Well-posedness of strong solutions for the Vlasov equation coupled to non-Newtonian fluids in dimension three

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Abstract. We consider the Cauchy problem for coupled system of Vlasov and non-Newtonian fluid equations. We establish local well-posedness of the strong solutions, provided that the initial data are regular enough.

Mathematics Subject Classification. 35D35, 35Q30, 76A05, 35Q83.

Keywords. Non-Newtonian fluid, Navier–Stokes equations, Vlasov equation, Strong solution.

1. Introduction

We study the following coupled system of Vlasov and non-Newtonian fluid equations in phase space $\mathbb{R}^3 \times \mathbb{R}^3$:

$$
\begin{align*}
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(u - v)f] &= 0, \\
\partial_t u - \nabla \cdot (G[|Du|^2]Du) + (u \cdot \nabla)u + \nabla p &= -\int_{\mathbb{R}^3} (u - v)f dv, \\
\text{div } u &= 0,
\end{align*}
$$

(1.1)

where $Du$ is the symmetric part of the velocity gradient, namely,

$$
Du = D_{ij} := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2, 3.
$$

In (1.1), we indicate by $u : \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}^3$, $p : \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}$ and $f : \mathbb{R}^3 \times \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}$ the flow velocity vector, the scalar pressure and the density function of particles, respectively. In this article, we study the Cauchy problem of (1.1) with

$$
f(x, v, 0) = f_0(x, v), \quad u(x, 0) = u_0(x), \quad x, v \in \mathbb{R}^3,
$$

(1.2)

and in particular, $u_0$ holds the compatibility condition, that is, $\text{div } u_0 = 0$.

Here, we make some assumptions on the viscous part of the stress tensor, $G[|Du|^2]$.

Assumption 1.1. We suppose that $G : [0, \infty) \rightarrow [0, \infty)$ is a smooth function and satisfies the following the structure conditions: There is a positive constant $m_0$ such that for any $s \in [0, \infty)$

$$
\begin{align*}
G[s] &\geq m_0, & G[s] + 2G'[s]s &\geq m_0, \\
|G^{(k)}[s]s^\alpha| &\leq C_k |G^{(k-1)}[s]|, & \alpha &\in \{0, 1\},
\end{align*}
$$

(1.3)

where $G^{(k)}[\cdot]$ is the $k$–th derivative of $G$, and $m_0$ and $C_k$ are positive constants.
We remark that some typical types of $G[s]$ satisfying Assumption 1.1 are of the following power-law models, e.g.,

$$G[s] = \left( m^{\frac{2}{q} - 2} + s \right)^{\frac{q - 2}{2}} , \quad 2 < q < \infty,$$

or

$$G[s] = m_0 + (\sigma + s)^{\frac{q - 2}{2}} , \quad 1 < q < \infty , \quad \sigma > 0 .$$

We recall some known results for the case $G[s] = 1$, namely, the fluid equations are of Newtonian case. Hamdache [10] proved the global existence of weak solutions to the time-dependent Stokes system coupled with the Vlasov equation in a bounded domain. Later, existence of weak solution was extended to the Vlasov–Navier–Stokes system by Boudin et al. [6] in a periodic domain (refer also to [7, 8] for hydrodynamic limit problems). When the fluid is inviscid, Baranger and Desvillettes established the local existence of solutions to the compressible Vlasov–Euler equations [4]. We report briefly some known results related to the existence of solutions for the non-Newtonian fluids. In the case that $G[s] = (\mu_0 + \mu_1 s)^{\frac{q - 2}{2}}$, $\mu_0, \mu_1 > 0$, Málek, Nečas, Rokyta and Růžička proved in [15] that a strong solution exists globally in time in periodic domains $\mathbb{T}^3$ for $q \geq \frac{11}{5}$ (see [17] for the whole space case). Also, they established the small data global existence of a strong solution for $q > \frac{5}{3}$ in $\mathbb{T}^3$ (refer to the local in time existence for [5] for shear thinning case, $\frac{5}{7} < q < 2$ and [3] for shear thickening case).

In case of non-Newtonian fluid, recently, Mucha et al. [16] investigated the Cucker–Smale flocking model coupled with an incompressible viscous generalized Navier–Stokes equations with $G$ given in (1.4), for $q \geq \frac{11}{5}$ in a periodic spatial domain $\mathbb{T}^3$. To be more precise, the following equations are considered:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(F_{CS}(f) + (u - v)) f] = 0 ,$$

$$\partial_t u - \nabla \cdot (G[|Du|^2]Du) + (u \cdot \nabla)u + \nabla p = - \int_{\mathbb{R}^3} (u - v) f dv ,$$

where

$$F_{CS}(f)(t, x, v) = \int_{\mathbb{T}^3 \times \mathbb{R}^3} (w - v) \psi(|x - y|) f(t, y, w) dy dw$$

and $\psi(\cdot)$ the communication weight is positive, decreasing and $\|\psi\|_{C^1} < \infty$.

The viscosity part of stress tensor $G$ in [16] was assumed to satisfy the structure conditions, which are given as

$$G[s] s \geq C(\mu_0 s + s^2) , \quad |G[s]| \leq C(\mu_0 + s)^{\frac{q - 2}{2}} , \quad \mu_0 > 0 , \quad (1.7)$$

$$\left( G[|Du|^2]Du - G[|Dw|^2]Dw : Du - Dw \right) \geq C \left( \mu_0 |Du - Dw|^2 + |Du - Dw|^q \right) , \quad (1.8)$$

$$G[s] + G'[s] s \approx (\mu_0 + s)^{\frac{q}{2}} . \quad (1.9)$$

Under the assumption that a non-negative $f_0 \in (L^\infty \cap L^1)(\mathbb{T}^3 \times \mathbb{R}^3)$ with compact support and $u_0 \in W^{1,2}(\mathbb{T}^3)$, existence of weak solutions was established for system (1.5)–(1.6) in the class (see [16, Theorem 2.1])

$$f \in L^\infty \left( \left[0, T\right]; L^1(\mathbb{T}^3 \times \mathbb{R}^3) \right) , \quad |v|^2 f \in L^\infty \left( \left[0, T\right]; L^1(\mathbb{T}^3 \times \mathbb{R}^3) \right) ,$$

$$u \in C \left( \left[0, T\right]; L^2(\mathbb{T}^3) \right) \cap L^\infty \left( \left[0, T\right]; W^{1,2}(\mathbb{T}^3) \right) \cap L^2 \left( \left[0, T\right]; W^{2,2}(\mathbb{T}^3) \right) ,$$

$$\nabla u \in L^\infty \left( \left[0, T\right]; L^q(\mathbb{T}^3) \right) \cap L^q \left( \left[0, T\right]; L^{3q}(\mathbb{T}^3) \right) , \quad u_t \in L^2 \left( \left[0, T\right]; L^2(\mathbb{T}^3) \right) .$$

It was also shown via a Lyapunov functional approach that convergence of flocking states is made exponentially fast in time as well.

The second and third authors with their collaborators also proved, independently, global-in-time existence of weak solutions for system (1.5)–(1.6) for the case of $G[s] = \mu_0 + \mu_1 s^{q-2}$, with $\mu_0 \geq 0 , \quad \mu_1 > 0$.
and \( q > 2 \). More precisely, if the initial data \( f_0 \in L^\infty \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3) \) with the compact support and \( u_0 \in L^2(\mathbb{T}^3) \), weak solutions exist in the class
\[
\begin{align*}
    f &\in L^\infty \left( (0, T); L^1 \cap L^\infty(\mathbb{T}^3 \times \mathbb{R}^3) \right), \quad |v|^2 f \in L^\infty \left( [0, T]; L^1(\mathbb{T}^3 \times \mathbb{R}^3) \right), \\
u &\in L^\infty \left( [0, T]; L^2(\mathbb{T}^3) \right) \cap L^q \left( 0, T; W^{1,q}(\mathbb{T}^3) \right), \quad L^2 \left( 0, T; W^{1,2}(\mathbb{T}^3) \right), \quad \text{if } \mu_0 > 0.
\end{align*}
\]

As following almost the same arguments in [9], we also can establish existence of global-in-time weak solutions to (1.1)–(1.2), provided that \( G \) satisfies structure conditions (1.3), (1.7) and (1.8) with \( q > 2 \). Since its verification is rather tedious repetitions of the arguments in [9], we just give the statement without proof.

**Theorem 1.2.** Let \( T > 0 \). Suppose that initial data \( f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^3 \times \mathbb{R}^3) \) and \( u_0 \in L^2(\mathbb{R}^3) \) satisfy
\[
\begin{align*}
    (i) &\quad \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f_0 \, dv \, dx = 1, \quad \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f_0 \, dv \, dx < \infty, \quad f_0 \geq 0, \\
    (ii) &\quad \text{supp}_x f_0 \text{ is bounded in } \mathbb{R}^3 \text{ for each } x \in \mathbb{R}^3.
\end{align*}
\]
Assume further that \( G \) satisfies (1.3), (1.7) and (1.8) with \( q > 2 \). Then, there exists a global weak solution \((f, u)\) in class (1.10)–(1.12) to Eqs. (1.1)–(1.2).

**Remark 1.3.** In [9], the authors considered system (1.5)–(1.6) for the specific case of \( G[s] = \mu_0 + \mu_1 s^{\frac{q-2}{2}} \), where \( q > 2 \), \( \mu_0 \geq 0 \) and \( \mu_1 > 0 \). Based on uniform estimates for solutions of approximated equations and the monotonicity property of stress tensor, they exploited the Schauder fixed point theorem to show the global existence of weak solutions.

On the other hand, the equations under our consideration deal with general types of strain tensors satisfying conditions (1.3), (1.7) and (1.8) with \( q > 2 \). However, the proof for existence of weak solutions could be performed almost the same way as what was done in [9], since structure conditions (1.3), (1.7) and (1.8) are essential ingredients to allow us to have such existence.

Our main objective is to prove existence of strong solutions for system (1.1)–(1.2) where strong solutions are defined below (see Definition 1.4). More precisely, we establish the well-posedness of strong solutions locally in time (Theorem 1.5).

One of main tools is to use Schauder fixed point theorem via the weighted estimate for \( f \) (Lemma 3.2 below). In particular, we use the optimal transport technique to show the stability of solutions to the Vlasov equation with respect to the velocity vector \( u \) as well as the existence of solutions to the Vlasov equation. To prove the stability of solutions, we borrow the idea of the paper of Han-Kwan et al. [11]. However, unlike [11], we exploit estimates on the Wasserstein distance instead of Loeper’s functional.

We introduce the notion of strong solutions to (1.1)–(1.2).

**Definition 1.4.** Let \( T > 0 \) and \( k \geq 3 \). We say that \((f, u)\) is a strong solution to (1.1)–(1.2) with (1.3) if the following conditions are satisfied:
\[
\begin{align*}
    (i) &\quad f(x, \xi, t) \geq 0 \text{ for all } (x, \xi, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times (0, T], \\
    (ii) &\quad (1 + |v|^k) \nabla u \cdot \nabla \beta f \in L^2(\mathbb{R}^3 \times \mathbb{R}^3), \quad \text{where } 0 \leq |\alpha| + |\beta| \leq 2, \\
    (iii) &\quad u \in C(0, T; H^3(\mathbb{R}^3)) \cap L^2(0, T; H^4(\mathbb{R}^3)), \\
    (iv) &\quad (f, u) \text{ solves Eqs. (1.1)–(1.2) in the pointwise sense.}
\end{align*}
\]

Now, we are ready to state the main result.

**Theorem 1.5.** Let \( k \geq 3 \). Suppose that
\[
(1 + |x|^2 + |v|^2) f_0(x, v) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3), \quad u_0 \in H^3(\mathbb{R}^3).
\]
Assume further that \( f_0(x, v) \geq 0 \) satisfies for \( p > \max \{5, \frac{2k+3}{2}\} \)

\[
(1 + |v|^p) \nabla_v^\alpha \nabla_x^\beta f_0 \in L_v^2 L_x^2(\mathbb{R}^3 \times \mathbb{R}^3), \quad 0 \leq |\alpha| + |\beta| \leq 2. 
\]

Then, there exists \( T^* > 0 \) such that system (1.1)–(1.2) has a unique strong solution \((f, u)\) on the time-interval \((0, T^*)\) in the sense of Definition 1.4.

Remark 1.6. We note that the local time \( T^* \) in Theorem 1.5 is dependent on the size of data in (1.13) and \( \|u_0\|_{H^3(\mathbb{R}^3)} \).

Remark 1.7. One can observe that assumption (1.13) implies the following:

- \( f_0(x, v) \leq \frac{C_0}{(1 + |v|^p)} \) due to the embedding w.r.t x-variable (see Sect. 3.1.2).
- \( (1 + |v|^k) \nabla_v^\alpha \nabla_x^\beta f_0 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \) if \( k < (2p - 3)/2 \) by the Hölder inequality (see Proposition 3.3 below).

These will be used later in the proof of Theorem 1.5.

Remark 1.8. The novelty of this paper is that, compared to known results, one of main difficulties is of course caused by nonlinear structure of viscous part of fluid, which we successfully controlled somehow. Another improvement is that initial data of Vlasov equation are not assumed to be compactly supported, which makes arguments a bit complicated for solvability of the Vlasov equation. To the knowledge of authors, previous results seem to suppose compactly supported initial data, and we do not know, however, if decay condition (1.13) is optimal or not.

This paper is organized as follows: In Sect. 2, we review some preliminary results. Section 3 is devoted to the study of Vlasov equation. In Sect. 4, we present the proof of Theorem 1.5. In Sect. 5, the convergence of two strong solutions of the fluid equation involving the drag force is shown.

2. Preliminary

We first introduce some notations. Let \((X, \| \cdot \|)\) be a normed space. By \( L^q(0, T; X) \), we denote the space of all Bochner measurable functions \( \varphi : (0, T) \to X \) such that

\[
\left\{ \begin{align*}
\| \varphi \|_{L^q(0, T; X)} := \left( \int_0^T \| \varphi(t) \|^q \, dt \right)^{\frac{1}{q}} < \infty, & \text{ if } 1 \leq q < \infty, \\
\| \varphi \|_{L^\infty(0, T; X)} := \sup_{t \in (0, T)} \| \varphi(t) \| < \infty, & \text{ if } q = \infty.
\end{align*} \right.
\]

For \( 1 \leq q \leq \infty \), we mean by \( W^{k, q}(\mathbb{R}^3) \) the usual Sobolev space. In particular, for \( q = 2 \), we write \( W^{k, 2}(\mathbb{R}^3) \) as \( H^k(\mathbb{R}^3) \). Let \( A = (a_{ij})_{i,j=1}^3 \) and \( B = (b_{ij})_{i,j=1}^3 \) be matrix valued maps, we then denote

\[
A : B = \sum_{i,j=1}^3 a_{ij} b_{ij}, \quad \nabla A \cdot \nabla B = \sum_{i,j=1}^3 \nabla a_{ij} \cdot \nabla b_{ij}, \quad \nabla^2 A : \nabla^2 B = \sum_{i,j=1}^3 \nabla^2 a_{ij} : \nabla^2 b_{ij}.
\]

The letter \( C \) is used to represent a generic constant, which may change from line to line.

Next we recall the Aubin-Lions lemma, which will be used for compactness (see, e.g., [18]).

Lemma 2.1. Let \( B \) be a Banach space and \( B_i, i = 0, 1 \), separable, reflexive Banach spaces, and suppose that

\[
B_0 \hookrightarrow \hookrightarrow B \hookrightarrow B_1,
\]

where \( \hookrightarrow \) denotes continuous imbedding and \( \hookrightarrow \hookrightarrow \) compact imbedding. Let

\[
W := \left\{ v \in L^{p_0}(I; B_0) \quad \text{with} \quad \frac{dv}{dt} \in L^{p_1}(I; B_1) \right\}
\]
for some finite interval $I \subset \mathbb{R}$, where $p_i$ satisfying $1 < p_i < \infty$. Then, we have

$$W \hookrightarrow L^{p_0}(I; \mathcal{B}).$$

We also remind a version of Schauder’s fixed point theorem.

**Lemma 2.2.** Let $V$ be a Banach space and $K$ be a non-empty convex closed subset of $V$. If $T : K \to K$ is a continuous mapping and $T(K)$ is contained in a compact subset of $K$, then $T$ has a fixed point.

Next, we introduce a priori estimate, which is one of key estimates involving third derivatives of $u$ (refer [12, Lemma 2.1]).

**Lemma 2.3.** Let $l$ be a positive integer, $\bar{\sigma}_l : \{1, 2, \cdots, l\} \to \{1, 2, \cdots, l\}$ a permutation of $\{1, 2, \cdots, l\}$, and $\pi_l$ a mapping from $\{1, 2, \cdots, l\}$ to $\{1, 2, 3\}$. Suppose that $u \in C^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$. Assume further that $G : [0, \infty) \to [0, \infty)$ is infinitely differentiable and satisfies properties given in (1.3). Then, the multi-derivative of $G$ can be rewritten as the following decomposition:

$$\partial^{(l_1)} \partial^{(l_2)} \cdots \partial^{(l_l)} G \left[|Du|^2\right] = 2(G' \left[|Du|^2\right] Du : \partial_{x \sigma_l(1)} \partial_{x \sigma_l(l-1)} \cdots \partial_{x \sigma_l(1)} Du) + E_l,$$

where $\sigma_l := \pi_l \circ \bar{\sigma}_l$ and

$$E_l = 2(\partial_{x \sigma_l(1)} (G' \left[|Du|^2\right] Du) \partial^{(l-1)} Du + \partial_{x \sigma_l(1)} E_{(l-1)}), \quad E_1 = 0,$$

where $\partial^{(l-1)} = \partial_{x \sigma_l(l-1)} \cdots \partial_{x \sigma_l(1)}$. Furthermore, we obtain the following.

1. $E_2$ and $E_3$ satisfy

$$|E_2| \leq CG \left[|Du|^2\right] |\nabla Du|^2,$$

$$|E_3| \leq CG \left[|Du|^2\right] (|\nabla Du|^3 + |\nabla^2 Du| |\nabla Du|).$$

2. For $1 \leq \alpha \leq l$

$$\left\| \partial^{(1)} G \left[|Du|^2\right] \partial^{(l-\alpha)} Du \right\|_{L^2} + \left\| E_\alpha \partial^{(l-\alpha)} Du \right\|_{L^2} \leq C \left\| G\left[|Du|^2\right] \right\|_{L^\infty} \left\| Du \right\|_{L^\infty} + \left\| Du \right\|_{L^\infty} \left\| \nabla^l Du \right\|_{L^2}. \quad (2.15)$$

Next, we also introduce a monotonicity property of the viscous part of the stress tensor, which is useful for the uniqueness of strong solutions for fluid equations (see [12, Lemma 2.2]).

**Lemma 2.4.** Let $v, w \in W^{1,2}(\mathbb{R}^3)$. Under Assumptions 1.1 on $G$, we have

$$m_0 \left\| Dv - Dw \right\|_{L^2(\mathbb{R}^3)}^2 \leq \int_{\mathbb{R}^3} (G \left[|Dv|^2\right] Dv - G \left[|Dw|^2\right] Dw) : (Dv - Dw) \, dx,$$

where $m_0$ is a positive constant in (1.3).

### 2.1. Review on maximal functions

Let us first remind the notion of maximal functions. For every $g \in L^1_{loc}(\mathbb{R}^3)$, the associated maximal function, denoted $Mg$, is defined by

$$Mg(x) := \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y)| \, dy,$$

for a.e. $x \in \mathbb{R}^3$, where $B_r(x)$ is the ball of radius $r$ centered at $x$.

It is well known that if $g \in L^p(\mathbb{R}^3)$ ($1 < p \leq \infty$) then so is $Mg$, and there is a constant $C_p > 0$ such that

$$\left\| Mg \right\|_p \leq C_p \left\| g \right\|_p. \quad (2.16)$$

We will use the following property of the maximal function (see [1, Lemma 3]).
Lemma 2.5. If \( g \in W^{1,p}(\mathbb{R}^3) \) for \( p \geq 1 \) then, for a.e. \( x, y \in \mathbb{R}^3 \) one has
\[
|g(x) - g(y)| \lesssim |x - y| (M \nabla g(x) + M \nabla g(y)).
\] (2.17)

2.2. Preliminary results on optimal mass transportation and Wasserstein space

In this subsection, we introduce the Wasserstein space and remind some properties of it. For more detail, readers may refer \cite{2}.

Definition 2.6. Let \( \mu \) be a probability measure on \( \mathbb{R}^d \) and \( T : \mathbb{R}^d \mapsto \mathbb{R}^d \) a measurable map. Then, \( T \) induces a probability measure \( \nu \) on \( \mathbb{R}^d \) which is defined as
\[
\int_{\mathbb{R}^d} \varphi(y) d\nu(y) = \int_{\mathbb{R}^d} \varphi(T(x)) d\mu(x) \quad \forall \varphi \in C(\mathbb{R}^d).
\]

We denote \( \nu := T \# \mu \) and say that \( \nu \) is the push-forward of \( \mu \) by \( T \).

Definition 2.7. Let us denote by \( P_2(\mathbb{R}^d) \) the set of all Borel probability measures on \( \mathbb{R}^d \) with a finite second moment. For \( \mu, \nu \in P_2(\mathbb{R}^d) \), we consider
\[
W_2(\mu, \nu) := \left( \inf_{\gamma \in \Gamma(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\gamma(x, y) \right)^{\frac{1}{2}},
\] (2.18)
where \( \Gamma(\mu, \nu) \) denotes the set of all Borel probability measures on \( \mathbb{R}^d \times \mathbb{R}^d \) which has \( \mu \) and \( \nu \) as marginals, i.e.,
\[
\gamma(A \times \mathbb{R}^d) = \mu(A) \quad \text{and} \quad \gamma(\mathbb{R}^d \times A) = \nu(A)
\]
for every Borel set \( A \subset \mathbb{R}^d \).

Equation (2.18) defines a distance on \( P_2(\mathbb{R}^d) \) which is called the Wasserstein distance. Equipped with the Wasserstein distance, \( P_2(\mathbb{R}^d) \) is called the Wasserstein space. It is known that the infimum in the right-hand side of Eq. (2.18) always achieved. We will denote by \( \Gamma_o(\mu, \nu) \) the set of all \( \gamma \) which minimize the expression.

Definition 2.8. Let \( \sigma : [a, b] \mapsto P_2(\mathbb{R}^d) \) be a curve. We say that \( \sigma \) is absolutely continuous and denote it by \( \sigma \in AC_2([a, b]; P_2(\mathbb{R}^d)) \), if there exists \( m \in L^2([a, b]) \) such that
\[
W_2(\sigma(s), \sigma(t)) \leq \int_s^t m(r) dr \quad \forall \ a \leq s \leq t \leq b.
\] (2.19)
If \( \sigma \in AC_2([a, b]; P_2(\mathbb{R}^d)) \), then the limit
\[
|\sigma'|(t) := \lim_{s \to t} \frac{W_2(\sigma(s), \sigma(t))}{|s - t|}
\]
exists for \( L^1 \)-a.e \( t \in [a, b] \). Moreover, the function \( |\sigma'| \) belongs to \( L^2(a, b) \) and satisfies
\[
|\sigma'|(t) \leq m(t) \quad \text{for } L^1 \text{-a.e. } t \in [a, b],
\]
for any \( m \) satisfying (2.19). We call \( |\sigma'| \) by the metric derivative of \( \sigma \).
Lemma 2.9. ([2], Theorem 8.3.1) If \( \sigma \in AC_2(a, b; \mathcal{P}_2(\mathbb{R}^d)) \), then there exists a Borel vector field \( v : \mathbb{R}^d \times (a, b) \mapsto \mathbb{R}^d \) such that

\[
v_t \in L^2(\sigma_t) \text{ for } L^1 - \text{a.e } t \in [a, b],
\]

and the continuity equation

\[
\partial_t \sigma_t + \nabla \cdot (v_t \sigma_t) = 0
\]

holds in the sense of distribution.

Conversely, if a weak* continuous curve \( \sigma : [a, b] \mapsto \mathcal{P}_2(\mathbb{R}^d) \) satisfies the continuity equation for some Borel vector field \( v_t \) with \( ||v_t||_{L^2(\sigma_t)} \in L^2(a, b) \) then \( \sigma : [a, b] \mapsto \mathcal{P}_2(\mathbb{R}^d) \) is absolutely continuous and \( |\sigma_t| \leq ||v_t||_{L^2(\sigma_t)} \text{ for } L^1 - \text{a.e } t \in [a, b] \).

**Notation:** In Lemma 2.9, we use notation \( v_t := v(\cdot, t) \) and \( \sigma_t := \sigma(t) \). Throughout this paper, we keep this convention, unless any confusion is to be expected, and a usual notation \( \partial_t \) is adopted for temporal derivative, i.e., \( f_t := f(\cdot, t) \) and \( \partial_t f := \frac{\partial f}{\partial t} \).

Lemma 2.10. For \( i = 1, 2 \), let \( \sigma^i \in AC_2(a, b; \mathcal{P}_2(\mathbb{R}^d)) \) and \( \sigma^i \) satisfies

\[
\partial_t \sigma^i_t + \nabla \cdot (v^i_t \sigma^i_t) = 0.
\]

We set

\[
Q(t) := \frac{1}{2} W^2\mathcal{V}_2(\sigma^1_t, \sigma^2_t).
\]

Then, we have \( Q \in W^{1, 2}(a, b) \) and, for a.e \( t \in (a, b) \)

\[
\frac{dQ}{dt} \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} (v^1_t(x) - v^2_t(y)) \cdot (x - y) d\gamma_t(x, y), \quad \gamma_t \in \Gamma_0(\sigma^1_t, \sigma^2_t).
\]

Proof. Refer to Lemma 2.4.1 of [14]. \( \square \)

3. Vlasov equation

In this section, for given \( \bar{u} \in L^\infty(0, T; H^3(\mathbb{R}^3)) \cap L^2(0, T; H^4(\mathbb{R}^3)), \) we consider the following linearized system of the Vlasov-type equation of (1.1):

\[
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot ((\bar{u} - v)f) = 0, \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3, \quad t > 0,
\]

which requires initial condition \( f(x, v, 0) = f_0(x, v) \).

3.1. Solutions of Vlasov equation as a curve in Wasserstein space

3.1.1. ODE. For given \( \bar{u} \in L^\infty(0, T; H^3(\mathbb{R}^3)) \cap L^2(0, T; H^4(\mathbb{R}^3)), \) we define a vector field \( \Xi : [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \times \mathbb{R}^3 \) by

\[
\Xi(t, x, v) := (v, \bar{u}(t, x) - v).
\]

We also define a flow map \( \Phi : [0, T] \times [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3 \times \mathbb{R}^3 \) corresponding to the vector field \( \Xi \) by

\[
\Phi(t; s, x, v) := (X(t; s, x, v), V(t; s, x, v)),
\]

where

\[
\begin{Bmatrix}
\frac{d}{dt}X(t; s, x, v) = V(t; s, x, v), \\
\frac{d}{dt}V(t; s, x, v) = \bar{u}(t, X(t; s, x, v)) - V(t; s, x, v), \\
X(s; s, x, v) = x, \quad V(s; s, x, v) = v.
\end{Bmatrix}
\]
We note
\[ \frac{d}{dt} \| V(t; s, x, v) \|^2 = 2V' \cdot V = 2(\bar{u} - V) \cdot V \leq 0, \]
for \(|V| \geq |\bar{u}|\). Hence, if \(|v| \geq \| \bar{u} \|_{L^\infty} \) then \(|V(t; s, x, v)| \leq |v|\) for all \((t, x) \in [s, T] \times \mathbb{R}^3\). Let \(M\) be a positive number such that
\[ \| \bar{u} \|_{L^\infty} + \| \bar{u} \|_{L^2} \leq M. \]
Then we know \(\| \bar{u} \|_{L^\infty} \leq C_1 M\) for some \(C_1 > 0\). Hence, we have
\[ |V(t; s, x, v)| \leq \max \{ C_1 M, |v| \} \quad \text{for all} \quad (t, x) \in [s, T] \times \mathbb{R}^3. \quad (3.21) \]
This implies
\[ |X(t; s, x, v)| \leq |x| + (t - s) \sup_{s \leq \tau \leq t} |V(\tau; s, x, v)| \leq |x| + (t - s) \max \{ C_1 M, |v| \}. \quad (3.22) \]

3.1.2. Flow map generating a solution of Vlasov equation. Let \(f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)\) be a non-negative function and we may assume \(\| f_0 \|_{L^1} = 1\) without loss of generality. That is, \(f_0 \in \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)\). We define \(f : [0, T] \mapsto \mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)\) by
\[ f(t) := \Phi(t; 0, \cdot, \cdot) \# f_0. \quad (3.23) \]
Then, \(f\) is the unique solution of (refer to \([2]\))
\[ \partial_t f + \nabla_x \cdot (vf) + \nabla_v \cdot ((\bar{u} - v)f) = 0. \]
We note that the change of variable formula combined with (3.23) gives
\[ f(t, X(t; 0, x, v), V(t; 0, x, v)) = e^{3t} f_0(x, v). \quad (3.24) \]
We also see that (3.24) can be written as
\[ f(t, x, v) = e^{3t} f_0(X(0; t, x, v), V(0; t, x, v)). \quad (3.25) \]
Suppose \(f_0\) satisfies (1.13), that is there exists \(C_2 > 0\) such that
\[ f_0(x, v) \leq \frac{C_2}{1 + |v|^p}, \quad \forall (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3, \quad p > 5. \quad (3.26) \]
Then we have
\[ \rho_f(t, x) := \int_{\mathbb{R}^3} f(t, x, v)dv \]
\[ = \int_{\mathbb{R}^3} e^{3t} f_0(X(0; t, x, v), V(0; t, x, v))dv. \quad (3.27) \]
First of all, we note that if \(|v| \geq C_1 M \geq \| \bar{u} \|_{L^\infty} \), then \(|V(0; t, x, v)| \geq |v|\). Hence,
\[ \int_{|v| \geq C_1 M} e^{3t} f_0(X(0; t, x, v), V(0; t, x, v))dv \leq \int_{|v| \geq C_1 M} \frac{C_2}{1 + |V(0; t, x, v)|^p} e^{3t} dv \]
\[ \leq \int_{|v| \geq C_1 M} \frac{C_2}{1 + |v|^p} e^{3t} dv. \quad (3.28) \]
On the other hand, we get
\[ \int_{|v| < C_1 M} e^{3t} f_0(V(0; t, x, v), V(0; t, x, v))dv \leq \pi e^{3t} \| f_0 \|_{L^\infty} (C_1 M)^3. \quad (3.29) \]
Combining (3.27), (3.28) and (3.29), we have
\[ \rho_f(t, x) \leq \tilde{C}e^{3t}, \]
where
\[ \tilde{C} := \pi \|f_0\|_{L^\infty}(C_1 M)^3 + \int_{|v| \geq C_1 M} \frac{1}{1 + |v|^p} dv. \]
That is \( \rho_f \in L^\infty(0, T; L^\infty(\mathbb{R}^3)) \). Similarly, we have \( m_2 f \in L^\infty(0, T; L^\infty(\mathbb{R}^3)) \) and
\[ \|m_2 f(t)\|_{L^\infty} \leq \pi e^{3t} \|f_0\|_{L^\infty}(C_1 M)^5 + e^{3t} \int_{|v| \geq R} |v|^2 \frac{1}{1 + |v|^p} dv, \]
where
\[ m_2 f(t, x) := \int_{\mathbb{R}^3} |v|^2 f(t, x, v) dv. \]
Suppose \( f_0 \in \mathcal{P}_2(\mathbb{R}^3 \times \mathbb{R}^3) \), that is
\[ \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|x|^2 + |v|^2) f_0(x, v) dx dv < \infty. \]
Then, we have
\[
\begin{align*}
\iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|x|^2 + |v|^2) f(t, x, v) dx dv &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|X(t; 0, x, v)|^2 + |V(t; 0, x, v)|^2) f_0(x, v) dx dv \\
&\leq 2(1 + t^2) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|x|^2 + |v|^2 + C_1^2 M^2) f_0(x, v) dx dv < \infty,
\end{align*}
\]
where we exploit (3.21) and (3.22) in the first inequality. This says that \( t \mapsto f_t \) is a curve in the Wasserstein space \( \mathcal{P}_2(\mathbb{R}^3 \times \mathbb{R}^3) \). Exploiting Lemma 2.9, we can show that it is an absolutely continuous curve in \( \mathcal{P}_2(\mathbb{R}^3 \times \mathbb{R}^3) \) as follows:
\[
\|\Xi(t, \cdot, \cdot)\|_{L^2_{f_t}}^2 = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |\Xi(t, x, v)|^2 f(t, x, v) dx dv
\]
\[
\leq 4 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|v|^2 + M^2) f(t, x, v) dx dv
\]
\[
= 4 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|V(t; 0, x, v)|^2 + M^2) f_0(x, v) dx dv
\]
\[
\leq 8 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|v|^2 + M^2) f_0(x, v) dx dv.
\]
which implies
\[
\int_0^T \| \Xi(t, \cdot, \cdot) \|_{L^2_t} dt \leq T \left( 8 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|v|^2 + M^2) f_0(x, v) dv \right)^{\frac{1}{2}} < \infty.
\]

For convenience, we set \( f_t := f(t) \) and thus, we have \( t \mapsto f_t \in AC^2(0, T; \mathcal{P}_2(\mathbb{R}^3 \times \mathbb{R}^3)) \).

### 3.1.3. Estimation of Wasserstein distance.

Compared to [11] in which authors used Loeper’s functional to estimate the distance between two solutions to the Vlasov equation, we estimate the Wasserstein distance between two solutions to the Vlasov equation.

**Lemma 3.1.** Let \( \bar{u}_i \in L^\infty(0, T; H^3(\mathbb{R}^3)) \cap L^2(0, T; H^4(\mathbb{R}^3)) \) for \( i = 1, 2 \). Suppose that \( f_i \) is a solution of the following equation associated with \( \bar{u}_i \):

\[
\partial_t f_i + \nabla_x \cdot (v f_i) + \nabla_v \cdot ((\bar{u}_i - v) f_i) = 0,
\]

\( f_i(0) = f_0 \),

where \( f_0 \in \mathcal{P}_2(\mathbb{R}^3 \times \mathbb{R}^3) \) is given. We set \( Q(t) := \frac{1}{2} W_2^2(f_1(t), f_2(t)) \). Then, we have

\[
Q(t) \leq e^{(2 + \| \bar{u}_2 \|_{L^\infty(0, T; H^3(\mathbb{R}^3))})^t} \| \rho_1 \|_{L^\infty(0, T; L^\infty)} \int_0^t \| (\bar{u}_1 - \bar{u}_2)(s) \|^2_{L^2} ds.
\]

Here, \( \rho_i(t, x) := \iint_{\mathbb{R}^3} f_i(t, x, v) dv \) for \( i = 1, 2 \).

**Proof.** From Lemma 2.10, we have

\[
\frac{d}{dt} Q(t) \leq \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} (v_1 - v_2, \bar{u}_1(x_1, t) - \bar{u}_2(x_2, t) - v_1 + v_2) \cdot
\]

\[
(x_1 - x_2, v_1 - v_2) d\gamma_t(x_1, v_1, x_2, v_2),
\]

where \( d\gamma_t \in \Gamma_\alpha(f_1(t), f_2(t)) \). Hence, we have

\[
\frac{d}{dt} Q(t) \leq \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \|x_1 - x_2\|^2 + |v_1 - v_2|^2 d\gamma_t(x_1, v_1, x_2, v_2)
\]

\[
+ 2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} |\bar{u}_1(x_1, t) - \bar{u}_2(x_1, t)|^2 + |\bar{u}_2(x_1, t) - \bar{u}_2(x_2, t)|^2 d\gamma_t(x_1, v_1, x_2, v_2)
\]

\[
\leq 2Q(t) + 2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |\bar{u}_1(x_1, t) - \bar{u}_2(x_1, t)|^2 f_1(x_1, v_1, t) dx_1 dv_1 + \| \nabla \bar{u}_2 \|^2_{L^\infty} Q(t)
\]

\[
\leq (2 + \| \nabla \bar{u}_2 \|^2_{L^\infty}) Q(t) + \| \rho_1(t) \|_{L^\infty} \| (\bar{u}_1 - \bar{u}_2)(t) \|^2_{L^2}.
\]

Exploiting Gronwall’s inequality to (3.32) with \( Q(0) = 0 \), we have

\[
Q(t) \leq \int_0^t h(s) e^{\int_0^s g(\theta) d\theta} ds,
\]

where

\[
g(s) := 2 + \| \nabla \bar{u}_2(s) \|^2_{L^\infty} \quad \text{and} \quad h(s) := \| \rho_1(s) \|_{L^\infty} \| (\bar{u}_1 - \bar{u}_2)(s) \|^2_{L^2}.
\]

This completes the proof. □
3.2. Some estimates on the Vlasov equation

In this subsection, we present the proof of solvability of linear Eq. (3.20) and provide some estimates in a weighted Lebesgue space. In the next lemma, we consider the case that $f_0$ is compactly supported and smooth.

Lemma 3.2. Let $T > 0$, $k \in \mathbb{N}$ and $f_0 \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\bar{u} \in L^\infty(0; (H^3 \cap C^\infty)(\mathbb{R}^3)) \cap L^2(0; (H^4 \cap C^\infty)(\mathbb{R}^3))$. Suppose that $f$ is the solution given in (3.25) to Eq. (3.20). Then, we have $$(1 + |v|^k) \nabla_v^\alpha \nabla_x^\beta f \in L^\infty(0, T; L^2(\mathbb{R}^3 \times \mathbb{R}^3))$$
and furthermore,

$$\sup_{0 < t \leq T} \sum_{0 \leq |\alpha| + |\beta| \leq 2} \|(1 + |v|^k) \nabla_v^\alpha \nabla_x^\beta f(t)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \leq \sum_{0 \leq |\alpha| + |\beta| \leq 2} \|(1 + |v|^k) \nabla_v^\alpha \nabla_x^\beta f_0\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \exp \left( C \int_0^T \left( 1 + \|\bar{u}(\tau)\|_{H^4(\mathbb{R}^3)} \right) d\tau \right),$$

(3.33)

where $C$ is a constant depending only on $k$.

Proof. Due to the assumptions of $f_0$ and $\bar{u}$, it is well-known that $f$ has compact support and smooth with respect to $(x, v)$ variables (see Sect. 3.1.1–3.1.2). Therefore, it suffices to prove estimate (3.33). Indeed, we observe first that

$$- \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla_v \cdot [(\bar{u} - v) \nabla_v^\alpha \nabla_x^\beta f] \nabla_v^\alpha \nabla_x^\beta f \, dv \, dx$$

$$= 3 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dv \, dx - \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (\bar{u} - v) \cdot \nabla_v [\nabla_v^\alpha \nabla_x^\beta f] \nabla_v^\alpha \nabla_x^\beta f \, dv \, dx$$

$$= 3 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dv \, dx + \frac{3}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla_v \cdot [(\bar{u} - v) [\nabla_v^\alpha \nabla_x^\beta f] [\nabla_v^\alpha \nabla_x^\beta f]] \, dv \, dx$$

$$= \frac{3}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dv \, dx. \quad (3.34)$$

On the other hand, we compute that

$$- \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{2k} \nabla_v \cdot [(\bar{u} - v) \nabla_v^\alpha \nabla_x^\beta f] \nabla_v^\alpha \nabla_x^\beta f \, dv \, dx$$

$$= 3 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{2k} |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dv \, dx - \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{2k} (\bar{u} - v) \cdot \nabla_v [\nabla_v^\alpha \nabla_x^\beta f] \nabla_v^\alpha \nabla_x^\beta f \, dv \, dx$$

$$= 3 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{2k} |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dv \, dx + \frac{3}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla_v \cdot [(|v|^{2k} (\bar{u} - v)) [\nabla_v^\alpha \nabla_x^\beta f] [\nabla_v^\alpha \nabla_x^\beta f]] \, dv \, dx$$

$$= \frac{3}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{2k} \nabla_v^\alpha \nabla_x^\beta f|^2 \, dv \, dx + \frac{3}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{2k} \nabla_v^\alpha \nabla_x^\beta f|^2 \, dv \, dx$$

$$= \frac{k}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{2k-2} \cdot (\bar{u} - v) |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dv \, dx + \frac{3}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{2k} |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dv \, dx$$

$$\leq -(k - \frac{3}{2}) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{2k} |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dv \, dx + k \|\bar{u}\|_{L^\infty} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{2k-1} |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dv \, dx
Exploiting (3.36), we estimate

Using Hölder and Young’s inequalities, we have

\[ I \le \left( -\left( k - \frac{3}{2}\right) + \frac{2k - 1}{2}\right) \int |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dx + \frac{1}{2} \|\bar{u}\|_{L^\infty} \int |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dx \]

Hence, combining (3.20) by \((1 + \|\bar{u}\|_{L^\infty})\), we obtain

\[ 2k \int |v|^{2k} |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dx \le \int |\nabla_v^\alpha \nabla_x^\beta f|^2 \, dx \]

This is a constant depending on \(d\).

Next, testing (3.20) by \((1 + |v|^2)\) and integrating over \(\mathbb{R}^3 \times \mathbb{R}^3\) with estimate (3.36), we obtain

\[ \frac{1}{2} \frac{d}{dt} \int (1 + |v|^2) |f|^2 \, dx = - \int (1 + |v|^2) \nabla_v \cdot ((\bar{u} - v) f) \, dx. \]

Taking the differential operator \(\nabla_x\) to (3.20), the equation can be rewritten as

\[ \nabla_x f_t + v \cdot \nabla_x \nabla_x f + \nabla_v \cdot ((\bar{u} - v) \nabla_x f) + \nabla_v \cdot (\nabla_x (\bar{u} - v) - f) = 0. \]

Testing (3.38) by \((1 + |v|^2)\partial_x f\), we obtain

\[ \frac{1}{2} \frac{d}{dt} \int (1 + |v|^2) |\nabla_x f|^2 \, dx = - \int (1 + |v|^2) \nabla_v \cdot ((\bar{u} - v) \nabla_x f) \nabla_x f \, dx \]

Exploiting (3.36), we estimate \(I_1\) as follows:

\[ I_1 \le C(1 + \|\bar{u}\|_{L^\infty}) \int (|v|^{2k} + 1)|\nabla_x f|^2 \, dx. \]

Using Hölder and Young’s inequalities, we have

\[ I_2 \le C\|\nabla \bar{u}\|_{L^\infty} \int (1 + |v|^2) \left( |\nabla_v f|^2 + |\nabla_x f|^2 \right) \, dx. \]

Hence, combining \(I_1\) with \(I_2\), we obtain
\[
\frac{d}{dt} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k})|\nabla_x f|^2 \, dv \, dx \\
\leq C(1 + \|\bar{u}\|_{L^\infty} + \|\nabla \bar{u}\|_{L^\infty}) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|v|^{2k} + 1)\left(|\nabla_v f|^2 + |\nabla_x f|^2\right) \, dv \, dx.
\]
\[
\leq C(1 + \|\bar{u}\|_{H^1}) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|v|^{2k} + 1)\left(|\nabla_v f|^2 + |\nabla_x f|^2\right) \, dv \, dx,
\]
where we use Sobolev embedding in last inequality. Again, taking the differential operator \(\nabla_x^2\) to (3.38), it is rewritten as
\[
\nabla_x^2 f_t + v \cdot \nabla_x \nabla_x^2 f + \nabla_v \cdot ((\bar{u} - v)\nabla_x^2 f) + 2\nabla_v \cdot (\nabla_x (\bar{u} - v)\nabla_x f) + \nabla_v \cdot (\nabla_x^2 (\bar{u} - v)f) = 0.
\]
Multiplying (3.40) with \((1 + |v|^{2k})\nabla_x^2 f\) and integrating over the phase variables, we obtain
\[
\frac{d}{dt} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k})|\nabla_x^2 f|^2 \, dv \, dx = - \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k})\nabla_v \cdot [(\bar{u} - v)\nabla_x^2 f] |\nabla_x^2 f| \, dv \, dx
\]
\[
- 2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k})\nabla_v \cdot [\nabla_x (\bar{u} - v)\nabla_x f] |\nabla_v f| \, dv \, dx
\]
\[
- \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k})\nabla_v \cdot [\nabla_x^2 (\bar{u} - v)f] |\nabla_x^2 f| \, dv \, dx := II_1 + 2II_2 + II_3.
\]
Using (3.36), Hölder and Young’s inequalities, the terms \(II_1, II_2\) and \(II_3\) are estimated, respectively, as follows.
\[
II_1 \leq C(1 + \|\bar{u}\|_{L^\infty}) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k})|\nabla_x^2 f|^2 \, dv \, dx,
\]
\[
II_2 \leq \|\nabla \bar{u}\|_{L^\infty} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k})\left(|\nabla_v \nabla_x f|^2 + |\nabla_x f|^2\right) \, dv \, dx,
\]
and
\[
II_3 \leq \|\nabla^2 \bar{u}\|_{L^\infty} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k})\left(|\nabla_v f|^2 + |\nabla^2_x f|^2\right) \, dv \, dx.
\]
Combining estimates \(II_1-II_3\), we get
\[
\frac{1}{2} \frac{d}{dt} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k})|\nabla_x^2 f|^2 \, dv \, dx \\
\leq C(1 + \|\bar{u}\|_{H^1}) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k})\left(|\nabla_v f|^2 + |\nabla_v \nabla_x f|^2 + |\nabla_x^2 f|^2\right) \, dv \, dx.
\]
Taking \(\nabla_v\) to (3.20), we have
\[
\nabla_v f_t + v \cdot \nabla_x \nabla_v f + \nabla_v \cdot ((\bar{u} - v)\nabla_v f) + \nabla_v \cdot (\partial_v (\bar{u} - v)f) = -\nabla_x f.
\]
Again, testing \((1 + |v|^{2k})\nabla_v f\) and integrating over the phase variables, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k}) |\nabla_v f|^2 \, dv \, dx = - \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k}) \nabla_v \cdot [(\bar{u} - v) \nabla_v f] \nabla_v f \, dv \, dx \\
- \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k}) \nabla_v \cdot [\nabla_v (\bar{u} - v) f] \nabla_v f \, dv \, dx \\
- \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k}) \nabla_x f \nabla_v f \, dv \, dx \\
\leq C(1 + ||\bar{u}||_{L^\infty}) \int_{\mathbb{R}^3 \times \mathbb{R}^3} (|v|^{2k} + 1)(|\nabla_x f|^2 + |\nabla_v f|^2) \, dv \, dx,
\]
where Young’s inequality is used for last term. Therefore,
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k}) |\nabla_v f|^2 \, dv \, dx \\
\leq C(1 + ||\bar{u}||_{H^s}) \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k})(|\nabla_x f|^2 + |\nabla_v f|^2) \, dv \, dx. \quad (3.43)
\]
Taking \(\nabla_v\) to (3.38), we get
\[
\nabla_v \nabla_x f_t + v \cdot \nabla_x \nabla_v \nabla_x f + \nabla_v \cdot ((\bar{u} - v) \nabla_v \partial_t f) + \nabla_v \cdot (\nabla_v \nabla_x (\bar{u} - v) f) \\
= -\nabla_x \nabla_x f_t + \nabla_v \partial_v f - \nabla_v \cdot (\nabla_x \bar{u} \nabla_v f). \quad (3.44)
\]
Testing (3.44) by \((1 + |v|^{2k})\nabla_v \nabla_x f\) and using integration by parts, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (1 + |v|^{2k}) |\nabla_v \nabla_x f|^2 \, dv \\
\leq C(1 + ||\nabla\bar{u}||_{H^s}) \int_{\mathbb{R}^3} (1 + |v|^{2k})(|\nabla_v \nabla_x f|^2 + |\nabla_v^2 f|^2 + |\nabla_x^2 f|^2) \, dv. \quad (3.45)
\]
Lastly, taking \(\nabla_v\) to (3.42), we have
\[
\nabla_v^2 f_t + v \cdot \nabla_x \partial_v^2 f + \nabla_v \cdot ((\bar{u} - v) \nabla_v^2 f) = 2\nabla_v^2 f - 2\nabla_x \nabla_v f. \quad (3.46)
\]
Testing (3.46) by \((1 + |v|^{2k})\partial_v^2 f\) and integrating over the phase variables, by the direct calculations, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (1 + |v|^{2k}) |\nabla_v \nabla f|^2 \, dv \\
\leq C(1 + ||\nabla\bar{u}||_{L^\infty}) \int_{\mathbb{R}^3} (1 + |v|^{2k})(|\nabla_v^2 f|^2 + |\nabla_x \nabla_v f|^2) \, dv. \quad (3.47)
\]
Summing up estimates (3.37), (3.39), (3.41), (3.43), (3.45) and (3.47), we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (1 + |v|^{2k}) \left(|f|^2 + |\nabla_x f|^2 + |\nabla_v f|^2 + |\nabla_v \nabla_x f|^2 + |\nabla_v^2 f|^2 + |\nabla_v^2 f|^2\right) \\
\leq C(1 + ||\nabla\bar{u}||_{H^s}) \int_{\mathbb{R}^3} (1 + |v|^{2k}) \left(|f|^2 + |\nabla_x f|^2 + |\nabla_v f|^2 + |\nabla_v \nabla_x f|^2 + |\nabla_v^2 f|^2 + |\nabla_v^2 f|^2\right) \, dv. \]
Put \( X(t) := \iint (1 + |v|^2)(|f|^2 + |\nabla_x f|^2 + |\nabla_v f|^2 + |\nabla_v \nabla_x f|^2 + |\nabla_x^2 f|^2 + |\nabla_x^2 f|^2) \, dx \, dv \). Thus, we have
\[
\frac{1}{2} \frac{d}{dt} X(t) \leq C(1 + \|\bar{u}\|_{H^4}) X(t). \tag{3.48}
\]
Applying Gronwall’s inequality to (3.48) with \( \bar{u} \in L^\infty(0; T; H^3(\mathbb{R}^3)) \cap L^2(0; T; H^4(\mathbb{R}^3)) \), we finally get the desired result, that is,
\[
\sup_{0 \leq t \leq T} \sum_{0 \leq |\alpha| + |\beta| \leq 2} \|(1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta f\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \leq \sum_{0 \leq |\alpha| + |\beta| \leq 2} \|(1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta f_0\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \exp \left( C \int_0^T \left( 1 + \|\bar{u}(\tau)\|_{H^4(\mathbb{R}^3)} \right) d\tau \right).
\]
We complete the proof. \( \square \)

Using Lemma 3.2, we obtain similar results for more general data \( f_0 \) and \( \bar{u} \). In particular, compact support of \( f_0 \) is not necessary.

**Proposition 3.3.** Let \( T > 0 \) and \( k \geq 3 \). For a given \( \bar{u} \in L^\infty(0; T; H^3(\mathbb{R}^3)) \cap L^2(0; T; H^4(\mathbb{R}^3)) \) and the initial data \( f_0 \) satisfying
\[
\sum_{0 \leq |\alpha| + |\beta| \leq 2} (1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta f_0 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3), \tag{3.49}
\]
there exists a unique solution \( f \) to (3.20) such that \((1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta f \in L^\infty(0, T; L^2(\mathbb{R}^3 \times \mathbb{R}^3)) \). Furthermore, we obtain the following estimate:
\[
\sup_{0 \leq t \leq T} \sum_{0 \leq |\alpha| + |\beta| \leq 2} \|(1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta f\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \leq \sum_{0 \leq |\alpha| + |\beta| \leq 2} \|(1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta f_0\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \exp \left( C \int_0^T \left( 1 + \|\bar{u}(\tau)\|_{H^4(\mathbb{R}^3)} \right) d\tau \right),
\]
where \( C \) is a constant depending only on \( k \).

**Proof.** For a proof, we introduce a cutoff function \( \phi_\delta \) and mollifier function \( \eta_\varepsilon \) as follows. Let \( \phi \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R}^3) \) be a function such that
\[
\phi(x, v) = \begin{cases} 
1, & |x|^2 + |v|^2 \leq 1, \\
0, & |x|^2 + |v|^2 \geq 2,
\end{cases} \tag{3.50}
\]
and for \( \delta > 0 \), we set \( \phi_\delta(x, v) := \phi(\delta x, \delta v) \) for all \((x, v) \in \mathbb{R}^3 \times \mathbb{R}^3\). For \( \varepsilon > 0 \), we define a mollifier \( \theta_\varepsilon \) to satisfy
\[
\theta \in C^\infty_c, \quad \theta \geq 0, \quad \int_{\mathbb{R}^3} \theta = 1 \quad \text{and} \quad \theta_\varepsilon(x) = \varepsilon^3 \theta \left( \frac{x}{\varepsilon} \right).
\]
Thus we let \( \eta^\varepsilon(x, v) = \theta^\varepsilon(x) \theta^\varepsilon(v) \), and let
\[
\bar{u}^\varepsilon = \bar{u} * \theta^\varepsilon(x) \quad \text{and} \quad f_\delta^\varepsilon = (\phi_\delta f_0) * \eta^\varepsilon(x, v).
\]
Then, we define \( (f_\delta^\varepsilon, \bar{u}^\varepsilon) \) as the unique solution on \([0, T]\) to the approximated equation of linear Eq. (3.20):
\[
\partial_t f_\delta^\varepsilon + \nabla_x \cdot (v f_\delta^\varepsilon) + \nabla_v \cdot \left[ (\bar{u}^\varepsilon - v) f_\delta^\varepsilon \right] = 0, \quad f_\delta^\varepsilon \big|_{t=0} = f_\delta^\varepsilon.
\tag{3.51}
\]
obtained by the method of characteristics (see Sect. 3.1.1–3.1.2). We set
\[
\mathcal{X} := \{ g \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \mid (1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta g \in L^2(\mathbb{R}^3 \times \mathbb{R}^3), \ 0 \leq |\alpha| + |\beta| \leq 2 \}.
\]

We then claim that \( f_{\delta \epsilon}^x \) strongly converges to \( f_0 \) in \( \mathcal{X} \), that is,
\[
\|(1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta (f_{\delta \epsilon}^x - f_0)\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \to 0, \quad \text{as } \delta \to 0 \text{ and } \epsilon \to 0.
\]

The proof of (3.52) is as follows: As mentioned, due to decay assumption (1.13) in Theorem 1.5, we prove that \( (1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta f_0 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \). Indeed,
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |(1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta f_0|^2 \, dv \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |(1 + |v|^p)\nabla_v^\alpha \nabla_x^\beta f_0|^2 \frac{1 + |v|^k}{1 + |v|^p} \, dv \, dx \\
\leq \int_{\mathbb{R}^3} \frac{1 + |v|^k}{1 + |v|^p} \, dv \int_{\mathbb{R}^3} \|(1 + |v|^p)\nabla_v^\alpha \nabla_x^\beta f_0\|_{L^\infty(\mathbb{R}^3)}^2 \, dx \\
\leq C \int_{\mathbb{R}^3} \|(1 + |v|^p)\nabla_v^\alpha \nabla_x^\beta f_0\|_{L^\infty(\mathbb{R}^3)}^2 \, dx < \infty,
\]
where we use \( \int_{\mathbb{R}^3} \frac{1 + |v|^k}{1 + |v|^p} \, dv < \infty \) (since \( p > \frac{2k+3}{2} \)). Considering estimate (3.53), due to the definition of \( f_{\delta \epsilon}^x \), for any \( \varsigma \) we can choose \( R_0 > 0 \) such that for a sufficiently small \( \epsilon \) and \( \delta \)
\[
\int_{\mathbb{R}^3} \int_{|v| > R_0} |(1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta (f_{\delta \epsilon}^x - f_0)|^2 \, dv \, dx < \frac{\varsigma}{2},
\]
On the other hand, we note that
\[
\int_{\mathbb{R}^3} \int_{|v| \leq R_0} |(1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta (f_{\delta \epsilon}^x - f_0)|^2 \, dv \, dx < \frac{\varsigma}{2},
\]
Via relation (3.54) and (3.55), choosing sufficiently small \( \epsilon \) and \( \delta \), we have for any \( \varsigma \)
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} |(1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta (f_{\delta \epsilon}^x - f_0)|^2 \, dv \, dx < \varsigma,
\]
which implies (3.52). Also, due to Lemma 3.2, the approximated solution \( (f_{\delta \epsilon}^x, \bar{u}^x) \) for (3.51) satisfies the following estimate
\[
\sup_{0 < t \leq T} \sum_{0 \leq |\alpha| + |\beta| \leq 2} \|(1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta f_{\delta \epsilon}^x\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \\
\leq \sum_{0 \leq |\alpha| + |\beta| \leq 2} \|(1 + |v|^k)\nabla_v^\alpha \nabla_x^\beta f_0\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}^2 \exp \left( C \int_0^T \left( 1 + \|\bar{u}(\tau)\|_{W^{4,2}} \right) \, d\tau \right).
\]
When \( \delta \) and \( \epsilon \) go to 0, there exists a weak limit \( \bar{f} \) such that \( f_{\delta \epsilon}^x \rightharpoonup \bar{f} \) (at least up to a sequence) in the weighted space \( \mathcal{X} \) and \( \bar{f} \) is a solution to equation (3.20) in the sense of distribution and moreover the solution is unique by the standard argument. Hence, we briefly give a proof for a unique solvability of the solution \( f \) to linear Eq. (3.20) with initial data (3.49) for a given \( \bar{u} \).
4. Proof of Theorem 1.5

In this section, we prove the existence of a local solution of the second equation of (1.1). For given vector field \( U : \mathbb{R}^3 \to \mathbb{R}^3 \) with \( \text{div} \ U = 0 \) and tensor field \( F : \mathbb{R}^3 \to \mathbb{R}^3 \), we first consider the following non-Newtonian Stokes-type equations with drift term

\[
\begin{cases}
\partial_t u - \nabla \cdot (G \ |Du|^2) Du + (U \cdot \nabla)u + \nabla p = F, \\
\text{div} \ u = 0,
\end{cases}
\quad \text{in } Q_T := \mathbb{R}^3 \times (0, T)
\]

(4.56)

with the initial condition

\[
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3.
\]

(4.57)

We show the local well-posedness of (4.56)–(4.57) in the next lemma.

**Lemma 4.1.** Suppose that \( U \in L^\infty(0, T; H^3(\mathbb{R}^3)) \cap L^2(0, T; H^4(\mathbb{R}^3)) \) and \( F \in L^2(0, T; H^2(\mathbb{R}^3)) \). There exists \( T^* := T^*(\|u_0\|_{H^3}, \|U\|_{(L^\infty \cap H^3)\cap (L^2 \cap H^4)}(Q_T), \|F\|_{L^2 H^2(Q_T)}) > 0 \) such that there exist unique solutions \( u \in L^\infty(0, T^*; H^3(\mathbb{R}^3)) \) \( \cap \ L^2(0, T^*; H^4(\mathbb{R}^3)) \) to Eqs. (4.56)–(4.57). Moreover, \( u \) satisfies

\[
\frac{d}{dt} \|u\|_{H^3}^2 + \frac{m_0}{2} \int_{\mathbb{R}^3} (\|\nabla^2 Du\|^2 + \|\nabla Du\|^2 + \|Du\|^2) \, dx \\
\leq C\|\nabla U\|_{L^\infty} \|u\|_{H^3}^2 + C\|F\|_{H^2}^2 + C\|G\|_{H^3} \|Du\|_{H^2} \L_\infty (\|Du\|_{H^3}^2 + \|Du\|_{L^\infty})\|\nabla^2 Du\|_{L^2}^2 
\]

(4.58)

**Proof.** The proof is similar to that in [13, Proposition 3.1]. The only difference is the control of the class \( F \). For the sake of convenience, we give a proof in Appendix. \( \square \)

For proof of Theorem 1.5 using Schauder fixed point theorem (Lemma 2.2), we introduce a function space \( X_M \) defined as follows:

\[
X_M := \{ u : \mathbb{R}^3 \times [0, T) \to \mathbb{R}^3 : \|u\|_X < M \quad \text{and} \quad \|u_t\|_{L^2(0, T; L^2(\mathbb{R}^3))} < \infty \}.
\]

with the norm \( \|u\|_X^2 := \|u\|_{L^\infty(0, T; H^3(\mathbb{R}^3))} + \frac{m_0}{2} \|u\|_{L^2(0, T; H^4(\mathbb{R}^3))}^2 \).

**Proof of Theorem 1.5.** First, we define a map \( \Theta : X_M \to X_M \) by \( \Theta(\bar{u}) = u \):

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \left[ \left( \bar{u} - v \right) f \right] &= 0, \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3, \quad t > 0, \\
\partial_t u - \nabla \cdot (G |Du|^2) Du + (\bar{u} \cdot \nabla) u + \nabla p &= - \int_{\mathbb{R}^3} (\bar{u} - v) f \, dv, \\
\nabla_x \cdot u &= 0.
\end{align*}
\]

(Step A: \( \Theta \) is well defined): We set

\[
g(x) := C[G(x^2)(x^2 + x) + (G^2(x^2) + G^8(x^2))(x^2 + x^4)x^4] \quad \text{and} \quad g_M := \sup_{0 \leq x \leq M} g(x),
\]

\[
F := \int_{\mathbb{R}^3} (\bar{u} - v) f \, dv.
\]

From estimate (4.58) in Lemma 4.1, we know

\[
\frac{d}{dt} \|u\|_{H^3}^2 + \frac{m_0}{2} \int_{\mathbb{R}^3} (\|\nabla^2 Du\|^2 + \|\nabla Du\|^2 + \|Du\|^2) \, dx \\
\leq C\|\nabla \bar{u}\|_{L^\infty} \|u\|_{H^3}^2 + C\|F\|_{H^2}^2 + Cg(\|u\|_{H^3(\mathbb{R}^3)}) \|u\|_{H^3(\mathbb{R}^3)}^2 
\]

(4.59)

\[
\leq C(M\|u\|_{H^3}^2 + g_M\|u\|_{H^3(\mathbb{R}^3)}^2) + C\|F\|_{H^2}^2.
\]
Once we have $F \in L^2(0,T;H^2(\mathbb{R}^3))$, we can choose $0 < T_1 \leq T^*$ ($T^*$ given in Lemma 4.1) such that

$$\|u\|_{L^\infty(0,T_1;H^3(\mathbb{R}^3))} + \frac{m_0}{2} \|u\|_{L^2(0,T_1;H^4(\mathbb{R}^3))} \leq M.$$

Now, we check $F \in L^2(0,T;H^2(\mathbb{R}^3))$. Indeed, we note that

$$\left| \int_{\mathbb{R}^3} v f \, dv \right| = \left| \int_{|v| \leq 1} v f \, dv + \int_{|v| > 1} v f \, dv \right|$$

$$\leq \int_{|v| \leq 1} f \, dv + \left( \int_{|v| > 1} |v|^{2k} f^2 \, dv \right)^{\frac{1}{2}} \left( \int_{|v| > 1} \frac{1}{|v|^{2(k-1)}} \, dv \right)^{\frac{1}{2}}$$

$$\leq \int_{|v| \leq 1} f \, dv + C \left( \int_{|v| > 1} |v|^{2k} f^2 \, dv \right)^{\frac{1}{2}}.$$

Using Jensen and Hölder’s inequalities, we have

$$\int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} v f \, dv \right|^2 \, dx \leq C \int_{\mathbb{R}^3} \left( \int_{|v| \leq 1} |v|^{2k} f^2 \, dv \right) dx + C \int_{\mathbb{R}^3} \left( \int_{|v| > 1} |v|^{2k} f^2 \, dv \right) dx$$

$$\leq C \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2k}) f^2 \, dv \, dx.$$

Thus, we have

$$\left\| \hat{u} \int_{\mathbb{R}^3} v f \, dv \right\|_{L^2(\mathbb{R}^3)} \leq \|\hat{u}\|_{L^\infty(\mathbb{R}^3)} \left\| \int_{\mathbb{R}^3} v f \, dv \right\|_{L^2(\mathbb{R}^3)}$$

$$\leq C \|\hat{u}\|_{L^\infty(\mathbb{R}^3)} \left( \left\| (1 + |v|^{k}) f \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \right).$$

Similarly,

$$\left\| \hat{u} \int_{\mathbb{R}^3} f \, dv \right\|_{L^2(\mathbb{R}^3)} \leq C \|\hat{u}\|_{L^\infty(\mathbb{R}^3)} \left( \left\| (1 + |v|^{k}) f \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \right).$$

Using the same method, we can check that

$$\left\| \int_{\mathbb{R}^3} f \, dv \right\|_{H^2(\mathbb{R}^3)} + \left\| \int_{\mathbb{R}^3} v f \, dv \right\|_{H^2(\mathbb{R}^3)} \leq C \sum_{0 \leq \beta \leq 2} \left\| (1 + |v|^{k}) \partial_x^\beta f \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}.$$ (4.60)

Using estimate (4.60), we get

$$\left\| \hat{u} \int_{\mathbb{R}^3} f \, dv \right\|_{H^2(\mathbb{R}^3)} \leq \|\hat{u}\|_{H^2(\mathbb{R}^3)} \left\| \int_{\mathbb{R}^3} f \, dv \right\|_{L^2(\mathbb{R}^3)} + \|\hat{u}\|_{L^2(\mathbb{R}^3)} \left\| \int_{\mathbb{R}^3} f \, dv \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}$$

$$\leq C \sum_{0 \leq \beta \leq 2} \|\hat{u}\|_{H^2(\mathbb{R}^3)} \left\| (1 + |v|^{k}) \partial_x^\beta f \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)}.$$
Finally, using estimate (3.33), we obtain
\[
\int_0^{T_1} \left\| \int \bar{u} - v \, f \, dv \right\|^2_{H^2(\mathbb{R}^3)} \, dt \\
\leq C \sum_{0 \leq \beta \leq 2} \int_0^{T_1} \left( \| \bar{u} \|^2_{H^2(\mathbb{R}^3)} + 1 \right) \| (1 + |v|^k) \partial_x^\beta f \|^2_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \, dt \\
\leq C \sum_{0 \leq \beta \leq 2} \sup_{0 < \tau \leq T_1} \left( \| (1 + |v|^k) \partial_x^\beta f(\tau) \|^2_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \right) \int_0^{T_1} \left( \| \bar{u} \|^2_{H^2(\mathbb{R}^3)} + 1 \right) \, dt < \infty, \tag{4.61}
\]
which implies \( F \in L^2(0, T_1; H^2(\mathbb{R}^3)) \). Note that
\[
\int_0^T \| u \|^2_{H^4} \, dt \leq T \sup_{0 \leq t \leq T} \| u(t) \|^2_{H^3(\mathbb{R}^3)} + \int_0^T \| \nabla u \|^2_{H^3} \, dt \\
\leq T \sup_{0 \leq t \leq T} \| u(t) \|^2_{H^3(\mathbb{R}^3)} + C \int_0^T \| Du \|^2_{H^3} \, dt, \tag{4.62}
\]
where Korn’s inequality was used. Applying Gronwall’s inequality to (4.59), we get
\[
\sup_{0 \leq t \leq T_1} \| u(t) \|^2_{H^3(\mathbb{R}^3)} + \frac{m_0}{2} \int_0^{T_1} \| u \|^2_{H^4(\mathbb{R}^3)} \, dt \\
\leq C \left( 1 + \frac{T_1 m_0}{2} \right) \| u_0 \|^2_{H^3(\mathbb{R}^3)} e^{CT_1} + \int_0^{T_1} \| F \|^2_{H^2} e^{C(T_1-t)} \, dt \\
\leq C \left( 1 + \frac{T_1 m_0}{2} \right) e^{CT_1} \left[ \| u_0 \|^2_{H^3(\mathbb{R}^3)} + \sum_{0 \leq \beta \leq 2} \int_0^{T_1} \left( \| \bar{u} \|^2_{H^2(\mathbb{R}^3)} + 1 \right) \| (1 + |v|^k) \partial_x^\beta f \|^2_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \, dt \right] \\
\leq C \left( 1 + \frac{T_1 m_0}{2} \right) e^{CT_1 + C \sqrt{T_1 \sqrt{M}}} \left[ \| u_0 \|^2_{H^3(\mathbb{R}^3)} + \sum_{0 \leq \beta \leq 2} \| (1 + |v|^k) \partial_x^\beta f_0 \|^2_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} (1 + M) T_1 \right], \tag{4.63}
\]
where we use estimate (4.61) in second inequality with (4.62) and the result in Proposition 3.3 and Hölder’s inequality for \( \bar{u} \in X_M \) in third inequality. If we choose a sufficiently small \( 0 < \bar{T} \leq T_1 \) such that
\[
C \left( 1 + \frac{\bar{T} m_0}{2} \right) e^{C \bar{T} + C \sqrt{\bar{T} \sqrt{M}}} \left[ \| u_0 \|^2_{H^3(\mathbb{R}^3)} + \sum_{0 \leq \beta \leq 2} \| (1 + |v|^k) \partial_x^\beta f_0 \|^2_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} (1 + M) \bar{T} \right] < M,
\]
then \( \| u \|_{X_M} < M \). Lastly, we also prove that \( u_t \in L^2(0, \bar{T}; L^2(\mathbb{R}^3)) \). It is enough to show \( F := \int (\bar{u} - v) f \, dv \in L^2(0, \bar{T}; L^2(\mathbb{R}^3)) \) due to (A.90) and (A.91). Indeed, by estimate (4.61) for the control of the drag force and \( u \in L^\infty(0, \bar{T}; H^3(\mathbb{R}^3)) \cap L^2(0, T_1; H^4(\mathbb{R}^3)) \), it follows that
Assume Lemma 5.1. Let $\Theta$ is continuous w.r.t $X\Theta$: with initial condition $\rho$. We consider the equation for $\tilde{\rho}$.

The fixed point theorem gives us the existence of solutions for system (1.1) and contraction property (5.70)

which implies $u_t \in L^2(0, \tilde{T}; L^2(\mathbb{R}^3))$.

(Step B: $\Theta$ is continuous w.r.t $L^2(0, \tilde{T}; L^2(\mathbb{R}^3))$–topology): From Corollary 5.2 in Sect. 5, we know that $\Theta : X_M \hookrightarrow X_M$ is continuous w.r.t the topology inherited from $L^2(0, \tilde{T}; L^2(\mathbb{R}^3))$. Due to the Aubin-Lions lemma, $X_M$ is a compact subset of $L^2(0, \tilde{T}; L^2(\mathbb{R}^3))$ and $X_M$ is clearly convex. Hence, the Schauder’s fixed point theorem gives us the existence of solutions for system (1.1) and contraction property (5.70) says the uniqueness of solutions.

5. Contraction of the iteration with respect to $L^2(0, T; L^2(\mathbb{R}^3))$

Lemma 5.1. Assume $u_0 \in H^3(\mathbb{R}^3)$ and, for $i = 1, 2$, $\bar{u}_i \in L^\infty(0, T; H^3(\mathbb{R}^3)) \cap L^2(0, T; H^4(\mathbb{R}^3))$. Let $f_i \in AC_2(0, T; \mathcal{P}_2(\mathbb{R}^3 \times \mathbb{R}^3)$ be the solution of (3.30) with a given $f_0 \in \mathcal{P}_2$ satisfying (3.26) and set $\rho_i(t, x) := \int f_i(t, x, v)dv$ for $i = 1, 2$. Suppose that $u_i \in L^\infty(0, T; H^3(\mathbb{R}^3)) \cap L^2(0, T; H^4(\mathbb{R}^3))$ is the solution of

$$\partial_t u_i - \Delta u_i + (\bar{u}_i \cdot \nabla) u_i + \nabla_x p_i = - \int (\bar{u}_i - v) f_i dv, \quad \nabla_x u_i = 0 \quad \text{in} \quad Q_T := \mathbb{R}^3 \times (0, T)$$

with initial condition $u_i(0) = u_0$. Then, we have

$$\|u_1 - u_2\|_{L^\infty(0, T; L^2(\mathbb{R}^3))} \leq e^{A C_1 T} \|C_1 T + C_3\| \|\bar{u}_1 - \bar{u}_2\|_{L^2(0, T; L^2(\mathbb{R}^3))},$$

(5.64)

where

$$C_1 := C_1 \left(\|\rho_1\|_{L^\infty(Q_T)}, \|\bar{u}_2\|_{L^\infty H^3_\sigma(Q_T)}, \|u_1 - u_2\|_{L^\infty H^3_\sigma(Q_T)}\right)$$

and

$$C_3 := C_3 \left(\|\rho_1\|_{L^\infty(Q_T)}, \|\bar{u}_2\|_{L^\infty H^3_\sigma(Q_T)}\right).$$

Estimate (5.64) implies that there exists $T \ll 1$ such that

$$\|u_1 - u_2\|_{L^2(0, T; L^2(\mathbb{R}^3))} \leq \frac{1}{2} \|\bar{u}_1 - \bar{u}_2\|_{L^2(0, T; L^2(\mathbb{R}^3))}.$$ 

Proof. We consider the equation for $\tilde{\rho} := u_1 - u_2$ and $\tilde{p} := p_1 - p_2$:

$$\partial_t \tilde{u} - \nabla \cdot [G(|Du_1|^2)Du_1 - G(|Du_2|^2)Du_2] + \nabla \tilde{p} + (u_1 \cdot \nabla) \tilde{u}
= -((\bar{u}_1 - \bar{u}_2) \cdot \nabla) u_2 - \int f_1 dv (\bar{u}_1 - \bar{u}_2) + \int (f_2 - f_1) dv \bar{u}_2 + \int v (f_1 - f_2) dv,$$
and we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \tilde{u} \|^2_{L^2} + m_0 \| \nabla \tilde{u} \|^2_{L^2} = - \int_{\mathbb{R}^3} \tilde{u} \cdot ((\tilde{u}_1 - \tilde{u}_2) \cdot \nabla) u_2 \, dx - \int_{\mathbb{R}^3} \rho_1 (\tilde{u}_1 - \tilde{u}_2) \cdot \tilde{u} \, dx + A(t)
\]
\[
= I_1 + I_2 + A(t),
\]
where
\[
A(t) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \tilde{u} \cdot v (f_1 - f_2) \, dv \, dx - \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \tilde{u} \cdot \tilde{u}_2 (f_1 - f_2) \, dv \, dx.
\]

By Hölder and Young’s inequalities, we note,
\[
I_1 \leq C \| \nabla u_2 \|_{L^\infty} \left( \| \tilde{u} \|^2_{L^2} + \| \tilde{u}_1 - \tilde{u}_2 \|^2_{L^2} \right),
\]
and
\[
I_2 \leq C \| \rho_1 \|_{L^\infty} \left( \| \tilde{u} \|^2_{L^2} + \| \tilde{u}_1 - \tilde{u}_2 \|^2_{L^2} \right).
\]

On the other hand, we have
\[
A(t) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \tilde{u} \cdot v (f_1 - f_2) \, dv \, dx - \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \tilde{u} \cdot \tilde{u}_2 (f_1 - f_2) \, dv \, dx
\]
\[
:= A_1 + A_2.
\]

We note that
\[
A_1 = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} [\tilde{u}(x_1, t) \cdot v_1 - \tilde{u}(x_2, t) \cdot v_2] d\gamma_t(x_1, v_1, x_2, v_2)
\]
\[
= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \tilde{u}(x_1, t) \cdot (v_1 - v_2) + (\tilde{u}(x_1, t) - \tilde{u}(x_2, t)) \cdot v_2 d\gamma_t(x_1, v_1, x_2, v_2)
\]
\[
:= A_{11} + A_{12}.
\]

Now we estimate
\[
|A_{11}| \leq 2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |\tilde{u}(x_1, t)|^2 + |v_1 - v_2|^2 d\gamma_t(x_1, v_1, x_2, v_2)
\]
\[
\leq 2 \left( \| \rho_1 \|_{L^\infty} \| \tilde{u} \|^2_{L^2} + Q(t) \right),
\]
and we exploit (2.16) and (2.17) to get
\[
|A_{12}| \leq \iint_{\mathbb{R}^3} \iint_{\mathbb{R}^3} (M \nabla \bar{u}(x_1, t) + M \nabla \bar{u}(x_2, t)) |x_1 - x_2| |v_2| d\gamma_t(x_1, v_1, x_2, v_2)
\]
\[
\leq \varepsilon \iint_{\mathbb{R}^3} \iint_{\mathbb{R}^3} ((M \nabla \bar{u})^2(x_1, t) + (M \nabla \bar{u})^2(x_2, t)) |v_2|^2 d\gamma_t(x_1, v_1, x_2, v_2)
\]
\[+ \frac{1}{\varepsilon} \iint_{\mathbb{R}^3} \iint_{\mathbb{R}^3} |x_1 - x_2|^2 d\gamma_t(x_1, v_1, x_2, v_2)
\]
\[
\leq \varepsilon \iint_{\mathbb{R}^3} \iint_{\mathbb{R}^3} ((M \nabla \bar{u})^2(x_1, t)) |v_1 - v_2|^2 + |v_1|^2 d\gamma_t(x_1, v_1, x_2, v_2)
\]
\[+ \varepsilon \iint_{\mathbb{R}^3} (M \nabla \bar{u})^2(x, t) |v|^2 f_2(x, v, t) \, dx \, dv + \frac{1}{\varepsilon} Q(t)
\]
\[
\leq \varepsilon \left( \|\nabla \bar{u}\|_{L^\infty}^2 Q(t) + (\|m_2 f_1\|_{L^\infty} + \|m_2 f_2\|_{L^\infty}) \|M \nabla \bar{u}\|_{L^2}^2 \right) + \frac{1}{\varepsilon} Q(t)
\]
\[
\leq \varepsilon \left( \|\nabla \bar{u}\|_{L^\infty}^2 Q(t) + (\|m_2 f_1\|_{L^\infty} + \|m_2 f_2\|_{L^\infty}) \|\nabla \bar{u}\|_{L^2}^2 \right) + \frac{1}{\varepsilon} Q(t).
\]

Adding above estimates,
\[
|A_1| \leq 2(\|\rho_1\|_{L^\infty} \|\bar{u}\|_{L^2}^2 + Q(t)) + \varepsilon(\|\nabla \bar{u}\|_{L^\infty}^2 Q(t)
\]
\[+ (\|m_2 f_1\|_{L^\infty} + \|m_2 f_2\|_{L^\infty}) \|\nabla \bar{u}\|_{L^2}^2) + \frac{1}{\varepsilon} Q(t).
\]

On the other hand,
\[
A_2 = \iint_{\mathbb{R}^3} \iint_{\mathbb{R}^3} \bar{u}(x_1, t) \cdot (\bar{u}_2(x_1, t) - \bar{u}_2(x_2, t)) + (\bar{u}(x_1, t) - \bar{u}(x_2, t)) \cdot \bar{u}_2(x_2, t) d\gamma_t(x_1, v_1, x_2, v_2)
\]
\[
\leq \|\nabla \bar{u}_2\|_{L^\infty} \iint_{\mathbb{R}^3} \iint_{\mathbb{R}^3} (|\bar{u}(x_1, t)|^2 + |x_1 - x_2|^2) \, d\gamma_t(x_1, v_1, x_2, v_2)
\]
\[+ \|\bar{u}_2\|_{L^\infty} \iint_{\mathbb{R}^3} \iint_{\mathbb{R}^3} \varepsilon(|\nabla \bar{u}(x_1, t)|^2 + |\nabla \bar{u}(x_2, t)|^2) + \frac{1}{\varepsilon} |x_1 - x_2|^2 d\gamma_t(x_1, v_1, x_2, v_2)
\]
\[
\leq \|\nabla \bar{u}_2\|_{L^\infty} (\|\rho_1\|_{L^\infty} \|\bar{u}\|_{L^2}^2 + Q(t)) + \|\bar{u}_2\|_{L^\infty} (\varepsilon \|\nabla \bar{u}\|_{L^2}^2 + \frac{1}{\varepsilon} Q(t)).
\]

Combining (5.65) and (5.66), we have
\[
A(t) \leq C_1 (\|\bar{u}\|_{L^2}^2 + Q(t)) + \varepsilon C_2 \|\nabla \bar{u}\|_{L^2}^2,
\]
where
\[
C_1 = C_1 (\|\rho_1\|_{L^\infty(Q_T)}, \|\bar{u}_2\|_{L^\infty H^1_3(Q_T)}, \|\bar{u}\|_{L^\infty H^1_3(Q_T)}), \text{ and } C_2 \text{ depends on } (\|m_2 f_1\|_{L^\infty(Q_T)}, \\
\|m_2 f_2\|_{L^\infty(Q_T)}, \|\bar{u}_2\|_{L^\infty H^1_3(Q_T)}).\]

Hence, we have
\[
\frac{1}{2} \frac{d}{dt} \|\bar{u}\|_{L^2}^2 + (m_0 - \varepsilon C) \|\nabla \bar{u}\|_{L^2}^2
\]
\[
\leq (\|\nabla u_2\|_{L^\infty} + \|\rho_1\|_{L^\infty}) (\|\bar{u}\|_{L^2}^2 + \|\bar{u}_1 - \bar{u}_2\|_{L^2}^2) + C_1 (\|\bar{u}\|_{L^2}^2 + Q(t))
\]
\[\leq C_1 (\|\bar{u}\|_{L^2}^2 + Q(t)) + C_3 \|\bar{u}_1 - \bar{u}_2\|_{L^2}^2,
\]
where $C_3$ depends on $\|\rho_1\|_{L^\infty(Q_T)}$ and $\|\tilde{u}_2\|_{L^\infty H_0^2(Q_T)}$. Plugging (3.31) into (5.68), we get

$$
\frac{1}{2} \frac{d}{dt} \|\tilde{u}(t)\|_{L^2}^2 \leq C_1 \|\tilde{u}(t)\|_{L^2}^2 + C_3 \|\tilde{u}_1 - \tilde{u}_2\|_{L^2}^2 + C_1 e^{C_1 t} \int_0^t \|\tilde{u}_1 - \tilde{u}_2\|_{L^2}^2 ds.
$$

Using Gronwall’s lemma with $\tilde{u}(0) \equiv 0$, we obtain

$$
\|\tilde{u}(t)\|_{L^2}^2 \leq e^{2C_1 t} \int_0^t (g(s) + h(s)) ds,
$$

(5.69)

where

$$
g(t) := 2C_3 \|\tilde{u}_1 - \tilde{u}_2\|_{L^2}^2 \quad h(t) := 2C_1 e^{C_1 t} \int_0^t \|\tilde{u}_1 - \tilde{u}_2\|_{L^2}^2 ds.
$$

From (5.69), we have

$$
\|\tilde{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq e^{4C_1 T} (C_3 + C_1 T) \|\tilde{u}_1 - \tilde{u}_2\|_{L^2(0,T;L^2(\mathbb{R}^3))}^2,
$$

which gives us

$$
\|\tilde{u}\|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq T \|\tilde{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq T e^{4C_1 T} (C_3 + C_1 T) \|\tilde{u}_1 - \tilde{u}_2\|_{L^2(0,T;L^2(\mathbb{R}^3))}^2 \leq \frac{1}{4} \|\tilde{u}_1 - \tilde{u}_2\|_{L^2(0,T;L^2(\mathbb{R}^3))}^2
$$

for some small $T \ll 1$. \hfill \square

**Corollary 5.2.** Assume $u_0 \in H^3(\mathbb{R}^3)$ and $f_0 \in \mathcal{P}_2(\mathbb{R}^3 \times \mathbb{R}^3)$ satisfying (3.26). Let

$$
X_M = \{u : \|u\|_{L^\infty(0,T;H^3(\mathbb{R}^3))} + \|u\|_{L^2(0,T;H^4(\mathbb{R}^3))} \leq M \}
$$

and $\Theta : X_M \to X_M$ be defined by $u := \Theta(u)$ where

$$
\partial_t u - \Delta u + (\tilde{u} \cdot \nabla) u + \nabla_x p = -3 \int_\mathbb{R}^3 (\tilde{u} - v) f dv, \quad \nabla_x \cdot u = 0,
$$

$$
\partial_t f + \nabla_x \cdot (vf) + \nabla_x \cdot [(\tilde{u} - v)f] = 0,
$$

$$
u_1(0) = u_0, \quad f(0) = f_0.
$$

Then, for small $T \ll 1$, $\Theta$ is a contraction mapping with respect to the topology induced by the norm $L^2(0,T;L^2(\mathbb{R}^3))$. That is, we have

$$
\|u_1 - u_2\|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq \frac{1}{2} \|\tilde{u}_1 - \tilde{u}_2\|_{L^2(0,T;L^2(\mathbb{R}^3))},
$$

(5.70)

for small $T \ll 1$.

**Acknowledgements**

We would like to thank the anonymous referee for valuable comments that improved the quality of this paper.

**Author contributions** Each of the authors contributed to each part of this study equally.
Funding Kyungkeun Kang’s work is supported by NRF-2019R1A2C1084685. Hwa Kil Kim’s work is supported by NRF-2021R1F1A1048231. Jae-Myong Kim was supported by National Research Foundation of Korea Grant funded by the Korean Government (NRF-2020R1C1C1A01006521).

Declarations
Conflict of interest The authors declare no competing interests.

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Appendix A

In this section, as mentioned, we give a proof of Lemma 4.1.

Proof of Lemma 4.1. We suppose that $u$ is regular. We then compute certain a priori estimates. First of all, we note that by the $L^2$-energy estimate with (1.3),

$$
\frac{d}{dt} \|u\|^2_{L^2(\mathbb{R}^3)} + m_0 \|Du\|^2_{L^2(\mathbb{R}^3)} \leq C \|u\|^2_{L^2(\mathbb{R}^3)} + C \|F\|^2_{L^2(\mathbb{R}^3)}. 
$$

(A.71)

\begin{itemize}
  \item (\|\nabla u\|_{L^2}-estimate) Taking derivative $\partial_{x_i}$ to (4.56) and multiplying $\partial_{x_i}, u,$

  $$
  \frac{1}{2} \frac{d}{dt} \|\partial_{x_i} u\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \partial_{x_i} (G [\|Du\|^2] Du) : \partial_{x_i} Du \, dx
  $$

  $$
  = - \int_{\mathbb{R}^3} \partial_{x_i} ((U \cdot \nabla) u) \cdot \partial_{x_i} u \, dx - \int_{\mathbb{R}^3} F \cdot \partial_{x_i} \partial_{x_i} u \, dx.
  $$

\end{itemize}

Noting that

$$
\partial_{x_i} (G [\|Du\|^2] Du) : \partial_{x_i} Du = [\partial_{x_i} G [\|Du\|^2] Du + G [\|Du\|^2] \partial_{x_i} Du] : \partial_{x_i} Du
$$

$$
= 2G' [\|Du\|^2] (Du : \partial_{x_i} Du) (Du : \partial_{x_i} Du) + G [\|Du\|^2] |\partial_{x_i} Du|^2
$$

$$
= 2G' [\|Du\|^2] |Du : \partial_{x_i} Du|^2 + G [\|Du\|^2] |\partial_{x_i} Du|^2,
$$

we have

$$
\frac{1}{2} \frac{d}{dt} \|\partial_{x_i} u\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} G [\|Du\|^2] |\partial_{x_i} Du|^2 \, dx + \int_{\mathbb{R}^3} 2G' [\|Du\|^2] |Du : \partial_{x_i} Du|^2 \, dx
$$

$$
= - \int_{\mathbb{R}^3} \partial_{x_i} ((U \cdot \nabla) u) \cdot \partial_{x_i} u \, dx - \int_{\mathbb{R}^3} F \cdot \partial_{x_i} \partial_{x_i} u \, dx. \quad \text{(A.72)}
$$

Taking $A = Du$ and $B = \partial_{x_i} Du$ in the last inequality, we use the following inequality:

$$
G[|A|^2]|B|^2 + 2G'[|A|^2](A : B)^2 \geq m_0|B|^2. \quad \text{(A.73)}
$$

Indeed, if $G'[|A|^2] \geq 0$, then

$$
G[|A|^2]|B|^2 + 2G'[|A|^2](A : B)^2 \geq G[|A|^2]|B|^2 \geq m_0|B|^2.
$$
In case that \( G' \|A\|^2 < 0 \), we note that
\[
G(\|A\|^2) |B|^2 + 2G(\|A\|^2)(A : B)^2 \geq (G(\|A\|^2) + 2G'(\|A\|^2)\|A\|^2) |B|^2 \geq m_0 |B|^2.
\]
Applying inequality (A.73) to (A.72), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\partial_x, u\|^2_{L^2(\mathbb{R}^3)} + \int m_0 |\partial_x, Du|^2 \, dx 
\leq - \int \partial_x_i \left( ((U \cdot \nabla) u) \cdot \partial_{x_i} u \right) \, dx - \int F \cdot \partial_x, \partial_{x_i} u \, dx. 
\]
We will treat the term in right-hand side caused by convection together later.

- \(|\nabla^2 u|_{L^2}\)-estimate: Taking the derivative \( \partial_{x_j} \partial_{x_i} \), on (4.56) and multiplying it by \( \partial_x, \partial_{x_i} u \),
\[
\frac{1}{2} \frac{d}{dt} \|\partial_{x_j} \partial_{x_i} u\|^2_{L^2(\mathbb{R}^3)} + \int \partial_{x_j} \partial_{x_i} \left[ G(\|Du\|^2) \, Du \right] : \partial_{x_j} \partial_{x_i} Du \, dx 
\leq \int \partial_{x_j} \partial_{x_i} \left( ((U \cdot \nabla) u) \cdot \partial_{x_i} \partial_{x_j} u \right) \, dx - \int \partial_{x_j} \partial_{x_i} F : \partial_{x_j} \partial_{x_i} u \, dx. 
\]
We observe that
\[
\int \partial_{x_j} \partial_{x_i} \left[ G(\|Du\|^2) \, Du \right] : \partial_{x_j} \partial_{x_i} Du \, dx 
= \int G(\|Du\|^2) |\partial_{x_j} \partial_{x_i} Du|^2 + \sum_{\sigma} \int \partial_{x_{\sigma(i)}} G(\|Du\|^2) (\partial_{x_{\sigma(i)}} Du : \partial_{x_j} \partial_{x_i} Du) \, dx 
+ \int \partial_{x_j} \partial_{x_i} G(\|Du\|^2) (Du : \partial_{x_j} \partial_{x_i} Du) \, dx =: I_{22} + I_{23} + I_{24},
\]
where \( \sigma : \{i, j\} \rightarrow \{i, j\} \) is a permutation of \( \{i, j\} \). We separately estimate terms \( I_{22} \) and \( I_{23} \) in (A.76). Using Hölder, Young’s and Gagliardo-Nirenberg inequalities, we have for \( I_{22} \)
\[
|I_{22}| = \left| \int 2G' \|Du\|^2 (Du : \partial_{x_{\sigma(i)}} Du)(\partial_{x_{\sigma(i)}} Du : \partial_{x_j} \partial_{x_i} Du) \, dx \right| 
\leq C \|G \|_{L^\infty} \|\nabla Du\|^2_{L^4} \|\nabla^2 Du\|_{L^2} 
\leq C \|G \|_{L^\infty} \|Du\|_{L^\infty} \|\nabla^2 Du\|_{L^2},
\]
where we used condition (1.3).

For \( I_{23} \), using Lemma 2.3, we compute
\[
I_{23} = \int 2(\sqrt{G'} \|Du\|^2)(Du : \partial_{x_j} \partial_{x_i} Du) + E_2 (Du : \partial_{x_j} \partial_{x_i} Du) \, dx 
= \int E_2 (Du : \partial_{x_j} \partial_{x_i} Du) \, dx + 2 \int G' \|Du\|^2 |Du : \partial_{x_j} \partial_{x_i} Du|^2 \, dx 
:= I_{231} + I_{232}.
\]
The term \( I_{231} \) is estimated as
\[
|I_{231}| \leq C \|G \|_{L^\infty} \|Du\|_{L^\infty} \|\nabla Du\|^2_{L^4} \|\nabla^2 Du\|_{L^2} 
\leq C \|G \|_{L^\infty} \|Du\|_{L^\infty} \|\nabla^2 Du\|_{L^2},
\]
where we used the first inequality of (2.14). We combine estimates (A.75)–(A.77) to get
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\partial_x \partial_x u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} G\left[\|Du\|^2\right]|\partial_x \partial_x Du|^2 + \int_{\mathbb{R}^3} 2G'[\|Du\|^2]|Du : \partial_x \partial_x Du|^2 \\
\leq C\|G\left[\|Du\|^2\right]\|_{L^\infty} + \|Du\|_{L^2}^2 \|\nabla^2 Du\|_{L^2}^2 - \int_{\mathbb{R}^3 \mathbb{R}^3} \partial_{x_i} \partial_{x_j} ((u \cdot \nabla u)) \cdot \partial_{x_i} \partial_{x_j} u. 
\end{align*}
\]

Similarly as in (A.74), we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\partial_x \partial_x u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} m_0 |\partial_x \partial_x Du|^2 \\
\leq C\|G\left[\|Du\|^2\right]\|_{L^\infty} + \|Du\|_{L^2}^2 \|\nabla^2 Du\|_{L^2}^2 - \int_{\mathbb{R}^3} \partial_{x_i} \partial_{x_j} ((U \cdot \nabla u)) \cdot \partial_{x_i} \partial_{x_j} u - \int_{\mathbb{R}^3} \partial_{x_i} F \cdot \partial_{x_i} \partial_{x_j} \partial_{x_i} u dx. 
\end{align*}
\]  

(A.78)

• (\|\nabla^3 u\|_{L^2}-estimate) For convenience, we denote \(\partial^3 := \partial_{x_i} \partial_{x_j} \partial_{x_k}\). Similarly as before, taking the derivative \(\partial^3\) on (4.56) and multiplying it by \(\partial^3 u\),
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\partial^3 u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \partial^3 G\left[\|Du\|^2\right] Du : \partial^3 Du dx \\
- \int_{\mathbb{R}^3} \partial^3 ((U \cdot \nabla u)) \cdot \partial^3 u dx - \int_{\mathbb{R}^3} \partial^3 F : \partial^3 u dx. 
\end{align*}
\]  

(A.79)

Direct computations show that
\[
\begin{align*}
\int_{\mathbb{R}^3} \partial^3 G\left[\|Du\|^2\right] Du : \partial^3 Du dx \\
= \int_{\mathbb{R}^3} G\left[\|Du\|^2\right] |\partial^3 Du|^2 dx \\
+ \sum_{\sigma_3} \int_{\mathbb{R}^3} \partial_{x_{\sigma_3(i)}} G\left[\|Du\|^2\right] (\partial_{x_{\sigma_3(k)}} \partial_{x_{\sigma_3(j)}} Du : \partial^3 Du) dx \\
+ \int_{\mathbb{R}^3} \partial^3 G\left[\|Du\|^2\right] (Du : \partial^3 Du) dx = I_{31} + I_{32} + I_{33} + I_{34}, 
\end{align*}
\]

where \(\sigma_3 = \pi_3 \circ \tilde{\sigma}_3\) such that \(\tilde{\sigma}_3 : \{i, j, k\} \rightarrow \{i, j, k\}\) is a permutation of \(\{i, j, k\}\) and \(\pi_3\) is a mapping from \(\{i, j, k\}\) to \(\{1, 2, 3\}\).

We separately estimate terms \(I_{32}, I_{33}\) and \(I_{34}\). We note first that
\[
\begin{align*}
|I_{32}| \leq \int_{\mathbb{R}^3} 2(G\left[\|Du\|^2\right]\|Du\|\|\partial_{x_{\sigma_3(i)}} Du\|\|\partial_{x_{\sigma_3(k)}} \partial_{x_{\sigma_3(j)}} Du\| |\partial^3 Du| dx \\
\leq C\|G\left[\|Du\|^2\right]\|_{L^\infty} \|\nabla Du\|_{L^8} \|\nabla^2 Du\|_{L^3} \|\nabla^3 Du\|_{L^2} 
\end{align*}
\]  

(A.80)
For $I_{33}$, we have

$$
|I_{33}| = \int_{\mathbb{R}^3} (2G'[|Du|^2]Du : \partial_{x_3(j)} \partial_{x_3(i)} Du + E_2)(\partial_{x_3(k)} Du : \partial^3 Du) \, dx
$$

$$
\leq \int_{\mathbb{R}^3} 2(G'[|Du|^2]|Du|\nabla^2 Du| + G[|Du|^2]|\nabla Du|^2)|\nabla Du|\nabla^3 Du| \, dx
$$

$$
\leq C\|G'[|Du|^2]Du\|_{L^\infty}\|\nabla^2 Du\|_{L^3}\|\nabla Du\|_{L^6}\|\nabla^3 Du\|_{L^2} + \|G[|Du|^2]\|_{L^\infty}\|\nabla Du\|_{L^6}\|\nabla^3 Du\|_{L^2}
$$

$$
\leq C\|G'[|Du|^2]\|_{L^\infty}\|\nabla Du\|_{L^6}\|\nabla^2 Du\|_{L^6} + C\|G[|Du|^2]\|_{L^\infty}\|\nabla^2 Du\|_{L^6}^2 + 2\epsilon\|\nabla^3 Du\|_{L^2}^2
$$

$$
\leq C(\|G'[|Du|^2]\|_{L^\infty}\|\nabla Du\|_{L^6} + \|G(|Du|)\|_{L^\infty})\|\nabla^2 Du\|_{L^6}^2 + 2\epsilon\|\nabla^3 Du\|_{L^2}^2, 
$$

(A.81)

where we use same argument as (A.80) in the fourth inequality. Finally, for $I_{34}$, using Lemma 2.3, we note that

$$
I_{34} = \int_{\mathbb{R}^3} (2G'[|Du|^2]Du : \partial^3 Du) + E_3(\partial^3 Du) \, dx
$$

$$
= 2\int_{\mathbb{R}^3} G'[|Du|^2]|Du|\partial^3 Du|^2 \, dx + \int_{\mathbb{R}^3} E_3(\partial^3 Du) \, dx. 
$$

(A.82)

The second term in (A.82) is estimated as follows:

$$
\int_{\mathbb{R}^3} E_3(\partial^3 Du) \, dx \leq \int_{\mathbb{R}^3} |E_3||Du|\nabla^3 Du| \, dx
$$

$$
\leq C\int_{\mathbb{R}^3} G[|Du|^2]|(\nabla Du)^3 + |\nabla^2 Du|\nabla Du|)|Du|\nabla^3 Du| \, dx
$$

$$
\leq C\|G[|Du|^2]\|_{L^\infty}\|Du\|_{L^\infty}\|\nabla Du\|_{L^6}^3 + \|\nabla^2 Du\|_{L^6}\|\nabla^3 Du\|_{L^2}^3
$$

$$
\leq C\|G[|Du|^2]\|_{L^\infty}\|Du\|_{L^6}^2\|\nabla^2 Du\|_{L^6}\|\nabla^3 Du\|_{L^2}^2
$$

$$
+ C\|G[|Du|^2]\|_{L^\infty}\|Du\|_{L^6}\|\nabla^2 Du\|_{L^6}\|\nabla^3 Du\|_{L^2}^2 + 2\epsilon\|\nabla^3 Du\|_{L^2}^2
$$

$$
\leq C(\|G[|Du|^2]\|_{L^\infty}\|Du\|_{L^6}^2 + \|G(|Du|)\|_{L^\infty})\|\nabla^2 Du\|_{L^6}^2 + 2\epsilon\|\nabla^3 Du\|_{L^2}^2, 
$$

(A.83)

where we use same argument as (A.81) in the third inequality. Adding up estimates (A.79)–(A.83), we obtain

$$
\frac{d}{dt}\|\partial^3 u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} G[|Du|^2]|\partial^3 Du|^2 \, dx + \int_{\mathbb{R}^3} 2G'[|Du|^2]|Du : \partial^3 Du|^2 \, dx
$$

$$
\leq C(\|G[|Du|^2]\|_{L^\infty}^2 + \|G[|Du|^2]\|_{L^\infty})((\|Du\|^2_{L^\infty} + \|Du\|_{L^4\infty})^2 + \|\nabla^2 Du\|_{L^2}^6 + 5\epsilon\|\nabla^3 Du\|_{L^2}^2
$$

$$
- \int_{\mathbb{R}^3} \partial^3((U \cdot \nabla u)) \cdot \partial^3 u \, dx - \int_{\mathbb{R}^3} \partial^2 F : \partial^4 u \, dx.
$$

Hence, we have

$$
\frac{d}{dt}\|\partial^3 u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} m_0|\partial^3 Du|^2 \, dx
$$

$$
\leq C(\|G[|Du|^2]\|_{L^\infty}^2 + \|G[|Du|^2]\|_{L^\infty})((\|Du\|^2_{L^\infty} + \|Du\|_{L^4\infty})^2 + \|\nabla^2 Du\|_{L^2}^6 + 5\epsilon\|\nabla^3 Du\|_{L^2}^2
$$
Furthermore, we have

\begin{equation}
\int_{\mathbb{R}^3} \partial^3 ((U \cdot \nabla u)) \cdot \partial^3 u \, dx - \int_{\mathbb{R}^3} \partial^2 F \cdot \partial^4 u \, dx.
\end{equation}

Next, we estimate the terms caused by convection terms in (A.74), (A.78) and (A.84).

\begin{equation}
\sum_{1 \leq |\alpha| \leq 3} \int_{\mathbb{R}^3} \partial^\alpha [(U \cdot \nabla) u] \cdot \partial^\alpha u \, dx = \sum_{1 \leq |\alpha| \leq 3} \int_{\mathbb{R}^3} [\partial^\alpha ((U \cdot \nabla) u) - U \cdot \nabla \partial^\alpha u] \partial^\alpha u \, dx
\end{equation}

\begin{equation}
\leq \sum_{1 \leq |\alpha| \leq 3} \|\partial^\alpha ((U \cdot \nabla) u) - U \cdot \nabla \partial^\alpha u\|_{L^2} \|\partial^\alpha u\|_{L^2}
\end{equation}

\begin{equation}
\leq \sum_{1 \leq |\alpha| \leq 3} \|\nabla U\|_{L^\infty} \|u\|_{H^3} \|\partial^\alpha u\|_{L^2}
\leq C \|\nabla U\|_{L^\infty} \|u\|_{H^3}^2,
\end{equation}

where we use the following inequality:

\begin{equation}
\sum_{|\alpha| \leq m} \int_{\mathbb{R}^3} \|\nabla^\alpha (fg) - (\nabla^\alpha f)g\|_{L^2} \leq C (\|f\|_{H^{m-1}} \|\nabla f\|_{L^{\infty}} \|g\|_{H^m}).
\end{equation}

For the external force $F$,

\begin{equation}
\sum_{0 \leq |\alpha| \leq 2} \int_{\mathbb{R}^3} \partial^\alpha F \cdot \partial^\alpha u \, dx = C \|F\|^2_{H^2} + \frac{m_0}{64} \sum_{1 \leq |\beta| \leq 2} \|\partial^\beta u\|^2_{L^2}.
\end{equation}

We combine (A.71), (A.74), (A.78) and (A.84) with (A.85) and (A.86) to conclude

\begin{equation}
\frac{d}{dt} \|u\|^2_{H^3(\mathbb{R}^3)} + \frac{m_0}{2} \int_{\mathbb{R}^3} (|\nabla^3 Du|^2 + |\nabla^2 Du|^2 + |\nabla Du|^2 + |Du|^2) \, dx
\leq C \|\nabla U\|_{L^\infty} \|u\|^2_{H^3} + C \|G\|^2_{H^2} + C \|G\|_{L^{\infty}} (\|Du\|^2_{L^\infty} + \|Du\|_{L^\infty}) \|\nabla^2 Du\|^2_{L^2}
\end{equation}

\begin{equation}
+ C (\|G\|_{L^2} \|Du\|^2_{L^\infty} + \|G\|_{L^2} \|Du\|^2_{L^\infty}) (\|Du\|^2_{L^\infty} + \|Du\|^4_{L^\infty}) \|\nabla^2 Du\|^2_{L^2}.
\end{equation}

Furthermore, we have

\begin{equation}
\|G\|_{L^\infty} \leq \max_{0 \leq s \leq \|Du\|_{L^\infty}} G[s] \leq \max_{0 \leq s \leq C \|u\|_{H^3}} G[s] := g(\|u\|_{H^3}),
\end{equation}

where $g : [0, \infty) \mapsto [0, \infty)$ is a non-decreasing function. We set $X(t) := \|u(t)\|_{H^3(\mathbb{R}^3)}$ and it then follows from (A.87) and (A.88) that

\begin{equation}
\frac{d}{dt} X^2 \leq f_3(X) X^2 + C \|F\|_{H^2(\mathbb{R}^3)}^2,
\end{equation}

for some non-decreasing continuous function $f_3$, which immediately implies that there exists $T_3 > 0$ such that $\sup_{0 \leq t \leq T_3} X(t) < \infty$.

We note that $\partial_t u \in L^2((0, T); L^2(\mathbb{R}^3))$. Indeed, we introduce the antiderivative of $G$, denoted by $\tilde{G}$, i.e., $\tilde{G}[s] = \int_0^s G[\tau] \, d\tau$. Multiplying $\partial_t u$ to (4.56), integrating it by parts and using Hölder and Young’s inequalities, we have

\begin{equation}
\frac{1}{2} \int_{\mathbb{R}^3} |\partial_t u|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \tilde{G}[|Du|^2] \, dx \leq C \int_{\mathbb{R}^3} |\bar{u}|^2 |\nabla u|^2 \, dx + C \int_{\mathbb{R}^3} |F|^2 \, dx.
\end{equation}
Again, integrating estimate (A.89) over the time interval $[0, T]$, we obtain
\[
\int_0^T \int_{\mathbb{R}^3} |\partial_t u|^2 \, dx \, dt + \int_{\mathbb{R}^3} \hat{G} ||Du(\cdot, T)||^2 \, dx \\
\leq \int_{\mathbb{R}^3} \hat{G} ||Du_0||^2 \, dx + C \int_0^T \int_{\mathbb{R}^3} |\bar{u}|^2 |\nabla u|^2 \, dx \, dt + C \int_0^T \int_{\mathbb{R}^3} |F|^2 \, dx \, dt. \tag{A.90}
\]
Using Sobolev embedding, the second term in (A.90) is estimated as follows:
\[
\int_0^T \int_{\mathbb{R}^3} |\bar{u}|^2 |\nabla u|^2 \, dx \, dt \leq \int_0^T \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \, dt \\
\leq C \sup_{0 < \tau \leq T} \|u(\tau)\|_{H^2} \int_0^T \|\nabla u\|_{L^2}^2 \, dt < \infty, \tag{A.91}
\]
and due to the assumption for the external force $F$, we also get $\int_0^T \int_{\mathbb{R}^3} |F|^2 \, dx \, dt < \infty$. Therefore, we obtain $\partial_t u \in L^2(0, T; L^2(\mathbb{R}^3))$.

For the uniqueness of a solution, we let $u_1$ and $u_2$ be strong solutions for system (4.56). First of all, we rewrite the equation for $\tilde{u} := u_1 - u_2$ and $\bar{p} := p_1 - p_2$.
\[
\bar{u}_t - \nabla \cdot (G(|Du_1|^2) - G(|Du_2|^2)) + (U \cdot \nabla)\bar{u} + \nabla \bar{p} = 0,
\]
with $\nabla \cdot \bar{u} = 0$ and $\nabla \cdot U = 0$. Multiplying $\bar{u}$ on the both sides of the equation above and integrating on $\mathbb{R}^3$, we obtain
\[
\frac{d}{dt} \|\bar{u}\|_{L^2(\mathbb{R}^3)}^2 + m_0 \|\nabla \bar{u}\|_{L^2(\mathbb{R}^3)}^2 \leq 0, \tag{A.92}
\]
where we use Lemma 2.4 and the divergence-free condition. Applying Gronwall’s inequality to estimate (A.92), we get $\bar{u}(x, 0) = 0$ in $L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$. Hence this implies the uniqueness of a solution. Finally, we introduce Galerkin approximation procedure of equation (4.56)–(4.57) to make up construction of solution. We omit this part (see refer to [13, Proposition 3.1] for detailed proof). \qed

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(Received: June 16, 2023; revised: September 27, 2023; accepted: October 4, 2023)