Scaling transition and edge effects for negatively dependent linear random fields on $\mathbb{Z}^2$

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Abstract. We obtain a complete description of anisotropic scaling limits and the existence of scaling transition for a class of negatively dependent linear random fields on $\mathbb{Z}^2$ with moving-average coefficients $a(t,s)$ decaying as $|t|^{-q_1}$ and $|s|^{-q_2}$ in the horizontal and vertical directions, $q_1^{-1} + q_2^{-1} < 1$. The scaling limits are taken over rectangles whose sides increase as $\lambda$ and $\lambda^\gamma$ when $\lambda \to \infty$, for any $\gamma > 0$. We prove that the scaling transition in this model is closely related to the presence or absence of the edge effects.

1 Introduction

A stationary random field (RF) $X = \{X(t); t \in \mathbb{Z}^\nu\}$ on $\nu$-dimensional lattice $\mathbb{Z}^\nu, \nu \geq 1$ with finite variance is said (covariance) long-range dependent (LRD) if $\sum_{t \in \mathbb{Z}^\nu} |\operatorname{Cov}(X(t), X(0))| = \infty$, short-range dependent (SRD) if $\sum_{t \in \mathbb{Z}^\nu} |\operatorname{Cov}(X(t), X(0))| < \infty$, $\sum_{t \in \mathbb{Z}^\nu} \operatorname{Cov}(X(t), X(0)) \neq 0$, and negatively dependent (ND) if $\sum_{t \in \mathbb{Z}^\nu} |\operatorname{Cov}(X(t), X(0))| < \infty$, $\sum_{t \in \mathbb{Z}^\nu} \operatorname{Cov}(X(t), X(0)) = 0$. The above definitions apply per se to RFs with finite 2nd moment; related albeit not equivalent definitions of LRD, SRD, and ND properties are discussed in [6], [9], [19] and other works. For linear (moving-average) RFs, Lahiri and Robinson [9] define similar concepts through summability properties of the moving-average coefficients. The last paper also discusses the importance of spatial LRD in applied sciences, including the relevant literature.

The above classification plays an important role in limit theorems. Consider the sum $S_{K_\lambda}^X := \sum_{t \in K_\lambda} X(t)$ of the values of RF $X$ over large ‘sampling region’ $K_\lambda \subset \mathbb{Z}^\nu$ with $|K_\lambda| = \sum_{t \in K_\lambda} 1 \to \infty$ ($\lambda \to \infty$). Under additional conditions, the variance $\operatorname{Var}(S_{K_\lambda}^X)$ grows faster than $|K_\lambda|$ under LRD, as $O(|K_\lambda|)$ under SRD, and slower than $|K_\lambda|$ under ND. In the latter case, $\operatorname{Var}(S_{K_\lambda}^X)$ may grow as slow as the ‘volume’ $|\partial K_\lambda|$ of

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the boundary $\partial K_\lambda$, or even slower than $|\partial K_\lambda|$, giving rise to ‘edge effects' which may affect or dominate the limit distribution of $S_{K_\lambda}$; see [9], [20].

Probably, the most studied case of limit theorems for RFs deal with rectangular summation regions, which allows for partial sums and limit RFs, similarly as in the case $\nu = 1$. Let $X = \{X(t); t \in \mathbb{Z}^d\}$ be a stationary random field (RF) on $\mathbb{Z}^\nu$, $\nu \geq 1$, $\gamma = (\gamma_1, \cdots, \gamma_\nu) \in \mathbb{R}_+^\nu$ be a collection of positive numbers (exponents), and

$$K_{\lambda, \gamma}(x) := \prod_{i=1}^\nu [1, [\lambda^{\gamma_i} x_i]], \quad x = (x_1, \cdots, x_\nu) \in \mathbb{R}_+^\nu, \quad (1.1)$$

be a family of $\nu$-dimensional ‘rectangles’ indexed by $\lambda > 0$, whose sides grow at generally different rate $O(\lambda^{\gamma_i}), i = 1, \cdots, \nu$ as $\lambda \to \infty$, and

$$S_{\lambda, \gamma}^X(x) := \sum_{t \in K_{\lambda, \gamma}(x)} X(t), \quad x \in \mathbb{R}_+^\nu \quad (1.2)$$

be the corresponding partial sums RF. [18], [19], [17], [23] discussed the anisotropic scaling limits for any $\gamma \in \mathbb{R}_+^\nu$ of some classes of LRD RFs $X$ in dimension $\nu = 2, 3$, viz.,

$$A_{\lambda, \gamma}^{-1} S_{\lambda, \gamma}^X(x) \xrightarrow{fdd} V_{\gamma}^X(x), \quad x \in \mathbb{R}_+^\nu \quad (1.3)$$

as $\lambda \to \infty$, where $A_{\lambda, \gamma} \to \infty$ is a normalization. Following [16], [23] the family $\{V_{\gamma}^X; \gamma \in \mathbb{R}_+^\nu\}$ of all scaling limits in (1.3) will be called the scaling diagram of RF $X$. [16] noted that the scaling diagram provides a more complete ‘large-scale summary’ of RF $X$ compared to (usual) isotropic or anisotropic scaling at fixed $\gamma \in \mathbb{R}_+^\nu$ as discussed in [1], [4], [9], [10], [12], [22] and other works.

[18], [19], [17] observed that for a large class of LRD RFs $X$ in dimension $\nu = 2$, the scaling diagram essentially consists of three points: $V^X = \{V_0^X, V_+^X, V_-^X\}$, $V_0^X$ termed the well-balanced and $V_{\pm}^X$ the unbalanced scaling limits of $X$. More precisely (assuming $\gamma_1 = 1$ w.l.g.) the exists a (nonrandom) $\gamma_0^X > 0$ such that $V_{\gamma}^X \equiv V_{(1, \gamma)}^X$ do not depend on $\gamma$ for $\gamma > \gamma_0^X$ and $\gamma < \gamma_0^X$, viz.,

$$V_{\gamma}^X = \begin{cases} V_+^X, & \gamma > \gamma_0^X, \\ V_-^X, & \gamma < \gamma_0^X, \\ V_0^X, & \gamma = \gamma_0^X \end{cases} \quad (1.4)$$

and $V_+^X \neq a V_-^X (\forall a > 0)$. The above fact was termed the scaling transition [18], [19]. It was noted in the above-mentioned works that the scaling transition constitutes a new and general feature of spatial dependence which occurs in many spatio-temporal models including telecommunications and economics ([3], [7], [13], [14], [16], [11]). However, as noted in [17], [23], these studies were limited to LRD models and the existence of the scaling transition under ND remained open.
The present paper discusses the scaling transition for linear ND RFs on $\mathbb{Z}^2$ having a moving-average representation

$$X(t, s) = \sum_{(u,v)\in\mathbb{Z}^2} a(t-u, s-v)\varepsilon(u, v), \quad (t, s) \in \mathbb{Z}^2, \quad (1.5)$$

in standardized i.i.d. sequence $\{\varepsilon(u, v); (u, v) \in \mathbb{Z}^2\}, E\varepsilon(u, v) = 0, E\varepsilon^2(u, v) = 1$ with deterministic moving-average coefficients

$$a(t, s) = \frac{1}{(|t|^2 + |s|^{2q_2/q_1})^{q_1/2}} \left(L_0 \left(\frac{t}{(|t|^2 + |s|^{2q_2/q_1})^{1/2}}\right) + o(1)\right), \quad |t| + |s| \to \infty, \quad (1.6)$$

$(t, s) \neq (0, 0)$, where $q_i > 0, i = 1, 2$ satisfy

$$0 < Q := \frac{1}{q_1} + \frac{1}{q_2} < 2 \quad (1.7)$$

and $L_0(u), u \in [-1, 1]$ is a bounded piece-wise continuous function on $[-1, 1]$. The form of moving-average coefficients in (1.6) is the same as in [17] and can be generalized to some extent but we prefer to use (1.6) for better comparison with the results of [17]. Condition $Q < 1$ guarantees that $\sum_{(t,s)\in\mathbb{Z}^2} |a(t, s)| < \infty$ and the ND property of $X$ in (1.5) is a consequence of the zero-sum condition

$$\sum_{(t,s)\in\mathbb{Z}^2} a(t, s) = 0, \quad \text{for} \quad Q < 1. \quad (1.8)$$

In contrast, [17] assumes $1 < Q < 2$ implying $\sum_{(t,s)\in\mathbb{Z}^2} |a(t, s)| = \infty$ and the LRD property of the corresponding linear RF $X$ in (1.5). The main results of this paper and [17] are illustrated in Fig. 1 showing 8 regions $R_{11}, \cdots, R_{33}$ of the parameter set $\{(1/q_1, 1/q_2) : 0 < Q < 2\}$ of the linear RF $X$ in (1.5)-(1.8) with different unbalanced limits.

![Figure 1. Regions in the parameter set $0 < Q < 2$ with different unbalanced limits of RF $X$ in (1.5)-(1.8). The dashed segment separates the LRD and ND regions.](image-url)
The regions $R_{11}, \cdots, R_{33}$ in Fig. 1 are described in Table 1. Recall the definition of fractional Brownian sheet (FBS) $B_{H_1, H_2} = \{B_{H_1, H_2}(x, y); (x, y) \in \mathbb{R}^2_+\}$ with Hurst parameters $0 < H_1, H_2 \leq 1$ as a Gaussian process with zero mean and covariance

$$EB_{H_1, H_2}(x_1, y_1)B_{H_1, H_2}(x_2, y_2) = (1/4)(x_1^{2H_1} + x_2^{2H_1} - |x_1 - x_2|^{2H_1})(y_1^{2H_2} + y_2^{2H_2} - |y_1 - y_2|^{2H_2}), \quad (1.9)$$

see [2]. FBS $B_{H_1, H_2}$ with one of the parameters $H_i$ equal to $1/2$ or $1$ have a very specific dependence structure (either independent or completely dependent (invariant) increments in one direction, see [18], [19]) and play a particular role in our work. With a slight abuse of notation, we also introduce two RFs

$$B_{0,1/2} = \{B_{0,1/2}(x, y) := B(y); (x, y) \in \mathbb{R}^2_+\}, \quad B_{1/2,0} = \{B_{1/2,0}(x, y) := B(x); (x, y) \in \mathbb{R}^2_+\} \quad (1.10)$$

which depend on one of the two coordinates on the plane (either $x$, or $y$) only, where $B = \{B(x); x > 0\}$ is a standard Brownian motion.

| Parameter region | Critical $\gamma_0^{X}$ | $V^{X}_+$ | $V^{X}_-$ | Range of Hurst parameters |
|------------------|-------------------------|-----------|-----------|---------------------------|
| $R_{11}$         | $q_1/q_2$               | $B_{1, H_2}$ | $B_{H_2,1}$ | $1/2 < \tilde{H}_1, \tilde{H}_2 < 1$ |
| $R_{12}$         | $q_1/q_2$               | $B_{H_1,1/2}$ | $B_{H_2,1}$ | $1/2 < \tilde{H}_1, H_1 < 1$ |
| $R_{21}$         | $q_1/q_2$               | $B_{1, H_2}$ | $B_{1/2, H_2}$ | $1/2 < \tilde{H}_2, H_2 < 1$ |
| $R_{22}^+$       | $q_1/q_2$               | $B_{H_1,1/2}$ | $B_{1/2, H_2}$ | $1/2 < H_1, H_2 < 1$ |
| $R_{22}$         | $q_1/q_2$               | $B_{H_1,1/2}$ | $B_{1/2, H_2}$ | $0 < H_1, H_2 < 1/2$ |
| $R_{23}$         | $2q_1(1 - Q)$           | $B_{H_2,1}$ | $B_{1/2,0}$ | $0 < H_1 < 1/2$ |
| $R_{32}$         | $1/2q_1(1 - Q)$         | $B_{0,1/2}$ | $B_{1/2, H_2}$ | $0 < H_2 < 1/2$ |
| $R_{33}$         | $1$                     | $B_{0,1/2}$ | $B_{1/2,0}$ | |

Table 1: Unbalanced scaling limits $V^{X}_\pm$ in regions $R_{11}, \cdots, R_{33}$ in Fig. 1

Parameters $\tilde{H}_i, H_i, i = 1, 2$ in Table 1 (expressed in terms of $q_1, q_2$) are specified in the beginning of Sec. 2. The description in Table 1 is not very precise since it omits various asymptotic constants which may vanish in some cases, meaning that some additional conditions on $a(t, s)$ are needed for the validity of the results in this table. The rigorous formulations including the normalizing constants $A_{\lambda, \gamma}$ are presented in Sec. 2. The limit distributions in regions $R_{11}, R_{12}, R_{21}, R_{22}^+$ refer to LRD set-up and are part of [17].

The new results under ND refer to regions $R_{22}, R_{23}, R_{32}, R_{33}$ of Table 1. Note $2q_1(1 - Q) > q_1/q_2$ in $R_{23}$ and $1/(2q_2(1 - Q)) < q_1/q_2$ in $R_{32}$. The limit $B_{1/2,0}$ in region $R_{23}$ can be related to the ‘horizontal edge effect’ which dominates the limit distribution of $S^{X}_{\lambda, \gamma}$ unless the vertical length $O(\lambda^{\gamma})$ of $K_{[\lambda x, \lambda^{\gamma}y]}$ grows fast enough vs. its horizontal length $O(\lambda)$, or $\gamma > 2q_1(1 - Q)$ holds, in which case FBS $B_{H_1,1/2}$ dominates. Similarly, $B_{0,1/2}$ in region $R_{32}$ can be related to the ‘vertical edge effect’ appearing in the
limit of $S_{\lambda,\gamma}^X$ unless the vertical length $O(\lambda^\gamma)$ increases sufficiently slow w.r.t the horizontal length $O(\lambda)$, or $\gamma < 1/2q_2(1 - Q)$ holds, in which case FBS $B_{1/2,H_2}$ dominates. Finally, $R_{33}$ can be characterized as the parameter region where the edge effects (either horizontal, or vertical) completely dominate the limit behavior of $S_{\lambda,\gamma}^X$. The above interpretation of $R_{32}, R_{23}$ and $R_{33}$ is based on the approximations of $S_{\lambda,\gamma}^X$ by suitable ‘edge terms’ which are discussed in Sec. 3.

The results of the present work are related to the work Lahiri and Robinson [9] which discussed the limit distribution of sums of linear LRD, SRD and ND RFs over homothetically inflated or isotropically rescaled (i.e., $\gamma_1 = \cdots = \gamma_\nu = 1$) star-like regions $K_{\lambda}$ of very general form. This generality of $K_{\lambda}$ does not seem to allow for a natural introducing of partial sums, restricting the problem to the convergence of one-dimensional distributions in contrast to finite-dimensional distributions in the present paper. While [9] consider several forms of moving-average coefficients, the only case when $a(t, s)$ in (1.6) satisfy the assumptions in [9] seems to be the ‘isotropic’ case $q_1 = q_2$. As explained in Remark 2.2 in the latter case and $\gamma = 1$ our limit results agree with [9], including the ‘edge effect’. We also note Damarackas and Paulauskas [3] who discussed partial sums limits of linear LRD, SRD and ND RFs, possibly with infinite variance and moving-average coefficients which factorize along coordinate axes (i.e., different from (1.6)) in which case the scaling limits in (1.3) do not depend on $\gamma$ and the scaling transition does not exist. See also [23], Remark 4.1.

The following comments are in order. We expect that our results can be generalized to linear ND RFs in higher dimensions $\nu \geq 3$, as well as to linear ND RFs with infinite variance, although the scaling diagram in [23] for LRD RFs and $\nu = 3$ is more complicated. We also expect that our results remain valid, in some sense, for more general sampling regions similar to [9] and ‘anisotropically inflated’ by scaling factors $\lambda$ and $\lambda^\gamma$ in the horizontal and vertical directions.

The rest of the paper is organized as follows. Sec. 2 contains the main results (Theorems 2.1 - 2.3). The proofs of these results are given in Sec. 3. The last Sec. 4 presents two examples of fractionally integrated ND RFs, extending the examples of fractionally integrated LRD RFs in [17].

**Notation.** In what follows, $C$ denote generic positive constants which may be different at different locations. We write $\overset{\text{fdd}}{\longrightarrow}$, $\overset{\text{fdd}}{=}$, and $\not\overset{\text{fdd}}{=}$. for the weak convergence, equality and inequality of finite-dimensional distributions, respectively. $\mathbb{R}_+^\nu := \{ \mathbf{x} = (x_1, \cdots, x_\nu) \in \mathbb{R}^\nu : x_i > 0, i = 1, \cdots, \nu \}$, $\mathbb{R}_+ := \mathbb{R}_+^1$, $\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}$, $\lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\}, x \in \mathbb{R}$. $\mathbf{1}(A)$ stands for the indicator function of a set $A$. 

##
2 Main results

We use the notation:

\[ H_1 := \frac{3}{2} + \frac{q_1}{q_2} - q_1 = \frac{1}{2} + q_1(Q - 1), \quad H_2 := \frac{3}{2} + \frac{q_2}{q_1} - q_2 = \frac{1}{2} + q_2(Q - 1), \]  
\[ \tilde{H}_1 := \frac{3}{2} - q_1 + \frac{q_1}{2q_2} = 1 - \frac{q_1}{2}(2 - Q), \quad \tilde{H}_2 := \frac{3}{2} - q_2 + \frac{q_2}{2q_1} = 1 - \frac{q_2}{2}(2 - Q), \]

\[ \gamma^0 := \frac{q_1}{q_2}, \quad \gamma_{\text{edge},1}^0 := \frac{1}{2q_2(1 - Q)}, \quad \gamma_{\text{edge},2}^0 := 2q_1(1 - Q), \]

\[ Q_{\text{edge},1} := \frac{3}{2q_1} + \frac{1}{q_2}, \quad Q_{\text{edge},2} := \frac{1}{q_1} + \frac{3}{2q_2}, \quad \tilde{Q}_1 := \frac{1}{2q_1} + \frac{1}{q_2}, \quad \tilde{Q}_2 := \frac{1}{q_1} + \frac{1}{2q_2}. \]

Note the partition in Fig. 1 is formed by segments belonging to the lines \( Q_{\text{edge},i} = 1, \tilde{Q}_i = 1, i = 1, 2 \) and \( Q = 1, Q = 2 \). Let

\[ a_{\infty}(t, s) := (|t|^2 + |s|^{2q_2/q_1})^{-q_1/2}L_0\left(t/(|t|^2 + |s|^{2q_2/q_1})^{1/2}\right), \quad (t, s) \in \mathbb{R}^2. \]  
\[ \text{(2.2)} \]

Note \( a_{\infty} \) is the scaling limit of \( a(t, s) \) in (1.6): \( \lim_{\lambda \to \infty} \lambda^{-1/2}a(\langle \lambda t \rangle, \langle \lambda^{q_1/q_2} s \rangle) = a_{\infty}(t, s) \) for any \( (t, s) \in \mathbb{R}^2, (t, s) \neq (0, 0) \). We also use the following notation for kernels of the limit RFs expressed as stochastic integrals w.r.t. Gaussian white noise \( W(du, dv), (u, v) \in \mathbb{R}^2, EW(du, dv) = 0, E(W(du, dv))^2 = du dv \) on \( \mathbb{R}^2 \). Namely, for \( (u, v) \in \mathbb{R}^2, (x, y) \in \mathbb{R}^2 \), let

\[ h_0(x, y; u, v) := \begin{cases} \int_{(0, x) \times (0, y)} a_{\infty}(t - u, s - v)dt ds, & (u, v) \not\in (0, x) \times (0, y), \\ -\int_{\mathbb{R}\setminus(0, x) \times (0, y)} a_{\infty}(t - u, s - v)dt ds, & (u, v) \in (0, x] \times (0, y], \end{cases} \]

\[ h_1(x, y; u, v) := 1(0 < v \leq y) \begin{cases} \int_{(0, x) \times \mathbb{R}} a_{\infty}(t - u, s)ds, & u \not\in (0, x], \\ -\int_{\mathbb{R}\setminus(0, x) \times \mathbb{R}} a_{\infty}(t - u, s)ds, & u \in (0, x], \end{cases} \]

\[ h_2(x, y; u, v) := 1(0 < u \leq x) \begin{cases} \int_{\mathbb{R}\setminus(0, y) \times \mathbb{R}} a_{\infty}(t, s - v)ds, & v \not\in (0, y], \\ -\int_{\mathbb{R}\times(0, y) \setminus(0, y)} a_{\infty}(t, s - v)ds, & v \in (0, y]. \end{cases} \]

We point out that definitions (2.3) apply under ND condition \( Q < 1 \) only. Under the LRD condition \( 1 < Q < 2 \) the definition of the corresponding kernels take a somewhat simpler form:

\[ h_0(x, y; u, v) := \int_{(0, x) \times (0, y]} a_{\infty}(t - u, s - v)dt ds, \]

\[ h_1(x, y; u, v) := 1(0 < v \leq y) \int_{(0, x) \times \mathbb{R}} a_{\infty}(t - u, s)ds, \]

\[ h_2(x, y; u, v) := 1(0 < u \leq x) \int_{\mathbb{R}\times(0, y]} a_{\infty}(t, s - v)ds, \]  
\[ \text{(2.4)} \]
Proposition 2.1\footnote{hold for any $(u, v) \in \mathbb{R}^2$. In \cite{2.12} the kernels $h_i, i = 1, 2$ of \cite{2.3} are explicitly written. For the sake of completeness, we also introduce the kernels}

$$
\tilde{h}_1(x, y; u, v) := x \int_{(0, y]} a_\infty(u, s - v)ds, \quad \tilde{h}_2(x, y; u, v) := y \int_{(0,x]} a_\infty(t - u, v)dt \tag{2.5}
$$

used in the definition of the unbalanced limits in regions $R_{11}, R_{12}, R_{21}$ of Table 1. Using the above notation we define the well-balanced limit

$$
V_0^X(x, y) := \int_{\mathbb{R}^2} h_0(x, y; u, v)W(du, dv), \quad \text{for} \quad Q_{\text{edge},1} \land Q_{\text{edge}2} > 1, 0 < Q < 2, Q \neq 1 \tag{2.6}
$$

in both cases $0 < Q < 1$ (ND) and $1 < Q < 2$ (LRD), with respective $h_0$ \cite{2.3} and \cite{2.4}. Similarly, we define RFs

$$
V^X_+(x, y) := \int_{\mathbb{R}^2} h_1(x, y; u, v)W(du, dv), \quad \text{for} \quad \begin{cases}
Q < 1, & Q_{\text{edge},1} > 1, \\
1 < Q < 2, & Q_1 < 1
\end{cases} \tag{2.7}
$$

and

$$
V^X_-(x, y) := \int_{\mathbb{R}^2} h_2(x, y; u, v)W(du, dv), \quad \text{for} \quad \begin{cases}
Q < 1, & Q_{\text{edge},2} > 1, \\
1 < Q < 2, & Q_2 < 1
\end{cases} \tag{2.8}
$$

Finally, we define RFs

$$
V^X_+(x, y) := \int_{\mathbb{R}^2} \tilde{h}_1(x, y; u, v)W(du, dv), \quad \text{for} \quad 1 < Q < 2, \tilde{Q}_1 > 1 \tag{2.9}
$$

$$
V^X_-(x, y) := \int_{\mathbb{R}^2} \tilde{h}_2(x, y; u, v)W(du, dv), \quad \text{for} \quad 1 < Q < 2, \tilde{Q}_2 > 1
$$

Recall that the stochastic integral $I(h) := \int_{\mathbb{R}^2} h(u, v)W(du, dv)$ w.r.t. the white noise $W$ is well-defined for any $h \in L^2(\mathbb{R}^2)$ and has a Gaussian distribution with zero mean and variance $EI(h)^2 = ||h||^2 := \int_{\mathbb{R}^2} h(u, v)^2dudv$.

**Proposition 2.1** The Gaussian RFs $V_0^X = \{V_0^X(x, y); (x, y) \in \mathbb{R}^2_x\}, V^X_+ = \{V^X_+(x, y); (x, y) \in \mathbb{R}^2_x\}$ are well-defined in respective regions of parameters $q_i > 0, i = 1, 2$ as indicated in \cite{2.6}, \cite{2.7}, \cite{2.8} and \cite{2.9}. Moreover,

$$
V^X_+ \overset{\text{fdd}}{=} \begin{cases}
\sigma_1B_{H_1,1/2}, & Q_{\text{edge},1} > 1, \tilde{Q}_1 < 1, Q \neq 1, \\
\tilde{\sigma}_1B_{1,\tilde{H}_2}, & \tilde{Q}_1 > 1
\end{cases} \tag{2.10}
$$

and

$$
V^X_- \overset{\text{fdd}}{=} \begin{cases}
\sigma_2B_{1/2,\tilde{H}_2}, & Q_{\text{edge},2} > 1, \tilde{Q}_2 < 1, Q \neq 1, \\
\tilde{\sigma}_2B_{\tilde{H}_1,1}, & \tilde{Q}_2 > 1
\end{cases} \tag{2.11}
$$

where $\sigma_i := ||h_i(1,1;\cdot,\cdot)|| < \infty, \tilde{\sigma}_i := ||\tilde{h}_i(1,1;\cdot,\cdot)|| < \infty, i = 1, 2.$
We note that for $1 < Q < 2$ the statement of Proposition 2.1 is part of (17, Thms. 3.1-3.3). For $Q < 1$ and $h_0$ in (2.3) the fact that $h_0 \in L^2(\mathbb{R}^2)$ (i.e., that $V_0^X$ is well-defined) follows from Lemma 3.1 (3.22)-(3.23). The statements (2.10) and (2.11) for $Q < 1$ follow by rewriting the kernels $h_i, i = 1, 2$ in (2.3) as

$$h_1(x, y; u, v) = 1(0 < v \leq y) \begin{cases} \int_{[0,x]} |t-u|^{-q_1(1-\frac{1}{Q})} L_{1,\text{sign}(t-u)} dt, & u \notin (0, x], \\ - \int_{\mathbb{R}\setminus[0,x]} |t-u|^{-q_1(1-\frac{1}{Q})} L_{1,\text{sign}(t-u)} dt, & u \in (0, x], \end{cases}$$

$$h_2(x, y; u, v) = 1(0 < u \leq x) \begin{cases} \int_{[0,y]} |s-v|^{-q_2(1-\frac{1}{Q})} L_{2,\text{sign}(s-v)} ds, & v \notin (0, y], \\ - \int_{\mathbb{R}\setminus[0,y]} |s-v|^{-q_2(1-\frac{1}{Q})} L_{2,\text{sign}(s-v)} ds, & v \in (0, y], \end{cases}$$

where

$$L_{1,\pm} := \int_{\mathbb{R}} a_\infty(\pm 1, s) ds, \quad L_{2,\pm} := \int_{\mathbb{R}} a_\infty(t, \pm 1) dt$$

(note $L_{2,+} = L_{2,-}$ since $a_\infty(t, s) = a_\infty(t, -s)$ is symmetric in $s$). Furthermore, the above integrals can be rewritten as

$$h_1(x, y; u, v) = \frac{1(0 < v \leq y)}{(1/2) - H_1} \left( L_{1,+} \left( (x-u)^{H_1-(1/2)} - (-u)^{H_1-(1/2)} \right) + L_{1,-} \left( (x-u)^{H_1-(1/2)} - (-u)^{H_1-(1/2)} \right) \right),$$

$$h_2(x, y; u, v) = \frac{1(0 < u \leq x)}{(1/2) - H_2} \left( L_{2,+} \left( (y-v)^{H_2-(1/2)} - (-v)^{H_2-(1/2)} \right) + L_{2,-} \left( (y-v)^{H_2-(1/2)} - (-v)^{H_2-(1/2)} \right) \right),$$

with $H_i, i = 1, 2$ defined in (2.1). Whence we immediately see that $h_i, i = 1, 2$ factorize into a product of kernels of (fractional) Brownian motions with one-dimensional time, see [21], implying facts (2.10) and (2.11) for $Q < 1$.

The main object of this paper is the scaling transition for ND RF $X$, viz.,

$$A_{\lambda,\gamma}^{-1}S_{\lambda,\gamma}^X(x, y) \overset{\text{fdd}}{\longrightarrow} \begin{cases} V_+^X(x, y), & \gamma > \gamma_0^X, \\ V_-^X(x, y), & \gamma < \gamma_0^X, \\ V_0^X(x, y), & \gamma = \gamma_0^X, \end{cases}$$

where $A_{\lambda,\gamma} \to \infty (\lambda \to \infty)$ is normalization and

$$S_{\lambda,\gamma}^X(x, y) = \sum_{(t,s) \in K_{[\lambda x,\lambda^\gamma y]}} X(t, s), \quad (x, y) \in \mathbb{R}_+^2$$

is the partial sum of RF $X$ on rectangle $K_{[\lambda x,\lambda^\gamma y]} = \{(t, s) \in \mathbb{Z}^2 : 1 \leq t \leq [\lambda x], 1 \leq s \leq [\lambda^\gamma y]\}$. This question is treated in Theorems 2.2, 2.4. In these theorems, $X$ is a linear RF as in (1.5) with moving-average coefficients $a(t, s)$ satisfying (1.6), (1.8), where the parameters $q_1, q_2$ satisfy $0 < Q < 1$ and belong to different regions of Fig. 1.
Theorem 2.2 [Region $R^{-}_{22}$: ‘no edge effects’] Let $Q_{\text{edge},1} \cap Q_{\text{edge},2} > 1$. Then the convergence in (2.13) holds with $\gamma_0^X = \gamma^0 = q_1/q_2$, $A_{\lambda,\gamma} = \lambda^{H(\gamma)}$ and the limit RFs specified in (2.7), (2.8), viz.,

\[
\begin{align*}
V_+^X &= \sigma_1 B_{H_1,1/2}, \\
V_-^X &= \sigma_2 B_{1/2,H_2}, \\
V_0^X &= V_0,
\end{align*}
\]

\[
H(\gamma) = \begin{cases} 
H_1 + (\gamma/2), & \gamma > \gamma^0, \\
\gamma H_2 + (1/2), & \gamma < \gamma^0,
\end{cases}
\]

where $H_i, \sigma_i, i = 1, 2$ are given in Proposition 2.1

Define

\[
\sigma_{\text{edge},1}^2 := 2 \sum_{v \geq 0} \left( \sum_{t \in \mathbb{Z}, s \geq 0} a(t, s + v) \right)^2 + 2 \sum_{t \leq 0} \left( \sum_{t \in \mathbb{Z}, s \leq 0} a(t, s + v) \right)^2,
\]

\[
\sigma_{\text{edge},2}^2 := 2 \sum_{u \geq 0} \left( \sum_{t \geq 0, s \in \mathbb{Z}} a(t + u, s) \right)^2 + 2 \sum_{u \leq 0} \left( \sum_{t \leq 0, s \in \mathbb{Z}} a(t + u, s) \right)^2.
\]

The convergence of the above series in the corresponding regions of parameters $q_1, q_2$ will be established later.

Theorem 2.3 [Regions $R_{23}$ and $R_{32}$: ‘one-sided edge effects’]

(i) Let $Q_{\text{edge},2} < 1 < Q_{\text{edge},1}$. Then the convergence in (2.13) for $\gamma \neq \gamma_0^X$ holds with $\gamma_0^X = \gamma_{\text{edge},2}^0$, $A_{\lambda,\gamma} = \lambda^{H(\gamma)}$ and the limit RFs specified in (2.7), (2.8), viz.,

\[
\begin{align*}
V_+^X &= \sigma_1 B_{H_1,1/2}, \\
V_-^X &= \sigma_{\text{edge},1} B_{1/2,0}, \\
V_0^X &= V_0,
\end{align*}
\]

\[
H(\gamma) = \begin{cases} 
H_1 + (\gamma/2), & \gamma > \gamma_{\text{edge},2}^0, \\
1/2, & \gamma < \gamma_{\text{edge},2}^0,
\end{cases}
\]

with $H_1, \sigma_1$ as in Proposition 2.1 and $\sigma_{\text{edge},1} < \infty$ defined in (2.14).

(ii) Let $Q_{\text{edge},1} < 1 < Q_{\text{edge},2}$. Then the convergence in (2.13) for $\gamma \neq \gamma_0^X$ holds with $\gamma_0^X = \gamma_{\text{edge},1}^0$, $A_{\lambda,\gamma} = \lambda^{H(\gamma)}$ and the limit RFs specified in (2.7), (2.8), viz.,

\[
\begin{align*}
V_+^X &= \sigma_2 B_{1/2,H_2}, \\
V_-^X &= \sigma_{\text{edge},2} B_{0,1/2}, \\
V_0^X &= V_0,
\end{align*}
\]

\[
H(\gamma) = \begin{cases} 
\gamma H_2 + (1/2), & \gamma < \gamma_{\text{edge},1}^0, \\
\gamma/2, & \gamma > \gamma_{\text{edge},1}^0,
\end{cases}
\]

with $H_2, \sigma_2$ as in Proposition 2.1 and $\sigma_{\text{edge},2} < \infty$ defined in (2.14).

Theorem 2.4 [Region $R_{33}$: ‘two-sided edge effects’] Let $Q_{\text{edge},1} \lor Q_{\text{edge},2} < 1$. Then the convergence in (2.13) holds with $\gamma_0^X = 1$, $A_{\lambda,\gamma} = \lambda^{H(\gamma)}$ and the limit RFs specified in (2.7), (2.8), viz.,

\[
\begin{align*}
V_+^X &= \sigma_{\text{edge},2} B_{0,1/2}, \\
V_-^X &= \sigma_{\text{edge},1} B_{1/2,0}, \\
V_0^X &= \sigma_{\text{edge},1} B_{1/2,0} + \sigma_{\text{edge},2} B_{0,1/2}
\end{align*}
\]

\[
H(\gamma) = \begin{cases} 
\gamma/2, & \gamma > 1, \\
1/2, & \gamma < 1, \\
1/2, & \gamma = 1,
\end{cases}
\]

where $B_{1/2,0}$ and $B_{0,1/2}$ are independent RFs and $\sigma_{\text{edge},i}, i = 1, 2$ are given in (2.14).
Remark 2.1 The ‘edge effects’ in (i) and (ii) of Theorem 2.3 may be also respectively labelled as ‘horizontal’ and ‘vertical’. Note the normalizations $\lambda^H(\gamma) = \lambda^{1/2}$ in (i), $\gamma < \gamma_{\text{edge},2}^0$ (respectively, $\lambda^H(\gamma) = \lambda^{\gamma/2}$ in (ii), for $\gamma > \gamma_{\text{edge},1}^0$) is proportional to the the square root of the horizontal (respectively, vertical) length of the rectangle $K_{[\lambda x, \lambda y]}$ suggesting that the sum $S_{\lambda, \gamma}^N(x, y)$ behaves as a sum of weakly dependent r.v.’s indexed by points on the horizontal (respectively, vertical) edges of this rectangle.

Remark 2.2 In the ‘isotropic’ case $q_1 = q_2 = \beta$ we have that $Q = 2/\beta \in (0, 1)$ is equivalent to $\beta \in (2, \infty)$ and $Q_{\text{edge},1} = Q_{\text{edge},2} = 5/(2\beta)$. In this case either Theorem 2.2 (for $2 < \beta < 5/2$) or Theorem 2.4 (for $5/2 < \beta < \infty$) applies and our results can be related to ([9], Theorems 4.1 and 4.3). In the case $2 < \beta < 5/2$ Theorem 2.2 for $\gamma = \gamma^0 = 1$ yields the Gaussian limit with the variance $\|h_0(x, y; \cdot, \cdot)\|^2$ which coincides with the variance in ([9], (4.2)). In the interval $5/2 < \beta < \infty$ the coincidence of the limiting variances in Theorem 2.2 $\gamma = 1$ and ([9], (4.5)) is less straightforward and can be verified by noting that for any $\delta > 0$

$$x\sigma_{\text{edge},1}^2 + y\sigma_{\text{edge},2}^2 = \lim_{\lambda \to \infty} \lambda^{-1} \sum_{(u,v) \in \mathbb{Z}^2 : \text{dist}((u,v), \partial K_{[\lambda x, \lambda y]}) \leq \delta \lambda} \left( \sum_{(t,s) \in K_{[\lambda x, \lambda y]}} a(t-u, s-v) \right)^2,$$

(2.15)

where $\partial K_{[\lambda x, \lambda y]} = \{(t,s) \in K_{[\lambda x, \lambda y]} : \text{dist}((t,s), K_{[\lambda x, \lambda y]}^c) = 1\}$ is the boundary of $K_{[\lambda x, \lambda y]}$, $K_{[\lambda x, \lambda y]}^c = \mathbb{Z}^2 \setminus K_{[\lambda x, \lambda y]}$ and $\text{dist}((u,v), A) = \inf_{(t,s) \in A} \| (t,s) - (u,v) \|$ is the distance between $(u,v) \in \mathbb{Z}^2$ and $A \subset \mathbb{Z}^2$. (2.15) follows by using the summability properties of $|a(t,s)|$ guaranteeing the convergence of the series in (2.14) and the zero-sum condition (1.8).

Remark 2.3 In Theorem 2.3 the convergence in (2.13) at $\gamma = \gamma_0^X$ is an open question. We conjecture that the corresponding limits agree with $B_{1/2,0}$ for $\gamma_0^X = \gamma_{\text{edge},2}^0$ and $B_{0,1/2}$ for $\gamma_0^X = \gamma_{\text{edge},1}^0$ however the normalizations might involve additional logarithmic factors making the proofs more difficult. A similar difficulty may accompany the attempts to extend (2.13) to values $q_1, q_2$ belonging to the boundaries of the partition in Fig.1, particularly to the segments $Q_i = 1, Q_{\text{edge},i} = 1, i = 1, 2$. On the other hand, the normalizing exponent $H(\gamma) \equiv H(\gamma, q_1, q_2)$ in Theorems 2.2, 2.3, and in ([17], Thms. 3.1-3.3, case $k = 1$) extends by continuity to a (jointly) continuous function on the set $\{ \gamma > 0, 0 < Q < 2 \}$ suggesting that the above extension of the scaling transition is plausible.

Remark 2.4 Note $\sigma_{\text{edge},1}$ and/or $\sigma_{\text{edge},2}$ in (2.14) may vanish provided

$$\sum_{t \in \mathbb{Z}} a(t,s) = 0 \quad (\forall t \in \mathbb{Z}) \quad \text{and/or} \quad \sum_{s \in \mathbb{Z}} a(t,s) = 0 \quad (\forall s \in \mathbb{Z})$$

(2.16)

hold. Clearly, each of the conditions in (2.16) imply (1.8) and the ND property of the corresponding RF, but the converse implication is not true. Relations (2.16) may be termed the vertical (respectively, horizontal) ND property. Under (2.16) the corresponding limits in Theorems 2.3 and 2.4 are trivial, while nontrivial limits may exists under different conditions on $\gamma$ and $q_1, q_2$, eventually changing Fig. 1.
3 Proofs of Theorems 2.2-2.4

3.1 Outline of the proof and preliminaries

Let us explain the main steps of the proof of Theorems 2.2-2.4. By definition, $S^X_{\lambda,\gamma}(x, y)$ can be rewritten as a weighted sum

$$S^X_{\lambda,\gamma}(x, y) = \sum_{(u, v) \in \mathbb{Z}^2} \varepsilon(u, v) G_{\lambda,\gamma}(u, v), \quad G_{\lambda,\gamma}(u, v) := \sum_{(t, s) \in K[\lambda x, \lambda \gamma y]} a(t - u, s - v) \quad (3.1)$$

in i.i.d.r.v.s $\varepsilon(u, v), (u, v) \in \mathbb{Z}^2$. It happens that different regions of ‘noise locations’ $(u, v)$ contribute to different limit distributions in our theorems.

Figure 2. Partition of $\mathbb{Z}^2$ by $K = K[\lambda x, \lambda \gamma y]$ and 8 ‘outer’ sets $K^c_{i,j}, i, j = 1, 0, -1, (i, j) \neq (0, 0)$. The shaded regions $\partial_{0,1} K, \partial_{0,-1} K$ (respectively, $\partial_{1,0} K, \partial_{-1,0} K$) are related to the ‘horizontal (respectively, vertical) edge effect’.
The basic decomposition

$$\mathbb{Z}^2 = K \cup \bigcup_{(i,j) = -1,0,1,(i,j) \neq (0,0)} K_{i,j}^c,$$  \hspace{1cm} (3.2)

of $\mathbb{Z}^2$ into 9 sets is shown in Fig. 2. Following (3.2) we decompose (3.1) as

$$S_{\lambda,\gamma}(x,y) = S(K) + \sum_{i,j = -1,0,1,(i,j) \neq (0,0)} S(K_{i,j}^c), \hspace{1cm} (3.3)$$

where, with zero-sum condition [1.8] in mind,

$$S(K) := \sum_{(u,v) \in K} \varepsilon(u,v)G_{\lambda,\gamma}(u,v) = - \sum_{(u,v) \in K} \varepsilon(u,v)G_{\lambda,\gamma}^c(u,v), \hspace{1cm} (3.4)$$

$$S(K_{i,j}^c) := \sum_{(u,v) \in K_{i,j}^c} \varepsilon(u,v)G_{\lambda,\gamma}(u,v), \hspace{1cm} G_{\lambda,\gamma}^c(u,v) := \sum_{(t,s) \not\in K} a(t-u, s-v).$$

As shown in Lemma 3.2,

- terms $S(K), S(K_{-1,0}^c), S(K_{0,1}^c)$ in (3.3) contribute to the FBS limit $\sigma_1 B_{1,2}^{H_1,1} / 2$ for $\gamma > \gamma_0^X$ and the remaining terms in (3.3) are negligible;

- terms $S(K), S(K_{0,-1}^c), S(K_{0,1}^c)$ in (3.3) contribute to the FBS limit $\sigma_2 B_{1/2,2}^{1/2}$ for $\gamma < \gamma_0^X$ and the remaining terms in (3.3) are negligible;

- all terms on the r.h.s. of (3.3) are relevant for proving the well-balanced limit $V_0^X$ in Theorem 2.2;

- the main contribution to the unbalanced limit $\sigma_{edge,1} B_{1/2,0}$ in Theorems 2.3-2.4 comes from innovations ‘close’ to the horizontal edges of $K$ (the shaded regions $\partial_{0,1} K$ and $\partial_{0,-1} K$ in Fig. 2);

- the main contribution to the unbalanced limit $\sigma_{edge,2} B_{0,1/2}$ in Theorems 2.3-2.4 comes from innovations ‘close’ to the vertical edges of $K$ (the shaded regions $\partial_{1,0} K$ and $\partial_{-1,0} K$ in Fig. 2).

After identification of the main terms, the limit distributions of these terms is obtained using a general criterion for the convergence of linear forms in i.i.d.r.v.s towards stochastic integral w.r.t. white noise. Consider a linear form

$$S(g) := \sum_{(u,v) \in \mathbb{Z}^2} g(u,v)\varepsilon(u,v)$$  \hspace{1cm} (3.5)

with real coefficients $\sum_{(u,v) \in \mathbb{Z}^2} g(u,v)^2 < \infty$. The following proposition is a version of ([6], Prop. 14.3.2) and ([23], Prop. 5.1) and we omit the proof.

**Proposition 3.1** Let $S(g_\lambda, \lambda > 0$ be as in (3.5). Suppose $g_\lambda(u,v)$ are such that for a real-valued function $h \in L^2(\mathbb{R}^2)$ and some $m_i = m_i(\lambda) \to \infty, \lambda \to \infty, i = 1,2$ the functions

$$\tilde{g}_\lambda(u,v) := (m_1 m_2)^{1/2} g_\lambda(\lfloor m_1 u \rfloor, \lfloor m_2 v \rfloor), \hspace{1cm} (u,v) \in \mathbb{R}^2$$  \hspace{1cm} (3.6)
tend to $h$ in $L^2(\mathbb{R}^2)$, viz.,

$$
\|\tilde{g}_\lambda - h\|^2 = \int_{\mathbb{R}^2} |\tilde{g}_\lambda(u,v) - h(u,v)|^2 du dv \to 0, \quad \lambda \to \infty.
$$

(3.7)

Then

$$
S(g_\lambda) \overset{d}{\rightarrow} I(h) := \int_{\mathbb{R}^2} h(u,v)W(du, dv), \quad \lambda \to \infty.
$$

(3.8)

### 3.2 Two auxiliary lemmas

The proofs of our results requires evaluation of various multiple series involving the coefficients $a(t, s)$. Similarly as in [17], [23] these series are estimated by corresponding multiple integrals involving the function $a_\infty(t, s)$ in (2.2) or

$$
\rho(t, s) := (|t|^{q_1} + |s|^{q_2})^{-1}, \quad (t, s) \in \mathbb{R}^2.
$$

(3.9)

Indeed, by elementary inequalities

$$
|a(t, s)| \leq C \rho(t, s) \quad ((t, s) \in \mathbb{Z}^2), \quad |a_\infty(t, s)| \leq C \rho(t, s) \quad ((t, s) \in \mathbb{R}^2)
$$

(3.10)

see ([23], (2.16)) the function $\rho$ in (3.11)-(3.26) can be replaced by $|a_\infty|$ while most multiple sums involving $a(t, s)$ in the proofs below can be evaluated by corresponding multiple integrals in Lemma 3.1.

**Lemma 3.1** Let $\rho(t, s)$ be defined as in (3.9), where $q_i > 0, i = 1, 2$. Then for any $\delta > 0, h > 0$

$$
\int_{\mathbb{R}^2} \rho(t, s)^h 1(\rho(t, s) > \delta) dt ds < \infty, \quad Q > h,
$$

(3.11)

$$
\int_{\mathbb{R}^2} \rho(t, s)^h 1(\rho(t, s) < \delta) dt ds < \infty, \quad Q < h.
$$

(3.12)

In addition, let $Q < 1$. Then

$$
\int_0^\infty du \left( \int_0^1 dt \int_0^\infty \rho(t + u, s) ds \right)^2 < \infty, \quad Q_{\text{edge}, 1} < 1,
$$

(3.13)

$$
\int_0^\infty du \left( \int_0^1 dt \int_0^\infty \rho(t + u, s) ds \right)^2 < \infty, \quad Q_{\text{edge}, 1} > 1,
$$

(3.14)

$$
\int_0^\infty dv \left( \int_0^\infty dt \int_0^1 \rho(t, s + v) ds \right)^2 < \infty, \quad Q_{\text{edge}, 2} < 1,
$$

(3.15)

$$
\int_0^\infty dv \left( \int_0^\infty dt \int_0^1 \rho(t, s + v) ds \right)^2 < \infty, \quad Q_{\text{edge}, 2} > 1,
$$

(3.16)

$$
\int_0^\infty du \left( \int_0^1 dt \int_0^1 \rho(t + u, s - v) ds \right)^2 < \infty, \quad Q_{\text{edge}, 1} > 1,
$$

(3.17)

$$
\int_0^1 du \int_0^\infty dv \left( \int_0^1 dt \int_0^1 \rho(t - u, s + v) ds \right)^2 < \infty, \quad Q_{\text{edge}, 2} > 1,
$$

(3.18)

$$
\int_0^\infty du \int_0^\infty dv \left( \int_0^1 dt \int_0^1 \rho(t + u, s + v) ds \right)^2 < \infty, \quad Q > 2/3,
$$

(3.19)
Proof of (3.13). Write the l.h.s. of (3.13) as
\[
\int_0^\infty \int_0^\infty \int_0^1 dt \int_0^\infty \rho(t+u,s+v) ds < \infty, \quad 2/3 < Q < \frac{1}{q_1} + \frac{3}{q_2} < 2, \tag{3.20}
\]
\[
\int_0^\infty \int_0^\infty \int_0^1 dt \int_0^\infty \rho(t+u,s+v) ds < \infty, \quad 2/3 < Q < \frac{3}{q_1} + \frac{1}{q_2} < 2, \tag{3.21}
\]
\[
\int_{\mathbb{R}^2 \setminus [0,1]^2} du dv \int_0^\infty dt \int_0^1 \rho(t-u,s-v) dt ds < \infty, \quad 1 < Q_{\text{edge},1} \wedge Q_{\text{edge},2}, \tag{3.22}
\]
\[
\int_{[0,1]^2} du dv \int_{\mathbb{R}^2 \setminus [0,1]^2} \rho(t-u,s-v) dt ds < \infty, \quad 1 < Q_{\text{edge},1} \wedge Q_{\text{edge},2}, \tag{3.23}
\]

Furthermore, as \( \mu \to \infty \)
\[
\int_0^\infty \int_0^\infty \int_0^1 dt \int_0^\mu \rho(t+u,s+v) ds < \infty \quad \text{and using (3.12)}.
\]
\[
\int_0^\infty \int_0^\infty \int_0^1 dt \int_0^\mu \rho(t+u,s+v) ds < \infty, \quad 1 < \frac{1}{2q_1} + \frac{3}{2q_2}; \tag{3.24}
\]
\[
\int_0^\infty \int_0^\infty \int_0^1 dt \int_0^\mu \rho(t+u,s+v) ds < \infty, \quad 1 < \frac{3}{2q_1} + \frac{1}{2q_2}; \tag{3.25}
\]
\[
\int_0^\infty dv \left( \int_0^1 dt \int_0^\mu \rho(t,s+v) ds \right)^2 < \infty, \quad \mu = (3-2q_2)^{1/2}. \tag{3.26}
\]

Proof. Relations (3.11) and (3.12) are proved in [17], Prop. 5.1.

Proof of (3.13). Write the l.h.s. of (3.13) as \( I = \int_0^\infty du \int_0^\infty \int_1^\infty dt \int_0^\infty \rho(t-u,|u|+v|s|^q) ds < \infty \) where \( I_1 \) is analogous.

Proof of (3.14). By integrating the inner integral w.r.t. \( t \in (0,\infty) \) the l.h.s. of (3.14) can be written as \( I = C \int_0^\infty dv \left( \int_0^\infty (s+v)^{(q_2/q_1)-q_2} ds \right)^2 \). Then \( I_1 \leq C \int_0^1 \frac{v}{u} dv < \infty \) since \( Q_{\text{edge},2} > 1 \) and \( I_2 \leq C \int_1^\infty v u^{(q_2/q_1)-q_2} dv < \infty \) since \( (1/q_1) < 1 \). The proof of (3.14) is analogous.

Proof of (3.17). Write the l.h.s. of (3.17) as \( I = I_1 + I_2 \) where \( I_1 := \int_0^1 u dv \), \( I_2 := \int_0^\infty u dv \). Then \( I_1 \leq C \int_0^\infty \rho(u,0)^2 du < \infty \) and, by change of variables \( t \to ut, s \to u^{|q_1/q_2|} s \), \( I_2 \leq \int_0^1 u^{2+2(q_1/q_2)-2q_2} du \). The proof of (3.18) is analogous.

Proof of (3.19). Split the l.h.s. of (3.19) as \( I = I_1 + I_2 \), where \( I_1 := \int_0^\infty du \int_0^\infty dv \), \( I_2 := \int_0^\infty du \int_0^\infty dv \). Then \( I_1 \leq C \int_0^\infty \rho(u,v)^2 du dv \leq C \) due to (3.12). Next by change of variables: \( t \to \rho(u,v)^{-1/q_1} t, s \to \rho(u,v)^{-1/q_2} s \). Then \( I_2 \leq C \int_{(0,\infty)^2 \setminus (0,1)^2} \rho(u,v)^2 du dv \leq C \) due to (3.12).
where the first integral converges due to (3.11) and \( Q > 2/3 \), and the second due to (3.12) and \( Q < 1 \). This proves (3.19).

Proof of (3.20). Split the l.h.s. of (3.20) as \( I = I_1 + I_2 \), where \( I_1 := \int_{(0,1) \times D} dv \cdots, I_2 := \int_{(1,\infty) \times D} dv \cdots \). Then \( I_1 \leq C \int_0^\infty dv \left( \int_0^\infty (s+v)^{-q_2} ds \right)^2 \leq C \int_0^\infty dv \left( 1 \wedge v^{-2(q_2-1)} \right) < \infty \) since \( 2(q_2-1) > 1 \) follows from \( 3/q_2 < 2 \). Next, \( I_2 \leq C \int_1^\infty du \left( \int_0^\infty (u^{q_1} + (s+v)^q)^{-1} ds \right)^2 \leq C J_1 J_2 \), where \( J_1 := \int_1^\infty u^{(q_1/q_2) - 2q_2} du < \infty \) for \( \frac{1}{q_1} + \frac{3}{q_2} < 2 \) and

\[
J_2 := \int_0^\infty dv \left( \int_0^\infty (1 + (s+v)^q)^{-1} ds \right)^2 \leq C \int_0^1 dv \left( \int_0^\infty (1 + s^{q_2})^{-1} ds \right)^2 + C \int_1^\infty dv \left( \int_0^v s^{-q_2} ds + \int_v^\infty s^{-q_2} ds \right)^2 \leq C + C \int_1^\infty v^{2-2q_2} dv < \infty
\]

since \( q_2 > 3/2 \). This proves (3.20) and (3.21) is analogous.

Proof of (3.22). Split \( \mathbb{R}^2 \setminus [0,1]^2 = \bigcup_{i=1}^8 D_i \) similarly as in Fig. 2 and then the l.h.s. of (3.21) as \( I = \sum_{i=1}^8 I_i \) where \( I_i := \int_{D_i} \left( \int_{[0,1]^2} \rho(t-u,s-v) dt ds \right)^2 dv, D_1 := (-\infty,0)^2, D_2 := (-\infty,0) \times (0,1), D_3 := (0,1) \times (-\infty,0) \) etc. Here, \( I_1 < \infty \) according to (3.19). Similarly, \( I_2 = \int_{(0,\infty) \times (0,1)} dv \left( \int_{[0,1]^2} \rho(t+u,s-v) dt ds \right)^2 < \infty \) and \( I_3 = \int_{(0,1) \times (0,\infty)} dv \left( \int_{[0,1]^2} \rho(t-u,s+v) dt ds \right)^2 < \infty \) follow from (3.17) and (3.18), respectively. The remaining relations \( I_i < \infty, i = 3, \ldots, 8 \) follow from (3.17)-(3.19) in a similar way, proving (3.22).

Proof of (3.23). Split \( \mathbb{R}^2 \setminus [0,1]^2 = \bigcup_{i=1}^8 D_i' \), where \( D_0' := \mathbb{R}^2 \setminus [-1,2]^2, D'_1 := (-1,0)^2, D'_2 := (-1,0) \times (0,1), D'_3 := (-1,0) \times (1,2) \) etc. Accordingly, for the l.h.s. \( J \) of (3.23) we have \( J \leq C \sum_{i=0}^8 J_i' \), where \( J_i' := \int_{D_i'} dv \left( \int_{D_i} \rho(t-u,s-v) dt ds \right)^2, i = 0, 1, \ldots, 8 \). Note \( \rho(t-u,s-v) \leq \delta \) for some \( \delta > 0 \) and all \( (s,v) \in [0,1]^2, (t,s) \in D_0' \), implying \( J_i' \leq \left( \int_{[0,1]^2} \rho(t,s) dt ds \right)^2 < \infty \) according to (3.12). The proof of \( J_i' < \infty, i = 1, \ldots, 8 \) uses (3.17)-(3.19) and is analogous to that of \( I_i < \infty, i = 1, \ldots, 8 \) in the proof of (3.22) above. For instance, \( J_2' = \int_{[0,1]^2} dv \left( \int_{(-1,0) \times (0,1)} \rho(t-u,s-v) dt ds \right)^2 = \int_{[0,1]^2} dv \left( \int_{(0,1)^2} \rho(t+u,s-v) dt ds \right)^2 < \infty \) according to (3.17). This proves (3.23).

Proof of (3.24). Split the l.h.s. of (3.24) as \( J_\mu = \int_{\mathbb{R}^2 \setminus [0,1]^2} dv \cdots + \int_{[0,1]^2} dv \cdots =: J'_\mu + J''_\mu \). Then \( J''_\mu \leq \left( \int_{[0,1]^2} \rho(0,s) ds \right)^2 \leq C \) since \( q_2 > 1 \). Next, consider \( J'_\mu = \int_0^1 dv \int_0^1 dv \cdots + \int_0^\infty dv \cdots =: I'_\mu + I''_\mu \). We have \( I'_\mu \leq \int_0^\infty dv \left( \int_0^1 \rho(0,s+v) ds \right)^2 = \int_0^1 dv \left( \int_0^1 \rho(s+v)^{-q_2} ds \right)^2 + \int_0^\infty (\mu/v^{q_2})^2 dv \) and hence

\[
I''_\mu = O(\mu^{(3-2q_2)v_0})(1 + 1(3 = 2q_2) \log \mu)
\]

satisfies the bound in (3.24).

Consider \( I''_\mu = \int_1^\infty dv \left( \int_0^1 dv \cdots + \right) \leq L'_\mu + L''_\mu \). Here, \( L''_\mu \leq \int_1^\infty dv \int_1^\infty (\mu \rho(u,v))^2 dv = \mu^2 \int_1^\infty dv \int_1^\infty (u^{q_1} + v^{q_2})^{-2} dv = \mu^2 \int_1^\infty u^{(q_1/q_2) - 2q_1} du \int_{u^{q_1/q_2}}^{1 + v^{q_2}} (1 + v^{q_2})^{-2} dv \) where the last integral is
evaluated by splitting it over $u < \mu^{q_2/q_1}$ and $u > \mu^{q_2/q_1}$ yielding

$$L''_\mu = O(\mu^{3-2q_2+(q_2/q_1)}).$$

Finally, $L'_\mu = \int_{\mu^{q_2/q_1}}^{1} du \int_{0}^{\mu} dv \cdots + \int_{\mu^{q_2/q_1}}^{\infty} du \int_{0}^{\mu} dv \cdots =: L'_{\mu,1} + L'_{\mu,2}$ where

$$L'_{\mu,1} \leq \int_{1}^{\mu^{q_2/q_1}} u^{3(q_1/q_2)-2q_1} du \int_{0}^{\mu^{q_1/q_2}} dv (\int_{0}^{\infty} ds \frac{1}{1+(s+v)^{q_2}})^2 \leq C \int_{1}^{\mu^{q_2/q_1}} u^{3(q_1/q_2)-2q_1} du \int_{0}^{\mu^{q_1/q_2}} dv (\int_{0}^{\infty} ds \frac{1}{1+(s+v)^{q_2}}) = O(\mu^{3-2q_2+(q_2/q_1)}(1 + 1(q_2 = 3/2)\log \mu))$$

and $L'_{\mu,2} \leq C \mu^3 \int_{\mu^{q_2/q_1}}^{\infty} u^{-2q_1} du = O(\mu^{3-2q_2+(q_2/q_1)}),$ proving (3.24). The proof of (3.25) is completely analogous, by exchanging $t$ and $s$.

Proof of (3.26). Split the l.h.s. of (3.26) as $I_\mu = \int_{0}^{1} du (\cdots)^2 + \int_{1}^{\infty} du (\cdots)^2 =: I_{1,\mu} + I_{2,\mu}$. Then $I_{1,\mu} \leq (\int_{0}^{\infty} \rho(0,s)ds)^2 < C$ and $I_{2,\mu} \leq \int_{1}^{\infty} du (\int_{1}^{\infty} (s+v)^{-q_2} ds)^2$ satisfies the bound in (3.26) since $q_2 > 1$. This proves (3.26) and completes the proof of the lemma.

We use the above lemma for evaluation of ‘remainder terms’ in (3.4), where

$$K_{c_{1,-1}} := \{(u,v) \in \mathbb{Z}^2 : u \leq 0, v \leq 0\}, \ K_{c_{1,-1}} := \{(u,v) \in \mathbb{Z}^2 : u > [\lambda x], v \leq 0\},$$

$$K_{0_{-1}} := \{(u,v) \in \mathbb{Z}^2 : 1 \leq u \leq [\lambda x], v \leq 0\};$$

the remaining sets analogously defined, viz., $K_{c_{1,0}} := \{(u,v) \in \mathbb{Z}^2 : u \leq 0, 1 \leq v \leq [\lambda^2 y]\}, K_{c_{1,0}} := \{(u,v) \in \mathbb{Z}^2 : u > [\lambda x], 1 \leq v \leq [\lambda^2 y]\}; K_{c_{-1,1}} := \{(u,v) \in \mathbb{Z}^2 : u \leq 0, v > [\lambda^2 y]\}, K_{0_{-1}} := \{(u,v) \in \mathbb{Z}^2 : 1 \leq u \leq [\lambda x], v > [\lambda^2 y]\}, K_{c_{1,1}} := \{(u,v) \in \mathbb{Z}^2 : u > [\lambda x], v > [\lambda^2 y]\}.$

**Lemma 3.2** Let $0 < Q < 1, Q_{edge,i} \neq 1, i = 1, 2$. Then

$$E(S(K_{i,j}^c))^2 = o(\lambda^{2H(\gamma)}), \quad \lambda \to \infty$$

(3.28)

in the following cases:

$$\gamma > \gamma_0^X, \quad (i, j) = (-1, -1), (1, -1), (1, 1), (1, -1), (0, -1), (0, 1)$$

(3.29)

and

$$\gamma < \gamma_0^X, \quad (i, j) = (-1, -1), (1, -1), (1, 1), (1, -1), (-1, 0), (1, 0).$$

(3.30)

**Proof.** In view of the bound in (3.10) and the reflection symmetry of the function $\rho(t,s)$ w.r.t. the coordinate axes, it suffices to prove (3.29) for $(i, j) = (-1, -1)$ and $(i, j) = (-1, 0)$. Moreover, it suffices to consider $x = y = 1$. 

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Case \((i,j) = (-1, -1)\). Denote \(J_{\lambda, \gamma}\) the l.h.s. of (3.28) for \(x = y = 1, i = j = -1\). Then using (3.10)

\[
J_{\lambda, \gamma} = \sum_{u,v \geq 0} \left( \sum_{t=1}^{[\lambda]} \sum_{s=1}^{[\gamma]} a(t+u,s+v) \right)^2 \\
\leq C \int_{\mathbb{R}^2_+} dudv \left( \int_0^\lambda \int_0^{\lambda_0} \rho(t+u,s+v) dtds \right)^2 \\
= C \begin{cases} 
\lambda^{3+3\gamma^0 - 2q_1} \int_{\mathbb{R}^2_+} dudv \left( \int_0^1 \int_0^{\gamma - \gamma_0} \rho(t+u,s+v) dtds \right)^2, & \gamma \geq \gamma_0, \\
\lambda^{(3+3\gamma^0 - 2q_1)(\gamma^0/\gamma_0)} \int_{\mathbb{R}^2_+} dudv \left( \int_0^{\gamma^1 - (\gamma^0)} \int_0 \rho(t+u,s+v) dtds \right)^2, & \gamma \leq \gamma_0.
\end{cases}
\]

(3.31)

Then from (3.31), (3.21), (3.22), (3.24) with \(\mu = \lambda^{\gamma - \gamma_0}\) we obtain that

\[
J_{\lambda, \gamma} \leq C \begin{cases} 
\lambda^{\phi_+(\gamma)}, & \gamma \geq \gamma_0, \\
\lambda^{\phi_-(\gamma)}, & \gamma \leq \gamma_0,
\end{cases}
\]

where

\[
\phi_+(\gamma) := 3 + 3\gamma^0 - 2q_1 + (\gamma - \gamma^0)(3 + (1/\gamma^0) - 2q_2)_+,
\]

\[
\phi_-(\gamma) := \gamma((3/\gamma^0) + 3 - 2q_2) + (\gamma^0 - \gamma)((3/\gamma^0) + 1 - 2q_2)_+.
\]

Then (3.28) follows from

\[
2H(\gamma) > \begin{cases} 
\phi_+(\gamma), & \gamma > \gamma_0, \\
\phi_-(\gamma), & \gamma < \gamma_0, \\
\gamma \neq \gamma_0^X,
\end{cases}
\]

(3.32)

with \(H(\gamma)\) defined in Theorems 2.2 and 2.4. The latter theorems determine four of cases of \(Q_{\text{edge},i}, i = 1, 2\) in which \(\gamma_0^X\) takes different values.

(C1) Case \(Q_{\text{edge},1} > 1\). Then \(\gamma_0^X \geq \gamma_0^0, 2H(\gamma) = 3 + 2\gamma^0 - 2q_1 + \gamma\) (see Theorems 2.2 and 2.3 (i)) and \(\phi_+(\gamma) = 3 + 3\gamma^0 - 2q_1 + (\gamma - \gamma_0)(3 + (1/\gamma^0) - 2q_2)_+ < 3 + 2\gamma^0 - 2q_1 + \gamma = 2H(\gamma)\) follows by \(\gamma > \gamma_0^0, Q > 1\);

(C2) Case \(Q_{\text{edge},1} < 1 < Q_{\text{edge},2}\). Then \(\gamma_0^X = \gamma_0^0, 2H(\gamma) = \gamma\). For \(\gamma_0^{\text{edge},1} < \gamma < \gamma_0^0\) we have that

\((3/\gamma^0) + 1 - 2q_2)_+ = 0, \phi_-(\gamma) = \gamma((3/\gamma^0) + 3 - 2q_2)\) and \(\phi_-(\gamma) < 2H(\gamma)\) reduces to \(Q_{\text{edge},1} < 1\); for \(\gamma \geq \gamma_0^0\) relation \(\phi_+(\gamma) = 3 + 3\gamma^0 - 2q_1 + (\gamma - \gamma_0)(3 + (1/\gamma^0) - 2q_2)_+ < 2H(\gamma) = \gamma\) follows similarly;

(C3) Case \(Q_{\text{edge},1} \vee Q_{\text{edge},2} < 1\). Then \(\gamma_0^X = 1, 2H(\gamma) = \gamma(\gamma > \gamma_0^X)\) (see Theorem 2.4), \((3 + (1/\gamma^0) - 2q_2)_+ = 0, (3/\gamma^0) + 1 - 2q_2)_+ = 0\) and \(\phi_+(\gamma) = 3 + 3\gamma^0 - 2q_1, \phi_-(\gamma) = \gamma((3/\gamma^0) + 3 - 2q_2)\). Then \(\phi_+(\gamma) < \gamma (\gamma \geq \gamma_0 \vee 1), \phi_-(\gamma) < \gamma (1 < \gamma \leq \gamma_0 \vee 1)\) follow by \(Q_{\text{edge},i} < 1, i = 1, 2\).

Case \((i,j) = (0,-1)\). Write \(\tilde{J}_{\lambda, \gamma}\) for the l.h.s. of (3.28) for the above \(i, j\) and \(x = y = 1\). Then similarly
to (3.31)
\[
\bar{J}_{\lambda,\gamma} \leq C \int_0^\lambda du \int_1^\infty dv \left( \int_0^\lambda \int_0^{\lambda y} \rho(t-u, s+v) dt ds \right)^2 
\leq C\lambda \int_0^\infty dv \left( \int_0^\lambda \int_0^{\lambda y} \rho(t, s+v) dt ds \right)^2
\]
(3.33)
\[
= C \left\{ \begin{array}{ll}
\lambda^3 + 3\gamma^0 - 2q_1 \int_0^\infty dv \left( \int_0^1 \int_0^{\lambda y} \rho(t, s+v) dt ds \right)^2, & \gamma \geq \gamma^0, \\
\lambda^1 + (2 + 3\gamma^0 - 2q_1)(2/\gamma^0) \int_0^\infty dv \left( \int_0^{\lambda^1} \rho(t, s+v) dt ds \right)^2, & \gamma \leq \gamma^0.
\end{array} \right.
\]
(3.34)

follow from (3.33), (3.34) and (3.15), (3.16). Then (3.28) for \((i, j) = (0, -1)\) follows from
\[
2H(\gamma) > \begin{cases}
\bar{\phi}_+(\gamma), & \gamma > \gamma^0, \\
\bar{\phi}_-(\gamma), & \gamma < \gamma^0,
\end{cases} \quad \gamma \neq \gamma^0.
\]
(3.36)

with \(H(\gamma)\) defined in Theorems 2.2, 2.4

(C1) Case \(Q_{\text{edge}, 1} > 1\). Then \(\gamma^0_X \geq \gamma^0, 2H(\gamma) = 3 + 2\gamma^0 - 2q_1 + \gamma\) and \(\bar{\phi}_+(\gamma) = 3 + 3\gamma^0 - 2q_1 + (\gamma - \gamma^0)(3 - 2q_2)_+ < 2H(\gamma)\) as in (C1);

(C2) Case \(Q_{\text{edge}, 1} < 1 < Q_{\text{edge}, 2}\). Then \(\gamma^0_X = \gamma^0_{\text{edge}, 1} < \gamma^0, 2H(\gamma) = \gamma\). For \(\gamma^0_{\text{edge}, 1} < \gamma < \gamma^0\) we have that \(\bar{\phi}_-(\gamma) = 1 + (2 + 3\gamma^0 - 2q_1)(2/\gamma^0) < \gamma = 2H(\gamma)\) is equivalent to \(\gamma > \gamma^0_{\text{edge}, 1}\) (condition 1 > \((1/q_1) + (1/2q_2)\) in (3.35) is satisfied by \(Q_{\text{edge}, 1} < 1\); for \(\gamma \geq \gamma^0\) relation \(\bar{\phi}_+ (\gamma) < \gamma\) follows by \(Q_{\text{edge}, 1} < 1\);

(C3) Case \(Q_{\text{edge}, 1} \lor Q_{\text{edge}, 2} < 1\). Then \(\gamma^0_X = 1, 2H(\gamma) = \gamma\ (\gamma > \gamma^0_X)\). Let \(\gamma > \gamma^0 \lor 1\) then \(\bar{\phi}_+(\gamma) < \gamma\) as in (C2). Next, let \(1 < \gamma \leq \gamma^0\) then \(\bar{\phi}_-(\gamma) = 1 < \gamma = 2H(\gamma)\).

The above discussion proves (3.28) for (3.29). The proof of (3.28) for (3.30) is analogous and omitted. Lemma 3.2 is proved.

3.3 Proof of Theorem 2.2

Similarly as in [17], [23] we restrict the proof of (2.13) in Theorems 2.2, 2.4 to one-dimensional convergence at a given point \((x, y) \in \mathbb{R}^2_+\). We use Proposition 3.1 Write \(\lambda^{-H(\gamma)} S_X^{\lambda, \gamma} (x, y) = Q(g_{\lambda, \gamma})\) as a linear form.
in \([3.5]\) with coefficients

\[
g_{\lambda,\gamma}(x,y;u,v) := \lambda^{-H(\gamma)} \sum_{(t,s) \in K_{[\lambda x,\lambda y]}} a(t-u,s-v), \quad (u,v) \in \mathbb{Z}^2. \tag{3.37}
\]

Accordingly, it suffices to prove

\[
\|\tilde{g}_{\lambda,\gamma}(x,y;\cdot,\cdot) - h_j(x,y;\cdot,\cdot)\|^2 \to 0, \quad j = 0, 1, 2 \tag{3.38}
\]

in respective cases \(\gamma = \gamma^0, \gamma > \gamma^0, \gamma < \gamma^0\), with suitably chosen \(m_i = m_i(\lambda), i = 1, 2\), see \([3.6], [3.7]\).

For this, it is convenient to rewrite \(g_{\lambda,\gamma}\) and \(\tilde{g}_{\lambda,\gamma}\) as integrals of step functions and change the variables \((t,s) \to (\tilde{m}_1 t, \tilde{m}_2 s) : \mathbb{R}^2 \to \mathbb{R}^2\), with \(\tilde{m}_i, i = 1, 2\) defined below. Then using \([1.8]\)

\[
\tilde{g}_{\lambda,\gamma}(x,y;u,v) = \kappa_{\lambda,\gamma} \begin{cases} 
\int_{K_{\lambda,\gamma}(x,y)} a(\tilde{m}_1 t, \tilde{m}_2 s) \text{d}t \text{d}s, & (u,v) \notin \tilde{K}_{\lambda,\gamma}(x,y), \\
-\int_{K_{\lambda,\gamma}^c(x,y)} a(\tilde{m}_1 t, \tilde{m}_2 s) \text{d}t \text{d}s, & (u,v) \in \tilde{K}_{\lambda,\gamma}(x,y),
\end{cases} \tag{3.39}
\]

where

\[
\tilde{K}_{\lambda,\gamma}(x,y) := (0, [\lambda x]/\tilde{m}_1) \times (0, [\lambda y]/\tilde{m}_2), \quad \tilde{K}_{\lambda,\gamma}^c(x,y) := \mathbb{R}^2 \setminus \tilde{K}_{\lambda,\gamma}(x,y), \quad \kappa_{\lambda,\gamma} := \frac{1}{2} \tilde{m}_1 \tilde{m}_2 \lambda^{-H(\gamma)}.
\]

Consider \([3.38]\) for \(\gamma = \gamma^0\). Let \(m_1 = \tilde{m}_1 := \lambda, m_2 = \tilde{m}_2 := \lambda^0\). Then \(\tilde{K}_{\lambda,\gamma^0}(x,y) = (0, [\lambda x]/\lambda) \times (0, [\lambda y]/\lambda^0), \kappa_{\lambda,\gamma^0} = \lambda^{\gamma_1}\) and

\[
\tilde{g}_{\lambda,\gamma^0}(x,y;u,v) = \begin{cases} 
\int_{\tilde{K}_{\lambda,\gamma^0}(x,y)} a(\lambda t, \lambda^0 s) \text{d}t \text{d}s, & (u,v) \notin \tilde{K}_{\lambda,\gamma^0}(x,y), \\
-\int_{\tilde{K}_{\lambda,\gamma^0}^c(x,y)} a(\lambda t, \lambda^0 s) \text{d}t \text{d}s, & (u,v) \in \tilde{K}_{\lambda,\gamma^0}(x,y),
\end{cases} \tag{3.41}
\]

where

\[
a(\lambda t, \lambda^0 s) := \lambda^{\gamma_1} a((\lambda t) - [\lambda t], [\lambda^0 s] - [\lambda^0 v]) \to a_\infty(t-u, s-v) \tag{3.42}
\]

for any \((t,s) \neq (s,v)\). Moreover,

\[
|a_\lambda(t,s,u,v)| \leq C \rho(t-u,s-v), \quad (t,s) \in \mathbb{R}^2, \quad (s,v) \in \mathbb{R}^2 \tag{3.43}
\]

see \([17], (7.28)\), with \(C > 0\) independent of \(\lambda \geq 1\) and \(\rho\) as in \([3.9]\). Whence, the point-wise convergence

\[
\tilde{g}_{\lambda,\gamma^0}(x,y;u,v) \to h_0(x,y;u,v), \quad (u,v) \in \mathbb{R}^2, \quad (u,v) \notin \partial K(x,y) (= \text{the boundary of } K(x,y) := (0,x) \times (0,y))
\]

easily follows. Moreover, since \(\tilde{K}_{\lambda,\gamma^0}(x,y) \subset K(x,y)\) we see from \([3.43]\) that \(\tilde{g}_{\lambda,\gamma^0}(x,y;u,v) \leq \bar{h}(u,v) + \bar{h}_\lambda(u,v)\) where

\[
\bar{h}_0(x,y;u,v) := C \begin{cases} 
\int_{K(x,y)} \rho(t-u,s-v) \text{d}t \text{d}s, & (u,v) \notin K(x,y), \\
\int_{K^c(x,y)} \rho(t-u,s-v) \text{d}t \text{d}s, & (u,v) \in K(x,y),
\end{cases}
\]

\[
\bar{h}_\lambda(u,v) := C \begin{cases} 
\int_{K(x,y)} \rho(t-u,s-v) \text{d}t \text{d}s, & (u,v) \notin K(x,y), \\
\int_{K^c(x,y)} \rho(t-u,s-v) \text{d}t \text{d}s, & (u,v) \in K(x,y),
\end{cases}
\]
and \( \tilde{h}_\lambda(u,v) := 1((u,v) \in K(x,y) \setminus \tilde{K}_{\lambda,\gamma}(x,y)) \int_{\tilde{K}_{\lambda,\gamma}(x,y)} \rho(t-u,s-v)dt ds \). Then \( \|\tilde{h}\| < \infty, \|	ilde{h}_\lambda\| \to 0 \) follow from Lemma 3.1 (3.22–3.23). The above facts together with the dominated convergence theorem prove (3.38) for \( j = 0 \) or \( \gamma = \gamma^0 \).

Next, consider (3.38) for \( j = 1 \) or \( \gamma > \gamma^0 \). Let \( m_1 = \tilde{m}_1 := \lambda, m_2 := \lambda^\gamma, \tilde{m}_2 := \lambda^\gamma \). Using Lemma 3.2 in (3.38) we can replace \( \tilde{g}_{\lambda,\gamma}(x,y,\cdot,\cdot) \) of (3.39) by its restriction on \( \mathbb{R} \times (0, \lambda^\gamma / \lambda^\gamma] \), or the function

\[
\begin{align*}
h_{\lambda,\gamma}(x,y;u,v) & := 1(0 < v \leq |\lambda^\gamma / \lambda^\gamma|) \left\{ \begin{array}{ll}
\int_{[0,|\lambda^\gamma / \lambda|] \times \mathbb{R}} a_{\lambda,\gamma}(t,s,u,v)dt ds, & u \in (0, |\lambda^\gamma / \lambda|], \\
- \int_{\mathbb{R}^2 \setminus [0,|\lambda^\gamma / \lambda|] \times \mathbb{R}} a_{\lambda,\gamma}(t,s,u,v)dt ds, & u \not\in (0, |\lambda^\gamma / \lambda|],
\end{array} \right.
\end{align*}
\]

(3.44)

where

\[
a_{\lambda,\gamma}(t,s,u,v) := \kappa_{\lambda,\gamma} a([\lambda t] - [\lambda u], [\lambda^\gamma s])1([-\lambda^\gamma v / \lambda^\gamma 0 < s \leq [\lambda^\gamma y - \lambda^\gamma v / \lambda^\gamma 0]
\]

tends to \( a_\infty(t - u, s) \) point-wise for any \( t \neq u, s \in \mathbb{R}, 0 < v < y \). Hence as \( -[\lambda^\gamma v / \lambda^\gamma 0 \to -\infty, [\lambda^\gamma y - \lambda^\gamma v / \lambda^\gamma 0 \to \infty (0 < v < y) \) we see that \( h_{\lambda,\gamma}(x,y;u,v) \to h_{1}(x,y;u,v) \) in (2.4) point-wise for any \( (u,v) \in \mathbb{R}^2, u \not\in \{0\}, v \not\in \{0\} \). Then the \( L^2(\mathbb{R}^2) \)-convergence in (3.38) for \( j = 1 \) follows similarly as in the case \( j = 0 \) above. The proof of (3.38) for \( j = 2 \) or \( \gamma < \gamma^0 \) is similar using \( m_1 = \lambda, \tilde{m}_1 = \lambda / \gamma, m_2 = \tilde{m}_2 = \lambda^\gamma \) and Lemma 3.2. Theorem 2.2 is proved.

3.4 Proof of Theorem 2.3

(i) The convergence in (2.13) for \( \gamma > \gamma^0_{\text{edge},2} \) follows from the the proof of Theorem 2.2 or the convergence in (3.38) with \( \tilde{g}_{\lambda,\gamma}(x,y,\cdot,\cdot) \) replaced by \( h_{\lambda,\gamma}(x,y;u,v) \) of (3.44).

Let us prove (2.13) for \( \gamma < \gamma^0_{\text{edge},2} \). Using Lemma 3.2 we can there replace \( S^{X}_{\lambda,\gamma}(x,y) \) by

\[
S^0_{\lambda,\gamma}(x,y) := \sum_{1 \leq t \leq |\lambda^\gamma|, v \in \mathbb{Z}} \varepsilon(u,v) \sum_{(t,s) \in K_{|\lambda^\gamma|,\lambda^\gamma y]} a(t-u,s-v),
\]

Split \( S^0_{\lambda,\gamma}(x,y) = R^0_{\lambda,\gamma}(x,y) + R^+_{\lambda,\gamma}(x,y) + R^-_{\lambda,\gamma}(x,y) \) where

\[
R^0_{\lambda,\gamma}(x,y) := \sum_{1 \leq t \leq |\lambda^\gamma|, v \leq 0} \varepsilon(u,v) \sum_{(t,s) \in K_{|\lambda^\gamma|,\lambda^\gamma y]} a(t-u,s-v),
\]

\[
R^+_{\lambda,\gamma}(x,y) := \sum_{1 \leq t \leq |\lambda^\gamma|, 1 \leq v \leq |\lambda^\gamma y|} \varepsilon(u,v) \sum_{(t,s) \in K_{|\lambda^\gamma|,\lambda^\gamma y]} a(t-u,s-v),
\]

\[
R^-_{\lambda,\gamma}(x,y) := \sum_{1 \leq t \leq |\lambda^\gamma|, v > |\lambda^\gamma y|} \varepsilon(u,v) \sum_{(t,s) \in K_{|\lambda^\gamma|,\lambda^\gamma y]} a(t-u,s-v).
\]
where $R_{\lambda,\gamma}^0(x,y)$ is further rearranged as

$$R_{\lambda,\gamma}^0(x,y) := \sum_{(u,v) \in K_{[x,\lambda\gamma]}} \varepsilon(u,v) \sum_{(t,s) \in K_{[x,\lambda\gamma]}} a(t - u, s - v)$$

$$= - \sum_{(u,v) \in K_{[x,\lambda\gamma]}} \varepsilon(u,v) \sum_{(t,s) \notin K_{[x,\lambda\gamma]}} a(t - u, s - v)$$

$$= M_{\lambda,\gamma}^0(x,y) + M_{\lambda,\gamma}^+(x,y) + M_{\lambda,\gamma}^-(x,y)$$

where

$$M_{\lambda,\gamma}^-(x,y) := - \sum_{1 \leq u \leq |\lambda x|, 1 \leq v \leq |\lambda y|} \varepsilon(u,v) \sum_{1 \leq t, 1 \leq s \leq |\lambda y|} a(t - u, s - v),$$

$$M_{\lambda,\gamma}^0(x,y) := - \sum_{1 \leq u \leq |\lambda x|, 1 \leq v \leq |\lambda y|} \varepsilon(u,v) \sum_{1 \leq t \leq |\lambda x|, s \notin [1,|\lambda y|]} a(t - u, s - v),$$

$$M_{\lambda,\gamma}^+(x,y) := - \sum_{1 \leq u \leq |\lambda x|, 1 \leq v \leq |\lambda y|} \varepsilon(u,v) \sum_{t > |\lambda x|, 1 \leq v \leq |\lambda y|} a(t - u, s - v).$$

Let us prove that $M_{\lambda,\gamma}^\pm(x,y)$ are negligible, viz.,

$$E(M_{\lambda,\gamma}^\pm(x,y))^2 = \sum_{1 \leq u \leq |\lambda x|, 1 \leq v \leq |\lambda y|} \left( \sum_{1 \leq t, 1 \leq s \leq |\lambda y|} a(t - u, s - v) \right)^2 = o(\lambda), \quad \gamma < \gamma_{\text{edge},2}$$  \(3.45\) follows by integral approximation as in the proof of Lemma 3.2. Indeed, using (3.10) the l.h.s. of (3.45) can be evaluated as

$$C \int_1^\lambda du \int_0^{\lambda y} dv \left( \int_0^\infty dt \int_0^{\lambda y} \rho(t + u, s - v) ds \right)^2 \leq C \lambda^\gamma \int_1^\lambda du \left( \int_0^\infty dt \int_0^{\lambda y} \rho(t + u, s) ds \right)^2$$

$$= C \lambda^\gamma \int_1^\lambda du \left( \int_0^\infty (t + u)^{(q_1/q_2) - q_1} dt \right)^2 = C \lambda^\gamma \int_1^\lambda u^{2(1+(q_1/q_2)-q_1)} du$$

$$= C \lambda^\gamma + 3 + 2(q_1/q_2) - 2q_1 = o(\lambda), \quad \text{for} \quad \gamma < \gamma_{\text{edge},2}.$$

We finally arrive at the main term:

$$M_{\lambda,\gamma}(x,y) := R_{\lambda,\gamma}^-(x,y) + R_{\lambda,\gamma}^+(x,y) + M_{\lambda,\gamma}^0(x,y)$$

$$= \sum_{1 \leq u \leq |\lambda x|, 1 \leq v \leq |\lambda y|} \varepsilon(u,v) \sum_{1 \leq t \leq |\lambda x|, s \in [1,|\lambda y|]} a(t - u, s - v)$$

$$- \sum_{1 \leq u \leq |\lambda x|, 1 \leq v \leq |\lambda y|} \varepsilon(u,v) \sum_{1 \leq t \leq |\lambda x|, s \notin [1,|\lambda y|]} a(t - u, s - v)$$

$$= \sum_{1 \leq u,t \leq |\lambda x|} \left\{ \sum_{s \in [1,|\lambda y|], v \in [1,|\lambda y|]} \varepsilon(u,v) a(t - u, s - v) \right\}$$

Let us show the convergence

$$\lambda^{-1/2} M_{\lambda,\gamma}(x,y) \xrightarrow{fdd} \sigma_{\text{edge},1} B(x), \quad \gamma < \gamma_{\text{edge},2}$$  \(3.47\)
Note $|s - v| \geq 1$ in (3.46) and $\min\{|s - v|; s, v \in \mathbb{Z}\}$ is achieved at the lower and upper rows $\{1 \leq t \leq \lfloor \lambda x \rfloor, s = 1\}$ and $\{1 \leq t \leq \lfloor \lambda x \rfloor, s = \lceil \lambda y \rceil\}$ of the rectangle $K_{\lfloor \lambda x, \lambda y \rfloor}$. For a large but fixed $L \geq 1$ decompose the l.h.s. of (3.47) as $M_{\lambda, \gamma}(x, y) = \sum_{i=0}^{4} V_{L, \lambda, \gamma}^{(i)}(x, y)$, where

$$V_{L, \lambda, \gamma}^{(1)}(x, y) := \sum_{1 \leq u, t \leq \lfloor \lambda x \rfloor} \sum_{-L < s \leq 0 < v \leq L} \varepsilon(u, v) a(t - u, s - v),$$

$$V_{L, \lambda, \gamma}^{(2)}(x, y) := - \sum_{1 \leq u, t \leq \lfloor \lambda x \rfloor} \sum_{-L < s \leq 0 < v \leq L} \varepsilon(u, v) a(t - u, s - v),$$

$$V_{L, \lambda, \gamma}^{(3)}(x, y) := - \sum_{1 \leq u, t \leq \lfloor \lambda x \rfloor} \sum_{L \leq s \leq \lfloor \lambda x \rfloor} \varepsilon(u, v) a(t - u, s - v),$$

$$V_{L, \lambda, \gamma}^{(4)}(x, y) := \sum_{1 \leq u, t \leq \lfloor \lambda x \rfloor} \sum_{L < v \leq \lfloor \lambda y \rfloor \leq 0 < s \leq \lfloor \lambda y \rfloor} \varepsilon(u, v) a(t - u, s - v),$$

$V_{L, \lambda, \gamma}^{(0)}(x, y) := M_{\lambda, \gamma}(x, y) - \sum_{i=1}^{4} V_{L, \lambda, \gamma}^{(i)}(x, y)$. Note $V_{L, \lambda, \gamma}^{(i)}(x, y), i = 1, 2, 3, 4$ are independent and have similar structure, each consisting of a finite number $L$ of (independent) weighted sums of noise variables $\varepsilon(u, v), 1 \leq t \leq \lfloor \lambda x \rfloor$ in a neighborhood of the lower ($v = 0$) or upper ($v = \lceil \lambda y \rceil$) boundaries of the rectangle $K_{\lfloor \lambda x, \lambda y \rfloor}$; moreover, the distribution of these sums does not depend on $y$. We claim that

$$\lambda^{-1/2} V_{L, \lambda, \gamma}^{(i)}(x, y) \xrightarrow{\text{fdd}} \sigma_{L}^{(i)} B(x), \quad i = 1, 2, 3, 4, \quad 1 \leq L < \infty, \quad \lambda \to \infty, \quad L \to \infty$$

$$\lim_{L \to \infty} \limsup_{\lambda \to \infty} \lambda^{-1} \mathbb{E}(V_{L, \lambda, \gamma}^{(0)}(x, y))^2 = 0, \quad (3.49)$$

$$\lim_{L \to \infty} \sum_{i=1}^{4} (\sigma_{L}^{(i)})^2 = \sigma_{\text{edge}, 1}^2. \quad (3.50)$$

The asymptotic variances in (3.48) are given by

$$(\sigma_{L}^{(i)})^2 := \begin{cases} \sum_{v=1}^{L} \left( \sum_{t \in \mathbb{Z}, 1 \leq s \leq L} a(t, s + v) \right)^2, & i = 2, 4, \\ \sum_{v=1}^{L} \left( \sum_{t \in \mathbb{Z}, 1 \leq s \leq L} a(t, s - v) \right)^2, & i = 1, 3. \end{cases} \quad (3.51)$$

We omit the proof of (3.48)-(3.50) which reduces to a standard application of Lindeberg’s theorem and the dominated convergence theorem and relies on the fact that the series for $\sigma_{\text{edge}, 1}^2$ in (2.14) absolutely converges:

$$\sum_{v \geq 0} \left( \sum_{t \in \mathbb{Z}, s \geq 0} |a(t, s + v)|^2 \right) + \sum_{v \leq 0} \left( \sum_{t \in \mathbb{Z}, s \leq 0} |a(t, s + v)|^2 \right) < \infty. \quad (3.52)$$

The convergence in (3.52) follows from (3.17) and the assumption $Q_{\text{edge}, 2} < 1$. By Slutsky’s theorem, (3.48)-(3.50) imply (3.47), thereby completing the proof of Theorem 2.3(i). The proof of Theorem 2.3(ii) is analogous and is omitted. \qed
3.5 Proof of Theorem 2.4

For \( \gamma < \gamma_0^X = 1 \) the convergence in (3.47) and the approximation \( S^X_{\lambda,\gamma}(x, y) = M_{\lambda,\gamma}(x, y) + o_p(\lambda^{1/2}) \) follow as in the proof of Theorem 2.3 (i); particularly, (3.45) holds for \( \gamma < 1 \) in view of (3.13) and \( Q_{\text{edge,1}} < 1 \). For \( \gamma > 1 \) the proof of the theorem is analogous using the approximation \( S^X_{\lambda,\gamma}(x, y) = N_{\lambda,\gamma}(x, y) + o_p(\lambda^{\gamma/2}) \), where

\[
N_{\lambda,\gamma}(x, y) := \sum_{1 \leq v, s \leq \lfloor \lambda^{-\gamma}y \rfloor} \left\{ \sum_{t \in [1, \lfloor \lambda x \rfloor]} \sum_{u \in [1, \lfloor \lambda x \rfloor]} \varepsilon (t, u) a(t - u, s - v) \right\}
\]

is the analog of \( M_{\lambda,\gamma}(x, y) \) in (3.46) and satisfies

\[
\lambda^{-\gamma/2} N_{\lambda,\gamma}(x, y) \xrightarrow{\text{fdd}} \sigma_{\text{edge,2}} B_2(y)
\]

where \( B_2 \) is a standard Brownian motion. Finally, for \( \gamma = 1 \) we have the approximation \( S^X_{\lambda,\gamma}(x, y) = M_{\lambda,\gamma}(x, y) + N_{\lambda,\gamma}(x, y) + o_p(\lambda^{1/2}) \) and the joint convergence in (3.47) and (3.53) towards independent BM \( B = B_1 \) and \( B_2 \), thus proving Theorem 2.4.

4 Fractionally integrated negative dependent RFs

1. Isotropic fractionally integrated RF. Introduce the (discrete) Laplacian \( \Delta Y(t, s) := (1/4) \sum_{|u| + |v| = 1} Y(t + u, s + v) - Y(t, s) \) and a lattice isotropic linear RF

\[
X(t, s) = \sum_{(u, v) \in \mathbb{Z}^2} a(u, v) \varepsilon (t - u, s - v), \quad (t, s) \in \mathbb{Z}^2
\]

where \( \{\varepsilon (t, s), (t, s) \in \mathbb{Z}^2\} \) are standard i.i.d. r.v.s,

\[
a(u, v) := \sum_{j=0}^{\infty} \psi_j (-d) p_j (u, v), \quad \psi_j (d) := \Gamma (j - d) / \Gamma (j + 1) \Gamma (-d), \quad |d| < 1/2
\]

and \( p_j (u, v) \) are \( j \)-step transition probabilities of a symmetric nearest-neighbor random walk \( \{W_j; j = 0, 1, \cdots\} \) on \( \mathbb{Z}^2 \) with equal 1-step probabilities \( P(W_1 = (u, v) | W_0 = (0, 0)) = 1/4, |u| + |v| = 1 \). Note \( \sum_{j=0}^{\infty} |\psi_j (d)| < \infty, \sum_{j=0}^{\infty} \psi_j (d) = 0 (-1/2 < d < 0), \sum_{j=0}^{\infty} \psi_j (d)^2 < \infty (0 < d < 1/2) \) implying

\[
\sum_{(t, s) \in \mathbb{Z}^2} |a(t, s)| < \infty, \quad \sum_{(t, s) \in \mathbb{Z}^2} a(t, s) = 0 \quad (-1/2 < d < 0),
\]

\[
\sum_{(t, s) \in \mathbb{Z}^2} a(t, s)^2 < \infty \quad (0 < d < 1/2),
\]

see also [8], [17]. As shown in the latter papers, for \( 0 < d < 1/2 \) the linear RF \( X \) in (4.1) is the unique stationary solution of the fractional equation

\[
(-\Delta)^d X(t, s) = \varepsilon (t, s),
\]

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where the operator on the l.h.s. is defined as
\[
(-\Delta)^d Y(t, s) = \sum_{j=0}^{\infty} \psi_j(d)(1 + \Delta)^j Y(t, s) = \sum_{(u,v) \in \mathbb{Z}^2} b(u, v) Y(t - u, s - v),
\]
\[
b(u, v) := \sum_{j=0}^{\infty} \psi_j(d)p_j(u, v), \quad (u, v) \in \mathbb{Z}^2.
\]

Moreover, for \(0 < d < 1/2\) the RF \(X\) in (4.4) is LRD with moving-average coefficients satisfying (1.6) with \(q_1 = q_2 = 2(1 - d) \in (1, 2)\) and a constant angular function \(L_0\). By stationary solution of (4.1) we mean a covariance stationary RF \(Y = \{Y(t, s); (t, s) \in \mathbb{Z}^2\}\), with finite second moment \(EY(t, s)^2 < \infty\) such that the series \((-\Delta)^d Y(t, s) = \sum_{(u,v) \in \mathbb{Z}^2} b(u, v) Y(t - u, s - v)\) converges in mean square and \((-\Delta)^d Y(t, s) = \varepsilon(t, s)\) holds, for each \((t, s) \in \mathbb{Z}^2\). The following proposition extends the above-mentioned result to negative \(d \in (-1/2, 0)\).

**Proposition 4.1** Let \(-1/2 < d < 0\). Then:

(i) RF \(X\) in (4.1) is a stationary solution of equation (4.5). Moreover, this solution is unique among the class of all linear RFs \(Y = \{Y(t, s); (t, s) \in \mathbb{Z}^2\}\) with \(\sum_{(u,v) \in \mathbb{Z}^2} |c(u, v)| < \infty\) and \(\mu = EY(t, s) = 0\).

(ii) \(X\) in (4.1) is a ND RF and
\[
a(t, s) = (A + o(1))(t^2 + s^2)^{-(1-d)}, \quad t^2 + s^2 \to \infty,
\]

where \(A := \pi^{-1}\Gamma(1 - d)/\Gamma(d) < 0\). Particularly, \(a(t, s)\) satisfy (1.6) with \(q_1 = q_2 = 2(1 - d) \in (2, 3), Q = 1/(1 - d) \in (2/3, 1)\) and a constant angular function \(L_0(z) = A\).

**Proof.** We use the spectral representation \(\varepsilon(t, s) = \int_{\Pi^2} e^{i(tx+sy)} Z(dx, dy), (t, s) \in \mathbb{Z}^2\), where \(\Pi := [-\pi, \pi], Z(dx, dy)\) is a (random) complex-valued spectral measure, \(\overline{Z}(dx, dy) = Z(-dx, -dy), EZ(dx, dy) = 0, E|Z(dx, dy)|^2 = (2\pi)^{-2}dx dy\). Then \(X(t, s)\) in (4.1) can be written as
\[
X(t, s) = \int_{\Pi^2} e^{i(tx+sy)} \tilde{a}(x, y) Z(dx, dy),
\]
where \(\tilde{a}(x, y)\) is the Fourier transform:
\[
\tilde{a}(x, y) := \sum_{(u,v) \in \mathbb{Z}^2} e^{-i(ux+vy)} a(u, v) = (1 - \tilde{\rho}_1(x, y))^{-d},
\]
\[
\tilde{\rho}_1(x, y) := (1/4) \sum_{|u|+|v|=1} e^{-i(ux+vy)} = (\cos x + \cos y)/2, \text{ see } (\ref{5.5}).
\]
Applying the operator \((-\Delta)^d\) to \(X\) in (4.7) we obtain \((-\Delta)^d X(t, s) = \int_{\Pi^2} e^{i(tx+sy)} (\sum_{(u,v) \in \mathbb{Z}^2} e^{-i(ux+vy)} b(u, v)) \tilde{a}(x, y) Z(dx, dy) = \int_{\Pi^2} e^{i(tx+sy)} \tilde{b}(x, y) \tilde{a}(x, y) Z(dx, dy) = \varepsilon(t, s)\) since \(\tilde{b}(x, y) = 1/\tilde{a}(x, y), \) the series \(\sum_{(u,v) \in \mathbb{Z}^2} e^{-i(ux+vy)} b(u, v) = \tilde{b}(x, y)\) converges in \(L^2(\Pi^2)\) and \(\tilde{a}(x, y)\) is bounded on \(\Pi^2\). To show the uniqueness part, let \(Y(t, s) = \)
\[ \sum_{(u,v) \in \mathbb{Z}^2} c(u,v) \in (t-u, s-v) \text{ be another solution having spectral representation } Y(t, s) = \int_{\Pi^2} e^{i(tx+sy)} \tilde{c}(x, y) Z(dx, dy) \text{ where } \tilde{c}(x, y) \text{ is a bounded function on } \Pi^2. \]

Then \((-\Delta)^d(X(t, s) - Y(t, s)) = \int_{\Pi^2} e^{i(tx+sy)} \tilde{b}(x, y) \tilde{c}(x, y) Z(dx, dy) = 0 \text{ or } \int_{\Pi^2} \tilde{b}(x, y)^2 \tilde{c}(x, y) - \tilde{a}(x, y)^2 dx dy = 0 \text{ implying } \tilde{c}(x, y) = \tilde{a}(x, y) \text{ a.e. on } \Pi^2 \text{ since } \tilde{b}(x, y)^2 = |1 - \hat{\rho}_1(x, y)|^{2d} > 0 \text{ a.e. on } \Pi^2. \] Therefore, \(c(t, s) = a(t, s)\) and \(Y(t, s) = X(t, s), (t, s) \in \mathbb{Z}^2\).

(ii) The ND property of \(X\) follows from the zero-sum condition in (4.3). The proof of (4.6) using the Moivre-Laplace theorem carries over from ([S], proof of Prop. 5.1) to the case \(-1/2 < d < 0\) virtually without any change. 

**Remark 4.1** It follows from (4.7) that RF \(X\) has an explicit spectral density \(f(x, y) = (2\pi)^{-2}|\tilde{a}(x, y)|^2 = 2^{-2d}|(1 - \cos x) + (1 - \cos y)|^{-2d}, (x, y) \in \Pi^2\) which vanishes as const \((x^2 + y^2)^{-2d} \rightarrow 0\) as \(x^2 + y^2 \rightarrow 0\).

2. Anisotropic ND fractionally integrated RF. Following [17] consider the ‘discrete heat operator’ \(\Delta_{1,2} Y(t, s) := Y(t, s) - \theta Y(t - 1, s) - \frac{1 - \theta}{2}(Y(t - 1, s + 1) + Y(t - 1, s - 1)), 0 < \theta < 1\) and a corresponding ‘fractional power’ defined as

\[ \Delta_{1,2}^d Y(t, s) := \sum_{(u,v) \in \mathbb{Z}^2} b(u, v) Y(t - u, s - v), \quad (4.9) \]

where

\[ b(u, v) = \psi_u(d) g_\theta(u, v) 1(u \geq 0), \quad -3/4 < d < 0, \quad (4.10) \]

where \(g_\theta(u, v)\) are \(u\)-step transition probabilities of a random walk \(\{W_u; u = 0, 1, \cdots\}\) on \(\mathbb{Z}\) with 1-step probabilities \(P(W_1 = v|W_0 = 0) = \theta\) if \(v = 0\), \((1 - \theta)/2\) if \(v = \pm 1\). As shown in [17], \(\sum_{(u,v) \in \mathbb{Z}^2} b(u, v)^2 < \infty\) and

\[ b(u, v) = \frac{1}{(|u|^2 + |v|^{2q_2/q_1})^{q_1/2} L_0 \left( \frac{|u|}{(|u|^2 + |v|^{2q_2/q_1})^{1/2}}; d, \theta \right) + o(1) \}, \quad |u| + |v| \rightarrow \infty, \quad (4.11) \]

where

\[ q_1 = (3/2) + d, \quad q_2 = 2q_1 = 2((3/2) + d), \quad (4.12) \]

\[ L_0(z; d, \theta) = \begin{cases} \frac{z^{-(d-(3/2))}}{\Gamma(-d)2\pi^{(1-\theta)/2}} \exp \{- \frac{\sqrt{(1/2)^2 - 1}}{2(1-\theta)} \}, & 0 < z \leq 1, \\ 0, & -1 \leq z \leq 0 \end{cases} \]

and \(L_0(z; d, \theta), z \in [-1, 1]\) is a bounded continuous function. Similarly to (4.1), the inverse operator can be defined by \(\Delta_{1,2}^{-d} Y(t, s) := \sum_{(u,v) \in \mathbb{Z} \times \mathbb{Z}} a(u, v) Y(t - u, s - v), \) where

\[ a(u, v) = \psi_u(-d) g_\theta(u, v) 1(u \geq 0), \quad -3/4 < d < 0, \quad (4.13) \]

are obtained from \(b(u, v)\) in (4.10) with \(d\) replaced by \(-d\). By direct inspection of the proof of Prop. 4.1 in [17] we find that \(a(t, s)\) also satisfy (4.11)-(4.12) with \(d\) replaced by \(-d\), for \(-3/4 < d < 0\), in other
words, the linear RF $X$ in (4.1) with moving-average coefficients in (4.13) satisfies the assumptions in (4.6) with $q_1 = (3/2) - d, q_2 = 2q_1 = 2((3/2) - d)$ and

$$Q = \frac{2}{3 - 2d} + \frac{1}{3 - 2d} \in \left(\frac{2}{3}, 1\right), \quad -\frac{3}{4} < d < 0.$$  

Moreover, the above $X$ can be regarded as a stationary solution (in the sense as explained before Proposition 4.1) of the fractional equation:

$$\Delta_{1,2}^{d}X(t, s) = \varepsilon(t, s), \quad (t, s) \in \mathbb{Z}^2. \quad (4.14)$$

We arrive at the following proposition whose proof is similar to that of Proposition 4.1 and we omit the details.

**Proposition 4.2** Let $-3/4 < d < 0$. Then:

(i) RF $X$ in (4.1) with coefficients $a(u, v)$ in (4.13) is a stationary solution of equation (4.14). Moreover, this solution is unique among the class of all linear RFs $Y(t, s) = \mu + \sum_{(u,v) \in \mathbb{Z}^2} c(u,v)\varepsilon(t-u, s-v), (t, s) \in \mathbb{Z}^2$ with $\sum_{(u,v) \in \mathbb{Z}^2} |c(u,v)| < \infty$ and $\mu = EY(t, s) = 0$.

(ii) The above $X$ is a ND RF and $a(u, v)$ satisfy (1.6) with $q_1 = (3/2) - d, q_2 = 2q_1 = 2((3/2) - d)$ and a continuous angular function $L_0(z) = L_0(z; \theta, -d), z \in [-1, 1]$ as defined in (4.12).

**References**

[1] Anh, V.V., Leonenko, N.N. and Ruiz-Medina, M.D. (2013) Macroscale limit theorems for filtered spatiotemporal random fields. Stochastic Anal. Appl. 31, 460–508.

[2] Ayache, A., Leger, S. and Pontier, M. (2002) Drap Brownien fractionnaire. Potential Anal. 17, 31–43.

[3] Damarackas, J. and Paulauskas, V. (2017) Spectral covariance and limit theorems for random fields with infinite variance. J. Multiv. Anal. 153, 156-175.

[4] Dobrushin, R.L. and Major, P. (1979) Non-central limit theorems for non-linear functionals of Gaussian fields. Probab. Th. Rel. Fields 50, 27–52.

[5] Gaigalas, R. and Kaj, I. (2003) Convergence of scaled renewal processes and a packet arrival model. Bernoulli 9, 671–703.

[6] Giraitis, L., Koul, H.L. and Surgailis, D. (2012) *Large Sample Inference for Long Memory Processes*. Imperial College Press, London.

[7] Kaj, I. and Taqqu, M.S. (2008) Convergence to fractional Brownian motion and to the Telecom process: the integral representation approach. In: Vares, M.E. and Sidoravicius, V. (eds.) *An Out of Equilibrium 2*. Progress in Probability, vol. 60, pp. 383–427. Birkhäuser, Basel.
[8] Koul, H.L., Mimoto, N. and Surgailis, D. (2016) Goodness-of-fit tests for marginal distribution of linear random fields with long memory. Metrika 79, 165–193.

[9] Lahiri, S.N. and Robinson, P.M. (2016) Central limit theorems for long range dependent spatial linear processes. Bernoulli 22, 345–375.

[10] Lavancier, F. (2007) Invariance principles for non-isotropic long memory random fields. Statist. Inference Stoch. Process. 10, 255–282.

[11] Leipus, R., Philippe, A., Pilipauskaitė, V. and Surgailis, D. (2018) Sample autocovariances of random-coefficient AR(1) panel model. Preprint.

[12] Leonenko, N.N. (1999) Random Fields with Singular Spectrum. Kluwer, Dordrecht.

[13] Mikosch, T., Resnick, S., Rootzén, H. and Stegeman, A. (2002) Is network traffic approximated by stable Lévy motion or fractional Brownian motion? Ann. Appl. Probab. 12, 23–68.

[14] Pilipauskaitė, V. and Surgailis, D. (2014) Joint temporal and contemporaneous aggregation of random-coefficient AR(1) processes. Stochastic Process. Appl. 124, 1011–1035.

[15] Pilipauskaitė, V. and Surgailis, D. (2015) Joint aggregation of random-coefficient AR(1) processes with common innovations. Statist. Probab. Lett. 101, 73–82.

[16] Pilipauskaitė, V. and Surgailis, D. (2016) Anisotropic scaling of random grain model with application to network traffic. J. Appl. Probab. 53, 857–879.

[17] Pilipauskaitė, V. and Surgailis, D. (2017) Scaling transition for nonlinear random fields with long-range dependence. Stochastic Process. Appl. 127, 2751–2779.

[18] Puplinskaitė, D. and Surgailis, D. (2015) Scaling transition for long-range dependent Gaussian random fields. Stochastic Process. Appl. 125, 2256–2271.

[19] Puplinskaitė, D. and Surgailis, D. (2016) Aggregation of autoregressive random fields and anisotropic long-range dependence. Bernoulli 22, 2401–2441.

[20] Ripley, B.D. (1988) Statistical Inference for Spatial Processes. Cambridge Univ. Press, Cambridge.

[21] Samorodnitsky, G. and Taqqu, M.S. (1994) Stable non-Gaussian random variables. Chapman and Hall, London.

[22] Surgailis, D. (1982) Zones of attraction of self-similar multiple integrals. Lithuanian Math. J. 22, 185–201.

[23] Surgailis, D. (2019) Anisotropic scaling limits of long-range dependent linear random fields on $\mathbb{Z}^3$. J. Math. Anal. Appl. 472, 328–351.