Symmetries of complexified space-time and algebraic structures (in quantum theory).

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Abstract

In the paper is discussed description of some algebraic structures in quantum theory by using formal recursive constructions with “complex Poincaré group” $ISO(4, \mathbb{C})$.

I. INTRODUCTION

At 1989, on the occasion of celebration of famous E. Wigner’s 1939 paper [1], S. Weinberg wrote [2]:

... I want here to take up a challenge in footnote 1 of Wigner’s 1939 paper. The first sentence of Wigner’s paper is as follows:

“It is perhaps the most fundamental principal of quantum mechanics that the system of states forms a linear manifold."...

Looking down at the bottom of the page, one finds that footnote 1 says:

“The possibility of a future non-linear character of the quantum mechanics must be admitted of course.”

In the succeeding half-century little has been done with the idea that quantum mechanics may have small non-linearities. ...

The fundamental principal mentioned above by E. Wigner may be considered as yet another example of miracle effectiveness of algebraic structures in quantum mechanics, because it is related with possibility of representation of a group as a subset of an algebra of linear operators on the linear manifold of states. In such approach linearity is consequence of distributivity of algebraic operations “+” and “·”: $a(b + c) = ab + ac$.

On the other hand, the Wigner’s construction of representations of Poincaré group developed in [1] may be considered as an example of the theory of induced representation of semidirect products [4], i.e. the existence of a linear manifold here is a part of a structure of representation of the particular class of groups with an abelian subgroup (translations).

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So maybe success of algebraic approach in quantum mechanics is close related with a structure of the symmetry group of space-time, or otherwise, the structure of space-time follows from basic principles of quantum theory?

It is not possible to answer such a difficult question in a short article and here is simply discussed some formal structures with groups, rings and algebras, possibly relevant with such kind of questions.

II. FROM ALGEBRAS TO GROUPS AND BACK AGAIN

A. From algebras to groups ...

In this paper are not discussed Lie algebras of Lie groups and instead of it are used more simple structures like group of invertible elements of given algebra, but even such approach produces some interesting construction. Let us accept view on an algebra as a group with operators. More generally, we can consider the algebra even as a set \( A \) with four kinds of operators:

\[
\begin{align*}
S_b &: a \mapsto a + b; \quad a, b \in A, \\
L_l &: a \mapsto la; \quad a, l \in A, \\
R_r &: a \mapsto ra; \quad a, r \in A, \\
C_\lambda &: a \mapsto \lambda a; \quad a \in A, \lambda \in \mathbb{k}.
\end{align*}
\]

where \( \mathbb{k} \) is field of coefficients of the algebra \( A \), for example real or complex numbers, \( \mathbb{R} \), \( \mathbb{C} \), but the last family of operator is written here for completeness and is not discussed in detail because of some difficulties mentioned below.

For an associative algebra \( A \) all operators Eq. (1) — are transformations and if to consider only invertible \( l, r \) (and \( \lambda \neq 0 \)), then all possible compositions of the operators \( S_b, L_l, R_r \) (and \( C_\lambda \)) — are some group of transformations of set \( A \). Let us denote \( \hat{A} \) group of invertible elements of algebra \( A \), i.e. for \( l, r \in \hat{A}, \exists l^{-1}, r^{-1} \in \hat{A} \).

Let us consider first structure of group \( D(A) \) of transformations generated by elements \( S_b, L_l \); \( b \in A, l \in \hat{A} \) — the group is simpler, but encapsulate full structure of the algebra \( A \), i.e. laws of addition and multiplication. If the algebra \( A \) may be represented as subset of algebra of \( n \times n \) matrices any element of \( D(A) \) may be represented as \((2n \times 2n)\) matrix. It is enough to represent elements of \( a \in A \) as \((n \times 2n)\) matrix:

\[
a \longleftrightarrow \begin{pmatrix} a \\ 1 \end{pmatrix},
\]

then an operator \( S_b \) may be written as \((2n) \times (2n)\) matrix:

\[
\begin{pmatrix} a \\ 1 \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix}
\]

__1__I.e. here we are considering only structure of ring \( A \).

__2__For arbitrary \( l, r \) it is semigroup.
\begin{align}
S_b &\leftrightarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} a + b \\ 1 \end{pmatrix}, \\
L_l &\leftrightarrow \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}; \quad \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} la \\ 1 \end{pmatrix},
\end{align}
and operator $L_l$ may be written as $(2n) \times (2n)$ matrix
\begin{align}
D(B,L) &\equiv \begin{pmatrix} L & B \\ 0 & 1 \end{pmatrix}; \quad L \in \hat{A}, \ B \in A.
\end{align}
More precisely:
(A) Operators $S_b, L_l$ — are simply two special cases of the elements Eq. (3):
\begin{align}
S_b &= D(h,1); \quad L_l = D(a,l).
\end{align}
(B) A composition of two matrices Eq. (3) again has the same structure:
\begin{align}
\begin{pmatrix} L_1 & B_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L_2 & B_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} L_1 L_2 & L_1 B_2 + B_1 \\ 0 & 1 \end{pmatrix},
\end{align}
i.e.,
\begin{align}
D_{(L_1L_2, B_2 + B_1L_1)} = D_{(L_1, L_2)}D_{(B_2, L_2)}.
\end{align}
(C) And, finally, any element Eq. (3) itself is simply expressed as a product of two basic transformations:
\begin{align}
\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} L & B \\ 0 & 1 \end{pmatrix} \Rightarrow D_{(B,L)} = S_B L_L.
\end{align}
It should be mentioned also, that the order of elements in last expression does matter because they are not commute. The noncommutativity also related with an important expression for adjoint action of $L_L$:
\begin{align}
\begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & LB \\ 0 & 1 \end{pmatrix},
\end{align}
i.e.,
\begin{align}
L_L P_B L_L^{-1} = P_{LB}, \quad \text{or} \quad \text{Ad}[L_L](P_B) = P_{LB},
\end{align}
where $\text{Ad}[a](b) \equiv aba^{-1}$ is adjoint action of some element of a group.
Here was used an identity $L_L^{-1} = L_{L^{-1}}$. An expression for inversion for an arbitrary element of $\mathcal{D}(A)$ is also can be simply found and checked:
\begin{align}
\begin{pmatrix} L & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L^{-1} & -L^{-1}B \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\end{align}
i.e.,
\begin{align}
D_{(B,L)}^{-1} = D_{(-L^{-1}B, L^{-1})}.
\end{align}
1. Semidirect products

It is possible to describe the same things in more general way without a matrix representation by using *semidirect product* of groups. Let $B$ is abelian group with additive notation: $b_1, b_2, b_1 + b_2 \in B$ and each element of other group $L$ generates some transformation of $B$, then semidirect product $B \rtimes L$ is set of pairs $(b, l)$, $b \in B$, $l \in L$ with composition law:

$$(b_1, l_1) (b_2, l_2) = (b_1 + l_1(b_2), l_1l_2).$$

(14)

The Eq. (14) coincides with Eq. (8), i.e. group $D(A)$ can be expressed as semidirect product:

$$D(A) = A_+ \rtimes \hat{A},$$

(15)

where algebra $A$ considered as a *group with operators* mentioned earlier, i.e. $A_+$ is the algebra $A$ considered as an abelian group with respect to addition $(a, b) \mapsto a + b$ and elements of $\hat{A}$ are operators i.e. transformations of $A_+$ represented as left multiplications $a \mapsto l a$; $l \in \hat{A} \subset A$, $a \in A_+ \cong A$.

It may be useful also to consider separately actions $C_\lambda$ of scalars and to use subgroup $\hat{A}_o \equiv \hat{A}/k_*$, $\hat{A} \cong \hat{A}_o \times k_*$ where $k_* = k - \{0\}$ is a multiplicative group consisting of all element of field $k$ except zero. It is possible to construct another semidirect product with the group $\hat{A}_o$ instead of $\hat{A}$:

$$D_o(A) \equiv A_+ \rtimes \hat{A}_o.$$  

(16)

Simple examples of semidirect products are affine groups of motions of Euclidean and pseudo-Euclidean spaces $ISO(n)$ and $ISO(l, m)$, classical Galilean and Poincaré groups, an affine group of the general relativity. It should be mentioned, that $D(A)$ also can be represented as an affine group. If $A$ is $N$-dimensional algebra, then instead of $(n \times n)$-matrix representation of $A$ with $n^2 \geq N$ and $(2n \times 2n)$ matrices Eq. (2) acting on $(n \times 2n)$ matrix Eq. (2), it is possible to use an affine group of $(N + 1) \times (N + 1)$ matrices acting on vectors with $N + 1$ components.

Really, let us consider the algebra $A$ as a vector space $A_+$ and use notation $\vec{a}$ for an element of $A_+$, considered as a vector with $N$ components. Let us now consider an action of $\hat{A}$ via left multiplications: $a' = l a$. Because it is linear transformation for any $l$, it can be

3 In more general case $B$ is not necessary abelian group and multiplicative notation is used: $g_1, g_2, g_1g_2 \in B$, any element $\Lambda \in L$ generates an automorphism of $B$: $\Lambda(g_1g_2) = \Lambda(g_1)\Lambda(g_2)$, then instead of Eq. (14) in definition is used:

$$(g_1, \Lambda_1)(g_2, \Lambda_2) = (g_1\Lambda_1(g_2), \Lambda_1\Lambda_2)$$

(14')

4 $A_+$ produced from $A$ via “forgetting operation of multiplication”.

4
described by $N \times N$ matrix $\hat{l}$, $\vec{a}' = \hat{l} \vec{a}$. So instead of action of matrix Eq. (5) on a rectangular matrix Eq. (2) here is an affine group with $(N + 1) \times (N + 1)$ matrices:

$$
\begin{pmatrix}
\hat{l} & \vec{b} \\
\vec{0} & 1
\end{pmatrix}
\begin{pmatrix}
\vec{a} \\
1
\end{pmatrix}
= 
\begin{pmatrix}
\hat{l} \vec{a} + \vec{b} \\
1
\end{pmatrix}, \quad \text{(17)}
$$

where $\vec{a}, \vec{b}$ are $N$-dimensional vectors (i.e. $1 \times N$ “matrices”), $\hat{l}$ is $N \times N$ matrix, $\vec{0}$ is $N \times 1$ “matrix” (transposed vector) with $N$ zeros, and 1 is unit (scalar).

A classical Poincaré group is an example of an affine group $ISO(3, 1) = R^4 \rtimes SO(3, 1)$. A spinor Poincaré group [1, 4, 5] of quantum mechanics is also may be represented as a semidirect product $R^4 \rtimes SL(2, \mathbb{C})$, but in a less direct way, than for affine groups. For example, it is possible to represent the additive abelian group $R^4$ as a four-dimensional space of Hermitian $2 \times 2$ matrices $H$ using Pauli matrices $\sigma_i$:

$$
\vec{v} \rightarrow H = \sum_{i=0}^{3} v_i \sigma_i, \quad \det(H) = \|\vec{v}\|_{Mink} \equiv v_0^2 - v_1^2 - v_2^2 - v_3^2 \quad \text{(18)}
$$

then a transformation generated by some element $M \in SL(2, \mathbb{C})$ and used in definition of the semidirect product is represented as:

$$
H \mapsto M HM^*, \quad M \in SL(2, \mathbb{C}), \quad H = H^* \quad \text{(19)}
$$

The action Eq. (19) is the same for $M$ and $-M$, it is the well known representation of $SL(2, \mathbb{C})$ with a two-fold covering homomorphism of Lorentz group $SO(3, 1)$ [4, 5].

A matrix representation of the spinor Poincaré group is discussed below.

Let us consider now all three operators $S_b, L_l, R_r$ from Eq. (2). First two operators was used in construction of $D(\mathcal{A})$ and now it is necessary to include also $R_r$. The operators have property:

$$
R_{r_1} R_{r_2} a = R_{r_1} (ar_2) r_1 = R_{r_{2} r_1} a, \quad \text{(20)}
$$

i.e., the opposite order of elements in comparison with $L_l$. To avoid such a problem, it is possible instead of $R_r$ to use $R'_r \equiv R_{r^{-1}} = R^{-1}_r$. It should be mentioned also, that instead of a map $r \rightarrow r^{-1}$, here could be used any other antiisomorphism, for example, if $\mathcal{A}$ is $*$-algebra it may be $R'_r \equiv R_{r^*}$.

Let us denote as $\mathcal{T}(\mathcal{A})$ a group of transformations (of $\mathcal{A}$) generated by arbitrary compositions of operators $S_b, L_l, R'_r$, where $l, r \in \hat{\mathcal{A}}$. Any such transformation can be expressed as a combined action:

$$
T_{(B, L, R)} a = S_B L_L R'_R a = L a R^{-1} + B \quad \text{(21)}
$$

for some $L, R \in \hat{\mathcal{A}}$, $B \in \mathcal{A}$.

It is clear, that the composition of two such elements is again in $\mathcal{T}(\mathcal{A})$:

$$
T_{(B_1, L_1, R_1)} T_{(B_2, L_2, R_2)} = T_{(L_1 B_2 R^{-1}_1 + B_1, L_1 L_2, R_1 R_2)} \quad \text{(22)}
$$
and coincides with definition of composition for semidirect product of the additive abelian group $\mathcal{A}_+ \cong \mathcal{A}$ and a group $\hat{\mathcal{A}} \times \hat{\mathcal{A}}$ represented as direct product of two commuting subgroups of operators $L_l$ and $R_r$, where action of an element $(l, r) \in \hat{\mathcal{A}} \times \hat{\mathcal{A}}$ on $\mathcal{A}_+$ is defined simply as: $a \mapsto lar^{-1}$, i.e.

$$\left( b_1, (l_1, r_1) \right) \left( b_2, (l_2, r_2) \right) = \left( l_1 b_2 r_1^{-1} + b_1, (l_1 l_2, r_1 r_2) \right)$$  \hspace{1cm} (23)$$

and so the abstract set of triples with law of composition Eq. (22), Eq. (23) is semidirect product:

$$\tilde{T}(\mathcal{A}) \equiv \mathcal{A}_+ \rtimes (\hat{\mathcal{A}} \times \hat{\mathcal{A}}).$$  \hspace{1cm} (24)$$

It should be mentioned, that there is some difference between $\tilde{T}(\mathcal{A})$ and group of transformation $T(\mathcal{A})$ of $\mathcal{A}$ defined above. Action of $\tilde{T}(\mathcal{A})$ on $\mathcal{A}$ is defined as:

$$\tilde{T}(B, L, R): a \mapsto LaR^{-1} + B$$  \hspace{1cm} (25)$$

and so for any element of the center $c \in \mathcal{C}(\hat{\mathcal{A}})$ (i.e. $c \in \hat{\mathcal{A}}$; $ca = ac, \forall a \in \mathcal{A}$) the action of $\tilde{T}(B, L, R) \in \tilde{T}(\mathcal{A})$ coincides with action of $\tilde{T}(B, cL, cR)$.

So $\mathcal{T}(\mathcal{A})$ is the quotient group of $\tilde{T}(\mathcal{A})$ with respect to an equivalence relation $\tilde{T}(B, L, R) \sim \tilde{T}(B, cL, cR)$ discussed above. It will be shown that the spinor Poincaré group used in quantum mechanics can be represented formally as a subgroup of $\tilde{T}(\mathcal{S})$ (where $\mathcal{S}$ is Pauli algebra of $2\times2$ complex matrices).

Let us denote $\hat{\mathcal{A}}_c = \hat{\mathcal{A}}/\mathcal{C}(\hat{\mathcal{A}})$, $\hat{\mathcal{A}} \cong \hat{\mathcal{A}}_c \times \mathcal{C}(\hat{\mathcal{A}})$, then it is possible to write:

$$\mathcal{T}(\mathcal{A}) = \mathcal{A}_+ \rtimes (\hat{\mathcal{A}} \times \hat{\mathcal{A}}_c) = \mathcal{A}_+ \rtimes (\mathcal{C}(\hat{\mathcal{A}}) \times \hat{\mathcal{A}}_c \times \hat{\mathcal{A}}_c).$$  \hspace{1cm} (26)$$

Here is also justified to consider construction with $\hat{\mathcal{A}}_c$ instead of $\hat{\mathcal{A}}$:

$$\tilde{T}_c(\mathcal{A}) \equiv \mathcal{A}_+ \rtimes (\hat{\mathcal{A}}_c \times \hat{\mathcal{A}}_c).$$  \hspace{1cm} (27)$$

It is also close related with consideration of field of scalars (operators $\mathcal{C}(\lambda$ in Eq. (1)), because in many algebras discussed in the paper all elements of the center are simply $c = \lambda 1$, $\lambda \in \mathcal{K}$, or $\hat{\mathcal{A}}_c = \hat{\mathcal{A}}_c$.

If an algebra $\mathcal{A}$ is represented via $n \times n$ matrices, there is interesting representation of $\tilde{T}(B, L, R) \in \tilde{T}(\mathcal{A})$ as $(2n) \times (2n)$ matrix:

$$\tilde{T}(B, L, R) \leftrightarrow \begin{pmatrix} L & BR \\ 0 & R \end{pmatrix}. \hspace{1cm} (28)$$

Verification of Eq. (22) is straightforward:

$$\begin{pmatrix} L_1 & B_1 R_1 \\ 0 & R_1 \end{pmatrix} \begin{pmatrix} L_2 & B_2 R_2 \\ 0 & R_2 \end{pmatrix} = \begin{pmatrix} L_1 L_2 & (L_1 B_2 R_1^{-1} + B_1) R_1 R_2 \\ 0 & R_1 R_2 \end{pmatrix}. \hspace{1cm} (29)$$

If $\mathcal{A}$ is *-algebra: $(ab)^* = b^* a^*$, there is an involute automorphism of $\tilde{T}(\mathcal{A})$:

$$*: \tilde{T}(B, L, R) \mapsto \tilde{T}(B^*, R^{*-1}, L^{*-1}). \hspace{1cm} (30)$$

There is an invariant subgroup of the involution $\hat{\mathcal{D}}(\mathcal{A}) \subset \tilde{T}(\mathcal{A})$:

$$\hat{\mathcal{D}}(H, G) \equiv \tilde{T}(H, G, G^{*-1}), \quad H \in \mathcal{A}_+, H = H^*, G \in \hat{\mathcal{A}}. \hspace{1cm} (31)$$
2. Pauli algebra

As an example let us consider the Pauli algebra $\mathcal{S} = \mathbb{C}(2 \times 2)$ of all $2 \times 2$ complex matrices. In such a case the group of invertible elements is represented by matrices with nonzero determinant $\tilde{\mathcal{S}} = GL(2, \mathbb{C})$ and

$$\tilde{T}(\mathcal{S}) = \mathbb{C}^4 \rtimes (GL(2, \mathbb{C}) \times GL(2, \mathbb{C})).$$

(32)

Here is possible to show some difficulties with description of scalars. It is clear, that any matrix $M \in GL(2, \mathbb{C})$ can be represented as product $M = \lambda M'$ of scalar $\lambda = \det(M)$ and some matrix with unit determinant $M' = M/\det(M)$, $M' \in SL(2, \mathbb{C})$. But such decomposition is not unique, because the matrix $-M'$ also has unit determinant and so $M = (-\lambda)(-M')$.

Let us denote $S_\circ = PGL(2, \mathbb{C}) = GL(2, \mathbb{C})/C_* \cong SL(2, \mathbb{C})/\mathbb{Z}_2$, where $C_* = \mathbb{C} - \{0\}$ is multiplicative group of complex numbers without zero and $\mathbb{Z}_2$ is used to express a discrete group with two elements $\mathbb{Z}_2 \cong \{1, -1\}$. Construction of the last term also coincides with two-fold representation of the Lorentz group discussed in Eq. (19) and so $PGL(2, \mathbb{C}) \cong SO(3, 1)$.

It is possible to write

$$T(\mathcal{S}) = \mathbb{C}^4 \rtimes (\mathbb{C} \times PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C}))$$

(33)

and

$$T_\circ(\mathcal{S}) = \mathbb{C}^4 \rtimes (PGL(2, \mathbb{C}) \times PGL(2, \mathbb{C})).$$

(34)

The problem with $PGL(2, \mathbb{C})$ group is because of representation as a quotient group, i.e. a set of an equivalence classes, the pairs of two matrices $M, -M \in SL(2, \mathbb{C})$.

Spinor Poincaré group is a subgroup of $\tilde{D}(\mathcal{S})$, where in definition Eq. (11) is used an element $G \in SL(2, \mathbb{C}) \cong S_\circ \times \mathbb{Z}_2$.

The Eq. (11) for $\tilde{D}(\mathcal{S})$ and matrix representation Eq. (28) of $\tilde{T}(\mathcal{S})$ produce a known matrix representation of the spinor Poincaré group $\mathbb{R}^4 \rtimes SL(2, \mathbb{C})$ [5]:

$$(H, \Lambda) \longleftrightarrow \begin{pmatrix} \Lambda & H\Lambda^*-1 \\ 0 & \Lambda^*-1 \end{pmatrix}, \quad H \in \mathbb{C}(2 \times 2), \quad H = H^*, \quad \Lambda \in SL(2, \mathbb{C}).$$

(35)

3. Symmetries of “complex space-time”

The action of $T(\mathcal{S})$ group on $a \in \mathbb{C}^4$ is composition of shifts $a \mapsto a + b$, scaling (dilations) $a \mapsto \lambda a$, $\lambda \in \mathbb{C}$ and $SO(4, \mathbb{C})$ “rotations” $a \mapsto LaR^{-1}$, for some $L, R \in GL(2, \mathbb{C})$, $\det(L)/\det(R) = \pm 1$. The relation of the last transformation with “complex rotations” described by correspondence $\mathbb{C}^4 \leftrightarrow \mathbb{C}(2 \times 2)$ represented by analogue of Eq. (18) for vector $\vec{v} \in \mathbb{C}^4$ with complex coefficients $v_i$. 
B. ... and back again

In the previous part were discussed groups expressed via semidirect products like $D(\mathcal{A})$ or $T(\mathcal{A})$ for some algebra $\mathcal{A}$. Here is considered an “opposite process” of construction of a specific algebraic structure associated with a group represented as a semidirect product $G = B \rtimes L$. Let us again do not discuss field of scalars and describe construction of a ring first.

1. Nonlinear quasi-ring structure

For any group $G$ it is possible to consider a space of transformation of the group, i.e. functions $f: G \to G$, or $f \in G^G$. Such transformation is called endomorphism $e \in \text{End}(G) \subset G^G$ if $e(g_1g_2) = e(g_1)e(g_2)$, $\forall g_1, g_2 \in G$. Let us use notation $f \circ h$ for composition $f(h(g))$, $g \in G$ and also introduce an operation $f \bowtie h$:

$$(f \bowtie h)(g) \equiv f(g)h(g), \quad g, f(g), h(g) \in G \quad (f, g \in G^G).$$

The operation has the following properties:

$$(f \bowtie g)h = (fh) \bowtie (gh) \quad (\forall f, g, h \in G^G). \quad (37a)$$

**Proof:**

$$(f \bowtie g)(h(g)) = f(h(g))g(h(g)) = (fh \bowtie gh)(g).$$

$$(\forall e \in \text{End}(G), f, g \in G^G). \quad (37b)$$

**Proof:**

$$e(f \bowtie g)(g) = e(f(g)g(g)) = e(f(g))e(g(g)) = (ef) \bowtie (eg)(g).$$

$$(\forall g \in G, f, g \in G^G) \Rightarrow f \bowtie g = g \bowtie f \in \text{End}(G). \quad (37c)$$

The endomorphism should not be mixed with identity map $1(g) = g$.

Due to Eq. $(37a)$ (left distributivity), Eq. $(37b)$ (right distributivity) and Eq. $(37c)$ here is some “nonlinear analogue” of ring with zero is $\mathcal{O}$, unit is $1$.

Let us introduce $f^{-} \in G^G$:

$$f^{-}(g) \equiv (f(g))^{-1}, \quad f \bowtie f^{-} = \mathcal{O},$$

where $\mathcal{O} \in \text{End}(G)$ is the trivial endomorphism of any element to unit $1 \in G$, $\mathcal{O}(g) = 1$, $\forall g \in G$. The endomorphism should not be mixed with identity map $1(g) = g$.

Due to Eq. $(37a)$ (left distributivity), Eq. $(37b)$ (right distributivity) and Eq. $(37c)$ here is some “nonlinear analogue” of ring with zero is $\mathcal{O}$, unit is $1$. 
Let the group $G$ has a commutative subgroup $B$ and we are working only with $B$-invariant endomorphisms of $G$. It is possible to consider restrictions of such functions on $B$, i.e. endomorphisms of $B$ “recursively defined” using expressions with “constants” from group $G$ (see examples below). But for endomorphisms of $B$ Eq. (37b), Eq. (37a) show distributivity of “$\sim$” with respect to composition. If $f, g$ are endomorphisms, $fg$ is also endomorphism. For commutative group $f \sim g = g \sim f$ is endomorphism due to Eq. (37c).

So compositions of any given set of endomorphisms $B$ together with addition introduced by commutative operation “$\sim$” and negation Eq. (38) generate a ring of endomorphisms.

If $B$ is isomorphic with a vector space, it is possible to introduce complete structure of algebra on the ring by definition of action of scalars $f \mapsto \lambda f$ on the space of endomorphisms $B$ simply as:

$$(\lambda f)(b) \equiv f(\lambda b); \ f, \lambda f \in \text{End}(B). \tag{39}$$

The action of scalars and “$\sim$” satisfies to necessary identities:

$$(\alpha f \sim \beta f)(b) = (\alpha f)(b) (\beta f)(b) = f(\alpha b) f(\beta b) \forall f \in \text{End}(B) = f(\alpha b + \beta b) = ((\alpha + \beta)f)(b)$$

$$(\alpha f \sim \alpha g)(b) = (\alpha f)(b) (\alpha g)(b) = f(\alpha b) g(\alpha b) = (f \sim g)(\alpha b) = (\alpha (f \sim g))(b)$$

It was already mentioned, that for the group $G = B \times L$ the additive abelian subgroup $B$ can be considered as subspace of pairs $(b, 1) \in G$ and the subgroup $L$ corresponds to $(0, l) \in G$.

If the additive abelian group $B$ is isomorphic with vector space, the endomorphisms — are simply linear transformations (by definition).

A natural example of endomorphism of $B$ is automorphism expressed by inner automorphism of $G$, $\text{Ad}[g]: b \mapsto gb^{-1}$. For arbitrary semidirect product it is also possible to check:

$$\text{Ad}[(a, l)](b, 1) = (l(b), 1); \ (a, l) \in G; \ (b, 1), (l(b), 1) \in B \subset G. \tag{40}$$

It was already discussed in relation with Eq. (11) and matrix representation Eq. (10). It is clear from Eq. (40), the restriction of the inner automorphism on $B$ does not depends on $a$ and an element $(0, l) \in L$ may be used instead, and if $B$ is the vector space, then Eq. (11) is simply a linear transformation $b \mapsto lb, b \in B, l \in L$ similarly with expression Eq. (11) for algebras used before.

Let us use shortenings:

$^{5}$An abelian group may also be not isomorphic with a vector space, for example $U(1)$ and torus, but further in this paper are discussed only additive abelian groups isomorphic with $\mathbb{C}^n$ or $\mathbb{R}^n$.

$^{6}$I.e. invertible endomorphism.
\[ [l](b) \equiv \text{Ad}([0,l])(b,1) = (l(b),1) \equiv l(b), \quad l \in L, \ b \in B, \ [l] \in \text{End}(B) \quad (41) \]

It is possible to check, that the composition of such automorphisms is simply \([l_1][l_2] = [l_1l_2]\). Now let us consider \([l_1] \sim [l_2] \in \text{End}(B)\):

\[
([l_1] \sim [l_2])(b) = (l_1(b),1)(l_2(b),1) = (l_1(b) + l_2(b),1) = l_1(b) + l_2(b). \quad (42)
\]

It was not possible simply write \((l_1 + l_2)b\) instead of \(l_1(b) + l_2(b)\), because for elements \(l_1, l_2\) of the group \(L\) was defined only product, not addition, but anyway Eq. (42) defines some commutative operation “\(\sim\)” distributive with compositions and we have a ring of endomorphisms.

If \(B\) is isomorphic with a vector space, then the ring is a subset of an algebra of linear operators on the space and it is possible to introduce a structure of the algebra on the ring by definition action of scalars as in Eq. (39). Let us denote such algebra \(\tilde{A}G\) or \(\tilde{A}(B \rtimes L)\).

**Note:** It should be mentioned also, that if group \(L\) has some matrix (or algebraic) representation, then it is possible to write \(\hat{l}_1 + \hat{l}_2\) or \(\hat{\lambda}\), but anyway in general case \([\hat{l}_1] \sim [\hat{l}_2] \neq [\hat{l}_1 + \hat{l}_2]\) and even \(\lambda[\hat{l}] \neq [\lambda\hat{l}]\). An example is \(\tilde{A}(\hat{D}(A))\) (relevant to spinor Poincaré group) there \(\lambda[\hat{l}] = [\lambda|^{2}\lambda]\).

### 3. “Restoring” of algebraic structures

Let us consider particular cases of algebra \(\tilde{A}\) for groups \(\hat{D}(A), \hat{T}(A)\) and \(\hat{A}(A)\).

The \(\hat{D}(A)\) is represented as \(A \times \hat{A}\). Let us consider a pair \((a,l) \in \hat{D}(A), \ a \in A, \ l \in \hat{A}\), because \(\hat{A} \subset A\), it is possible to consider \(l\) as element of algebra \(A\) and write \(l_1 + l_2\) and \(\lambda l\). On the other hand, all invertible elements of \(A\) can be considered also as elements of group \(\hat{A}\) and because action of element \(l \in \hat{A}\) on \(b \in A\) is defined as left multiplication, it is simple to check for \(\hat{D}(A)\):

\[
\lambda[l] = [\lambda l], \quad \forall \lambda \neq 0, \quad (43)
\]

and also from Eq. (12) for \(\hat{D}(A)\) follows:

\[
l_1 + l_2 \in \hat{A} \implies [l_1] \sim [l_2] = [l_1 + l_2]. \quad (44)
\]

and so, if span of all invertible elements \(\hat{A}\) is full algebra \(A\), then \(\tilde{A}\Delta(A) \cong A\).

Although \(\hat{D}(A)\) is also may be represented as \(A \times \hat{A}\), the Eq. (13) here is not true, because action of an element \(l \in \hat{A}\) on \(b \in A\) is defined as \(l : b \mapsto \hat{b}l\) and so

\[
\lambda[l](b) \equiv [l](\lambda b) = \hat{\lambda}\hat{b}\lambda^* \neq [\lambda l](b) = \lambda\hat{b}\lambda^*l
\]

and Eq. (44) also is not true:

\[
([l_1] \sim [l_2])(b) = \hat{\lambda_1}\hat{b_1}l_1 + \hat{\lambda_2}\hat{b_2}l_2 \neq [l_1 + l_2](b) = (\hat{\lambda_1} + \hat{\lambda_2})b(\hat{l_1} + \hat{l_2}) \quad (46)
\]

and here really we see two different algebraic structures.
And, finally, for “biggest” group \( \mathcal{T}(A) \), \( \mathcal{D}(A) \subset \mathcal{T}(A) \), \( \tilde{\mathcal{A}}(A) \subset \mathcal{T}(A) \) it is possible to show, that if span of all invertible elements \( \tilde{\mathcal{A}} \) is full algebra \( A \), then \( \tilde{\mathcal{A}}(\mathcal{T}(A) \cong A \otimes A \).

It is clear also, that instead of \( \tilde{\mathcal{A}} \) in constructions \( \mathcal{D}(A) \) or \( \mathcal{T}(A) \) it is possible to use any subgroup \( \tilde{\mathcal{A}}' \subset \tilde{\mathcal{A}} \) with property: span\( (\tilde{\mathcal{A}}') = A \) and for groups \( \mathcal{D}'(A) \), \( \mathcal{T}'(A) \) constructed by using \( \tilde{\mathcal{A}}' \) is again true:

\[
\tilde{\mathcal{A}}\mathcal{D}'(A) \cong A, \quad \tilde{\mathcal{A}}\mathcal{T}'(A) \cong A \otimes A.
\] (47)

4. Pauli algebra and complex Poincaré group

Let us now consider the Pauli algebra \( S \). A group of invertible elements \( \tilde{S} = \text{GL}(2, \mathbb{C}) \) and a subgroup \( \tilde{S}' = \text{SL}(2, \mathbb{C}) \) satisfy the condition span\( (\tilde{A}) = \text{span}(\tilde{A}') = A \) and so structure of the Pauli algebra may be “recovered” using an affine group of \( \mathbb{C}^2 \), i.e. \( \mathcal{D}(S) = \text{IGL}(2, \mathbb{C}) \), or subgroup \( \mathcal{D}'(S) = \text{ISL}(2, \mathbb{C}) \):

\[
S = \tilde{\mathcal{A}}\mathcal{D}(S) = \tilde{\mathcal{A}}(\text{IGL}(2, \mathbb{C})) = \tilde{\mathcal{A}}\mathcal{D}'(S) = \tilde{\mathcal{A}}(\text{ISL}(2, \mathbb{C})),
\]

but the groups seem do not have some interesting applications in physics.

On the other hand, it was shown, that the spinor Poincaré group is a subgroup of \( \tilde{\mathcal{D}}(S) \), but as it was discussed in relation with Eq. (46), the algebra \( \tilde{\mathcal{A}}(\tilde{\mathcal{D}}(S)) \) is not the Pauli algebra.

So it is convenient to consider “a complex conformal Poincaré group” \( \tilde{T}(S) \) Eq. (32) or its subgroup like \( \text{ISO}(4, \mathbb{C}) \). The groups contain classical and spinor Poincaré group together with affine group \( \text{ISL}(2, \mathbb{C}) \) as subgroups.

III. CONCLUSION

For establishing examples of relations of the group of space-time symmetries and algebraic structures on the space of quantum states here was used the Pauli algebra, usually related with half-spinor representations of spin-1/2 particles. It was shown, that using simple constructions based on formal treatment of algebras as groups with operators it is possible “to recover” structure of the Pauli algebra and a linear manifold of quantum states from the group of symmetries of complex space-time, but application of similar construction \( \tilde{\mathcal{A}} \) to the spinor Poincaré group generates another algebra with additive structure Eq. (46), more appropriate for manipulations with density matrices (and probabilities).

On the other hand, complex Poincaré group \( \text{ISO}(4, \mathbb{C}) \) includes classical and spinor Poincaré groups, an affine group of a two-dimensional complex space together with a conjugated group defined by Eq. (30), — it is compatible with structure of the 2D Hilbert space. Application of the discussed constructions \( \tilde{\mathcal{A}} \) to complex Poincaré group “recovers” additive structure of the 2D Hilbert space of quantum states related with composition of wave vectors, but also associated with structures induced by its real spinor Poincaré subgroup discussed above.
These last properties of complex Poincaré group may be interesting in relation with questions of nonlocality in quantum mechanics — really, it just was mentioned that additive probabilistic picture compatible in such approach with structure of (spinor) Poincaré group, but for consistency with description of wave vectors and Hilbert spaces may be relevant more wide group of symmetries, like complex Poincaré group.

A question about nonlinearities affected in Weinberg’s paper [2] was practically not discussed here, but possibly a quasi-ring discussed in section II B 1 may be interesting from such point of view. It should be mentioned also, that although operation “⌣” defined here coincides with addition for semidirect products like the Poincaré group and defines a linear structure, from a “recursive” point of view it is nonlinear operation, as it may be clear from definition Eq. (36). So despite of “reduction” to the standard algebraic structure of quantum mechanics for (complex) Poincaré group, the quasi-algebra $\mathfrak{A}$ may generate some non-algebraic structures for more general groups like symmetry group $SO(4, 1)$ of de Sitter space. On the other hand, even for this case (or for more difficult models of general relativity) locally we have usual Poincaré group and so there are no difficulties with introduction of algebraic structures locally using the constructions like above.

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