THE DECORATION THEOREM FOR MANDELBROT AND MULTIBROT SETS

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Abstract. We prove the decoration theorem for the Mandelbrot set (and Multibrot sets) which says that when a “little Mandelbrot set” is removed from the Mandelbrot set, then most of the resulting connected components have small diameters.

1. Introduction

The Mandelbrot set $M$ is defined as the set of quadratic polynomials $z^2 + c$ with connected Julia sets. It is a compact, connected, and full set, and in addition, it has a rich combinatorial structure. Furthermore, the Mandelbrot set is self-similar in a certain sense: there are infinitely many subsets that, together with the induced combinatorics, are canonically homeomorphic to $M$; these are called small copies of $M$ within $M$.

There is a classification of all small copies of the Mandelbrot set; every copy has particular dynamical properties, hence all copies can be defined and distinguished in the dynamical planes (see Definition 3.3).

In this paper, we prove the following theorem which was conjectured in the mid-1990 by Mikhail Lyubich and Dierk Schleicher, as well as by Carsten Lunde Petersen:

Decoration Theorem. Let $M$ be the Mandelbrot set and let $M_s$ be a small copy of the Mandelbrot set. Then for any $\varepsilon > 0$, there are at most finitely many connected components of $M \setminus M_s$ with diameter at least $\varepsilon$.

The countably many components of $M \setminus M_s$ are called the decorations of $M_s$, and the claim is that most of them are small.

Our main tool will be puzzle and parapuzzle theory.

Remark. The entire construction and the proof will be carried out for the Mandelbrot set, but they work just the same for all Multibrot sets $M_d := \{ c \in \mathbb{C} : \text{the Julia set of } z \mapsto z^d + c \text{ is connected} \}$ for $d \geq 2$. We refrained from working out the details for simplicity of notation. More precisely, everything we are doing uses conformal pull-backs of a single annulus (in two different cases); we do not encounter
problems where the combinatorics grows more slowly than the shrinking of moduli for high-degree pull-backs. However, we do use the result that all parameters \( c \in \mathcal{M}_d \) that are not infinitely renormalizable have trivial fibers (so that \( \mathcal{M}_d \) is locally connected at these parameters). This was proved by Yoccoz for \( d = 2 \) and by Kahn and Lyubich for all \( d \geq 2 \).

The Decoration Theorem thus holds for all degree \( d \geq 2 \).

**Remark.** The entire construction is local in the sense that it not only works for the Mandelbrot set, but also for full families of quadratic-like maps (and similarly for Multibrot-like maps). The details are quite similar to the text as written and are omitted.

1.1. **Terminology and Notation:** \( f_c(z) = z^2 + c \) is a quadratic polynomial.

The level of equipotentials will be called *height*, and the *depth* of a puzzle piece is the number of iterations it takes to map the puzzle piece to a piece of the initial puzzle.

For every puzzle piece the upper index is its depth. If a puzzle piece is “unique”, then the subindex will be 0 or 1 depending on the context. For example, \( Y^n_0 \) contains the critical point while \( Y^n_1 \) contains the critical value.

*We will use the following conventions:* objects in the parameter plane will be denoted by calligraphic capital letters (such as \( \mathcal{M}, Z_{n}^{r} \)) while those in dynamical planes will be denoted by Roman capitals (such as \( Z_{m}^{n}, Y_{p}^{p} \)).

We will use floor brackets (for example, \( [Y^n_{1}] \) or \( [Z_{m}^{n}] \)) to untruncuate the corresponding puzzle or parapuzzle piece.

Following tradition, we slightly abuse (and thus simplify) notation and use the modulus of an annulus \( A \) even when \( A \) is not open, provided its boundary is piecewise smooth (which will always be the case for annuli constructed by puzzle pieces).

By “combinatorics” we mean the angles and heights of rays and equipotentials. In particular, two puzzle pieces in different planes are (combinatorially) the same if there is a homeomorphism of their boundaries sending rays and equipotentials to rays and equipotentials with equal angles and heights.

In the paper all renormalizations are simple; we will not consider crossed renormalizations.

1.2. **Outline of the paper.** In Section 2 we will briefly review the puzzle (and parapuzzle) construction. We also will fix some conventions.

Section 3 contains the combinatorics. We will discuss the relation between small copies of the Mandelbrot set and their decorations, as well as with puzzle pieces in the dynamical planes.
First we reformulate the problem in terms of puzzle and parapuzzle pieces. Every decoration is inside a parapuzzle piece $Z^n_i$ associated to that decoration.

If the Decoration Theorem was not true, then big decorations must accumulate at some point $c_0$ from the Mandelbrot set. The aim is to show that for every $Z^n_i$ sufficiently close to $c_0$ there exists an annulus $A^n_i$ such that the following properties hold:

- $A^n_i$ surrounds $Z^n_i$, but neither contains nor surrounds $c_0$;
- the moduli of $A^n_i$ are uniformly bounded below.

This will conclude the proof.

The first observation is that $c_0$ must be so that the fiber of $M$ at $c_0$ is not trivial. Therefore, by Yoccoz’s results it is enough to consider the case when $z^2 + c_0$ is an infinitely renormalizable polynomial. Hence $c_0 \in \mathcal{M}' \subset M$, where $\mathcal{M}'$ is a small copy of $M$ within $\mathcal{M}$.

Every $Z^n_i$ is inside some decoration’s parapuzzle piece $Z_{jm}^m$ associated with $\mathcal{M}'$. We will show that $A^n_i := Z_{jm}^m \setminus Z^n_i$ are annuli that satisfy the above requirement.

By Lemma 3.7 every $Z^n_i$ corresponds to dynamical puzzle pieces $Z^n_i$. All puzzle pieces $Z^n_i$ are preimages of a single puzzle piece $Z_0^0$, which itself is inside a puzzle piece $\mathcal{Z}_0^0$ (Proposition 3.9).

In Section 4.1 we will consider a particular case that we call “simple”. We pull back conformally the annulus $\mathcal{Z}_0^0 \setminus Z_0^0$ to the annulus $\mathcal{Z}_0^0 \setminus Z^n_i$ around $Z^n_i$. For $\mathcal{Z}_0^0 \setminus Z^n_i$ there exists the corresponding annulus $\mathcal{Z}_0^0 \setminus Z^n_i$ in the parameter plane with comparable modulus.

Lemma 4.4 (Section 4.2) shows the existence of a big collection of annuli with bounded below modulus. Pulling back conformally these annuli we obtain a collection of annuli $\tilde{A}$ within $Z_{jm}^m$, where every annulus has a corresponding annulus in parameter space. If the decoration is “unsimple” and sufficiently close to $c_0$ (Section 4.3), then $Z^n_i$ at $c = c_0$ is surrounded by an annulus $A^n_i \subset \tilde{A}$.

Acknowledgements. I am very grateful to Carsten Lunde Petersen, Mikhail Lyubich, Pascale Roesch, Davoud Cheraghi and “Bremen dynamical group”, in particular Dierk Schleicher, Vladlen Timorin, Nikita Selinger, Yauhen Mikulich for very useful discussions.

I am very grateful to Dierk Schleicher for his invaluable assistance in writing this paper.

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1Actually, we will denote by $A^n_i$ subannuli of $Z_{jm}^m \setminus Z^n_i$ and prove that these satisfy the conditions stated above; this also would imply that $Z_{jm}^m \setminus Z^n_i$ satisfy the above conditions.
2. Puzzle and parapuzzle pieces

Let \( \mathcal{R} \) be a finite collection of periodic and preperiodic rays such that:

- every ray lands;
- every landing point is the landing point of at least two rays in \( \mathcal{R} \);
- \( \mathcal{R} \) is forward invariant: \( f_c(R) \) is in \( \mathcal{R} \) for every \( R \in \mathcal{R} \).

The family \( \mathcal{R} \) together with an equipotential gives a partition of a neighborhood of the Julia set (we assume that all sets in this partition are closed; their boundaries may intersect). Any bounded component \( X^n_0 \) of the partition is called a puzzle piece of depth 0. We say that \( X^n_i \) is a puzzle piece of depth \( n \) if it is a preimage of a puzzle piece of depth 0 under \( f^n_c \).

Denote by \( F_{\mathcal{R}} \) the family of all puzzle pieces associated with \( \mathcal{R} \) (of all depths).

**Property 2.1.** All puzzle pieces from \( F_{\mathcal{R}} \) are closed topological discs. If \( X^n_i, X^m_j \in F_{\mathcal{R}} \), then either their interiors do not intersect or \( X^n_i \subseteq X^m_j \); the latter can happen only if \( n \geq m \).

The following criterion is useful in order to determine when a topological disc is a puzzle piece (in an appropriate family).

**Proposition 2.2.** Suppose a closed topological disc \( X \) is bounded by periodic and preperiodic rays and truncated by an equipotential. Then there exists a family \( F \) so that \( X \) is a puzzle piece for \( F \) if and only if the forward orbit of \( \partial X \) does not intersect the interior of \( X \).

Indeed, assume that \( X \) is bounded by \( R_1, \ldots, R_s \) and consider \( \mathcal{R} = \bigcup_{k \geq 0} \bigcup_j f^k(R_j) \). If the assumption in Proposition 2.2 is satisfied, then \( X \) is a puzzle piece in the family \( F = F_{\mathcal{R}} \). The converse is obvious.

Assume that the critical value \( c \) is in the interior of a puzzle piece \( X^0_1 \) of depth 0. It is well known that there exists a topological disc \( \mathcal{X}^0_1 \) in the parameter plane such that the boundary \( \partial \mathcal{X}^0_1 \) has the same combinatorial structure as \( \partial X^0_1 \). In addition, for every parameter \( c \in \text{int} \mathcal{X}^0_1 \), all puzzle pieces from \( F_{\mathcal{R}} \) of depth 0 “exist” and depend continuously on \( c \). We will say that the family \( F_{\mathcal{R}} \) exists in \( \mathcal{X}^0_1 \). Depending on properties of \( \mathcal{R} \), the family \( F_{\mathcal{R}} \) may exist in a bigger domain.

In general, the family \( F_{\mathcal{R}} \) depends on \( c \). Let \( X^n_i \in F_{\mathcal{R}} \) be a puzzle piece in the dynamical plane of \( z^2 + c_1 \), where \( c_1 \in \text{int} \mathcal{X}^0_1 \). We will say that \( X^n_i \) exists for \( c_2 \in \text{int} \mathcal{X}^0_1 \) if there exists a puzzle piece \( \tilde{X}^n_i \) in the dynamical plane of \( z^2 + c_2 \) such that the boundaries of \( \tilde{X}^n_i \) and \( X^n_i \) are combinatorially equivalent. To simplify, we will write \( \tilde{X}^n_i = X^n_i \); this convention allows us to use the notion of puzzle pieces without referring to a particular dynamical plane.
Let $X^n_i \subseteq X^n_0$ be a topological disc in the parameter plane bounded by parameter rays and an equipotential. Then $X^n_i$ is called the paranormal puzzle piece associated to $X^n_i$ for the family $F$ if there exists a puzzle piece $X^n_i \in F^n_R$ with the same combinatorics and the following equality holds:

\[ \text{int } X^n_i = \{ c \in \mathbb{C} | \text{int } X^n_i \text{ exists for } c \text{ and } c \in \text{int } X^n_i \}. \]

The next property shows a relation between puzzle pieces in different families.

**Property 2.3.** Suppose a puzzle piece $X^n_0 \in F_1$ is inside a puzzle piece $Y^n_0 \in F_2$ of depth 0. Then any preimage of $X^n_i$ under $f^n_{c_0}$ is inside a puzzle piece of depth $n$ from $F_2$.

3. **The Combinatorial Construction**

Let $\mathcal{M}_s$ be a small copy of the Mandelbrot set. Then the component of $\mathcal{M} \setminus \mathcal{M}_s$ containing the main cardioid is “big,” and any other component is a part of $\mathcal{M}$ cut off by two external rays landing at some tip of $\mathcal{M}_s$ (such a component is called a decoration [KL1]; and a tip of $\mathcal{M}_s$ must be a Misiurewicz point).

To be more precise, each $\mathcal{M}_s$ has an integer $q \geq 2$ so that each tip of $\mathcal{M}_s$ is the landing point of exactly $q$ parameter rays. They chop off $q - 1$ decorations $\mathcal{L}_k$ (closures of the components of $\mathcal{M} \setminus \{t\}$ that do not intersect the main cardioid) from $\mathcal{M}$.

If the decoration conjecture was not true, then there would be some $\epsilon > 0$ and infinitely many decorations $\mathcal{L}_1, \mathcal{L}_2, \ldots$ with diameters at least $2\epsilon$. Denote by $a_i$ the tip of $\mathcal{L}_i$ (defined as the point of intersection of $\mathcal{L}_i$ and $\mathcal{M}_s$). We may extract a subsequence so that all $a_i$ are different. Now let us choose in each $\mathcal{L}_i$ any point $b_i$ such that $|a_i - b_i| \geq \epsilon$; let $c_0$ be an accumulation point of the sequence $\{b_i\}$; we may assume that all $|b_i - c_0| < \epsilon/2$.

**Proposition 3.1.** The quadratic map $f_{c_0}(z) = z^2 + c_0$ is an infinitely renormalizable polynomial. Moreover, $c_0$ belongs to $\mathcal{M}_s$.

**Proof.** Every decoration $\mathcal{L}_i$ is separated from $\mathcal{M} \setminus \mathcal{L}_i$ by two parameter rays landing at $a_i \in \mathcal{M}_s$. In addition, $\mathcal{L}_j \subset \mathcal{M} \setminus \mathcal{L}_i$ for $i \neq j$; therefore the accumulation point $c_0$ of decorations can not be in any $\mathcal{L}_i$ unless $c_0 = a_i$. Thus $c_0 \in \mathcal{M}_s$.

Since $|a_i - b_i| \geq \epsilon$ but $|b_i - c_0| < \epsilon/2$, the sets

\[ N_{\epsilon/2}(c_0) \cap \mathcal{M}_s \quad \text{and} \quad N_{\epsilon/2}(c_0) \cap \mathcal{L}_i \]

are all disjoint (where $N_{\epsilon/2}(c_0)$ denotes the $\epsilon/2$-neighborhood of $c_0$). This implies that $\mathcal{M}$ is not locally connected at $c_0$. By Yoccoz’s results (see [Hu]) $z^2 + c_0$ is an infinitely renormalizable polynomial. □
3.1. **Small copies of the Mandelbrot set.** A repelling periodic cycle \( \overline{\alpha} = \{\alpha_k\}_{k=0}^{n-1} \) is called *dividing* if there are at least two rays landing at each \( \alpha_k \). By \( R = R(\overline{\alpha}) \) we denote the configuration of rays landing at \( \overline{\alpha} \).

**Proposition 3.2 (see [Mi]).** Let \( \overline{\alpha} = \{\alpha_k\}_{k=0}^{n-1} \) be a dividing repelling periodic cycle.

- Let \( Y_1 \) be the component of \( \mathbb{C} \setminus R(\overline{\alpha}) \) containing the critical value \( c \). Then \( Y_1 \) is a sector bounded by two external rays.
- Let \( Y_0 \) be the component of \( \mathbb{C} \setminus f_c^{-1}R(\overline{\alpha}) \) containing the critical point \( 0 \). Then \( Y_0 \) is bounded by four external rays: two of them land at a periodic point \( \alpha_k \), and two others land at the symmetric point \( -\alpha_k \).
- The rays of \( R(\overline{\alpha}) \) form either one or two cycles under iterates of \( f_c \). All cycles have the same period.

Let \( p \geq 2 \) be the period of the rays \( R(\overline{\alpha}) \), let \( \alpha_0 \) be the periodic point on the boundary of \( Y_0 \), and let \( \alpha_1 \) be the point on the boundary of \( Y_1 \). Define \( Y_1^0 \) to be \( Y_1 \) truncated by the equipotential of height 1. Let \( Y_1^p \) be the unique component of \( f_c^{-p}(Y_1^0) \) attached to \( \alpha_1 \). By construction we have \( Y_1^p \supseteq Y_1^0 \).

The following definition is one of several equivalent ways of defining (simple) renormalization and small Mandelbrot sets:

**Definition 3.3.** A quadratic map \( f_c \) is called **DH renormalizable** of period \( p \) if there exists a cycle \( \overline{\alpha} \) as above such that \( c \) will not escape from \( Y_1^p \) under iteration of \( f^p : Y_1^p \to Y_1^p \) (in particular \( c \in Y_1^p \)). The associated small copy of the Mandelbrot set is the closure of the set of parameters \( c \) such that \( z^2 + c \) is DH renormalizable with the fixed ray pattern \( R(\overline{\alpha}) \).

It is known [DH] that a small copy of the Mandelbrot set is indeed canonically homeomorphic to \( \mathcal{M} \).
For the rest of the paper, we will fix the cycle $\overline{\alpha}$ and the corresponding small copy $M_s$ of the Mandelbrot set.

We now give a detailed description of the construction of $Y^p_1$. By Proposition 3.2 the sector $Y^1_1$ is bounded by two rays; denote them by $R^{\phi_1}$ and $R^{\phi_2}$. The strip $Y^0_0$ is bounded by two pairs of rays. One of them lands at the periodic point $\alpha_0$; we can assume that this ray pair has the angles $R^{\phi_1/2+1/2}$, $R^{\phi_2}$ (possibly by interchanging $\phi_1$ and $\phi_2$). The other pair is then $R^{\phi_1/2}$, $R^{\phi_2+1/2}$ and lands at $-\alpha_0$. The strip $[Y^p_1]$ is defined as $f^{-p+1}(Y^0_0)$, where the pullback is taken along the orbit of the periodic rays $R^{\phi_1}$ and $R^{\phi_2}$. Further, the strip $[Y^1_1]$ is bounded by two pairs of rays and one of them consists of $R^{\phi_1}$, $R^{\phi_2}$ landing at $\alpha_1$; denote by $R^{\psi_1}$ and $R^{\psi_2}$ the other pair. The last two rays depend continuously on the parameter $c$ whenever $R^{\phi_1}$ and $R^{\phi_2}$ do (the critical value cannot cross the forward orbit of $R^{\psi_1}$ and $R^{\psi_2}$).

It is known [Mi] that the parameter rays $R^{\phi_1}$ and $R^{\phi_2}$ land together at the root of $M_s$. We define $\mathcal{Y}_1$ to be the sector bounded by $R^{\phi_1} \cup R^{\phi_2}$ and containing $M_s$. Whenever $c \in \text{int} \mathcal{Y}_1$ the ray configuration $R(\overline{\alpha})$ depends holomorphically on $c$. The next statement implies that “small Julia sets do not intersect” (except at points of the orbit $\overline{\alpha}$); the proof follows from Proposition 3.2 and the definition of the strip $Y^p_1$.

**Property 3.4.** If $c$ belongs to the sector $\mathcal{Y}_1$, then for all $0 \leq k < p$ the interiors of strips $f^k_c([Y^p_1])$ are disjoint and do not intersect $Z^0_0$.

Let the parapuzzle piece $\mathcal{Y}^0_0$ be the sector $\mathcal{Y}_1$ truncated by the equipotential of height 1. If $c$ belongs to int $\mathcal{Y}^0_0$, then the puzzle pieces $Y^p_1$ and $Y^0_1$ depend continuously on $c$. From now we assume that $c \in \text{int} \mathcal{Y}^0_0$.

By $Z^0_0$ we denote the sector bounded by the rays $R^{\psi_1}$ and $R^{\psi_2}$, not containing 0, and truncated by the equipotential of height 1/2 (Figure 2).

Let $\overline{R} = \bigcup_{k \geq 0} \{f^k_c(R^{\psi_1}), f^k_c(R^{\psi_2})\}$ be the forward orbit of the rays $R^{\psi_1}$ and $R^{\psi_2}$; it contains $R(\overline{\alpha})$ but perhaps not all the rays that land
at the same point as $R^{\psi_1}, R^{\psi_2}$. Using $R$ and the equipotential of height 1/2, we get a partition of a neighborhood of the Julia set. Denote by $F$ the corresponding puzzle family. From the construction and Proposition 3.2, it follows that $F$ exists in $\mathcal{Y}_0^0$ and the puzzle piece $Z_0^0$ is in $F$. (Note that with respect to this family, $Y_1^0$ is not a puzzle piece but the union of two such pieces, separated by the rays $R^{\psi_1}, R^{\psi_2}$. The set $Y_1^p$ is not a puzzle piece in $F$ either because it is bounded by the equipotential at height $2^{-p}$; but there is a puzzle piece of depth 0 bounded by the same rays as $Y_1^p$ and the equipotential at height 1/2.)

**Definition 3.5.** We define the collection $\{Z^n_i\} \subset F$ as the set of maximal conformal pullbacks of $Z_0^n$. This means that $Z^n_i \not\subseteq Z^n_l$ and $f^n$ maps conformally $Z^n_i$ to $Z_0^0$ for any $i,l$ and $m < n$.

Let $\mathcal{L}_{i,1}, \ldots, \mathcal{L}_{i,q-1}$ be the group of $q-1$ decorations touching a common Misiurewicz point $a_i$ which is a tip of $\mathcal{M}_s$. There exists a pair of parameter rays $\mathcal{R}_1, \mathcal{R}_2$ that land together at $a_i$ and separate the group $\bigcup_{j=1}^{q-1} \mathcal{L}_{i,j}$ from $\mathcal{M} \setminus \bigcup_{j=1}^{q-1} \mathcal{L}_{i,j}$. Also, if $c$ belongs to $\bigcup_{j=1}^{q-1} \mathcal{L}_{i,j}$, then there exists a constant $n = n(i)$ such that $f^n(0) \in Z_0^0$ but $f^k(0) \not\in Z_0^0$ for $0 \leq k < n$ (the constant $n(i)$ is called the escaping time $[L1]$).

**Definition 3.6.** Denote by $Z^n_i$ the parapuzzle piece containing $\bigcup_{j=1}^{q-1} \mathcal{L}_{i,j}$, bounded by $\mathcal{R}_1$ and $\mathcal{R}_2$, and truncated by the equipotential of height $1/2^n$.

The parapuzzle pieces $Z^n_i$ correspond to dynamical puzzle pieces $Z^n_i$:

**Lemma 3.7 (Parapuzzles Pieces).** For any $n$ the set

$$\mathcal{J}_n := \text{int} \mathcal{Y}_0^0 \bigcup_{m \leq n-1} \bigcup_j \mathcal{Z}_j^m$$

is an open Jordan disk. When $c$ moves in $\mathcal{J}_n$, the boundaries of all $Z^n_i$ are disjoint and move holomorphically; moreover there is a one to one correspondence between $Z^n_i \subset \mathcal{Y}_1^0$ and $\mathcal{Z}_i^n \subset \mathcal{Y}_0^1$ such that:

$$c \in Z^n_i \text{ if and only if } c \in \mathcal{Z}_i^n.$$ 

**Proof.** This is a classical result (see for example [ALS, Lemma 3.3, 10]).

### 3.2. Secondary Decorations.

As $z^2 + c_0$ is infinitely renormalizable, there exists a small copy $\mathcal{M}'_s$ of the Mandelbrot set such that $\mathcal{M}'_s$ contains $c_0$ and $\mathcal{M}'_s \subset \mathcal{M}_s$. Let $\mathcal{C}$ be the dividing periodic cycle associated with $\mathcal{M}'_s$, and let $p'$ be the period of the rays. In general, we will use an apostrophe to mark that some structure is associated with $\mathcal{M}'_s$ (except in Subsection 4.3). All above results for $\mathcal{M}_s$ are true for $\mathcal{M}'_s$. For instance, the puzzle and parapuzzle pieces $\mathcal{Y}_1^0, Z_j^m, \mathcal{Z}_j^m$ are defined in the same way as $\mathcal{Y}_0^1, Z^n_i, \mathcal{Z}_i^n$. 

Figure 3. Two possibilities for the strip $[Y_1^{p'}]$. Right: $[Y_1^{p'}]$ separates $Z_0^0$ and 0. Left: $[Y_1^{p'}]$ does not separate $Z_0^0$ and 0.

More precisely, the sector $Y'$ is bounded by a periodic dynamic ray pair $R^{\phi_1}$ and $R^{\phi_2}$. Pulling back this sector for $p'$ iterations, we obtain a strip within $Y_1'$ that is bounded by two dynamic ray pairs consisting of $R^{\phi_1}$ and $R^{\phi_2}$ respectively of $R^{\phi_1'}$ and $R^{\phi_2'}$. The latter two rays, together with their forward orbits, define the family $F'$ of puzzle pieces associated to $\mathcal{M}'$. In particular, $Z_0^0 \in F'$ is the puzzle piece bounded by the two rays $R^{\phi_1}$ and $R^{\phi_2}$ and the equipotential at height 1. The parapuzzle piece $Y_0^0$ is bounded by the parameter ray pair $R^{\phi_1'}$, $R^{\phi_2'}$ and truncated at height 1. The family $F'$ exists for all $c \in Y_1^0$. The puzzle pieces $Z_j^m$ are maximal conformal pull-backs of $Z_0^0$, and if $Z_j^m \subset Y_1^0$, then $Z_j^m$ exists.

Without loss of generality we can assume that all decorations $L_i$ are inside $Y_1^0$. From now on we will only consider parameters $c \in \text{int} Y_1^0$ (observe that $\text{int} Y_1^0 \subset Y_1^0$). The next proposition is “equivalent” to $\mathcal{M}' \subset \mathcal{M}_s$.

**Proposition 3.8.** For every $k' < p'$ there exists a $k < p$ such that

$$f_c^k \left( [Y_1^{p'}] \right) \subset \text{int} f_c^k \left( [Y_1^p] \right).$$

**Proof.** Let $x_1$ and $x_2$ be the two landing points of the two ray pairs bounding $[Y_1^{p'}]$. These belong to the “small Julia set” associated with $\mathcal{M}_s$ (for the parameter $c_0$), hence $\{x_1, x_2\} \subset Y_1^p$.

Then for any $k'$ there exists a $k \in \{0, 1, \ldots, p - 1\}$ (in fact, $k \equiv k' \pmod{p}$) such that $f_c^k \left( \{x_1, x_2\} \right) \subset \text{int} f_c^k \left( [Y_1^p] \right)$.

We need to show that $f_c^{k'} \left( [Y_1^{p'}] \right)$ does not intersect $f_c^k \left( \partial [Y_1^p] \right)$ if $k' < p'$. Recall that we have the following proper surjective maps (of

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$^2$As before, $Y_1^0$ is not in $F'$.
degree 2):

\[ f_p : [Y_1^p] \to Y_1, \]
\[ f'_p : [Y_1'^p] \to Y_1', \]

where \( Y_1' \subset \text{int} \, Y_1 \). Therefore if the above statement is not true, then \( Y_1' \) would have non trivial intersection with \( \partial Y_1 \).

We will refer to decorations associated to \( \mathcal{M}_s \) as primary decorations, and to those associated to \( \mathcal{M}_s' \) as secondary decorations (both in dynamical and in parameter spaces), and similarly for puzzle pieces. In addition, we need another collection of puzzle pieces associated with \( \mathcal{M}_s' \) that will be denoted \( \hat{Z}_i^s \).

**Definition and Proposition 3.9** (The Fundamental Annulus). For every \( c \in \mathcal{Y}_0^s \), there exists a puzzle piece \( \hat{Z}_0^s \in F' \) of depth 0 the interior of which contains \( Z_0^0 \). For \( k' < p' \) we have \( \text{int} \, \hat{Z}_0^s \cap f'_c (\{Y_1'^p\}) = \emptyset \). The boundary of the annulus \( \hat{Z}_0^s \setminus Z_0^0 \) depends holomorphically on \( c \).

Note that \( \hat{Z}_0^s \) is inside \( Y_1 \) but may or may not be inside \( Y_1' \). In fact, \( \hat{Z}_0^s \) is inside \( Y_1' \) if and only if \( \hat{Z}_0^s = Z_0^0 \) (see Figure 3).

**Proof.** We need to show that for any puzzle piece \( X' \in F' \) of depth 0, we have \( \partial X' \cap Z_0^0 = \emptyset \). But any \( X' \in F' \) at depth 0 is truncated by the equipotential of greater height than \( Z_0^0 \) (height 1 for \( F' \) and height 1/2 for \( F \)). Therefore it is enough to verify that if a ray \( R' \) bounds \( X' \), then \( R' \) does not intersect \( Z_0^0 \). Note that \( R' \) belongs to \( \partial f'^k_c (\{Y_1'^p\}) \) for some \( k' < p' \), where \( f'^k_c (\{Y_1'^p\}) \subset \text{int} \, f'^k_c (\{Y_1^p\}) \) for some \( k < p' \) (see Proposition 3.8). Hence \( R' \) does not intersect \( Z_0^0 \) (see Property 3.4); this proves the existence of \( \hat{Z}_0^s \). Holomorphic dependence of the boundary is clear as long as all pieces exist, which is the case for all \( c \in \mathcal{Y}_0^s \).

For \( k' < p' \) the sets \( f'^k_c (\{Y_1'^p\}) \) are untruncated puzzle pieces in \( F' \) of depth 0 by construction, and so is \( \hat{Z}_0^s \). But \( f'^k_c (\{Y_1'^p\}) \) do not intersect \( Z_0^0 \) as we just showed, so these puzzle pieces have disjoint interiors.

By definition, any \( Z_i^s \) is a conformal pullback of \( Z_0^0 \) under a branch of \( (f^n_c)^{-1} \). It may or may not be possible to conformally pull back the larger domain \( \hat{Z}_0^s \) along the same orbit (i.e., choosing the same preimage branches). If it is, we denote the puzzle piece thus obtained by \( \hat{Z}_i^s \supset Z_i^s \). It is clear that \( \hat{Z}_i^s \in F' \).

The following proposition illustrates the relation of puzzle and parapuzzle pieces associated with \( \mathcal{M}_s \) and \( \mathcal{M}_s' \):

**Proposition 3.10.** For any \( Z_i^s \) and any \( c \in \text{int} \, \mathcal{Y}_1^0 \), the following properties hold:
(1) if \( Z_i^n \) and \( \text{int} Y^{p'}_1 \) intersect, then \( Z_i^n \subset \text{int} Y^{p'}_1 \);
(2) if \( Z_i^n \) and \( Z_j^{m'} \) intersect with \( m \leq n \), then \( Z_i^n \subset \text{int} Z_j^{m'} \); in addition, if \( Z_i^n \) exists, then \( Z_i^n \subset Z_j^{m'} \);
(3) if \( Z_i^n \) and \( Z_j^{m'} \) intersect within \( Y^{p'}_1 \), then \( m \leq n \) and \( Z_i^n \subset \text{int} Z_j^{m'} \);
(4) if \( c = c_0 \), then \( \hat{Z}_i^n \) exists and is contained in some \( Z_j^{m'} \) with \( m \leq n \).

Proof. The set \( Z_0^n \) is contained in a puzzle piece \( \hat{Z}_i^n \) of \( F' \) of depth 0 (Proposition 3.9), so by induction \( Z_j^n \) is contained in a puzzle piece (say \( X^m \)) of \( F' \) of depth \( n \). Hence if it intersects a puzzle piece \( Z_j^{m'} \) of depth \( m \leq n \), then it is contained in that piece. If \( \hat{Z}_i^n \) exists, then it equals \( X^m \) by construction. This proves (2).

Similarly, if \( Z_i^n \) and \( \text{int} Y^{p'}_1 \) intersect, then \( Z_i^n \subset \text{int} X^m \subset \text{int} Y^{p'}_1 \); this is (1).

Let us now show that if \( Z_i^n \) and \( Z_j^{m'} \) intersect within \( Y^{p'}_1 \), then \( m/p' \) is the “escaping time” for \( Z_i^n \) under \( f_{c_0}' : [Y^{p'}_1] \to Y_1^p \), i.e., \( m/p' \) is the least iterate so \( (f_{c_0}')^{m/p'}(Z_i^n) \nsubseteq [Y^{p'}_1] \); this will imply that \( m \leq n \).

Observe that, up to truncation, the two domains \( Y_1^{p'} \), \( Z_0^n \) are puzzle pieces in \( F' \) of depth 0. By Proposition 3.9 we have \( n \geq p' \), so the map \( f_{c_0}' \) sends \( X^m \) to a puzzle piece (say \( X^{m-p'} \)) in \( Y_1^p \). Either \( X^{m-p'} \) is a subset of \( Z_0^n \) (in this case \( m = p' \)) or \( X^{m-p'} \) is a subset of \( Y_1^{p'} \); in the second case we use induction. This proves (3) (using (2)).

Assume that \( c = c_0 \) (so that \( f_{c_0} \) is renormalizable with respect to \( \mathcal{M}_s \)); thus all \( Z_j^{m'} \) exist. Therefore if \( Z_i^n \) is in \( Y^{p'}_1 \), then \( Z_i^n \) is in some \( Z_j^{m'} \), where \( c_0 \notin Z_j^{m'} \). By induction on \( n \) it is easy to prove that \( \hat{Z}_i^n \) exists and has conformal pullbacks.

If \( c = c_0 \), then for every \( Z_i^n \subset Y_1^{0} \) there is a \( Z_j^{m} \) so that \( Z_i^n \subset \hat{Z}_i^n \subset Z_j^{m} \) (the depth \( m \) is unique because of the condition that \( Z_j^{m} \) be maximal). Let us fix \( Z_i^n \subset Y_1^{0} \) and the corresponding \( Z_j^{m} \).

We will distinguish the following two cases (depending on \( c_0 \)): the decoration \( Z_i^n \) is

- **simple:** if \( f_{c_0}^k \left( [\hat{Z}_i^n] \right) \nsubseteq [Z_j^{m}] \) for all \( k \in \{1, 2, \ldots, n\} \);
- **unsimple:** otherwise; i.e., if there is a \( k \in \{1, 2, \ldots, n\} \) with \( f_{c_0}^k \left( [\hat{Z}_i^n] \right) \subseteq [Z_j^{m}] \).

In the simple case, the proof will be based on the fact that \( \hat{Z}_i^n \) exists, which will provide a fundamental annulus of uniform modulus. In the unsimple case, \( Z_i^n \) is surrounded by an annulus that is a conformal pullback of an annulus in Lemma 4.4.
4. The Proof

4.1. The Simple Case. The purpose of this section is to prove the following statement:

**Proposition 4.1** (Fundamental Annulus). Let $Z^n_i$ be puzzle piece of a simple decoration. Then $\tilde{Z}^n_i \subset Z^m_j \subset \text{int} \, \mathcal{Y}_{1}^0$ exists, the annulus $\tilde{Z}^n_i \setminus Z^n_i$ exists and moves holomorphically (with the base point $c_0$) in some open Jordan disc $\mathcal{J} \subset \mathcal{Y}_{1}^0$ containing $Z^m_j$, and the quasiconformal dilatation of the holomorphic motion for any parameter $c \in \mathcal{J}$ is bounded in terms of the conformal distance of $c$ to the boundary of $\mathcal{Y}_{1}^0$.

The main point of this proposition is that the quasiconformal dilatation of the motion is bounded in terms of the distance to the boundary of $\mathcal{Y}_{1}^0$ (not $\mathcal{J}$).

**Proof.** From the construction we see that the boundary of the annulus $\tilde{Z}^n_0 \setminus Z^0_0$ moves holomorphically whenever $c \in \text{int} \, \mathcal{Y}_{1}^0$ (Proposition 3.9). By the $\lambda$-lemma we can extend this motion to a holomorphic motion $h_c$ of the closure of the annulus itself, and so that the quasiconformal dilatation is bounded above in terms of the distance to the boundary of $\mathcal{Y}_{1}^0$. Whenever $\tilde{c} \not\in f^k_c (\tilde{Z}^n_i)$ for all $k = 1, \ldots, n$, then $\tilde{c}$ has a neighborhood so that for all $c$ from this neighborhood we can pull back the holomorphic motion $h_c$ along the orbit $\tilde{Z}^n_i, f_c (\tilde{Z}^n_i), \ldots, f^m_c (\tilde{Z}^n_i) = \tilde{Z}^n_0$ by the formula $h^n_c = f^{−n} \circ h_c \circ f^n_c$. Such a pull-back does not change the quasiconformal dilatation.

Define $T := \{ k \in \{1, \ldots, n\} : f^k_{c_0} (\tilde{Z}^n_i) \subset \mathcal{Y}_{1}^0 \}$. For each $k \in T$, there exist $m_k \leq n - k$ and $j_k$ so that $f^k_{c_0} (\tilde{Z}^n_i) = \tilde{Z}^{n-k}_{i'} \subset Z^{m_k}_{j_k}$ for some $i'$ (Proposition 3.10 (4)).

Recall from Lemma 3.7 that all parapuzzle pieces $Z^n_i$ exist and, similarly, all $Z^m_j$ exist and are contained in $\text{int} \, \mathcal{Y}_{1}^0$. Define

$$\mathcal{J} := \text{int} \, \mathcal{Y}_{1}^0 \setminus \bigcup_{k \in T} \tilde{Z}^{m_k}_{j_k}.$$

By the definitions of $(m_k, j_k)$ and of simplicity, we have $(m, j) \neq (m_k, j_k)$ for all $k \in T$; therefore, $Z^{m_j} \subset \mathcal{J}$ (because all secondary decoration parapuzzle pieces $Z^{m_j'}$ are disjoint).

Since for the parameter $c_0$, we have $Z^n_i \subset \tilde{Z}^n_i \subset Z^m_j$ by Proposition 3.10 (4), it follows that $Z^n_i \subset Z^m_j \subset \mathcal{J}$.

The set $\mathcal{J}$ has been constructed so that for all parameters $c \in \mathcal{J}$, the dynamical puzzle piece $\tilde{Z}^n_i$ exists and depends holomorphically on $c$ (those parameter puzzle pieces for which problems could occur are exactly the ones that are removed in the definition of $\mathcal{J}$). But $\tilde{Z}^n_i \subset \tilde{Z}^n_0$, so $\tilde{Z}^n_i$ also moves holomorphically, and thus also the annulus $\tilde{Z}^n_i \setminus Z^n_i$. 

Observe that the quasiconformal dilatation was introduced only for the annulus \( \hat{Z}_i^n \setminus Z_i^n \); everywhere else a conformal preimage of this dilatation was used.

By definition, when \( c \in Z_i^n \), then the critical value \( c \) must be inside \( Z_i^n \subset \hat{Z}_i^n \); but since \( Z_j^m \subset J \), there are parameters \( c \in J \) where the critical value is not in \( Z_j^m \supset \hat{Z}_i^n \). Therefore, the dynamic rays that land together and form the boundary of \( \lfloor \hat{Z}_i^n \rfloor \) have counterparts in parameter space at the same angles, and these form the boundary of \([\hat{Z}_i^n]\).

Therefore the parapuzzle piece \( \hat{Z}_i^n \) exists and is compactly contained in \( J \subset Y_0^\alpha \).

Corollary 4.2 (Annuli in Parameter Space, Simple Case). The sets \( \hat{Z}_i^n \setminus Z_i^n \) (for all \( n \) and \( i \) that correspond to simple cases) are non-degenerate annuli within \( Z_j^m \), and their moduli are bounded below in terms of their distance to \( \partial Y_1^\alpha \).

Proof. If \( Z_i^n \) exists and corresponds to a simple decoration, then \( \hat{Z}_i^n \) and \( Z_i^n \) exist by Proposition 4.1. We have \( Z_i^n \subset \hat{Z}_i^n \) by construction, and their boundaries are disjoint. Thus \( \hat{Z}_i^n \setminus Z_i^n \) is a non-degenerate annulus. Its modulus is at least as big as the modulus of \( \hat{Z}_i^n \setminus Z_i^n \) for the parameter \( c_0 \), up to the quasiconformal distortion between the \( \hat{Z}_i^n \setminus Z_i^n \) for the various parameters \( c \) (see [ALS] or [L2]), and this depends only on distance of \( \hat{Z}_i^n \) to the boundary of \( Y_1^\alpha \).

Theorem 4.3 (The Decoration Theorem, Simple Case). Suppose that \( L_\nu \) is a sequence of decorations of \( M_s \) and \( c_0 \) is a limiting parameter of them. If all these decorations are simple (with respect to \( c_0 \)), then diameters of \( L_\nu \) tend to zero.

Proof. Each decoration \( L_\nu \) is contained in a primary parapuzzle piece \( Z_i^n \) (where \( n = n(\nu) \) and \( i = i(\nu) \)), which in turn is contained in a secondary parapuzzle piece \( Z_j^m \), with \( c_0 \notin Z_j^m \). The \( Z_j^m \) must accumulate at \( c_0 \) and thus cannot accumulate at the boundary of the wake of \( M_s \), which is \( Y_1^\alpha \) (\( c_0 \) is infinitely renormalizable and thus separated from \( \partial Y_1^\alpha \) by infinitely many parameter ray pairs, and these ray pairs separate \( Z_j^m \) from the wake boundary for sufficiently large \( m \)).

For sufficiently large \( \nu \), the moduli of the annuli \( \hat{Z}_i^n \setminus Z_i^n \) are bounded uniformly for all \( \nu \) by Corollary 4.2. But \( c_0 \notin Z_j^m \supset \hat{Z}_i^n \), so the diameters of the \( Z_i^n \) must tend to zero.

This concludes the proof in the simple case.

4.2. Constructing More Annuli. In the following lemma \( X \) and \( V \) may be associated to arbitrary dividing periodic cycles:
Lemma 4.4. In the dynamical plane of $z^2 + c_0$ there are two puzzle pieces $X$ and $V$ with the following properties:

- $c_0 \in V \subset X \subset Y^0_1$;
- any iterated preimage of $X$ is either inside $Z^0_0$ or has non empty intersection with it;
- the parapuzzle pieces $X$ and $V$ corresponding to $X$ and $V$ exist;
- there exists an $\varepsilon_0 = \varepsilon_0(X,V) > 0$ such that any iterated preimage $V^n$ of $V$ which is inside $X$ satisfies:
  \[ \text{mod} \left( X \setminus V^n \right) \geq \varepsilon_0. \]

Proof. Since $c_0$ is infinitely renormalizable, there are infinitely many nested renormalization domains, and these are contained in $[Y^0_{1\beta}]$ provided the level $N$ of the renormalization is sufficiently large.

We shall prove that if $X$ and $V$ are two renormalization domains around $c_0$ of levels $N$ and $N+2$ for sufficiently large $N$ and truncated at sufficiently small heights, then all four properties are satisfied. Both domains are bounded by two pairs of dynamic rays and one equipotential; the landing points of these ray pairs will be called the vertices of $\partial X$ or $\partial V$. One of the vertices will be a periodic point, the other one preperiodic on the same orbit.

Then the first condition is satisfied, and the second follows because $Z^0_0$ is bounded by a dynamic ray pair outside of the secondary renormalization domain, as well as a fixed equipotential (see Property 2.3 for illustration).

The third claim also follows by standard results.

The last claim is similar to \cite{L1} Lemma 4.5. We will give a sketch of the argument.

Let $V^n$ be an iterated preimage of $V$ with $V^n \subset X$. For every $\eta > 0$ there is a $\delta(\eta) > 0$ so that if $V^n$ has distance at least $\eta$ to both vertices of $\partial X$, then $V^n$ must have distance at least $\delta(\eta)$ from $\partial X$ (see Figure 4 and its caption). This implies that $\text{mod}(X \setminus V^n) \geq \varepsilon_1(X,V) > 0$.

The only case left is when $V^n$ is very close to one of the two vertices of $\partial X$. The small Julia set corresponding to $X$ has two fixed points; we call them $\alpha$ and $\beta$ (in analogy to standard notation) so that $\alpha$ is the dividing fixed point of the small Julia set. The non-dividing fixed point $\beta'$ is the periodic vertices of $X$; denote the non-periodic vertex by $\beta''$. Let $q$ be the period of renormalization of $X$. We may assume that $V$ is very close to $\beta$ (possibly by replacing $V$ with $f^q(V)$).

Denote by $\alpha' \in X$ the non-periodic preimage of $\alpha$ under $f^q_{c_0} : X \to f^q_{c_0} (X)$. Let $R_1$ and $R_2$ be the two rays that land at $\alpha'$ that separate $\beta$ from all other rays landing at $\alpha'$ (if any). These rays have the following two properties:

- the rays $R_1$, $R_2$ separate $\{\beta\}$ from $\{\beta', \alpha, c_0\}$;
- $f^q_{c_0}(R_1)$, $f^q_{c_0}(R_2)$ land at $\alpha$ and separate $\{\beta, \alpha'\}$ from $\{\beta', c_0\}$.
There are two vertices in \( \partial X \); let \( D \) and \( D' \) be \( \eta \)-disks around these two vertices. All rays sufficiently close to the boundary rays of \( X \) have their entire limit sets within \( D \) or \( D' \) (because the vertices of \( X \) are repelling periodic and preperiodic points and have trivial fibers). Therefore, these rays, together with equipotentials close to those on the boundary of \( X \), cover a definite neighborhood of \( \partial X \). Right: The construction of the strip \( S \).

Let \( S \subset X \) be the strip bounded by the two ray pairs \( R_1 \) and \( R_2 \) as well as \( f_{c_0}^q(R_1) \) and \( f_{c_0}^q(R_2) \). Then there exists a \( k > 0 \) so that \( f_{c_0}^{qk}(V^n) \subset S \); if we assume \( k \) to be minimal with that property, then \( f_{c_0}^{ql}(V^n) \) is contained in the same component of \( X \setminus S \) as \( \beta \) for all \( l = 0, 1, \ldots, k - 1 \). (Note that the only part of this construction that depends on \( n \) is \( k \).)

Let \( S' \) be \( S \) truncated at some equipotential, say at the same height as \( V \); we have \( f_{c_0}^{qk}(V^n) \subset S' \). There is an annulus \( A \subset X \) for which \( S' \) is the bounded complementary component, and \( c_0 \) is contained in the unbounded complementary component; it can be chosen so that \( A \) has a conformal preimage under \( f_{c_0}^q \) that is separated from \( c_0 \) by the ray pair \( f_{c_0}^q(R_1), f_{c_0}^q(R_2) \). Note again that \( A \) does not depend on \( n \).

Pulling back this annulus under \( f_{c_0}^{qk} \) along the orbit of \( V^n \), we obtain the annulus around \( V^n \) with the same modulus.

4.3. The Unsimple Case. Let us fix \( X, V, \) and \( \mathcal{V} \) as in Lemma 4.4. By \( X' \) we denote the pullback of \( X \) under \( f_{c_0} \), so that \( f_{c_0} : X' \to X \) is two to one. We will work in \( X' \) so that the critical value is not in the way of further pull-backs.

Let \( V^k \subset X' \) be a maximal pullback of \( V \). Then \( X' \setminus V^k \) is an annulus, and its boundary moves holomorphically whenever \( c \in \text{int } \mathcal{V} \). By the \( \lambda \)-lemma we have a holomorphic motion \( h_c^k \) of the annulus \( X' \setminus V^k \) with the quasiconformal dilatation depending on the distance of \( c \) to \( \partial \mathcal{V} \).
Proposition 4.5 (Parameter Annuli in Unsimple Case). For the parameter $c_0$, let $Z^n_1$ be a puzzle piece corresponding to an unsimple decoration and let $Z^n_j$ be the secondary puzzle piece with $Z^n_j \supset Z^n_1$. Let $V \in X$ be as in Lemma 4.4. If $Z^n_j \in V$, then $Z^n_j \in V$, and there are

- an open Jordan disk $J$ containing $Z^n_j$ and $c_0$
- a domain $V^k$
- and, for every $c \in J$, an annulus $A^n_c$ the boundary of which depends holomorphically on $c$

so that $A^n_c \subset Z^n_j$ and $Z^n_j$ is contained in the bounded complementary component of $A^n_c$ whenever $Z^n_j$ exists, and $f_{c}^{m+n'}: A^n_c \to X' \setminus V^k$ is a conformal isomorphism for some $n' \geq 0$.

For all $c \in J$, there is a holomorphic motion from the closed annulus $X' \setminus V^k(c_0)$ to the closed annulus $X' \setminus V^k(c)$, and its dilatation is bounded in terms of the conformal distance from $c$ to $\partial V$. This holomorphic motion can be pulled back conformally, with the same dilatation, to a holomorphic motion from $A^n_{c_0}$ to $A^n_c$.

For all $c \in J$, the annuli $A^n_c$ are bounded by dynamic rays at the same angles, and by equipotentials at equal heights. The corresponding parameter rays at the same angles, and equipotentials at the same heights, bound an annulus $A^n$ in parameter space. Its modulus is bounded below by the distance of $A^n$ to $\partial V$.

Proof. Let us consider

$$J = \text{int} V \setminus \bigcup_{q<m} \bigcup_{t} [Z^q_t] .$$

It is clear that $Z^n_j \subset J$ (because all $Z^n_j$ are maximal) and $J$ is a Jordan disc. We have $Z^n_j \subset V \subset X$ for $c_0$ and thus for $c \in J$ by construction, hence $Z^n_j \in V$.

We have $f_{c_0}^s(Z^n_1) \subset Z^{m-s}_j$ for $s \leq m$. By maximality of $Z^n_j$, we have $f_{c_0}^s(Z^n_j) \cap Z^n_j = \emptyset$ for $s \leq m$; but by definition of “unsimple”, there is an $n'' \leq n - m$ such that

$$f_{c_0}^{m+n''}(Z^n_1) \subset Z^n_j \subset V \subset X .$$

Therefore, $f_{c_0}^{m+n''-1}(Z^n_1) \subset X'$; let $n'$ be minimal so that $f_{c_0}^{m+n'}(Z^n_1)$ has non-empty intersection with $\text{int} X'$. Hence there exists a maximal pull-back $V^k$ of $V$ so that

$$f_{c_0}^{m+n'}(Z^n_1) \subset V^k \subset X'$$

(in fact, $k \leq n'' - n' + 1$: the pull-back $V^{n''-n'+1}$ always satisfies (4), and the maximal pull-back may have smaller value of $k$).

We will now construct open annuli $A^n_c$ for all $c \in J$ so that $f_{c}^{m+n'}: A^n_c \to (\text{int} X') \setminus V^k$ is a conformal isomorphism. We will describe the construction for $c_0$ explicitly, but the rays and equipotentials that define these annuli exist for all $c \in J$. 
Figure 5. Construction of the annulus $A^n_{c_0}$.

For the parameter $c_0$, there exists a domain $X'' \supset f_{c_0}^m(Z^n_i)$ so that $f_{c_0}^m: X'' \to X'$ is a conformal isomorphism, by minimality of $n'$. This domain is bounded by certain dynamic rays and equipotentials, and an analogous domain $X''$ thus exists for all $c \in \text{int } \mathcal{X}$. Similarly, a domain $V'' \subset X''$ with $f_{c_0}^m: V'' \to V^k$ exists for all $c \in \text{int } \mathcal{V}$ (because $V^k$ exists for $c \in \mathcal{V}$). (Note that for some $c \in \text{int } \mathcal{V}$, the puzzle piece $Z^n_i$ may not exist; but if it does, then $f_{c_0}^m(Z^n_i) \subset V''$ because this is so for $c_0$, by (4)).

Now we have annuli $\text{int}(X'') \setminus V''$, and we want to pull them back $m$ more iterations. This will work for all $c \in J$. Indeed, for these $c$, the set $Z^n_j$ exists, and $f_{c_0}^m(Z^n_j) = Z^n_0$. For the parameter $c_0$, the puzzle piece $Z^n_0 = f_{c_0}(Z^n_j) \supset f_{c_0}^m(Z^n_i)$ intersects $X''$. The way $X$ was constructed, it follows that $X'' \subset Z^n_0$ (this is the second condition in Lemma 4.4). For all $c \in \text{int } \mathcal{X}$, the combinatorics of the boundaries of $X''$ and of $Z^n_0$ are the same, so these properties remain true for all $c \in \text{int } \mathcal{X}$. For every $c \in J$, we have a conformal isomorphism $f_{c_0}^m: Z^n_j \to Z^n_0$, and this yields an open annulus $A^n_c \subset Z^n_j$ so that $f_{c_0}^{m+n'}: A^n_c \to \text{int}(X') \setminus V^k$ is a conformal isomorphism.

We have $f_{c_0}^{m+n'}(Z^n_i) \subset V^k$, so $Z^n_i$ is contained in the bounded complementary component of $A^n_{c_0}$. This property persists for all parameters $c \in J$ for which $Z^n_i$ exists.

The outer boundary of $X'$ consists of pieces of eight dynamic rays and four equipotentials, and the same is true for the inner boundary, which is $\partial V^k$. The boundary thus depends holomorphically on $c$. As before, by the $\lambda$-lemma this yields a holomorphic motion from $X'(c_0) \setminus V^k(c_0)$ to $X'(c) \setminus V^k(c)$ the dilatation of which is bounded above by the distance of $c$ to $\partial \mathcal{V}$. Since all pull-backs were conformal, we obtain a holomorphic motion from $A^n_{c_0}$ to $A^n_c$ the dilatation of which is bounded again by the conformal distance of $c$ to $\partial \mathcal{V}$. Note that this is independent of $m$ and thus of $n$ (even though $J$ depends on $m$).
Recall from Lemma 4.4 that the modulus of $X' \setminus V^k$, and thus of $A^n_{c_0}$, is bounded below by some constant $\varepsilon_0/2$ that depends only on $X$ and $V$, and thus on $c_0$ alone but not on $n$, $m$, or $k$.

As before, there is thus an annulus $A^n := \{ c \in \mathbb{C} : c \in A^n_c \}$ in parameter space. The modulus of $A^n$ depends on $c_0$ and on the conformal distance from $A^n$ to $\partial V$.

This concludes the proof. □

**Corollary 4.6.** For large $n$, the moduli of $Z^m_j \setminus Z^n_i$ are bounded below by a constant that depends only on $c_0$.

More precisely, if $Z^m_j \subset V$, then $Z^m_j \setminus Z^n_i \supset A^n$, and hence the modulus of the annulus $Z^m_j \setminus Z^n_i$ is bounded below in terms of its conformal distance to $\partial V$.

**Proof.** In Proposition 4.5, we proved that $A^n_c \subset Z^m_j$ and that $Z^n_i$ is contained in the bounded complementary component of $A^n_c$. Therefore $A^n \subset Z^m_j$, and $Z^n_i$ is contained in the bounded complementary component of $A^n$. Therefore $\text{mod}(Z^m_j \setminus Z^n_i) \geq \text{mod}A^n$. As $n$ tends to $\infty$, this conformal distance is bounded below, so that all $A^n$ have their moduli bounded below. □
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