1}\]
from $\mathscr{I}$ to $S$ is a similar morphism for $\mathbb{S}\text{Pic}^0_+(\mathscr{I}_S/S)$. It follows that the image of $\alpha$ is contained in $\mathbb{S}\text{Pic}^0_+(\mathscr{I}/\mathscr{I})$, so $\mathbb{S}\text{Pic}^0_+(\mathscr{I}/\mathscr{I})$ is a subgroup.

It follows from Theorem 5.4 that $\mathbb{S}\text{Pic}^0_+(\mathscr{I}/\mathscr{I})$ is locally of finite type. Since we also know that $\mathbb{S}\text{Pic}^0_+(\mathscr{I}_S/S)$ is proper over $S$, we deduce that $\mathbb{S}\text{Pic}^0_+(\mathscr{I}/\mathscr{I})$ is proper over $\mathscr{I}$.

Finally, applying Proposition A.26 we see that $\mathbb{S}\text{Pic}^0_+(\mathscr{I}/\mathscr{I})$ is obtained by the base change from a smooth and proper scheme $A$ over $\mathscr{I}/\Gamma$. Furthermore, by the same theorem, the group structure on $\mathbb{S}\text{Pic}^0_+(\mathscr{I}/\mathscr{I})$ also comes from a group structure on $A$. Thus, $A$ is an abelian superscheme.

### 6.2.2. From a family of SUSY-curves to a polarized abelian superscheme.

We can apply Theorem 6.3 to a family of SUSY curves $f: \mathscr{X} \to \mathscr{I}$. Recall that this is a proper smooth morphism of relative dimension $(1, 1)$ equipped with a certain distribution of rank $(0, 1)$ (see e.g., [42] for details). We claim that in this case the obtained abelian superscheme over $\mathscr{I}/\Gamma$ carries a natural polarization.

**Theorem 6.4.** Let $f: \mathscr{X} \to \mathscr{I}$ be a relative SUSY curve. There exists a natural polarized abelian superscheme over $\mathscr{I}/\Gamma$ that extends the relative Jacobian of the corresponding family of usual curves over $f_{\text{bos}}: X \to S$.

**Proof.** Let us set for brevity $\mathbb{S}P^0 = \mathbb{S}\text{Pic}^0_+(\mathscr{X}/\mathscr{I})$. As we have seen in Theorem 6.3, $\mathbb{S}P^0 = \mathscr{I} \times_{\mathscr{I}/\Gamma} A$, for an abelian scheme $A$ over $\mathscr{I}/\Gamma$.

First, assume that $f$ admits a section $\sigma: \mathscr{I} \to \mathscr{X}$. Viewing this section as a Neveu-Schwarz (NS) puncture we can construct the corresponding relative effective divisor $\mathscr{L}_\sigma \hookrightarrow \mathscr{X}$ supported on $\sigma(\mathscr{I})$ (see [24, Lemma 2.9] and also [17]). On the other hand, considering the diagonal $\delta$ as an NS puncture on the family $\mathscr{X} \times_{\mathscr{I}} \mathscr{X}$, we get a relative effective divisor $\mathscr{L}_\delta \hookrightarrow \mathscr{X} \times_{\mathscr{I}} \mathscr{X}$, so that we have a line bundle $\mathcal{O}_{\mathscr{X} \times_{\mathscr{I}} \mathscr{X}}(\mathscr{L}_\delta - p_1^* \mathscr{L}_\sigma)$ on $\mathscr{X} \times_{\mathscr{I}} \mathscr{X}$ which restricts to the trivial line bundle over $\mathscr{X} \times_{\mathscr{I}} \sigma(\mathscr{I})$. Hence, it defines a morphism of superschemes over $\mathscr{I}$,

$$\alpha_\sigma: \mathscr{X} \to \mathbb{S}P^0$$

depending on $\sigma$, which restricts to the standard Abel morphism on the underlying curve $X$ over $S$.

Set $\mathbb{P}^V := \mathscr{I} \times_{\mathscr{I}/\Gamma} A^V$, where $A^V$ is the dual abelian scheme to $A$ over $\mathscr{I}/\Gamma$. Note that the dual abelian scheme $A^V$ exists by a theorem of Raynaud (see [21, Ch. I, Sec. 1]). Let $\mathcal{P}$ be the pull-back of the Poincaré line bundle from $A \times A^V$ to $\mathbb{P} \times \mathbb{P}^V$ (normalized along the zero sections). Consider the line bundle

$$\mathcal{L}_\sigma := (\alpha_\sigma \times \text{Id})^* \mathcal{P}$$

on $\mathscr{X} \times \mathbb{P}^V$. Note that it is trivialized along $\sigma(\mathscr{I}) \times \mathbb{P}^V$ and along $\mathscr{X} \times 0$. Hence, it defines a morphism

$$\mathbb{P}^V \to \mathbb{S}P^0,$$

sending the zero section to the zero section. By Proposition A.26, this morphism comes from a homomorphism of abelian schemes over $\mathscr{I}/\Gamma$

$$\lambda_\sigma: A^V \to A$$

(recall that any morphism between abelian schemes preserving the zero section is a homomorphism, see [51, Cor. 6.4]). Furthermore, the induced morphism of abelian schemes over $S$ corresponds to the standard principal polarization of $J(X)^V \simeq J(X)$. Hence, $(A^V, \lambda)$ is a polarized abelian scheme over $\mathscr{I}/\Gamma$ extending the relative Jacobian of $X$ over $S$. 
We claim that if $\sigma' : \mathcal{S} \to \mathcal{X}$ is another section, then $\lambda_{\sigma'} = \lambda_{\sigma}$. Indeed, first we observe that $\alpha_{\sigma'}$ differs from $\alpha_{\sigma}$ by the translation map $t_\xi : \mathbb{S} \mathcal{P}^0 \to \mathbb{S} \mathcal{P}^0$, corresponding to the $\mathcal{S}$-point $\xi \in \mathbb{S} \mathcal{P}^0(\mathcal{S})$ coming from the relative divisor $\mathcal{Z}_{\sigma'} - \mathcal{Z}_{\sigma}$. Then, using the standard isomorphism

$$(x + y, z)^* \mathcal{P} \simeq (x, z)^* \mathcal{P} \otimes (y, z)^* \mathcal{P},$$

we get an isomorphism

$$\mathcal{L}_{\sigma'} \simeq \mathcal{L}_{\sigma} \otimes \mathbb{P}_2(\mathcal{P}_{(x, \mathcal{Y})}).$$

Hence $\mathcal{L}_{\sigma'}$ defines the same element in the relative Picard functor, so the corresponding homomorphisms are the same.

In general, $f : \mathcal{X} \to \mathcal{S}$ has a section locally in the étale topology, so we define the homomorphism from $A^\vee$ to $A$ locally and then use descent (Proposition A.28) to show that it exists globally.

□

Appendix A. Miscellaneous results

A.1. Associated primes of a module. The definition of associated prime ideal of a module over an arbitrary ring as well as the primary decomposition of an ideal are defined and studied in many places; see for instance [40]. For the case of a noetherian superring $\mathbb{A}$, if we restrict ourselves to $\mathbb{Z}_2$-graded ideals and modules, the theory is very much like the usual one in commutative algebra.

If $M$ is a ($\mathbb{Z}_2$-graded) $\mathbb{A}$-module, the associated ($\mathbb{Z}_2$-graded) primes of $M$ are the prime ideals that occur among the annihilators of the elements of $M$ (see [64, Coro. 4.3.6]). Since $\mathbb{A}$ is noetherian, each finitely generated module has associated primes and there are only finitely many of them. By the very definition one has [64, Thm. 6.4.3]:

**Proposition A.1.**

1. A prime ideal $p$ of $\mathbb{A}$ is an associated prime to $M$ if and only if there is either an injective morphism of $\mathbb{A}$-modules $\mathbb{A}/p \hookrightarrow M$ or an injective morphism of $\mathbb{A}$-modules $(\mathbb{A}/p)^\Pi \hookrightarrow M$.

2. If $M$ is finitely generated, there is a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_s = M$$

such that either $M_j/M_{j-1} \simeq \mathbb{A}/p_j$ or $M_j/M_{j-1} \simeq (\mathbb{A}/p_j)^\Pi$, where $p_j$ is a prime ideal for every index $j = 1, \ldots, s$.

3. If $M$ is finitely generated and $f \in \mathbb{A}$ is a homogeneous element that does not belong to any of the associated primes of $M$, then

$$\text{Tor}_1^\mathbb{A}(A/(f), M) = 0,$$

so that there is an exact sequence

$$0 \to f \cdot \mathbb{A} \otimes_\mathbb{A} M \to M \to M/f \cdot M \to 0.$$

□

**Remark A.2.** If $\mathcal{X}$ is a noetherian superscheme, the above notions can be globalized for any coherent sheaf $M$ on $\mathcal{X}$. There is a particular situation that is used in the extension to the “super” setting of Castelnuovo-Mumford properties of $m$-regularity (Subsection 4.3). Let

$$\mathbb{E} = \mathbb{E}_k(m, n) = k[x_0, \ldots, x_m, \theta_1, \ldots, \theta_n]$$

be a polynomial ring in $x_0, \ldots, x_m$ and $\theta_1, \ldots, \theta_n$ with $x_i$ even, $\theta_j$ odd, and $\mathbb{E}/\mathbb{E}^\vee$ a superscheme.

We have

$$\text{Tor}_1^\mathbb{E}(\mathbb{E}/\mathbb{E}^\vee, \mathbb{E}) = 0,$$

so that there is an exact sequence

$$0 \to \mathbb{E}^\vee \otimes_\mathbb{E} \mathbb{E} \to \mathbb{E} \to \mathbb{E}/\mathbb{E}^\vee \to 0.$$
be a free polynomial $k$-algebra and $\mathcal{X}' = \mathbb{P}^m_{k,n} = \text{Proj} \mathbb{B}$ the projective superspace over $k$. Given a $\mathbb{Z}$-homogeneous polynomial $f \in k[x_0, \ldots, x_m]$ of $\mathbb{Z}$-degree 1, we consider $f$ as an even element of $\mathbb{B}$. If $\mathbb{B}' = \mathbb{B}/(f)$ and $\mathcal{X}' = \text{Proj} \mathbb{B}' \simeq \mathbb{P}^m_{k,n}$, there is a closed immersion

$$\mathbb{P}^{m-1,n} \simeq \mathcal{X}' \hookrightarrow \mathcal{X} = \mathbb{P}^m_{k,n},$$

which identifies $\mathcal{X}'$ with the closed super hyperplane defined by the ideal $\mathcal{J} \simeq f \cdot \mathbb{B} \simeq \mathcal{O}_{\mathcal{X}}(-1)$ generated by $f$.

Let $\mathcal{M}$ be a coherent sheaf on $\mathcal{X}$. If $\mathcal{X}'$ does not contain any of the points of $\mathcal{X}'$ corresponding to the associated primes of $\mathcal{M}$, then the 1-Tor sheaf vanishes:

$$\text{Tor}_1^{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}'}, \mathcal{M}) = 0,$$  \hspace{1cm} (A.1)

and one has an exact sequence

$$0 \rightarrow \mathcal{M}(-1) \simeq \mathcal{J} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{M} \rightarrow \mathcal{M} \rightarrow \mathcal{M}|_{\mathcal{X}'} \rightarrow 0.$$  \hspace{1cm} (A.2)

$\triangle$

A.2. Filtrations of a graded module and super Artin-Rees theorem. Let $\mathbb{A}$ be a superring, $I$ a $\mathbb{Z}_2$-graded ideal and $M$ a $\mathbb{Z}_2$-graded $\mathbb{A}$-module.

**Definition A.3.** A filtration

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$$

is an $I$-filtration if $I \cdot M_n \subseteq M_{n+1}$ for every $n$, and it is a stable filtration if there exists an integer $s \geq 0$ such that $I \cdot M_n = M_{n+1}$ for every $n \geq s$.

Let us consider the bigraded $\mathbb{A}$-algebra

$$S_I(\mathbb{A}) = \bigoplus_{n \geq 0} \lambda^n I^n,$$

where $\lambda$ is an even indeterminate of $\mathbb{Z}$-degree 1 which we use to keep track of the $\mathbb{Z}$-degree. For any $I$-filtration $\{M_n\}$ of an $\mathbb{A}$-module $M$ we have a bigraded $S_I(\mathbb{A})$-module

$$S(M) = \bigoplus \lambda^n M_n.$$

**Lemma A.4.** Assume that $\mathbb{A}$ is noetherian.

1. $S_I(\mathbb{A})$ is an $\mathbb{A}$-algebra of finite type, so that it is noetherian.
2. If $M$ is a finitely generated $\mathbb{A}$-module, then an $I$-filtration $\{M_n\}$ of $M$ is stable if and only if the associated $S_I(\mathbb{A})$-module $S(M)$ is finitely generated.

**Proof.** (1). If $I = (a_1, \ldots, a_s)$, then $S_I(\mathbb{A}) = \mathbb{A}[\lambda a_1, \ldots, a_s \lambda]$.

(2). Every $M_s$ is finitely generated over $\mathbb{A}$, and the same happens for $N_n = \bigoplus_{s=0}^n M_s$, so that it generates a submodule of $S(M)$ which is finitely generated over $S_I(\mathbb{A})$. This submodule is given by

$$P_n = M_0 \oplus \cdots \oplus \lambda^n M_n \oplus \lambda^{n+1} I M_n \oplus \cdots \oplus \lambda^{n+s} I^s M_n \oplus \cdots.$$ 

One then has an ascendent chain

$$P_0 \subseteq \cdots \subseteq P_n \subseteq \cdots$$

whose union is $S(M)$. Since $S_I(\mathbb{A})$ is noetherian, $S(M)$ is finitely generated if and only there is $n_0$ such that $P_{n_0} = S(M)$. This is equivalent to $\lambda^n M_{n_0} = M_{n_0+n_0}$ for every $n \in \mathbb{N}$, that is, to the stability of the filtration. $\square$
We can now prove the super version of the Artin-Rees Lemma (see [49, Lemma 7.8]). Our proof is based on [3, Prop. 10.9].

**Proposition A.5** (Artin-Rees). Let $\mathbb{A}$ be a noetherian superring, $I$ an ideal in it, $M$ a finitely generated $\mathbb{A}$-module, and $\{M_n\}$ a stable $I$-filtration of $M$. If $M' \hookrightarrow M$ is a submodule, the induced filtration $\{M' \cap M_n\}$ is stable.

**Proof.** Since

$$I(M' \cap M_n) \subseteq I M' \cap IM_n \subseteq M' \cap M_{n+1},$$

$\{M' \cap M_n\}$ is an $I$-filtration. The associated bigraded $S_I(\mathbb{A})$-module is a submodule of $S(M)$.

We now apply Lemma A.4 repeatedly: $S(M)$ is finitely generated, and then $S(M')$ is finitely generated as well, since $S_I(\mathbb{A})$ is noetherian, so that the $I$-filtration $\{M' \cap M_n\}$ is stable. □

**Corollary A.6** (Artin-Rees Lemma). There exists an integer $s \geq 0$ such that

$$(I^n M) \cap M' = I^{n-s}(I^s M) \cap M'$$

for every $n \geq s$. □

### A.3. Local criterion for flatness

We now review the extension to superrings of the local criterion for flatness, which for the ordinary case can be found for instance in [1, 48]. We only consider a simpler version which suffices for these notes (see [49, Lemma 7.7]).

**Lemma A.7** (Local criterion for flatness). Let $\phi: \mathbb{A} \to \mathbb{B}$ be a morphism of finite type of local noetherian superrings and $I$ an ideal of $\mathbb{A}$. Then $\phi$ is flat if and only if the induced morphism $\bar{\phi}: \mathbb{A}/I \to \mathbb{B}/IB$ is flat and $I \otimes_{\mathbb{A}} \mathbb{B} \cong IB$.

**Proof.** The proof of [48, Thm. 22.3] applies straightforwardly to our situation as $IB$ is contained in the Jacobson radical of $\mathbb{B}$, since $\mathbb{B}$ is local, and we can use the super version of the Artin-Rees lemma (Corollary A.6). □

**Corollary A.8.** Let $\psi: \mathbb{A} \to \mathbb{B}$, $\phi: \mathbb{B} \to \mathbb{B}'$ be morphisms of finite type of local noetherian superrings and write $\psi' = \phi \circ \psi: \mathbb{A} \to \mathbb{B}'$ for the composition. Assume that $\psi$ and $\psi'$ are flat. Then $\phi$ is flat if and only if the induced morphism $\bar{\phi}: \mathbb{B}/m_{\mathbb{A}}\mathbb{B} \to \mathbb{B}'/m_{\mathbb{A}}\mathbb{B}'$ is flat, where $m_{\mathbb{A}}$ is the maximal ideal of $\mathbb{A}$.

**Proof.** One has $m_{\mathbb{A}} \otimes_{\mathbb{A}} \mathbb{B} \simeq m_{\mathbb{A}} \mathbb{B}$ and $m_{\mathbb{A}} \otimes_{\mathbb{A}} \mathbb{B}' \simeq m_{\mathbb{A}} \mathbb{B}'$ since $\psi$ and $\psi'$ are flat. Then, $(m_{\mathbb{A}} \mathbb{B}) \otimes_{\mathbb{B}} \mathbb{B}' \simeq (m_{\mathbb{A}} \mathbb{B})\mathbb{B}'$ and one concludes by Lemma A.7. □

### A.4. Nakayama’s lemma for half exact functors

We prove here a super extension of Nakayama’s lemma for half exact functors given in [56]. Let $\mathbb{A}$ be a local noetherian superring, $m$ its maximal ideal and $\kappa = \mathbb{A}/m$. Let $\mathfrak{M}(\mathbb{A})$ be the category of $\mathbb{Z}_2$-graded finitely generated $\mathbb{A}$-modules and $T$ a half exact functor from $\mathfrak{M}(\mathbb{A})$ to itself which is linear, that is, such that for every pair of $\mathbb{Z}_2$-graded finitely generated $\mathbb{A}$-modules $M$, $N$ the natural map

$$\text{Hom}_{\mathbb{A}}(M, N) \to \text{Hom}_{\mathbb{A}}(T(M), T(N))$$

is a morphism of $\mathbb{Z}_2$-graded modules.

**Lemma A.9.** If $T(\kappa) = 0$, then $T = 0$.

**Proof.** We have to prove that $T(M) = 0$ for every $\mathbb{Z}_2$-graded finitely generated $\mathbb{A}$-module $M$. By Proposition A.1 $M$ has a filtration whose successive quotients are of the form $\mathbb{A}/p$ for certain prime ideals $p$. Then, it is enough to prove that $T(\mathbb{A}/p) = 0$ for every ($\mathbb{Z}_2$-graded) prime ideal $p$. We proceed by induction on the dimension of the ordinary ring $\mathbb{A}/p$. If
\( \mathbb{A}/\mathfrak{p} = 1 \), then \( \mathfrak{p} = \mathfrak{m} \) and this is our assumption. If \( \mathfrak{p} \neq \mathfrak{m} \), take an even element \( a \in \mathfrak{m} - \mathfrak{p} \). For every prime ideal \( \mathfrak{q} \) containing \( \mathfrak{p} + \langle a \rangle \) one has \( \dim \mathbb{A}/\mathfrak{q} < \dim \mathbb{A}/\mathfrak{p} \), and then \( T(\mathbb{A}/\mathfrak{q}) = 0 \) by induction. Applying this to the successive quotients of the above mentioned filtration in the case \( M = \mathbb{A}/\mathfrak{p} + \langle a \rangle \) we obtain \( T(\mathbb{A}/\pi(\mathfrak{p}) + \langle a \rangle) = 0 \).

Now, from the exact sequence

\[
0 \to \mathbb{A}/\mathfrak{p} \xrightarrow{\partial} \mathbb{A}/\mathfrak{p} \to \mathbb{A}/\mathfrak{p} + \langle a \rangle \to 0,
\]

we get \( a \cdot T(\mathbb{A}/\mathfrak{p}) = 0 \), so that \( \mathbb{A}/pf = 0 \) by the super Nakayama Lemma ([5, 11] or [64, 6.4.5]). \( \square \)

The natural morphism \( M = \text{Hom}_\mathbb{A}(\mathbb{A}, M) \to \text{Hom}_\mathbb{A}(T(\mathbb{A}), T(M)) \), defined for every \( \mathbb{Z}_2 \)-graded finitely generated \( \mathbb{A} \)-module \( M \), induces an morphism of functors \( T(\mathbb{A}) \otimes_\mathbb{A} \to T \).

**Proposition A.10.** Under the above hypothesis, the following conditions are equivalent:

1. the above morphism of functors is an isomorphism,
   \[ T(\mathbb{A}) \otimes_\mathbb{A} \cong T; \]
2. the morphism \( T(\mathbb{A}) \to T(\mathbb{A}/\mathfrak{m}) \) is surjective;
3. \( T \) is right exact.

**Proof.** It is clear that (1) implies (2).

(2) \( \implies \) (3). Let \( Q \) be the half exact functor given by \( Q(M) = T(M)/\text{Im}((T(\mathbb{A}) \otimes_\mathbb{A} M)) \).

Since \( Q(\mathbb{A}/\mathfrak{m}) = 0 \), one has \( Q = 0 \) by Lemma A.9 and then, \( T(\mathbb{A}) \otimes_\mathbb{A} M \to T(M) \) is a surjection for every \( M \). It follows that \( T \) is right exact.

(3) \( \implies \) (1). Since \( \mathcal{M} \) is finitely generated and \( \mathbb{A} \) is noetherian, there is an exact sequence

\[ \mathbb{A}^{r,s} \to \mathbb{A}^{p,q} \to M \to 0. \]

Moreover, \( T(\mathbb{A}^{r,s}) \cong T(\mathbb{A}) \otimes_\mathbb{A} \mathbb{A}^{r,s} \) and \( T(\mathbb{A}^{p,q}) \cong T(\mathbb{A}) \otimes_\mathbb{A} \mathbb{A}^{p,q} \) as \( T \) is half exact, and there is an exact sequence

\[ T(\mathbb{A}) \otimes_\mathbb{A} \mathbb{A}^{r,s} \to T(\mathbb{A}) \otimes_\mathbb{A} \mathbb{A}^{p,q} \to T(M) \to 0 \]

as \( T \) is right exact. This proves that \( T(M) \cong T(\mathbb{A}) \otimes_\mathbb{A} M \). \( \square \)

**A.5. Separated and proper morphisms of superschemes.** For convenience we mention a few of types of morphisms of superschemes.

**Definition A.11.** A morphism \( f : \mathcal{X} \to \mathcal{Y} \) of superschemes is:

1. affine, if for every affine open sub-superscheme \( \mathcal{U} \subset \mathcal{Y} \) the inverse image \( f^{-1}(\mathcal{U}) \) is affine;
2. finite, if it is affine, and for \( \mathcal{U} = \text{Spec} \mathbb{A} \subset \mathcal{Y} \), then \( f^{-1}(\mathcal{U}) = \text{Spec} \mathbb{B} \), where \( \mathbb{B} \) is a finitely generated graded \( \mathbb{A} \)-module;
3. locally of finite type, if \( \mathcal{Y} \) has an affine open cover \( \{ \mathcal{U}_i = \text{Spec} \mathbb{A}_i \} \) such that every inverse image \( f^{-1}(\mathcal{U}_i) \) has an open affine cover \( \mathcal{V}_i = \{ \mathcal{V}_{ij} = \text{Spec} \mathbb{B}_{ij} \} \), where each \( \mathbb{B}_{ij} \) is a finitely generated graded \( \mathbb{A}_i \)-algebra.
4. \( f \) is of finite type if in addition each \( \mathcal{V}_i \) can be taken to be finite. In other words, if it is locally of finite type and quasi-compact.
5. separated, if the diagonal morphism \( \delta_f : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X} \) is a closed immersion (actually it is enough to ask that it is a closed morphism);
(6) proper, if it is separated, of finite type and universally closed;
(7) flat, if for every point \( x \in X \), \( \mathcal{O}_{\mathcal{X},x} \) is flat over \( \mathcal{O}_{\mathcal{Y},f(x)} \);
(8) faithfully flat, if it is flat and surjective.

The following proposition is standard.

**Proposition A.12.** The properties of being flat and faithfully flat are stable under base change and composition.

The separateness and properness of a morphism depend only on the associated morphism between the underlying ordinary schemes.

**Proposition A.13.** A morphism \( f : \mathcal{X} \to \mathcal{Y} \) of superschemes which is locally of finite type is proper (resp. separated) if and only if the induced scheme morphism \( f_{\text{bos}} : X \to S \) is proper (resp. separated).

**Proof.** Since the bosonic reduction of the diagonal morphism of \( f \) is the diagonal morphism of \( f_{\text{bos}} \), \( \delta_f \) is closed if and only if \( \delta_{f_{\text{bos}}} \) is closed. This proves the separatedness. For the properness we have to see first that \( f \) is of finite type if and only if \( f_{\text{bos}} \) is of finite type, which is true because we are assuming that \( f \) is locally of finite type, and secondly that \( f \) is universally closed if and only \( f_{\text{bos}} \) is universally closed, which is true as this is a topological question.

Thus, one has:

**Corollary A.14.** The valuative criteria for separatedness and properness [34, Thm. II.4.3 and II.4.7] are still valid in the graded setting.

We can then extend to superschemes many of the properties of proper and separated morphisms of schemes.

**Proposition A.15.**

1. The properties of being separated and proper are stable under base change.
2. Every morphism of affine superschemes is separated.
3. Open immersions are separated and closed immersions are proper.
4. The composition of two separated (resp. proper) morphisms of superschemes is separated (resp. proper).
5. If \( g : \mathcal{Y} \to \mathcal{X} \) and \( f : \mathcal{X} \to \mathcal{Y} \) are morphisms of superschemes and \( f \circ g \) is proper and \( f \) is separated, then \( g \) is proper. If \( f \circ g \) is separated, then \( g \) is separated.

### A.6. Smooth Morphisms

A definition of smooth morphism of superschemes was given in [9, 2.16] for superschemes that are locally of finite type over an algebraically closed field. We would like to remove the requirement that the base field is algebraically closed. Let us review the standard definitions in the case of ordinary schemes. A scheme \( X \) over \( k \) is smooth if it is locally of finite type and geometrically regular, that is, if for every extension \( k \to K \) where \( K \) is an algebraically closed field, the local rings of the scheme \( X \times_{\text{Spec } k} \text{Spec } K \) are regular. A scheme morphism \( f : X \to S \) is smooth when it is locally of finite presentation (or simply locally of finite type if the schemes are locally noetherian), flat, and its fibres \( X_s \) are smooth over the residue field \( \kappa(s) \) for every point \( s \in S \). By [31, 17.15.5], a flat morphism \( f : X \to S \)
locally of finite presentation is smooth if and only if the sheaf of relative differentials $\Omega_f$ is a locally free $\mathcal{O}_X$-module of rank equal to the relative dimension $\dim f$. This suggests the following definition.

**Definition A.16.** A morphism $f: \mathcal{X} \to \mathcal{S}$ of superschemes of relative dimension $(m,n)$ is smooth if:

1. $f$ is locally of finite presentation (if the superschemes are locally noetherian, it is enough to ask that $f$ is locally of finite type);
2. $f$ is flat;
3. the sheaf of relative differentials $\Omega_{\mathcal{X}/\mathcal{S}}$ is locally free of rank $(m,n)$.

A morphism $f: \mathcal{X} \to \mathcal{S}$ of superschemes is étale if it is smooth of relative dimension $(0,0)$.

When $\mathcal{S} = \text{Spec} \ k$ is a single point one obtains the definitions of smooth or étale superscheme over a field.

One has the following criterion for smoothness over a field, which extends Proposition 2.14 in [9].

**Proposition A.17.** A superscheme $\mathcal{X}$ of dimension $(m,n)$, locally of finite type over a field $k$, is smooth if and only if

1. $\mathcal{X}$ is a smooth scheme over $k$ of dimension $m$;
2. the $\mathcal{O}_\mathcal{X}$-module $E = \mathcal{J}/\mathcal{J}^2$ is locally free of rank $n$ and the natural map $\Lambda_{\mathcal{O}_\mathcal{X}} \mathcal{E} \to Gr_1(\mathcal{O}_\mathcal{X})$ is an isomorphism.

If $\mathcal{X}$ is smooth of dimension $(m,n)$ then it is locally split, and for every closed point $x \in X$ there exist graded local coordinates, that is, $m$ even functions $(z_1, \ldots, z_m)$ which generate the maximal ideal $m_x$ of $\mathcal{O}_{\mathcal{X},x}$ and $n$ odd functions $(\theta_1, \ldots, \theta_n)$ generating $\mathcal{E}_x$, such that $(dz_1, \ldots, dz_m, d\theta_1, \ldots, d\theta_n)$ is a basis for $\Omega_{\mathcal{X},x}$.

**Proof.** Assume that the two conditions hold. Since the question is local, we can assume that $\mathcal{X}$ is affine and $E$ is a free $\mathcal{O}_\mathcal{X}$-module with generators $\theta_1, \ldots, \theta_n$. Let us choose some lifts $\bar{\theta}_i \in \mathcal{J}$ of $\theta_i$ for $i = 1, \ldots, n$. Since $X$ is smooth, there exists a splitting $\mathcal{O}_X \to \mathcal{O}_{\mathcal{X},+}$ of the nilpotent extension $\mathcal{O}_{\mathcal{X},+} \to \mathcal{O}_X$. Thus, we get a homomorphism

$$\mathcal{O}_X[\theta_1, \ldots, \theta_n] \to \mathcal{O}_{\mathcal{X}}.$$  

Since it induces an isomorphism of the associative graded algebras, it is an isomorphism. Hence, $\mathcal{X}$ is locally split and we have $\Omega_{\mathcal{X}/X} \simeq \mathcal{O}_X d\theta_1 \oplus \ldots \oplus \mathcal{O}_X d\theta_n$, and the exact sequence of $\mathcal{O}_{\mathcal{X}}$-modules

$$\Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_\mathcal{X} \to \Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}/X} \to 0$$

splits:

$$\Omega_{\mathcal{X}} \simeq (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_\mathcal{X}) \oplus \mathcal{O}_X d\theta_1 \oplus \ldots \oplus \mathcal{O}_X d\theta_n.$$  

Since $X$ is smooth, $\Omega_{\mathcal{X},x}$ is free of rank $m$ for every point $x$, and then $\Omega_{\mathcal{X}}$ is locally free, so that $\mathcal{X}$ is smooth.

For the converse, assume that $X$ is smooth and consider a local ring, which we still denote by $\mathcal{O}_X$. The exact sequence induced by $i: X \to \mathcal{X}$ gives an exact sequence

$$\mathcal{E} = \mathcal{J}/\mathcal{J}^2 \to \Omega_{\mathcal{X}} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_X \to \Omega_X \to 0.$$
where $\delta(\theta)$ is the class of $d\theta$ modulo $\mathcal{J}$. Taking even and odd parts we get an isomorphism
\[
\Omega_X \simeq (\Omega_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X)_+ \simeq (\Omega_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X)_-.
\]
and a surjection
\[
E \to (\Omega_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X)_-.
\] (A.3)

Since $(\Omega_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X)_+$ is free of rank $m$ and $X$ has dimension $m$, we get that $X$ is smooth by [31, 17.15.5]. On the other hand, by definition of odd dimension, there exists a surjection $\mathcal{O}_X \to \mathcal{O}_E$. Since $(\Omega_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X)_-$ is free of rank $n$, this implies that (A.3) is an isomorphism, so $E$ is also free $\mathcal{O}_X$-modules of rank $n$.

Since $X$ is smooth, as before, we can choose a splitting $\mathcal{O}_X \to \mathcal{O}_{\mathcal{X}'}$. Let also $(\theta_1, \ldots, \theta_n)$ be sections of $\mathcal{J}$ projecting to a basis of $E$, and let $(z_1, \ldots, z_m)$ be a set of minimal generators of the maximal ideal $m$ of $\mathcal{O}_X$. Then $(dz_1, \ldots, dz_m, d\theta_1, \ldots, d\theta_n)$ is a basis of $\Omega_{\mathcal{X}'}$. It remains to prove that the natural epimorphism
\[
\rho : \mathcal{O}_X[t_1, \ldots, t_n] \to \mathcal{O}_{\mathcal{X}'} : t_i \mapsto \theta_i,
\]
where $(t_i)$ are formal odd variables, is an isomorphism. We proceed by induction on $n$, the case $n = 0$ being trivial. Let $\mathcal{I}$ be the closed sub-superscheme of $\mathcal{X}'$ defined by the ideal $(\theta_n)$. Then $\Omega_{\mathcal{X}'}$ is a free $\mathcal{O}_{\mathcal{I}}$-module with the basis $(dz_1, \ldots, dz_m, d\theta_1, \ldots, d\theta_{n-1})$, and by induction assumption we have an isomorphism
\[
\mathcal{O}_X[t_1, \ldots, t_{n-1}] \to \mathcal{O}_{\mathcal{I}} = \mathcal{O}_{\mathcal{X}'}/(\theta_n).
\] (A.4)

Now, if an element $f \in \mathcal{O}_X[t_1, \ldots, t_n]$ is in the kernel of $\rho$, reducing modulo $\theta_n$ and applying the above isomorphism, we see that
\[
f = \sum a_{j_1, \ldots, j_{n-1}} t_{j_1} \cdots t_{j_{n-1}} \cdot t_n,
\]
where the sum runs over $1 \leq j_1 < \cdots < j_{n-1} \leq n-1$ and the coefficients are in $\mathcal{O}_X$. Let us take differentials in the identity
\[
0 = \rho(f) = \sum a_{j_1, \ldots, j_{n-1}} \theta_{j_1} \cdots \theta_{j_{n-1}} \cdot \theta_n.
\]
Since $(dz_1, \ldots, dz_m, d\theta_1, \ldots, d\theta_n)$ is a basis of $\Omega_{\mathcal{X}'}$, looking at the coefficient of $d\theta_n$, we get
\[
0 = \sum a_{j_1, \ldots, j_{n-1}} \theta_{j_1} \cdots \theta_{j_{n-1}}.
\]
Hence, using isomorphism (A.4) we get that all the coefficients should be zero, so $f = 0$. \qed

A.6.1. Formally smooth morphisms. As in the commutative case, there are other possible approaches to defining smoothness of morphisms. We will prove the equivalence between smoothness and formal smoothness plus local finite type in the locally noetherian case.

**Definition A.18.** A morphism of superschemes $f : \mathcal{X} \to \mathcal{I}$ is formally smooth if for every affine $\mathcal{I}$-superscheme $\text{Spec}(B)$ and every nilpotent ideal $N \subset B$, any $\mathcal{I}$-morphism $\text{Spec}(B/N) \to \mathcal{X}$ extends to an $\mathcal{I}$-morphism $\text{Spec}(B) \to \mathcal{X}$.

**Example A.19.** For any superring $\mathcal{A}$ the polynomial superalgebra $\mathcal{B}(m,n)$ defined in Equation (2.1) is formally smooth over $\mathcal{A}$, that is, $\text{Spec} \mathcal{B} \to \text{Spec} \mathcal{A}$ is formally smooth.

**Proposition A.20.** Let $f : \mathcal{X} \to \mathcal{I}$ be a formally smooth morphism locally of finite type of relative dimension $(m,n)$ of locally noetherian superschemes.

1. $\Omega_{\mathcal{X}/\mathcal{I}}$ is locally free of rank $(m,n)$.
(2) Let $\mathcal{X} \rightarrow \mathcal{Y}$ be a closed immersion of $\mathcal{X}$-superschemes locally of finite type, there is a natural exact sequence of sheaves of $\mathcal{O}_\mathcal{X}$-modules
\[
0 \rightarrow I/I^2 \rightarrow (\Omega^1_{\mathcal{Y}/\mathcal{X}})|_{\mathcal{X}} \rightarrow \Omega^1_{\mathcal{X}/\mathcal{Y}} \rightarrow 0,
\]
where $I$ is the ideal sheaf of $\mathcal{X}$ in $\mathcal{Y}$.

(3) Assume that $\mathcal{X}$ and $\mathcal{Y}$ are affine, $\mathcal{X} = \text{Spec } \mathcal{B}$, $\mathcal{Y} = \text{Spec } \mathcal{A}$. Then there exists an affine open covering $\mathcal{B} = \cup_i \text{Spec } \mathcal{B}_i$, where each $\text{Spec } \mathcal{B}_i \rightarrow \text{Spec } \mathcal{A}$ is a standard smooth morphism, i.e.,
\[
\mathcal{B}_i = \mathcal{A}[x_1, \ldots, x_p, \theta_1, \ldots, \theta_q]/(f_1, \ldots, f_c, \phi_1, \ldots, \phi_d),
\]
where $x_i$ and $f_j$ are even, while $\theta_r$ and $\phi_s$ are odd, such that the matrices $(\partial f_i/\partial x_j)_{i,j \leq c}$ and $(\partial \phi_i/\partial \theta_j)_{i,j \leq d}$ are invertible in $\mathcal{A}_i$.

Proof. We prove (1) and (2) together. The statements are local, so we can assume $\mathcal{X} = \text{Spec } \mathcal{B}$, $\mathcal{Y} = \text{Spec } \mathcal{A}$, where $\mathcal{B}$ is a finitely generated $\mathcal{A}$-algebra. Let $\mathcal{B}'$ be another superalgebra over $\mathcal{A}$ such that $\mathcal{B} \simeq \mathcal{B}'/I$. By smoothness of $\text{Spec } \mathcal{B}$ there exists a homomorphism $\mathcal{B} \rightarrow \mathcal{B}'/I^2$ lifting the isomorphism $\mathcal{B} \simeq \mathcal{B}'/I$. Hence, we have a splitting $\mathcal{B}'/I^2 = \mathcal{B} \oplus I/I^2$. Let $x \mapsto x_1$ denote the corresponding projection
\[
\mathcal{B}' \rightarrow \mathcal{B}/I^2 \rightarrow I/I^2.
\]
It is easy to check that it is derivation. Hence it induces a well defined splitting
\[
\Omega^1_{\mathcal{B}/\mathcal{A}} \otimes_{\mathcal{B}'} \mathcal{B} \rightarrow I/I^2 : \quad dx \mapsto x_1
\]
of the the standard exact sequence
\[
I/I^2 \rightarrow \Omega^1_{\mathcal{B}'/\mathcal{A}} \otimes_{\mathcal{B}'} \mathcal{B} \rightarrow \Omega^1_{\mathcal{B}/\mathcal{A}} \rightarrow 0.
\]
Applying this in the case when $\mathcal{B}'$ is a free polynomial superalgebra $\mathcal{B}(m', n')$ over $\mathcal{A}$ (Equation (2.1)), we deduce that $\Omega^1_{\mathcal{B}/\mathcal{A}}$ is a direct summand of the free $\mathcal{B}$-module $\Omega^1_{\mathcal{B}'/\mathcal{A}} \otimes_{\mathcal{B}'} \mathcal{B}$. Therefore, $\Omega^1_{\mathcal{B}/\mathcal{A}}$ is locally free.

(3) is proved similarly to [63, Lemma 10.136.10].

\[ \square \]

Remark A.21. One can check as in the even case that formal smoothness for finitely generated superalgebras $\mathcal{B}/\mathcal{A}$ is equivalent to smoothness understood in terms of the naive cotangent complex $N\mathcal{L}_{\mathcal{B}/\mathcal{A}} := \tau_{\leq 1} L_{\mathcal{B}/\mathcal{A}}$, i.e., $H_1(L_{\mathcal{B}/\mathcal{A}}) = 0$ and $H_0(L_{\mathcal{B}/\mathcal{A}})$ is a finitely generated projective module (see [63, Prop. 10.137.8]).

\[ \triangle \]

Example A.22. If $k$ is a field, and $(\mathcal{B}, m)$ is a local Noetherian superalgebra over $k$ with $\mathcal{B}/m = k$, obtained as a localization of a finitely generated $k$-superalgebra, the formal smoothness of $\mathcal{B}$ over $k$ implies that the completion $\hat{\mathcal{B}}$ is isomorphic to the formal power ring over $k$ in even variables $x_1, \ldots, x_m$ and odd variables $\theta_1, \ldots, \theta_n$ (see also [25]). In particular, if $\Omega^1_{\mathcal{B}/k}$ has rank $(m, 0)$ then $\hat{\mathcal{B}}$ is purely even, hence, so is $\mathcal{B}$.

We are now going to prove the equivalence between smoothness and formal smoothness for morphisms locally of finite type. Recall that all the superschemes are supposed to be locally noetherian.

Lemma A.23. Let $f: \mathcal{X} \rightarrow \mathcal{Y}$ be a smooth morphism of relative dimension $(m, n)$, and let $f$ be a function on $\mathcal{X}$ such that $df$ generates a subbundle (of rank $(1, 0)$ or $(0, 1)$) in $\Omega^1_{\mathcal{Y}/\mathcal{X}}|_{\mathcal{Y}}$, \(^{6}\) where $\mathcal{Y}$ is the closed sub-superscheme corresponding to the ideal sheaf $(f)$. Then $\mathcal{Y}$ is smooth of dimension $(m - 1, n)$ or $(m, n - 1)$ for $f$ even or odd, respectively.

\(^{6}\)By a subbundle we mean a locally free subsheaf such that the corresponding quotient is locally free
Proof. First, let us check that $\mathcal{Y} \to \mathcal{X}$ is flat. This is a local question, so we can assume $S = \text{Spec}(A)$ and $\mathcal{X} = \text{Spec}(B)$ for some local noetherian superrings, and that $f$ is in the maximal ideal $m \subset A$. We know that $B/mB$ admits generators in the maximal ideal $\bar{z}_1, \ldots, \bar{z}_m, \theta_1, \ldots, \theta_n$ (with $\bar{z}_j$ even and $\theta_j$ odd) such that the image $f$ of $f$ in $B/mB$ is one of them (see Proposition A.17), and the map

$$\mathbb{A}/m[t_1, \ldots, t_m, \psi_1, \ldots, \psi_n] \to B/mB, \quad t_i \mapsto \bar{z}_i, \quad \psi_j \mapsto \bar{\theta}_j$$

from the free polynomial superring is étale.

Let us lift $(\bar{z}_i)$ and $(\bar{\theta}_j)$ to some elements $(z_i)$ and $(\theta_j)$ in the maximal ideal of $B$. We claim that the morphism

$$\mathbb{A}[t_1, \ldots, t_m, \psi_1, \ldots, \psi_n] \to B : \quad t_i \mapsto z_i, \quad \psi_j \mapsto \theta_j$$

is étale. Indeed, it is clear that it is unramified (i.e., relative differentials vanish). Also, it is flat, as follows immediately from Corollary A.8. Now let us consider $B/(f)$. Note that $f$ is one of the coordinates $(t_1, \ldots, t_m, \psi_1, \ldots, \psi_n)$. Hence, the induced map

$$\mathbb{A}[t_1, \ldots, t_m, \psi_1, \ldots, \psi_n]/(\bar{f}) \to B/(f)$$

is again flat. But $\mathbb{A}[t_1, \ldots, t_m, \psi_1, \ldots, \psi_n]/(\bar{f})$ is still a polynomial superring, so it is flat over $\mathbb{A}$. Thus, $B/(f)$ is flat over $\mathbb{A}$.

The exact sequence

$$0 \to \mathcal{I}_{\mathcal{Y}/\mathcal{X}}/\mathcal{I}_{\mathcal{Y}/\mathcal{X}}^2 \to (\Omega_{\mathcal{X}/\mathcal{Y}})|_{\mathcal{Y}} \to \Omega_{\mathcal{Y}/\mathcal{X}} \to 0$$

shows that $\Omega_{\mathcal{Y}/\mathcal{X}}$ is locally free of rank $(m-1, n)$ or $(m, n-1)$ for $f$ even or odd, respectively.

Thus, it remains to compute the dimension of the fibres of $\mathcal{Y} \to \mathcal{X}$. To this end we can work over a field $k$ and use the characterization of smooth superschemes $\mathcal{X}$ of dimension $(m, n)$ as locally split superschemes with smooth bosonization of dimension $m$ and the conormal bundle of $X$ in $\mathcal{X}$ of rank $(0, n)$ (see Proposition A.17). Note that $df$ generates a subbundle of $\Omega_{\mathcal{X}/k}$ in an open neighbourhood of $\mathcal{Y}$. If $f$ is even then $Y$ is a smooth divisor given by the function $f|_X$ on $X$, while the odd dimension does not change. If $f$ is odd then $Y = X$ and $f$ can be locally taken as one of the odd coordinates on $\mathcal{X}$, so the odd dimension of $\mathcal{Y}$ is $n - 1$. □

Remark A.24. The standard argument for an analogous statement in the even case is based on the fact that an even function $f$ as in Lemma A.23 would not be a zero divisor. This is not true for odd functions, so the argument had to be modified.

We now obtain the equivalence we were seeking for.

Proposition A.25. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of superschemes, locally of finite type. Then $f$ is formally smooth if and only if it is smooth.

Proof. If $f$ is formally smooth then it is locally standard (Proposition A.20 (3)). Hence, we can assume that $\mathcal{X}$ is a sub-superscheme in a relative affine space $P = k\mathcal{P}_{\mathcal{Y}} \to \mathcal{Y}$ given by $(f_1, \ldots, f_c)$, where $df_1, \ldots, df_c$ generate a subbundle in $\Omega_{P/\mathcal{Y}}|_{\mathcal{X}}$. Replacing $P$ by an open neighbourhood of $\mathcal{X}$ we can assume that $df_1, \ldots, df_c$ generate a subbundle of rank $(a, b) = (p-m, q-n)$ in $\Omega_{P/\mathcal{Y}}$, where $a$ (resp., $b$) is the number of even (resp., odd) functions among $(f_i)$. Now applying Lemma A.23 we derive that $f$ is smooth.

Conversely, starting with a smooth morphism $f : \mathcal{X} \to \mathcal{Y}$, let us embed locally $\mathcal{X}$ into a relative affine space $P$. Locally we can choose functions $(f_1, \ldots, f_c) \in \mathcal{I}_\mathcal{X}$ such that the kernel of $\Omega_{P/\mathcal{Y}}|_{\mathcal{X}} \to \Omega_{\mathcal{X}/\mathcal{Y}}$ (which is locally free) is generated by $df_1, \ldots, df_c$. The ideal
(f_1, \ldots, f_e) defines a standard formally smooth closed sub-superscheme \( X' \hookrightarrow \mathcal{U} \subset P \) such that \( X' \hookrightarrow X' \) (here \( \mathcal{U} \) is some open neighbourhood of \( X' \) in \( P \)). By the first part of the proof, \( X' \) is smooth over \( \mathcal{I} \) of the same relative dimension as \( X' \).

Now by flatness, it is enough to check that \( X_s = (\mathcal{U} \cap X')_s \) for every \( s \in S \), so we can work over a field. Since \( X \) and \( X' \) are smooth of the same dimension, we have \( X = X' \). Furthermore, the conormal sheaves of \( X \) in \( X' \) and \( X' \) are locally free of the same rank, and one surjects onto the other, hence they are equal. Since both \( X \) and \( X' \) are smooth, they are locally split (see Prop. A.17), and it follows that \( X = X' \), so \( X' \) is formally smooth over \( \mathcal{I} \). \( \square \)

A.6.2. Structure of the smooth morphisms. All the superschemes are again locally noetherian. We shall freely use the equivalence between smoothness and formal smoothness for morphisms locally of finite type (Proposition A.25).

**Proposition A.26.** Let \( f: X \to \mathcal{I} \) be a smooth morphism of relative dimension \((m, 0)\). Then the morphism \( f/\mathcal{I}: X/\mathcal{I} \to \mathcal{I}/\mathcal{I} \) induced between the bosonic quotients is a smooth morphism of relative dimension \((m, 0)\), and \( f \) is obtained from \( f/\mathcal{I} \) by the base change with respect to \( \mathcal{I} \to \mathcal{I}/\mathcal{I} \). Furthermore, the base change with respect to \( \mathcal{I} \to \mathcal{I}/\mathcal{I} \) yields an equivalence between the categories of smooth \( \mathcal{I}/\mathcal{I} \)-schemes of relative dimension \( m \) and smooth \( \mathcal{I} \)-superschemes of relative dimension \( m \).

**Proof.** This is a local question, so we can consider a homomorphism of local superalgebras \( f: A \to \mathcal{B} \) instead. We have the induced homomorphisms \( f_+: A_+ \to \mathcal{B}_+ \) and

\[ \alpha: \mathcal{B}' := \mathcal{B}_+ \otimes_{A_+} A \to \mathcal{B}. \]

We would like to show that \( \alpha \) is an isomorphism.

Let \( \mathfrak{m}_- \subset A \) (resp., \( \mathfrak{m}_- \subset \mathcal{B} \)) denote the maximal ideal. First, we note that \( \mathcal{B}/\mathfrak{m}\mathcal{B} \) is smooth of even dimension over the field \( k = A/\mathfrak{m}_A \), so it is purely even (see Example A.22). Since \( A_- \subset \mathfrak{m}_A \), this implies that \( \mathcal{B}_- = A_- \cdot \mathcal{B}_+ \). Hence, the homomorphism \( \alpha \) is surjective. Let \( I = \ker \alpha \). Then we have an exact sequence

\[ 0 \to I/I^2 \to \mathcal{B} \otimes_{A} \Omega_{\mathcal{B}/A} \to \Omega_{\mathcal{B}/A} \to 0 \quad (A.5) \]

(see Proposition A.20 for the exactness on the left due to smoothness of \( \mathcal{B} \) over \( A \)). But \( \Omega_{\mathcal{B}/A} \cong A \otimes_{A_+} \Omega_{\mathcal{B}_+/A_+}. \) Hence, \( \Omega_{\mathcal{B}/A} \otimes_{A} k \cong \Omega_{\mathcal{B}_+/A_+} \otimes_{A_+} k \), so tensoring the composed map

\[ \Omega_{\mathcal{B}/A} \to \mathcal{B} \otimes_{A} \Omega_{\mathcal{B}/A} \to \Omega_{\mathcal{B}/A} \quad (A.6) \]

by \( k \) over \( A \) one has

\[ A \otimes_{A_+} \Omega_{\mathcal{B}_+/A_+} \to \Omega_{\mathcal{B}/A} \]

which is an isomorphism as \( \mathcal{B}_+ \otimes_{A_+} k \to \mathcal{B} \otimes_{A} k \). Since both maps in Equation (A.6) are surjective, they become isomorphisms upon tensoring with \( k \) over \( A \). Thus, tensoring the sequence A.5 by \( k \) and using the fact that \( \Omega_{\mathcal{B}/A} \) is flat over \( A \), we deduce that \( I/I^2 \otimes_{A} k = 0 \). Since \( \mathfrak{m}_A \subset \mathfrak{m}_B \), this implies that \( I/I^2 = 0 \), so \( I = 0 \). This proves the claim that \( \alpha \) is an isomorphism.

Now let us check that \( \mathcal{B}_+ \) is formally smooth over \( A_+ \) (in the category of purely even commutative rings). Given a square-zero extension \( S_+ \to R_+ \) fitting into a commutative
let us form the corresponding commutative square

\[
\begin{array}{c}
A \quad \rightarrow \\
\downarrow \\
S \quad \downarrow \\
\downarrow \\
B \quad \rightarrow \quad R
\end{array}
\]

where \( S := A \otimes_{A_+} S_+, R = B \otimes_{B_+} R_+ \). Note that taking even components we recover the original square. By the formal smoothness of \( B \) over \( A \) there exists a lifting homomorphism \( f: B \rightarrow S \). Now the even component of \( f \) is the required lifting \( B_+ \rightarrow S_+ \).

For the last statement, it suffices to check that for any scheme \( X_0 \) and \( Y_0 \), smooth over \( S/\Gamma \), and for any morphism \( f: S \times_{\Gamma} X_0 \rightarrow S \times_{\Gamma} Y_0 \), one has \( f = S \times_{\Gamma} (f/\Gamma) \). Again, this is a local statement, so we can assume that \( S = \text{Spec}(A) \), \( S/\Gamma = \text{Spec}(A_+) \), and \( X_0 \) and \( Y_0 \) correspond to some \( A_+ \)-algebras \( B \) and \( C \). Then our statement is that any (even) homomorphism of \( \mathbb{Z}_2 \)-graded \( A \)-algebras

\[
\phi: B \otimes_{A_+} A \rightarrow C \otimes_{A_+} A
\]

is determined by the corresponding homomorphism of \( A_+ \)-algebras \( \phi_+: B \rightarrow C \), and this is clearly true. \( \square \)

**Corollary A.27.** If \( \mathcal{X} \) is smooth of dimension \((m,0)\) over \( \mathcal{I} \) and \( \mathcal{I} \) is even, then \( \mathcal{X} \) is even.

### A.7. Faithfully flat descent for superschemes.

Grothendieck’s proof of the faithfully flat descent in [32, Exp. VIII] can be straightforwardly extended to superschemes, as the key Lemma 1.4 there is also true for superrings and \( \mathbb{Z}_2 \)-graded modules. In particular, we have the analogue of Grothendieck’s Theorem 1.1 (faithfully flat descent for quasi-coherent sheaves on superschemes), which has two parts, the analogues of Corollaries 1.2 and 1.3, respectively. We state here the results for reference.

**Proposition A.28** (Descent for homomorphisms). Let \( p: \mathcal{I} \rightarrow \mathcal{J} \) be a faithfully flat quasi-compact morphism of superschemes, \( \mathcal{R} = \mathcal{I} \times_{\mathcal{J}} \mathcal{J} \) and \((p_1, p_2): \mathcal{R} \rightarrow \mathcal{I} \) the projections. Let \( \mathcal{M}, \mathcal{N} \) be (graded) quasi-coherent sheaves on \( \mathcal{J} \), \( \mathcal{M}' = p^* \mathcal{M}, \mathcal{N}' = p_1^* \mathcal{N} \) and \( \mathcal{M}'' = p_1^* \mathcal{M}' = p_2^* \mathcal{M}', \mathcal{N}'' = p_1^* \mathcal{N}' = p_2^* \mathcal{N}' \). The sequence

\[
\text{Hom}_{\mathcal{J}}(\mathcal{M}, \mathcal{N}) \rightarrow \text{Hom}_{\mathcal{J}}(\mathcal{M}', \mathcal{N}') \rightarrow \text{Hom}_{\mathcal{J}}(\mathcal{M}'', \mathcal{N}'')
\]

of (homogeneous) homomorphisms induced by the projections is exact, that is, it establishes a one-to-one correspondence between \( \text{Hom}_{\mathcal{J}}(\mathcal{M}, \mathcal{N}) \) and the coincidence locus of the pair of arrows. \( \square \)

**Proposition A.29** (Descent for modules). Let \( p: \mathcal{I} \rightarrow \mathcal{J} \) be a faithfully flat quasi-compact morphism of superschemes, and let \( \mathcal{R} = \mathcal{I} \times_{\mathcal{J}} \mathcal{J} \) and \((p_1, p_2): \mathcal{R} \rightarrow \mathcal{J} \) the projections. Let \( \mathcal{M} \) be a graded quasi-coherent sheaf on \( \mathcal{J} \) with a descent data, that is, an isomorphism \( p_1^* \mathcal{M} \cong p_2^* \mathcal{M} \). Then, there is a graded quasi-coherent sheaf \( \mathcal{N} \) on \( \mathcal{J} \) such that \( \mathcal{M} \cong p^* \mathcal{N} \). \( \square \)
One also has a statement corresponding to Grothendieck’s Corollary 1.9. For every superscheme $\mathcal{X}$ let us denote by $H(\mathcal{X})$ the set of all closed sub-superschemes of $\mathcal{X}$.

**Proposition A.30** (Descent for closed sub-superschemes). Let $p: \mathcal{I} \to \mathcal{J}$ be a faithfully flat quasi-compact morphism of superschemes. The sequence of sets

$$H(\mathcal{J}) \to H(\mathcal{I}) \Rightarrow H(\mathcal{J})$$

is exact. \[ \square \]

A.8. **Local representability of a group functor.** Let $\mathcal{I}$ be a superscheme and $\iota: \mathcal{S}G \hookrightarrow \mathcal{S}F$ an open morphism of étale sheaves on $\mathcal{I}$-superschemes. We also say that $\mathcal{S}G \hookrightarrow \mathcal{S}F$ is representable by open immersions. This means that for every $\mathcal{I}$-superscheme $\mathcal{J} \to \mathcal{I}$ and every sheaf morphism $\lambda: \mathcal{J} \to \mathcal{S}F$ (or, equivalently, every $\mathcal{J}$-valued point $\lambda$ of $\mathcal{S}F$, or every $\lambda \in \mathcal{S}F(\mathcal{J})$), the fibre product $\mathcal{S}G \times_{\mathcal{S}F, \lambda} \mathcal{J}$ is representable by a superscheme $\mathcal{S}\mathcal{G}_{\lambda} \to \mathcal{J}$ and the the induced morphism $\mathcal{S}\mathcal{G}_{\lambda} \to \mathcal{J}$ is an open immersion. We visualize this by the cartesian diagram

$$\begin{array}{ccc}
\mathcal{S}G & \xrightarrow{\iota} & \mathcal{S}F \\
\lambda \downarrow & & \lambda \downarrow \\
\mathcal{S}\mathcal{G}_{\lambda} := \mathcal{S}G \times_{\mathcal{S}F, \lambda} \mathcal{J} & \xrightarrow{\xi_{\mathcal{J}}} & \mathcal{J}
\end{array}$$

Assume that $\mathcal{S}F$ is a sheaf in groups. For every $\mathcal{J}$-valued point $\xi: \mathcal{J} \to \mathcal{S}F$, one defines the translated open subfunctor $\iota_{\xi}: \mathcal{S}G \cdot \xi \hookrightarrow \mathcal{S}F$ by setting $\mathcal{S}G \cdot \xi(\mathcal{J})$ as the image of the composition $\mathcal{S}G(\mathcal{J}) \xrightarrow{\mathrm{Id} \times \xi_{\mathcal{J}}} \mathcal{S}G(\mathcal{J}) \times \mathcal{S}F(\mathcal{J}) \to \mathcal{S}F(\mathcal{J})$. Notice that $\iota_{\xi}: \mathcal{S}G \cdot \xi \hookrightarrow \mathcal{S}F$ is also representable by open immersions, as for every $\mathcal{J}$-superscheme $\mathcal{J} \to \mathcal{I}$ and every sheaf morphism $\lambda: \mathcal{J} \to \mathcal{S}F$ one has

$$\mathcal{S}\mathcal{G}_{\xi, \lambda} \simeq \mathcal{S}\mathcal{G}_{\lambda^{-1}}$$

Proceeding as in [33, Chap. 0, Prop. 4.5.4], one obtains:

**Proposition A.31.** Under the above hypotheses, suppose also that:

1. There exists a set $I$ of $\mathcal{J}$-valued points $\xi_{i} \in \mathcal{S}F(\mathcal{J})$ of $\mathcal{S}F$ such that the open subfunctor $\bigcup_{i \in I} \mathcal{S}G \cdot \xi_{i} \hookrightarrow \mathcal{S}F$ equals $\mathcal{F}$. This is equivalent to saying that for every $\mathcal{I}$-superscheme $\mathcal{J} \to \mathcal{I}$ and every sheaf morphism $\lambda: \mathcal{J} \to \mathcal{S}F$ the open sub-superschemes $\mathcal{S}\mathcal{G}_{\xi_{i}, \lambda} \hookrightarrow \mathcal{J}$, for $i \in I$, form an open covering of $\mathcal{J}$;

2. $\mathcal{S}G$ is representable.

Then $\mathcal{S}F$ is representable as well.

**Proof.** Let us denote by $\mathcal{G}$ the $\mathcal{I}$-superscheme that represents $\mathcal{S}G$. Since the translated functors $\mathcal{S}G \cdot \xi_{i}$ are isomorphic to $\mathcal{S}G$, they are representable by $\mathcal{I}$-superschemes, which we denote by $\mathcal{G} \cdot \xi_{i}$. Moreover, the condition $\bigcup_{i \in I} \mathcal{S}G \cdot \xi_{i} \hookrightarrow \mathcal{S}F$ yields glueing conditions for the superschemes $\mathcal{G} \cdot \xi_{i}$, which therefore glue to yield an $\mathcal{I}$-superscheme $\bigcup_{i \in I} \mathcal{G} \cdot \xi_{i}$ whose functor of points is $\bigcup_{i \in I} \mathcal{S}G \cdot \xi_{i} = \mathcal{S}F$. \[ \square \]

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