A COMBINATORIAL APPROACH TO THE LITTLEWOOD CONJECTURE IN A FIELD OF FORMAL SERIES

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Abstract. A long-standing conjecture of Littlewood about simultaneous Diophantine approximation has an analogous problem for a field of formal Laurent series \( \mathbb{F}(t^{-1}) \). That is, we can ask whether for any series \( \Theta, \Phi \) and any \( \epsilon > 0 \), there is a polynomial \( \alpha \) such that
\[
|\alpha \Theta x \alpha \Phi y| < \epsilon \text{ where } x \inf_{\beta \in \mathbb{F}^{\times}} |\Theta - \beta|.
\]
If the base field \( \mathbb{F} \) is finite, then this problem is still open and we explore this problem via group action on a Bruhat-Tits building.

Contents

1. Introduction 1
2. Connection to homogeneous dynamics 3
3. The building associated to \( SL(3, \mathbb{K}) \) 5
4. The quotient complex \( \Gamma \setminus \Delta \) and stabilizers 8
5. Further discussion 11
References 12

1. Introduction

For a given real number \( x \), let us denote by \( \langle x \rangle \) the difference \( \inf_{n \in \mathbb{Z}} |x - n| \) between \( x \) and the nearest integer. A well-known conjecture of Littlewood concerning simultaneous Diophantine approximation can be stated as follows: for any real numbers \( x, y \) and any \( \epsilon > 0 \), do there exist infinitely many positive integers \( n \) such that
\[
n\langle nx \rangle \langle ny \rangle < \epsilon?
\]
Pollington and Velani showed in [PV00] that for each \( x \in \mathbb{R} \), the intersection of \( y \in \mathbb{R} \) satisfying \( \lim_{n \to \infty} n\langle nx \rangle \langle ny \rangle = 0 \) with the set of badly approximable numbers has Hausdorff dimension one. The authors in [EKL06] achieved a remarkable progress on this problem by classifying ergodic invariant measures of diagonal actions on \( SL(k, \mathbb{Z}) \setminus SL(k, \mathbb{R}) \) with positive entropy. As a result, they have shown that the exceptional set of pairs \( (x, y) \in \mathbb{R}^2 \)
must have Hausdorff dimension zero. Meanwhile, Einsidler, Mohammadi and Lindenstrauss (2017) proved the similar measure rigidity theorems in positive characteristic setting.

The analogous problem for a field of formal Laurent series was first considered by Davenport and Lewis (1963). They gave a negative answer to this problem when the field is \( \mathbb{F}(\!\!(t^{-1})\!) \) with \(|\mathbb{F}| = \infty\). In this article, we consider the case when \( \mathbb{F} \) is a finite field.

Denote by \( \mathbb{F}_q \) the finite field of order \( q \). Let \( \mathbb{K} = \mathbb{F}_q(\!\!(t^{-1})\!) \), \( \mathbb{Z} = \mathbb{F}_q[\![t]\!] \), and \( \mathcal{O} = \mathbb{F}_q[\![t^{-1}]\!] \). The absolute value \(| \cdot |\) on \( \mathbb{K} \) is given by \(|\Theta| = q^{\deg(\Theta)}\). Define \( \langle \Theta \rangle = \inf_{\beta \in \mathbb{Z}} |\Theta - \beta| \), the distance between \( \Theta \) and \( \mathbb{Z} \). For each badly approximable series \( \Theta \), the authors in (2007) explicitly constructed uncountably many badly approximable power series \( \Phi \) such that \( p(\Theta, \Phi) \) satisfies the Littlewood conjecture, using the theory of continued fraction. We give a connection between the combinatorics and topology of group action and diophantine approximation.

Let 
\[
U = \left\{ u_{\Theta, \Phi} = \begin{pmatrix} 1 & \Theta & \Phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \Theta, \Phi \in \mathcal{O} \right\}
\]
be a unipotent subgroup of \( \text{SL}(3, \mathbb{K}) \) and let \( A^+ \) be the semigroup given by
\[
\{ \text{diag}(t^{-r-s}, t^r, t^s) : r, s \geq 0 \}.
\]
Let \( \Delta \) be the Bruhat-Tits building associated to \( \text{SL}(3, \mathbb{K}) \) and \( \mathcal{A} \) be the standard apartment of \( \Delta \). Denote by \( \pi \) the natural projection \( \Delta \to \text{SL}(3, \mathbb{Z}) \setminus \Delta \).

See Section 3 for the detail.

**Theorem 1.1.** A pair \((\Theta, \Phi) \in \mathbb{K}^2\) satisfies
\[
(1.1) \quad \inf_{\alpha \in \mathbb{Z}\setminus\{0\}} |\alpha| \langle \alpha \Theta \rangle \langle \alpha \Phi \rangle > 0
\]
if and only if the image of the vertices in the \( A^+ \)-cone of \( \mathcal{A} \) under \( \pi \circ u_{\Theta, \Phi} \) is finite.

**Sketch of proof.** It is enough for us to consider \((\Theta, \Phi)\in \mathcal{O}^2\). Let
\[
U = \left\{ u_{\Theta, \Phi} = \begin{pmatrix} 1 & \Theta & \Phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \Theta, \Phi \in \mathcal{O} \right\}.
\]
A pair \((\Theta, \Phi) \in \mathcal{O}^2\) satisfies (1.1) if and only if the orbit of \( \text{SL}(3, \mathbb{Z}) u_{\Theta, \Phi} \) under the semigroup \( A^+ \) given by \( \{ \text{diag}(t^{-r-s}, t^r, t^s) : r, s \geq 0 \} \) is bounded.
in $SL(3,\mathbb{Z}) \setminus SL(3,\mathbb{K})$. We will discuss these details in Section 2. The idea is similar to the real number case, which was already used in [EKL06].

Let $G = SL(3,\mathbb{K})$ and $\Gamma = SL(3,\mathbb{Z})$. Let $K$ be a maximal open compact subgroup $SL(3,\mathcal{O})$ in $SL(3,\mathbb{K})$. We will show that there is a pair $(\Theta, \Phi) \in \mathcal{O}^2$ such that $\Gamma u_{\Theta, \Phi} A^+$ is contained in $\Gamma K$ as follows.

Let $\Delta$ be the Bruhat-Tits building associated to the group $PGL(3,\mathbb{K})$. The set of vertices $\text{Vert}(\Delta)$ can be identified with $PGL(3,\mathbb{K})$ which is also identified with the similarity classes of $\mathcal{O}$-lattices via $gPGL(3,\mathbb{O}) \leftrightarrow [ge_1\mathcal{O} \oplus ge_2\mathcal{O} \oplus ge_3\mathcal{O}], \quad g \in PGL(3,\mathbb{K}).$

In particular, there is a standard apartment in $\Delta$, which we will denote by $\mathcal{A}$. The vertices of $\mathcal{A}$ are the classes of the form $[[t^l, t^m, 1]] = [t^l e_1 \mathcal{O} \oplus t^m e_2 \mathcal{O} \oplus e_3 \mathcal{O}]$, which we will denote by $\psi_{l,m}$, for $(l, m) \in \mathbb{Z}^2$. Moreover, the semigroup $A^+$ acts as translation on $\mathcal{A}$. In Section 3 we review some of the properties of $\Delta$ and $\mathcal{A}$.

The group $\Gamma$ acts on $\Delta$ as type-preserving automorphisms and there are exactly three $\Gamma$-orbits in $\text{Vert}(\Delta)$. Let us denote by $\pi: \Delta \rightarrow \Gamma \setminus \Delta$ the natural projection. Let $S$ be the set of vertices in $\Gamma \setminus \Delta$. Each vertex in $S$ is the class of the form $[[t^a, t^b, 1]] = [0 \leq b \leq a]$ and we will denote it by $s_{a,b}$. Section 4 describes the quotient complex $\Gamma \setminus \Delta$.

In fact, we have $\Gamma \setminus G = \bigsqcup_{s \in S} sK$ where $sK = \{s\psi_k | k \in K\} \subset \Gamma \setminus G$. For any given $a \in A^+$ and $g \in G$, the map $\pi \circ g \circ a: \text{Vert}(\mathcal{A}) \rightarrow S$ is well-defined. Let $V^+$ be the set of vertices contained in $A^+$-cone of $\mathcal{A}$. If $\pi \circ g(V^+) = \{s_1, \ldots, s_k\}$ and if each $s_i$ corresponds to the double coset $\Gamma g_i K$, then this implies that $\Gamma g A^+ \subset \bigsqcup_{i=1}^k \Gamma g_i K$. As a finite union of compact sets, $\bigsqcup_{i=1}^k \Gamma g_i K$ is compact in $\Gamma \setminus G$. This completes the proof. \hfill \Box

2. Connection to homogeneous dynamics

Let $G = SL(3,\mathbb{K})$, $\Gamma = SL(3,\mathbb{Z})$ and let $A$ be the full diagonal subgroup of $G$. Let $T$ be the subgroup of $A$ containing the elements $\text{diag}(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_i \in \mathcal{O}$.

For given $r, s \in \mathbb{Z}$, let $a_{r,s}$ be the $3 \times 3$ diagonal matrix $\text{diag}(t^{-r-s}, t^r, t^s)$ and $A^+$ be the semigroup $\{a_{r,s} | r, s \geq 0\}$. Given a pair of elements $\Theta$ and $\Phi$
in $K$, denote by $x_{\Theta, \Phi}$ the element $\Gamma u_{\Theta, \Phi}$ in $\Gamma \backslash G$, where

$$u_{\Theta, \Phi} = \begin{pmatrix} 1 & \Theta & \Phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Let us recall the following simple observation which is analogous to the Dirichlet principle of the Diophantine approximation in real numbers. In [GG17], even more general statement is proved.

**Lemma 2.1** (Dirichlet principle). For any $\Theta \in K$ and $n \in \mathbb{N}$, there exists a non-zero polynomial $\alpha \in \mathbb{Z}$ such that $|\alpha| \leq q^n$ and $\langle \alpha \Theta \rangle < q^{-n}$. 

**Proof.** Without loss of generality, we may assume that $\Theta \in t^{-1} \mathcal{O}$. Given $\Theta = \theta_1 t^{-1} + \theta_2 t^{-2} + \cdots$ and $\alpha = a_0 + a_1 t + \cdots + a_n t^n$, we have $|\langle \alpha \Theta \rangle| < q^{-n}$ if and only if $a_0 \theta_j + \cdots + a_n \theta_{j+n} = 0$ for all $1 \leq j \leq n$. Equivalently,

$$\begin{pmatrix} \theta_1 & \cdots & \theta_{n+1} \\ \vdots & & \vdots \\ \theta_n & \cdots & \theta_{2n} \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = 0$$

has a nontrivial solution $(a_0, \ldots, a_n)$. As the matrix has more columns than rows, the non-trivial solution exists and this completes the proof. \hfill $\square$

Recall that the Littlewood’s conjecture says

$$\inf_{\alpha \in \mathbb{Z} \backslash \{0\}} |\alpha| |\langle \alpha \Theta \rangle\langle \alpha \Phi \rangle| = 0$$

holds for all $(\Theta, \Phi) \in K^2$. The following proposition explains the connection between simultaneous Diophantine approximation problem and dynamics on $SL(3, \mathbb{Z}) \backslash SL(3, K)$. It is similar to the real number case which is presented in [EKL06].

**Proposition 2.2.** Let us consider the right multiplication action of $A^+$ on $\Gamma \backslash G$. A pair of elements $(\Theta, \Phi) \in K^2$ satisfies (2.1) if and only if the orbit of $x_{\Theta, \Phi}$ under the semigroup $A^+$ is unbounded in $\Gamma \backslash G$.

**Proof.** ($\Leftarrow$) Let $\epsilon > 0$ be arbitrary. Suppose $\Gamma u_{\Theta, \Phi} A^+$ is unbounded in $\Gamma \backslash G$. Then, there is a pair $(r, s)$ of nonnegative natural number such that in the $\mathbb{Z}$-lattice in $K^3$ generated by the row vectors of $u_{\Theta, \Phi} a_{r,s}$ there exists a vector $v$ with $\|v\|_{\sup} < \epsilon$. Since

$$\begin{pmatrix} 1 & \Theta & \Phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-r-s} & 0 & 0 \\ 0 & t^r & 0 \\ 0 & 0 & t^s \end{pmatrix} = \begin{pmatrix} t^{-r-s} & t^r \Theta & t^s \Phi \\ 0 & t^r & 0 \\ 0 & 0 & t^s \end{pmatrix},$$
this vector \( v \) is of the form \((\alpha t^{-r-s}, (\alpha \Theta - \beta)t^r, (\alpha \Phi - \delta)t^s)\), for some \( \alpha, \beta, \delta \in \mathbb{Z}\setminus \{0\} \). If \( \alpha = 0 \), then \( |\beta t^r| \) and \( |\delta t^s| \) cannot be small unless both \( \beta \) and \( \delta \) is 0. Hence, we have \( \alpha \neq 0 \) and \( |\alpha| |\alpha \Theta - \beta| |\alpha \Phi - \delta| < \epsilon^3 \).

(\( \Rightarrow \)) Suppose that for a given \( n > 0 \), a triple \((\alpha, \beta, \delta) \in \mathbb{Z}^3\) with \( \alpha \neq 0 \) satisfies
\[
\|\alpha(\alpha \Theta - \beta)(\alpha \Phi - \delta)\|_\infty < q^{-5n}.
\]
If we have
\[
\max(|\alpha \Theta - \beta|, |\alpha \Phi - \delta|) < q^{-n},
\]
then there is nothing to prove. Assume on the contrary, without losing generality, that \( |\alpha \Theta - \beta| \geq q^{-n} \) and \( |\alpha(\alpha \Phi - \delta)| < q^{-4n} \). Now Lemma 2.1 implies that there exists \( \alpha' \in \mathbb{Z} \) satisfying \( |\alpha'| \leq q^n \) such that \( \langle \alpha' \alpha \Theta \rangle < q^{-n} \).

In other words, \( |\alpha' \alpha \Theta - \beta'| < q^{-n} \) for some \( \beta' \in \mathbb{Z} \). This implies that we have
\[
|\alpha' \alpha \Theta - \beta'| < q^{-n}, \quad |\alpha' \alpha(\alpha' \alpha \Phi - \alpha' \delta)| < q^{-2n},
\]
and
\[
|\alpha' \alpha(\alpha' \alpha \Theta - \beta')(\alpha' \alpha \Phi - \alpha' \delta)| < q^{-3n}.
\]
There exist \( r, s \in \mathbb{N} \) such that \( |t^r(\alpha' \alpha \Theta - \beta)| = q^{-n}, \ |t^s(\alpha' \alpha \Phi - \alpha' \delta)| = q^{-n} \) and in this case, \( |t^{-r-s} \alpha' \alpha| < q^{-3n} \). Thus,
\[
(\alpha' \alpha t^{-r-s}, (\alpha' \alpha \Theta - \beta)t^r, (\alpha' \alpha \Phi - \alpha' \delta)t^s)
\]
will be a desired short vector in \( \mathbb{Z}^3 \mathbb{u}_{\Theta, \Phi} A^+ \). \[\Box\]

3. The Building Associated to \( SL(3, \mathbb{K}) \)

In this section, we review the definition of the Bruhat-Tits building of \( G \) and briefly summarize some of its properties we are going to use. We follow the book of Abramenko and Brown [AB08].

A maximal cell in a simplicial complex is called a chamber. A finite dimensional simplicial complex is called chamber complex if all the maximal cells have the same dimension and any two chambers are connected by a sequence of chambers whose consecutive ones have a common face of codimension one.

Let \((W, S)\) be a Coxeter group. A Coxeter complex is a complex isomorphic to the one obtained from the partially ordered set whose elements are the cosets \( w(S'), S' \subset S \), ordered by \( A < B \) if \( B \subset A \).

**Definition 3.1.** A building is a complex \( \Delta \) together with a collection of subcomplexes called apartments satisfying the following properties:

(B0) Every apartment is a Coxeter complex.
(B1) For each pair of cells $A, B \in \Delta$ there exists an apartment containing it.

(B2) If $\Sigma, \Sigma'$ are two apartments containing $A$ and $B$, then there exists an isomorphism $\varphi : \Sigma \to \Sigma'$ which stabilizes $A, B$ pointwise.

Let $G$ be a group of automorphisms of a building $\Delta$. We say that $G$ acts strongly transitively if for any apartment $\Sigma$ and a chamber $C \in \Sigma$ and an apartment $\Sigma'$ and $C' \in \Sigma'$ there exists $g \in G$ so that $g\Sigma = \Sigma'$ and $gC = C'$.

The Bruhat-Tits building $\Delta$ of the group $PGL(3, K)$ is defined as follows. It is a two-dimensional flag complex defined as follows. The set of vertices $\text{Vert}(\Delta)$ of $\Delta$ is defined by

$$\text{Vert}(\Delta) = \{ \Lambda \subset K^3 | \mathcal{O}\text{-submodule of rank 3} \}/ \sim$$

where $\Lambda_1 \sim \Lambda_2$ if and only if there exists $\alpha \in K^*$ such that $\Lambda_1 = \alpha \Lambda_2$. Further, $[\Lambda_1], [\Lambda_2], [\Lambda_3]$ form a simplex if and only if we can choose a representative $L_i \in [\Lambda_i]$ such that $t^{-1}L_1 \subset L_2 \subset L_3 \subset L_1$. Figure 1 depicts the neighborhood of each vertex $v$ in $\Delta$ when $q = 2$.

In particular, when $q = 2$ the link of each vertex in $\Delta$ is the well-known Heawood graph. See Figure 2.

Figure 1. The neighborhood of a vertex in $\Delta$ of $PGL(3, K)$

Figure 2. The link of each vertex in $\Delta$, $(q = 2)$
The coset space $PGL(3, \mathbb{K})/PGL(3, \mathcal{O})$ can be identified with $\text{Vert}(\Delta)$ via
\[ gPGL(3, \mathbb{K}) \leftrightarrow [ge_1\mathcal{O} \oplus ge_2\mathcal{O} \oplus ge_3\mathcal{O}], \quad g \in PGL(3, \mathbb{K}). \]
Every apartment is isomorphic to a plane tesselated by equilateral triangles. Among those, we fix a standard apartment $\mathcal{A}$ in $\Delta$ whose vertices are the classes of the form $[[t^l,t^m,1]] = [t^l e_1\mathcal{O} \oplus t^m e_2\mathcal{O} \oplus e_3\mathcal{O}]$, for $(l, m) \in \mathbb{Z}^2$. Let us denote such a vertex by $v_{l,m}$.

![Figure 3. The standard apartment $\mathcal{A}$](image1.png)

The semigroup $A^+$ acts as translation on $\mathcal{A}$. More precisely, an element $\text{diag}(t^{-r-s}, t^r, t^s)$ maps $v_{l,m}$ to $v_{l-r-2s, m+r-s}$. The $A^+$-orbit of $v_{0,0}$ is the set of vertices in the red region in Figure 4. We call this region the $A^+$-cone.

![Figure 4. $A^+$-cone](image2.png)

We give a following simple observation. Recall that $\pi: \Delta \to \Gamma \backslash \Delta$ denotes the projection map.

**Proposition 3.2.** The orbit of $\Gamma g$ under the semigroup $A^+$ is bounded in $\Gamma \backslash G$ if the image of the vertices in the $A^+$-cone of $\mathcal{A}$ under $\pi \circ g$ is finite.

**Proof.** Let $V^+$ be the set of vertices contained in $A^+$-cone of $\mathcal{A}$. If $\pi \circ g(V^+) = \{s_1, \ldots, s_k\}$ and if each $s_i$ corresponds to the double coset $\Gamma g_i K$, then this implies that $\Gamma g A^+ \subset \bigcup_{i=1}^k \Gamma g_i K$. As a finite union of compact sets, $\bigcup_{i=1}^k \Gamma g_i K$ is compact in $\Gamma \backslash G$. □

Therefore, the problem reduces to finding an element $u$ in $U$ such that $\pi \circ u$ maps $V^+$ to a finite subset of the set of vertices in $\Gamma \backslash \Delta$. 
4. THE QUOTIENT COMPLEX $\Gamma \backslash \Delta$ AND STABILIZERS

Recall that the set of vertices in $\Delta$ can be identified with the left coset space $PGL(3, K)/PGL(3, \mathcal{O})$. Let $\tau : \text{Vert}(\Delta) \to \{1, 2, 3\}$ be the type function so that the vertices of every chamber are mapped bijectively onto $\{1, 2, 3\}$. Let us denote by $\Gamma$ the group $SL(3, \mathbb{Z})$. Then, $\Gamma$ acts on $\Delta$ by left translation. It preserves the type of the vertex and acts transitively on the set of vertices of the same type. Hence, there are exactly three mutually disjoint $\Gamma$-orbits in the set of vertices in $\Delta$.

![Figure 5. The simplicial cone $\mathcal{C}$](image)

Let $\mathcal{C}$ be the upper right simplicial cone based at the vertex $v_{0,0}$, which consists of the vertices $v_{l,m}$ with $0 \leq m \leq l$. It is a fundamental domain for the action of $\Gamma$ on $\Delta$. In other words, any simplex of $\Delta$ is $\Gamma$-equivalent to a unique simplex in $\mathcal{C}$. Thus, the quotient complex $\Gamma \backslash \Delta$ is the simplicial cone $\mathcal{C}$ itself, although the group $\Gamma$ does not act freely on $\Delta$. Let $\pi : \Delta \to \Gamma \backslash \Delta$ be the natural projection. Let $S$ be the set of vertices of $\Gamma \backslash \Delta$ and denote by $s_{a,b}$ the vertex $\pi(v_{a,b})$ in $\Gamma \backslash \Delta$. See Figure 5 for the structure of the quotient complex $\Gamma \backslash \Delta$. We refer to [So13] in which the details are given.

![Figure 6. Chambers in $\Gamma \backslash \Delta$ with assigned number](image)

For convenience, we assign a number to each chamber as follows. Give $a^2 + 2(a - b) + 1$ to the chamber in $\Gamma \backslash \Delta$ whose vertices are $s_{a,b}, s_{a+1,b}$ and $s_{a+1,b+1}$. The chamber in $\Gamma \backslash \Delta$ whose vertices are $s_{a,b}, s_{a+1,b}$ and $s_{a,b-1}$ will be assigned by $a^2 + 2(a - b) + 2$. See Figure 6.
Consider the neighborhood of the vertex $v_{1,1}$ in $\Delta$. The assigned number under the projection $\pi$ of each chamber is given in Fig. 7.

Figure 7. The neighborhood of the vertex $v_{1,1}$ in $\Delta$, $(q = 2)$

On the other hand, figure 8 describes the image of the chambers of the standard apartment $\mathcal{A}$ under the projection $\pi$. To determine $\pi(v_{l,m})$ for all $(l, m) \in \mathbb{Z}^2$, we need to consider each of six cases according to the order of $l, m$ and 0. For instance, if $l \leq 0 < m$, then $SL(3, \mathbb{Z})[[t^l, t^m, 1]] = [[t^{m-l}, t^{-l}, 1]]$ and hence $\pi(v_{l,m}) = s_{m-l, -l}$. For the other five cases the images under $\pi$ can be determined similarly.

Figure 8. Chambers in $\mathcal{A}$ with assigned numbers

As we mentioned before, the group $\Gamma$ does not act freely on $\Delta$, and so is not the case for any finite index subgroup of $\Gamma$. Let us denote by $\Gamma_{a,b}$ the stabilizer of the vertex $v_{a,b}$ in $\Gamma$. Then we have

$$\Gamma_{a,b} = \Gamma \cap \left( \begin{array}{ccc} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & 1 \end{array} \right) SL(3, \mathcal{O}) \left( \begin{array}{ccc} t^{-a} & 0 & 0 \\ 0 & t^{-b} & 0 \\ 0 & 0 & 1 \end{array} \right)$$
Therefore, we can calculate the index between the stabilizer groups of vertices, edges, and faces. Although these are not fully used in the rest of the paper, we present all the indices for future reference. First, the indices of subgroups corresponding to edges and the face containing $v_{0,0}$ are

\[
|\Gamma_{0,0} : \Gamma_{0,0} \cap \Gamma_{1,0}| = q^2 + q + 1, \quad |\Gamma_{0,0} : \Gamma_{0,0} \cap \Gamma_{1,1}| = q^2 + q + 1,
\]

\[
|\Gamma_{0,0} : \Gamma_{0,0} \cap \Gamma_{1,0} \cap \Gamma_{1,1}| = (q^2 + q + 1)(q + 1).
\]

When $b = 0$, we have

\[
|\Gamma_{a,0} : \Gamma_{a-1,0} \cap \Gamma_{a,0}| = q^2, \quad |\Gamma_{a,0} : \Gamma_{a,0} \cap \Gamma_{a,1}| = q^2 + q,
\]

\[
|\Gamma_{a,0} : \Gamma_{a,0} \cap \Gamma_{a+1,1}| = q + 1, \quad |\Gamma_{a,0} : \Gamma_{a,0} \cap \Gamma_{a+1,0}| = t,
\]

\[
|\Gamma_{a,0} : \Gamma_{a,0} \cap \Gamma_{a+1,1}| = q^2 + q, \quad |\Gamma_{a,0} : \Gamma_{a,0} \cap \Gamma_{a+1,0} \cap \Gamma_{a+1,1}| = q + 1.
\]

When $a = b$, we have

\[
|\Gamma_{a,a} : \Gamma_{a-1,a-1} \cap \Gamma_{a,a}| = q^2, \quad |\Gamma_{a,a} : \Gamma_{a,a-1} \cap \Gamma_{a,a}| = q^2 + q,
\]

\[
|\Gamma_{a,a} : \Gamma_{a,a} \cap \Gamma_{a+1,a}| = q + 1, \quad |\Gamma_{a,a} : \Gamma_{a,a} \cap \Gamma_{a+1,a+1}| = q + 1,
\]

\[
|\Gamma_{a,a} : \Gamma_{a,a-1} \cap \Gamma_{a,a} \cap \Gamma_{a+1,a}| = q^2 + q, \quad |\Gamma_{a,a} : \Gamma_{a,a} \cap \Gamma_{a+1,a} \cap \Gamma_{a+1,a+1}| = q + 1.
\]

Finally, for the case $0 < b < a$, we have

\[
|\Gamma_{a,b} : \Gamma_{a-1,b} \cap \Gamma_{a,b}| = q^2, \quad |\Gamma_{a,b} : \Gamma_{a-1,b-1} \cap \Gamma_{a,b}| = q^2,
\]

\[
|\Gamma_{a,b} : \Gamma_{a,b-1} \cap \Gamma_{a,b}| = q, \quad |\Gamma_{a,b} : \Gamma_{a,b+1} \cap \Gamma_{a,b}| = q,
\]

\[
|\Gamma_{a,b} : \Gamma_{a+1,b} \cap \Gamma_{a,b}| = 1, \quad |\Gamma_{a,b} : \Gamma_{a+1,b+1} \cap \Gamma_{a,b}| = 1,
\]

\[
|\Gamma_{a,b} : \Gamma_{a,b} \cap \Gamma_{a-1,b} \cap \Gamma_{a-1,b-1}| = q^3, \quad |\Gamma_{a,b} : \Gamma_{a,b} \cap \Gamma_{a-1,b-1} \cap \Gamma_{a,b-1}| = q^2,
\]

\[
|\Gamma_{a,b} : \Gamma_{a,b} \cap \Gamma_{a,b-1} \cap \Gamma_{a+1,b}| = q, \quad |\Gamma_{a,b} : \Gamma_{a,b} \cap \Gamma_{a+1,b} \cap \Gamma_{a+1,b+1}| = 1,
\]

\[
|\Gamma_{a,b} : \Gamma_{a,b} \cap \Gamma_{a+1,b+1} \cap \Gamma_{a,b+1}| = q, \quad |\Gamma_{a,b} : \Gamma_{a,b} \cap \Gamma_{a,b+1} \cap \Gamma_{a-1,b}| = q^2.
\]
5. Further discussion

In the study of [1.1] without loss of generality, we may assume both Θ and Φ are contained in $t^{-1}O$. Let $v_{l,m}$ be the vertex of $\Delta$ corresponding to the class $[[t^l, t^m, 1]]$, or equivalently, $\text{diag}(t^l, t^m, 1)PGL(3, O)$ in view of the element in $PGL(3, K)/PGL(3, O)$. Let us recall that

$$u_{\Theta, \Phi} = \begin{pmatrix} 1 & \Theta & \Phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$U$ is the group of $\{u_{\Theta, \Phi} : \Theta, \Phi \in O\}$, and $S$ is the set of vertices of $\Gamma \backslash \Delta$. As we mentioned in the introduction and discussed in the Section 3, we are interested in finding an element $u$ in $U$ for which $\pi \circ u$ maps the vertices in the $A^+$-cone of $A$ to a finite subset of $S$.

![Figure 9. $A^+$-cone](image)

We give a necessary condition for the images of $\pi \circ u|_A$ in $\Gamma \backslash \Delta$. Let $\mathcal{D}_n$ be the union of two upper right and lower right sectors $\mathcal{C}_n^+$ and $\mathcal{C}_n^-$ based at the vertex $v_{-n,0}$. Then, the set of vertices of $\mathcal{D}_n$ is $\{v_{l,m} : l \geq \max\{-n, m-n\}\}$, $(n \geq 0)$. Recall that $T$ is the group of $\{\text{diag}(\alpha_1, \alpha_2, \alpha_3) \in G : \alpha_i \in O\}$.

**Lemma 5.1.** An element $g \in SL(3, K)$ fixes all the vertices in $\mathcal{D}_n$ pointwise if and only if $g = u_{\Theta, \Phi}a$ for some $\Theta, \Phi \in t^{-n}O$ and $a \in T$. In particular, $g$ fixes all the vertices $v_{l,m}$ with $\max\{0, m\} \leq l$ pointwise if and only if $g \in UT$.

**Proof.** ($\Leftarrow$) It follows from the definition that $T$ fixes $\text{Vert}(A)$ pointwise. Moreover, if we assume that $\Theta, \Phi \in t^{-n}O$ and $\max\{-n, m-n\} \leq l$, then we have

$$\begin{pmatrix} 1 & \Theta & \Phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t^l & 0 & 0 \\ 0 & t^m & 0 \\ 0 & 0 & 1 \end{pmatrix} PGL(3, O) = \begin{pmatrix} t^l & t^m \Theta & \Phi \\ 0 & t^m & 0 \\ 0 & 0 & 1 \end{pmatrix} PGL(3, O)$$

$$= \begin{pmatrix} t^l & 0 & 0 \\ 0 & t^m & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t^{m-l} \Theta & t^{-l} \Phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} PGL(3, O) = v_{l,m}$$
since $t^{m-l}\Theta, t^{-l}\Phi \in \mathcal{O}$. Hence, $u_{\Theta, \Phi}$ fixes $v_{l,m}$.

$(\Rightarrow)$ Suppose $g$ fixes all the classes $[[t^l, t^m, 1]]$ with $\max\{-n, m-n\} \leq l$. Letting $D$ the set of diagonal matrices $\text{diag}(t^l, t^m, 1)$ with $\max\{-n, m-n\} \leq l$, we have $g \in \bigcap_{d \in D} d \cdot SL(3, \mathcal{O}) \cdot d^{-1}$. This intersection is the group of upper-triangular matrices whose $1 \times 2$ and $1 \times 3$ entries are in $t^{-n}\mathcal{O}$ and $2 \times 3$ entry is 0, which is equal to the product of $\{u_{\Theta, \Phi} : \Theta, \Phi \in t^{-n}\mathcal{O}\}$ and $T$. \hfill $\square$

Figure 10 describes a necessary condition for the image of $\pi \circ u|_{\mathcal{A}}$ for $u \in U$ (Compare the assigned numbers with those of the standard apartment $\mathcal{A}$ in Figure 8).

![Figure 10. Image of $\pi \circ u|_{\mathcal{A}}$](image)

Since the field $K$ with the absolute value $|\cdot|$ is complete, by Proposition 11.105 of [AB08] the apartment system of $\Delta$ is maximal. Since the group $SL(3, K)$ acts strongly transitively on $\Delta$, every apartment $\Sigma$ can be written as $g\mathcal{A}$ for some $g \in SL(3, K)$.

If $\{v_{l,m} : \max\{0, m\} \leq l\} \subseteq \Sigma \cap \mathcal{A}$, then due to Lemma 5.1 $\Sigma = u\mathcal{A}$ for some $u \in U$. Therefore, the problem reduces to the question whether there is an apartment $\Sigma$ in $\Delta$ of which all the chambers in $A^+\text{-cone}$ are mapped to finitely many chambers and $\Sigma \cap \mathcal{A}$ contains the set of vertices $\{v_{l,m} : \max\{0, m\} \leq l\}$.

**References**

[AB07] B. Adamczewski and Y. Bugeaud, On the Littlewood conjecture in fields of power series, *Adv. Stud. Pure Math.*, **49**, Math. Soc. Japan, Tokyo (2007), 1-20

[AB08] P. Abramenko and K. S. Brown, *Buildings - Theory and Applications*, Springer-Verlag, Berlin, (2008)

[ALN17] F. Adiceam, W. F. Lunnon and E. Nesharim, *A counterexample to $t$-adic Littlewood*, Preprint
[BPP16] A. Broise-Alamichel, J. Parkkonen and F. Paulin, *Equidistribution and counting under equilibrium states in negatively curved spaces and graphs of groups. Applications to non-Archimedean Diophantine approximation*, preprint, arXiv:1612.06717

[Ch11] Y. Cheung, Hausdorff dimension of the set of singular pairs, *Ann. of Math.*, **173** (2011), 127-167

[DL63] H. Davenport and D.J. Lewis, An analogue of a problem of Littlewood, *Michigan Math. J.*, **10** (1963), 157-160

[EKL06] M. Einsiedler, A. Katok and E. Lindenstrauss, Invariant measures and the set of exceptions to Littlewood’s conjecture, *Ann. of Math. (2)*, **164** (2006), 513-560

[EML17] M. Einsiedler, A. Mohammadi and E. Lindenstrauss, *Diagonal actions in positive characteristic*, preprint, arXiv:1705.10418

[GG17] A. Ganguly and A. Ghosh, Dirichlet’s theorem in function fields, *Canad. J. Math.*, **69** (2017), 532-547

[Mo95] S. Mozes, Actions of Cartan subgroup, *Israel J. Math.*, **90** (1995), 253-294

[So13] C. Soulé, Chevalley groups over polynomial rings, *Homological Group Theory*, London Mathematical Society Lecture Note Series **36** (Cambridge University Press), (2013), 359-368

[PV00] A. Pollington and S. Velani, On a problem in simultaneous Diophantine approximation: Littlewood’s conjecture, *Acta Math.*, **185** (2000), 287-306

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