Hyperons and $\Theta^+_8$ in Holographic QCD

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We revisit the holographic description of strange baryons in the context of the Sakai-Sugimoto construction, by considering the strange quark mass as heavy. Hyperons are described by a massive ($K, K^*$) multiplet, bound to a light-flavor instanton in bulk, much in the spirit of the Callan-Klebanov construction. The modular Hamiltonian maps onto the Landau problem, a charged particle in a 2-dimensional external magnetic field, induced by the bulk Chern-Simons interaction, plus spin-orbit coupling. The ensuing holographic hyperon spectrum compares fairly with the empirical one. The holographic strange pentaquark baryon $\Theta^+_8$ is shown to be unbound.

I. INTRODUCTION

The holographic principle in general [1, 2], and the D4-D8-D$\bar{8}$ holographic set-up in particular [3] provide a framework for addressing QCD in the infrared in the double limit of a large number of colors and strong ’t Hooft gauge coupling $\lambda = g^2_{YM} N_c$. It is confining and exhibits spontaneous chiral symmetry breaking geometrically. The light meson sector is well described by an effective action with manifest chiral symmetry and very few parameters, yet totally in line with more elaborate effective theories of QCD [4]. The same set-up can be minimally modified to account for the description of heavy-light mesons, with manifest heavy quark symmetry [5–9].

Light and heavy-light baryons are dual to instantons and instanton-heavy meson bound states in bulk [10–15], providing a robust geometrical approach to the multi-body bound state problem. The holographic construction provides a dual realization of the chiral soliton approach and its bound states variants [16, 17], without the shortcomings of the derivative expansion. It is a geometrical realization of the molecular approach [18, 19], without the ambiguities of the nature of the meson exchanges, and the arbitrariness in the choice of the many couplings and form factors [20]. Alternative holographic models for the description of heavy hadrons, have been developed in [21, 22].

Chiral symmetry constrains the light quark interactions, while heavy quark symmetry restricts the spin interactions between heavy quarks [23, 24]. Both symmetries are inter-twined by the phenomenon of chiral doubling [25–27] as shown experimentally in [28, 29]. A theoretical approach to the multiquark states should have manifest chiral and heavy quark symmetry, a clear organizational principle in the confining regime, and should address concisely the multi-body bound state problem. The holographic construction provides this framework.

In [7] two of us have analyzed the holographic baryon spectrum by considering three massless flavors $u, d, s$. The strange quark mass was introduced through a bulk instanton holonomy, assumed small and treated in perturbation theory. However, the strange quark mass is intermediate between heavy and light, and may require a treatment beyond perturbation theory. In this work, we will propose such a treatment.

We will consider the kaon mass as large, and identify the strangeness brane as heavy in the formulation outlined in [6, 7, 30]. Hyperons (baryons with strangeness $-1$) will be sought as bound states of massive kaons, to a bulk flavor instanton made of only the light $u, d$ flavors. The ensuing modular Hamiltonian for the hyperons, will be diagonalized without recourse to perturbation theory. This con-
We show that the modular Hamiltonian maps to the Landau problem of a particle in a magnetic field in 2-dimensions, plus spin-orbit coupling. In the analysis to follow for the hyperon spectrum, in comparison to the one in [7].

The organization of the paper is as follows: In section II, we detail the modular Lagrangian for hyperons (baryons with strangeness \(-1\)) in leading order in the heavy meson mass expansion. We retain the exact kaon mass contribution, and the non-local contributions stemming from the Coulomb back-reaction and the bulk form of Gauss law. In section III we show that the modular Hamiltonian maps onto the Landau problem of a particle in a magnetic field in 2-dimensions, plus spin-orbit coupling. In section IV we detail the hyperon spectra, including the strange pentaquark exotic \(\Theta^*\), for two different approximations of the Gauss law contribution. Our conclusions are in V. A number of Appendices are added to complement the derivations in the text.

II. THE MODULAR LAGRANGIAN

The modular Lagrangian for the holographic description of heavy-light mesons bound to a bulk flavor instanton, has been discussed in [6, 7] for standard baryons, and for their exotics in [30, 33-35]. Here we propose to use it for kaons, assuming the strangeness to be a heavy flavor.

In brief, we identify the strange heavy-light flavor field \(\Phi\) in bulk with the \((0^-, 1^-)\) kaon multiplet as \(\Phi = (K, K^*)\), and proceed to bind it to a light flavor instanton as in [6, 30]. The ensuing modular Lagrangian is composed of the collective variables \((\chi_i, a_I, \rho)\) for the instanton collective position, SU(2) orientation and size.

In addition, when binding to the core instanton in bulk, the kaon multiplet transmutes to a 2-component complex modular coordinate \(\chi\), via a zero-mode. In the analysis to follow for the hyperon spectrum, this coordinate will be quantized as a boson, much in the spirit of the bound state approach in the dual analysis in [31, 32]. The fermionic statistics was considered in [6, 30], in the analysis of the much heavier baryons and their exotics, as it captures the key features of the heavier quark, and heavy quark symmetry.

A. General

The full holographic modular Lagrangian for heavy-light kaons bound to a bulk flavor instanton, is given by (see Eq. 23 in [30])

\[
\mathcal{L} = + \frac{1}{2} \dot{\chi}^\dagger \dot{\chi} + \frac{3i}{\rho^2} \chi^\dagger \chi - \frac{37 + 12 \frac{2^2}{192}}{3} \chi^\dagger \chi \\
+ \frac{78i}{5 \rho^2} \chi^\dagger \tau^a \chi a^a - \frac{12}{5 \rho^2} (\chi^\dagger \tau^a \chi)^2 \\
+ \left( \frac{1}{4 \rho^2} + \frac{\dot{\rho}^2}{4} + \frac{\dot{\chi}^2}{4 \rho^2} \right) \chi^\dagger \chi - \frac{1}{2} m_{H}^2 \chi^\dagger \chi \\
+ \mathcal{L}_{\Phi_0}[m_H] + \mathcal{L}_{\text{Coulomb}} ,
\]

where \(\rho\) is the size of the instanton with \(\tilde{\rho} = 16 \pi a_c N_c \rho\). The rescaling \(\chi \rightarrow e^{-imHt/\sqrt{m_H}}\chi\) with the heavy and bare mass \(m_H\) of the \((K, K^*)\) multiplet, subsumed. The moduli of \(SU(2)\) rotation reads

\[
\chi^a = Tr(\tau^a a_I^{-1} a_I) .
\]

The first two lines are standard, with the first term in the third line following from the coupling \(tr \Phi^\dagger \chi \chi\), and leading to a non-vanishing correction to the metric in the space \(y_I = (\rho, a_I)\).

The constraint field contribution \(\mathcal{L}_{\Phi_0}[m_H]\) in the last line was analyzed in Appendices A.3 and A.4 of Ref. [30], and is given by

\[
\mathcal{L}_{\Phi_0}[m_H] = -\frac{1}{8} J_0^\dagger \frac{1}{-D_M + m_H^2} J_0 ,
\]

with the non-local source \(J_0\) given in (A10). It follows from the Gauss constraint on the flavor gauge field in bulk, and is by far the most involved to unravel. For convenience, we detail its analysis in Appendix A.B. Aside from the explicit mass dependence in (1), there is an implicit mass dependence in \(\mathcal{L}_{\Phi_0}[m_H]\) which we have noted in the argument. Since the strange mass is intermediate between light \(u, d\) and heavy \(c, b\), we will address the implicit mass dependence in \(\mathcal{L}_{\Phi_0}[m_H]\) both in the light \(m_H \rightarrow 0\), and heavy \(m_H \rightarrow \infty\).

The Coulomb contribution \(\mathcal{L}_{\text{Coulomb}}\) was originally detailed in Appendix B in [30], and for convenience, briefly reviewed in Appendix C, with the result

\[
\mathcal{L}_{\text{Coulomb}} = -J_C^\dagger \frac{1}{2 (-aN_c \nabla^2 + f^2 \chi^\dagger \chi)} J_C ,
\]
The non-local source $J_C = (\rho^2 + \rho)$ is given in (C2-C3). Throughout, the Coulomb contribution which is small, will be mostly ignored. It is a correction to be added in perturbation theory to the modular Hamiltonian, and assessed only at the end. The holographic heavy kaon mass in the large mass limit, is given by

$$M_K = m_H + \frac{M_{KK}}{\sqrt{2}},$$

(4)

with $m_H$ the bare mass of the kaon doublet, and the Kaluza-Klein scale $M_{KK} = 475$ MeV. In what will follow, $m_H \sim M_K$, unless specified otherwise.

Note that a naive expansion of the Coulomb and Gauss constraint contributions in (1) as shown in Appendix D, leads to a degenerate but stable hyperon spectrum to order $m_H^0$, but unstable at subleading order. The unexpanded constraints produce a stable hyperon spectrum as we detail below.

### B. $L_{\Phi_0}[0]$ and no Coulomb

We start the analysis of (1) by considering the simple case with $m_H = 0$ only in the Gauss constraint or $L_{\Phi_0}[0]$, and no Coulomb back-reaction. Both approximations will be revisited below. With this in mind, the modular Lagrangian simplifies

$$L_{\text{qua}} = \frac{1}{2} \dot{\chi}^T \chi + \frac{3i}{\rho^2} \dot{\chi}^T \chi - \frac{37 + 12\rho^2}{192} \chi^T \chi + \frac{99i}{40\rho^2} \chi^T \chi^a - \frac{75}{8\rho^2} \chi^T \chi .$$

(5)

Note that without the spin-orbit coupling, we have

$$L_0 = \frac{1}{2} \dot{\chi}^T \chi + \frac{3i}{\rho^2} \dot{\chi}^T \chi - \frac{37 + 12\rho^2}{192} \chi^T \chi - \frac{75}{8\rho^2} \chi^T \chi .$$

(6)

By setting the kaon modular variable $\chi$ as

$$\chi = \left( x_1 + iy_1 \right),$$

(7)

(6) can be written as two harmonic oscillators coupled to magnetic field

$$L_0 = \frac{1}{2} \left( \dot{x}_1^2 + \dot{x}_2^2 \right) + \omega_c(y_1 \dot{x}_1 - x_1 y_1 + y_2 \dot{x}_2 - x_2 y_2) - \frac{m_H^2 + \Omega^2}{2} \left( x_1^2 + x_2^2 \right),$$

(8)

where we have defined

$$\Omega^2 = \frac{75}{4\rho^2} + \frac{37 + 6\sqrt{6}}{96} \frac{1}{\rho^2}, \quad \omega_c = \frac{3}{\rho^2} .$$

(9)

This observation will be exploited next.

### III. HYPERON SPECTRUM

Following on the preceding arguments, we now analyze the modular Hamiltonian stemming from (5). Without the spin-orbit contributions as we noted in (8), it maps on the well-known Landau problem in 2-dimensions. In this regime, the hyperons are stable but degenerate. The spin-orbit contribution modifies the potential in the holographic $\rho$-direction, and lifts the hyperon degeneracy.

#### A. Landau problem

For the modular Lagrangian (8), the pertinent Schroedinger equation reads

$$H \phi_n(x) = E \phi_n ,$$

(10)

with the modular Hamiltonian

$$H = \frac{1}{2} D_i D_i + \frac{\omega^2}{2} \bar{\chi} \chi ,$$

(11)

with $z = x + iy$. The long derivative is $D_i = \partial_i - iA_i$, with the $U(1)$ gauge field $A_i = \omega_c(y, -x)$. We now define the operators

$$a = \frac{i}{2\sqrt{\omega_c}} (D_x - iD_y)$$

$$b = \frac{i}{2\sqrt{\omega_c}} (-\partial_x - i\partial_y - \omega_c(x + iy)) ,$$

(12)

which diagonalizes the kinetic contribution

$$\frac{1}{2} D_i D_i = \omega_c(2a^\dagger a + 1) ,$$

(13)

For the harmonic contribution, we note that

$$b^\dagger - a = -i\sqrt{\omega_c}(x - iy) ,$$

(14)

hence the Hamiltonian can be written as

$$H = \omega_c(2a^\dagger a + 1) + \frac{\omega^2}{2\omega_c}(b^\dagger a)(b - a^\dagger) .$$

(15)
The Hamiltonian (15) can then be diagonalized with the help of the following Bogoliubov transformation

\[ a^\dagger = \cosh \theta A^\dagger + \sinh \theta B , \]
\[ b^\dagger = \cosh \theta B^\dagger + \sinh \theta A . \]  

Using \([A, A^\dagger] = [B, B^\dagger] = 1\) and \([A, B] = [A, B^\dagger] = 0\), which preserves the commutation relations, we can fix the value of \(\theta\) as

\[ \tanh 2\theta = \frac{2\alpha}{1 + 2\alpha} , \quad \alpha = \frac{\omega^2}{4\omega_c^2} . \]  

The modular Hamiltonian without spin-orbit coupling is then diagonalized as

\[ H = \Omega^+ + \Omega^- + \Omega^+ A^\dagger A + \Omega^- B^\dagger B , \]  

with

\[ \Omega_{\pm} = \sqrt{m_H^2 + \Omega^2 + \omega_c^2} \pm \omega_c . \]  

In the next subsection we will explore the spin-orbital contribution.

### B. Spin-orbit

For fixed modular variable \(\tilde{\rho}\), the holographic spectrum without spin-orbit following from (19) is harmonic. Since the modular coordinate \(\chi\) is quantized as a boson, the net spin and isospin of the hyperon core is determined by the instanton quantum moduli with \([J^P] = [\frac{1}{2} \pm \frac{1}{2}]\) assignment, in the absence of spin-orbit effects. With spin-orbit contributions, the resulting hyperon states carry \([\frac{1}{2} \pm \frac{1}{2}, (\frac{1}{2} \pm l)]\) assignments, for even \(l\). We now proceed to analyze the dynamical effects of the spin-spin and spin-orbit effects.

#### 1. The \(l = 0\) state

For \(l = 0\) and \(J = \frac{1}{2}\), the energy level with \(n B^\dagger\) excitations is

\[ E_n - E_0 = n \left( \sqrt{m_H^2 + \frac{111}{4\tilde{\rho}^2}} + \frac{37 + 12\frac{\tilde{\rho}}{\rho}}{96} - \frac{3}{\tilde{\rho}^2} \right) , \]  

and the lowest one is archived for \(n = 1\). To proceed we need to fix the \(\rho\) wave function. For that, the induced potential is given by \(\Omega^-\) to which we add the harmonic oscillator potential term \(\frac{1}{2}\omega_c^2 \tilde{\rho}^2\), plus the quartic term \(-\frac{3}{\tilde{\rho}^2} \chi^\dagger \tilde{\tau}^a \chi \chi^\dagger \tilde{a}^\dagger \chi\) as in Eq. 43 in [30].

The result is

\[ V(\tilde{\rho}) = \frac{1}{2} \omega_c^2 \tilde{\rho}^2 + \sqrt{m_H^2 + \frac{111}{4\tilde{\rho}^2}} + \frac{37 + 12\frac{\tilde{\rho}}{\rho}}{96} - \frac{3}{\tilde{\rho}^2} + \frac{9(\sqrt{m_H^2 + \frac{111}{4\tilde{\rho}^2}} + \frac{37 + 12\frac{\tilde{\rho}}{\rho}}{96} - \frac{3}{\tilde{\rho}^2})}{5(m_H^2 + \frac{111}{4\tilde{\rho}^2} + \frac{37 + 12\frac{\tilde{\rho}}{\rho}}{96})\tilde{\rho}^4} . \]  

We note that (22) is stable for small \(\rho\). The additional parameter \(\delta\) captures a spin-spin ordering ambiguity to be discussed below. With this in mind, and using the estimate

\[ \frac{Z^2}{\rho^2} \approx \sqrt{\frac{3}{2}} \frac{1}{\rho^2} \]  

the splitting between \(\Lambda^0\) and nucleon, can be solved numerically for \(\delta = 1\). The result is

\[ M_{\Lambda^0} - M_N = 0.237M_{KK} . \]  

#### 2. The \(l = 2\) state

For \(M_{KK} = 0.475\) GeV, the splitting is about 112.7 MeV, smaller than the empirical splitting of 177 MeV. This is reasonable, since the omitted Coulomb back-reaction is positive (see below).

For the \(l \neq 0\) cases, the quantization needs to be considered more carefully, as operator ordering issues arise. Indeed, we note that the spin operator in the Bogoliubov transformed basis, reads
\[ \chi^+ \tau^a \chi = \frac{1}{\sqrt{m_H^2 + \Omega^2 + \omega_c^2}} (A^l - B_i) \tau^a_i (A - B^i)_j. \]  
\[ (25) \]

The spin-spin and spin-orbit effects will be treated in first order perturbation theory. When evaluating the average of \( \chi^+ \tau^a \chi \), one recovers the standard Schwinger representation of a \( \frac{1}{2} \)-spin

\[ S^a = \frac{1}{2} B^i \tau^a_i B^j, \]  
\[ (26) \]

with \( A_1, A_2 \) constructed using \( (x_1,y_1) \) and \( (x_2,y_2) \), respectively. When evaluating \( (\chi^+ \tau^a \chi)^2 \), without normal ordering, gives

\[ (0|B_i(A^l - B)\tau^a(A^l - B)|0) = 12, \]  
\[ (A^l - B)\tau^a_i (B^l - A) B^i_j|0) = 6. \]  
\[ (27) \]

for \( i = 1, 2 \). With normal ordering, the result is different

\[ (0|B_i : (A^l - B)\tau^a_i (B^l - A) : B^i_j|0) = 3. \]  
\[ (28) \]

which is ordering free. This choice corresponds to \( \delta = 1 \), and will be subsumed throughout.

For \( l = 2, 4, \ldots \), one has \( J = (l \pm 1)/2 \). We first consider the \( J = (l-1)/2 \) case. Following our recent arguments in \[30\] (e.g. Eqs. 44-45), the effective potential reads

\[ V \left( J = \frac{l - 1}{2}, \rho \right) = \frac{1}{2\rho^2} \left( l(l + 2) - \frac{(l + 2)\alpha N_c}{\sqrt{m_H^2 + \Omega^2 + \omega_c^2 \rho^2}} + \frac{3\alpha^2 N_c^2}{4(m_H^2 + \Omega^2 + \omega_c^2)\rho^4} \right) \]  
\[ + \frac{\omega_c^2}{2\rho^2} + \sqrt{m_H^2 + \Omega^2 + \omega_c^2} - \omega_c + \frac{9}{5} \left( \frac{m_H^2 + 11\rho^2}{4\rho^4} + \frac{37 + 12\rho^2}{96} - \frac{3}{\rho^2} \right), \]  
\[ (29) \]

with \( \alpha = \frac{31}{10} \). The \( 1/m_H^2 \) term due to the spin-orbit coupling is kept to maintain stability at small \( \rho \). The change of the potential as one increases \( m_H \) tends to decrease for larger \( l \). For \( l = 2 \), the the potentials at \( m_H = 2 \) and \( m_H = \infty \) differ moderately, but the net difference is small.

Similarly, in the \( J = \frac{l+1}{2} \) case the effective potential is

\[ V \left( J = \frac{l + 1}{2}, \rho \right) = \frac{1}{2\rho^2} \left( l(l + 2) + \frac{l\alpha N_c}{\sqrt{m_H^2 + \Omega^2 + \omega_c^2 \rho^2}} + \frac{3\alpha^2 N_c^2}{4(m_H^2 + \Omega^2 + \omega_c^2)\rho^4} \right) \]  
\[ + \frac{\omega_c^2}{2\rho^2} + \sqrt{m_H^2 + \Omega^2 + \omega_c^2} - \omega_c + \frac{9}{5} \left( \frac{m_H^2 + 11\rho^2}{4\rho^4} + \frac{37 + 12\rho^2}{96} - \frac{3}{\rho^2} \right). \]  
\[ (30) \]

For \( \delta = 1 \) and \( m_H = M_{KK} \), a numerical analysis for the hyperon states gives

\[ [J = \frac{1}{2}, l = 2, I = 1] : M(S_3(1\frac{1}{2}^+)) - M_N = 302 \text{MeV} \]  
\[ [J = \frac{3}{2}, l = 2, I = 1] : M(S_3(1\frac{3}{2}^+)) - M_N = 501 \text{MeV} \]  
\[ (31) \]
which are to be compared to the measured values of 254 MeV and 444 MeV. The splitting between the centroid is much more accurate
\[
M(\Sigma_s(\frac{3}{2}^+)) - M(\Sigma_s(\frac{1}{2}^+)) = 199\text{MeV},
\]
compared to 191 MeV, empirically.

### IV. HYPERON SPECTRUM REVISITED

We now consider the hyperon spectrum with spin-orbit effect, but with \( L_0[m_H] \) in the opposite limit of large \( m_H \) for comparison. The details of \( L_0[m_H] \) are presented in Appendix A, including its closed form results in the heavy mass limit.

#### A. Without Coulomb

In this case the potentials in the holographic \( \rho \)-direction are modified as follows
\[
V_l(\vec{\rho}) = \frac{1}{2} \phi_\rho^2 \rho^2 + \left( m_H^2 + \frac{9}{\rho^4} \left( 1 + \frac{4.11}{m_H^2 \rho^2} + \frac{37 + 12 \rho^2}{96} \right) \right)^{1/2} - \frac{3}{\rho^2}
\]
\[
+ \frac{9}{\rho^4} \left( \frac{m_H^2 + \frac{9}{\rho^4} \left( 1 + \frac{4.11}{m_H^2 \rho^2} + \frac{37 + 12 \rho^2}{96} \right) \right)^{1/2} - \frac{3}{\rho^2}
\]
\[
+ 5 \left( m_H^2 + \frac{9}{\rho^4} \left( 1 + \frac{4.11}{m_H^2 \rho^2} + \frac{37 + 12 \rho^2}{96} \right) \right) \rho^2 \right) .
\]

for \( l = 0 \), and for \( l = 2 \)
\[
V_l(J = \frac{l + 1}{2}, \vec{\rho}) = \frac{1}{2(1 + \frac{1}{2 m_H \rho^2})} \rho^2 \left( l(l+2) - \frac{(l+2)\tilde{\alpha} N_c}{\sqrt{m_H^2 + \tilde{\Omega}^2 + \omega_c^2 \rho^2}} + \frac{3\tilde{\alpha}^2 N_c^2}{4 (m_H^2 + \tilde{\Omega}^2 + \omega_c^2 \rho^2)} \right)
\]
\[
V_l(J = \frac{l - 1}{2}, \vec{\rho}) = \frac{1}{2(1 + \frac{1}{2 m_H \rho^2})} \rho^2 \left( l(l+2) - \frac{(l+2)\tilde{\alpha} N_c}{\sqrt{m_H^2 + \tilde{\Omega}^2 + \omega_c^2 \rho^2}} + \frac{3\tilde{\alpha}^2 N_c^2}{4 (m_H^2 + \tilde{\Omega}^2 + \omega_c^2 \rho^2)} \right)
\]

#### TABLE I. Hyperon and exotic spectrum

| \( B \) | \( 1J^P \) | \( n_\nu \) | \( n_s \) | Mass(small) | Mass(small with Coulomb) | Mass(large) | Mass(large with Coulomb) | Exp-MeV |
|---|---|---|---|---|---|---|---|---|
| \( \Lambda_s \) | \( 0\frac{1}{2}^+ \) | 0 | 0 | 962 | 1182 | 974 | 1152 | 1115 |
| \( \Sigma_s \) | \( 1\frac{1}{2}^+ \) | 2 | 0 | 1134 | 1315 | 1149 | 1306 | 1192 |
| \( \Theta_s \) | \( 1\frac{1}{2}^+ \) | 2 | 0 | 1346 | 1472 | 1254 | 1398 | 1387 |
| \( \Theta^+ \) | \( 0\frac{1}{2}^+ \) | 0 | 0 | - | 1617 | - | 1599 | |

with
\[
\tilde{\alpha} = \frac{13}{10} + \frac{162}{35 m_H^2 \rho^4},
\]
\[
\tilde{\Omega}^2 = \frac{37 + 6\sqrt{6} \rho^2}{96} + \frac{9 \times 4.11}{m_H^2 \rho^2}.
\]

Also, there is a modification to the curvature in the \( \rho \) direction. We should also include the leading warp-
ing contribution at large $m_H$, the details of which are identical to those presented in [30]. With this in mind, the hyperon spectrum is now given by

$$
[J = \frac{1}{2}, l = 0, I = 0] : M(\Lambda) - M_N = 68.1\text{MeV},
$$

$$
[J = \frac{1}{2}, l = 2, I = 1] : M(\Sigma_s(\frac{1}{2}^+)) - M_N = 289\text{MeV},
$$

$$
[J = \frac{3}{2}, l = 2, I = 1] : M(\Sigma_s(\frac{3}{2}^+)) - M_N = 400\text{MeV}
$$

(36)

The $J = \frac{1}{2} \Sigma$ state is pushed up, and the $J = \frac{3}{2} \Sigma$ state is pushed down, with a split in the centroid

$$
\frac{M_\Sigma(\frac{1}{2}^+) + M_\Sigma(\frac{3}{2}^+)}{2} - M_N = 344\text{MeV},
$$

(37)

which is close to the empirical value of 349 MeV.

### B. With Coulomb

As we indicated earlier, throughout we assumed $m_H \sim M_{KK}$ in (4). Here we correct for this shortcoming, with

$$
m_H = 0.68M_{KK}.
$$

(38)

and $M_{KK} = 475$ MeV fixed by the light baryon spectrum [14].

Also, the neglected Coulomb contribution can be estimated in perturbation theory, and in the heavy meson mass limit, it is about

$$
V_C \approx \frac{83}{30\rho^2}.
$$

(39)

This provides for an upper bound estimate.

With in mind, the modified hyperon masses are

$$
[J = \frac{1}{2}, l = 0, I = 0] : M(\Lambda) - M_N = 214\text{MeV},
$$

$$
[J = \frac{1}{2}, l = 2, I = 1] : M(\Sigma_s(\frac{1}{2}^+)) - M_N = 368\text{MeV}
$$

$$
[J = \frac{3}{2}, l = 2, I = 1] : M(\Sigma_s(\frac{3}{2}^+)) - M_N = 460\text{MeV}
$$

(40)

The experimental values are 177 MeV, 254 MeV and 440 MeV respectively, with 37 MeV, 133 MeV and 20 MeV differences.

Using the corrected value of $m_H$ above, and the upper estimate for the Coulomb correction, in Table I, we collect all hyperon masses for the three approximations presented earlier. The chief observation is that the large mass analysis without Coulomb corrections appear closer to the empirical values of the lowest three empirical hyperons, without any adjustable parameter. These results are to be compared to those reported by Callan and Klebanov using the Skyrme model [31, 32], with also no Coulomb corrections.

We recall that in the present holographic construction, the relative splitting between the hyperons, and the splitting of the hyperon centroid from the nucleon, which eliminate much of the uncertainty in $M_{KK}$, are in remarkable agreement with the empirically reported splittings.

### C. Exotics

This approach extends to light multiquark exotics with open or hidden strangeness, much like the heavier multiquark exotics with open or hidden charm and bottom discussed in [30, 33–35]. In particular, an estimate of the mass of the strange pentaquark $\Theta^+_0$ (the exotic $uudd\bar{s}$), is given in Table I. The mass of about 1600 MeV, stems mainly from the $\Omega^+$ frequency (anti-particle) which is $\frac{1}{3}$ higher than the $\Omega^−$ (particle). (Recall that the effective magnetic field induced by the bulk Chern-Simons interaction, is repulsive for particles, and attractive for anti-particles). An additional repulsion of about $\frac{1}{3}$ stems from the Coulomb back-reaction in the heavy mass estimate. A $\Theta^+_0$ of about 1600 MeV lies above the $nK^+$ threshold of 1434 MeV, and is unstable against strong decay. This result is consistent with the fact that the proposed $\Theta^+_s$ state [36–39], is in so far unaccounted for experimentally.

### V. CONCLUSIONS

In the holographic construction presented in [5–7], heavy hadrons are described in bulk using a set of degenerate $N_f$ light D8-D ¯\text{8} branes plus one heavy probe brane in the cigar-shaped geometry that spontaneously breaks chiral symmetry. This construction enforces both chiral and heavy-quark symmetry and describes well the low-lying heavy-light
mesons, baryons and multi-particle exotics [30, 33–35]. Heavy hadrons whether standard or exotics, are composed of heavy-light mesons bound to a core instanton in bulk.

In [7] the analysis of the hyperon spectrum was carried to order $m_H^0$ where spin effects are absent. In this analysis, we have now carried the analysis at next to leading order in $1/m_H$ where the spin-orbit and spin corrections are manifest. In contrast to [7], the modular fields were quantized as bosons and not fermions. The quantized Hamiltonian describes a particle in an external 2-dimensional magnetic field, with spin-orbit coupling.

The hyperon spectrum with the Gauss constraint treated in both the heavy and light kaon mass limit, shows very small changes. It is in overall agreement with the empirical hyperon spectrum, and is much improved in comparison to the analysis in [7], where the strange mass was analysed perturbatively. This construction allows for the description of multiquark exotics with strangeness, and shows that the contentious exotic $\Theta^+_s$ is unbound. In a way, this construction should be regarded as the dual of the improved Callan-Klebanov construction for hyperons, as bound kaon-Skyrmions [31, 32].

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Appendix A:
Derivation of $L_{\Phi_0}[m_H]$

For a generic kaon mass of order $m_H$, we must include its contribution in the Gauss law constraint as captured by the time component $\Phi_{M=0}$ of the heavy-light vector field. This is the most difficult term to unravel to order $1/m_H$. For that, we recall from Appendices A.3 in [30], that the constraint equation for $\Phi_0$ is

$$(-D^2_\lambda + m_H^2)\Phi_0 + 2F_{M0}\Phi_M$$

$$- \frac{i}{16\pi^2 a} F_{PQ}(\partial_P + A_P)\Phi_Q = 0 .$$

(A1)

after using the self-dual condition for $F$. Using the standard relations for $\tilde{\sigma}_{MN}$, we have for the last two contributions in (A1)

$$F_{PQ}\partial_P\Phi_Q = \frac{6\rho_0^2}{(X^2 + \rho^2)^2} \frac{df}{dr} \tilde{\sigma} \cdot X \chi ,$$

$$F_{PQ}A_P\Phi_Q = - \frac{6\rho_0^2}{(X^2 + \rho^2)^2} f \tilde{\sigma} \cdot X \chi .$$

(A2)

For the first contribution in (A1) we have

$$F_{M0}\Phi_M = \frac{6f}{(X^2 + \rho^2)^2} \left( \rho^2 \tilde{\sigma} \cdot \dot{X} + \tilde{\sigma} \cdot X \rho \rho \right) \chi$$

$$+ \chi^a D_M \Phi^a \tilde{\sigma}_{MN} \chi f .$$

(A3)

with

$$\Phi^a = \frac{1}{2(X^2 + \rho^2)} \tilde{\sigma} \cdot X \tau^a \sigma \cdot X ,$$

(A4)

or more explicitly

$$\chi^a D_M \Phi^a \tilde{\sigma}_{MN} \chi f = \frac{3\rho_0^2 f}{(X^2 + \rho^2)^2} \tilde{\sigma} \cdot X \tau^a \chi \chi^a .$$

(A5)

Inserting (A5-A6) into (A1) we have

$$(-D^2_\lambda + m_H^2)\Phi_0 + J_0 = 0 ,$$

(A6)

with

$$J_0 = \frac{12f}{(X^2 + \rho^2)^2} \left( \rho^2 \tilde{\sigma} \cdot \dot{X} + \tilde{\sigma} \cdot X \rho \rho \right) \chi$$

$$+ \frac{6\rho_0^2}{(X^2 + \rho^2)^2} \tilde{\sigma} \cdot X \tau^a \chi \chi^a$$

$$+ \frac{3i}{2\pi^2 a} \frac{\rho^2 f}{(X^2 + \rho^2)^3} \tilde{\sigma} \cdot X \chi + \frac{2f}{r} \frac{\partial \hat{A}_0}{\partial r} \tilde{\sigma} \cdot X \chi$$

(A7)

the source for $\Phi_0$

$$L_{\Phi_0} = \frac{1}{8} \int d^4X J^4_0(X) \Phi_0(X) .$$

(A8)
In this equation the Abelian part of $F_{N0}$ has been included. Since
\[ \frac{1}{r} \frac{\partial A_0}{\partial r} = \frac{i}{4\pi^2 a (X^2 + \rho^2)^2} \left( 1 + \frac{2\rho^2}{X^2 + \rho^2} \right) \] (A9)
one finally has
\[ J_0 = \frac{12f}{(X^2 + \rho^2)^2} \left( \rho^2 \sigma \cdot \dot{X} + \dot{\sigma} \cdot X \dot{\rho} \right) \chi + \frac{6f^2}{(X^2 + \rho^2)^2} \tilde{g} \cdot X \tau^a \chi^a + \frac{i}{2\pi^2 a (X^2 + \rho^2)^2} \left( 1 + \frac{5\rho^2}{X^2 + \rho^2} \right) \tilde{g} \cdot \dot{X} \chi \] (A10)

To solve (A6), we need the massive spin-0 propagator in the instanton background
\[ G_2(X, Y) = \langle X | \frac{1}{-D_M^2 + m_H^2} | Y \rangle , \] (A11)
in terms of which the Gauss law constraint yields the modular Lagrangian contribution (A6) in the form
\[ \mathcal{L}_{\phi_0} = -\frac{1}{8} \int d^4 X d^4 Y J_0^\dagger(X) \langle X | \frac{1}{-D_M^2 + m_H^2} | Y \rangle J_0(Y) , \] (A12)

**Appendix B:**
**Expansion of $\mathcal{L}_{\phi_0}[m_H]$**

The spin-0 Greens function (A11) is not known for arbitrary $m_H$, except for $m_H = 0$. Here, we provide a general expression for the different modular contributions in $\mathcal{L}_{\phi_0}[m_H]$, and then specialize to the two extreme cases of $m_H = 0$ and large $m_H$, for which analytical expressions can be obtained. More specifically, we have
\[ \mathcal{L}_{\phi_0} = -\chi^\dagger \chi \left( \frac{\alpha}{\rho^2} \dot{X}^2 + \frac{\beta}{\rho^2} (\dot{\rho}^2 + \rho^2 a_1^2) \right) + \frac{i\gamma}{\rho^2} \chi^\dagger \tau^a \chi \chi^a - \frac{\delta}{\rho^2} \chi^\dagger \chi , \] (B1)
with the coefficients
\[ \alpha = \frac{12^2}{4} \int d^4 X d^4 Y \frac{f(X) g_1(X, Y) f(Y)}{(X^2 + 1)^2(Y^2 + 1)^2} , \]
\[ \beta = \frac{12^2}{16} \int d^4 X d^4 Y \frac{f(X) g_2(X, Y) f(Y)}{(X^2 + 1)^2(Y^2 + 1)^2} , \]
\[ \gamma = \frac{48N_c}{8} \int d^4 X d^4 Y \frac{f(X)}{(X^2 + 1)^2} \times \left( 1 + \frac{5}{X^2 + 1} \right) g_2(X, Y) , \]
\[ \delta = \frac{64N_c^2}{16} \int d^4 X d^4 Y \frac{f(X)}{(X^2 + 1)^2} \times \left( 1 + \frac{5}{X^2 + 1} \right) g_2(X, Y) \times \frac{f(Y)}{(Y^2 + 1)^2} \left( 1 + \frac{5}{Y^2 + 1} \right) . \] (B2)
Here the scalar functions trace over the spin-0 propagator
\[ g_1(X, Y) = \text{tr} G_2(X, Y) , \]
\[ g_2(X, Y) = \text{tr} (\sigma \cdot X G_2(X, Y) \tilde{\sigma} \cdot Y) , \] (B3)
after the re-scaling $\rho \rightarrow 1$ and $m_H \rightarrow m_H \rho^2$. For $m_H = 0$, the expressions will be quoted explicitly below. For $m_H$ large, spin-0 propagator is zero-mode free, and can be approximated by its free part
\[ G_2(X, Y) \rightarrow \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik \cdot (X - Y)}}{k^2 + m_H^2} , \] (B4)

Using the Fourier transforms
\[ \frac{f(X)}{(X^2 + 1)^2} = \int \frac{d^4k}{(2\pi)^4} g_1(k) e^{ik \cdot X} , \]
\[ \frac{f(X)}{(X^2 + 1)^2} \left( 1 + \frac{5}{X^2 + 1} \right) = \int \frac{d^4k}{(2\pi)^4} g_2(k) e^{ik \cdot X} , \] (B5)
we have
\[ g_1(k) = \frac{4\sqrt{\pi}}{15} e^{-|k|(1 + |k|)} , \]
\[ g_2(k) = \frac{4\sqrt{\pi}}{105} e^{-|k|(5|k|^2 + 22|k| + 22)} , \] (B6)
so that
\[ \alpha = \frac{12^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{|g_1(k)|^2}{k^2 + m_H^2} , \]
\[ \beta = \frac{12^2}{16} \int \frac{d^4k}{(2\pi)^4} \frac{|\nabla g_2(k)|^2}{k^2 + m_H^2} , \]
\[ \gamma = \frac{48N_c}{4} \int \frac{d^4k}{(2\pi)^4} \frac{\nabla g_1(k) \cdot \nabla g_2(k)}{m_H^2 + m_H^2} , \]
\[ \delta = \frac{64N_c^2}{8} \int \frac{d^4k}{(2\pi)^4} \frac{\nabla g_2(k) \cdot \nabla g_2(k)}{m_H^2 + m_H^2} . \] (B7)
Large $m_H$:

\[
\alpha = \frac{12^2}{\pi^2 m_H^4} \int d^4X \frac{1}{(X^2 + 1)^2} = \frac{24}{5 m_H^4},
\]
\[
\beta = \frac{12^2}{4\pi^2 m_H^4} \int d^4X \frac{X^2}{(X^2 + 1)^2} = \frac{6}{5 m_H^4},
\]
\[
\gamma = \frac{48 N_c}{2\pi^2 m_H^4} \int d^4X \frac{X^2}{(X^2 + 1)^7} \times \left(1 + \frac{5}{X^2 + 1}\right)^2 = \frac{54 N_c}{35 m_H^4},
\]
\[
\delta = \frac{64 N_c^2}{4\pi^2 m_H^4} \int d^4X \frac{X^2}{(X^2 + 1)^7} \times \left(1 + \frac{5}{X^2 + 1}\right)^2 = \frac{146 N_c^2}{35 m_H^4}.
\]

This will actually contribute to order $\frac{1}{m_H}$ after the re-scaling in $\chi$.

Zero $m_H$:

\[
\alpha = \beta = \frac{1}{4},
\]
\[
\gamma = \frac{N_c}{2},
\]
\[
\delta = \frac{25 N_c^2}{24}.
\]

Appendix C:
Coulomb correction

Here we provide a complete treatment of the Coulomb back-reaction contribution (with more details in Appendix B in [30]). After re-scaling the U(1) flavor gauge field in bulk $A_0 \rightarrow iA_0$, the Lagrangian for $A^0$ reads

\[
\mathcal{L}[A_0] = \frac{a N_c}{2} (\bar{\chi} A_0)^2 + \frac{f^2}{2} \chi^\dagger \chi A_0^2 + A_0 (\rho^{cl} + \rho)
\]

(C1)

where $\rho^{cl}$ is the classical source (without the modular field $\chi$)

\[
\rho^{cl} = a N_c \nabla^2 A_0^{cl} = -\frac{3 N_c}{\pi^2 (x^2 + \rho^2)^4}
\]

and $\rho$ the quantum source with the modular field

\[
\rho = \frac{f^2}{2} i(\chi^\dagger \dot{\chi} - \dot{\chi}^\dagger \chi) + \frac{3}{16 \pi^2 a} \frac{2 \rho^2 - X^2}{(X^2 + \rho^2)^2} f^2 \chi^\dagger \chi.
\]

(C3)

Note that the contribution

\[
\frac{3}{16 \pi^2 a} \frac{2 \rho^2 - X^2}{(X^2 + \rho^2)^2} f^2 \chi^\dagger \chi
\]

(C4)

originates solely from the Chern-Simons term in bulk.

Given the action for $A_0$, at the minimum we have

\[
\mathcal{L}_{\text{Coulomb}} = -J_C \frac{1}{2} \left(-a N_c \bar{\chi}^2 + f^2 \chi^\dagger \chi\right) J_C,
\]

(C5)

with $J_C = (\rho^{cl} + \rho)$, which is a complicated function of the scalar $\chi^\dagger \chi$. More importantly, it yields always a positive mass correction. Note that the $f^2/m_H$ term in the denominator plays the role of a screening mass, which can be made more manifest through a coordinate transformation.

For a general analysis of the Coulomb correction, we need the Green's function in the background field,

\[
\mathcal{G}_1(X,Y) = \langle X | -a N_c \nabla^2 + f^2 \chi^\dagger \chi | Y \rangle,
\]

(C6)

In the text, we provide an estimate of this contribution in perturbation theory, with the replacement $\chi^\dagger \chi \rightarrow 1$, for a single bound kaon.

Appendix D:
Naive $1/m_H$ analysis

In (1) both the Gauss law constraint and the Coulomb back-reaction are complicated functions of the modular coordinate $\chi$ and $m_H$. Naively, a standard quantum analysis would require expanding them in $1/m_H$. This expansion, leads to an unstable hyperon spectrum at next-to-leading order, as we now demonstrate. In a way the charge constraint and screening should not be expanded, to guarantee quantum stability.

Consider (1) with all terms expanded to order to order $O(1/m_H^2)$

\[
\mathcal{L} = \mathcal{L}_{\text{quadratic}} + \mathcal{L}_{\text{int}}
\]

(D1)
where the quadratic part reads

\[
\mathcal{L}_{\text{quadratic}} = i\chi^4 \dot{\chi} + \frac{1}{2m_H} \chi^4 \dot{\chi} + \frac{9}{2\rho^2} \chi^4 \chi + \frac{9}{2m_H\rho^2} j
\]

\[
+ \frac{78}{5m_H\rho^2} i\chi^4 \tau^a \chi^a + \frac{102}{m_H\rho^2} \chi^4 \chi - \frac{37 + 12\sqrt{2}}{192m_H} \chi^4 \chi ,
\]

and the “high-order contribution” \(\mathcal{L}_{\text{int}}\) reads

\[
\mathcal{L}_{\text{int}} = \frac{12}{5m_H\rho^2} \hat{\mathbf{S}}^2 - \frac{2}{3\rho^2} n^2
\]

\[
+ \frac{1}{m_H\rho^2} \left( - \frac{56}{5} n^2 + \frac{4}{3} n^3 - \frac{4}{3} j n\rho^2 \right)
\]

\[
+ \frac{1}{m_H^2\rho^6} \left( - \frac{128n^4}{45} + \frac{376n^3}{45} - 4017n^2 \right) - \frac{70}{n\rho^2} \left( \frac{56}{5} - \frac{8}{3} n \right) - \frac{2}{3} j^2 \hat{\rho}^4 ,
\]

with

\[
j = \frac{i}{2} \left( \chi^4 \chi - \chi^4 \chi \right) , \quad n = \chi^4 \chi ,
\]

We now focus on the quadratic part, by replacing \(\chi \rightarrow e^{im_H t} \sqrt{m_H} \), so that

\[
\mathcal{L}_{\text{quadratic}} = \frac{1}{2} \chi^4 \dot{\chi} + \frac{9i}{2\rho^2} \chi^4 \dot{\chi} - \frac{m_H^2}{2} \chi^4 \chi
\]

\[
+ \left( \frac{102}{5\rho^2} - \frac{37 + 12\sqrt{2}}{192} \right) \chi^4 \chi
\]

\[
+ \frac{78i}{5\rho^2} \chi^4 \tau^a \chi^a .
\]

Again, this can be interpreted as a system with two harmonic oscillators in a \(\rho\) dependent background magnetic field, coupled with each other by the spin-orbital term. In terms of (7), we have

\[
\mathcal{L} = \frac{1}{2} (\vec{x}_1^2 + \vec{x}_2^2) + \frac{9}{2\rho^2} (y_1 \dot{x}_1 - x_1 \dot{y}_1 + y_2 \dot{x}_2 - x_2 \dot{y}_2)
\]

\[
- \frac{m_H^2 + \Omega^2(\rho)}{2} (x_1^2 + x_2^2) + \text{Spin-Orbit} .
\]

with \(\vec{x}_1 = (x_1, y_1), \vec{x}_2 = (x_2, y_2)\) and

\[
\Omega^2(\rho) = -\frac{204}{5\rho^4} + \frac{37 + 12\sqrt{2}}{96} .
\]

We proceed to quantize (D6) in the Born-Oppenheimer approximation. We fix \(y_1\) and \(Z\) and first quantize \(\vec{x}_1\) and \(\vec{x}_2\). This is justified in the large \(m_H\) limit, where \(\chi\) is fast-moving at frequency \(m_H\), while the other degrees of freedom are slow moving with a typical frequency \(\omega_y = \frac{1}{\sqrt{6}} M_{KK}\).

We first look at the \(l = 0\) state where the spin-orbit coupling vanishes. In this case \(\vec{x}_1\) and \(\vec{x}_2\) decouple, and we have two identical harmonic oscillators in the background field

\[
\vec{A} = \omega_c(y, -x), \omega_c = \frac{9}{2\rho^2} .
\]

This is the famed Landau problem, with a spectrum

\[
E = (n_+ + \frac{1}{2})\Omega_+ + (n_- + \frac{1}{2})\Omega_- ,
\]

with

\[
\Omega_\pm = \sqrt{m_H^2 + \Omega^2 + \omega_c^2 \pm \omega_c} .
\]

At large \(m_H\), one has

\[
\Omega_\pm = m_H \pm \omega_c + \frac{\Omega^2(\rho) + \omega_c^2}{2m_H} + \mathcal{O} \left( \frac{1}{m_H^2} \right) .
\]

Clearly, the \(\pm\) solutions can be interpreted as particle/antiparticles. To leading order in \(\mathcal{O}(1/m_H)\), the two frequencies agrees with the case where \(\chi\) is quantized as fermion. Unfortunately,

\[
\Omega^2(\rho) + \omega_c^2 = \frac{81}{4\rho^4} - \frac{204}{5\rho^4} < 0 ,
\]

indicating an instability at the quadratic order. We conclude, that the screening effect in the Coulomb part should not be expanded, as it causes a charge instability.

[1] J. M. Maldacena, *Int. J. Theor. Phys.* **38**, 1113 (1999), arXiv:hep-th/9711200.

[2] J. Erlich, E. Katz, D. T. Son, and M. A. Stephanov, *Phys. Rev. Lett.* **95**, 261602 (2005), arXiv:hep-ph/0501128.

[3] T. Sakai and S. Sugimoto, *Prog. Theor. Phys.* **113**, 843 (2005), arXiv:hep-th/0412141.

[4] T. Fujiwara, T. Kugo, H. Terao, S. Uehara, and K. Yamawaki, *Prog. Theor. Phys.* **73**, 926 (1985).
[5] Y. Liu and I. Zahed, Phys. Rev. D 95, 056022 (2017), arXiv:1611.03757 [hep-ph].

[6] Y. Liu and I. Zahed, Phys. Rev. D 95, 116012 (2017), arXiv:1704.03412 [hep-ph].

[7] Y. Liu and I. Zahed, Phys. Rev. D 96, 056027 (2017), arXiv:1705.01397 [hep-ph].

[8] S.-w. Li, Phys. Rev. D 96, 106018 (2017), arXiv:1707.06439 [hep-th].

[9] D. Fujii and A. Hosaka, Phys. Rev. D 101, 126008 (2020), arXiv:2003.13415 [hep-ph].

[10] H. Hata, T. Sakai, S. Sugimoto, and S. Yamato, Prog. Theor. Phys. 117, 1157 (2007), arXiv:hep-th/0701280.

[11] K. Hashimoto, T. Sakai, and S. Sugimoto, Prog. Theor. Phys. 120, 1093 (2008), arXiv:0806.3122 [hep-th].

[12] K.-Y. Kim and I. Zahed, JHEP 09, 007 (2008), arXiv:0807.0033 [hep-th].

[13] H. Hata and M. Murata, Prog. Theor. Phys. 119, 461 (2008), arXiv:0710.2579 [hep-th].

[14] K. Hashimoto, N. Iizuka, T. Ishii, and D. Kadoh, Phys. Lett. B 691, 65 (2010), arXiv:0910.1179 [hep-th].

[15] P. H. C. Lau and S. Sugimoto, Phys. Rev. D 95, 126007 (2017), arXiv:1612.09503 [hep-th].

[16] I. Zahed and G. E. Brown, Phys. Rept. 142, 1 (1986).

[17] M. Rho, D. O. Riska, and N. N. Scoccola, Z. Phys. A 341, 343 (1992).

[18] J.-J. Wu, R. Molina, E. Oset, and B. S. Zou, Phys. Rev. Lett. 105, 232001 (2010), arXiv:1007.0573 [nucl-th].

[19] M. Karlmer and J. L. Rosner, Phys. Rev. Lett. 115, 122001 (2015), arXiv:1506.06386 [hep-ph].

[20] Y.-H. Lin and B.-S. Zou, Phys. Rev. D 100, 056005 (2019), arXiv:1908.05309 [hep-ph].

[21] H. G. Dosch, G. F. de Teramond, and S. J. Brodsky, Phys. Rev. D 92, 074010 (2015), arXiv:1504.05112 [hep-ph].

[22] J. Sonnenschein and D. Weisssman, Eur. Phys. J. C 79, 326 (2019), arXiv:1812.01619 [hep-ph].

[23] E. V. Shuryak, Nucl. Phys. B 198, 83 (1982).

[24] N. Isgur and M. B. Wise, Phys. Rev. Lett. 66, 1130 (1991).

[25] M. A. Nowak, M. Rho, and I. Zahed, Phys. Rev. D 48, 4370 (1993), arXiv:hep-ph/9209272.

[26] W. A. Bardeen and C. T. Hill, Phys. Rev. D 49, 409 (1994), arXiv:hep-ph/9304265.

[27] M. A. Nowak, M. Praszalowicz, M. Sadzikowski, and J. Wasiluk, Phys. Rev. D 70, 031503 (2004), arXiv:hep-ph/0403184.

[28] B. Aubert et al. (BaBar), Phys. Rev. Lett. 90, 242001 (2003), arXiv:hep-ex/0304021.

[29] D. Besson et al. (CLEO), Phys. Rev. D 68, 032002 (2003), [Erratum: Phys.Rev.D 75, 119908 (2007)], arXiv:hep-ex/0305100.

[30] Y. Liu, M. A. Nowak, and I. Zahed, Phys. Rev. D 104, 114021 (2021), arXiv:2108.04334 [hep-ph].

[31] C. G. Callan, Jr. and I. R. Klebanov, Nucl. Phys. B 262, 365 (1985).

[32] C. G. Callan, Jr., K. Hornbostel, and I. R. Klebanov, Phys. Lett. B 202, 269 (1988).

[33] Y. Liu, M. A. Nowak, and I. Zahed, Phys. Rev. D 100, 126023 (2019), arXiv:1904.05189 [hep-ph].

[34] Y. Liu, M. A. Nowak, and I. Zahed, Phys. Rev. D 104, 114022 (2021), arXiv:2108.07074 [hep-ph].

[35] R. L. Jaffe, Conf. Proc. C 760705, 455 (1976).

[36] M. Praszalowicz, in Workshop on Skyrmions and Anomalies (1987).

[37] D. Diakonov, V. Petrov, and M. V. Polyakov, Z. Phys. A 359, 305 (1997), arXiv:hep-ph/9703373.

[38] M. Praszalowicz, Phys. Lett. B 575, 234 (2003), arXiv:hep-ph/0308114.