Resistance Distances and Kirchhoff Indices 
Under Graph Operations

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ABSTRACT The resistance distance between any two vertices of a connected graph \( G \) is defined as the net effective resistance between them in the electrical network constructed from \( G \) by replacing each edge with a unit resistor. The Kirchhoff index of \( G \) is defined as the sum of resistance distances between all pairs of vertices. In this paper, two unary graph operations on \( G \) are taken into consideration, with the resulted graphs being denoted by \( RT(G) \) and \( H(G) \). Using electrical network approach and combinatorial approach, we derive explicit formulae for resistance distances and Kirchhoff indices of \( RT(G) \) and \( H(G) \). It turns out that resistance distances and Kirchhoff indices of \( RT(G) \) and \( H(G) \) could be expressed in terms of resistance distances and graph invariants of \( G \). Our result generalizes the previously known result on the Kirchhoff index of \( RT(G) \) for a regular graph \( G \) to the Kirchhoff index of \( RT(G) \) for an arbitrary graph \( G \).

INDEX TERMS Resistance distance, Kirchhoff index, multiplicative degree-Kirchhoff index, additive degree-Kirchhoff index, Foster’s formula.

I. INTRODUCTION
Let \( G \) be a connected graph with vertex set \( V(G) \) and edge set \( E(G) \). Suppose that \( V(G) = \{v_1, v_2, \ldots, v_n\} \). It is well known that distance functions are of fundamental to a graph. The most natural and best known distance function defined on a graph is the (shortest-path) distance, where the distance between any two vertices of \( G \) is defined as the length of a shortest path connecting them. In 1993, a new novel distance function, resistance distance, was identified by Klein and Randić [1]. The concept of resistance distance originates from electrical circuit theory. If we view \( G \) as an electrical network \( N \) by replacing each edge of \( G \) with a unit resistor, then the resistance distance [1] between \( v_i \) and \( v_j \), denoted by \( \Omega_G(v_i, v_j) \), is defined as the net effective resistance between the corresponding nodes in the electrical network \( N \). In contrast to the shortest path distance, the resistance distance has a notable feature that if \( v_i \) and \( v_j \) are connected by more than one paths, then they are closer than they are connected by the only shortest path.

Beside being a distance function on graphs and an important component of electrical circuit theory, resistance distance has been found to have significant applications in chemistry. In comparison with shortest-path distance, resistance distance is more suitable to describe the fluid or wave-like communications in molecules. In particular, resistance distance-based graph invariants, turn out to play important roles in the study of QSAR (quantitative structure-activity relationship) and QSPR (quantitative structure-property relationship). The most widely used resistance distance-based graph invariant is the Kirchhoff index [1], which is defined as the sum of resistance distances between all pairs of vertices. In other words, the Kirchhoff index \( Kf(G) \) of \( G \) is defined as:

\[
Kf(G) = \sum_{i<j} \Omega_G(v_i, v_j).
\]

Later, two modifications of the Kirchhoff index were introduced, which take the degrees of graphs into consideration. One is the multiplicative degree-Kirchhoff index, which is defined by Chen and Zhang [2] as:

\[
Kf^*(G) = \sum_{i<j} d_i d_j \Omega_G(v_i, v_j),
\]

where \( d_i \) is the degree (i.e., the number of neighbors) of the vertex \( v_i \). The other one is the additive degree-Kirchhoff index...
defined by Gutman et al. [3] as:

$$Kf^+(G) = \sum_{i<j}(d_i + d_j)\Omega_G(v_i, v_j). \quad (2)$$

In recent years, the computation of resistance distances and Kirchhoff indices of graphs under unary or binary operations has attracted much attention. In [4], Xu computed the Kirchhoff index of product and lexicographic product of two graphs. In [5], Zhang et al. derived explicit formulae for Kirchhoff indices of join, corona and cluster of two graphs. Then, Arauz [6] obtained the Kirchhoff index for generalized corona and cluster of networks. In [7], bounds for the degree Kirchhoff index of the line and para-line graphs were determined. Later, the Kirchhoff index in a composition of a rooted tree $T$ and a graph $G$ were studied in [8]. In [9], Yang and Klein obtained resistance distances in various composite graphs, such as join, product, composition, direct product, strong product, corona and rooted product. In [10], Bu et al. investigated resistance distance in subdivision-vertex join and subdivision-edge join of graphs. Then, Chen [11] obtained resistance distances and Kirchhoff indices of generalized join of graphs. Liu et al. [12] gave resistance distances and Kirchhoff indices of $R$-vertex join and $R$-edge join of two graphs. In [13], resistance distances for subdivision-vertex and subdivision-edge coronae were obtained. Then Kirchhoff indices of subdivision-vertex and subdivision-edge neighbourhood corona were obtained in [14] and [15]. In [16], Kirchhoff indices of $n$-prism networks (i.e. the graph obtained by the product of the path graph and the cycle graph) were obtained. After that, resistance distances and Kirchhoff indices of corona and neighborood corona were obtained in [17]. Then Kirchhoff indices of subdivision-vertex and subdivision-edge corona were obtained. In [18], ordinary corona and neighbourood corona were obtained in [19]. Then Kirchhoff indices of $R$-vertex and $R$-edge corona were obtained in [20]. Then, resistance distances and Kirchhoff indices of $Q$-double join graphs were obtained in [21] and other novel graph operations were obtained in [22]. Besides these binary operations, special attention has been paid to resistance distances and related topological indices under unary operations, such as subdivision, triangulation, vertex-face operation, and so on. In [23], resistance distances in subdivision of a graph were determined. In [24], Gao et al. obtained the formula for the Kirchhoff index of subdivision of a regular graph. Then their result was generalized to the subdivision of general graphs in [25]. In [26], Wang et al. determined the Kirchhoff indices for triangulation $T(G)$ (denoted by $R(G)$ in their paper) and $Q(G)$ of a regular graph $G$. Then Yang and Klein [27], Huang et al. [28] independently generalized their result to the Kirchhoff index of $T(G)$ of a general graph $G$. In [29], Liu et al. studied a new graph operation $RT(G)$ and obtained Kirchhoff index of $RT(G)$ for a regular graph $G$. Later, they also obtained the Hosoya index of $RT(G)$ in [30]. In [31], Shangguan and Chen obtained resistance distances in the vertex-face graph of a planar graph. In [32], resistance distances and Kirchhoff indices of stellated graphs were obtained.

Motivated by these results, in this paper, we take two unary graph operations into consideration. For a graph $G$, let $RT(G)$ and $H(G)$ (detailed definitions will be given in the later) be the resulted graphs obtained from $G$ by the two unary operations. First, resistance distances and the Kirchhoff index of $RT(G)$ for a general graph $G$ are determined, which generalized the result obtained by Liu et al. in [29]. Then, formulae for resistance distances and the Kirchhoff index of $H(G)$ are derived. It turns out that resistance distances in $RT(G)$ and $H(G)$ could be expressed in terms of resistance distances of $G$, and Kirchhoff indices of $RT(G)$ and $H(G)$ could be expressed in terms of Kirchhoffian graph invariants (i.e. Kirchhoff index, multiplicative degree-Kirchhoff index, additive degree-Kirchhoff index) and parameters of $G$.

Now we give detailed definitions of $RT(G)$ and $H(G)$. For a connected graph $G$, let $T(G)$ be the triangulation of $G$, i.e. $T(G)$ is obtained from $G$ by changing each edge of $G$ into a triangle $v_iv_jv_k$ with $v_i$ being the new vertex associated with $v_iv_j$. Let $RT(G)$ be the graph obtained from $T(G)$ by adding a new edge corresponding to every vertex of $G$, and by joining each new edge to the corresponding vertex of $G$. For example, the graph $G$ and corresponding $RT(G)$ are given in Figure 1. Let $\Delta$ be the maximum degree of $G$. Then the graph $H(G)$ is defined to be the graph obtained from $G$ by adding $\Delta - d_i$ pendant vertices to every vertex $v_i$ of $G$. For instance, the graph $G$ and corresponding $H(G)$ are depicted in Figure 2.

**II. RESISTANCE DISTANCES AND KIRCHHOFF INDICES UNDER GRAPH OPERATIONS**

In this section, we compute resistance distances and Kirchhoff indices of $RT(G)$ and $H(G)$ for an arbitrary graph $G$. For convenience, we divide this section into two subsections.
A. RESISTANCE DISTANCES AND THE Kirchhoff INDEX OF $RT(G)$

Recall that $V(G) = \{v_1, v_2, \ldots, v_n\}$. We label the vertices of $RT(G)$ in the following way: for each edge $v_kv_l \in E(G)$, we label the vertex in $RT(G)$ that associated with $v_kv_l$ by $v_{kl}$; for each vertex $v_i \in V(G)$, we label the end-vertices of the edge in $RT(G)$ that associate with $v_i$ (the edge newly added to $G$ with respect to $v_i$) by $v_{1i}^1$ and $v_{1i}^2$. Now we let $V' = \{v_{kl} | v_kv_l \in E(G)\}$ and $V'' = \{v_{1i}^1, v_{1i}^2 | v_i \in V(G)\}$. Then clearly,

$$V(T(G)) = V(G) \cup V',$$

and

$$V(RT(G)) = V(G) \cup V' \cup V''.$$

To obtain resistance distances in $RT(G)$, we first introduce the cut-vertex property in electrical circuit theory. For a connected graph $G$, a vertex $v_k$ of $G$ is called a cut-vertex of $G$ if the deletion of $v_k$ disconnects $G$.

**Lemma 1:** (The Cut-Vertex Property): Let $G$ be a connected graph with $v_k$ being a cut vertex of $G$. If $v_i$ and $v_j$ are vertices which belong to different components in $G - v_k$, then

$$\Omega_G(v_i, v_j) = \Omega_G(v_i, v_k) + \Omega_G(v_k, v_j).$$

By the structure of $RT(G)$, it is obvious that for any two vertices in $V(T(G))$, the resistance distance between them in $T(G)$ is the same as that in $RT(G)$. Thus to obtain resistance distances in $RT(G)$, it is needed to introduce resistance distances in $T(G)$, as given in the following Lemma. For simplicity, we denote the resistance distance functions of $G$, $T(G)$, and $RT(G)$ by $\Omega$, $\Omega^T$ and $\Omega^R$, respectively.

**Lemma 2** ([23], [27]): Let $G$ be a connected graph. Then resistance distances in $T(G)$ can be computed as follows:

1) For $v_i, v_j \in V(G)$,

$$\Omega^T(v_i, v_j) = \frac{2}{3} \Omega(v_i, v_j).$$

2) For $v_i \in V(G), v_pq \in V'$,

$$\Omega^T(v_i, v_{pq}) = \frac{1}{2} + \frac{\Omega(v_i, v_p) + \Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6}.\quad (4)$$

3) For $v_{pq}, v_{st} \in V'$,

$$\Omega^T(v_{pq}, v_{st}) = 1 + \frac{\Omega(v_{pq}, v_t) + \Omega(v_{pq}, v_i) + \Omega(v_q, v_s)}{6} + \frac{\Omega(v_q, v_t) - \Omega(v_p, v_q) - \Omega(v_s, v_t)}{6}.\quad (5)$$

On the basis of resistance distances in $T(G)$, resistance distances in $RT(G)$ could be given in the following result.

**Theorem 1:** Let $G$ be a connected graph. Then resistance distances in $RT(G)$ can be computed as follows:

1) For $v_i, v_j \in V(G)$,

$$\Omega^R(v_i, v_j) = \frac{2}{3} \Omega(v_i, v_j).$$

2) For $v_i \in V(G), v_{pq} \in V'$,

$$\Omega^R(v_i, v_{pq}) = \frac{1}{2} + \frac{\Omega(v_i, v_p) + \Omega(v_i, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6}.\quad (7)$$

3) For $v_{pq}, v_{st} \in V'$,

$$\Omega^R(v_{pq}, v_{st}) = 1 + \frac{\Omega(v_{pq}, v_t) + \Omega(v_{pq}, v_i) + \Omega(v_q, v_s)}{6} + \frac{\Omega(v_q, v_t) - \Omega(v_p, v_q) - \Omega(v_s, v_t)}{6}.\quad (8)$$

4) For $v_i \in V(G), v_j \in V'' (a \in \{1, 2\})$,

$$\Omega^R(v_i, v_j) = \Omega^R(v_i, v_j^1) = \frac{2}{3} + \frac{2}{3} \Omega(v_i, v_j).\quad (9)$$

5) For $v_{pq} \in V', v_{pq}^a \in V'' (a \in \{1, 2\})$,

$$\Omega^R(v_{pq}, v_{pq}^a) = \Omega^R(v_{pq}, v_{pq}^1) = \frac{7}{6} + \frac{\Omega(v_{pq}, v_p) + \Omega(v_{pq}, v_q)}{3} - \frac{\Omega(v_p, v_q)}{6}.\quad (10)$$

6) For $v_i^a, v_j^b \in V'' (a \in \{1, 2\}, b \in \{1, 2\}),$ if $i = j$, then

$$\Omega^R(v_i^a, v_i^b) = \frac{2}{3},$$

otherwise,

$$\Omega^R(v_i^a, v_j^b) = \frac{4}{3} + \frac{2}{3} \Omega(v_i, v_j).\quad (11)$$

**Proof:** Clearly, for any two vertices $x, y$ belonging to $V(T(G)) = V(G) \cup V'$, $\Omega^T(x, y) = \Omega^R(x, y)$, Thus Eqs. (6), (7) and (8) follows directly from Eqs. (3), (4) and (5) in Lemma 2. For any $v_i \in V(G)$, it is easily seen that

$$\Omega^R(v_i, v_i^1) = \Omega^R(v_i, v_i^2) = \frac{2}{3}.$$

Then, by the cut-vertex property, we have

$$\Omega^R(v_i^1, v_i^1) = \Omega^R(v_i^2, v_i^2) = \Omega^R(v_i, v_j) + \frac{2}{3}.\quad \square$$

Thus Eq. (9) is derived by substituting the result in Eq. (6) into the above equation. In the same way, Eq. (10) could be derived from Eq. (7), and Eq. (11) could be derived from Eq. (6).

In the following, according to Theorem 1, we compute the Kirchhoff index of $RT(G)$.

**Theorem 2:** Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges. Then

$$Kf(RT(G)) = 6Kf(G) + Kf^+(G) + \frac{1}{6}Kf^*(G) + \frac{m^2}{2} + \frac{7n^2}{2} + \frac{8mn}{3} - \frac{3n}{2} - \frac{m}{3}.\quad (12)$$
Proof: Since $V(\text{RT}(G)) = V(T(G)) \cup V''$, and any two vertices in $T(G)$ has the same resistance distance as in $\text{RT}(G)$.

Hence

$$
Kf(\text{RT}(G)) = \sum_{\{u,v\} \subset V(\text{RT}(G))} \Omega^R(u,v)
$$

$$
= \sum_{\{u,v\} \subset V(T(G))} \Omega^R(u,v) + \sum_{\{u,v\} \subset V''} \Omega^R(u,v)
+ \sum_{u \in V(T(G)) \cap v \in V''} \Omega^R(u,v)
= \sum_{\{u,v\} \subset V(T(G))} \Omega^R(u,v) + \sum_{\{u,v\} \subset V''} \Omega^R(u,v)
= Kf(T(G)) + \sum_{\{u,v\} \subset V''} \Omega^R(u,v). \quad (13)
$$

For the Kirchhoff index of $T(G)$, it has been shown in [27] that

$$
Kf(T(G)) = \frac{2}{3} Kf(G) + \frac{1}{3} Kf^+(G) + \frac{1}{3} Kf^+(G)
+ \frac{3m^2 - n^2 + 2mn - 2m + n}{6}. \quad (14)
$$

We proceed to compute the second term in the summation of Eq. (13). Since it is understood that $V'' = \{v_1, v_2, v_3, v_4, \ldots, v_i, v_j\}$, and for any $i \neq j$,

$$
\Omega^R(v_i, v_j) = \Omega^R(v_i, v_j) = \Omega^R(v_j, v_j) = \Omega^R(v_i, v_j).
$$

Thus it follows by Theorem 1 that

$$
\sum_{\{u,v\} \subset V''} \Omega^R(u,v) = \sum_{i=1}^{n} \Omega^R(v_i, v_j) + 4 \sum_{i<j} \Omega^R(v_i, v_j)
= \sum_{i=1}^{n} \frac{2}{3} + 4 \sum_{i<j} \frac{4}{3} \Omega(v_i, v_j)
= \frac{2n}{3} + \frac{8n(n-1)}{3} + \frac{8}{3} Kf(G). \quad (15)
$$

For the third term in the summation of Eq. (13), we divide it into two terms:

$$
\sum_{u \in V(T(G)) \cap v \in V''} \sum_{u \in V(T(G)) \cap v \in V''} \Omega^R(u,v)
= \sum_{u \in V(T(G)) \cap v \in V''} \sum_{u \in V(T(G)) \cap v \in V''} \Omega^R(u,v) + \sum_{u \in V(T(G)) \cap v \in V''} \Omega^R(u,v). \quad (16)
$$

For one thing, note that $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then by Theorem 1, we have

$$
\sum_{u \in V(G) \cap v \in V''} \sum_{u \in V(G) \cap v \in V''} \Omega^R(u,v) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \Omega^R(v_i, v_j) + \Omega^R(v_i, v_j) \right]
= 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{2}{3} + \frac{2}{3} \Omega(v_i, v_j) \right]
= 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{2}{3} + \frac{2}{3} \Omega(v_i, v_j) \right]
= 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \frac{2}{3} + \frac{2}{3} \Omega(v_i, v_j) \right]
= \frac{8}{3} Kf(G) + \frac{2}{3} Kf^+(G) + n^2 + \frac{7mn}{3} + \frac{n}{3}. \quad (22)
$$

For another, by Theorem 1, we get

$$
\sum_{v \in V''} \sum_{v \in V''} \Omega^R(u,v) = \sum_{v \in V''} \sum_{v \in V''} \left[ \Omega^R(v_i, v_j) + \Omega^R(v_i, v_j) \right]
= 2 \sum_{v \in V''} \sum_{v \in V''} \left[ \frac{2}{3} + \frac{2}{3} \Omega(v_i, v_j) \right]
= 2 \sum_{v \in V''} \sum_{v \in V''} \left[ \frac{2}{3} + \frac{2}{3} \Omega(v_i, v_j) \right]
= \frac{8}{3} Kf(G) + \frac{2}{3} Kf^+(G) + n^2 + \frac{7mn}{3} + \frac{n}{3}. \quad (22)
$$

If we let $\Omega(v)$ be the sum of resistance distances between $v$ and all the other vertices of $G$, then the second term in the summation of Eq. (18) becomes

$$
\sum_{v \in V''} \sum_{v \in V''} \left[ \Omega(v_i, v_j) + \Omega(v_i, v_j) \right]
= \sum_{v \in V''} \sum_{v \in V''} \left[ \Omega(v_i, v_j) + \Omega(v_i, v_j) \right]
= \sum_{v \in V''} \sum_{v \in V''} \left[ \Omega(v_i, v_j) + \Omega(v_i, v_j) \right]
= \frac{8}{3} Kf(G) + \frac{2}{3} Kf^+(G) + n^2 + \frac{7mn}{3} + \frac{n}{3}. \quad (22)
$$

In addition, by the famous Foster’s formula [33], which states that the sum of resistance distances between all pairs of adjacent vertices in a connected graph of order $n$ is equal to $n - 1$, we have

$$
\sum_{v \in V''} \sum_{v \in V''} \Omega(v_i, v_j) = n - 1. \quad (20)
$$

Substituting Eqs. (20) and (19) back into Eq. (18), we have

$$
\sum_{v \in V''} \sum_{v \in V''} \Omega^R(u,v) = \frac{7mn}{3} + \frac{2}{3} Kf^+(G) - \frac{n(n-1)}{3}. \quad (21)
$$

Then substituting Eqs. (17) and (21) back into Eq. (16), we get

$$
\sum_{u \in V(T(G)) \cap v \in V''} \sum_{u \in V(T(G)) \cap v \in V''} \Omega^R(u,v)
= \frac{8}{3} Kf(G) + \frac{2}{3} Kf^+(G) + n^2 + \frac{7mn}{3} + \frac{n}{3}. \quad (22)
$$
Finally, substituting Eqs. (14), (15) and (22) back into Eq. (13), we get

\[ Kf(\mathcal{R}T(G)) = 6Kf(G) + Kf^+(G) + \frac{1}{6}Kf^*(G) \]
\[ + \frac{m^2}{2} + \frac{7n^2}{2} + \frac{8mn}{3} - \frac{3n - m}{2}. \]

as required.

In particular, if \( G \) is \( r \)-regular, then \( Kf^+(G) = 2rKf(G) \), \( Kf^*(G) = r^2Kf(G) \), and \( m = \frac{bn}{2} \). Thus, as a straightforward consequence of Theorem 2, if \( G \) is a regular graph, then the Kirchhoff index of \( \mathcal{R}T(G) \) could be expressed in a much simpler way, which is expressed only in terms of the Kirchhoff index, the number of vertices and the regularity degree of \( G \).

Corollary 1: Let \( G \) be a \( r \)-regular graph of order \( n \). Then

\[ Kf(\mathcal{R}T(G)) = \frac{(r + 6)^2}{6}Kf(G) + \frac{(r + 5)n}{2} \]
\[ + \frac{(r + 6)(5n - 4)n}{6} + \frac{(r - 2)(r + 6)n^2}{8}. \]

It is easily verified that our result which was given in Corollary 1 coincides with their result. Thus our result given in Theorem 2 generalizes their result.

B. RESISTANCE DISTANCES AND THE KIRCHHOFF INDEX OF \( H(G) \)

As before, suppose that \( V(G) = \{ v_1, v_2, \ldots, v_n \} \). Let \( d_i \) be the degree of \( v_i \) and let \( \Delta \) be the maximum degree of \( G \). For each \( v_i \in V(G) \), if \( \Delta - d_i > 0 \), we label the pendent vertices that attached to \( v_i \) in \( H(G) \) by \( v_i^1, v_i^2, \ldots, v_i^{\Delta - d_i} \). Let

\[ V^* = \{ v_i^1, v_i^2, \ldots, v_i^{\Delta - d_i}, v_{i+1}^1, \ldots, v_n^\Delta \}. \]

Then,

\[ V(H(G)) = V(G) \cup V^*. \]

For the sake of simplicity, we use \( \Omega(u, v) \) and \( \Omega^H(u, v) \) to denote the resistance distance between vertices \( u \) and \( v \) in \( G \) and \( H(G) \), respectively. According to the structure of \( H(G) \) and the cut-vertex property, it is straightforward to obtain resistance distances in \( H(G) \), which are expressed in terms of resistance distances of \( G \) as follows.

Theorem 3: Let \( G \) be a connected graph. Then resistance distances in \( H(G) \) are given as follows.

1) For \( v_i, v_j \in V(G) \),

\[ \Omega^H(v_i, v_j) = \Omega(v_i, v_j). \]  \hspace{2cm} (24)

2) For \( v_i \in V(G) \), \( v_j^k \in V^* \),

\[ \Omega^H(v_i, v_j^k) = 1 + \Omega(v_i, v_j). \]  \hspace{2cm} (25)

3) For \( v_i^k, v_j^l \in V^* \), if \( i = j \), then

\[ \Omega^H(v_i^k, v_j^l) = 2. \]  \hspace{2cm} (26)

otherwise,

\[ \Omega^H(v_i^k, v_j^l) = 2 + \Omega(v_i, v_j). \]  \hspace{2cm} (27)

By Theorem 3, we are able to give exact expression for the Kirchhoff index of \( H(G) \). It turns out that the Kirchhoff index of \( H(G) \) could be expressed in a quite neat way, only involves the Kirchhoffian indices, the maximum degree, the number of vertices and the number of edges of \( G \).

Theorem 4: Let \( G \) be a connected graph with \( n \) vertices, \( m \) edges, and maximum degree \( \Delta \). Then the Kirchhoff index of \( H(G) \) can be computed as follows.

\[ Kf(H(G)) = (\Delta + 1)^2Kf(G) - (\Delta + 1)Kf^+(G) \]
\[ + Kf^*(G) + (n\Delta - 2m)(n\Delta - 2m + n - 1). \]  \hspace{2cm} (28)

Proof: Notice that \( V(H(G)) = V(G) \cup V^* \). Then

\[ Kf(H(G)) = \sum_{[u,v] \subseteq V(H(G))} \Omega^H(u, v) \]
\[ = \sum_{[u,v] \subseteq V(G)} \Omega^H(u, v) + \sum_{u \in V(G)} \sum_{v \in V^*} \Omega^H(u, v) \]
\[ + \sum_{[u,v] \subseteq V^*} \Omega^H(u, v) \]
\[ = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq \Delta - d_i} \Omega^H(v_i^k, v_j^l) \]
\[ + \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq \Delta - d_i} \Omega^H(v_i^k, v_j^l) \]
\[ + \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq \Delta - d_i} \Omega^H(v_i^k, v_j^l). \]  \hspace{2cm} (29)

By Theorem 3, we have

\[ Kf(H(G)) = \sum_{1 \leq i \leq \Delta} \Omega(v_i, v_j) \]
\[ + \sum_{i=1}^{n} \sum_{k=1}^{\Delta - d_i} \sum_{j=1}^{n} [1 + \Omega(v_i, v_j)] + \sum_{i=1}^{n} \sum_{1 \leq k < l \leq \Delta - d_i} 2 \]
\[ + \sum_{1 \leq i \leq \Delta} \sum_{1 \leq j \leq \Delta - d_i} [2 + \Omega(v_i, v_j)] \]
\[ = Kf(G) + \sum_{i=1}^{n} \sum_{j=1}^{n} (\Delta - d_i)[1 + \Omega(v_i, v_j)] \]
\[ + \sum_{i=1}^{n} 2 \times \binom{\Delta - d_i}{2} \]
\[ + \sum_{1 \leq i \leq \Delta} \sum_{1 \leq j \leq \Delta} (\Delta - d_i)(\Delta - d_j)[2 + \Omega(v_i, v_j)]. \]  \hspace{2cm} (30)
For the second term in the summation of Eq. (30), we have
\[\sum_{i=1}^{n} \sum_{j=1}^{n} (\Delta - d_i)(\Delta - d_j)[1 + \Omega(v_i, v_j)]\]
\[= \sum_{i=1}^{n} \sum_{j=1}^{n} (\Delta - d_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} (\Delta - d_i)\Omega(v_i, v_j)\]
\[= n \sum_{i=1}^{n} (\Delta - d_i)\left[\sum_{j=1}^{n} \Omega(v_j, v_j)\right] + \Delta \sum_{i=1}^{n} \sum_{j=1}^{n} \Omega(v_i, v_j)\]
\[\quad - \sum_{i=1}^{n} \sum_{j=1}^{n} d_i\Omega(v_j, v_j)\]
\[= n(n\Delta - \sum_{i=1}^{n} d_i) + 2\Delta Kf(G) - \sum_{i=1}^{n} \sum_{j=1}^{n} d_i\Omega(v_j, v_j)\]
\[= \sum_{i=1}^{n} (n\Delta - 2m) + 2\Delta Kf(G)\]
\[\quad - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (d_i + d_j)\Omega(v_j, v_j)\]
\[= n^2\Delta - 2mn + 2\Delta Kf(G) - Kf^+(G). \quad (31)\]

For the third term in the summation of Eq. (30), noticing that \(\sum_{i=1}^{n} d_i = 2m\), it follows that
\[\sum_{i=1}^{n} 2 \times \left(\Delta - \frac{d_i}{2}\right) = \sum_{i=1}^{n} (\Delta - d_i)(\Delta - d_i - 1)\]
\[= n\sum_{i=1}^{n} [\Delta^2 - (2d_i + 1)\Delta + d_i(d_i + 1)]\]
\[= n\Delta^2 - (4m + n)\Delta + \sum_{i=1}^{n} d_i^2 + 2m. \quad (32)\]

For the fourth term in the summation of Eq. (30), we have
\[\sum_{1 \leq i < j \leq n} (\Delta - d_i)(\Delta - d_j)[2 + \Omega(v_i, v_j)]\]
\[= 2 \sum_{1 \leq i < j \leq n} (\Delta - d_i)(\Delta - d_j)\]
\[\quad + \sum_{1 \leq i < j \leq n} (\Delta - d_i)(\Delta - d_j)\Omega(v_i, v_j)\]
\[= \sum_{i=1}^{n} \sum_{j=1}^{n} (\Delta - d_i)(\Delta - d_j) - \sum_{i=1}^{n} (\Delta - d_i)^2\]
\[\quad + \sum_{1 \leq i < j \leq n} (\Delta - d_i)(\Delta - d_j)\Omega(v_i, v_j)\]
\[= \left[\sum_{i=1}^{n} (\Delta - d_i)\right] \left[\sum_{j=1}^{n} (\Delta - d_j)\right]\]
\[\quad - \sum_{i=1}^{n} (\Delta^2 - 2d_i\Delta + d_i^2)\]
\[\quad + \sum_{1 \leq i < j \leq n} (\Delta - d_i)(\Delta - d_j)\Omega(v_i, v_j)\]
\[= (n\Delta - 2m)^2 - n\Delta^2 + 4m\Delta - \sum_{i=1}^{n} d_i^2\]
\[\quad + \sum_{1 \leq i < j \leq n} \left[\Delta^2 - (d_i + d_j)\Delta - d_i d_j\right]\Omega(v_i, v_j)\]
\[= (n\Delta - 2m)^2 - n\Delta^2 + 4m\Delta - \sum_{i=1}^{n} d_i^2\]
\[\quad + \Delta^2 Kf(G) - \Delta Kf^+(G) - Kf^+(G). \quad (33)\]

Substituting results in Eqs. (31), (32), (33) back into Eq. (30), we get
\[Kf(H(G))\]
\[= Kf(G) + n^2\Delta - 2mn + 2\Delta Kf(G)\]
\[\quad - Kf^+(G) + n\Delta^2 - (4m + n)\Delta + \sum_{i=1}^{n} d_i^2 + 2m\]
\[\quad + (n\Delta - 2m)^2 - n\Delta^2 + 4m\Delta - \sum_{i=1}^{n} d_i^2 + \Delta^2 Kf(G)\]
\[\quad - \Delta Kf^+(G) - Kf^+(G)\]
\[= (\Delta + 1)^2 Kf(G) - (\Delta + 1)Kf^+(G) + Kf^+(G)\]
\[\quad + (n\Delta - 2m)^2 + (n\Delta - 2m)(n - 1)\]
\[= (\Delta + 1)^2 Kf(G) - (\Delta + 1)Kf^+(G)\]
\[\quad + Kf^+(G) + (n\Delta - 2m)(n\Delta - 2m + n - 1).\]

The proof is completed. \(\square\)

III. CONCLUSION

In this paper, resistance distances and Kirchhoff indices under two kinds of unary graph operations are determined. It turns out that resistance distances and Kirchhoff indices of graphs under these operations could be expressed in terms of resistance distances and graph invariants of the original graph in a neat way. These formulae not only establish nice relations between resistance distances and Kirchhoff indices of the resulted complex graph and those of the original graph, but also make the computation of resistance distances and Kirchhoff indices of these complex graph become greatly simplified. Along this line, further study on resistance distances and Kirchhoff indices for some other unary graph operations, such as total graph, medial graph, Mycielskian graph, is greatly anticipated.

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