Dynamic Potential Games in Communications: Fundamentals and Applications

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Abstract—In a noncooperative dynamic game, multiple agents operating in a changing environment aim to optimize their utilities over an infinite time horizon. Time-varying environments allow to model more realistic scenarios (e.g., mobile devices equipped with batteries, wireless communications over a fading channel, etc.). However, solving a dynamic game is a difficult task that requires dealing with multiple coupled optimal control problems. We focus our analysis on a class of problems, named dynamic potential games, whose solution can be found through a single multivariate optimal control problem. Our analysis generalizes previous studies by considering that the set of environment’s states and the set of players’ actions are constrained, as it is required by most of the applications. And the theoretical results are the natural extension of the analysis for static potential games. We apply the analysis and provide numerical methods to solve four key example problems, with different features each: energy demand control in a smart-grid network, network flow optimization in which the relays have bounded link capacity and limited battery life, uplink multiple access communication with users that have to optimize the use of their batteries, and two optimal scheduling games with nonstationary channels.

Index Terms—Dynamic games, dynamic programming, game theory, multiple access, network flow, optimal control, resource allocation, scheduling, smart grid.

I. INTRODUCTION

GAME theory is a field of mathematics that studies conflict and cooperation between intelligent decision makers [1]. It has become a useful tool for modeling communication and networking problems, such as power control and resource sharing (see, e.g., [2]), wherein the strategies followed by the users (i.e., players) influence each other, and the actions have to be taken in a decentralized manner. However, one main assumption of classic game theory is that the users operate in a static environment, which neither changes with time, nor it is influenced by the players’ actions. This assumption is unrealistic in many communication and networking problems. For instance, wireless devices have to maximize throughput while facing time-varying fading channels, and mobile devices may have to control their transmitter power while saving their battery level. These time-varying scenarios can be better modeled by dynamic games.

In a noncooperative dynamic game, the players compete in a time-varying environment, which we assume can be characterized by a deterministic discrete-time dynamical system equipped with a set of states and a Markovian state-transition equation. Each player has its utility function, which depends on the current state of the system and the players’ current actions. Both the state and action sets are subject to constraints. Since the state-transitions induce a notion of time-evolution in the game, we consider the general case wherein utilities, state-transition function and constraints can be nonstationary. A dynamic game starts at an initial state. Then, the players take some action, based on the current state of the game, and receive some utility values. Then, the game moves to another state. This sequence of state-transitions is repeated at every time step over a (possibly) infinite time horizon. We consider the case in which the aim of each player is to find the sequence of actions that maximizes its long term cumulative utility, given other players’ sequence of actions. Thus, a game can be represented as a set of coupled optimal-control-problems (OCP), which are difficult to solve in general. Fortunately, there is a class of dynamic games, named dynamic potential games (DPG), that can be solved through a single multivariate-optimal-control-problem (MOCP). The benefit of DPG is that solving a single MOCP is generally simpler than solving a set of coupled OCP (see [3] for a recent survey on DPG).

The pioneering work in the field of DPG is that of [4], later extended by [5] and [6]. There have been two main approaches to study DPG: the Euler-Lagrange equations and the Pontryagin’s maximum (or minimum) principle. Recent analysis by [3] and [7] used the Euler-Lagrange with DPG in its reduced form, that is when it is possible to isolate the action from the state-transition equation, so that the action is expressed as a function of the current and future (i.e., after transition) states. However, in many cases, it is not possible to find such reduced form of the game (i.e., we cannot isolate the action). The more general case of DPG in nonreduced form was studied with the Pontryagin’s maximum principle approach by [5] and [8] for discrete and continuous time models, respectively. However, in all these studies [3–8], the games have been analyzed without explicitly considering constraints for the state and action sets.

The main theoretical contribution of this work is to analyze DPG with constrained action and state sets, as it is required by most of applications (e.g., in a network flow problem, the aggregated throughput of multiple users is bounded by the maximum link capacity; or in cognitive radio, the aggregated power of all secondary users is bounded by the maximum...
interference allowed by the primary users). To do so, we apply the Euler-Lagrange equation to the Lagrangian (as it is customary in the MOCP literature [9]), rather than to the utility function (as done by earlier works [3] and [4]). Using the Lagrangian, we can formulate the optimality condition in the general nonreduced form (i.e., it is not necessary to isolate the action in the transition equation). In addition, we establish the existence of a suitable conservative vector field as an easily verifiable condition for a dynamic game to be of the potential type. To the best of our knowledge, this is a novel extension of the conditions established for static games by [10] and [11].

The second main contribution of this work is to show that the the proposed framework can be applied to several communication and networking problems in a unified manner. We present four examples with increasing complexity level. First, we model the energy demand control in a smart grid network as a linear-quadratic-dynamic-game (LQDG). This scenario is illustrative because the analytical solution of an LQDG is known. The second example is an optimal network flow problem, in which there are two levels of relay nodes equipped with finite batteries. The users aim to maximize their flow while optimizing the use of the nodes’ batteries. This problem illustrates that, when the utilities have some separable form, it is straightforward to establish that the problem is a DPG. However, the analytical solution for this problem is unknown and we have to solve it numerically. It turns out that, since all batteries will deplete eventually, the game will get stuck in this depletion-state. Hence, we can approximate the infinite-horizon MOCP by an effective finite-horizon problem, which simplifies the numerical computation. The third example is an uplink multiple access channel wherein the users’ devices are also equipped with batteries (this example was introduced in the preliminary paper [12]). Again, the simple—but more realistic—extension of battery-usage optimization makes the game dynamic. In this example, instead of rewriting the utilities in a separable form, we perform a very general analysis to establish that the problem is a DPG. The fourth example studies two decentralized scheduling problems: proportional fair and equal rate scheduling, where multiple users share a time-varying channel (see the preliminary paper [13]). This example shows how to use the proposed framework in its most general form. The utilities are nonconcave, with no obvious separable form. The problem is nonstationary, with state-transition equation changing with time. And there is no reason that justifies a finite horizon approximation of the problem, so we have to use optimal control methods (e.g., dynamic programming) to solve it numerically.

Outline: Sec. II introduces the problem setting, formally defining the problem, its solution and the assumptions in which we base our analysis. In Sec. III we review static potential games together with the instrumental notion of conservative vector field. In Sec. IV we provide sufficient conditions for a dynamic game with constrained state and action sets to be a DPG, and show that a DPG can be solved through and equivalent MOCP. Sections V-VII deal with the application examples, the methods for solving them, and some illustrative simulations. We provide some conclusions in Sec. IX.

II. Problem Setting

Let \( Q \triangleq \{1, \ldots, Q\} \) denote the set of players and let \( X \subseteq \mathbb{R}^2 \) denote the set of states of the game. Note that the dimensionality of the state set can be different from the number of players (i.e., \( S \neq Q \)). At every time step \( t \), the state-vector of the game is represented by \( x_t \triangleq (x_t^S)_{k=1}^S \in X \). Every player \( i \in Q \) can be influenced only by a subset of states \( X_i^i \subseteq X \). The partition of the state space \( X \) among players is done in the component domain. We define \( X(i) \subseteq \{1, \ldots, S\} \) as the subset of indexes of state-vector components that influence player \( i \), then \( x_t^i \triangleq (x_t^m)_{m \in X(i)} \) indicates the value of the state-vector for player \( i \) at time \( t \).

This generality allows for games in which multiple players are affected by common components of the state vector (e.g., when they share a common resource), and includes the particular case wherein they share no components. We also define \( x_i^{−i} \triangleq (x_t^m)_{m \notin X(i)} \) for the vector of components that do not influence player \( i \), for some subset \( X^{−i} \subseteq X \).

Let \( U \subseteq \mathbb{R}^Q \) denote the set of actions of all players, and let \( U^i \subseteq \mathbb{R} \) stand for the subset of actions of player \( i \), such that \( U \triangleq \prod_{i=1}^Q U^i \). The extension to higher dimensional action sets is straightforward (i.e., when \( U^i \subseteq \mathbb{R}^X \)), but we restrict to scalar actions in order to simplify notation (the general case will be introduced for some of the application examples). We write \( u_t^i \in U^i \) the action variable of player \( i \) at time \( t \), such that the vector \( u_t \triangleq (u_t^1, \ldots, u_t^Q) \in U \) contains the actions of all players. We also define \( u_t^{−i} \triangleq (u_t^1, \ldots, u_t^{i−1}, u_t^{i+1}, \ldots, u_t^Q) \in U^{−i} \triangleq \prod_{i \neq j} U^j \) as the vector of all players’ actions except that of player \( i \). Hence, by slightly abusing notation, we can rewrite \( u_t = (u_t^1, u_t^{−1}) \).

The state transitions are determined by \( f : X \times U \times N \rightarrow X \), such that the nonstationary Markovian dynamic equation of the game is \( x_{t+1} = f(x_t, u_t) \), which can be split among components: \( x_{t+1}^k = f^k(x_t, u_t) \) for \( k = 1, \ldots, S \), such that \( f \triangleq \{f^k\}_{k=1}^S \). The dynamic is Markovian because the state transition to \( x_{t+1} \) depends on the current state-action pair \((x_t, u_t)\), rather than on the whole history of state-action pairs \( \{(x_0, u_0), \ldots, (x_t, u_t)\}\).

We include a vector of \( C \) extra constraints, \( g \triangleq (g^C)_{c=1}^C \), as it is required by most applications. In order to simplify notation, when we write \( x_t \in X \) or \( u_t \in U \), we assume that the vector of constraints \( g(x_t, u_t, t) \) is satisfied.

Each player has its utility function \( \pi^i : X^i \times U \times N \rightarrow \mathbb{R} \), such that, at every time \( t \), each player receives an utility value equal to \( \pi^i(x_t^i, u_t^i, u_t^{−i}, t) \).

The aim of player \( i \) is to find the sequence of actions \( \{u^i_0, u^i_1, \ldots\} \) that maximizes its long term cumulative utility, given other players’ sequence of actions \( \{u^j_0, u^j_1, \ldots\} \). Thus, a discrete-time infinite-horizon noncooperative Markovian dynamic game can be represented as a set of \( Q \) coupled optimal control problems:

\[
\mathcal{G}_t : \begin{array}{c}
\text{maximize} \\
\sum_{t=0}^{\infty} \beta^t \pi^i(x_t^i, u_t^i, u_t^{−i}, t) \\
\text{s.t.} \\
x_{t+1}^i = f(x_t^i, u_t^i, t) \quad x_0 \text{ given} \\
g(x_t^i, u_t^i, t) \leq 0
\end{array}
\]
where $0 < \beta < 1$ is the discount factor that bounds the cumulative utility. Note that, since the players can share state-vector components, we have included all constraints from every other player in each player’s OCP. Problem (1) is infinite-horizon because the reward is accumulated over infinite time steps.

The solution concept of problem (1) in which we are interested is the Nash Equilibrium (NE) of the game, which is defined as follows.

**Definition 1.** A solution of problem (1), known as a Nash Equilibrium (NE), is a feasible sequence of actions $\{u^i_t\}_{t=0}^\infty$ that satisfies the following condition for every player $i \in Q$:

$$
\sum_{t=0}^\infty \beta^t \pi^i(x^i_t, u^i_t, u^{−i}_t, t) \geq \sum_{t=0}^\infty \beta^t \pi^i(x^i_t, u^i_t, u^{−i}_t, t),
$$

\[ \forall x^i \in X^i, \forall u^i_t \in U^i. \quad (2) \]

In this work, we consider the following assumptions:

**Assumption 1.** The utilities $\pi^i$ are twice continuously differentiable in $X^i \times U^i$, while the state-transition function $f$ and the constraints $g$ are continuously differentiable in $X \times U$.

**Assumption 2.** State-transition function $f$ and constraints $g$ satisfy some regularity conditions, such as the Mangasarian-Fromovitz constraint qualifications [14, Sec. 3.2].

**Assumption 3.** The state and action spaces, $X^i$ and $U^i$, are open and convex subsets of a real vector space.

**Assumption 4.** There is at least one feasible point that is a NE of the game, i.e., the set of solutions of problem (1) is nonempty.

Under these rather general assumptions, finding a NE of problem (1) is a difficult task because the utilities, dynamic equation and constraints of the individual optimal control problems (OCP) are coupled among players. However, when problem (1) is a DPG, we can solve it through an equivalent MOCP—as opposed to a set of coupled univariate OCP. In Section IV, we introduce conditions that guarantee this equivalence for DPG with constrained state and action sets. These conditions generalize the well studied but simpler case of static potential games [10], [11], which is reviewed in the following section.

### III. Overview of Static Potential Games

Static games are a simplified version of dynamic games in the sense that there are neither states, nor system dynamics. The aim of each player $i$, given other players’ actions $u^{−i}$, is to choose an action $u^i \in U^i$ that maximizes its utility function:

$$
\mathcal{G}_2 : \forall i \in Q \maximize_{u^i \in U^i} \pi^i(u^i, u^{−i}) \quad \text{s.t.} \quad g(u) \leq 0
$$

\[ \text{(3)} \]

where (similar to dynamic games but removing the time-dependence subscript) $u^i \in U^i$ refers to the action of player $i$; and $u^{−i} = \{u^j \in Q, j \neq i\}$ is the set of actions of the rest of agents, such that $u = (u^i, u^{−i}) \in U$ denotes the set of all players’ actions. We assume $U \subseteq \mathbb{R}^q$ to be open and convex.

In general, finding or even characterizing the set of equilibrium points (e.g., in terms of existence or uniqueness) of problem (3) is difficult. Fortunately, there are particular cases of this problem for which the analysis is greatly simplified. Potential games is one of these cases.

**Definition 2.** Problem (3) is called a static potential game if there is a function $\Pi : U \rightarrow \mathbb{R}$, named the potential, that satisfies the following condition for every player $t$:

$$
\pi^i(u^i, u^{−i}) = \pi^i(v^i, u^{−i}) - \Pi(u^i, u^{−i}) - \Pi(v^i, u^{−i})
$$

\[ \forall u^i, v^i \in U^i, \forall i \in Q. \quad (4) \]

It can be shown that a necessary and sufficient condition for a static game to be potential is the following:

$$
\frac{\partial \pi^i(u)}{\partial u^i} = \frac{\partial \Pi(u)}{\partial u^i}, \quad \forall i \in Q
$$

\[ \text{(5)} \]

In addition, if the utilities and the potential are twice continuously differentiable, a necessary and sufficient condition for the existence of $\Pi$ is given by

$$
\frac{\partial^2 \pi^i(u)}{\partial u^i \partial u^j} = \frac{\partial^2 \pi^i(u)}{\partial u^j \partial u^i}, \quad \forall i, j \in Q
$$

\[ \text{(6)} \]

Once we have established that the game is of the potential type, we can find its potential function by solving the following line integral:

$$
\Pi(u) = \sum_{i=1}^Q \int_0^1 \frac{\partial \pi^i(\xi^i(\lambda), u^{−i})}{\partial u^i} d\xi^i(\lambda) d\lambda
$$

\[ \text{(7)} \]

We can gain insight on potential games by relating (5), (6) and (7) to the concept of conservative vector field.

**Lemma 1.** Let $F$ be a vector field with continuous derivatives defined over an open convex set. The following conditions on $F$ are equivalent:

(i) There exists a scalar potential function $\Pi$ such that $F = \nabla \Pi$, where $\nabla$ is the gradient.

(ii) $\nabla \times F = 0$.

(iii) For any oriented simple closed curve $C$, $\int_C F \cdot ds = 0$.

(iv) For any two oriented simple curves $C_1$ and $C_2$ with the same end points, $\int_{C_1} F \cdot ds = \int_{C_2} F \cdot ds$.

A vector field satisfying one (and, hence, all) of the conditions (i)–(iv) is called a conservative vector field.

**Proof:** See, e.g., [15, Th. 7].

Let us define a vector field with components the partial derivatives of the utilities:

$$
F \triangleq \left( \frac{\partial \pi^1(u)}{\partial u^1}, \ldots, \frac{\partial \pi^Q(u)}{\partial u^Q} \right)
$$

\[ \text{(8)} \]

If this field is conservative, from (5) and Lemma 1(i), we have that $F = \nabla \Pi(u)$, and the scalar potential function $\Pi$ can be found by the line integral of the field, which yields an equation similar to (7). Once we have found $\Pi$, it can be seen that necessary conditions for $u^*$ to be an equilibrium of the game are also necessary conditions for the following optimization problem [10]:

$$
\mathcal{P}_1 : \maximize_{u \in U} \Pi(u) \quad \text{s.t.} \quad g(u) \leq 0
$$

\[ \text{(9)} \]

Indeed, optimization theorems concerning existence and con-
verge can now be applied to game \(3\). In particular, reference \[10\] showed that the local maxima of the potential function are a subset of the local equilibria of the game, and that equilibria that are not local maxima are usually unstable. Furthermore, in the case that all players’ utilities are quasi-concave, the maximum is unique and coincides with the stable equilibrium of the game.

This same approach can be extended to dynamic games. Nevertheless, instead of obtaining an analogous optimization beforehand. Fortunately, there are cases in which the player’s utilities have some separable structure that allows us to easily deduce that the game is of the potential type, as it is explained in the following lemma.

**Lemma 3.** Problem \(1\) is a dynamic potential game if the utility function of every player \(i \in \mathcal{Q}\) can be expressed as the sum of a term that is common to all players plus another term that depends neither on its own action, nor on its own state-components:

\[
\pi^i(x_t^i, u_t^i, u_t^{-i}, t) = \Pi(x_t, u_t^i, u_t^{-i}, t) + \Theta(x_t^{-i}, u_t^{-i}, t) \quad \forall u_t^i \in U_t^i \tag{16}
\]

By subtracting \(15\) from \(14\) and summing over all \(t\), we conclude the proof.

Condition \(11\) is usually difficult to check in practice because we do not know \(\Pi\) beforehand. Fortunately, there are cases in which the player’s utilities have some separable structure that allows us to easily deduce that the game is of the potential type, as it is explained in the following lemma.

**Lemma 4.** Problem \(1\) is a dynamic potential game if all players’ utilities satisfy the following conditions \(\forall i, j \in \mathcal{Q}\):

\[
\frac{\partial^2 \pi^i(x_t, u_t, t)}{\partial x_t^i \partial u_t^j} = \frac{\partial^2 \pi^i(x_t, u_t, t)}{\partial x_t^j \partial u_t^i} = \frac{\partial^2 \pi^i(x_t, u_t, t)}{\partial u_t^i \partial u_t^j} = \frac{\partial^2 \pi^i(x_t, u_t, t)}{\partial u_t^j \partial u_t^i} \tag{19}
\]

**Proof:** Under Assumption \(1\) we can introduce the following vector field:

\[
F \triangleq \left( \nabla_{x_t^i} \pi^i(x_t, u_t, t)^T, \ldots, \nabla_{x_t^i} \pi^Q(x_t^Q, u_t, t)^T, \frac{\partial \pi^i(x_t^i, u_t, t)}{\partial u_t^i}, \ldots, \frac{\partial \pi^Q(x_t^Q, u_t, t)}{\partial u_t^i} \right) \tag{20}
\]

where \(\nabla_{x_t^i} \pi^i(x_t^i, u_t, t) = \left( \frac{\partial \pi^i(x_t^i, u_t, t)}{\partial x_t^i} \right) m \in \mathcal{X}(i)\). From Lemma \(2\) we can express \(20\) as

\[
F = \nabla \Pi(x_t, u_t, t) \tag{21}
\]

From Assumption \(3\) and Lemma \(1\)(i), we know that \(F\) is conservative. Hence, Lemma \(1\)(ii) establishes that the second partial derivatives must satisfy \(17\)–\(19\).

Now we derive the main theoretical result. Introduce the
following MOCP:

\[ P_2 : \begin{align*}
\text{maximize} & \quad \sum_{t=0}^{\infty} \beta^t \Pi(x_t, u_t, t) \\
\text{s.t.} & \quad x_{t+1} = f(x_t, u_t, t), \ x_0 \text{ given} \\
& \quad g(x_t, u_t, t) \leq 0
\end{align*} \tag{22} \]

**Theorem 1.** If problem (1) is a dynamic potential game, under Assumptions [1][4] the solution of the MOCP (22) is a NE of (1) when the objective function of the MOCP is given by

\[ \Pi(x_t, u_t, t) = \sum_{i=1}^{Q} \left( \int_0^1 \sum_{m \in X(i)} \partial \pi^i(\eta(\lambda), u_t, t) \frac{d\eta_i^m(\lambda)}{d\lambda} d\lambda + \int_0^1 \sum_{m \in X(i)} \partial \pi^i(x_t, \xi(\lambda), t) \frac{d\xi_i^m(\lambda)}{d\lambda} d\lambda \right) \tag{23} \]

where \( \eta(\lambda) \triangleq (\eta^k(\lambda))_{k=1}^S, \xi(\lambda) \triangleq (\xi^k(\lambda))_{k=1}^Q, \) and \( \eta(0)-\xi(0) \) and \( \eta(1)-\xi(1) \) correspond to the initial and final state-action conditions, respectively.

**Proof:** The proof is structured in five steps. First, we compute the Euler equation of the Lagrangian of the dynamic game and derive a set of necessary optimality conditions. Second, we study when the necessary optimality conditions of the game become equal to those of the MOCP. Third, we show that the solution of the MOCP is a NE of the game, whose existence is guaranteed by Assumption [4]. Finally, we derive the objective of the MOCP as the potential function of the field. We proceed to explain the details.

First, for problem (1) introduce each player’s Lagrangian \( \forall i \in Q \):

\[ L^i(x_t, u_t, t, \lambda_t^i, \mu_t^i) = \sum_{t=0}^{\infty} \beta^t \left( \pi^i(x_t, u_t, t) + \lambda_t^i f(x_t, u_t, t) - x_{t+1} + \mu_t^i g(x_t, u_t, t) \right) \]

\[ = \sum_{t=0}^{\infty} \beta^t \Phi^i(x_t, u_t, t, \lambda_t^i, \mu_t^i) \tag{24} \]

where \( \lambda_t^i \triangleq (\lambda_t^k)_{k=1}^S \) and \( \mu_t^i \triangleq (\mu_t^k)_{k=1}^C \) are the corresponding vectors of multipliers, and we introduced the shorthand:

\[ \Phi^i(x_t, u_t, t, \lambda_t^i, \mu_t^i) \triangleq \pi^i(x_t, u_t, t) + \lambda_t^i f(x_t, u_t, t) - x_{t+1} + \mu_t^i g(x_t, u_t, t) \]

\[ + \lambda_t^i f(x_t, u_t, t) - x_{t+1} + \mu_t^i g(x_t, u_t, t) \tag{25} \]

The discrete time Euler-Lagrange equations [9] Sec. 6.1 [3] applied to each player’s Lagrangian are given by:

\[ \frac{\partial \Phi^i(x_{t-1}, u_{t-1}, t-1, \lambda_{t-1}^i, \mu_{t-1}^i)}{\partial x_t^m} + \frac{\partial \Phi^i(x_t, u_t, t, \lambda_t^i, \mu_t^i)}{\partial x_t^m} = 0, \quad \forall m \in X(i) \tag{26} \]

\[ \frac{\partial \Phi^i(x_{t-1}, u_{t-1}, t-1, \lambda_{t-1}^i, \mu_{t-1}^i)}{\partial u_t^i} + \frac{\partial \Phi^i(x_t, u_t, t, \lambda_t^i, \mu_t^i)}{\partial u_t^i} = 0, \quad \forall m \in X(i) \tag{27} \]

Actually, note that (26)–(27) are the Euler-Lagrange equations in a more general form than the standard reduced form. In the standard reduced form (see, e.g., [9] Sec. 6.1, [3]), the current action can be posed as a function of the current and future states: \( u_t = \varphi(x_t, x_{t+1}) \), for some function \( \varphi : X \times X \rightarrow U \).

The reason why we introduced this general form of the Euler-Lagrange equations is that such function \( \varphi \) may not exist for an arbitrary state-transition function \( f \). By substituting (25) into (26)–(27), and adding the corresponding constraints, we obtain the KKT conditions of the game for every player \( i \in Q \), the state-components \( m \in X(i) \), and all extra constraints:

\[ \frac{\partial \pi^i(x_t, u_t, t)}{\partial x_t^m} + \sum_{k=1}^{S} \lambda_t^k \frac{\partial f^k(x_t, u_t, t)}{\partial x_t^m} = 0 \tag{28} \]

\[ + C \sum_{c=1}^{C} \mu_t^c \frac{\partial g^c(x_t, u_t, t)}{\partial x_t^m} - \lambda_t^{im} - \gamma_t^{im} = 0 \]

\[ + C \sum_{c=1}^{C} \mu_t^c \frac{\partial g^c(x_t, u_t, t)}{\partial u_t^i} = 0 \tag{29} \]

\[ x_{t+1} = f(x_t, u_t, t), \ g(x_t, u_t, t) \leq 0 \tag{30} \]

\[ \mu_t^i \leq 0, \quad \mu_t^i \gamma(x_t, u_t, t) = 0 \tag{31} \]

Second, we find the KKT conditions of the MOCP. To do so, we obtain the Lagrangian of (22):

\[ \ell^\Pi(x_t, u_t, t, \gamma, \delta_t) = \sum_{t=0}^{\infty} \beta^t \left( \Pi(x_t, u_t, t) + \gamma_t f(x_t, u_t, t) - x_{t+1} + \delta_t \right) \]

where \( \gamma_t \triangleq (\gamma_k)_{k=1}^S \) and \( \delta_t \triangleq (\delta_t^i)_{i=1}^C \) are the corresponding multipliers. Again, from (32) we derive the Euler-Lagrange equations, which, together with the corresponding constraints, yield the KKT system of optimality conditions for all state-components, \( m = 1, \ldots, S \), and all actions, \( i = 1, \ldots, Q \):

\[ \frac{\partial \Pi(x_t, u_t, t)}{\partial x_t^m} + \sum_{k=1}^{S} \gamma_t^k \frac{\partial f^k(x_t, u_t, t)}{\partial x_t^m} = 0 \tag{33} \]

\[ + C \sum_{c=1}^{C} \delta_t^c \frac{\partial g^c(x_t, u_t, t)}{\partial x_t^m} - \gamma_t^{im} = 0 \]

\[ \frac{\partial \Pi(x_t, u_t, t)}{\partial u_t^i} + \sum_{k=1}^{S} \gamma_t^k \frac{\partial f^k(x_t, u_t, t)}{\partial u_t^i} = 0 \tag{34} \]

\[ + C \sum_{c=1}^{C} \delta_t^c \frac{\partial g^c(x_t, u_t, t)}{\partial u_t^i} = 0 \]

\[ x_{t+1} = f(x_t, u_t, t), \ g(x_t, u_t, t) \leq 0 \tag{35} \]

\[ \delta_t \leq 0, \quad \delta_t \gamma(x_t, u_t, t) = 0 \tag{36} \]

In order for the MOCP (22) to have the same optimality conditions as the game (1), by comparing (28)–(31) with (33)–(36), we conclude that the following conditions must be
satisfied \( \forall i \in Q \):

\[
\frac{\partial \pi^i}{\partial x^m_i} (x_i, u_i, t) = \frac{\partial \Pi (x_i, u_i, t)}{\partial x^m_i}, \quad \forall m \in \mathcal{X}(i) \tag{37}
\]

\[
\frac{\partial \pi^i}{\partial u^i} (x_i, u_i, t) = \frac{\partial \Pi (x_i, u_i, t)}{\partial u^i}, \quad \forall u^i \tag{38}
\]

\[
\lambda^i = \gamma^i, \quad \mu^i = \delta^i \tag{39}
\]

Note that condition (39) represents a feasible point of the game. The reason is that existence of dual variables in the MOCP is guaranteed by suitable constraint qualifications. Therefore, under Assumption 2 there exists some \( \gamma^i \) and \( \delta^i \) that satisfy the KKT conditions of the MOCP, altogether with some suitable primal variables. Substituting these dual variables of the MOCP in place of the individual \( \lambda^i \) and \( \mu^i \) in (28)–(31), for all \( i \in Q \), results in a system of equations where the only unknowns are the user strategies. This system has exactly the same structure as the one already presented for the MOCP in the primal variables. Therefore, the MOCP primal solution also satisfies the KKT conditions of the game. Finally, it is straightforward to see that an optimal solution of the MOCP that satisfies the KKT conditions will be also be NE of the game. Let \( \{ u^i_t \}_{t=0}^\infty \) denote the solution of the MOCP so that it satisfies the following inequality \( \forall u^i_t \in \mathcal{U}^i \):

\[
\sum_{t=0}^\infty \beta^t \Pi (x_t, u^i_t, u^{i-1}_t, t) \geq \sum_{t=0}^\infty \beta^t \Pi (x^*_t, u^*_t, u^{i-1}_t, t) \tag{40}
\]

When conditions (37)–(39) are satisfied, Lemma 2 states that problem (1) is a DPG. Therefore, from Definition 2 we conclude that the optimal solution of MOCP (22) is also a NE of game (1). The opposite may not be true in general. We remark that this solution, in which dual variables are shared between players, is only a subclass of the possible NE of the game. Nevertheless, other NE that do not share this property have been referred to as unstable by [10] for static games.

Although we have shown that we can find a NE of the dynamic game by solving a MOCP, we still need to find the actual objective of the MOCP. In order to find \( \Pi \), we deduce from (37), (38), (20) and (21) that the vector field (20) can be expressed as

\[
F \triangleq \nabla \Pi (x_t, u_t, t) \tag{41}
\]

Therefore, Lemma 1(i) establishes that \( F \) is conservative. Thus, the objective of the MOCP is the potential of the field, which can be computed through the line integrals in (22). \( \square \)

The usefulness of Theorem 1 is that, in order to find a NE of (1), instead of solving several coupled control problems, we can check whether (1) is a dynamic potential game (i.e., Lemmas 2–4 hold) and, if so, we can compute the objective (23) and solve the equivalent MOCP (22). In the next sections, we show how to apply this methodology to different practical problems.

V. ENERGY DEMAND IN THE SMART GRID AS A LINEAR QUADRATIC DYNAMIC GAME

Our first example consists in a linear-quadratic-dynamic-game (LQDG) that solves a smart grid resource allocation problem. LQDG are convenient because they are amenable to analytical and closed form solutions [16, Ch. 6]. Our analysis is novel though. To the best of our knowledge, LQDG have not been studied under the easier DPG framework before.

A. Energy demand control dynamic game and equivalent MOCP

Consider a community of \( Q \) users (i.e., players) that use the smart grid resources in different activities (like communications, heating, lighting, home appliances or production needs). Suppose that the electrical grid has \( S \) types of energy resources (such as rechargeable batteries, coal, fuel, hydroelectric power or biomass). The state of the game \( x_t \in \mathbb{R}^S \) is the total amount of overall resources in the smart grid at time \( t \). All players share all components of the state-vector (i.e., \( x^i = x \) and \( \mathcal{X}(i) = \{1, \ldots, S\} \), \( \forall i \in Q \)). The amount of resources consumed or contributed by player \( i \) at time \( t \) is denoted by the action vector \( u^i_t \in \mathbb{R}^A^i \), where \( A^i \) is the number of activities.

The expenditure and contribution of each player \( i \) is weighted by matrix \( B^i \in \mathbb{R}^{S \times A^i} \). Also, resources can be autonomously recharged/depleted, which is modeled by a shared matrix \( C \in \mathbb{R}^{S \times S} \). Thus, the state transition of the system is \( f = C x_t + \sum_{i \in Q} B^i u^i_t \).

We consider two cost terms: unsatisfied demand and unbalanced resources. Given the available resources \( x_t \), every player \( i \) will have a target demand \( D^i x_t \) that it wants to satisfy, for some demand matrix \( D^i \in \mathbb{R}^{A^i \times S} \). The disutility from an unsatisfied demand is modelled by the quadratic form \( (D^i x_t - u^i_t) \top Q^i (D^i x_t - u^i_t) \), with demand cost matrix \( Q^i \in \mathbb{R}^{A^i \times A^i} \). In addition, the available resources should be just enough to satisfy the demand. There is a cost for having too little (e.g., productivity decrease) or too much (e.g., storage costs) resources. This cost can be modeled as another quadratic form: \( (x_t - x_t-1) \top R (x_t - x_t-1) \), with unbalanced resources cost matrix \( R \in \mathbb{R}^{S \times S} \). In order to pose the game as a maximization problem, we assume \( \{Q^i\}_{i \in Q} \) to be negative definite matrices, and \( R \) a negative semidefinite matrix (this is represented by \( Q^i < 0 \) and \( R \preceq 0 \)).

The dynamic energy demand control game is given by the following coupled optimal control problems:

\[
\max_{\{u^i_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \left( (x_t - x_t-1)^\top R (x_t - x_t-1) \right. \left. + (D^i x_t - u^i_t)^\top Q^i (D^i x_t - u^i_t) \right) \tag{42}
\]

s.t. \( x_{t+1} = C x_t + \sum_{i=1}^Q B^i u^i_t, \ x_0 \) given

By defining augmented state and action vectors:

\[
\bar{x}_t \triangleq [x^T_t, x^T_{t-1}, \ldots]^\top, \ \bar{u}_t \triangleq [D^i x_t - u^i_t]^\top \tag{43}
\]

we can rewrite (42) in the standard linear-quadratic form:

\[
\max_{\{\bar{u}_t\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \left( \bar{x}_t^\top R \bar{x}_t + \bar{u}_t^\top Q \bar{u}_t \right) \tag{44}
\]

s.t. \( \bar{x}_{t+1} = A \bar{x}_t - \sum_{i=1}^Q B^i \bar{u}_t, \ \bar{x}_0 \) given
where
\[ A \triangleq \begin{bmatrix} C + \sum_{i \in Q} B^i D^i & 0_{S \times S} \\ I_S & 0_{S \times S} \end{bmatrix}, \quad \tilde{B}^i \triangleq \begin{bmatrix} B^i \\ 0_{S \times A^i} \end{bmatrix} \] (45)
\[ \tilde{R} \triangleq \begin{bmatrix} R & -R \\ -R & R \end{bmatrix} \] (46)
and where \( I_S \) and \( 0_{S \times S} \) denote the identity and null matrices of size \( S \times S \), respectively. LQDG games in the form (44) have been presented by reference [16, Ch. 6], where a NE is found by i) solving the system of coupled finite horizon OCP; ii) finding the limit of this solution as the horizon tends to infinity, and then iii) verifying that this limiting solution provides a NE solution for the infinite-horizon game. Here we follow a different and simpler approach. First, we show that problem (44) can be expressed in the separable form [16]:
\[ \pi^i(\tilde{x}_t, \tilde{u}_t) = \tilde{x}_t^T \tilde{R} \tilde{x}_t + \sum_{p \in Q} \tilde{u}_t^p Q^p \tilde{u}_t^p - \sum_{j \in Q : j \neq i} \tilde{u}_t^j Q^j \tilde{u}_t^j \] (47)
whence we identify the potential and separable functions:
\[ \Pi(\tilde{x}_t, \tilde{u}_t, t) = \tilde{x}_t^T \tilde{R} \tilde{x}_t + \sum_{p \in Q} \tilde{u}_t^p Q^p \tilde{u}_t^p \] (48)
\[ \Theta(\tilde{u}_t, t) = -\sum_{j \in Q : j \neq i} \tilde{u}_t^j Q^j \tilde{u}_t^j \] (49)
From Lemma 3, we conclude that problem (42) is a dynamic potential game. Hence, from Theorem 4, we can find a NE by solving the following MOCP:
\[ \text{maximize}_{\{u_t\} \in \prod_{t=0}^{\infty} U_t} \ V(\tilde{x}_0) := \sum_{t=0}^{\infty} \beta^t \left( \tilde{x}_t^T \tilde{R} \tilde{x}_t + \sum_{p \in Q} \tilde{u}_t^p Q^p \tilde{u}_t^p \right) \] (50)
\[ \text{s.t.} \quad \tilde{x}_{t+1} = A \tilde{x}_t + \sum_{i=1}^{Q} B^i \tilde{u}^i_t, \quad \tilde{x}_0 \text{ given} \]
where the cumulative objective function \( V \) is usually named value function in the optimal control literature (see, e.g., [17]). Let \( \tilde{u}_t \triangleq (\tilde{u}^i_t)_{i=1}^{Q} \) be the vector of all players’ augmented actions. Aggregate all players’ demand matrices in a block diagonal matrix \( Q \triangleq \text{diag} (Q^1, \ldots, Q^Q) \) of size \( \sum_{i=1}^{Q} A^i \times \sum_{i=1}^{Q} A^i \). and aggregate all players’ expenditure weighing matrices in a \( S \times \sum_{i=1}^{Q} A^i \) thick matrix \( \tilde{B} \triangleq (\tilde{B}^1, \ldots, \tilde{B}^Q) \). Then, we can rewrite the value and transition functions:
\[ V(\tilde{x}_0) = \sum_{t=0}^{\infty} \beta^t \left( \tilde{u}^i_t Q \tilde{u}^i_t + \tilde{x}^T_t \tilde{R} \tilde{x}_t \right) \] (51)
\[ \tilde{x}_{t+1} = A \tilde{x}_t - B \tilde{u}_t \] (52)

B. Analytical solution to the MOCP and simulation results

It is well known that the value function satisfies a recursive relationship, known as Bellman equation (see, e.g., [17]):
\[ V(\tilde{x}_t) = \beta^t \left( \tilde{u}^i_t Q \tilde{u}^i_t + \tilde{x}^T_t \tilde{R} \tilde{x}_t \right) + \beta^{t+1} V(\tilde{x}_{t+1}) \] (53)
Moreover, for an LQ control problem, it is known [16, Ch. 6] that the optimal value function (51) can be expressed as a quadratic form of the state:
\[ V(\tilde{x}_t) = \tilde{x}^T_t P \tilde{x}_t \] (54)
for some negative semidefinite matrix \( P \). We can use (54) to find a closed form expression for the sequence of optimal actions as follows. Expand (52) and (54) into (53):
\[ V(\tilde{x}_t) = \beta^t \left( \tilde{u}^i_t Q \tilde{u}^i_t + \tilde{x}^T_t \tilde{R} \tilde{x}_t \right) + \beta^{t+1} (A \tilde{x}_t - B \tilde{u}_t)^T P (A \tilde{x}_t - B \tilde{u}_t) \] (55)
Now, we just have to maximize (55) over \( \tilde{u}_t \). Since \( Q \) and \( P \) are negative definite and semidefinite matrices, respectively, a necessary and sufficient condition for the maximum is
\[ \nabla_{\tilde{u}_t} V(\tilde{x}_t) = \beta Q \tilde{u}_t - \beta^{t+1} B^T P (A \tilde{x}_t - B \tilde{u}_t) = 0 \] (56)
From (56), we obtain an analytical expression for the optimal action at any time step:
\[ \tilde{u}_t = \beta (Q + \beta B^T P B)^{-1} B^T PA \tilde{x}_t \] (57)
If we are also interested in finding the optimal value, we can expand (57) into (55) and isolate \( P \):
\[ P_{n+1} = \tilde{R} + \beta A^T P_n A \] (58)
Note that (58) is a discrete algebraic Riccati equation, which is known to be a contraction mapping if \( Q \prec 0 \), \( \tilde{R} \preceq 0 \) and the spectral radius of \( A \) is smaller than one [18, Ch. 5] (the analysis can be performed under weaker conditions though [17], [19]). When (58) is a contraction, it has a unique solution \( P^* \) that can be approximated by iterating the following fixed point equation, such that \( \lim_{n \to \infty} P_n = P^* \):
\[ P_{n+1} = \tilde{R} + \beta A^T P_n A \] (59)
We have simulated the smart grid model for \( Q = 8 \) players, \( S = 4 \) resources, \( A^i = 6 \) activities for every player, random negative definite matrices \( Q^i, \forall i \in Q \), and random negative semidefinite matrix \( \tilde{R} \) (to build these negative matrices we build an intermediate matrix, e.g., \( R_{\text{ran}} \), by drawing random numbers from a uniform distribution, with support \([0, 10]\) for \( Q^i \) and \([0, 5]\) for \( \tilde{R} \), and compute \( R = -R_{\text{ran}} R_{\text{ran}}^T \)). Matrices \( C \), \( B \) and \( D \) are also random with elements drawn from the spherical normal distribution. Finally, the initial state was set to a vector of ones, i.e., \( \tilde{x}_0 = 1_S \), and discount factor \( \beta = 0.9 \).
that the agents adjust their actions \( u_i^t \) to satisfy the target demand. The equilibrium between target demand and players’ activities is an expected consequence of the stability of the LQ game in infinite horizon [16 Ch. 6].

\[
\text{Fig. 1. Dynamic smart grid scenario with } Q = 8 \text{ players. (Top) Instant utility values of players. (Bottom) Players’ decision coefficients evolution in time.}
\]

VI. NETWORK FLOW CONTROL: INFINITE HORIZON APPROXIMATED BY A FINITE HORIZON DYNAMIC GAME

Several works (see, e.g., [20 – 23]) have considered network flow control as an optimization problem wherein each source is characterized by a utility function that depends on the transmission rate, and the goal is to maximize the aggregated utility. We generalize the standard model by considering that the nodes are equipped with batteries that are depleted proportionally to the outgoing flow. In addition we consider several layers of relay nodes, each one with multiple links, so there are several paths between source and destination. When the batteries are completely depleted, no more transmissions are allowed and the game is over. Hence, although we formulate the problem as an infinite horizon dynamic game, the effective time horizon—before the batteries deplete—is finite. This problem has no known analytical solution, but the utilities are concave. Therefore, the finite horizon approximation is convenient because we can solve an equivalent concave optimization problem, significantly reducing the computational load with respect to other optimal control algorithms (e.g., dynamic programming).

A. Network flow control dynamic game and equivalent MOCP

Let \( u_i^{t,a} \) denote the flow along path \( a \) for user \( i \) at time \( t \). Suppose there are \( A_i \) possible paths for each player \( i \in Q \), so that \( u_i^t = (u_i^{t,a})_{a=1}^{A_i} \) denotes the \( i \)-th player’s action vector. Let \( A = \sum_{i=1}^{Q} A_i \) denote the total number of available paths.

Suppose there are \( S \) relay nodes. Let \( x_k^t \) denote the battery level of relay node \( k \). The state of the game is given by \( x_t \equiv x_t^S \) such that all players share all components of the state-vector (i.e., \( X^t = X \) and \( X(i) = \{1, \ldots, S\}, \forall i \in Q \)). The battery level evolves with the following state-transition equation for all components \( k = 1, \ldots, S \):

\[
x_{t+1}^k = x_t^k - \delta \sum_{a=1}^{Q} \sum_{i=1}^{S} u_{i,a}^{t} , \quad x_0^k = B_{\text{max}}^k \quad \text{(60)}
\]

where \( F_k \) denotes the subset of flows through node \( k \), \( B_{\text{max}}^k \) is a positive scalar that stands for the maximum battery level of node \( k \), and \( \delta \) is a proportional factor.

Similar to the standard static flow control problem, each player intends to maximize a concave function \( \Gamma : \mathcal{U}^t \rightarrow \mathbb{R} \) of the sum of rates across all available paths. This function \( \Gamma \) can take different forms depending on the scenario under study, like the square root [24], \( \Gamma(\cdot) \equiv \sqrt{\cdot} \), or a capacity form, \( \Gamma(\cdot) \equiv \log(1+\cdot) \). In addition to the transmission rate, we include the relay nodes’ battery level in each player’s utility, weighted by some positive parameter \( \alpha \). The combination of these two objectives can be understood as the player aiming to maximize its total transmission rate, while saving the batteries of the relays.

For all links in the network, there is some capacity constraint. Let \( c_{\text{max}} \in \mathbb{R}^L \) denote the vector with maximum capacity for each link, where \( L \) is the total number of available links. Let \( \mathbf{M} = [m_{la}] \) denote the \( L \times A \) connectivity-matrix between links and players’ paths, such that the element \( m_{la} = 1 \) if link \( l \) can be used by path \( a \), and \( m_{la} = 0 \) otherwise.

The dynamic network flow control game is given by the following set of coupled OCP:

\[
\begin{align*}
\text{maximize} & \quad \sum_{t=0}^{\infty} \beta^t \left( \Gamma \left( \sum_{a=1}^{A_i} u_{i,a}^{t} \right) + \alpha \sum_{k=1}^{S} x_t^k \right) \\
\text{subject to} & \quad \forall i \in Q : \quad x_{t+1}^k = x_t^k - \delta \sum_{a=1}^{Q} \sum_{i=1}^{S} u_{i,a}^{t} , \quad \text{(61)}
\end{align*}
\]

\[
\begin{align*}
\mathcal{G}_i : \quad & x_0^k = B_{\text{max}}^k, \quad 0 \leq x_t^k \leq B_{\text{max}}^k \\
\mathbf{Mu} & \leq \mathbf{c}_{\text{max}}, \quad u_{i,a}^{t} \geq 0 \\
& k = 1, \ldots, S, \quad a = 1, \ldots, A_i
\end{align*}
\]

Note that each player’s utility can be expressed in separable form:

\[
\begin{align*}
\pi^i(x_{t+1}^i, u_t, t) & = \Gamma \left( \sum_{a=1}^{A_i} u_{i,a}^{t} \right) + \alpha \sum_{k=1}^{S} x_t^k \\
& = \sum_{p \in Q} \Gamma \left( \sum_{a=1}^{A_p} u_{i,a}^{p} \right) + \alpha \sum_{k=1}^{S} x_t^k - \sum_{j \in Q, j \neq i} \Gamma \left( \sum_{a=1}^{A_j} u_{i,a}^{j} \right) \quad \text{(62)}
\end{align*}
\]

Therefore, Lemma 3 establishes that problem (61) is a dy-
A dynamic potential game, with potential function given by:

$$\Pi(x_t, u_t, t) = \sum_{p \in Q} \Gamma \left( \sum_{a=1}^{A^p} u_{ta}^p \right) + \alpha \sum_{k=1}^{S} x_k^t$$ (63)

Furthermore, Theorem 1 establishes that we can find a NE of (61) by solving the following MOCP:

$$\begin{align*}
\text{maximize} & \quad \sum_{i=0}^{\infty} \beta^t \left( \sum_{p \in Q} \Gamma \left( \sum_{a=1}^{A^p} u_{ta}^p \right) + \alpha \sum_{k=1}^{S} x_k^t \right) \\
\text{s.t.} & \quad x_{t+1}^k = x_t^k - \delta \sum_{i \in Q} u_{ia}^{\pi x} F_k \\
& \quad x_0^k = B_{\text{max}}^k, \quad 0 \leq x_t^k \leq B_{\text{max}}^k \\
& \quad M_{u_t} \leq c_{\text{max}}, \quad u_t \geq 0 \\
& \quad k = 1, \ldots, S
\end{align*}$$ (64)

**B. Finite horizon approximation to the MOCP and simulation results**

As opposed to the LQ smart-grid problem, there is not known closed form solution for problem (64). Thus, we have to rely on numerical methods to solve the MOCP. Note that we have not introduced any mechanism to recharge the node batteries in (64). Suppose that we set the weight parameter $\alpha$ in $\Pi$ low enough to incentivize some positive transmission. Eventually, the nodes’ batteries will be depleted, so the system will get stuck in an equilibrium state, with no further state transitions. Thus, we can approximate the infinite-horizon problem (64) as a finite-horizon problem, with horizon bounded by the time-step at which all batteries have been depleted. Moreover, in our setting, we have assumed $\Gamma$ to be concave. Therefore, we can effectively solve (64) with convex optimization solvers (we use the software described in [25]). The benefit of using a convex optimization solver is that standard optimal control algorithms are computationally demanding when the state and action spaces are subsets of vector spaces.

For our numerical experiment, we consider $Q = 2$ players that share a network of $S = 4$ relay nodes, organized in two layers (see Figure 2). In this particular setting, each player is allowed to use four paths, $A^1 = A^2 = 4$. The connectivity matrix $M$ can be obtained from Figure 2. The battery is initialized to $B_{\text{max}} = 1$ for the four relay nodes, we set the depleting factor $\delta = 0.05$, and the discount factor $\beta = 0.9$. The function $\Gamma(\cdot)$ is chosen to be the square root. And the constraint vector of maximum capacities is $c = [0.5, 0.25, 1, 0.5, 1, 0.5, 1, 0.8, 0.75, 1, 0.7, 1]^T$.

Figure 3 shows the evolution of the eight flows along with the battery evolution of each of the relaying nodes. As expected, as long as the battery level decreases, the maximum flow in each of the paths is also reduced, until the battery of every node is depleted and none of the users is allowed to transmit anymore. Clearly, the system enters in an equilibrium state and the finite horizon equals the infinite horizon solution.

**VII. DYNAMIC MULTIPLE ACCESS CHANNEL: NONSEPARABLE UTILITIES**

In this section, we consider an uplink scenario in which every user $i \in Q$ independently chooses its transmitter power, $u_i^t$, aiming to achieve the maximum rate allowed by the channel [12]. If multiple users transmit at the same time, they will interfere each other, which will decrease their rate, so that they have to find an equilibrium. Let $R_i^t$ denote the rate achieved by user $i$ with normalized noise at time $t$:

$$R_i^t \triangleq \log \left( 1 + \frac{|h_i|^2 u_i^t}{1 + \sum_{j \in Q: j \neq i} |h_j|^2 u_j^t} \right)$$ (65)

where $h_i$ denotes the fading channel coefficient of user $i$.

**A. Multiple access channel dynamic potential game and equivalent MOCP**

Let $x_i^t \in [0, B_{i,\text{max}}^t]$ denote the battery level for each player $i \in Q$, which is discharged proportionally to the transmitted
power $u_i$. The state of the system is given by the vector with all individual battery levels: $x_i = \{x_i^t\}_{t \in \mathcal{Q}} \in \mathcal{X}$, thus each player is only affected by its own battery and, therefore, $S = \mathcal{Q}$ and $\mathcal{X}(i) = \{i\}$. Suppose the agents aim to maximize its transmission rate, while also saving their battery. This scenario yields the following dynamic game:

$$
\max_{\{u_i\} \in \prod_{t=0}^{\infty} \mathcal{U}} \sum_{t=0}^{\infty} \beta^t \left( R_i^t + \alpha x_i^t \right)
$$

s.t. $x_{i+1}^t = x_i^t - \delta u_i^t$, $x_i^0 = B_{\max}^i$

where $\alpha$ is the weight given for saving the battery, $\delta$ is the discharging factor, and $P_{\max}^i$ and $B_{\max}^i$ denote the maximum transmitter power and maximum available battery level for node $i$, respectively. Problem (66) is a dynamic infinite-horizon extension of the static problem proposed in [26].

Instead of looking for a separable structure in the players’ utilities, we show that conditions (17)–(19) are satisfied:

$$
\frac{\partial^2 \pi^i(x_t, u_t, t)}{\partial x_t^2 \partial u_t^t} = \frac{\partial^2 \pi^j(x_t, u_t, t)}{\partial x_t^2 \partial u_t^t} = 0
$$

(67)

$$
\frac{\partial^2 \pi^i(x_t, u_t, t)}{\partial x_t \partial x_t^t} = \frac{\partial^2 \pi^j(x_t, u_t, t)}{\partial x_t \partial x_t^t} = 0
$$

(68)

$$
\frac{\partial^2 \pi^i(x_t, u_t, t)}{\partial u_t \partial u_t^t} = \frac{\partial^2 \pi^j(x_t, u_t, t)}{\partial u_t \partial u_t^t} = \frac{-|h_i^t|^2 |h_j^t|^2}{\left(1 + \sum_{p \in \mathcal{Q}} |h_p|^2 u_p^t \right)^2}
$$

(69)

Now, we can apply Lemma 4 to establish that problem (66) is a dynamic potential game. Therefore, from Theorem 1 we know that we can find a NE of (66) by solving the following MOCP, with objective potential $\Pi$ given by (23):

$$
\max_{\{u_i\} \in \prod_{t=0}^{\infty} \mathcal{U}} \sum_{t=0}^{\infty} \beta^t \left( \log (1 + \sum_{i=1}^{Q} |h_i^t|^2 u_i^t) \right) + \alpha \sum_{i=1}^{Q} x_i^t
$$

(70)

s.t. $x_{i+1}^t = x_i^t - \delta u_i^t$, $x_i^0 = B_{\max}^i$

$0 \leq u_i^t \leq P_{\max}^i$, $0 \leq x_i^t \leq B_{\max}^i$

$\forall i \in \mathcal{Q}$

B. Simulation results

Similar to the network flow problem (Sec. VII), when the batteries have been completely depleted, the system enters into an equilibrium state. Thus, the solution can be approximated by solving a finite horizon problem. Moreover, since the problem is concave, we can use convex optimization software, like [25]. Alternatively, we could solve the KKT system with an efficient ad-hoc distributed algorithm, like in [12].

We simulated an scenario with $Q = 4$ users. We set the maximum battery level $B_{\max}^i = 33$ for all users, the maximum power allowed per user $P_{\max}^i = 5$ for all users, the weight battery utility factor $\alpha = 0.001$, the transmitter power battery depletion factor $\delta = 1$, and the discount factor $\beta = 0.95$. The channel gains are $|h_1^t| = 2.019$, $|h_2^t| = 1.002$, $|h_3^t| = 0.514$, and $|h_4^t| = 0.308$.

Figure 4 shows appealing results: the solution of the MOCP—which is a NE of the game—is actually a schedule. In other words, instead of creating interference among users, they wait until the users with higher channel-gain have depleted their batteries.

Fig. 4. Dynamic multiple access scenario with $Q = 4$ users. (Top) Sequence of transmitter power chosen by every user. (Bottom) Evolution of the transmission rates.

VIII. OPTIMAL SCHEDULING: NONSTATIONARY PROBLEM WITH DYNAMIC PROGRAMMING SOLUTION

In this section we present the most general form of the proposed framework for constrained DPG, and show its applicability to two decentralized scheduling problems. First, the utilities are nonseparable, so we have to verify second order conditions (17)–(19). Second, neither the equivalent MOCP can be approximated by a finite horizon problem, nor the utilities are concave. Thus, we cannot rely upon convex optimization software and we have to use optimal control methods, like dynamic programming [17]. Finally, we consider a nonstationary scenario, in which the channel coefficients evolve with time. This makes the state-transition equations (and the utility for the equal rate problem) depend not only on the current state, but also on time. This problem was introduced in the preliminary paper [13].

A. Proportional fair and equal rate scheduling games and their equivalent MOCP

The rate achieved by user $i$ at time $t$ for a nonstationary scenario is given by:

$$
R_i^t \triangleq \log \left( 1 + \frac{|h_i^t|^2 u_i^t}{1 + \sum_{j \in \mathcal{Q}, j \neq i} |h_j^t|^2 u_j^t} \right)
$$

(71)

where $u_i^t$ is the transmitter power of player $i$, and $|h_i^t|$ is its time-varying channel coefficient.

We propose two different decentralized scheduling problems, namely, proportional fair and equal rate scheduling.
1) Proportional fair scheduling: Proportional fair is a compromise-based scheduling algorithm. It aims to maintain a balance between two competing interests: trying to maximize total throughput while, at the same time, guaranteeing a minimal level of service for all users [27]–[29].

In order to achieve this tradeoff, we propose the following game:

\[
\text{maximize } \sum_{t=0}^{\infty} \beta^t x_t^i \quad \forall i \in Q
\]

\[
\text{s.t. } x_{t+1}^i = \left(1 - \frac{1}{t}\right) x_t^i + \frac{R_t^i}{t}
\]

where \( x_t^i \) is the cumulative rate of user \( i \) at time \( t \). The state of the system is the vector \( x_t = (x_t^i)_{i \in Q} \). Since each player aims to maximize its own average rate, the state-components are unshared among players: \( S = Q \) and \( \mathcal{X}(i) = \{i\} \).

In order to show that problem (72) is a DPG, we evaluate Lemma 3 with positive result, and obtain \( \Pi \) from Theorem 1. Thus, from Theorem 1, we can find a NE of dynamic game (72) by solving the following MOCP:

\[
\text{maximize } \sum_{t=0}^{\infty} \beta^t \sum_{i \in Q} x_t^i \quad \forall i \in Q
\]

\[
\text{s.t. } x_{t+1}^i = \left(1 - \frac{1}{t}\right) x_t^i + \frac{R_t^i}{t}
\]

\[
x_0^i = 0, \quad 0 \leq u_t^i \leq P_{\text{max}}^i
\]

where the state of the system is the vector of all players’ average rates \( x_t = (x_t^i)_{i \in Q} \). Since each player aims to maximize its own average rate, the state-components are unshared among players: \( S = Q \) and \( \mathcal{X}(i) = \{i\} \).

2) Equal rate scheduling: In this problem, the aim of each user is to maximize its rate, while at the same time keeping the users’ cumulative rates as close as possible. Let \( x_t^i \) denote the cumulative rate of user \( i \). The state of the system is the vector \( x_t = (x_t^i)_{i \in Q} \). Again \( S = Q \) and \( \mathcal{X}(i) = \{i\} \). This problem is modeled by the following game:

\[
\text{maximize } \sum_{i \in Q} \text{maximize } \sum_{t=0}^{\infty} \beta^t \left(1 - \alpha\right) R_t^i
\]

\[
\text{s.t. } x_{t+1}^i = x_t^i + R_t^i
\]

\[
x_0^i = 0, \quad 0 \leq u_t^i \leq P_{\text{max}}^i
\]

where parameter \( \alpha \) weights the contribution of both terms.

Similar to the proportional fair problem, it is easy to verify that conditions (67)–(69) are satisfied and, hence, we can find a NE of dynamic game (74) by solving the following MOCP, where we obtained the potential \( \Pi \) by integrating (23):

\[
\text{maximize } \sum_{i \in Q} \text{maximize } \sum_{t=0}^{\infty} \beta^t \left(1 - \alpha\right) \log \left(1 + \sum_{i=1}^{Q} |h_t^i|^2 u_t^i\right)
\]

\[
\text{s.t. } x_{t+1}^i = x_t^i + R_t^i
\]

\[
x_0^i = 0, \quad 0 \leq u_t^i \leq P_{\text{max}}^i
\]

B. Solving the MOCP with dynamic programming and simulation results

MOCP (73) and (75) cannot be approximated by finite horizon problems. Moreover, the problems are not concave. Therefore, we cannot rely on convex optimization software. In order to solve these problems, we will use dynamic programming methods [17].

Introduce the optimal policy \( \phi : \mathcal{X} \times \mathbb{N} \rightarrow \mathcal{U} \) as the sequence of optimal actions \( u_t^i = \phi(x_t^i, t)\). This provides the solution to the MOCP. Introduce also the optimal value function:

\[
V^*(x_0) = \max_{(u_t^i) \in \mathcal{U}} \sum_{t=0}^{\infty} \beta^t \Pi(x_t^i, u_t^i, t)
\]

\[
= \sum_{i=0}^{\infty} \beta^t \Pi(x_t^i, u_t^*, t) = (76)
\]

One standard method to cope with nonstationary MOCP is to augment the state space so that it includes the time as an extra dimension for some time interval \( t \in [0, T] \). Let the augmented state-vector at time \( t \) be denoted by \( \tilde{x}_t = (x_t^i, t) \in \tilde{X} = \mathcal{X} \times \mathbb{N} \). Then, the Bellman optimality equation in the augmented state space is given by

\[
V^*(\tilde{x}_t) = \Pi(\tilde{x}_t, u_t^*) + \beta V^*(f(\tilde{x}_t, u_t^*))
\]

where \( f : \mathcal{X} \times \{0, \ldots, T\} \rightarrow \mathcal{X} \) is the state-transition function in the augmented state space. Since we are tackling an infinite horizon problem, when augmenting the state space with the time dimension, it is convenient to impose a periodic time variation:

\[
f(\tilde{x}_t, u_t^*) = \begin{cases} f(x_t, u_t, t) & \text{if } t < T \\ f(x_T, u_T, 0) & \text{if } t = T \end{cases}
\]

(78)

Otherwise, \( T \) should large enough to make \( \beta^T \approx 0, \) so that (76) is well approximated.

Under some standard assumptions, (77) is a contraction mapping (see, e.g., [17]). Therefore, we can numerically solve the MOCP (i.e., find the optimal policy \( \phi \) for every state \( \tilde{x}_t \)) with an iterative procedure. One further difficulty for solving MOCP with continuous state and action spaces is that dynamic programming methods are mainly derived for discrete state-action spaces. Two common approaches to overcome this limitation are i) to use a parametric approximation of the value function (e.g., consider a neural network with inputs the continuous state action variables that is trained by minimizing the error in the Bellman equation), or ii) to discretize the continuous spaces, so the value function is approximated in a set of points. For the sake of clarity, we follow the discretization approach here. We remark that it may be problematic to finely discretize the state-action spaces in practice though, since by augmenting the state-space, we are increasing the number of states, and the computational load will increase exponentially with the number of states. These and other approximation techniques, usually known as approximate dynamic programming, are still an active area of research (see, e.g., [17] Vol 2. Ch. 6), [30].

Among the available dynamic programming methods, we
choose value iteration for its reduced complexity per iteration (with respect to policy iteration), which is especially relevant when the state-grid has fine resolution (i.e., large number of states). Value iteration is summarized in Algorithm 1, where the operator $\lceil x \rceil$ denotes the closest point to $\bar{x}$ in the discrete grid.

**Algorithm 1:** Value Iteration for the non-stationary MOCP

**Inputs:** number of states $S$, threshold $\epsilon$

**Discretize** the augmented space $\bar{X}$ into a grid of $S$ states

**Initialize** $\Delta = \infty$, $k = 0$ and $V_0(\bar{x}_s) = 0$ for $s = 1 \ldots S$

**while** $\Delta > \epsilon$

**for** every state $s = 1$ to $S$

$\bar{x}_s \leftarrow$ the $s$-th point on the grid

$\phi(\bar{x}_s) = \arg\max_u \Pi(\bar{x}_s, u) + \beta V_k(\lceil f(\bar{x}_s, u) \rceil)$

$V_{k+1}(\bar{x}_s) = \Pi(\bar{x}_s, \phi(\bar{x}_s)) + \beta V_k(\lceil f(\bar{x}_s, \phi(\bar{x}_s)) \rceil)$

**end for**

$k = k + 1$

$\Delta = \max_s |V_{k+1}(\bar{x}_s) - V_k(\bar{x}_s)|$

**end while**

**Return:** $\phi(\bar{x}_s)$ and $V_{k+1}(\bar{x}_s)$ for $s = 1, \ldots, S$

We simulate a simple scenario with $Q = 2$ users. The channel coefficients are sinusoids with different frequency and different amplitude for each user (see Fig. 5). The maximum transmitter power is $P_{\text{max}}^1 = P_{\text{max}}^2 = 5$, with 20 possible power levels per user, which amounts to 400 possible actions. We discretize the state-space (i.e., the users’ rates) into a grid of 30 points per user. The nonstationarity of the environment is surmounted by augmenting the state-space with $T = 20$ time steps. Hence, the augmented state space has a total of $30^2 \times 20 = 18,000$ states. For the equal-rate problem, the utility function uses $\alpha = 0.9$.

The solution of the proportional fair game leads to an efficient scheduler (see Figure 6), in which both users try to minimize interference so that they approach their respective maximum rates.

For the equal rate problem, we observe that the agents achieve much lower rate, but very similar between them (see Figure 7). The trend is that the user with a channel with less gain (User 2, red-dashed line) tries to achieve its maximum rate, while the user with higher gain channel (User 1, blue-continuous line) reduces its transmitter power to match the rate of the other user. In other words, the user with poorest channel sets a bottleneck for the other user.

**IX. CONCLUSIONS**

DPG provide a useful tradeoff for competitive multiagent applications under time-varying environments. On one hand, DPG allows nonstationary scenarios, thus, more realistic models. On the other hand, the analysis and solution of DPG is affordable through an equivalent MOCP. We presented a complete description of DPG and provided conditions for a dynamic game with constrained state and action sets to be of the potential type. To the best of our knowledge, previous works have not dealt with DPG with constraints explicitly.

We also introduced a range of communication and networking examples: energy demand control in a smart-grid network, network flow with relays that have bounded link capacity and limited battery life, multiple access communication in which users have to optimize the use of their batteries, and two optimal scheduling games with nonstationary channels. Although these problems have different features each—including utilities in separable and nonseparable form, convex and non-convex objectives, closed-form and numerical solutions, and solution methods based on convex optimization and dynamic programming algorithms—the proposed framework allowed us to analyze and solve them in a unified manner.
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