ON THE DECAY RATES FOR A ONE-DIMENSIONAL POROUS
ELASTICITY SYSTEM WITH PAST HISTORY

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ABSTRACT. This paper studies a porous elasticity system with past history
\begin{align*}
\rho u_{tt} - \mu u_{xx} - b\phi_x &= 0, \\
J \phi_{tt} - \delta \phi_{xx} + bu_x + \xi \phi + \int_0^\infty g(s)\phi_{xx}(t-s)ds &= 0.
\end{align*}

By introducing a new variable, we establish an explicit and a general decay of
energy for the case of equal-speed wave propagation as well as for the nonequal-
speed case. To establish our results, we mainly adopt the method developed
by Guesmia, Messaoudi and Soufyane [Electron. J. Differ. Equa. 2012(2012),
1-45] and some properties of convex functions developed by Alabau-Boussouira
and Cannarsa [C. R. Acad. Sci. Paris Ser. I, 347(2009), 867-872], Lasiecka
and Tataru [Differ. Inte. Equa., 6(1993), 507-533]. In addition we remove the
assumption that \( b \) is positive constant in [J. Math. Anal. Appl., 469(2019),
457-471] and hence improve the result.

1. Introduction. In this paper we are concerned with the following porous elastic-
ity system with past history
\begin{align*}
\rho u_{tt} - \mu u_{xx} - b\phi_x &= 0, \\
J \phi_{tt} - \delta \phi_{xx} + bu_x + \xi \phi + \int_0^\infty g(s)\phi_{xx}(t-s)ds &= 0, \\
u(x,0) &= u_0, \quad u_t(x,0) = u_1, \quad \phi(x,-t) = \phi_0, \quad \phi_t(x,0) = \phi_1, \quad x \in (0,1), \\
u_x(0,t) = \phi_x(0,t) &= \phi(0,t) = \phi(1,t) = 0, \quad t > 0,
\end{align*}

where \((x,t) \in (0,1) \times \mathbb{R}^+\). The viscoelastic materials exhibit a natural weak damping
which is related to their special property of retaining a long time range memory of
their past histories. The function \( g(t) \) is generally called relaxation function.

The materials often arise in many practical problems, for instance, soil mechanics,
engineering, power technology, biology, material science and so on. The theory of
porous elastic materials has been proposed by Cowin and Nunziato [24], where the
authors established a nonlinear theory of elastic materials with voids. See also [7, 8].
The equations for one-dimensional theories of porous materials are given by
\begin{align*}
\rho u_{tt} &= T_x + \gamma u_t, \\
J \phi_{tt} &= H_x + G,
\end{align*}

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where $T$ is the stress, $H$ is the equilibrated stress and $G$ is the equilibrated body force. The function $u(x,t)$ represents the displacement of the solid elastic material, $\phi(x,t)$ is the volume fraction. The constitutive equations are

$$
\begin{align*}
T &= \mu u_x + b\phi, \\
H &= \delta \phi_x, \\
G &= -bu_x - \xi \phi - \tau \phi_t,
\end{align*}
$$

where the constitutive coefficients $\rho, J, \mu, \gamma, b, \delta, \xi$ and $\tau$, in one-dimensional case, satisfy

$$\xi > 0, \quad \delta > 0, \quad \mu > 0, \quad \rho > 0, \quad J > 0, \quad \text{and} \quad \mu \xi \geq b^2. \quad (6)$$

In recent years, there are so many mathematical researchers paid their attentions to study asymptotic behavior of solutions to the equations proposed to study elastic materials. From a mathematical point of view, some of this interest comes from the need to establish general results, which are useful to clarify the empirical observations of engineers. First we mention the work [28]. In this paper, Quintanilla considered (5) only with porous dissipation, i.e., $\gamma = 0$, and proved that the weak damping $\tau \phi_t$ is not strong enough to obtain an exponential decay of the solutions. In addition the author established a slow decay. Recently Apalara [3] proved that the dissipation is strong enough to exponentially stabilize the system in the case of equal-speed wave propagations, i.e.,

$$\frac{\mu}{\rho} = \frac{\delta}{J}. \quad (7)$$

Furthermore, Apalara [4] considered the system when the weak damping $\tau \phi_t$ is of nonlinear term, i.e., $\alpha(t)g(\phi_t)$, and obtained an explicit and general decay rate provided (7) holds. Santos, Campelo and Almeida Júnior [32] studied two cases of (5): the porous elastic system with porous dissipation ($\gamma = 0, \tau > 0$) and the porous elastic system with elastic dissipation ($\gamma > 0, \tau = 0$). They proved that the corresponding semigroup is exponentially stable if and only if the wave speeds of the system are equal. In the case of lack of exponential stability they showed that the solution decays polynomially and proved that the rate of decay is optimal. Magaña and Quintanilla [19] studied a porous elastic solids with viscoelastic damping

$$
\begin{align*}
\rho u_{tt} - \mu u_{xx} - b\phi_x - \gamma u_{xxt} &= 0, \quad x \in (0, L), \quad t > 0, \\
J \phi_{tt} - \delta \phi_{xx} + bu_x + \xi \phi &= 0, \quad x \in (0, L), \quad t > 0,
\end{align*}
$$

They proved that the viscoelastic damping ($-\gamma u_{xxt}$) is not enough to get exponential stability. With respect to porous elastic solids with memory term, the only one we find was due to Apalara [2]. In [2], the author considered

$$
\begin{align*}
\rho u_{tt} - \mu u_{xx} - b\phi_x &= 0, \\
J \phi_{tt} - \delta \phi_{xx} + bu_x + \xi \phi + \int_0^t g(t-s)\phi_{xx}(x,s)ds &= 0.
\end{align*}
$$

By assuming $g'(t) \leq -\eta(t)g(t)$, the author established a general decay result in the case of the equal-speed wave propagation case under the assumption that the constant $b$ is positive. Recently, the present author and his co-author, Feng and Yin [11], extended the result to the case of the non-equal wave speeds, and the result also holds for $b < 0$. For more results concerning the stability to porous elastic solids, one can refer to [12, 23, 26, 29, 30, 31] and so on.
The model of porous thermoelasticity can be obtained from the classical work of Iesan\cite{15,16}. In\cite{5}, the authors considered the following porous thermoelasticity of the form
\[
\begin{aligned}
\rho u_{tt} &= \mu u_{xx} + b\phi_x - \beta \theta_x, \\
J \phi_{tt} &= \alpha \phi_{xx} - bu_x - \xi \phi + m \theta - \tau \phi_t, \\
c\theta_t &= k \theta_{xx} - \beta u_{xt} - m \phi_t,
\end{aligned}
\]
where the function $\theta$ is the temperature difference. They established the exponential decay of the system based on the methods developed by Liu and Zheng\cite{18}. If the porous dissipation is absence, i.e., $\tau = 0$, the heat effect alone is not strong enough to exponentially stabilize the system but only a slow decay is established, see\cite{6}. However, Santos et. al.\cite{33}, proved that the system is exponentially stable if and only if (7) holds. In\cite{34}, Soufyane investigated a thermoelasticity system with finite memory term acting on porous equation
\[
\begin{aligned}
\varphi_{tt} - (\varphi_x + \psi)_x + \theta_x &= 0, \\
\psi_{tt} - \psi_{xx} + (\varphi_x + \psi) - \theta + \int_0^t g(t-s)\psi_{xx}(x,s)ds &= 0, \\
\theta_t - \theta_{xx} + \varphi_{xt} + \psi_t &= 0.
\end{aligned}
\]
Under the assumption $g'(t) \leq -\kappa g^p(t)$, the author proved the system is exponential decay if $p = 1$ and polynomial decay if $1 < p < 3/2$. Messaoudi and Fareh\cite{20,21} considered a more general system than the one in\cite{34}:
\[
\begin{aligned}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \theta_x &= 0, \\
\rho_2 \psi_{tt} - \alpha \psi_{xx} + k(\varphi_x + \psi) - \theta + \int_0^t g(t-s)\psi_{xx}(x,s)ds &= 0, \\
\rho_3 \theta_t - k \theta_{xx} + \varphi_{xt} + \psi_t &= 0.
\end{aligned}
\]
Under the assumption $g'(t) \leq -\xi g(t)$, they established a general decay for the energy in the case of equal wave speeds in\cite{20}, and in the case of nonequal speeds in\cite{21}. The author of the present work, Feng\cite{10}, studied the general model (8) with past history, i.e., the memory term in the second equation is replaced by $\int_0^\infty g(s)\psi_{xx}(t-s)ds$. Under the assumption $g'(t) \leq -\kappa g^p(t)$, he extended the results in\cite{34} to the case of nonequal speeds. For more results concerning porous thermoelasticity, one can refer to\cite{25,35,36} and so on.

In this paper, we established an explicit and a general decay result of energy by introducing suitable energy and perturbed Lyapunov functionals for system (1)-(4) under equal wave speeds and non-equal wave speeds cases. And the case of non-equal wave speeds is more realistic from the physics point of view. Here we mainly adopt the method developed by Guesmia, Messaoudi and Soufyane\cite{14}, Guesmia\cite{13} and some properties of convex functions developed by Alabau-Boussouira and Cannarsa\cite{1}, Lasiecka and Tataru\cite{17}. In the present work we remove the assumption that $b$ is positive constant in Apalara\cite{2}. As coupling is considered, the constant $b$ must be different from 0. Hence we extend and improve the result in Apalara\cite{2}.

Following the same arguments as in Dafermos\cite{9}, we introduce a new variable $\eta$ to deal with the infinite memory, defined by
\[
\eta(x,t,s) = \phi(x,t) - \phi(x,t-s).
\]
It is easy to verify
\[
\eta_t(x,t,s) + \eta_t(x,t,s) = \phi_t(x,t).
\]
Then we can obtain the following system, which is equivalent to problem (1)-(4),
\[
\begin{align*}
\rho u_{tt} - \mu u_{xx} - b\phi_x &= 0, \\
J\phi_{tt} - \delta_1 \phi_{xx} + bu_x + \xi \phi - \int_0^\infty g(s)\eta_{xx}(s)ds &= 0, \\
\eta_t + \eta_s &= \phi_t, \\
u(x, 0) &= u_0, \quad u_t(x, 0) = u_1, \quad \phi(x, -t) = \phi_0, \quad \phi_t(x, 0) = \phi_1, \quad x \in (0, 1), \\
\eta_0 &= \phi_0(x) - \phi_0(x, -s), \quad x \in (0, 1), \\
u_t(0, t) &= u_1(1, t) = \phi(0, t) = \phi(1, t) = \eta(0, t, s) = \eta(1, t, s) = 0, \quad s, t > 0.
\end{align*}
\]
where \(\delta_1 = \delta - \int_0^\infty g(s)ds\).

The outline of this paper is as follows: In Section 2, we give some assumptions and main results. In Section 3, we establish the decay rates of energy to the system (9)-(14).

2. Preliminaries and main results. In this section, we give some assumptions for our consideration and our main results. In the following, \(c\) is used to denote a generic positive constant.

For the relaxation function \(g(t)\), we assume

**Assumption H1.** \(g(t) : \mathbb{R}^+ \to \mathbb{R}^+\) is a decreasing \(C^1\) function satisfying
\[
g(0) > 0 \quad \text{and} \quad \delta - \int_0^\infty g(s)ds = \delta_1 > 0.
\]

**Assumption H2.** There exists an increasing strictly convex function \(G : \mathbb{R}^+ \to \mathbb{R}^+\) of class \(C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+)\) satisfying
\[
G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t \to +\infty} G'(t) = +\infty
\]
such that
\[
\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))}ds + \sup_{s \in \mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty.
\]

With respect to the new variable \(\eta\), as in Pata and Zucchi [27], we define the operator \(T\) as
\[
T\eta = -\eta_s, \quad \eta \in D(T),
\]
which is the infinitesimal generator of a translation semigroup with domain
\[
D(T) = \{\eta \in \mathcal{M}|\eta_s \in \mathcal{M}, \eta(0) = 0\}.
\]
The history space
\[
\mathcal{M} = L^2_g(\mathbb{R}^+; H^1(0, 1)) = \left\{ \eta : \int_0^\infty g(s) \int_0^1 \eta^2_x(s)dxds < +\infty \right\},
\]
is a Hilbert space with inner-product
\[
(\eta, \zeta)_{\mathcal{M}} = \int_0^\infty g(s) \int_0^1 \eta_x(s)\zeta_x(s)dxds,
\]
and norm
\[
\|\eta\|_{\mathcal{M}}^2 = \int_0^\infty g(s) \int_0^1 \eta^2_x(s)dxds.
\]
Let \( U(t) = (u, \phi, \tilde{u}, \tilde{\phi}, \eta)^T \), and \( U_0 = U(0) = (u_0, \phi_0, u_1, \phi_1, \eta_0)^T \). We define the following spaces:
\[
\mathcal{H} = H^1_u(0,1) \times H^1(0,1) \times L^2_u(0,1) \times L^2(0,1) \times \mathcal{M},
\]
and
\[
\mathcal{H}_1 = \{ U \in \mathcal{H} | u \in H^2_u(0,1), \phi \in H^2(0,1) \cap H^1(0,1), \eta \in D(T), \tilde{\phi} \in H^1(0,1), \tilde{u} \in H^1_u(0,1), u_x(0) = u_x(1) = 0 \},
\]
where
\[
H^1_u(0,1) = \left\{ y \in H^1(0,1) \left| \int_0^1 y(x)dx = 0 \right\} \right.,
\]
\[
L^2_u(0,1) = \left\{ y \in L^2(0,1) \left| \int_0^1 y(x)dx = 0 \right\} \right.,
\]
and
\[
H^2_u(0,1) = H^2(0,1) \cap H^1_u(0,1).
\]

By using semigroup method, see, for example, [14], we can prove the following well-posedness result to the problem (9)-(14) and (6).

**Theorem 2.1.** For any initial data \( U_0 \in \mathcal{H} \), problem (9)-(14) and (6) has a unique weak solution
\[
U(t) \in C([0,\infty), \mathcal{H}).
\]
In addition, if \( U_0 \in \mathcal{H}_1 \), the solution is strong solution, and
\[
U(t) \in C([0,\infty), \mathcal{H}_1) \cap C^1([0,\infty), \mathcal{H}).
\]

Now we define the energy functional to problem (9)-(14) and (6) by
\[
E(t) = E(u, \phi, \eta) = \frac{1}{2} \int_0^1 (\rho u_t^2 + J \phi_t^2 + \mu u_x^2 + \delta_1 \phi_x^2 + \xi \phi^2 + 2b u_x \phi)dx
\]
\[
+ \frac{1}{2} \int_0^\infty g(s) \int_0^1 \eta_{xx}^2(s)dxds. \tag{18}
\]

The stability results of the present work are given in the following theorems.

**Theorem 2.2.** Suppose that Assumption H1 and Assumption H2 hold. Let \( \frac{\rho}{\mu} = \frac{\xi}{\delta_1} \).

For any initial data \( U_0 \in \mathcal{H} \) such that there exists some positive constant \( m_0 \),
\[
\int_0^1 \eta_{0xx}^2(s)dx \leq m_0, \quad \forall \ t > 0,
\]
then there exist positive constants \( \beta_1, \beta_2 \) and \( \epsilon_0 \) depending on \( \| U_0 \|_\mathcal{H} \) such that for any \( t \geq 0 \),
\[
E(t) \leq \beta_2 G^{-1}_1(\beta_1 t), \tag{19}
\]
where
\[
G_1(s) = \int_s^1 \frac{1}{\tau G^1(\epsilon_0 \tau)} d\tau, \quad s \in (0,1].
\]

**Theorem 2.3.** Suppose that Assumption H1 and Assumption H2 hold. Let \( \frac{\rho}{\mu} \neq \frac{\xi}{\delta_1} \).

For any initial data \( U_0 \in \mathcal{H}_1 \) such that there exists some positive constant \( m_0 \),
\[
\max \left\{ \int_0^1 \eta_{0xx}^2(s)dx, \int_0^1 \eta_{0xx}^2(s)dx \right\} \leq m_0, \quad \forall \ s > 0,
\]
then there exist positive constants $\beta_3$ and $\epsilon_0$ depending on $\|U_0\|_{H}$ such that for any $t \geq 0$,

$$E(t) \leq G_0^{-1}\left(\frac{\beta_3}{t}\right), \quad \text{(20)}$$

where

$$G_0(s) = sG'(\epsilon_0 s), \quad \forall \ s > 0.$$ 

**Remark 1.** (1). Since $\lim_{t \to 0^+} G_1(t) = +\infty$, we can obtain the strong stability of problem (9)-(14), i.e.,

$$\lim_{t \to +\infty} E(t) = 0. \quad \text{(21)}$$

(2). The decay rate (19) is weaker than the exponential decay

$$E(t) \leq c_1 e^{-c_2 t}, \quad \forall \ t \in \mathbb{R}^+.$$ 

The estimate (19) coincides with (22) with $G=\operatorname{Id}$.

**Remark 2.** We can get (21) from (20) but it is weaker than

$$E(t) \leq \frac{c}{t}, \quad \forall \ t \in \mathbb{R}^+,$$

which coincides with (20) when $G = \operatorname{Id}$.

**Remark 3.** Generally speaking, the assumption which the constant $b$ is positive is not required in the studies on porous-elasticity. Our results also hold that the constant $b$ is negative, but the proposed analysis can not be applied for $b = 0$. In fact, the case $b \neq 0$ means the existence of a coupling meanwhile when $b = 0$ the system is uncoupled and we can not expect decay.

Here we give three examples to illustrate several rates of energy decay. One can find in Guesmia [13].

**Example 2.1.** Let $g(t) = \frac{\mu}{(1+t)^p}$ with $p > 1$ and $\mu > 0$ so small that (15) holds. Assumption H2 holds for $G(t) = t^{1+\frac{q}{2}}$ with $q \in (0, \frac{p-1}{2})$. Then from (19) and (20) we obtain for some constant $\rho > 0$ such that for any $q \in (0, \frac{p-1}{2})$,

$$E(t) \leq \frac{\rho}{(1+t)^q},$$

and

$$E(t) \leq \frac{\rho}{(1+t)^{\frac{q}{p+q}}}.$$ 

**Example 2.2.** Let $g(t) = \mu e^{-(\ln(2+t))^p}$ with $p > 1$ and $\mu > 0$ small enough so that (15) holds. For

$$G(t) = \int_0^t (-\ln s)^{1-\frac{q}{2}} e^{-(\ln s)^{\frac{1}{2}}} ds,$$

when $t$ is near zero, assumption H2 holds with $q \in (1, p)$. Then from (19) we obtain there exist two constants $\rho_1 > 0$ and $\rho_2 > 0$ such that for any $q \in (1, p)$,

$$E(t) \leq \rho_1 e^{-\rho_2(\ln(1+t))^q}.$$ 

For any $q > 1$, assumption H2 also holds with $G(t) = t^q$. Then (20) gives us for some $\rho > 0$,

$$E(t) \leq \frac{\rho}{(1+t)^{\frac{q}{2}}}.$$
Example 2.3. Let \( g(t) = \mu e^{-(1+t)p} \) with \( p \in (0, 1) \) and \( \mu > 0 \) small enough so that (15) holds. For \( G(t) = \int_0^t (-\ln s)^{1-\frac{1}{q}} ds \), when \( t \) is near zero, assumption H2 holds with \( q \in (1, \frac{p}{2}) \). Then from (19), we can get there exist two constants \( \rho_1 > 0 \) and \( \rho_2 > 0 \) such that for any \( q \in (1, \frac{p}{2}) \),
\[
E(t) \leq \rho_1 e^{-\rho_2 t^q}.
\]
For any \( q > 1 \), assumption H2 also holds with \( G(t) = t^q \). Then (20) gives us for some \( \rho > 0 \),
\[
E(t) \leq \frac{\rho}{(1 + t)^{\frac{1}{q}}}.
\]

3. General decay. In this section, we shall prove Theorems 2.2 and 2.3, which will be divided into the following three subsections.

3.1. Technical Lemmas.

Lemma 3.1. The energy functional \( E(t) \) is non-increasing and satisfies
\[
E'(t) = \frac{1}{2} \int_0^\infty g'(s) \int_0^1 \eta_x^2(s) dx ds \leq 0. \tag{23}
\]

Proof. Multiplying (9) by \( u_t \) and (10) by \( \phi_t \), respectively, and using integration by parts, we can get
\[
\frac{1}{2} \frac{d}{dt} \left[ \int_0^1 \left( \rho u_t^2 + J \phi_t^2 + \mu u_x^2 + \delta_1 \phi_x^2 + \xi \phi^2 + 2bu_x \phi \right) dx \right] + \int_0^\infty g(s) \int_0^1 \phi_{xt}(t) \eta_x(s) dx ds = 0. \tag{24}
\]
Note that
\[
\int_0^\infty g(s) \int_0^1 \phi_{xt}(t) \eta_x(s) dx ds = \frac{1}{2} \frac{d}{dt} \int_0^\infty g(s) \int_0^1 \eta_x^2(s) dx ds - \frac{1}{2} \int_0^\infty g'(s) \int_0^1 \eta_x^2(s) dx ds. \tag{25}
\]
Inserting (25) into (24), we can get the desired estimate (23). The proof is done.

In the sequel we define the following functionals:
\[
I_1(t) = -\rho \int_0^1 u u_t dx, \quad I_2(t) = J \int_0^1 \phi \phi_t dx + \frac{b \rho}{\mu} \int_0^1 \phi \int_0^x u_t(y) dy dx,
\]
\[
I_3(t) = -J \int_0^\infty g(s) \int_0^1 \phi_t(t) \eta(s) dx ds,
\]
and
\[
I_4(t) = \text{sign}(b) \left[ \frac{\delta_1 \rho}{\mu} \int_0^1 u_t \phi_x dx + J \int_0^1 \phi_t u_x dx + \frac{\rho}{\mu} \int_0^\infty g(s) \int_0^1 u_t(t) \eta_x(s) dx ds \right].
\]
Lemma 3.2 ([2]). The functional $I_1(t)$ satisfies for any $t \geq 0$,

$$I_1'(t) \leq -\rho \int_0^1 u_t^2 dx + \frac{3\mu}{2} \int_0^1 u_x^2 dx + \frac{b^2 c_2^2}{2\mu} \int_0^1 \phi_x^2 dx,$$

(26)

where $c_2 > 0$ is Poincaré’s constant.

Lemma 3.3. For any $\varepsilon, \varepsilon_1 > 0$, the functional $I_2(t)$ satisfies

$$I_2'(t) \leq -(\delta_1 - \varepsilon_1) \int_0^1 \phi_2^2 dx - \left(\varepsilon - \frac{b^2}{\mu}\right) \int_0^1 \phi^2 dx + \varepsilon \int_0^1 u_t^2 dx + (J + c_0) \int_0^1 \phi_t^2 dx + \frac{\delta - \delta_1}{4\varepsilon_1} \int_0^\infty g(s) \int_0^1 \eta_x^2(s) dx ds,$$

(27)

where $c_0$ is a positive constant depending on $\varepsilon$.

Proof. We take the derivative of $I_2(t)$ with respect to $t$ and use (9)-(10) to get

$$I_2'(t) = \int_0^1 \phi_2^2 dx - \delta_1 \int_0^1 \phi_x^2 dx - \left(\varepsilon - \frac{b^2}{\mu}\right) \int_0^1 \phi^2 dx + \varepsilon \int_0^1 u_t^2 dx + (J + c_0) \int_0^1 \phi_t^2 dx + \frac{\delta - \delta_1}{4\varepsilon_1} \int_0^\infty g(s) \int_0^1 \phi_x(t) \eta_x(s) dx ds. \quad (28)$$

It follows from Hölder’s inequality and Young’s inequality that for any $\varepsilon, \varepsilon_1 > 0$,

$$D_1 \leq \varepsilon \int_0^1 \left( \int_0^x u_t(y) dy \right)^2 dx + \frac{b^2 \rho^2}{4\varepsilon \mu^2} \int_0^1 \phi_t^2 dx$$

$$\leq \varepsilon \int_0^1 u_t^2 dx + \frac{b^2 \rho^2}{4\varepsilon \mu^2} \int_0^1 \phi_t^2 dx,$$

(29)

$$D_2 \leq \varepsilon_1 \int_0^1 \phi_x^2 dx + \frac{1}{4\varepsilon_1} \int_0^1 \left( \int_0^\infty g(s) \eta_x(s) ds \right)^2 dx$$

$$\leq \varepsilon_1 \int_0^1 \phi_x^2 dx + \frac{\delta - \delta_1}{4\varepsilon_1} \int_0^\infty g(s) \int_0^1 \eta_x^2(s) dx ds. \quad (30)$$

Then (27) follows from (28)-(30). This completes the proof. □

Lemma 3.4. The functional $I_3(t)$ satisfies for any $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$,

$$I_3'(t) \leq -\frac{J}{2} (\delta - \delta_1) \int_0^1 \phi_2^2 dx + 2\varepsilon_2 \int_0^1 \phi_2^2 dx + \varepsilon_3 \int_0^1 u_t^2 dx$$

$$-c_1 \int_0^\infty g'(s) \int_0^1 \eta_x^2(s) dx ds + c_2 \int_0^\infty g(s) \int_0^1 \eta_x^2(s) dx ds,$$

(31)

where $c_1$ and $c_2$ are two positive constants, and $c_2$ depends on $\varepsilon_2$ and $\varepsilon_3$. 
Proof. By using (10), we can obtain
\[
I'_3(t) = -J \int_0^\infty g(s) \int_0^1 \phi_t(t) \eta(s) dx ds - J \int_0^\infty g(s) \int_0^1 \phi(t) \eta_t(s) dx ds
= \delta_1 \int_0^\infty g(s) \int_0^1 \phi_x(t) \eta_x(s) dx ds + b \int_0^\infty g(s) \int_0^1 u_x(t) \eta(s) dx ds
+ \xi \int_0^\infty g(s) \int_0^1 \phi(t) \eta(s) dx ds + \int_0^1 \left( \int_0^\infty g(s) \eta_x(s) ds \right)^2 dx
- J \int_0^\infty g(s) \int_0^1 \phi_t(t) \eta_t(s) dx ds.
\] (32)

Hölder’s inequality, Young’s inequality and Poincaré’s inequality give us for any \( \varepsilon_2 > 0 \) and \( \varepsilon_3 > 0 \),
\[
\delta_1 \int_0^\infty g(s) \int_0^1 \phi_x(t) \eta_x(s) dx ds
\leq \varepsilon_2 \int_0^1 \phi_x^2 dx + \frac{\delta_1^2 (\delta - \delta_1)}{4 \varepsilon_2} \int_0^\infty g(s) \int_0^1 \eta_x^2(s) dx ds,
\] (33)
\[
b \int_0^\infty g(s) \int_0^1 u_x(t) \eta(s) dx ds
\leq \varepsilon_3 \int_0^1 u_x^2 dx + \frac{b^2 \varepsilon_3^2 (\delta - \delta_1)}{4 \varepsilon_3} \int_0^\infty g(s) \int_0^1 \eta_x^2(s) dx ds,
\] (34)
\[
\xi \int_0^\infty g(s) \int_0^1 \phi(t) \eta(s) dx ds
\leq \frac{\varepsilon_2^2}{c_\phi^2} \int_0^1 \phi^2 dx + \frac{\xi \varepsilon_2^2}{c_\phi^2} \int_0^1 \left( \int_0^\infty g(s) \eta(s) ds \right)^2 dx
\leq \varepsilon_2 \int_0^1 \phi^2 dx + \frac{\xi \varepsilon_2^2}{c_\phi^2} (\delta - \delta_1) \int_0^\infty g(s) \int_0^1 \eta_x^2(s) dx ds,
\] (35)
and
\[
\int_0^1 \left( \int_0^\infty g(s) \eta_x(s) ds \right)^2 dx \leq (\delta - \delta_1) \int_0^\infty g(s) \int_0^1 \eta_x^2(s) dx ds.
\] (36)
Noting (11), we shall see that
\[
-J \int_0^\infty g(s) \int_0^1 \phi_t(t) \eta_t(s) dx ds
= -J(\delta - \delta_1) \int_0^1 \phi_t^2 dx + J \int_0^\infty g(s) \int_0^1 \phi(t) \eta_t(s) dx ds
= -J(\delta - \delta_1) \int_0^1 \phi_t^2 dx - J \int_0^\infty g'(s) \int_0^1 \phi_t(t) \eta(s) dx ds
\leq -\frac{1}{2} J(\delta - \delta_1) \int_0^1 \phi_t^2 dx
+ \frac{J}{2(\delta - \delta_1)} \int_0^1 \left( \int_0^\infty -g'(s) ds \right) \left( \int_0^\infty -g'(s) \eta^2(s) ds \right) dx
\]
Inserting (33)-(37) into (32), we can get (31) with
\[ c_1 = \frac{Jg(0)c_2^2}{2(\delta - \delta_1)} \quad \text{and} \quad c_2 = \frac{\delta_1^2(\delta - \delta_1)}{4\varepsilon_2} + \frac{b^2c_2^2(\delta - \delta_1)}{4\varepsilon_3} + \frac{c_2^4\xi^2(\delta - \delta_1)}{4\varepsilon_2}. \]

The proof is done. \(\square\)

**Lemma 3.5.** The functional \(I_4(t)\) satisfies that for any \(\varepsilon_4 > 0\),
\[
I'_4(t) \leq -\frac{|b|}{2} \int_0^1 u_x^2 dx + \varepsilon_4 \int_0^1 v_t^2 dx + c_3 \int_0^1 \phi_x^2 dx \\
+ \text{sign}(b) \left( \frac{\delta_1 \rho}{\mu} - J \right) \int_0^1 u_t \phi_x dx + \frac{|b|(\delta - \delta_1)}{2\mu} \int_0^\infty g(s) \int_0^1 \eta_x^2 dx ds \\
- c_4 \int_0^\infty g' (s) \int_0^1 \eta_x^2 dx ds,
\]
where \(c_3\) and \(c_4\) are two positive constants, and \(c_4\) depends on \(\varepsilon_4\).

**Proof.** First by using boundary conditions \(u_x(0, t) = u_x(1, t) = 0\), we have
\[
\int_0^1 u_{xx} \phi_x dx = - \int_0^1 \phi_{xx} u_x dx,
\]
and
\[
\int_0^\infty g(s) \int_0^1 u_x(t) \eta_{xx} dx ds = - \int_0^\infty g(s) \int_0^1 u_{xx}(t) \eta_x dx ds.
\]
Then differentiating \(I_4(t)\) with respect to \(t\) and using (9)-(10), we obtain that
\[
I'_4(t) = \text{sign}(b) \left[ \frac{\delta_1 \rho}{\mu} \int_0^1 u_t \phi_x dx + \frac{\delta_1 \rho}{\mu} \int_0^1 u_t \phi_x dx + J \int_0^1 \phi \phi_x dx \\
+ J \int_0^1 \phi_x u_x dx + \frac{\rho \mu}{\mu} \int_0^\infty g(s) \int_0^1 u_t(t) \eta_x dx ds \\
+ \frac{\rho \mu}{\mu} \int_0^\infty g(s) \int_0^1 u_t(t) \eta_x dx ds \right] \\
= \text{sign}(b) \delta_1 \int_0^1 u_x \phi_x dx + \frac{\delta_1 |b|}{\mu} \int_0^1 \phi_x^2 dx + \text{sign}(b) \frac{\delta_1 \rho}{\mu} \int_0^1 u_t \phi_x dx \\
+ \text{sign}(b) \delta_1 \int_0^1 \phi_x u_x dx - |b| \int_0^1 u_x^2 dx - \text{sign}(b) \xi \int_0^1 \phi u_x dx \\
+ \text{sign}(b) \int_0^\infty g(s) \int_0^1 u_x(t) \eta_x ds dx - \text{sign}(b) J \int_0^1 u_t \phi_x dx \\
+ \text{sign}(b) \int_0^\infty g(s) \int_0^1 u_x(t) \eta_x ds dx \\
+ \text{sign}(b) \frac{b}{\mu} \int_0^\infty g(s) \int_0^1 \phi(t) \eta_x ds dx \\
+ \text{sign}(b) \frac{b}{\mu} \int_0^\infty g(s) \int_0^1 u_t(t) \eta_x ds dx.
\[
\frac{\delta_1 |b| }{\mu} \int_0^1 \phi_x^2 dx + \text{sign}(b) \left( \frac{\delta_0 \rho}{\mu} - J \right) \int_0^1 u_t \phi_{xt} dx - |b| \int_0^1 u_x^2 dx \\
+ |b| \mu \int_0^\infty g(s) \int_0^1 \phi_x(t) \eta_x(s) ds - \text{sign}(b) \xi \int_0^1 u_x \phi dx \\
+ \text{sign}(b) \frac{\rho}{\mu} \int_0^\infty g(s) \int_0^1 u_t(t) \eta_{xt}(s) dx ds.
\]

Noting (11), we see that
\[
\text{sign}(b) \frac{\rho}{\mu} \int_0^\infty g(s) \int_0^1 u_t(t) \eta_{xt}(s) dx ds
\]
\[
= \text{sign}(b) \frac{\rho}{\mu} \int_0^\infty g(s) \int_0^1 u_t(t) [\phi_{xt}(t) - \eta_{xs}(s)] dx ds
\]
\[
= \text{sign}(b) \frac{\rho}{\mu} \int_0^\infty g(s) ds \int_0^1 u_t \phi_{xt} dx - \text{sign}(b) \frac{\rho}{\mu} \int_0^\infty g(s) \int_0^1 u_t(t) \partial_x \eta_x(s) dx ds
\]
\[
= \text{sign}(b) \frac{\rho}{\mu} \int_0^\infty g(s) ds \int_0^1 u_t \phi_{xt} dx + \text{sign}(b) \frac{\rho}{\mu} \int_0^\infty g'(s) \int_0^1 u_t(t) \eta_x(s) ds dx ds.
\]

Replacing (40) in (39) and recalling \( \delta = \int_0^\infty g(s) ds + \delta_1 \), we can get
\[
I'_4(t) = \frac{\delta_1 |b| }{\mu} \int_0^1 \phi_x^2 dx + \text{sign}(b) \left( \frac{\delta_0 \rho}{\mu} - J \right) \int_0^1 u_t \phi_{xt} dx
\]
\[
- |b| \int_0^1 u_x^2 dx + \frac{|b|}{\mu} \int_0^\infty g(s) \int_0^1 \phi_x(t) \eta_x(s) ds dx ds
\]
\[
+ \text{sign}(b) \frac{\rho}{\mu} \int_0^\infty g'(s) \int_0^1 u_t(t) \eta_x(s) ds dx ds - \text{sign}(b) \xi \int_0^1 u_x \phi dx.
\]

In view of Hölder’s, Young’s and Poincaré’s inequalities, we shall see that for any \( \varepsilon > 0 \),
\[
D_3 \leq \frac{|b|}{2\mu} \int_0^1 \phi_x^2 dx + \frac{|b|}{2\mu} \int_0^1 \left( \int_0^\infty g(s) \eta_x(s) ds \right)^2 dx
\]
\[
\leq \frac{|b|}{2\mu} \int_0^1 \phi_x^2 dx + \frac{|b|}{2\mu} (\delta - \delta_1) \int_0^\infty g(s) \int_0^1 \eta_x^2(s) ds dx ds,
\]
\[
D_4 \leq \varepsilon_4 \int_0^1 u_x^2 dx + \frac{\rho^2}{4\varepsilon_4 \mu^2} \int_0^1 \left( \int_0^\infty g'(s) \eta_x(s) ds \right)^2 dx
\]
\[
\leq \varepsilon_4 \int_0^1 u_x^2 dx - \frac{\rho^2 g(0)}{4\varepsilon_4 \mu^2} \int_0^1 g'(s) \int_0^1 \eta_x^2(s) ds dx ds,
\]

and
\[
D_5 \leq \frac{|b|}{2} \int_0^1 u_x^2 dx + \frac{\varepsilon_4^2 \rho^2}{2|b|} \int_0^1 \phi_x^2 dx,
\]

which, together with (41)-(43), gives us (38) with
\[
c_3 = \frac{\delta_1 |b| }{\mu} + \frac{|b|}{2\mu} \quad \text{and} \quad c_4 = \frac{\rho^2 g(0)}{4\varepsilon_4 \mu^2}.
\]
The proof is hence complete.

In the sequel, we define the functional $\mathcal{L}(t)$ by

$$
\mathcal{L}(t) = N E(t) + I_1(t) + N_2 I_2(t) + N_3 I_3(t) + N_4 I_4(t),
$$

(44)

where $N, N_2, N_3$ and $N_4$ are positive constants will be chosen later. It is easy to show that for $N > 0$ large enough, the functional $\mathcal{L}(t)$ is equivalent to $E(t)$.

**Lemma 3.6.** It holds that for any $t \geq 0$,

$$
\mathcal{L}'(t) \leq -c E(t) + c \int_0^\infty g(s) \int_0^1 \eta_3^2(s) dx ds
$$

$$
+ \text{sign}(b) N_4 \left( \frac{\delta \rho}{\mu} - J \right) \int_0^1 u_t \phi_{xt} dx.
$$

(45)

**Proof.** It follows from (23), (26)-(27), (31) and (38) that for any $t \geq 0$,

$$
\mathcal{L}'(t) \leq - \left( \rho - N_2 \varepsilon - N_4 \varepsilon_4 \right) \int_0^1 u_t^2 dx - \left[ \frac{N_3}{2} J(\delta - \delta_1) - (J + c_0) N_2 \right] \int_0^1 \phi_t^2 dx
$$

$$
- \left( \frac{b}{2} N_4 - \frac{3 \mu}{2} - N_3 \varepsilon_3 \right) \int_0^1 u_x^2 dx - N_2 \left( \xi - \frac{b^2}{\mu} \right) \int_0^1 \phi_x^2 dx
$$

$$
- \left[ N_2(\delta_1 - \varepsilon_1) - 2 N_3 \varepsilon_2 - \frac{b^2 c_2^2}{2 \mu} - N_4 \varepsilon_3 \right] \int_0^1 \phi_x^2 dx
$$

$$
+ \left[ \frac{N}{2} - N_3 c_1 - N_4 c_3 \right] \int_0^\infty g'(s) \int_0^1 \eta_x^2(s) dx ds
$$

$$
+ \left[ N_3 c_2 + \frac{N_4 b(\delta - \delta_1)}{2 \mu} + \frac{N_2(\delta - \delta_1)}{4 \varepsilon_1} \right] \int_0^\infty g(s) \int_0^1 \eta_x^2(s) dx ds
$$

$$
+ \text{sign}(b) N_4 \left( \frac{\delta \rho}{\mu} - J \right) \int_0^1 u_t \phi_{xt} dx.
$$

(46)

In (46) we take

$$
\varepsilon = \frac{\rho}{2 N_2}, \quad \varepsilon_1 = \frac{\delta_1}{2}, \quad \varepsilon_2 = \frac{\delta_1}{4 N_3}, \quad \varepsilon_3 = \frac{\mu}{2 N_3}, \quad \varepsilon_4 = \frac{\rho}{4 N_4},
$$

to get

$$
\mathcal{L}'(t) \leq - \frac{\rho}{4} \int_0^1 u_t^2 dx - \left[ \frac{N_3}{2} J(\delta - \delta_1) - (J + c_0) N_2 \right] \int_0^1 \phi_t^2 dx
$$

$$
- \left( \frac{b}{2} N_4 - 2 \mu \right) \int_0^1 u_x^2 dx - N_2 \left( \xi - \frac{b^2}{\mu} \right) \int_0^1 \phi_x^2 dx
$$

$$
- \left( \frac{\delta_1}{2} N_2 - \frac{\delta_1}{2} - \frac{b^2 c_2^2}{2 \mu} - N_4 \varepsilon_3 \right) \int_0^1 \phi_x^2 dx
$$

$$
+ \left[ \frac{N}{2} - N_3 c_1 - N_4 c_3 \right] \int_0^\infty g'(s) \int_0^1 \eta_x^2(s) dx ds
$$

$$
+ \left[ N_3 c_2 + \frac{N_4 b(\delta - \delta_1)}{2 \mu} + \frac{N_2(\delta - \delta_1)}{2 \delta_1} \right] \int_0^\infty g(s) \int_0^1 \eta_x^2(s) dx ds
$$

$$
+ \text{sign}(b) N_4 \left( \frac{\delta \rho}{\mu} - J \right) \int_0^1 u_t \phi_{xt} dx.
$$

(47)
At this point we take $N_4 > 0$ such that $N_4 > 4\mu |b|$, which implies
\[
\frac{|b|}{2} N_4 - 2\mu > 0.
\]
Then we choose $N_2$ large such that
\[
\frac{\delta_1}{2} N_2 - \frac{\delta_1}{2} - \frac{b^2 c^2}{2\mu} - N_4 c_3 > 0.
\]
At last we pick $N_3 > 0$ large enough so that
\[
\frac{N_3}{2} J(\delta - \delta_1) - (J + c_0) N_2 > 0.
\]
In view of $\mu \xi > b^2$ and (47), we know that
\[
\mathcal{L}'(t) \leq -c \int_0^1 (u_t^2 + \phi_t^2 + u_x^2 + \phi_x^2 + \phi^2) dx + c \int_0^\infty g(s) \int_0^1 \eta^2_2(s) dx ds
\] 
\[+ \text{sign}(b) N_4 \left( \frac{\delta \rho}{\mu} - J \right) \int_0^1 u_t \phi dx,
\]
which, together with (18), gives us (45). The proof is now complete. \(\square\)

**Lemma 3.7.** Under the assumptions of Theorem 2.2, there exists a positive constant $\gamma_1 > 0$ such that for any $\epsilon_0 > 0$
\[
G'(\epsilon_0 E(t)) \int_0^\infty g(s) \int_0^1 \eta^2_2(s) dx ds \leq -\gamma_1 E'(t) + \gamma_1 \epsilon_0 E(t) G'(\epsilon_0 E(t)).
\] (49)

**Proof.** This lemma can easily be proved by repeating the arguments of Guesmia [13]. \(\square\)

**Lemma 3.8.** Under the assumptions of Theorem 2.2, we obtain for any $\tau > 0$ and $\epsilon_0 > 0$,
\[
E'(t) \leq -c\tau E(t) G'(\epsilon_0 E(t)) + \text{sign}(b) N_4 \frac{\delta \rho}{\mu} - J \int_0^1 u_t \phi dx.
\] (50)

where $E(t) = \tau \left( G'(\epsilon_0) \mathcal{L}(t) + c \gamma_1 E(t) \right)$.

**Proof.** Multiplying (45) by $G'(\epsilon_0 E(t))$ and using (49), we have
\[
G'(\epsilon_0 E(t)) \mathcal{L}'(t) \leq -cG'(\epsilon_0 E(t)) E(t) + cG'(\epsilon_0 E(t)) \int_0^\infty g(s) \int_0^1 \eta^2_2(s) dx ds
\] 
\[+ \text{sign}(b) N_4 \left( \frac{\delta \rho}{\mu} - J \right) \int_0^1 u_t \phi dx
\] 
\[\leq -c \epsilon_0 G'(\epsilon_0 E(t)) E(t) - cE'(t)
\] 
\[+ \text{sign}(b) N_4 \left( \frac{\delta \rho}{\mu} - J \right) \int_0^1 u_t \phi dx.
\]
Taking $\epsilon_0 > 0$ small enough so that $c - c\epsilon_0 > 0$, and then we obtain
\[
G'(\epsilon_0 E(t)) \mathcal{L}'(t) + c E'(t)
\] 
\[\leq -c G'(\epsilon_0 E(t)) E(t) + \text{sign}(b) N_4 \left( \frac{\delta \rho}{\mu} - J \right) G'(\epsilon_0 E(t)) \int_0^1 u_t \phi dx.
\]
For any $\tau > 0$, we find that $E(t)$, defined by
\[ E(t) = \tau \left( G'(\epsilon_0) L(t) + c_1 E(t) \right) \]
is equivalent to $E(t)$, and
\[ E'(t) \leq -c\tau E(t) G'(\epsilon_0 E(t)) + \text{sign}(b) N_4 \tau \int_0^1 u_t \phi_{xt} dx, \]
where we used the fact $G'(\epsilon_0 E(t))$ is non-increasing.

3.2. Proof of Theorem 2.2. Since $\frac{\rho}{\delta} = \frac{\delta}{2}$, then (50) gives us
\[ E'(t) \leq -c\tau E(t) G'(\epsilon_0 E(t)). \]
We take $\tau > 0$ so small that $E(t) \leq E(t)$ and $E(0) \leq 1$. (51)
Noting that $s \mapsto s G'(\epsilon_0 s)$ is non-increasing, we can get
\[ E'(t) \leq -c E(t) G'(\epsilon_0 E(t)). \]
Let
\[ G_1(t) = \int_t^1 \frac{1}{s G'(\epsilon_0 s)} ds, \]
then we have
\[ (G_1(E(t)))' \geq c. \] (52)
Integrating (52) over $[0, t]$, we have
\[ G_1(E(t)) \geq ct + G_1(E(0)), \]
which, together with (51) and $G_1(1) = 0$, implies
\[ G_1(E(t)) \geq ct. \]
Since $G_1^{-1}$ is decreasing, we obtain
\[ E(t) \leq G_1^{-1}(ct). \]
Using $E(t) \sim E(t)$, we get the desired result (19).

3.3. Proof of Theorem 2.3. In this subsection, we consider the case $\frac{\rho}{\delta} \neq \frac{\delta}{2}$, which is more realistic from the physics point of view.

Differentiating system (9)-(13) with respect to time, we get a new system
\[ \begin{aligned}
\rho u_{ttt} - \mu u_{xxt} - b \phi_{xt} &= 0, \\
J \phi_{ttt} - \delta_1 \phi_{xxt} + bu_{xt} + \xi \phi_t - \int_1^\infty g(s) \eta_{xxt}(s) ds &= 0, \\
\eta_{tt} + \eta_{xt} &= \phi_t, \\
u_{xxt}(0, t) &= u_{xxt}(1, t) = \phi_t(0, t) = \phi_t(1, t) = \eta(0, t, s) = \eta(1, t, s) = 0.
\end{aligned} \] (53)
For $U_0 \in H_1$, system (53) is well posed. Next we introduce second-order energy functional to problem (9)-(13) by
\[ \tilde{E}(t) = E(u_t, \phi_t, \eta_t) = \frac{1}{2} \int_0^1 \left( \rho u_{tt}^2 + J \phi_{tt}^2 + \mu u_{xxt}^2 + \delta_1 \phi_{xxt}^2 + \xi \phi_t^2 + 2bu_{xt} \phi_t \right) dx \\
+ \frac{1}{2} \int_0^\infty g(s) \int_0^1 \eta_{xxt}^2(s) dxds. \] (54)
By using the same arguments as in Lemma 3.1, we have the following lemma.
Lemma 3.9. Under the assumptions in Theorem 2.3, the second-order energy \( \tilde{E}'(t) \) defined by (54) is non-increasing and satisfies

\[
\tilde{E}'(t) = \frac{1}{2} \int_0^\infty g'(s) \int_0^1 \eta_{xt}^2(s) ds dx \leq 0.
\]  

(55)

Now we employ the method in [22], see also [14], to estimate the last term in the right hand side of (45).

Lemma 3.10. It holds for any \( \nu > 0 \) and \( t \geq 0 \),

\[
sign(b)N_4 \tau \left( \frac{\delta \rho}{\mu} - J \right) \int_0^1 u_t \phi_{xt} dx \leq \nu \int_0^1 u_t^2 dx + C_\nu \int_0^\infty g(s) \int_0^1 \eta_{xt}^2(s) ds dx
\]

\[
- C_\nu \int_0^\infty g'(s) \int_0^1 \eta_{xt}^2(s) ds dx.
\]  

(56)

Proof. Since \( \int_0^\infty g(s) ds = \delta - \delta_1 \), it is obvious that for any \( t > 0 \),

\[
sign(b)N_4 \tau \left( \frac{\delta \rho}{\mu} - J \right) \int_0^1 u_t \phi_{xt} dx
\]

\[
= \frac{\sign(b)N_4 \tau \left( \frac{\delta \rho}{\mu} - J \right) \int_0^\infty g(s) \int_0^1 u_t(t) \phi_{xt}(t) - \phi_{xt}(t - s) ds dx}{\delta - \delta_1}
\]

\[
:=D_7
\]

\[
+ \frac{\sign(b)N_4 \tau \left( \frac{\delta \rho}{\mu} - J \right) \int_0^\infty g(s) \int_0^1 u_t(t) \phi_{xt}(t - s) ds dx}{\delta - \delta_1}
\]

(57)

By using Young’s inequality and Hölder’s inequality, we infer that for any \( \nu > 0 \),

\[
D_7 \leq \frac{\nu}{4} \int_0^1 u_t^2 dx + C_\nu \int_0^\infty g(s) \int_0^1 \eta_{xt}^2(s) ds dx
\]

\[
\leq \frac{\nu}{2} E(t) + C_\nu \int_0^\infty g(s) \int_0^1 \eta_{xt}^2(s) ds dx,
\]  

(58)

and

\[
D_8 = \frac{\sign(b)N_4 \tau \left( \frac{\delta \rho}{\mu} - J \right) \int_0^1 u_t \left( g(0) \phi_x + \int_0^\infty g'(s) \phi_x(t - s) ds \right) dx}{\delta - \delta_1}
\]

\[
= \frac{\sign(b)N_4 \tau \left( \frac{\delta \rho}{\mu} - J \right) \int_0^1 u_t \int_0^\infty (-g'(s)) \phi_x(t - s) ds dx}{\delta - \delta_1}
\]

\[
\leq \frac{\nu}{2} E(t) - C_\nu \int_0^\infty g'(s) \int_0^1 \eta_{xt}^2(s) ds dx,
\]

which, together with (57)-(58), gives us (56).

Proof of Theorem 2.3. It follows from (50) and (56) that for any \( t > 0 \),

\[
\tilde{E}'(t) \leq -(c - \nu) E(t) G'(\epsilon_0 E(t)) + c G'(\epsilon_0 E(t)) \int_0^\infty g(s) \int_0^1 \eta_{xt}^2(s) ds dx
\]

\[
- c G'(\epsilon_0 E(t)) \int_0^\infty g'(s) \int_0^1 \eta_{xt}^2(s) ds dx.
\]
Taking $\nu > 0$ small enough so that $c - \nu > 0$, we can obtain
\[
\mathcal{E}'(t) \leq -c E(t) G'(\epsilon_0 E(t)) + c G'(\epsilon_0 E(t)) \int_0^\infty g(s) \int_0^1 \eta_{E_t}^2(s) ds ds
\]
\[
- c G'(\epsilon_0 E(t)) \int_0^\infty g(s) \int_0^1 \eta_{E_t}^2(s) ds ds.
\] (59)

Using (23), then we infer from (59) that
\[
E(t) G'(\epsilon_0 E(t)) \leq -c E'(t) G'(\epsilon_0 E(t)) - c \mathcal{E}'(t)
\]
\[
+ c G'(\epsilon_0 E(t)) \int_0^\infty g(s) \int_0^1 \eta_{E_t}^2(s) ds ds.
\] (60)

By using the same arguments as (49) for $\int_0^\infty g(s) \int_0^1 \eta_{E_t}^2(s) ds ds$, we have there exists a positive constant $\gamma_2$ such that for any $\epsilon_0 > 0$,
\[
G'(\epsilon_0 E(t)) \int_0^\infty g(s) \int_0^1 \eta_{E_t}^2(s) ds ds \leq -\gamma_2 \dot{E}'(t) + \gamma_2 \epsilon_0 E(t) G'(\epsilon_0 E(t)).
\] (61)

Choosing $\epsilon_0 > 0$ small, then we can get from (60)-(61)
\[
E(t) G'(\epsilon_0 E(t)) \leq -c E'(t) G'(\epsilon_0 E(t)) - c \mathcal{E}'(t) - c \dot{E}'(t).
\]

Recalling that $\mathcal{E}(t) \sim E(t)$ and $E(t) G'(\epsilon_0 E(t))$ is non-increasing, we can obtain for any $T > 0$,
\[
G_0(E(T)) T \leq \int_0^T G_0(E(t)) dt \leq c(G'(\epsilon_0 E(0)) + 1) E(0) + c \dot{E}(0),
\]
which implies (20) with $\beta_3 = c(G'(\epsilon_0 E(0)) + 1) E(0) + c \dot{E}(0)$. The proof of Theorem 2.3 is complete.

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