Verification of recursive parallel systems.

L. Bozzelli\textsuperscript{1} M. Benerecetti\textsuperscript{2} and A. Peron\textsuperscript{2}

\textsuperscript{1}Dept. of Mathematics and Applications
Università di Napoli “Federico II”
Napoli, Italy
laura.bozzelli@dma.unina.it

\textsuperscript{2}Dept. of Physical Sciences
Università di Napoli “Federico II”
Napoli, Italy
\{bene,peron\}@na.infn.it

Abstract

In this paper we consider the problem of proving properties of infinite behaviour of formalisms suitable to describe (infinite state) systems with recursion and parallelism. As a formal setting, we consider the framework of Process Rewriting Systems (PRSs). For a meaningful fragment of PRSs, allowing to accommodate both Pushdown Automata and Petri Nets, we state decidability results for a class of properties about infinite derivations (infinite term rewritings). The given results can be exploited for the automatic verification of some classes of linear time properties of infinite state systems described by PRSs. In order to exemplify the assessed results, we introduce a meaningful automaton based formalism which allows to express both recursion and multi–treading.

1 Introduction

Automatic verification of systems is nowadays one of the most investigated topic. A major difficulty to face when considering this problem comes to the fact that, reasoning about systems in general may require to deal with infinite state models. For instance, software systems may introduce infinite states both manipulating data ranging over infinite domains and having unbounded control structures such as recursive procedure calls and/or dynamic creation of concurrent processes (e.g. multi–treading). Many different formalisms have been proposed for the description of infinite state systems. Among the most popular are the well known formalisms of Context Free Process, Pushdown Processes, Petri Nets, and Process Algebras. The first two are models of sequential computation, whereas Petri Nets and Process Algebra explicitly take into account concurrency. The model checking problem for these infinite state formalisms have been studied in the literature. As far as context free processes and Pushdown Automata are concerned, decidability of the modal

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\(\mu\)-calculus, the most powerful of the modal and temporal logics used for verification, has been established (e.g. see [13]). As far as models of concurrency, certain linear time properties are already undecidable for small classes of Petri Nets (see [10], for a systematic picture of decidability issues about model checking infinite states concurrent models for both linear and branching time logics).

In order to exemplify our decidability results for formalisms involving recursion and parallelisms, in this paper we shall consider automata–based formalisms enriched with hierarchy (procedure call) concurrency and communication among concurrent components, which have gained popularity in recent years (e.g. see Statecharts [12], ROOM [20], and UML [4]). Pure formalisms underlying the above mentioned specification languages have been recently proposed and studied. In [2] Communicating Hierarchical Automata (CHAs) have been introduced which extend Finite State Machines (FSMs) with hierarchy and concurrency (though remaining in the finite–state case). In [3], the finite state formalism CHAs has been strengthened, precisely to capture the expressive power of Pushdown Automata (Recursive State Machine). Efficient algorithms for model checking and reachability analysis have been studied for Pushdown Automata (e.g. see [5] [11]), for CHAs [1] and for Recursive State Machines [3]. In [14], an extension of both CHAs and RSMs, called Dynamical Hierarchical Machines, is presented, which allow to model code mobility via communication and dynamic activation of state-transition machines. This formalism turns out to be Turing equivalent, since unbounded hierarchy (i.e. recursive call) together with the unrestricted ability of dynamically activating parallel components at any hierarchical level allows for easily simulating a double Pushdown Automaton. In this paper we shall consider a restricted version of Dynamic Hierarchical Machines, where recursion and dynamic activation of parallel components (multi–treads) is allowed, while communication among parallel components is restricted in such a way to prevent Turing equivalence.

Verification of formalisms which accommodate both parallelism and recursion is a challenging problem. To formally study this kind of systems, recently the formal framework of Process Rewrite Systems (PRSs) has been introduced [17]. This framework, which based on term rewriting, subsumes many common infinite states models such us Pushdown Systems, Petri Nets, Process Algebra, etc. As we shall see, also restricted Dynamic Hierarchical Automata can be easily encoded into PRSs. The decidability results already known in the litterature for the general framework of PRSs concern reachability, i.e. properties of finite sequences of term rewriting (derivations).

In this paper we extend the known decidability results for a relevant syntactic fragment of PRSs to properties of infinite derivations, thus allowing for automatic verification of some classes of linear time properties. Since this result is obtained within the general formalism of the considered fragment of PRSs, it applies not only to the specific context of restricted Dynamic Hierarchical Automata, but, more significantly, also to any specification formalism which can be accommodated within this fragment. The fragment we consider is that of PRSs in normal form, where every rewrite rule either only deal with procedure calls (this kind of rules allows to capture Pushdown Processes), or only deal with dynamic activation of processes and synchronization (this kind of rules allows to capture Petri Nets). A PRS in normal form is extended with a notion of acceptance a la Büchi. A subset of
rewrite rules is labelled as 'accepting' and an infinite derivation is accepting if there is an accepting rewriting rule which is applied infinitely often in that derivation. We prove that it is effectively decidable the problem whether, for a given set of rewriting rules, there is an infinite accepting derivation. We prove also that it is decidable whether there is an infinite derivation devoid of any application of an accepting rewrite rule, and whether there is an infinite derivation involving a positive (finite) number of applications of accepting rewriting rules.

The rest of the paper is structured as follows. In Section 2, we introduce the formalism of Dynamic Hierarchical Automata. In Section 3, we recall the framework of Process Rewriting Systems, we summarize some decidability results for reachability problems in the context of PRSs, and show how DHAs can be embedded in PRS. In Section 4, it is shown how decidability results about infinite derivations can be used to check properties about infinite executions of infinite state systems modelled by PRSs. In Section 5, we prove decidability of the three problems about infinite derivations in PRSs in normal form, mentioned above.

2 Dynamic Hierarchical Automata

In \[2\] Communicating Hierarchical Automata (CHAs) have been introduced which extend Finite State Machines (FSMs) with hierarchy, concurrency and communication among concurrent components. A CHA is a collection of Finite State Machines (FSMs). Hierarchy is achieved by injecting FSMs into states of other FSMs. Whenever a FSM state \(s\) is entered, if such a state contains a FSM \(M\), then \(M\) starts running. The state \(s\) can be left when \(M\) reaches a final state. From this perspective, entering the state \(s\) can be viewed as a procedure call, with \(M\) acting the part of the procedure. Finiteness of states of CHAs is guaranteed by syntactically forbidding recursive injection of FSMs into states. Concurrency is achieved by composing FSMs in parallel and by letting them run contemporaneously. Concurrent machines communicate by synchronizing on transitions with the same input label. The form of communication is a form of global synchronization: if a parallel component performs a transition labelled by an input symbol \(a\), all of the other components having \(a\) in their input alphabet must perform a transition labelled by \(a\). In \[3\], the finite state formalism CHAs has been strengthened, from the expressive power viewpoint, in such a way that the expressive power of Pushdown Automata is precisely reached. Such a formalism is called Recursive State Machine (RSM, for short). The additional expressive power is obtained by allowing recursive injection of FSMs into states (i.e. by admitting a form of recursive ‘procedure call’). Actually, with respect to CHAs, RSMs does not allow explicit representation of parallelism.

In \[14\] an extension of both CHAs and RSMs is presented called Dynamical Hierarchical Machines (DHMs). DHMs allow the explicit representation of hierarchy, parallelism and communication. Moreover, in order to model aspects of code mobility the form of communication between parallel components allows sending and receiving FSMs (communication in CHAs takes the form of pure communication). A FSM received from a parallel
component can be dynamically activated in different ways: either in parallel with the receiving component or in parallel with a component at a lower/upper level (with respect to the receiving component). Hierarchy and dynamical activation allows to easily simulate a Pushdown Automaton. Moreover, parallelism and the unrestricted ability of dynamically activating parallel components at any hierarchical level allows to easily simulate a double Pushdown Automaton thus reaching the expressive power of Turing Machines.

In this section we introduce an extension of RSMs, called Restricted Dynamical Hierarchical Automata (RDHAs for short), which allows recursive injection of FSMs into states (as in RSMs) and dynamical parallel activations of FSMs (as in DHMs). As in CHAs and RSMs, a RDHA is a collection of FSMs, and FSMs can be (recursively) injected into states. A transition of a FSM is decorated by a pair of symbols, the former being an input symbol (belonging to an alphabet $\Upsilon$), the latter representing an action. The possible actions are: $NIL$, representing the null action; $HALT$, representing the termination action; a channel symbol in $\Gamma$, representing a synchronization request on a channel name; $NEW(i,p)$, representing the dynamic activation of the $i$-th FSM in its initial state $p$. A transition is triggered by the input symbol and, when performed, produces the corresponding action.

In order to obtain a formalism which is less expressive than DHMs, parallelism communication and dynamic activation are presented in a restricted form. Actually, there is no explicit syntactical construct for parallelism in RDHAs. Parallelism is the result of dynamic activation of sequential machines. During its evolution, a FSM can dynamically activate in parallel with itself a (possibly unbound) number of FSMs. When a FSM $A$ activates another FSM $A'$, $A$ and $A'$ are put in parallel at the same hierarchical level (the father of $A'$ is the father of $A$).

Synchronization has the form of handshaking between two parallel components. A transition in a FSM labelled by a synchronization request on a channel $\alpha$ can be performed only if there is a FSM, activated in parallel with it, able to perform a transition with a synchronization request on the same channel name. Synchronization between parallel components of different hierarchical level (i.e. interlevel communication) is not allowed. Moreover, synchronization is allowed only if the involved parallel components are not waiting for return of a procedure call (i.e. if they are leaves in the hierarchy of activations).

A FSM $A'$ injected into a box $b$ of a FSM $A$ is disactivated either when it is in a final state associated with a transition departing from $b$ (a kind of procedure termination with value return) or when it performs the $HALT$ action (procedure termination without value return).

**Definition 2.1.** Let $\Upsilon$ and $\Gamma$ be finite alphabets for input symbols and channels, respectively. A Restricted Dynamic Hierarchical Automaton (RDHA for short) over $\Upsilon$ is a collection of sequential machines $A_1, \ldots, A_n$, with

$$A_i = \langle Q_i \cup B_i, Y_i, Q_i^0, Q_i^T, \delta_i \rangle,$$

where

- $Q_i$ is the finite set of nodes;
- $B_i$ is the finite set of boxes (we assume $B_i \cap Q_i = \emptyset$);
• $Q_i^0 \subseteq Q_i$ is the set of initial nodes;
• $Q_i^T \subseteq Q_i$ is the set of exit nodes;
• $Y_i : B_i \to \{1, \ldots, n\}$ is the hierarchy function associating boxes with sequential machines $A_1, \ldots, A_n$;
• $\delta_i \subseteq ((Q_i \setminus Q_i^T) \cup (B_i \times \bigcup_{j=1}^n Q_j)) \times \Gamma \times Sy \times (Q_i \cup (B_i \times \bigcup_{j=1}^n Q_j))$ is the transition relation, with

$$Sy = \{\text{NIL, HALT}\} \cup \Gamma \cup \{\text{NEW}(i, p) \mid i = 1, \ldots, n \text{ and } p \in Q_i^0\};$$

for any $\langle u, a, \xi, v \rangle \in \delta_i$, the following constraints are fulfilled:

- if $\xi \not\in \{\text{NIL, HALT}\}$, then $u, v \in Q_i$;
- if $\xi = \text{HALT}$, then $v \in Q_i^T$;
- if $u \not\in Q_i$, then $v \in Q_i$;
- if $v \not\in Q_i$, then $u \in Q_i$;
- if $u$ has the form $\langle b, q \rangle$, then $q \in Q_j^T$, with $j = Y_i(b)$;
- if $v$ has the form $\langle b, q \rangle$, then $q \in Q_i^0$, with $j = Y_i(b)$.

As a further constraint we require that $Q_i \cap Q_j = \emptyset$ and $B_i \cap B_j = \emptyset$ for all $1 \leq i < j \leq n$.

When an activation of a FSM $A_i$ in a state $u$ performs a transition $\langle u, a, \text{NIL}, \langle b, q \rangle \rangle$, it enters the box $b$ and activates an instance of the FSM $A_{Y_i(b)}$ (a procedure call) in its initial node $q$, and waits for the termination of $A_{Y_i(b)}$. The FSM $A_{Y_i(b)}$ terminates when it reaches an exit node $q_i$ and can be deallocated if either $q_i$ is reached by a transition of the form $\langle u', a', \text{HALT}, q_i \rangle$ (termination without value return), or if $A_i$ has a transition of the form $\langle \langle b, q_i \rangle, a', \text{NIL}, u' \rangle$ (termination with value return).

When an activation of a FSM $A_i$ in a state $u$ performs a transition $\langle u, a, \text{NEW}(j, q), v \rangle$, it enters the node $v$ and activates in parallel with itself an instance of $A_j$ in its initial node $q$ (a dynamic activation). The two activations of $A_i$ and $A_j$ can run asynchronously in parallel.

An activation of a FSM $A_i$ in a state $u$ can perform a transition $t_1 = \langle u, a, \gamma, v \rangle$, with $\gamma$ a channel name in $\Gamma$, only if there is a parallel activation of a FSM $A_j$ in a state $u'$ which can perform a transition of the form $t_2 = \langle u', a, \gamma, v' \rangle$ (synchronization on the same channel name and the same input symbol). Both $A_i$ and $A_j$ have to perform transitions $t_1$ and $t_2$, simultaneously.

Notice that, as a consequence of constraints imposed on the transition relations, actions of synchronization and dynamic creation can be performed only starting from nodes (i.e. leaves in the hierarchy of FSM activations).

In order to give a formal semantics of RDHAs we have to introduce the notion of configuration. A configuration is a tree which describes the collection of instances of
FSMs instantaneously activated, together with the hierarchy of activations (caller - called relationship). In particular, a node of the configuration tree is either a box (non leaf node of the tree) or a node (leaf node of the tree) of the considered RDHA, representing the current state of a FSM activation instance. A configuration tree is described by an algebra of terms composed of node and box symbols by means of a binary operation of sequential composition (i.e. procedure call) denoted by \((\_)\) and a binary operation of parallel composition denoted by \(_\|\_\). (We recall that a NEW action results in a parallel composition.)

**Definition 2.2.** Let \( \mathcal{A} = \{A_1, \ldots, A_n\} \), be a RDHA with \( A_i(Q_i \cup B_i, Y_i, Q_i^0, Q_i^T, \delta_i) \). The set of configuration terms of \( \mathcal{A} \), written \( Conf(\mathcal{A}) \), is inductively defined as follows:

- \( \varepsilon \in Conf(\mathcal{A}) \) (the empty configuration);
- \( \bigcup_{i=1}^{n} Q_i \subseteq Conf(\mathcal{A}) \);
- \( b.(t) \in Conf(\mathcal{A}) \), for \( b \in \bigcup_{i=1}^{n} B_i \) and \( t \in Conf(\mathcal{A}) \);
- \( t_1 \parallel t_2 \in Conf(\mathcal{A}) \), for \( t_1, t_2 \in Conf(\mathcal{A}) \).

In the following we shall consider equal configuration terms of \( Conf(\mathcal{A}) \) up to commutativity and associativity of parallel composition. Moreover, the configuration term \( \varepsilon \) will be treated as the identity element for parallel and sequential composition. More precisely, for a RDHA \( \mathcal{A} \), configuration terms of \( \mathcal{A} \) are considered equal up to a notion of equivalence \( \approx_{\mathcal{A}} \) defined as the least equivalence fulfilling the following requirements:

- \( t_1 \parallel t_2 \approx_{\mathcal{A}} t_2 \parallel t_1 \), for all \( t_1, t_2 \in Conf(\mathcal{A}) \);
- \( t_1 \parallel (t_2 \parallel t_3) \approx_{\mathcal{A}} (t_1 \parallel t_2) \parallel t_3 \), for all \( t_1, t_2, t_3 \in Conf(\mathcal{A}) \);
- \( t \parallel \varepsilon \approx_{\mathcal{A}} t \), for any \( t \in Conf(\mathcal{A}) \);
- \( b.(\varepsilon) \approx_{\mathcal{A}} b; \)
- \( b.(t_1) \approx_{\mathcal{A}} b.(t_2) \), for all \( t_1, t_2 \in Conf(\mathcal{A}) \) such that \( t_1 \approx_{\Phi} t_2 \).

In the following we shall define the semantics of RDHAs in terms of Labelled Transition Systems (LTSSs) defined in the well known style of Structured Operational Semantics (see [19]). The LTS for a RDHA \( \mathcal{A} \) is the triple \( \langle Conf_{\mathcal{A}}, \Upsilon, \xrightarrow{\cdot} \rangle \), where the states are the configuration terms, the set of labels is \( \Upsilon \) and the transition relation \( \xrightarrow{\cdot} \subseteq Conf_{\mathcal{A}} \times \Upsilon \times Conf_{\mathcal{A}} \) is defined by the following set of rules. (In the following \( \delta \) stands for \( \bigcup_{i=1}^{n} \delta_i \).)
\[ q \rightarrow q' \quad \langle q, a, \text{NIL}, q' \rangle \in \delta \]  
\[ q \rightarrow b.(q') \quad \langle q, a, \text{NIL}, \langle b, q' \rangle \rangle \in \delta \]  
\[ b.(q) \rightarrow q' \quad \langle \langle b, q \rangle, a, \text{NIL}, q' \rangle \rangle \in \delta \]  
\[ q \rightarrow \varepsilon \quad \langle q, a, \text{HALT}, q' \rangle \in \delta \]  
\[ q \rightarrow q' \parallel p \quad \langle q, a, \text{NEW}(j, p), q' \rangle \in \delta \]  
\[ q_1 \parallel q_2 \rightarrow q'_1 \parallel q'_2 \quad \langle q_1, a, \alpha, q'_1 \rangle, \langle q_2, a, \alpha, q'_2 \rangle \in \delta \]  

Axiom 1 considers the case of a basic transition inside a FSM. Axiom 2 considers the case of a procedure call: the FSM \( A_{Y_i(b)} \), with \( b \in B_i \), is activated in its initial state \( q' \). Axiom 3 (resp. 4) considers the case of a procedure termination with (resp. without) value return. Axiom 5 considers the case of a transition with dynamic activation; notice that the newly activated FSM is put in parallel with the activating one. Rule 6 deals with synchronization: two requests of synchronization on a common channel name are synchronized. Rule 7 allows a parallel component, which does not perform synchronization requests, to freely (asynchronously) evolve. Rule 8 allows a machine, which does not perform synchronization requests, to freely evolve in the context of procedural call.

3 Process Rewriting Systems and RDHAs

In this section we recall the framework of Process Rewriting Systems (PRSs). In this setting we rephrase the LTS semantics of RDHAs given in the previous section. We conclude the section by summarizing some decidability results, known in the literature, for the problem of model checking of systems described by PRSs.

3.1 Process Rewrite Systems

In this section we recall the notion of Process Rewrite System, as introduced in [17]. The idea is that a process (and its current state) is described by a term. The behaviour of a
process is given by rewriting the corresponding term by means of a finite set of rewriting rules.

**Definition 3.1 (Process Term).** Let \( \text{Var} \) be a finite set of process variables. The set \( T \) of process terms over \( \text{Var} \) is inductively defined as follows:

- \( \text{Var} \subseteq T \);
- \( \varepsilon \in T \);
- \( t_1 \parallel t_2 \in T \), for all \( t_1, t_2 \in T \);
- \( X.(t) \in T \), for all \( X \in \text{Var} \) and \( t \in T \),

where \( \varepsilon \) denotes the empty term, \( \parallel \) denotes parallel composition, and \( .() \) denotes sequential composition\(^1\).

We denote with \( T_{\text{SEQ}} \) the subset of terms in \( T \) devoid of any occurrence of parallel composition operator, and with \( T_{\text{PAR}} \) the subset of terms in \( T \) devoid of any occurrence of the sequential composition operator. Notice that we have \( T_{\text{PAR}} \cap T_{\text{SEQ}} = \text{Var} \cup \{ \varepsilon \} \).

In the rest of the paper we only consider process terms modulo commutativity and associativity of \( \parallel \), moreover \( \varepsilon \) will act as the identity for both parallel and sequential composition. Therefore, we introduce the relation \( \approx_T \), which is the smallest equivalence relation on \( T \) such that for all \( t_1, t_2, t_3 \in T \) and \( X \in \text{Var} \):

- \( t_1 \parallel t_2 \approx_T t_2 \parallel t_1 \), \( t_1 \parallel (t_2 \parallel t_3) \approx_T (t_1 \parallel t_2) \parallel t_3 \), and \( t_1 \parallel \varepsilon \approx_T t_1 \);
- \( X.(\varepsilon) \approx_T X \), and if \( t_1 \approx_T t_2 \), then \( X.(t_1) \approx_T X.(t_2) \).

In the paper, we always confuse terms and their equivalence classes (w.r.t. \( \approx_T \)). In particular, \( t_1 = t_2 \) (resp., \( t_1 \neq t_2 \)) will be used to mean that \( t_1 \) is equivalent (resp., not equivalent) to \( t_2 \).

**Definition 3.1 (Process Rewrite System).** A Process Rewrite System (or PRS, or Rewrite System) over the alphabet \( \Sigma \) and the set of process variables \( \text{Var} \) is a finite set of rewrite rules \( \mathcal{R} \subseteq T \times \Sigma \times T \) of the form \( t \overset{a}{\rightarrow} t' \), where \( t \) (\( \neq \varepsilon \)) and \( t' \) are terms in \( T \), and \( a \in \Sigma \).

The semantics of a PRS \( \mathcal{R} \) is given by a Labelled Transition System \( \langle T, \Sigma, \rightarrow \rangle \), where the set of states is the set of terms \( T \) of \( \mathcal{R} \), the set of actions is the alphabet \( \Sigma \) of \( \mathcal{R} \), and the transition relation \( \rightarrow \subseteq T \times \Sigma \times T \) is the smallest relation satisfying the following inference rules:

\[
\frac{t \overset{a}{\rightarrow} t'}{t \overset{a}{\rightarrow} t'} \quad \frac{t_1 \overset{a}{\rightarrow} t_1'}{t_1 \parallel t \overset{a}{\rightarrow} t_1' \parallel t} \quad \frac{t_1 \overset{a}{\rightarrow} t_1'}{X.(t_1) \overset{a}{\rightarrow} X.(t_1')} \forall X \in \text{Var}
\]

\(^1\)\[17\] also allows terms of the form \( t_1.(t_2) \), where \( t_1 \) is a parallel composition of variables. In the current context this generalization is not relevant.
For a PRS $\mathcal{R}$ with set of terms $T$ and LTS $\langle T, \Sigma, \rightarrow \rangle$, a path in $\mathcal{R}$ from $t \in T$ is a path in $\langle T, \Sigma, \rightarrow \rangle$ from $t$, i.e. a (finite or infinite) sequence of LTS edges $t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} t_2 \xrightarrow{a_2} \ldots$ such that $t_0 = t$ and $t_j \xrightarrow{a_j} t_{j+1} \in \rightarrow$ for any $j$. A run in $\mathcal{R}$ from $t$ is a maximal path from $t$, i.e. a path from $t$ which is either infinite or has the form $t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} \ldots \xrightarrow{a_{n-1}} t_n$ and there is no edge $t_n \xrightarrow{a_n} t' \in \rightarrow$, for any $a_n \in \Sigma$ and $t' \in T$. We write $\text{runs}_\mathcal{R}(t)$ (resp., $\text{runs}_\mathcal{R,\infty}(t)$) to refer to the set of runs (resp., infinite runs) in $\mathcal{R}$ from $t$, and $\text{runs}(\mathcal{R})$ to refer to the set of all the runs in $\mathcal{R}$.

The LTS semantics induces, for a rule $r \in \mathcal{R}$, the following notion of one-step derivation by $r$. The one-step derivation by $r$ relation, $\xrightarrow{\sigma}_r$, is the least relation such that:

- $t \xrightarrow{\sigma}_r t'$, for $r = t \xrightarrow{a} t'$
- $t_1 \parallel t \xrightarrow{\sigma}_r t_2 \parallel t$, if $t_1 \xrightarrow{\sigma}_r t_2$ and $t \in T$
- $X.(t_1) \xrightarrow{\sigma}_r X.(t_2)$, if $t_1 \xrightarrow{\sigma}_r t_2$ and $X \in \text{Var}$

A finite derivation in $\mathcal{R}$ from a term $t$ to a term $t'$ (through a finite sequence $\sigma = r_1 r_2 \ldots r_n$ of rules in $\mathcal{R}$), is a sequence $d$ of one-step derivations $t_0 \xrightarrow{r_1}_r t_1 \xrightarrow{r_2}_r t_2 \xrightarrow{r_3}_r \ldots \xrightarrow{r_{n-1}}_r t_n$, with $t_0 = t$, $t_n = t'$ and $t_i \xrightarrow{r_{i+1}}_r t_{i+1}$ for all $i = 0, \ldots, n - 1$. The derivation $d$ is a $n$-step derivation (or a derivation of length $n$), and for succinctness is denoted by $t \xrightarrow{\sigma}_r^* t'$. Moreover, we say that $t'$ is reachable in $\mathcal{R}$ from term $t$ (through derivation $d$). If $\sigma$ is empty, we say that $d$ is a null derivation.

A infinite derivation in $\mathcal{R}$ from a term $t$ (through an infinite sequence $\sigma = r_1 r_2 \ldots$ of rules in $\mathcal{R}$), is an infinite sequence of one step derivations $t_0 \xrightarrow{r_1}_r t_1 \xrightarrow{r_2}_r t_2 \ldots$ such that $t_0 = t$ and $t_i \xrightarrow{r_{i+1}}_r t_{i+1}$ for all $i \geq 0$. For succinctness such derivation is denoted by $t \xrightarrow{\sigma}_r^\infty$. Notice that there is a strict correspondence between the notion of derivation from a term $t$ and that of path from the term $t$. In fact, we have that there is a path $t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} t_2 \ldots$ from $t_0$ in $\mathcal{R}$ iff there exists a derivation $t_0 \xrightarrow{r_1}_r t_1 \xrightarrow{r_2}_r t_2 \ldots$ from $t_0$ in $\mathcal{R}$, with $a_i = \text{label}(r_i)$, for any $i$ (where for a rule $r \in \mathcal{R}$ with $r = t \xrightarrow{a} t'$, $\text{label}(r)$ denotes the label $a$ of $r$).

We rephrase now the semantics of RDHAs in the setting of PRSs. Let $\mathcal{A} = \{ A_1, \ldots, A_n \}$, be a RDHA with $A_i = \langle Q_i \cup B_i, Y_i, Q^0_i, Q^T_i, \delta_i \rangle$. The PRS for $\mathcal{A}$, written $\text{PRS}_\mathcal{A}$, is given as follows:

1. $\text{Var} = \{ X_q : q \in \bigcup_{i=1}^{n} Q_i \} \cup \{ X_b : b \in \bigcup_{i=1}^{n} B_i \}$ is the set of variables indexed over the set of nodes and boxes;
2. the alphabet $\Sigma$ equals the alphabet $\Upsilon$ of $\mathcal{A}$
3. the set of rules $\mathcal{R}$ is the union of the following sets:

   (a) $\{ X_q \xrightarrow{a} X_q' : \langle q, a, \text{NIL}, q' \rangle \in \bigcup_{i=1}^{n} \delta_i \}$

   (b) $\{ X_q \xrightarrow{\varepsilon} : \langle q, a, \text{HALT}, q' \rangle \in \bigcup_{i=1}^{n} \delta_i \}$
(c) \( \{X_q \xrightarrow{a} X_b.(X_p) : \langle q, a, NIL, \langle b, p \rangle \rangle \in \bigcup_{i=1}^{n} \delta_i \} \)

(d) \( \{X_b.(X_p) \xrightarrow{a} X_q : \langle \langle b, p \rangle, a, NIL, q \rangle \rangle \in \bigcup_{i=1}^{n} \delta_i \} \)

(e) \( \{X_{q_1} \parallel X_{q_2} \xrightarrow{a} X_{q'_1} \parallel X_{q'_2} : \langle q_1, a, \gamma, q'_1 \rangle, \langle q_2, a, \gamma, q'_2 \rangle \rangle \in \bigcup_{i=1}^{n} \delta_i \} \)

It is easy to show that the given translation of a RDHA \( \mathcal{A} \) into PRS\( \mathcal{A} \), is correct in the sense that the LTS for a \( \mathcal{A} \) and the LTS for PRS\( \mathcal{A} \) are isomorphic.

The embedding of RDHAs into PRSs suggests an immediate interpretation of PRS format rules. Rules involving sequential composition allow one to model procedure call and termination: in particular a rule of the form \( X \xrightarrow{a} Y.\langle t \rangle \) allows to model procedure call, and a rule of the form \( Y.\langle t \rangle \xrightarrow{a} Z \) allows to model procedure termination (possibly with value return if \( t \neq \varepsilon \)). Rules involving parallel composition allow to model dynamic process activation and synchronization among parallel process: the former can be expressed by rules having the form \( t_1 \xrightarrow{a} t_1 \parallel t_2 \), whereas the latter by rules having the form \( t_1 \parallel t_2 \xrightarrow{a} t'_1 \parallel t'_2 \).

In the following, we shall consider PRS in a syntactical restricted form called normal form.

**Definition 3.2 (Normal Form).** A PRS \( \mathcal{R} \) is said to be in normal form if every rule \( r \in \mathcal{R} \) has one of the following forms:

**PAR rules:** Any rule devoid of sequential composition;

**SEQ rules:** \( X \xrightarrow{a} Y.\langle Z \rangle, X.\langle Y \rangle \xrightarrow{a} Z \) or \( X \xrightarrow{a} Y \), or \( X \xrightarrow{a} \varepsilon \).

with \( X, Y, Z \in \text{Var} \). A PRS where all the rules are SEQ rules is called sequential PRS. Similarly, a PRS where all the rules are PAR rules is called parallel PRS.

With reference to our embedding of RDHA into PRS, notice that the PRS\( \mathcal{A} \) for a RDHA \( \mathcal{A} \) is in normal form and consists of both sequential and parallel rules.

The sequential and parallel fragments of PRS are significant: in [14] it is shown that sequential PRSs are semantically equivalent (via bisimulation equivalence) to Pushdown Automata (PDA), while parallel PRSs are semantically equivalent to Petri Nets (PN). Moreover, from the fact that Pushdown systems and Petri Nets are not comparable (see [13, 7]) it follows that PRSs in normal form are strictly more expressive than both their sequential and parallel fragment.

### 3.2 Decidability results for PRSs

In this section we will summarize decidability results on PRSs which are known in the literature and which will be exploited in further sections of the paper.

**Verification of ALTL (Action–based LTL)**

Given a finite set \( \Sigma \) of atomic propositions, the set of formulae \( \varphi \) of ALTL over \( \Sigma \) is defined as follows:

\[
\varphi ::= \text{true} \mid \neg \varphi \mid \varphi_1 \land \varphi_2 \mid \langle a \rangle \varphi \mid \varphi_1 U \varphi_2 \mid G \varphi \mid F \varphi
\]
where \(a \in \Sigma\).

In order to give semantics to ALTL formulæ on a PRS \(R\), we need some additional notation. Given a path \(\pi = t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} t_2 \xrightarrow{a_2} \ldots\) in \(R\), \(\pi^i\) denotes the suffix of \(\pi\) starting from the \(i\)-th term in the sequence, i.e. the path \(t_i \xrightarrow{a_i} t_{i+1} \xrightarrow{a_{i+1}} \ldots\). The set of all the suffixes of \(\pi\) is denoted by \(\text{suffix}(\pi)\) (notice that if \(\pi\) is a run in \(R\), then \(\pi^i\) is also a run in \(R\), for each \(i\).) If the path \(\pi = t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} \ldots\) is non–trivial (i.e., the sequence contains at least two terms) \(\text{firstact}(\pi)\) denotes \(a_0\), otherwise we set \(\text{firstact}(\pi)\) to an element non in \(\Sigma\).

ALTL formulæ over a PRS \(R\) are interpreted in terms of the set of PRS runs satisfying the given ALTL formulæ. The *denotation of a formulæ \(\varphi\) relative to \(R\), in symbols \([[\varphi]]_R\), is defined inductively as follows:

- \([[\text{true}}]]_R = \text{runs}(R)
- \([[\neg \varphi]]_R = \text{runs}(R) \setminus [[\varphi]]_R
- \([[\varphi_1 \land \varphi_2]]_R = [[\varphi_1]]_R \cap [[\varphi_2]]_R
- \([[\varphi_1 \lor \varphi_2]]_R = \{ \pi \in \text{runs}(R) \mid \text{firstact}(\pi) = a \text{ and } \pi^i \in [[\varphi_1]]_R \}
- \([[\varphi_1 \land \varphi_2]]_R = \{ \pi \in \text{runs}(R) \mid \text{for some } i \geq 0, \pi^i \text{ is defined and } \pi^i \in [[\varphi_2]]_R, \text{ and for all } j < i, \pi^j \in [[\varphi_1]]_R \}
- \([[\text{true}}]]_R = \{ \pi \in \text{runs}(R) \mid \text{suffix}(\pi) \subseteq [[\varphi]]_R\}
- \([[\neg \varphi]]_R = \{ \pi \in \text{runs}(R) \mid \text{suffix}(\pi) \cap [[\varphi]]_R \neq \emptyset\}

For any term \(t \in T\) and ALTL formulæ \(\varphi\), we say that \(t\) satisfies \(\varphi\) (resp., satisfies \(\varphi\) restricted to infinite runs) (w.r.t \(R\)), in symbols \(t \models_R \varphi\) (resp., \(t \models_{R,\infty} \varphi\)), if \(\text{runs}_R(t) \subseteq [[\varphi]]_R\) (resp., \(\text{runs}_{R,\infty}(t) \subseteq [[\varphi]]_R\)).

The model–checking problem (resp., model–checking problem restricted to infinite runs) for ALTL and PRSs is the problem of deciding if, given a PRS \(R\), an ALTL formulæ \(\varphi\) and a term \(t\) of \(R\), \(t \models_R \varphi\) (resp., \(t \models_{R,\infty} \varphi\)). The following are well–known results:

**Proposition 3.1 (see [17]).** The model–checking problem for ALTL and parallel PRSs, possibly restricted to infinite runs, is decidable.

**Proposition 3.2 (see [3, 5, 17]).** The model–checking problem for ALTL and sequential PRSs, possibly restricted to infinite runs, is decidable.

**Verification of the reachable property**

A *state property* of a PRS \(R\) over the alphabet \(\Sigma\), is a formulæ of the propositional language over the set of atomic propositions of the form \(EN(a)\) for each \(a \in \Sigma\), defined as follows:

\[
\varphi ::= EN(a) \mid \neg \varphi \mid \varphi_1 \land \varphi_2 \mid \varphi_1 \lor \varphi_2
\]
where $a \in \Sigma$

The intuitive meaning of the atomic proposition $EN(a)$ is that action $a$ is currently enabled. The semantics of a formula in this language is given in terms of the set of process terms satisfying the formula. Therefore, the denotation $[[\varphi]]_\mathcal{R}$ of the state formula $\varphi$ is defined as follows:

- $[[EN(a)]]_\mathcal{R} = \{ t \in T \mid r : t \xrightarrow{a} t' \in \mathcal{R}, \text{ for some } t' \in T \}$
- $[[\varphi_1 \land \varphi_2]]_\mathcal{R} = [[\varphi_1]]_\mathcal{R} \cap [[\varphi_2]]_\mathcal{R}$
- $[[\varphi_1 \lor \varphi_2]]_\mathcal{R} = [[\varphi_1]]_\mathcal{R} \cup [[\varphi_2]]_\mathcal{R}$
- $[[\neg \varphi]]_\mathcal{R} = T \setminus [[\varphi]]_\mathcal{R}$

For any term $t \in T$ and state formula $\varphi$, we say that $t$ satisfies $\varphi$ (w.r.t $\mathcal{R}$), in symbols $t \models_\mathcal{R} \varphi$, if $t \in [[\varphi]]_\mathcal{R}$.

Given a state formula $\varphi$ and a process term $t$, the reachable state property problem in $\mathcal{R}$ w.r.t $t$ and $\varphi$ is the problem of deciding whether there exists a process term $t'$ reachable from $t$ in $\mathcal{R}$, with $t' \models \varphi$.

**Proposition 3.3 (see [17]).** The reachable state property problem for PRS is decidable.

4 Verification of properties about infinite runs in PRS.

Our goal is to show decidability of some problems about infinite derivations of PRSs in normal form, and show how decidability of these problems can be used to check interesting properties of infinite state systems modelled by PRSs in normal form. For this reason we introduce the notion of Büchi Rewrite System (BRS). Intuitively, a BRS is a PRS where we can distinguish between non-accepting rules and accepting rules.

**Definition 4.1 (Büchi Rewrite System).** A Büchi Rewrite System (BRS) over a finite set of process variables $Var$ and an alphabet $\Sigma$ is a pair $\langle \mathcal{R}, \mathcal{R}_F \rangle$, where $\mathcal{R}$ is a PRS over $Var$ and $\Sigma$, and $\mathcal{R}_F \subseteq \mathcal{R}$ is the set of accepting rules.

A Büchi Rewrite System $\langle \mathcal{R}, \mathcal{R}_F \rangle$ is called a BRS in normal form (resp., sequential BRS, parallel BRS), if the underlying PRS $\mathcal{R}$ is a PRS in normal form (resp., parallel PRS, sequential PRS).

**Definition 4.2 (Acceptance in Büchi Rewrite Systems).** Let us consider a BRS $M = \langle \mathcal{R}, \mathcal{R}_F \rangle$. An infinite derivation $t \xrightarrow{\sigma} t'$ in $\mathcal{R}$ from $t$ is said to be accepting (in $M$) if $\sigma$ contains infinite occurrences of accepting rules.

A finite derivation $t \xrightarrow{\sigma^* \mathcal{R}} t'$ in $\mathcal{R}$ from $t$ is said to be accepting (in $M$) if $\sigma$ contains some occurrence of accepting rule.
The main result of this paper is the following:

Given a BRS \((\mathcal{R}, \mathcal{R}_F)\) in normal form and a process variable \(X\) it is decidable whether:

**Problem 1:** there exists an infinite accepting derivation from \(X\);

**Problem 2:** there exists an infinite derivation from \(X\), not containing occurrences of accepting rules;

**Problem 3:** there exists an infinite derivation from \(X\), containing a finite non-null number of occurrences of accepting rules.

Before proving this result in Section 5.2, we show how a solution to these problems can be effectively employed to perform model checking of some linear time properties of infinite runs (from process variables) in PRSs in normal form. In particular we consider the following small ALTL fragment

\[ \varphi ::= F \psi \mid GF \psi \mid \neg \varphi \]  

(1)

where \(\psi\) denotes an ALTL propositional formula\(^2\). For succinctness we denote a ALTL propositional formula of the form \(<a>\) true (with \(a \in \Sigma\)) simply by \(a\).

The fragment allows us to express some useful properties on infinite runs. Examples are simple safety properties such as \(G \bigvee_{i=1}^n a_i\) (resp., \(G \bigwedge_{i=1}^n \neg a_i\)) meaning that the system only executes (resp., never executes) actions from the set \(\{a_1, ..., a_n\}\); guarantee properties such as \(F \bigvee_{i=1}^n a_i\) (resp., \(F \bigwedge_{i=1}^n \neg a_i\)), meaning that the system eventually executes [resp., does not only execute] actions from the set \(\{a_1, ..., a_n\}\); response properties such as \(GF \bigvee_{i=1}^n a_i\) (resp., \(GF \bigwedge_{i=1}^n \neg a_i\)), meaning that the system infinitely often executes actions from the set (resp., outside the set) \(\{a_1, ..., a_n\}\); and persistence properties such as \(GF \bigvee_{i=1}^n a_i\) (resp., \(GF \bigwedge_{i=1}^n \neg a_i\)), meaning that the system executes almost always (resp., finitely often) some actions in the set \(\{a_1, ..., a_n\}\).

To prove the decidability of the model–checking problem restricted to infinite runs for this fragment of ALTL we need some definitions.

Given a propositional formula \(\psi\) over \(\Sigma\) we denote by \([\psi]_\Sigma\) the subset of \(\Sigma\) inductively defined as follows

- \(\forall a \in \Sigma \ [a]_\Sigma = \{a\}\)
- \([\neg \psi]_\Sigma = \Sigma \setminus ([\psi]_\Sigma)\)
- \([\psi_1 \land \psi_2]_\Sigma = ([\psi_1]_\Sigma \cap [\psi_2]_\Sigma)\)

\(^2\)The set of ALTL propositional formulæ \(\psi\) over the set \(\Sigma\) of atomic propositions (or actions) is so defined:

\[ \psi ::= <a> \text{ true} \mid \psi \land \psi \mid \neg \psi \ (\text{where } a \in \Sigma) \]
Evidently, given a PRS $\mathcal{R}$ over $\Sigma$, a ALTL propositional formula $\psi$ and an infinite run $\pi$ of $\mathcal{R}$ we have that $\pi \in \left[\left[\psi\right]\right]_{\mathcal{R}}$ iff $\operatorname{firstact}(\pi) \in \left[\left[\psi\right]\right]_{\Sigma}$.

Given a rule $r = t \rightarrow t' \in \mathcal{R}$ we say that $r$ satisfies the propositional formula $\psi$ if $a \in \left[\left[\psi\right]\right]_{\Sigma}$.

We denote by $AC(\psi)$ the set of the rules in $\mathcal{R}$ that satisfy $\psi$.

The following is a model–checking procedure for the fragment of $ALTL$ defined above with input a PRS $\mathcal{R}$ in normal form, a temporal formula $\varphi$, and a process variable $X$. Let us denote by $\psi$ the propositional formula associated to $\varphi$.

- Build the $BRS \langle \mathcal{R}, \mathcal{R}_F \rangle$, where $\mathcal{R}_F = AC(\psi)$;
- Then if $\varphi$ is of the form:
  
  $F\psi$: $X \models_{\mathcal{R},\infty} F\psi$ if, and only if there does not exist an infinite derivation in $\langle \mathcal{R}, \mathcal{R}_F \rangle$ starting from $X$ not containing occurrences of accepting rules. This amounts to solving Problem 2.
  
  $\neg F\psi$: $X \models_{\mathcal{R},\infty} \neg F\psi$ if, and only if there does not exist an infinite derivation in $\langle \mathcal{R}, \mathcal{R}_F \rangle$ starting from $X$ containing occurrences of accepting rules. This amounts to solving a combination of Problem 1 and Problem 3.
  
  $GF,\psi$: $X \models_{\mathcal{R},\infty} GF\psi$ if, and only if there does not exist an infinite derivation in $\langle \mathcal{R}, \mathcal{R}_F \rangle$ starting from $X$ containing a finite number of occurrences of accepting rules. This amounts to solving a combination of Problem 2 and Problem 3.
  
  $\neg GF\psi$: $X \models_{\mathcal{R},\infty} \neg GF\psi$ if, and only if there does not exist an infinite derivation starting from $X$ containing an infinite number of occurrences of accepting rules. This amounts to solving Problem 1.

So, we obtain the following result.

**Theorem 4.1.** The model–checking problem for PRSs in normal form and the fragment $ALTL$ restricted to infinite runs from process variables is decidable.

## 5 Decidability results on infinite derivations

In this section we prove the main results of the paper, namely the decidability of the problems about infinite derivations stated in Section 4. Therefore, in Subsection 5.1 we report some preliminary results on the decidability of some properties about derivations of parallel and sequential $BRS$s which are necessary to carry out the proof of the main result, which is given in Subsection 5.2.

### 5.1 Decidability results on derivations of parallel and sequential $BRS$s

In this section we establish simple decidability results on derivations of parallel and sequential $PRS$s. These results are the basis for the decidability proof of the problems 1-3.
Proposition 5.1. Given a parallel BRS $\langle \mathcal{R}'', \mathcal{R}'_F \rangle$ over $\text{Var}$ and the alphabet $\Sigma$, and two variables $X, Y \in \text{Var}$, it is decidable whether:

1. there exists a derivation in $\mathcal{R}'$ of the form $X \overset{\sigma}{\rightarrow}^* X' \parallel Y$ for some term $t$.
2. there exists an accepting (resp., non-accepting) finite derivation in $\mathcal{R}'$ of the form $X \overset{\sigma}{\rightarrow}^* X' \parallel Y$, for some term $t$, with $|\sigma| > 0$.
3. there exists an accepting (resp., non-accepting) finite derivation in $\mathcal{R}'$ of the form $X \overset{\varepsilon}{\rightarrow}^* \varepsilon$.
4. there exists an accepting (resp., non-accepting) finite derivation in $\mathcal{R}'$ of the form $X \overset{\varepsilon}{\rightarrow}^* Y$.

Proof. Parallel PRSs are semantically equivalent to Petri Nets [21]. The first problem is, therefore, reducible to the partial reachability problem for Petri Nets, which has been proved to be decidable in [16, 15].

To prove decidability of the remaining problems, we exploit decidability of the model-checking problem for full Action–based LTL in parallel PRSs (see Proposition 3.1).

Let us consider the second problem. To show decidability of this problem, we start from $\mathcal{R}'$ and build a new parallel PRS $\mathcal{R}''$ over the alphabet $\Sigma = \{f, nf, Y\}$, in the following way. We substitute every accepting (resp., non–accepting) rule in $\mathcal{R}'$ of the form $t \xrightarrow{a} t'$, with the rule $t \xrightarrow{f} t'$ (resp., $t \xrightarrow{nf} t'$). Finally, we add the rule $r = Y \overset{Y}{\rightarrow} Y$.

The reason to add the rule $Y \overset{Y}{\rightarrow} Y$ is to allow us to express reachability of variable $Y$ as an ALTL formula. Similarly, the addition of the rules of the form $t \xrightarrow{f} t'$ [$t \xrightarrow{nf} t'$] allows us to express in ALTL the application of accepting [non–accepting] rules along a run. The second problem is, therefore, reducible to the problem of checking whether there exists a run $\pi \in \text{runs}_{\mathcal{R}''}(X)$ satisfying the following LTL formula:

$$\varphi := F (f) F (Y) \text{true} \quad \text{[resp., } \varphi := (nf) (nf) \text{true U (Y)true } \text{]}$$

The formula $F (f) F (Y) \text{true}$ intuitively means that, at least one accepting rule is eventually applied, and, after that, the rule labelled $Y$ is eventually applied (in other words, $Y$ is reachable after some accepting rule application). On the other hand, the formula $(nf) (nf) \text{true U (Y)true }$ means that, after a non-accepting rule (here we look for derivations in $\mathcal{R}''$ with length strictly greater than 0), $Y$ is reached by applying only non–accepting rules. In terms of ALTL model–checking, the second problem corresponds to checking whether, for all $\pi \in \text{runs}_{\mathcal{R}''}(X)$, $\pi \notin [\varphi]_{\mathcal{R}''}$, or, in other words, to checking whether $X \models_{\mathcal{R}''} \neg \varphi$. If the result of this check is true, the second problem has a negative answer, otherwise, the answer is positive.

Let us now consider the third problem. Similarly to the problem above, starting from $\mathcal{R}'$, we build a new PRS $\mathcal{R}''$, this time on the new alphabet $\Sigma = \{f, nf\} \cup \text{Var}$, as follows. We substitute every accepting (resp., non–accepting) rule in $\mathcal{R}'$ of the form $t \xrightarrow{a} t'$ with the rule $t \xrightarrow{f} t'$ (resp., $t \xrightarrow{nf} t'$). Finally, for all $Y \in \text{Var}$ we add the rule $Y \overset{Y}{\rightarrow} Y$. Notice that,
by construction, a term \( t \) has no successor in \( \mathcal{R}'' \) if and only if \( t = \varepsilon \). Let now \( \varphi_1 \) be the following LTL formula:

\[
\varphi_1 = \bigvee_{Y \in \text{Var}} (\langle Y \rangle \text{true}) \lor (\langle nf \rangle \text{true}) \lor (\langle f \rangle \text{true})
\]

The negation of \( \varphi_1 \) (namely, \( \neg \varphi_1 \)) means that no rule can be applied, in other words the system has terminated. It is now easy to see that the problem is reducible to the following LTL model-checking problem in \( \mathcal{R}'' \):

\[
X \models_{\mathcal{R}''} \neg F (\langle f \rangle (\neg \varphi_1)) \quad \text{[resp., } X \models_{\mathcal{R}''} \neg (\langle nf \rangle \text{true} U \neg \varphi_1)]
\]

whose intuitive meaning is that it can never be the case that from \( X \) the system can eventually reach termination after some application of accepting rules [resp., the system cannot reach termination by applying only non–accepting rule].

Finally, let us consider the fourth problem. Starting from \( \mathcal{R}' \), we build a new PRS \( \mathcal{R}'' \) over the alphabet \( \Sigma = \{ f, nf, \varepsilon \} \cup \text{Var} \), as follows. We substitute every accepting (resp., non–accepting) rule in \( \mathcal{R}' \) of the form \( t \xrightarrow{a} t' \) with the rule \( t \xrightarrow{f} t' \) (resp., \( t \xrightarrow{nf} t' \)). For every \( Z \in \text{Var} \), we add the rule \( Z \xrightarrow{f} Z \). Finally, we add the rule \( Y \xrightarrow{\varepsilon} \varepsilon \). Again, by construction, in \( \mathcal{R}'' \) a term \( t \) has no successor if, and only if, \( t = \varepsilon \). Let now \( \varphi_2 \) be the following LTL formula,

\[
\varphi_2 = \bigvee_{Y \in \text{Var}} (\langle Y \rangle \text{true}) \lor (\langle f \rangle \text{true}) \lor (\langle nf \rangle \text{true}) \lor (\langle \varepsilon \rangle \text{true})
\]

It is easy to see that the problem is now reducible to the following LTL model-checking problem in \( \mathcal{R}'' \):

\[
X \models_{\mathcal{R}''} \neg (F (\langle f \rangle \text{true}) \land (\langle nf \rangle \text{true} \lor (\langle f \rangle \text{true}) U (\langle Y \rangle \langle \varepsilon \rangle \neg \varphi_2)))
\]

\[
[\text{resp., } X \models_{\mathcal{R}''} \neg (\langle nf \rangle \text{true} U (\langle Y \rangle \langle \varepsilon \rangle \neg \varphi_2))]
\]

meaning that it not the case that some accepting rule is eventually applied \( (F (\langle f \rangle \text{true})) \), while only rules in \( \mathcal{R}' \) (either accepting or non–accepting rules) are applied \( (\langle nf \rangle \text{true} \lor (\langle f \rangle \text{true})) \) until \( Y \) is eventually reached and followed by immediate termination \( (\langle Y \rangle \langle \varepsilon \rangle \neg \varphi_2)) \) [resp., it is not the case that \( Y \) is eventually reached and followed by immediate termination by applying only non–accepting rules].

Let us now define an additional notion of reachability in a sequential PRS, and show that it is decidable whether two terms are reachable according to this notion. As we shall see in the next section, this decidability result will be needed to prove decidability of the problems on infinite derivations we are interested in.

**Definition 5.1.** Given a sequential PRS \( \mathcal{R} \) over \( \text{Var} \), and variables \( X,Y \in \text{Var} \), we say that \( Y \) is reachable from \( X \) in \( \mathcal{R} \), if there exists a term \( t \in T \setminus \{ \varepsilon \} \) of the form \( X_1.(X_2.(\ldots X_n.(Y) \ldots)) \) (with \( n \) possibly equals to zero) such that \( X \Rightarrow^{t}_{\mathcal{R}} t \).
Proposition 5.2. Let \( \langle \mathcal{R}_{SEQ}, \mathcal{R}_{SEQ,F} \rangle \) be a sequential BRS over \( \text{Var} \) and the alphabet \( \Sigma \). Given any two process variables \( X \) and \( Y \) in \( \text{Var} \), it is decidable whether:

1. \( Y \) is reachable from \( X \) in \( \mathcal{R}_{SEQ} \).
2. \( Y \) is reachable from \( X \) in \( \mathcal{R}_{SEQ} \) through a non accepting derivation.

Proof. The proof relies on the decidability of the model-checking problem for LTL and sequential PRSs (see Proposition 3.2).

First, we construct a new sequential PRS \( \mathcal{R}'_{SEQ} \) over the alphabet \( \Sigma = \{ f, nf, Y \} \), in the following way. We replace every accepting (resp., not accepting) rule in \( \mathcal{R}_{SEQ} \) of the form \( t \xrightarrow{a} t' \) with the rule \( t \xrightarrow{f} t' \) (resp., \( t \xrightarrow{nf} t' \)). Finally, we add the rule \( r = Y \xrightarrow{Y} Y \).

Now, the first problem can be restated as the problem of deciding, given two variables \( X \) and \( Y \), whether the following property is satisfied:

A. There exists a derivation in \( \mathcal{R}_{SEQ} \) of the form \( X \Rightarrow^* \mathcal{R}_{SEQ} t \) for some term \( t \in T_{SEQ} \setminus \{ \varepsilon \} \) with \( t = X_1.(X_2.(\ldots X_n.(Y)\ldots)) \).

Satisfaction of Property A can be expressed by the following LTL satisfaction problem: Property A is satisfied if, and only if, there exists a run \( \pi \in \text{runs}_{\mathcal{R}_{SEQ}}(X) \) satisfying the following LTL formula:

\[
\varphi := F(\langle Y \rangle true)
\]

Therefore, Property A is not satisfied if, and only if, for all \( \pi \in \text{runs}_{\mathcal{R}_{SEQ}}(X) \), \( \pi \not\in [[\varphi]]_{\mathcal{R}_{SEQ}} \), that is if, and only if, \( X \models_{\mathcal{R}_{SEQ}} \neg \varphi \).

Finally, consider the second problem. This problem can be restated as the problem of deciding, given two variables \( X \) and \( Y \), whether the following property is satisfied:

B. There exists a finite non–accepting derivation in \( \mathcal{R}_{SEQ} \) of the form \( X \Rightarrow^* \mathcal{R}_{SEQ} t \), for some term \( t \in T_{SEQ} \setminus \{ \varepsilon \} \), with \( t = X_1.(X_2.(\ldots X_n.(Y)\ldots)) \).

As it was the case for Property A, the satisfaction of Property B is reducible to the following LTL satisfaction problem in \( \mathcal{R}_{SEQ} \):

\[
X \models_{\mathcal{R}_{SEQ}} \neg (\langle nf \rangle true U (\langle Y \rangle true))
\]

Proposition 5.3. Let us consider a sequential (resp., parallel) BRS \( \langle \mathcal{R}', \mathcal{R}'_F \rangle \) over \( \text{Var} \) and the alphabet \( \Sigma \). Given \( X \in \text{Var} \), it is decidable whether the following condition is satisfied:

- there exists in \( \mathcal{R}' \) an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation containing a finite non–null number of accepting rule occurrences) from \( X \).
Proof. The proof relies on decidability of the model-checking problem for LTL and sequential PRSs (resp. parallel PRSs) restricted to infinite runs (see Propositions 3.2 and 3.1).

We first construct a new sequential (resp., parallel) PRS $\mathcal{R}'$ over the alphabet $\Sigma = \{f, nf\}$ as follows. We replace every accepting (resp., non–accepting) rule in $\mathcal{R}$ of the form $t \xrightarrow{\alpha} t'$ with the rule $t \xrightarrow{f} t'$ (resp., $t \xrightarrow{nf} t'$).

Let us first consider the problem of deciding whether

A. There exists an accepting infinite derivation in $\mathcal{R}'$ from $X$.

The negation of Property A can be expressed by the following ALTL formula

$$\varphi := \neg GF (\langle f \rangle true)$$

Therefore, Property A is not satisfied if, and only if, $X \models_{\mathcal{R}', \infty} \varphi$.

Now, let us consider the problem of deciding whether

B. There exists an infinite derivation in $\mathcal{R}'$ from $X$ devoid of accepting rules.

The negation of Property B can be expressed by the following formula LTL

$$\varphi := \neg G (\langle nf \rangle true)$$

Property B is, therefore, not satisfied if, and only if, $X \models_{\mathcal{R}', \infty} \varphi$.

Finally, let us consider the problem of deciding whether

C. There exists an infinite derivation in $\mathcal{R}'$ from $X$ containing a finite non–null number of accepting rule occurrences.

The negation of Property C can be expressed by the following formula LTL

$$\varphi := \neg F (\langle f \rangle (G (\langle nf \rangle true)))$$

Again, Property C is not satisfied if, and only if, $X \models_{\mathcal{R}', \infty} \varphi$.

Theorem 5.1. Let us consider a sequential BRS $\langle \mathcal{R}_{SEQ}, \mathcal{R}_{SEQ,F} \rangle$ and a parallel BRS $\langle \mathcal{R}_{PAR}, \mathcal{R}_{PAR,F} \rangle$ over Var and the alphabet $\Sigma$. Given $X \in \text{Var}$, it is decidable whether one of the following conditions is satisfied:

- there exists a variable $Y \in \text{Var}$ reachable (resp., reachable through a non–accepting derivation, reachable) from $X$ in $\mathcal{R}_{SEQ}$, and there exists in $\mathcal{R}_{PAR}$ an infinite accepting derivation (resp., an infinite derivation devoid of accepting rule occurrences, an infinite derivation containing a finite non–null number of accepting rule occurrences) from $Y$.

- there exists in $\mathcal{R}_{SEQ}$ an infinite accepting derivation (resp., an infinite derivation devoid of accepting rule occurrences, an infinite derivation containing a finite non–null number of accepting rule occurrences) from $X$.

Proof. The result follows directly from Propositions 5.2 and 5.3. 

\[\square\]
5.2 Decidability of properties about infinite derivations

To prove decidability of Problems 1–3 stated in the previous section, we show that each of those problems can be reduced to (a combination of) two similar, but simpler, problems: the first is a decidability problem on infinite derivations restricted to parallel BRSs; the second is a decidability problem on infinite derivations restricted to sequential BRSs. Since each of those restricted problems is decidable (see theorem 5.1), decidability of Problems 1–3 is entailed.

In particular, we show that, given a BRS $\langle \mathcal{R}, \mathcal{R}_F \rangle$ in normal form over $\text{Var}$ and the alphabet $\Sigma$, it is possible to effectively construct two BRSs, a parallel BRS $\langle \mathcal{R}_{\text{PAR}}, \mathcal{R}_{\text{PAR},F} \rangle$ and a sequential BRS $\langle \mathcal{R}_{\text{SEQ}}, \mathcal{R}_{\text{SEQ},F} \rangle$, in such a way that:

1. Problem 1 is reducible to the problem of deciding, given a process variable $X$, if one of the following conditions is satisfied:
   - There exists a variable $Y \in \text{Var}$ reachable from $X$ in $\mathcal{R}_{\text{SEQ}}$, and there exists an infinite accepting derivation in $\mathcal{R}_{\text{PAR}}$ from $Y$.
   - There exists an infinite accepting derivation in $\mathcal{R}_{\text{SEQ}}$ from $X$.

2. Problem 2 is reducible to the problem of deciding, given a process variable $X$, if one of the following conditions is satisfied:
   - There exists a variable $Y \in \text{Var}$ reachable from $X$ in $\mathcal{R}_{\text{SEQ}}$ through a non–accepting derivation, and there exists an infinite derivation in $\mathcal{R}_{\text{PAR}}$ from $Y$ not containing accepting rule occurrences.
   - There exists an infinite derivation in $\mathcal{R}_{\text{SEQ}}$ from $X$ not containing accepting rule occurrences.

3. Problem 3 is reducible to the problem of deciding, given a process variable $X$, if one of the following conditions is satisfied:
   - There exists a variable $Y \in \text{Var}$ reachable from $X$ in $\mathcal{R}_{\text{SEQ}}$, and there exists an infinite derivation in $\mathcal{R}_{\text{PAR}}$ from $Y$ containing a finite non–null number of accepting rule occurrences.
   - There exists an infinite derivation in $\mathcal{R}_{\text{SEQ}}$ from $X$ containing a finite non–null number of accepting rule occurrence.

In the following, $\mathcal{R}^P$ (resp., $\mathcal{R}_F^P$) denotes the set $\mathcal{R}$ (resp., the set $\mathcal{R}_F$) restricted to PAR rules.

Before illustrating the main idea underlying our approach, we need few additional definitions and notation, which allows us to look more in detail at the structure of derivations. The following definition introduces the notion of level of application of a rule in a derivation:
Definition 5.2. Let \( \frac{r}{s} t' \) be a single–step derivation in \( \mathcal{R} \). We say that \( r \) is applied at level 0 in \( \frac{r}{s} t' \), if \( t = t \parallel s, t' = t \parallel s' \) (for some \( t, s, s' \in T \)), and \( r = s \overset{a}{\rightarrow} s' \), for some \( a \in \Sigma \).

We say that \( r \) is applied at level \( k > 0 \) in \( \frac{r}{s} t' \), if \( t = t \parallel X.(s), t' = t \parallel X.(s') \) (for some \( t, s, s' \in T \)), \( s \overset{r}{\rightarrow} s' \), and \( r \) is applied at level \( k - 1 \) in \( s \overset{r}{\rightarrow} s' \).

The definition above extends in the obvious way to \( n \)-step derivations and to infinite derivations. The next definition introduces the notion of subderivation starting from a term.

Definition 5.1 (Subderivation). Let \( \frac{r}{s} t \parallel X.(s) \overset{\sigma}{\rightarrow} s' \) be a finite or infinite derivation in \( \mathcal{R} \) starting from \( t \). The subderivation \( \frac{r}{s} t' \) is defined as follows:

1. if \( d \) is the null derivation or \( s = \varepsilon \), then \( d' \) is the null derivation;
2. if \( \sigma = r\sigma' \), and \( d \) is of the form \( t \parallel X.(Z) \overset{r}{\rightarrow} Y \), then \( d' \) is the null derivation.
3. if \( \sigma = r\sigma' \), and \( d \) is of the form \( t \parallel X.(s) \overset{r}{\rightarrow} s' \parallel X.(s') \overset{\sigma'}{\rightarrow} s' \) (with \( r = X.(Z) \overset{a}{\rightarrow} Y \)) then \( d' \) is of the form \( t \parallel X.(s) \overset{r}{\rightarrow} s' \parallel X.(s') \overset{\sigma'}{\rightarrow} s' \);
4. if \( \sigma = r\sigma' \), and \( d \) is of the form \( t \parallel X.(s) \overset{r}{\rightarrow} t' \parallel X.(s) \overset{\sigma'}{\rightarrow} t' \) then \( d' \) is of the form \( t \parallel X.(s) \overset{r}{\rightarrow} \).

Moreover, we say that \( d' \) is a subderivation of \( \frac{r}{s} t \parallel X.(s) \overset{\sigma}{\rightarrow} s' \).

Clearly, in the definition above \( \mu \) is a subsequence of \( \sigma \). Moreover, if \( k \) is the level of application of a rule occurrence of \( \mu \) in the derivation \( d \) then, \( k > 0 \), and this occurrence is applied in the subderivation \( d' = s \overset{\mu}{\rightarrow} s' \) at level \( k - 1 \).

Moreover, we say that a subderivation \( s \overset{\sigma}{\rightarrow} \) of \( \frac{r}{s} t \parallel X \overset{\sigma}{\rightarrow} s' \) is a maximal subderivation in \( \frac{r}{s} t \parallel X \overset{\sigma}{\rightarrow} s' \), if there is no subderivation \( \frac{r}{s} t \parallel X \overset{\rho}{\rightarrow} s' \) of \( \frac{r}{s} t \parallel X \overset{\sigma}{\rightarrow} s' \), with \( \rho \) a proper subsequence of \( \sigma \).

Given a sequence \( \sigma = r_1r_2 \ldots r_n \ldots \) of rules in \( \mathcal{R} \), and a subsequence \( \sigma' = r_{k_1}r_{k_2} \ldots r_{k_m} \ldots \) of \( \sigma \) \( \sigma \setminus \sigma' \) denotes the sequence obtained by removing from \( \sigma \) all and only the occurrences of rules in \( \sigma' \) (namely, those \( r_i \) for which it exists a \( j = 1, \ldots, |\sigma'| \), with \( k_j = i \)).
Definition 5.3. The class \( \Xi_{PAR} \) (resp. \( \Pi_{PAR} \)) is the class of derivations \( t \Rightarrow^*_\mathcal{R} \) in \( \mathcal{R} \) not satisfying the following property:

- The derivation \( t \Rightarrow^*_\mathcal{R} \) can be written in the form \( t \Rightarrow^*_\mathcal{R} t' \parallel X.(s) \Rightarrow^*_\mathcal{R} \) such that the subderivation of \( t' \parallel X.(s) \Rightarrow^*_\mathcal{R} \) from \( s \) is an infinite derivation (resp., an accepting infinite derivation).

Let us sketch the main ideas at the basis of our technique. To fix the ideas, let us consider the problem 1. Moreover, let us focus first on the class of derivations \( \Pi_{PAR} \), showing how it is possible to mimic accepting infinite derivations in \( \mathcal{R} \) from a variable, belonging to this class, by using only PAR rules belonging to an extension of the parallel BRS \( \langle \mathcal{R}^P, \mathcal{R}^F \rangle \) denoted by \( \langle \mathcal{R}_{PAR}, \mathcal{R}_{PAR,F} \rangle \). More precisely, we show that

A. if \( p \Rightarrow^*_\mathcal{R} \) (with \( p \in T_{PAR} \)) is an infinite accepting derivation in \( \Pi_{PAR} \) then, there exists an infinite accepting derivation in \( \langle \mathcal{R}_{PAR}, \mathcal{R}_{PAR,F} \rangle \) from \( p \), and vice versa.

With reference to Problems 2 (resp., 3), within \( \langle \mathcal{R}_{PAR}, \mathcal{R}_{PAR,F} \rangle \) it will also be possible to simulate an infinite derivation in \( \mathcal{R} \) from \( p \in T_{PAR} \) belonging to \( \Xi_{PAR} \), and not containing accepting rule occurrences (resp., containing a finite non-null number of accepting rule occurrences), through an infinite derivation in \( \mathcal{R}_{PAR} \) from \( p \), not containing accepting rule occurrences (resp., containing a finite non-null number of accepting rule occurrences), and vice versa.

Suppose now that the accepting infinite derivation \( p \Rightarrow^*_\mathcal{R} \) belongs to \( \Pi_{PAR} \). Then, all its possible subderivations contain all, and only, the rule occurrences in \( \sigma \) applied at a level \( k \) greater than 0 in \( p \Rightarrow^*_\mathcal{R} \). If \( \sigma \) contains only PAR rule occurrences the statement A is evident since \( \langle \mathcal{R}_{PAR}, \mathcal{R}_{PAR,F} \rangle \) is an extension of \( \langle \mathcal{R}^P, \mathcal{R}^F \rangle \). Otherwise, \( p \Rightarrow^*_\mathcal{R} \) can be written in the form:

\[
p \Rightarrow^*_\mathcal{R} t \parallel Z' \Rightarrow^*_\mathcal{R} t \parallel Y.(Z) \Rightarrow^*_\mathcal{R}
\]

where \( r = Z' \Rightarrow^*_\mathcal{R} Y.(Z) \), \( \lambda \) contains only occurrences of rules in \( \mathcal{R}_P \), and \( t \in T_{PAR} \). Let \( Z \Rightarrow^*_\mathcal{R} \) be the subderivation of \( t \parallel Y.(Z) \Rightarrow^*_\mathcal{R} \) from \( Z \).

Since \( p \Rightarrow^*_\mathcal{R} \) is in \( \Pi_{PAR} \), \( Z \Rightarrow^*_\mathcal{R} \) does not contain infinite occurrences of accepting rules. Thus, only one of the following three cases may occur:

A. \( Z \Rightarrow^*_\mathcal{R} \) leads to the term \( \varepsilon \), and \( p \Rightarrow^*_\mathcal{R} \) is of the form

\[
p \Rightarrow^*_\mathcal{R} t \parallel Z' \Rightarrow^*_\mathcal{R} t \parallel Y.(Z) \Rightarrow^*_\mathcal{R} \parallel Y \Rightarrow^*_\mathcal{R}
\]

where \( \rho \) is a subsequence of \( \omega_1 \) and \( t \Rightarrow^*_\mathcal{R} \). The infinite derivation above is accepting if, and only if, the following infinite derivation, obtained by anticipating (by interleaving) the application of the rules in \( \rho \) before the application of the rules in \( \xi = \omega_1 \setminus \rho \), is accepting

\[
p \Rightarrow^*_\mathcal{R} t \parallel Z' \Rightarrow^*_\mathcal{R} t \parallel Y.(Z) \Rightarrow^*_\mathcal{R} t \parallel Y \Rightarrow^*_\mathcal{R}
\]
where \( t \xrightarrow{\xi} \mathcal{T} \).

The idea is to collapse the derivation \( \mathcal{Z} \xrightarrow{\rho} \mathcal{Y} \) into a single accepting PAR rule of the form \( \mathcal{Z} \xrightarrow{\rho} \mathcal{Y} \), if \( r\rho \) is accepting, or into a non–accepting PAR rule of the form \( \mathcal{Z} \xrightarrow{\rho} \mathcal{Y} \), if \( r\rho \) is non–accepting.

Notice that in the step from (2) to (3), we exploit the fact that the properties on infinite derivations we are interested in are insensitive to permutation of rule applications within a derivation.

Now, we can apply recursively the same reasoning to the infinite accepting derivation in \( \mathcal{R} \) from \( t\|Y \in T_{PAR} \)

\[
\begin{align*}
t\|Y \xrightarrow{\xi} \mathcal{T}\|Y \xrightarrow{\omega_2} \mathcal{T}
\end{align*}
\]

which belongs to \( \Pi_{PAR} \).

The subderivation \( \mathcal{Z} \xrightarrow{\rho} \mathcal{Y} \) leads to a variable \( \mathcal{W} \) and \( p \xrightarrow{\sigma} \mathcal{T} \) can be written as:

\[
\begin{align*}
p \xrightarrow{\lambda} \mathcal{T} \| Z' \xrightarrow{r} \mathcal{T} \| Y.(Z) \xrightarrow{\omega_1} \mathcal{T} \| Y.(W) \xrightarrow{r'} \mathcal{T} \| W' \xrightarrow{\omega_2}
\end{align*}
\]

where \( r' = Y.(W) \xrightarrow{b} W' \) (with \( W' \in Var \)), \( \rho \) is a subsequence of \( \omega_1 \) and \( t \xrightarrow{\omega_1/\rho} \mathcal{T} \).

The derivation above is accepting if and only if the following derivation is accepting

\[
\begin{align*}
p \xrightarrow{\lambda} \mathcal{T} \| Z' \xrightarrow{r} \mathcal{T} \| Y.(Z) \xrightarrow{\omega_1} \mathcal{T} \| Y.(W) \xrightarrow{r'} \mathcal{T} \| W' \xrightarrow{\omega_2}
\end{align*}
\]

with \( \xi = \omega_1/\rho \).

In this case we shall collapse the derivation \( \mathcal{Z} \xrightarrow{\rho} \mathcal{Y} \) into a single accepting PAR rule of the form \( \mathcal{Z} \xrightarrow{\rho} \mathcal{Y} \), if \( r\rho \) is accepting, or into a non–accepting PAR rule of the form \( \mathcal{Z} \xrightarrow{\rho} \mathcal{Y} \), otherwise.

Now, we can apply recursively the same reasoning to the infinite accepting derivation in \( \mathcal{R} \) from \( t\|W' \in T_{PAR} \)

\[
\begin{align*}
t\|W' \xrightarrow{\xi} \mathcal{T}\|W' \xrightarrow{\omega_2}
\end{align*}
\]

which belongs to \( \Pi_{PAR} \).

In this case \( \mathcal{Z} \xrightarrow{\rho} \mathcal{W} \) does not influence the applicability of rules in \( \omega \setminus \rho \) in the derivation \( t\|Y.(Z) \xrightarrow{\rho} \mathcal{T} \) (i.e. the rule applications occurring in \( \rho \) can be arbitrarily interleaved with any rule application in \( \omega/\rho \)). In other terms, we have \( t \xrightarrow{\omega/\rho} \mathcal{T} \) that is still an infinite accepting derivation in \( \Pi_{PAR} \). On the other hand, if \( \mathcal{Z} \xrightarrow{\rho} \mathcal{W} \) contains some occurrence of accepting rule or \( r \) is an accepting rule, we cannot abstract this information away. In fact, it might be the case that the infinite number of accepting rules occurring in \( p \xrightarrow{\sigma} \mathcal{T} \) is due to an infinite number of occurrences of subderivations like
Lemma 5.1. The parallel $\parallel_{\mathcal{R}}$. For this reason, we keep track of a possible occurrence of accepting rule in the sequence $r\rho$ by adding a new variable $Z_{\text{ACC}}$ and an accepting rule of the form $Z' \rightarrow Z_{\text{ACC}}$.

In other words, we are going to build a parallel $\langle \mathcal{R}_{\text{PAR}}, \mathcal{R}_{\text{PAR,F}} \rangle$ where all the maximal finite accepting [resp., non-accepting] subderivations and the maximal infinite subderivations containing a finite number of accepting rule occurrences are abstracted away by PAR rules not occurring in $\mathcal{R}$, according to the intuitions given above.

The extended $\langle \mathcal{R}_{\text{PAR}}, \mathcal{R}_{\text{PAR,F}} \rangle$ is constructed in two steps. In the first step, $\langle \mathcal{R}^p, \mathcal{R}^p_F \rangle$ is extended with PAR rules of the form $X \overset{a}{\rightarrow} Y$, where $X,Y \in \text{Var}$, and $a \in \{\$,\#\}$, in such a way that it is possible to keep track of subderivations of the forms $A$ and $B$. In the second step, rules of the form $X \rightarrow Z_{\text{ACC}}$ or of the form $X \rightarrow Z_{\text{NOT-ACC}}$ (with $X \in \text{Var}$) are added, so as to be able to keep track of derivations of the form $C$.

The first extension of $\langle \mathcal{R}^p, \mathcal{R}^p_F \rangle$, denoted by $\langle \overline{\mathcal{R}}_{\text{PAR}}, \overline{\mathcal{R}}_{\text{PAR,F}} \rangle$, is defined as follows.

**Definition 5.4.** The BRS $\langle \overline{\mathcal{R}}_{\text{PAR}}, \overline{\mathcal{R}}_{\text{PAR,F}} \rangle$ is the least parallel BRS over $\text{Var}$ and the alphabet $\Sigma' = \Sigma \cup \{\$,\#\}$ satisfying the following properties:

1. $\overline{\mathcal{R}}_{\text{PAR}} \supseteq \mathcal{R}^p$, and $\overline{\mathcal{R}}_{\text{PAR,F}} \supseteq \mathcal{R}_F^p$;
2. $X \overset{\#}{\rightarrow} Y \in \overline{\mathcal{R}}_{\text{PAR}}$, if there is a rule $X \overset{a}{\rightarrow} Y.(Z) \in \mathcal{R} \setminus \mathcal{R}_F$, and $Z \overset{\sigma^*}{\rightarrow}_{\overline{\mathcal{R}}_{\text{PAR}}} \epsilon$ with $\sigma$ non-accepting in $\overline{\mathcal{R}}_{\text{PAR}}$;
3. $X \overset{\$}{\rightarrow} Y \in \overline{\mathcal{R}}_{\text{PAR,F}}$, if there is a rule $r = X \overset{a}{\rightarrow} Y.(Z) \in \mathcal{R}$, and $Z \overset{\sigma^*}{\rightarrow}_{\overline{\mathcal{R}}_{\text{PAR}}} \epsilon$ and, either $\sigma$ is accepting in $\overline{\mathcal{R}}_{\text{PAR}}$, or $r \in \mathcal{R}_F$;
4. $X \overset{\#}{\rightarrow} W' \in \overline{\mathcal{R}}_{\text{PAR}}$, if there are rules $X \overset{a}{\rightarrow} Y.(Z) \in \mathcal{R} \setminus \mathcal{R}_F$ and $Y.(W) \overset{b}{\rightarrow} W' \in \mathcal{R} \setminus \mathcal{R}_F$, and $Z \overset{\sigma^*}{\rightarrow}_{\overline{\mathcal{R}}_{\text{PAR}}} W$, with $\sigma$ non-accepting;
5. $X \overset{\$}{\rightarrow} W' \in \overline{\mathcal{R}}_{\text{PAR,F}}$, if there are rules $r = X \overset{a}{\rightarrow} Y.(Z) \in \mathcal{R}$ and $r' = Y.(W) \overset{b}{\rightarrow} W' \in \mathcal{R}$, and $Z \overset{\sigma^*}{\rightarrow}_{\overline{\mathcal{R}}_{\text{PAR}}} W$ and, either $\sigma$ is accepting in $\overline{\mathcal{R}}_{\text{PAR}}$, or $r \in \mathcal{R}_F$, or $r' \in \mathcal{R}_F$.

**Lemma 5.1.** The parallel BRS $\langle \overline{\mathcal{R}}_{\text{PAR}}, \overline{\mathcal{R}}_{\text{PAR,F}} \rangle$ can be effectively constructed.

**Proof.** Figure\[\square\] reports the procedure $\text{BUILD-PARALLEL-BRS}(\langle \mathcal{R}, \mathcal{R}_F \rangle)$, which, starting from $\langle \mathcal{R}, \mathcal{R}_F \rangle$, builds a parallel BRS $\langle \mathcal{R}_{\text{PAR,AUX}}, \mathcal{R}_{\text{PAR,AUX,F}} \rangle$. The algorithm $\text{BUILD-PARALLEL-BRS}(\langle \mathcal{R}, \mathcal{R}_F \rangle)$ employs three auxiliary sets of rules $\overline{\mathcal{R}}$, $\overline{\mathcal{R}}_F$ and $\text{RuleSEQ}$, and a flag.

From Proposition 5.1 follows that the conditions checked in each of the if statements in lines 9, 16, 22 and 29 are decidable, therefore, the procedure is effective.

Let us show that the algorithm terminates. To see this, it suffices to prove that the number of iterations of the repeat loop is finite. Recall that the termination condition of this loop is $\text{flag} = \text{false}$. At the beginning of every iteration the flag is set to false.
Algorithm BUILD-PARALLEL-BRS(⟨ℜ, ℜ_F⟩)
1 \( ℜ_{PAR,AUX} := \{ r \in ℜ \mid r \text{ is a PAR rule} \}; \)
2 \( ℜ_{PAR,AUX,F} := \{ r \in ℜ_F \mid r \text{ is a PAR rule} \}; \)
3 \( ℜ := ℜ_F := \emptyset; \)
4 RuleSEQ := \( \{ X \xrightarrow{a} Y.(Z) \in ℜ \}; \)
5 repeat
6 \( \text{flag:=false;} \)
7 while RuleSEQ \( \neq \emptyset \) do
8 extract a rule \( X \xrightarrow{a} Y.(Z) \) from RuleSEQ;
9 if \( Z \xrightarrow{\sigma}^{*}_{PAR,AUX} \varepsilon, \) and (\( \sigma \) is accepting or \( X \xrightarrow{a} Y.(Z) \in ℜ_F \)) then
10 if \( X \xrightarrow{\$} Y \notin ℜ \) then
11 \( ℜ_{PAR,AUX} := ℜ_{PAR,AUX} \cup \{ X \xrightarrow{\$} Y \}; \)
12 \( ℜ := ℜ \cup \{ X \xrightarrow{\$} Y \}; \)
13 \( ℜ_{PAR,AUX,F} := ℜ_{PAR,AUX,F} \cup \{ X \xrightarrow{\$} Y \}; \)
14 \( ℜ_F := ℜ_F \cup \{ X \xrightarrow{\$} Y \}; \)
15 \( \text{flag:=true;} \)
16 if \( X \xrightarrow{a} Y.(Z) \notin ℜ_F \) and \( Z \xrightarrow{\sigma}^{*}_{PAR,AUX} \varepsilon \) and \( \sigma \) is not accepting then
17 if \( X \xrightarrow{\#} Y \notin ℜ \) then
18 \( ℜ_{PAR,AUX} := ℜ_{PAR,AUX} \cup \{ X \xrightarrow{\#} Y \}; \)
19 \( ℜ := ℜ \cup \{ X \xrightarrow{\#} Y \}; \)
20 \( \text{flag:=true;} \)
21 for each \( Y.(W) \xrightarrow{b} W' \in ℜ \) do
22 if \( Z \xrightarrow{\sigma}^{*}_{PAR,AUX} W \) and (\( \sigma \) is accepting or \( X \xrightarrow{a} Y.(Z) \in ℜ_F \) or \( Y.(W) \xrightarrow{b} W' \in ℜ_F \)) then
23 if \( X \xrightarrow{\$} W' \notin ℜ \) then
24 \( ℜ_{PAR,AUX} := ℜ_{PAR,AUX} \cup \{ X \xrightarrow{\$} W' \}; \)
25 \( ℜ := ℜ \cup \{ X \xrightarrow{\$} W' \}; \)
26 \( ℜ_{PAR,AUX,F} := ℜ_{PAR,AUX,F} \cup \{ X \xrightarrow{\$} W' \}; \)
27 \( ℜ_F := ℜ_F \cup \{ X \xrightarrow{\$} W' \}; \)
28 \( \text{flag:=true;} \)
29 if \( X \xrightarrow{a} Y.(Z) \notin ℜ_F \) and \( Y.(W) \xrightarrow{b} W' \notin ℜ_F \) and \( Z \xrightarrow{\sigma}^{*}_{PAR,AUX} W \) and \( \sigma \) is not accepting then
30 if \( X \xrightarrow{\#} W' \notin ℜ \) then
31 \( ℜ_{PAR,AUX} := ℜ_{PAR,AUX} \cup \{ X \xrightarrow{\#} W' \}; \)
32 \( ℜ := ℜ \cup \{ X \xrightarrow{\#} W' \}; \)
33 \( \text{flag:=true;} \)
34 done \( \triangleright \) for
35 done \( \triangleright \) while
36 RuleSEQ := \( \{ X \xrightarrow{a} Y.(Z) \in ℜ \}; \)
37 until flag = false

Figure 1: Algorithm to turn a BRS in normal form into a parallel BRS.
Moreover flag is reset to true either when a new rule of the form $X \xrightarrow{\$} Y$ is added to $\mathcal{R}$ (lines 10–15, or lines 23–28), or when a new rule of the form $X \xrightarrow{\#} Y$ is added to $\mathcal{R}$ (lines 17–20, or lines 30–33). Since the set of rules of the form $X \xrightarrow{\$} Y$ or $X \xrightarrow{\#} Y$ (with $X,Y$ in $\text{Var}$) is finite (being $\text{Var}$ finite), termination immediately follows.

We now prove that $\langle \mathcal{R}_{\text{PAR,AUX}}(\mathcal{R}_{\text{PAR,AUX,F}}) \rangle$ is a parallel BRS satisfying the properties of Definition 5.4. The following properties are clearly satisfied:

a) After initialization (lines 1–4) $\mathcal{R}_{\text{PAR,AUX}} = \mathcal{R} \cup \{ r \in \mathcal{R} \mid r$ is a PAR rule} (where for simplicity we consider the lines 11–12, 18–19, 24–25, 31–32 a single atomic instruction).

Moreover, $\mathcal{R}_{\text{PAR,AUX,F}} = \mathcal{R}_F \cup \{ r \in \mathcal{R}_F \mid r$ is a PAR rule} (where for simplicity we consider the lines 13–14, 26–27 a single atomic instruction).

b) $\mathcal{R}_F \subseteq \mathcal{R}$.

With $\mathcal{R}_{\text{PAR}}$ (resp., $\mathcal{R}_{\text{PAR}}$) we denote the set $\mathcal{R}_{\text{PAR,AUX}}$ (resp. $\mathcal{R}_{\text{PAR,AUX,F}}$) at termination of the algorithm. We show that $\langle \mathcal{R}_{\text{PAR}},\mathcal{R}_{\text{PAR,F}} \rangle$ satisfies Properties 1-5 of Definition 5.4 Property 1 is clearly satisfied as a consequence of lines 1 and 2 of the algorithm.

Let us prove Property 2. Let $X \xrightarrow{\sigma} Y$. $(Z) \in \mathcal{R} \setminus \mathcal{R}_F$ and $Z \xrightarrow{\ast}_{\mathcal{R}_{\text{PAR}}} \varepsilon$, with $\sigma$ non accepting (i.e., not containing occurrences of rules in $\mathcal{R}_{\text{PAR,F}}$). We have to show that $X \xrightarrow{\#} Y \in \mathcal{R}_{\text{PAR}}$.

In particular, we show that $X \xrightarrow{\#} Y \in \mathcal{R}$. Let us consider the last iteration of the repeat loop. Since any update of the sets $\mathcal{R}_{\text{PAR,AUX}}, \mathcal{R}_{\text{PAR,AUX,F}}, \mathcal{R}_F$ (the flag is set true) involves a new iteration of this loop, it follows that at this step $\mathcal{R}_{\text{PAR,AUX}} = \mathcal{R}_{\text{PAR}}, \mathcal{R}_{\text{PAR,AUX,F}} = \mathcal{R}_{\text{PAR,F}}$, and they will not be updated anymore. Now the rule $X \xrightarrow{\sigma} Y$. $(Z)$ is examined during an iteration of the inner while loop. During this iteration, since $\mathcal{R}_{\text{PAR,AUX}} = \mathcal{R}_{\text{PAR}}$ and $\mathcal{R}_{\text{PAR,AUX,F}} = \mathcal{R}_{\text{PAR,F}}$, the condition of the if statement in line 16 must be satisfied. On the other hand, since $\mathcal{R}_{\text{PAR,AUX}}$ and $\mathcal{R}$ cannot be update anymore, the condition of the if statement in line 17 cannot be satisfied. Therefore, $X \xrightarrow{\#} Y \in \mathcal{R}$, and Property 2 is proved. Following a similar reasoning, we can easily prove that also Properties 3–5 are satisfied.

Finally, it is easy to see that $\langle \mathcal{R}_{\text{PAR}},\mathcal{R}_{\text{PAR,F}} \rangle$ is the least parallel BRS over $\text{Var}$ and $\Sigma'$ satisfying Properties 1-5 of Definition 5.4.

Now, let us consider the parallel BRS $\langle \mathcal{R}_{\text{PAR}},\mathcal{R}_{\text{PAR,F}} \rangle$ computed by the algorithm of Lemma 5.1. As anticipated, in order to simulate subderivations of the form $C$, we need to add additional PAR rules in $\langle \mathcal{R}_{\text{PAR}},\mathcal{R}_{\text{PAR,F}} \rangle$. We need of the following decidability result.

**Proposition 5.4.** Given a BRS $\langle \mathcal{R},\mathcal{R}_F \rangle$ in normal form, and a variable $X \in \text{Var}$, it is decidable whether there exists a finite accepting derivation in $\mathcal{R}$ from $X$.

**Proof.** We show that the problem is reducible to the reachable property problem for PRSs, which is decidable (see Proposition 3.3).
Starting from $\mathfrak{R}$, we build a new $\text{PRS} \mathfrak{R'}$ in the following way. Consider the alphabet $\Sigma = \{f, nf\}$, containing only two symbols. Substitute every accepting (resp., not accepting) rule in $\mathfrak{R}$ of the form $t \xrightarrow{a} t'$ with the rule $t \xrightarrow{f} t'$ (resp., $t \xrightarrow{nf} t'$).

Clearly, there exists a finite accepting derivation in $\mathfrak{R}$ from $X$ if and only if there exists a term $t$ reachable from $X$ in $\mathfrak{R'}$ satisfying the state property $EN(f)$. This concludes the proof.

Now, we consider the set of variables $\hat{\text{Var}} = \text{Var} \cup \{Z_{\text{ACC}}, Z_{\text{NOT-ACC}}\}$. Moreover, with $T$ [resp., $T_{\text{PAR}}, T_{\text{SEQ}}$] we refer to the set of process terms [resp., the set of terms in which no sequential composition occurs, the set of terms in which no parallel composition occurs] built over $\hat{\text{Var}}$.

The following definition provides an extension of $\langle \mathfrak{R}_{\text{PAR}}, \mathfrak{R}_{\text{PAR},F} \rangle$ suitable to our purposes.

**Definition 5.5 (Rewrite System $\mathfrak{R}_{\text{PAR}}$).** The BRS $\langle \mathfrak{R}_{\text{PAR}}, \mathfrak{R}_{\text{PAR},F} \rangle$ is the parallel BRS defined from $\langle \mathfrak{R}, \mathfrak{R}_F \rangle$ and $\langle \mathfrak{R}_{\text{PAR}}, \mathfrak{R}_{\text{PAR},F} \rangle$ as follows:

- $\mathfrak{R}_{\text{PAR}} = \mathfrak{R}_{\text{PAR}} \cup \{X \rightarrow Z_{\text{ACC}} | \exists r = X \xrightarrow{a} Y. (Z) \in \mathfrak{R} \text{ such that either } r \in \mathfrak{R}_F \text{ or there exists a finite accepting derivation in } \mathfrak{R} \text{ from } Z\} \cup \{X \rightarrow Z_{\text{NOT-ACC}} | \exists r = X \xrightarrow{a} Y. (Z) \in \mathfrak{R} \setminus \mathfrak{R}_F\}$

- $\mathfrak{R}_{\text{PAR},F} = \mathfrak{R}_{\text{PAR},F} \cup \{X \rightarrow Z_{\text{ACC}} \in \mathfrak{R}_{\text{PAR}}\}$

**Lemma 5.2.** $\langle \mathfrak{R}_{\text{PAR}}, \mathfrak{R}_{\text{PAR},F} \rangle$ can be built effectively.

**Proof.** It follows directly from Proposition 5.4. □

**Remark 5.1.** Notice that $\mathfrak{R}_{\text{PAR}} \setminus \mathfrak{R}_{\text{PAR}}$ contains rules of the form $X \rightarrow Z_{\text{ACC}}$ or of the form $X \rightarrow Z_{\text{NOT-ACC}}$, and every rule in $\mathfrak{R}_{\text{PAR}}$ does not contain in the left-hand side any occurrence of $Z_{\text{ACC}}$ and $Z_{\text{NOT-ACC}}$. Therefore, it immediately follows that for all $t \in T$:

$$t \xrightarrow*[\gamma_{\text{PAR}}]{} \varepsilon \iff t \xrightarrow*[\mathfrak{R}_{\text{PAR}}]{} \varepsilon$$

(1)

and for all $X,Y \in \text{Var}$

$$X \xrightarrow*[\gamma_{\text{PAR}}]{} Y \iff Y \xrightarrow*[\mathfrak{R}_{\text{PAR}}]{} Y$$

(2)

From (1)–(2), it follows immediately that $\langle \mathfrak{R}_{\text{PAR}}, \mathfrak{R}_{\text{PAR},F} \rangle$ still satisfies properties 2-5 of Definition 5.4.

Now, let us go back to Problem 1 and consider an infinite accepting derivation of the form $X \xrightarrow*[\gamma]{}$, with $X \in \text{Var}$. If $X \xrightarrow*[\gamma]{}$ belongs to $\Pi_{\text{PAR}}$, as we have seen, it is possible to mimic that derivation with an accepting infinite derivation in $\langle \mathfrak{R}_{\text{PAR}}, \mathfrak{R}_{\text{PAR},F} \rangle$, and vice versa.

Let us now assume that $X \xrightarrow*[\gamma]{}$ does not belong to the class $\Pi_{\text{PAR}}$. In this case, the derivation $X \xrightarrow*[\gamma]{}$ can be written in the form $X \Rightarrow t\|Y.(Z) \xrightarrow*[\gamma]{}$, with $Z \in \text{Var}$, and where
the subderivation of \( t\| Y.(Z) \xrightarrow{\rho}^* \) from \( Z \) is an infinite accepting derivation in \( \mathcal{R} \) from \( Z \).

To mimic this kind of derivations, we build, starting from the BRSs \( \langle \mathcal{R}_{\text{PAR}}, \mathcal{R}_{\text{PAR,F}} \rangle \) and \( \langle \mathcal{R}, \mathcal{R}_F \rangle \), a sequential BRS \( \langle \mathcal{R}_{\text{SEQ}}, \mathcal{R}_{\text{SEQ,F}} \rangle \) according to the following definition:

**Definition 5.6 (Rewrite System \( \mathcal{R}_{\text{SEQ}} \)).** The BRS \( \langle \mathcal{R}_{\text{SEQ}}, \mathcal{R}_{\text{SEQ,F}} \rangle \) is the sequential BRS defined from \( \langle \mathcal{R}, \mathcal{R}_F \rangle \) over the alphabet \( \Sigma' = \Sigma \cup \{\$,\#\} \) as follows:

- \( \mathcal{R}_{\text{SEQ}} = \{X \xrightarrow{\alpha} Y.(Z) \in \mathcal{R}\} \cup \{X \xrightarrow{\#} Y \mid X,Y \in \text{Var} \text{ and there exists a non-accepting derivation } X \xrightarrow{\alpha}^* \mathcal{R}_{\text{PAR}} p\|Y \text{ in } \mathcal{R}_{\text{PAR}} \text{ for some } p \in \mathcal{T}_{\text{PAR}} \text{ with } |\sigma| \geq 1\} \cup \{X \xrightarrow{\$} Y \mid X,Y \in \text{Var} \text{ and there exists an accepting derivation } X \xrightarrow{\alpha}^* \mathcal{R}_{\text{PAR}} p\|Y \text{ in } \mathcal{R}_{\text{PAR}} \text{ for some } p \in \mathcal{T}_{\text{PAR}}\}

- \( \mathcal{R}_{\text{SEQ,F}} = \{X \xrightarrow{\alpha} Y.(Z) \in \mathcal{R}_F\} \cup \{X \xrightarrow{\$} Y \in \mathcal{R}_{\text{SEQ}}\} \)

**Lemma 5.3.** \( \langle \mathcal{R}_{\text{SEQ}}, \mathcal{R}_{\text{SEQ,F}} \rangle \) can be built effectively.

**Proof.** Follows directly from the definition of \( \langle \mathcal{R}_{\text{SEQ}}, \mathcal{R}_{\text{SEQ,F}} \rangle \) and Proposition 5.1.

Soundness and completeness of the procedure described above is stated by the following theorem, whose proof is reported in the appendix.

**Theorem 5.2 (Soundness and Completeness).** Given \( X \in \text{Var} \), there exists an infinite accepting derivation in \( \mathcal{R} \) from \( X \) (resp., an infinite derivation devoid of accepting rule occurrences, an infinite derivation with a finite non-null number of accepting rule occurrences) if, and only if, one of the following conditions is satisfied:

1. there exists a variable \( Y \in \text{Var} \) reachable (resp., reachable through a non-accepting derivation, reachable) from \( X \) in \( \mathcal{R}_{\text{SEQ}} \), and there exists in \( \mathcal{R}_{\text{PAR}} \) an infinite accepting derivation (resp., an infinite derivation devoid of accepting rule occurrences, an infinite derivation containing a finite non-null number of accepting rule occurrences) from \( Y \).

2. there exists in \( \mathcal{R}_{\text{SEQ}} \) an infinite accepting derivation (resp., an infinite derivation devoid of accepting rule occurrences, an infinite derivation containing a finite non-null number of accepting rule occurrences) from \( X \).

This result, together with Theorem 5.1 allow us to conclude that Problems 1-3, stated at the beginning of this section, are decidable.

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APPENDIX

A Definitions and simple properties

In this section we give some definitions and deduce simple properties that will be used in Appendices B and C for the proof of Theorem 5.2.

In the following Var denotes the set of variables Var ∪ {ZACC, ZNOT_ACC}, T denotes the set of terms over Var, and TPAR (resp., TSEQ) the set of terms in T not containing sequential (resp., parallel) composition.

Definition A.1. The set of subterms of a term t ∈ T, denoted by SubTerms(t), is defined inductively as follows:

- SubTerms(ε) = {ε};
- SubTerms(X) = {X}, for all X ∈ Var;
- SubTerms(X(t)) = SubTerms(t) ∪ {X(t)}, for all X(t) ∈ T with t ≠ ε;
- SubTerms(t1∥t2) = U(t′1,t′2)∈S(SubTerms(t′1) ∪ SubTerms(t′2)) ∪ {t1∥t2},
  with S = {(t′1,t′2) ∈ T × T | t′1, t′2 ≠ ε and t1∥t2 = t′1∥t′2} and t1, t2 ∈ T \ {ε}.

Definition A.2. The set of terms obtained from a term t ∈ T substituting an occurrence of a subterm st of t with a term t′ ∈ T, denoted by t[st → t′], is defined inductively as follows:

- t[t → t′] = {t′};
- X(t)[st → t′] = {X(s) | s ∈ t[st → t′]}, for all terms X(t) ∈ T with t ≠ ε and st ∈ SubTerms(X(t)) \ {X(t)};
- t1∥t2[st → t′] = {t″∥t′2 | (t′1, t′2) ∈ T × T, t′1, t′2 ≠ ε, t′1∥t′2 = t1∥t2, st ∈ SubTerms(t′1), t″ ∈ t′1[st → t′]}, for all t1, t2 ∈ T \ {ε} and st ∈ SubTerms(t1∥t2) \ {t1∥t2}.

Definition A.3. For a term t ∈ T, the set of terms SEQ(t) is the subset of TSEQ \ {ε} defined inductively as follows:

- SEQ(ε) = ∅;
- SEQ(X) = {X}, for all X ∈ Var;
- SEQ(X(t)) = {X(t′) | t′ ∈ SEQ(t)}, for all X ∈ Var and t ∈ T \ {ε};
- SEQ(t1∥t2) = SEQ(t1) ∪ SEQ(t2).

Definition A.4. Let σ1 and σ2 be finite sequences of rules in R (the empty sequence is denoted by ε). The interleaving of σ1 and σ2, in symbols Interleaving(σ1, σ2), is the set of rule sequences in R inductively defined as follows:
\begin{itemize}
  \item \textit{Interleaving}(\epsilon, \sigma) = \textit{Interleaving}(\sigma, \epsilon) = \{\sigma\};
  \item \textit{Interleaving}(r_1\sigma_1, r_2\sigma_2) = \{r_1\sigma \mid \sigma \in \textit{Interleaving}(\sigma_1, r_2\sigma_2)\} \cup \{r_2\sigma \mid \sigma \in \textit{Interleaving}(r_1\sigma_1, \sigma_2)\}
\end{itemize}

where \(r_1\) and \(r_2\) are rules in \(\mathcal{R}\).

For a term \(t \in T_{\text{SEQ}} \setminus \{\varepsilon\}\) having the form \(t = X_1.(X_2.(\ldots X_n.(Y).(\ldots))\), with \(n \geq 0\), we denote the variable \(Y\) by \(\text{last}(t)\). Given two terms \(t, t' \in T_{\text{SEQ}} \setminus \{\varepsilon\}\), with \(t = X_1.(X_2.(\ldots X_n.(Y).(\ldots))\) and \(t' = X'_1.(X'_2.(\ldots X'_{n'}.(Y').(\ldots))\), we denote with \(t \circ t'\) the term \(X_1.(X_2.(\ldots X_n.(X'_1.(X'_2.(\ldots X'_{n'}.(Y').(\ldots))).\ldots))\). Notice that \(t \circ t'\) is the only term in \(t[Y \rightarrow t']\) and that the operation \(\circ\) on terms in \(T_{\text{SEQ}} \setminus \{\varepsilon\}\) is associative.

\textbf{Remark A.1.} \textit{For terms} \(t, t' \in T\), \textit{with} \(t' \neq \varepsilon\) \textit{and} \(s \in \text{SubTerms}(t)\), \textit{it holds that if} \(s \in t[st \rightarrow t']\), \textit{then} \(t' \in \text{SubTerms}(s)\).

\textbf{Proposition A.1.} \textit{The following properties hold:}

\textbf{P1.} \textit{If} \(t \xrightarrow{r} t'\) \textit{and} \(t \in \text{SubTerms}(s)\), \textit{for some} \(s \in T\), \textit{then it holds} \(s \xrightarrow{r} s'\), \textit{for all} \(s' \in s[t \rightarrow t']\);

\textbf{P2.} \textit{If} \(t \xrightarrow{r} t'\) \textit{is an infinite derivation in} \(\mathcal{R}\) \textit{and} \(t \in \text{SubTerms}(s)\), \textit{for some} \(s \in T\), \textit{then it holds} \(s \xrightarrow{r} s'\).

\textbf{Proof.} \textit{In the proof we exploit the following property (which can be easily checked):}

\textbf{A.} \textit{If} \(t \xrightarrow{r} t'\) \textit{and} \(t \in \text{SubTerms}(s)\), \textit{for some} \(s \in T\), \textit{then it holds} \(s \xrightarrow{r} s'\) \textit{for all} \(s' \in s[t \rightarrow t']\).

Let us prove the property P1 reasoning by induction on the length of \(\sigma\).

\textbf{Base Step:} \(|\sigma| = 0\). In this case Property P1 is obvious.

\textbf{Induction Step:} \(|\sigma| > 0\). The derivation \(t \xrightarrow{r} t'\) can be written in the form

\[ t \xrightarrow{r} t' \quad \text{with} \quad |\sigma'| = |\sigma| - 1. \]

Let \(s \in T\) be a term with \(t \in \text{SubTerms}(s)\). By inductive hypothesis, it holds that \(s \xrightarrow{r} \bar{s}\), for all \(\bar{s} \in s[t \rightarrow t']\). Since \(t \xrightarrow{r} t'\) and \(t \in \text{SubTerms}(\bar{s})\) (see Remark A.1), from Property A we deduce that, for all \(s' \in \bar{s}[t \rightarrow t']\), it holds \(s \xrightarrow{r} s'\). Moreover, one can easily prove that, for all \(s' \in s[t \rightarrow t']\), there exists a \(\bar{s} \in s[t \rightarrow t']\) such that \(s' \in \bar{s}[t \rightarrow t']\). This immediately proves the thesis.

Now, let us prove Property P2. The infinite derivation \(t \xrightarrow{r} t'\) can be written in the form

\[ t \xrightarrow{r_1} t_1 \xrightarrow{r_2} t_2 \xrightarrow{r_3} \ldots \]

Now, let \(s \in T\) be a term with \(t \in \text{SubTerms}(s)\). From Property A, it follows that \(s \xrightarrow{r} s_1\), for all \(s_1 \in s[t \rightarrow t]\). Moreover, from Remark A.1 we deduce that \(t_1 \in \text{SubTerms}(s_1)\). Therefore, by repeating the reasoning above, it is possible to define a sequence of terms, \((s_n)_{n \in \mathbb{N}}\), such that

\[ s \xrightarrow{r} s_1 \quad \text{and} \quad s_n \xrightarrow{r} s_{n+1}, \quad \text{for all} \ n > 0, \]

thus proving the thesis.
By observing that $t \rightarrow^*_{\mathcal{R}} t'$, we have the thesis. Let us prove Property P2. By Property P1, we have that $t \rightarrow^*_{\mathcal{R}} t' \circ t = t'' \circ t$ for all $t'' \in T_{SEQ} \setminus \{\varepsilon\}$.

**Proof.** Let us prove Property P1. We know that $\text{last}(t) \rightarrow^*_{\mathcal{R}} t'$ and $\text{last}(t) \in \text{SubTerms}(t)$. By observing that $t[\text{last}(t) \rightarrow t'] = \{t \circ t'\}$, from Property A.1 we obtain the thesis. Let us prove Property P2. By Property P1, we have that $t \rightarrow^*_{\mathcal{R}} t' \circ t$. Moreover, for all $t'' \in T_{SEQ} \setminus \{\varepsilon\}$ we have that $t \in \text{SubTerms}(t'' \circ t)$ and $t'' \circ t[t \rightarrow t'] = \{t'' \circ t \circ t'\}$. From Property A.1 we have the thesis.

**B Proof of the sufficient condition of Theorem 5.2**

In order to prove the if direction of Theorem 5.2 we need the following Lemmata B.1 B.5

**Lemma B.1.** If $r = t \rightarrow^* \mathcal{R}_{\text{PAR}} \setminus \mathcal{R}$, then there exists a finite derivation in $\mathcal{R}$ of the form $t \rightarrow^*_{\mathcal{R}} t'$, with $|\sigma| > 0$. Moreover, $\sigma$ is accepting (resp., non-accepting), if $r \in \mathcal{R}_{\text{PAR,F}}$ (resp., $r \notin \mathcal{R}_{\text{PAR,F}}$).

**Proof.** Let $r = t \rightarrow^* t' \in \mathcal{R}_{\text{PAR,F}} \setminus \mathcal{R}$, then $c \in \# \setminus \$ and $t, t' \in \text{Var}$. Moreover, $c = \#$ (resp., $c = \$)$ if, and only if, $r \notin \mathcal{R}_{\text{PAR,F}}$ (resp., $r \notin \mathcal{R}_{\text{PAR,F}}$). Let us consider the Algorithm Build–parallel–BRS (see Lemma 5.1). Suppose that $r$ is the $n$-th rule added to $\mathcal{R}$ during the execution of the algorithm. Rule $r$ is added to $\mathcal{R}$ during an execution of the repeat loop, where a rule $r'$ of the form $X \rightarrow Y.(Z) \in \mathcal{R}$ is considered. The proof is by induction on $n$.

**Base Step** $n = 1$. At this step of the algorithm the following holds:

1. $\mathcal{R}_{\text{PAR, AUX}} = \{r \in \mathcal{R} \mid r$ is a PAR rule$\}$ and $\mathcal{R}_{\text{PAR, AUX,F}} = \{r \in \mathcal{F} \mid r$ is a PAR rule$\}$

First assume that $r \notin \mathcal{R}_{\text{PAR, F}}$. Then, $c = \#$, and there are two cases:

- $r$ is added to $\mathcal{R}$ in Line 19. Then $r = X \rightarrow^* Y$, and the condition in the if statement of Line 16 is satisfied. Therefore, $r' \notin \mathcal{F}$ and $Z \rightarrow^*_{\mathcal{R}_{\text{PAR, AUX}}} \varepsilon$, with $\rho$ devoid of (accepting) rules in $\mathcal{R}_{\text{PAR, AUX,F}}$. From Property A.1 above, $\rho$ must be a sequence of non-accepting rules in $\mathcal{R}$. Therefore, $X \rightarrow^*_{\mathcal{R}} Y.(Z) \rightarrow^*_{\mathcal{R}} Y$ is a non-accepting derivation in $\mathcal{R}$.

- $r$ is added to $\mathcal{R}$ by the inner for loop in Lines 21–34, when a rule $r''$ of the form $Y.(W) \rightarrow^* W' \in \mathcal{R}$ is considered. Then, $r = X \rightarrow^* W'$, and $r$ is added to $\mathcal{R}$ in Line 32. Hence, the condition of the if statement in Line 29 is satisfied. Therefore, $r', r'' \notin \mathcal{F}$ and $Z \rightarrow^*_{\mathcal{R}_{\text{PAR, AUX}}} W$, with $\rho$ devoid of (accepting) rules...
in $\mathcal{R}_{PAR,AUX,F}$. Again, from Property 1, $\rho$ must be a sequence of non-accepting rules in $\mathcal{R}$. It follows that $X \xrightarrow{r'} Y(Z) \xrightarrow{b} Y(W) \xrightarrow{r''} W'$ is a non-accepting derivation in $\mathcal{R}$.

Assume now that $r \in \mathcal{R}_{PAR,F}$. Then, $c = \$$, and there are two cases:

- $r$ is added to $\mathcal{R}$ in Line 12. Thus, $r = X \xrightarrow{\$$} Y$, and the condition of the if statement in Line 9 must be satisfied. In particular, we have that $Z \xrightarrow{\star_{\mathcal{R}_{PAR,AUX}}} \epsilon$.

  By Property 1 it holds that $Z \Rightarrow_{\mathcal{R}} \epsilon$. Therefore, if $r' \in \mathcal{R}_F$, the thesis is proved. Otherwise, $Z \xrightarrow{b}_{\mathcal{R}_{PAR,AUX}} \epsilon$, with $\rho$ containing occurrences of (accepting) rules in $\mathcal{R}_{PAR,AUX,F}$. In this case, the thesis immediately follows from Property 1.

- $r$ is added to $\mathcal{R}$ by the inner for loop in Lines 22–34, when a rule $r''$ of the form $Y(W) \xrightarrow{b} W' \in \mathcal{R}$ is considered. Then, $r = X \xrightarrow{\$$} W'$, and $r$ is added to $\mathcal{R}$ in Line 25. Hence, the condition of the if statement in Line 22 must be satisfied. In particular, it holds that $Z \Rightarrow_{\mathcal{R}_{PAR,AUX}} W$. By Property 1, we have $Z \Rightarrow_{\mathcal{R}} W$. Therefore, if $r' \in \mathcal{R}_F$ or $r'' \in \mathcal{R}_F$, we obtain the thesis.

  Otherwise, it holds that $Z \xrightarrow{b}_{\mathcal{R}_{PAR,AUX}} W$, with $\rho$ containing occurrences of (accepting) rules in $\mathcal{R}_{PAR,AUX,F}$. In this case, $\rho$ is a sequence of accepting rules in $\mathcal{R}$ by Property 1 and the thesis immediately follows.

**Induction Step** $n > 1$. Let $\mathcal{R}'$ be the set of the rules in $\mathcal{R}$ after $n - 1$ rules have been added. Then the following condition holds:

2. $\mathcal{R}_{PAR,AUX} = \{r \in \mathcal{R} \mid r \text{ is a PAR rule}\} \cup \mathcal{R}'$, and

   $\mathcal{R}_{PAR,AUX,F} = \{r \in \mathcal{R}_F \mid r \text{ is a PAR rule}\} \cup \{X \xrightarrow{\$$} \epsilon \in \mathcal{R} \}.$

By inductive hypothesis, the thesis holds for every rule in $\mathcal{R}'$. Let us consider the case where $r \notin \mathcal{R}_{PAR,F}$ (the proof is similar in the case where $r \in \mathcal{R}_{PAR,F}$). Then, $c = \#$, and there are two cases:

- $r$ is added to $\mathcal{R}$ in Line 19. Thus, $r = X \xrightarrow{\#} Y$, and the condition of the if statement in Line 16 must be satisfied. Therefore, $r' \notin \mathcal{R}_F$ and $Z \xrightarrow{\star_{\mathcal{R}_{PAR,AUX}}} \epsilon$ with $\rho$ devoid of (accepting) rules in $\mathcal{R}_{PAR,AUX,F}$. From Property 2 either $\rho$ contains occurrences of non-accepting rules in $\mathcal{R}$, or it contains occurrences of rules in $\mathcal{R} \setminus \mathcal{R}_{PAR,F}$. By inductive hypothesis, for every rule in $\mathcal{R} \setminus \mathcal{R}_{PAR,F}$ of the form $t_1 \xrightarrow{\epsilon} t_2$ there exists a non accepting derivation in $\mathcal{R}$ of the form $t_1 \Rightarrow_{\mathcal{R}} \epsilon$. As a consequence, there exists a non accepting derivation in $\mathcal{R}$ of the form $Z \xrightarrow{\#} \epsilon$, with $\rho'$ non-accepting. Therefore, $X \xrightarrow{\epsilon} Y(Z) \xrightarrow{\star_{\mathcal{R}}} Y$ is a non-accepting derivation in $\mathcal{R}$.

- $r$ is added to $\mathcal{R}$ by the inner for loop, when a rule $r''$ of the form $Y(W) \xrightarrow{b} W' \in \mathcal{R}$ is considered. Then, $r = X \xrightarrow{\#} W'$, and $r$ is added to $\mathcal{R}$ in Line 32. Therefore, the
condition of the if statement in Line 29 must be satisfied. Hence, \( r', r'' \notin \mathcal{R}_F \)
and \( Z \overset{\rho}{\rightarrow}_{\mathcal{R}_{\text{P AR, AUX}}} W \), with \( \rho \) devoid of (accepting) rules in \( \mathcal{R}_{\text{P AR, AUX}}. \) Again, from Property 2, either \( \rho \) contains occurrences of non-accepting rules in \( \mathcal{R} \), or it contains occurrences of rules in \( \mathcal{R} \setminus \mathcal{R}_{\text{P AR, F}}. \) By inductive hypothesis, it follows that \( Z \overset{\rho}{\rightarrow}_{\mathcal{R}} W \), with \( \rho' \) non-accepting. Therefore, the derivation \( X \overset{\rho}{\rightarrow}_{\mathcal{R}} Y. (Z) \overset{\rho}{\rightarrow}_{\mathcal{R}} Y. (W) \overset{\rho'}{\rightarrow}_{\mathcal{R}} W \) is a non-accepting derivation in \( \mathcal{R}. \)

Lemma B.2. Let \( p, p', p'' \in T_{\text{P AR}} \), where \( p' \) does not contain occurrences of \( Z_{\text{ACC}} \) and \( Z_{\text{NOT, ACC}} \), and \( p'' \) does not contain occurrences of variables in \( \text{Var}. \)

If \( p \overset{\varphi}{\rightarrow}_{\text{P AR}} p' \parallel p'' \), then there exists a term \( t \in T \) and a derivation \( p \overset{\varphi}{\rightarrow}_{\mathcal{R}} p' \parallel t \) in \( \mathcal{R} \), with \( |\varphi| > 0 \), if \( |\sigma| > 0 \). Moreover, if \( \sigma \) is accepting (resp., non-accepting) then, \( \rho \) can be chosen accepting (resp., non-accepting).

Proof. The proof is by induction on the length of the rule sequence \( \sigma \).

Base Step \( |\sigma| = 0 \). In this case the conclusion immediately follows.

Induction Step \( |\sigma| > 0 \). In this case the derivation \( p \overset{\varphi}{\rightarrow}_{\text{P AR}} p' \parallel p'' \) can be written in the following form:

\[
p \overset{\varphi}{\rightarrow}_{\text{P AR}} p' \parallel p''
\]

with \( |\varphi'| < |\sigma| \), \( r \in \mathcal{R}_{\text{P AR}} \) and \( p', p'' \in T_{\text{P AR}} \). Moreover, \( p' \) does not contain occurrences of \( Z_{\text{ACC}} \) and \( Z_{\text{NOT, ACC}} \), and \( p'' \) does not contain occurrences of variables in \( \text{Var}. \)

By inductive hypothesis, there exists a term \( t \in T \), and a derivation \( p \overset{\varphi}{\rightarrow}_{\mathcal{R}} p' \parallel t \), with \( |\varphi'| > 0 \), if \( |\sigma'| > 0 \), and \( \rho' \) accepting (resp., non-accepting), if \( \sigma' \) is accepting (resp., non-accepting). Then, there are three possible cases:

1. \( r \) is a PAR rule in \( \mathcal{R} \). From the definition of \( \mathcal{R}_{\text{P AR}} \), \( r \in \mathcal{R}_{\text{P AR}} \), and \( r \in \mathcal{R}_F \) if, and only if, \( r \in \mathcal{R}_{\text{P AR, F}}. \) Moreover, \( p' = p'' \) and \( p \overset{\varphi}{\rightarrow}_{\mathcal{R}} p' \). Therefore, \( p \overset{\varphi}{\rightarrow}_{\mathcal{R}} p' \parallel t \), with \( \rho' \) accepting (resp., non-accepting), if \( \sigma' \) is accepting (resp., non-accepting).

2. \( r \in \mathcal{R}_{\text{P AR}} \setminus \mathcal{R} \). Therefore, \( r = X \overset{a}{\rightarrow} Y \), with \( X, Y \in \text{Var} \), \( a \in \{\#, $\} \) and \( r \) accepting (resp., non-accepting), if \( a = $ \) (resp., \( a = \# \)). From Lemma B.1, we have that \( X \overset{\rho}{\rightarrow}_{\mathcal{R}} Y \), with \( \rho'' \) accepting (resp., non-accepting), if \( r \in \mathcal{R}_{\text{P AR, F}} \) (resp., \( r \notin \mathcal{R}_{\text{P AR, F}} \)), and \( |\rho''| > 0 \). Moreover, \( p' = p'' \) and \( p \overset{\varphi}{\rightarrow}_{\mathcal{R}} p' \). Hence, there exists in \( \mathcal{R} \) the derivation \( p \overset{\varphi}{\rightarrow}_{\mathcal{R}} p' \parallel t \), where the rule sequence \( \rho' \rho'' \) is accepting (resp., non-accepting), if \( \sigma' \) is accepting (resp., non-accepting).

3. \( r \in \mathcal{R}_{\text{P AR}} \setminus \mathcal{R}_{\text{P AR}} \). Therefore, \( r = X \overset{a}{\rightarrow} Y \), with \( X \in \text{Var} \), \( Y \in \{Z_{\text{ACC}}, Z_{\text{NOT, ACC}}\} \) and \( r \) accepting (resp., non-accepting), if \( Y = Z_{\text{ACC}} \) (resp., \( Y = Z_{\text{NOT, ACC}} \)). From the definition of \( \mathcal{R}_{\text{P AR}} \), it follows that \( X \overset{\rho}{\rightarrow}_{\mathcal{R}} t \), with \( \rho'' \) accepting (resp.,
not accepting), if \( r \in \mathbb{R}_{PAR,F} \) (resp., \( r \notin \mathbb{R}_{PAR,F} \)), and \(|\rho''| > 0\). Clearly, \( p'' = \overrightarrow{Y} \), and \( \overrightarrow{Y} = p' \parallel X \). Hence, \( p \xrightarrow{\rho''}_{\mathbb{R}} \overrightarrow{Y} = p' \parallel X \parallel \overrightarrow{Y} \), and the rule sequence \( \rho' \rho'' \) is accepting (resp., non–accepting), if \( \sigma' r \) is accepting (resp., non–accepting).

Lemma B.3. For every \( p \in T_{PAR} \), if \( p \xrightarrow{\sigma}_{\mathbb{R}_{PAR}} \) is an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules) in \( \mathbb{R}_{PAR} \), then there exists in \( \mathbb{R} \) an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules) from \( p \).

Proof. To prove the lemma, we use the following property:

A If \( p' \parallel p'' \xrightarrow{\sigma}_{\mathbb{R}_{PAR}} \) (with \( p', p'' \in T_{PAR} \)), and \( p'' \) does not contain variables in \( \text{Var} \), then \( p' \xrightarrow{\sigma}_{\mathbb{R}_{PAR}} \).

Property A easily follows from the observation that the left-hand side of each rule in \( \mathbb{R}_{PAR} \) does not contain occurrences of \( Z_{\text{ACC}} \) and \( Z_{\text{NOT,ACC}} \).

Let now \( p \in T_{PAR} \), and \( p \xrightarrow{\sigma}_{\mathbb{R}_{PAR}} \) be an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules) in \( \mathbb{R}_{PAR} \). We prove that there exists a sequence of terms \( (p_n)_{n \in \mathbb{N}} \) in \( T_{PAR} \), and a sequence of terms \( (t_n)_{n \in \mathbb{N} \setminus \{0\}} \) satisfying the following properties:

i. \( p_0 = p \).

ii. for all \( n \in \mathbb{N} \), there exists in \( \mathbb{R}_{PAR} \) an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules) from \( p_n \).

iii. for all \( n \in \mathbb{N} \), \( p_n \xrightarrow{p_{n+1}} t_{n+1} \), with \( p_n \) non–null and accepting (resp., non–accepting).

Since, by setting \( p_0 = p \), Property ii is satisfied for \( n = 0 \), it suffices to prove that the following property holds for any \( p \in T_{PAR} \):

B If \( p \xrightarrow{\sigma}_{\mathbb{R}_{PAR}} \) is an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules) in \( \mathbb{R}_{PAR} \), then, the following hold:

1. there exists a term \( p' \in T_{PAR} \), and a term \( t \), such that \( p \xrightarrow{\rho}_{\mathbb{R}} p' \parallel t \), with \( \rho \) non–null and accepting (resp., non–accepting), and

2. there exists in \( \mathbb{R}_{PAR} \) an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules) from \( p' \).

Let us prove Property B. The infinite derivation \( p \xrightarrow{\sigma}_{\mathbb{R}_{PAR}} \) can be written in the form:

\[
p \xrightarrow{\lambda}_{\mathbb{R}_{PAR}} \overline{p} \xrightarrow{\omega}_{\mathbb{R}_{PAR}}
\]

where \( \overline{p} \xrightarrow{\omega}_{\mathbb{R}_{PAR}} \) is an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules) in \( \mathbb{R}_{PAR} \) from \( \overline{p} \in T_{PAR} \), and \( p \xrightarrow{\lambda}_{\mathbb{R}_{PAR}} \overline{p} \) is a non–null finite accepting (resp.,

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non–accepting) derivation in \( R_{PAR} \). Now, \( \overline{p} \) can be written in the form \( p' \parallel p'' \), where \( p' \) does not contain occurrences of \( Z_{ACC} \) and \( Z_{NOT,ACC} \), and \( p'' \) does not contain occurrences of variables in \( Var \). From Property A, \( p' \stackrel{\lambda}{\Rightarrow}_{R_{PAR}} \) is an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules) in \( R_{PAR} \), hence Property B.2 holds.

\( \downarrow \) From Lemma B.2 applied to the accepting derivation (resp., non–accepting derivation) \( p \stackrel{\lambda}{\Rightarrow}_{R_{PAR}} \overline{p} = p' \parallel p'' \), there exists a term \( t \) and a derivation \( p \stackrel{\rho}{\Rightarrow}_{R} t \), with \( \rho \) non–null and accepting (resp., non–accepting), hence Property B.1 holds.

Now, let \( (p_n)_{n \in \mathbb{N}} \) and \( (t_n)_{n \in \mathbb{N} \setminus \{0\}} \) be the sequence satisfying Properties i–iii. By Property iii, the derivation

\[
p_0 \stackrel{\rho_1}{\Rightarrow}_{R} p_1 \parallel t_1 \stackrel{\rho_2}{\Rightarrow}_{R} p_2 \parallel t_1 \parallel t_2 \stackrel{\rho_2}{\Rightarrow}_{R} p_3 \parallel t_1 \parallel \ldots \parallel t_n \stackrel{\rho_n}{\Rightarrow}_{R} p_{n+1} \parallel t_1 \parallel \ldots \parallel t_n \parallel t_{n+1} \stackrel{\rho_{n+1}}{\Rightarrow}_{R} \ldots
\]

is an infinite accepting derivation (resp. an infinite derivation devoid of accepting rules) in \( R \) from \( p \). Hence the thesis. \hfill \Box

**Lemma B.4.** If \( p \stackrel{\sigma}{\Rightarrow}_{R_{PAR}} \) is an infinite derivation in \( R_{PAR} \) from \( p \in T_{PAR} \) containing a finite non–null number of accepting rule occurrences, then there exists an infinite derivation in \( R \) from \( p \) containing a finite non–null number of accepting rule occurrences.

**Proof.** The infinite derivation \( p \stackrel{\sigma}{\Rightarrow}_{R_{PAR}} \) can be written in the form:

\[
p \stackrel{\lambda}{\Rightarrow}_{R_{PAR}} \overline{p} \stackrel{\sigma}{\Rightarrow}_{R_{PAR}}
\]

where \( \overline{p} \stackrel{\sigma}{\Rightarrow}_{R_{PAR}} \) is an infinite derivation in \( R_{PAR} \) from \( \overline{p} \in T_{PAR} \) devoid of accepting rule occurrences, and \( p \stackrel{\lambda}{\Rightarrow}_{R_{PAR}} \overline{p} \) is an accepting finite derivation in \( R_{PAR} \). Now \( \overline{p} \) can be written in the form \( p' \parallel p'' \), where \( p' \) does not contain occurrences of \( Z_{ACC} \) and \( Z_{NOT,ACC} \), and \( p'' \) does not contain occurrences of variables in \( Var \). By Property A in the proof of Lemma B.3 we have that \( p' \stackrel{\sigma}{\Rightarrow}_{R_{PAR}} \) is an infinite derivation in \( R_{PAR} \) devoid of accepting rule occurrences. From Lemma B.3, there exists an infinite derivation \( p' \stackrel{\rho}{\Rightarrow}_{R} \) in \( R \) from \( p' \) devoid of accepting rule occurrences. Finally, from Lemma B.2 applied to the accepting derivation \( p \stackrel{\lambda}{\Rightarrow}_{R_{PAR}} \overline{p} = p' \parallel p'' \), there exists a term \( t \) and a derivation \( p \stackrel{\rho}{\Rightarrow}_{R} t \), with \( \rho \) accepting. Hence, the derivation \( p \stackrel{\rho}{\Rightarrow}_{R} p' \parallel t \) is an infinite derivation in \( R \) containing a finite non–null number of accepting rule occurrences. \hfill \Box

**Lemma B.5.** Let \( t, t' \in T_{SEQ} \) and \( s \) be any term in \( T \) such that \( t \in SEQ(s) \). The following results hold:

1. If \( t \stackrel{\rho}{\Rightarrow}_{R_{SEQ}} t' \), then there exists a term \( s' \in T \), with \( t' \in SEQ(s') \), such that \( s \stackrel{\rho}{\Rightarrow}_{R} s' \), and \( |\sigma| > 0 \). Moreover, if \( r \in R_{SEQ,F} \) (resp., \( r \notin R_{SEQ,F} \)), then \( \sigma \) can be chosen accepting (resp., non–accepting).

2. If \( t \stackrel{\sigma}{\Rightarrow}_{R_{SEQ}} t' \) and \( t \neq \varepsilon \), then there exists a \( s' \in T \), with \( t' \in SEQ(s') \), such that \( s \stackrel{\sigma}{\Rightarrow}_{R} s' \), and \( |\sigma'| > 0 \), if \( |\sigma| > 0 \). Moreover, if \( \sigma \) is accepting (resp., non–accepting), then \( \sigma' \) is accepting (resp., non–accepting).
3. If $t \xrightarrow{\sigma^*_\text{SEQ}}$ is an accepting infinite derivation (resp., an infinite derivation devoid of accepting rules) in $\mathcal{R}_{\text{SEQ}}$ from $t \in T_{\text{SEQ}}$, then there exists in $\mathcal{R}$ an accepting infinite derivation (resp., an infinite derivation devoid of accepting rules) from $s$.

4. If $t \xrightarrow{\sigma^*_\text{SEQ}}$ is an infinite derivation in $\mathcal{R}_{\text{SEQ}}$ from $t \in T_{\text{SEQ}}$ containing a finite non-null number of accepting rule occurrences, then there exists an infinite derivation in $\mathcal{R}$ from $s$ containing a finite non-null number of accepting rule occurrences.

Proof. Let us first prove Property 1. We use the following two properties, whose proofs are immediate. Let $t \in SEQ(s)$, $s \in T$ and $t = X_1.(X_2.(\ldots X_n.(Y)\ldots))$, with $n \geq 0$. Then:

A. if $st \in T_{\text{SEQ}} \setminus \{\varepsilon\}$ and $t' = X_1.(X_2.(\ldots X_n.(st)\ldots))$, then there exists a $s' \in s[Y \rightarrow st]$ (notice that $Y$ is a subterm of $s$) such that $t' \in SEQ(s')$.

B. if $Z \in Var$, $st' \in T$, and $st = st'\|Z$, then there exists a $s' \in s[Y \rightarrow st]$ such that $X_1.(X_2.(\ldots X_n.(Z)\ldots)) \in SEQ(s')$.

We can now distinguish the following two cases:

- $r = Y \xrightarrow{a} Z_1.(Z_2) \in \mathcal{R}$. From the definition of $\mathcal{R}_{\text{SEQ}}$, it follows that $r \in \mathcal{R}_{\text{SEQ}}$, and $r \in \mathcal{R}_F$ if, and only if, $r \in \mathcal{R}_{\text{SEQ},F}$. Moreover, $t = X_1.(X_2.(\ldots X_n.(Y)\ldots))$ and $t' = X_1.(X_2.(\ldots X_n.(Z_1.(Z_2))\ldots))$. Let $s \in T$ be such that $t \in SEQ(s)$. From Property A above, there exists a $s' \in s[Y \rightarrow Z_1.(Z_2)]$ such that $t' \in SEQ(s')$. Since $Y \xrightarrow{a} Z_1.(Z_2)$, by Proposition A.1 it follows that $s \xrightarrow{\rho} s'$, with $r \in \mathcal{R}_F$ if, and only if, $r \in \mathcal{R}_{\text{SEQ},F}$, and the thesis is proved.

- $r = Y \xrightarrow{a} Z$ with $Y, Z \in Var$ and $a \in \{\#, \$\}$. Moreover $t = X_1.(X_2.(\ldots X_n.(Y)\ldots))$ and $t' = X_1.(X_2.(\ldots X_n.(Z)\ldots))$. From the definition of $\mathcal{R}_{\text{SEQ}}$ there exists a derivation in $\mathcal{R}_{\text{PAR}}$ of the form $Y \xrightarrow{a^*_{\text{PAR}}} p\|Z$ for some $p \in T_{\text{PAR}}$, with $|\sigma| > 0$. Moreover, if $r \in \mathcal{R}_{\text{SEQ},F}$ (resp., $r \notin \mathcal{R}_{\text{SEQ},F}$), then $Y \xrightarrow{a^*_{\text{PAR}}} p\|Z$ can be chosen accepting (resp., non-accepting). From Lemma B.2 there exists a term $st$ such that $Y \xrightarrow{\rho^*_{\mathcal{R}}} st\|Z$, with $|\rho| > 0$ and $\rho$ accepting (resp., non-accepting) if $\sigma$ is accepting (resp., non-accepting). Let $s \in T$ be such that $t \in SEQ(s)$. From Property B above, there exists a term $s' \in s[Y \rightarrow st\|Z]$ such that $t' \in SEQ(s')$. Now, $Y \xrightarrow{\rho^*_{\mathcal{R}}} st\|Z$. From Proposition A.1 we conclude that $s \xrightarrow{\rho^*_{\mathcal{R}}} s'$, with $|\rho| > 0$, and $\rho$ accepting (resp., non-accepting), if $r \in \mathcal{R}_{\text{SEQ},F}$ (resp., $r \notin \mathcal{R}_{\text{SEQ},F}$). Hence the thesis.

Property 2 can easily be proved by induction on the length of $\sigma$, and using Property 1 above.

Let us now consider Property 3. The infinite accepting derivation (resp., the infinite derivation devoid of accepting rules) $t \xrightarrow{\sigma^*_{\text{SEQ}}}$ can be written in the form:

$$t \xrightarrow{\rho^*_{\text{SEQ}}} T \xrightarrow{\omega^*_{\text{SEQ}}}$$
with \( t \xrightarrow{\rho_{\mathcal{SEQ}}} \bar{t} \) a non–null finite accepting derivation (resp., finite non–accepting derivation), and \( \bar{t} \xrightarrow{\omega_{\mathcal{SEQ}}} \) an infinite accepting derivation (resp., infinite derivation devoid of accepting rules) from \( \bar{t} \in T_{\mathcal{SEQ}} \). Let \( s \in T \) be such that \( t \in SEQ(s) \). From Property 2 of the lemma, there exists a term \( \bar{s} \in T \), with \( \bar{t} \in SEQ(\bar{s}) \) and such that \( s \xrightarrow{\lambda_{\mathcal{R}}} \bar{s} \), \( |\lambda| > 0 \), and \( \lambda \) accepting (resp., non–accepting). By repeating the reasoning above, it follows that there exists a sequence of terms, \((s_n)_{n \in \mathbb{N}}\), such that for all \( n \in \mathbb{N} \):

- \( s_n \xrightarrow{\lambda_{\mathcal{R}}} s_{n+1} \), with \( \lambda_n \) accepting (resp., non–accepting), \( |\lambda_n| > 0 \) and \( s_0 = s \).

Therefore, the following derivation

\[
s = s_0 \xrightarrow{\lambda_0} s_1 \xrightarrow{\lambda_1} \ldots s_n \xrightarrow{\lambda_n} s_{n+1} \ldots
\]

is an accepting infinite derivation (resp., an infinite derivation devoid of accepting rules) in \( \mathcal{R} \) from \( s \). This proves the thesis.

We now prove Property 4. The infinite derivation \( t \xrightarrow{\sigma_{\mathcal{SEQ}}} \), containing a finite non–null number of accepting rule occurrences, can be written in the form:

\[
t \xrightarrow{\rho_{\mathcal{SEQ}}} \bar{t} \xrightarrow{\omega_{\mathcal{SEQ}}} \]

where \( t \xrightarrow{\rho_{\mathcal{SEQ}}} \bar{t} \) is a finite accepting derivation, and \( \bar{t} \xrightarrow{\omega_{\mathcal{SEQ}}} \) is an infinite derivation from \( \bar{t} \in T_{\mathcal{SEQ}} \) devoid of accepting rules. Let \( s \in T \) be such that \( t \in SEQ(s) \). From Property 2 of the lemma, there exists a term \( \bar{s} \in T \), with \( \bar{t} \in SEQ(\bar{s}) \), such that \( s \xrightarrow{\lambda_{\mathcal{R}}} \bar{s} \), with \( \lambda \) accepting. From Property 3 of the lemma, there exists an infinite derivation in \( \mathcal{R} \) from \( \bar{s} \) devoid of accepting rules. From this observation the thesis immediately follows.

We are now ready to prove the if direction of Theorem 5.2. Let \( X \in Var \) and assume that one of the following conditions holds:

- **C1** there exists a variable \( Y \) reachable (resp., reachable through a non–accepting derivation, reachable) from \( X \) in \( \mathcal{R}_{\mathcal{SEQ}} \), and there exists in \( \mathcal{R}_{\mathcal{PAR}} \) an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation containing a finite non–null number of accepting rule occurrences) from \( Y \).

- **C2** there exists in \( \mathcal{R}_{\mathcal{SEQ}} \) an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation containing a finite non–null number of accepting rule occurrences) from \( X \).

We have to prove that there exists in \( \mathcal{R} \) an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation with a finite non–null number of accepting rules) from \( X \).

First, assume that Condition **C2** holds. In this case the thesis follows from Property 3 of Lemma B.5 (resp., Property 3 of Lemma B.5, Property 4 of Lemma B.5), since \( X \in SEQ(X) \).
Assume that Condition C1 holds instead. Then, by Lemma B.3 (resp., Lemma B.3, Lemma B.4), there exists a term $t \in T_{SEQ}$ of the form $X_1.(X_2.(\ldots X_n.(Y) \ldots))$ (with $n \geq 0$), and a variable $Y$ such that:

- $X \xrightarrow{\rho_{SEQ}}^* t$, for some rule sequence $\rho$ (resp., with $\rho$ non–accepting, for some rule sequence $\rho$)
- $Y \xrightarrow{\sigma^*}$, with $\sigma$ infinite and accepting (resp., devoid of accepting rules, containing a finite non–null number of accepting rule occurrences).

From Property 2 of Lemma B.3 and the fact that $X \in SEQ(X)$, there exists a term $s \in T$, with $t \in SEQ(s)$ and $X \xrightarrow{\lambda^*} s$, for some rule sequence $\lambda$ (resp., with $\lambda$ non–accepting, for some rule sequence $\lambda$). Since $Y \in SubTerms(s)$, from Proposition A.1 it follows that $s \xrightarrow{\sigma^*}$ is an infinite derivation in $\mathcal{R}$, with $\sigma$ accepting (resp., devoid of accepting rules, containing a finite non–null number of accepting rule occurrences), hence the thesis.

C Proof of the necessary condition of Theorem 5.2

In order to prove only if direction of Theorem 5.2 we need the following Lemmata C.1–C.6.

Lemma C.1. Let $t \parallel X.(s) \xrightarrow{\sigma^*}_{\mathcal{R}}$ be a derivation in $\mathcal{R}$, and let $s \xrightarrow{\sigma^*}_{\mathcal{R}}$ be the subderivation of $t \parallel X.(s) \xrightarrow{\sigma^*}_{\mathcal{R}}$ from $s$. Then, the following properties are satisfied:

1. If $s \xrightarrow{\sigma^*}_{\mathcal{R}}$ is infinite, then it holds that $t \xrightarrow{\sigma \setminus \sigma^*}_{\mathcal{R}}$. Moreover, if $t \parallel X.(s) \xrightarrow{\sigma^*}_{\mathcal{R}}$ is in $\Pi_{PAR}$ (resp., in $\Xi_{PAR}$), then also $t \xrightarrow{\sigma \setminus \sigma^*}_{\mathcal{R}}$ is in $\Pi_{PAR}$ (resp., in $\Xi_{PAR}$).

2. If $s \xrightarrow{\sigma^*}_{\mathcal{R}}$ leads to $\varepsilon$ then, the derivation $t \parallel X.(s) \xrightarrow{\sigma^*}_{\mathcal{R}}$ can be written in the form

$$t \parallel X.(s) \xrightarrow{\sigma^*}_{\mathcal{R}} t' \parallel X \xrightarrow{\sigma^*}_{\mathcal{R}}$$

with $t \xrightarrow{\lambda^*}_{\mathcal{R}} t'$ and $\sigma_1 \in \text{Interleaving}(\lambda, \sigma')$.

3. If $s \xrightarrow{\sigma^*}_{\mathcal{R}}$ leads to a term $s' \neq \varepsilon$, then one of the following conditions is satisfied:

- There is a derivation $t \xrightarrow{\sigma \setminus \sigma^*}_{\mathcal{R}}$. Moreover, if $t \parallel X.(s) \xrightarrow{\sigma^*}_{\mathcal{R}}$ is in $\Pi_{PAR}$ (resp., in $\Xi_{PAR}$), then also $t \xrightarrow{\sigma \setminus \sigma^*}_{\mathcal{R}}$ is in $\Pi_{PAR}$ (resp., in $\Xi_{PAR}$). If $t \parallel X.(s) \xrightarrow{\sigma^*}_{\mathcal{R}}$ is finite and leads to $t'$, then $\overline{t} = X.(s') \parallel t'$ with $t \xrightarrow{\sigma \setminus \sigma^*}_{\mathcal{R}} t'$.
- $s' = W \in \text{Var}$ and the derivation $t \parallel X.(s) \xrightarrow{\sigma^*}_{\mathcal{R}}$ can be written in the form

$$t \parallel X.(s) \xrightarrow{\sigma^*}_{\mathcal{R}} t' \parallel X.(W) \xrightarrow{r}_{\mathcal{R}} t' \parallel W' \xrightarrow{\sigma^*}_{\mathcal{R}}$$

where $r = X.(W) \xrightarrow{\alpha} W' \in \mathcal{R}$. Moreover, $t \xrightarrow{\lambda^*}_{\mathcal{R}} t'$ with $\sigma_1 \in \text{Interleaving}(\lambda, \sigma')$. 

Proof. The assertion follows directly from the definition of subderivation. □

Lemma C.2. Let \( t \xrightarrow{\sigma^*}_r \) be a derivation in \( \Pi_{PAR} \) (resp., in \( \Xi_{PAR} \)). The following properties hold:

1. If \( t \xrightarrow{\sigma^*}_r \) can be written in the form \( t \xrightarrow{\sigma_1^*}_{r_1} t' \xrightarrow{\sigma_2^*}_{r_2} \), then \( t' \xrightarrow{\sigma_2^*}_{r_2} \) is in \( \Pi_{PAR} \) (resp., in \( \Xi_{PAR} \)).

2. For every finite derivation of the form \( t' \xrightarrow{\sigma^*}_r t \), the derivation \( t' \xrightarrow{\sigma^*}_r t \xrightarrow{\sigma^*}_r \) is in \( \Pi_{PAR} \) (resp., in \( \Xi_{PAR} \)).

3. For every term \( t \in T_{PAR} \) \( t \parallel p \xrightarrow{\sigma^*}_r \) is in \( \Pi_{PAR} \) (resp., in \( \Xi_{PAR} \)).

Proof. The assertion follows directly from the definition of subderivation. □

Lemma C.3. Let \( p \xrightarrow{\sigma^*}_r Y.(s) \xrightarrow{\omega^*}_r \) be a derivation with \( s \neq \varepsilon \) and \( p \in T_{PAR} \). Then, \( p \xrightarrow{\sigma^*}_r Y.(s) \) can be written in the form

\[
p \xrightarrow{\sigma_1^*}_r t' \| Z \xrightarrow{\sigma'_r} t' \| Y.(Z') \xrightarrow{\sigma_2^*}_r t \| Y.(s)
\]

with \( r = Z \rightarrow Y.(Z') \), and

\[
Z' \xrightarrow{\sigma_2'_r} s \quad \text{and} \quad t' \xrightarrow{\sigma_2^*}_r t
\]

with \( \sigma_2 \in \text{Interleaving}(\sigma'_2, \sigma''_2) \). Moreover, the following property is satisfied:

A. The subderivation of \( t' \| Y.(Z') \xrightarrow{\sigma_2^*}_r t \| Y.(s) \xrightarrow{\omega^*}_r \) from \( Z' \) can be written in the form

\[
Z' \xrightarrow{\sigma_2^*}_r s \xrightarrow{\omega^*}_r
\]

where \( s \xrightarrow{\omega^*}_r \) is the subderivation of \( t \| Y.(s) \xrightarrow{\omega^*}_r \) from \( s \).

Proof. The proof is by induction on the length of \( \sigma \).

Base Step \( |\sigma| = 1 \). In this case, there exists a rule \( r = Z \rightarrow Y.(Z') \in \mathcal{R} \) with \( p = t \parallel Z \) and \( Z' = s \). So, the first part of the assertion holds, with \( \sigma_1 \) and \( \sigma_2 \) the empty sequences. As far as Property A is concerned, it suffices to observe that in this case \( \sigma_2 \) is the empty sequence.

Induction Step \( |\sigma| > 1 \). The derivation \( p \xrightarrow{\sigma^*}_r Y.(s) \) can be written in the form

\[
p \xrightarrow{\sigma'_r} \bar{t} \xrightarrow{r'_r} t \| Y.(s),
\]

with \( r' \in \mathcal{R} \) and \( |r'| = |\sigma| - 1 \).

There are three cases:

- \( \bar{t} = t \| Y.(\bar{t}) \), with \( \bar{t} \xrightarrow{r'_r} s \). It immediately follows that \( \bar{t} \neq \varepsilon \). By inductive hypothesis, \( p \xrightarrow{\sigma^*_r} Y.(\bar{t}) \) can be written in the form \( p \xrightarrow{\sigma_1^*}_r t' \parallel Z \xrightarrow{r'_r} t' \parallel Y.(Z') \xrightarrow{\sigma_2^*}_r t \parallel Y.(\bar{t}) \), with \( r' = Z \rightarrow Y.(Z') \). Moreover, he have that \( Z' \xrightarrow{\sigma_2^*}_r \bar{t} \) and \( t' \xrightarrow{\sigma_2^*}_r t \), with \( \rho_2 \in \text{Interleaving}(\rho'_2, \rho''_2) \). As a consequence, we have that \( p \xrightarrow{\sigma^*}_r Y.(s) \).
Lemma C.4. There exists a term \( B \).

Proof. By inductive hypothesis, the subderivation of \( t' \parallel Y.(Z') \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \parallel t' \parallel Y.(Z') \) can be written in the form \( Z' \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} s \), and \( t' \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} t \) and \( \sigma \in \text{Interleaving}(\sigma', \sigma'') \). The first part of the assertion is proved.

We consider now Property \( A \). By inductive hypothesis, the subderivation of \( t' \parallel Y.(Z') \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \parallel t' \parallel Y.(Z') \) can be written in the form \( Z' \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \parallel s \parallel Y.(\bar{t}) \parallel t' \parallel Y.(\bar{t}) \), with \( \bar{t} \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \bar{t} \), where \( \bar{t} \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \bar{t} \) is the subderivation of \( t' \parallel Y.(\bar{t}) \parallel t' \parallel Y.(\bar{t}) \). Notice that \( t' \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \bar{t} \) can be written in the form \( t' \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} s \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \bar{t} \), where \( s \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \bar{t} \) is the subderivation of \( t' \parallel Y.(s) \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \bar{t} \) from \( s \). Considering that \( \alpha' = \rho \alpha' \), the thesis holds.

- \( t' = t \parallel Z' \parallel r' = Z \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} Y.(Z') \) and \( s = Z' \). In this case the first part of the assertion holds taking \( \sigma_1 = \sigma', \sigma_2 = \varepsilon \) and \( r = r' \). As far as Property \( A \) is concerned, it suffices to observe that \( \sigma_2 \) is the empty sequence.

- \( t' = t \parallel Y.(s), \) with \( t \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} t \). By inductive hypothesis, \( p \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \parallel t' \parallel Y.(s) \) can be written in the form \( p \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \parallel t' \parallel Y.(s) \), and \( r = Z \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} Y.(Z') \). Moreover, it holds that \( Z' \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} s \) and \( t' \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} t \), with \( \rho \in \text{Interleaving}(\rho, \rho') \). As a consequence, it holds that \( p \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} p \parallel Y.(s) \) can be written in the form \( p \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \parallel t' \parallel Y.(s) \), with \( \sigma_2 = \rho \alpha' \). Moreover, taking \( \sigma_2 = \rho \) and \( \sigma'' = \rho' \), it holds that \( Z' \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} s \), \( t' \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} t \), and \( \sigma_2 \in \text{Interleaving}(\sigma', \sigma'') \). This proves the first part of the assertion.

We consider now the property \( A \). By inductive hypothesis, the subderivation of \( t' \parallel Y.(Z') \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \parallel Y.(s) \) can be written in the form \( Z' \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} s \parallel s \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} \parallel Y.(s) \), where \( s \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} \) is the subderivation of \( t' \parallel Y.(s) \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \parallel Y.(s) \) from \( s \). Considering that \( \alpha' = \rho \alpha' \), the thesis holds.

Lemma C.4. If \( p \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} \parallel p' \), with \( p, p' \in T_{PAR} \), then the following properties hold:

\textbf{A.} There exists a term \( s \in T_{PAR} \) such that \( p \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} p' \parallel s \), with \( s \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} t \), \( \mu \) is non–accepting if \( \sigma \) is not accepting, and \( s = \varepsilon \) if \( t = \varepsilon \).

\textbf{B.} There exists a term \( s \in T_{PAR} \) such that \( p \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} p' \parallel s \), with \( |\rho| > 0 \) if \( |\sigma| > 0 \), \( s = \varepsilon \) if \( t = \varepsilon \), and \( \rho \) is accepting (resp., non–accepting) if \( \sigma \) is accepting (resp., non–accepting).

\textbf{Proof.} The proof is by induction on the length of finite derivations \( p \overset{\alpha'}{\Rightarrow}_{\mathcal{R}} \) in \( \mathcal{R} \) from terms in \( T_{PAR} \).

\textbf{Base Step} \( |\sigma| = 0 \). In this case the assertion is obvious.

\textbf{Induction Step} \( |\sigma| > 0 \). The derivation \( p \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \) can be written in the form

\begin{equation}
 p \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} \bar{t} \overset{\sigma'}{\Rightarrow}_{\mathcal{R}} t \parallel p'
\end{equation}

with \( r \in \mathcal{R} \), and \( |\sigma'| < |\sigma| \). There are two cases:
1. r is a PAR rule. Then, we have that $\overline{t} \in T_{\text{PAR}}$ and $r \in R_{\text{PAR}}$. Moreover, it holds that $r \in R_F$ if $r \in R_{\text{PAR,F}}$. Let us consider property A. By inductive hypothesis, there exists a $s \in T_{\text{PAR}}$ such that $\overline{t} \xrightarrow{r}_{*} s \parallel p'$ where $s \xrightarrow{t}_{\text{PAR}} \mu$ is non–accepting if $\sigma'$ is not accepting, and $s = \varepsilon$ if $t = \varepsilon$. Therefore, it holds that $p \xrightarrow{t}_{*} \overline{t} \xrightarrow{r}_{*} s \parallel p'$, where $s \xrightarrow{t}_{\text{PAR}} \mu$ is non accepting if $\sigma$ is not accepting, and $s = \varepsilon$ if $t = \varepsilon$. Let us consider B. By inductive hypothesis, there exists a $s \in T_{\text{PAR}}$ such that $\overline{t} \xrightarrow{r}_{*} s \parallel p'$ where $r \rho'$ is accepting (resp., non–accepting) if $\sigma'$ is accepting (resp., non accepting), and $s = \varepsilon$ if $t = \varepsilon$. Therefore, it holds that $p \xrightarrow{t}_{*} \overline{t} \xrightarrow{r}_{*} s \parallel p'$, where $r \rho'$ is accepting (resp., non–accepting) if $\sigma = r \sigma'$ is accepting (resp., non–accepting), and $s = \varepsilon$ if $t = \varepsilon$. Thus, the assertion is proved.

2. $r = Z \cdot A Y. (Z')$. In this case, we have that $p = p'' \parallel Z$ and $\overline{t} = p'' \parallel Y. (Z')$, with $p'' \in T_{\text{PAR}}$. From Equation (1), let $Z \xrightarrow{t_1}_{*}$ be the subderivation of $\overline{t} = p'' \parallel Y. (Z') \xrightarrow{Z}{\_} \overline{t}_* Z$ from $Z'$. By Lemma C.1, we can distinguish three subcases:

- $t_1 \neq \varepsilon$ and $p'' \xrightarrow{t_1}_{*} t'$. Moreover, we have that $t \parallel p' = t' \parallel Y. (t_1)$, $t' = p'' \parallel t''$, for some term $t''$, and $t = t'' \parallel Y. (t_1)$ (in particular, $t \neq \varepsilon$). Let us consider Property A. Since $|\sigma' \setminus \lambda| < |\sigma|$, by inductive hypothesis, there exists a term $s \in T_{\text{PAR}}$ such that $p'' \xrightarrow{t''}_{*} s \parallel p'$, where $s \xrightarrow{t''}_{\text{PAR}} \mu$ is non–accepting if $\sigma' \setminus \lambda$ is not not accepting. Therefore, we have $p'' \parallel Z \xrightarrow{t''}_{*} s \parallel Z \parallel p'$, where $s \parallel Z \xrightarrow{t''}_{\text{PAR}} s \parallel Z \xrightarrow{\lambda}_{*} t'' \parallel Y. (Z') \xrightarrow{\lambda}_{*} t'' \parallel Y. (t_1) = t$, and $\mu r \lambda$ is non accepting if $\sigma$ is not accepting, thus proving the assertion.

Let us consider now Property B. By inductive hypothesis there exists a term $s \in T_{\text{PAR}}$ such that $p'' \xrightarrow{t''}_{*} s \parallel p'$ where $r \rho'$ is accepting (resp., non–accepting) if $\sigma' \setminus \lambda$ is accepting (resp., non–accepting). Now, by definition of $R_{\text{PAR}}$, it holds that $r' = Z \rightarrow Z \in R_{\text{PAR}}$ with $Z \in \{Z_{\text{ACC}}, Z_{\text{NOT-ACC}}\}$, and $Z = Z_{\text{ACC}}$ (resp., $Z = Z_{\text{ NOT-ACC}}$) if $r \lambda$ is accepting (resp., non–accepting). Then, we have that $p = p'' \parallel Z \xrightarrow{t''}_{*} R_{\text{PAR}} p'' \parallel Z \xrightarrow{r}_{*} s \parallel p' \parallel Z$ where $r \rho'$ is accepting (resp., non–accepting) if $r \lambda (\sigma' \setminus \lambda)$ is accepting (resp., non–accepting). Since $\sigma$ is a reordering of $r \lambda (\sigma' \setminus \lambda)$, we obtain the assertion.

- $t_1 = \varepsilon$ and the derivation $p'' \parallel Y. (Z') \xrightarrow{t_1}_{*} t \parallel p'$ can be written in the following form:
  \begin{equation}
  p'' \parallel Y. (Z') \xrightarrow{t_1}_{*} t' \parallel Y \xrightarrow{t'_1}_{*} t \parallel p',
  \end{equation}
  with $p'' \xrightarrow{t_1}_{*} t'$, and $\sigma_1 \in \text{Interleaving}(\lambda, \sigma_1')$. Now, it holds that $Z' \xrightarrow{t_1}_{*} Z \varepsilon$, with $|\lambda| < |\sigma|$. By inductive hypothesis, we have $Z' \xrightarrow{t_1}_{*} Z \varepsilon$ where $\rho$ is accepting (resp., non–accepting) if $\lambda$ is accepting (resp., non–accepting). By Remark C.1 it follows that $r' = Z \varepsilon \parallel Y \in R_{\text{PAR}}$, with $c = \#$ (resp., $c = \#$) if $r \lambda$ is accepting (resp., non–accepting). So, it holds that $r' \in R_{\text{PAR,F}}$ (resp., $r' \notin R_{\text{PAR,F}}$) if $r \lambda$ is accepting (resp., non–accepting). Now, $p'' \parallel Y \xrightarrow{t_1}_{*}$
Lemma C.5. For \( p \in T_{PAR} \), let \( p \overset{\omega_1^*}_{\omega_2} \) be an accepting infinite derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation containing a finite non-null number of accepting rules) in \( \mathcal{R} \) from \( p \) belonging to \( \Pi_{PAR} \) (resp., \( \Xi_{PAR} \)). Then, there exists

t'||Y \overset{\omega_1^*}_{\omega_2} t||p' \) with \( |\sigma'_1\sigma_2| < |\sigma| \). Let us consider Property A. By inductive hypothesis, there exists a \( s \in T_{PAR} \) such that \( p''||Y \overset{\omega_1^*}_{\omega_2} s||p' \), where \( s \overset{\omega_1^*}_{\omega_2} t, \mu \) is non-accepting if \( \sigma'_1\sigma_2 \) is not accepting, and \( s = \varepsilon \) if \( t = \varepsilon \). So, we have \( p = p''||Z \overset{\omega_1^*}_{\omega_2} p''||Y \overset{\omega_1^*}_{\omega_2} s||p' \), where \( s \overset{\omega_1^*}_{\omega_2} t, \mu \) is non-accepting if \( \sigma \) is not accepting, and \( s = \varepsilon \) if \( t = \varepsilon \). Thus, the assertion is proved. Let us consider Property B. By inductive hypothesis, there exists a \( s \in T_{PAR} \) such that \( p''||Y \overset{\omega_1^*}_{\omega_2} s||p' \), where \( \rho' \) is accepting (resp., non-accepting) if \( \sigma'_1\sigma_2 \) is accepting (resp., non-accepting), and \( s = \varepsilon \) if \( t = \varepsilon \). Therefore, we have \( p = p''||Z \overset{\omega_1^*}_{\omega_2} p''||Y \overset{\omega_1^*}_{\omega_2} s||p' \) with \( \rho' \rho \) accepting (resp., non-accepting) if \( r\rho\sigma'_1\sigma_2 \) is accepting (resp., non-accepting), and \( s = \varepsilon \) if \( t = \varepsilon \). Since \( \sigma \) is a reordering of \( r\rho\sigma'_1\sigma_2 \), the assertion is proved.

- \( t_1 = W \in \text{Var} \) and the derivation \( p''||Y.(Z') \overset{\omega_1^*}_{\omega_2} t||p' \) can be written in the form

\[
p''||Y.(Z') \overset{\omega_1^*}_{\omega_2} t'||Y.(W) \overset{\omega_1^*}_{\omega_2} t''||W' \overset{\omega_1^*}_{\omega_2} t||p',
\]

with \( p'' \overset{\omega_1^*}_{\omega_2} t', \ r' = Y.(W) \overset{\omega_1^*}_{\omega_2} W' \) and \( \sigma_1 \in \text{Interleaving}(\lambda, \sigma'_1) \). Now, we have that \( Z' \overset{\omega_1^*}_{\omega_2} W \), with \( |\lambda| < |\sigma| \). By inductive hypothesis, it holds that \( Z' \overset{\omega_1^*}_{\omega_2} W \) with \( \rho \) accepting (resp., non-accepting) if \( \lambda \) is accepting (resp., non-accepting). Now, \( r = Z \overset{\omega_1^*}_{\omega_2} Y.(Z') \in \mathcal{R} \) and \( r' = Y.(W) \overset{\omega_1^*}_{\omega_2} W' \in \mathcal{R} \). By remark 5.1, it follows that \( r'' = Z \overset{\omega_1^*}_{\omega_2} W' \in \mathcal{R}_{PAR} \), with \( c = \$ \) (resp., \( c = \# \)) if \( rr' \rho' \) accepting (resp., non-accepting). So, it follows that \( r'' \in \mathcal{R}_{PAR,F} \) (resp., \( r' \notin \mathcal{R}_{PAR,F} \)) if \( rr' \rho' \) is accepting (resp., non-accepting). Now, we have a derivation \( p''||W' \overset{\omega_1^*}_{\omega_2} t'||W' \overset{\omega_1^*}_{\omega_2} t||p' \), with \( |\sigma'_1\sigma_2| < |\sigma| \). Let us consider Property A. By inductive hypothesis, there exists a term \( s \in T_{PAR} \) such that \( p''||W' \overset{\omega_1^*}_{\omega_2} s||p' \), where \( s \overset{\omega_1^*}_{\omega_2} t, \mu \) is non-accepting if \( \sigma'_1\sigma_2 \) is not accepting, and \( s = \varepsilon \) if \( t = \varepsilon \). So, we have \( p = p''||Z \overset{\omega_1^*}_{\omega_2} p''||W' \overset{\omega_1^*}_{\omega_2} s||p' \), where \( s \overset{\omega_1^*}_{\omega_2} t, \mu \) is non-accepting if \( \sigma \) is not accepting, and \( s = \varepsilon \) if \( t = \varepsilon \), thus proving the assertion. Let us consider Property B. By inductive hypothesis, there exists a term \( s \in T_{PAR} \) such that \( p''||W' \overset{\omega_1^*}_{\omega_2} s||p' \), with \( \rho' \) accepting (resp., non-accepting) if \( \sigma'_1\sigma_2 \) is accepting (resp., non-accepting), and \( s = \varepsilon \) if \( t = \varepsilon \). As a consequence, we have \( p = p''||Z \overset{\omega_1^*}_{\omega_2} p''||W' \overset{\omega_1^*}_{\omega_2} s||p' \) with \( rr' \rho' \) accepting (resp., non-accepting) if \( rr' \rho' \) is accepting (resp., non-accepting), and \( s = \varepsilon \) if \( t = \varepsilon \). Since \( \sigma \) is a reordering of \( rr' \rho' \), the assertion is proved. \( \square \)

Lemma C.5. For \( p \in T_{PAR} \), let \( p \overset{\omega_1^*}_{\omega_2} \) be an accepting infinite derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation containing a finite non-null number of accepting rules) in \( \mathcal{R} \) from \( p \) belonging to \( \Pi_{PAR} \) (resp., \( \Xi_{PAR} \)). Then, there exists
an accepting infinite derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation containing a finite non–null number of accepting rules) in $\mathcal{R}_{PAR}$ from $p$.

**Proof.** We give the proof in the case in which $p \xrightarrow{\xi_{\mathcal{R}}}^{*} \in \Pi_{PAR}$. The proof for the other two cases is similar. We have to prove that there exists an accepting infinite derivation in $\mathcal{R}_{PAR}$ from $p$. First of all, we show that there exists a term $p' \in T_{PAR}$ satisfying the following conditions:

1. $p \xrightarrow{\rho}^{*}_{PAR} p'$, with $\rho$ accepting;

2. There exists an accepting infinite derivation in $\mathcal{R}$ from $p'$ belonging to $\Pi_{PAR}$.

The derivation $p \xrightarrow{\xi_{\mathcal{R}}}^{*}$ can be written in the form

$$p \xrightarrow{\xi_{\mathcal{R}}}^{*} t_{1} \xrightarrow{r_{\mathcal{R}}} t_{2} \xrightarrow{\lambda_{\mathcal{R}}}^{*}$$

where $r$ is an accepting rule in $\mathcal{R}$ and $t_{2} \xrightarrow{\lambda_{\mathcal{R}}}^{*}$ is an accepting infinite derivation. By Property 1 of Lemma C.2, the derivation $t_{2} \xrightarrow{\lambda_{\mathcal{R}}}^{*}$ belongs to $\Pi_{PAR}$.

Let us consider the case in which $r$ is a PAR rule applied at level zero in the one–step derivation $t_{1} \xrightarrow{r} t_{2}$. In this case, we have that $t_{1} = t\parallel s$, $t_{2} = t\parallel s'$, with $s, s' \in T_{PAR}$ and $r = s \rightarrow s'$. By Property A of Lemma C.4 applied to the derivation $p \xrightarrow{\xi_{\mathcal{R}}}^{*} t_{1} = t\parallel s$, there exists a term $\overline{t} \in T_{PAR}$ such that $p \xrightarrow{\xi_{\mathcal{R}}}^{*} \overline{t}\parallel s$ and $\overline{t} \xrightarrow{\lambda_{\mathcal{R}}}^{*} t$. Then, we have a derivation $p \xrightarrow{\xi_{\mathcal{R}}}^{*} \overline{t}\parallel s \xrightarrow{r_{\mathcal{R}}} \overline{t}\parallel s'$, where $r$ is an accepting rule in $\mathcal{R}_{PAR}$. By taking $p' = \overline{t}\parallel s' \in T_{PAR}$, we obtain $p \xrightarrow{\xi_{\mathcal{R}}}^{*} p'$, with $\rho$ accepting. Moreover, the following derivation from $p'$

$$p' \xrightarrow{t'} s' \xrightarrow{r_{\mathcal{R}}} s' = t_{2} \xrightarrow{\lambda_{\mathcal{R}}}^{*}$$

is an infinite accepting derivation. Considering that $t_{2} \xrightarrow{\lambda_{\mathcal{R}}}^{*}$ is in $\Pi_{PAR}$, from Property 2 of Lemma C.2 it follows that the derivation of Eq. 2 is in $\Pi_{PAR}$. As a consequence, we have that $p'$ satisfies the desired properties.

Let us consider now the case in which $r$ is not a PAR rule applied at level zero in the derivation $t_{1} \xrightarrow{r_{\mathcal{R}}} t_{2}$. Then, we deduce that $t_{1} = t\parallel w$, $t_{2} = t\parallel X.(s)$, with $w \xrightarrow{r_{\mathcal{R}}} X.(s)$ (with $s$ possibly equal to $\varepsilon$). Moreover, either $w = X.(s')$ with $s' \xrightarrow{r_{\mathcal{R}}} s$, or $r = w\rightarrow X.(s)$ and $r$ is a $SEQ$ rule. Let us consider the first case. (The second case can be dealt with analogously.)

From Lemma C.3 applied to the derivation $p \xrightarrow{\xi_{\mathcal{R}}}^{*} t_{1} = t\parallel X.(s')$, it follows that there are two variables $Z, Z'$ such that

$$Z \xrightarrow{a} X.(Z') \in \mathcal{R}, p \xrightarrow{\mathcal{R}}^{*} t\parallel Z, \text{ and } Z' \xrightarrow{r_{\mathcal{R}}}^{*} s'.$$

By Property A of Lemma C.4 applied to the derivation $p \xrightarrow{\mathcal{R}}^{*} t\parallel Z$, there exists a term $\overline{t} \in T_{PAR}$ such that

$$p \xrightarrow{\mathcal{R}}^{*} \overline{t}\parallel Z \text{ and } \overline{t} \xrightarrow{\mathcal{R}}^{*} t.$$

With reference to Eq. 1 let $s \xrightarrow{\lambda_{\mathcal{R}}}^{*}$ be the subderivation of $t_{2} = t\parallel X.(s) \xrightarrow{\lambda_{\mathcal{R}}}^{*}$ from $s$. Notice that $s \xrightarrow{\lambda_{\mathcal{R}}}^{*}$ is not an accepting infinite derivation. By Lemma C.1 we distinguish the following cases:
• $t \Rightarrow_{\mathcal{R}}^{\lambda|\lambda'}$, and this derivation is in $\Pi_{PAR}$. Since $s \Rightarrow_{\mathcal{R}}^{\lambda'}$ is not an accepting infinite derivation, it follows that $t \Rightarrow_{\mathcal{R}}^{\lambda|\lambda'}$ is an accepting infinite derivation (belonging to $\Pi_{PAR}$). From Eq. (3), we have that $Z' \Rightarrow_{\mathcal{R}}^{r'} s' \Rightarrow_{\mathcal{R}}^{r} s$. From the definition of $\mathcal{R}_{PAR}$, we have that $r' = Z \Rightarrow Z_{ACC} \in \mathcal{R}_{PAR}$ (namely, an accepting rule in $\mathcal{R}_{PAR}$). By Eq. (4), we have a derivation $p \Rightarrow_{\mathcal{R}_{PAR}}^{*} \mathcal{T} \Leftarrow Z \Rightarrow_{\mathcal{R}_{PAR}}^{r'} \mathcal{T} \Rightarrow Z_{ACC}$. Taking $p' = \mathcal{T} \Rightarrow Z_{ACC}$, we obtain that $p \Rightarrow_{\mathcal{R}_{PAR}}^{*} p'$, with $\rho$ accepting. Moreover, from Eq. (4) we have a derivation

$$p' = \mathcal{T} \Rightarrow Z_{ACC} \Rightarrow_{\mathcal{R}}^{*} \mathcal{T} \Rightarrow Z_{ACC} \Rightarrow_{\mathcal{R}}^{\lambda|\lambda'}$$

which is an infinite accepting derivation. Considering that the derivation $t \Rightarrow_{\mathcal{R}}^{\lambda|\lambda'}$ belongs to $\Pi_{PAR}$, by properties 2 and 3 of Lemma C.2 it follows that the derivation of Eq. (5) belongs to $\Pi_{PAR}$.

This shows that $p'$ satisfies the required properties.

• $s \Rightarrow_{\mathcal{R}}^{\lambda'}$ leads to term $st \neq \varepsilon$, and the second condition of Property 3 of Lemma C.1 holds. Therefore, the derivation $t\Rightarrow_{\mathcal{R}}^{X.(s)}\Rightarrow_{\mathcal{R}}^{\lambda'}$ can be written in the form

$$t\Rightarrow_{\mathcal{R}}^{X.(s)} \Rightarrow_{\mathcal{R}}^{\lambda'} \mathcal{T} \Rightarrow Y \Rightarrow_{\mathcal{R}}^{\lambda'}$$

with $Y \in \text{Var}$ and

$$t \Rightarrow_{\mathcal{R}}^{\lambda'} \mathcal{T}, X.(s) \Rightarrow_{\mathcal{R}}^{\lambda''} Y \text{ with } \lambda' \text{ subsequence of } \lambda_1$$

(7)

By Property 1 of Lemma C.2 the derivation $\mathcal{T} \Rightarrow_{\mathcal{R}}^{Y} \Rightarrow_{\mathcal{R}}^{\lambda_{2}'}$ belongs to $\Pi_{PAR}$.

By Eq. (3) and Eq. (7), we have an accepting derivation $Z \Rightarrow_{\mathcal{R}}^{*} X.(s') \Rightarrow_{\mathcal{R}}^{s} X.(s) \Rightarrow_{\mathcal{R}}^{\lambda''}$ $Y$. By Property B of Lemma C.4 we obtain

$$Z \Rightarrow_{\mathcal{R}_{PAR}}^{*} Y \text{ with } \eta \text{ accepting}$$

(8)

From Eq. (4) and Eq. (8), we have a derivation $p \Rightarrow_{\mathcal{R}_{PAR}}^{*} \mathcal{T} \Rightarrow Z \Rightarrow_{\mathcal{R}_{PAR}}^{\eta} \mathcal{T} \Rightarrow Y$. Taking $p' = \mathcal{T} \Rightarrow Y$, we obtain $p \Rightarrow_{\mathcal{R}_{PAR}}^{*} p'$, with $\rho$ accepting. Moreover, from Eq. (4), (6) and (7) we have an infinite accepting derivation

$$p' = \mathcal{T} \Rightarrow Y \Rightarrow_{\mathcal{R}}^{*} Y \Rightarrow_{\mathcal{R}}^{\lambda'} \mathcal{T} \Rightarrow Y \Rightarrow_{\mathcal{R}}^{\lambda_{2}'}$$

(9)

Considering that the derivation $\mathcal{T} \Rightarrow_{\mathcal{R}}^{Y} \Rightarrow_{\mathcal{R}}^{\lambda_{2}'}$ belongs to $\Pi_{PAR}$, from Property 2 of Lemma C.2 it follows that the derivation of Eq. (9) belongs to $\Pi_{PAR}$.

This shows that $p'$ satisfies the required properties.

• $s \Rightarrow_{\mathcal{R}}^{\lambda'}$ leads to $\varepsilon$, and the derivation $t\Rightarrow_{\mathcal{R}}^{X.(s)}\Rightarrow_{\mathcal{R}}^{\lambda'}$ can be written in the form

$$t\Rightarrow_{\mathcal{R}}^{X.(s)} \Rightarrow_{\mathcal{R}}^{\lambda'} \mathcal{T} \Rightarrow \mathcal{T} \Rightarrow_{\mathcal{R}}^{\lambda'}$$

(10)
where \( t \xRightarrow{\lambda_1^t} \mathfrak{F} \), with \( \lambda_1^t \) subsequence of \( \lambda_1 \).

Moreover, from Property 1 of Lemma C.2, the derivation \( t \parallel X \mathrel{\xRightarrow{\lambda_1^t}} \mathfrak{F} \) belongs to \( \Pi_{\text{PAR}} \).

From Eq. (3) we have an accepting derivation \( Z \mathrel{\xRightarrow{\eta}} X.s' \mathrel{\xRightarrow{\tau}} X.s \mathrel{\xRightarrow{\lambda_1^t}} X \). From Property B of Lemma C.4 we obtain a derivation

\[
Z \mathrel{\xRightarrow{\eta}} X \text{ with } \eta \text{ accepting}
\]  

(11)

From Eq. (4) and Eq. (11), we have a derivation

\[
p \mathrel{\xRightarrow{p'}} \mathfrak{F} t \parallel Z \mathrel{\xRightarrow{\eta}} X \text{ with } \eta \text{ accepting.}
\]

Moreover, from Eq. (4) and Eq. (10) we have the following infinite accepting derivation

\[
p' = t \parallel X \mathrel{\xRightarrow{\lambda_2^t}} \mathfrak{F} \mathrel{\xRightarrow{\lambda_2^t}} X
\]  

(12)

Considering that the derivation \( t \parallel X \mathrel{\xRightarrow{\lambda_2^t}} \mathfrak{F} \) belongs to \( \Pi_{\text{PAR}} \), from Property 2 of Lemma C.2 it follows that the derivation of Eq. (12) belongs to \( \Pi_{\text{PAR}} \).

This proves that \( p' \) satisfies the required properties.

Now, by exploiting Properties 1 and 2, we can prove the thesis of the lemma. By Properties 1 and 2, it follows that there exists a sequence of terms in \( T_{\text{PAR}} \), \((p_n)_{n \in \mathbb{N}}\), satisfying the following properties:

i. \( p_0 = p \);

ii. \( p_n \xRightarrow{\rho_n} p_{n+1} \), with \( \rho_n \) accepting, for all \( n \in \mathbb{N} \);

iii. there exists an accepting infinite derivation in \( \mathfrak{R} \) from \( p_n \) belonging to \( \Pi_{\text{PAR}} \), for all \( n \in \mathbb{N} \).

The existence of such a sequence \((p_n)_{n \in \mathbb{N}}\) immediately implies the thesis.

\[\Box\]

Lemma C.6. Let \( X \in \text{Var} \) and \( X \xRightarrow{\sigma} X \) be an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation containing a finite non-null number \( n \) of accepting rule occurrences) in \( \mathfrak{R} \) from \( X \). Then, one of the following conditions is satisfied:

1. there exists a variable \( Y \) reachable (resp., reachable through a non–accepting derivation, reachable) from \( X \) in \( \mathfrak{R}_{\text{SEQ}} \), and there exists an accepting infinite derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation containing a finite non–null number \( n \) of accepting rule occurrences) in \( \mathfrak{R}_{\text{PAR}} \) from \( Y \).

2. there exists a term \( t \in T_{\text{SEQ}} \setminus \{\varepsilon\} \) with \( t = X_1.(X_2.(\ldots X_k.(Y) \ldots)) \) (with \( k \geq 0 \)) such that \( X \xRightarrow{\sigma} X \xRightarrow{\varepsilon \mathrel{\xRightarrow{t}}} \mathfrak{F} \), with \( \rho \) accepting (resp., non–accepting, accepting) in \( \mathfrak{R}_{\text{SEQ}} \), and there exists an accepting infinite derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation containing a finite number \( m \), with \( 0 \leq m < n \), of accepting rule occurrences) in \( \mathfrak{R} \) from \( Y \).
Proof. We give the proof for the case where $X \xrightarrow{\alpha} \xi$ is an infinite accepting derivation (the proof for the other two cases is similar). We have to prove that one of the following properties is satisfied:

A there exists a variable $Y$ reachable from $X$ in $\mathcal{R}_{SEQ}$, and there exists an accepting infinite derivation in $\mathcal{R}_{PAR}$ from $Y$.

B there exists a term $t \in T_{SEQ} \setminus \{\varepsilon\}$ with $t = X_1.(X_2.(\ldots X_k.(Y)\ldots))$ (with $k \geq 0$) such that $X \xrightarrow{\alpha} \xi$, with $\rho$ accepting in $\mathcal{R}_{SEQ}$, and there exists an accepting infinite derivation in $\mathcal{R}$ from $Y$.

The proof is by induction on the level $k$ of application of the first occurrence of an accepting rule $r$, in an infinite accepting derivation in $\mathcal{R}$ from a variable.

Base Step $k = 0$. If $X \xrightarrow{\alpha}$ belongs to the class $\Pi_{PAR}$, from Lemma C.3, Property A follows, setting $Y = X$. Otherwise, from Lemma C.3 it follows that the derivation $X \xrightarrow{\alpha}$ can be written in the form

$$X \xrightarrow{\alpha} t || Z \xrightarrow{\alpha} t || Y.(Z') \xrightarrow{\alpha}$$

where $r' = Z \xrightarrow{\alpha} Y.(Z')$, and the subderivation of $t || Y.(Z') \xrightarrow{\alpha}$ from $Z'$, namely $Z' \xrightarrow{\alpha}$, is an infinite accepting derivation. By noticing that every rule occurrence in $\sigma_2'$ is applied to level greater then zero in $X \xrightarrow{\alpha}$, and that we are considering the case where $k = 0$, it follows that $r$ must occur in the rule sequence $\sigma_1 r' \xi$, where $\xi = \sigma_2 \setminus \sigma_2'$. From Lemma C.1 we have that $t \xrightarrow{\alpha} \xi$. Therefore, there exists in $\mathcal{R}$ a derivation of the form $X \xrightarrow{\lambda} t' || Z \xrightarrow{\lambda} t' || Y.(Z')$, with $\lambda$ accepting, if $r'$ is not accepting. From Property B of Lemma C.3 applied to the derivation $X \xrightarrow{\lambda} t' || Z$, there exists a term $p \in T_{PAR}$, and a derivation $X \xrightarrow{\alpha} p || Z$, with $\rho$ accepting, if $\lambda$ is accepting. From the definition of $\mathcal{R}_{SEQ}$, we have that $X \xrightarrow{\alpha} Z \xrightarrow{\alpha} Y.(Z')$, with $\mu$ accepting if $\rho$ is accepting. Now, either $r'$ is an accepting rule in $\mathcal{R}$, and is an accepting rule in $\mathcal{R}_{SEQ}$ as well, or $\lambda$ is accepting, and $\mu$ is accepting as well. Therefore, variable $Z'$ is reachable from $X$ in $\mathcal{R}_{SEQ}$ through an accepting derivation, and there exists an accepting infinite derivation in $\mathcal{R}$ from $Z'$. This is exactly what Property B states.

Induction Step $k > 0$. If $X \xrightarrow{\alpha}$ belongs to the class $\Pi_{PAR}$, from Lemma C.5 Property A follows, by setting $Y = X$.

Otherwise, by Lemma C.3 the derivation $X \xrightarrow{\alpha}$ can be written in the form

$$X \xrightarrow{\alpha} t || Z \xrightarrow{\alpha} t || Y.(Z') \xrightarrow{\alpha}$$

where $r' = Z \xrightarrow{\alpha} Y.(Z')$, and the subderivation of $t || Y.(Z') \xrightarrow{\alpha}$ from $Z'$, namely $Z' \xrightarrow{\alpha}$ is an infinite accepting derivation. Let $\xi$ be the rule sequence $\sigma_2 \setminus \sigma_2'$. There can be two cases:
• the rule sequence $\sigma_1 r' \xi$ contains an occurrence of the accepting rule $r$. In this case, the thesis follows by reasoning as in the base step.

• $\sigma'_2$ contains the first occurrence of $r$ in $\sigma$. Clearly, this occurrence is the first accepting rule occurrence in the infinite derivation $Z' \Rightarrow^*_{R}$, and it is applied to level $k - 1$ in $Z' \Rightarrow^*_R$. By inductive hypothesis, the thesis holds of the derivation $Z' \Rightarrow^*_{R}$. Therefore, it suffices to prove that $Z'$ is reachable from $X$ in $R_{SEQ}$. By Property A of Lemma C.4, applied to derivation $X \equiv_\xi t \| Z$, there exists a term $p \in T_{PAR}$, and a derivation of the form $X \Rightarrow^*_{PAR} p \| Z$ in $R_{PAR}$. From the definition of $R_{SEQ}$, we finally have that $X \Rightarrow^*_\xi Z' \Rightarrow^*_\xi Y$. Hence the thesis.

Now, we can prove the only if direction of Theorem 5.2. Let $X \in Var$ and $X \Rightarrow^*_R$ be an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation with a finite non-null number of accepting rules) in $R$ from $X$. We have to prove that one of the following conditions holds:

• there exists a variable $Y$ reachable (resp., reachable through a non-accepting derivation, reachable) from $X$ in $R_{SEQ}$, and there exists in $R_{PAR}$ an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation containing a finite non-null number of accepting rule occurrences) from $Y$.

• there exists in $R_{SEQ}$ an infinite accepting derivation (resp., an infinite derivation devoid of accepting rules, an infinite derivation containing a finite non-null number of accepting rule occurrences) from $X$.

In the following, we give the proof for the case where $X \Rightarrow^*_R$ is an infinite accepting derivation (the proof for the other two cases is similar). We have to prove that one of the following conditions holds:

C1 there exists a variable $Y$ reachable from $X$ in $R_{SEQ}$, and there exists an accepting infinite derivation in $R_{PAR}$ from $Y$.

C2 there exists an accepting infinite derivation in $R_{SEQ}$ from $X$.

It suffices to prove that, assuming that Condition C1 does not hold, Condition C2 must hold. Under this hypothesis, we show that there exists a sequence of terms $(t_n)_{n \in N}$ in $T_{SEQ} \setminus \{\varepsilon\}$, satisfying the following properties:

i. $t_0 = X$

ii. for all $n \in N$, $last(t_n) \Rightarrow^*_\xi t_{n+1}$, with $\rho_n$ accepting.

iii. for all $n \in N$, there exists an infinite accepting derivation in $R$ from $last(t_n)$.
iv. for all $n \in \mathbb{N}$, the term $last(t_n)$ is reachable from $X$ in $\mathcal{R}_{SEQ}$.

For $n = 0$, Properties i, iii and iv are satisfied, by setting $t_0 = X$.

Assume now the existence of a finite sequence of terms $t_0, t_1, \ldots, t_n$ in $T_{SEQ} \setminus \{\varepsilon\}$, satisfying Properties i–iv. It suffices to prove that there exists a term $t_{n+1}$ in $T_{SEQ} \setminus \{\varepsilon\}$ satisfying the iii and iv, and a derivation $last(t_n) \xrightarrow{\rho_n}^{*} \mathcal{R}_{SEQ} t_{n+1}$, with $\rho_n$ accepting.

By inductive hypothesis, $last(t_n)$ is reachable from $X$ in $\mathcal{R}_{SEQ}$, and there exists in $\mathcal{R}$ an infinite accepting derivation from $last(t_n)$. From Lemma C.6 applied to the variable $last(t_n)$, and the fact that Condition C1 does not hold, it follows that there exists a term $t \in T_{SEQ} \setminus \{\varepsilon\}$ such that $last(t_n) \xrightarrow{\rho_n}^{*} t$, with $\rho_n$ accepting, and there exists an infinite accepting derivation in $\mathcal{R}$ from $last(t)$. The term $last(t)$ is reachable in $\mathcal{R}_{SEQ}$ from $last(t_n)$, and $last(t_n)$ is reachable from $X$ in $\mathcal{R}_{SEQ}$. Therefore, $last(t)$ is reachable in $\mathcal{R}_{SEQ}$ from $X$.

Thus, by setting $t_{n+1} = t$, we obtain the result.

Let now $(t_n)_{n \in \mathbb{N}}$ be the sequence of terms in $T_{SEQ} \setminus \{\varepsilon\}$ satisfying Properties i–iv. Then, by Property P1 of Proposition A.2 we have that for every $n \in \mathbb{N}$:

$$t_n \xrightarrow{\rho_n}^{*} \mathcal{R}_{SEQ} t_n \circ t_{n+1}$$

that is an accepting derivation. Moreover, by Property P2 of Proposition A.2 we have that, for all $n \in \mathbb{N}$:

$$t_0 \circ t_1 \circ \ldots \circ t_n \xrightarrow{\rho_n}^{*} \mathcal{R}_{SEQ} t_0 \circ t_1 \circ \ldots \circ t_n \circ t_{n+1}$$

that is an accepting derivation. Therefore, the following derivation

$$X = t_0 \xrightarrow{\rho_0}^{*} \mathcal{R}_{SEQ} t_0 \circ t_1 \xrightarrow{\rho_1}^{*} \mathcal{R}_{SEQ} t_0 \circ t_1 \circ t_2 \xrightarrow{\rho_2}^{*} \mathcal{R}_{SEQ} \ldots \xrightarrow{\rho_n}^{*} \mathcal{R}_{SEQ} t_0 \circ t_1 \circ \ldots \circ t_n$$

$$\xrightarrow{\rho_n}^{*} \mathcal{R}_{SEQ} t_0 \circ t_1 \circ \ldots \circ t_n \circ t_{n+1} \xrightarrow{\rho_{n+1}}^{*} \mathcal{R}_{SEQ} \ldots$$

is an infinite accepting derivation in $\mathcal{R}_{SEQ}$ from $X$. Hence Condition C2 holds.