Study about fuzzyω-paracompact space in fuzzy topological space

Prof.Dr.Munir Abdul Khalik AL-Khafaji and Gazwan Haider Abdulhusein

Department of Mathematics, AL-Mustinsiryah University, Baghdad, Iraq

mnraziz@uomustansiriyah.edu.iq and haider.gazwan@uomustansiriyah.edu.iq

Abstract: The purpose of this paper is to introduce a new class of fuzzy paracompact space is named fuzzy ω-paracompact space on fuzzy topological space also study the relationships with fuzzy ω-separation axioms and we give some characterization on fuzzy ω-paracompact space by using fuzzy countable set also we study the fuzzy ω-paracompact subspace and consider some relationship between fuzzy paracompact space and fuzzy ω-paracompact space by using certain types of fuzzy ω-continuous functions.

1. Introduction

The concept, which we will be considered in this paper, is the so called “fuzzy sets” which is totally different from the classical concept which is called “a crisp set”. The recent concept is introduced by Zadeh in 1965 [15], in which he defines fuzzy sets as a class of objects with a continuum of grades of membership and such a set is characterized by a membership function that assigns to each object a grade of membership ranging between zero and one, In (1968) Chang [2] introduced the definition of fuzzy topological spaces and extended in a straightforward manner some concepts of crisp topological spaces to fuzzy topological spaces. Later Lowen [10] (1976) redefined what is now known as stratified fuzzy topology. While Wong [13] in 1974 discussed and generalized some properties of fuzzy topological spaces. The note on paracompact space has been introduced by Ernest Michael [4] in (1953). Qutaiba Ead Hassan [9] in (2005) introduced characterizations of fuzzy paracompactness. In this paper we introduce the concepts of fuzzy ω-open set and fuzzy ω-paracompact space and fuzzy ω-paracompact subspace on fuzzy topological space, and studied the relationships with fuzzy ω-separation axioms also we presented some types of fuzzy ω-continuous function and we give some characterization. And we obtained several properties.

2. Preliminaries

2.1 Definition [15]

Let X be a non empty set, and let I be the unit interval i.e I=[0,1], a fuzzy set in X is a function from X into the unit interval I, ̃: X → I be a function A fuzzy set ̃ in X can be represented by the set of pairs: ̃ = {(x, μ̃(x)) : x ∈ X} the family of all fuzzy sets in X is denoted by I X.

2.2 Definition [6]

A fuzzy point x r is a fuzzy set such that :

μ x r (y) = r > 0 if x = y , ∀ y ∈ X and
μ x r (y) = 0 if x ≠ y , ∀ y ∈ X, The family of all fuzzy points of ̃ will be denoted by FP(̃).

2.3 Definition [13]

A fuzzy point x r is said to belong to a fuzzy set ̃ in X (denoted by : x r ∈ ̃) if and only if μ x r ≤ μ ̃ (x) ̃

2.4 Proposition[13]

Let ̃ and ̃ be two fuzzy sets in X with membership functions μ ̃ and μ ̃ respectively, then for all x ∈ X:

1. ̃ ⊆ ̃ if and only if μ ̃ (x) ≤ μ ̃ (x).
2. ̃ = ̃ if and only if μ ̃ (x) = μ ̃ (x).
3. ̃ = ̃ if and only if μ ̃ (x) = min{ μ ̃ (x) , μ ̃ (x) }.
2.5 Definition [7]

The support of a fuzzy set \( \tilde{A} \), \( \text{Supp} (\tilde{A}) \), is the crisp set of all \( x \in X \), such that \( \mu_{\tilde{A}} (x) > 0 \). 

2.6 Definition [2]

A fuzzy topology is a family \( \mathcal{T} \) of fuzzy subsets in \( X \), satisfying the following conditions:
(a) \( \emptyset, 1_X \in \mathcal{T} \).
(b) If \( \tilde{A}, \tilde{B} \in \mathcal{T} \), then \( \tilde{A} \cap \tilde{B} \in \mathcal{T} \).
(c) If \( \tilde{A}_i \in \mathcal{T}, \forall i \in J \), where \( J \) is any index set, then \( \bigcup_{i \in J} \tilde{A}_i \in \mathcal{T} \).

\( \mathcal{T} \) is called fuzzy topology for \( \tilde{X} \), and the pair \( (X, \mathcal{T}) \) is a fuzzy topological space. Every member of \( \mathcal{T} \) is called open fuzzy set (\( \mathcal{T} \)-open fuzzy set). A fuzzy set \( \tilde{C} \) in \( 1_X \) is called closed fuzzy set (\( \mathcal{T} \)-closed fuzzy set) if and only if its complement \( \tilde{C}^c \) is \( \mathcal{T} \)-open fuzzy set.

2.7 Definition [8]

If \( T \), the complement of \( 1_X \) referred to \( 1_X \) denoted by \( 1_X \), is defined by \( 1_X = 1_X - \tilde{B} \).

2.8 Definition [1]

An fuzzy open set \( \tilde{A} \) in a fuzzy topological space \( (X, \mathcal{T}) \) is said to be clopen if its complement \( 1_X - \tilde{A} \) is an fuzzy open.

2.9 Definition [2]

A fuzzy set \( \tilde{B} \) in a fuzzy topological space \( (\tilde{A}, \tilde{T}) \) is said to be a fuzzy neighborhood of a fuzzy point \( x \) in \( \tilde{A} \) if there is a fuzzy open set \( \tilde{G} \) in \( \tilde{A} \) such that \( \mu_{\tilde{x}} (x) \leq \mu_{\tilde{G}} (x) \leq \mu_{\tilde{B}} (x), \forall x \in X \).

2.10 Definition [11]

Let \( (X, \tilde{T}) \) be a fuzzy topological space and \( \tilde{B} \in \mathcal{P}(1_X) \), then the relative fuzzy topology for \( \tilde{B} \) defined by \( \tilde{T}_B = \{ \tilde{B} \cap \tilde{G} : \tilde{G} \in \tilde{T} \} \). The corresponding \( (\tilde{B}, \tilde{T}_B) \) is called fuzzy subspace of \( (X, \tilde{T}) \).

2.11 Definition [3]

Let \( (X, \tilde{T}) \) be a fuzzy topological space a family \( \tilde{Z} \) of fuzzy sets is open cover of a fuzzy set \( \tilde{A} \) if and only if \( \tilde{A} \subseteq \bigcup \{ \tilde{G} : \tilde{G} \in \tilde{Z} \} \) and each member of \( \tilde{Z} \) is a fuzzy open set.

2.12 Definition [12]

Let \( B = \{ \tilde{B}_\alpha : \alpha \in \Lambda \} \), \( C = \{ \tilde{C}_\beta : \beta \in \Lambda \} \) (\( \beta < \alpha \)) be any two collection of fuzzy sets in \( (X, \tilde{T}) \), then \( C \) is a refinement of \( B \) if for each \( \beta \in \Lambda \) there exist \( \alpha \in \Lambda \) such that \( \mu_{\tilde{C}_\beta} (x) \leq \mu_{\tilde{B}_\alpha} (x) \).

2.13 Definition [5]

A fuzzy topological space \( (X, \tilde{T}) \) is said to be fuzzy connected, if it has no proper fuzzy clopen set.

Otherwise it is called fuzzy disconnected.

2.14 Definition [15]

Let \( f \) be a function from universal set \( X \) to universal set \( Y \). Let \( \tilde{B} \) be a fuzzy subset in \( 1_Y \) with membership function \( \mu_{\tilde{B}} (Y) \). Then, the inverse of \( \tilde{B} \), written as \( f^{-1}(\tilde{B}) \), is a fuzzy subset of \( 1_X \) whose membership function is defined by \( \mu_{f^{-1}(\tilde{B})} (x) = \mu_{\tilde{B}} (f(x)) \), for all \( x \) in \( X \). If \( \tilde{A} \) be a fuzzy subset in \( 1_X \) with membership function \( \mu_{\tilde{A}} (x) \). The image of \( \tilde{A} \), written as \( f(\tilde{A}) \), is a fuzzy subset in \( 1_Y \) whose membership function is defined by

\[
\mu_{f(\tilde{A})}(y) = \begin{cases} 
\sup_{z \in f^{-1}(y)} \mu_{\tilde{A}}(z) & \text{if } f^{-1}(y) \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]

for all \( y \) in \( Y \), where \( f^{-1}(y) = \{ x | f(x) = y \} \).

From the above it is clear that:

4. \( \tilde{D} = \tilde{A} \cup \tilde{B} \) if and only if \( \mu_D (x) = \max \{ \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x) \} \).
1. If \( f \) is injective then \( \mu_{f(A)}(y) = \left\{ \sup_{z \in f^{-1}(y)} \mu_A(z) \right\} \) if \( f^{-1}(y) \neq \emptyset \)

2. If \( f \) is surjective then \( \forall x \in X \) then \( \mu_B(f(x)) = \mu_B(y) \forall y \in Y, x \in f^{-1}(y) \)

3. If \( f \) is bijective then \( \mu_{f(A)}(y) = \mu_A(x) \forall x = f^{-1}(y) \mu_{f^{-1}(B)}(x) = \mu_B(y), \forall y \in Y, y = f(x) \)

3. Fuzzy \( \omega \)-Open Set In Fuzzy Topological Space

3.1 Definition [14]
A fuzzy set \( \tilde{A} \) in a fuzzy topological space \((X, \tilde{T})\) is called a fuzzy uncountable if and only if \( \text{supp}(\tilde{A}) \) is an uncountable subset of \( X \)

3.2 Definition
A fuzzy point \( x_r \) of a fuzzy topological space \((X, \tilde{T})\) is called a fuzzy condensation point of \( \tilde{A} \subseteq 1_X \) if \( B \cap \tilde{A} \) is fuzzy uncountable for each fuzzy open set \( B \) containing \( x_r \). And the set of all fuzzy condensation point of \( \tilde{A} \) is denoted by \( \text{Cond} (\tilde{A}) \)

3.3 Definition
A fuzzy subset \( \tilde{A} \) in a fuzzy topological space \((X, \tilde{T})\) is called a fuzzy \( \omega \)-closed set if it contains all its fuzzy condensation point. The complement fuzzy \( \omega \)-closed sets are called fuzzy \( \omega \)-open sets.

3.4 Theorem
A fuzzy subset \( \tilde{G} \) of a fuzzy topological space \((X, \tilde{T})\) is fuzzy \( \omega \)-open set if and only if \( x_r \in \tilde{G} \) there exist a fuzzy open set \( \tilde{U} \) such that \( x_r \in \tilde{U} \) and \( \tilde{U} - \tilde{G} \) is countable.

**Proof:** \( \tilde{G} \) is fuzzy \( \omega \)-open set if and only if \( 1_X - \tilde{G} \) is fuzzy \( \omega \)-closed set. And \( 1_X - \tilde{G} \) is fuzzy \( \omega \)-closed set if and only if \( \text{Cond}(1_X - \tilde{G}) \subseteq 1_X - \tilde{G} \). And \( \text{Cond}(1_X - \tilde{G}) \subseteq 1_X - \tilde{G} \) if and only if each \( x_r \in \tilde{G} \), \( x_r \notin \text{Cond}(1_X - \tilde{G}) \). Thus \( x_r \notin \text{Cond}(1_X - \tilde{G}) \) there exist a fuzzy open set \( \tilde{U} \) such that \( x_r \in \tilde{U} \) and \( \tilde{U} \cap (1_X - \tilde{G}) = \emptyset \) is countable.

3.5 Theorem
A fuzzy subset \( \tilde{G} \) of a fuzzy topological space \((X, \tilde{T})\) is \( \omega \)-open set if and only if for each \( x_r \notin \tilde{G} \) there exist an fuzzy open set \( \tilde{U} \) containing \( x_r \) and countable fuzzy subset \( \tilde{C} \) of \( 1_X \) such that \( \tilde{U} - \tilde{C} \subseteq \tilde{G} \).

**Proof:** \((\Rightarrow)\) suppose \( \tilde{G} \) is fuzzy \( \omega \)-open set and let \( x_r \in \tilde{G} \). Then there exist a fuzzy open set \( \tilde{U} \) and \( x_r \in \tilde{U} \) and \( \tilde{U} - \tilde{G} \) is countable. Set \( \tilde{C} = \tilde{U} - \tilde{G} \), then \( \tilde{C} \) is countable and \( x_r \in \tilde{U} - \tilde{C} = \tilde{U} - (\tilde{U} - \tilde{G}) \subseteq \tilde{G} \)

\((\Leftarrow)\) let \( x_r \in \tilde{G} \) then by assumption there exist fuzzy open set \( \tilde{U} \) containing \( x_r \) and countable fuzzy subset \( \tilde{C} \) of \( 1_X \) such that \( \tilde{U} - \tilde{C} \subseteq \tilde{G} \), since \( \tilde{U} - \tilde{G} \subseteq \tilde{C} \) then \( \tilde{U} - \tilde{G} \) is countable, hence \( \tilde{G} \) is fuzzy \( \omega \)-open set.

3.6 Proposition
Every fuzzy open set is fuzzy \( \omega \)-open set.

**Proof:** Let \( \tilde{G} \) be fuzzy open set and \( x_r \notin \tilde{G} \). Set \( \tilde{U} = \tilde{G} \), \( \tilde{C} = \emptyset \), then \( \tilde{U} \) is fuzzy open set and \( \tilde{C} \) countable set. Such that \( x_r \in \tilde{U} - \tilde{C} \subseteq \tilde{G} \), thus \( \tilde{G} \) is fuzzy \( \omega \)-open set.

**Remark**
The converse of (3.6 proposition) is not true in general as the following examples show:

3.8 Example: let \( X = \{ a, b, c \} \) and \( \tilde{A}, \tilde{B} \) are fuzzy subset in \( 1_X \) where
\( 1_X = \{ (a, 1), (b, 1), (c, 1) \}, \tilde{A} = \{ (a, 0.6), (b, 0.6), (c, 0.7) \} \)
\( \tilde{B} = \{ (a, 0.5), (b, 0.5), (c, 0.4) \}, \tilde{T} = \{ 0 \}, 1_X, \tilde{A} \) be a fuzzy topology on \( X \), Then the fuzzy set \( \tilde{B} \) is a fuzzy \( \omega \)-open set but not fuzzy open set.

3.9 Definition
Let \( \tilde{B} \) be a fuzzy set in a fuzzy topological space \((X, \tilde{T})\) then the \( \omega \)-interior of \( \tilde{B} \) is denoted by \( \omega\text{-Int}(\tilde{B}) \) and defined by \( \omega\text{-Int}(\tilde{B}) = \cup \{ \tilde{G} : \tilde{G} \) is a fuzzy \( \omega \)-open set in \( 1_X, \tilde{G} \subseteq \tilde{B} \} \)
3.10 Definition
Let $\mathbb{B}$ be a fuzzy set in a fuzzy topological space $(X, \mathcal{T})$ then, The $\omega$-closure of $\mathbb{B}$ is denoted by $\omega \text{-cl} (\mathbb{B})$ and defined by $\omega \text{-cl} (\mathbb{B}) = \bigcap \{ \mathcal{G} : \mathcal{G} \text{ is a fuzzy } \omega \text{-closed set in } 1_X , \mathbb{B} \subseteq \mathcal{G} \}$

3.11 Theorem
Let $\mathbb{A}$ be fuzzy subset of a fuzzy topological space $(X, \mathcal{T})$ then $(\mathbb{T}_\mathbb{A})^\omega = \mathbb{T}_\mathbb{A}^\omega$

**Proof:** To prove $(\mathbb{T}_\mathbb{A})^\omega \subseteq \mathbb{T}_\mathbb{A}^\omega$, let $\mathbb{B} \in (\mathbb{T}_\mathbb{A})^\omega$ and $x_r \in \mathbb{B}$, by (3.5 Theorem), There exist fuzzy open set $\mathbb{V}$ of $\mathbb{T}_\mathbb{A}$ and $\mathbb{C}$ countable subset of $\mathbb{T}_\mathbb{A}$ such that $x_r \in \mathbb{V} - \mathbb{C} \subseteq \mathbb{B}$, choose $\mathbb{U} \in \mathbb{T}$ such that $\mathbb{V} = \mathbb{U} \cap \mathbb{A}$, Then $\mathbb{U} - \mathbb{C} \subseteq \mathbb{T}_\mathbb{A}^\omega$, $x_r \in \mathbb{U} - \mathbb{C}$ and $\mathbb{U} - \mathbb{C} \cap \mathbb{A} = \mathbb{V} - \mathbb{C} \subseteq \mathbb{B}$, Therefore $\mathbb{B} \in \mathbb{T}_\mathbb{A}^\omega$, To prove $\mathbb{T}_\mathbb{A}^\omega \subseteq (\mathbb{T}_\mathbb{A})^\omega$, let $\mathbb{G} \in \mathbb{T}_\mathbb{A}^\omega$ then there exist $\mathbb{H} \in \mathbb{T}_\mathbb{A}$ such that $\mathbb{H} = \mathbb{G} \cap \mathbb{A}$ if $x_r \in \mathbb{G}$ then $x_r \in \mathbb{H}$ and there exist fuzzy open set $\mathbb{U}$ of $\mathbb{T}$ and $\mathbb{D}$ countable subset of $\mathbb{T}$ such that $x_r \in \mathbb{U} - \mathbb{D} \subseteq \mathbb{H}$, We put $\mathbb{V} = \mathbb{U} \cap \mathbb{A}$, then $\mathbb{V} \in \mathbb{T}_\mathbb{A}$ and $x_r \in \mathbb{V} - \mathbb{D} \subseteq \mathbb{G}$, It follows that $\mathbb{G} \in (\mathbb{T}_\mathbb{A})^\omega$.

3.12 Definition
The fuzzy family $\{\mathbb{B}_\alpha : \alpha \in \Lambda\}$ of subset of a fuzzy topological space $(X, \mathcal{T})$ is called
1- Fuzzy $\omega$-locally finite if for each $x_r \in 1_X$ there exist an fuzzy $\omega$-open set $\mathcal{G}$ containing $x_r$ such that the set $\{\mathcal{G} \cap \mathbb{B}_\alpha \neq \emptyset : \alpha \in \Lambda\}$ is finite
2- Fuzzy $\omega$-discrete if for each $x_r \in 1_X$ there exist an fuzzy $\omega$-open set $\mathcal{G}$ containing $x_r$ such that the set $\{\mathcal{G} \cap \mathbb{B}_\alpha \neq \emptyset : \alpha \in \Lambda\}$ has at most one member

3.13 Proposition
Every fuzzy locally finite (resp.fuzzy discrete) family of any fuzzy topological space $(X, \mathcal{T})$ is fuzzy $\omega$-locally finite (resp.fuzzy $\omega$-discrete)

**Proof:** Follows from the fact (every fuzzy open set is fuzzy $\omega$-open set)

3.14 Definition
A fuzzy topological space $(X, \mathcal{T})$ is called a fuzzy anti-locally-countable if each nonempty fuzzy open subset of $1_X$ is uncountable.

3.15 Definition
A fuzzy topological space $(X, \mathcal{T})$ is said to be
1- $\omega$-$\mathcal{T}_1$ if for each pair of distinct fuzzy point $x_t$ and $y_t$ of $1_X$ there exist fuzzy $\omega$-open set $\mathcal{G}$ such that either $x_t \in \mathcal{G}$ and $y_t \notin \mathcal{G}$ or $y_t \in \mathcal{G}$ and $x_t \notin \mathcal{G}$.
2- $\omega$-$\mathcal{T}_2$ if for each pair of distinct fuzzy point $x_t$ and $y_t$ of $1_X$ there exist fuzzy $\omega$-open sets $\mathcal{G}$ and $\mathcal{H}$ such that $x_t \in \mathcal{G}$ and $y_t \notin \mathcal{G}$ and $y_t \in \mathcal{H}$ and $x_t \notin \mathcal{H}$.
3- $\omega$-$\mathcal{T}_2$ if for each pair of distinct fuzzy point $x_t$ and $y_t$ of $1_X$ there exist disjoint fuzzy $\omega$-open sets $\mathcal{G}$ and $\mathcal{H}$ containing $x_t$ and $y_t$ respectively.

3.18 Definition
A fuzzy topological space $(X, \mathcal{T})$ is called a fuzzy $\omega$-regular space if for each fuzzy $\omega$-closed subset $\mathbb{B}$ of $1_X$ and a fuzzy point $x_t$ in $1_X$ such that $x_t \notin \mathbb{B}$, there exist disjoint fuzzy $\omega$-open sets $\mathbb{U}$ and $\mathbb{V}$ containing $x_t$ and $\mathbb{B}$ respectively.

3.19 Definition
A fuzzy topological space $(X, \mathcal{T})$ is called a fuzzy $\omega$-Normal space if for each pair of disjoint fuzzy $\omega$-closed sets $\mathbb{A}$ and $\mathbb{B}$ in $1_X$ there exist disjoint fuzzy $\omega$-open sets $\mathbb{U}$ and $\mathbb{V}$ containing $\mathbb{A}$ and $\mathbb{B}$ respectively.

3.20 Theorem
A fuzzy topological space $(X, \mathcal{T})$ is fuzzy $\omega$-Normal if for each pair of fuzzy $\omega$-open sets $\mathcal{G}$ and $\mathcal{H}$ in $1_X$ such that $1_X = \mathcal{G} \cup \mathcal{H}$ there are fuzzy $\omega$-closed sets $\mathbb{U}$ and $\mathbb{V}$ contained in $\mathcal{G}$ and $\mathcal{H}$ respectively such that $1_X = \mathbb{U} \cup \mathbb{V}$

**Proof:** Obvious
3.21 Theorem
Every fuzzy $\omega$-closed subspace of fuzzy $\omega$-Normal space is fuzzy $\omega$-Normal space.
Proof: Obvious

3.22 Proposition
Every fuzzy $\omega$-regular space is fuzzy $\omega$-$\bar{T}_2$ space
Proof: Let $x_r$ and $y_i$ be pair of fuzzy distinct points in a fuzzy $\omega$-regular space $1_X$. Then $x_r$ is a fuzzy point of $1_X$ which is not in the fuzzy $\omega$-closed subset $\{y_i\}$ of $1_X$ so by fuzzy $\omega$-regularity of $1_X$ there exist fuzzy disjoint $\omega$-open sets $\bar{U}$ and $\bar{V}$ containing $x_r$ and $y_i$ respectively. Hence $1_X$ is fuzzy $\omega$-$\bar{T}_2$ space.

3.23 Proposition
If $(X,\bar{T})$ is fuzzy anti-locally countable topological space and $\bar{A}$ fuzzy $\omega$-open subset of $1_X$ then $\omega$-cl($\bar{A}$) = cl($\bar{A}$).
Proof: Clearly $\omega$-cl($\bar{A}$) $\subseteq$ cl($\bar{A}$). On the other hand, let $x_r \in$ cl($\bar{A}$) and $\bar{G}$ be an fuzzy $\omega$-open subset containing $x_r$ then by (3.5 Theorem) There exist an fuzzy open set $\bar{H}$ containing $x_r$ and countable set $\bar{C}$ such that $\bar{H} - \bar{C} \subseteq \bar{G}$, thus $(\bar{H} - \bar{C}) \cap \bar{A} \subseteq \bar{G} \cap \bar{A}$ and so $\bar{H} \cap \bar{A} - \bar{C} \subseteq \bar{G} \cap \bar{A}$. As $x_r \in \bar{H}$ and $x_r \in$ cl($\bar{A}$), $\bar{H} \cap \bar{A} \neq \emptyset$. And then as $\bar{H}$ and $\bar{A}$ are fuzzy $\omega$-open sets, $\bar{H} \cap \bar{A}$ is fuzzy $\omega$-open set and as $1_X$ is fuzzy anti-locally countable, $\bar{H} \cap \bar{A}$ is fuzzy uncountable and so is $(\bar{H} \cap \bar{A}) - \bar{C}$. Thus $\bar{G} \cap \bar{A}$ is uncountable therefore $\bar{G} \cap \bar{A} \neq \emptyset$ which means that $x_r \in \omega$-cl($\bar{A}$).

3.24 Corollary
If $(X,\bar{T})$ is fuzzy anti-locally countable topological space and $\bar{A}$ fuzzy $\omega$-open subset of $1_X$ then $\omega$-Int($\bar{A}$) = Int($\bar{A}$).
Proof: Obvious

3.25 Theorem
If a fuzzy topological space $(X,\bar{T})$ is fuzzy anti-locally-countable space then every fuzzy $\omega$-Normal space is fuzzy Normal space.
Proof: Let $\bar{F}$ and $\bar{H}$ be two disjoint fuzzy closed subset of fuzzy anti-locally-countable $\omega$-Normal space $1_X$, then there are fuzzy $\omega$-open sets $\bar{U}$ and $\bar{V}$ such that $\bar{F} \subseteq \bar{U}$ and $\bar{H} \subseteq \bar{V}$ and $\bar{U} \cap \bar{V} = \emptyset$ this implies that $\omega$-cl($\bar{U}$) $\cap$ $\bar{V}$ = $\emptyset$ and $\bar{U} \cap \omega$-cl($\bar{V}$) = $\emptyset$ since $1_X$ is fuzzy anti-locally-countable so by (3.23 Proposition) we get cl($\bar{U}$) $\cap$ $\bar{V}$ = $\emptyset$ and $\bar{U} \cap$ cl($\bar{V}$) = $\emptyset$ since Int(cl($\bar{U}$)) $\subseteq$ cl($\bar{U}$) and Int(cl($\bar{V}$)) $\subseteq$ cl($\bar{V}$) then Int(cl($\bar{U}$)) $\cap$ $\bar{V}$ = $\emptyset$ and $\bar{U} \cap$ Int(cl($\bar{V}$)) = $\emptyset$. And this implies that Int(cl($\bar{U}$)) $\cap$ cl($\bar{V}$) = $\emptyset$ and Int(cl($\bar{U}$)) $\cap$ cl($\bar{V}$) = $\emptyset$ thus Int(cl($\bar{U}$)) $\cap$ Int(cl($\bar{V}$)) = $\emptyset$, hence Int(cl($\bar{U}$)) and Int(cl($\bar{V}$)) are disjoint fuzzy open sets in $1_X$ containing $\bar{F}$ and $\bar{H}$ respectively hence $(X,\bar{T})$ is fuzzy Normal space.

3.26 Definition
Two fuzzy families $\{\bar{A}_\lambda\}_{\lambda \in \Delta}$ and $\{\bar{B}_\lambda\}_{\lambda \in \Delta}$ of subset of a fuzzy space $1_X$ are said to be similar if for every finite subset $\Delta$ of $\Lambda$ the fuzzy sets $\bigcap_{\lambda \in \Delta} \bar{A}_\lambda$ and $\bigcap_{\lambda \in \Delta} \bar{B}_\lambda$ are either empty or nonempty.

3.27 Definition
Let $(X,\bar{T})$ be a fuzzy topological space a family $W$ of fuzzy sets is $\omega$-open cover of a fuzzy set $\bar{A}$ if and only if $\bar{A} \subseteq \cup\{\bar{G} : \bar{G} \in W\}$ and each member of $W$ is a fuzzy $\omega$-open set. A sub cover of $W$ is a sub family which is also cover.

3.28 Definition
A function $f : (X,\bar{T}) \rightarrow (Y,\bar{\sigma})$ is said to be fuzzy $\omega$-continuous at a fuzzy point $x_r \in 1_X$ if for each fuzzy open subset $\bar{V}$ in $1_Y$ containing $f(x_r)$ there exists an fuzzy $\omega$-open subset $\bar{U}$ of $1_X$ that containing $x_r$ such that $f(\bar{U}) \subseteq \bar{V}$ and $f$ is called fuzzy $\omega$-continuous if it is fuzzy $\omega$-continuous at each fuzzy point.
3.29 Definition
A function \( f: (X, \tilde{T}) \rightarrow (Y, \tilde{\sigma}) \) is said to be
1- fuzzy pre- \( \omega \)-open, if image of each fuzzy \( \omega \)-open set is fuzzy \( \omega \)-open
2- fuzzy \( \omega \)-irresolute if \( f^{-1}(\tilde{F}) \) is fuzzy \( \omega \)-closed in \( 1_X \) for each fuzzy \( \omega \)-closed subset \( \tilde{F} \) of \( 1_Y \)

4. Fuzzy \( \omega \)-Paracompact space

4.1 Definition
A fuzzy topological space \((X, \tilde{T})\) is said to be:
Fuzzy paracompact space if for each fuzzy open covering of \( 1_X \) has a fuzzy locally finite open refinement. [9]
Fuzzy \( \omega \)-paracompact space if for each fuzzy \( \omega \)-open covering of \( 1_X \) has a fuzzy \( \omega \)-locally finite \( \omega \)-open refinement

4.2 Propositions
If a fuzzy topological space \((X, \tilde{T})\) is a fuzzy locally countable space then \((X, \tilde{T}_\omega)\) is fuzzy paracompact space.
\[ \text{Proof:} \text{ Follows from the fact every fuzzy discrete space is fuzzy locally finite and A fuzzy topological space \((X, \tilde{T})\) is fuzzy locally countable if and only if } \tilde{T}_\omega = \tilde{T}_{\text{dis}} \]

4.3 Propositions
If a fuzzy covering \( \{ \tilde{U}_\lambda \}_{\lambda \in \Lambda} \) of a fuzzy topological space \((X, \tilde{T})\) has a fuzzy locally-finite (fuzzy \( \omega \)-locally finite) \( \omega \)-open refinement then there exist a fuzzy locally-finite (fuzzy \( \omega \)-locally finite) \( \omega \)-open covering \( \{ \tilde{G}_\lambda \}_{\lambda \in \Lambda} \) of \( 1_X \) such that \( \tilde{G}_\lambda \subseteq \tilde{U}_\lambda \) for each \( \lambda \in \Lambda \).
\[ \text{Proof:} \text{ Let } \{ \tilde{V}_\gamma \}_{\gamma \in \Gamma} \text{ be the fuzzy locally-finite (fuzzy \( \omega \)-locally finite) \( \omega \)-open refinement } \{ \tilde{U}_\lambda \}_{\lambda \in \Lambda} \text{ therefore there exist a function } \beta: \Gamma \rightarrow \Lambda \text{ such that } \tilde{V}_\gamma \subseteq \tilde{U}_{\beta(\gamma)} \text{ for each } \gamma \in \Gamma \text{. Let } \tilde{G}_\lambda = \bigcup_{\gamma: \beta(\gamma) = \lambda} \tilde{V}_\gamma \text{ then the family } \{ \tilde{G}_\lambda \}_{\lambda \in \Lambda} \text{ is fuzzy } \omega \text{-open covering of } 1_X \text{ with the property that } \tilde{G}_\lambda \subseteq \tilde{U}_\lambda \text{ for each } \lambda \in \Lambda . \]
Also \( \{ \tilde{G}_\lambda \}_{\lambda \in \Lambda} \) is fuzzy locally-finite (fuzzy \( \omega \)-locally finite).

If \( x_r \in 1_X \) there is an fuzzy open \( (\omega \)-open) set \( \tilde{W} \) containing \( x_r \) such that the set \( \Gamma_0 = \{ \gamma \in \Gamma : \tilde{W} \cap \tilde{V}_\gamma \neq \emptyset \} \) is finite. But since \( \tilde{W} \cap \tilde{G}_\lambda \neq \emptyset \)
\[ \text{If and only if } \lambda = \beta(\gamma) \text{ for some } \gamma \in \Gamma_0 \text{ so the set } \{ \lambda \in \Lambda : \tilde{W} \cap \tilde{G}_\lambda \neq \emptyset \} \text{ is finite } \]

4.4 Corollary
A fuzzy topological space \((X, \tilde{T})\) is fuzzy \( \omega \)-paracompact space if and only if for every fuzzy \( \omega \)-open covering \( \{ \tilde{U}_\lambda \}_{\lambda \in \Lambda} \) of \( 1_X \) there exist an fuzzy \( \omega \)-locally finite \( \omega \)-open covering \( \{ \tilde{V}_\lambda \}_{\lambda \in \Lambda} \) of \( 1_X \) such that \( \tilde{V}_\lambda \subseteq \tilde{U}_\lambda \) for each \( \lambda \in \Lambda \).

4.5 Propositions
Let \((X, \tilde{T})\) be a fuzzy \( \omega \)-paracompact space and let \( \tilde{H} \) be a fuzzy subset of \( 1_X \) and \( \tilde{F} \) be an fuzzy \( \omega \)-closed of \( 1_X \) which disjoint from \( \tilde{H} \); if for every \( x_r \in \tilde{F} \) there exist disjoint fuzzy \( \omega \)-open set \( \tilde{U}_{x_r} \) and \( \tilde{V}_{x_r} \) containing \( x_r \) and \( \tilde{H} \) respectively then there are disjoint \( \omega \)-open set \( \tilde{U} \) and \( \tilde{V} \) containing \( \tilde{H} \) respectively.
\[ \text{Proof:} \text{ Consider the fuzzy } \omega \text{-open covering } \{ \tilde{U}_{x_r} \}_{x_r \in \tilde{F}} \cup \{ 1_X - \tilde{F} \} \text{ of an fuzzy } \omega \text{-paracompact space } (X, \tilde{T}) \text{ then by (4.4 Corollary) there exist an fuzzy } \omega \text{-locally finite } \omega \text{-open covering } \{ \tilde{G}_{x_r} \}_{x_r \in \tilde{F}} \cup \tilde{G} \text{ of } 1_X \text{ such that } \tilde{G} \subseteq 1_X - \tilde{F} \text{ and } \tilde{G}_{x_r} \subseteq \tilde{U}_{x_r} \text{ for each } x_r \in \tilde{F} \text{. if } \tilde{U}_{x_r} \cap \tilde{V}_{x_r} = \emptyset \text{ then } \tilde{G}_{x_r} \cap \tilde{V}_{x_r} = \emptyset \text{ so } \omega - \text{cl}(\tilde{G}_{x_r}) \cap \tilde{V}_{x_r} = \emptyset \text{ for each } x_r \in \tilde{F} \text{ then the fuzzy sets } \tilde{U} = \bigcup_{x_r \in \tilde{F}} \tilde{G}_{x_r} \text{ and } \tilde{V} = 1_X - \bigcup_{x_r \in \tilde{F}} \omega - \text{cl}(\tilde{G}_{x_r}) \text{ are the required } \omega \text{-open sets of } 1_X \]

4.6 Propositions
Each fuzzy $\omega$-paracompact fuzzy $\omega$-regular (resp. fuzzy $\omega$-$\tilde{T}_2$) space is fuzzy $\omega$-Normal space.

Proof: Let $(X,\tilde{T})$ be an fuzzy $\omega$-paracompact fuzzy-$\tilde{T}_2$ space and let $x_r$ be any fuzzy point in $1_x$ which is not in an arbitrary fuzzy $\omega$-open set $\tilde{F}$ of $1_x$ therefore for each $y_1 \in \tilde{F}$ there are disjoint fuzzy $\omega$-open sets $\tilde{U}_{y_1}$ and $\tilde{V}_{x_r}$ containing $y_1$ and $\{x_r\}$ respectively so by (4.5 Propositions) there exist disjoint fuzzy $\omega$-open sets $\tilde{U}$ and $\tilde{V}$ containing $\tilde{F}$ and $x_r$ respectively this shows that $(X,\tilde{T})$ is fuzzy $\omega$-regular space, thus we have $(X,\tilde{T})$ fuzzy-$\omega$-paracompact fuzzy $\omega$-regular. Let $\tilde{F}$ and $\tilde{H}$ be any fuzzy two disjoint fuzzy $\omega$-closed subset of $1_x$, since $\tilde{H}$ is fuzzy $\omega$-closed so by fuzzy $\omega$-regularity of $1_x$ for each $y_1 \in \tilde{F}$ there exist disjoint fuzzy $\omega$-open sets $\tilde{U}_{y_1}$ and $\tilde{V}_{y_1}$ containing $y_1$ and $\tilde{H}$ respectively therefore By (4.5 Propositions) there exist disjoint fuzzy $\omega$-open sets $\tilde{U}$ and $\tilde{V}$ containing $\tilde{F}$ and $\tilde{H}$ this showed that $(X,\tilde{T})$ is fuzzy $\omega$-Normal space.

4.7 Corollary
Every fuzzy $\omega$-paracompact $\tilde{T}_2$ space is an fuzzy $\omega$-Normal space.

Proof: Follows by the fact (Every fuzzy $\tilde{T}_2$ space is an fuzzy $\omega$-$\tilde{T}_2$ space) and (4.5 Propositions)

4.8 Proposition
If $(X,\tilde{T})$ is an fuzzy anti-locally countable fuzzy $\omega$-paracompact $\tilde{T}_2$-(resp. $\omega$-$\tilde{T}_2$, $\omega$-regular, $\omega$-Normal) space then it is fuzzy paracompact.

Proof: From 4.6 Propositions and 4.7 Corollary we have only to assume that $1_x$ is an fuzzy $\omega$-paracompact $\omega$-Normal space. Therefore by 3.24 Corollary and 3.25 Theorem $(X,\tilde{T})$ is fuzzy paracompact.

4.9 Theorem
A fuzzy topological space $(X,\tilde{T})$ is fuzzy $\omega$-paracompact $\omega$-Normal space if and only if every fuzzy $\omega$-open covering of $1_x$ has a fuzzy $\omega$-locally finite $\omega$-closed refinement.

Proof: $(\Rightarrow)$ Let $\{\tilde{U}_\lambda\}_{\lambda \in \Lambda}$ be a fuzzy $\omega$-open covering of a fuzzy $\omega$-paracompact $\omega$-Normal space $(X,\tilde{T})$ so by (4.4 Corollary) there exist an fuzzy $\omega$-locally finite $\omega$-open covering $\{\tilde{V}_\lambda\}_{\lambda \in \Lambda}$ of $1_x$ such that $\tilde{V}_\lambda \subseteq \tilde{U}_\lambda$ for each $\lambda \in \Lambda$, since $(X,\tilde{T})$ is fuzzy $\omega$-Normal space then there exist an fuzzy $\omega$-locally finite $\omega$-closed refinement of $\{\tilde{V}_\lambda\}_{\lambda \in \Lambda}$ which also fuzzy covers of $1_x$. $(\Leftarrow)$ Let $(X,\tilde{T})$ be a fuzzy topological space with the property that every fuzzy $\omega$-open covering of it has fuzzy $\omega$-locally finite $\omega$-closed refinement, thus $(X,\tilde{T})$ is fuzzy $\omega$-Normal space, it remains only to show $(X,\tilde{T})$ is fuzzy $\omega$-paracompact. For this let $\{\tilde{W}_\lambda\}_{\lambda \in \Lambda}$ be a fuzzy $\omega$-open covering of $1_x$ and $\{\tilde{F}_\lambda\}_{\lambda \in \Lambda}$ be fuzzy $\omega$-locally finite $\omega$-closed refinement of $\{\tilde{W}_\lambda\}_{\lambda \in \Lambda}$ therefore for each $x_r \in 1_x$ there exist fuzzy $\omega$-open set $\tilde{U}_{x_r}$ containing $x_r$ such that the fuzzy set $\{\gamma \in \Gamma : \tilde{U}_{x_r} \cap \tilde{F}_\gamma \neq \emptyset\}$ is finite. Consider $\{\tilde{E}_\gamma\}_{\gamma \in \Theta}$ is fuzzy $\omega$-locally finite $\omega$-closed refinement of the fuzzy $\omega$-open covering $\{\tilde{U}_{x_r}\}_{x_r \in 1_x}$ of $1_x$ then for each $\gamma \in \emptyset$ the fuzzy set $\{\gamma \in \Gamma : \tilde{E}_\gamma \cap \tilde{F}_\gamma \neq \emptyset\}$ is finite so there exist fuzzy $\omega$-locally finite family $\{\tilde{G}_\gamma : \gamma \in \Gamma\}$ of fuzzy $\omega$-open set of $1_x$ such that $\tilde{F}_\gamma \subseteq \tilde{G}_\gamma$ for each $\gamma \in \Gamma$ which also fuzzy cover of $1_x$, since $\{\tilde{F}_\lambda\}_{\lambda \in \Lambda}$ is fuzzy refinement of $\{\tilde{W}_\lambda\}_{\lambda \in \Lambda}$ so for each $\gamma \in \Gamma$ there is $\lambda(\gamma) \in \Lambda$ such that $\tilde{F}_\gamma \subseteq \tilde{W}_{\lambda(\gamma)}$ therefore $\{\tilde{G}_\gamma \cap \tilde{W}_{\lambda(\gamma)} : \gamma \in \Gamma\}$ is fuzzy $\omega$-locally finite $\omega$-open refinement of $\{\tilde{W}_\lambda\}_{\lambda \in \Lambda}$ Hence $(X,\tilde{T})$ fuzzy $\omega$-paracompact space.

4.10 Proposition
Let $\{\tilde{H}_\lambda\}_{\lambda \in \Lambda}$ be an fuzzy $\omega$-locally finite family of fuzzy $\omega$-closed sets of fuzzy $\omega$-paracompact $\omega$-Normal space $(X,\tilde{T})$ then there exists an fuzzy $\omega$-locally finite family $\{\tilde{U}_\lambda\}_{\lambda \in \Lambda}$ of fuzzy $\omega$-open subset of $1_x$ such that $\tilde{U}_\lambda \subseteq \tilde{H}_\lambda$ for each $\lambda \in \Lambda$ and the fuzzy families $\{\tilde{H}_\lambda\}_{\lambda \in \Lambda}$ and $\{\omega-cl(\tilde{U}_\lambda)\}_{\lambda \in \Lambda}$ are similar.

Proof: Let $\{\tilde{H}_\lambda\}_{\lambda \in \Lambda}$ be fuzzy $\omega$-locally finite family of fuzzy $\omega$-closed sets of fuzzy $\omega$-paracompact $\omega$-Normal space $(X,\tilde{T})$. for each $x_r \in 1_x$ there exist fuzzy $\omega$-open set $\tilde{G}_x$ containing $x_r$ such that $\tilde{G}_x$, 

\[\{\tilde{U}_\lambda\}_{\lambda \in \Lambda}\] 

\[\{\omega-cl(\tilde{U}_\lambda)\}_{\lambda \in \Lambda}\] 

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intersects only finite number of \( H \lambda \) and clearly the fuzzy family \( \{ \tilde{G}_{\lambda} \}_{\lambda \in \Lambda} \) forms fuzzy \( \omega \)-open covering of \( 1_X \), therefore by \textbf{(4.9 Theorem)} \( \{ G_{\lambda} \}_{\lambda \in \Lambda} \) has a fuzzy \( \omega \)-locally finite \( \omega \)-closed refinement \( \{ \tilde{F}_{\gamma} \}_{\gamma \in \Gamma} \) and \( \tilde{F} \), it intersects only finite number of \( \{ H_{\lambda} \}_{\lambda \in \Lambda} \) for each \( \gamma \in \Gamma \), so there exist fuzzy \( \omega \)-locally finite family \( \{ \tilde{E}_{\lambda} : \lambda \in \Lambda \} \) of fuzzy \( \omega \)-open sets of \( 1_X \) such that \( H_{\lambda} \subseteq \tilde{E}_{\lambda} \) for each \( \lambda \in \Lambda \), hence there exist an fuzzy \( \omega \)-locally finite family \( \{ \tilde{U}_{\lambda} : \lambda \in \Lambda \} \) of fuzzy \( \omega \)-open sets such that \( H_{\lambda} \subseteq \tilde{U}_{\lambda} \subseteq \omega \text{-cl}(\tilde{U}_{\lambda}) \subseteq \tilde{E}_{\lambda} \) for each \( \lambda \in \Lambda \) and the fuzzy families \( \{ H_{\lambda} \}_{\lambda \in \Lambda} \) and \( \{ \omega \text{-cl}(\tilde{U}_{\lambda}) \}_{\lambda \in \Lambda} \) are similar.

5. \textbf{Fuzzy \( \omega \)-Paracompact subset}

\textbf{5.1 Proposition}

Every fuzzy \( \omega \)-paracompact subset of a fuzzy topological space \( (X, \tilde{T}) \) is fuzzy \( \omega \)-paracompact subspace.

\textbf{Proof:} Let \( H \) be a fuzzy \( \omega \)-paracompact subset of a fuzzy topological space \( (X, \tilde{T}) \) and let \( \{ H_{\lambda} \}_{\lambda \in \Lambda} \) be fuzzy covering of \( H \) by fuzzy \( \omega \)-open subset of \( H \). By \textbf{(3.11 Theorem)} there exist an fuzzy \( \omega \)-open subset \( \tilde{V}_{\lambda} \) of \( 1_X \) such that \( \tilde{V}_{\lambda} = \tilde{V}_{\lambda} \cap \tilde{H} \) for each \( \lambda \in \Lambda \), then \( \{ \tilde{V}_{\lambda} \}_{\lambda \in \Lambda} \) is fuzzy covering of \( H \) by fuzzy \( \omega \)-open subset of \( 1_X \). So by hypothesis there exist fuzzy \( \omega \)-locally finite \( \omega \)-open refinement \( \{ G_{\gamma} \}_{\gamma \in \Gamma} \) of the fuzzy family \( \{ \tilde{V}_{\lambda} \}_{\lambda \in \Lambda} \) which covers \( H \). Therefore \( \{ G_{\gamma} \cap \tilde{H} \}_{\gamma \in \Gamma} \) is fuzzy \( \omega \)-locally finite \( \omega \)-open refinement of \( \{ \tilde{U}_{\lambda} \}_{\lambda \in \Lambda} \) in \( \tilde{H} \). Thus \( H \) is fuzzy \( \omega \)-paracompact subspace of \( (X, \tilde{T}) \)\( \blacksquare \)

\textbf{5.2 Proposition}

An fuzzy \( \omega \)-closed subset of fuzzy \( \omega \)-paracompact space is fuzzy \( \omega \)-paracompact subspace.

\textbf{Proof:} Let \( F \) be fuzzy \( \omega \)-closed subset of fuzzy \( \omega \)-paracompact space \( 1_X \) and let \( \{ U_{\lambda} \}_{\lambda \in \Lambda} \) be fuzzy covering of \( F \) by fuzzy \( \omega \)-open set of \( 1_X \). Then \( \{ U_{\lambda} \}_{\lambda \in \Lambda} \cup \{ 1_X - F \} \) is fuzzy covering of \( 1_X \) then by hypothesis and in virtue of \textbf{(4.4 Corollary)} there exist an fuzzy \( \omega \)-locally finite \( \omega \)-open covering \( \{ G_{\lambda} \}_{\lambda \in \Lambda} \cup \tilde{G} \) of \( 1_X \) such that \( \tilde{G} \subseteq 1_X - F \) and \( \tilde{G}_{\lambda} \subseteq U_{\lambda} \) for each \( \lambda \in \Lambda \) therefore \( \{ G_{\lambda} \}_{\lambda \in \Lambda} \) is fuzzy \( \omega \)-locally finite \( \omega \)-open refinement of \( \{ U_{\lambda} \}_{\lambda \in \Lambda} \) which cover \( F \). This show that \( F \) fuzzy \( \omega \)-paracompact subset to \( 1_X \) and by \textbf{(5.1 Proposition)} we obtain \( F \) fuzzy \( \omega \)-paracompact subspace \( \blacksquare \)

\textbf{5.3 Proposition}

If a fuzzy topological space \( (X, \tilde{T}) \) is fuzzy \( \omega \)-\( T_2 \) space and has a fuzzy subset \( \tilde{F} \) which is fuzzy \( \omega \)-paracompact subset to \( 1_X \) then for each \( x_r \in 1_X - \tilde{F} \) there exist two disjoint fuzzy \( \omega \)-open sets of \( 1_X \) containing \( x_r \) and \( \tilde{F} \).

\textbf{Proof:} Let \( \tilde{F} \) be fuzzy \( \omega \)-paracompact subset of fuzzy \( \omega \)-\( T_2 \) space \( (X, \tilde{T}) \) and let \( x_r \) be any fuzzy point of \( 1_X - \tilde{F} \) then for each \( y_{\tilde{t}} \in \tilde{F} \) there exist fuzzy \( \omega \)-open sets \( \tilde{V}_{y_{\tilde{t}}} \) and \( \tilde{V}_{x_r} \) such that \( y_{\tilde{t}} \in \tilde{V}_{y_{\tilde{t}}} \) and \( x_r \in \tilde{V}_{x_r} \) and \( \tilde{V}_{y_{\tilde{t}}} \cap \tilde{V}_{x_r} = \emptyset \) this implies that \( \omega \text{-cl}(\tilde{V}_{y_{\tilde{t}}} \cap \tilde{V}_{x_r}) = \emptyset \) hence \( x_r \not\in \omega \text{-cl}(\tilde{V}_{y_{\tilde{t}}}) \) for each \( y_{\tilde{t}} \in \tilde{F} \).

Now \( \{ \tilde{V}_{y_{\tilde{t}}} \}_{y_{\tilde{t}} \in \tilde{F}} \) is fuzzy cover of \( \tilde{F} \) by fuzzy \( \omega \)-open subset of \( 1_X \) thus by hypothesis and in virtue of \textbf{(4.4 Corollary)} there exist an fuzzy \( \omega \)-locally finite covering \( \{ G_{y_{\tilde{t}}} \}_{y_{\tilde{t}} \in \tilde{F}} \) of \( \tilde{F} \) such that for each \( y_{\tilde{t}} \in \tilde{F} \), \( G_{y_{\tilde{t}}} \) is fuzzy \( \omega \)-open set in \( 1_X \) and \( G_{y_{\tilde{t}}} \subseteq \tilde{V}_{y_{\tilde{t}}} \) therefore \( x_r \not\in \omega \text{-cl}(\tilde{G}_{y_{\tilde{t}}}) \) for each \( y_{\tilde{t}} \in \tilde{F} \). Hence \( \tilde{U}= \bigcup_{y_{\tilde{t}} \in \tilde{F}} \tilde{G}_{y_{\tilde{t}}} \) and \( \tilde{V}= 1_X - \bigcup_{y_{\tilde{t}} \in \tilde{F}} \omega \text{-cl}(\tilde{G}_{y_{\tilde{t}}}) \). Therefore there exist two disjoint fuzzy \( \omega \)-open sets of \( 1_X \) containing \( x_r \) and \( \tilde{F} \)\( \blacksquare \)

\textbf{5.4 Corollary}

If \( \tilde{F} \) is fuzzy \( \omega \)-paracompact subset of a fuzzy topological \( \omega \)-\( T_2 \) space \( (X, \tilde{T}) \) then \( \tilde{F} \) is fuzzy \( \omega \)-Normal subspace of \( 1_X \).

\textbf{Proof:} Obvious
5.5 Proposition
If a fuzzy topological space \((X, \tilde{T})\) is fuzzy \(\omega\)-regular space and \(\tilde{F}\) is fuzzy subset of \(1_x\) which is fuzzy \(\omega\)-paracompact subset of \(1_x\) then for each fuzzy \(\omega\)-open set \(\tilde{U}\) containing \(\tilde{F}\) there exist fuzzy \(\omega\)-closed set \(\tilde{H}\) containing \(\tilde{F}\) and it is contained in \(\tilde{U}\) furthermore \(\tilde{F}\) is is fuzzy \(\omega\)-Normal subspace of \(1_x\).

Proof: Since a fuzzy topological space \((X, \tilde{T})\) is fuzzy \(\omega\)-regular space so by (3.22 Proposition) and (5.3 Proposition) \(\tilde{F}\) fuzzy \(\omega\)-closed subset of \(1_x\). And by (5.4 Corollary) it is fuzzy \(\omega\)-Normal subspace of \(1_x\), therefore for each \(x_r \in \tilde{F}\) there exist fuzzy \(\omega\)-open set \(\tilde{U}_{x_r}\) such that \(x_r \in \tilde{U}_{x_r} \subseteq \omega-cl(\tilde{U}_{x_r}) \subseteq \tilde{U}\) since \(\tilde{F}\) is fuzzy \(\omega\)-paracompact subset of \(1_x\) so there exist an fuzzy \(\omega\)-locally finite family \(\{\tilde{G}_y\}_{y \in \Gamma}\) of \(\tilde{F}\) by fuzzy \(\omega\)-open sets of \(1_x\) which refines \(\{\tilde{U}_{x_r}\}_{x_r \in \tilde{F}}\) and covers \(\tilde{F}\) therefore \(\tilde{H} = \bigcup\omega-cl(\tilde{G}_y)\) is the required fuzzy \(\omega\)-closed set.

5.6 Theorem
Let \((X, \tilde{T})\) be a fuzzy \(\omega\)-disconnected space then the statements are equivalent:
1- \((X, \tilde{T})\) is fuzzy \(\omega\)-paracompact space
2- Every fuzzy proper \(\omega\)-closed subset of \(1_x\) is fuzzy \(\omega\)-paracompact subset of \(1_x\)
3- Every fuzzy proper \(\omega\)-closed subset of \(1_x\) is fuzzy \(\omega\)-paracompact subspace
4- Every proper fuzzy \(\omega\)-closed subset of \(1_x\) is fuzzy \(\omega\)-paracompact
5- There exist a fuzzy proper \(\omega\)-closed subset \(\tilde{F}\) of \(1_x\) such that both \(\tilde{F}\) and \(1_x - \tilde{F}\) are fuzzy \(\omega\)-paracompact.

Proof: (1\(\Rightarrow\)2) Follows from 5.2 Proposition
(2\(\Rightarrow\)3) Follows from 5.1 Proposition
(3\(\Rightarrow\)4) Obvious
(4\(\Rightarrow\)5) Clear.
(5\(\Rightarrow\)1) let \((X, \tilde{T})\) be a fuzzy topological space contains a fuzzy proper \(\omega\)-closed subset \(\tilde{F}\) in which both \(\tilde{F}\) and \(1_x - \tilde{F}\) are fuzzy \(\omega\)-paracompact and let \(\{\tilde{G}_y\}_{y \in \Gamma}\) be any fuzzy \(\omega\)-open cover of \(1_x\), then \(\tilde{F} \cap \tilde{G}_y \subseteq \tilde{F}\) and \(\{1_x \cap \tilde{G}_y\}_{y \in \Gamma}\) Covering \(\tilde{F}\) and \(1_x - \tilde{F}\) respectively therefore there exist fuzzy \(\omega\)-locally finite refinement \(\{\tilde{V}_{\lambda}\}_{\lambda \in \Lambda}\) and \(\{\tilde{V}_u\}_{u \in \theta}\) of \(\tilde{F} \cap \tilde{G}_y \subseteq \tilde{F}\) and \(\{1_x \cap \tilde{G}_y\}_{y \in \Gamma}\) Covering \(\tilde{F}\) and \(1_x - \tilde{F}\) respectively such that \(\tilde{V}_{\lambda}\) is fuzzy \(\omega\)-open set in \(\tilde{F}\) for each \(\lambda \in \Lambda\) and \(\tilde{V}_u\) is fuzzy \(\omega\)-open set in \(1_x - \tilde{F}\) for each \(u \in \theta\), then both \(\tilde{V}_{\lambda}\) and \(\tilde{V}_u\) are fuzzy \(\omega\)-open sets in \(1_x\) for each \(\lambda \in \Lambda\) and \(u \in \theta\).

Therefore \(\{\tilde{V}_\beta\}_{\beta \in \Lambda \cup \theta}\) is fuzzy \(\omega\)-locally finite \(\omega\)-open refinement of \(\{\tilde{G}_y\}_{y \in \Gamma}\) which covers \(1_x\), hence \((X, \tilde{T})\) is fuzzy \(\omega\)-paracompact space.

Remark
In the above theorem if \((X, \tilde{T})\) is fuzzy \(\omega\)-connected space then the only fuzzy \(\omega\)-closed subset of \(1_x\) are fuzzy empty set and \(1_x\) itself so the condition that \((X, \tilde{T})\) is fuzzy \(\omega\)-disconnected space is essential.

5.8 Proposition
Let \(\tilde{G}\) be a fuzzy \(\omega\)-closed subset of a fuzzy topological space \((X, \tilde{T})\) then \(\tilde{G}\) is fuzzy \(\omega\)-paracompact subset if and only if \(\tilde{G}\) is fuzzy \(\omega\)-paracompact subspace.

Proof: In view of (5.1 Proposition), we need only to prove the only if part.
Let \(\tilde{G}\) be a fuzzy \(\omega\)-closed \(\omega\)-paracompact subspace of a fuzzy topological space \((X, \tilde{T})\) and let \(\{\tilde{U}_\lambda\}_{\lambda \in \Lambda}\) be fuzzy covering of \(\tilde{G}\) by fuzzy \(\omega\)-open subset of \(1_x\), then \(\{\tilde{G} \cap \tilde{U}_\lambda\}_{\lambda \in \Lambda}\) is a fuzzy covering of \(\tilde{G}\) by fuzzy \(\omega\)-open subset of \(\tilde{G}\), since \(\tilde{G}\) be a fuzzy \(\omega\)-paracompact subspace of a fuzzy topological space \((X, \tilde{T})\) therefore by (4.4 Corollary) there exist an fuzzy \(\omega\)-locally finite \(\omega\)-open covering \(\{\tilde{V}_{\lambda}\}_{\lambda \in \Lambda}\) of \(\tilde{G}\) such that for each \(\lambda \in \Lambda\), \(\tilde{V}_{\lambda} \subseteq \tilde{G} \cap \tilde{U}_\lambda \subseteq \tilde{U}_\lambda\) and \(\tilde{V}_{\lambda}\) is fuzzy \(\omega\)-open set in \(\tilde{G}\) so for each \(\lambda \in \Lambda\), \(\tilde{V}_{\lambda}\) is fuzzy \(\omega\)-open set in \(1_x\), since \(\tilde{G}\) and \(1_x - \tilde{G}\) are fuzzy \(\omega\)-open sets in \(1_x\) this implies that \(\{\tilde{V}_{\lambda}\}_{\lambda \in \Lambda}\) is fuzzy \(\omega\)-locally finite in \(1_x\).
5.9 Proposition
Let $\tilde{G}$ and $\tilde{H}$ be two fuzzy subset of a fuzzy topological space $(X,\tilde{T})$ if $\tilde{G}$ is fuzzy $\omega$-closed and $\tilde{H}$ is fuzzy $\omega$-paracompact subset to $1_X$ then $\tilde{G} \cap \tilde{H}$ is fuzzy $\omega$-paracompact subset to $1_X$ furthermore it is fuzzy $\omega$-paracompact subset to $\tilde{H}$

Proof: Let $\{\tilde{V}_\lambda\}_{\lambda \in \Lambda}$ be any fuzzy covering of $\tilde{G} \cap \tilde{H}$ by fuzzy $\omega$-open subset of $1_X$ since $1_X - \tilde{G}$ is fuzzy $\omega$-open set in $1_X$ and $\tilde{H} - \tilde{G} \subseteq 1_X - \tilde{G}$ then for each $x_\rho \in \tilde{H} - \tilde{G}$ there exist fuzzy $\omega$-open set $\tilde{W}$ in $1_X$ such that $x_\rho \in \tilde{W} \subseteq \tilde{H} - \tilde{G}$ and $\{\tilde{V}_\lambda\}_{\lambda \in \Lambda} \cup \{\tilde{W}\}_{x_\rho \tilde{H} - \tilde{G}}$ is a fuzzy covering of $\tilde{H}$ by fuzzy $\omega$-open subset of $1_X$, since $\tilde{H}$ is fuzzy $\omega$-paracompact subset to $1_X$, therefore this cover has fuzzy $\omega$-locally finite refinement $\{\tilde{Z}_\gamma\}_{\gamma \in \Gamma_1}$. Which covers $\tilde{H}$ and $\tilde{Z}_\gamma$ is fuzzy $\omega$-open set in $1_X$ for each $\gamma \in \Gamma_1$ that is the fuzzy $\omega$-locally finite subfamily $\{\tilde{Z}_\gamma\}_{\gamma \in \Gamma_1}$ where $\Gamma_1 = \{\gamma \in \Gamma; \tilde{Z}_\gamma \subseteq \tilde{V}_\lambda$ for some $\lambda \in \Lambda\}$ is fuzzy $\omega$-open refinement of $\{\tilde{V}_\lambda\}_{\lambda \in \Lambda}$ which covers $\tilde{G} \cap \tilde{H}$ too, thus $\tilde{G} \cap \tilde{H}$ is fuzzy $\omega$-paracompact subset to $1_X$, since $\tilde{H}$ is fuzzy $\omega$-paracompact subset to $1_X$ so by 5.1 Proposition it is fuzzy $\omega$-paracompact subspace of $1_X$ since $\tilde{G}$ fuzzy $\omega$-closed in $1_X$ hence $\tilde{G} \cap \tilde{H}$ is fuzzy $\omega$-closed subset of $\tilde{H}$ and then by 5.2 Proposition $\tilde{G} \cap \tilde{H}$ is fuzzy $\omega$-paracompact subset to $\tilde{H}$. ■

5.10 Proposition
Let $f: (X,\tilde{T}) \rightarrow (Y,\tilde{\sigma})$ be a fuzzy $\omega$-continuous surjection which maps Fuzzy $\omega$-open sets onto Fuzzy open sets, if $\tilde{G}$ is fuzzy $\omega$-paracompact subset to $1_X$ then $f(\tilde{G})$ is fuzzy paracompact subset to $1_Y$.

Proof: Let $\{\tilde{V}_\lambda\}_{\lambda \in \Lambda}$ be any fuzzy covering of $f(\tilde{G})$ by fuzzy open sets of $1_Y$ since $f$ is fuzzy $\omega$-continuous function then $\{f^{-1}(\tilde{V}_\lambda)\}_{\lambda \in \Lambda}$ is fuzzy covering of $\tilde{G}$ by fuzzy $\omega$-open subset of $1_X$. But $\tilde{G}$ is fuzzy $\omega$-paracompact subset to $1_X$ therefore there exist a fuzzy $\omega$-locally finite $\omega$-open family $\{\tilde{Z}_\gamma\}_{\gamma \in \Gamma}$ of subset of $1_X$ which refines $\{f^{-1}(\tilde{V}_\lambda)\}_{\lambda \in \Lambda}$ and cover $\tilde{G}$ since $f$ surjection and maps fuzzy $\omega$-open sets onto fuzzy open sets then $\{f(\tilde{Z}_\gamma)\}_{\gamma \in \Gamma}$ is fuzzy locally finite open family $\{Z_\gamma\}_{\gamma \in \Gamma}$ of subset of $1_Y$ which refines $\{\tilde{V}_\lambda\}_{\lambda \in \Lambda}$ and cover $f(\tilde{G})$ this shows that $(\tilde{G})$ is fuzzy paracompact subset to $1_Y$ ■

5.11 Corollary
Let $f: (X,\tilde{T}) \rightarrow (Y,\tilde{\sigma})$ be a fuzzy $\omega$-continuous surjection which maps fuzzy open sets onto fuzzy open sets, if $(X,\tilde{T})$ is fuzzy $\omega$-paracompact space then $(Y,\tilde{\sigma})$ is fuzzy paracompact space.

Proof: Obvious

5.12 Proposition
Let $f: (X,\tilde{T}) \rightarrow (Y,\tilde{\sigma})$ be a fuzzy $\omega$- irresolute pre- $\omega$-open surjection function if $\tilde{G}$ is fuzzy $\omega$-paracompact subset to $1_X$ then $f(\tilde{G})$ is fuzzy $\omega$-paracompact subset to $1_Y$.

Proof: Similar to the proof of 5.10 Proposition.

5.13 Corollary
Let $f: (X,\tilde{T}) \rightarrow (Y,\tilde{\sigma})$ be a fuzzy $\omega$- irresolute open surjection function if $\tilde{G}$ is fuzzy $\omega$-paracompact subset to $1_X$ then $f(\tilde{G})$ is fuzzy $\omega$-paracompact subset to $1_Y$.

Proof: Obvious

5.14 Corollary
Let $f: (X,\tilde{T}) \rightarrow (Y,\tilde{\sigma})$ be a fuzzy $\omega$- irresolute (pre- $\omega$-open) open surjection function if $(X,\tilde{T})$ is fuzzy $\omega$-paracompact space then $(Y,\tilde{\sigma})$ is fuzzy $\omega$-paracompact space.

Proof: Obvious

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