Analysis of the dynamics of a switched circuit from the bifurcation theory

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Abstract. This document shows an analysis of the stability of a Buck type power converter, which, being a switched electronic circuit, has complex dynamics. The analysis consists of determining the influence on its stability of the variations of each of its parameters and determining the points where these changes occur, known as the bifurcation point. This analysis allows investigating these types of problems from the bifurcation theory allowing to understand the stability and operation of the converters.

1. Introduction

Electric power systems are dynamic systems that have great complexity to be analyzed dynamically and that integrate elements such as generators, motors, non-linear loads, transmission lines, transformers, power converters, etc. Each of them with its own model in state space. The great interest of studying the dynamics of electrical systems is to be able to determine the future behavior (to predict) of the system against variations in its states (disturbances), summarizing this objective in stability and control analysis.

This document will analyze the dynamics that describe a specific power converter, called Buck converter; this interest is due to the fact that it is currently impossible to think of an electrical power system without the inclusion of these electronic devices, which have highly complex dynamics because they are characterized by being switched electronic circuits, that is, their dynamics have discontinuities. Another very important objective for the analysis of qualitative stability is to observe the variations of the system against the changes of its parameters, since in many systems such stability will depend strictly on the selected parameters. If the system has variations in its stability against these parameters, it is said that there is a fork for the system given a parameter. For this reason, the analysis that will be carried out in this document will focus on the variations of the dynamics of the Buck converter versus its constructive parameters, which will be described in the preceding sections; this initiative has been studied in recent years by different authors in the area of engineering and mathematics, such as [1-3].

The objective of the document will be to verify through simulations in Matlab the influence of the bifurcation theory on the dynamics of the Buck converter. For this we have organized the document as follows: a second section where the mathematical concepts that will be used in In the preceding sections, in section three the mathematical model in state space of the Buck converter is described where the discontinuous dynamics is shown, finally in section four the results generated by varying each of the converter parameters are illustrated.
2. Mathematical concepts
The mathematical model of a dynamic system can be represented in the state space or a set of first-order differential equations as follows Equation (1).

\[ \dot{x} = f(x, \mu), \]  

where \( x \in \mathbb{R}^n \), being \( x \) the vector of system states \( y \in \mathbb{R}^k \) with \( \mu \) as the status parameter. The solution or trajectory of Equation (1) \( x(t) \) will depend on the initial conditions defined. For nonlinear dynamic systems, these are mostly calculated by computer integration techniques [4].

2.1. Balance points
The equilibrium points for the system Equation (2) are given by \( \dot{x} = 0 \), that is Equation (2).

\[ f(x, \mu) = 0. \]  

For a given value of \( \mu \), the solution of Equation (2) will be an equilibrium point for the system Equation (1). Once the equilibrium points have been defined, it is necessary to study the classification of their stability through linearization around them, as shown. Taking \( x_0 \) and \( \mu_0 \) as equilibrium points, the system Equation (2) can be rewritten as Equation (3).

\[ \dot{x} = Ax, \]  

where \( A \) is the Jacobian matrix (Equation (4)) of the defined system Equation (2),

\[ A = \left[ a_{ij} \right] = \left[ \frac{\partial f_i}{\partial x_j} \right] \text{ with } x = x_0. \]  

The stability of each of the equilibrium points of the system Equation (2), will depend on the eigenvalues of matrix \( A \); if all eigenvalues have a negative real part, the equilibrium point is stable known as attractor, in addition a function of Lyapunov can be associated for each attractor point [5].

2.2. Hyperbolicity and structural stability
If none of the eigenvalues of matrix \( A \) have zero real part, it is said that the point \( x_0 \) is a hyperbolic point; two consequences of a hyperbolic equilibrium point are the following:

If matrix \( A \) does not have zero eigenvalues, then \( x_0 \) is a simple transverse zero of Equation (2). Therefore, by the implicit function theorem, the existence of a soft function \( x(\mu) \) is guaranteed with \( x(\mu_0) = x_0 \) which shows the variation of \( x_0 \) when the \( \mu \) parameter is varied. In addition, there is no variation in the number of equilibrium points when \( \mu = \mu_0 \). The qualitative structure of the phase diagram for the nonlinear system is the same as that of the linearized system, this because of the Hartman-Grodman theorem [6,7].

2.3. Bifurcation theory
Bifurcation theory is a branch of applied mathematics where its main interest is the analysis of Equation (2), where \( x \) is an equilibrium solution and \( \mu \) is a scalar parameter, that is, determining what the variation of \( x \) is \( (\mu) \) for when \( \mu \) varies. The \( \mu \) parameter is called a fork parameter [8].

Chair-node bifurcation: for one dimension the general shape of this type of fork must be given by Equation (5).

\[ \dot{x} = \mu \pm x^2. \]  

For a second order system, the general form must be given by Equation (6).
\[
\begin{align*}
\dot{x} &= \mu \pm x^2 \\
\dot{y} &= -y
\end{align*}
\]  \quad (6)

Fork bifurcation: for first order systems by Equation (7).

\[
\dot{x} = \mu x \pm x^3
\]  \quad (7)

For second order systems by Equation (8).

\[
\begin{align*}
\dot{x} &= \mu x \pm x^3 \\
\dot{y} &= -y
\end{align*}
\]  \quad (8)

Trans-critical bifurcation: for first order systems by Equation (9).

\[
\dot{x} = \mu x - x^2
\]  \quad (9)

For second order systems by Equation (10).

\[
\begin{align*}
\dot{x} &= \mu x - x^2 \\
\dot{y} &= -y
\end{align*}
\]  \quad (10)

3. Buck converter dynamic model

Although the Buck converter is one of the simplest power electronics converters, it is also one of the most used in this field.

For the analysis of the circuit of Figure 1 two instants of time must be considered, the first one when the switch S is closed, the equivalent circuit of Figure 2 is generated, which has the following in state spaces as the model system [9], Equation (11).

\[
\begin{align*}
L \frac{di}{dt} &= V_{in} - v \\
C \frac{dv}{dt} &= i - \frac{v}{R}
\end{align*}
\]  \quad (11)

The Buck converter model can be summarized in a single Equation (5), where \( u = 1 \) indicates that the switch is closed and \( u = 0 \) that the switch is open. These systems are also called Filippov Systems (systems that have a discontinuous vector field). Equation (13).
\[
\dot{V}_i = \begin{bmatrix}
\frac{1}{RC} & 1 \\
-\frac{1}{LC} & 0
\end{bmatrix} V + \begin{bmatrix}
0 \\
\frac{V_{in}}{L}
\end{bmatrix} u.
\]  
(13)

A common control law is through state feedback, so \( V_{con} = v(t) + Zi(t) \) which should be compared with a defined voltage \( v_{low} \) so that when \( v(t) + Zi(t) > v_{low} \) so \( u = 0 \) and when \( v(t) + Zi(t) < v_{low} \), \( u = 1 \).

\[
F = \lambda f_1 + (1 - \lambda) f_2, \quad 0 \leq \lambda = \frac{H_s f_2}{H_s (f_2 - f_1)} \leq 1,
\]  
(14)

where \( F \) is the linear combination of the fields \( f_1 \) and \( f_2 \), \( H \) is the function of the switching region and \( H_s \) is the variation of the switching region with respect to the state variables \( H_s = \left( \frac{\partial H}{\partial v} \frac{\partial H}{\partial i} \right) \). In the case of the Buck converter [12], the fields \( f_1 \) and \( f_2 \) are defined as follows Equation (15).

\[
f_1 = \left( -\frac{1}{CR}v + \frac{1}{C}i \quad -\frac{1}{L}v \right),
\]
\[
f_2 = \left( -\frac{1}{CR}v + \frac{1}{C}i \quad -\frac{1}{L}v + \frac{V_{in}}{L} \right),
\]  
(15)

and the switching region \( H \) will be Equation (16).

\[
H(v, i) = v + Zi - V_{low} = 0,
\]  
(16)

where \( H_s = \begin{bmatrix} 1 & Z \end{bmatrix}, \) is obtained, which when being replaced in Equation (16) the vector field for the Buck converter on the switching region can be obtained Equation (17).

\[
F = \left( -\frac{1}{CR}v + \frac{1}{C}i \quad -\frac{2}{L} + \frac{1}{2CR} \right) v - \frac{1}{CZ} i.
\]  
(17)

3.1. Buck converter as a Filippov system

When an analysis of the orbits described in the phase plane of the Buck converter is performed [10], it can be noted that it follows a discontinuous behavior, typical of Filippov systems [11] where the discontinuity region is defined by Equation (14).

\[
F = \lambda f_1 + (1 - \lambda) f_2, \quad 0 \leq \lambda = \frac{H_s f_2}{H_s (f_2 - f_1)} \leq 1,
\]  
(14)

where \( F \) is the linear combination of the fields \( f_1 \) and \( f_2 \), \( H \) is the function of the switching region and \( H_s \) is the variation of the switching region with respect to the state variables \( H_s = \left( \frac{\partial H}{\partial v} \frac{\partial H}{\partial i} \right) \). In the case of the Buck converter [12], the fields \( f_1 \) and \( f_2 \) are defined as follows Equation (15).

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and the switching region \( H \) will be Equation (16).

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F = \left( -\frac{1}{CR}v + \frac{1}{C}i \quad -\frac{2}{L} + \frac{1}{2CR} \right) v - \frac{1}{CZ} i.
\]  
(17)
so it can be said that \( u = f(\theta) \), for the case where \( \theta > 0 \) then \( u = 0 \) and \( u = 1 \) in any other case, that is Equation (18).

\[
\begin{align*}
    u &= \begin{cases} 
        0, & \theta > 0 \\ 
        1, & \text{otro caso}.
    \end{cases} 
\end{align*}
\]

(18)

The approximation of the variable \( u \) is made through functions such as the arc tangent, hyperbolic tangent and sigmoidal function, among others Equation (19).

\[
\begin{align*}
    u &= \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(\delta \theta) \\
    u &= \frac{1}{2} (-\tanh(\delta \theta) + 1), \\
    u &= 1 - \frac{1}{1 + e^{-\delta \theta}}
\end{align*}
\]

(19)

where \( \delta \) is an idealization factor for each of the proposed functions, a comparison of the three functions with two different values of \( \delta \) is made in Figure 3. The three functions to approximate the variable \( u \) show very good results, however the hyperbolic tangent and the sigmoidal function have a better approximation for very large \( \delta \) values.

![Figure 3. Approximation of the variable \( u \) (a) with \( \delta = 10 \) and (b) with \( \delta = 100 \).](image)

4. Results

The simulations corresponding to the Buck converter that are presented in this section were carried out using Matlab, which has a specialized toolbox to perform bifurcation analysis in dynamic systems. The nominal values of the parameters used for the respective simulations are the following: Resistance \( R = 22 \, \Omega \), Capacitance \( C = 47 \, \mu F \), Inductance \( L = 20 \, mH \), Input voltage \( V_{\text{in}} = 8V \) and low voltage \( V_{\text{low}} = 5V \). For each of these parameters, the phase diagrams of the system can be constructed, however, in this document we carry out the analysis varying only 3 of these parameters; these were \( R \), \( Z \) and \( V_{\text{in}} \). The results are shown in the Figure 4 to Figure 6.

The first result obtained from the simulation corresponding to the graph in Figure 4(a) is that when the converter works with the nominal values of its parameters, the behavior of the system is stable. Figure 4(a) shows the existence of a stable focus, for which the system before reaching that point is on a limit cycle, that is, there is the presence of oscillations to reach the stable focus from any initial condition of the system. For resistance values between \( 22 \, \Omega < R < 26.2 \, \Omega \) the presence of two limit cycles is observed, one stable and one unstable, for which the system behavior will depend on its initial conditions, in Figure 4(b) you can see the phase diagram. For values of \( R > 26.2 \, \Omega \), the system again has a stable focus that, regardless of the initial condition, will reach it for a time \( t > 0 \).
The variations of the impedance $Z$ are associated with the commutation, for this case when $Z$ takes $Z$ values $Z < -10 \, \Omega$ the dynamics of the system behaves completely as a stable focus Figure 5(a). For values between $-10 \, \Omega < Z < 0 \, \Omega$ there are two limit cycles, one stable and the other unstable. For initial conditions outside the stable cycle, it can be observed in Figure 5(b) that the system tends to reach the stable cycle. And for values of $Z > 0 \, \Omega$, a new equilibrium point appears which continues to be stable as the previous one Figure 5(c).

For the variations of the input voltage, it is observed that the dynamics change considerably around the value of $V_{\text{in}} = 9 \, \text{V}$ this behavior can be observed in Figure 6. For values of $V_{\text{in}} < 9 \, \text{V}$ it can be witnessing the existence of a single stable focus to which the system converges regardless of the equilibrium condition Figure 6(a). For $V_{\text{in}} = 9 \, \text{V}$ limit cycles appear, one stable (blue) and the other unstable (red) Figure 6(b). Finally, for values of $V_{\text{in}} > 9 \, \text{V}$ the limit cycles increase their amplitude, however their dynamics remains stable Figure 6(c).
5. Conclusions
Switched electronic circuits contain an interesting set of non-linear behaviors in their dynamics, which is evidenced through the modifications that phase diagrams undergo when their parameters are varied. These are very attractive for branch analysis in non-smooth systems; specifically, this occurred in the Buck configuration circuit which was the case study in this work. In addition to the changes in the qualitative dynamics caused by the circuit parameters R, C, L and V_{in}, it was possible to observe those incorporated by means of the control law through the Z and V_{low} parameters. In this way, for the analysis, the Buck controlled by a state feedback control law was taken into account. For the diagram obtained with variations in R, it is observed how the Hopf bifurcation is recorded approximately at R = 26.2 Ω, a value that defines the behavior of the equilibrium point as a stable focus R < 26.2 Ω and the formation of limit cycles. However, since R = 22 Ω limit cycles occur, but there are two regions of attraction towards the limit cycle and towards the equilibrium point that behaves as a stable focus. In this way, what the bifurcation indicates is the change in the dynamics of the equilibrium point. The limit cycles observed in Figure 4b correspond to the unstable limit cycles that delimit the regions of attraction.

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