Characterizations of left orders in left Artinian rings

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$R$ is a ring with 1,

$\mathcal{C} = \mathcal{C}_R$ is the set of regular elements of $R$,

$Q = Q_{l,cl}(R) := \mathcal{C}^{-1}R$ is the \textbf{left quotient ring} (the \textbf{classical left ring of fractions}) of $R$ (if it exists),

$n$ is a prime radical of $R$ and $\nu$ is its nilpotency degree ($n^\nu \neq 0$ but $n^{\nu+1} = 0$),

$\overline{R} := R/n$ and $\pi : R \to \overline{R}$, $r \mapsto \overline{r} = r + n$,

$\overline{\mathcal{C}} := \mathcal{C}_{\overline{R}}$ is the set of regular elements of $\overline{R}$,

$\overline{Q} := \overline{\mathcal{C}}^{-1}\overline{R}$ is its left quotient ring,

$\mathcal{C}' := \pi^{-1}(\overline{\mathcal{C}}) := \{ c \in R \mid c + n \in \overline{\mathcal{C}} \}$,

$Q' := \mathcal{C}'^{-1}R$. 

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A ring $R$ is a **left Goldie ring** if

(i) $R$ satisfies ACC for left annihilators,

(ii) $R$ contains no infinite direct sums of left ideals.

**Thm (Goldie, 1958, 1960).** A ring $R$ is a semiprime left Goldie ring iff it has an Artinian left quotient ring which is semi-simple.

Lessieur and Croisot (1959): prime case.

**Question:** When $Q$ does exist and is a left Artinian ring?

Answer: Small (1966), Robson (1967), Tachikawa (1971), Hajarnavis (1972) and Bavula (2012).

In all the proofs of the criteria above Goldie’s Thm is used.
Theorem. Let \( A \) be a left Artinian ring and \( \tau \) be its radical. Then

1. The radical \( \tau \) of \( A \) is a nilpotent ideal.

2. The factor ring \( A/\tau \) is semi-simple.

3. An \( A \)-module \( M \) is semi-simple iff \( \tau M = 0 \).

4. There is only finite number of non-isomorphic simple \( A \)-modules.

5. The ring \( A \) is a left noetherian ring.
Robson’s Criterion.

Let $W$ be the sum of all the nilpotent ideals of the ring $R$.

**Theorem (Robson, 1967).** TFAE

1. The ring $R$ has a left Artinian left quotient ring $Q$.

2. (a) The ring $R$ is $W$-reflective,
(b) the ring $R$ is $W$-quorite,
(c) $R/W$ is a left Goldie ring,
(d) $W$ is a nilpotent ideal of $R$, and
(e) the ring $R$ satisfies ACC on $\mathcal{C}$-closed left ideals.
R is \textit{W-reflective} if, for \( c \in R \), then \( c \in C \) iff \( c + W \in C_{R/W} \left( \Leftrightarrow C' = C \right) \).

R is \textit{W-quorite} if, given \( w \in W \) and \( c \in C \), there exist \( c' \in C \) and \( w' \in W \) s.t. \( c'w = w'c \).

A l.ideal \( I \) of \( R \) is \textit{C-closed} if \( cr \in I \), where \( c \in C \) and \( r \in R \), then \( r \in I \).

Robson’s Criterion is based on the work of Feller and Swokowski (1961, 1961, 1961) and Talintyre (1963).
Thm (Small’s Criterion, 1966, 1966) \( TFAE \)

1. \( R \) has a left Artinian left quotient ring \( Q \).

2. (a) \( R \) is a left Goldie ring,

(b) \( W \) is a nilpotent ideal of \( R \),

(c) for all \( k \geq 1 \), \( R/(r(W^k) \cap W) \) is a left Goldie ring,

(d) \( r + W \in C_{R/W} \implies r \in C \) (i.e. \( C' \subseteq C \)).
Thm (Hajarnavis, 1972) TFAE

1. $R$ has a left Artinian left quotient ring $Q$.

2. (a) $R$ and $R/W$ are left Goldie rings,
   (b) $W$ is a nilpotent ideal of $R$,
   (c) for all $k \geq 1$, $R/r(W^k)$ has finite left uniform dimension,
   (d) $r + W \in C_{R/W} \implies r \in C$ (i.e. $C' \subseteq C$).

His approach is very close to Small’s but improvement has been done by using some of the results of Goldie and Talintyre.
Suppose that \( \overline{R} := R/n \) is a (semiprime) left Goldie ring.

By Goldie’s Thm, \( \overline{Q} := \overline{C}^{-1} \overline{R} \) is a semi-simple (Artinian) ring.

The \( n \)-adic filtration: \( R \supset n \supset \cdots \supset n^i \supset \cdots \)

\[ \text{gr } R = \overline{R} \oplus n/n^2 \oplus \cdots \oplus n^i/n^{i+1} \oplus \cdots \]

For \( i \geq 1 \), \( \tau_i := \text{tor}_C(n^i/n^{i+1}) := \{ u \in n^i/n^{i+1} \mid \overline{c}u = 0 \text{ for some } \overline{c} \in \overline{C} \} \) is the \( \overline{C} \)-torsion submodule of the left \( \overline{R} \)-module \( n^i/n^{i+1} \).

\( \tau_i \) is an \( \overline{R} \)-bimodule. Then the \( \overline{R} \)-bimodule \( f_i := (n^i/n^{i+1})/\tau_i \) is a \( \overline{C} \)-torsion free, left \( \overline{R} \)-module.

There is a unique ideal \( t_i \) of \( R \) s. t. \( n^{i+1} \subseteq t_i \subseteq n^i \) and \( t_i/n^{i+1} = \tau_i \). Clearly, \( f_i \cong n^i/t_i \).
Thm (B., 2012) \textit{TFAE}

1. The ring $R$ has a left Artinian left quotient ring $Q$.

2. (a) The ring $R$ is a left Goldie ring,

(b) $n$ is a nilpotent ideal,

(c) $C' \subseteq C$,

(d) the left $R$-modules $f_i$, where $i \geq 1$, contain no infinite direct sums of nonzero submodules, and

(e) for every element $c \in C$, the map $\cdot c : f_i \to f_i$, $f \mapsto fc$, is an injection.

If one of the equivalent conditions holds then $C = C'$, $C^{-1}n$ is the prime radical of the ring $Q$ which is a nilpotent ideal of nilpotency degree $\nu$, and the map $Q/C^{-1}n \to \overline{Q}$, $c^{-1}r \mapsto \overline{c^{-1}r}$, is a ring isomorphism with the inverse $\overline{c^{-1}r} \mapsto c^{-1}r$. 

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Corollary. Let \( R \) be a left Noetherian ring. TFAE

1. \( R \) has a left Artinian left quotient ring.

2. \( C' \subseteq C \).

3. For each element \( \alpha \in \overline{C} \), there exists an element \( c = c(\alpha) \in C \) such that \( \alpha = c + n \).

1 \( \Leftrightarrow \) 2 is due to Small (1966).
Corollary. Let $R$ be a commutative ring. TFAE

1. The ring $R$ has an Artinian quotient ring.

2. (a) The ring $\bar{R}$ is a Goldie ring.

   (b) $n$ is a nilpotent ideal.

   (c) $C' \subseteq C$.

   (d) The $\bar{R}$-modules $\bar{f}_i, 1 \leq i \leq \nu$, contain no infinite direct sums of nonzero submodules.

3. (a) The ring $\bar{R}$ is a Goldie ring.

   (b) $n$ is a nilpotent ideal.

   (c) For each element $\alpha \in \bar{C}$, there exists an element $c = c(\alpha) \in C$ such that $\alpha = c + n$.

   (d) The $\bar{R}$-modules $\bar{f}_i, 1 \leq i \leq \nu$, contain no infinite direct sums of nonzero submodules.
4. \( R \) is a Goldie ring and \( C' \subseteq C \).

5. \( R \) is a Goldie ring and, for each element \( \alpha \in \overline{C} \), there exists an element \( c = c(\alpha) \in C \) such that \( \alpha = c + n \).

1 \iff 4 \text{ P. F. Smith (1972).}
Theorem (B., 2012) Let $R$ be a ring. TFAE

1. The ring $R$ has a left Artinian ring left quotient ring $Q$.

2. The set $\mathcal{C}$ is a left denominator set in the ring $\text{gr} \ R$, $\mathcal{C}^{-1}\text{gr} \ R$ is a left Artinian ring, $\frak{n}$ is a nilpotent ideal and $\mathcal{C}' \subseteq \mathcal{C}$.

3. The set $\mathcal{C}$ is a left denominator set in the ring $\text{gr} \ R$, the left quotient ring $Q(\text{gr} \ R/\tau)$ of the ring $\text{gr} \ R/\tau$ is a left Artinian ring, $\frak{n}$ is a nilpotent ideal and $\mathcal{C}' \subseteq \mathcal{C}$.

If one of the equivalent conditions holds then $\text{gr} \ Q \simeq Q(\text{gr} \ R/\tau) \simeq \mathcal{C}^{-1}\text{gr} \ R$ where $\text{gr} \ Q$ is the associated graded ring with respect to the prime radical filtration.
Criteria similar to Robson’s criterion

Theorem (B., 2012) Let $R$ be a ring. TFAE

1. The ring $R$ has a left Artinian left quotient ring $Q$.

2. (a) The ring $\overline{R}$ is a left Goldie ring.

(b) $n$ is a nilpotent ideal.

(c) $C' \subseteq C$.

(d) If $c \in C'$ and $n \in n$ then there exist elements $c_1 \in C'$ and $n_1 \in n$ such that $c_1n = n_1c$.

(e) The ring $R$ satisfies ACC for $C'$-closed left ideals.
A left quotient ring of a factor ring

Theorem (B., 2012) Let $R$ be a ring with a left Artinian left quotient ring $Q$, and $I$ be a $C$-closed ideal of $R$ such that $I \subseteq n$. Then

1. The set $C_{R/I}$ of regular elements of the ring $R/I$ is equal to the set \{c + I \mid c \in C\}.

2. The ring $R/I$ has a left Artinian left quotient ring $Q(R/I)$ and the map $Q/C^{-1}I \rightarrow Q(R/I)$, $c^{-1}r + C^{-1}I \mapsto (c + I)^{-1}(r + I)$, is a ring isomorphism with the inverse $(c + I)^{-1}(r + I) \mapsto c^{-1}r + C^{-1}I$. 

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