The Quantum Brachistochrone Problem can be formulated as follows:

*Given two quantum states \(|\psi_i\rangle\) and \(|\psi_f\rangle\), we want to find the (time-independent) Hamiltonian \(H\) that performs the transformation

\[
|\psi_i\rangle \rightarrow |\psi_f\rangle = e^{-i\frac{\hbar}{\omega} H \tau} |\psi_i\rangle
\]

in the minimal time \(\tau\), for a fixed value of the difference of eigenvalues of \(H\).

This problem was solved in [2] where it was shown that, if we put \(\omega = |E_+ - E_-|\), the minimal time to perform the required transformation is

\[
\tau = \frac{2\hbar}{\omega} \arccos |\langle \psi_i | \psi_f \rangle|.
\]

The same problem can be formulated allowing the use of pseudo Hermitian Hamiltonians [3] and its solution gives the apparently paradoxical possibility of operating a computational process in an arbitrarily small time and with limited energy costs [4].

In section III of [5] we showed that the analysis of the so-called transition problem, introduced in [6], induces the impossibility to realize a pseudo Hermitian dynamics of an Hermitian quantum system. This last observation implies that only simulations of pseudo Hermitian (PH) dynamics by means of open quantum dynamics are possible (note that in the finite dimensional case PT symmetric is equivalent to pseudo Hermiticity (see [7] and [8])).

In [1] the authors propose a physical apparatus, whose schematic representation is given in figure 1, that simulates a PT symmetric dynamics (see [1] for the explicit expression of the unitary operators \(V\), \(U_1\) and \(U_2\)). For our remarks, we need only to note that the apparatus realizes a unitary transformation between the initial and final states (see figure 2).

This apparatus apparently contradicts our analysis in section IV of [5], where we showed that a simulation of a pseudo Hermitian dynamics by an Hermitian open dynamics inevitably introduces dissipative effects. In fact the subdynamics on the work qubit \(e\), obtained tracing over the auxiliary space \(a\), in the apparatus represented in figure 1 is a trace preserving completely positive map. But using Proposition IV.1 of [5], this last observation has as immediate consequence that the subdynamics is a pseudo Hermitian evolution only on a restricted subset of states. Indeed this simulation is only partial and this contradiction is solved.

More explicitly, we note that the subdynamics proposed in [1] is not deterministic and so a non zero probability that the desired evolution is not realized exists. In particular we observe that

\[
U |0_e\rangle \otimes |0_a\rangle = a e^{-\frac{i}{\gamma} H T} |0_e\rangle \otimes |0_a\rangle + \beta |\psi_e\rangle \otimes |1_a\rangle
\]

where \(\gamma = \|e^{-\frac{i}{\gamma} H T} |0_e\rangle\|\) and \(|\beta|^2\) is the probability to obtain the wrong transformation.

Moreover, equation (2) can be used to establish the connection between the physical quantities involved in the transformation:

\[
T \geq \frac{2\hbar}{\omega} \arccos |\alpha(\langle 0_e | e^{-\frac{i}{\gamma} H T} |0_e\rangle)|.
\]

where \(T\) is the time needed to operate the transformation. If after a time \(T\) we obtain that

\[
\frac{e^{-\frac{i}{\gamma} H T}}{\gamma} |0_e\rangle = |1_e\rangle
\]

we have that the faster evolution is due to an increment of \(\omega\), and not to the pseudo Hermitian nature of \(H\). Moreover because \(|\alpha|\) is necessarily < 1 we obtain that this
simulation is not efficient with respect to a purely closed Hermitian dynamics. This means that we can realize a pseudo Hermitian evolution, but, according to the analysis in sections IV–V of [5], this simulation has an energy and/or efficiency cost.

FIG. 2. A more general apparatus.

[1] C. Zheng, L. Hao, and G. L. Long, ArXiv e-prints (2011), arXiv:1105.6157 [quant-ph].
[2] A. Carlini, A. Hosoya, T. Koike, and Y. Okudaira, Phys. Rev. Lett. 96, 060503 (2006).
[3] C. M. Bender, D. C. Brody, H. F. Jones, and B. K. Meister, Phys. Rev. Lett. 98, 040403 (2007).
[4] A. Mostafazadeh, Phys. Rev. Lett. 99, 130502 (2007).
[5] F. Masillo, ArXiv e-prints (2011), arXiv:1105.3332 [quant-ph].
[6] A. Mostafazadeh, Phys. Lett. B 650, 208 (2007), arXiv:0706.1872 [quant-ph].
[7] G. Scolarici and L. Solombrino, J. Math. Phys. 44, 4450 (2003).
[8] A. Mostafazadeh, J. Math. Phys. 43, 3944 (2002).