Comments on A,B,C Chains
of Heterotic and Type II Vacua

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ABSTRACT

We construct, as hypersurfaces in toric varieties, Calabi–Yau manifolds corresponding to F-theory vacua dual to $E_8 \times E_8$ heterotic strings compactified to six dimensions on $K3$ surfaces with non-semisimple gauge backgrounds. These vacua were studied in the recent work of Aldazabal, Font, Ibáñez and Uranga. We extend their results by constructing many more examples, corresponding to enhanced gauge symmetries, by noting that they can be obtained from previously known Calabi–Yau manifolds corresponding to $K3$ compactification of heterotic strings with simple gauge backgrounds by means of extremal transitions of the conifold type.

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1. Introduction

F-theory [1] has proved to be a powerful tool for analysing string dualities. In particular it provides the basis for a geometric understanding of string dualities. For example, it was argued in [2] that F-theory compactified on an elliptic Calabi–Yau manifold is dual to the heterotic $E_8 \times E_8$ theory on a $K3$ surface. Furthermore it was shown that the base of the elliptic Calabi–Yau manifold is a Hirzebruch surface $F_n$, or a blowup thereof. Upon further toroidal compactification to four dimensions, we obtain Type II/heterotic duality which has been well studied recently [3]. Many examples of Calabi–Yau manifolds corresponding to six dimensional F-theory/heterotic dual pairs were obtained as hypersurfaces in toric varieties in [4], where the gauge group in the effective theory appeared as a result of embedding simple gauge backgrounds in each $E_8$ factor on the heterotic side. These will be referred to as A models. These examples were shown to be organized in the form of chains obtained on the heterotic side by sequential Higgsing of the commutant of the gauge background in $E_8 \times E_8$. Furthermore, the polyhedra encoding the toric data of these manifolds were shown to exhibit a regular structure related in a simple way to the unbroken gauge symmetry of their heterotic duals.

In a recent work, Aldazabal et al. [5] discuss an interesting class of models, which they call the B, C and D series, in which the gauge backgrounds are non-semisimple (i.e., the backgrounds included one or more $U(1)$ factors). However, only a few members of each chain in this class were obtained. In this paper, we realise a large number of members of these chains as hypersurface in toric varieties by relating them to previously known models by extremal transitions of the conifold type. The fact that the models are related by extremal transitions was also noted in [5] for their original examples, though it was not used to construct them. The point emphasized here, as in [4], is that toric geometry is the natural arena in which to discuss these webs of vacua.

This paper is organized as follows. In §2, we give a brief review of simple and non-semisimple gauge backgrounds on $K3$, followed in §3 by a review of the construction of the A, B and C chains on the heterotic side. In §4, we demonstrate our method for constructing Calabi–Yau manifolds corresponding to the B and C chains, and compare the Hodge numbers of various members obtained using the methods of Batyrev [6,7] with the heterotic results. We also construct Calabi–Yau manifolds corresponding to models with extra tensor multiplets. §5 concludes with a brief discussion of our results, and an Appendix contains figures of the polyhedra that we discuss.
2. Simple and Non-Semisimple Backgrounds on $K3$

In this section we give a brief review, following [5], of the construction of heterotic models in six and four dimensions by compactifying $E_8 \times E_8$ heterotic string on $K3$ and $K3 \times T^2$, respectively. First, consider the case of simple gauge backgrounds. Let $H_{1,2}$ be the background gauge groups (simple subgroups of $E_8$) and $k_{1,2}$ the corresponding instanton numbers (second Chern classes of the background gauge bundles on the $K3$). The contribution of each $E_8$ to the unbroken gauge group in six dimensions is then the commutant $G_{1,2}$ of $H_{1,2}$ respectively. The number of hypermultiplets in the representation $R_a$ of $G$ is then

$$N(R_a) = kT(M_a) - \dim(M_a),$$

where the adjoint of $E_8$ decomposes under $G \times H$ as $248 = \sum_a (R_a, M_a)$, and $T(M_a)$ is given by $\text{tr}(T^i_a T^j_a) = T(M_a)\delta_{ij}$, $T^i_a$ being a generator of $H$ in the representation $M_a$.

Anomaly cancellation requires $k_1 + k_2 = 24$, so it is convenient to define

$$n = k_1 - 12 = 12 - k_2$$

and take $n \geq 0$ (i.e., $k_1 \geq k_2$). If $n \leq 8$ we can take $H_1 = H_2 = SU(2)$ and obtain $E_7 \times E_7$ gauge symmetry in six dimensions with the following matter content:

$$\frac{1}{2}(8 + n)(56, 1) + \frac{1}{2}(8 - n)(1, 56) + 62(1, 1)$$

If $9 \leq n \leq 12$, $k_2$ cannot support an $SU(2)$ background, and the instantons in the second $E_8$ are necessarily small producing an unbroken $E_8$. The gauge group in six dimensions is thus $E_7 \times E_8$ with matter content

$$\frac{1}{2}(8 + n)(56, 1) + (53 + n)(1, 1)$$

Models with subgroups of the above can be obtained by gauge symmetry breaking via Higgs mechanism, or, equivalently, by taking the subgroups of $E_8$ other than $SU(2)$ as $H_{1,2}$.

There exists another possibility for constructing heterotic models in six dimensions which was first considered in ref. [8]. It consists of considering $U(1)$ (i.e., nonsemisimple) backgrounds in each $E_8$ and proceeds as follows. The instanton number of the $U(1)$ configuration is taken to be

$$m_i = \frac{1}{16\pi^2} \int_{K3} \frac{1}{30} \text{Tr} F_{U(1),i}^2, \quad i = 1, 2$$
and anomaly cancellation again requires $m_1 + m_2 = 24$.

Now the adjoint of $E_8$ decomposes under $E_7 \times U(1)$ as

$$248 = (133, 0) + (56, q) + (56, -q) + (1, 2q) + (1, -2q) + (1, 0)$$

The generator $Q$ of $U(1)$ is normalized as a generator of $E_8$ in the adjoint, so $q = \frac{1}{2}$. The index theorem applied to this case gives

$$N_q = mq^2 - 1,$$

for the number, $N_q$, of hypermultiplets of charge $q$.

Thus the matter content of the resulting $E_7 \times U(1) \times E_7 \times U(1)$ turns out to be

$$\left\{ \frac{1}{4}(m_1 - 4)(56, \frac{1}{2}; 1, 0) + \frac{1}{4}(m_2 - 4)(1, 0; 56, \frac{1}{2}) + (m_1 - 1)(1, 1; 1, 0) + (m_2 - 1)(1, 0; 1, 1) + \text{c.c.} \right\} + 20(1, 0; 1, 0)$$

It was further noticed in [5] that these $U(1)$'s are such that the anomaly 8-form does not in general factorize into a product of two 4-forms and hence the residual anomaly cannot be cancelled by the Green-Schwarz mechanism. The anomaly can, however, be cancelled completely for a certain linear combination of the above $U(1)$'s. The orthogonal combination is still anomalous, and must be spontaneously broken if we want to obtain a consistent low energy theory. The mechanism for such breaking was described in refs. [8] and [9]. One linear combination of the two photons becomes massive by swallowing a $B$-field zero mode. Moreover, in our case, the combination which gets mass is the anomalous one. So the actual gauge group is $E_7 \times E_7 \times U(1)$, and the anomaly is absent.

We can now combine both abelian and non-abelian backgrounds and obtain $H \times U(1)$ bundles with instanton numbers $(k, m)$ in each $E_8$. The gauge group is then $G \times U(1)$, with $G$ the commutant of $H \times U(1)$ in $E_8$, and the adjoint of $E_8$ decomposes as $248 = \sum_a (R_a, q_a, M_a)$ under $G \times U(1) \times H$. The number of hypermultiplets in the $(R_a, q_a)$ representation of $G \times U(1)$ is again given by the index theorem

$$N(R_a, q_a) = kT(M_a) + (mq_a^2 - 1) \dim M_a$$

where we again normalize $\text{Tr}Q^2 = 30$. This may now be generalized in a straightforward way to the case of $H \times U(1)^{8-d}$ bundles with $d \leq 6$. In the following section, we consider two choices leading to the B and C type models.
3. Construction of the B and C Chains on the Heterotic Side

This section summarises the construction of the B and C chains on the heterotic side as described in [5].

The B type chains can be constructed by embedding \((3, 3)\) \(U(1)\) instantons and \((9 + n, 9 - n)\) \(SU(2)\) instantons in the two \(E_8\)'s. The commutant in each \(E_8\) of this background is \(E_6 \times U(1)\). The unbroken \(U(1)\) gauge group is then the diagonal combination, \(U(1)_D\), of the two \(U(1)\)'s.

The \(E_6 \times E_6 \times U(1)_D\) spectrum is given in [5] as:

\[
\left\{ \frac{1}{2}(k_1 - 3)(27, 1, \frac{1}{2\sqrt{6}}) + \frac{1}{2}(k_2 - 3)(1, 27, \frac{1}{2\sqrt{6}}) + \frac{1}{2}(k_1 + k_2 + 10)(1, 1, \frac{3}{2\sqrt{6}}) + \text{c.c.} \right\} + (2k_1 + 2k_2 + 13)(1, 1, 0)
\]

where \(k_1 = 9 + n\) and \(k_2 = 9 - n\). The Hodge numbers of the Calabi–Yau manifold for the F-theory dual of this model would be* \((h_{21}, h_{11}) = (48, 16)\). Given this spectrum, we can systematically Higgs the \(E_6\) gauge groups as mentioned in [5] and [10]. Upon Higgsing the first \(E_6\), the Hodge numbers change as shown in the first part of Table 3.1.

The C type chains are constructed by embedding \(SU(2) \times U(1)_1 \times U(1)_2\) instantons in each \(E_8\), with instanton numbers \((7 + n, 3, 2; 7 - n, 3, 2)\). The commutant in each \(E_8\) of this background is \(SO(10) \times U(1)^2\). The unbroken \(U(1) \times U(1)\) gauge group consists of the diagonal combinations of the \(U(1)^2\) groups from each \(E_8\).

*These are correct only for \(n \leq 6\). For \(n > 6\), the group is no longer \(E_6 \times E_6 \times U(1)\), but \(E_6 \times E_7 \times U(1)\). For \(n = 7\), the Hodge numbers are \((48, 18)\), the extra vector multiplet in the 4D Coulomb branch coming from a small instanton. For \(n = 8\), there is no small instanton, and the Hodge numbers are \((49, 17)\).
Table 3.1: The change in the Hodge numbers of the B and C models obtained upon sequential Higgsing of the first $E_6 (SO(10)$ for the C models) to subgroups. The entry in each row gives the change in the Hodge numbers from the Hodge numbers corresponding to the group in the previous row. Here $k_1$ is the number of $SU(2)$ instantons in the first $E_8$. Higgsing is possible only when $\Delta h_{21}$ is nonnegative. For the second $E_6$ (or $SO(10)$), replace replace $k_1$ by $k_2$.

The $SO(10) \times SO(10) \times U(1) \times U(1)$ spectrum is given in [5] as:

\[
\begin{align*}
\frac{1}{2}(k_1 - 3)(16, 1, 0, -\frac{1}{4}) + \frac{1}{2}(k_2 - 3)(1, 16, 0, -\frac{1}{4}) \\
+ \frac{1}{2}(k_1 - 1)(10, 1, \frac{1}{2}\sqrt{2}, 0) + \frac{1}{2}(k_2 - 1)(1, 10, \frac{1}{2}\sqrt{2}, 0) \\
+ \frac{1}{2}(k_1 + k_2 + 3)[(1, 1, \frac{1}{2}\sqrt{2}, -\frac{1}{2}) + (1, 1, \frac{1}{2}\sqrt{2}, \frac{1}{2})] + 4(1, 1, \frac{1}{2}\sqrt{2}, 0) + \text{c.c.}
\end{align*}
\]

with $k_1 = 7 + n$ and $k_2 = 7 - n$. The Hodge numbers of the Calabi–Yau manifold for the F-theory dual of this model would be* $(h_{21}, h_{11}) = (39, 15)$ [5]. Upon Higgsing the first $SO(10)$, the Hodge numbers change as shown in the second part of Table 3.1.

*These are correct only for $n \leq 4$. For $n > 4$, the group is no longer $SO(10) \times SO(10) \times U(1)^2$, but $SO(10) \times E_6 \times U(1)^2$. For $n = 5$, the Hodge numbers are $(39, 17)$, the extra vector multiplet in the 4D Coulomb branch coming from a small instanton. For $n = 6$, there is no small instanton, and the Hodge numbers are $(40, 16)$.  

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4. Calabi–Yau Manifolds corresponding to the B and C Chains

4.1. Constructing enhanced gauge groups

We give a brief review of the procedures used to construct Calabi–Yau manifolds corresponding to enhanced gauge symmetries in the A chains. The starting point is the construction of the lowest members of these chains as elliptic fibrations over the Hirzebruch surface $\mathbb{F}_n$. These may be described as follows. We have homogeneous coordinates $s, t, u, v, x, y, w$ acted on by three $\mathbb{C}^*$'s $\lambda, \mu, \nu$ with weights as in Table 4.1, and the manifold is realised as the vanishing locus of a polynomial of the indicated degrees. Higher members of the chains, i.e., those corresponding to enhanced gauge symmetry, can be obtained by introducing A-D-E singularities in the generic fibres using Tate’s algorithm [11]. The exact correspondence between the singularity type and the coefficients of the Weierstrass equation has been worked out by Bershadsky et al. [12].

|   | $s$ | $t$ | $u$ | $v$ | $x$ | $y$ | $w$ | degrees |
|---|-----|-----|-----|-----|-----|-----|-----|---------|
| $\lambda$ | 1   | 1   | $n$ | 0   | 2$n+4$ | 3$n+6$ | 0   | $6n+12$ |
| $\mu$    | 0   | 0   | 1   | 1   | 4   | 6   | 0   | 12      |
| $\nu$    | 0   | 0   | 0   | 0   | 2   | 3   | 1   | 6       |

**Table 4.1:** The scaling weights of the elliptic fibration over $\mathbb{F}_n$.

Consider the Weierstrass equation,

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

(4.1)

where the $a_i$’s are locally defined polynomial functions on the base. Denote the affine coordinates of the $\mathbb{P}_1$ base of the Calabi–Yau manifold and the $\mathbb{P}_1$ base of the elliptic $K3$ fibres by $z'$ and $z$, respectively. The singularities are then encoded in the degrees of vanishing of the $a_i$’s. This has been worked out in detail in [12]. We reproduce some of their results in Table 4.2. For example, if we want an enhanced perturbative gauge group on the heterotic side, then we locate the singularities at $z = 0$ and $z = \infty$, corresponding to the first and second $E_8$’s, respectively. It is easiest to explain this using a specific example. Consider the $n = 0$ model, which is dual to the heterotic theory with 12 instantons in
The Newton polyhedron\cite{6,7} describing our Calabi–Yau manifold has 335 points, and the Hodge numbers are $(h_{11}, h_{21}) = (3, 243)$. If we now want to unhiggs an $SU(2)$ subgroup of the first $E_8$, we proceed as follows:

The $SU(2)$ singularity corresponds to coefficients $a_i$ with $\deg(a_1, a_2, a_3, a_4, a_6) = (0, 0, 1, 1, 2)$, where the numbers indicate the degrees of vanishing of the $a_i$’s near $z = 0$. We can introduce this singularity by noting that when the Weierstrass equation (4.1) is written in terms of the homogeneous coordinates, the terms labelled by $a_i$ contain a factor of $w^i$. Also, noting that $z = u/v$, we see that if the degree of vanishing of $a_i$ near $z = 0$ is $n$, then the lowest power of $u$ in $a_i$ is also $n$. Thus the $SU(2)$ singularity is obtained by discarding from the Newton polyhedron, the points with powers $(n, i)$ of $u$ and $w$, respectively, with $n < (0, 0, 1, 1, 2)$ for $i = (1, 2, 3, 4, 6)$. The Hodge numbers are now $(4, 214)$. The generalization to other gauge groups, as well as unhiggsing to subgroups of the second $E_8$, is straightforward. It is worth mentioning here that the new polyhedron does not describe a singular manifold. Rather, it describes the smooth Calabi–Yau manifold resulting from the resolution of the above singularity.

| Group     | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_6$ |
|-----------|-------|-------|-------|-------|-------|
| $SU(2)$   | 0     | 0     | 1     | 1     | 2     |
| $Sp(k)$   | 0     | 0     | $k$   | $k$   | $2k$  |
| $SU(2k)$  | 0     | 1     | $k$   | $k$   | $2k$  |
| $SU(2k+1)$| 0     | 1     | $k$   | $k+1$ | $2k+1$|
| $G_2$     | 1     | 1     | 2     | 2     | 3     |
| $SO(4k+1)$| 1     | 1     | $k$   | $k+1$ | $2k$  |
| $SO(4k+2)$| 1     | 1     | $k$   | $k+1$ | $2k+1$|
| $SO(4k+3)$| 1     | 1     | $k+1$ | $k+1$ | $2k+1$|
| $SO(4k+4)^*$| 1     | 1     | $k+1$ | $k+1$ | $2k+1$|
| $F_4$     | 1     | 2     | 2     | 3     | 4     |
| $E_6$     | 1     | 2     | 2     | 3     | 5     |
| $E_7$     | 1     | 2     | 3     | 3     | 5     |
| $E_8$     | 1     | 2     | 3     | 4     | 5     |

Table 4.2: The relation between the degrees of vanishing of terms in the Weierstrass Polynomial and the enhanced Gauge Groups.
4.2. Calabi–Yau manifolds for the B and C series

The Calabi–Yau manifolds of the B and C chains were constructed in a similar way in ref [5]. The Calabi–Yau manifolds of the B chains were described as elliptic fibrations over \( \mathbb{F}_n \), with the fibre the torus in \( \mathbb{P}^{(1,1,2)}_2 [4] \). Similarly, the Calabi–Yau manifolds of the C chains were described as elliptic fibrations with the fibre the torus in \( \mathbb{P}_2 [3] \). However, this fibration structure was not actually used to construct the Calabi–Yau manifolds in ref [5], but was only observed to hold in the examples studied. The examples themselves were obtained by matching the expected hodge numbers with those in a list of Calabi–Yau manifolds.

We first attempted to construct higher members of these chains by requiring the Calabi–Yau manifolds to be elliptic fibrations as described above, and then attempting to find reflexive polyhedra satisfying these conditions. However, we were not able to find many examples. In particular, for the B series with \( n = 2 \), although we found about 25 reflexive polyhedra, we were only able to identify gauge groups \( U(1) \) and \( G \times U(1) \), for \( G = SU(4), SO(8), E_6, E_7 \). We were not able to identify the other reflexive polyhedra corresponding to elliptic Calabi–Yau manifolds (with fibres a \( \mathbb{P}^{(1,1,2)}_2 [4] \)) with any heterotic model. It turns out, however, that we can identify more gauge groups if we consider not the torus in \( \mathbb{P}^{(1,1,2)}_2 \) but rather in a blowup of this space. The polyhedron of this torus, together with the polyhedra of the A and C tori, are shown in fig 4.1. Then it is possible to identify gauge groups \( G \times U(1) \), with

\[
G = \{ SU(1), SU(2), SU(3), SU(4), SO(8), SO(10), E_6, E_7 \}.
\]

There were quite a few models which we could not identify with gauge groups. Furthermore, some gauge groups were “missing”, such as \( SU(5) \times U(1) \). We list the the Hodge numbers obtained with this torus in Table 4.7 along with the corresponding gauge groups whenever they could be identified. This table also lists hodge numbers computed using the torus of the B series from Fig 4.2 (refer to section 4.3), for comparison.

Similarly, for the C series with \( n = 2 \), we only found the model with gauge group \( U(1)^2 \). The only other reflexive polyhedron with fibre corresponding to the torus in \( \mathbb{P}_2 [3] \) had Hodge numbers corresponding to the B model with gauge group \( SU(4) \times U(1) \). Thus while the procedure for obtaining models by constructing reflexive polyhedra produces a large number of members of the B series, we were only able to construct the lowest member of the C series.
These polyhedra were constructed by requiring that the polyhedra corresponding to the $K3$ fibres project onto the polyhedron of the torus, a criterion which was also used to construct the polyhedra of ref[4]. This restriction is not actually necessary, and it is probable that the number of reflexive polyhedra obtained may be increased by relaxing this condition, though we have not investigated this in a systematic way. In the next section, we present a different approach for constructing a rich class of models, especially in the C series.

4.3. Conifold Transitions

It was pointed out already in [5] that the A,B,C and D chains are connected to each other by conifold transitions. This was shown explicitly for the lowest members of the A and B chains for the case $n = 4$. It was argued that the chains should all be related by such transitions. This now provides a useful way to actually construct the higher members (those corresponding to enhanced gauge groups) of the chains. Since the A chains are well understood, if we can induce an extremal transition to the B,C and D chains by means of singularities (by restricting the form of the monomials that define the manifolds in the A chains), we will then have constructed candidate duals for the heterotic B, C and D chains. By comparing the Hodge numbers of the Calabi–Yau manifolds thus obtained with the data on the heterotic side, we will be able to verify the conjectured duality.

In terms of the homogeneous coordinates $s, t, u, v, x, y, w$ defined above, the transition from the A model to the B model is effected by adjusting the complex structure of the
manifold so that the equation may be written (with G and F polynomials in $s, t, u, v, x, y, w$) as

$$xG - yF = 0.$$  

Noting that the relative scalings of $x, y, w$ are 2:3:1 respectively, and that the manifold condition implies that the powers of $x, y$ and $w$ add up to 6, we see that the B model is obtained by restricting the Newton polyhedron of the A model to points whose last coordinate (i.e., the power of $w$) is less than or equal to 4.

Similarly, it turns out that it is possible to induce an extremal transition that takes the A model to the C model. This happens when we adjust the complex structure of the manifold so that the equation reads

$$x^2 G - yF = 0.$$  

This is equivalent to restricting the Newton Polyhedron of the A model to points whose last coordinate is less than or equal to 3.

As with the Calabi–Yau manifolds corresponding to enhanced gauge groups, it is important to note that the Newton polyhedron obtained by the above method describes the smooth manifold obtained by resolving the singularities introduced into the original manifold.

The Hodge numbers of the Calabi–Yau manifolds obtained by the above construction precisely match the values for the spaces given in [5]. That is, for $n = 0, \ldots, 8$, the lowest members of the A series map to the lowest members of the B series given in [5], while for $n > 8$, the polyhedra obtained are non-reflexive, hence do not describe a manifold. Similarly, for $n = 0 \ldots 6$, the lowest members of the A chains map to the lowest members of the C chains (except for $n = 4$ — see the second footnote to Table 4.6), while for $n > 6$, the polyhedra obtained are non-reflexive$^1$.

4.4. Reflexive Polyhedra

We now describe the sequences of reflexive polyhedra corresponding to the above models and discuss the qualitative differences between the polyhedra for the A, B and C series

$^1$Actually, it is only the Hodge numbers that match. The polyhedra obtained are very different. The manifolds obtained by the above method are birationally equivalent [13] to those described in [5].
constructed as in the previous section. (Note that the polyhedra described below are the polyhedra dual to the Newton polyhedra of the previous section. We hope that this will not cause confusion.)

The polyhedra of the A series were described in [4]. Apart from two points, which for each member of the $n$’th chain of the A series are $(-1, 0, 2, 3)$ and $(1, n, 2, 3)$, the points of the polyhedron lie in the plane $x_1 = 0$ forming the polyhedron $^3\nabla$ of the K3. For each member of a chain, the polyhedron of the K3 is again divided into a top and a bottom by the polyhedron $^2\nabla$ of the torus and we may write

$$^3\nabla_{n,H} = ^3\nabla^n_{\text{top}} \cup ^3\nabla^H_{\text{bot}},$$

where $^3\nabla^n_{\text{top}}$ depends only on $n$ while $^3\nabla^H_{\text{bot}}$ depends only on the group $H$ that is perturbatively un-Higgsed in the heterotic side. In particular then, the dual polyhedron $^4\nabla_{n,SU(1)}$ of the lowest member of each chain can be written as

$$^4\nabla_{n,SU(1)} = ^3\nabla_{n,SU(1)} \cup \{(-1, 0, 2, 3), (1, n, 2, 3)\}.$$

We can describe the tops and bottoms of $^3\nabla_{n,H}$ quite simply. There is an interesting symmetry between the tops and bottoms. The tops of the polyhedra describe the terminal (unbroken) gauge groups, which are subgroups of the first $E_8$ and are listed below. The bottoms describe the enhanced gauge groups which are subgroups of the second $E_8$.

| $n$ | A Series | B Series | C Series |
|-----|----------|----------|----------|
| 0, 1, 2 | $SU(1)$ | $U(1)$ | $U(1)^2$ |
| 3 | $SU(3)$ | $SU(3)\times U(1)$ | $SU(3)\times U(1)^2$ |
| 4 | $SO(8)$ | $SO(8)\times U(1)$ | $SO(10)\times U(1)^2$ |
| 5 | $F_4$ | $E_6\times U(1)$ | $E_6\times U(1)^2$ |
| 6 | $E_6$ | $E_6\times U(1)$ | $E_6\times U(1)^2$ |
| 7, 8 | $E_7$ | $E_7\times U(1)$ | $-$ |
| 9, 10, 11, 12 | $E_8$ | $-$ | $-$ |

Table 4.3: The terminal groups for the A, B and C series. Note that the groups do not in all cases simply acquire extra $U(1)$ factors.
The bottoms $3\nabla^H_{\text{bot}}$, may be specified by giving the points that lie below the torus $2\nabla$. We label the points of the torus as indicated in figure 4.2. Also, let $pt^{(j)}_r$ represent $j$ points of the lattice directly below the corresponding points of $2\nabla$. Thus, $pt^{(2)}_1$ represents the points $(0, -1, 2, 3)$ and $(0, -2, 2, 3)$. Then the bottoms for each group can be specified by giving the set of $pt^{(j)}_r$'s corresponding to the gauge group, as shown in Table 4.4. One might naively expect that the top for a given (terminal) gauge group consists of the same points as the bottom, except placed above the torus instead of below it. While this is not true in general, it turns out that we can modify the tops so that they are indeed symmetric with the corresponding bottoms, without changing the hodge numbers of the manifold. It turns out that the number of extra points added is accompanied by a decrease in the number of non-toric deformations of the polyhedra, so that the Calabi–Yau manifold is unchanged.

Having described the structure of the reflexive polyhedra in the A series, it is now easy to describe the structure of the polyhedra in the B and C series.

![Figure 4.2: The tori for the A series and its conifolds.](image)

Consider first the B series, constructed as described in section 4.3. We have found that the general structure of the polyhedra of the B series is similar to that of the A series. That is, there are always exactly two 4-dimensional points, which in our basis are $(1,0,2,3)$ and $(-1,0,2,3)$. Also, the three dimensional polyhedron formed by the points with the first coordinate equal to zero form a reflexive polyhedron, which we recognise as encoding the
Furthermore, the points with first two coordinates zero are always those shown in fig. 4.2.

| Group   | Points in the Bottom                                      |
|---------|----------------------------------------------------------|
| $SU(2)$ | $\{pt_1^{(1)}, pt_2^{(1)}\}$                           |
| $SU(3)$ | $\{pt_1^{(1)}, pt_2^{(1)}, pt_3^{(1)}\}$               |
| $SU(4)$ | $\{pt_1^{(1)}, pt_2^{(1)}, pt_3^{(1)}, pt_4^{(1)}\}$   |
| $SU(5)$ | $\{pt_1^{(1)}, pt_2^{(1)}, pt_3^{(1)}, pt_4^{(1)}, pt_5^{(1)}\}$ |
| $SO(10)$| $\{pt_1^{(2)}, pt_2^{(2)}, pt_3^{(1)}, pt_4^{(2)}, pt_5^{(1)}\}$ |
| $E_6$  | $\{pt_1^{(3)}, pt_2^{(2)}, pt_3^{(2)}, pt_4^{(1)}, pt_5^{(1)}\}$ |
| $E_7$  | $\{pt_1^{(4)}, pt_2^{(3)}, pt_3^{(2)}, pt_4^{(2)}, pt_5^{(1)}\}$ |
| $E_8$  | $\{pt_1^{(6)}, pt_2^{(4)}, pt_3^{(3)}, pt_4^{(2)}, pt_5^{(1)}\}$ |
| $SO(9)$| $\{pt_1^{(2)}, pt_2^{(2)}, pt_3^{(1)}, pt_4^{(1)}\}$     |
| $F_4$  | $\{pt_1^{(3)}, pt_2^{(2)}, pt_3^{(1)}, pt_4^{(1)}\}$     |

**Table 4.4:** The relation between the bottoms and the enhanced gauge groups for the A series.

While we find that the point labeled $pt_8$ is always present in the polyhedra of the B series, this is not the only change in going from the A to the B series. The polyhedra of the $K3$ fibres are also different — extra points are added to the tops, but the bottoms remain the same. This is because the terminal groups change upon going from the A series to the B series. Just as in the A series, we can modify the tops so that they become symmetric with the bottoms of the same gauge groups. The terminal groups are listed in Table 4.3.

We have also studied Calabi–Yau manifolds of the B series corresponding to enhanced gauge groups. We find that, in general, the conifold transition described in Sec. 3.2 maps a member of the A series corresponding to enhanced gauge group $G$ to a member of the B series corresponding to enhanced gauge group $G \times U(1)$. There are, however, some exceptions. For instance, the A model with gauge group $F_4$ maps to the B model with gauge group $E_6 \times U(1)$. (The gauge groups on the B series side are identified by their hodge numbers as predicted by the heterotic calculations of Sec. 2). We list the gauge groups in the B series obtained by conifolding the A series in Table 4.5. Notice that these are consistent with the terminal gauge groups listed in Table 4.3.

We list the tables of Hodge numbers of the B models in Table 4.6 at the end of this section. We also present views of the bottoms of the polyhedra corresponding to different
enhanced gauge groups in figures A.1 and A.2. We find that the Dynkin diagrams of the
gauge groups are visible in the polyhedra, as observed in [4] for the A series.

Consider next the C series, constructed as described in section 4.3. We have found
that the general structure of the polyhedra of the C series is also similar to that of the
A series. That is, there are always exactly two 4-dimensional points, which in our basis
are (1,0,2,3) and (-1,0,2,3). Also, the three dimensional polyhedron formed by the points
with the first coordinate equal to zero form a reflexive polyhedron, which we recognise as
encoding the $K3$ of the fibration. Furthermore, the points with first two coor dinates zero
are always those shown in Fig. 4.2.

While we find that the points labeled $pt_8$ and $pt_9$ are always present in the polyhedra
of the C series, this is again not the only change in going from the A to the C series. The
polyhedra of the $K3$ fibres are also different — extra points are added to the tops, but the
bottoms remain the same. Once again, this is because the terminal gauge groups change
upon going from the A series to the C series. Furthermore, it is possible to modify the
tops without changing the hodge numbers, so that they are symmetric with the bottoms
of the same gauge groups. The terminal gauge groups are listed in Table 4.3.

| A series groups | B series groups | C series groups |
|-----------------|-----------------|-----------------|
| $SU(2)$         | $SU(2) \times U(1)$ | $SU(2) \times U(1)^2$ |
| $SU(3)$         | $SU(3) \times U(1)$ | $SU(3) \times U(1)^2$ |
| $SU(2k)$        | $SU(2k) \times U(1)$ | $SU(2k) \times U(1)^2$ |
| $SU(2k + 1)$    | $SU(2k + 1) \times U(1)$ | $SU(2k) \times U(1)^2$ |
| $Sp(k)$         | $Sp(k) \times U(1)$ | $Sp(k) \times U(1)^2$ |
| $G_2$           | $SO(7) \times U(1)$ | $SO(10) \times U(1)^2$ |
| $SO(7)$         | $SO(7) \times U(1)$ | $SO(10) \times U(1)^2$ |
| $SO(8)$         | $SO(8) \times U(1)$ | $SO(10) \times U(1)^2$ |
| $SO(9)$         | $SO(10) \times U(1)$ | $SO(10) \times U(1)^2$ |
| $SO(10)$        | $SO(10) \times U(1)$ | $SO(10) \times U(1)^2$ |
| $F_4$           | $E_6 \times U(1)$ | $E_6 \times U(1)^2$ |
| $E_6$           | $E_6 \times U(1)$ | $E_6 \times U(1)^2$ |
| $E_7$           | $E_7 \times U(1)^*$ | (not reflexive) |

Table 4.5: The correspondence between gauge groups in the A, B and C series. Asterisks indicate the presence of extra tensor multi-plets.
We have also studied Calabi–Yau manifolds of the C series corresponding to enhanced gauge groups. We find that, in general, the extremal transition described in Sec. 3.2 maps a member of the A series corresponding to enhanced gauge group $G$ to a member of the C series corresponding to enhanced gauge group $G \times U(1)^2$. There are again some exceptions. For instance, the A model with gauge group $SU(4)$ maps to the C model with gauge group $SU(5) \times U(1)^2$. We list the gauge groups in the C series obtained by extremal transitions of the A series in Table 4.5. The hodge numbers of the C series are listed in Table 4.6.

| $n$ | $SU(1)$ | $SU(2)$ | $SU(3)$ | $SU(4)$ | $SU(5)$ | $SO(10)$ | $E_6$ | $E_7$ |
|-----|---------|---------|---------|---------|---------|---------|------|------|
| 0   | (148, 4) | (133, 5) | (122, 6) | (115, 7) | (106, 8) | (103, 9) | (98, 10) | (91, 19) |
| 1   | (148, 4) | (127, 5) | (112, 6) | (103, 7) | (92, 8)  | (88, 9)  | (82, 10) | (74, 20) |
| 2   | (148, 4) | (121, 5) | (102, 6) | (91, 7)  | (78, 8)  | (73, 9)  | (66, 10) | (57, 21) |
| 3   | (152, 6) | (119, 7) | (96, 8)  | (83, 9)  | (68, 10) | (62, 11) | (54, 12) | (44, 24) |
| 4   | (164, 8) | (125, 9) | (98, 10) | (83, 11) | (66, 12) | (59, 13) | (50, 14) | (39, 27) |
| 5   | (178, 10)| (133, 11)| (102, 12)| (85, 13) | (66, 14) | (58, 15) | (48, 16) | (36, 30) |
| 6   | (194, 10)| (143, 11)| (108, 12)| (89, 13) | (68, 14) | (59, 15) | (48, 16) | (35, 31) |
| 7   | (210, 12)| (153, 13)| (114, 14)| (93, 15) | (70, 16) | (60, 17) | (48, 18) | (34, 34) |
| 8   | (227, 11)| (164, 12)| (121, 13)| (98, 14) | (73, 15) | (62, 16) | (49, 17) | (34, 34) |

| $n$ | $SU(1)$ | $SU(2)$ | $SU(3)$ | $SU(4)$ | $SU(5)$ | $SO(10)$ | $E_6$ | $E_7$ |
|-----|---------|---------|---------|---------|---------|---------|------|------|
| 0   | (101, 5) | (90, 6)  | (85, 7)  | (78, 8)* | (73, 9)  | (70, 10)| (65, 17)|
| 1   | (101, 5) | (86, 6)  | (79, 7)  | (70, 8)* | (64, 9)  | (60, 10)| (54, 18)|
| 2   | (101, 5) | (82, 6)  | (73, 7)  | (62, 8)* | (55, 9)  | (50, 10)| (43, 19)|
| 3   | (103, 7) | (80, 8)  | (69, 9)  | (56, 10)*| (48, 11) | (42, 12)| (34, 22)|
| 4† | (110, 10)† | (83, 11)† | (70, 12)† | (55, 13)*†| (46, 14)†| (39, 15)†| (30, 26)†|
| 5   | (120, 12)| (89, 13) | (74, 14) | (57, 15)*| (47, 16) | (39, 17)| (29, 29)|
| 6   | (131, 11)| (96, 12) | (79, 13) | (60, 14)*| (49, 15) | (40, 16)| (29, 29)|

Table 4.6: The Hodge numbers $(h_{21}, h_{11})$ for the B and C series.

Note that we do not observe the spaces marked with a * by the methods described in Sec. 3.2. The Hodge numbers listed here have not been observed by us. Rather, they were calculated using the heterotic data. If these spaces were indeed to exist, they would have these Hodge numbers. Furthermore, the Hodge numbers for the $n = 4$ case (marked with a †) are those obtained by extremal transitions. However, this method yields terminal gauge group $SO(10) \times U(1)^2$ (see Table 4.5), but the terminal gauge group obtained in [5] is
| $(h_{21}, h_{11})$ | $(h_{21}, h_{11})$ | Gauge Groups          | $(h_{21}, h_{11})$ | $(h_{21}, h_{11})$ | Gauge Groups          |
|-------------------|-------------------|-----------------------|-------------------|-------------------|-----------------------|
| (100, 4)          | (100, 4)          | $SU(1) \times U(1)$   | (66, 8)           | (66, 8)           |                        |
| (118, 4)          | (118, 4)          | $SU(2) \times U(1)$   | (70, 8)           | (70, 8)           | $SU(5) \times U(1)$   |
| (148, 4)          | (148, 4)          | $SU(2) \times U(1)$   | (72, 8)           | (72, 8)           | $SO(8) \times U(1)$   |
| (75, 5)           | (75, 5)           | $SU(2) \times U(1)$   | (78, 8)           | (78, 8)           |                        |
| (89, 5)           | (89, 5)           | $SU(2) \times U(1)$   | (80, 8)           | (80, 8)           |                        |
| (93, 5)           | (101, 5)          | $SU(2) \times U(1)$   | (57, 9)           | (57, 9)           |                        |
| (111, 5)          | (121, 5)          | $SU(2) \times U(1)$   | (61, 9)           | (61, 9)           |                        |
| (121, 5)          | (121, 5)          | $SU(2) \times U(1)$   | (63, 9)           | (63, 9)           |                        |
| (72, 6)           | (70, 6)           | $SU(2) \times U(1)$   | (65, 9)           | (65, 9)           |                        |
| (78, 6)           | (82, 6)           | $SU(2) \times U(1)$   | (69, 9)           | (69, 9)           |                        |
| (82, 6)           | (82, 6)           | $SU(2) \times U(1)$   | (73, 9)           | (73, 9)           | $SO(10) \times U(1)$  |
| (88, 6)           | (88, 6)           | $SU(2) \times U(1)$   | (56, 10)          | (56, 10)          |                        |
| (90, 6)           | (98, 6)           | $SU(3) \times U(1)$   | (58, 10)          | (58, 10)          |                        |
| (102, 6)          | (102, 6)          | $SU(2) \times U(1)$   | (64, 10)          | (64, 10)          |                        |
| (90, 6)           | (98, 6)           | $SU(3) \times U(1)$   | (66, 10)          | (66, 10)          | $E_6 \times U(1)$     |
| (102, 6)          | (102, 6)          | $SU(2) \times U(1)$   | (55, 11)          | (55, 11)          |                        |
| (65, 7)           | (67, 7)           | $SU(3) \times U(1)$   | (62, 12)          | (62, 12)          |                        |
| (67, 7)           | (67, 7)           | $SU(3) \times U(1)$   | (49, 13)          | (49, 13)          |                        |
| (69, 7)           | (69, 7)           | $SU(3) \times U(1)$   | (54, 14)          | (54, 14)          |                        |
| (75, 7)           | (79, 7)           | $Sp(3) \times U(1)$   | (59, 15)          | (59, 15)          |                        |
| (75, 7)           | (81, 7)           | $Sp(3) \times U(1)$   | (61, 15)          | (61, 15)          |                        |
| (85, 7)           | (85, 7)           | $SU(4) \times U(1)$   | (53, 17)          | (53, 17)          | $E_6 \times U(1) + 8T$ |
| (91, 7)           | (91, 7)           | $SU(4) \times U(1)$   | (58, 18)          | (58, 18)          | $E_7 \times U(1) + 10T$|
| (64, 8)           | (64, 8)           | $Sp(4) \times U(1)$   | (57, 21)          | (57, 21)          |                        |
| (64, 8)           | (64, 8)           | $Sp(4) \times U(1)$   | (57, 21)          | (57, 21)          |                        |

Table 4.7: Hodge numbers for the B series with $n = 2$ obtained by two different means. The first column gives the values obtained using the torus of the B series from fig 4.1 and the second gives the values using the torus of the B series from fig 4.2. The third column lists the gauge groups, where known. Blank entries in either of the first two columns indicate missing hodge numbers, while in the third column they indicate unidentified gauge groups. In the last two entries, the presence of extra tensor multiplets is noted.
$SO(8) \times U(1)^2$. There is thus a discrepancy of one for each of the $(h_{21}, h_{11})$ in this chain. For instance, the lowest member of the chain should have Hodge numbers $(111, 9)$ rather than $(110, 10)$ as listed here.

Views of the polyhedra corresponding to different enhanced gauge groups are presented in figures A.3 and A.4. Once again, we find that the Dynkin diagrams of the gauge groups are visible in the polyhedra, as observed in [4] for the A series.

4.5. Nonperturbative Vacua — Tensor Multiplets

The previous sections described F-theory vacua dual to perturbative heterotic vacua. It is also straightforward to describe vacua corresponding to extra tensor multiplets, which are nonperturbative vacua on the heterotic side. Briefly, the extra tensor multiplets in the A series are obtained by blowing up the base $\mathcal{F}_n$ of the Calabi–Yau manifold [2]. This is achieved by adding extra lines of weights to the weight systems that describe the Newton polyhedron [14]. This results in extra points being added to the fan of the $\mathcal{F}_n$ in the dual polyhedron in such a way that the change in the $h_{11}$ of the $\mathcal{F}_n$ (which can be computed easily — see ref.[15] ) is precisely the required number of extra tensor multiplets.

Combining this method with the conifold transitions of Sec. 3.2, we can now construct B and C series vacua with extra tensor multiplets. We simply introduce a given number of extra tensor multiplets on the A series and then apply the conifold transitions to obtain the B and C series vacua with the same number of extra tensor multiplets. We have found that the Hodge numbers of these Calabi–Yau manifolds follow a simple pattern. For the B series, the creation of an extra tensor multiplet increases $h_{11}$ by 1, while reducing $h_{21}$ by 17, which is one less than the dual Coxeter number of $E_7$. This agrees with the results of [5]. Of course, creation of an extra tensor multiplet removes an $SU(2)$ instanton, so that the terminal gauge group may change, thus, the change in the Hodge numbers may not be simply given by $(\Delta h_{21}, \Delta h_{11}) = (-17, 1)$. The correction, when it exists, may be obtained by working out the new terminal groups using the data listed in Table 3.1.

Additional subtleties arise when there would be less than three $SU(2)$ instantons in any $E_8$. These have been discussed in detail in [5]. The result is that these situations cannot be realised. Where we would expect two $SU(2)$ instantons to be present, it turns out that one of these is replaced by a $U(1)$ instanton, and the other by a tensor multiplet, leaving an unbroken $E_7$ gauge group. Also, where we would expect one $SU(2)$ instanton, it is replaced by a $U(1)$ instanton, again leaving an unbroken $E_7$ gauge group. The Hodge numbers can be worked out easily from the heterotic side by the methods discussed here.
and in [5] and we have indeed found that the Calabi–Yau manifolds constructed as above have precisely these Hodge numbers. For example starting with the \( n = 2 \) model, which has \( SU(2) \) instanton numbers \((11, 7)\) in the two \( E_8 \)’s respectively, removal of 9 or 10 instantons from the first \( E_8 \) yields Hodge numbers \((57, 21)\) — the Calabi–Yau manifolds are actually the same because creation of 9 tensor multiplets results in the creation of a tenth one, by the above considerations. Furthermore, it is not possible to remove all the \( SU(2) \) instantons from any \( E_8 \) since this yields a member of the A series [5]. In fact, what happens is that introduction of the conifold singularity destroys the condition of vanishing first Chern class.

In Table 4.8 we list the Hodge numbers for the Calabi–Yau manifolds corresponding to extra tensor multiplets in the \( n = 2 \) model. These tensor multiplets are obtained by removing \( SU(2) \) instantons from the first \( E_8 \).

| Tensors | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9, 10 |
|---------|----|----|----|----|----|----|----|----|----|-------|
| B series | (148, 4) | (131, 5) | (114, 6) | (97, 7) | (80, 8) | (67, 11) | (62, 14) | (59, 17) | (58, 18) | (57, 21) |
| C series | (101, 5) | (90, 6) | (79, 7) | (68, 8) | (57, 9) | (48, 12) | (44, 16) | (43, 19) | (43, 19) | —     |

**Table 4.8:** The Hodge numbers for the \( n = 2 \) B and C models with extra tensor multiplets.

For the C series, creation of an extra tensor multiplet increases \( h_{11} \) by 1, while reducing \( h_{21} \) by 11, which is one less than the dual Coxeter number of \( E_6 \). This agrees with the results of [5]. Of course, creation of an extra tensor multiplet removes an \( SU(2) \) instanton, so that the terminal gauge group may change, thus, the change in the Hodge numbers may not simply be given by \((\Delta h_{21}, \Delta h_{11}) = (-11, 1)\). Again, the correction, if any, may be obtained by working out the new terminal groups using the data listed in Table 3.1.

Subtleties again arise when there would be less than three \( SU(2) \) instantons in any \( E_8 \). These situations cannot be realised. Where we would expect two \( SU(2) \) instantons to be present, it turns out that one of these is replaced by a \( U(1) \) instanton, and the other by a tensor multiplet, leaving an unbroken \( E_6 \) gauge group. Also, where we would expect one \( SU(2) \) instanton, it is replaced by a \( U(1) \) instanton, again leaving an unbroken \( E_6 \) gauge group. The Hodge numbers can be worked out easily from the heterotic side by the methods discussed here and in [5] and we have indeed found that the Calabi–Yau manifolds constructed as above have precisely the expected Hodge numbers. For example starting with the \( n = 2 \) model, which has \( SU(2) \) instanton numbers \((9, 5)\) in the two \( E_8 \)’s
respectively, removal of 7 or 8 instantons from the first $E_8$ yields Hodge numbers $(43, 19)$ — the Calabi–Yau manifolds are actually the same because creation of 7 tensor multiplets results in the creation of another one. Once again, it is not possible to remove all the $SU(2)$ instantons from any $E_8$ since we would then fall back on the B series [5] — it is not possible to take the conifold of this model without destroying the condition of vanishing first Chern class.

In Table 4.8 we list the Hodge numbers for the Calabi–Yau manifolds corresponding to extra tensor multiplets in the $n = 2$ C model. These tensor multiplets are obtained by removing $SU(2)$ instantons from the first $E_8$.

5. Discussion

In this article we have studied F-theory duals of heterotic $E_8 \times E_8$ compactifications on $K3$ with non-semisimple backgrounds of the type $H \times U(1)$ or $H \times U(1)^2$ in each $E_8$, which were originally studied in [5], and were called B and C type models respectively. Our paper extends the previous results by using toric geometry to construct many more examples of B and C models corresponding to enhanced gauge symmetry, as well as models with extra tensor multiplets. This description also has the interesting consequence that the Dynkin diagrams of the gauge groups are visible in the polyhedra, as observed in [4] for the A series. We find that the Calabi–Yau manifolds corresponding to such vacua are most easily constructed by applying conifold-type extremal transitions to the A models of [4], and that construction of the B and C models can be systematised easily.

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A. Appendix: Figures

We append below figures of some of the bottoms of the polyhedra of the B and C series. In the electronic version of this article the figures are in color. The color coding is as in ref [4].
Figure A.1: Two views of some of the polyhedra for the B series with $n = 2$. 
Figure A.2: Two views of some of the polyhedra for the B series with $n = 2$. 
Figure A.3: Two views of some of the polyhedra for the C series with $n = 2$. 
Figure A.4: Two views of some of the polyhedra for the C series with $n = 2$. 
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