Recovering quantum information through partial access to the environment

Laleh Memarzadeh\textsuperscript{1,4}, Chiara Macchiavello\textsuperscript{2} and Stefano Mancini\textsuperscript{3}

\textsuperscript{1} Department of Physics, Sharif University of Technology, Teheran, Iran
\textsuperscript{2} Department of Physics ‘A Volta’, University of Pavia, I-27100 Pavia, Italy
\textsuperscript{3} School of Science and Technology, University of Camerino, I-62032 Camerino, Italy
E-mail: memarzadeh@sharif.ir

\textit{New Journal of Physics} 13 (2011) 103031 (16pp)
Received 27 June 2011
Published 24 October 2011
Online at http://www.njp.org/
doi:10.1088/1367-2630/13/10/103031

\textbf{Abstract.} We investigate the possibility of correcting errors occurring on a multipartite system through a feedback mechanism that acquires information through partial access to the environment. A partial control scheme of this type might be useful in dealing with correlated errors. In fact, in such a case, it could be enough to gather local information to decide what kind of global recovery to perform. Then, we apply this scheme to the depolarizing and correlated errors and quantify its performance by means of entanglement fidelity.

\textsuperscript{4} Author to whom any correspondence should be addressed.
1. Introduction

Quantum noise is the main obstacle in realizing quantum information tasks. It results from the errors introduced on the system’s state by the unavoidable interaction with the surrounding environment [1]. As a consequence, the quantum coherence features of the system’s state are washed out. To restore them, one could think of measuring the system (gathering information about its state) and then applying a correction procedure. This is the idea underlying the quantum feedback control mechanism [2]. Actually, also quantum error correcting codes can be thought of as belonging to this kind of strategy [3].

Indeed, one could make a measurement on the final state of the environment and consider its classical result to recognize what kind of error has occurred on the system due to the interaction with the environment. Then, a proper correction should be performed on the system to reduce the effect of quantum noise [4]. In recent years, much attention has been devoted to this scheme from different aspects. In [5–7], the capacity for this scenario has been studied, and in [8], it has been shown that in certain cases repeated application of this scheme allows one to remove the effects of quantum noise completely. For a given measurement, the optimal recovery scheme (the recovery necessary for restoring the maximum value of quantum information) has been derived in [4], while in [9] it has been shown that the optimal measurement depends on the dimension of the system’s Hilbert space.

In extending this quantum control strategy to multipartite systems, we must deal with a more intricate scenario. For instance, access to all subsystems’ environments may not be available. Then we will address the problem of recovering quantum information by feedback partial control; that is, the measurement is only made on some of the subsystems’ environments, while the error correction is performed on all subsystems. In this case, the feedback scheme will be effective if errors occurring on different subsystems are somehow correlated, so that gaining information on the measured subsystems also means to indirectly gain information about non-measured ones. This will help in designing the recovery operation on the whole system. We will consider quite a general kind of correlated error on qubits and determine the optimal recovery depending on the degree of the errors’ correlation. We will also find the scaling of the performance versus the number of qubits (subsystems) while monitoring the error on just one of them.

This paper is organized as follows. In section 2, we briefly present the main conceptual and computational tools needed to recover quantum information by means of a quantum feedback control scheme. To get some insights, we apply, in section 3, this strategy to the correlated

References
depolarizing channel for two qubits when only one is monitored. We then derive the main result for the system of \( n \) qubits in section 4 and present our conclusions in section 5.

2. Recovering quantum information by feedback control

The evolution of a system interacting with an environment can be described by a completely positive and trace-preserving map \( T: \mathcal{L}(\mathcal{H}_{\text{initial}}) \rightarrow \mathcal{L}(\mathcal{H}_{\text{final}}) \) transforming the initial system’s density operator in Hilbert space \( \mathcal{H}_{\text{initial}} \) to a final density operator in Hilbert space \( \mathcal{H}_{\text{final}} \) (\( \mathcal{L}(\mathcal{H}) \) is the space of linear operators on \( \mathcal{H} \)). At the same time, the initial state of the environment in Hilbert space \( \mathcal{K}_{\text{initial}} \) is mapped into a final one in \( \mathcal{K}_{\text{final}} \). The evolution of the system can be described as the unitary evolution of the system and the environment given by the unitary operator \( U: \mathcal{H}_{\text{initial}} \otimes \mathcal{K}_{\text{initial}} \rightarrow \mathcal{H}_{\text{final}} \otimes \mathcal{K}_{\text{final}} \). By denoting by \( \rho \) and \( \sigma \) the initial state of the system and the environment, respectively, the map of the system evolution reads

\[
T(\rho) = \text{tr}_{\mathcal{K}_{\text{final}}} [U(\rho \otimes \sigma)U^\dagger],
\]

where \( \text{tr}_\bullet \) denotes the trace on the space \( \bullet \).

To acquire some information about the errors occurring on the system, one can carry out a measurement on the environment after the interaction with the system has taken place. In general, this is described by a positive operator-valued measure on \( \mathcal{K}_{\text{final}} \), namely a set of operators \( M_\alpha \in \mathcal{L}(\mathcal{K}_{\text{final}}) \) satisfying

\[
\sum_\alpha M_\alpha = I, \quad M_\alpha \geq 0.
\]

(1)

The index \( \alpha \) labels the classical measurement outcomes. Considering an arbitrary observable \( A \in \mathcal{L}(\mathcal{H}_{\text{final}}) \), the expectation value of this observable is

\[
\langle A \rangle = \text{tr}_{\mathcal{H}_{\text{final}}} \text{tr}_{\mathcal{K}_{\text{final}}} [U(\rho \otimes \sigma)U^\dagger (A \otimes I)],
\]

(2)

where \( I \) is the identity on \( \mathcal{L}(\mathcal{K}_{\text{final}}) \).

**Definition 1.** We define by \( T_\alpha: \mathcal{L}(\mathcal{H}_{\text{initial}}) \rightarrow \mathcal{L}(\mathcal{H}_{\text{final}}) \),

\[
T_\alpha(\rho) := \text{tr}_{\mathcal{K}_{\text{final}}} [U(\rho \otimes \sigma)U^\dagger (I \otimes M_\alpha)],
\]

the selected channel output corresponding to the outcome \( \alpha \).

Then, replacing \( I \) in (2) with the identity resolution (1), we obtain

\[
\langle A \rangle = \sum_\alpha \text{tr}_{\mathcal{H}_{\text{final}}} (T_\alpha(\rho)A).
\]

Rewriting the expectation value of \( A \) in the following way:

\[
\langle A \rangle = \sum_\alpha p_\alpha \text{tr} \left[ \frac{1}{p_\alpha} T_\alpha(\rho)A \right],
\]

we can conclude that \( p_\alpha = \text{tr}(T_\alpha(\rho)) \) is the probability of getting \( \alpha \) as the result of the measurement and the density matrix \( \frac{1}{p_\alpha} T_\alpha(\rho) \) as the selected state of the system after performing the measurement on the environment.

We can also define the most informative measurement [4] in terms of Kraus operators [10] composing the channel \( T_\alpha \).
Definition 2. Given a channel $T = \sum_\alpha T_\alpha$, the most informative measurement on the environment is such that we can describe the selected output of the channel $T_\alpha$ by a single Kraus operator $T_\alpha(\rho) = t_\alpha \rho t_\alpha^\dagger$.

Therefore

$$T = \sum_\alpha t_\alpha \rho t_\alpha^\dagger, \quad \sum_\alpha t_\alpha^\dagger t_\alpha = I. \quad (3)$$

In order to correct the errors due to the interaction with the environment, we have to introduce a recovery operation.

Definition 3. Let $R_\alpha : \mathcal{L}(\mathcal{H}_{\text{final}}) \to \mathcal{L}(\mathcal{H}_{\text{initial}})$ be the recovery operator that acts on the selected output of the channel $T_\alpha(\rho)$ and depends on the classical outcome of the measurement $\alpha$. Then, the overall corrected channel takes the form

$$T_{\text{corr}} := \sum_\alpha R_\alpha \circ T_\alpha. \quad (4)$$

Using (3) and a Kraus representation [10] for the recovery channel $R_\alpha$,

$$R_\alpha(\rho') = \sum_\beta r^{(\alpha)}_\beta \rho'^{r^{(\alpha)}}_\beta, \quad \sum_\beta r^{(\alpha)}_\beta r^{(\alpha)}_\beta^\dagger = I,$$

we can decompose the corrected channel as

$$T_{\text{corr}}(\rho) = \sum_{\alpha,\beta} t^{(\alpha)}_\beta t_\alpha \rho t^{(\alpha)}_\beta^\dagger. \quad (5)$$

To quantify the performance of the correction scheme, we use the entanglement fidelity [11, 12].

Definition 4. For a general map $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ with Kraus operators $A_k$, the entanglement fidelity is defined as

$$F(\Phi) := \langle \Psi | \Phi \otimes I (|\Psi\rangle\langle\Psi|) |\Psi\rangle = \frac{1}{d^2} \sum_k |\text{tr}(A_k)|^2, \quad (6)$$

where $d = \dim \mathcal{H}$ and $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ is a maximally entangled state.

We are interested in $F(T_{\text{corr}})$, the entanglement fidelity of the corrected map (4). As a consequence of (5) and (4), we have

$$F(T_{\text{corr}}) = \frac{1}{d^2} \sum_{\alpha,\beta} |\text{tr}(r^{(\alpha)}_\beta t_\alpha)|^2. \quad (7)$$

The entanglement fidelity reaches its maximum value if quantum information is completely recovered or in other words if the corrected channel becomes an identity map. In [4], it has been shown that there exists a family of operators that completely recover quantum information if and only if

$$t_\alpha^\dagger t_\alpha = c_\alpha I, \quad \forall \alpha, \quad (8)$$

with $c_\alpha \in \mathbb{R}_+$ and $\sum_\alpha c_\alpha = 1$.

These results are obtained with the assumption that full access to the environment is available and it is possible to carry out a measurement on the whole environment after the
interaction with the system. However, more generally we should assume that our access to the environment is partial. Here, we want to investigate how the performance of this correction scheme behaves in this case and to see if we can still completely retrieve quantum information. To shed light on this problem, we study a map for which the complete recovery of quantum information is possible, provided that we have complete access to the environment.

Specifically we are going to consider the depolarizing quantum channel. In the following we will consider

$$\mathcal{H} := \mathbb{C}^2, \quad \mathcal{K} := \mathbb{C}^2 \otimes \mathbb{C}^2.$$

**Definition 5.** The single-qubit depolarizing channel is defined by

$$\mathcal{H}_{\text{initial}} = \mathcal{H}_{\text{final}} = \mathcal{H},$$

$$\mathcal{K}_{\text{initial}} = \mathcal{K}_{\text{final}} = \mathcal{K},$$

and

$$t_\alpha = \sqrt{p_\alpha} \sigma_\alpha,$$

where the operators $\sigma_\alpha$, with $\alpha = 0, 1, 2, 3$ (the Greek indices go from 0 to 3 while Latin indices go from 1 to 3), denote the Pauli operators (including the identity operator), while $p_0 = 1 - p$ and $p_1 = p_2 = p_3 = \frac{p}{3}$.

**Remark 1.** Since the Pauli operators satisfy the condition (8), the quantum information in this case can be completely recovered. To achieve this it is enough to consider a recovery channel described by a single Kraus operator $\sigma_\alpha$, where $\alpha$ is the classical outcome of the measurement. Hence from (7) we have

$$F(T_{\text{corr}}) = \sum_{\alpha=0}^{3} |\sqrt{p_\alpha}|^2 = 1.$$

However, the situation will be different when we enlarge the Hilbert spaces of the system and the environment while carrying out a measurement just on a subsystem of the environment. In the following sections, we show how the performance of this scheme behaves when our access to the environment is partial.

### 3. Depolarizing channel for two qubits

To study the feedback control scheme with partial access to the environment, we start by analyzing the depolarizing channel $T : \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ acting on two qubits. In the following, we assume that we can carry out a measurement on $\mathcal{L}(\mathcal{K})$ while the state of the environment belongs to $\mathcal{L}(\mathcal{K} \otimes \mathcal{K})$. Since the access to the environments is partial, the measurement cannot be the most informative measurement (see definition 2) and therefore the selected output of the channel is, in general, given by

$$T_\alpha(\rho) = \sum_\beta t_{\alpha,\beta} \rho t_{\alpha,\beta}^\dagger.$$
The Kraus operators $t_{\alpha,\beta}$ will be

\[ t_{\alpha,\beta} = \sqrt{p_{\alpha}p_{\beta}} \sigma_{\alpha} \otimes \sigma_{\beta} \]

in the case of no correlations and

\[ t_{\alpha,\beta} = \sqrt{p_{\alpha}\delta_{\alpha,\beta}} \sigma_{\alpha} \otimes \sigma_{\beta} \]

in the case of perfect correlations. In the first case, the outcome of the measurement does not give any information about the error occurring on the second qubit; therefore the selected output of the channel is

\[ T_{uc}^{\alpha} = \sum_{\beta} p_{\alpha} p_{\beta} (\sigma_{\alpha} \otimes \sigma_{\beta}) \rho (\sigma_{\alpha} \otimes \sigma_{\beta})^\dagger. \]  

(9)

In the second case, the measurement of the environment is the most informative one (according to definition 2); hence the selected output of the channel is

\[ T_{cc}^{\alpha}(\rho) = p_{\alpha} (\sigma_{\alpha} \otimes \sigma_{\alpha}) \rho (\sigma_{\alpha} \otimes \sigma_{\alpha})^\dagger. \]

(10)

We will now consider a more general situation, which interpolates between the above two situations. More explicitly, we consider a correlated noise model that is a convex combination of the two cases described above, namely the uncorrelated noise and the completely correlated noise for two qubits [13].

**Definition 6.** The following convex combination of channels (9) and (10),

\[ T_{\alpha} := (1 - \mu) T_{uc}^{\alpha} + \mu T_{cc}^{\alpha}, \]

where $\mu \in [0, 1]$ quantifies the amount of correlation in noise, defines the selected output.

Our aim now is to design the recovery channel in order to achieve the maximum value of the entanglement fidelity for the corrected channel.

**Lemma 1.** The recovery map

\[ R_{\alpha}(\rho') := \sum_{\gamma} q_{\alpha,\gamma} (\sigma_{\alpha} \otimes \sigma_{\gamma}) \rho' (\sigma_{\alpha} \otimes \sigma_{\gamma})^\dagger, \quad \sum_{\gamma} q_{\alpha,\gamma} = 1, \]

is optimal for the channel of definition 6.

**Proof.** The recovery map can be described, without loss of generality, as

\[ R_{\alpha}(\rho') = \sum_{\gamma} (\sigma_{\alpha} \otimes A_{\gamma}^\alpha) \rho' (\sigma_{\alpha} \otimes A_{\gamma}^\alpha)^\dagger, \]

where the single-qubit operators $A_{\gamma}^\alpha$ can be expressed in terms of the identity and the Pauli operators as

\[ A_{\gamma}^\alpha = \sum_{\delta} c_{\gamma,\delta}^\alpha \sigma_{\delta}. \]

The completeness condition $\sum_{\gamma} (\sigma_{\alpha} \otimes A_{\gamma}^\alpha)^\dagger (\sigma_{\alpha} \otimes A_{\gamma}^\alpha) = I$ gives the following normalization condition for the coefficients $c_{\gamma,\delta}^\alpha$:

\[ \sum_{\gamma,\delta} |c_{\gamma,\delta}^\alpha|^2 = 1. \]
Note that this is the most general map we can use as recovery. Actually, on the first qubit the optimal action is to invert the action of $\sigma_\alpha$ by $\sigma_\alpha$ itself, while on the second one we consider a generic operator $A_{\alpha \gamma}$ possibly correlated with that on the first qubit (this is the reason for the presence of the index $\alpha$ on $A_{\alpha \gamma}$). Then the entanglement fidelity of the corrected channel, using (7), takes the form

$$F(T_{corr}) = (1 - \mu) \sum_{\alpha, \beta, \gamma} p_\alpha p_\beta |c_{\gamma, \beta}^\alpha|^2 + \mu \sum_{\alpha, \gamma} p_\alpha |c_{\gamma, \alpha}^\alpha|^2.$$ 

Note that this can be rewritten as

$$F(T_{corr}) = (1 - \mu) \sum_{\alpha, \beta} p_\alpha p_\beta q_{\alpha, \beta} + \mu \sum_{\alpha} p_\alpha q_{\alpha, \alpha}, \quad (11)$$

where we defined the probabilities

$$q_{\alpha, \beta} = \sum_\gamma |c_{\gamma, \beta}^\alpha|^2.$$ 

Equation (11) is the same expression that we would obtain by assuming the recovery as a Pauli channel, namely with the Kraus operators

$$\sigma_\alpha \otimes A_{\gamma}^\alpha = \sqrt{q_{\gamma, \alpha, \gamma}} \sigma_\alpha \otimes \sigma_\gamma$$

with $\sum_\gamma q_{\gamma, \alpha, \gamma} = 1$ for all $\gamma$. \(\square\)

By virtue of lemma 1, the corrected channel can be written as

$$T_{corr}(\rho) = (1 - \mu) \sum_{\alpha, \beta, \gamma} p_\alpha p_\beta q_{\alpha, \gamma} (I \otimes \sigma_\gamma \sigma_\beta) \rho (I \otimes \sigma_\beta \sigma_\gamma) + \mu \sum_{\alpha, \gamma} p_\alpha q_{\alpha, \gamma} (I \otimes \sigma_\gamma \sigma_\alpha) \rho (I \otimes \sigma_\alpha \sigma_\gamma),$$

and its entanglement fidelity becomes

$$F(T_{corr}) = \frac{1 - \mu}{16} \sum_{\alpha, \beta, \gamma} p_\alpha p_\beta q_{\alpha, \gamma} |\text{tr}(I \otimes \sigma_\gamma \sigma_\beta)|^2 + \frac{\mu}{16} \sum_{\alpha, \gamma} p_\alpha q_{\alpha, \gamma} |\text{tr}(I \otimes \sigma_\alpha \sigma_\gamma)|^2$$

$$= (1 - \mu) \sum_{\alpha, \beta} p_\alpha p_\beta q_{\alpha, \beta} + \mu \sum_{\alpha} p_\alpha q_{\alpha, \alpha}. \quad (12)$$

Taking into account that $\sum_\gamma q_{\alpha, \gamma} = 1$ for all values of $\alpha$, the above equation is simplified as follows:

$$F(T_{corr}) = (1 - \mu) \frac{p}{3} + (1 - p) \left((1 - \mu) \left(1 - \frac{4p}{3}\right) + \mu \right) q_{0,0} + \frac{p}{3} \sum_{i=1}^{3} \left((1 - \mu) \left(1 - \frac{4p}{3}\right) q_{i,0} + \mu q_{i,i}\right). \quad (12)$$

Then, the following theorem holds.

**Theorem 1.** Upon recovery, the maximum achievable entanglement fidelity for the channel of definition 6 is:

- **Region A**

  $$F_{\text{max}}^A(T_{corr}) = 1 - p,$$

  for $0 < \mu < \mu_{AB} = \frac{3 - 4p}{6 - 4p}.$

New Journal of Physics 13 (2011) 103031 (http://www.njp.org/)
• **Region B**

\[
F_{\text{max}}^{B}(T_{\text{corr}}) = (1 - \mu) \left( 1 - 2p + \frac{4p^2}{3} \right) + \mu,
\]

for

\[
\mu > \mu_{AB}
\]

and

\[
\mu > \mu_{BC} = \frac{4p - 3}{4p}.
\]

• **Region C**

\[
F_{\text{max}}^{C}(T_{\text{corr}}) = (1 - \mu) \frac{p}{3} + \mu p,
\]

for \(0 < \mu < \mu_{BC}\).

**Proof.** The optimal recovery channel is achieved by maximizing expression (12) over the parameters \(q_{\alpha,\gamma}\). Our strategy to maximize the entanglement fidelity is to optimize the correction performance for each channel component

\[
T_{\text{corr}}^{(\alpha)} = R_\alpha \circ T_\alpha.
\]

When the outcome of the measurement is \(\alpha = 0\), the entanglement fidelity of the corrected map \(T_{\text{corr}}^{(\alpha=0)}\) is

\[
F_{\text{corr}}^{(0)} = (1 - \mu)(1 - p) \frac{p}{3} + (1 - p) \left[ (1 - \mu) \left( 1 - \frac{4p}{3} \right) + \mu \right] q_{00}.
\]

For \((1 - \mu)(1 - \frac{4p}{3}) + \mu > 0\) the coefficient of \(q_{00}\) is positive; therefore the maximum of \(F_{\text{corr}}^{(0)}\) is attained by choosing \(q_{0,0} = 1\). For \((1 - \mu)(1 - \frac{4p}{3}) + \mu < 0\) the maximum is achieved for \(q_{0,0} = 0\). This means that for \(\mu < \frac{4p-3}{4p}\), if the outcome of the measurement is 0 (no error on the first qubit), the most appropriate recovery is to perform a Pauli channel on the second qubit and leave the first qubit unchanged. For \(\mu > \frac{4p-3}{4p}\), if the outcome of the measurement is 0, the amount of correlation on noise is large enough to ensure that the second qubit has passed through the channel safely and no correction is required on either of them.

To find the optimum recovery for the other possible outcomes of the measurement, we have to maximize the following expressions:

\[
F_{\text{corr}}^{(i)} = (1 - \mu) \frac{p^2}{9} + (1 - \mu) \frac{p}{3} \left( 1 - \frac{4p}{3} \right) q_{i,0} + \mu \frac{p}{3} q_{i,i}.
\]

(13)

Note that the probabilities \(q_{i,j}\) with \(j \neq i\) do not appear in (13). Moreover, since \(F_{\text{corr}}^{(i)}\) is linear in the parameters \(q_{i,j}\) and at least one of the coefficients is positive, remembering the normalization condition \(\sum_{\gamma} q_{\alpha,\gamma} = 1\), we set \(q_{i,j} = 0\) for \(j \neq i\) to achieve the maximum value for \(F_{\text{corr}}^{(i)}\). Hence we can write \(q_{i,0} = 1 - q_{i,i}\). Substituting it into equation (13), we obtain

\[
F_{\text{corr}}^{(i)} = (1 - \mu) \frac{p}{3} (1 - p) + \frac{p}{3} \left( \mu - (1 - \mu) \left( 1 - \frac{4p}{3} \right) \right) q_{i,i}.
\]
Therefore if the outcome of the measurement is \(i = 1, 2, 3\), for \(\mu > \frac{3-4p}{6-4p}\) the optimum correction can be performed by taking \(q_{i,i} = 1\) and for \(\mu < \frac{3-4p}{6-4p}\) the best performance of the recovery is attainable by taking \(q_{i,i} = 1\). Therefore, the optimum correction varies depending on the values of \(p\) and \(\mu\), and can be summarized as:

**Region A:** In this region, \(0 < \mu < \mu_{AB} = \frac{3-4p}{6-4p}\). The optimum correction is achieved by choosing \(q_{0,0} = 1\):
\[
R_0(\rho') = (\sigma_0 \otimes I) \rho' (\sigma_a \otimes I).
\]

Therefore, the maximum entanglement fidelity in this region is given by
\[
F_{\text{max}}^A(T_{\text{corr}}) = 1 - p.
\]

**Region B:** In this region, \(\mu > \mu_{AB}\) and \(\mu > \mu_{BC} = \frac{4p-3}{4p}\). For the measurement outcome \(\alpha\), the optimum recovery is given by \(q_{a,a} = 1\):
\[
R_\alpha(\rho') = (\sigma_a \otimes \sigma_a) \rho' (\sigma_a \otimes \sigma_a).
\]

Therefore, the maximum entanglement fidelity in this region is given by
\[
F_{\text{max}}^B(T_{\text{corr}}) = (1 - \mu) \left(1 - 2p + \frac{4p^2}{3}\right) + \mu.
\]

**Region C:** In this region \(0 < \mu < \mu_{BC}\). If the outcome of the measurement is \(\alpha = 0\) the optimal recovery is given by \(q_{0,0} = 0\):
\[
R_0(\rho') = \sum_{i=1}^{3} q_i (I \otimes \sigma_i) \rho' (I \otimes \sigma_i), \quad \sum_{i=1}^{3} q_i = 1,
\]
and for the measurement outcome \(i = 1, 2, 3\), the optimal recovery is given by \(q_{i,i} = 1\):
\[
R_i(\rho') = (\sigma_i \otimes \sigma_i) \rho' (\sigma_i \otimes \sigma_i), \quad i = 1, 2, 3.
\]

The maximum entanglement fidelity takes the form
\[
F_{\text{max}}^C(T_{\text{corr}}) = (1 - \mu) \frac{p}{3} + \mu p.
\]

Based on theorem 1, we can identify three different regions for the optimum correction in the plane of \(p\) and \(\mu\), as shown in figure 1.

**Remark 2.** It is interesting to note that the critical value \(\mu_{AB}\) for the correlation parameter \(\mu\) has the same form as the one characterizing the correlated depolarizing channel in terms of classical information transmission [13]. In that context, the critical value \(\mu_{AB}\) gives a threshold value for the optimal input states: the mutual information along the channel is maximized with product states for \(\mu \leq \mu_{AB}\), while it achieves its maximum value with maximally entangled states for \(\mu \geq \mu_{AB}\) [13].
Figure 1. Different parameter regions for optimal recovery in the case of a two-qubit channel. In region A, the best correction strategy is to not act at all on the second qubit. In region B, the best correction strategy is to act on the second qubit in the same way as in the first qubit. In region C, the best correction strategy is to act on the second qubit with a Pauli channel or an identity map, depending on the measurement outcome.

4. Depolarizing channel for $n$ qubits

In the previous section, we have seen how the performance of the quantum feedback control scheme behaves if we can carry out measurements over $\mathcal{L}(K)$ while the total Hilbert space of the environment is $K \otimes K$. If we had full access to the environment, we could completely retrieve quantum information. However, we have shown that having only partial access to the environment and exploiting the correlation in noise we are still capable of partially recovering quantum information. Now we want to see how the performance of the correction behaves if we keep increasing the Hilbert space of the environment without increasing our access to it. To do so, we consider a correlated depolarizing channel defined by $T: \mathcal{L}(\mathcal{H}^\otimes n) \rightarrow \mathcal{L}(\mathcal{H}^\otimes n)$, resembling the long-term memory channels introduced in [14], and we carry out measurement on $\mathcal{L}(K)$ while the state of the environment belongs to $\mathcal{L}(K^\otimes n)$.

**Definition 7.** Let us define the selected output of the channel corresponding to the classical outcome of the measurement, $\alpha$, as

$$T_\alpha(\rho) := (1 - \mu) T^{uc}_\alpha(\rho) + \mu T^{cc}_\alpha(\rho),$$

with

$$T^{uc}_\alpha(\rho) := \sum_{\beta_1, \ldots, \beta_n} p_\alpha \left( \prod_{i=2}^n p_{\beta_i} \right) \left[ \sigma_\alpha \bigotimes_{i=2}^n \sigma_{\beta_i} \right] \rho \left[ \sigma_\alpha \bigotimes_{i=2}^n \sigma_{\beta_i} \right]$$

and

$$T^{cc}_\alpha(\rho) := p_\alpha \sigma_\alpha^\otimes n \rho \sigma_\alpha^\otimes n.$$
Similarly to the previous section, we have the following:

**Lemma 2.** The recovery operator

\[ R_\alpha (\rho') = \sum_{y_2, \ldots, y_n} q_{\alpha, y_2, \ldots, y_n} \left[ \sigma_\alpha \bigotimes \left( \bigotimes_{i=2}^{n} \sigma_{y_i} \right) \right] \rho \left[ \sigma_\alpha \bigotimes \left( \bigotimes_{i=2}^{n} \sigma_{y_i} \right) \right], \]

with the constraint

\[ \sum_{y_2, \ldots, y_n} q_{\alpha, y_2, \ldots, y_n} = 1, \quad \forall \alpha, \]

is optimal for the channel of definition 7.

**Proof.** Having the classical outcome of the measurement \( \alpha \), we know that error \( \sigma_\alpha \) has occurred on the first qubit and the effect of error can be completely removed by performing \( \sigma_\alpha \) on the first qubit for correction. Therefore, the recovery map should have the following form:

\[ R_\alpha (\rho') = \sum_{y_2, \ldots, y_n} \left( \sigma_\alpha \otimes A_{y_2, \ldots, y_n}^{(a)} \right) \rho' \left( \sigma_\alpha \otimes A_{y_2, \ldots, y_n}^{(a)} \right)^\dagger, \]

where \( A_{y_2, \ldots, y_n}^{(a)} \) is an operator acting on \( n - 1 \) qubits. Expanding it in terms of products of Pauli matrices, we have

\[ A_{y_2, \ldots, y_n}^{(a)} = \sum_{\alpha, \beta} c^{y_2, \ldots, y_n}_{\alpha, \beta} \sigma_\alpha \otimes \sigma_\beta \otimes \cdots \otimes \sigma_\beta. \]

The completeness condition

\[ \sum_{y_2, \ldots, y_n} \left( \sigma_\alpha \otimes A_{y_2, \ldots, y_n}^{(a)} \right) \rho' \left( \sigma_\alpha \otimes A_{y_2, \ldots, y_n}^{(a)} \right)^\dagger = I \]

imposes the following constraint on the coefficients \( c^{y_2, \ldots, y_n}_{\alpha, \beta} \):

\[ \sum_{y_2, \ldots, y_n} |c^{y_2, \ldots, y_n}_{\alpha, \beta}|^2 = 1. \]

Considering this general form of Kraus operators for the recovery map and using (7), the entanglement fidelity of the corrected channel takes the form

\[ F(T_{\text{corr}}) = (1 - \mu) \sum_{\alpha, \beta_2, \ldots, \beta_n} \left| \sum_{y_2, \ldots, y_n} c^{y_2, \ldots, y_n}_{\alpha, \beta_2, \ldots, \beta_n} \right|^2 p_\alpha \prod_{i=2}^{n} p_{\beta_i} + \mu \sum_{\alpha, y_2, \ldots, y_n} \left| c^{y_2, \ldots, y_n}_{\alpha, \alpha, \ldots, \alpha} \right|^2 p_\alpha. \]

The above expression can be written as

\[ F(T_{\text{corr}}) = (1 - \mu) \sum_{\alpha, \beta_2, \ldots, \beta_n} p_\alpha q_{\alpha, \beta_2, \ldots, \beta_n} \prod_{i=2}^{n} p_{\beta_i} + \mu \sum_{\alpha} p_\alpha q_{\alpha, \alpha, \ldots, \alpha}. \]

By defining

\[ q_{\alpha, \beta_2, \ldots, \beta_n} = \sum_{y_2, \ldots, y_n} \left| c^{y_2, \ldots, y_n}_{\alpha, \beta_2, \ldots, \beta_n} \right|^2 \]
the same value of entanglement fidelity in equation (19) will be obtained by assuming the recovery map with following Kraus operators:

\[ \sqrt{q_{a_1\ldots a_n}} \sigma_{a_1} \otimes \sigma_{a_2} \otimes \cdots \otimes \sigma_{a_n} \]  

with the constraint that \( \sum_{\gamma_2\ldots\gamma_n} q_{a_1\gamma_2\ldots\gamma_n} = 1 \) for all \( \alpha \).

The entanglement fidelity (7) of the channel in definition 7 corrected using theorem 2 results in

\[ F(T_{\text{corr}}) = (1 - \mu) \sum_{\alpha,\beta_2\ldots\beta_n} p_\alpha p_{\beta_2} \cdots p_{\beta_n} q_{a,\beta_2\ldots\beta_n} + \mu \sum_\alpha p_\alpha q_{a,a\ldots a}. \]  

(22)

Then we have the following theorem.

**Theorem 2.** Upon recovery, the maximum achievable entanglement fidelity for the channel of definition 7 is:

- **Region A**

  \[ F^A_{\text{max}}(T_{\text{corr}}) = (1 - \mu)(1 - p)^{n-1} + \mu(1 - p), \]

  for

  \[ 0 < \mu < \frac{(1 - p)^{n-1} - \left(\frac{p}{3}\right)^{n-1}}{(1 - p)^{n-1} - \left(\frac{p}{3}\right)^{n-1} + 1}. \]

- **Region B**

  \[ F^B_{\text{max}}(T_{\text{corr}}) = (1 - \mu) \left((1 - p)^n + 3 \left(\frac{p}{3}\right)^n\right) + \mu, \]

  for

  \[ \mu > \frac{(1 - p)^{n-1} - \left(\frac{p}{3}\right)^{n-1}}{(1 - p)^{n-1} - \left(\frac{p}{3}\right)^{n-1} + 1} \]

  and

  \[ \mu > -\frac{(1 - p)^{n-1} - \left(\frac{p}{3}\right)^{n-1}}{(1 - p)^{n-1} - \left(\frac{p}{3}\right)^{n-1} - 1}. \]

- **Region C**

  \[ F^C_{\text{max}}(T_{\text{corr}}) = (1 - \mu)(\frac{p}{3})^{n-1} + \mu p, \]

  for

  \[ 0 < \mu < -\frac{(1 - p)^{n-1} - \left(\frac{p}{3}\right)^{n-1}}{(1 - p)^{n-1} - \left(\frac{p}{3}\right)^{n-1} - 1}. \]

**Proof.** To maximize the entanglement fidelity in (22) over its parameters, we maximize it for each value of the measurement outcome \( \alpha \). If the outcome of the measurement is zero, then the
entanglement fidelity is given by
\[
F^{(0)}_{\text{corr}} = \[(1 - \mu)(1 - p)^n + \mu(1 - p)]q_{0, \ldots, 0} + (1 - \mu)(1 - p)^{n-1} \times \frac{p}{3} \sum_{j, \text{perm}} q_{0, j, 0, \ldots, 0} + (1 - \mu)(1 - p)^{n-2} \left( \frac{p}{3} \right)^2 \\
\times \sum_{j, k, \text{perm}} q_{0, j, k, 0, \ldots, 0} + \cdots + (1 - \mu)(1 - p) \left( \frac{p}{3} \right)^{n-3} \sum_{i_{2}, \ldots, i_{n}} q_{0, i_{2}, \ldots, i_{n}},
\]
(23)
where the notation \( \text{perm} \) in the summations above refers to all possible permutations of the last \( n - 1 \) indices. Note that for \( p < 3/4 \) the largest coefficient in the above expression is that in front of \( q_{0,0,0,\ldots,0} \) and therefore in this case, for any value of \( \mu \), \( F^{(0)}_{\text{corr}} \) is always maximized by \( q_{0,0,0,\ldots,0} = 1 \). In the case of \( p > 3/4 \) the largest coefficient, excluding the first, is given by the last one in (23). Therefore, in this case the optimal solution can be searched by setting to zero all values of \( q \) that contain at least one value 0 among the last \( n - 1 \) indices. The expression for the entanglement fidelity that we are going to maximize is now simplified as
\[
F^{(0)}_{\text{corr}} = \[(1 - \mu)(1 - p)^n + \mu(1 - p)]q_{0, \ldots, 0} + (1 - \mu)(1 - p) \left( \frac{p}{3} \right)^{n-1} \sum_{i_{2}, \ldots, i_{n}} q_{0, i_{2}, \ldots, i_{n}}.
\]
(24)
Using the condition in equation (14), we know that
\[
\sum_{i_{2}, \ldots, i_{n}} q_{0, i_{2}, \ldots, i_{n}} = 1 - q_{0, \ldots, 0}.
\]
Replacing it in equation (24), we find that
\[
F^{(0)}_{\text{corr}} = (1 - \mu)(1 - p) \left( \frac{p}{3} \right)^{n-1} + (1 - \mu) \left[ (1 - p)^n - (1 - p) \left( \frac{p}{3} \right)^{n-1} \right] + \mu(1 - p) q_{0, \ldots, 0}.
\]
It is easy to see that for
\[
0 < \mu < -\frac{(1 - p)^{n-1} - \left( \frac{p}{3} \right)^{n-1}}{(1 - p)^{n-1} + \left( \frac{p}{3} \right)^{n-1} - 1},
\]
the coefficient of \( q_{0, \ldots, 0} \) is negative; therefore the maximum of \( F^{(0)}_{\text{corr}} \) is attainable for \( q_{0, \ldots, 0} = 0 \). For
\[
\mu > -\frac{(1 - p)^{n-1} - \left( \frac{p}{3} \right)^{n-1}}{(1 - p)^{n-1} + \left( \frac{p}{3} \right)^{n-1} - 1},
\]
the coefficient of \( q_{0, \ldots, 0} \) is positive, so the maximum of \( F^{(0)}_{\text{corr}} \) is achieved by taking \( q_{0, \ldots, 0} = 1 \).

When the outcome of the measurement is \( i = 1, 2, 3 \), then the entanglement fidelity of the corrected channel is
\[
F^{(i)}_{\text{corr}} = (1 - \mu)(1 - p)^n \frac{p}{3} q_{i, 0, \ldots, 0} + (1 - \mu)(1 - p)^{n-2} \left( \frac{p}{3} \right)^2 \times \sum_{j, \text{perm}} q_{i, j, 0, \ldots, 0} + (1 - \mu)(1 - p)^{n-3} \left( \frac{p}{3} \right)^3 \times \sum_{j, k, \text{perm}} q_{i, j, k, 0, \ldots, 0} + \cdots + \frac{p}{3} \sum_{i_{2}, \ldots, i_{n}} q_{i, i_{2}, \ldots, i_{n}} \left[ (1 - \mu) \left( \frac{p}{3} \right)^{n-1} + \mu \prod_{k=k}^{n} \delta_{i, k} \right].
\]
(25)
Note that for $p > 3/4$ the largest coefficient in the above expression is that in front of $q_i, i, \ldots, i$; therefore in this case the optimal solution corresponds to $q_i, i, \ldots, i = 1$ for any value of $\mu$. Note also that in the last line the coefficient in front of $q_i, i, \ldots, i$ is always larger than the other ones, so in order to look for the maximum we can always set $q_i, i, \ldots, n = 0$ for all cases except $q_i, i, \ldots, i$. Moreover, for $p < 3/4$, the largest coefficients in $(25)$, with the exclusion of the last line, are the ones in front of $q_i, 0, \ldots, 0$. Therefore, by these considerations, we can restrict our search for the maximum values to the case of vanishing $q$ except for $q_i, 0, \ldots, 0$ and $q_i, i, \ldots, i$. We can then write

$$F^{(i)}(T_{\text{corr}}) = (1 - \mu)(1 - p)^{n-1} p^3 q_i, 0, \ldots, 0 + \left[ (1 - \mu) \left( \frac{p^n}{3} \right) + \mu \frac{p}{3} \right] q_i, i, \ldots, i.$$ 

From equation (14), we know that

$$q_i, i, \ldots, i = 1 - q_i, 0, \ldots, 0.$$

Therefore

$$F^{(i)}(T_{\text{corr}}) = (1 - \mu) \left( \frac{p^n}{3} \right) + \mu \frac{p}{3} + \frac{p}{3} \left[ (1 - p)^{n-1} - \left( \frac{p^n}{3} \right)^{n-1} \right] - \mu q_i, 0, \ldots, 0.$$

The coefficient of $q_i, 0, \ldots, 0$ is positive for

$$0 < \mu < \frac{(1 - p)^{n-1} - \left( \frac{p^n}{3} \right)^{n-1}}{(1 - p)^{n-1} + \left( \frac{p^n}{3} \right)^{n-1} + 1}.$$

Therefore in this region we should take $q_i, 0, \ldots, 0 = 1$ and for

$$\mu > \frac{(1 - p)^{n-1} - \left( \frac{p^n}{3} \right)^{n-1}}{(1 - p)^{n-1} + \left( \frac{p^n}{3} \right)^{n-1} + 1},$$

the coefficient of $q_i, 0, \ldots, 0$ is negative and therefore we should take $q_i, i, \ldots, i = 1$. 

Figure 2 shows regions A, B and C of theorem 2 for different values of $n$ (including the previously analyzed case of $n = 2$). We can see that by increasing $n$, regions A and C become smaller. This can be understood by noting that region A corresponds to the case when although the measurement outcome shows that an error has occurred on the first qubit, we expect that the other qubits have not experienced any error. However, the chance of this holding true is lowered by increasing $n$. The same reasoning can be applied to region C.

Figure 3 shows the entanglement fidelity versus the number of qubits in the system for $p = 0.4$ and for different values of $\mu$. It is interesting to note that for large $n$ the entanglement fidelity does not go to zero, due to the role of noise correlation in performing the recovery operation.

5. Conclusions

The main result presented in this paper is the possibility of recovering quantum information on a multipartite system by using limited access to the environment. In particular, we have addressed the important question of how well quantum information can be recovered on a multiple qubit
Figure 2. Different parameter regions for optimal recovery in the case of $n = 2$ (dashed-dotted line), $n = 3$ (dotted line), $n = 4$ (dashed line) and $n = 5$ (solid line) qubit channels. In region A, the best correction strategy is to not act at all on the $n - 1$ qubits. In region B, the best correction strategy is to act on the $n - 1$ qubits in the same way as in the first qubit. In region C, the best correction strategy is to act on each of the $n - 1$ qubits with a Pauli channel or an identity map, depending on the measurement outcome.

Figure 3. Entanglement fidelity for the feedback-corrected channel versus the number of qubits $n$. The values of parameters are $p = 0.4$ and $\mu = 0.9$ (solid line), $\mu = 0.7$ (dashed line) and $\mu = 0.5$ (dotted line) lying in the region B (see figure 2).

system by carrying out a measurement on one environmental subsystem. We have considered quite a general kind of correlated error on qubits and we have determined the optimal recovery depending on the degree of error correlation. We have also found the scaling of the performance.
versus the number of qubits while monitoring the error on just one of them. Interestingly enough, for a finite degree of error correlation, the recovery ability is preserved by increasing the number of non-measured subsystems of the environment.

As a final remark, we point out that when considering partial control, one could exploit correlations residing on the system’s state itself rather than on the errors. This would be more in the spirit of [15] and is left for future investigations.

Acknowledgment

We acknowledge financial support from the European Commission under the FET-Open grant agreement CORNER (number FP7-ICT-213681).

References

[1] Zurek W H 1991 Phys. Today 44 36
[2] Wiseman H M and Milburn G J 2010 Quantum Measurement and Control (New York: Cambridge University Press)
[3] Knill E and Laflamme R 1997 Phys. Rev. A 55 900
[4] Gregoratti M and Werner R F 2003 J. Mod. Opt. 50 915
[5] Hayden P and King Ch 2005 Quantum Inf. Comput. 5 156
[6] Smolin J A, Verstraete F and Winter A 2005 Phys. Rev. A 72 052317
[7] Winter A 2005 arXiv:quant-ph/0507045
[8] Buscemi F, Chiribella G and D’Ariano G M 2005 Phys. Rev. Lett. 95 090501
[9] Memarzadeh L, Cafaro C and Mancini S 2011 J. Phys. A: Math. Theory 44 045304
[10] Kraus K 1983 States, Effects, and Operations: Fundamental Notions of Quantum Theory (Berlin: Springer)
[11] Schumacher B 1996 Phys. Rev. A 54 2614
[12] Nielsen M A 1996 arXiv:quant-ph/9600612
[13] Macchiavello C and Palma G M 2002 Phys. Rev. A 65 050301
[14] Datta N and Dorlas T 2007 J. Phys. A: Math. Theory 40 8147
[15] Mancini S 2006 Phys. Rev. A 73 010304
   Serafini A and Mancini S 2010 Phys. Rev. Lett. 104 220501