Maximum entropy change and least action principle for nonequilibrium systems

Q.A. Wang
Institut Supérieur des Matériaux et Mécaniques Avancés,
44, Avenue F.A. Bartholdi, 72000 Le Mans, France

Abstract

A path information is defined in connection with different possible paths of irregular dynamic systems moving in its phase space between two points. On the basis of the assumption that the paths are physically differentiated by their actions, we show that the maximum path information leads to a path probability distribution in exponentials of action. This means that the most probable paths are just the paths of least action. This distribution naturally leads to important laws of normal diffusion. A conclusion of this work is that, for probabilistic mechanics or irregular dynamics, the principle of maximization of path information is equivalent to the least action principle for regular dynamics.

We also show that an average path information between the initial phase volume and the final phase volume can be related to the entropy change defined with natural invariant measure of dynamic system. Hence the principles of least action and maximum path information suggest the maximum entropy change. This result is used for some chaotic systems evolving in fractal phase space in order to derive their invariant measures.
1 Introduction

This work is an investigation, in connection with the concepts of information, entropy and least action principle of Maupertuis, of the thermostatics of dynamic system arbitrarily away from equilibrium. It is hoped that the results may have some connections with quantum gravity theories, quantum cosmology and quantum chromodynamics in particle physics because, firstly, most, if not all, of the systems treated in these fields are in nonequilibrium state and, secondly, the statistical concepts of probability, information, entropy and least (stationary) action solutions (instantons) to Feynman path integrals are so essential and widely used in these disciplines[1]. An important character of the path integral methods is that the probability of certain evolution is determined by an exponential functional of action which is postulated for the first time by Feynman[2]. The reader will find that this is a natural consequence of the principle of maximum path information combined with action of the trajectories of nonequilibrium evolution.

Although the theoretical formalism of equilibrium statistics is well founded on the physical laws with the help of the algorithm of maximum entropy[3, 4], the present statistical methodology for nonequilibrium systems seems to some extent uncertain[5]. For some nonequilibrium systems, the extension of the concept of Boltzmann and Gibbs entropies are in progress[5, 6, 7]. For stationary state sufficiently close to equilibrium, the principle of minimum entropy production is proposed[8]. For certain systems far from equilibrium, e.g., in the context of the climate of earth, there is the method of maximum entropy production (or minimum entropy exchange)[9, 10]. Here the entropy is always in the sense of Clausius, so the entropy production $dS_i \geq 0$ according to the second law. Recently, we have witnessed the development of nonextensive statistics theories based on the Tsallis entropy which has been maximized for nonequilibrium systems[11, 12] in chaotic motions or on the edge of chaos according to actual understanding. This is nothing but an extension of Jaynes principle for nonequilibrium systems[13] to another entropy which is not necessarily an entropy in the sense of Clausius.

This work is intended to study the behavior of irregular dynamic systems from the point of view of information theory. It is expected that the reasoning of this work concerning the principle of maximum information and entropy in connection with the concept of action and the principle of Maupertuis, can be helpful for understanding the physics of dynamic processes. In the second part of this paper, we specify some definitions and assumptions used...
in this work. In the third part, we describe a path information\cite{14} defined in connection with different possible paths of nonequilibrium systems moving in its phase space between two points (cells). On the basis of the assumption that the paths are physically differentiated by their actions, we show that the maximum path information yields an exponential path probability distribution depending on action which implies the most probable paths are just the paths of least action\footnote{During the preparation of this manuscript, Dr. Touchette wrote me that this connection was known in the field of random perturbation of dynamic systems and large deviation. The possible references are: M.I. Freidlin, A.D.Wentzell, Random Perturbations of Dynamic Systems, Springer-Verlag, 1983; and Y. Oono, Large deviation and statistical physics, Progress of Theoretical Physics Supplement 99, 165-205, 1989. I am most indebted to him for pointing out that.}. Finally, this \textit{least action distribution} is differentiated with time and position in order to derive, in a quite general manner without additional assumptions, the Fokker-Planck equation, the Fick’s laws of normal diffusion, the Ohm’s law of electrical conduction and the Fourier’s law of heat conduction.

In the forth part, an averaged path information is related to the difference of entropy between two phase volumes. The entropy is defined with the natural invariant measure of the nonequilibrium system. This relationship suggests that, if one maximizes path information in order to know the probability distribution of paths for dynamic systems whose motion forms geodesic flows in phase space, the entropy change of the dynamic process must be maximized in order to derive the invariant measures of the process.

Finally, the above result is applied to some chaotic systems evolving in fractal phase space for which a relative entropy change is given by $R = \sum_i \mu_i - \sum_i \mu_i^q$, where $q$ is a positive real parameter characterizing the geometrical features of the phase space and $\mu_i$ a natural invariant measure. Its maximization (or extremization) with appropriate constraints can yield several power law invariant measures.

\section{Assumptions and definitions}

Let us begin by some assumptions and definitions which specify the range of validity of this work.

1. Phase space and partition. As usual the phase space $\Gamma$ of a thermodynamic system is defined such that a physical state of the system is
represented by a point in that space. For a $N$-body system moving in
three dimensional ordinary configuration space, the $\Gamma$-space is of $6N$
dimension ($3N$ positions and $3N$ momenta) if it can be smoothly occu-
pied. The position of the state point at $t = 0$ of a movement is called
initial condition of that movement. We suppose that a phase volume
$\Omega$ accessible to the system can be partitioned into $v$ cells of volume $s_i$
with $i = 1, 2, ...v$ in such a way that $s_i \cap s_j = \emptyset$ ($i \neq j$) and $\bigcup_{i=1}^{v} s_i = \Omega$.
A state of the system can be represented by a sufficiently small phase
cell in coarse graining way. The movement of a dynamic system can
be represented by its trajectories (in the sense of classical mechanics)
in $\Gamma$ space.

2. The natural invariant measure\cite{17} $\mu_i$ (probability distribution for a
nonequilibrium system to visit different phase cells of given partitions)
is defined for the cell $i$ as follows :

$$
\mu_i = \int_{s_i} d\rho(x)
$$

(1)

where $\rho(x)$ is the probability density and $x$ represents the phase po-

tion. For dynamic systems viewed as geodesic flow in phase space,
the phase trajectories is uniformly distributed in the accessible phase
volume\cite{15}. In this case, $\rho(x)$ can be constant over all the phase space
and $\mu_i$ becomes proportional to $s_i$, a phase cell of a given partition.
This property will be useful for calculating entropy change of a system
moving in fractal phase space (see below).

3. The ergodicity for nonequilibrium system is described by

$$
\overline{Q} = \lim_{T \to \infty} \frac{1}{T} \int_0^T Q(t) dt = \lim_{v \to \infty} \sum_{i=1}^{v} \mu_i Q_i.
$$

(2)

which means that the average over time is equal to the ensemble average
over the occupied phase space volume, where $T$ is the period of the
evolution, $Q(t)$ is the value of a quantity $Q$ at time $t$ and $Q_i$ is the
value of $Q$ in the cell $i$. In this way, the evolution of information
over time can be estimated by the time evolution of the phase space
volume occupied by the statistical ensemble of identical systems under
consideration.
4. The information we address in this work is our ignorance about the system under consideration associated with some uncertainty or probability distribution. According to Shannon, this information can be measured by the formula

$$H = - \sum_{i=1}^{v} p_i \ln p_i$$  \hspace{1cm} (3)$$

with respect to certain probability distribution $p_i$ and the index $i$ is summed over all the possible situations or events.

5. In this work, entropy $S$ of a nonequilibrium system is defined with the natural invariant measure $\mu_i$ either by Eq.(3) or by other formula, i.e.,

$$S = - \sum_{i=1}^{v} \mu_i \ln \mu_i = - \int_{\Omega} \rho(x) \ln \rho(x) dx$$  \hspace{1cm} (4)$$
or, more generally, $S = \int_{\Omega} \sigma[\rho(x)] dx$, where $\Omega$ is the total occupied phase volume and $\sigma$ is the entropy density. In the special case of Eq.(4), $\sigma[\rho(x)] = \rho(x) \ln \rho(x)$.

6. According to the ergodic assumption mentioned above, the entropy change $\Delta I(T)$ of a dynamic system during a long time process can be estimated by the ensemble average of the entropy change during the scale refinement,

$$\Delta I(T) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \delta i(t) dt$$  \hspace{1cm} (5)$$

$$\Rightarrow \lim_{v_T \to \infty} \sum_{i=1}^{v_T} \mu_i \int_{s_i} \sigma_i ds - \sum_{j=1}^{v_0} \mu_j \int_{s_j} \sigma_j ds = \lim_{v_k \to \infty} \sum_{i=1}^{v_k} \mu_i \int_{s_i} \sigma_i ds - \sum_{j=1}^{v_0} \mu_j \int_{s_j} \sigma_j ds$$

where $\Delta I(T)$ is the average entropy change during the time $T$, $\delta i(t)$ is the entropy change at time $t$, $\sigma_i$ is the density of information on the phase cell $i$ of volume $s_i$, $v_T$ and $v_0$ are the total numbers of the cells of phase space accessible to the system and visited by the trajectories at time $t = T$ and $t = 0$, respectively. According to our assumption, the $v_T$
cells visited by the system form a geometry which can be reproduced by the \( v_k \) cells yielded from the \( v_0 \) cells by certain map (scale refinement) of \( k \) iterations. We may put \( v_k = v_T \) when \( k \) and \( T \) are large.

3 About a path information

3.1 Uncertainties of irregular dynamics

In a previous work\cite{14}, we defined a path information for irregular dynamic systems moving in the \( \Gamma \)-space between two points, \( a \) and \( b \), which are in two elementary cells of a given partition of the phase space. It is known that if the motion of the system is regular, there will be only a fine bundle of paths which track each other between the initial and the final cells and minimize action according to the principle of least action\cite{15}. However, if the dynamics is irregular due to large number of degrees of freedom or strong sensitivity to initial conditions, we can have the following two dynamic uncertainties:

1. Between any two fixed cells \( a \) and \( b \), there may be different possible paths (labelled by \( k=1,2,...,w \)) each having a probability \( p_k(b|a) \) to be followed by the system. This is the uncertainty we studied in \cite{14}.

2. In a fixed period of time, we can observe different possible paths leaving the cell \( a \) and leading to different final cells \( b \) in a final phase volume \( B \), each having a probability to be followed by the system.

In this section, the discussions will be made with the first uncertainty. By definition, the path probability distribution \( p_k(b|a) \) is a transition probability from state \( a \) to state \( b \) via path \( k \) for all systems between these two states. We have \( \sum_{k=1}^{w} p_k(b|a) = 1 \). The dynamic uncertainty associated with \( p_k(b|a) \) is measured by the Shannon information:

\[
H(a, b) = - \sum_{k=1}^{w} p_k(b|a) \ln p_k(b|a).
\]

(6)

\( H(a, b) \) is a path information, i.e., the missing information necessary for predicting which path a system takes from \( a \) to \( b \).
3.2 Least action distribution

The path probability distribution $p_k(b|a)$ due to dynamic irregularity can be studied in connection with information theory and action integral on the basis of the assumption that the different paths are uniquely differentiated by their actions. For classical mechanical systems, action is given by

$$A_{ab}(k) = \int_{t_{ab}(k)} L_k(t) dt$$

(7)

where $L_k(t)$ is the Lagrangian of the system at time $t$ along the path $k$. For other systems, the action may be given by different calculations, e.g., Yang-Mills action and Euclidean action for quantum field theory and quantum cosmology[1]. The average action between state $a$ and state $b$ is given by

$$A_{ab} = \sum_{k=1}^{w} p_k(b|a) A_{ab}(k).$$

(8)

The maximization of $H(a, b)$ under the constraints associated with the normalization of $p_k(b|a)$ and the average action leads to

$$p_k(b|a) = \frac{1}{Q} \exp[-\eta A_{ab}(k)]$$

(9)

Putting this distribution into $H(a, b)$ of Eq.(6), we get

$$H(a, b) = \ln Q + \eta A_{ab} = \ln Q - \eta \frac{\partial}{\partial \eta} \ln Q$$

(10)

where $Q$ is given by $Q = \sum_{k=1}^{w} \exp[-\eta A_{ab}(k)]$ and $A_{ab} = -\frac{\partial}{\partial \eta} \ln Q$.

It is proved that[14] the distribution Eq.(9) is stable with respect to the fluctuation of action, and that Eq.(9) is a least (stationary) action distribution, i.e., the most probable paths are just the paths of least action. We have indeed $\delta p_k(b|a) = -\eta p_k(b|a) \delta A_{ab}(k) = 0$, so that $\delta A_{ab}(k) = 0$ leading to Euler-Lagrange equation $\frac{\partial}{\partial x} \frac{\partial L_k(t)}{\partial \dot{x}} - \frac{\partial L_k(t)}{\partial x} = 0[15]$ and to Newton’s second law which are satisfied by the most probable paths. For stochastic process like Brownian motion[16], it can be proved[14] that the Lagrange multiplier $\eta$ is positive and given by the inverse of the diffusion coefficient $\eta = \frac{1}{2mD}$. So the action of Brownian particles has a minimum along the most probable paths.
The physical content of $\eta$ can be made clearer if we use the general relationship $D = \frac{l^2}{2\tau}$ for Brownian motion leading to

$$\eta = \frac{\tau}{ml^2},$$

(11)

where $l$ is the mean free path and $\tau$ the mean free time of the Brownian particles. If we still consider detailed balance\[16\], we get $D = \mu k_B T$ where $\mu$ is the mobility of the diffusing particles, $k_B$ the Boltzmann constant and $T$ the temperature. This means

$$\eta = \frac{1}{2m\mu K_B T} = \frac{\gamma}{2K_B T},$$

(12)

where $m\gamma = \frac{1}{\mu}$ is the friction constant of the particles in the diffusion mixture.

Eq.(9) can now be written as

$$p_k(b|a) = \frac{1}{Z} \exp[-\frac{\gamma A_{ab}(k)}{2K_B T}],$$

(13)

3.3 A calculation of transition probability

Now let us look at a particle of mass $m$ diffusing along a given path from $a$ to a cell $b$ of its $\mu$-space (one particle phase space). The path is cut into $N$ infinitesimally small segments each having a spatial length $\Delta x_i = x_i - x_{i-1}$ with $i = 1...N$ ($x_0 = x_a$ and $x_N = x_b$). $t = t_i - t_{i-1}$ is the time interval spent by the system on every segment. The Lagrangian on the segment $i$ is given by

$$L(x, \dot{x}, t) = \frac{m(x_i - x_{i-1})^2}{2(t_i - t_{i-1})^2} - \left(\frac{\partial U}{\partial x}\right)_i \frac{(x_i - x_{i-1})}{2} - U(x_{i-1})$$

(14)

where the first term on the right hand side is the kinetic energy of the particle, the second is the average increment of its potential energy and the third is its potential energy at the point $x_{i-1}$. The action on the segment $i$ is just

$$A_i = \frac{m(\Delta x_i)^2}{2t} + F_i \frac{\Delta x_i}{2} - U(x_{i-1})t,$$

(15)

where $F_i = -\left(\frac{\partial U}{\partial x}\right)_i$ is the force on the segment $i$. According to Eq.(9), the transition probability $p_{i/i-1}$ from $x_{i-1}$ to $x_i$ on the path $k$ is given by

$$p_{i/i-1} = \frac{1}{Z_i} \exp\left(-\eta \left[\frac{m}{2t} \Delta x_i^2 + F_i \frac{t}{2} \Delta x_i\right]\right)$$

(16)
where \( Z_i \) is calculated as follows

\[
Z_i = \int_{-\infty}^{\infty} dx_i \exp \left( -\eta \left[ \frac{m}{2t} \Delta x_i^2 + F_i \frac{t}{2} \Delta x_i \right] \right) \quad (17)
\]

\[
= \exp \left[ F_i^2 \frac{\eta t^3}{8m} \right] \sqrt{\frac{2\pi t}{m\eta}}.
\]

The potential energy at the point \( x_{i-1} \) disappears in the expression of \( p_{i/i-1} \) because it does not depend on \( x_i \).

The total action is given by

\[
A_{ab}(k) = \sum_{i=1}^{N} A_i = \sum_{i=1}^{N} \left[ \frac{m(\Delta x_i)^2}{2t} + F_i \frac{t}{2} \Delta x_i - U(x_{i-1}) \right]_k \quad (18)
\]

so the transition probability from \( a \) to \( b \) via the path \( k_b \) is the following:

\[
p_k(b|a) = \frac{1}{Z} \exp \left[ -\eta \sum_{i=1}^{N} \left[ \frac{m(\Delta x_i)^2}{2t} + F_i \frac{t}{2} \Delta x_i \right] \right] \quad (19)
\]

\[
= \prod_{i=1}^{N} p_{i/i-1}
\]

where

\[
Z = \prod_{i=1}^{N} Z_i = \prod_{i=1}^{N} \exp \left[ F_i^2 \frac{\eta t^3}{8m} \right] \sqrt{\frac{2\pi t}{m\eta}}. \quad (20)
\]

### 3.4 A derivation of diffusion laws

#### 3.4.1 Fokker-Planck equation

The Fokker-Planck equation describes the time evolution of the probability density function of position and velocity of a particle. This equation can be derived on the basis of the assumptions of Brownian motion and Markovian process[16]. We will show here that this equation can be derived from the distributions given by Eq.(16) and Eq.(19) without any assumptions.

The calculation of the derivatives \( \frac{\partial p_{i/i-1}}{\partial t}, \frac{\partial F_{i}p_{i/i-1}}{\partial x_i} \) and \( \frac{\partial^2 p_{i/i-1}}{\partial x_i^2} \) straightforwardly leads to

\[
\frac{\partial p_{i/i-1}}{\partial t} = -\frac{\tau}{m} \frac{\partial (F_{i}p_{i/i-1})}{\partial x_i} + D \frac{\partial^2 p_{i/i-1}}{\partial x_i^2}. \quad (21)
\]
This is the Fokker-Planck equation, where $D$ is given by $D = \frac{1}{2m\eta}$ and $\tau$ is the mean free time supposed to be the time interval $t$ of the particle on each segment of its path. In view of the Eq.(19), it is easy to show that this equation is also satisfied by $p_k(b|a)$ if $x_i$ is replaced by $x_b$ or $x$, the final position. Now let $n_a$ and $n_b$ be the particle density at $a$ and $b$, respectively. The following relationship holds

$$n_b = \sum_{k=1}^{w} n_ap_k(b|a)$$

(22)

which is valid for any $n_a$. This means (let $n = n_b$ and $x = x_b$):

$$\frac{\partial n}{\partial t} = -\frac{\tau}{m}\frac{\partial (F_n)}{\partial x} + D\frac{\partial^2 n}{\partial x^2},$$

(23)

which describes the time evolution of the particle density.

### 3.4.2 Fick’s laws of diffusion

If the external force $F$ is zero, we get

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}$$

(24)

This is the second Fick’s law of diffusion. The first Fick’s law

$$J(x,t) = -D\frac{\partial n}{\partial x}$$

(25)

can be easily derived if we consider matter conservation $\frac{\partial n(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x}$, where $J(x,t)$ is the flux of the particle flow.

The diffusion constant $D$ can be related to the partition function $Z_a$ by combining the relationship $D = \frac{1}{2m\eta}$ and Eq.(10), i.e.,

$$D = \frac{1}{2m} \frac{\partial A_a}{\partial H_a} = -\frac{1}{2m} \frac{\partial^2 (\ln Z_a)}{\partial H_a \partial \eta}.$$ 

(26)

### 3.4.3 Ohm’s law of electrical conduction

Considering the charge conservation $\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x}$, where $\rho(x,t) = qn(x,t)$ is the charge density, $j(x,t) = qJ(x,t)$ is the flux of electrical currant and $q$ is the charge of the currant carriers, we have, from Eq.(23),

$$j = \frac{\tau}{m} F\rho - D\frac{\partial \rho}{\partial x}$$

(27)
where $F = qE$ is the electrostatic force on the carriers and $E$ is the electric field. If the carrier density is uniform everywhere, i.e., $\frac{\partial n}{\partial x} = 0$, we get

$$j = \tau qE = \sigma E,$$

(28)

where $\sigma = \frac{q \tau}{m} = \frac{na^2 \tau}{m}$ is the formula of electrical conductivity widely used for metals.

### 3.4.4 Fourier’s law of heat conduction

For heat conduction, let the external force $F$ be zero. We consider a crystal idealized by a lattice of identical harmonic oscillators each having an energy $e_k = N\nu(x, t)h\nu$ where $h$ is the Planck constant, $\nu$ is the frequency of a mode and $N\nu(x, t)$ is the number of phonons of that mode situated at $x$ at time $t$ in the intervals $x \rightarrow x + dx$ and $\nu \rightarrow \nu + d\nu$. Suppose that there is no mass flow and other mode of energy transport in the crystal. Heat is transported only through the phonon flow. The phonons of frequency $\nu$ diffuse in the crystal lattice, among the lattice imperfections, impurities and other phonons with in addition anharmonic effects[19], just like a particle of mass $m = h\nu/c^2$ having transition probability $p_k(b|a)$. Let $n_\nu = N\nu/dx$ be the density of phonons which must satisfy

$$n_\nu(x_b, t) = \sum_k n_\nu(x_a, t)p_k(b|a)$$

(29)

and also Eq.(24).

The total energy density $\rho(x, t)$ of phonons at $x$ and time $t$ is given by

$$\rho(x, t) = \int_0^{\nu_m} h\nu n_\nu(x, t)\rho(\nu)d\nu,$$

(30)

where $\rho(\nu)d\nu$ is the mode number in the interval $\nu \rightarrow \nu + d\nu$, and $\nu_m$ is the maximal frequency of the lattice vibration. This relationship holds for any state density $\rho(\nu)$ and frequency $\nu$. This implies

$$\frac{\partial \rho(x, t)}{\partial t} = D\Delta_x \rho(x, t)$$

(31)

On the other hand, the variation of energy density $\delta \rho(x, t)$ can be related to temperature change $\delta T(x, t)$ by

$$\delta \rho(x, t) = c\delta T(x, t)$$

(32)
where $c$ is the heat capacity per unit volume supposed constant everywhere in the crystal. This leads to

$$\frac{\partial \rho}{\partial t} = \kappa \Delta_x T(x, t)$$  \hspace{1cm} (33)

where $\kappa = Dc$ is the heat conductivity. This equation can be recast into

$$c \frac{\partial T(x, t)}{\partial t} = \kappa \Delta_x T(x, t)$$  \hspace{1cm} (34)

which describes the evolution of temperature distribution due to the heat flow. When a stationary state is reached, temperature is everywhere constant, i.e., $\frac{\partial T(x, t)}{\partial t} = \Delta_x T(x, t) = 0$, and the temperature distribution is given by $\nabla_x T(x) = \text{constant}$.

Now considering the energy conservation in an elementary volume between $x$ and $x + dx$ in which we have $-\frac{\partial \rho(x, t)}{\partial t} = \nabla_x \cdot J(x, t)$, Fourier’s law of heat conduction follows

$$J(x, t) = -\kappa \nabla_x T(x, t).$$  \hspace{1cm} (35)

Summarizing this section, we have introduced a path information à la Shannon for irregular dynamic systems and assumed that the different paths between any two fixed points in phase space are characterized by their actions. This starting points lead to a least action transition probability in exponentials of action which maximizes the path information and implies that the most probable paths are just the paths (geodesics) required by the least action principle of classical mechanics. From this least action distribution, we can derive important physical laws for normal diffusion without any additional assumptions about the dynamic process (Brownian and Markovian or not). This result can be considered as a robust support to this approach of probabilistic mechanics or statistical dynamics. In what follows, the above method concerning only two fixed phase cells will be extended to arbitrary final cell $b$ in order to study the variation of entropy of dynamic process from the point of view of path information.

4 Maximize entropy change for nonequilibrium systems

Now the above result concerning only two fixed cells will be generalized to arbitrary destination $b$. Let us consider an ensemble of $N$ identical chaotic
systems leaving the initial cell $a$ for some destinations. The travelling time is specified, say, $t_{ab} = t_b - t_a$. After $t_{ab}$, all the phase points occupied by the systems form the final phase volume $B$ partitioned into cells labelled by $b$. We observe $N_b$ systems travelling along an ensemble of paths labelled also by $b$ leading to certain cell $b$. A path probability can be defined by $p_{b|a} = N_b/N$ which is normalized by $\sum_b p_{b|a} = 1$. We always suppose each ensemble of paths $b$ is characterized by the average action $A_{ab}$ over the ensemble given by Eq.(8). Then a total average action can be defined by $A_a = \sum_b p_{b|a} A_{ab}$. The uncertainty concerning the choice of final cells is just the second uncertainty mentioned in section 3.1. It is measured by the following Shannon information

$$H_a = - \sum_b p_{b|a} \ln p_{b|a}$$

(36)

and can be maximized just as in section 3.2 to give

$$p_{b|a} = \frac{1}{Z} \exp[-\theta A_{ab}]$$

(37)

where $\theta$ is the Lagrange multiplier associated with $A_a$. As has been done for the distribution of Eq.(9) in [14], it can be proved that this distribution is stable with respect to the fluctuation of the average action $A_{ab}$. It has also been proved in [14] that $A_{ab}$ has a stationary (minimum if $\eta > 0$ and maximum if $\eta < 0$) when the uncertainty (path information) $H(a, b)$ is maximized. This means that the ensemble of paths with stationary $A_{ab}$ are the most probable, i.e., $\delta A_{ab} = 0 \iff \delta p_{b|a} = 0$.

Our aim here is to derive the invariant measures $\mu_i$ which define the entropy $S$ for dynamic systems by Eq.(4) or, more generally, by certain functional $\sigma[\rho(x)]$. Our idea is to “complement” the method of maximum entropy by another method for dynamic systems. This is the method of maximum entropy change.

In what follows, we suppose that entropy changes when an irregular dynamic system moves from the initial cell $a$ to the final cells $b$ in $B$. The initial entropy of the systems at time $t = 0$ is given by Shannon formula:

$$S_a = - \sum_{a}^{\text{over } A} \mu_a \ln \mu_a$$

(38)

where $\mu_a$ is the invariant measure of the systems in the initial phase volume $A$ at $t = 0$ and the cell index $a$ is summed over all $A$. After a time $t_{ab}$, the
systems from the cell $a$ travelling along an ensemble $b$ of paths are found in a cell $b$ with invariant measure $\mu_b(a)$. Supposing Markovian process, we have

$$\mu_b(a) = \mu_a p_b | a.$$  \hspace{1cm} (39)

We calculate first the contribution of the cell $a$ to the final entropy of the systems in $B$. It is given by

$$S_b(a) = - \sum_b \mu_b(a) \ln \mu_b(a)$$

$$= - \mu_a \ln \mu_a - \mu_a \sum_b p_b | a \ln p_b | a.$$  \hspace{1cm} (40)

The total final entropy is then

$$S_b = - \sum_a S_b(a)$$

$$= - \sum_a \mu_a \ln \mu_a - \sum_a \mu_a \sum_b p_b | a \ln p_b | a$$

$$= S_a + \sum_a \mu_a H_a = S_a + \overline{H}_a,$$  \hspace{1cm} (41)

where $\overline{H}_a$ is the average of the path information $H_a$ over all the initial phase volume $A$. So if $H_a$ is maximized, $\overline{H}_a$ should also be maximized.

Eq. (41) is an essential result of this work. It states that, if $H_a$ as well as $\overline{H}_a$ must be maximized by the least action distributions, then the entropy change $\Delta S = S_b - S_a = \overline{H}_{b|a}$ must be also maximized for dynamic process.

We would like to emphasize here that maximum (extremum) entropy change does not exclude the possibility of maximum (extremum) entropy. In fact, if the entropy change is maximized at any moment of an evolution, the entropy must also be at maximum all the time. This can be considered as a proof of the maximum entropy method derived from action principle for stochastic process. The choice of these two methods for a given process of course depends on which of entropy and entropy change is available for the probability functionals to be derived. Following section is an example of the availability of entropy change without even knowing the exact entropy functionals.
5 The entropy change due to fractal geometry

Now let us use the method of extremum entropy change (an extension of maximum entropy change) to derive invariant measure of dynamic systems. We consider a statistical ensemble of identical dynamic systems all moving in fractal phase space. The outcome of this movement is not specified. This may be a relaxation process towards equilibrium or a long evolution from one nonequilibrium state to another. For this kind of systems, an entropy change has been derived in our previous work\cite{18} in term of invariant measures. We give here a brief description of the method of maximum entropy change and some of its consequences.

In reference\cite{18}, we supposed that all state points in the phase space were equiprobable and uniformly distributed in the initial condition volume. This is true if the ensemble of studied systems forms geodesic flows on phase manifolds\cite{15}. In this case, the invariant measure $\mu_i$ of a phase cell $i$ of a given partition is just proportional to the volume $s_i$ of the cell. We supposed in addition the state density is scale invariant, which is in accordance with the time independent invariant measure if we consider the ergodic assumption. In this case, the relative entropy change from zeroth to $\lambda^{th}$ iteration of the scale refinement (finer and finer partition of the phase space) is given by

$$ R = \sum_{i=1}^{v_\lambda} \left( \mu_i - \mu_i^q \right) $$

(42)

where $i$ is the index and $v_\lambda$ is the total number of the cells of a phase partition. $q$ should be considered as a parameter characterizing the topological feature of the phase space. If the phase space is a simple fractal with dimension $d_f$, we have $q = d_f/d$ where $d$ is the dimension of the phase space when it is smoothly occupied. Eq.(42) is originally derived with the hypothesis of incomplete information and incomplete probability distribution for fractal phase space\cite{18}. But its form remains the same for complete probability distribution $p_i$ (i.e. $\sum_{i=1}^{v} p_i = 1$). The following presentation is made within this formalism.

$R$ has the following properties

1. $R$ is a relative change, so $-1 \leq R \leq 1$. 

15
2. For a system composed of two sub-systems $A$ and $B$ with $q_A$ and $q_B$ satisfying the following rule of product joint probability

$$\mu_{i_{A^B}}^{q_{A}+B} = \mu_{i_A}^{q_A}\mu_{i_B}^{q_B},$$  \hspace{1cm} (43)

it is easy to show the following nonadditivity:

$$R(A + B) = R(A) + R(B) + R(A)R(B).$$  \hspace{1cm} (44)

3. $R$ is positive and concave for $q > 1$, and negative and convex for $q < 1$. If $q = 1$, there is no fractal, so $R = 0$. In other words, the fractal feature of the phase space is responsible for the entropy change. This is in accordance with the numerical result for some chaotic maps[20]. In the case of a simple fractal, $q = d_f/d > 1$ ($q = d_f/d < 1$) implies phase space expansion (contraction) during scale refinement. So for chaotic dynamic systems, $q > 1$ corresponds to positive Lyapunov exponent (we lose knowledge about the system so that $R > 0$) and $q < 1$ to negative Lyapunov exponent (we gain knowledge about the system so that $R < 0$).

4. We see that $R$ is a function of the dimension difference $d_f - d$. If we divide $R$ by $q - 1 = \frac{d_f - d}{d}$, we get

$$H_q = \frac{R}{q - 1} = -\frac{\sum_{i=1}^{w}(\mu_i - \mu_i^q)}{1 - q}. \hspace{1cm} (45)$$

This is the form of Tsallis entropy[11]. If $q = 1$ or $d_f = d$, the phase space is uniformly occupied and the ratio $H_1$ becomes Shannon entropy. Other extended entropies can be obtained by defining entropy change rate in different manners[18].

6 Some invariant measures for chaotic systems

According to our method, $R$ can be extremized under the constraints associated with the normalization $\sum_{i=1}^{v}\mu_i = 1$ and with our knowledge about the random variables $x^{(\sigma)}$ ($\sigma = 1, 2...w$ is the index of these variables) concerned in the nonequilibrium process, with $w$ multipliers $\gamma_{\sigma}$ each connected with
an expectation $x^{(\sigma)} = \sum_{i=1}^{w} \mu_{i} x_{i}^{(\sigma)} \mu_{i} = 1$ where $x_{i}^{(\sigma)}$ is the value of $x^{(\sigma)}$ when the system is at the state $i$. The probability distribution is given by

$$\mu_{i} = \frac{1}{Q} (1 - \sum_{\sigma=1}^{w} \gamma_{\sigma} x_{i}^{(\sigma)})^{1/(1-q)}.$$ 

(46)

This distribution can be applied to many chaotic maps. Following are some examples.

### 6.1 Un model for population evolution

The logistic map $y_{n+1} = Ay_{n} - By_{n}^2$ $(0 < y_{n} < y_{\text{max}})$[21] is often used for modelling the biological population evolution, where $y_{n}$ is the population of the $n^{\text{th}}$ year or of $n^{\text{th}}$ order of iteration, $y_{\text{max}} = A/B$ is the maximal population (for given living conditions) not to exceed in order that the concerned species do not die out next year, $A$ is a positive constant connected with population growth and $B$ is a positive constant connected with the population decrease.

In this model, the two variables for the population evolution is $x_{n}$ and $x_{n}^2$. So if in Eq.(46) one puts $x^{(1)} = y$ and $x^{(2)} = y^2$, one gets,

$$\mu(x) = \frac{1}{Q} (1 - \sum_{\sigma=1}^{2} \gamma_{\sigma} x_{i}^{(\sigma)})^{1/(1-q)} = \frac{1}{Q} (1 - \gamma_{1} y - \gamma_{2} y^2)^{1/(1-q)}.$$ 

(47)

According to the assumption of the roles of the $y$-term and $y^2$-term in the population evolution, we can suppose that $\gamma_{1}$ is related to population increase and $\gamma_{2}$ to population decrease. When $\gamma_{1} y + \gamma_{2} y^2 \gg 1$, Eq.(47) becomes

$$\mu(x) = \frac{1}{Q} (-\gamma_{1} y - \gamma_{2} y^2)^{1/(1-q)}.$$ 

(48)

By comparison with the distribution given for $A = 4$[21] : $\mu(x) = \frac{1}{\pi[x(1-x)]^{1/2}}$ where $x = y/y_{\text{max}}$, we obtain $\gamma_{1} = -y_{\text{max}}$, $\gamma_{2} = 1$, $Q = \pi/y_{\text{max}}$ and $q = 3$. So $R > 0$. This implies an increasing entropy process with positive Lyapunov exponent.

### 6.2 The continued fraction map

The continued fraction map[17] is a chaotic map given by $x_{n+1} = 1/x_{n} - [1/x_{n}]$ where $[1/x_{n}]$ is the integer part of $1/x_{n}$ and $x_{n}$ is a real number
between 0 and 1. The probability distribution is given by $\mu(x) = 1/(1+x) \ln 2$ and satisfies $\sum_x \mu(x) = 1$. This distribution can be obtained from Eq. (46) by using $w = 1$, $x^{(1)} = x$, $\gamma_1 = -1$, $Q = \ln 2$ and $q = 2$. This is also an increasing entropy process with Lyapunov exponent $\lambda = \ln 2$.[17]

Other application can be found in [18], e.g., for Ulam map $x_{n+1} = 1 - \mu x_n^2$ ($-1 < x_n < 1$) with $\mu = 2$, $q = 3$ and for Zipf-Mandelbrot’s law $\mu(x) = \frac{A}{(1-Bx)^\tau}$, $q = 1 + 1/\tau$.

7 Concluding remarks

A path information is defined in connection with different possible paths of chaotic system moving in its phase space between two cells. On the basis of the assumption that the paths are physically differentiated by their actions, we show that the maximum path information leads to a path probability distribution in exponentials of action from which the well known Fokker-Planck equation, the Fick’s laws, the Ohm’s law and the Fourier’s law can be easily derived in a general way. This result strongly suggests that, for the probabilistic case of irregular dynamics, that the principle of least action for regular dynamics should be replaced by the principle of maximization of path information or dynamic uncertainty.

We show that an extended path information between two phase volumes can be related to the entropy change of the processes linking the two phase volumes. Hence the principles of least action and maximum path information suggest the maximum entropy change which can be used to derive the most probable invariant measures for nonequilibrium systems. This result is used to derive invariant measures for some chaotic systems evolving in fractal phase space. We would like to emphasize that the method of maximum entropy change is in accordance with the principle of maximum entropy which can be used for nonequilibrium systems when available. If the system of interest occupying the two phase volumes is in equilibrium, then maximum entropy change of dynamic process linking the phase volumes naturally leads to the Jaynes’ principle of maximum entropy for equilibrium systems, as discussed in [22].
Acknowledgments

I acknowledge with great pleasure the useful discussions with Professors A. Le Méhauté, F. Tsobnang, L. Nivanen, M. Pezeril and Dr. W. Li.

References

[1] S.W. Hawking, T. Hertog, Phys. Rev. D, 66(2002)123509;
    S.W. Hawking, Gary.T. Horowitz, Class.Quant.Grav., 13(1996)1487
    S. Weinberg, Quantum field theory, vol.II, Cambridge University Press,
    Cambridge, 1996 (chapter 23: extended field configurations in particle
    physics and treatments of instantons)

[2] R.P. Feynman, Quantum mechanics and path integrals, McGraw-Hill
    Publishing Company, New York, 1965

[3] E.T. Jaynes, The evolution of Carnot’s principle, The opening talk at
    the EMBO Workshop on Maximum Entropy Methods in x-ray crystal-     
    logical and biological macromolecule structure determination, Orsay,
    France, April 24-28, 1984; Reprinted in Ericksen & Smith, Vol 1, pp.
    267-282; Phys. Rev., 106,620(1957)

[4] M. Tribus, Décisions Rationelles dans l’incertain, (Paris, Masson et Cie,
    1972)P14-26; or Rational, descriptions, decisions and designs, (Perga-
    mon Press Inc., 1969)

[5] D. Ruelle, Is there a unified theory of nonequilibrium statistical me-
    chanics? Proceedings of the International Conference on Theoretical
    Physics Th-2002 (Paris, July 22-27, 2002), Birkhäuser Verlag, Berlin,
    2004, p.489;
    D. Ruelle, Extending the definition of entropy to nonequilibrium steady
    state, Proc. Nat. Acad. Sci., 100,30054(2003)

[6] G. Gallavotti, E.G.D. Cohen, Note on nonequilibrium stationary states
    and entropy, cond-mat/0312306

[7] P.L. Garrido, S. Goldstein and J.L. Lebowitz, The Boltzmann entropy
    for dense fluid not in local equilibrium, cond-mat/0310575
S. Goldstein and J.L. Lebowitz, *On the (Boltzmann) Entropy of Nonequilibrium Systems*, cond-mat/0304251

[8] I. Prigogine, *Bull. Roy. Belg. Cl. Sci.*, 31, 600(1945)

[9] G. Paltridge, *Quart. J. Roy. Meteor. Soc.*, 101, 475(1975)

[10] Tim Lenton, *Maximum entropy production in Edinburgh, a report of the second 'Daisyworld and beyond' workshop*, (2002), document obtained from [http://www.cogs.susx.ac.uk/daisyworld/ws2002_overview.html](http://www.cogs.susx.ac.uk/daisyworld/ws2002_overview.html)

[11] C. Tsallis, *J. Stat. Phys.*, 52, 479(1988); C. Tsallis, F. Baldovin, R. Cerbino and P. Pierobon, *Introduction to Nonextensive Statistical Mechanics and Thermodynamics*, cond-mat/0309093

[12] Q.A. Wang, *Euro. Phys. J. B*, 26(2002)357

Q.A. Wang, *Chaos, Solitons & Fractals*, 12(2001)1431, Erranta : cond-mat/0009343

[13] E.T. Jaynes, Gibbs vs Boltzmann entropies, *American Journal of Physics*, 33, 391(1965);

E.T. Jaynes, Where do we go from here? *Maximum entropy and Bayesian methods in inverse problems*, pp.21-58, edited by C. Ray Smith and W.T. Grandy Jr., D. Reidel Publishing Company (1985)

[14] Q.A. Wang, Maximum path information and the principle of least action for chaotic system, *Chaos, Solitons & Fractals*, (2004), in press; cond-mat/0405373 and ccsd-00001549

[15] V.L. Arnold, *Mathematical methods of classical mechanics*, Springer-Verlag, New York, 1989

[16] R. Kubo, M. Toda, N. Hashitsume, *Statistical physics II, Nonequilibrium statistical mechanics*, Springer, Berlin, 1995

[17] C. Beck and F. Schlägl, *Thermodynamics of chaotic systems*, Cambridge University Press, 1993
[18] Q.A. Wang, *Chaos, Solitons & Fractals*, **19**, 639(2004)
Q. A. Wang and A. Le Méhauté, *Chaos, Solitons & Fractals*, **21**, 893(2004)

[19] F. Bonetto, J.L. Lebowitz, L. Rey-Bellet, Fourier’s Law: a Challenge for Theorists, math-ph/0002052

[20] T. Gilbert, J.R. Dorfman, and P. Gaspard, *Phys. Rev. Lett.*, **85**, 1606(2000);
I. Claus and P. Gaspard, The fractality of the relaxation modes in deterministic reaction-diffusion systems, cond-mat/0204264

[21] Robert C. Hilborn, *Chaos and Nonlinear Dynamics*, Oxford University press, New York, 1994

[22] Q.A. Wang, Action principle and Jaynes’ guess method, cond-mat/0407515