Abstract

The similarity between classical and quantum physics is large enough to make an investigation of quantization methods a worthwhile endeavour. As history has shown, Dirac’s canonical quantization method works reasonably well in the case of conventional quantum mechanics over \( \mathbb{R}^n \) but it may fail in non-trivial phase spaces and also suffer from ordering problem. Affine quantization is an alternative method, similar to the canonical quantization, that may offer a positive result in situations for which canonical quantization fails. In this paper we revisit the affine quantization method on the half line. We formulate and solve some simple models, the free particle and the harmonic oscillator.

1 Introduction

Although non-relativistic quantum mechanics stands as a well-established theory and a well experimentedly test theory, the question of how to pass from classical to quantum theory and a better understanding of the relation between classical and quantum mechanics are still of particular interest. Indeed,

- ongoing attempts to quantize general relativity where a definitive answer to the question of the correct quantum theory of gravitation is still missing,

- a quantization method that is able to take the nonlinear structure into account right from the outset is a useful tool to construct and study possible candidates for a theory of gravity,

- a better knowledge of quantization in a situation when physical systems satisfy constraints or boundary conditions is needed.

In physics, quantization is generally understood as a correspondence between a classical and a quantum theory. The question is how can we construct a quantum theory if a classical system is given? If we consider quantum theory to be more fundamental theory and classical mechanics to be only approximatively correct, the very concept of quantization seems pointless or appears to be ill-founded since it attempts to construct a "correct" theory from a theory which is only approximatively correct. There are quantum systems for which no classical
counterpart exists: Example: He-II superfluidity or even a spin 1/2 particle. For these systems a quantization method would not make sense.

Quantum mechanics, like any other physical theory, classical mechanics, electrodynamics, relativity, thermodynamics, cannot be derived. The laws of quantum mechanics, expressed in mathematical form, are the results of deep physical intuition, as indeed, are all other physical theories. Their validity can only be checked experimentally. From this point of view, quantization is not a method for deriving quantum mechanics, rather is is a way to understand the deeper physical reality which underlies the structure of both the classical and quantum mechanics and which unifies the two from geometrical perspectives.

It is conceptually very difficult to describe a quantum theory from scratch, without the help of a reference classical theory. The similarity between classical and quantum physics is large enough to make quantization a worthwhile approach. There is a certain mathematical richness in the various theories of quantization where the method does make sense. Quantization in its modern sense is therefore often understood as construction of a quantum theory with help of a classical reference, not necessarily as a strict mapping. Quantization is studied not only for the sake of novel predictions: it is equally rewarding to reproduce existing results in a more illuminating manner.

Originally P. A. M. Dirac introduced the canonical quantization in his 1926 doctoral thesis, The method of classical analogy for quantization [1]. The canonical quantization or correspondence principle is an attempt to take a classical theory described by the phase space variables, let’s say $p$ and $q$, and a Hamiltonian $H(q,p)$ to define or construct its corresponding quantum theory. The following simple technique for quantizing a classical system is used. Let $q^i, p_i, i = 1, 2 \ldots n$, be the canonical positions and momenta for a classical system with $n$ degrees of freedom. Their quantized counterparts $\hat{q}^i, \hat{p}_i$ are to be realized as operators on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n, dx)$ by the prescription

\[
(\hat{q}^i \psi)(x) = x^i \psi(x); \quad (\hat{p}_i \psi)(x) = -i\hbar \frac{\partial}{\partial x^i} \psi(x); \quad i = 1, 2, \ldots n, \quad x \in \mathbb{R}^n. \quad (1)
\]

This method is known as canonical quantization and is the basic method of quantization of a classical mechanics model [2, 4, 3]. More general quantities, such as the Hamiltonians, become operators according to the rule

\[
H(p, q) \to \hat{H}(\hat{p}, \hat{q}), \quad (2)
\]

an expression that may have ordering ambiguities [5, 6]. In which canonical coordinates system does such a quantization method works?

1. According to Dirac replacing classical canonical coordinates by corresponding operators is found in practice to be successful only when applied with the dynamical coordinates and momenta referring to a cartesian system of axes and not to more general curvilinear coordinates.

2. Cartesian coordinates can only exist on a flat space.

3. The canonical quantization seems to depend on the choice of coordinates.

4. Beyond the ordering problem, one should keep in mind that $[\hat{q}, \hat{p}] = i\hbar I_d$ holds true with self adjoint operators $\hat{q}$, $\hat{p}$, only if both have continuous spectrum ($-\infty, +\infty)$, and there is uniqueness of the solution, up to unitary equivalence (von Neumann).
There are two attitudes that may be taken towards this apparent dependence of the method of the canonical quantization on the choice of coordinates. The first view would be to acknowledge the cartesian character that is seemingly part of the method. The second view would be to regard it as provisional and seek to find a quantization formulation that eliminates this apparently unphysical feature of the current approaches.

The aim of eliminating the dependence on cartesian coordinates in the standard approaches is no doubt one of the motivations for several methods such as the geometric quantization [7, 8, 9, 10], the path integral quantization [11], the deformation quantization [8, 12, 13, 14, 15, 16], the Klauder-Berezin-Toeplitz quantization [17, 18, 19, 20]. There is no general theory of quantization presently available which is applicable in all cases, and indeed, often the techniques used to quantize has to be tailored to the problem in question.

As history has shown, Dirac’s canonical quantization method works reasonably well in the case of conventional quantum mechanics over \( \mathbb{R}^n \) due to the following reason:

1. the underlying configuration space \( \mathbb{R}^n \) is so well behaved;
2. when we try to quantize classical systems with phase spaces other than the cotangent bundle \( T^*\mathbb{R}^n \), the situation changes drastically;
3. already in classical mechanics, phase spaces different from \( T^*\mathbb{R}^n \) require a more elaborated mathematical formalism;
4. global and topological aspects play a much bigger role in quantum theory than in classical physics.

Although quite successful in applications, the canonical quantization method has some severe shortcoming from a theoretical point of view. A number of question arise in connection with the scheme of canonical quantization.

1. Let \( Q \) be the position space manifold of the classical system and \( q \) any point in it. Geometrically, the phase space of the system is the cotangent bundle \( \Gamma = T^*Q \). If \( Q \) is linear, means \( Q \approx \mathbb{R}^n \), then the replacement \( q^i \rightarrow x^i, p_j \rightarrow -i\hbar \frac{\partial}{\partial x^j} \) works fine. But what happen if \( Q \) is not linear?
2. How do we quantize observables which involves higher powers of \( q^i, p_j \), as for example \( f(q^i, p_j) = (q^i)^n(p_j)^m \) when \( n + m \geq 3 \)
3. How should we quantize a more general phase spaces, which are the symplectic manifolds not necessarily cotangent bundles?

As we are currently interested in new developments in quantization methods [21, 22, 23], the goal of this paper is to highlight through simple models the benefits of affine quantization.

In section (2), we revisit a method of quantization by John Klauder, the affine quantization, then in section (3) we formulate and solve the free particle and the harmonic oscillator. Concluding remarks are given in section (4).
2 Affine Quantization

While in confinement due to covid-19, our attention has been drawn on a recent published paper of John Klauder on The benefits of Affine Quantization [24]. Our motivation is due to the fact that there is a difficulty with canonical quantization when it comes to configuration spaces other then $\mathbb{R}^n$. Consider, for example, a particle that is restricted to move on the positive real line. The configuration space is $Q = \mathbb{R}^+$. It seems reasonable to use the position $q$ and momentum $p$ as classical observables, which satisfy the usual commutations relations. However, when we try to represent these by operators $\hat{q} = q$ and $\hat{p} = -i\hbar \frac{\partial}{\partial q}$, it turns out that the momentum operator $\hat{p}$ is not self-adjoint on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^+, dq)$. Thus a straightforward application of Dirac’s canonical quantization recipe is impossible.

Our goal is to apply the method of affine quantization to study the free particle and the harmonic oscillator that would serve as a test in order to study a toy model of a massive Klein Gordon field coupled to an harmonic oscillator at the the boundary considering the half line. In this section, we revisit the affine quantization method from some previous works of the author [25, 26, 27].

Let us start with a single degree of freedom, the classical phase space variables $p$ and $q$ are real satisfying a standard Poisson bracket

$$\{q, p\} = 1,$$

(3)
multiply by $q$ the equation (3) we get

$$q\{q, p\} = q,$$

(4)
that is equivalent to $\{q, pq\} = q$, setting $d = pq$, we have

$$\{q, d\} = q.$$  

(5)
The two variables $d$ and $q$ form a lie-algebra and are worthy of consideration as new pair of classical variables even though they are not canonical coordinates. It is also possible to restrict $q$ to $q > 0$ or $q < 0$ consistent with $d$. The variable $d$ acts to dilate $q$ and not to translate $\hat{q}$ as the variable $p$ does.

For the case of the single degree of freedom above, the canonical quantization involves $\hat{q}$ and $\hat{p}$ which are self adjoint operators that satisfy the canonical commutation relation

$$[\hat{q}, \hat{p}] = i\hbar I_d.$$  

(6)
From the canonical quantization, it follows that

$$\hat{q}[\hat{q}, \hat{p}] = [\hat{q}, \hat{q}\hat{p}] = [\hat{q}, \frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2}] = [\hat{q}, \hat{q}] = i\hbar \hat{q},$$

(7)
where the dilation operator is define as $\hat{d} \equiv (\hat{q}\hat{p} + \hat{p}\hat{q})/2$ is self adjoint. The operator $\hat{d}$ is called the dilation operator because it dilates $\hat{q}$ rather than translates $\hat{q}$ as $\hat{p}$ does, in particular

$$e^{\frac{i\hbar}{\hbar} \hat{d}} \hat{q} e^{-\frac{i\hbar}{\hbar} \hat{d}} = \hat{q} + q I_d,$$

(8)
while

$$e^{i\ln(|q|) \hat{d}/\hbar} \hat{q} e^{-i\ln(|q|) \hat{d}/\hbar} = |q| \hat{q} = q|\hat{q}|.$$  

(9)
In the second relation in equation (9), \( q \neq 0 \), and \( q \) as well as \( \hat{q} \) are normally chosen to be dimensionless. According to equation (7), the existence of canonical operators guarantees the existence of affine operators. If \( \hat{q} > 0 \) or \( \hat{q} < 0 \), then the operator \( \hat{p} \) cannot be made self-adjoint, however in that case, both \( \hat{q} \) and \( \hat{d} \) are self-adjoint. As usual \( \hat{q} \) and \( \hat{p} \) are irreducible, but \( \hat{q} \) and \( \hat{d} \) are reducible. There are three inequivalent irreducible representations; one with \( \hat{q} > 0 \), one with \( \hat{q} < 0 \) and one with \( \hat{q} = 0 \) and all three involve representations that are self-adjoint. The first two irreducible choices are the most interesting and, for the present, we focus on the choice \( \hat{q} > 0 \).

3 Testing some models

3.1 The free particle

The simplest Hamiltonian on can envisage is the free particle on the half line. The Hamiltonian reads

\[
\mathcal{H}_f(x, p_x) = \frac{1}{2m} p_x^2,
\]

(10)

with \( \{x, p_x\} = 1 \), \( x > 0 \), in term of the affine variables as described in section (2) we may rewrite

\[
\mathcal{H}_f(x, d_x) = \frac{1}{2m} d_x^2 x^{-2} d_x^2,
\]

(11)

where the variable \( d_x \) is the dilation variable \( d_x = p_x x \) and \( \{x, d_x\} = x \). By mean of canonical quantization where the affine variables \( x, p_x \) are respectively promoted to operators \( \hat{x}, \hat{p}_x \), the corresponding Hamiltonian for the free particle read

\[
\hat{\mathcal{H}}_f(\hat{x}, \hat{d}_x) = \frac{1}{2m} \hat{d}_x (\hat{x})^{-2} \hat{d}_x,
\]

(12)

where \( \hat{d}_x \) stands for the dilation operator and \( [\hat{x}, \hat{d}_x] = i\hbar \hat{x} \). We have the representation \( \hat{d} \equiv (1/2)x \partial_x + 1 \) and \( \hat{x} \equiv x \), with \( x > 0 \). The time independent eigenvalue equation can be written as

\[
(\frac{1}{2m} \hat{d}(\hat{x})^{-2} \hat{d}) \phi(x) = E \phi(x),
\]

(13)

that is equivalent to

\[
\left( \frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\hbar^2}{2m} \frac{3}{4} x^{-2} \right) \phi(x) = E \phi(x).
\]

(14)

We are then interested in solving the problem (14). If we divide \( -\frac{\hbar^2}{2m} \) and setting \( k^2 = \frac{2mE}{\hbar^2} \), where we assume \( E > 0 \) and label \( \alpha = 3/4 \), the equation (14) takes the form

\[
\phi''(x) = \left( \frac{\alpha}{x^2} - k^2 \right) \phi(x).
\]

(15)

Let’s consider the change of variable variable \( x = k^{-1} y \). The equation in (15) is rewritten in terms of the new variable \( y \) as follows

\[
\phi''(y) = \left( \frac{\alpha}{y^2} - 1 \right) \phi(y).
\]

(16)
Setting \( \phi(y) = y^{1/2} \varphi(y) \) and reminding that \( \alpha = 3/4 \), we obtain the ordinary differential equation

\[
\varphi''(y) + \frac{1}{y} \varphi'(y) + \left( 1 - \frac{1}{y^2} \right) \varphi(y) = 0,
\]

(17)

that is a variant of the Bessel’s equation and the solutions defined the Bessel functions \( J_1(x) \) and \( Y_1(x) \). A continuum of eigenfunctions exist for the problem

\[
\phi_k(x) = (kx)^{\frac{3}{4}} J_{\pm 1}(kx), \quad E_k = \frac{k^2 \hbar^2}{2m}, \quad k \in \mathbb{R}.
\]

(18)

Both signs satisfy the important closure relation

\[
\int_0^\infty \phi_k(x) \phi_k(y) dk = \delta(x - y),
\]

(19)

since the order of the bessel function is greater than \(-1/2\), this is explain in reference [36]. Since both positive and negative orders give independent representations of the identity operator, there is the freedom to use either sign in forming physical states.

### 3.2 The harmonic oscillator

We consider the one dimensional harmonic oscillator represented by the classical Hamiltonian

\[
H_o(x, p_x) = \frac{1}{2m} p_x^2 + \frac{1}{2} m \omega^2 x^2,
\]

(20)

where \((p_x, x) \in \mathbb{R} \times \mathbb{R}^+\), that means \( x > 0 \), with \( \{ x, p_x \} = 1 \). Our aim is to test the affine quantization that has already been partially considered in [27].

Let’s first determine the classical affine variables as in the section (2). We set \( d_x = p_x x \), also called the dilation variable and the new variables called affine variables are \( x \) and \( d_x \) that satisfy \( \{ x, d_x \} = x \). The Hamiltonian in equation (20) can be then rewritten in term of the affine variables as

\[
H_o(x, d_x) = \frac{1}{2m} d_x (x^{-2}) d_x + \frac{1}{2} m \omega^2 x^2.
\]

(21)

By mean of the canonical quantization, the classical affine variables are promote as operators

\[
d_x \to \hat{d}_x; \quad x \to \hat{x},
\]

(22)

and the corresponding Hamiltonian quantized is

\[
\hat{H}_o(\hat{x}, \hat{d}_x) = \frac{1}{2m} \hat{d}_x (\hat{x})^{-2} \hat{d}_x + \frac{1}{2} m \omega^2 \hat{x}^2.
\]

(23)

The affine fundamental operators satisfies the commutation relations

\[
[\hat{x}, \hat{d}_x] = i \hbar \hat{x},
\]

(24)

and act as follows

\[
\hat{x} \psi(x, t) = x \psi(x, t); \quad x > 0, \quad \hat{x} > 0,
\]

(25)
\[ \hat{d}_x \psi(x,t) = -i\hbar \left( \frac{\partial}{\partial x} + 1 \right) \psi(x,t), \] (26)

where the wave functions are normalized

\[ \int_0^\infty |\psi(x,t)|^2 \, dx = 1. \] (27)

We can then solve for the Schrödinger equation

\[ i\hbar \frac{\partial \psi(x,t)}{\partial t} = \hat{H}_o \psi(x,t). \] (28)

Since we are in presence of autonomous system we may set the Ansatz

\[ \psi(x,t) = e^{-iEt/\hbar} \phi(x), \] (29)

and then the corresponding time-independent eigenvalue equation is

\[ \hat{H}_o (\hat{d}_x, \hat{d}_x) \phi(x) = E \phi(x), \] (30)

that is explicitly the equation

\[ \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{3\hbar^2}{8m} \frac{1}{x^2} + \frac{1}{2} m\omega^2 x^2 \right) \phi(x) = E \phi(x), \] (31)

and we can rewrite as

\[ \left[ -\frac{d^2}{dx^2} + \frac{3}{4} \frac{1}{x^2} + \frac{m^2\omega^2}{\hbar^2} x^2 \right] \phi(x) = \frac{2mE}{\hbar^2} \phi(x). \] (32)

For matter of simplification, let’s set the following parameters

\[ \lambda = \frac{m\omega}{\hbar}; \quad k^2 = \frac{2mE}{\hbar^2}; \quad \alpha = \frac{3}{4}, \] (33)

and the equation (32) becomes

\[ \left[ -\frac{d^2}{dx^2} + \lambda^2 x^2 + \frac{\alpha}{x^2} \right] \phi(x) = k^2 \phi(x). \] (34)

A standard asymptotic analysis for \( x \to \infty, x \to 0 \), require the following Ansatz for the wave function

\[ \phi(x) = x^{\beta+1} e^{-\frac{\lambda}{2}x^2} v(x), \] (35)

where the constant \( \beta \) is related to \( \alpha \) by \( \alpha = \beta(\beta + 1) \) and \( v(x) \) is unknown function. The relation between \( \alpha \) and \( \beta \) gives two possible values of \( \beta \) that are \( \beta_+ = 1/2, \beta_- = -3/2 \). If not necessary we keep \( \beta \) as it is. From the Ansatz in equation (35), an equation for the unknown function \( v(x) \) is given by

\[ v''(x) + \left[ 2(\beta + 1)x^{-1} - 2\lambda x \right] v'(x) + \left[ k^2 - \lambda(2\beta + 3) \right] v(x) = 0. \] (36)

By change of variable \( y = \lambda x^2 \) on equation (36), we obtain the differential equation

\[ yv''(y) + \left[ (\beta + 3/2) - y \right] v'(y) + \left[ \frac{k^2}{4\lambda} - \frac{1}{4}(2\beta + 3) \right] v(y) = 0. \] (37)
The general solution of the equation \((37)\), also known as Kummer’s differential equation, can be expressed in terms of confluent hypergeometric functions

\[
v(y) = A_1 F_1 \left( \frac{1}{2} (\beta + 3/2) - \frac{1}{2} \mu, \beta + 3/2, y \right) + B y^{-(\beta+1/2)} 1 F_1 \left( \frac{1}{2} (-\beta - \mu + 1/2), \frac{1}{2} - \beta, y \right),
\]

where \(\mu = k^2/(2\lambda)\), and \(1 F_1(a, c, y)\) denotes the confluent hypergeometric function which has the following series representation

\[
1 F_1(a, c, y) = 1 + \frac{a}{c} y + \frac{a(a+1)}{c(c+1)} \frac{y^2}{2!} + \ldots
\]

Due to the asymptotic behavior of the confluent hypergeometric function given by

\[
1 F_1(a, c, y) \sim e^y y^{a-c},
\]

which implies divergencies of both of the terms in equations \((38)\) and then the impossibility to normalize the wave function, we impose the following conditions

\[
\frac{1}{2} (\beta + 3/2) - \frac{1}{2} \mu = -n \quad \text{or} \quad -\frac{1}{2} (-\beta - \mu + 1/2) = -n
\]

As we already said the condition

\[
\alpha = \beta (\beta + 1)
\]

gives two values of \(\beta\) that are \(\beta_+ = 1/2\) or \(\beta_- = -3/4\).

The condition \(\frac{1}{2} (\beta + 3/2) - \frac{1}{2} \mu = -n\) with \(\beta_+ = 1/2\) gives the solution

\[
\phi_n(x) = A_n x^{3/2} e^{-m\omega x^2/\hbar} 1 F_1(-n, 2, \frac{m\omega}{\hbar} x^2); \quad E_n = 2\hbar \omega (n + 1),
\]

where \(A_n\) is a constant to be determined after normalization.

The second condition \(-\frac{1}{2} (-\beta - \mu + 1/2) = -n\) with \(\beta_- = -3/4\) gives the solution

\[
\phi_n(x) = B_n x^{3/2} e^{-m\omega x^2/\hbar} 1 F_1(-n, 2, \frac{m\omega}{\hbar} x^2); \quad E_n = 2\hbar \omega (n + 1),
\]

where \(B_n\) is a constant to be determined after normalization.

In conclusion, the solution of the eigenvalue equation \((32)\) is the normalizable wave function

\[
\phi_n(x) = C_n x^{3/2} e^{-m\omega x^2/\hbar} 1 F_1(-n, 2, \frac{m\omega}{\hbar} x^2), \quad n = 0, 1, 2, \ldots
\]

where \(C_n\) is the constant to be determined after normalization and the energy value are \(E_n = 2\hbar \omega (n + 1)\). The constant \(C_n\) is determined from the normalization condition

\[
\int_0^\infty |\phi_n(x)|^2 dx = 1.
\]

4 Concluding remarks

We have revisited the procedure of affine quantization introduced by John R. Klauder. Our motivation is to better understand the quantization method on non trivial phase spaces where canonical quantization fails. We have tested the procedure for the simple case of the free particle and the case of the harmonic oscillator, both on the half line.
For the case of the free particle in the upper half line, the equivalent of the Hamiltonian in the affine coordinate turns out to be the case of a particle in a square inverse potential \( V(x) = \alpha/x^2 \). This kind of problem has been considered in the literature \([28, 29, 30]\) and the case of \(-1/4 < \alpha < 0\) has been explicitly discussed in \([28]\). There is a richness in the inverse square potential in quantum mechanics due to the connections to diverse physical phenomena \([31, 32, 33]\).

For the case of the harmonic oscillator on the half line, the equivalent of the Hamiltonian in terms of the affine coordinates is the case of model with the potential \( V(x) = (1/2)m\omega^2x^2 + \alpha/x^2, x > 0 \), that is a well known model in the literature. In fact, family of quantum Hamiltonians known as spiked harmonic oscillators is given by the general Hamiltonian operator \( H = -d^2/dx^2 + x^2 + \alpha/x^2 \) acting on the Hilbert space \( L^2(0, \infty) \). The name of the operator derives from the graphical shape of the full potential \( V(x) = x^2 + \alpha/x^2 \) which shows a pronounced peak near the origin for \( \alpha > 0 \). The spiked harmonic oscillators has drawn the attention of many authors because it represents the simplest model of certain realistic interaction potentials in atomic, molecular and nuclear physics, and second and also due to its interesting intrinsic properties from the viewpoint of mathematical physics \([34, 35]\).

We are interested in considering more toy models for instance the case of toy model that consists of a massive Klein Gordon field in 1+1 dimensions restricted to the left half line by a boundary and coupled to a harmonic oscillator at that boundary, thus introducing some additional degrees of freedom. The classical Hamiltonian is defined by

\[
H = \int_{-\infty}^{0} dx \left( \frac{1}{2} \pi(x, t)^2 + \frac{1}{2} \left( \partial_x \phi(x, t) \right)^2 + \frac{1}{2} \mu^2 \phi(x, t)^2 \right) + \beta \phi(0, t) q(t) + \frac{1}{2} m \omega^2 q(t)^2 + \frac{1}{2} m p(t)^2. \tag{45}
\]

We hope to report on that problem soon \([37]\).

As pointed out in \([27]\), gravity does not fit well with canonical quantization and affine quantization may be an alternative procedure. So we are also interested in understanding how the affine quantization can help in quantum gravity.

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