Constructing $\mathbb{Q}$-Fano 3-folds à la Prokhorov & Reid

Tom Ducat

Abstract

We generalise a construction by Prokhorov & Reid of two families of $\mathbb{Q}$-Fano 3-folds of index 2 to obtain five more families of $\mathbb{Q}$-Fano 3-folds; four of index 2 and one of index 3. Two of the families constructed have the same Hilbert series and we study these cases in more detail.

Introduction

$\mathbb{Q}$-Fano 3-folds. In this paper a $\mathbb{Q}$-Fano 3-fold $X$ will be a normal projective three-dimensional variety over $\mathbb{C}$ with $-K_X$ ample and at worst $\mathbb{Q}$-factorial terminal singularities. Unless otherwise stated, we will assume that $X$ has Picard rank $\rho_X = 1$ (so that $X$ appears as the end product of a Minimal Model Program).

The ($\mathbb{Q}$-Fano) index of $X$ is given by:

$$q_X := \max \{ q \in \mathbb{Z}_{\geq 1} : -K_X = qA \text{ for some } A \in \text{Cl}(X) \}$$

and any Weil divisor $A$ for which $-K_X = q_X A$ is called a primitive ample divisor on $X$. We usually consider $X$ to be polarised by $A$, that is, with an embedding into weighted projective space given by $\text{Proj}$ of the graded ring $R(X,A) = \bigoplus_{k \geq 0} H^0(X,\mathcal{O}_X(kA))$. Some of the basic numerical invariants for $(X,A)$ are the codimension of this embedding, the index $q_X$, the degree $A^3$ and the basket of singularities $B_X$.

Prokhorov & Reid’s construction. Starting from either $X = \mathbb{P}^3$ or $X = X_2 \subset \mathbb{P}^4$, Prokhorov & Reid [7, §6.3.3] construct a $\mathbb{Q}$-Fano 3-fold $Y$ by writing down a Sarkisov link which makes a divisorial extraction from a certain kind of singular curve $\Gamma \subset X$ followed by the Kawamata blowdown of a divisor $E \cong \mathbb{P}(1,1,2)$ to a (polarised) $\frac{1}{5}(2,2,1)$ cyclic quotient singularity $Q \in Y$.

The two families of $\mathbb{Q}$-Fano 3-folds $(Y,B)$ constructed in this way correspond to the cases A.1 and A.2 of Table 1, which lists their numerical data as well as their ID in the Graded Ring Database [1]. In particular they both have index 2.

The main result. We generalise Prokhorov & Reid’s construction to obtain families for all of the other cases of Table 1.

Theorem 1. For each of the cases A.1–A.4 of Table 1 there is a Sarkisov link starting from the corresponding $\mathbb{Q}$-Fano 3-fold $X$ of Table 3 and ending with $Y$. This Sarkisov link starts with a divisorial extraction from an irreducible singular curve $\Gamma \subset X$ followed by the Kawamata blowdown to a cyclic quotient singularity $Q \in Y$ (the $\frac{1}{5}(1,3,4)$ point in case A.4).

The cases B.1 and B.2 can also be constructed by exactly the same method, however we must start with a (non-extremal) divisorial extraction from a reducible curve $\Gamma \subset X$. Therefore the $Y$ that are constructed in this manner have Picard rank $\rho_Y > 1$.

In case A.3 we construct two different families corresponding to divisorial extractions from two types of non-isomorphic curve singularity $P \in \Gamma \subset X$. For all of the other cases the family constructed is unique.

Received 11 October 2016; revised 9 November 2017; published online 6 April 2018.

2010 Mathematics Subject Classification 14E30 (primary).

This work was completed whilst the author was an International Research Fellow of the Japanese Society for the Promotion of Science and was supported by Grant-in-Aid for JSPS Fellows, No. 15F15771.
These are all of the possible examples that can be obtained by this method in its current form.

Contents. In § 1 we generalise Prokhorov & Reid’s main construction of a Sarkisov link which begins with a special kind of divisorial extraction from a singular curve contained in a smooth $\mathbb{Q}$-Fano 3-fold.

In § 2 we switch to the local setting and use the unprojection construction developed in [4] to classify the curve singularities for which this special divisorial extraction exists. This comprises the main part of the proof of Theorem 1.

In § 3 we use a global version of this unprojection construction to study the two families of case A.3 in more detail. The results of that section are summarised in Proposition 3.1.

1. Prokhorov & Reid’s construction

We now outline our generalisation of Prokhorov & Reid’s construction [7, § 6.3.3], following their paper very closely. The original construction was for the weighted projective plane $E = \mathbb{P}(1, 1, 2)$.

The Sarkisov link. Starting from a $\mathbb{Q}$-Fano 3-fold $X$ the general idea is to construct a new $\mathbb{Q}$-Fano 3-fold $Y$ by writing down a simple Sarkisov link of type II:

$$\sigma \xymatrix{ X' \ar[rd] \ar[rr]_{\pi} & & Y \ar[ld] }$$

where $\sigma : X' \to X$ is a Mori extraction from a curve $\Gamma \subset X$ and $\pi : X' \to Y$ is the Kawamata blowup of a terminal cyclic quotient singularity $Q \in Y$. Prokhorov & Reid construct such a Sarkisov link by the following method:

Step 1. Take the weighted projective plane $E = \mathbb{P}(1, r, ra - 1)$ for some $r \geq 2$, $a \geq 1$. We consider weights of this form for the following two reasons:

- $E$ has a $\frac{1}{2}(1, -1)$ singularity (that is, a type $A_{r-1}$ Du Val singularity),
- $E$ is isomorphic to the exceptional divisor in the Kawamata blowup of the terminal cyclic quotient singularity $\frac{1}{ra+r-1}(1, r, ra - 1)$. (In fact $E$ will become the exceptional divisor for $\pi$.)

Step 2. Consider the embedding of $E$ given by $\phi = \phi|_{\mathcal{O}_E(r)}$, that is,

$$\phi : E \hookrightarrow \mathbb{P}(1, 1, a, ra - 1), \quad \phi(e_0 : e_1 : e_2) = (e_1 : e_0^r : e_0 e_2 : e_2^r)$$

where the image $\phi(E) \subset \mathbb{P}(1, 1, a, ra - 1)_{x,y,z}$ is given by the equation $xz = y^r$. We consider $\mathbb{Q}$-Fano 3-folds $(X, A)$ with an embedding $E \hookrightarrow X$ such that $A|_E = \mathcal{O}_E(r)$. The point of

Table 1. The $\mathbb{Q}$-Fano 3-folds $(Y, B)$ constructed in this paper.

| Case | $Y \subset \mathbb{P}(r^3)$ | $\rho_Y$ | codim | $q_Y$ | $B^3$ | $B_Y$ | GRDB |
|------|------------------|--------|-------|------|------|------|------|
| A.1  | $Y \subset \mathbb{P}(1^4, 2^2, 3)$ | 1      | 3     | 2    | $\frac{7}{3}$ | $\frac{1}{3}(1, 2, 2)$ | #40836 |
| A.2  | $Y \subset \mathbb{P}(1^5, 2^2, 3)$ | 1      | 4     | 2    | $\frac{10}{3}$ | $\frac{1}{3}(1, 2, 2)$ | #40933 |
| A.3  | $Y \subset \mathbb{P}(1^3, 2^2, 3, 4, 5)$ | 1      | 4     | 2    | $\frac{10}{3}$ | $\frac{1}{3}(1, 2, 4)$ | #40663 |
| A.4  | $Y \subset \mathbb{P}(1^2, 2^2, 3^2, 4, 5)$ | 1      | 4     | 3    | $\frac{5}{3}$ | $\frac{2}{3}(1, 1, 1), \frac{1}{3}(1, 3, 4)$ | #41200 |
| B.1  | $Y \subset \mathbb{P}(1^4, 2^2, 3, 4, 5)$ | 4      | 5     | 2    | $\frac{12}{7}$ | $\frac{1}{3}(1, 2, 4)$ | #40837 |
| B.2  | $Y \subset \mathbb{P}(1^3, 2^2, 3, 4, 5, 6, 7)$ | 5      | 6     | 2    | $\frac{10}{7}$ | $\frac{1}{3}(1, 2, 6)$ | #40664 |
considering such embeddings is that the $\frac{1}{r}(1, -1)$ singularity of $E$ is supported at a smooth point $P = P_u \in X$.

**Lemma 1.1.** Let $(X, A)$ be a $\mathbb{Q}$-Fano 3-fold which admits an embedding $E \subset X$ as a $\mathbb{Q}$-Cartier divisor such that $E \in |eA|$ and $\mathcal{O}_E(A|_E) = \mathcal{O}_E(r)$. Then $X$ is of the form:

(i) $\mathbb{P}(1, 1, a, ra - 1)$ or (ii) $X_{ra} \subset \mathbb{P}(1, 1, a, ra - 1, e)$.

Moreover all possible choices are explicitly listed in Table 2.

**Proof.** Let $q$ be the index of $X$, that is, $K_X = -qA$. We have $K_E = (a + 1)A|_E$ and so, by the adjunction formula, we get

$$(a + 1)A|_E = (K_X + E)|_E = -(q - e)A|_E$$

and hence $q = a + e + 1$. Since $q > e$, by the Kodaira vanishing theorem we get that $H^1(X, (m - e)A) = H^1(X, K_X + (m + q - e)A) = 0$ for all $m \geq 0$. It follows from the standard short exact sequence:

$$0 \to \mathcal{O}_X((m - e)A) \to \mathcal{O}_X(mA) \to \mathcal{O}_E((mA)|_E) \to 0$$

that any such $(X, A)$ has Hilbert series:

$$P_{X,A}(t) = \frac{P_{E,\mathcal{O}_E(r)}(t)}{1 - t^e} = \frac{1 - t^{ra}}{(1 - t)^2(1 - t^a)(1 - t^{ra-1})(1 - t^e)}.$$  

(i) If $e = ra$, then $X = \mathbb{P}(1, 1, a, ra - 1)$ and $X$ is terminal if and only if $\frac{1}{ra-1}(1, 1, a)$ is at worst a terminal singularity. This happens if and only if

$$a + 1 \equiv 0 \mod ra - 1 \Rightarrow a + 1 \geq ra - 1 \Rightarrow a(r - 1) \leq 2$$

and the only solutions for $(r, a)$ are $(2, 1)$, $(3, 1)$ and $(2, 2)$.

(ii) If $e \neq ra$, then $X$ can be written as a hypersurface

$$X_{ra} \subset \mathbb{P}(1, 1, a, ra - 1, e)_{u,x,y,z,t}$$

and a necessary condition for $X$ to have terminal singularities at $P_e$ is that $e < ra$. Therefore bounding which numerical cases occur is equivalent to bounding $ra$. Note that if $ra > 2$, then $P_e \in X$ is necessarily a $\frac{1}{ra}(1, a, e)$ cyclic quotient singularity and we must impose conditions to ensure it is terminal. One of the following must occur.

(i) $a + 1 = ra - 1 \Rightarrow a(r - 1) = 2$ and hence $r \leq 3, a \leq 2$.

| $X$ | $(r, a)$ | $q$ | $e$ | $q'$ | $l$ | $d$ | $(ra - 1) \mid d$ |
|-----|---------|----|-----|------|----|-----|----------------|
| $\mathbb{P}^3$ | $(2, 1)$ | 4 | 2 | 2 | 3 | 7 | ✓ |
| $\mathbb{P}(1^3, 2)$ | $(3, 1)$ | 5 | 3 | 2 | 5 | 14 | ✓ |
| $\mathbb{P}(1^2, 2, 3)$ | $(2, 2)$ | 7 | 4 | 3 | 5 | 13 | × |
| $\mathbb{X}_2 \subset \mathbb{P}^4$ | $(2, 1)$ | 3 | 1 | 2 | 3 | 5 | ✓ |
| $\mathbb{X}_3 \subset \mathbb{P}(1^4, 2)$ | $(3, 1)$ | 3 | 1 | 2 | 5 | 8 | ✓ |
| $\mathbb{X}_4 \subset \mathbb{P}(1^3, 2, 3)$ | $(2, 2)$ | 4 | 1 | 3 | 5 | 7 | × |
| $\mathbb{X}_4 \subset \mathbb{P}(1^3, 2, 3)$ | $(4, 1)$ | 4 | 2 | 2 | 7 | 15 | ✓ |
| $\mathbb{X}_4 \subset \mathbb{P}(1^2, 2^2, 3)$ | $(2, 2)$ | 5 | 2 | 3 | 5 | 9 | ✓ |
| $\mathbb{X}_5 \subset \mathbb{P}(1^2, 2, 3, 5)$ | $(2, 3)$ | 6 | 2 | 4 | 7 | 11 | × |
| $\mathbb{X}_6 \subset \mathbb{P}(1^2, 2, 3, 5)$ | $(3, 2)$ | 6 | 3 | 3 | 8 | 17 | × |
(ii) $e + 1 = ra - 1 \Rightarrow e = ra - 2$. We claim that $e \leq \frac{1}{2}ra$ and $ra \leq 4$. If not then $e > \frac{1}{2}ra > 2$ implying $P_1 \in X$, which is a hyperquotient singularity with weights $\frac{1}{e}(1, 1, 1; 2)$. This is never terminal for $e > 2$.

(iii) $a + e = ra - 1 \Rightarrow e = ra - a - 1$. By a similar calculation we get $e \leq \frac{1}{2}ra$, which implies $(r - 2)a \leq 2$ and $r \leq 4$. If $r = 3$ or $4$, then $a \leq 2$. If $r = 2$ then $X$ has a $\frac{1}{a-1}(1, 1, a; 2)$ hyperquotient singularity which is only terminal if $a \leq 3$.

We have reduced to either $(r \leq 4, a \leq 2)$ or $(r, a) = (2, 3)$. Checking all of these possible cases gives the list in Table 2.

In particular we note that $X$ has a $\frac{1}{ra-1}$ quotient singularity, where the $\frac{1}{ra-1}(1, r)$ point of $E$ is supported, and that $X$ is smooth along $E$ elsewhere.

Step 3. We let $\Gamma \subset E \subset X$ be an irreducible curve of degree $d$ (where $d$ will be chosen in Step 5) passing through the point $P \in X$ such that:

(i) $\Gamma$ is contained in the smooth locus of $X$.

If $ra > 2$ then, since $E \subset X$ passes through exactly one singular point of $X$ (the $\frac{1}{ra-1}$-quotient point), this holds if and only if $\Gamma$ avoids the $\frac{1}{ra-1}$-quotient point of $E$. A necessary condition is that $(ra - 1) \mid d$ (which is also trivially true if $ra = 2$).

(ii) $\Gamma$ is smooth apart from an ‘appropriately singular’ point at $P \in X$.

Here ‘appropriately singular’ means that there should exist a Mori extraction:

$$\sigma : (F \subset X') \rightarrow (\Gamma \subset X)$$

(that is, a divisorial extraction in the Mori category of terminal 3-folds) with exceptional divisor $F$, such that:

- $X'$ has exactly one singularity of index $> 1$ which is of the form $\frac{1}{e}(1, 1, -1)$,
- $\sigma$ induces an isomorphism $E' \cong E$, where $E'$ is the birational transform of $E$.

The extraction $\sigma$ is given by the blowup of the symbolic power algebra of the ideal sheaf $\mathcal{I}_{\Gamma/X}$ [4, Proposition 1.4]. The type of curve singularities that satisfy these two conditions will be explained more carefully in Proposition 2.3, but for now we assume that such a divisorial extraction exists.

We take $\sigma : X' \rightarrow X$ to be the left hand side of our Sarkisov link.

**Remark 1.2.** Even if $\Gamma \subset X$ is a reducible curve then a Mori extraction $\sigma : X' \rightarrow X$ with these properties still may exist. However $\sigma$ will not be an extremal extraction. Indeed the relative Picard rank $\rho_{X'/X}$ will equal the number of irreducible components of $\Gamma$.

Step 4. Let $q$ be the index of $X$, that is, $K_X = -qA$, and let $E \in |eA|$. According to the proof of Lemma 1.1 we have $q = a + e + 1$.

We define $q' = q - e = a + 1$. Then, writing $A' = \sigma^* A$, we get

$$K_{X'} = \sigma^* K_X + F = -qA' + F \quad \text{and} \quad E' = \sigma^* E - F = eA' - F.$$ 

In particular it follows that $K_{X'} = -(q'A' + E')$. We let $l = ra + r - 1$ and define $B' = A' + \frac{q'}{l} E'$. Note that $\frac{l+1}{q'} = \frac{ra+e}{a+1} = r$ and hence we can write $K_{X'} = -q'B' + \frac{1}{r} E'$.

Step 5. We now make the clever choice $d = l + re$ so that $B'$ will become numerically trivial when restricted to $E'$.
On $E$ we have $dA|_E = r\Gamma$ since $P \in E$ has index $r$ and $\Gamma$ has degree $d$ (which is coprime to $r$). Moreover since $E \cong E'$ we must have $(F \cap E') \cong \Gamma$ and therefore it follows that $(dA - rF)|_{E'} = 0$. Now we see that $O_{E'}(lB') = O_{E'}$ for this choice of $d$, since:
\[
(lB')_{E'} = (lA' + rE')_{E'} = ((l + re)A' - rF)_{E'} = (dA' - rF)_{E'} = 0.
\]

**Step 6.** Finally, by the Kodaira vanishing theorem we have that
\[
H^1(X', lB' - E') = H^1(X', K_{X'} + aA' - rK_{X'}) = 0
\]
which implies that the restriction map
\[
H^0(X', lB') \to H^0(E', O_{E'})
\]
is surjective. Thus $|lB'|$ is a free linear system which is ample outside $E'$ and numerically trivial along $E'$, hence $B'$ is nef and the morphism $\pi: (X', B') \to (Y, B)$, defined by $Y = \text{Proj} \, R(X', B')$, contracts the divisor $E' \cong \mathbb{P}(1, r, ra - 1)$ to a point $Q \in Y$ and is otherwise an isomorphism. By the following Lemma this point is a $\frac{1}{4}(1, r, ra - 1)$ cyclic quotient singularity and, since the discrepancy of $E'$ over $Y$ is $\frac{1}{4}$, $\pi$ is the Kawamata blowdown.

**Lemma 1.3.** $Q \in Y$ is locally analytically isomorphic to a $\frac{1}{4}(1, r, ra - 1)$ quotient singularity.

**Proof.** Let $o \in U$ denote the $\frac{1}{4}(1, r, ra - 1)$ cyclic quotient singularity. Note that $E' \subset X'$ has conormal bundle $O_{E'}(-E') = O_{E'}(F - eA') = O_{E'}(d - re) = O_{E'}(l)$. The case in which $E' \cong \mathbb{P}^2$ with conormal bundle $O_{\mathbb{P}^2}(2)$ contracts to a $\frac{1}{4}(1, 1, 1)$ singularity is due to Mori [6].

By following the proof of [6, Lemma 3.32] in our setting, we see that (possibly after replacing $Y$ by its normalisation) our map $\pi$ is given by the blowup of the graded sheaf of ideals $I = \bigoplus_{k \geq 0} I_k$ supported at $Q \in Y$, where $I_k = \pi_* O_{X'}(-kE') \subset \pi_* O_{X'} = O_Y$. Moreover $R^1 \pi_*(O_{X'}(-kE')) = 0$ for all $k \geq 0$ and hence, by applying $\pi_*$ to the short exact sequence
\[
0 \to O_{X'}(-kE') \to O_{X'}(-kE') \to O_{E'}(kl) \to 0,
\]
we obtain
\[
\text{gr}_{T}(O_{Y,Q}) \cong \bigoplus_{k \geq 0} I_k/I_{k+1} \cong \bigoplus_{k \geq 0} H^0(E, O_{E'}(kl)) \cong \text{gr}_{\mathbb{Q},o}(O_{U,o}).
\]
Since cyclic quotient singularities of codimension $\geq 3$ are rigid [9, Theorem 3] we can apply a Theorem of Gerstenhaber [5, p. 418] to obtain an isomorphism of complete local rings $\hat{O}_{Y,Q} \cong \hat{O}_{U,o}$ and hence a local analytic isomorphism of singularities. \hfill \Box

**Conclusion.** All of the steps in this construction are valid provided we can show the existence of a divisorial extraction $\sigma: X' \to X$ with the properties that were claimed in condition (ii) of Step 3. In this case we will have constructed a Sarkisov link from $(X, A)$ to a $\mathbb{Q}$-Fano $3$-fold $(Y, B)$ of index $q'$ and degree $B^3 = A'^2(A' + \frac{3}{2}E') = \frac{4}{r}A^3$.

2. **Divisorial extractions from singular curves**

We consider the six remaining cases of Table 2 which satisfy $(ra - 1) \mid d$ and check that the construction actually does work in these cases. We will call these six cases A.1–A.4 and B.1 and B.2 according to the order in which they appear in Table 3.
2.1. Curves in type A Du Val singularities

We now change focus slightly and assume that we are in the local setting. We take \( P \in S \) to be the germ of a type \( A_{r-1} \) Du Val surface singularity and we fix isomorphisms
\[
(P \in S) \cong (0 \in \mathbb{C}^2_{a_\beta})/\mathbb{Z}(1,-1) \cong (0 \in V(xz - y')) \subset \mathbb{C}^3_{x,y,z}
\]
where \((x, y, z) = (\alpha^r, \alpha\beta, \beta^r)\) are the invariant generators for the \( \frac{1}{r}(1, -1) \) cyclic group action.

As is well known, \( P \in S \) has minimal resolution
\[
\mu: (D \subset \tilde{S}) \rightarrow (P \in S)
\]
where the exceptional divisor \( D = \bigcup_{i=1}^{r-1} D_i \) is a chain of \(-2\)-curves of length \( r - 1 \). In particular \( D_i \cong \mathbb{P}^1 \) for all \( i \) and coordinates on \( D_i \) can be given in terms of the ratio \( x/y^i = y^{r-1}/z \) or equivalently in terms of the orbinites \( \alpha^{r-1}/\beta^i \) (see [8, Theorem 3.2, Exercise 4.6]). Given a curve \( P \in \Gamma \subset S \) we write \( \Gamma \) for the birational transform of \( \Gamma \) on \( \tilde{S} \).

**Definition 2.1.** We call a curve germ \( P \in \Gamma \subset S \) a singularity of type \( \Gamma_{(a_1, \ldots, a_{r-1})} \) if \( \tilde{\Gamma} \) has \( a_i \) branches intersecting \( D_i \) transversely, for all \( i \). (In particular \( \tilde{\Gamma} \) avoids the intersection points \( D_{i-1} \cap D_i \).

The orbifold equation. The orbifold equation of \( \Gamma \) is the equation \( \gamma \in \mathbb{C}[\alpha, \beta] \) defining \( q^{-1}\tilde{\Gamma} \subset \mathbb{C}^2 \), the preimage of \( \Gamma \) under the quotient map \( q: \mathbb{C}^2 \rightarrow S \). If \( P \in \Gamma \) is a singularity of type \( \Gamma_{(a_1, \ldots, a_{r-1})} \), then the orbifold equation of \( \Gamma \) factors analytically:
\[
\gamma(\alpha, \beta) = \prod_{i=1}^{r-1} \prod_{j=1}^{a_i} (\lambda_{ij} \alpha^{-i} - \mu_{ij} \beta^i)
\]
for some functions \( \lambda_{ij}, \mu_{ij} \in \mathbb{C}[[x, y, z]] \) according to the branches of \( \tilde{\Gamma} \). We will usually multiply out this expression and collect together terms to write \( \gamma \) as:
\[
\gamma(\alpha, \beta) = \sum_{j=0}^{a_1 + \ldots + a_{r-1}} c_j \alpha^{m_j} \beta^{n_j}
\]
where \( c_j \in \mathbb{C}[[x, y, z]] \) are invariant polynomials with constant term \( c_{j,0} \neq 0 \) and the points \((m_j, n_j)\) lie on the boundary of the Newton polygon \( \text{Newt}(\gamma) \). The faces of \( \text{Newt}(\gamma) \) have slope \(-\frac{r}{i}\) for \( i = 1, \ldots, r - 1 \). If we let \( A_i := \sum_{j \leq i} a_j \), then restricting \( \gamma \) to a face gives
\[
\sum_{j=A_{i-1}}^{A_i} c_{j,0} \alpha^{m_j} \beta^{n_j} = \alpha^{m_{A_{i-1}}} \beta^{n_{A_i}} \gamma_i(\alpha, \beta)
\]

**Table 3.** The \( \mathbb{Q} \)-Fano 3-folds \((X, A)\) which give genuine Sarkisov links.

| Case | \( X \) | codim | \( q_X \) | \( A^3 \) | \( B_X \) | \( (r, a) \) | \( d \) | Singularity type |
|------|--------|-------|------|------|------|-------|------|------------------|
| A.1  | \( \mathbb{P}^3 \) | 0     | 4    | 1    | \( \emptyset \) | (2, 1) | 7    | \( \Gamma_{(3)} \) |
| A.2  | \( X_2 \subset \mathbb{P}^4 \) | 1     | 3    | 2    | \( \emptyset \) | (2, 1) | 5    | \( \Gamma_{(3)} \) |
| A.3  | \( \mathbb{P}(1^3, 2) \) | 0     | 5    | \( \frac{1}{2} \) | \( \frac{1}{2}(1, 1, 1) \) | (3, 1) | 14   | \( \Gamma_{(1, 3)} \) or \( \Gamma_{(4, 0)} \) |
| A.4  | \( X_4 \subset \mathbb{P}(1^2, 2^2, 3) \) | 1     | 5    | \( \frac{1}{3} \) | \( 2 \times \frac{1}{2}(1, 1, 1), \frac{1}{3}(1, 2, 2) \) | (2, 2) | 9    | \( \Gamma_{(3)} \) |
| B.1  | \( X_3 \subset \mathbb{P}(1^4, 2) \) | 1     | 3    | \( \frac{3}{2} \) | \( \frac{1}{2}(1, 1, 1) \) | (3, 1) | 8    | \( \Gamma_{(4, 0)} \) |
| B.2  | \( X_4 \subset \mathbb{P}(1^3, 2, 3) \) | 1     | 4    | \( \frac{4}{3} \) | \( \frac{1}{2}(1, 1, 2) \) | (4, 1) | 15   | \( \Gamma_{(5, 0, 0)} \) |
where $\gamma_i$ is a homogeneous polynomial in $\alpha^{r-i}, \beta^i$ of degree $a_i$ whose roots give the intersection points of $\Gamma \cap D_i$. By assumption these roots are distinct and both $\alpha \nmid \gamma_i$ and $\beta \nmid \gamma_i$. We can consider degenerations of $\Gamma$ by allowing the $\gamma_i$ to pick up multiple roots and by allowing some of the coefficients $c_{j,0}$ to vanish.

**Example 2.2.** For the $A_2$ singularity, the singularity types $\Gamma_{(1,3)}$ and $\Gamma_{(4,0)}$ have the resolutions given in Figure 1. In both cases a format for the orbifold equation is given in Proposition 2.3.

**Weighted $A_{r-1}$ singularities.** Going back to Table 3, the orbirates $\alpha, \beta$ at $P \in E$ are naturally weighted with weights $\frac{1}{r}, \frac{ra-r}{r}$ (and hence $x, y, z$ have weights $1, a, ra - 1$). If $\Gamma \subset E$ is a curve of degree $d$ then each term appearing in the orbinate equation $\gamma$ must have degree $\leq \frac{d}{r}$ with respect to these weights. Since we chose $d = l + re = qr - 1 \equiv -1 \mod r$, it follows that $\Gamma$ has an orbifold equation of the form $\gamma = \alpha^{r-1} \phi + \beta \psi$, where $\phi, \psi \in \mathbb{C}[x, y, z]$ have degree $\leq q - 1$.

2.2. ‘Appropriately singular’ curves

Proposition 2.3 describes precisely what is meant by the statement ‘appropriately singular’ in Step 3(ii) of the construction. Since only row B.2 of Table 3 has $r = 4$ and $d = 15$ in this case, when we come to consider curves contained in an $A_3$ Du Val singularity we will assume that $\alpha, \beta$ have weights $\frac{1}{4}, \frac{3}{4}$ and that $\deg(\gamma) \leq \frac{15}{4}$. This helps us to exclude some additional cases very easily (see Remark 2.4).

**Proposition 2.3.** Suppose $P \in \Gamma \subset S \subset U$ where $P \in S$ is an $A_{r-1}$ Du Val singularity with $r \leq 4$, $P \in U$ is a smooth 3-fold and $\Gamma$ is a curve of degree $d \equiv r - 1 \mod r$. Suppose that there exists a Mori extraction $\sigma: (F \subset U') \to (\Gamma \subset U)$ where $U'$ has a single high index singularity of type $\frac{1}{r}(1,1,-1)$. Then the birational transform $S' = \sigma^{-1}S$ is isomorphic to $S$ and $P \in \Gamma$ is one of the following singularity types (up to a degeneration).

*Type A1.* The only possibility is $P \in \Gamma_{(3)}$ with multiplicity 3 and orbinate equation: 
\[ \gamma_{(3)} = a\alpha^3 + b\alpha^2\beta + c\alpha\beta^2 + d\beta^3 \]

*Type A2.* There are two possibilities.

(i) $P \in \Gamma_{(1,3)}$ with multiplicity 4 and orbinate equation: 
\[ \gamma_{(1,3)} = a\alpha^5 + b\alpha^4\beta + c\alpha^3\beta^2 + d\alpha\beta^3 + e\beta^4. \]

(ii) $P \in \Gamma_{(4,0)}$ with multiplicity 4 and orbinate equation: 
\[ \gamma_{(4,0)} = a\alpha^8 + b\alpha^6\beta + c\alpha^4\beta^2 + d\alpha^2\beta^3 + e\beta^4. \]

*Type A3.* Suppose moreover that $\alpha, \beta$ have weights $\frac{1}{4}, \frac{3}{4}$ and that $\Gamma$ has degree $\leq 15$. Then the only possibility is $P \in \Gamma_{(5,0,0)}$ with multiplicity 5 and orbinate equation: 
\[ \gamma_{(5,0,0)} = a\alpha^{15} + b\alpha^{12}\beta + c\alpha^9\beta^2 + d\alpha^6\beta^3 + e\alpha^3\beta^4 + f\beta^5. \]
REMARK 2.4. If we remove the condition on the degree in the $A_3$ case then, with more work, it is possible to show that the curve singularities satisfying the conditions of the Theorem are precisely $\Gamma(5,0,0)$, $\Gamma(1,0,4)$, $\Gamma(0,3,1)$ and their degenerations. However the last two singularity types cannot occur in case B.2 since $\gamma(1,0,4)$ and $\gamma(0,3,1)$ both contain terms of degree $>\frac{15}{4}$.

Proof. These statements can be checked by constructing divisorial extractions by unprojection, as in the style of [4]. Prokhorov & Reid treat the $A_1$ case [7, §6.1], and both the $A_1$ and $A_2$ cases appear in [4, §3]. Therefore we only consider the $A_3$ case.

Since we are assuming that $\deg \gamma \leq\frac{15}{4}$, we can write the orbifold equation of $\Gamma$ as

$$\gamma(\alpha, \beta) = \alpha^3 \phi(x, y) + \beta \psi(y, z)$$

where $\phi, \psi$ are functions of degree $\leq 3$ and the variables $x = \alpha^4, y = \alpha \beta, z = \beta^4$ have weights $1, 1, 3$. Therefore we can write $\phi$ and $\psi$ as

$$\phi = ax + by + c x^2 + d x y + e y^2 + \phi_3, \quad \psi = f y + g y^2 + h z + \psi_3$$

where $a, \ldots, h \in \mathbb{C}$ are constants and $\phi_3, \psi_3$ only contain terms of degree 3. We can assume $h \neq 0$, else $\alpha \mid \gamma$ and hence $\gamma$ is reducible. To show that $\Gamma$ has the form stated in the Theorem we must show that $a, \ldots, g = 0$.

By [4, Lemma 2.1] the equations defining our curve $\Gamma$ as a subvariety of $U$ can be written as the minors of a $2 \times 3$ matrix

$$\begin{vmatrix}
  x & y^3 & -\psi(y, z) \\
  y & z & \phi(x, y)
\end{vmatrix} = 0$$

where the first minor $x z - y^4$ is the equation defining $P \in S$, the $A_3$ hypersurface section containing $\Gamma$. For $\lambda, \mu \in \mathbb{C}$, let $H_{\lambda, \mu}$ be the hyperplane section containing $\Gamma$ which is given by the equation:

$$h_{\lambda, \mu} := x z - y^4 + \lambda (x \phi + y \psi) + \mu (y^3 \phi + z \psi) = 0.$$ 

If, for some $\lambda, \mu$, $P \in H_{\lambda, \mu}$ is Du Val singularity of type $A_{\leq 2}$ then, by changing coordinates and following the constructions in the $A_{\leq 2}$ cases, the divisorial extraction from $\Gamma$ will have a singularity of index $\leq 3$. Therefore in order for $U'$ to have a $\frac{1}{3}(1, 1, 3)$ singularity we require that $P \in H_{\lambda, \mu}$ is a type $A_3$ singularity or worse $\forall \lambda, \mu \in \mathbb{C}$. By the finite determinacy of Du Val singularities, this happens if and only if $h_{\lambda, \mu}$ has a factorisation as a product $h_{\lambda, \mu} \equiv XZ \mod m^3$ for some $X, Z$, $\forall \lambda, \mu \in \mathbb{C}$.

First we check which conditions are needed for $h_{\lambda, \mu}$ to factor as a product $\mod m^3$:

$$h_{\lambda, \mu} \equiv x z + \lambda x (a x + b y) + \lambda y (f y + h z) + \mu (y^3 \phi + z \psi) \mod m^3.$$ 

This factors if and only if the discriminant of this quadratic form vanishes identically, that is:

$$\lambda (a(h \lambda - f \mu)^2 + b^2 h \lambda \mu - b(h \lambda + f \mu) + f) = 0, \quad \forall \lambda, \mu \in \mathbb{C}.$$ 

Since we are assuming $h \neq 0$ this implies $a = b = f = 0$.

Now it is possible to check that $h_{\lambda, \mu}$ can factorised $\mod m^4$ as follows:

$$h_{\lambda, \mu} \equiv x z + \lambda x (c x^2 + d x y + e y^2) + \lambda y (g y^2 + h z) + \mu z (g y^2 + h z)$$

$$\equiv X Z - \lambda (c h^3 \beta^3 - d h^2 \beta^2 + e h \lambda - g) \beta^3 \mod m^4$$

where

$$X = x + h(\lambda y + \mu z) + g y^2 - h \lambda \mu (c x^2 + d x y + e y^2) + h^2 \lambda^2 \mu y (c x + d y) - c h^3 \lambda^3 \mu y^2$$

$$Z = z + \lambda (c x^2 + d x y + e y^2) - h \lambda^2 y (c x + d y) + c h^2 \lambda^3 y^2.$$
Therefore \( h_{\lambda,\mu} \) factorises mod \( m^4 \) if and only if

\[
ch^3 \lambda^3 - dh^2 \lambda^2 + eh \lambda - g = 0, \quad \forall \lambda \in \mathbb{C}
\]

and this implies \( c = d = e = g = 0 \).

We have proved that it is necessary for \( \Gamma \) to be of the form stated in the Theorem. To prove that the symbolic blowup of such a curve \( \Gamma \) actually does have a \( \frac{1}{2}(1,1,3) \) singularity and that \( S' \cong S \), we can construct \( U' \) explicitly using the serial unprojection method of [4].

Since the coefficient of \( z \) in \( \psi \) is non-zero, we may rescale and reassign the names of the coefficients \( a,b,c,... \) so that \( \phi(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 \) and \( \psi(y,z) = ey^3 + z \). The orbifold equation \( \gamma(\alpha,\beta) \) can now be written as a homogeneous quintic polynomial in \( \alpha^3 \) and \( \beta \), \( \gamma(\alpha,\beta) = \gamma'(\alpha^3,\beta) \) where

\[
\gamma'(X,Y) = aX^5 + bX^4Y + cX^3Y^2 + dX^2Y^3 + eXY^4 + Y^5
\]

and \( a,...,e \in \mathbb{C} \).

Let \( I_\Gamma \subset O_U \) be the ideal defining \( \Gamma \subset U \) and let \( \nu,\xi,\eta \in I_\Gamma \) be the three equations defining \( \Gamma \). Then, as in [4, §2.4], the ordinary blowup of \( \Gamma \subset U \) can be written as a codimension 2 complete intersection \( \sigma_1: U_1 \subset U \times \mathbb{P}^2_{(\nu: \xi: \eta)} \rightarrow U \) where the equations defining \( U_1 \) are the two syzygies coming from Cramer’s rule, given by the matrix product:

\[
\begin{pmatrix}
x & y^3 & -(ey^3 + z) \\
y & z & ax^3 + bx^2y + cxy^2 + dy^3
\end{pmatrix}
\begin{pmatrix}
\nu \\
-\xi \\
\eta
\end{pmatrix} = 0.
\]

In particular \( \eta = xz - y^4 \) is the minor obtained by deleting the third column of the matrix, which is the equation of the \( A_3 \) singularity \( P \in S \).

Starting with \( U_1 \), we now construct a sequence of serial unprojections \( u_i: (D_i \subset U_i) \rightarrow U_{i+1} \) for \( i = 1,2,3 \), by identifying a sequence of unprojection divisors \( D_i \subset U_i \). Since the divisorial extraction is given by the blowup of the symbolic power algebra \( U' = \text{Proj}_U \bigoplus I_\Gamma^{[n]} \) [4, Proposition 1.4], we can view the unprojection variables as ‘missing generators’ contained in the higher symbolic powers of \( I_\Gamma \). Calculating the unprojection equations can be done either by hand or by computer. Unfortunately they get a little bit messy, so we introduce the following notation in order to simplify them:

\[
\begin{align*}
\Xi_1 &:= \xi + e\eta, \\
\Xi_2 &:= \xi^2 + e\xi\eta + d\eta^2, \\
\Xi_3 &:= \xi^3 + e\xi^2\eta + d\xi\eta^2 + e\eta^3.
\end{align*}
\]

For the first unprojection, note that the equations (2.1) are contained in the ideal \( (x,y,z) \) which implies that the central fibre \( \sigma_1^{-1}(P) \) contains the divisor \( D_1 = \mathbb{P}^2_{\nu: \xi: \eta} \subset U_1 \) defined by the ideal \( I_{D_1} = (x,y,z) \subset O_{U_1} \). We can unproject \( D_1 \) to get a new generator \( \zeta \in I_\Gamma^{[2]} \) with the following unprojection equations.

\[
\begin{align*}
x\zeta &= y^2\Xi_2 + u\eta \\
y\zeta &= \xi\nu - (ax^2 + bxy + cy^2)\eta^2 \\
z\zeta &= \nu(\nu + dy^2\eta) + y^2(ax^2 + bxy + cy^2)\Xi_1\eta.
\end{align*}
\]

 Altogether the five equations (2.1) and (2.2) give the unprojection to a codimension 3 variety \( u_1: (D_1 \subset U_1) \rightarrow (P_\zeta \subset U_2) \) which contracts \( D_1 \) to a \( \frac{1}{2}(1,1,1) \) quotient singularity at the coordinate point \( P_\zeta \subset U_2 \). The unprojection \( U_2 \) also comes with a birational morphism to \( U \):

\[
\sigma_2: U_2 \subset U \times \mathbb{P}(1,1,1,2)_{\nu: \xi: \eta: \zeta} \rightarrow U.
\]
The reduced central fibre $\sigma_2^{-1}(P)_{\text{red}}$ now contains a second divisor $D_2 = \mathbb{P}(1, 1, 2)_{\eta: \xi; \nu}$ given by the ideal $I_{D_2} = (x, y, z, \nu, \zeta) \subset \mathcal{O}_{U_2}$. Unprojecting $D_2$ gives a new generator $\theta \in I_1^{[3]}$ with unprojection equations:

\begin{align}
x\theta &= y\Xi_3 + \zeta\eta \\
y\theta &= \xi\zeta - (ax + by)\eta^3 \\
z\theta &= \zeta(\nu + y(cx + dy)\eta) + y(ax + by)\eta(x\Xi_2 + y\Xi_1) \\
\nu\theta &= \zeta(\xi + cy\eta^2) + y(ax + by)\eta^2\Xi_2.
\end{align} \tag{2.3}

Similarly, the nine equations (2.1), (2.2) and (2.3) taken altogether give the unprojection to a codimension 4 variety $u_2: (D_2 \subset U_2) \dasharrow (P_0 \in U_3)$, which contracts $D_2$ to a $\frac{1}{3}(1, 1, 2)$ quotient singularity at the coordinate point $P_0 \in U_3$. Moreover $U_3$ comes with a birational morphism to $U$:

$$\sigma_3: U_3 \subset U \times \mathbb{P}(1, 1, 2, 3)_{\nu: \xi; \eta: \zeta; \theta} \to U.$$

The reduced central fibre $\sigma_3^{-1}(P)_{\text{red}}$ contains a third divisor $D_3 = \mathbb{P}(1, 1, 3)_{\xi; \eta; \theta}$ given by the ideal $I_{D_3} = (x, y, z, \nu, \zeta) \subset \mathcal{O}_{U_3}$. Unprojecting $D_3$ gives a new generator $\kappa \in I_1^{[4]}$ with unprojection equations:

\begin{align}
x\kappa &= \xi\Xi_3 + \eta(\theta + b\eta^3); \\
y\kappa &= \xi\theta - a\eta^4; \\
z\kappa &= \theta(\nu + (bx^2 + cxz + dy^2)\eta) + a\eta(x^2\Xi_3 + xyz\Xi_2 + y^2\Xi_1\eta^2); \\
\nu\kappa &= \theta(\xi + (bx + cy)\eta^2) + a\eta^2(x\Xi_3 + y\Xi_2); \\
\zeta\kappa &= \theta(\nu + b\eta^3) + a\Xi_3\eta^3.
\end{align} \tag{2.4}

Finally, the fourteen equations (2.1), (2.2), (2.3) and (2.4) together define a codimension 5 variety $u_3: (D_3 \subset U_3) \dasharrow (P_\kappa \in U_4)$, which contracts $D_3$ to a $\frac{1}{4}(1, 1, 3)$ quotient singularity at the coordinate point $P_\kappa \in U_4$. This $U_4$ also comes with a birational morphism to $U$:

$$\sigma_4: U_4 \subset U \times \mathbb{P}(1, 1, 1, 2, 3, 4)_{\nu: \xi; \eta: \zeta; \theta; \kappa} \to U.$$

We claim that this is the divisorial extraction from $\Gamma$, that is, that $U' = U_4$ and $\sigma = \sigma_4$. Indeed substituting $x = y = z = 0$ into these fourteen equations, we see that the reduced central fibre $Z := \sigma_4^{-1}(P)_{\text{red}} \subset P \times \mathbb{P}(1, 1, 1, 2, 3, 4)$ is cut out by the equations

$$Z = \{\nu = \zeta = \xi\theta - a\eta^4 = \theta^2 + b\theta\eta^3 + a\Xi_3\eta^3 = \theta\eta + b\eta^4 + \xi\Xi_3 = 0\}.$$

It follows that $Z$ is (generically) the union of five lines meeting at $P_\kappa \in U_4$ corresponding to the five roots of $\gamma'$, since the equations defining $Z$ imply

$$\xi(\theta\eta + b\eta^4 + \xi\Xi_3) = a\eta^5 + b\xi\eta^4 + c\xi^2\eta^3 + d\xi^3\eta^2 + e\xi^4\eta + \xi^5 = 0$$

that is, $\gamma'(\eta, \xi) = 0$.

If the roots of $\gamma$ are distinct then we can check that $U_4$ is smooth along $Z$ outside of $P_\kappa \in U_4$. However if $\gamma'$ degenerates to obtain a root of multiplicity $m$, then $m$ of the components of $Z$ come together and $U_4$ picks up an isolated $cA_{m-1}$ singularity at the point where $\kappa = 0$ along this line.

Finally we must check that the birational transform $S' = \sigma^{-1}S$ is isomorphic to $S$. Since $S'$ is given by $\eta = xz - y^4 = 0$ we can check that $S' \cap Z = P_\kappa$ is a single point. Moreover at this point, $S'$ is given by the hyperplane section $(\eta = 0) \subset \frac{1}{4}(1, 1, 3)_{\xi; \eta; \theta} = \frac{1}{4}(1, 3)_{\xi; \theta}$, which is an $A_3$ singularity and hence $S' \cong S$. \qed
2.3. Proof of Theorem 1

The last thing left to check in the proof of Theorem 1 is the existence in each case of a divisorial extraction \( \sigma : X' \to X \), which has the properties claimed in Step 3(ii) of the construction. This is equivalent to checking that a degree \( d \) curve \( \Gamma \subset E \subset X \) can have one of the singularity types appearing in Proposition 2.3.

Possible singularity types. We simply have to check whether it is possible to take coefficients of degree \( \geq 0 \) in the relevant orbifold equation of Proposition 2.3. Only case B.1 and the singularity type \( \Gamma_{(1,3)} \) is impossible. In the other cases we get the following singularity types by taking coefficients of the following degrees:

(A.1) Singularity type \( \Gamma_{(3)} \) if we take \( a_2, b_2, c_2, d_2 \in \mathbb{C}[u_1, x_1, y_1, z_1] \).

(A.2) Singularity type \( \Gamma_{(3)} \) if we take \( a_1, b_1, c_1, d_1 \in \mathbb{C}[u_1, x_1, y_1, z_1] \).

(A.3.i) Singularity type \( \Gamma_{(1,3)} \) if we take \( a_3, b_3, c_2, d_1, e_0 \in \mathbb{C}[u_1, x_1, y_1, z_2] \).

(A.3.ii) Singularity type \( \Gamma_{(4,0)} \) if we take \( a_2, b_2, c_2, d_2, e_2 \in \mathbb{C}[u_1, x_1, y_1, z_2] \).

(A.4) Singularity type \( \Gamma_{(3)} \) if we take \( a_3, b_2, c_1, d_0 \in \mathbb{C}[u_1, x_1, y_2, z_3] \).

(B.1) Singularity type \( \Gamma_{(4,0)} \) if we take \( a_0, b_0, c_0, d_0, e_0 \in \mathbb{C}[u_1, x_1, y_1, z_2] \).

(B.2) Singularity type \( \Gamma_{(5,0,0)} \) if we take \( a_0, b_0, c_0, d_0, e_0, f_0 \in \mathbb{C}[u_1, x_1, y_1, z_3] \).

Irreducibility. It is not hard to check that the generic curve with such a choice of coefficients is irreducible in each of the cases A.1–A.4. However in the B.1 case, since the coefficients are all constant, the orbifold equation for \( \Gamma \) is a quartic polynomial in \( \alpha^2 \) and \( \beta \). Thus \( \Gamma \) decomposes as a union of four irreducible smooth curves (which meet non-transversely) corresponding to the roots of this quartic. Since \( \rho_N = 5 \) we must have \( \rho_Y = 4 \). Similarly in the B.2 case \( \Gamma \) has five irreducible smooth components and \( \rho_Y = 5 \).

3. Constructing the two A.3 cases by unprojection

In their treatment of the A.1 and A.2 cases, Prokhorov & Reid [7, §6.4] also explain how to construct these two families explicitly by unprojection. Similar unprojection constructions are also possible for the remaining cases of Table 1, which are essentially global versions of the local unprojection construction used to prove Proposition 2.3. We will only construct and study the two families in the A.3 case. The enthusiast will enjoy working out the remaining cases A.4, B.1, B.2 for his or herself.

Since we know the Hilbert series of the \( \mathbb{Q} \)-Fano 3-fold \( Y \) which we are trying to construct and we know how to construct \( Y \) using unprojection, we can guess the degrees of the masked generators and relations which are obscured in the Hilbert numerator.

3.1. The two A.3 families

By following the method in §1 for the A.3 case, we construct a \( \mathbb{Q} \)-Fano \( (Y, B) \) of index \( q' = 2 \), degree \( B^3 = \frac{7}{5} \) and \( \dim |B| = 3 \) which has a single \( \frac{1}{2}(1, 2, 4) \) quotient singularity. By the Ice Cream formula [3] the Hilbert series of \( Y \) is:

\[
P_{Y,B}(t) = \frac{1 + t^2}{(1 - t)^3} + \frac{t^2 + t^4}{(1 - t)^3(1 - t^6)} = \frac{N(t)}{(1 - t)^3(1 - t^3) (1 - t^5)(1 - t^4)(1 - t^5)}
\]

where the numerator is the degree 17 polynomial:

\[
N(t) = 1 - 2t^4 - 2t^5 - 2t^6 + 2t^7 + 3t^8 + 3t^9 + 2t^{10} - 2t^{11} - 2t^{12} - 2t^{13} + t^{17}
\]

\[
= 1 - (2t^4 + 2t^5 + 3t^6 + t^7 + t^8) + (t^6 + 3t^7 + 4t^8 + 4t^9 + 3t^{10} + t^{11}) - \cdots
\]

where \( \cdots \) denotes the usual palindromic Gorenstein symmetry. We claim that \( Y \) has three masked relations in degrees 6, 7, 8 in addition to the six relations in degrees \( 4^2, 5^2, 6^2 \) which
can be read off from \( N(t) \). This will be justified by the unprojection calculation below and it suggests that \( Y \) has a presentation in the familiar Gorenstein codimension 4 format with nine equations and sixteen syzygies:

\[
Y_{42,52,63,7,8} \subset \mathbb{P}(1^3, 2^2, 3, 4, 5)_{u, x, y, z, \xi, \nu, \zeta, \theta}.
\]

We can assume that the orbinates at the \( \frac{1}{5}(1,2,4) \) point \( P_\theta \in Y \) are given by \( u_1, \xi_2, \zeta_4 \). Then \( Y \) has four equations of the form \( x \theta = A_6, y \theta = B_6, z \theta = C_7, \nu \theta = D_8 \) which have degrees 6, 7, 8 and eliminate the four variables \( x, y, z, \nu \) at the point \( P_\theta \). We consider the projection from \( P_\theta \in Y \):

\[
\tilde{\theta}: (P_\theta \in Y) \longrightarrow (\Pi \subset Y') \subset \mathbb{P}(1^3, 2^2, 3, 4).
\]

This corresponds to the \( (1, 2, 4) \)-weighted blowup of \( P_\theta \) followed by a small contraction of some curves in the tangent space \( T_y Y \cap Y \). Indeed \( Y \) does not contain any point along \( \mathbb{P}(2,4)_{\xi, \zeta} \), as the only high index point of \( Y \) is \( P_\theta \), and since

\[
T_y Y \cap Y = \{ A_6 = B_6 = C_7 = D_8 = 0 \} \subset \mathbb{P}(1,2,4,5)_{u, \xi, \zeta, \theta}
\]

this can define at most a set of isolated lines passing through \( P_\theta \). The projection takes us out of the Mori category since \( Y' \) contains a line of \( \frac{1}{3}(1,1) \) singularities with a dissident \( \frac{1}{3}(1,2,3) \) point at \( P_\zeta \in Y' \) (in addition to the non-\( \mathbb{Q} \)-factorial singularities given by the small contraction). In particular the orbinates at \( P_\zeta \in Y' \) are \( u_1, \xi_2, \nu_3 \) and hence \( Y' \) has equations \( x\zeta, y\zeta, z\zeta = \cdots \) in degrees 5, 5, 6 which eliminate \( x, y, z \) at \( P_\zeta \in Y \). We have found the masked equations in degrees 6, 7, 8 as claimed.

\( Y' \) is a special 3-fold in codimension 3 containing the plane \( \Pi = \mathbb{P}(1,2,4)_{(u,\xi,\zeta)} \) which is the exceptional divisor of the projection. We reverse this process to construct \( Y \) by unprojecting \( \Pi \subset Y' \).

The special 3-fold \( Y' \). The projection gives a codimension 3 variety \( Y' \) defined by the maximal Pfaffians of a \( 5 \times 5 \) skew matrix \( M \) with entries of the following weights:

\[
Y' = \text{Pf} \begin{pmatrix} 4 & 3 & 3 & 2 \\ 3 & 3 & 2 \\ 2 & 1 \\ 1 \end{pmatrix} = 0 \subset \mathbb{P}(1^3, 2^2, 3, 4)_{u, x, y, z, \xi, \nu, \zeta}.
\]

Without loss of generality we can use row and column operations and rename variables to write \( M \) in the form:

\[
M = \begin{pmatrix} \xi & \nu & t_1 \xi + d_3' & -c_2' \\ -a_3' & \nu + c_3' & \xi \\ t_2 & y \\ z & x \end{pmatrix}
\]

where \( a' \in \mathbb{C}[u, x, y], c' \in \mathbb{C}[u, y, z] \) and \( b', c', t \in \mathbb{C}[u, y] \) are polynomials of the indicated degree. Now we must impose some further conditions to ensure that \( Y' \) contains \( \Pi \), or equivalently that the equations \( \text{Pf} M = 0 \) are contained in the ideal \( I_{\Pi} = (x, y, z, \nu) \). The two ways of doing this are given by the Tom and Jerry formats of [2]. In our case \( M \) is in Tom2 format for the ideal \( I_{\Pi} \) if \( d', e', t \in I_{\Pi} \) (that is, all entries except those in the second row and column are in \( I_{\Pi} \)) and in Jer34 format if \( a', c', d', t \in I_{\Pi} \) (that is, all entries in the third and fourth row and column are in \( I_{\Pi} \)). In either case we have \( t = t_1(u, y) \in I_{\Pi} \) and hence we can take \( t = y \).
Recovering $\Gamma \subset X$. In direct analogy to Prokhorov & Reid’s example (see [7, §6.4, (6.4.4)]), by projecting from $P_1 \in Y'$ we see that we can construct $Y$ as a double unprojection, starting from the complete intersection:

$$
\begin{pmatrix}
  x & y^2 & -(d'y + e'z) \\
  y & z & a'x + c'y \\
  \nu & -\xi & 1
\end{pmatrix} = 0
$$

with first unprojection ideal $(x, y, z)$. Taking the minors of the $2 \times 3$ matrix appearing in this format recovers the equations of the curve $\Gamma \subset X = \mathbb{P}(1, 1, 2, 3)_{u, x, y, z}$, which was blown up in the construction of the Sarkisov link.

3.1.1. The Tom$_2$ family $\mathcal{T}$. For $M$ to be in Tom$_2$ format we must take $d', e' \in I_{\Pi}$ in addition to $t = y$. We can take $d'_3 = d_2y$ and $e'_2 = e_1y + f_0z$. Then $M$ and the unprojection equations for the ideal $I_{\Pi}$ are:

$$
\begin{pmatrix}
  \zeta & \nu & y(\xi + d) & -(ey + fz) \\
  -a' & \nu + c' & \xi \\
  z & y & x
\end{pmatrix}
\begin{pmatrix}
  x \theta \\
  y \theta \\
  z \theta \\
  \nu \theta
\end{pmatrix} = \zeta^3(\xi + d) + e\zeta(\nu + c') + f(\nu + c')^2
$$

(3.1)

We note that in this case the curve $P \in \Gamma \subset X$ has the orbifold equation:

$$
\gamma = a'_4\alpha^5 + c'_2\alpha^3\beta + d_2\alpha^2\beta^3 + e_1\alpha^3 + f_0\beta^7
$$

which is the orbifold equation for a singularity of type $\Gamma_{(1,3)}$.

3.1.2. The Jer$_{34}$ family $\mathcal{J}$. For $M$ to be in Jer$_{34}$ format we must take $a', c', d' \in I_{\Pi}$ in addition to $t = y$. This time we can take $d'_3 = a_2x + b_2y$, $c'_3 = c_2y$ and $d'_3 = d_2y$. Then $M$ and the unprojection equations for the ideal $I_{\Pi}$ are:

$$
\begin{pmatrix}
  \zeta & \nu & y(\xi + d) & -c' \\
  -(ax + by) & \nu + cy & \xi \\
  z & y & x
\end{pmatrix}
\begin{pmatrix}
  x \theta \\
  y \theta \\
  z \theta \\
  \nu \theta
\end{pmatrix} = \zeta^3 + d\xi^2 + ce'\xi + e'(\xi + be')
$$

(3.2)

In this case the orbifold equation of the curve $P \in \Gamma \subset X$ is

$$
\gamma = a_2\alpha^8 + b_2\alpha^6\beta + c_2\alpha^4\beta^2 + d_2\alpha^2\beta^3 + e'_2\beta^4
$$

which is the orbifold equation for a singularity of type $\Gamma_{(4,0)}$.

3.1.3. Conclusion. We sum up the results of this section in the following Proposition.

**Proposition 3.1.** Suppose that $(Y, B)$ is a $\mathbb{Q}$-Fano 3-fold with $-K_Y = 2B$, degree $B^3 = \frac{7}{5}$, $\dim |B| = 3$ and a $\frac{1}{2}(1, 2, 4)$ quotient singularity $Q \subset Y$. Then $Y$ is embedded as a subvariety in weighted projective space of codimension 4 $Y \subset \mathbb{P}(1, 1, 1, 2, 2, 3, 4, 5)_{u, x, y, z, \xi, \nu, \zeta, \theta}$ and belongs to one of the following two families.

(i) The Tom$_2$ family $\mathcal{T}$, with equations given by (3.1) for some choice of functions $a'_3, c'_3, d_2, e_1, f_0 \in \mathbb{C}[u, x, y, z]$ of the specified degree. In this case the Sarkisov link
Y \longrightarrow X = \mathbb{P}(1,1,1,2) \text{ ends with a divisorial contraction to a curve } \Gamma \subset X \text{ with a singularity of type } \Gamma_{(1,3)}.

(ii) The Jer_{34} family \mathcal{J}, with equations given by (3.2) for some choice of functions \(a_2, b_2, c_2, d_2, e'_2 \in \mathbb{C}[u,x,y,z]\) of the specified degree. In this case the Sarkisov link \(Y \longrightarrow X = \mathbb{P}(1,1,1,2)\) ends with a divisorial contraction to a curve \(\Gamma \subset X\) with a singularity of type \(\Gamma_{(4,0)}\).

3.2. A common degeneration

By Proposition 3.1 we see that every \(\mathbb{Q}\)-Fano 3-fold \(Y\) with the numerical invariants given in §3.1 belongs to one of the two families \(\mathcal{T}\) and \(\mathcal{J}\). Moreover these correspond precisely to two families of \(\mathbb{Q}\)-Fano 3-folds which admit a Sarkisov link to \(X = \mathbb{P}(1,1,1,2)\) ending in the divisorial contraction to \(\Gamma \subset X\), a degree 14 curve either with orbifold equation \(\gamma_{\mathcal{T}}\) and a singularity of type \(\Gamma_{(1,3)}\), or with orbifold equation \(\gamma_{\mathcal{J}}\) and a singularity of type \(\Gamma_{(4,0)}\).

The format for \(Y\) is determined by the curve \(\Gamma \subset X\), which in turn is determined by the choice of coefficients in the orbifold equation, up to rescaling by a constant. Counting the number of monomials in \(\gamma_{\mathcal{T}}\) and \(\gamma_{\mathcal{J}}\) we see that the dimension of each family is given by:

\[
\dim \mathcal{T} = 37 - 1 = 36 \quad \text{and} \quad \dim \mathcal{J} = 35 - 1 = 34.
\]

Moreover these two families have a common degeneration corresponding to the curve singularity \(P \in \Gamma\) of type \(\Gamma_{3,2}\) with orbifold equation:

\[
\gamma_{3,2}(\alpha, \beta) = a_2 \alpha^8 + b_2 \alpha^6 \beta + c_2 \alpha^4 \beta^2 + d_2 \alpha^2 \beta^3 + e_1 \alpha \beta^5 + f_0 \beta^7
\]

which is obtained when \(M\) is simultaneously in both Tom_{32} and Jer_{34} format, that is, take either \(a'_3 = a_2 x + b_2 y, \quad c'_3 = c_2 y\) in (3.1) or \(e'_2 = e_1 y + f_0 z\) in (3.2). Similarly the dimension of this intersection is \(\dim(\mathcal{T} \cap \mathcal{J}) = 31\) and these families fit together as in Figure 2.

Now suppose that \(Y\) is a general member of the intersection \(\mathcal{T} \cap \mathcal{J}\). Since \(M\) is in both Tom_{32} and Jer_{34} format simultaneously it follows from looking at either format, \(\mathcal{T}\) or \(\mathcal{J}\), that all of the nine equations defining \(Y\) are contained in \(m_u^2\), where \(m_u\) is the maximal ideal of the point \(P_u \in Y\). Hence \(P_u \in Y\) is an index 1 singularity, which has embedding dimension 7 and therefore cannot be terminal. It would be interesting to know if such a singularity is canonical.

Remark 3.2. These two deformation families sharing a common intersection is a common feature of Tom and Jerry constructions. For example all classes of \(\mathbb{Q}\)-Fano 3-folds in codimension 4, which have index 1 and admit a type I projection, have at least two deformation families corresponding to Tom and Jerry families [2].

Acknowledgement. I would like to thank Miles Reid for all his help and advice as well as the referee, whose comments have helped improve the layout of the paper.
References

1. G. Brown and A. Kasprzyk, ‘The graded ring database’, 2015. http://www.grdb.co.uk
2. G. Brown, M. Kerber and M. Reid, ‘Fano 3-folds in codimension 4, Tom and Jerry, Part I’, Compos. Math. 148 (2012) 1171–1194.
3. A. Buckley, M. Reid and S. Zhou, ‘Ice cream and orbifold Riemann–Roch’, Izv. Math. 77 (2013) 461–486.
4. T. Ducat, ‘Divisorial extractions from singular curves in smooth 3-folds’, Int. J. Math. 27 (2016) 23.
5. S. Kleiman and J. Landolfi, ‘Geometry and deformation of special Schubert varieties’, Compos. Math. 23 (1971) 407–434.
6. S. Mori, ‘Threefolds whose canonical bundles are not numerically effective’, Ann. of Math. (2) 116 (1982) 133–176.
7. Y. Prokhorov and M. Reid, ‘On Q-Fano threefolds of Fano index 2’, Minimal models and extremal rays, Advanced Studies in Pure Mathematics 70 (Mathematical Society of Japan, Tokyo, 2016) 397–420.
8. M. Reid, ‘Surface cyclic quotient singularities and Hirzebruch-Jung resolutions’, 2018, http://homepages.warwick.ac.uk/~masda/surf/more/cyclic.pdf.
9. M. Schlessinger, ‘Rigidity of quotient singularities’, Invent. Math. 14 (1971) 17–26.

Tom Ducat
Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
Japan

Current address:
School of Mathematics
University of Bristol and the Heilbronn Institute for Mathematical Research
Bristol BS8 1TW
United Kingdom

Tom.Ducat@bristol.ac.uk