A Classification Theorem for Varieties Generated by Wreath Products of Groups

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To Professor Alexander Yu. Ol’shanskii, my teacher, on his 70’th anniversary

Abstract. We suggest a criterion classifying all the cases when for a nilpotent group \( A \) of a restricted exponent and for any abelian group \( B \) the variety \( \text{var}(A \text{Wr} B) \) generated by the wreath product \( A \text{Wr} B \) is equal to the product variety \( \text{var}(A) \text{var}(B) \). This continues our previous research on varieties generated by wreath products of other classes of groups (abelian groups, finite groups, etc.). The obtained theorem generalizes some known results in the literature considering the same problem for more restricted cases. Some applications of the criterion also are discussed.

1. Introduction

The main aim of this work is to suggest a criterion under which for a nilpotent group of restricted exponent \( A \) and for an abelian group \( B \) the wreath product \( A \text{Wr} B \) generates the product of varieties \( \text{var}(A) \) and \( \text{var}(B) \) respectively, or in other notation, a condition under which the equality

\[
\text{var}(A \text{Wr} B) = \text{var}(A) \text{var}(B)
\]

holds for the given groups \( A \) and \( B \). By default all the wreath products here are assumed to be Cartesian (complete) wreath products, although the analogs of statements are true for direct (restricted) wreath products also. Denote by \( C_n \) the cyclic group of order \( n \), and for any group \( G \) denote by \( G^k \) or by \( G^\infty \) the direct product of \( k \) or of countably many copies of \( G \) respectively. In these notations our main result is:

Theorem 1.1. Let \( A \) be any nilpotent group of finite exponent \( m \), and let \( B \) be any abelian group. Then the equality \((\ast)\) holds for \( A \) and \( B \) if and only if:

a) either the group \( B \) is not of finite non-zero exponent;

b) or \( B \) is of some finite exponent \( n > 0 \), and it contains a subgroup isomorphic to the direct product \( C_d^c \times C_{n/d}^\infty \), where \( c \) is the nilpotency class of \( A \), and \( d \) is the largest divisor of \( n \) coprime with \( m \).

If the abelian group \( B \) is of finite exponent, then by Prüfer’s theorem \( [25] \) it is a direct product of some finite cyclic subgroups. So the point (b) above just states that in that direct product the cycles of order \( d \) and of order \( n/d \) are present “sufficiently many” times: \( B \) should contain at least \( c \) copies of \( C_d \) and infinitely many copies of \( C_{n/d} \). In other words, if for a prime divisor \( q \) of \( n \) we denote by \( q^n \) the highest degree of \( q \) dividing \( n \), then the group \( B \) should contain \( c \) copies of \( C_{q^c} \), if \( q \) is coprime with \( m \), and infinitely many copies of \( C_{q^m} \), if \( q \) divides \( m \) (see also Remark 5.1).

Theorem 1.1 continues our research on classification of cases when \((\ast)\) holds for groups \( A \) and \( B \) of certain classes of groups. In particular, in \([15, 16]\) we gave a
full classification for \((\ast)\) holding for any abelian groups \(A\) and \(B\), and in \([17, 18]\) we classified the cases when \(A\) and \(B\) are any finite groups.

One of the oldest occurrence of the equality \((\ast)\) is due to G. Higman who considered \((\ast)\) holding for \(A = C_p\) and \(B = C_n\) \([9]\). C. Haughton covered the case of any cyclic \(A, B\) (see \([21]\) and \([4]\)). In \S5 of \([22]\) B.H. Neumann, H. Neumann, P.M. Neumann ask: “If the groups \(A, B\) belong to the varieties \(\mathfrak{U}, \mathfrak{V}\), respectively, then \(A \wr B\) (and hence also \(A \varpi B\)) belongs to the product variety \(\mathfrak{U} \mathfrak{V}\). If \(A\) generates \(\mathfrak{U}\) and \(B\) generates \(\mathfrak{V}\), then one might hope that \(A \wr B\) generates \(\mathfrak{U} \mathfrak{V}\); but this is in general not the case”. Then they bring examples where \((\ast)\) does or does not hold for some specific groups such as \(p\)-groups, free groups, discriminating groups, infinite direct powers, etc. For these and some other earliest results see Hanna Neumann’s monograph \([21]\) and \([22, 2, 4, 3, 9]\). Our criterion generalizes these and also some other known results in literature in full or in part.

And in general, motivation of such study is explained by importance of wreath product as a key tool to study the product varieties of groups. For, the product \(\mathfrak{U} \mathfrak{V}\) consists of extensions of all groups \(A \in \mathfrak{U}\) by all groups \(B \in \mathfrak{V}\), and if the equality \((\ast)\) holds for some fixed groups \(A\) and \(B\) generating the varieties \(\mathfrak{U}\) and \(\mathfrak{V}\) respectively, then we can restrict ourselves to consideration of \(\varpi(A \wr B)\), which is easier to study rather than to explore all the extensions inside \(\mathfrak{U} \mathfrak{V}\). There are very many examples, when this approach is used (we listed some of them in \([16]\)).

Technique of the arguments below is based both on traditional theory of varieties of groups and on some newer approaches. In Section \(3\) we suggest the idea of application of \(K_p\)-series and of D. Shield’s formula \([26, 27]\) as tools to describe some non-nilpotent varieties via their finitely-generated nilpotent groups (see more details in \([19]\)). Also, our usage of Fitting and Frattini subgroups is based not only on their well known properties reflected, for example, in \([21, 25]\) but also on initial articles of W. Gaschütz and H. Fitting \([7, 8, 6]\): in their original work some results are formulated in sharper forms for certain specific cases, which are relevant in our study (compare constructions with critical groups and the application of Theorem of Clifford in Section \(2\) below with constructions in \([21\) Section 2 in Chapter 5\] or with \([4]\)).

Below we without any definitions use the basic notions on varieties of groups, relatively free groups, verbal subgroups, discriminating groups, wreath products, etc. The background information can be found in \([21, 25]\).

Acknowledgments. In initial steps of this work the case of wreath products of \(p\)-groups, which currently is covered by Section \(3\) was handled using the \(K_p\)-series. Later we discussed the topic with Prof. A.Yu. Ol’shanskii, who suggested a shorter proof for Lemma \(3.1\) by the arguments of \([24]\). In Section \(3\) we present both proofs, and the most part of considerations with \(K_p\)-series is not included into the current text, we just provide references to \([19]\) where closer discussion on \(K_p\)-series is presented.

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2. A characterization of the critical groups in \(\varpi(A) \varpi(B)\)

This section prepares the tools required to prove the sufficiency part of Theorem \(1.1\). We without any definitions use the notions of Frattini subgroup \(\Phi(G)\) and
the Fitting subgroup $F(G)$ of the given group $G$. Their definitions and basic properties can be found in W. Gaschütz’s important article [7]. For constructions of semisimple groups see H. Fitting’s work [6]. Concise summary of these concepts also is presented in [25]. For definitions and basic properties of critical groups we refer to [21, 4].

Assume $A$ is a nilpotent group of class $c$ and of exponent $m \neq 1$, and $B$ is an abelian group of exponent $n \neq 1$. Take any critical group $K$ of the variety $\text{var}(A) \text{var}(B)$. It is an extension of a group $N \in \text{var}(A)$ by some group $S \in \text{var}(B)$, where $S$ as a finite abelian group has a direct decomposition $S = U \times U^*$, with $U$ being a subgroup generated by all the elements of exponents coprime to $m$. Our main technical goal is to establish some bounds of the number of direct cyclic summands of $U$ in terms of the nilpotency class $c$. Below we without loss of generality may assume that both $S$ and $N$ are non-trivial, for, (a) if $S = \{1\}$, then $K \in \text{var}(A)$ and $U = \{1\}$; (b) if $N = \{1\}$, then $K \in \text{var}(B)$ and according to [21, 51.36] $K$ is a cyclic group of prime power order (as any critical abelian group), and so $U$ either is a non-trivial cyclic group or is trivial, so the bounds are evident.

We will twice use the result of W. Gaschütz proved in [2] as Satz 7: in any finite group $G$ any abelian normal subgroup $H$, intersecting with $\Phi(G)$ trivially, causes a factorization $G = H \tilde{G}$, where $\tilde{G} \neq G$, $H \cap \tilde{G} = \{1\}$ and $H$ is a direct product of factors which are some minimal normal subgroups in $G$.

Denote $\Phi = \Phi(K)$ and $F = F(K)$ and notice that the Fitting subgroup $F$ is a $p$-group for some prime $p$ because (as a finite nilpotent group) $F$ is a direct product of its Sylow subgroups for distinct primes. These Sylow subgroups intersect trivially, and each of them is characteristic in $F$, and thus is normal in $K$. If $F$ contained more then one such Sylow subgroups, $K$ would be isomorphic to the subdirect product of its factor groups by those Sylow subgroups. Since $K$ is a critical group, that option is ruled out.

Since $N$ is a normal nilpotent subgroup in $K$, it is contained in $F$. For the same reason $F$ also contains the normal (in fact also characteristic) subgroup $\Phi$. By [7, Satz 2] the factor group

$$\Phi(K/\Phi) = \Phi(K)/\Phi = \Phi/\Phi \cong \{1\}$$

is trivial (in terminology of W. Gaschütz $K/\Phi$ is a $\Phi$-free group). Our first application of Satz 7 is for the $\Phi$-free group $G = K/\Phi$ and $H = F/\Phi$. To apply it we yet have to show that $F/\Phi$ is abelian. By [7, Satz 10] the Fitting subgroup $F(K/\Phi)$ of $K/\Phi$ is equal to $F(K)/\Phi = F/\Phi$. Since $F/\Phi$ is normal in $K/\Phi$, by [7, Satz 10] the Frattini subgroup of $F/\Phi$ is in the Frattini subgroup of $K/\Phi$. So $F/\Phi$ also is a $\Phi$-free group. But a nilpotent finite group is $\Phi$-free only if it is a direct product of cycles of prime order (see [5] or [7, Abschnitt 3]). Thus by Satz 7 the factor-group $F/\Phi$ is a direct product of copies of a cycle $C_p$ for some prime $p$, and the factors of this direct product can be grouped so that each group is a minimal normal subgroup in the whole $K/\Phi$. 

To approach to the second application of Satz 7 denote by $I$ the intersection $I = N \cap \Phi$, and notice that, since the subgroup $I$ is normal in $K$, we can apply Satz 2 from [7] to get the Frattini subgroup of the factor-group $K/I$:

$$\Phi(K/I) = \Phi(K)/I = \Phi/I.$$

Clearly, the factor $N/I$ has a trivial intersection with $\Phi/I$. Also, $N/I$ is abelian, since

$$N/I = N/(N \cap \Phi) \cong N\Phi/\Phi \leq F/\Phi,$$
where \( F/\Phi \) is an elementary abelian group. \( N/I \) is normal in \( K/I \), since \( N \) is normal in \( K \). These preparations allows us to apply Satz 7 for \( G = K/I \) and \( H = N/I \). We get that \( N/I \) is a direct product

\[
N/I = V_1/I \times \cdots \times V_s/I,
\]

where each \( V_i/I \) is a minimal normal subgroup in \( K/I \), and \( V_i/I \) also is a direct product of some number of copies of the finite cycle \( C_p \) (since \( N/I \) is isomorphic to a subgroup of the elementary abelian group \( F/\Phi \)). Denote by \( \tilde{N}/I \) the compliment of \( N/I \) in \( K/I \) (here \( \tilde{N} \) is taken to be the full pre-image of that compliment under natural homomorphism with kernel \( I \)). We have:

\[
\tilde{N}/I \neq K/I, \quad N/I \cdot \tilde{N}/I = K/I \quad \text{and} \quad N/I \cap \tilde{N}/I \cong \{1\}.
\]

For now let us leave aside the case when \( N/I \) is trivial, that is, when \( N \leq \Phi \) (we will consider this case a little later).

The constructions above allow us to apply a theorem of S. Oates and M.B. Powell, from \([23]\) mentioned by Hanna Neumann in \([21]\) as Theorem 51.37 and stressed as a “much deeper result”. Namely, if we additionally assume \( s > c \), then we will have:

i) \( K = \langle \tilde{N}, V_1, \ldots, V_s \rangle \) because \( K \) is generated by \( \tilde{N} \cup I \cup N \), and where \( I \) can be omitted as it lies in the Frattini subgroup (the set of non-generators of \( K \));

ii) no proper subset of the set \( \{V_1, \ldots, V_s\} \) suffices, together with \( N \), to generate \( K \) because each \( V_i/I \) admits the actions by elements of \( \tilde{N} \);

iii) every mutual commutator group \( [V_{\pi(1)}, \ldots, V_{\pi(s)}] \), where \( \pi \) is some permutation of the integers \( 1, \ldots, s \), is trivial because all factors \( V_i \) are inside the nilpotent subgroup \( N \) of class at most \( c \).

Then by Theorem 51.37 the group \( K \) is not critical. This contradiction implies that \( s \leq c \) (still with assumption \( N \leq \Phi \) made above).

Notice that we used Theorem 51.37 just for briefness of the argument, for, in current circumstances it would not be very complicated to directly show \( K \) is not a critical group by application of methods with special commutators to the group \( K \) (see Section 3 in Chapter 3 in \([21]\), in particular, Lemma 33.35, Lemma 33.37 and Lemma 33.43).

The group \( N/I \) and each factor \( V_i/I \) of \( \langle 2.1 \rangle \) can be considered to be finite-dimensional vector spaces over the field \( \mathbb{F}_p \). Actions of elements of \( \tilde{N}/I \) on \( N/I \) by conjugations form a linear presentation \( \psi \) of this group over the Galois field \( \mathbb{F}_p \). In other words \( N/I \) can be considered to be an \( \mathbb{F}_p\tilde{N}/I \)-module.

Since the direct factor \( U \) of the group \( S \) is of order coprime to \( m \) and to \( p \), then by Schur’s Lemma \( U \) can be identified by its copy in \( K \). And since its order is coprime to \( |I| \), we can identify that copy with its image in the factor group \( \tilde{N}/I \). For simplicity let us use the same notation \( U \) for this copy \( UI/I \) also.

Let us show that no non-trivial element \( u \in U \) may centralize the space \( N/I \), that is, the restriction of \( \psi \) to \( U \) is a faithful presentation. Since \( K/N \cong S \) is abelian, the factor-group \( (K/I)/(N/I) \) also is abelian. By \( \langle 2.2 \rangle \) that factor-group is isomorphic to \( \tilde{N}/I \). Thus if \( u \) centralizes \( N/I \), then it is easy to check that \( \langle u \rangle \) is a normal subgroup in the whole group \( K/I = N/I \cdot \tilde{N}/I \). But a normal and abelian subgroup \( \langle u \rangle \) should lie in the Fitting subgroup \( F(K/I) = F/I \), which is impossible since \( \langle u \rangle \) contains elements of orders coprime to \( p \).
Fix an index \( i = 1, \ldots, s \), and consider the linear presentation \( \psi_i \) of \( U \) in the subspace \( V_i/I \) (the restriction of \( \psi \) in \( V_i/I \)), that is, the \( \mathbb{F}_p U \)-module \( V_i/I \). Although \( V_i/I \) is irreducible under actions (conjugations) of \( \tilde{N}/I \) (or even of the whole group \( K/I \)), it may no longer be irreducible under actions of a smaller subgroup \( U \). Let us show, that \( V_i/I \) is a direct sum of isomorphic copies of a certain irreducible \( \mathbb{F}_p U \)-submodule.

Since \( \tilde{N}/I \) is abelian, \( U \) is normal in it, and by Theorem of Clifford [25] Theorem 8.1.3 \( V_i/I \) can be presented as a direct sum of some irreducible \( \mathbb{F}_p U \)-submodules. Moreover, if \( \{ W_{i,1}, \ldots, W_{i,r_i} \} \) is a maximal subsystem of these submodules inducing all the pairwise non-isomorphic irreducible \( \mathbb{F}_p U \)-submodules of \( V_i/I \), then one may group all the \( \mathbb{F}_p U \)-submodules in \( V_i/I \) into blocks (called homogeneous components) \( \overline{W}_{i,1}, \ldots, \overline{W}_{i,r_i} \) such that each \( \overline{W}_{i,j} \), \( j = 1, \ldots, r_i \), is a sum of all irreducible \( \mathbb{F}_p U \)-submodules of \( V_i/I \) isomorphic to \( W_{i,j} \). By Theorem of Clifford the components \( \overline{W}_{i,j} \) are conjugated modules, moreover, \( \tilde{N}/I \) acts on them as a transitive permutation group. But since \( U \) and \( \tilde{N}/I \) are abelian, conjugation is identical operation here, and we have just one component: \( r = 1 \) and

\[
V_i/I = \overline{W}_{i,1} = W_{i,1} \oplus \cdots \oplus W_{i,1}.
\]

Denote by \( \rho_i \) the restriction of the presentation \( \psi_i \) of \( U \) in the subspace \( W_{i,1} \). By our construction \( \rho_i \) is irreducible. Denoting its kernel by \( U_i \) we get that \( U/U_i \) has an irreducible and faithful presentation in \( W_{i,1} \). But an abelian group may have such a presentation only if it is cyclic. By Theorem of Remak:

\[
U/V \cong U/U_1 \times \cdots \times U/U_s,
\]

where \( V = \bigcap \{ U_i \mid i = 1, \ldots, s \} \). If an element \( v \) is in \( V \), it centralizes each of \( W_{i,1} \). By decomposition (2.3) it also centralizes each of the direct summands \( W_{i,1} \) in \( \overline{W}_{i,1} \). Thus \( v \) centralizes the sum of all \( V_i/I = \overline{W}_{i,1} \), \( i = 1, \ldots, r \), which is the whole space \( N/I \). But, as we mentioned above, none of the non-trivial elements of \( U \) may centralize \( N/I \), and so \( V \cong \{ 1 \} \). It follows from this and from (2.4) that \( U \) is a direct product of \( s \) cyclic groups \( U/U_1, \ldots, U/U_s \). We earlier proved that \( s \leq c \), that is, \( U \) is a direct product of at most \( c \) cyclic groups, provided that the assumption \( N \not\subseteq \Phi \) holds.

So it remains to cover the case when \( N \leq \Phi \). Since \( K/N \) is abelian, \( K/\Phi \) also is abelian. Then \( K \) is nilpotent by [7] Satz 10]. As such, \( K \) is a direct product of its Sylow subgroups by distinct primes. Since \( K \) is a critical group, it consist of just one Sylow \( p \)-subgroup for the \( p \) obtained from \( N \) earlier. Thus \( U \cong \{ 1 \} \), and the number of non-trivial direct cyclic factors is zero.

Assembling the bounds obtained in this section we get:

**Lemma 2.1.** Let \( A \) be a nilpotent group of class \( c \) and of exponent \( m \neq 1 \), and \( B \) be an abelian group of exponent \( n \neq 1 \). Then any critical group \( K \) of the variety \( \text{var}(A) \text{var}(B) \) is an extension of a group from the variety \( \text{var}(A) \) by some direct product \( U \times U^* \) from the variety \( \text{var}(B) \), such that

1. the orders of non-trivial elements from \( U \) are coprime to \( m \);
2. \( U \) can be presented as a direct product of at most \( c \) cyclic groups;
3. the orders of non-trivial elements from \( U^* \) can only be divided by primes, which are common divisors of \( m \) and \( n \).
3. The $K_p$-series and the case with wreath products of $p$-groups

In this section we consider the equality (3) for the case when $A$ is a nilpotent $p$-group of exponent $p^u$, and $B$ is an abelian group of exponent $p^v$, with $u, v > 0$. This case is covered by Lemma 3.1 below, and as we mentioned in the Introduction, we present two independent proofs for it.

The notations and technique with $K_p$-series in this section are just for the first proof only, and they are not needed for understanding the rest of this paper. So we recommend not to focus on them and skip to Lemma 3.1 unless the reader is interested to see our application of Shield’s formula to varieties of groups (see details in [19]).

G. Baumslag showed that a wreath product of non-trivial groups $A$ and $B$ is nilpotent if and only if $A$ is a nilpotent $p$-group of finite exponent $p$, and $B$ is a finite $p$-groups [1]. The exact nilpotency class was computed by D. Shield in [26, 27] (for background information and a survey see Chapter 4 of J.D.P. Meldrum’s monograph [12]). The $K_p$-series $K_{i,p}(B)$ of $B$ is defined for $i = 1, 2, \ldots$ as $K_{i,p}(B) = \prod \{\gamma_r(B)p^j |$ for all $r, j$ with $rp^j \geq i\}$, where $\gamma_r(B)$ is the $r$th term of the lower central series of $B$. Let $d$ be the maximal integer, such that $K_{d,p}(B) \neq \{1\}$. Then for each $s = 1, \ldots, d$ define $c(s)$ by $p^{d(s)} = [K_{s,p}/K_{s+1,p}]$, and also set: $\alpha = 1 + (p - 1)\sum_{s=1}^{d} s \cdot c(s)$ and $b = (p - 1)d$. By D. Shield’s theorem the nilpotency class of $A \text{Wr} B$ is the maximum $\max_{h=1,\ldots,c} \{a h + (s(h) - 1)b\}$, where $s(h)$ is chosen so that $p^{d(h)}$ is the exponent of the $h$th term $\gamma_h(A)$ of the lower central series of $A$.

Denote by $\beta$ the cardinality of $B$ and by $A^\beta$ the Cartesian product of $\beta$ copies of $A$. For the given fixed positive integer $l$ and for the integer $t \geq l$ denote $Z(l, t) = C_{p^l}^t \times C_{p^l}^{t-1}$. As we prove in [19] there is a positive integer $t_0$, such that for all $t > t_0$ the nilpotency class of the wreath product $A \text{Wr} Z(l, t)$ is equal to:

$$ (3.1) \quad c + ct(p - 1) \left(\frac{1-p^{t-1}}{1-p} + l/t \cdot p^{t-1}\right) + (\alpha - 1)(p - 1)p^{t-1}, $$

where the exponent of $\gamma_c(A)$ is $p^\alpha$ ($\alpha \neq 0$, since the class of $A$ is $c$).

In $A$ we can construct a finite subgroup $\hat{A}$ such that the exponents of terms $\gamma_h(\hat{A})$ and $\gamma_h(A)$ are equal for each $h = 1, \ldots, c$ [19]. If $\hat{A}$ is a $z$-generator group, denote $Y(z, t) = C_p^{t-z}$. We proved in [19] that there is a positive integer $t_1$ such that for all $t > t_1$ the nilpotency class of the wreath product $\hat{A} \text{Wr} Y(z, t)$ is equal to:

$$ (3.2) \quad c + ct(z - p - 1) \left(\frac{1-p^{t-1}}{1-p} + (\alpha - 1)(p - 1)p^{t-1}\right). $$

**Lemma 3.1.** Let $A$ be a nilpotent $p$-group of exponent $p^u$, and $B$ be an abelian group of exponent $p^v$, with $u, v > 0$. Then the wreath product $A \text{Wr} B$ generates the variety $\text{var}(A) \text{var}(B) = \text{var}(A, B)$ if and only if the group $B$ contains a subgroup isomorphic to the infinite direct power $C_{p^\infty}$ of the cycle $C_{p^\infty}$.

**Proof based on $K_p$-series.** Sufficiency of the condition is easy to deduce from the discriminating property of $C_{p^\infty}$ (see [3] or Corollary 17.44 in [21]).

To prove the necessity part, assume the group $B$ does not contain a subgroup isomorphic to $C_{p^\infty}$. By Prüfer’s theorem [25] $B$ is a direct product of copies of some cyclic subgroups the orders of which are bounded by $p^u$. There are only finitely many, say $l$, such factors of order $p^u$, and collecting them together we get $B = B_1 \times B_2$, where $B_1 = C_{p^l}$, and where $\exp B_2 \leq p^{u-1}$.

Any $t$-generator group $G \in \text{var}(A \text{Wr} B)$ by [21, 16.31] is in the variety generated by all the $t$-generator subgroups of $A \text{Wr} B$. Assume $T$ is one of such subgroups, and denote by $H$ its intersection with the base subgroup $A^B$ of $A \text{Wr} B$. Then $T/H \cong \ldots$
\((T^{AB})/A^B \leq (A \operatorname{Wr} B)/A^B \cong B\), and so \(T\) is an extension of \(H\) by an at most \(t\)-generator subgroup \(B'\) of \(B = B_1 \times B_2\). By [10] \(G\) is embeddable into \(H \operatorname{Wr} B'\). \(B'\) is a direct product of at most \(t\) cycles, of which at most \(l\) are of order \(p^v\) and the rest are of lower orders. So \(B' \leq Z(l, t)\) for a suitable \(t\). Since \(H \leq A^3\), we get \(H \operatorname{Wr} B \in \var(A^2 \operatorname{Wr} Z(l, t))\). Thus the nilpotency class of \(H \operatorname{Wr} B\) (and of \(T\)) are bounded by formula (3.1) for all \(t > t_0\). Let us find a \(t\)-generator group in \(\var(A) \var(B)\), which is of higher class, at lest for some \(t\). The group \(\hat{A} \operatorname{Wr} Y(z, t)\) is \(t\)-generator. For \(t > t_1\) its nilpotency class is given by formula (3.2). Notice that (3.1) and (3.2) both consist of three summands of which the first and the third are the same. An easy comparison of the second summands (see [19] for details) shows that the nilpotency class of the \(t\)-generator group \(\hat{A} \operatorname{Wr} Y(z, t)\) from the variety \(\var(A) \var(B)\) is higher than the maximum of the nilpotency classes of the \(t\)-generator groups in \(\var(A \operatorname{Wr} B)\) for all large enough \(t\). Thus \(\hat{A} \operatorname{Wr} Y(z, t) \not\in \var(A \operatorname{Wr} B)\). \(\square\)

Notice that the above proof explicitly constructs the group that belongs to the variety \(\var(A) \var(B)\) but not to the variety \(\var(A \operatorname{Wr} B)\) when (27) fails to hold.

**Proof based on Ol’shanskiǐ’s theorem.** The sufficiency part of the proof for Lemma 5.1 can again be covered by the same remark about the discriminating property of \(C_p^\infty\):

Again by Prüfer’s theorem [25] the abelian group \(B\) is a direct product of copies of some cyclic subgroups of \(p\)-power orders. If in this direct product the number of factors of order \(p^v\) is finite, then the \(p^{v-1}\)-th power

\[ C = B^{p^{v-1}} = \langle b^{p^{v-1}} \mid b \in B \rangle \]

of \(B\) is a finite group.

Thus the group \(A \operatorname{Wr} B\) is an extension of a nilpotent \(p\)-group \(A^B\) of finite exponent by means of the group \(B\), which in turn is an extension of the finite \(p\)-group \(C\) by the abelian \(p\)-group \(B/C\) of exponent \(p^{v-1}\). The full pre-image \(\hat{C}\) of \(C\) under natural homomorphism \(A \operatorname{Wr} B \rightarrow (A \operatorname{Wr} B)/A^B\) is a nilpotent \(p\)-group by G. Baumslag’s theorem [11].

Therefore \(A \operatorname{Wr} B\) belongs to the product of the nilpotent variety \(N = \var(\hat{C})\) and the abelian variety \(\mathfrak{A}_{p^{v-1}}\). Assuming now that \(\var(A \operatorname{Wr} B) = \var(A) \var(B)\) we have the inclusion

\[ \var(A) \var(B) \subset N \mathfrak{A}_{p^{v-1}}. \]

By A.Yu. Olshanskii’s theorem [24], if \(\mathcal{U} \subseteq \mathcal{V}\) are varieties of groups with decompositions (into indecomposable factors) \(\mathcal{U} = \mathcal{U}_1 \cdots \mathcal{U}_k\) and \(\mathcal{V} = \mathcal{V}_1 \cdots \mathcal{V}_l\), then there exists an intermediate variety \(\mathcal{M}\) (in the sense \(\mathcal{U} \subseteq \mathcal{M} \subseteq \mathcal{V}\)) with two decompositions \(\mathcal{M} = \mathcal{M}_1 \cdots \mathcal{M}_k = \mathcal{M}_1 \cdots \mathcal{M}_l\) such that \(\mathcal{U}_i \subseteq \mathcal{M}_i\) for each \(i\) and \(\mathcal{V}_j \subseteq \mathcal{M}_j\) for each \(j\).

By (3.3) we can apply this theorem for \(\mathcal{U} = \mathcal{U}_1 \mathcal{U}_2 = \var(A) \var(B)\) and \(\mathcal{V} = \mathcal{V}_1 \mathcal{V}_2 = \mathcal{N} \mathfrak{A}_{p^{v-1}}\). In particular, \(\mathcal{N}_1 \subseteq \mathcal{M}\) and \(\mathcal{N}_2 \subseteq \mathfrak{A}_{p^{v-1}}\). Since nilpotent varieties are indecomposable, then \(\mathcal{M}_1 = \mathcal{N}_1\) and \(\mathcal{M}_2 = \mathcal{N}_2\). So \(\var(B) \subseteq \mathcal{M}_2 \subseteq \mathfrak{A}_{p^{v-1}}\). The later inclusion contradicts the assumption that \(\exp B = p^v\). \(\square\)

4. Sylow and Hall subgroups of groups in \(\var(A \operatorname{Wr} B)\)

This section contains some technical results and routine computations for Sylow and Hall subgroups in groups generated by wreath products. They will be used in the proof of Theorem 1.1 in Section 5. Following the conventional notation we denote by \(QX, SX, CX\) and \(DX\) the classes of all homomorphic images, subgroups, cartesian
products and direct products of finitely many groups of $\mathfrak{X}$ respectively. By Birkhoff’s Theorem [21] for any class of groups $\mathfrak{X}$ the variety $\text{var}(\mathfrak{X})$ generated by it can be obtained by these three operations: $\text{var}(\mathfrak{X}) = QSC \mathfrak{X}$. For the given classes of groups $\mathfrak{X}$ and $\mathfrak{Y}$ denote $\text{var}(\mathfrak{X}) = \{X \mid X \in \mathfrak{X}, Y \in \mathfrak{Y} \}$. 

In two technical lemmas below we group a few statements which either are known facts in the literature, or are proved by us earlier (see Proposition 22.11 and Proposition 22.13 in [21], Lemma 1.1 and Lemma 1.2 in [15] and also [28]). Their proofs can be found in [15]. Both of these lemmas will be repeatedly used below.

**Lemma 4.1.** For arbitrary classes $\mathfrak{X}$ and $\mathfrak{Y}$ of groups and for arbitrary groups $X^*$ and $Y^*$, where either $X^* \in \mathfrak{QX}$, or $X^* \in \mathfrak{SX}$, or $X^* \in \mathfrak{CX}$, and where $Y \in \mathfrak{Y}$, the group $X^* \text{ Wr } Y$ belongs to the variety $\text{var}(\mathfrak{X} \text{ Wr } \mathfrak{Y})$.

**Lemma 4.2.** For arbitrary classes $\mathfrak{X}$ and $\mathfrak{Y}$ of groups and for arbitrary groups $X$ and $Y^*$, where $X \in \mathfrak{X}$ and where $Y^* \in \mathfrak{QY}$, the group $X \text{ Wr } Y^*$ belongs to the variety $\text{var}(\mathfrak{X} \text{ Wr } \mathfrak{Y})$. Moreover, if $\mathfrak{X}$ is a class of abelian groups, then for each $Y^* \in \mathfrak{QY}$ the group $X \text{ wr } Y^*$ also belongs to $\text{var}(\mathfrak{X} \text{ Wr } \mathfrak{Y})$.

Let $A$ be a nilpotent group of class $c$ and of exponent $m \neq 1$, and $B$ be an abelian group of exponent $n \neq 1$. Denote by $\mathfrak{U}$ the variety generated by $A$, and assume $p_1, \ldots, p_s$ are all the prime divisors of $m$. Since $\mathfrak{U} \subseteq \mathfrak{N}_{c,m} = \mathfrak{N}_c \cap \mathfrak{B}_m$ is a nilpotent variety, according to [21, Corollary 35.12] it is generated by its $c$-generator free group $F = F_c(\mathfrak{U})$. Being a finite nilpotent group, $F$ is a direct product of its Sylow $p_i$-subgroups, $i = 1, \ldots, s$:

\[(4.1) \quad F = S_{p_1} \times \cdots \times S_{p_s}.
\]

Notice that all the $p_i$’s actually participate: none of the Sylow subgroups is trivial, since $\exp(F) = \exp(\mathfrak{U}) = \exp(A)$.

Assume there is a common prime divisor $p$ of $m$ and $n$ (it is one of the primes $p_i$), and let $p^\nu$ be the highest power of $p$ dividing $n$. Further, denote by $B(p)$ the $p$-primary component of the abelian group $B$ (the subgroup of all elements whose order is a power of the prime $p$). The following lemma allows to localize the Sylow $p$-subgroups of groups in the variety generated by $A \text{ Wr } B$:

**Lemma 4.3.** In the above notations the following equality holds:

\[(4.2) \quad \text{var}(A \text{ Wr } B) \cap \text{var}(S_p) = \text{var}(S_p \text{ Wr } B(p)) \, .
\]

**Proof.** The group $S_p \text{ Wr } B(p)$ evidently is in the product variety $\text{var}(S_p) \mathfrak{A}_{p^\nu}$. Since $S_p$ belongs to the variety $\text{var}(F) = \text{var}(A)$ and since $B(p)$ is a subgroup of $B$, then by Lemma 4.1 and Lemma 4.2 the group $S_p \text{ Wr } B(p)$ is in variety $\text{var}(A \text{ Wr } B)$. So the right-hand side of (4.2) lies in the left-hand side.

To show the opposite inclusion notice that all groups of $\text{var}(S_p) \mathfrak{A}_{p^\nu}$ are $p$-groups, and it will be sufficient to take any $p$-group $P$ in $\text{var}(A \text{ Wr } B)$, and to show that it is in $\text{var}(S_p \text{ Wr } B(p))$. Since $\text{var}(A \text{ Wr } B)$ is locally finite, we may assume $P$ is finite. By [21, 16.31] $\text{var}(A \text{ Wr } B)$ is generated by all the finitely generated subgroups $\{R_i \mid i \in I\}$ of the group $A \text{ Wr } B$, which, clearly, all are finite also.

By G. Higman’s result [9, Lemma 4.3, point (ii)] $P \in \mathfrak{QSD}\{R_i \mid i \in I\}$, that is, there is a finite sequence of groups $R_1, \ldots, R_l$ (repetition of groups is allowed), such that $P$ is a surjective image of a subgroup $R$ of the direct product $R_1 \times \cdots \times R_l$ of the groups $R_i$ under a homomorphism $\rho : R \to P$. It is easy to show that $P$ is an image of a suitably chosen Sylow $p$-subgroup $\tilde{P}$ of $R$. This is true for even more
general situation: for any finite group $G$ and for its homomorphic image $H = \varphi(G)$ any Sylow $p$-subgroup $P$ of $H$ is an image of a suitable Sylow $p$-subgroup $\bar{P}$ of $G$. Indeed, the image $\varphi(Q)$ of any Sylow $p$-subgroup $Q$ of $G$ is a Sylow $p$-subgroup of $H$, and thus $h^{-1}\varphi(Q)h = \varphi(Q)^h = P$ for some $h \in H$, so we can choose $\bar{P} = Q^h$. For the purposes of the current proof set $\bar{P}$ be the respective Sylow $p$-subgroup of $R$ such that $\rho(\bar{P}) = P$.

$\bar{P}$ is the sub-direct product of its projections $\bar{P}_i$ on direct factors $R_i$, $i = 1, \ldots, l$. Since each $R_i$ is some subgroup of $A \wr B$, denote by $M_i$ the intersection of $\bar{P}_i$ with the base group $A^B$. $M_i$ is normal in $\bar{P}_i$, and one can consider the factor group $\bar{P}_i/M_i$, which is isomorphic to some subgroup $N_i$ of $B$ because:

\begin{equation}
\bar{P}_i/M_i = \bar{P}_i/(\bar{P}_i \cap A^B) \cong (\bar{P}_i \cdot A^B)/A^B \leq A \wr B/A^B \cong B.
\end{equation}

By the Kaloujnine-Krasner theorem [10] $\bar{P}_i$ can be embedded into the wreath product $M_i \wr N_i$.

Being a $p$-subgroup in $B$, the group $N_i$ is a subgroup also in $B(p)$, and thus by Lemma 4.2 $\bar{P}_i \in \var{M_i \wr N_i} \subseteq \var{M_i \wr B(p)}$ holds. Further, since $M_i \in \var{F}$, using G. Higman’s result again, we get $M_i \in \QSD{F}$, that is, $M_i$ is a surjective image of a subgroup $L$ of the direct product $F_1 \times \cdots \times F_k$ of finitely many copies of the finite group $F$ under a homomorphism $\mu_i : L \to M_i$. The above used argument about the pre-image of a Sylow $p$-subgroup shows here that the $p$-group $M_i$ also is an image of a Sylow $p$-subgroup $M_i$ of $L$. The group $\hat{M}_i$ is a sub-direct product of its projections $\hat{M}_{i,r}$ on direct factors $F_r$, $r = 1, \ldots, k$. It follows from (4.1) that each of $\hat{M}_{i,r}$ is embeddable into $S_p$. Thus $M_i \in \var{S_p}$ and $M_i \wr B(p) \in \var{S_p \wr B(p)}$ by Lemma 4.1.

If $n$ is a divisor of $m$, then Lemma 4.2 together with Lemma 3.1 already allows to classify all cases, when (7) holds for a nilpotent group $A$ of finite exponent $m$ and for an abelian group of exponent $n$.

The open case (when $n$ has prime divisors, not dividing $m$) can be covered by the following lemma, the proof of which is sketched, since it is similar to the previous proof. Let $p$ be a prime divisor of $m$ not dividing $n$, let $q$ be a prime divisor of $n$ not dividing $m$, and assume $q^r$ is the highest power of $q$ dividing $n$. Let $B(q)$ be the $p$-primary component of $B$, and let $F$ and $S_{p_1}, \ldots, S_{p_s}$ denote the same as in Lemma 4.3. Then the following lemma localizes the Hall $\{p, q\}$-subgroups in groups of $\var{A \wr B}$:

**LEMMA 4.4.** In the above notations the following equality holds:

\begin{equation}
\var{A \wr B} \cap \var{S_p} \mathfrak{A}_{q^r} = \var{S_p \wr B(q)}.
\end{equation}

**PROOF SKETCH.** By arguments similar to the first paragraph of the previous proof the right side of (4.4) lies in the left side.

To show the opposite inclusion denote $\pi = \{p, q\}$ and notice that it will be sufficient to take any $\pi$-group $P$ in $\var{A \wr B}$, and to show it is in $\var{S_p \wr B(q)}$. Again, it is enough to consider finite groups $P$ only, and to assume by [11 16.31] that $P$ is a surjective image of a subgroup $R$ of the direct product $R_1 \times \cdots \times R_l$ under a homomorphism $\rho : R \to P$ for some finite subgroups $R_i$ of $A \wr B$.

For the next step we need some modifications, since the Hall $\pi$-subgroups not always satisfy the properties we used above for Sylow $p$-subgroups. The groups we deal with are at most nilpotent-by-abelian, and thus they are soluble. For the finite soluble groups the theorems of P. Hall [25] provide the tools we need: for any finite
soluble group $G$ and for its homomorphic image $H = \varphi(G)$ any Hall $\pi$-subgroup $P$ of $H$ is an image of a Hall $\pi$-subgroup $\hat{P}$. For the image $\varphi(Q)$ of any Hall $\pi$-subgroup $Q$ of $G$ is a Hall $\pi$-subgroup of $H$, and thus $h^{-1}\varphi(Q)h = \varphi(h^q)$ for some $h \in H$, and we can choose $\hat{P} = \varphi(Q)$. For the current proof set $\hat{P}$ be the respective Hall $\pi$-subgroup of $R$, such that $\rho(\hat{P}) = P$.

$\hat{P}$ is the sub-direct product of its projections $\hat{P}_i$ on direct factors $R_i$, $i = 1, \ldots, l$. Like in previous proof denote $M_i = \hat{P}_i \cap A^B$. Calculations similar to \([1,3]\) show that the factor group $\hat{P}_i/M_i$ is isomorphic to some subgroup $N_i$ of $B$. Thus the $\pi$-group $\hat{P}_i$ can be embedded into the wreath product $M_i \Wr N_i$.

Being an image of a $\pi$-group, $N_i$ also is a $\pi$-group. Since $p$ does not divide $n$, the group $N_i$ is inside $B(q)$, and thus by Lemma \([12]\) $\hat{P}_i \in \var(M_i \Wr N_i) \subseteq \var(M_i \Wr B(q))$ holds.

Since $M_i \in \var(F)$, using G. Higman’s result for one more time we get that $M_i$ is a surjective image of a subgroup $L \leq F_1 \times \cdots \times F_k$ (of copies of $F$) under a homomorphism $\mu : L \to M_i$. The above used argument about the pre-image of a Hall $\pi$-subgroup shows that the $\pi$-group $M_i$ also is an image of a Hall $\pi$-subgroup $\hat{M}_i$ of $L$. Since $q$ does not divide $n$, the group $M_i$ actually is a $p$-group. $\hat{M}_i$ is a sub-direct product of its projections $\hat{M}_{i,r}$ on direct factors $F_r$, $r = 1, \ldots, k$, and each of $\hat{M}_{i,r}$ is embeddable into $S_p$. Thus $M_i \in \var(S_p)$ and $M_i \Wr B(q) \in \var(S_p \Wr B(q))$ by Lemma \([4,1]\) \(\square\)

Notice that these two lemmas do not cover the case when the exponent of $A$ has a prime divisor $p$, which does not divide the exponent of $B$. In this connection see Remark \([5,1]\) on “three roles” of prime divisors of $m$ and $n$ below.

5. The proof of Theorem \([1.1]\)

The facts collected in previous sections allow to assemble the proof for our main result:

**Proof of Theorem \([1.1]\)** Let us start with statement (a) of Theorem \([1.1]\) when the group $B$ is not of finite exponent. If $B$ contains an element $z$ of infinite exponent, then it contains the infinite cyclic group $\langle z \rangle = C \cong \mathbb{Z}$, which is a discriminating group \([21, 17.6]\). Since $\var(\langle z \rangle) = \var(B) = \mathfrak{A}$, then by \([21, 17.5]\) the group $B$ also discriminates $\mathfrak{A}$. It remains to apply \([21, 22.42]\) to get that $A \Wr B$ generates $\var(A) \var(B)$.

Now assume all the elements of $B$ are of finite non-zero exponents, which in this case have no common upper bound. $B$ generates $\mathfrak{A}$, and it will be enough to show that $B$ again is a discriminating group. Since $C$ discriminates $\mathfrak{A}$, any finite system of non-identities $w_1, \ldots, w_d$ of $\mathfrak{A}$ can simultaneously be falsified on some set $z_1, \ldots, z_s \in C$. Take an element $z \in B$ with exponent larger than the values of all words $w_i$ on variables $z_1, \ldots, z_s$. The non-identities will simultaneously be falsified on some elements in the subgroup $\langle z \rangle \leq B$.

Turning to the proof of necessity for the condition at statement (b) of Theorem \([1.1]\) notice that it will be enough to consider two cases: either if $B$ fails to contain $C_d^c$, or if it fails to contain $C_{n/d}^\infty$.

**Case 1.** If $B$ does not contain a subgroup isomorphic to $C_d^c$, there is at least one prime divisor $q$ of $n$, such that $B$ does not contain $C_q^c$, where $q^v$ is the exponent of $B(q)$, that is, the highest power of $q$ dividing $n$. Using the notation of \([4,1]\), the
exponent $m$ has at least one prime divisor $p = p_i$, such that the nilpotency class of the Sylow $p$-subgroup $S_p = S_{p_i}$ is equal to $c$ (otherwise the class of $A$ would be lower than $c$).

We have $\text{var}(A \wr B) \cap \text{var}(S_p) \mathfrak{A}_{q^v} = \text{var}(S_p \wr B(q))$ by Lemma 4.4. By Lemma 2.1 any critical group of $\text{var}(S_p)$ $\mathfrak{A}_{q^v}$ is an extension of some group from $\text{var}(S_p)$ by a direct product of at most $c$ cyclic groups from $\mathfrak{A}_{q^v}$. Recalling the steps of the proof of Lemma 4.4 we see that this direct product can be taken to be a subgroup of $B(q)$. Thus in (4.4) we can replace $B(q)$ by some finite (at most $c$-generator) subgroup $B^*$ of $B(q)$.

According to Theorem 1 in [18] (which we also bring in Section 6 below as Theorem 6.1) the variety $\text{var}(S_p \wr B(q)) = \text{var}(S_p \wr B^*)$ is strictly less than the product $\text{var}(S_p \wr B(q)) = \text{var}(S_p) \mathfrak{A}_{q^v}$ because $B, B(q)$ and $B^*$ do not possess a subgroup, isomorphic to $C_{q^v}^c$, where $c$ is the nilpotency class of $S_p$ (that is, the class of $A$). Therefore the equality (2) does not hold, since the product $\text{var}(A) \text{var}(B)$ does contain $\text{var}(S_p) \text{var}(B(q))$ as a subvariety.

Case 2. If $B$ does not contain a subgroup isomorphic to $C_{n/d}^\infty$, there is at least one common prime divisor $p$ of $m$ and of $n$, such that $B$ does not possess $C_p^{p^v}$, where $p^v$ is the exponent of $B(p)$ and the highest power of $p$ dividing $n$. In this case the nilpotency class of the Sylow $p$-subgroup $S_p = S_{p_i}$ may be less than $c$, but $S_p$ certainly is non-trivial, otherwise the exponent $m$ would not be divisible by $p$.

This time we use the equality $\text{var}(A \wr B) \cap \text{var}(S_p) \mathfrak{A}_{p^v} = \text{var}(S_p \wr B(p))$ of Lemma 4.3. According to Lemma 3.1 in Section 3 the variety $\text{var}(S_p \wr B(p))$ is strictly less than the product $\text{var}(S_p) \text{var}(B(p)) = \text{var}(S_p) \mathfrak{A}_{p^v}$ because $B$ and $B(p)$ do not possess a subgroup isomorphic to $C_p^\infty$. Therefore like in Case 1 above, the equality (2) does not hold, since $\text{var}(A) \text{var}(B)$ does contain $\text{var}(S_p) \text{var}(B(p))$.

It remains to prove sufficiency of the condition at statement (b) in Theorem 1.1. Assume $B$ contains a subgroup isomorphic to $C_d \times C_{n/d}^\infty$, and take an arbitrary critical group $K$ in $\text{var}(A) \text{var}(B)$. By Lemma 2.1 $K$ is an extension of some group $N \in \text{var}(A)$ by some group $S \in \text{var}(B)$, and $S = U \times U^*$, where $U$ and $U^*$ meet the points (1)–(3) of Lemma 2.1. In particular, by points (1) and (2) $U$ is a subgroup of $C_d^c$, and by point (3) $U^*$ is a subgroup of $C_{n/d}^\infty$. So we have

$$S = U \times U^* \leq C_d^c \times C_{n/d}^\infty \leq B,$$

and therefore by Lemma 4.3 and Lemma 4.2 the group $K$ belongs to the variety $\text{var}(A \wr B)$.

The product $\text{var}(A) \text{var}(B)$ is a locally finite variety, and it is generated by its critical groups [21, 51.41]. Since they all belong to the variety $\text{var}(A \wr B)$, the equality (2) does hold for the considered case.

The proof of Theorem 1.1 is completed.  

**Remark 5.1.** Notice an interesting pattern about “three roles” of the prime divisors of the exponents $m$ and $n$ according to Theorem 1.1.

a) The most important role for feasibility of equality (2) belongs to the primes $p$ dividing both $m$ and $n$: for each of them the group $B$ should contain the infinite direct power $C_p^{p^v}$ of the cycle $C_p^{p^v}$, where $p^v$ is the highest power of $p$ dividing $n$.

b) Less demanding are the primes $q$ that divide $n$ but not $m$: for each of them the group $B$ should contain the direct power $C_q^{c_q}$, with $c$ being the class of $A$, and $q^{v_q}$ being the highest power of $q$ dividing $n$.  

c) And, finally, the primes \( p \) that divide \( m \) but not \( n \) have no impact on feasibility of (2) (they just participate in determination of the exponent of \( \text{var}(A) \)).

6. Some comparison with earlier results

Let us restate two main theorems of earlier research [15]–[18] and show how Theorem 1.1 generalizes them for wide classes of groups. Firstly, restrict our consideration to finite groups (recall that we denoted by \( C_n \) the cycle of order \( n \)):

Theorem 6.1 (Theorem 1 in [18]). For finite non-trivial groups \( A \) and \( B \) the equality (2) holds if and only if:

- a) the exponents of group \( A \) and \( B \) are coprime;
- b) \( A \) is a nilpotent group, \( B \) is an abelian group;
- c) \( B \) contains a subgroup isomorphic to the direct power \( C_c^n \), where \( c \) is the nilpotency class of \( A \), and \( n \) is the exponent of \( B \).

That the condition (b) of Theorem 6.1 is necessary, follows from A.L. Shmel’kin’s theorem [28, Theorem 6.3]. But if \( A \) is nilpotent (and of finite exponent, since it is finite) and \( B \) is abelian, then the point (b) of Theorem 1.1 requires that \( B \) contains the direct product \( C_d \times C_{n/d}^\infty \), where \( d \) is the largest divisor of \( n \), coprime with \( m \). But since \( B \) is finite, it contains no nontrivial subgroup \( C_{n/d}^\infty \). So we get \( n/d = 1 \), that is, the condition (a) of Theorem 6.1.

The second direction for restriction is to consider Theorem 1.1 for abelian groups. We had proved:

Theorem 6.2 (Theorem 6.1 in [15] or Theorem A in [16]). For abelian non-trivial groups \( A \) and \( B \) the equality (2) holds if and only if:

- a) either at least one of the groups \( A \) and \( B \) is not of finite exponent;
- b) or if \( \exp A = m \) and \( \exp B = n \) are both finite, and for every common prime divisor \( p \) of \( m \) and \( n \) a direct decomposition of the \( p \)-primary component \( B(p) \) of \( B \) contains infinitely many direct summands \( C_p^v \), where \( p^v \) is the highest power of \( p \) dividing \( n \).

To make this theorem more intact with Theorem 1.1 and Theorem 6.1, let us reformulate it slightly differently. For every common prime divisor \( p \) of \( m \) and \( n \) the component \( B(p) \) contains infinitely many direct summands \( C_p^v \) if and only if \( B(p) \) contains the direct product of all of them, that is, if \( B \) contains the direct power \( C_{n/d}^\infty \), with \( d \) defined in Theorem 1.1. Thus:

Theorem 6.3 (an equivalent form for Theorem 6.2). For abelian non-trivial groups \( A \) and \( B \) the equality (2) holds if and only if:

- a) either at least one of the groups \( A \) and \( B \) is not of finite exponent;
- b) or if \( \exp A = m \) and \( \exp B = n \) are both finite, and \( B \) contains a subgroup isomorphic to the infinite direct power \( C_d \times C_{n/d}^\infty \), where \( d \) is the largest divisor of \( n \) coprime with \( m \).

Since here \( A \) is nilpotent, we have \( c = 1 \), and the point (b) in Theorem 1.1 would require that \( B \) contains the product \( C_d \times C_{n/d}^\infty \) (just one factor \( C_d \)). So this may seem to be a stronger requirement than the point (b) in Theorem 6.3. However, by elementary properties of abelian groups, for any prime divisor \( p \) of \( n \) the group \( B \) contains at least one subgroup isomorphic to \( C_p^v \). The direct product of some of these...
Theorem 1.1 is easy-to-use: it just evolves the nilpotency class of $A$. So for abelian groups Theorem 1.1 is a partial generalization of this case also.

7. Examples and applications of the criterion

Some number of the examples, in which \((\text{\#})\) holds or does not hold for wreath products of abelian groups or of finite groups, can be obtained by Theorem 6.1 in \([16]\), Theorem A in \([16]\) or Theorem 1 in \([18]\). Here we just give references to them: examples 4.6, 5.4, 6.3, 6.4 in \([15]\), examples 6.9, 7.5, 8.5, 8.6 in \([16]\), examples 1, 2 in \([18]\) (repetitions allowed). The listed examples can be handled and generalized by Theorem 1.1. In particular:

**Example.** In \([11]\) L.G. Kovács has computed the variety generated by dihedral group $D_4$ of order 8: $\text{var}(D_4) = \mathfrak{A}_2^2 \cap \mathfrak{N}_2$. In \([18]\) we have seen that for any odd $n$ the equality \((\text{\#})\) holds for $A = D_4$ and for finite abelian group $B$ of exponent $n$ if and only if $B$ contains $C_n^2$. And, if \((\text{\#})\) holds, then $\text{var}(A \, \text{Wr} \, B) = (\mathfrak{A}_2^2 \cap \mathfrak{N}_2)\mathfrak{A}_n$.

Now we can generalize this in two directions. First, if $n$ is even, present it in the form $n = d \cdot 2^k$, where $d$ is odd (we can assume $d \neq 1$ since that is the case of odd $n$ covered in \([18]\)). Then by Theorem 1.1 \((\text{\#})\) holds for $A = D_4$ and for any abelian group $B$ of finite exponent if and only if $B$ contains the direct product $C_d^2 \times C_{2^k}$ (which, clearly, is isomorphic to $C_n^2 \times C_{2^k}$). So, for instance, \((\text{\#})\) does not hold if $B = C_6 = C_3 \times C_2$, if $B = C_{2^k} \times C_2$ or if $B = C_2^k \times C_2^2$ (for any $k \geq 1$), but \((\text{\#})\) does hold if $B = C_{2^k}^2$.

As a second direction for generalization we can replace $A = D_4$ by its finite or infinite direct power $D_4^k$ or $D_4^\infty$. Then the requirement for $B$ remains unchanged.

**Example.** The quaternion group $Q_8$ of order 8 also is nilpotent of class 2, and it generates the same variety $\text{var}(Q_8) = \text{var}(D_4) = \mathfrak{A}_2^2 \cap \mathfrak{N}_2$ (see \([21]\)). Thus examples similar to the points of Example 7 can be constructed for the group $A = Q_8$.

**Example.** We above denoted $\mathfrak{N}_{c,m} = \mathfrak{N}_c \cap \mathfrak{B}_m$. Using simple properties of critical groups from \([4]\) and Proposition 2 from \([17]\) we saw in \([17]\) that \((\text{\#})\) does not hold when $A = F_2(\mathfrak{N}_{2,3})$ and $B = C_2$, and \((\ddagger)\) holds if we take $B = C_{2^k}$ (for any $k \geq 2$) instead. Using the technique of R.G. Burns with critical groups in \([4]\) Section 3 one could build analogs of this example for any variety $\mathfrak{N}_{2,p}$ and $C_q$, where prime numbers $p$ and $q$ are chosen so that $q$ divides $p - 1$.

But much more general cases were covered by Theorem 1 in \([18]\). Namely let $m,n > 1$ be any coprime integers, and let $s \geq c$. Then for $A = F_s(\mathfrak{N}_{c,m})$ and for an abelian group $B$ of exponent $n$ the wreath product $A \, \text{Wr} \, B$ generates $\mathfrak{N}_{c,m} \mathfrak{A}_n$ if and only if $B$ contains the direct product $C_n^s$ (we required $s \geq c$ to ensure $\text{var}(A) = \mathfrak{N}_{c,m}$).

Again, we can generalize this example in two directions. Firstly, if $n$ is not coprime to $m$, present it as $n = d \cdot n/d$, where $d$ is the largest divisor of $n$, coprime to $m$ (clearly $n/d \neq 1$). In these circumstances \((\text{\#})\) holds with an abelian group $B$ of exponent $n$, $A \, \text{Wr} \, B$ generates $\mathfrak{N}_{c,m} \mathfrak{A}_n$, and if only if $B$ contains $C_d^2 \times C_{n/d}^s$.

Secondly, we can consider relatively free groups of $\mathfrak{N}_{c,m}$ of infinite rank, which were not covered in \([18]\). By Theorem 1.1 the equality \((\text{\#})\) holds for $A = F_\infty(\mathfrak{N}_{c,m})$ and for the group $B$ mentioned above if and only if $B$ contains $C_{d}^2 \times C_{n/d}^\infty$.

Since there is no shortage in examples of nilpotent groups which are either finite or have finite exponents, one may continue this list of examples, as the criterion of Theorem 1.1 is easy-to-use: it just evolves the nilpotency class of $A$ and the direct decomposition of the abelian group $B$. 

$C_{pr}$ (for all $p$ coprime to $m$) does provide the direct summand $C_d$. So for abelian groups Theorem 1.1 is a partial generalization of this case also.
Theorem 1.1 also allows to add something our old research topic on ◦-products $\mathfrak{V} \circ B$ and $A \odot \mathfrak{U}$ (see the Ph.D. thesis [13], advisor A.Yu. Ol’shanskii, M.S.U.). For the given variety $\mathfrak{V}$ and the group $B$ the variety $\mathfrak{V} \circ B$ is that generated by extensions of all groups from $\mathfrak{V}$ by the group $B$, and $A \odot \mathfrak{U}$ is the variety, generated by all extensions of the group $A$ by all groups from $\mathfrak{U}$. In [13] we used these ◦-products as tools to study the product varieties via consideration of cases, when $\mathfrak{V} \circ B = \mathfrak{V} \var(B)$ or $A \odot \mathfrak{U} = \var(A) \mathfrak{U}$. Now we have one more case:

**Theorem 7.1.** Let $\mathfrak{V}$ be any non-trivial nilpotent variety of finite exponent and let $B$ be any abelian group. Then the equality $\mathfrak{V} \circ B = \mathfrak{V} \var(B)$ holds if and only if either the group $B$ is not of finite non-zero exponent, or if $B$ is of some finite exponent $n > 0$, and it contains a subgroup isomorphic to the direct product $C_d^c \times C_\infty^{n/d}$, where $c$ is the nilpotency class of $\mathfrak{V}$, and $d$ is the largest divisor of $n$ coprime with $m$.

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