Can chaotic quantum energy levels statistics be characterized using information geometry and inference methods?

C. Cafaro and S. A. Ali

Department of Physics, State University of New York at Albany-SUNY, 1400 Washington Avenue, Albany, NY 12222, USA

In this paper, we review our novel information geometrodynamical approach to chaos (IGAC) on curved statistical manifolds and we emphasize the usefulness of our information-geometrodynamical entropy (IGE) as an indicator of chaoticity in a simple application. Furthermore, knowing that integrable and chaotic quantum antiferromagnetic Ising chains are characterized by asymptotic logarithmic and linear growths of their operator space entanglement entropies, respectively, we apply our IGAC to present an alternative characterization of such systems. Remarkably, we show that in the former case the IGE exhibits asymptotic logarithmic growth while in the latter case the IGE exhibits asymptotic linear growth.

At this stage of its development, IGAC remains an ambitious unifying information-geometric theoretical construct for the study of chaotic dynamics with several unsolved problems. However, based on our recent findings, we believe it could provide an interesting, innovative and potentially powerful way to study and understand the very important and challenging problems of classical and quantum chaos.

PACS numbers: 02.50.Tt, 02.50.Cw, 02.40.-k, 05.45.-a, 05.45.Mt, 03.65.Ta

Keywords: Inductive inference, information geometry, statistical manifolds, entropy, chaos and entanglement.

I. INTRODUCTION

In classical and quantum dynamics there is no unified characterization of chaos. In the Riemannian and Finslerian (a Finsler metric is obtained from a Riemannian metric by relaxing the requirement that the metric be quadratic on each tangent space) geometrodynamical approach to chaos in classical Hamiltonian systems, an active field of research concerns the possibility of finding a rigorous relation among the sectional curvature, the Lyapunov exponents, and the Kolmogorov-Sinai dynamical entropy (i.e. the sum of positive Lyapunov exponents). The largest Lyapunov exponent characterizes the degree of chaoticity of a dynamical system and, if positive, it measures the mean instability rate of nearby trajectories averaged along a sufficiently long reference trajectory. Moreover, it is known that classical chaotic systems are distinguished by their exponential sensitivity to initial conditions and that the absence of this property in quantum systems has lead to a number of different criteria being proposed for quantum chaos. Exponential decay of fidelity, hypersensitivity to perturbation and the Zurek-Paz quantum chaos criterion of linear von Neumann’s entropy growth are some examples. These criteria accurately predict chaos in the classical limit, but it is not clear that they behave the same far from the classical realm.

The present work makes use of the so-called Entropic Dynamics (ED). ED is a theoretical framework that arises from the combination of inductive inference (Maximum relative Entropy Methods) and Information Geometry (Riemannian geometry applied to probability theory). As such, ED is constructed on statistical manifolds. It is developed to investigate the possibility that laws of physics - either classical or quantum - might reflect laws of inference rather than laws of nature.

This article is a follow up of a series of the authors works. Especially the work presented in will be discussed in more detail. The ED theoretical framework is used to explore the possibility of constructing a unified characterization of classical and quantum chaos. The general formalism of the IGAC is presented by investigating a system with $3l$ degrees of freedom (microstates), each one described by two pieces of relevant information, its mean expected value and its variance (Gaussian statistical macrostates). This leads to consider an ED model on a non-maximally symmetric $6l$-dimensional statistical manifold $M_s$. It is shown that $M_s$ possesses a constant negative Ricci curvature that is proportional to the number of degrees of freedom of the system, $R_{M_s} = -3l$. It is shown that the system explores statistical volume elements on $M_s$ at an exponential rate. We define an information geometrodynamical entropy (IGE) $S_{M_s}$ of the system and we show it increases linearly in time.

*Electronic address: carlocafaro2000@yahoo.it
†Electronic address: alis@alum.rpi.edu
(statistical evolution parameter) and is moreover proportional to the number of degrees of freedom of the system. The geodesics on \( M_s \) are hyperbolic trajectories. Using the Jacobi-Levi-Civita (JLC) equation for geodesic spread, it is shown that the Jacobi vector field intensity \( J_{M_s} \) diverges exponentially and is proportional to the number of degrees of freedom of the system. Thus, \( R_{M_s}, S_{M_s} \) and \( J_{M_s} \) are proportional to the number of Gaussian-distributed microstates of the system. This proportionality leads to conclude there is a substantial link among these information-geometric indicators of chaoticity. We emphasize that our IGE provides an information-geometric analog of the Zurek-Paz quantum chaos criterion \[10\]. As a physical application of our general theoretical scheme that we have called the Information Geometrodynamical Approach to Chaos (IGAC), we provide an information-geometric analogue of quantum energy level statistics for integrable and chaotic quantum spin chains. It is known \[17\] that in the integrable case, the antiferromagnetic Ising chain is immersed in a transverse homogeneous magnetic field and the level spacing distribution of its spectrum is the Poisson distribution. Instead, in the chaotic case, the antiferromagnetic Ising chain is immersed in a tilted homogeneous magnetic field and the level spacing distribution of its Hamiltonian spectrum is the Wigner-Dyson distribution. The antiferromagnetic Ising chain in external magnetic field is one example of order-to-chaos transition in quantum many-body context and it is used here as a demonstrating example of the conjectured connection between the Wigner-Dyson (Poisson) statistics and nonintegrability (integrability) in quantum mechanics. Moreover, it is known that integrable and chaotic quantum antiferromagnetic Ising chains are characterized by asymptotic logarithmic and linear growths of their operator space entanglement entropies \[17\], respectively.

Following the results provided by Prosen, we study the information-geometrodynamics of a Poisson distribution coupled to an Exponential bath (regular case) and that of a Wigner-Dyson distribution coupled to a Gaussian bath (chaotic case). Remarkably, we show that in the former case the IGE exhibits asymptotic logarithmic growth while in the latter case it exhibits asymptotic linear growth.

The layout of this paper is as follows. In Section II, the general formalism of the IGAC is applied to study a simple example, an ED Gaussian statistical model. In Section III, the main indicators of chaoticity within our novel theoretical construct are introduced by studying the ED Gaussian model. In Section IV, special focus is devoted to the role of the IGE as an indicator of temporal complexity on curved statistical manifolds. In Section V, after presenting the basics of the IG of Poisson and Wigner-Dyson distributions, we briefly review the conventional approach suitable to study the energy level statistics of integrable and chaotic antiferromagnetic Ising chains immersed in external magnetic fields. In Section VI, we present our IGAC-based novel characterization of the quantum energy level statistics of such Ising chains. Finally, in Section VII, we present our final remarks.

II. THEORETICAL STRUCTURE OF THE IGAC: A SIMPLE EXAMPLE

The IGAC arises as a theoretical framework to study chaos in informational geodesic flows describing physical, biological or chemical systems. A geodesic on a curved statistical manifold represents the maximum probability path a complex dynamical system explores in its evolution between the initial and the final macrostates. Each point of the geodesic is parametrized by the macroscopic dynamical variables defining the macrostate of the system. Furthermore, each macrostate is in a one-to-one relation with the probability distribution representing the maximally probable description of the system being considered. The set of macrostates forms the parameter space while the set of probability distributions forms the statistical manifold. The parameter space is homeomorphic to the statistical manifold. IGAC is the information-geometric analogue of conventional geometrodynamical approaches \[1,2\] where the classical configuration space \( \Gamma_E \) is being replaced by a statistical manifold \( M_S \) with the additional possibility of considering chaotic dynamics arising from non conformally flat metrics (the Jacobi metric is always conformally flat, instead). It is an information-geometric extension of the Jacobi geometrodynamics (the geometrization of a Hamiltonian system by transforming it to a geodesic flow \[18\]). The reformulation of dynamics in terms of a geodesic problem allows the application of a wide range of well-known geometrical techniques in the investigation of the solution space and properties of the equation of motion. The power of the Jacobi reformulation is that all of the dynamical information is collected into a single geometric object in which all the available manifest symmetries are retained- the manifold on which geodesic flow is induced.

Using information-geometric methods, we have investigated in some detail the still open problem of finding a unifying description of classical and quantum chaos \[10\]. One of our goals in this paper is that of representing an additional step forward in that research direction.

A. The ED Gaussian Model

Maximum relative Entropy (ME) methods are used to construct an ED model that follows from an assumption about what information is relevant to predict the evolution of the system. Given a known initial macrostate (probability
distribution) and that the system evolves to a final known macrostate, the possible trajectories of the system are examined. A notion of distance between two probability distributions is provided by IG. As shown in [19, 20] this distance is quantified by the Fisher-Rao information metric tensor.

In the following example, we consider an ED model whose microstates span a 3l-dimensional space labelled by the variables \[ \{X, \bar{X}^{(1)}, \bar{X}^{(2)}, \ldots, \bar{X}^{(l)} \} \] with \[ \bar{X}^{(\alpha)} = (x^{(\alpha)}_1, x^{(\alpha)}_2, x^{(\alpha)}_3) \], \( \alpha = 1, \ldots, l \) and \( x^{(\alpha)}_a \in \mathbb{R} \) with \( a = 1, 2, 3 \). We assume the only testable information pertaining to the 3l degrees of freedom \( \{ x^{(\alpha)}_a \} \) consists of the expectation values \[ \langle x^{(\alpha)}_a \rangle \] and variances \( \Delta x^{(\alpha)}_a = \sqrt{\langle (x^{(\alpha)}_a - \langle x^{(\alpha)}_a \rangle)^2 \rangle} \). The set of these expectation values define the 6l-dimensional space of macrostates of the system. A measure of distinguishability among the macrostates of the ED model is obtained by assigning a probability distribution \( P(\bar{X} | \Theta) \) to each macrostate \( \Theta \) where \( \Theta = \{ (1) \theta^{(\alpha)}_a, (2) \theta^{(\alpha)}_a \} \) with \( \alpha = 1, 2, \ldots, l \) and \( a = 1, 2, 3 \). The process of assigning a probability distribution to each state endows a metric structure. Specifically, the Fisher-Rao information metric is a measure of distinguishability among macrostates. It assigns an IG to the space of states. Each macrostate may be viewed as a point of a 6l-dimensional statistical manifold with coordinates given by the numerical values of the expectations \[ \langle x^{(\alpha)}_a \rangle = \theta^{(\alpha)}_a \] and \( \Delta x^{(\alpha)}_a = \theta^{(\alpha)}_a \). The available information can be written in the form of the following 6l information constraint equations,

\[
\begin{align*}
\langle x^{(\alpha)}_a \rangle &= \int_{-\infty}^{+\infty} dx^{(\alpha)}_a P^{(\alpha)}_a (x^{(\alpha)}_a | (1) \theta^{(\alpha)}_a, (2) \theta^{(\alpha)}_a), \\
\Delta x^{(\alpha)}_a &= \left[ \int_{-\infty}^{+\infty} dx^{(\alpha)}_a \left( x^{(\alpha)}_a - \langle x^{(\alpha)}_a \rangle \right)^2 P^{(\alpha)}_a (x^{(\alpha)}_a | (1) \theta^{(\alpha)}_a, (2) \theta^{(\alpha)}_a) \right]^{\frac{1}{2}}.
\end{align*}
\]

The probability distributions \( P^{(\alpha)}_a \) in (1) are constrained by the conditions of normalization,

\[
\int_{-\infty}^{+\infty} dx^{(\alpha)}_a P^{(\alpha)}_a (x^{(\alpha)}_a | (1) \theta^{(\alpha)}_a, (2) \theta^{(\alpha)}_a) = 1.
\]

Information theory identifies the Gaussian distribution as the maximum entropy distribution if only the expectation value and the variance are known [21]. ME methods allow us to associate a probability distribution \( P(\bar{X} | \Theta) \) to each point in the space of states \( \Theta \). The distribution that best reflects the information contained in the prior distribution \( m(\bar{X}) \) updated by the information \( \{ \langle x^{(\alpha)}_a \rangle, \Delta x^{(\alpha)}_a \} \) is obtained by maximizing the relative entropy

\[
S(\Theta) = - \int_{\{\bar{X}\}} d\bar{X} P(\bar{X} | \Theta) \log \left( \frac{P(\bar{X} | \Theta)}{m(\bar{X})} \right).
\]

As a working hypothesis, the prior \( m(\bar{X}) \) is set to be uniform since we assume the lack of prior available information about the system (postulate of equal a priori probabilities). Upon maximizing (3), given the constraints (1) and (2), we obtain

\[
P(\bar{X} | \Theta) = \prod_{\alpha=1}^{l} \prod_{a=1}^{3} P^{(\alpha)}_a (x^{(\alpha)}_a | \mu^{(\alpha)}_a, \sigma^{(\alpha)}_a)
\]

where

\[
P^{(\alpha)}_a (x^{(\alpha)}_a | \mu^{(\alpha)}_a, \sigma^{(\alpha)}_a) = \left(2\pi \sigma^{(\alpha)}_a \right)^{\frac{-1}{2}} \exp \left[ -\frac{(x^{(\alpha)}_a - \mu^{(\alpha)}_a)^2}{2\sigma^{(\alpha)}_a^2} \right].
\]
and, in the standard notation for Gaussians, $(1)\dot{\theta}_i^{(a)} \equiv \langle x_i^{(a)} \rangle \equiv \mu_i^{(a)}$, $(2)\ddot{\theta}_i^{(a)} \equiv \Delta x_i^{(a)} \equiv \sigma_i^{(a)}$. The probability distribution $(4)$ encodes the available information concerning the system. Note we assumed uncoupled constraints among microvariables $x_i^{(a)}$. In other words, we assumed that information about correlations between the microvariables need not to be tracked. This assumption leads to the simplified product rule $(4)$. However, coupled constraints would lead to a generalized product rule in $(4)$ and to an information metric tensor with non-trivial off-diagonal elements (covariance terms). For instance, the total probability distribution $P(x, y|\mu_x, \sigma_x, \mu_y, \sigma_y)$ of two dependent Gaussian distributed microvariables $x$ and $y$ reads

$$P(x, y|\mu_x, \sigma_x, \mu_y, \sigma_y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \times$$

$$\times \exp \left\{ \frac{1}{2(1-r^2)} \left[ \frac{(x-\mu_x)^2}{\sigma_x^2} - 2r\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2} \right] \right\},$$

where $r \in (-1, +1)$ is the correlation coefficient given by

$$r = \frac{\langle (x-\langle x \rangle)(y-\langle y \rangle) \rangle}{\sqrt{\langle (x-\langle x \rangle)^2 \rangle} \sqrt{\langle (y-\langle y \rangle)^2 \rangle}} = \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\sigma_x\sigma_y}.$$

The information metric tensor induced by $(6)$ is $(10)$,

$$g_{ij} = \begin{pmatrix}
-\frac{1}{\sigma_x^2(r^2-1)} & 0 & 0 \\
0 & -\frac{2}{\sigma_x^2(r^2-1)} & 0 \\
0 & 0 & -\frac{2}{\sigma_x^2(r^2-1)}
\end{pmatrix},$$

where $i, j = 1, 2, 3, 4$. The Ricci curvature scalar associated with manifold characterized by $(7)$ is given by

$$R = g^{ij}R_{ij} = -\frac{8(r^2-2)+2r^2(3r^2-2)}{8(r^2-1)}.$$

It is clear that in the limit $r \to 0$, the off-diagonal elements of $g_{ij}$ vanish and the scalar $R$ reduces to the result obtained in $(10)$, namely $R = -2 < 0$. We could have in principle considered a correlated Gaussian process characterized by two correlated Gaussian microvariables $x$ and $y$. Such a process would lead to an ED on a five-dimensional statistical manifold whose elements are probability distributions of the form $P(x, y|\mu_x, \sigma_x, \mu_y, \sigma_y, \sigma_{xy})$ coordinatized by the expectation values $\mu_x$ and $\mu_y$ as well as the square-root of the three independent elements of the symmetric covariance matrix, namely, $\sigma_x, \sigma_y$ and $\sigma_{xy}$. However, in view of the computational difficulty in obtaining analytical expressions for the elements of a $5 \times 5$ information metric with $\sigma_{xy}$ playing the role of a macro-dynamical variable, we have chosen to consider the ED on a four-dimensional statistical manifold whose elements are given in $(10)$. Correlation terms may be fictitious. They may arise for instance from coordinate transformations. On the other hand, correlations may arise from external fields in which the system is immersed. In such situations, correlations among $x_i^{(a)}$ effectively describe interaction between the microvariables and the external fields. Such generalizations would require more delicate analysis.

We cannot determine the evolution of microstates of the system since the available information is insufficient. Not only is the information available insufficient but we also do not know the equation of motion. In fact there is no standard "equation of motion". Instead we can ask: how close are the two total distributions with parameters $(\mu_a^{(a)}, \sigma_a^{(a)})$ and $(\mu_a^{(a)} + d\mu_a^{(a)}, \sigma_a^{(a)} + d\sigma_a^{(a)})$? Once the states of the system have been defined, the next step concerns the problem of quantifying the notion of change from the macrostate $\Theta$ to the macrostate $\Theta + d\Theta$. A convenient measure of change is distance. The measure we seek is given by the dimensionless distance $ds$ between $P\left(\vec{X} \mid \Theta\right)$ and $P\left(\vec{X} \mid \Theta + d\Theta\right),$

$$ds^2 = g_{\mu\nu}d\Theta^\mu d\Theta^\nu$$

with $\mu, \nu = 1, 2, \ldots, 6l$

where

$$g_{\mu\nu} = \int d\vec{X} P\left(\vec{X} \mid \Theta\right) \frac{\partial \log P\left(\vec{X} \mid \Theta\right)}{\partial \Theta^\mu} \frac{\partial \log P\left(\vec{X} \mid \Theta\right)}{\partial \Theta^\nu}$$

(11)
is the Fisher-Rao information metric. Substituting (11) into (11), the metric \( g_{\mu\nu} \) on \( M_s \) becomes a \( 6l \times 6l \) matrix \( M \) made up of 3\( l \) blocks \( M_{2 \times 2} \) with dimension 2 \( \times 2 \) given by,

\[
M_{2 \times 2} = \begin{pmatrix}
\left( \sigma_a^{(\alpha)} \right)^{-2} & 0 \\
0 & 2 \times \left( \sigma_a^{(\alpha)} \right)^{-2}
\end{pmatrix}
\]  

(12)

with \( \alpha = 1, 2, \ldots, l \) and \( a = 1, 2, 3 \). From (11), the "length" element (10) reads,

\[
ds^2 = l \sum_{\alpha=1}^{3} \sum_{a=1}^{l} \left[ \frac{1}{\left( \sigma_a^{(\alpha)} \right)^2} d\mu_a^{(\alpha)^2} + \frac{2}{\left( \sigma_a^{(\alpha)} \right)^2} d\sigma_a^{(\alpha)^2} \right].
\]  

(13)

We bring attention to the fact that the metric structure of \( M_s \) is an emergent (not fundamental) structure. It arises only after assigning a probability distribution \( P \left( \hat{X} \mid \hat{\Theta} \right) \) to each state \( \hat{\Theta} \).

III. INFORMATION-GEOMETRIC INDICATORS OF CHAOS WITHIN THE IGAC

The relevant indicators of chaoticity within the IGAC are the Ricci scalar curvature \( R_{M_s} \) (or, more correctly, the sectional curvature \( K_{M_s} \)), the Jacobi vector field intensity \( J_{M_s} \) and the IGE \( S_{M_s} \) once the line element on the curved statistical manifold \( M_s \) underlying the entropic dynamics has been specified.

A. Ricci Scalar Curvature, Anisotropy and Compactness

Given the Fisher-Rao information metric, we use standard differential geometry methods applied to the space of probability distributions to characterize the geometric properties of \( M_s \). Recall that the Ricci scalar curvature \( R \) is given by,

\[
R = g^{\mu\nu} R_{\mu\nu},
\]  

(14)

where \( g^{\mu\nu} g_{\nu\rho} = \delta^\rho_\mu \) so that \( g^{\mu\nu} = (g_{\mu\nu})^{-1} \). The Ricci tensor \( R_{\mu\nu} \) is given by,

\[
R_{\mu\nu} = \partial_\gamma \Gamma^\gamma_{\mu\nu} - \partial_\nu \Gamma^\gamma_{\mu\lambda} + \Gamma^\gamma_{\mu\nu} \Gamma^\rho_\gamma - \Gamma^\gamma_{\mu\rho} \Gamma^\rho_\nu.
\]  

(15)

The Christoffel symbols \( \Gamma^\rho_{\mu\nu} \) appearing in the Ricci tensor are defined in the standard manner as,

\[
\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right).
\]  

(16)

Using (12) and the definitions given above, we can show that the Ricci scalar curvature becomes

\[
R_{M_s} = R^\alpha_{\alpha} = \sum_{\rho \neq \sigma} K \left( e_\rho, e_\sigma \right) = -3l < 0.
\]  

(17)

The scalar curvature is the sum of all sectional curvatures of planes spanned by pairs of orthonormal basis elements \( \{ e_\rho = \partial_{\theta^\rho(p)} \} \) of the tangent space \( T_p M_s \) with \( p \in M_s \),

\[
K \left( a, b \right) = \frac{R_{\mu\nu\rho\sigma} a^\mu b^\nu a^\rho b^\sigma}{\left( g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} \right) a^\mu b^\nu a^\rho b^\sigma}, a = \sum_{\rho} \left( a, h^\rho \right) e_\rho,
\]  

(18)

where \( \left( e_\rho, h^\rho \right) = \delta^\rho_\mu \). Notice that the sectional curvatures completely determine the curvature tensor. From (17) we conclude that \( M_s \) is a 6\( l \)-dimensional statistical manifold of constant negative Ricci scalar curvature. A detailed analysis on the calculation of Christoffel connection coefficients using the ED formalism for a four-dimensional manifold of Gaussians can be found in [10].
It can be shown that $\mathcal{M}_s$ is not a pseudosphere (maximally symmetric manifold). The first way this can be understood is from the fact that the Weyl Projective curvature tensor \( \mathcal{R} \) (or the anisotropy tensor) $W_{\mu\nu\rho\sigma}$ defined by

$$ W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{R_{\mu\nu\rho\sigma}}{n(n-1)} (g_{\sigma\sigma}g_{\mu\nu} - g_{\nu\nu}g_{\mu\sigma}), $$

with $n = 6l$ in the present case, is non-vanishing. In (19), the quantity $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor defined in the usual manner by

$$ R^\alpha_{\beta\rho\sigma} = \partial_\sigma \Gamma^\alpha_{\rho\beta} - \partial_\rho \Gamma^\alpha_{\beta\sigma} + \Gamma^\alpha_{\lambda\sigma} \Gamma^\lambda_{\rho\beta} - \Gamma^\alpha_{\lambda\rho} \Gamma^\lambda_{\beta\sigma}. $$

Considerations regarding the negativity of the Ricci curvature as a strong criterion of dynamical instability and the necessity of compactness of $\mathcal{M}_s$ in "true" chaotic dynamical systems would require additional investigation.

The issue of symmetry of $\mathcal{M}_s$ can alternatively be understood from consideration of the sectional curvature. In view of (13), the negativity of the Ricci scalar implies the existence of expanding directions in the configuration space manifold $\mathcal{M}_s$. Indeed, from (14) one may conclude that negative principal curvatures (extrema of sectional curvatures) dominate over positive ones. Thus, the negativity of the Ricci scalar is only a sufficient (not necessary) condition for local instability of geodesic flow. For this reason, the negativity of the scalar provides a strong criterion of local instability. Scenarios may arise where negative sectional curvatures are present, but the positive ones could prevail in the sum so that the Ricci scalar is non-negative despite the instability in the flow in those directions. Consequently, the signs of the sectional curvatures are of primary significance for the proper characterization of chaos.

Yet another useful way to understand the anisotropy of the $\mathcal{M}_s$ is the following. It is known that in $n$ dimensions, there are at most $\frac{n(n+1)}{2}$ independent Killing vectors (directions of symmetry of the manifold). Since $\mathcal{M}_s$ is not a pseudosphere, the information metric tensor does not admit the maximum number of Killing vectors $K_\mu$, defined as

$$ \mathcal{L}_K g_{\mu\nu} = D_\mu K_\nu + D_\nu K_\mu = 0, $$

where $D_\mu$, defined as

$$ D_\mu K_\nu = \partial_\mu K_\nu - \Gamma^{\rho}_{\mu\nu} K_\rho $$

is the covariant derivative operator with respect to the connection $\Gamma$ defined in (10). The Lie derivative $\mathcal{L}_K g_{\mu\nu}$ of the tensor field $g_{\mu\nu}$ along a given direction $K$ measures the intrinsic variation of the field along that direction (that is, the metric tensor is Lie transported along the Killing vector) (22). Locally, a maximally symmetric space of Euclidean signature is either a plane, a sphere, or a hyperboloid, depending on the sign of $\mathcal{R}$. In our case, none of these scenarios occur. As will be seen in what follows, this fact has a significant impact on the integration of the geodesic deviation equation on $\mathcal{M}_s$. At this juncture, we emphasize it is known that the anisotropy of the manifold underlying system dynamics plays a crucial role in the mechanism of instability. In particular, fluctuating sectional curvatures require also that the manifold be anisotropic. However, the connection between curvature variations along geodesics and anisotropy is far from clear and is currently under investigation.

Krylov was the first to emphasize (24) the use of $\mathcal{R} < 0$ as an instability criterion in the context of an $N$-body system (a gas) interacting via Van der Waals forces, with the ultimate hope to understand the relaxation process in a gas. However, Krylov neglected the problem of compactness of the configuration space manifold which is important for making inferences about exponential mixing of geodesic flows (25). Why is compactness so significant in the characterization of chaos? True chaos should be identified by the occurrence of two crucial features: 1) strong dependence on initial conditions and exponential divergence of the Jacobi vector field intensity, i.e., stretching of dynamical trajectories; 2) compactness of the configuration space manifold, i.e., folding of dynamical trajectories. Compactness is required in order to discard trivial exponential growths due to the unboundedness of the "volume" available to the dynamical system. In other words, the folding is necessary to have a dynamics actually able to mix the trajectories, making practically impossible, after a finite interval of time, to discriminate between trajectories which were very nearby each other at the initial time. When the space is not compact, even in presence of strong dependence on initial conditions, it could be possible in some instances (though not always), to distinguish among different trajectories originating within a small distance and then evolved subject to exponential instability.

As a final remark, we emphasize that it is known from IG (8) that there is a one-to-one relation between elements of the statistical manifold and the parameter space. More precisely, the statistical manifold $\mathcal{M}_s$ is homeomorphic to the parameter space $\mathcal{D}_\Theta$. This implies the existence of a continuous, bijective map $h_{\mathcal{M}_s, \mathcal{D}_\Theta}$:

$$ h_{\mathcal{M}_s, \mathcal{D}_\Theta} : \mathcal{M}_s \ni P \left( \vec{X} \right| \hat{\Theta} ) \rightarrow \hat{\Theta} \in \mathcal{D}_\Theta $$

(23)
The inverse image $h_{\mathcal{M}_s; D_\alpha}^{-1}(\vec{\Theta}) = P\left( X \mid \vec{\Theta} \right)$. The inverse image $h_{\mathcal{M}_s; D_\alpha}^{-1}$ is the so-called homeomorphism map. In addition, since homeomorphisms preserve compactness, it is sufficient to restrict ourselves to a compact subspace of the parameter space $D_\alpha$ in order to ensure that $\mathcal{M}_s$ is itself compact.

B. Canonical Formalism

The geometrization of a Hamiltonian system by transforming it to a geodesic flow is a well-known technique of classical mechanics associated with the name of Jacobi [18]. Transformation to geodesic motion is obtained in two steps: 1) conformal transformation of the metric; 2) rescaling of the time parameter [27]. The reformulation of dynamics in terms of a geodesic problem allows the application of a wide range of well-known geometrical techniques in the investigation of the solution space and properties of equations of motions. The power of the Jacobi reformulation is that all of the dynamical information is collected into a single geometric object - the manifold on which geodesic flow is induced - in which all the available manifest symmetries are retained. For instance, integrability of the system is connected with the existence of Killing vectors and tensors on this manifold [28, 29].

In this Section we study the trajectories of the system on $\mathcal{M}_s$. We emphasize ED can be derived from a standard principle of least action (of Maupertuis-Euler-Lagrange-Jacobi type) [6, 30]. The main differences are that the dynamics being considered here, namely ED, is defined on a space of probability distributions $\mathcal{M}_s$, not on an ordinary linear space $V$ and the standard coordinates $q_\mu$ of the system are replaced by statistical macrovariables $\Theta^\mu$. The geodesic equations for the macrovariables of the Gaussian ED model are given by,

$$\frac{d^2 \Theta^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{d\Theta^\nu}{d\tau} \frac{d\Theta^\rho}{d\tau} = 0 \quad (24)$$

with $\mu = 1, 2, \ldots, 6l$. Observe the geodesic equations are nonlinear second order coupled ordinary differential equations. They describe a reversible dynamics whose solution is the trajectory between an initial and a final macrostate. The trajectory can be equally well traversed in both directions.

1. Geodesics on $\mathcal{M}_s$

We determine the explicit form of (24) for the pairs of statistical coordinates $(\mu_\alpha^{(a)}, \sigma_\alpha^{(a)})$. Substituting the expression of the Christoffel connection coefficients into (24), the geodesic equations for the macrovariables $\mu_\alpha^{(a)}$ and $\sigma_\alpha^{(a)}$ associated to the microstate $x_\alpha^{(a)}$ become,

$$\frac{d^2 \mu_\alpha^{(a)}}{d\tau^2} - \frac{2}{\sigma_\alpha^{(a)}} \frac{d\mu_\alpha^{(a)}}{d\tau} \frac{d\sigma_\alpha^{(a)}}{d\tau} = 0, \quad \frac{d^2 \sigma_\alpha^{(a)}}{d\tau^2} - \frac{1}{\sigma_\alpha^{(a)}} \left( \frac{d\sigma_\alpha^{(a)}}{d\tau} \right)^2 + \frac{1}{2\sigma_\alpha^{(a)}} \left( \frac{d\mu_\alpha^{(a)}}{d\tau} \right)^2 = 0, \quad (25)$$

with $\alpha = 1, 2, \ldots, l$ and $a = 1, 2, 3$. This is a set of coupled ordinary differential equations, whose solutions are

$$\mu_\alpha^{(a)}(\tau) = \frac{\left( w_\alpha^{(a)} \right)^2}{\cosh(2\beta_\alpha^{(a)} \tau) - \sinh(2\beta_\alpha^{(a)} \tau)}, \quad \sigma_\alpha^{(a)}(\tau) = \frac{B_\alpha^{(a)} \exp(-\beta_\alpha^{(a)} \tau)}{\exp(-2\beta_\alpha^{(a)} \tau) + \left( w_\alpha^{(a)} \right)^2} + C_\alpha^{(a)} \quad (26)$$

The quantities $B_\alpha^{(a)}$, $C_\alpha^{(a)}$, $\beta_\alpha^{(a)}$ are real integration constants and they can be evaluated once the boundary conditions are specified. We observe that since every geodesic is well-defined for all temporal parameters $\tau$, $\mathcal{M}_s$ constitutes a geodesically complete manifold [31]. It is therefore a natural setting within which one may consider global questions and search for a weak criterion of chaos [2]. Furthermore, since $\left| \mu_\alpha^{(a)}(\tau) \right| < +\infty$ and $\left| \sigma_\alpha^{(a)}(\tau) \right| < +\infty \forall \tau \in \mathbb{R}^+$, $\forall a = 1, 2, 3$ and $\forall \alpha = 1, \ldots, N$, the parameter space $\{ \vec{\Theta} \}$ (homeomorphic to $\mathcal{M}_s$) is compact. The compactness of the configuration space manifold $\mathcal{M}_s$ assures the folding mechanism of information-dynamical trajectories (the folding mechanism is a key-feature of true chaos, [2]).
It is known [30] that the Riemannian curvature of a manifold is intimately related to the behavior of geodesics on it. If the Riemannian curvature of a manifold is negative, geodesics (initially parallel) rapidly diverge from one another. For the sake of simplicity, we assume very special initial conditions: \( B_{\alpha}^{(\alpha)} \equiv \Xi, \beta_{\alpha}^{(\alpha)} \equiv \lambda \in \mathbb{R}^+, \ C_{\alpha}^{(\alpha)} = 0, \forall \alpha = 1, 2, ..., l \) and \( a = 1, 2, 3 \). However, the conclusions drawn can be generalized to more arbitrary initial conditions. We observe that since every maximal geodesic is well-defined for all temporal parameters \( \tau \), \( \mathcal{M}_s \) constitute a geodesically complete manifold [31]. It is therefore a natural setting within which one may consider global questions and search for a weak criterion of chaos [2].

C. Exponential divergence of the Jacobi field intensity

The actual interest of the Riemannian formulation of the dynamics stems from the possibility of studying the instability of natural motions through the instability of geodesics of a suitable manifold, a circumstance that has several advantages. First of all a powerful mathematical tool exists to investigate the stability or instability of a geodesic flow: the Jacobi-Levi-Civita equation for geodesic spread [32]. The JLC-equation describes covariantly how nearby geodesics locally scatter. It is a familiar object both in Riemannian geometry and theoretical physics (it is of fundamental interest in experimental General Relativity). Moreover the JLC-equation relates the stability or instability of natural motions through the instability of geodesics of a suitable manifold, a circumstance that has

Consider the behavior of the one-parameter family of neighboring geodesics \( \mathcal{F}_{G_\mathcal{M}_s} (\lambda) \equiv \{ \Theta^\mu_\mathcal{M}_s (\tau; \lambda) \}_{\lambda \in \mathbb{R}^+} \), where

\[
\mu_\alpha^{(\alpha)} (\tau; \lambda) = \Xi^2 \frac{1}{2\lambda \cosh (2\lambda \tau) - \sinh (2\lambda \tau) + \frac{\Xi}{8\lambda^2}},
\]

\[
\sigma_\alpha^{(\alpha)} (\tau; \lambda) = \Xi \frac{\cosh (\lambda \tau) - \sinh (\lambda \tau)}{\cosh (2\lambda \tau) - \sinh (2\lambda \tau) + \frac{\Xi}{8\lambda^2}},
\]

with \( \alpha = 1, 2, ..., l \) and \( a = 1, 2, 3 \). The relative geodesic spread on a (non-maximally symmetric) curved manifold as \( \mathcal{M}_s \) is characterized by the Jacobi-Levi-Civita equation, the natural tool to tackle dynamical chaos [23, 32],

\[
\frac{D^2 \delta \Theta^\mu}{D\tau^2} + R_{\mu \nu \rho \sigma} \frac{\partial \Theta^\nu}{\partial \tau} \delta \Theta^\rho \frac{\partial \Theta^\sigma}{\partial \tau} = 0
\]

(28)

where the covariant derivative \( \frac{D^2 \delta \Theta^\mu}{D\tau^2} \) in [28] is defined as [33],

\[
\frac{D^2 \delta \Theta^\mu}{D\tau^2} = \frac{d^2 \delta \Theta^\mu}{d\tau^2} + 2\Gamma_{\alpha \beta}^{\mu} \frac{d\delta \Theta^\alpha}{d\tau} \frac{d\delta \Theta^\beta}{d\tau} + \Gamma_{\alpha \beta}^{\mu} \frac{d\Theta^\alpha}{d\tau} \frac{d^2 \delta \Theta^\beta}{d\tau^2} + \Gamma_{\alpha \beta \gamma}^{\mu} \frac{d\Theta^\alpha}{d\tau} \frac{d\Theta^\beta}{d\tau} \frac{d\delta \Theta^\gamma}{d\tau},
\]

(29)

and the Jacobi vector field \( J^\mu \) is given by [34],

\[
J^\mu \equiv \delta \Theta^\mu \overset{\text{def}}{=} \delta_\lambda \Theta^\mu = \left( \frac{\partial \Theta^\mu (\tau; \lambda)}{\partial \lambda} \right) \bigg|_{\tau=\text{const}} \delta \lambda.
\]

(30)

Notice that the JLC-equation appears intractable already at rather small \( l \). For isotropic manifolds, the JLC-equation can be reduced to the simple form [32],

\[
\frac{D^2 J^\mu}{D\tau^2} + K J^\mu = 0, \mu = 1, ..., 6l
\]

(31)

where \( K \) is the constant value assumed throughout the manifold by the sectional curvature. The sectional curvature of manifold \( \mathcal{M}_s \) is the \( 6l \)-dimensional generalization of the Gaussian curvature of two-dimensional surfaces of \( \mathbb{R}^3 \). If \( K < 0 \), unstable solutions of the equation [31] are of the form

\[
J (\tau) = \frac{1}{\sqrt{-K}} \omega (0) \sinh \left( \sqrt{-K} \tau \right)
\]

(32)
once the initial conditions are assigned as \( J(0) = 0 \), \( \frac{dJ(0)}{dt} = \omega(0) \) and \( K < 0 \). Equation (28) forms a system of \( 6l \) coupled ordinary differential equations linear in the components of the deviation vector field (30) but nonlinear in derivatives of the metric (11). It describes the linearized geodesic flow: the linearization ignores the relative velocity of the geodesics. When the geodesics are neighboring but their relative velocity is arbitrary, the corresponding nonlinear geodesic deviation equation is the so-called generalized Jacobi equation [35, 36]. The nonlinearity is due to the existence of velocity-dependent terms in the system. Neighboring geodesics accelerate relative to each other with a rate directly measured by the curvature tensor \( R_{\alpha\beta\gamma\delta} \). Substituting (27) in (28) and neglecting the exponentially decaying terms in \( \delta \Theta^\mu \) and its derivatives, integration of (28) leads to the following asymptotic exponential growth of the Jacobi vector field intensity (a classical feature of chaos),

\[
J_{\mathcal{M}_s} = \|J\| = (g_{\mu\nu} J^\mu J^\nu)^{\frac{1}{2}} \approx 3t e^{\lambda \tau}.
\]

Finally, we point out that in our approach the quantity \( \lambda_j \),

\[
\lambda_j \overset{\text{def}}{=} \lim_{\tau \to \infty} \frac{1}{\tau} \ln \left[ \frac{\|J_{\mathcal{M}_s}(\tau)\|}{\|J_{\mathcal{M}_s}(0)\|} \right]
\]

would play the role of the conventional Lyapunov exponents.

IV. LINEARITY OF THE INFORMATION GEOMETRODYNAMICAL ENTROPY

We investigate the stability of the trajectories of the ED model considered on \( \mathcal{M}_s \). It is known [30] that the Riemannian curvature of a manifold is closely connected with the behavior of the geodesics on it. If the Riemannian curvature of a manifold is negative, geodesics (initially parallel) rapidly diverge from one another. For the sake of simplicity, we assume very special initial conditions: \( B^\alpha \equiv \Xi, \beta^\alpha \equiv \lambda \in \mathbb{R}^+, C^\alpha = 0, \forall \alpha = 1, 2, \ldots, l \) and \( a = 1, 2, 3 \). However, the conclusion we reach can be generalized to more arbitrary initial conditions. Recall \( \mathcal{M}_s \) is the space of probability distributions \( \{P(\tilde{X} | \tilde{\Theta})\} \) labeled by \( 6l \) statistical parameters \( \tilde{\Theta} \). These parameters are the coordinates for the point \( P \), and in these coordinates a volume element \( dV_{\mathcal{M}_s} \), reads,

\[
dV_{\mathcal{M}_s} = \sqrt{g} d^6 \tilde{\Theta} = \prod_{a=1}^{l} \prod_{a=1}^{3} \frac{\sqrt{2}}{\sigma_a} \mu_a^2 a^a d\mu_a d\sigma_a.
\]

The volume of an extended region \( \Delta V_{\mathcal{M}_s}(\tau; \lambda) \) of \( \mathcal{M}_s \) is defined by,

\[
\Delta V_{\mathcal{M}_s}(\tau; \lambda) \overset{\text{def}}{=} \prod_{\alpha=1}^{l} \prod_{a=1}^{3} \int_{\mu_a^0(0)}^{\mu_a(\tau)} \int_{\sigma_a^0(0)}^{\sigma_a(\tau)} \frac{\sqrt{2}}{\sigma_a^2} \mu_a^2 d\mu_a d\sigma_a.
\]

where \( \mu_a^0(\tau) \) and \( \sigma_a^0(\tau) \) are given in (27) and where the scalar \( \lambda \) is the chosen quantity used to define the one-parameter family of geodesics \( \mathcal{F}_{G_{\mathcal{M}_s}}(\lambda) = \{X_{\mathcal{M}_s}(\tau; \lambda)\}^\mu_{\alpha=1,..,6l}_{\lambda \in \mathbb{R}^+} \). The quantity that encodes relevant information about the stability of neighboring volume elements is the the average volume \( V_{\mathcal{M}_s}(\tau; \lambda) \),

\[
V_{\mathcal{M}_s}(\tau; \lambda) \overset{\text{def}}{=} \langle \Delta V_{\mathcal{M}_s}(\tau; \lambda) \rangle_\tau = \frac{1}{\tau} \int_0^\tau \Delta V_{\mathcal{M}_s}(\tau'; \lambda) d\tau' \approx e^{3\lambda \tau}.
\]

The ratio \( \frac{V_{\mathcal{M}_s}(\tau)}{V_{\mathcal{M}_s}(0)} \) with \( V_{\mathcal{M}_s}(\tau) \) in (37) representing the temporal average of the \( 3l \)-fold over trajectories of maximum probability (geodesics) is a measure of the number of the accessible macrostates in configuration (statistical) manifold \( \mathcal{M}_s \) after a finite temporal increment \( \tau \). In other words, \( V_{\mathcal{M}_s}(\tau) \) can be interpreted as the temporal evolution of the system’s uncertainty volume \( V_{\mathcal{M}_s}(0) \). For instance \( V_{\mathcal{M}_s}(0) \) may be a spherical volume of initial points whose center is a given point on the attractor and whose surface consists of configuration points from nearby trajectories. An attractor is a subset of the manifold \( \mathcal{M}_s \) toward which almost all sufficiently close trajectories converge asymptotically, covering it densely as the time goes on. Strange attractors are called chaotic attractors. Chaotic attractors have at least one finite positive Lyapunov exponent [37]. As the center of \( V_{\mathcal{M}_s}(0) \) and its surface points evolve in time, the
spherical volume becomes an ellipsoid with principal axes in the directions of contraction and expansion. The average rates of expansion and contraction along the principal axes are the Lyapunov exponents [38].

The asymptotic regime of diffusive evolution in (37) describes the exponential increase of average volume elements on $M$. The exponential instability characteristic of chaos forces the system to rapidly explore large areas (volumes) of the statistical manifolds. From equation (37), we notice that the parameter $\lambda$ characterizes the exponential growth rate of average statistical volumes $\mathcal{V}_M(\tau; \lambda)$ in $M$. This suggests that $\lambda$ may play the same role ordinarily played by Lyapunov exponents [39]. It is interesting to note that this asymptotic behavior appears also in the conventional description of quantum chaos where the von Neumann entropy increases linearly at a rate determined by the Lyapunov exponents. The linear increase of entropy as a quantum chaos criterion was introduced by Zurek and Paz [4]. In our information-geometric approach a relevant quantity that can be useful to study the degree of instability characterizing the ED model is the information-geometrodynamical entropy (IGE) defined as [10],

$$S_{M_c} \equiv \lim_{\tau \to \infty} \log \mathcal{V}_{M_c}(\tau; \lambda).$$

The IGE is intended to capture the temporal complexity (chaoticity) of ED theoretical models on curved statistical manifolds $M$ by considering the asymptotic temporal behaviors of the average statistical volumes occupied by the evolving macrovariables labelling points on $M$. Substituting (37) in (38), we obtain

$$S_{M_c} = \lim_{\tau \to \infty} \log \left\{ \frac{1}{\tau} \int_0^\tau \left[ \prod_{\alpha=1}^3 \prod_{\alpha=1}^3 \int_{\rho^{(\alpha)}_a(0)} \int_{\sigma^{(\alpha)}_a(0)} \sqrt{2} d\rho^{(\alpha)}_a d\sigma^{(\alpha)}_a \right] d\tau \right\} \approx 3\lambda \tau. \quad (39)$$

Before discussing the meaning of (39), recall that in conventional approaches to chaos the notion of entropy is introduced, in both classical and quantum physics, as the missing information about the systems fine-grained state [4, 10]. For a classical system, suppose that the phase space is partitioned into very fine-grained cells of uniform volume $\Delta v$, labelled by an index $j$. If one does not know which cell the system occupies, one assigns probabilities $p_j$ to the various cells; equivalently, in the limit of infinitesimal cells, one can use a phase-space density $\rho(X_j) = \frac{p_j}{\Delta v}$. Then, in a classical chaotic evolution, the asymptotic expression of the information needed to characterize a particular coarse-grained trajectory out to time $\tau$ is given by the Shannon information entropy (measured in bits),

$$S_{\text{classical}}^{(\text{chaotic})} = - \int dX \rho(X) \log_2 (\rho(X) \Delta v) = - \sum_j p_j \log_2 p_j \sim K \tau. \quad (40)$$

where $\rho(X)$ is the phase-space density and $p_j = \frac{\Delta v}{\Sigma}$ is the probability for the corresponding coarse-grained trajectory. $S_{\text{classical}}^{(\text{chaotic})}$ is the missing information about which fine-grained cell the system occupies. The quantity $K$ represents the linear rate of entropy increase and it is called the Kolmogorov-Sinai entropy (or metric entropy) ($K$ is the sum of positive Lyapunov exponents, $K = \sum_j \lambda_j$). $K$ quantifies the degree of classical chaos. The Kolmogorov-Sinai entropy provides a measure of the rate at which information is lost by an evolving chaotic system ($K$ has dimension entropy/time) and has its roots in the definition of the Shannon entropy. It is worthwhile emphasizing that the quantity that grows asymptotically as $K \tau$ is really the average of the information on the left side of equation (40). This distinction can however be ignored provided we assume the chaotic system has roughly constant Lyapunov exponents over the accessible region of phase space. In quantum mechanics the fine-grained alternatives are normalized state vectors in Hilbert space. From a set of probabilities for various state vectors, one can construct a density operator

$$\hat{\rho} = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j|, \quad \hat{\rho} |\psi_j\rangle = \lambda_j |\psi_j\rangle. \quad (41)$$

The normalization of the density operator, $tr(\hat{\rho}) = 1$, implies that the eigenvalues make up a normalized probability distribution. The von Neumann entropy (natural generalization of both Boltzmann’s and Shannon’s entropy) of the density operator $\hat{\rho}$ (measured in bits) [11],

$$S_{\text{quantum}}^{(\text{chaotic})} = - tr (\hat{\rho} \log_2 \hat{\rho}) = - \sum_j \lambda_j \log_2 \lambda_j \sim K_q \tau \quad (42)$$

can be thought of as the missing information about which eigenvector the system is in. Entropy quantifies the degree of unpredictability about the system’s fine-grained state. In quantum mechanics, the von Neumann entropy plays a role analogous to that played by the Shannon entropy in classical probability theory. They are both functionals of the state, are both monotone under a relevant kind of mapping, and can both be singled out uniquely by natural
requirements. von Neumann’s entropy reduces to the Shannon entropy for diagonal density matrices. However, in general the von Neumann entropy is a subtler object than its classical counterpart. The quantity $\mathcal{K}_q$ in (42) can be interpreted as the non-commutative (quantum theory is a non-commutative probability theory) quantum analog of the Kolmogorov-Sinai dynamical entropy, the so-called quantum dynamical entropy [43]. Examples of quantum dynamical entropies applied to quantum chaos and quantum information theory are the Alicki-Fannes (AF) [44] entropy and the Connes-Narnhofer-Thirring (CNT) [45] entropy. Both the AF and CNT entropy coincide with the KS entropy on finite-dimensional quantum systems. However, they differ when moving from finite to infinite quantum systems.

Recall that decoherence is the loss of phase coherence between the set of preferred quantum states in the Hilbert space of the system due to the interaction with the environment. Moreover, decoherence induces transitions from quantum to classical systems. Therefore, classicality is an emergent property of an open quantum system. Motivated by such considerations, Zurek and Paz investigated implications of the process of decoherence for quantum chaos. They considered a chaotic system, a single unstable harmonic oscillator characterized by a potential $V(x) = -\frac{\lambda x^2}{2}$ ($\lambda$ is the Lyapunov exponent), coupled to an external environment. In the reversible classical limit [42], the von Neumann entropy of such a system increases linearly at a rate determined by the Lyapunov exponent,

$$S_{\text{quantum}}^{(\text{chaotic})} (\text{Zurek-Paz}) \sim \frac{\lambda}{2} \tau.$$  

Notice that the consideration of $3l$ uncoupled identical unstable harmonic oscillators characterized by potentials $V_i(x) = -\frac{\lambda_i x^2}{2}$ ($\lambda_i = \lambda_j$; $i, j = 1, 2, ..., 3l$) would simply lead to

$$S_{\text{quantum}}^{(\text{chaotic})} (\text{Zurek-Paz}) \sim \frac{3}{2} \lambda \tau.$$  

The resemblance of equations (39) and (44) is remarkable and a more detailed discussion about this analogy is presented in [16] where an information-geometric analogue of the Zurek-Paz quantum chaos criterion in the classical reversible limit is proposed [4]. This analogy is illustrated applying the IGAC to a set of $n$-uncoupled three-dimensional anisotropic inverted harmonic oscillators characterized by a Ohmic distributed frequency spectrum.

The entropy-like quantity $S_{M_s}$ in (39) is the asymptotic limit of the natural logarithm of the statistical weight $\langle \Delta V_{M_s} \rangle_s$ defined on $M_s$ and it grows linearly in time, a quantum feature of chaos. Indeed, equation (39) may be considered the information-geometric analog of the Zurek-Paz chaos criterion. In our chaotic ED Gaussian model, the IGE production rate is determined by the information-geometric parameter $\lambda$ characterizing the exponential growth rate of average statistical volumes $V_{M_s}(\tau; \lambda)$ in $M_s$.

In conclusion, for the example under investigation, we have

$$R_{M_s} = -3l, \; S_{M_s} \sim \frac{3}{2} \lambda \tau, \; J_{M_s} \sim \frac{3}{2} \lambda e^{\lambda \tau}.$$  

The IGE grows linearly as a function of the number of Gaussian-distributed microstates of the system. This supports the fact that $S_{M_s}$ may be a useful measure of temporal complexity [16]. Furthermore, these three indicators of chaoticity, the Ricci scalar curvature $R_{M_s}$, the information-geometric entropy $S_{M_s}$ and the Jacobi vector field intensity $J_{M_s}$ are proportional to $3l$, the dimension of the microspace with microstates $\{ \vec{X} \}$ underlying our chaotic ED Gaussian model. This proportionality leads to the conclusion that there is a substantial link among these information-geometric measures of chaoticity since they are all extensive functions of the dimensionality of the microspace underlying the macroscopic chaotic entropic dynamics (see [15]). Curvature, information-geometrodynamical entropy and Jacobi field intensity are linked within our formalism. We are aware that our findings are reliable in the restrictive assumption of Gaussianity. However, we believe that with some additional technical machinery, more general conclusions can be achieved and this connection among indicators of chaoticity may be strengthened.

V. INFORMATION GEOMETRY OF QUANTUM ENERGY LEVEL STATISTICS: AN APPLICATION TO ENTANGLEMENT IN QUANTUM SPIN CHAINS

In what follows, we apply the IGAC to study the entropic dynamics on curved statistical manifolds induced by classical probability distributions of common use in the study of regular and chaotic quantum energy level statistics. In doing so, we suggest an information-geometric characterization of a special class of regular and chaotic quantum energy level statistics. More precisely, we present an information-geometric analogue of the logarithmic and linear entanglement entropy growth in regular and quantum chaotic spin chains, respectively.
A. The Information Geometry of the Poisson and Wigner-Dyson Distributions

The theory of quantum chaos (quantum mechanics of systems whose classical dynamics are chaotic) is not primarily related to few-body physics. Indeed, in real physical systems such as many-electron atoms and heavy nuclei, the origin of complex behavior is the quite strong interaction among many particles. To deal with such systems, a famous statistical approach has been developed which is based upon the Random Matrix Theory (RMT) [47]. The main idea of this approach is to neglect the detailed description of the motion and to treat these systems statistically bearing in mind that the interaction among particles is so complex and strong that generic properties are expected to emerge. The simplest models of RMT are full random matrices of a given symmetry. One of the main results of RMT is the prediction of a specific kind of correlations of the energy spectra of complex quantum systems. Among many characteristics of these correlations, the most popular one is the distribution of spacings between nearest energy levels in the spectra. The exact analytical expression of this distribution is very complicated; instead, one uses the so-called Wigner-Dyson surmise (a very simple expression which gives a very good approximation to the exact result). The known manifestation of quantum chaos is the so-called Wigner-Dyson (WD) distribution for spacings between neighboring levels in the spectrum. In the other limiting case of completely integrable (regular) systems, the distribution turns out to be very close to the Poissonian one. A distinctive property of the WD distribution is the repulsion between neighboring levels in the spectra; the degree of this repulsion (linear, quadratic or quartic) depends on the symmetry of random matrices. For systems without time reversal invariance the relevant ensemble of random matrices is the Gaussian Unitary Ensemble (GUE) [17], characterized by the probability distribution

$$p_{\text{GUE}}(\theta) = \frac{32}{\pi^2} \theta^2 \exp\left(-\frac{4}{\pi} \theta^2\right), \quad \text{(quadratic repulsion)} \quad (46)$$

where $\theta$ is the average spacing of the energy levels. For systems invariant with respect to time reversal the ensemble is the Gaussian Orthogonal Ensemble (GOE) [17],

$$p_{\text{GOE}}(\theta) = \frac{\pi}{2} \theta \exp\left(-\frac{\pi}{4} \theta^2\right), \quad \text{(linear repulsion)} \quad (47)$$

For systems with time reversal invariance but with half-integer spin, the energy is described by the Gaussian Symplectic Ensemble (GSE) of random matrices [17],

$$p_{\text{GSE}}(\theta) = \frac{2^{18}}{3^9 \pi^3} \theta^4 \exp\left(-\frac{64}{9 \pi} \theta^2\right), \quad \text{(quartic repulsion)} \quad (48)$$

Equations (46), (47) and (48) are standard accepted conjectures. Besides energy level statistics in the extreme integrable (Poisson) and chaotic (Wigner-Dyson) regimes, there is also energy level statistics in the mixed regime, i.e., such having a mixed classical dynamics where regular and chaotic regions coexist in the phase space. A convenient and often successful parametrization of the correct probability distribution in the transition region between Poisson and WD distributions is provided by the Brody interpolation formula [48],

$$p^{(\text{Brody})}_\beta(\theta) = \gamma (\beta + 1) \exp\left(-\gamma \theta^{\beta+1}\right), \quad (49)$$

where $\gamma = \left\{ \Gamma \left[ \frac{\beta+2}{\beta+1} \right] \right\}^{\beta+1}$ and $\Gamma(\beta)$ is the Euler Gamma function. This distribution is normalized and, by construction, has mean spacing $\langle \theta \rangle = 1$. We recover the Poisson case by taking $\beta = 0$ while the Wigner case is recovered for $\beta = 1$. However, a criticism of the Brody distribution is the lack of a first principles justification for its validity. The fact remains that it does fit the specific results found when considering explicit model systems. It is essentially an ad hoc one-parameter family of distributions and has no deep physical background, but it does interpolate between Poisson and Wigner-Dyson in a simple, effective manner. Our objective here is to apply our information-geometric formalism (based on statistical inference methods) to Wigner-Dyson and Poisson probability distributions.

Most of the probability distributions arise from the maximum entropy formalism as a result of some simple statements concerning averages. Not all distribution are generated in this way. Some distributions are generated by combining the results of simple cases (multinomial from a binomial). Other distributions are found as a result of a change of variable (Cauchy distribution). For instance, the Weibull distribution [49] can be obtained from an exponential distribution as a result of a power law transformation. Assume our knowledge of the microstate $x$ is encoded in an exponential distribution,

$$p(x|\theta) = \frac{1}{\theta} e^{-x^\theta}, \quad (50)$$
where \( x \) may be considered the spacing of the energy levels while \( \theta \) is the average spacing, \( \theta = \langle x \rangle \). Note that the study of probability distributions could, in principle, be restricted to the exponential type since an arbitrary distribution can be represented in exponential form. It is said that the exponential family of distributions is dense in the totality of probability distributions \([50]\). We can re-express \( x \in X \) in \( p(x|\theta) \) in terms of another random variable \( y = f(x) \in Y \), assuming \( f \) is an invertible mapping. For instance, consider the power law transformation

\[
x \rightarrow y = f(x) = \left( \frac{x}{\zeta} \right)^\beta.
\]  

(51)

We clearly have,

\[
p_{\text{old}}(x) \rightarrow \hat{p}_{\text{new}}(y) = \int_x dp_{\text{old}}(x) \delta(y - f(x))
\]

\[
= \int_x dp_{\text{old}}(x) \left[ \frac{1}{|\frac{\partial f}{\partial x}|} \right] \delta(f^{-1}(y) - x)
\]

\[
= \left[ \frac{1}{|\frac{\partial f}{\partial x}|} \right] p_{\text{old}}(x) \bigg|_{x=f^{-1}(y)}.
\]

(52)

Therefore, considering (50) and (51), equation (52) leads to

\[
\hat{p}_{\text{new}}(y) = n \zeta e^{-y^\beta} y^{n-1}.
\]

(53)

It is worthwhile emphasizing that since \( |\frac{\partial f}{\partial x}| \) does not depend on \( \theta \) and since \( \int_y dy = \int_x \frac{dx}{|\frac{\partial f}{\partial x}|} \), we have

\[
\int_y dy \hat{p}_{\text{new}}(y) \partial_\mu \log \hat{p}_{\text{new}}(y) \partial_\nu \log \hat{p}_{\text{new}}(y) = \int_x dp_{\text{old}}(x) \partial_\mu \log p_{\text{old}}(x) \partial_\nu \log p_{\text{old}}(x).
\]

(54)

Equation (54) leads to conclude that the Fisher-Rao information metric \( g_{\mu\nu} \) is invariant under transformations of the random variable. For the sake of completeness, let us show that the information metric is also covariant under reparametrization. Suppose that \( \left( \hat{\theta}_\mu \right) \) is a new set of coordinates, specified in terms of the old set through the invertible relationship \( \hat{\theta}_\mu = \hat{\theta}_\mu (\theta) \). Defining \( \hat{p}_\mu (x) \equiv p_{\mu(\hat{\theta})}(x) \), we are then able to compute the new metric tensor \( \hat{g}_{\mu\nu} \left( \hat{\theta} \right) \) in terms of \( g_{\mu\nu} \left( \theta \right) \). Indeed, since \( \frac{\partial}{\partial \theta^\mu} \hat{p}_\mu = \frac{\partial}{\partial \hat{\theta}^\mu} p_{\mu(\hat{\theta})} \), we obtain

\[
\hat{g}_{\mu\nu} \left( \hat{\theta} \right) = \left[ \frac{\partial \hat{\theta}^\rho \partial \hat{\theta}^\sigma}{\partial \theta^\mu \partial \theta^\nu} g_{\rho\sigma} \left( \theta \right) \right]_{\theta = \hat{\theta} \left( \hat{\theta} \right)}.
\]

(55)

Letting \( \hat{\theta} = \frac{\hat{\theta}}{\Lambda^\nu} \), from (54) we obtain the Weibull probability distribution,

\[
p_{\text{Weibull}}(y|\Lambda) = \frac{n}{\Lambda} \left( \frac{y}{\Lambda} \right)^{n-1} e^{-\left( \frac{y}{\Lambda} \right)^n}, \quad \Lambda = \left( \frac{\theta}{\zeta} \right)^\frac{1}{\beta}.
\]

(56)

Moreover, letting \( n = 2, y = \Delta \) and \( \Lambda = \frac{2D}{\sqrt{\Delta}} \), from (50) we obtain the standard Wigner-Dyson distribution,

\[
p_{\text{Wigner-Dyson}}(\Delta|D) = \frac{\pi \Delta}{2D^2} e^{-\frac{\pi \Delta^2}{2D^2}}, \quad D = \sqrt{\frac{\pi}{2}} \left( \frac{\theta}{\zeta} \right)^\frac{1}{\beta}.
\]

(57)

In conventional notations, \( \Delta \) is the spacing between two neighboring energy levels and \( D \) is the average spacing \([51]\). Recall that the Fisher-Rao information metric \( G^{(P)}_{\mu\nu} (\theta) \) of a Poissonian probability distribution \( p(x|\theta) \) is defined as,

\[
G^{(P)}_{\mu\nu} (\theta) = \int dx p(x|\theta) \partial_\mu \log p(x|\theta) \partial_\nu \log p(x|\theta) \quad \text{with} \quad \partial_\mu = \frac{\partial}{\partial \theta^\mu}.
\]

(58)

where \( p(x|\theta) \) is given by,

\[
p(x|\theta) = \frac{1}{\theta} \exp \left( -\frac{x}{\theta} \right).
\]

(59)
The Poisson line element \((ds^2)_{\text{Poisson}}\) is defined as,
\[
(ds^2)_{\text{Poisson}} = G^{(P)}_{\mu \nu} (\theta) \, d\theta^\mu d\theta^\nu = \frac{1}{\theta^2} d\theta^2. \tag{60}
\]
The Fisher-Rao information metric \(G^{(\text{WD})}_{\mu \nu} (\phi)\) of a Wigner-Dyson probability distribution \(q(y|\phi)\) is defined as,
\[
G^{(\text{WD})}_{\mu \nu} (\phi) = \int dy \, q(y|\phi) \partial_\mu \log q(y|\phi) \partial_\nu \log q(y|\phi) \text{ with } \partial_\mu = \frac{\partial}{\partial \phi^\mu}, \tag{61}
\]
where \(q(y|\phi)\) is given by,
\[
q(y|\phi) = \frac{\pi y}{2 \phi^2} \exp\left(-\frac{\pi y^2}{4 \phi^2}\right), \quad \phi = \frac{\sqrt{\pi}}{2} \left(\frac{\theta}{\lambda}\right)^{\frac{1}{2}}. \tag{62}
\]
Notice that \(q(y|\phi)\) in (62) is equivalent to \(p_{\text{Wigner-Dyson}}(\Delta|D)\) in (57) with \(y = \Delta\) and \(\phi = D\). The Wigner-Dyson line element \((ds^2)_{\text{Wigner-Dyson}}\) is defined as,
\[
(ds^2)_{\text{Wigner-Dyson}} = G^{(\text{WD})}_{\mu \nu} (\phi) \, d\phi^\mu d\phi^\nu. \tag{63}
\]
Notice that the Poisson distribution and the Wigner-Dyson distributions are related through the combination of a change of random variable and a new reparametrization, namely
\[
q(y|\phi) = p(x(y)|\theta(\phi)) \, J(y) \tag{64}
\]
where,
\[
x(y) = \lambda y^2, \quad \theta(\phi) = \frac{4 \phi^2}{\pi} \lambda, \quad J(y) = \left| \frac{\partial x(y)}{\partial y} \right|. \tag{65}
\]
Considering equations (54) and (55), the Wigner-Dyson line element \((ds^2)_{\text{Wigner-Dyson}}\) becomes
\[
(ds^2)_{\text{Wigner-Dyson}} = G^{(\text{WD})}_{\mu \nu} (\phi) \, d\phi^\mu d\phi^\nu = G^{(P)}_{\mu \nu} (\theta(\phi)) \, d\theta(\phi)^\mu d\theta(\phi)^\nu = \frac{4}{\phi^2} d\phi^2. \tag{66}
\]
Equations (60) and (66) will be used in our IGAC in quantum spin chains systems. Before considering such information-geometric characterization of quantum energy level statistics for regular and chaotic spin chains immersed in an external magnetic field, we briefly review the main points of the more standard approach to these topics.

### B. Entanglement in quantum spin chains: standard formalism

One of the most important concepts in quantum information theory is that of entanglement, an intrinsic property of composite quantum systems. Entanglement plays an essential role in many-body quantum phenomena, such as superconductivity \[52\] and quantum phase transitions \[53\]. Moreover, it is an important concept in quantum computation and information processing \[54\]. An excellent theoretical framework for investigating entanglement properties is offered by spin chains. Quantum spin chains belong to the most studied models of quantum statistical mechanics. However, only for a few types of models have the thermal and ground state structures been determined. This is mainly a consequence of the complicated correlations that can arise among quantum states. These strong correlations can even be present in pure quantum states, while classical pure states can only have a trivial product state structure. Unlike the classical case, the restrictions of pure states on the quantum spin chain to local subsystems are typically mixed states. This type of correlation between subsystems is commonly referred to as entanglement. The von Neumann entropy, defined as,
\[
S_{\text{von Neumann}} = -tr(\rho \log \rho), \tag{67}
\]
is a standard measure of the nonpurity of the reduced density matrix \(\rho\), thus it is a very useful quantity in the description of entanglement \[55\]. Several simple models of spin chains can be studied analytically and there also exist
efficient numerical techniques. There are two widely used methods of characterizing entanglement in spin chains. The first of these describes the entanglement between two spins in the chain with a quantity called concurrence [50]. The other one measures entanglement of a block of spins with the rest of the chain with the von Neumann entropy when the chain is in its ground state [57]. As a side remark, we emphasize that entanglement entropy does not refer exclusively to the characterization of quantum systems in their ground states, but, more generally, it refers to any many particles quantum dynamical states undergoing a unitary time evolution. The method used in [57] is known as the density matrix renormalization group method (DMRG) [58]. It is based on the fact that many degrees of freedom are redundant in quantum state description; therefore, the system is adequately described by taking into account maximally entangled components only. von Neumann entropy is supposed to play an important role in quantifying the essential subspace of a reduced density matrix. The possibility of compressing such density matrices from its full dimension to a much smaller subspace without significant loss of information is the starting point of the DMRG analysis. Classical complexity of quantum states can be characterized by a mixed state entanglement entropy. For instance, the von Neumann entropy of a block of \( L \) neighboring spins in a XX chain, describing entanglement of the block with the rest of the chain is given by [59],

\[
S_L = -tr (\rho_L \log \rho_L) \propto L \log L.
\]

The reduced density matrix \( \rho_L \) is obtained from the ground state \( |\Psi_g\rangle \) of the chain by tracing out external degrees of freedom,

\[
\rho_L = tr_{N-L} |\Psi_g\rangle \langle \Psi_g|, \quad H |\Psi_g\rangle = E_g |\Psi_g\rangle.
\]

The Hamiltonian \( H \) in (69) is given by [57, 59],

\[
H = -\sum_{l=1}^{N} (s_l^x s_{l+1}^x + s_l^y s_{l+1}^y) - h \sum_{l=1}^{N} s_l^z,
\]

where \( s_l^\alpha (\alpha = x, y, z) \) are the Pauli spin matrices at sites \( l = 1, 2, ..., N \) of a periodic chain and \( h \) is the magnetic field. The logarithmic growth of the entanglement entropy is a general consequence of the fact that in one dimensional systems near quantum phase transitions, the entropy is a logarithmic function of the size of the system [60]. Furthermore, there is also a time dependent version of this DMRG, known as \( \tau \)-DMRG [61]. This method is used to study the evolution of pure states, density matrices and operators. Note that the classical complexity of quantum operators can be characterized using the operator space entanglement entropy of a density operator, not the state entanglement entropy of a mixed state. For the evolution of density matrices and operators, a superket corresponding to an operator \( O \) expanded in the computational basis of products of local operators is considered. For instance, for a chain of \( n \)-qubits, a basis of \( 4^n \) Pauli operators is used, \( \sigma^0 \otimes ... \otimes \sigma^{n-1} \), with \( s_j \in \{0, x, y, z\} \) and \( \sigma^0 = I \). The key idea of \( \tau \)-DMRG is to represent any operator in a matrix product form [62],

\[
O = \sum_{s_j} tr (A_0^{s_0} ... A_{n-1}^{s_{n-1}}) \sigma^{s_0} \otimes ... \otimes \sigma^{s_{n-1}}
\]

in terms of the \( 4n \) matrices \( A_j^{s_j} \) of fixed dimension \( D \). The number of parameters in the matrix product (MPO) representation of the operator is \( 4nD^2 \). The minimal \( D \) required, \( D_c (\tau) \), is equal to the maximal rank of the reduced super density matrix over bipartitions of the chain. The way entropy can be computed in the spaces of operators can be found in [62]. The \( \tau \)-DMRG method is very efficient in classical simulations of many body quantum dynamics requiring that the computational costs grow polynomially in time and, consequently, that the entanglement entropy grows no faster than logarithmically. However, it is known that the asymptotic behavior of computational costs and entanglement entropies of integrable and chaotic Ising spin chains are very different [17]. Here Prosen considered the question of time efficiency implementing an up-to-date version of the \( \tau \)-DMRG for a family of Ising spin \( 1/2 \) chains in arbitrary oriented magnetic field, which undergoes a transition from integrable (transverse Ising) to nonintegrable chaotic regime as the magnetic field is varied. An integrable (regular) Ising chain in a general homogeneous transverse magnetic field is defined through the Hamiltonian \( H(\text{regular}) \equiv H(0, 2) \), where

\[
H (h_x, h_y) = \sum_{j=0}^{n-2} \sigma^x_{j+1} \sigma^x_j + \sum_{j=0}^{n-1} (h^x \sigma^x_j + h^y \sigma^y_j).
\]

In this case, the computational cost shows a polynomial growth in time, \( D_c(\text{regular}) (\tau) \propto \tau^c \), while the entanglement entropy is characterized by logarithmic growth,

\[
S(\text{regular}) \propto c \log \tau + c'.
\]
The quantity $K$ represents initial state observable or error measures, and can be interpreted as a kind of quantum dynamical entropy.

Instead, a quantum chaotic Ising chain in a general homogeneous tilted magnetic field is defined through the Hamiltonian $H_{\text{chaotic}} \equiv H(1, 1)$, where $H$ is defined in (72). In this case, the computational cost shows an exponential growth in time, $D_{x}^{(\text{chaotic})}(\tau) \sim \exp(K_{q} \tau)$ while the entanglement entropy is characterized by linear growth,

$$S^{(\text{chaotic})} \sim K_{q} \tau.$$  \hspace{1cm} (74)

The quantity $K_{q}$ is a constant, asymptotically independent of the number of indexes of the initial local operators used to calculate the operator space entropy, that depends only on the Hamiltonian evolution and not on the details of the initial state observable or error measures, and can be interpreted as a kind of quantum dynamical entropy.

It is well known the quantum description of chaos is characterized by a radical change in the statistics of quantum energy levels [63]. The transition to chaos in the classical case is associated with a drastic change in the statistics of the nearest-neighbor spacings of quantum energy levels. In the regular regime the distribution agrees with the Poisson statistics while in the chaotic regime the Wigner-Dyson distribution works very well. Uncorrelated energy levels are characteristic of quantum systems corresponding to a classically regular motion while a level repulsion (a suppression of small energy level spacing) is typical for systems which are classically chaotic. A standard quantum example is provided by the study of energy level statistics of an Hydrogen atom in a strong magnetic field. It is known that level spacing distribution (LSD) is a standard indicator of quantum chaos [64]. It displays characteristic level suppression of small energy level spacing) is typical for systems which are classically chaotic. A standard quantum levels are characteristic of quantum systems corresponding to a classically regular motion while a level repulsion (a Poisson statistics while in the chaotic regime the Wigner-Dyson distribution works very well. Uncorrelated energy levels are characteristic of quantum systems corresponding to a classically regular motion while a level repulsion (a suppression of small energy level spacing) is typical for systems which are classically chaotic.

The transition to chaos in the classical case is associated with a drastic change in the statistics of the nearest-neighbor spacings of quantum energy levels. In the regular regime the distribution agrees with the Poisson statistics while in the chaotic regime the Wigner-Dyson distribution works very well. Uncorrelated energy levels are characteristic of quantum systems corresponding to a classically regular motion while a level repulsion (a suppression of small energy level spacing) is typical for systems which are classically chaotic. A standard quantum example is provided by the study of energy level statistics of an Hydrogen atom in a strong magnetic field. It is known that level spacing distribution (LSD) is a standard indicator of quantum chaos [64]. It displays characteristic level suppression of small energy level spacing) is typical for systems which are classically chaotic. A standard quantum levels are characteristic of quantum systems corresponding to a classically regular motion while a level repulsion (a Poisson statistics while in the chaotic regime the Wigner-Dyson distribution works very well. Uncorrelated energy levels are characteristic of quantum systems corresponding to a classically regular motion while a level repulsion (a suppression of small energy level spacing) is typical for systems which are classically chaotic.

In the next paragraph, we will encode the relevant information about the spin-chain in a suitable composite-probability distribution taking into account the quantum spin chains and the configurations of the external magnetic field in which they are immersed.

VI. AN INFORMATION GEOMETRIC MODEL OF REGULAR AND CHAOTIC QUANTUM SPIN CHAINS

A. Integrable Statistical Model: Poisson coupled to an Exponential Bath

Recall that in the ME method [7], the selection of relevant variables is made on the basis of intuition guided by experiment; it is essentially a matter of trial and error. The variables should include those that can be controlled or experimentally observed, but there are cases where others must also be considered. Our objective here is to choose the relevant microvariables of the system and select the relevant information concerning each one of them. In the integrable case, the Hamiltonian $H_{\text{regular}}$ describes an antiferromagnetic Ising chain immersed in a transverse magnetic field $B_{\text{transverse}} = B_{1, 1}$ with the level spacing distribution of its spectrum given by the Poisson distribution

$$p_{A}^{(\text{Poisson})}(x_{A}|\mu_{A}) = \frac{1}{\mu_{A}} \exp \left( -\frac{x_{A}}{\mu_{A}} \right),$$  \hspace{1cm} (75)

where the microvariable $x_{A}$ is the spacing of the energy levels and the macrovariable $\mu_{A}$ is the average spacing. The chain is immersed in the transverse magnetic field which has just one component $B_{1}$ in the Hamiltonian $H_{\text{regular}}$. We translate this piece of information in our IGA formalism, coupling the probability (75) to an exponential bath $p_{B}^{(\text{exponential})}(x_{B}|\mu_{B})$ given by

$$p_{B}^{(\text{exponential})}(x_{B}|\mu_{B}) = \frac{1}{\mu_{B}} \exp \left( -\frac{x_{B}}{\mu_{B}} \right),$$  \hspace{1cm} (76)

where the microvariable $x_{B}$ is the intensity of the magnetic field and the macrovariable $\mu_{B}$ is the average intensity. More correctly, $x_{B}$ should be the energy arising from the interaction of the magnetic field with the spin $\frac{1}{2}$ particle magnetic moment, $x_{B} = -\mu \cdot \vec{B} = -\mu B \cos \varphi$ where $\varphi$ is the tilt angle. For the sake of simplicity, let us set $\mu = 1$, then in the transverse case $\varphi = 0$ and therefore $x_{B} = B \equiv B_{1}$. This is our best guess and we justify it by
noticing that the magnetic field intensity is indeed a relevant quantity in this experiment (see equation (73)) and its components (intensity) are quantities that are varied during the transitions from integrable to chaotic regimes. In the regular regime, we say the magnetic field intensity is set to a well-defined value $\langle x_B \rangle = \mu_B$. Furthermore, notice that the exponential distribution is identified by information theory as the maximum entropy distribution if only one piece of information (the expectation value) is known. Finally, the chosen composite probability distribution $P^{\text{integrable}}(x_A, x_B| \mu_A, \mu_B)$ encoding relevant information about the system is given by,

$$P^{\text{integrable}}(x_A, x_B| \mu_A, \mu_B) = \frac{1}{\mu_A \mu_B} \exp \left( - \frac{x_A + x_B}{\mu_A + \mu_B} \right). \quad (77)$$

Again, we point out that our probability (77) is our best guess and, of course, must be consistent with numerical simulations and experimental data in order to have some merit. We point out that equation (77) is not fully justified from a theoretical point of view, a situation that occurs due to the lack of a systematic way to select the relevant microvariables of the system (and to choose the appropriate information about such microvariables). Let us denote $\mathcal{M}_S^{\text{integrable}}$ the two-dimensional curved statistical manifold underlying our information geometrodynamics. The line element $(ds^2)_{\text{integrable}}$ on $\mathcal{M}_S^{\text{integrable}}$ is given by,

$$(ds^2)_{\text{integrable}} = \frac{1}{\mu_A^2} d\mu_A^2 + \frac{1}{\mu_B^2} d\mu_B^2. \quad (78)$$

Applying our IGAC to the line element in (75) and following the steps provided in the ED Gaussian model of Sections II and III of this paper, we obtain polynomial growth in $\nu_{\mathcal{M}_S}^{\text{integrable}}$ and logarithmic IGE growth,

$$\nu_{\mathcal{M}_S}^{\text{integrable}}(\tau) \sim \infty \exp(c_{IG} \tau), \quad S_{\mathcal{M}_S}^{\text{integrable}}(\tau) \sim \infty \ c_{IG} \log \tau + c_{IG}'. \quad (79)$$

The quantity $c_{IG}$ is a constant proportional to the number of exponential probability distributions in the composite distribution used to calculate the IGE and $c_{IG}'$ is a constant that depends on the values assumed by the statistical macrovariables $\mu_A$ and $\mu_B$. Equations (79) may be interpreted as the information-geometric analogue of the computational complexity $D_\varepsilon^{\text{regular}}(\tau)$ and the entanglement entropy $S^{\text{regular}}$ defined in standard quantum information theory, respectively. We cannot state they are the same since we are not fully justifying, from a theoretical standpoint, our choice of the composite probability (77).

\section*{B. Chaotic Statistical Model: Wigner-Dyson coupled to a Gaussian Bath}

In the chaotic case, the Hamiltonian $\mathcal{H}^{\text{chaotic}}$ describes an antiferromagnetic Ising chain immersed in a tilted homogeneous magnetic field $\vec{B}_{\text{tilted}} = B_\perp \vec{e}_\perp + B_\parallel \vec{e}_\parallel$ with the level spacing distribution of its spectrum given by the Poisson distribution $p^{\text{Wigner-Dyson}}_A(x'_A | \mu'_A)$,

$$p^{\text{Wigner-Dyson}}_A(x'_A | \mu'_A) = \frac{\pi x'_A}{2 \mu'_A^2} \exp \left( - \frac{\pi x'_A^2}{4 \mu'_A^2} \right). \quad (80)$$

where the microvariable $x'_A$ is the spacing of the energy levels and the macrovariable $\mu'_A$ is the average spacing. The chain is immersed in the tilted magnetic vector field which has two components $B_\perp$ and $B_\parallel$ in the Hamiltonian $\mathcal{H}^{\text{chaotic}}$. We translate this piece of information in our IGAC formalism, coupling the probability (80) to a Gaussian $p^{\text{Gaussian}}_B(x_B | \mu'_B, \sigma'_B)$ given by,

$$p^{\text{Gaussian}}_B(x_B | \mu'_B, \sigma'_B) = \frac{1}{\sqrt{2\pi \sigma'_B^2}} \exp \left( - \frac{(x_B - \mu'_B)^2}{2 \sigma'_B^2} \right). \quad (81)$$

where the microvariable $x'_B$ is the intensity of the magnetic field, the macrovariable $\mu'_B$ is the average intensity, and $\sigma'_B$ is its covariance: during the transition from the integrable to the chaotic regime, the magnetic field intensity is being varied (experimentally). It is being tilted and its two components ($B_\perp$ and $B_\parallel$) are being varied as well. Our best guess based on the experimental mechanism that drives the transitions between the two regimes is that
magnetic field intensity ( actually the microvariable $\mu B \cos \varphi$) is Gaussian-distributed (two macrovariables) during this change. In the chaotic regime, we say the magnetic field intensity is set to a well-defined value $\langle x'_B \rangle = \mu'_B$ with covariance $\sigma_B = \sqrt{\langle (x'_B - \langle x'_B \rangle)^2 \rangle}$. Furthermore, the Gaussian distribution is identified by information theory as the maximum entropy distribution if only the expectation value and the variance are known. Therefore, the chosen composite probability distribution $P^{(\text{chaotic})}(x'_A, x'_B | \mu'_A, \mu'_B, \sigma'_B)$ encoding relevant information about the system is given by,

$$P^{(\text{chaotic})}(x'_A, x'_B | \mu'_A, \mu'_B, \sigma'_B) = P^{(\text{Wigner-Dyson})}_A(x'_A | \mu'_A) \cdot P^{(\text{Gaussian})}_B(x'_B | \mu'_B, \sigma'_B) = \frac{\pi (2\pi \sigma_B^2)^{-\frac{1}{2}}}{2 \mu'_B^2} x'_A \exp \left[ - \left( \frac{\pi x'_A^2}{4 \mu'_B^2} + \frac{(x'_B - \mu'_B)^2}{2 \sigma_B^2} \right) \right].$$

(82)

Let us denote $\mathcal{M}^{(\text{chaotic})}_S$ the three-dimensional curved statistical manifold underlying our information geometrodynamics. The line element $(ds^2)_{\text{chaotic}}$ on $\mathcal{M}^{(\text{chaotic})}_S$ is given by,

$$(ds^2)_{\text{chaotic}} = \frac{4}{\mu'_A^2} d\mu'_A^2 + \frac{1}{\sigma'_B^2} d\sigma'_B^2 + \frac{2}{\sigma'_B^2} \frac{d\sigma'_B}{\mu'_B}.$$

(83)

Applying our IGAC to the line element in (83) and following the steps provided in the ED Gaussian model of Sections II and III of this paper, we obtain exponential growth in $V^{(\text{chaotic})}_{\mathcal{M}_s}$ and linear IGE growth,

$$V^{(\text{chaotic})}_{\mathcal{M}_s} (\tau) \xrightarrow{\tau \to \infty} C_{IG} \exp(\mathcal{K}_{IG} \tau), \quad S^{(\text{chaotic})}_{\mathcal{M}_s} (\tau) \xrightarrow{\tau \to \infty} \mathcal{K}_{IG} \tau.$$

(84)

The constant $C_{IG}$ encodes information about the initial conditions of the statistical macrovariables parametrizing elements of $\mathcal{M}^{(\text{chaotic})}_S$. The constant $\mathcal{K}_{IG}$,

$$\mathcal{K}_{IG} \xrightarrow{\tau \to \infty} \lim \frac{dS_{\mathcal{M}_s}(\tau)}{d\tau} = \frac{1}{\tau} \log \left( \frac{\| J_{\mathcal{M}_s} (\tau) \|}{\| J_{\mathcal{M}_s} (0) \|} \right) \overset{\text{def}}{=} \lambda_J,$$

(85)

is the model parameter of the chaotic system and depends on the temporal evolution of the statistical macrovariables.

One of the major limitations of our findings is the lack of a detailed account of the comparison of the theory with experiment. This point will be one of our primary concerns in future works. However, some considerations may be carried out at the present stage. The experimental observables in our theoretical models are the statistical macrovariables characterizing the composite probability distributions. In the integrable case, where the coupling between a Poisson distribution and an exponential one is considered, $\mu_A$ and $\mu_B$ are the experimental observables. In the chaotic case, where the coupling between a Wigner-Dyson distribution and a Gaussian is considered, $\mu'_A$ and $\mu'_B$ play the role of the experimental observables. We believe one way to test our theory may be that of determining a numerical estimate of the leading Lyapunov exponent $\lambda_{\text{max}}$ or the Lyapunov spectrum for the Hamiltonian systems under investigation directly from experimental data (measurement of a time series) and compare it to our theoretical estimate for $\lambda_J$. However, we are aware that it may be extremely hard to evaluate Lyapunov exponents numerically. Otherwise, knowing that the mean values of the positive Lyapunov exponents are related to the Kolmogorov-Sinai (KS) dynamical entropy, we suggest to measure the KS entropy $\mathcal{K}$ directly from a time signal associated to a suitable combination of our experimental observables and compare it to our indirect theoretical estimate for $\mathcal{K}_{IG}$ from the asymptotic behaviors of our statistical macrovariables. We are aware that the ground of our discussion is quite qualitative. However, we hope that with additional study, especially in clarifying the relation between the IGE and the entanglement entropy, our theoretical characterization presented in this paper will find experimental support in the future. Therefore, the statement that our findings may be relevant to experiments verifying the existence of chaoticity and related dynamical properties is known at a macroscopic level in energy level statistics in chaotic and regular quantum spin chains is purely a conjecture at this stage.
VII. FINAL REMARKS

In this paper, we reviewed our novel information-geometrodynamical approach to chaos (IGAC) on curved statistical manifolds and we emphasized the usefulness of our information-geometrodynamical entropy (IGE) as an indicator of chaoticity in a simple application. Furthermore, knowing that integrable and chaotic quantum antiferromagnetic Ising chains are characterized by asymptotic logarithmic and linear growths of their operator space entanglement entropies, respectively, we applied our IGAC to present an alternative characterization of such systems. Remarkably, we have shown that in the former case the IGE exhibits asymptotic logarithmic growth while in the latter case the IGE exhibits asymptotic linear growth.

It is worthwhile emphasizing the following points: the statements that spectral correlations of classically integrable systems are well described by Poisson statistics and that quantum spectra of classically chaotic systems are universally correlated according to Wigner-Dyson statistics are conjectures, known as the BGS (Bohigas-Giannoni-Schmit, \[68\]) and BTG (Berry-Tabor-Gutzwiller, \[69\]) conjectures, respectively. These two conjectures are very important in the study of quantum chaos, however their validity finds some exceptions. Several other cases may be considered. For instance, chaotic systems having a spectrum that does not obey a Wigner-Dyson distribution may be considered. A chaotic system can also have a spectrum following a Poisson, semi-Poisson, or other types of critical statistics \[70\]. Moreover, integrable systems having a spectrum that does not obey a Poisson distribution may be considered as well. For instance, the Harper model would represent such a situation. Moreover, it is worthwhile pointing out that not every chaotic system characterized by entropy-like quantities growing linearly in time has a spectrum described by a Wigner-Dyson distribution. Well-known examples presenting such a situation are the cat maps \[71\] and the famous kicked rotator \[72\] where its spectrum follows a Poisson distribution in cylinder representation and a Wigner-Dyson in torus representation but the properties of entropy-like quantities are the same in both representations (at least classically). All these cases are not discussed in our characterization.

Therefore, at present stage, because of the above considerations and because of the lack of experimental evidence in support of our theoretical construct, we can only conclude that the IGAC might find some potential applications in certain regular and chaotic dynamical systems and this remains only a conjecture. However, we hope that our work convincingly shows that this information-geometric approach may be considered a serious effort trying to provide a unifying criterion of chaos of both classical and quantum varieties, thus deserving further research.

Acknowledgments

The authors are grateful to Prof. Ariel Caticha and Dr. Adom Giffin for useful comments. We thank an anonymous Referee for constructive criticism that lead to concrete improvement of this work.

[1] L. Casetti, C. Clementi, and M. Pettini, "Riemannian theory of Hamiltonian chaos and Lyapunov exponents", Phys. Rev. E 54, 5969-5984 (1996).
[2] M. Di Bari and P. Cipriani, "Geometry and Chaos on Riemann and Finsler Manifolds", Planet. Space Sci. 46, 1543 (1998).
[3] T. Kawabe, "Indicator of chaos based on the Riemannian geometric approach", Phys. Rev. E71, 017201 (2005); T. Kawabe, "Chaos based on Riemannian geometric approach to Abelian-Higgs dynamical system", Phys. Rev. E67, 016201 (2003).
[4] W. H. Zurek and J. P. Paz, "Decoherence, Chaos, and the Second Law", Phys. Rev. Lett. 72, 2508 (1994); "Quantum Chaos: a decoherent definition", Physica D83, 300 (1995).
[5] C. M. Caves and R. Schack, "Unpredictability, Information, and Chaos", Complexity 3, 46-57 (1997); A. J. Scott, T. A. Brun, C. M. Caves, and R. Schack, "Hypersensitivity and chaos signatures in the quantum baker's map", J. Phys. A39, 13405 (2006).
[6] A. Caticha, "Entropic Dynamics", in Bayesian Inference and Maximum Entropy Methods in Science and Engineering, ed. by R.L. Fry, AIP Conf. Proc. 617, 302 (2002).
[7] A. Caticha, "Relative Entropy and Inductive Inference", Bayesian Inference and Maximum Entropy Methods in Science and Engineering, ed. by G. Erickson and Y. Zhai, AIP Conf. Proc. 707, 75 (2004); A. Caticha and A. Giffin, "Updating Probabilities", in Bayesian Inference and Maximum Entropy Methods in Science and Engineering, ed. by Ali Mohammad-Djafari, AIP Conf. Proc. 872, 31-42 (2006); A. Caticha and R. Preuss, "Maximum entropy and Bayesian data analysis: Entropic prior distributions", Phys. Rev. E70, 046127 (2004).
[8] S. Amari and H. Nagaoka, Methods of Information Geometry, American Mathematical Society, Oxford University Press, 2000; S. Amari, Differential-Geometrical Methods in Statistics, Springer-Verlag (1985).
[9] C. Cafaro, S. A. Ali and A. Giffin, "An Application of Reversible Entropic Dynamics on Curved Statistical Manifolds", in Bayesian Inference and Maximum Entropy Methods in Science and Engineering, ed. by Ali Mohammad-Djafari, AIP Conf. Proc. 872, 243-251 (2006).
C. Cafaro, "Information Geometry and Chaos on Negatively Curved Statistical Manifolds", Physica D234, 70-80 (2007).

C. Cafaro, "Information Geometry and Chaos on Negatively Curved Statistical Manifolds", in Bayesian Inference and Maximum Entropy Methods in Science and Engineering, ed. by K. Knuth, et al., AIP Conf. Proc. 954, 175 (2007).

A. Caticha and C. Cafaro, "From Information Geometry to Newtonian Dynamics", in Bayesian Inference and Maximum Entropy Methods in Science and Engineering, ed. by K. Knuth, et al., AIP Conf. Proc. 954, 165 (2007).

C. Cafaro, "Works on an Information Geometrodynamical Approach to Chaos", DOI: 10.1016/j.chaos.2008.04.017, Chaos, Solitons & Fractals (2008).

C. Cafaro, "Information-Geometric Indicators of Chaos in Gaussian Models on Statistical Manifolds of Negative Ricci Curvature", DOI: 10.1007/s10773-008-9726-x, Int. J. Theor. Phys. (2008).

C. Cafaro, "Information Geometry, Inference Methods and Chaotic Energy Levels Statistics", accepted for publication in Mod. Phys. Lett. B (2008).

C. Cafaro and S. A. Ali, "Geometrodynamics of Information on Curved Statistical Manifolds and its Applications to Chaos", EJTP 5, 139-162 (2008).

T. Prosen and M. Znidaric, "Is the efficiency of classical simulations of quantum dynamics related to integrability?", Phys. Rev. E75, 015202 (2007); T. Prosen and I. Pizorn, "Operator space entanglement entropy in transverse Ising chain", Phys. Rev. A76, 032316 (2007).

C. G. J. Jacobi, "Vorlesungen uber Dynamik", Reimer, Berlin (1866).

R.A. Fisher, "Theory of statistical estimation", Proc. Cambridge Philos. Soc. 122, 700 (1925).

C.R. Rao, "Information and accuracy attainable in the estimation of statistical parameters", Bull. Calcutta Math. Soc. 37, 81 (1945).

E. T. Jaynes, "Probability Theory: The Logic of Science", Cambridge University Press (2003).

S. I. Goldberg, "Curvature and Homology", Academic Press Inc. (1962).

F. De Felice and J. S. Clarke, "Relativity on Curved Manifolds", Cambridge University Press (1990); M. P. do Carmo, Riemannian Geometry, Birkhauser, Boston, 1992.

N. S. Krylov, "Works on the Foundations of Statistical Physics", Princeton University Press, Princeton, 1979.

M. Pellicott, "Exponential mixing for the Geodesic Flow on Hyperbolic Three-Manifolds, Journal of Statistical Physics 67, 667 (1992).

J. Jost, "Compact Riemann Surfaces: An Introduction to Contemporary Mathematics", Springer-Verlag (1997).

M. Biesiada, "The Power of the Maupertuis-Jacobi Principle- Dreams and Reality", Chaos, Solitons & Fractals 5, 869 (1994).

M. Biesiada, "Searching for an invariant description of chaos in general relativity", Class. Quantum Grav. 12, 715 (1995).

C. Uggl, K. Rosquist and R. T. Jantzen, "Geometrizing the dynamics of Bianchi cosmology", Phys. Rev. D42, 404 (1990).

V.I. Arnold, Mathematical Methods of Classical Physics, Springer-Verlag, 1989.

J. M. Lee, "Riemannian Manifolds: An Introduction to Curvature", Springer Verlag (1997).

M. P. do Carmo, Riemannian Geometry, Birkhauser, Boston, 1992.

H. C. Ohanian and R. Ruffini, "Gravitation and Spacetime", W.W. Norton & Company (1994).

F. De Felice and J. S. Clarke, "Relativity on curved manifolds", Cambridge University Press (1990).

C. Chicone and B. Mashhoon, "The generalized Jacobi equation", Class. Quantum Grav. 19 4231-4248 (2002).

D. E. Hodgkinson, "A modified equation of geodesic deviation", Gen. Rel. Grav. 3, 351 (1972).

T. Tel and M. Gruz, "Chaotic Dynamics: An Introduction Based on Classical Mechanics", Cambridge University Press (2006).

A. Wolf, "Quantifying chaos with Lyapunov exponents", in Chaos, ed. A. V. Holden, Princeton University Press, Princeton, pp. 273-290 (1986).

J. P. Eckmann and D. Ruelle, "Expanding modes of strange attractors", Rev. Mod. Phys. vol. 57, 617-656 (1985).

E. T. Jaynes, "Information theory and statistical mechanics, I", Phys. Rev. 106, 620 (1957); E. T. Jaynes, "Information theory and statistical mechanics, II", Phys. Rev. 108, 171 (1957).

S. Stenholm and K. Suominen, "Quantum Approach to Informatics", Wiley-Interscience (2005).

W. H. Zurek, Phys. Today 44, No. 10, 36 (1991); 46, No. 12, 81 (1993); Prog. Theor. Phys. 89, 281 (1993).

F. Benatti, "Deterministic Chaos in Infinite Quantum Systems", Springer-Verlag Berlin (1993); F. Benatti, "Classical and Quantum Entropies: Dynamics and Information", in Entropy edited by A. Greven et. al., Princeton Series in Applied Mathematics (2003).

R. Alicki and M. Fannes, "Defining Quantum Dynamical Entropy", Lett. Math. Phys. 32, 75-82 (1994); R. Alicki and M. Fannes, "Quantum Dynamical Systems", Oxford University Press (2001).

A. Connes et. al., "Dynamical Entropy of C * Algebras and von Neumann Algebras", Commun. Math. Phys. 112, 691-719 (1987).

D. P. Feldman and J. P. Crutchfield, "Measures of complexity: Why?", Phys. Lett. A238, 244-252 (1998); A. Manning, "Topological entropy for geodesic flows", Annals of Mathematics 110, 567-573 (1979).

C. E. Porter, "Statistical Theories of Spectra: Fluctuations", Academic Press, New York (1965); M. L. Mehta, "Random Matrices and the Statistical Theory of Energy Levels", Academic Press, New York (1991).

T. A. Brody et. al., "Random-matrix physics: spectrum and strength fluctuations", Rev. Mod. Phys. 53, 385 (1981); T. Prosen and M. Robnik, "Semiclassical energy level statistics in the transition region between integrability and chaos: transition from Brody-like to Berry-Robnik behavior", J. Phys. A27, 8059-8077 (1994); T. Prosen and M. Robnik, "Energy level statistics in the transition region between integrability and chaos", J. Phys. A26, 2371-2387 (1993).

M. Tribus, "Rational Descriptions, Decisions and Designs", Pergamon Press Inc., New York (1969).
[50] D. C. Brody, "Notes on exponential families of distributions", arXiv: cond-mat/0705.2173 (2007).
[51] T. S. Biro et al., "Chaos and Gauge Field Theory", World Scientific Publishing Co., Singapore (1994).
[52] M. Tinkham, "Introduction to Superconductivity", Mc.Graw-Hill, New York (1996).
[53] S. Sachdev, "Quantum Phase Transitions", Cambridge University Press, Cambridge (2001).
[54] M. A. Nielsen and I. L. Chuang, "Quantum Computation and Quantum Communication", Cambridge University Press, Cambridge (2000).
[55] C. H. Bennett et al., "Concentrating partial entanglement by local operations", Phys. Rev. A53, 2046 (1996).
[56] T. J. Osborne and M. A. Nielsen, "Entanglement in a simple quantum phase transition", Phys. Rev. A66, 032110 (2002).
[57] J. P. Keating and F. Mezzadri, "Random Matrix Theory and Entanglement in Quantum Spin Chains", Commun. Math. Phys. 252, 543 (2004).
[58] S. R. White, "Density Matrix Formulation for Quantum Renormalization Groups", Phys. Rev. Lett. 69, 2863 (1992).
[59] V. Eisler and Z. Zimboras, "Entanglement in the XX spin chain with an energy current", Phys. Rev. A71, 042318 (2005).
[60] P. Calabrese and J. Cardy, "Entanglement Entropy and Quantum Field Theory", J. Stat. Mech. Theor. Exp. P06002 (2004).
[61] S. R. White and A. E. Feiguin, "Real-Time Evolution using the Density Matrix Renormalization Group", Phys. Rev. Lett. 93, 076401 (2004); G. Vidal, "Efficient Classical Simulations of Slightly Entangled Quantum Computations", Phys. Rev. Lett. 91, 147902 (2003).
[62] Tomaz Prosen, "Chaos and complexity of quantum motion", J. Phys. A40, 7881-7918 (2007).
[63] G. Casati and B. Chirikov, Quantum Chaos, Cambridge University Press (1995); M. V. Berry, "Chaotic Behavior in Dynamical Systems", ed. G. Casati (New York, Plenum), 1985; M. Robnik and T. Prosen, "Comment on energy level statistics in the mixed regimes", arXiv: chao-dyn/9706023 (1997).
[64] F. Haake, "Quantum Signatures of Chaos", Springer-Verlag, Berlin (1991) (2nd enlarged edition, 2000).
[65] A. Wolf et. al, "Determining Lyapunov Exponents from Time Series", Physica D16, 285-317 (1985); J. Wright, "Method for calculating a Lyapunov exponent", Phys. Rev. A29, 2924-2927 (1984).
[66] P. Grassberger and I. Procaccia, "Estimation of the Kolmogorov entropy from a chaotic signal", Phys. Rev. A28, 2591-2593 (1983).
[67] B. Efron, "Defining the curvature of a statistical problem", Annals of Statistics 3, 1189 (1975).
[68] O. Bohigas et. al, "Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws", Phys. Rev. Lett. 52, 1 (1984).
[69] M. C. Gutzwiller, "Chaos in Classical and Quantum Mechanics", Springer-Verlag, New York (1990).
[70] A. M. Garcia-Garcia and J. Wang, "Universality in quantum chaos and the one parameter scaling theory", arXiv: 0707.3964 (2007); "Anderson Localization in Quantum Chaos: Scaling and Universality", Acta Physica Polonica A112, 635-653 (2007).
[71] Y. Gu, "Evidences of classical and quantum chaos in the time evolution of nonequilibrium ensembles", Phys. Lett. A149, 95-100 (1990); J. P. Keating, "Asymptotic properties of the periodic orbits of the cat maps", Nonlinearity 4, 277-307 (1991); J. P. Keating, "The cat maps: quantum mechanics and classical motion", Nonlinearity 4, 309-341 (1991).
[72] F. M. Izrailev, "Simple models of quantum chaos: spectrum and eigenfunctions", Phys. Rep. 196, 299-392 (1990).