Number of partitions of \( n \) into parts not divisible by \( m \)

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Abstract

In this note, we obtain a formula which leads to a practical and efficient method to calculate the number of partitions of \( n \) into parts not divisible by \( m \) for given natural numbers \( n \) and \( m \). Our formula is a generalization of Euler’s recurrence for integer partitions which can be viewed as the case \( m = 1 \) of our formula. Our approach primarily involves the principle of inclusion and exclusion. We also use our approach to obtain a natural combinatorial proof of a identity of Glaisher which generalizes a classical theorem of Euler.

1 Introduction

Euler [1, Corollary 1.2] proved that the number of partitions of \( n \) into distinct parts is equal to the number of partitions into distinct parts. Glaisher [4, 5] generalized Euler’s result by proving that for any positive integers \( m \) and \( n \), the number of partitions of \( n \) into parts not divisible by \( m \) is equal to the number of partitions of \( n \) with each part appearing less than \( m \) times. In this note, we obtain a formula that helps us to quickly calculate these numbers. Our formula generalizes Euler’s recurrence relation for integer partitions [1, Corollary 1.8].

We use the principle of inclusion and exclusion (PIE) to study these objects, an approach used by the present author and Rattan in the author’s PhD Thesis [2, Section 5.1] to obtain a natural combinatorial proof of Euler’s recurrence. This approach was also used by the present author [3] to obtain a formula for the number of partitions of \( n \) with a given parity of the smallest part.

Note that the calculation of \( p(n) \) is very easy compared to finding all the partitions of \( n \) because of the availability of formulae such as Hardy-Ramanujan-Rademacher formula and recurrences such as Euler’s recurrence. Therefore, Theorem 1 below describes an efficient method to calculate the number of partitions of \( n \) into parts not divisible by \( m \), as demonstrated by examples in Section 4.
2 Main Theorem

Theorem 1. Let $P_m(n)$ denote the number of partitions of $n$ into parts not divisible by $m$. Then, $P_m(n)$ is given by the following formula.

$$P_m(n) = p(n) + \sum_{k \geq 1} (-1)^k \left( p \left( n - \frac{mk(3k - 1)}{2} \right) + p \left( n - \frac{mk(3k + 1)}{2} \right) \right).$$

Remark 2. For $m = 1$, $P_m(n) = 0$, and Theorem 1 immediately yields Euler’s recurrence.

Proof of Theorem 1. We recall some notation defined in [2, Section 5.1] and define some new notation.

- $A_{j,k}(n)$ is the set of partitions of $n$ having exactly $k$ parts of size $j$;
- $B_{j,k}(n)$ is the set of partitions of $n$ having at least $k$ parts of size $j$;
- $C_{j,k}(n)$ is the set of partitions of $n$ having at most $k$ parts of size $j$.

The following properties of these sets are immediate.

1. $|B_{j,k}(n)| = p(n - jk)$.
2. If $j \neq j'$, then $|B_{j,k}(n) \cap B_{j',k'}(n)| = p(n - jk - j'k')$.
3. $C_{j,k}(n) = B_{j,k+1}(n)$, where the complementation is with respect to the set $\text{Par}(n)$, consisting of all partitions of $n$.
4. In particular, $A_{j,0}(n) = C_{j,0}(n) = B_{j,1}(n)$.

We also need the following notation.

- $\mathcal{D}$ denotes the set of nonempty distinct partitions.
- For $s \in \mathbb{N}$, $T_s$ denotes the set of partitions into $s$ distinct parts.
- For a partition $\pi$, $n(\pi)$ denotes the number of parts in $\pi$. 
Then using PIE, we have

\[ P_m(n) = |A_{m,0}(n) \cap A_{2m,0}(n) \cap A_{3m,0}(n) \cap \cdots | \]
\[ = | \cap_{i \geq 1} A_{im,0}(n) | \]
\[ = | \cup_{i \geq 1} B_{im,1}^c(n) | \]
\[ = \sum_{s \geq 0} (-1)^s \sum_{(i_1, i_2, \ldots, i_s) \in T_s} |B_{i_1m,1}(n) \cap B_{i_2m,1}(n) \cap \cdots \cap B_{i_sm,1}(n)| \]
\[ = \sum_{s \geq 0} (-1)^s \sum_{(i_1, i_2, \ldots, i_s) \in T_s} p(n - i_1m - i_2m - \cdots - i_sm) \]
\[ = \sum_{s \geq 0} \sum_{\pi \in T_s} (-1)^s p(n - m|\pi|) \]
\[ = p(n) + \sum_{s \geq 1} \sum_{\pi \in T_s} (-1)^s p(n - m|\pi|) \]
\[ = p(n) + \sum_{\pi \in D} (-1)^{n(\pi)} p(n - m|\pi|), \]

which completes the proof by Euler’s pentagonal number theorem \cite[Theorem 1.6]{1}. Note that here we are able to use PIE even though there are infinitely many sets because all except finitely many are empty. \hfill \Box

3 A proof of Glaisher’s identity

Let \( Q_m(n) \) denote the number of partitions of \( n \) with each part appearing less than \( m \) times. Then using PIE, we have

\[ Q_m(n) = | \cap_{i \geq 1} C_{i,m-1}(n) | \]
\[ = | \cup_{i \geq 1} B_{i,m}^c(n) | \]
\[ = \sum_{s \geq 0} (-1)^s \sum_{(i_1, i_2, \ldots, i_s) \in T_s} |B_{i_1m,1}(n) \cap B_{i_2m,1}(n) \cap \cdots \cap B_{i_sm,1}(n)| \]
\[ = \sum_{s \geq 0} \sum_{(i_1, i_2, \ldots, i_s) \in T_s} p(n - i_1m - i_2m - \cdots - i_sm) \]
\[ = \sum_{s \geq 0} \sum_{\pi \in T_s} (-1)^s p(n - m|\pi|) \]
\[ = p(n) + \sum_{s \geq 1} \sum_{\pi \in T_s} (-1)^s p(n - m|\pi|) \]
\[ = p(n) + \sum_{\pi \in D} (-1)^{n(\pi)} p(n - m|\pi|), \]

completing the proof of Glaisher’s generalization of Euler’s identity.
4 Examples

For computational purposes, it is convenient to note a few terms and compare the pattern with the terms appearing in Euler’s recurrence for partitions, which is given as

\[ p(n) = p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) \]
\[ + p(n - 12) + p(n - 15) - p(n - 22) - p(n - 26) + \cdots \]

Then, by Theorem 1, we have the following formula for \( P_m(n) \).

\[ P_m(n) = p(n) - p(n - m) - p(n - 2m) + p(n - 5m) + p(n - 7m) \]
\[ - p(n - 12m) - p(n - 15m) + p(n - 22m) + p(n - 26m) - \cdots \]

First suppose \( n = 17 \) and \( m = 3 \). Then, we have

\[ P_3(17) = p(17) - p(14) - p(11) + p(2) \]
\[ = 297 - 135 - 56 + 2 \]
\[ = 108. \]

Thus, 108 out of 297 partitions of 17 have parts not divisible by 3 and the remaining 189 partitions have at least one part that is divisible by 3.

Next, suppose \( n = 164 \) and \( m = 7 \). Then, we have

\[ P_7(164) = p(164) - p(157) - p(150) + p(129) + p(115) - p(80) - p(59) + p(10) \]
\[ = 156919475295 - 80630964769 - 4085325313 \]
\[ + 4835271870 + 1064144451 - 15796476 - 831820 + 42 \]
\[ = 41318063280. \]

Thus, 41318063280 out of 156919475295 partitions of 164 have parts not divisible by 7 and the remaining 115601412015 partitions have at least one part that is divisible by 7.

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