The Limiting Distribution of the Hook Length of a Randomly Chosen Cell in a Random Young Diagram

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Abstract
Let \( p(n) \) be the number of all integer partitions of the positive integer \( n \) and let \( \lambda \) be a partition, selected uniformly at random from among all such \( p(n) \) partitions. It is known that each partition \( \lambda \) has a unique graphical representation, composed by \( n \) non-overlapping cells in the plane called Young diagram. As a second step of our sampling experiment, we select a cell \( c \) uniformly at random from among the \( n \) cells of the Young diagram of the partition \( \lambda \). For large \( n \), we study the asymptotic behavior of the hook length \( Z_n = Z_n(\lambda, c) \) of the cell \( c \) of a random partition \( \lambda \). This two-step sampling procedure suggests a product probability measure, which assigns the probability \( 1/np(n) \) to each pair \( (\lambda, c) \). With respect to this probability measure, we show that the random variable \( \pi Z_n/\sqrt{6n} \) converges weakly, as \( n \to \infty \), to a random variable whose probability density function equals \( 6y/\pi^2(e^y - 1) \) if \( 0 < y < \infty \), and zero elsewhere.

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1 Introduction and Statement of the Main Result

For a natural number \( n \), we say that \( \lambda \) is a partition of \( n \) if \( \lambda \) is a sequence \((\lambda_1, \lambda_2, ..., \lambda_k)\) of positive integers satisfying \( \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \) and such that
\[
\sum_{j=1}^{k} \lambda_j = n.
\]
The summands $\lambda_j$ in (1) are usually called parts of $\lambda$. The Young diagram of a partition is an array of square boxes, or cells, in the first quadrant of the plane, left-justified, with $\lambda_t$ cells in the $t$-th row counting from the bottom. We label these cells $(t, s)$, with $t$ denoting the row number of the cell and $s$ the column number in the Young diagram. For example, in the Young diagram of $\tilde{\lambda} = (5, 4, 3, 2, 2, 2, 2, 1)$ of the partition of $n = 22$ as $22 = 5 + 4 + 3 + 3 + 2 + 2 + 2 + 1$, the cell $(3, 2)$ is the square whose vertices in the $(s, t)$-plane have coordinates $(1, 2), (2, 2), (2, 3), (1, 3)$. Reading consecutively the numbers of cells in the columns of the array of the partition $\lambda$, beginning from the most left column, we get the conjugate partition $\lambda^* = (\lambda^*_1, \lambda^*_2, ..., \lambda^*_l)$, where $l = \lambda_1$. The Young diagram of $\lambda^*$ is called conjugate of the Young diagram of $\lambda$. In our example, we have $\tilde{\lambda}^* = (8, 7, 4, 2, 1)$, which is the conjugate partition of 22, namely, $22 = 8 + 7 + 4 + 2 + 1$. The hook length of the cell $c = (t, s)$ in the partition $\lambda$ is defined by the formula

$$h(\lambda, c) := \lambda_t - s + \lambda^*_s - t + 1,$$

that is, $h(\lambda, c)$ is the number of cells in the hook comprised by the $(t, s)$-cell itself and by the cells in the $t$-th row right of $(t, s)$ and $s$-th column above $(t, s)$. Recalling our example given above, we get $h(\tilde{\lambda}, (3, 2)) = 6$ since $t = 3, \lambda_3 = 3, s = 2$ and $\lambda^*_5 = 7$.

Let $\Lambda(n)$ be the set of all partitions of $n$ and let $p(n) = |\Lambda(n)|$ (further on, by $|T|$ we denote the cardinality of the set $T$). The number $p(n)$ is determined asymptotically by the famous partition formula of Hardy and Ramanujan [14]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right), \quad n \to \infty.$$  

A precise asymptotic expansion for $p(n)$ was found later by Rademacher [25] (more details may be also found in [2, Chapter 5]). Further on, we assume that, for fixed integer $n \geq 1$, a partition $\lambda \in \Lambda(n)$ is selected uniformly at random (uar). In other words, we assign the probability $1/p(n)$ to each $\lambda \in \Lambda(n)$. In this way, each numerical characteristic of $\lambda$ can be regarded as a random variable defined on $\Lambda(n)$ (or, a statistic in the sense of the random generation of partitions of $n$). The study of the asymptotic behavior of various partition statistics for large $n$ is a subject of intensive research in combinatorics, number theory and statistical physics. Erdős and Lehner [5] were apparently the first who established a probabilistic limit theorem related to integer partitions. As a matter of fact, they found an appropriate normalization for the number of parts in a random partition of $n$ and showed that it converges weakly to the extreme value (Gumbel) distribution as $n \to \infty$. By conjunction of the corresponding Young diagram, the same limit theorem holds true for the largest part of a random integer partition. For other typical distributional results and limit theorems of various integer partition statistics, we refer the reader, e.g., to [26], [27], [28], [6], [9], [24], [20], [31], [10].

Another subject of study in the asymptotic theory of integer partitions is related to the limit shape of the underlying Young diagrams. Here is a simple
setting of the problem. Let

\[ l_j = l_j(\lambda) := |\{q : \lambda_q = j\}| \quad (4) \]

be the multiplicity of the part equal to \( j \) in a partition \( \lambda \in \Lambda(n), \ j = 1, 2, \ldots, n \).

The upper boundary of the Young diagram of \( \lambda \) is a piecewise constant function

\[ X_\lambda : [0, \infty) \to \mathbb{Z}_+ := \{0, 1, \ldots\} \]

given by

\[ X_\lambda(t) := \sum_{j \geq t} l_j. \]

If the set \( \Lambda(n) \) is endowed with a probability measure \( \mu_n \) (e.g., \( \mu_n \) is the uniform measure \( \mathbb{P} \) or the Plancherel distribution on the set of Young diagrams with \( n \) cells), the limit shape, with respect to \( \mu_n \) as \( n \to \infty \), is understood as a function (curve) \( s = s(t) \) in the plane \((t, s)\) such that for every \( \delta > 0 \) and any \( \epsilon > 0 \),

\[ \lim_{n \to \infty} \mu_n \{ \lambda \in \Lambda(n) : \sup_{t \geq \delta} |A_n^{-1}X_\lambda(tB_n) - s(t)| > \epsilon \} = 0, \quad (5) \]

where \( A_n \) and \( B_n \) are suitable scaling constants satisfying \( A_nB_n = 1 \). With respect to the uniform distribution \( \mathbb{P} \) on \( \Lambda(n) \), the limit shape in (5) exists under the scaling \( A_n = B_n = \sqrt{n} \) and is determined by the function

\[ s(t) = -\log (1 - e^{-\pi t/\sqrt{6}}), \quad (6) \]

or, in a more symmetric form, by the equation

\[ e^{-\pi s/\sqrt{6}} + e^{-\pi t/\sqrt{6}} = 1. \quad (7) \]

The limit shape (6) for the uniform distribution on \( \Lambda(n) \) was first identified by Temperley \[29\] in relation to a model of a growing crystal. Afterward Szalay and Turán \[26\] obtained essentially analogous estimates of those in \(5\) and \(6\), which were used later by Vershik \[30\] to establish \(7\) and generalize it to other types of probability measures and models. In the same paper \[30\], Vershik gave also an important probabilistic frame of the relationship between problems from statistical mechanics (models of ideal gas) and problems related to the asymptotic theory of partitions. An alternative proof of \(7\) was also given by Pittel in \[24\]. Recent developments in this area are presented in \[3\].

In the present paper we study a statistic produced by a random selection of a cell in a random Young diagram. The sampling procedure combines the outcomes of two experiments: first, we select a partition \( \lambda \in \Lambda(n) \) at random, and then we select a cell \( c = (t, s) \in \lambda \) at random. A similar type of sampling was first proposed and studied in the context of part multiplicities of a random integer partition in \[4\], and then in the context of part sizes in \[22\] and \[23\]. The sampling procedure that we introduced leads to the product probability space

\[ \Omega(n) = \Lambda(n) \times \{c \in \lambda : \lambda \in \Lambda(n)\} \quad (8) \]
and the product probability measure $\mathbb{P}(\cdot)$, defined by

$$
\mathbb{P}((\lambda, c) \in \Omega(n)) = \frac{1}{np(n)},
$$

(9)

(For more details on the theory of product probability spaces, we refer the reader, e.g., to [18, Part I, Section 4.2]). We also denote by $\mathbb{E}(\cdot)$ the expectation with respect to the probability measure $\mathbb{P}$.

Our aim in this paper is to determine asymptotically, as $n \to \infty$, the distribution of the hook length of a randomly chosen cell of the Young diagram of a partition $\lambda \in \Lambda(n)$ chosen at random. More precisely, for any pair $(\lambda, c) \in \Omega(n)$, we denote the underlying hook length by

$$
Z_n = Z_n(\lambda, c) := h(\lambda, c),
$$

(10)

where the function $h(\lambda, c)$ was defined earlier by (2). We organize the paper as follows. Section 2 contains some auxiliary facts that we need further. In Section 3 we prove the following limit theorem for $Z_n$.

**Theorem 1** Let $0 < y < \infty$. Then, we have

$$
\lim_{n \to \infty} \mathbb{P}(\pi Z_n/\sqrt{6n} \leq y) = \frac{6}{\pi^2} \int_0^y \frac{u}{e^u - 1} du.
$$

**Remark 1.** The hook lengths of cells in the Young diagram of an integer partition play an important role in algebraic combinatorics thanks to the famous hook-length formula due to Frame et al. [8]. It represents the number of standard Young tableaux $d(\lambda)$ of shape $\lambda \in \Lambda(n)$ by the hook lengths of the cells $c \in \lambda$ in the following way:

$$
d(\lambda) = \frac{n!}{\prod_{c \in \lambda} h(\lambda, c)}.
$$

(11)

A standard Young tableau of shape $\lambda$ is a labelling of the cells of the Young diagram of $\lambda$ with numbers 1 to $n$ so that the labels are strictly increasing from bottom to top along columns and from left to right along rows. There exists a bijection between the partitions from the set $\Lambda(n)$ and the irreducible representations of the symmetric group $S_n$ of $n$ letters (see, e.g., [17, Chapters 1 and 4]). The work of Young [32, 33] shows that $d(\lambda)$ gives the dimension of the irreducible representation of $S_n$ related to the partition $\lambda \in \Lambda(n)$. A probabilistic proof of formula (11) was given by Greene et al. [12]. It is based on a construction of a random walk on the cells of the Young diagram (called also a hook walk). In its first step, a cell of a Young diagram containing $n$ cells is selected at random. If this cell has coordinates $(t, s)$, then the walk continues on the cells $\neq (t, s)$ in the hook of the cell $(t, s)$. A detailed discussion on the enumerational and algorithmic aspects of formula (11) may be found in [10, Section 5.1.4].

**Remark 2.** Our method of proof is based on:
• some combinatorial identities due to Han [13] (see Lemmas 1 and 2 in the next section)

• the saddle point method in terms of admissibility in the sense of Hayman (see [7, Chapter VIII.5]);

• the method of moments in probability theory (the Fréchet-Shohat limit theorem; see, e.g., [18, Chapter IV, Section 11.4]).

Another possible approach to the problem could be built on Vershik’s ideas for the limit shape of a random Young diagram (see [30]). In this context, one has to deal with a family of probability measures \( \mu_w, w \in (0,1) \), defined on the set of all partitions \( \Lambda = \bigcup_{n \geq 0} \Lambda(n) \). The key feature in a study of this type is that, under certain conditions, the uniform probability measure \( P \) on \( \Lambda(n) \) is recovered as a conditional distribution \( \mu_w(\cdot | \Lambda(n)) \). Moreover, \( \mu_w \) can be constructed as a product measure, resulting into mutually independent random part multiplicities \( l_j, j = 1, 2, ... \) (see their definition given by (4)).

In the context of random partitions, this phenomenon was first observed and applied by Fristedt [9]. The computation of the expectation of the sum of the part sizes \( n = n(\lambda) = \lambda_1 + \lambda_2 + ... \) of a partition \( \lambda = (\lambda_1, \lambda_2, ...) \in \Lambda \) with respect to the measure \( \mu_w \) suggests the proper choice of the parameter \( w = w(n) \) that approaches 1 as \( n \) becomes large. By the above property of the conditional distribution \( \mu_w(\cdot | \Lambda(n)) \), we can consider \( \lambda \in \Lambda(n) \subset \Lambda \) as selected from the set \( \Lambda(n) \). Dealing with the family of measures \( \mu_w \) on the space \( \Lambda \) avoids some technical difficulties because, under this setting, the multiplicities \( l_j \) are independent. One can now introduce the usual scaling for random Young diagrams with \( n \) cells (\( A_n = B_n = \sqrt{n} \)) in order to obtain their limit shape \( 5 \)-\( 7 \) in the \((t, s)\)-plane as \( n \to \infty \). (For more details, we refer the reader to [30] and [3]). Next, we can choose a point at random from the region bounded by the lines \( t = 0, s = 0 \) and by the curve \( 7 \). The computation of the horizontal and vertical distances between this point and the curve is easy. Their sum shows the typical hook length of a cell in a random Young diagram for large \( n \). Further on, a possible way to identify the required limiting distribution is, e.g., to apply some known asymptotic results from [24] or [22] for the part sizes of a random partition \( \lambda \in \Lambda(n) \) and of its conjugate partition \( \lambda^* \). In our further study we prefer, however, to follow an approach, which, in our opinion, is more direct: we apply the Cauchy formula to a generating function identity and then employ the saddle point method for the asymptotic analysis of the coefficients representing the moments of the underlying statistic.

2 Preliminaries

2.1 Combinatorial Identities and Generating Functions

We start with the definition of another partition statistic, defined on the set \( \Lambda(n) \) of all partitions of \( n \). We equip this set with the uniform probability
measure and, for any \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \in \Lambda(n) \), we define its \( m \)-th moment by

\[
Y_{m,n} = \sum_{j=1}^{k} \lambda_j^m,
\]

(12)

In the case \( m = 0 \) the sum in the right-hand side of (12) clearly counts the number of parts in \( \lambda \), while, by (1), \( Y_{1,n} = n \). Further on, \( E(.) \) will stand for the expectation with respect to the uniform probability measure on \( \Lambda(n) \). From (12) it obviously follows that

\[
E(Y_{m,n}) = \frac{1}{p(n)} \sum_{\lambda \in \Lambda(n)} \sum_{j} \lambda_j^m, \quad m = 1, 2, ....
\]

(13)

Furthermore, the definition of the product probability measure \( \mathbb{P} \) on the space \( \Omega(n) \) (see (9) and (8), respectively) and definition (10) of the random variable \( Z_n \) imply that

\[
E(Z_n^m) = \frac{1}{np(n)} \sum_{\lambda \in \Lambda(n)} \sum_{c \in \lambda} h(\lambda, c), \quad m = 1, 2, ....
\]

(14)

Our first preliminary facts are related to identities that are due to Han [13]. We present the first one in terms of the expectations introduced by (13) and (14).

**Lemma 1** [13, Corollary 6.5]. For \( m = 1, 2, .... \), we have

\[
E(Z_n^m) = \frac{1}{n} E(Y_{m+1,n}).
\]

Han [13, Theorem 6.6] also obtained an identity for the generating function of \( E(Y_{m,n}) \). It will be the starting point in our asymptotic analysis. Let

\[
g(x) = \prod_{j=1}^{\infty} (1 - x^j)^{-1} = 1 + \sum_{n=1}^{\infty} p(n)x^n
\]

(15)

be the Euler partition generating function.

**Lemma 2** For any \( m \geq 1 \) and \( |x| < 1 \), we have

\[
1 + \sum_{n=1}^{\infty} p(n)E(Y_{m,n})x^n = g(x)F_m(x),
\]

where

\[
F_m(x) = \sum_{j=1}^{\infty} \frac{j^m x^j}{1 - x^j}.
\]

(16)

**Remark 3.** In fact, Han [13] has proved the following combinatorial identity:

\[
\sum_{\lambda \in \Lambda(n)} \sum_{c \in \lambda} h^m(\lambda, c) = \sum_{\lambda \in \Lambda(n)} \sum_{j} \lambda_j^{m+1}.
\]

Lemma 1 translates it in terms of the expectations (13) and (14).
2.2 Analytic Combinatorics Background: Meinardus Theorem on Weighted Partitions and Hayman Admissibility of Generating Functions

Lemma 2 implies that the coefficient \( p(n)E(Y_{m,n}) \) of \( x^n \) can be expressed by a Cauchy integral whose integrand contains the product \( g(x)F_m(x) \). Its behavior heavily depends on the properties of the partition generating function \( g(x) \) whose infinite product representation (15) shows that its main singularity is at the point \( x = 1 \) (see also \cite[Chapter 5]{2}). A complete asymptotic expansion of \( E(Y_{m,n}) \), for fixed and odd \( m > 1 \), was obtained by Grabner et al. \cite{10}. We need here only the main asymptotic term of \( E(Y_{m,n}) \), \( m = 1, 2, ... \), which can be obtained using the Hayman-Meinardus version of the saddle point method. Here, we will give a brief account on this subject. This approach was also demonstrated in \cite[Sections 5 and 6]{22}.

We start with some general remarks on the Hardy-Ramanujan formula (3). The Cauchy integral stemming from (15) and representing \( p(n) \) can be analyzed with the aid of the Hardy, Ramanujan and Rademacher’s circle method developed in \cite{14} and \cite{25}. Subsequent generalizations of these results are related to extensions of the class of generating functions, for which an infinite product representation similar to (15) is valid. An important result in this direction is due to Meinardus \cite{19} (see also \cite[Chapter 6]{2}) who obtained the asymptotic of the Taylor coefficients of infinite products of the form:

\[
\prod_{j=1}^{\infty} (1 - x^j)^{-b_j}
\]

under certain general assumptions on the sequence of non-negative numbers \( \{b_j\}_{j \geq 1} \). Meinardus’ approach is based on considering the Dirichlet generating series

\[
D(z) = \sum_{j=1}^{\infty} b_j z^{-j}, \quad z = u + iv, \quad u, v \in \mathbb{R}.
\]

We briefly describe here Meinardus’ assumptions avoiding their precise statements as well as some extra notations and concepts.

- \((M_1)\) The first assumption \((M_1)\) specifies the domain \( \mathcal{H} = \{z = u + iv : u \geq -C_0\}, 0 < C_0 < 1 \), in the complex plane, in which \( D(z) \) has an analytic continuation.

- \((M_2)\) The second one \((M_2)\) is related to the asymptotic behavior of \( D(z) \), whenever \( |v| \to \infty \). A function of the complex variable \( z \) which is bounded by \( O(|3(z)|^{C_1}), 0 < C_1 < \infty \), in certain domain of the complex plane is called a function of finite order. Meinardus’ second condition requires that \( D(z) \) is of finite order in the whole domain \( \mathcal{H} \).

- \((M_3)\) Finally, the Meinardus third condition \((M_3)\) implies a bound on the ordinary generating function of the sequence \( \{b_j\}_{j \geq 1} \). It can be stated in
a way simpler than the Meinardus’ original expression by the inequality:
\[
\sum_{j \geq 1} b_j e^{-j \alpha} \sin^2(\pi j y) \geq C_2 \alpha^{-\epsilon_1}, \quad 0 < \frac{\alpha}{2\pi} < |y| < \frac{1}{2},
\]
for sufficiently small \(\alpha\) and some constants \(C_2, \epsilon_1 > 0\) \((C_2 = C_2(\epsilon_1))\) (see [11]).

It is known that the Euler partition generating function \(g(x)\) (which is obviously of the form (17) satisfies the Meinardus scheme of conditions \((M_1) - (M_3)\) (see, e.g., [2, Theorem 6.3]).

In the asymptotic analysis of the Cauchy integral stemming from Lemma 2 we shall use a variant of the saddle point method given by Hayman’s theorem for admissible power series [15] (for more details, see, e.g., [7, Chapter VIII.5]). To present Hayman’s idea and show how it can be applied to the proof of Theorem 1, we need to introduce some auxiliary notations.

We consider here a function \(G(x) = \sum_{n=1}^{\infty} G_n x^n\) that is analytic for \(|x| < \rho\), \(0 < \rho < \infty\). For \(0 < r < \rho\), we let
\[
a(r) = r \frac{G'(r)}{G(r)}, \quad b(r) = r \frac{G'(r)}{G(r)} + r^2 \frac{G''(r)}{G(r)} - r^2 \left( \frac{G'(r)}{G(r)} \right)^2.
\]

In the statement of the Hayman’s result, we shall use the terminology given in [7, Chapter VIII.5]. We assume that \(G(x) > 0\) for \(x \in (R_0, \rho) \subset (0, \rho)\) and satisfies the following three conditions.

- **Capture condition.** \(\lim_{r \to \rho} a(r) = \infty\) and \(\lim_{r \to \rho} b(r) = \infty\).
- **Locality condition.** For some function \(\delta = \delta(r)\) defined over \((R_0, \rho)\) and satisfying \(0 < \delta < \pi\), one has
  \[G(re^{i\theta}) \sim G(r)e^{i\delta(a(r) - \theta^2 b(r)/2)}\]
  as \(r \to \rho\), uniformly for \(|\theta| \leq \delta(r)\).
- **Decay condition.**
  \[G(re^{i\theta}) = o \left( \frac{G(r)}{\sqrt{b(r)}} \right)\]
  as \(r \to \rho\), uniformly for \(\delta(r) \leq |\theta| < \pi\).

**Definition 1** A function \(G(x)\) satisfying the capture, locality and decay conditions is called a Hayman admissible function.
**Hayman Theorem.** Let $G(x)$ be a Hayman admissible function and $r = r_n$ be the unique solution in the interval $(R_0, \rho)$ of the equation

$$a(r) = n. \quad (21)$$

Then the Taylor coefficients of $G(x)$ satisfy, as $n \to \infty$,

$$G_n \sim \frac{G(r_n)}{r_n \sqrt{2\pi b(r_n)}}$$

with $a(r_n)$ and $b(r_n)$ given by (19) and (20), respectively.

### 3 Proof of Theorem 1

The proof is divided into two parts.

(A) Proof of Hayman admissibility for $g(x)$.

(B) Obtaining an asymptotic estimate for the Cauchy integral stemming from Lemma 2.

#### 3.1 Part (A)

This part of the proof follows the same line of reasoning given in [22, pp. 338-340]. For the sake of completeness, we present it here.

First, we need to show how Hayman theorem can be applied to find the asymptotic behavior of the Taylor coefficients of the partition generating function $g(x)$. Since in (17) we have $b_j = 1, j \geq 1$, the Dirichlet generating series (18) is $D(z) = \zeta(z)$, where $\zeta$ denotes the Riemann zeta function: $\zeta(z) = \sum_{j=1}^{\infty} j^{-z}$, $z = u + iv$. We set in (21) $r = r_n = e^{-d_n}, d_n > 0$, where $d_n$ is the unique solution of the equation

$$a(e^{-d_n}) = n. \quad (22)$$

Granovsky et al. [11] showed that the first two Meinardus conditions imply that the unique solution of (22) has the following asymptotic expansion:

$$d_n = \sqrt{\zeta(2)/n} + \frac{\zeta(0)}{2n} + O(n^{-1-\beta})$$

$$= \frac{\pi}{\sqrt{6n}} - \frac{1}{4n} + O(n^{-1-\beta}), \quad (23)$$

where $\beta > 0$ is fixed constant (here we have also used that $\zeta(0) = -1/2$; see [11], Chapter 23.2). We also notice that (20) and (23) imply that

$$b(e^{-d_n}) = 2\zeta(2)d_n^{-3} + O(d_n^{-2}) \sim \frac{2\sqrt{6}}{\pi} n^{3/2}. \quad (24)$$
(This is a particular case of Lemma 2.2 from [21] with \( D(z) = \zeta(z) \).) Hence, by (22) and (24), \( a(e^{-dn}) \to \infty \) and \( b(e^{-dn}) \to \infty \) as \( n \to \infty \), that is, Hayman’s ”capture” condition is satisfied with \( r = r_n = e^{-dn} \). To show next that Hayman’s ”decay” condition is satisfied, we set

\[
\delta_n = \frac{d_n^{4/3}}{\omega(n)} = \frac{\pi^{4/3}}{(6n)^{2/3}\omega(n)} \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right),
\]

(25)

where \( \omega(n) \) is a function satisfying \( \omega(n) \to \infty \) as \( n \to \infty \) arbitrarily slowly and in the second equality we have used (23). We can apply now an estimate for \( c \) positive constants \( d_1 \) and \( d_2 \), respectively. Hence, all conditions of Hayman’s theorem hold, and we can apply it with \( G_n = p(n) \), \( G(x) = g(x) \), \( r_n = e^{-dn} \) and \( \rho = 1 \) to find that

\[
p(n) \sim \frac{e^{nd_n} g(e^{-dn})}{\sqrt{2\pi b(e^{-dn})}}, \quad n \to \infty.
\]

(28)

**Remark 4.** To show that formula (28) yields (3), one has to replace (23) and (24) in the right-hand side of (28). The asymptotic of \( g(e^{-dn}) \) is determined by a general lemma due to Meinardus [19] (see also [2] Lemma 6.1). Since \( \zeta(0) = -1/2 \) and \( \zeta'(0) = -\frac{1}{12} \log(2\pi) \) (see [1] Chapter 23.2), in the particular case of \( g(e^{-dn}) \), this lemma implies that

\[
g(e^{-dn}) = \exp \left( \zeta(2)d_n^{-1} - \zeta(0) \log d_n + \zeta'(0) + O(d_n^{c_1}) \right)
\]

\[
= \exp \left( \frac{\pi^2}{6d_n} + \frac{1}{2} \log(2\pi) + O(d_n^{c_1}) \right), \quad n \to \infty,
\]

where \( 0 < c_1 < 1 \). The rest of the computation leading to (3) is based on simple algebraic manipulations and cancellations.
3.2 Part (B)

We are now ready to apply the Cauchy coefficient formula to the generating function identity of Lemma 2. We use the circle \( x = e^{-d_n+i\theta}, -\pi < \theta \leq \pi \), as a contour of integration and obtain, for any fixed \( m \geq 1 \), that

\[
p(n)E(Y_{m,n}) = \frac{e^{nd_n}}{2\pi} \int_{-\pi}^{\pi} g(e^{-d_n+i\theta})F_m(e^{-d_n+i\theta})e^{-i\theta n} d\theta.
\]

Then, we break up the range of integration as follows:

\[
p(n)E(Y_{m,n}) = J_1(m, n) + J_2(m, n),
\]

where

\[
J_1(m, n) = \frac{e^{nd_n}}{2\pi} \int_{-\delta_n}^{\delta_n} g(e^{-d_n+i\theta})F_m(e^{-d_n+i\theta})e^{-i\theta n} d\theta,
\]

\[
J_2(m, n) = \frac{e^{nd_n}}{2\pi} \int_{\delta_n < |\theta| \leq \pi} g(e^{-d_n+i\theta})F_m(e^{-d_n+i\theta})e^{-i\theta n} d\theta,
\]

where \( m = 1, 2, \ldots \), and \( \delta_n \) is defined by (25).

To estimate \( J_2(m, n) \), we notice that, for fixed \( m \), by the definition of Riemann integrals, (16) and (23),

\[
|F_m(e^{-d_n+i\theta})| = \left| \sum_{j=1}^{\infty} j^m e^{-jd_n+i\theta} \right| \leq \sum_{j=1}^{\infty} \frac{j^m e^{-jd_n}}{1-e^{-jd_n}} = \sum_{j=1}^{\infty} \frac{(jd_n)^m e^{-jd_n}}{1-e^{-jd_n}} d_n^{-m-1}
\]

\[
\sim d_n^{-m-1} \int_0^\infty \frac{u^m e^{-u}}{1-e^{-u}} du = O(d_n^{-m-1}) = O(n^{(m+1)/2}).
\]

The last two equalities follow from the estimate

\[
d_n^{-1} = \frac{\sqrt{6n}}{\pi} + O(1),
\]

which is a simple consequence of (24). Combining (29) - (33), and (28) with the asymptotic equivalence (25) for the numbers \( p(n) \), we obtain

\[
|J_2(m, n)| \leq \frac{e^{nd_n}}{2\pi} \int_{\delta_n < |\theta| < \pi} |g(e^{-d_n+i\theta})||F_m(e^{-d_n+i\theta})| d\theta
\]

\[
= O(e^{nd_n} g(e^{-d_n}) n^{(m+1)/2} e^{-cd_n^{m/2}})
\]

\[
O \left( \frac{e^{nd_n} g(e^{-d_n})}{\sqrt{b(e^{-d_n})}} n^{(m+1)/2} n^{3/4} e^{-c2n^{m/2}} \right)
\]

\[
= O(p(n) n^{(m+1)/2} n^{3/4} e^{-c2n^{m/2}}) = o(p(n) n^{(m+1)/2}),
\]
where \( c_2 > 0 \).

For the asymptotic estimate of \( J_1(m, n) \), we need to expand \( F_m(x) \) around the point \( x = e^{-d_n} \). Thus, for any fixed \( m = 1, 2, \ldots \) and uniformly for any \( |\theta| \leq \delta_n \), we have

\[
F_m(e^{-d_n + i\theta}) = F_m(e^{-d_n}) + O\left(|\theta| \frac{d}{dx} F_m(x)|_{x=e^{-d_n}}\right)
\]

As previously, we can consider the sum representing \( F_m(e^{-d_n}) \) as a Riemann sum. So, for large \( n \), we can replace it by the value of the corresponding integral (see, e.g., [1, Section 27.1]). Hence, by (16), (23) and (33), we have

\[
F_m(e^{-d_n}) = \sum_{j=1}^{\infty} j^m e^{-jd_n} \sim d_n^{m-1} \int_0^{\infty} \frac{u^m e^{-u}}{1-e^{-u}} du = \zeta(m+1) m! \zeta(m+1), \quad m = 1, 2, \ldots (36)
\]

In the same way, we can estimate the first derivative of \( F_m \):

\[
\frac{d}{dx} F_m(x)|_{x=e^{-d_n}} = \sum_{j=1}^{\infty} \frac{j^{m+1} e^{-jd_n} e^{d_n}}{(1-e^{-jd_n})^2} \sim d_n^{-m-2} \int_0^{\infty} \frac{u^{m+1} e^{-u}}{(1-e^{-u})^2} du = O(n^{-m-2}) = O(n^{(m+2)/2}),
\]

where the last \( O \)-estimate follows from (33). Hence, by (25) and (37), the error term in (36) becomes

\[
O(\delta_n n^{(m+2)/2}) = O(n^{(m+1)/2} n^{1/2} n^{-2/3}/\omega(n)) = O(n^{(m+1)/2} n^{-1/6}/\omega(n)) = o(n^{(m+1)/2}).
\]

Consequently, (34) - (38) imply that

\[
F_m(e^{-d_n + i\theta}) = \left(\frac{n}{\zeta(2)}\right)^{(m+1)/2} m! \zeta(m+1) + o(n^{(m+1)/2}).
\]

Inserting this estimate and (27) into (30) and applying the asymptotic of the
partition function $p(n)$ from (28), we obtain

$$J_1(m, n) = e^{nd_n} g(e^{-dn})\int_{-\delta_n}^{\delta_n} \frac{g(e^{-dn} + i\theta)}{g(e^{-dn})} \left( \left( \frac{n}{\zeta(2)} \right)^{(m+1)/2} m!\zeta(m+1) + o(n^{(m+1)/2}) \right) e^{-i\theta n} d\theta$$

$$= e^{nd_n} g(e^{-dn}) \left( \left( \frac{n}{\zeta(2)} \right)^{(m+1)/2} m!\zeta(m+1) + o(n^{(m+1)/2}) \right)$$

$$\times \int_{-\delta_n}^{\delta_n} e^{-\theta^2 b(e^{-dn})/2} (1 + 1/\omega^3(n)) d\theta$$

$$= e^{nd_n} g(e^{-dn}) \left( \left( \frac{n}{\zeta(2)} \right)^{(m+1)/2} m!\zeta(m+1) + o(n^{(m+1)/2}) \right)$$

$$\times \int_{-\delta_n}^{\delta_n} \frac{e^{-\theta^2/2}}{\sqrt{b(e^{-dn})}} d\theta$$

$$\sim e^{nd_n} g(e^{-dn}) \left( \left( \frac{n}{\zeta(2)} \right)^{(m+1)/2} m!\zeta(m+1) \right)$$

$$\times \int_{-\infty}^{\infty} e^{-y^2/2} dy$$

$$\sim p(n) \left( \frac{n}{\zeta(2)} \right)^{(m+1)/2} m!\zeta(m+1).$$

(39)

In the first asymptotic equivalence we have used (24) and (25) in order to get

$$\delta_n \sqrt{b(e^{-dn})} \sim \frac{\pi^{5/6} \sqrt{2}}{6^{1/6} \omega(n)} n^{1/12} \to \infty$$

if $\omega(n) \to \infty$ as $n \to \infty$ not too fast, so that $n^{1/12}/\omega(n) \to \infty$. Combining (29) - (31), (34) and (39), we get

$$p(n) \mathcal{E}(Y_{m,n}) = p(n) \left( \frac{n}{\zeta(2)} \right)^{(m+1)/2} m!\zeta(m+1) + o(n^{(m+1)/2} p(n)),$$

which in turn shows that

$$\mathcal{E}(Y_{m,n}) \sim \left( \frac{n}{\zeta(2)} \right)^{(m+1)/2} m!\zeta(m+1), \ m = 1, 2, ...$$

(40)

Now, we recall Lemma 1, (13) and (14). Combining these observations with
we obtain
\[
E(Z_n^m) = \frac{1}{np(n)} \sum_{\lambda \in \Lambda(n)} \sum_j \lambda_j^{m+1} = \frac{1}{n} \mathcal{E}(Y_{n,m+1})
\]
\[
\sim \frac{1}{n} \left( \frac{n}{\zeta(2)} \right)^{m/2+1} (m+1)! \zeta(m+2), \quad m = 1, 2, ...\]

Consequently,
\[
\lim_{n \to \infty} E\left( \left( \sqrt{\frac{\zeta(2)}{n}} Z_n \right)^m \right) = \lim_{n \to \infty} E\left( \left( \frac{\pi}{\sqrt{6n}} Z_n \right)^m \right) = \frac{(m+1)! \zeta(m+2)}{\zeta(2)} = \frac{6}{\pi^2} \int_0^\infty \frac{u^{m+1}}{e^u - 1} du, \quad m = 1, 2, ....
\]

Hence, the Frechet-Shohat limit theorem [18, Chapter IV, Section 11.4] implies that the sequence \(\{\pi Z_n/\sqrt{6n}\}_{n \geq 1}\) converges in distribution to a random variable with probability density function \(\frac{6}{\pi^2} \int_0^\infty \frac{u^{m+1}}{e^u - 1} du, u \geq 0,\) and zero elsewhere, which completes the proof of the theorem.

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