COLLECTIVE BEHAVIORS OF A WINFREE ENSEMBLE ON AN INFINITE CYLINDER

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Abstract. The Winfree model is the first phase model for synchronization and it exhibits diverse asymptotic patterns that cannot be observed in the Kuramoto model. In this paper, we propose a Winfree type model describing the aggregation of particles on the surface of an infinite cylinder. For a special case, our proposed model is in fact equivalent to the complex Winfree model. For the proposed model, we present a sufficient framework leading to the complete oscillator death and uniform $\ell^p$-stability in a large coupling regime. We also derive the corresponding kinetic model via uniform-in-time mean-field limit. In addition, we also provide several numerical simulations for the particle and compare them with analytical results.

1. Introduction. Collective dynamics of many-body systems often appears in biological, chemical and physical complex systems in our nature [1, 2, 5, 4, 8, 10, 9, 33, 35]. Despite of its ubiquity, its systematic research is a quite recent event only in a half century ago by two pioneers, Y. Kuramoto and A. Winfree [24, 35]. Recently, modeling and analysis for collective dynamics have received lots of attention due to recent applications in the control problem of multi-agent systems such as drones, driverless cars etc [10, 30]. In this paper, we are interested in a generalized Winfree model on the infinite cylinder. A Kuramoto type particle model on an infinite cylinder was proposed in [13], and its emergent dynamics has been investigated in the aforementioned work. To motivate our work, we begin with a brief introduction on the Winfree model. In 1967, Arthur Winfree proposed a phase-coupled model
for synchronization which can exhibit a synchronous behaviors of phase-coupled oscillators.

Let \( \theta_j = \theta_j(t) \) be the phase of the \( j \)-th Winfree oscillator, and let \( I = I(\theta) \) and \( S = S(\theta) \) be the influence and sensitivity functions, respectively. Then, the Winfree model reads as follows.

\[
\dot{\theta}_j = \nu_j + \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} S(\theta_j) I(\theta_k), \quad j = 1, \ldots, N,
\] (1)

where \( \nu_j \) is the natural frequency of the \( j \)-th oscillator and \( \kappa_{jk} \) is coupling strength between \( j \)-th and \( k \)-th oscillators. For definiteness of presentation, one can take the following explicit pair for \( I \) and \( S \) which has been used in \([3, 15, 16, 31, 32]\):

\[
I(\theta) = 1 + \cos \theta, \quad S(\theta) = -\sin \theta.
\] (2)

Compared to the extensive studies \([1, 6, 7, 10, 11, 20]\) for the Kuramoto model, the Winfree model did not receive enough attention in the past. One of reason for that is the lack of conserved quantities other than the number of oscillators. So far, we do not have a nontrivial conserved quantity for (1) except the total number of oscillators. However, the lack of conserved quantity yields diverse asymptotic patterns which can not be observed in the Kuramoto model, e.g., existence of oscillator death or chimera state. Recently, in \([12, 14, 15, 16, 18, 19, 22, 28]\), emergent asymptotic patterns for the Winfree model have been extensively studied by providing explicit sufficient frameworks leading to complete / partial oscillator deaths and phase-locking via dynamical systems theory. In this work, we are interested in the following questions:

- (Q1): Can we lift particle model (1) onto the the infinite cylinder \( \mathbb{T}^N \times \mathbb{R}^N \)?
- (Q2): If then, can we identify some sufficient frameworks leading to the collective dynamics?

As mentioned before, the above questions were already addressed in \([13]\) for the Kuramoto model. The authors believe that this work provides the first step forward higher dimensional Winfree models which are not known at present.

The main results of this paper are three-fold. First, we propose a Winfree type model on the infinite cylinder \( \mathbb{T}^N \times \mathbb{R}^N \) by lifting the Winfree model. Let \( \{(x_j, y_j)\}_{j=1}^{N} \) be a Winfree ensemble on the infinite cylinder. Then, our proposed particle model describes the temporal evolution of the state \( \{(x_j, y_j)\}_{j=1}^{N} \):

\[
\begin{align*}
\dot{x}_j &= \nu_j - \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} \left[ \delta \sin x_j \cosh y_j 
+ (1 - \gamma) \sin(x_j + x_k) \cosh(y_j + y_k) + \gamma \sin(x_j - x_k) \cosh(y_j - y_k) \right], \\
\dot{y}_j &= \omega_j - \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} \left[ \delta \cos x_j \sinh y_j 
+ (1 - \gamma) \cos(x_j + x_k) \sinh(y_j + y_k) + \gamma \cos(x_j - x_k) \sinh(y_j - y_k) \right],
\end{align*}
\] (3)

where \( \Omega_j := (\nu_j, \omega_j) \) is the natural velocity of the \( j \)-th particle which is assumed to be random vector extracted from laws \( g_1(\nu) \) and \( g_2(\omega) \), respectively, and a \( N \times N \) matrix \( K = (\kappa_{jk}) \) represents a coupling matrix between particles. For the case
formulated in terms of system parameters and initial configuration: existence of a unique equilibrium. More precisely, we propose a sufficient framework by extending the real-valued phases into the complex-valued phases presented as in [26].

Second, we present a sufficient framework for the complete oscillator death and existence of a unique equilibrium. More precisely, we propose a sufficient framework formulated in terms of system parameters and initial configuration:

\[ 0 < \delta \leq 1, \ 0 \leq \gamma \leq 1, \ \alpha \in \left(0, \frac{\pi}{4}\right), \ \beta > 0, \ \kappa \gg 1, \ (X^{in}, Y^{in}) \in \mathcal{R}(\alpha) \times \mathcal{R}(\beta), \ (4) \]

where the set \( \mathcal{R}(\alpha) \) is defined by the following relation:

\[ \mathcal{R}(\alpha) := \{ X = (x_1, \cdots, y_N) : x_j \in (-\alpha, \alpha), \ j = 1, \cdots, N \}. \]

Then, under the proposed frameworks in terms of parameters in (4), one has the uniform boundedness and complete oscillator death (see Proposition 1 and Proposition 2):

\[ (X(t), Y(t)) \in \overline{\mathcal{R}(\alpha)} \times \overline{\mathcal{R}(\beta)}, \]

\[ \| \dot{X}(t) \|_p + \| \dot{Y}(t) \|_p \leq e^{-\rho t \Lambda_1(\alpha, \beta)} \left( \| X^{in} \|_p + \| Y^{in} \|_p \right), \]

where \( t \geq 0, \ \Lambda_1(\alpha, \beta) > 0, \ p \in \mathbb{N}, \ \| \cdot \|_p \) is the \( \ell_p \)-norm and

\[ \dot{X} := (\dot{x}_1, \cdots, \dot{x}_N), \ \dot{Y} := (\dot{y}_1, \cdots, \dot{y}_N). \]

In addition, above inequality implies the existence of a unique equilibrium: there exists a unique equilibrium \((X^\infty, Y^\infty)\) such that

\[ \lim_{t \to \infty} \left( \| X(t) - X^\infty \|_p + \| Y(t) - Y^\infty \|_p \right) = 0, \]

(see Theorem 3.4).

Third, we provide a rigorous derivation of the kinetic equation from particle model (3) with a uniform coupling strength (i.e., \( k_{jk} = k, 1 \leq j, k \leq N \)) in the mean-field limit \( N \to \infty \). For simplicity, we set

\[ z = (x, y) \in \mathbb{T} \times \mathbb{R}, \ \Omega = (\nu, \omega) \in \mathbb{R}^2, \ dz = dx dv, \ d\Omega = dv d\omega, \]

and let \( f = f(t, z, \Omega) \) be the one-oscillator probability density function at phase position \((x, y)\), at time \( t \). Then, the standard BBGKY hierarchy yields

\[
\begin{aligned}
\partial_t f + \nabla_{(x,y)} \cdot \left( (L_1[f], L_2[f]) f \right) &= 0, \quad (x, y, \nu, \omega) \in \mathbb{T} \times \mathbb{R}^3, \quad t > 0, \\
L_1[f] := &\nu - \kappa \delta \sin x \cosh y \\
L_2[f] := &\omega - \kappa \delta \cos x \sinh y
\end{aligned}
\]

\[
\begin{aligned}
-\kappa \int_{\mathbb{T} \times \mathbb{R}^3} \left[ (1 - \gamma) \sin(x + x_*) \cosh(y + y_*) + \gamma \sin(x - x_*) \cosh(y - y_*) \right] f_* dz_* d\Omega_*, \\
-\kappa \int_{\mathbb{T} \times \mathbb{R}^3} \left[ (1 - \gamma) \cos(x + x_*) \sinh(y + y_*) + \gamma \cos(x - x_*) \sinh(y - y_*) \right] f_* dz_* d\Omega_*,
\end{aligned}
\]

(5)
subject to suitable periodicity and constraints:

$$f(t, x, y, \nu, \omega) = f(t, x + 2\pi, y, \nu, \omega) \geq 0, \quad (t, x, y, \nu, \omega) \in \mathbb{T} \times \mathbb{R}^3,$$

$$\int_{\mathbb{T} \times \mathbb{R}^2} f(t, x, y, \nu, \omega) dx dy d\omega = g_1(\nu), \quad \text{for all } t, \nu,$$

$$\int_{\mathbb{T} \times \mathbb{R}^2} f(t, x, y, \nu, \omega) dx dy d\nu = g_2(\omega), \quad \text{for all } t, \omega.$$

Associated with the solution $$\{(z_j, \Omega_j)\}_{j=1}^N$$ to particle system (3), one has an empirical measure $$\mu^N_t$$:

$$\mu^N_t(dz, d\Omega) := \frac{1}{N} \sum_{j=1}^N \delta(z - z_j(t)) \otimes \delta(\Omega - \Omega_j(t)),$$

and the limiting measure $$\mu_t$$:

$$\mu_t(dz, d\Omega) = f(t, z, \Omega) dz d\Omega.$$

We derive a uniform stability estimate of the particle system (see Theorem 3.5) and use it with contradiction argument to derive a uniform-in-time mean-field limit:

$$\lim_{N \to \infty} \sup_{t \in [0, \infty)} W_p(\mu^N_t, \mu_t) = 0,$$

where $$W_p$$ is $$p$$-Wasserstein distance (see Theorem 4.3). For the derived uniform mean-field limit (5), we also provide an asymptotic behavior of measure-valued solution $$\mu_t$$.

The rest of this paper is organized as follows. In Section 2, we briefly recall the Winfree model and propose our new particle model on the infinite cylinder which can be viewed as a generalized Winfree model. In Section 3, we provide sufficient frameworks for uniform boundedness, complete oscillator death, a unique equilibrium and uniform $$\ell_p$$-stability. In Section 4, we study the uniform mean-field limit from particle system (3) to kinetic equation (5) in $$N \to \infty$$. In Section 5, we present numerical simulations for the particle model, and compare them with analytical results. In Section 6, we briefly summarize our main results and discuss some remaining issues to be covered in future work.

2. Preliminaries. In this section, we briefly review the Winfree model and introduce a particle model on the infinite cylinder $$\mathbb{T} \times \mathbb{R}$$ by lifting the Winfree model on the circle $$\mathbb{T}$$.

2.1. The Winfree model. This subsection provides an overview of known results for the Winfree model. First, we briefly explain why the Winfree model can enforce a phase synchrony. An ensemble of pulse-coupled oscillators firing at $$\theta = 0$$ can be realized as rotators on the unit circle. When field oscillators fire at $$\theta = 0$$, the test oscillator tries to be entrained to the field particles who have just fired. To be precise, let $$\theta_j = \theta_j(t)$$ be the phase of the $$j$$-th Winfree oscillator and its velocity be governed by the Winfree model combined with (1) - (2):

$$\dot{\theta}_j = \nu_j + \frac{1}{N} \sum_{k=1}^N \kappa_{jk}(-\sin \theta_j)(1 + \cos \theta_k), \quad j = 1, \cdots, N.$$  \hfill (6)

When we consider the situation in which a test oscillator lies on the first quadrant, say $$\theta_j \in (0, \frac{\pi}{2})$$ and neighboring field particle is firing at the origin $$\theta_k = 0$$. In this
case, for synchrony to occur, the test particle should decelerate its phase velocity and be attracted to the origin \( \theta = 0 \). This is achieved by the fact \((- \sin \theta_j)/(1 + \cos \theta_j) < 0\) in the nonlinear coupling term. Thus, the nonlinear coupling term in (6) is built on such observation. A general modeling of \( S \) and \( I \) based on the explicit example (6) was introduced and studied in [18, 19, 22]. We will not deal with it in detail here, but will briefly review the emergent dynamics of (6) which studied therein.

For this, we define the rotation number of the \( j \)-th oscillator \( \theta_j \) as
\[
\rho_j := \lim_{t \to \infty} \frac{\theta_j(t)}{t}, \quad \text{if R.H.S. exists},
\]
and recall concepts for the oscillator death and phase-locking as follows.

**Definition 2.1.** [18, 19] Let \( \Theta := (\theta_1, \cdots, \theta_N) \) be a Winfree ensemble whose dynamics is governed by (6).

1. \( \Theta \) tends to “complete oscillator death (COD)” state, if the rotation numbers of all oscillators are zero:
\[
|\{i : \rho_i = 0\}| = N,
\]
where \( |A| \) is the cardinality of set \( A \).
2. \( \Theta \) tends to “partial oscillator death (POD)” state, if at least two rotation numbers are zero:
\[
2 \leq |\{i : \rho_i = 0\}| < N.
\]
3. \( \Theta \) tends to “complete phase-locking” state, if the rotation numbers of all oscillators are the same with the nonzero value \( \rho \):
\[
|\{i : \rho_i = \rho\}| = N.
\]
4. \( \Theta \) tends to “partial phase-locking” state, if at least two rotation numbers take the same nonzero value:
\[
2 \leq |\{i : \rho_i = \rho\}| < N.
\]

For homogeneous Winfree oscillators (i.e., \( \nu_j = \nu, 1 \leq j \leq N \)), if we can observe the mixed hybrid state consisting of phase-locking oscillators and running oscillators, then we say that \( \Theta \) exhibits the “chimera” state. For \( 1 \leq n \leq N \), we set
\[
C_n := \{1, \cdots, n\}.
\]

In the following theorem, we quote the result on the emergence of the COD and POD states without a proof.

**Theorem 2.2.** (Formation of COD and POD states [19, 22]) Suppose that system parameters and initial data satisfy
\[
1 \leq n \leq N, \quad 0 < \alpha < \pi, \quad \theta_j^{\alpha} \in (-\alpha, \alpha), \quad \forall j \in C_n,
\]
and let \( \Theta \) be a solution to (6) with the initial data. Then, there exists a coupling strength \( \kappa_c(\nu_1, \cdots, \nu_n, \alpha, n) > 0 \) such that if \( \kappa > \kappa_c \), we have
\[
\rho_j = 0, \quad \forall j \in C_n.
\]

**Remark 1.** In [18, 19], the existence of some chimera states have been studied on a locally coupled networks. Recently, in [14], the authors present a rigorous study on the existence of bump states (chimera-like states) in homogeneous ensemble under heterogeneous frustration effects. In [15, 16], the robustness of the COD state has been investigated in general network under the time-delayed interactions, and the
interplay of adaptive coupling and random effect, respectively. The existence of periodic locked orbit and complete phase-locking were studied in \cite{28, 29}.

2.2. The Winfree model on $T \times \mathbb{R}$. In this subsection, we introduce the Winfree model on the infinite cylinder. We assume that the phase of $j$-th particle $z_j = z_j(t)$ moving on the infinite cylinder is governed by the Winfree model:

$$\dot{z}_j = \Omega_j - \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} \sin z_j(1 + \cos z_k), \quad j = 1, \ldots, N, \quad (7)$$

where $\Omega_j = (\nu_j, \omega_j)$ is the natural velocity of $j$-th particle. Note that the infinite cylinder can be identified as $T \times \mathbb{R} \subset \mathbb{C}$. We set

$$z_j(t) = x_j(t) + iy_j(t), \quad \Omega_j := \nu_j + i\omega_j \quad j = 1, \ldots, N. \quad (8)$$

We use the trigonometric formula:

$$\sin z_j = \sin(x_j + iy_j) = \sin x_j \cosh y_j + i \cos x_j \sinh y_j,$$

$$\cos z_k = \cos(x_k + iy_k) = \cos x_k \cosh y_k - i \sin x_k \sinh y_k,$$

to get

$$\sin z_j(1 + \cos z_k) = \sin x_j \cosh y_j + \sin x_j \cosh y_j \cos x_k \cosh y_k + \cos x_j \sin x_k \sinh y_j \sinh y_k$$

$$+ i(\cos x_j \sinh y_k + \cos x_j \sinh y_j \cos x_k \cosh y_k - \sin x_j \cosh y_j \sin x_k \sinh y_k). \quad (9)$$

In (7), we use the relations (8) and (9) to derive the equations for $x_j$ and $y_j$:

$$\begin{align*}
\dot{x}_j &= \nu_j - \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} \left( \sin x_j \cosh y_j(1 + \cos x_k \cosh y_k) + \cos x_j \sin x_k \sinh y_j \sinh y_k \right), \\
\dot{y}_j &= w_j - \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} \left( \cos x_j \sinh y_j(1 + \cos x_k \cosh y_k) - \sin x_j \cosh y_j \sin x_k \sinh y_k \right),
\end{align*}$$

or equivalently,

$$\begin{align*}
\dot{x}_j &= \nu_j - \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} \left[ \sin x_j \cosh y_j \\
&\quad + \frac{1}{2} \sin(x_j + x_k) \cosh(y_j + y_k) + \frac{1}{2} \sin(x_j - x_k) \cosh(y_j - y_k) \right], \\
\dot{y}_j &= w_j - \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} \left[ \cos x_j \sinh y_j \\
&\quad + \frac{1}{2} \cos(x_j + x_k) \sinh(y_j + y_k) + \frac{1}{2} \cos(x_j - x_k) \sinh(y_j - y_k) \right]. \quad (10)
\end{align*}$$

Based on above model (10), we propose the following generalized model:

$$\begin{align*}
\dot{x}_j &= \nu_j - \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} \left[ \delta \sin x_j \cosh y_j \\
&\quad + (1 - \gamma) \sin(x_j + x_k) \cosh(y_j + y_k) + \gamma \sin(x_j - x_k) \cosh(y_j - y_k) \right], \\
\dot{y}_j &= w_j - \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} \left[ \delta \cos x_j \sinh y_j \\
&\quad + (1 - \gamma) \cos(x_j + x_k) \sinh(y_j + y_k) + \gamma \cos(x_j - x_k) \sinh(y_j - y_k) \right]. \quad (11)
\end{align*}$$
Here, $\kappa_{jk}$ is the nonnegative coupling strength between the $j$-th and $k$-th particles and $0 \leq \delta, \gamma \leq 1$ are system parameters.

Note that the generalized model \((11)\) reduces to the previous well-known particle models as below.

**Example A** (Winfree model): Consider the case $\delta = 1$, $\gamma = \frac{1}{2}$, $\kappa_{jk} = \kappa$, $y_j^{in} = 0$, $\omega_j = 0$, $j, k = 1, \cdots, N$.

Then, it is easy to see that $y_j(t) = 0$, $t \geq 0$, $j = 1, \cdots, N$.

Thus, system \((11)\) becomes

$$\dot{x}_j = \nu_j - \frac{\kappa}{N} \sum_{k=1}^{N} \sin x_j (1 + \cos x_k), \quad x_j \in \mathbb{T}, \quad t > 0, \quad j = 1, \cdots, N.$$  

**Example B** (The complex Kuramoto model): Consider the case $\delta = 0$, $\gamma = 1$, $\kappa_{jk} = \kappa$, $j, k = 1, \cdots, N$.

Thus, system \((11)\) becomes

$$\begin{cases} 
\dot{x}_j = \nu_j - \frac{\kappa}{N} \sum_{k=1}^{N} \sin(x_k - x_j) \cosh(y_k - y_j), & t > 0, \quad j = 1, \cdots, N, \\
\dot{y}_j = w_j - \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} \cos(x_k - x_j) \sinh(y_k - y_j). 
\end{cases}$$

This model was first suggested in [13] to generalize the Kuramoto model and authors studied on the collective dynamic of the model on the infinite cylinder. This shows one motivation of our model; we want to find a generalization covering a complex variant of the Winfree model at the same time as the Kuramoto model.

3. **Emergent dynamics of the particle model.** In this section, we study emergent behaviors of the generalized Winfree model proposed in previous section:

$$\begin{cases} 
\dot{x}_j = \nu_j - \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} \left[ \delta \sin x_j \cosh y_j + (1 - \gamma) \sin(x_j + x_k) \cosh(y_j + y_k) + \gamma \sin(x_j - x_k) \cosh(y_j - y_k) \right], \\
\dot{y}_j = w_j - \frac{1}{N} \sum_{k=1}^{N} \kappa_{jk} \left[ \delta \cos x_j \sinh y_j + (1 - \gamma) \cos(x_j + x_k) \sinh(y_j + y_k) + \gamma \cos(x_j - x_k) \sinh(y_j - y_k) \right], \\
(x_j, y_j)|_{t=0^+} = (x_j^{in}, y_j^{in}). 
\end{cases}$$  \((12)\)

For $X = (x_1, \cdots, x_N)$, we set $\ell_p$-norm $\| \cdot \|_p$:

$$\|X\|_p := \left( \sum_{k=1}^{N} |x_k|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \|X\|_\infty := \max_{1 \leq k \leq N} |x_k|.$$

For the $j$-th Winfree particle $(x_j, y_j)$, we define the oscillator death as

$$\lim_{t \to \infty} \frac{x_j(t)}{t} = 0 \quad \text{and} \quad y_j(t) \in (-\beta, \beta), \quad t \geq T,$$
Lemma 3.1. Suppose that system parameters and initial data satisfy set for generalized Winfree flow (12).

The following lemmas are presented several estimates, we set positively invariant set and an attractor along generalized Winfree flow (12).

Existence of attractors.

Proof. We will show that the interval $I := [-\alpha, \alpha]$ is a positively invariant set of $X(t)$. It suffices to show that once the flow hits the boundary points $-\alpha$ or $\alpha$ at some finite time $t^*$, then it will flow into the interval $I$ again so that the flow is confined in the closed interval $I$ afterwards. Suppose that there is $\ell$ such that $x_\ell(t^*) = -\alpha$ and $x_j(t^*) \in [-\alpha, \alpha], \quad 1 \leq \ell \neq j \leq N.$
By direct calculation, we observe that the right-hand limit of $\dot{x}_\ell$ at time $t^*$ satisfies
\[
\dot{x}_\ell(t^*) = \nu_\ell + d_\ell \delta \sin \alpha \cosh y_\ell \\
- \frac{1}{N} \sum_{k=1}^{N} \kappa_{\ell k} \left( (1 - \gamma) \sin(-\alpha + x_k) \cosh(y_\ell + y_k) + \gamma \sin(-\alpha - x_k) \cosh(y_\ell - y_k) \right) \\
\geq -\nu^\infty + d_\ell \delta \sin \alpha > 0.
\]
(13)

On the other hand, we suppose that there exists $\ell$ such that
\[
x_\ell(t^*) = \alpha \quad \text{and} \quad x_j(t^*) \in [-\alpha, \alpha], \quad 1 \leq \ell \neq j \leq N.
\]
Similarly, one has
\[
\dot{x}_\ell(t^*) = \nu_\ell - d_\ell \delta \sin \alpha \cosh y_\ell \\
- \frac{1}{N} \sum_{k=1}^{N} \kappa_{\ell k} \left( (1 - \gamma) \sin(\alpha + x_k) \cosh(y_\ell + y_k) + \gamma \sin(\alpha - x_k) \cosh(y_\ell - y_k) \right) \\
\leq \nu^\infty - d_\ell \delta \sin \alpha < 0.
\]
(14)

Finally, we combine (13) and (14) to conclude that $I$ is a positively invariant set.

**Lemma 3.2.** Suppose that system parameters and initial data satisfy
\[
0 < \delta \leq 1, \quad 0 \leq \gamma \leq 1, \quad \alpha \in \left(0, \frac{\pi}{4}\right), \quad \beta > 0, \\
d_{\min} > \max \left\{ \frac{\nu^\infty}{\delta \sin \alpha}, \frac{\omega^\infty}{\delta \cos \alpha \sinh \beta} \right\}, \quad (X^{in}, Y^{in}) \in \overline{R(\alpha)} \times \overline{R(\beta)},
\]
and let $(X(t), Y(t))$ be a corresponding smooth global solution of (12). Then, we have
\[
(X(t), Y(t)) \in \overline{R(\alpha)} \times \overline{R(\beta)}, \quad t > 0.
\]

**Proof.** We will show that the interval $I := [-\beta, \beta]$ is a positively invariant set of $Y(t)$ following the same argument used in Lemma 3.1. Suppose that there are index $\ell$ and time $t^*$ such that
\[
y_\ell(t^*) = -\beta \quad \text{and} \quad y_j(t^*) \in [-\beta, \beta], \quad 1 \leq \ell \neq j \leq N.
\]
Since
\[
\cos(x_\ell \pm x_k) \geq \cos 2\alpha > 0, \quad \sinh(-\beta \pm y_k) \geq 0, \quad \text{for all} \quad 1 \leq \ell, j \leq N,
\]
we have
\[
\dot{y}_\ell(t^*) = \omega_\ell + d_\ell \delta \cos x_\ell \sinh \beta \\
- \frac{1}{N} \sum_{k=1}^{N} \kappa_{\ell k} \left( (1 - \gamma) \cos(x_\ell + x_k) \sinh(-\beta + y_k) + \gamma \cos(x_\ell - x_k) \sinh(-\beta - y_k) \right) \\
\geq -\omega^\infty + d_\ell \delta \cos \alpha \sinh \beta > 0.
\]
(15)
Similarly, we suppose that there are index $\ell$ and time $t_0$ such that
\[ y_\ell(t_0) = \beta \quad \text{and} \quad y_j(t_0) \in [-\beta, \beta], \quad 1 \leq \ell \neq j \leq N. \]
Then, we have
\[
y_\ell'(t^*_\ast) = \omega_\ell - d_\ell \delta \cos x_\ell \sinh \beta - \frac{1}{N} \sum_{k=1}^{N} \kappa_{\ell k} (1 - \gamma) \cos(x_\ell + x_k) \sinh(\beta + y_k) + \gamma \cos(x_\ell - x_k) \sinh(\beta - y_k) \\
\leq \omega_{\infty} - d_\ell \delta \cos \alpha \sinh \beta < 0.
\]
Finally we combine (15) and (16) to conclude that $I$ is a positively invariant set.

Now, we are ready to show existence of an attractor for generalized Winfree flow (12).

**Proposition 1.** (Existence of an attractor) Suppose that system parameters and initial data satisfy
\[ 0 < \delta \leq 1, \quad 0 \leq \gamma \leq 1, \quad \alpha \in \left(0, \frac{\pi}{4}\right), \quad \beta^* > 0, \quad \beta \in (0, \beta^*), \]
\[ d_{\min} > \max \left\{ \frac{\nu_{\infty}}{\delta \sin \alpha}, \frac{\omega_{\infty}}{\delta \cos \alpha \sinh \beta}, \frac{\sqrt{2} \omega_{\infty}}{\delta \sinh \beta^*} \right\}, \quad (X^{in}, Y^{in}) \in \mathcal{R}\left(\frac{\pi}{4}\right) \times \mathcal{R}(\beta^*). \]
Then, the Cauchy problem (12) has a global unique solution satisfying
\[ (X(t), Y(t)) \in \mathcal{R}(\alpha) \times \mathcal{R}(\beta), \quad t \geq t^*_\ast, \]
for some time $t^*_\ast$.

**Proof.** Since
\[ d_{\min} > \max \left\{ \frac{\sqrt{2} \nu_{\infty}}{\delta}, \frac{\sqrt{2} \omega_{\infty}}{\delta \sinh \beta^*} \right\}, \]
Lemma 3.2 and the standard Cauchy-Lipschitz theory imply the existence of a unique global solution such that
\[ (X(t), Y(t)) \in \mathcal{R}\left(\frac{\pi}{4}\right) \times \mathcal{R}(\beta^*) \quad \text{for all} \ t \geq 0. \]
For a given $t > 0$, we define
\[ M = M(t) := \arg\max_{1 \leq j \leq N} |x_j(t)|, \quad \bar{M} = \bar{M}(t) := \arg\max_{1 \leq j \leq N} |y_j(t)|. \]
Then extremal trajectories $x_M(t)$ and $x_{\bar{M}}(t)$ are Lipschitz continuous and piecewise differentiable. We will show that $x_M$ and $y_M$ enter into $[-\alpha, \alpha]$ and $[-\beta, \beta]$ in some finite-time $t^*_\ast$ through the next two steps.

- **Step A** (Estimates for $X(t)$): If $x_M(0) \in [-\alpha, \alpha]$, then Lemma 3.2 yields
\[ X(t) \in \mathcal{R}(\alpha) \quad \text{for all} \ t \geq 0. \]
Suppose that
\[ |x_M(0)| \in \left[\alpha, \frac{\pi}{4}\right]. \]
Then one has
\[
\frac{d}{dt}|x_M(t)| = \text{sgn}(x_M)\nu_M - d_M\delta \sin|x_M| \cosh y_M \\
- \frac{1}{N} \sum_{k=1}^{N} \kappa_{Mk}(1 - \gamma) \sin(|x_M| + \text{sgn}(x_M)x_k) \cosh(y_M + y_k) \\
- \frac{1}{N} \sum_{k=1}^{N} \kappa_{Mk} \gamma \sin(|x_M| - \text{sgn}(x_M)x_k) \cosh(y_M - y_k) \\
\leq \nu_{\infty} - d_{\min}\delta \sin \alpha < 0.
\]

Hence, \(|x_M|\) enters \([0, \alpha]\) in some \(t_1\):
\[
t_1 \geq \frac{\frac{\pi}{4} - \alpha}{d_{\min}\delta \sin \alpha - \nu_{\infty}}.
\]

**Step B** (Estimates for \(Y(t)\)): If \(y_{\bar{M}}(t_1) \in [-\beta, \beta]\), then Lemma 3.2 implies
\[
Y(t) \in \overline{R}(\beta) \quad \text{for all } t \geq t_1.
\]

This concludes the desired result with \(t_* = t_1\). Suppose that
\[
|y_{\bar{M}}(t_1)| \in [\beta, \beta^*].
\]

Then it follows from Step A that
\[
X(t) \in \overline{R}(\alpha) \quad \text{for } t \geq t_1.
\]

Note that for \(t \geq t_1\),
\[
\frac{d}{dt}|y_{\bar{M}}(t)| = \omega_{\bar{M}} - d_{\bar{M}}\delta \cos x_{\bar{M}} \sinh |y_{\bar{M}}| \\
- \frac{1}{N} \sum_{k=1}^{N} \kappa_{\bar{M}k}(1 - \gamma) \cos(x_{\bar{M}} + x_k) \sinh(|y_{\bar{M}}| + \text{sgn}(y_{\bar{M}})y_k) \\
- \frac{1}{N} \sum_{k=1}^{N} \kappa_{\bar{M}k} \gamma \cos(x_{\bar{M}} - x_k) \sinh(|y_{\bar{M}}| - \text{sgn}(y_{\bar{M}})y_k) \\
\leq \omega_{\infty} - d_{\min}\delta \cos \alpha \sinh \beta < 0.
\]

The same argument used in Step 1 shows that \(|y_{\bar{M}}|\) enters \([0, \beta]\) in some time \(t_*\):
\[
t_* \geq \frac{\beta^* - \beta}{d_{\min}\delta \cos \alpha \sinh \beta - \omega_{\infty}} + t_1.
\]

\[\Box\]

**Remark 2.** Proposition 3.1 says that for any initial data
\[
(X^{in}, Y^{in}) \in \overline{R}\left(\frac{\pi}{4}\right) \times \overline{R}(\beta^*), \quad \beta^* > 0,
\]

there exists a smaller region \(\overline{R}(\alpha) \times \overline{R}(\beta)\) with \(\sinh \beta^* > \sqrt{2} \cos \alpha \sinh \beta\) such that can attract the phase of a solution after finite time \(t_*\) if coupling strength is large enough to meet
\[
d_{\min} > \max\left\{\frac{\nu_{\infty}}{\delta \sin \alpha}, \frac{\omega_{\infty}}{\delta \cos \alpha \sinh \beta}\right\}.
\]
Since we will focus on the large time behavior in the large coupling strength regime, from now on we assume that for a given \( \alpha \in (0, \frac{\pi}{4}) \) and \( \beta > 0 \), system parameters satisfy

\[
d_{\text{min}} > \max \left\{ \frac{\nu^\infty}{\delta \sin \alpha}, \frac{\omega^\infty}{\delta \cos \alpha \sinh \beta} \right\}, \quad (X^\infty, Y^\infty) \in \mathcal{R}(\alpha) \times \mathcal{R}(\beta).
\]

From this point of view, theorems to be studied from now on is valid for any initial data \((17)\) in the large coupling regime.

2. We note that \(d_{\text{min}} = \kappa\) for the uniform coupling strength \(\kappa_{ij} \equiv \kappa\).

3.2. \textbf{Exponential relaxation to a unique equilibrium.} In this subsection, we show the emergence of a unique locked state \((X^\infty, Y^\infty)\) for generalized Winfree model \((12)\) with the uniform coupling strength such that

\[
\lim_{t \to \infty} |x_j(t) - x_j^\infty| = 0, \quad \lim_{t \to \infty} |y_j(t) - y_j^\infty| = 0, \quad j = 1, \ldots, N.
\]

For a given configuration \((X(t), Y(t))\), we introduce diameters:

\[
\mathcal{D}(X(t)) := \max_{1 \leq j, k \leq N} |x_j(t) - x_k(t)|, \quad \mathcal{D}(Y(t)) := \max_{1 \leq j, k \leq N} |y_j(t) - y_k(t)|,
\]

\[
\mathcal{D}(X(t), Y(t)) = \mathcal{D}(X(t)) + \mathcal{D}(Y(t)),
\]

and set

\[
\Lambda_0(\alpha, \beta) := \delta \cos \alpha + \cos 2\alpha - \delta \sin \alpha \sinh \beta - \sin 2\alpha \sinh 2\beta.
\]

In next lemma, we present a practical aggregation result by following the uniform upper bound with the order of \(O(\kappa^{-1})\).

**Lemma 3.3.** Suppose that system parameters and initial data satisfy

\[
0 < \delta \leq 1, \quad 0 \leq \gamma \leq 1, \quad \alpha \in \left(0, \frac{\pi}{4}\right), \quad \beta > 0, \quad \Lambda_0(\alpha, \beta) > 0,
\]

\[
\kappa_{jk} \equiv \kappa > \max \left\{ \frac{\nu^\infty}{\delta \sin \alpha}, \frac{\omega^\infty}{\delta \cos \alpha \sinh \beta} \right\}, \quad 1 \leq j, k \leq N, \quad (X^\infty, Y^\infty) \in \mathcal{R}(\alpha) \times \mathcal{R}(\beta).
\]

Then, corresponding global solution \((X(t), Y(t))\) to \((12)\) satisfies for \(t \geq 0\),

\[
\mathcal{D}(X(t), Y(t)) \leq \left( \mathcal{D}(X^\infty, Y^\infty) - \frac{\mathcal{D}(\nu) + \mathcal{D}(\omega)}{\kappa \Lambda_0} \right) e^{-\kappa \Lambda_0 t} + \frac{\mathcal{D}(\nu) + \mathcal{D}(\omega)}{\kappa \Lambda_0}.
\] (18)

**Proof.** For given \(t > 0\), we set

\[
M = M(t) := \arg \max_{1 \leq j \leq N} |x_j(t)|, \quad m = m(t) := \arg \min_{1 \leq j \leq N} |x_j(t)|,
\]

\[
\bar{M} = \bar{M}(t) := \arg \max_{1 \leq j \leq N} |y_j(t)|, \quad \bar{m} = \bar{m}(t) := \arg \min_{1 \leq j \leq N} |y_j(t)|.
\]

\textbf{Step A.} (Estimates for \(\mathcal{D}(X(t))\)): Note that

\[
\dot{x}_M(t) = \nu_M - \kappa \delta \sin x_M \cosh y_M
\]

\[
- \frac{\kappa}{N} \sum_{k=1}^{N} \left( (1 - \gamma) \sin(x_M + x_k) \cosh(y_M + y_k) + \gamma \sin(x_M - x_k) \cosh(y_M - y_k) \right),
\]

\[
\dot{x}_m(t) = \nu_m - \kappa \delta \sin x_m \cosh y_m
\]

\[
- \frac{\kappa}{N} \sum_{k=1}^{N} \left( (1 - \gamma) \sin(x_m + x_k) \cosh(y_m + y_k) + \gamma \sin(x_m - x_k) \cosh(y_m - y_k) \right).
\]
We subtract $\dot{x}_m$ from $\dot{x}_M$ and use Taylor’s theorem for the function $(x, y) \mapsto \sin x \cosh y$ to obtain
\[
\frac{d}{dt} D(X) = (\nu_M - \nu_m) - \kappa \delta \cos \alpha \cosh b(x_M - x_m) - \kappa \delta \sin \alpha \sinh b(y_M - y_m)
\]
\[
- \frac{\kappa (1 - \gamma)}{N} \sum_{k=1}^{N} \left( \cos c_k \cosh d_k(x_M - x_m) + \sin c_k \cosh d_k(y_M - y_m) \right)
\]
\[
- \frac{\kappa \gamma}{N} \sum_{k=1}^{N} \left( \cos e_k \cosh f_k(x_M - x_m) + \sin e_k \cosh f_k(y_M - y_m) \right),
\]
for some $a \in (\alpha, \alpha), \ b \in (\beta, \beta), \ c_k, d_k \in (-2\alpha, 2\alpha), \ e_k, f_k \in (-2\beta, 2\beta)$.

Then, we have
\[
\frac{d}{dt} D(X) = (\nu_M - \nu_m)
\]
\[
- \kappa \left( \delta \cos \alpha \cosh b + \frac{1}{N} \sum_{k=1}^{N} ((1 - \gamma) \cos c_k \cosh d_k + \gamma \cos e_k \cosh f_k) \right) |x_M - x_m|
\]
\[
- \kappa \left( \delta \sin \alpha \sinh b + \frac{1}{N} \sum_{k=1}^{N} ((1 - \gamma) \sin c_k \cosh d_k + \sin e_k \cosh f_k) \right) |y_M - y_m|
\]
\[
\leq D(\nu) - \kappa (\delta \cos \alpha + \cos 2\alpha) D(X) + \kappa (\delta \sin \alpha \sinh \beta + \sin 2\alpha \sinh 2\beta) D(Y).
\]

\textbf{Step B. (Estimates for $D(Y(t))$): Note that}
\[
\dot{y}_M = \omega_M - \kappa \delta \cos x_M \sinh y_M
\]
\[
- \frac{\kappa}{N} \sum_{k=1}^{N} \left( (1 - \gamma) \cos(x_M + x_k) \sinh(y_M + y_k) + \gamma \cos(x_M - x_k) \sinh(y_M - y_k) \right),
\]
\[
\dot{y}_m = \omega_m - \kappa \delta \cos x_m \sinh y_m
\]
\[
- \frac{\kappa}{N} \sum_{k=1}^{N} \left( (1 - \gamma) \cos(x_m + x_k) \sinh(y_m + y_k) + \gamma \cos(x_m - x_k) \sinh(y_m - y_k) \right).
\]

As in Step A, one has
\[
\frac{d}{dt} D(Y) = (\omega_M - \omega_m) + \kappa \delta \sin \bar{a} \sinh \bar{b}(x_M - x_m) - \kappa \delta \cos \bar{a} \cosh \bar{b}(y_M - y_m)
\]
\[
+ \frac{\kappa (1 - \gamma)}{N} \sum_{k=1}^{N} \left( \sin \tilde{c}_k \sinh \tilde{d}_k(x_M - x_m) - \cos \tilde{c}_k \cosh \tilde{d}_k(y_M - y_m) \right)
\]
\[
+ \frac{\kappa \gamma}{N} \sum_{k=1}^{N} \left( \sin \tilde{e}_k \sinh \tilde{f}_k(x_M - x_m) - \cos \tilde{e}_k \cosh \tilde{f}_k(y_M - y_m) \right),
\]
for some $\bar{a} \in (\alpha, \alpha), \ \bar{b} \in (\beta, \beta), \ \tilde{c}_k, \tilde{d}_k \in (-2\alpha, 2\alpha), \ \tilde{e}_k, \tilde{f}_k \in (-2\beta, 2\beta)$.

Thus, we have
\[
\frac{d}{dt} D(Y) = (\omega_M - \omega_m)
\]
The estimate (18) yields the practical aggregation estimate:

$$
\nu
$$

Remark 3.

1. The estimate (18) yields the practical aggregation estimate:

2. For an identical ensemble with $X, Y$ and the emergence of phase synchronization. For an non-identical ensemble, Lemma 3.3 says the uniform bound of the diameter of $X, Y$ and the emergence of practical synchronization.

Next, we show the frequencies of all oscillator tend to zero exponentially fast which implies the oscillator death of all oscillators, namely, complete oscillator death state. For this, we first set

$$
\Lambda_1(\alpha, \beta) := \delta \cos \alpha + \cos 2\alpha - \cosh 2\beta - \delta \sin \alpha \sinh \beta - 2 \sin 2\alpha \sinh 2\beta.
$$

Proposition 2. (Emergence of complete oscillator death) Suppose that system parameters and initial data satisfy

$$
0 < \delta \leq 1, \quad 0 \leq \gamma \leq 1, \quad \alpha \in \left(0, \frac{\pi}{4}\right), \quad \beta > 0, \quad \Lambda_1(\alpha, \beta) > 0,
$$

and let $(X(t), Y(t))$ be corresponding global solution to (12). Then, for $p \geq 1$ one has

$$
\|\dot{X}(t)\|_p + \|\dot{Y}(t)\|_p \leq e^{-\kappa \lambda_0 t} \left(\|X^{in}\|_p + \|Y^{in}\|_p\), \quad t \geq 0.
$$

Proof. For notational simplicity, we set

$$
V(t) = (v_1(t), \ldots, v_N(t)), \quad W(t) = (w_1(t), \ldots, w_N(t)),
$$

$$
v_j(t) = \dot{x}_j(t), \quad w_j(t) = \dot{y}_j(t), \quad \sigma_j := \text{sgn}(v_j), \quad \varsigma_j := \text{sgn}(w_j), \quad j = 1, \ldots, N.
$$

Finally, we combine (19) and (20) to obtain a differential inequality:

$$
\frac{d}{dt} \mathcal{D}(X, Y)(t) \leq (\mathcal{D}(\nu) + \mathcal{D}(\nu)) - \kappa \lambda_0 \mathcal{D}(X, Y), \quad \text{for all } t > 0.
$$

Then Gronwall’s lemma yields

$$
\mathcal{D}(X, Y)(t) \leq \mathcal{D}(X^{in}, Y^{in}) e^{-\kappa \lambda_0 t} + \frac{\mathcal{D}(\nu) + \mathcal{D}(\omega)}{\kappa \lambda_0} \left(1 - e^{-\kappa \lambda_0 t}\right)
$$

$$
= \left(\mathcal{D}(X^{in}, Y^{in}) - \frac{\mathcal{D}(\nu) + \mathcal{D}(\omega)}{\kappa \lambda_0}\right) e^{-\kappa \lambda_0 t} + \frac{\mathcal{D}(\nu) + \mathcal{D}(\omega)}{\kappa \lambda_0}.
$$

Remark 3. 1. The estimate (18) yields the practical aggregation estimate:

$$
\lim_{\kappa \to \infty} \limsup_{t \to \infty} \mathcal{D}(X(t), Y(t)) = 0.
$$

2. For an identical ensemble with $\nu_j = \nu, \omega_j = \omega, 1 \leq j \leq N$, Lemma 3.3 implies the emergence of phase synchronization. For an non-identical ensemble, Lemma 3.3 says the uniform bound of the diameter of $X, Y$ and the emergence of practical synchronization.
Below, we estimate the term $I_{1\ell}$, $\ell = 1, \ldots, 10$ one by one.

**Case A.1** (Estimate of $I_{11} + I_{13} + I_{17}$): By direct estimate, we have

\[
I_{11} + I_{13} + I_{17} = -p\kappa\delta \sum_{j=1}^{N} \cos x_j \cosh y_j |v_j|^p - \frac{p\kappa}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \left( (1 - \gamma) \cos (x_j + x_k) \cosh (y_j + y_k) + \gamma \cos (x_j - x_k) \cosh (y_j - y_k) \right) |v_j|^p \\
\leq -p\kappa (\delta \cos \alpha + \cos 2\alpha) \|V\|_p^p. \tag{22}
\]

**Case A.2** (Estimate of $I_{14} + I_{18}$): We use Jensen’s inequality to obtain

\[
I_{14} + I_{18} = -\frac{p\kappa}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sigma_j \sigma_k (1 - \gamma) \cos (x_j + x_k) \cosh (y_j + y_k) |v_j|^{p-1} |v_k| \\
+ \frac{p\kappa}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sigma_j \sigma_k \gamma \cos (x_j + x_k) \cosh (y_j + y_k) |v_j|^{p-1} |v_k| \\
\leq \frac{p\kappa \cosh 2\beta}{N} \sum_{j=1}^{N} |v_j|^{p-1} \sum_{k=1}^{N} |v_k| \\
\leq p\kappa \cosh 2\beta \left( \sum_{j=1}^{N} |v_j|^p \right)^{1 - \frac{1}{p}} \left( \sum_{k=1}^{N} |v_k|^p \right)^{\frac{1}{p}} = p\kappa \cosh 2\beta \|V\|_p^p. \tag{23}
\]

**Case A.3** (Estimate of $I_{12} + I_{15} + I_{19}$): By Hölder’s inequality, we have

\[
I_{12} + I_{15} + I_{19}
\]
Finally, we insert (22)–(25) into (21) to obtain

\[ = -p\kappa \delta \sum_{j=1}^{N} \sigma_j \zeta_j \sin x_j \sinh y_j |v_j|^{p-1} |w_j| \]

\[ - \frac{p\kappa}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sigma_j \zeta_j (1 - \gamma) \sin(x_j + x_k) \sinh(y_j + y_k) |v_j|^{p-1} |w_j| \]

\[ - \frac{p\kappa}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sigma_j \zeta_j \gamma \sin(x_j + x_k) \sinh(y_j + y_k) |v_j|^{p-1} |w_j| \]

\[ \leq p\kappa (\delta \sin \alpha \sinh \beta + \sin 2\alpha \sinh 2\beta) \sum_{j=1}^{N} |v_j|^{p-1} |w_j| \]

\[ \leq p\kappa (\delta \sin \alpha \sinh \beta + \sin 2\alpha \sinh 2\beta) \left( \sum_{j=1}^{N} |v_j|^{p} \right)^{\frac{p-1}{p}} \left( \sum_{j=1}^{N} |w_j|^{p} \right)^{\frac{1}{p}} \]

\[ = p\kappa (\delta \sin \alpha \sinh \beta + \sin 2\alpha \sinh 2\beta) ||V||_p^{p-1} ||W||_p. \]

\[ \circ \text{ Case A.4 (Estimate of } \mathcal{I}_{16} + \mathcal{I}_{10} \text{): Similar to Case A.2, one has} \]

\[ \mathcal{I}_{14} + \mathcal{I}_{10} = -\frac{p\kappa}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sigma_j \zeta_k (1 - \gamma) \sin(x_j + x_k) \sinh(y_j + y_k) |v_j|^{p-1} |w_k| \]

\[ + \frac{p\kappa}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sigma_j \zeta_k \gamma \sin(x_j + x_k) \sinh(y_j + y_k) |v_j|^{p-1} |w_k| \]

\[ \leq p\kappa \sin 2\alpha \sinh 2\beta \left( \sum_{j=1}^{N} |v_j|^{p} \right)^{\frac{p-1}{p}} \left( \sum_{k=1}^{N} |w_k|^{p} \right)^{\frac{1}{p}} \]

\[ = p\kappa \sin 2\alpha \sinh 2\beta ||V||_p^{p-1} ||W||_p. \]

Finally, we insert (22)–(25) into (21) to obtain

\[ \frac{d}{dt} ||V||_p^{p} \leq -p\kappa (\delta \cos \alpha + \cos 2\alpha - \cosh 2\beta) ||V||_p^{p} \]

\[ + p\kappa (\delta \sin \alpha \sinh \beta + \sin 2\alpha \sinh 2\beta) ||V||_p^{p-1} ||W||_p. \]  

\[ \bullet \text{ Case B (Estimates for } ||W||_p \text{): Basically, we follow the same estimates for Case A.} \]

\[ \text{Note that} \]

\[ \frac{d}{dt} ||W||_p^{p} = p \sum_{j=1}^{N} |w_j|^{p-1} \frac{d}{dt} |w_j| \]

\[ = p\kappa \delta \sum_{j=1}^{N} (\zeta_j \sigma_j \sin x_j \sinh y_j |w_j|^{p-1} |v_j| - \cos x_j \cosh y_j |w_j|^{p}) \]

\[ + p\kappa \left( 1 - \gamma \right) \sum_{j=1}^{N} \sum_{k=1}^{N} \zeta_j \sin(x_j + x_k) \sinh(y_j + y_k) (\sigma_j |w_j|^{p-1} |v_j| + \sigma_k |w_j|^{p-1} |v_k|) \]

\[ - p\kappa \left( 1 - \gamma \right) \sum_{j=1}^{N} \sum_{k=1}^{N} \cos(x_j + x_k) \cosh(y_j + y_k) (|w_j|^{p} + \zeta_j \sigma_k |w_j|^{p-1} |v_k|) \]
Then, for any system parameters and initial data satisfy
\[ \text{(Exponential relaxation to a unique equilibrium)} \]
\[ \text{Theorem 3.4.} \]

Let \( \kappa \) satisfy
\[ \kappa \equiv \kappa > \max \left\{ \frac{\nu}{\delta \sin \alpha}, \frac{\omega}{\delta \cos \alpha \sinh \beta} \right\}, \]
with \( 1 \leq j, k \leq N, (X^{in}, Y^{in}) \in \mathcal{R}(\alpha) \times \mathcal{R}(\beta), \)
and let \( (X(t), Y(t)) \) be corresponding global solution to \( (12) \). Then, there exist a unique equilibrium \( (X^{\infty}, Y^{\infty}) \) and a positive constant \( C \) such that for \( j = 1, \cdots, N, \)
\[ |x_j(t) - x_j^{\infty}| \leq Ce^{-\kappa_1 t} \text{ and } |y_j(t) - y_j^{\infty}| \leq Ce^{-\kappa_1 t}, \quad t \geq 0. \]

**Proof.** Let \( \{X(t), Y(t)\} \) be a solution to \( (12) \) with initial data \( (X^{in}, Y^{in}) \). Then, for \( j = 1, \cdots, N, \) we apply Proposition 1 to obtain
\[ |x_j(t) - x_j(t)| + |y_j(t) - y_j(t)| = \left| \int_t^i v_j(s) \right| + \left| \int_t^i w_j(s) \right| \]
\[ \leq \left( \|V(0)\|_p + \|W(0)\|_p \right) \int_t^i e^{-\kappa_1 s} ds \]
\[ \leq \left( \|V(0)\|_p + \|W(0)\|_p \right) \frac{e^{-\kappa_1 t} - e^{-\kappa_1 \tilde{t}}}{\kappa_1}. \]

Then, for any \( \varepsilon > 0 \), we can find a positive time \( M \) such that if \( t, \tilde{t} \geq M, \) then
\[ |x_j(t) - x_j(t)| < \varepsilon, \quad |y_j(t) - y_j(t)| < \varepsilon, \quad j = 1, \cdots, N. \]

This immediately implies that there exists a unique asymptotic limit \( (x_j^{\infty}, y_j^{\infty}) \) such that
\[ \lim_{\tilde{t} \to \infty} x_j(\tilde{t}) = x_j^{\infty}, \quad \lim_{\tilde{t} \to \infty} y_j(\tilde{t}) = y_j^{\infty}, \quad j = 1, \cdots, N. \]

In (30), we let \( \tilde{t} \to \infty \) and use the relation (31) to derive the estimate: for \( j = 1, \cdots, N, \)
Finally, we set \( C := \frac{\|V(0)\|_p + \|W(0)\|_p}{\kappa_A} \) to derive the desired result. \( \square \)

**Remark 4.** The fact \( \Lambda_0(\alpha, \beta) > \Lambda_1(\alpha, \beta) > 0 \) implies
\[
\{ (\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ : \Lambda_1(\alpha, \beta) > 0 \} \subset \{ (\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ : \Lambda_0(\alpha, \beta) > 0 \}.
\]

### 3.3. Uniform \( \ell_p \)-stability

In this subsection, we derive uniform \( \ell_p \)-stability of generalized Winfree model (12) with respect to initial data and natural frequency for later use.

**Theorem 3.5.** Suppose that system parameters and initial data satisfy
\[
0 < \delta \leq 1, \quad 0 \leq \gamma \leq 1, \quad \alpha \in (0, \frac{\pi}{4}), \quad \beta > 0, \quad \Lambda_1(\alpha, \beta) > 0,
\]
\[
\kappa_{jk} = \kappa > \max \left\{ \frac{\nu^\infty}{\delta \sin \alpha}, \frac{\omega^\infty}{\delta \cos \alpha \sinh \beta} \right\}.
\]
and let \((X(t), Y(t))\) and \((\tilde{X}(t), \tilde{Y}(t))\) be two global solutions of (12) with natural frequencies \(\{\nu_j, \omega_j\}\) and \(\{\bar{\nu}_j, \bar{\omega}_j\}\), and the initial data \((X^{in}, Y^{in})\) and \((\tilde{X}^{in}, \tilde{Y}^{in})\), respectively. Then, there exists a positive constant \(\tilde{C}\) such that for \(p \geq 1\) and for \(t > 0\):
\[
\|X(t) - \tilde{X}(t)\|_p + \|Y(t) - \tilde{Y}(t)\|_p \leq \tilde{C} \left[ \|X^{in} - \tilde{X}^{in}\|_p + \|Y^{in} - \tilde{Y}^{in}\|_p + \left( \sum_{j=1}^{N} |\nu_j - \bar{\nu}_j|^p \right)^{\frac{1}{p}} + \left( \sum_{j=1}^{N} |\omega_j - \bar{\omega}_j|^p \right)^{\frac{1}{p}} \right].
\]

**Proof.** Let \((X, Y) = (x_j, y_j)\) and \((\tilde{X}, \tilde{Y}) = (\tilde{x}_j, \tilde{y}_j)\) be two global smooth solutions to (12) corresponding to the initial data \((X^{in}, Y^{in})\) and \((\tilde{X}^{in}, \tilde{Y}^{in})\), respectively. We also set
\[
\sigma_j := \text{sgn}(x_j - \tilde{x}_j) \quad \text{and} \quad \varsigma_j := \text{sgn}(y_j - \tilde{y}_j)
\]
and use Taylor’s theorem for the function \((x, y) \mapsto \sin x \cosh y\) to obtain
\[
\frac{d}{dt} \|X(t) - \tilde{X}(t)\|_p^p = p \sum_{j=1}^{N} |x_j - \tilde{x}_j|^{p-1} \frac{d}{dt} |x_j - \tilde{x}_j|
\]
\[
= p \sum_{j=1}^{N} |x_j - \tilde{x}_j|^{p-1} \sigma_j (\nu_j - \bar{\nu}_j) - p\kappa \delta \sum_{j=1}^{N} \cos a_j \cosh b_j |x_j - \tilde{x}_j|^p
\]
\[
- p\kappa \delta \sum_{j=1}^{N} \sigma_j \varsigma_j \sin a_j \sinh b_j |x_j - \tilde{x}_j|^{p-1} |y_j - \tilde{y}_j|
\]
\[
- p\kappa \frac{(1 - \gamma)}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \cos c_{kj} \cosh d_{kj} |x_j - \tilde{x}_j|^p + \sigma_j \varsigma_k |x_j - \tilde{x}_j|^{p-1} |x_k - \tilde{x}_k|
\]
\[
- p\kappa \frac{(1 - \gamma)}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sigma_j \varsigma_k \cosh d_{kj} |x_j - \tilde{x}_j|^{p-1} (\varsigma_j |y_j - \tilde{y}_j| + \varsigma_k |y_k - \tilde{y}_k|)
\]

...
\[
- \rho_k \sum_{j=1}^{N} \sum_{k=1}^{N} \cos e_{kj} \cosh f_{kj}(|x_j - \tilde{x}_j|^p - \sigma_j \sigma_k |x_j - \tilde{x}_j|^{p-1}|x_k - \tilde{x}_k|)
\]
\[
- \rho_k \sum_{j=1}^{N} \sum_{k=1}^{N} \sigma_j \sin e_{kj} \sinh f_{kj} |x_j - \tilde{x}_j|^{p-1}(\varsigma_j |y_j - \tilde{y}_j| - \varsigma_k |y_k - \tilde{y}_k|),
\]
and
\[
\frac{d}{dt} \|Y(t) - \tilde{Y}(t)\|_p = \rho \sum_{j=1}^{N} |y_j - \tilde{y}_j|^{p-1} \sin \omega_j - \sin \tilde{\omega}_j - \rho \delta \sum_{j=1}^{N} \cos \tilde{a}_j \cosh \tilde{b}_j |y_j - \tilde{y}_j|^{p} 
\]
\[
+ \rho \delta \sum_{j=1}^{N} \sin \tilde{\omega}_j \sinh \tilde{a}_j |x_j - \tilde{x}_j| |y_j - \tilde{y}_j|^{p-1} 
\]
\[
+ \rho \delta \left(1 - \gamma \right) \sum_{j=1}^{N} \sum_{k=1}^{N} \sin \tilde{c}_{kj} \sinh \tilde{d}_{kj} |y_j - \tilde{y}_j|^{p-1}(\sigma_j |x_j - \tilde{x}_j| + \sigma_k |x_k - \tilde{x}_k|) \tag{34}
\]
\[
- \rho \delta \left(1 - \gamma \right) \sum_{j=1}^{N} \sum_{k=1}^{N} \cos \tilde{c}_{kj} \cosh \tilde{d}_{kj} (|y_j - \tilde{y}_j|^p + \varsigma_j \varsigma_k |y_j - \tilde{y}_j|^{p-1}|y_k - \tilde{y}_k|) 
\]
\[
+ \rho \delta \sum_{j=1}^{N} \sum_{k=1}^{N} \cos \tilde{c}_{kj} \cosh \tilde{d}_{kj} (|y_j - \tilde{y}_j|^p - \varsigma_j \varsigma_k |y_j - \tilde{y}_j|^{p-1}|y_k - \tilde{y}_k|),
\]
where
\[
a_j, \tilde{a}_j \in (-\alpha, \alpha), \ b_j, \tilde{b}_j \in (-\beta, \beta),
\]
\[
ce_{kj}, \tilde{c}_{kj}, c_{kj}, \tilde{c}_{kj} \in (-2\alpha, 2\alpha), \ d_{kj}, \tilde{d}_{kj}, f_{kj}, \tilde{f}_{kj} \in (-2\beta, 2\beta).\]

As in the proof of Proposition 2, we apply Hölder’s inequality and Jensen’s inequality to (33) and (34) to derive
\[
\frac{d}{dt} \|X(t) - \tilde{X}(t)\|_p \leq \rho \|X - \tilde{X}\|_{p}^{p-1} \left(\sum_{j=1}^{N} |\nu_j - \tilde{\nu}_j|^p\right)^{\frac{1}{2}} - \rho \delta \cos \alpha
\]
\[
+ \cos 2\alpha) \|X - \tilde{X}\|_{p} + \rho \delta (\cosh 2\beta \|X - \tilde{X}\|_{p}
\]
\[
+ \delta \sin \alpha \sinh \beta + 2 \sin 2\alpha \sinh 2\beta \|Y - \tilde{Y}\|_{p} \|X - \tilde{X}\|_{p}^{p-1} \tag{35}
\]
and
\[
\frac{d}{dt} \|Y(t) - \tilde{Y}(t)\|_p \leq \rho \|Y - \tilde{Y}\|_{p}^{p-1} \left(\sum_{j=1}^{N} |\omega_j - \tilde{\omega}_j|^p\right)^{\frac{1}{2}}
\]
\[
- \rho \delta (\cos \alpha + \cos 2\alpha) \|Y - \tilde{Y}\|_{p} + \rho \delta (\cosh 2\beta \|Y - \tilde{Y}\|_{p}
\]
\[
+ \delta \sin \alpha \sinh \beta + 2 \sin 2\alpha \sinh 2\beta \|X - \tilde{X}\|_{p} \|Y - \tilde{Y}\|_{p}^{p-1}. \tag{36}
\]
Finally, we use (35) and (36) to obtain a Grönwall’s inequality:
\[
\frac{d}{dt}(\|X - \tilde{X}\|_p + \|Y - \tilde{Y}\|_p) \\
\leq -\kappa \Lambda_1(\alpha, \beta)(\|X - \tilde{X}\|_p + \|Y - \tilde{Y}\|_p) + \left[\left(\sum_{j=1}^{N} |\nu_j - \tilde{\nu}_j|^p\right)^\frac{1}{p}\right] \\
+ \left(\sum_{j=1}^{N} |\omega_j - \tilde{\omega}_j|^p\right)^\frac{1}{p}.
\]

This yields
\[
\|X - \tilde{X}\|_p + \|Y - \tilde{Y}\|_p \\
\leq e^{-\kappa \Lambda_1 t}\left[\|X^{in} - \tilde{X}^{in}\|_p + \|Y^{in} - \tilde{Y}^{in}\|_p\right] + \frac{1}{\kappa \Lambda_1}\left[\left(\sum_{j=1}^{N} |\nu_j - \tilde{\nu}_j|^p\right)^\frac{1}{p}\right] \\
+ \left(\sum_{j=1}^{N} |\omega_j - \tilde{\omega}_j|^p\right)^\frac{1}{p}.
\]

\[\square\]

**Remark 5.** For general coupling strength \(K = (\kappa_{ij})\), we can also derive analogous results in Proposition 2, Theorem 3.4 and Theorem 3.5 by replacing \(\kappa\) and \(\Lambda_1\) with \(d_{\text{min}}\) and \(\Lambda_2\), where
\[
d_{\text{min}} > \max\left\{\frac{\nu^\infty}{\delta \sin \alpha}, \frac{\omega^\infty}{\delta \cos \alpha \sinh \beta}\right\},
\]
\[
\Lambda_2(\alpha, \beta) \\
:= \delta \cos \alpha + \cos 2\alpha - \frac{d_{\text{max}}}{d_{\text{min}}}\left(\cosh 2\beta + \delta \sin \alpha \sinh \beta + 2 \sin \alpha \sinh 2\beta\right) > 0,
\]
respectively. We omit the detailed proof.

4. **Uniform-in-time mean-field limit for the kinetic model.** In this section, we study the uniform-in-time mean-field limit of particle model (3) in the presence of a uniform coupling strength \(\kappa_{jk} = \kappa\) using the arguments in [17, 23]. Since the mean-coupled vector field in the R.H.S. of (37) is Lipschitz and locally bounded, Neunzert’s arguments [25, 27] can be directly applied and produced the finite-time mean-field limit which is valid in any finite-time window and for any initial data. For simplicity, we set
\[
z = (x, y) \in \mathbb{T} \times \mathbb{R}, \quad \Omega = (\nu, \omega) \in \mathbb{R}^2, \quad dz = dx dy, \quad d\Omega = d\nu d\omega.
\]

Consider the particle generalized Winfree model:
\[
\begin{aligned}
\dot{x}_j &= \nu_j - \frac{\kappa}{N} \sum_{k=1}^{N} \left[\delta \sin x_j \cosh y_j \\
&+ (1 - \gamma) \sin(x_j + x_k) \cosh(y_j + y_k) + \gamma \sin(x_j - x_k) \cosh(y_j - y_k)\right], \\
\dot{y}_j &= w_j - \frac{\kappa}{N} \sum_{k=1}^{N} \left[\delta \cos x_j \sinh y_j \\
&+ (1 - \gamma) \cos(x_j + x_k) \sinh(y_j + y_k) + \gamma \cos(x_j - x_k) \sinh(y_j - y_k)\right], \\
\dot{\nu}_j &= 0, \quad \dot{\omega}_j = 0,
\end{aligned}
\] (37)
and the corresponding kinetic model:

\[
\begin{cases}
\partial_t f + \nabla_{(x,y)} \cdot (L_1[f], L_2[f]) f = 0, & (x,y,\nu, \omega) \in \mathbb{T} \times \mathbb{R}^3, \quad t > 0, \\
L_1[f] := \nu - \kappa \delta \sin x \cosh y \\
L_2[f] := \omega - \kappa \delta \cos x \sinh y \\
- \kappa \int_{\mathbb{T} \times \mathbb{R}^3} \left[ (1 - \gamma) \sin(x + x_*) \cosh(y + y_*) + \gamma \sin(x - x_*) \cosh(y - y_*) \right] f_* dz_* d\Omega_*, \\
- \kappa \int_{\mathbb{T} \times \mathbb{R}^3} \left[ (1 - \gamma) \cos(x + x_*) \sinh(y + y_*) + \gamma \cos(x - x_*) \sinh(y - y_*) \right] f_* dz_* d\Omega_*.
\end{cases}
\]

(38)

Note that the variable \( \Omega = (\nu, \mu) \) acts like a parameter.

Recall that our goal is to verify some sort of convergence from the empirical measure:

\[
\mu_1^n := \frac{1}{N} \sum_{j=1}^{N} \delta(z - z_j(t)) \otimes \delta(\Omega - \Omega_j(t)),
\]

where \( \{(z_j, \Omega_j) = (x_j, y_j, \nu_j, \omega_j)\}_{j=1}^{N} \) is a solution to particle system (37) to the measure

\[
\mu_t(dz, d\Omega) = f(t, z, \Omega) dz d\Omega
\]

in suitable weak sense uniformly in time.

4.1. A measure-theoretic framework. In this subsection, we present a measure-theoretic framework and introduce the concept of measure-valued solution and finite-time mean field limit. Let \( \mathcal{P}(\mathbb{T} \times \mathbb{R}^3) \) be the set of all Radon probability measures and for \( \mu \in \mathcal{P}(\mathbb{T} \times \mathbb{R}^3) \), we set

\[
\langle \mu, h \rangle := \int_{(\mathbb{T} \times \mathbb{R}^3)} h(x,y,\nu,\omega) \mu(dx,dy,d\nu,d\omega), \quad h \in C_0(\mathbb{T} \times \mathbb{R}^3),
\]

where \( C_0 \) denotes the set of continuous functions vanishing at infinity. Then, we denote the space of all weakly continuous time dependent measures by \( C_w((0,T); \mathcal{P}(\mathbb{T} \times \mathbb{R}^3)) \), i.e., for any \( \mu_t \in C_w([0,T); \mathcal{P}(\mathbb{T} \times \mathbb{R}^3)) \) we have

\[
\langle \mu_t, h \rangle \text{ is continuous as a function of } t, \quad \forall h \in C_0^1(\mathbb{T} \times \mathbb{R}^3).
\]

Next, we present a definition of the measure-valued solution to kinetic equation (38).

**Definition 4.1.** [21, 25] For \( T \in [0, \infty) \), \( \mu_t \in L^\infty([0,T); \mathcal{P}(\mathbb{T} \times \mathbb{R}^3)) \) is a measure-valued solution to (38) with the initial datum \( \mu_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R}^3) \), if the following three conditions hold:

1. Total mass is normalized: \( \langle \mu_t, 1 \rangle = 1 \).
2. \( \mu \) is weakly continuous in \( t \):

\[
\langle \mu_t, f \rangle \text{ is continuous in } t, \quad \forall f \in C^1_0(\mathbb{T} \times \mathbb{R}^3 \times [0,T)).
\]

3. \( \mu \) satisfies the equation (38) in a weak sense: \( \forall \varphi \in C^1_0(\mathbb{T} \times \mathbb{R}^3 \times [0,T)), \)

\[
\langle \mu_t, \varphi(\cdot,t) \rangle - \langle \mu_0, \varphi(\cdot,0) \rangle = \int_0^t \langle \mu_s, \partial_s \varphi + L_1[\mu_s] \partial_x \varphi + L_2[\mu_s] \partial_y \varphi \rangle ds,
\]
where \( L_1[\mu_s] \) and \( L_2[\mu_s] \) are defined as follows.

\[
\begin{aligned}
L_1[\mu_s] &:= \nu - \kappa \delta \sin x \cosh y - \kappa \int_{\mathbb{T} \times \mathbb{R}^3} \left[(1 - \gamma) \sin(x + x_*) \cosh(y + y_*) + \gamma \sin(x - x_*) \cosh(y - y_*)\right] \mu_s(\mathcal{dz}_s, d\Omega_s), \\
L_2[\mu_s] &:= \omega - \kappa \delta \cosh x \sinh y - \kappa \int_{\mathbb{T} \times \mathbb{R}^3} \left[(1 - \gamma) \cos(x + x_*) \sinh(y + y_*) + \gamma \cos(x - x_*) \sinh(y - y_*)\right] \mu_s(\mathcal{dz}_s, d\Omega_s).
\end{aligned}
\]

**Remark 6.** Note that for a solution \( \{(z_j, \Omega_j)\}_{j=1}^N \) to particle system (37), the empirical measure

\[
\mu^N_t := \frac{1}{N} \sum_{j=1}^N \delta(z - z_j(t)) \otimes \delta(\Omega - \Omega_j(t)),
\]

is a measure-valued solution to kinetic equation (38) in the sense of Definition 4.1. Thus, ODE solution to (37) can be understood as a measure-valued solution for (38). Hence, we can treat the particle and kinetic models in the same framework. Likewise, the classical solution to kinetic equation (38) is also a measure-valued solution as well.

Next, we discuss how to measure the distance between the solution to (37) and (38) by equipping a metric to the probability measure space \( \mathcal{P}(\mathbb{T} \times \mathbb{R}^3) \), and the concept of finite-in-time mean-field limit. In fact, we can endow \( p \)-Wasserstein distance \( \mathcal{W}_p \) in the probability space \( \mathcal{P}(\mathbb{T} \times \mathbb{R}^3) \).

**Definition 4.2.** \([27, 34]\)

1. For \( p \in \mathbb{N} \), let \( \mathcal{P}_p(\mathbb{T} \times \mathbb{R}^3) \) be a collection of all probability measures with finite \( p \)-th moment: for some \( \xi_0 \in \mathbb{T} \times \mathbb{R}^3 \),

\[
\langle \mu, \|\xi - \xi_0\|_p^p \rangle < \infty.
\]

Then, \( p \)-Wasserstein distance \( \mathcal{W}_p(\mu, \tilde{\mu}) \) is defined for any \( \mu, \tilde{\mu} \in \mathcal{P}_p(\mathbb{T} \times \mathbb{R}^3) \) as

\[
\mathcal{W}_p(\mu, \tilde{\mu}) := \inf_{\nu \in \Upsilon(\mu, \tilde{\mu})} \left( \int_{\mathbb{T}^2 \times \mathbb{R}^6} \|\xi - \xi_*\|_p^p \, d\nu(\xi, \xi_*) \right)^{\frac{1}{p}},
\]

where \( \Upsilon(\mu, \tilde{\mu}) \) denotes the collection of all probability measures on \( \mathbb{T}^2 \times \mathbb{R}^6 \) with marginals \( \mu \) and \( \tilde{\mu} \).

2. If \( \lim_{p \to \infty} \mathcal{W}_p \) exists, then we define \( \mathcal{W}_\infty \) as the limit.

3. For any \( T \in (0, \infty] \), the kinetic equation (38) is derivable from particle system (37) in \( [0, T) \), or equivalent to say the mean-field limit from particle system (37) to kinetic equation (38), which is valid in \( [0, T) \), if for every solution \( \mu_t \) of kinetic equation (38) with initial data \( \mu_0 \), the following condition holds: for some \( p \in \mathbb{N} \) and \( t \in [0, T) \),

\[
\lim_{N \to \infty} \mathcal{W}_p(\mu^N_t, \mu_0) = 0 \iff \lim_{N \to \infty} \mathcal{W}_p(\mu^N_t, \mu_t) = 0,
\]

where \( \mu^N_t \) is a measure-valued solution of the particle model (37) with initial data \( \mu^N_0 \).

For a later use, we quote result on the approximation of a measure by empirical measures without a proof.
**Proposition 3.** For any given \( p \in \mathbb{N} \) and \( \mu \in \mathcal{P}_p(\mathbb{T} \times \mathbb{R}^3) \) with compact support, there exists a sequence of empirical measures \( \mu^N \in \mathcal{P}_p(\mathbb{T} \times \mathbb{R}^3) \) such that

\[
\mu^N \text{ has a common compact support with } \mu \text{ and } \lim_{N \to \infty} W_p(\mu^N, \mu) = 0.
\]

**Remark 7.** The construction of the approximation can be followed by the method of Theorem 6.18 in the book [34] by finding a sequence of atomic measures \( \sum_{j=1}^{N} a_j \delta_j \) with rational numbers \( a_j \) such that \( \sum_{j=1}^{N} a_j = 1 \).

4.2. **A uniform mean-field limit.** In this subsection, we present a uniform mean-field limit which is valid in the whole time \((0, \infty)\). We basically follow the approach given in [17, 23].

Note that the probability density functions \( g_1(\nu), g_2(\omega) \) for natural frequencies appear as \( \nu \)- and \( \mu \)-marginal density functions of \( f \):

\[
\int_{\mathbb{T} \times \mathbb{R}^2} f(t, x, y, \nu, \omega) dx dy d\omega = g_1(\nu), \quad \text{for all } t, \nu,
\]

\[
\int_{\mathbb{T} \times \mathbb{R}^2} f(t, x, y, \nu, \omega) dx dy d\nu = g_2(\omega), \quad \text{for all } t, \omega.
\]

For \( t \geq 0 \), we set \( R_1(\mu_t) \) and \( R_2(\mu_t) \) be the projections of \( \text{supp} \mu_t \) to \( x \) and \( y \) spaces, respectively:

\[
R_1(\mu_t) : = \mathbb{P}_x \text{supp} \mu_t = \{ x \in \mathbb{T} : (x, y, \nu, \omega) \in \text{supp} \mu_t \},
\]

\[
R_2(\mu_t) : = \mathbb{P}_y \text{supp} \mu_t = \{ y \in \mathbb{R} : (x, y, \nu, \omega) \in \text{supp} \mu_t \}.
\]

**Theorem 4.3.** (Uniform mean-field limit) Suppose that system parameters and initial data satisfy

\[
0 < \delta \leq 1, \quad 0 \leq \gamma \leq 1, \quad \alpha \in \left( 0, \frac{\pi}{4} \right), \quad \beta > 0, \quad \Lambda_1(\alpha, \beta) > 0,
\]

\[
\kappa > \max \left\{ \frac{\nu}{\delta \sin \alpha}, \frac{\omega}{\delta \cos \alpha \sinh \beta} \right\}, \quad R_1(\mu_0) \subseteq (-\alpha, \alpha), \quad R_2(\mu_0) \subseteq (-\beta, \beta),
\]

\[
\int_{\mathbb{T} \times \mathbb{R}^3} (1 + |x|^p + |y|^p + |\nu|^p + |\omega|^p) \mu_0(dx, dy, d\nu, d\omega) < \infty, \quad \text{for some } p \in \mathbb{N}.
\]

Then, the following assertions hold.

1. There exists a unique measure-valued solution \( \mu \in L^\infty([0, \infty); \mathcal{P}(\mathbb{T} \times \mathbb{R}^3)) \) to kinetic equation (38) with initial data \( \mu_0 \): \( \mu_t \) is approximated by empirical measure \( \mu^N_t \) in \( p \)-Wasserstein distance uniformly in time:

\[
\lim_{N \to \infty} \sup_{t \in [0, \infty)} W_p(\mu^N_t, \mu_t) = 0,
\]

where \( \mu^N_t \) is the solution of (37) with the initial data \( \mu^N_0 \) and \( \lim_{N \to \infty} W_p(\mu^N_0, \mu_0) = 0. \)

2. If \( \tilde{\mu}_t \) is another measure-valued solution to (38) with the initial data \( \tilde{\mu}_0 \) satisfying initial conditions in (39), then there exists a nonnegative constant \( G_0 \) independent of \( t \) such that

\[
W_p(\mu_t, \tilde{\mu}_t) \leq G_0 W_p(\mu_0, \tilde{\mu}_0), \quad t \in [0, \infty).
\]
Proof. (i) Since the distribution of natural frequency $\Omega = (\nu, \omega)$ does not have its own dynamics, i.e., it does not change in time. It suffices to estimate the variance with respect to $(x, y)$ variable. Let $\mu_0 \in \mathcal{P}(\mathbb{T} \times \mathbb{R}^3)$ be a probability measure with

$$R_1(\mu_0) \subseteq (-\alpha, \alpha), \quad R_2(\mu_0) \subseteq (-\beta, \beta).$$

Then, we claim that there exists a measure-valued solution $\mu_t \in L^\infty([0, \infty); \mathcal{P}(\mathbb{T} \times \mathbb{R}^3))$ to (38) with the initial data $\mu_0$:

1. $\mu_t$ is approximated by $\mu_t^N$ in $p$-Wasserstein distance uniformly in time:

$$\lim_{N \to \infty} \sup_{t \in [0, \infty)} W_p(\mu_t^N, \mu_t) = 0.$$

2. $\mu_t$ is unique in the class of measure-valued solution with initial data $\mu_0$ and satisfies $R_1(\mu_t) \subseteq (-\alpha, \alpha), R_2(\mu_t) \subseteq (-\beta, \beta)$.

For the simplicity of presentation, we split its proof into the following three steps:

- **Step A** (Extraction of Cauchy approximation for $\mu_0$ in $W_p$): Let $\mu_0^N$ be an approximation of $\mu_0$ satisfying

$$\lim_{N \to \infty} W_p(\mu_0^N, \mu_0) = 0. \quad (40)$$

The existence of such approximation is guaranteed by Proposition 3. Then, due to (40), for any $\varepsilon$, there is a positive integer $N = N(\varepsilon)$ such that

$$W_p(\mu_n^0, \mu_m^0) < \varepsilon, \quad \text{for} \quad n, m > N(\varepsilon).$$

Since the empirical measures $\mu_n^0$ and $\mu_m^0$ are both concentrated at finite points, we denote them by

$$\mu_n^0 := \frac{1}{n} \sum_{i=1}^{n} \delta((x_i^{in}, y_i^{in}, \nu_i, \omega_i)), \quad \mu_m^0 := \frac{1}{m} \sum_{j=1}^{m} \delta((x_j^{in}, y_j^{in}, \nu_j, \omega_j)).$$

Then, we can find an optimal plan $(a_{ij})$ whose entries are nonnegative real numbers and satisfying

$$W_p(\mu_n^0, \mu_m^0) = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} (|x_i^{in} - x_j^{in}|^p + |y_i^{in} - y_j^{in}|^p + |\nu_i - \nu_j|^p + |\omega_i - \omega_j|^p), \quad (41)$$

where $a_{ij}$ satisfies the constraints:

$$\sum_{i=1}^{n} a_{ij} = n, \quad \sum_{j=1}^{m} a_{ij} = m.$$

- **Step B** (Approximation of $W_p(\mu_n^0, \mu_m^0)$): To associate (41) with $\ell_p$-distance between $\{x_i^{in}, y_i^{in}, \nu_i, \omega_i\}$ and $\{x_j^{in}, y_j^{in}, \nu_j, \omega_j\}$, we approximate (41) with rational coefficients $r_{ij}$ instead of real ones $a_{ij}$ with some small error as follows. More precisely, we find proper rational numbers $r_{ij}$ such that they have the same denominator $D_{mn}$ and

$$|r_{ij} - a_{ij}| \leq \frac{e^p}{d_x(0)^p + d_y(0)^p + \epsilon^p + d_\nu^p + d_\omega^p}, \quad \sum_{i=1}^{n} r_{ij} = n, \quad \sum_{j=1}^{m} r_{ij} = m, \quad (42)$$

where
where the nonnegative numbers \( d_x(0), d_y(0), d_v, \) and \( d_\omega \) are given as follows:

\[
\begin{align*}
d_x(0) & := \max_{i,j} |x_i^0 - \bar{x}_j^0|, \\
d_y(0) & := \max_{i,j} |y_i^0 - \bar{y}_j^0|, \\
d_v & := \max_{i,j} |\nu_i - \bar{\nu}_j|, \\
d_\omega & := \max_{i,j} |\omega_i - \bar{\omega}_j|.
\end{align*}
\]

Then, it follows from (41) and (42) that

\[
\begin{align*}
& \left| W_p^p(\mu_0^n, \mu_0^m) - \frac{1}{mn} \sum_{i=1}^n \sum_{j=1}^m r_{ij} (|x_i^n - \bar{x}_j^n|^p + |y_i^n - \bar{y}_j^n|^p + |\nu_i - \bar{\nu}_j|^p + |\omega_i - \bar{\omega}_j|^p) \right| \\
& \leq \frac{1}{mn} \sum_{i=1}^n \sum_{j=1}^m |a_{ij} - r_{ij}| (|x_i^n - \bar{x}_j^n|^p + |y_i^n - \bar{y}_j^n|^p + |\nu_i - \bar{\nu}_j|^p + |\omega_i - \bar{\omega}_j|^p) \\
& \leq (d_x(0))^p + d_y(0))^p + d_v^p + d_\omega^p) \frac{1}{mn} \sum_{i=1}^n \sum_{j=1}^m |a_{ij} - r_{ij}| \leq \varepsilon^p.
\end{align*}
\]

(43)

We set

\[
r_{ij} := \frac{N_{ij}}{D_{mn}}, \quad N_{ij} \in \mathbb{Z}_+, \quad M_{mn} := D_{mn}mn.
\]

Then, we can rewrite

\[
\begin{align*}
& \frac{1}{mn} \sum_{i=1}^n \sum_{j=1}^m r_{ij} (|x_i^n - \bar{x}_j^n|^p + |y_i^n - \bar{y}_j^n|^p + |\nu_i - \bar{\nu}_j|^p + |\omega_i - \bar{\omega}_j|^p) \\
& = \frac{1}{M_{mn}} \sum_{i=1}^n \sum_{j=1}^m N_{ij} (|x_i^n - \bar{x}_j^n|^p + |y_i^n - \bar{y}_j^n|^p + |\nu_i - \bar{\nu}_j|^p + |\omega_i - \bar{\omega}_j|^p) \\
& = \frac{1}{M_{mn}} \sum_{k=1}^{M_{mn}} (|x_k^n - \bar{x}_k^n|^p + |y_k^n - \bar{y}_k^n|^p + |\nu_k - \bar{\nu}_k|^p + |\omega_k - \bar{\omega}_k|^p),
\end{align*}
\]

by reindexing for each summand. Here, we note that \( \sum_{i,j} N_{ij} = M_{mn}, \) and \( \{(x_i^n, y_i^n, \nu_i, \omega_i) : 1 \leq i \leq n\} \) and \( \{(\bar{x}_j^n, \bar{y}_j^n, \bar{\nu}_j, \bar{\omega}_j) : 1 \leq j \leq m\}, \) respectively. We combine (43) and (44) to obtain

\[
\begin{align*}
& \left| W_p^p(\mu_0^n, \mu_0^m) - \frac{1}{M_{mn}} \sum_{k=1}^{M_{mn}} (|x_k^n - \bar{x}_k^n|^p + |y_k^n - \bar{y}_k^n|^p + |\nu_k - \bar{\nu}_k|^p + |\omega_k - \bar{\omega}_k|^p) \right| \leq \varepsilon^p.
\end{align*}
\]

(45)

\( \diamond \) **Step C** (Lifting the information at 0 to \( t > 0 \)): We use the uniform \( \ell_p \)-stability in Theorem 3.5 to show that \( \mu_0^N \) is also Cauchy. We recall the term

\[
\sum_{i=1}^n \sum_{j=1}^m \frac{N_{ij}}{D_{mn}} (|x_i^n - \bar{x}_j^n|^p + |y_i^n - \bar{y}_j^n|^p + |\nu_i - \bar{\nu}_j|^p + |\omega_i - \bar{\omega}_j|^p),
\]

\[
\sum_{i=1}^n \frac{N_{ij}}{D_{mn}} = n, \quad \sum_{j=1}^m \frac{N_{ij}}{D_{mn}} = m.
\]
It corresponds to a plan between $\mu_0^n$ and $\mu_m^m$, which we denote by $v(\mu_0^n, \mu_m^m)$:

$$
\int_{\mathbb{T}^2 \times \mathbb{R}^6} (|\tilde{x} - \bar{x}|^p + |\tilde{y} - \bar{y}|^p + |\nu - \bar{\nu}|^p + |\omega - \bar{\omega}|^p)dv(\mu_0^n, \mu_m^m)
= \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{N_{ij}}{M_{mn}} ((|x_i^{in} - \bar{x}_j^{in}|^p + |y_i^{in} - \bar{y}_j^{in}|^p + |\nu_i - \bar{\nu}_j|^p + |\omega_i - \bar{\omega}_j|^p)

= \frac{1}{M_{mn}} \sum_{k=1}^{M_{mn}} ((|x_k^{in} - \bar{x}_k^{in}|^p + |y_k^{in} - \bar{y}_k^{in}|^p + |\nu_k - \bar{\nu}_k|^p + |\omega_k - \bar{\omega}_k|^p).
$$

Then, it follows from (32), (45) and Jensen’s inequality that

$$
W_p^p(\mu_0^n, \mu_m^m) \leq \int_{\mathbb{T}^2 \times \mathbb{R}^6} (|\tilde{x} - \bar{x}|^p + |\tilde{y} - \bar{y}|^p + |\nu - \bar{\nu}|^p + |\omega - \bar{\omega}|^p)dv(\mu_0^n, \mu_m^m)
= \frac{1}{M_{mn}} \sum_{k=1}^{M_{mn}} ((|x_k(t) - \bar{x}_k(t)|^p + |y_k(t) - \bar{y}_k(t)|^p + |\nu_k - \bar{\nu}_k|^p + |\omega_k - \bar{\omega}_k|^p)

\leq \frac{2^{p-1}C^p + 1}{M_{mn}} \sum_{k=1}^{M_{mn}} ((|x_k^{in} - \bar{x}_k^{in}|^p + |y_k^{in} - \bar{y}_k^{in}|^p + |\nu_k - \bar{\nu}_k|^p + |\omega_k - \bar{\omega}_k|^p

\leq 2^{p-1}G^p(W_p^p(\mu_0^n, \mu_m^m) + \varepsilon^p) \leq 2^pG^p\varepsilon^p,
$$

where $G := (\tilde{C} + 2^{1-p})^\frac{1}{p}$. Now for any $\varepsilon > 0$, we can find a positive integer $L$ such that for any $n, m > L$, we have

$$
W_p^p(\mu_0^n, \mu_m^m) \leq 2G\varepsilon.
$$

This shows that the sequence $\mu_i^N$ is a Cauchy sequence in $W_p$ metric, thus we can find a limit measure $\mu_i$. We next apply similar arguments in [21] to show that the limit $\mu_i$ is the unique measure-valued solution of the kinetic equation (38) with initial data $\mu_0$. Moreover, because our estimates (46), we can conclude that for any $\varepsilon$, there exists a positive constant $L$, such that

$$
\sup_{t \in (0, \infty)} W_p(\mu_i^N, \mu_i) \leq 4G\varepsilon, \quad \text{for} \quad N > L.
$$

This yields

$$
\lim_{N \to \infty} \sup_{t \in (0, \infty)} W_p(\mu_i^N, \mu_i) = 0.
$$

This uniform convergence also guarantees the uniform compact support of $\mu_i$.

(ii) We will show the uniform $W_p$-stability of the measure-valued solutions whose exitsences are guaranteed by the first assertion. For measures $\mu_0$ and $\tilde{\mu}_0$ in $\mathcal{P}(\mathbb{T} \times \mathbb{R}^3)$, let $\mu$ and $\tilde{\mu}$ be corresponding measure-valued solution to (38). Then, it follows from (40) that for any $\varepsilon \leq 1$, there is $N_0(\varepsilon) \in \mathbb{N}$ such that

$$
W_p(\mu, \mu_i^N) < \frac{\varepsilon}{2}, \quad W_p(\tilde{\mu}, \mu_i^N) < \frac{\varepsilon}{2}, \quad \text{for} \quad N \geq N_0(\varepsilon).
$$

Then, we use (46) and (47) to obtain

$$
W_p^p(\mu, \tilde{\mu}_i) \leq (W_p(\mu, \mu_i^N) + W_p(\mu_i^N, \tilde{\mu}_i)) + W_p(\tilde{\mu}_i, \tilde{\mu}_0)\varepsilon^p
\leq (\varepsilon + W_p(\mu_i^N, \tilde{\mu}_i))\varepsilon^p \leq 2^{p-1}(\varepsilon\varepsilon^p + W_p(\mu_i^N, \tilde{\mu}_i))
\leq 4^{p-1}(2\varepsilon^2 + G^pW_p^p(\mu_0^n, \tilde{\mu}_0^N)).
$$
Letting $N \to \infty$, we obtain
\[ \mathcal{W}_p^N(\mu_t, \tilde{\mu}_t) \leq 4^p \varepsilon^p + 4^{p-1} G \mathcal{W}_p^N(\mu_0, \tilde{\mu}_0). \]
Since $\varepsilon$ is arbitrary, we have the uniform $\mathcal{W}_p$-stability:
\[ \mathcal{W}_p(\mu_t, \tilde{\mu}_t) \leq 4^{1/p} G \mathcal{W}_p(\mu_0, \tilde{\mu}_0), \quad t \geq 0. \]

Now, for an asymptotic behavior of measure-valued solution $\mu$ to kinetic equation (38), we can apply Theorem 3.4 to the approximate solution $\mu^N_t$. Since the relaxation to an equilibrium in Theorem 3.4 is independent of $N$, the mean-field limit preserves the result.

**Corollary 1.** Suppose system parameters and initial data satisfy
\[ 0 < \delta \leq 1, \quad 0 \leq \gamma \leq 1, \quad \alpha \in \left(0, \frac{\pi}{4}\right), \quad \beta > 0, \quad \Lambda_1(\alpha, \beta) > 0, \]
\[ \kappa > \max \left\{ \frac{\nu^\infty}{\delta \sin \alpha}, \frac{\omega^\infty}{\delta \cos a \sinh \beta} \right\}, \quad R_1(\mu_0) \subseteq (-\gamma, \alpha), \quad R_2(\mu_0) \subseteq (-\beta, \beta), \]
\[ \int_{\mathbb{T} \times \mathbb{R}^3} \left(1 + |x|^p + |y|^p + |\nu|^p + |\omega|^p\right)\nu_0(\nu, \omega) \mu_0(\nu, \omega) < \infty, \quad \text{for some} \quad p \in \mathbb{N}, \]
and let $\mu$ be a measure-valued solution to (38) whose existence is guaranteed by Theorem 3.3. Then, there exist a unique measure-valued function $\mu_\infty$ and a nonnegative constant $C$ such that
\[ \mathcal{W}_p(\mu_t, \mu_\infty) \leq 2Ce^{-\kappa \Lambda_1 t}, \quad t \geq 0. \]

**Proof.** Consider an empirical measure $\mu^N_t$. According to Theorem 3.4, for each $\mu^N_t$, we have a unique asymptotic equilibrium $\mu^N_\infty$. The uniform stability in Theorem 3.5 implies that the sequence $\{\mu^N_\infty\}$ is a Cauchy sequence and thus generates a unique limit measure $\mu_\infty$. Moreover, estimate (29) gives
\[ \mathcal{W}_p(\mu^N_t, \mu^N_\infty) \leq 2Ce^{-\kappa \Lambda_1 t}. \]
We note that the $p$-th moment of $(\nu, \omega)$ would be canceled because $\mu^N_t$ and $\mu^N_\infty$ has the same natural frequency distribution. For any $\varepsilon > 0$, we can find $N_0 \in \mathbb{N}$ such that, for $N \geq N_0$ we obtain
\[ \mathcal{W}_p(\mu_t, \mu_\infty) \leq \mathcal{W}_p(\mu_t, \mu^N_t) + \mathcal{W}_p(\mu^N_t, \mu^N_\infty) + \mathcal{W}_p(\mu^N_\infty, \mu_\infty) \leq 2\varepsilon + 2Ce^{-\kappa \Lambda_1 t}. \]
Since $\varepsilon$ is arbitrary, we have
\[ \mathcal{W}_p(\mu_t, \mu_\infty) \leq 2Ce^{-\kappa \Lambda_1 t}. \]

5. **Numerical simulations.** In this section, we present several numerical simulations for the particle model. In all simulations, we used the 4th-order Runge-Kutta method and set system parameters in (12) as
\[ \delta = 1 \quad \text{and} \quad \gamma = \frac{1}{2}, \]
to observe a behavior of the solution to particle model (10), which can be considered as the complex Winfree model.

First, we present several numerical simulations for the particle model to observe the asymptotic behavior of $\frac{X(t)}{t}$ for $X^\infty \notin \mathcal{R}(\frac{\pi}{4})$. We can provide the validity of the range restriction for initial data through this observation, and for this purpose, we employ two global solutions of (12) with all conditions except initial
configurations are set to be identical. In simulations, natural frequencies $\nu_j$ and $\omega_j$ are chosen randomly in the interval $[0,1]$, the number of oscillators is set by 10, coupling strengths $k_{jk}$ are set by 5, and two different initial phase configurations $(X_1^{in}, Y_1^{in})$ and $(X_2^{in}, Y_2^{in})$ are chosen randomly in $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]^N \times \mathcal{R}(10)$ and $\mathcal{R}\left(\frac{\pi}{4}\right) \times \mathcal{R}(10)$ respectively.

![Figure 1](image1.png)

(a) Time evolution of $\frac{X_1(t)}{t}$

(b) Time evolution of $\frac{X_2(t)}{t}$

**Figure 1.** Time evolution of $\frac{X(t)}{t}$ for two different initial data

In Figure 1, one can observe that $\frac{X_1(t)}{t}$ is not vanishing until $t = 2000$, while $\frac{X_2(t)}{t}$ approaches to zero within short time compared to $\frac{X_1(t)}{t}$, which implies that the dynamical model (10) represents that the different emergent dynamics depends on the initial data.

Next, we present several numerical simulations for the particle model to observe the existence of a unique equilibrium in Theorem 3.4 and uniform $\ell_1$-stability in Theorem 3.5. In simulations, we assume that $\alpha = \beta = 0.25$ to satisfy

$$\Lambda_1(\alpha, \beta) = \delta \cos \alpha + \cos 2\alpha - \cosh 2\beta - \delta \sin \alpha \sinh \beta - 2 \sin 2\alpha \sinh 2\beta \approx 0.1567 > 0.$$

Then, we employ two global solutions to (12) with all conditions except initial configurations are set identically to check the uniqueness of the equilibrium and uniform $\ell_1$-stability. Natural frequencies $\nu_j$ and $\omega_j$ are chosen randomly in the interval $[0,1]$, the number of oscillators $N$ is set by 20, and two different initial phase configurations $(X_1^{in}, Y_1^{in})$ and $(X_2^{in}, Y_2^{in})$ are chosen randomly in $\mathcal{R}\left(\frac{\pi}{4}\right) \times \mathcal{R}(100)$. In these settings, lower bound of homogeneous coupling strength $\kappa_{jk} \equiv \kappa$ in Theorem 3.4 and Theorem 3.5 is given by

$$\kappa > \max \left\{ \frac{\nu_\infty}{\delta \sin \alpha}, \frac{\omega_\infty}{\delta \cos \alpha \sin \beta} \right\} \approx 4.0856.$$

Hence, for simulation, we set $\kappa = 5$ and observe the dynamics in the time interval $[0,1]$. 
First, we observe the configuration of \((X_1, Y_1)\) and \((X_2, Y_2)\) to show the existence of a unique equilibrium. In Figure 2, (a) shows the configuration of \((X_1^n, Y_1^n)\) and \((X_2^n, Y_2^n)\) and (b) shows the terminal configuration of \((X_1, Y_1)\) and \((X_2, Y_2)\), where we can observe the same terminal configuration of \((X_1, Y_1)\) and \((X_2, Y_2)\) independent of initial configuration. Moreover, in Figure 2, (a) also represents the fact that equilibrium state exists in the trapping set \(R(\alpha) \times R(\beta) = R(0.25) \times R(0.25)\).

Next, we observe the graph of \(\|X_1 - X_2\|_1 + \|Y_1 - Y_2\|_1\) and \(\log(\|X_1 - X_2\|_1 + \|Y_1 - Y_2\|_1)\) with respect to time. In Figure 3, (a) shows the \(\ell_1\)-difference between two solutions and (b) shows the linear decay of logarithm value of the \(\ell_1\)-difference between two solutions, which implies the exponential decay of the \(\ell_1\)-difference.

6. **Conclusion.** In this paper, we have proposed a Winfree type model on the infinite cylinder. For the derivation of model, we employed a complex phase from the real-valued phase and lift the Winfree model to the complex plane. By comparing the real and imaginary parts of the resulting system, we can derive our Winfree
model on the infinite cylinder. For the proposed model, we study the existence of a unique equilibrium and uniform $\ell_p$-stability for the particle model. Using the uniform stability estimate of particle model, we provide a uniform mean-field limit from the particle model to the kinetic equation. Following the usual approach, one can derive a mean-field limit in any finite-time interval for any initial configurations, but this finite-in-time mean-field limit result is not valid for the whole time. We use the aggregation estimate for the particle model to derive the uniform mean-field limit for an admissible set of initial configurations. Of course, there are several issues that were not discussed in this work. For example, we considered only the complete oscillator death state in a large coupling regime, but in an intermediate coupling strength regime, partial oscillator death state can emerge, and complete / partial oscillator phase-locking. These interesting issues will be discussed in a future work.

REFERENCES

[1] J. A. Acebron, L. L. Bonilla, C. J. P. Pérez Vicente, F. Ritort and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, Rev. Mod. Phys., 77 (2005), 137–185.
[2] G. Albi, N. Bellomo, L. Fermo, S.-Y. Ha, J. Kim, L. Pareschi, D. Poyato and J. Soler, Vehicular traffic, crowds, and swarms: From kinetic theory and multiscale methods to applications and research perspectives, Math. Models Methods Appl. Sci., 29 (2019), 1901–2005.
[3] J. T. Ariaratnam and S. H. Strogatz, Phase diagram for the Winfree model of coupled nonlinear oscillators, Phys. Rev. Lett., 86 (2001), 4278–4281.
[4] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Proccaccini, M. Viale and V. Zdravkovic, Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study, Proc. Natl. Acad. Sci. USA, 105 (2008), 1232–1237.
[5] J. Buck and E. Buck, Biology of synchronous flashing of fireflies, Nature, 211 (1966), 562.
[6] Y.-P. Choi, S.-Y. Ha, S. Jung and Y. Kim, Asymptotic formation and orbital stability of phase-locked states for the Kuramoto model, Physica D, 241 (2012), 735–754.
[7] N. Chopra and M. W. Spong, On exponential synchronization of Kuramoto oscillators, IEEE Trans. Automatic Control, 54 (2009), 353–357.
[8] F. Cucker and S. Smale, Emergent behavior in flocks, IEEE Trans. Automat. Control, 52 (2007), 852–862.
[9] P. Degond and S. Motsch, Large scale dynamics of the persistent turning walker model of fish behavior, J. Stat. Phys., 131 (2008), 989–1021.
[10] F. Dörfler and F. Bullo, Synchronization in complex network of phase oscillators: A survey, Automatica J. IFAC, 50 (2014), 1539–1564.
[11] F. Dörfler and F. Bullo, On the critical coupling for Kuramoto oscillators, SIAM J. Appl. Dyn. Syst., 10 (2011), 1070–1099.
[12] F. Giannuzzi, D. Marinazzo, G. Nardulli, M. Pellicoro and S. Stramaglia, Phase diagram of a generalized Winfree model, Physical Review E, 75 (2007), 051104.
[13] S.-Y. Ha, M.-J. Kang, C. Lattanzio and B. Rubino, A class of interacting particle systems on the infinite cylinder with flocking phenomena, Math. Models Methods Appl. Sci., 22 (2012), 1250008, 25 pp.
[14] S.-Y. Ha, M. Kang and B. Moon, On the emerging asymptotic patterns for the Winfree model with frustrations, submitted.
[15] S.-Y. Ha and D. Kim, Robustness and asymptotic stability for the Winfree model on a general network under the effect of time-delay, J. Math. Phys., 59 (2018), 112702, 20 pp.
[16] S.-Y. Ha, D. Kim and B. Moon, Interplay of random inputs and adaptive couplings in the Winfree model, submitted.
[17] S.-Y. Ha, J. Kim and X. T. Zhang, Uniform stability of the Cucker-Smale model and its application to the mean-field limit, Kinet. Relat. Models, 11 (2018), 1157–1181.
[18] S.-Y. Ha, D. Ko, J. Park and S. W. Ryoo, Emergence of partial locking states from the ensemble of Winfree oscillators, Quart. Appl. Math., 75 (2017), 39–68.
[19] S.-Y. Ha, D. Ko, J. Park and S. W. Ryoo, Emergent dynamics of Winfree oscillators on locally coupled networks, J. Differential Equations, 260 (2016), 4203–4236.
[20] S.-Y. Ha, D. Ko, J. Park and X. T. Zhang, Collective synchronization of classical and quantum oscillators, EMS Surv. Math. Sci., 3 (2016), 209–267.
[21] S.-Y. Ha and J.-G. Liu, A simple proof of the Cucker-Smale flocking dynamics and mean-field limit, Commun. Math. Sci., 7 (2009), 297–325.
[22] S.-Y. Ha, J. Park and S. W. Ryoo, Emergence of phase-locked states for the Winfree model in a large coupling regime, Discrete Contin. Dyn. Syst., 35 (2015), 3417–3436.
[23] S.-Y. Ha, J. Park and X. T. Zhang, A global well-posedness and asymptotic dynamics of the kinetic Winfree equation, Discrete Contin. Dyn. Syst. Ser. B, 25 (2020), 1317–1344.
[24] Y. Kuramoto, International symposium on mathematical problems in mathematical physics, Lecture Notes in Theoretical Physics, 30 (1975), 420.
[25] C. Lancellotti, On the Vlasov limit for systems of nonlinearly coupled oscillators without noise, Transport Theory Statist. Phys., 34 (2005), 523–535.
[26] P. V. Mieghem, A complex variant of the Kuramoto model, preprint, (2009), available at: https://www.nas.ewi.tudelft.nl/people/Piet/papers.
[27] H. Neunzert, An introduction to the nonlinear Boltzmann-Vlasov equation, Kinetic Theories and the Boltzmann Equation, Lecture Notes in Math., Springer, Berlin, 1048 (1984), 60–110.
[28] W. Oukil, A. Kessi and Ph. Thieullen, Synchronization hypothesis in the Winfree model, Dyn. Syst., 32 (2017), 326–339.
[29] W. Oukil, Ph. Thieullen and A. Kessi, Invariant cone and synchronization state stability of the mean field models, Dyn. Syst., 34 (2019), 422–433.
[30] D. A. Paley, N. E. Leonard, R. Sepulchre, D. Grunbaum and J. K. Parrish, Oscillator models and collective motion, IEEE Control Systems, 27 (2007), 89–105.
[31] D. D. Quinn, R. H. Rand and S. Strogatz, Singular unlocking transition in the Winfree model of coupled oscillators, Physical Review E, 75 (2007), 036218, 10 pp.
[32] D. D. Quinn, R. H. Rand and S. Strogatz, Synchronization in the Winfree model of coupled nonlinear interactions, A. ENOC 2005 Conference, Eindhoven, Netherlands (CD-ROM), (2005).
[33] J. Toner and Y. Tu, Flocks, herds, and schools: A quantitative theory of flocking, Phys. Rev. E, 58 (1998), 4828–4858.
[34] C. Villani, Optimal Transport: Old and New, Grundlehren der Mathematischen Wissenschaften, 338, Springer-Verlag, Berlin, 2009.
[35] A. Winfree, Biological rhythms and the behavior of populations of coupled oscillators, J. Theoret. Biol., 16 (1967), 15–42.

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