THE GLOBAL STABILITY OF 2-D VISCOUS AXISYMMETRIC CIRCULATORY FLOWS

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Abstract. In this paper, we study the global existence and stability problem of a perturbed viscous circulatory flow around a disc. This flow is described by two-dimensional Navier-Stokes equations. By introducing some suitable weighted energy space and establishing a priori estimates, we show that the 2-D circulatory flow is globally stable in time when the corresponding initial-boundary values are perturbed sufficiently small.

1. Introduction. In this paper, we are concerned with the axisymmetric circulatory flow problem for the two-dimensional compressible Navier-Stokes equations (see Figure 1 and Figure 2 below). The compressible Navier-Stokes equations in two space dimensions are

\[
\begin{aligned}
\frac{\partial \rho}{\partial t} + \text{div}(\rho u) &= 0, \\
\rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u + \nabla P(\rho) &= \mu \Delta u + (\mu + \lambda)\nabla \text{div}u,
\end{aligned}
\]

where \( \rho \geq 0 \) is the density, \( u = (u_1, u_2) \) is the velocity, \( \mu > 0 \) and \( \lambda \) are the first and second viscosity coefficient (or shear viscosity and bulk viscosity) respectively. In addition, \( \mu + \lambda > 0 \) holds, and the state equation is \( P(\rho) = A \rho^\gamma \) with \( A > 0 \) and \( \gamma > 1 \).

We now give a mathematical description on the 2-D viscous axisymmetric flow around a disc \( \{x = (x_1, x_2) \in \mathbb{R}^2 : r = |x| = \sqrt{x_1^2 + x_2^2} \leq 1\} \). Set \( \Omega = \{x \in \mathbb{R}^2 : r > 1\} \) and \( (\rho(t, x), u(t, x)) = (\rho(t, r), u_r(t, r)\frac{x}{r} + u_\theta(t, r)\frac{x^\perp}{r}) \), where \( u_r(t, r) \) and \( u_\theta(t, r) \) are the radial speed and angular speed of gases respectively, and \( x^\perp = (-x_2, x_1) \).
In this case, (1) admits the following equivalent form in $[0, \infty) \times \Omega$:

$$\frac{\partial t}{\partial t} + \frac{1}{r} \partial_r (r \rho u_r) = 0,$$

$$\rho \partial_t u_r + \rho (u_r \partial_r u_r - \frac{u_r^2}{r}) + \partial_r P(\rho) = (2\mu + \lambda) \partial_r \left( \frac{1}{r} \partial_r (ru_r) \right),$$

$$\rho \partial_t u_\theta + \rho (u_r \partial_r u_\theta + \frac{u_\theta u_r}{r}) = \mu \partial_r \left( \frac{1}{r} \partial_r (ru_\theta) \right).$$

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**Figure 1.** Subsonic case of a viscous flow around a disc

**Figure 2.** Supersonic-sonic-subsonic case of a viscous flow around a disc

Note that we can look for a special steady circulatory flow $(\bar{\rho}(r), \bar{u}_r(r), \bar{u}_\theta(r))$ of (2)-(4) around the disc $\{ x : r \leq 1 \}$. Such a flow is called the background solution.
of (2)-(4) in the whole paper. By (2)-(4), we have that for \((\bar{\rho}(r), \bar{u}_r(r), \bar{u}_\theta(r))\),

\[
\begin{align*}
\frac{1}{r} \partial_r (r \bar{p} \bar{u}_r) &= 0, \\
\bar{p}(\bar{u}_r, \partial_r \bar{u}_r - \frac{\bar{u}_\theta^2}{r}) + \partial_r P(\bar{p}) &= (2\mu + \lambda) \partial_r (\frac{1}{r} \partial_r (r \bar{u}_r)), \\
\bar{p}(\bar{u}_r, \partial_r \bar{u}_\theta + \frac{\bar{u}_\theta \bar{u}_r}{r}) &= \mu \partial_r (\frac{1}{r} \partial_r (r \bar{u}_\theta)).
\end{align*}
\]

(5)

From the physical point of view for the axisymmetric circulatory flow, \([5]\) shall admit a natural boundary condition on \(\Sigma = \{ x : r = 1 \}\)

\[
(\bar{\rho}, \bar{u}_r, \bar{u}_\theta)|_{r=1} = (\bar{\rho}_0, 0, M_0),
\]

(6)

where \(\bar{\rho}_0 > 0\) and \(M_0 > 0\) are given constants.

Moreover, one imposes that at infinity

\[
\lim_{r \to \infty} \bar{u}_r(r) = 0, \quad \lim_{r \to \infty} \bar{u}_\theta(r) = 0.
\]

(7)

It is not difficult to know that \([5,6,8]\) have the unique solution in \(\Omega\)

\[
(\bar{\rho}(r), \bar{u}_r(r), \bar{u}_\theta(r)) = \left( \bar{\rho}_0 r^{-1} + \frac{(\gamma - 1)M_0^2}{2A\gamma} (1 - \frac{1}{r^2}) \right)^{\frac{\gamma}{\gamma-1}}, \quad 0, \quad \frac{M_0}{r}.\]

(8)

If \(M_0 < c(\bar{\rho}_0) = \sqrt{P'/\bar{\rho}}(\bar{\rho}_0)\), which means that the flow is subsonic on the boundary \(\Sigma\), then the circulatory flow is always subsonic in \(\Omega\) (i.e., \(u_\theta(r < r_0) < c(\bar{\rho}(r))\) holds for any \(r > 1\), one can see Figure [1]: if \(M_0 > c(\bar{\rho}_0)\), which means that the flow is supersonic on the boundary \(\Sigma\), then the flow is supersonic for \(1 < r < r_0\), subsonic for \(r > r_0\), and sonic on \(r = r_0\), respectively (see Figure [2], where \(r_0 = \sqrt{\frac{\gamma + 1}{2c^2(\bar{\rho}_0) + (\gamma + 1)M_0^2}} > 1\). Here we point out that the subsonic or supersonic-subsonic properties of viscous circulatory flows do not play the essential roles in the present paper. However, for the inviscid flow, the subsonic or supersonic-subsonic properties are crucial in order to analyze the blowup mechanism and determine the blowup time of the small perturbed solutions to the 2-D compressible Euler system (see [5]). In addition, one can see more mathematical descriptions on the steady circulatory flows of compressible Euler equations in \([2,8]\) and references therein.

Next we focus on the global stability problem of the background solution \([8]\). Namely, the global solution of (2)-(4) is required to be studied under the following perturbed initial-boundary value conditions:

\[
\begin{align*}
\rho(0, r) &= \bar{\rho}(r) + \rho_0(r), \quad u_r(0, r) = u_r^0(r), \quad u_\theta(0, r) = \bar{u}_\theta(r) + u_\theta^0(r), \quad \text{for } r \geq 1, \\
\lim_{r \to \infty} u_r(t, r) &= 0, \quad \lim_{r \to \infty} u_\theta(t, r) = 0, \quad \text{for } t \geq 0, \\
(u_r(t, r), u_\theta(t, r))|_{r=1} &= (g_r^0(t), M_0 + g_\theta^0(t)), \quad \text{for } t \geq 0,
\end{align*}
\]

(9)

where \(\rho_0(r) \in H_\text{loc}^2(\Omega), (u_r^0(r), u_\theta^0(r)) \in H_\text{loc}^1(\Omega), (g_r^0(t), g_\theta^0(t)) \in H_\text{loc}^3([0, \infty))\).

Moreover, the initial-boundary values in (9) are compatible on the circle \(S_0 = \{(t, x) : t = 0, r = 1\}\).

Choosing a function \(\varphi(r) \in C_0^\infty[0, +\infty)\) such that \(\varphi(1) = 1\) and \(\text{supp} \varphi \subset [\frac{1}{2}, 2]\).

Denote by

\[
(u_r^\varphi(t, r), u_\theta^\varphi(t, r)) = (\varphi(r)g_r^0(t), \varphi(r)g_\theta^0(t)).
\]
Let
\[
\begin{align*}
\rho(t, r) &= \phi(t, r) + \bar{\rho}(r), \\
u_r(t, r) &= v_r(t, r) + u^b_r(t, r), \\
\nu_\theta(t, r) &= v_\theta(t, r) + \bar{u}_\theta(r) + u^b_\theta(t, r)
\end{align*}
\]
and
\[
(v^0_r, v^0_\theta)(r) = (u^b_r(0, r), u^b_\theta(r) - u^b_\theta(0, r)).
\]

Then equations (2)-(4) together with (9) are respectively written as
\[
\begin{align*}
\partial_t \phi + \frac{1}{r} \partial_r (r \bar{\rho} v_r) &= f, \\
\partial_r v_r - \frac{2\bar{u}_\theta}{r} v_\theta + \frac{\lambda}{\bar{\rho}} \partial_r \left( \frac{1}{r} \partial_r (r v_r) \right) &= g_1, \\
\partial_r v_\theta - \frac{\mu}{\bar{\rho}} \partial_r \left( \frac{1}{r} \partial_r (r v_\theta) \right) &= g_2,
\end{align*}
\]
and
\[
\begin{align*}
\phi(0, r) &= \phi(0, r), (v_r, v_\theta)(0, r) = (v^0_r, v^0_\theta)(r), \\
(v_r, v_\theta)|_{r=1} &= 0, \lim_{r \to \infty} (v_r, v_\theta) = 0,
\end{align*}
\]
where
\[
\begin{align*}
f &= -\frac{1}{r} \partial_r (r \phi v_r) - \frac{1}{r} \partial_r (r \bar{\rho} u^b_r + r \phi u^b_r), \\
g_1 &= \frac{(v_\theta + u^b_\theta)^2}{r} - (v_r + u^b_r) \partial_r (v_r + u^b_r) - \partial_r Q(\bar{\rho}, \phi) - \frac{(2\mu + \lambda)}{(\bar{\rho} + \phi)} \partial_r \left( \frac{1}{r} \partial_r (r v_r) \right) \\
&\quad + \frac{2\mu + \lambda}{\bar{\rho}} \partial_r \left( \frac{1}{r} \partial_r (r u^b_\theta) \right) - \partial_r u^b_\theta + \frac{2\bar{u}_\theta u^b_\theta}{r}, \\
g_2 &= - (v_r + u^b_r) \partial_r (v_\theta + u^b_\theta) - \frac{(v_\theta + u^b_\theta)(v_r + u^b_r)}{r} - \frac{\mu \phi}{(\bar{\rho} + \phi)} \partial_r \left( \frac{1}{r} \partial_r (r v_\theta) \right) \\
&\quad + \frac{\mu}{\bar{\rho}} \partial_r \left( \frac{1}{r} \partial_r (r u^b_\theta) \right) - \partial_r u^b_\theta
\end{align*}
\]
with
\[
Q(\bar{\rho}, \phi) = \frac{1}{2} A \gamma (\gamma - 2) \phi^2 \int_0^1 (\bar{\rho} + \phi s)^{\gamma - 3} s ds.
\]

To state our main results conveniently, we now introduce the following notations:

For $w_1(r), w_2(r) \in L^2[1, \infty)$, set
\[
(w_1, w_2) = \int_1^{+\infty} w_1 w_2 dr
\]
and
\[
\|w_1\|_{L^2} = \|w_1\|_2 = \left( \int_1^{+\infty} w_1^2 dr \right)^{\frac{1}{2}}.
\]

For $k \in \mathbb{N}$, we define
\[
\|w\|_{\tilde{H}^k}^2 = \sum_{j=0}^k \| \sqrt{r} \partial_r^j w \|_2^2
\]
and
\[
\tilde{H}^k = \{ w \in \mathcal{D}'(1, +\infty) : \|w\|_{\tilde{H}^k} < +\infty \}.
\]

The main conclusion in our paper is:
Theorem 1.1. There exist a constant $\varepsilon > 0$ such that
\[
\|\rho_0\|_{H^2} + \|u_0^0\|_{H^3} + \|v_0^0\|_{H^3} \leq \varepsilon,
\]
and
\[
\sum_{k=0}^{1} \left( \sup_{0 \leq \tau \leq \infty} |\partial_t^k g_0^0(\tau)| + \sup_{0 \leq \tau \leq \infty} |\partial_t^k g_0^0(\tau)| \right)^2 + 2 \int_0^\infty \left( (\partial_t^k g_0^0)^2 + (\partial_t^k g_0^0)^2 \right) d\tau \leq \varepsilon^2,
\]
then problem (10) together with (13) has a unique global solution $(\phi, v) \in C([0, \infty), H^2 \times H^4)$, which fulfills for any $t \in [0, \infty)\$
\[
\|\phi\|_{H^2}^2 + \|v\|_{H^3}^2 + \int_0^t \left( \|\partial_r (\rho^{-2} \phi)\|_{H^1}^2 + \|\partial_r v\|_{H^3}^2 \right) ds \leq C \varepsilon^2,
\]
where $v = (v_r, v_0)$, and the constant $C > 0$ depends only on $\mu$, $\lambda$, $M_0$, $\bar{\rho}_0$, $A$ and $\gamma$.

Remark 1. For the Navier-Stokes equations (1), one knows from Theorem 1.1 that the global stability of the perturbed axisymmetric circulatory flows holds both for the subsonic background solution and the subsonic-sonic-supersonic background solution. This is somewhat different from the case of compressible Euler system, where the blowup result is shown and the shock wave is formed from the blowup point (see [5]). On the other hand, even if for the small initial data problem of 2-D compressible isentropic Euler equations, the classical axisymmetric solution will develop singularity in finite time (see the details in [1] and [13]).

Remark 2. So far there have been extensive results on the global spherically symmetric weak/strong/classical solutions to the compressible Navier-Stokes equations (in this case, the solution admits a form $(\rho(t, x), u(t, x)) = (\rho(t, r), U(t, r)\frac{X}{r})$, one can see [15, 14, 4], [12, 17] and the references therein. Here, we point out that even if $u_0 \equiv 0$, the initial data in [9] does not satisfy the assumptions of the symmetric initial data in [15], where $\int_1^\infty r |u_0(x)|^2 dr < \infty$ is demanded. However, in our case, $\int_1^\infty \frac{r |u_0(x)|^2}{1} dr = M_0^2 \int_1^\infty \frac{dr}{r} = \infty$.

Remark 3. For the general Cauchy problem or initial-boundary value problem in exterior domain of 2-D compressible Navier-Stokes equations, when the initial data are in some suitably weighted energy spaces or are of the small perturbations of constant states, many authors have established the local/global existence of weak/strong/classical solutions in appropriate function spaces, one can find the details in [20], [5], [11], [6], [13, 14], [16] and so on. If we want to study the general (not axisymmetric) global perturbation problem of the 2-D circulatory flows for (1), the methods applied in the above references cannot be applied directly since our perturbed initial data are rather different from those (for examples, our initial data have not finite energies or are not the small perturbation of the constant states). On the other hand, motivated by the results and methods in [8] and [9, 10], where the global stabilities and large time behaviors of the perturbed plane Couette flow, of the perturbed constant equilibrium on the half space or of the parallel flow in an infinite layer of $\mathbb{R}^2$ are studied respectively when the Reynolds and Mach numbers are sufficiently small, we hope that it will be probably useful for our future research on the global stability of generally perturbed viscous circulatory flows.
Let’s recall some previous works which are closely related to our results. For the initial-boundary value problem of (1), the local classical solution of compressible Navier-Stokes equations is obtained in [16] with \( \rho_0 \) being positive and bounded. Applying the energy methods in Sobolev spaces, the authors in [14] established the global existence of classical solutions to (1) when the initial data are of small perturbations for a non-vacuum constant state and no slip boundary conditions are given. There are also some interesting results about the existence of global strong solutions to 2-D Navier-Stokes equations [1], for instance, one can see [18] and references therein. For the arbitrary initial data with the finite total energy, the global existence of weak solutions to compressible Navier-Stokes equations has been established by P. L. Lions in [6] for suitably large adiabatic exponent \( \gamma \), and this result was improved to the case of any \( \gamma > 1 \) for the spherically symmetric solution in [7]. In addition, D. Hoff in [4] showed the existence of spherically symmetric weak solutions for \( \gamma = 1 \) and the discontinuous initial data. It is worth mentioning that the global radially symmetric strong solutions of the Navier-Stokes equations was established in [8]. Here we point out that our problem (10)-(12) has a nontrivial rotation and the related background solution is not a constant state, which are different from those cases mentioned in the previous references.

To prove Theorem 1.1, we require to establish some global weighted energy estimates of the solution \((\phi, v)\) to (10)-(12). Thanks to the delicate analysis on system (10)-(12), the uniform weighted estimates of \((\phi, v)\) are obtained by making full use of the properties of the background solution and choosing suitable multipliers. Based on this and the local existence result of classical solution to (10)-(12) with (13), Theorem 1.1 is shown by the continuity induction method.

The paper is organized as follows: In §2 we derive some uniform energy estimates from the linearized parts of (10)-(12). Based on this, some uniform weighted energy inequalities of \((\phi, v)\) are obtained in §3 and subsequently the proof of Theorem 1.1 is completed.

2. Some elementary estimates. In this section, we will derive some useful inequalities on the solution \((\phi, v)\) to (10)-(12) together with (13). For convenience, we set \( c_0 = \sup_{r \in [1, \infty)} \bar{\rho}(r) + \sup_{r \in [1, \infty)} \bar{\phi}(r) \) through the whole section.

**Lemma 2.1.** (Weighted \( L^2 \)-estimate of \((\phi, v)\) and time-integral estimate of \((v, \partial_r v)\)). For the solution \((\phi, v) \in C([0, \infty), \tilde{H}^2 \times \tilde{H}^3)\) of problem (10)-(12) with (13), we have that for any \( t \geq 0 \),

\[
\frac{1}{2} \left( \| \sqrt{r} v \|_{L^2}^2 + A \gamma \| \sqrt{r} \phi \|_{L^2}^2 \right) + \int_0^t \left( \frac{\mu}{r^{1/2}} \| v \|_{L^2}^2 + \mu \| \phi \|_{L^2}^2 \right) d\tau \\
\leq C \left( \| \sqrt{r} v^0 \|_{L^2}^2 + \| \sqrt{r} \phi^0 \|_{L^2}^2 \right) + C \int_0^t A_1 d\tau,
\]

where \( C > 0 \) is a generic constant, \( v^0 = (v_0^\theta, v_0^\rho) \), and \( A_1 = (g, \bar{\rho} v) + (f, \bar{\rho} \gamma^{-2} r \phi) \).

**Proof.** Computing \( \int_1^\infty (12) \times \bar{\rho} v \, dr \) and \( \int_1^\infty (12) \times \bar{\rho} v \, dr \) yields respectively,

\[
(\partial_t v, \bar{\rho} v) - \left( \frac{2 u^\theta}{r} v_\theta, \bar{\rho} v \right) + (A \gamma \gamma^{-2} \phi, \bar{\rho} v) \\
-(\frac{2 \mu + \lambda}{\bar{\rho}} \partial_r (\frac{1}{r} \partial_r (v \bar{\rho})), \bar{\rho} v) = (g_1, \bar{\rho} v)
\]

(14)
and
\[(\partial_t v_\theta, r \partial_r v_\theta) - (\frac{\mu}{\rho} \partial_t (\frac{1}{r} \partial_r (r v_\theta)), r \partial_r v_\theta) = (g_2, r \partial_r v_\theta).\] (15)

Note that
\[
(\partial_t v_r, r \partial_r v_r) = \frac{1}{2} \frac{d}{dt} \|r \partial_r v_r\|_2^2,
\]
\[
-\left(\frac{2 \tilde{u}}{r} v_\theta, r \partial_r v_r\right) = -(2M_0 \tilde{\rho} \frac{v_\theta}{r^{1/2}}, v_r \frac{v_r}{r^{1/2}}) \leq \frac{2M_0^2 \tilde{\gamma}^2}{2\mu + \lambda} \|v_\theta\|_2^2 + \frac{2\mu + \lambda}{2} \|v_r \frac{v_r}{r^{1/2}}\|_2^2,
\]
and
\[
-(\frac{\mu}{\rho} \partial_t (\frac{1}{r} \partial_r (r v_r)), \rho \partial_r v_r) = \mu \left(\frac{1}{r} \partial_r (r v_r), \partial_r (r v_r)\right) = \mu \left(\frac{v_r}{r^{1/2}} \|\|v_r \|_2^2 + \|r^{1/2} \partial_r v_r\|_2^2\right).
\]

Then from (14) we arrive at
\[
\frac{1}{2} \frac{d}{dt} \|r \partial_r v_r\|_2^2 + (\partial_t (A\tilde{\gamma} \tilde{\rho}^{-2} \phi), \rho \partial_r v_r) + \frac{2\mu + \lambda}{2} \|v_r \frac{v_r}{r^{1/2}}\|_2^2 + (2\mu + \lambda) \|r^{1/2} \partial_r v_r\|_2^2 \leq \frac{2M_0^2 \tilde{\gamma}^2}{2\mu + \lambda} \|v_\theta\|_2^2 + (g_1, \rho \partial_r v_r).
\] (16)

Also, by (15) we see that
\[
\frac{1}{2} \frac{d}{dt} \|r \partial_r v_\theta\|_2^2 + \mu \left(\|v_r \|_2^2 + \|r^{1/2} \partial_r v_\theta\|_2^2\right) = (g_2, \rho \partial_r v_\theta).
\] (17)

It follows from
\[
\int_0^t \left(17\right) \times \left(1 + \frac{2M_0^2 \tilde{\gamma}^2}{\mu(2\mu + \lambda)}\right) \frac{d}{d\tau} \left(\frac{1}{2} \|r \partial_r v_\theta\|_2^2 + \int_0^\tau \left(\partial_t (A\tilde{\gamma} \tilde{\rho}^{-2} \phi), \rho \partial_r v_r\right) + \frac{\mu}{2} \|v_r \frac{v_r}{r^{1/2}}\|_2^2 + \mu \|r^{1/2} \partial_r v_r\|_2^2 \right) d\tau \leq \frac{1}{2} \|r \partial_r v_\theta\|_2^2 + C \int_0^t (g, \rho \partial_r v_\theta) d\tau.
\] (18)

In addition, by
\[
\int_1^\infty \left(10\right) \times A\gamma \tilde{\rho}^{-2} r \phi dr, \text{ one immediately obtains}
\]
\[
\frac{A\gamma}{2} \frac{d}{dt} \left(\|r^{-2} \tilde{r} \phi\|_2^2 + (A \gamma \tilde{\rho}^{-2} \phi, \partial_r (r \partial_r v_r)) = (f, A \gamma \tilde{\rho}^{-2} r \phi).\right)
\] (19)

Adding (18) to (19) derives
\[
\frac{1}{2} \left(\|r \partial_r v_\theta\|_2^2 + A\gamma \|r \tilde{\rho}^{-2} \phi\|_2^2\right) + \int_0^t \left(\frac{\mu}{2} \|v_r \|_2^2 + \mu \|r^{1/2} \partial_r v_\theta\|_2^2\right) d\tau \leq C \left(\|r \partial_r v_\theta\|_2^2 + \|r^{-2} \tilde{r} \phi\|_2^2\right) + C \left|\int_0^t (g, \rho \partial_r v_\theta) + (f, \tilde{\rho}^{-2} r \phi)\right| \frac{d\tau}{d\tau}.
\] (20)

Thus, the proof of Lemma 2.1 is completed. 

**Lemma 2.2.** (Weighted $L^2$-estimate of $(v, \partial_t v, \partial_r v)$ and time-integral estimate of $(\partial_t \phi, \partial_r \phi)$). For the solution $(\phi, v) \in C([0, \infty), H^2 \times H^3)$ of problem (11) with (13), we have that for any $t \geq 0$,
\[
\|v\|_{r^{1/2}}^2 + \|r^{1/2} \partial_r v\|_2^2 \leq C \left(\|\tilde{u}\|_{r^{1/2}}^2 + \|r^{1/2} \partial_r v\|_2^2\right) + C \left|\int_0^t (A_1 + A_2) d\tau\right|.
\]
where $A_1$ is defined in Lemma 2.1, and $A_2 = (g, \tilde{\rho} \tilde{v}_r \tilde{\partial}_r v) + (f, A \gamma \tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi)$.

**Proof.** It follows from $\int_{1}^{+\infty} (11) \times \tilde{\rho} \tilde{v}_r \tilde{\partial}_t v_r, dr$ and $\int_{1}^{+\infty} (12) \times \tilde{\rho} \tilde{v}_r \tilde{\partial}_t v_r, dr$ that

\[
(\partial_t v_r, \tilde{\rho} \tilde{v}_r) - \left(\frac{2\tilde{u}}{r} \tilde{v}_r, \tilde{\rho} \tilde{v}_r\right) + (\partial_r (A \gamma \tilde{\rho} \tilde{v}_r \tilde{\partial}_r \phi), \tilde{\rho} \tilde{v}_r) - \left(\frac{2\mu + \lambda}{\rho} \partial_r \left(\frac{1}{r} \partial_r (\tilde{v}_r \tilde{\rho} \tilde{v}_r)\right), \tilde{\rho} \tilde{v}_r \tilde{\partial}_t v_r\right) = (g_1, \tilde{\rho} \tilde{v}_r \tilde{\partial}_t v_r)
\]

(21)

and

\[
(\partial_t v_\theta, \tilde{\rho} \tilde{v}_\theta) - \left(\frac{\mu}{\rho} \partial_r \left(\frac{1}{r} \partial_r (\tilde{v}_\theta \tilde{\rho} \tilde{v}_\theta)\right), \tilde{\rho} \tilde{v}_r \tilde{\partial}_t v_\theta\right) = (g_2, \tilde{\rho} \tilde{v}_r \tilde{\partial}_t v_\theta).
\]

(22)

Note that

\[
(\partial_t v_r, \tilde{\rho} \tilde{v}_r) = \|\sqrt{\tilde{\rho}} \partial_t v_r\|_2^2,
\]

(23)

\[-\left(\frac{2\tilde{u}}{r} \tilde{v}_r, \tilde{\rho} \tilde{v}_r\right) \leq \frac{1}{2} \|\sqrt{\tilde{\rho}} \partial_t v_r\|_2^2 + 2c_0 M_0^2 \|\tilde{v}_r\|_{r^{1/2}}^2,
\]

(24)

\[-\left(\frac{2\mu + \lambda}{\rho} \partial_r \left(\frac{1}{r} \partial_r (\tilde{v}_r \tilde{\rho} \tilde{v}_r)\right), \tilde{\rho} \tilde{v}_r \tilde{\partial}_t v_r\right) = \frac{2\mu + \lambda}{2} \frac{d}{dt} \left(\frac{v_r}{r^{1/2}}\right)^2 + \|r^{1/2} \partial_r v_r\|_2^2,
\]

(25)

Then by (21), (22) and (23)-(25), we obtain

\[
\frac{1}{2} \|\sqrt{\tilde{\rho}} \partial_t v_r\|_2^2 + (\partial_r (A \gamma \tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi), \tilde{\rho} \tilde{v}_r \tilde{\partial}_t v_r) + \frac{2\mu + \lambda}{2} \frac{d}{dt} \left(\frac{v_r}{r^{1/2}}\right)^2 + \|r^{1/2} \partial_r v_r\|_2^2
\]

\[
\leq (g, \tilde{\rho} \tilde{v}_r \tilde{\partial}_t v_r) + 2c_0 M_0^2 \|\tilde{v}_r\|_{r^{1/2}}^2.
\]

(26)

Computing $\int_{1}^{+\infty} (10) \times A \gamma \tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi, dr$ yields

\[
A \gamma \sqrt{\tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi} \left(\frac{1}{r} \partial_r (r \tilde{v}_r \tilde{\rho} v_r), A \gamma \tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi\right) = (f, A \gamma \tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi).
\]

(27)

We see that

\[
(\partial_r (A \gamma \tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi), \tilde{\rho} \tilde{v}_r \tilde{\partial}_t v_r) + \left(\frac{1}{r} \partial_r (r \tilde{v}_r \tilde{\rho} v_r), A \gamma \tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi\right) = -\partial_r (A \gamma \tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi) + 2(\tilde{v}_r \tilde{\rho} \tilde{v}_r, A \gamma \tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi).
\]

(28)

Adding (26) to (27) and using equality (28), one has

\[
\frac{1}{2} \|\sqrt{\tilde{\rho}} \partial_t v_r\|_2^2 + \|A \gamma \sqrt{\tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi}\|_2^2
\]

\[-A \gamma \partial_t (\tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi) + 2A \gamma (\tilde{v}_r \tilde{\rho} \tilde{v}_r, \tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi) + \frac{2\mu + \lambda}{2} \frac{d}{dt} \left(\frac{v_r}{r^{1/2}}\right)^2 + \|r^{1/2} \partial_r v_r\|_2^2
\]

\[
\leq (g, \tilde{\rho} \tilde{v}_r \tilde{\partial}_t v_r) + (f, A \gamma \tilde{\rho} \tilde{v}_r \tilde{\partial}_t \phi) + 2c_0 M_0^2 \|\tilde{v}_r\|_{r^{1/2}}^2.
\]

(29)
On the other hand, direct computation yields
\[
\frac{d}{dt} \left( \frac{2\mu + \lambda}{2} \parallel \frac{v}{r^{1/2}} \parallel^2 + \frac{2\mu + \lambda}{2} \parallel r^{1/2} \partial_r v \parallel^2 - (A\gamma \bar{\rho}^{\gamma - 2} \phi, \partial_r (\bar{p} v_r)) \right) \\
+ \frac{1}{2} \parallel \sqrt{r} \partial_r v \parallel^2 + A\gamma \parallel r^{\gamma - 2} \partial_t \phi \parallel^2 \\
\leq -2(\partial_r (r \bar{p} v_r), A\gamma \bar{\rho}^{\gamma - 2} \partial_t \phi) + 2CM_0^2 \parallel \frac{v}{r^{1/2}} \parallel^2 + (g, \bar{\rho} \partial_r v) + (f, A\gamma r \bar{\rho}^{\gamma - 2} \partial_t \phi).
\]

(30)

It is noted that
\[
-2(\partial_r (r \bar{p} v_r), A\gamma \bar{\rho}^{\gamma - 2} \partial_t \phi) \leq 2A\gamma \parallel \sqrt{\bar{\rho}^{\gamma - 2} \partial_r (r \bar{p} v_r)} \parallel^2 + \frac{A\gamma}{2} \parallel r^{\gamma - 2} \partial_t \phi \parallel^2 \\
\leq C(\parallel r^{1/2} \partial_r v_r \parallel^2 + \parallel \frac{v}{r^{1/2}} \parallel^2) + \frac{A\gamma}{2} \parallel r^{\gamma - 2} \partial_t \phi \parallel^2
\]

and
\[
-(A\gamma \bar{\rho}^{\gamma - 2} \phi, \partial_r (\bar{p} v_r)) = -(A\gamma \bar{\rho}^{\gamma - 2} \phi, \partial_r \bar{p} v_r) - (A\gamma \bar{\rho}^{\gamma - 2} \phi, \bar{p} \partial_r v_r) \\
\leq C \parallel r^{\gamma - 2} \partial_t \phi \parallel^2 + \frac{\mu}{4} \parallel \frac{v}{r^{1/2}} \parallel^2 + \frac{\mu}{4} \parallel \partial_r v_r \parallel^2.
\]

(31)

Combining (30)-(32), we arrive at
\[
\frac{d}{dt} \left( \frac{\mu}{4} \parallel \frac{v}{r^{1/2}} \parallel^2 + \frac{\mu}{4} \parallel r^{1/2} \partial_r v \parallel^2 - C \parallel \sqrt{r} \bar{\rho}^{\gamma - 2} \phi \parallel^2 \right) \\
+ \frac{1}{2} \parallel \sqrt{r} \partial_r v \parallel^2 + A\gamma \parallel r^{\gamma - 2} \partial_t \phi \parallel^2 \\
\leq C(\parallel r^{1/2} \partial_r v_r \parallel^2 + \parallel \frac{v}{r^{1/2}} \parallel^2) + (g, \bar{\rho} r \partial_r v) + (f, A\gamma r \bar{\rho}^{\gamma - 2} \partial_t \phi).
\]

(33)

This, together with Lemma 2.1, yields
\[
\parallel \frac{v}{r^{1/2}} \parallel^2 + \parallel r^{1/2} \partial_r v \parallel^2 + \int_0^t \left( \parallel \sqrt{r} \partial_r v \parallel^2 + \parallel \sqrt{r} \bar{\rho}^{\gamma - 2} \partial_t \phi \parallel^2 \right) d\tau \\
\leq C(\parallel \frac{v}{r^{1/2}} \parallel^2 + \parallel r^{1/2} \partial_r v \parallel^2) \\
+ C \int_0^t \left( (g, \bar{\rho} r \partial_r v) + (f, A\gamma r \bar{\rho}^{\gamma - 2} \partial_t \phi) + (g_1, \bar{p} v_r) + (f, A\gamma r \bar{\rho}^{\gamma - 2} \partial_t \phi) \right) d\tau.
\]

(34)

Then Lemma 2.2 is shown.

Taking \( \partial_t \) to equations (10)-(12), we then have
\[
\partial_t (\partial_t \phi) + \frac{1}{r} \partial_r (\gamma \bar{p} \partial_r v_r) = \partial_t f, \\
\partial_t^2 v_r - \frac{2\bar{u}}{r} \partial_t v_0 + A\gamma \partial_r (\bar{\rho}^{\gamma - 2} \partial_t \phi) - \frac{2\mu + \lambda}{\bar{\rho}} \partial_r (\frac{1}{r} \partial_r (r \partial_r v_r)) = \partial_t g_1, \\
\partial_t^2 v_0 - \frac{\mu}{\bar{\rho}} \partial_r (\frac{1}{r} \partial_r (r \partial_r v_0)) = \partial_t g_2.
\]

(35)
Lemma 2.3. (Weighted $L^2$-estimate of $(\partial_t \phi, \partial_t v)$ and time-integral estimate of $(\partial_t \phi, \partial_t^2 v)$). For the solution $(\phi, v) \in C([0, \infty), H^2 \times \tilde{H}^3)$ of problem (10)-(12) with (13), we have that for any $t \geq 0$,

$$
\|\sqrt{\rho r} \partial_t v\|_{L^2}^2 + \|\sqrt{\rho \gamma^{-2} r} \partial_t \phi\|_{L^2}^2 + \int_0^t \left( \frac{\mu}{2} \|\partial_r v\|_{L^2}^2 + \mu \|r^{1/2} \partial_r^2 v\|_{L^2}^2 \right) d\tau \\
\leq C(\|v(0)\|_{H^2}^2 + \|\rho_0\|_{H^1}^2) + C \int_0^t (A_1 + A_2 + A_3) d\tau,
$$

where $A_1$ and $A_2$ are defined in Lemma 2.1 and Lemma 2.2 respectively, and $A_3 = (\partial_t g_1, \sqrt{\rho r} \partial_t \phi) + (\partial_t f, \gamma \sqrt{\rho r} \gamma^{-2} \partial_t^2 \phi)$.

Analogously, as Lemma 2.2 we have the following higher order weighted energy estimates:

Lemma 2.4. (Weighted $L^2$-estimate of $(\partial_t v, \partial_t^2 v)$ and time-integral estimate of $(\partial_t^2 \phi, \partial_t^3 v)$). For the solution $(\phi, v) \in C([0, \infty), H^2 \times \tilde{H}^3)$ of problem (10)-(12) with (13), we have that for any $t \geq 0$,

$$
\|\partial_t v\|_{L^2}^2 + \|r^{1/2} \partial_r v\|_{L^2}^2 + \int_0^t \left( \|\sqrt{\rho r} \partial_r^2 v\|_{L^2}^2 + \|\sqrt{r \rho \gamma^{-2}} \partial_r^2 \phi\|_{L^2}^2 \right) d\tau \\
\leq C(\|v(0)\|_{H^3}^2 + \|\rho_0\|_{H^1}^2) + C \int_0^t (A_1 + A_2 + A_3 + A_4) d\tau,
$$

where $A_1 - A_4$ are defined in Lemma 2.1-Lemma 2.3 respectively, and

$$
A_4 = (\partial_t g, \sqrt{\rho r} \partial_t^2 v) + (\partial_t f, \gamma \sqrt{\rho r} \gamma^{-2} \partial_t^2 \phi).
$$

Next we start to derive the a priori estimates on $\phi$. It follows from $\partial_r (\tilde{\rho}^{-2} \times (10))$ and direct computation that

$$
\partial_r \partial_r (\tilde{\rho}^{-1} \partial_r^2 v) + \tilde{\rho}^{-1} \partial_r^2 \partial_r v = \tilde{\rho}^{-1} \partial_r^2 v + \partial_r (\tilde{\rho}^{-2} f - \tilde{\rho}^{-2} \frac{1}{r} \partial_r (r \tilde{\rho}) v_r).
$$

In addition, computing $\tilde{\rho}^{-1} \times (11)$ yields

$$
A_1 \tilde{\rho}^{-1} \tilde{\rho}^{-1} \partial_r (\tilde{\rho}^{-2} \phi) - (2\mu + \lambda) \tilde{\rho}^{-1} \partial_r^2 v_r = \tilde{\rho}^{-1} \left( -\partial_r v_r + \frac{2\tilde{\mu}}{\tilde{\rho}} v_\theta + \frac{2\mu + \lambda}{\tilde{\rho}} \partial_r \left( \frac{1}{r} v_r \right) + g_1 \right).
$$

Adding $(2\mu + \lambda) \times (38)$ to (39), we arrive at

$$
(2\mu + \lambda) \partial_t \partial_t (\tilde{\rho}^{-2} \phi) + A_1 \tilde{\rho}^{-1} \partial_r (\tilde{\rho}^{-2} \phi) = h,
$$

where

$$
h = (2\mu + \lambda) \left( \tilde{\rho}^{-1} \partial_r^2 v_r + \partial_r (\tilde{\rho}^{-2} f - \tilde{\rho}^{-2} \frac{1}{r} \partial_r (r \tilde{\rho}) v_r) \right) \\
+ \tilde{\rho}^{-1} \left( -\partial_r v_r + \frac{2\tilde{\mu}}{\tilde{\rho}} v_\theta + \frac{2\mu + \lambda}{\tilde{\rho}} \partial_r \left( \frac{1}{r} v_r \right) + g_1 \right).
$$
By \( \int_1^\infty (40) \times r \partial_r (\bar{\rho}^{-2} \phi) dr \), one has

\[
\frac{2\mu + \lambda}{2} \frac{d}{dt} ||r^{1/2} \partial_r (\bar{\rho}^{-2} \phi) ||_2^2 + A\gamma \sqrt{\bar{\rho} r} \partial_r (\bar{\rho}^{-2} \phi) ||_2^2 \leq (\partial_r (\bar{\rho}^{-2} \phi), rh). \tag{41}
\]

On the other hand, we have from (40) that

\[
(2\mu + \lambda) \partial_t \partial_r^2 (\bar{\rho}^{-2} \phi) + A\gamma \bar{\rho} \partial_r^2 (\bar{\rho}^{-2} \phi) = \partial_r h - A\gamma \partial_r (\bar{\rho}^{-2} \phi). \tag{42}
\]

In addition, by \( \int_1^\infty (42) \times r \partial_r^2 (\bar{\rho}^{-2} \phi) dr \), we have

\[
\frac{2\mu + \lambda}{2} \frac{d}{dt} ||r^{1/2} \partial_r^2 (\bar{\rho}^{-2} \phi) ||_2^2 + A\gamma \sqrt{\bar{\rho} r} \partial_r^2 (\bar{\rho}^{-2} \phi) ||_2^2 \\
\leq (\partial_r^2 (\bar{\rho}^{-2} \phi), r \partial_r h) - (\partial_r^2 (\bar{\rho}^{-2} \phi), r \partial_r (\bar{\rho}^{-2} \phi)). \tag{43}
\]

Based on the preparations above, we can prove

**Lemma 2.5.** (Weighted \( H^2 \)-estimate of \( \phi \) and time-integral estimate of \( (\partial_r^2 \phi, \partial_r^2 v_r) \)). For the solution \((\phi, v) \in C([0, \infty), H^2 \times H^3) \) of problem (10)-(12) with (13), we have that for any \( t \geq 0 \),

\[
\| \bar{\rho}^{-2} \phi \|_{H^2}^2 + \int_0^t \left( \| \partial_r (\bar{\rho}^{-2} \phi) \|_{H^1}^2 + \| r^{1/2} \partial_r^2 v_r \|_2^2 \right) dr \\
\leq C(\| \rho_0 \|_{H^2}^2 + \| v_0 \|_{H^2}^2) + C \int_0^t \left( A_1 + A_2 + A_3 + g_1 \| \partial_r h \|_{H^1}^2 + \| f \|_{H^1}^2 \right) d\tau,
\]

where \( A_1 - A_3 \) are defined in Lemma 2.4 Lemma 2.2 respectively.

**Proof.** At first, we see that

\[
(\partial_r (\bar{\rho}^{-2} \phi), rh) \leq \frac{1}{2} \| \sqrt{\bar{\rho} r} \partial_r (\bar{\rho}^{-2} \phi) \|_2^2 + C \| r^{1/2} h \|_2^2. \tag{44}
\]

Note that

\[
\| r^{1/2} h \|_2^2 \leq C(\| r^{1/2} f \|_2^2 + \| r^{-1/2} v \|_2^2 + \| r^{1/2} \partial_r v_r \|_2^2 + \| r^{1/2} \partial_r^2 v_r \|_2^2 + \| r^{1/2} g_1 \|_2^2). \]

This, together with (41), yields

\[
\frac{2\mu + \lambda}{2} \frac{d}{dt} ||r^{1/2} \partial_r (\bar{\rho}^{-2} \phi) ||_2^2 + \frac{A\gamma}{2} \sqrt{\bar{\rho} r} \partial_r (\bar{\rho}^{-2} \phi) ||_2^2 \\
\leq C(\| r^{1/2} f \|_2^2 + \| r^{-1/2} v \|_2^2 + \| r^{1/2} \partial_r v_r \|_2^2 + \| r^{1/2} \partial_r^2 v_r \|_2^2 + \| r^{1/2} g_1 \|_2^2). \tag{45}
\]

Similarly, we see that

\[
(\partial_r^2 (\bar{\rho}^{-2} \phi), r \partial_r h) \leq \frac{1}{4} \| \sqrt{\bar{\rho} r} \partial_r^2 (\bar{\rho}^{-2} \phi) \|_2^2 + C \| r^{1/2} \partial_r h \|_2^2,
\]

\[
(\partial_r^2 (\bar{\rho}^{-2} \phi), r \partial_r (\bar{\rho}^{-2} \phi)) \leq \frac{1}{4} \| \sqrt{\bar{\rho} r} \partial_r^2 (\bar{\rho}^{-2} \phi) \|_2^2 \\
+ C(\| r^{1/2} \partial_r f \|_2^2 + \| r^{-1/2} \partial_r v_r \|_2^2 + \| r^{1/2} \partial_r^2 v_r \|_2^2 + \| r^{1/2} \partial_r^2 g_1 \|_2^2). \tag{46}
\]

In addition, by applying Holder inequality, one has

\[
\| r^{1/2} \partial_r h \|_2^2 \\
\leq C(\| r^{1/2} f \|_2^2 + \| r^{-1/2} v \|_2^2 + \| r^{1/2} \partial_r v_r \|_2^2 + \| r^{1/2} \partial_r^2 v_r \|_2^2 + \| r^{1/2} \partial_r g_1 \|_2^2) + C(\| r^{1/2} \partial_r f \|_2^2 + \| r^{-1/2} \partial_r v_r \|_2^2 + \| r^{1/2} \partial_r^2 v_r \|_2^2 + \| r^{1/2} \partial_r^2 g_1 \|_2^2).
\]

Then combining (46) with (43) yields

\[
\frac{2\mu + \lambda}{2} \frac{d}{dt} \| r^{1/2} \partial_r^2 (\rho \gamma - 2 \phi) \|^2_2 + \frac{A\gamma}{2} \| \sqrt{r} \tau \partial_r^2 (\rho \gamma - 2 \phi) \|^2_2 \\
\leq C(\| r^{1/2} f \|^2_2 + \| r^{-1/2} v \|^2_2 + \| r^{1/2} \partial_r v \|^2_2 + \| r^{1/2} \partial_r^2 v \|^2_2 + \| r^{1/2} g_1 \|^2_2) \\
+C(\| r^{1/2} \partial_r f \|^2_2 + \| r^{-1/2} \partial_r v \|^2_2 + \| r^{1/2} \partial_r^2 v \|^2_2 + \| r^{1/2} \partial_r g_1 \|^2_2).
\]

(47)

On the other hand, one can rewrite (10) as

\[
(2\mu + \lambda) \| r^{1/2} \partial_r v_r \|^2_2 = \rho \partial_r v_r - \frac{2\mu}{r} v_0 + A\gamma \rho \partial_r (\rho \gamma - 2 \phi) - \mu \partial_r (\frac{v_r}{r}) - \rho g_1,
\]

which implies

\[
(2\mu + \lambda) \| r^{1/2} \partial_r^2 v_r \|^2_2 \leq C(\| r^{1/2} \rho \partial_r v_r \|^2_2 + \| r^{1/2} \rho \partial_r^2 (\rho \gamma - 2 \phi) \|^2_2 \\
+ \| r^{1/2} \rho \partial_r (\rho \gamma - 2 \phi) \|^2_2 + \| r^{1/2} \partial_r (\frac{v_r}{r}) \|^2_2 + \| r^{1/2} \rho g_1 \|^2_2).
\]

(48)

From (45) and Lemma 2.1, 2.2 we see that

\[
\frac{2\mu + \lambda}{2} \| r^{1/2} \partial_r (\rho \gamma - 2 \phi) \|^2_2 + \frac{A\gamma}{2} \int_0^t \| \sqrt{r} \tau \partial_r (\rho \gamma - 2 \phi) \|^2_2 d\tau \\
\leq C(\| \rho \theta \|^2_{\dot{H}^1} + \| v_0 \|^2_{\dot{H}^2}) + C \int_0^t (A_1 + A_2 + \| r^{1/2} g_1 \|^2_2 + \| r^{1/2} f \|^2_2) d\tau |. (50)
\]

This, together with (46)-(47), (49), and Lemma 2.3 yields

\[
\frac{2\mu + \lambda}{2} \| r^{1/2} \partial_r^2 (\rho \gamma - 2 \phi) \|^2_2 + \int_0^t \left( \frac{A\gamma}{2} \| \sqrt{r} \tau \partial_r^2 (\rho \gamma - 2 \phi) \|^2_2 + (2\mu + \lambda) \| r^{1/2} \partial_r^2 v_r \|^2_2 \right) d\tau \\
\leq C(\| \rho \theta \|^2_{\dot{H}^1} + \| v_0 \|^2_{\dot{H}^2}) + C \int_0^t \left( A_1 + A_2 + A_3 + \| g_1 \|^2_{\dot{H}^1} + \| f \|^2_{\dot{H}^1} \right) d\tau |. (51)
\]

Then Lemma 2.3 is proved.

3. Global estimates of $(\phi, v)$ and proof of Theorem 1.1. For the solution $(\phi, v) \in C([0, \infty), \dot{H}^2 \times \dot{H}^3)$ of problem (10)-(12) with (13), by Lemma 2.1, 2.5, we arrive at

\[
\| v \|^2_{\dot{H}^1} + \| \partial_r v \|^2_{\dot{H}^1} + \| \rho \gamma - 2 \phi \|^2_{\dot{H}^2} + \| \sqrt{r} \tau \partial_r \phi \|^2_2 \\
+ \int_0^t \left( \| \frac{v}{r^{1/2}} \|^2_2 + \| \partial_r v \|^2_{\dot{H}^1} + \| \sqrt{r} \tau v_0 \|^2_2 + \| \sqrt{r} \tau \partial_r \phi \|^2_2 \\
+ \| \sqrt{r} \tau \partial_r^2 \phi \|^2_2 + \| \partial_r (\rho \gamma - 2 \phi) \|^2_{\dot{H}^1} \right) d\tau \\
\leq C(\| \rho \theta \|^2_{\dot{H}^1} + \| v_0 \|^2_{\dot{H}^1}) + C \int_0^t (A_1 + A_2 + A_3 + A_4 + \| f \|^2_{\dot{H}^1} + \| g_1 \|^2_{\dot{H}^1}) d\tau |
\]

(52)

By equation (48), one arrives at

\[
\| \partial_r^2 v_r \|^2_{\dot{H}^1} \leq C\| \rho \partial_r v_r - \frac{2\mu}{r} v_0 + A\gamma \rho \partial_r (\rho \gamma - 2 \phi) - (2\mu + \lambda) \partial_r (\frac{v_r}{r}) \|^2_{\dot{H}^1} + \| g_1 \|^2_{\dot{H}^1}.
\]

(53)
This, together with (52) and direct integration with respect to variable \( t \), yields
\[
\|v\|^2_{H^3} + \|\nabla^2 v_r\|^2_{H^1} + \|\partial_t v\|^2_{H^2} + \|\sqrt{\gamma - 2}\phi\|^2_{H^2} + \|\sqrt{\gamma - 2}\phi\|^2_{H^2} \\
+ \int_0^t \left( \|\nabla^3 v_r\|^2_{H^1} + \|\partial_t^2 v_r\|^2_{H^1} + \|\sqrt{\gamma} \partial_t v_r\|^2_{H^1} + \|\sqrt{\gamma} \partial_t^2 v_r\|^2_{H^1} + \|\sqrt{\gamma} \partial_t^3 v_r\|^2_{H^1} \right) d\tau \\
\leq C \left( \|v_0\|^2_{H^3} + \|\rho_0\|^2_{H^2} + \sup_{0 \leq \tau \leq t} \|g_1\|^2_{H^2} \right) \\
+ \left| \int_0^t \left( A_1 + A_2 + A_3 + A_4 + \|f\|^2_{H^1} + \|g_1\|^2_{H^2} \right) d\tau \right|.
\] (54)

In addition, we rewrite (12) as
\[
\mu \partial_t^2 v_\theta = \bar{\rho} \partial_t v_\theta - \mu \partial_t \left( \frac{v_\theta}{r} \right) - \bar{\rho} g_2,
\] (55)
which implies that for \( k \geq 1 \)
\[
\|\partial_t^k v_\theta\|_{H^k} \leq C(\|\sqrt{\rho} \partial_t v_\theta - \mu \partial_t \left( \frac{v_\theta}{r} \right)\|_{H^k} + \|g_2\|_{H^k}).
\] (56)

Then by (56) and (54), we get
\[
\|v\|^2_{H^3} + \|\nabla^2 v_r\|^2_{H^1} + \|\sqrt{\gamma - 2}\phi\|^2_{H^2} + \|\sqrt{\gamma} \partial_t \phi\|^2_{H^2} \\
+ \int_0^t \left( \|\nabla^3 v_r\|^2_{H^1} + \|\partial_t^2 v_r\|^2_{H^1} + \|\sqrt{\gamma} \partial_t v_r\|^2_{H^1} + \|\sqrt{\gamma} \partial_t^2 v_r\|^2_{H^1} \right) d\tau \\
\leq C \left( \|v_0\|^2_{H^3} + \|\rho_0\|^2_{H^2} + \sup_{0 \leq \tau \leq t} \|g_1\|^2_{H^2} \right) \\
+ \left| \int_0^t \left( A_1 + A_2 + A_3 + A_4 + \|f\|^2_{H^1} + \|g_2\|^2_{H^2} \right) d\tau \right|. \] (57)

Define the energy
\[
N(t) = \|v\|^2_{H^3} + \|\partial_t v\|_{H^2} + \|\sqrt{\gamma - 2}\phi\|_{H^2} + \|\sqrt{\gamma} \partial_t \phi\|_{H^1} + \int_0^t \left( \|\nabla^3 v_r\|^2_{H^1} + \|\partial_t^2 v_r\|^2_{H^1} \right) d\tau.
\]

Next, we show that \( N(t) \) is uniformly bounded for any \( t \geq 0 \). In the proof procedure, we will employ the following Gagliardo-Nirenberg’s inequality repeatedly:

**Lemma 3.1.** Let \( 2 \leq p \leq +\infty \) and let \( j \) and \( k \) be integers satisfying
\[
0 \leq j < k, \quad k > j + \left( \frac{1}{2} - \frac{1}{p} \right).
\]
Then there exists a constant \( C > 0 \) such that
\[
\|\partial_t^j f\|_{L^p(1, +\infty)} \leq C \|f\|_{L^2(1, +\infty)}^{1 - \frac{n}{2}} \|\partial_t^k f\|_{L^2(1, +\infty)}^{\frac{n}{2}},
\]
where \( a = \frac{1}{k} \left( j + \frac{1}{2} - \frac{1}{p} \right) \).
We now state the main result in this section.

**Proposition 1.** Assume that \( N(t) \leq 1 \). Then there exists a constant \( G_b > 0 \) such that if

\[
\sum_{k=0}^{\infty} \left( \sup_{0 \leq \tau \leq \infty} |\partial_t^k g_0(\tau)| + \sup_{0 \leq \tau \leq \infty} |\partial_t^k g_r(\tau)| \right) + \sum_{k=0}^{\infty} \int_0^\infty ((\partial_t^k g_0)^2 + (\partial_t^k g_r)^2) d\tau \leq G_b,
\]

we then have

\[
N(t) \leq C(\|v_0\|_{H^3}^2 + \|\rho_0\|_{H^2}^2) + CG_b + CN(t)^{3/2}.
\]

**Proof.** At first, we deal with the terms \( A_i \) \((i = 1, 2, 3, 4)\) in (57). For this purpose, we require to treat the terms \( f \) and \( g \) \((g_1, g_2)\) in (10)-(12).

We now rewrite \( f \) and \( g_i \) \((i = 1, 2)\) as

\[
f = f_1 + f_2, \quad g_i = g_{i1} + g_{i2},
\]

where

\[
f_1 = -\frac{1}{r} \partial_r(r \phi v_r), \quad f_2 = -\frac{1}{r} \partial_r(r \rho u_r) + r \phi u_r,
\]

\[
g_{11} = \frac{v_0^2}{r} - v_r \partial_r v_r - \partial_r Q(\bar{\rho}, \phi) \quad \frac{(2 \mu + \lambda) \phi}{(\phi + \bar{\rho}) \bar{\rho}} \partial_r \left( \frac{1}{r} \partial_r (rv_r) \right),
\]

\[
g_{12} = 2v_0 + 2\bar{u} + u_0^b - u_0^b \partial_r (v_r + v_0) - v_r \partial_r u_r + \frac{2 \mu + \lambda}{\bar{\rho}} \partial_r \left( \frac{1}{r} \partial_r (ru_r^b) \right) \partial_r u_r,
\]

\[
g_{21} = -v_r \partial_r v_r - \frac{v_0 v_r}{r} - \frac{\mu \phi}{(\phi + \bar{\rho}) \bar{\rho}} \partial_r \left( \frac{1}{r} \partial_r (rv_0) \right),
\]

\[
g_{22} = -v_r \partial_r u_0^b - u_0^b \partial_r (v_0 + u_0^b) - \frac{v_0 u_r^b + u_0^b v_r + (u_0^b)^2}{r} + \frac{\mu}{\bar{\rho}} \partial_r \left( \frac{1}{r} \partial_r (ru_0^b) \right) \partial_r u_0^b.
\]

To estimate \( \int_0^t A_1 d\tau \), by the expression of \( A_1 \), we first estimate the term \( \int_0^t (g_1, \bar{\rho}rv_r) d\tau \). Note that

\[
\int_0^t (g_1, \bar{\rho}rv_r) d\tau = \int_0^t (g_{11}, \bar{\rho}rv_r) d\tau + \int_0^t (g_{12}, \bar{\rho}rv_r) d\tau.
\]

In addition, one has

\[
\int_0^t (g_{11}, \bar{\rho}rv_r) d\tau = \int_0^t \int_1^\infty \left( \frac{v_0^2}{r} - v_r \partial_r v_r - \partial_r Q(\bar{\rho}, \phi) \quad \frac{(2 \mu + \lambda) \phi}{(\phi + \bar{\rho}) \bar{\rho}} \partial_r \left( \frac{1}{r} \partial_r (rv_r) \right) \right) r \bar{\rho} v_r dr d\tau.
\]

(58)

We start to treat each integrand in (58). It follows from Lemma \(3.1\) with \( p = +\infty \), \( j = 0, k = 1 \) that

\[
\|\sqrt{r} f\|_{L^\infty} \leq C \|\sqrt{r} f\|_{L^2} \|\partial_r (\sqrt{r} f)\|_{L^2} \leq C \|\sqrt{r} f\|_{L^2} \|\sqrt{r} \partial_r f\|_{L^2} + \|\sqrt{r} f\|_{L^2}.
\]
By direct computation, we have
\[
\left| \int_0^t \int_1^\infty \frac{v^2}{r} r \rho v_r drd\tau \right| \\
\leq C \int_0^t \int_1^\infty |\sqrt{r}v| |\frac{v}{\sqrt{r}}| drd\tau \leq C \int_0^t \|\sqrt{r}v\|_2 \|\frac{v}{\sqrt{r}}\|_2 \|v\|_{L^\infty} d\tau \\
\leq CN(t)^{3/2},
\]
and
\[
\left| \int_0^t \int_1^\infty v_r \partial_r v_r r \rho v_r drd\tau \right| \\
\leq C \int_0^t \int_1^\infty |\sqrt{r} \partial_r v_r| |\sqrt{r} v_r| |v_r| drd\tau \leq C \int_0^t \|\sqrt{r} \partial_r v_r\|_2 \|\sqrt{r} v_r\|_2 \|v_r\|_{L^\infty} d\tau \\
\leq CN(t)^{3/2},
\]
and
\[
\left| \int_0^t \int_1^\infty \partial_t Q(\bar{\rho}, \phi)r \rho v_r drd\tau \right| \\
\leq C \int_0^t \int_1^\infty \left( |\phi \partial_t (\bar{\rho}^{-2} \phi) v_r| + |\phi^2 \partial_t \bar{\rho} v_r| \right) drd\tau \\
\leq C \int_0^t \|\sqrt{r} \phi\|_2 \|\sqrt{r} \partial_t (\bar{\rho}^{-2} \phi)\|_2 \|v_r\|_{L^\infty} d\tau + \int_0^t \|\phi\|_2 \|\frac{v_r}{\sqrt{r}}\|_2 \|\partial_t \bar{\rho}^{-3/2}\|_2 d\tau \\
\leq CN(t)^{3/2} \quad \text{(By } \partial_t \bar{\rho} = 0 \text{ and } \|\bar{\rho}^{-2} \phi\|_2 \leq C \|\phi\|_2 \|\partial_t (\bar{\rho}^{-2} \phi)\|_2). \quad \text{(61)}
\]

In addition,
\[
\left| \int_0^t \int_1^\infty \frac{2\mu + \lambda}{\phi + \bar{\rho}} \phi \partial_r (\frac{1}{r} \partial_r (rv_r)) r \rho v_r drd\tau \right| \\
\leq C \int_0^t \int_1^\infty \left( |\phi \frac{v_r}{\sqrt{r} \sqrt{r}} v_r| + |\phi \partial_r v_r| + |\sqrt{r} \phi \partial_r^2 v_r| \right) drd\tau \\
\leq CN(t)^{3/2}. \quad \text{(62)}
\]

With respect to the term \( \int_0^t (g_{12}, r \rho v_r) d\tau \), we have
\[
\int_0^t (g_{12}, r \rho v_r) d\tau = \int_0^t \int_1^\infty \left( \frac{2v_\theta + 2\bar{u} + u^b_\theta}{r} u^b_\theta - u^b_\phi \partial_r (v_r + u^b_r) \\
- v_r \partial_r u^b_r + \frac{2\mu + \lambda}{\bar{\rho}} \partial_r (\frac{1}{r} \partial_r (ru^b_r)) - \partial_t u^b_r \right) r \rho v_r drd\tau. \quad \text{(63)}
\]

Next, we analyze each integrand in \( \textbf{[63]} \). For the term \( \int_0^t \int_1^\infty \frac{2v_\theta + 2\bar{u} + u^b_\theta}{r} u^b_\theta r \rho v_r drd\tau \), we have that for any \( \eta \in (0, 1) \),
\[
\left| \int_0^t \int_1^\infty \frac{2v_\theta + 2\bar{u} + u^b_\theta}{r} u^b_\theta r \rho v_r drd\tau \right| \\
\leq C \sup_{0 \leq \tau \leq t} |g^b_\theta(\tau)| N(t) + \eta N(t) + C(\eta) \int_0^t (g^b_\theta(\tau))^2 d\tau, \quad \text{(64)}
\]
where \( C(\eta) > 0 \) is a constant depending only on \( \eta \).
Also, we can obtain for $\eta \in (0, 1)$,
\[
|\int_0^t \int_1^\infty \partial_t u^1, r \bar{\rho} v, dr d\tau| \leq \eta N(t) + C(\eta) \int_0^t (\partial_t g_0^0) d\tau. \tag{65}
\]
Similarly to (61), then
\[
|\int_0^t (f_1, \bar{\rho}^{\gamma - 2} r \phi) d\tau|
\]
\[
= |\int_0^t \int_1^\infty r \phi v, \partial_r (\bar{\rho}^{\gamma - 2} \phi) dr d\tau|
\]
\[
\leq C \int_0^t \int_1^\infty \left( |r \phi^2 v, \partial_r \bar{\rho}| + |r \phi v, \partial_r \phi| \right) dr d\tau
\]
\[
\leq CN(t)^{3/2}. \tag{66}
\]
The other left terms in $\int_0^t (g_{12}, r \bar{\rho} v_r) d\tau$ can be treated analogously. Combining all analysis above, we eventually arrive at
\[
|\int_0^t A_1 d\tau| \leq C \left( \sup_{0 \leq \tau \leq t} |g_0^0(\tau)| + \sup_{0 \leq \tau \leq t} |g_0^0(\tau)| N(t) + \eta N(t) \right)
\]
\[
+ CN(t)^{3/2} + C(\eta) \int_0^t ((g_0^0)^2 + (\partial_t g_0^0)^2 + (g_r^0)^2 + (\partial_t g_r^0)^2) dr. \tag{67}
\]
Simultaneously, $A_i$ $(i = 2, 3, 4)$ can be estimated similarly as $A_1$. For example, we show the estimate of the term $\partial_t g_1, \bar{\rho} r \partial_t v_r)$ in $A_4$. We see that
\[
\int_0^t (\partial_t g_1, \bar{\rho} r \partial_t v_r) d\tau = \int_0^t (\partial_t g_{11}, \bar{\rho} r \partial_t v_r) d\tau + \int_0^t (\partial_t g_{12}, \bar{\rho} r \partial_t v_r) d\tau.
\]
For the $\int_0^t (\partial_t g_{11}, \bar{\rho} r \partial_t v_r) d\tau$, one has
\[
\int_0^t (\partial_t g_{11}, \bar{\rho} r \partial_t v_r) d\tau
\]
\[
= \int_0^t \int_1^\infty \left( \frac{2 v_0}{r} \partial_t v_0 - \partial_t (v_r, \partial_r v_r) - \partial_t Q(\bar{\rho}, \phi)
\]
\[
- \partial_t \left( \frac{2 \mu + \lambda}{(\phi + \bar{\rho}) \bar{\rho}} \partial_r \left( \frac{1}{r} \partial_r (v_r) \right) \right) r \bar{\rho} \partial_t v_r, dr d\tau. \tag{68}
\]
We start to treat each integrand in (68). It follows from Lemma 3.1 and direct computation, we have
\[
|\int_0^t \int_1^\infty \frac{2 v_0}{r} \partial_t v_0 r \bar{\rho} \partial_t v_r, dr d\tau|
\]
\[
\leq C \int_0^t \int_1^\infty |v||\partial_t v| |\partial_t v_r| dr d\tau \leq C \int_0^t \|\partial_t v\|_2 \|\partial_t v\|_2 \|v\|_{L^\infty} d\tau
\]
\[
\leq CN(t)^{3/2}. \tag{69}
\]
By Lemma 3.1 with $p = +\infty$, $j = 1$, $k = 2$ and direct computation, we can obtain that
\[
|\int_0^t \int_1^\infty \partial_t (v, \partial_r v_r) r \bar{\rho} \partial_t v_r, dr d\tau|
\]
Next, we analyze each integrand in (73). 

\[
\leq C \int_0^t \int_1^\infty |\sqrt{r} \partial_r v| |\sqrt{r} v_r| |\partial_{tt} v| drd\tau + \int_0^t \int_1^\infty |\sqrt{r} \partial_r v| |\sqrt{r} \partial_{tt} v| |\partial_{tt} v| drd\tau \\
\leq C \int_0^t \|\sqrt{r} v\|_{L^\infty} \|\sqrt{r} \partial_{tt} v\|_{L^2} dr + \int_0^t \|\sqrt{r} \partial_r v\|_{L^\infty} \|\sqrt{r} \partial_{tt} v\|_{L^2} dr \\
\leq C N(t)^{3/2},
\]

and 

\[
\left| \int_0^t \int_1^\infty \partial_t Q(\tilde{\rho}, \phi) r \tilde{\rho} \partial_{tt} v \, drd\tau \right| \\
\leq C \int_0^t \int_1^\infty \left( |\phi \partial_r (\tilde{\rho}^{-2} \phi) r \partial_{tt} v_r| + |\partial_r \phi \partial_r (\tilde{\rho}^{-2} \phi) r \partial_{tt} v_r| + |\phi \partial_r \partial_r (\tilde{\rho} \tilde{\rho} \partial_{tt} v_r)\right) drd\tau \\
\leq C \int_0^t \|\sqrt{r} \phi\|_{L^\infty} \|\sqrt{r} \partial_r (\tilde{\rho}^{-2} \phi)\|_{L^\infty} \|\partial_{tt} v\|_{L^2} dr \\
+ \int_0^t \|\sqrt{r} \partial_r \phi\|_{L^\infty} \|\partial_r (\tilde{\rho}^{-2} \phi)\|_{L^\infty} \|\partial_{tt} v\|_{L^2} dr \\
+ \int_0^t \|\sqrt{r} \partial_r \phi\|_{L^\infty} \|\phi\|_{L^\infty} \|\sqrt{r} \partial_{tt} v\|_{L^2} dr \\
\leq C N(t)^{3/2} \quad \text{(By } \partial_r \tilde{\rho} = O(r^{-3}) \text{ and } \|\tilde{\rho}^{-2} \phi\|_2 \leq C \|\phi\|_2 \|\partial_r (\tilde{\rho}^{-2} \phi)\|_2\).
\]

In addition, 

\[
\left| \int_0^t \int_1^\infty \partial_t \left( \frac{2\mu + \lambda}{\tilde{\rho}} \partial_r \left( \frac{1}{r} \partial_r (rv_r) \right) \right) r \tilde{\rho} \partial_{tt} v \, drd\tau \right| \\
\leq \left| \int_0^t \int_1^\infty \partial_t \left( \frac{2\mu + \lambda}{\tilde{\rho}} \partial_r \left( \frac{1}{r} \partial_r (rv_r) \right) \right) r \tilde{\rho} \partial_{tt} v \, drd\tau \right| \\
+ \left| \int_0^t \int_1^\infty \left( \frac{2\mu + \lambda}{\tilde{\rho}} \partial_t \left( \frac{1}{r} \partial_r (rv_r) \right) \right) r \tilde{\rho} \partial_{tt} v \, drd\tau \right| \\
\leq C \int_0^t \left( \|\sqrt{r} \phi\|_{H^\infty} + \|\sqrt{r} \partial_r \phi\|_{H^\infty} \right) \|v_r\|_{L^2} \|\sqrt{r} \partial_{tt} v_r\|_{L^2} dr \\
+ C \int_0^t \|\sqrt{r} \partial_r \phi\|_{H^\infty} \|\partial_r v_r\|_{L^2} \|\sqrt{r} \partial_{tt} v_r\|_{L^2} dr \\
\leq C N(t)^{3/2}.
\]

With respect to the term \( \int_0^t (\partial_t g_{12}, r \tilde{\rho} \partial_{tt} v_r) d\tau \), we have 

\[
\int_0^t (\partial_t g_{12}, r \tilde{\rho} \partial_{tt} v_r) d\tau \\
= \int_0^t \int_1^\infty \left( \frac{2v_r + 2\tilde{u}_b + u_b}{r} \partial_t u_b^b \partial_r (v_r + u_b^b) - v_r \partial_t u_b^b \partial_r (v_r + u_b^b) \right. \\
\left. - v_r \partial_t u_b^b + \frac{2\mu + \lambda}{\tilde{\rho}} \partial_r \left( \frac{1}{r} \partial_r (rv_r) \right) - \partial_t u_b^b \right) r \tilde{\rho} \partial_{tt} v_r drd\tau \\
+ \int_0^t \int_1^\infty \left( \frac{2\partial_t v_r + \partial_t u_b^b}{r} u_b^b - u_b^b \partial_r (v_r + u_b^b) - \partial_t v_r \partial_t u_b^b \right) r \tilde{\rho} \partial_{tt} v_r drd\tau.
\]

Next, we analyze each integrand in (73).
For the term \( \int_0^t \int \frac{2v_\theta + 2u + u_\theta^b}{r} \partial_t u_\theta^b + \frac{2\partial_t v_\theta + \partial_t u_\theta^b}{r} u_\theta^b \partial_t u rv dr d\tau \), we have that for any \( \eta \in (0, 1) \),

\[
\left| \int_0^t \int \frac{2v_\theta + 2u + u_\theta^b}{r} \partial_t u_\theta^b + \frac{2\partial_t v_\theta + \partial_t u_\theta^b}{r} u_\theta^b \partial_t u rv dr d\tau \right| \leq C \sup_{0 \leq \tau \leq t} |\partial_t u_\theta^0(\tau)| N(t) + \eta N(t) + C(\eta) \int_0^t \left( (g_\theta^0(\tau))^2 + (\partial_t g_\theta^0(\tau))^2 \right) d\tau,
\]

where \( C(\eta) > 0 \) is a constant depending only on \( \eta \).

Also, we can obtain for \( \eta > 0 \),

\[
\left| \int_0^t \int \partial_t u_r^b \partial_t u rv dr d\tau \right| \leq \eta N(t) + C(\eta) \int_0^t (\partial_t u_\theta^0(\tau))^2 d\tau.
\]

It follows from (68)-(77) that

\[
\int_0^t (\partial_t g_1, \rho \partial_t u rv) d\tau \\
\leq C \sup_{0 \leq \tau \leq t} |\partial_t g_\theta^0(\tau)| N(t) + \eta N(t) + C(\eta) \int_0^t \left( (g_\theta^0(\tau))^2 + (\partial_t g_\theta^0(\tau))^2 \right) d\tau.
\]

Thus, we obtain that for any small \( \eta > 0 \)

\[
\sum_{i=1}^4 \left| \int_0^t A_i d\tau \right| \leq C \left( \sup_{0 \leq \tau \leq t} |g_\theta^0(\tau)| + \sup_{0 \leq \tau \leq t} |g_\theta^0(\tau)| \right) N(t) + \eta N(t) \\
+ C N(t)^{3/2} + C(\eta) \sum_{k=0}^2 \int_0^t (\partial_t^k g_\theta^0(\tau))^2 + (\partial_t^k g_\theta^0(\tau))^2 d\tau.
\]

Meanwhile,

\[
\sup_{0 \leq \tau \leq t} \|g\|_{H^2}^2 + \int_0^t (\|f\|_{H^2}^2 + \|g\|_{H^2}^2) d\tau \\
\leq C \sum_{k=0}^1 \left( \sup_{0 \leq \tau \leq t} |\partial_t^k g_\theta^0(\tau)| + \sup_{0 \leq \tau \leq t} |\partial_t^k g_\theta^0(\tau)| \right) N(t) + \eta N(t) \\
+ C N(t)^{3/2} + C(\eta) \sum_{k=0}^2 \int_0^t (\partial_t^k g_\theta^0(\tau))^2 + (\partial_t^k g_\theta^0(\tau))^2 d\tau.
\]

Consequently, by (57) and (76)-(77), we have that

\[
N(t) \leq C(\|v_0\|_{H^3}^2 + \|\rho_0\|_{H^3}^2) + C \sum_{k=0}^1 \left( \sup_{0 \leq \tau \leq t} |\partial_t^k g_\theta^0(\tau)| + \sup_{0 \leq \tau \leq t} |\partial_t^k g_\theta^0(\tau)| \right) N(t) \\
+ \eta N(t) + C N(t)^{3/2} + C(\eta) \sum_{k=0}^2 \int_0^t (\partial_t^k g_\theta^0(\tau))^2 + (\partial_t^k g_\theta^0(\tau))^2 d\tau,
\]

which completes the proof of Proposition 1.1.

Based on Proposition 1.1 we now start to prove Theorem 1.1.

Proof of Theorem 1.1.
Proof. By Proposition 1 when \( \|v_0\|^2_{L^2} + \|\rho_0\|^2_{L^2} + \sum_{k=0}^{2} \int_{0}^{t} ((\partial_t^k g_0)^2 + (\partial_t^k g_0)^2) d\tau \leq \varepsilon^2 \)
and \( \varepsilon > 0 \) is small, then \( N(t) \leq C\varepsilon^2 \) holds uniformly for any \( t \geq 0 \). This, together with the local existence of classical solution to (10)-(12) (one can be referred to [16]) and continuity argument, we can obtain the global solution of problem (10)-(12) with (13). Thus we complete the proof of Theorem 1.1.

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