INFLATION: WHERE DO WE STAND?

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In this short review, the predictions of inflation are presented and compared to the most recent measurements of the Cosmic Microwave Background (CMB) anisotropy. It is argued that inflation is compatible with these observations but that these ones are not yet accurate enough to probe the details of the scenario.

Keywords: Cosmology; inflation; cosmological perturbations

1. Introduction

Inflation is presently our most convincing scenario of the very early universe, not only because it is physically attractive, but also because it makes a series of definite predictions that can be tested concretely by means of astrophysical observations. In this short article, we quickly review what are these predictions and describe how the on-going high accuracy Cosmic Microwave Background (CMB) anisotropy measurements can probe the physics of inflation. It is argued that inflation is presently compatible with all the available data but that these ones are not yet accurate enough to test the details of the inflationary scenario. This review is organized as follows. In the next section, we present the inflationary predictions and, in particular, describe the theory of cosmological perturbations of quantum-mechanical origin. In the next section, we compare these predictions with the recently released CMB anisotropy observations. Finally, we quickly present our conclusions.

2. Basic Equations and Basic Predictions

2.1. Background Evolution and Cosmological Perturbations

Inflation is a phase of the cosmic evolution which took place before the hot Big Bang era. This phase allows us to solve many problems (horizon, flatness, monopoles etc ...) plaguing the standard cosmological model. The main idea is that the evolution of the Friedman-Lemaître-Robertson-Walker (FLRW) scale factor $a(t)$ was accelerated, $\frac{d^2a}{dt^2} > 0$. This acceleration is caused by a fluid the pressure of which is negative as required by the Einstein equations of motion. Typically, since
at very high energy a fluid description of matter is likely to be inadequate, one uses
(quantum) field theory and models matter by one or many scalar fields. Therefore,
typically, the theory describing matter during inflation possesses the Lagrangian
\[ L_{\text{matter}} = -\frac{1}{2} \sum_{i=1}^{N} g^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_j - V(\varphi_1, \cdots, \varphi_N). \] (1)

At the end of the accelerated phase, the scalar field oscillates at the bottom of its
potential and decays into radiation. This is how one smoothly connects inflation
with the radiation dominated era.

In order to go further, one must consider the theory of cosmological perturbations
of quantum-mechanical origin. This theory aims at explaining the origin and the evolution of the inhomogeneities that are observed in our Universe. One
of the most interesting aspects of this theory is that it makes accurate predictions which are currently under observational scrutiny. Technically, one uses the
fact that, in the very early universe, the inhomogeneities were small as revealed by the COsmic Background Explorer (COBE) measurement of the CMB anisotropy, \( \delta T/T \approx 10^{-5} \). This permits to work with a linear theory which is of course a great
technical simplification. Concretely, one writes that the metric tensor is given by
\[ \gamma_{\mu\nu}(\eta, \vec{x}) = \left[ g_{\mu\nu}(\eta) + \epsilon h_{\mu\nu}(\eta, \vec{x}) + \epsilon^2 \ell_{\mu\nu}(\eta, \vec{x}) + \cdots \right] \, dx^\mu \, dx^\nu, \] (2)
where \( g_{\mu\nu}(\eta) \) is the standard homogeneous and isotropic FLRW metric. In the same
manner, the matter sector can be expressed as
\[ \varphi_i(\eta, \vec{x}) = \varphi_i(\eta) + \epsilon \delta \varphi_i(\eta, \vec{x}) + \epsilon^2 \delta^{(2)} \varphi_i(\eta, \vec{x}) + \cdots, \] (3)
where \( i = 1, \cdots, N \) with \( N \) the total number of scalar fields. In Eqs. and , \( \epsilon \) is a small parameter in which the expansion is performed. The equations linking the perturbed metric \( h_{\mu\nu} \) and the perturbed scalar fields \( \delta \varphi_i \) are just the linearized Einstein equations.

### 2.2. Density Perturbations

Let us consider that two scalar fields, say \( \varphi_1 \) and \( \varphi_2 \), are present in the early universe
(the following formalism can be easily generalized to the case where there are more
fields). We now introduce the so-called perturbed adiabatic field and perturbed
entropy field defined by
\[ \delta \sigma = (\cos \theta) \, \delta \varphi_1 + (\sin \theta) \, \delta \varphi_2, \quad \delta s = - (\sin \theta) \, \delta \varphi_1 + (\cos \theta) \, \delta \varphi_2, \] (4)
where
\[ \cos \theta = \frac{\varphi_1'}{\sqrt{(\varphi_1')^2 + (\varphi_2')^2}}, \quad \sin \theta = \frac{\varphi_2'}{\sqrt{(\varphi_1')^2 + (\varphi_2')^2}}, \] (5)
where a primes denotes a derivative with respect to the conformal time. The case
where there is only one field corresponds to \( \varphi_2 = 0 \) or \( \theta = 0 \). In this case, it is
obvious that there is no entropy field, \( \delta s = 0 \). Instead of working with \( \delta \sigma \) and \( \delta s \), it is more convenient to use \( \delta s \) and the generalized Mukhanov variable defined by

\[
v \equiv a \left[ \delta \sigma + \frac{1}{H} \sqrt{(\varphi'_1)^2 + (\varphi'_2)^2 \Phi} \right] = \cos \theta v_1 + \sin \theta v_2 ,
\]

where \( \Phi \) is the Bardeen potential and \( v_1, v_2 \) the Mukhanov variables associated to the first and second scalar fields respectively. The quantity \( H \) is defined by \( a'/a \), \( a(\eta) \) being the FLRW scale factor. Then, in the Fourier space, the equations of motion of the system can be written as

\[
v'' + v \left[ k^2 - \frac{(a\sqrt{\gamma})''}{a\sqrt{\gamma}} \right] = 2a (\theta' \delta s)' + 4a \theta' \left( \mathcal{H} + \frac{\gamma'}{4\gamma} \right) \delta s ,
\]

\[
\delta s'' + 2\mathcal{H} \delta s' + (k^2 + a^2 V_{ss} + 3\theta'^2) \delta s = \frac{4\theta'}{\kappa \sqrt{(\varphi'_1)^2 + (\varphi'_2)^2}} k^2 \Phi ,
\]

where the function \( \gamma \) is defined by \( \gamma = 1 - \mathcal{H}'/\mathcal{H}^2 \) and the quantity \( V_{ss} \) is given by \( V_{ss} = \sin^2 \theta V_{\varphi_1 \varphi_1} - \sin 2\theta V_{\varphi_1 \varphi_2} + \cos^2 \theta V_{\varphi_2 \varphi_2} \). The quantity \( k \) denotes the wavenumber of the corresponding Fourier modes. We see that the system can be reduced to equations describing parametric oscillators (the time-dependent frequency coming from the “interaction” between the perturbations and the background) together with an acting force term, originating from the non-adiabatic nature of the fluctuations. The integration of the above equations lead to the power spectra of the adiabatic and non-adiabatic components during inflation (there is also a mixed term). Each component is characterized by an amplitude and a spectral index that can be evaluated for instance if the slow-roll approximation holds. In the single field case, the spectrum is almost scale-invariant, the deviations from scale invariance being small and given by the derivative of the inflaton potential.

The next question is now to propagate these spectra to the radiation-domination era. For this purpose, let us introduce the purely geometrical quantity \( \zeta \) defined by

\[
\zeta = \Phi + 2 \frac{\mathcal{H}^{-1} \Phi'}{3} + \frac{\Phi}{1 + \omega} ,
\]

where \( \omega \equiv \rho/p \) is the equation of state parameter. One can also work in terms of \( \zeta_{\text{BST}} \) related to \( \zeta \) by

\[
\zeta_{\text{BST}} = -\Phi - 2 \frac{\mathcal{H}^{-1} \Phi'}{3} + \frac{k^2}{3\gamma \mathcal{H}^2} \Phi ,
\]

On super-Hubble scales, one has \( \zeta_{\text{BST}} = -\zeta \). The importance of the quantity \( \zeta \) lies in the fact that, under certain circumstances that we are going to discuss, it is conserved on super-Hubble scales regardless of what happens during the complicated phase where one goes from inflation to the radiation-dominated era. Therefore, \( \zeta \) can be viewed as a “tracer” for density perturbations. Conservation of the perturbed
stress-energy tensor implies the following equation

\[
\zeta'_{\text{BST}} = - \frac{\mathcal{H}}{\rho + p} \delta p_{\text{nad}} - \frac{1}{3} \partial_i \partial^i (\delta s) .
\] (11)

On super-Hubble scales, the last term is negligible but the first one is still important. This first term is the so-called non-adiabatic pressure and is defined by the following expression:

\[
\delta p_{\text{nad}} \equiv \delta p - c_s^2 \delta \rho,
\]

where \( c_s \equiv \rho'/\rho' \) is the sound velocity. Expressing the perturbed energy density and the perturbed pressure explicitly, one has

\[
\delta p_{\text{nad}} = (\delta p_1 - c_{s_1}^2 \delta \rho_1) + (\delta p_2 - c_{s_2}^2 \delta \rho_2) + (c_{s_1}^2 - c_{s_2}^2) \left( \frac{\rho_1}{\rho} + \frac{p_1}{p} \right) S_{12},
\] (12)

where the quantity \( S_{12} \) is given by

\[
S_{12} = \frac{\delta \rho_1}{\rho_1 + p_1} - \frac{\delta \rho_2}{\rho_2 + p_2}.
\] (13)

We see that the non-adiabatic pressure contains two contributions. The terms \( \delta p_i - c_{s_i}^2 \delta \rho_i \) originate from intrinsic entropy perturbations (if any) of the fluids under consideration while the term proportional to \( S_{12} \) represents the entropy of mixing. The previous expressions are valid for any type of matter. In the case of scalar fields, one obtains

\[
\delta p_{\text{nad}} = -\frac{2}{\kappa} \left( 1 - c_{s_i}^2 \right) \frac{k^2}{a^2} \Phi + \frac{2\theta'}{a^2} \sqrt{\left( \varphi'_1 \right)^2 + \left( \varphi'_2 \right)^2} \delta s,
\] (14)

where \( \kappa = 8\pi/m_{\nu_i}^2 \). On super-Hubble scales the non-adiabatic pressure only comes from the entropy field \( \delta s \), i.e. is sourced by the entropy of mixing and not by the intrinsic entropy perturbations (which exist for a scalar field).

Therefore, if only one field is present, \( \zeta \) is conserved and this implies that the scale-invariant spectrum generated during inflation is “transferred” to the perturbations of the various components in the post-inflationary phase. Moreover, these fluctuations are adiabatic, i.e. they satisfy

\[
\frac{\delta \rho}{\rho}_{\text{cdm}} = \frac{\delta \rho}{\rho}_{\text{bayrons}} = \frac{3}{4} \frac{\delta \rho}{\rho}_{\text{neutrinos}} = \frac{3}{4} \frac{\delta \rho}{\rho}_{\text{photons}}.
\] (15)

These relations are then used to calculate observables like, for instance, the multipole moments \( C_\ell \) which characterize the CMB anisotropies.

If more than one field is present the situation is more complicated since one can no longer use the conservation of \( \zeta \) to predict the spectra in the post-inflationary phase (in addition, as already mentioned above, there are several of them, \( P_{\zeta}, P_{\delta s} \) and \( P_{\zeta - \delta s} \)). Furthermore, the relations (15) are violated and, hence, the \( C_\ell \) can strongly differ from the single field case.
2.3. Gravitational Waves

Gravitational waves are unavoidably produced in the early universe. Since the tensor modes do not couple to matter, the gravitational wave spectrum is independent of the type of matter (or of the number of scalar fields) present during inflation. It reads

$$ P_h = \frac{16 H_{\text{inf}}^2}{\pi m^2_{\text{Pl}} \left[ 1 - 2 (C + 1) \epsilon - 2 \epsilon \ln \frac{k}{k_*} \right]}, $$

where $\epsilon = m^2_{\text{Pl}} \left( V'/V \right)^2 / (16 \pi^2)$ is the first slow-roll parameter. It would be of utmost importance to detect the primordial gravitational waves since this would provide a direct access to the energy scale of inflation $H_{\text{inf}}$. Unfortunately, this is observationally quite challenging.

Another important check of the inflationary scenario would be to measure the tensor spectrum to scalar spectrum ratio. Under quite general circumstances, this can be expressed as

$$ \frac{T}{S} = -8 n_T \sin^2 \Delta, $$

where $n_T = -2 \epsilon$ is the tensor spectral index and $\Delta$ is a quantity which measures the correlation between adiabatic and non-adiabatic perturbations. In absence of non-adiabatic perturbations (i.e. in the single field case), $\sin \Delta = 1$. Eq. (17) tells us that tensor modes are sub-dominant since the spectral index is expected to be small. A direct check of this consistency relation would be a direct proof of inflation.

2.4. Other Possibilities

There are other effects that could modify the basic predictions of inflation presented above. Here, we discuss two possibilities. Firstly, there is the so-called trans-Planckian problem of inflation. It consists in the following. In a typical model of inflation, the wavelength of the modes of astrophysical interest today were, at the beginning of inflation, when the initial conditions are chosen, smaller than the Planck length. In this regime, the framework used to perform the calculations, i.e. quantum field theory in curved space-time, is likely to break down. In other words, the predictions of inflation could be modified by short distance physics. It is not easy to predict what these modifications could be but a quite generic prediction is that superimposed oscillations in the primordial power spectra should appear

$$ P_{\text{tpl}} = P_\zeta \left\{ 1 - 2 |x| \sigma_0 \cos \left[ \frac{2 \epsilon}{\sigma_0} \ln \left( \frac{k}{k_*} \right) + \psi \right] + \ldots \right\}, $$

where $P_\zeta$ is the standard spectrum. The oscillations in the power spectra are usually transferred to the multipole moments $C_\ell$ which therefore also exhibit superimposed oscillations (of course the amplitude and the frequency of these oscillations have nothing to with the acoustic oscillations).
A second interesting possibility is the presence of topological defects that would be produced at the end of inflation (in the case where the underlying model possesses several fields). This has recently been studied in Ref. 13 in the case of hybrid inflation. Interestingly enough, it has been shown that, unless some fine-tuning of the model parameters is present, the inflationary multipole moments would be significantly changed.

3. So Where Do We Stand?

We now quickly discuss what are the consequences of the recently released Wilkinson Microwave Anisotropy Probe (WMAP) data for inflation 14. First of all, these data are compatible with a spatially flat universe, $\Omega_0 \simeq 1$. Secondly, the initial (scalar) power spectrum is found to be compatible with scale invariance but a small deviation from scale invariance cannot yet be established with enough statistical confidence. No signal of non-adiabatic fluctuations has been found. The first slow-roll parameter $\epsilon$ is constrained to be $\epsilon \lesssim 0.03$ and we have an upper bound on the scale of inflation $15$

$$\frac{H_{\text{inf}}}{m_{\text{pl}}} \lesssim 1.4 \times 10^{-5}.$$  (19)

Some single field models are already ruled out, for instance the ones with a potential of the form $V(\phi) \propto \phi^p$, with $p \geq 6$ 16. The case of the quartic potential $p = 4$ is on the border line while the massive potential $p = 2$ is still compatible 15. Thirdly, no gravitational waves have been detected. There is only an upper bound on the ratio $T/S$, namely $T/S \lesssim 0.3$ 16. Because there is no detection of gravitational waves, the consistency check of inflation has obviously not been verified. Fourthly, the statistical properties of the fluctuations seem to be Gaussian which is compatible with single field inflation. Recently, it has also been shown that there is a hint for wiggles in the multipole moments but that the standard slow-roll model remains the most probable one 15. In addition, this hint is linked to the presence of the so-called “cosmic variance outliers” which could very well disappeared with new data or be of non-primordial origin (i.e. linked to some astrophysical foregrounds). Finally, no sign of topological defects has been detected.

As a general conclusion, one can say that the predictions of inflation are compatible with the currently available data. Interestingly enough, the data already allow us to exclude some models of inflation. However, these data are not yet accurate enough to provide us with something which could be considered as a definite proof of inflation like a small deviation from scale invariance or, even better but much more difficult, a detection of a background of stochastic gravitational waves satisfying the consistency check of inflation. Probably, we will have to wait for the next generation of observations to achieve this ambitious goal.
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