Fulde–Ferrell superfluids in spinless ultracold Fermi gases

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Abstract

The Fulde–Ferrell (FF) superfluid phase, in which fermions form finite momentum Cooper pairings, is well studied in spin-singlet superfluids in past decades. Different from previous works that engineer the FF state in spinful cold atoms, we show that the FF state can emerge in spinless Fermi gases confined in optical lattice associated with nearest-neighbor interactions. The mechanism of the spinless FF state relies on the split Fermi surfaces by tuning the chemistry potential, which naturally gives rise to finite momentum Cooper pairings. The phase transition is accompanied by changed Chern numbers, in which, different from the conventional picture, the band gap does not close. By beyond-mean-field calculations, we find the finite momentum pairing is more robust, yielding the system promising for maintaining the FF state at finite temperature. Finally we present the possible realization and detection scheme of the spinless FF state.

1. Introduction

Cold atoms in optical lattices provide an ideal experimental plateau for quantum simulation of the quantum many-body system. Compared with conventional solid-state systems, it possesses remarkable advantages such as the well controllability and tunability of the system parameters and free of disorder [1]. Furthermore, by utilizing recently developed technique with laser-assisted tunneling [2–5] or periodic-driven external fields [6–9], cold atoms show promising potential in synthesizing exotic optical lattice models and artificial gauge fields. It paves the way to quantum simulate various condensed-matter systems, and search possible unconventional phases that are not ever detected in solid-state systems. Among them, the Fulde–Ferrell (FF) phase [10, 11] attracts tremendous research interest.

The FF state is an unconventional superfluid state with spatially oscillating order parameters. It originates from Cooper pairings with finite center-of-mass momentum, which is the prominent feature distinguished from Bardeen–Cooper–Schrieffer (BCS) state. The FF state provides a central concept for understanding exotic phenomena in different physics branches [12, 13]. It is predicted to emerge in systems with large spin polarization [14–17]. Due to the stringent conditions on materials, the evidence of the FF state in condensed matters is still pending. On the other hand, in the past few years, it opens an alternative way in synthesizing the FF superfluids in cold atoms, by taking advantages of anisotropic optical lattices [18], spin-dependent optical lattices [19], spin–orbital couplings (SOC) [20–25], periodic-driven optical lattices [26], multi-orbital interactions [27], the optical control of Feshbach resonances [28], and instantaneous spin imbalance via radio-frequency fields [29]. The series of investigations in cold atoms reveals that the FF superfluids can originate from the distortion of Fermi surfaces instead of large spin imbalance [15–17, 29], which is expected to facilitate its observation in cold-atom experiments. As the results, cold atoms exhibit a potential candidate to realize and study the FF superfluids.

So far in searching FF superfluids in cold atoms, the earlier advances [18–28] bear similarities that they all focus on a spinful system. In these systems, the contact interaction between opposite pseudo-spin atoms plays the key role for the superfluid phases. In cold-atom experiments, the interaction is induced via Feshbach resonance, and conventionally spatial homogeneous. In order to engineer FF superfluids, the earlier works...
design SOC via current laser techniques to break the homogeneity of the band dispersion, and thus a distorted Fermi surface is engineered. However, the idea based on SOC is no longer valid for a spinless system. For that sake, an interesting question motivates us whether it is possible to explore FF superfluids in a spinless system.

In this paper, different from previous works that focus on spinful systems, we show that FF superfluids can emerge in spinless ultracold Fermi gases trapped in a two-dimensional (2D) optical lattice. The paper is organized as follows. In section 2, we present the model Hamiltonian and the mean-field framework. In section 3 we show the phase diagram and topological features of the system. The stability of the emergent FF state against fluctuations is estimated by the Berezinskii–Kosterlitz–Thouless (BKT) transition temperature [30–32], yielding the system promising for maintaining the FF state at finite temperature. In section 4, we give a possible scheme to detect the FF state via the pair correlation, and discuss the experimental realization of the spinless lattice model. In section 5, we summarize the work.

2. Model Hamiltonian

We start with spinless Fermi gases trapped in a 2D square lattice. The lattice model is illustrated in figure 1(a), and can be described by the following Hubbard Hamiltonian

$$H = -t_0 \sum_{i,j} c_i^\dagger c_j - t \sum_{i,j} n_i c_j - \mu \sum_i n_i - U \sum_{i,j} n_i n_j.$$  (1)

Here $c_i^\dagger$ ($c_i$) are the fermionic creation (annihilation) operators on site $i$. The summations $\sum_i$ and $\sum_j$ range over all nearest-neighbor (NN) and all next-nearest-neighbor (NNN) sites, respectively. The corresponding tunneling amplitudes are $t_0$ and $t$. Hereafter we set $\hbar$ as the energy unit. $\mu$ is the chemistry potential. $U$ characterizes the attractive interaction strength. A candidate system described by this model is the fully-spin-polarized Fermi gas with dipole–dipole interactions. Due to the Pauli exclusion, each site in a spinless lattice system is occupied by a maximum of one fermion. Therefore the onsite interaction that gives rise to $s$-wave Cooper pairings is prohibited, while the long-range one is still valid. In the tight binding approximation, our focus here is the NN interaction, which will lead to superfluid order parameters with $p$-wave symmetry [33, 34].

We firstly investigate the single-particle properties. The Hamiltonian without interactions in the momentum space is given by

$$H_0(k) \equiv \xi_k = -\mu - 2t_0 \cos(k_x a) - 2t_0 \cos(k_y a) - 2t \cos(k_x a + k_y a) - 2t \cos(k_x a - k_y a).$$  (2)

Figures 1(b) and (c) show the band structures of the single-particle system. We find that the band hosts five valleys in the center and corners of the first Brillouin zone (BZ) when $t_0 < 2t$, by contrast, only one valley is present when $t_0 > 2t$. This implies that, by changing $\mu$, the Fermi surface can be split from a single enclosed curve into disjoint lines if $t_0 < 2t$. It reveals a possibility for rich Cooper pairing types, inspiring us to search the possible FF state. For simplicity, we set $t_0 = t/2$ in the calculations of the whole paper.

Then we study the main features of the interacting system at zero temperature. In order to capture qualitative understanding of the interacting Fermi gas, we take the mean-field Bogoliubov-de Gennes (BdG) approach to study the superfluid phases. The order parameter can be introduced by

$$-U \eta_j n_j = \Delta_c c_j^\dagger + \Delta^*_c c_j - |\Delta|^2/U.$$  

Thus the Hamiltonian (1) of a $L \times L$ lattice can be diagonalized by employing the Bogoliubov transformation $c_i = \sum_\eta (u_\eta^i \gamma_\eta + v_\eta^i \gamma_\eta^\dagger)$. Here $u_\eta = (u_\eta^0, \ldots, u_\eta^{L-1})^T$ and $v_\eta = (v_\eta^0, \ldots, v_\eta^{L-1})^T$ satisfy the following BdG equations

![Figure 1.](image-url)
In the high filling regime, we see that the increase of $\mu$ drives a transition from the BCS to FF states. It yields in the large-$\mu$ regime, the finite momentum pairing dominates over the zero-momentum one.

3.2. Topological phase transition

The phase diagram shown in figure 2(a) is accompanied by the topological transition. It has been well studied that the chiral $p$-wave superfluids can exhibit features of a Chern insulator, and harbor the topological edge
states protected by the particle-hole symmetry \[39-42\]. The topological phase transition can be characterized by the Chern number \(\mathcal{C}\) defined by

\[
\mathcal{C} = \frac{1}{2\pi} \int \! dk \, \text{Tr} \, F_{\sigma}(k),
\]

where the gauge field \(F_{\sigma}(k) = \partial_k A_\sigma(k) - \partial_\sigma A_k(k)\), the Berry connection \(A_\mu(k) = -i(\alpha_k |\partial_\mu| |\alpha_k\rangle\rangle\), and \(|\alpha_k\rangle = (|\alpha_{1k}\rangle, \ldots, |\alpha_{nk}\rangle, \ldots)^T\) with \(|\alpha_{nk}\rangle\) as the base of the \(n\)th occupied band. \(|\alpha_{nk}\rangle\) can be obtained by the BdG Hamiltonian, which, in the base \(\Psi = (c_{Q/2+k}, c_{Q/2-k}^\dagger)^T\), is expressed as

\[
H_{\text{BdG}}(k) = \begin{pmatrix}
\xi_{Q/2+k} & \Delta k e^{i\varphi/2} \\
\Delta k e^{-i\varphi/2} & -\xi_{Q/2-k}
\end{pmatrix},
\]

where \(\Delta_k = i2\Delta \sin(k_x a) + 2\Delta \sin(k_y a)\). For simplicity, we have denoted \(\varphi = |Q_{x+y}| a = \pi l\) with \(l = 0(1)\) for the BCS(FF) state, respectively.

In figure 4 we plot the order parameter \(\Delta\), the band gap \(\Gamma\), and the Chern number \(\mathcal{C}\) as a function of \(n\). Results are obtained by setting \(U = 6.0t\).

Figure 3. Illustration of the Fermi surfaces in the 1st BZ at (a) \(\mu = -2.0t\) and (b) \(\mu = 3.0t\). Fermions occupy the gray regions enclosed by the Fermi surfaces. In the low filling regime (a), the system hosts a single Fermi surface (green solid boundaries of the gray region), and the BCS-type pairing (red dashed-dotted arrows) is dominant. In the high filling regime (b), the Fermi surfaces are split into four disjoint lines in the 1st BZ. It leads to a competition between the BCS-type pairing, and the FF-type one (blue dashed arrows) with a finite momentum \(Q\) (black solid arrows).

Figure 4. The band gap \(\Gamma\), the order parameter \(\Delta\), and the Chern number \(\mathcal{C}\) as a function of \(n\). Results are obtained by setting \(U = 6.0t\).
of the superfluid order $\Delta$. The phase transition also undergoes a change of $\xi$, however, the gap $\Gamma$ is still open during the transition. This is because the transition is of first order, which is revealed by the discontinuous behavior of $\Delta$ with respect to $\mu$. The evolution from the topologically nontrivial BCS ($\zeta = 0$) to topologically trivial FF phase ($\zeta = 0$) is thus not an adiabatic continuum deformation.

### 3.3. Stability of FF superfluids

At zero temperature, in figure 5(a), we calculate the difference of the thermodynamic potential $\Omega$ between the FF and possible BCS states (see appendix A). We can see that the FF state hosts lower energy than the possible BCS state. It reveals the dominance of the finite momentum pairing over the zero-momentum one in the high filling regime, thus the FF state is the ground state.

At finite temperature, the long-range superfluid order in a 2D system is destroyed by its phase fluctuations. Instead, states with quasi-long-range order, which are characterized by the vortex–antivortex pairs [44–47], drive a BKT-type phase transition. The superfluids are formed below the critical temperature, which is known as the BKT transition temperature $T_{\text{BKT}}$. When the temperature exceeds $T_{\text{BKT}}$, the ground state of the system turns to the pseudo-gap phase, in which the superfluid components are destroyed even though the pairings $\{\Delta\}$ do not vanish. The stability of the FF superfluids at finite temperature can therefore be estimated by $T_{\text{BKT}}$.

In order to study the phase fluctuations, we impose a phase $\theta$ in the superfluid order parameters $\Delta = \Delta e^{i\theta}$. After making the standard Hubbard–Stratonovich transformation and integrating out the fermion fields $\{c, \bar{c}\}$ (see appendix A), the effective action can be expressed as $S_{\text{eff}} = S_0 + S_\theta$. $S_0$ describes the mean-field action independent from $\theta$. $S_\theta$ characterizes the $\theta$-dependent action originated from the phase fluctuations. Its form is written as

$$S_\theta = \frac{i}{2} \int d\tau \sum_{\mu,\nu=x,y} [i\mu \partial_\tau \theta_\mu \bar{\theta}_\nu + i\nu \partial_\tau \bar{\theta}_\mu \theta_\nu] + \frac{1}{2} \mu_i \partial_\tau \theta_i \bar{\theta}_i \partial_\tau \theta_i \bar{\theta}_i \] + P \left( \partial_\tau \theta \right)^2 - i\Lambda \partial_\tau \theta \right] \right].$$

The detailed derivations of $I_{\mu\nu}$, $P$, and $A$ are presented in appendix B. The BKT transition temperature is then determined by [47]

$$T_{\text{BKT}} = \frac{\pi}{2} \sqrt{\frac{J_{xx}}{J_{yy}}}. \tag{6}$$

In figure 5(b) we plot $T_{\text{BKT}}$ of the FF and possible BCS states by changing the interaction strength $U$. We see that the FF superfluids still exist and remain robust against the phase fluctuations below $T_{\text{BKT}}$, yielding the system promising for maintaining the FF superfluids at finite temperature. In the BCS regime, $T_{\text{BKT}}$ increases monotonically to the interaction strength $U$. However, in the BEC regime, $T_{\text{BKT}}$ approaches a constant independent from $U$. This is because the system behaves like a condensation of tightly-bound bosonic dimers due to the strong attractive interaction [44]. Since $T_{\text{BKT}}$ of the FF state is higher than the possible BCS state, it implies the finite momentum pairing can enhance the superfluids robust against the fluctuations. We gives the finite temperature phase diagram in figure 5(c). Compared with the zero temperature one (see figure 2(a)), it displays the BCS phase region shrinks obviously, while the FF phase region changes slightly.

### 4. Discussions

#### 4.1. Pair correlation

The signature of the FF superfluids can be detected by the pair correlations [48]. At the critical transition point, the pair correlations will exhibit a discontinuous behavior by tuning $\mu$. In figure 4(a), we have known that the
order parameter $\Delta$ hosts a discontinuous evolution during the transition from the BCS to FF states. This discontinuous behavior will influence on the pair correlations [48].

For a spinless Fermi gas, the pair wave function between the $i$th and $j$th sites is expressed as

$$\Delta(r_i, r_j) = (c(r_i)c(r_j)).$$

By making the following transformation

$$r_c = (r_i + r_j)/2, \quad \delta r = r_i - r_j,$$

the pair wave function $\Delta(r_c, \delta r)$ can be rewritten as $\Delta(r_c, r_c + \delta r)$ in the center-of-mass frame. The mean pair correlation function can thus be obtained by [48, 49]

$$P = \frac{1}{N_L} \sum_{r_c, \delta r} |\Delta(r_c, \delta r)|^2,$$

where $N_L = L \times L$ is the total number of the 2D lattice sites.

In figure 6, we plot $P$ with respect to $\mu$ and see that $P$ behaves a sudden jump at the transition critical points. It implies a possible way to detect the phase transition from the BCS to FF states via current experimental techniques [17].

4.2. Experimental realization

The $p$-wave superfluids in a 2D optical lattice can be readily designed by various proposals in cold atoms. The NN and NNN tunnelings can be constructed by the laser-assisted tunneling protocol [50]. Here we give two possible schemes. The first scheme is that we can introduce a magnetic gradient field to generate adjacent-site detuning $\delta_{ij} = j_x \delta_x + j_y \delta_y$. Here $j = (j_x, j_y)$ denotes the site index in the 2D lattice. Then we can implement Raman transitions with detuning $\delta_{xy}$ to generate the NN tunneling $t_0$, and ones with detuning $\delta_x \pm \delta_y$ to generate the NNN tunneling $t$. The tunneling amplitudes can be changed separately. An alternative scheme takes advantages of the magnetic gradient field with checkerboard structure $\delta_{ij} = (-1)^i+j \cdot \delta$. The NN tunneling $t_0$ is prohibited since the adjacent-site detuning is $2\delta$, making the NNN tunneling $t$ prominent because of no detuning between two NNN sites. Then $t_0$ is reconstructed by a Raman transition with detuning $2\delta$.

The spin-triplet interaction in the spinless system can be introduced directly via $p$-wave Feshbach resonances [51, 52]. Several works provide alternative ways for synthesizing $p$-wave superfluids by implementing artificially induced molecules [53], higher orbital atoms [54], or Bose–Fermi mixture [55].

4.3. Large-$U$ limit

In this paper, we focus on the BCS–BEC crossover with $\Delta \sim t$ because the mean-field method can give a good picture within the interaction strength we choose [56]. However, when $U$ is extremely large with $\Delta \gg t$, fermions are tightly bounded into bosonic molecules [57]. The system will exhibit a hard-core bosonic gas with tiny tunneling and strong long-range attractive interaction, yielding a Mott insulator [58] whose dispersion is nearly a flat band. At this time, it should be noted that the mean-field method is no longer appropriate to describe the system.

5. Conclusions

In summary, we investigate the emergent FF superfluids in a spinless Fermi gas. The novelty of our work is highlighted as follows: (i) the FF state is supported by the split Fermi surfaces other than the spin imbalance.
Different from the spinful case, the Cooper pairing momentum does not depend on external polarized fields. (ii) The order parameters of the FF state stems from the p-wave symmetric pairing, and forms a checkerboard spatial structure. (iii) The topological phase transition between the BCS and FF states is of first order, thus can occur even without gap closing. (iv) By employing the beyond-mean-field analysis, we find the finite momentum pairing is more robust against phase fluctuations than the zero-momentum one. (v) The lattice model and the associated FF state are readily realized and detected via current experimental techniques in cold atoms. These features above distinguish our work from the conventional pictures for the mechanics and properties of the FF state, showing the lattice model as a promising candidate system for evidencing and investigating the FF state in cold atoms.

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Appendix A. Mean-field approach

The Hamiltonian of the 2D model in section 2 can be expressed as

$$H(r) = c^\dagger(r) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(r) \right] c(r) - U c^\dagger(r) c^\dagger(r + \delta r) c(r + \delta r) c(r).$$  \hspace{1cm} (A.1)

Here $V_{\text{trap}}(r) = V_0 \cos^2(x/a) + V_0 \cos^2(y/a)$ with the lattice constant $a$ is the lattice trap potential. Hereafter we set $\hbar = 1$ and the Boltzmann constant $k_B = 1$. The partition function is given by

$$Z = \int D[c, c^\dagger] e^{-S_{\text{full}}[c, c^\dagger]}$$ \hspace{1cm} (A.2)

with the effective action formulated as

$$S_{\text{eff}}[c, c^\dagger] = \int \! d\tau dx \left[ c^\dagger(r, \tau) \partial_\tau c(r, \tau) - H(r) \right].$$ \hspace{1cm} (A.3)

Here $\tau$ is the imaginary time. We employ the standard Hubbard–Stratonovich transformation with the following pairing fields

$$\Delta_x(r) = U \langle c(r) c(r + \delta x) \rangle = \Delta e^{iQ \cdot \vec{r}},$$ \hspace{1cm} (A.4)

$$\Delta_y(r) = U \langle c(r) c(r + \delta y) \rangle = \Delta e^{iQ \cdot \vec{r}},$$ \hspace{1cm} (A.5)

where

$$Q = (\pi/a, \pi/a), \quad \vec{r} = r + \delta \nu/2, \quad (\nu = x/y).$$ \hspace{1cm} (A.6)

Integrating out the fermion fields $[c, c^\dagger]$, we obtain the effective action expressed as

$$S_{\text{eff}}[\Delta_x, \Delta_y] = \int \! d\tau dr \left[ \epsilon_0(r) - \frac{1}{2} \text{Tr} \ln G^{-1}(r, \tau) \right].$$ \hspace{1cm} (A.7)

Here $G^{-1}(r, \tau) = -\partial_\tau - H_{\text{BdG}}(r)$. $\epsilon_0 = |\Delta_x(r)|^2/U + |\Delta_y(r)|^2/U - \mu/2$. $G^{-1}(r, \tau)$ is the inverse Green’s function.

We make the Fourier transformation from the $(r, \tau)$ space to the $(k, i\omega_n)$ space. Here $\omega = (2n+1)\pi/\beta$ ($n \in \mathbb{Z}$) is the fermionic Matsubara frequency, and $\beta \equiv 1/T$ with the temperature $T$. By choosing the base $\Psi_k = (\epsilon_{Q/2+k}, \epsilon_{Q/2-k})^T$, the BdG Hamiltonian $H_{\text{BdG}}$ in the tight binding approximation with is expressed as

$$H_{\text{BdG}}(k) = \begin{pmatrix} \xi_{Q/2+k} & \Delta_k e^{i\varphi/2} \\ \Delta_k e^{-i\varphi/2} & -\xi_{Q/2-k} \end{pmatrix}. \hspace{1cm} (A.8)$$

The thermodynamical potential is written as

$$\Omega = \epsilon_0 - \frac{1}{2\beta} \sum_{k, \omega_n} \ln \left[ -\beta (\omega_n - E_k^\pm) \right] = \epsilon_0 - \frac{1}{2\beta} \sum_{k, \omega_n} \ln \left( 1 + e^{-\beta E_k^\pm} \right), \hspace{1cm} (A.9)$$

where $E_k^\pm$ is the $\alpha$th eigenvalue of the Hamiltonian (A.8), and $\epsilon_0 = \sum_k (|\Delta_x|^2/U + |\Delta_y|^2/U + \xi_k/2)$. The filling factor $n$ can be obtained by
Appendix B. Phase fluctuation

In order to calculate the phase fluctuation in the 2D system, we impose a variable phase in the order parameter $\Delta \to \Delta e^{i\theta}$ in the Hamiltonian (A.6). By making the following unitary transformation

$$U = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix},$$

the inverse Green’s function $G^{-1}(r, \tau, \theta)$ in the new representation can be divided into two items

$$G^{-1}(r, \tau, \theta) = U^\dagger[-\partial_\tau - H_{\text{BdG}}(r, \theta)]U = G^{-1}(r, \tau) - \Sigma(r, \tau, \theta).$$

The first item $G$ is the original $\theta$-independent Green’s function. Its form in the momentum space is given by

$$G^{-1}(k, \omega_n) = i\omega_n - H_{\text{BdG}}(k).$$

The second item $\Sigma$ is the $\theta$-dependent self energy expressed as

$$\Sigma = [i\partial\tau \theta/2 + (\nabla\theta)^2/8 m] \sigma_z - (i\nabla^2 \theta/4 m + i\nabla \theta \cdot \nabla/2 m) \mathbb{I}.$$ 

Here $\sigma_i (i = x, y, z)$ are Pauli matrices, and $\mathbb{I}$ is the $2 \times 2$ identical matrix. The effective action (A.7) now becomes

$$S_{\text{eff}} = \int d\tau d\mathbf{r} \left[ \epsilon_0(r) - \frac{1}{2} \text{Tr} \ln G^{-1} \right] \equiv S_0 + S_{\text{fl}}$$

with

$$S_0 = \int d\tau d\mathbf{r} \left[ \epsilon_0(r) - \frac{1}{2} \text{Tr} \ln G^{-1} \right],$$

$$S_{\text{fl}} = -\frac{1}{2} \int d\tau d\mathbf{r} \ \text{Tr} \ln(1 - G\Sigma).$$

We expand $S_0$ to the second order and obtain

$$S_0 \approx \frac{1}{2} \int d\tau d\mathbf{r} \left[ \text{Tr}(G\Sigma) + \frac{1}{2} \text{Tr}(G\Sigma G\Sigma) \right]$$

$$= \frac{1}{2} \int d\tau d\mathbf{r} \left[ \sum_{\mu, \nu = x, y} \left( J_{\mu\nu} \partial_\mu \theta \partial_\nu \theta + iJ_{\tau\nu} \partial_\nu \theta \partial_\tau \theta \right) + P(\partial_\tau \theta)^2 - iA \partial_\tau \theta \right],$$

where

$$J_{\mu\nu} = \frac{n}{4m} + \frac{\beta}{8} \sum_{k, \alpha} \frac{k_\alpha^2}{m^2} f(E^0_k) [f(E^0_k) - 1], \quad J_{\tau\nu} = 0,$$

$$J_{\tau\nu} = \frac{1}{(2\pi)^2} \sum_{k, \alpha} \frac{k_\alpha}{m} \text{Tr} \left[ G(k, \omega_n) G(k, \omega_n) \sigma_z \right],$$

$$P = \frac{1}{(2\pi)^2} \sum_{k, \alpha} \text{Tr} \left[ G(k, \omega_n) G(k, \omega_n) \sigma_z \right],$$

$$A = n,$$

and $f(E) = \frac{1}{e^{E/kT} + 1}$ is the Fermi–Dirac distribution. The BKT transition temperature $T_{\text{BKT}}$ is determined by self-consistently solving the following equations

$$T_{\text{BKT}} = \frac{\pi}{2} \sqrt{f(E_k) f(E_k) - 1} \sim 0$$

Therefore $J_{\tau\tau} = J_{\nu\nu} \approx \frac{n}{4m}$ and hence

$$T_{\text{BKT}} \approx \frac{\pi}{8m} n.$$ 

This is different from the three-dimensional (3D) case, in which the superfluid transition temperature $T_c \sim n^{1/3}$ [63].

It should be noted that in the lattice calculations at nonzero temperature, since we have assume $\hbar = k_B = m = 1$, it is no longer appropriate to use the tunneling amplitude $\tau$ as the temperature unit. Instead,
we use the Fermi temperature $T_F$ as the temperature unit. In particular, for the calculation of figure 5, we have set $t = 0.08 E_R$ with the lattice recoil energy $E_R = \hbar^2 k_f^2 / 2m$ ($k_f = \pi / a$).

$T_F$ can be defined as following. In the 2D system, the Fermi wave vector $k_F$ is obtained by

$$
n = \frac{N}{S} = \frac{2 \times \pi k_F^2}{(2\pi)^2}.
$$

(B.15)

The Fermi temperature $T_F$ is given by

$$
T_F = \frac{k_F^2}{2m}.
$$

(B.16)

Combining equations (B.15) and (B.16), we obtain the final expression of the Fermi temperature:

$$
T_F = \pi n / m.
$$

(B.17)

Inserting equation (B.17) into equation (B.14), we can obtain the relation between $T_{\text{BKT}}$ in the BEC regime and $T_F$:

$$
T_{\text{BKT}} \approx T_F / 8.
$$

(B.18)

By contrast, the ratio $T_c / T_F \approx 0.218$ for the 3D case [63]. The practical temperature in cold-atom experiments is typically of order $10^{-2} T_F$ [64]. Therefore, it is expected that the FF superfluids, or the nonzero pairing momentum state, can exist in real experiments.

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