A unifying approach on bias and variance analysis for classification

Cemre Zor* · Terry Windeatt

Abstract Standard bias and variance (B&V) terminologies were originally defined for the regression setting and their extensions to classification have led to several different models / definitions in the literature. In this paper, we aim to provide the link between the commonly used frameworks of Tumer & Ghosh (T&G) and James. By unifying the two approaches, we relate the B&V defined for the 0/1 loss to the standard B&V of the boundary distributions given for the squared error loss. The closed form relationships provide a deeper understanding of classification performance, and their use is demonstrated in two case studies.

Keywords Bias and Variance · Classification · Performance Analysis · Regression

1 Introduction

Bias and variance (B&V) analysis is one of the main approaches that provide useful insights into the underlying theories behind classification performance. The analysis is borrowed from the regression setting and aims to decompose the prediction error of a given classifier into the terms of B&V to evaluate their effects on the performance. Therefore, it can help answer questions such as “How can we compare the accuracy of two different types of classifiers?”, “What is it that makes stronger classifiers perform well? Is it the reduction in the bias they bring about, or in variance, or both?”. Other than being theoretically interesting, the answers to these questions are also meant to provide better classifier design strategies which bring about improved prediction performance.

After the initial decomposition of the prediction error into the standard B&V terms in the regression setting by [1], different studies have attempted to carry over this analysis into the classification setting while preserving the meanings of the terms and the additive property of the decomposition. These studies had to address two main difficulties: 1) In classification, the predictor (estimator) responses (labels/targets) are obtained as hard outputs instead of the soft/continuous responses of regression 2) The associated loss function is 0/1 in contrast to the

*Corresponding author

C. Zor
Centre for Medical Image Computing (CMIC)
University College London
London, WC1E 7JE, UK E-mail: c.zor@ucl.ac.uk

T. Windeatt
Centre for Vision, Speech and Signal Processing (CVSSP)
University of Surrey
Guildford, GU2 7XH, UK E-mail: t.windeatt@surrey.ac.uk
squared error. In order to circumvent the difficulties, different frameworks have proposed significantly different ways to define the classification B&V, yet, almost all frameworks have brought about multiple drawbacks. Some examples for these drawbacks can be given as the limitations on the use of these frameworks with general loss functions other than 0/1, and the existence of impractical and non-standard characteristics like negative variance and non-zero bias assigned to the Bayes classifier.

Standing out from the rest of the frameworks, [2] and [3] have overcome most shortcomings and managed to provide coherent decompositions applicable for general loss functions while adhering to the characteristics of the standard B&V of the regression setting. In particular, James has defined a generalised B&V in line with the original meaning, such that, bias quantifies the average distance between the aggregate predictor output and the aggregate data response, and the variance quantifies the variability of the predictor. James has also proposed two new notions, systematic/bias effect (SE) and variance effect (VE), in order to retain the additive decomposition of the error for all loss functions.

From a different point of view, [4,5,6] have assumed the existence of soft classifier outputs and directly inherited the B&V terms from the regression setting rather than proposing new definitions. In other words, they have shown that it is possible to decompose the classification error into the standard B&V of the estimation errors belonging to the underlying probability distributions, using squared error loss.

In the literature, there have been studies focusing on the relationship between the various definitions of classification B&V by formulating them in unified frameworks [7,2,8]. However, there has not yet been any attempt to link the Tumer & Ghosh (T&G) analytical model to any other classification B&V framework. The aim of this paper is to establish this missing connection by relating T&G to James’ model, chosen as a representative of the classification B&V frameworks. Specifically, after presenting the frameworks of Geman et al., James and T&G in detail, we reformulate James’ SE and VE in terms of the T&G parameters. We also provide two case studies to highlight the importance of the theoretical derivations established in this paper.

The links established in this study not only provide insights to understanding classification performance, but also lead to closed form expressions of classification B&V for determining when and how performance improves or deteriorates. The findings are expected to be useful to the pattern recognition and machine learning community, as the B&V decomposition of 0/1 loss function and the B&V trade-off are tools employed by many researchers. A few examples of recent studies addressing the issues raised in this paper can be given as follows: In [9], a theoretical and experimental analysis of the T&G framework is utilised for analysing the performance of linear combination rules and bagging ensembles. In [10], the first attempt at a unified framework for hyperspectral image classification from a B&V decomposition point of view is made, while considering all steps of the classification process: feature extraction, feature selection, classification, and post-processing. The proposed decision tree ensemble in [11], is shown to achieve increased accuracy by guaranteeing a significant reduction in variance and a lower bias compared to the standard ensemble methods. The B&V decomposition used in [12] shows how their proposed ensemble classifiers decrease error due to a reduction in bias, a reduction in unbiased variance, an increase in biased variance or, a combination of these factors. In [13], an extensive experimental comparison of the B&V trade-off is made for sixteen ensemble methods. A useful discussion about the B&V decomposition problem is provided in [14], and a novel attempt is made at estimating B&V. In [15], the proposed novel classifier combining rule is shown to be superior to other fixed combining rules by demonstrating that it has noticeably lower bias and slightly higher variance. In [16], the B&V decomposition is used in cryptographic systems to determine the impacts of the error rate of template and stochastic attacks, and also to extract the best profiled attack. In order to achieve a trade-off between fairness and accuracy in sensitive applications such as healthcare or criminal justice, in [17] cost-based metrics of discrimination are decomposed into bias, variance, and noise, and methods are proposed for estimating and reducing each term.
2 Bias-Variance Frameworks

The bias and variance decomposition of the prediction error was initially performed by [1] in the regression setting using squared-error loss, and can be summarised as follows.

Let the estimate of the response, $\hat{Y}$, for each input pattern (feature vector) $X$ be $\hat{Y}$, which is a variable depending on the training set. Different training sets consist of different patterns sampled from the same underlying distribution, or same patterns represented by different modalities. Given a particular pattern $X$, the effectiveness of the prediction can be measured using the mean squared error as

$$E_{Y} \left[ (Y - \hat{Y})^2 | X \right] = E_{Y} \left[ \left( \hat{Y} - E_{Y} [\hat{Y} | X] \right)^2 | X \right] + E_{Y} \left[ (Y - E_{Y} [Y | X])^2 | X \right]$$

(1)

In Eq. (1), the first term on the r.h.s., indicating the expected difference between the estimate and its expectation, is named as variance of the predictor; whereas the second term, indicating the squared difference between the regression and the expected estimation as its bias. The third term is the variance of the response, which can be referred to as the irreducible noise.

Subsequently, [18], [19], [20], [7], [21], [22], [23], [3] and [2] have extended the analysis for the classification setting; however, a universally accepted framework still does not exist. Although all frameworks aim at preserving the meanings of the standard B&V terms and the additive property of the error decomposition, they have difficulties maintaining other crucial B&V characteristics and/or bring about undesirable properties, such that: 1) Dietterich and Kong allow the existence of negative variance, and moreover, allow the Bayes classifier to have positive bias. 2) For each input pattern, Breiman separates the predictors into two sets as biased and unbiased; and accordingly, each test pattern has solely bias or variance. 3) Kohavi and Wolpert assign a non-zero bias to the Bayes classifier. 4) The definitions of Tibshirani, Heskes and Breiman are difficult to generalise and extend to general loss functions.

Among all definitions, the framework of James overcomes the listed disadvantages, and offers advantages in the sense of characterising consistent B&V decompositions for arbitrary loss functions. This framework is investigated in detail in Section 2.1. Note that in [7,2,8], the links between the majority of these definitions have been established under unified representations.

2.1 Bias-Variance Analysis of James

[2] extends the B&V analysis, originally proposed for squared error under regression setting by [1], to all symmetric loss functions including 0/1. James argues that it is not possible to both preserve the additive property of the prediction error decomposition and maintain the terms’ standard meanings. Hence, in his decomposition, two sets of formulations are proposed to satisfy these requirements separately. First, by adhering to the original meanings, the bias is defined as the average distance between the systematic parts of the response and the predictor; and the variance as the variability of the predictor. Second, in addition to the bias and variance terms, the notions of systematic effect (SE) and variance effect (VE) are proposed. These new terms satisfy the additive error decomposition for all symmetric loss functions, and characterise the effects of bias and variance on the prediction error. For example, having (positive) variance might actually trigger a reduction in the prediction error and hence bring about a negative VE. It should be noted that in the standard setting, under squared error loss, SE and VE simplify into bias and variance respectively, satisfying the additive decomposition.

Using the B&V of the standard setting from Eq. (1), the generalised B&V of an estimator $\hat{Y}$ for a given pattern $X$ is given by

$$Bias \left( \hat{Y}_X \right) = L(SY_X, S\hat{Y}_X) \quad \text{and} \quad Var \left( \hat{Y}_X \right) = E_{Y} [L(\hat{Y}_X, S\hat{Y}_X)]$$

(2)
where $L(u, w)$ is the loss when $w$ is used in predicting $u$, and $S \hat{Y}$ and $SY$ are called the systematic parts of $\hat{Y}$ and $Y$, respectively (Note that we will drop the conditioning on the input pattern $X$ for simplicity). For squared error loss ($L_2$), we know that $S \hat{Y} = E_\omega[Y]$, which is the regression or the best predictor of $Y$ under $L_2$ [1]. Hence, during the extension for general loss functions ($L$) as given in Eq. [2] $S \hat{Y}$ can be re-written as $S \hat{Y} = \arg\min_w E[Y|L(\hat{Y}, w)]$, and likewise, $SY = \arg\min_{\omega} E[Y|L(Y, w)]$.

The decomposition of the prediction error is interpreted as the sum of three terms: the variance of the response, the systematic (bias) effect ($S\hat{Y}$) and the variance effect ($VE$):

$$E_{\hat{Y}, Y}[L(\hat{Y}, Y)] = Var(Y) + SE(\hat{Y}, Y) + VE(\hat{Y}, Y)$$

where $SE(\hat{Y}, Y) = E_\hat{Y}[L(Y, \hat{Y})] - L(Y, SY)$, and $VE(\hat{Y}, Y) = E_\hat{Y}[L(Y, \hat{Y})] - L(Y, S\hat{Y})$.

For the specific case of classification with 0/1 loss, we have $L(u, w) = I(u \neq w)$, where $I(q)$ is the indicator function. Assume that we are working on a $k$ class problem, such that instead of operating in a continuous domain as in regression, $Y$ takes on discrete values: $Y \in \{\omega_1, \omega_2, \omega_3, \ldots, \omega_k\}$. If we define

$$P(\omega_i|X) = p(Y = \omega_i|X) \quad \text{and} \quad \hat{P}(\omega_i|X) = p(\hat{Y} = \omega_i|X)$$

where $P$ is the posterior probability distribution function for the response, $Y$, and $\hat{P}$ is the distribution function obtained over multiple training sets for the estimation, $\hat{Y}$. Then,

$$SY = \arg\min_{\omega_i} E_Y[I(Y \neq \omega_i)|X] = \arg\max_{\omega_i} P(\omega_i|X)$$

$$S\hat{Y} = \arg\max_{\omega_i} \hat{P}(\omega_i|X).$$

(5)

It can be observed in Eq. [5] that $SY$ is the class assigned by the Bayes rule; and $S\hat{Y}$ can be interpreted as the aggregate classifier decision. Therefore

$$Bias_J(S\hat{Y}) = I(S\hat{Y} \neq SY|X)$$

$$Var_J(Y) = p(Y \neq SY|X) = 1 - \max_{\omega_i} P(\omega_i|X)$$

$$Var_J(\hat{Y}) = p(\hat{Y} \neq S\hat{Y}|X) = 1 - \max_{\omega_i} \hat{P}(\omega_i|X)$$

$$SE(\hat{Y}, Y) = p(Y \neq S\hat{Y}|X) - p(Y \neq SY|X) = P(SY|X) - P(S\hat{Y}|X)$$

$$VE(\hat{Y}, Y) = p(Y \neq \hat{Y}|X) - p(Y \neq S\hat{Y}|X) = P(S\hat{Y}|X) - \sum_{i} P(\omega_i|X) \hat{P}(\omega_i|X).$$

(6)

Note that, the irreducible error, $Var_J(Y)$, is equal to the Bayes error.

2.2 Tumer and Ghosh Model (T&G Model)

In contrast to the other B&V classification frameworks, Tumer and Ghosh (T&G) provide an analytic model [4,5,6] by utilising the class posterior probability distributions of the data and their estimates given by the predictors, assuming these distributions are known. By decomposing the squared estimation errors belonging to these distributions using Geman’s formulation, the added classification error (on top of the Bayes error) for a non-optimal learning algorithm is measured. It is important to note that this framework does not propose new terminologies of bias and variance to analyse the classification error under 0/1 loss, rather uses the standard terminology in an unconventional way to address the new setting. The analysis is confined to the data points (patterns) $X \in R^n$ instead of $X \in R^n$, but the conclusions derived are expected to hold for $X \in R^n$.
The T&G theory assumes that in a localised decision (transition) region, only two classes possess significant posterior probabilities, which will leave the contributions of other classes as negligible. Consider a multi-class problem and the posterior probability functions belonging to classes $\omega_i$ and $\omega_j$ for a data point $X$, given as $P(\omega_i|X)$ and $P(\omega_j|X)$ as illustrated in Figure 1. The summation of the posterior probabilities for the rest of the classes can be referred to as $P(\omega_r|X)$. An instance of the imperfect predictor, namely a base classifier which is trained on one realisation of a given data set, would approximate each posterior probability with an error (depicted by dashed lines in Figure 1) $\tilde{P}(\omega_i|X) = P(\omega_i|X) + \epsilon_i(X)$ where $\epsilon_i(X)$ is a random variable denoting the approximation (prediction) error for the class $\omega_i$ given input $X$.

The optimal decision boundary given by the Bayes classifier can be depicted as the intersection line crossing through $X^*$ in Figure 1. It is then possible to denote any pattern in the decision region in terms of its distance to the optimal boundary, such that $X_a = X^* + a$, and name the set of all possible distance values, $a$, in the decision region as $A$. Hence, a pattern located at the estimate decision boundary of a base classifier can be given by $X_b = X^* + b$. Therefore, $\tilde{P}(\omega_i|X^* + b) = P(\omega_i|X^* + b)$ and $\tilde{P}(\omega_j|X^* + b) = P(\omega_j|X^* + b) + \epsilon_j(X_b)$. (7)

By making the assumption that the posterior probabilities are locally monotonic functions around the decision boundaries in the transition regions, a linear approximation can be given:

$$P(\omega|X^* + b) + \epsilon_i(X_b) = P(\omega|X^* + b) + \epsilon_j(X_b)$$

where $P'$ denotes the derivative of $P$. Using $P(\omega|X^*) = P(\omega|X^*)$ to rewrite Eq. 8

$$b = \frac{\epsilon_i(X_b) - \epsilon_j(X_b)}{s}$$

where $s = P'(\omega_i|X^*) - P'(\omega_j|X^*)$. Note again that $\epsilon_i(X_b), \epsilon_j(X_b)$ and $b$ are random variables dependent on the posterior probability estimations obtained from the given learning algorithm. It is assumed that $\epsilon_i(X)$ can be decomposed into a class-specific bias, $\beta_i$; and a zero-mean, $(\sigma_i)^2$ class-specific variance noise term, $n_i$, which has the same distribution characteristics across all $X$: $\epsilon_i(X) = \beta_i + n_i$. Thus,

$$b = \frac{n_i - n_j}{s} + \frac{\beta_i - \beta_j}{s}$$

(10)
In Figure[1] the Bayes error region is depicted by the areas shaded by light grey (note the double addition of the areas under \(P(\omega_i|X)\)). The added error region (on top of the Bayes error) associated with the given realisation of the non-optimal classifier is the dark shaded area, which can be approximated by a triangle whose height is \(b\), and base length is given by

\[
P(\omega_i|X^* + b) - P(\omega_i|X^*) = P(\omega_i|X^*) + bP(\omega_i|X^*) - P(\omega_i|X^*) - bP(\omega_i|X^*) = bs. \tag{11}
\]

Accordingly, the area of this triangular region is equal to \(b^2s/2\); with expected value, \(R_{\text{add}}:\)

\[
R_{\text{add}} = \frac{s}{2} \int_{b \in A} b^2 p(b)db \tag{12}
\]

where \(p(b)\) is the probability distribution function of \(b\), which is defined within the decision region \(A\). It will now be shown that \(R_{\text{add}}\) can be written in terms of the B&V associated with the classifier estimation errors. For that, let us first define the variance for the random variable \(b\):

\[
\sigma_b^2 = \int_{b \in A} (b - E[b])^2 p(b)db = \int_{b \in A} b^2 p(b)db - 2E[b] \int_{b \in A} b p(b)db + (E[b])^2 = \int_{b \in A} b^2 p(b)db - (E[b])^2 \tag{13}
\]

Eq. [13] can be combined with Eq. [12] such that

\[
\frac{s}{2}\sigma_b^2 = \frac{s}{2} \left( \int_{b \in A} b^2 p(b)db - (E[b])^2 \right) = R_{\text{add}} - \frac{s}{2}E[b]. \tag{14}
\]

Therefore,

\[
R_{\text{add}} = \frac{s}{2} \left( \sigma_b^2 + (E[b])^2 \right) = \frac{s}{2} \left( \sigma_b^2 + \beta^2 \right) \tag{15}
\]

where \(\sigma_b^2\) is the T&G overall variance (\(\text{Var}_{TG} = \sigma_b^2\)) and \(\beta\) is the T&G overall bias (\(\text{Bias}_{TG} = \beta\)).

In Eq. [15] the expected added area is shown to be a function of the standard B&V in terms of the variable \(b\), which is itself a function of the classifier approximation errors as given in Eq. [9]. Therefore, the overall B&V utilised in the T&G model relates to the difference of the classifier approximation errors. Using Eq. [10] the terms in Eq. [15] can now be further expanded. Initially, we have

\[
\text{Bias}_{TG} = \beta = E[b] = \frac{\beta_i - \beta_j}{s} \tag{16}
\]

An alternative expression for \(\sigma_b^2\) can be derived using Eq. [10]

\[
\text{Var}_{TG} = \sigma_b^2 = \text{var}\left(\frac{n_i - n_j}{s}\right) \tag{17}
\]

\[
= \frac{1}{s^2} E\left[\left(n_i - n_j - E(n_i - n_j)\right)^2\right] = \frac{(\sigma_i)^2 + (\sigma_j)^2 - 2\text{cov}(n_i, n_j)}{s^2} \tag{18}
\]

where \((\sigma_j)^2\) is the class-specific variance of \(n_j\). Finally, substituting Eq. [16] and Eq. [18] in Eq. [15] leaves us with

\[
R_{\text{add}} = \frac{(\sigma_i)^2 + (\sigma_j)^2 - 2\text{cov}(n_i, n_j)}{2s} + \frac{s\beta^2}{2} \tag{19}
\]
which explicitly defines the expected added error region in terms of the B&V of the individual class approximation errors and is the main result of the T&G analysis.

As a summary, T&G obtains an additive B&V decomposition of the classification error because its underlying assumption on the posterior estimates allows the added error to be expressed as a function of the location of the decision boundary, which is dependent on the estimation noise. This assumption, which allows the added error region to be expressed as a triangle, implicitly converts the classification error problem into a regression problem. Refer to [4,5,6] for further details about the T&G model, to [8] for an analysis of the shortcomings of the T&G model assumptions, and to [24,25] for an extension of the T&G model where assumptions such as the existence of the ideal boundary between the same classes in the vicinity of the estimate boundary are relaxed.

3 Connections Between the Frameworks of Geman et al., James and T&G

In this study, we establish the relation between the frameworks of Geman et al., James and T&G in order to reveal when and how the classification B&V affect the prediction error, i.e. the components that trigger an increase or decrease in \( SE \) and \( VE \) are given in closed form. Specifically, after setting up the connection between the T&G model and the decomposition of Geman et al. in Section 3.1, we formulate \( SE \) and \( VE \) of James’ framework in terms of B&V of T&G in Section 3.2. A discussion on the established connections between the two frameworks are given in Section 3.3, followed by two case studies underlining the findings of this theoretical study in Section 4.

3.1 T&G Model in terms of Bias-Variance of Geman et al.

By decomposing the squared posterior probability prediction error, \( \epsilon_i(X) \), into terms of bias and a zero-mean noise for a given pattern \( X \) and class \( \omega_i \), T&G model is similar to the formulations of Geman et al. given in the regression framework.

From T&G model we know that \( \epsilon_i(X) = \beta_i + n_i \). Therefore,

\[ \beta_i^2 = (E[\epsilon_i(X)])^2 = \left(E[\tilde{P}(\omega_i|X) - P(\omega_i|X)]\right)^2. \tag{20} \]

where \( \beta_i \) is the class-specific bias for the class \( \omega_i \). For the class-specific variance:

\[ \sigma_i^2 = E \left[ (n_i - E[n_i])^2 \right] = E \left[ \left((\epsilon_i(X) - \beta_i) - E[\epsilon_i(X) - \beta_i]\right)^2 \right] = E \left[ (\epsilon_i(X) - E[\epsilon_i(X)])^2 \right] = E \left[ (P(\omega_i|X) - \tilde{P}(\omega_i|X) - E[P(\omega_i|X) - \tilde{P}(\omega_i|X)])^2 \right] = E \left[ (\tilde{P}(\omega_i|X) - E[\tilde{P}(\omega_i|X)])^2 \right]. \tag{21} \]

In contrast to the original framework of Geman et al. and its extension for classification by James, where the B&V terms are defined for the response and its estimator (\( Y \) and \( \tilde{Y} \)), T&G define the B&V on the estimate posterior probabilities produced by the prediction functions for each class (using \( P \) and \( \tilde{P} \)). Let us now compare Eq. 20 and Eq. 21 from T&G formulation with the original B&V definitions of Geman et al. From Eq. 17 we have,
Bias$(\hat{Y}) = \left( E_F \left[ \hat{Y}|X \right] - E_F \left[ Y|X \right] \right)^2$

$Var(\hat{Y}) = E_F \left[ \left( \hat{Y} - E_F \left[ \hat{Y}|X \right] \right)^2 \right] |X|.$

By interchanging $Y|X$ given in Eq. 22 with $P(\omega_i|X)$, and $\hat{P}(\omega_i|X)$ with $\hat{Y}|X$, it can be observed that the class-specific B&V terms of T&G as formulated in Eq. 20 and Eq. 21 can be obtained. Remember that the relation of the class-specific terms to the T&G overall B&V are provided in Eq. 16 and Eq. 18. Note that although $Y|X$ given in the Geman framework is a random variable, its T&G substitute $P(\omega_i|X)$ is a constant. Also, the class-specific bias of T&G is not squared ($\beta_i$ instead of $\beta_i^2$); leading to the possibility of a positive or negative value.

3.2 Bias-Variance Analysis of James using T&G Model

While establishing the connection between the B&V decompositions of James and T&G, we will still be relying on the assumption made in T&G model that the added error region for a multiclass problem would mainly be composed of the contributions of two classes, as specified in Section 2.2. Remembering that $a$ is the distance between the pattern $X_a$ located at $X^* + a$ and $X^*$, Eq. 11 can explicitly be written for all patterns lying within the decision region $A$ (i.e. $\forall a, t_1 < a < t_2$), such that

$$R_{add} = \int_{atA} SE(\hat{Y}_{X_a}, Y_{X_a})da + \int_{atA} VE(\hat{Y}_{X_a}, Y_{X_a})da.$$

Since under 0/1 loss $Var_f(Y_{X_a})$ is equal to the Bayes error for the pattern $X_a$, Eq. 23 can be re-written as

$$R_{add} = \int_{atA} SE(\hat{Y}_{X_a}, Y_{X_a})da + \int_{atA} VE(\hat{Y}_{X_a}, Y_{X_a})da.$$

For patterns outside the decision region, $SE$ and $VE$ are zero, making the total expected error equal to the Bayes error, and leaving $R_{add}$ as zero.

In Section 3.2.2 and Section 3.2.3, we will derive $SE$ and $VE$ using T&G terminology. However, first, we dedicate Section 3.2.1 for the expansions of the probability terms $P$ and $\hat{P}$ to be later used in the $SE$ and $VE$ calculations.

3.2.1 Probability Distribution Analysis

Using Figure 1 and Eq. 11 for any $a > 0$,

$$P(\omega_j|X_a) - P(\omega_j|X_a) = sa$$

where $\omega_j$ is the Bayes class. If the Bayes error rate for $X_a$ is expressed by $z(X_a)$, then

$$P(\omega_j|X_a) = 1 - z(X_a).$$
A unifying approach on bias and variance analysis for classification

After denoting the summation of the posterior probabilities belonging to classes with negligible contributions (which are considered as noise) by \( P(\omega_j | X_a) = \eta(X_a) \), we also have

\[
P(\omega_i | X_a) = z(X_a) - \eta(X_a).
\]

Eq. 27 can be used to rewrite Eq. 25 such that

\[
P(\omega_j | X_a) = sa + (z(X_a) - \eta(X_a)).
\]

Similarly, for \( a < 0 \) we have

\[
P(\omega_i | X_a) - P(\omega_j | X_a) = -sa
\]

(a)

| \( P(\omega_i | X_a) \) | \( P(\omega_j | X_a) \) | \( P(\omega_r | X_a) \) |
|----------------|----------------|----------------|
| \( 1 - z(X_a) \) | \( z(X_a) - \eta(X_a) \) | \( 0 \) |
| \( z(X_a) - \eta(X_a) \) | \( 1 - z(X_a) \) | \( 0 \) |
| \( \eta(X_a) \) | \( \eta(X_a) \) | \( 0 \) |
| \( S \ Y \) | \( \omega_i \) | \( \omega_j \) |
| \( S \ Z \) | \( \omega_j \) | \( \omega_i \) |

(b)

Table 1 Derivations of \( P \) and \( \hat{P} \) for different \( a \) values.

After denoting the summation of the posterior probabilities belonging to classes with negligible contributions (which are considered as noise) by \( P(\omega_j | X) = \eta(X_a) \), we also have

\[
P(\omega_i | X_a) = z(X_a) - \eta(X_a).
\]

Eq. 27 can be used to rewrite Eq. 25 such that

\[
P(\omega_j | X_a) = sa + (z(X_a) - \eta(X_a)).
\]

Similarly, for \( a < 0 \) we have

\[
P(\omega_i | X_a) - P(\omega_j | X_a) = -sa
\]

for which the Bayes class is \( \omega_i \). The remaining calculations for \( a < 0 \) follow in accordance with those given for \( a > 0 \) above, and a summary of the findings is provided in Table 1-a. Here, for each \( a \) value, the Bayes class is indicated as \( S \ Y \), and the second dominant class is named as \( S \ Z \).

In Table 1-b, the distribution function obtained over training sets for the prediction, \( \hat{P} \), is analysed. We know that the assignment of a pattern \( X_a \) into the classes \( \omega_i \) or \( \omega_j \) depends on the location of the estimation boundary \( X_b = x^* + b \), with \( b \) being a random variable. Hence, while calculating \( \hat{P} \), the distribution of \( b \) is utilised. From Figure 1 for the input pattern \( X_a \) to be assigned to the class \( \omega_i \) by a classifier \( c_1 \), the boundary of the classifier has to be located at \( b_{c_1} > a \). Hence, the probability of the aggregate decision for \( X_a \) being \( \omega_i \) is equal to the sum of the probabilities of all such boundaries, i.e. \( \hat{P}(\omega_i | X_a) = \int_{b_{c_1}}^{b_{c_2}} p(b) db \). Similar analysis follows for \( \hat{P}(\omega_j | X_a) \). As for \( \hat{P}(\omega_r | X_a) = 0 \), we know that due to the T&G assumption of having only two predominant classes within \( A \), the classifiers are not expected to assign a pattern into any other class.

Making use of the expansions provided in this section, \( SE \) and \( VE \) will be derived employing the T&G terminology in the following sections. During the derivations, \( da \) will be dropped for simplicity.

3.2.2 Calculation of \( SE \)

Using Eq. 6 we have,

\[
\]
\[
\int_{\omega \in \Lambda} \hat{P}(\omega|X_a) \geq \int_{\omega \in \Lambda} P(\omega|X_a).
\]

We will now analyse the terms within Eq. (30) separately. Since \( S \hat{Y} = \arg \max_{\omega \in \Lambda} P(\omega|X_a) \) under 0/1 loss (see Eq. (5)), \( P(S \hat{Y}|X_a) \) denotes the posterior probability of the class that the Bayes classifier assigns for \( X_a \). Thus,

\[
\int_{\omega \in \Lambda} P(S \hat{Y}|X_a) = \int_{\omega = t_1}^0 P(\omega|X_a) + \int_{\omega = a}^{t_2} P(\omega|X_a).
\]

The calculation of \( P(S \hat{Y}|X_a) \) on the other hand is not as trivial. From Eq. (5) \( S \hat{Y} = \arg \max_{\omega \in \Lambda} \hat{P}(\omega|X_a) \), which suggests that \( P(S \hat{Y}|X_a) \) stands for the underlying posterior probability of the most probable class according to the base classifier decisions. To analyse \( S \hat{Y} \), let us first determine when it is equal to the Bayes decision class, \( S Y \). It can be observed from Table 1-a that for \( a > 0 \), \( S Y = \omega_j \). Hence,

\[
\begin{cases}
S \hat{Y} = S Y & \text{if } \hat{P}(\omega_j|X_a) > \hat{P}(\omega_i|X_a) \\
& \quad \Rightarrow \int_{b = t_1}^{a} p(b)db > \int_{b = a}^{t_2} p(b)db.
\end{cases}
\]

Conversely, for patterns with \( a < 0 \),

\[
\begin{cases}
S \hat{Y} = S Y & \text{if } \hat{P}(\omega_j|X_a) > \hat{P}(\omega_i|X_a) \\
& \quad \Rightarrow \int_{b = t_1}^{a} p(b)db < \int_{b = a}^{t_2} p(b)db.
\end{cases}
\]

The point \( m \) where \( \int_{b = a}^{m} p(b)db = \int_{b = a}^{t_2} p(b)db = 0.5 \) is the median of the probability distribution of \( b \). For the case of \( 0 < m \), Eq. (32) and Eq. (33) can be simplified such that

\[
\begin{cases}
S \hat{Y} = S Y & \text{if } 0 < m < a \quad \text{or} \quad a < 0 < m
\end{cases}
\]

Bearing in mind the assumption that within the localised decision region only two classes are likely to have considerable posterior probabilities, we have \( S \hat{Y} = SZ \) (second dominant class) for all other \( a \) than those defined in Eq. (34) \( P(S \hat{Y}|X_a) \) can now be reformulated as

\[
\int_{\omega \in \Lambda} P(S \hat{Y}|X_a) = \int_{\omega = 0}^{a < 0} P(S Y|X_a) \\
+ \int_{0 < a < m} P(S Z|X_a) + \int_{a = m}^{a > 0} P(S Y|X_a).
\]

Using Table 1-a, Eq. (35) can be expanded to

\[
\int_{\omega \in \Lambda} P(S \hat{Y}|X_a) = \int_{\omega = t_1}^{a = 0} P(\omega_j|X_a) \\
+ \int_{a = 0}^{m} P(\omega_j|X_a) + \int_{a = m}^{t_2} P(\omega_j|X_a).
\]
Similar analysis shows that the result of Eq. 36 also holds for \( m < 0 \).

Combining the derivations for \( P_{SY} \) and \( P_{\hat{S}Y} \) from Eq. 31 and 36 and referring to Table 1a, Eq. 30 becomes

\[
\int_{a \in A} S E(\hat{Y}_{X_a}, Y_{X_a}) = \int_{a=0}^{t_2} P(\omega_j|X_a) - \int_{a=m}^{t_z} P(\omega_j|X_a) = \int_{a=0}^{m} \left( P(\omega_j|X_a) - P(\omega_j|X_a) \right) = \int_{a=0}^{m} sa = m^2 s / 2. \tag{37}
\]

Note that in Eq. 37, \( s \) is positive.

3.2.3 Calculation of VE

Using the given definition of VE from Eq. 6, we have

\[
\int_{a \in A} VE(\hat{Y}_{X_a}, Y_{X_a}) = \int_{a \in A} P(S \hat{Y}|X_a) - \int_{a \in A} \left[ \sum_{j} P(\omega_j|X_a) \hat{P}(\omega_j|X_a) \right]. \tag{38}
\]

The first term of Eq. 38, \( \int_{a \in A} P(S \hat{Y}|X_a) \), has been derived during the calculation of \( SE \) in Eq. 36. Using Table 1a, Eq. 36 can further be expanded to take the following form:

\[
\int_{a \in A} P(S \hat{Y}|X_a) = \int_{a=0}^{t_2} \left[ 1 - z(X_a) \right] + \int_{a=0}^{m} \left[ z(X_a) - \eta(X_a) \right] + \int_{a=m}^{t_z} \left[ 1 - z(X_a) \right] = \int_{a=0}^{t_2} \left[ -sa + (z(X_a) - \eta(X_a)) \right] + \int_{a=m}^{t_z} \left[ sa + (z(X_a) - \eta(X_a)) \right] \tag{39}
\]
For the second term of Eq. 38, Table 1 is used again (For convenience, $da$ and $db$ will be dropped in the intermediate steps):

\[
\int_{a \in A} \left[ \sum_i P(\omega_i | X_a) \hat{P}(\omega_i | X_a) \right] da = \int_{a=a_1}^{0} \left[ -sa + (z(X_a) - \eta(X_a)) \right] \int_{b=a}^{t_2} p(b) db da + \int_{a=a_1}^{0} \int_{b=a}^{t_2} \left[ sa + (z(X_a) - \eta(X_a)) \right] p(b) db da \\
+ \int_{a=a_1}^{t_2} (z(X_a) - \eta(X_a)) \int_{b=a}^{t_2} p(b) db da + \int_{a=a_1}^{t_2} \left[ sa + (z(X_a) - \eta(X_a)) \right] \int_{b=a}^{t_2} p(b) db da \\
= \int_{a=a_1}^{t_2} (z(X_a) - \eta(X_a)) + \int_{a=a_1}^{t_2} -sa \int_{b=a}^{t_2} p(b) + \int_{a=a_1}^{t_2} sa \int_{b=a}^{t_2} p(b) \\
= \int_{a=a_1}^{t_2} (z(X_a) - \eta(X_a)) + \int_{b=b_1}^{t_2} \left[ \int_{a=a_1}^{t_2} p(b) + \int_{b=b_1}^{t_2} \int_{a=a_1}^{t_2} p(b) - \int_{b=b_1}^{t_2} \int_{a=a_1}^{t_2} p(b) \right] \\
= \int_{a=a_1}^{t_2} (z(X_a) - \eta(X_a)) + \frac{s}{2} \left[ (t_1^2 + t_2^2) - b_2^2 \right] p(b) db. \tag{40}
\]

Using Eq. 12 for added error from Section 2.2, Eq. 40 can further be simplified.

\[
\int_{a \in A} \left[ \sum_i P(\omega_i | X_a) \hat{P}(\omega_i | X_a) \right] = \int_{a \in A} (z(X_a) - \eta(X_a)) \\
+ \frac{s(t_1^2 + t_2^2)}{2} - R_{add} \tag{41}
\]

By combining the two terms derived in Eq. 36 and Eq. 41, the formulation of \( VE \) given in Eq. 38 can finally be expanded as follows:

\[
\int_{a \in A} VE(\tilde{Y}_{X_a}, Y_{X_a}) = \int_{a \in A} (z(X_a) - \eta(X_a)) + \frac{s}{2} \left[ (t_1^2 + t_2^2) - m^2 \right] \\
- \int_{a \in A} (z(X_a) - \eta(X_a)) - \frac{s(t_1^2 + t_2^2)}{2} + R_{add} \\
= R_{add} - \frac{m^2 s}{2}. \tag{42}
\]

By writing \( R_{add} \) in terms of the T&G overall bias and variance (\( \beta \) and \( \sigma_b^2 \)) as given in Eq. 15, Eq. 42 can be reformulated as

\[
\int_{a \in A} VE(\tilde{Y}_{X_a}, Y_{X_a}) = R_{add} - \frac{m^2 s}{2} \\
= \frac{(\beta^2 + \sigma_b^2 - m^2)}{2}. \tag{43}
\]
where $s$ is positive.

The verification for the expected added error calculation provided in Eq. 24 can be obtained by adding $SE$ and $VE$ given in Eq. 37 and 42 to give

$$R_{add} = \int_{a \in A} SE(\hat{Y}_X, Y_X) da + \int_{a \in A} VE(\hat{Y}_X, Y_X) da.$$

$$= \frac{m^2 s}{2} + R_{add} - \frac{m^2 s}{2}$$

$$= R_{add}.$$  \hspace{1cm} (44)

### 3.3 Discussion

The systematic effect ($SE$) of James has been reformulated in terms of T&G model parameters in Eq. 37, which shows that the effect of classification bias on prediction performance depends on the median of the decision boundary distribution belonging to the base classifiers: $SE$ increases with the squared value of the median of the decision boundary distribution. In a symmetric distribution like the normal distribution, the median is equal to the mean and $SE$ is directly linked to the squared T&G overall bias, $\beta$.

The T&G overall bias term, which is equal to the mean of the decision boundary distribution, ($\beta = E[\beta]$), was shown in Eq. 16 to be the difference between the class-specific biases of the individual estimation errors belonging to the two classes of interest ($\beta_i - \beta_j$) scaled by a positive constant. Therefore, any circumstance causing the absolute value of this difference to decrease brings about decreased $SE$. Examples are when both estimation error biases, $\beta_i$ and $\beta_j$, are positive or negative in equal amounts. On the other hand, having $\beta_i$ and $\beta_j$ of opposite sign would create larger confusion regions, shift the expected value of the decision boundaries ($E[\beta]$) away from the Bayes boundary, and cause an increase in $SE$.

It has been demonstrated in Eq. 43 that $VE$ is directly related to the T&G overall variance, but also depends on the shape of the decision boundary distribution. The latter is represented by ($\beta^2 - m^2$), which shows that when the difference between the mean and the median of the decision boundary distribution increases, $VE$ increases as well. On the other hand, for cases with symmetric boundary distributions, $VE$ only depends on the T&G overall variance, as the difference between the mean and the median of the boundary distribution vanishes.

In Eq. 17, the T&G overall variance is defined as the variance of the decision boundary distribution, which is equal to the variance of the difference of estimation noise measured on the two classes of interest scaled by a positive constant. In Eq. 18 this was shown to be equal to the summation of the class-specific variance terms belonging to the individual estimation errors ($\sigma_i^2$ and $\sigma_j^2$), which are accompanied by a covariance term that vanishes for independent classifier outputs. Hence, an increase in any of the class-specific variance terms belonging to individual estimation errors always causes an increase in the T&G overall variance, and therefore $VE$.

### 4 Case Studies

In order to demonstrate the applicability of the T&G and James frameworks on real problems and provide practical insight, we experimentally evaluate two example scenarios in Section 4.1-4.2, by using a benchmark data set. In Section 4.1 we use the T&G theory to design more accurate ensembles and demonstrate the links established in this paper between the T&G and James frameworks. In Section 4.2 we analyse the effect of James variance on classification performance by relating it to the T&G model parameters. Since the T&G model is based on assumptions such as the existence of only two dominant classes at the decision region and the single dimensionality of the data, we will assume that the theory is applicable to multiple dimensions, and make appropriate approximations during the calculations.
4.1 Case Study 1

In this case study, we analyse and compare the performance of a single classifier and an ensemble by utilising the B&V frameworks of T&G and James. The ensemble rule selected for this study is the average, which is the multiple classifier system with mean combination. After summarising the theoretical derivations provided for the ensemble average under the T&G model in [4,5,8], we provide an experimental analysis on a benchmark data set by employing both of the frameworks.

First, we define $b_{ave}$ to be the decision boundary of the ensemble by using Eq. [10] such that

$$b_{ave} = \frac{\bar{n}_i - \bar{n}_j}{s} + \tilde{\beta}_i - \tilde{\beta}_j$$

(45)

where $\bar{n}_i = \frac{1}{N} \sum_{m=1}^N n_i^m$ and $\tilde{\beta}_i = \frac{1}{N} \sum_{m=1}^N \beta^m_i$, with $N$ being the total number of base classifiers. Here, $n_i^m$ and $\beta^m_i$ are noise and bias terms belonging to the base classifier $m$ of the ensemble for the class $\omega_i$. From Eq. [45]

$$\bar{\beta}^{ave} = E[b_{ave}] = \frac{\tilde{\beta}_i - \tilde{\beta}_j}{s}$$

(46)

Following the derivations of [8] for the expansion of the terms in Eq. [47] the added error of the mean combination rule takes the following form:

$$P^{add}_{ave} = \frac{s}{2} \left( (\sigma^{ave})^2 + (E[b_{ave}])^2 \right) = \frac{1}{N^2} \left( \sum_{m=1}^N \left( \frac{(\sigma^m_i)^2 + (\sigma^m_j)^2 - 2 \text{cov}(n_i^m, n_j^m)}{s^2} \right) + \frac{s}{2} (\bar{\beta}^{ave})^2 \right. + \left. \frac{1}{2N^2s} \left( \sum_{i=1}^N \sum_{j=1, j \neq m}^N \text{cov}(n_i^m, n_i^m) + \text{cov}(n_j^m, n_j^m) - 2 \text{cov}(n_i^m, n_j^m) \right) \right).$$

(48)

T&G analyse the added error for the specific case where there is no bias, under two assumptions:

1. The noise between classes are i.i.d. and have the same variance for all $m$, i.e. $\text{cov}(n_i^m, n_j^m) = 0$ and $\sigma^m_i = \sigma^m_j = \sigma$. This assumption leads to each classifier having the same expected added error, i.e. $P^{add}_{ave} = P^{add}, \forall m$.
2. For different classes, noise between two classifiers are i.i.d. i.e. $\text{cov}(n_i^m, n_j^m) = 0$.

It is shown in [5,8] that under these assumptions, Eq. [48] can be simplified into

$$P^{add}_{ave} = P^{add} \left( 1 + \frac{C(N - 1)}{N} \right)$$

(49)

where $P^{add}$ is the added error of a single classifier, $C = \sum_{i} P_i C_i$, $P_i$ is the prior probability of the class $\omega_i$ and $C_i$ is the average correlation coefficient among classifiers for this class.

When identical classifiers are used for building the mean classifier ensemble, i.e. $C = 1$, Eq. [49] takes the form $P^{add}_{ave} = P^{add}$. Thus, the difference between the average added errors of the ensemble and the single classifier becomes zero, which means combining classifiers does not provide any benefit. On the other hand, when there is independence between the errors of any pair of base classifiers, i.e. $C = 0$, the error of the ensemble becomes equal to the error of the single classifier divided by the number of classifiers.

We aim to simulate the impact of variance in classification performance and the effect of correlation on the efficiency of the averager ensemble, by employing a benchmark data set, “Image Segmentation” [26]. The data set
consists of 7 classes and 19 dimensions, and comprises 210 training and 2100 test samples. To attain bias close to zero, we work with strong base classifiers, which are chosen as Neural Networks (NNs) with a single hidden layer of 16 nodes and 8 epochs.

Firstly, 200 different groups of classifiers, each of which is formed of 6 NN classifiers with random initial weights, are created. For each group, the expected error rate, mean correlation coefficient \( C \), variance \( \sigma \) as given by T&G, VE as formulated by James and ensemble average are recorded. Note that T&G make the assumption of fixed variance for all data and therefore the same expected added error for all classifiers. We try to comply with these assumptions by creating classifiers with similar variance when averaged over all test samples.

In Figure 2 (a), in order to investigate the relation between the frameworks of T&G and James, the T&G variance for each of the 200 groups is plotted against its corresponding VE. A correlation coefficient of value 0.749 is measured between these variables, in line with the novel theoretical findings presented in Eq. 43. After making the assumption that for strong base classifiers the mean \( \beta \) and the median \( m \) of the decision boundary approach zero, in Eq. 43 T&G variance becomes directly related to James VE. Moreover, this assumption simplifies Eq. 15 such that the added error (and therefore the error) becomes a direct consequence of the T&G variance. This can be visualised in Figure 2 (b), where the correlation coefficient between the expected error rate of the base classifiers and the T&G variance is 0.633.

Secondly, the behaviour of the correlation coefficient \( C \) against the ensemble performance is depicted in Figure 2 (c). The performance metric used here is the difference between the ensemble and the average base classifier accuracy. Supporting the T&G findings given in Eq. 49, where decrease in \( C \) is shown to be related to gain in the average ensemble performance, the negative correlation coefficient between these two variables is measured as -0.512.

This case study validates the theory for powerful ensemble construction based on reduced average T&G correlation between the base classifiers, on experimental data. Classification performance, T&G and VE have also been shown to be directly linked to each other under an approximately zero bias and zero median scenario. An important finding demonstrated here is that although T&G framework depends on many assumptions, the theoretical outcomes generally hold in practice.

4.2 Case Study 2

The aim of this case study is to visualise the effect of VE on the prediction performance. It can be deduced from Eq. 43 that when the mean of the decision boundary distribution deviates from its median, VE can actually take negative values. This circumstance, which is expected to occur when the classifier of interest is weak, implies that having James variance may have a positive impact on the generalisation error. In order to investigate, we use 50 NN classifiers and 5 node/epoch combinations on the Image Segmentation data set. The complexity of the NNs...
is decreased with the set of nodes and/or epochs: [(16/32), (8/8), (2/4), (1/1)]. As in Section 4.1, the classifier perturbation is achieved by using random initial weights.

In Figure 3, the error rates and VE corresponding to the 5 combinations are plotted. As the error rate increases (i.e. as the complexity of the classifiers decreases), VE can be observed to increase with the error until the error reaches a certain value (0.6). When the error rate is equal to 0.8, the corresponding VE drops below zero, which corresponds to a very weak classifier: a NN with 1 node and 1 epoch. The decision boundary for this classifier is expected to have a skewed distribution due to its inability to correctly identify the real boundary. Hence, having increased variance helps the classifier obtain the correct label sporadically by diverging from the aggregate false decision. In other words, we obtain a negative VE, which means that James variance actually boosts the prediction performance. This explanation is also in line with the experimental evidence provided in [2] and [27]. Finally, note that when the error is as low as 0.07, that is for NNs with 16 nodes and 32 epochs, the VE is expected to be directly related to T&G variance similar to the case study in Section 4.1 as the mean and the medians of the decision boundary is expected to be similar and close to zero.

5 Conclusions

The relationships established in this paper not only provide deeper insight into classification theory, but also present the effects of B&V (SE and VE) defined by James in a closed form, which is useful for explaining the behaviours of these terms, such as when they cause performance increase or decrease. Understanding B&V of T&G in terms of James terminology provides further advantages in scenarios where it is not possible to measure the underlying class a posteriori probability distributions. Although the framework of James requires knowledge of the label distribution for an input pattern, this information can usually be approximated more accurately than the lower level probability distributions. This duality provides the user with the flexibility to be able to switch between the models while working on classifier design, and its use is highlighted on two case studies.

To date, the focus of research has been on classification B&V and T&G frameworks separately, but there has been no previous attempt to relate the two. It has to be remembered that James framework has been chosen in this paper as a representative among many classification B&V formulations due to its advantages, and the established T&G linkage can be expanded to other B&V frameworks by utilising the unified notations given in [7,2,8].

There is no doubt that when there is a requirement to understand why a classifier or ensemble performs well, or when an experimental comparison of classifiers needs to be made, it is common practice for many researchers to refer to bias and variance analysis, examples of which are given in the Introduction. Recently, [28] hypothesised that the reason deep networks do not over-fit is that, as complexity is increased, two bias-variance curves appear: variance begins to increase, and then decrease at the transition from the first to second bias/variance trade-off. As
further work, we intend to investigate how the two models described in this paper may be used to explain this over-fitting anomaly.

References

1. S. Geman, E. Bienenstock, and R. Doursat, “Neural networks and the bias/variance dilemma,” Neural Computation, vol. 4, no. 1, pp. 1–58, 1992.
2. G. James, “Variance and bias for general loss functions,” Machine Learning, vol. 51, no. 2, pp. 115–135, 2003.
3. P. Domingos, “A unified bias-variance decomposition for zero-one and squared loss,” in AAAI/IAAI, pp. 564–569, AAAI Press, 2000.
4. K. Tumer and J. Ghosh, “Analysis of decision boundaries in linearly combined neural classifiers,” Pattern Recognition, vol. 29, no. 2, pp. 341–348, 1996.
5. K. Tumer and J. Ghosh, “Error correlation and error reduction in ensemble classifiers,” Connection science, vol. 8, no. 3–4, pp. 385–404, 1996.
6. K. Tumer and J. Ghosh, “Linear and order statistics combiners for pattern classification,” in Combining Artificial Neural Nets, pp. 127–162, 1999.
7. J. H. Friedman, “On bias, variance, 0/1-loss, and the curse-of-dimensionality,” Data Min. Knowl. Discov., vol. 1, no. 1, pp. 55–77, 1997.
8. L. I. Kuncheva, Combining Pattern Classifiers: Methods and Algorithms, Wiley-Interscience, 2004.
9. G. Fumera, F. Roli, and A. Serrau, “A theoretical analysis of bagging as a linear combination of classifiers,” IEEE Trans. Pattern Anal. Mach. Intell., vol. 30, pp. 1293–1299, July 2008.
10. A. Merentitis, C. Debes, and R. Heremans, “Ensemble learning in hyperspectral image classification: Toward selecting a favorable bias-variance tradeoff,” IEEE Journal of Selected Topics in Applied Earth Observations and Remote Sensing, vol. 7, pp. 1089–1102, April 2014.
11. L. Zhang and P. N. Suganthan, “Oblique decision tree ensemble via multisurface proximal support vector machine,” IEEE Transactions on Cybernetics, vol. 45, pp. 2165–2176, Oct 2015.
12. R. Youusi and A. Bagnall, “Ensembles of random sphere cover classifiers,” Pattern Recognition, vol. 49, pp. 213 – 225, 2016.
13. A. Narassiguin, M. Bibimoune, H. Elghazel, and A. Aussem, “An extensive empirical comparison of ensemble learning methods for binary classification,” Pattern Anal. Appl., vol. 19, pp. 1093–1128, Nov 2016.
14. D. Mahajan, V. Gupta, S. S. Keerthi, S. Sundararajan, S. Narayananmurthy, and R. Kidambi, “Efficient estimation of generalization error and bias-variance components of ensembles,” 2017.
15. T. T. Nguyen, X. C. Pham, A. W. Liew, and W. Pedrycz, “Aggregation of classifiers: A justifiable information granularity approach,” IEEE Transactions on Cybernetics, pp. 1–10, 2018.
16. L. Lerman, N. Veshchikov, O. Markowitch, and F. Standaert, “Start simple and then refine: Bias-variance decomposition as a diagnosis tool for leakage profiling,” IEEE Transactions on Computers, vol. 67, pp. 268–283, Feb 2018.
17. I. Chen, F. D. Johansson, and D. Sontag, “Why is my classifier discriminatory?” 2018.
18. L. Breiman, “Arcing classifiers,” The Annals of Statistics, vol. 26, no. 3, pp. 801–849, 1998.
19. R. Kohavi and D. Wolpert, “Bias plus variance decomposition for zero-one loss functions,” in International Conference on Machine Learning, pp. 275–283, 1996.
20. T. E. Dietterich and E. B. Kong, “Error-correcting output coding corrects bias and variance,” in International Conference on Machine Learning, pp. 313–321, 1995.
21. D. Wolpert, “On bias plus variance.” Neural Computation, vol. 9, no. 6, pp. 1211–1243, 1997.
22. T. Heskes, “Bias/variance decompositions for likelihood-based estimators,” Neural Computation, vol. 10, no. 6, pp. 1425–1433, 1998.
23. R. Tibshirani, “Bias, variance and prediction error for classification rules,” tech. rep., Department of Statistics, University of Toronto, 1996.
24. B. Biggio, G. Fumera, and F. Roli, “Bayesian analysis of linear combiners,” in Proceedings of the 7th International Conference on Multiple Classifier Systems, Multiple Classifier Systems 07, (Berlin, Heidelberg), pp. 292–301, Springer-Verlag, 2007.
25. B. Biggio, G. Fumera, and F. Roli, Bayesian Linear Combination of Neural Networks, pp. 201–230. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009.
26. M. Lichman, “Uci machine learning repository,” 2013.
27. C. Zor, T. Windeatt, and B. A. Yanikoglu, “Bias-variance analysis of ecoc and bagging using neural nets,” in Ensembles in Machine Learning Applications, pp. 59–73, Springer, 2011.
28. M. Belkin, D. Hsu, S. Ma, and S. Mandal, “Reconciling modern machine learning practice and the classical bias variance trade off,” Proceedings of the National Academy of Sciences, vol. 116, p. 201903070, 07 2019.