Bound on global error of the fast multipole method for Helmholtz equation in 2-D

Wenhui Meng*

School of Mathematics, Northwest University, Xi'an, 710127, China

Abstract

This paper analyze the global error of the fast multipole method (FMM) for two-dimensional Helmholtz equation. We first propose the global error of the FMM for the discretized boundary integral operator. The error is caused by truncating Graf’s addition theorem, according to the limiting forms of Bessel and Neumann functions, we provide sharper and more precise estimates for the truncations of Graf’s addition theorem. Finally, using the estimates we derive the explicit bound and convergence rate for the global error of the FMM for Helmholtz equation, numerical experiments show that the results are valid. The method in this paper can also be applied to the FMM for other problems such as potential problems, elastostatic problems, Stokes flow problems and so on.

Keywords: Fast multipole method, Global error, Convergence rate, Helmholtz equation, Graf’s addition theorem

1 Introduction

The fast multipole method (FMM) proposed by Rokhlin [1] that has been widely applied in solving particle interaction problems and boundary integral equations. For solving a dense linear system with \(N\) unknowns by an iterative method, it will require \(O(N^2)\) operations for storing the matrix and computing the matrix-vector product. FMM can reduce the computing time and memory requirement to \(O(N)\). Some applications of the FMM for solving the Helmholtz equations can be found in [2]-[8].

Few studies were devoted so far to a serious estimation of the error of the FMM, especially for the global error. Most of the existing works only focused on the estimation of the truncation error of the multipole and local expansions, see [1]-[3], [5], [9]-[11], some of them got the bounds which involve unknown constants. In addition, some predictions and control methods for the global errors of FMM were proposed, see [4], [7] and [12], those articles give empirical formulas to determine the truncation numbers of the multipole and local expansions.

However, in FMM for solving the Helmholtz equations, the errors are produced in each step of the algorithm, not only multipole and local expansions, but also M2M, M2L and L2L translations.

*E-mail address: yonkey.mwh@163.com.
For such a complex algorithm, one expects to study its global error by theoretical method. The most interesting and important two problems are: how to describe the global error, and how to estimate its bound.

In the FMM for Helmholtz equation, the multipole and local expansions, M2L translations are based on Graf’s addition theorem for $H^{(1)}_m$, whereas M2M and L2L translations are based on Graf’s addition theorem for $J_m$. Graf’s addition theorem is\[13, 14\]:

$$B_m(|x - y|)e^{\pm i m \theta_{x - y}} = \sum_{n = -\infty}^{\infty} B_{m+n}(|x|)e^{\pm i (m+n) \theta_x} J_n(|y|)e^{\mp in \theta_y}, \quad |y| < |x|, \quad (1)$$

an alternative form is:

$$B_m(|x + y|)e^{\pm i m \theta_{x + y}} = \sum_{n = -\infty}^{\infty} B_{m-n}(|x|)e^{\pm i (m-n) \theta_x} J_n(|y|)e^{\pm in \theta_y}, \quad |y| < |x|, \quad (2)$$

where $m \in \mathbb{Z}$, $B$ denotes $J, Y, H^{(1)}, H^{(2)}$ or any linear combination of these functions. When $B = J$, the restriction $|y| < |x|$ is unnecessary. In FMM solver, the infinite sums (1) and (2) were truncated, we denote the remainder term of (1) as

$$R_{B_{m,p}}(x, y) := \left( \sum_{n = p+1}^{\infty} B_{m+n}(|x|)e^{\pm i (m+n) \theta_x} J_n(|y|)e^{\mp in \theta_y} \right),$$

and for (2) it is $R_{B_{m,p}}(x, -y)$, where $p$ is truncation number. In [20], we have studied the bounds for $R_{m,p}^J$ and $R_{m,p}^Y$, but the bound for $R_{m,p}^Y$ is not satisfactory for smaller $p$ and larger $m$, we will improve the result in this paper.

The aim of this paper is to study the global error of the FMM for the Helmholtz equation. In Section 2, we describe the global error of the FMM as the formula of $R_{m,p}^\partial$. In Section 3, according to the limiting form of $Y_n(z)$ as $z \to 0$, we propose two novel bounds for $R_{m,p}^Y$, one of them is extremely sharp for any $m$ and $p$. In Section 4, by using the bounds for $R_{m,p}^Y$ and $R_{m,p}^J$ given in Section 3 and [20], we estimate the bounds and convergence rates of the global errors of FMM for different tree structure, and those bounds do not involve unknown constant. Finally, the results are tested by numerical experiments and show that the derived bounds are valid.

2 Error of FMM for Helmholtz equation

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a connected boundary $\partial \Omega$, the solution of Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0, \quad x \in \mathbb{R}^2 \setminus \Omega$$

satisfies the Sommerfeld radiation condition

$$\lim_{R \to \infty} \sqrt{R} \left( \frac{\partial u(x)}{\partial R} - iku(x) \right) = 0, \quad R = |x|$$

can be represented as the single- or double-layer potentials\[15\]:

$$(S\varphi)(x) := 2 \int_{\partial \Omega} \Phi(x, y)\varphi(y)ds(y), \quad x \in \mathbb{R}^2 \setminus \Omega,$$
When an element can be found in \[16\]. Since the product $Az$ and some translations. The FMM algorithm is described briefly below.

The groups of nodes, which can be accomplished by the multipole and local expansions of the integrals with cell-to-cell interactions by a hierarchical tree structure (quadtree for 2-D) of cells containing $L = 0$, and then divide it into four equal cells of level 1. Continue dividing in this way, that is, take a parent cell of level $L$ and divide it into four child cells of level $L + 1$. Stop dividing a cell if it only includes $m$ points. A cell having no child cells is called a leaf cell. In addition, for each cell in the tree structure, determine the list of well separated cells.

### Construct the quadtree

Choose a square that covers the domain $\Omega$, call this square the cell of level 0, and then divide it into four equal cells of level 1. Continue dividing in this way, that is, take a parent cell of level $L$ and divide it into four child cells of level $L + 1$. Stop dividing a cell if it only includes $m$ points. A cell having no child cells is called a leaf cell. In addition, for each cell in the tree structure, determine the list of well separated cells.

### Upward pass

Compute the multipole moments at all leaves for every source point $y$, and then calculate the multipole moments of the parent cell with the M2M translation. Continue calculating the multipole moments of the cells up to level 2.

### Downward pass

Compute the local moments on all cells starting from level 2 and downward to all the leaves. The local moments of a cell are the sum of two parts. One is the contributions of

\[
(K\varphi)(x) := 2\int_{\partial\Omega} \frac{\partial\Phi(x, y)}{\partial n(y)} \varphi(y) ds(y), \quad x \in \mathbb{R}^2 \setminus \partial \Omega,
\]

where the density $\varphi$ is an integrable function, $\Phi(x, y)$ is the fundamental solution to the Helmholtz equation, in 2-D which is given by

\[
\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|), \quad x \neq y.
\]

Let the potentials satisfy boundary conditions, the integral equations about $(S\varphi)(x)$ and $(K\varphi)(x)$ for $x \in \partial \Omega$ are obtained.

Assume that the boundary curve $\partial \Omega$ is analytic, with a regular parametric representation of the form

\[
y(t) = (x_1(t), x_2(t)), \quad 0 \leq t \leq 2\pi.
\]

Choose an equidistant set of knots $y_j = y(t_j), j = 1, \ldots, 2N$, and use the quadrature rule, we have

\[
(S\varphi)(x) = 2\int_{\partial\Omega} \Phi(x, y) \varphi(y) ds(y) \approx \frac{i\pi}{2N} \sum_{j=1}^{2N} H_0^{(1)}(k|x - y_j|) \varphi(y_j)s(y_j), \quad x \in \partial \Omega,
\]

where $s(y_j) = \sqrt{x_1^2(t_j) + x_2^2(t_j)}$. For $x_i \in \partial \Omega (i = 1, \ldots, 2N)$, the discretized integral can be written as the product of matrix

\[
A = \begin{pmatrix} H_0^{(1)}(k|x_1 - y_1|) & \cdots & H_0^{(1)}(k|x_1 - y_{2N}|) \\
\vdots & \ddots & \vdots \\
H_0^{(1)}(k|x_{2N} - y_1|) & \cdots & H_0^{(1)}(k|x_{2N} - y_{2N}|) \end{pmatrix},
\]

and vector

\[
z = \left( \varphi(y_1)s(y_1), \varphi(y_2)s(y_2), \ldots, \varphi(y_{2N})s(y_{2N}) \right)^T.
\]

When $x_i = y_j$, the function $H_0^{(1)}(k|x_i - y_j|)$ has logarithmic singularity, the proper numerical treatment can be found in \[16\]. Since $A$ is a dense matrix, it will require $O(N^2)$ operation for computing the product $Az$.

FMM can accelerate the computation of $Az$. In FMM, the node-to-node interactions are replaced with cell-to-cell interactions by a hierarchical tree structure (quadtree for 2-D) of cells containing groups of nodes, which can be accomplished by the multipole and local expansions of the integrals and some translations. The FMM algorithm is described briefly below.

### Construct the quadtree

Choose a square that covers the domain $\Omega$, call this square the cell of level 0, and then divide it into four equal cells of level 1. Continue dividing in this way, that is, take a parent cell of level $L$ and divide it into four child cells of level $L + 1$. Stop dividing a cell if it only includes $m$ points. A cell having no child cells is called a leaf cell. In addition, for each cell in the tree structure, determine the list of well separated cells.

### Upward pass

Compute the multipole moments at all leaves for every source point $y$, and then calculate the multipole moments of the parent cell with the M2M translation. Continue calculating the multipole moments of the cells up to level 2.

### Downward pass

Compute the local moments on all cells starting from level 2 and downward to all the leaves. The local moments of a cell are the sum of two parts. One is the contributions of
its well separated cells, which can be calculated by the M2L translation. Another is the contributions of all the far cells, which can be calculated by the L2L translation. For a level 2 cell, only the M2L translation used.

**Final product** For a field point $x$, compute the contributions from the source points close to $x$ by direct integrations, and the contributions from other source points are computed by using the local expansions.

In this section, based on the expansions and translations of the fast multipole method for the single-layer potential $(S\varphi)(x)$, we analyze the local truncation error of each step of the algorithm, and then derive the global error of the FMM. In the following analysis, both the expansions and translations are truncated from $-p$ to $p$.

### 2.1 Errors of the multipole moments

Suppose $D$ is a cell that covers the field point $x$, $C_L$ is a well separated cell of $D$ and located in the $L$ level of the tree structure, the centroid of $C_L$ is $O_{CL}$. By Graf’s addition theorem (1), for each $x \in D$ and all $y_j \in C_L$, when $|y_j - O_{CL}| < |x - O_{CL}|$, we obtain the following multipole expansion:

$$
\sum_{y_j \in C_L} H_0^{(1)}(k|x - y_j|)\varphi(y_j)s(y_j)
= \sum_{y_j \in C_L} \left( \sum_{n=-\infty}^{\infty} H_n^{(1)}(k|x - O_{CL}|)e^{in\theta_{xy} - \alpha_{CL}} J_n(k|y_j - O_{CL}|)e^{-in\theta_{xy} - \alpha_{CL}} \right) \varphi(y_j)s(y_j)
= \sum_{n=-\infty}^{\infty} H_n^{(1)}(k|x - O_{CL}|)e^{in\theta_{xy} - \alpha_{CL}} M_n(O_{CL}),
$$

where

$$M_n(O_{CL}) = \sum_{y_j \in C_L} J_n(k|y_j - O_{CL}|)e^{-in\theta_{xy} - \alpha_{CL}} \varphi(y_j)s(y_j) \quad n \in \mathbb{Z}.
$$

are the multipole moments about $C_L$.

**Fig.1.** The M2M translations.

Let $C_{L+1,i}(1 \leq i \leq n_S)$ denote the nonempty children of $C_L$, $O_{CL+1,i}$ is the centroid of $C_{L+1,i}$, and $n_S$ is the number of $C_{L+1,i}$. For a quadtree, it is obvious that $1 \leq n_S \leq 4$. By Graf’s addition theorem
(2), we have the following M2M translation:

\[ M_n(O_{CL}) = \sum_{y_j \in C_L} J_n(k|y_j - O_{CL}|) e^{-in\theta_{y_j} - O_{CL}} \varphi(y_j)s(y_j) \]

\[ = \sum_{i=1}^{n_s} \sum_{y_j \in C_{L+1,i}} \left( \sum_{l=-\infty}^{\infty} J_l(k|y_j - O_{C_{L+1,i}}|) e^{-il\theta_{y_j} - O_{CL}} \right) \varphi(y_j)s(y_j) \]

\[ \times J_{n-l}(k|O_{C_{L+1,i}} - O_{CL}|) e^{-i(n-l)\theta_{O_{C_{L+1,i}} - O_{CL}}} \varphi(y_j)s(y_j) \]

\[ = \sum_{i=1}^{n_s} \sum_{l=-p}^{p} M_l(O_{C_{L+1,i}}) J_{n-l}(k|O_{C_{L+1,i}} - O_{CL}|) e^{-i(n-l)\theta_{O_{C_{L+1,i}} - O_{CL}}} + E_{M2M}(O_{CL}, n, p), \]

where

\[ E_{M2M}(O_{CL}, n, p) = \sum_{i=1}^{n_s} \sum_{l=-p}^{p} R_n^l(k(O_{C_{L+1,i}} - O_{CL}), -k(y_j - O_{C_{L+1,i}})) \varphi(y_j)s(y_j) \]  \hspace{1cm} (5)

is the truncation error of M2M translation. See Fig.1 for the M2M translations. Suppose

\[ M_l(O_{C_{L+1,i}}) = \tilde{M}_l(O_{C_{L+1,i}}) + E_{M1}(O_{C_{L+1,i}}, p), \]

where \( \tilde{M}_l(O_{C_{L+1,i}}, p) \) are the approximations of \( M_l(O_{C_{L+1,i}}) \) and \( E_{M1}(O_{C_{L+1,i}}, p) \) are the errors. Thus, we have

\[ M_n(O_{CL}) = \sum_{i=1}^{n_s} \sum_{l=-p}^{p} \tilde{M}_l(O_{C_{L+1,i}}, p) J_{n-l}(k|O_{C_{L+1,i}} - O_{CL}|) e^{-i(n-l)\theta_{O_{C_{L+1,i}} - O_{CL}}} + E_{Mn}(O_{CL}, p), \]

where

\[ E_{Mn}(O_{CL}, p) = \sum_{i=1}^{n_s} \sum_{l=-p}^{p} EM_l(O_{C_{L+1,i}}, p) J_{n-l}(k|O_{C_{L+1,i}} - O_{CL}|) e^{-i(n-l)\theta_{O_{C_{L+1,i}} - O_{CL}}} \]

\[ + E_{M2M}(O_{CL}, n, p). \]

are the errors of the multipole moments \( M_n(O_{CL}) \). We have the following theorem.

**Theorem 1** Suppose \( C_{L+1,i} \) \((1 \leq i \leq n_s)\) are the children of \( C_L \), \( O_{CL} \) and \( O_{C_{L+1,i}} \) are the centroid of \( C_L \) and \( C_{L+1,i} \) respectively. For each cell \( C_L \) and \( n \in \mathbb{Z} \),

\[ E_{Mn}(O_{CL}, p) = \sum_{i=1}^{n_s} \left( \sum_{l=-p}^{p} EM_l(O_{C_{L+1,i}}, p) J_{n-l}(k|O_{C_{L+1,i}} - O_{CL}|) e^{-i(n-l)\theta_{O_{C_{L+1,i}} - O_{CL}}} \right) \]

\[ \times \sum_{y_j \in C_{L+1,i}} R_n^l(k(O_{C_{L+1,i}} - O_{CL}), -k(y_j - O_{C_{L+1,i}})) \varphi(y_j)s(y_j). \]

In particular, if \( C_L \) is a leaf cell, then

\[ E_{Mn}(O_{CL}, p) = 0, \quad n \in \mathbb{Z}. \] \hspace{1cm} \( \Box \)

It is obvious that the multipole moments for a leaf cell are calculated by (4) directly, then they are exact.
2.2 Errors of the expansions and translations

Suppose the field point \( x \in D_{L_{\text{max}}} \subset D_{L_{\text{max}}-1} \subset \cdots \subset D_L \subset D_{L-1} \subset \cdots \subset D_2 \), where \( D_L \) is a cell of level \( L \), \( D_{L-1} \) is the parent of \( D_L \), and \( D_{L_{\text{max}}} \) is the leaf cell.

Let \( D_i \) (1 \( \leq i \leq n_{D_L} \)) denote the nonempty well separated cells of cell \( D_L \), where \( n_{D_L} \) is the number of \( D_i \), \( O_{D_i} \) and \( O_{D_i} \) are the centroid of \( D_L \) and \( D_i \) respectively. It is easy to see that 1 \( \leq n_{D_L} \leq 27 \). From the multipole expansion (3), for each field point \( x \in D_L \) and all source points \( y_j \in D_i \), when \( |y_j - O_{D_i}| < |x - O_{D_i}| \), we have

\[
\sum_{i=1}^{n_{D_L}} \sum_{y_j \in D_i} H_n^{(1)}(k|x - y_j|) \varphi(y_j) s(y_j)
\]

\[
= \sum_{i=1}^{n_{D_L}} \sum_{y_j \in D_i} \left( \sum_{n=-\infty}^{\infty} H_n^{(1)}(k|x - O_{D_i}|) e^{i \theta_k - \varphi_{D_i}} J_n(k|y_j - O_{D_i}|) e^{-i \theta_y - \varphi_{D_i}} \right) \varphi(y_j) s(y_j)
\]

\[
= \sum_{i=1}^{n_{D_L}} \sum_{n=-p}^{p} H_n^{(1)}(k|x - O_{D_i}|) e^{i \theta_k - \varphi_{D_i}} M_n(O_{D_i}) + E_{ME}(x, O_{D_L}, p),
\]

where

\[
E_{ME}(x, O_{D_L}, p) = \sum_{i=1}^{n_{D_L}} \sum_{y_j \in D_i} R_{0,p}(k(x - O_{D_i}), k(y_j - O_{D_i})) \varphi(y_j) s(y_j)
\]

is the truncation error of the multipole expansion.

Since \( M_n(O_{D_i}) = \tilde{M}_n(O_{D_i}, p) + E_{Mn}(O_{D_i}, p) \), it follows that

\[
\sum_{i=1}^{n_{D_L}} \sum_{n=-p}^{p} H_n^{(1)}(k|x - O_{D_i}|) e^{i \theta_k - \varphi_{D_i}} M_n(O_{D_i})
\]

\[
= \sum_{i=1}^{n_{D_L}} \sum_{n=-p}^{p} H_n^{(1)}(k|x - O_{D_i}|) e^{i \theta_k - \varphi_{D_i}} \tilde{M}_n(O_{D_i}, p) + E_{MM}(x, O_{D_L}, p),
\]

where

\[
E_{MM}(x, O_{D_L}, p) = \sum_{i=1}^{n_{D_L}} \sum_{n=-p}^{p} H_n^{(1)}(k|x - O_{D_i}|) e^{i \theta_k - \varphi_{D_i}} EM_n(O_{D_i}, p)
\]

is the error produced by the M2M translation.

In addition, for the main part of (8), by Graf’s addition theorem (2), when \( |x - O_{D_i}| < |O_{D_L} - O_{D_i}| \), we obtain the following local expansion:

\[
\sum_{i=1}^{n_{D_L}} \sum_{n=-p}^{p} H_n^{(1)}(k|x - O_{D_i}|) e^{i \theta_k - \varphi_{D_i}} \tilde{M}_n(O_{D_i}, p)
\]

\[
= \sum_{i=1}^{n_{D_L}} \sum_{n=-p}^{p} \left( \sum_{m=-\infty}^{\infty} J_m(k|x - O_{D_i}|) e^{i \theta_y - \varphi_{D_i}} \right) H_n^{(1)}(k|O_{D_L} - O_{D_i}|) e^{i(n-m)\theta_{O_{D_L} - \varphi_{D_i}}}
\]

\[
\times \tilde{M}_n(O_{D_i}, p)
\]

\[
= \sum_{m=-p}^{p} \tilde{L}M_m(O_{D_L}, p) J_m(k|x - O_{D_i}|) e^{i \theta_k - \varphi_{D_i}} + E_{M2L}(x, O_{D_L}, p),
\]

(10)
where

\[
\tilde{\mathbf{L}}_m(\mathbf{O}_{DL}, p) = \sum_{i=1}^{n_{DL}} \sum_{n=-p}^{p} \tilde{\mathbf{M}}_n(\mathbf{O}_{DL}, p) H^{(1)}_{n-m}(k|\mathbf{O}_{DL} - \mathbf{O}_{DL}|) e^{i(n-m)\theta_{\mathbf{O}_{DL} - \mathbf{O}_{DL}}} \tag{11}
\]

is the M2L translation and

\[
E_{M2L}(x, \mathbf{O}_{DL}, p) = \sum_{i=1}^{n_{DL}} \sum_{n=-p}^{p} R^H_{n,p} (k(\mathbf{O}_{DL} - \mathbf{O}_{DL}), -k(x - \mathbf{O}_{DL})) \tilde{\mathbf{M}}_n(\mathbf{O}_{DL}, p) \tag{12}
\]

is the error of the M2L translation. We call \(\tilde{\mathbf{L}}_m(\mathbf{O}_{DL}, p)\) the approximate local moments about \(\mathbf{D}_{L}\) translated by \(\tilde{\mathbf{M}}_n(\mathbf{O}_{DL}, p)\).

Suppose \(\mathbf{O}_{DL-1}\) is the centroid of \(\mathbf{D}_{L-1}\), \(\tilde{\mathbf{L}}_l(\mathbf{O}_{DL-1}, p)\) are the approximate local moments about \(\mathbf{D}_{L-1}\), by (2), we have

\[
\sum_{l=-p}^{p} \tilde{\mathbf{L}}_l(\mathbf{O}_{DL-1}, p) J_l(k|x - \mathbf{O}_{DL-1}|) e^{i\theta_{\mathbf{x} - \mathbf{O}_{DL-1}}}
\]

\[
= \sum_{l=-p}^{p} \tilde{\mathbf{L}}_l(\mathbf{O}_{DL-1}, p) \left( \sum_{m=-\infty}^{\infty} J_m(k|x - \mathbf{O}_{DL}|) e^{i\theta_{\mathbf{x} - \mathbf{O}_{DL}}} \right) \times J_{l-m}(k|\mathbf{O}_{DL} - \mathbf{O}_{DL-1}|) e^{i(l-m)\theta_{\mathbf{O}_{DL} - \mathbf{O}_{DL-1}}}
\]

\[
= \sum_{m=-p}^{p} \tilde{\mathbf{L}}_m(\mathbf{O}_{DL}, p) J_m(k|x - \mathbf{O}_{DL}|) e^{i\theta_{\mathbf{x} - \mathbf{O}_{DL}}} + E_{L2L}(x, \mathbf{O}_{DL}, p), \tag{13}
\]

where

\[
\tilde{\mathbf{L}}_m(\mathbf{O}_{DL}, p) = \sum_{l=-p}^{p} \tilde{\mathbf{L}}_l(\mathbf{O}_{DL-1}, p) J_{l-m}(k|\mathbf{O}_{DL} - \mathbf{O}_{DL-1}|) e^{i(l-m)\theta_{\mathbf{O}_{DL} - \mathbf{O}_{DL-1}}}, \tag{14}
\]
is the L2L translation and
\[ E_{L2L}(x, O_{DL}, p) = \sum_{l=-p}^{p} R_l^p(k(O_{DL} - O_{DL-1}), -k(x - O_{DL})) \tilde{L}_l(O_{DL-1}, p) \] (15)
is the error of the L2L translation. We call \( \tilde{LL}_m(O_{DL}, p) \) the approximate local moments about \( DL \) translated by \( \tilde{L}_l(O_{DL-1}, p) \). See Fig.2 for the M2L and L2L translations.

2.3 Errors of the local moments

Since \( \tilde{L}_m(O_{DL}, p) = \tilde{LM}_m(O_{DL}, p) + \tilde{LL}_m(O_{DL}, p) \), adding (11) and (14), we obtain
\[
\tilde{L}_m(O_{DL}, p) = \sum_{i=1}^{n_{DL}} \sum_{n=-p}^{p} \tilde{M}_n(O_{DL}, p) H_{n-m}^{(1)}(k|O_{DL} - O_{DL}|) e^{i(n-m)\theta_{O_{DL}} - \theta_{O_{DL-1}}}
+ \sum_{l=-p}^{p} \tilde{L}_l(O_{DL-1}, p) J_{l-m}(k|O_{DL} - O_{DL-1}|) e^{i(l-m)\theta_{O_{DL}} - \theta_{O_{DL-1}}}.
\]

In addition, since there are no L2L translation in the level 2, it follows that \( \tilde{LL}_m(O_{D2}, p) = 0 \) and
\[
\tilde{L}_m(O_{D2}, p) = \sum_{i=1}^{n_{D2}} \sum_{n=-p}^{p} \tilde{M}_n(O_{D2}, p) H_{n-m}^{(1)}(k|O_{D2} - O_{D2}|) e^{i(n-m)\theta_{O_{D2}} - \theta_{O_{D2}}}. \]

What are the exact values of the local moments \( L_m(O_{DL}) \)? From the M2L translation and Graf’s addition theorem, for each \( m \in \mathbb{Z} \) and \( 2 \leq L \leq L_{max} \), we have
\[
LM_m(O_{DL}) = \sum_{i=1}^{n_{DL}} \sum_{n=-\infty}^{\infty} M_n(O_{DL}, p) H_{n-m}^{(1)}(k|O_{DL} - O_{DL}|) e^{i(n-m)\theta_{O_{DL}} - \theta_{O_{DL}}} = \sum_{i=1}^{n_{DL}} \sum_{y_j \in D_{L_i}} H_{m}^{(1)}(k|y_j - O_{DL}|) e^{-im\theta_{y_j} - \theta_{O_{DL}}} \varphi(y_j)s(y_j). \] (16)

In addition,
\[
L_m(O_{D2}) = LM_m(O_{D2}) = \sum_{i=1}^{n_{D2}} \sum_{y_j \in D_{L_i}} H_{m}^{(1)}(k|y_j - O_{D2}|) e^{-im\theta_{y_j} - \theta_{O_{D2}}} \varphi(y_j)s(y_j).
\]

Now, by the L2L translation
\[
LL_m(O_{DL}) = \sum_{l=-\infty}^{L} L_l(O_{DL-1}, p) J_{l-m}(k|O_{DL} - O_{DL-1}|) e^{i(l-m)\theta_{O_{DL}} - \theta_{O_{DL-1}}} \] (17)
and Graf’s addition theorem, we have
\[
LL_m(O_{DL}) = \sum_{L_i=2}^{L} \sum_{i=1}^{n_{DL_i}} \sum_{y_j \in D_{L_i}} H_{m}^{(1)}(k|y_j - O_{DL}|) e^{-im\theta_{y_j} - \theta_{O_{DL}}} \varphi(y_j)s(y_j) \] (18)
and
\[
L_m(O_{DL}) = \sum_{L_i=2}^{L} \sum_{i=1}^{n_{DL_i}} \sum_{y_j \in D_{L_i}} H_{m}^{(1)}(k|y_j - O_{DL}|) e^{-im\theta_{y_j} - \theta_{O_{DL}}} \varphi(y_j)s(y_j). \] (19)
The formulas (18) and (19) can be easily derived by the mathematical induction.

Suppose \( L_m(O_{DL}) = \tilde{L}_m(O_{DL}, p) + EL_m(O_{DL}, p) \), where \( EL_m(O_{DL}, p) \) are the errors of the local moments. We have the following theorem.
Theorem 2 Suppose $\mathcal{D}_i (1 \leq i \leq n_{DL})$ are the well separated cells of $\mathcal{D}_L$. For each cell $\mathcal{D}_L$ and $m \in \mathbb{Z}$, when $L = 2$,

$$
EL_m(\mathbf{O}_{D_2}, p) = \sum_{i=1}^{n_{DL}} \left( \sum_{n=-p}^{p} EM_n(\mathbf{O}_{DL}, i) H_{n-m}(k|\mathbf{O}_{D_2} - \mathbf{O}_{DL_i}|) e^{i(n-m)\theta_{\mathcal{D}_L} - \mathcal{D}_L_i,}
+ \sum_{y_j \in \mathcal{D}_L_i} R^H_{m,p}(k(\mathbf{O}_{D_2} - \mathbf{O}_{DL_i}), k(y_j - \mathbf{O}_{DL_i})) \varphi(y_j) s(y_j) \right),
$$

when $L \geq 3$,

$$
EL_m(\mathbf{O}_{DL}, p) = \sum_{i=1}^{n_{DL}} \left( \sum_{n=-p}^{p} EM_n(\mathbf{O}_{DL}, i) H_{n-m}(k|\mathbf{O}_{DL} - \mathbf{O}_{DL_i}|) e^{i(n-m)\theta_{\mathcal{D}_L} - \mathcal{D}_L_i,}
+ \sum_{y_j \in \mathcal{D}_L_i} R^H_{m,p}(k(\mathbf{O}_{DL} - \mathbf{O}_{DL_i}), k(y_j - \mathbf{O}_{DL_i})) \varphi(y_j) s(y_j) \right) + \sum_{l=2}^{L-1} \sum_{i=1}^{n_{DL}} \sum_{y_j \in \mathcal{D}_L_i} R^H_{m,p}(k(\mathbf{O}_{DL} - \mathbf{O}_{DL_{l-1}}), k(\mathbf{O}_{DL} - \mathbf{O}_{DL_{l-1}})) \varphi(y_j) s(y_j),
$$

where

$$
R^H_{m,p}(x,y) := \left( \sum_{n=-p}^{m} H_m(|x|) e^{-i(m+n)\theta} J_n(|y|) e^{in\theta} \right).
$$

Proof. Since $M_n(\mathbf{O}_{DL_i}, p) = \tilde{M}_n(\mathbf{O}_{DL_i}, p) + EM_n(\mathbf{O}_{DL_i}, p)$, from (4), (11) and (16), we have

$$
LM_m(\mathbf{O}_{DL}) = \sum_{i=1}^{n_{DL}} \sum_{n=-\infty}^{\infty} M_n(\mathbf{O}_{DL_i}, i) H_{n-m}(k|\mathbf{O}_{DL} - \mathbf{O}_{DL_i}|) e^{i(n-m)\theta_{\mathcal{D}_L} - \mathcal{D}_L_i,}
= \sum_{i=1}^{n_{DL}} \sum_{n=-p}^{p} \left( \tilde{M}_n(\mathbf{O}_{DL_i}, p) + EM_n(\mathbf{O}_{DL_i}, p) \right) H_{n-m}(k|\mathbf{O}_{DL} - \mathbf{O}_{DL_i}|) e^{i(n-m)\theta_{\mathcal{D}_L} - \mathcal{D}_L_i,}
+ \sum_{i=1}^{n_{DL}} \sum_{y_j \in \mathcal{D}_L_i} R^H_{m,p}(k(\mathbf{O}_{DL} - \mathbf{O}_{DL_i}), k(y_j - \mathbf{O}_{DL_i})) \varphi(y_j) s(y_j)
= \text{LM}_m(\mathbf{O}_{DL}, p) + \sum_{i=1}^{n_{DL}} \sum_{n=-p}^{p} EM_n(\mathbf{O}_{DL_i}, i) H_{n-m}(k|\mathbf{O}_{DL} - \mathbf{O}_{DL_i}|) e^{i(n-m)\theta_{\mathcal{D}_L} - \mathcal{D}_L_i,}
+ \sum_{i=1}^{n_{DL}} \sum_{y_j \in \mathcal{D}_L_i} R^H_{m,p}(k(\mathbf{O}_{DL} - \mathbf{O}_{DL_i}), k(y_j - \mathbf{O}_{DL_i})) \varphi(y_j) s(y_j),
$$

we let

$$
ELM_m(\mathbf{O}_{DL}, p) := \sum_{i=1}^{n_{DL}} \sum_{n=-p}^{p} EM_n(\mathbf{O}_{DL_i}, i) H_{n-m}(k|\mathbf{O}_{DL} - \mathbf{O}_{DL_i}|) e^{i(n-m)\theta_{\mathcal{D}_L} - \mathcal{D}_L_i,}
+ \sum_{i=1}^{n_{DL}} \sum_{y_j \in \mathcal{D}_L_i} R^H_{m,p}(k(\mathbf{O}_{DL} - \mathbf{O}_{DL_i}), k(y_j - \mathbf{O}_{DL_i})) \varphi(y_j) s(y_j).
$$

It is obvious that $EL_m(\mathbf{O}_{D_2}, p) = ELM_m(\mathbf{O}_{D_2}, p)$.  

In addition, from (14), (17) and (19), we have

\[
\begin{align*}
\mathbf{LL}_m(O_{D_L}) &= \sum_{l=-\infty}^{\infty} L_l(O_{D_{L-1}}) J_{l-m}(k|O_{D_L} - O_{D_{L-1}}|) e^{i(l-m)\theta_{O_{D_L}}-\theta_{O_{D_{L-1}}}} \\
&= \widetilde{LL}_m(O_{D_L}) + \sum_{l=-p}^{p} EL_l(O_{D_{L-1}}, p) J_{l-m}(k|O_{D_L} - O_{D_{L-1}}|) e^{i(l-m)\theta_{O_{D_L}}-\theta_{O_{D_{L-1}}}} \\
&\quad + \sum_{L=2}^{L-1} \sum_{i=1}^{n_{D_{L}}} \sum_{l_{i}=2}^{L_{i}} \mathcal{R}_{m,p}^{H}(k(y_j - O_{D_{L-1}}), k(O_{D_{L}} - O_{D_{L-1}})) \varphi(y_j) s(y_j),
\end{align*}
\]

where

\[
\mathcal{R}_{m,p}^{H}(x,y) := \left( \sum_{n=-\infty}^{\infty} \sum_{n=-m+1}^{n} H_{m+n}(|x|) e^{-i(m+n)\theta_s J_n(|y|)} e^{i\theta_{x}}, \right.
\]

we write

\[
\begin{align*}
\mathbf{ELL}_m(O_{D_L}, p) &= \sum_{l=-p}^{p} EL_l(O_{D_{L-1}}, p) J_{l-m}(k|O_{D_L} - O_{D_{L-1}}|) e^{i(l-m)\theta_{O_{D_L}}-\theta_{O_{D_{L-1}}}} \\
&\quad + \sum_{L=2}^{L-1} \sum_{i=1}^{n_{D_{L}}} \sum_{l_{i}=2}^{L_{i}} \mathcal{R}_{m,p}^{H}(k(y_j - O_{D_{L-1}}), k(O_{D_{L}} - O_{D_{L-1}})) \varphi(y_j) s(y_j).
\end{align*}
\]

From \( EL_m(O_{D_L}, p) = ELM_m(O_{D_L}, p) + \mathbf{ELL}_m(O_{D_L}, p) \), we complete the proof of Theorem 2. \( \square \)

### 2.4 Error of the final product

Suppose \( D_A \) is the set of adjacent cells of \( D_{L_{\text{max}}} \), then we can write the final product as

\[
\sum_{j=1}^{2N} H_0^{(1)}(k|x - y_j|) \varphi(y_j)s(y_j) = \left( \sum_{y_j \in D_A} + \sum_{y_j \notin D_A} \right) H_0^{(1)}(k|x - y_j|) \varphi(y_j)s(y_j).
\]

Since the part for \( y_j \in D_A \) is directly calculated, it is exact. The part for \( y_j \notin D_A \) is calculated by FMM. By the analysis in Section 2.1 and 2.2, we can derive the error of final product as follows:

**Theorem 3** Suppose \( D_L \) is a cell of level \( L(2 \leq L \leq L_{\text{max}}) \), \( D_{L-1} \) is the parent cell of \( D_L \), \( D_{L_{\text{max}}} \) is the leaf cell of \( D_L \). For each field point \( x \in D_{L_{\text{max}}} \),

\[
\sum_{y_j \notin D_A} H_0^{(1)}(k|x - y_j|) \varphi(y_j)s(y_j) = \sum_{m=-p}^{p} \mathbf{\tilde{L}}_m(O_{D_{L_{\text{max}}}}, p) J_{m}(k|x - O_{D_{L_{\text{max}}}}|) e^{im\theta_{x} - \theta_{O_{D_{L_{\text{max}}}}}} + \mathbf{E_S}(x, p),
\]

the error \( \mathbf{E_S}(x, p) \) can be described as

\[
\begin{align*}
\mathbf{E_S}(x, p) &= \sum_{L=2}^{L_{\text{max}}} \sum_{i=1}^{n_{D_L}} \sum_{l_{i}=2}^{L_{i}} \mathcal{R}_{0,p}^{H}(k(x - O_{D_L}), k(y_j - O_{D_L})) \varphi(y_j)s(y_j) \\
&\quad + \sum_{L=2}^{L_{\text{max}}} \sum_{i=1}^{n_{D_L}} \sum_{l_{i}=1}^{L_{i}} H_0^{(1)}(k|x - O_{D_L|}) e^{in\theta_{x} - \theta_{O_{D_L}}}, \mathbf{E_M}(O_{D_I}, p) \\
&\quad + \sum_{L=2}^{L_{\text{max}}} \sum_{i=1}^{n_{D_L}} \sum_{l_{i}=1}^{L_{i}} \mathcal{R}_{m,p}^{H}(k(O_{D_L} - O_{D_I}), -k(x - O_{D_L})) \tilde{M}_n(O_{D_I}, p) \\
&\quad + \sum_{L=3}^{L_{\text{max}}} \sum_{n=-p}^{p} \mathcal{R}_{m,n}^{L}(k(O_{D_L} - O_{D_{L-1}}), -k(x - O_{D_L})) \tilde{L}_n(O_{D_{L-1}}, p),
\end{align*}
\]

\( \square \)
From (20) and (21), we conclude that

\[ \sum_{y_j \notin \mathcal{D}_i} H_0^{(1)}(k|y_j|) \varphi(y_j) s(y_j) \]

in addition, from the L2L translation (14), we have

\[ \sum_{L=2}^{L_{max}} \sum_{i=1}^{n_{\mathcal{D}_i}} \sum_{y_j \in \mathcal{D}_i} H_0^{(1)}(k|y_j|) \varphi(y_j) s(y_j) \]

Proof. From the multipole expansion (6), (8) and M2L translation (10), we have

\[ \sum_{L=2}^{L_{max}} \sum_{i=1}^{n_{\mathcal{D}_i}} \sum_{p} H_n^{(1)}(k|x - \mathcal{O}_{\mathcal{D}_i, p}) e^{i \theta_k - \mathcal{O}_{\mathcal{D}_i, p}} \tilde{M}_n(\mathcal{O}_{\mathcal{D}_i, p}) + \sum_{L=2}^{L_{max}} [E_{ME}(x, \mathcal{O}_{\mathcal{D}_L, p}) + E_{MM}(x, \mathcal{O}_{\mathcal{D}_L, p})] \]

in addition, from the L2L translation (14), \( \tilde{L}_L m(\mathcal{O}_{\mathcal{D}_L, p}) = 0 \) and

\[ \tilde{L}_m(\mathcal{O}_{\mathcal{D}_L, p}) = \tilde{L}_L m(\mathcal{O}_{\mathcal{D}_L, p}) + \tilde{L}_L m(\mathcal{O}_{\mathcal{D}_L, p}) \]

it follows that

\[ \sum_{L=2}^{L_{max}} \sum_{m=-p}^{p} \tilde{L}_L m(\mathcal{O}_{\mathcal{D}_L, p}) J_m(k|x - \mathcal{O}_{\mathcal{D}_L, p}) e^{i \theta_k - \mathcal{O}_{\mathcal{D}_L, p}} \]

\[ = \sum_{m=-p}^{p} \tilde{L}_m(\mathcal{O}_{\mathcal{D}_2, p}) J_m(k|x - \mathcal{O}_{\mathcal{D}_2, p}) e^{i \theta_k - \mathcal{O}_{\mathcal{D}_2, p}} + \sum_{L=3}^{L_{max}} \sum_{m=-p}^{p} \tilde{L}_L m(\mathcal{O}_{\mathcal{D}_L, p}) J_m(k|x - \mathcal{O}_{\mathcal{D}_L, p}) e^{i \theta_k - \mathcal{O}_{\mathcal{D}_L, p}} \]

\[ = \sum_{m=-p}^{p} \tilde{L}_m(\mathcal{O}_{\mathcal{D}_3, p}) J_m(k|x - \mathcal{O}_{\mathcal{D}_3, p}) e^{i \theta_k - \mathcal{O}_{\mathcal{D}_3, p}} + E_{L2L}(x, \mathcal{O}_{\mathcal{D}_3, p}) \]

\[ + \sum_{L=4}^{L_{max}} \sum_{m=-p}^{p} \tilde{L}_L m(\mathcal{O}_{\mathcal{D}_L, p}) J_m(k|x - \mathcal{O}_{\mathcal{D}_L, p}) e^{i \theta_k - \mathcal{O}_{\mathcal{D}_L, p}} \]

\[ = \sum_{m=-p}^{p} \tilde{L}_m(\mathcal{O}_{\mathcal{D}_{L_{max}}, p}) J_m(k|x - \mathcal{O}_{\mathcal{D}_{L_{max}}, p}) e^{i \theta_k - \mathcal{O}_{\mathcal{D}_{L_{max}}, p}} + \sum_{L=3}^{L_{max}} E_{L2L}(x, \mathcal{O}_{\mathcal{D}_L, p}). \] (21)

From (20) and (21), we conclude that

\[ \sum_{y_j \notin \mathcal{D}_i} H_0^{(1)}(k|y_j|) \varphi(y_j) s(y_j) = \sum_{m=-p}^{p} \tilde{L}_m(\mathcal{O}_{\mathcal{D}_{L_{max}}, p}) J_m(k|x - \mathcal{O}_{\mathcal{D}_{L_{max}}, p}) e^{i \theta_k - \mathcal{O}_{\mathcal{D}_{L_{max}}, p}} + E_S(x, p), \]

where

\[ E_S(x, p) = \sum_{L=2}^{L_{max}} [E_{ME}(x, \mathcal{O}_{\mathcal{D}_L, p}) + E_{MM}(x, \mathcal{O}_{\mathcal{D}_L, p}) + E_{M2L}(x, \mathcal{O}_{\mathcal{D}_L, p})] + \sum_{L=3}^{L_{max}} E_{L2L}(x, \mathcal{O}_{\mathcal{D}_L, p}). \]
By (7), (9), (12), (15), we prove the conclusion. \( \square \)

In fact, we can derive another description of \( E_S(x,p) \). Since the final product is computed by using the local expansion, it follows that

\[
\sum_{y_j \not\in D_A} H_0^{(1)}(k|x - y_j|)\varphi(y_j)s(y_j)
\]

\[
= \sum_{L=2}^{L_{max}} \sum_{i=1}^{n_{DL}} \sum_{y_j \in D_{L,i}} H_0^{(1)}(k|x - y_j|)\varphi(y_j)s(y_j)
\]

\[
= \sum_{m=-\infty}^{L_{max}} L_m(O_{D_{L_{max}}})J_m(k|x - O_{D_{L_{max}}}|)e^{im\theta_x - O_{D_{L_{max}}}}
\]

\[
= \sum_{m=-p}^{L_{max}} L_m(O_{D_{L_{max}}, p})J_m(k|x - O_{D_{L_{max}}}|)e^{im\theta_x - O_{D_{L_{max}}}}
\]

\[
+ \sum_{L=2}^{L_{max}} \sum_{i=1}^{n_{DL}} \sum_{y_j \in D_{L,i}} R_{0,p}^H(k(y_j - O_{D_{L_{max}}}, k(x - O_{D_{L_{max}})))\varphi(y_j)s(y_j)
\]

\[
= \sum_{m=-p}^{L_{max}} L_m(O_{D_{L_{max}}, p})J_m(k|x - O_{D_{L_{max}}}|)e^{im\theta_x - O_{D_{L_{max}}}}
\]

\[
+ \sum_{m=-p}^{L_{max}} E L_m(O_{D_{L_{max}}, p})J_m(k|x - O_{D_{L_{max}}}|)e^{im\theta_x - O_{D_{L_{max}}}}
\]

\[
+ \sum_{L=2}^{L_{max}} \sum_{i=1}^{n_{DL}} R_{0,p}^H(k(y_j - O_{D_{L_{max}}}, k(x - O_{D_{L_{max}}}))\varphi(y_j)s(y_j),
\]

thus we have

\[
E_S(x, p) = \sum_{m=-p}^{L_{max}} E L_m(O_{D_{L_{max}}, p})J_m(k|x - O_{D_{L_{max}}}|)e^{im\theta_x - O_{D_{L_{max}}}}
\]

\[
+ \sum_{L=2}^{L_{max}} \sum_{i=1}^{n_{DL}} R_{0,p}^H(k(y_j - O_{D_{L_{max}}}, k(x - O_{D_{L_{max}}}))\varphi(y_j)s(y_j).
\]  \( (22) \)

Fig. 3. The case of unconvergent multipole expansion.
It is worth pointing out that, by Graf’s addition theorem, the multipole expansion (6) is divergent when \(|y_j - O_{DL_i}| \geq |x - O_{DL_i}|\). For domain \(\Omega\) with corners, a uniform mesh yields poor convergence and has to be replaced by a graded mesh. In this case, the cell \(DL_i\) and its well separated cell \(DL_i^l\) may be in different level. If \(DL_i\) is a leaf cell and in the level \(L_{DL_i}\), then \(L - L_{DL_i} \geq 0\). When \(L - L_{DL_i} \geq 3\), \(|y_j - O_{DL_i}|\) is either very close to \(|x - O_{DL_i}|\) or larger than it (see Fig.3), it follows that the multipole expansion is either poor convergence or not convergence. Thus, in order to ensure the convergence of the FMM, we should limit \(L - L_{DL_i} \leq 2\).

3 Bounds for \(R_{m,p}(x,y)\)

We first give the monotonicity of \(J_n\) and \(|H_n|\) in the following lemmas.

**Lemma 1** Suppose \(n \in \mathbb{N}\) and \(z \in \mathbb{R}\). For fixed \(n\), when \(0 < z \leq n\), \(J_n(z)\) is positive and strictly increasing function of \(z\).

**Proof.** From [14], we have

\[
J_n(z) > 0, \quad z < j_{n,1},
\]

\[
J'_n(z) > 0, \quad z < j'_{n,1},
\]

where \(j_{n,1}\) and \(j'_{n,1}\) are the first zero of \(J_n(z)\) and \(J'_n(z)\) respectively. Since \(n \leq j'_{n,1} < j_{n,1}\), when \(z \leq n\), it follows that \(J_n(z) > 0\) and \(J'_n(z) > 0\), which proves the conclusion.

**Lemma 2** Suppose \(n \in \mathbb{Z}\), \(z \in \mathbb{R}\) and \(z > 0\). For fixed \(n\), \(|H_n(z)|\) is a strictly decreasing function of \(z\). And for fixed \(z\), with the increase of \(|n|\), \(|H_n(z)|\) is strictly increasing.

**Proof.** When \(n \geq 0\), the conclusions were proven by Amini [10]. Since \(H_{-n}(z) = (-1)^n H_n(z)\), it follows that \(|H_{-n}(z)| = |H_n(z)|\), thus the conclusions also hold for \(n < 0\).

By Lemma 1 and 2, we can derive the monotonicity of \(B_{m,p}^J(x,y)\) and \(B_{m,p}^H(x,y)\) in the following lemma.
Lemma 3 Suppose \( m, p \in \mathbb{N}, x, y, a, b \in \mathbb{R} \). When \( 0 < y \leq b \) and \( p \geq y \),
\[
B^I_{m,p}(x,y) \leq B^I_{m,p}(x,b).
\]
When \( 0 < y \leq b < a \leq x \) and \( p \geq y \),
\[
B^H_{m,p}(x,y) \leq B^H_{m,p}(a,b).
\]

Proof. By Lemma 1, when \( n \geq y > 0 \), \( J_n(y) \) is a positive and increasing function of \( y \). Hence, when \( p \geq y \) and \( 0 < y \leq b \), we have
\[
B^I_{m,p}(x,y) = \sum_{n=p+1}^{\infty} |J_n(y)||J_{n+m}(x)| + |J_{n-m}(x)|
\leq \sum_{n=p+1}^{\infty} |J_n(b)||J_{n+m}(x)| + |J_{n-m}(x)| = B^I_{m,p}(x,b).
\]

By Lemma 2, \( |H_n(x)| \) is a strictly decreasing function of \( x \). Thus, when \( 0 < y \leq b < a \leq x \) and \( p \geq y \),
\[
B^H_{m,p}(x,y) = \sum_{n=p+1}^{\infty} |J_n(y)||H_{n+m}(x)| + |H_{n-m}(x)|
\leq \sum_{n=p+1}^{\infty} |J_n(b)||H_{n+m}(a)| + |H_{n-m}(a)| = B^H_{m,p}(a,b).
\]
The proof is complete. \( \square \)

We study the upper bounds for \( J_n(z) \) and \( Y_n(z) \) below. For each \( n \in \mathbb{N} \) and real number \( z \geq 0 \), the following upper bounds for \( J_n(z) \) hold [13, 14]:
\[
|J_n(z)| \leq \frac{1}{\Gamma(n+1)} \left( \frac{z}{2} \right)^n,
\]
\[
|J_n(z)| \leq \sqrt{\frac{1}{2\pi n}} \left( \frac{e^z}{2n} \right)^n, \quad n \geq 1,
\]
\[
|J_n(z)| \leq \begin{cases} 
1, & n = 0, \\
\frac{1}{\sqrt{2}}, & n \geq 1.
\end{cases}
\]

We now study the bound for \( Y_n(z) \) by its limiting form [13, 14]:
\[
Y_n(z) \sim -\left( \frac{2}{z} \right)^n \frac{\Gamma(n)}{\pi}, \quad z \to 0,
\]
where \( n \) is a positive integer.

Lemma 4 Suppose \( n \in \mathbb{N}, z \in \mathbb{R} \). When \( 0 < z \leq n \), \( Y_n(z) < 0 \).

Proof. From [14], when \( n \geq 0 \) and \( 0 < z < y_{n,1}, Y_n(z) < 0 \), where \( y_{n,1} \) is the first positive zero of \( Y_n(z) \). In addition, when \( n \geq 0, n < y_{n,1} \), Thus \( Y_n(z) < 0 \) when \( 0 < z \leq n \). \( \square \)

Lemma 5 Suppose \( n \in \mathbb{N}^+, z \in \mathbb{R} \) and \( z > 0 \). Let \( C_n(z) \) be defined by
\[
C_n(z) := -Y_n(z) \left( \frac{z}{2} \right)^n \frac{\pi}{\Gamma(n)}.
\]

When \( n \geq z + 1, C_n(z) > 1 \) and
\begin{itemize}
  \item For fixed \( n \), \( C_n(z) \) is a strictly increasing function of \( z \), and as \( z \to 0 \), \( C_n(z) \to 1 \);
  \item For fixed \( z \), \( C_n(z) \) is a strictly decreasing function of \( n \), and as \( n \to \infty \), \( C_n(z) \to 1 \).
\end{itemize}

**Proof.** First, by the definition of \( C_n(z) \), we have

\[
Y'_n(z) + Y_n(z) \frac{n}{z} = -C'_n(z) \left( \frac{2}{z} \right)^n \frac{\Gamma(n)}{\pi}.
\]  \hfill (23)

In addition, the recurrence relations \[14\]

\[
Y_{n-1}(z) + Y_{n+1}(z) = \frac{2n}{z} Y_n(z),
\]  \hfill (24)

\[
Y_{n-1}(z) - Y_{n+1}(z) = 2Y'_n(z)
\]  \hfill (25)

hold. Now, adding (24) and (25), by (23), we have

\[
Y_{n-1}(z) = -C'_n(z) \left( \frac{2}{z} \right)^n \frac{\Gamma(n)}{\pi}.
\]

By Lemma 4, when \( 0 < z \leq n - 1 \), \( Y_{n-1}(z) < 0 \), hence

\[
C'_n(z) > 0, \quad 0 < z \leq n - 1.
\]

In addition, by the limiting form of \( Y_n(z) \) as \( z \to 0 \), for each \( n \geq 1 \), \( C_n(z) \to 1(z \to 0) \). Thus,

\[
C_n(z) > 1, \quad 0 < z \leq n - 1.
\]

Next, by (24) and the definition of \( C_n(z) \), we have

\[
Y_{n-1}(z) = -Y_{n+1}(z) + \frac{2n}{z} Y_n(z) = \left( \frac{2}{z} \right)^{n+1} \frac{\Gamma(n+1)}{\pi} \left( C_{n+1}(z) - C_n(z) \right),
\]

by Lemma 4, when \( 0 < z \leq n - 1 \), \( Y_{n-1}(z) < 0 \), thus

\[
C_{n+1}(z) < C_n(z), \quad 0 < z \leq n - 1.
\]

From Stirling’s approximation\[17\],

\[
\Gamma(n + 1) \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n, \quad n \to \infty.
\]

it follows that

\[
Y_n(z) = -C_n(z) \left( \frac{2}{z} \right)^n \frac{\Gamma(n)}{\pi}
\]

\[
\sim -C_n(z) \left( \frac{2}{z} \right)^n \frac{\sqrt{2\pi(n-1)}}{\pi} \left( \frac{n-1}{e} \right)^{n-1}
\]

\[
= -eC_n(z) \left( \frac{n-1}{n} \right)^{n-\frac{1}{2}} \sqrt{\frac{2}{\pi n}} \left( \frac{2n}{ez} \right)^n
\]

\[
\sim -C_n(z) \sqrt{\frac{2}{\pi n}} \left( \frac{2n}{ez} \right)^n.
\]

By asymptotic expansion of \( Y_n(z) \) \[14\], we have \( C_n(z) \to 1 \) as \( n \to \infty \). \hfill \( \square \)

From the upper bounds for \( J_n \) and \( Y_n \) given in this section, we derive some bounds for \( B^\text{ref}_{m,p}(x, y) \) in the following lemmas.
Lemma 6 Suppose \( m, p \in \mathbb{N}, x, y \in \mathbb{R} \) with \( x, y \geq 0 \). When \( p \geq \max\{ex/2, ey/2\} \),
\[
\sum_{n=p+1}^{\infty} |J_n(y)||J_{n+m}(x)| \leq \frac{x^{m}t^{p+1}}{\pi(2p + 2)^{m+1}(1 - t)},
\]
when \( p \geq \max\{ex/2, ey/2, m\} \),
\[
\sum_{n=p+1}^{\infty} |J_n(y)||J_{n-m}(x)| \leq \frac{1.05(2p + 2)^{m-1}t^{p+1}}{\pi x^{m}(1 - t)},
\]
where
\[
t = \frac{e^2xy}{4(p + 1)^2}.
\]
Proof. The proof can be found in [20]. \( \square \)

Lemma 7 Suppose \( m, p \in \mathbb{N}, x, y \in \mathbb{R} \) with \( x > y \geq 0 \). When \( p + m \geq x \),
\[
\sum_{n=p+1}^{\infty} |J_n(y)||Y_{n+m}(x)| \leq \frac{\alpha_{m,p}(r)C_{p+m+1}(x)r^{p+1}}{\pi x^{m}(1 - r)},
\]
where \( r = y/x \) and
\[
\alpha_{m,p}(r) = \begin{cases} 
1/(p + 1), & m = 0, \\
2(b_{m,p} - c_m(r)), & m \geq 1,
\end{cases}
\]
b_{m,p} = 2p + 2m + 1, \( c_m(r) = \frac{(2m - 1)x}{1 - r} \),
the function \( C_n(x) \) is defined in Lemma 5.
Proof. From the upper bound of \( J_n \) and Lemma 5, when \( p + m \geq x \),
\[
\sum_{n=p+1}^{\infty} |J_n(y)||Y_{n+m}(x)| \leq \sum_{n=p+1}^{\infty} y^n \frac{2^n\Gamma(n+1)\Gamma(n+m)(n+m)!}{\pi x^{m}} C_{n+m}(x) \leq \frac{2^m}{\pi x^{m}} \sum_{n=p+1}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n+1)} r^n = \frac{2^mC_{p+m+1}(x)}{\pi x^{m}} \sum_{n=p+1}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n+1)} r^n.
\]
where \( r = y/x < 1 \). When \( m = 0 \),
\[
\sum_{n=p+1}^{\infty} |J_n(y)||Y_n(x)| \leq \frac{C_{p+1}(x)}{\pi} \sum_{n=p+1}^{\infty} \frac{r^n}{n} \leq \frac{C_{p+1}(x)r^{p+1}}{\pi(p + 1)(1 - r)},
\]
when \( m \geq 1 \),
\[
\sum_{n=p+1}^{\infty} \frac{\Gamma(n+m)}{\Gamma(n+1)} r^n = \sum_{n=p+1}^{\infty} \left( \frac{r^{n+m-1}(m-1)}{1 - r} \right) = \left( \frac{r^{p+m}}{1 - r} \right)^{(m-1)} \leq \frac{r^{p+1}}{1 - r} \sum_{i=0}^{m-1} \frac{\Gamma(m)}{\Gamma(m-i)} \frac{\Gamma(p+m+1)}{\Gamma(p+i+2)} \left( \frac{r}{1 - r} \right)^i \leq \frac{r^{p+1}}{1 - r} \sum_{i=0}^{m-1} \left( \frac{p + m + 1 - \frac{1}{2} m - \frac{1}{2} i}{1 - r} \right)^{m-i-1} \left( \frac{r}{1 - r} \right)^i \leq \frac{r^{p+1}}{(1 - r)^{2m-1}} \sum_{i=0}^{m-1} (2m - 1)^i (2p + 2m + 1)^{m-i-1} \left( \frac{r}{1 - r} \right)^i = \frac{r^{p+1}(b_{m,p} - c_m(r))}{(1 - r)^{2m-1}(b_{m,p} - c_m(r))},
\]
when \( m \geq 1 \).
The proof is complete. □

In the next lemma, we will give a more concise bound for the infinite sum in Lemma 7. However, the conclusion only hold for the case $0 \leq r < 1/2$.

**Lemma 8** Suppose $m, p \in \mathbb{N}, x, y \in \mathbb{R}$ with $x > 2y \geq 0$. When $p \geq \max\{m - 2, x\}$,

$$
\sum_{n=p+1}^{\infty} |J_n(y)||Y_n+m(x)| \leq \frac{2(2p + m + 2)^{m-1}C_{p+m+1}(x)r^{p+1}}{\pi x^m(1 - 2r)},
$$

where $r = y/x$ and the function $C_n(x)$ is defined in Lemmas 5.

**Proof.** From the proof of Lemma 7, when $p \geq x$ and $m \geq 1$,

$$
\sum_{n=p+1}^{\infty} |J_n(y)||Y_n+m(x)| \leq \frac{2^m C_{p+m+1}(x)}{\pi x^m} \sum_{n=p+1}^{\infty} \frac{\Gamma(n + m)}{\Gamma(n + 1)} r^n
\leq \frac{2^m C_{p+m+1}(x)r^{p+1}}{\pi x^m(1 - r)} \sum_{i=0}^{m-1} (2m - i - 1)^i (2p + m + 2 + i)^{m-i-1} \left(\frac{r}{1 - r}\right)^i.
$$

We define

$$
S_i := (2m - i - 1)^i (2p + m + 2 + i)^{m-i-1}, \quad 0 \leq i \leq m - 1,
$$

when $p \geq m - 2$,

$$
\frac{S_i}{S_{i+1}} = \frac{(2m - i - 1)^i (2p + m + 2 + i)^{m-i-1}}{(2m - i - 2)^{i+1} (2p + m + 3 + i)^{m-i-2}}
= \frac{2p + m + 3 + i}{2m - i - 2} \left(\frac{2m - i - 1}{2m - i - 2}\right)^i \left(\frac{2p + m + 3 + i}{2p + m + 2 + i}\right)^{m-i-1}
> \frac{3}{2} \left(\frac{2p + m + 2 + i}{2p + m + 3 + i}\right)^{m-1},
$$

(26)

in addition, from the Binomial Theorem, we have

$$
\left(\frac{2p + m + 3 + i}{2p + m + 2 + i}\right)^{m-1} = \left(1 + \frac{1}{2p + m + 2 + i}\right)^{m-1}
= 1 + \sum_{j=1}^{m-1} \frac{\Gamma(m)}{\Gamma(m-j)\Gamma(j+1)} \frac{1}{(2p + m + 2 + i)^j}
\leq 1 + \sum_{j=1}^{m-1} \frac{1}{2^j-1 (2p + m + 2 + i)^j}
\leq 1 + \frac{2m - 2}{4p + m + 5 + 2i} < \frac{7}{5},
$$

(27)

From (26) and (27), we see that

$$
\frac{S_i}{S_{i+1}} > \frac{15}{14} > 1.
$$

Thus, when $m \geq 1$,

$$
\sum_{n=p+1}^{\infty} |J_n(y)||Y_n+m(x)| \leq \frac{2^m C_{p+m+1}(x)r^{p+1}}{\pi x^m(1 - r)} \sum_{i=0}^{m-1} S_i \left(\frac{r}{1 - r}\right)^i
\leq \frac{2^m C_{p+m+1}(x)r^{p+1} S_0}{\pi x^m(1 - r)} \sum_{i=0}^{m-1} \left(\frac{r}{1 - r}\right)^i
= \frac{2^m C_{p+m+1}(x)r^{p+1}(2p + m + 2)^{m-1}}{\pi x^m(1 - 2r)} \left[1 - \left(\frac{r}{1 - r}\right)^m\right],
$$

17
since \(0 \leq 2y < x\), it follows that
\[
0 \leq \frac{r}{1 - r} < 1,
\]
hence, we have
\[
\sum_{n=p+1}^{\infty} |J_n(y)||Y_{n+m}(x)| \leq \frac{2(2p + m + 2)^{m-1}C_{p+m+1}(x)r^{p+1}}{\pi x^m(1 - 2r)}.
\]
When \(m = 0\), the conclusion obviously holds. \(\square\)

For the bound of \(B_{m,p}^H(x, y)\), by the monotonicity of \(|H_n|\) (Lemma 2), we have
\[
|H_{n-m}(x)| \leq |H_{n+m}(x)|, \quad m, n \in \mathbb{N},
\]
it follows that
\[
B_{m,p}^H(x, y) = \sum_{n=p+1}^{\infty} |J_n(y)||H_{n+m}(x)| + |H_{n-m}(x)|
\leq 2 \sum_{n=p+1}^{\infty} |J_n(y)||H_{n+m}(x)|
\leq 2 \sum_{n=p+1}^{\infty} |J_n(y)|(|J_{n+m}(x)| + |Y_{n+m}(x)|). \tag{28}
\]
In addition, by Lemma 5, when \(n + m \geq x + 1\),
\[
|Y_{n+m}(x)| > 1 \geq |J_{n+m}(x)|,
\]
thus we also have
\[
B_{m,p}^H(x, y) \leq 4 \sum_{n=p+1}^{\infty} |J_n(y)||Y_{n+m}(x)|, \quad p + m \geq x. \tag{29}
\]
In fact, by Lemma 5 and the Stirling’s inequality\cite{17}:
\[
\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \leq \Gamma(n + 1) \leq e\sqrt{n} \left(\frac{n}{e}\right)^n,
\]
we can give another upper bound for \(|H_{n}(z)|\). When \(n \geq z + 1\), it follows that
\[
|H_{n}(z)| \leq 2|Y_{n}(z)| \leq 2C_{n}(z) \left(\frac{2}{z}\right)^n \left(\frac{\Gamma(n)}{\pi}\right) \leq \frac{2\epsilon C_{n}(z)(2n/ez)^n}{\pi\sqrt{n}}. \tag{30}
\]
At the end of this section, we performed some numerical experiments to test the bounds given in Lemma 7 and 8. Since the exact value of the infinite sum
\[
s_m(x, y) := \sum_{n=0}^{\infty} |J_n(y)||Y_{n+m}(x)|
\]
is unknown, we approximate it by the finite sum:
\[
s_m(x, y) \approx \sum_{n=0}^{10^3} |J_n(y)||Y_{n+m}(x)|.
\]
Therefore, the relative truncation error can be approximated by
\[
\epsilon_{m,p}(x, y) = \frac{\sum_{n=p+1}^{10^3} |J_n(y)||Y_{n+m}(x)|}{s_m(x, y)}.
\]
Fig. 4. The approximation of $\epsilon_{m,p}(3, 1)$ and their bound with respect to $p$.

Fig. 5. The approximation of $\epsilon_{m,p}(3, 1)$ and their bound with respect to $m$.

Fig. 6. The approximation of $\epsilon_{p,p}(3, 1)$ and their bound with respect to $p$. 
It is obvious that the approximations of $s_m(x,y)$ and $\epsilon_{m,p}(x,y)$ are all less than their exact values.

In Fig.4, fix $x = 3, y = 1$ and $m = 10$, the approximation and bounds for $\epsilon_{m,p}(x,y)$ are plotted as functions of $p$. In Fig.5, fix $p = 40$, the approximation and bounds are plotted as functions of $m$. From the results, we see that the bounds given in Lemma 7 and 8 are all sharper than those from [20]. In addition, the bound in Lemma 7 is sharper for larger $p$ and smaller $m$, the bound in Lemma 8 is extremely sharp for each $p$ and $m$.

In Fig.6, the approximation and bounds for $\epsilon_{p,p}(x,y)$ are plotted as functions of $p$. From the results, we see that the bound in Lemma 8 is in close agreement with the approximation.

From the above analysis, when $0 \leq r < 1/2$, the result in Lemma 8 is more satisfactory, but when $1/2 \leq r < 1$, the result in Lemma 7 is useful.

4 Bound for $E_S(x, p)$

We will estimate the bound and convergence rate for $E_S(x, p)$ in this section. Assume the boundary curve $\partial \Omega$ is divided into $2N$ points and covered by a square with length $d$.

For a level $L$ cell $C_L$ with centroid $O_{C_L}$, we let $|O_{C_{L+1,i}} - O_{C_L}| = d_L$, where $O_{C_{L+1,i}}$ is the centroid.
of $C_{L+1,i}$, and $C_{L+1,i}$ is the child of $C_L$. It is obvious that

$$d_L = \frac{\sqrt{2}d}{2^{L+2}}.$$  

In addition, for each source point $y \in C_{L+1,i}$, the inequality $|y - O_{C_{L+1,i}}| \leq d_L$ hold (See Fig.7).

Suppose the field point $x \in D_{L_{\text{max}}}$, $D_{L_{\text{max}}-1} \subset \cdots \subset D_L \subset D_{L-1} \subset \cdots \subset D_2$, where $D_L$ is a cell of level $L$, $D_{L-1}$ is the parent of $D_L$. From Fig.8, we see that

$$|x - O_{D_L}| \leq 2d_L,$$

$$|O_{D_L} - O_{D_{L-1}}| = 2d_L,$$

where $O_{D_L}$ and $O_{D_{L-1}}$ are the centroid of $D_L$ and $D_{L-1}$ respectively.

Let $D_{I_i}$ be the well separated cell of $D_L$, which is located in the $L_{D_{I_i}}$ level of the tree structure and $0 \leq L - L_{D_{I_i}} \leq 2$. From Fig.9, we have

$$|x - O_{D_{I_i}}| \geq \sqrt{2}\varepsilon_{L - L_{D_{I_i}}}d_L,$$

$$|O_{D_L} - O_{D_{I_i}}| \geq \sqrt{2}\zeta_{L - L_{D_{I_i}}}d_L,$$

and for each source point $y \in D_{I_i}$,

$$|y - O_{D_L}| \geq 3\sqrt{2}d_L,$$

$$|y - O_{D_{I_i}}| \leq \eta_{L - L_{D_{I_i}}}d_L,$$

where $O_{D_{I_i}}$ is the centroid of $D_{I_i}$. The values of $\varepsilon_{L - L_{D_{I_i}}}, \zeta_{L - L_{D_{I_i}}}$ and $\eta_{L - L_{D_{I_i}}}$ are shown in the following Table 1.
Lemma 9 Suppose $C_L$ is a level $L$ cell that covers $M_{C_L}$ source points $y$, $C_{L_{\max}}$ is a leaf cell and $C_{L'_{\max}} \subset C_L$. For $2 \leq L \leq L'_{\max} - 1$ and $-p \leq n \leq p$, when $p \geq \sqrt{2ekd/32}$,
\[
|EM_n(O_{C_L}, p)| \leq \frac{1.05AM_{C_L}}{\pi(p + 1)} \left[ 2 \left( \frac{p + 1}{kd_L} \right)^p + \left( \frac{2p + 2}{kd_L} \right)^{\frac{p}{n}} \right] \frac{\zeta^{p+1}}{1 - \zeta_L},
\]
where
\[
\zeta_L = \left( \frac{ekd_L}{2p + 2} \right)^2.
\]

**Proof.** Since $|O_{C_{L+1,i}} - O_{C_L}| = d_L$, and $|y_j - O_{C_{L+1,i}}| \leq d_L(y_j \in C_{L+1,i})$, from Theorem 1 and Lemma 3, when $p \geq kd_L$, we have
\[
|EM_n(O_{C_L}, p)| \leq \sum_{i=1}^{n_s} \left( \sum_{l=-p}^{p} |EM_l(O_{C_{L+1,i}}, p)||J_{n-l}(kd_L)| \right)
\]
\[
+ A \sum_{y_j \in C_{L+1,i}} B_{l_{n_i}, l}(kd_L, k|y_j - O_{C_{L+1,i}}|) \right)
\]
\[
\leq \sum_{i=1}^{n_s} \sum_{l=-p}^{p} |EM_l(O_{C_{L+1,i}}, p)| + AM_{C_L}B_{l_{n_i}, p}(kd_L, kd_L), \tag{31}
\]
where $C_{L+1,i}$ is the child of $C_L$. In addition, for the leaf cell $C_{L'_{\max}}$,
\[
EM_n(O_{C_{L'_{max}}}, p) = 0, \quad -p \leq n \leq p. \tag{32}
\]
By Lemma 6, when \( p \geq ekd_L/2 \), we have

\[
AM_{CL} B_{p|L(D)}^J (kd_L, kd_L) \leq \frac{AM_{CL}}{2\pi(p + 1)} \left[ \frac{(kd_L)^{[n]}}{(2p + 2)^{[n]}} + \frac{1.05(2p + 2)^{[n]}}{(kd_L)^{[n]}} \right] \frac{s_{L}^{p+1}}{1 - s_{L}},
\]

where

\[
s_{L} = \left( \frac{ekd_L}{2p + 2} \right)^2.
\]

When \( L = L_{max} - 1 \), by (31) and (32), we have

\[
|EM_n(0,C_L,p)| \leq AM_{CL} B_{p|L(D)}^J (kd_L, kd_L) \leq \frac{1.05AM_{CL}(2p + 2)^{[n]}}{\pi(p + 1)(kd_L)^{[n]}} \frac{s_{L}^{p+1}}{1 - s_{L}}.
\]

When \( L = L_{max} - 2 \),

\[
|EM_n(0,C_L,p)| \leq \sum_{i=1}^{n_{S}} \sum_{i=0}^{p} |EM_i(0,C_{L+1,1},p)| + AM_{CL} B_{p|L(D)}^J (kd_L, kd_L)
\]

\[
\leq \sum_{i=1}^{n_{S}} 2.1AM_{CL+1,i} s_{L+1}^{p+1} \frac{1}{\pi(p + 1)(1 - s_{L+1})} \sum_{i=0}^{p} \left( \frac{2p + 2}{kd_{L+1}} \right)^{i} + AM_{CL} B_{p|L(D)}^J (kd_L, kd_L)
\]

\[
\leq \frac{2.1AM_{CL}}{4^{p+1}\pi(p + 1)} \left\{ \frac{2p + 2}{2p + 2 - k_{d_{L+1}}} \left( \frac{2p + 2}{kd_{L+1}} \right)^{p} \right\} \frac{s_{L}^{p+1}}{1 - s_{L}} + AM_{CL} B_{p|L(D)}^J (kd_L, kd_L)
\]

\[
\leq \frac{1.05AM_{CL}}{\pi(p + 1)} \left[ \left( \frac{p + 1}{kd_L} \right)^{p} + \frac{2p + 2}{kd_{L+1}} \right] \frac{s_{L}^{p+1}}{1 - s_{L}}.
\]

where \( M_{CL+1,i} \) is the number of source points in \( C_{L+1,i} \). When \( L = L_{max} - 3 \),

\[
|EM_n(0,C_L,p)| \leq \sum_{i=1}^{n_{S}} 2.1AM_{CL+1,i} s_{L+1}^{p+1} \frac{1}{\pi(p + 1)(1 - s_{L+1})} \sum_{i=0}^{p} \left( \frac{p + 1}{kd_{L+1}} \right)^{p} + \frac{2p + 2}{kd_{L+1}} \right\} \left( \frac{2p + 2}{kd_{L+1}} \right)^{p} \right\} \frac{s_{L}^{p+1}}{1 - s_{L}} + AM_{CL} B_{p|L(D)}^J (kd_L, kd_L)
\]

\[
\leq \frac{1.05AM_{CL}}{\pi(p + 1)} \left[ \left( \frac{p + 1}{2^{p+1} + 1} \right)^{p} + \left( \frac{2p + 2}{kd_L} \right)^{[n]} \right] \frac{s_{L}^{p+1}}{1 - s_{L}}.
\]

By this method, we can induce that

\[
|EM_n(0,C_L,p)| \leq \frac{1.05AM_{CL}}{\pi(p + 1)} \left[ \sum_{j=0}^{L_{max} - L - 2} \left( \frac{p + 1}{2^{p+1}} \right)^{j} \left( \frac{p + 1}{kd_L} \right)^{p} + \left( \frac{2p + 2}{kd_L} \right)^{[n]} \right] \frac{s_{L}^{p+1}}{1 - s_{L}},
\]

since \( (p + 1)/2^{p+1} \leq 1/2 \), it follows that

\[
\sum_{j=0}^{L_{max} - L - 2} \left( \frac{p + 1}{2^{p+1}} \right)^{j} \leq \sum_{j=0}^{L_{max} - L - 2} \frac{1}{2^j} = 2 - \frac{1}{2^{L_{max} - L - 2}} < 2.
\]

The proof is complete. \( \square \)

We also want to estimate the bound for \( EL_m(O_{DL},p) \). However, since the formula of \( EL_m(O_{DL},p) \) is a very complex recurrence relation, it is very difficult to give a sharp bound. Thus, it is almost impossible to estimate the bound for \( E_S(x,p) \) by (22). From Section 2.3, we see that

\[
\lim_{p \to \infty} L_m(O_{DL},p) = L_m(O_{DL}), \quad m \in \mathbb{Z}.
\]

it follows that \( EL_m(O_{DL},p) \to 0(p \to \infty) \).
4.2 Bound for $E_S(x, p)$

In this section, we will estimate the bound for $E_S(x, p)$ proposed in Theorem 3. For simplicity of presentation, we denote the four parts of $E_S(x, p)$ briefly by

$$
E_{S,1}(x, p) := \sum_{L=2}^{L_{\max}} \sum_{i=1}^{n_{DL}} \sum_{y_j \in D_{yi}} R^H_{0,p}(k(x - O_{DL}), k(y_j - O_{DL})) \varphi(y_j) s(y_j),
$$

$$
E_{S,2}(x, p) := \sum_{L=2}^{L_{\max}} \sum_{i=1}^{n_{DL}} \sum_{n=1}^{p} H_n^{(1)}(k|\cdot - O_{DL}|) e^{i n \theta(x - O_{DL})} E M_n(O_{DL}, p),
$$

$$
E_{S,3}(x, p) := \sum_{L=2}^{L_{\max}} \sum_{i=1}^{n_{DL}} \sum_{n=1}^{p} R^H_{n,p}(k(O_{DL} - O_{DL}), -k(x - O_{DL})) M_n(O_{DL}, p),
$$

$$
E_{S,4}(x, p) := \sum_{L=3}^{L_{\max}} \sum_{i=1}^{n_{DL}} \sum_{n=1}^{p} R^H_{n,p}(k(O_{DL} - O_{DL-1}), -k(x - O_{DL})) M_n(O_{DL-1}, p).
$$

We will give the bounds for $E_{S,1}(x, p), E_{S,2}(x, p), E_{S,3}(x, p), E_{S,4}(x, p)$ in the following theorems.

**Theorem 4** Suppose the boundary curve $\partial \Omega$ is divided into $2N$ points and covered by a square with length $d$. For each field point $x \in \partial \Omega$, when $p \geq \varepsilon_k d/16$, $\varepsilon_k$, $\eta_L$, we have

$$
|E_{S,1}(x, p)| \leq \frac{A}{\pi(p+1)} \sum_{i=0}^{I} \to N_i(x) \left[ \frac{r_i^{p+1}}{1 - r_i} + 2C_{p+1} \left( \frac{\varepsilon_k d^2}{8} \right) \frac{r_i^{p+1}}{1 - r_i} \right],
$$

where $A = \| \varphi(y) s(y)\|_{\infty}$, $I = \max \{i|N_i(x) \neq 0\}$,

$$
t_i = \frac{\sqrt{2} \varepsilon_k e^2 k^2 d^2}{28^i(p+1)^2}, \quad r_i = \frac{2^{i+1/2}}{\varepsilon_i},
$$

and $C_p$ is defined in Lemmas 5.

**Proof.** Since for $x \in D_L$ and $y_j \in D_{yi}$, $|y_j - O_{DL}| = \eta_L - L_D, d_L$ and $|x - O_{DL}| \geq \sqrt{2} \varepsilon_L - L_D, d_L$, by Lemma 3, when $p \geq k \eta i d_2$, we have

$$
|E_{S,1}(x, p)| \leq \sum_{L=2}^{L_{\max}} \sum_{i=1}^{n_{DL}} \sum_{y_j \in D_{yi}} B^H_{0,0}(k|\cdot - O_{DL}|, k|y_j - O_{DL}|) |\varphi(y_j) s(y_j)|
$$

$$
\leq A \sum_{L=2}^{L_{\max}} \sum_{i=1}^{n_{DL}} M_{DL} B^H_{0,0}(\sqrt{2} \varepsilon_L - L_D, d_L, \eta_L - L_D, d_L),
$$

in addition, by Lemma 6 and 7, when $p \geq \sqrt{2} \varepsilon_k k d_2 /2$,

$$
\sum_{L=2}^{L_{\max}} \sum_{i=1}^{n_{DL}} M_{DL} B^H_{0,0}(\sqrt{2} \varepsilon_L - L_D, d_L, \eta_L - L_D, d_L) \\
\leq \sum_{L=2}^{L_{\max}} \sum_{i=0}^{I} \to M_{DL}(x) \left[ \frac{r_i^{p+1}}{1 - r_i} + 2C_{p+1}(\sqrt{2} \varepsilon_i k d_L) \frac{r_i^{p+1}}{1 - r_i} \right],
$$

where $I = \max \{i|N_i(x) \neq 0\}$ and

$$
t_{L,i} = \frac{\sqrt{2} \varepsilon_k \eta_L e^2 k^2 d_L^2}{4(p+1)^2} = \frac{2^i \varepsilon_k e^2 k^2 d_L^2}{\sqrt{2}(p+1)^2}, \quad r_i = \frac{\eta_L}{\sqrt{2} \varepsilon_i} = \frac{2^{i+1/2}}{\varepsilon_i}.
$$

Since $d_L \leq d_2$, when $p \geq \sqrt{2} \varepsilon_k k d_2$, $0 < t_{L,i} \leq t_{L,i} < 1$ and by Lemma 5

$$
C_{p+1}(\sqrt{2} \varepsilon_i k d_L) \leq C_{p+1}(\sqrt{2} \varepsilon_i k d_2), \quad i = 0, 1, 2,
$$

24
it follows that
\[
|E_{S,1}(x, p)| \leq A \sum_{L=2}^{L_{\text{max}}} \sum_{i=0}^{L_{\text{max}}} \frac{1}{\pi(p+1)} \left[ \frac{t_{L,i}^{p+1}}{1-t_{L,i}} + 2C_{p+1}(\sqrt{2} \epsilon_i k d_{L,i}) \frac{r_{L,i}^{p+1}}{1-r_{L,i}} \right],
\]
\[
\leq A \sum_{L=2}^{L_{\text{max}}} \sum_{i=0}^{L_{\text{max}}} \frac{1}{\pi(p+1)} \left[ \frac{t_{2,i}^{p+1}}{1-t_{2,i}} + 2C_{p+1}(\sqrt{2} \epsilon_i k d_{2,i}) \frac{r_{2,i}^{p+1}}{1-r_{2,i}} \right],
\]
\[
= A \sum_{L=2}^{L_{\text{max}}} \sum_{i=0}^{L_{\text{max}}} N_i(x) \left[ \frac{t_{2,i}^{p+1}}{1-t_{2,i}} + 2C_{p+1} \left( \frac{\epsilon_i k d_{2,i}}{8} \right) \frac{r_{2,i}^{p+1}}{1-r_{2,i}} \right],
\]
which proves the theorem. □

**Theorem 5** For each field point \(x \in \partial \Omega\), when \(p \geq 3kd_2/8 + 1\),
\[
|E_{S,2}(x, p)| \leq \frac{2.1e^{3k^2d^2ANC_p (3kd_2)(2^p + p + 1)(\sqrt{2}e)}^p}{48\pi^2(p+1)^3 \sqrt{p} (1-\epsilon_2)},
\]
where \(A = \|\phi(y)s(y)\|_{\infty}\) and \(\epsilon_2\) is defined in Lemma 9.

**Proof.** If \(L_{D_i} < L\), then \(D_i\) is a leaf cell and
\[
EM_n(O_{D_i}, p) = 0, \quad -p \leq n \leq p.
\]
Since \(|x - O_{D_i}| \geq \sqrt{2}e_{L - L_{D_i}} d_L\) and \(\epsilon_0 = 3\), by Lemma 2 and 9, when \(p \geq k d_2/2\),
\[
|E_{S,2}(x, p)| \leq 2.1A \sum_{L=2}^{L_{\text{max}}} \sum_{i=0}^{L_{\text{max}}} \left[ \frac{2}{\pi(p+1)} \left( \frac{p+1}{kd_L} \right)^p \left( \frac{2p+2}{kd_L} \right) \right] \left( s_{L,i}^{p+1} \right) \left| H_p^{(2)}(3\sqrt{2}kd_L) \right|.
\]

Since \(M_{L,0}(x) \leq N_0(x) < 2N\), and from (30), it follows that
\[
|E_{S,2}(x, p)| \leq \frac{8.4e^{AC_p (3\sqrt{2}kd_L)(2^p + p + 1)} \sum_{L=2}^{L_{\text{max}}} M_{L,0}(x) \left( \frac{p+1}{kd_L} \right)^p \left( s_{L}^{p+1} \right) \left( \sqrt{2}e \right)^p}{48\pi^2(2^p+1)^3 \sqrt{p} (1-\epsilon_2)^2},
\]
\[
\leq \frac{8.4e^{ANC_p (3\sqrt{2}kd_L)(2^p + p + 1)} \left( \sqrt{2}e \right)^p}{48\pi^2(2^p+1)^3 \sqrt{p} (1-\epsilon_2)^2},
\]
This proves the theorem. □

We will give the bound for \(E_{S,3}(x, p)\), since \(\widetilde{M}_n(O_{D_i}, p) = M_n(O_{D_i}) - EM_n(O_{D_i}, p)\), we write
\[
E_{S,31}(x, p) := \sum_{L=2}^{L_{\text{max}}} \sum_{i=1}^{n_{D_L}} \sum_{n=-p}^{n} R_{n,p}^H(k(O_{D_L} - O_{D_i}), -k(x - O_{D_L}))M_n(O_{D_L}),
\]
\[
E_{S,32}(x, p) := \sum_{L=2}^{L_{\text{max}}} \sum_{i=1}^{n_{D_L}} \sum_{n=-p}^{n} R_{n,p}^H(k(O_{D_L} - O_{D_i}), -k(x - O_{D_L}))EM_n(O_{D_i}, p).
\]
Theorem 6 For each field point \( x \in \partial \Omega \), when \( p \geq \zeta_1 k d / 16 \),
\[
|E_{S,31}(x,p)| \leq \frac{A}{\pi} \sum_{l=0}^{L_{\text{max}}} N_l(x) \left[ \frac{2 C_{p+1} (\zeta i d)^{p+1}}{p+1} + \frac{8 C_{p+2} (\zeta i d)^{2 p+1}}{3 p+2} \right],
\]
where \( \gamma_i = \sqrt{2} / \zeta_i \) and \( \lambda_i = \gamma_i e^{\sqrt{2} \zeta_i} \).

Proof. Since \( |x - O_{D_L}| \leq 2d_L \) and \( |O_{D_L} - O_{D_I}| \geq \sqrt{2} \zeta_L - L_{D_I} d_L \), by Lemma 3, we have
\[
|E_{S,31}(x,p)| \leq \sum_{L=2}^{L_{\text{max}}} \sum_{n=1}^{n_{D_L}} \sum_{p=1}^{p_{D_L}} B_{n,p}^H (k|O_{D_L} - O_{D_I}|, k|x - O_{D_L}|) |M_n(O_{D_I})|,
\]
substituting the multipole moment (4) into the above formula, and by \( |J_\theta(z)| \leq 1 \), we obtain
\[
\sum_{L=2}^{L_{\text{max}}} \sum_{n=1}^{n_{D_L}} \sum_{p=1}^{p_{D_L}} B_{n,p}^H (\sqrt{2} \zeta_L - L_{D_I} |k|, 2k d_L) |M_n(O_{D_I})|,
\]
\[
\leq \sum_{L=2}^{L_{\text{max}}} \sum_{n=1}^{n_{D_L}} \sum_{p=1}^{p_{D_L}} B_{n,p}^H (\sqrt{2} \zeta_L - L_{D_I} |k|, 2k d_L) \sum_{y_j \in D_I} |J_n(k|y_j - O_{D_I}|)| \varphi(y_j) s(y_j)|
\]
\[
\leq A \sum_{L=2}^{L_{\text{max}}} \sum_{n=1}^{n_{D_L}} \sum_{y_j \in D_I} \left[ 2 \sum_{p=1}^{p_{D_L}} B_{n,p}^H (\sqrt{2} \zeta_L - L_{D_I} |k|, 2k d_L) |J_n(k|y_j - O_{D_I}|)|
\]
\[
+ B_{0,p}^H (\sqrt{2} \zeta_L - L_{D_I} |k|, 2k d_L) \right].
\]
Since \( |y_j - O_{D_I}| \leq \eta_{L-D_I} d_L \), from (28) and the upper bounds of \( J_n(z) \), for each \( 1 \leq n \leq p \),
\[
B_{n,p}^H (\sqrt{2} \zeta_L - L_{D_I} |k|, 2k d_L) |J_n(k|y_j - O_{D_I}|)|
\]
\[
\leq 2 \sum_{l=p+1}^{\infty} |J_l(2k d_L)| \left( |J_{l+1}(\sqrt{2} \zeta_L - L_{D_I} |k|)| + |Y_{l+1}(\sqrt{2} \zeta_L - L_{D_I} |k|)| \right) |J_n(k|y_j - O_{D_I}|)|
\]
\[
\leq \left( \frac{\eta_{L-D_I} |k|}{2^{n-1}(n+1)} \right) \sum_{l=p+1}^{\infty} |J_l(2k d_L)| |Y_{l+1}(\sqrt{2} \zeta_L - L_{D_I} |k|)|
\]
\[
+ 2 \sum_{l=p+1}^{\infty} |J_l(2k d_L)| |J_{l+1}(\sqrt{2} \zeta_L - L_{D_I} |k|)|,
\]
since \( \zeta_{L-D_I} \geq 4 \), by Lemma 6 and 8, when \( p \geq \sqrt{2} \zeta_1 k d_2 / 2 \),
\[
|E_{S,31}(x,p)| \leq A \sum_{L=2}^{L_{\text{max}}} \sum_{n=1}^{n_{D_L}} \sum_{y_j \in D_I} \left[ B_{0,p}^Y (\sqrt{2} \zeta_L - L_{D_I} |k|, 2k d_L) + B_{0,p}^Y (\sqrt{2} \zeta_L - L_{D_I} |k|, 2k d_L) \right]
\]
\[
+ 2 \sum_{n=1}^{p_{D_L}} \left( \frac{\eta_{L-D_I} |k|}{2^{n-1}(n+1)} \right) \sum_{l=p+1}^{\infty} |J_l(2k d_L)| |Y_{l+1}(\sqrt{2} \zeta_L - L_{D_I} |k|)|
\]
\[
+ 4 \sum_{n=1}^{p_{D_L}} \sum_{l=p+1}^{\infty} |J_l(2k d_L)| |J_{l+1}(\sqrt{2} \zeta_L - L_{D_I} |k|)|
\]
\[
\leq A \sum_{L=2}^{L_{\text{max}}} \sum_{n=1}^{n_{D_L}} \left[ 2 C_{p+1} (\sqrt{2} \zeta_i k d_L)^{p+1} \pi(p+1)(1 - 2 \zeta_i) + \right.
\]
\[
\left. + 1 + 2 \sum_{n=1}^{p_{D_L}} \left( \frac{\sqrt{2} \zeta_i k d_L}{2p+2} \right)^n \frac{u_{L,i}^{p+1}}{\pi(p+1)(1 - u_{L,i})} \right]
\]
\[
+ 8 \sum_{n=1}^{p_{D_L}} \left( \frac{\eta_i}{2 \sqrt{2} \zeta_i} \right)^n (2p + n + 2)^{n-1} C_{p+n+1} (\sqrt{2} \zeta_i k d_L)^{p+1} \pi(n+1)(1 - 2 \zeta_i) \right] M_{L,i}(x)
\]
where
\[ \gamma_i = \frac{\sqrt{2}}{\zeta_i} \leq \frac{\sqrt{2}}{4}, \quad u_{L,i} = \frac{\zeta_i e^2 k^2 d_L^2}{\sqrt{2}(p+1)^2} < \gamma_i, \]
since \( u_{L,i} < \gamma_i \) and
\[ \sum_{n=1}^{p} \left( \frac{\sqrt{2} \zeta_i k d_L}{2p+2} \right)^n < 1, \]
by Lemma 5, we have
\[ |E_{S,31}(x,p)| \leq A \sum_{i=0}^{f} N_i(x) \left\{ \frac{2C_{p+1}(\sqrt{2} \zeta_i k d_2) + 3}{\pi(p+1)(1-2\gamma_i)} \right\} \frac{8C_{p+2}(\sqrt{2} \zeta_i k d_2)\gamma_i^{p+1}}{\pi(1-2\gamma_i)} \times \sum_{n=1}^{p} \left( \frac{\eta_i}{2\sqrt{2} \zeta_i} \right)^n \frac{(2p+n+2)^{n-1}}{\Gamma(n+1)} \times \left( \frac{3p+2}{\pi(1-2\gamma_i)} \right)^{\frac{3p+2}{n}}. \] (33)

Now, by the inequality
\[ \sum_{n=0}^{p} \frac{x^n}{\Gamma(n+1)} \leq e^x, \]
we obtain
\[ \sum_{n=1}^{p} \left( \frac{\eta_i}{2\sqrt{2} \zeta_i} \right)^n \frac{(2p+n+2)^{n-1}}{\Gamma(n+1)} \leq \sum_{n=1}^{p} \left( \frac{\eta_i}{2\sqrt{2} \zeta_i} \right)^n \frac{(3p+2)^{n-1}}{\Gamma(n+1)} \leq \frac{e^{\frac{3p+2}{n}}}{3p+2}, \] (34)
where \( \eta_i = 2^{i+1} \). For simplicity of notation, we let
\[ \lambda_i := \gamma_i e^{\frac{1}{2\sqrt{2} \zeta_i}}, \]
an easy computation shows that
\[ \lambda_0 \approx 0.6009, \quad \lambda_1 \approx 0.6374, \quad \lambda_2 \approx 0.6640. \]

From (33) and (34), we prove the theorem. \( \square \)

**Theorem 7** For each field point \( x \in \partial \Omega \), when \( p \geq kd/2 \),
\[ |E_{S,32}(x,p)| \leq \frac{2.1\sqrt{2} e^2 k^2 d^2 ANC_{p+1}(kd_2/2)(2^{1-p} + 1)}{96\pi^2(p+1)^4(1-2\gamma_0)(1-s_2)} \left( \frac{3e^2}{32} \right)^p, \]
where \( \gamma_0 \) and \( \zeta_2 \) are defined in Theorem 6 and Lemma 9 respectively.

**Proof.** Since \( |x - O_{DL}| \leq 2d_L, |O_{DL} - O_{DL'}| \geq \sqrt{2} \zeta_{L-L_{DL}}, d_L \), and \( \zeta_0 = 4 \), by Lemma 8, Lemma 9 and (29), when \( p \geq 4\sqrt{2}kd_2 \), we have
\[ |E_{S,32}(x,p)| \leq \sum_{L=2}^{L_{max}} \sum_{n=0}^{p} \sum_{i=1}^{n_{DL}} |R_{n,p}^H(k(O_{DL} - O_{DL}), k(O_{DL} - x))| |EM_n(O_{DL'}, p)| \leq \sum_{L=2}^{L_{max}} \sum_{n=0}^{p} \sum_{i=1}^{n_{DL}} B_{n,p}^H(\sqrt{2} \zeta_{L-L_{DL}}, kd_L, 2kd_L) |EM_n(O_{DL'}, p)| \leq \frac{8.4 ANC_{p+1}(4\sqrt{2}kd_2)}{\pi^2(p+1)^2(1-2\gamma_0)} \sum_{L=2}^{L_{max}} M_{L,0}(x) \frac{\zeta_{L}^{p+1}p^{p+1}}{1-s_L} \times \sum_{n=0}^{p} \left[ \left( \frac{p+1}{kd_L} \right)^p + \left( \frac{2p+2}{kd_L} \right)^n \right] \frac{2p+n+2}{(4\sqrt{2}kd_L)^n}, \]
since

\[
\sum_{n=0}^{p} \left[ 2 \left( \frac{p + 1}{kd_L} \right)^p + \left( \frac{2p + 2}{kd_L} \right)^n \right] \frac{(2p + n + 2)^n}{(4\sqrt{2}kd_L)^n} \\
\leq 2 \left( \frac{p + 1}{kd_L} \right)^p \sum_{n=0}^{p} \left( \frac{3p + 3}{4\sqrt{2}kd_L} \right)^n + \sum_{n=0}^{p} \left[ \frac{3(p + 1)^2}{2\sqrt{2}kd_L^2} \right]^n \\
\leq 4 \left[ \frac{3(p + 1)^2}{4\sqrt{2}kd_L^2} \right]^p + 2 \left[ \frac{3(p + 1)^2}{2\sqrt{2}kd_L^2} \right]^p,
\]

it follows that

\[
|E_{S,32}(x, p)| \leq \frac{16.8AC_{p+1}(4\sqrt{2}kd_2)}{\pi^2(p + 1)^2(1 - 2\gamma_0)(1 - \varsigma_2)} \sum_{L=2}^{L_{max}} \frac{M_{L,0}(x)c_L^{p+1}\gamma_0^{p+1}}{2 \left[ \frac{3(p + 1)^2}{4\sqrt{2}kd_L^2} \right]^p + \left[ \frac{3(p + 1)^2}{2\sqrt{2}kd_L^2} \right]^p}
\]

\[
\leq \frac{8.4\sqrt{2}ANC_{p+1}(4\sqrt{2}kd_2)}{\pi^2(p + 1)^2(1 - 2\gamma_0)(1 - \varsigma_2)} \left[ \frac{3\epsilon_2^2}{64} + \frac{3\epsilon_2^2}{32} \right] \sum_{L=2}^{L_{max}} \varsigma_L \\
\leq \frac{2.1\sqrt{2}\epsilon_2^2d^2ANC_{p+1}(4\sqrt{2}kd_2)(2^{1-p} + 1)}{96\pi^2(p + 1)^2(1 - 2\gamma_0)(1 - \varsigma_2)} \left( \frac{3\epsilon_2^2}{32} \right)^p.
\]

This proves the theorem. □

**Theorem 8** For each field point \( x \in \partial \Omega \), when \( p \geq 3kd/8 + 1 \),

\[
|E_{S,4}(x, p)| \leq \frac{e^3k^2d^2c_{\mathcal{N}}AC_p(3kd_2)}{96\pi^2(p + 1)^2\sqrt{p(1 - \tau)}} \left[ \left( \frac{\epsilon}{36} \right)^p + \frac{33.6}{32p + 32 - \sqrt{2kd}} \left( \frac{\sqrt{2}e}{6} \right)^p \right],
\]

where \( c \) is a constant and

\[
\tau = \frac{(\sqrt{2}kd)^2}{32p + 32}.
\]

**Proof.** Since \( |y_j - O_{D_L}| \geq 3\sqrt{2}d_L \) and

\[
\lim_{p \to \infty} \overline{L}_m(O_{D_L}, p) = L_m(O_{D_L}) = \sum_{L=2}^{L_{max}} \sum_{L_{max}^{D_L}} \sum_{L=2}^{L_{max}} H_m^{(1)}(k|y_j - O_{D_L}|)e^{-im\delta_{y_j} - O_{D_L}}\varphi(y_j)s(y_j),
\]

by Lemma 2, for each \( m \in \mathbb{Z} \), it follows that

\[
|L_m(O_{D_L}, p)| \leq c_m \sum_{L=2}^{L_{max}} \sum_{L=2}^{L_{max}^{D_L}} \sum_{L=2}^{L_{max}} |H_m^{(1)}(k|y_j - O_{D_L}|)| |\varphi(y_j)s(y_j)| \\
\leq Ac_m \left[ H_m^{(1)}(3\sqrt{2}kd_L) \right] \sum_{L=2}^{L_{max}} \sum_{L=2}^{L_{max}} M_{L,i},
\]

where \( c_m \) is a positive constant, which is independent of \( O_{D_L}, p \) and \( y_j \). In addition, for each \( L \),

\[
\sum_{L=2}^{L_{max}} \sum_{L=2}^{L_{max}} M_{L,i}(x) < 2N.
\]

Now, from \( O_{D_L} - O_{D_L-1} = 2d_L \), \( |x - O_{D_L}| \leq 2d_L \), Lemma 2 and 3, when \( p \geq 2kd_3 \),

\[
|E_{S,4}(x, p)| \leq \sum_{L=3}^{L_{max}} \sum_{n=-p}^{p} B_{[n],p}(k|O_{D_L} - O_{D_L-1}|, k|x - O_{D_L}|) \left| \overline{L}_n(O_{D_L-1}, p) \right| \\
\leq 2NA \sum_{L=3}^{L_{max}} \sum_{n=-p}^{p} c_nB_{[n],p}(2kd_L, 2kd_L) \left| H_n^{(1)}(3\sqrt{2}kd_{L-1}) \right| \\
\leq 4cNA \sum_{L=3}^{L_{max}} \left| H_n^{(1)}(3\sqrt{2}kd_{L-1}) \right| \sum_{n=0}^{p} B_{[n],p}(2kd_L, 2kd_L),
\]

28
where \( c = \max\{c_n | -p \leq n \leq p \} \). From Lemma 6 and (30), when \( p \geq 3\sqrt{2kd} + 1 \),

\[
\left| E_{S,4}(x, p) \right| \leq \frac{8cN A}{\pi^2(1 - \tau_1)} \sum_{L=3}^{L_{\text{max}}} C_p(3\sqrt{2kd}L^{-1})\tau_{L_0}^{p+1} \left( \frac{\sqrt{2p}}{3ekdL_{-1}} \right) \left( \frac{\sqrt{2p}}{6ekdL_{-1}} \right)^{p-1} \left[ \frac{(2kdL)^n}{(2p+2)^{p+1}} \right] \frac{(2kdL)^n}{(2p+2)^{p+1}}
\]

\[
\leq \frac{4cN A}{\pi^2(1 - \tau_1)} \sum_{L=3}^{L_{\text{max}}} C_p(3\sqrt{2kd}L^{-1})\tau_{L_0}^{p+1} \left( \frac{\sqrt{2p}}{6ekdL_{-1}} \right)^{p-1} \left[ \frac{1 + 0.05}{p+1 - kdL} \left( \frac{p+1}{kdL} \right)^p \right] d_L^2
\]

where

\[
\tau_L = \left( \frac{ekdL}{p+1} \right)^2,
\]

in addition, by Lemma 5 and \( d_L = \sqrt{2d}/2^{L+2} \), we have

\[
\left| E_{S,4}(x, p) \right| \leq \frac{4k^2c^2NAC_p(3\sqrt{2kd})}{\pi_2(1 - \tau_1)} \sum_{L=3}^{L_{\text{max}}} \left( \frac{ekdL}{p+1} \right)^2 \left( \frac{\sqrt{2e}}{6ekd} \right)^{p-1} \left[ \frac{1 + 0.05}{p+1 - kdL} \left( \frac{p+1}{kdL} \right)^p \right] d_L^2
\]

\[
\leq \frac{4k^2c^2NAC_p(3\sqrt{2kd})}{\pi_2(1 - \tau_1)} \sum_{L=3}^{L_{\text{max}}} \left( \frac{\sqrt{2e}}{6p+6} \right)^p + \left( \frac{1}{p+1 - kdL} \left( \frac{\sqrt{2e}}{6} \right)^p \right) d_L^2
\]

\[
\leq \frac{e^3k^2d^2cNAC_p(3\sqrt{2kd})}{96\pi_2(1 - \tau_1)} \left[ \left( \frac{e}{36} \right)^p + \left( \frac{1}{p+1 - kd} \left( \frac{\sqrt{2e}}{6} \right)^p \right) \right],
\]

which completes the proof. \( \square \)

From Theorem 4 to 8, we can give the bound for \( E_S(x, p) \) by

\[
\left| E_S(x, p) \right| \leq \left| E_{S,1}(x, p) \right| + \left| E_{S,2}(x, p) \right| + \left| E_{S,3}(x, p) \right| + \left| E_{S,4}(x, p) \right|
\]

Furthermore, we can also derive the estimate for the 2-norm of the vector

\[
E_S(x, p) = (E_S(x_1, p), E_S(x_2, p), \ldots, E_S(x_{2N}, p))
\]
as follows.

**Theorem 9** Suppose the boundary curve \( \partial \Omega \) is divided into \( 2N \) points and covered by a square with length \( d \). For the field point \( x_j \in \partial \Omega (1 \leq j \leq 2N) \), when \( p \geq \max\{3kd/8 + 1, \zeta_{\lambda}kd/16 \} \),

\[
\left\| E_{S,1}(x,p) \right\|_2 \leq \frac{2\sqrt{2\pi}A^{1/2}}{\pi p} \sum_{i=0}^{I} \frac{N_i}{1 - r_i},
\]

\[
\left\| E_{S,2}(x,p) \right\|_2 \leq \frac{2.1\sqrt{2\pi}A^{3/2}}{48\pi^2 p^{7/2}} \left( \frac{\sqrt{2e}}{6} \right)^p,
\]

\[
\left\| E_{S,31}(x,p) \right\|_2 \leq \frac{8\sqrt{2\pi}A^{1/2}}{3\pi p} \sum_{i=0}^{I} \frac{N_i}{1 - 2\gamma_i},
\]

\[
\left\| E_{S,32}(x,p) \right\|_2 \leq \frac{2.1A^{3/2}}{48\pi^2 p^{4}(1 - 2\gamma_0)} \left( \frac{3e^2}{32} \right)^p
\]

\[
\left\| E_{S,34}(x,p) \right\|_2 \leq \frac{1.05\sqrt{2\pi}A^{3/2}}{96\pi^2 p^{7/2}} \left( \frac{\sqrt{2e}}{6} \right)^p,
\]

where \( N_i = \max\{N_i(x_j) | 1 \leq j \leq 2N \} \) and \( I = \max\{i | N_i \neq 0 \} \).

**Proof.** We only give the proof for \( E_{S,31}(x,p) \). From Theorem 6, for each \( x_j \in \partial \Omega \), when \( p \geq \zeta_{\lambda}kd/16 \),

\[
\left| E_{S,31}(x_j,p) \right| \leq \frac{A}{\pi} \sum_{i=0}^{I} \frac{N_i}{1 - 2\gamma_i} \left\{ \frac{2C_1+1}{3p+2} \left( \frac{\zeta_{\lambda}kd}{8} \right)^{p+1} + \frac{3}{p+1} \gamma_i \left( \frac{\zeta_{\lambda}kd}{8} \right)^{p+1} + \frac{8C_{p+2}}{3p+2} \gamma_i \left( \frac{\zeta_{\lambda}kd}{8} \right)^{p+1} \right\},
\]

29
where \( N_i = \max\{N_i(x_j)|1 \leq j \leq 2N\} \). Since \( \gamma_i < \lambda_i \) and \( C_p(z) \to 1(p \to +\infty) \), it follows that

\[
\|E_{S,31}(x, p)\|_2 = \left( \sum_{j=1}^{2N} |E_{S,31}(x_j, p)|^2 \right)^{\frac{1}{2}} 
\leq A\sqrt{2N} \sum_{i=0}^{I} \frac{N_i}{1-2\gamma_i} \left\{ \left[ 2C_{p+1}(\frac{\zeta_{kd}}{\lambda_i}) + 3 \right] \gamma_i^{p+1} + \frac{8C_{p+2}(\frac{\zeta_{kd}}{\lambda_i})}{3p+2} \right\} \lambda_i^{p+1},
\]

which proves the conclusion. \( \square \)

From Theorem 9, we can derive the convergence rate of \( \|E_S(x, p)\|_2 \) in Table 2.

| \( I = 0 \) | \( I = 1 \) | \( I = 2 \) |
| --- | --- | --- |
| \( \|E_{S,1}(x, p)\|_2 \) | \( p^{-1}(0.4714)^p \) | \( p^{-1}(0.7071)^p \) | \( p^{-1}(0.9428)^p \) |
| \( \|E_{S,2}(x, p)\|_2 \) | \( p^{-1}(0.6407)^p \) | \( p^{-1}(0.6407)^p \) | \( p^{-1}(0.6407)^p \) |
| \( \|E_{S,31}(x, p)\|_2 \) | \( p^{-1}(0.6009)^p \) | \( p^{-1}(0.6374)^p \) | \( p^{-1}(0.6640)^p \) |
| \( \|E_{S,32}(x, p)\|_2 \) | \( p^{-4}(0.6927)^p \) | \( p^{-4}(0.6927)^p \) | \( p^{-4}(0.6927)^p \) |
| \( \|E_{S,4}(x, p)\|_2 \) | \( p^{-4}(0.6407)^p \) | \( p^{-4}(0.6407)^p \) | \( p^{-4}(0.6407)^p \) |
| \( \|E_S(x, p)\|_2 \) | \( p^{-4}(0.6927)^p \) | \( p^{-4}(0.7071)^p \) | \( p^{-4}(0.9428)^p \) |

According to Table 2, we see that for large \( p \), when \( I = 0 \),

\[
\|E_{S,1}(x, p)\|_2 \ll \|E_{S,3}(x, p)\|_2, \quad \|E_{S,2}(x, p)\|_2 \ll \|E_{S,3}(x, p)\|_2, \quad \|E_{S,4}(x, p)\|_2 \ll \|E_{S,3}(x, p)\|_2,
\]

when \( I = 1 \) and \( 2 \),

\[
\|E_{S,3}(x, p)\|_2 \ll \|E_{S,3}(x, p)\|_2 \ll \|E_{S,1}(x, p)\|_2, \quad \|E_{S,4}(x, p)\|_2 \ll \|E_{S,3}(x, p)\|_2 \ll \|E_{S,1}(x, p)\|_2.
\]

Thus, we can estimate the bound of \( \|E_S(x, p)\|_2 \) by that of \( \|E_{S,1}(x, p)\|_2 \) and \( \|E_{S,3}(x, p)\|_2 \).

## 5 Error of FMM for \((K\varphi)(x)\)

In this section, we will study the bound for error of FMM for the double-layer potential \((K\varphi)(x)\). Use the quadrature rule, we have

\[
(K\varphi)(x) = 2 \int_{\partial\Omega} \frac{\partial \Phi(x, y)}{\partial n(y)} \varphi(y) ds(y) \approx \frac{i\pi}{2N} \sum_{j=1}^{2N} \frac{\partial H_0^{(1)}(k|x - y_j|)}{\partial n(y_j)} \varphi(y_j)s(y_j), \quad x \in \partial\Omega.
\]

Most of the expansions and translations about \((K\varphi)(x)\) are similar to that of \((S\varphi)(x)\), we only show the differences.

The multipole moments about \( C_L \) are:

\[
M_n(O_{C_L}) = \sum_{y_j \in C_L} \frac{\partial J_n(k|y_j - O_{C_L}|)}{\partial n(y_j)} e^{-in\theta_j} \varphi(y_j)s(y_j), \quad n \in \mathbb{Z}.
\]
and the errors are:

\[ EM_n(O_{CL}, p) = \sum_{i=1}^{n_d L} \left( \sum_{l=1}^{p} EM_l(O_{CL+1, i}, p) J_{n-1}(k|O_{CL+1, i} - O_{CL}|) e^{-i(n-l)\theta} O_{CL+1, i} - O_{CL} \right) e^{-i\theta} |O_{CL+1, i} - O_{CL}| \]

\[ + \sum_{y_j \in CL+1, i} \frac{\partial R^0_{l,p}(k(O_{CL+1, i} - O_{CL}), -k(y_j - O_{CL+1, i}))}{\partial \nu(y_j)} \varphi(y_j) s(y_j) \].

The multipole expansion:

\[ \sum_{i=1}^{n_d P_L} \sum_{y_j \in D_L} \frac{\partial H_0^{(1)}(k|y - y_j|)}{\partial \nu(y_j)} \varphi(y_j) s(y_j) \]

\[ = \sum_{i=1}^{n_d P_L} \sum_{n=-p}^{p} H_n^{(1)}(k|y - O_{D_L}|) e^{i\theta} \nu \varphi(y_j) s(y_j), \]

in which

\[ E_{ME}(x, O_{D_L}, p) = \sum_{i=1}^{n_d L} \sum_{y_j \in D_L} \frac{\partial R^H_{0, p}(k(x - O_{D_L}), k(y_j - O_{D_L}))}{\partial \nu(y_j)} \varphi(y_j) s(y_j). \]

It follows that

\[ E_K(x, p) = L_{max} \sum_{L=2}^{n_d L} \sum_{y_j \in D_L} \sum_{y_j \in D_L} \frac{\partial R^H_{0, p}(k(x - O_{D_L}), k(y_j - O_{D_L}))}{\partial \nu(y_j)} \varphi(y_j) s(y_j) \]

\[ + \sum_{L=2}^{L_{max}} \sum_{i=1}^{n_d P_L} \sum_{n=-p}^{p} H_n^{(1)}(k|y - O_{D_L}|) e^{i\theta} \nu \varphi(y_j) s(y_j) \]

\[ + \sum_{L=2}^{L_{max}} \sum_{i=1}^{n_d P_L} \sum_{n=-p}^{p} R_n^{H}(k(O_{D_L} - O_{D_L}), k(y_j - O_{D_L})) \varphi(y_j) s(y_j) \]

\[ + \sum_{L=3}^{L_{max}} \sum_{n=-p}^{p} R_n^{H}(k(O_{D_L} - O_{D_L}), k(y_j - O_{D_L})) \varphi(y_j) s(y_j) \].

The local moments about \( D_L \) are:

\[ L_m(O_{D_L}) = \sum_{L=2}^{L_{max}} \sum_{i=1}^{n_d L} \sum_{y_j \in D_L} \frac{\partial H_0^{(1)}(k|y_j - O_{D_L}|) e^{-i\theta} \nu}{\partial \nu(y_j)} \varphi(y_j) s(y_j). \]

The lemmas in Section 3 can also be used to estimate the bound for \( E_K(x, p) \). Suppose \( \mathcal{B} \) denotes \( J, Y \) or \( H \), by the recurrence relations \[ \tag{35} \]

\[ 2\mathcal{B}_n(z) = \mathcal{B}_{n-1}(z) - \mathcal{B}_{n+1}(z), \]

\[ 2n \mathcal{B}_n(z) = \mathcal{B}_{n-1}(z) + \mathcal{B}_{n+1}(z), \]

we have

\[ \frac{\partial \mathcal{B}_n(k|y - y_c|) e^{i\theta}}{\partial \nu(y)} = \frac{k}{2} \left[ \mathcal{B}_{n-1}(k|y - y_c|) e^{i\theta} - \mathcal{B}_{n+1}(k|y - y_c|) e^{-i\theta} \right] e^{i\theta}, \]

where \( \theta \) is the angle between the vector \( y - y_c \) and the outward normal \( \nu(y) \). It follows that

\[ \left| \frac{\partial \mathcal{B}_n(k|y - y_c|) e^{i\theta}}{\partial \nu(y)} \right| \leq \frac{k}{2} \left[ |\mathcal{B}_{n-1}(k|y - y_c|)| + |\mathcal{B}_{n+1}(k|y - y_c|)| \right]. \]
Moreover, we have

\[
\frac{\partial R_{m,p}(k(x - x_c), k(y - y_c))}{\partial \nu(y)} \leq \left( \sum_{n=p+1}^{\infty} + \sum_{n=-\infty}^{-p-1} \right) |\mathcal{B}_{m+n}(k|x - x_c|)| \left| \frac{\partial J_n(k|y - y_c|)e^{i\pi n \theta y - y_c}}{\partial \nu(y)} \right|
\]

\[
\leq k \left( \sum_{n=p+1}^{\infty} + \sum_{n=-\infty}^{-p-1} \right) |\mathcal{B}_{m+n}(k|x - x_c|)| \left[ |J_{n-1}(k|y - y_c|)| + |J_{n+1}(k|y - y_c|)| \right]
\]

\[
\leq \frac{k}{2} \left( \sum_{n=p+1}^{\infty} \right) \left[ |\mathcal{B}_{n+m}(k|x - x_c|)| + |\mathcal{B}_{n-m}(k|x - x_c|)| \right] \left[ |J_{n-1}(k|y - y_c|)| + |J_{n+1}(k|y - y_c|)| \right],
\]

by the proof of Lemma 1 and the recurrence relation (35), when \( n - 1 \geq k|y - y_c| \), it follows that

\[ 0 \leq J_{n+1}(k|y - y_c|) \leq J_{n-1}(k|y - y_c|), \]

thus when \( p \geq k|y - y_c| \), we obtain

\[
\frac{\partial R_{m,p}(k(x - x_c), k(y - y_c))}{\partial \nu(y)} \leq k \sum_{n=p}^{\infty} \left[ |\mathcal{B}_{n+m+1}(k|x - x_c|)| + |\mathcal{B}_{n-m+1}(k|x - x_c|)| \right] |J_n(k|y - y_c|)|.
\]

We can give the bound for \( E_K(x, p) \) in the following theorem.

**Theorem 10** Suppose the boundary curve \( \partial \Omega \) is divided into \( 2N \) points and covered by a square with length \( d \). For the field point \( x_j \in \partial \Omega(1 \leq j \leq 2N) \), when \( p \geq \max\{3kd/8 + 1, \zeta_1ekd/16 + 1\} \),

\[
\|E_{K,1}(x, p)\|_2 \lesssim \frac{2^{L_{\max} + 4} AN^{1/2}}{\pi d} \sum_{i=0}^{I} N_i \tau_i^p \frac{1}{2^i (\sqrt{2 \varepsilon_i} - \eta_i)},
\]

\[
\|E_{K,2}(x, p)\|_2 \lesssim \frac{2.1 AN^{3/2} ek^2 d}{\pi^2 p^{5/2}} \left( \frac{\sqrt{2 \varepsilon}}{6} \right)^p,
\]

\[
\|E_{K,31}(x, p)\|_2 \lesssim \frac{2^{L_{\max} + 4} AN^{1/2}}{\sqrt{2 \pi d}} \sum_{i=0}^{I} \frac{N_i \gamma_i (\sqrt{2 \varepsilon_i - 2})^{p}}{2^i},
\]

\[
\|E_{K,32}(x, p)\|_2 \lesssim \frac{1.05 \sqrt{2} AN^{3/2} ek^2 d}{\pi^2 p^{3/2} (1 - 2 \gamma_0)} \left( \frac{3c^2}{32} \right),
\]

\[
\|E_{K,4}(x, p)\|_2 \lesssim \frac{1.05 \sqrt{2} AN^{3/2} ek^2 d}{12 \pi^2 p^{5/2}} \left( \frac{\sqrt{2 \varepsilon}}{6} \right)^p,
\]

where \( N_i = \max\{N_i(x_j)|1 \leq j \leq 2N\} \) and \( I = \max\{|i|N_i \neq 0\} \). \( \square \)

The proof of this theorem is quite similar to that given earlier for \( E_2(x, p) \) and so is omitted.

### 6 Numerical experiments

In this section, we performed some numerical experiments to test the errors and their bounds derived in this paper. The numerical experiments were realized as FORTRAN programs.
Consider the boundary integral operator

\[(S \varphi)(x) = \int_{\partial \Omega} \Phi(x, y) \varphi(y) ds(y), \quad x \in \partial \Omega,\]

the boundary curve \(\partial \Omega\) is kite-shaped, with the parametric representation

\[
\partial \Omega : (\cos t + 0.65 \cos 2t - 0.65, 1.5 \sin t). \quad 0 \leq t \leq 2\pi.
\]

We choose a square with length \(d = 4\) covering \(\partial \Omega\), see Fig.10 for the geometry.

![Fig.10. The kite-shaped domain.](image)

The boundary is divided into \(2N = 1000\) points, and the leaf cells of the tree structure cover up to \(\ln N \approx 6\) points. Thus, an adaptive quadtree structure is constructed. In addition, we fixed the wave number \(k = 5\) and the density function \(\varphi(y) \equiv 1\). An easy computation shows that

\[A = \|\varphi(y)s(y)\|_\infty \approx 2.2718.\]

We first test \(E_S(x, p)\) and its bound for the fixed field point \(x\). From Theorem 4 and 6, the bounds for \(E_{S,1}(x, p)\) and \(E_{S,31}(x, p)\) are dependent on \(I\) and \(N_i(x)\), thus we choose three representative points \(x_1, x_{32}\) and \(x_{94}\), see Table 3 for details.

| Table 3. Three representative points |
|--------------------------------------|
| \(x\) | \(N_0(x)\) | \(N_1(x)\) | \(N_2(x)\) | \(I\) |
| \(x_1(0.999, 0.009)\) | 980 | 0 | 0 | 0 |
| \(x_{32}(0.928, 0.300)\) | 971 | 8 | 0 | 1 |
| \(x_{94}(0.427, 0.835)\) | 979 | 0 | 4 | 2 |

In what follows, the exact values of \(E_S(x, p)\) are computed by the formula given in Theorem 2, and we fix the unknown constant \(c = 1.2\) in the bound of \(E_{S,4}(x, p)\). In Fig.11, 12 and 13, the four parts of \(|E_S(x_1, p)|, |E_S(x_{32}, p)|, |E_S(x_{94}, p)|\) and also their bounds are plotted as functions of \(p\) respectively. From the results, we see that the bounds and exact values are in good agreement. More concretely, when \(I = 0\) (Fig.11), \(|E_{S,3}(x, p)|\) and its bound are all larger than the other three, when \(I = 1\) (Fig.12) and 2 (Fig.13), \(|E_{S,1}(x, p)|\) is the largest.
Fig. 11. The exact values of $|E_{S,j}(x_1, p)|(j = 1, \cdots, 4)$ (left) and their bounds (right).

Fig. 12. The exact values of $|E_{S,j}(x_{32}, p)|(j = 1, \cdots, 4)$ (left) and their bounds (right).

Fig. 13. The exact values of $|E_{S,j}(x_{84}, p)|(j = 1, \cdots, 4)$ (left) and their bounds (right).
Next, in Fig. 14, $\|E_S(x, p)\|_2$ and its bound are shown as functions of $p$. We see that the bound is not sharp, and with the increase of $p$, it is far away from the exact value. In fact, from Theorem 9, when $I = 2$,

$$\|E_S(x, p)\|_2 \lesssim O(p^{-1}r_2^p),$$

where

$$r_2 = \sup \left\{ \frac{|y - O_{DL}|}{|x - O_{DL}|} \mid x \in D, y \in DL \right\} \approx 0.9428.$$ 

However, in this experiment,

$$\max \left\{ \frac{|y - O_{DL}|}{|x - O_{DL}|} \mid x \in D, y \in DL \right\} \approx 0.7682,$$

for larger $p$, it is obvious that $0.7682^p \ll 0.9428^p$. In Fig. 15, we replace $r_2$ by 0.7682, the satisfactory result is obtained. The same reasons cause the unsatisfactory bounds in Fig. 11-13.

From the above analysis, we conclude that the bounds for the global errors of FMM for Helmholtz equation given in this paper are valid.
7 Conclusions

This article focuses on the global error of fast multipole method for Helmholtz equation. Explicit bounds and convergence rates of the global errors were derived. From those results, we see that the two main parts of the global errors are $E_1$ and $E_3$, that is, the errors of multipole expansion and M2L translation. Thus, for fixed error $\varepsilon$, we can estimate the smallest truncation number $p$ by the bounds for $E_1$ and $E_3$.

In FMM, two kinds of tree structures are available, namely symmetric and asymmetric tree structures. It is well known that the asymmetry tree structure is more compact and efficient than the symmetric one. However, in the symmetric tree, all the leaves are in the same level, it follows that $I = L - L_{DL_i} = 0$ for each cell $D_L$. Our results show that the FMM with symmetric tree has higher convergence rate than that with asymmetric one.

In this paper, the global error of the FMM for Helmholtz equation was described as the expression of $R_{m,p}$, and we estimate the bounds for $R_{m,p}$ by the limiting forms of $Y_n(z)$ and $J_n(z)$ as $z \to 0$, which have the same forms with those in potential problems. Thus, the proposed method and results can be easily applied to study the global errors of the FMM for potential problems, elastostatic problems and Stokes flow problems.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (11201373) and Natural Science Foundation of Shaanxi Provincial Department of Education (14JK1747).

References

[1] V.Rokhlin, Rapid solution of integral equations of classical potential theory, J.Comp.Phys., 60(1985):187-207.

[2] V.Rokhlin, Rapid solution of integral equations of scattering theory in two dimensions, J. Comput. Phys., 86(1990):414-439.

[3] J.Rahola, Diagonal forms of the translation operators in the fast multipole algorithm for scattering problems, BIT 36(2)(1996):333-358.

[4] H.Cheng, J.Huang, T.J.Leiterman, An adaptive fast solver for the modified Helmholtz equation in two dimensions, J. Comput. Phys., 211(2006):616-637.

[5] S.Amini, A.T.J.Profit, Multi-level fast multipole solution of the scattering problem, Eng. Anal. Bound. Elem., 27 (2003) 547-564.

[6] N.A.Gumerov, R.Duraiswami, Fast multipole methods for the Helmholtz equation in three dimensions, Elsevier: Oxford, 2004.
[7] N.Nishimura, Fast multipole accelerated boundary integral equation methods. Applied Mechanics Reviews, 55(4)(2002):299-324.

[8] Y.J.Liu, Fast Multipole Boundary Element Method-Theory and Applications in Engineering, Cambridge University Press, Cambridge, 2009.

[9] L.F.Greengard, J.Huang, A new version of the fast multipole method for screened coulomb interactions in three dimensions, J. Comput. Phys., 180(2002):642-658.

[10] S.Amini, A.Profit, Analysis of the truncation errors in the fast multipole method for scattering problems, J. Comput. Appl. Math., 115(2000):23-33.

[11] Eric Darve, The fast multipole method I: error analysis and asymptotic complexity, SIAM J. Numer. Anal., 38(1)(2000):98-128.

[12] Seiya KISHIMOTO and Shinichiro OHNUKI, Error analysis of multilevel fast multipole algorithm for electromagnetic scattering problems, IEICE TRANS.ELECTRON., 95(1)(2012):71-78.

[13] Milton Abramowitz and Irene A. Stegun, eds., Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover: New York, 1972.

[14] Frank W.J. Olver, NIST Handbook of Mathematical Functions, Cambridge University Press, 2010.

[15] D.Colton and R.Kress, Integral Equation in Scattering Theory. John Wiley and sons, New York, 1983.

[16] R.Kress, Boundary integral equation in time-harmonic acoustic scattering. Math.Comput. Modelling, 15(3-5)(1991):229-243.

[17] G.Nemes, New asymptotic expansion for the Gamma function, Arch. Math., 95(2)(2010):161-169.

[18] G.Hardy, J.E.Littlewood and G.Polya, Inequalities(Second Edition), Cambridge University Press, 2011.

[19] Steele, J. Michael, The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities. MAA Problem Books Series. Cambridge University Press, 2004.

[20] W.Meng, L.Wang, Bounds for truncation errors of Graf’s and Neumann’s addition theorems, Numer.Algor., (2016)72:91-106.