Max-min dependence coefficients for Multivariate Extreme Value Distributions

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Abstract: We measure the dependence among sub vectors of a random vector with Multivariate Extreme Value distribution by using the expected value of a range and relate this coefficient of dependence with the multivariate tail dependence and extremal coefficients. The introduced coefficient extends the concept of madogram for several locations and several regions. The results are illustrated with some usual distributions and applied to financial data.

Keywords: multivariate extreme value theory, dependence coefficients, range

1 Introduction

The dependence structure of a Multivariate Extreme Value (MEV) distribution is completely characterised by its dependence function (Resnick (1987), Beirlant et al. (2004)). Since this function cannot be easily inferred from data the dependence coefficients are useful, despite the fact that one coefficient cannot preserve all the information about this function.

The most popular of the dependence coefficients are those based on the tail dependence (Sybyuya (1960), Li (2009)). They summarise the probability of occurrence of extreme values for one or more random variable given that another(s) assumes extreme values too. For the MEV distributions the extremal coefficient (Tiago de Oliveira (1962-63), Smith (1990)) is certainly a crucial and perhaps insurmountable tool when we have to summarise the dependence. For a $d$-dimensional random vector we have $2^d - d$ extremal coefficients which consistency properties are discussed in Schlather and Tawn (2002). For an overview of other dependence measures for dependence see, for instance, Joe (1997).

To the best of our knowledge, there is no extremal dependence coefficients for $p \geq 3$ subvectors $X_1, \ldots, X_p$ of a random vector $X = (X_1, \ldots, X_d)$ with MEV distribution. The need of to evaluate the strength of dependence among sub vectors arises for instance in the setting of max-stable random fields. Let $\{X_i\}_{i \in \mathbb{R}^2}$ be a max-stable random field and $I_1, \ldots, I_p$ sets of locations in $\mathbb{R}^2$. The joint distribution of $X_i, i \in \bigcup_{j=1}^p I_j$, is a MEV distribution and we want to resume the dependence among the grouped values $\{X_i, i \in I_j\}$ at different regions $I_j, j = 1, \ldots, p$. This problem is treated by several authors for two variables $X_1$ and $X_2$ corresponding to two locations $i$ and $j$ (see Naveau et al. (2009) and references therein) and the obtained results are extended for two regions $I_1$ and $I_2$ in Fonseca et al. (2012).

In finance, we are frequently interested in assessing the dependence among several big world markets, considering each one as a random subvector. For an application with grouped financial stock markets see for instance Ferreira and Ferreira (2012a).

We propose to evaluate the degree of dependence among sub-vectors $X_1, \ldots, X_p$ of $X$ with MEV distribution by using an expected range, which will be referred as a "max-min coefficient". It is a summary measure that takes into account the whole group of the extremal coefficients $\epsilon_{X_j}$ of $X_j, j = 1, \ldots, p$. Our approach is an extension of the modeling for pairwise dependence throughout the madogram (Poncet et al. (2006), Naveau et al. (2009)), an extreme-value analogue of the variogram (Cressie (1993)), since it enables to resume the spatial dependence structure for several locations or regions of locations.

The proposed moment-based dependence tool takes into account the spread and dependence among the subvectors and can be easily estimated.
The paper is organized as follows. We introduce in section 2 the dependence coefficient whose is well defined for any random vector with MEV distribution, is a function of its copula and is invariant with respect permutations of the variables. Its relations with the multivariate tail dependence and the extremal coefficients are presented. Based on the expected range coefficient considered we compare a MEV distribution with others more concordant distributions and state some bounds. In section 3, we compute the max-min coefficients for the marginal distribution of the Multivariate Maxima of Moving Maxima process and the Symmetric Logistic distribution. We refer briefly an estimator for the max-min coefficients and apply it to grouped financial stock markets.

2 Max-min dependence coefficients

Let \( \mathbf{X} = (X_1, ..., X_d) \) be a vector of unit Fréchet random variables, that is, with marginal distribution function \( F(x) = \exp(-x^{-1}) \), \( x > 0 \), and \( G \) denote the Multivariate Extreme Value distribution of \( \mathbf{X} \).

The tail dependence function (Huang (1992), Schmidt and Stadtmuller (2006)) of \( G \) is defined by

\[
l(x_1, ..., x_d) = \lim_{t \to \infty} \left( -t \log P \left( F(X_1) \leq 1 - \frac{x_1}{t}, ..., F(X_d) \leq 1 - \frac{x_d}{t} \right) \right) = \lim_{t \to \infty} t P \left( F(X_1) > 1 - \frac{x_1}{t} \lor ... \lor F(X_d) > 1 - \frac{x_d}{t} \right)
\]

and, in particular,

\[
l(x, ..., x) = x(1, ..., 1) = x \lim_{t \to \infty} P \left( \bigvee_{j=1}^{d} F(X_j) > 1 - \frac{1}{t} \mid F(X_i) > 1 - \frac{1}{t} \right).
\]

For \( \delta_i(S) = 1 \) if \( i \in S \) and \( \delta_i(S) = 0 \) if \( i \notin S \), it holds that

\[
l(\delta_1(S), ..., \delta_d(S)) = \epsilon_{\mathbf{X}_S}, \tag{1}
\]

where \( \epsilon_{\mathbf{X}_S} \) is the extremal coefficient of the subvector \( \mathbf{X}_S \) of \( \mathbf{X} \) with indices in \( S \) (Tiago de Oliveira (1962-63), Smith (1990)). It takes values in \([1, \vert S \vert]\), with \( \epsilon_{\mathbf{X}_S} = 1 \) when \( \mathbf{X}_S \) has the minimum copula \( C_{\mathbf{X}_S}(u_1, ..., u_d)_S = \bigwedge_{j \in S} u_j \) and \( \epsilon_{\mathbf{X}_S} = \vert S \vert \) when \( \mathbf{X}_S \) has the product copula \( C_{\mathbf{X}_S}(u_1, ..., u_d)_S = \prod_{j \in S} u_j \).

Let \( \mathcal{I} = \{I_1, ..., I_d\} \) be a partition of \( D = \{1, ..., d\} \), \( M(I_j) = \bigvee_{i \in I_j} X_i \) and \( \mathbf{X}_{I_j} \) the subvector of \( \mathbf{X} \) with indices in \( I_j \). We resume the extremal dependence among \( \mathbf{X}_{I_j} \), \( j = 1, ..., p \), by the coefficient \( R(\mathbf{X}, \lambda, \mathcal{I}) \) defined as follows.
Definition 2.1. Let $X$ be a vector of unit Fréchet random variables and Multivariate Extrem Value distribution. For each $\lambda = (\lambda_1, ..., \lambda_p) \in (0, \infty)^p$ and partition $I$ of $D$, we define

$$R(X, \lambda, I) = E \left( \bigvee_{j=1}^{p} F^{\lambda_j} (M(I_j)) - \bigwedge_{j=1}^{p} F^{\lambda_j} (M(I_j)) \right).$$

We remark the relations

$$R(X, \lambda, I) = E \left( \bigvee_{\{i,j\} \subset \{1, ..., p\}} |F^{\lambda_i} (M(I_j)) - F^{\lambda_j} (M(I_i))| \right) \quad (2)$$

and

$$R(X, \lambda, I) = E \left( \bigvee_{j=1}^{p} \bigvee_{i \in I_j} F^{\lambda_i} (X_i) - \bigwedge_{j=1}^{p} \bigvee_{i \in I_j} F^{\lambda_i} (X_i) \right). \quad (3)$$

By taking $p = 2$ in (2), we find in $\frac{1}{2} R(X, \lambda, I)$ the generalized madogram introduced in Fonseca et al. (2012), which in turns is the the $\lambda$-madogram (Naveau et al. (2009)) when $d = 2 = p$ and $1 - \lambda_2 = \lambda_1 \in (0, 1)$.

The max-min coefficient and the generalized madograms for pairs of sets $I_i$ and $I_j$ can be related throughout

$$R(X, \lambda, I) \geq \bigvee_{\{i,j\} \subset \{1, ..., p\}} R((X_{I_i}, X_{I_j}), (\lambda_i, \lambda_j), \{I_i, I_j\}).$$

We first present a key result that relates the expectation of $\bigvee_{j=1}^{d} F^{\lambda_j} (X_j)$ with the tail dependence function of $G$, which enables the derivation of the main properties of $R(X, \lambda, I)$. The result also points out that in this work we can assume that the MEV distributions have unit Fréchet margins without loss of generality.

Proposition 2.1. Let $X$ be a random vector with MEV distribution $G$, unit Fréchet $F$ marginals and tail dependence function $l$. If $Y$ has MEV distribution with marginal distributions $F_j, j = 1, ..., d$, and the same copula as $X$, then for each $\lambda \in (0, \infty)^d$, it holds that

$$E \left( \bigvee_{j=1}^{d} F^{\lambda_j} (Y_j) \right) = E \left( \bigvee_{j=1}^{d} F^{\lambda_j} (X_j) \right) = \frac{l(\lambda_1^{-1}, ..., \lambda_d^{-1})}{1 + l(\lambda_1^{-1}, ..., \lambda_d^{-1})}. \quad (4)$$

Proof. We first deduce the distribution of $\bigvee_{j=1}^{d} F^{\lambda_j} (Y_j)$. Denoting the copula of $X$ by $C_X$, we have, for each $u \in [0, 1]$,

$$P \left( \bigvee_{j=1}^{d} F^{\lambda_j} (Y_j) \leq u \right) = C_X \left( u^{\lambda_1^{-1}}, ..., u^{\lambda_d^{-1}} \right) = G \left( - \frac{1}{\lambda_1^{-1}} \log u, ..., - \frac{1}{\lambda_d^{-1}} \log u \right) = \exp \left( - l \left( (- \log u) \lambda_1^{-1}, ..., (- \log u) \lambda_d^{-1} \right) \right) = u^{l(\lambda_1^{-1}, ..., \lambda_d^{-1})}. $$
Then
\[
E \left( \bigvee_{j=1}^{d} F_{j}^{\lambda_{i}}(Y_{j}) \right) = \int_{0}^{1} u^{l(\lambda_{1}^{-1}, \ldots, \lambda_{d}^{-1})} l(\lambda_{1}^{-1}, \ldots, \lambda_{d}^{-1}) \, du = \frac{l(\lambda_{1}^{-1}, \ldots, \lambda_{d}^{-1})}{1 + l(\lambda_{1}^{-1}, \ldots, \lambda_{d}^{-1})}.
\]

The next result shows that the max-min coefficient takes into account the tail dependence function of all subvectors \(X_{\cup j \in T} I_{j}\) of \(X\) with indices in \(\cup j \in T, \emptyset \neq T \subseteq \{1, \ldots, p\}\).

From the Proposition 2.1 it holds that, for each \(\emptyset \neq T \subseteq \{1, \ldots, p\}\),

\[
\epsilon(I_{j}, j \in T) \equiv E \left( \bigvee_{j \in T} \bigvee_{i \in I_{j}} F^{\lambda_{i}}(X_{i}) \right) = \frac{l \left( \sum_{j \in T} \lambda_{j}^{-1} \delta_{1}(I_{j}), \ldots, \sum_{j \in T} \lambda_{j}^{-1} \delta_{d}(I_{j}) \right)}{1 + l \left( \sum_{j \in T} \lambda_{j}^{-1} \delta_{1}(I_{j}), \ldots, \sum_{j \in T} \lambda_{j}^{-1} \delta_{d}(I_{j}) \right)},
\]

leading to the following relations of the max-min coefficients with the tail dependence and the extremal coefficients.

**Proposition 2.2.** If \(X\) has MEV distribution then, for each partition \(\mathcal{I} = \{I_{1}, \ldots, I_{p}\}\) of \(D\) and \(\lambda \in (0, \infty)^{p}\), it holds that

\[
R(X, \lambda, \mathcal{I}) = \epsilon(I_{j}, j \in \{1, \ldots, p\}) - \sum_{\emptyset \neq T \subseteq \{1, \ldots, p\}} (-1)^{|T|+1} \epsilon(I_{j}, j \in T)
\]

and

\[
R(X, 1, \mathcal{I}) = \frac{\epsilon_{X}}{1 + \epsilon_{X}} - \sum_{\emptyset \neq T \subseteq \{1, \ldots, p\}} (-1)^{|T|+1} \frac{\epsilon_{X_{\cup j \in T} I_{j}}}{1 + \epsilon_{X_{\cup j \in T} I_{j}}}.
\]

**Proof.** To obtain the first equality we first apply in (3) the relation

\[
\bigwedge_{j=1}^{p} \bigvee_{j \in I_{j}} F^{\lambda_{i}}(X_{i}) = \sum_{\emptyset \neq T \subseteq \{1, \ldots, p\}} (-1)^{|T|+1} \bigvee_{j \in T} \bigvee_{i \in I_{j}} F^{\lambda_{i}}(X_{i})
\]

and then the Proposition 2.1 with (5). The statement in (7) is a consequence of (6) and (1). \(\Box\)

For the particular case of \(I_{j} = \{j\}, j = 1, \ldots, p = d\), we denote \(R(X, \lambda, \mathcal{I})\) simply by \(R(X, \lambda)\) and we have, as a consequence of the above result, that

\[
R(X, \lambda) = \frac{l(\lambda_{1}^{-1}, \ldots, \lambda_{d}^{-1})}{1 + l(\lambda_{1}^{-1}, \ldots, \lambda_{d}^{-1})} - \sum_{\emptyset \neq S \subseteq \{1, \ldots, d\}} (-1)^{|S|+1} \frac{l(\lambda_{1}^{-1} \delta_{1}(S), \ldots, \lambda_{d}^{-1} \delta_{d}(S))}{1 + l(\lambda_{1}^{-1} \delta_{1}(S), \ldots, \lambda_{d}^{-1} \delta_{d}(S))}
\]

and

\[
R(X, 1) = \frac{\epsilon_{X}}{1 + \epsilon_{X}} - \sum_{\emptyset \neq S \subseteq \{1, \ldots, d\}} (-1)^{|S|+1} \frac{\epsilon_{X_{S}}}{1 + \epsilon_{X_{S}}}.
\]
These two relations extend the result of the Proposition 1 in Naveau et al. (2009) and equation (14) in Cooley et al. (2006), where, for \( d = 2, 1 = \lambda_2 = \lambda_1 = \lambda \in (0, 1) \), we have

\[
\frac{1}{2} R(X, \lambda) = \frac{I(\lambda, \lambda - \lambda)}{1 + I(\lambda, \lambda - \lambda)} - \frac{3}{2} \frac{1}{(1 + \lambda)(2 - \lambda)}
\]

and \( R(X, 1) = \frac{\epsilon_{X, 1}}{\epsilon_{X, 1}} \).

Our next step is to compare the value of the max-min coefficient \( R(X, \lambda, T) \) with the corresponding coefficient in the two boundary cases of independent or totally dependent \( X_{I_j}, j = 1, \ldots, p \).

**Proposition 2.3.** Let \( X, \hat{X} = (\hat{X}_1, ..., \hat{X}_d) \) and \( \hat{X} = (\hat{X}_1, ..., \hat{X}_d) \) be vectors of unit Fréchet random variables with MEV distributions such that \( X_{I_j}, j = 1, ..., p \), are independent, \( X_{I_j}, j = 1, ..., p \), are totally dependent and, for each \( j = 1, ..., p \), \( \hat{X}_{I_j}, X_{I_j} \) and \( X_{I_j} \) are identically distributed. Then, for each \( \lambda \in (0, \infty)^d \) and partition \( T \) of \( D \), it holds that

(a) \( R(\hat{X}, \lambda, T) \leq R(X, \lambda, T) \leq R(\hat{X}, \lambda, T) \),

(b) \( R(\hat{X}, 1, T) = 0 \),

(c) \( R(\hat{X}, 1, T) = \sum_{j=1}^{p} \frac{\epsilon_{X_{I_j}}}{1 + \sum_{j=1}^{p} \epsilon_{X_{I_j}}} - \sum_{T \subseteq \{1, \ldots, p\}} (-1)^{|T|+1} \sum_{j \in T} \frac{\epsilon_{X_{I_j}}}{1 + \sum_{j \in T} \epsilon_{X_{I_j}}} \).

**Proof.** If \( X \) has MEV distribution then it is associated (Marshall and Olkin (1983)) and then the variables \( M(I_j), j = 1, ..., p \), are also associated (Esary et al. (1967)). For these associated variables it holds that

\[
\bigwedge_{j=1}^{p} P(M(I_j) > x_j) \geq P \left( \bigcap_{j=1}^{p} M(I_j) > x_j \right) \geq \prod_{j=1}^{p} P(M(I_j) > x_j)
\]

and

\[
\bigwedge_{j=1}^{p} P(M(I_j) \leq x_j) \geq P \left( \bigcap_{j=1}^{p} M(I_j) \leq x_j \right) \geq \prod_{j=1}^{p} P(M(I_j) \leq x_j).
\]

By taking \( \hat{M}(I_j) = \bigvee_{i \in I_j} \hat{X}_i \) and \( \hat{M}(I_j) = \bigvee_{i \in I_j} \hat{X}_i, j = 1, ..., p, \) we can rewrite the above inequalities as

\[
P \left( \bigcap_{j=1}^{p} \hat{M}(I_j) > x_j \right) \geq P \left( \bigcap_{j=1}^{p} M(I_j) > x_j \right) \geq P \left( \bigcap_{j=1}^{p} \hat{M}(I_j) > x_j \right)
\]

and

\[
P \left( \bigcap_{j=1}^{p} \hat{M}(I_j) \leq x_j \right) \geq P \left( \bigcap_{j=1}^{p} M(I_j) \leq x_j \right) \geq P \left( \bigcap_{j=1}^{p} \hat{M}(I_j) \leq x_j \right).
\]
This concordance order implies (Shaked and Shanthikumar (2007)) that

\[
E\left(f\left(\bigvee_{j=1}^{p} M(I_j)\right)\right) \leq E\left(f\left(\bigvee_{j=1}^{p} \bar{M}(I_j)\right)\right) \leq E\left(f\left(\bigvee_{j=1}^{p} \tilde{M}(I_j)\right)\right),
\]

and

\[
E\left(f\left(\bigwedge_{j=1}^{p} M(I_j)\right)\right) \geq E\left(f\left(\bigwedge_{j=1}^{p} \bar{M}(I_j)\right)\right) \geq E\left(f\left(\bigwedge_{j=1}^{p} \tilde{M}(I_j)\right)\right)
\]

for all non-decreasing functions \(f\). The result in (a) then follows by taking \(f = F\) and replacing \(X_i\) by \(\frac{X_i}{\lambda_i}, i \in I_j\), for each \(j = 1, \ldots, p\).

The equalities in (b) and (c) follows from (7) and, in particular for (b), we recall that if \(X_I\), \(j = 1, \ldots, d\), are totally dependent vectors then the copula of \(X\) is also the copula of the minimum (Nelsen, 2006). \(\square\)

For the particular case of \(I_j = \{j\}, j = 1, \ldots, d\), the equality in (c) leads to

\[
R(\bar{X}, 1) = \frac{d}{d+1} - \sum_{\emptyset \neq S \subseteq \{1, \ldots, d\}} (-1)^{|S|+1} \frac{|S|}{1 + |S|} = \frac{d}{d+1} - \sum_{k=1}^{d} (-1)^{k+1} \frac{(d)}{k+1} = \frac{d-1}{d+1}
\]

which extends the result for the case of \(d = 2\), where \(R(\bar{X}, 1) = \frac{1}{4}\).

3 Examples and an application

In order to illustrate the previous results, we consider two families of Multivariate Extreme Value distributions and we compute the expressions for \(R(X, \lambda, T)\), which can be easily implemented.

**Example 3.1.** If \(X\) is the MEV marginal distribution of the Multivariate Maxima of Moving Maxima processes considered in Smith and Wissman (1996) then \(l(x_1, \ldots, x_d) = \sum_{t=1}^{\infty} \sum_{j=1}^{d} x_{t,j}^{\alpha_{t,j}}\).

For this tail dependence function we obtain

\[
R(X, \lambda, T) = \sum_{\emptyset \neq T \subseteq \{1, \ldots, p\}} \left(\frac{(-1)^{|T|+1}}{1 + \sum_{k=1}^{\infty} \sum_{t \in T} \sum_{j \in T} \lambda_{t,j}^{-1} \alpha_{t,j}}\right) - \frac{1 + (-1)^p}{1 + \sum_{t=1}^{\infty} \sum_{k=1}^{\infty} \sum_{j=1}^{d} \lambda_{t,j}^{-1} \alpha_{t,j}}, \quad \square
\]

and, in particular,

\[
R(X, \lambda) = \sum_{\emptyset \neq S \subseteq \{1, \ldots, d\}} \left(\frac{(-1)^{|S|+1}}{1 + \sum_{t=1}^{\infty} \sum_{k=1}^{\infty} \sum_{j \in S} \lambda_{t,j}^{-1} \alpha_{t,j}}\right) - \frac{1 + (-1)^d}{1 + \sum_{t=1}^{\infty} \sum_{k=1}^{\infty} \sum_{j=1}^{d} \lambda_{t,j}^{-1} \alpha_{t,j}}.
\]

**Example 3.2.** For the Symmetric Logistic model \(l(x_1, \ldots, x_d) = \left(\sum_{j=1}^{d} x_j^{1/\theta}\right)^{\theta}\) we have

\[
R(X, \lambda, T) = \sum_{\emptyset \neq T \subseteq \{1, \ldots, p\}} \left(\frac{(-1)^{|T|+1}}{1 + \left(\sum_{t \in T} \lambda_{t}^{-1/\theta} |I_t|\right)^{\theta}}\right) - \frac{1 + (-1)^p}{1 + \left(\sum_{t=1}^{p} \lambda_{t}^{-1/\theta} |I_t|\right)^{\theta}}.
\]
In the table 1 we present the estimates corresponding to
from January 1993 to March 2004.

| Indexes           | Closing Values                  |
|-------------------|---------------------------------|
| CAC 40 (France)   |                                 |
| FTSE100 (UK)      |                                 |
| SMI (Swiss)       |                                 |
| XDAX (Germany)    |                                 |
| Dow Jones (USA)   |                                 |
| Nasdaq (USA)      |                                 |
| SP500 (USA)       |                                 |
| HSI (China)       |                                 |
| Nikkei (Japan)    |                                 |

Ferreira and Ferreira (2012b). The data are monthly maximums of the negative log-returns of
As an application of this estimation procedure we consider for
in Fermanian et al. (2004).

A natural estimator for $R(X, \lambda, T)$ is

$$R(X, \lambda, T) = \frac{1}{n} \sum_{k=1}^{n} \left( \prod_{j=1}^{p} \bigvee_{i \in I_j} \hat{F}_{ij}^{\lambda_j}(X_{ij}^{(k)}) - \prod_{j=1}^{d} \bigvee_{i \in I_j} \hat{F}_{ij}^{\lambda_j}(X_{ij}^{(k)}) \right),$$

and, in particular for $p = d$,

$$R(X, \lambda) = \frac{1}{n} \sum_{k=1}^{n} \left( \bigvee_{j=1}^{d} \bigvee_{i \in I_j} \hat{F}_{ij}^{\lambda_j}(X_{ij}^{(k)}) - \bigvee_{j=1}^{d} \bigvee_{i \in I_j} \hat{F}_{ij}^{\lambda_j}(X_{ij}^{(k)}) \right).$$

If we denote $\bar{M}_k(I_j)^{\lambda_j} = \bigvee_{i \in I_j} \hat{F}_{ij}^{\lambda_j}(X_{ij}^{(k)})$ then we can write

$$R(X, \lambda, T) = \frac{1}{n} \sum_{k=1}^{n} \bigvee_{i \in I_j} \bar{M}_k(I_j)^{\lambda_j} - \sum_{T \subseteq \{1, \ldots, p\}} (-1)^{|T|+1} \frac{1}{n} \sum_{k=1}^{n} \bigvee_{i \in A} \bar{M}_k(I_j)^{\lambda_j}. $$

The strong consistency of the terms of this sum is stated in the proof of the Proposition 3.8 of Ferreira and Ferreira (2012b) and the asymptotic normality can be deduced from the Theorem 6 in Fermanian et al. (2004).

As an application of this estimation procedure we consider for $X$ some financial stock markets grouped in the tree big world markets $I_1 = $Europe, $I_2 = $USA and $I_3 = $Far East, as considered in Ferreira and Ferreira (2012b). The data are monthly maximums of the negative log-returns of the closing values of the stock market indexes CAC 40 (France), FTSE100 (UK), SMI (Swiss), XDAX (German), Dow Jones (USA), Nasdaq (USA), SP500 (USA), HSI (China) and Nikkei (Japan), from January 1993 to March 2004.

In the table 1 we present the estimates corresponding to $\bar{M}(A) = \bigvee_{k=1}^{n} \bigvee_{i \in A} \hat{F}_{ij}(X_{ij}^{(k)})$ which we need to compute $R(X, 1, T)$ for $T = \{I_1, I_2, I_3\}$.

We obtain $R(X, 1, T) = 0.321$, $R((X_{I_1}, X_{I_2}), 1, \{I_1, I_2\}) = 0.172$, $R((X_{I_1}, X_{I_3}), 1, \{I_1, I_3\}) = 0.222$ and $R((X_{I_2}, X_{I_3}), 1, \{I_2, I_3\}) = 0.247$. 

Several parametric and non-parametric estimators for the tail dependence function are available in the literature (Beirlant et al. (2004), Schmidt and Stadtmüller (2006), Krajina (2010)) which can be applied to the terms in $R(X, 1, T)$. The comparison of estimation procedures is out of our purposes in this paper and we simply remark that the definition of the max-min dependence coefficient suggests a non-parametric estimator based on sample means.
Table 1: $I_1 =$Europe, $I_2 =$USA and $I_3 =$Far East

References

[1] Beirlant, J., Goegebeur, Y., Segers, J. and Teugels, J. (2004) Statistics of Extremes: Theory and Applications. John Wiley & Sons.

[2] Cressie, N. A. C. 1993 Statistics for spatial data. (revised edn) New York, NY: Wiley.

[3] Esary, J. D., Proschan, F. and Walkup, D. W. (1967). Association of Random Variables, with Applications. Ann. of Math. Stat., Vol. 38, No. 5, pp. 1466-1474.

[4] Falk, M., Hüsler, J. and Reiss, R.-D. (2010) Laws of Small Numbers: Extremes and Rare Events. 3rd ed. Birkhäuser, Basel.

[5] Fermanian, J.-D., Radulović, D., Wegkamp, M. (2004). Weak convergence of empirical copula processes. Bernoulli 10(5), 847-860.

[6] Ferreira, H. and Ferreira, M. (2012a) Fragility Index of block tailed vectors. J. Stat. Plan. Infer. 142 (7), 1837-1848

[7] Ferreira, H. and Ferreira, M. (2012b). On extremal dependence of block vectors, Kybernetika 48(5), 988-1006.

[8] Fonseca, C., Pereira, L., Ferreira, H. and Martins, A.P. (2012) Generalized madogram and pairwise dependence of maxima over two regions of a random field. arXiv:1104.2637v2.

[9] Huang, X. (1992). Statistics of Bivariate Extreme Values. Ph. D. thesis, Tinbergen Institute Research Series 22, Erasmus University Rotterdam.

[10] Joe, H. (1997). Multivariate Models and Dependence Concepts. Chapman & Hall, London.

[11] Krajina, A. (2010). An M-Estimator of Multivariate Tail Dependence. Ph. D. thesis, Tilburg: Tilburg University Press.

[12] Li, H. (2009). Orthant tail dependence of multivariate extreme value distributions, J. Mult. Anal., 100(1), 243-256.

[13] Marshall, A.W., Olkin, I. (1983) Domains of attraction of multivariate extreme value distributions. Ann. Prob., 11:168-177.

[14] Naveau, P., Guillou, A., Cooley, D. and Diebolt, J. (2009). Modelling pairwise dependence of maxima in space, Biometrika, vol. 96, issue 1, pages 1-17.

[15] Nelsen, R.B. (2006). An Introduction to Copulas. Second Edition. Springer, New York.

[16] Poncet, P., Cooley, D. and Naveau, P. (2006) Variograms for max-stable random fields. In Dependence in probability and statistics (eds P. Bertail, P. Doukhan and P. Soulier). Lecture Notes in Statistics, no. 187, pp. 373-390. New York, NY: Springer.

[17] Resnick, S. (1987). Extreme Values, Regular Variation and Point Processes. Springer-Verlag, New York.

[18] Schlather, M. and Tawn, J.A. (2002). Inequalities for the extremal coefficients of multivariate extreme value distributions. Extremes, 5(1), 87-102.
[19] Schlather, M. and Tawn, J. A. (2003) A dependence measure for multivariate and spatial extreme values: Properties and inference. Biometrika 90, 139-156.

[20] Schmidt, R., Stadtmüller, U. (2006). Nonparametric estimation of tail dependence, Scand. J. of Stat. 33, 307-335.

[21] Shaked, M. and Shanthikumar, J.G. (2007) Stochastic orders. Springer-Verlag, New York.

[22] Sibuya, M. (1960). Bivariate extreme statistics. Ann. Inst. Statist. Math. 11, 195-210.

[23] Smith, R.L. (1990). Max-stable processes and spatial extremes. Preprint, Univ. North Carolina, USA.

[24] Tiago de Oliveira, J. (1962/63). Structure theory of bivariate extremes, extensions. Est. Mat., Estat. e Econ. 7, 165-195.