Construction of the noncommutative rank I
Bergman domain

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Abstract In this paper we present a harmonic oscillator realization of the most
degenerate discrete series representations of the $SU(2, 1)$ group and the deformation
quantization of the coset space $D = SU(2, 1)/U(2)$ with the method of coherent
state quantization. This short article is based on a talk given at the 9-th International
Workshop, Varna "Lie Theory and Its Applications in Physics" (LT-9).

1 Introduction

It is believed that ordinary differential geometry should be replaced by noncommu-
tative geometry [1] when we are approaching the Planck scale and quantum field
theories defined on noncommutative space time (NCQFT) [2, 3] are considered
as the right way to explore the effects of quantum gravity.

The simpliest noncommutative space is the Moyal space, which is a symplec-
tic manifold generated by the noncommutative coordinates $x_\mu$, such that
$[x_\mu, x_\mu] = i\theta_{\mu\nu}$, where $\theta_{\mu\nu}$ is a constant. The first well defined quantum field theory on 4
dimensional Moyal space is the Grosse-Wulkenhaar model [5]. It is not only pertur-
bative renormalisable to all orders but also asymptotically safe, namely the beta
function for the coupling constant is zero at the fixed point of this model. Hence
this model is a candidate to be constructed nonperturbatively, namely it’s possible
to obtain the exact Green’s function which is unique and analytic in the coupling
constant, by resumming the perturbation series [6]. Recently the two dimensional
Grosse-Wulkenhaar model has been constructed in [7].

Since the noncommutative quantum fields theories are better behaved than their
commutative counterparts, it is very natural to construct other noncommutative man-
ifolds and physics models over them.
In this paper we construct the noncommutative coset space $D = SU(2, 1) / (SU(2) \times U(1))$, with the method of coherent state quantization. For doing this we also introduce a harmonic oscillator realization of the most degenerate discrete series representation of the group $SU(2, 1)$ which is a generalization for the $SU(1, 1)$ case introduced by H. Grosse and P. Presnajder [13]. The interested reader could look at [13, 12] for more details about the coherent state quantization and [10, 11, 17] for more details about the representation theory of noncompact Lie group. In [8] and [9] we have studied the harmonic oscillator realization of the maximal degenerate discrete series representations for an arbitrary $SU(m, n)$ group.

The construction of the noncommutative coset space $SU(2, 1) / U(2)$ has been also studied by [14], [15] with the method of Berezin-Toeplitz quantization and by [16] with the method of “WKB quantization”. The interested reader could go to the references for details.

2 The $SU(m, 1)$ group and its Lie algebra

The group $G = SU(m, 1)$ is defined as a subgroup of the matrix group $SL(m+n, C)$:

$$g = \begin{pmatrix} a_{m \times m} & b_{m \times 1} \\ c_{1 \times m} & d \end{pmatrix} \in G$$

satisfies the constraint

$$g^\dagger \Gamma g = g \Gamma g^\dagger = \Gamma, \quad \Gamma = \begin{pmatrix} I_{m \times m} & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

Here $I$’s represents unit matrices and 0’s the blocks of zeros.

The maximal compact subgroup is defined by matrices

$$K = S(U(m) \times U(1)) = \{ \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}, \ det(K_1K_2) = 1 \}. \quad (3)$$

The Bergman domain is defined as the coset space $D = G/K$:

$$D = \{ Z | 1 - |Z|^2 > 0 \} = \{ z | 1 - |z_1|^2 - |z_2| - \cdots - |z_m|^2 > 0 \}, \quad (4)$$

where $Z = (z_1, \cdots, z_m)$ are the coordinates of the coset space $D$. It is a pseudo-convex domain over which we could define a holomorphic Hilbert spaces [14] with the reproducing Bergman kernel:

$$K(W^\dagger, Z) = (1 - W^\dagger Z)^{-N}, \quad (5)$$

where $Z$ and $W$ are complex $m$-columns, and $N = m + 1, m + 2, \cdots$ is a natural number characterizing the representation.
\( D \) is also a Kähler manifold with the Kähler metric defined by the derivations of the Bergman kernel:
\[
g_{ij} = \frac{1}{N} \partial_z \partial_{\bar{z}} \log K(Z, Z).
\]
More explicitly we have:
\[
g_{ij} = [\delta_{ij} - \frac{|Z|^2}{1 - |Z|^2} + z_i \bar{z}_j], \quad g^{ij} = (1 - |Z|^2)(\delta_{ij} - \bar{z}_i z_j).
\]
We could easily calculate that the Ricci tensor:
\[
R_{ij} = -(m + 1) g_{ij}
\]
and the curvature
\[
R = -(m + 1)
\]
and verify that the metric \( g_{ij} \) is a solution to the Einstein’s equation in the vacuum:
\[
R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{ij} = 0 \quad (8)
\]
with the cosmological constant \( \Lambda = \frac{m+1}{2} \).

The Lie algebra \( g = \text{Lie}(G) = su(m, 1) \) is defined by \( M = \left\{ \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix} \right\} \in g, \)
\( M^\dagger \Gamma = -\Gamma M, \) where \( A^\dagger = -A, D^\dagger = -D, \text{tr}(A + D) = 0. \)

Consider the Cartan decomposition of the Lie algebra \( g = 1 + p \) and let \( a \in p \) be a maximal Abelian subalgebra. We could choose for \( a \) the set of all matrices of the form
\[
H_t = \begin{pmatrix}
O_{(m-1)\times(m-1)} & O_{(m-1)\times1} & O_{(m-1)\times1} \\
O_{1\times(m-1)} & 0 & t \\
O_{1\times(m-1)} & t & 0
\end{pmatrix}
\]

where \( t \) is a real number.

Define the linear functional over \( H_t \) by \( \alpha(H_t) = t \), the roots of \( (g, a) \) are given by
\[
\pm \alpha, \pm 2\alpha,
\]
with multiplicities \( m_\alpha = 2 \) and \( m_{2\alpha} = 1. \)

Define
\[
\delta := \{ a_t | a_t = \exp H_t, H_t \in a \}.
\]
so we have
\[
a_t = \begin{pmatrix} I & O & 0 \\ O & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix},
\]
where the symbol \( I \) stands for the identity matrix and \( O \) is the matrix with entries 0.
3 The holomorphic discrete series of representations of the
\(SU(m, 1)\) group

The unitary irreducible representations for \(G = SU(m, 1)\) are the principal series, the discrete series and the supplementary series. We consider only the discrete series of representations, which are realized in the Hilbert space \(L^2_D\) of holomorphic functions with the inner product defined by:

\[
(f, g)_N = \int d\mu_N(Z, \bar{Z}) \tilde{f}(\bar{Z})g(Z),
\]

(13)

where \(d\mu_N(Z, \bar{Z}) = c_N|\det(E - Z\bar{Z})|^{N-\frac{m+1}{2}}|dZ|\) is the normalised measure and 
\(c_N = \pi^{-2}(N-2)(N-1)\).

The discrete series of representations \(T_N\) is defined by:

\[
T_N f(Z) = [\det(CZ + d)]^{-N} f(Z'), \quad N = m + 1, m + 2, \cdots
\]

(14)

where

\[
Z' = (AZ + B)(CZ + d)^{-1}
\]

(15)

In the following we consider only the \(m = 2\) case and construct the harmonic oscillator realization of the most degenerate discrete series representation \(T_N\). We introduce a \(3 \times 1\) matrix \(\hat{Z} = (\hat{z}_a)\), \(a = 1, 2, 3\), of bosonic oscillators acting in Fock space and satisfying commutation relations

\[
[\hat{z}_a, \hat{z}_b^\dagger] = \Gamma_{ab}, \quad \alpha, \beta = 1, 2, 3
\]

(16)

\[
[\hat{z}_a, \hat{z}_b] = [\hat{z}_a^\dagger, \hat{z}_b^\dagger] = 0,
\]

(17)

where \(\Gamma\) is a \(3 \times 3\) matrix defined in (2). It can be easily seen that for all \(g \in SU(2, 1)\) these commutation relations are invariant under transformations:

\[
\hat{Z} \mapsto g \hat{Z}, \quad \hat{Z}^\dagger \mapsto \hat{Z}^\dagger g^\dagger.
\]

(18)

We could define the creation and annihilation operators \(\hat{a}_\alpha\) and \(\hat{b}\) as:

\[
\hat{Z} = \left( \begin{array}{c} \hat{a} \\hat{b}^\dagger \end{array} \right) \quad [\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}, \alpha, \beta = 1, 2, \quad [\hat{b}, \hat{b}^\dagger] = 1,
\]

(19)

and all other commutation relations among oscillator operators vanish.

The Fock space \(\mathcal{F}\) in question is generated from a normalized vacuum state \(|0\rangle\), satisfying \(\hat{a}_\alpha |0\rangle = \hat{b} |0\rangle = 0\), by repeated actions of creation operators:

\[
|m_\alpha, n\rangle = \prod_\alpha \frac{(\hat{a}_\alpha^\dagger)^{m_\alpha} (\hat{b}^\dagger)^n}{\sqrt{m_\alpha! n!}} |0\rangle.
\]

(20)
We shall use the terminology that the state $|m_\alpha, n\rangle$ contains $m = \sum m_\alpha$ particles $a$ and $n$ particles $b$.

Consider a basis of $su(2, 1)$ Lie algebra $X = X_{\alpha a}^A$, $A = 1, \cdots, 8$, $a, b = 1, 2, 3$, we assign the operator

$$\hat{X} = -\hat{Z}^\dagger \Gamma X \hat{Z} = -z_{\alpha a}^A X_{\alpha a}^A \hat{z}.$$ \hfill (21)

Their anti-hermicity follows directly:

$$\hat{X}^\dagger = -\text{tr}(\hat{Z}^\dagger X \Gamma \hat{Z}) = +\text{tr}(\hat{Z}^\dagger \Gamma X \hat{Z}) = -\hat{X}.$$

Using commutation relations for annihilation and creation operators we have:

$$[\hat{X}, \hat{Y}] = [\hat{Z}^\dagger \Gamma X \hat{Z}, \hat{Z}^\dagger \Gamma Y \hat{Z}] = -\hat{Z}^\dagger \Gamma [X, Y] \hat{Z}. \hfill (22)$$

So that the operators $\hat{X}_a$ satisfy in Fock space the $su(2, 1)$ commutation relations. The assignment

$$g = e^{\xi A_{X_\alpha}} \in SU(2, 1) \Rightarrow \hat{T}(g) = e^{\xi A_{\hat{X}_\alpha}} \hfill (23)$$

then defines a unitary $SU(2, 1)$ representation in Fock space.

### 4 The Coherent States Quantization of $D = SU(2, 1)/U(2)$ and the Star Product

We briefly describe the construction of coherent states on coset space of a Lie group following [12]. Let $T_g$ be an unitary irreducible representation of an arbitrary Lie group $G$ in a Hilbert space $\mathcal{H}$, $|z_0\rangle \in \mathcal{H}$ is a normalized state in the Garding space of $T_g$. Let $K$ be the stability group of the $|z_0\rangle$, for which $T_k|z_0\rangle = e^{i\omega(k)}|z_0\rangle$, for $k \in K$. Then for each $z = g z_0 \in D = G/K$ we could assign a coherent states $|z\rangle = \psi_c = T(g)|z_0\rangle$. Define the functions $\omega_0(g) = <z_0 | T(g) | z_0 >$ and $\omega(g, z) = <z | T_g | z > = \omega_0(g z_0^{-1} g z)$. As $|z_0\rangle$ is in the Garding space, $\omega(g)$ is a smooth function in $g$.

For $G = SU(2, 1)$, the state $|z_0\rangle$ is defined in the Fock space as:

$$|z_0\rangle = (\hat{b}^\dagger)^N |0\rangle = \frac{1}{\sqrt{N}}|0, 0; N\rangle. \hfill (24)$$

Here $N$ is a natural number that specifies the representation: $\hat{N}|z_0\rangle = N|z_0\rangle$. All other states in the representation space are obtained by the action of rising operators given in [20]. The stability group for $|z_0\rangle$ is $K = S(U(2) \times U(1))$.

Using the $K\delta K$ decomposition of $g = k^\dagger \delta q$ [17], for which $k, q \in K$, we obtain:

$$\omega_0(g) = <z_0 | \hat{T}(g) | z_0 > = \frac{1}{\cosh r} [(1 + \ln \cosh r) e^{i((\alpha(q) - \alpha(k)) \Gamma)}]^N \hfill (25)$$

Consider an operator acting on $\mathcal{H}$:
\[ \hat{F} = \int dg \hat{F}(g) T(g) = \int dg \hat{F}(g) \omega(g, z) \]  

(26)

where \( \hat{F}(g) \) is a distribution on a group \( G \) with compact support. We also define for each \( \hat{F} \) a biholomorphic function:

\[ F(z, \bar{z}) = \langle \Psi_1 | \hat{F} | \Psi_2 \rangle. \]  

(27)

The star product of two such bi-holomorphic functions \( F \) and \( G \) is then defined by (13):

\[ (F \star G)(z, \bar{z}) = \langle \Psi_1 | \hat{F} \hat{G} | \Psi_2 \rangle = (F \star G)(z, \bar{z}) \]

\[ = \int dg_1 dg_2 \hat{F}(g_1) \hat{G}(g_2) \omega(g_1 g_2, z). \]  

(28)

Obviously the star product defined above is noncommutative, associative and is invariant under the action of the group \( G \). The noncommutative algebra of functions \( \{ F(z) \} \) induces a noncommutative structure on the coset space \( D \). That’s how we construct the noncommutative version of the Bergman domain, which is noted as \( \hat{D} \).

Now we shall study the explicit form of the star product for \( G = SU(2, 1) \). Using the explicit form of the group element \( g = e^{\xi A} \) and integration by parts we have:

\[ F_{A_1 \ldots A_n}(z) = (-1)^n (\partial_{\xi_{A_1}} \cdots \partial_{\xi_{A_n}} \omega)(e^{\xi A} z)|_{\xi=0} = (-1)^n \langle z | \hat{X}_{A_1} \cdots \hat{X}_{A_n} | z \rangle. \]  

(29)

Here \( \hat{X}_A, (A = 1 \cdots, 8) \) are the left-invariant vector field on group \( G \) corresponding to the Lie algebra basis \( X_A \) whose explicit form is given in (9).

From the definition of the star product (28) it follows that:

\[ (F_{A_1 \ldots A_n} \star F_{B_1 \ldots B_m})(z) \]

\[ = (-1)^{n+m} \langle z | \hat{X}_{A_n} \hat{X}_{B_m} \cdots \hat{X}_{A_1} \hat{X}_{B_1} \omega | z \rangle \mid_{g=e}. \]  

(30)

We define the function \( \xi_A \) as the expectation value of the operator \( \hat{X}_A \) between the coherent states as:

\[ \xi_A(z) = \frac{1}{N} \langle z | \hat{X}_A | z \rangle = \frac{1}{N} \langle z_0 | \hat{T}^A(g, z) \hat{X}_A \hat{T}(g, z_0) | z_0 \rangle. \]  

(31)

The star product between these coordinates functions reads:

\[ (\xi_A \star \xi_B)(z) = \frac{1}{N^2} \langle z | \hat{X}_A \hat{X}_B | z \rangle = (1 + A_N) \xi_A(z) \xi_B(z) + \frac{1}{2N} f^C_{A,B} \xi_C(z) + B_N \delta_{A,B}, \]  

(32)

where \( A_N \) and \( B_N \) depend on the Bernoulli numbers coming from the Baker-Campbell-Hausdorff formula and are of order \( 1/N \). We see that the parameter of the non-commutativity is \( \lambda_N = 1/N \). For \( N \to \infty \) we recover the commutative product.
According to the Harish-Chandra imbedding theorem, we could always imbed the commutative maximal Hermitian symmetric space into the noncompact part of the Cartan subalgebra. So the coordinates of the noncommutative Bergman domain \( \hat{D} \) can be identified as the coordinate functions corresponding to the noncompact Cartan subalgebra.

5 Conclusions and prospectsives

In this paper we have constructed the noncommutative Bergman domain \( \hat{D} \) whose commutative counterpart is the coset space \( D = G/K \), where \( G = SU(2,1) \) and \( K = S(U(2) \times U(1)) \). This result could be generalized to an arbitrary type one rank one Cartan domain \( D = G = SU(m,1)/S(U(m) \times U(1)) \) straightforwardly.

In [9] we have build a model of quantum theory of real scalar fields on the noncommutative manifold \( \hat{D} \) and find that the one loop quantum correction to the 2 point function is finite. This is a hint of the finiteness of quantum field theory on \( \hat{D} \) and this deserves further studies.

Acknowledgements The author is very grateful to Harald Grosse and Peter Prešnajder for useful discussions.

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