Propagation invariance and dark hollow structures of sinh-Gaussian beams with small complex parameters

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Abstract. Through investigating the sinh-Gaussian beam with complex beam parameters, it is the first time to find that such beams can carry vortices and exhibit dark hollow intensity distributions when the complex beam parameters are sufficiently small. A closed-form propagation equation for sinh-Gaussian beams through paraxial ABCD optical systems is derived based on the Collins formula and illustrated with numerical methods. It is shown that the perfect hollow configuration can retain a quite long propagating distance under small complex beam parameters. The analytical discussions affirm the numerical conclusions.

1. Introduction
A dark hollow beam (DHB) is an enclosure-shaped light beam with a null intensity center on the beam axis. DHBs have interesting physical properties and a large range of potential applications in atom guiding, trapping and focusing [1, 2]. Therefore, DHBs have always received intensive attention and extensive investigation during past decades. Some known examples of DHBs include high-order Bessel and Bessel-Gauss beams [3–10], Laguerre-Gaussian beams [11-18], hollow Gaussian beams [19,20], helical Mathieu and Mathieu-Gauss beams [21–23], among others [24-29]. So far, how to build good mathematical models to describe DHBs is still an actively studied subject in theory. Very recently, Sun et al. [30] and Li [31] used a sinh-Gaussian function to mimic DHBs with circular or elliptic pattern configurations.

As is well known, the Hermite–sinusoidal–Gaussian beams are ones of the exact solutions of the paraxial wave equation in the rectangular coordinate system [32-34]. Being the special cases of the Hermite–sinusoidal–Gaussian beams, Sinh-Gaussian beams or sine-Gaussian beams have been introduced and their propagation properties have been studied widely [35-40]. In this paper, we again investigate the sinh-Gaussian beams with arbitrary complex beam parameters. It is found that the sinh-Gaussian beams with small complex parameters can be used as mathematical models to describe DHBs carrying vortices. In Section 2, by virtue of numerical calculations, we first demonstrate that the intensity pattern of a sinh-Gaussian beam with small complex parameters exhibits a central dark hollow surrounded by a bright enclosure whose configuration can be controlled by adjusting the complex beam parameters. In Section 3 a closed-form propagation equation of sinh-Gaussian beams through an ABCD optical system has been derived based on the Collins formula, and numerical simulations of sinh-Gaussian beams propagating in free space are performed. Finally, some discussions and conclusions are presented in Section 4.
2. Hollow structure of sinh-Gaussian beams with complex parameters

In the present paper the field $E(x, y, 0)$ of a sinh-Gaussian beam at the position of $z = 0$ is assumed to be

$$E(x, y, 0) = \exp \left( -\frac{x^2 + y^2}{w_0^2} \right) \sinh \left( \frac{\beta_x x + \beta_y y}{w_0} \right)$$

(1)

where $w_0$ is the waist width related to the Gaussian beam, $\beta_x$ and $\beta_y$ are the beam parameters associated with the sinh-function part. For the well-known sinh- or sine-Gaussian beams previously investigated, $\beta_x$ and $\beta_y$ are always considered to be pure real or imaginary simultaneously [35-40]. Note that the beams only carry edge dislocations and represent two-lobe patterns of the intensity distributions.

In fact, the discussions given in [32-34] also admit the exact solution existence for the other complex values of $\beta_x$ and $\beta_y$. In this paper the intensity and phase distributions of the sinh-Gaussian beam (1) with arbitrary complex parameters will be analyzed with numerical simulations and analytical means. Without any loss of generality and for convenience, in the following treatments, the intensity distribution will be normalized by their individual maximum values of light intensities and the scaled traversal coordinates $p_w = p/w_0$ ($p = x$ or $y$) will be used. Therefore the intensity distribution of beam (1) will be only determined by the complex parameters $\beta_x$ and $\beta_y$.

In the following we mainly consider two cases for the beam (1), that is, $\beta_y = i \beta_x$ with $\beta_x$ being real and $\beta_y = \beta_x^*$ with $\beta_x$ being complex. The calculating results indicate that, when $|\beta_x| = |\beta_y|$ is sufficiently small, the pattern of the intensity distribution in the source plane exhibits a perfect dark hollow configuration, that is, along the bright enclosure the maximum intensity distribution is almost identical, as shown in Figs.1 and 2. We point out that the doughnut intensity profile becomes more perfect with decreasing the values of $|\beta_x| = |\beta_y|$. For example, for $\beta_y = i \beta_x$ with $\beta_x \leq 0.5$ along the bright enclosure the maximum intensity distributions are quite homogeneous. On the other hand, when $|\beta_x| = |\beta_y|$ is appropriately large, the intensity distribution may be approximately circular, elliptic or even square-shaped patterns, which is shown in Fig.3.

Figure 1. Intensity (upper) and corresponding phase distribution (bottom) of a sinh-Gaussian beam in the source plane for $\beta_y = i \beta_x$ with $\beta_x = 0.1$(A), 0.4(B) and 0.6(C).
Figure 2. Same as Figure 1 but for $\beta_y = \beta_x^*$ with $\beta_x/(1+i) = 0.1(A)$, $0.3(B)$ and $0.6(C)$.

For other small values of $|\beta_x| \neq |\beta_y|$, the dark hollow intensity distributions can still occur but lose the perfection, which can be seen from Fig. 4. Obviously, enlarging the difference $|\beta_x| - |\beta_y|$ enhances the deviation of the intensity distribution from the perfect configuration.

Figure 3. Intensity distributions of a sinh-Gaussian beam in the source plane for $\beta_y = \beta_x^*$ with $\beta_x = 0.7+0.85i$ (Left), $0.9+1.35i$ (Middle) and $1.0+1.65i$ (Right).

Figure 4. Same as Figure 3 but for $\beta_x = 0.2$ and $\beta_y = 0.2i(A)$, $0.21i(B)$ and $0.22i(C)$.

Therefore, to mimic the dark hollow beam using a sinh-Gaussian form with complex beam parameters, it is necessary to let $\beta_y = i\beta_x$ with $\beta_x$ being real or $\beta_y = \beta_x^*$ with $\beta_x$ being complex and $|\beta_x| = |\beta_y|$ being sufficiently small. Finally, it should be pointed out that the beam (1) with
\( \beta_y = \pm i \beta \) represents a special case corresponding to \( f(z) = \sinh(z) \) with \( z = x \pm iy \) investigated in Ref.[29].

### 3. Paraxial propagation of sinh-Gaussian beams with complex parameters

According to the Collins formula, the field distribution of a sinh-Gaussian beam propagating through a paraxial ABCD system in the \( z \)-plane can be expressed as [35]

\[
E(x, y, z) = i \frac{\exp \left(-\frac{ikD(x^2 + y^2)}{2B} \right)}{\pi z_B} \iint du dv \exp \left[-\frac{1+IA}{z_B}(u^2 + v^2) \right] \\
\exp \left[\frac{2ixw}{z_B} + \beta_x u + \frac{2iyw}{z_B} + \beta_y v \right] \\
\exp \left[\frac{2ixw}{z_B} - \beta_x u + \frac{2iyw}{z_B} - \beta_y v \right] \\
\]

where \( k = 2\pi/\lambda \) is the wave number with \( \lambda \) being the wavelength. Substituting Eq. (1) into Eq. (2) and completing a bit of algebra, we have

\[
E(x, y, z) = i \frac{\exp \left(-\frac{ikD(x^2 + y^2)}{2B} \right)}{\pi z_B} \iint du dv \exp \left[-\frac{1+IA}{z_B}(u^2 + v^2) \right] \\
\exp \left[\frac{2ixw}{z_B} + \beta_x u + \frac{2iyw}{z_B} + \beta_y v \right] \\
\exp \left[\frac{2ixw}{z_B} - \beta_x u + \frac{2iyw}{z_B} - \beta_y v \right] \\
\neq 0
\]

where \( z_B = B/z_R \) with \( z_R = kw_R^2/2 \) being the Rayleigh distance associated with the Gaussian beam. Making use of the Gaussian integral formula [41]

\[
\int_{-\infty}^{\infty} \exp(-\alpha x^2 + \beta x) dx = \exp \left(\frac{\beta^2}{4\alpha} \right) \quad \text{Re}(\alpha) > 0
\]

and performing some mathematical manipulation, one obtains

\[
E(x, y, z) = E(z_B) \exp \left[-\frac{D(x^2 + y^2)}{A-iz_B} \right] \sinh \left(\frac{\beta_x x + \beta_y y}{A-iz_B} \right)
\]

where \( E(z_B) = \frac{2}{A-iz_B} \exp \left[\frac{z_B (\beta_x^2 + \beta_y^2)}{4(z_B + iA)} \right] \) is a global and unimportant factor in the discussion of optical intensity distribution on a specified observation plane. This indicates that the sinh-Gaussian beam is propagation variable or, formally, maintains a sinh-Gaussian beam structure during propagation.

From Eq. (5) we can see that the propagating field distributions mainly depend on the parameters \( z_B \) and \( \beta \). For the free-space propagation with \( A = D = 1 \), \( B = z \) and \( z_B = z/z_R \) being the scaled propagation distance, Figs. 5 - 7 give the variations of the intensity and phase distributions for a sinh-Gaussian beam with different propagation distance. Figure 5 confirms that, when \( xy = \) sufficiently small, the bright enclosure surrounding the dark hollow region can well retain its homogenization with increasing propagation distance. For instance, for the case \( \beta_y = i\beta \), with \( \beta_y \leq 0.25 \) the numerical results show that the intensity patterns can still maintain the perfect dark hollow structure even at \( z_B = 40 \). Similar conclusions also hold true for \( \beta_y = \beta^* \), with small complex
values. However, for appropriately large values of $\beta_y$, the bright enclosure loses its homogenization with increasing propagation distance, which is plotted in Figs. 6 and 7. In fact, the calculations also indicate that the intensity distribution patterns finally evolve into specified configurations after propagating a long distance besides the pattern seemly rotates around the vortex center or the propagation axis during the propagation process.

**Figure 5.** Evolutions of the intensity (upper) and corresponding phase distribution (bottom) of the propagating beam (8) for $\beta_y = i\beta_z = i/4$ at $z_B = z/z_R = 0$ (A) and 40 (B).

**Figure 6.** Evolutions of the intensity (upper) and the corresponding phase distribution (bottom) of the propagating beam (8) for $\beta_y = i\beta_z = i$ at $z_B = z/z_R = 0$ (A), 2 (B) and 40 (C).
4. Discussions and conclusions

In fact, for the special case of $\beta_\gamma \rightarrow i\beta_\gamma$ and the free space propagation, from Eq.(1) we have

$$\sinh \left( \frac{\beta_\gamma x_w + \beta_\gamma y_w}{1 - iz_B} \right) = \sinh \left( \frac{\beta_\gamma re^{i\theta}}{1 - iz_B} \right) = \sum_{n=0}^{\infty} \frac{2}{(2n+1)!} \left( \frac{\beta_\gamma re^{i\theta}}{1 - iz_B} \right)^{2n+1}$$

and

$$E(r, z) = \exp \left( \frac{-r^2}{1 - iz_B} \right) \sum_{n=0}^{\infty} \frac{2}{(2n+1)!} \left( \frac{\beta_\gamma re^{i\theta}}{1 - iz_B} \right)^{2n+1} \quad \text{for small } \beta_\gamma \rightarrow r \exp \left( i\theta - \frac{r^2}{1 - iz_B} \right)$$

It approximately represents a vortex beam with topological charge index one. Therefore, the analytical result clearly demonstrates that the vortices for sinh-Gaussian beams with small complex parameters can occur. To the best of our knowledge, the fact has not previously been reported in the literatures.

In summary, we have again investigated the sinh-Gaussian beams with complex beam parameters. It is found that, for such beams with sufficiently small complex beam parameters, the intensity patterns can exhibit dark hollow configurations and the phase distributions reveal the vortex occurrence. The configurations of the bright enclosure can be controlled by carefully choosing the complex parameters of the beams. Based on the Collins propagation formula, the analytical propagation equation of sinh-Gaussian beams with complex parameter through the paraxial $ABCD$ system has been derived. Numerical stimulations have also been completed for the evolution of sinh-Gaussian beams with complex parameter propagating in free space. Moreover, their propagation characteristics have been illustrated graphically. It is found that, for sufficiently small complex values of $\beta_\gamma (\beta_\gamma)$, the sinh-Gaussian beam is propagation invariant and the perfect dark hollow structure can retain a quite long propagating distance.

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