ON QUANTIZATION OF A NILPOTENT ORBIT CLOSURE IN $G_2$

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(Communicated by Kailash C. Misra)

Abstract. Let $G$ be the complex exceptional Lie group of type $G_2$. Among the five nilpotent orbits in its Lie algebra $\mathfrak{g}$, only the 8-dimensional orbit $O_8$ has non-normal orbit closure $\overline{O}_8$. In this manuscript, we will give a quantization model of $\overline{O}_8$, verifying a conjecture of Vogan made in 1984.

1. Introduction

Let $G$ be a complex simple Lie group. The $G$-conjugates of a nilpotent element $X \in \mathfrak{g}$ form a nilpotent orbit $O \subset \mathfrak{g}$. Following the ideas in [19] or [21], one would like to attach unitary representations to all such orbits along with their finite $G$-equivariant covers. More precisely, let $V$ be a finite $G$-equivariant cover of an affine Poisson $G$-variety containing a nilpotent orbit $O$ as an open set, with its ring of regular functions $R(V)$; then one would like to find a $(\mathfrak{g}_C, K_C)$-module $X_V$ such that we have the $G$-module isomorphism

$$X_V|_{K_C} \cong R(V)$$

(note that $K \leq G$ is the maximal compact subgroup of $G$, hence its complexification $K_C$ is isomorphic to $G$). Throughout this work, we will call $X_V$ a quantization of $V$.

As hinted in [19], one needs to pay special attention when the orbit closure $\overline{O}$ is not normal. One reason is due to the algebro-geometric fact that $R(\overline{O}) \cong R(O)$ if and only if $\overline{O}$ is normal. Following the spirit of the orbit method, one needs to give a quantization model for $V = O$ and $V = \overline{O}$ separately when $\overline{O}$ is not normal.

Here is a summary on the current progress of the above quantization scheme. In [3], Barbasch constructs such models for a large class of classical nilpotent orbits. Using a completely different method in [5], Ranee Brylinski constructs a Dixmier algebra for all classical nilpotent orbit closures. The reconciliation between the two models is the main theme of the Ph.D. thesis of the author [22].

Contrary to the classical setting, very little is known about the scheme for exceptional groups. We now focus on the case for $G = G_2$. Let $\{\alpha, \beta\}$ be the simple roots of $\mathfrak{g}$, with $\alpha$ being the short root. The fundamental weights of $\mathfrak{g}$ are therefore given by

$$\{\omega_1, \omega_2\} = \{2\alpha + \beta, 3\alpha + 2\beta\}.$$
By the Bala-Carter classification, we have five nilpotent orbits $O_0$, $O_6$, $O_8$, $O_{10}$ and $O_{12}$ in $\mathfrak{g}$. Following the study of completely prime primitive ideals of Joseph in [9], Vogan in [19] conjectured a quantization model for $O_8$ and $\overline{O}_8$ for $G_2$:

**Conjecture 1.1** ([19 Conjecture 5.6]). Let $\lambda \in \mathfrak{h}^*$ and $J(\lambda)$ be the maximal primitive ideal in $U(\mathfrak{g})$ with infinitesimal character $\lambda$. Then the $(\mathfrak{g}_C, K_C) \cong (\mathfrak{g} \times \mathfrak{g}, G)$-modules

$$U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2)), \quad U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2)),$$

are quantizations of $O_8$ and $\overline{O}_8$ respectively. In particular,

$$U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))|_{K_C} \cong R(O_8), \quad U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))|_{K_C} \cong R(\overline{O}_8).$$

As a consequence, $O_8$ has non-normal closure.

Interestingly, by the classification of spherical unitary dual of complex $G_2$ given by Duflo in [8], $U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))$ is unitarizable while $U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))$ is not (this fact is also observed by Vogan in p. 226 of [21]. Later, Levasseur and Smith in [10] proved that $U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))|_{K_C} \cong R(O_8)$ and $\overline{O}_8$ are not normal, but were unable to prove the rest of the conjecture. The main result of this manuscript is the following:

**Theorem 1.2.** As $K_C \cong G$ modules,

$$U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))|_{K_C} \cong R(\overline{O}_8).$$

**Remark 1.3.** This quantization model of nilpotent orbit closure is very different from the classical model given in [5]. Namely, the Brylinski model is not necessarily of the form $U(\mathfrak{g})/J(\lambda)$. In particular, when the classical nilpotent orbit closure $\overline{O}$ is not normal (the classification of all such orbit closures is given in [13]), the infinitesimal character of the Brylinski model $\lambda_O$ always yields associated variety $AV(U(\mathfrak{g})/J(\lambda_O)) = O'$, where $O'$ is strictly smaller than $O$.

In fact, it can be shown that the Brylinski model always contains the composition factor $U(\mathfrak{g})/J(\lambda_O)$. This is part of the on-going work of Barbasch and the author [4].

Before going to the proof of Theorem 1.2 it is worthwhile to mention the orbits other than $O_8$ in $\mathfrak{g}$. Indeed, Kraft in [12] confirmed that $O_8$ is the only nilpotent orbit with non-normal closure. So we just need to consider quantizations of the orbits (and their covers) only. For the zero orbit $O_0$ the quantization is trivial, and the quantization of the minimal orbit $O_6$ is $U(\mathfrak{g})/J(\frac{1}{3}(3\omega_1 + \omega_2))$, where $J(\frac{1}{3}(3\omega_1 + \omega_2))$ is the Joseph ideal. The 10-dimensional orbit $O_{10}$ is a special orbit with fundamental group $S_3$. It is a simple exercise to compare the formulas in [2] and [14] that the spherical unipotent representation attached to $O_{10}$ is a quantization of $O$ (as a bonus, the other two unipotent representations attached to $O_{10}$ essentially give quantization of all covers of $O_{10}$ as well). Finally, the quantization of the principal orbit $O_{12}$ is well known to be the principal series representation with zero infinitesimal character. In conclusion, we completed the picture of quantization for all nilpotent orbits of $\mathfrak{g}$ and their closures.
2. Proof of the theorem

As mentioned in the Introduction, the non-normality of $\mathcal{O}_8$ implies that $R(\mathcal{O}_8) \subsetneq R(\mathcal{O}_8)$. In fact, Costantini in [7] gives the discrepancies in terms of $G$-modules:

**Theorem 2.1** ([7, Theorem 5.6]). Let $V_{(a,b)}$ be the finite-dimensional irreducible representation of $G_2$ with highest weight $a\omega_1 + b\omega_2$, where $a$ and $b$ are non-negative integers. Then

$$R(\mathcal{O}_8) \cong R(\mathcal{O}_8) \oplus \bigoplus_{n \in \mathbb{N} \cup \{0\}} V_{(1,n)}.$$ 

The following lemma gives another expression of the discrepancies between $R(\mathcal{O})$ and $R(\mathcal{O})$:

**Lemma 2.2.** As virtual $G$-modules,

$$\bigoplus_{n \in \mathbb{N} \cup \{0\}} V_{(1,n)} = \text{Ind}^G_T(1,0) - \text{Ind}^G_T(0,1) - \text{Ind}^G_T(2,0) + \text{Ind}^G_T(1,1) + \text{Ind}^G_T(0,2) - \text{Ind}^G_T(2,1),$$

where $\text{Ind}^G_T(a,b)$ is the shorthand for the induced module $\text{Ind}^G_T(e^{a\omega_1 + b\omega_2})$.

**Proof.** The lemma can be derived from the Weyl character formula. Namely, by the $W(G_2)$-symmetry of weights of $V_{(a,b)}$, we have

$$V_{(a,b)} = \sum_{w \in W(G_2)} \det(w) \text{Ind}^G_T(\lambda_w)$$

with $\lambda_w$ being the unique $W(G_2)$-conjugate of $w[(a,b) + (1,1)] - (1,1)$ lying in the dominant chamber. In fact, we have

$$V_{(1,n)} = \text{Ind}^G_T(1,n) - \text{Ind}^G_T(1,n+3) - \text{Ind}^G_T(2,n) + \text{Ind}^G_T(2,n+2) - \text{Ind}^G_T(3,n-1) + \text{Ind}^G_T(3,n) - \text{Ind}^G_T(3,n+1) + \text{Ind}^G_T(3,n+2) + \text{Ind}^G_T(6,n-2) - \text{Ind}^G_T(6,n) - \text{Ind}^G_T(7,n-2) + \text{Ind}^G_T(7,n-1)$$

for $n > 1$, and

$$V_{(1,0)} = \text{Ind}^G_T(1,0) - \text{Ind}^G_T(1,3) - \text{Ind}^G_T(2,0) + \text{Ind}^G_T(2,2) - \text{Ind}^G_T(0,1) + \text{Ind}^G_T(3,0) - \text{Ind}^G_T(3,1) + \text{Ind}^G_T(3,2) + \text{Ind}^G_T(0,2) - \text{Ind}^G_T(6,0) - \text{Ind}^G_T(1,2) + \text{Ind}^G_T(4,1);$$

$$V_{(1,1)} = \text{Ind}^G_T(1,1) - \text{Ind}^G_T(1,4) - \text{Ind}^G_T(2,1) + \text{Ind}^G_T(2,3) - \text{Ind}^G_T(3,0) + \text{Ind}^G_T(3,1) - \text{Ind}^G_T(3,2) + \text{Ind}^G_T(3,3) + \text{Ind}^G_T(3,1) - \text{Ind}^G_T(6,1) - \text{Ind}^G_T(4,1) + \text{Ind}^G_T(7,0).$$

The lemma is proved by adding up the terms. \hfill \Box

We now study the two Harish-Chandra bi-modules $U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))$ and $U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))$:

**Proposition 2.3.** As $K_C \cong G$-modules

$$U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2))|_{K_C} = \text{Ind}^G_T(0,0) - \text{Ind}^G_T(0,1) - \text{Ind}^G_T(2,0) + \text{Ind}^G_T(1,1);$$

$$U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2))|_{K_C} = \text{Ind}^G_T(0,0) - \text{Ind}^G_T(1,0) - \text{Ind}^G_T(0,2) + \text{Ind}^G_T(2,1).$$
Proof. To cater for subsequent calculations, we let \( \mathfrak{h}^* = \{(x, y, z) \in \mathbb{C}^3 | x + y + z = 0\} \), with short simple root \( \alpha = (1, -1, 0) \) and long simple root \( \beta = (-1, 2, -1) \). Then
\[
\lambda_1 = \frac{1}{2}(\omega_1 + \omega_2) = (1, 1/2, -3/2); \quad \lambda_2 = \frac{1}{2}(5\omega_1 - \omega_2) = (2, -1/2, -3/2).
\]

The character formulas of \( U(\mathfrak{g})/J(\lambda) \) for regular \( \lambda \) are well known by the work of Barbasch and Vogan [2]: Consider the subgroup \( W_\lambda \) of \( W(G_2) \) generated by roots \( \alpha \) satisfying \( 2 \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \). Then the formula is given by
\[
U(\mathfrak{g})/J(\lambda) = \sum_{w \in W_\lambda} \det(w) X(\lambda, w\lambda),
\]
where \( X(\mu, \nu) = K\text{-finite part of } \text{Ind}_T^G(e^{(\mu, \nu)} \otimes 1) \) is the principal series representation with character \( (\mu, \nu) \in \mathfrak{h}_C \), the complexification of the maximal torus \( \mathfrak{h} \) in \( \mathfrak{g} \) (here we treat \( G \) as a real Lie group). In particular, the \( G \cong K_C \)-types of \( X(\mu, \nu) \) are equal to \( \text{Ind}_T^G(e^{\mu - \nu}) \) (see Theorem 1.8 of [2] for more details on the principal series representations).

Now apply the above recipe for \( \lambda_1 = (1, 1/2, -3/2) \): With the above notation, \( W_{\lambda_1} \) is isomorphic to \( W(A_1 \times \tilde{A}_1) \), generated by the roots \( \{(0, 1, -1), (2, -1, -1)\} \). Hence the character formula of \( U(\mathfrak{g})/J(\lambda_1) \) is given by
\[
U(\mathfrak{g})/J(\lambda_1) = X((1, 1/2, -3/2), (1, 1/2, -3/2)) - X((1, 1/2, -3/2), (1, -3/2, 1/2)) \nonumber
- X((1, 1/2, -3/2), (-1, 3/2, -1/2)) + X((1, 1/2, -3/2), (-1, -1/2, 3/2)).
\]
Upon restricting to \( K_C \), we have
\[
U(\mathfrak{g})/J(\lambda_1)|_{K_C} \cong \text{Ind}_T^G(e^{(0,0,0)}) - \text{Ind}_T^G(e^{(0,2,-2)})
- \text{Ind}_T^G(e^{(2,-1,-1)}) + \text{Ind}_T^G(e^{(2,1,-3)}).
\]
Again, by \( W(G_2) \)-symmetry of finite-dimensional irreducible \( G \)-modules, the above expression can be written in the form as in the proposition. The calculations for \( U(\mathfrak{g})/J(\lambda_2) \) are identical to the one above. We omit the calculations here. \( \square \)

Proof of Theorem 1.2 By the result of Levasseur and Smith in [10],
\[
U(\mathfrak{g})/J(\frac{1}{2}(\omega_1 + \omega_2)|_{K_C} \cong R(O_8)
\]
as \( G \)-modules. Therefore the first equation of Proposition 2.3 gives
\[
R(O_8) \cong \text{Ind}_T^G(0,0) - \text{Ind}_T^G(0,1) - \text{Ind}_T^G(2,0) + \text{Ind}_T^G(1,1)
\]
as virtual \( G \)-modules. By Theorem 2.1 and Lemma 2.2, we need to show that
\[
U(\mathfrak{g})/J(\frac{1}{2}(5\omega_1 - \omega_2)|_{K_C} = R(O_8) - \bigoplus_{n} V(1,n)
= R(O_8) - (\text{Ind}_T^G(1,0) - \text{Ind}_T^G(0,1) - \text{Ind}_T^G(2,0)
+ \text{Ind}_T^G(1,1) + \text{Ind}_T^G(0,2) - \text{Ind}_T^G(2,1)).
\]
This is readily seen to be true from equation (2.1) and the second equation of Proposition 2.3. \( \square \)
3. Final remarks

In [18], Sommers gives some conjectures on the multiplicities of small representations of $R(O)$ for the exceptional groups. In particular, given that his conjecture is true, one can show the non-normality of some orbit closures.

To describe more explicitly which orbits $O$ are conjectured to have non-normal closures, recall that Lusztig in [11] partitioned all nilpotent orbits in $\mathfrak{g}$ by special pieces, i.e., for all nilpotent orbit $O'$, it must belong to exactly one of the special pieces

$$S_O := \{O' \subseteq \overline{O}|O' \nsubseteq \overline{O}_{\text{spec}} \text{ for any other special orbit } \overline{O}_{\text{spec}} \subseteq \overline{O}\},$$

where $O$ runs through all special orbits in $\mathfrak{g}$.

For each $O' \in S_O$, Lusztig assigned a Levi subgroup $H(O',O)$ of the Lusztig's quotient $\overline{A}(O)$. For example, the largest orbit in the special piece $O \in S_O$ has $H(O,O) = 1$, and the smallest orbit $O'' \in S_O$ has $H(O'',O) = \overline{A}(O)$.

By the conjecture of Sommers, if $O$ has a non-abelian Lusztig's quotient, i.e., $\overline{A}(O) = S_3$, $S_4$ or $S_5$, then all $O' \in S_O$ with $H(O',O)$ not equal to 1 or $\overline{A}(O)$ (that is, not equal to $O$ or $O''$) have non-normal closures.

For example, in the case of $G_2$ we studied above, we have $O_8 \in S_{O_{10}}$ and $H(O_8,O_{10}) = S_2 \leq S_3 = \overline{A}(O_{10})$. So $\overline{O}_8$ is conjectured to have non-normal closure, which has been shown to be true.

We would like to end our manuscript with the following:

**Conjecture 3.1.** Suppose $O$ is a nilpotent orbit with $\overline{A}(O) = S_3$, $S_4$ or $S_5$, and $O' \in S_O$ satisfies $H(O',O) \neq 1$, $\overline{A}(O)$. Then there exists two distinct completely prime primitive ideals $J(\lambda_1)$, $J(\lambda_2)$ such that

$$U(\mathfrak{g})/J(\lambda_1)|_{K_C} \cong R(O), \quad U(\mathfrak{g})/J(\lambda_2)|_{K_C} \cong R(\overline{O}).$$

**References**

[1] Jens Carsten Jantzen and Karl-Hermann Neeb, *Lie theory*, Progress in Mathematics, vol. 228, Birkhäuser Boston, Inc., Boston, MA, 2004. Lie algebras and representations; Edited by Jean-Philippe Anker and Bent Orsted. MR2042688

[2] Dan Barbasch and David A. Vogan Jr., *Unitary representations of complex semisimple groups*, Ann. of Math. (2) 121 (1985), no. 1, 41–110, DOI 10.2307/1971193. MR782556

[3] D. Barbasch, *Regular Functions on Covers of Nilpotent Coadjoint Orbits*, http://arxiv.org/abs/0810.0688v1, 2008

[4] D. Barbasch and K. Wong, *Regular Functions of Symplectic Nilpotent Orbit Closures and their Normality*, in preparation.

[5] Ranee Brylinski, *Dixmier algebras for classical complex nilpotent orbits via Kraft-Procesi models. I*, The orbit method in geometry and physics (Marseille, 2000), Progr. Math., vol. 213, Birkhäuser Boston, Boston, MA, 2003, pp. 49–67. MR1995374

[6] David H. Collingwood and William M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993. MR1251060

[7] Mauro Costantini, *On the coordinate ring of spherical conjugacy classes*, Math. Z. 264 (2010), no. 2, 327–359, DOI 10.1007/s00209-008-0468-5. MR2574980

[8] Michel Duflo, *Réprésentation unitaires irréductibles des groupes simples complexes de rang deux* (French, with English summary), Bull. Soc. Math. France 107 (1979), no. 1, 55–96. MR532562

[9] A. Joseph, *Goldie rank in the enveloping algebra of a semisimple Lie algebra. I, II*, J. Algebra 65 (1980), no. 2, 269–283, 284–306, DOI 10.1016/0021-8693(80)90217-3. MR585721

[10] T. Levasseur and S. P. Smith, *Primitive ideals and nilpotent orbits in type $G_2$*, J. Algebra 114 (1988), no. 1, 81–105, DOI 10.1016/0021-8693(88)90214-1. MR931902
[11] G. Lusztig, *Notes on unipotent classes*, Asian J. Math. 1 (1997), no. 1, 194–207, DOI 10.4310/AJM.1997.v1.n1.a7. MR1480994

[12] Hanspeter Kraft, *Closures of conjugacy classes in G_2*, J. Algebra 126 (1989), no. 2, 454–465, DOI 10.1016/0021-8693(89)90313-X. MR1025000

[13] Hanspeter Kraft and Claudio Procesi, *On the geometry of conjugacy classes in classical groups*, Comment. Math. Helv. 57 (1982), no. 4, 539–602, DOI 10.1007/BF02565876. MR694606

[14] William M. McGovern, *Rings of regular functions on nilpotent orbits and their covers*, Invent. Math. 97 (1989), no. 1, 209–217, DOI 10.1007/BF01850661. MR99319

[15] William M. McGovern, *A branching law for Spin(7, C) → G_2 and its applications to unipotent representations*, J. Algebra 130 (1990), no. 1, 166–175, DOI 10.1016/0021-8693(90)90106-X. MR1045742

[16] William M. McGovern, *Rings of regular functions on nilpotent orbits. II. Model algebras and orbits*, Comm. Algebra 22 (1994), no. 3, 765–772, DOI 10.1080/00927879408824874. MR1261003

[17] William M. McGovern, *Completely prime maximal ideals and quantization*, Mem. Amer. Math. Soc. 108 (1994), no. 519, viii+67, DOI 10.1090/memo/0519. MR1191608

[18] Eric Sommers, *Conjectures for small representations of the exceptional groups*, Comment. Math. Univ. St. Paul. 49 (2000), no. 1, 101–104. MR1777157

[19] David A. Vogan Jr., *The orbit method and primitive ideals for semisimple Lie algebras, Lie algebras and related topics* (Windsor, Ont., 1984), CMS Conf. Proc., vol. 5, Amer. Math. Soc., Providence, RI, 1986, pp. 281–316. MR832204

[20] David A. Vogan Jr., *Unitary representations of reductive Lie groups*, Annals of Mathematics Studies, vol. 118, Princeton University Press, Princeton, NJ, 1987. MR908075

[21] David A. Vogan Jr., *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups (Brunswick, ME, 1989), Progr. Math., vol. 101, Birkhäuser Boston, Boston, MA, 1991, pp. 315–388. MR1168491

[22] Kayue Wong, *Dixmier algebras on complex classical nilpotent orbits and their representation theories*, ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–Cornell University. MR3217989

[23] Kayue Daniel Wong, *Regular functions of symplectic spherical nilpotent orbits and their quantizations*, Represent. Theory 19 (2015), 333–346, DOI 10.1090/ert/474. MR3434893

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