Gaussian phase sensitivity of boson-sampling-inspired strategies

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In this work we study the phase sensitivity of generic linear interferometric schemes using Gaussian resources and measurements. Our formalism is based on the Fisher information. This allows us to separate the contributions of the measurement scheme, the experimental imperfections, and auxiliary systems. We demonstrate the strength of this formalism using a broad class of multimode Gaussian states that includes well-known results from single- and two-mode metrology scenarios. Using this, we prove that input coherent states or squeezing beat the non-classical states proposed in preceding boson-sampling-inspired phase-estimation schemes. We also develop a novel polychromatic interferometric protocol, demonstrating an enhanced sensitivity with respect to two-mode squeezed-vacuum states, for which the ideal homodyne detection is formally shown to be optimal.

I. INTRODUCTION

During the last decade a considerable attention has been devoted to figure out the optimal phase-estimation scheme for a (linear) photonic interferometer using Gaussian states and ideal quadrature measurements [11-15] by means of the celebrated parameter estimation theory [6]. For instance, the so-called quantum Fisher information (QFI), which dictates the ultimate phase sensitivity under generic measurements [7-9, 16], has been intensively studied for noisy and lossy two-mode Mach-Zehnder interferometers (MZI) pumped by either a cross-product of coherent and squeezed-vacuum state [17, 18], or a two-mode squeezed-vacuum state [19, 21].

The phase sensitivity of the multimode scenario is less understood [22-31]. Recent work suggests that, in the case of decoherence-free Gaussian resources with fixed average number of photons, the optimal Heisenberg limit is reached with a trivial squeezed-vacuum state [24, 28]. Unfortunately, these results display an intricate dependence on the desired parameter which represents a major obstacle for treatments based on the Symmetric Logarithmic Derivative (SLD) [2-5]. Moreover, the optimal Gaussian scheme involves non-trivial technical challenges [29-34], such as engineering the passive transformation and generating high-intensity, highly-squeezed light beams.

In view of these problems, when working with experimental constraints we must focus on the Fisher Information (FI) [32, 33] for the resources at hand—families of states, transformations and measurements. This task has completed in the single-modeMZI scenario [34], and in some cases also for the multimode setup [35]. In this work we compute the Fisher information of arbitrary multimode interferometers working with Gaussian input states and quadrature measurements. The FI approach is a versatile treatment to study the phase resolution of general circuits, such as reconfigurable photonic circuits [29, 31, 36, 37] with homodyne measurements.

II. GAUSSIAN PHASE SENSITIVITY

Our work is devoted to studying an interferometric setup 11, 13, 15 such as the one in Fig. 1. This phase-estimation scheme consists of: (i) an N-mode input state of light φ with m principal modes and N−m auxiliary degrees of freedom that will be eventually discarded 17, 18, (ii) an interferometer L that prepares the state of light prior to interaction, (iii) the actual phase transformation ϕ that we wish to detect, (iv) a final measurement stage that combines a linear transformation with local homodyne measurement on m modes.
contains the MZI, and the vast majority of Gaussian (single) phase-estimation previously treated as particu-
lar instances \[17, 18, 24, 34, 49–51\]. Our work focuses on the (single) phase-rotation \(\varphi\), such that the whole propagation is described by \(S(\varphi)\). The output modes of the probe system are finally assessed by a generic quadrature measurement determined by \(\Sigma_g\).

FIG. 1. (color online). Sketch of a generic N-mode Gaussian phase-estimation strategy consisting of a probe m-mode state, characterized by \(\langle R_S \rangle\) and \(V_S\) (orange), and an ancilla \((N-m)\)-mode state, characterized by \(\langle R_A \rangle\) and \(V_A\) (green). Both probe and ancillary systems interact via the interferometer modeled by \(L\), whereafter the first probe mode undergoes the (single) phase rotation \(\varphi\), such that the whole interferometric evolution, local phase rotation \(\varphi\), and local homodyne measurements. Furthermore, without loss of generality, we assume that the measurement is described by \(\rho\), the (single) phase rotation \(\varphi\) acts as a local rotation \(\hat{\alpha}_\beta\) acting on one of the modes.

We will now proceed in three steps. The following section will introduce the phase-space formalism, explaining how to express states \(\hat{\rho}\), interferometers, local phase rotations and measurements. Later in Sect. II B we will introduce the Cramer-Rao bound and how the Fisher Information determines the maximum achievable sensitivity of our interferometer. Finally, Sect. II C connects both formalisms, providing an explicit formula for the Fisher information and the phase sensitivity of our setup, expressed in terms of the first and second moments of the input state, the covariance matrix of the measurement and the passive transformations \(L\) and \(K\).

### A. Phase-space formalism

We model the light using two quadratures per mode, \(\hat{q}_i\) and \(\hat{p}_i\), which satisfy the canonical commutation relations \([\hat{q}_i, \hat{p}_i] = i [J_N]_{i i}\). Here we have introduced the symplectic form \([44, 46]\) \(J_N = \bigoplus_{i=1}^{N} J_1\), expressed in terms of \([J_{\alpha \beta}] = \varepsilon_{\alpha \beta}\), the Levi-Civita symbol in two dimensions \(\varepsilon_{\alpha \beta}\). Any operator \(\hat{O}\) is described in terms of the Weyl Symbol \(W_O(R)\) spanned by the phase-space basis \(R = (q_1, p_1, \ldots, q_N, p_N)^T \in \mathbb{R}^{2N}\) with support in the real symplectic space \((\mathbb{R}^{2N}, J_N)\) \([15, 46]\). Gaussian states are those whose density matrix has a Weyl symbol \(W(\rho)\) that will be extensively used throughout this work.

The initial state undergoes a multimode interferometer transformation, given by an \(2N \times 2N\) orthogonal, symplectic matrix \([44]\). For convenience, we split this matrix into system and ancilla

\[
L = \begin{pmatrix} L_S & L_{SA} \\ L_{AS} & L_A \end{pmatrix},
\]

where \(L_{SA}\) is a \(2m \times 2(2N-m)\) isometry, while \(L_S\) is a non-orthogonal \(2m \times 2m\) matrix which satisfies a symplectic-like relation \(L_{SA} = J_m^T L_S L_m\) as well.

After this preparation, the bosonic system suffers an unknown phase shift \(U(\varphi)\), generated by the operator

\[
\hat{H} = \frac{1}{4}(\hat{q}_1^2 + \hat{p}_1^2) - \frac{1}{2}.
\]

The phase shift \(\hat{U}\) induces a rotation in phase space \(U_N(\varphi) = U(\varphi) \otimes I_{N-1}\), with \(U(\varphi)\) given by Eq. (4). The combined N-mode transformation \(S(\varphi)\) is composed of an \(2m \times 2m\) non-orthogonal (non-singular) matrix \(S_A\) acting solely upon the probe system, and an isometry \(S_{SA}(\varphi)\) describing the interference between the system and the ancillas

\[
S_S(\varphi) = U_m(\varphi)L_S, \quad S_{SA}(\varphi) = U_m(\varphi)L_{SA}.
\]

Since \(S(\varphi)\) describes a passive interferometric evolution, the following relations must hold \([1]\)

\[
S_S(\varphi) S_{SA}^T(\varphi) = I_m - S_{SA}(\varphi) S_{SA}^T(\varphi), \quad S_S(\varphi) = J_m^T S_{SA}(\varphi) J_m, \quad \text{for } \varphi \in \mathbb{R},
\]

which also shall be used in the subsequent derivation.

The phase-estimation task is finally accomplished by performing a \(m\)-mode Gaussian measurement with outcome \(A \in \mathbb{R}^{2m}\). Any \(m\)-mode general-dyne measurement acting as a Gaussian POVM \(\Pi_A\) is characterized by a
In particular, for any unbiased estimator function with $F$ the mean square error \cite{7, 11, 14} we now consider the ideal $N$-mode homodyne detection scheme consisting of identical local quadrature measurements, with $K = I_N$ and $r_i = r$.

We can compute the probability $p(\lambda | \varphi)$ of obtaining a measurement outcome $\lambda$ conditioned to a phase shift $\varphi$. This is a Gaussian function characterized by the first-moment vector and the CV matrix \cite{52}

\begin{equation}
\langle \lambda | \varphi \rangle = S_{\lambda S}(\varphi) R_j + S_{SA}(\varphi) \langle R_A \rangle,
\end{equation}

\begin{equation}
\sigma(\varphi) = \sum_{\lambda} S_{\lambda S}(\varphi) V_{\lambda S} S_{\lambda S}^T(\varphi) + S_{SA}(\varphi) V_{\lambda A} S_{\lambda A}^T(\varphi),
\end{equation}

\begin{equation}
\sigma_{\lambda S}(\varphi).
\end{equation}

Note how the probe system statistics $\hat{p}_S$ only appears in $\langle \lambda_S(\varphi) \rangle$ and $\sigma_{\lambda S}$.

### B. Basics of phase estimation theory

Using the so-called maximum likelihood and Bayesian estimators \cite{11, 19, 59}, we can approximate an unknown phase shift $\varphi$ from a set of measurement outcomes $\lambda$. The precision of this method will be determined by the conditional probability $p(\lambda | \varphi)$, as well as the estimator strategy $p_{est}(\varphi | \lambda)$. The statistical inference process is described by the probability distribution $\pi | \lambda \rangle \geq F(\varphi)$. As shown in App. A there is a closed-form formula for the QFI when working with isothermal Gaussian input states and passive linear transformations $\pi \pi'$. The first-moment $\langle \lambda' \rangle$ and CV $V_{\lambda}$ belong to the probe mode immediately before undergoing the phase-shift rotation. This expression is independent of $\varphi$ because of the phase-shift generator $H(\varphi) = H \pi H'$. The limit requires implementing a measurement strategy that can depend on the estimator $\varphi$. This can involve elaborate transformations $L$ and $K$ and measurements of second or higher order moments of the quadrature. For this reason, unlike the vast majority of the previous works $\pi \pi' \pi'$, we will center on discussing the FI and the attainable limits of phase sensitivity under given experimental setups and constraints. As we will show below, this is not a severe restriction. We can compute the sensitivity of protocols that are experimentally feasible [cf. Fig. 1].

\section{C. FI analysis}

We now present the main result which is the basis of the future analysis. Starting from the identity \cite{14}, in
App. [13] we decompose the FI as follows,

$$F(\varphi) = F_{\text{S}} + F_{\text{Anc}}(\varphi) + F_{\text{Int}}(\varphi) - F_{\text{Meas}}(\varphi)$$

(18)

$$+ \frac{(2n_c + 1)^2}{1 + (2n_c + 1)^2} \left( \text{Tr}(V_i^T V_i) - \text{Tr}(P_i L_S L_S^T) \right),$$

where $F_{\text{S}}$ is the QFI associated to the $m$-mode probe system alone (which is obtained from [17] in the absence of the ancilla), and $P_\varphi = U(\varphi) \otimes 0_{m-1}$ is a $2m \times 2m$ projection matrix. The new functions $F_{\text{Anc}}(\varphi)$, $F_{\text{Meas}}(\varphi)$, and $F_{\text{Int}}(\varphi)$ respectively encode the influence of the input ancilla state, the $m$-mode quadrature measurement, and the interference between the ancilla and system. The measurement contribution reads

$$F_{\text{Meas}}(\varphi) = \langle R^T_i \rangle \partial_\varphi \sigma_i \Sigma \partial_\varphi \sigma_i \langle R_i \rangle$$

(19)

$$- \frac{1}{2} \text{Tr} \left( \partial_\varphi \Sigma \partial_\varphi \sigma_i \left( S_{SA} V_T S_{SA}^T \right) \right).$$

The symmetric and symplectic $2m \times 2m$ matrix

$$\tilde{\Sigma}_S = (S_{SA}^T)^{-1} V_S^{-1} S_{SA}^{-1} (\Sigma_i^{-1} - \langle \lambda^T \rangle \tilde{V}_A \partial_\varphi \langle \lambda \rangle + 2 \text{Tr}(\partial_\varphi \tilde{V}_A \partial_\varphi \sigma_i)),$$

$$+ (S_{SA}^T)^{-1} V_S^{-1} S_{SA}^{-1} (S_{SA}^T)^{-1} V_S^{-1} S_{SA}^{-1},$$

(20)

is manifestly independent of the input ancilla state. The influence of the ancilla is fully contained in

$$F_{\text{Anc}}(\varphi) = 2 \langle R_i^T \rangle \partial_\varphi \sigma_i \tilde{\Sigma}_S \partial_\varphi \sigma_i \langle R_i \rangle$$

(21)

$$+ \langle R_i^T \rangle \partial_\varphi \sigma_i \tilde{\Sigma}_S \partial_\varphi \sigma_i \langle R_i \rangle$$

$$- \partial_\varphi \langle \lambda^T \rangle \tilde{V}_A \partial_\varphi \langle \lambda \rangle + 2 \text{Tr}(\partial_\varphi \tilde{V}_A \partial_\varphi \sigma_i)$$

$$- \frac{1}{2} \text{Tr} \left( \partial_\varphi \sigma_i^{-1} \partial_\varphi \left( S_{SA} V_A S_{SA}^T \right) \right).$$

with

$$\tilde{V}_A = \sigma_i^{-1} S_{SA} (V_A^{-1} + S_{SA} \sigma_i^{-1} S_{SA})^{-1} S_{SA} \sigma_i^{-1}.$$

(22)

Similarly, the function $F_{\text{Int}}(\varphi)$ only depends on the input system state and system-ancilla interference $S_{SA}$ (see Eq. [19] in App. [3]). Note that both $F_{\text{Anc}}(\varphi)$ and $F_{\text{Int}}(\varphi)$ vanish when the system-ancilla interference cancels (which corresponds to the non-assisted scenario without ancilla system).

Let us give a brief overview about the derivation of the expression [18]. From Eqs. [9] and [10] we may separate the contribution of the ancilla state. Indeed, using the so-called Woodbury identity (cf. Eq. [B1] in App. [3] [35] [59],

$$F(\varphi) = F_S(\varphi) + F_{\text{Anc}}(\varphi),$$

(23)

we can separate the contribution $F_S(\varphi)$ from the first-moment $\langle \lambda(\varphi) \rangle$ and CV $\sigma_S(\varphi)$. Collecting all remaining terms that depend on the auxiliary system, $F_{\text{Anc}}$ adopts the form in Eq. [21]. This procedure may be repeated, using the symplectic-like identities [2] and [7], to separate from $F_S(\varphi)$ the interference $F_{\text{Int}}(\varphi)$ and measurement terms $F_{\text{Meas}}(\varphi)$, as shown in Eq. [B12]. Finally, using property [B13], one may group the remaining terms into the QFI $F_S$ (see Eq. [B14]) plus additional corrections, as show in Eq. [18].

The closed-form expression [18] is valid for any probe isothermal Gaussian state $W(V_S, (R_S)) \in \mathcal{G}(m, n_c)$, and for arbitrary interferometric schemes, Gaussian ancilla states as well as measurements. In particular, we pay special attention to input coherent resources and the so-called Quantum Uniform Multimode Interferometer (QUMI) recently studied in the context of boson-sampling inspired phase-estimation strategies [10] [32]. These are further discussed in the following section.

1. Coherent ancilla state and QUlMI

In the simple scenario in which the ancillary system are coherent states $V_A = I_{N-m}$, that interfere with the system through a QUlMI device—cf. the linear transformation $L$ from Eq. (C1) —, the FI simplifies to

$$F(\varphi) = F_S(\varphi) + F_{\text{Anc}}(\varphi) + F_{\text{Int}}(\varphi)$$

$$+ 2 \langle R^T_i \rangle \partial_\varphi \sigma_i \tilde{\Sigma}_S \partial_\varphi \sigma_i \langle R_i \rangle,$$

(24)

with

$$\sigma = \left( \Sigma + I_m \right) + S_S \left( V_S - I_m \right) S_S^T,$$

(25)

The term $F_S(\varphi)$ is the FI of a phase estimation scheme that uses a Gaussian input state with first-moment $\langle R_S \rangle$ and CV $V_S$, along with a Gaussian measurement with a white background noise $\tilde{\Sigma}_S$.

For a state with homogeneous input intensity $n_c$—i.e. $\langle R_i \rangle = \sqrt{2m} n_c$, for $i \in [1, N]$—, the QUlMI device concentrates all photons on the first probe mode. In this scenario, $S_{SA} \langle R_A \rangle = (N-m)/m S_S \langle R_S \rangle$ (which follows from the transformation (C1)). This means that the ancilla terms in Eq. (21) are positive and increase the FI—provided $V_S$ is a positive semi-definite matrix. The auxiliary coherent state improves the phase sensitivity, although it introduces some background noise in the measurement outcome.

This result simplifies in the ideal homodyne detection in which the system is also in a coherent state $V_S = I_m$. The ancilla proves beneficial still increases phase sensitivity, since $F_S(\varphi)$ becomes $4m n_c$ in the optimal operating points $\varphi_{\text{opt}} = \mp \pi/4$ (see the discussion around Eq. (35) in Sect. III A). Using Eq. (24) we obtain the phase sensitivity for the QUlMI assisted coherent setup

$$\langle \delta \varphi \rangle^2 = \frac{1}{4n_c N},$$

(26)

This coincides with the phase sensitivity of a single-mode coherent state with input intensity $n_c$. Moreover, Eq. (26) shows that input coherent resources outperform earlier QUlMI-based phase-estimation using single-photon states [10] [45], for any size of the interferometer. For
more general assisted phase-estimation schemes, it is less clear to see the influence owing to the interferometer $F_{\text{int}}(\varphi)$ and ancilla $F_{\text{Anc}}(\varphi)$ contributions at first sight, instead they deserve a more profound analysis that is beyond the scope of the present treatment \[47\] \[48\].

III. APPLICATION: N-MODE HOMODYNE DETECTION WITHOUT ANCILLA SYSTEM

We will now compare the strength of our treatment with earlier Gaussian phase-estimation analysis \[47\] \[48\] \[24\] \[33\], using no auxiliary modes ($N = m$), Gaussian pure input states ($\bar{n}_i = 0$) and an ideal N-mode homodyne measurement. Since there are no ancillas, we can eliminate the subscript $S$, $S_S(\varphi) \rightarrow S(\varphi)$, $\langle R_k^S \rangle \rightarrow \langle R^T \rangle, V_S \rightarrow V$, and $F_S \rightarrow F$. Both the F1 \[13\]

$$F(\varphi) = F - F_{\text{Meas}}(\varphi) + \frac{1}{2} \left( \text{Tr} \left( V^T V' \right) - 2 \right),$$

and the contribution from the measurement radically simplify [cf. Eq. (B15) in App. B].

$$F_{\text{Meas}}(\varphi) = \langle R^T \rangle L^T P^*_x S V S^T \tilde{\Sigma} S V S^T P_x L \langle R \rangle$$

$$- \frac{1}{2} \text{Tr} \left( \tilde{\Sigma} \partial_\varphi \left( S V S^T \right) \right)^2 + \text{Tr} \left( \tilde{\Sigma} \tilde{\Sigma} \partial_\varphi \left( J_N S V S^T \right) \right)^2. \quad (28)$$

The matrix $\Sigma$ that characterizes the Gaussian measurement appears in the new matrix $\tilde{\Sigma} = \left( \Sigma + S V S^T \right)$.

For an ideal homodyne detection in either position or momentum quadrature, $\Sigma$ effectively becomes a projection matrix, $\pi^{(x)} = \text{diag}(1,0,1,0,\cdots,0,1)$ or $\pi^{(p)} = \text{diag}(0,1,0,1,\cdots,0,1)$ respectively. In this case $\tilde{\Sigma}$ must be understood as a Moore-Penrose (MP) inverse \[46\] \[53\] \[60\], computed as follows \[53\]

$$\tilde{\Sigma}^{(x/p)} = \left( \pi^{(x/p)} S V S^T \pi^{(x/p)} \right)^{\text{MP}}. \quad (29)$$

Note also that the CV matrix of the chosen measurement remains invariant $\Sigma = K \Sigma K^T$ under any interferometric transformation $K$; rendering this choice irrelevant \[61\].

A. Coherent and one-mode squeezed resources

Let us analyze a collection of independent single-mode squeezed states, characterized by an arbitrary displacement $\langle R \rangle \in \mathbb{R}^{2N}$ and the CV matrix,

$$V = V_1(s_1) \bigoplus V_{N-1}(s_2), \quad (30)$$

with

$$V_i(s) = \bigoplus_{j=1}^{N} \begin{pmatrix} 1 & 0 \ 0 & \frac{s}{s_j} \end{pmatrix},$$

The squeezing of the first and of the remaining $N - 1$ modes are given by the parameters $s_1, s_2 \in \mathbb{R}^+$. When $N = 2$, this state reduces to the vast majority of non-entangled Gaussian states previously studied: when $s_1 = s_2 = s$, it maps to studies of single-mode squeezed states \[3\] \[17\] \[24\] \[31\] \[62\] \[66\], when $s_2 = 1$ we have the squeezed mode combined with a coherent state from Refs. \[13\] \[21\] \[51\] \[60\] \[67\] \[68\], and for $s_1 = s_2 = 1$ we recover the coherent phase-estimation scenario and the SNI scaling.

The QFI depends of the average number of photons on the mode that undergoes the phase rotation. We can therefore concentrate on the QUMI setup, which maximizes this intensity. For this we find

$$S_{\text{QUMI}} V S_{\text{QUMI}}^T = \left( \begin{array}{cc} \Omega_N(\varphi, s_1, s_2) & 0 \\ 0 & V_{N-2}(s_2) \end{array} \right), \quad (31)$$

where $\Omega_N$ is a $4 \times 4$ real, symmetric matrix whose representation does not affect the discussion [cf. App. C and Eq. (C2)]. Note how the size of \[31\] grows as $2(N_{s_1} + 1) \times 2(N_{s_1} + 1)$ for a large number $N_{s_1}$ of states with squeezing $s_1$. Replacing \[31\] in \[29\], we further obtain

$$\tilde{\Sigma}^{(x/p)} = A^{(x/p)} \oplus \text{diag}(1,0,\cdots,1,0). \quad (32)$$

Here, $A^{(x/p)}$ is a $4 \times 4$ real, matrix given by Eqs. \[C10\] and \[C11\]. By paying attention to \[31\], it is clear to see that the matrices within the trace in the expression \[28\] effectively play the role of a projection operator in the phase space supporting the mode undergoing the rotation, i.e. $\partial_\varphi(S_{\text{QUMI}} V S_{\text{QUMI}}^T) = \partial_\varphi \Omega_N \oplus 0_{N-1}$. Having evaluated the quantities \[31\] and \[32\], after substitution in \[28\] one obtains the F1

$$F^{(x/p)}(\varphi) = F - \frac{1}{2N^2} \left( a_N(s_2, s_1) + \frac{a_N^2(s_1, s_2)}{(s_1 s_2)^2} - 2N^2 \right) + f^{(x/p)}(\sin^2 \varphi, s_1, s_2) - \langle R'_1 \rangle W_{N}^{(x/p)}(\varphi, s_1, s_2) \langle R'_1 \rangle, \quad (33)$$

where we have introduced $a_N(s_1, s_2) = (N - 1)s_1 + s_2$, two auxiliary functions $f^{(x)}_N, f^{(p)}_N$ [cf. \(C19\)] and a real symmetric matrix $W_{N}^{(x/p)} \in \mathbb{R}^{2N \times 2N}$ [cf. Eq. \(C14\)].

The optimal phase-estimation strategy for a given $\varphi$ must saturate the QCRB \[10\]. In that case the last three terms in Eq. \[33\] cancel each other and $F = F$. To illustrate, let us consider an input beam with the same low-intensity $\bar{n}_c \ll N$ in each mode. For this choice, the QFI takes the form

$$F = 2\bar{n}_c N \left[ \frac{1}{2N^2} + \frac{1}{2} \left( \frac{4\bar{n}_c}{s_2} \left( \frac{1}{s_1} + \frac{1}{s_2} - \frac{1}{s_2} \right) + \frac{s_2}{s_2} - 2 \right) + O(N^{-1}) \right]. \quad (34)$$

The first term, proportional to $N$, reproduces the QFI of coherent input state, and the second and third term cancel precisely for that type of input $s_1 = s_2 = 1$. Moreover,
In the particular situation of homogeneous squeezing with coefficients $s_1 = s_2 = 1$, the QCRB for a position quadrature measurement is of the form of the interferometer size and distinct input probe states: the black-solid, blue-dashed, and orange-dot-dashed lines correspond to the tensor product of coherent (i.e. $s_1 = 1$, $s_2 = e^{2\phi}$), one-mode squeezed ($s_1 = e^{2\phi}$ and $s_2 = 1$), and single-mode squeezed (i.e. $s_1 = s_2 = e^{2\phi}$) states, respectively. For a fair comparison, we have fixed the input mean photon number per mode to an identical value for all input states, i.e. $\bar{n} \approx 1.38$, as well as we have chosen the unknown phase shift $\varphi = \pi/3$ and the squeezing parameter $s' = 1/2$. (Right) Similarly, the FI as a function of the mean photon number per mode for a fixed squeezing parameter. We have taken the same values for the rest of parameters.

we may expand Eq. (33) in the limit of large interferometers $1 \ll N$ with finite energy $1/N \ll s_{1/2} \ll N$

$$F^{(s/p)}(\varphi) = 4\text{s} N s_2 (1 \mp \sin(2\varphi)) + 2s_1^{-1}(s_1 (1 \pm 1) + s_2 (1 \pm 1)) \left( s_1 (1 - s_2)^2 \sin^2(2\varphi) \left( s_1 (1 \mp 1) + s_2 (1 \pm 1) \right) + \bar{n}_c (s_1 - s_2) (1 \mp \sin(2\varphi)) (1 - s_2^2 \mp (1 + s_2^2) \cos(2\varphi)) \right) + \mathcal{O}(N^{-1}).$$

Here the signs $\mp$ correspond to the use of position and momentum quadratures, respectively.

Inspecting Eq. (35) reveals that the leading sensitivity in $F^{(s/p)}$ resembles the QFI of coherent states [34] around the optimal working points $\psi^{(s/p)}_{\text{opt}} = \mp \pi/4$. In other words, the combination of ideal homodyne detection and squeezed input resources with $s_1 \neq s_2$ can approach the QCRB for large interferometers, though it never saturates the QFI except in the strict coherent limit. On the other hand, if we use displaced single-mode squeezed states $s_1 = s_2 = s$, the ideal homodyne detection is never an optimal measurement scheme: the three last terms in the right-hand side of Eq. (35) never cancel each other if $0 < |\langle R \rangle|$ and $0 < s$.

For input resources with vanishing displacement, the optimal working point $\psi^{(s/p)}_{\text{opt}}$ is found by solving second order equations in the variable $y = \sin^2 \varphi \to \sqrt{y} \to \text{sin}^2 \varphi$. For instance, the condition to saturate the QCRB for a position quadrature measurement is

$$y^2 \alpha_N^2(s_1, s_2) + y \beta_N^2(s_1, s_2) + \delta_N^2(s_1, s_2) = 0,$$

with coefficients $\alpha_N^2$, $\beta_N^2$, and $\delta_N^2$ given by Eq. (C22).

In the particular situation of an homogeneous squeezing $s_1 = s_2 = e^{-2s'}$ with $s' \in \mathbb{R}$, the QCRB is saturated for

$$\cos \left( 2\psi^{(s/p)}_{\text{opt}} \right) = \pm \tanh(2s').$$

This coincides with the single-mode Gaussian state results, found with alternative methods based on the fidelity [39, 40, 42] or the SLD [34].

The subsidiary condition (36) proves that a quadrature detection is no longer optimal for a tensor product of zero-displacement states with $s_1 = s$ and $s_2 = 1$. We see this in the left panel of Fig. 2 which shows the real roots of (36) as a function of $N$ for two fixed squeezing values $s$. Note how these roots are always above or at most equal to 1 for all problem sizes $N$. Consequently, there is no value $\psi^{(s/p)}_{\text{opt}}$ for which the QCRB is saturated except for the single-mode Gaussian metrology setup $N = 1$. This observation is also confirmed by computing the roots in the limits of extreme squeezing in either position or momentum, i.e. $\lim_{s \to \infty} \sin^2 \varphi^{(s)} = N^2/(2N - 1)$. All these findings are consistent with results obtained in the single- and two-mode phase-estimation analysis based on the SLD [17, 25, 31, 54], for displaced squeezed states the SLD is a quadratic operator in terms of the quadrature operators $\hat{X}$, $\hat{P}$ (which means that the optimal measurement scheme is non-Gaussian), however it becomes linear when dealing with either coherent or squeezed-vacuum resources [34].

The FI is also plotted in Fig. 2 for the purpose of comparison. The central panel depicts this in terms of the interferometer size $N$ for a given homogeneous intensity $\bar{n}$, while the right panel illustrates it as a function of $\bar{n}$ at a fixed interferometer size $N = 100$. In summary, these figures outline the main conclusion from Eq. (35): that is, none of the non-entangled Gaussian states along with the QUMI architecture provide a better scaling than the SNL (see, the black solid line) in the finite energetic regime and for large interferometer sizes. In other words,
our analysis indicates that QUMI-based phase-estimation strategies provide no real advantage w.r.t. the resolution-energy tradeoff [10, 43].

B. Two-mode squeezed resources and polychromatic phase generator

Let us now study a metrology setup using two-mode non-degenerate squeezed states as resources. These states have been shown to overcome the SNL in estimation errors or phase sensitivities when using homodyne [20, 63], intensity [69], or parity measurements [70, 72]. The input state will be described by the first-moment vector \( \langle \mathbf{R} \rangle \in \mathbb{R}^4 \) and a CV matrix \([14, 19, 73]\):

\[
V = \begin{pmatrix}
\cosh 2s' & 0 & \sinh 2s' & 0 \\
0 & \cosh 2s' & 0 & -\sinh 2s' \\
\sinh 2s' & 0 & \cosh 2s' & 0 \\
0 & -\sinh 2s' & 0 & \cosh 2s'
\end{pmatrix}
\]

(38)

that depends on the squeezing parameter \( s' \in \mathbb{R}^+ \). Notice that one obtains the relation related to the coherent resources discussed previously for the choice \( s' = 0 \).

These states can be generated using the well-established procedure of pumping a non-degenerate optimal parametric amplifier (OPA) with a strong coherent beam, say at frequency \( 2\omega_0 \). These input photons are split into highly correlated pairs that conserve the total energy \( \omega_{1,2} = \omega_0 \pm \Omega \). Here \( \Omega < \omega_0 \) is a small modulation frequency that renders the photons distinguishable [45, 70, 71]. To the best of our knowledge, there is no previous treatment that studied the influence of such modulation from the metrological point of view (for instance, see Refs. [23, 63, 71]).

We will now go beyond previous phase-estimation analysis, addressing a polychromatic metrology scenario in which each port of the two-mode interferometric setup is fed with beams at two different frequencies. We label these modes with the annihilation operators \( \hat{a}_{\omega_1} \) and \( \hat{a}_{\omega_2} \), and consider that different frequencies may experience a different single-mode phase-shift, generated by

\[
\hat{H}_{\text{pol}}(\epsilon) = (1 + \epsilon)\hat{n}_{\omega_1} + (1 - \epsilon)\hat{n}_{\omega_2}.
\]

(39)

The parameter \(-1 \leq \epsilon \leq 1\) can be regarded as a frequency-dependent index of refraction or optical path, and \(\hat{n}_{\omega_i} = \hat{a}_{\omega_i}^\dagger \hat{a}_{\omega_i} \). As the total average energy \( \langle \hat{H}_{\text{pol}}(\epsilon) \rangle \) remains constant for distinct \( \epsilon \), we can compare the resolution-energy trade-off retrieved by polychromatic Gaussian phase-estimation scenarios. The choice (39) returns an extension of the phase-shift generator that is

\[
U_{\text{pol}}(\varphi, \epsilon) = U((1 + \epsilon)\varphi) \oplus U((1 - \epsilon)\varphi),
\]

(40)

which reduces to the conventional generator (A5) for the choices \( \epsilon = \pm 1 \). Further, we shall consider that the transformations \( L \) represents a beam splitter with transmissivity \( \tau \).

Returning to the phase space formalism, the polychromatic QFI can be expressed as follows,

\[
F_{\text{pol}}(\epsilon) = (1 + \epsilon)^2 F_1 + (1 - \epsilon)^2 F_2 + 4(1 - \epsilon^2)\left( \text{Tr}(V_2' V_{12}) + 2 \langle \mathbf{R}^T_1 \mathbf{V}_{12}' \mathbf{R}_2 \rangle \right),
\]

(41)

where \( F_1 = \mathcal{F}_2 = (1 + 4\tau(1 - \tau))\tilde{n}_s(\tilde{n}_s + 2) \) with \( \tilde{n}_s \) denoting the total average number of photons, whereas \( \langle \mathbf{R}^T_1 \rangle = (L \mathbf{R}_1) \), and \( V_{12}' = \text{diag}(1 - 2\tau)\sinh 2s', -(1 - 2\tau)\sinh 2s' \).

The left panel of Fig. 3 shows a log-log plot of the QFI in terms of the squeezing parameter, for distinct choices of the frequency modulation. The polychromatic QFI is larger than the monochromatic counterpart for sufficient high squeezing (1 \( \ll \) \( s' \)), and the highest sensitivity is obtained for \( \epsilon = 0 \). Interestingly, the sensitivity grows with the squeezing with an identical power for all values of \( \epsilon \), so that the polychromatic QFI may be approximately expressed as \( F_{\text{pol}}(\epsilon) \approx c(\epsilon, \tau)F_1 \) with \( c \) being a multiplicative enhancement independent of \( s' \). This factor is found to take values \( 2 \lesssim c \lesssim 10 \) for the available modulation frequencies and transmissivity, implying that a polychromatic setup can provide a significant improvement of the resolution-energy trade-off compared to the monochromatic MZI, e.g. \( F_{\text{pol}} \sim 10\tilde{n}_s^2 \) for \( 1 \ll s', \epsilon = 0 \) and \( \tau = 0 \).

The treatment about the FI presented in Sect. II C holds for very general phase generators beyond (44) and can be adapted to the polychromatic scenario. Going back to the general expression (B12) and replacing the phase-shift generator (40), we obtain a closed-form expression of the FI associated to the polychromatic strategy by following a similar procedure as to compute the expression (27) discussed in Sect. II C. The result is

\[
F_{\text{pol}}(\varphi, \epsilon) = F_{\text{pol}}(\epsilon) - F_{\text{Meas}}(\varphi, \epsilon) - 2(1 + \epsilon^2) + \frac{1}{2} \text{Tr}(V_1' V_2')
\]

(42)

\[
- 2(1 - \epsilon^2)\left( \text{Tr}(V_{12} V_{12}') + 3 \langle \mathbf{R}^T_1 \mathbf{V}_{12}' \mathbf{R}_2 \rangle \right),
\]

where \( F_{\text{Meas}}(\varphi, \epsilon) \) is obtained from (28) after substituting the CV matrix [38].

The central panel of Fig. 3 displays the deviation of the Fisher information from the Quantum limit \( F_{\text{pol}}(\varphi, \epsilon) - F_{\text{pol}}(\epsilon) \), in the case of position quadrature measurements, for a fixed unknown phase shift. As expected, the deviation is always negative or zero. However, it also remains close to zero for a growing squeezing around \( \epsilon \approx \pm 1/2 \). This indicates that an ideal N-mode quadrature detection may constitute an optimal measurement scheme. We can verify this for two-mode squeezed vacuum states and \( \epsilon = \pm 1 \). After a 50:50 beam splitter transformation (i.e. \( \tau = 1/2 \)), the probe system is in the ten-mode quadrature detection (\( i = 1 \)).

In agreement with the discussion in the previous section, we may expect to recover an identical relation for the operating point as Eq. (57). Indeed, after some manipulation Eq. (12) boils down to a simple algebraic ex-
pression in the argument $y = \cos(4\phi)$ (see Eqs. (C23) and (C24)), from which follows the subsidiary condition: 
\[ \cos \left( \frac{4\phi}{\cos(\epsilon)} \right) = \mp \epsilon/|\epsilon| \tanh(2s'). \] 
This is complementary to earlier findings for homodyne or intensity detection schemes combined with active interferometry [21, 51, 60].

The right panel in Fig. 3 also illustrates the saturation of the QCRB, as zeros of the difference $F_{\text{pol}}(\phi, \epsilon) - F_{\text{pol}}(\epsilon)$, for a strategy based on position measurements. Note how this deviation is an oscillating function of the phase, with an amplitude that grows with the squeezing parameter $\epsilon$. Unfortunately, the quadrature measurement is no longer optimal in the case of a vanishing modulation frequency $\epsilon = 0$, which is when the polychromatic scheme obtains the largest improvement over the conventional strategy. In this case, the optimal operating point is determined by an algebraic equation $J_0(\epsilon)(\cos(4\phi), s') = 0$ [cf. discussion around Eq. (C25) in App. C], as in preceding sections. The closed-form expressions for these roots given in Eq. (C28), shows that no value of squeezing $0 < s'$ can saturate the QCRB for a given phase shift $\phi$.

\section{Photons-analysis and nonunit-efficiency detection}

Finally, we address the degrading effects owning to the experimental imperfections, extending our treatment to include these in the analysis of the FI. In most interesting cases, the photon-loss process, determined by a given strength $\eta_{\text{loss}}$, and the nonunit-efficiency detection, designated by $\eta_{\text{eff}}$, can be regarded as the major limits to interferometric precision [17, 21, 23, 51, 58, 85]. Furthermore, it is customary to assume that the environmental noise and photon-loss mechanism act identically and independently upon each probe mode [29], as well as the environment is in a thermal state at a temperature determined by the mean photon number $n_{\text{th}}$. Under this considerations, the light interferometric propagation is modified in the presence of decoherence as $S(\phi)V S^T(\phi) \to \eta_{\text{loss}} S(\phi)V S^T(\phi)+(1-\eta_{\text{loss}})(1+n_{\text{th}})I_N$ [18, 73, 80, 81]. Combining this result with Eqs. (9) and (10), we directly obtain

\[ \langle \Lambda(\phi) \rangle = \sqrt{\eta_{\text{loss}}} S(\phi) \langle R \rangle , \]

\[ \sigma(\phi) = (1 - \eta_{\text{eff}} + (1 - \eta_{\text{loss}})(1 + n_{\text{th}})) I_N \]

\[ + \eta_{\text{eff}} \Sigma + \eta_{\text{loss}} S(\phi)V S^T(\phi) , \]

\[ \sigma_{\text{deco}}(\eta_{\text{loss}}, \eta_{\text{eff}}) \]

where the CV matrix $\sigma_{\text{deco}}$ solely regards photon-loss effects. By replacing (43) into the general equation (14) and doing some manipulation as illustrated in Sect. II C, we obtain a closed-form expression of the FI in presence of these decoherence effects, say $F_{\text{deco}}$, similar in structure to (18) (see Eq. (C34) in App. C). In the particular case we assume the propagation photon losses and nonunit efficiency contribute equally, i.e. $\eta_{\text{loss}} = \eta_{\text{eff}} = \eta$, the FI can be cast as follows,

\[ F_{\text{deco}}(\phi, \eta, n_{\text{th}}) = \eta^2 F(\phi) \]

\[ - \eta \langle R^T \rangle L^T P_{\phi}^{-1} \Sigma_{\text{deco}}(\eta, n_{\text{th}}) P_{\phi} L \langle R \rangle \]

\[ + \frac{1}{2} \text{Tr} \left( \partial_\phi \Sigma_{\text{deco}}(\eta, n_{\text{th}}) \partial_\phi \sigma_{\text{deco}}(\eta) \right) \]

\[ - (1 - \eta^2) \left( 2 + \frac{1}{2} \text{Tr} \left( \Sigma_{\phi} \langle S V S^T \rangle^2 \right) \right) , \]

where $\Sigma_{\text{deco}}(\eta, n_{\text{th}})$ is a $2N \times 2N$ real, symmetric matrix (given by Eq. (C35)) that fully contains the influence owning to the environmental thermal noise. Recall $F(\phi)$ denotes the FI in the ideal scenario.
TABLE I. Summary of the phase sensitivity retrieved by the distinct Gaussian interferometric phase-estimation strategies involving an ideal homodyne detection, expressed in terms of the interferometer size \(N\) and the input average photon number per mode \(\bar{n}\). The particular choice of the probe Gaussian state and the interferometric scheme are specified in the first two columns. The third column illustrate the scaling of the FI with respect to \(\bar{n}\) at a fixed value of the interferometer size \(N\). The fourth column represents instead the scaling in terms of \(N\) and for a given input intensity \(\bar{n}\), whilst the fifth column determines which strategies are enabled to attain the QCRB. Accordingly, the SNL-type scaling in terms of the input intensity must be understood as \(F \sim 4\bar{n}\), whereas the SNL-type scaling is similarly defined as \(F \sim 4N\) in terms of the interferometer size.

| Input resources | Interferometric transformation | Scaling per mode energy | Scaling with interferometer size | QCRB |
|-----------------|-------------------------------|-------------------------|-------------------------------|------|
| Coherent \((s_1 = s_2 = 1)\) | QUMI                          | SNL                     | SNL                           | Yes  |
| single-mode squeezed vacuum \((s_1 = s_2 = s)\) | Any                           | HL                      | Constant                      | Yes  |
| one-mode squeezed| QUMI                          | sub-SNL                 | sub-SNL                       | No   |
| coherent \(s_1 = e^{-2s'} \text{and } s_2 = 1\) | Any                           | sub-SNL                 | sub-SNL                       | No   |
| two-mode squeezed vacuum \((N = 2)\) | 50:50 beam splitter           | HL                      | -                             | Yes  |

\[ (\delta\phi)^2 = (4\bar{n})^{-1} \]

\[ (\delta\phi)^2 = (8\bar{n}(\bar{n} + 1))^{-1} \]

\[ (\delta\phi)^2 \approx ((4\bar{n})^2N)^{-1} \]

**IV. OUTLOOK AND CONCLUDING REMARKS**

In this work we have presented a theoretical framework to explore the metrological potential of generic Gaussian interferometric schemes accessible with current photonic technology. Our treatment proves convenient to address the optimal phase-estimation scheme and operating point: in particular, we recover the vast majority of previous well-known results in the single- and two-mode Gaussian metrology scenarios. In Table I we summarize the phase sensitivity provided by the choice of different input states and interferometric schemes in the finite energetic regime. To a large extent this table contains most of previous results related to Gaussian phase resolution in the absence of photon loss and for perfect detection schemes [17, 13, 34].

Interestingly, input coherent resources were shown to outperform the probe non-classical states used in previous QUMI-based phase-estimation proposals. Moreover, our analysis revealed that in the low-intensity regime (e.g. when squeezing parameter is small compared to the interferometer size \(N\)) the QUMI architecture along with probe single-mode squeezed states is unable to provide a real metrological advantage with respect to the best classical strategy for a large \(N\).

Additionally, we also developed a polychromatic version of the well-established MZI setup endowed with probe two-mode non-degenerate squeezed-vacuum states. We show that this setup can significantly improve the resolution-energy trade-off with optimal (ideal) quadrature measurements. Besides our treatment is a versatile approach to address the impact of experimental imperfections on the phase sensitivity unlike the analysis based on the complex SLD: e.g., we show that the optimal working point associated to coherent resources is significantly shifted by both the photon losses and the
nonunit-efficiency detection.

Remarkably, the recent developments on the fabrication and manipulation of integrated photonic circuits [36, 37, 82] makes them more resilient to phase stability, or photon losses and noise effects, which opens new avenues to implement higher sophisticated phase-estimation experiments with relatively little effort [35] (e.g. endowed with current photon sources and measurement detection schemes). In particular this prospect highlights the demand for further theoretical tools enable to explore its feasible metrological power. In this sense, the present treatment could render a valuable theoretical support to envisage a new series of experiments in the realm of quantum phase estimation.

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Appendix A: Basics of phase estimation

In this section we briefly sketch the derivation of the general expressions (14) and (17) by using results from matrix analysis theory [58, 59] and the matrix identities (2) and (7) just relying on the interested set of probe iso-thermal Gaussian states. We start from the formal definition of the FI, which reads [8, 9, 15]

$$F(\phi) = \int d^{2m} \lambda \frac{1}{p(\lambda|\phi)} \left( \frac{\partial p(\lambda|\phi)}{\partial \phi} \right)^2.$$  \hspace{1cm} (A1)

Thanks to the probability distribution characterizing the Gaussian phase-estimation scheme is a Gaussian function, the result of the integral involved in (A1) is a Gaussian function as well. This can be seen more clearly once computed the derivative of the probability distribution, i.e.,

$$\frac{\partial p(\lambda|\phi)}{\partial \phi} = \frac{1}{2} p(\lambda|\phi) \left( (\lambda - \langle \lambda \rangle)^T \sigma^{-1} \partial_\phi \sigma \sigma^{-1} (\lambda - \langle \lambda \rangle) + 2(\lambda - \langle \lambda \rangle)^T \sigma^{-1} \partial_\phi \langle \lambda \rangle) - Tr(\sigma^{-1} \partial_\phi \sigma) \right),$$  \hspace{1cm} (A2)

where the last term of the right-hand side appears due to the dependence of the probability distribution normalization-constant with the desired phase shift [58]. Hence, one may realize that the classical Fisher information (A1) reduces to carry out the integral of a quadratic polynomial (in the variable $\lambda$) weighted by $p(\lambda|\phi)$. After substituting Eq. (A2) in (A1), it is convenient to swap from $p(\lambda|\phi)$ to a zero-mean Gaussian probability distribution $p(\tilde{\lambda}|\phi)$, with the CV $\sigma$, by making the change of variables $\tilde{\lambda} = \sigma^{-1}(\lambda - \langle \lambda \rangle)$. Upon doing this, we obtain

$$F(\phi) = \frac{1}{4} \int d^{2m} \tilde{\lambda} p(\tilde{\lambda}|\phi) \left( Tr(\sigma^{-1} \partial_\phi \sigma) \right)^2 - 2Tr(\sigma^{-1} \partial_\phi \sigma)(\tilde{\lambda}^T \partial_\phi \sigma \tilde{\lambda} - 2\partial_\phi \langle \lambda^T \rangle \tilde{\lambda})$$

$$+ \left( (\tilde{\lambda}^T \partial_\phi \sigma \tilde{\lambda})(\tilde{\lambda}^T \partial_\phi \sigma \tilde{\lambda}) + 4(\partial_\phi \langle \lambda \rangle \tilde{\lambda})(\partial_\phi \langle \lambda \rangle \tilde{\lambda}) - 4(\partial_\phi \langle \lambda \rangle \tilde{\lambda})(\tilde{\lambda}^T \partial_\phi \sigma \tilde{\lambda}) \right).$$  \hspace{1cm} (A3)

Since $p(\tilde{\lambda}|\phi)$ is centered with respect to the origin $\tilde{\lambda} = 0$, the contribution coming from the linear and third-order terms in the right-hand side of (A3) must cancel. Additionally, the rest of the contributions can be readily computed by using standard results holding for multivariable Guassian integrals [58], e.g.

$$\int d^{2m} \lambda \ |\lambda|^4 \exp \left( -\frac{1}{2} (\lambda - \langle \lambda \rangle)^T \sigma^{-1} (\lambda - \langle \lambda \rangle) \right) = |\langle \lambda \rangle|^4 + 4 \langle \lambda^T \rangle \sigma \langle \lambda \rangle + Tr^2(\sigma) + 2Tr(\sigma)|\langle \lambda \rangle|^2 + 2Tr(\sigma \sigma^T),$$  \hspace{1cm} (A4)

with $|x|$ denoting the usual Euclidean norm of a $2m$-dimensional vector $x$. The obtained expression can be further manipulated by employing matrix identities (e.g. $\partial_\phi \sigma^{-1} = -\sigma^{-1} \partial_\phi \sigma \sigma^{-1}$), so we are left with the Eq. (14), as wanted.

Now we turn the attention to the formula (17) of the QFI by virtue of the phase generator $U_N(\phi) = U(\phi) \oplus I_{N-1}$ with

$$U(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.$$  \hspace{1cm} (A5)
This is readily worked out from the general expression of the QFI valid for any pure or mixed single-mode Gaussian state provided in [2,38,39]. The latter takes the following form for the set $G(1, n_t)$ of interesting states and the phase-shift generator $A\delta$.

$$F = \langle R_{1}^{T} \rangle \partial_{\phi} U^{T}(\phi)V_{1}\partial_{\phi} U(\phi)\langle R_{1} \rangle - \frac{1}{2(1 + (2n_t + 1)^{-2})} \text{Tr} \left( \partial_{\phi} (U^{T}(\phi)V_{1}^{-1}U(\phi)) (\partial_{\phi} (U^{T}(\phi)V_{1} U(\phi)) \right),$$

(A6)

By replacing the matrix identity [2], we obtain upon some straightforward manipulation

$$\begin{align*}
F &= \frac{1}{(2n_t + 1)^{2}} \langle R_{1}^{T} \rangle \partial_{\phi} U^{T}(\phi)V_{1}\partial_{\phi} U(\phi)\langle R_{1} \rangle + \frac{(2n_t + 1)^{-2}}{2(1 + (2n_t + 1)^{-2})} \text{Tr} \left( \partial_{\phi} (JU^{T}(\phi)V_{1} U(\phi)) \right) \\
&= \frac{1}{(2n_t + 1)^{2}} \left( \langle R_{1}^{T} \rangle \partial_{\phi} U^{T}(\phi)V_{1}\partial_{\phi} U(\phi)\langle R_{1} \rangle + \frac{1}{1 + (2n_t + 1)^{-2}} \text{Tr} \left( \partial_{\phi} (JU^{T}(\phi)V_{1} U(\phi)) \right) \right) \\
&- (2n_t + 1)^{2} \text{Tr} \left( \partial_{\phi} U^{T}(\phi)\partial_{\phi} U(\phi) \right),
\end{align*}$$

(A7)

which after substituting $\partial_{\phi} U(\phi) = JU(\phi)$ leads to the expression [17]. Notice that in the pure case (i.e. $n_t = 0$) the expression [17] identically coincides with the result independently obtained from the standard expression of the QFI $F = 4(\Delta \hat{H})^{2}$.

Appendix B: Gaussian phase estimation

In this appendix we extensively illustrate the derivation of Eq. (18) appearing in Sect. II C. We firstly write down the so-called Woodbury identity [58,59] to conveniently express the inverse of $\Sigma$ in terms of the CV matrix $\sigma$, 

$$\sigma^{-1} = \sigma_{S}^{-1} - \sigma_{S}^{-1} S_{SA} (V_{A}^{-1} + S_{SA}^{T} \sigma_{S}^{-1} S_{SA})^{-1} S_{SA}^{T} \sigma_{S}^{-1},$$

(B1)

which always holds as $V_{A}^{-1} + S_{SA}^{T} \sigma_{S}^{-1} S_{SA}$ is expected to be an invertible matrix. Notice that we have omitted the explicit dependence of the matrices with $\phi$ for seek of clarity. It is easy to see that the above identity allows us to separate the FI contribution in [14] that is completely independent of the ancilla CV matrix. This yields the expression [14]. Furthermore, since $\Sigma_{S}^{-1} + (S_{S}^{-1})^{-1} V_{S}^{-1} S_{S}^{-1}$ must be an invertible matrix as well (see remark 2.16.21 in [59]), we can repeat this procedure,

$$\sigma_{S}^{-1} = (S_{S}^{-1})^{-1} V_{S}^{-1} S_{S}^{-1} - (S_{S}^{-1})^{-1} V_{S}^{-1} S_{S}^{-1} (\Sigma_{S}^{-1} + (S_{S}^{-1})^{-1} V_{S}^{-1} S_{S}^{-1})^{-1} (S_{S}^{-1})^{-1} V_{S}^{-1} S_{S}^{-1},$$

(B2)

to obtain

$$F_{S}(\phi) = \langle R_{S}^{T} \rangle \partial_{\phi} (S_{S}^{T} S_{S} V_{S}^{-1} S_{S}^{-1}) \partial_{\phi} \langle R_{S} \rangle - \frac{1}{2} \text{Tr} \left( \partial_{\phi} \left( (S_{S}^{-1})^{-1} V_{S}^{-1} S_{S}^{-1} \right) \partial_{\phi} \left( S_{S} V_{S} S_{S}^{T} \right) \right) - F_{\text{Meas}}(\phi),$$

(B3)

where we have identified $F_{\text{Meas}}(\phi)$ as the residual contribution given by Eq. (19). We can further simplify (B3) by substituting the corresponding identities for the inverse of the sub-block matrices of $S$ (see proposition 2.8.7 in [59]), i.e.

$$S_{S}^{-1} = S_{S}^{T} - \Delta S_{S},$$

(B4)

$$S_{S}^{T}^{-1} = S_{S} - \Delta S_{S}^{T},$$

(B5)

with

$$\Delta S_{S} = S_{S}^{-1} S_{SA} (S / S_{S})^{-1} S_{AS} S_{S}^{-1},$$

(B6)

where $S / S_{S}$ stands for the Schur complement of $S_{S}$ in $S$, i.e. $S / S_{S} = S_{A} - S_{AS} S_{S}^{-1} S_{SA}$, which is non-singular for a realistic transformation $S$. By plugging the relations (B4) into (B3), after some tedious calculation we obtain

$$F_{S}(\phi) = \langle R_{S}^{T} \rangle \partial_{\phi} S_{S} V_{S}^{-1} S_{S}^{T} \partial_{\phi} \langle R_{S} \rangle - \frac{1}{2} \text{Tr} \left( \partial_{\phi} \left( S_{S} V_{S}^{-1} S_{S}^{T} \right) \partial_{\phi} \left( S_{S} V_{S} S_{S}^{T} \right) \right) + \tilde{F}_{\text{int}}(\phi) - F_{\text{Meas}}(\phi),$$

(B7)
where

\[ F_{\text{Int}}(\varphi) = F_{\text{Int}}(\varphi) - \Tr(\partial_\varphi S_S^T \partial_\varphi S S^{-1} S_S A S_S^T V_S), \]  

(B8)

and \( F_{\text{Int}}(\varphi) \) is given by

\[
F_{\text{Int}}(\varphi) = \Tr(\partial_\varphi S_S^T \partial_\varphi S S^{-1} S_S A S_S^T V_S) - 2 \left( R_S^T \right) \partial_\varphi S_S^T S_S V_S^{-1} \Delta S_S \partial_\varphi S \langle R_S \rangle \\
+ \left( R_S^T \right) \partial_\varphi S_S^T \Delta S_S^T V_S^{-1} \Delta S_S \partial_\varphi S \langle R_S \rangle + \frac{1}{2} \Tr \left( \partial_\varphi \left( 2 S_S V_S^{-1} \Delta S_S - \Delta S_S V_S^{-1} \Delta S_S \right) \partial_\varphi \left( S_S V_S S_S^T \right) \right).
\]

(B9)

Now one may substitute the inverse matrix \( V_S^{-1} \) in (B7) according to the symplectic-like identity (2), and further, use the relation (7) to cast the Eq. (B7) in the following form

\[ F_S(\varphi) = \frac{1}{(2\eta_t + 1)^2} \left( \langle R_S^T \rangle \partial_\varphi (J_m S_S)^T S_S V_S S_S^T \partial_\varphi (J_m S_S) \langle R_S \rangle + \frac{1}{2} \Tr \left( \left( \partial_\varphi (J_m S_S) V_S S_S^T \right)^2 \right) \right) \]

\[ + \tilde{F}_{\text{Int}}(\varphi) - F_{\text{Meas}}(\varphi). \]

(B10)

Let us now focus on the trace term of the above equation. Upon straightforward manipulation, this takes a simpler form, i.e.

\[
\Tr \left( \left( \partial_\varphi \left( J_m S_S V_S S_S^T \right) \right)^2 \right) = \Tr \left( \left( J_m \partial_\varphi S_S V_S S_S^T + J_m S_S V_S \partial_\varphi S_S^T \right)^2 \right) \\
= 2\Tr \left( \left( J_m \partial_\varphi S_S V_S S_S^T \right)^2 \right) + 2\Tr \left( \partial_\varphi S_S J_m V_S J_m S_S^T S_S V_S \partial_\varphi S_S^T \right) \\
= 2\Tr \left( \left( \partial_\varphi (J_m S_S) V_S S_S^T \right)^2 \right) - 2(2\eta_t + 1)^2 \left( \Tr \left( \partial_\varphi S_S^T \partial_\varphi S_S \right) \right) \\
- \Tr \left( \partial_\varphi S_S^T \partial_\varphi S_S V_S^{-1} S_S A S_S^T V_S \right),
\]

(B11)

where once again we have made use of the linear properties of the trace, as well as the identities (2) and (6). Replacing the result (B11) in the Eq. (B10) directly returns the expression

\[ F_S(\varphi) = \left( \langle R_S^T \rangle \partial_\varphi (J_m S_S)^T S_S V_S S_S^T \partial_\varphi (J_m S_S) \langle R_S \rangle + \frac{1}{2} \Tr \left( \left( \partial_\varphi (J_m S_S) V_S S_S^T \right)^2 \right) \right) \frac{1}{(2\eta_t + 1)^2} \]

\[ - \Tr \left( \partial_\varphi S_S^T \partial_\varphi S_S \right) - F_{\text{Meas}}(\varphi) + F_{\text{Int}}(\varphi), \]  

(B12)

after rearranging the contribution \( \Tr \left( \partial_\varphi S_S^T \partial_\varphi S_S V_S^{-1} S_S A S_S^T V_S \right) \) into the definition of \( F_{\text{Int}}(\varphi) \). One can proceed by noticing from (A5) that \( \partial_\varphi U(\varphi) = U(\varphi) \dot{J} \). The latter combined with the Eq. (5) directly yields

\[
\partial_\varphi S_S(\varphi) = P_\varphi J_m L_S,
\]

(B13)

where \( P_\varphi = U(\varphi) \oplus 0_{m-1} \), which is a \( 2m \times 2m \) projection matrix (i.e., \( P_\varphi P_\varphi^T = P_\varphi^T P_\varphi = I_1 \oplus 0_{m-1} \) as well as \( J_m^T P_\varphi J_m = P_\varphi \)). Here \( 0_{m-1} \) stands for the \( 2(\text{m-1}) \times 2(\text{m-1}) \) null matrix (i.e. all its entries are zero), so that the effect of \( P_\varphi \) through the subsequent computation is to drop the explicit dependence with the slice of matrix that is not supported by the phase space of the phase-shifted mode \( (\hat{q}_1, \hat{p}_1) \); for instance, for value \( \varphi = 0 \), \( (R_S) \) and \( V_S \) get projected into the displacement vector \( (R_1') \) and the CV matrix \( V_1' \) of the first probe mode immediately before undergoing the phase rotation, that is \( (R_1') = P_0 L_S (R_S) \) and \( V_1' = P_0 L_S V_S L_S^T P_0^T \). By virtue of the latter, after some straightforward manipulation once replaced Eq. (B13) in (B12), one gets

\[ F_S(\varphi) = \frac{1}{(2\eta_t + 1)^2} \left( \langle R_1'^T \rangle V_1'^T (R_1') + \Tr \left( V_1'^T V_1' \right) \right) - F_{\text{Meas}}(\varphi) + F_{\text{Int}}(\varphi) - \Tr \left( P_0 L_S L_S^T \right), \]

(B14)

from which is readily to identify the QFI characteristic of the \( m \)-mode probe system upon close inspection. By conveniently manipulating (B14) once plunged into \( \{23\} \), we arrive at the desired expression \( \{15\} \) for the FI.

In the particular case of non-assisted phase-estimation schemes and pure input Gaussian states, the expression \( \{18\} \) boils down to \( \{27\} \). In this scenario, the aforementioned auxiliary matrix \( \Sigma \) further becomes \( S V^{-1} S^T (\Sigma^{-1} + S V^{-1} S^T)^{-1} S V^{-1} S^T \), so that the measurement contribution \( F_{\text{Meas}} \), given by Eq. \( \{19\} \), can be substantially simplified.
as well. More specifically, by substituting this observation we obtain the first term in the right-hand side of (28), whereas the second terms may be further simplified by using the symplectic-like identities for $\Sigma$, $S(\varphi)$ and $V$ as before (as well as $\partial_\varphi A^{-1} = -A^{-1}\partial_\varphi AA^{-1}$),

$$
\begin{align*}
\text{Tr}
\left(
\partial_\varphi \Sigma \partial_\varphi \left(SVST^T\right)
\right)
&= \text{Tr}
\left(
\partial_\varphi \left(\Sigma^{-1} + SV^{-1}S^T\right)^{-1}SV^{-1}S^T \partial_\varphi \left(SVST^T\right)SV^{-1}S^T
\right) \\
&+ 2 \text{Tr}
\left(
\partial_\varphi \left(SV^{-1}S^T\right) \left(\Sigma^{-1} + SV^{-1}S^T\right)^{-1} \left(SV^{-1}S^T\right) \partial_\varphi \left(SVST^T\right)
\right) \\
&= -\text{Tr}
\left(J_N^T \partial_\varphi \left(\Sigma + SVST^T\right) \partial_\varphi \left(SVST^T\right)J_N \partial_\varphi \left(SVST^T\right)\right) \\
&+ 2 \text{Tr}
\left(\left(\Sigma^{-1} + SV^{-1}S^T\right)^{-1} J_N \left(SVST^T\right)J_N^T \partial_\varphi \left(SVST^T\right)J_N^T \partial_\varphi \left(SVST^T\right)\right) \\
&= \text{Tr}
\left(\left(\Sigma + SVST^T\right)^{-1} \partial_\varphi \left(SVST^T\right)\left(\Sigma + SVST^T\right)^{-1} \partial_\varphi \left(SVST^T\right)\right) \\
&- 2 \text{Tr}
\left(\left(\Sigma + SVST^T\right)^{-1} \partial_\varphi \left(SVST^T\right)\partial_\varphi \left(J_NSVST^T\right)\partial_\varphi \left(J_NSVST^T\right)\right),
\end{align*}
$$

where once again we have employed the linearity properties of the trace and $J_N = -J_N^T$. By substituting (B15) in (19), it is clear to see that we arrive at the desired expression (28) for the measurement contribution.

**Appendix C: N-mode homodyne detection without ancilla system**

1. **Explicit expressions from Sect. III A**

In this section we provide the explicit form corresponding to the QUMI transformation introduced in Sect. III A, as well as the functions and matrices involved in the expressions from (31) to (36) appearing in Sect. III B. Formally, the unitary evolution describing the QUMI is for which the transformation between the mode 1 and $j$ is determined by the transmissivity amplitude $\tau_j = 1/N$. More specifically, the associated orthogonal matrix, say $L_{\text{QUMI}}$, takes the form

$$
L_{\text{QUMI}} = \begin{pmatrix}
\sqrt{\frac{1}{N}} & 0 & \sqrt{\frac{1}{N}} & 0 & \sqrt{\frac{1}{N}} & 0 & \cdots & 0 \\
0 & \sqrt{\frac{1}{N}} & 0 & \sqrt{\frac{1}{N}} & 0 & \sqrt{\frac{1}{N}} & \cdots & 0 \\
-\sqrt{\frac{N-1}{N}} & 0 & \sqrt{\frac{1}{N(N-1)}} & 0 & \sqrt{\frac{1}{N(N-1)}} & 0 & \cdots & 0 \\
0 & -\sqrt{\frac{N-1}{N}} & 0 & \sqrt{\frac{1}{N(N-1)}} & 0 & \sqrt{\frac{1}{N(N-1)}} & \cdots & 0 \\
0 & 0 & -\sqrt{\frac{N-2}{N-1}} & 0 & \sqrt{\frac{1}{(N-1)(N-2)}} & 0 & \cdots & 0 \\
0 & 0 & 0 & -\sqrt{\frac{N-2}{N-1}} & 0 & \sqrt{\frac{1}{(N-1)(N-2)}} & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & -\sqrt{2} & 0 & \sqrt{\frac{1}{2}} \\
0 & 0 & 0 & 0 & \cdots & 0 & -\sqrt{\frac{1}{2}} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \sqrt{\frac{1}{2}}
\end{pmatrix}, \quad (C1)
$$

By computing the Eq. (31) once replaced (C1) for different small values of $N$, an induction procedure for greater $N$ reveals that

$$
\Omega_N(\varphi, s_1, s_2) = \begin{pmatrix}
d_1 & c_1 & c_2 & \frac{c_1}{s_1s_2} \\
c_1 & d_2 & c_3 & -\frac{c_2}{s_1s_2} \\
c_2 & c_3 & d_3 & 0 \\
\frac{c_3}{s_1s_2} & -\frac{c_2}{s_1s_2} & 0 & d_4
\end{pmatrix}, \quad (C2)
$$
whose diagonal entries are determined by
\[
\begin{align*}
    d_1 &= \frac{1}{s_1 s_2 N} (s_1 s_2 a_N(s_2, s_1) \cos^2 \varphi + a_N(s_1, s_2) \sin^2 \varphi), \\
    d_2 &= \frac{1}{s_1 s_2 N} (a_N(s_1, s_2) \cos^2 \varphi + s_1 s_2 a_N(s_2, s_1) \sin^2 \varphi), \\
    d_3 &= \frac{a_N(s_1, s_2)}{N}, \\
    d_4 &= \frac{a_N(s_2, s_1)}{s_1 s_2 N},
\end{align*}
\]  

(C3)  

(C4)  

(C5)  

(C6)

whereas the non-diagonal elements are given by,
\[
\begin{align*}
    c_1 &= \frac{a_N(s_1, s_2) - s_1 s_2 a_N(s_2, s_1)}{2 s_1 s_2 N} \sin(2 \varphi), \\
    c_2 &= \frac{(s_2 - s_1) \sqrt{N - 1}}{N} \cos \varphi, \\
    c_3 &= \frac{(s_1 - s_2) \sqrt{N - 1}}{N} \sin \varphi.
\end{align*}
\]

(C7)  

(C8)  

(C9)

On the other side, by replacing the generic form (C2) in (32) and using results borrowed from matrix analysis to compute the Moore-Penrose pseudoinverse 53, one obtains the auxiliary matrix (32) with
\[
A^{(z)} = 
\begin{pmatrix}
    d_{11} & 0 & -d_{12} & 0 \\
    0 & 0 & 0 & 0 \\
    -d_{21} & 0 & d_{22} & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix},
\]

(C10)  

whereas for the momentum quadrature measurement
\[
A^{(p)} = 
\begin{pmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{pmatrix}
\]

(C11)

which reduces to the expected results \(A^{(z)} = \text{diag}(1, 0, 0, 0)\) or \(A^{(p)} = \text{diag}(0, 1, 0, 1)\) when the initial squeezing vanishes \(i\) i.e. \(s_1 = s_2 = 1\).

Moreover, by substituting (C2) in (27), one obtains the expression \(33\) for the FI for position quadrature measurement after a long tedious calculation where we have introduced
\[
\begin{align*}
    (W^{(z)}_N)_{11} &= -\frac{i N s_1 s_2}{2 N s_1 s_2 \cot(\varphi) + 2i((N - 1)s_1 + s_2)} + \frac{i N s_1 s_2 \sin(\varphi)}{2 N s_1 s_2 \cos(\varphi) - 2i \sin(\varphi)((N - 1)s_1 + s_2)} + \frac{s_1 - s_2}{N}, \\
    (W^{(z)}_N)_{22} &= \frac{((N - 1)s_1 + s_2)^3}{N^3 s_1^3 s_2^3 \cot^2(\varphi) + N s_1 s_2 ((N - 1)s_1 + s_2)^2}, \\
    (W^{(z)}_N)_{12} &= (W^{(z)}_N)_{21} = \frac{N s_1 s_2 \sin(2\varphi)((N - 1)s_1 + s_2)}{2 N^2 s_1^2 s_2^2 \cos^2(\varphi) + 2 \sin^2(\varphi)((N - 1)s_1 + s_2)^2},
\end{align*}
\]

(C12)  

(C13)  

(C14)

or for the momentum quadrature measurement
\[
\begin{align*}
    (W^{(p)}_N)_{11} &= \frac{((N - 1)s_2 + s_1)^3}{N^3 \cot^2(\varphi) + N((N - 1)s_2 + s_1)^2}, \\
    (W^{(p)}_N)_{22} &= -\frac{1}{N^2 \cot^2(\varphi) + ((N - 1)s_2 + s_1)^2} + \frac{1}{N} - \frac{1}{s_2}, \\
    (W^{(p)}_N)_{12} &= (W^{(p)}_N)_{21} = -\frac{N \sin(2\varphi)((N - 1)s_2 + s_1)}{2 (N^2 \cos^2(\varphi) + \sin^2(\varphi)((N - 1)s_2 + s_1)^2)},
\end{align*}
\]

(C15)  

(C16)  

(C17)
as well as, the auxiliary functions determining the influence of the second-moment resources,

\[
\begin{aligned}
f_N^{(2)}(\sin^2 \varphi, s_1, s_2) &= \left( (N-1)s_1 s_2^2 (N^2 s_1^2 + 1) + 2s_1^2 (N^2((N-1)N + 1)s_1^2 - N(N + 3) + 2) + 1 \right) \\
&- 2(N-1)^2s_1^2 s_2^2 ((2N^2 + N - 2) s_1^2 - 6) + 2(N-1)^4s_1^4 \\
&+ (N-1)s_1 s_2^3 \left( s_1^2 (N (N s_1^2 - 7) + 6) + 8 \right) \\
&+ \cos(2\varphi)(N s_1 (s_2 - 1) + s_1 - s_2) (2s_2^2 ((N-1)N + 1)s_1^2 - 1) + (N-1)s_1 (s_1^2 - 4) s_2 \\
&- 2(N-1)^2 s_1^2 + (N-1)s_1 s_2^2) (s_1 s_2 + N - 1) + s_2) + (N-1)^3s_1^3 (s_1^2 + 8) s_2^2 \sin^2 \varphi \right) \frac{1}{2y_s^2(\varphi)},
\end{aligned}
\]

or

\[
\begin{aligned}
f_N^{(2)}(\sin^2 \varphi, s_1, s_2) &= \left( 2N^4s_1 s_2 (s_2^2 - 1)^2 + N^3(s_1 - s_2) (8s_1 s_2^4 - 7s_1 s_2^2 + s_1 - s_2^3 - s_2) \\
&+ \cos(2\varphi)(N s_1 (s_2 - 1) + s_1 - s_2) ((N-1)s_2 + N + s_1) (2s_1 s_2 (-N^2 + N + s_1^2 - 1) \\
&+ (N-1) (4s_1^2 - 1) s_2^2 + (1 - N)s_1^2 + 2(N-1)^2s_1 s_2^2 + N^2(s_1 - s_2)^2 (12s_1 s_2^2 - 2s_1 s_2 - 3s_2^2 - 1) \\
&+ N(s_1 - s_2)^3 (8s_1 s_2^2 + s_1 - 3s_2) + (s_1 - s_2^4 (2s_1 s_2 - 1)) \right) \sin^2 \varphi \frac{1}{2s_1 s_2 y_s(\varphi)},
\end{aligned}
\]

with \(y_s(\varphi) = (N s_1 s_2)^2 \cos^2 \varphi + a_2^2 N(s_1, s_2) \sin^2 \varphi \) and \(y_p(\varphi) = (N^2 \cos^2 \varphi + a_2^2 N(s_2, s_1) \sin^2 \varphi \).

When addressing the optimal working point, as stated in the main text (by demanding the second and third terms in the right-hand side of Eq. (C3) cancel) we find out the second-order polynomial \(R(\varphi)\) with real coefficients given by,

\[
\begin{aligned}
\alpha_N^{(2)}(s_1, s_2) &= -2N^2 s_1^2 s_2^2 ((N-1)s_1 + s_2)^2 \left( 2N^2 - \left( \frac{N-1}{s_2} + \frac{1}{s_1} \right)^2 - ((N-1)s_2 + s_1)^2 \right) \\
&- 2N^2((N-1)s_1 (s_2 - 1) + s_1 - s_2) (2s_2^2 ((N-1)N + 1)s_1^2 - 1) + (N-1)s_1 (s_1^2 - 4) s_2 - 2(N-1)^2 s_1^2 \\
&+ (N-1)s_1 s_2^2) (s_1 s_2 + N - 1) + s_2) + ((N-1)s_1 + s_2)^4 \left( 2N^2 - \left( \frac{N-1}{s_2} + \frac{1}{s_1} \right)^2 - ((N-1)s_2 + s_1)^2 \right) \\
&+ N^4 s_1 s_2^4 \left( 2N^2 - \left( \frac{N-1}{s_2} + \frac{1}{s_1} \right)^2 - ((N-1)s_2 + s_1)^2 \right),
\end{aligned}
\]

\[
\begin{aligned}
\beta_N^{(2)}(s_1, s_2) &= 2N^2 ((N-1)^2 s_1^2 s_2^2 (N^2 s_1^2 - 1) - (N-1)s_1 s_2^3 (2N^2 s_1^2 + (N-2)^2 s_1^4 - 4) \\
&- (N-1)^2 s_1^2 s_2^2 (N^2 s_1^2 + s_1^2 + 4) + s_1^2 (N^2 s_1^2 - N^2 s_1^2 - (N(N-2)N + 8) + 12) + 6) s_1^4 + 1) \\
&+ (N-1)^4 s_1^4 + 4(N-1)^3 s_1^3 s_2 + (N-1)^3 s_1^3 s_2^2 \left( \frac{N}{2N^2 - 1} + 4 - 4 \right),
\end{aligned}
\]

\[
\begin{aligned}
\delta_N^{(2)}(s_1, s_2) &= N^4 s_1^2 s_2^4 \left( 2N^2 - \left( \frac{N-1}{s_2} + \frac{1}{s_1} \right)^2 - ((N-1)s_2 + s_1)^2 \right).
\end{aligned}
\]

2. Explicit expressions from Sec.III B

Now we turn the attention to the polychromatic scenario described in Sect. III B As stated in the discussion about the Fisher information, after some readily manipulation one can show that for the choices \(c = \pm 1\) the expression (12) boils down to

\[
F^{(p)}(\varphi, c) = F^{(p)}(\varphi) - \frac{2(\frac{\sin(\varphi) \sinh(2\varphi)}{\cosh(2\varphi) + \sin(\varphi) \cosh(2\varphi)})^2}{\left( \frac{\cosh(2\varphi) - \sin(\varphi) \cosh(2\varphi)}{\cosh(2\varphi) + \sin(\varphi) \cosh(2\varphi)} \right)^2},
\]

for a measurement quadrature in position, or

\[
F^{(p)}(\varphi, c) = F^{(p)}(\varphi) - \frac{2(\frac{\sin(\varphi) \sinh(2\varphi)}{\cosh(2\varphi) - \sin(\varphi) \cosh(2\varphi)})^2}{\left( \frac{\cosh(2\varphi) + \sin(\varphi) \cosh(2\varphi)}{\cosh(2\varphi) - \sin(\varphi) \cosh(2\varphi)} \right)^2},
\]

for a measurement quadrature in momentum. By paying attention to Eqs. (C23) and (C24), it is clear that the optimal operating point is obtained by demanding the numerator of the second term in the right-hand side cancels.
Upon doing this, one straightforwardly arrives to the relation determining the optimal angle for the choice $\epsilon = \pm 1$. For the most general case of modulation frequency (i.e. $\epsilon \neq \pm 1$), one obtains the following subsidiary condition from a perturbative analysis

$$(3 + \cosh(4s') - 2\cos(4\phi)\sinh^2(2s'))f_0^{(x)}(\cos(4\phi), s') + \epsilon f_1^{(x)}(\phi, s') \approx 0,$$  \hspace{1cm} (C25)

where we have introduced the auxiliary functions

$$f_0^{(x)}(\cos(4\phi), s') = \sinh^2(2r) \left(-2\sinh^2(2r)\cos(4\phi) + \cosh(4r) + 3\right) \left(2\sinh^2(4r) \left(2\cosh^2(4\phi) - 1\right) - 4(\cosh(8r) - 9)\cos(4\phi) + 3\cosh(8r) + 29\right),$$

$$f_1^{(x)}(\phi, s') = 16\sinh^3(2r) \left(2\phi\sinh(2r)\cosh(4r)\sin(10\phi) + \cosh(6r)(-14\phi\sin(2\phi) + 3\phi\sin(6\phi) - 3\cosh(6\phi)) + \cosh(2r)(4(\cosh(4r) + 3)\cos(2\phi) + 16\sinh^2(r)\cosh^2(r)\cos(10\phi) - 2\cosh(2\phi) + 45\phi\sin(6\phi) - 13\cos(6\phi))\right),$$ \hspace{1cm} (C26)

Clearly, from the Eq. (C26) follows that in the particular case $\epsilon = 0$ the optimal operating point $\varphi_{opt}^{(x)}$ is figure out from solving the second-order polynomial $f_0^{(x)}(y, s') = 0$ with argument understood as $y = \cos(4\phi)$. Doing this, one directly obtains

$$y = \frac{1}{2} \left(\cosh(8r) \pm 4\sqrt{6 - 2\cosh(8r) - 9}\right)\cosh^2(4r),$$ \hspace{1cm} (C28)

which is greater than the unit except for the choice of coherent resources $s' = 0$ when one of the roots becomes $x \to -1$, retrieving in turn the same result $\varphi_{opt}^{(x)} = -\pi/4$ as previously obtained in Sect. III A as expected.

3. Explicit expressions from Sec. III C

In this appendix, we briefly illustrate the derivation of Eqs. (43) and (45) appearing in Sect. III C. Firts, the nonunit efficiency of a single-mode homodyne measurement mainly resides in the use of photon-detectors suffering from a limited resolution $\eta_{eff} \in [0, 1]$, which results in a vacuum noise contribution proportional to $\sqrt{1 - \eta_{eff}}$ in the measurement outcomes, i.e.

$$|\lambda| = \frac{1}{2} \left(\sqrt{\frac{1+r}{2}}q' + \sqrt{\frac{1+r}{2}}p'\right) + \sqrt{1 - \eta_{eff}(q_{vac}, p_{vac})}.$$ \hspace{1cm} (C29)

Without loss of generality, this source of noise may be well approximated by the combination of an ideal Gaussian detector (described by the CV matrix $\Sigma$) preceding by a beam splitter with transmission coefficient identical to the photon-detector resolution factor, where the probe mode would fictiously interfere with an input vacuum beam representing $(q_{vac}, p_{vac})$. In our framework, this corresponds to take the CV matrix determining the non-ideal Gaussian measurement scheme as

$$\Sigma = \eta_{eff}\Sigma + (1 - \eta_{eff})I_N,$$ \hspace{1cm} (C30)

which returns the lossless homodyne detection scenario for $\eta_{eff} = 1$. On the other side, decoherence effects of the probe $N$-mode system taking place during the light field propagation through the interferometer can be straightforwardly formulated in terms of the interaction with an environment modelled by a continuum of oscillators [81]. When the system-environment interaction is essentially linear, the time evolution of our probe $N$-mode system is governed by the Fokker-Plank (or diffusion) equation expressed in the interaction picture [80],

$$\frac{\partial W(R, t)}{\partial t} = \left(\frac{\partial}{\partial R}\right)^T \Gamma(t)R + \left(\frac{\partial}{\partial R}\right)^T D \left(\frac{\partial}{\partial R}\right)W(R, t),$$ \hspace{1cm} (C31)

with $(\frac{\partial}{\partial R})^T = \bigoplus_{i=1}^N \left(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i}\right)$; $\Gamma(t)$ and $D$ are $2N \times 2N$ real, symmetric matrices that essentially encrypt the photon-losses and thermal noise effects, respectively. In the interesting dissipative scenario the above matrices take
the following simple form
\[
\Gamma(t) = \frac{\gamma(t)}{2} \sum_{i=1}^{N} \mathbf{I}_1, \\
D = \frac{\gamma(t)(1 + 2n_{th})}{4} \sum_{i=1}^{N} \mathbf{I}_1,
\]
(C32)
(C33)
where \(\gamma(t)\) is the usual dissipative coefficient. Equation (C31) is a linear Fokker-Plank equation that can be fairly straightforwardly solved by using the Green function method \[80\]. Furthermore, thanks to the diagonal form of the above dissipative and noise matrices, the decoherence evolution commutes with the phase shift rotation \[18, 34, 75\], and we obtain the result \[13\]. Substituting this in the Eq. (27), and following a similar procedure as to compute the expression \[23\], we obtain
\[
F_{\text{deco}}(\varphi, \eta_{\text{loss}}, \eta_{\text{eff}}, n_{th}) = F(\varphi, \eta_{\text{loss}}, \eta_{\text{eff}}) - \eta_{\text{loss}} \langle R^T \rangle \mathbf{L}^T \mathbf{P}_\varphi^T \Sigma_{\text{deco}}^{-1} \mathbf{P}_\varphi \mathbf{L}(R) + \frac{1}{2} \text{Tr}(\partial_\varphi \Sigma_{\text{deco}}^{-1} \partial_\varphi \Sigma_{\text{deco}}),
\]
with
\[
\Sigma_{\text{deco}}(\eta_{\text{loss}}, \eta_{\text{eff}}, n_{th}) = \sigma_{\text{deco}} \left( (1 - \eta_{\text{eff}} + (1 - \eta_{\text{loss}})(1 + n_{th}) \right)^{-1} \mathbf{I}_N + \sigma_{\text{deco}}^{-1} \right) \sigma_{\text{deco}}^{-1},
\]
(C34)
(C35)
where \(F(\varphi, \eta_{\text{loss}}, \eta_{\text{eff}})\) comprises the Fisher information obtained from the noiseless expression \[27\] after substituting \(S(\varphi) \rightarrow \sqrt{\eta_{\text{loss}}}S(\varphi)\), and \(\Sigma \rightarrow \eta_{\text{eff}} \Sigma\). It is worthwhile to realize that the second contribution in the right-hand side of \(\Sigma_{\text{deco}}\) will always be negative for any \(\varphi \in \mathbb{R}\) and \(\langle R \rangle \in \mathbb{R}^{2N}\), since the CV matrix \(\Sigma_{\text{deco}}^{-1}\) is positive-semidefinite by construction, and further, it asymptotically cancels in the limit of an ideal phase-estimation scenario (i.e. \(F_{\text{deco}} \rightarrow F\) when \(\eta_{\text{eff}}, \eta_{\text{loss}} \rightarrow 1\), as expected. Notice that the corresponding QFI is formally obtained from Eq. (17) (with \(n_t = 0\) by replacing \(\langle R' \rangle \rightarrow \sqrt{\eta_{\text{loss}}} \mathbf{P}_\varphi \mathbf{L}(R)\), and \(V_1 \rightarrow \mathbf{P}_0 (\eta_{\text{loss}} \mathbf{L} \mathbf{V}_1 \mathbf{L}^T + (1 - \eta_{\text{loss}})(1 + n_{th}) \mathbf{I}_N) \mathbf{P}_0^T\).

In particular, in the dissipative scenario \(\eta_{\text{eff}} = \eta_{\text{loss}} = \eta\), we find the FI for the previously-studied coherent resources and homodyne detection, i.e.
\[
F^{(\varphi, \eta, n_{th})}(\varphi, \eta, n_{th}) = 2\bar{n}_c N (1 \mp \sin(2\varphi)) \left( \frac{2 + n_{th}(1 - \eta)}{\eta(1 - \eta)n_{th} - 2} \right) = 2\bar{n}_c N (1 \mp \sin(2\varphi)).
\]
(C36)
By comparing with the ideal result \[26\] for \(F\), it is clear from the above equation \(\text{(C36)}\) that the “optimal” working point defined in Sect. IIIC, is obtained from demanding \(\sin(2\varphi^{(\varphi, \eta)}) = \mp(2(\eta/\bar{n}_c)^2 - 1)\), which returns a result that substantially differs from the ideal case (i.e. \(\varphi = \mp\pi/4\)).

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