Ramsey numbers of path-matchings, covering designs and 1-cores

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Abstract

A path-matching of order \( p \) is a vertex disjoint union of nontrivial paths spanning \( p \) vertices. Burr and Roberts [3], and Faudree and Schelp [6] determined the 2-color Ramsey number of path matchings. In this paper we study the multicolor Ramsey number of path matchings. We formulate the following conjecture which, if true, is best possible:

For all integers \( p_1 \geq \cdots \geq p_r \geq 4 \), if \( n \geq p_1 - (r - 1) + \sum_{i=2}^{r} \left\lceil \frac{p_i}{3} \right\rceil \), then every \( r \)-coloring of \( K_n \) contains a path matching of color \( i \) and order at least \( p_i \) for some \( i \in [r] \).

Our main result is the proof of this conjecture with a smaller error term which depends only on \( r \). We also prove the conjecture exactly when \( n \) is sufficiently large compared to the number of colors, and when the number of colors is at most 3. As a corollary we get that in every \( r \)-coloring of \( K_n \) there is a monochromatic path matching of order at least \( 3 \left\lfloor \frac{n}{r} \right\rfloor \), which is essentially best possible.

The proof of the main result is based on two other results, interesting on their own. One is a minimax theorem for path matchings derived from a result of Las Vergnas (extending Tutte’s 1-factor theorem). The other is an estimate on block sizes of covering designs (which can also be formulated as an estimate on the sizes of monochromatic 1-cores in colored complete graphs). Block sizes in covering designs have been studied intensively before, but only for the uniform case (when all block sizes are equal). For our purposes we established the following estimate that allows arbitrary block sizes:

For all integers \( p_1 \geq \cdots \geq p_r \geq 2 \), if \( n \geq \max \{ p_1, \frac{2p_1}{3} - \frac{1}{r_3} + \sum_{i=2}^{r_3} \frac{p_i}{3} \} \) and \( B_1, \ldots, B_r \) are subsets of \( [n] \) such that each pair of \( [n] \) is covered by at least one \( B_i \), then \( |B_i| \geq p_i \) for some \( i \in [r] \).

1 Introduction

We denote by \( P_k \) the path with \( k \) vertices and define a path-matching as a vertex disjoint union of paths, each with at least 2 vertices. The order of a path-matching \( P \) is \( |V(P)| \); i.e. the number of vertices spanned by \( P \). A path matching can clearly be written as vertex disjoint union of \( P_2 \)-s and \( P_3 \)-s. Thus the maximum order of a path matching in a graph is equal to the maximum order of a path matching containing only \( P_2 \) and \( P_3 \) components. We note that sometimes a path matching is called a linear forest in the literature [3, 6].

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Here we study the Ramsey problem for path matchings: what is the order of the largest monochromatic path matching we can find in every \(r\)-coloring of the edges of \(K_n\)? Note that this belongs to the part of Ramsey theory where the target graph is a large monochromatic member of a family instead of a specified graph. Many other families have been investigated, for example the family of connected graphs, graphs without isolated vertices, highly connected graphs, graphs of small diameter, etc. A survey on problems of this flavor is \([9]\).

Burr and Roberts \([3]\) proved that if \(n \geq \left\lfloor \frac{4r}{3} \right\rfloor - 1\), then in every \(2\)-coloring of \(K_n\) there is a monochromatic path matching of order \(p\). Later, Faudree and Schelp \([6]\) proved a non-symmetric version: that is if \(p_1 \geq p_2 \geq 2\) and

\[
n \geq p_1 + \left\lceil \frac{p_2}{3} \right\rceil - 1,
\]

then in every \(2\)-coloring of \(K_n\), there is a path matching of color \(i\) and order at least \(p_i\) for some \(i \in [r]\). (In fact, in both cases above the authors prove a stronger statement where the formula takes into account the number of paths of odd length.) We extend these results to \(r\)-colorings with \(r \geq 3\).

We formulate the following conjecture for the off-diagonal multicolor Ramsey number of path matchings.

**Conjecture 1.1.** Let \(r \geq 2\) and let \(p_1 \geq p_2 \geq \cdots \geq p_r \geq 4\) be integers. If

\[
n \geq p_1 - (r - 1) + \sum_{i=2}^{r} \left\lceil \frac{p_i}{3} \right\rceil,
\]

then every \(r\)-coloring of \(K_n\) contains a path matching of color \(i\) and order at least \(p_i\) for some \(i \in [r]\).

Conjecture 1.1 would be sharp as shown by the extremal coloring

\([p_1 - 1, \left\lfloor \frac{p_2}{3} \right\rfloor - 1, \ldots, \left\lfloor \frac{p_r}{3} \right\rfloor - 1]\),

where the notation \([m_1, m_2, \ldots, m_r]\) represents the \(r\)-coloring obtained by partitioning \(V(K_n)\) into \(r\) parts \(A_i, 1 \leq i \leq r\) so that \(|A_i| = m_i\) for \(i = 1, 2, \ldots, r\), and the color of any edge \(e = (x, y)\) is the maximum \(j\) for which \(\{x, y\}\) has a non-empty intersection with \(A_j\).

First note that it is an easy exercise to check that if \(p_i = 4\) for all \(i \in [r]\), then Conjecture 1.1 is true (and also follows from Theorem 1.2 below).

The condition \(p_i \geq 4\) can be weakened, but in particular, for \(r \geq 3\), we must exclude the case where \(p_i = 3\) for all \(i \in [r]\), since in that case \(p_1 - (r - 1) + \sum_{i=2}^{r} \left\lceil \frac{p_i}{3} \right\rceil = p_1 = 3\), and clearly \(n \geq 3\) is not sufficient. More generally, suppose we have \(p_1 \geq \ldots \geq p_r \geq 4\) and \(n = p_1 - (r - 1) + \sum_{i=2}^{r} \left\lceil \frac{p_i}{3} \right\rceil\). Choose \(s > r\) large enough so that \(s \geq \binom{r}{2}\) and let \(p_{r+1} = \cdots = p_s = 3\). Now set \(N = p_1 - (s - 1) + \sum_{i=1}^{s} \left\lceil \frac{p_i}{3} \right\rceil = n\). Since \(s \geq \binom{r}{2}\), we may color each edge of \(K_N\) with a different color and thus we have no path matchings of the desired order. Because of these examples, we don’t hazard a more precise conjecture, but it seems as if placing a restriction on the number of \(p_i\)-s which are equal to 3 would suffice.

We also note that to prove Conjecture 1.1 it would suffice to prove the following conjecture which, on the surface, seems weaker (since the values of the \(p_i\)-s are more restricted).

**Conjecture 1.1’.** Let \(r \geq \ell \geq 2\) and let \(p_1 \geq p_2 \geq \cdots \geq p_r \geq 4\) be integers where \(p_1 = \cdots = p_{\ell}\), and \(p_{\ell+1}, \ldots, p_r\) are divisible by 3. If

\[
n \geq p_1 - (r - 1) + \sum_{i=2}^{r} \left\lceil \frac{p_i}{3} \right\rceil,
\]

then every \(r\)-coloring of \(K_n\) contains a path matching of color \(i\) and order at least \(p_i\) for some \(i \in [r]\).
This implies Conjecture 1.1 since if we are given \( p'_1 \geq p'_2 \geq \cdots \geq p'_r \geq 4 \), then we can round each \( p'_i \) up to the nearest multiple of 3 or \( p'_i \), whichever is smaller, without changing the value of the formula; i.e. we get a sequence \( p_1 \geq \cdots \geq p_r \geq 4 \) and an index \( \ell \) such that \( p_1 = \cdots = p_\ell \), and \( p_{\ell+1}, \ldots, p_r \) are all divisible by 3 where \( p_i \geq p'_i \) for all \( i \in [r] \) and \( p'_1 - (r - 1) + \sum_{i=2}^{r} \lfloor p_i/3 \rfloor = p_1 - (r - 1) + \sum_{i=2}^{r} \lfloor p_i/3 \rfloor \).

Finally, it is worth comparing Conjecture 1.1 with the well-known result of Cockayne and Lorimer [4] which gives the \( r \)-color Ramsey number of a matching.

**Theorem 1.2** (Cockayne, Lorimer [4]). Let \( r \geq 2 \) and let \( p_1 \geq p_2 \geq \cdots \geq p_r \geq 4 \) be positive even integers. If

\[
   n \geq p_1 - (r - 1) + \sum_{i=2}^{r} \frac{p_i}{2},
\]

then every \( r \)-coloring of \( K_n \) contains a matching of color \( i \) and order at least \( p_i \) for some \( i \in [r] \).

Theorem 1.2 is sharp, shown by the coloring \([p_1 - 1, p_2/2 - 1, \ldots, p_r/2 - 1]\).

### 1.1 Results

Our main result is the proof of Conjecture 1.1 (allowing for \( p_i < 4 \)) with a small error term (we have \( \frac{r}{2} \) in place of \( r - 1 \); however, we don’t have ceilings on each of the \( \frac{p_i}{3} \) terms). In fact, this implies that Conjecture 1.1 holds when \( p_i \equiv 1 \mod 3 \) for all \( 2 \leq i \leq r \).

**Theorem 1.3.** Let \( r \geq 2 \) and let \( p_1 \geq p_2 \geq \cdots \geq p_r \geq 2 \) be integers. If

\[
   n \geq p_1 - \frac{r}{3} + \sum_{i=2}^{r} \frac{p_i}{3},
\]

then every \( r \)-coloring of \( K_n \) contains a path matching of color \( i \) and order at least \( p_i \) for some \( i \in [r] \).

We also prove Conjecture 1.1 (allowing for \( p_i < 4 \)) in the case where \( p_1 \) is large enough in terms of \( r \) (which suffices to prove Conjecture 1.1 when \( r \leq 3 \), as we will see in Section 1.2).

**Theorem 1.4.** Let \( r \geq 2 \) and let \( p_1 \geq p_2 \geq \cdots \geq p_r \geq 2 \) be integers. If

\[
   p_1 \geq 4r - 6 - \sum_{i=2}^{r} \left( \left\lfloor \frac{p_i}{3} \right\rfloor - \frac{p_i}{3} \right) \quad \text{and} \quad n \geq p_1 - (r - 1) + \sum_{i=2}^{r} \left\lfloor \frac{p_i}{3} \right\rfloor,
\]

then every \( r \)-coloring of \( K_n \) contains a path matching of color \( i \) and order at least \( p_i \) for some \( i \in [r] \).

We get the following corollary of Theorem 1.3 in the case where all the \( p_i \)-s are equal (stated here using the inverse formulation).

**Corollary 1.5.** Let \( r \geq 2 \). Every \( r \)-coloring of \( K_n \) contains a monochromatic path matching of order at least \( 3 \left\lfloor \frac{n}{r+2} \right\rfloor \).

This is sharp if \( n \) is divisible by \( r + 2 \) as shown by the extremal coloring

\[
   \left\lfloor \frac{3n}{r+2}, \frac{n}{r+2}, \ldots, \frac{n}{r+2} \right\rfloor.
\]

Also note that by combining the previous two results, we get that Conjecture 1.1 (allowing for \( p_i < 4 \)) holds when \( 3 \left\lfloor \frac{n}{r+2} \right\rfloor \geq 4r - 6 \); i.e. \( n \geq (4r - 2)(r + 2) + r + 1 \) (see the discussion following Conjecture 1.1 to see why such an assumption may be necessary).
Corollary 1.6. Let \( r \geq 2 \) and let \( p_1 \geq p_2 \geq \cdots \geq p_r \geq 2 \) be integers. If \( n \geq (\frac{1}{3} - 2)(r + 2) + r + 1 \) and \( n \geq p_1 - (r - 1) + \sum_{i=2}^{r} \frac{p_i}{3} \), then every \( r \)-coloring of \( K_n \) contains a path matching of color \( i \) and order at least \( p_i \) for some \( i \in [r] \).

The proof of Theorem 1.4 is based on Corollary 2.6, a minimax theorem on path matchings derived from a result of Las Vergnas, Theorem 2.5 (extending Tutte’s 1-factor theorem). Interestingly, when we apply Corollary 2.6 to \( r \)-colored complete graphs, we need a suitable estimate on block sizes in covering designs (which can be also formulated as an estimate on block sizes on 1-cores). For our purposes we established the following estimate that allows arbitrary block sizes.

Theorem 1.7. Let \( r \geq 2 \) and let \( p_1 \geq p_2 \geq \cdots \geq p_r \geq 2 \) be integers. If

\[
 n \geq \max \left\{ p_1, \frac{5p_1}{6} - \frac{r}{3} + \sum_{i=2}^{r} \frac{p_i}{3} \right\}
\]

and \( B_1, \ldots, B_r \) are subsets of \( [n] \) such that each pair of \([n]\) is covered by at least one \( B_i \), then \( |B_i| \geq p_i \) for some \( i \in [r] \).

The proof of Theorem 1.7 and the supporting background material (covering designs, 1-cores, minimax theorem) are described in Section 2.

### 1.2 2-color case and 3-color case

From Theorem 1.3, we get the following corollary for \( r \leq 3 \).

Corollary 1.8.

(i) Let \( p_1 \geq p_2 \geq 2 \) be integers. If \( n \geq p_1 + \left\lceil \frac{6}{5} \right\rceil - 1 \), then every 2-coloring of \( K_n \) contains a path matching of color \( i \) and order at least \( p_i \) for some \( i \in [2] \).

(ii) Let \( p_1 \geq p_2 \geq p_3 \geq 2 \) be integers such that \((p_1, p_2, p_3) \neq (3, 3, 3) \). If \( n \geq p_1 + \left\lceil \frac{6}{5} \right\rceil + \left\lceil \frac{4}{5} \right\rceil - 1 \), then every 3-coloring of \( K_n \) contains a path matching of color \( i \) and order at least \( p_i \) for some \( i \in [3] \).

Proof.

(i) When \( r = 2 \), we may apply Theorem 1.4 since \( p_1 \geq 2 = 4(2) - 6 \).

(ii) When \( r = 3 \), suppose \((p_1, p_2, p_3) \neq (3, 3, 3) \). We claim that unless \((p_1, p_2, p_3) \in \{(3, 3, 2), (4, 3, 3), (5, 3, 3)\} \) we may apply Theorem 1.4. If \( p_1 = 2 \), then \( p_1 = 2 = 4(3) - 6 - 6(1/3 + 1/3) \geq 4(3) - 6 - 6 \sum_{i=2}^{3} \left( \left\lceil \frac{6}{5} \right\rceil - \frac{p_i}{3} \right) \). If \( p_1 = 3 \) and \( p_2 + p_3 \leq 4 \), then \( p_1 = 3 \geq 4(3) - 6 - 6(1/3 + 1/3) \geq 4(3) - 6 - 6 \sum_{i=2}^{3} \left( \left\lceil \frac{6}{5} \right\rceil - \frac{p_i}{3} \right) \). If \( p_1 \geq 4 \) and it is not the case that \( p_2 = 3 = p_3 \), then \( p_1 \geq 4 = 4(3) - 6 - 6(1/3 + 1/3) \geq 4(3) - 6 - 6 \sum_{i=2}^{3} \left( \left\lceil \frac{6}{5} \right\rceil - \frac{p_i}{3} \right) \). If \( p_1 \geq 6 \), then \( p_1 \geq 6 = 4(3) - 6 \).

We are left to deal with the cases \((p_1, p_2, p_3) \in \{(3, 3, 2), (4, 3, 3), (5, 3, 3)\} \) each of which can be easily seen by direct inspection.

We note that Corollary 1.8(i) can be derived from the above mentioned result of Faudree and Schelp [6]. However, the proof of their result is difficult and relies on the 2-color Ramsey number of paths determined in [8]. Therefore the short proof of Corollary 1.8(i) is justified. Scobee [13] determined the Ramsey number \( R(m_1 P_3, m_2 P_3, m_3 P_3) = m_2 + m_3 + 3m_1 - 2 \) for \( m_1 \geq m_2 \geq m_3 \) and \( m_1 \geq 2 \). This implies Corollary 1.8(ii) when all \( p_i \)’s are divisible by 3. However, the proof of Scobee’s result is difficult (20 pages). Although it is conceivable
that one can derive Corollary 1.8(ii) from Scobee’s result, Corollary 1.8(ii) provides a direct (and much shorter) proof.

Finally, we note that extremal colorings are not unique for Corollary 1.8. For example, assume $p_2 = 6$ in Corollary 1.8(i). A 2-coloring of $K_{p_1}$, where color 2 is a star with $p_1 - 1$ edges plus an edge, provides an extremal coloring. A more sophisticated example for Corollary 1.8(ii) occurs in the case where $p_1 = p_2 = p_3 = 9$. In addition to the general example $8, 2, 2$ described above, another example is obtained by partitioning the vertices of $K_{12}$ into $A_1, A_2, A_3$ with $|A_i| = 4$ and coloring edges in $A_i$ and in $[A_i, A_{i+1}]$ with color $i$.

2 Covering designs, 1-cores, and a deficiency formula for path matchings

2.1 Ramsey numbers of covering designs

A covering design is a family of sets called blocks in an $n$-element set $V$ such that each pair of $V$ is covered by at least one block. It is generally assumed that all blocks have the same size $p$. Then $C(n, p, 2) = C(n, p) = r$ is used to denote the minimum number of blocks in a covering design. The asymptotics of $C(n, p)$ for fixed $p$ was determined by Erdős and Hanani [5] and the breakthrough of R. M. Wilson [17] provided equality with constructing block designs for every admissible $n \geq n_0(p)$.

One can formulate the inverse problem of finding $C(n, p)$ as a Ramsey problem. For given $r, p$ find the smallest $n = R_r(p)$ such that every covering design on $n$ vertices with $r$ blocks must contain a block of size at least $p$. Mills [13] determined the asymptotic of $R_r(p)/p$ for $r \leq 13$ and this ratio is also known for values of $r$ in the form $q^2 + q + 1$ or $q^2 + q$ when $PG(2, q)$ exists (see the excellent survey of Füredi [7, Chapter 7]). This problem was also studied, using a different formulation by Horák and Sauer [10]. However, there is no conjecture for the limit of $R_r(p)/p$ for general $r$.

For our goals we consider covering designs with variable block sizes, which leads to the off-diagonal case of the Ramsey number $R_r(p)$. We prove the following upper bound for this Ramsey number (Theorem 1.7 restated in terms of covering designs).

Theorem 2.1. Let $r \geq 2$ and let $p_1 \geq p_2 \geq \cdots \geq p_r \geq 2$ be integers. If

$$n \geq \max \left\{ p_1, \frac{5p_1}{6} - \frac{r}{3} + \sum_{i=2}^{r} \frac{p_i}{3} \right\},$$

then every covering design on $[n]$ with $r$ blocks $B_1, \ldots, B_r$ contains a block $B_i$ such that $|B_i| \geq p_i$ for some $i \in [r]$.

2.2 Ramsey numbers of 1-cores

Theorem 2.1 can be reformulated in terms of the Ramsey number of one specific family, graphs with minimum degree at least one, i.e. graphs without isolated vertices. With a slight abuse of the original definition, we call 1-cores the graphs of this family. (The $k$-core of a graph $G$ was defined by Seidman [16] as the largest connected subgraph of $G$ with minimum degree at least $k$, subsequently many papers [1] and textbooks [2] define it without the connectivity condition.) To see that the Ramsey number of the family of 1-cores is the same as the Ramsey number of a covering design, given an $r$-coloring of $K_n$, we can replace the 1-core of color $i$ with a clique of color $i$ (allowing for edges to have multiple colors) without changing the size of the 1-core and thus each clique corresponds to a block in the covering design language.

While we don’t use the following result directly, it will be interesting to compare what is known about the diagonal case with the off-diagonal case. Also, while this result is
essentially contained in [13, 10] (albeit in a different language), we think its short proof is worth sharing here.

**Proposition 2.2** (c.f. [13, 10]). Let \( n, r, \) and \( q \) be positive integers where \( r = q^2 + q + \ell \) and \( \ell \in \{0,1\} \). Every \( r \)-coloring of the edges of \( K_n \) contains a monochromatic 1-core of order at least \( \left( \frac{q+1}{q^2+q+\ell} \right) n \). Furthermore, this is best possible when \( q \) is a prime power.

We say that a vertex sees a color if it is incident with an edge of that color.

*Proof.* Suppose for contradiction that this is not the case; that is, for each color \( i \), there are more than \( (1 - \frac{q+1}{q^2+q+\ell})n \) vertices which do not see color \( i \). So on average, the number of colors a vertex does not see is more than

\[
\sum_{i=1}^{r} (1 - \frac{q+1}{q^2+q+\ell})n = r \left( 1 - \frac{q + 1}{q^2 + q + \ell} \right) = (q^2 + q + \ell) \left( 1 - \frac{q + 1}{q^2 + q + \ell} \right) = q^2 - 1 + \ell,
\]

which implies that some vertex \( v \) sees fewer than \( r - (q^2 - 1 + \ell) = q + 1 \) colors, i.e. \( v \) sees at most \( q \) colors. So \( v \) is incident with at least \( \frac{n-1}{q} \) edges of the same color, which gives a monochromatic 1-core on at least \( \frac{n-1}{q} + 1 \) vertices. But since \( \frac{n-1}{q} + 1 > \frac{n}{q} \geq \left( \frac{q+1}{q^2+q+\ell} \right) n \), this is a contradiction.

When \( q \) is a prime power, we can construct a tightness example using an affine plane of order \( q \) (when \( \ell = 0 \)) or a projective plane of order \( q \) (when \( \ell = 1 \)).

Note that the above very roughly says that in any \( r \)-coloring of the edges of \( K_n \) there is a monochromatic 1-core of order at least \( \frac{n}{q} \).

In Theorem 2.3 the emphasis is on the off-diagonal case, which is what we need for our purposes in the upcoming proof of Theorem 3.1. As far as we know, the off-diagonal case has not been studied.

**Theorem 2.3.** Let \( r \geq 2 \) and let \( p_1 \geq p_2 \geq \cdots \geq p_r \geq 2 \) be integers. If

\[
n \geq \max \left\{ p_1, \frac{5p_1}{6} - \frac{r}{3} + \sum_{i=2}^{r} \frac{p_i}{3} \right\}, \tag{2}
\]

then every \( r \)-coloring of \( K_n \) contains a 1-core of color \( i \) and order at least \( p_i \) for some \( i \in [r] \).

*Proof.* Consider an \( r \)-edge coloring of a complete graph on a set \( V \) of vertices where

\[
|V| = n = \max \left\{ p_1, \left\lfloor \frac{5p_1}{6} - \frac{r}{3} + \sum_{i=2}^{r} \frac{p_i}{3} \right\rfloor \right\}. \tag{3}
\]

Assume indirectly that for every \( i, 1 \leq i \leq r \) in color \( i \) for the size of the largest 1-core \( S_i \) we have \( |S_i| \leq p_i - 1 \). This will lead to a contradiction.

Consider the partition \( V = X_1 \cup X_2 \cup X_3 \), where for \( i = 1, 2 \) \( X_i \) is the set of vertices which are incident with \( i \)-distinct colors \( X_3 \) is the set of vertices incident with \( 3 \)-distinct colors. Note that every vertex is incident with at least \( 2 \)-distinct colors (i.e. \( X_1 = \emptyset \)); otherwise there is a monochromatic 1-core on \( n \) vertices. By (2) and the indirect assumption, we have \( p_1 \leq n \leq p_1 - 1 \), a contradiction.

Note that

\[
|V| = n \leq \sum_{i=1}^{r} |S_i| - |X_2| - 2|X_3|, \tag{4}
\]

since the vertices in \( X_2 \) are counted twice in the sum \( \sum_{i=1}^{r} |S_i| \), and the vertices in \( X_3 \) are counted at least three times in the sum \( \sum_{i=1}^{r} |S_i| \).
Claim 1. $|X_2| < \frac{3p_1}{2}$.

Proof. Since the graph induced by $X_2$ is locally 2-colored (i.e. each vertex sees exactly two colors), either there exists a color $i$ such that every vertex in $X_2$ sees color $i$, in which case $|X_2| \leq p_i - 1 \leq p_i$, or there are a total of at most three colors used on $X_2$ and since every vertex sees two colors, there is a monochromatic 1-core on at least $2|X_2|/3$ vertices. This implies $2|X_2|/3 < p_i \leq p_1$, i.e. $|X_2| < \frac{3p_1}{2}$.

Now by Claim 1 we have

$$|X_3| = n - |X_2| > n - \frac{3p_1}{2}.	ag{5}$$

Then from (4) and (5) we get

$$n \leq \sum_{i=1}^{r} |S_i| - |X_2| - 2|X_3| \leq \sum_{i=1}^{r} p_i - n - |X_3| - r < \sum_{i=1}^{r} p_i - n - (n - \frac{3p_1}{2}) - r$$

which implies

$$3 \left[ \frac{5p_1}{6} - \frac{r}{3} + \sum_{i=2}^{r} \frac{p_i}{3} \right] \leq 3n < \frac{5p_1}{2} + \sum_{i=2}^{r} p_i - r,$$

contradicting (3).

Finally we raise the problem of improving the bounds in Theorem 2.3, which (as we will see) would have the effect of improving Theorem 1.4.

Problem 2.4. Improve the bounds in Theorem 2.3, or give an example to show that they cannot be improved.

2.3 Deficiency formula for path matchings

The deficiency formula for path matchings can be derived from a special case of a result of Las Vergnas [11]. Let $f, g$ be integer-valued functions on the vertex set $V$ of a graph $G$ such that $0 \leq g(v) \leq 1 \leq f(v)$ for all $v \in V$. A $(g, f)$-factor is a subgraph $F$ of $G$ satisfying $g(v) \leq d_F(v) \leq f(v)$ for all $v \in V$. Las Vergnas [11] gave a necessary and sufficient condition for the existence of a $(g, f)$ factor of a graph. If $g \equiv 1, f \equiv 2$ then an $(g, f)$-factor is a perfect path matching, a path matching covering all vertices of $G$. In this case the condition simplifies and can be stated as follows. Let $q_G(S)$ denote the number of isolated vertices of a graph $G$ in a set $S \subset V(G)$.

Theorem 2.5 (Las Vergnas [11]). There exists a perfect path matching in $G$ if and only if

$$2|X| \geq q_G(V(G) \setminus X)$$

for all $X \subset V(G)$.

This result is “self-refining” in the sense that one can easily derive from it the minimax formula for the deficiency of path matchings (see [12] Exercise 3.1.16 in which Berge’s formula is derived from Tutte’s theorem). Let $pd(G)$ be the path matching deficiency of $G$, the number of vertices uncovered by any path matching of maximum order in $G$.

Corollary 2.6. $pd(G) = \max\{q_G(V(G) \setminus X) - 2|X| : X \subset V(G)\}$.

We call a set $X$ achieving the maximum in Corollary 2.6 an LV set.
3 Large monochromatic path matchings

Theorem 1.3 and Theorem 1.4 are implied by the following theorem.

**Theorem 3.1.** Let \( r \geq 2 \) and let \( p_1 \geq p_2 \geq \cdots \geq p_r \geq 2 \) be integers. If \( n \) satisfies (1) and (2); i.e.

\[
 n \geq \max \left\{ p_1 - (r - 1) + \sum_{i=2}^{r} \left\lceil \frac{p_i}{3} \right\rceil, \frac{5p_1}{6} - \frac{r}{3} + \sum_{i=2}^{r} \frac{p_i}{3} \right\},
\]

then every \( r \)-coloring of \( K_n \) contains a path matching of color \( i \) and order at least \( p_i \) for some \( i \in [r] \).

To see that Theorem 1.3 follows from Theorem 3.1, first note that (1) trivially holds, \( n \geq p_1 - \frac{r}{3} + \sum_{i=2}^{r} \frac{p_i}{3} \) immediately implies \( n \geq \frac{5p_1}{6} - \frac{r}{3} + \sum_{i=2}^{r} \frac{p_i}{3} \). If there exists \( 2 \leq i \leq r \) such that \( p_i \not\equiv 1 \mod 3 \), then

\[
 n \geq p_1 - \frac{r}{3} + \sum_{i=2}^{r} \frac{p_i}{3} \geq p_1 - \frac{r}{3} - \frac{2}{3}(r-2) - \frac{1}{3} \sum_{i=2}^{r} \left\lceil \frac{p_i}{3} \right\rceil = p_1 - (r - 1) + \sum_{i=2}^{r} \frac{p_i}{3},
\]

and (1) is satisfied. Otherwise \( p_i \equiv 1 \mod 3 \) for all \( 2 \leq i \leq r \). Since \( n \) is an integer, we have

\[
 n \geq \left\lceil \frac{p_1}{3} + \sum_{i=2}^{r} \frac{p_i}{3} \right\rceil.
\]

To see that Theorem 1.4 follows from Theorem 3.1, first note that (1) trivially holds, and \( n \geq p_1 - (r - 1) + \sum_{i=2}^{r} \left\lceil \frac{p_i}{3} \right\rceil \geq p_1 \). Furthermore, we have \( n \geq p_1 - (r - 1) + \sum_{i=2}^{r} \left\lceil \frac{p_i}{3} \right\rceil \geq \frac{5p_1}{6} - \frac{r}{3} + \sum_{i=2}^{r} \frac{p_i}{3} \) precisely when \( p_1 \geq 4r - 6 - 6 \left( \sum_{i=2}^{r} \left\lceil \frac{p_i}{3} \right\rceil - \frac{p_i}{3} \right) \), and thus (2) holds.

**Proof of Theorem 3.1.** Our proof is inspired by Petrov’s non-inductive proof of Theorem 1.2 (for another similar proof see [18]).

Let us consider an \( r \)-edge coloring of a complete graph on a set \( V \) of vertices with

\[
 |V| = n = \max \left\{ p_1 - (r - 1) + \sum_{i=2}^{r} \left\lceil \frac{p_i}{3} \right\rceil, \frac{5p_1}{6} - \frac{r}{3} + \sum_{i=2}^{r} \frac{p_i}{3} \right\}.
\]

Assume indirectly that for every \( i, 1 \leq i \leq r, \) the order of the largest path matching in color \( i \) is less than \( p_i \). This will lead to a contradiction. Let us denote by \( G_i \) the subgraph containing the edges in color \( i, 1 \leq i \leq r \). We apply Corollary 2.6 for each \( G_i \), and denote the corresponding LV sets by \( X_i \).

Using the indirect assumption and Corollary 2.6, we have

\[
 n - |X_i| \geq q_{G_i}(V \setminus X_i) > 2|X_i| + (n - p_i),
\]

which gives

\[
 p_i' := p_i - 3|X_i| > 0,
\]

and thus

\[
 \left\lceil \frac{p_i}{3} \right\rceil = \left\lfloor \frac{p_i'}{3} \right\rfloor + |X_i|.
\]

Let \( p' = \max_{1 \leq i \leq r} p_i' \) and let \( \ell \in [r] \) such that \( p = p_i' \). Since \( |X_i| \geq 0 \) for all \( i \in [r] \), we have

\[
 p' = p_\ell - 3|X_\ell| \leq p_\ell,
\]
Denote $X = \cup_{i=1}^r X_i$ and let $Y = V \setminus X$. Then for all $i \in [r]$ we have

$$q_{G_i}(Y) \geq q_{G_i}(V \setminus X_i) - |X \setminus X_i|$$
$$> 2|X_i| + (N - p_i) - |X \setminus X_i| = 3|X_i| + (N - p_i) - |X| = |Y| - p'_i,$$

or equivalently, the largest 1-core in $G_i[Y]$ has size less than $p'_i$.

Thus by Theorem 2.3 we get

$$|Y| < \max \left\{ p', \left[ \frac{5p' - r}{3} + \sum_{i \in [r] \setminus \{\ell\}} \frac{p'_i}{3} \right] \right\}$$
$$\leq \max \left\{ p' - (r - 1) + \sum_{i \in [r] \setminus \{\ell\}} \left[ \frac{p'_i}{3} \right], \left[ \frac{5p' - r}{3} + \sum_{i \in [r] \setminus \{\ell\}} \frac{p'_i}{3} \right] \right\}$$
$$= \max \left\{ p' - \left[ \frac{p'}{3} \right] - (r - 1) + \sum_{i=1}^r \left[ \frac{p'_i}{3} \right], \left[ \frac{5p' - p'}{3} - \frac{r}{3} + \sum_{i=1}^r \frac{p'_i}{3} \right] \right\}. \quad (9)$$

On the other hand we have

$$|Y| = n - |X|$$
$$\geq n - \sum_{i=1}^r |X_i|$$
$$\overset{9\circ} = \max \left\{ p_1 - \left[ \frac{p_1}{3} \right] - (r - 1) + \sum_{i=1}^r \left[ \frac{p'_i}{3} \right], \left[ \frac{5p_1 - p_1}{6} - \frac{p}{3} - \frac{r}{3} + \sum_{i=1}^r \frac{p'_i}{3} \right] \right\}$$
$$\overset{9\circ} \geq \max \left\{ p' - \left[ \frac{p'}{3} \right] - (r - 1) + \sum_{i=1}^r \left[ \frac{p'_i}{3} \right], \left[ \frac{5p' - p'}{6} - \frac{p'}{3} - \frac{r}{3} + \sum_{i=1}^r \frac{p'_i}{3} \right] \right\},$$

a contradiction with (9), finishing the proof of Theorem 3.1.

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