A brief history of hidden quantum symmetries in Conformal Field Theories*

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Abstract

We review briefly a stream of ideas concerning the role of quantum groups as hidden symmetries in Conformal Field Theories, paying particular attention to the field theoretical representation of quantum groups based on Coulomb gas methods. An extensive bibliography is also included.

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Integrability, Yang-Baxter equation and Quantum Groups

In the last few years it has become clear the close relationship between conformal field theories (CFT’s), specially the rational theories, and the theory of quantum groups. From a more general point of view this is just one aspect of the connection between integrable systems and quantum groups. In fact if one recalls the history, quantum groups originated from the study of integrable models and more precisely from the quantum inverse scattering approach to integrability. The basic relation:

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v)$$  \hspace{1cm} (1)

which is at the core of this approach may also be taken as a starting point in the definition of quantum groups [FRT].

In statistical mechanics the operator \(T(u)\) is nothing but the monodromy matrix which depends on the spectral parameter \(u\) and whose trace gives the transfer matrix of a vertex model. Meanwhile the matrix \(R(u)\) stands for the Boltzmann weights of this vertex model. The \(RTT\) equation guarantees the commutativity of a one parameter family of transfer matrices:

$$[\text{tr } T(u), \text{tr } T(v)] = 0$$  \hspace{1cm} (2)

This equation is equivalent to the existence of an infinite number of conserved quantities when taking the thermodynamic limit. In these setting the Yang-Baxter equation:

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)$$  \hspace{1cm} (3)

which underlies the integrability of the model appears as the associativity condition of the ”quantum algebra” (1).

RCFT’s, exchange algebras and the braid group : the polynomial equations

Let us turn now to conformal field theories. There, it was soon realized that the braiding matrices of the conformal blocks of a RCFT also satisfy a Yang-Baxter like equation. In fact these matrices provide a representation of the braid group in the space of conformal blocks. The braid group and the exchange algebras appeared suddenly to play an important role in CFT’s and more generally in 2D quantum field theories [TK,Ko,Fr,FRS,Rh,MS]. However there are some structural differences with regard to the vertex models in statistical mechanics. In CFT the braiding matrices do not depend on any spectral parameter while in statistical mechanics the Boltzmann weights certainly do depend on it and only when taking the limits \(u \to \pm \infty\) one gets non trivial solutions of the Yang-Baxter equation without spectral parameter. In addition ,the
Boltzmann weights of a vertex model $R_{ij}^{kl}(u)$ depend on four labels while the braiding matrices $B_{pp'}^{kl}_{ij}$ depend on six labels which are the primary fields of a RCFT and consequently are subject to certain constraints given by the fusion rules. These later conditions are equivalent to the restrictions on the heights of a restricted solid on solid model (RSOS) where the Coxeter diagram is playing the role of the fusion rules in RCFT.

Having in mind the $RTT$ equation of integrable models as well as the braiding properties of the conformal blocks it was proposed in reference [AGS1] that quantum groups should be present in RCFT’s as ”hidden symmetries” underlying and explaining the origin of the polynomial equations of Moore and Seiberg. The motivation of these authors in writing the polynomial equations was to give an axiomatic definition of RCFT’s which would, among another things, give a rigorous proof of the Verlinde conjecture concerning the diagonalization of the fusion rules $N_{ij}^k$ by means of the modular matrix $S_{ij}$ which acts on the Virasoro characters [V].

Reference [AGS1] was in this way a first attempt to lay down a ”quantum group” interpretation of the duality properties of RCFT’s, which are encoded in the polynomial equations. Indeed one of these equations is nothing but the Yang-Baxter equation for the braiding matrices of the conformal blocks mentioned above. A second attempt in the same direction was taken by Moore and Reshetikhin in reference [MR].

A first clue of the problem is suggested by the classical limit of a WZW model based on a group $G$ which is obtained when taking the level $k$ going to infinity. This model has an infinite number of primary fields labelled by their spin $j = 0, 1/2, 1, \ldots$ whose braiding and fusion properties can be computed entirely from the representation theory of the classical group $G$ [MS]. Thus for example the fusion matrices $F$ of this classical WZW model become the 6-j symbols of the group $G$, while the fusion rules are nothing but the usual Clebsch-Gordan decomposition of the tensor product of irreps of $G$. More generally, what one calls the duality data of a RCFT, namely: fusion rules, braiding matrices, fusion matrices, modular matrices, etc. are computable in the classical limit using ordinary group theory. Of course not all the RCFT’s known admit such a classical limit where all the conformal weights of the primary fields vanish and consequently the braiding properties become almost trivial. One example of this is provided by the $c < 1$ theories. However what this limit suggested was that some ”deformation” of the ”classical” group theory could in principle explain the whole duality data of genuine quantum RCFT’s.

**Quantum group phenomenology**

At that time quantum groups have already taken off from the land of integrable systems. The general notion of quantum groups were developed by Drinfeld and Jimbo in references [Dr1,Ji]. The first non classical example of this definition was a $q$-deformation of the Lie algebra $SL(2)$
which was found earlier by Kulish and Reshetikhin [KuR]. Drinfeld and Jimbo generalized this quantum deformation to include all the classical Lie groups affine or not. Later on in a series of papers Kirillov and Reshetikhin [KR,R] constructed the representation theory of \( U_q(Sl(2)) \) and others for a generic value of the deformation parameter \( q \).

After these works it was rather clear that quantum groups were the correct "deformation" of classical groups needed to explain the duality properties of RCFT’s. In reference [AGS2] it was considered the case of the WZW model based on the Kac-Moody algebra \( \widehat{SU}(2) \) at level \( k \), and it was shown that the braiding matrices, computed by Tsuchiya and Kanie [TK] solving the Knizhnik-Zamolodchikov equation for the conformal blocks [KZ], were in fact given by the representation theory of \( U_q(Sl(2)) \) with \( q = e^{\pi i/k+2} \). This work was later generalized to other groups, where the previous relation between the quantum deformation parameter and the data of the WZW model reads:

\[
q = e^{i\pi/k+g}
\]

with \( k \) the level and \( g \) the dual Coxeter number of \( G \). It was also possible to understand the modular properties and in particular the Verlinde theorem using exclusively quantum-group tools [AGS3]. In what concerns modular properties, these lead to the concept of ribbon Hopf algebras [RT].

Similar results were obtained for the minimal models of type \((p,p')\), where the braiding matrices essentially factorize into the product of two quantum 6-j symbols with deformation parameters: \( q = e^{i\pi p/p'} \) and \( q' = e^{i\pi p'/p} \) [AGS1, FFK]. Also some orbifold models where shown to have a quantum group structure [DPR] given by quasi-Hopf algebras [Dr2].

The previous examples showed that the deformation parameters coming from RCFT’s were given by roots of unity. The representation theory of quantum groups for generic values of \( q \) is essentially the same as the classical one [Ro]. However quantum groups at roots of unity were shown to display unexpected and interesting properties [L,RA]. The trouble with roots of unity is that many of the formulas in the representation theory for \( q \) generic break down or become ambiguous. A solution to this problem, which may look a bit ad hoc, is to restrict the representations to the good ones, so that one has a well defined highest weight theory, and throwing away the bad representations [L,PS,AGS2-3]. For example for \( U_q(Sl(2)) \) the good representations, also called regular or of type II, have spin \( j \) between zero and \( k/2 \), and are in one to one correspondence with the integrable irreps of the WZW model \( \widehat{SU}(2)_k \). This truncation was also seen to occur in the representation theory of the Hecke algebra \( H_N(q) \) which is the centralizer of the tensor product of the spin 1/2 representation of \( U_q(Sl(2)) \) [Wz,AGS2]. The relation between fusion rules of RCFT’s and quantum groups have been further studied in [GP,FGP,FD,T].

These efforts of trying to match bits of data from different pieces of mathematics and physics
were in a way "phenomenological", because it was missing a neat understanding of the close relationship between RCFT’s and quantum groups. Further research has deepen and broaden the knowledge of the topic, which cannot yet considered as being closed.

One astonishing aspect of the connection between RCFT’s and quantum groups is that on one side one is dealing with an infinite dimensional symmetry i.e. the Virasoro algebra, Kac-Moody algebra, etc. and on the other the symmetries, although a bit peculiar, are finite dimensional. How could it be possible that some finite dimensional structure would know so much about an infinite dimensional one?. Making a very rough analogy we could say that given a function of the form: \( f(z) = z^\alpha \sum a_n z^n \) the information that quantum groups are telling us is the integer part of \( \alpha \), which is what determines the monodromy of the function, i.e. \( f(e^{2\pi i} z) = e^{2\pi i \alpha} f(z) \). In RCFT’s the role of the function \( f(z) \) is played by the conformal blocks while the phase factors \( e^{2\pi i\alpha} \) or \( e^{\pi i\alpha} \) become the monodromy or braiding matrices respectively. To complete this analogy we could say that the chiral algebra (i.e. Virasoro, Kac-Moody, etc) would imply a differential equation for the function \( f(z) \), equation that would contain in a disguised way the monodromy of their solutions.

In this context the connection between RCFT’s and quantum groups is the relation between a differential equation and the monodromy or braiding properties of their solutions. The later problem is deeply connected to the classic and well known Riemann-Hilbert problem, which consist in the characterization of a differential equation and the set of the solutions from the knowledge of their monodromy properties. What the polynomial equations establish are the consistency conditions to be satisfied by the braiding and fusion matrices associated to a set of differential equations which are in principle unknown. The "reconstruction program", advocated by Moore and Seiberg, which tries to determine from every solution to the polynomial equations a set of differential equations whose solutions give the conformal blocks of the theory, is nothing else but a Riemann-Hilbert problem. This program has only been sketched but, as one can imagine, the task of finding a solution to the polynomial equations seems already quite formidable without the help of some guiding principle.

An important thing that we must not forget is the meaning of the differential equations satisfied by the conformal blocks. They guarantee the decoupling of null vectors of the degenerate primary fields. The very existence of these fields is what makes the theory rational (finite number of primary fields) and solvable. We may expect that quantum groups would also have a lot to say about the decoupling of null vectors of the corresponding primary fields.

**Vertex formulation of CFT’s**

From the previous discussion it seemed natural that the ordinary formulation of RCFT’s as the representation of a chiral algebra (or rather the product of a holomorphic \( \mathcal{A}_L \) times an antiholomorphic chiral algebra \( \mathcal{A}_R \)) should be enlarged in order to accommodate the quantum
group symmetry, which otherwise remains hidden or invisible. A signal of this hidden nature was that objects like 3j-symbols or the $R$-matrix itself did not show up in RCFT’s while for example 6j-symbols did. In general the IRF content of quantum groups was clearly present in RCFT’s. This suggested that what was needed, to uncover the quantum group structure of RCFT’s, was a ”vertex formulation” of conformal field theories. In statistical mechanics it is well known that some models admit both kinds of formulations, either as vertex or as IRF models, in which case there exist a vertex-IRF map which establishes the correspondences [Ba]. It happens also, when going from the vertex to the IRF formulation, that one misses along the way the degrees of freedom on which the quantum group is acting. These kind of arguments were also used in [Wi] in an attempt to derive a quantum group structure from the 3D Chern-Simon theory.

In group theoretical language the vertex formulation is like working with tensor products of representations of a group, while the IRF formulation is like performing the tensor product of these irreps and keeping only the spaces on which the centralizer of the group is acting:

$$V_{1/2} \otimes \cdots \otimes V_{1/2} = \bigoplus_j (W_j^N \otimes V_j)$$

In the vertex formulation the group or q-group is acting on each individual vector space, $V_{1/2}$ in the example above, by means of the comultiplication. The vector space $W_j^N$ is formed by the invariant tensors of the q-group and serves as the representation space of the centralizer $C_{1/2}^N$ which is defined as the set of endomorphisms of $V_{1/2}^{\otimes N}$ that commute with the action of the group. The dimension of $W_j^N$ equals the multiplicity of the spin $j$ irrep into the tensor product $V_{1/2}^{\otimes N}$. In RCFT’s $W_j^N$ can be identified with the space of conformal blocks of $N$ legs of spin $1/2$ and one leg of spin $j$, and its dimension can be computed using the fusion rules of the theory. From the previous identification we see that the centralizer of the quantum group is precisely realized on the space of conformal blocks, while they remain invariant under the action of the quantum group. This explains in a neat way the meaning of hidden quantum symmetries when talking about the role of quantum groups in RCFT’s.

In more physical terms the centralizer has the structure of a braid group when acting on the space of conformal blocks. In the case of a WZW model based on the group $SU(2)$ it was shown by Kanie and Tsuchiya that the braid group representation that one obtains from the conformal blocks with spin $1/2$ primary fields at the external legs gives a representation of the Hecke algebra $H_N(q)$ or more precisely of the Temperley-Lieb-Jones algebra. On the other hand Kirillov and Reshetikhin showed that the centralizer of the quantum group $U_q(Sl(2))$ in the tensor product of spin $1/2$ representations was also a Hecke algebra $H_N(q)$ [KR]. This connection was again part of the phenomenology that we mention before, however this time one could see that the missing objects to complete the link with quantum groups were the q-group spaces $V_{1/2}$ and $V_j$ of equation(5). These spaces should not be confused with the Verma modules
$H_j$ of the Kac-Moody algebra. If this were the case then the q-group $U_q(Sl(2))$ would already be contained in the Kac-Moody algebra $\hat{SU}(2)$. This suggest that the full symmetry algebra should be something like a tensor product $\hat{SU}(2) \otimes U_q(Sl(2))$. For a general conformal field theory the symmetry would be $A_L \otimes Q_L$ for the holomorphic degrees of freedom and $A_R \otimes Q_R$ for the antiholomorphic ones. $Q_L$ and $Q_R$ denote the quantum groups associated to the chiral algebras $A_L$ and $A_R$ respectively. Later on we shall discuss in more detail the structure of the tensor products $A_L \otimes Q_L$ which turns out to be a bit more subtle than just a direct tensor product. The key idea in this respect is that the q-group spaces $V_j$ contain fields or states that cannot be obtained from the primary fields by the action of an operator in the chiral algebra but by that of a genuine quantum group operator. We should then distinguish between two kind of descendants, those of the chiral algebra $A_L$ and those of the quantum group $Q_L$.

Having settled these conceptual matters the problem was then to give an "explicit" construction of these quantum group fields which could not be conformal (either primary or descendants) in the usual sense. This technical problem was overcome by the use of the Feigin-Fuchs or Coulomb gas formalism which has proved to be extremely important in the study of conformal field theories [FF].

The Coulomb Gas formalism of CFT’s

A Coulomb gas is nothing but a free field realization of a conformal field theory. The first known CFT’s admitting a Coulomb gas version are the BPZ $c < 1$ theories [BPZ], which were studied by Dotsenko and Fateev using only one free scalar field [DF]. The CFT’s based on the $W_n$ algebras [Z] require the introduction of $n-1$ scalar fields [FZ,FL], while the WZW models need both scalar fields and $\beta - \gamma$ systems [Wa,FFr]. An open problem is to know whether any conformal field theory admits a free field realization. The fact that a highly interacting field theory allows a free field description is on the other hand a mistery with very deep consequences. Some of them are the ability of a systematic computation of the conformal blocks, the characterization of the null vectors, a straightforward derivation of the Kac’s formula, the construction of a BRST formulation à la Felder [Fe], and, as we shall show next, the construction of quantum groups.

The basic feature of a Coulomb gas is that the chiral algebra generators can be constructed entirely in terms of a set of free fields which have very simple operator product expansions. Moreover the primary fields, which are essentially vertex operators as in string theory, also satisfy simple OPE’s. All this reduces enormously the task of computing correlators which are given by integrals of products of monomials. If we are studying the $c < 1$ or a WZW model we know that these integrals come from the solution of hypergeometric type equations. However in the Coulomb gas these integrals come from the screening charge that one has to introduce in order to balance the background charge. A screening charge is an integral $\int dt J_a(t)$
which has the property of commuting with all the operators of the chiral algebra $O_n$ up to a total derivative:

$$[ O_n , \int dt J_a(t) ] = \int dt \frac{\partial}{\partial t} (X_n) \quad (6)$$

If there are not boundary contributions to (6) then the screening charges fully commute with the chiral algebra.

Another ingredient of this construction is the use of Fock spaces of free fields instead of Verma modules. The later are recovered as cohomological classes associated to a BRST charge constructed using screening operators [Fe]. The philosophy adopted in references [GS1-2] was to use these Fock spaces to enlarge, rather than to restrict, the number of fields in the theory in order to give room to q-groups. We could say in this sense that quantum groups are hidden in a RCFT in much the same manner that the corresponding free field version is hidden, which lead us in the long run to ask why the solution to the null vector decoupling equations of a RCFT admit an integral representation. This is a highly non trivial fact and a mistery which certainly needs more study.

**Field theory formulation of quantum groups**

The starting point of the work of reference [GS1] is the definition of a special kind of screened vertex operators which form the basis for the representation spaces of quantum groups. They are defined integrating a collection of screening vertex operators $J_a$ around a primary field $\phi_\alpha$:

$$e^\alpha_{a_1, \ldots, a_n}(z_P, z_\infty) = \int_{C_1} dt_1 J_{a_1}(t_1) \cdots \int_{C_n} dt_n J_{a_n}(t_n) \phi_\alpha(z_P) \quad (7)$$

In the choice of the integration contours $C_1, \cdots, C_n$ we have taken into account the fact that there are branch cuts in the integrand coming from the OPE’s between the screening operators with the primary field as well as among the screening themselves. They are choosen as a nested set of Hankel’s contours, similar to the ones that appear in the definition of the $\Gamma$ function, i.e. they go from infinity to infinity encercling the point where the primary field is inserted. Strictly speaking the operators $e^\alpha_{a_1, \ldots, a_n}$ depend on two points, one is the point $z_P$ where the primary field is located and the other is the point at infinity $z_\infty$, which plays the role of a base point.

In this construction the primary field itself $\phi_\alpha$, is identified with the highest weight vector of a q-group space $V_\alpha$. The remaining vertices obtained integrating screening operators are identified with q-group descendants. This leads naturally to the interpretation of the screening charges as quantum lowering operators: $F_a = \int_C dt J_a(t)$. This interpretation was already anticipated in references [Sa,BMP]. In the $c < 1$ theory there are two screening operators $J_+$ and $J_-$ hence we have two lowering operators $F_+$ and $F_-$ [GS2]. In the Wakimoto construction
of $SU(n)$ there are $n-1$ screenings operators which then lead to $n-1$ lowering operators each one associated to a positive simple root $[RRR]$. Under this interpretation each screening charge $J_a$ give rise to a quantum group $U_{q_a}(Sl(2))$, where the deformation parameter $q_a$ is given by the braiding factor among two screenings of the same type. If there are various screening operators they will define in general larger $q$-groups whose defining relations would follow from their braiding properties. For example for a WZW theory the braiding between screening operators: satisfies:

$$J_a(z) J_b(w) = q^{C_{ab}} J_b(w) J_a(z)$$

(8)

where $C_{ab}$ is the symmetrized Cartan matrix and $q$ is given by eq.(4).

Equation (8) reflects the non-local nature of the screening operators and in that sense the quantum deformation parameter $q$ acquires a conformal field theory meaning.

From this equation one can also derive the quantum deformation of the Serre relations satisfied by the lowering operators $F_a$ [BMP, RRR].

After applying a sufficient number of screening operators to a primary field one eventually gets zero or a null vector. To understand why this happens we have to recall that the q-vertex operators do depend in general on the point at infinity $z_\infty$ and the point $z_P$ where the primary field is inserted, however under certain circumstances the branch cut effectively disappears and the contour integrals shrink to the neighbour of the point $z_P$. This property holds for primary fields within the Kac’s formula and it is extensively used in the BRST construction. In our q-group interpretation this means that the primary field give rise to a finite dimensional representation. In other words, a null vector in the CFT is at the same time a null vector for the q-group. This is the core of the connection between quantum groups and RCFT’s.

It would be of some interest to find a more intrinsic definition of the q-vertex operators (7), i.e. a definition independent of the use of the Coulomb gas formalism. A possibility is to consider these vertices as the solutions of the null vector decoupling equations. The primary fields themselves would of course be a solution, the one with conformal properties, while the others will span the rest of a representation space. The order of the equation, which is $nm$ for the primary field $\phi_{nm}$ for a $c < 1$ theory, would then give the dimension of the associated representation space of the quantum group.

We conclude that screening charges are lowering operators, similarly the Cartan operators are identified with the Coulomb charges. These two sets of operators form the Borel subalgebra which is traditionally associated with the lower triangular matrices. To complete the RCFT picture of q-groups one needs a definition of the raising operators $E_a$. These cannot be represented as integrals of screening operators with opposite charge because they do not have the correct conformal properties. What one really needs is an operator that would destroy integrals of screening operators and this can be achieved with the action of the chiral algebra.
Take for example the Virasoro operator $L_{-1}$, acting on a q-vertex operator it integrates each of the screening charges leaving a boundary term as can be seen in eq.(6). Explicit computations show that eq.(6) implies the q-group relation:

$$[O_n, \int dt J_a(t)] = \int dt \frac{\partial}{\partial t} (X_n) \Rightarrow [E_a, F_b] = \delta_{a,b} \frac{K_a - K_a^{-1}}{q_a - q_a^{-1}}$$  \hspace{1cm} (9)

In this sense the q-group generators $E_a$ are contained in the chiral algebra generators $O_n$, which shows that the relation between $A_L$ and $Q_L$ is more intricated than expected. From equation (9) we also see that the shrinking condition for the contours which define a q-vertex operator is equivalent to the statement that this q-vertex operator is a highest weight vector of the quantum group:

$$\frac{\partial}{\partial z_\infty} e_{a_1,\cdots,a_n}^\alpha = 0 \Rightarrow E_a e_{a_1,\cdots,a_n}^\alpha = 0 \forall a$$  \hspace{1cm} (10)

It is also curious to observe some parallelism between this construction and the construction of the quantum double by Drinfeld [GS1,BL1].

Having defined the q-group spaces and the q-group generators, it is easy to obtain the comultiplication, the R-matrix, the 3j-symbols, the 6j-symbols, that is all the ingredients of quantum groups in a field theoretical framework. These ideas and techniques, which were first applied to the $c < 1$ theories, have been extended successfully to WZW models [RRR], $W_n$ algebras [C], N=1 and 2 superconformal field theories [Jz] and a WZW model based on the supergroup $Gl(1,1)$ [RSa]. Moreover the whole construction has a geometrical and topological flavor [GS3]. Everything follows from contour manipulations involving elementary complex analysis and q-combinatorics. Group theory when quantum deformed becomes a topology of contours (for a more elaborated version of this idea see [FeW,SV]).

**Towers of algebras**

Until now we have proceed to unravel the quantum group structure of RCFT’s from the integral representation of the decoupling equations, we can refer to this line of thought as the analytic approach. A complementary point of view partially inspired in Pasquier’s ethiology of IRF-models [P] consist in starting the construction with the fusion rules and from that piece of information to obtain the duality (braiding and fusion) matrices in the very same way one associates in the Jone’s fundamental construction a Temperley-Lieb-Jones algebra to a given graph [GHJ]. In this algebraic approach the RCFT defines a model of the Bratelli diagram in terms of the fusion rules[GS4]. Recent progress by Ocneanu [Oc] indicates, as it should be expected, that associated with the RCFT tower of algebras it is possible to define in a unique way a quantum group structure. For a different version of the algebraic approach based on operator algebras see [FrK]. We believe that it would be fruitful both from a mathematical
and a physical point of view to get a deeper understanding of the interplay between these two apparently different mathematical approaches, namely the monodromy of differential equations with their related Riemann-Hilbert problem and the theory of subfactors.

Cannonical quantization approach to CFT’s

The discussions above about the role of quantum groups in CFT’s have been done in the general framework of the bootstrap program of BPZ, where actions for the local fields are not strictly needed. In doing so we have omitted another important approach to the role of quantum groups in quantum field theories, namely the one based on the use of local actions or hamiltonians which are quantized in a cannonical way. The traditional methods were applied to Liouville theory which was shown to exhibit quantum group attributes in a time when quantum groups were not known [GN,FT]. In fact in reference [FT] first appeared the defining relations of the quantum Lie group $SL_q$, not to be confused with the quantum Lie algebra $sl_q$ which is the same as the quantized universal enveloping algebra $U_q(Sl(2))$.

After quantum groups came to fashion this approach was renewed again in connection with Liouville theory [Bb,Ge,ST], Toda field theory [BG,HM] and the Wess-Zumino-Witten model [AS,Fa,Ga,AF,AFS,AFSV]. The use of a different language has made a bit difficult to establish the connections with the, let’s say, bootstrap approach although they certainly do exist. Particularly interesting are the lattice formulations of the Liouville and the WZW theories which may perhaps allow a Bethe ansantz analysis analogous to the one applied in the study of magnetic chains. We refer to the lectures of prof. L.D. Fadeev for a discussion of these matters.

Integrable quantum field theories

So far we have considered applications of the quantum groups to conformal field theories but their range of applicability includes also massive integrable 2D field theories. An archetype of the later is the well known sine-Gordon theory. In references [RS,BL2] it was shown that there exist a hidden quantum symmetry which governs the scattering of the solitons, antisolitons and bound states of the model, which explains in particular the S-matrix obtained in [ZZ]. This symmetry is affine and can also be given a field theory realization in terms of non-local conserved charges [BL3,FL,BFL].

$q$-RCFT’s: New hidden symmetries ?.Elliptic quantum groups

We would like now to discuss briefly some new directions of research. As we have seen, a particular interesting class of RCFT’s are the WZW models based on Kac-Moody algebras. A centrally extended affine Lie algebras can also be deformed in a very non trivial way. These
q-RCFT's preserve much of the structure of the ordinary RCFT's and can be interpreted as some kind of massive field theories. In reference [FR] Frenkel and Reshetikhin have found a q-deformation of the Kniniznik-Zamolodchikov equation which governs the correlators of these q-WZW model. This is a finite difference equation whose solutions are q-hypergeometric functions. The connection matrices, or in CFT language, the braiding matrices of this "q-conformal blocks" turns out to be given by the elliptic solution to the Yang-Baxter equation of the RSOS model of Andrews, Baxter and Forrester [ABF]. These authors also pose the question of the possible existence of a hidden quantum symmetry which would underlie the elliptic Boltzmann weights, much in the same way that quantum groups underlie the trigonometric solutions. It has also been suggested as a candidate for this "elliptic quantum symmetry" the Sklyanin algebra which is intimately related to the Baxter's eight vertex model [Sk].

The solution to these questions is not known to us. We would like however to propose a way to attack the problem which consist in following the same steps that were pursued in the unravelling of quantum groups in RCFT's, namely : find a free field realization of the q-Kac-Moody algebra, then look for the q-screening operators and finally define in terms of q-integrals the "elliptic version" of quantum groups. The first two steps of this program have already been achieved in references [FJ,M,Sh,KQS,ABG] where various free field versions of the q deformed Kac-Moody algebra have been given. In [Sh] it is also constructed a q-screening operator whose q-integral à la Jackson commutes with the generators of q-Kac-Moody. Therefore one has in principle all the ingredients to unravel a new hidden quantum symmetry.

**Spin chains at roots of unity**

To finish this brief history we would like to make some comments in connection to recent progress by De Concini and Kac [DK] in the study of the representation theory of quantum groups at roots of unity (see also [DKP]. As we have already mentioned the quantum deformation parameter of the quantum group associated to a RCFT is a root of unity , for example $q = e^{\pi i/k+2}$ for a SU(2) WZW model. This implies that strictly speaking the quantum group associated to this RCFT should be $U_q(SU(2))$ mode out by the central Hopf which appears when $q$ is a root of unity [McSo].

Another phenomena that appears for quantum groups at roots of unity is the existence of non restricted representations, i.e. representations which transform non trivially under the central Hopf subalgebra. In a series of papers [GRS,GS5,BGS,CGS] we have started the study of a family of these representations called semicyclic and nilpotents, which have the property that their tensor product is decomposable [GS5,A] so that one can define 3j and 6j symbols. An important question is wether this new class of braiding and fusion matrices may become the duality data of a new class of decoupling equations defining a new hierarchy of conformal field theories.
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