THEORY OF BINARY NONLINEARIZATION AND ITS APPLICATIONS TO SOLITON EQUATIONS

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Binary symmetry constraints are applied to the nonlinearization of spectral problems and adjoint spectral problems into so-called binary constrained flows, which provide candidates for finite-dimensional Liouville integrable Hamiltonian systems. The resulting constraints on the potentials of spectral problems give rise to a kind of involutive solutions to zero curvature equations, and thus the integrability by quadratures can be shown for zero curvature equations once the corresponding binary constrained flows are proved to be integrable. The whole process to carry out binary symmetry constraints is called binary nonlinearization. The principal task of binary nonlinearization is to expose the Liouville integrability for the resulting binary constrained flows, which can usually be achieved as a consequence of the existence of hereditary recursion operators. The theory of binary nonlinearization is applied to the multi-wave interaction equations associated with a $4 \times 4$ matrix spectral problem as an illustrative example. The Backlund transformations resulted from symmetry constraints are given for the multi-wave interaction equations, and thus a kind of involutive solutions is presented and the integrability by quadratures is shown for the multi-wave interaction equations.

1 Introduction

The nonlinearization technique arose in the theory of soliton equations ten years ago. One gradually realizes that it provides a powerful approach for analyzing soliton equations, in both continuous and discrete cases, especially for showing the integrability by quadratures for soliton equations. The manipulation of nonlinearization leads to finite-dimensional Liouville integrable systems (for example, see [1-3]) and also establish a bridge between infinite-dimensional soliton equations and finite-dimensional Liouville integrable systems [1-3]. What is more, it paves a method of separation of variables for soliton equations [4-6] and thus the determination of finite-gap solutions to soliton equations is transformed into solving the corresponding Jacobi inversions problems [4-6].

Mathematically speaking, much excitement in the theory of nonlinearization comes from a kind of specific symmetry constraints, engendered from the variational derivative of the spectral parameter of spectral problems [4-6], which has a close relation with eigenfunctions of hereditary recursion operators [4-6]. Symmetry constraints are themselves a common conceptional umbrella...
under which one can manipulate both mono-nonlinearization and binary nonlinearization, and the study of symmetry constraints is a crucial part of the kernel of the mathematical theory of nonlinearization.

Although it is not clear how to determine pairs of canonical variables to obtain Hamiltonian structures of constrained flows in a general case of mono-nonlinearization, there exists a natural way for determining symplectic structures to exhibit Hamiltonian forms for binary constrained flows while doing binary nonlinearization. Therefore, one can expect to establish a rigorous theory for binary nonlinearization. In this paper, we would only like to discuss the procedure of manipulating binary nonlinearization. More considerations on the topics related to binary nonlinearization will be made elsewhere.

The paper is structured as follows. In section 2, a procedure for doing binary nonlinearization is depicted for general matrix spectral problems. Then in section 3, an illustrative example is carried out to make an application of the procedure to the multi-wave interaction equations associated with a 4 × 4 matrix spectral problem. Finally in section 4, a summary is given along with some concluding remarks.

2 Binary Nonlinearization

In this section, we would like to describe the procedure of manipulating binary nonlinearization for soliton equations (see, for example, [10,19]). Let us start from a general matrix spectral problem

\[ \phi_x = U\phi = U(u, \lambda)\phi, \quad U = (U_{ij})_{r \times r}, \quad \phi = (\phi_1, \cdots, \phi_r)^T \]

(2.1)

where \( \lambda \) is a spectral parameter and \( u = (u_1, \cdots, u_q)^T \) is a \( q \)-dimensional vector potential. Associated with the spectral problem (2.1), suppose that for each integer \( m \geq 0 \), we have the evolution law for the eigenfunction \( \phi \):

\[ \phi_{t_m} = V^{(m)}\phi = V^{(m)}(u, u_x, \cdots; \lambda)\phi, \quad V^{(m)} = (V_{ij}^{(m)})_{r \times r}. \]

(2.2)

If the Gateaux derivative \( U' \) of \( U \) with the potential \( u \) is injective, then the isospectral (\( \lambda_{t_m} = 0 \)) zero curvature equation

\[ U_{t_m} - V_x^{(m)} + [U, V^{(m)}] = 0 \]

(2.3)

will determine an evolution equation

\[ u_{t_m} = X_m(u) = JG_m = \frac{\delta H_m}{\delta u}, \quad u = u(x, t_m), \]

(2.4)
where $X_m$ depends on the potential $u$ and its spatial derivatives of $u$ up to some finite order, $J(u)$ is a Hamiltonian operator, and $	ilde{H}_m(u)$ is a Hamiltonian functional. Evidently, the compatibility condition of the adjoint spectral problem and the adjoint associated spectral problem

$$
\psi_x = -U^T(u, \lambda)\psi, \quad \psi_{t_m} = -V^{(m)}T(u, \lambda)\psi, \quad \psi = (\psi_1, \cdots, \psi_r)^T
$$

(2.5)
is still the same as the zero curvature equation (2.3), guaranteed by the evolution equation

$$
u_{t_m} = X_m(u) \text{ defined by (2.4).}
$$

It has been pointed out \cite{10,16} that

$$
J \frac{\delta \lambda}{\delta u} = E^{-1} J \psi^T \frac{\partial U(u, \lambda)}{\partial u} \phi, \quad E = -\int_\Omega \psi^T \frac{\partial U(u, \lambda)}{\partial \lambda} \phi \, dx,
$$

is a symmetry of the evolution equation (2.4), where $E$ is called the normalized constant, and $\Omega = (0, T)$ if $u$ is assumed to be periodic with period $T$ and $\Omega = (-\infty, \infty)$ if $u$ is assumed to belong to the Schwartz space. Let us now introduce $N$ distinct eigenvalues $\lambda_1, \cdots, \lambda_N$, and so we have

$$
\phi_2^{(s)} = U(u, \lambda_s)\phi^{(s)}, \quad \psi_2^{(s)} = -U^T(u, \lambda_s)\psi^{(s)}, \quad 1 \leq s \leq N;
$$

(2.6)

$$
\phi_2^{(s)} = V^{(m)}(u, \lambda_s)\phi^{(s)}, \quad \psi_2^{(s)} = -V^{(m)}T(u, \lambda_s)\psi^{(s)}, \quad 1 \leq s \leq N;
$$

(2.7)

where the corresponding eigenfunctions and adjoint eigenfunctions are denoted by $\phi^{(s)}$ and $\psi^{(s)}$, $1 \leq s \leq N$. Suppose that the covariant $G_{m_0}$ is Lie point, and then the so-called binary Bargmann symmetry constraint reads as

$$
X_{m_0} = \sum_{s=1}^N E_s \mu_s J \lambda_s \frac{\delta \lambda_s}{\delta u}, \quad \text{i.e.,} \quad JG_{m_0} = J \sum_{s=1}^N \mu_s \psi^{(s)} \phi^{(s)} \frac{\partial U(u, \lambda_s)}{\partial u} \phi^{(s)},
$$

(2.8)

where $\mu_s$, $1 \leq s \leq N$, are arbitrary non-zero constants, and $E_s$, $1 \leq s \leq N$, are $N$ normalized constants. The right-hand side of the binary symmetry constraint (2.8) is a linear combination of $N$ symmetries $E_s J \delta \lambda_s / \delta u$, $1 \leq s \leq N$, and the requirement of $\mu_s \neq 0$, $1 \leq s \leq N$, is natural, since the symmetry $J \delta \lambda_s / \delta u$ is just not involved in our symmetry constraint (2.8) if some constant $\mu_s = 0$. It is also worthy to mention that $\phi^{(s)}$ and $\psi^{(s)}$ can not be expressed in terms of $x, u$ and spatial derivatives of $u$ to some finite order, and thus such symmetries (or the corresponding covariants $\delta \lambda_s / \delta u$, $1 \leq s \leq N$) are not Lie point, contact or Lie-Bäcklund symmetries.

Let us now assume that we can solve the symmetry constraint (2.8) for $u$:

$$
u = \tilde{u}(\phi^{(1)}, \cdots, \phi^{(N)}; \psi^{(1)}, \cdots, \psi^{(N)}).
$$

(2.9)
Replacing $u$ with $\tilde{u}$ in the system (2.6) and the system (2.7), we obtain the so-called spatial binary constrained flow:

$$
\phi^{(s)}_x = U(\tilde{u}, \lambda_s) \phi^{(s)}, \quad \psi^{(s)}_x = -U^T(\tilde{u}, \lambda_s) \psi^{(s)}, \quad 1 \leq s \leq N,
$$

(2.10)

and the so-called temporal binary constrained flow

$$
\phi^{(s)}_{t_m} = V^{(m)}(\tilde{u}, \lambda_s) \phi^{(s)}, \quad \psi^{(s)}_{t_m} = -V^{(m)T}(\tilde{u}, \lambda_s) \psi^{(s)}, \quad 1 \leq s \leq N.
$$

(2.11)

Note that the spatial and temporal binary constrained flows (2.10) and (2.11) are nonlinear with respect to eigenfunctions and adjoint eigenfunctions, but the original spectral problems (2.6) and (2.7) are linear. Moreover, the spatial binary constrained flow (2.10) is a system of ordinary differential equations, and the temporal binary constrained flow (2.11) is usually a system of partial differential equations but it can be transformed into a system of ordinary differential equations under the control of (2.10).

The principal task in the theory of binary nonlinearization is to show that the spatial binary constrained flow (2.10) and the temporal binary constrained flow (2.11) under the control of (2.10) are Liouville integrable, and thus the integrability by quadratures can be shown for evolution equations possessing zero curvature representations. The main tools to do this are Lax representations generated from the stationary zero curvature equations and $\tau$-matrix formulations of Lax operators. These will form the important part of the theory of binary nonlinearization and will be shed right on in our example.

Now if $\phi^{(s)}$ and $\psi^{(s)}$, $1 \leq s \leq N$, solve two binary constrained flows (2.10) and (2.11) simultaneously, then $u = \tilde{u}$ will present a solution to the evolution equation (2.4), since the evolution equation (2.4) with the potential $u = \tilde{u}$ is the compatibility of two binary constrained flows (2.10) and (2.11). It also follows that the evolution equation $u_{t_m} = X_m(u)$ is decomposed into two finite-dimensional Liouville integrable systems, and the constraint $u = \tilde{u}$ gives rise to a Bäcklund transformation between the evolution equation and the resulting finite-dimensional Liouville integrable systems. Furthermore, two binary constrained flows (2.10) and (2.11) can be solved by separation of variables and all what we need to do upon obtaining separated variables is to solve the Jacobi inversion problems. This whole process to carry out symmetry constraints is called binary nonlinearization.

3 Applications to Soliton Equations

By a soliton hierarchy we means a hierarchy of evolution equations with a recursion relation as follows

$$
u_{t_m} = X_m(u) = \Phi^m X_0 = JG_m = J\frac{\delta H_m}{\delta u}, \quad m \geq 0,
$$

(3.1)
where $u$ is the vector potential $u = u(x, t_m)$, $X_m$, $m \geq 0$, depend on the potential $u$ and its spatial derivatives of $u$ up to some finite order, $\Phi$ is a hereditary common recursion operator, $J(u)$ is a Hamiltonian operator, and $\tilde{H}_m(u)$, $m \geq 0$, are the Hamiltonian functionals. Moreover, we have

$$[X_m, X_n] := \frac{\partial}{\partial \varepsilon} \left( X_m(u + \varepsilon X_n - X_n(u + \varepsilon X_m)) \right)_{\varepsilon=0} = 0, \quad m, n \geq 0, \quad (3.2)$$

$$\{\tilde{H}_m, \tilde{H}_n\}_J := \int (\frac{\delta \tilde{H}_m}{\delta u})^T J \frac{\delta \tilde{H}_n}{\delta u} \ dx = 0, \quad m, n \geq 0, \quad (3.3)$$

where $[\cdot, \cdot]$ and $\{\cdot, \cdot\}_J$ are called the commutator of vector fields and the Poisson bracket associated with the Hamiltonian operator $J$, respectively. It follows that each equation $u_{t_m} = X_m$ in the soliton hierarchy (3.1) has infinitely many symmetries $\{X_n\}_{n=0}^{\infty}$ and infinitely many conserved functionals $\{\tilde{H}_n\}_{n=0}^{\infty}$. This property can be obtained from a bi-Hamiltonian formulation, and its natural bi-Hamiltonian formulation for the hierarchy (3.1) is formed by $J$ and $M = J\Psi$ if they are a Hamiltonian pair, where $\Psi$ denotes the adjoint operator of $\Phi$, i.e., $\Psi = \Phi^\dagger$. The above property can also be obtained from a weaker condition that $M$ is a symmetric operator ($M^\dagger = M$), i.e., $\Phi J = J\Psi$.

A soliton hierarchy can usually be constructed from a hierarchy of zero curvature equations as the compatibility conditions of a given matrix spectral problem (2.1) and its associated spectral problems (2.2). The hereditary recursion operator can be generated from the matrix spectral problem (2.1), and the Hamiltonian formulations can be determined by a trace identity (3.4). All this information shows that we can have everything for manipulating binary nonlinearization for our soliton hierarchy (3.1). Applications of binary nonlinearization have been made to the KdV, the MKdV, the AKNS, the Kaup-Newell, the Dirac, the WKh soliton hierarchies and so on. In what follows, an illustrative example of the multi-wave interaction equations associated with a $4 \times 4$ matrix spectral problem will be carried out for making applications of the procedure of binary nonlinearization.

### 3.1 Multi-wave Interaction Equations

Let us start from a $4 \times 4$ matrix spectral problem:

$$\phi_x = U(u, \lambda) \phi, \quad U = \begin{pmatrix} \alpha_1 \lambda & u_{12} & u_{13} & u_{14} \\ u_{21} & \alpha_2 \lambda & u_{23} & u_{24} \\ u_{31} & u_{32} & \alpha_3 \lambda & u_{34} \\ u_{41} & u_{42} & u_{43} & \alpha_4 \lambda \end{pmatrix}, \quad \lambda U_0 + U_1, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}, \quad (3.4)$$

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where $U_0 = \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are distinct constants, and the potential $u$ is defined by

$$u = \rho(U_1) = (u_{21}, u_{12}, u_{31}, u_{13}, u_{41}, u_{23}, u_{14}, u_{42}, u_{24}, u_{43}, u_{34})^T. \quad (3.5)$$

The associated spectral problem is chosen as

$$\phi_t = V(u, \lambda) \phi, \quad V = \begin{pmatrix}
\beta_1 \lambda & v_{12} & v_{13} & v_{14} \\
v_{21} & \beta_2 \lambda & v_{23} & v_{24} \\
v_{31} & v_{32} & \beta_3 \lambda & v_{34} \\
v_{41} & v_{42} & v_{43} & \beta_4 \lambda
\end{pmatrix} = \lambda V_0 + V_1, \quad (3.6)$$

where $V_0 = \text{diag}(\beta_1, \beta_2, \beta_3, \beta_4)$, and $\beta_1, \beta_2, \beta_3, \beta_4$ are distinct constants. Then the isospectral ($\lambda_t = 0$) compatibility condition $U_t - V_t + [U, V] = 0$ of the spectral problem (3.4) and the associated spectral problem (3.6) gives rise to

$$U_{1t} - V_{1x} + [U_0, V_1] = [U_1, V] = 0. \quad (3.7)$$

These two matrix equations lead to the following multi-wave interaction equations

$$u_{ij,t} = \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j} u_{ij,x} + \sum_{k=1, k \neq i,j}^4 \left( \frac{\beta_k - \beta_i}{\alpha_k - \alpha_i} - \frac{\beta_k - \beta_j}{\alpha_k - \alpha_j} \right) u_{ik} u_{kj}, \quad 1 \leq i \neq j \leq 4, \quad (3.8)$$

which contain a couple of physically important nonlinear models as special reductions. Note that the compatibility condition of the adjoint spectral problem and adjoint associated spectral problem:

$$\psi_x = -U^T(u, \lambda)\psi, \quad \psi_t = -V^T(u, \lambda)\psi, \quad \psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T \quad (3.9)$$

still gives rise to the above multi-wave interaction equations. It is easy to find that the multi-wave interaction equations (3.8) have the Hamiltonian structure

$$u_{ij,t} = \frac{\delta H}{\delta u_{ij}}, \quad 1 \leq i \neq j \leq 4, \quad (3.10)$$

where the Hamiltonian operator $J$ reads as

$$J = \text{diag}
\begin{pmatrix}
(\alpha_1 - \alpha_2)\sigma_0, (\alpha_1 - \alpha_3)\sigma_0, (\alpha_1 - \alpha_4)\sigma_0, \\
(\alpha_2 - \alpha_3)\sigma_0, (\alpha_2 - \alpha_4)\sigma_0, (\alpha_3 - \alpha_4)\sigma_0
\end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.11)$$
The Hamiltonian function $\tilde{H}$ in (3.10) can directly be computed but it is omitted here due to its complicated form. We remark that the soliton hierarchy containing higher-order multi-wave interaction equations can also be generated from the spectral problem (3.6), but in this report we focus on the multi-wave interaction equations (3.8).

### 3.2 Binary Constrained Flows

Let us now construct binary constrained flows of the multi-wave interaction equations (3.8). Based on the general procedure of binary nonlinearization, the symmetry of (3.8) generated from the conserved functional $\lambda(u)$ reads as

\[
EJ \frac{\delta \lambda}{\delta u} = E\rho([U_0, \rho^{-1}(\frac{\delta \lambda}{\delta u})]) = \rho([U_0, \rho^{-1}(\psi^T \frac{\partial U(u, \lambda)}{\partial u})]), \tag{3.12}
\]

where $E$ is the normalized constant and $\rho$ is the mapping defined by (3.5).

Upon introducing $N$ distinct eigenvalues $\lambda_1, \cdots, \lambda_N$, we obtain $N$ replicas of Lax systems:

\[
\phi^{(s)} = U(u, \lambda_s)\phi^{(s)}, \quad \psi^{(s)} = -U^T(u, \lambda_s)\psi^{(s)}, \quad 1 \leq s \leq N; \tag{3.13}
\]

\[
\phi^{(s)} = V(u, \lambda_s)\phi^{(s)}, \quad \psi^{(s)} = -V^T(u, \lambda_s)\psi^{(s)}, \quad 1 \leq s \leq N; \tag{3.14}
\]

where the eigenfunctions $\phi^{(s)}$ and the adjoint eigenfunctions $\psi^{(s)}$ are assumed to be denoted by

\[
\phi^{(s)} = (\phi_{1s}, \phi_{2s}, \phi_{3s}, \phi_{4s})^T, \quad \psi^{(s)} = (\psi_{1s}, \psi_{2s}, \psi_{3s}, \psi_{4s})^T, \quad 1 \leq s \leq N. \tag{3.15}
\]

These $N$ eigenvalues lead to a more general symmetry of (3.8)

\[
Z_0 := \rho([U_0, \rho^{-1}(\sum_{s=1}^{N} \mu_s \psi^{(s)}T \frac{\partial U(u, \lambda_s)}{\partial u})]) = \rho([U_0, \sum_{s=1}^{N} \mu_s \phi^{(s)}\psi^{(s)}T]),
\]

where the $\mu_s$’s are arbitrary non-zero constants. On the other hand, we can find by inspection that the multi-wave interaction equations (3.8) have a Lie point symmetry

\[
Y_0 := \rho([\Gamma, U_1]), \quad \Gamma = \text{diag}(\gamma_1, \gamma_2, \gamma_3, \gamma_4), \quad 1 \leq i \neq j \leq 4,
\]

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are distinct constants. In fact, we can directly prove that both $Y_0$ and $Z_0$ are two symmetries of the multi-wave interaction equations (3.8). Define

\[
\delta U_1 = [\Gamma, U_1] \text{ or } [U_0, \sum_{s=1}^{N} \mu_s \phi^{(s)}\psi^{(s)}T],
\]
then \([U_0, \delta V_1] = [V_0, \delta U_1]\) uniquely determines
\[
\delta V_1 = [\Gamma, V_1] \text{ or } [V_0, \sum_{s=1}^{N} \mu_s \phi^{(s)} \psi^{(s)} T].
\]

Now it follows from (3.7) that the symmetry problem only requires to verify that
\[
(\delta U_1, \delta V_1) = ([\Gamma, U_1], [\Gamma, V_1]) \text{ or } ([U_0, \sum_{s=1}^{N} \mu_s \phi^{(s)} \psi^{(s)} T], [V_0, \sum_{s=1}^{N} \mu_s \phi^{(s)} \psi^{(s)} T])
\]
satisfies the linearized system of the multi-wave interaction equations (3.8):
\[
(\delta U_1)_t - (\delta V_1)_x + [\delta U_1, V_1] + [U_1, \delta V_1] = 0,
\]
which can easily be proved.

Therefore, a binary symmetry constraint of (3.8) can be taken as
\[
Y_0 = Z_0, \text{ i.e., } [\Gamma, U_1] = [U_0, \sum_{s=1}^{N} \mu_s \phi^{(s)} \psi^{(s)} T].
\] (3.17)

Solving this equation for \(u\), we obtain the required constraints on the potentials
\[
u_{ij} = \bar{u}_{ij} := \frac{\alpha_i - \alpha_j}{\gamma_i - \gamma_j} \langle \Phi_i, B \Psi_j \rangle, \quad 1 \leq i \neq j \leq 4,
\] (3.18)
where \(\langle \cdot, \cdot \rangle\) denotes the standard inner-product of \(\mathbb{R}^N\), \(B\) is defined by
\[
B = \text{diag}(\mu_1, \mu_2, \cdots, \mu_N),
\] (3.19)
and \(\Phi_i, \Psi_i\) are defined by
\[
\Phi_i = (\phi_{i1}, \cdots, \phi_{iN})^T, \quad \Psi_i = (\psi_{i1}, \cdots, \psi_{iN})^T, \quad 1 \leq i \leq 4.
\] (3.20)

The substitution of \(u\) with
\[
\tilde{u} = (\tilde{u}_{21}, \tilde{u}_{12}, \tilde{u}_{31}, \tilde{u}_{13}, \tilde{u}_{41}, \tilde{u}_{32}, \tilde{u}_{23}, \tilde{u}_{14}, \tilde{u}_{42}, \tilde{u}_{24}, \tilde{u}_{43}, \tilde{u}_{34})
\]
into the binary Lax systems (3.13) and (3.14) yields the so-called binary constrained flows:
\[
\phi^{(s)}_x = U(\tilde{u}, \lambda_s) \phi^{(s)}_x, \quad \psi^{(s)}_x = -U^T (\tilde{u}, \lambda_s) \psi^{(s)}_x, \quad 1 \leq s \leq N; \quad \text{(3.21)}
\]
\[
\phi^{(s)}_t = V(\tilde{u}, \lambda_s) \phi^{(s)}_t, \quad \psi^{(s)}_t = -V^T (\tilde{u}, \lambda_s) \psi^{(s)}_t, \quad 1 \leq s \leq N; \quad \text{(3.22)}
\]
which are two systems of ordinary differential equations, since \(V\) doesn’t involve any spatial derivative of \(\tilde{u}\).
3.3 Liouville Integrability

In order to analyze the Liouville integrability of the above two binary constrained flows, let us first introduce a symplectic structure over \( \mathbb{R}^{8N} \):

\[
\omega^2 = \sum_{i=1}^{4} B d\Phi_i \wedge d\Psi_i = \sum_{i=1}^{4} \sum_{s=1}^{N} \mu_s d\phi_{is} \wedge d\psi_{is}, \tag{3.23}
\]

where \( B \) is defined by (3.19) and \( \Phi_i \) and \( \Psi_i \) are defined by (3.20). Then the associated Poisson bracket reads as

\[
\{f, g\} = \omega^2(Idg, Idf) = \sum_{i=1}^{4} \left( \frac{\partial f}{\partial \Psi_i} B^{-1} \frac{\partial g}{\partial \Phi_i} - \frac{\partial f}{\partial \Phi_i} B^{-1} \frac{\partial g}{\partial \Psi_i} \right) = \sum_{i=1}^{4} \sum_{s=1}^{N} \mu_s^{-1} \left( \frac{\partial f}{\partial \psi_{is}} \frac{\partial g}{\partial \phi_{is}} - \frac{\partial f}{\partial \phi_{is}} \frac{\partial g}{\partial \psi_{is}} \right), \tag{3.24}
\]

where the vector field \( Idf \) is uniquely determined by

\[\omega^2(X, Idf) = df(X), \quad X \in T(\mathbb{R}^{8N}).\]

A Hamiltonian system with a Hamiltonian \( H \) defined over this symplectic manifold \( (\mathbb{R}^{8N}, \omega^2) \) is given by

\[
\Phi_i t = \{\Phi_i, H\} = -B^{-1} \frac{\partial H}{\partial \Phi_i}, \quad \Psi_i t = \{\Psi_i, H\} = B^{-1} \frac{\partial H}{\partial \Phi_i}, \quad 1 \leq i \leq 4, \quad \tag{3.25}
\]

where \( t \) is assumed to be the evolution variable.

In order to prove that two binary constrained flows are Liouville integrable, we need generate their Hamiltonian structures and integrals of motion. A direct computation can verify the following theorem.

**Theorem 3.1** The spatial binary constrained flow (3.21) and the temporal binary constrained flow (3.22) are Hamiltonian systems with the evolution variables \( x, t \) and the Hamiltonian functions

\[
H^x = -\sum_{k=1}^{4} \alpha_k \langle A\Phi_k, B\Psi_k \rangle - \sum_{1 \leq k < l \leq 4} \frac{\alpha_k - \alpha_l}{\beta_k - \beta_l} \langle \Phi_k, B\Psi_l \rangle \langle \Phi_l, B\Psi_k \rangle, \tag{3.26}
\]

\[
H^t = -\sum_{k=1}^{4} \beta_k \langle A\Phi_k, B\Psi_k \rangle - \sum_{1 \leq k < l \leq 4} \frac{\beta_k - \beta_l}{\gamma_k - \gamma_l} \langle \Phi_k, B\Psi_l \rangle \langle \Phi_l, B\Psi_k \rangle, \tag{3.27}
\]

respectively, where the matrix \( A \) is given by

\[A = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N).\]
It is well known that Lax representations can engender integrals of motion. For the binary constrained flows (3.21) and (3.22), let us introduce a Lax operator $L(\lambda)$ as follows

$$L(\lambda) = (L_{ij}(\lambda))_{4 \times 4} = \Gamma + D(\lambda) = \text{diag}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) + D(\lambda),$$  \hspace{1cm} (3.29)

$$D(\lambda) = (D_{ij}(\lambda))_{4 \times 4} = \sum_{i=1}^{N} \frac{\mu_i \phi_i \psi_j}{\lambda - \lambda_i}, \quad 1 \leq i, j \leq 4.$$  \hspace{1cm} (3.30)

Now if we define

$$\tilde{U}(\lambda) = U(\tilde{u}, \lambda), \quad \tilde{V}(\lambda) = V(\tilde{u}, \lambda),$$  \hspace{1cm} (3.31)

then we obtain the following result.

**Theorem 3.2** The binary constrained flows (3.21) and (3.22) have the following Lax representations:

$$(L(\lambda))_x = [\tilde{U}(\lambda), L(\lambda)], \quad (L(\lambda))_t = [\tilde{V}(\lambda), L(\lambda)],$$  \hspace{1cm} (3.32)

respectively.

An $\mathcal{r}$-matrix formulation can also be directly shown for the Lax operator $L(\lambda)$ defined by (3.29) and (3.30).

**Theorem 3.3** The Lax operator $L(\lambda)$ defined by (3.29) and (3.30) has the following $\mathcal{r}$-matrix formulation

$$\{L(\lambda) \otimes L(\mu)\} = \left[ \frac{1}{\mu - \lambda} \mathcal{P}, L_1(\lambda) + L_2(\mu) \right], \quad \mathcal{P} = \sum_{i,j=1}^{4} E_{ij} \otimes E_{ji},$$  \hspace{1cm} (3.33)

where $L_1(\lambda) = L(\lambda) \otimes I_4$, $L_2(\mu) = I_4 \otimes L(\mu)$, and

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}, \quad \{L(\lambda) \otimes L(\mu)\}_{ij,kl} = \{L_{ik}(\lambda), L_{jl}(\mu)\}, \quad 1 \leq i, j, k, l \leq 4.$$  

Now first from the Lax representations in (3.32), we obtain

$$(\nu I_4 - L(\lambda))_x = [\tilde{U}(\lambda), \nu I_4 - L(\lambda)], \quad (\nu I_4 - L(\lambda))_t = [\tilde{V}(\lambda), \nu I_4 - L(\lambda)],$$

where $\nu$ is a parameter, and thus

$$(\det(\nu I_4 - L(\lambda)))_x = 0, \quad (\det(\nu I_4 - L(\lambda)))_t = 0.$$  \hspace{1cm} (3.34)

Second from the $\mathcal{r}$-matrix formulation (3.33), we can obtain

$$\{\text{tr} L^k(\lambda), \text{tr} L^l(\mu)\} = 0, \quad k, l \geq 1.$$  \hspace{1cm} (3.35)
Expand the determinant of the matrix $\nu I_4 - L(\lambda)$ as

$$\det(\nu I_4 - L(\lambda)) = \nu^4 - \mathcal{F}_1(\lambda)\nu^3 + \mathcal{F}_2(\lambda)\nu^2 - \mathcal{F}_3(\lambda)\nu + \mathcal{F}_4(\lambda),$$  \hspace{1cm} (3.34)

where by Newton’s identities on elementary symmetric polynomials, we have

$$\begin{align*}
\mathcal{F}_1(\lambda) &= \text{tr}L(\lambda), \\
\mathcal{F}_2(\lambda) &= \frac{1}{6}(\text{tr}L(\lambda))^2 - \frac{1}{2}\text{tr}L^2(\lambda), \\
\mathcal{F}_3(\lambda) &= \frac{1}{3}\text{tr}L^3(\lambda) - \frac{1}{12}\text{tr}(L(\lambda))\text{tr}L^2(\lambda), \\
\mathcal{F}_4(\lambda) &= \det L(\lambda) = \frac{1}{24}(\text{tr}L(\lambda))^4 - \frac{1}{4}\text{tr}L^4(\lambda) + \frac{1}{3}(\text{tr}L^2(\lambda))^2 \\
&\quad - \frac{1}{4}(\text{tr}L(\lambda))^2\text{tr}L^3(\lambda) + \frac{1}{6}(\text{tr}L(\lambda))\text{tr}L^3(\lambda).
\end{align*}$$

Therefore, it follows that

$$\begin{align*}
(F_i(\lambda))_x &= 0, \\
(F_i(\lambda))_t &= 0, \\
\{F_i(\lambda), F_j(\mu)\} &= 0, \quad 1 \leq i, j \leq 4. \hspace{1cm} (3.35)
\end{align*}$$

Make a further expansion of $F_i(\lambda)$ as follows

$$F_i(\lambda) = \sum_{k \geq 0} F_{ik} \lambda^{-k}, \quad 1 \leq i \leq 4,$$  \hspace{1cm} (3.36)

where obviously the $F_{i0}$’s are all constants. Then it follows from (3.35) that

$$\begin{align*}
(F_{ik})_x &= 0, \\
(F_{ik})_t &= 0, \\
\{F_{ik}, F_{jl}\} &= 0, \quad 1 \leq i, j \leq 4, \quad k, l \geq 0. \hspace{1cm} (3.37)
\end{align*}$$

Therefore, two binary constrained flows (3.21) and (3.22) have the common integrals of motion: $F_{il}$, $1 \leq i \leq 4$, $l \geq 1$, which are in involution in pairs under the Poisson bracket (3.24). Now it is direct to show the following theorem on the Liouville integrability of two binary constrained flows (3.21) and (3.22).

**Theorem 3.4** Two binary constrained flows (3.21) and (3.22) have the common integrals of motion: $F_{il}$, $1 \leq i \leq 4$, $l \geq 1$, which are in involution in pairs under the Poisson bracket (3.24), and of which the functions $F_{is}$, $1 \leq i \leq 4$, $1 \leq s \leq N$, are functionally independent over a dense open subset of $\mathbb{R}^{8N}$. Thus, two binary constrained flows (3.21) and (3.22) are Liouville integrable.

### 3.4 Involutive Solutions

Note that under the constraints on the potentials (3.18), the compatibility condition of (3.13) and (3.14) is still only the multi-wave interaction equations (3.8). Thus, solutions $(\Phi_i(x, t), \Psi_i(x, t))$ to two binary constrained flows (3.21)
and (3.22) will present solutions to the multi-wave interaction equations (3.8), namely,

\[ u_{ij}(x,t) = \frac{\alpha_i - \alpha_j}{\gamma_i - \gamma_j}(\Phi_i(x,t), B\Psi_j(x,t)), \quad 1 \leq i \neq j \leq 4, \quad (3.38) \]

which also shows the integrability by quadratures for the multi-wave interaction equations (3.8) because \( \Phi_i \) and \( \Psi_i \), \( 1 \leq i \leq 4 \), can be determined by quadratures.

It is also easy to observe that under the Poisson bracket (3.24), two Hamiltonians \( H^x \) and \( H^t \) defined by (3.27) and (3.27) commute, namely

\[ \{H^x, H^t\} = 4 \sum_{i=1}^{4} \left( \frac{\partial H^x}{\partial \Phi_i}, B^{-1} \frac{\partial H^t}{\partial \Phi_i} - \frac{\partial H^x}{\partial \Psi_i}, B^{-1} \frac{\partial H^t}{\partial \Psi_i} \right) = 0. \quad (3.39) \]

It follows that the above solutions determined by (3.38) give rise to a kind of involutive solutions to the multi-wave interaction equations (3.8). Let us denote two Hamiltonian flows of (3.21) and (3.22) by \( g_{H^x}^t \) and \( g_{H^t}^x \) respectively. Then the resulting involutive solutions can be written as

\[ u_{ij}(x,t) = \frac{\alpha_i - \alpha_j}{\gamma_i - \gamma_j}(g_{t} g_{x}, g_{t} \Phi_{i_0}, B g_{x} g_{t} \Psi_{j_0}) = \frac{\alpha_i - \alpha_j}{\gamma_i - \gamma_j}(g_{t} g_{x}, g_{t} \Phi_{i_0}, B g_{x} g_{t} \Psi_{j_0}), \quad 1 \leq i \neq j \leq 4, \]

where the initial values \( \Phi_{i_0} \) and \( \Psi_{j_0} \) of \( \Phi_i \) and \( \Psi_i \), \( 1 \leq i \leq 4 \), can be taken to be any arbitrary constant vectors of \( \mathbb{R}^N \).

In a word, solutions of two binary constrained flows (3.21) and (3.22) can lead to involutive solutions of the multi-wave interaction equations (3.8). More importantly, such involutive solutions show us an explicit Bäcklund transformation from the multi-wave interaction equations (3.8) to two finite-dimensional Liouville integrable Hamiltonian systems determined by (3.26) and (3.27) and thus the integrability by quadratures for the multi-wave interaction equations (3.8).

4 Summary and Conclusion

The multi-wave interaction equations (3.8) have been taken as an illustrative example of binary nonlinearization. Binary symmetry constraints (3.17), containing arbitrary distinct constants \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \), and arbitrary non-zero constants \( \mu_1, \mu_2, \mu_3, \mu_4 \), were proposed for the multi-wave interaction equations (3.8). Two finite-dimensional Liouville integrable Hamiltonian systems
determined by (3.26) and (3.27), resulted from two binary constrained flows, determine a kind of involutive solutions to the multi-wave interaction equations (3.8). If we take $B = I_N$, and

$$\Gamma = V_0, \ i.e., \ \text{diag}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = \text{diag}(\beta_1, \beta_2, \beta_3, \beta_4),$$

then the result established here will cover all results presented in Ref. [30]. This also implies that our result leads to a larger class of finite-dimensional integrable Hamiltonian systems.

The theory of binary nonlinearization shows that binary symmetry constraints in the continuous case decompose soliton equations (PDEs) into binary constrained flows (ODEs). The $r$-matrix formulation can be used to expose the Liouville integrability of binary constrained flows. The resulting constraints on the potentials also present a kind of involutive solutions to soliton equations and thus eventually show the integrability by quadratures for soliton equations. The whole theory can also be applied to many other types of soliton equations.

We point out that in the case on $2 \times 2$ traceless matrix spectral problems, binary nonlinearization will be reduced to mono-nonlinearization, if we take the reduction $$(\psi_{1j}, \psi_{2j})^T = (-\phi_{2j}, \phi_{1j})^T, \ 1 \leq j \leq N.$$ That is to say, in such a case, all results achieved through mono-nonlinearization can be obtained from the results achieved through binary nonlinearization. Therefore, in this sense, binary nonlinearization is much broader and more systematic.

We have seen that symmetry constraints yield nonlinear constraints on potentials of soliton equations, and put linear spectral problems (linear with respect to eigenfunctions) into nonlinear binary constrained flows (nonlinear again with respect to eigenfunctions), which seemly makes it more complicated to solve soliton equations. However, since spectral problems are overdetermined, to guarantee the existence of eigenfunctions of spectral problems, one needs compatibility conditions which are often nonlinear partial differential equations. The symmetry property of the constraints brings us the Liouville integrability for nonlinear binary constrained flows. Therefore, the symmetry property makes up for the complexity of nonlinearization while manipulating binary nonlinearization.

Of particular interest in the study of binary symmetry constraints are to create new classical integrable Hamiltonian systems which supplement the known classes of classical integrable systems and to expose the integrability by quadratures for soliton equations by using binary constrained flows. We mention that high-order symmetry constraints with involved Lie-Bäcklund symmetries having non-degenerate Hamiltonians can also be carried out for soliton
equations without much difficulty, but the case of degenerate Hamiltonians needs some particular consideration. The rigorous considerations on binary symmetry constraints and two-binary symmetry constraints, especially on variable symplectic structures of binary constrained flows, are being expected to be made in a future publication.

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