Double resonant processes in $\chi^{(2)}$ nonlinear periodic media

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Abstract

In a one-dimensional periodic nonlinear $\chi^{(2)}$ medium, by choosing a proper material and geometrical parameters of the structure, it is possible to obtain two matching conditions for simultaneous generation of second and third harmonics. This leads to new diversity of the processes of the resonant three-wave interactions, which are discussed within the framework of slowly varying envelope approach. In particular, we concentrate on the fractional conversion of the frequency $\omega \rightarrow (2/3)\omega$. This phenomenon occurs by means of intermediate energy transfer to the first harmonic at the frequency $\omega/3$ and can be controlled by this mode. By analogy the same medium allows ”nondirect” second harmonic generation, controlled by the cubic harmonic. Propagation of localized pulses in the form two coupled bright solitons on first and third harmonics and a dark soliton on the second harmonic is possible.

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1. Introduction

In recent years a concept of photonic band gap (PBG) materials attracted much attention because of their potential applications in many technical and scientific areas. The most of the theoretical studies of PBG materials focused on the problem of the solution of the dispersion relation that contains an information on the propagation of electromagnetic waves through periodic dielectric structures in the linear regime, when the dielectric constant is independent of the field intensity. On the other hand, it is well known that the introduction of an intensity-dependent refractive index can significantly change the transmission properties of a medium. A considerable activity devoted to the investigation of the phenomena associated with the propagation of the electromagnetic waves in nonlinear media provided a bulk of evidence that PBG structures possessing nonlinearity imply the existence of many interesting and useful phenomena which can occur via nonlinear interaction and therefore are believed to be equally fruitful as PBG systems based on the linear regime. For example, it has been found that the existence of the PBG in which linear electromagnetic effects are absent does not exclude the possibility of nonlinear wave propagation. It has been demonstrated that for certain values of the input power the radiation with the frequency in the stop gap can be transmitted due to the excitation of nonlinear solitary waves which may provide practical way of coupling large amounts of optical energy into otherwise unpenetrable photonic bandgap. If the dielectric has either a positive or negative Kerr coefficient the transmission properties of a medium are dramatically changed. For instance, the range of the forbidden wavelengths can be altered interactively by the nonlinear medium response and this effect can induce an intensity-dependence which has important application in optical bistable switching. Recently, the study of bistability in periodic $\chi^{(2)}$ materials has been reported in Ref. 8.

On the other hand, the possibility of multiplication or division by an integer of a frequency of an electromagnetic wave in a nonlinear medium, in particular the second and third harmonic generation is of great practical importance. In this paper we examine the pos-
sibility of the fractional transformation of the frequency which occurs in the photonic crystal represented by 1D periodic multilayer system possessing $\chi^{(2)}$ nonlinearity. This phenomenon as well as any scheme of nonlinear conversion must include phase matching mechanism which is usually achieved by use of an appropriate birefringent nonlinear crystal or by employing some kind of the phase matching which corrects the relative phase between the interacting waves by modulating the sign or magnitude of the nonlinear coefficient on the scale of the coherence length while the linear part of the refractive index is continuous. In contrast, the phase matching in photonic crystals is accomplished by periodic modulation of the linear refractive index or by introducing a defect mode. The enhanced gain in such structures comes about as a result of the interplay of the high electromagnetic mode density in the vicinity of the band edge due to the modified boundary conditions and strong confinement of both the pump and higher harmonic signals. Simultaneously, low group velocity leads to larger interaction of the interacting waves. The process of the fractional frequency conversion which we consider in this paper represents a particular case of three wave interaction in systems with $\chi^{(2)}$ nonlinearity. Namely, we show that a simultaneous combination of $\omega = 3\omega - 2\omega$ process and the second harmonic generation in a 1D periodic nonlinear medium possessing only $\chi^{(2)}$ nonlinearity results in the output frequency conversion $\omega \rightarrow (2/3)\omega$.

In the Section II propose 1D periodic PBG structure in which the resonant conditions for simultaneous SHG and THG and derive a set of coupled-mode equations which allow to determine the evolution of the intensities of the interacting modes. In Section III we analyze stability of the second harmonic of a constant amplitude in terms of the system of the evolution equations and determine the conditions which define regions of stability. In Section IV a stationary process is considered in more detail when the initial energy is arbitrarily distributed among the modes for phase-matched waves. In Section V we identify different types of the evolution of the intensities in terms of the potential governing newtonian particle and present the results of numerical simulations which correspond to the different regimes. In Section VI a particular solitary wave solution is presented in the case when the coefficients $\gamma_1$ and $\gamma_3$ are of the same order. In Section VII we summarize the
results and discuss both the material the structural parameters which affect the efficiency of the processes considered and conditions under which these phenomena can be described in the frame of the parabolic approximation.

2. Statement of the problem

A. Resonant conditions and phase-matching in 1D Photonic Band Gap Structure

Let us consider a TE polarized plane wave incident on a periodic medium with the dielectric permittivity $\varepsilon(x, \omega)$, $\varepsilon(x, \omega) = \varepsilon(x + L, \omega)$, where $L$ is a period of the structure. We assume that the medium possesses a $\chi^{(2)}$ nonlinearity which is also periodic with the period $L$: $\chi^{(2)}(x) = \chi^{(2)}(x + L)$. The inclusion of the isotropic second order nonlinearity leads to the nonlinear polarization of the form

$$P_{NL}(x, t) = \chi^{(2)}(x)E^2(x, t).$$

(1)

The Maxwell equations for the TE-polarized wave $\mathbf{E} = (0, E, 0)$ propagating along the $x$-axis, i.e. $E \equiv E(x, t)$, lead to the wave equation

$$-c^2\frac{\partial^2 E(x, t)}{\partial x^2} + \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} \varepsilon(x, t - t')E(x, t') \, dt' = -4\pi \frac{\partial^2}{\partial t^2} P_{NL}(x, t)$$

(2)

where

$$\varepsilon(x, t - t') = \delta(t - t') + 4\pi \chi(x, t - t')$$

(3)

and $\chi(x, t)$ is the linear permittivity [while $\varepsilon(x, \omega)$ is the Fourier transform of $\varepsilon(x, t)$].

Specifically, we will consider a periodic multilayered structure consisting of two different components denoted as $a$ and $b$ and fabricated from the materials possessing $\chi^{(2)}$ nonlinearity (as the matter of fact it is enough that only one of the slabs is nonlinear). Then we define the filling fraction $f = a/(a+b)$ as an important geometrical parameter which characterizes the stack.
Let us assume that the filling fraction \( f \) and the dielectric permitivities of the two types of layers \( \epsilon_a, \epsilon_b \) are chosen in such a way that there exist three frequencies \( \omega_j \equiv \omega(q_j) \) \((j = 1, 2, 3)\) such that

\[
\omega_3 = 3\omega_1 + \Delta \omega_3, \quad \omega_2 = 2\omega_1 + \Delta \omega_2 \tag{4}
\]

where \(|\Delta \omega_j| \ll \omega_1\) and

\[
q_3 = 3q_1 + Q_1, \quad q_2 = 2q_1 + Q_2, \tag{5}
\]

where \(Q_j \ (j = 1, 2)\) are vectors of the reciprocal lattice, are satisfied. In such a system one has two resonant conditions for the SHG and third harmonic generation (THG) fulfilled simultaneously (if \(\Delta \omega_2 = \Delta \omega_3 = 0\) one has the exact phase matching). We notice that one of the possibilities to satisfy the mentioned resonant conditions which we call the double resonance hereafter, is to choose all \(\omega_j\) bordering stop gaps such that (i) all \(\omega_j\) belong to the center of the Brillouin zone (BZ) or (ii) \(\omega_1\) and \(\omega_3\) are at the boundary and \(\omega_2\) is at the center of the BZ. However, both possibilities are not very interesting from the point of view of practical applications, since it is essential to have energy transfer among modes in space, while the group velocities of modes bordering a stop gap are zero. Therefore we choose the frequencies \(\omega_j\) to be shifted towards allowed bands and therefore possessing non-zero group velocities \(v_j\).

We have shown that such resonant conditions can be achieved in the 1D periodic structure consisting of the alternating slabs of \(\text{Al}_{0.1}\text{Ga}_{0.9}\text{As}\) and InSb characterized by the dielectric constant \(\epsilon_{\text{GaAs}} = 10.97\) and \(\epsilon_{\text{InSb}} = 16.4\) at \(\lambda = 2\mu m\). Both \(\text{Al}_{0.1}\text{Ga}_{0.9}\text{As}\) and InSb possess the second order nonlinearity with \(\chi^{(2)} = 1.68 \cdot 10^{-10} \text{ m/V}\) and \(\chi^{(2)} = 1.84 \cdot 10^{-10} \text{ m/V}\), respectively. In Fig. 1(a) we present a photonic band structure for 1D periodic system consisting of the alternating layers of \(\text{Al}_{0.1}\text{Ga}_{0.9}\text{As}\) and InSb when the filling fraction \(f = 0.1\). It is demonstrated that in such a system the resonant conditions given by Eqs. (4) and (5) can be satisfied for both wave vectors and frequencies of the fundamental and the final states of the second and third harmonics. The band structure shown in Fig. 1(a) is determined
by using standard plane wave method for an 1D infinite system and the frequencies are expressed in normalized units. In order to correlate our simulations with the experimentally accessible systems we choose the lattice constant $a = 0.7334\mu m$. The frequencies are taken such that the fundamental signal represented by a laser source operating at $2\mu m$ corresponds to the frequency close to the band gap edge of the third lowest band and the second and third harmonic signals with the wavelengths $1\mu m$ and $0.667\mu m$, respectively, correspond to the frequencies from the vicinity of the 7. and 10. lowest gap, respectively. In Fig. 1(a) we indicate the dispersion curve at $\lambda = 2\mu m$ which corresponds to the fundamental signal by a solid line for six lowest bands (as it would be without material dispersion). The broken lines refer to the 7. lowest band at $\lambda = 1\mu m$, which corresponds to the second harmonic and to the 10. lowest band at $\lambda = 0.667\mu m$ which belongs to the third harmonic signal. In evaluating the photonic band structure for the fundamental and the final higher harmonic states we include the material dispersion in the range of the frequencies considered by substituting the tabulated values of the dielectric constants at $\lambda = 2\mu m$: $\epsilon_{AlGaAs} = 10.97$, $\epsilon_{InSb} = 16.4$; at $\lambda = 1\mu m$: $\epsilon_{AlGaAs} = 11.86$, $\epsilon_{InSb} = 18.23$ and at $\lambda = 0.5\mu m$: $\epsilon_{AlGaAs} = 17.77$, $\epsilon_{InSb} = 13.32$. The phase matching between the interacting waves was accomplished by varying of the filling fraction of the slabs and choosing the components with suitable material parameters that constitute a geometrical configuration in which the resonant conditions for both nonlinear processes are satisfied. By taking into account the material dispersion of AlGaAs and InSb associated with the wavelengths corresponding to the fundamental signal, the second and third harmonic we have found that the phase matching between the fundamental and the second harmonic and between the fundamental and the third harmonic occurs for the opposite propagating waves. In what follows without restriction of generality the geometry is taken such that the fundamental (i.e. having the lowest frequency wave) is always forward-propagating. In Fig. 1(b) we show in detail the dispersion curves in the extended scheme in the range of the wave vectors and the frequencies in the vicinity of the first lowest band where the regions of the curves having positive and negative derivatives correspond to forward- and backward-traveling waves, respectively. The phase matching between the forward-traveling
fundamental and the second harmonic waves is determined by the intersection of the two curves which represent the fundamental signal indicated by the solid line and a dashed curve which represents the second harmonic divided by a factor 2. We show that the exact phase matching between the fundamental wave and the second harmonic is possible when the wave vector \( q = q_{\text{SHG}} = 0.468 \) and the frequency \( \omega a / 2 \pi c = 0.3667 \), while the exact phase matching between the fundamental wave and the third harmonic is possible when the wave vector \( q = q_{\text{THG}} = 0.471 \) and the frequency \( \omega a / 2 \pi c = 0.3680 \). The phase matching between the forward-traveling fundamental and third harmonic wave is determined by the intersection of the two curves which represent the fundamental signal indicated by the solid line and a dash-dotted curve which represents the third harmonic divided by a factor 3. Since in the system considered the exact phase matching between the fundamental and second harmonic and the fundamental and third harmonic occur for the wave vectors \( q_{\text{SHG}} = 0.468 \) and \( q_{\text{THG}} = 0.471 \) that are not identical, we introduce parameters \( \Delta q_{\text{SHG}}, \Delta q_{\text{THG}}, \Delta \omega_{\text{SHG}}, \) and \( \Delta \omega_{\text{THG}} \), that characterize the conditions under which approximate phase matching occurs.

Namely we choose the wave vector \( q = 0.47 \) at which the frequency of the fundamental signal is \( \omega a / 2 \pi c = 0.3677 \) and corresponding frequency of the second harmonic divided by factor 2 yields \( \omega a / 2 \pi c = 0.3667 \) while the frequency of the third harmonic divided by the factor 3 yields \( \omega a / 2 \pi c = 0.3680 \). Then the mismatch in frequency between the fundamental and the second the harmonic wave at \( q = 0.47 \) is \( \Delta \omega_2 = \Delta \omega_{\text{SHG}} = -0.001 \) in respect to the wave vector \( q = 0.468 \) at which an exact phase matching between the fundamental wave and the second harmonic occurs. The mismatch in frequency between the fundamental and third harmonic wave at \( q = 0.47 \) is \( \Delta \omega_3 = \Delta \omega_{\text{THG}} = 0.0003 \) in respect to the wave vector \( q = 0.471 \) at which an exact phase matching between the fundamental wave and the third harmonic takes place.
B. Evolution equations

The wave evolution in the processes of higher harmonic generation is well described in terms of the so-called envelope function approach. In the system described above one has to take into account three resonant waves, i.e. to express the electric field in the form

\[ E = \sum_{j=1}^{3} A_j \phi_j(x) e^{i\omega_j t} + \text{c.c.} \]  

(6)

where \( \phi_j(x) \) \((j = 1, 2, 3)\) are orthogonal and normalized eigenfunction of the eigenvalue problem

\[ [\epsilon^2 (d^2/dx^2) + \hat{\epsilon}_0(x; \omega_j) \omega_j^2] \phi_j(x) = 0 \]  

(7)

\( A_j(x) \) \((j = 1, 2, 3)\) is a slowly varying, compared with \( \phi_j(x) e^{i\omega_j t} \), amplitude of \( j \)-th mode, and \text{c.c.} stands for the complex conjugate. Using the conventional and well elaborated procedure (see e.g.) one arrives at the system of equations

\[ i \frac{\omega_1}{\omega_1} \left( \frac{\partial A_1}{\partial t} + v_1 \frac{\partial A_1}{\partial x} \right) + \gamma_3 \bar{A}_2 A_3 e^{i(\Delta\omega_3 - \Delta\omega_2)t} + 2\gamma_1 \bar{A}_1 A_2 e^{i\Delta\omega_2 t} = 0 \]  

(8)

\[ i \frac{\omega_2}{\omega_2} \left( \frac{\partial A_2}{\partial t} + v_2 \frac{\partial A_2}{\partial x} \right) + \gamma_3 \bar{A}_1 A_3 e^{i(\Delta\omega_3 - \Delta\omega_2)t} + \bar{\gamma}_1 A_1^2 e^{-i\Delta\omega_2 t} = 0 \]  

(9)

\[ i \frac{\omega_3}{\omega_3} \left( \frac{\partial A_3}{\partial t} + v_3 \frac{\partial A_3}{\partial x} \right) + \bar{\gamma}_3 A_1 A_2 e^{-i(\Delta\omega_3 - \Delta\omega_2)t} = 0 \]  

(10)

where the coefficients are given by

\[ \gamma_1 = 2\pi \int_0^L \chi^{(2)}(x; -\omega, 2\omega) \bar{\phi}_1(x) \bar{\phi}_1(x) \phi_2(x) \, dx \]  

(11)

\[ \gamma_3 = 2\pi \int_0^L \chi^{(2)}(x; -\omega, 3\omega) \bar{\phi}_1(x) \bar{\phi}_2(x) \phi_3(x) \, dx \]  

(12)

\( L \) is a length of the stack, and we assume the following symmetry relations.

\[ \chi^{(2)}(x; \omega_1, \omega_2) = \chi^{(2)}(x; -\omega_3, \omega_2) = \chi^{(2)}(x; \omega_1, -\omega_3) \]
are taken into account and bar denotes the complex conjugation.

We note that the system of Eqs. (8)-(10) is derived for a finite structure subject to periodic boundary conditions. The outcomes, however, can be extended to an infinite structure by computing the limit $L \to \infty$. The difference between the results obtained for finite and infinite structures is estimated to be of $O(L^{-1})$. Bearing this in mind in what follows analytical results are derived for an infinite (or semi-infinite) structure and for numerical estimates of the coefficients $\gamma_j$ a finite structure are used. Another important comment to be made is that we consider (8)-(10) also on a semi-line $x \geq 0$. Formally to do this one should use proper eigenfunctions $\phi_j(x)$ of the boundary value problem (rather than conventional Bloch functions). For special types of solutions, however, (say possessing a symmetry with respect to origin) this leads to a negligible error which very roughly can be estimated to be of order of $\lambda/l$, where $\lambda$ is a length of the carrier wave and $l$ is a characteristic scale of spatial variations of $A_j$ (in other words to be of order of the small parameter of the structure).

It is worth to point out that in the case of pulse propagation (or more precisely when $\lim_{t \to \pm \infty} A_j(x, t) = 0$) one of the integrals of the evolution system (8)-(10) is given by

$$u_1 W_1(x) + u_2 W_2(x) + u_3 W_3(x) = 0$$

(13)

where

$$W_j(x) = \int_{-\infty}^{\infty} |A_j(x,t)|^2 dt.$$  

(14)

3. Stability of the second harmonic

One of the solutions of the Eqs. (8)-(10) which is of a special importance for the next consideration is that of possessing the second harmonic having a constant amplitude the and first and third harmonics having zero amplitude. We refer such a solution as background, hereafter. Then, considering an infinite system, the question about stability of such a state
appears. We consider this problem within the framework of system of the equations (8)-(10) for slowly varying amplitudes which are rescaled as follows

\[ a_1 = \sqrt{\frac{2}{w_2}} A_1 \exp(-i\delta_1 \omega_1 t), \quad a_2 = A_2 \exp(i\varphi_1), \quad a_3 = \frac{\Gamma_3}{\Gamma_1} \sqrt{\frac{2}{w_2}} A_3 \exp[i(-\delta_3 \omega_3 t + \varphi_3 + \varphi_1)]. \] (15)

In the case of perfect phase matching (\(\delta_2 = \delta_3 = 0\)), we have

\[ i \left( \frac{\partial a_1}{\partial \tau} + \frac{\partial a_1}{\partial \xi} \right) + \bar{a}_2 a_3 + 2\bar{a}_1 a_2 = 0, \] (16)

\[ i \left( \frac{1}{w_2} \frac{\partial a_2}{\partial \tau} + \sigma_2 \frac{\partial a_2}{\partial \xi} \right) + \bar{a}_1 a_3 + a_1^2 = 0, \] (17)

\[ i \left( \frac{1}{w_3} \frac{\partial a_3}{\partial \tau} + \sigma_3 \frac{\partial a_3}{\partial \xi} \right) + \mu a_1 a_2 = 0. \] (18)

Here \(\tau = \omega_1 \Gamma_1 t, \xi = (\omega_1 \Gamma_1/v_1)x, \sigma_j = \text{sign}(v_j) w_j = |v_j|/v_1, \Gamma_j = |\gamma_j|, \mu = \frac{3v_1 \Gamma_1}{|v_3| \Gamma_2},\) and \(\phi_j = \arg(\gamma_j).\) (Recall that in accordance with our convention \(v_1 > 0.\))

We are interested in the stability properties of a solution of the Eqs. (16)-(18) as follows: \(a_{1,3} = 0\) and \(a_2 = \rho \exp[i(\omega \tau - k \xi)],\) where \(\omega = \sigma_2 w_2 k\) and \(\rho\) is a positive constant playing the role of the amplitude of the background. Following the conventional procedure we consider perturbations \(\alpha_j\) of the mentioned solution, i.e. we substitute \(a_n = \alpha_n \exp[ni(\omega \tau - k \xi + \phi)]\) for \((n = 1, 3)\) and \(a_2 = (\rho + \alpha_2) \exp[2i(\omega \tau - k \xi + \phi)],\) where \(|\alpha_j| \ll \rho\) into the Eqs. (16)-(18). Next we linearize the system of equations with respect to the \(\alpha_j\) and represent \(\alpha_j = \alpha_{0j} \exp[i(\Omega \tau - K \xi)]\) where \(\alpha_{0j}\) are constants. Then equations for \(\alpha_1\) and \(\alpha_3\) are singled out and we obtain the dispersion relation \(f(\Omega, K) = 0\) for small excitations against the background. The complex roots of this equation indicate instability. After straightforward analysis, one obtains the conditions for \(f(\Omega, K) = 0\) having no complex roots. They read

\[ k^2 \left( 1 - \frac{v_2}{v_1} \right)^2 > 4 \] (19)

\[ 3 \left( k \left| 1 - \frac{v_2}{v_1} \right| + 2 \right) \left| k \left( 1 - \frac{v_2}{v_3} \right) \right| < \mu \] (20)
\[(v_1 - v_2)(v_2 - v_3) > 0\]  \hspace{1cm} (21)

The first consequence comes out from (19): only a background in a form of a plane wave, i.e. with \(k \neq 0\), is stable. The respective boundary conditions to (16)-(18) read

\[
\lim_{\xi \to \infty} a_{1,3} = 0, \quad \lim_{\xi \to \infty} a_2 = \rho e^{i(k\xi - w_2\tau)}
\]  \hspace{1cm} (22)

An important consequence of (21) is that the stable second harmonic can be achieved in a structure where phase matching takes place for waves with "ordered" group velocities: either \(v_1 > v_2 > v_3\) or \(v_1 < v_2 < v_3\). In particular, an unstable case is when the first and third harmonics are forward propagating, and the second harmonic propagates in the opposite direction.

As it has been mentioned above in a real configuration it is almost impossible to avoid phase mismatches \(\Delta \omega_j\). They, however, result in the same terms of the evolution equations as the "rotation" of the background. Hence phase mismatch can either enhance or suppress the instability of a static background. Below it will be convenient to consider \(\omega_2\) as a reference frequency, and define \(\delta \omega_1 = -\Delta \omega_2/2\) and \(\delta \omega_3 = \Delta \omega_3 - \frac{3}{2} \Delta \omega_2\) as a mismatch of the frequencies of the first and third harmonic, respectively. Then, considering the Eqs. (16)-(18) subject to the boundary conditions (22), taking into account the phase mismatch, and introducing new dependent variables \(\tilde{a}_n = a_n \exp[i \frac{\pi}{2}(k\xi - w_1\tau)]\), one arrives at the system of equations

\[
i \left( \frac{\partial \tilde{a}_1}{\partial \tau} + \frac{\partial \tilde{a}_1}{\partial \xi} \right) + \tilde{a}_2 \tilde{a}_3 + 2 \tilde{a}_1 a_2 + \nu_1 \tilde{a}_1 = 0 \]  \hspace{1cm} (23)

\[
i \left( \frac{1}{w_2} \frac{\partial \tilde{a}_2}{\partial \tau} + \sigma_2 \frac{\partial \tilde{a}_2}{\partial \xi} \right) + \tilde{a}_1 \tilde{a}_3 + \tilde{a}_1^2 = 0 \]  \hspace{1cm} (24)

\[
i \left( \frac{1}{w_3} \frac{\partial \tilde{a}_3}{\partial \tau} + \sigma_3 \frac{\partial \tilde{a}_3}{\partial \xi} \right) + \mu \tilde{a}_1 a_2 + \tilde{v}_3 \tilde{a}_3 = 0 \]  \hspace{1cm} (25)

where \(\nu_j = \frac{\Delta \omega_j}{\omega_1}, \quad \tilde{\nu}_j = \delta_j - \nu_j, \quad \delta_1 = \frac{1}{2} k(1 - w_2),\) and \(\delta_3 = \frac{3k(w_3 - w_2)}{2w_3}\).  \hspace{1cm} 11
It follows from the above discussion, that in the particular case when it is possible simultaneously satisfy the two conditions $\delta_j = \nu_j$ the case of quasi-phase matching is mathematically equivalent to the case of exact phase matching with the background in a form of a plane wave. Thus, one can associate an effective wave vector of the background with a phase mismatch and if this wave vector belongs to the stability region then one can speak about stabilization of the background by the phase mismatch.

4. Stationary process

Let us consider in more detail a stationary process when $A_2$ does not depend on slow time while the temporal dependence of $A_j$ ($j = 1, 3$) is harmonic $\sim e^{-i \delta \omega_j t}$. Then the system of the evolution equations (8)-(10) can be rewritten in the form

$$i \frac{d a_1}{d \xi} + \bar{a}_2 a_3 + 2 \bar{a}_1 a_2 - \nu_1 a_1 = 0$$  \hspace{1cm} (26)

$$i \sigma_2 \frac{d a_2}{d \xi} + \bar{a}_1 a_3 + a_1^2 = 0$$  \hspace{1cm} (27)

$$i \sigma_3 \frac{d a_3}{d \xi} + \mu a_1 a_2 - \nu_3 a_3 = 0$$  \hspace{1cm} (28)

First we concentrate on the case of exact phase matching: $\Delta \omega_2 = \Delta \omega_3 = 0$ ($\nu_1 = \nu_3 = 0$). Then system of the Eqs. (26)-(28) allows a particular exact solution describing the energy transfer from the first and third modes to the second one. That takes place, however, only for a specific relation between the amplitudes of the first and third modes. In the present paper we focus on a more generic situation when the input energy is distributed among modes arbitrarily. In particular, we are interested in the case when initially the second mode is not excited and in the effect of each of modes on the energy transfer to the second harmonic.

To this end we note that the system of the Eqs. (8)-(10) possesses two integrals:

$$N = |a_1|^2 + 2 \sigma_2 |a_2|^2 + \frac{3 \sigma_3}{\mu} |a_3|^2$$  \hspace{1cm} (29)
\[ \frac{dN}{d\xi} = 0, \text{ it substitutes the relation (13) in the case of monochromatic waves} \] and

\[ H = a_1^2 \bar{a}_2 + \bar{a}_1^2 a_2 + a_1 a_2 \bar{a}_3 + \bar{a}_1 \bar{a}_2 a_3 \]

(30)

\( \frac{dH}{d\xi} = 0 \).

For the next consideration it is convenient to separate real and imaginary parts of the amplitudes, \( a_1 = q_1 + ip_1, a_2 = q_2 + i\sigma_2 p_2, a_3 = q_3 + i\sigma_3 p_3 \). Then one can consider the system of the Eqs. (26)-(28) as a dynamical one generated by the Hamiltonian

\[ H = \sigma_2 \sigma_3 \mu q_1 p_2 p_3 + 2\sigma_2 q_1 p_1 p_2 - \sigma_2 p_1 p_2 q_3 + q_1 q_2 q_3 + \sigma_3 \mu p_1 q_2 p_3 + q_1^2 q_2 - p_1^2 q_2 \]

(31)

with the canonical Poisson brackets and conjugated variables \( q_j \) and \( p_j \). A remarkable property of this system is that the Hamiltonian admits another representation:

\[ H = q_2 \frac{dp_2}{d\xi} - p_2 \frac{dq_2}{d\xi} \]

(32)

which gives the relation between the amplitude and the phase of the second harmonic.

In a generic case, when \( H \neq 0 \) the dynamics is characterized by permanent energy exchange among modes. In what follows, however, we will be especially interested in situations when all energy is concentrated in one or two of higher harmonics. This can happen only if \( H = 0 \): indeed, if the field of the first or the second harmonic is zero, then \( H \) is zero as well.

The peculiarity of the frequency conversion among modes in periodic media is that the modes involved in the process can be either co-propagating or counter-propagating. Naturally, the behavior of the system strongly depends on the case under consideration, as we already have seen on example of the stability. We can distinguish four different cases: (i) forward propagating waves (\( \sigma_2 = \sigma_3 = 1 \)), (ii) backward propagating second harmonic (\( \sigma_2 = -1 \) and \( \sigma_3 = 1 \)), (iii) backward propagating third harmonic (\( \sigma_2 = 1 \) and \( \sigma_3 = -1 \)), and (iv) backward propagating second and third harmonics (\( \sigma_2 = \sigma_3 = -1 \)). Notice that the last case occurs in the structure depicted in Fig. 1.

In order to study the dynamical system for \( q_j \) and \( p_j \) we observe that it follows from (32) that when \( H = 0 \) one obtains \( q_2/p_2 = \cot \theta_2 \) where \( \theta_2 \) is the phase of the second harmonic.
(hereafter we speak about the phases of $a_j$) which does not depend on the $\xi$ and thus is an integral of motion. On the other hand, by computing $p_2$ from (31) we obtain

$$\cot \theta_2 = \frac{p_1 q_3 - \mu q_1 p_3 - 2q_1 p_1}{q_1 q_3 + \mu p_1 p_3 + q_1^2 - p_1^2}$$

(33)

The last formula allows one to express the functions $q_1, p_1, q_3,$ and $p_3$ through the other ones.

The next consideration is devoted to the case when the second harmonic has phase shift $\pi/2$ with respect to the first and third ones:

$$\theta_2 = \pm \pi/2.$$  

(34)

This means that $q_2 \equiv 0$ and $p_2 \neq 0$ at any point in space. Then by introducing a new variable

$$X_p = \sigma_2 \int_{0}^{\xi} p_2(\xi') d\xi'$$

(35)

one can linearize the dynamical system which now takes the form

$$\frac{dq_1}{dX_p} - q_3 + 2q_1 = 0$$

(36)

$$\frac{dq_3}{dX_p} + \sigma_3 \mu q_1 = 0$$

(37)

$$\frac{dp_1}{dX_p} - \frac{q_3 p_1}{q_1} = 0$$

(38)

The first two equations are linear and thus are trivially resolved with respect to $q_1$ and $q_3$:

$$q_1 = C_1 e^{\omega_+ X_p} + C_2 e^{\omega_- X_p}$$

(39)

$$q_3 = -C_1 \omega_- e^{\omega_+ X_p} - C_2 \omega_+ e^{\omega_- X_p}$$

(40)

where
\[ \omega_\pm = \begin{cases} 
-1 \pm \sqrt{1 - \sigma_3 \mu} & \text{if } \sigma_3 \mu < 1 \\
-1 \pm i \sqrt{\sigma_3 \mu - 1} & \text{if } \sigma_3 \mu > 1 
\end{cases} \tag{41} \]

Then for \( p_1 \) we obtain

\[ p_1 = p_{01} e^{2X_p} \left( C_1 e^{\omega_+ X_p} + C_2 e^{\omega_- X_p} \right) \tag{42} \]

The constants \( C_1, C_2, \) and \( p_{01} \) are determined by the amplitude of the field at the input of the structure.

5. Fractional frequency conversion

Let us look for stationary points of the Eqs. (36)-(38), i.e. for solutions tending to constant at \( \xi \to \infty \). In the case when they exist, stationary points correspond to situation where the energy of the electromagnetic field is either distributed among modes or concentrated in one of them and energy exchange among modes does not occurs any more. Then one can distinguish two cases: (i) \( X_p \to \pm \infty \) what happens when \( \pm \sigma_2 p_2^{(st)} > 0 \), and (ii) \( X_p \to X^{(st)} = \text{const.} \) what happens when \( p_2^{(st)} = 0 \), where \( p_2^{(st)} = \lim_{\xi \to \infty} p_2(\xi) \). The first situation corresponds to the case when the energy is converted to the second harmonic and the second one corresponds to the case when the energy output of the second harmonic is zero. In this section we concentrate on the former case.

It follows from the Eqs. (36), (37), and (38) that in the case of forward propagating third harmonic (\( \sigma_3 = 1 \)) a bounded nontrivial solution can exists only if \( p_{01} = 0 \). Moreover, a convergent solution of the Eqs. (39), (40) can exist only when \( \sigma_2 p_2^{(st)} > 0 \) what corresponds to the \( q_1^{(st)} = q_3^{(st)} = 0 \). In physical terms this conclusion means that the energy initially distributed among three modes is totally transferred during the process into the second harmonic. The necessary condition for such process requires the first, \( a_1 \), and third \( a_3 \), harmonics to be forward propagating waves and have phase difference either 0 or \( \pi \) while the second one, \( a_2 \) has a relative phase shift \( \pm \pi/2 \) [the actual phases of the modes are defined by the last conditions and the relations(15)].
In order to understand which parameters of the input signal result in stationary points we observe that \( X_p \) as a function of \( \xi \) can be treated as a coordinate of an effective newtonian particle governed by the equation (the case \( \sigma_3 = -1 \) is included in the next consideration, as well)

\[
\frac{d^2 X_p}{d\xi^2} = -\frac{\partial U(X_p)}{\partial X_p}
\]

(43)

The potential \( U(X_p) \) is easily computed from the equation for \( q_2 \) and (39), (40) in terms of elementary functions for arbitrary value of \( \mu \). It has, however, a rather cumbersome form to be represented here. Instead, we mention that the dynamics of the effective particle essentially depends on whether (a) \( 0 < \sigma_3\mu < 1 \), (b) \( \sigma_3\mu > 1 \) or (c) \( \sigma_3\mu < 0 \) and \( \mu > 1 \).

In order to simplify the discussion of these cases we assume that there is no input second harmonic. This means that initially (i.e. at \( \xi = 0 \)) \( X_p = 0, \frac{dX_p}{d\xi} = 0 \) i.e. the energy of the effective particle is determined by the point of intersection of the energy curve with \( U \)-axis, and that the type of the motion depends on the value of the potential energy at \( X_p = 0 \). In Fig. 2 we display three typical configurations of \( U(X_p) \) which correspond to the above cases. They represent different kinds of motion: Fig. 2(a) corresponds to motion of the particle to the right which ends up in the steady state motion. This type of motion means that \( X_p \to \sigma_2 p_2^{(st)} \xi \) and thus corresponds to the total energy transfer from the first and third harmonics to the second one. In Fig. 2(b) the particle is initially in a potential well and thus undergoes oscillations around the local minimum of the effective potential. This type of motion corresponds to periodic energy exchange among modes. Finally, Fig. 2(c) corresponds to exponentially growing solution which indicates the instability within the framework of the parabolic approximation.

The first two types of the behavior described above are represented in Figs. 3(a) and 3(b), where we show the results of the numerical simulations obtained by solving the dynamical equations (26)-(28) for slowly varying amplitudes \( A_j \) \( (j = 1, 2, 3) \). We display the evolution of the intensities \( |A_j|^2 \) \( (j = 1, 2, 3) \) as a function on \( \xi \). In Figs. 3(a), (b) we have chosen the same initial values of the amplitudes of the input signal and the same structure parameters as those
used in Figs. 2(a) and 2(b). In Fig. 3(a) energy transfer from the first and the third harmonic to the second one is shown. The time of the energy transfer is inversely proportional to the square root of the field intensity which is a direct consequence of the evolution equations. Taking into account that the amplitude of the first harmonic is relatively small, this situation can be viewed as frequency down-conversion $\omega_3 \rightarrow (2/3)\omega_3$ – in the case when the input energy is dominated by the pump signal with the frequency $\omega (= \omega_3)$. Then after the transition time the energy is transferred into the mode with the fractional frequency $\frac{2}{3}\omega$ ($= \omega_2$). The fractional down-conversion is observed even when the first harmonic is negligibly small (but nonzero) at the input. The existence of the first harmonic in this process is fundamental. Indeed, $A_3 \equiv \text{const.}, A_1 \equiv 0,$ and $A_2 \equiv 0$ is a solution of the dynamical system (8)-(10) and thus the first harmonic is necessary in order to initiate the process of the down conversion. Moreover, at small enough intensities of the first harmonic we have observed local amplification of the first harmonic.

The case which corresponds to the periodic exchange of the energy among modes is shown in Fig. 3(b). Here we are close to the minimum of the potential energy and the interaction among modes is rather weak (the second harmonic is not zero but has a very small amplitude). Almost the whole energy of the electric field is concentrated in the third harmonic. However it is possible to find parameters at which interactions among modes are strong and even locally the whole energy is concentrated in one or two modes.

As it is evident, by a proper choice of initial parameters in the case of a “potential” depicted in Fig. 2(b) an effective particle can be placed outside the potential well. Then the fractional frequency conversion occurs. This situation is given in Fig. 3(c). Finally, in Fig. 3(d) it is shown the exchange of the energy among modes in the presence of mismatch. Although the second harmonic is still generated (i.e. fractional down conversion occurs) at specific distances which are placed almost periodically one observes enhancement of the first harmonic.

Returning to the example of a real structure described in Section 2 one cannot provide frequency conversion to one of the frequencies because of strong instability (this instability
however is an artifact of the parabolic approximation: direct numerical simulations of this case will be reported elsewhere). Meantime, when the third harmonic has an amplitude much higher than the first and second harmonic, one can achieve a stable propagation of the modes, which, however, manifest very weak interaction. This situation is shown in Fig. 4. The stability of the picture represented is very sensitive to the relative phases and the amplitudes of the waves.

6. One solitary wave solution

Let us now return to evolution equations (23)-(25). As it has been mentioned, the phenomenon described above is essentially based on the geometry of the structure. As a consequence the effective nonlinear coefficients $\gamma_1$, and $\gamma_3$ can be of different orders. In particular, one can achieve the relation $|\gamma_1| \gg |\gamma_3|$. Then the equations (23)-(25) in the absence of phase mismatch are reduced to the conventional system for the SHG which is integrable by means of the inverse scattering technique. Another integrable case occurs when $|\gamma_3| \gg |\gamma_1|$. Then the Eqs. (23)-(25) are reduced to the system describing decay of the pump wave, which is the third harmonic, and the back process: $A_3 \leftrightarrow A_1 + A_2$. A characteristic feature of the last process is that the frequency of the three waves are related by (4).

In the case when $\gamma_1$ and $\gamma_3$ are of the same order it is still possible to find a particular solitary wave solution having the form of the coupled bright solitons of the first and third harmonics and a dark soliton associated with the second harmonic. It reads

\[ \tilde{a}_1 = \frac{\alpha_{\pm}^2 u_1}{\cosh(\beta \zeta)} \] (44)

\[ \tilde{a}_2 = i \frac{\alpha_{\pm}^2 \beta u_1^2}{1 - \mu \alpha_{\pm}^2 u_1} \left( i \tilde{\nu}_3 + \tanh(\beta \zeta) \right) \] (45)

\[ \tilde{a}_3 = \frac{\mu \alpha_{\pm}^2 \beta u_1^2}{1 - \mu \alpha_{\pm}^2 u_1 \cosh(\beta \zeta)} \] (46)

Here
ζ = \frac{(\xi - v\tau)v_2}{v_2 - vv_1},

v is a velocity of the pulse,

u = \frac{v_3\sigma_3 - vv_1}{(1 - v)v_3}, \quad u_1 = \sigma_3 - \frac{vv_1}{v_3}

\alpha^2 = \frac{\mu(\mu + u) \pm \sqrt{u(u - \mu)}}{\mu u_1(\mu + 3u)}

and the phases of the frequencies of the modes must be chosen to satisfy the relation

\frac{\nu_3}{\nu_1} = \frac{(1 - \mu\alpha^2 u_1)^2}{\alpha^2 u_1(\mu\alpha^2 u_1 - 2)}, \quad (47)

7. Discussion and Conclusion

Within the slowly varying amplitude approximation we derived a set of coupled mode equations which describe the evolution of the intensities of the resonant waves. We used these equations to study double resonant processes in $\chi^{(2)}$ nonlinear periodic media in which two matching conditions for the generation of the second and third harmonic are simultaneously satisfied. We show that such conditions can be achieved in one-dimensional photonic band gap structure by choosing a proper combination of geometrical and material parameters.

Speaking about possibilities of experimental fabrication of structures with double phase matching we notice that it is more difficult to match parameters, than say for the second harmonic generation only. In particular it is intriguing when actual experimental values of the refractive index are taken into account. Meantime it seems that there are no restriction for possibility to find a proper geometry almost for any $\chi^{(2)}$ material. Another possible technological difficulty could be a lattice mismatch of the layers. In this context it is worth noting that a periodic structure is rather flexible in the sense that it allows inclusion of new components which being much thinner than the main nonlinear slabs do not affect the phenomenon of frequency conversion. Namely in the case when chosen layers possess large
lattice mismatch, slabs of the third kind having width much smaller than \(a\) or \(b\) could be included between nonlinear layers.

It has been shown that the use of double resonances allows one to obtain difference frequency generation. A particular example is \(\omega \rightarrow \left(\frac{2}{3}\right)\omega\). This however is not direct conversion but it occurs with participation of the mode with the frequency \(\omega/3\).

Parabolic approximation which for a given scaling results in a dynamical system is generated by Hamiltonian \(H\). The non-zero Hamiltonian corresponds to the permanent energy exchange among the modes, while \(H = 0\) corresponds to the case when the energy can be (subject to definite conditions) concentrated in one or two higher harmonics (that is the case of fractional conversion). In processes of the frequency conversion in periodic media the modes can be either co-propagating or counter-propagating and consequently the behavior observed is very different. In particular, one can distinguish the following types of the dynamics: (a) the total energy is transferred from the first and third harmonic into the second one, (b) a periodic exchange between the modes occurs and (c) exponentially growing solution, which represents instability within the parabolic approximation. The qualitative picture based on the analysis of the motion of the effective particle in the potential which indicates different types regimes is very well reflected by the results obtained from the numerical simulation carried out to solve dynamical equations.

The evolution equations governing three-wave processes in the presence of double resonances allow a solitary wave solution in a form of coupled two bright solitons (they correspond to the first and third harmonics) and a dark soliton on the second harmonic.

Finally, we also notice that periodic structures, involving geometrical factors rather than only material properties can provide us with a large diversity of matching conditions. In particular one can obtain double resonances as follows [c.f. the Eqs. (4) and (5)]

\[
\omega_3 = 2\omega_2 + \Delta \omega_3, \quad \omega_2 = 2\omega_1 + \Delta \omega_2, \tag{48}
\]

and

\[
q_3 = 2q_2 + Q_1, \quad q_2 = 2q_1 + Q_2, \tag{49}
\]
which result is a different kind of three wave interactions. Solitonic solutions generated by
the respective interactions in the presence of dispersion have been recently considered in\(^2\).
Naturally, the dispersion can lead also to solitons in the case of double resonances defined
in the Eqs. (4), (5).

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FIGURES

Fig. 1. (a) Photonic band structure of 1D periodic structure consisting of alternating slabs of Al$_{0.1}$Ga$_{0.9}$As with $\epsilon_a = 10.97$ and InSb with $\epsilon_b = 16.4$ at $\lambda = 2\mu m$. (solid curve) in which fractional frequency conversion takes place. The broken curves refer to the 7. lowest band at $\lambda = 1\mu m$ which corresponds to the second harmonic and the 10. lowest band at $\lambda = 0.667\mu m$ which corresponds to the third harmonic signal. $q$ is measured in the units $2\pi/(a + b)$. (b) The detailed picture of the region of the photonic band structure shown in Fig. 1a in extended zone scheme in which both frequencies and the wave vectors that satisfy the resonant conditions for simultaneous SHG and THG. The solid curve indicates the calculated dispersion curve near $2\mu m$, the dashed line refers to the region near $1\mu m$ with values divided by a factor 2 and the dash-dotted line refers to the region near $0.667\mu m$ with values divided by a factor 3. The exact phase matching between the forward-travelling fundamental and oppositely travelling second harmonic occurs when $q = q_{SHG} = 0.468$, while the exact phase matching between the fundamental wave and third harmonic is possible for oppositely propagating waves when $q = q_{THG} = 0.471$. $q$ is measured in the units $2\pi/(a + b)$.

Fig. 2. Examples of the effective potential $U(X_p)$ for the structure parameters and initial amplitudes of waves as follows: $\gamma_1 = 82.14 + 78.48i$, $\gamma_3 = -14.57 - 5.07i$, (a) $v_1 = 0.5382$, $v_2 = 0.2808$, $v_3 = 0.2433$, $A_1 = 0.13 \cdot 10^9$ V/m, $A_3 = 0.31 \cdot 10^9$ V/m, $A_2 = 0$, $\mu \approx 0.1225$; (b) $v_1 = 2.382$, $v_2 = 0.2808$, $v_3 = 0.06433$, $A_1 = 0.1 \cdot 10^9$ V/m, $A_3 = -0.9 \cdot 10^9$ V/m, $A_2 = 0$, $\mu \approx 2.051$; (c) $v_1 = 0.5382$, $v_2 = 0.2808$, $v_3 = -0.2433$, $A_1 = 0.13 \cdot 10^9$ V/m, $A_3 = 0.31 \cdot 10^9$ V/m, $A_2 = 0$, $\mu \approx 0.1225$. 

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Fig. 3. Evolution of intensities of the first (solid line), the second- (broken line) and third-harmonic (dashed line) signals. (a) the same structure parameters and input amplitudes as in Figs. 2a; (b) the same structure parameters and input amplitudes as in Figs. 2b; (c) the same structure parameters as in Fig. 2b and the input amplitudes $A_1 = 0.1 \cdot 10^9$ V/m, $A_3 = 0.4 \cdot 10^9$ V/m, $A_2 = 0$; (d) the same structure parameters and input amplitudes as in Fig. 2a except the phase mismatch $\Delta \omega_2 \approx -0.002$ and $\Delta \omega_3 \approx 0.0009$ was included. The intensities represented are normalized to $[\chi^{(2)}]^{-2}$ and time is measured in $\omega_1^{-1}$ units.

Fig. 4. Evolution of intensities of the first (solid line), the second- (broken line) and third-harmonic (dashed line) signals in a structure depicted in Fig. [1], where the input amplitudes $A_1 = 0.255 \cdot 10^9$ V/m, $A_3 = -1.996 \cdot 10^9$ V/m, $A_2 = 0$. 
Fractional frequency conversion in 1D Al\textsubscript{0.1}Ga\textsubscript{0.9}As/InSb stack

(a) \( \varepsilon_a=10.97 \) \( \varepsilon_b=16.4 \) \( \lambda=2\mu m \) \( f=0.1 \)

THG: \( q_3=3q_1+Q_1, \omega_3=3\omega_1 \)

3.harmonic \( \lambda\approx0.667\mu m \)

SHG: \( q_2=2q_1+Q_2, \omega_2=2\omega_1 \)

2.harmonic \( \lambda\approx1\mu m \)

fundamental signal \( \lambda\approx2\mu m \)
SHG and THG in 1D Al$_{0.1}$Ga$_{0.9}$As/InSb stack

$\varepsilon_a = 10.97$  $\varepsilon_b = 16.4$  $f = 0.1$

$2\omega \sim \lambda = 1\mu m$  (SHG)

$3\omega \sim \lambda = 0.667\mu m$  (THG)

$\omega \sim \lambda = 2\mu m$  (PUMP)

$q_{\text{SHG}}/2 \sim |q_{\text{THG}}/3|$

forward-wave  backward-wave
