On a generalized fractional boundary value problem based on the thermostat model and its numerical solutions via Bernstein polynomials

Sina Etemad\textsuperscript{1}, Brahim Tellab\textsuperscript{2}, Chernet Tuge Deressa\textsuperscript{3}, Jehad Alzabut\textsuperscript{4,5,*}, Yongkun Li\textsuperscript{6} and Shahram Rezapour\textsuperscript{1,7,*}

\textsuperscript{1}Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
\textsuperscript{2}Department of Mathematics and General Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
\textsuperscript{3}Onaght model, Càdiz, Spain
\textsuperscript{4}Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
\textsuperscript{5}Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt
\textsuperscript{6}Department of Mathematics, Faculty of Science, Minia University, Minia, Egypt
\textsuperscript{7}Department of Mathematics, Faculty of Science, University of Mazandaran, Babol, Iran

\textsuperscript{*}Correspondence: jalzabut@psu.edu.sa; sh.rezapour@azaruniv.ac.ir; rezapourshahram@yahoo.ca

Abstract

In this paper, we introduce a new structure of the generalized multi-point thermostat control model motivated by its standard model. By presenting integral solution of this boundary problem, the existence property along with the uniqueness property are investigated by means of a special version of contractions named $\mu\phi$-contractions and the Banach contraction principle. Then, on the given nonlinear generalized BVP of thermostat, the Bernstein polynomials are introduced and numerical solutions obtained by them are presented. At the end, three different structures of nonlinear thermostat models are designed and the results are examined.

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1 Introduction

Fractional calculus and the existing notions in it are of high interest in different aspects of applied sciences, and one can find some instances of applications like signal and image processing, control theory, economics, optical systems, thermal materials, aerodynamics, mechanical systems, biology, and bio-mathematics [1–10]. The starting point for such a topic can be seen in many published papers in which mathematicians deal with some properties such as the existence of solution, uniqueness property, the property of stability, positivity, etc., and establish these properties to various abstract boundary value problems. Such a diversity and importance led to the publication of many research papers in this field, which revealed the flexibility of fractional calculus theory in designing various mathematical models. The main methods conducted in these articles are by terms of fixed point techniques [11–33].

Along with the investigation of existence theory, the approximation of solution and numerical methods are also of interest. Al-Smadi \textit{et al.} [34] utilized a method based on the
homotopy analysis for finding approximate solutions of a SEIR epidemic model in the fractional settings. Dhage, Dhage, and Ntouyas in [35] applied two notions of partial compactness and continuity to develop the method of Kranoselskii theorem and found the approximation of solutions regarding a hybrid ODE. Chadha et al. published a paper on the Faedo–Galerkin approximation for solutions of a neutral nonlocal FDE in a separable Hilbert space [36]. Other applications of approximate techniques can be found in [37–40].

The Bernstein polynomials are one of the strongest numerical techniques which possess some important properties such as the unity partition and continuity on [0, 1] [41]. In recent years, due to the importance and accuracy of this technique, the numerical solutions of a wide range of linear and nonlinear BVPs have been obtained for Riccati type FDEs, Bessel FDE, Lane–Emden equations, etc., [11–13, 42–44]. We here use these polynomials to find approximate solution of our given multi-point FBVP introduced in the sequel.

In 2006, two mathematicians, Infante and Webb, simulated a mathematical model of a mechanical instrument named thermostat in the form of a second-order boundary problem on the interval [0, 1] which is insulated at $s = 0$ via the controller for $s = 1$ [45]. This model has the following mathematical structure:

$$\begin{cases}
u''(s) = g(s, u(s)), & (s \in J := [0, 1]), \\
u'(0) = 0, & \mu u'(1) + u(\zeta) = 0,
\end{cases}$$

(1)

where $\zeta \in J$ and the parameter $\mu > 0$. Further, two other mathematicians, Nieto and Pimentel, converted the above problem to a similar version of arbitrary order [46] which takes the following structure:

$$\begin{cases}^cD^p u(s) = g(s, u(s)), & (s \in J := [0, 1]), \\
u'(0) = 0, & \mu^cD^{p-1} u(1) + u(\zeta) = 0, & \mu > 0, p \in (1, 2], p - 1 \in (0, 1],
\end{cases}$$

(2)

where $^cD^p$ is the Caputo derivative and $\zeta \in J$. In fact, at the time $s = \zeta$ and based on the existing temperature, the sensor detects that the thermostat discharges or adds heat.

In this manuscript, we concentrate on this aim in which some existence and uniqueness aspects and numerical analysis of solutions for a generalized fractional boundary value problem (GFBVP) based on thermostat model are investigated. Indeed, we formulate the following structure of a generalized multi-point thermostat control model motivated by the standard model (1):

$$\begin{cases}^cD^p u(s) = g(s, \beta u(s), ^cD^\sigma u(s), ^cT^p u(s)), & (s \in J := [0, 1]), \\
^cD^1 u(0) = \epsilon_1 \int_0^1 u(r) \, dr, \quad \sum_{j=1}^m u(\xi_j) + k^cD^{p-1} u(1) = \epsilon_2 \int_0^1 u(r) \, dr,
\end{cases}$$

(3)

in which $p \in (1, 2]$, $\sigma \in (0, 1)$, $p - 1 \in (0, 1]$, $k, \beta, \rho > 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $m \in \mathbb{N}$, $\epsilon_1, \epsilon_2 \in \mathbb{R}$, and $^cD^1 = \frac{d}{ds}$. Along with these, the mapping $g : J \times (\mathbb{R}^\geq 0)^3 \to \mathbb{R}^\geq 0$ is continuous and $^cD^\ell$ and $^cT^\ell$ display the derivation and integration operators of order $\ell \in \{1, p, p - 1\}, \sigma, \rho$ in the sense of Caputo and Riemann–Liouville.

The novelty, motivation, and objective of this research are:

- The multi-point nonlinear system (3) is a generalized form of the mathematical model of thermostat that by assuming $p = 2$, $\mu = k > 0$, $m = 1$,
\( \zeta_1 = \cdots = \zeta_m = \zeta \in (0,1), \varepsilon_1 = 0, \varepsilon_2 = 0, \) and \( \beta = 1, \) we obtain the second-order integro-differential FBVP (1);

- The existence property of solutions to the generalized nonlinear GFBVP (3) is derived by terms of a special case of contractions entitled \( \mu - \varphi \)-contraction and \( \mu \)-admissible maps;

- The approximate solution of this generalized nonlinear GFBVP of thermostat is obtained via Bernstein polynomials;

- The accuracy and absolute errors of the mentioned numerical technique are examined in different examples of thermostat model.

The rest of the manuscript is provided as follows: The primitive notions are indicated and recalled in Sect. 2. The existence property along with the uniqueness property are investigated in Sect. 3. Bernstein polynomials and numerical solutions obtained by them are presented in Sect. 4. In Sect. 5, three different structures of nonlinear thermostat GFBVP are designed, and the results are examined. We end the manuscript by giving conclusions in Sect. 6.

2 Basic notions

In this section, we provide some general basic tools and results of fractional calculus that allow us to achieve our desired results. For more details on this subject, we advise the authors to consult, for example, [47, 48].

**Definition 2.1** The Riemann–Liouville fractional integral (FRL-integral) of order \( \nu > 0 \) for a continuous function \( f : \mathbb{R}^+ \to \mathbb{R} \) is defined by

\[
I^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) \, d\tau,
\]

such that integral (4) converges.

**Definition 2.2** The Caputo derivative of order \( \nu > 0 \) for a continuous function \( f : \mathbb{R}^+_\alpha \to \mathbb{R} \) is defined by

\[
c^{D}_\nu f(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-\tau)^{n-\nu-1} f^{(n)}(\tau) \, d\tau,
\]

such that integral (5) exists, where \( n = [\nu] + 1.\)

**Lemma 2.3** ([49]) For \( p > 0 \) and \( \alpha > 0, \)

- \( c^{D}_0^p \alpha^\nu = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-p)} \alpha^\nu \) if \( \alpha \in [0] \cup \mathbb{N} \) and \( \alpha \geq |p| \) or \( \alpha \notin \mathbb{N} \) and \( \alpha > |p|, \)

- \( c^{D}_0^p \alpha^\nu = 0 \) if \( \alpha \in [0] \cup \mathbb{N} \) and \( \alpha < |p|, \)

- \( c^{D}_0^p \alpha^\nu = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-p)} \alpha^\nu, \)

- \( c^{D}_0^p C = 0 \) for all constant \( C.\)

Now, consider the family \( \Phi \) of all nondecreasing functions \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) which satisfy \( \sum_{n=1}^{\infty} \varphi^n(t) < +\infty, \forall t > 0. \) Consider \( (H,d) \) as a metric space, \( F \) as a self-map on \( H, \) and \( \mu : H \times H \to \mathbb{R}^+. \)
Definition 2.4 ([50]) $F$ is said to be a $\mu - \varphi$-contraction if
\[ \mu(x,y)d(Fx,Fy) \leq \varphi(d(x,y)), \quad \forall x, y \in H. \]

Definition 2.5 ([50]) $F$ is said to be $\mu$-admissible if
\[ \mu(x,y) \geq 1 \quad \Rightarrow \quad \mu(Fx,Fy) \geq 1, \quad \forall x, y \in H. \]

Theorem 2.6 ([50]) Let $(H,d)$ be a complete metric space and $F : H \to H$ be a $\mu - \varphi$-contraction. Assume:
(i) $F$ is $\mu$-admissible,
(ii) There exists $x^* \in H$ such that $\mu(x^*,Fx^*) \geq 1$,
(iii) For each sequence $x_n$ in $H$ which converges to $x \in H$ such that $\mu(x_n,x_{n+1}) \geq 1$ for all $n$, we have $\mu(x_n,x) \geq 1$ for all $n$.
Then there exists $z \in H$ satisfying the operator equation $Fz = z$.

3 The existence property
It is known that $X = \{ u : u, \mathcal{C}^p u \in C_{\mathbb{R}}(J) \}$ is a Banach space with the sup norm $\|u\|_X := \sup_{s \in J} |u(s)| + \sup_{s \in J} |\mathcal{C}^p u(s)|$, where $C_{\mathbb{R}}(J)$ denotes the collection of all continuous real-valued functions on $J$. In the following, we characterize the structure of the solutions for given GFBVP caused by thermostat model (3) which plays a key role in our required method. Before it, we introduce some notations for simplicity:

\[
\begin{align*}
V_1 & := \varepsilon_1, \\
V_2 & := 1 - \frac{\varepsilon_1}{2}, \\
V_3 & := m - \varepsilon_2, \\
V_4 & := \left( \frac{k}{1+(3-p)} - \frac{\varepsilon_2}{2} + \sum_{j=1}^{m} \xi_j \right), \\
V & := V_2 V_3 + V_1 V_4.
\end{align*}
\]

(6)

Proposition 3.1 Let $p \in (1,2], p - 1 \in (0,1], k > 0, 0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_m < 1, m \in \mathbb{N}, \varepsilon_1, \varepsilon_2 \in \mathbb{R}$, and $h \in C_{\mathbb{R}}(J)$. Then the solution of the linear thermostat GFBVP

\[
\begin{align*}
\mathcal{C}^p u(s) &= h(s), \quad (s \in J := [0,1]), \\
\mathcal{C}^1 u(0) &= \varepsilon_1 \int_0^1 u(r) \, dr, \\
\sum_{j=1}^{m} u(\xi_j) + k \mathcal{C}^{p-1} u(1) &= \varepsilon_2 \int_0^1 u(r) \, dr,
\end{align*}
\]

is given by

\[
u(s) = \int_0^s \frac{(s-r)^{p-1}}{\Gamma(p)} h(r) \, dr - \frac{kA(s)}{V} \int_0^1 h(r) \, dr - \frac{A(s)}{V} \sum_{j=1}^{m} \int_0^{\xi_j} \frac{(\xi_j-r)^{p-1}}{\Gamma(p)} h(r) \, dr + \frac{G(s)}{V} \int_0^1 \int_0^r \frac{(r-q)^{p-1}}{\Gamma(p)} h(q) \, dq \, dr,
\]

(8)

where $A, G \in C_{\mathbb{R}}(J)$ are introduced as

\[
A(s) := V_2 + V_1 s, \quad G(s) := (\varepsilon_2 V_2 - \varepsilon_1 V_4) + (\varepsilon_1 V_3 + \varepsilon_2 V_1) s.
\]

(9)
Proof. We assume that $u_\ast$ satisfies the linear thermostat GFBVP (7). Then $^cD^p u_\ast(s) = h(s)$. By integrating of order $1 < p \leq 2$ on both sides of it, we get

$$u_\ast(s) = \frac{1}{\Gamma(p)} \int_0^s (s - r)^{p-1} h(r) \, dr + c_0 + c_1 s, \quad (10)$$

where we try to obtain the constant values of the coefficients $c_0, c_1 \in \mathbb{R}$. On the other hand, we have

$$^cD^1 u_\ast(s) = \frac{1}{\Gamma(p-1)} \int_0^s (s - r)^{p-2} h(r) \, dr + c_1 \quad (11)$$

and

$$^cD^{p-1} u_\ast(s) = \int_0^s h(r) \, dr + c_1 \frac{s^{2-p}}{\Gamma(3-p)} \quad (12)$$

and

$$\int_0^1 u_\ast(r) \, dr = \frac{1}{\Gamma(p)} \int_0^1 \int_0^r (r-q)^{p-1} h(q) \, dq \, dr + c_0 + \frac{1}{2} c_1. \quad (13)$$

Now, in view of notations (6) and by using boundary conditions (7) and by invoking relations (11), (12), and (13), we reach

$$c_0 = - \frac{kV_2}{V} \int_0^1 h(r) \, dr - \frac{V_2}{V} \sum_{j=1}^m \frac{c_j}{\Gamma(p)} (\zeta_j - r)^{p-1} h(r) \, dr$$

$$+ \frac{(\epsilon_2 V_2 - \epsilon_1 V_4)}{V} \int_0^1 \int_0^r (r-q)^{p-1} h(q) \, dq \, dr$$

$$+ \frac{(\epsilon_1 V_3 + \epsilon_2 V_1)}{V} \int_0^1 \int_0^r (r-q)^{p-1} h(q) \, dq \, dr \quad (14)$$

and

$$c_1 = - \frac{kV_1}{V} \int_0^1 h(r) \, dr - \frac{V_1}{V} \sum_{j=1}^m \frac{c_j}{\Gamma(p)} (\zeta_j - r)^{p-1} h(r) \, dr$$

$$+ \frac{(\epsilon_1 V_3 + \epsilon_2 V_1)}{V} \int_0^1 \int_0^r (r-q)^{p-1} h(q) \, dq \, dr \quad (15)$$

Eventually, by (14) and (15), we substitute the obtained values for the coefficients $c_0$ and $c_1$ in (10) and it becomes

$$u_\ast(s) = \frac{1}{\Gamma(p)} \int_0^s (s - r)^{p-1} h(r) \, dr - \frac{kA(s)}{V} \int_0^1 h(r) \, dr$$

$$- \frac{A(s)}{V} \sum_{j=1}^m \frac{c_j}{\Gamma(p)} (\zeta_j - r)^{p-1} h(r) \, dr + \frac{G(s)}{V} \int_0^1 \int_0^r (r-q)^{p-1} h(q) \, dq \, dr,$$

which confirms that $u_\ast$ satisfies (8), and accordingly, the proof is finished. \qed
To follow the procedure of the paper, we introduce the operator $\mathbb{K} : X \rightarrow X$ associated with the nonlinear thermostat GFBVP which takes the form

\[
(\mathbb{K}u)(s) = \int_0^s (s-r)^{p-1} \frac{1}{\Gamma(p)} g(r, \beta u(r), \psi^p u(r), \mathcal{I}^p u(r)) \, dr
\]

\[
- \frac{kA(s)}{V} \int_0^s g(r, \beta u(r), \psi^p u(r), \mathcal{I}^p u(r)) \, dr
\]

\[
- \frac{A(s)}{V} \sum_{j=1}^{m} \int_0^s (\zeta_j - r)^{p-1} \frac{1}{\Gamma(p)} g(r, \beta u(r), \psi^p u(r), \mathcal{I}^p u(r)) \, dr
\]

\[
+ \frac{G(s)}{V} \int_0^s \int_0^r (r-q)^{p-1} \frac{1}{\Gamma(p)} g(q, \beta u(q), \psi^p u(q), \mathcal{I}^p u(q)) \, dq \, dr,
\]

where the functions $A, G \in C_{\mathbb{R}}(I)$ are introduced by (9).

Before presenting our main theorems, we equip the space $X$ with the metric $d$ formulated as $d(x, y) = \|x - y\|_X$.

It is well known that $(X, d)$ is a complete metric space (see [51]).

**Theorem 3.2** Consider a continuous function $g : J \times \mathbb{R}^3 \rightarrow \mathbb{R}$ and assume that the following assumptions hold:

(ASS1) There are a map $\varphi \in \Phi$ ($\Phi$ is the family defined in Section 2) and a function $w : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for all $x, \hat{x}, y, \hat{y}, z, \hat{z} \in \mathbb{R}$ we have $w(x, y) \geq 0$ and

\[
|g(s, x, y, z) - g(s, \hat{x}, \hat{y}, \hat{z})| \leq \frac{1}{\vartheta_1 + \vartheta_2} \varphi\left(|x - \hat{x}| + |y - \hat{y}| + |z - \hat{z}|\right),
\]

where $\vartheta_1$ and $\vartheta_2$ are two positive real constants which satisfy the following inequalities:

\[
\vartheta_1 > \frac{1}{\Gamma(p + 1)} + \frac{k(\|V_2\| + \|V_1\|)}{|V|} + \frac{|V_2| + |V_1|}{|V| \Gamma(p + 1)} \sum_{j=1}^{m} s_j^p + \frac{|e_2 V_2 - e_1 V_4| + |e_1 V_3 + e_2 V_1|}{|V| \Gamma(p + 2)}
\]

and

\[
\vartheta_2 > \frac{1}{\Gamma(p + 1)} + \frac{k(\|V_2\| + \|V_1\|)}{|V| \Gamma(2 - \sigma)} + \frac{|V_2| + |V_1|}{|V| \Gamma(p - \sigma + 1)} \sum_{j=1}^{m} s_j^{p-\sigma}
\]

\[
+ \frac{|e_2 V_2 - e_1 V_4| + |e_1 V_3 + e_2 V_1|}{|V| \Gamma(p - \sigma + 2)}.
\]

(ASS2) $\exists x^* \in X$ s.t. $w(x^*(s), \mathbb{K}x^*(s)) \geq 0, \forall s \in J$.

(ASS3) $\forall x, y \in X$, we have

\[
w(x(s), y(s)) \geq 0 \Rightarrow w(\mathbb{K}x(s), \mathbb{K}y(s)) \geq 0 \text{ for all } s \in J.
\]

(ASS4) For each sequence $x_n \rightarrow x$ in $X$ which converges to $x$ in $X$ and $w(x_n(s), x_{n+1}(s)) \geq 0, \forall s \in J$ and $\forall n \in \mathbb{N}$, we have $w(x_n(s), x(s)) \geq 0$.

(ASS5) The constants $\beta$ and $\rho$ are linked by the relation $\beta + \frac{1}{\Gamma(p + 1)} < 1$.

Then problem (3) has a solution.
Proof. Let us define a map $\mu : X \times X \to \mathbb{R}^+$ as

$$
\mu(x, y) = \begin{cases} 
1, & \text{if } w(x(s), y(s)) \geq 0, \\
0, & \text{otherwise}.
\end{cases}
$$

For all $x, y \in X$ and $w(x(s), y(s)) \geq 0$ for each $s \in J$, we have

$$
|Kx(s) - Ky(s)| \\
= \frac{1}{\Gamma(p)} \int_0^s |s - r|^{p-1} \left| g(r, \beta x(r), ^cD^\sigma x(r), ^I\sigma x(r)) - g(r, \beta y(r), ^cD^\sigma y(r), ^I\sigma y(r)) \right| \, dr \\
+ \frac{k|A(s)|}{|V|} \\
\times \int_0^s \left| g(r, \beta x(r), ^cD^\sigma x(r), ^I\sigma x(r)) - g(r, \beta y(r), ^cD^\sigma y(r), ^I\sigma y(r)) \right| \, dr \\
+ \frac{|A(s)|}{|V| \Gamma(p)} \\
\times \sum_{j=1}^m \int_0^s |s - r|^{p-1} \left| g(r, \beta x(r), ^cD^\sigma x(r), ^I\sigma x(r)) \\
- g(r, \beta y(r), ^cD^\sigma y(r), ^I\sigma y(r)) \right| \, dr \\
+ \frac{|G(s)|}{|V| \Gamma(p)} \\
\times \int_0^1 \int_0^s |r - q|^{p-1} \left| g(r, \beta x(r), ^cD^\sigma x(r), ^I\sigma x(r)) \\
- g(r, \beta y(r), ^cD^\sigma y(r), ^I\sigma y(r)) \right| \, dq \, dr \\
\leq \vartheta_1 \sup_{s \in J} \left| g(s, \beta x(s), ^cD^\sigma x(s), ^I\sigma x(s)) - g(s, \beta y(s), ^cD^\sigma y(s), ^I\sigma y(s)) \right|

\leq \vartheta_1 \sup_{s \in J} \left| g(s, \beta x(s), ^cD^\sigma x(s), ^I\sigma x(s)) - g(s, \beta y(s), ^cD^\sigma y(s), ^I\sigma y(s)) \right|
$$

(17)

and

$$
| ^cD^\sigma Kx(s) - ^cD^\sigma Ky(s)| \\
= \frac{1}{\Gamma(p - \sigma)} \int_0^s |s - r|^{p-\sigma-1} \left| g(r, \beta x(r), ^cD^\sigma x(r), ^I\sigma x(r)) - g(r, \beta y(r), ^cD^\sigma y(r), ^I\sigma y(r)) \right| \, dr \\
+ \frac{k|A(s)|}{|V| \Gamma(1 - \sigma)} \\
\times \sum_{j=1}^m \int_0^s |s - r|^{p-\sigma-1} \left| g(r, \beta x(r), ^cD^\sigma x(r), ^I\sigma x(r)) \\
- g(r, \beta y(r), ^cD^\sigma y(r), ^I\sigma y(r)) \right| \, dr \\
+ \frac{|A(s)|}{|V| \Gamma(1 - \sigma)} \\
\times \int_0^1 \int_0^s |r - q|^{p-\sigma-1} \left| g(r, \beta x(r), ^cD^\sigma x(r), ^I\sigma x(r)) \\
- g(r, \beta y(r), ^cD^\sigma y(r), ^I\sigma y(r)) \right| \, dq \, dr \\
\leq \vartheta_2 \sup_{s \in J} \left| g(s, \beta x(s), ^cD^\sigma x(s), ^I\sigma x(s)) - g(s, \beta y(s), ^cD^\sigma y(s), ^I\sigma y(s)) \right|
$$

(18)
\begin{align}
&\times \int_{0}^{s} |1 - \sigma|^{-\alpha} \left| g(r, \beta x(r), \mathcal{D}^\alpha x(r), \mathcal{I}^\rho x(r)) - g(r, \beta y(r), \mathcal{D}^\alpha y(r), \mathcal{I}^\rho y(r)) \right| \, dr \\
&+ \frac{|A(s)|}{|V| \Gamma(p - \sigma)} \\
&\times \sum_{j=1}^{m} \int_{0}^{q} |\xi_j - r|^{p - \sigma - 1} \left| g(r, \beta x(r), \mathcal{D}^\alpha x(r), \mathcal{I}^\rho x(r)) - g(r, \beta y(r), \mathcal{D}^\alpha y(r), \mathcal{I}^\rho y(r)) \right| \, dr \\
&\leq \varphi \left( \beta \sup_{s \in J} |x(s) - y(s)| + \sup_{s \in J} |\mathcal{D}^\alpha x(s) - \mathcal{D}^\alpha y(s)| + \varphi \left( \beta \sup_{s \in J} |x(s) - y(s)| + \sup_{s \in J} |\mathcal{D}^\alpha x(s) - \mathcal{D}^\alpha y(s)| + \sup_{s \in J} |\mathcal{I}^\rho x(s) - \mathcal{I}^\rho y(s)| \right) \right) \\
&\leq \varphi \left( \beta \sup_{s \in J} |x(s) - y(s)| + \sup_{s \in J} |\mathcal{D}^\alpha x(s) - \mathcal{D}^\alpha y(s)| + \varphi \left( \beta \sup_{s \in J} |x(s) - y(s)| + \sup_{s \in J} |\mathcal{D}^\alpha x(s) - \mathcal{D}^\alpha y(s)| + \sup_{s \in J} |\mathcal{I}^\rho x(s) - \mathcal{I}^\rho y(s)| \right) \right) \\
&\leq \varphi \left( \beta \sup_{s \in J} |x(s) - y(s)| + \sup_{s \in J} |\mathcal{D}^\alpha x(s) - \mathcal{D}^\alpha y(s)| + \sup_{s \in J} |\mathcal{I}^\rho x(s) - \mathcal{I}^\rho y(s)| \right) \\
&\leq \varphi \left( \beta \sup_{s \in J} |x(s) - y(s)| + \sup_{s \in J} |\mathcal{D}^\alpha x(s) - \mathcal{D}^\alpha y(s)| + \beta \sup_{s \in J} |x(s) - y(s)| + \sup_{s \in J} |\mathcal{D}^\alpha x(s) - \mathcal{D}^\alpha y(s)| \right) \\
&\leq \varphi \left( \beta + \frac{1}{\Gamma(p + 1)} \right) \sup_{s \in J} |x(s) - y(s)| + \sup_{s \in J} |\mathcal{D}^\alpha x(s) - \mathcal{D}^\alpha y(s)| \\
&\leq \varphi \sup_{s \in J} |x(s) - y(s)| + \sup_{s \in J} |\mathcal{D}^\alpha x(s) - \mathcal{D}^\alpha y(s)|.
\end{align}
This means that 
\[ d(\mathbb{K}x, \mathbb{K}y) \leq \varphi(d(x, y)) \]  
Consequently, from the definition of the map \( \mu \) it follows that

\[ \mu(x, y) d(\mathbb{K}x, \mathbb{K}y) \leq \varphi(d(x, y)), \quad \forall x, y \in X, \]

which means that \( \mathbb{K} \) is a \( \mu - \varphi \) contraction. Furthermore, in view of the definition of the map \( \mu \) and assumption (ASS3), we can easily verify that \( \mathbb{K} \) is \( \mu \)-admissible.

Let now \( x_n \) be a sequence in \( X \) which approaches to \( x \) in \( X \) and satisfies \( \mu(x, x_{n+1}) \geq 1 \), \( \forall n \in \mathbb{N} \) and \( \omega(x_n(s), x_{n+1}(s)) \geq 0 \). Then from the definition of the map \( \mu \) together with assumption (ASS5), we can directly verify that \( \mu(x_n, x) \geq 1 \). At this time, all the assumptions of Theorem 2.6 are fulfilled. Consequently, the operator \( \mathbb{K} \) admits a fixed point which is solution of our nonlinear thermostat GFBVP (3). \( \square \)

**Theorem 3.3** Let \( g : J \times \mathbb{R}^3 \to \mathbb{R} \) be continuous and the following assumptions hold:

(ASS6) \( \exists R > 0, \text{ s.t. } \forall s \in J \), \( \forall x, \bar{x}, y, \bar{y}, z, \bar{z} \in \mathbb{R}, \)

\[
|g(s, x, y, z) - g(s, \bar{x}, \bar{y}, \bar{z})| \leq R(|x - \bar{x}| + |y - \bar{y}| + |z - \bar{z}|),
\]

(ASS7) The constants \( \beta \) and \( \rho \) are linked by the relation \( \beta + \frac{1}{\Gamma(p+1)} \rho < 1 \) and

\[
\gamma = R \left( \frac{1}{\Gamma(p+1)} + \frac{k(|V_2| + |V_1|)}{|V|} + \frac{|V_2| + |V_1|}{|V|\Gamma(p+1)} \sum_{j=1}^{m} \xi_j^\rho \right.
\]

\[
+ \frac{|\varepsilon_2 V_2 - \varepsilon_1 V_4| + |\varepsilon_1 V_3 + \varepsilon_2 V_1|}{|V|\Gamma(p+2)} + \frac{1}{\Gamma(p-\sigma + 1)} + \frac{k(|V_2| + |V_1|)}{|V|\Gamma(2-\sigma)}
\]

\[
+ \frac{|V_2| + |V_1|}{|V|\Gamma(p-\sigma + 1)} \sum_{j=1}^{m} \xi_j^{\rho-\sigma} + \frac{|\varepsilon_2 V_2 - \varepsilon_1 V_4| + |\varepsilon_1 V_3 + \varepsilon_2 V_1|}{|V|\Gamma(p-\sigma + 2)} \right) < 1. \tag{19}
\]

Then the nonlinear thermostat GFBVP (3) has exactly one solution.

**Proof** By following the same arguments of the calculations used in Theorem 3.2 together with the hypotheses of Theorem 3.3, we write

\[
\|\mathbb{K}x(s) - \mathbb{K}y(s)\|
\]

\[
\leq R \left[ \frac{1}{\Gamma(p+1)} + \frac{k(|V_2| + |V_1|)}{|V|} + \frac{|V_2| + |V_1|}{|V|\Gamma(p+1)} \sum_{j=1}^{m} \xi_j^\rho \right.
\]

\[
+ \frac{|\varepsilon_2 V_2 - \varepsilon_1 V_4| + |\varepsilon_1 V_3 + \varepsilon_2 V_1|}{|V|\Gamma(p+2)} \right]
\]

\[
\times \sup_{s \in \mathbb{J}} \left| g(s, \beta x(s), ^c D^\rho x(s), \mathcal{I}^\rho x(s)) - g(s, \beta y(s), ^c D^\rho y(s), \mathcal{I}^\rho y(s)) \right|
\]

\[
\leq R \left[ \frac{1}{\Gamma(p+1)} + \frac{k(|V_2| + |V_1|)}{|V|} + \frac{|V_2| + |V_1|}{|V|\Gamma(p+1)} \sum_{j=1}^{m} \xi_j^\rho \right.
\]

\[
+ \frac{|\varepsilon_2 V_2 - \varepsilon_1 V_4| + |\varepsilon_1 V_3 + \varepsilon_2 V_1|}{|V|\Gamma(p+2)} \right].
\]
Thus,\[
\parallel \beta |x(s) - y(s)| + |cD^\sigma x(s) - cD^\sigma y(s)| + |I^p x(s) - I^p y(s)|\parallel
\]
\[
\leq R \left[ \frac{1}{\Gamma(p + 1)} + \frac{k(|V_2| + |V_1|)}{|V|} + \frac{|V_2| + |V_1|}{|V|}\Gamma(p + 1) \sum_{j=1}^{m} \zeta_j^p 
+ \frac{|\varepsilon_2 V_2 - \varepsilon_1 V_4| + |\varepsilon_1 V_3 + \varepsilon_2 V_1|}{|V|}\Gamma(p + 2) \right] 
\times \left( \sup_{s \in I} |x(s) - y(s)| + \sup_{s \in I} |cD^\sigma x(s) - cD^\sigma y(s)| \right).
\]

Similarly, we obtain
\[
|cD^\sigma kx(s) - cD^\sigma ky(s)|
\]
\[
\leq R \left[ \frac{1}{\Gamma(p - \sigma + 1)} + \frac{k(|V_2| + |V_1|)}{|V|}\Gamma(2 - \sigma) 
+ \frac{|V_2| + |V_1|}{|V|}\Gamma(p - \sigma + 1) \sum_{j=1}^{m} \zeta_j^{p - \sigma} 
+ \frac{|\varepsilon_2 V_2 - \varepsilon_1 V_4| + |\varepsilon_1 V_3 + \varepsilon_2 V_1|}{|V|}\Gamma(p - \sigma + 2) \right] 
\times \left( \sup_{s \in I} |x(s) - y(s)| + \sup_{s \in I} |cD^\sigma x(s) - cD^\sigma y(s)| \right).
\]

Thus,
\[
\|kx - ky\|_\mathcal{X} \leq R \left( \frac{1}{\Gamma(p + 1)} + \frac{k(|V_2| + |V_1|)}{|V|} 
+ \frac{|V_2| + |V_1|}{|V|}\Gamma(p + 1) \sum_{j=1}^{m} \zeta_j^p 
+ \frac{|\varepsilon_2 V_2 - \varepsilon_1 V_4| + |\varepsilon_1 V_3 + \varepsilon_2 V_1|}{|V|}\Gamma(p + 2) 
\right) \|x - y\|_\mathcal{X}.
\]
It yields \( \|Kx - Ky\|_X \leq \gamma \|x - y\|_X \), \( \forall x, y \in X \). Now, from the assumption \( \gamma < 1 \) and the Banach contraction principle, we conclude that \( K \) admits a unique fixed point which represents the unique solution of our nonlinear thermostat GFBVP (3).

\[ \square \]

### 4 Numerical solutions via Bernstein polynomials

In the first place, we describe the basic formulation of Bernstein polynomials which are necessary to derive our developed results.

**Definition 4.1** [41] The \( m + 1 \) Bernstein basis polynomials of degree \( m \) are defined on \([0, 1]\) by

\[
b_{j,m}(z) = \binom{m}{j} z^j (1 - z)^{m-j}, \quad 0 \leq j \leq m,
\]

where \( \binom{m}{j} = \frac{m!}{j!(m-j)!} \).

A recursive expression also can be used to formulate the Bernstein basis polynomials on the interval \([0, 1]\) such that the Bernstein polynomials of \((j, m)\)th degree can be rewritten as

\[
b_{j,m}(z) = (1 - z)b_{j,m-1}(z) + zb_{j-1,m-1}(z).
\]

We can easily show that each of Bernstein basis polynomials is positive and also the sum of all Bernstein basis polynomials is equal to unity for any real \( z \) belonging to the interval \([0, 1]\); in other words, \( \sum_{j=0}^{m} b_{j,m}(z) = 1 \).

It is easy to show that any given polynomial of degree \( m \) can be developed as a linear combination of basic functions

\[
\phi(z) = \sum_{j=0}^{m} c_j b_{j,m}(z) = C^T B(z), \quad m \geq 1,
\]

in which the Bernstein vector \( B(z) \) and the Bernstein coefficient vector \( C \) are defined as

\[
B(z) = [b_{0,m}(z), b_{1,m}(z), \ldots, b_{m,m}(z)],
\]

and \( C^T = [c_0, c_1, \ldots, c_m] \) with

\[
c_j = \int_0^1 \phi(z) d_{j,m}(z) \, dz, \quad 0 \leq j \leq m. \tag{20}
\]

In [52], Juttler has explicitly represented \( d_{j,m}(z) \) by the following formula:

\[
d_{j,m}(z) = \sum_{k=0}^{m} \lambda_{j,k} b_{k,m}(z), \quad 0 \leq j \leq m, \tag{21}
\]

where for \( 0 \leq j, k \leq m \),

\[
\lambda_{j,k} = (-1)^{j+k} \binom{m+j}{j} \sum_{i=0}^{\min(j,k)} \binom{m+i+1}{m-j+1} \binom{m+i+1}{m-k} \binom{m-i}{m-j} \binom{m-i}{m-k}. \tag{22}
\]
Furthermore,

\[ I^\nu B(z) \simeq I^{(\nu)} B(z), \quad \nu > 0, \quad (23) \]

where \( I^\nu \) denotes the \( \nu^{th} \)-FRL-integral and \( I^{(\nu)} \) represents the \((m+1) \times (m+1)\) operational matrix of the \( \nu^{th} \)-FRL-integral.

Also, by using the binomial expansion of \((1 - z)^{m-j}\), we can write

\[
\begin{align*}
  b_{j,m}(z) &= \binom{m}{j} z^j (1 - z)^{m-j} \\
  &= \binom{m}{j} z^j \left( \sum_{k=0}^{m-j} (-1)^k \binom{m-j}{k} z^k \right) \\
  &= \sum_{k=0}^{m-j} (-1)^k \binom{m}{j} \binom{m-j}{k} z^{j+k} \\
  &= \sum_{i=j}^{m} (-1)^{i-j} \binom{m}{j} \binom{m-j}{i-j} z^i, \quad 0 \leq j \leq m.
\end{align*}
\]

As we can find in [52], an approach for the direct least squares approximation with the help of Bernstein polynomials is based on the construction of the basis \( \{d_{0,m}(z), d_{1,m}(z), \ldots, d_{m,m}(z)\} \) which represents the dual in Bernstein basis of \( m^{th} \)-degree on \([0,1]\). It is specified as

\[ \int_0^1 b_{j,m}(z) d_{k,m}(z) dz = \delta_{jk}, \quad 0 \leq j, k \leq m, \quad (25) \]

where \( \delta_{jk} \) denotes the Kronecker symbol.

4.1 Fractional matrix of integration

The conclusion of the next theorem is useful for us.

**Theorem 4.2** Let \( B(z) \) be the Bernstein vector introduced in (23) and \( I^{(\nu)} \) be the \((m+1) \times (m+1)\) operational matrix of the \( \nu^{th} \)-FRL-integral which is formulated by

\[
I^{(\nu)} = \begin{pmatrix}
  w_{0,0} & w_{0,1} & \cdots & w_{0,i} & \cdots & w_{0,m} \\
  w_{1,0} & w_{1,1} & \cdots & w_{1,i} & \cdots & w_{1,m} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  w_{i,0} & w_{i,1} & \cdots & w_{i,i} & \cdots & w_{i,m} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  w_{m,0} & w_{m,1} & \cdots & w_{m,i} & \cdots & w_{m,m}
\end{pmatrix}. \quad (26)
\]

Then we have

\[
  w_{i,j} = \sum_{q=i}^{m} \sum_{k=0}^{m} (-1)^{q-i} \binom{m}{i} \binom{m-i}{q-i} \lambda_{jk} \mu_{qk} \frac{\Gamma(q+1)}{\Gamma(q+v+1)}. \quad (27)
\]
where
\[\mu_{qk} = \sum_{p=k}^{m} \frac{(-1)^{p-k}}{q + p + v + 1} \binom{m}{k} \binom{m-k}{p-k},\]

and \(\lambda_{jk}\) are the coefficients expressed by (22).

**Proof** From expression (23) and by exploiting relationship (25) which connects Bernstein polynomials with their dual, we can write
\[I(\nu) = \langle I_{\nu} B(z), D^T \rangle,\]

where \(D\) denotes the vector of dual polynomials of Bernstein polynomials and \(\langle I_{\nu} B(z), D^T \rangle\) stands for the \((m+1) \times (m+1)\)-matrix formulated as
\[\langle I_{\nu} B(z), D^T \rangle = \langle I_{\nu} b_{im}(z), d_{jm}(z) \rangle, \quad 0 \leq i,j \leq m.\]

Here, we have
\[w_{ij} = \langle I_{\nu} b_{im}(z), d_{jm}(z) \rangle = \int_0^1 I_{\nu} b_{im}(z) d_{jm}(z) \, dz.\]

By applying (24), we get
\[I_{\nu} b_{im}(z) = \sum_{q=i}^{m} (-1)^{q-i} \binom{m}{i} \binom{m-i}{q-i} \frac{\Gamma(q+1)}{\Gamma(q+v+1)} \gamma_{q+i}.\]

Afterwards, in view of (21) we conclude that
\[w_{ij} = \sum_{q=i}^{m} \sum_{k=0}^{m} (-1)^{q-i} \binom{m}{i} \binom{m-i}{q-i} \lambda_{jk} \frac{\Gamma(q+1)}{\Gamma(q+v+1)} \int_0^1 z^{q+i} b_{k,m}(z) \, dz, (28)\]

but we have
\[\int_0^1 z^{q+i} b_{k,m}(z) \, dz = \sum_{p=k}^{m} \frac{(-1)^{p-k}}{q + p + v + 1} \binom{m}{k} \binom{m-k}{p-k} = \mu_{qk}. (29)\]

Therefore, a combination of (28) and (29) ends the proof of our Theorem 4.2. \(\square\)

**4.2 Fractional matrix of derivative**

We can write the derivative of \(B(z)\) as
\[\frac{dB(z)}{dz} = D^{(1)} B(z),\]

where \(D^{(1)}\) stands for the \((m+1) \times (m+1)\)-operational matrix of derivative given in the following format:
\[D^{(1)} = AWB^*.\]

For more details, we refer to [43, 53–55].
By applying (30), it is obvious that for each \( n \in \mathbb{N} \) we have

\[
\frac{d^n B(z)}{dz^n} = (D^{(1)})^n B(z).
\]

Consequently,

\[
D^{(n)} = (D^{(1)})^n, \quad n = 1, 2, \ldots
\]

Now, in order to generalize the operational matrix of derivative, we indicate the following formulations.

For \( \nu > 0 \), the \( \nu \)-th Caputo derivative of \( B(z) \) is given as

\[
cD^{\nu} B(z) \simeq cD^{(\nu)} B(z), \quad (32)
\]

in which \( cD^{(\nu)} \) stands for the \((m + 1) \times (m + 1)\)-operational matrix of the \( \nu \)-th Caputo derivative which is given by

\[
cD^{(\nu)} = \begin{pmatrix}
\sum_{j=1}^{m} \omega_{0,j,0} & \sum_{j=1}^{m} \omega_{0,j,1} & \cdots & \sum_{j=1}^{m} \omega_{0,j,2} & \cdots & \sum_{j=1}^{m} \omega_{0,j,m} \\
\sum_{j=1}^{m} \omega_{1,j,0} & \sum_{j=1}^{m} \omega_{1,j,1} & \cdots & \sum_{j=1}^{m} \omega_{1,j,2} & \cdots & \sum_{j=1}^{m} \omega_{1,j,m} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\sum_{j=1}^{m} \omega_{m,j,0} & \sum_{j=1}^{m} \omega_{m,j,1} & \cdots & \sum_{j=1}^{m} \omega_{m,j,2} & \cdots & \sum_{j=1}^{m} \omega_{m,j,m}
\end{pmatrix}, \quad (33)
\]

where

\[
\omega_{i,j,p} = (-1)^{j-i} {m \choose i} {m-i \choose j-i} \frac{\Gamma(j + 1)}{\Gamma(j + 1 - \nu)} \sum_{k=0}^{m} \lambda_{pk} \mu_{kj},
\]

\( \lambda_{pk} \) is defined as (22) and

\[
\mu_{kj} = \sum_{s=k}^{m} \frac{(-1)^{s-k}}{j - s + 1} {m-s \choose s-k}.
\]

5 Some simulative examples

Before illustrating our theoretical results by some numerical examples, we present, in general, the principle of the Bernstein collocation method applied to our problem (3) to obtain an accurate numerical solution. For this fact, let us consider for all \( s \in J \)

\[
cD^{p} u(s) = g(s, \beta u(s), cD^{\sigma} u(s), I^{p} u(s)), \quad (34)
\]

with the following conditions:

\[
cD^{1} u(0) = \varepsilon_1 \int_0^1 u(r) \, dr, \quad \sum_{j=1}^{m} u(\zeta_j) + k^{c}cD^{p-1} u(1) = \varepsilon_2 \int_0^1 u(r) \, dr. \quad (35)
\]
Now, to determine an approximation of the exact solution $u(s)$ by Bernstein polynomials, we utilize the FRL-integral of Bernstein polynomials with the operational matrix of the Caputo derivative used in [53].

We know that the approximate solution of $u(s)$ by Bernstein polynomials is defined by

$$u(s) \approx \phi(s) = \sum_{j=0}^{m} c_j b_j(s) = C^T B(s),$$  \hspace{1cm} (36)$$

such that $C$ is an indeterminate vector. By replacing the approximate solution given by (36) in (34) and (35), we can write respectively

$$C^T \epsilon D^{(p)} B(s) = g(s, \beta C^T B(s), C^T \epsilon D^{(o)} B(s), C^T I^{(o)} B(s))$$  \hspace{1cm} (37)$$

and

$$- c_0 m + c_1 m = \frac{E_1}{m + 1} C^T 1,$$  \hspace{1cm} (38)$$

$$C^T \sum_{j=1}^{3} B(\zeta_j) + k C^T \epsilon D^{(p-1)} B(1) = \frac{E_2}{m + 1} C^T 1,$$  \hspace{1cm} (39)$$

where $1 = [1, 1, \ldots, 1]_m^T$.

For convenience of computations, we consider equidistant points and the roots of the Legendre polynomial of degree $(m-1)$ in $[0,1]$. So, to obtain the solution $u(s)$, we collocate equation (37) at $(m-1)$ points together with equations (38) and (39). Therefore, we get $(m+1)$ equations with $(m+1)$ indeterminate coefficients. Consequently, the approximate solution can be determined.

At the moment, we are ready to illustrate the Bernstein collocation method with some simulative examples.

**Example 5.1** According to the nonlinear thermostat GFBVP (3), consider

$$\left\{ \begin{align*}
\epsilon D^{\frac{5}{2}} u(s) &= \phi(s) + \frac{s}{10} u(s) - \frac{\sin(s)}{20} \epsilon D^{\frac{1}{2}} u(s) - \frac{1}{11} \frac{2}{3} u(s), \\
\epsilon D^{1} u(0) &= 0, \quad \sum_{j=1}^{3} u(\frac{j}{3}) + \frac{1111}{118} \epsilon D^{4} u(1) = \frac{107}{720} \int_0^1 u(r) \, dr.
\end{align*} \right.$$  \hspace{1cm} (40)$$

In this example, we have $p = \frac{5}{2}$, $\sigma = \frac{1}{4}$, $\rho = \frac{7}{3}$, $\epsilon_1 = 0$, $\epsilon_2 = \frac{107}{720}$, $\beta = \frac{1}{10}$, $k = \frac{1111}{118}$, and $\zeta_j = \frac{j}{3}$ for $j = 1, 2, 3$, and

$$\phi(s) = \frac{3}{16} \frac{s^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} + \frac{24}{11} \frac{s^{\frac{7}{2}}}{\Gamma\left(\frac{7}{2}\right)} - \frac{1}{10} \left( s^3 + 2s^4 \right) \sin(s) \left( \frac{1}{10} \frac{1}{\Gamma\left(\frac{11}{2}\right)} s^{\frac{11}{2}} + \frac{6}{5} \frac{1}{\Gamma\left(\frac{13}{2}\right)} s^{\frac{13}{2}} \right)$$

$$+ \frac{2}{11} \frac{1}{\Gamma\left(\frac{13}{2}\right)} s^{\frac{13}{2}} + \frac{48}{111} \frac{1}{\Gamma\left(\frac{16}{2}\right)} s^{\frac{16}{2}}.$$  \hspace{1cm} (41)$$

We take $w(x, y) = 1$, $\forall x, y \in X$ and $\psi(s) = \frac{1}{2} s$ for any $s \in J$. Therefore, it is easy to verify that our nonlinear thermostat GFBVP (40) satisfies all assumptions of Theorem 3.2, and its exact solution is $u(s) = s^2 + 2s^3$. 

Table 1  Legendre polynomials and exact solution at different values of variable with absolute error

| s   | Legendre polynomials | Exact solution | Absolute error |
|-----|----------------------|----------------|----------------|
| 0.0 | 0.000006             | 0.000000       | $6.0000 \times 10^{-6}$ |
| 0.1 | 0.121003             | 0.012000       | $1.0300 \times 10^{-4}$ |
| 0.2 | 0.055900             | 0.056000       | $1.0000 \times 10^{-5}$ |
| 0.3 | 0.144110             | 0.144000       | $1.1000 \times 10^{-4}$ |
| 0.4 | 0.287800             | 0.288000       | $2.0000 \times 10^{-4}$ |
| 0.5 | 0.500190             | 0.500000       | $1.9000 \times 10^{-4}$ |
| 0.6 | 0.800012             | 0.792000       | $0.8000 \times 10^{-5}$ |
| 0.7 | 0.176100             | 1.176000       | $1.0000 \times 10^{-4}$ |
| 0.8 | 1.664020             | 1.664000       | $2.0000 \times 10^{-5}$ |
| 0.9 | 2.267900             | 2.268000       | $1.0000 \times 10^{-4}$ |
| 1.0 | 3.000130             | 3.000000       | $1.3000 \times 10^{-4}$ |

Now, we substitute $u(s)$ by $C^T B(s)$ in the nonlinear thermostat GFBVP (40), from which we get the following system:

\[
\begin{align*}
C^T \mathcal{D}^{\frac{3}{2}} B(s) &= \phi(s) + C^T \frac{s}{10} B(s) - C^T \frac{\sin(s)}{10} \mathcal{D}^{\frac{1}{2}} B(s) - C^T \frac{1}{11} I^{\frac{1}{2}} B(s), \\
-c_0 + c_1 &= 0, \\
C^T \sum_{j=1}^{3} B(\sqrt{\frac{4}{11}(5-j)(7-j)}) + \frac{11\Gamma(\frac{11}{3})}{118} C^T \mathcal{D}^{\frac{1}{2}} B(1) &= \frac{3407}{720(m+1)} C^T 1.
\end{align*}
\]

In Table 1 we list the absolute errors $|u_m(s) - u(s)|$ of approximate solution $u_m(s)$ computed via the roots of shifted Legendre polynomials at $m = 4$. Also, the graphs are plotted in Fig. 1.

Now, we investigate the next example.

Example 5.2  According to the nonlinear thermostat GFBVP (3), consider

\[
\begin{align*}
\mathcal{D}^{\frac{3}{2}} u(s) &= \phi(s) + \frac{s}{10} u(s) + \frac{\sin(s)}{2} \mathcal{D}^{\frac{1}{2}} u(s) - \frac{7s}{100} I^{\frac{1}{2}} u(s), \\
\mathcal{D}^{1} u(0) &= 0, \\
\sum_{j=1}^{3} u(\sqrt{\frac{3}{4}(5-j)(7-j)}) + \frac{11\Gamma(\frac{11}{3})}{6} \mathcal{D}^{\frac{1}{2}} u(1) &= 7 \int_{0}^{1} u(r) \, dr,
\end{align*}
\]

(41)

where

\[
\phi(s) = \frac{6}{\Gamma(\frac{3}{2})} s^{\frac{1}{2}} - \frac{s^4}{10} - \frac{3 \sin(s)}{\Gamma(\frac{11}{2})} s^{\frac{1}{2}} + \frac{21s}{50\Gamma(\frac{19}{3})} s^{\frac{1}{3}}.
\]
Table 2  Legendre polynomials and exact solution at different values of variable with absolute error

| s   | Legendre polynomials | Exact solution | Absolute error |
|-----|----------------------|----------------|----------------|
| 0.0 | 0.000002             | 0.000000       | 2.0000 × 10⁻⁶ |
| 0.1 | 0.001005             | 0.001000       | 5.0000 × 10⁻⁶ |
| 0.2 | 0.008004             | 0.008000       | 4.0000 × 10⁻⁶ |
| 0.3 | 0.027011             | 0.027000       | 1.1000 × 10⁻⁶ |
| 0.4 | 0.064010             | 0.064000       | 1.0000 × 10⁻⁶ |
| 0.5 | 0.125012             | 0.125000       | 1.2000 × 10⁻⁶ |
| 0.6 | 0.216003             | 0.216000       | 3.0000 × 10⁻⁶ |
| 0.7 | 0.343011             | 0.343000       | 1.1000 × 10⁻⁶ |
| 0.8 | 0.512003             | 0.512000       | 1.0000 × 10⁻⁵ |
| 0.9 | 0.729010             | 0.729000       | 1.0000 × 10⁻⁵ |
| 1.0 | 1.000012             | 1.000000       | 1.2000 × 10⁻⁵ |

In the present example, we have \( p = \frac{5}{3}, \quad \sigma = \frac{1}{5}, \quad \rho = \frac{7}{3}, \quad \varepsilon_1 = 0, \quad \varepsilon_2 = 7, \quad \beta = \frac{1}{10}, \quad k = \frac{m(10)}{6}, \) and \( \zeta_j = \frac{3}{\sqrt{(4-j)(s-j)}} \) for \( j \in \{1, 2, 3\}. \)

We take \( w(x,y) = x, \forall x, y \in X \) and \( \phi(s) = \frac{1}{2}s, \forall s \in J. \) Hence, all assumptions of Theorem 3.2 are satisfied and the exact solution of the nonlinear thermostat GFBVP (41) is given by \( u(s) = s^3. \) By the same arguments used in problem (40), we get the following system:

\[
\begin{align*}
C^T \mathbf{D}^{\frac{3}{2}} B(s) &= \phi(s) + C^T \frac{1}{10} B(s) + C^T \frac{\sin(s)}{2} \mathbf{D}^{\frac{1}{2}} B(s) - C^T \frac{7}{100} I^{\frac{3}{2}} B(s), \\
-c_0 + c_1 &= 0, \\
C^T \sum_{j=1}^{3} B(\sqrt{\frac{1}{(4-j)(s-j)}}) + \frac{\Gamma(\frac{1}{2})}{6} C^T \mathbf{D}^{\frac{1}{2}} B(1) &= \frac{7}{m+1} C^T \mathbf{1}.
\end{align*}
\]

Now, we list the absolute errors \(|u_m(s) - u(s)|\) of approximate solution \(u_m(s)\) by utilizing the roots of shifted Legendre polynomials at \( m = 7 \) in Table 2, and the graphs are plotted in Fig. 2.

Example 5.3  According to the nonlinear thermostat GFBVP (3), consider

\[
\begin{align*}
\mathcal{D}^\frac{3}{2} u(s) &= \phi(s) + \frac{3u(s)}{15+2u^2(s)} + \frac{\sin(s)}{10} \mathcal{D}^\frac{1}{2} u(s) + \frac{1}{14(1+e^{-s})} \mathcal{D}^\frac{3}{2} u(s), \\
\mathcal{D}^1 u(0) &= 3 \int_0^1 u(r) \, dr, \\
\sum_{j=1}^{2} u\left(\frac{j}{m+1}\right) + \frac{131(\frac{1}{2})!^2(\frac{3}{2})!^2}{72(\Gamma(\frac{1}{2})^2-\Gamma(\frac{3}{2})^2)} \mathcal{D}^\frac{3}{2} u(1) &= 3 \int_0^1 u(r) \, dr, 
\end{align*}
\]

(42)
where
\[
\phi(s) = \frac{2}{\Gamma\left(\frac{3}{2}\right)} s^{\frac{3}{2}} - \frac{0.3(s^2 - 2s)}{15 + 2(s^2 - 2s)} - \frac{1}{5\Gamma\left(\frac{5}{3}\right)} s^{\frac{5}{3}} + \frac{1}{5\Gamma\left(\frac{3}{3}\right)} s^{\frac{3}{3}} - \frac{1}{7\Gamma\left(\frac{5}{4}\right)} \times \frac{s^{\frac{15}{4}}}{1 + e^{-s}} + \frac{1}{7\Gamma\left(\frac{15}{4}\right)} \times \frac{s^{\frac{11}{4}}}{1 + e^{-s}}.
\]

In this example, we have \( p = \frac{7}{52}, \sigma = \frac{1}{3}, \rho = \frac{7}{4}, \epsilon_1 = 3, \epsilon_2 = 3, \beta = 0.3, k = \frac{13\Gamma\left(\frac{3}{2}\right)^{\frac{3}{2}}}{72\Gamma\left(\frac{5}{4}\right)^{\frac{5}{4}}}, \) and \( \zeta_j = \frac{j}{11} \) for \( j \in \{1, 2\} \). Take \( R = 0.03 \). Then, by direct calculation, we find
\[
\beta + \frac{1}{\Gamma(\rho + 1)} \approx 0.9218... < 1,
\]
and \( \gamma \approx 0.8922... < 1 \). Consequently, Theorem 3.3 ensures the existence of a unique solution of the nonlinear thermostat GFBVP (42).

6 Conclusions
In this work, we introduced a new generalized version of the mathematical model of the thermostat in the form of the nonlinear GFBVP given as (3). The existence property for its solutions was established via a special form of contractions and \( \mu \)-admissible maps. The uniqueness property was verified by the Banach principle. Further, we used the Bernstein operational matrix of the Caputo fractional derivative and the Bernstein operational matrix of FRL-integral which are necessary to obtain accurate numerical solutions to the nonlinear thermostat GFBVP (3) via Bernstein polynomials. We have designed two examples to illustrate the accuracy of the numerical method in finding the exact and approximate solutions. Then we checked the uniqueness property in the third example. For the next works, we will apply these methods on different mathematical models designed by nonsingular operators.

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The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.
Author details
1 Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran. 2 Laboratory of Applied Mathematics, Kordi Merbaha University, B.P. 511, 30000, Ouargla, Algeria. 3 College of Natural Sciences, Department of Mathematics, Jimma University, Jimma, Ethiopia. 4 Department of Mathematics and General Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia. 5 Group of Mathematics, Faculty of Engineering, Ostum Technical University, Ankara, 06374, Turkey. 6 Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China. 7 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan.

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References
1. Tuan, N.H., Mohammadi, H., Rezapour, S.: A mathematical model for Covid-19 transmission by using the Caputo fractional derivative. Chaos Solitons Fractals 140, 110107 (2020). https://doi.org/10.1016/j.chaos.2020.110107
2. Tan, X., Yuan, L., Zhou, J., Zheng, Y., Yang, F.: Modeling the initial transmission dynamics of influenza A/H1N1 in Guangdong province. Int. J. Infect. Dis. 17(7), 479–484 (2013). https://doi.org/10.1016/j.ijid.2012.11.018
3. Baleaun, D., Etemad, S., Rezapour, S.: A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions. Bound. Value Probl. 2020, 64 (2020). https://doi.org/10.1186/s13662-020-01361-0
4. Mahjani, M.G., Neshati, J., Masha, H.P., Ghanbarzadeh, A., Jafarian, M.: Evaluation of corrosion behaviour of organic coatings with electrochemical noise and electrochemical impedance spectroscopy. Surf. Eng. 22(4), 229–234 (2006). https://doi.org/10.1177/1743294806126752
5. Neshati, J., Masha, H.P., Jafarian, M.: Electrochemical noise analysis for estimation of corrosion rate of carbon steel in crude oil. Anti-Corros. Methods Mater. 54(1), 27–33 (2007). https://doi.org/10.1108/00035590710717366
6. Thabet, S.T., Etemad, S., Rezapour, S.: On a coupled Caputo conformable system of pantograph problems. Turk. J. Math. 45(1), 496–519 (2021). https://doi.org/10.3906/mat-2010-70
7. Mohammadi, H., Kumar, S., Rezapour, S., Etemad, S.: A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control. Chaos Solitons Fractals 144, 110668 (2021). https://doi.org/10.1016/j.chaos.2021.110668
8. Neshati, J., Masha, H.P., Mahjani, M.G., Jafarian, M.: Electrochemical noise analysis for estimation of corrosion rate of carbon steel in crude oil. Corros. Eng. Sci. Technol. 42(4), 371–376 (2007). https://doi.org/10.1179/174327807X124879
9. Baleaun, D., Mohammadi, H., Rezapour, S.: Analysis of the model of HIV-1 infection of CD4+ T-cell with a new approach of fractional derivative. Adv. Differ. Equ. 2020, 71 (2020). https://doi.org/10.1186/s13662-020-02544-w
10. Rezapour, S., Tabet, S.T., Etemad, S., Mohammadi, H.: A mathematical analysis of a system of Caputo-Fabrizio fractional differential equations for the anthrax disease model in animals. Adv. Differ. Equ. 2020, 481 (2020). https://doi.org/10.1186/s13662-020-02937-x
11. Karapinar, E., Fulga, A.: An admissible hybrid contraction with an Ulam type stability. Demonstr. Math. 52, 428–436 (2019). https://doi.org/10.1515/dema-2019-0037
12. Alqthamani, B., Fulga, A., Karapinar, E.: Fixed point results on δ-symmetric quasi-metric space via simulation function with an application to Ulam stability. Mathematics 6(10), 208 (2018). https://doi.org/10.3390/math60100208
13. Brzdek, J., Karapinar, E., Petrusel, A.: A fixed point theorem and the Ulam stability in generalized δ-$\phi$-metric spaces. J. Math. Anal. Appl. 467, 501–520 (2018). https://doi.org/10.1016/j.jmaa.2018.07.022
14. Adiguzel, RS., Aksoy, U., Karapinar, E., Erhan, I.M.: Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 2021, 155 (2021). https://doi.org/10.1007/s13398-021-01095-3
15. Adiguzel, RS., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solution of a boundary value problem associated with a fractional differential equation. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6652
16. Adiguzel, RS., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solutions of fractional differential equations via Geraghty type hybrid contractions. Appl. Comput. Math. 20(2), 313–333 (2021)
17. Bachir, F.S., Abbas, S., Benbachir, M., Benchora, M.: Hilfer-Hadamard fractional differential equations; existence and attractivity. Adv. Theory Nonlinear Anal. Appl. 5(1), 49–57 (2021). https://doi.org/10.11391/ata.849828
18. Lazreg, J.E., Abbas, S., Benchohra, M., Karapinar, E.: Impulsive Caputo-Fabrizio fractional differential equations in b-metric spaces. Open Math. 19(1), 363–372 (2021). https://doi.org/10.1515/math-2021-0040
19. Rezapour, S., Imran, M., Hussain, A., Martinez, F., Etemad, S., Kaabar, M.K.A.: Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs. Symmetry 13(3), 469 (2021). https://doi.org/10.3390/sym13030469
20. Zada, A., Alam, M., Irazu, U.: Analysis of q-fractional implicit boundary value problem having Stekletjes integral conditions. Math. Methods Appl. Sci. 44(6), 4381–4413 (2021). https://doi.org/10.1002/mma.7038
21. Wang, J., Shah, K., Ali, A.: Existence and Hyers-Ulam stability of fractional nonlinear impulsive switched coupled evolution equations. Math. Methods Appl. Sci. 41(6), 2392–2402 (2018). https://doi.org/10.1002/mma.4748
22. Rezapour, S., Ntouyas, S.K., Iqbal, M.Q., Hussain, A., Etemad, S., Tariboon, J.: An analytical survey on the solutions of the generalized double-order $\phi$-integrodifferential equation. J. Funct. Spaces 2021, Article ID 667757 (2021). https://doi.org/10.1155/2021/667757
23. Ahmad, B., Ntouyas, S.K., Alsaedi, A., Alnadihi, M.: Existence theory for fractional-order neutral boundary value problems. Fract. Differ. Calc. 8(1), 111–126 (2018). https://doi.org/10.7155/fdc-2018-0007
24. Ntouyas, S.K., Etemad, S.: On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions. Appl. Math. Comput. 266, 235–243 (2015). https://doi.org/10.1016/j.amc.2015.05.036
25. Matar, M.M., Abbas, M.I., Alzubab, J., Kaabar, M.K.A., Etemad, S., Rezapour, S.: Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives. Adv. Differ. Equ. 2021, 68 (2021). https://doi.org/10.1186/s13662-021-03228-9
26. Baleanu, D., Rezapour, S., Saberpour, Z.: On fractional integro-differential inclusions via the extended fractional Caputo-Fabrizio derivation. Bound. Value Probl. 2019, 79 (2019). https://doi.org/10.1186/s13661-019-1194-0

27. Aydogan, S.M., Baleanu, D., Mousalou, A., Rezapour, S.: On high order fractional integro-differential equations including the Caputo-Fabrizio derivative. Bound. Value Probl. 2018, 90 (2018). https://doi.org/10.1186/s13661-018-1008-9

28. Baleanu, D., Etemad, S., Rezapour, S.: On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators. Alex. Eng. J. 59(5), 3019–3027 (2020). https://doi.org/10.1016/j.aej.2020.04.053

29. Rezapour, S., Samei, M.E.: On the existence of solutions for a multi-singular pointwise defined fractional q-integro-differential equation. Bound. Value Probl. 2020, 38 (2020). https://doi.org/10.1186/s13661-020-01342-3

30. Sabergahdamin, F., Mashia, H.P.: Fixed-point results for multi-valued operators in quasi-ordered metric spaces. Appl. Math. Lett. 25(11), 1856–1861 (2012). https://doi.org/10.1016/j.aml.2012.02.046

31. Mashia, H.P., Sabergahdamin, F., Shahzad, N.: Fixed point theorems in partial metric spaces with an application. Filomat 27(4), 617–624 (2013)

32. Sabergahdamin, F., Mashia, H.P., Altun, I.: Fixed-point theorems for integral-type contractions on partial metric spaces. Ukr. Math. J. 68, 940–949 (2016). https://doi.org/10.1007/s11253-016-1267-5

33. Rezapour, S., Azzaouci, B., Tellab, B., Etemad, S., Mashia, H.P.: An analysis on the positive solutions of the Caputo multiterm semilinear differential equation. J. Funct. Spaces 2021, Article ID 6022941 (2021). https://doi.org/10.1155/2021/6022941

34. Al-Smadi, M.H., Gumah, G.: On the homotopy analysis method for fractional SIR epidemic model. Res. J. Appl. Sci. Eng. Technol. 7(18), 3809–3820 (2014). https://doi.org/10.19026/RJSET.7.738

35. Dhage, B.C., Dugg, B., Ntouyas, S.K.: Approximating solutions of nonlinear hybrid differential equations. Appl. Math. Lett. 34(18), 76–80 (2014). https://doi.org/10.1016/j.aml.2014.04.002

36. Chadha, A., Pandey, D.N.: Faedo-Galerkin approximation of solutions for a neutral fractional differential equation with deviating argument. Mediterr. J. Math. 13, 3041–3067 (2016). https://doi.org/10.1007/s00009-015-0671-7

37. Kamenskii, M., Obukhovskii, V., Petrosyan, G., Yao, J.C.: Existence and approximation of solutions to nonlocal boundary value problems for fractional differential inclusions. Fixed Point Theory Appl. 2019, 2 (2019). https://doi.org/10.1186/s13663-018-0562-1

38. Muslim, M., Aghar, R.P.: Existence, uniqueness and convergence of approximate solutions of nonlocal functional differential equations. C. R. Math. 270(2), 249–259 (2011)

39. Sontakke, B.R., Shaikh, A.: Approximate solutions of time fractional Kawahara and modified Kawahara equations by fractional complex transform. Commun. Numer. Anal. 2016(2), 218–229 (2016). https://doi.org/10.5899/2016/cna-00277

40. Pathak, H.K., Rodriguez-Lopez, R.: Existence and approximation of solutions to nonlinear hybrid ordinary differential equations. Appl. Math. Lett. 39, 101–106 (2015). https://doi.org/10.1016/j.aml.2014.08.018

41. Farouki, R.T.: The Bernstein polynomial basis: a centennial retrospective. Comput. Aided Geom. Des. 29(6), 379–419 (2012). https://doi.org/10.1016/j.cagd.2012.03.001

42. Yuzbasli, S.: Numerical solutions of fractional Riccati type differential equations by means of the Bernstein polynomials. Appl. Math. Comput. 219(11), 6328–6343 (2013). https://doi.org/10.1016/j.amc.2012.12.006

43. Yousefi, A.A., Behroozifar, M.: Operational matrices of Bernstein polynomials and their applications. Int. J. Syst. Sci. 41(6), 709–716 (2010). https://doi.org/10.1080/00207720903154783

44. I, O.R., Sezer, M.: Bernstein series solution of a class of Lane-Emden type equations. Math. Probl. Eng. 2013, Article ID 423797 (2013). https://doi.org/10.1155/2013/423797

45. Infante, G., Webb, J.: Loss of positivity in a nonlinear scalar heat equation. Nonlinear Differ. Equ. Appl. 13, 249–261 (2006). https://doi.org/10.1007/s00030-005-0039-y

46. Nieto, J.J., Pimentel, J.: Positive solutions of a fractional thermostat model. Bound. Value Probl. 2013, 5 (2013). https://doi.org/10.1186/1687-2770-2013-5

47. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)

48. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)

49. Diethelm, K.A., Ford, N.J., Freed, A.D., Luchko, Y.: Algorithms for the fractional calculus: a selection of numerical methods. Comput. Methods Appl. Mech. Eng. 194(6–8), 743–773 (2005). https://doi.org/10.1016/j.cma.2004.06.006

50. Samet, B., Vetro, C., Vetro, P.: Fixed point theorems for α-ψ-contractive type mappings. Nonlinear Anal., Theory Methods Appl. 75(4), 2154–2165 (2012). https://doi.org/10.1016/j.na.2011.10.014

51. Su, Y.: Boundary value problem for coupled system of nonlinear fractional differential equations. Appl. Math. Lett. 22(1), 64–69 (2009). https://doi.org/10.1016/j.aml.2008.03.001

52. Juttler, B.: Boundary value problem for coupled system of nonlinear fractional differential equations. Adv. Comput. Math. 8, 345–352 (1998). https://doi.org/10.1023/A:1018912801267

53. Saadatmandi, A.: Bernstein operational matrix of fractional derivatives and its applications. Appl. Math. Model. 38(4), 1365–1372 (2014). https://doi.org/10.1016/j.apm.2013.08.007

54. Razzaghi, M., Yousefi, S.: The Legendre wavelets operational matrix of integration. Int. J. Syst. Sci. 32(4), 495–502 (2001). https://doi.org/10.1080/002077201202227

55. Saadatmandi, A., Dehghan, M.: A new operational matrix for solving fractional-order differential equations. Comput. Math. Appl. 59(3), 1326–1336 (2010). https://doi.org/10.1016/j.camwa.2009.07.006