Delocalisation phenomena in one-dimensional models with long-range correlated disorder: a perturbative approach

L. Tessieri

Department of Chemistry, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6

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Abstract

We study the nature of electronic states in one-dimensional continuous models with weak correlated disorder. Using a perturbative approach, we compute the inverse localisation length (Lyapunov exponent) up to terms proportional to the fourth power of the potential; this makes possible to analyse the delocalisation transition which takes place when the disorder exhibits specific long-range correlations. We find that the transition consists in a change of the Lyapunov exponent, which switches from a quadratic to a quartic dependence on the strength of the disorder. Within the framework of the fourth-order approximation, we also discuss the different localisation properties which distinguish Gaussian from non-Gaussian random potentials.

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1 Introduction

In recent years, the interest for one-dimensional (1D) models with correlated disorder has been steadily growing, as it has become progressively clear that correlations of the random potential can deeply affect the structure of
the electronic eigenstates and endow 1D disordered models with far richer transport properties than it was previously thought.

For a long time it was believed that 1D systems could not display complex features like the metal-insulator transition which takes place in three-dimensional (3D) models, since it was known that all one-electron states are localised in 1D systems with totally random potentials, regardless of the strength of disorder (see [1] and references therein). Further research, however, led to the study of some variants of the standard 1D Anderson model which exhibited a discrete set of extended states (see, e.g., [2]). All these systems were characterised by random potentials with short-range spatial correlations, a feature absent from the original Anderson model where the site energies are totally uncorrelated.

Eventually, the role of long-range correlations was also investigated and it was shown that this kind of correlations can produce a continuum of extended states [3]. This finding was corroborated by the work of Izrailev and Krokhin who, using second-order perturbation theory, managed to establish an analytical relation between localisation length and potential pair correlators, and used this result to show how specific long-range disorder correlations lead to the appearance of mobility edges in 1D discrete models [4]. An experimental confirmation of this result was obtained by studying the transmission of microwaves in a single-mode waveguide with a random array of correlated scatterers [5]. The results established in [3] and [4] for discrete lattices were subsequently extended to continuous models and related to parallel phenomena occurring in different fields like the propagation of waves in random media [6] and the dynamics of classical stochastic oscillators [7].

The discovery that 1D disordered systems can display a metal-insulator transition analogous to the one which takes place in 3D models constituted a crucial advancement for the understanding of anomalous transport properties of 1D random models. This theoretical progress is also relevant from the technological point of view, because it paves the way for the construction of 1D devices with pre-determined mobility edges which can be used as window filters in electronic, acoustic, and photonic structures [8]. The importance of the result makes therefore highly desirable to reach a complete comprehension of the link between long-range correlations of the disorder and the appearance of a continuum of extended electronic states in 1D models.

At the analytical level, our understanding of this problem rests on the results which were first derived in [4]. Unfortunately, both this work and the ones that have followed in its wake suffered from two main limitations: the
disorder was supposed to be weak, so that a perturbative approach could be
applied, and all the analytical results were obtained in the second-order ap-
proximation. (These constraints were absent from the work [3] of de Moura
and Lyra, but they drew their conclusions from numerical calculations and
did not provide any analytical insight on the delocalisation mechanism.) An-
alytical calculations are greatly simplified by truncating the expansion of the
inverse localisation length to the second-order term; the main drawback of
this choice is that it leaves open the question of the true nature of the states
of the “extended” phase. In fact, a vanishing second-order inverse localisa-
tion length can be related to various physical phenomena: it can indicate
that the electronic states are completely delocalised, but it can also be a sign
of weak forms of localisation, characterised by power-law decay of the elec-
tronic probability distribution; finally, it is the result to be expected when the
spatial ranges of electronic states in the “extended” and “localised” phases
differ by orders of magnitude.

The present work constitutes an attempt to partially remove the limits of
the previous analyses; in this study we still focus our attention on the case of
weak disorder but, with the help of a systematic perturbative technique, we
manage to go beyond the second order approximation and obtain analytical
results correct to the fourth order of perturbation theory. By this way we can
shed light on the true nature of the states that are classified as “e xtended” in
the framework of the second-order theory and we are also able to discuss the
differences which emerge at this refined level of description between Gaussian
and non-Gaussian random potentials.

In the case of Gaussian disorder with long-range correlations, it turns
out that the electronic states are exponentially localised on both sides of
the mobility edge identified in [3]; however, the dependence of their inverse
localisation length on the disorder strength changes from a quadratic to a
quartic form upon crossing the critical threshold. For weak disorder, this
implies that electrons in the “extended” states can actually move over far
longer distances than the electrons which find themselves in a “localised”
state. The use of terms like “mobility edge” and “delocalisation transition”
is therefore legitimate, provided that one keeps in mind that the qualifica-
tion of “extended” must be understood as “localised on a large spatial scale”
rather than “completely delocalised”. The analysis of the localisation length
in the fourth-order approximation also reveals that, upon increasing the en-
ergy of the electrons, a second threshold appears, which separates the states
whose spatial extension scales with the inverse of the fourth power of the
potential from states that are characterised by an even weaker localisation (whose exact form cannot be ascertained within the limits of the fourth-order approximation).

The distinction between Gaussian and non-Gaussian disorder cannot be discussed within the framework of the second-order approximation, because the second-order inverse localisation length only depends on the second moment of the random potential, and differences between Gaussian and non-Gaussian distributions only show up in the higher moments. A systematic computation of the higher-order terms of the inverse localisation length, however, establishes a connection between the $n$-th term of the expansion and the corresponding $n$-th moment of the potential: therefore, the fourth-order results obtained in this work allow us a first analysis of the differences between Gaussian and non-Gaussian disorder. Lifting the Gaussian requirement seems to strengthen the localisation of electronic states, because the fourth-order inverse localisation length is increased by a new term proportional to the fourth cumulant of the random potential (cumulants are specific combinations of the moments which vanish in the Gaussian case but are otherwise non-zero). Specifically, we analyse in detail the difference between a Gaussian potential with long-range binary correlations and a non-Gaussian potential with the same pair correlators but with a fourth-order cumulant having exponential decreasing form: we find that the non-zero cumulant produces quartic localisation of all electronic states that lie beyond the second-order mobility edge, thus wiping out the additional threshold that appears in the Gaussian case.

This paper is organised as follows: in Sec. 2 we give an exact formulation of the problem and we define the model under study; in Sec. 3 we expose the perturbative method used and we give the general results for the second and fourth-order terms of the localisation length. We devote Sec. 4 to the case of random potential with short-range correlations; the problem of long-range correlation is then discussed in Sec. 5. Finally, we summarise the main results and express our conclusions in Sec. 6.

2 Formulation of the problem
2.1 Definition of the model

We consider the 1D disordered model defined by the stationary Schrödinger equation

\[-\frac{\hbar^2}{2m} \psi''(x) + \varepsilon U(x) \psi(x) = E \psi(x) \quad (1)\]

where \(\psi(x)\) represents the state of a quantum particle ("electron") of mass \(m\) and energy \(E\) moving in a continuous random potential \(U(x)\). The dimensionless parameter \(\varepsilon\) is introduced to measure the strength of the disorder; for the sake of simplicity, in the rest of this paper we will adopt a system of units in which \(\hbar^2/2m = 1\). We will set the zero of the energy scale at the average value of the potential and we will assume that \(E > 0\) on this scale.

To complete the definition of the system under study, the statistical properties of the random potential must be precised. In the first place, we will assume that model \((1)\) is spatially homogeneous in the mean; from the mathematical point of view, this requirement implies that the moments of \(U(x)\) satisfy the invariance condition

\[\langle U(x_1 + \delta) U(x_2 + \delta) \ldots U(x_n + \delta) \rangle = \langle U(x_1) U(x_2) \ldots U(x_n) \rangle \quad (2)\]

for every spatial translation \(\delta\). In Eq. \((2)\), and throughout this paper, the angular brackets denote the average over disorder realisations, or ensemble average. Property \((2)\) implies that the \(n\)-th moment of the random potential can depend only on the relative positions of the points \((x_1, x_2, \ldots, x_n)\): this allows one to represent the \(n\)-point correlator as a function of \(n - 1\) relative coordinates according to the identity

\[\chi_n(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{n-1}) = \langle U(x) U(x + \tilde{x}_1) U(x + \tilde{x}_2) \ldots U(x + \tilde{x}_{n-1}) \rangle \quad (3)\]

Our second basic assumption on model \((1)\) is that its average features do not depend on the sign of the random potential, i.e., that the physical properties of the model are invariant under the transformation \(U(x) \rightarrow -U(x)\). This is ensured by assuming that all odd moments of the potential vanish

\[\langle U(x_1) U(x_2) \ldots U(x_{2n+1}) \rangle = 0.\]

Finally, for reasons of mathematical convenience, we will suppose that a finite length scale \(l_c\) exists for the model \((1)\) such that the statistical correlations between the values \(U(x_1)\) and \(U(x_2)\) of the potential become negligible when
the distance separating the points $x_1$ and $x_2$ exceeds $l_c$. For the particular case of the two-point correlation function this assumption translates in the condition

$$\langle U(x_1)U(x_2) \rangle \simeq 0 \quad \text{for} \quad |x_1 - x_2| \gg l_c; \quad (4)$$

more generally, the existence of a correlation length $l_c$ implies that the $n$-point correlation functions $\langle U(x_1)U(x_2) \ldots U(x_n) \rangle$ must satisfy the so-called “product property”, meaning that, if the points $x_1, x_2, \ldots, x_n$ can be divided in two groups with $|x_i - x_j| \gg l_c$ for $x_i$ and $x_j$ belonging to different groups, the average $\langle U(x_1)U(x_2) \ldots U(x_n) \rangle$ takes the value obtained by averaging the two different groups separately. To deal with the case of long-range correlations, we will first make use of the previous hypothesis and derive results valid for any finite $l_c$; when possible, we will then take the limit $l_c \to \infty$.

Having thus defined the statistical properties of the random potential, we can proceed to enunciate the most specific assumption about the model under investigation. In the following, we will restrict our considerations to the case of weak disorder, as defined by the relation

$$\varepsilon^2 \langle U(x)U(x') \rangle \ll E^2. \quad (5)$$

and by the analogous conditions on the higher moments of the random potential. Notice that, to ensure that condition (5) is valid independently of the specific form of the function $U(x)$, the parameter $\varepsilon$ must satisfy the condition $\varepsilon \ll 1$.

Eq. (1), together with the specified statistical properties of the random potential and the assumption of weak disorder, completely defines the model under study. Our goal consists in determining the spatial behaviour of the solutions of Eq. (1); to this end it is necessary to give a precise definition of the key parameter known as localisation length.

### 2.2 Localisation length

To shed light on the localisation properties of model (1), we will use as an indicator of the localised or extended nature of the electronic states the quantity

$$\lambda = \lim_{x \to \infty} \frac{1}{4x} \ln \langle \psi^2(x)k^2 + \psi'^2(x) \rangle, \quad (6)$$

where the parameter $k$ is related to the electronic energy by the relation $E = k^2$. In definition (6) the wavefunction $\psi(x)$ can be supposed to be real,
so that no need arises to consider the complex extension of the logarithm function.

In this paper we will always refer to the quantity (6) as the inverse localisation length or Lyapunov exponent, although this parameter is more usually defined through the expression

$$\tilde{\lambda} = \lim_{x \to \infty} \frac{1}{2x} \langle \ln \left( \psi^2(x) k^2 + \psi'^2(x) \right) \rangle.$$  \hfill (7)

The parameters (6) and (7) differ in principle, but both can be used effectively to ascertain the localised or extended character of the electronic states: in fact, they belong to the same family of generalised Lyapunov exponents

$$L_q = \lim_{x \to \infty} \frac{1}{x} \ln \left( \langle |\psi(x)|^q \rangle^{1/q} \right),$$

which were introduced long ago for the study of localisation in 1D disordered models (see [9] and references therein). In technical terms, Eq. (7) defines the standard Lyapunov exponent $L_0$, whereas the parameter (6) is equal to the “generalised Lyapunov exponent of order two” divided by a factor two, $\lambda = L_2/2$. Our preference for the definition (6) of the inverse localisation length over the possible alternative (7) stems mainly from the fact that the first choice makes the study of model (1) more amenable to analytical treatment. The exact relation between the Lyapunov exponents (6) and (7) is not a completely solved problem; however, it is clear that the two affine indicators can be used equivalently to determine whether an eigenfunction of Eq. (1) is localised or not. Indeed, for delocalised states the alternative definitions (6) and (7) give the same result, since both parameters vanish when the solutions of Eq. (1) are extended. As for exponentially localised states, for the case of interest here, i.e., that of weak disorder, one can prove that the Lyapunov exponents (6) and (7) coincide at least to the second order of perturbation theory [7]. In Sec. 4 we will come back to this point to show that the identity seems to hold also beyond the second-order approximation.

From the physical point of view, the difference between the alternative parameters (6) and (7) is perhaps best understood if the problem of electronic localisation in model (1) is re-defined in terms of the energetic instability of a stochastic oscillator. In fact, the physical comprehension of the model (1) can be enhanced by observing that solving the stationary Schrödinger equation (1) is completely equivalent to studying the dynamics of the classical
stochastic oscillator defined by the Hamiltonian

\[ H = k \left( \frac{p^2}{2} + \frac{q^2}{2} \right) + \varepsilon k \frac{q^2}{2} \xi(t). \tag{8} \]

The Hamiltonian (8) represents an oscillator with frequency \( k \) perturbed by a noise \( \xi(t) \); the equivalence of this system with model (1) can be seen by considering that the dynamical equation of the oscillator

\[ \ddot{q}(t) + k^2 (1 + \varepsilon \xi(t)) q(t) = 0 \tag{9} \]

coincides with Eq. (1) if one identifies the electronic wavefunction \( \psi(x) \) with the orbit \( q(t) \) of the oscillator and if the noise \( \xi(t) \) is related to the random potential \( U(x) \) by the identity

\[ \xi(x) = -U(x)/E. \tag{10} \]

Underlying the mathematical equivalence of Eqs. (1) and (9) is the physical parallelism between the localisation of the quantum states of model (1) and the energetic instability of the stochastic oscillator (8) \cite{7,10}. The connection between the two phenomena emerges clearly if one writes the inverse localisation length (6) in terms of the dynamical variables of the random oscillator. The Lyapunov exponent (6) can then be expressed as

\[ \lambda = \lim_{t \to \infty} \frac{1}{4t} \ln \langle k^2 (p^2(t) + q^2(t)) \rangle \]

and interpreted as the growth rate of the average energy of the random oscillator (8). A positive value of the parameter (6) can therefore be read both as a sign of exponential localisation of the eigenstates of model (1) and as an indication that the stochastic oscillator (8) is energetically unstable.

As for parameter (7), it can be rewritten in terms of the oscillator variables as

\[ \tilde{\lambda} = \lim_{T \to \infty} \lim_{\delta \to 0} \int_0^T \ln \frac{x(t + \delta)}{x(t)} dt. \]

In this form, the parameter (7) reveals itself as the exponential divergence rate of nearby orbits, i.e., as the Lyapunov exponent for the trajectories of the oscillator (8). We can conclude that the parameters (6) and (7) are both growth rates, measuring respectively the increase of the oscillator energy and that of the distance between initially nearby orbits.
3 Perturbative expansion and general results

To compute the localisation length (3), one needs to determine the asymptotic behaviour of $\langle \psi^2(x) \rangle$ and $\langle \psi'^2(x) \rangle$. A convenient way to achieve this goal consists in deriving a deterministic differential equation for the average squares of the wavefunction and of its derivative. To this end, we introduce the scaled derivative

$$\phi(x) = \psi'(x)/k$$

and the vector

$$u(x) = \begin{pmatrix} \psi^2(x) \\ \phi^2(x) \\ \psi(x)\phi(x) \end{pmatrix}.$$  

Taking the Schrödinger equation (1) as starting point, it is easy to prove that $u(x)$ obeys the stochastic equation

$$\frac{du}{dx} = (A + \varepsilon \xi(x) B) u,$$

(11)

where $\xi(x)$ is the scaled potential (10) and the symbols $A$ and $B$ stand for the matrices

$$A = \begin{pmatrix} 0 & 0 & 2k \\ 0 & 0 & -2k \\ -k & k & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2k \\ -k & 0 & 0 \end{pmatrix}.$$  

We now need to replace the stochastic equation (11) with an ordinary differential equation for $\langle u(x) \rangle$; this can be done in the following way. For a fixed initial condition $u(0) = u_0$, the average solution of Eq. (11) can be formally expressed as

$$\langle u(x) \rangle = Z(x) u_0,$$

(12)

with

$$Z(x) = \exp(Ax) \left[ 1 + \langle T\exp \int_0^x \varepsilon M(x') dx' \rangle \right],$$

where $\exp$ indicates the T-ordered exponential and $M(x)$ is the random matrix

$$M(x) = \xi(x) \exp(-Ax) B \exp(Ax).$$

(13)

Differentiating both sides of Eq. (12) one obtains

$$\frac{d\langle u \rangle}{dx} = Z'(x) u_0;$$

(14)
on the other hand, by inverting the relation (12), one arrives at
\[ u_0 = Z^{-1}(x)\langle u(x) \rangle. \]

Substituting this result in the right-hand side (r.h.s.) of Eq. (14), one finally obtains the desired equation for the average of \( u(x) \)
\[ \frac{d\langle u(x) \rangle}{dx} = Z'(x)Z^{-1}(x)\langle u(x) \rangle = K(x)\langle u(x) \rangle. \] (15)

For weak disorder (\( \varepsilon \to 0 \)), the generator \( K(x) \) that appears in Eq. (15) can be represented by an expansion in powers of the perturbative parameter \( \varepsilon \)
\[ K(x) = \sum_{n=0}^{\infty} \varepsilon^n K_n(x). \] (16)

Van Kampen [11] and Terwiel [12] have shown that all partial generators \( K_n(x) \) can be expressed in terms of specific combinations of the moments of the random matrix (13) known as “ordered cumulants” and they have provided a systematic set of rules to construct them. Applying Van Kampen’s prescriptions to the present case, the zeroth-order generator turns out to have the simple form
\[ K_0 = A \]
which corresponds to neglecting altogether the effects of the weak random potential. As for the first-order term, it vanishes like all odd-order generators \( K_{2n+1}(x) \) because of the condition that the odd moments of \( U(x) \) be null. The second-order generator is less trivial and represents the first term where the effects of the random potential appear
\[ K_2(x) = \exp(Ax) \int_0^x dx_1 (M(x)M(x_1)) \exp(-Ax) . \] (17)

Straightforward passages allow to derive from this compact expression the matrix elements of \( K_2(x) \), whose explicit form is
\[
\begin{align*}
(K_2)_{11}(x) &= (K_2)_{12}(x) = (K_2)_{13}(x) = (K_2)_{23}(x) = (K_2)_{32}(x) = 0 \\
(K_2)_{21}(x) &= \frac{1}{k^2} \int_0^x \chi_2(y) [1 + \cos(2ky)] dy \\
(K_2)_{22}(x) &= (K_2)_{33}(x) = \frac{1}{k^2} \int_0^x \chi_2(y) [-1 + \cos(2ky)] dy \\
(K_2)_{31}(x) &= \frac{1}{k^2} \int_0^x \chi_2(y) \sin(2ky) dy
\end{align*}
\] (18)
where the symbol $\chi_2$ represents the two-point correlator of the potential $U(x)$ defined by Eq. (3). The fourth-order generator has a more complex form and depends both on the four- and the two-point correlators of the potential

$$K_4(x) = \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \exp(-Ax)$$
$$\times \left[ \langle M(x)M(x_1)M(x_2)M(x_3) \rangle - \langle M(x)M(x_1) \rangle \langle M(x_2)M(x_3) \rangle 
- \langle M(x)M(x_2) \rangle \langle M(x_1)M(x_3) \rangle 
- \langle M(x)M(x_3) \rangle \langle M(x_1)M(x_2) \rangle \right] \exp(Ax)$$

(19)

The integrand in Eq. (19) is a non-trivial example of ordered cumulant; notice that it differs from an ordinary cumulant because the matrices $M(x_1)$ and $M(x_2)$ do not commute for $x_1 \neq x_2$. The explicit form of the matrix elements of $K_4(x)$ can be worked out from Eq. (19); the expressions, however, are quite lengthy and do not add much to the physical understanding of the problem, so that we will omit them here. In theory, the Van Kampen scheme allows one to compute the terms of the expansion (16) to any desired order; in practice, however, the involved calculations become rather cumbersome beyond the second order. In this paper we will not consider terms of higher order than the fourth; within this approximation, we will be able to derive the standard second-order localisation length and the first correction to this result.

In principle, the generators $K_n(x)$ are functions of the spatial coordinate $x$ (and of the initial condition $u(0)$ chosen to solve Eq. (11)), but the dependence on these factors dies out as soon as the condition $x \gg l_c$ is fulfilled \[1\]. To determine the asymptotic behaviour of $\langle u(x) \rangle$, therefore, we need not bother with the space-dependent generators $K_n(x)$ but we have only to consider their constant asymptotic limits

$$K_n = \lim_{x \to \infty} K_n(x).$$

(20)

Notice that the existence of the asymptotic generators (20) is mathematically ensured by the sufficient condition that the random potential $U(x)$ possess a finite correlation length $l_c$. This implies that, when we relax this condition in order to study the case of long-range correlations, we will have to pay attention on whether the asymptotic generators (20) continue to be well-defined in the limit $l_c \to \infty$.

To sum up, on the basis of the previous considerations we can conclude that, in order to analyse with fourth-order accuracy the asymptotic behaviour
of \langle u(x) \rangle, we can legitimately replace in Eq. (15) the exact generator with its asymptotic and truncated form

\[ K(x) \simeq A + \varepsilon^2 K_x + \varepsilon^4 K_4. \]  

(21)

The eigenvalue of the matrix (21) with the largest real part determines the exponential growth rate of the vector \langle u(x) \rangle and, consequently, the inverse localisation length (6). The original problem is thus reduced to that of solving the secular equation of the matrix (21); determining the largest root of this equation, one obtains that the desired inverse localisation length has the form

\[ \lambda = \varepsilon^2 \lambda_2 + \varepsilon^4 \lambda_4 + o(\varepsilon^4) \]

and that the general expressions for the terms \lambda_2 and \lambda_4 are

\[ \lambda_2(k) = \frac{1}{4k^2} \int_0^\infty \chi_2(x) \cos(2kx) dx \]  

(22)

and

\[
\begin{align*}
\lambda_4(k) &= \frac{1}{8k^4} \int_0^\infty dx_1 \chi_2(x_1) \int_0^\infty dx_2 \chi_2(x_2) \sin(2kx_2) \\
&+ \frac{1}{4k^4} \lim_{x \to \infty} \int_{\omega(x)} dx_1 dx_2 dx_3 \left\{ [\chi_4(x_1, x_1 + x_2, x_1 + x_2 + x_3) - \chi_2(x_1)\chi_2(x_3)] \right\} \\
&- \chi_2(x_1)\chi_2(x_3) \left[ \cos(2kx_1) \cos(2kx_3) - \cos(2kx_1 + 2kx_2 + 2kx_3) \right] \\
&+ \chi_2(x_1 + x_2)\chi_2(x_2 + x_3) \left[ \cos(2kx_1 + 2kx_2) \cos(2kx_2 + 2kx_3) - \cos(2kx_1 + 2kx_2 + 2kx_3) \right] \\
&- \chi_2(x_1 + x_2 + x_3) \chi_2(x_2) \\
&\times \left[ \cos(2kx_1 + 2kx_2 + 2kx_3) \cos(2kx_2) - \cos(2kx_1 + 2kx_2) \right]
\end{align*}
\]  

(23)

where \( \omega(x) \) is the integration domain

\[ \omega(x) = \{(x_1, x_2, x_3) : 0 \leq x_1, 0 \leq x_2, 0 \leq x_3, x_1 + x_2 + x_3 \leq x \}. \]

Expression (22) reproduces the standard second-order formula for the Lyapunov exponent of model (I) (see, e.g., [1]); the result shows that the second-order inverse localisation length for an eigenstate with wavevector \( k \) is proportional to the cosine transform of the binary potential correlator taken at twice the value of the wave vector. Not surprisingly, the fourth-order Lyapunov exponent (23) has a more complicated form, involving both second and fourth moments of the potential. The increasing complexity of the mathematical formulae reflects the different degree of sophistication of the second- and fourth-order approximation schemes. From the physical point
of view, this difference is perhaps better understood if one visualises localisation as an interference effect (see, e.g., [13]). In the second-order (or Born) approximation, localisation is seen as an interference phenomenon generated by double scatterings of the electron at two points separated by a distance over which the random potential has non-negligible correlations: hence the connection between the second-order localisation length and the two-point correlator established by formula (22). The picture corresponding to the second-order approximation, therefore, includes an electronic wavefunction (with wavevector $k$) which is scattered first backward, generating a wave of intensity $O(\varepsilon)$ and then is rescattered in the forward direction, with intensity $O(\varepsilon^2)$. To go beyond the Born approximation, one must include the interference effects produced by multiple scatterings in the analysis of localisation. The fourth-order approximation, for instance, takes into account both quadruple scatterings and couples of double scatterings. In the first case, an electronic wavefunction is scattered back and forth four times, generating a forward-scattered wave of intensity $O(\varepsilon^4)$ which interferes constructively with the incoming wave; in the case of double scatterings, one considers the interference between two twice-scattered waves, each of intensity $O(\varepsilon^2)$. This is the reason why the term (23) involves the four-point potential correlator as well as various combinations of double products of two-points correlators.

To make the result for the fourth-order term (23) mathematically simpler and physically more transparent, it is opportune to express the fourth moment of the potential in terms of the second moments and of the fourth-order cumulant, writing

$$
\chi_4(x_1, x_2, x_3) = \chi_2(x_1)\chi_2(x_3 - x_2) + \chi_2(x_2)\chi_2(x_3 - x_1) + \chi_2(x_3)\chi_2(x_2 - x_1) + \Delta_4(x_1, x_2, x_3)
$$

(24)

where we have used the symbol $\Delta_4$ to represent the fourth cumulant. Eq. (24) represents the generalisation for non-Gaussian disorder of the familiar identity which allows one to express the fourth moment in terms of those of order two in the Gaussian case. In fact, in expression (24) the fourth moment is given as the sum of two parts: the Gaussian combination of the second moments and the cumulant, which represents the non-Gaussian contribution and vanishes in the Gaussian case. This interpretation of Eq. (24) makes quite natural to distinguish two components in the fourth-order Lyapunov exponent and to write

$$
\lambda_4(k) = \lambda_4^{(G)}(k) + \lambda_4^{(NG)}(k)
$$

(25)
where the first term in the r.h.s. of Eq. (25) represents the Gaussian part which depends only on the second moments of the random potential, whereas the second term is proportional to the cumulant and constitutes the specific manifestation of the non-Gaussian nature of disorder on the localisation properties of model (1). Inserting (24) in formula (23) one obtains that the explicit forms of the Gaussian and non-Gaussian terms are

\[ \lambda^{(G)}_{4}(k) = \frac{1}{8k^5} \int_{0}^{\infty} dx_1 \chi_2(x_1) \int_{0}^{\infty} dx_2 \chi_2(x_2) \sin(2kx_2) + \frac{1}{4k^4} \lim_{x \to \infty} \int_{\omega(x)} dx_1 dx_2 dx_3 \left\{ \chi_2(x_1 + x_2) \chi_2(x_2 + x_3) \right\} \]

and

\[ \lambda^{(NG)}_{4}(k) = \frac{1}{4k^4} \lim_{x \to \infty} \int_{\omega(x)} dx_1 dx_2 dx_3 \Delta_4(x_1, x_1 + x_2, x_1 + x_2 + x_3) \left[ \cos(2kx_1) \cos(2kx_3) - \cos(2kx_1 + 2kx_2) \cos(2kx_2 + 2kx_3) \right] \]

The Gaussian term (26) can be further simplified with some calculus work, provided that the binary correlation function decays quickly enough at infinity, which is ensured by condition (1). In this way one can lower the dimensionality of the integral (26), reducing it to a much more manageable one-dimensional integral. Introducing the functions

\[ \varphi_c(k, x) = \int_{0}^{\infty} \chi_2(y) \cos(2ky) dy, \quad \varphi_s(k, x) = \int_{0}^{\infty} \chi_2(y) \sin(2ky) dy, \]

the final expression for the Gaussian part of the fourth-order Lyapunov exponent can be written in the form

\[ \lambda^{(G)}_{4}(k) = \frac{1}{16k^5} \varphi_s(k, 0) \left[ \varphi_c(k, 0) + 4 \varphi_c(0, 0) \right] - \frac{1}{8k^4} \left[ \varphi_c(k, 0) + 2 \varphi_c(0, 0) \right] \int_{0}^{\infty} \chi_2(x) x \cos(2kx) dx + \frac{1}{8k^4} \varphi_s(k, 0) \int_{0}^{\infty} \chi_2(x) x \sin(2kx) dx + \frac{1}{4k^4} \int_{0}^{\infty} \left[ \varphi_c(0, x) \varphi_c(k, x) - \varphi_c^2(0, x) \cos(2kx) \right] dx \]

As for the non-Gaussian term (27), the dimensionality of the integral cannot be lowered without formulating some specific hypothesis on the form of
the cumulant $\Delta_4$. A change of variable, however, allows one to cast expression (27) in the form

$$
\lambda_4^{(NG)}(k) = \frac{1}{k^4} \int_0^\infty dx \int_0^x dy \int_x^\infty dz \Delta_4(y,x,z) \times \left[ \cos(2ky) \cos(2kz - 2kx) - \cos(2kz) \right]
$$

which is more convenient for our later use.

Formulae (29) and (30) constitute the central result of this section. They represent the general form of the Gaussian and non-Gaussian parts of the fourth-order Lyapunov exponent (25); by applying them to specific cases, we will be able to perform an accurate analysis of the localisation properties of model (1).

4 Disorder with short range correlations

In this section, we analyse the localisation properties of random potentials characterised by two different kinds of short-range correlations: a delta-correlated potential (white noise) and a potential with exponentially decaying correlations.

4.1 White noise

A white noise is defined as a random potential whose cumulants have the form

$$
\Delta_n(x_1, x_2, \ldots, x_{n-1}) = \sigma_n \delta(x_1) \delta(x_2) \ldots \delta(x_{n-1})
$$

(31)

(see, e.g., [4]). In Eq. (31) the $\sigma_n$ are constants which determine the strength of the cumulants, while $(x_1, x_2, \ldots, x_{n-1})$ represent relative coordinates of the points $(x, x + x_1, x + x_2, \ldots, x + x_{n-1})$.

Let us consider the Gaussian case first. In this case, all cumulants vanish except the second, which coincides with the correlation function

$$
\Delta_2(x) = \chi_2(x) = \sigma_2 \delta(x).
$$

Substituting the binary correlator (32) in Eq. (22), one recovers the standard result for the second-order inverse localisation length

$$
\lambda_2(k) = \frac{\sigma_2}{8k^2}.
$$

(33)
To determine the fourth-order correction to this result, the first step consists in observing that in the case of white noise the functions \((28)\) take the form
\[
\phi_c(x) = \begin{cases} 
0 & \text{for } x > 0 \\
\frac{\sigma^2}{2} & \text{for } x = 0 \\
\sigma^2 & \text{for } x < 0 
\end{cases}
\]
\[
\phi_s(x) = 0 \tag{34}
\]
Substituting (32) and (34) in expression (29), one arrives at the conclusion that the fourth-order Gaussian correction is zero
\[
\lambda^{(G)}_4(k) = 0.
\]
This peculiar result is a consequence of the delta-like nature of the correlation function. To understand this point, one needs to go back to Eq. (19) which shows that the partial generator \(K_4(x)\) is an integral of the fourth ordered cumulant of the random matrix \((13)\). As pointed out in Sec. 3, an ordered cumulant differs from an ordinary one because the matrices \(M(x)\) do not commute for different values of \(x\). In the special case when the correlation function has the form \((32)\), however, the integrand in Eq. (19) vanishes unless all integration variables are equal to \(x\). Consequently, the matrices in (19) commute and the ordered cumulant reduces to an ordinary one, which is zero because of the Gaussian assumption. Actually, this reasoning can be extended to all generators \(K_n(x)\) with \(n > 2\) and one is thus led to the conclusion that, in the special case of Gaussian white noise, the asymptotic limit of the generator \((16)\) is exactly equal to
\[
K = A + \varepsilon^2 K_2. \tag{35}
\]
The fact that \(K_4(x)\) vanishes does not automatically imply that the fourth-order Lyapunov exponent must be zero; however it is possible to prove that this is actually the case by solving the secular equation for the exact generator \((33)\). In this way one arrives at the result
\[
\lambda = \varepsilon^2 \frac{\sigma_2}{8k^2} - \varepsilon^6 \frac{\sigma_2^3}{128k^6} + o(\varepsilon^6)
\]
which shows that no term of order \(O(\varepsilon^4)\) arises from the second-order generator \(K_2\) in the case of Gaussian white noise.

If we drop the Gaussian assumption, the fourth-order generator no longer vanishes; however, assuming that the fourth cumulant has the form \((31)\), one
finds that the non-Gaussian term (30) is zero. We are thus led to the conclusion that, in the case of white noise, all eigenstates of Eq. (1) are localised, because the second-order Lyapunov exponent (33) is strictly positive for all values of the wavevector \( k \). However, the specific nature of the correlations prevents quadruple scatterings of electrons from producing any net destructive or constructive interference effect on the wavefunction, so that neither an enhancement nor a reduction of localisation show up in the fourth-order approximation.

4.2 Exponentially decaying correlations

We now turn our attention to the case of disorder with exponentially decaying correlations. Specifically, we assume that the two-point correlation function and the fourth-order cumulant have the form

\[
\chi_2(x) = \sigma_2 \exp(-\beta_2 |x|) \tag{36}
\]

and

\[
\Delta_4(x,y,z) = \sigma_4 \exp\left[-\beta_4 (|x| + |y| + |z|)\right]. \tag{37}
\]

In Eqs. (36) and (37), the constants \( \beta_2^{-1} \) and \( \beta_4^{-1} \) represent the range of the correlation function and of the fourth cumulant, respectively; note that the two parameters may be different.

Inserting the binary correlator (36) in Eq. (22), one obtains that the second-order Lyapunov exponent is

\[
\lambda_2(k) = \frac{\sigma_2}{4k^2} \frac{\beta_2}{4k^2 + \beta_2^2}. \tag{38}
\]

This result confirms the general rule that random potentials with short-range correlations produce localisation of all the electronic eigenstates (with the possible exception of a discrete set of extended states). Note that the Lyapunov exponent (38) tends to zero in the limit \( \beta_2 \to 0 \): physically, this means that the localisation of the electronic states becomes weaker and weaker as the range of the disorder correlations stretches over increasingly larger distances. In the opposite case, i.e. when the range of the correlation function tends to zero, one can recover from Eq. (38) the result (33) already derived for the white-noise case. More precisely, one obtains the white-noise expression if the limit \( \beta_2 \to \infty \) is taken while keeping constant the integral of the
second moment: \( \int_{-\infty}^{+\infty} \chi_2(x)dx = \tilde{\sigma}_2 \). This condition implies that the correlation strength \( \sigma_2 \) must scale as \( \sigma_2 = \tilde{\sigma}_2 \beta_2^2 / 2 \); when this relation is satisfied, the second-order Lyapunov exponent \( \tilde{\sigma}_2 \) reduces to the white-noise form \( \sigma_2 = \tilde{\sigma}_2 \beta_2^2 / 2 \) for \( \beta_2 \to \infty \).

The binary correlator \( \tilde{\sigma}_2 \) also determines the form of the Gaussian part of the fourth-order Lyapunov exponent. For \( x > 0 \) the functions \( \varphi_c(k, x) \) take the form

\[
\varphi_c(k, x) = \frac{\sigma_2}{4k^2 + \beta_2^2} [\beta_2 \cos(2kx) - 2k \sin(2kx)] \exp(-\beta_2 x)
\]

\[
\varphi_s(k, x) = \frac{\sigma_2}{4k^2 + \beta_2^2} [2k \cos(2kx) + \beta_2 \sin(2kx)] \exp(-\beta_2 x)
\]

correspondingly, Eq. (29) becomes

\[
\lambda_{4}^{(G)}(k) = \frac{\sigma_2^2 \beta_2}{4} \left\{ \frac{\beta_4^2}{\beta_2^2 + 22k^2 \beta_2^2 + 48k^4} \right\} \left( \beta_2^2 + 4k^2 \right)^{3/2}
\]

This result coincides with the expression obtained in Ref. [15] for the fourth-order term of the inverse localisation length \( \tilde{\eta} \); an important consequence of Eq. (39), therefore, is that the identity of the localisation lengths \( \tilde{\eta} \) and \( \eta \) is not restricted to the second order, but holds at least up to the fourth order when disorder is weak and Gaussian and the correlation function has the exponential form \( \tilde{\sigma}_2 \). This coincidence further corroborates the conclusions of subsection 2.2 about the substantial equivalence of the generalised Lyapunov exponents \( \tilde{\eta} \) and \( \eta \).

To complete the discussion of the fourth-order approximation, we have to compute the non-Gaussian term \( \tilde{\eta}_4 \) when the fourth cumulant has the form \( \beta_4 \). A straightforward calculation leads to

\[
\lambda_{4}^{(NG)}(k) = \frac{15\sigma_4}{8} \frac{\beta_4}{k^2(\beta_4^2 + 4k^2)(\beta_4^2 + 4k^2)(9\beta_4^2 + 4k^2)}
\]

An important conclusion which can be drawn from Eq. (40) is that the non-Gaussian character of the disorder produces an enhancement of localisation. This increase may be negligible in the present case, because the second-order Lyapunov exponent \( \tilde{\sigma}_2 \) never vanishes; that is not the case, however, when long-range correlations of the potential come into play, as we will discuss in the next section.
Considered together, Eqs. (39) and (40) show that the Gaussian and non-Gaussian parts of the fourth-order Lyapunov exponent are both decreasing functions of the wavevector $k$, with the same asymptotic behaviour for large values of $k$. This implies that the relative importance of the two terms is determined only by the relative magnitude of the cumulant strengths $\sigma_2$ and $\sigma_4$ and of the decay constants $\beta_2$ and $\beta_4$; when the corresponding parameters are of the same order of magnitude, neither component of the fourth-order Lyapunov exponent dominates the other. Comparing Eq. (38) with (39) and (40), one can also observe that for large values of $k$ the second-order Lyapunov exponent decays as $\lambda_2(k) \propto 1/k^4$, whereas the fourth-order correction vanishes as $\lambda_4 \propto 1/k^8$. This shows how the Born approximation becomes more and more accurate as the energy of the electrons is increased, in agreement with the general principle that electrons of higher energy are less sensitive to the details of the random potential.

Eqs. (39) and (40) also show that the both parts of the fourth-order Lyapunov exponent tend to vanish when the correlation lengths $\beta_2^{-1}$ and $\beta_4^{-1}$ tend simultaneously to infinity, as one can expect considering that the localisation effects become less pronounced as the range of disorder correlations is increased. Note, however, that the two localisation lengths need not be equal; in particular, it is possible to consider the case of a strongly correlated potential for which the correlation function is constant in space (i.e., $\beta_2 = 0$) and the spatial variation of disorder only shows up through the fourth cumulant (37) with $\beta_4 > 0$. In this case the terms (38) and (39) vanish and the inverse localisation length coincides with the non-Gaussian component (10).

Finally, we observe that the two parts (39) and (40) of the fourth-order Lyapunov exponent tend to zero in the white-noise limit, in agreement with the results of the previous subsection, as can be seen by taking the limits $\beta_2 \to \infty$ and $\beta_4 \to \infty$ (with the constraints $\int \chi_2(x) dx = \tilde{\sigma}_2$ and $\int \Delta_4(x_1, x_2, x_3) d^3x = \tilde{\sigma}_4$).

5 Disorder with long-range correlations

In this section, we focus our attention on the case of disorder with long-range correlations; more precisely, we consider a potential with correlation function of the form

$$\chi_2(x) = \sigma_2 \frac{\sin(2k_c x)}{x}. \quad (41)$$
The correlator \((41)\) is substantially different from those considered in the previous section, because it lacks any finite length scale beyond which it becomes negligible; as we will see, this feature strongly alters the localisation properties of the electronic wavefunctions. To enhance the physical understanding of the problem, we will analyse first the physical properties which descend from the specific form \((41)\) of the two-point correlation function, postponing the discussion of non-Gaussian effects until the end of this section.

As a consequence of the slow decay of the function \((41)\), great attention must be paid when applying the formalism developed in Sec. 3 to the present case. In fact, in the derivation of the general results of Sec. 3 we avoided any mathematical inconsistency by conveniently assuming the existence of a finite correlation length; one could therefore question the chevalier application of the general formulae \((22)\) and \((29)\) to a case in which no correlation length can be properly individuated and condition \((4)\) is not satisfied. To avoid these difficulties, we will study long-range correlators of the form \((41)\) as the limit case for \(\beta \to 0^+\) of the more general correlation function

\[
\chi_2(x) = \sigma_2 \frac{\sin(2k_c x)}{x} \exp(-\beta x). \quad (42)
\]

Since the correlator \((42)\) decays exponentially for \(\beta > 0\), the results of Sec. 3 can be safely used for this model; the localisation properties for the particular case \((41)\) can then be deduced by discussing the limit form of the Lyapunov exponent for \(\beta \to 0^+\).

Having defined our approach, we can proceed to derive the second-order Lyapunov exponent for the correlation functions \((42)\) and \((41)\). To simplify the mathematical form of the results and get rid of absolute value signs, in the rest of this section we will suppose that \(k > 0\). This is not a restrictive hypothesis, because the wavevector \(k\) enters model \((4)\) only via the energy \(E = k^2\) which is an even function of \(k\); consequently, the Lyapunov exponent is also even in \(k\).

Inserting \((42)\) in Eq. \((22)\), one obtains

\[
\lambda_2(\beta, k) = \frac{\sigma_2}{8k^2} \left\{ \arctan \left[ \frac{4\beta k_c}{\beta^2 + 4(k^2 - k_c^2)} \right] + \pi \theta \left( 4k_c^2 - 4k^2 - \beta^2 \right) \right\} \quad (43)
\]

where \(\theta(x)\) is the step function defined as

\[
\theta(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x \leq 0
\end{cases}
\]
In the limit $\beta \to 0^+$ expression (43) becomes

$$\lambda_2(k) = \begin{cases} 
\frac{\sigma_2 \pi}{8k^2} & \text{for } 0 < k < k_c \\
0 & \text{for } k_c < k 
\end{cases}$$

(44)

The behaviour of the second-order Lyapunov exponent (43) and (44) is represented in Fig. 1.

![Figure 1: Second-order Lyapunov exponent](image)

Note that the limit $\beta \to 0^+$ entails a sharp qualitative change in both the correlation function and the second-order Lyapunov exponent. For the correlation function, the limit $\beta \to 0^+$ brings about a transition from an exponential to a power-law decay; correspondingly, the strictly positive Lyapunov exponent (43) is converted in the inverse localisation length (44) which vanishes when the wavevector $k$ exceeds the critical value $k_c$. Comparing expressions (43) and (44), therefore, one can conclude that the passage from a disorder with finite-range correlations to a disorder with long-range correlations causes the emergence of a continuum of delocalised states, separated by a mobility edge from the localised ones [4].

This is obviously an important conclusion, because it shows that a kind of Anderson transition can occur in 1D disordered models; however, it is
not completely satisfactory, because it is valid only within the limits of the second-order approximation. More precisely, one cannot label as “extended” the states for \( k > k_c \) without a certain margin of ambiguity, because the fact that the second-order term \( (44) \) vanishes does not guarantee that the same be true for all higher-order terms of the Lyapunov exponent. It is therefore impossible to ascertain on the basis of Eq. \( (14) \) whether the electronic states for \( k > k_c \) are really extended or, rather, localised in a different form or on a much larger spatial scale than the states for \( k < k_c \). To shed light on this point, it is useful to determine the fourth-order Lyapunov exponent \( (29) \) in the case of long-range correlations of the form \( (41) \). Once more, we will first compute the fourth-order term \( (29) \) for the correlator \( (42) \) and then we will use this result to study the limit \( \beta \to 0^+ \).

When the two-point correlation function has the form \( (42) \), the functions \( (28) \) become

\[
\varphi_c(k, x) = \frac{\sigma^2}{2} \text{Im} \left\{ E_1 \left[ \beta x - i2(k + k_c)x \right] - E_1 \left[ \beta x - i2(k - k_c)x \right] \right\}
\]

\[
\varphi_s(k, x) = \frac{\sigma^2}{2} \text{Re} \left\{ -E_1 \left[ \beta x - i2(k + k_c)x \right] + E_1 \left[ \beta x - i2(k - k_c)x \right] \right\},
\]

where \( E_1(z) \) is the exponential integral defined in the complex plane by

\[
E_1(z) = \int_{z}^{\infty} \frac{e^{-t}}{t} \, dt \quad \text{with} \quad |\arg z| < \pi
\]

(see, e.g., [16]). Correspondingly, the Gaussian part of the fourth-order Lyap-
punov exponent takes the form

\[ \lambda_4^{(G)}(\beta, k) = \frac{\sigma_2^2}{4k^2} \left\{ \frac{1}{16} \left[ \frac{\beta}{\beta^2 + 4(k - k_c)^2} - \frac{\beta}{\beta^2 + 4(k + k_c)^2} \right] + \frac{1}{k} \left( \alpha_1(k) + \frac{\pi}{2} s_3(k) \right) \right\} \ln \frac{\beta^2 + 4(k + k_c)^2}{\beta^2 + 4(k - k_c)^2} \]

\[ - \frac{\beta}{8} \left[ \frac{1}{\beta^2 + 4k_c^2} + \frac{1}{\beta^2 + 4(k + k_c)^2} \right] \ln \frac{\beta^2 + (k + 2k_c)^2}{\beta^2 + k^2} \]

\[ - \frac{\beta}{8} \left[ \frac{1}{\beta^2 + 4k_c^2} + \frac{1}{\beta^2 + 4(k - k_c)^2} \right] \ln \frac{\beta^2 + (k - 2k_c)^2}{\beta^2 + k^2} \]

\[ - \frac{1}{2} \frac{\beta^2 + 4(k + k_c)^2}{k - k_c} \left[ \alpha_1(k) + \alpha_3(k) + \frac{\pi}{2} s_3(k) \right] \]

\[ + \frac{1}{2} \frac{\beta^2 + 4(k - k_c)^2}{k - k_c} \left[ \alpha_1(k) + \alpha_4(k) + \frac{\pi}{2} s_3(k) + \pi s_2(k) \right] \]

\[ + \frac{k_c}{\beta^2 + 4k_c^2} \left[ 2\alpha_1(k) - \alpha_2(k) + \pi s_3(k) - \frac{\pi}{2} s_1(k) \right] + F_\beta(k) \right\} \]

where the functions \( \alpha_i(k) \) are defined as

\[ \alpha_1(k) = \frac{1}{2} \arctan \frac{4\beta k_c}{\beta^2 + 4(k^2 - k_c^2)} \]

\[ \alpha_2(k) = \frac{1}{2} \arctan \frac{4\beta k_c}{\beta^2 + k^2 - 4k_c^2} \]

\[ \alpha_3(k) = \arctan \frac{2\beta k_c}{\beta^2 + k^2 + 2kk_c} \]

\[ \alpha_4(k) = \arctan \frac{2\beta k_c}{\beta^2 + k^2 - 2kk_c} \]

and the symbols \( s_i(k) \) represent the step-functions

\[ s_1(k) = \begin{cases} 1 & \text{if } \beta^2 + k^2 - 4k_c^2 < 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ s_2(k) = \begin{cases} 1 & \text{if } \beta^2 + k^2 - 2kk_c < 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ s_3(k) = \begin{cases} 1 & \text{if } \beta^2 + 4(k^2 - k_c^2) < 0 \\ 0 & \text{otherwise} \end{cases} \]
Finally, the function \( F_\beta(k) \) which appears in the r.h.s. of Eq. (43) is defined by the integral representation

\[
F_\beta(k) = \frac{1}{\sigma^2 k} \int_0^\infty \chi_2(x) \left[ \varphi_c(0,0) - \varphi_c(0,x) \right] \sin(2kx) dx.
\] (46)

For \( k > 2k_c \) the function (46) can be expressed through the series

\[
F_\beta(k) = \frac{1}{2k} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left\{ \frac{\beta^2 + 4k_c^2}{\beta^2 + 4(k + k_c)^2} \right\}^{\frac{n}{2}} \cos(n\delta_1) - \left\{ \frac{\beta^2 + 4k_c^2}{\beta^2 + 4(k - k_c)^2} \right\}^{\frac{n}{2}} \cos(n\delta_2) \sin(n\delta_0) \] (47)

where

\[
\delta_0 = \arctan \left( \frac{2k_c}{\beta} \right), \quad \delta_1 = \arctan \left( \frac{2(k + k_c)}{\beta} \right), \quad \delta_2 = \arctan \left( \frac{2(k - k_c)}{\beta} \right). \]

For \( k < \sqrt{k_c^2 + \beta^2 - k_c} \), the function \( F(k) \) can be represented in the alternative form

\[
F_\beta(k) = \frac{1}{4k} \arctan \left( \frac{2k_c}{\beta} \right) \ln \frac{\beta^2 + 4(k + k_c)^2}{\beta^2 + 4(k - k_c)^2} + G_\beta(k)
\]

where \( G_\beta(k) \) is the series

\[
G_\beta(k) = \frac{1}{2k} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} c_n \left\{ \frac{(k + k_c)^2}{\beta^2 + k_c^2} \right\}^n - \left\{ \frac{(k - k_c)^2}{\beta^2 + k_c^2} \right\}^n \] (48)

with coefficients

\[
c_n = \sum_{l=0}^{\infty} \frac{1}{2n + l} \left( \frac{\beta}{2\sqrt{\beta^2 + k_c^2}} \right)^l \sin \left( (2n + l) \arctan \left( \frac{k_c}{\beta} \right) \right). \]

Note that, when \( \beta \geq \sqrt{8k_c} \), the convergence regions of the series (47) and (48) overlap, so that there is no need to use the integral representation (46) to
compute the values of the function $F_\beta(k)$; the integral (46) must be evaluated only if $\beta < \sqrt{8k_c}$ and for those values of $k$ comprised in the interval $\sqrt{k_c^2 + \beta^2} - k_c < k < 2k_c$.

The behaviour of the fourth-order Lyapunov exponent (47) is represented for $k < 2k_c$ in Figs. 2 and 3, and for $k > 2k_c$ in Fig. 4. As shown by the

Figure 2: Fourth-order Lyapunov exponent for $k < 2k_c$ and $\beta \geq k_c$

figures, for large values of $\beta$ the fourth-order Lyapunov exponent is a smooth and decreasing function of $k$, assuming small but positive values everywhere (except, of course, than in the low-energy limit $k \to 0$, where the results are not valid because the weak disorder condition (5) is no longer satisfied). Physically this means that, when the correlation range of the disordered potential is short, the fourth-order Lyapunov exponent (13) is a negligible correction to the second-order term (13). The behaviour of $\lambda_4^{(\mathcal{G})}(\beta, k)$ becomes more complex for small values of $\beta$. First, the function looses its monotonic behaviour and even starts to assume negative values in an interval comprised within the range $0 < k < k_c$. Note that the fourth-order term can become negative as long as the second-order term is positive; physically, this means that the fourth-order term reduces the localisation effect. Finally, for $\beta \ll k_c$, the function $\lambda_4^{(\mathcal{G})}(\beta, k)$ develops a pronounced negative minimum in a left
Figure 3: Fourth-order Lyapunov exponent for $k < 2k_c$ and $\beta \leq k_c$

![Graph showing fourth-order Lyapunov exponent for $k < 2k_c$ and $\beta \leq k_c$.]

Figure 4: Fourth-order Lyapunov exponent for $k > 2k_c$

![Graph showing fourth-order Lyapunov exponent for $k > 2k_c$.]
neighbourhood of \( k_c \) and a sharp positive peak in a right neighbourhood \( k_c \). For values of \( k \) close to \( k_c \), therefore, the fourth-order term can represent a conspicuous correction to the leading term when \( \beta \ll k_c \); since in this limit the second-order term tends to assume the discontinuous form (44), one can conclude that the effect of the fourth-order correction is that of smoothing the Lyapunov exponent in the region \( k \simeq k_c \).

The expression of the fourth-order Lyapunov exponent (45) is quite complicated, but it simplifies significantly in the case of greatest interest, i.e., in the limit \( \beta \to 0^+ \), when it takes the form

\[
\lambda_4^{(C)}(k) = \begin{cases} 
\frac{\sigma_2^2}{4k^4} \left[ \frac{\pi}{32k} \ln \left( \frac{k + k_c}{k - k_c} \right)^2 \right] & \text{for } 0 < k < k_c \\
\frac{\pi k^2 + kk_c + k_c^2}{8k_c(k_c^2 - k^2)} + F_0(k) & \text{for } k_c < k < 2k_c \\
\frac{\pi}{8k_c(k_c^2 - k^2)} + F_0(k) & \text{for } 2k_c < k
\end{cases}
\]

(49)

The function (49) is plotted in Figs. 5 and 6. In Fig. 5 the graph of \( \lambda_4^{(C)}(\beta, k) \)
with $\beta = 10^{-2}k_c$ is also plotted for comparison. As can be seen from Fig. 5, the fourth-order Lyapunov exponent (19) displays a qualitative behaviour close to that of the corresponding term (15) for small values of $\beta$. There are two main differences, though: first, the fourth-order Lyapunov exponent (19) is strictly zero for $k > 2k_c$ and, second, it diverges for $k \to k_c$.

This divergence must be disregarded, however, because the limit $\beta \to 0^+$ cannot be legitimately taken in a neighbourhood of $k = k_c$. To understand this point we recall that, as pointed out in Sec. 3, the existence of the asymptotic generators (20) cannot be taken for granted when one drops the assumption that a finite correlation length exists. In the present case, a problem of this kind arises for the second-order asymptotic generator (17) which cannot be defined for $k = k_c$ in the limit $\beta \to 0^+$. In fact, as can be deduced from formula (18), the matrix element $(K_2)_{31}$ of the second-order asymptotic generator is proportional to the sine transform of the correlation function. The sine transform of the correlator (18) is a well-behaved function as long as $\beta > 0$; in the limit $\beta \to 0^+$, however, it diverges for $k = k_c$. As a consequence, the general formulae (29) and (30) can be applied to the study of disorder with long-range correlations (41) only for wavevectors $k$ which are not close to the critical value $k_c$. We can thus conclude that formula (49)
describes the correct behaviour of the fourth-order Lyapunov exponent everywhere, except that in a small neighbourhood of $k_c$. (Strictly speaking, the previous analysis also invalidates the second-order result (44) for $k \approx k_c$; this expression, however, descends from formula (22), which has a broader range of validity since it can be derived with alternative methods [1] that do not require the existence of the asymptotic generator $K_2$.)

Having established the range of validity of the expression (49), we can now comment the physical meaning of the result. The first remarkable aspect of the fourth-order Lyapunov exponent (49) is that it takes positive values in the interval $k_c < k < 2k_c$ where the second-order term (44) vanishes. This means that the inverse localisation length, which is a quadratic function of the potential for $k < k_c$, assumes a quartic form when the wavevector exceeds the critical value $k_c$. This feature clarifies the nature of the transition which occurs at $k = k_c$: when the threshold is crossed, the electronic states continue to be exponentially localised, but over spatial scales which are much larger than those typical of the states with $k < k_c$. The transition at $k_c$, therefore, does not bring a change from exponential localisation to complete delocalisation, but an increase of order $O(1/\varepsilon^2)$ in the spatial extension of the electronic states. Since we are considering the case of weak disorder, $\varepsilon \ll 1$, this increase can be huge: the bold use of terms like “mobility edge” and “delocalisation transition” adopted in Ref. [4] is thus fully justified.

A second relevant feature of the fourth-order Lyapunov exponent (49) is that it vanishes for $k > 2k_c$. This implies that, when the potential displays long-range correlations of the form (44), a second transition takes place at $k = 2k_c$, with the electrons becoming even more weakly localised. The spatial extension of the electronic states beyond this second threshold cannot be estimated within the fourth-order approximation, which allows the only conclusion that for $k > 2k_c$ the inverse localisation length must be an infinitesimal of order $O(\varepsilon^6)$ or higher.

The emergence of a second threshold at $k = k_c$ is an effect of the Gaussian nature of the potential that we have been considering up to now. The situation is different for non-Gaussian potentials, because in this case localisation is enhanced by the cumulant-generated term (30). The specific form of the non-Gaussian term (30) depends on that of the fourth cumulant of the disorder; to discuss a concrete example, we will consider the case of a random potential with correlation function and fourth cumulant of the forms (11) and (7), respectively. The fact that the correlation function has the form (11) implies that all moments of the potential decay slowly, so that
it is correct to speak of disorder with long-range correlations at all orders. On the other hand, the non-zero fourth cumulant \((37)\) makes the fourth moment of the potential different from what it would be in the Gaussian case. Since the cumulant \((37)\) decays exponentially, the fourth moment of the potential significantly differs from its Gaussian form only in a limited spatial range (which can actually be very small if \(\beta_4\) is large); yet this difference, no matter how small, is enough to induce the localisation of all electronic eigenstates with \(k > 2k_c\) with inverse localisation length equal to \((40)\). We can therefore conclude that, in the case of disorder with long-range correlations of the form \((41)\), non-Gaussian potentials generate a stronger localisation of the electrons that their Gaussian counterparts.

To conclude this section, we remark all the results obtained so far for the localisation of electrons in a disordered solid can be transposed to any other model described by Eq. \((1)\): in particular, the results of this section can be applied to the dynamics of the stochastic oscillator \((8)\). In the language appropriate to this model, the results of this section can be expressed by saying that, when the stochastic perturbation of the frequency displays long-range temporal correlations of the form \((14)\), the behaviour of the oscillator can be dramatically altered by letting the unperturbed frequency \(k\) cross some critical thresholds. Specifically, the oscillator is energetically unstable, with energy growing with the rate \((44)\), for frequencies lower than \(k_c\). As soon as the unperturbed frequency exceeds the value \(k_c\), however, the energy growth rate drops by a factor \(O(\varepsilon^2)\) so that the energetic instability of the oscillator manifests itself only on long time scales. Finally, for \(k > 2k_c\) the oscillator becomes even more stable, with a further reduction of the rate of energy growth which may be at least of order \(O(\varepsilon^2)\) or of a simple factor \(O(\varepsilon^0)\) depending on whether the noise is Gaussian or not.

6 Conclusions

In this paper we studied the localisation properties of 1D models with weak disorder, focusing our attention on the role played by spatial correlations of the random potential in shaping the structure of the electronic states. Using a perturbative approach based on a cumulant expansion technique, we were able to derive both the standard second-order expression for the inverse localisation length and a new result for the fourth-order correction. The knowledge of the fourth-order term of the Lyapunov exponent makes pos-
sible to investigate the delocalisation transition which takes place when the disorder exhibits specific long-range correlations. The analysis of Sec. 4 reveals that this transition consists in a sharp change of the scaling law for the inverse localisation length: when the electronic wavevector reaches a critical value, the Lyapunov exponent switches from a quadratic to a quartic dependence on the disorder strength $\varepsilon$ and, correspondingly, the spatial extension of the electronic states increases by a factor $O(1/\varepsilon^2)$. In the case of Gaussian potentials, a second critical value of the wavevector exists, beyond which the electronic states become even more delocalised, with Lyapunov exponent $\lambda = o(\varepsilon^4)$ for $\varepsilon \to 0$.

This additional transition is absent in the case of non-Gaussian potentials, which cannot be distinguished from their Gaussian counterparts in the second-order approximation but whose specific features emerge when the description is brought to the refined fourth-order level. Specifically, non-Gaussian potentials show their effect through the addition to the Gaussian Lyapunov exponent of an extra term proportional to the fourth cumulant of the random potential. This cumulant-generated term strengthens the localisation effect and maintains the quartic scaling of the Lyapunov exponent. Loosely speaking, the requirement that the disorder be Gaussian seems to act as a constraint which reduces the degree of disorder and therefore favours the delocalisation of the electronic states, in agreement with the general principle which links the degree of randomness with that of localisation of the electronic states.

From this point of view, this paper can be seen as a further step towards the construction of a hierarchy of disorders according to the strength of the localisation effects they produce in 1D models. In totally disordered systems, for which the potential has no spatial correlation, all electronic states are exponentially localised; when the degree, if not the strength, of the disorder is somewhat reduced by the presence of short-range correlations of the potential, a discrete set of extended states may appear; finally, when the potential exhibits specific long-range spatial correlations and thus the randomness of the model is further diminished, a continuum of states with large spatial extension emerges. This paper clarifies the features of the extended states which appear in the last case and shows that, in the class of disorders with long-range correlations, Gaussian potentials seem to be “less random” than non-Gaussian ones.
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