ASPECT RATIO AND SLOPE OF ALGEBRAIC RECTANGLES INSCRIBED IN LINES OVER FIELDS

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Abstract. Let \( k \) be a field. By an algebraic rectangle in \( k^2 \) we mean four points in \( k^2 \) subject to certain conditions that in the case where \( k \) is the field of real numbers yield four vertices of a rectangle. We study algebraic rectangles inscribed in lines in \( k^2 \) by parametrizing these rectangles in two ways, one involving slope and the other aspect ratio. This produces two paths, one that finds rectangles with specified slope and the other rectangles with specified aspect ratio. We describe the geometry of these paths and its dependence on the choice of four lines.

1. Introduction

Given four lines in the real plane that are not all parallel, there is a rectangle whose vertices lie on these lines; i.e., the rectangle is inscribed on the lines. There are always more inscribed rectangles nearby and all of these rectangles appear as part of a flow of inscribed rectangles through the configuration of lines. The purpose of this article is to account for all these rectangles by describing the paths they take through the configuration, as well as locating them by their slopes and aspect ratios. Our methods are algebro-geometric, elementary and work over an arbitrary field. While it is possible to prove some of the results of the paper computationally, the equations in raw form are often unwieldy and opaque, so we have sought to minimize the computational approach and instead give algebraic insight into how the geometry of the solution depends on the initial four lines.

The present article continues our study but is mostly independent of this previous work, where the problem of finding inscribed rectangles was recast as that of finding the intersection of cones in \( \mathbb{R}^3 \), namely the “hyperbolically rotated” cones. We are motivated by Schwartz’s recent work on rectangles inscribed in lines, in which the rectangle inscription problem is treated in the case in which none of the lines involved are parallel or perpendicular to each other. Schwartz’s approach is a mix of computational and topological methods based on the geometry of the real plane, and so while his methods don’t extend to arbitrary fields (or because they don’t extend to fields), the difference of his methods when contrasted with ours shows some of the richness involved in locating rectangles inscribed in lines, as well as indicating why the problem is a bit of an orphan when trying to seat it among traditional research areas.

The indirect motivation for both Schwartz’s work and ours is the so-called square peg problem—a problem that remains open in full generality—of finding a square inscribed in

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The outlier case in which all four lines are parallel is described in Remark 8.7.
By a theorem of Vaughn [3, p. 71], every simple closed curve in the plane has a rectangle inscribed it. In particular, every polygon has a rectangle inscribed in it (and in fact has a square inscribed in it [1]). While Vaughn’s proof guarantees that an inscribed rectangle must exist, it does not give a means for finding the rectangle, even when the curve in question is a polygon. To find a rectangle inscribed in a polygon amounts to finding a rectangle inscribed on segments of four lines, which brings us to the current problem of finding all rectangles inscribed in four lines, as well as locating these rectangles by their slopes and aspect ratios.

For the purpose of giving intuition for the results that follow, we briefly describe the behavior of the rectangles inscribed in four lines \(A,B,C,D\). Figure 1 shows two sets of rectangles inscribed on four lines in \(\mathbb{R}^2\). These examples suggest a “flow” of rectangles, and it is with the description of this flow that the paper is concerned. We do not assume anything in general about the four lines in the configuration other than that all four lines are not parallel to each other and do not all go through the same point, and that none of the four lines are parallel to the \(y\)-axis (which for a formally real field \(\mathbb{K}\) can always be arranged by rotation). As discussed in Section 2, the search for inscribed rectangles can then be reduced to the search for rectangles whose vertices lie in sequence on the lines \(A,B,C,D\), where \(C\) and \(D\) meet each other in exactly one point. We can view the \(x\)-coordinates of the first and second vertices (those vertices on \(A\) and \(B\)) as points in \(\mathbb{K}^2\). Better, we can view them as points in the projective closure \(\overline{\mathbb{K}^2}\) of \(\mathbb{K}^2\) and use a curve in the projective plane to track the path of the rectangles. While this is once removed from the configuration of four lines and the actual rectangles inscribed in them, it is easy to map this inscription data into the real plane and inscribe the rectangles in the original configuration of four lines. Thus we shift focus to the projective plane and work out the geometry of the inscribed rectangles there.
The points in the projective plane that represent the inscribed rectangles comprise a conic in $\mathbb{k}^2$ that we call the inscription curve for the configuration. The inscription curve is the union of two paths: a slope path given by a regular map $\mathbb{P}^1(\mathbb{k}) \to \mathbb{P}^2(\mathbb{k})$ that sends a slope represented as a point on the projective line $\mathbb{P}^1(\mathbb{k})$ to a point representing a rectangle having this slope, and an aspect path given by a regular map $\mathbb{P}^1(\mathbb{k}) \to \mathbb{P}^2(\mathbb{k})$ that sends an aspect “ratio” in $\mathbb{P}^1(\mathbb{k})$ to a rectangle having this aspect ratio. The slope and aspect paths solve the problem of finding the rectangles of specified slope and aspect ratio. In Sections 5 and 6 we give succinct versions of the defining polynomials for these paths in order to exhibit some of the internal symmetry of the algebra of the paths; the formulations of these polynomials belong to our main results. The two paths share a number of formal features that suggest more could be learned about the relationship between them.

In Section 7 we show that the slope and aspect paths either (a) have exactly the same image in $\mathbb{P}^2(\mathbb{k})$, and thus find the same rectangles, or (b) they are distinct lines that find different sets of rectangles. Case (b) occurs precisely if the diagonals of the configuration, when not undefined, are orthogonal (Theorem 7.1). In case (a), the affine piece of the inscription curve in $\mathbb{k}^2$ is a non-degenerate conic (Theorem 7.1), which if $\mathbb{k} = \mathbb{R}$ is a hyperbola (Corollary 7.9). Otherwise, in case (b) each path is a line, so that the two lines form a degenerate conic. (It can happen that one of these lines is the line at infinity.) Case (b) is illustrated by the examples in Figure 2.

![Figure 2](image)

**Figure 2.** A degenerate configuration. The dotted line in the left figure is the slope side locus (see Section 8) while the dotted line in the right figure is the aspect side locus (also in Section 8). Along the slope side locus, aspect ratio remains constant while slope changes, and along the aspect side locus, slope remains constant while aspect ratio changes. This phenomenon only happens in the degenerate case. (Compare to Figure 1.)

The slope and aspect paths reside in the space of parallelograms inscribed in the configuration, where each such parallelogram is represented by a point in the affine plane $\mathbb{k}^2$. In Section 8 an affine transformation is used to map these paths onto the rectangle side locus.
for the configuration; this locus consists of the midpoints of the sides of the rectangles that join the lines \( A \) and \( B \) in the configuration. If some of the lines in the transformation are parallel, this affine transformation may not be invertible, in which case some collapsing of information occurs. But in any case, the side locus tracks the rectangles through the configuration in a more direct way than the slope and aspect paths, which exist in a parameter space. In Section 8 we relate this locus to the rectangle loci studied in [4] and [5].

A final word on generality: Even though our main motivation is the case in which \( k \) is the field of real numbers, we work with an arbitrary field for two reasons. First, doing so comes at no extra expense since our arguments are algebro-geometric and need very little modification to be cast in the general setting of fields. So while we have no specific application in mind for, say, rectangles over finite fields, our approach does apply to such rectangles so it seems worthwhile to note this. Second, working over a field allows us to find rectangles whose vertices are restricted to a subfield of the real numbers. For example, by applying the results of the paper to the field of rational numbers, we find inscribed rectangles in the real plane whose vertices have rational coordinates.

Figures 1, 2, 5, and 6 were created in Maple\textsuperscript{TM}, version 2018.2, using the parameterizations in Corollaries 5.11 and 6.10.

## 2. Inscribed Rectangles

Throughout the paper \( k \) denotes a field. In only a few instances the choice of field matters, and in these cases we put additional hypotheses on \( k \). Otherwise, \( k \) is assumed to be an arbitrary field.

**Definition 2.1.** A configuration \( \mathcal{C} \) of four lines in \( k^2 \) consists of two pairs of lines \( A,C \) and \( B,D \) such that \( A \neq C \) and \( B \neq D \). We assume that none of these lines are vertical, i.e., that none are of the form \( \{(x,y) : y \in k\} \), where \( x \) is fixed in \( k \). Throughout the paper, we represent the lines by the equations

\[
A : y = m_A x + b_A, \quad B : y = m_B x + b_B, \quad C : y = m_C x + b_C, \quad D : y = m_D x + b_D,
\]

where the \( m_A, b_A \), etc., are elements of \( k \). Since we only consider one configuration at a time, we simply write \( \mathcal{C} \) rather than, say, \( \mathcal{C} = (A,C; B,D) \). The lines \( A,C \) are always understood to be a pair, as are the lines \( B,D \). Any two of the four lines in the configuration are allowed to be parallel (with the exception of \( C \) and \( D \) in the standard configuration, defined below).

Since we work over a field, and hence do not have recourse to the notion of a line segment between two points, we define parallelograms and rectangles as sets of four vertices with appropriate properties.

**Definition 2.2.** An (algebraic) parallelogram inscribed in the configuration \( \mathcal{C} \) is a set of points

\[
(x_A, y_A) \in A, \ (x_B, y_B) \in B, \ (x_C, y_C) \in C, \ (x_D, y_D) \in D
\]

such that \( x_A - x_B = x_D - x_C \) and \( y_A - y_B = y_D - y_C \). These points are the vertices of the parallelogram. Equivalently, the set of four points in sequence on \( A,B,C,D \) is a parallelogram if the line through a pair of adjacent points in the sequence (with \( A \) and \( D \) considered adjacent also) is parallel to the line through the other two points. We allow the possibility that two or more of the vertices are the same point; in this case, we say that the parallelogram
is degenerate. Note that if all four vertices are the same point, then the parallelogram is a point through which all four lines $A, B, C, D$ pass, and thus a parallelogram that is a point can only occur when $C$ is a configuration for which all four lines pass through a point.

**Definition 2.3.** An (algebraic) rectangle inscribed in $C$ is a set $R$ of vertices $(x_A, y_A) \in A, (x_B, y_B) \in B, (x_C, y_C) \in C, (x_D, y_D) \in D$ such that $R$ is a parallelogram inscribed in $C$ subject to the condition

$$(x_C - x_B)(x_B - x_A) + (y_C - y_B)(y_B - y_A) = 0.$$  

A rectangle inscribed in $C$ then is a parallelogram whose vertices lie in sequence on the lines in the configuration and satisfy an “orthogonality” condition. If $k$ is the field of real numbers, then this condition says that the line passing through the vertices on lines $A$ and $B$ is perpendicular to the line passing through the vertices on $B$ and $C$. However, interpreting this condition as an orthogonality condition for fields that are not formally real is dubious: If $k$ is the field of complex numbers, then the same line can be “orthogonal” to itself under this definition (e.g., $y = ix$). Thus for fields such as the field of complex numbers, what we are calling algebraic rectangles may not match with other natural notions of rectangles defined using inner products more typical for the choice of such a field. But our primary interest is in formally real fields[\footnote{A field is formally real if $-1$ is not a sum of squares in the field.}] including $\mathbb{R}$ itself, and in these cases, our algebraic rectangles reflect an obvious choice of orthogonality relation.

A parallelogram inscribed in $C$ has an explicit ordering of its vertices that is compatible with the pairings in $\mathcal{C}$. In searching for parallelograms or rectangles inscribed on four lines, there is no loss of generality in the approach via configurations. Suppose we begin with four lines $L_1, L_2, L_3, L_4$. To find all rectangles inscribed on these four lines, we may break this search into cases. For example, if we want the parallelograms having one vertex on $L_1$, the next on $L_2$, the next on $L_3$ and the next on $L_4$, then we set $A = L_1, B = L_2, C = L_3, D = L_4$ and deal with the configuration formed from the two pairs $A, C$ and $B, D$. The set of parallelograms inscribed in this configuration includes those whose vertices are in the desired sequence. Or, if we want the parallelograms that have a vertex on $L_2$, the next two vertices on $L_1$ and a final vertex on $L_4$, then we set $A = L_2, B = L_1, C = L_1, D = L_4$. The configuration that results from the pairs $A, C$ and $B, D$ then has these parallelograms inscribed in it. As discussed in [4, Section 4], finding all rectangles inscribed on four lines requires finding all the rectangles inscribed in 21 configurations involving these four lines. (In some of these configurations, two pairs of lines share a line; these configurations are needed to find the rectangles having two vertices on the same line.)

Our methods for finding inscribed rectangles work best when not all of the four lines in the configuration are parallel. The assumption that not all four lines are parallel is, of course, a very mild restriction, and we leave a discussion of it for Remark [7]. Thus our focus is on configurations $\mathcal{C}$ in which at least two lines are not parallel. In this case, we may always relabel the lines in $\mathcal{C}$ so that if we are seeking rectangles inscribed in $\mathcal{C}$, we can do so under the assumption that $C$ is not parallel to $D$. For example, if $A$ and $D$ are not parallel but $C$ and $D$ are, then by switching the labels for $A$ and $C$ we have that the resulting configuration shares the same inscribed rectangles as that of $\mathcal{C}$.  

\footnote{A field is formally real if $-1$ is not a sum of squares in the field.}
To simplify calculations we also assume that the two lines \( C \) and \( D \) meet at the origin. Since having two non-parallel lines meet at the origin can be accomplished by translation, there is no loss of generality in assuming this. This requirement that \( C \) and \( D \) meet at the origin is part of the notion of a standard configuration, defined next. We also make an additional assumption that the four lines do not go through a single point. The case of four lines through a point is not ruled out because of a wish to restrict to lines in general position (we don’t ever require general position) or because it is somehow obscure. Instead, this case is excluded from our notion of a standard configuration because of its centrality to these configurations, but we leave this idea for a future paper in which we view the case of four lines through a point as the configuration \( C \) viewed from “infinity.”

**Definition 2.4.** The configuration \( C \) is standard if \( C \) and \( D \) are distinct lines that meet at the origin and the four lines in \( C \) do not go through the same point. Thus \( C \) is standard if and only if \( m_C \neq m_D, b_C = b_D = 0 \) and at least one of \( b_A \) and \( b_B \) is nonzero.

**Remark 2.5.** If \( k \) is formally real and \( A, B, C, D \) are four lines in \( k^2 \), then there exists an orthogonal linear transformation of the plane such that the images of \( A, B, C, D \) are not vertical lines. If furthermore \( C \) and \( D \) are not parallel, then there is an invertible affine transformation of \( k^2 \) that results in the images of \( C \) and \( D \) meeting in the origin. Thus if \( k \) is formally real, there is no loss of generality in assuming that \( C \) is standard whenever the four lines in \( C \) are not all parallel and do not all go through the same point.

Despite our focus on rectangles inscribed in \( C \), and hence occurring in the affine plane, it is useful to work in the projective closure of the plane. This becomes essential in later sections when formulating the slope and aspect paths, as well as the inscription curve for the configuration.

**Notation 2.6.** We let \( \overline{k^2} \) denote the projective closure of \( k^2 \):
\[
\overline{k^2} = \{ [x : y : z] : \text{not all of } x, y, z \text{ are zero} \},
\]
where \( [x : y : z] \) is a point in homogeneous coordinates. The line at infinity for \( k^2 \) is
\[
\{ [0 : y : z] : y, z \in k, y, z \text{ not both 0} \}.
\]
As usual, \( k^2 \) can be identified with the points of the form \([1 : y : z] \).

In the projective closure of \( k^2 \), the four lines \( A, B, C, D \) can be viewed as a complete quadrilateral, meaning that we have four lines and six points of intersection, where some of these six points are possibly at infinity. The six points give rise to three “diagonals,” two of which we will need in Section 7. Such a notion is made more complicated by the presence of parallel lines in the configuration, but working with the projective closure removes this obstacle. However, the admittance of non-distinct lines in the configuration (the cases \( A = B, B = C, A = D \)) results in a configuration of three lines, in which case at least one diagonal is undefined.

**Definition 2.7.** Let \( C \) be a standard configuration. We define the diagonals \( E \) and \( F \) of the configuration as follows.

1. If \( A \neq B \), then \( E \) is the line in \( \overline{k^2} \) through the intersection of \( A \) and \( B \), which may be the point at infinity, and that of \( C \) and \( D \). Otherwise, if \( A = B \), then \( E \) is undefined.
(2) If \( A \neq D \) and \( B \neq C \), then \( F \) is the line in \( \overrightarrow{E} \) through the intersection of \( A \) and \( D \) and that of \( B \) and \( C \). (Either or both of these points of intersection may be points at infinity.) Otherwise, if \( A = D \) or \( B = C \), then \( F \) is undefined.

Remark 2.8. Let \( \mathcal{C} \) be a standard configuration.

(1) It cannot happen that both diagonals \( E \) and \( F \) are undefined, or that \( E \) is undefined and \( F \) is line at infinity. This is because \( A \neq C \), \( B \neq D \) and \( C \) is not parallel to \( D \).

(2) If \( A \parallel B \) and \( A \neq B \), then \( E \) is the line through the intersection of \( C \) and \( D \) that is parallel to \( A \) and \( B \); see the second configuration in Figure 4.

(3) If \( A \parallel D \) and \( B \parallel C \), then \( F \) is the line at infinity; see the third configuration in Figure 4. If \( A \) and \( D \) intersect in a single point and \( B \) and \( C \) are parallel and distinct, then \( F \) is the line through this point that is parallel to \( B \) and \( C \). Similarly, if \( B \) and \( C \) intersect in a single point and \( A \) and \( D \) are parallel and distinct, then \( F \) is the line through the point that is parallel to \( A \) and \( D \); see the first configuration in Figure 4.
3. Slope and aspect ratio

In order to treat slope and aspect ratio of inscribed rectangles formally in the cases where these values are infinite, we define “slope points” and “aspect points” of rectangles as points on the projective line \( \mathbb{P}^1(k) := \{ [x : y] : x, y \in k, x \neq 0 \text{ or } y \neq 0 \} \). If \( R \) is a non-degenerate rectangle inscribed in \( A \) and the slope of the line passing through the vertex on \( A \) and the vertex on \( B \) has slope \( m \in k \), then under Definition 3.1 the “slope point” of this rectangle is \([m : 1]\). If this same line instead has infinite slope, then the rectangle has slope point \([1 : 0]\).

The following definition is motivated by the idea that the slope of a rectangle inscribed in \( A \) is the slope of the line through the vertices of the rectangle that lie on \( A \) and \( B \). However, since we permit degenerate rectangles, these two vertices may coincide, and so we define the slope in this case to be the slope of a line that is orthogonal to the line through the vertices that lie on \( B \) and \( C \). This last complication is the reason why two equations rather than one are needed to define the slope of a rectangle.

**Definition 3.1.** A (possibly degenerate) rectangle in \( k^2 \) whose vertices are \((x_A, y_A) \in A\), \((x_B, y_B) \in B\), \((x_C, y_C) \in C\), \((x_D, y_D) \in D\) has slope point \( \alpha = [s : t] \in \mathbb{P}^1(k) \) if \( s, t \) is a solution to the system of equations

\[
\begin{align*}
(x_B - x_A)S - (y_B - y_A)T &= 0 \\
(y_C - y_B)S + (x_C - x_B)T &= 0
\end{align*}
\]

The slope of the rectangle is \( \frac{s}{t} \in k \cup \{\infty\} \), where this slope is \( \infty \) if \( t = 0 \).

**Remark 3.2.** The slope point of the inscription is whichever of \([y_A - y_B : x_A - x_B]\) and \([x_B - x_C : y_C - y_B]\) is defined. When both are defined, these two points are equal since the line through \((x_A, y_A)\) and \((x_B, y_B)\) is orthogonal to the line through \((x_B, y_B)\) and \((x_C, y_C)\), where orthogonality here is as in Definition 3.3.

**Definition 3.3.** Two slope points \( \sigma_1 = [s_1 : t_1], \sigma_2 = [s_2 : t_2] \) are orthogonal if \( s_1s_2 + t_1t_2 = 0 \). A line \( L_1 \) with slope \( m_1 \) is orthogonal to a line \( L_2 \) with slope \( m_2 \) (written \( L_1 \perp L_2 \)) if the slope point \([m_1 : 1]\) of \( L_1 \) is orthogonal to the slope point \([m_2 : 1]\) of \( L_2 \); i.e., \( m_1m_2 = -1 \).

We adopt the convention that the line at infinity for \( k^2 \) is orthogonal to every line in \( k^2 \).

As with slope, the aspect ratio of a rectangle is made more complicated by degenerate rectangles, a case we handle similarly to that of slope through a pair of homogeneous linear equations. As with slope, we represent aspect ratio as a point in \( \mathbb{P}^1(k) \), and for this reason we work with an “aspect point” rather than an aspect ratio.

**Definition 3.4.** A rectangle in \( k^2 \) with coordinates \((x_A, y_A) \in A\), \((x_B, y_B) \in B\), \((x_C, y_C) \in C\), \((x_D, y_D) \in D\) has aspect point \( \alpha = [u : v] \) if \( u, v \) is a solution to the system of equations

\[
\begin{align*}
(x_B - x_C)U - (y_A - y_B)V &= 0 \\
(y_B - y_C)U + (x_A - x_B)V &= 0
\end{align*}
\]

The aspect ratio of the rectangle is \( \frac{u}{v} \in k \cup \{\infty\} \), where this aspect ratio is \( \infty \) if \( v = 0 \).

The aspect ratio of an inscribed rectangle is well defined because the orthogonality condition in Definition 3.3 guarantees that the system \(2\) has a nonzero solution. It is a simple
consequence of the equations in Definition 3.4 that in the case where \( k \) is a real closed field\(^3\), the absolute value of the aspect ratio in Definition 3.4 coincides with the usual definition of aspect ratio:

\[
\left| \frac{u}{v} \right| = \frac{\sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}}{\sqrt{(x_B - x_C)^2 + (y_B - y_C)^2}}
\]

Also, if the two vertices \((x_A, y_A)\) and \((x_B, y_B)\) are the same, the aspect point of the rectangle is \([0 : 1]\); i.e., this rectangle has aspect ratio 0. If instead, \((x_B, y_B) = (x_C, y_C)\), then the aspect point is “infinite”; i.e., it is \([1 : 0]\). The former degenerate rectangle lies on the diagonal \(E\) while the latter lies on \(F\).

4. The inscription curve

The purpose of this section is to highlight the essential data for tracking rectangles inscribed in a configuration \(\mathcal{C}\) of four lines. For example, since by assumption none of the lines in \(\mathcal{C}\) are vertical, simply knowing the \(x\)-coordinates of the four vertices of a rectangle is enough to specify a rectangle. Better, it follows from Theorem 4.6 that under the assumption that \(C\) and \(D\) are not parallel (a natural assumption for us; see Section 2), knowing only the \(x\)-coordinates \(x_A\) and \(x_B\) of the vertices that lie on the lines \(A\) and \(B\) completely determines the rectangle.

Thus tracking all the rectangles amounts to finding the points \((x_A, x_B) \in \mathbb{k}^2\) such that \(x_A\) and \(x_B\) are the \(x\)-coordinates of the vertices on \(A\) and \(B\) of an inscribed rectangle. We show in Theorem 4.6 that the relationship between these \(x\)-coordinates \(x_A\) and \(x_B\) is given by an algebraic curve, a conic that we call the inscription curve for the configuration. It will be convenient for several reasons to view the inscription curve as a curve in the projective plane \(\mathbb{P}^2\) and to thus projectivize the problem by expanding the search for rectangles inscribed in \(\mathcal{C}\) to rectangles inscribed in uniformly scaled versions of the configuration \(\mathcal{C}\) also. The advantage of doing this will become apparent in Section 5 when we parameterize the rectangles using the notion of a slope path. Finding rectangles inscribed in a scaled version of \(\mathcal{C}\) is equivalent to finding rectangles inscribed in \(\mathcal{C}\) (simply scale the rectangle until its vertices lie on \(\mathcal{C}\)), so the ambiguity that homogeneous coordinates introduces is easily resolved because of projective features of the rectangle inscription problem.

More notation is needed to accommodate some of our arguments in projective space because we need flexibility in allowing the configuration to scale up or down. The idea is that a representative of the equivalence class of a rectangle inscription in homogeneous coordinates may not yield a rectangle on the configuration \(\mathcal{C}\) but instead one on a uniformly scaled version of \(\mathcal{C}\).

**Notation 4.1.** Let \(\mathcal{C}\) be a configuration. For each \(r \in \mathbb{k}\), let \(\mathcal{C}(r)\) be the configuration whose four lines are \(L(r) : y = m_L x + b_L r\), where \(L \in \{A, B, C, D\}\).

We first consider parallelograms since this allows us to exploit the simple idea that to find inscribed rectangles, one may locate inscribed parallelograms and then find the rectangles among this collection as those having a pair of adjacent sides that are orthogonal.

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\(^3\)A field is real closed if it is formally real, every polynomial of odd degree has at least one root in \(F\), and every element or its additive inverse has a square root in the field.
Throughout the rest of the article, we use the following notation to streamline our algebraic expressions.

**Notation 4.2.** Let $\mathcal{C}$ be a configuration. Because we frequently need to consider the differences between various instances of the constants $m_A, m_B, m_C, m_D$ as well as $b_A, b_B, b_C, b_D$, we write $m_{AB}$ for $m_A - m_B$, $m_{BC}$ for $m_B - m_C$, etc., and $b_{AB}$ for $b_A - b_B$, etc.

**Lemma 4.3.** Let $\mathcal{C}$ be a configuration such that $C$ is not parallel to $D$. Then points $(x_A, y_A) \in A, (x_B, y_B) \in B, (x_C, y_C) \in C, (x_D, y_D) \in D$ are the vertices of a parallelogram inscribed in $\mathcal{C}$ if and only if

$$x_C = \frac{1}{m_{DC}}(m_{AD}x_A + m_{DB}x_B + b_{AB} - b_{DC}) \quad \text{and} \quad x_D = x_A - x_B + x_C.$$  

**Proof.** Suppose that $(x_A, y_A) \in A, (x_B, y_B) \in B, (x_C, y_C) \in C, (x_D, y_D) \in D$ are the vertices of a parallelogram $P$ inscribed in $\mathcal{C}$. Since $P$ is a parallelogram,

$$x_A - x_B = x_D - x_C \quad \text{and} \quad m_Ax_A - m_Bx_B + b_{AB} = m_Dx_D - m_Cx_C + b_{DC}.$$  

Rewriting, we have

$$\begin{bmatrix} 1 & -1 \\ m_A & -m_B \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + \begin{bmatrix} 0 \\ b_{AB} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ m_D & -m_C \end{bmatrix} \begin{bmatrix} x_D \\ x_C \end{bmatrix} + \begin{bmatrix} 0 \\ b_{DC} \end{bmatrix}.$$  

By assumption, $m_C \neq m_D$ since the lines $C$ and $D$ are not parallel, so

$$\begin{bmatrix} x_D \\ x_C \end{bmatrix} = \frac{1}{m_{DC}} \begin{bmatrix} -m_C & 1 \\ -m_D & 1 \end{bmatrix} \left( \begin{bmatrix} 1 & -1 \\ m_A & -m_B \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + \begin{bmatrix} 0 \\ b_{AB} - b_{DC} \end{bmatrix} \right).$$  

Thus $x_C$ is as in the statement of the theorem. Since $x_A - x_B = x_D - x_C$, we have also that $x_D = x_C + x_A - x_B$.

Conversely, if $x_A, x_B, x_C, x_D \in \mathbb{k}$ and $x_C$ and $x_D$ satisfy the equations in the statement of the theorem, then the above matrix calculations show

$$y_A - y_B = m_Ax_A - m_Bx_B + b_{AB} = m_Dx_D - m_Cx_C + b_{DC} = y_D - y_C.$$  

From this it follows that the four points in the theorem define a parallelogram inscribed in $\mathcal{C}$. \hfill \square

In light of the lemma, every pair of elements in $\mathbb{k}$ defines a parallelogram in $\mathcal{C}$:

**Definition 4.4.** Let $\mathcal{C}$ be a configuration such that $C$ is not parallel to $D$, let $x_A, x_B \in \mathbb{k}$ and let $x_C$ and $x_D$ be defined as in Lemma 4.3. The parallelogram inscribed in $\mathcal{C}$ and defined by $x_A, x_B$ is the parallelogram with vertices of the form

$$(x_L, m_la_L + b_L) \in \mathcal{L}, \text{ where } L \in \{A, B, C, D\}.$$  

We define now the essential data for representing inscribed rectangles.

**Definition 4.5.** Let $\mathcal{C}$ be a configuration such that $C$ is not parallel to $D$. Then a point $p = [x : x_A : x_B] \in \mathbb{P}^2(\mathbb{k})$ is a rectangle inscription for $\mathcal{C}$ if and only if the parallelogram inscribed in $\mathcal{C}(x)$ and defined by $x_A, x_B$ is a rectangle. We say that this rectangle is specified by $p$. If $x = 0$, then $p$ is a rectangle inscription at infinity for $\mathcal{C}$. The slope point of a rectangle
Theorem 4.6. Let $C$ be a configuration such that $C$ is not parallel to $D$, and let $X, X_A, X_B$ be indeterminates for $k$. Define a polynomial $f$ by

$$f(X, X_A, X_B) = \frac{1}{m_{DC}} \cdot (m_{AD}X_A + m_{DB}X_B + (b_{AB} - b_{DC})X).$$

Then the rectangle inscriptions for $C$ are the zeroes in $\mathbb{P}^2(k)$ of the polynomial

$$h(X, X_A, X_B) = (m_BX_B - m_AX_A + b_{BA}X)(m_CY(X, X_B) - m_BX_B + b_{CB}X) + (X_B - X_A)(f(X, X_A, X_B) - X_B).$$

Proof. Let $[x : x_A : x_B] \in \mathbb{P}^2(k)$, and write the vertices for the parallelogram inscribed in $C(x)$ and defined by $x_A, x_B$ as $(x_L, y_L) \in L(x)$, where $L \in \{A, B, C, D\}$. Then $x_C = f(x, x_A, x_B)$, $x_D = x_A - x_B + x_C$ and $y_L = m_Lx_L + b_Lx$, where $L \in \{A, B, C, D\}$. By definition, this parallelogram is a rectangle if and only if $(x_A - x_B)(x_B - x_C) + (y_A - y_B)(y_B - y_C) = 0$. After substituting for $x_C, y_A, y_B, y_C$, this last equation yields $h(x, x_A, x_B) = 0$. The theorem now follows. □

Definition 4.7. Let $C$ be a configuration such that $C$ is not parallel to $D$. The inscription curve for $C$ is the projective conic in $\mathbb{P}^2$ defined by the polynomial $h(X, X_A, X_B)$ in Theorem 4.6, while the affine inscription curve for $C$ is the affine curve in $k^2$ defined by $h(1, X_A, X_B) = 0$.

The zeros of the polynomial $h(1, X_A, X_B) = 0$ completely determine the rectangles inscribed in $C$, as in Figure 5. We will use this fact to parameterize the inscription curve using slope (Section 5) and aspect ratio (Section 6) in order to locate rectangles of specified slope and aspect ratio.

5. The slope path

While the inscription curve locates the rectangles inscribed in a standard configuration $C$, these curves do not directly find rectangles of a specified slope. To remedy this, we develop the notion of a slope path that gives a regular map from $\mathbb{P}^1(k)$ to the inscription curve for $C$ so that an element $\sigma \in \mathbb{P}^1(k)$ is sent to a rectangle inscription with slope point $\sigma$. In this way, we locate for each point in $\mathbb{P}^1(k)$ a rectangle inscription with slope point $\sigma$. We see that at most two of these inscriptions are at infinity, and thus for all but at most two slope points $\sigma$ there is a rectangle inscribed in $C$ with slope point $\sigma$.

In Notation 5.3 we propose candidates for a parameterization of the slope path, and in Theorem 5.9 we prove that these candidates work and give the desired slope path. Much of what follows in this and later sections depends on how these polynomials are written in terms of two polynomials $E(S, T)$ and $F(S, T)$ that encode information about the diagonals $E$ and $F$. Thus it is not so much the existence of the equations in Notation 5.3 that is the issue—raw but unwieldy versions of parameterizing polynomials can be found through
Figure 5. On the left is the inscription curve corresponding to the configuration on the right. A point \((x_A, x_B)\) on the inscription curve determines an inscribed rectangle by specifying the first coordinate \(x_A\) of the rectangle vertex on \(A\) and the first coordinate \(x_B\) of the vertex on \(B\), thus specifying one side of what is guaranteed to be an inscribed rectangle by the fact that \((x_A, x_B)\) lies on the inscription curve. For a standard configuration, the vertices on \(A\) and \(B\) uniquely determine the rectangle.

computational means— but the form of the polynomials and how they encode geometrical information that matters for the ability to give conceptual rather than purely computational proofs of the results in this and later sections. In this sense, Notation 5.6 is one of our main “theorems.”

The parameterization in Notation 5.6 and Theorem 5.9 is built from the polynomials \(\mathcal{E}(S,T)\) and \(\mathcal{F}(S,T)\) given in the following notation. These polynomials depend on the diagonals \(E\) and \(F\), as we show in Lemma 5.3.

Notation 5.1. For a standard configuration \(C\) and indeterminates \(S\) and \(T\) for \(k\), let

\[
\begin{align*}
\mathcal{E}^*(S,T) &= (b_A m_B - b_B m_A) S + b_A B T \\
\mathcal{F}^*(S,T) &= (b_B m_D A + b_A m_{BC}) S + (b_B m_{CDB} + b_B m_{AD} m_C) T \\
\mathcal{G}(S,T) &= \text{first polynomial in the list } 1, \mathcal{E}^*(S,T), \mathcal{F}^*(S,T) \text{ that is the gcd of } \mathcal{E}^*(S,T) \text{ and } \mathcal{F}^*(S,T) \text{ in } k[S,T].
\end{align*}
\]

We say \(\mathcal{G}(S,T)\) is the slope factor of \(\mathcal{C}\), and we define

\[
\begin{align*}
\mathcal{E}(S,T) &= \frac{\mathcal{E}^*(S,T)}{\mathcal{G}(S,T)} \\
\mathcal{F}(S,T) &= \frac{\mathcal{F}^*(S,T)}{\mathcal{G}(S,T)}.
\end{align*}
\]

Remark 5.2. Either \(\mathcal{E}(S,T)\) and \(\mathcal{F}(S,T)\) are both constant polynomials or both are linear polynomials, depending on the degree of \(\mathcal{G}\). The polynomial \(\mathcal{G}(S,T)\) is \(\mathcal{F}^*(S,T)\) only in the case where \(\mathcal{E}^*(S,T) = 0\).

In the following lemma, by the slope point of a line \(L\) we mean the point \([s : t] \in \mathbb{P}^1(k)\) such that the equation for \(L\) is \(sX - tY + b = 0\) for some \(b \in k\). Thus if \(L\) has slope \(m_L \in k\), then \([m_L : 1]\) is the slope point for \(L\).
Lemma 5.3. Let $\mathfrak{C}$ be a standard configuration.

(1) $\mathcal{E}(S, T) = 0$ iff $E$ is undefined. Otherwise, the slope point of $E$ is orthogonal to the zero of $\mathcal{E}^*(S, T)$ in $\mathbb{P}^1(k)$.

(2) $\mathcal{F}(S, T) = 0$ iff $F$ is undefined or $F$ is the line at infinity. Otherwise, the slope point of $F$ is the zero of $\mathcal{F}^*(S, T)$ in $\mathbb{P}^1(k)$.

Proof. (1) It is clear from the relevant definitions that $\mathcal{E}^*(S, T) = 0$ iff $\mathcal{E}(S, T) = 0$. Suppose $\mathcal{E}^*(S, T) = 0$. Then $b_A m_B - b_B m_A = 0$ and $b_A = b_B$. This last condition and the fact that all four lines do not go through the origin imply (since $\mathfrak{C}$ is standard) that $b_A \neq 0$ and $0 = b_A m_B - b_B m_A = b_A m_B$. Thus $m_B = m_A$. This proves that $A = B$ and hence that $E$ is undefined. The converse is clear since if $A = B$, then $m_A = m_B$ and $b_A = b_B$, which implies that $\mathcal{E}^*(S, T) = 0$.

To prove the next assertion, suppose that $E$ is defined. Then $A \neq B$. If $A \parallel B$, then by Remark 2.8, $m_E = m_A = m_B$. Since $A \neq B$, this implies that $b_{AB} \neq 0$. Thus

$$[m_E : 1] = [m_A b_{AB} : b_{AB}] = [m_B b_A - m_A b_B : b_{AB}],$$

which is orthogonal to the zero of $\mathcal{E}^*$. Otherwise, if $A$ is not parallel to $B$, then $A$ and $B$ intersect in the point

$$\left(\frac{b_{BA}}{m_{AB}}, m_A \left(\frac{b_{BA}}{m_{AB}}\right) + b_A\right) = \left(\frac{b_{BA}}{m_{AB}}, \frac{m_A b_B - m_B b_A}{m_{AB}}\right).$$

Thus since $C$ and $D$ intersect at the origin, the slope point for $E$ is $[m_A b_B - m_B b_A : b_{BA}]$, which is orthogonal to the zero of $\mathcal{E}^*$.

(2) It is clear that $\mathcal{F}^*(S, T) = 0$ iff $\mathcal{F}(S, T) = 0$. To prove the first assertion in (1), we show that $\mathcal{F}^*(S, T) = 0$ iff $A = D$, $B = C$, or $A \parallel D$ and $B \parallel C$. Suppose that $\mathcal{F}^*(S, T) = 0$. Then $b_B m_{DA} + b_A m_{BC} = 0$. This along with the fact that the coefficient of $T$ in $\mathcal{F}^*$ is 0 yields

$$0 = b_A m_{CB} m_D + b_B m_{AD} m_C = b_B m_{DA} m_D - b_B m_{DA} m_C = b_B m_{DA} m_{DC}.$$

Since $C$ is not parallel to $D$, we conclude that $b_B = 0$ or $m_A = m_D$. In either case, we have $0 = b_B m_{DA} + b_A m_{BC} = b_A m_{BC}$.

If $b_B = 0$, the assumption that all four lines do not go through the origin implies that $b_A \neq 0$. Thus if $b_B = 0$, we have $m_A = m_D$, so that since $C$ goes through the origin, it must be that $B = C$.

If $b_B \neq 0$, then $m_A = m_D$ so that if $b_A \neq 0$, then $m_B = m_C$ and $A \parallel D$ and $B \parallel C$; otherwise, if $b_A = 0$, then $A = D$. This proves that if $\mathcal{F}(S, T) = 0$, then $F$ is undefined or $F$ is the line at infinity.

The converse is straightforward, so it remains to prove the last assertion. The assumption that $F$ is neither undefined nor the line at infinity implies that $A \neq D$, $B \neq C$, or at most one of the pairs $\{A, D\}$, $\{B, C\}$ consists of parallel lines.

Suppose that $A \parallel D$. Then $m_A = m_D$ and the slope of $F$ is $m_D$. Since $A \neq D$, it must be then that $b_A \neq 0$, and since $B$ is not parallel to $C$, $m_B \neq m_C$. Therefore, the zero of $\mathcal{F}^*$ in $\mathbb{P}^1(k)$ is

$$[b_A m_{CB} m_D + b_B m_{AD} m_C : -(b_B m_{DA} + b_A m_{BC})] = [b_A m_{CB} m_D : -b_A m_{BC}] = [m_D : 1].$$
Thus, if \( A \parallel D \), then the zero of \( F^*(S,T) \) is the slope point of \( D \), which is the slope point of \( F \). A similar argument shows that if \( B \parallel C \), then the zero of \( F^*(S,T) \) is the slope point of \( B \), which in this case is the slope point of \( F \).

Finally, suppose that \( A \) is not parallel to \( D \) and \( B \) is not parallel to \( C \). Since \( A \neq D \) and \( B \neq C \), the pairs \( A,D \) and \( B,C \) each meet in a single point. The two points
\[
\left( \frac{b_A}{m_{DA}}, \frac{m_{DB}b_A}{m_{DA}} \right), \left( \frac{b_B}{m_{CD}}, \frac{m_{CB}b_B}{m_{CB}} \right)
\]
are the points of intersection of \( A \) and \( D \) and \( B \) and \( C \), respectively. The slope point of the line through these two points is
\[
[m_{DA}m_Cb_B - m_{CB}m_Db_A : m_{DA}b_B - m_{CB}b_A],
\]
which is also the zero of \( F^*(S,T) \). This proves the lemma.

We define another polynomial that is needed for the parameterization of the slope path. It amounts to a scaling factor for rectangle inscriptions and, as will be evident in Theorem 5.9, is zero when the rectangle inscription is at infinity.

**Notation 5.4.** Let \( S,T \) be indeterminates for the field \( k \). For a standard configuration \( \mathcal{C} \), we define a polynomial in \( k[S,T] \) by
\[
\mathcal{X}^*(S,T) = (m_Am_C - m_Bm_D)S^2 - \beta ST + (m_Am_C - m_Bm_D)T^2.
\]
where \( \beta = (m_Am_B + 1)m_{CD} + (m_Cm_D + 1)m_{AB} \). With \( G(S,T) \) the slope factor of \( \mathcal{C} \), we let
\[
\mathcal{X}(S,T) = \frac{\mathcal{X}^*(S,T)}{G(S,T)}.
\]

The polynomial \( \mathcal{X}^*(S,T) \) can be identically zero; see Theorem 5.12. That \( \mathcal{X}(S,T) \) is a polynomial in \( k[S,T] \) follows from the next lemma.

**Lemma 5.5.** Let \( \mathcal{C} \) be a standard configuration. Then \( \mathcal{X}(S,T) \) is the unique solution in \( k[S,T] \) to the equations
\[
\begin{align*}
\frac{b_A}{m_{DA}}\mathcal{X}(S,T) &= m_{AD}(S - m_CT) \cdot \mathcal{E} - (T + m_AS) \cdot \mathcal{F} \\
\frac{b_B}{m_{CD}}\mathcal{X}(S,T) &= m_{BC}(S - m_DT) \cdot \mathcal{E} - (T + m_BS) \cdot \mathcal{F}.
\end{align*}
\]

**Proof.** Direct calculation shows that the two equations in the lemma are valid with \( \mathcal{X}, \mathcal{E} \) and \( \mathcal{F} \) replaced by \( \mathcal{X}^*, \mathcal{E}^* \) and \( \mathcal{F}^* \). Since \( \mathcal{C} \) is standard, \( b_A \) and \( b_B \) are not both 0 and so the lemma follows after removing \( G(S,T) \) from the resulting equations.

**Notation 5.6.** For the standard configuration \( \mathcal{C} \), we define the following homogeneous polynomials in \( k[S,T] \).
\[
\begin{align*}
\mathcal{X}_A(S,T) &= (m_CT - S) \cdot \mathcal{E} + S \cdot \mathcal{F} \\
\mathcal{X}_B(S,T) &= (m_DT - S) \cdot \mathcal{E} + S \cdot \mathcal{F} \\
\mathcal{X}_C(S,T) &= (m_CT - S) \cdot \mathcal{E} \\
\mathcal{X}_D(S,T) &= (m_DT - S) \cdot \mathcal{E}.
\end{align*}
\]
We define also \( \mathcal{Y}_L(S,T) = m_L \cdot \mathcal{X}_L + b_L \cdot \mathcal{X} \), where \( L \in \{ A, B, C, D \} \).

These polynomials ultimately give the coordinates of inscribed rectangles after they have been reduced by their gcd and scaled accordingly using \( \mathcal{X}(S,T) \); see Corollary 5.11. The following lemma is useful for calculating the slope and aspect ratio of these rectangles.
Lemma 5.7. For a standard configuration $E$, we have
\[
Y_A - Y_B = m_{CD} S \cdot E, \quad X_A - X_B = m_{CD} T \cdot E, \quad Y_B - Y_C = -T \cdot F, \quad X_B - X_C = S \cdot F
\]

Proof. The lemma is verified by direct calculation. For the calculation of $Y_A - Y_B$ and $Y_B - Y_C$, use the expression for $X$ in Lemma 5.5. □

Lemma 5.8. If $E$ is a standard configuration, then the polynomials $X, X_A, X_B$ are relatively prime in $k[S,T]$.

Proof. Let $G(S, T)$ be a gcd of $X(S,T), X_A(S,T)$ and $X_B(S,T)$. If $G$ is a unit the proof is complete, so suppose that $G$ is not a unit. In the ring $k[S,T]$, $G$ divides $X_A - X_B = m_{CD} TE$. Since $E$ is a standard configuration, $m_{CD} \neq 0$. Thus $G$ divides $T$ or $E$. We show that both cases lead to a contradiction to the assumption that $G$ is not a unit.

Case 1: $G$ divides $E$.

In this case, $G$ divides $SF$ since $G$ divides $X_A = (m_C T - S)E + SF$. Since $E$ and $F$ are relatively prime, $G$ divides $S$, which since $G$ is not a unit means that $G$ and $S$ are associates, so that $S$ divides $E$ and $X$. Thus, by Lemma 5.5 $S$ divides either $(T + m_A S)F$ or $(T + m_B S)F$, which implies that $S$ divides $F$ since $S$ does not divide either of $T + m_A S$ or $T + m_B S$. Since $S$ divides $E$, this is a contradiction to the fact that $E$ and $F$ are relatively prime. Thus Case 1 cannot occur.

Case 2: $G$ divides $T$.

Since $G$ is not a unit, $G$ and $T$ are associates, so $T$ divides $X, X_A,$ and $X_B$. The fact that $T$ divides $X_A$ implies that $T$ divides $F - E$. Since also $T$ divides $X$, we have from Lemma 5.5 that if $b_A \neq 0$, then $T$ divides
\[
m_{AD} E - m_A F = m_A (E - F) - m_D E
\]
while if $b_B \neq 0$, then $T$ divides
\[
m_{BC} E - m_B F = m_B (E - F) - m_C E.
\]
Since $T$ divides $F - E$, we conclude that if $b_A \neq 0$, then $T$ divides $m_D E$ while if $b_B \neq 0$, then $T$ divides $m_C E$.

Now $T$ does not divide $E$ since that along with the fact that $T$ divides $F - E$ would imply $T$ divides $F$, a contradiction to the assumption that $E$ and $F$ are relatively prime. This means that if $b_A \neq 0$, then $m_D = 0$, while if $b_B \neq 0$, then $m_C = 0$. The fact that $T$ divides $E - F$ implies that $T$ divides $E^* - F^*$, which implies that the coefficient of $S$ in $E^* - F^*$ is 0; i.e.,
\[
0 = b_A m_B - b_B m_A - b_D m_D + b_B m_A - b_A m_B + b_A m_C = b_A m_C - b_B m_D.
\]
We have shown that if $b_A \neq 0$, then $m_D = 0$. In this case, from $0 = b_A m_C - b_B m_D$ we conclude that $m_C = 0$, a contradiction to the fact that $m_C \neq m_D$. Similarly, if $b_B \neq 0$, then $m_C = 0$ so that $m_D = 0$, again a contradiction. Therefore, $T$ does not divide $E - F$, and hence Case 2 cannot occur. This proves the lemma. □

The polynomials from Notation 5.6 can now be used to find rectangles with specified slope. In keeping with our approach, this is done by finding the rectangle inscriptions with specified slope point rather than the rectangles themselves. In Corollary 5.11 we give an explicit expression for the vertices of the rectangles.
Theorem 5.9. If \( \mathcal{C} \) is a standard configuration, then the map
\[
\pi: \mathbb{P}^1(k) \to \mathbb{P}^2(k) : \sigma \mapsto [\mathcal{X}(\sigma) : \mathcal{X}_A(\sigma) : \mathcal{X}_B(\sigma)]
\]
sends \( \sigma = [s : t] \) to a rectangle inscription with slope \( \frac{s}{t} \) and aspect ratio \( m_{CD} \cdot \frac{\mathcal{E}(\sigma)}{\mathcal{F}(\sigma)} \).

Proof. If there is \( \sigma \in \mathbb{P}^1(k) \) such that \( \mathcal{X}(\sigma) = \mathcal{X}_A(\sigma) = \mathcal{X}_B(\sigma) = 0 \), then since each of these equations is homogeneous in two variables, \( \mathcal{X}, \mathcal{X}_A \) and \( \mathcal{X}_B \) have a common linear factor, contrary to Lemma 5.8. Therefore, for each \( \sigma \in \mathbb{P}^1(k) \), the point \( p = [\mathcal{X}(\sigma) : \mathcal{X}_A(\sigma) : \mathcal{X}_B(\sigma)] \) is in \( \mathbb{P}^2(k) \).

Observe next that \( \mathcal{X}_A + \mathcal{X}_C = \mathcal{X}_B + \mathcal{X}_D \) and \( \mathcal{Y}_A + \mathcal{Y}_C = \mathcal{Y}_B + \mathcal{Y}_D \). Thus, for each \( \sigma \in \mathbb{P}^1(k) \), the points \( (\mathcal{X}_L(\sigma), \mathcal{Y}_L(\sigma)) \), \( L \in \{A, B, C, D\} \), are the vertices of a parallelogram inscribed in \( \mathcal{C}(\mathcal{X}(\sigma)) \). Moreover, by Lemma 5.7
\[
(\mathcal{X}_A - \mathcal{X}_B)(\mathcal{X}_B - \mathcal{X}_C) = m_{CD}m_{ST}m_{EF} = -(\mathcal{Y}_A - \mathcal{Y}_B)(\mathcal{Y}_B - \mathcal{Y}_C),
\]
and so \( p \) is a rectangle inscription and hence lies on the inscription curve for \( \mathcal{C} \).

Let \( \sigma = [s : t] \in \mathbb{P}^1(k) \). That the slope point of the inscription \( p \) is \( \sigma \) follows from the observation via Lemma 5.7 that
\[
(\mathcal{X}_B(\sigma) - \mathcal{X}_A(\sigma))s = (\mathcal{Y}_B(\sigma) - \mathcal{Y}_A(\sigma))t,
\]
\[
(\mathcal{Y}_C(\sigma) - \mathcal{Y}_B(\sigma))s = -(\mathcal{X}_C(\sigma) - \mathcal{X}_B(\sigma))t.
\]
The aspect point of the rectangle inscription is calculated similarly from Lemma 5.7
\[
(\mathcal{Y}_B - \mathcal{Y}_A)\mathcal{F} = -m_{CD}m_{ST}m_{EF} = (\mathcal{X}_B - \mathcal{X}_C)m_{CD}\mathcal{E},
\]
\[
(\mathcal{Y}_C - \mathcal{Y}_B)m_{CD}\mathcal{E} = Tm_{CD}m_{EF} = (\mathcal{X}_A - \mathcal{X}_B)\mathcal{F}.
\]
From these two equations we deduce that for each \( \sigma \in \mathbb{P}^1(k) \), the aspect point of the inscription \( p \) is \([m_{CD}m_{EF}(\sigma) : \mathcal{F}(\sigma)]\). (It cannot be that \( \mathcal{E}(\sigma) = \mathcal{F}(\sigma) = 0 \) since \( \mathcal{E} \) and \( \mathcal{F} \) are relatively prime in \( k[S, T] \).) This proves the theorem.

Definition 5.10. The image of the map \( \pi \) is the slope path for the configuration \( \mathcal{C} \).

The next corollary gives the affine piece of the slope path in \( k^2 \). But more important for present purposes, it gives for all but at most two choices of \( \sigma \in \mathbb{P}^1(k) \) the coordinates of the vertices of the unique inscribed rectangle with slope point \( \sigma \).

Corollary 5.11. Suppose that \( \mathcal{C} \) is a standard configuration. If \( \sigma = [s : t] \in \mathbb{P}^1(k) \) such that \( \mathcal{X}(\sigma) \neq 0 \), then there is a rectangle inscribed in \( \mathcal{C} \) with slope \( \frac{s}{t} \). Its vertices are
\[
\left( \frac{\mathcal{X}_L(\sigma)}{\mathcal{X}(\sigma)}, \frac{\mathcal{Y}_L(\sigma)}{\mathcal{X}(\sigma)} \right) \in L, \text{ where } L \in \{A, B, C, D\}.
\]

Proof. Let \( \sigma = [s : t] \in \mathbb{P}^1(k) \) such that \( \mathcal{X}(\sigma) \neq 0 \). Then
\[
\begin{bmatrix}
1 \quad \mathcal{X}_A(\sigma) \\
\mathcal{X}(\sigma) \quad \mathcal{X}_B(\sigma) \\
\mathcal{X}(\sigma) 
\end{bmatrix}
\]
is by Theorem 5.9 a rectangle inscription for \( \mathcal{C} \) with slope \( \frac{s}{t} \). By Lemma 5.7 and the definitions of \( \mathcal{X}_D \) and \( \mathcal{Y}_D \), we see that \( \mathcal{X}_A + \mathcal{X}_C = \mathcal{X}_B + \mathcal{X}_D \) and \( \mathcal{Y}_A + \mathcal{Y}_C = \mathcal{Y}_B + \mathcal{Y}_D \). Moreover, direct calculation using Lemma 5.5 shows that \( m_{CD}\mathcal{X}_C = m_{AD}\mathcal{X}_A + m_{DB}\mathcal{X}_B + b_{AB}\mathcal{X} \), so by Theorem 4.6 the points in the statement of the theorem are the vertices of a rectangle inscribed in \( \mathcal{C} \) with slope point \( \sigma \).
It can happen that the slope path is the line at infinity. The significance of this is that if the slope path is the line at infinity, then there is only one element of \( k \cup \{ \infty \} \) that can occur as the slope of a rectangle inscribed in \( \mathcal{C} \). This is a consequence of the next theorem and Theorem 5.12. However, Theorem 5.12 shows that the slope path is the line at infinity only in two very special situations.

**Theorem 5.12.** The following are equivalent for a standard configuration \( \mathcal{C} \).

1. \( A \parallel D \) and \( B \parallel C \) or \( A \perp C \) and \( B \perp D \).
2. The polynomial \( \chi(S,T) \) is identically 0.
3. The slope path is the line at infinity.

**Proof.** The equivalence of (2) and (3) is clear. To see that (2) implies (1), suppose \( \chi \) is identically zero. Then so is \( X \). We work in the polynomial ring \( \mathbb{P}(k) \) points \( \alpha \) and hence \( m \) lead to these same conclusions.

Since \( m \neq m_D \), we have \( m_B = m_C \) or \( m_B = -\frac{1}{m_D} \). If \( m_B = m_C \), then since \( m_A = \frac{m_B m_D}{m_C} \), we have \( m_A = m_D \). In this case, \( B \parallel C \) and \( A \parallel D \). Otherwise, if \( m_B = -\frac{1}{m_D} \), then \( B \perp D \) and \( m_A = \frac{m_B m_D}{m_C} \) implies that \( A \perp C \). If instead of \( m_C \neq 0 \) we assume \( m_D \neq 0 \), similar arguments lead to these same conclusions.

Conversely, to prove (1) implies (2), assume (1). We have either \( m_A m_C = -1 = m_B m_D \) or \( m_A = m_D \) and \( m_B = m_C \). Thus \( m_B m_C = m_B m_D \). The calculation above shows that \( \beta = 0 \) and hence \( \chi^*(S,T) = 0 \), from which it follows that \( \chi(S,T) = 0 \). \( \square \)

### 6. Aspect path

We develop the aspect path along similar lines to the slope path by first proposing in Notation 6.5 the parameterizing polynomials that are needed to find for each \( \alpha = [u,v] \in \mathbb{P}(k) \) a rectangle inscribed with aspect ratio \( \frac{u}{v} \). We show in Theorem 6.8 that this parameterization does indeed what we intend. Along the way we define an aspect factor that just as with slope factor must be removed from the coordinate polynomials in order to achieve the aspect path. While for the slope path we worked in the ring \( k[S,T] \) and evaluated homogeneous polynomials at slope points \( \sigma = [s:t] \in \mathbb{P}(k) \), for the aspect path we work in the polynomial ring \( k[U,V] \) and evaluate homogeneous polynomials at aspect points \( \alpha = [u:v] \in \mathbb{P}(k) \).

The slope path coordinate polynomials are ultimately dependent on the two polynomials \( \mathcal{E}(S,T) \) and \( \mathcal{F}(S,T) \). There are similar polynomials for the aspect path:
Notation 6.1. For a standard configuration $\mathfrak{C}$, we define
\[
\mathcal{M}^*(U,V) = (b_A m_{C} m_{D} + b_B m_{AD} m_{C}) U + m_{CD} b_{B} V
\]
\[
\mathcal{N}^*(U,V) = (b_B m_{AD} + b_A m_{CB}) U + m_{CD} (b_{AM} - b_{BM}) V
\]
\[
\mathcal{H}(S,V) = \text{first polynomial in the list } 1, \mathcal{M}^*(U,V), \mathcal{N}^*(U,V) \text{ that is the aspect factor for } \mathcal{M}^*(U,V) \text{ and } \mathcal{N}^*(U,V).
\]
We say $\mathcal{H}(U,V)$ is the aspect factor of $\mathfrak{C}$ and $\mathcal{P}(U,V)$ is the aspect polynomial for $\mathfrak{C}$. Using the aspect factor we reduce the preceding polynomials and define
\[
\mathcal{M}(U,V) = \frac{\mathcal{M}^*(U,V)}{\mathcal{H}(U,V)} \quad \mathcal{N}(U,V) = \frac{\mathcal{N}^*(U,V)}{\mathcal{H}(U,V)}.
\]

The polynomial defined next is the analogue of the polynomial $X^*(S,T)$ defined in Section 5. It detects the aspect ratios of the rectangle inscriptions at infinity and serves as a scale factor for rectangles along the aspect path.

Notation 6.2. For a standard configuration $\mathfrak{C}$, we define a polynomial in $k[U,V]$ by
\[
\mathcal{P}^*(U,V) = m_{BC} m_{AD} U^2 - \beta UV + m_{AB} m_{CD} V^2.
\]
where $\beta = (m_{A} m_{B} + 1) m_{CD} + (m_{CM} m_{D} + 1) m_{AB}$ is as in Notation 5.4. With $\mathcal{H}(U,V)$ the aspect factor for $\mathfrak{C}$, we also set
\[
\mathcal{P}(U,V) = \frac{\mathcal{P}^*(U,V)}{\mathcal{H}(U,V)}.
\]

Remark 6.3. If $k$ is a formally real field, then in a standard configuration $\mathfrak{C}$ we have that the equivalent conditions of Theorem 5.12 hold if and only if the discriminant of $\mathcal{P}^*(U,V)$ is 0. This follows from the fact that the discriminant of the quadratic $\mathcal{P}^*(U,V)$ simplifies to $4(m_{AB} m_{C} - m_{BM} m_{D})^2 + \beta^2$. Thus the discriminant is 0 if and only if $X^*(S,T)$ is identically 0. The claim now follows from Theorem 5.12.

That $\mathcal{P}(U,V)$ is a polynomial follows from the next lemma.

Lemma 6.4. Let $\mathfrak{C}$ be a standard configuration. Then $\mathcal{P}(U,V)$ is the unique solution in $k[U,V]$ to the equations
\[
\begin{aligned}
b_A \mathcal{P} &= (m_{AD} U - m_{CD} m_{A} V) \mathcal{M} - (m_{AB} m_{C} U + m_{CD} V) \mathcal{N} \\
b_B \mathcal{P} &= (m_{BC} U - m_{CD} m_{B} V) \mathcal{M} - (m_{B} m_{CD} U + m_{CD} V) \mathcal{N}
\end{aligned}
\]

Proof. As in the proof of Lemma 5.3, direct calculation shows that the equations in the lemma are valid with $\mathcal{P}, \mathcal{M}$ and $\mathcal{N}$ replaced by $\mathcal{P}^*, \mathcal{M}^*$ and $\mathcal{N}^*$. Since $b_A$ and $b_B$ are not both 0, the lemma follows after removing the aspect factor from the resulting equations. \qed

The coordinate polynomials for the aspect path are now defined using $\mathcal{M}$ and $\mathcal{N}$.

Notation 6.5. To a standard configuration $\mathfrak{C}$ we associate the following homogeneous polynomials in $k[U,V]$.
\[
\begin{aligned}
\mathcal{P}_A(U,V) &= (m_{CD} V - U) \cdot \mathcal{M} + m_{C} U \cdot \mathcal{N} \\
\mathcal{P}_B(U,V) &= (m_{CD} V - U) \cdot \mathcal{M} + m_{D} U \cdot \mathcal{N}
\end{aligned}
\]
\[
\begin{aligned}
\mathcal{P}_C(U,V) &= m_{D} U \cdot \mathcal{N} - U \cdot \mathcal{M} \\
\mathcal{P}_D(U,V) &= m_{C} U \cdot \mathcal{N} - U \cdot \mathcal{M}
\end{aligned}
\]

We also define $Q_L(U,V) = m_L \mathcal{P}_L(U,V) + b_L \mathcal{P}(U,V)$, where $L \in \{A, B, C, D\}$. 

The next lemma is the analogue of Lemma 5.7 and as in that case allows us to calculate slope and aspect ratio for the rectangles along the aspect path.

**Lemma 6.6.** For a standard configuration $C$, we have
\[
\begin{align*}
\mathcal{P}_A - \mathcal{P}_B &= m_{CD}U\mathcal{N}, \\
\mathcal{Q}_A - \mathcal{Q}_B &= m_{CD}U\mathcal{M}, \\
\mathcal{Q}_B - \mathcal{Q}_C &= -m_{CD}NV, \\
\mathcal{P}_B - \mathcal{P}_C &= m_{CD}V\mathcal{M}.
\end{align*}
\]

*Proof.* The lemma is verified by direct calculation. \hfill \Box

**Lemma 6.7.** The polynomials $\mathcal{P}, \mathcal{P}_A, \mathcal{P}_B$ are relatively prime in $k[U, V]$.

*Proof.* Let $g(U, V)$ be the gcd of $\mathcal{P}, \mathcal{P}_A$ and $\mathcal{P}_B$. We assume that $g$ is not a unit and we show this leads to a contradiction. By Lemma 6.6, $g$ divides $\mathcal{P}_A - \mathcal{P}_B = m_{CD}U\mathcal{N}$. Since $\mathcal{C}$ is standard, $m_C \neq m_D$, so $g$ divides $U$ or $\mathcal{N}$. Suppose first that $g$ divides $U$. Since $g$ is not a unit, $g$ and $U$ are associates. Thus $U$ divides $\mathcal{P}, \mathcal{P}_A$ and $\mathcal{P}_B$. Since $U$ divides $\mathcal{P}_A$, we have that $U$ divides $m_{CD}V\mathcal{M}$, and so $U$ divides $\mathcal{M}$. Also, since $U$ divides $\mathcal{P}$ and $U$ divides $\mathcal{M}$, Lemma 6.4 implies that $U$ divides $m_{CD}V\mathcal{N}$, so that $U$ divides $\mathcal{N}$. However, this is impossible since $\mathcal{M}$ and $\mathcal{N}$ are relatively prime. This contradiction shows that $g$ does not divide $U$, which means $g$ divides $\mathcal{N}$.

We show that this also leads to a contradiction. Since $g$ is not a unit, and since $\mathcal{M}$ and $\mathcal{N}$ are relatively prime and $g$ divides $\mathcal{P}_A$, it follows that $g$ divides $m_{CD}V - U$. Since $g$ is not a unit, $g$ and $m_{CD}V - U$ are associates.

If $b_A \neq 0$, then using the fact that $g$ divides $\mathcal{P}$ and $\mathcal{N}$ but not $\mathcal{M}$, Lemma 6.4 implies that $U - m_{CD}V$ divides $m_{AD}U - m_{CD}m_AV$. Since $m_{CD} \neq 0$, this implies that $m_A = m_{AD}$, and hence $m_D = 0$. But also $m_{CD}V - U$ divides
\[
\mathcal{N}^*(U, V) = (b_{BM}m_{AD} + b_Am_{CB})U + m_{CD}(b_{Am} - b_Bm_A)V.
\]

A zero of former polynomial is therefore a zero of the latter, so that
\[
(b_{BM}m_{AD} + b_Am_{CB})m_{CD} + m_{CD}(b_{Am} - b_Bm_A) = 0.
\]
Since $m_{CD} \neq 0$,
\[
b_{BM}m_{AD} + b_Am_{CB} + b_{Am} - b_Bm_A = 0,
\]
so that $-b_Bm_D + b_Am_C = 0$. Since $m_D = 0$ and $b_A \neq 0$, we have then that $m_C = 0$, a contradiction to the fact that $m_C \neq m_D$.

Similarly, if $b_B \neq 0$, then $U - m_{CD}V$ divides $m_{BC}U - m_{CD}m_BV$, and this implies that $m_C = 0$. Using that $U - m_{CD}V$ divides $\mathcal{N}^*$, we have as above that $-b_Bm_D + b_Am_C = 0$. Since $b_B \neq 0$, this implies that $m_D = 0$, contrary to the assumption that $m_C \neq m_D$. In all cases, if $g$ is not a unit, we arrive at a contradiction, so we conclude that $\mathcal{P}, \mathcal{P}_A$ and $\mathcal{P}_B$ are relatively prime. \hfill \Box

**Theorem 6.8.** If $C$ is a standard configuration, then the map
\[
\phi: P^1(k) \to P^2(k): \alpha \mapsto [\mathcal{P}(\alpha): \mathcal{P}_A(\alpha): \mathcal{P}_B(\alpha)]
\]

sends $\alpha = [u:v] \in P^1(k)$ to a rectangle inscription with aspect ratio $\frac{u}{v}$ and slope $\frac{M(\alpha)}{N(\alpha)}$.

*Proof.* We omit details since the proof follows the same lines as that of Theorem 5.9. As in that proof, $\phi$ is well defined. Similarly, Lemma 6.6 implies that for each $\alpha \in P^1(k)$, the point $[\mathcal{P}(\alpha): \mathcal{P}_A(\alpha): \mathcal{P}_B(\alpha)]$ is a rectangle inscription with aspect point $\alpha$ and slope point $[M(\alpha): N(\alpha)]$. \hfill \Box
Definition 6.9. The image of the map \( \phi \) is the aspect path for \( \mathcal{C} \).

Corollary 6.10. Suppose that \( \mathcal{C} \) is a standard configuration. If \( \alpha = [u : v] \in \mathbb{P}^1(\mathbb{k}) \) such that \( \mathcal{P}(\alpha) \neq 0 \), then there is a rectangle inscribed in \( \mathcal{C} \) with aspect ratio \( \frac{2}{v} \). Its vertices are of the form

\[
\left( \frac{\mathcal{P}(\sigma)}{\mathcal{P}(\sigma) + \mathcal{Q}(\sigma)} \right) \in L, \text{ where } L \in \{A, B, C, D\}.
\]

Proof. The proof is similar to that of Corollary 6.12. Let \( \alpha = [u : v] \in \mathbb{P}^1(\mathbb{k}) \) such that \( \mathcal{P}(\alpha) \neq 0 \). By Lemma 6.6 and the definitions of \( \mathcal{P}_C, \mathcal{P}_D \) and \( \mathcal{Q}_C, \mathcal{Q}_D \), this fact along with Theorem 6.8 implies as in the proof of Theorem 5.9 that the four points in the statement of the theorem are the vertices of a rectangle inscribed in \( \mathcal{C} \) with aspect point \( \alpha \).

Theorem 6.11. The following are equivalent for a standard configuration \( \mathcal{C} \).

1. The aspect path is the line at infinity.
2. The polynomial \( \mathcal{P}(U, V) \) is identically 0.
3. \( m^2_A = -1 \) and either \( m_A = m_B = m_C \) or \( m_A = m_B = m_D \).

Proof. The equivalence of (1) and (2) follows from Theorem 6.8. To prove (2) implies (3), suppose that \( \mathcal{P}(U, V) \) is identically 0. Then \( \mathcal{P}(U, V) = 0 \). Since the coefficients \( m_{ABC}, m_{AD} \) and \( m_{ABC}, m_{CD} \) are 0, one of the following sets consists of parallel lines: \( \{A, C, D\}, \{A, B, D\}, \{B, C, D\}, \{A, B, C\} \). Since \( \mathcal{C} \) is a standard configuration, \( C \) is not parallel to \( D \). Thus we have either that \( m_A = m_B = m_D \) or \( m_A = m_B = m_C \). In both cases, since the coefficient of \( UV \) is 0, we have

\[
0 = \beta = (m_A m_B + 1)(m_C - m_D) + (m_C m_D + 1)(m_A - m_B) = (m_A^2 + 1)(m_C - m_D),
\]

which since \( m_C \neq m_D \) yields that \( m_A^2 + 1 \). The converse, that (3) implies (2), is straightforward.

Corollary 6.12. If \( \mathcal{C} \) is a standard configuration, then the slope path and aspect path cannot both be the line at infinity.

Proof. Suppose by way of contradiction that the slope path and aspect path are both the line at infinity. By Theorem 6.12, one of the following two cases must occur: (a) \( A \parallel D \) and \( B \parallel C \) or (b) \( A \perp C \) and \( B \perp D \). Similarly, by Theorem 6.11, we have \( m_A^2 = -1 \) and either (c) \( m_A = m_B = m_D \) or (d) \( m_A = m_B = m_C \). Since \( C \) is not parallel to \( D \), it cannot be that (a) and (c) hold, nor can (a) and (d) hold. Similarly, if (b) holds, then \( m_A m_C = -1 = m_B m_D \). By either (c) or (d), \( m_A = m_B \), so \( m_C = m_D \), contrary to the assumption that \( C \) is not parallel to \( D \). This proves the corollary.

Corollary 6.13. If \( \mathcal{C} \) is a standard configuration, and \( |\mathbb{k}| \) is the cardinality of the field \( \mathbb{k} \), then there are at least \( |\mathbb{k}| - 1 \) rectangles inscribed in \( \mathcal{C} \), and so the affine inscription curve is not the empty set.

Proof. By Corollary 6.12 and Theorems 5.12 and 6.11, at least one of \( \mathcal{X}(S, T) \) and \( \mathcal{P}(U, V) \) is not identically zero and hence has at most two zeros in \( \mathbb{P}^1(\mathbb{k}) \). Since the cardinality of \( \mathbb{P}^1(\mathbb{k}) \) is \( |\mathbb{k}| + 1 \), Corollary 5.11 or 6.10 implies, depending on whether \( \mathcal{X} \) or \( \mathcal{P} \) is not identically zero, that there are at least \( |\mathbb{k}| - 1 \) rectangles inscribed in \( \mathcal{C} \). Each such rectangles has a rectangle inscription on the affine inscription curve and so this curve is not the empty set.
7. Degeneracy of the slope and aspect paths

The slope path and aspect path are either conics or lines because they are rational curves parameterized by polynomials of degree \( \leq 2 \). We investigate this situation in detail with a goal of fully describing the behavior of the slope and aspect paths, as well as the inscription curve, which as we show in Corollary 7.8 is the union of the two paths. In the first theorem we give criteria for when the aspect and slope paths are lines. The criterion in (6) involving the orthogonality of the diagonals of the configuration is important in how it connects “degenerate” behavior of the slope and aspect paths to a very elementary property of the configuration. In the case where \( k = \mathbb{R} \) and no two lines in the configuration are parallel or perpendicular, Schwartz in [5, Theorem 3.3] gave a partial version of this criterion by showing in this case that statement (6) implies that under these same assumptions, (6) implies that there are two sets of rectangles inscribed in \( \mathcal{C} \), one set of which consists of rectangles with the same slope and the other rectangles with the same aspect ratio. Schwartz also shows that under these same assumptions, (6) implies that the rectangle locus (see Section 8 for a definition) is a degenerate hyperbola. Since the rectangle locus is the image of the affine inscription curve under an affine transformation (see Section 8), this follows also from Theorem 7.1.

Theorem 7.1. The following are equivalent for a standard configuration \( \mathcal{C} \).

1. The slope path is a line.
2. The aspect path is a line.
3. The inscription curve is the union of two distinct lines, one of which is the slope path and the other the aspect path.
4. Every rectangle inscription on the slope path has the same aspect ratio.
5. Every rectangle inscription on the aspect path has the same slope.
6. If the diagonals \( E \) and \( F \) of \( \mathcal{C} \) are defined, they are orthogonal.
7. \( (m_{BM} + 1)(m_{BC}b_A - m_{AC}b_B)b_A = (m_{AM} + 1)(m_{BD}b_A - m_{AD}b_B)b_B \).

Proof. (4) \( \iff \) (7): Suppose every rectangle inscription on the slope path has the same aspect point \( \alpha \). Let \( \Phi : \mathbb{P}^1(k) \to \mathbb{P}^1(k) \) be given by \( \Phi(\sigma) = [m_{CD}E(\sigma) : F(\sigma)] \) for each \( \sigma \in \mathbb{P}^1(k) \). The matrix

\[
N = \begin{bmatrix}
m_{CD}(b_A b_B - b_B m_A) & m_{CD}(b_A - b_B) \\
b_B m_A b_B + b_A m_B & b_A m_B + b_B m_A
\end{bmatrix},
\]

which is formed from the coefficients of the indeterminates \( S \) and \( T \) in \( m_{CD}E \) and \( F \), defines the mapping \( \Phi \). By Theorem 5.9 \( [m_{CD}E(\sigma) : F(\sigma)] = \alpha \) for all \( \sigma \in \mathbb{P}^1(k) \). Therefore, the image of the linear transformation given by the matrix \( N \) is a line, and hence \( \det(N) = 0 \). Calculating \( \det(N) \) leads to the equation in (7).

Conversely, assuming the equation in (7) holds, we have \( \det(N) = 0 \), and so the image of the linear transformation given by \( N \) is a line through the origin or the zero vector in \( k^2 \). This image is the zero vector only if each entry in the matrix is 0, which cannot occur since \( [m_{CD}E(\sigma) : F(\sigma)] \) is a well-defined point in \( \mathbb{P}^1(k) \) for each \( \sigma \). Thus the image is a line through the origin, from which it follows that \( [m_{CD}E(\sigma) : F(\sigma)] \) is the same for all choices of \( \sigma \). Hence (4) holds by Theorem 5.9.

(7) \( \Rightarrow \) (1): As in the proof that (7) implies (4), the image of the mapping \( \Phi \) is a point and the matrix \( N \) has determinant 0. This implies that one row of the matrix is a scalar multiple of the other, so that the two homogeneous polynomials \( \mathcal{E}^* \) and \( \mathcal{F}^* \) (which are either
0 or linear) share a common zero. Thus the slope factor is not 1. Since \( \deg \mathcal{E}^* = \deg \mathcal{F}^* = 1 \), this implies that \( \mathcal{E} \) and \( \mathcal{F} \) are constants, and hence the polynomials \( \mathcal{X}, \mathcal{X}_A, \mathcal{X}_B \) all have degree \( \leq 1 \). A projective rational plane curve that is parameterized by three polynomials has order equal to the highest degree of these polynomials \([6] \text{ Exercise 3, p. 151}\), so the slope path is a line by Theorem 5.9.

(1) \( \Rightarrow \) (6): If \( E \) or \( F \) is undefined, there is nothing to prove. If \( F \) is the line at infinity, then \( E \) is orthogonal to every line distinct from it, and hence to \( F \) also, and so (6) holds in this case. Now assume that \( E \) and \( F \) are defined and \( F \) is not the line at infinity. Then \( \mathcal{E}^* \neq 0 \) and \( \mathcal{F}^* \neq 0 \) by Lemma 5.3. If the slope factor is 1, then \( \deg \mathcal{E} = 1 \) and \( \deg \mathcal{F} = 1 \) and so the polynomials \( \mathcal{X}, \mathcal{X}_A, \mathcal{X}_B \) all have degree 2. But, as in the proof that (7) implies (1), we have then that the slope path is not a line since it is a projective rational plane curve that by Theorem 5.9 is parameterized by polynomials of degree 2. However, this contradicts (1), so we conclude that the slope factor must be either \( \mathcal{E}^* \) or \( \mathcal{F}^* \). Since \( \deg \mathcal{E}^* = \deg \mathcal{F}^* = 1 \), this implies that \( \mathcal{E}^* \) and \( \mathcal{F}^* \) have the same zero in \( \mathbb{P}^1(\mathbb{k}) \). By Lemma 5.3 the slope point of \( E \) is orthogonal to the slope point of \( F \), which verifies (6).

(6) \( \Rightarrow \) (2): If \( E \) is undefined, then \( \mathcal{E}^* = 0 \) by Lemma 5.3, so that \( \mathcal{F}^* \) must be the slope factor for \( \mathcal{E} \). Similarly, if \( F \) is undefined or \( F \) is the line at infinity, then Lemma 5.3 implies that \( \mathcal{F}^* = 0 \) and so \( \mathcal{E}^* \) must be the slope factor for \( \mathcal{E} \). Either way, 1 is not the slope factor. On the other hand, if \( E \) and \( F \) are defined and \( F \) is not the line at infinity, then Lemma 5.3 and the assumption in (6) that \( E \) and \( F \) are orthogonal imply that \( \mathcal{E}^* \) and \( \mathcal{F}^* \) share the same zero. Since \( \deg \mathcal{E}^* = \deg \mathcal{F}^* = 1 \), this implies that \( \mathcal{E} \) and \( \mathcal{F} \) are constants, and hence \( \mathcal{X}, \mathcal{X}_A, \mathcal{X}_B \) all have degree \( \leq 1 \), which by Theorem 5.9 implies that the slope path is a line.

(2) \( \Rightarrow \) (5): Assume that the aspect path is a line. As in the proof that (1) implies (6), this implies that the aspect factor \( \mathcal{H}(U, V) \) is not 1, and so \( \mathcal{H} \) is either \( \mathcal{M}^* \) or \( \mathcal{N}^* \). Suppose \( \mathcal{H} = \mathcal{M}^* \). Then \( \mathcal{M}^* \) divides \( \mathcal{N}^* \) and \( \mathcal{M}(U, V) = 1 \) while \( \mathcal{N}(U, V) = \delta \) for some \( \delta \in \mathbb{k} \), where this last assertion follows from the fact that \( \mathcal{M}^* \) and \( \mathcal{N}^* \) are homogeneous of degree \( \leq 1 \). Therefore, for each \( \sigma \in \mathbb{P}^1(\mathbb{k}) \), Theorem 6.8 implies that the slope point of the rectangle inscription \( [\mathcal{P}(\sigma) : \mathcal{P}_A(\sigma) : \mathcal{P}_B(\sigma)] \) on the aspect path is \( [\mathcal{M}(\sigma) : \mathcal{N}(\sigma)] = [1 : \delta] \), which proves that if \( \mathcal{H} = \mathcal{M}^* \), then every rectangle inscription on the aspect path has the same slope. On the other hand, if \( \mathcal{H} = \mathcal{N}^* \), then \( \mathcal{M} = \mathcal{M}^* = 0 \) and \( \mathcal{N} = 1 \). Therefore, for each \( \sigma \in \mathbb{P}^1(\mathbb{k}) \), the slope point of the rectangle inscription \( [\mathcal{P}(\sigma) : \mathcal{P}_A(\sigma) : \mathcal{P}_B(\sigma)] \) is \( [\mathcal{M}(\sigma) : \mathcal{N}(\sigma)] = [0 : 1] \), which completes the proof that (2) implies (5).

(5) \( \Rightarrow \) (7): The proof is similar to the proof that (4) implies (7). Let \( \Psi : \mathbb{P}^1(\mathbb{k}) \to \mathbb{P}^1(\mathbb{k}) \) be given by \( \Psi(\alpha) = [\mathcal{M}(\alpha) : \mathcal{N}(\alpha)] \), \( \alpha \in \mathbb{P}^1(\mathbb{k}) \). The adjoint matrix \( \text{Adj}(N) \) of the matrix \( N \) in the proof of the equivalence of (4) and (7) defines \( \Psi \). Since the determinant of \( \text{Adj}(N) \) is the same as that of \( N \), the proof now proceeds along the same lines as that of (4) \( \Rightarrow \) (7).

(1) \( \Leftrightarrow \) (3): That (3) implies (1) is clear. Conversely, assume (1). We have established that (1) is equivalent to (5), so both the slope path and the aspect path are lines. Thus there are linear functions \( \ell_1, \ell_2 \in \mathbb{k}[X, X_A, X_B] \) such that the slope path is the zero set of \( \ell_1 \) in \( \mathbb{P}^2(\mathbb{k}) \) and the aspect path is the zero set of \( \ell_2 \) in \( \mathbb{P}^2(\mathbb{k}) \). Let \( h(X, X_A, X_B) \) be the defining equation of the inscription curve, as in Theorem 4.6. Let \( \overline{\mathbb{k}} \) denote the algebraic closure of \( \mathbb{k} \). By the Nullstellensatz, the fact that \( \ell_1 \) and \( \ell_2 \) are irreducible imply that there are \( f, g \in \overline{\mathbb{k}}[X, X_A, X_B] \) such that \( h = f\ell_1 = g\ell_2 \). Since \( h \) has degree \( \leq 2 \) and \( \ell_1 \) and \( \ell_2 \) have
degree 1, this implies that $h = \lambda_1 \ell_2$ for some $\lambda \in \mathbb{K}$. Thus the inscription curve is the union of the zero sets of $\ell_1$ and $\ell_2$, hence the union of the slope path and the aspect path.

Finally, to see that the slope path and the aspect path are not the same, use the fact that we have established already that (1) and (4) are equivalent. Thus, if the slope path is the aspect path, then every rectangle inscription on the aspect path has the same aspect ratio, contrary to Theorem 6.8. From this we conclude that the slope path and aspect path are distinct from each other.

Definition 7.2. A standard configuration is degenerate if it satisfies the equivalent conditions of Theorem 7.1.

Corollary 7.3. Any of the following cases are sufficient for a standard configuration to be degenerate: $A = B; B = C; A = D; A \parallel D$ and $B \parallel C; \text{ or } A \perp C$ and $B \perp D$.

Proof. This follows from Theorem 7.1(7). □

Corollary 7.4. If $C$ is a standard configuration consisting of four distinct lines, three of which are parallel, then $C$ is non-degenerate.

Proof. Since $C$ is standard, the lines $A$ and $B$ along with one of $C$ and $D$ are parallel. Suppose that $A, B, C$ are parallel and distinct. Then $D$ meets all three lines, and it follows that $E$ and $F$ are parallel to $A, B, C$. By Theorem 7.1, $C$ is not degenerate. On the other hand, if $A, B, D$ are parallel instead, then a similar argument shows again that $C$ is not degenerate. □

Remark 7.5. Let $C$ be a standard configuration, and let $\Phi, \Psi : \mathbb{P}^1(k) \to \mathbb{P}^1(k)$ be as in the proof of Theorem 7.1. The matrix $N$ in the proof of this theorem defines the mapping $\Phi$, while the adjoint matrix $\text{Adj}(N)$ of $N$ defines $\Psi$. The proof of the theorem shows that $\det(N) = 0$ if and only if $C$ is degenerate, so $\Phi$ and $\Psi$ are isomorphisms (and hence induce a fractional linear transformation on $k$) if and only if $C$ is not degenerate. It thus follows from Theorems 5.9 and 6.8 that if $C$ is not degenerate, then the aspect ratio of a rectangle inscribed in $C$ depends entirely on its slope, and vice versa.

If $b_A = b_B = 1$ in a standard configuration, then the equations for the slope and aspect paths (and specifically, the polynomials $E^*, F^*, M^*, N^*$) become more streamlined and the degeneracy criterion in Theorem 7.1(7) simplifies, as the next corollary shows. As long as three lines in a standard configuration $C$ in the real plane do not go through the same point, then $C$ can be rotated and scaled to achieve $b_A = b_B = 1$. More generally, it is straightforward to see that if $k$ is a formally real field and $C$ is a standard configuration for which three lines do not go through the same point, then there is invertible affine transformation from $k^2$ to $k^2$ that maps $C$ onto a standard configuration for which $b_A = b_B = 1$. Thus over a formally real field, the assumption that $b_A = b_B = 1$ is really no restriction at all.

Corollary 7.6. If $C$ is a standard configuration such that $b_A = b_B$, then $C$ is degenerate iff $m_A m_C = m_B m_D$.

Proof. Suppose that $b_A = b_B$. Then $b_A \neq 0$, since all four lines in $C$ do not go through the origin. Now apply Theorem 7.1(7). □

Theorem 7.7. The following are equivalent for a standard configuration $C$. 

1. $C$ is degenerate
2. $b_A = b_B = 1$
(1) \( \mathcal{C} \) is non-degenerate.
(2) The slope path is the inscription curve.
(3) The aspect path is the inscription curve.
(4) The slope path is the aspect path.
(5) No two rectangle inscriptions on the aspect path have the same slope.
(6) No two rectangle inscriptions on the slope path have the same aspect ratio.
(7) No two inscribed rectangles have the same slope.
(8) No two inscribed rectangles have the same aspect ratio.

Proof. To see that (1) implies (2), suppose that \( \mathcal{C} \) is non-degenerate. By Theorem 7.1, the slope path is not a line. Let \( \overline{k} \) be the algebraic closure of \( k \), let \( g(X,X_A,X_B) \) be the defining equation for the slope path, and let \( h(X,X_A,X_B) \) be the defining equation for the inscription curve as in Theorem 4.6. Since the slope path lies on the inscription curve, the Nullstellensatz implies that \( h \) is in the radical of the ideal \( g(X,X_A,X_B) \) in the ring \( \overline{k}[X,X_A,X_B] \). Since a projective rational plane curve that is parameterized by polynomials has order equal to the highest degree of these polynomials [6, Exercise 3, p. 151], this implies that \( g \) has degree 2. If \( g \) is irreducible in this ring, then since \( h \) has degree at most 2, \( h = \lambda g \) for some \( \lambda \in \overline{k} \). In this case, the slope path is the inscription curve, as claimed. Otherwise, if \( g \) is not irreducible, then \( g \) is a product of two linear homogeneous polynomials \( \ell_1, \ell_2 \) in \( \overline{k}[X,X_A,X_B] \). It follows that \( h = \mu \ell_1 \ell_2 = \mu g \) for some \( \mu \in \overline{k} \). Thus \( h \) and \( g \) have the same zeroes in \( \mathbb{P}^2(k) \), which proves that the slope path is the inscription curve. For the proof that (1) implies (3), apply the same argument to the aspect path. It follows from this also that (1) implies (4).

That (2) implies (5) follows from the fact that every rectangle inscription on the slope path has a different slope by Theorem 5.9. Similarly, (3) implies (6) since every rectangle inscription on the aspect path has a different aspect ratio by Theorem 6.8. Also, that (4) implies (5) follows from Theorem 5.9. That (5) implies (1) and (6) implies (1) follows from Theorem 7.1. This proves that (1) –(6) are equivalent.

That (4) implies (7) and (8) follows from the already established fact that (4) implies (5) and (6). Also, that (7) implies (1) and (8) implies (1) follow from (4) and (5) of Theorem 7.1. Thus (1)–(8) are all equivalent.

Corollary 7.8. The inscription curve for a standard configuration is the union of the slope path and the aspect path.

Proof. If the configuration \( \mathcal{C} \) is degenerate, then this follows from Theorem 7.1(3), while if \( \mathcal{C} \) is not degenerate, then this follows from Theorem 7.7.

It is shown in [4] and [5, Theorem 1.1], each with different methods, that for a configuration in the real plane that has no parallel or perpendicular lines, the curve in the plane consisting of the midpoints of the rectangles inscribed in the configuration (i.e., the rectangle locus) is a hyperbola. By working with the affine inscription curve instead, we prove in the next corollary that in all cases, regardless of the presence of parallel or perpendicular lines, this curve is a hyperbola. As we discuss in Section 8, the rectangle locus is the image of the affine inscription curve under an affine transformation that is invertible when no lines in the configuration are parallel or perpendicular to each other. Thus we recover the theorem that the rectangle locus is a hyperbola with an entirely different proof from those in [4] and [5].
Corollary 7.9. If $C$ is a standard configuration and $k = \mathbb{R}$, then the affine inscription curve is a hyperbola.

Proof. If the slope and aspect paths are lines, then these are distinct by Theorem 7.1(3), and so in this case the affine inscription curve is a degenerate hyperbola. It remains to consider the case that the slope path or the aspect path is not a line (and so neither are lines by Theorem 7.1). By Theorem 7.7 the inscription curve is the slope path. To show that the affine inscription curve is a hyperbola in this case, it suffices since the inscription curve is a projective conic to show that the slope path has two points at infinity. For $\sigma \in \mathbb{P}^1(k)$, we have that $[\mathcal{X}(\sigma) : \mathcal{X}_A(\sigma) : \mathcal{X}_B(\sigma)]$ is a point at infinity if and only if $\mathcal{X}(\sigma) = 0$. Since $C$ is non-degenerate, Theorem 7.1 implies that the slope path is not linear, and so the slope factor is 1 (see the proof of (6) implies (1) in Theorem 7.1). Hence $\mathcal{X}(S,T) = \mathcal{X}^*(S,T)$. Since also $C$ is non-degenerate, combining Theorem 5.12 and Corollary 7.3 shows that $\mathcal{X}^*(S,T) \neq 0$. With $\beta$ as in Notation 5.4, the discriminant of the quadratic $\mathcal{X}^*(S,T)$ is $\beta^2 + 4(m Am_c - m Bm_D)^2$. Since $\mathcal{X}^*(S,T) \neq 0$, it cannot be that both $\beta = 0$ and $m Am_c - m Bm_D = 0$. Thus, since $k = \mathbb{R}$, there are two distinct zeroes $\sigma_1, \sigma_2 \in \mathbb{P}^1(k)$ of $\mathcal{X}^*(S,T) = \mathcal{X}(S,T)$. Theorem 5.9 implies $[\mathcal{X}(\sigma_i) : \mathcal{X}_A(\sigma_i) : \mathcal{X}_B(\sigma_i)]$ has slope point $\sigma_i$ for each of $i = 1, 2$, so there are two distinct points at infinity on the slope path. Since the projective closure of the affine inscription curve is a conic with two points at infinity, the affine inscription curve is a hyperbola. □

8. The side locus

Although the inscription curve completely determines the set of rectangles inscribed in a standard configuration $C$, it is sometimes useful to have a more direct way of representing these rectangles within the configuration itself. In 4 and 5, this is done using the rectangle locus of the configuration, the set of centers of the inscribed rectangles. If neither of the pairs of lines in the configuration consists of parallel lines, then each point on the rectangle locus uniquely determines a rectangle inscribed in $C$, and so the rectangle locus gives a good way to track the path of inscribed rectangles through the configuration. It is shown in 4 and 5 that in the case in which $k = \mathbb{R}$ and neither pair in $C$ consists of parallel or perpendicular lines, the rectangle locus is a hyperbola. However, if at least one of the pairs in the standard configuration $C$ consists of parallel lines or both pairs consist of perpendicular lines, then the locus can be a line, a line with a segment missing, or a point; see 4.

We show in this section how the inscription curve helps explain these observations. Rather than being restricted to the case $k = \mathbb{R}$, our methods work for any field of characteristic different from 2. Also, while the rectangle locus finds the inscribed rectangles by tracking their centers, we describe another type of locus, the side locus, that tracks the midpoints of one of the sides of the inscribed rectangles. For a standard configuration $C$, the side locus has the advantage over the rectangle locus of never being simply a point. Moreover, as long as sides $A$ and $B$ are not parallel, the rectangles inscribed in $C$ are uniquely determined by the points on the side locus.

Throughout this section we assume that $k$ is a field of characteristic other than 2. Such a restriction allows us to consider the midpoint of two points in $k^2$: If $(x_1, y_1), (x_2, y_2) \in k^2$, then the midpoint of these two points is $(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2))$. 


Definition 8.1. Let $\mathcal{C}$ be a standard configuration. The rectangle locus for $\mathcal{C}$ is the set of points $p \in \mathbb{k}^2$ that occur as the center of a rectangle inscribed in $\mathcal{C}$. If $(x_L, y_L) \in L$ with $L \in \{A, B, C, D\}$ are the vertices of the rectangle, then the center is the point
\[
\left(\frac{1}{2}(x_A + x_C), \frac{1}{2}(y_A + y_C)\right) = \left(\frac{1}{2}(x_B + x_D), \frac{1}{2}(y_B + y_D)\right).
\]
The side locus for $\mathcal{C}$ is the set of points in $\mathbb{k}^2$ that occur as midpoints of the vertices $(x_A, y_A) \in A$ and $(x_B, y_B) \in B$ of a rectangle inscribed in $\mathcal{C}$.

The rectangle locus is the subject of [4] and [5], and we discuss it in our context at the end of this section. Our main focus here is the side locus for $\mathcal{C}$, and we show first that it is the image under an affine transformation of the affine inscription curve for $\mathcal{C}$.

Lemma 8.2. Let $\mathcal{C}$ be a standard configuration, and let $A : \mathbb{k}^2 \to \mathbb{k}^2$ be the affine transformation defined for all $x_A, x_B \in \mathbb{k}$ by
\[
A(x_A, x_B) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ m_A & m_B \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + \frac{1}{2}(b_A + b_B) + \frac{1}{2}(b_A b_B).
\]
Then the side locus for $\mathcal{C}$ is the image under $A$ of the affine inscription curve. The transformation $A$ is invertible if the lines $A$ and $B$ are not parallel; otherwise, the image of $A$ is a line.

Proof. The affine inscription curve consists of the set of points $(x_A, x_B)$ such that $[1 : x_A : x_B]$ is a rectangle inscription for $\mathcal{C}$, while the side locus is the set of points $(x, y)$ such that $2x = x_A + x_B$ and $2y = y_A + y_B$, where $(x_A, y_A) \in A$ and $(x_B, y_B) \in B$ are adjacent vertices of a rectangle inscribed in $\mathcal{C}$. To see that the image of the affine inscription curve under $A$ is the side locus, first let $x_A, x_B \in \mathbb{k}$ such that $[1 : x_A : x_B]$ is a rectangle inscription. Let $y_A = m_A x_A + b_A$ and $y_B = m_B x_B + b_B$. The side midpoint of the rectangle having inscription $[1 : x_A : x_B]$ is
\[
\left(\frac{1}{2}(x_A + x_B), \frac{1}{2}(y_A + y_B)\right) = \left(\frac{1}{2}(x_A + x_B), \frac{1}{2}(m_A x_A + b_A + m_B x_B + b_B)\right).
\]
It follows that $A(x_A, x_B) = \left(\frac{1}{2}(x_A + x_C), \frac{1}{2}(y_A + y_C)\right)$, and so $A$ maps the affine inscription curve into the side locus. To see that the image of the affine inscription curve is all of the side locus, let $(x, y) \in \mathbb{k}^2$ be the midpoint of the vertices $(x_A, y_A) \in A$ and $(x_B, y_B) \in B$ of a rectangle inscribed in $\mathcal{C}$. Then $2x = x_A + x_B$, $2y = y_A + y_B$, and $A(x_A, x_B) = (x, y)$, so the side locus is the image under $A$ of the affine inscription curve.

To prove the last sentence of the lemma, let
\[
M = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ m_A & m_B \end{bmatrix}.
\]
Then $\det(M) = \frac{1}{2} (m_B - m_A)$, so $A$ is invertible if and only if $A$ and $B$ are parallel. Otherwise, rank $M = 1$, and the image of $\mathbb{k}^2$ under $A$ is a line. □

Since the inscription curve is the union of the slope path and the aspect path, we can use Lemma 8.2 to view the side locus as a union of the images of the slope path and the aspect paths. We use the following terminology for these images.
Definition 8.3. Let \( \mathcal{C} \) be a standard configuration. The side aspect locus for \( \mathcal{C} \) is the image in \( k^2 \) of the affine aspect path under \( A \), while the side slope locus for \( \mathcal{C} \) is the image in \( k^2 \) of the affine slope path under \( A \).

Applying Corollary 5.11 and Lemma 8.2, we see that the side slope locus can be parameterized as the set of points
\[
\left( \frac{X_A(\sigma) + X_B(\sigma)}{2X(\sigma)}, \frac{Y_A(\sigma) + Y_B(\sigma)}{2X(\sigma)} \right),
\]
where \( \sigma \in \mathbb{P}^1(k) \) and \( X(\sigma) \neq 0 \).

Each such point is the midpoint of a side of an inscribed rectangle having slope point \( \sigma \).

Similarly, Corollary 6.10 implies that the side aspect locus is the set of all points
\[
\left( \frac{P_A(\alpha) + P_B(\alpha)}{2P(\alpha)}, \frac{Q_A(\alpha) + Q_B(\alpha)}{2P(\alpha)} \right),
\]
where \( \alpha \in \mathbb{P}^1(k) \) and \( P(\alpha) \neq 0 \).

Theorem 8.4. The side locus is the union of the side slope locus and the side aspect locus.

Proof. Apply Corollary 7.8 and Lemma 8.2. □

Remark 8.5. If \( A \parallel D \) and \( B \parallel C \) or \( A \perp C \) and \( B \perp D \), then the slope side locus is the empty set since the slope path is by Theorem 5.12 the line at infinity in this case.

We can describe the side locus using Lemma 8.2.

Theorem 8.6. Let \( \mathcal{C} \) be a standard configuration. If \( A \parallel D \) and \( B \parallel C \) or \( A \perp C \) and \( B \perp D \), then the side locus is a line. Otherwise:

1. If \( A \) and \( B \) are parallel, then the side locus is a subset of the line midway between \( A \) and \( B \) (see Figure 6).
2. If \( A \) and \( B \) are not parallel, then the side locus is a conic. The conic is degenerate if and only if \( \mathcal{C} \) is degenerate; if it is degenerate, then the side locus is the union of two lines, one of which is the side aspect locus and the other the side slope locus.

Each point on the side locus is the center of a unique rectangle inscribed in \( \mathcal{C} \).

Proof. The first claim follows from Theorem 7.1, Corollaries 7.3 and 7.8, Lemma 8.2 and Remark 8.5. For (1), observe that by Lemma 8.2 the image of \( A \) is a line, so the side locus is a subset of this line. For (2), use the fact that if the lines \( A \) and \( B \) are not parallel, then by Lemma 8.2, \( A \) is an invertible transformation, and so (2) follows from the fact that the affine inscription curve is a conic and Theorems 5.9 and 7.7. □

We omit details, but it is easy to modify the results in this section to obtain analogous results for the rectangle locus. In particular, we may parameterize the rectangle locus using slope or aspect ratio. For example, the image of the affine slope path is the set of points
\[
\left( \frac{X_A(\sigma) + X_C(\sigma)}{2X(\sigma)}, \frac{Y_A(\sigma) + Y_C(\sigma)}{2X(\sigma)} \right),
\]
where \( \sigma \in \mathbb{P}^1(k) \) and \( X(\sigma) \neq 0 \).

We can modify Lemma 8.2 to obtain the rectangle locus as the image of the affine inscription curve. Along with Corollary 7.9, this allows us to recover one of the main results in [3] and [5] with an entirely different proof, namely that if \( k = \mathbb{R} \) and the pairs in \( \mathcal{C} \) do not consist of parallel lines, then the rectangle locus is a hyperbola.
Figure 6. Two sets of rectangles inscribed in a configuration in which $A$ is parallel to $B$. The side locus (the dotted line in the figures) is a line with a gap. In the first figure the rectangles travel down the side locus while in the second figure they have begun to travel back up. The configuration is non-degenerate, and so no two rectangles have the same slope or the same aspect ratio.

If, on the other hand, one of the pairs in $\mathcal{C}$ consists of parallel lines, then this hyperbola is mapped under an affine transformation into a line, which helps explain the phenomenon discussed in [4] of a rectangle locus being a line with a missing segment in some cases. In particular, this case occurs only when $\mathcal{C}$ is non-degenerate, and then the gap in the locus is a consequence of the gap between branches of the hyperbola being mapped to the gap in the locus.

In any case, for $\mathbb{k} = \mathbb{R}$ the nature of the rectangle locus and the side locus is explained by the fact that these loci are each the image of a hyperbola under an affine transformation.

Remark 8.7. As discussed in the introduction, our approach throughout the paper assumes that at least two lines in a configuration are not parallel. The situation in which all four lines are parallel can be described as follows. If the pairs $A,C$ and $B,D$ do not share the same midline (i.e., the line consisting of midpoints of the points on either line), then there are no rectangles inscribed in $\mathcal{C}$. Otherwise, if the pairs share the same midline, then the midline is the rectangle locus, and for each choice of $x_A, x_B \in \mathbb{k}$ with $x_A \neq x_B$, there is a unique rectangle inscribed in $\mathcal{C}$ whose vertices on $A$ and $B$ are $(x_A, m_A x_A + b_A)$ and $(x_B, m_B x_B + b_B)$. Therefore, the affine inscription “curve” for this configuration is not a curve at all but the entire plane $\mathbb{k}^2$ with the line $y = x$ deleted. In any case, this accounts for all the rectangles inscribed in $\mathcal{C}$.

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