Factoriality of Hecke-von Neumann Algebras of Right-Angled Coxeter Groups

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Abstract. The Hecke algebra $C_q[W]$ of a Coxeter group $W$, associated to parameter $q$, can be completed to a von Neumann algebra $N_q(W)$. We study such algebras in case where $W$ is right-angled. We determine the range of $q$ for which $N_q(W)$ is a factor, i.e. has trivial center. Moreover, in case of nontrivial center, we prove a result allowing to decompose $N_q(W)$ into a finite direct sum of factors.

1. Introduction

For $q > 0$ the Hecke algebra $C_q[W]$ of a Coxeter group $W$ is a deformation of the group algebra of $W$, consisting of finitely supported functions on $W$ with a modified product, yielding the ordinary group algebra for $q = 1$. In case where $q$ is an integer, the Hecke algebra has a nice geometric interpretation. Recall that a building of type $W$ can be thought of as a space endowed with a $W$-valued metric [1, Chapter 5]. If it is locally finite, i.e. for every $w$ there are finitely many points $y$ at distance $w$ from a fixed point $x$, then for every $w \in W$ we may consider an operator $A_w$ on the space of finitely supported functions on the building, which averages the functions over spheres of radius $w$. If all spheres of radius $s$, where $s$ is any of the standard generators of $W$, have cardinality $q$, the operators $A_w$ generate an algebra isomorphic to $C_q[W]$ [7]. For $q = 1$ the averages are taken over 1-element sets, and are in fact translations, so one indeed gets the group algebra.

The Hecke algebra has a natural action on the Hilbert space of square-integrable functions on $W$, obtained by extending its action on itself by left multiplication. It therefore completes to a von Neumann algebra—the Hecke-von Neumann algebra $N_q(W)$. Its importance stems from the theory of weighted $L^2$-cohomology of Coxeter groups [4], where the cohomology spaces are modules over $N_q(W)$.

A generalization of the Singer Conjecture [4, Conjecture 14.7] deals with vanishing of certain weighted cohomology spaces of $W$. The motivation for this work was to try to approach this problem using the central decomposition of $N_q(W)$ to better understand the structure of these spaces. As it turns out, although the centers of $N_q(W)$ can be nontrivial, they contribute nothing new in the subject of decomposing the weighted cohomology of $W$; the decompositions described...
in [4, Theorem 11.1] are finer than those induced by the central decomposition of \( N_q(W) \).

Although our results are not fit for the applications we initially had in mind, they are interesting in their own right. Namely, we show that the Hecke-von Neumann algebras of irreducible right-angled Coxeter groups are factors, up to a 1-dimensional direct summand. The main result of the paper reads as follows.

**Main Theorem** (Theorem 5.3). Suppose that \((W, S)\) is an irreducible right-angled Coxeter system with \(|S| \geq 3\). Then the Hecke-von Neumann algebra \( N_q(W) \) is a factor if and only if

\[
q \in [\rho, \rho^{-1}],
\]

where \( \rho \) is the convergence radius of \( W(t) \), the spherical growth series of \( W \). Moreover, for \( q \) outside this interval, \( N_q(W) \) is a direct sum of a factor and \( \mathbb{C} \).

Since taking centers commutes with tensor products, and the Hecke-von Neumann algebra \( N_q(W) \) of an arbitrary right-angled Coxeter group \( W \) is the tensor product of the Hecke-von Neumann algebras of irreducible factors of \( W \), the central decomposition of \( N_q(W) \) follows in the general case.

The paper is divided into two parts. The first part consists of Sections 2 and 3, and is purely group-theoretic. Section 2 introduces the basic notions related to Coxeter groups, and in Section 3 we analyze certain double cosets in a right-angled Coxeter group, and use them to define the graph \( \Gamma(W, S) \), whose connectivity is related to restrictions satisfied by elements of the center of \( N_q(W) \). We show that for irreducible \( W \), this graph consists of a single connected component and at most two isolated vertices. The remaining two sections constitute the analytic part. In Section 4 we define the Hecke-von Neumann algebras as algebras of operators on the Hilbert space \( \ell^2(W) \), the symbols of its elements in \( \ell^2(W) \), and the isomorphism \( j \), which allows us to restrict to the case where \( q \leq 1 \). Section 5 puts the graph \( \Gamma(W, S) \) to work, and characterizes the symbols of elements in the center of \( N_q(W) \) for irreducible \( W \). It turns out that, depending on \( q \), the potential symbol of a non-trivial element in the center has either infinite \( \ell^2 \)-norm, or defines a multiple of a unique one-dimensional projection. This gives the two cases of the Main Theorem. Finally, in Section 6 we describe how our results are related to the paper [5] of Dykema, dealing with free products of certain von Neumann algebras.

We would like to thank Adam Skalski, our mentor during the internship at the Warsaw Center of Mathematics and Computer Science, for his support and many stimulating discussions. We are also grateful to Jan Dymara for introducing us to the subject of Hecke-von Neumann algebras.

2. **Coxeter groups**

A group \( W \) generated by a finite set \( S \), subject to the presentation

\[
W = \langle S \mid (st)^{m(s,t)} = 1 : s, t \in S, m(s, t) \neq \infty \rangle,
\]

where the exponents \( m(s, t) \in \{1, 2, \ldots, \infty\} \) satisfy \( m(s, s) = 1, m(s, t) \geq 2 \) for \( s \neq t \), and \( m(s, t) = m(t, s) \), is called a Coxeter group. Together with the fixed generating set, the pair \((W, S)\) is called a Coxeter system. One usually encodes the data from the definition of a Coxeter system into a graph, called the Coxeter diagram, having \( S \) as the vertex set, and an edge with label \( m(s, t) \) between every two vertices \( s, t \) with
\(m(s, t) \geq 3\) (i.e. the edges between commuting generators are omitted). One also considers the odd Coxeter diagram, which encodes information about conjugacy of generators, and is obtained from the Coxeter diagram by removing all the edges with even labels, including \(\infty\). Two generators \(s, t \in S\) are conjugate in \(W\) if and only if they belong to the same connected component of the odd Coxeter diagram of \((W, S)\) \cite[Lemma 3.3.3]{3}.

It is an important observation that for any \(T \subseteq S\) the subgroup of \(W\) generated by \(T\) is also a Coxeter group, with the same exponents as \(W\) \cite[Theorems 4.1.6(i) and 3.4.2(i)]{3}. Such a subgroup is called special. The group \(W\) is called irreducible if it does not decompose into a nontrivial direct product of special subgroups. This is equivalent to the connectedness of its Coxeter diagram.

Any Coxeter group carries a word length, given by

\[
[w] = \min\{n : (\exists s_1, \ldots, s_n \in S) \ w = s_1 \cdots s_n\}.
\]

By \cite[Corollary 4.1.5]{3}, the word length in a special subgroup agrees with the word length in \(W\). If \([w] = n\) and \(w = s_1 \cdots s_n\), then the word \(s_1 \cdots s_n\) is called a reduced expression for \(w\). By \cite[Proposition 4.1.1]{3}, the set of generators appearing in any reduced expression for \(w\) is independent of the choice of expression. We will denote it by \(S(w)\). With this notation, the special subgroup \(W_T\) consists exactly of the elements \(w \in W\) with \(S(w) \subseteq T\).

The power series

\[
W(t) = \sum_{w \in W} t^{[w]} = \sum_{n=0}^{\infty} |\{w \in W : |w| = n\}| t^n
\]

is called the spherical growth series of \(W\). Its radius of convergence will be denoted by \(\rho\). If \(W\) is infinite, then \(\rho \leq 1\). Since the coefficients of \(W(t)\) are non-negative real numbers, the series diverges at \(t = \rho\).

The map \(w \mapsto (-1)^{|w|}\) is a group homomorphism, and hence \(|sw| = |w| \pm 1\) and \(|ws| = |w| \pm 1\) for \(w \in W\) and \(s \in S\). Among all groups generated by elements of order 2, Coxeter groups are characterized by the following equivalent conditions \cite[Theorems 3.2.16 and 3.3.4]{3}.

**Theorem 2.1.** Let \((W, S)\) be a Coxeter system, \(w \in W, s_s, s_1, \ldots, s_n, t \in S\).

- **Deletion condition:** If \(w = s_1 \cdots s_n\) is not a reduced word then there exist \(1 \leq i < j \leq n\) such that \(w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_n\).

- **Exchange condition:** If \(w = s_1 \cdots s_n\) is a reduced expression and \(sw\) is not, then \(sw = s_1 \cdots \hat{s}_i \cdots s_n\) for some \(i\); a similar statement holds with \(ws\) in place of \(sw\).

- **Folding condition:** If \(sw\) and \(wt\) are reduced words, then either \(swt = w\) or \(swt\) is reduced.

A special case of a Coxeter system is a right-angled Coxeter system, in which the exponents \(m(s, t)\) for \(s \neq t\) are either 2 or \(\infty\). They interpolate between the free and direct products of copies of \(\mathbb{Z}_2\). From now on, we will assume that \((W, S)\) is right-angled. Its odd Coxeter diagram has no edges, and thus conjugate generators are equal. This has consequences for the Deletion condition (and the Exchange condition, which is a special case of the Deletion condition). If we have a cancellation of the form \(usvtw = uvw\) with \(u, v, w \in W\) and \(s, t \in S\), then \(v^{-1}sv = t\), so \(s = t\) and \(v\) commutes with \(s\). By the next lemma, this means that \(S(v) \subseteq C(r)\), where \(C(r)\) is the set of all generators commuting with \(r\). We will frequently use the following lemma, describing centralizers of generators of a right-angled Coxeter group.
Lemma 2.2. In a right-angled Coxeter system \((W, S)\) the centralizer of a generator \(r \in S\) is the special subgroup \(W_{C(r)}\).

Proof. The main theorem of [2] gives the description of the centralizer of a generator \(r \in S\) in an arbitrary Coxeter group as a semidirect product \(W_T \rtimes F\), where \(W_T\) is a special subgroup, and \(F\) is the fundamental group of the connected component of the odd Coxeter diagram of \(W\) containing \(r\). The odd Coxeter diagram of a right-angled Coxeter group has no edges, so the centralizer is just \(W_T\). Clearly, \(T = C(r)\). \qed

3. The Graph \(\Gamma(W, S)\)

Let \((W, S)\) be an infinite irreducible right-angled Coxeter system, and let \(s, t \in S\) satisfy \(m(s, t) = \infty\). Denote by \(D\) the special subgroup of \(W\) generated by \(s\) and \(t\). It is isomorphic to \(\mathbb{Z}_2 \ast \mathbb{Z}_2\). Reduced expressions for elements of \(D\) are just alternating words in \(s\) and \(t\), and in particular they are unique.

To any \(w \in W\) there corresponds a double coset of \(D\), defined as \(DwD^\prime = \{ dwd' : d, d' \in D \}\). By [3, Lemma 4.3.1], it contains a unique element \(w_0\) of minimal length, such that any \(u \in DwD\) can be written as \(dw_0d'\) with \(d, d' \in D\) and \(|u| = |d| + |w_0| + |d'|\).

We will call the double coset \(DwD\) non-degenerate if its shortest element \(w_0\) does not centralize \(D\), i.e. it does not commute with either \(s\) or \(t\). We are ready to define the main tool of this paper, the graph \(\Gamma(W, S)\). The vertex set of \(\Gamma(W, S)\) is \(W\), and for \(w \in W\) and \(s \in S\) there are edges joining \(w\) to both \(ws\) and \(sw\), provided that there exists \(t \in S\) such that \(m(s, t) = \infty\) and the double coset \(DwD^\prime\) of \(D = \langle s, t \rangle\) is non-degenerate. In Section 4 we will show that each edge in this graph corresponds to a restraint on the symbol of an operator in the center of the Hecke-von Neumann algebra. Now we will study connectivity of \(\Gamma(W, S)\). We will show that under some mild assumptions it consists of an isolated vertex \(1\) and a single connected component.

For \(w \in W\) its right and left descent sets are defined as

\[
D_R(w) = \{ s \in S : |ws| < |w| \} \quad \text{and} \quad D_L(w) = \{ s \in S : |sw| < |w| \}.
\]

The right (resp. left) descent set of \(w\) is the set of all letters in which a reduced expression can end (resp. begin). By the discussion above, the generators in \(D_R(w)\) pairwise commute, and thus if \(W\) is infinite, the descent sets are proper subsets of \(S\). Observe, that \(|vw| = |v| + |w|\) if and only if \(D_R(v) \cap D_L(w) = \emptyset\). Indeed, if the intersection is nonempty, \(v\) and \(w\) have reduced expressions \(v\) and \(w\), respectively ending and starting with the same generator, which cancels out in their concatenation. If on the other hand a cancellation of a pair of generators occurs in \(vw\), then the subword of \(vw\) lying between the innermost pair of canceling generators is reduced, and all its letters commute with the deleted generator, so in fact it can be moved to the end and lies in both \(D_R(v)\) and \(D_L(w)\). Yet another useful observation is the following. Suppose that \(|ws| > |w|\) for some \(w \in W\) and \(s \in S\). In this case, we have

\[
D_R(ws) = (D_R(w) \cap C(s)) \cup \{ s \},
\]

since the set on the right is contained in \(D_R(ws)\), and if \(t \in D_R(ws) \setminus \{ s \}\), then \(t\) commutes with \(s\) and lies in \(D_R(w)\).
**Lemma 3.1.** Suppose that \((W,S)\) is an irreducible infinite right-angled Coxeter system, and let \(v, w \in W\). Then there exists \(u \in W\) such that \(|vuw| = |v| + |u| + |w|\).

**Proof.** First, suppose that \(D_L(w) \subseteq D_R(v)\). Since \(W\) is infinite, the set \(S \setminus D_R(v) = \{s_1, \ldots, s_n\}\) is nonempty. Set \(v_k = vs_1 \cdots s_k\) for \(k = 0, 1, \ldots, n\). By (3.2) we have \(D_R(v_k) \subseteq D_R(v)\) and thus \(|v_{k+1}| = |v_k| + 1\). Moreover,

\[
D_R(v_{k+1}) \cap D_R(v) = D_R(v_k) \cap C(s_{k+1}) \cap D_R(v) = \bigcap_{i=1}^{k+1} C(s_i) \cap D_R(v),
\]

and therefore

\[
D_R(v_n) \cap D_L(w) \subseteq D_R(v_n) \cap D_R(v) = \bigcap_{s \notin D_R(v)} C(s) \cap D_R(v) = \emptyset,
\]

since otherwise we would find a generator commuting with all the others, which contradicts irreducibility. Hence, we can use \(u = s_1 \cdots s_n\).

Now, suppose that \(D_L(w) \not\subseteq D_R(v)\). Let \(D_L(w) \setminus D_R(v) = \{s_1, \ldots, s_n\}\) and denote \(v_k = vs_1 \cdots s_k\). As before, we conclude that \(|v_{k+1}| = |v_k| + 1\) and

\[
D_R(v_{k+1}) \cap D_L(w) = (D_R(v_k) \cap D_L(w)) \cup \{s_{k+1}\},
\]

since \(D_L(w) \subseteq C(s_{k+1})\). Therefore, \(D_L(w) \subseteq D_R(v_n)\), and \(|v_n| = |v| + n\). We may now apply the first case with \(v_n\) in place of \(v\) to obtain the conclusion. \(\square\)

**Lemma 3.2.** For an irreducible infinite right-angled Coxeter system \((W,S)\) with \(|S| \geq 3\) all elements \(w \in W\) with \(S(w) = S\) lie in the same component of \(\Gamma(W,S)\).

**Proof.** Let \(v, w \in W\) satisfy \(S(v) = S(w) = S\). Denote their reduced expressions by \(v\) and \(w\). By Lemma 3.1 there exists \(u \in W\) with reduced expression \(u\) such that \(vuw\) is reduced. Let \(uw = s_1 \cdots s_n\) with \(s_i \in S\), and define \(v_i = vs_1 \cdots s_i\). We have \(S(v_i) = S\), and by irreducibility, there exists a generator \(t \in S\) not commuting with \(s_{i+1}\). Moreover, for \(D = \langle s_{i+1}, t \rangle\) the double coset \(Dv_tD\) is non-degenerate. Indeed, its shortest element contains all generators from \(S \setminus \{s_{i+1}, t\}\) \(\not\subseteq \emptyset\), and by irreducibility they can not all commute with \(D\). Thus \(v_i\) and \(v_{i+1}\) are connected by an edge in \(\Gamma(W,S)\). The same holds for the analogous path from \(w\) to \(vuw\). Therefore \(v\) and \(w\) lie in the same component. \(\square\)

**Lemma 3.3.** Suppose that \((W,S)\) is an irreducible infinite right-angled Coxeter system, \(w \in W \setminus \{1\}\) and \(S(w) \neq S\). Then either

1. \(W \cong \mathbb{Z}_2 * \mathbb{Z}_2^k\) and \(w\) is the generator of the \(\mathbb{Z}_2\) free factor, or
2. there exists \(s \in S \setminus S(w)\) such that \(w\) is connected by an edge in \(\Gamma(W,S)\) with \(ws\) and \(sw\).

**Proof.** Suppose that situation (1) does not hold. We need to find \(s \in S \setminus S(w)\) and \(t \in S \setminus \{s\}\) such that \(D = \langle s, t \rangle\) is infinite and the double coset \(DwD\) is non-degenerate. We will consider three cases.

**Case 1.** The subgroup generated by \(S \setminus S(w)\) is infinite. We may thus find \(s, t \in S \setminus S(w)\) generating infinite \(D\). Since \(s, t \notin S(w)\), the shortest element in \(DwD\) is \(w\). Assume that for all choices of \(s\) and \(t\) the double coset \(DwD\) is degenerate, so that both \(s\) and \(t\) commute with all generators in \(S(w)\).

Denote by \(T\) the subset of \(S \setminus S(w)\) consisting of all generators not commuting with all of \(S \setminus S(w)\). It is nonempty since \((S \setminus S(w))\) is infinite. By assumption, every \(s \in T\) commutes with \(S(w)\), and, by definition of \(T\), with \(S \setminus (S(w) \cup T)\).
This gives a decomposition of $S$ into two non-empty commuting subsets $T$ and $S \setminus T$, which contradicts irreducibility of $(W, S)$.

**Case 2.** $S(w)$ generates an infinite subgroup of $W$. It follows that there exist non-commuting $s, r \in S(w)$. If we can find $t \in S \setminus S(w)$ such that $D = \langle s, t \rangle$ is infinite, then the double coset $DwD$ is non-degenerate. Indeed, its shortest element $w_0$ is obtained from $w$ by canceling some occurrences of $s$ and $t$, and we still have $r \in S(w_0)$. Thus, $s$ does not commute with $w_0$.

But if each $s \in S(w)$ which does not commute with all of $S(w)$, commutes with $S \setminus S(w)$, we get a contradiction similar to the one in Case 1.

**Case 3.** Both $S \setminus S(w)$ and $S(w)$ consist of pairwise commuting generators. For $s \in S \setminus S(w)$ denote

$$N(s) = \{ t \in S(w) : m(s, t) = \infty \}. \tag{3.6}$$

If $N(s)$ contains at least two distinct elements $r, t$, then for $D = \langle s, t \rangle$ the shortest element of $DwD$ contains the generator $r$, and does not commute with $s$, so $DwD$ is non-degenerate. If $N(s)$ is empty, then $s$ commutes with all other generators, which contradicts irreducibility.

What remains is the case where for all $s \in S \setminus S(w)$ we have $N(s) = \{ f(s) \}$ for some function $f : S \setminus S(w) \to S(w)$. But for any $s \in S(w)$ all generators in the set $\{ s \} \cup f^{-1}(s)$ commute with its complement. This contradicts irreducibility, provided that $S(w) \neq \{ s \}$, which is excluded by assuming that the situation (1) from the statement of the Lemma does not hold. \hfill \ensuremath{\Box}

**Proposition 3.4.** Let $(W, S)$ be an irreducible infinite right-angled Coxeter system. Then all elements of $W$ except 1, and, in case where $W \cong \mathbb{Z}_2 * \mathbb{Z}_2$, the generator of the $\mathbb{Z}_2$ free factor, lie in the same connected component of $\Gamma(W, S)$.

**Proof.** We already know by Lemma 3.2 that all elements $w \in W$ with $S(w) = S$ lie in the same connected component of $\Gamma(W, S)$. It remains to show that for $w$ as required in the statement with $S(w) \neq S$ we may find a path in $\Gamma(W, S)$ from $w$ to some element $v$ with $S(v) = S$. But this can be done by inductively applying Lemma 3.3. \hfill \ensuremath{\Box}

### 4. HECKE-VON NEUMANN ALGEBRAS

Let $(W, S)$ be a Coxeter system, and let $q$ be a positive real number. By [3, Proposition 19.1.1] there exists a unique $\ast$-algebra $C_q[W]$ with basis $\{ \hat{T}_w : w \in W \}$, such that for any $s \in S$ and $w \in W$

$$
\begin{align*}
\hat{T}_s \hat{T}_w &= \begin{cases} 
\hat{T}_{sw} & \text{if } |sw| > |w|, \\
q \hat{T}_{sw} + (q-1) \hat{T}_w & \text{otherwise}; 
\end{cases} \\
\hat{T}_w^n &= \hat{T}_{w^{-1}}.
\end{align*}
$$

It is called the Hecke algebra of $W$ associated to $q$. It satisfies also the analogue of the first of formulas (4.1) with the order of $s$ and $w$ reversed.

If $q$ is an integer, this algebra can be interpreted as an algebra generated by averaging operators on a certain space of functions associated to a regular building of type $(W, S)$ and thickness $q$ (for details consult [7]). When $q = 1$, we get the group algebra of $W$. 
Denote $T_w = q^{-|w|/2} T_w$. These normalized elements satisfy for all $s \in S$ and $w \in W$ the identity

$$T_s T_w = \begin{cases} T_{sw} & \text{if } |sw| > |w| \\ T_{sw} + p T_w & \text{otherwise,} \end{cases}$$

(4.2)

where

$$p = \frac{q - 1}{q^{1/2}}.$$ (4.3)

The map $j : C_q[W] \to C_q^{-1}[W]$ defined as the linear extension of

$$j(T_w) = (-1)^{|w|} T_w$$

satisfies $j(T_s T_w) = j(T_s) j(T_w)$, and hence is an isomorphism of $*$-algebras.

Now, consider the Hilbert space $\ell^2(W)$ of square-summable complex functions on $W$. Using the embedding of $C_q[W]$ into $\ell^2(W)$ given by $T_w \mapsto \delta_w$ we can uniquely extend the left and right actions of $C_q[W]$ on itself to bounded $*$-representations on $\ell^2(W)$. We will identify $C_q[W]$ with the subspace of $\mathcal{B}(\ell^2(W))$ consisting of left multiplication operators. Also, for $T \in C_q[W]$, the corresponding right multiplication operator on $\ell^2(W)$ will be denoted by $T^r$. The Hecke-von Neumann algebra $\mathcal{N}_q(W)$ is the von Neumann algebra generated by $C_q[W]$ in $\mathcal{B}(\ell^2(W))$. The right-hand variant of $\mathcal{N}_q(W)$, generated by the operators $T^r$ with $T \in C_q[W]$ will be denoted by $\mathcal{N}_q^r(W)$. It follows from [3, Proposition 19.2.1] that the commutant of $\mathcal{N}_q$ is $\mathcal{N}_q^r$ and vice versa. The main object of study in this paper is the center of the Hecke-von Neumann algebra $\mathcal{N}_q(W)$.

As in the case of the group von Neumann algebra, to any element $T \in \mathcal{N}_q(W)$ we may associate its symbol $T \delta_1 \in \ell^2(W)$. If the symbol of $T$ is 0, then we have

$$T \delta_w = T_{tw} \delta_1 = T_{tw} T \delta_1 = 0,$$

(4.5)

and thus $T = 0$, so the mapping $T \mapsto T \delta_1$ is injective. If $\tilde{\xi} \in \ell^2(W)$ is a symbol of an operator in $\mathcal{N}_q(W)$, it will be denoted by $T(\tilde{\xi})$. The same reasoning applies to $\mathcal{N}_q^r(W)$. In this case the operator with symbol $\tilde{\xi}$ will be denoted by $T^r(\tilde{\xi})$.

Recall that there is an isomorphism $j : C_q[W] \to C_q^{-1}[W]$. Through the natural embedding of $C_q[W]$ it extends to an isometry of $\ell^2(W)$. Therefore, it also extends to an isometric isomorphism of the Hecke-von Neumann algebras $\mathcal{N}_q(W)$ and $\mathcal{N}_q^{-1}(W)$.

5. Description of the Center

Let $(W, S)$ be a right-angled Coxeter system. Take $q > 0$ and let

$$p = \frac{q - 1}{q^{1/2}}.$$ (5.1)

We may assume that $q \leq 1$, since the case of $q > 1$ can be reduced to the latter using the isomorphism $j$. Furthermore, we may assume that $(W, S)$ is irreducible. Indeed, if there is a decomposition $(W, S) = (W_T \times W_U, T \cup U)$, then we have a decomposition of the corresponding Hecke algebra into an algebraic tensor product,

$$C_q[W] \cong C_q[W_T] \otimes C_q[W_U].$$ (5.2)
resulting in a tensor product decomposition of the Hecke-von Neumann algebra

\[ \mathcal{N}_q(W) \cong \mathcal{N}_q(W_T) \otimes \mathcal{N}_q(W_u). \]  

By [6, Corollary 11.2.17] the center of a tensor product of von Neumann algebras is the tensor product of their centers, and so the general case can be reduced to the irreducible one.

We will begin by describing the conditions satisfied by the symbol of an operator in \( \mathcal{N}_q(W) \) commuting with a single \( T_s \).

**Lemma 5.1.** Let \( s \in S \) and \( T(\xi) \in \mathcal{N}_q(W) \). Then \( T(\xi) \) commutes with \( T_s \) if and only if for all \( w \in W \) such that \( |sws| = |w| + 2 \) we have

\[
\xi(sw) = \xi(ws), \\
\xi(sws) = \xi(w) + p\xi(sw).
\]

**Proof.** We start by assuming that \( T(\xi) \) commutes with \( T_s \). If \( |sws| = |w| + 2 \), then \( |sw| = |ws| = |w| + 1 \) and we have

\[
\xi(sw) = \langle T(\xi)\delta_1, T_s^*\delta_w \rangle = \langle T_s^*T(\xi)\delta_1, \delta_w \rangle = \langle T(\xi)T_s^*\delta_1, \delta_w \rangle = \langle T(\xi)\delta_1, T_s^*\delta_w \rangle = \xi(sw)
\]

and

\[
\xi(sws) = \langle T(\xi), T_sT_s^*\delta_w \rangle = \langle T(\xi)T_sT_s^*\delta_1, \delta_w \rangle = \langle T(\xi)\delta_1, T_s^*\delta_w \rangle = \xi(w) + p\xi(sw).
\]

Now, assume that for any \( w \in W \) with \( |sws| = |w| + 2 \) conditions (5.4) hold. It suffices to show that \( T_sT(\xi) \) and \( T(\xi)T_s \) have the same symbol, i.e. that

\[
T_s\xi = T_sT(\xi)\delta_1 = T(\xi)T_s\delta_1 = T_s^*\xi.
\]

The group \( W \) has a decomposition into a disjoint sum of double cosets \( \langle s \rangle w \langle s \rangle \), which induces a decomposition of \( L^2(W) \) into a direct sum of subspaces invariant under both \( T_s \) and \( T_s^* \). Therefore, we just need to check whether equation (5.7) holds for the restriction of \( \xi \) to any such double coset.

Let \( \{ w, sw, ws, sws \} \) be a double coset of \( \langle s \rangle \). It has a unique shortest element, which we assume to be \( w \), and its cardinality is either 4 or 2. In the latter case \( sw = ws \) and \( sws = w \). A direct calculation shows that in both cases the restriction of \( \xi \) satisfies (5.7).

Now, we find the restrictions on the symbol of an operator in \( \mathcal{N}_q(W) \) commuting with two elements \( T_s \) and \( T_t \).

**Proposition 5.2.** Let \( (W,S) \) be a right-angled Coxeter system. Suppose that \( q \leq 1 \) and \( s,t \in S \) satisfy \( m(s,t) = \infty \). Consider an operator \( T(\xi) \in \mathcal{N}_q(W) \) which commutes with both \( T_s \) and \( T_t \). Then the restriction of its symbol \( \xi \) to any non-degenerate double coset \( DwD \) of \( D = \langle s,t \rangle \) with shortest element \( w \) is given by the formula

\[
\xi(dw^d) = \xi(w)q^{(|dw^d| - |w|)/2}.
\]

**Proof.** There are two kinds of non-degenerate double cosets of \( D \), those in which the shortest element does not commute with both generators of \( D \), and those where it commutes with exactly one of them. We will consider these two cases separately.
If \( w \) does not commute with both generators of \( D \), the mapping \( D \times D \to DwD \) given by \((d, d') \mapsto d^{-1}wd'\) is bijective. Otherwise we would have \( w = dwd' \) for some \((d, d') \neq (1, 1)\), which is impossible, as no cancellation may occur in \( dwd' \). This map induces the structure of the Cayley graph of \( D \times D \) on \( DwD \), whose fragment is presented in Fig. 1a. An important observation is that the path length of this graph faithfully reproduces word lengths of elements of \( DwD \), by which we mean that for \( v \in DwD \) its length is the sum of \(|w|\) and the distance between \( v \) and \( w \) in the graph. In particular, Fig. 1b shows quadruples of elements of \( DwD \) for which Lemma 5.1 gives relations on the values of \( \xi \). Every gray square corresponds to a single application of the Lemma to a quadruple \( v, vr, rv, rvr \) with \( v \in DwD \) and \( r \in \{s, t\} \). The arrow is pointing from the shortest element \( v \) to the longest one, \( rvr \), and corresponds to the relation

\[
(5.9) \quad \xi(rvr) = \xi(v) + p\xi(rv),
\]

while diagonal lines arising from the relations

\[
(5.10) \quad \xi(rv) = \xi(vr)
\]

are joining elements on which the values of \( \xi \) are equal.

Now, take some \( v_0 \in DwD \) in which an arrow starts, and define inductively \( v_{i+1} \) as the element at which the arrow starting in \( v_i \) ends. Also, let \( \lambda_i \) be the value of \( \xi \) on the elements joined by the line crossing the arrow from \( v_i \) to \( v_{i+1} \). This is illustrated in Fig. 2a. We have, by passing to infinity,

\[
(5.11) \quad \xi(v_0) = \xi(v_k) - p \sum_{i=0}^{k-1} \lambda_i = -p \sum_{i=0}^{\infty} \lambda_i.
\]

In particular, the value \( \xi(v_0) \) depends only on the \( \lambda_k \), which imposes new equalities, corresponding to adding to Fig. 1b diagonal lines as in Fig. 2b. It follows that...
\( \xi(\text{dwd'}) \) depends only on the distance between \( \text{dwd'} \) and \( w \) in the graph, i.e. on \( |\text{dwd'}| - |w| \).

Now, suppose that \( w \) commutes with one of the generators of \( D \), say \( t \). The vertices of the graph in Fig. 1a still correspond to elements of \( DwD \), but no longer in a one-to-one manner. However, with the exception of the quarter containing \( twt \) (the lower left quarter in Fig. 3), the length of elements of the double coset is faithfully reflected by the path length in the graph. Indeed, if a pair of generators in \( \text{dwd'} \) is canceled, one of them has to lie in \( \text{d} \), while the other in \( \text{d}' \). It follows that they must be equal to \( t \), and be adjacent to \( w \).
As in the previous case we obtain a system of relations for values of \( \xi \) on \( DwD \), represented in Fig. 3, only this time distinct vertices may correspond to the same element of \( DwD \), so there are additional equalities not indicated in the diagram. The same reasoning as before leads to the conclusion that \( \xi(dwd') \) depends only on \( |dwd'|-|w| \).

We have therefore shown that for a non-degenerate double coset \( DwD \), we have

\[
(5.12) \quad \xi(dwd') = f(|dwd'|-|w|)
\]

for some function \( f: \mathbb{N} \to \mathbb{C} \). By Lemma 5.1, it satisfies the recurrence relation

\[
(5.13) \quad f(n+2) = pf(n+1) + f(n).
\]

Its general solution is

\[
(5.14) \quad f(n) = \alpha q^{n/2} + \beta (-q)^{-n/2}.
\]

Since \( q \leq 1 \) and \( \xi \in \ell^2(W) \), we have \( \beta = 0 \), which ends the proof.

We are now ready to describe the center of the Hecke-von Neumann algebra of an irreducible right-angled Coxeter system with at least 3 generators. For 2 generators we get the group \( \mathbb{Z}_2 \rtimes \mathbb{Z}_2 \), and it suffices to apply Lemma 5.1 to describe the large center of its Hecke-von Neumann algebra.

**Theorem 5.3.** Suppose that \((W, S)\) is an irreducible right-angled Coxeter system with \( |S| \geq 3 \). Then the Hecke-von Neumann algebra \( \mathcal{N}_q(W) \) is a factor if and only if

\[
(5.15) \quad q \in [\rho, \rho^{-1}],
\]

where \( \rho \) is the convergence radius of \( W(t) \), the spherical growth series of \( W \). Moreover, for \( q \) outside this interval, \( \mathcal{N}_q(W) \) is a direct sum of a factor and \( \mathbb{C} \).

**Proof.** It is sufficient to consider the case where \( q \leq 1 \). Suppose that \( T(\xi) \in Z(\mathcal{N}_q(W)) \). If \( v, w \in W \) are connected by an edge in the graph \( \Gamma(W, S) \), they lie in the same non-degenerate double coset. Hence, by Proposition 5.2,

\[
(5.16) \quad \xi(w)q^{-|w|/2} = \xi(v)q^{-|v|/2}.
\]

By Proposition 3.4, all nontrivial elements of \( W \), except possibly one, lie in the same connected component \( K \) of \( \Gamma(W, S) \), and thus there exists \( \lambda \in \mathbb{C} \) such that for \( w \in K \)

\[
(5.17) \quad \xi(w) = \lambda q^{w/2}.
\]

If \( K \neq W \setminus \{1\} \), the additional element outside \( K \) is a generator \( s \in S \). Since \( |S| \geq 3 \) and \( (W, S) \) is irreducible, there is another generator \( t \in S \) not commuting with \( s \), and by Lemma 5.1,

\[
(5.18) \quad \xi(s) = \xi(tst) - p\xi(ts) = \lambda q^{3/2} - (q-1)q^{-1/2} \lambda q = \lambda q^{1/2},
\]

so (5.16) extends to \( W \setminus \{1\} \). It follows that \( \xi \) is a linear combination of \( \delta_1 \) and \( \xi \) defined by

\[
(5.19) \quad \xi(w) = q^{w/2}.
\]

But the \( \ell^2 \)-norm of \( \xi \) can be expressed in terms of the growth series of \( W \), namely

\[
(5.20) \quad ||\xi||_2^2 = \sum_{w \in W} q^{w/2} = W(q).
\]

This is finite only for \( q < \rho \), so for \( q \geq \rho \) the Hecke-von Neumann algebra \( \mathcal{N}_q \) is a factor.
To finish the proof we will show that for $q < \rho$, the vector $ζ$ is a symbol of an operator in $N_q(W)$ which is proportional to a one-dimensional projection. This has been done in [3, Lemma 19.2.5], and we will reproduce the argument. First, observe that

$$T_s^qζ = \sum_{w ∈ W} q^{|w|}/2T_s^qδ_w = \sum_{|w| < |w|} q^{|w|}/2T_s^q(δ_w + q^{1/2}δ_{ws})$$

$$= \sum_{|w| < |w|} q^{|w|}/2(δ_w) + q^{1/2}δ_{ws} + pq^{1/2}δ_{ws}$$

$$= \sum_{|w| < |w|} q^{|w|}/2δ_w + q^{|w|}/2(δ_{ws}) + q^{|w|}/2(δ_{ws}) + pq^{1/2}δ_{ws} = q^{1/2}ζ,$$

and therefore for finitely supported $ζ = \sum a_wδ_w ∈ ℓ^2(W)$ we obtain

$$∥T^q(ζ)∥_2 = \left|\sum a_wq^{|w|}/2ζ\right|_2 = \left|\sum a_wq^{|w|}/2\right| ≤ W(ζ) ≤ W(ζ)∥ζ∥_2.$$

This means that the map $ζ ↦ T^q(ζ)$ extends to a bounded operator $Q$ on $ℓ^2(W)$. Take $T^q(η) ∈ N_q^q = N_q^q$. For finitely supported $ζ ∈ ℓ^2(W)$ we have

$$T^q(η)Qζ = T^q(η)T^q(ζ)ζ = T^q(η)ζ = QT^q(η)ζ,$$

and thus $Q ∈ N_q^q = N_q^q$. But the symbol of $Q$ is $ζ$. It remains to observe that $T(ζ)$ is self-adjoint and by (5.20)

$$T(ζ)ζ = \sum_{w ∈ W} q^{|w|}/2T_w^qζ = \sum_{w ∈ W} q^{|w|}ζ = W(q)ζ,$$

so $W(q)^{−1}T(ζ)$ is an orthogonal projection onto the space spanned by $ζ$. □

6. Final remarks

Adam Skalski brought to our attention the fact that the results of this article are related to those of [5], where decompositions of free products of certain von Neumann algebras are studied.

Let $(M, φ)$ be a pair consisting of a von Neumann algebra $A$ and a faithful normal state $φ$. Furthermore, let $\{M_i\}$ be a family of unital subalgebras of $M$. A reduced word is then a product $x_1 \cdots x_k$ where $x_i ∈ M_{j(i)}$ and $j(i + 1) ≠ j(i)$ for all $i < k$. The family $\{M_i\}$ is free if any reduced word $x_1 \cdots x_k$ with $φ(x_i) = 0$ for all $i$ satisfies $φ(x_1 \cdots x_k) = 0$. Now, if $(M_i, φ_i)$ are von Neumann algebras endowed with faithful normal states, their free product is the unique von Neumann algebra $M = \bigstar(M_i, φ_i)$ with a faithful normal state $φ$ and embeddings $(M_i, φ_i) ↪ (M, φ)$ such that the embedded copies of $M_i$ are free and together generate $M$. For more details on free products consult [6].

The Hecke-von Neumann algebra $N_q(W)$ comes equipped with a faithful normal state $φ(T) = ⟨T δ_1, δ_1⟩$ and we will implicitly assume that all Hecke-von Neumann algebras are endowed with this natural state. Also, when considering commutative algebras $L^∞(Ω, µ)$ for a measure space $(Ω, µ)$, we assume that they are endowed with the integral with respect to $µ$, which is a state when $µ(Ω) = 1$.

**Lemma 6.1.** Suppose that $W$ decomposes into a free product of special subgroups $W_i$. Then $N_q(W)$ is the free product of the Hecke-von Neumann algebras $N_q(W_i)$ associated to the free factors $W_i$ of $W$. 

Proof. It is enough to consider a free product $W = U \ast V$. Denote the states on the corresponding Hecke-von Neumann algebras by $\phi_W, \phi_U$, and $\phi_V$. Clearly, $\mathcal{N}_q(W)$ is generated by embedded copies of $\mathcal{N}_q(U)$ and $\mathcal{N}_q(V)$, on which $\phi_W$ restricts to $\phi_U$ and $\phi_V$, respectively.

Now, a nontrivial reduced word $T = T_1 \cdots T_k$ in elements of ker $\phi_U$ and ker $\phi_V$ can be approximated by linear combinations of nontrivial reduced words of the form $T(w_1) \cdots T(w_k)$, where $w_i$ are nontrivial elements of $U$ and $V$. But such a word is equal to $T(w_1 \cdots w_k)$, where $w_1 \cdots w_k$ is a nontrivial reduced word in $W$, so $\phi_W(T) = 0$. \qed

In [5], among others, the free products of finite-dimensional commutative von Neumann algebras were studied. We may thus apply these results to Hecke-von Neumann algebras of right-angled Coxeter groups which decompose into free products of abelian groups $\mathbb{Z}^2$. Let us first study the Hecke-von Neumann algebras of these groups in more detail.

**Lemma 6.2.** There exists an isomorphism of the Hecke-von Neumann algebra $\mathcal{N}_q(\mathbb{Z}^2)$ and the commutative algebra $L^\infty(\mathbb{Z}^2_k, \mu_k)$, where

$$\mu_k(w) = \frac{q^{|w|}}{(q + 1)^k},$$

preserving their natural states.

**Proof.** We have $\mathcal{N}_q(\mathbb{Z}^2_2) \cong \mathcal{N}_q(\mathbb{Z}_2)^{\otimes k}$, so it is enough to consider the group $\mathbb{Z}_2$. Commutativity is clear, and $\dim \mathcal{N}_q(\mathbb{Z}_2) = 2$, so we just need to exhibit a decomposition of the unit of $\mathcal{N}_q(\mathbb{Z}_2)$ into two projections. A simple calculation shows that this decomposition is

$$1 = \left( \frac{\sqrt{q}}{q + 1} T_s + \frac{1}{q + 1} \right) + \left( -\frac{\sqrt{q}}{q + 1} T_s + \frac{q}{q + 1} \right),$$

and thus $\mathcal{N}_q(\mathbb{Z}_2)$ is isomorphic to $\mathbb{C}^2$ with the state

$$(z_1, z_2) \mapsto \frac{z_1 + q z_2}{1 + q},$$

i.e. to $L^\infty(\mathbb{Z}_2, \mu_1)$. Finally, we see that $L^\infty(\mathbb{Z}_2, \mu_1)^{\otimes k} \cong L^\infty(\mathbb{Z}^2_k, \mu_k)$. \qed

Now we will recall Theorem 2.3 and Proposition 2.4 of [5], reformulated to better suit our usage. For $r > 1$ by $L(F_r)$ we will denote the interpolated free group factor endowed with its standard state. Also, $L(F_1)$ will stand for the group von Neumann algebra $L(\mathbb{Z})$. Thus, we have $L(F_r) \ast L(F_s) = L(F_{r+s})$.

**Theorem 6.3 ([5]).** Let $X$ and $Y$ be finite (possibly empty) sets endowed with measures $\mu_X$ and $\mu_Y$ of full support, satisfying $\mu_X(X) \leq 1$ and $\mu_Y(Y) \leq 1$. Consider the von Neumann algebras $M$ and $N$ defined as follows

1. If $\mu_X(X) = \mu_Y(Y) = 1$, the cardinalities $|X|, |Y| \geq 2$, and $|X| + |Y| \geq 5$, we set $M = L^\infty(X, \mu_X)$ and $N = L^\infty(Y, \mu_Y)$,

2. If $\mu_X(X) < 1$ and $\mu_Y(Y) = 1$, we set $M = L(F_r) \oplus L^\infty(X, \mu_X)$ and $N = L^\infty(Y, \mu_Y)$,
for some \( r \geq 1 \), and endow \( M \) with the state
\[
\phi_M(x, f) = (1 - \mu_X(X))\phi_{L(F)}(x) + \int_X f \, d\mu_X.
\]

(3) If \( \mu_X(X) < 1 \) and \( \mu_Y(Y) < 1 \), we put
\[
M = L(F_r) \oplus L^\infty(X, \mu_X) \quad \text{and} \quad N = L(F_t) \oplus L^\infty(Y, \mu_Y)
\]
for some \( r, t \geq 1 \), endow \( M \) with the state \( \phi_M \) defined above, and \( N \) with the analogous state
\[
\phi_N(x, f) = (1 - \mu_Y(Y))\phi_{L(F)}(x) + \int_Y f \, d\mu_Y.
\]

With these definitions, we have
\[
(6.4) \quad M \ast N \cong L(F_s) \oplus L^\infty(Z, \nu),
\]
where \( s \geq 1 \),
\[
(6.5) \quad Z = \{(x, y) \in X \times Y : \mu_X(x) + \mu_Y(y) > 1\},
\]
and \( \nu(x, y) = \mu_X(x) + \mu_Y(y) - 1 \).

In particular, under the assumptions of Theorem 6.3, the free product \( M \ast N \) is a factor if and only if
\[
(6.6) \quad \max_{x \in X} \mu_X(x) + \max_{y \in Y} \mu_Y(y) \leq 1.
\]

Now, consider the irreducible right-angled Coxeter group
\[
(6.7) \quad W = \mathbb{Z}_2^{k_1} \ast \mathbb{Z}_2^{k_2} \ast \cdots \ast \mathbb{Z}_2^{k_t}
\]
with \( n \geq 2 \). If we assume that \( k_1 \geq 2 \), we may inductively apply Theorem 6.3 to obtain the description of \( \mathcal{N}_q(W) \) as a direct sum of an interpolated free group factor and a finite-dimensional commutative algebra:
\[
(6.8) \quad \mathcal{N}_q(W) \cong L(F_2) \oplus L^\infty(Z, \nu),
\]
where, using the notation of Lemma 6.2,
\[
(6.9) \quad Z = \left\{(w_1, \ldots, w_n) \in \mathbb{Z}_2^{k_1} : \sum_{i=1}^n \mu_{k_i}(w_i) > n - 1\right\}.
\]

Suppose that \( q \geq 1 \). We have
\[
(6.10) \quad \sum_{i=1}^n \mu_{k_i}(w_i) = \sum_{i=1}^n q^{|w_i| - k_i} \left( \frac{q}{q + 1} \right)^{k_i} \leq \frac{q}{q + 1} \sum_{i=1}^n q^{|w_i| - k_i},
\]
and if at least for one \( i \) the inequality \( |w_i| < k_i \) holds, this yields
\[
(6.11) \quad \sum_{i=1}^n \mu_{k_i}(w_i) \leq \frac{q}{q + 1} \left( n - 1 + \frac{1}{q} \right) = n - 1 - \frac{q - 2}{q + 1} \leq n - 1,
\]
since \( n \geq 2 \). Therefore, depending on whether
\[
(6.12) \quad \sum_{i=1}^n \left( \frac{q}{q + 1} \right)^{k_i} > n - 1,
\]
the set \( Z \) is either empty or contains exactly one element \((w_1, \ldots, w_n)\), where \( w_i \) is the unique longest element of \( \mathbb{Z}_2^{k_i} \). Calculating the exponential growth rate of \( W \) by
considering the spherical growth series of its free factors would lead to exactly the same condition. Hence, Theorem 6.3 allows to find the range of \( q \) for which \( \mathcal{N}_q(W) \) is a factor, additionally exhibiting it to be the interpolated free group factor—but only in the case where \( W \) is a free product of finite right-angled Coxeter groups. Our result works for arbitrary right-angled Coxeter groups, but gives less information about the structure of the obtained factors.

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