1. Introduction

In the weak-field and slow-motion approximation of General Relativity, a test particle orbiting a central slowly rotating body of mass $M$ and angular momentum $\mathbf{J}$, assumed to be constant, is acted upon by a non-central acceleration\(^1\) of the form

$$\mathbf{a}_{\text{GM}} = \frac{\mathbf{v}}{c} \times \mathbf{B}_g,$$  \hspace{1cm} (1)

in which $\mathbf{v}$ is the velocity of the test particle, $c$ is the speed of light in vacuum and $\mathbf{B}_g$ is the gravitomagnetic field given by\(^2\)

$$\mathbf{B}_g = \frac{2G}{c} \frac{[\mathbf{J} - 3(\mathbf{J} \cdot \hat{r}) \hat{r}]}{r^3}. \hspace{1cm} (2)$$

In it $\hat{r}$ is the unit position vector of the test particle and $G$ is the Newtonian gravitational constant. The gravitomagnetic field $\mathbf{B}_g$ can be derived\(^2\) from the gravitomagnetic potential $\mathbf{A}_g$

$$\mathbf{A}_g = \frac{G}{c} \frac{\mathbf{J} \times \mathbf{r}}{r^3}, \hspace{1cm} (3)$$

according to

$$\nabla \times \mathbf{A}_g = -\frac{\mathbf{B}_g}{2}. \hspace{1cm} (4)$$
Equation (2) and Equation (3) hold far from the central body supposed to be spherically symmetric and rigidly rotating. An inertial frame $K\{x, y, z\}$ with its origin located at the center of mass of the rotating body, the $z$ axis directed along $J$ and the $\{x, y\}$ plane equal to the equatorial plane of the gravitating source is adopted.

This effect, which affects the longitude of the ascending node $\Omega$ and the argument of pericenter $\omega$ of the orbit of a test particle$^3$, is currently under measurement by analyzing the laser-ranged data to LAGEOS satellites in the gravitational field of the Earth$^4$.

According to the gravitational analogue of the Larmor theorem$^5$, we could obtain Equation (1) by considering an accelerated frame rotating with angular velocity

$$\Omega_L = \frac{B_g}{2c}.$$ (5)

Indeed, in it an inertial Coriolis acceleration

$$a_{\text{Cor}} = 2v \times \Omega_L$$ (6)

is experienced by the proof mass.$^3$

It seems natural to pose the following question. In an accelerated frame, apart from the Coriolis and centrifugal inertial forces$^b$, a particle feels an acceleration

$$r \times \frac{d\Omega_L}{dt}$$ (7)

as well if the angular velocity vector $\Omega_L$ is time-dependent. Does a gravitomagnetic analogue of such term exist? If so, it should be induced by temporal variations of the angular momentum of the central body $J$. In the case of the Earth we know that $J_\oplus$ changes in time due to the luni-solar torques which generate, among other things, the precession of equinoxes$^6$.

In this paper we will try to investigate this feature. The plan of the work is as follows. In Section 2 we derive the full expression of the gravitomagnetic acceleration experienced by the test mass when the gravitomagnetic potential due to the central rotating mass is time-dependent. In Section 3 the orbital effects on the semimajor axis, the inclination and the node of the orbit of an Earth satellite are derived and the possibility of measuring such effect in the terrestrial field with LAGEOS-like satellites are discussed. Section 4 is devoted to the conclusions.

In Table 1 the numerical values of the parameters used in the text are quoted. For the Earth, the Moon and the Sun reference$^7$ has been used.

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$^a$It should be noticed that the equivalence is useful only in spatial regions in which the field considered is uniform, so that$^5$ $-4A_k = B_g \times r$.

$^b$The centrifugal force is proportional to the square of $\Omega_L$, so that the impact of a possible gravitomagnetic analogue, proportional to the square of $B_g$, can be neglected because we are in the weak-field approximation.
2. The gravitomagnetic acceleration

In the weak-field and slow-motion linearized approximation of General Relativity, the Lagrangian $\mathcal{L}$ of a test particle of mass $m$ moving in a metric $ds^2 = g_{00}(dx^0)^2 + g_{ij}(dx^i dx^j)$ is given by

$$\mathcal{L} = \mathcal{L}_{(GE)} - \frac{2m}{c} A_g \cdot \mathbf{v}, \quad (8)$$

where the gravitomagnetic potential $A_g$ is generated by the off-diagonal components $g_{0k}$, $k = 1, 2, 3$ of the space-time metric while $\mathcal{L}_{(GE)}$ denotes the gravitoelectric Schwarzschild terms. The Lagrange equations of motion are

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \right) - \frac{\partial \mathcal{L}}{\partial r} = 0. \quad (9)$$

From Equation (8) it follows

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \mathbf{v}} \right) = m \frac{d\mathbf{v}}{dt} - \frac{2m}{c} (\mathbf{v} \cdot \nabla) \mathbf{A}_g = \frac{2m}{c} \frac{\partial \mathbf{A}_g}{\partial t} + (GE), \quad (10)$$

$$\frac{\partial \mathcal{L}}{\partial r} = -\frac{2m}{c} \nabla (\mathbf{A}_g \cdot \mathbf{v}) + (GE). \quad (11)$$

Then, the equations of motion of a test particle are

$$m \frac{d\mathbf{v}}{dt} = -\frac{2m}{c} \nabla (\mathbf{A}_g \cdot \mathbf{v}) + \frac{2m}{c} (\mathbf{v} \cdot \nabla) \mathbf{A}_g + \frac{2m}{c} \frac{\partial \mathbf{A}_g}{\partial t} + (GE) =$$

$$-\frac{2m}{c} [\mathbf{v} \times (\nabla \times \mathbf{A}_g)] + \frac{2m}{c} \frac{\partial \mathbf{A}_g}{\partial t} + (GE). \quad (12)$$

In Equation (12) (GE) represents the post-Newtonian gravitoelectric acceleration due to the Schwarzschild part of the metric, of order $O(c^{-2}) + O(c^0)$, which reduces to the Newtonian central term, of order $O(c^0)$, in the limit $c \to \infty$. For $-4\mathbf{A}_g = \mathbf{B}_g \times \mathbf{r}$, i.e. when the field is uniform, the gravitomagnetic part of Equation (12) becomes

$$\mathbf{a}_{GM} = \frac{\mathbf{v}}{c} \times \mathbf{B}_g - \frac{1}{2c} \frac{\partial \mathbf{B}_g}{\partial t} \times \mathbf{r}. \quad (13)$$
If we use Equation (6), Equation (13) becomes
\[ a_{GM} = 2v \times \Omega_L + r \times \frac{\partial \Omega_L}{\partial t}. \] (14)
Equation (14) shows that the gravitational analogue of the Larmor theorem extends exactly also to the case in which the gravitomagnetic field is explicitly time-dependent.

### 3. The consequences of the variability of Earth’s angular momentum

We will focus on the second term of Equation (13). When the mass-energy distribution is a central spherically symmetric rigidly rotating body, so that Equation (2) and Equation (3) can be applied, Equation (13) can be written
\[ a_{Lar} = \frac{2G}{c^2r^3} \frac{dJ}{dt} \times r = \frac{2G}{c^2r^3} M \times r, \] (15)
in which \( M = \frac{dJ}{dt} \) is the momentum of the external forces. Indeed, in a body-fixed rotating frame with an angular velocity \( \tilde{\omega}_0 \) the dynamical Euler equations hold
\[ \frac{dJ}{dt} = \frac{\partial J}{\partial t} + \tilde{\omega}_0 \times J = M. \] (16)
However, we are using an inertial frame so that
\[ \frac{dJ}{dt} = \frac{\partial J}{\partial t} = M. \] (17)
In the case of the Earth \( M \) is the external torque exerted on its equatorial centrifugal bulge by the other bodies of the Solar System\(^8\). We will consider only the Moon and the Sun. In the frame \( K \) previously defined and by assuming the Earth as an oblate spheroid whose rotation is affected by the presence of a disturbing body of mass \( M_B \) (e.g. the Sun) moving around it on an approximately circular orbit of radius \( d \) lying on the ecliptic plane, the external torque can be written as\(^8\)
\[ M_x = \frac{3GM_B(C-A)_{\oplus}}{d^3} \sin^2 \lambda \sin \varepsilon \cos \varepsilon \] (18)
\[ M_y = -\frac{3GM_B(C-A)_{\oplus}}{d^3} \sin \lambda \cos \lambda \sin \varepsilon \] (19)
\[ M_z = 0, \] (20)
where \((C - A)_{\oplus} = J_2M_B\,R^2_{\oplus}\) in which \( J_2 \) is the first even zonal coefficient of the multipolar expansion of the terrestrial gravitational field and \( M_{\oplus} \) and \( R_{\oplus} \) are the

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\(^8\)Recall that the other planets induce secular effects on the ecliptical plane: the planetary precession and a secular change in the obliquity \( \varepsilon \). The luni-solar torque affects the equatorial plane causing the Earth’s spin to precess and nutate about the ecliptic pole. While the precession does not affect \( \varepsilon \) but only the position of the equinox in the ecliptic, the nutation does affect \( \varepsilon \) periodically.
A gravitomagnetic effect

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mass and the equatorial radius, respectively, of the Earth. Moreover, \( \lambda \) is the ecliptical longitude of the perturbing body and \( \varepsilon \) is the inclination of the ecliptic to the equator.

The next steps in order to evaluate the observable consequences on the orbit of an Earth artificial satellite consist of projecting the perturbing acceleration onto the usual radial, along and cross-track directions, evaluating the so obtained components \( R, T \) and \( N \) of \( \mathbf{a_{Lar}} \) in terms of the osculating Keplerian elements of the orbiter and calculating their rates of changes by averaging over an orbital revolution of the satellite. In performing the orbital averages \( d \) and \( \varepsilon \) will be considered constant.

From Equation (15) and Equation (18)–Equation (20) it is straightforward to obtain

\[
R = 0
\]

\[
T = 6 \left( \frac{G}{c} \right)^2 \frac{M_B(C - A)\oplus}{d^3r^2} \mathcal{T}
\]

\[
N = -6 \left( \frac{G}{c} \right)^2 \frac{M_B(C - A)\oplus}{d^3r^2} [A \cos(\omega + f) + B \sin(\omega + f)],
\]

where

\[
\mathcal{T} = \frac{\sin 2\varepsilon \sin \Omega(1 - \cos 2\lambda)}{4} + \frac{\sin \varepsilon \cos \Omega \sin 2\lambda}{2}
\]

\[
A = \frac{\sin 2\varepsilon \cos i \sin \Omega(\cos 2\lambda - 1)}{4} - \frac{\sin \varepsilon \cos i \cos \Omega \sin 2\lambda}{2}
\]

\[
B = \frac{\sin 2\varepsilon \cos \Omega(\cos 2\lambda - 1)}{4} + \frac{\cos \varepsilon \sin \Omega \sin 2\lambda}{2},
\]

in which \( i \) is the inclination of the satellite’s orbit and \( f \) is its true anomaly. From Equation (21)–Equation (23) it can be noticed that the disturbing acceleration is non-central.

Among the Keplerian orbital elements of a typical Earth satellites the semimajor axis \( a \), the inclination \( i \) and the node \( \Omega \) are, in general, well measured, as in the case of LAGEOS laser-ranged satellites. Then, we will calculate the rates of change of such elements according to

\[
\frac{da}{dt} = \frac{2}{n\sqrt{1 - e^2}} \left[ R e \sin(\omega + f) + T \frac{a(1 - e^2)}{r} \right],
\]

\[
\frac{di}{dt} = \frac{1}{na\sqrt{1 - e^2}} N \frac{r}{a} \cos(\omega + f),
\]

\[
\frac{d\Omega}{dt} = \frac{1}{na\sqrt{1 - e^2}} N \frac{r}{a} \sin(\omega + f),
\]

in which \( e \) is the satellite’s orbital eccentricity, \( n = \sqrt{GMa^{-3}} \) is the satellite mean motion and

\[
r = \frac{a(1 - e^2)}{1 + e \cos f}
\]
on the unperturbed Keplerian ellipse.

By inserting Equation (21)–Equation (23) in Equation (27)–Equation (29), averaging over an orbital revolution of the satellite and neglecting terms of order $O(e)$, it is possible to obtain the long-term evolutions of $a$, $i$ and $\Omega$

\[
\frac{da}{dt} = 6 \left( \frac{G}{c} \right)^2 \frac{M_B(C - A)_\oplus \sin i \sin \varepsilon}{n a^2 d^3} \left[ \cos \varepsilon \sin \Omega - \frac{1 + \cos \varepsilon}{2} \sin(\Omega - 2\lambda) + \frac{1 - \cos \varepsilon}{2} \sin(\Omega + 2\lambda) \right], \tag{31}
\]

\[
\frac{di}{dt} = 6 \left( \frac{G}{c} \right)^2 \frac{M_B(C - A)_\oplus \cos i \sin \varepsilon}{4na^3d^3} \left[ \cos \varepsilon \sin \Omega - \frac{1 + \cos \varepsilon}{2} \sin(\Omega - 2\lambda) + \frac{1 - \cos \varepsilon}{2} \sin(\Omega + 2\lambda) \right], \tag{32}
\]

\[
\frac{d\Omega}{dt} = 6 \left( \frac{G}{c} \right)^2 \frac{M_B(C - A)_\oplus \sin \varepsilon}{4na^3d^3 \sin i} \left[ \cos \varepsilon \cos \Omega + \frac{1 - \cos \varepsilon}{2} \cos(\Omega - 2\lambda) - \frac{1 + \cos \varepsilon}{2} \cos(\Omega + 2\lambda) \right]. \tag{33}
\]

From an inspection of Equation (31)–Equation (33) it can be noticed that there are no secular, linear trends but only long-period harmonic perturbations whose frequencies are linear combinations of the longitude of the satellite’s ascending node $\Omega$ and the ecliptical longitude $\lambda$ of the disturbing body $B$. E.g., for the Sun and LAGEOS we have

\[
P(\Omega) = 1043.67 \text{ days}, \tag{34}
\]

\[
P(\Omega - 2\lambda) = -221.34 \text{ days}, \tag{35}
\]

\[
P(\Omega + 2\lambda) = 155.42 \text{ days}. \tag{36}
\]

Notice also the $d^{-3}$ dependence, typical of a tidal effect.

Concerning the possibility of measuring such effects in the terrestrial field, unfortunately there is no hope of detecting them: indeed, they are $c^{-2}$ effects multiplied by $G^2$. For LAGEOS the experienced acceleration due to the solar torque is of the order of

\[
a_{\text{Lar}}^\odot = 6 \left( \frac{G}{c} \right)^2 \frac{M_\odot(C - A)_\oplus}{d_{\odot}^3 a^2} \sim 10^{-17} \text{ cm s}^{-2}, \tag{37}
\]

while the present level of sensitivity is $9 \times 10^{-10} \text{ cm s}^{-2}$. Consequently, the amplitudes of the periodic signals of the semimajor axis, the inclination and the node amounts to $10^{-7} \text{ cm}$ and $10^{-8} \text{ mas}$, respectively. If the action of the lunar torque is considered, it turns out that

\[
\frac{a_{\text{Lar}}^m}{a_{\text{Lar}}^\odot} \sim 2.1. \tag{38}
\]

\[^d\text{In integrating Equation (31)–Equation (33) the corrections } \Delta \lambda \text{ and } \Delta \varepsilon \text{ due to precession and nutation have been neglected.}\]
A posteriori, this result justifies the previous choice of neglecting terms of order $O(e)$ in the satellite’s eccentricity and the contribution of the torques of the other planets of the Solar System.

4. Conclusions

Motivated by the gravitational analogue of the Larmor theorem, in this paper we have investigated some consequences of the variability of the proper angular momentum of a central slowly rigidly rotating body due to external torques on the gravitomagnetic equations of motion of a test mass in the weak-field and slow-motion approximation of General Relativity. The test particle turns out to be affected by a post-Newtonian $O(c^{-2})$ non-central acceleration analogue to the inertial acceleration due to the temporal derivative of the angular velocity vector arising in a non-inertial rotating frame. In view of a possible measurement of such effect with LAGEOS-like satellites in the gravitational field of the Earth, we have calculated the perturbations induced on the semimajor axis, the inclination and the node of an orbiting proof mass. We have found no secular terms but only long-term periodic effects: their magnitude is far too small to be detected.

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