Investigating the homogeneous Bethe-Salpeter equation in Minkowski space

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Abstract. A short review of a novel approach, based on the Nakanishi perturbation-theory integral representation and devised for solving the Bethe-Salpeter equation directly in Minkowski space, is presented. The two ingredients for constructing such a new tool are: i) the Nakanishi perturbation-theory integral representations of the $N$-leg transition amplitudes and ii) the Light-front framework, that also allows one to gain insights into the Fock content of the states of an interacting system. Numerical results, corresponding to the Bethe-Salpeter equation in ladder approximation, for a simple system, composed by two massive scalars interacting through the exchange of a massive scalar, are provided for illustrating the effectiveness of the approach.

1. Introduction
To achieve a fully covariant description of a bound system in Minkowski space, still represents a challenge, though the field theoretical framework for dealing with was settled up in the early fifties [1, 2]. In particular, one should recall that a bound system appears as a pole in the analytic structure of the T-matrix, that also has cuts, related to the scattering states. It is very important to notice that such a pole cannot be reconstructed within a perturbative framework, and therefore, in order to reproduce a bound state one has to resort to non perturbative tools, like e.g. path integrals or integral equations. The homogeneous Bethe-Salpeter equation (BSE) [1] is an integral equation, obtained within a field-theory framework, that allows one to describe bound states in Minkowski space (see [3] for an extended review and [4] for a textbook presentation). It should be pointed out that the scattering states fulfill an inhomogeneous integral equation, in strict analogy with the non relativistic approach (see Ref. [5] for a recent introduction to the topic). Summarizing, the BSE yields the possibility to take properly into account the dynamics, within a field-theoretical framework, directly in Minkowski space, differently, e.g., from the path-integral approach carried out on the lattice, that exploits a Euclidean space. Indeed, the vast literature on BSE illustrates the non trivial efforts for getting solutions. In particular, roughly speaking, the largest amount of efforts was put on devising numerical methods in Euclidean space, e.g. by applying a Wick rotation to BSE (see, e.g., Ref. [6] for a general review and also Ref. [7] for a recent application to the spinor-spinor BSE) or on reducing from 4D to 3D the formal expression of BSE, introducing suitable constraints (see, e.g., Ref. [8] for an example of a covariant reduction). Only in the last years, solutions of BSE in Minkowski space have been presented, as shown in Refs. [9, 10], and more recently in Ref. [11]. Such an improvement was made feasible by applying the so-called perturbation-theory integral representation (PTIR) introduced by N Nakanishi [12, 13] with the aim to give a novel formal description of a generic
N-leg transition amplitude, but within the Feynman-diagram framework. If one adopts the PTIR of the 3-leg amplitude, i.e. the so-called vertex function, as a suitable Ansatz for the BS amplitude, after properly taking into account the formal relation between the BS amplitude itself and the vertex function, one can proceed toward an elaboration that leads to obtain solutions of the BSE directly in Minkowski space.

In what follows, it will be briefly illustrated the formalism for solving the homogeneous BSE developed in Refs. [5, 11] (see also Ref. [14]), where both the Nakanishi PTIR and the non explicitly covariant version of the Light-front (LF) approach (see Ref. [15] for a recent review of this specific LF framework) were adopted. Moreover, for the sake of concreteness, some relevant numerical results are also reviewed. It should be pointed out that results for the inhomogeneous case, obtained through a direct integration of the singularities present in the inhomogeneous BSE in ladder approximation, can be found in Ref. [16], while in Ref. [17] the scattering length and the phase-shifts, obtained within the Nakanishi PTIR approach, will be shown.

2. The BSE in a nutshell

In this section, the derivation of the homogeneous BSE is shortly reviewed (for a more detailed presentation see, e.g., Refs. [3] and [4]) in order to introduce some of the relevant issues of the field and the general formalism. For the sake of simplicity, let us consider a simple system composed by two massive scalars that interacts through the exchange of a massive scalar, i.e. a model with interacting Lagrangian like $L_{int} = g \phi^2 \chi$ (see Ref. [18] for the caveats about this model), but without self-energy and vertex corrections.

The path for obtaining the homogeneous BSE, that, we repeat, is the proper equation determining a bound state within a field-theory framework, starts with the 4-point Green’s Function, given by

$$G(x_1, x_2; y_1, y_2) = \langle 0 | T\{\phi_1(x_1)\phi_2(x_2)\phi_1^+(y_1)\phi_2^+(y_2)\} | 0 \rangle .$$

(1)

The four-point Green’s function fulfills an integral equation, pictorially described in figure 1 and schematically written as follows

$$G = G_0 + G_0 I G ,$$

(2)

where $G_0$ is the free Green’s function of the two interacting scalars, viz

$$G_0(k, p) = G_0^{(1)} G_0^{(2)} = \frac{i}{(\frac{p}{2} + k)^2 - m^2 + i\epsilon} \frac{i}{(\frac{p}{2} - k)^2 - m^2 + i\epsilon} ,$$

(3)

where $p = p_1 + p_2$ is the total momentum of the interacting system with total mass $p^2 = M^2$ and $k = (p_1 - p_2)/2$. In equation (2), the kernel $I$ is given by the infinite sum of irreducible Feynman graphs, namely the ones that cannot be split into two more simple graphs by drawing a line that does not cut any exchanged-scalar lines. As an example, in figure 2, the first diagrams contributing to the irreducible kernel are depicted. It has to point out that all the diagrams contributing to the 4-point Green’s function, i.e. the reducible and the irreducible ones, can be generated by infinitely iterating each diagram present in the irreducible kernel [1]. Inserting a complete Fock basis in $G$, one can single out the bound state contribution (assuming that only one bound state be present, for the sake of simplicity). In the Fourier space, the contribution appears as a pole, i.e.

$$G_B(k, q; p_B) \simeq \frac{i}{(2\pi)^4} \frac{\phi(k; p_B) \bar{\phi}(k; p_B)}{2\omega_B(p_0 - \omega_B + i\epsilon)} .$$

(4)
where \( \omega_B = \sqrt{M_B^2 + |p|^2} \) and \( \phi(k; p_B) \) is the Bethe-Salpeter amplitude for a bound state, after factorizing the "trivial" global motion of the two-particle system (i.e. a 4D delta function). In configuration space, the full BS amplitude reads as follows

\[
\Phi(X, x; p_B) = \langle 0|T\{\phi_1(x_1)\phi_2(x_2)\}|p_B, \beta\rangle
\]

(5)

where \( X = (x_1 + x_2)/2 \) and \( x = x_1 - x_2 \).

In conclusion, for \( p_0 \to \omega_B \), i.e. in the neighborhood of the bound-state pole, the 4-point Green’s function can be approximated by \( G \simeq G_B + \text{regular terms} \). Inserting in the integral equation (2), such an approximate expression (that becomes exact in the limit \( p_0 \to \omega_B \)), one can deduces the integral equation that determines the BS amplitude for a bound state, i.e. the homogeneous BSE. The equation reads

\[
\phi(k; p_B, \beta) = G_0(k; p_B, \beta) \int d^4q' \ I(k, q'; p_B) \ \phi(q'; p_B, \beta)
\]

(6)

with nor self-energy neither vertex corrections, at the present stage. It is worth noting that \( I(k, q'; p_B) \), the irreducible kernel in the BSE, is exactly the same appearing in the integral equation for the 4-point Green’s function, equation (2).
3. The Nakanishi perturbation-theory integral representation

The Nakanishi PTIR of the $N$-leg transition amplitude (depicted in figure 3) is the first ingredient for constructing a very effective tool for solving the homogeneous BSE (for the treatment of the inhomogeneous BSE, see Ref. [5]). In the sixties, Nakanishi [12, 13] proposed the PTIR, after a suitable elaboration of the well-known parametric formula for any Feynman diagram. In particular, for $N$ external legs, a generic contribution to the transition amplitude can be written as follows

$$f_{\mathcal{G}}(p_1, p_2, \ldots, p_N) \propto \prod_{r=1}^{k} \int d^4q_r \frac{1}{(\ell_1^2 - m_1^2)(\ell_2^2 - m_2^2) \cdots (\ell_n^2 - m_n^2)}$$

(7)

where $n$ propagators and $k$ loops are present. The label $\mathcal{G}$ in (7) is a shorthand notation for indicating a graph with a given pair $(n, k)$. Following the standard elaboration (see, e.g., [4]), one can write

$$f_{\mathcal{G}}(s) \propto \prod_{i=1}^{n} \int_0^1 d\alpha_i \frac{\delta(1 - \sum_{j=1}^{n} \alpha_j)}{U^2(\alpha)} \left[ F(n, N, \alpha, s) + i\epsilon \right]^{n-2k}$$

(8)

where $U^2(\alpha)$ is a well defined function, not relevant for the following discussion, and

$$F(n, N, \alpha, s) = -\sum_{j=1}^{n} \alpha_j m_j^2 + \sum_{h} \eta_h s_h$$

with the dependence upon the external momenta, $p_1, p_2, \ldots, p_N$, traded off in favor of all the independent scalar products $s \equiv \{s_1, s_2, \ldots, s_h, \ldots\}$ that one can construct.

In order to get rid of the dependence upon $(n, k)$ in the denominator of equation (8), Nakanishi proposed a simple, but effective, change of variable, obtaining a compact and elegant expression of the full $N$-leg amplitude $f_N(s) = \sum_{\mathcal{G}} f_{\mathcal{G}}(s)$. It is relevant to emphasize that the sum on $\mathcal{G}$ is infinite. In more detail, by introducing the following identity

$$1 = \prod_{h} \int_0^1 dz_h \delta \left( z_h - \frac{\eta_h}{\beta} \right) \int_0^\infty d\gamma \delta \left( \gamma - \sum_{l} \frac{\alpha_l m_l^2}{\beta} \right)$$

(9)

with $\beta = \sum \eta_h$ and after integrating by parts $n - 2k - 1$ times one gets

$$f_{\mathcal{G}}(s) \propto \prod_{h} \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1 - \sum_h z_h)}{(\gamma - \sum_h z_h s_h)}$$

(10)
where $\tilde{\phi}_G(z, \gamma)$ is a proper weight function. The dependence upon the details of the diagram, i.e. $(n, k)$, has moved from the denominator to the numerator. This simple achievement can be obtained for any diagram $G$ that contributes to $f_N(s)$, with the final result that one meets the same denominator in all the expressions. Therefore the full $N$-leg transition amplitude can be formally written as

$$f_N(s) = \sum_G f_G(s) \propto \prod_h \int_0^1 dz_h \int_0^\infty d\gamma \frac{\delta(1 - \sum_h z_h)}{(\gamma - \sum_h z_h \delta_h)}$$

where the weight function $\phi_N(z_h, \gamma)$ is

$$\phi_N(z_h, \gamma) = \sum_G \tilde{\phi}_G(z_h, \gamma)$$

It is very important to emphasize that the Nakanishi weight function is a real function that depends upon real variables, $z_h \in [-1, 1]$ and $\gamma \in [0, \infty]$ ($\gamma$ is positive by definition, see equation (9)).

Within the BS framework, the Nakanishi integral representation (11) can be usefully exploited for obtaining an expression of the 3-leg transition amplitude, that in turn is related to the BS amplitude through the proper multiplication by the inverse of $G_0$, (see Refs. [9, 10] for an early application of the Nakanishi representation to the bound state and Ref. [5] for the extension to the scattering states). In particular, the benefit of applying the Nakanishi representation, that has been devised within the Feynman-diagram framework in other words within a perturbative scheme, is given by its explicit analytic structure. This greatly helps the formal manipulations of the BSE that one has to carry out for making feasible the calculations of the BS amplitude in Minkowski space.

In particular, the PTIR vertex function (see figure 4) is given by

$$f_3(s) = \int_0^1 dz \int_0^\infty d\gamma \frac{\phi_3(z, \gamma)}{\gamma - \frac{p^2}{4} - k^2 - zk \cdot p - i\epsilon} .$$

In what follows, the following questions will be addressed: (i) how can the Nakanishi weight function, $\phi_3$, be determined for an actual, dynamical model? (ii) Can the Nakanishi representation of the vertex function, elaborated in perturbation theory, be used in a non perturbative realm, as the BS framework does?

**Figure 4.** Pictorial representation of the 3-leg transition amplitude, namely the so-called vertex function.
4. The projection of the BSE onto the null plane and the Nakanishi PTIR

The second ingredient for obtaining a viable equation that allows one to calculate the BS amplitude in Minkowski space is represented by the so-called projection onto the null-plane (see Ref. [15] for a recent review).

Let us consider the PTIR vertex function as an Ansatz for the BS amplitude, by taking into account that a factor $G_0$ has to be included. This increases the power of the denominator in equation (12) by a factor of two, namely the BS amplitude is written [9, 10, 5, 11]

$$\Phi_b(k, p) = i \int_{-1}^{+1} dz' \int_0^\infty d\gamma' \frac{g_b(\gamma', z'; \kappa^2)}{[\gamma' + \kappa^2 - k^2 - p \cdot k z' - i\epsilon]^3}$$

(13)

where $g_b(\gamma', z'; \kappa^2)$ is the Nakanishi weight function and $\kappa^2 = m^2 - M^2/4$.

Since the analytic structure of the PTIR Ansatz for the BS amplitude is explicit, one can integrate equation (13) on the LF variable $k^- = k^0 + k_z$ (see, e.g., [15]), obtaining the valence component of the state of the interacting system, once the state itself has been expanded onto the Fock basis (see [5, 11] for more details). The valence component reads in terms of the Nakanishi weight function $g_b$ as follows

$$\psi_{n=2}(\xi, k_\perp) = \frac{p^+}{\sqrt{2}} \frac{1}{(1 - \xi)} \int \frac{dk^-}{2\pi} \Phi_b(k, p) =$$

$$= \frac{1}{\sqrt{2}} \frac{1}{(1 - \xi)} \int_0^\infty d\gamma' \frac{g_b(\gamma', 1 - 2\xi; \kappa^2)}{[\gamma' + k_\perp^2 + \kappa^2 + (2\xi - 1)^2 \frac{M^2}{4} - i\epsilon]^2}$$

(14)

When we consider a 4D plane wave and integrate on $k^-$, it is easily seen that the conjugate variable $x^+ = t + z$ has to be vanishing, and therefore the null plane $x^+ = 0$ becomes the hypersurface where the initial dynamics takes place ($x^+$ plays the role of LF time, see Ref. [20] for an introduction to the Hamiltonian dynamics onto the null-plane). This observation explains why in the LF jargon the $k^-$-integration is dubbed LF projection onto the null plane (see [15] for a review of relevant formal results obtained by exploiting the $k^-$-integration within the so-called non explicitly covariant LF approach).

The LF projection onto the null plane can be applied to the homogeneous BSE, equation (6), obtaining [5, 11] (see Ref. [10] for the explicitly covariant LF treatment)

$$\int_0^\infty d\gamma' \frac{g_b(\gamma', z; \kappa^2)}{[\gamma' + \gamma + z^2m^2 + (1 - z^2)\kappa^2 - i\epsilon]^2} =$$

$$= \int_0^\infty d\gamma' \int_{-1}^{+1} dz' V_b^{LF}(\gamma, z; \gamma', z')g_b(\gamma', z'; \kappa^2).$$

(15)

with $V_b^{LF}(\gamma, z; \gamma', z')$ determined by the irreducible kernel $I(k, k', p)$ that appears both in equations (2) and (6).

It should be pointed out that equation (15) is an integral equation, and therefore it allows one to take into account non-perturbative effects in the BS amplitude, once the Ansatz (13) is assumed. On the other hand, within a perturbative framework, at any order, the Ansatz is exact. Therefore the problem is: does the equation in (15), formally obtained from the homogeneous BSE, have solutions (that, indeed, are non perturbative solutions)? In Refs. [9, 10, 11], a positive answer has been given through a careful numerical investigation of the so-called ladder approximation of the irreducible kernel $I(k, k', p)$, represented by the first diagram in figure 2. For the calculations within a a cross-ladder approximation (i.e. the first and second diagrams in figure 2) and an explicitly covariant LF approach see Ref. [19].

Nakanishi enriched his theoretical investigation of the $N$-leg transition amplitudes by demonstrating a theorem on the uniqueness of the weight function $\phi_N(z_h, \gamma)$ [13]. If such a
theorem holds also in the non perturbative context of the BSE, then a simpler integral equation for the weight function can be deduced. In particular, within the LF framework one gets [5, 11]

\[ g_b(\gamma, z; \kappa^2) = \int_0^\infty d\gamma' \int_{-1}^1 dz' \mathcal{V}_b(\gamma, z; \gamma'; \kappa^2) g_b(\gamma', z'; \kappa^2) \]

(16)

where \( \mathcal{V}_b(\gamma, z; \gamma', z'; \kappa^2) \) is a new kernel, properly related to \( V_b^{LF}(\gamma, z; \gamma', z') \), viz

\[ V_b^{LF}(\gamma, z; \gamma'', z') = \int_0^\infty d\gamma' \frac{\mathcal{V}_b(\gamma', z; \gamma'', z'; \kappa^2)}{[\gamma + \gamma + z^2m^2 + (1 - z^2)\kappa^2 - i\epsilon]^2} \]

(17)

An equation analogous to (16) was adopted in Ref. [9], but using standard canonical variables.

5. Quantitative studies

We have carried out a comprehensive investigation [11], in ladder approximation, of the simple scalar model, \( \mathcal{L} = g\phi^2\chi \), (i) varying both binding energies \( 0 < B/m \leq 2 \) (with \( B = 2m - M \)) and the mass of the exchanged scalar, \( \mu \), (ii) using the two eigen-equations, equations (15) and (16), both obtained within the non-explicitly covariant LF approach. In particular, numerical results for eigenvalues and weight functions \( g_b \), in ladder approximation, have been also obtained in Refs. [9, 10], but using different numerical techniques, and different frameworks, in Ref. [9] the standard variables and the uniqueness theorem and in Ref. [10] the explicitly covariant LF and equation (15). In our work [11] (see also Ref. [14]), in order to solve both equations (15) and (16), we have expanded the weight function \( g_b \) onto a basis given by the product of Gegenbauer (for the compact variable \( z \)) and Laguerre polynomials. It was very gratifying to obtain substantially the same results from both equations. Before illustrating the numerical results, it is necessary to emphasize what the results themselves indicate. First of all, the different numerical techniques adopted in Refs. [9, 10, 11] lead to the same eigenvalues, within the numerical accuracy adopted. Furthermore, as shown in Ref. [11], the uniqueness theorem by Nakanishi holds, as numerically checked at the level of a few percent.

In table 1, there is an example of the accurate agreement one can achieve (for the final results see Ref. [11]). As it can be immediately realized, the ladder approximation of the irreducible kernel depends linearly upon \( \alpha = g^2/(16\pi m^2) \) and it is customary, to assign a value to the binding energy of the system and solve an eigenvalue problem, where the eigenvalues are the numerical values of the coupling \( \alpha \) and the eigenvectors are the corresponding weight functions \( g_b \). It should be pointed out that equation (15) is a generalized eigenequation, while equation (16) is a standard one. In figure 5 (see Ref. [11]), the comparison between the eigenvectors corresponding to the solutions of equations (15) and (16), for a given value of the binding energy and the mass of the exchanged scalar, is shown.

5.1. Valence Probabilities and LF Distributions

Once the Nakanishi weight function is evaluated, one can straightforwardly obtain the BS amplitude and normalize it. Then, the probability of the valence wave function, \( \psi_{n=2}(\xi, k_\perp) \), results properly determined [11]. Moreover, one can calculate the LF distributions of the valence state, that represent relevant observables, e.g. in Hadron Physics.

The probability of the valence component is given by [11]

\[ P_{val} = \frac{1}{(2\pi)^3} \int_0^1 \frac{d\xi}{2\xi(1-\xi)} \int d\xi \left[ \psi_{n=2}^2(\xi, k_\perp) \right] = \frac{1}{(16\pi)^2} \int_{-1}^1 dz (1-z^2) \int_0^\infty d\gamma \int_0^\infty d\gamma' \frac{g_b(\gamma', z; \kappa^2)}{[\gamma + \gamma + z^2m^2 + (1 - z^2)\kappa^2]^2} \]

(18)
Table 1. Preliminary values of $\alpha = g^2/(16\pi m^2)$ (see Ref. [11] for the final ones) by solving the generalized eigenequation (15) (indicated by LF-V (FSV) in the third column) and the standard eigenequation (16) (indicated by LF-U (FSV) in the fourth column). In the second column, for the sake of comparison, there are the results obtained in Ref. [10]. The chosen value for the mass of the exchanged scalar is $\mu/m = 0.50$.

| $B/m$ | $\alpha$ LF-V (CK) | $\alpha$ LF-V (FSV) | $\alpha$ LF-U (FSV) |
|-------|---------------------|---------------------|---------------------|
| 0.01  | 1.440               | 1.44                | 1.44                |
| 0.10  | 2.498               | 2.50                | 2.50                |
| 0.20  | 3.251               | 3.25                | 3.25                |
| 0.50  | 4.901               | 4.90                | 4.90                |
| 1.00  | 6.712               | 6.71                | 6.71                |

Figure 5. Left panel: the Nakanishi weight function $g_b(\gamma,z;\kappa^2)$, with $\kappa^2 = m^2 - M^2/4$, for $B/m = 1$ and $\mu/m = 0.5$ vs the adimensional variable $\gamma/m^2$ ($\gamma = k^2$) and two different values of the variable $z$. Right panel: the same as in the left panel, but vs the variable $z$ and two values of $\gamma$ (see Ref. [11]).

Table 2 shows preliminary results of the valence probability (see Ref. [11] for the final results and more details) for the case with $\mu/m = 0.5$. It is worth noting that such a calculation is a fully dynamical evaluation and that, as expected when $B \to 0$ (i.e. the continuum is approached), the two-particle state becomes dominated by the valence component, i.e. $P_{val} \to 1$.

The longitudinal and transverse-momentum LF distributions are defined in terms of the valence wave function as follows [11]

$$
\phi(\xi) = 2 \frac{(1 - z^2)}{(16\pi^2)^2} \int_0^\infty d\gamma \left[ \int_0^\infty d\gamma' \frac{g_b(\gamma',z;\kappa^2)}{[\gamma' + \gamma + z^2 m^2 + (1 - z^2)\kappa^2]^2} \right]^2
$$

$$
\mathcal{P}(\gamma) = \frac{1}{(16\pi^2)^2} \int_{-1}^1 dz (1 - z^2) \left[ \int_0^\infty d\gamma' \frac{g_b(\gamma',z;\kappa^2)}{[\gamma' + \gamma + z^2 m^2 + (1 - z^2)\kappa^2]^2} \right]^2
$$

In figure 6 an example of the previous distributions is shown. It is very interesting that the calculations performed by using the Nakanishi weight function obtained from equation (15) and...
Table 2. Valence probability for $\mu/m = 0.50$.

| $B/m$ | $\alpha$ | $P_{val}$ |
|-------|----------|-----------|
| 0.001 | 1.167    | 0.98      |
| 0.01  | 1.440    | 0.96      |
| 0.10  | 2.498    | 0.87      |
| 0.20  | 3.251    | 0.83      |
| 0.50  | 4.900    | 0.77      |
| 1.00  | 6.711    | 0.74      |
| 2.00  | 8.061    | 0.72      |

Figure 6. Left panel: the longitudinal LF-distribution, $\phi(\xi) = \int dk_{\perp} |\psi_{n=2}(\xi,k_{\perp})|^2$, vs the longitudinal-momentum fraction $\xi = k^+/M$. Dash-double-dotted line: $B/m = 0.20$. Dotted line: $B/m = 0.50$. Solid line: $B/m = 1.0$. Dashed line: $B/m = 2.0$. Note that $\int_0^1 d\xi \phi(\xi) = P_{val}$. Right panel: the same as the left panel, but for the transverse LF-distribution $P(\gamma) = \int d\xi |\psi_{n=2}(\xi,k_{\perp})|^2$ vs the adimensional variable $\gamma/m^2$ ($\gamma = k_{\perp}^2$). Note that $\int_0^{\infty} d\gamma P(\gamma) = P_{val}$ (see Ref. [11]).

the one from equation (16) are indistinguishable in the figure. The right panel of figure (6) puts in evidence the similar behavior of the transverse distributions, corresponding to different values of the binding energy (notice that the same holds for different values of $\mu$, as well). In the log plot, the tails can be easily made overlapping by a suitable shift, showing a universal behavior. This feature suggests that it is possible to extract information on the dynamics (clearly dictated by the adopted kernel) by analyzing the tail of the transverse distribution at high momentum.

It is worth noting that the strong agreement between the results from equations (15) and (16) holds also for the valence probability, that, after all, represents the normalization of the LF distributions.
6. Conclusions and perspectives

The cross-fertilization between the Light-Front framework and the Nakanishi perturbation-theory integral representation paves the path toward a new class of non perturbative calculations, carried out within a rigorous field-theoretical approach, namely the solutions of the Bethe-Salpeter equation in Minkowski space, both for bound and scattering states.

The LF framework has well-known advantages, with respect to the canonical approach, very helpful in performing the analytical integrations. This is a useful feature that can be exploited for making more easy to find the solutions of the BSE.

Our numerical investigations, performed in ladder approximation at the present stage (the cross-ladder contributions will be investigated in [21]), confirm both the robustness of the Nakanishi Ansatz for the BS amplitude and the uniqueness theorem for the weight function $g_b$. Moreover, we extended the numerical analysis of an actual dynamical model by studying the valence probability and the LF distributions, that represent appealing observables for a deep insight into the dynamics of an interacting system.

Calculations are in progress for both (i) scattering length and (ii) phase-shifts, still in ladder approximation [17]. This task will make complete the first stage of the investigation of this new tool. In the meantime, in order to face with the crossed-box contribution, a simple symmetry transformation has been explored, that in turn should allow one to resum an infinite set of cross-ladder diagrams. A sketch of the main steps of this new investigation are given in figures 8, 7 and 9. Staring from the T-matrix in ladder approximation, as shown in figure 7, one could apply the transformation illustrated in figure 8, and obtain the cross-ladder series, as depicted in figure 9, that contribute to the irreducible kernel I in equations (2) and (6).
Figure 9. The infinite cross-ladder terms that contribute to the irreducible kernel $I$ in equations (2) and (6)

the previous schematic pattern within the PTIR framework will be the aim of a forthcoming work [21].

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