The Three F’s for Bicategories I:
Localization by Fractions is Exact

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Abstract

We study the interaction between the notions of filteredness, fractions and fibrations in the theory of bicategories, generalizing classical results for categories. We give an explicit formula for filtered pseudo-colimits of categories indexed by a bicategory, and we use it to compute the hom-categories of a bicategory of fractions. As a consequence, we show that the canonical pseudo-functor into a bicategory of fractions is exact.

Introduction

The rich interaction between the three F notions of filtered categories, categories of fractions and Grothendieck fibrations goes back to [GZ67] and to SGA4 [SGDV72, Exp. VI, §6], where the following results are shown.

(A) For any pseudo-functor \( F: C^{op} \to \text{Cat} \) from a category \( C \), its pseudo-colimit is given by the localization at the Cartesian arrows of the fibration determined by its Grothendieck construction \( \mathcal{E} F \).

(B) For any fibration \( P: E \to B \), if \( B^{op} \) is pseudofiltered, then its family of Cartesian arrows satisfies the axioms for a calculus of fractions.

(C) The hom-sets of a category of right fractions \( C[W^{-1}] \) are filtered colimits of hom-sets of \( C \), and the localization functor \( C \to C[W^{-1}] \) is left exact.

Throughout this introduction, we will refer to these results as (A), (B), and (C).

Since then, these three F notions have been generalized to 2-dimensional category theory:

1. In [Ken92] a set of axioms for filtered 2-categories is given, and the same axioms also make sense for bicategories. We define pseudofiltered bicategories in Section 2.

2. In [Pro96] the theory of bicategories of fractions is developed, generalizing the axioms and the construction in [GZ67].

3. Fibrations of 2-categories and bicategories are introduced in [Her99] [Bak] [Buc14].

In this paper we show that the results ([A], [B]) and ([C]) also hold for these three notions.

The authors of [Buc14] [Bak] generalize the Grothendieck construction \( \mathcal{E} F \) to trihomomorphisms \( F \) from a bicategory into the tricategory \( \text{Bicat} \) of bicategories. Result (A) becomes in this context:

Theorem 4.13. For any trihomomorphism \( F: B \to \text{Bicat} \) from a bicategory \( B \), its tricolimit in \( \text{Bicat} \) is given by the localization of its Grothendieck construction \( \mathcal{E} F \) at both the Cartesian arrows and the Cartesian 2-cells.
We show how, with the proper setup, the proof of this theorem can be given as a one-line computation using the fact, shown in [Buc14 Prop 3.3.12], that the Grothendieck construction provides an equivalence between trihomomorphisms into \( \text{Bicat} \) and fibrations between bicategories. Note that Theorem 4.13 involves the localization of a bicategory at both arrows and 2-cells, which we define but won’t use at this level of generality (in [BVS], the sequel to this paper, we will compute this tricofiltrant for the case when \( \mathcal{E} \) is filtered). The less general situation that is relevant to us here is that of a pseudo-functor \( F : \mathcal{B} \to \text{Cat} \) from a bicategory \( \mathcal{B} \), for which we show that it suffices to localize only at the Cartesian arrows (in what follows \( \pi_0 \) denotes the left adjoint to the inclusion \( d \) of \( \text{Cat} \) into \( \text{Bicat} \)).

**Corollary 4.15** For any pseudo-functor \( F : \mathcal{B} \to \text{Cat} \) from a bicategory \( \mathcal{B} \), its pseudo-colimit is given by the localization of \( \pi_0(\text{el}dF) \) at the family of equivalence classes of Cartesian arrows.

This result generalizes \((A)\) to the case in which \( C \) is a bicategory.

We also show (see Corollary 3.14) that \((B)\) still holds when each of the three \( F \) notions is replaced by their bicategorical analogues introduced in items 1, 2, and 3 above. This follows from the following more general Lifting Fractions Lemma (since in a co-pseudofiltered bicategory, the collection of all arrows satisfies right fractions).

**Lemma 3.12** (Lifting Fractions Lemma). For a fibration of bicategories \( \mathcal{E} \to \mathcal{B} \), if a family of arrows in \( \mathcal{B} \) satisfies right fractions, then so does the family of Cartesian arrows over it.

We prove this lemma using a new set \( \text{Frc} \) of fractions axioms for a family of arrows of a bicategory that we introduce. This set of axioms is simpler but equivalent to the original one in [Pro96]. Working with the new set of axioms instead of the original one makes a significant difference here. This proof provided the main motivation for the new formulation of the axioms.

Since applying \( \pi_0 \) to a bicategorical calculus of fractions yields a 1-dimensional calculus of fractions (see Lemma 3.7), it follows that if \( \mathcal{B} \) is pseudofiltered then the localization in Corollary 4.13 can be computed as a category of fractions. This yields an explicit formula for filtered pseudo-colimits of categories indexed by a bicategory in terms of the data of \( \mathcal{B} \) and the image of \( F \). We give the details in Proposition 4.17. We note that a similar formula is given in [DS06] for the case in which \( \mathcal{B} \) is a (strict) 2-category, and \( F \) is a (strict) 2-functor; see also [Dat14] for the relation with the bicategory of fractions. Our formula in Proposition 4.17 is the generalization of the formula in [DS06] to the non-strict case, and can immediately be seen to match theirs when the coherence datum is dropped. We sketch now how we apply this formula in Section 5 to establish result \((C)\) for bicategories of fractions, where both \( \mathcal{B} := (\mathcal{W}/A)^{op} \) and \( F := F^B_A \) are non-strict.

Recall from [GZ67] that the hom-sets of the category of fractions are given as filtered colimits of sets. More precisely, for a family \( \mathcal{W} \) of arrows of a category \( \mathcal{C} \), and a pair of objects \( A, B \), we can define a slice category \( \mathcal{W}/A \), whose objects are the arrows in \( \mathcal{W} \) with codomain \( A \) and whose arrows are given by commutative triangles, and a functor

\[
F^B_A : \mathcal{W}/A \overset{op} \rightarrow \mathcal{C}^{op} \overset{C(-,B)} \rightarrow \text{Set}
\]

where \( U \) is the functor that maps each arrow to its domain. When \( \mathcal{W} \) satisfies fractions, this is a filtered diagram of sets whose colimit is \( C[\mathcal{W}^{-1}](A,B) \). We show in Section 5.1 that a similar description exists for the hom-categories of a bicategory of fractions. For a family \( \mathcal{W} \) of arrows of a bicategory \( \mathcal{B} \), and a pair of objects \( A, B \), we define a slice bicategory \( \mathcal{W}/A \) and a pseudo-functor

\[
F^B_A : (\mathcal{W}/A)^{op} \overset{U} \rightarrow (\mathcal{B})^{op} \overset{\mathcal{B}(-,B)} \rightarrow \text{Cat}
\]

similarly to \((0.1)\). As a further application of the Lifting Fractions Lemma 3.12 to the fibration given by \( U \), we show that if \( \mathcal{W} \) satisfies fractions then \((\mathcal{W}/A)^{op}\) is a filtered bicategory. Furthermore, we show that the formula for the pseudo-colimit of \( F^B_A \), which one gets from Proposition 4.17, can be seen to match the construction in [Pro96] of the hom-categories \( B[\mathcal{W}^{-1}](A,B) \). This is the content of Proposition 5.4 establishing the first part of the result \((C)\) for bicategories of fractions.
Since composition of arrows in a 1-category of fractions does not depend on the choice of the Ore squares used in the construction, we obtain as a first consequence of Proposition 5.6 that the vertical composition of 2-cells in \[ \text{Pro96, p.258} \] does not depend on the choice of Ore squares in \( \mathcal{B} \). The fact that the vertical composition of 2-cells is independent of these choices was also proved directly in \[ \text{Tom16, Prop. 5.1} \].

As another application of Proposition 5.6, we show in Section 5.2 how this result can be combined with an exactness property of \( \text{Cat} \) to show the second part of result (C) for bicategories. In \[ \text{Can16, see also DDS18a} \], the commutativity in \( \text{Cat} \) of filtered pseudo-colimits and finite weighted limits is shown, when these limits are indexed by strict 2-functors with 2-categories as domain. We use the facts that any bicategory is biequivalent to a 2-category and that any \( \text{Cat} \)-valued pseudo-functor from a 2-category is equivalent to a 2-functor to deduce that this commutativity still holds for pseudo-functors from bicategories. Finally, following the same method of the 1-dimensional original proof in \[ \text{GZ67,§I.3} \], we combine this commutativity result with Proposition 5.6 to prove the following theorem, that finishes to establish (C) for bicategories:

**Theorem 5.17.** Let \( \mathcal{B} \) be a bicategory and \( \mathcal{W} \) be a right bicalculus of fractions. Then the localization pseudo-functor \( \mathcal{B} \to \mathcal{B}[\mathcal{W}^{-1}] \) commutes with finite weighted bilimits.

**Organization**

The paper is organized as follows. In Sections 1.1 and 1.2, we introduce the notation we will be using and recall basic properties of bicategory theory. In Section 1.3, we introduce some classical properties of fibrations of bicategories that we will need in the paper.

In Section 2, we introduce a new set of axioms for a pseudofiltered bicategory, that can be related to the filtered axioms for a 2-category in \[ \text{Ken92} \] just like the axioms for a pseudofiltered category are related to the axioms for a filtered category in \[ \text{SGA 4, GSDV72} \] (see Propositions 2.8 and 2.9). We show how these axioms are equivalent to (a subset of) the ones introduced in \[ \text{DS06} \] and corrected in \[ \text{DS21} \].

In Section 3, we introduce a new set \( \text{Frc} \) of fractions axioms for a family \( \mathcal{W} \) of arrows of a bicategory and we show that these axioms are equivalent to the original ones in \[ \text{Pro96} \] (and to the modified version that can be found in \[ \text{PS21} \]). Using these axioms, we show the Lifting Fractions Lemma 3.12 mentioned in the introduction, and its Corollary 3.14 that establishes the analogue of result (B) for bicategories.

In Section 4, Theorem 4.13, we generalize result (A) to an arbitrary trihomomorphism \( F \) into \( \text{Bicat} \). We then show in Corollary 4.15 how, when \( F \) takes its values in \( \text{Cat} \), we obtain a simpler formula for its pseudo-colimit as a localization of a 1-category, which can be computed by fractions when the indexing bicategory is pseudofiltered. We give the explicit formula of this pseudo-colimit in Proposition 4.17.

Finally, in Section 5, we show result (C) for bicategories, as detailed above.

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1 Preliminaries

1.1 Notation

Throughout the paper, in addition to the usual hyperref links, we also use links to tie together the different axioms that are introduced: clicking on the name of an axiom (that we denote with boldface letters) will take the reader to its definition.
1. Abstract categories are written using a bold font style (\textbf{A, B, C}), and bicategories using a calligraphic font style (\calligra{A, B, C}). Set denotes the category of sets, \textbf{Cat} denotes the 2-category of categories, and \textbf{Bicat} denotes the tricategory of bicategories [GPS95].

2. To introduce elements in a (bi)category, we use the type-theory inspired notation “\(A :: \mathcal{A}\)” rather than the usual set-theory inspired notation “\(A \in \mathcal{A}\)”. We use a classical “trickle down” abbreviation of typing declarations, for example \(f : A \to B : \mathcal{A}\) is short for \(A, B : \mathcal{A} \text{ and } f : A \to B\).

3. We write \(\text{f} : A \simeq B : \mathcal{A}\) to denote an equivalence \(f\) from \(A\) to \(B\) in the bicategory \(\mathcal{A}\). We write \(\alpha : f \simeq g : A \to B : \mathcal{A}\) to denote an invertible 2-cell \(\alpha\) from \(f\) to \(g\). Isomorphisms of (bi)categories are denoted by \(\cong\).

4. As in [JFS17], we use \(\mathcal{Y}\) (the first letter of “Yoneda” when written in hiragana) to denote Yoneda embeddings.

5. For \(\mathcal{A}, \mathcal{B} : \text{Bicat}\), we denote by \([\mathcal{A}, \mathcal{B}]\) the bicategory of pseudo-functors, pseudo-natural transformations and modifications. This is the hom-bicategory \(\text{Bicat}(\mathcal{A}, \mathcal{B})\) of the tricategory \(\text{Bicat}\).

1.2 A few bicategorical facts

We study here bicategories as introduced by Bénabou [Bén67], following mostly the notation from the book “2-Dimensional Categories” written by Niles Johnson and Donald Yau [JY21]. We recall the main definitions of bicategory theory to fix our notation for the rest of this paper. We omit details which are ubiquitous in the literature, referring the reader to [JY21], among many other choices.

**Definition 1.1** (Bicategory). A bicategory \(\mathcal{B} : \text{Bicat}\) is the data of objects, arrows between objects and 2-cells between parallel arrows together with:

- A **vertical composition** that we denote by \(\circ\), which makes \(\mathcal{B}(A, B)\) into a category for all \(A, B : \mathcal{B}\). For each arrow \(f : A \to B : \mathcal{B}\) we denote its identity 2-cell by \(1_f : f \to f\).

- A **horizontal composition** which is functorial in the vertical composition:

  \[c_{A,B,C} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C) : \text{Cat}\]

  We denote horizontal composition of arrows by \(\circ\), and horizontal composition of 2-cells by \(\ast\). By a usual abuse of notation, we use the same symbol \(\ast\) for the whiskering of arrows with 2-cells: \(\beta \ast f := \beta \ast 1_f\) and \(g \ast \alpha := 1_g \ast \alpha\).

  We allow ourselves to omit the symbol \(\circ\) both for the horizontal composition of arrows and the vertical composition of 2-cells.

- An **identity arrow** \(1_A : A \to A\) for each object \(A : \mathcal{B}\), **right** and **left unitors** \(r_u : u1_A \simeq u\) and \(l_u : 1_Bu \simeq u\) respectively for each \(u : A \to B : \mathcal{B}\), and **associators** \(a_{u,v,w} : (uv)w \simeq u(vw)\) for each triple \(u, v, w\) of composable arrows. Unitors and associators are required to be natural, and the triangle and pentagon axioms are required to hold.

  We will assume that the reader is familiar with the idea of pasting diagrams and with the coherence theorems of bicategory theory.

**Definition 1.2** (Functors between Bicategories). A **pseudo-functor** \(F : \mathcal{A} \to \mathcal{B} : \text{Bicat}\) is the data of a mapping of objects to objects, a mapping of arrows to arrows and a mapping of 2-cells to 2-cells, compatible with domain and codomain, together with:

- A natural **functoriality constraint** \(F_{u,v}^2 : Fu \circ Fv \simeq F(u \circ v)\) for each pair \(u, v\) of composable arrows.

- A **unity constraint** \(F_A^0 : 1_{FA} \simeq F(1_A)\) for each object \(A : \mathcal{B}\).
This data is subject to four axioms: local functoriality, the lax associativity, lax left and lax right unity axioms. For each \( A, B : \mathcal{A} \) we denote the local hom-functors by \( F_{A,B} : \mathcal{A}(A,B) \to \mathcal{B}(FA,FB) \).

**Lemma 1.7.** The properties of a Cartesian arrow, equivalent to the ones in [Buc14, Def. 3.1.1], respectively essentially surjective, full, and faithful, we have the following detailed description of is an equivalence of categories. Writing explicitly what it means for the functor in \((1.6)\) to be canonical diagram exhibits the category \( E \) is Cartesian if and only if, for every \( Z : \mathcal{E} \), the canonical diagram

\[
\begin{align*}
\mathcal{E}(Z,X) & \xrightarrow{f_Z} \mathcal{E}(Z,Y) \\
\mathcal{B}(PZ,PX) & \xrightarrow{(Pf)_*} \mathcal{B}(PZ,PY)
\end{align*}
\]

exhibits the category \( \mathcal{E}(Z,X) \) as a bipullback of \( \mathcal{E}(Z,Y) \) and \( \mathcal{B}(PZ,PX) \) over \( \mathcal{B}(PZ,PY) \). We denote by \( C_{\mathcal{L}} \) the family of Cartesian arrows of \( P \).

Recall also that \( \textbf{Cat} \) has pseudo-pullbacks, which are computed as follows: given a cospan of functors \( A \xrightarrow{F} B \xleftarrow{G} C \), the objects in the pseudo-pullback \( A \times_B C \) are given by triples \((A,C,\alpha)\), where \( FA \xrightarrow{\alpha} GC \) is an invertible arrow, and the arrows \((A,C,\alpha) \to (A',C',\alpha')\) by pairs \( A \xrightarrow{f} A' \), \( C \xrightarrow{g} C' \) such that \( \alpha' (Ff) = (Gg) \alpha \). Then, we can express the statement above that “the canonical diagram exhibits the category \( \mathcal{E}(Z,X) \) as a bipullback...” equivalently as “the comparison morphism

\[
\begin{align*}
\mathcal{E}(Z,X) \xrightarrow{(f_*PZ,X,F_{Z,Y}^2)} \mathcal{E}(Z,Y) \times_{\mathcal{B}(PZ,PY)} \mathcal{B}(PZ,PX)
\end{align*}
\]

is an equivalence of categories”. Writing explicitly what it means for the functor in \((1.6)\) to be respectively essentially surjective, full, and faithful, we have the following detailed description of the properties of a Cartesian arrow, equivalent to the ones in [Buc14] Def. 3.1.1].

**Lemma 1.7.** An arrow \( f : X \to Y : \mathcal{E} \) is Cartesian if and only if it satisfies

0. For each \( Z : \mathcal{E}, g : Z \to Y, h : PZ \to PX \) with an isomorphism \( \alpha : Pf \circ h \simeq Pg \)

\[ X \xrightarrow{f} Y \]

\[ PZ \xrightarrow{PZ,PX} PY \]
there exists an \( \hat{h} : Z \to X \) and isomorphisms \( \hat{\alpha} : f \hat{h} \simeq g, \hat{\beta} : P \hat{h} \simeq h \) such that \( \alpha \circ (P \hat{f} * \hat{\beta}) = P \hat{\alpha} \circ P^2 \hat{h} \). We say that \( (\hat{h}, \hat{\alpha}, \hat{\beta}) \) is a lift of \( (h, \alpha) \), or that it factors \( g \) through (the Cartesian arrow) \( \hat{f} \) above \( \alpha \).

1. If \( g, h : Z \to X \) are arrows in \( \mathcal{E} \), and there are 2-cells \( \alpha : fg \Rightarrow fh \) in \( \mathcal{E} \), \( \beta : Pg \Rightarrow Ph \) in \( \mathcal{B} \), such that \( P \alpha \) equals the composition \( P(fg) \simeq Pf \circ Pg \overset{Pf \star \beta}{\Rightarrow} P \circ Ph \simeq P(fh) \), then there is a 2-cell \( \beta : g \Rightarrow h \) in \( \mathcal{E} \) such that \( P \hat{\beta} = \beta \), \( f \star \beta = \alpha \).

2. If \( \alpha, \beta : g \Rightarrow h : Z \to X \) are 2-cells in \( \mathcal{E} \), then the equalities \( P \alpha = P \beta \) and \( f \star \alpha = f \star \beta \) together imply \( \alpha = \beta \). In particular, this implies that the 2-cell \( \hat{\beta} \) in item 1 is unique. \( \square \)

We begin by introducing a notion of 1-fibration, which is weaker than the notion of fibration introduced in [Buc14] Def. 3.1.5 and does not involve Cartesian 2-cells (see Definition 1.10). This is done here mainly for a matter of convenience, because we noticed that Cartesian 2-cells are not needed for proving the Lifting Fractions Lemma 3.12. In addition, we mention that some preliminary computations seem to indicate that 1-fibrations could be related to a kind of lax homomorphism into \( \text{Bicat} \), with an equivalence given by a Grothendieck construction, just like fibrations are related to trihomomorphisms in [Buc14].

**Definition 1.8 (1-Fibration).** We say that \( P \) is a 1-fibration if for any \( E : \mathcal{E} \) and \( f : B \to PE \), there exists an object \( \hat{B} : \mathcal{B} \) and a Cartesian arrow \( \hat{f} : \hat{B} \to E \) with \( P \hat{B} = B \) and \( P \hat{f} = f \):

\[
\hat{B} \xrightarrow{\hat{f}} E \quad \rightsquigarrow \quad B \xrightarrow{f} PE
\]

We refer to \( \hat{f} \) as a Cartesian lift of \( f \) (at \( E \)).

It is natural to ask in Definition 1.8 for an invertible 2-cell \( P \hat{f} \simeq f \) instead of the equality (see [Buc14] Rem. 3.1.6), or even for an equivalence \( PB \simeq B \) and an invertible 2-cell filling the appropriate triangle. A careful check of the proof of Lemma 3.12 shows that the results still hold under those hypotheses.

We recall now from [Buc14] 3.1.4, 3.1.5 what it means for the pseudo-functor \( P : \mathcal{E} \to \mathcal{B} \) to be a fibration.

**Definition 1.9.** We say that \( \alpha : f \Rightarrow g : X \to Y : \mathcal{E} \) is Cartesian if it is Cartesian for the local functor \( P_{X,Y} : \mathcal{E}(X,Y) \to \mathcal{B}(PX,PY) \). We denote by \( \mathcal{C}_2 \) the family of Cartesian 2-cells of \( P \).

**Definition 1.10.** We say that \( P \) is a fibration when it is a locally fibred 1-fibration such that the horizontal composition of Cartesian 2-cells is Cartesian.

The following are some basic properties of Cartesian arrows that hold when \( P \) is a fibration (see [Buc14] 3.1.8 to 3.1.12) and that we will be using in this paper:

**Proposition 1.11.** Equivalences are Cartesian, and Cartesian arrows are closed both under composition and under invertible 2-cells.

**Proposition 1.12.** Given composable arrows \( f \) and \( g \), if \( g \) and \( gf \) are Cartesian then so is \( f \).

**Proposition 1.13.** If \( f \) is Cartesian and \( Pf \) is an equivalence, then so is \( f \).

## 2 Filtered and Pseudofiltered Bicategories

In this section we consider the filtered axioms for a 2-category that can be found in [Ken92] and the pseudofiltered axioms introduced in [DS06] and corrected in [DS21]. We give here the definitions of filtered (2.2) and pseudofiltered (2.7) bicategories in two equivalent ways: as a set of axioms and as a general statement about existence of pseudo-cocones. By showing that the axioms introduced here are equivalent to the ones in the literature, it follows from our work that some of the requirements can be dropped from these axioms (see Lemma 2.3 and Proposition 2.13).

We begin by recalling the definition of pseudo-cocone of a diagram in a bicategory (a version for 2-categories and 2-functors can be found, for example, in [DS21] p.13]).
Definition 2.1 (Pseudo-Cocones). By a diagram (in \(B\)) we mean a pseudo-functor \(F: C \to B\), where \(C\) is a bicategory. The diagram is said to be finite when \(C\) is (that is, it has finitely many objects, arrows, and 2-cells). A pseudo-cocone of a diagram, with apex \(E: B\) is given, as usual, by a pseudo-natural transformation \(\theta: F \Rightarrow \Delta E\) into the constant pseudo-functor. As such, it is given by two families of arrows and invertible 2-cells -

\[
\begin{array}{c}
F A \\
\downarrow \theta_A \\
E
\end{array}
\begin{array}{c}
F A \\
\downarrow \theta_A \\
E
\end{array}
\begin{array}{c}
F A \\
\downarrow \theta_A \\
E
\end{array}
\begin{array}{c}
F A \\
\downarrow \theta_A \\
E
\end{array}
\]

- \(PC0\) For \(A: C\), we have a unity axiom

PC1 For \(A \xrightarrow{f} B \xrightarrow{g} C: C\), we have a functoriality axiom

PC2 For \(A \xrightarrow{f} B : C\), we have a vertical naturality axiom

The dual notion is called pseudo-cone.

2.1 On filtered bicategories

Now that we have explicitly stated the definition of pseudo-cocone, we can define and characterize filtered bicategories as follows:

Definition 2.2 (Filtered Bicategory). A non-empty bicategory \(B\) is said to be filtered if every finite diagram has a pseudo-cocone. \(B\) is said to be cofiltered if \(B^{op}\) is filtered; that is, if any finite diagram in \(B\) has a pseudo-cone.

Proposition 2.3 (Filtered Bicategory). A non-empty bicategory is filtered if and only if it satisfies the following three axioms:

0-Flt For any objects \(A, B\), there exist an object \(C\) and arrows \(u: A \to C\), \(v: B \to C\),

\[
\begin{array}{c}
A \\
\downarrow u \\
B \\
\downarrow v \\
C
\end{array}
\]
1-Flt For any pair of parallel arrows \( f, g : A \to B \), there exist an object \( C \), an arrow \( u : B \to C \), and a 2-cell \( \gamma : uf \Rightarrow ug \),

\[
\begin{array}{c}
\xymatrix{
A \ar[r]_{f} \ar@{=>}[r]_{\gamma} & B \ar[r]_{g} \ar@{=>}[r]_{\delta} & C
}
\end{array}
\]

2-Flt For any pair of parallel 2-cells \( \alpha, \beta : f \Rightarrow g : A \to B \), there exist an object \( C \) and an arrow \( u : B \to C \) such that \( u \ast \alpha = u \ast \beta \),

\[
\begin{array}{c}
\xymatrix{
A \ar[r]^{f} \ar@{=>}[r]_{\alpha} & B \ar[r]_{g} \ar@{=>}[r]_{\beta} & C
}
\end{array}
\]

We denote the dual cofiltered axioms with the names \( 0-\text{Flt}^{\text{op}}, 1-\text{Flt}^{\text{op}} \), and \( 2-\text{Flt}^{\text{op}} \).

We note that these three axioms appear in [Ken92] for the case of 2-categories except that the 2-cell in Axiom 1-Flt was required to be invertible. Before providing the proof of Proposition 2.3, we show that this requirement can be omitted in the presence of Axiom 2-Flt. A similar situation was observed for an axiom for the right bicalculus of fractions (see [RS21] Prop. 2.3). We mention in passing to the interested reader that we have realized that with the same reasoning one can show that, in [DDS18a] Def. 3.1.2, the sentence “If \( f \in \Sigma \), we may choose \( \alpha \) invertible” can be omitted as well.

**Lemma 2.4.** Assuming 2-Flt, the 2-cell \( \gamma \) in Axiom 1-Flt can be taken to be invertible.

**Proof.** Consider a pair of parallel arrows \( f, g : A \to B \). We use Axiom 1-Flt twice. First we apply this axiom to \( f \) and \( g \), which yields an arrow \( u^{1} : B \to C^{1} \) and a 2-cell \( \gamma^{1} : u^{1}f \Rightarrow u^{1}g \). Then, we apply Axiom 1-Flt to \( u^{1}g \) and \( u^{1}f \), which yields an arrow \( u^{2} : C^{1} \to C^{2} \) and, after composing with the associators, a 2-cell \( \delta' : (u^{2}u^{1})g \Rightarrow (u^{2}u^{1})f \). Taking \( u' = u^{2}u^{1} : B \to C' \), and defining \( \gamma' : u'f \Rightarrow u'g \) as a composition of \( u^{2} \ast \gamma^{1} \) with the associators, we have a pair of 2-cells \( \gamma' : u'f \Rightarrow u'g \) and \( \delta' : u'g \Rightarrow u'f \).

We then use Axiom 2-Flt also twice. First we apply it to the 2-cells \( \delta' \gamma' \), and \( 1_{u'f} \), which yields an arrow \( v^{1} : C' \to C^{2} \) that co-equifies these 2-cells. We then apply Axiom 2-Flt to the 2-cells \( v^{1} \ast (\gamma' \delta') \) and \( 1_{v^{1}(u'g)} \), this yields an arrow \( v^{2} : C^{2} \to C \) that co-equifies these 2-cells. Taking \( v = v^{2}u^{1} : C' \to C \), it follows that \( \gamma = v \ast \gamma' \) and \( \delta = v \ast \delta' \) are the inverses of each other, and taking \( u = vu' \) concludes the proof. \( \square \)

**Proof of Proposition 2.3.** The same ideas from [DDS18a] Prop. 3.1.5] work, but for the sake of completeness we give this brief proof here. For the \((\Rightarrow)\) direction, each of the three axioms in Proposition 2.3 follows by considering pseudo-cones of the diagrams

\[
\begin{array}{c}
\{ \bullet \bullet \} \to \{ A \ B \} \quad \{ \bullet \bullet \bullet \} \to \{ A \ B \} \quad \{ \bullet \bullet \bullet \bullet \} \to \{ A \ B \}
\end{array}
\]

For the opposite direction \((\Leftarrow)\), given a finite pseudo-diagram \( F : C \to B \), we proceed as follows:

0. First, we build a pseudo-cocone on the objects of the diagram; that is, a collection of arrows \( FA \to E \), one for each \( A : C \), by using 0-Flt successively.

\(^{1}\)One of our reasons not to keep the names BF0-BF2 from [Ken92] is to avoid confusion with the names of the axioms for a bicategory of fractions [Pro96].
1. We consider then each arrow of $C$. Given such an arrow $A \xrightarrow{f} B$, we use axiom 1-Flt with the two parallel arrows $\theta_A$ and $\theta_B F f$, this gives an arrow $E \xrightarrow{u} E'$ and a 2-cell $\theta_f$ (that can be taken invertible by Lemma 2.4). We then “update” our cocone by composing it with $u$, in other words we rename $E := E'$, $\theta_A := u \theta_A$ for all $A$, and $\theta_g := u \theta_g$ for any previously constructed 2-cell $\theta_g$. After considering all the arrows of the diagram one by one, we obtain a pseudo-cocone on the objects and arrows of the diagram.

2. Finally we consider each of the equations coming from the axioms PC0-2 in Definition 2.1. We proceed in a similar manner to step 1, but using 2-Flt instead of 1-Flt to make our pseudo-cocone satisfy these finitely many equations.

Remark 2.5 (Why not weighted cocones?). The notion of conical cocone in bicategory theory is not sufficient for all applications. Hence it is a natural question to wonder if one could define other forms of filteredness using other, more exhaustive, types of cocones. Even though this level of generalization might be useful to consider, in this specific case, the conical cocones are enough. Indeed any conical cocone is also a weighted cocone on the same diagram for any weight, as the conical weight is terminal. Hence, if we were to define a filtered bicategory with the family of all finite weighted diagrams having a cocone, we would get an equivalent definition.

2.2 On pseudofiltered bicategories

A bicategory is said to be connected if it is connected as a graph; that is, for any pair of objects in it, there is a "zig-zag" (a finite sequence of forward and backward arrows) linking them. A diagram $C \to B$ is said to be connected if $C$ is.

Warning 2.6. The use of the prefix “pseudo” (without hyphen) in the following definition of pseudofiltered bicategory comes from the 1-dimensional case [GSD, SGA 4, Tome 2]. It weakens the notion of filtered by requiring the diagram to be connected. This should not be confused with the use of the same prefix (with hyphen) in the word “pseudo-cocone” in the same definition. Recall that the notion of pseudo-cocone is given in Definition 2.1, this is the usual use of the prefix “pseudo-” in bicategory theory, as in pseudo-functor and pseudo-natural transformation.

Definition 2.7 (Pseudofiltered Bicategory). A non-empty bicategory $B$ is said to be pseudofiltered if every finite connected diagram has a pseudo-cocone. $B$ is said to be pseudocofiltered if $B^{\text{op}}$ is pseudofiltered.

Almost by definition, we have

Proposition 2.8 (Filtered = Pseudofiltered + Connected). A non-empty bicategory is filtered if and only if it is pseudofiltered and connected.

Proposition 2.9 (Pseudofiltered Bicategory). A bicategory is pseudofiltered if and only if it satisfies the following three axioms:

0-pFlt For any objects $A, B, C$ and arrows $u: C \to A$, $v: C \to B$, there exist an object $D$ and arrows $r: A \to D$, $s: B \to D$:

1-pFlt Axiom 1-Flt in Proposition 2.5

2-pFlt Axiom 2-Flt in Proposition 2.5

As we only changed the names of 1-Flt and 2-Flt, Lemma 2.4 still applies here and we have:

Corollary 2.10. Assuming 2-pFlt the 2-cell $\gamma$ in Axiom 1-pFlt can be taken to be invertible.
And we also have, with an evident application of 1-pFlt:

**Lemma 2.11.** Under the assumption of 1-pFlt and 2-pFlt, one can add the existence of an (invertible) 2-cell $\gamma: ru \Rightarrow sv$ to the consequence of 0-pFlt. $\Box$

**Remark 2.12** (A few other definitions and axioms).

- In [DS06], the authors give a definition of a pre-2-filtered 2-category using two axioms $F_1$ and $F_2$. This is a notion that, while weaker than that of pseudo-2-filtered, still admits a construction of pre-2-filtered pseudo-colimits of categories that is analogous to the construction of (pseudo)filtered colimits of sets. The notion of pseudo-2-filtered 2-category is however needed in order to show that these pseudo-colimits commute with finite connected cotensors [DS06, Th. 2.4].

- In [DS06] the authors define pseudo-2-filtered 2-categories by strengthening $F_1$ to $FF_1$, while keeping $F_2$. In [DS21], they correct this definition so that [DS06, Th. 2.4] holds, by adding a third axiom, $F_3$.

- All these axioms can be considered for a bicategory instead of a strict 2-category. We outline below how one can show that the notion of pseudofiltered bicategory in Definition 2.7 is equivalent to the one in [DS21], obtaining along the way that axiom $F_2$ can in fact be dropped from this definition.

**Exercise 2.13.**

1. Assume that $B$ is pseudofiltered. Show that each of the axioms $FF_1$, $F_2$, and $F_3$ given in [DS21] holds in $B$, by using a pseudo-cocone of a finite connected diagram.

2. Show that axiom $FF_1$ immediately implies both 0-pFlt and 1-pFlt. Note, as mentioned in [DS21, Remark in p.240] that axiom $F_3$ immediately implies 2-pFlt.

After doing the exercise above, the reader will have shown:

**Proposition 2.14.** The following are equivalent sets of axioms, each expressing the property that $B$ is pseudofiltered:

(i) 0-pFlt, 1-pFlt and 2-pFlt in Proposition 2.9.

(ii) $FF_1$, $F_2$, and $F_3$ in [DS21].

(iii) $FF_1$ and $F_3$ in [DS21]. $\Box$

### 3 Diagrammatic Axioms of Fractions

In this section, we introduce a new set of axioms for a right bicalculus of fractions for a family $W$ of arrows of a bicategory $B$, a notion originally introduced in [Pro96]. This new set of axioms is then used to prove a series of lemmas, that are important for the rest of this paper, with simpler computations than the ones that would be required with the original axioms. We show in Proposition 3.6 that both sets of axioms are equivalent, so that when these axioms are satisfied the localization of $B$ at $W$ can be constructed as in [Pro96]. We begin by recalling the results as can be found in [Pro96], except that we give the following more general definition of a localization simultaneously at a family of 1-cells and at a family of 2-cells, since this notion will be used in the present paper (see Theorem 3.13).

**Definition 3.1** (Localization by a family of arrows and 2-cells). Let $B$ be a bicategory, $W_1$ be a family of arrows of $B$ and $W_2$ be a family of 2-cells of $B$. We say that a pseudo-functor $L: B \rightarrow C$ is a localization of $B$ with respect to $W_{1,2}$ if:

---

2In private correspondence, E. Dubuc has made the following clarifications to the content of [DS21]: the 2-cell $\varepsilon$ in axiom $F_3$ is required to be invertible, and axiom $F_2$ is not needed for proving the Remark on p.240.
• it maps the arrows of $W_1$ to equivalences and the 2-cells of $W_2$ to invertible 2-cells, and furthermore,

• for each bicategory $D$, the precomposition with $L$,

$|C, D| \xrightarrow{L^*} |B, D|_{W_{1,2}}$

is a biequivalence of bicategories.

Here $|B, D|_{W_{1,2}}$ stands for the full sub-bicategory of $|B, D|$ on those pseudo-functors that send the arrows of $W_1$ to equivalences and 2-cells of $W_2$ to invertible 2-cells. If such a localization exists, it is unique up to biequivalence and pseudo-natural isomorphism. We usually denote $C$ by $B|W^{-1}$ and $L$ by $L_{W_{1,2}}$. Similarly, we define a localization of a bicategory $B$ by a family of arrows $W$, using the empty family of 2-cells in the above (or any subfamily of the family of invertible 2-cells). For simplicity, we then write $|B, D|_{W}$ and $L_{W}$ omitting the empty family of 2-cells.

The first time a localization for bicategories was considered is in [Pro96], where a set of conditions $BF1$-$BF5$ is given that allows for a construction of the localization of a family of arrows by a “right bicalculus of fractions”, generalizing the one in [GZ67]. We record this result below:

**Definition 3.2 (Calculus of Fractions).** We say that the pair $(B, W)$ (or the family $W$) admits a bicalculus of right fractions, or satisfies right fractions, when the conditions $BF1$-$BF5$ in [Pro96] are satisfied. We say that $(B, W)$ satisfies left fractions if $(B^{op}, W^{op})$ satisfies right fractions, and we say that $W$ satisfies fractions if it satisfies right or left fractions.

**Theorem 3.3.** Let $B$ be a bicategory and $W$ be a family of arrows admitting a bicalculus of fractions. Then the localization of $B$ by $W$ exists.

Note that in the original [Pro96], the universal property of the bicategory of fractions is stated slightly differently from the one we give in Definition 3.1 above, but in [PS21] both properties are seen to coincide, see [PS21, Remarks 3.7 (1), (2)].

### 3.1 A new set of diagrammatic axioms of fractions

We fix throughout this section a class $W$ of arrows of a bicategory $B$ that contains the equivalences $BF1$, is stable under composition $BF2$, and is closed under invertible 2-cells $BF5$. We use the symbol $\Rightarrow$ to denote arrows in $W$. We present now a set of three axioms that can replace the remaining axioms 3 and 4 from [Pro96] (or from [PS21]), that is the diagrammatic axioms.

**0-Frc** Given objects $A, B, C$ and arrows $w: A \Rightarrow B \in W$, $f: C \Rightarrow B$, there exist an object $D$, arrows $u: D \Rightarrow C \in W$, $h: D \Rightarrow A$, and an invertible 2-cell $\alpha: fu \simeq wh$:

![Diagram](https://via.placeholder.com/150)

**1-Frc** Given objects $B, C, D$, arrows $w: C \Rightarrow D \in W$, $f, g: B \Rightarrow C$, and a 2-cell $\alpha: wf \Rightarrow wg$, there exist an object $A$, an arrow $u: A \Rightarrow B \in W$, and a 2-cell $\beta: fu \Rightarrow gu$ such that $a_{w,g,u} \circ (\alpha \ast u) = (w \ast \beta) \circ a_{w,f,u}$:

![Diagram](https://via.placeholder.com/150)
2-Frc  For any objects $B, C, D$, arrows $w: C \to D \in W$, $f, g: B \to C$, and 2-cells $\alpha, \beta: f \Rightarrow g$, such that $w \circ \alpha = w \circ \beta$, there exist an object $A$ and an arrow $u: A \to B \in W$, such that $\alpha \circ u = \beta \circ u$:

$$
\begin{align*}
A \xrightarrow{w} B \xrightarrow{ug} D & = \quad A \xrightarrow{fu} C \xrightarrow{ug} D
\end{align*}
$$

Remark 3.4. Note that the axiom \([0\text{-Frc}]\) is the same as axiom \([\text{BF3}]\) in \([\text{Pre90}]\). Also, the content of axiom \([2\text{-Frc}]\) is shown to follow from the \([\text{BF}]\) set of axioms in \([\text{Tom16}]\) (see also \([\text{PS21}]\) Lemma 2.5]). Finally, note that axiom \([\text{BF4}]\) has a first part which is exactly the content of axiom \([1\text{-Frc}]\), a statement about the invertibility of $\beta$ that can be omitted (as shown in \([\text{PS21}]\) Lemma 2.3]), and a final part that we recall now for convenience:

Furthermore, the collection of triples $(A, u, \beta)$ such that $\alpha \circ u = w \circ \beta$ as in \([1\text{-Frc}]\) satisfies the following property: for any two such triples $(A_1, u_1, \beta_1)$, $(A_2, u_2, \beta_2)$, there exists an object $X: B$, arrows $s: X \to A_1$ and $t: X \to A_2$, and an invertible 2-cell $\varepsilon: u_1 \circ s \simeq u_2 \circ t$ such that $u_1$s and $u_2$t are in $W$ and such that we have the following equality of pastings:

$$
\begin{align*}
\begin{array}{c}
\begin{array}{ccc}
& & u_2 \\
& s & \\
\stackrel{u_1}{\downarrow} & & \beta \\
\end{array} & & \begin{array}{ccc}
& & \beta_2 \\
\stackrel{t}{\downarrow} & & u_1 \\
\end{array}
\end{array} & = & \begin{array}{c}
\begin{array}{ccc}
& & u_2 \\
& s & \\
\stackrel{u_1}{\downarrow} & & \beta \\
\end{array} & & \begin{array}{ccc}
& & \beta_2 \\
\stackrel{t}{\downarrow} & & u_1 \\
\end{array}
\end{array}
\end{align*}
$$

(3.5)

**Proposition 3.6 (Equivalence of Diagrammatic Axioms).** Assuming the closure Axioms \([\text{BF1}]\), \([\text{BF2}]\), and \([\text{BF5}]\), we have the equivalence

$$
[\text{BF3} + \text{BF4}] \iff [0\text{-Frc} + 1\text{-Frc} + 2\text{-Frc}]
$$

**Proof.** In view of Remark 3.4 it only remains to show that the final part of the axiom \([\text{BF4}]\) follows from the \([\text{Frc}]\) set of axioms. Pick two triples $(\alpha_1, u_1, \beta_1)$, $(\alpha_2, u_2, \beta_2)$ as in \([1\text{-Frc}]\). Applying first \([0\text{-Frc}]\) to $u_1$ and $u_2$, we get

such that both $u_1\cdot s'$ and $u_2\cdot t'$ are in $W$. As we have the following equalities of pasting:

$$
\begin{align*}
\begin{array}{c}
\begin{array}{ccc}
& & u_2 \\
& s & \\
\stackrel{u_1}{\downarrow} & & \beta \\
\end{array} & & \begin{array}{ccc}
& & \beta_2 \\
\stackrel{t}{\downarrow} & & u_1 \\
\end{array}
\end{array} & = & \begin{array}{c}
\begin{array}{ccc}
& & u_2 \\
& s & \\
\stackrel{u_1}{\downarrow} & & \beta \\
\end{array} & & \begin{array}{ccc}
& & \beta_2 \\
\stackrel{t}{\downarrow} & & u_1 \\
\end{array}
\end{array}
\end{align*}
$$

12
we get, by $2$-$\text{Frc}$ that the two 2-cells in (3.3) can be made equal by precomposing with an arrow $u \in W$, that is

![Diagram](image)

We then take $s = s'\circ u$, $t = t'\circ u$ and $\varepsilon = a_{u_2, u_1} (\varepsilon' \circ u) \circ a_{u_1, s'}^{-1}$ and the equation we get is precisely the one in (3.3).

3.2 On one- and two-dimensional calculi of fractions

We introduce here the following lemma, that will allow us to compute pseudofiltered pseudocolimits of categories using only the ordinary calculus of fractions from [GZ67]. We will denote these well-known axioms for right fractions by $\text{R0}$ (wide subcategory), $\text{R1}$ (Ore condition) and $\text{R2}$ (existence of equalizing arrows), we refer to [GZ67] or [Fri11 §3] for details. We consider $\pi_0$, the left adjoint to the inclusion $d: \text{Cat} \rightarrow \text{Bicat}$. We note that $2$-$\text{Frc}$ plays no role in the following proof.

**Lemma 3.7.** Let $B$ be a bicategory and $W$ a family of arrows of $B$ satisfying right fractions. Let $W_0 = [W]$ be the family of arrows of $\pi_0 B$ given by the equivalence classes containing arrows of $W$. Then $W_0$ admits a (one-dimensional) calculus of fractions in the category $\pi_0 B$.

**Proof.** We will write $[f]: x \rightarrow y$ for the equivalence class of an arrow $f: x \rightarrow y$ in $\pi_0 B$. An arrow of $W_0$ is, by definition, of the form $[w]$ for some $w \in W$. However, the reader will notice that even if $f$ is not in $W$, it’s possible that $[f]$ is in $W_0$, $W_0$ satisfies $\text{R0}$ as $W$ contains identities and is stable under composition. Also, $\text{0-Frc}$ implies that $W_0$ satisfies $\text{R1}$.

Hence the only technical step is proving $\text{R2}$. Let $f, g: b \rightarrow c$ and $w: c \rightarrow d \in W$ be arrows of $B$ such that

$$[wf] = [wg]$$

We want to show that there is an arrow $u: a \rightarrow b \in W$ such that $[fu] = [gu]$. We know that the relation $[wf] = [wg]$ corresponds to the existence of a zig-zag of 2-cells from $wf$ to $wg$. More precisely, there is an integer $n \geq 1$, a sequence of parallel arrows $h_0, \ldots, h_n: b \rightarrow d$ with $h_0 = wf$, $h_n = wg$ and a sequence of 2-cells each oriented either as $\alpha_i: h_i \Rightarrow h_{i+1}$ or as $\alpha_i: h_{i+1} \Rightarrow h_i$, for $i \in \{0, \ldots, n-1\}$. If all arrows $h_i$ were of the form $wk_i$, then we could easily conclude by applying $\text{1-Frc}$ successively, but this is not necessarily the case. We solve this problem by using $\text{0-Frc}$ in the following proof by induction on $n$:

If $n = 1$, as stated above, we’re done by $\text{1-Frc}$. If $n > 1$, we apply the axiom $\text{0-Frc}$ to the arrows $w \in W$ and $h_1$ to obtain a square,

![Diagram](image)

This will allow us, in the rest of this proof, to replace $h_1$ with $h_1v$, which, using the 2-cell above, can itself be seen as being of the form $wh'$ as desired. We do this by whiskering the 2-cells $\alpha_i$ with $v$, for $i \in \{1, \ldots, n-1\}$, and we construct in this way a zig-zag of length $n - 1$ between $(wg)v$ and $h_1v$. By adding the invertible 2-cell $h_1v \simeq wh'$ on one side and the associator on the other side
of this zig-zag, we then get a zig-zag of length \( n - 1 \) between \( w(gv) \) and \( wh' \). By the induction hypothesis, we have an object \( a_1 : B \) and an arrow \( u_1 : a_1 \to p \in W \) such that \([(gv)u_1] = [h'u_1] = whh' \\
\begin{tikzpicture}

\node (B) at (-1,0) \( B \);
\node (C) at (1,0) \( C \);
\node (D) at (2,0) \( D \);
\node (S) at (-1,-1) \( S \);
\node (T) at (0,-1) \( T \);
\node (U) at (2,-2) \( U \);
\node (V) at (3,-2) \( V \);

\draw[->] (B) to node[auto] \( w \) (C);
\draw[->] (C) to node[auto] \( f \) (D);
\draw[->] (C) to node[auto] \( g \) (S);
\draw[->] (S) to node[auto] \( v \) (T);
\draw[->] (T) to node[auto] \( u \) (D);
\draw[->] (B) to node[auto] \( f \) (S);
\draw[->] (S) to node[auto] \( v \) (U);
\draw[->] (U) to node[auto] \( w \) (D);
\draw[->] (D) to node[auto] \( u \) (V);
\end{tikzpicture}
\\
\text{(3.8)}
\]

We are now only left to deal with the 2-cell \( \alpha_0 \) and, to do so, we construct first a 2-cell \( \gamma \) between \( wh' \) and \( w(fv) \) using \( \alpha_0 \ast v \) and \( \alpha \) as follows. Note that \( \alpha_0 \) is oriented in an arbitrary direction, so \( \gamma \) can either be \( a_{w,f,v} \circ (\alpha_0 \ast v) \circ \alpha \) or \( \alpha^{-1} \circ (\alpha_0 \ast v) \circ a_{w,f,v}^{-1} \), depending on the direction of \( \alpha_0 \). Now by \[1\text{-Frc}\] on \( \gamma \) (independently of its direction, which below is drawn downwards but could be upwards), we have an object \( a_0 : B \) and \( u_0 : a_0 \to p \in W \) such that \([(fv)u_0] = [h'u_0] = whh' \\
\begin{tikzpicture}

\node (B) at (-1,0) \( B \);
\node (C) at (1,0) \( C \);
\node (D) at (2,0) \( D \);
\node (S) at (-1,-1) \( S \);
\node (T) at (0,-1) \( T \);
\node (U) at (2,-2) \( U \);
\node (V) at (3,-2) \( V \);

\draw[->] (B) to node[auto] \( w \) (C);
\draw[->] (C) to node[auto] \( v \) (D);
\draw[->] (C) to node[auto] \( u \) (S);
\draw[->] (S) to node[auto] \( g \) (T);
\draw[->] (T) to node[auto] \( f \) (U);
\draw[->] (U) to node[auto] \( w \) (D);
\draw[->] (D) to node[auto] \( u \) (V);
\end{tikzpicture}
\\
\text{(3.9)}
\]

Finally, applying \[0\text{-Frc}\] to \( u_0 \) and \( u_1 \), we get an object \( a \) and an arrow \( a : a \to p \in W \) that factors, up to invertible 2-cells, through both \( u_0 \) and \( u_1 \) and hence such that

\[ [g(vu)] = [(gv)u] \vDash \vDash [h'u] \vDash \vDash [(fv)u] = [f(vu)] \]

Noting that \( vu \in W \), this finishes the proof.

\section{Lifting fractions and filtered axioms through fibrations}

We are now ready to combine the three \( F \)'s. We will show in Lemma \ref{lem:lift-fractions} that a family of arrows satisfying right fractions can be lifted through a 1-fibration of bicategories. We obtain as a corollary that the Cartesian arrows of a fibration over a pseudocofiltered bicategory satisfy fractions, as shown in \[\text{Cartesian arrows of fibrations over pseudocofiltered bicategories satisfy fractions}
\]

\begin{lemma}[Pseudocofiltered implies Fractions]
If \( B \) is a pseudocofiltered bicategory as in Definition \ref{def:pscofiltered}, then the collection of all arrows of \( B \) satisfies right fractions.
\end{lemma}

\begin{proof}
When \( W \) is the collection of all arrows of \( B \), certainly the closure properties are satisfied. Also, \[0\text{-Frc}\] is given by \[0\text{-pFlt}\] (see Lemma \ref{lem:1-frc-implies-0-frc}), and \[2\text{-Frc}\] is given by \[2\text{-pFlt}\]. To show \[1\text{-Frc}\], consider the (connected) diagram:

\begin{tikzpicture}

\node (B) at (-1,0) \( B \);
\node (C) at (1,0) \( C \);
\node (D) at (2,0) \( D \);
\node (S) at (-1,-1) \( S \);
\node (T) at (0,-1) \( T \);
\node (U) at (2,-2) \( U \);
\node (V) at (3,-2) \( V \);

\draw[->] (B) to node[auto] \( w \) (C);
\draw[->] (C) to node[auto] \( f \) (D);
\draw[->] (C) to node[auto] \( g \) (S);
\draw[->] (S) to node[auto] \( v \) (T);
\draw[->] (T) to node[auto] \( u \) (D);
\draw[->] (B) to node[auto] \( f \) (S);
\draw[->] (S) to node[auto] \( v \) (U);
\draw[->] (U) to node[auto] \( w \) (D);
\draw[->] (D) to node[auto] \( u \) (V);
\end{tikzpicture}

The existence of the 2-cell \( \beta : fu \Rightarrow gu \) such that \( a_{w,g,u} \circ (\alpha \ast u) = (w \ast \beta) \circ a_{w,f,u} \) follows either by considering a pseudo-cone of this diagram, or as follows. First we apply \[1\text{-pFlt}\] to \( f \) and \( g \), we get thus an arrow \( u' : A \to B \) and a 2-cell \( \gamma : fu' \Rightarrow gu' \), and then we apply \[2\text{-pFlt}\] to the 2-cells \( a_{w,g,u'} \circ (\alpha \ast u') \) and \( (w \ast \gamma) \circ a_{w,f,u'} \).
\end{proof}

\begin{remark}[Conditions for Fractions to imply (Pseudo)cofilteredness]
Recall that an object 1 in a bicategory \( B \) is called (bi)terminal if it is the bilimit of the empty diagram; that is, if for all \( E : B \) the category \( B(E,1) \) is a contractible groupoid. This immediately implies that for any \( E : B \), for any \( u : E \to 1 \) and for any \( E' : B \), we have
\end{remark}
1. for any pair of parallel arrows $E' \xrightarrow{f} E$, there is an (invertible) 2-cell $uf \simeq ug$, and

2. for any pair of parallel 2-cells $E' \xrightarrow{f} E$, their whiskerings by $u$ are equal.

Assume that $B$ has a (bi)terminal object 1, and let $W$ be a family of arrows of $B$ admitting a calculus of fractions, such that $W$ contains all the arrows into 1. Note that in this case $B$ is connected and hence the notions of cofilteredness and co-pseudofilteredness are equivalent via Proposition 2.8. Then, using item 1 above, $1\text{-Flt}^{op}$ (for $B$ and $W$) implies $1\text{-Frc}$ using item 2. Finally, under these hypotheses $0\text{-Frc}$ implies $0\text{-Flt}^{op}$ at once, so $B$ is cofiltered.

Note that, considering the family $W$ of all the arrows of $B$, we have in particular that when $B$ has a terminal object the converse of Lemma 3.10 holds.

Lemma 3.12 (Lifting Fractions Lemma). Let $P: E \rightarrow B$ be a 1-fibration of bicategories (in particular, $P$ could be a fibration). Let $W$ admit a calculus of right fractions on $B$. Then $C_W$ admits a calculus of right fractions on $E$, where $C_W$ is the family of Cartesian arrows over $W$.

Proof. The required closure properties of Cartesian arrows are shown in Proposition 1.11. In view of Proposition 3.6, we can work with the $\text{Frc}$ set of axioms instead of $\text{BF3}$ and $\text{BF4}$. Throughout this proof we denote both the arrows in $W$ and those in $C_W$ by $\circ \rightarrow$ (but not all Cartesian arrows).

$0\text{-Frc}$ Consider objects $A, B, C: E$ and arrows $w: A \rightarrow B$ in $C_W$ and $f: C \rightarrow B$. Use first $0\text{-Frc}$ in $B$, to get a diagram of the form

$$
\begin{array}{c}
D \xrightarrow{h} PA \\
\phantom{D} \downarrow \simeq \downarrow Pw \\
PC \xrightarrow{pf} PB
\end{array}
$$

Now choose a Cartesian lift $\hat{u}: \hat{D} \rightarrow C$ of $u$ (at $C$) and then consider the 2-cell $\gamma = P^2_{f,\hat{u}} \circ \alpha^{-1}$ and a lift of $(h, \gamma)$, as in item 0 in Lemma 1.7.

The diagram on the left then shows $0\text{-Frc}$ as required.

$1\text{-Frc}$ Let us consider objects $B, C, D: E$, arrows $w: C \rightarrow D$ in $C_W$, $f, g: B \rightarrow C$, and a 2-cell $\alpha: wf \Rightarrow wg$. We define the 2-cell $P\alpha$ as the composition $PwPf = P(wf) \Rightarrow P(wg) \simeq PwPg$, where the unnamed isomorphisms are structural 2-cells of $F$. We proceed now in three steps which are outlined in the diagram below.
In step (1), we use \textbf{1-Frc} in \( B \). We get then \( u \) and \( \hat{\beta} \) such that the following equation holds:

\[
 a_{Pw,Pg,u} \circ (\widehat{P\alpha} \star u) = ((Pw) \star \hat{\beta}) \circ a_{Pw,P\beta,u} \tag{3.13}
\]

In step (2), we take a Cartesian lift \( \hat{u} \) of \( u \) (at \( B \)).

In step (3), we apply item 1 in Lemma \textbf{1.7} to construct \( \hat{\beta} \) using: “\( f := w \), “\( g := f \hat{u} \), “\( h := g\hat{u} \), “\( \alpha \)” as the composition \( w(f\hat{u}) \simeq (wf)\hat{u} \) and “\( \beta \)” as the composition \( P(f\hat{u}) \simeq (Pf)u \Rightarrow (Pg)u \simeq P(g\hat{u}) \) (where the unnamed isomorphisms are either structural 2-cells of \( F \) or associators). To verify the hypothesis in the Lemma amounts to checking that \( P(\alpha \star \hat{u}) \) equals the following composition (where for convenience we write \( \circ \) for the composition of arrows in \( B \) but we omit this symbol for the composition in \( E \))

\[
P((wf)\hat{u}) \simeq Pw \circ (Pf \circ u) \overset{(Pw)\star \hat{\beta}}{\Rightarrow} Pw \circ (Pg \circ u) \simeq P((wg)\hat{u}),
\]

where each of the two unnamed isomorphisms are given by the structural 2-cells of \( E \) and the associators, or equivalently that the two dashed paths in the cube below are equal.

\[
\begin{array}{ccc}
P((wf)\hat{u}) & \overset{Pw,f,\hat{u}}{\longrightarrow} & P(w(f\hat{u})) \\
P(wf) \circ u & \overset{P(wg)\hat{u}}{\longrightarrow} & Pw \circ (Pf \circ u) \\
(Pw \circ Pf) \circ u & \overset{Pw,F,\hat{u}}{\longrightarrow} & Pw \circ (Pg \circ u)
\end{array}
\]

Since the top part of the left face, the front and the back faces are all commutative by the definition of pseudo-functor, the bottom part of the left side is commutative by the definition of \( \widehat{P\alpha} \) above, and the commutativity of the bottom face is precisely equation \textbf{(3.13)}, these two paths are indeed equal as desired.

The equation “\( f \star \hat{\beta} = \alpha \)” in the conclusion of item 1 in Lemma \textbf{1.7} when applied with the definitions above, becomes then \( a_{w,g,\hat{u}} \circ (\alpha \star \hat{u}) = (w \star \hat{\beta}) \circ a_{w,f,\hat{a}} \), as required in the axiom \textbf{1-Frc}.

\textbf{2-Frc} Let us consider objects \( B,C,D : E \), arrows \( w : C \to D \) in \( C_{\mathcal{W}} \), \( f,g : B \to C \), and 2-cells \( \alpha,\beta : f \Rightarrow g \), such that \( w \star \alpha = w \star \beta \). We apply \( P \) and use \textbf{2-Frc} in \( B \), in this way we have \( u : A \Rightarrow PB \) in \( W \) such that \( (P\alpha) \star u = (P\beta) \star u \), and we lift \( u \) to a Cartesian arrow \( \hat{u} : A \to B \). Then, since by pre- and post-composing by structural cells of \( P \) we get \( P(\alpha \star \hat{u}) = P(\beta \star \hat{u}) \) and since \( w \star (\alpha \star \hat{u}) = w \star (\beta \star \hat{u}) \), by item 2 in Lemma \textbf{1.7} \( \alpha \star \hat{u} = \beta \star \hat{u} \) and we are done.

\[ \square \]

Combining Lemmas \textbf{3.10} and \textbf{3.12} we have a bicategorical version of \textbf{GSDV72} (SGA 4, Tome 2) exp. VI, Prop. 6.4]:
Corollary 3.14. Let \( \mathcal{E} \to \mathcal{B} \) be a fibration of bicategories. If \( \mathcal{B} \) is pseudocofiltered, then the Cartesian arrows satisfy right fractions.

Of course, we also have the dual results. Dualizing a calculus of fractions by \(-^{co}\) doesn't change the definition, and dualizing a right/left calculus of fractions by \(-^{op}\) makes it into a left/right calculus of fractions. Hence a co-fibration (see [Buc14, Remark 2.2.14]) lifts left calculi, an op-fibration lifts right calculi, and a coop-fibration lifts left calculi. We record here the following version of Corollary 3.14, which is the one we will actually use to compute colimits in this paper.

Corollary 3.15. Let \( \mathcal{E} \to \mathcal{B} \) be a co-fibration of bicategories. If \( \mathcal{B} \) is pseudofiltered, then the co-Cartesian arrows satisfy left fractions.

4 Bicategory-Indexed Tricolimits of Bicategories

In ordinary 1-category theory, a pseudo-colimit of categories can be computed by localizing, at the Cartesian arrows, the fibration associated to the diagram by its Grothendieck construction [GSDV72, (SGA 4, Tome 2) Exposé VI Section 6]. In this section we show, using fibrations and localizations of bicategories as described in Sections 1.3 and 3 respectively, how conical tricolimits of bicategories (as in Definition 4.10) can similarly be computed by localizing the associated fibration at the Cartesian arrows and 2-cells. As an application, we show in Section 4.2 that we can compute bicategory-indexed pseudofiltered pseudo-colimits of categories by using only the ordinary one-dimensional calculus of fractions.

4.1 Computing colimits in Bicat

Let \( \mathcal{B} \) be a bicategory. In this subsection we will consider a trihomomorphism \( F: \mathcal{B} \to \text{Bicat} \), as originally defined in [GPS95] and developed in more detail in [Gur09], and show that one can compute its conical tricolimit by localizing the bicategory given by its Grothendieck construction (or bicategory of elements, \( \text{el} F \)).

We recall that for arbitrary tricategories \( \mathcal{A} \) and \( \mathcal{B} \), there is a tricategory \([\mathcal{B}, \mathcal{A}]\) of trihomomorphisms, trinatural transformations, trimodifications and perturbations, defined in [GPS95, Gur09]. A trihomomorphism \( F \) as above can be seen as an object of \([\mathcal{B}, \text{Bicat}]\), when \( \mathcal{B} \) is interpreted as a tricategory with trivial 3-cells.

Remark 4.1 (Variance and Duality). Since we have chosen to work with a covariant \( F: \mathcal{B} \to \text{Bicat} \), it is convenient for us to consider the bicategory of elements \( \text{el} F \), with objects given by pairs \((C, x)\), with \( x: FC \) as usual, (that we will denote here as \((x, x_-)\) for the sake of comparison with [Buc14]) and the following arrows and 2-cells

Throughout the paper, \( \text{el} F \) will refer to this bicategory, that can be traced back to at least [Str76] for the case of 2-categories. We found this particular choice of directions to be convenient for doing computations in the covariant case, since it induces a covariant trihomomorphism \( \text{el}: [\mathcal{B}, \text{Bicat}] \to \text{Bicat}/\mathcal{B} \), without any appearance of dual bicategories. We remark however that this is done only for convenience, and we could get similar results in this paper working with the construction \( \text{el}' \) below instead (this is similar to how one can think of a pseudocone as either a lax or oplax cone whose structural 2-cells are invertible).

The bicategory \( \text{el} F \) can be compared with the construction in [Bak] and [Buc14], where (following the classical correspondence between fibrations and pseudo-functors) one starts from a
trihomomorphism $B^{coop} \to \text{Bicat}$ and defines the arrows and 2-cells of its Grothendieck construction as:

\[
\begin{align*}
(f, f^-): (x, x^-) & \to (y, y^-) \\
x_- \xrightarrow{f} Ff(y_-) & \quad \alpha^-: (f, f^-) \Rightarrow (g, g^-) \\
g^- \xleftarrow{\alpha^- \downarrow} (F\alpha)_y & \xrightarrow{\alpha^- \downarrow} Fg(y_-)
\end{align*}
\]

If we denote this construction by $\text{el}'$, we observe that we have

\[
\text{el} F := (\text{el}'((D_{op} \circ F)^{co}))^{op}
\]

where $D_{op}: \text{Bicat} \to \text{Bicat}^{co}$ is the dual operator sending a bicategory to its op-dual bicategory $\mathcal{X} \mapsto \mathcal{X}^{op}$, and where the $\text{el}'$ construction is taken over the base bicategory $B$ instead of over $B$. As it is defined as a (fibration)$^{op}$, we refer to it (and to the bicategories having the corresponding lifting properties) as a co-fibration.

Using (4.2) we can translate definitions and results from [Buc14] to our setting, as in the following proposition.

**Proposition 4.3** ([Buc14 Prop. 3.3.4]). For each trihomomorphism $F: B \to \text{Bicat}$, its Grothendieck construction yields a co-fibration of bicategories $P_F: \text{el} F \to B$ whose co-Cartesian arrows (resp. 2-cells) are those whose second coordinate is an equivalence (resp. invertible).

**Definition 4.4** (Constant Trihomomorphism). Let $B, A$ be tricategories. The trihomomorphism $\Delta: A \to [B, A]$ maps $A$ to the constant trihomomorphism that maps all objects of $B$ to $A$, and all arrows, 2- and 3-cells of $B$ to identities.

Consider the Cartesian product $B \times \mathcal{X}$ of bicategories, whose structure is defined pairwise. Then the projection $\pi_1: B \times \mathcal{X} \to B$ is a fibration of bicategories whose Cartesian arrows (resp. 2-cells) are those pairs of arrows (resp. 2-cells) whose second coordinate is an equivalence (resp. invertible) in $\mathcal{X}$. The following results follows from [Buc14 Constr. 3.3.3].

**Remark 4.5** (Constant Trihomomorphism). For any bicategory $\mathcal{X}$, there is a biequivalence of bicategories (that is actually an isomorphism) $\text{el}(\Delta \mathcal{X}) \cong B \times \mathcal{X}$, making the following triangle commute strictly

\[
\begin{tikzcd}
\text{el}(\Delta \mathcal{X}) \ar{r}{\cong} \ar{d}[swap]{P_{\Delta \mathcal{X}}} & B \times \mathcal{X} \ar{d}{\pi_1} \\
B \ar[rightarrow, bend left=15]{ur}
\end{tikzcd}
\]

We will now consider the following strict slice tricategory $\text{Bicat}/B$, in which Buckley’s generalisation of the Grothendieck construction naturally lands:

0. its objects are pairs $(E, P)$ where $E$ is a bicategory and $P: E \to B$ is a pseudo-functor - when it is clear we omit the pseudo-functor and denote these by $E$,
1. its arrows $F: (E, P) \to (E', P')$ are pseudo-functors $F: E \to E'$ such that $P' \circ F = P$,
2. its 2-cells $\alpha: F \to F'$ are pseudo-natural transformation such that $P' \ast \alpha = 1_P$,
3. its 3-cells $\Gamma: \alpha \Rightarrow \alpha'$ are modifications such that $P' \ast \Gamma = 1_{P'}$ (where $\ast$ stands for the whiskering of a pseudo-functor and a modification).

**Remark 4.6** (Product of Bicategories). For any bicategory $\mathcal{A}$, and any pseudo-functor $P: E \to B$, there is a natural biequivalence of bicategories (that is actually an isomorphism)

\[
(\text{Bicat}/B)(E, B \times \mathcal{A}) \cong [E, \mathcal{A}]
\]
The tricategory $\text{Fib}(B)$ is defined in \cite[Def. 3.2.4]{Buc14} as a sub-tricategory of $\text{Bicat}/B$, full on 2- and 3-cells, whose objects are those pseudo-functors that are fibrations of bicategories, and whose arrows are those pseudo-functors that are Cartesian as defined in \cite[Def. 3.2.4]{Buc14}, that is such that they respect both Cartesian arrows and 2-cells. We define $\text{coFib}(B)$ analogously. The biequivalence of bicategories considered in Remark \ref{rem:cartesian} is clearly Cartesian in this sense. Recalling then the notation in Definition \ref{def:tricategory} we conclude:

**Remark 4.8 (Constant Trihomomorphism).** For any bicategory $A$, and any pseudo-functor $P: E \to B$, there is a natural biequivalence of bicategories (that is actually an isomorphism)

$$\text{coFib}(B)(E, \text{el}(\Delta A)) \cong [E, A]_{C_{1,2}}$$

(given by postcomposition with the biequivalence of bicategories considered in Remark \ref{rem:cartesian}, together with (4.7)) where $[E, A]_{C_{1,2}}$ is the bicategory introduced in Definition \ref{def:tricategory} and $C_1$, resp. $C_2$, are the families of co-Cartesian arrows, resp. 2-cells, of $E$.

**Definition 4.10 (Triclimits).** Let $F: B \to A$ be a trihomomorphism between tricategories. We consider the trihomomorphism

$$A \xrightarrow{\Delta} [B, A] \xrightarrow{[B, A](F, \_, \_)} \text{Bicat}$$

mapping $A$ to the bicategory $[B, A](F, \Delta A)$. If this trihomomorphism is representable (in the sense of tricategory theory), we say that $F$ has a triclimit, and we refer to the object $L$ representing it as the triclimit of $F$. Explicitly, this amounts to saying that there are biequivalences of bicategories, natural in $A$:

$$[B, A](F, \Delta A) \simeq A(L, A)$$

**Proposition 4.11 ([Buc14] Prop. 3.3.12, Local Biequivalence).** For $F, G: B \to \text{Bicat}$, the Grothendieck construction yields a biequivalence of bicategories

$$[B, \text{Bicat}](F, G) \simeq \text{coFib}(B)(\text{el } F, \text{el } G)$$

(4.12)

that is natural in $F$ and $G$.

We are now ready to prove:

**Theorem 4.13 (Conical Triclimits in $\text{Bicat}$).** Let $B$ be a bicategory, and $F: B \to \text{Bicat}$ be a trihomomorphism. Then the triclimit of $F$ in $\text{Bicat}$ is given by the localization $\left(\text{el } F\right)\left[C_{1,2}^{-1}\right]$ as defined in Definition \ref{def:tricategory}, where $C_1$, resp. $C_2$ are the families of co-Cartesian arrows, resp. co-Cartesian 2-cells of $\text{el } F$.

**Proof.** Using in turn (4.12) and (4.9), we have the desired natural biequivalence of bicategories

$$[B, \text{Bicat}](F, \Delta (-)) \simeq \text{coFib}(B)(\text{el } F, \text{el}(\Delta (-))) \simeq \text{el } F, -\left[C_{1,2}\right]$$

(Note that the triclimit of $F$, resp. the desired localization, is defined as a trirepresentation of the trihomomorphism on the left hand side, resp. right hand side.)

As far as we know, the construction of a localization of a bicategory at both arrows and non-trivial 2-cells has never been considered. In this paper, we will apply this result in a case when we won’t need to localize by the 2-cells: the discrete case, that is the computation of pseudofiltered pseudo-colimits of categories in the following section. As will be discussed in a follow-up paper \cite{BVPS}, pseudofiltered triclimits of bicategories can also be computed without localizing at the 2-cells, because the condition in the following corollary holds when $B$ is pseudofiltered:

**Corollary 4.14.** Let $B$ be a bicategory, and $F: B \to \text{Bicat}$ be a trihomomorphism. Suppose that the localization $\left(\text{el } F\right)\left[C_{1}^{-1}\right]$ exists and that $L_{C_{1}}: \text{el } F \to \left(\text{el } F\right)\left[C_{1}^{-1}\right]$ sends co-Cartesian 2-cells to invertible ones, where $C_1$ is the families of co-Cartesian arrows of $\text{el } F$. Then the triclimit of $F$ in $\text{Bicat}$ exists and is given by the localization $\left(\text{el } F\right)\left[C_{1}^{-1}\right]$. 
Proof. For \( D \) a bicategory, as all pseudo-functors in \([\text{el} F, D]_{C_1}\) factor through \( L_{C_1} \) up to natural equivalence, they all send co-Cartesian 2-cells to invertible ones. Hence \([\text{el} F, D]_{C_1} \subseteq [\text{el} F, D]_{C_{1,2}}\). We also have by definition \([\text{el} F, D]_{C_{1,2}} \subseteq [\text{el} F, D]_{C_1}\). This implies that both universal properties coincide: a localization by \( C_1 \) is equivalently a localization by \( C_{1,2} \). The result is then a direct consequence of Theorem 4.13 above. \( \square \)

4.2 Pseudofiltered pseudo-colimits in \( \text{Cat} \)

We consider now a pseudo-functor \( F : B \rightarrow \text{Cat} \) from \( B \) a bicategory, and view it as a discrete trihomomorphism \( dF : B \rightarrow \text{Bicat} \). Looking at the proof of Theorem 4.13 in this case, we notice that we have equivalences of categories (natural in \( E : \text{Cat} \))

\[
[B, \text{Cat}](F, \Delta E) \cong [B, \text{Bicat}](dF, \Delta dE) \cong \text{coFib}(B)(\text{el}(dF), \text{el}(\Delta dE)) \cong [\text{el}(dF), dE]_{C_1, C_2}
\]

By looking explicitly at the proof of the local equivalence of the Grothendieck construction in [Buc14 Prop. 3.3.12], we can observe that the only mere equivalence (\( \simeq \)) left in this chain is, like the others, actually an isomorphism (\( \cong \)). Indeed, in the discrete case it is immediate that the inverse Grothendieck construction is an actual inverse, rather than only a quasi-inverse. Noting that \( E \) has only trivial 2-cells, we can continue this chain of natural isomorphisms as follows:

\[
[\text{el}(dF), dE]_{C_1, C_2} \cong [\text{el}(dF), dE]_{C_1} \cong [\pi_0 \text{el}(dF), E]_{C_1} \cong [(\pi_0 \text{el}(dF))[[C_1]^{-1}], E].
\]

where \([C_1]\) is the family of equivalence classes of co-Cartesian arrows after applying \( \pi_0 \), as in Lemma 3.7. Noting that a pseudo-colimit of a pseudo-functor between bicategories is defined as a strict representation of the pseudo-functor \([B, \text{Cat}](F, \Delta E)\), we conclude:

**Corollary 4.15** (Discrete Case). Let \( F : B \rightarrow \text{Cat} \) be a pseudo-functor. Then the pseudo-colimit of \( F \) is given by the category \( \pi_0(\text{el } dF)[W^{-1}] \), where \( W = [C_1] \) is the family of equivalence classes of co-Cartesian arrows. \( \square \)

**Remark 4.16.** Note that when \( B \) is a pseudofiltered bicategory, by Proposition 4.13, Corollary 3.15 and Lemma 3.7 the category \( (\pi_0 \text{el } dF)[W^{-1}] \) can be computed as a category of fractions. We fix a pseudofiltered bicategory \( B \) and a pseudo-functor \( F : B \rightarrow \text{Cat} \). We will now give an explicit formula of its pseudo-colimit, by describing the category \( (\pi_0 \text{el } dF)[W^{-1}] \). An idea to have in mind, in order to understand the content of the following proposition and its proof, is that the computation of this category as a category of fractions has redundant information, and the presentation of Proposition 4.17 is what we get by discarding this redundancy. When \( B \) is a (strict) 2-category, and \( F \) is a (strict) 2-functor, this construction can immediately be seen to match the one in [DS06].

**Proposition 4.17.** Let \( B \) be a pseudofiltered bicategory and \( F : B \rightarrow \text{Cat} \) a pseudo-functor. The pseudo-colimit of \( F \) can be constructed as the category \( \text{colim}_B F \) defined by the following data:

**Objects:** Pairs \((A, x)\) where \( A : B \) is an object and \( x : FA \) is an object of \( FA \).

**Premorphisms:** Quadruples \((C, u, v, \xi) : (A, x) \rightarrow (B, y)\) where \( C : B \) is an object, \( u : A \rightarrow C \) and \( v : C \rightarrow A \) are arrows and \( \xi : Fu(x) \rightarrow Fv(y) \) is an arrow of \( FC \).

**Homotopies:** Quintuples \((C, w_1, w_2, \alpha, \beta) : (C_1, u_1, v_1, \xi) \equiv (C_2, u_2, v_2, \xi)\), where \( C : B \) is an object, \( w_1 : C_1 \rightarrow C \) and \( w_2 : C_2 \rightarrow C \) are arrows and \( \alpha : w_1 u_1 \simeq w_2 u_2 \) and \( \beta : w_1 v_1 \simeq w_2 v_2 \) are invertible 2-cells such that

\[
\begin{align*}
Fw_1 \circ Fv_1(x) & \xrightarrow{(Fw_1, u_1, v_1)} F(w_1 u_1)Fx \xrightarrow{(F\alpha)} F(w_1 u_1)Fw_2 \circ Fv_2(x) \\
Fw_1(x) & \xrightarrow{(Fw_1, \xi)} Fw_2(x)
\end{align*}
\]

\[
\begin{align*}
Fw_1(y) & \xrightarrow{(Fw_1, \xi)} F(w_1 u_1)Fw_1(y) \xrightarrow{(F\alpha)} Fw_2(y) \\
Fw_1(y) & \xrightarrow{(\beta)} F(w_2 v_2)Fw_2(y)
\end{align*}
\]

(4.18)

We conclude.
**Arrows:** Equivalence classes of premorphisms under the homotopy relation in which two premorphisms are said to be homotopic if there is a homotopy between them.

**Identities:** For \((A, x)\) an object we define \(1_{(A, x)} = [(A, 1_A, 1_A, 1_{F1_A(x)})]\).

**Composition:** For \((A, x)\) \([C_1, u_1, v_1, \xi_1](B, y)\) \([C_2, u_2, v_2, \xi_2](C, z)\) we define the composite as follows. For any \(D: B\) and \(f: C_1 \to D\), \(g: C_2 \to D\) such that we have an invertible 2-cell \(\gamma: f v_1 \simeq g v_2\), which can be constructed using for example \(0\text{-}pFl\) (and Lemma 2.11), we define

\[
[(C_2, u_2, v_2, \xi_2)] \circ [(C_1, u_1, v_1, \xi_1)] = [(D, f u_1, g v_2, (F^2_{g,v_2}) \circ F g(\xi_2) \circ (F^2_{f,\xi_1}) y \circ (F^2_{f,v_1}) y \circ F f(\xi_1) \circ (F^2_{f,u_1}) z)] \quad (4.19)
\]

**Proof.** The category \((\pi_0 \mathcal{E}dF)[W^{-1}]\), which we know from Corollary 4.15 to be the pseudo-colimit of \(F\), can be rather complicated to describe. Indeed, two quotient-like constructions interfere: an identification of arrows \((\pi_0)\) and a free inversion of arrows (localization by calculus of fractions).

In order to sort out this interplay, we introduce the above category \(\text{colim} F\) which can be seen as a “sub-category” of \((\pi_0 \mathcal{E}dF)[W^{-1}]\): it has the same objects, but \(a \text{ priori}\) fewer arrows and \(a \text{ priori}\) fewer homotopies between these arrows. The inclusion of \(\text{colim} F \hookrightarrow (\pi_0 \mathcal{E}dF)[W^{-1}]\) is denoted by \(K\) in the proof. The notion of identities and composition chosen for \(\text{colim} F\) are sent to the corresponding notions in \((\pi_0 \mathcal{E}dF)[W^{-1}]\) by this inclusion (as detailed in Remark 4.23).

As we later prove that existence of homotopies in \((\pi_0 \mathcal{E}dF)[W^{-1}]\) implies existence of homotopies in \(\text{colim} F\), we get for free that \(\text{colim} F\) has a well-defined category.

The proof is then structured as follows: we first unfold the definition of \((\pi_0 \mathcal{E}dF)[W^{-1}]\), then we construct the functors \(H: \text{colim} F \leftrightarrow (\pi_0 \mathcal{E}dF)[W^{-1}]\): \(K\) and finally we prove that the pair forms an isomorphism of categories.

We first construct the category \((\pi_0 \mathcal{E}dF)[W^{-1}]\) as a category of fractions. Its objects are then pairs \((A, x)\), just as in the Proposition. Its arrows \((A, x) \to (B, y)\) are equivalence classes of roofs as follows:

**Roof:**

\[
(A, x) \xrightarrow{(u, \mu)} (C, z) \xleftarrow{(v, \nu)} (B, y)
\]

with \(C, u, v\) as in the Proposition, \(z: FC, \mu: Fu(x) \to z\) an arrow of \(FC\), and \(v: Fu(y) \simeq z\) an isomorphism of \(FC\).

**Equivalence of roofs:** Two roofs are in the same class, as usual, if they can be made part of a third common roof. That is, if there exists a cospan between their “ridges”

\[
\begin{array}{c}
(C_1, z_1) \\
(u_1, \mu_1) \\
(A, x) \\
\end{array} 
\quad \quad \quad 
\begin{array}{c}
(C_2, z_2) \\
(w_2, v_1) \\
(B, y) \\
\end{array} 
\quad \quad \quad 
\begin{array}{c}
(C', z') \\
(w_1' \circ \kappa, \nu_1) \\
(C, z) \\
\end{array}
\]

in \(\mathcal{E}dF\) such that the equivalence class of the right leg is in \(W\) and such that these two equations hold in \(\pi_0 \mathcal{E}dF\):

\[
[[w_1', \kappa](u_1, \mu_1)] = [[w_2', \lambda](u_2, \mu_2)] \quad [[w_1', \kappa](v_1, \nu_1)] = [[w_2', \lambda](v_2, \nu_2)] \in W. \quad (4.20)
\]

**Construction of \(H\).** We construct an assignment \(H: (\pi_0 \mathcal{E}dF)[W^{-1}] \to \text{colim} F\) on objects and arrows, which is the identity on objects, and is defined on the arrows as follows. Note that starting with a roof \(((C, z), (u, \mu), (v, \nu))\) we can construct a premorphism \((C, u, v, \xi)\) by defining...
\( \xi = \nu^{-1}\mu \). We will show now that two equivalent roofs yield homotopic premorphisms (with the notion of homotopy defined in the statement of the proposition), so that \( H \) is well defined.

Let’s describe explicitly what \((4.20)\) means. By the first of these two equations, in \( \text{el} dF \) we have arrows \((h_0, \sigma_0), \ldots, (h_n, \sigma_n)\): \((A, x) \rightarrow (C', z')\) such that \((h_0, \sigma_0) = (w'_1, \kappa)(v_1, \mu_1)\) and \((h_n, \sigma_n) = (w'_2, \lambda)(u_2, \mu_2)\), and 2-cells \(\{\alpha_i\}\) linking those arrows together in a “zig-zag” of length \(n\). Since these are 2-cells in \( \text{el} dF \), we have a commutative diagram in \( FC' \) given on the left of \((4.21)\) below (the orientation of the arrows \(\{\alpha_i\}\) is not fixed, so their direction in the diagram is arbitrary).

Looking at the second equation in \((4.20)\), by definition, we have an arrow \((k, \tau)\) of \( W \) such that \([(w'_1, \kappa)(v_1, \nu_1)] = [(k, \tau)] = [(w'_2, \lambda)(v_2, \nu_2)]\), hence with \(\tau\) an invertible arrow of \( FC' \). These two equalities lead to two zig-zags of 2-cells as above, and by concatenating them we have arrows \((k_0, \tau_0), \ldots, (k_m, \tau_m)\) such that \((k_0, \tau_0) = (w'_1, \kappa)(v_1, \nu_1)\) and \((k_m, \tau_m) = (w'_2, \lambda)(v_2, \nu_2)\), and 2-cells \(\{\beta_j\}\) linking them such that the diagram on the right of \((4.21)\) is commutative, with the further assumption that one of the \(\tau_j\) is invertible.

\[
\begin{align*}
F(w'_1u_1)(x) &\xrightarrow{(F\alpha_1)_x} F(h_1)(x) & F(w'_1u_1)(y) &\xrightarrow{(F\beta_1)_y} F(k_1)(y) \\
F(h_2)(x) &\xrightarrow{(F\alpha_2)_x} F(h_2)(x) & F(k_2)(y) &\xrightarrow{(F\beta_2)_y} F(k_2)(y) \\
F(w'_2u_2)(x) &\xrightarrow{(F\alpha_n)_x} F(w'_2u_2)(x) & F(w'_2u_2)(y) &\xrightarrow{(F\beta_m)_y} F(w'_2u_2)(y)
\end{align*}
\]

We now consider all the connected data given by the 2-cells \(\{\alpha_i\}, \{\beta_j\}\) in the pseudofiltered bicategory \( B \), as on the left of the diagram below, and we take a pseudo-cocone \((C, \phi)\) on it. We will define a homotopy using the data of the pseudo-cocone. More precisely (see the diagram below on the right), we define \(w_1 = \phi C_1, w_2 = \phi C_2, \alpha = \phi^{-1}_w \phi u_1 : w_1 u_1 \simeq w_2 u_2 : A \rightarrow C \) and \(\beta = \phi^{-1}_w \phi v_1 : w_1 v_1 \simeq w_2 v_2 : B \rightarrow C\).

\[
\begin{align*}
C' &\xrightarrow{\sim} C \\
C_1 &\xrightarrow{u_1} A \quad \quad C_2 \xrightarrow{v_2} B \\
C_1 &\xrightarrow{u_2} A \quad \quad C_2 \xrightarrow{v_2} B
\end{align*}
\]

We will now show that equation \((4.13)\) holds for \(\alpha\) and \(\beta\). As \( \text{el} dF \rightarrow B \) is a co-fibration, we can lift \( \phi^{C'} \) to the co-Cartesian arrow \((\phi^{C'}, 1_{\phi^{C'}(z')})\) (as described in Buckley’s cleavage introduced in \([\text{Buc}14\text{ Prop. 3.3.4}]\)). Furthermore, as \( \text{el} dF \rightarrow B \) is discrete, there exist unique lifts of \( \phi^{-1}_{w_1} \) and \( \phi^{-2}_{w_2} \), which are necessarily co-Cartesian and invertible, that can be denoted by the same names.
We then have the following diagrams:

\[
\begin{array}{c}
\text{(C, } F\phi_C(z')) \\
(w_1, \square_1) \\
\end{array}
\begin{array}{c}
\text{(C', } F\phi_C(z')) \\
(w_2, \square_2)
\end{array}
\begin{array}{c}
\text{(C, } F\phi_C(z')) \\
(v_1, v_1)
\end{array}
\begin{array}{c}
\text{(C', } F\phi_C(z')) \\
(v_2, v_2)
\end{array}
\end{array}
\]

where $\square_1$ is defined by the lifting of the co-Cartesian 2-cell $\phi^{-1}_{w_1}$, or more explicitly (with notation as in Remark [4.1]),

\[
Fw_1(z_1) \xrightarrow{F\phi^{-1}_{w_1}(z_1)} F(\phi_C')(w'_1)(z_1) \xrightarrow{(F^2\phi_C, w'_1)\lambda_1} F(\phi_C'F(w'_1))(z_1) \xrightarrow{F\phi_C'(\kappa)} F\phi_C'(z')
\]

and similarly $\square_2$ is defined by replacing all appearances of the subindex 1 by 2, and $\kappa$ by $\lambda$ in the above. Now we note that, since $(C, \phi)$ is a pseudo-cocone, in particular it satisfies axiom [PC2] in Definition [2.1] for each of the 2-cells $\alpha_i$ and $\beta_j$. This means that, once whiskered by $\phi_C'$, these 2-cells can be expressed with coherence 2-cells of $\phi$. Firstly, this implies that all the whiskerings $\phi_C' \circ \alpha_i$ and $\phi_C' \circ \beta_j$ are invertible. Secondly, this implies that the pastings of the diagrams in (4.22) can be computed and are the liftings of respectively $\alpha$ and $\beta$ to $\text{el}_DF$.

The diagrams in (4.22), and the observation above, show that $(\phi_{C'} \circ 1_{F\phi_C(z')})(w'_1, \kappa)(v_1, v_1) \simeq (\phi_{C'} \circ 1_{F\phi_C(z')})(k_j, \tau_j)$ for all $j$, and we consider it for the value of $j$ such that $\tau_j$ is invertible (that exists as shown above (4.14)). As both arrows on the right hand side of this equality are co-Cartesian arrows, using in turn Propositions [1.11] and [1.12] we conclude that so is $(\phi_{C'} \circ 1_{F\phi_C(z')})(w'_1, \kappa)$. By definition of the co-fibration $\text{el}_DF$ (see Proposition [4.3]), this means that $F\phi_{C'}(\kappa)$ is invertible (since it is an equivalence in a discrete bicategory), and hence so is the arrow $\square_1$ as defined above.

Now, by definition of the 2-cells of $\text{el}_DF$ (as in Remark [4.1]), the fact that the pastings of the diagrams in (4.22) are respectively $\alpha$ and $\beta$ means that we have the following commutative diagrams in $FC$:

\[
\begin{array}{c}
F(w_1u_1)(x) \xrightarrow{\square_1} Fw_1(z_1) \\
\end{array}
\begin{array}{c}
F(w_2u_2)(x) \xrightarrow{\square_2} Fw_2(z_2)
\end{array}
\begin{array}{c}
F(w_1z_1) \xrightarrow{\square_1} F(w_1v_1)(y) \\
\end{array}
\begin{array}{c}
F(w_2z_2) \xrightarrow{\square_2} F(w_2v_2)(y)
\end{array}
\]

with $\square_1^\mu$ (resp. with $1 \leftrightarrow 2$, $u \leftrightarrow v$, $x \leftrightarrow y$ and $\mu \leftrightarrow \nu$) defined as the composition

\[
F(w_1u_1)(x) \xrightarrow{(F_{w_1u_1})\nu} F(w_1)(u_1) \xrightarrow{F\phi_C'(\nu)} F\phi_C'(z') \xrightarrow{(F\phi_C')(\kappa)} F\phi_C'(z')
\]

We can then combine the two commutative diagrams above in the following commutative
As before, we first consider a premorphism \( \kappa \) where all the arrows are invertible:

\[
\begin{array}{c}
F(w_1u_1)(x) \xrightarrow{(F\alpha)} F(w_2u_2)(x) \\
\square_1 \xrightarrow{\square_1} \square_2 \\
Fw_1(z_1) \xrightarrow{\square_1^{-1} \circ \square_1} Fw_2(z_2) \\
\square_1^{-1} \xrightarrow{\square_1^{-1}} \square_2^{-1} \\
F(w_1v_1)(y) \xrightarrow{(F\beta)_y} F(w_2v_2)(y) \\
\end{array}
\]

Taking \( \xi_1 = \nu_1^{-1} \mu_1 \) and \( \xi_2 = \nu_2^{-1} \mu_2 \), the diagram on the right above is precisely the diagram \([4.18]\).

**Construction of \( K \).** We now construct an assignement \( K: \text{colim}_B F \to (\pi_0 \circ dF)[W^{-1}] \) on objects and arrows, which is the identity on objects, and maps a premorphism \((C, u, v, \xi)\) to the roof

\[
\begin{array}{c}
(C, Fv(y)) \\
(A, x) \\
\end{array} \xrightarrow{(u, \xi)} \begin{array}{c}
(C, Fv(y)) \\
(B, y) \\
\end{array}
\]

Note that \( K \) is well-defined since any homotopy \((C, w_1, w_2, \alpha, \beta)\) as in the proposition yields the roof

\[
\begin{array}{c}
(C, Fw_2 \circ Fv_2(y)) \\
(C_1, Fv_1(y)) \\
(A, x) \\
\end{array} \xrightarrow{(u_1, \xi_1)} \begin{array}{c}
(C, Fv_2(y)) \\
B, y \\
\end{array}
\]

where \( \kappa = F^{-2}_{w_2, v_2}(y) \circ (F\beta)_y \circ F^2_{w_1, v_1}(y) \).

**Isomorphism of Categories.** Starting with a premorphism, and applying \( K \) and \( H \) consecutively, we get the same premorphism we started with. In particular, note that two premorphisms are homotopic if and only if the associated roofs are equivalent, so the homotopy relation is indeed an equivalence relation. Starting instead with a roof and applying \( H \) and \( K \) consecutively, we get a new roof which is in the same class, furthermore they have not only a common roof but what’s called in [Fr11] Rem. 3.6] an *elementary equivalence*:

\[
\begin{array}{c}
(C, z) \\
(A, x) \\
\end{array} \xrightarrow{[(1, \nu^{-1} \circ F^{-1}(\gamma)_x)]} \begin{array}{c}
(C, Fv(y)) \\
(B, y) \\
\end{array}
\]

This establishes the existence of the isomorphism of categories

\[
\text{colim} F \simeq (\pi_0 \circ dF)[W^{-1}]
\]

**Remark 4.23 (Composition).** The formula in \([4.19]\) for the composition of arrows in the pseudo-colimit is independent of the choices of \( f, g, \) and \( \gamma \). Indeed, this formula is no other than the one coming from the composition of roofs in the category of fractions, using the assignements
$K$ and $H$ constructed in the proof of Proposition 4.17. We make this explicit as follows: given $[(C_1, u_1, v_1, \xi_1)], [(C_2, u_2, v_2, \xi_2)]$ as in (4.19) we apply $K$ and we have two roofs,

$$
\begin{align*}
&\xymatrix{(A, x) \ar[r]^{(u_1, \xi_1)} & (C_1, Fv_1(y)) \ar[r]^{(v_1, Fv_1(y))} & (B, y) \ar[r]^{(u_2, \xi_2)} & (C, z)} \\
&\xymatrix{(D, FgFv_2(z)) \ar[u]^{(f, \boxempty)} \ar[r]^{(g, 1_{FgFv_2(z)})} & (C_2, Fv_2(z)) \ar[u]^{(v_2, 1_{Fv_2(z)})} \ar[r] & (C, z) \ar[u]^{(1_{C}, \xi_2)}}
\end{align*}
$$

and $f, g$, and $\gamma$ provide the 2-cell $\gamma$ in $\text{el} \, dF$ in the diagram below, so that the square commutes in $\pi_0 \, \text{el} \, dF$

$$
\begin{align*}
&\xymatrix{(D, FgFv_2(z)) \ar[r]^{(f, \boxempty)} & (C_2, Fv_2(z)) \ar[r]^{(v_2, 1_{Fv_2(z)})} & (C, z) \ar[u]^{(1_{C}, \xi_2)}}
\end{align*}
$$

(4.24)

The arrow $\boxempty$, that exists by the fibration properties of $\text{el} \, dF$, can be defined as the composition

$$
FfFv_1(y) \xrightarrow{(F^2_{f,v_1})_y} F(fv_1(y)) \xrightarrow{(F\gamma)_y} F(gu_2)(y) \xrightarrow{(F^{-2}_{g,u_2})_y} FgFv_2(z) \xrightarrow{Fg(\xi_2)} FgFv_2(z).
$$

The formula in (4.19) is the one obtained by applying $H$ to the roof in (4.24) formed by the four outer arrows.

**Remark 4.25** (Elementary Homotopy). As in Fritz’s work [Fri11, Rem. 3.6], we have a notion of elementary homotopy here too: a triple $(w, \alpha, \beta)$: $(c_1, u_1, v_1, \xi_1) \equiv (C_2, u_2, v_2, \xi_2)$ where $w: C_1 \to C_2$ is an arrow and $\alpha$: $wu_1 \simeq u_2$ and $\beta$: $vw_1 \simeq v_2$ are invertible 2-cells such that

$$
\begin{align*}
&\xymatrix{Fw \circ Fv_1(x) \ar[r]^{(F^2_{w,v_1})_x} & F(wv_1)(x) \ar[r]^{(F\alpha)_x} & F(u_2)(x) \ar[l]_{Fw(\xi_1)}} \\
&\xymatrix{Fw \circ Fv_1(y) \ar[r]^{(F^2_{w,v_1})_y} & F(wv_1)(y) \ar[r]^{(F\beta)_y} & F(v_2)(y) \ar[l]_{Fw(\xi_2)}}
\end{align*}
$$

Those elementary homotopies then generate the whole equivalence relation. In particular, for any premorphism $(C, u, v, \xi)$ and any arrow $w: C \to C'$, we have an elementary homotopy generated by $w$:

$$
(w, 1_{wu}, 1_{wv}): (C, u, v, \xi) \equiv (C', wu, wv, \text{F\hat{w}(\xi)})
$$

where $\text{F\hat{w}(\xi)} = (F^2_{w,v})_y \circ Fw(\xi) \circ (F^{-2}_{w,u})_x$.

## 5 Two Basic Properties of Bicategories of Fractions

We show in this section the generalization of two classical results from [GZ67] to bicategories:

1. In [GZ67] it is shown that each hom-set of the category of fractions can be constructed as a filtered colimit of sets, that is indexed over a slice category. We show in Section 5.1 that for a bicategory of fractions one has a similar diagram (5.3) given by a $\text{Cat}$-valued pseudo-functor, whose domain is filtered (see Lemma 5.3), and finally that the hom-categories of the bicategory of fractions can be constructed as the pseudo-colimit of this pseudo-functor (see Proposition 5.3).
2. In Section 5.2 we generalize to bicategories another basic result from GZ67 Chapter I.3: the fact that the localization by fractions is exact.

We consider in this section a family $W$ of arrows of a bicategory $B$, containing the identities and closed under composition and invertible 2-cells, whose arrows we denote by $\to$.

### 5.1 Hom-categories of the localization are filtered colimits

**Definition 5.1** (Slice $W/A$). For $A: B$, we have a bicategory $W/A$ defined as the full sub-bicategory of the pseudo-slice bicategory $B/\alpha \to A$ of $B/A$, whose objects are given by the arrows in $W$. More explicitly, it has

- **Objects:** $(C, w)$, where $C: B$ and $w: C \to A$ in $W$.
- **Arrows:** $(f, \alpha): (C_1, w_1) \to (C_2, w_2)$, where $f: C_1 \to C_2$ is an arrow of $B$ and $\alpha: w_1 \Rightarrow w_2 f$ is an invertible 2-cell.
- **2-cells:** $\xi: (f_1, \alpha_1) \Rightarrow (f_2, \alpha_2)$ where $\xi: f_1 \Rightarrow f_2$ is a 2-cell of $B$ such that we have the equality of pasting diagrams

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\xi} & C_2 \\
\downarrow f_1 & \nearrow \alpha_1 & \downarrow w_2 \\
\downarrow w_1 & \nearrow \alpha_2 & \downarrow w_2 \\
A & \to & A
\end{array}
= 
\begin{array}{ccc}
C_1 & \xrightarrow{f_2} & C_2 \\
\downarrow w_1 & \nearrow \alpha_2 & \downarrow w_2 \\
A & \to & A
\end{array}
$$

$W/A$ comes equipped with a forgetful pseudo-functor (that is in fact a strict functor) $U: W/A \to B$.

**Remark 5.2.** We can also define the lax slice bicategory $B//A$, just as above, but without asking for the 2-cell “$\alpha$” appearing in the arrows to be invertible. This is a particular case of a lax comma bicategory of a diagram $C \xleftarrow{F} B \xrightarrow{G} D$, as constructed for example in Bucl14 4.2.1 (take $F = 0$, $D = 1$, and $G = A$). It follows from Bucl14 4.2.5 that $U: B//A \to B$ is a fibration. Note that the Cartesian arrows in $B//A$ are precisely those for which the 2-cell $\alpha$ is invertible. Also note that, since we assume $W$ to be closed:

- for any Cartesian arrow $(f, \alpha): (C_1, w_1) \to (C_2, w_2)$ of $B//A$, if $f$ and $w_2$ are in $W$, then so is $w_1$ (that is, $(C_1, w_1)$ is an object of $W/A$).

**Lemma 5.3** (Cofiltered Slices). Let $A: B$. If $W$ satisfies right fractions, then $W/A$ is a cofiltered bicategory as in Definition 2.2.

**Proof.** In view of Remark 5.2 applying Lemma 3.12 to the fibration $U: B//A \to B$, we get that the family $C_W$ of Cartesian arrows over $W$ (that is the arrows $(f, \alpha)$ as in Definition 5.1 such that $f$ is in $W$ and $\alpha$ is invertible) satisfies fractions in $B//A$. We show how this implies that this same family $C_W$ also satisfies fractions when restricted to $W/A$ (note that $W/A$ is locally full in $B//A$).

Consider axiom [1-Frc] for $C_W$ in $B//A$:

$$
\begin{array}{ccc}
D & \xrightarrow{h} & A \\
\downarrow u & \nearrow \alpha & \downarrow w \\
C & \xrightarrow{f} & B
\end{array}
$$

If $A, B, C$, and $f$ are in $W/A$, then by the statement marked $(*)$ in Remark 5.2 so is $D$, and by Propositions 1.11 and 1.12 so is $h$. It also follows from $(*)$ that $C_W$ satisfies the axioms [1-Frc] and [2-Frc] in $W/A$. Finally, since $W/A$ has $(A, id_A)$ as a (bi)terminal object, and the arrows into it are in $C_W$, we conclude by Remark 5.11. \qed
A proof of the following result could also be obtained using the properties of the fibration $U: \mathcal{B}/\mathcal{A} \to \mathcal{B}$, without asking $\gamma$ to be invertible, but we consider a direct explicit proof of this case to be clearer.

**Lemma 5.4.** $U: \mathcal{W}/\mathcal{A} \to \mathcal{B}$ has the following lift of squares property: given a cospan

$$(C, w) \xrightarrow{(u, \alpha)} (C_2, w_2) \xleftarrow{(v, \beta)} (C', w')$$

an object $D: \mathcal{B}$, arrows $h: D \to C$, $g: D \to C'$ and an invertible 2-cell $\gamma: vg \simeq uh$ as on the left, the diagram on the left in $\mathcal{B}$ can be lifted to a diagram in $\mathcal{W}/\mathcal{A}$:

\[
\begin{array}{ccc}
D & \xrightarrow{g} & C' \\
\downarrow{h} & & \downarrow{v} \\
C & \xrightarrow{\gamma} & C' \\
\downarrow{u} & & \downarrow{w} \\
C_2 & \xrightarrow{\alpha h} & \mathcal{A} \\
\end{array}
\]

\[
\begin{array}{ccc}
D & \xrightarrow{g} & C' \\
\downarrow{h} & & \downarrow{v} \\
C & \xrightarrow{\gamma} & C' \\
\downarrow{u} & & \downarrow{w} \\
C_2 & \xrightarrow{\alpha h} & \mathcal{A} \\
\end{array}
\]

**Proof.** We simply note that the lift of $g$, whose second coordinate is denoted by $\square$ in the diagram above, can be (uniquely) defined to make $\gamma$ into a 2-cell of $\mathcal{W}/\mathcal{A}$. Indeed, for this to happen the equation in Definition 5.1 has to be satisfied, that is:

\[
\begin{array}{ccc}
D & \xrightarrow{v g} & C_2 \\
\downarrow{w h} & & \downarrow{w_2} \\
\mathcal{A} & \xrightarrow{\gamma \cdot \alpha h} & \mathcal{A} \\
\end{array}
\]

We observe that $\square$ is uniquely defined as the pasting of the 2-cell on the left and the inverse of $\beta \ast g$. $\square$

Let $A, B: \mathcal{B}$ and define a pseudo-functor

$$F_A^B: (\mathcal{W}/\mathcal{A})^{\text{op}} \xrightarrow{U} \mathcal{B}^{\text{op}} \xrightarrow{\mathcal{B}(-, B)} \text{Cat} \quad (5.5)$$

**Proposition 5.6.** When $\mathcal{W}/\mathcal{A}$ is cofiltered, the pseudo-colimit of $F_A^B$ as constructed in Proposition 4.17 is a category that is isomorphic to the following one (that we denote as $\mathcal{B}[\mathcal{W}^{-1}](A, B)$, since it matches the one constructed in [Pro96, §2.3] as the hom-categories of the bicategory of fractions $\mathcal{B}[\mathcal{W}^{-1}]$):

**Objects (arrows in $\mathcal{B}[\mathcal{W}^{-1}]$):** Triples $(C, w, f): A \to B$ where $C: \mathcal{B}$ is an object, $w: C \to A$ is an arrow of $\mathcal{W}$, and $f: C \to B$ is an arrow

\[
\begin{array}{ccc}
C & \xrightarrow{w} & A \\
\downarrow{f} & & \downarrow{} \\
B & \xleftarrow{\ } & \mathcal{B} \\
\end{array}
\]

**Arrows (2-cells in $\mathcal{B}[\mathcal{W}^{-1}]$):** Equivalence classes of quintuples

$$(C, u, v, \alpha, \xi): (C_1, w_1, f_1) \to (C_2, w_2, f_2)$$
where \( C: B \) is an object, \( u: C \to C_1, v: C \to C_2 \) are arrows, \( \alpha: w_1u \simeq w_2v \) is an invertible 2-cell, and \( \xi: f_1u \to f_2v \) is a 2-cell such that \( w_1u \) (and therefore also \( w_2v \)) is isomorphic to an arrow in \( \mathcal{W} \).

\[
\text{(5.7)}
\]

Two such quintuples, \((C, u, v, \alpha, \xi), (\bar{C}, \bar{u}, \bar{v}, \bar{\alpha}, \bar{\xi}): (C_1, w_1, f_1) \to (C_2, w_2, f_2)\) are equivalent if there exists a homotopy

\[
(\bar{C}, h, \bar{h}, \gamma, \delta): (C, u, v, \alpha, \xi) \to (\bar{C}, \bar{u}, \bar{v}, \bar{\alpha}, \bar{\xi})
\]

where \( \bar{C}: B \) is an object, \( h: \bar{C} \to C, \bar{h}: \bar{C} \to \bar{C} \) are arrows, \( \gamma: uh \simeq \bar{u}h \) and \( \delta: vh \simeq \bar{v}h \) are invertible 2-cells such that \( w_1uh \) is isomorphic to an arrow in \( \mathcal{W} \) and we have the equalities of pastings of 2-cells:

\[
\text{(5.8)}
\]

\[
\text{(5.9)}
\]

Identities: For \((C, w, f)\) an object, we have the identity \( 1_{(C,w,f)} = [(C,1_C,1_C,1_{w1C},1_{f1C})] \).

Composition (vertical composition in \( \mathcal{W}^{-1} \)): As defined in [Pro96, p.258], and recalled in the proof of this proposition.

Proof. Computing the pseudo-colimit of \( P_A^B \) using the formula in Proposition 4.17 gives the following category

Objects: Pairs \((C, w, f)\), which we can consider as triples \((C, w, f)\) as in Proposition 5.6.

Premorphisms: Quadruples \((C, w), (u, \alpha_u), (v, \alpha_v), \xi): (C_1, w_1, f_1) \to (C_2, w_2, f_2)\), with \( C, u, v \) and \( \xi \) as in Proposition 5.6. \( w: C \Rightarrow A \) an arrow of \( \mathcal{W} \), and \( \alpha_u: w \simeq w_1u \) and \( \alpha_v: w \simeq w_2v \) invertible 2-cells.

Note that such a premorphism yields a quintuple as in Proposition 5.6 by taking \( \alpha = \alpha_v\alpha_u^{-1} \).

\[\text{In [Pro96, §2.3], since \( \mathcal{W} \) is assumed to be closed under invertible 2-cells, the words “isomorphic to” can be omitted, but note that this result holds without that assumption.}\]
Homotopies: Quintuples

\[
((\tilde{C}, \tilde{w}), (h, \varepsilon), (\tilde{h}, \tilde{\varepsilon}), \gamma, \delta) : ((C, w), (u, \alpha_u), (v, \alpha_v), \xi) \equiv ((\tilde{C}, \tilde{w}), (\tilde{u}, \alpha_{\tilde{u}}), (\tilde{v}, \alpha_{\tilde{v}}), \tilde{\xi}),
\]

with \( \tilde{C}, h, \tilde{h}, \gamma, \delta \) as in Proposition 5.6; \( \tilde{w} : \tilde{C} \to C \) is an arrow in \( \mathcal{W}, \varepsilon : \tilde{w} \simeq w \) and \( \tilde{\varepsilon} : \tilde{w} \simeq \tilde{w} \tilde{h} \) are invertible 2-cells and the following three equations hold (the first two state that \((h, \varepsilon), (\tilde{h}, \tilde{\varepsilon})\) are morphisms in \( \mathcal{W}/A \), and the third one is \((\ref{eq:1.15})\))

Note that, taking \( \alpha = \alpha_v \alpha_{\tilde{u}}^{-1} \) and \( \tilde{\alpha} = \tilde{\alpha}_v \tilde{\alpha}_{\tilde{u}}^{-1} \), and combining the first two equations, \((\ref{eq:5.3})\) follows and \((\ref{eq:5.3})\) is the third equation. This shows that any homotopy between two premorphisms yields an equivalence between the assigned quintuples, and in this way we have an assignment on objects and arrows \( \mathcal{H} : \operatorname{colim} F_B^A \to \mathcal{B}[\mathcal{W}^{-1}](A, B) \) that we will show is in fact an isomorphism of categories.

First, we note that any quintuple \((C, u, v, \alpha, \xi)\) as in Proposition 5.6 defines a premorphism by choosing an arrow \( w : C \to A \in \mathcal{W} \), an isomorphism \( \alpha_u : w \simeq w_1 u \) and putting \( \alpha_v \) as the composition \( w \simeq w_1 u \simeq w_2 v \). And finally, we let the reader check that any quintuple \((\tilde{C}, \tilde{h}, \tilde{\varepsilon}, \gamma, \delta)\) as in Proposition 5.6 defines a homotopy between the so-defined premorphisms, by choosing an arrow \( \tilde{w} : \tilde{C} \to A \in \mathcal{W} \), an isomorphism \( \tilde{w} \simeq \tilde{w}_1 \tilde{h} \) and putting

\[
\varepsilon : \tilde{w} \simeq \tilde{w}_1 \tilde{h} \tilde{h}, \quad \tilde{\varepsilon} : \tilde{w} \simeq \tilde{w}_1 \tilde{h} \tilde{h} \simeq \tilde{w}_2 \tilde{h} \tilde{h}.
\]

These constructions are clearly mutually inverse, so we have an assignment on objects and arrows \( K : \mathcal{B}[\mathcal{W}^{-1}](A, B) \to \operatorname{colim} F_B^A \), strictly inverse to \( \mathcal{H} \). \( K \) maps identities to identities by definition, so we will finish the proof by showing that \( K \) preserves the composition of the categories.

We consider thus two composable arrows of \( \mathcal{B}[\mathcal{W}^{-1}](A, B) \),

\[
(C_1, w_1, f_1) \xrightarrow{((C, u, v, \alpha, \xi) \mapsto (C_2, w_2, f_2) [(C', u', v', \alpha', \xi') \mapsto (C_3, w_3, f_3),}
\]

\]}
and an invertible 2-cell $\gamma$ fitting as follows:

In the situation in [Pro96], when defining the vertical composition in the bicategory of fractions, $\gamma$ is a chosen square, and the new quintuple whose class gives the composition is defined by pasting respectively the $\alpha$’s with $\gamma$, and the $\xi$’s with $\gamma$ (see [Pro96, p.258] for details).

We can apply $K$ to this new arrow of $B[\mathcal{W}^{-1}](A, B)$, and we claim that we get the same premorphism if we lift $\gamma$ to a 2-cell of $\mathcal{W}/A$ and use this 2-cell to compute the composition between the two induced premorphisms of $\text{colim} \rightarrow F_{\mathcal{B}} A B$, this is just a matter of following these constructions: we first apply $K$ to each of the two arrows above:

\[
\begin{align*}
((C_1, w_1), f_1) & \rightarrow ((C_2, w_2), f_2) & \rightarrow ((C_3, w_3), f_3).
\end{align*}
\]

We see then that we can compose these premorphisms of $\text{colim} \rightarrow F_{\mathcal{B}} A B$ as in the item Composition in Proposition 4.17 by choosing an invertible 2-cell in $(\mathcal{W}/A)^{\text{op}}$ of the following form

and in fact the composition in the category $\text{colim} F_{\mathcal{B}} A B$ is independent of this choice, as shown in Remark 4.23. If we choose this 2-cell by lifting $\gamma$ to $\mathcal{W}/A$ as below, using Lemma 5.4:

then we can compute the composition of the premorphisms by replacing the values in the formula (4.19). We let the reader check that applying the assignment $H : \text{colim} F_{\mathcal{B}} A B \rightarrow B[\mathcal{W}^{-1}](A, B)$ described above to this premorphism gives back precisely the formulas in [Pro96, p.258].

**Remark 5.10.** We note that this proof shows, in particular, that the vertical composition of 2-cells in $B[\mathcal{W}^{-1}](A, B)$ does not depend on the choice of the 2-cell $\gamma$ in $B$. This result is proved directly in [Tom16, Prop. 5.1]. Note that Propositions 5.6 (the homs of the bicategory of fractions are filtered pseudo-colimits of categories) and 4.17 (filtered pseudo-colimits of categories are 1-categories of fractions), combined, state that the homs of the bicategory of fractions are 1-categories.

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of fractions. In view of this, and as explained in Remark 1.23 we show here that the basic reason why this vertical composition is independent of this choice is that the composition of arrows in the 1-category of fractions is independent of the choice of the commutative square.

5.2 On the exactness of the calculus of fractions

We show now how Proposition 5.6 is one of the two ingredients needed in order to generalize the exactness of the localization by fractions from categories ([GZ 67, I, Prop 3.1]) to bicategories. The other required result is the commutativity in Cat of filtered pseudo-colimits with finite limits. This has been shown in [Can16], [DDS18a] for the strict case, and we generalize it here to the bicategorical case.

Let B, C be bicategories, F: B × C → Cat, W: C → Cat be pseudo-functors. We denote W-weighted bilimits by \( \lim_W \) and conical bicolimits, with indexing bicategory B, by \( \colim_B \). There is a canonical functor

\[
\diamond: \colim_B \lim_W F \to \lim_W \colim_B F
\]  

and the (W-weighted) bilimit is said to commute with the (conical) bicolimit when this functor is an equivalence of categories (of course, this makes sense for any bicategory D in place of Cat, considering equivalences in D instead). In [Can16], [DDS18a], this is shown to be the case when B is filtered, W is a finite weight (which implies in particular that C is finite, see [DDS18a, Def. 3.1]), and assuming furthermore when B, C, W, and F are required to be strict:

**Theorem 5.12.** Let B, C be (strict) 2-categories such that B is filtered, let W: C → Cat be a (strict) 2-functor that is a finite weight, and let F: B × C → Cat be a (strict) 2-functor. Then the canonical functor \( \diamond \) in (5.11) is an equivalence of categories.  

**Remark 5.13.** Let C, D be bicategories and F: C → D, W: C → Cat be pseudo-functors. Recall that the W-weighted bilimit \( \lim_W F \) is defined as a birepresentation of the pseudo-functor (pseudo-presheaf) given by the composition

\[
\text{Cones}_W^D(−): D^{op} \xrightarrow{k_{D^{op}}} [D, \text{Cat}] \xrightarrow{F^*} [C, \text{Cat}] \xrightarrow{[C, \text{Cat}](W,−)} \text{Cat}
\]

where \( k \) denotes the Yoneda embedding and \( F^* \) denotes the pre-composition by F. Note that this pseudofunctor maps D: D to the category of pseudo-natural transformations

\[
\text{Cones}_W^D(D) = [C, \text{Cat}](W, D(D, F(−)))
\]

from W to \( D(D, F(−)): C → \text{Cat} \), which is the formula that can be found in the original definition of weighted bilimit in [Str80] (1.12).

We recall that, by definition, being birepresentable is a property that is stable under equivalences in [D^{op}, \text{Cat}] (see [Str80] (1.11)]. Hence, if another indexing bicategory C', another weight \( W': C' → \text{Cat} \), and another pseudo-functor \( F': C' → D \) induce an equivalent pseudo-presheaf, then the bilimit \( \lim_W F \) will coincide with \( \lim_{W'} F' \). We will use this to show in (A) and (B) how equivalent choices of C, W and F lead to equivalent pseudo-presheaves and hence to the same bilimit. Recall also (see Remark 1.23 and for example [PW14, 1.10] for a proof) that equivalences are pointwise in functor bicategories.

**A** For each pseudo-functor \( S: C' → C \), we can consider \( S^*: [C, \text{Cat}] → [C', \text{Cat}] \), and we have an induced pseudo-natural transformation given by the pasting

\[
\begin{array}{c}
D^{op} \xrightarrow{k_{D^{op}}} [D, \text{Cat}] \xrightarrow{F^*} [C, \text{Cat}] \xrightarrow{[C, \text{Cat}](W,−)} \text{Cat} \\
\xrightarrow{(FS)^*} [C', \text{Cat}] \xrightarrow{[C', \text{Cat}](W,−)} \text{Cat}
\end{array}
\]  

(5.14)
where the triangle on the left-hand side is strictly commutative and the 2-cell on the right-hand side is given by the local hom-functors of the pseudofunctor $S^*$,

$$S^*_{W,-} : [C, \mathbf{Cat}](W, -) \to [C', \mathbf{Cat}](S^*(W), S^*(-)).$$

If $S$ is a biequivalence, then so is $S^*$ and, since biequivalences are locally equivalences and natural pointwise equivalences are equivalences, so is the pseudo-natural transformation in (5.14). As we explained above, we then have $\varprojlim_w F = \varprojlim_{W'S} FS$.

(B) For each pair of pseudo-natural transformations $\alpha : W' \Rightarrow W$ and $\beta : F \Rightarrow F'$, we have an induced pseudo-natural transformation given by the pasting

$$
\begin{array}{ccc}
D^{op} & \xrightarrow{k \circ \eta} & [D, \mathbf{Cat}] \\
\downarrow & \swarrow \downarrow \circ \eta & \\
[C, \mathbf{Cat}] & \xrightarrow{\hat{\alpha}} & \mathbf{Cat}
\end{array}

(5.15)

$$

where $\hat{\alpha}$ denotes the pseudo-natural transformation $\hat{k}[C, \mathbf{Cat}] \circ \eta(\alpha) = [C, \mathbf{Cat}](\alpha, -)$. Similarly to (A), if $\alpha$ and $\beta$ are equivalences, then so is the pseudo-natural transformation in (5.15) and we have $\varprojlim_w F = \varprojlim_{W'} F$.

Similarly, one can do the same operations (A) and (B) for (weighted) colimits.

Using this remark, we can now generalize the commutativity result in Theorem 5.12 to the bicategorical setting:

**Corollary 5.16.** The result in Theorem 5.12 also holds for $\mathcal{B}, \mathcal{C}$ bicategories, and $W, F$ pseudo-functors.

**Proof.** We first note that, since finite weighted bilimits can be constructed using bicotensors with a finite category, biproducts and biequalizers (this result goes back to [Str80], see also [DDS18a, p.208] and [Can16, Cor. 6.12] for a detailed explanation and proof), it suffices to prove Corollary 5.16 in these three cases (which is in fact precisely how Theorem 5.12 is proved in [DDS18b] and [Can16]). Observe that in these three cases $\mathcal{C}$ and $W$ are already strict, so we can assume this for this proof, and we do so in what follows.

We consider the 2-category $\text{st}(\mathcal{B})$, the biequivalence $\text{st}(\mathcal{B}) \simeq \mathcal{B}$, and the induced biequivalence $\text{st}(\mathcal{B}) \times \mathcal{C} \simeq \mathcal{B} \times \mathcal{C}$. Note that, since they are equivalent bicategories, if $\text{st}(\mathcal{B})$ is filtered, then so is $\mathcal{B}$. Composing $F$ with this pseudo-functor, and applying item (A) in Remark 5.13 we see that we can assume without loss of generality $\mathcal{B}$ to be a 2-category. We can then use the fact (see [Pow89, 4.2], or the nLab entry on pseudo-functors) that any $\mathbf{Cat}$-valued pseudo-functor with domain a 2-category is equivalent to a 2-functor. Using item (B) in Remark 5.13 (taking $\beta$ to be this equivalence, and $\alpha$ the identity), we can then also assume without loss of generality that $F$ is a 2-functor, as in Theorem 5.12.

Combining Corollary 5.16 with Proposition 5.6, we have:

**Theorem 5.17.** Let $\mathcal{B}$ be a bicategory and $\mathcal{W}$ be a (right) calculus of fractions. Then the localization pseudo-functor $L_{\mathcal{W}} : \mathcal{B} \to \mathcal{B}[W^{-1}]$ commutes with finite weighted bilimits. In other words, for $\mathcal{C}$ a finite bicategory, $W : \mathcal{C} \to \mathbf{Cat}$ a finite weight, and $F : \mathcal{C} \to \mathcal{B}$ a pseudo-functor, such that the finite weighted limit of $F$ by $W$ exists, the finite weighted limit of $L_{\mathcal{W}} \circ F$ by $W$ exists and the canonical arrow

$$L_{\mathcal{W}}(\varprojlim_w F) \to \varprojlim_w (L_{\mathcal{W}} \circ F)$$

is an equivalence in $\mathcal{B}[W^{-1}]$. 


Proof. The proof is formally similar to the one in [GZ67, I, Prop 3.1]. We want to show that for each \( A: B \) the canonical functor

\[
B[W^{-1}](A, \lim_{W} F) \longrightarrow \lim_{W} B[W^{-1}](A, F(-))
\]

is an equivalence of categories. By Proposition 5.8, this amounts to showing that so is

\[
\colim_{(W/A)^{op}} B(U_A(-), \lim_{W} F) \longrightarrow \lim_{W} \colim_{(W/A)^{op}} B(U_A(-), F(-))
\]

or, equivalently, since representables preserve limits [Str80 (1.21)],

\[
\colim_{(W/A)^{op}} \lim_{W} B(U_A(-), F(-)) \longrightarrow \lim_{W} \colim_{(W/A)^{op}} B(U_A(-), F(-))
\]

where \( U_A \) denotes the forgetful strict functor from \( W/A \). This is precisely the content of Corollary 5.16 (for the pseudo-functor \( W/A^{op} \times \mathcal{C} \xrightarrow{U_A \times F} \mathcal{B}^{op} \times \mathcal{B} \xrightarrow{\mathcal{B}(-,-)} \text{Cat} \)).

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