Path integral quantization for massive vector bosons

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Abstract

A parity-conserving and Lorentz-invariant effective field theory of self-interacting massive vector fields is considered. For the interaction terms with dimensionless coupling constants the canonical quantization is performed. It is shown that the self-consistency condition of this system with the second-class constraints in combination with the perturbative renormalizability leads to an SU(2) Yang-Mills theory with an additional mass term.

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I. INTRODUCTION

Vector mesons play an important role in the phenomenological description of hadronic processes (see, for example, Ref. [1]). Phenomenological low-energy chiral Lagrangians with vector mesons were already constructed in the 1960s [2–4]. Details of the construction of chirally invariant effective Lagrangians describing the interaction of vector mesons with pseudoscalars and baryons can be found, e.g., in Refs. [5–12]. Besides a knowledge of the most general effective Lagrangian, an effective field theory (EFT) program requires a systematic power counting. In Refs. [13, 14], it was shown how to consistently include virtual (axial-) vector mesons in EFT, if they appear only as internal lines in Feynman diagrams involving soft external pions and nucleons with small three-momenta. This is possible if a suitable renormalization condition such as the extended on-mass-shell renormalization scheme or the infrared renormalization is chosen (see also Refs. [15, 16]). In this case the masses of the vector mesons are treated as a large scale. For energies for which the vector mesons can be generated, the problem of the power counting is solved by using the complex-mass renormalization scheme [17–22], which is an extension of the on-mass-shell renormalization scheme for unstable particles [23]. Finally, it would also be possible to consider an EFT with a heavy scale much larger than the masses of the vector bosons. For example, if the Higgs boson were never discovered then such an EFT would appear as a possible candidate for a theory of the (electro-)weak interaction.

A covariant Lagrangian formalism for massive vector fields entails a special feature: they are described by Lagrangians with constraints. The self consistency of a system with constraints imposes conditions on the form of the Lagrangian.

In Ref. [24] the effective Lagrangian of Ref. [4] describing the interaction among ρ mesons, pions, and nucleons was considered. Requiring the perturbative renormalizability in the sense of effective field theory [25], the universality of the vector-meson coupling was derived. It has been argued [11, 26] that the effective Lagrangian of Ref. [4] is not the most general one because it starts with the Yang-Mills structure for the self-interacting part of the vector mesons and takes the coupling to the fermion doublet equal to the coupling of the vector-meson self interaction.

In this work we will close this gap. We start with the most general effective Lagrangian of a system of three interacting massive vector fields. In principle, the Lagrangian contains all interaction terms which respect Lorentz invariance, hermiticity, parity, and charge conservation. Assuming that the interaction terms with a higher number of derivatives and/or fields are suppressed by higher orders of some large scale we consider only interactions with dimensionless coupling constants. We analyze the quantization in the Hamilton formalism. To have a constrained system with the right number of degrees of freedom the determinant of the matrix of constraints should be non-vanishing. This nontrivial condition leads to relations among the coupling constants of the Lagrangian. Furthermore, demanding the perturbative renormalizability in the sense of EFT we arrive at additional consistency conditions for the parameters of the Lagrangian. As will be shown below, we are eventually led to the Yang-Mills structure incorporated in the effective Lagrangian of Ref. [4].

II. LAGRANGIAN

We consider a system of three interacting massive vector fields respecting Lorentz invariance and parity conservation. We assume that two vector fields correspond to a pair of
charged particles, \( V^\pm_\mu = (V^1_\mu \mp iV^2_\mu)/\sqrt{2} \), while the third component, \( V^3_\mu \), is neutral. The most general effective Lagrangian respecting all given symmetries contains an infinite number of terms and can be split as

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_C. \tag{1}
\]

Here, \( \mathcal{L}_A \) refers to the free Lagrangian plus a finite number of interaction terms with dimensionless coupling constants, \( \mathcal{L}_B \) contains an infinite number of “non-renormalizable” interactions, and \( \mathcal{L}_C \) refers to interactions with other degrees of freedom. In this work we only treat the vector-meson self-interaction terms with dimensionless coupling constants, assuming that the “non-renormalizable” interactions are suppressed by powers of some large scale,

\[
\mathcal{L}_A = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4. \tag{2}
\]

The free Lagrangian reads

\[
\mathcal{L}_2 = -\frac{1}{4} V^a_{\mu\nu} V^{a\mu\nu} + \frac{M^2_a}{2} V^a_{\mu} V^{a\mu}. \tag{3}
\]

Here, \( V^a_{\mu\nu} = \partial_\mu V^a_\nu - \partial_\nu V^a_\mu \), \( M_a \) is the mass of \( a \)-th vector field \((M_1 = M_2 = M)\), and a summation over \( a \) from 1 to 3 is implied. The interaction terms with dimensionless coupling constants involve either three or four vector fields. The interaction Lagrangian with three vector fields is of the form \[27\]

\[
\mathcal{L}_3 = -g^{abc} V^a_\mu V^b_\nu \partial^\mu V^{c\nu}, \tag{4}
\]

where \( g^{abc} \) \((a, b, c = 1, 2, 3)\) are coupling constants.

We use charge conservation and hermiticity to restrict the coupling constants \( g^{abc} \). Hermiticity requires that all of these couplings are real. The invariance of the Lagrangian of Eq. (4) under a global U(1) transformation with an infinitesimal parameter \( \varepsilon \),

\[
V^a_\mu \rightarrow V^a_\mu - \varepsilon \varepsilon^{abc} V^b_\mu, \tag{5}
\]

leads to

\[
(g^{abd} \varepsilon^{3dc} + g^{adc} \varepsilon^{3db} + g^{dbc} \varepsilon^{3da}) V^a_\mu V^b_\nu \partial^\mu V^{c\nu} = 0. \tag{6}
\]

Using Eq. (6) we express the couplings in terms of seven real parameters,

\[
\begin{align*}
g^{333} &= g_1, & g^{113} &= g_2, & g^{123} &= -g_3, & g^{213} &= g_3, \\
g^{223} &= g_2, & g^{311} &= g_4, & g^{321} &= -g_5, & g^{312} &= g_5, \\
g^{322} &= g_4, & g^{131} &= g_6, & g^{231} &= -g_7, & g^{132} &= g_7, \\
g^{232} &= g_6. & & & &
\end{align*}
\tag{7}
\]

All other constants vanish. Note that at this stage we did not demand invariance under a charge-conjugation transformation (C), \( V^a_\mu \rightarrow (-)^a V^a_\mu \), which would, in addition, eliminate all constants with \( a + b + c \) odd, namely \( g_1, g_2, g_4, \) and \( g_6 \).

The interaction Lagrangian involving four vector fields has the form

\[
\mathcal{L}_4 = -h^{abcd} V^a_\mu V^b_\nu V^c_\rho V^d_\sigma, \tag{8}
\]
where $h_{abcd}$ ($a, b, c, d = 1, 2, 3$) are real coupling constants satisfying the permutation symmetries

$$
h_{abcd} = h_{bida} = h_{dabc} = h_{bade} = h_{acbd} = h_{bcda} = h_{dabc}.
$$

(9)

Invariance under the U(1) transformation leads to

$$
(h_{ebcd} \epsilon^{3ea} + h_{aecd} \epsilon^{3eb} + h_{abed} \epsilon^{3ec} + h_{abce} \epsilon^{3ed}) V_\mu V^\nu V^{e\mu} V^{d\nu} = 0.
$$

(10)

Combining Eqs. (9) and (10) allows us to write the couplings $g_{abcd}$ in terms of five real parameters,

$$
h_{1111} = \frac{d_1 + d_2}{4},
$$

$$
h_{1122} = h_{2112} = h_{1221} = h_{2211} = \frac{d_2}{4},
$$

$$
h_{1313} = h_{3131} = \frac{d_1}{4},
$$

$$
h_{1333} = h_{3113} = h_{1331} = h_{3311} = \frac{d_3}{8},
$$

$$
h_{2121} = h_{1212} = \frac{d_1 - d_2}{4},
$$

$$
h_{2233} = h_{3223} = h_{2332} = h_{3322} = \frac{d_3}{8},
$$

$$
h_{2222} = \frac{d_1 + d_2}{4},
$$

$$
h_{2323} = h_{3232} = \frac{d_1}{4},
$$

$$
h_{3333} = d_5.
$$

(11)

All other coupling constants vanish. In this case, charge conservation also implies charge conjugation invariance, i.e., $a + b + c + d$ even.

### III. THE HAMILTONIAN METHOD

To quantize the above theory of massive vector fields we use the canonical formalism following Ref. [28]. The momenta conjugated to the fields $V^a_0$ and $V^a_i$ are defined as

$$
\pi_0^a = \frac{\partial L}{\partial \dot{V}^a_0} = -g^{bca}V^b_0 V^c_0,
$$

(12)

$$
\pi_i^a = \frac{\partial L}{\partial \dot{V}^a_i} = V^a_{0i} + g^{bca}V^b_0 V^c_i.
$$

(13)

Note that we will not use a fully covariant notation in deriving the Hamiltonian. The velocities $\dot{V}_0^a$ cannot be solved from Eq. (12), i.e. the corresponding momenta $\pi_0^a$ need to satisfy the primary constraints

$$
\phi_1^a = \pi_0^a + g^{bca}V^b_0 V^c_0 \approx 0.
$$

(14)
Here, \( \phi_a^1 \approx 0 \) denotes a weak equation in Dirac’s sense, namely that one must not use one of these constraints before working out a Poisson bracket \[29\]. On the other hand, from Eq. (13) we solve
\[
\dot{V}_i^a = \pi_i^a + \partial_0 V_i^a - g^{bca} V_0^b V_i^c.
\] (15)

Next we construct the so-called total Hamiltonian density:
\[
\mathcal{H}_1 = \phi_1^a z^a + \mathcal{H},
\] (16)
where
\[
\mathcal{H} = \frac{\pi_i^a \pi_i^a}{2} + \pi_i^a \partial_i V_i^a + \frac{1}{4} V_{ij}^a V_{ij}^a - \frac{M^2}{2} V_0^a V^a_{0\mu} \\
- g^{abc} V_0^a V_i^b \pi_i^c - g^{abc} V_0^a V_i^b \partial_i V_0^c - g^{abc} V_i^a V_0^b \partial_i V_0^c \\
+ g^{abc} V_i^a V_j^b \partial_i V_j^c + \frac{1}{2} g^{abc} g^{a'b'} c V_0^a V_0^b V_0^{a'} V_0^{b'} \\
+ h^{abcd} V_0^a V_0^b V_0^c V_0^{d\nu}.
\] (17)

In Eq. (16) \( z^a \) are arbitrary functions which have to be determined.

The primary constraints have to be conserved in time. Therefore, for each \( a \), we calculate the Poisson bracket of \( \phi_1^a \) with the Hamiltonian
\[
H_1 = \int d^3 x \mathcal{H}_1(x),
\] (18)
and obtain
\[
\{ \phi_1^a, H_1 \} = (g^{bca} + g^{cba} - g^{abc}) V_0^c V_i^b \\
+ \partial_i \pi_i^a + g^{abc} V_i^b \pi_i^c + (g^{abc} + g^{bac}) V_i^b \partial_i V_0^c \\
- g^{bca} \partial_i (V_0^b V_i^c) - g^{cba} \partial_i (V_0^c V_i^b) + M_0^2 V_i^a \\
- g^{abc} g^{a'b'} c V_i^b V_0^a V_0^{a'} V_0^{b'} - 4 h^{abcd} V_0^a V_0^b V_0^c V_0^{d\mu} \\
\equiv A^{ab} z^b + \chi^a \approx 0, \quad a = 1, 2, 3.
\] (19)

Using Eq. (17), defining \( \gamma_1 = g_5 + g_7 \) and \( \gamma_2 = g_4 + g_6 - 2 g_2 \), the matrix \( A \) is given by
\[
A = \begin{pmatrix}
0 & -2\gamma_1 V_0^3 & \gamma_2 V_0^1 - \gamma_1 V_0^2 \\
2\gamma_1 V_0^3 & 0 & \gamma_1 V_0^1 + \gamma_2 V_0^2 \\
-(\gamma_2 V_0^1 - \gamma_1 V_0^2) & -(\gamma_1 V_0^1 + \gamma_2 V_0^2) & 0
\end{pmatrix}.
\] (20)

Since the determinant of \( A \) vanishes, the three expressions on the left-hand side of the equations
\[
A^{ab} z^b = -\chi^a
\] (21)
are not independent. As a result, the system of equations (21) can be satisfied only if the right-hand sides satisfy the secondary constraint
\[
\phi_2 = \chi^1 (\gamma_1 V_0^1 + \gamma_2 V_0^2) + \chi^2 (\gamma_1 V_0^2 - \gamma_2 V_0^1) - \chi^3 2 \gamma_1 V_0^3 \approx 0.
\] (22)
Let us consider Eq. (21) for the case where at least one of $\gamma_1$ or $\gamma_2$ does not vanish. For non-vanishing $V^1_0$ and/or $V^2_0$ we obtain

$$z^1 = \frac{X_3 + \gamma_1 z^2 V^1_0 + \gamma_2 z^2 V^2_0}{\gamma_1 V^2_0 - \gamma_2 V^1_0},$$

$$z^3 = \frac{X_1 + 2 \gamma_1 z^2 V^3_0}{\gamma_2 V^1_0 - \gamma_1 V^2_0}. \quad (23)$$

Finally, $z^2$ can be solved from the time conservation of the constraint of Eq. (22), $\{\phi_2, H_1\} \approx 0$. However, this chain leads to the wrong number of constraints of the second class [30] for our system of three massive vector fields, namely $3 + 1 = 4$ rather than $3 + 3 = 6$ constraints. In other words, for a self-consistent theory we have to require

$$g_7 = -g_5, \quad 2g_2 = g_4 + g_6. \quad (24)$$

In this case none of the $z^b$ can be solved from Eq. (19) and

$$\{\phi^a_1, H_1\} = \partial_i \pi^a_i + g^{abc} V^b_i \pi^c_i + (g^{abc} + g^{bac}) V^b_i \partial_i V^c_i \nonumber$$

$$- g^{bca} \partial_i (V^b_0 V^c_i) - g^{cba} \partial_i (V^b_0 V^c_i) + M^a_0 V^a_i \nonumber$$

$$- g^{abc} g^{a'b'} V^b_0 V^{a'}_i V^b_0 - 4 h^{abcd} V^b_0 V^d_0 \nonumber$$

$$\equiv \phi^a_2 \approx 0, \quad a = 1, 2, 3, \quad (25)$$

are the secondary constraints. They also have to be conserved in time, i.e. their Poisson brackets with the Hamiltonian have to vanish. To obtain the right number of degrees of freedom for massive vector bosons, the $z^a$ have to be solvable from this condition. In such a case no more constraints occur and the Lagrangian describes the system with constraints of the second class.

Taking Eq. (24) into account, we obtain a system of three linear equations for the $z^a$,

$$\{\phi^a_2, H_1\} = M^{ab} z^b + Y^a \approx 0, \quad a = 1, 2, 3. \quad (26)$$

The $3 \times 3$ matrix $M$ is given by

$$M^{ab} = M^{2}_{0} \delta^{ab} - (g^{bca} + g^{cba}) \partial_i V^c_i \nonumber$$

$$- (g^{ace} g^{bde} - 4 h^{abcd}) V^c_0 V^d_0 \nonumber$$

$$- 4 (2 h^{abcd} + h^{a'b'd}) V^c_0 V^d_0, \quad (27)$$

and the $Y^a$ are some functions of the fields and conjugated momenta, the particular form of which plays no role in the following discussion. If the determinant of $M$ vanishes for any values of the fields, then the $z^a$ cannot generally be determined and additional constraints have to be imposed [28]. However, then one generates the wrong number of degrees of freedom. This problem, in its various appearances, is known as the Johnson-Sudarshan [31] and the Velo-Zwanziger [32] problem. To obtain a self-consistent field theory we have to demand that $\text{det} M$ does not vanish for any values of the fields. We refrain from displaying the lengthy expression of the determinant for arbitrary fields. In the appendix we provide the analysis leading to the following conditions for the coupling constants:

$$g_1 = g_2 = 0,$$
\[ d_2 = -d_1, \]
\[ d_1 \geq \frac{g_3^2}{2}, \]
\[ d_4 = -d_3, \]
\[ d_3 \leq -g_4^2 - g_5^2, \]
\[ d_5 = 0. \]  \hspace{1cm} (28)

Note that Eqs. (7) together with Eqs. (24) and (28) imply
\[ g_{abc} = -g_{bac}, \]  \hspace{1cm} (29)
i.e. the three-vector vertex couplings are antisymmetric in the first two indices.

**IV. QUANTIZATION**

In the following, we assume that the coupling constants are related to each other in such a way that \( \det M \) does not vanish and proceed with the quantization. Note that, for small values of the fields and their derivatives, the \( \varepsilon^a \) can be solved from Eq. (26) as a perturbative expansion in the coupling constants. According to a general theorem for systems with second-class constraints [28] there always exists a canonical set of dynamical variables (fields and conjugated momentum fields) with the following properties: the set of variables may be divided into two subsets \( \{\omega\} \) and \( \{\Omega\} \), each consisting of canonically conjugate pairs such that (a) the constraints only appear among conjugate pairs of \( \{\Omega\} \) and (b) the original and the new set of constraints are equivalent. The \( \omega \) are dynamical variables and the dynamics is described by the physical Hamiltonian
\[ \mathcal{H}^{ph} = \mathcal{H}(\omega, \Omega)|_{\Omega = 0} = \mathcal{H}^{ph}(\omega). \]  \hspace{1cm} (30)

As soon as the dynamical canonical variables and the Hamiltonian have been identified, the quantization can be performed using the path integral method. Let \( \omega^1 \) and \( \omega^2 \) denote the canonical fields and the respective conjugate momentum fields, both belonging to the set \( \{\omega\} \). The generating functional reads
\[ Z[J^\omega] = \int \mathcal{D}\omega e^{i \int d^4x \left[ \omega^2 \dot{\omega}^1 - \mathcal{H}^{ph}(\omega) + J^\omega \omega \right]} . \]  \hspace{1cm} (31)

By introducing a product of functional delta functions, \( \delta(\Omega) \), involving the constraints in terms of the set \( \{\Omega\} \), Eq. (31) is expressed as
\[ Z[J^\omega] = \int \mathcal{D}\omega \mathcal{D}\Omega \delta(\Omega) e^{i \int d^4x \left[ \omega^2 \dot{\omega}^1 + \Omega^2 \dot{\Omega}^1 - \mathcal{H}(\omega, \Omega) + J^\omega \omega + J^\Omega \Omega \right]} = Z[J], \]  \hspace{1cm} (32)

where \( J = (J^\omega, J^\Omega) \), in addition, generates a coupling to the variables \( \Omega^1 \) and \( \Omega^2 \) of \( \{\Omega\} \). Now we switch to the original variables. The functional \( \delta \) function in Eq. (32) can be written as
\[ \delta(\Omega) = \delta(\phi) \left[ \det\{\phi, \phi\} \right]^{1/2}, \]  \hspace{1cm} (33)

where \( \phi \) denotes the original system of constraints and
\[ \{\phi, \phi\}_{(a), (b)} = \{\phi^a_i, \phi^b_k\} \]  \hspace{1cm} (34)
is, in our specific case, the 6 × 6 matrix of the Poisson brackets of constraints [see Eqs. (14) and (23)]. The Poisson brackets \( \{ \phi^a, \phi^b \} \) vanish, once Eq. (24) is taken into account. As a consequence, the entries \( \{ \phi^a_2, \phi^b_2 \} \) do not contribute to the determinant. The square root is, thus, of the form
\[
[\det \{ \phi, \phi \}]^{1/2} = \det \{ \phi^a_1, \phi^b_1 \} = \det \{ \mathcal{M}^{ab} \} ,
\]
where the 3 × 3 matrix \( \mathcal{M} \) is given by Eq. (27) supplemented with the relations among coupling constants obtained above.

As \((\omega, \Omega)\) are canonical variables, the Jacobian corresponding to a change of variables to the original ones, is equal to one. Also the action is a canonically invariant quantity,
\[
S = \int d^4x \left[ \omega^2 \dot{\omega} + \Omega^2 \dot{\Omega} - \mathcal{H}(\omega, \Omega) \right] = \int d^4x \left[ \pi_{\omega}^a \dot{\omega}^a + \pi_{\Omega}^a \dot{\Omega}^a - \mathcal{H}(V, \pi) \right] .
\]
In the following, \( \{ J^{a\mu} \} \) will symbolically denote the set of external sources coupling to the vector fields of the original Lagrangian. After the change of variables, we trade the generating functional of Eq. (32) for a generating functional containing \( \{ J^{a\mu} \} \) only,
\[
Z[\{ J^{a\mu} \}] = \int \mathcal{D}V \mathcal{D}\pi \delta(\phi) [\det \{ \phi, \phi \}]^{1/2} e^{i \int d^4x \left[ \pi_{\omega}^a \dot{\omega}^a + \pi_{\Omega}^a \dot{\Omega}^a - \mathcal{H}(V, \pi) + J^{a\mu} V_{\mu}^a \right] } .
\]
Next we write \( \delta(\phi_2) \) and \( \det \{ \phi, \phi \} \) as functional integrals
\[
\delta(\phi_2) \sim \int \mathcal{D}\lambda e^{i \int d^4x \lambda^a \phi_2^a} ,
\]
\[
[\det \{ \phi, \phi \}]^{1/2} \sim \int \mathcal{D}c \mathcal{D}\bar{c} e^{i \int d^4x \mathcal{L}_{\text{ghost}}},
\]
where the ghost Lagrangian reads
\[
\mathcal{L}_{\text{ghost}} = M^2_a c^a c^a - \left( g^{ace} g^{bde} - 4 h_{abcd} \right) V^c_i V^d_i c^a c^b - 4 \left( 2 h_{bacd} + h_{bcad} \right) V^0_a V^a_i c^a c^b .
\]
By substituting Eqs. (38) with (39) in Eq. (37) and shifting the integration variable \( V_{\mu}^a \rightarrow V_{\mu}^a - \lambda^a \), we obtain
\[
Z[\{ J^{a\mu} \}] = \int \mathcal{D}V \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}\lambda \delta(\bar{\phi}_1) e^{i \int d^4x \left( \mathcal{K} + J^{a\mu} V_{\mu}^a \right) },
\]
where
\[
\bar{\phi}_1^a = \pi_{\omega}^a + g^{bec} (V_{0}^b - \lambda^b) (V_{0}^c - \lambda^c) ,
\]
\[
\mathcal{K} = \pi_{\omega}^a \dot{V}_{0}^a + \pi_{\Omega}^a \dot{V}_{0}^a - \mathcal{H}(V, \pi) + \lambda^a \phi_2^a + \mathcal{L}_{\text{ghost}}
\]
\[
= \pi_{\omega}^a \dot{V}_{0}^a + \pi_{\Omega}^a \dot{V}_{0}^a - \frac{\pi_{\omega}^a \lambda^a}{2} - \pi_{\Omega}^a \lambda^a - \frac{1}{4} \delta_{ij} V_{ij}^a + \frac{M^2_a}{2} V_{\mu}^a V_{a\mu}
\]
\[
+ g^{abc} V_{0}^c V_{0}^b \pi^a - g^{abc} V_{0}^c V_{0}^b \dot{V}_{ij} \pi_{ij}^a - \frac{1}{2} g^{abc} g^{a'b'c'} V_{0}^a V_{0}^b V_{0}^c V_{0}^d V_{\mu}^a V_{\mu}^b V_{\nu}^c V_{\nu}^d
\]
\[
- \frac{\pi_{\omega}^a}{2} \dot{\lambda}^a + \frac{M^2_a}{2} \dot{\lambda}^a \lambda^a + g^{abc} g^{a'b'c'} \lambda^a V_{0}^b \lambda^c V_{0}^d
\]
\[
+ 2 h_{abcd} \left( 2 \lambda^a \lambda^c V_{0}^b V_{\mu}^d \mu + 4 \lambda^c \lambda^d V_{0}^a V_{0}^b \right) - 8 V_{0}^a \lambda^b \lambda^c \lambda^d + 3 \lambda^a \lambda^b \lambda^c \lambda^d
\]
\[
+ M^2_a c^a c^a - \left( g^{ace} g^{bde} - 4 h_{abcd} \right) V_{ij} \pi_{ij} c^a c^b
\]
\[
- 4 \left( 2 h_{bacd} + h_{bcad} \right) \left( V_{0}^a V_{0}^d - \lambda^a V_{0}^d - \lambda^c \lambda^d \right) c^a c^b .
\]
Integrating over $\pi_\mu^a$ we obtain for the generating functional

$$Z[J^a] = \int \mathcal{D}V \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}\lambda e^{i \int d^4x \left( \mathcal{L}_{\text{eff}} + J^a V^a + \bar{J}^a \lambda^a \right)},$$

(42)

where

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_A - \frac{M^2}{2} \lambda^a \lambda^a + \left( \frac{1}{2} g_{abc} g^{cde} - 2 h_{abcd} \right) \lambda^a V^b \lambda^c V^d$$

$$+ 2 \left( h_{bcad} + h_{bcad} \right) V^c V^d \lambda^a \lambda^b - h_{abcd} \left( 8 V^a V^b V^c V^d - 3 \lambda^a \lambda^b \lambda^c \lambda^d \right)$$

$$+ M^2 \bar{c}^a c^a - (g_{ace} g^{bde} - 4 h_{bdac}) V^c V^d \bar{c}^a c^b$$

$$- 4 \left( h_{bcad} + h_{bcad} \right) (V^c V^d - \lambda^c V^d - V^c \lambda^d + \lambda^c \lambda^d) \bar{c}^a c^b,$$

(43)

and the Lagrangian $\mathcal{L}_A$ is given by Eqs. (2), (3), (4), and (8).

V. PERTURBATIVE RENORMALIZABILITY

Before turning to the relations which originate from demanding perturbative renormalizability, let us recall the constraints which have been obtained so far from the analysis of the determinant of $\mathcal{M}$ and the corresponding symmetry input. Table I contains a summary of the number of parameters of the unconstrained Lagrangian together with the number of independent parameters after the constraint analysis. Equations (28) already put severe restrictions on the coupling constants of the Lagrangian. For example, if, in addition, we demand an invariance under global SU(2) isospin transformations, then the Lagrangian of Eq. (2) contains only four real constants, namely, one mass parameter $M$, one three-vector-interaction coupling constant $g$, and two four-vector-interaction coupling constants $h_1$ and $h_2$:

$$\mathcal{L}_{\text{SU(2)}} = - \frac{1}{4} V_{\mu \nu}^{a} V^{a \mu \nu} + \frac{M^2}{2} V_{\mu}^{a} V^{a \mu} - g \epsilon^{abc} V_{\mu}^{a} V_{\nu}^{b} \partial^{\mu} V^{c \nu} - h_1 V_{\mu}^{a} V^{a \mu} V^{b \nu} - h_2 V_{\mu}^{a} V_{\nu}^{a} V^{b \mu} V^{b \nu}. $$

(44)

Also note that enforcing isospin symmetry in the present case entails charge-conjugation invariance. Applying the above constraint analysis results in the two inequalities

$$h_1 \geq \frac{g^2}{4},$$

$$h_2 \leq - \frac{g^2}{4}.$$  

(45)

We will now show that further relations among the coupling constants result by demanding that the ultraviolet divergences of the loop diagrams can be absorbed in the redefinition of masses, coupling constants, and fields of the most general effective Lagrangian containing all terms consistent with the assumed underlying symmetries. In particular, starting from the classical theory with a U(1) symmetry only, we will be led to infer the SU(2) symmetry of the quantized theory from perturbative renormalizability, instead of using it as an input to the analysis. Moreover, the inequalities of Eqs. (45) will be replaced $h_1 = -h_2 = g^2/4$. In other words, we will obtain the standard Yang-Mills Lagrangian plus an additional mass term.
TABLE I: Number of parameters before and after the constraint analysis depending on the symmetry input. The Lagrangian without internal symmetry contains 51 real parameters, namely 3 masses ($M$), 27 cubic coupling constants ($g$), and 21 quartic coupling constants ($h$). In all cases, the constraint analysis provides two additional inequalities [see Eqs. (28)].

| Symmetry | Parameters before | Total number | Parameters afterwards | Total number |
|----------|-------------------|--------------|-----------------------|--------------|
| U(1)     | 2 $M$, 7 $g$, 5 $h$ | 14           | 2 $M$, 3 $g$, 2 $h$  | 7            |
| U(1) + C | 2 $M$, 3 $g$, 5 $h$ | 10           | 2 $M$, 2 $g$, 2 $h$  | 6            |
| SU(2)    | 1 $M$, 1 $g$, 2 $h$ | 4            | 1 $M$, 1 $g$, 2 $h$  | 4            |

FIG. 1: One-loop contributions to the three-vector vertex function. The wiggly line corresponds to the vector meson.

Let us discuss the logarithmic divergences. Although dimensional regularization ignores the power-law divergences, it keeps track of all logarithmic divergences. Because the fields $\lambda^a$, $c^a$, and $\bar{c}^a$ do not have kinetic parts in Eq. (13), their contributions to perturbative calculations vanish in dimensional regularization.

We start with the logarithmically divergent parts of the vertex functions involving neutral fields only. As $g^{333} = g_1 = 0$ [see Eqs. (28)], we require that the divergent part of the $V^3V^3V^3$ vertex function vanishes (see Fig. 1). This leads to

$$g_4 \left(5g_4^2 + 3g_5^2 + 3d_3\right) = 0.$$ \hspace{1cm} (46)

For the $V^3V^3V^3$ vertex function (see Fig. 2), the same reasoning results in

$$5g_4^4 + 2g_5^2g_4^2 + 5g_5^4 + 5d_3^2 + 2d_3 \left(g_4^2 + 5g_5^2\right) = 0.$$ \hspace{1cm} (47)

The unique solution to Eqs. (16) and (47) reads

$$g_4 = 0, \quad d_3 = -g_5^2.$$ \hspace{1cm} (48)

Note that one of the two inequalities of Eqs. (28) is thereby replaced in terms of an equality.

Next, we demand that the divergent part of the one-loop contribution to the $V^1V^2V^3$ vertex function can be absorbed in the renormalization of the corresponding tree-order vertex. From this requirement we obtain

$$-g_5 \left[\left(M_3^4 + 2M_3^2M_2^2\right) g_3^2 + 6M_3^2M_2^2g_3g_5 + \left(M_4^4 + 2M_2^2M_3^2\right) g_5^2\right] = 0.$$
The solution is either \( g_5 = 0 \) (with no constraints for the values of \( g_3, M, \) and \( M_3) \) or for \( g_5 \neq 0 \):

\[
M_3 = M \quad \text{and} \quad g_3 = -g_5. \tag{49}
\]

As there is no tree-order contribution to the vertex function \( V^1 V^1 V^1 V^1 \), we demand that the divergent part of the one-loop contribution vanishes. Taking Eq. (48) for \( g_4 \) and \( d_3 \) into account, we obtain

\[
60d_1^2 M^4 M_3^4 + 8g_3^3 g_5 M^2 \left( M_3^2 - M^2 \right) M_3^4 + g_3^4 \left( 15M_3^4 - 4M^2 M_3^2 + M^4 \right) M_3^4 \\
+ 4g_3 g_5^3 M^4 \left( M_3^2 - M^2 \right) M_3^2 + 4g_3 g_5 M^4 \left( 2M_3^2 + M^2 \right) M_3^2 + 5g_5^5 M^4 \left( 2M_3^4 + M^4 \right) \\
- 4d_1 \left[ 2g_3 g_5 M^2 \left( M_3^2 - 4M^2 \right) M_3^4 + g_5^3 M^4 \left( 5M_3^2 - 2M^2 \right) M_3^2 \\
+ g_5^2 \left( 5M_3^8 - 2M^2 M_3^6 + 3M^4 M_3^4 \right) \right] = 0. \tag{50}
\]

As a result of Eqs. (49) and (50) we obtain that either

\[
d_1 = \frac{g_3^2}{2}, \quad g_5 = -g_3, \quad M_3 = M, \tag{51}
\]

or

\[
d_1 = \frac{g_3^3}{30M^4} \left( 5M_3^4 - 2M^2 M_3^2 + 3M^4 \\
\pm \sqrt{25M_3^8 - 20M^2 M_3^6 - 191M^4 M_3^4 + 48M^6 M_3^2 - 6M^8} \right), \quad g_5 = 0. \tag{52}
\]

The solution given in Eq. (52) relates masses and dimensionless coupling constants. It is therefore not compatible with the assumption that parameters with different dimensions be independent. We are thus left with the solution of Eq. (51).

Defining \( g = g_3 \), all the relations among the coupling constants and masses can be summarized as

\[
g^{abc} = -g \epsilon^{abc}, \\
h^{abcd} = \frac{1}{4} g^{abc} g^{ade}, \\
M_1 = M_2 = M_3 = M. \tag{53}
\]

FIG. 2: One-loop contributions to the four-vector vertex function. The wiggly line corresponds to the vector meson.
For the couplings in Eq. (53) the determinant \( \det \{ \phi, \phi \} \) does not depend on the fields, i.e. the ghost fields completely decouple from the vector-meson fields. The matrix \( M^{ab} \) in Eq. (26) becomes \( M^2 \delta^{ab} \), i.e. all \( z^a \) are solved for all field configurations. After integrating over the ghost fields in Eq. (32), the resulting effective Lagrangian can be written in a compact form

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_A = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \frac{M^2}{2} V^a_{\mu} V^a_{\mu},
\]

where

\[
G^a_{\mu\nu} = V^a_{\mu\nu} - g \epsilon^{abc} V^b_{\mu} V^c_{\nu}.
\]

The final generating functional of the Green’s functions has the form

\[
Z[\{ J^{a\mu} \}] = \int D V e^{i \int d^4 x \left( \mathcal{L}_A + J^{a\mu} V^a_{\mu} \right)}
\]

and results in “naive” Feynman rules. The remarkable result is that we have ultimately been led to the standard SU(2) Yang-Mills Lagrangian with an additional mass term.

VI. CONCLUSIONS

In this work we have considered the interaction terms with dimensionless coupling constants of the most general parity-conserving and Lorentz-invariant effective-field-theory Lagrangian of self-interacting massive charged and neutral vector fields. We have analyzed the quantization procedure of this system with second-class constraints using the canonical formalism. By demanding that the quantized theory is self-consistent in the sense of constraints and perturbative renormalizability we obtained relations among the 12 originally available coupling constants [U(1) symmetry], resulting in a substantial reduction in the number of independent couplings. In practice quantum theories are usually obtained by quantizing classical theories. However, it is understood that the natural logic is exactly the other way around, i.e. classical theories should be obtained as limits of quantum theories. Following this logic, only those classical theories could be treated as self-consistent, which are obtained from self-consistent quantum theories. To be specific, in the present case the considered model reduces to a charge-conjugation-invariant SU(2) Yang-Mills vector field theory with an additional mass term. This suggests that the massive Yang-Mills effective Lagrangian is the most general one and therefore could be used in phenomenological applications.

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VII. APPENDIX

In this appendix we explicitly display suitable field configurations from which we have inferred the constraints for the coupling constants summarized in Eqs. (28). By restricting
ourselves to particular field configurations, the analysis is greatly simplified without spoiling
the general argument, because $\det \mathcal{M}$ should be non-vanishing for arbitrary values of
the fields. The entries of the matrix $\mathcal{M}$ are defined in Eq. (27). Note that, in some cases, at
a given step we explicitly make use of constraints which have been obtained in a previous
step.

1. For $V^a_i V^b_i = V_0^a = \partial_i V^1_i = \partial_i V^2_i = 0$, one obtains
\[
\det \mathcal{M} = (M^2 - 2 g_2 \partial_i V^3_i)^2 \left( M_3^2 - 2 g_1 \partial_i V^3_i \right).
\]
This determinant can only be non-vanishing for arbitrary $\partial_i V^3_i$, if
\[
g_1 = g_2 = 0.
\]

2. In combination with Eqs. (24), we are left with only three independent $g$ couplings,
say, $g_3$, $g_4$, and $g_5$:
\[
g_1 = g_2 = 0, \quad g_6 = -g_4, \quad g_7 = -g_5,
\]
which, together with Eqs. (7), also implies Eq. (29), i.e. $g^{abc} = -g^{bac}$. Therefore, we
can omit from now on the term proportional to $\partial_i V^c_i$ in $\mathcal{M}^{ab}$:
\[
\mathcal{M}^{ab} = M^2_{abc} \delta^{ab} - (g^{ace} g^{bde} - 4 h^{acbd}) V^c_i V^d_i - 4 \left( 2 h^{abcd} + h^{acbd} \right) V^c_0 V^d_0.
\]
Moreover, using the permutation symmetries of Eq. (9), we obtain $\mathcal{M}^{ab} = \mathcal{M}^{ba}$. Thus,
the evaluation of the determinant simplifies to
\[
\det \mathcal{M} = \mathcal{M}^{11} \mathcal{M}^{22} \mathcal{M}^{33} - \mathcal{M}^{11} \left( \mathcal{M}^{23} \right)^2 - \mathcal{M}^{22} \left( \mathcal{M}^{13} \right)^2 - \mathcal{M}^{33} \left( \mathcal{M}^{12} \right)^2 + 2 \mathcal{M}^{12} \mathcal{M}^{13} \mathcal{M}^{23}.
\]

3. Next, we will investigate field configurations resulting in a diagonal matrix $\mathcal{M}$.

(a) Let us consider arbitrary $x := V^1_i V^1_i \geq 0$ and $y := V^1_0 V^1_0 \geq 0$, and set all the
remaining fields to zero. We than have
\[
\begin{align*}
\mathcal{M}^{11} &= M^2 + (d_1 + d_2) x - 3(d_1 + d_2) y, \\
\mathcal{M}^{22} &= M^2 - (g^2_3 - d_1 + d_2) x - (d_1 + d_2) y, \\
\mathcal{M}^{33} &= M_3^2 - (g^2_4 + g^2_5 - d_4) x - (d_3 + d_4) y.
\end{align*}
\]
From $\mathcal{M}^{11}$, we infer for $x = 0$ and arbitrary $y$ the inequality $d_1 + d_2 \leq 0$ and for
$y = 0$ and arbitrary $x$ the inequality $d_1 + d_2 \geq 0$. Both results combine into
\[
d_2 = -d_1.
\]
Using this result we infer from $\mathcal{M}^{22}$ for arbitrary $x$ the inequality
\[
d_1 \geq \frac{g^2_3}{2}.
\]
Similarly, from $\mathcal{M}^{33}$ we infer the inequalities
\[
\begin{align*}
d_3 + d_4 &\leq 0, \\
g^2_4 + g^2_5 &\leq d_4.
\end{align*}
\]
(b) Let us consider arbitrary \( x := V_i^3 V_i^3 \geq 0 \) and \( y := V_0^3 V_0^3 \geq 0 \), and set all the remaining fields to zero. We then have
\[
\mathcal{M}^{11} = M^2 - (g_4^2 + g_5^2 - d_4)x - (d_3 + d_4)y = \mathcal{M}^{22},
\]
\[
\mathcal{M}^{33} = M_3^2 + 4d_5x - 12d_5y.
\]

While \( \mathcal{M}^{11} \) does not provide any new information, we infer from \( \mathcal{M}^{33} \) the condition
\[
d_5 = 0. \tag{64}
\]

4. We now turn to a configuration involving also off-diagonal elements of \( \mathcal{M} \). Let us consider \( x := V_0^3 V_0^3 = V_0^3 V_0^3 \geq 0 \) and set all the remaining fields to zero. We then obtain
\[
\det \mathcal{M} = \mathcal{M}^{11}[\mathcal{M}^{22} \mathcal{M}^{33} - (\mathcal{M}^{23})^2]
\]
\[
= [M^2 - (d_3 + d_4)x][M^2 - (d_3 + d_4)x][M_3^2 - (d_3 + d_4)x] - 4(d_3 + d_4)^2 x^2.
\]

We already know that \( d_3 + d_4 \leq 0 \). The term in the curly braces has a root for a sufficiently large value of \( x \), unless the condition
\[
d_4 = -d_3 \tag{65}
\]
holds.

5. With the constraint analysis performed so far, we end up with
\[
\mathcal{M}^{11} = M^2 + \alpha V_i^2 V_i^2 + \beta V_i^3 V_i^3,
\]
\[
\mathcal{M}^{22} = M^2 + \alpha V_i^1 V_i^1 + \beta V_i^3 V_i^3,
\]
\[
\mathcal{M}^{33} = M_3^2 + \beta (V_i^1 V_i^1 + V_i^2 V_i^2),
\]
\[
\mathcal{M}^{12} = -\alpha V_i^1 V_i^2,
\]
\[
\mathcal{M}^{13} = -\beta V_i^1 V_i^3,
\]
\[
\mathcal{M}^{23} = -\beta V_i^2 V_i^3,
\]
where
\[
\alpha := 2d_1 - g_3^2 \geq 0,
\]
\[
\beta := -d_3 - g_4^2 - g_5^2 \geq 0.
\]

In particular, all the terms involving the fields \( V_0^a \) have disappeared from \( \mathcal{M} \). We finally want to verify that we have reached a point, beyond which we cannot obtain any further constraints from the analysis of the determinant. We will show that, for any field configuration, \( \det \mathcal{M} \geq M_4^4 M_3^4 \) as long as \( \alpha \geq 0 \) and \( \beta \geq 0 \). For that purpose we introduce the abbreviations
\[
u := V_i^1 V_i^1 \geq 0,
\]
\[
v := V_i^2 V_i^2 \geq 0,
\]
\[
w := V_i^3 V_i^3 \geq 0,
\]
\[
x := V_i^1 V_i^3,
\]
\[
y := V_i^2 V_i^3,
\]
\[
z := V_i^1 V_i^2,
\]
\[
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in terms of which the determinant reads
\[
\det \mathcal{M} = (M^2 + \alpha v + \beta w)(M^2 + \alpha u + \beta w)[M^2_3 + \beta (u + v)]
- (M^2 + \alpha v + \beta w)\beta^2 y^2 - (M^2 + \alpha u + \beta w)\beta^2 x^2 - [M^2_3 + \beta (u + v)]\alpha^2 z^2
- 2\alpha\beta^2 xyz.
\]
Making use of \(x^2 \leq uw, y^2 \leq vw, z^2 \leq uv,\) and \(xyz \leq uvw,\) we obtain as a lower bound for \(\det \mathcal{M},\)
\[
\det \mathcal{M} \geq (M^2 + \alpha v + \beta w)(M^2 + \alpha u + \beta w)[M^2_3 + \beta (u + v)]
- (M^2 + \alpha v + \beta w)\beta^2 uv
\]
\[
= M^4 M^2_3 + M^4 \beta (u + v) + M^2 M^2_3 [\alpha (u + v) + 2\beta w]
+ M^2 \beta (u + v) [\alpha (u + v) + \beta w] + M^2_3 \beta w [\alpha (u + v) + \beta w]
\]
\[
\geq M^4 M^2_3.
\]
This completes the analysis leading to the constraints given in Eqs. (28).

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Structures of the type $\partial^{\mu}V_{\mu}^{a}V^{b}V_{\nu}^{c}$ need not be included, because they are, up to a total derivative, equal to those given in Eq. (1). Moreover, structures of the type $\epsilon^{\mu\nu\alpha\beta}\partial_{\mu}V_{\nu}^{a}V^{b}V_{\alpha}^{c}$ are odd under parity.

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I.e. with a non-singular matrix $B$ of Poisson brackets $B_{uv} = \{\phi_{u}, \phi_{v}\}$.

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