SONIC-SUPERSONIC SOLUTIONS FOR THE TWO-DIMENSIONAL PSEUDO-STEADY FULL EULER EQUATIONS

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ABSTRACT. This paper is focused on the existence of classical sonic-supersonic solutions near sonic curves for the two-dimensional pseudo-steady full Euler equations in gas dynamics. By introducing a novel set of change variables and using the idea of characteristic decomposition, the Euler system is transformed into a new system which displays a transparent singularity-regularity structure. With a choice of weighted metric space, we establish the local existence of smooth solutions for the new system by the fixed-point method. Finally, we obtain a local classical solution for the pseudo-steady full Euler equations by converting the solution from the partial hodograph variables to the original variables.

1. Introduction. The two-dimensional full (non-isentropic) Euler equations for an ideal compressible fluid takes the form [14]:

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\
(\rho u)_x + (\rho u^2 + p)_x + (\rho u v)_y &= 0, \\
(\rho v)_y + (\rho u v)_x + (\rho v^2 + p)_y &= 0, \\
(\rho E)_t + (\rho Eu + pu)_x + (\rho Ev + pv)_y &= 0,
\end{align*}
\]

(1)

where \( \rho, (u, v), p \) and \( E \) are, respectively, the density, the velocity, the pressure and the specific total energy. For polytropic gases, \( E = \frac{u^2 + v^2}{2} + \frac{1}{\gamma-1} \frac{p}{\rho} \), where \( \gamma > 1 \) is the adiabatic gas constant.

In this paper, we are primarily interested in the so-called pseudo-steady case of system (1), that is, the solutions depend on the self-similar variables \( (\xi, \eta) = (x/t, y/t) \). Self-similar solutions occur when the initial data of (1) are given by constant states along each ray starting at the origin in the \((x, y)\) plane, for example,
the well-known two-dimensional Riemann problem [46]. In terms of variables $(\xi, \eta)$, system (1) changes to

\[
\begin{cases}
-\xi \rho_\xi - \eta \rho_\eta + (\rho u)_\xi + (\rho v)_\eta = 0, \\
-\xi (\rho u)_\xi - \eta (\rho u)_\eta + (\rho u^2 + p)_\xi + (\rho uv)_\eta = 0, \\
-\xi (\rho v)_\xi - \eta (\rho v)_\eta + (\rho v^2 + p)_\eta = 0, \\
-\xi (\rho E)_\xi - \eta (\rho E)_\eta + (\rho Eu + pu)_\xi + (\rho Ev + pv)_\eta = 0,
\end{cases}
\]

and the initial value problem of (1) changes to a boundary value problem of (2) at the infinity. System (2) has four eigenvalues

\[
\Lambda_0 = \Lambda_1 = \frac{V}{U}, \quad \Lambda_\pm = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2},
\]

where $(U, V) = (u - \xi, v - \eta)$ is the pseudo-velocity, $c = \sqrt{\gamma p/\rho}$ is the sound speed and $q = \sqrt{U^2 + V^2}$ is the pseudo-flow speed. The flow is supersonic $(q > c)$ at the infinity for bounded solutions and may change type to subsonic $(q < c)$ near the origin. Indeed, except for an extreme case which the solution is vacuum near the origin [29], the numerical simulations showed that the configurations of global solutions for the two-dimensional Riemann problem of (1) typically contain transonic structures [3, 25, 34]. In general, the supersonic and subsonic regions are separated by a boundary curve composed by sonic curves and transonic shocks which are free boundaries to be determined together with the solutions, see the monographs [26, 49] for details.

For the transonic problems of (2), iterative methods seem to be the most likely choices for constructing global solutions. As a first step, in the present paper we investigate the structure of solutions near a sonic curve for the pseudo-steady full Euler equations (2). This is an essential step for using the iterative process to establish a global transonic solution. Many efforts have been made to understand the structure of solutions near sonic curves for the Euler equations and its simplified models. For the steady isentropic Euler equations, some explicit examples of transonic solutions were presented in [14, 23], the existence conditions of continuous sonic-supersonic flows were given in [1], the existence of weak solutions in the compensated-compactness framework was investigated in [4, 6, 7, 32], the existence of subsonic-sonic solutions was provided in [39, 43, 44], the existence and uniqueness of smooth transonic flows in Laval nozzles was established in [40, 41]. We also refer the reader to [5, 8, 9, 11, 16, 45] for the study of transonic shocks arising in supersonic flow past a blunt body or a bounded nozzle. There are more related references on classical methods for solutions [17, 18], on perturbation arguments and linear theory [10, 33, 36] and on asymptotic models [2, 13], etc. In the past few years, the semi-hyperbolic patch problems, a kind of degenerate hyperbolic problems with sonic curves as boundaries, have been explored by applying the idea of characteristic decomposition, see [20, 21, 30, 37] for the Euler system, [38, 42] for the pressure-gradient system and [22] for the nonlinear wave system. The characteristic decomposition is a powerful tool for studying some degenerate hyperbolic problems which was revealed by Dai and Zhang [15], see [12, 24, 27, 28, 35] for more applications.

It seems successfully to choose a pair of appropriate coordinates to overcome the difficulty caused by the sonic degeneracy of the Euler system. Many of previous results on the steady isentropic irrotational Euler equations used the hodograph
method which allows to linearize the equations, see, e.g., [13, 14, 17, 31]. However, it is well-known that the hodograph method is difficult in treating boundary conditions and in transforming back to the original independent variables. In [23], Kuz’mín took the stream function Ψ and the potential function Φ as the coordinate system to investigate a perturbation problem for a given transonic solution. Recently, Zhang and Zheng [47, 48] employed the coordinate system \((\sqrt{q^2 - c^2}, Φ)\) to construct local classical sonic-supersonic solutions for the steady and pseudo-steady isentropic irrotational Euler equations. Hu and Li [19] established the existence of sonic-supersonic solutions for the steady full Euler equations by using the angle functions as the coordinate system. The angle functions were first introduced as the dependent variables by Li and Zheng [28].

The purpose of this paper is to construct a classical sonic-supersonic solution for (2). In order to deal with the pseudo-steady full Euler equations, we introduce a novel set of dependent and independent change variables to transform (2) into a new hyperbolic system with a clear singularity-regularity structure. We describe the problem as follows.

**Problem 1.1.** Given a piece of smooth curve \(Γ : η = ϕ(ξ), ξ ∈ [ξ₁, ξ₂]\), we assign the boundary data for \((ρ, u, v, p)\) on \(Γ, (ρ, u, v, p)(ξ, ϕ(ξ)) = (ρ, ˆu, ˆv, ˆp)(ξ)\) such that \(\dot{ρ}(ξ) > 0, \dot{ϕ}(ξ) > 0\) and \(γ\dot{p}(ξ)/\dot{ρ}(ξ) = (\ddot{u}(ξ) - ξ^2 + (v(ξ) - ϕ(ξ))^2\) for any \(ξ ∈ [ξ₁, ξ₂]\). This means that \(Γ\) is a sonic curve. We look for a classical solution for (2) in the region \(q > c\) near \(Γ\).

The main result in this paper is stated in the following theorem.

**Theorem 1.1.** Let \(\dot{θ}\) be the pseudo-flow angle on \(Γ\) defined by \(\dot{θ} = \arctan((\dot{v} - ϕ)/\dot{u})\). Assume that the curve \(Γ\) and the boundary data \((ρ, ˆu, ˆv, ˆp)\) satisfy

\[
\begin{align*}
(A₁) & : ϕ(ξ) ∈ C^4([ξ₁, ξ₂]), \quad (ρ, ˆu, ˆv, ˆp)(ξ) ∈ C^4([ξ₁, ξ₂]), \\
(A₂) & : ϕ'\cos θ - ϕ\sin θ > 0, \quad \cos θ + ϕ'\sin θ > 0, \\
(A₃) & : ϕ' > 0, \quad \dot{p} ≤ \frac{\sqrt{2(1 + ϕ'^2) + (γ - 1)(ϕ'\cos θ - \sin θ)^2}}{γ + 1),} \quad ∀ ξ ∈ [ξ₁, ξ₂].
\end{align*}
\]

Then there exists a classical solution for Problem 1.1 in the region \(q > c\) near \(Γ\).

**Remark 1.** The regularity conditions \((A₁)\) can be relaxed somewhat by using the concept of modulus of continuity as was done in [48]. The conditions \((A₂)\) mean that the pseudo-flow direction \((\cos θ, \sin θ)\) is neither tangent nor normal to the curve \(Γ\). The conditions \((A₂)\) and \((A₃)\) are mainly used to ensure that on the curve \(Γ\) the pseudo-Mach number \(M_a := q/c\) is strictly increasing along the pseudo-flow direction, which seems reasonable since the classical solution is expected to be constructed from the curve \(M_a = 1\) to the region \(M_a > 1\). The condition for \(\dot{p}\) in \((A₃)\) can be properly relaxed, see Remark 2 in Section 2.

To prove Theorem 1.1, the key is to characterize the degeneracy of the system near the sonic curve by suitable independent and dependent changing variables. For the pseudo-steady isentropic irrotational Euler equations, Zhang and Zheng [48] chose \((\sqrt{q^2 - c^2}, Φ)\) as the independent quantities and \((\ddot{θ}^+ c, \ddot{θ}^- c)\) as the dependent quantities to obtain a tidy first-order hyperbolic system. However, for the full Euler system, it is impossible to use \((\sqrt{q^2 - c^2}, Φ)\) as the independent quantities due to the non-existence of potential function \(Φ\). Moreover, the characteristic decomposition of \(c\) is considerably formidable for the full Euler equations such that \((\ddot{θ}^+ c, \ddot{θ}^- c)\) are not suitable to be the dependent quantities any more. Inspired by
Hu and Li [19], we introduce the pseudo-Mach angle \( \omega \) and the pseudo-flow angle \( \theta \) and then choose \((\cos \omega, \theta)\) as the independent coordinate system. To get suitable dependent variables, we further introduce a novel variable \( \Xi \), which is a function of pseudo-Mach angle \( \omega \), entropy \( S \) and pseudo-Bernoulli number \( B \). We derive the characteristic decomposition of \( \Xi \) under a pair of new weighted directional derivatives \((\tilde{\partial}^+, \tilde{\partial}^-)\). Taking \((\tilde{\partial}^+\Xi, \tilde{\partial}^-\Xi, \tilde{\partial}^+B, \tilde{\partial}^+S)\) as the new dependent quantities, we reduce the pseudo-steady full Euler equations (2) to a new closed system (39) in the partial hodograph \((\cos \omega, \theta)\)-plane. This new system has the desired explicitly singularity-regularity structure. With a choice of weighted metric space, we establish the local existence of smooth solutions for the new system by the fixed-point method. Thanks to the choice of the independent variables, we can transform back the solution to the original self-similar coordinate system and thus construct a local classical sonic-supersonic solution to (2).

The rest of the paper is organized as follows. In Section 2, we introduce a new set of variables to reformulate the problem by the idea of characteristic decomposition. In Section 3, we transform the problem into a new problem under a partial hodograph plane and then solve the new problem by employing the fixed-point method in a weighted metric space. In Section 4, we complete the proof of the main theorem by converting the classical solution in the partial hodograph plane into that in terms of original self-similar plane. Appendix is provided for the algorithmic arguments of characteristic decompositions.

2. The reformulation of the problem. In order to analyze the nonlinear degenerate problem under consideration in this paper, we adopt the pseudo-Mach angle, the pseudo-flow angle, the entropy and the pseudo-Bernoulli quantity as dependent variables. The angle functions as the dependent variables was invented by Li and Zheng [28] and then was applied in many problems, see, e.g. [12, 20, 24, 29, 30, 35, 37]. In this section, we provide the characteristic decompositions in angle variables and reformulate the problem in this new framework.

2.1. Characteristic decompositions in angle variables. For smooth solutions, system (2) can be rewritten as

\[
AW_\xi + BW_\eta = 0, \tag{5}
\]

where the primitive variables and the coefficient matrices are

\[
W = \begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix}, \quad A = \begin{pmatrix} U & \rho & 0 & 0 \\ 0 & U & 0 & \frac{1}{\rho} \\ 0 & 0 & U & 0 \\ 0 & \gamma p & 0 & U \end{pmatrix}, \quad B = \begin{pmatrix} V & 0 & \rho & 0 \\ 0 & V & 0 & 0 \\ 0 & 0 & V & \frac{1}{\rho} \\ 0 & 0 & \gamma p & V \end{pmatrix}.
\]

The pseudo-velocity \((U, V) = (u - \xi, v - \eta)\) is defined as before. The four eigenvalues of (5) are expressed in (3) and the corresponding left eigenvectors are

\[
\ell_0 = (0, U, V, 0), \quad \ell_1 = (c^2, 0, 0, -1), \quad \ell_\pm = (0, -\Lambda \pm \gamma p, \gamma p, \Lambda \pm U - V).
\]

By a standard calculation, system (5) changes to

\[
\begin{cases}
US_\xi + VS_\eta = 0, \\
UB_\xi + VB_\eta = -q^2, \\
-cpVu_\xi + cpVu_\eta + \sqrt{q^2 - c^2p_\xi} + \Lambda_+(-cpVu_\eta + cpVu_\eta + \sqrt{q^2 - c^2p_\eta}) = 0, \\
-cpVu_\xi + cpVu_\eta - \sqrt{q^2 - c^2p_\xi} + \Lambda_-(cpVu_\eta - cpVu_\eta - \sqrt{q^2 - c^2p_\eta}) = 0,
\end{cases}
\tag{6}
\]
where $S = p \rho^{-\gamma}$ is the entropy function and $B = \frac{q^2}{2} + \frac{c^2}{\gamma - 1}$ is the pseudo-Bernoulli function.

We introduce the pseudo-flow angle $\theta$ and the pseudo-Mach angle $\omega$ as follows
\[
\tan \theta = \frac{V}{U}, \quad \sin \omega = \frac{c}{q}.
\] (7)

Denote the angle variables
\[
\alpha := \theta + \omega, \quad \beta := \theta - \omega. \tag{8}
\]

Then we combine (3), (7) and (8) to obtain
\[
\tan \alpha = \Lambda_+, \quad \tan \beta = \Lambda_-, \quad \tan \theta = \Lambda_0 = \Lambda_1,
\] (9)

which mean that $\alpha, \beta, \theta$ are, respectively, the inclination angles of positive, negative and zero characteristics. According to (7) and the expression of $B$, the functions $(c, u, v)$ can be expressed in terms of $B, \theta, \omega$
\[
c = \sqrt{\frac{2\kappa B \sin^2 \omega}{\kappa + \sin^2 \omega}}, \quad u = \xi + c \frac{\cos \theta}{\sin \omega}, \quad v = \eta + c \frac{\sin \theta}{\sin \omega}.
\] (10)

where $\kappa = (\gamma - 1)/2$. Moreover, we introduce the normalized directional derivatives
\[
\partial^+ = \cos \alpha \partial_\zeta + \sin \alpha \partial_\eta, \quad \partial^- = \cos \beta \partial_\zeta + \sin \beta \partial_\eta, \quad \partial^0 = \cos \theta \partial_\zeta + \sin \theta \partial_\eta,
\] (11)

from which we have
\[
\begin{cases}
\partial_\zeta = -\frac{\sin \beta \partial^+ - \sin \alpha \partial^-}{\sin(2\omega)}, \\
\partial_\eta = \frac{\cos \beta \partial^+ - \cos \alpha \partial^-}{\sin(2\omega)}, \\
\partial^0 = \frac{\partial^+ + \partial^-}{2 \cos \omega}, \\
\partial^\perp = \frac{\partial^- - \partial^+}{2 \sin \omega}.
\end{cases}
\] (12)

In terms of the variables $(S, B, \omega, \theta)$, system (6) can be transformed into a new form
\[
\begin{cases}
\tilde{\partial}^0 S = 0, \\
\tilde{\partial}^0 B = -\sqrt{\frac{2\kappa B \sin^2 \omega}{\kappa + \sin^2 \omega}}, \\
\tilde{\partial}^+ \omega = -\frac{\sin \omega \sqrt{\kappa + \sin^2 \omega}}{\sqrt{2\kappa B}} + \frac{\sin(2\omega)}{4\kappa} \left( \frac{1}{\gamma} \tilde{\partial}^+ \ln S - \tilde{\partial}^+ \ln B \right), \\
\tilde{\partial}^- \omega = \frac{\sin \omega \sqrt{\kappa + \sin^2 \omega}}{\sqrt{2\kappa B}} - \frac{\sin(2\omega)}{4\kappa} \left( \frac{1}{\gamma} \tilde{\partial}^- \ln S - \tilde{\partial}^- \ln B \right).
\end{cases}
\] (13)

The detailed derivation of (13) is given in Appendix A.1. In order to get rid of $B$ in the non-homogeneous terms in (13), we further introduce the following weighted directional derivatives
\[
\tilde{\partial}^i = \hat{B} \tilde{\partial}^i, \quad i = \pm, 0, \perp,
\] (14)

where $\hat{B} = \sqrt{2\kappa B}$. Then system (13) can be rewritten as
\[
\begin{cases}
\tilde{\partial}^0 S = 0, \\
\tilde{\partial}^0 \ln \hat{B} = -\frac{\kappa}{\sqrt{\kappa + \sin^2 \omega}},
\tilde{\partial}^+ \omega = -\frac{\sin \omega \sqrt{\kappa + \sin^2 \omega}}{\sqrt{2\kappa B}} + \frac{\sin(2\omega)}{2\kappa} \hat{B} \tilde{\partial}^+ \ln S - \frac{1}{2\gamma} \ln S - \frac{1}{2\gamma} \ln \hat{B}), \\
\tilde{\partial}^- \omega = \frac{\sin \omega \sqrt{\kappa + \sin^2 \omega}}{\sqrt{2\kappa B}} - \frac{\sin(2\omega)}{2\kappa} \hat{B} \tilde{\partial}^- \ln S - \frac{1}{2\gamma} \ln S - \frac{1}{2\gamma} \ln \hat{B}).
\end{cases}
\] (15)

To derive the characteristic decompositions about the angle variables, we now introduce a new variable
\[ \Xi = \frac{1}{4\kappa} \ln \left( \frac{\sin^2 \omega}{\kappa + \sin^2 \omega} \right) - \frac{1}{2\kappa} \left( \frac{1}{2\gamma} \ln S - \ln \hat{B} \right). \]  

(16)

Then the last two equations of (15) change to

\[ \begin{cases} 
\hat{\theta}^+ + \sin(2\omega)\hat{\theta}^+ = -\sin \omega \sqrt{\kappa + \sin^2 \omega}, \\
\hat{\theta}^- + \sin(2\omega)\hat{\theta}^- = \sin \omega \sqrt{\kappa + \sin^2 \omega}.
\end{cases} \]  

(17)

For obtaining a closed hyperbolic system, we take the weighted directional derivatives of \((\Xi, S, B)\) as the dependent variables. Denote

\[ \begin{align*}
X &= \hat{\theta}^+ \Xi, & Y &= \hat{\theta}^- \Xi, & G &= \hat{\theta}^+ \left( \frac{1}{2\gamma} \ln S \right), & H &= \hat{\theta}^+ \left( \frac{1}{4\kappa} \ln S - \frac{1}{2\kappa} \ln \hat{B} \right).
\end{align*} \]  

(18)

Applying (15) and (16) yields

\[ \begin{align*}
\hat{\theta}^- \left( \frac{1}{2\gamma} \ln S \right) &= -G, & \hat{\theta}^- \left( \frac{1}{4\kappa} \ln S - \frac{1}{2\kappa} \ln \hat{B} \right) &= -H + \frac{\cos \omega}{\sqrt{\kappa + \sin^2 \omega}},
\end{align*} \]  

(19)

and then

\[ \begin{align*}
\hat{\theta}^+ \omega &= \frac{2 \sin \omega (\kappa + \sin^2 \omega)}{\cos \omega} (X + H), \\
\hat{\theta}^- \omega &= \frac{2 \sin \omega (\kappa + \sin^2 \omega)}{\cos \omega} (Y - H + \frac{\cos \omega}{\sqrt{\kappa + \sin^2 \omega}}).
\end{align*} \]  

(20)

In addition, by direct calculations, we have the following commutator relations

\[ \begin{align*}
\hat{\theta}^- \hat{\theta}^+ - \hat{\theta}^+ \hat{\theta}^- &= \cos(2\omega) \frac{\hat{\theta}^+ \beta - \hat{\theta}^- \alpha}{\sin(2\omega)} - \frac{\hat{\theta}^+ \beta - \cos(2\omega) \hat{\theta}^- \alpha}{\sin(2\omega)} \hat{\theta}^+ \\
&+ \hat{\theta}^- \ln \hat{B} \hat{\theta}^+ - \hat{\theta}^+ \ln \hat{B} \hat{\theta}^-
\end{align*} \]

\[ \begin{align*}
\hat{\theta}^0 \hat{\theta}^+ - \hat{\theta}^+ \hat{\theta}^0 &= \cos \omega \frac{\hat{\theta}^+ \theta - \hat{\theta}^0 \alpha}{\sin \omega} - \frac{\hat{\theta}^+ \theta - \cos \omega \hat{\theta}^0 \alpha}{\sin \omega} \hat{\theta}^+
\end{align*} \]

\[ + \hat{\theta}^0 \ln \hat{B} \hat{\theta}^+ - \hat{\theta}^+ \ln \hat{B} \hat{\theta}^0. \]  

(21)

Thanks to (15), (17)-(20) and (21), we get a hyperbolic system in terms of the variables \((X, Y, G, H)\)

\[ \begin{align*}
\hat{\theta}^0 G &= \left\{ \frac{(\kappa + 1)(X + Y)}{\cos \omega} + \frac{\kappa + 2 \sin^2 \omega}{\sqrt{\kappa + \sin^2 \omega}} \right\} G, \\
\hat{\theta}^0 H &= \left\{ \frac{(\kappa + 1)(X + Y)}{\cos \omega} + \frac{2 \kappa + 1}{\sqrt{\kappa + \sin^2 \omega}} \right\} H \\
&- \frac{(\kappa + 1)(X + Y)}{2 \cos \omega \sqrt{\kappa + \sin^2 \omega}} - \frac{(1 + \cos^2 \omega)}{2 \cos \omega} - \frac{G}{2 \sqrt{\kappa + \sin^2 \omega}}, \\
\hat{\theta}^- X &= \frac{\kappa + \sin^2 \omega}{\cos \omega} (X + H)(X - \cos(2\omega)Y) + \frac{\cos(2\omega) \sqrt{\kappa + \sin^2 \omega}}{2 \cos \omega} (X + Y) \\
&+ X \left\{ X + \cos(2\omega)Y - \frac{(\kappa - \sin^2 \omega) \cos \omega}{\sqrt{\kappa + \sin^2 \omega}} + 2 \kappa H - G \right\}, \\
\hat{\theta}^+ Y &= \frac{\kappa + \sin^2 \omega}{\cos \omega} (Y - H + \frac{\cos \omega}{\sqrt{\kappa + \sin^2 \omega}}) (Y - \cos(2\omega)X) + \frac{\cos(2\omega) \sqrt{\kappa + \sin^2 \omega}}{2 \cos \omega} (X + Y) \\
&+ Y \left\{ Y + \cos(2\omega)X + \cos \omega \sqrt{\kappa + \sin^2 \omega} - 2 \kappa H + G \right\}.
\end{align*} \]  

(22)
The detailed derivation of (22) is presented in Appendix A.2.

2.2. The boundary conditions. We now transform the boundary data for system (22) from the boundary condition (4). Denote

\[
\dot{S}(\xi) = \dot{\rho}(\xi)\rho^{-\gamma}(\xi) > 0,
\]
\[
\hat{B}(\xi) = \frac{\left(\hat{\rho}(\xi) - \xi\right)^2 + (\hat{\varphi}(\xi) - \varphi(\xi))^2}{2} + \frac{\gamma\dot{\rho}(\xi)}{(\gamma - 1)\rho(\xi)} > 0
\]  

(23)

for all \(\xi \in [\xi_1, \xi_2]\). Noting \(S(\xi, \varphi(\xi)) = \dot{S}(\xi)\) on \(\Gamma\) and using the equation \(\hat{\partial}^0 S = 0\) gives

\[
S_2(\xi, \varphi(\xi)) = \frac{\sin \hat{\theta} \hat{S}'}{\sin \theta - \cos \theta \varphi'}, \quad S_3(\xi, \varphi(\xi)) = \frac{\cos \hat{\theta} \hat{S}'}{\cos \theta \varphi' - \sin \theta},
\]

from which one has

\[
\hat{\partial}^0 S|_{\Gamma} = \frac{\hat{S}'}{\cos \theta \varphi' - \sin \theta}. \tag{24}
\]

By the definition of \(G\) and (24), we see that the boundary data of \(G\) on \(\Gamma\) is

\[
G|_{\Gamma} = \hat{G}(\xi) := \frac{\sqrt{2\kappa \hat{B}(\xi)}}{2\gamma S(\xi)} \left(\hat{\partial}^0 S|_{\Gamma}\right) = \frac{\hat{S}' \sqrt{2\kappa \hat{B}(\xi)}}{2\gamma \hat{S}(\xi)(\cos \theta \varphi' - \sin \theta)}. \tag{25}
\]

From the definition of \(H\), it follows that

\[
H|_{\Gamma} = \hat{H}(\xi) := \left(\frac{2\kappa}{\gamma(\kappa + 1)}\right)^{-\gamma} \frac{\sqrt{\kappa + 1} \hat{\rho}^\gamma}{4\kappa \sqrt{\gamma \hat{p}}} \left(\hat{\partial}^0 \left(\frac{S}{\hat{B}^\gamma}\right)\right)|_{\Gamma}. \tag{26}
\]

Making use of the second equation of (13) and noting the fact \(\omega = \pi/2\) on \(\Gamma\) arrives at

\[
\hat{\partial}^0 B|_{\Gamma} = -\sqrt{\frac{2\kappa \hat{B}(\xi)}{\kappa + 1}},
\]

which together with the function \(B(\xi, \varphi(\xi)) = \hat{B}(\xi)\) on \(\Gamma\), one deduces

\[
B_2(\xi, \varphi(\xi)) = \frac{\sin \hat{\theta} \hat{B}'}{\sin \theta - \cos \theta \varphi'}, \quad B_3(\xi, \varphi(\xi)) = \frac{\cos \hat{\theta} \hat{B}'}{\cos \theta \varphi' - \sin \theta},
\]

from which one obtains

\[
\hat{\partial}^0 B|_{\Gamma} = \frac{\hat{B}' + \sqrt{\frac{2\kappa \hat{B}}{\kappa + 1}}(\cos \hat{\theta} + \varphi' \sin \hat{\theta})}{\cos \theta \varphi' - \sin \hat{\theta}}. \tag{27}
\]

Combining (24) and (27) and doing a simple rearrangement leads to

\[
\hat{\partial}^0 \left(\frac{S}{\hat{B}^\gamma}\right)|_{\Gamma} = \left(\frac{2\kappa}{\gamma(\kappa + 1)}\right)^{\gamma + 1} \frac{\gamma^2 \sqrt{\hat{p}}}{\sqrt{\gamma \hat{p} \hat{\rho}^{\gamma - 1}(\sin \theta - \cos \theta \varphi')}} \left\{\left(\kappa + 1\right)\hat{\rho}' \frac{\sqrt{\gamma \hat{p} \hat{\rho}^{\gamma - 1}(\sin \theta - \cos \theta \varphi')}}{\sqrt{\gamma \hat{p} \hat{\rho}^{\gamma - 1}(\sin \theta - \cos \theta \varphi')}} + (\cos \hat{\theta} + \varphi' \sin \hat{\theta})\right\}.
\]

Inserting the above into (26) gets

\[
H|_{\Gamma} = \hat{H}(\xi) = \frac{1}{2\sqrt{\kappa + 1}(\sin \theta - \cos \theta \varphi')} \left\{\left(\kappa + 1\right)\hat{\rho}' + (\cos \hat{\theta} + \varphi' \sin \hat{\theta})\right\}. \tag{28}
\]
Now we consider the boundary data \((X, Y)\) on \(\Gamma\). In view of (12) and (14), we obtain \(X + Y = 2\cos \omega \partial^0_\gamma \Xi\) which means that \(X = -Y\) on \(\Gamma\). Adding the two equations in (17) gives

\[
X - Y = -\frac{\partial^+ \theta + \partial^- \theta}{\sin(2\omega)} = -\frac{\partial^0 \theta}{\sin \omega},
\]

which indicates that \(X = -Y = -\partial^0 \theta / 2\) on \(\Gamma\). On the other hand, we subtract the two equations in (17) to find that

\[
\partial^\perp \theta = (X + Y) \cos \omega + \sqrt{\kappa + \sin^2 \omega},
\]

which implies that \(\partial^\perp \theta|_{\Gamma} = \sqrt{\kappa + 1}\) or \(\partial^\perp \theta|_{\Gamma} = \sqrt{(\kappa + 1)/2kB(\xi)}\). Applying the function \(\theta(\xi, \varphi(\xi)) = \hat{\theta}(\xi)\), one has

\[
\theta_\xi(\xi, \varphi(\xi)) = \frac{\hat{\theta}' \cos \hat{\theta} + \varphi' \sqrt{\frac{\kappa + 1}{2kB}}}{\cos \hat{\theta} + \sin \hat{\theta} \varphi'}, \quad \theta_\eta(\xi, \varphi(\xi)) = \frac{\hat{\theta}' \sin \hat{\theta} - \sqrt{\frac{\kappa + 1}{2kB}}}{\cos \hat{\theta} + \sin \hat{\theta} \varphi'},
\]

from which we have

\[
\partial^0 \theta|_{\Gamma} = \frac{\hat{\theta}' + \sqrt{\frac{\kappa + 1}{2kB}} (\cos \hat{\theta} \varphi' - \sin \hat{\theta})}{\cos \hat{\theta} + \sin \hat{\theta} \varphi'}.
\]

Thus

\[
Y|_{\Gamma} = -X|_{\Gamma} = \frac{B\partial^0 \theta}{2}|_{\Gamma} = \frac{\sqrt{\kappa + 1}}{2} \sqrt{\frac{2kB}{\rho}} \hat{\theta}' + (\cos \hat{\theta} \varphi' - \sin \hat{\theta}) \bigg|_{\Gamma} =: \hat{a}_0(\xi).
\]

We next calculate the value \(\partial^0(q/c)\) on \(\Gamma\). Adding the last two equations of (13) gets

\[
\partial^0 \theta - \frac{\sin \omega}{\kappa + \sin^2 \omega} \partial^\perp (\sin \omega) = -\frac{\sin^2 \omega}{2\kappa} \partial^\perp \left(\frac{1}{\gamma} \ln S - \ln B\right),
\]

from which and the facts \(\partial^+ S + \partial^- S = 0, \partial^+ B + \partial^- B = 0\) on \(\Gamma\), we have by (26) and (30)

\[
\partial^\perp (\sin \omega)|_{\Gamma} = (\kappa + 1) \left\{ \partial^0 \theta|_{\Gamma} - \frac{B^\gamma}{2\kappa^2 S} \partial^\perp \left(\frac{S}{B^\gamma}\right)|_{\Gamma} \right\} = \frac{2(\kappa + 1)}{\sqrt{2kB(\xi)}} [\hat{a}_0(\xi) - \hat{H}(\xi)],
\]

which along with the function \(\sin \omega(\xi, \varphi(\xi)) = 1\) gives

\[
(\partial_\xi \sin \omega)|_{\Gamma} = \frac{2(\kappa + 1) \varphi'}{\sqrt{2kB}} \frac{\hat{a}_0(\xi) - \hat{H}(\xi)}{\cos \hat{\theta} + \sin \hat{\theta} \varphi'},
\]

\[
(\partial_\eta \sin \omega)|_{\Gamma} = -\frac{2(\kappa + 1)}{\sqrt{2kB}} \frac{\hat{a}_0(\xi) - \hat{H}(\xi)}{\cos \hat{\theta} + \sin \hat{\theta} \varphi'},
\]

from which one has

\[
(\partial^0 \sin \omega)|_{\Gamma} = \frac{2(\kappa + 1)}{\sqrt{2kB}} \frac{\cos \hat{\theta} \varphi' - \sin \hat{\theta}}{\cos \hat{\theta} + \sin \hat{\theta} \varphi'} [\hat{a}_0(\xi) - \hat{H}(\xi)].
\]

Hence

\[
\partial^0\left(\frac{q}{c}\right)|_{\Gamma} = \partial^0\left(\frac{1}{\sin \omega}\right)|_{\Gamma} = -\partial^0 \sin \omega)|_{\Gamma} = \frac{2(\kappa + 1)}{\sqrt{2kB}} \frac{\cos \hat{\theta} \varphi' - \sin \hat{\theta}}{\cos \hat{\theta} + \sin \hat{\theta} \varphi'} [\hat{H}(\xi) - \hat{a}_0(\xi)],
\]
which means by the assumptions (A2) in (4) that \( \partial_t^0(q/c)|_r \) and \( \tilde{H}(\xi) - \tilde{a}_0(\xi) \) have the same symbol for all \( \xi \in [\xi_1, \xi_2] \). We employ (28) and (30) to compute
\[
\dot{H}(\xi) - \dot{a}_0(\xi) = -\frac{b(\xi)}{2\sqrt{\kappa + 1}(\cos \theta \varphi' - \sin \theta)},
\]
where
\[
b(\xi) = \frac{(\gamma + 1)\sqrt{\rho}(\cos \hat{\theta} \varphi' - \sin \hat{\theta})^{\gamma'}}{2\sqrt{\rho}(\cos \theta + \sin \theta \varphi')} + \frac{\gamma + 1}{2\sqrt{\gamma \rho \rho'}^{\gamma'}} + \frac{2(1 + \varphi'^2) + (\gamma - 1)(\cos \hat{\theta} \varphi' - \sin \hat{\theta})^2}{2(\cos \hat{\theta} + \sin \theta \varphi')}.
\]
Recalling the assumption (A3) in (4), we see that \( b(\xi) < 0 \) and then \( \dot{H}(\xi) - \dot{a}_0(\xi) > 0 \) for all \( \xi \in [\xi_1, \xi_2] \).

We combine (25), (28) and (30) to obtain the boundary data \( (G, H, X, Y) \) on \( \Gamma \) with
\[
(G, H, X, Y)|_{\Gamma} = (\Gi, H, -\tilde{a}_0, \tilde{a}_0)(\xi) \in C^3([\xi_1, \xi_2]),
\]
(32)
The constraints (4) on the boundary data become
\[
\cos \hat{\theta} + \sin \hat{\theta} \varphi' > 0, \quad \cos \hat{\theta} \varphi' - \sin \hat{\theta} > 0, \quad \hat{\theta}'' < 0, \quad \dot{H} - \dot{a}_0 > 0, \quad \forall \xi \in [\xi_1, \xi_2].
\]
Then Theorem 1.1 is restated in the next theorem.

**Theorem 2.1.** Let conditions (33) be satisfied. Then the boundary problem (22) (32) admits a classical solution in the region \( q > c \) near the sonic curve \( \Gamma \).

**Remark 2.** Since we only need to use the inequality \( \dot{H}(\xi) - \dot{a}_0(\xi) > 0 \), the condition for \( \hat{\rho} \) in (4) can be relaxed to \( b(\xi) < 0 \) for all \( \xi \in [\xi_1, \xi_2] \).

3. **The problem in a partial hodograph plane.** Since the degeneracy on the sonic curve may cause singularities, we need to single out the feature of governing equations near the sonic curve. For this purpose, we introduce a partial hodograph transformation to solve the problem in the new coordinate system.

3.1. **Reformulated problem in a partial hodograph plane.** In this subsection, we reformulate the problem into a new problem in the partial hodograph plane.

3.1.1. **A partial hodograph transformation.** We introduce a partial hodograph transformation \( (\xi, \eta) \to (t, r) \) by defining
\[
t = \cos \omega(\xi, \eta), \quad r = \theta(\xi, \eta).
\]
(34)
Making use of (17) and (20), the Jacobian of this transformation is
\[
J := \frac{\partial(t, r)}{\partial(\xi, \eta)} = \frac{\hat{\rho}^+ \omega \hat{\rho}^- \theta - \hat{\rho}^+ \theta \hat{\rho}^- \omega}{2B^2 \cos \omega} = \frac{2\sin^2 \omega(\kappa + \sin^2 \omega)}{B^2 \cos \omega} \tilde{J},
\]
where
\[
\tilde{J} = X(Y - H) + Y(X + H) + \sqrt{\kappa + \sin^2 \omega} \hat{\rho}^0 \Xi + \frac{\cos \omega}{\sqrt{\kappa + \sin^2 \omega}} X + \frac{1}{2}.
\]
By the definition of \( \Xi \) (16) and the first two equations of (15), we obtain
\[
\hat{\rho}^0 \Xi = \frac{\hat{\rho} \sin \omega}{2\sin \omega(\kappa + \sin^2 \omega)} - \frac{1}{2\sqrt{\kappa + \sin^2 \omega}}.
\]
(36)
Hence, due to (30) and (31), one has
\[ J_{\Gamma} = -\hat{a}_0(\hat{a}_0 - \hat{H}) + \dot{\hat{a}}_0(-\hat{a}_0 + \hat{H}) + \frac{(\hat{\theta}^0 \sin \omega)_{\Gamma}}{2\sqrt{\kappa + 1}} \]
\[ = \left(2\hat{a}_0 - \frac{\sqrt{\kappa + 1}(\cos \theta \varphi' - \sin \theta)}{\cos \theta + \sin \theta \varphi'}\right)(\hat{H} - \hat{a}_0) \]
\[ = \frac{\sqrt{(\kappa + 1)\gamma \hat{p}}}{\sqrt{\hat{p}^2(\cos \theta + \sin \theta \varphi')}}(\hat{H} - \hat{a}_0) < 0. \]  \hfill (37)
Thus \( \tilde{J} \neq 0 \) and then by (35) \( J \neq 0 \) away from the sonic curve.
In terms of the new coordinates \((t, r)\), we derive
\[ \tilde{\partial}^+ = -\frac{2h^2(X+H)}{t} \partial_t - (2\sqrt{1-t^2}X_t + h)\partial_r, \]
\[ \tilde{\partial}^- = -\frac{2h^2(Y-H+h)}{t} \partial_t + (2\sqrt{1-t^2}Y_t + h)\partial_r, \]
\[ \tilde{\partial}^0 = -\frac{h^2(X+Y+gt)}{t^2} \partial_t - \sqrt{1-t^2}(X-Y)\partial_r, \]  \hfill (38)
where
\[ h = h(t) = \sqrt{(1-t^2)(\kappa + 1 - t^2)}, \quad g = g(t) = \frac{1}{\sqrt{\kappa + 1 - t^2}}. \]
Then system (22) can be rewritten as a new system under the coordinates \((t, r)\)
\[ \begin{align*}
G_t + \frac{(1-t^2)(X-Y)^2}{h^2(X+Y+gt)}G_r &= -\frac{\kappa+1}{h^2}G_t - \frac{g(1-2t^2)}{h^2(X+Y+gt)}G_t^2, \\
H_t + \frac{(1-t^2)(X-Y)^2}{h^2(X+Y+gt)}H_r &= -\frac{\kappa+1}{h^2}H_t - \frac{g(1-2t^2)}{h^2(X+Y+gt)}H_t^2, \\
X_t + \frac{(h+2\sqrt{1-t^2}X_t)}{2h^2(Y-H+gt)}X_r &= -\frac{(\kappa+1)(X+H)}{t} \cdot X + \frac{1-(1-2t^2)}{4g\sqrt{h^2(Y-H+gt)}}(X + Y), \\
Y_t + \frac{(h+2\sqrt{1-t^2}Y_t)}{2h^2(X+H)}Y_r &= -\frac{(\kappa+1)(Y+H-G^t)}{t} \cdot Y + \frac{1-(1-2t^2)}{4g\sqrt{h^2(X+H)}}(Y + Y),
\end{align*} \hfill (39)
We comment that system (39) has an explicitly singularity-regularity structure.

3.1.2. Boundary data in the partial hodograph plane. We now find the boundary data for system (39) in the the \((t, r)\) coordinates from (32)-(33). According to the assumption \(\theta' < 0\) in (33), the smooth function \(r = \hat{\theta}(\xi)\) is strictly decreasing, which implies that it can be expressed as \(\xi = \xi(r)\) for \(r \in [r_1, r_2]\), where \(r_1 = \hat{\theta}(\xi_1)\) and \(r_2 = \hat{\theta}(\xi_2)\).
It is clear that the sonic curve \(\Gamma: \eta = \varphi(\xi), \xi \in [\xi_1, \xi_2]\) on the \((\xi, \eta)\) plane is transformed to a segment on \(t = 0\) with \(r \in [r_1, r_2]\) on the \((t, r)\) plane. Moreover, we define the functions
\[ G_0(r) = \hat{G}(\xi(r)), \quad H_0(r) = \hat{H}(\xi(r)), \quad a_0(r) = \hat{a}_0(\xi(r)). \]
In addition, we see from system (39) that each classical solution should satisfy the following requirements
\[ \lim_{t \to 0^+} G_t(t, r) = 0, \quad \lim_{t \to 0^+} H_t(t, r) = \frac{1}{2\sqrt{\kappa + 1}}, \]
\[ \lim_{t \to 0^+} X_t(t, r) = a_1(r), \quad \lim_{t \to 0^+} Y_t(t, r) = a_1(r), \]
where
\[ a_1(r) = \frac{(\cos \theta \psi' - \sin \theta)(a_0 - \hat{H})}{\cos \theta + \sin \theta \psi'}(\hat{\xi}(r)) - \frac{1}{2\sqrt{\kappa + 1}}, \]
which is the corresponding value of \( (\hat{\partial}^0 \Xi)_r \) in the \((t, r)\) plane.

Therefore, we solve system (39) with the following boundary conditions:
\[
\begin{align*}
G(0, r) &= G_0(r), \quad H(0, r) = H_0(r), \quad X(0, r) = -a_0(r), \quad Y(0, r) = a_0(r), \\
G_t(0, r) = 0, \quad H_t(0, r) = \frac{1}{\sqrt{2\kappa + 1}}, \quad X_t(0, r) = a_1(r), \quad Y_t(0, r) = a_1(r),
\end{align*}
\]
for \( r \in [r_1, r_2] \). It is easily checked by (32) and (33) that
\[
(G_0(r), H_0(r), a_0(r), a_1(r)) \in C^3([r_1, r_2]), \quad H_0(r) - a_0(r) \geq \varepsilon_0, \quad a_1(r) + \frac{1}{2\sqrt{\kappa + 1}} \leq -\varepsilon_0, \quad \forall \ r \in [r_1, r_2]
\]
for some constant positive \( \varepsilon_0 \). Then Problem 1.1 can be reformulated into the following new problem in the hodograph plane.

**Problem 3.1.** Assume conditions (41) hold. We seek a local classical solution for system (39) with boundary conditions (40) in the region \( t > 0 \).

### 3.2. Existence of solutions in the partial hodograph plane.

In this subsection, we solve Problem 3.1 by applying the fixed-point method in a weighted metric space.

#### 3.2.1. The homogeneous boundary value problem.

We homogenize the boundary conditions (40) of system (39) by defining
\[
\begin{align*}
\tilde{G} &= G - G_0, \quad \tilde{H} = H - H_0 - \frac{t}{2\sqrt{\kappa + 1}}, \\
\tilde{X} &= X + a_0 - a_1 t, \quad \tilde{Y} = Y - a_0 - a_1 t.
\end{align*}
\]
Then the boundary conditions (40) are corresponding to
\[
\begin{align*}
\tilde{G}(0, r) &= \tilde{H}(0, r) = \tilde{X}(0, r) = \tilde{Y}(0, r) = 0, \\
\tilde{G}_t(0, r) = \tilde{H}_t(0, r) = \tilde{X}_t(0, r) = \tilde{Y}_t(0, r) = 0, & r \in [r_1, r_2].
\end{align*}
\]
By performing a direct calculation, system (39) changes to
\[
\begin{align*}
\tilde{G}_t + \frac{\sqrt{1-t^2}(\tilde{X} - \tilde{Y} - 2a_0 t)^2}{h^2[\tilde{X} + \tilde{Y} + (g + 2a_1) t]} \tilde{G}_r &= b_1(t, r), \\
\tilde{H}_t + \frac{\sqrt{1-t^2}(\tilde{X} - \tilde{Y} - 2a_0)^2}{h^2[\tilde{X} + \tilde{Y} + (g + 2a_1) t]} \tilde{H}_r &= b_2(t, r), \\
\tilde{X}_t - \frac{[h + 2t(\sqrt{1-t^2}(\tilde{Y} + a_0 + a_1) + \sqrt{1-t^2}(\tilde{X} - a_0 + a_1))]}{2h^2(\tilde{X} + \tilde{Y} + \phi)} \tilde{X}_r &= \frac{\tilde{X} + \tilde{Y}}{2t} + b_3(t, r), \\
\tilde{Y}_t + \frac{[h + 2t(\sqrt{1-t^2}(\tilde{Y} + a_0 + a_1) + \sqrt{1-t^2}(\tilde{X} - a_0 + a_1))]}{2h^2(\tilde{X} + \tilde{Y} + \phi)} \tilde{Y}_r &= \frac{\tilde{X} + \tilde{Y}}{2t} + b_4(t, r),
\end{align*}
\]
where
\[
\psi = a_0 - H_0 + \left(a_1 + g - \frac{1}{2\sqrt{\kappa + 1}}\right) t, \quad \phi = H_0 - a_0 + \left(a_1 + \frac{1}{2\sqrt{\kappa + 1}}\right) t,
\]
and the detailed expressions of \(b_i(t, r)\) (\(i = 1, 2, 3, 4\)) are listed below. The functions \(b_1(t, r)\) and \(b_2(t, r)\) are

\[
b_1(t, r) = -\frac{\kappa + 1}{h^2} \frac{G + G_0 t - g(1 - 2t^2)(G + G_0) + \sqrt{1 - t^2}(\bar{X} - \bar{Y} - 2a_0)G_0 t^2}{h^2[X + Y + (g + 2a_1)t]},
\]

\[
b_2(t, r) = -\frac{\kappa + 1}{h^2} \left( \frac{\bar{H} + H_0 + \frac{t}{2\sqrt{\kappa + 1}}}{2} \right) t
+ \frac{g^2 + (1 + g^2)(1 + g\sqrt{\kappa + 1}) + g^4t^2}{2\sqrt{\kappa + 1}(1 + g\sqrt{\kappa + 1})(1 - t^2)} b_2(t, r)t,
\]

where

\[
b_{21}(t, r) = \frac{t}{2h^2[X + Y + (g + 2a_1)t]} \left\{ g^2(G + G_0 + \kappa gt - gt^3) - 2g(\kappa - t^2)(H + H_0 + \frac{t}{2\sqrt{\kappa + 1}}) - 2\sqrt{1 - t^2}(\bar{X} - \bar{Y} - 2a_0)H_0' \right\}.
\]

The functions \(b_3(t, r)\) and \(b_4(t, r)\) are

\[
b_3(t, r) = b_{31}(t, r) \left( a_1 + \frac{\bar{X} + Y}{2t} \right) b_{32}(t, r)t + b_{33}(t, r)t
+ \frac{[h + 2t\sqrt{1 - t^2}(\bar{Y} - a_0 + a_1t)](-a_0' + a_1't)}{2h^2(\bar{Y} - \bar{H} + \psi)} + \frac{(1 - 2t^2)(\bar{X} + \bar{Y} + 2a_1t)}{4gh^2(\bar{Y} - \bar{H} + \psi)},
\]

\[
b_4(t, r) = \frac{t}{2h^2(\bar{X} + \bar{H} + \phi)} \left( a_1 + \frac{\bar{X} + Y}{2t} \right) + b_{42}(t, r)t + b_{43}(t, r)t
- \frac{[h + 2t\sqrt{1 - t^2}(\bar{X} - a_0 + a_1t)](a_0' + a_1't)}{2h^2(\bar{X} + \bar{H} + \phi)} + \frac{(1 - 2t^2)(\bar{X} + \bar{Y} + 2a_1t)}{4gh^2(\bar{X} + \bar{H} + \phi)},
\]

where

\[
b_{31}(t, r) = -\frac{(\kappa + 1)[\bar{X} + \bar{Y} + (g + 2a_1)t] - (\kappa + 2 - t^2)(\bar{Y} - \bar{H} + \psi)t^2}{h^2(\bar{Y} - \bar{H} + \psi)}
\]

\[
b_{32}(t, r) = \frac{[\bar{X} + \bar{H} + \phi][\bar{X} + \bar{Y} + 2a_1t + 2(\kappa + 1 - t^2)(\bar{Y} + a_0 + a_1t)]}{2h^2(\bar{Y} - \bar{H} + \psi)}
- \frac{[\bar{X} - a_0 + a_1t]^2}{2h^2(\bar{Y} - \bar{H} + \psi)},
\]

\[
b_{33}(t, r) = \frac{[\bar{X} - a_0 + a_1t]}{2h^2(\bar{Y} - \bar{H} + \psi)} \left\{ (1 - 2t^2)(\bar{Y} + a_0 + a_1t) + g(\kappa - 1 + t^2)t
- 2\kappa(H + H_0 + \frac{t}{2\sqrt{\kappa + 1}}) + G + G_0 \right\},
\]

and

\[
b_{41}(t, r) = -\frac{(\kappa + 1)[\bar{X} + \bar{Y} + (g + 2a_1)t] - (\kappa + 2 - t^2)(\bar{X} + \bar{H} + \phi)t^2}{h^2(\bar{X} + \bar{H} + \phi)}.
\]
Moreover, the eigenvalues of system \((44)\) are given by the following equations
\[
\lambda_{1}(t, r) = \lambda_{2}(t, r) = \frac{\sqrt{1 - t^2}(\bar{X} - \bar{Y} - 2a_{0})}{h^2[X + Y + (g + 2a_{1})t]} t^2,
\]
\[
\lambda_{3}(t, r) = -\frac{h + 2t\sqrt{1 - t^2}(\bar{Y} + a_{0} + a_{1})}{2h^2(\bar{Y} - H + \psi)} t,
\]
\[
\lambda_{4}(t, r) = \frac{h + 2t\sqrt{1 - t^2}(\bar{X} - a_{0} + a_{1})}{2h^2(X + H + \phi)} t.
\]

Moreover, the eigenvalues of system \((44)\) are
\[
\lambda_{1}(t, r) = \lambda_{2}(t, r) = \frac{\sqrt{1 - t^2}(\bar{X} - \bar{Y} - 2a_{0})}{h^2[X + Y + (g + 2a_{1})t]} t^2,
\]
\[
\lambda_{3}(t, r) = -\frac{h + 2t\sqrt{1 - t^2}(\bar{Y} + a_{0} + a_{1})}{2h^2(\bar{Y} - H + \psi)} t,
\]
\[
\lambda_{4}(t, r) = \frac{h + 2t\sqrt{1 - t^2}(\bar{X} - a_{0} + a_{1})}{2h^2(X + H + \phi)} t.
\]  

We define a region in the plane \((t, r)\)
\[
D_{\delta} = \{(t, r) \mid 0 \leq t \leq \delta, \bar{r}_{1}(t) \leq r \leq \bar{r}_{2}(t)\},
\]  
where \(\bar{r}_{1}(t)\) and \(\bar{r}_{2}(t)\) are smooth functions satisfying \(\bar{r}_{1}(0) = r_{1}, \bar{r}_{2}(0) = r_{2}\) and
\(\bar{r}_{1}(t) < \bar{r}_{2}(t)\) for \(t \in [0, \delta]\). The functions \(\bar{r}_{1}(t)\) and \(\bar{r}_{2}(t)\) will be determined in the iteration below. Then Problem 3.1 is equivalent to the following problem.

**Problem 3.2.** Assume \((41)\) holds. We look for a classical solution to system \((44)\) with homogeneous boundary condition \((43)\) in the region \(D_{\delta}\) for some constant \(\delta > 0\).

3.2.2. A weighted metric space. We establish the local existence of classical solutions to the homogeneous boundary value problem \((44)\) \((43)\) in a weighted metric space inspired by Zhang and Zheng [47]. First we give the definition of admissible functions and strong determinate domain to system \((44)\).

**Definition 3.1** (Admissible functions). The vector function \(F = (f_{1}, f_{2}, f_{3}, f_{4})^{T}\), defined for all \((t, r) \in D_{\delta}\), is an admissible vector function if the following hold:
(i) The functions \(f_{i}(i = 1, 2, 3, 4)\) are continuous on the region \(D_{\delta}\).
(ii) The functions \(f_{i}(i = 1, 2, 3, 4)\) satisfy the boundary value conditions in \((43)\).
(iii) There holds \(\sum_{i=1}^{4} \|\frac{4}{\delta} r_{i}(t; \tau, z)\|_{\infty} \leq M\) for some positive constant \(M\).

The set of all admissible vector functions is denoted by \(W_{\delta}^{M}\). Let \((f_{1}, f_{2}, f_{3}, f_{4})^{T} \in W_{\delta}^{M}\) be an admissible vector function and \((\tau, z)\) be a point in \(D_{\delta}\). Denote \(r_{i}(t; \tau, z)\) \((i = 1, 2, 3, 4)\) the integral curves passing through the point \((\tau, z)\) defined by the following equations
\[
\begin{align*}
\frac{4}{\delta} r_{i}(t; \tau, z) &= \lambda_{i}(t, r_{i}(t; \tau, z)), \\
\frac{4}{\delta} r_{i}(\tau; \tau, z) &= z,
\end{align*}
\]
where \(\lambda_{i}(i = 1, 2, 3, 4)\) are given in \((45)\) but with \((f_{1}, f_{2}, f_{3}, f_{4})\) replacing \((\bar{G}, \bar{H}, \bar{X}, \bar{Y})\).
**Definition 3.2** (Strong determinate domains). We call \( D_\delta \) defined in (46) is a strong determinate domain for system (44) if for any admissible vector function \( \mathbf{F} = (f_1, f_2, f_3, f_4)^T \) and for any point \((\tau, z) \in D_\delta\), the curves \( r_i(t; \tau, z) (i = 1, 2, 3, 4) \) stay inside \( D_\delta \) for all \( 0 \leq t \leq \tau \) until the intersection with the line \( t = 0 \).

Moreover, we use the notation \( \mathcal{S}_\delta^{M} \) to denote the function class which incorporates all continuously differentiable vector functions \( \mathbf{F} = (f_1, f_2, f_3, f_4)^T : D_\delta \to \mathbb{R}^4 \) satisfying the following properties:

\[
(P_1) : \mathbf{F}(0, r) = F_i(0, r) = 0,
\]

\[
(P_2) : \left\| \frac{\mathbf{F}(t, r)}{t^2} \right\|_\infty \leq M,
\]

\[
(P_3) : \left\| \frac{\partial_t \mathbf{F}(t, r)}{t^2} \right\|_\infty \leq M,
\]

\[
(P_4) : \partial_r \mathbf{F}(t, r) \text{ is Lipschitz continuous with respect to } r \text{ and } \left\| \frac{\partial_{rr} \mathbf{F}(t, r)}{t^2} \right\|_\infty \leq M,
\]

where \( \| \cdot \|_\infty \) denotes the supremum norm on the domain \( D_\delta \). It is obvious that \( \mathcal{S}_\delta^{M} \) is a subset of \( \mathcal{W}_\delta^{M} \) and both \( \mathcal{S}_\delta^{M} \) and \( \mathcal{W}_\delta^{M} \) are subsets of \( C^0(D_\delta, \mathbb{R}^4) \). For any elements \( \mathbf{F} = (f_1, f_2, f_3, f_4)^T, \mathbf{G} = (g_1, g_2, g_3, g_4)^T \) in the set \( \mathcal{W}_\delta^{M} \), we now define the weighted metric

\[
d(\mathbf{F}, \mathbf{G}) := \sum_{i=1}^{4} \left\| \frac{f_i - g_i}{t^2} \right\|_\infty.
\]

One can check that \( (\mathcal{W}_\delta^{M}, d) \) is a complete metric space, while the subset \( (\mathcal{S}_\delta^{M}, d) \) is not closed in the space \( (\mathcal{W}_\delta^{M}, d) \).

The existence theorem for Problem 3.2 can be stated as follows.

**Theorem 3.1.** Let conditions (41) be fulfilled and \( D_{\delta_0} \) be a strong determinate domain for system (44). Then there exists positive constants \( \delta \in (0, \delta_0) \) and \( M \) such that the problem (44) (43) admits a classical solution in the function class \( \mathcal{S}_\delta^{M} \).

**3.2.3. The proof of Theorem 3.1.** We use the fixed-point method to show Theorem 3.1. The proof is divided into four steps. In Step 1, we construct an integration iteration mapping in the function class \( \mathcal{S}_\delta^{M} \) by linearizing the differential equations (44). In Step 2, we establish a series of a priori estimates for \( b_i \) and \( \lambda_i \) \((i = 1, 2, 3, 4)\). In Step 3, we demonstrate the mapping is a contraction by employing the above estimates, which implies that the iteration sequence converge to a vector function in the limit. Finally, in Step 4, we show that this limit vector function also belongs to \( \mathcal{S}_\delta^{M} \).

**Step 1 (The iteration mapping).** Let vector function \((\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y})^T(t, r) \in \mathcal{S}_\delta^{M}\). We consider the linearized system of (44)

\[
\begin{aligned}
\frac{d}{dt} \tilde{G} &= b_1(t, r), \\
\frac{d}{dt} \tilde{H} &= b_2(t, r), \\
\frac{d}{dt} \tilde{X} &= \frac{\tilde{x} + \tilde{y}}{2t} + b_3(t, r), \\
\frac{d}{dt} \tilde{Y} &= \frac{\tilde{x} + \tilde{y}}{2t} + b_4(t, r),
\end{aligned}
\]
where
\[
\frac{d}{dt} = \partial_t + \lambda_i(t,r) \partial_r,
\]
and \(\lambda_i(t,r)\) and \(b_i(t,r)\) \((i = 1, 2, 3, 4)\) are given in (45) and (44), respectively, but with \((\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y})\) replacing \((\tilde{G}, \tilde{H}, \tilde{X}, \tilde{Y})\). The integral form of (49) is
\[
\begin{align*}
\tilde{G}(\tau, z) &= \int_0^\tau b_1(t, r_1(t; \tau, z)) \, dt, \\
\tilde{H}(\tau, z) &= \int_0^\tau b_2(t, r_2(t; \tau, z)) \, dt, \\
\tilde{X}(\tau, z) &= \int_0^\tau \left( \frac{\bar{x} + \bar{y}}{2t} + b_3 \right) (t, r_3(t; \tau, z)) \, dt, \\
\tilde{Y}(\tau, z) &= \int_0^\tau \left( \frac{\bar{x} + \bar{y}}{2t} + b_4 \right) (t, r_4(t; \tau, z)) \, dt,
\end{align*}
\]
where \((\tau, z)\) is any point in a strong determinate domain \(D_\delta\) for system (44), \(r_i(t; \tau, z)\) \((i = 1, 2, 3, 4)\) are defined as in (47). Thus system (51) determines an iteration mapping \(T\):
\[
T \begin{pmatrix} \tilde{g} \\ \tilde{h} \\ \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \tilde{G} \\ \tilde{H} \\ \tilde{X} \\ \tilde{Y} \end{pmatrix}.
\]
Obviously, the solving Problem 3.2 is equivalent to establish a fixed point for the mapping \(T\) in the function class \(S^M_\delta\).

**Step 2 (Estimates of coefficients in \(S^M_\delta\)).** For further convenience, we hereinafter derive a series of estimates about \(b_i\) and \(\lambda_i\) \((i=1,2,3,4)\). Throughout the paper, the notation \(K\) will denote a constant depending only on \(\varepsilon_0, \kappa, \delta\), the bounds of \(h, g\) and the \(C^3\) norms of \(G_0, H_0, a_0, a_1\), which may change from one expression to another.

We first obtain by the assumption \((\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y})^T \in S^M_\delta\) that
\[
|\tilde{g}| + |\tilde{h}| + |\bar{x}| + |\bar{y}| \leq M t^2, \quad |\tilde{g}_r| + |\tilde{h}_r| + |\bar{x}_r| + |\bar{y}_r| \leq M t^2, \\
|\tilde{g}_{rr}| + |\tilde{h}_{rr}| + |\bar{x}_{rr}| + |\bar{y}_{rr}| \leq M t^2.
\]
Furthermore, it is easily seen by the definitions of \(h, g, a_1, \psi, \phi\) and (41) that there exists a small constant \(\delta_0 < 1\) such that there hold
\[
h^2 \geq \frac{1}{2}, \quad g + 2a_1 \leq -\frac{\varepsilon_0}{2}, \quad \psi \leq -\frac{\varepsilon_0}{2}, \quad \phi \geq \frac{\varepsilon_0}{2}, \quad \forall (t,r) \in D_{\delta_0}.
\]
The region \(D_{\delta_0}\) is defined as in (46). From above and (53), there exists a constant \(\delta_1 \leq \delta_0\) such that for \((t,r) \in D_{\delta_1}\)
\[
|h^2(\bar{x} + \bar{y} + (g + 2a_1)t)| \geq \frac{1}{2}(\varepsilon_0 t - M t^2) \geq \frac{1}{2}(\varepsilon_0 - M \delta_1) t \geq \frac{\varepsilon_0 t}{8}, \\
2h^2(\bar{y} - \psi) \geq \frac{\varepsilon_0}{4} - M t^2 \geq \frac{\varepsilon_0}{4}, \quad 2h^2(\bar{x} + \bar{h} + \phi) \geq \frac{\varepsilon_0}{8} - M t^2 \geq \frac{\varepsilon_0}{8},
\]
which mean that the denominators in system (49) away from zero.

We next estimate \(b_3\), and \(b_1, b_2, b_4\) can be discussed analogously. Denote \(b_3\) as
\[
b_3 = -\frac{I_1}{h^2(\bar{y} - \bar{h} + \psi)} \left( a_1 + \frac{\bar{x} + \bar{y}}{2t} \right) + \frac{I_2 + I_3 + I_4}{2h^2(\bar{y} - \bar{h} + \psi)} t,
\]
where
\[
I_1 = (\kappa + 1)[\hat{x} + \hat{y} + (g + 2a_1)t] - (\kappa + 2 - t^2)(\hat{y} - \hat{h} + \psi)t^2 - \frac{1 - 2t^2}{2g}t,
\]
\[
I_2 = [h + 2t\sqrt{1 - t^2}(\hat{y} + a_0 + a_1t)][(a_0' + a_1')t],
\]
\[
I_3 = (\hat{x} + \hat{h} + \phi)[\hat{x} + \hat{y} + 2a_1t + 2(\kappa + 1 - t^2)(\hat{y} + a_0 + a_1t)] - (\hat{x} - a_0 + a_1t)^2
\]
\[
I_4 = (\hat{x} - a_0 + a_1t)\left[(1 - 2t^2)(\hat{y} + a_0 + a_1t) + g(\kappa - 1 + t^2)t
\]
\[
- 2\kappa(\hat{h} + Ho + \frac{t}{2\sqrt{\kappa + 1}}) + \hat{y} + G_0\right]
\]

Due to (53), it follows that
\[
|I_1| \leq K(Mt^2 + Kt) + K(Mt^2 + K)t^2 + Kt \leq K(1 + Mt)t,
\]
\[
|I_2| \leq |K + Kt(Mt^2 + K)|(K + Kt) \leq K(1 + Mt),
\]
\[
|I_3| \leq (K + Mt^2)[Mt^2 + K] + Kt \leq K(1 + Mt)^2,
\]
\[
|I_4| \leq (M^2 + K)[K(Mt^2 + K) + K + K(Mt^2 + K) + Mt^2 + K] \leq K(1 + Mt)^2.
\]

Inserting (56) into (55) and applying (54) yields
\[
|b_3| \leq K|I_1|(K + Mt) + K(|I_2| + |I_3| + |I_4|)t \leq K(1 + Mt)^2t.
\]

Moreover, we differentiate \(b_3\) with respect to \(r\) to get
\[
\frac{\partial b_3}{\partial r} = -\frac{\partial_r I_1}{h^2(y - h + \psi)}(a_1 + \frac{\hat{x} + \hat{y}}{2t}) - \frac{I_1}{h^2(y - h + \psi)}\left(\frac{a_1'}{2t} + \frac{\partial_r \hat{x} + \partial_r \hat{y}}{2t}\right)
\]
\[
+ \frac{\partial_r I_2 + \partial_r I_3 + \partial_r I_4}{2h^2(y - h + \psi)} - b_3 \frac{\partial_r \hat{y} - \partial_r \hat{h} + \partial_r \psi}{y - h + \psi},
\]

subsequently,
\[
\frac{\partial^2 b_3}{\partial r^2} = -\frac{\partial_{rr} I_1}{h^2(y - h + \psi)}(a_1 + \frac{\hat{x} + \hat{y}}{2t}) - \frac{2\partial_r I_1}{h^2(y - h + \psi)}\left(\frac{a_1'}{2t} + \frac{\partial_r \hat{x} + \partial_r \hat{y}}{2t}\right)
\]
\[
- \frac{I_1}{h^2(y - h + \psi)}\left(\frac{a_1''}{2t} + \frac{\partial_{rr} \hat{x} + \partial_{rr} \hat{y}}{2t}\right) + \frac{\partial_{rr} I_2 + \partial_{rr} I_3 + \partial_{rr} I_4}{2h^2(y - h + \psi)}
\]
\[
- b_3 \frac{\partial_{rr} \hat{y} - \partial_{rr} \hat{h} + \partial_{rr} \psi}{y - h + \psi} - 2\partial_r b_3 \frac{\partial_r \hat{y} - \partial_r \hat{h} + \partial_r \psi}{y - h + \psi}.
\]

Making use of (53)-(54) and (56)-(57), we obtain
\[
\left|\frac{\partial b_3}{\partial r}\right| \leq K(1 + Mt)^3t + K(1 + Mt)|\partial_r I_1| + K(|\partial_r I_2| + |\partial_r I_3| + |\partial_r I_4|)t,
\]
and
\[
\left|\frac{\partial^2 b_3}{\partial r^2}\right| \leq K(1 + Mt)^3t + K(1 + Mt)(|\partial_{rr} I_1| + |\partial_r I_1|)
\]
\[
+ K(|\partial_{rr} I_2| + |\partial_{rr} I_3| + |\partial_{rr} I_4|)t + K|\partial_r b_3|(1 + Mt).
\]

By direct calculations, one derives
\[
|\partial_r I_1| \leq K(1 + Mt)t, \quad |\partial_r I_1| \leq K(1 + Mt)t,
\]
\[
|\partial_r I_1| \leq K(1 + Mt)^2, \quad |\partial_{rr} I_1| \leq K(1 + Mt)^2, \quad i = 2, 3, 4.
\]
Putting (60) into (58) and (59) gives

$$\left| \frac{\partial b_3}{\partial r} \right| \leq K(1 + Mt)^3t, \quad \left| \frac{\partial^2 b_3}{\partial r^2} \right| \leq K(1 + Mt)^4t. \quad (61)$$

Repetition of the same arguments for $b_1, b_2$ and $b_4$ leads to

$$\left| \frac{\partial b_i}{\partial r} \right| \leq K(1 + Mt)^3t, \quad \left| \frac{\partial^2 b_i}{\partial r^2} \right| \leq K(1 + Mt)^4t, \quad i = 1, 2, 4. \quad (62)$$

For further use, we provide here the estimates about the eigenvalues $\lambda_i$. By (45) and (53)-(54), it suggests that

$$|\lambda| = \left| \frac{\sqrt{1 - t^2(\bar{x} - \bar{y} - 2a_0)^2}}{h^2[\bar{x} + \bar{y} + (g + 2a_1)]^2} \right| \leq \frac{K(K + Mt^2)t^2}{\varepsilon_0t} \leq K(1 + Mt)t. \quad (63)$$

Moreover, we have the estimates for $\partial_r \lambda_1$ and $\partial_r \lambda_1$ as follows

$$\left| \frac{\partial \lambda_1}{\partial r} \right| \leq K(1 + Mt)^2t, \quad \left| \frac{\partial^2 \lambda_1}{\partial r^2} \right| \leq K(1 + Mt)^3t. \quad (64)$$

Similar estimates also hold for the eigenvalues $\lambda_3$ and $\lambda_4$.

Summing up (57) and (61)-(64), we have the following a priori estimates

$$|b_i| \leq K(1 + Mt)^2t, \quad \left| \frac{\partial b_i}{\partial r} \right| \leq K(1 + Mt)^3t, \quad \left| \frac{\partial^2 b_i}{\partial r^2} \right| \leq K(1 + Mt)^4t, \quad \left| \varepsilon_0 \right| \leq K(1 + Mt)$$

for $i = 1, 2, 3, 4$.

**Step 3 (Properties of the mapping).** We have the following lemma about the properties of the mapping $T$.

**Lemma 3.1.** Let the assumptions in Theorem 3.1 hold. Then there exists positive constants $\delta \in (0, \delta_1), M$ and $0 < \nu < 1$ depending only on $\varepsilon_0, \gamma$, the bounds of $h, g$ and the $C^3$ norms of $G_0, H_0, a_0, a_1$ such that

1. $T$ maps $S_\delta^M$ onto $S_\delta^M$;
2. For any vector functions $\tilde{F}, F$ in $S_\delta^M$, there holds

$$d\left(T(\tilde{F}), T(F)\right) \leq \nu d(\tilde{F}, F). \quad (66)$$

**Proof.** Let $\delta$ and $M$ be two constants and $\tilde{F} = (\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y})^T$ and $F = (g, h, x, y)^T$ be two elements in $S_\delta^M$. $G = T(\tilde{F}) = (\tilde{G}, \tilde{H}, \tilde{X}, \tilde{Y})^T$ and $\tilde{G} = T(F) = (G, H, X, Y)^T$.

From the integral system (51), one directly obtains $\tilde{G}(0, z) = \tilde{H}(0, z) = \tilde{X}(0, z) = \tilde{Y}(0, z) = 0$.

Based on (65), it follows by the integral system (51) that for $\tau \leq \delta$

$$|\tilde{G}(\tau, z)| = \left| \int_0^\tau b_1 \, dt \right| \leq \int_0^\tau K(1 + M\delta)^2t dt \leq K(1 + M\delta)^2\tau^2,$$

$$|\tilde{H}(\tau, z)| = \left| \int_0^\tau b_2 \, dt \right| \leq \int_0^\tau K(1 + M\delta)^2t dt \leq K(1 + M\delta)^2\tau^2,$$
\[ |X(\tau, z)| = \left| \int_0^{\tau} \left( \frac{\dddot{x} + \dddot{y}}{2t} + b_3 \right) dt \right| \leq \int_0^{\tau} \frac{M}{2} t + K(1 + M\delta)^2 t dt \\
\leq \left[ \frac{M}{4} + K(1 + M\delta)^2 \right] \tau^2, \]

\[ |\tilde{Y}(\tau, z)| = \left| \int_0^{\tau} \left( \frac{\dddot{x} + \dddot{y}}{2t} + b_4 \right) dt \right| \leq \int_0^{\tau} \frac{M}{2} t + K(1 + M\delta)^2 t dt \\
\leq \left[ \frac{M}{4} + K(1 + M\delta)^2 \right] \tau^2, \]

from which we see that

\[ \frac{|\tilde{G}(\tau, z)|}{\tau^2} + \frac{|\tilde{H}(\tau, z)|}{\tau^2} + \frac{|X(\tau, z)|}{\tau^2} + \frac{|\tilde{Y}(\tau, z)|}{\tau^2} \leq \frac{M}{2} + K(1 + M\delta)^2. \quad (67) \]

Moreover, we differentiate system (51) with respect to \( z \) to find that

\[ \frac{\partial \tilde{G}}{\partial z}(\tau, z) = \int_0^\tau \frac{\partial b_1}{\partial r} \cdot \frac{\partial r_1}{\partial z} dt, \quad \frac{\partial \tilde{X}}{\partial z}(\tau, z) = \int_0^\tau \left( \frac{\dddot{x} + \dddot{y}}{2t} + \frac{\partial b_3}{\partial r} \right) \cdot \frac{\partial r_3}{\partial z} dt, \]

\[ \frac{\partial \tilde{H}}{\partial z}(\tau, z) = \int_0^\tau \frac{\partial b_2}{\partial r} \cdot \frac{\partial r_2}{\partial z} dt, \quad \frac{\partial \tilde{Y}}{\partial z}(\tau, z) = \int_0^\tau \left( \frac{\dddot{x} + \dddot{y}}{2t} + \frac{\partial b_4}{\partial r} \right) \cdot \frac{\partial r_4}{\partial z} dt, \quad (68) \]

where

\[ \frac{\partial r_i}{\partial z}(t; \tau, z) = \exp \left( \int_\tau^t \frac{\partial \lambda_i}{\partial r}(\tau, \bar{r}(s; \tau, z)) ds \right). \]

Applying (65) again gives

\[ \left| \frac{\partial r_i}{\partial z} \right| \leq \exp \left( \int_0^\delta (1 + M\delta)^2 s \right) \leq e^{K(1 + M\delta)^2 \delta^2}, \quad (69) \]

and then

\[ \left| \frac{\partial \tilde{G}}{\partial z} \right| + \left| \frac{\partial \tilde{H}}{\partial z} \right| \leq \int_0^\tau K(1 + M\delta)^3 t \cdot e^{K(1 + M\delta)^2 \delta^2} dt \leq K(1 + M\delta)^3 e^{K(1 + M\delta)^2 \delta^2} \tau^2, \]

\[ \left| \frac{\partial \tilde{X}}{\partial z} \right| + \left| \frac{\partial \tilde{Y}}{\partial z} \right| \leq \int_0^\tau \left[ M + K(1 + M\delta)^3 t \cdot e^{K(1 + M\delta)^2 \delta^2} dt \right. \]

\[ \leq \left( \frac{M}{2} + K(1 + M\delta)^3 \right) e^{K(1 + M\delta)^2 \delta^2} \tau^2. \]

We add the above two inequalities to get

\[ \left| \frac{\partial \tilde{G}}{\partial z} \right| + \left| \frac{\partial \tilde{H}}{\partial z} \right| + \left| \frac{\partial \tilde{X}}{\partial z} \right| + \left| \frac{\partial \tilde{Y}}{\partial z} \right| \leq \left( \frac{M}{2} + K(1 + M\delta)^3 \right) e^{K(1 + M\delta)^2 \delta^2}. \quad (70) \]

Now, differentiating (68) with respect to \( z \) leads to

\[ \frac{\partial^2 F_i}{\partial z^2}(\tau, z) = \int_0^\tau \left\{ \left( \mu_i \frac{\dddot{x} + \dddot{y}}{2t} + \frac{\partial^2 b_i}{\partial r^2} \right) \cdot \left( \frac{\partial r_i}{\partial z} \right)^2 + \left( \mu_i \frac{\dddot{x} + \dddot{y}}{2t} + \frac{\partial b_i}{\partial r} \right) \cdot \frac{\partial^2 r_i}{\partial z^2} \right\} dt \quad (71) \]
for \(i = 1, 2, 3, 4\), where \((F_1, F_2, F_3, F_4) = (\tilde{G}, \tilde{H}, \tilde{X}, \tilde{Y})\), \(\mu_1 = \mu_2 = 0\), \(\mu_3 = \mu_4 = 1\), and
\[
\frac{\partial^2 r_i}{\partial z^2}(t; \tau, z) = \int_{\tau}^{t} \frac{\partial^2 \lambda_i}{\partial r^2} \cdot \frac{\partial r_i}{\partial z}(s; r_i(s; \tau, z)) \, ds \times \exp \left( \int_{\tau}^{s} \frac{\partial \lambda_i}{\partial r}(s; r_i(s; \tau, z)) \, ds \right).
\]
In view of (65) and (69), we observe
\[
\left| \frac{\partial^2 r_i}{\partial z^2} \right| \leq \int_{0}^{\delta} K(1 + M\delta)^3 e^{K(1 + M\delta)^2 \delta^2} \, ds \times e^{K(1 + M\delta)^2 \delta^2}
\leq K\delta^2(1 + M\delta)^3 e^{K(1 + M\delta)^2 \delta^2}.
\]
Putting the above into (71) and using (65) and (69) again yields
\[
\left| \frac{\partial^2 \tilde{G}}{\partial z^2} \right| + \left| \frac{\partial^2 \tilde{H}}{\partial z^2} \right| \leq \int_{0}^{\tau} \left\{ K(1 + M\delta)^4 t \cdot e^{K(1 + M\delta)^2 \delta^2}
+ K(1 + M\delta)^3 t \cdot K\delta^2(1 + M\delta)^3 e^{K(1 + M\delta)^2 \delta^2} \right\} \, dt
\leq K[1 + \delta^2(1 + M\delta)^2](1 + M\delta)^4 e^{K(1 + M\delta)^2 \delta^2} \tau^2,
\]
and
\[
\left| \frac{\partial^2 \tilde{X}}{\partial z^2} \right| + \left| \frac{\partial^2 \tilde{Y}}{\partial z^2} \right| \leq \int_{0}^{\tau} \left\{ [Mt + K(1 + M\delta)^4 t] \cdot e^{K(1 + M\delta)^2 \delta^2}
+ [Mt + K(1 + M\delta)^3 t] \cdot K\delta^2(1 + M\delta)^3 e^{K(1 + M\delta)^2 \delta^2} \right\} \, dt
\leq \left( \frac{M}{2} + K(1 + M\delta)^4 \right)[1 + \delta^2(1 + M\delta)^4] e^{K(1 + M\delta)^2 \delta^2} \tau^2.
\]
Combining (72) and (73), one acquires
\[
\left| \frac{\partial \tilde{G}}{\partial z^2}(\tau, z) \right| + \left| \frac{\partial \tilde{H}}{\partial z^2}(\tau, z) \right| + \left| \frac{\partial \tilde{X}}{\partial z^2}(\tau, z) \right| + \left| \frac{\partial \tilde{Y}}{\partial z^2}(\tau, z) \right|
\leq \left( \frac{M}{2} + K(1 + M\delta)^4 \right)[1 + \delta^2(1 + M\delta)^4] e^{K(1 + M\delta)^2 \delta^2}.
\]
We now choose \(M \geq 64K \geq 64\) and then take \(\delta \leq \min\{1/M^2, \delta_1\}\), where \(\delta_1\) is given in (54), to obtain
\[
\left( \frac{M}{2} + K(1 + M\delta)^4 \right)[1 + \delta^2(1 + M\delta)^4] e^{K(1 + M\delta)^2 \delta^2}
\leq \left( \frac{M}{2} + 16K \right)(1 + 16\delta^2) e^{4K\delta^2} \leq \left( \frac{M}{2} + \frac{M}{4} \right) \left( 1 + \frac{1}{8} \right) e^{\frac{1}{8}K} \leq \frac{27}{32} e^{\frac{1}{8}K} M < M,
\]
which implies by recalling (67), (70) and (74) that the properties \((P_2)-(P_4)\) are preserved by the mapping \(T\). Thus, in order to determine \(T(\tilde{F}) \in \mathcal{D}^{M}_{\delta}\), it suffices to check that \(\tilde{G}_r(0, z) = \tilde{H}_r(0, z) = \tilde{X}_r(0, z) = \tilde{Y}_r(0, z) = 0\). Towards this objective, we differentiate the integral system (51) with respect to \(\tau\)
\[
\frac{\partial F_i}{\partial \tau}(\tau, z) = \mu_i \frac{\tilde{x} + \tilde{y}}{2\tau} + b_i + \int_{0}^{\tau} \left( \mu_i \frac{\tilde{x} + \tilde{y}}{2t} + \frac{\partial b_i}{\partial r} \right) \cdot \frac{\partial r_i}{\partial \tau} \, dt, \quad i = 1, 2, 3, 4.
\]
where $F_i$ ($i = 1, 2, 3, 4$) and $\mu_i$ are as in (71), and
\[
\frac{\partial r_i(t; \tau, z)}{\partial \tau} = -\lambda_i \frac{\partial r_i(t; \tau, z)}{\partial z}.
\]
(77)

Recalling (53) and (65), one directly obtains by (76) that $\tilde{G}_r(0, z) = \tilde{H}_r(0, z) = \tilde{X}_r(0, z) = \tilde{Y}_r(0, z) = 0$. Thus we have proved $T(\tilde{F}) \in S^M$.

We next establish (66) for some positive constant $\nu < 1$. Denote $(F_1, F_2, F_3, F_4) = (\tilde{G}, \tilde{H}, \tilde{X}, \tilde{Y})$, $(F_1', F_2', F_3', F_4') = (\tilde{G}, \tilde{H}, \tilde{X}, \tilde{Y})$ and $\mu_1 = \mu_2 = 0$, $\mu_3 = \mu_4 = 1$. Then we have by (49)
\[
\frac{d}{dt} F_i = \mu_i \frac{\tilde{x} + \tilde{y}}{2t} + b_i(\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y}, t, r), \quad \frac{d}{dt} F_i' = \mu_i \frac{\tilde{x} + \tilde{y}}{2t} + b_i(\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y}, t, r)
\]
for $i = 1, 2, 3, 4$, which along with (50) suggests
\[
\frac{d}{dt}(F_i - F_i') = \left( \frac{d}{dt} F_i - \left( \frac{d}{dt} F_i' + [\lambda_i(\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y}, t, r) - \lambda_i(\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y}, t, r)]F_i' \right) \right)
\]
\[
= \mu_i \frac{\tilde{x} - \tilde{x}}{2t} + \frac{h_t(\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y}, t, r) - b_i(\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y}, t, r)}{2t} - [\lambda_i(\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y}, t, r) - \lambda_i(\tilde{g}, \tilde{h}, \tilde{x}, \tilde{y}, t, r)]F_i'
\]
\[
:= I_5 + I_6 + I_7.
\]
(78)

It is clear that
\[
|I_5| \leq \frac{\mu_i |\tilde{x} - \tilde{x}| + |\tilde{y} - \tilde{y}|}{2t} \leq \frac{\mu_i t}{2d(\tilde{F}, \tilde{F})}.
\]
(79)

For $I_6$ and $I_7$, we only consider the case $i = 3$, the other cases are treated similarly. By the expression of $\lambda_3$ in (45), it renders
\[
|I_7| \leq \left| \frac{h + 2t\sqrt{1 - t^2} (\tilde{g} + a_0 + a_1 t)}{2h^2 (\tilde{y} - \tilde{h} + \psi)} - \frac{h + 2t\sqrt{1 - t^2} (\tilde{g} + a_0 + a_1 t)}{2h^2 (\tilde{y} - \tilde{h} + \psi)} \right| \cdot |F_{ir}|
\]
\[
\leq \frac{Mt^2}{2h^2 |\tilde{y} - \tilde{h} + \psi|} \cdot \left\{ |h + 2t\sqrt{1 - t^2} (a_0 + a_1 t + \tilde{y})| \cdot |\tilde{h} - \tilde{h}| \right.
\]
\[
+ |h + 2t\sqrt{1 - t^2} (a_0 + a_1 t - \psi + \tilde{h})| \cdot |\tilde{y} - \tilde{y}|
\]
\[
\leq K(1 + M\delta)Mt^2d(\tilde{F}, \tilde{F}).
\]
(80)

To estimate $I_6$, we recall (55) and employ (54) and (56) to see that
\[
|I_6| \leq \frac{|I_1 - \tilde{I}_1|}{h^2 |\tilde{y} - \tilde{h} + \psi|} \cdot |a_1 + \frac{\tilde{x} + \tilde{y}}{2t}| + \frac{|\tilde{I}_1|}{h^2 |\tilde{y} - \tilde{h} + \psi|} \cdot \left| \frac{\tilde{x} - \tilde{x}}{2t} + \frac{|\tilde{y} - \tilde{y}|}{2t} \right|
\]
\[
+ \frac{|I_2 - \tilde{I}_2| + |I_3 - \tilde{I}_3| + |I_4 - \tilde{I}_4|}{2h^2 |\tilde{y} - \tilde{h} + \psi|}
\]
\[
+ \frac{2|\tilde{I}_1| \cdot |a_1 + \frac{\tilde{x} + \tilde{y}}{2t}| + (|I_2| + |I_3| + |I_4|)t (|\tilde{y} - \tilde{y}| + |\tilde{h} - \tilde{h}|)}{2h^2 |\tilde{y} - \tilde{h} + \psi|}
\]
\[
\leq K(1 + M\delta) |I_1 - \tilde{I}_1| + Kt(|I_2 - \tilde{I}_2| + |I_3 - \tilde{I}_3| + |I_4 - \tilde{I}_4|)
\]
\[
+ K(1 + M\delta)^2 t^2 d(\tilde{F}, \tilde{F}),
\]
(81)
where \( I_i \) \((i = 1, 2, 3, 4)\) are defined in (55) and \( \bar{I}_i \) \((i = 1, 2, 3, 4)\) are the terms obtained by replacing \((\bar{y}, \bar{h}, \bar{x}, \bar{y})\) with \((\bar{y}, \bar{h}, \bar{x}, \bar{y})\) in \( I_i \). From the detailed expressions of \( I_i \) \((i = 1, 2, 3, 4)\), we derive

\[
I_1 - \bar{I}_1 = (\kappa + 1)(\bar{x} - \bar{x}) + (\bar{y} - \bar{y}) - (\kappa + 2 - t^2)\bar{t}^2(\bar{y} - \bar{y}) - (\bar{h} - \bar{h}),
\]

\[
I_2 - \bar{I}_2 = 2t\sqrt{1 - t^2}(-a'_0 + a'_1t)(\bar{y} - \bar{y}),
\]

\[
I_3 - \bar{I}_3 = [\bar{x} + \bar{y} + 2a_1t + 2(\kappa + 1 - t^2)(\bar{y} + a_0 + a_1t)] \cdot [(\bar{x} - \bar{x}) + (\bar{h} - \bar{h})] + (\bar{x} + \bar{h} + \phi)[(\bar{x} - \bar{x}) + (\bar{y} - \bar{y}) + 2(\kappa + 1 - t^2)(\bar{y} - \bar{y}) - (\bar{x} + \bar{x} - 2a_0 + 2a_1t)(\bar{x} - \bar{x}),
\]

and

\[
I_4 - \bar{I}_4 = \left\{(1 - 2t^2)(\bar{y} + a_0 + a_1t) + g(\kappa - 1 + t^2)t - 2\kappa(\bar{h} + H_0 + \frac{t}{2\sqrt{\kappa + 1}}) + \bar{g} + G_0\right\}(\bar{x} - \bar{x}) + (\bar{x} - a_0 + a_1t)[(1 - 2t^2)(\bar{y} - \bar{y}) - 2\kappa(\bar{h} - \bar{h}) + (\bar{y} - \bar{y})],
\]

which follows that

\[
|I_1 - \bar{I}_1| + |I_2 - \bar{I}_2| + |I_3 - \bar{I}_3| + |I_4 - \bar{I}_4| \leq KM(1 + M\delta)t^2d(\bar{\bar{F}}, \bar{\bar{F}}).
\]

Inserting the above into (81) leads to

\[
|I_6| \leq KM(1 + M\delta)^2t^2d(\bar{\bar{F}}, \bar{\bar{F}}),
\]

which together with (78)-(80) concludes

\[
|F_1 - \bar{F}_1| \leq \int_0^T (|I_5| + |I_6| + |I_7|) \, dt
\]

\[
\leq \int_0^T \left\{ \frac{\mu}{2}t + KM(1 + M\delta)^2t^2 + KM(1 + M\delta)t^2 \right\} d(\bar{\bar{F}}, \bar{\bar{F}}) \, dt
\]

\[
\leq \tau^2 \left\{ \frac{\mu}{4} + KM\delta(1 + M\delta) \right\} d(\bar{\bar{F}}, \bar{\bar{F}}),
\]

from which one has

\[
d\left( \mathcal{T}(\bar{\bar{F}}), \mathcal{T}(\bar{\bar{F}}) \right) = \frac{|\bar{G} - \bar{G}|}{\tau^2} + \frac{|\bar{H} - \bar{H}|}{\tau^2} + \frac{|\bar{X} - \bar{X}|}{\tau^2} + \frac{|\bar{Y} - \bar{Y}|}{\tau^2}
\]

\[
\leq \left\{ \frac{1}{2} + KM\delta(1 + M\delta)^2 \right\} d(\bar{\bar{F}}, \bar{\bar{F}}) \leq \frac{9}{16}d(\bar{\bar{F}}, \bar{\bar{F}}),
\]

by choosing \( \delta \) as in (75), which finishes the proof of (66). Thus \( \mathcal{T} \) is a contraction under the metric \( d \).

**Step 4 (Properties of the limit function).** Notice that \( (S_\delta^M, d) \) is not a closed subset in the complete space \( (\mathcal{W}_\delta^M, d) \), we need to confirm that the limit vector function of the iteration sequence \( \{\bar{F}^{(n)}\} \), defined by \( \bar{F}^{(n)} = \mathcal{T}(\bar{F}^{(n-1)}) \), is in \( S_\delta^M \). This follows directly from Arzela-Ascoli Theorem and the following lemma.

**Lemma 3.2.** With the assumptions in Theorem 3.1, the iteration sequence \( \{\bar{F}^{(n)}\} \) has the property that \( \{\partial_r \bar{F}^{(n)}(t, r)\} \) and \( \{\partial_t \bar{F}^{(n)}(t, r)\} \) are uniformly Lipschitz continuous on \( D_\delta \).
Proof. Suppose that $(\tilde{y}, \tilde{h}, \tilde{x}, \tilde{y})^T \in S^M_t$. It follows by Lemma 3.1 that $(\tilde{G}, \tilde{H}, \tilde{X}, \tilde{Y})^T = \mathcal{T}(u, v, w)^T$ also in $S^M_t$. The proof of this lemma is separated into three steps.

**Step I. Proof of** $|\tilde{G}_t| + |\tilde{H}_t| + |\tilde{X}_t| + |\tilde{Y}_t| \leq 2Mt$. We recall (76) and apply (65), (69), (77) to get

$$
\frac{\partial F_i}{\partial \tau}(\tau, z) \leq \mu_i \frac{\tilde{x}_t + \tilde{y}_t}{2\tau} + |g_i| + \int_0^\tau \left( \mu_i \frac{\tilde{x}_r + \tilde{y}_r}{2t} + \left| \frac{\partial g_i}{\partial \tau} \right| \right) \, dt
$$

$$
\leq \frac{\mu_i}{2} M\tau + K (1 + M\delta)^2 \tau + K\tau^2 (1 + M\delta)^4 e^{K(1 + M\delta)^2\delta^2},
$$

from which one obtains

$$
\left| \frac{\partial \tilde{G}}{\partial \tau}(\tau, z) \right| + \left| \frac{\partial \tilde{H}}{\partial \tau}(\tau, z) \right| + \left| \frac{\partial \tilde{X}}{\partial \tau}(\tau, z) \right| + \left| \frac{\partial \tilde{Y}}{\partial \tau}(\tau, z) \right| \leq \left( M + K (1 + M\delta)^2 + K\delta (1 + M\delta)^4 e^{K(1 + M\delta)^2\delta^2} \right) \tau \leq 2Mt.
$$

**Step II. Proof of** $|\tilde{G}_{tt}| + |\tilde{H}_{tt}| + |\tilde{X}_{tt}| + |\tilde{Y}_{tt}| \leq 2Mt$. We differentiate (76) with respect to $z$ to know that

$$
\frac{\partial^2 F_i}{\partial z\partial \tau}(\tau, z) = \left( \mu_i \frac{\tilde{x}_r + \tilde{y}_r}{2t} + \frac{\partial h_i}{\partial \tau} \right) \frac{\partial r_i}{\partial z}
$$

$$
+ \int_0^\tau \left\{ \left( \mu_i \frac{\tilde{x}_{rr} + \tilde{y}_{rr}}{2t} + \frac{\partial^2 h_i}{\partial r^2} \right) \frac{\partial r_i}{\partial z} \frac{\partial r_i}{\partial \tau} + \left( \mu_i \frac{\tilde{x}_r + \tilde{y}_r}{2t} + \frac{\partial h_i}{\partial \tau} \right) \frac{\partial^2 r_i}{\partial r \partial \tau} \right\} \, dt,
$$

where

$$
\frac{\partial^2 r_i}{\partial r \partial \tau}(t, \tau, \eta) = \exp \left( \int_{\tau}^t \frac{\partial \lambda_i}{\partial \tau} \, ds \right) \cdot \left\{ \int_{\tau}^t \frac{\partial^2 \lambda_i}{\partial r^2} \cdot \frac{\partial r_i}{\partial \tau} \, ds - \frac{\partial \lambda_i}{\partial \tau} \right\}.
$$

Using (65), (69) and (77) yields

$$
\left| \frac{\partial^2 r_i}{\partial r \partial \tau}(\tau, z) \right| \leq e^{K(1 + M\delta)^2\delta^2} \cdot \left\{ \int_0^\delta K(1 + M\delta)^4 e^{K(1 + M\delta)^2\delta^2} \, ds + (1 + M\delta)^2 t \right\}
$$

$$
\leq K\delta (1 + M\delta)^4 e^{K(1 + M\delta)^2\delta^2}.
$$

Thus

$$
\left| \frac{\partial^2 F_i}{\partial z\partial \tau}(\tau, z) \right| \leq \left( \frac{\mu_i}{2} M\tau + K (1 + M\delta)^3 \tau \right) e^{K(1 + M\delta)^2\delta^2}
$$

$$
+ \int_0^\tau \left\{ \left( \frac{\mu_i}{2} Mt + K (1 + M\delta)^4 t \right) e^{K(1 + M\delta)^2\delta^2} \right\} \left( 1 + M\delta \right) t
$$

$$
+ \left( \frac{\mu_i}{2} Mt + K (1 + M\delta)^3 t \right) K\delta (1 + M\delta)^4 e^{K(1 + M\delta)^2\delta^2}
$$

$$
\leq \tau \left( \frac{\mu_i}{2} M + K (1 + M\delta)^4 \right) \left( 1 + K\delta^2 (1 + M\delta)^4 \right) e^{K(1 + M\delta)^2\delta^2},
$$

from which one has

$$
\left| \frac{\partial^2 \tilde{G}}{\partial \tau \partial \tau}(\tau, z) \right| + \left| \frac{\partial^2 \tilde{H}}{\partial \tau \partial \tau}(\tau, z) \right| + \left| \frac{\partial^2 \tilde{X}}{\partial \tau \partial \tau}(\tau, z) \right| + \left| \frac{\partial^2 \tilde{Y}}{\partial \tau \partial \tau}(\tau, z) \right| \leq \tau \left( M + K (1 + M\delta)^4 \right) \left( 1 + K\delta^2 (1 + M\delta)^4 \right) e^{K(1 + M\delta)^2\delta^2} \leq 2Mt.
$$
Step III. Proof of $|\vec{G}_{tt}| + |\vec{H}_{tt}| + |\vec{X}_{tt}| + |\vec{Y}_{tt}| \leq 7M$. Differentiating (76) with respect to $\tau$ gives

$$
\frac{\partial^2 F_i}{\partial \tau^2}(\tau, z) = \frac{\dot{x}_i + \dot{y}_i}{\tau} - \frac{\ddot{x}_i + \ddot{y}_i}{\tau^2} + 2\frac{\partial b_i}{\partial \tau} + \int_0^\tau \left\{ \left( \frac{\mu_1}{2t} + \frac{b_h}{t} \right) \frac{\partial^2 r_i}{\partial \tau^2} + \left( \frac{\mu_1}{2t} + \frac{\partial^2 b_i}{\partial \tau^2} \right) \left( \frac{\partial r_i}{\partial \tau} \right)^2 \right\} \, dt,
$$

from which and Step I we have

$$
\left| \frac{\partial^2 F_i}{\partial \tau^2} \right| \leq 3\mu_1 M + 2\left| \frac{\partial b_i}{\partial \tau} \right| + \int_0^\delta \left\{ t \left( \frac{\mu_1}{2} M + K(1 + M\delta)^3 \right) \left| \frac{\partial^2 r_i}{\partial \tau^2} \right| + t \left( \frac{\mu_1}{2} M + K(1 + M\delta)^4 \right) K(1 + M\delta)^2 e^{K(1+M\delta)^2 \tau^3} \right\} \, dt. \tag{85}
$$

we next estimate $\frac{\partial b_i}{\partial \tau}$ and $\frac{\partial^2 r_i}{\partial \tau^2}$ for $i = 3$, the other cases are analogous. Recalling the expression of $b_3$ in (55) and differentiating it with respect to $\tau$ obtains

$$
\frac{\partial b_3}{\partial \tau}(\tau, z) = -\frac{I_{1\tau}}{k^2(y - h + \psi)} \left( a_1 + \frac{\dot{x} + \dot{y}}{2\tau} \right) - \frac{I_1}{h^2(y - h + \psi)} \left( \frac{\dot{x} + \dot{y}}{2\tau^2} \right) + \frac{(I_{2\tau} + I_{3\tau} + I_{4\tau})}{2h^2(y - h + \psi)} - b_3 \frac{2h_\tau(y - h + \psi) + h(y_r - h_r + \psi_r)}{h(y - h + \psi)},
$$

which combined with (56) and (65) arrives at

$$
\left| \frac{\partial b_3}{\partial \tau} \right| \leq K(1 + M\delta)|I_{1\tau}| + K\delta(|I_{2\tau}| + |I_{3\tau}| + |I_{4\tau}|) + K(1 + M\delta)^3. \tag{86}
$$

By the expressions of $I_i$ ($i = 1, 2, 3, 4$) in (55), we use the result in Step I to get

$$
|I_{1\tau}| + |I_{2\tau}| + |I_{3\tau}| + |I_{4\tau}| \leq K(1 + M\delta)^2.
$$

Putting the above into (86) suggests

$$
\left| \frac{\partial b_3}{\partial \tau} \right| \leq K(1 + M\delta)^3. \tag{87}
$$

We now deal with $\frac{\partial^2 r_3}{\partial \tau^2}$. Recalling (77) and making use of (69) and (83) concludes

$$
\left| \frac{\partial^2 r_3}{\partial \tau^2} \right| = \left| -\frac{\partial \lambda_3}{\partial \tau} + \frac{\partial r_3}{\partial z} \right| \leq e^{K(1+M\delta)^2 \tau^2} \left| \frac{\partial \lambda_3}{\partial \tau} \right| + K\delta^2(1 + M\delta)^5 e^{K(1+M\delta)^2 \tau^2}. \tag{88}
$$

To estimate $\left| \frac{\partial \lambda_3}{\partial t} \right|$, one calculates by the expression of $\lambda_3$ in (45)

$$
\frac{\partial \lambda_3}{\partial t} = -\frac{h_\lambda + 2(t\sqrt{1 - t^2})_\tau(y + a_0 + a_1 t) + 2t\sqrt{1 - t^2}(\dot{y}_t + a_1)}{2h^2(y - h + \psi)} - \frac{h + 2t\sqrt{1 - t^2}(\dot{y} + a_0 + a_1 t)}{2h^2(y - h + \psi)} - \lambda_3 \cdot \frac{2h_\lambda(y - h + \psi) + h(y_r - h_r + \psi_r)}{h(y - h + \psi)},
$$
from which one has by employing (53)-(54), (65) and Step 1
\[ \left| \frac{\partial \lambda_3}{\partial t} \right| \leq K(1 + M\delta)\delta + K(1 + M\delta) + K(1 + M\delta)\delta \cdot K(1 + M\delta) \]
\[ \leq K(1 + M\delta)^2, \]
which along with (88) gives
\[ \left| \frac{\partial^2 r_3}{\partial t^2} \right| \leq K(1 + M\delta)^2e^{K(1+M\delta)^2\delta^2}. \tag{89} \]

Combining with (85), (87) and (89), we obtain
\[ \left| \frac{\partial^2 F_i}{\partial \tau^2} \right| \leq 3\mu_iM + K(1 + M\delta)^3 + K\delta^2[\mu_iM + K(1 + M\delta)^3](1 + M\delta)^2e^{K(1+M\delta)^2\delta^2} \]
\[ \leq 3\mu_iM + K(1 + M\delta)^4, \]
from which we finally have
\[ \left| \frac{\partial^2 \tilde{G}}{\partial \tau^2}(\tau, z) + \frac{\partial^2 \tilde{H}}{\partial \tau^2}(\tau, z) + \frac{\partial^2 \tilde{X}}{\partial \tau^2}(\tau, z) + \frac{\partial^2 \tilde{Y}}{\partial \tau^2}(\tau, z) \right| \]
\[ \leq 6M + K(1 + M\delta)^4 \leq 7M. \tag{90} \]

We sum up (82), (84) and (90) and apply Lemma 3.1 to end the proof of Lemma 3.2, and then the proof of Theorem 3.1 is complete. \( \square \)

4. Solutions in the self-similar plane. Since Problem 3.2 and Problem 3.1 are equivalent, then by Theorem 3.1 the boundary value problem (39)-(41) has a local classical solution \((G, H, X, Y)(t, r)\). In this section, we construct a local smooth solution for Problem 1.1 by converting the solution in the \((t, r)\)-plane to that in the original self-similar \((\xi, \eta)\)-plane.

From the coordinate transformation (34), we see that
\[ \frac{\partial \xi}{\partial r} = \frac{\theta_i}{J}, \quad \frac{\partial \xi}{\partial t} = \frac{\sin \omega \omega_i}{J}, \]
\[ \frac{\partial \eta}{\partial r} = -\frac{\theta_i}{J}, \quad \frac{\partial \eta}{\partial t} = -\frac{\sin \omega \omega_i}{J}, \tag{91} \]
where \( J \) is defined in (35). On the other hand, we use (12), (17) and (20) to derive
\[ \tilde{B}\theta_\xi = (t \sin r - \sqrt{1 - t^2 \cos r})X \]
\[ + (t \sin r + \sqrt{1 - t^2 \cos r})Y + \sin r \sqrt{K + 1 - t^2}, \]
\[ \tilde{B}\theta_\eta = -(t \cos r + \sqrt{1 - t^2 \sin r})X \]
\[ - (t \cos r - \sqrt{1 - t^2 \sin r})Y - \cos r \sqrt{K + 1 - t^2}, \tag{92} \]
and
\[ \tilde{B}\omega_\xi = -\frac{1}{g_{\tau\tau}} \{(t \sin r - \sqrt{1 - t^2 \cos r})X \]
\[ - (t \sin r + \sqrt{1 - t^2 \cos r})(Y + gt) + 2t \sin r H\}, \]
\[ \tilde{B}\omega_\eta = \frac{1}{g_{\tau\tau}} \{(t \cos r + \sqrt{1 - t^2 \sin r})X \]
\[ - (t \cos r - \sqrt{1 - t^2 \sin r})(Y + gt) + 2t \cos r H\}. \tag{93} \]

Thus we use (91) and (92) to construct the functions \( \xi(t, r) \) and \( \eta(\tau, \xi) \) by solving the following differential equations
\[ \left\{ \begin{array}{l}
\frac{\partial \xi}{\partial \xi} = -\frac{B(t, r)}{2h^2}(t \cos r + \sqrt{1 - t^2 \sin r}X + (t \cos r - \sqrt{1 - t^2 \sin r}Y) + \cos r \sqrt{K + 1 - t^2} \right),
\end{array} \right. \]
\[ \xi(0, r) = \theta^{-1}(r), \tag{94} \]
and
\[
\frac{\partial t}{\partial r} = -\frac{B(t, r)}{\sqrt{2h^2(Y(H) + Y(X + H) + \frac{X + Y}{2gt} - gXt - \frac{1}{2}}} \frac{\partial}{\partial t},
\]
\[
\eta(0, r) = \phi(\hat{\theta}^{-1}(r)),
\]
where \( \hat{\theta}^{-1} \) denotes the inverse of \( \hat{\theta} \), which exists by the strictly monotonic assumption of \( \hat{\theta} \). The function \( B(t, r) \) in (94) and (95) is determined by the linear problem
\[
\begin{cases}
\left(\ln B\right)_t + \frac{\sqrt{1 - t^2}(X - Y)^2}{2h^2(Y)}(\ln B)_r = \frac{\kappa}{\nu(X + Y + gt)}, \\
\ln B(0, r) = \ln \sqrt{2\kappa B(\hat{\theta}^{-1}(r))},
\end{cases}
\]
which is derived from (15), (38) and (23). We comment that equations (94) and (95) are well-defined since we have by (37)
\[
X(Y - H) + Y(X + H) + \frac{X + Y}{2gt} - gXt - \frac{1}{2} \leq -\varepsilon < 0, \quad \forall (t, r) \in D_\delta
\]
for some small constant \( \varepsilon > 0 \). Similarly, the entropy function \( S(t, r) \) can be obtained by solving the linear problem
\[
\begin{cases}
S_t + \frac{\sqrt{1 - t^2}(X - Y)^2}{2h^2(Y)}S_r = 0, \\
S(0, r) = \bar{S}(\hat{\theta}^{-1}(r)).
\end{cases}
\]
Moreover, it follows by (91)-(93) that the Jacobian of the map \((t, r) \mapsto (\xi, \eta)\) is
\[
j := \frac{\partial(\xi, \eta)}{\partial(t, r)} = \frac{\bar{B}^2}{2h^2\{X(Y - H) + Y(X + H) + \frac{X + Y}{2gt} - gXt - \frac{1}{2}\}},
\]
which is strictly less than zero when \( t \in (0, \delta) \). Therefore, the map \((t, r) \mapsto (\xi, \eta)\) is an one-to-one mapping for \( t \in (0, \delta) \). Thus we have the functions \( t = t(\xi, \eta) \) and \( r = r(\xi, \eta) \), and then define by (34)
\[
\theta = r(\xi, \eta), \quad \omega = \arccos t(\xi, \eta), \quad \tilde{B} = \bar{B}(t(\xi, \eta), r(\xi, \eta)), \quad S = S(t(\xi, \eta), r(\xi, \eta)), \\
\alpha = r(\xi, \eta) + \arccos t(\xi, \eta), \quad \beta = r(\xi, \eta) - \arccos t(\xi, \eta).
\]

We next check that the functions defined in (96) satisfy system (15). For the equation of \( S \), we use (91)-(93) to compute
\[
\begin{aligned}
\tilde{\partial}^0 S &= \tilde{B}[\cos r(S_t \xi + S_r \eta) + \sin r(S_t \eta + S_r \xi)] \\
&= \tilde{B} \left\{ \left[\cos r\eta r - \sin r\eta \right]S_t - \left[\cos r\xi r - \sin r\xi \right]S_r \right\} \\
&= -\sqrt{1 - t^2} \left[\cos r(\tilde{B} \omega_\eta) + \sin r(\tilde{B} \omega_\eta)\right]S_t + \left[\cos r(\tilde{B} \theta_\xi) + \sin r(\tilde{B} \theta_\eta)\right]S_r \\
&= -\frac{h^2(X + Y + gt)}{t^2} \left\{ S_t + \frac{\sqrt{1 - t^2}(X - Y)^2}{h^2(X + Y + gt)}S_r \right\} = 0.
\end{aligned}
\]
The equation for \( \bar{B} \) can be checked analogously. We now discuss the third equation of (15). We directly calculate
\[
\begin{aligned}
\tilde{\partial}^+ \theta + \frac{\cos^2 \omega}{\kappa + \sin^2 \omega} \tilde{\partial}^+ \omega &= \tilde{B} \left\{ \left(\cos \alpha r \xi + \sin \alpha r \eta\right) - \frac{t^2(1 - t^2)^2}{h^2} \left(\cos \alpha t \xi + \sin \alpha t \eta\right) \right\}
\end{aligned}
\]
Thus we finish the proof of Theorem 1.1.

\[
\begin{aligned}
- \cos \alpha \eta_l + \sin \alpha \xi_l - \frac{t^2 \sqrt{1 - t^2}}{h^2} (\cos \alpha \eta_r - \sin \alpha \xi_r)
\end{aligned}
\]

\[
= \frac{\tilde{B}}{f} \left\{ - \cos \alpha \eta_l + \sin \alpha \xi_l - \frac{t^2 \sqrt{1 - t^2}}{h^2} (\cos \alpha \eta_r - \sin \alpha \xi_r) \right\}
\]

\[
= [\cos \alpha (\tilde{B} \theta) + \sin \alpha (\tilde{B} \theta)] + \frac{t^2 (1 - t^2)}{h^2} [\cos \alpha (\tilde{B} \omega) + \sin \alpha (\tilde{B} \omega)].
\]

Notice by (92)-(93) and (96) that

\[
\cos \alpha (\tilde{B} \theta) + \sin \alpha (\tilde{B} \theta) = \left\{ \begin{array}{ll}
\cos \alpha (t \sin r - \sqrt{1 - t^2} \cos r) \\
- \sin \alpha (t \cos r + \sqrt{1 - t^2} \sin r) \end{array} \right\} X + \left\{ \begin{array}{ll}
\cos \alpha (t \sin r + \sqrt{1 - t^2} \cos r) \\
- \sin \alpha (t \cos r - \sqrt{1 - t^2} \sin r) \end{array} \right\} Y + (\cos \alpha \sin r - \cos r \sin \alpha) \sqrt{\kappa + 1 - t^2}
\]

\[
= - \sin (2 \omega) X - \sin (\alpha - r) \sqrt{\kappa + 1 - t^2},
\]

and

\[
\cos \alpha (\tilde{B} \omega) + \sin \alpha (\tilde{B} \omega) = \frac{1}{g^2 t^2} \left\{ \begin{array}{ll}
- \cos \alpha (t \sin r - \sqrt{1 - t^2} \cos r) \\
+ \sin \alpha (t \cos r + \sqrt{1 - t^2} \sin r) \end{array} \right\} X + \left\{ \begin{array}{ll}
\cos \alpha (t \sin r + \sqrt{1 - t^2} \cos r) \\
- \sin \alpha (t \cos r - \sqrt{1 - t^2} \sin r) \end{array} \right\} Y + \frac{t^2 (1 - t^2)}{h^2} \left[ \sin (2 \omega) X + 2 t H \sin (\alpha - r) \right]
\]

\[
= \frac{1}{g^2 t^2} [\sin (2 \omega) X + 2 t H \sin (\alpha - r)].
\]

Inserting the above into (97) yields

\[
\dot{\theta} + \frac{\cos \omega}{\kappa + \sin^2 \omega} \dot{\omega}
\]

\[
= - \sin (2 \omega) X - \sin (\alpha - r) \sqrt{\kappa + 1 - t^2}
\]

\[
+ \frac{t^2 (1 - t^2)}{h^2} \left[ \sin (2 \omega) X + 2 t H \sin (\alpha - r) \right]
\]

\[
= - \sin (\alpha - r) \sqrt{\kappa + 1 - t^2 + 2 t H \sin (\alpha - r)} = - \sin \omega \sqrt{\kappa + \sin^2 \omega + \sin (2 \omega) H},
\]

which reaches the desired equation by the definition of $H$. The checking of the fourth equation of (15) is similar. Hence the functions $(S, \tilde{B}, \theta, \omega)(\xi, \eta)$ constructed in (96) satisfy the equations (15).

According to the functions $(S, \tilde{B}, \theta, \omega)(\xi, \eta)$, we can define

\[
c = \frac{\sin \omega (\xi, \eta) \tilde{B} (\xi, \eta)}{\sqrt{\kappa + \sin^2 \omega (\xi, \eta)}}, \quad u = \xi + c (\xi, \eta) \frac{\cos \theta (\xi, \eta)}{\sin \omega (\xi, \eta)}, \quad v = \eta + c (\xi, \eta) \frac{\sin \theta (\xi, \eta)}{\sin \omega (\xi, \eta)},
\]

\[
\rho = \left( \frac{\tilde{B}^2 (\xi, \eta) \sin^2 \omega (\xi, \eta)}{\gamma [\kappa + \sin^2 \omega (\xi, \eta)] [\tilde{B}^2 (\xi, \eta)]} \right)^{\frac{1}{\gamma - 1}}, \quad p = S (x, y) \left( \frac{\tilde{B}^2 (\xi, \eta) \sin^2 \omega (\xi, \eta)}{\gamma [\kappa + \sin^2 \omega (\xi, \eta)] [\tilde{B}^2 (\xi, \eta)]} \right)^{\frac{1}{\gamma - 1}}.
\]

It is not difficult to check that the functions $(\rho, u, v, p)(\xi, \eta)$ defined as above satisfy system (5) subject to the boundary condition $(\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{p})(\xi)$ on the sonic curve $\Gamma$. Thus we finish the proof of Theorem 1.1.
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Appendix A. The derivation of formulations. In this Appendix A, we present the details of derivations of (13) and (22) for completeness of the paper. The calculations are tedious but mostly straightforward.

A.1. The derivation of (13). We here only derive the third equation of (13) since the first two equations of (13) are easy and the fourth equation is parallel. Rewriting the third equation of (6) as

\[- \gamma pV + \gamma pU \Lambda_+ - \gamma pV (U_\xi + \Lambda_+ U_\eta) + \gamma pU (V_\xi + \Lambda_+ V_\eta) + c\sqrt{q^2 - c^2} (p_\xi + \Lambda_+ p_\eta) = 0,\]

which combined with (9)-(11) gives

\[- \sin \theta \sin \omega c \cos \alpha + \cos \theta \sin \omega c \sin \alpha - c \sin \theta \sin \omega \left( \frac{\cos \theta}{\sin \omega} \right) \bar{\partial}^+ \left( \frac{\cos \theta}{\sin \omega} \right) + c \cos \theta \sin \omega \bar{\partial}^+ p = 0,\]

that is,

\[1 - c \frac{\sin \theta}{\sin \omega} \sin \omega \frac{\bar{\partial}^+ \left( \frac{\cos \theta}{\sin \omega} \right)}{\sin \omega} + c \frac{\cos \theta}{\sin \omega} \sin \omega \bar{\partial}^+ \left( \frac{\sin \theta}{\sin \omega} \right) + c \frac{\cos \omega}{\sin \omega} \gamma p \bar{\partial}^+ p = 0. \tag{98}\]

On the other hand, thanks to the definition of pseudo-Bernoulli function, one has

\[B = \frac{q^2}{2} + \frac{c^2}{\gamma - 1} = \left( \frac{1}{2 \sin^2 \omega} + \frac{1}{\gamma - 1} \right) c^2 = \frac{\gamma (\kappa + \sin^2 \omega)}{2 \kappa \sin^2 \omega} \cdot \frac{p}{\rho},\]

which along with the entropy function \(S = p \rho^{-\gamma}\) leads to

\[\ln p = \frac{\gamma}{2 \kappa} \left( \ln B - \frac{1}{\gamma} \ln S - \ln \frac{\gamma (\kappa + \sin^2 \omega)}{2 \kappa \sin^2 \omega} \right),\]

from which, it follows that

\[\frac{1}{\gamma p} \bar{\partial}^+ p = \frac{1}{2 \kappa} \bar{\partial}^+ \left( \ln B - \frac{1}{\gamma} \ln S \right) + \frac{\cot \omega}{\kappa + \sin^2 \omega} \bar{\partial}^+ \omega.\]

Putting the above into (98) yields

\[\frac{\sin \omega}{c \cos \omega} - \frac{\sin \theta - \sin \theta \sin \omega \bar{\partial}^+ \theta - \cos \theta \cos \omega \bar{\partial}^+ \omega}{\sin^2 \omega} + \frac{\cot \omega}{\kappa + \sin^2 \omega} \bar{\partial}^+ \omega + \frac{\cos \theta}{c \cos \omega} \cdot \frac{\cos \theta \sin \omega \bar{\partial}^+ \theta - \sin \theta \cos \omega \bar{\partial}^+ \omega}{\sin^2 \omega} + \frac{1}{2 \kappa} \bar{\partial}^+ \left( \ln B - \frac{1}{\gamma} \ln S \right) = 0,\]

which rearranges to

\[\bar{\partial}^+ \theta + \frac{c \omega}{\kappa + \sin^2 \omega} \bar{\partial}^+ \omega + \frac{\sin (2 \omega)}{4 \kappa} \bar{\partial}^+ \left( \ln B - \frac{1}{\gamma} \ln S \right) + \frac{\sin^2 \omega}{c} = 0.\]
A.2. The derivation of (22). We present the derivations of the second and third equations in (22), the other two equations are analogous. Making use of the definition of $H$ and the commutator relation between $\hat{\partial}^0$ and $\hat{\partial}^+$ in (21), one finds by the first two equations of (15) that

\[
\hat{\partial}^0 H = \hat{\partial}^+ \hat{\partial}^0 \left( \frac{\ln S - \ln \tilde{B}}{4\kappa \gamma} \right) + \frac{\cos \omega \hat{\partial}^+ \theta - \hat{\partial}^0 \alpha}{\sin \omega} \hat{\partial}^0 \left( \frac{\ln S - \ln \tilde{B}}{4\kappa \gamma} \right)
\]

\[
- \hat{\partial}^+ \theta - \cos \omega \hat{\partial}^0 \alpha \theta H + \hat{\partial}^0 \ln \tilde{B} H - \hat{\partial}^+ \ln \tilde{B} \hat{\partial}^0 \left( \frac{\ln S}{4\kappa \gamma} - \frac{\ln \tilde{B}}{2\kappa} \right)
\]

\[
= \frac{1}{2} \hat{\partial}^+ \left( \frac{1}{\sqrt{\kappa + \sin^2 \omega}} \right) + \frac{\cos \omega \hat{\partial}^+ \theta - \hat{\partial}^0 \alpha}{\sin \omega} \frac{1}{2\sqrt{\kappa + \sin^2 \omega}}
\]

\[
- \frac{\hat{\partial}^+ \theta - \cos \omega \hat{\partial}^0 \alpha}{\sin \omega} H - \frac{\kappa}{\sqrt{\kappa + \sin^2 \omega}} \left( H + \frac{\hat{\partial}^+ \ln \tilde{B}}{2\kappa} \right).
\]

(99)

Noticing the following relations

\[
\hat{\partial}^+ \ln \tilde{B} = -\hat{\partial}^+ \left( \frac{1}{2\gamma} \ln S - \ln \tilde{B} \right) + \hat{\partial}^+ \left( \frac{1}{2\gamma} \ln S \right) = G - 2\kappa H,
\]

\[
\cos \omega \hat{\partial}^+ \theta - \hat{\partial}^0 \alpha = \cos \omega \left( -\sin(2\omega)X - \sin \omega \sqrt{\kappa + \sin^2 \omega} \right) - \left[ \sin \omega (Y - X) + \hat{\partial}^0 \omega \right]
\]

\[
= -\left( \frac{\kappa + 1}{\cos \omega} \right) \left( X + Y \right) + 2 \sin^2 \omega X - \frac{\sin \omega \sqrt{\kappa + \sin^2 \omega}}{\cos \omega} \left( 1 + \cos^2 \omega \right),
\]

\[
\hat{\partial}^+ \theta - \cos \omega \hat{\partial}^0 \alpha = \left[ \sin(2\omega)X - \sin \omega \sqrt{\kappa + \sin^2 \omega} \right] - \cos \omega \left[ \sin \omega (Y - X) + \hat{\partial}^0 \omega \right]
\]

\[
= -\left( \frac{\kappa + 1}{\cos \omega} \right) \left( X + Y \right) - 2 \sin \omega \sqrt{\kappa + \sin^2 \omega},
\]

\[
\hat{\partial}^+ \left( \frac{1}{\sqrt{\kappa + \sin^2 \omega}} \right) = -\frac{2 \sin(2\omega)}{2\sqrt{(\kappa + \sin^2 \omega)^3}} \hat{\partial}^+ \omega = -\frac{2 \sin^2 \omega}{\sqrt{\kappa + \sin^2 \omega}} (X + H),
\]

we acquire from (99)

\[
\hat{\partial}^0 H = -\frac{\sin^2 \omega}{\sqrt{\kappa + \sin^2 \omega}} (X + H) + \frac{1}{2\sqrt{\kappa + \sin^2 \omega}} \left\{ -\frac{(\kappa + 1)(X + Y)}{\cos \omega} \right\}
\]

\[
+ 2 \sin^2 \omega X - \frac{\sqrt{\kappa + \sin^2 \omega}(1 + \cos^2 \omega)}{\cos \omega}
\]

\[
+ \left\{ \frac{(\kappa + 1)(X + Y)}{\cos \omega} + 2 \sqrt{\kappa + \sin^2 \omega} \right\} H - \frac{G}{2\sqrt{\kappa + \sin^2 \omega}} \left( \frac{\kappa + 1}{2 \cos \omega} \right) \frac{1 + \cos^2 \omega}{\sqrt{\kappa + \sin^2 \omega}} - \frac{G}{2 \cos \omega},
\]

which is the second equation of (22).
We next derive the equation for $X$. Applying the commutator relation between $\partial^-$ and $\partial^+$ in (21) gives
\[
\partial^- \partial^+ \theta - \partial^+ \partial^- \theta = \frac{\cos(2\omega) \partial^+ \beta - \partial^- \alpha \partial^- \theta - \partial^+ \beta - \cos(2\omega) \partial^- \alpha \partial^+ \theta}{\sin(2\omega)} \\
+ \partial^- \ln \tilde{B} \partial^+ \theta - \partial^+ \ln \tilde{B} \partial^- \theta, \tag{100}
\]
and
\[
\partial^- X - \partial^+ Y = \frac{\cos(2\omega) \partial^+ \beta - \partial^- \alpha Y - \partial^+ \beta - \cos(2\omega) \partial^- \alpha X}{\sin(2\omega)} \\
+ \partial^- \ln \tilde{B} \cdot X - \partial^+ \ln \tilde{B} \cdot Y, \tag{101}
\]
On the other hand, we differentiate (17) along the directions $\partial^-$ and $\partial^+$ to obtain
\[
\partial^- \partial^+ \theta = -\sin(2\omega) \partial^- X - \left(2 \cos(2\omega) X + \frac{\cos \omega (\kappa + 2 \sin^2 \omega)}{\sqrt{\kappa + \sin^2 \omega}}\right) \partial^- \omega, \\
\partial^+ \partial^- \theta = \sin(2\omega) \partial^+ Y + \left(2 \cos(2\omega) Y + \frac{\cos \omega (\kappa + 2 \sin^2 \omega)}{\sqrt{\kappa + \sin^2 \omega}}\right) \partial^+ \omega.
\]
Inserting the above into (100) and using (17) yields
\[
-\sin(2\omega) (\partial^- X + \partial^+ Y) = 2 \cos(2\omega) (X \partial^- \omega + Y \partial^+ \omega) \\
+ \frac{\cos \omega (\kappa + 2 \sin^2 \omega)}{\sqrt{\kappa + \sin^2 \omega}} (\partial^- \omega + \partial^+ \omega) + [\cos(2\omega) \partial^+ \beta - \partial^- \alpha] Y \\
+ [\partial^+ \beta - \cos(2\omega) \partial^- \alpha] X + \cos \omega \sqrt{\kappa + \sin^2 \omega} (\partial^+ \beta - \partial^- \alpha) \\
- \partial^- \ln \tilde{B} [\sin(2\omega) X + \sin \omega \sqrt{\kappa + \sin^2 \omega}] \\
- \partial^+ \ln \tilde{B} [\sin(2\omega) Y + \sin \omega \sqrt{\kappa + \sin^2 \omega}]
\]
which combined with (101) and (20) deduces
\[
\partial^- X = -\frac{\cos(2\omega) \partial^- \omega + \partial^+ \beta - \cos(2\omega) \partial^- \alpha}{\sin(2\omega)} X - \frac{\cos(2\omega) \partial^+ \omega}{\sin(2\omega)} Y \\
- \frac{\sqrt{\kappa + \sin^2 \omega} (\partial^+ \beta - \partial^- \alpha)}{4 \sin \omega} \partial^- \ln \tilde{B} X + \frac{\sqrt{\kappa + \sin^2 \omega} (\partial^+ \ln \tilde{B} + \partial^- \ln \tilde{B})}{4 \cos \omega} \\
- \frac{\sqrt{\kappa + \sin^2 \omega} (\kappa + 2 \sin^2 \omega)}{2 \cos \omega} [X + Y - \frac{1}{2\kappa} (\partial^+ \ln \tilde{B} + \partial^- \ln \tilde{B})]. \tag{102}
\]
Due to the following equations
\[
\partial^+ \ln \tilde{B} + \partial^- \ln \tilde{B} = - \frac{2 \kappa \cos \omega}{\sqrt{\kappa + \sin^2 \omega}}, \quad \partial^- \ln \tilde{B} = - \frac{2 \kappa \cos \omega}{\sqrt{\kappa + \sin^2 \omega}} - G + 2 \kappa H, \\
\partial^+ \beta - \partial^- \alpha = - \frac{2(\kappa + 1) \sin \omega}{\cos \omega} (X + Y) - 4 \sin \omega \sqrt{\kappa + \sin^2 \omega},
\]
then (102) reduces to
\[
\tilde{\partial}^{-} X = -\frac{\cos(2\omega)\tilde{\partial}^{-} \omega + \tilde{\partial}^{+} \beta - \cos(2\omega)\tilde{\partial}^{-} \alpha}{\sin(2\omega)} X - \frac{\cos(2\omega)\tilde{\partial}^{+} \omega Y}{\sin(2\omega)} + \frac{\cos(2\omega)\sqrt{\kappa + \sin^{2}\omega}}{2\cos\omega} (X + Y) + \left( -\frac{2\kappa \cos\omega}{\sqrt{\kappa + \sin^{2}\omega}} - G + 2\kappa H \right) X. \tag{103}
\]
By a direct calculation, one has by (17) and (20)
\[
\cos(2\omega)\tilde{\partial}^{-} \omega + \tilde{\partial}^{+} \beta - \cos(2\omega)\tilde{\partial}^{-} \alpha = \tilde{\partial}^{+} \theta - \tilde{\partial}^{-} \omega - \cos(2\omega)\tilde{\partial}^{-} \theta
\]
\[
= -\sin(2\omega)X - \cos(2\omega)\sin(2\omega)Y - 2\cos^{2}\omega \sin\omega \sqrt{\kappa + \sin^{2}\omega}
\]
\[
- \frac{2\sin\omega(\kappa + \sin^{2}\omega)}{\cos\omega}(X + H).
\]
We put the above into (103) to finally obtain
\[
\tilde{\partial}^{-} X = \left\{ X + \cos(2\omega)Y + \cos\omega \sqrt{\kappa + \sin^{2}\omega} + \frac{\kappa + \sin^{2}\omega}{\cos^{2}\omega}(X + H) \right\} X
\]
\[
- \frac{\cos(2\omega)(\kappa + \sin^{2}\omega)}{\cos^{2}\omega}(X + H)Y + \frac{\cos(2\omega)\sqrt{\kappa + \sin^{2}\omega}}{2\cos\omega}(X + Y)
\]
\[
+ \left( -\frac{2\kappa \cos\omega}{\sqrt{\kappa + \sin^{2}\omega}} - G + 2\kappa H \right) X
\]
\[
= \frac{\kappa + \sin^{2}\omega}{\cos^{2}\omega}(X + H)(X - \cos(2\omega)Y) + \frac{\cos(2\omega)\sqrt{\kappa + \sin^{2}\omega}}{2\cos\omega}(X + Y)
\]
\[
+ \left\{ X + \cos(2\omega)Y + \frac{\cos\omega(\sin^{2}\omega - \kappa)}{\sqrt{\kappa + \sin^{2}\omega}} + 2\kappa H - G \right\} X,
\]
which is the third equation of (22).

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