Abstract. We prove a new Bernstein type inequality in $L^p$ spaces associated with the normal and the tangential derivatives on the boundary of a general compact $C^2$-domain. We give two applications: Marcinkiewicz type inequality for discretization of $L^p$ norms and positive cubature formulas. Both results are optimal in the sense that the number of function samples used has the order of the dimension of the corresponding space of algebraic polynomials.

1. Introduction and Main Results

We start with the two most classical inequalities for algebraic polynomials: the Bernstein inequality

$$
\|\varphi' P_n^{(r)}\|_{L^p[-1,1]} \leq C n^r \|P_n\|_{L^p[-1,1]}, \quad 0 < p \leq \infty, \quad r = 1, 2, \ldots
$$

and the Markov inequality

$$
\|P_n^{(r)}\|_{L^p[-1,1]} \leq C n^{2r} \|P_n\|_{L^p[-1,1]}, \quad 0 < p \leq \infty, \quad r = 1, 2, \ldots,
$$

where $\varphi(x) = \sqrt{1 - x^2}$ for $x \in [-1,1]$, $P_n$ is any algebraic polynomial of degree at most $n$, and $C > 0$ is a constant depending only on $r$ and $p$, see [15, Ch. 4] and references therein. (Here and throughout the paper, we do not care about the exact values of the constants.) These inequalities have been playing crucial roles in proving various results in approximation theory, see, for instance, [15, 17]. As a result, they have been generalized and improved in many directions, see [3, 5, 17, 19, 20, 23, 24, 34, 38, 42]. In particular, they were generalized on domains related to the spheres for $0 < p \leq \infty$ in [2], and on more general compact domains for $1 \leq p < \infty$ in [20] and for $p = \infty$ in [11, 13, 30, 39, 40].

The aim of this paper is to establish $L^p$-Bernstein(-Markov) inequalities for polynomials on compact $C^2$-domains in $\mathbb{R}^d$. In this introduction, we will describe our main results, the organization of the paper, the structure of the proofs and the methods used including a brief discussion of relevant earlier works.

First, we give the definition of $C^2$-domains. Let $B_r(\xi) := \{\eta \in \mathbb{R}^d : \|\eta - \xi\| < r\}$ denote the open ball with center $\xi \in \mathbb{R}^d$ and radius $r > 0$ in $\mathbb{R}^d$, and $B_r(\xi)$ the closure of $B_r(\xi)$.

**Definition 1.1.** A bounded set $\Omega$ in $\mathbb{R}^d$ is called a $C^2$-domain if there exist a positive constant $\kappa_0$, and a finite cover of the boundary $\partial \Omega$ by open sets $\{U_j\}_{j=1}^J$ in $\mathbb{R}^d$ such that

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(i) for each $1 \leq j \leq J$, there exists a function $\Phi_j \in C^2(\mathbb{R}^d)$ such that
\[ U_j \cap \partial \Omega = \{ \xi \in U_j : \Phi_j(\xi) = 0 \} \quad \text{and} \quad \nabla \Phi_j(\xi) \neq 0, \quad \forall \xi \in U_j \cap \partial \Omega; \]
(ii) for each $\xi \in \partial \Omega$,
\[ B_{\kappa_0}(\xi - \kappa_0 n_\xi) \subset \Omega \quad \text{and} \quad B_{\kappa_0}(\xi + \kappa_0 n_\xi) \subset \Omega^c = \mathbb{R}^d \setminus \Omega, \]
where $n_\xi$ denotes the unit outer normal vector to $\partial \Omega$ at $\xi$.

Throughout this paper, unless otherwise stated, the greek letter $\Omega$ is always used to denote a compact $C^2$-domain with boundary $\Gamma := \partial \Omega$ in $\mathbb{R}^d$.

**Remark 1.2.** For the purposes of this paper, it was convenient to include the rolling ball property (ii) into the definition. However, under appropriate additional assumptions (e.g. if $\Omega$ and $\Omega^c$ have finitely many path-connected components) (ii) can be obtained as a consequence of (i), see [41, Th. 1(iii), (v), Rem. 3, p. 304].

Next, we introduce some necessary notations for the rest of this paper. Let $\mathbb{S}^{d-1} := \{ x \in \mathbb{R}^d : \| x \| = 1 \}$ denote the unit sphere of $\mathbb{R}^d$ with $d \geq 2$. Here and throughout the paper, $\| \cdot \|$ denotes the Euclidean norm of $\mathbb{R}^d$. Given a set $E \subset \mathbb{R}^d$, we denote by $|E|$ the Lebesgue measure of $E$ in $\mathbb{R}^d$, and $#E$ the cardinality of the set $E$. Furthermore, we define $\text{diam}(E) = \max_{\xi, \eta \in E} \| \xi - \eta \|$, and denote by $\text{dist}(\xi, E)$ the distance from a point $\xi \in \mathbb{R}^d$ to a set $E \subset \mathbb{R}^d$; that is, $\text{dist}(\xi, E) := \inf_{\eta \in E} \| \xi - \eta \|$. We denote by $L^p(E)$, $0 < p < \infty$ the usual Lebesgue $L^p$-space defined with respect to the $d$-dimensional Lebesgue measure $dx$ on the set $E \subset \mathbb{R}^d$.

In the case when $p = \infty$ and $E \subset \mathbb{R}^d$ is compact, we set $L^\infty(E) = C(E)$, the space of all continuous functions on $E$ with the uniform norm $\| \cdot \|_\infty$.

Let $\Pi_d^\mathbb{R}$ denote the space of all real algebraic polynomials in $d$ variables of total degree at most $n$. Given $\xi \in \mathbb{R}^d$, we denote by $\partial_\xi f = (\xi \cdot \nabla) f$ the directional derivative of $f$ along the direction of $\xi$, and for a positive integer $\ell$, we define $\partial^\ell_\xi := (\xi \cdot \nabla)^\ell$. Throughout this paper, the letter $c$ denotes a generic constant whose value may change from line to line, and the notation $A \sim B$ means that there exists a constant $c > 0$, called the constant of equivalence, such that $c^{-1}B \leq A \leq cB$.

The $L^p$-Bernstein-Markov inequalities on $\Omega$ are formulated in terms of the normal and tangential derivatives prescribed on the boundary $\Gamma$ of $\Omega$. To be precise, we denote by $n_\eta$ the outer unit normal vector to $\Gamma$ at $\eta \in \Gamma$. For $\xi \in \Omega$, $f \in C^\infty(\Omega)$ and nonnegative integers $l_1, l_2$, we define
\[ D^{l_1, l_2}_\eta f(\xi) := \max \left\{ \| \partial^\ell_\tau \partial^{l_2}_{n_\eta} f(\xi) \| : \tau \in \mathbb{S}^{d-1}, \tau \cdot n_\eta = 0 \right\}, \quad \eta \in \Gamma. \]

Furthermore, given a parameter $\mu \geq \sqrt{\text{diam}(\Omega)} + 1$, we define
\[ D^{l_1, l_2}_{n, \mu} f(\xi) := \max \left\{ D^{l_1, l_2}_\eta f(\xi) : \eta \in \partial \Omega, \| \eta - \xi \| \leq \mu \varphi_{n, \Gamma}(\xi) \right\}, \quad \xi \in \Omega, \]
where
\[ \varphi_{n, \Gamma}(\xi) := \sqrt{\text{dist}(\xi, \partial \Omega)} + n^{-1}, \quad n = 1, 2, \ldots, \xi \in \Omega. \]

Here, the inclusion of the term $n^{-1}$ in the distance function $\varphi_{n, \Gamma}$ allows us to combine the Bernstein and Markov type inequalities on a single line.

With the above notation, we can state our main result as follows:
Theorem 1.3. Let $\Omega \subseteq \mathbb{R}^d$ be a compact $C^2$-domain with boundary $\Gamma := \partial \Omega$, and let $\mu \geq \sqrt{\text{diam}(\Omega)} + 1$ be a given parameter. Then with the above notation, we have that for any $f \in \Pi_n^d$ and $0 < p \leq \infty$,

$$
(1.2) \quad \left\| \varphi_{n, \Gamma}^{r,j+l} f \right\|_{L^p(\Omega)} \leq C n^{r+j+2l} \left\| f \right\|_{L^p(\Omega)}, \quad r, j, l = 0, 1, \ldots,
$$

where the constant $C > 0$ depends only on $\Omega$, $\mu$ and $p$ as $p \to 0$.

Remark 1.4. (i) Note that the inequality (1.2) estimates not only the directional derivatives of $f$, but also a maximum of such derivatives in a certain neighborhood defined in (1.1). Therefore, the result can be viewed as a maximal function type inequality.

(ii) It can be easily seen that the order $n^{r+j+2l}$ on the right hand side of (1.2) is sharp as $n \to \infty$. Note that the standard one-dimensional Bernstein-Markov inequality applied along straight line segments from $\Omega$ would only give

$$
\left\| \varphi_{n, \Gamma}^{r,j+l} f \right\|_{L^\infty(\Omega)} \leq C n^{r+j+2l} \left\| f \right\|_{L^\infty(\Omega)}, \quad r, j, l = 0, 1, \ldots.
$$

The improvement from $n^{2r+j+2l}$ to $n^{r+j+2l}$ for the full range of $1 \leq p \leq \infty$ is exactly what is needed in many applications and reflects the correct order for the tangential component.

(iii) The same result is also true if we replace $\partial_\tau^j$ with mixed directional derivatives $\partial_{\tau_1} \ldots \partial_{\tau_{d-1}}$ in the definition of $D_{\eta}^{l_1, l_2}$, where $\{\tau_1, \ldots, \tau_{d-1}\}$ is an orthonormal basis of the space $\{\tau \in \mathbb{R}^d : \tau \cdot n_\eta = 0\}$ of tangent vectors to $\Gamma$ at $\eta$, and $\alpha_1 + \cdots + \alpha_{d-1} = l_1$. See Theorem 5.1 for the details.

(iv) Note that the operators $D_{n, \mu}^{l_1, l_2}$, $l_1, l_2 = 0, 1, \ldots$ are not commutative, which means that Theorem 1.3 for higher-order derivatives cannot be deduced from the corresponding result for lower order derivatives.

(v) Computationally, the quantity $\text{dist}(\xi, \partial \Omega)$ appearing in $\varphi_{n, \Gamma}$ may be inconvenient requiring certain non-linear minimization to be found. In Section 3, we introduce a readily computable equivalent metric defined in terms of the functions parametrizing the boundary of $\Omega$.

As an application of Theorem 1.3, we deduce Marcinkiewicz type inequalities on $C^2$-domains. These inequalities provide a basic tool for the discretization of the $L^p$-norm and are widely used in the study of the convergence properties of Fourier series, interpolation processes and orthogonal expansions. Interesting results on Marcinkiewicz type inequalities for univariate polynomials can be found in [19, 31, 34], while results for multivariate polynomials on some special multidimensional domains, such as the Euclidean balls, spheres, polytopes, cones, spherical sectors, toruses, were established in [9, 27, 35]. The Bernstein type inequality stated in Theorem 1.3 allows us to extend these results to general $C^2$-domains. For the special case $p = \infty$, such results have been shown to be valid under less restrictive smoothness requirements, see, e.g., [29, 34]. More recently, Marcinkiewicz type inequalities were studied in a more general setting for elements of finite dimensional spaces (see [10, 12, 25]).

We need some notations and a definition. Let $\rho : \Omega \times \Omega \to [0, \infty)$ be the metric on $\Omega$ defined by

$$
(1.3) \quad \rho(\xi, \eta) \equiv \rho_\Omega(\xi, \eta) := \|\xi - \eta\| + \sqrt{\text{dist}(\xi, \Gamma)} - \sqrt{\text{dist}(\eta, \Gamma)}, \quad \xi, \eta \in \Omega.
$$
For \( \xi \in \Omega \) and \( t > 0 \), we define
\[
U(\xi, t) := \{ \eta \in \Omega : \rho(\xi, \eta) \leq t \}, \quad U^o(\xi, t) := \{ \eta \in \Omega : \rho(\xi, \eta) < t \}.
\]
By Corollary 3.3 in Section 3, we have
\[
|U(\xi, t)| \sim t^d(\sqrt{\text{dist}(\xi, \Gamma)} + t), \quad \xi \in \Omega, \quad 0 < t \leq 2,
\]
with the constant of equivalence depending only on \( \Omega \). In particular, this implies
\[
|U(\xi, 2t)| \leq C|U(\xi, t)|, \quad \forall \xi \in \Omega, \quad \forall t > 0,
\]
where the constant \( C > 0 \) depends only on \( \Omega \). Moreover, according to Corollary 3.3(ii), every ball \( U(\xi, Lt) \) with \( L > 1 \) and \( t > 0 \) can be covered with \( m \leq C L^d \) balls \( U(\xi_j, t) \), \( j = 1, \ldots, m \).

**Definition 1.5.** A finite collection \( \mathcal{R} = \{R_1, \ldots, R_N\} \) of pairwise disjoint subsets of \( \Omega \) is called a partition of \( \Omega \) with norm \( \leq \delta \) if \( \Omega = \bigcup_{j=1}^N R_j \) and for each \( 1 \leq j \leq N \), \( R_j \subset U(\xi_j, \delta) \) for some \( \xi_j \in R_j \). A partition \( \{R_1, \ldots, R_N\} \) of \( \Omega \) is said to be a regular partition with norm \( \delta \) if it satisfies that
\[
U^o(\xi_j, \delta/2) \subset R_j \subset U(\xi_j, \delta), \quad \forall 1 \leq j \leq N.
\]

We give two remarks on this definition.

**Remark 1.6.** A regular partition of \( \Omega \) can be constructed through the so called maximal separated subsets of \( \Omega \). Indeed, for \( \delta > 0 \), a set \( \Lambda \subset \Omega \) is called \( \delta \)-separated (with respect to the metric \( \rho \)) if \( \rho(\omega, \omega') \geq \delta \) for any two distinct \( \omega, \omega' \in \Lambda \), while a \( \delta \)-separated subset \( \Lambda \subset \Omega \) is called maximal if \( \Omega = \bigcup_{\omega \in \Lambda} U(\omega, \delta) \). Given a maximal \( \delta \)-separated subset \( \{\xi_j\}_{j=1}^N \subset \Omega \), we define
\[
R_1 = \bigcap_{j=2}^N \left( U(\xi_1, \delta) \setminus U^o(\xi_j, \delta/2) \right),
\]
\[
R_j = \left( \bigcap_{1 \leq i \leq j} U(\xi_i, \delta) \setminus U^o(\xi_i, \delta/2) \right) \setminus \bigcup_{i=1}^{j-1} R_i, \quad j = 2, 3, \ldots, N.
\]
It is easily seen that \( \{R_1, \ldots, R_N\} \) is a regular partition of \( \Omega \) with norm \( \delta \).

**Remark 1.7.** If \( \{R_1, \ldots, R_N\} \) of \( \Omega \) is a regular partition of \( \Omega \) with norm \( \delta \in (0,1) \), then one must satisfy \( N \sim \delta^{-d} \), with the constants of equivalence depending only on \( \Omega \). To see this, let \( \xi_j \in R_j \), \( 1 \leq j \leq N \) be such that (1.4) is satisfied. Then the set \( \{\xi_j\}_{j=1}^N \) is \( \frac{1}{\delta} \)-separated, and hence by Corollary 3.3(iii), we have \( N \leq C_1 \delta^{-d} \).

On the other hand, by (1.4) and (1.5), we have
\[
|\Omega| = \sum_{j=1}^N |R_j| \leq \sum_{j=1}^N |U(\xi_j, \delta)| \leq C \delta^d \left( 1 + \sqrt{\text{diam}(\Omega)} \right) N,
\]
which implies the lower estimate \( N \geq C_1 \delta^{-d} \).

Now we can state the Marcinkiewicz type inequality on a \( C^2 \)-domain as follows:

**Theorem 1.8.** Given a compact \( C^2 \)-domain \( \Omega \subset \mathbb{R}^d \), there exists a constant \( \delta_0 \in (0,1) \) depending only on \( \Omega \) such that if \( \{R_1, \ldots, R_N\} \) is a partition of \( \Omega \) with norm
Theorem 1.10. Given a compact $C^\infty$-domain, there exists a constant $\delta_0 \in (0, 1)$ depending only on $\Omega$ such that if $\{R_1, \ldots, R_N\}$ is a partition of $\Omega$ with norm $\leq \frac{\delta_0}{n}$ for some positive integer $n$, then for any points $\xi_j \in R_j$, $1 \leq j \leq N$, and any $1 \leq q < \infty$, we have
\[
\frac{1}{2}\|f\|_{L^q(\Omega)} \leq \left( \sum_{j=1}^N \|f(\xi_j)\|^q |\partial R_j| \right)^{\frac{1}{q}} \leq \frac{2}{q} \|f\|_{L^q(\Omega)}, \quad \forall f \in \Pi_n^d,
\]
where we need to replace the $\ell^p$-norm with $\max_{1 \leq j \leq N} |f(\xi_j)|$ if $p = \infty$.

Remark 1.9. While the Bernstein inequality in Theorem 1.3 holds for the full range of $0 < p \leq \infty$, our proof of the Marcinkiewicz inequality (1.6), which relies on Hölder’s inequality, fails for $0 < p < 1$.

We will also apply Theorem 1.3 to show the existence of “good” positive cubature formulas on $C^2$-domains:

Theorem 1.10. Given a compact $C^2$-domain $\Omega \subset \mathbb{R}^d$, there exists a constant $\delta_0 \in (0, 1)$ depending only on $\Omega$ such that if $\{R_1, \ldots, R_N\}$ is a partition of $\Omega$ with norm $\leq \frac{\delta_0}{n}$ for some positive integer $n$, then for any points $\xi_j \in R_j$, $1 \leq j \leq N$, there exist weights $\lambda_j > 0$, $1 \leq j \leq N$ such that
\[
\frac{1}{4}|R_j| \leq \lambda_j \leq C \left| U \left( \xi_j, \frac{1}{n} \right) \right|, \quad j = 1, 2, \ldots, N
\]
and
\[
\int_{\Omega} f(\eta) \, d\eta = \sum_{j=1}^N \lambda_j f(\xi_j), \quad \forall f \in \Pi_n^d,
\]
where $C > 0$ is a constant depending only on $\Omega$.

Note that both Theorem 1.3 and Theorem 1.10 apply to general partitions of the domain $\Omega$. If, in addition, $\{R_1, \ldots, R_N\}$ is a regular partition of $\Omega$ with norm $\frac{\delta_0}{n}$, then the number of required points $\xi_j$ in both (1.6) and (1.7) has the optimal asymptotic order $n^d \sim \text{dim} \Pi_n^d$ as $n \to \infty$, and the weights in (1.6) and (1.7) satisfy
\[
\lambda_j \sim |R_j| \sim \left( \frac{\delta}{n} \right)^d \left( \frac{\delta}{n} \right)^d + \text{dist}(\xi_j, \partial \Omega), \quad j = 1, 2, \ldots, N.
\]

Now we present the structure of the paper and the methods used. In Section 2, we give the precise definition of $C^2$-domains, and introduce a certain class of $C^2$-domains, called domains of special type, which have simpler boundary structure. More importantly, we prove a decomposition proposition, Proposition 2.5, which asserts that every compact $C^2$-domain can be decomposed as a finite union of domains of special type.

After that, in Section 3, we introduce a new metric $\tilde{\rho}_G$ on a domain $G$ of special type, and prove that $\tilde{\rho}_G$ is equivalent to the restriction of the metric $\rho_\Omega$ on $G$ if $G \subset \Omega$ is attached to the boundary $\partial \Omega$. This new metric $\tilde{\rho}_G$ has the advantage that it is easier to deal with. Moreover, the equivalence of $\tilde{\rho}_G$ with $\rho_\Omega$ combined with the decomposition proposition for $C^2$-domains allows us to effectively reduce considerations near the boundary to certain problems on domains of special type.

After these preparations, we prove our main result, Theorem 1.3, in Sections 4.

The proof is long and rather involved, so we break it into several parts. A crucial part is given in Section 4, where we establish a Bernstein type inequality on domains of special type in $\mathbb{R}^2$. An important ingredient in our proof is to construct a family of parabolas touching the boundary $\Gamma = \partial \Omega$ and lying inside the domain.
\(\Omega\), for which every point \((x, y) \in \Omega\) near the boundary \(\Gamma\) can be connected with a unique boundary point through one of these parabolas. In many cases, we may use these parabolas to replace the usual line segments that are parallel to one of the coordinate axis. Indeed, performing a change of variables, we prove in Section 4 that every double integral near the boundary of \(\Omega\) can be expressed as iterated integrals along the family of parabolas. This technique plays a crucial role in the proof of the Bernstein inequality along tangential directions on the boundary \(\partial \Omega\).

In Section 5, we show how the Bernstein type inequality on higher-dimensional domains of special type can be deduced from the corresponding result for \(d = 2\). One of the main difficulties in the higher-dimensional case comes from the fact that we have to deal with certain non-commutative mixed directional derivatives along different tangential directions on the boundary.

In Section 6, we prove Theorem 1.3 using the decomposition proposition, and the results on domains of special type that have already been proven in Sections 4-5.

Let us now provide some brief comments regarding earlier relevant works. A different approach to the problem of estimating derivatives of algebraic polynomials is based on pluripotential theory, see, e.g. the fundamental work [1] and the paper [6] which also discusses the connection with the “real geometric” approach pursued here. The natural idea of reducing the multivariate problem to the univariate one by considering restrictions to certain smaller-dimensional subsets appears already in [42] where restrictions to segments yield Markov-type estimates for gradients of polynomials. This would not suffice for Bernstein-type estimates and considering higher-order restrictions, such as ellipses or parabolas, is a natural choice. For example, in [38], ellipsoids were used to obtain strong Bernstein-Markov type inequalities, while parametrization by curves of degree two was employed in [26]. In contrast to the above works, our primary focus in the current work is integral norms, where the associated family of degree two curves (parabolas in our case) needs to cover the domain in a uniform manner allowing an appropriate change of variable.

The proofs of Theorem 1.8 and Theorem 1.10 are given in the last section, Section 7. The Bernstein inequality stated in Theorem 1.3 plays a crucial role in these proofs.

Finally, we point out that the Bernstein inequality stated in Theorem 1.3 has many applications besides the applications presented in Section 7. These include the inverse theorem on the error of polynomial approximation in terms of moduli of smoothness, and the Chebyshev type cubature formulas on \(C^2\)-domains. We will return to these topics in an upcoming paper.

2. Decomposition of \(C^2\)-domains into domains of special type

Our aim in this section is to show that every compact \(C^2\)-domain can be decomposed as a finite union of domains of special type, whose definition is given as follows:

**Definition 2.1.** A set \(G \subset \mathbb{R}^d\) is called an upward \(x_d\)-domain with base size \(b > 0\) and parameter \(L \geq 1\) if there exist \(\xi \in \mathbb{R}^d\) and \(g \in C^2(\mathbb{R}^{d-1})\) such that

\[
G = \xi + \left\{ (x, y) \in \mathbb{R}^d : \ x \in (-b, b)^{d-1}, \ g(x) - Lb < y \leq g(x) \right\}.
\]
In this case, we define

\[(2.1) \quad G^* : = \xi + \left\{ (x, y) : x \in (-2b, 2b)^{d-1}, \min_{u \in (-2b, 2b)^{d-1}} g(u) - 4Lb < y \leq g(x) \right\}. \]

Furthermore, given a parameter \( \lambda > 0 \), we define

\[G(\lambda) : = \xi + \left\{ (x, y) : x \in (-\lambda b, \lambda b)^{d-1}, \quad g(x) - \lambda Lb < y \leq g(x) \right\},\]

and call the set

\[\partial^* G(\lambda) := \xi + \left\{ (x, g(x)) : x \in (-\lambda b, \lambda b)^{d-1} \right\}\]

the essential boundary of the set \( G(\lambda) \).

Several remarks on this definition are in order.

**Remark 2.2.** (a) With a possible change of the point \( \xi \in \mathbb{R}^d \), we may always assume that the function \( g \) satisfies

\[\min_{x \in [-2b, 2b]^{d-1}} g(x) = 4Lb.\]

(b) For technical reasons, sometimes we may need to choose the base size \( b \) as small as we wish, and the parameter \( L \) large enough so that

\[(2.2) \quad L \geq 4\sqrt{d} \quad \max_{x \in [-2b, 2b]^{d-1}} \|\nabla g(x)\| + 1.\]

In general, given \( 1 \leq j \leq d \), we may define an upward or downward \( x_j \)-domain \( G \subset \mathbb{R}^d \) and the corresponding sets \( G(\lambda), \partial^* G(\lambda), G^* \) in a similar way, using the reflections: for \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \),

\[\sigma_j(x) : = (x_1, \ldots, x_{j-1}, x_d, x_{j+1}, \ldots, x_{d-1}, x_j),\]

\[\tau_j(x) : = (x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_d).\]

**Definition 2.3.**

(i) A set \( G \subset \mathbb{R}^d \) is called an upward \( x_j \)-domain with base size \( b > 0 \) and parameter \( L \geq 1 \) if its reflection \( E : = \sigma_j(G) \) is an upward \( x_d \)-domain with base size \( b \) and parameter \( L \), in which case we define

\[G(\lambda) = \sigma_j(E(\lambda)), \quad \partial^* G(\lambda) = \sigma_j(\partial^* E(\lambda)), \quad G^* = \sigma_j(E^*).\]

(ii) A set \( G \subset \mathbb{R}^d \) is called a downward \( x_j \)-domain with base size \( b > 0 \) and parameter \( L \geq 1 \) if its reflection \( H : = \tau_j(G) \) is an upward \( x_d \)-domain with base size \( b \) and parameter \( L \geq 1 \), in which case we define

\[G(\lambda) = \tau_j(H(\lambda)), \quad \partial^* G(\lambda) = \tau_j(\partial^* H(\lambda)), \quad G^* = \tau_j(H^*).\]

Now we are in a position to define a domain of special type attached to the boundary of a \( C^2 \)-domain. Let \( e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d \) denote the standard canonical basis in \( \mathbb{R}^d \). A rectangular box in \( \mathbb{R}^d \) is a set of the form \([a_1, b_1] \times \cdots \times [a_d, b_d] \) with \( -\infty < a_j < b_j < \infty, \ j = 1, \ldots, d \). Here and throughout the paper, we always assume that the sides of a rectangular box are parallel with the coordinate axes.

**Definition 2.4.**

(i) A set \( G \subset \mathbb{R}^d \) is called a domain of special type if there exists \( 1 \leq j \leq d \) such that it is either an upward or a downward \( x_j \)-domain. In this case, we call \( \partial^* G(\lambda) \) the essential boundary of \( G(\lambda) \), and set

\[\partial^* G := \partial^* G(1) \quad \text{and} \quad \partial^* G^* := \partial^* G(2).\]
(ii) Let $\Omega$ be a compact $C^2$-domain in $\mathbb{R}^d$ and $G \subset \Omega$ a domain of special type.

We say $G$ is attached to the boundary $\partial \Omega$ of $\Omega$ if $\overline{G^c} \cap \partial \Omega = \partial G^c$ and there exists an open rectangular box $Q$ in $\mathbb{R}^d$ such that $G^* = Q \cap \Omega$.

With the above definitions, we can now state the main result in this section as follows:

**Proposition 2.5.** Given a compact $C^2$-domain $\Omega \subset \mathbb{R}^d$, there exists a finite cover of the boundary $\Gamma := \partial \Omega$ of $\Omega$ by domains of special type $G_1, \ldots, G_{m_0} \subset \Omega$ attached to $\Gamma$. Moreover, we may select these domains in a way such that the base size of each $G_j$ is as small as we wish, and the parameter of each $G_j$ satisfies the condition (2.2).

**Remark 2.6.** From the proof of Proposition 2.5 below, it is easily seen that for each parameter $\lambda_0 \in (0, 1]$, we can find $m_0 \leq C(\Omega, \lambda_0)$ domains $G_1, \ldots, G_{m_0} \subset \Omega$ of special type attached to the boundary $\Gamma$ of a given $C^2$-domain $\Omega$ such that $\Gamma \subset \bigcup_{j=1}^{m_0} G_j(\lambda_0)$. The point here is that we can choose the parameter $\lambda_0 \in (0, 1]$ as small as we wish.

**Remark 2.7.** By Definition 2.4, for each domain $G_j$ given in Proposition 2.5, we have $G_j(\lambda) \subset G_j^* \subset \Omega$ for all $0 < \lambda \leq 2$.

**Proof.** Since the essential boundary $\partial' G$ of a domain $G \subset \Omega$ of special type attached to $\Gamma$ is open relative to the topology of $\Gamma$, it suffices to show that for each fixed $\xi \in \Gamma$, there exists a domain $G_\xi$ of special type attached to $\Gamma$ such that $\xi \in \partial' G_\xi$, and such that its base size is as small as we wish, and its parameter satisfies (2.2).

Without loss of generality, we may assume that $\xi = 0 \in \Gamma$ since otherwise we may translate the domain $\Omega$ so that the origin coincides with the origin. Let $n_0$ denote the unit outer normal vector to $\Gamma$ at the origin. Without loss of generality, we may also assume that $n_0 \cdot e_d = \max_{1 \leq i \leq d} |n_0 \cdot e_i|$, since the other cases can be treated similarly. Indeed, if $1 \leq j \leq d$ is such that $|n_0 \cdot e_j| = \max_{1 \leq i \leq d} |n_0 \cdot e_i|$, then a slight modification of the construction of the set $G_0$ below would lead to an upward or a downward $x_j$-domain attached to $\Gamma$ according to whether $n_0 \cdot e_j > 0$ or $n_0 \cdot e_j < 0$.

Since $\Omega$ is a $C^2$-domain, there exists a number $r_0 \in (0, 1)$ depending only on $\Omega$ such that for each $\eta \in \Gamma$,

\begin{equation}
B_{8d r_0}([\eta - 8d r_0 n_\eta]) \subset \Omega \quad \text{and} \quad B_{8d r_0}([\eta + 8d r_0 n_\eta]) \subset \Omega^c = \mathbb{R}^d \setminus \Omega,
\end{equation}

where $n_\eta$ denotes the unit outer normal vector to $\Gamma$ at $\eta$.

Next, since $n_0 \in S^{d-1}$, we have $n_0 \cdot e_d = \max_{1 \leq i \leq d} |n_0 \cdot e_i| \geq \frac{1}{\sqrt{d}}$. Thus, by the implicit function theorem, there exist an open rectangular box $V_0 := I_0 \times (-a_0, a_0)$ centered at $0 \in \Gamma$ with $I_0 := (-\delta_0, \delta_0)^{d-1}$ and $a_0 > 0$, and a $C^2$-function $h$ on $\mathbb{R}^{d-1}$ such that $h(0) = 0$, $h(I_0) \subset (-a_0, a_0)$, and the surface $\Gamma_0 := \Gamma \cap V_0$ can be represented as

\[ \Gamma_0 = \{(x, h(x)) : x \in I_0\}. \]

By continuity, we may choose the constant $\delta_0$ small enough so that $0 < \delta_0 < r_0$, $n_\eta \cdot e_d \geq \frac{1}{2 \sqrt{d}}$ for every $\eta \in \Gamma_0$, and

\begin{equation}
\| \nabla h \|_{L^\infty(I_0)} < r_0 \delta_0^{-1}.
\end{equation}

In particular, (2.4) implies that

\begin{equation}
|h(x)| < \sqrt{d} r_0, \quad \forall x \in I_0.
\end{equation}
Using (2.3) with \( \eta = (x, h(x)) \in \Gamma_0 \) and \( x \in I_0 \), and taking into account the fact that \( n_\eta \cdot e_\delta \geq \frac{1}{2} / \sqrt{d} \), we obtain a simple geometric argument that
\[
\begin{align*}
\{(x, y) : x \in I_0, \ h(x) < y < h(x) + 8\sqrt{d}r_0 &\} \subset \Omega^c, \\
\end{align*}
\]
and
\[
\begin{align*}
\{(x, y) : x \in I_0, \ h(x) - 8\sqrt{d}r_0 \leq y < h(x) &\} \subset \Omega^c, \\
\end{align*}
\]
where \( \Omega^c \) denotes the interior of \( \Omega \). It then follows by (2.5) that
\[
\begin{align*}
\{(x, y) : x \in I_0, \ h(x) < y < 7\sqrt{d}r_0 &\} \subset \Omega^c, \\
\end{align*}
\]
and
\[
\begin{align*}
\{(x, y) : x \in I_0, \ -7\sqrt{d}r_0 \leq y < h(x) &\} \subset \Omega^c.
\end{align*}
\]

Now setting
\[
\begin{align*}
\varsigma_0 := (0, -7\sqrt{d}r_0), \ g_0(x) := h(x) + 7\sqrt{d}r_0, \\
\end{align*}
\]
we obtain from (2.0) and (2.7) that
\[
\begin{align*}
\varsigma_0 + \{(x, y) : x \in I_0, \ g(x) < y < 14\sqrt{d}r_0 &\} \subset \Omega^c, \\
\end{align*}
\]
and
\[
\begin{align*}
\varsigma_0 + \{(x, y) : x \in I_0, \ 0 \leq y < g_0(x) &\} \subset \Omega^c.
\end{align*}
\]
Furthermore, by (2.4), we have
\[
6\sqrt{d}r_0 < g_0(x) < 8\sqrt{d}r_0, \quad \forall x \in I_0.
\]

Third, we define
\[
G_0 := \varsigma_0 + \{(x, y) : x \in (-b_0, b_0)^{d-1}, \ g_0(x) - \delta_0 \leq y \leq g_0(x)\},
\]
where \( b_0 \in (0, \delta_0/2) \) is a constant such that
\[
\frac{\delta_0}{b_0} \geq 4\sqrt{d}/\|\nabla h\|_{L^\infty(I_0)} + 1.
\]
Clearly, \( G_0 \) is an upward \( x_d \)-domain with base size \( b_0 \) and parameter \( L_0 := \delta_0/b_0 \). Since \([-2b_0, 2b_0]^{d-1} \subset I_0 \) and \( \delta_0 < r_0 \), we obtain from (2.10) that
\[
\alpha_0 := \min_{x \in [-2b_0, 2b_0]^{d-1}} g_0(x) - 4L_0b_0 \geq 6\sqrt{d}r_0 - 4\delta_0 > 2r_0 > 0.
\]
It then follows from (2.9) that
\[
G_0^* = \varsigma_0 + \{(x, y) : x \in (-2b_0, 2b_0)^{d-1}, \ \alpha_0 < y \leq g_0(x)\}
\]
\[
\subset \varsigma_0 + \{(x, y) : x \in [-\delta_0, \delta_0]^{d-1}, \ 0 \leq y \leq g_0(x)\} \subset \Omega.
\]
Finally, (2.9) implies that
\[
\overline{G_0^*} \cap \Gamma = \overline{\partial G_0^*}
\]
while (2.0) together with (2.8) implies that
\[
G_0^* = \Omega \cap Q_0
\]
with \( Q_0 = \varsigma_0 + (-2b_0, 2b_0)^{d-1} \times (\alpha_0, 14\sqrt{d}r_0) \). Thus, \( G_0 \subset \Omega \) is an upward \( x_d \)-domain attached to \( \Gamma \). This completes the proof of Proposition 2.6. \( \square \)
3. Metrics on domains of special type

Let $\rho_G : \Omega \times \Omega \to [0, \infty)$ be the metric on $\Omega$ given in (1.3). In this section, we shall introduce a new metric $\hat{\rho}_G$ on a domain $G$ of special type, which is equivalent to the restriction of $\rho_G$ on $G$ if $G \subset \Omega$ is attached to $\Gamma := \partial \Omega$. This new metric $\hat{\rho}_G$ is easier to deal with in applications.

To be precise, let $G \subset \mathbb{R}^d$ be an $x_d$-upward domain with base size $b \in (0, 1)$ and parameter $L > 0$:

$$G := \varsigma + \{(x, y) : x \in (-b, b)^{d-1}, \ g(x) - Lb < y \leq g(x)\}, \ \varsigma \in \mathbb{R}^d,$$

where $g$ is a $C^2$-function on $\mathbb{R}^{d-1}$. Then

$$G^* = \varsigma + \{(x, y) : x \in (-2b, 2b)^{d-1}, \ \min_{u \in [-2b, 2b]^{d-1}} \ g(u) - 4Lb < y \leq g(x)\}$$

and we define a metric $\hat{\rho}_G : \overline{G^*} \times \overline{G^*} \to (0, \infty)$ by

$$(3.1) \quad \hat{\rho}_G(\varsigma + \xi, \varsigma + \eta) := \max\left\{ \|\xi_x - \eta_x\|, \sqrt{g(\xi_x) - \xi_y} - \sqrt{g(\eta_x) - \eta_y}\right\}$$

for all $\xi = (\xi_x, \xi_y), \eta = (\eta_x, \eta_y) \in \overline{G^*} - \varsigma$. Here and throughout this paper, we often use Greek letters $\xi, \eta, \alpha, \ldots$ to denote points in $\mathbb{R}^d$ and write $\xi \in \mathbb{R}^d$ as $\xi = (\xi_x, \xi_y)$ with $\xi_x \in \mathbb{R}^{d-1}$ and $\xi_y \in \mathbb{R}$. Finally, we can define the metric $\hat{\rho}_G$ on a more general $x_j$-domain $G \subset \mathbb{R}^d$ (upward or downward) in a similar way.

Our aim in this section is to show that the metric $\hat{\rho}_G$ defined above is equivalent to the restriction of $\rho_G$ on $G$ when $G \subset \Omega$ is attached to $\Gamma = \partial \Omega$.

**Proposition 3.1.** If $G \subset \Omega$ is a domain of special type attached to $\Gamma$, then

$$\hat{\rho}_G(\xi, \eta) \sim \rho_G(\xi, \eta), \quad \xi, \eta \in G$$

with the constants of equivalence depending only on $G$ and $\Omega$.

**Proof.** Without loss of generality, we may assume that

$$G := \{(x, y) : x \in (-b, b)^{d-1}, \ g(x) - Lb < y \leq g(x)\}, \ b \in (0, 1), \ L \geq 1,$$

and $g \in C^2(\mathbb{R}^{d-1})$ satisfies $\min_{x \in [-2b, 2b]^{d-1}} g(x) = 4Lb$. Indeed, a slight modification of the proof below works equally well for more general domains of special type.

Since $G$ is attached to $\Gamma$, it follows that

$$\Gamma' := \{(x, g(x)) : x \in [-2b, 2b]^{d-1}\} \subset \Gamma.$$

The following lemma plays an important role in the proof of Proposition 3.1.

**Lemma 3.2.** If $\xi = (\xi_x, \xi_y) \in G$, then

$$c_*(g(\xi_x) - \xi_y) \leq \text{dist}(\xi, \Gamma') \leq g(\xi_x) - \xi_y,$$

where $c_* = \frac{1}{3\sqrt{1 + \|\nabla g\|_{\infty}}}$ and $\|\nabla g\|_{\infty} = \max_{x \in [-2b, 2b]^{d-1}} \|\nabla g(x)\|$. 

**Proof.** Let $\xi = (\xi_x, \xi_y) \in G$. Since $(\xi_x, g(\xi_x)) \in \partial' G \subset \Gamma'$, we have

$$\text{dist}(\xi, \Gamma') \leq \|(\xi_x, \xi_y) - (\xi_x, g(\xi_x))\| = g(\xi_x) - \xi_y.$$

It remains to prove the inverse inequality,

$$(3.2) \quad \text{dist}(\xi, \Gamma') \geq c_*(g(\xi_x) - \xi_y).$$
Let \((x, g(x)) \in \Gamma^\prime\) be such that
\[
\text{dist}(\xi, \Gamma^\prime) = \|\xi - (x, g(x))\|.
\]
Since
\[
\text{dist}(\xi, \Gamma^\prime) \geq \|x - \xi_x\|,
\]
holds trivially if \(\|x - \xi_x\| \geq c_*(g(\xi_x) - \xi_y)\). Thus, without loss of generality, we may assume that \(\|x - \xi_x\| < c_*(g(\xi_x) - \xi_y)\). We then write
\[
\|\xi - (x, g(x))\|^2 = \|\xi_x - x\|^2 + |\xi_y - g(\xi_x)|^2 +
\]
\[
\|g(\xi_x) - g(x)\|^2 + 2(\xi_y - g(\xi_x)) \cdot (g(\xi_x) - g(x)).
\]
Since \(\|x - \xi_x\| \leq c_*(g(\xi_x) - \xi_y)\), we have
\[
\|\xi_x - x\|^2 + |\xi_y - g(\xi_x)|^2 + 2(\xi_y - g(\xi_x)) \cdot (g(\xi_x) - g(x))
\]
\[
\leq (1 + \|\nabla g\|_\infty^2)\|\xi_x - x\|^2 + 2\|\nabla g\|_\infty(g(\xi_x) - \xi_y)\|\xi_x - x\|
\]
\[
\leq \left[ c_*^2(1 + \|\nabla g\|_\infty^2) + 2\|\nabla g\|_\infty c_* \right] (g(\xi_x) - \xi_y)^2 \leq \frac{7}{9} (g(\xi_x) - \xi_y)^2.
\]
Thus, using \((3.3)\), we obtain
\[
\text{dist}(\xi, \Gamma^\prime)^2 = \|\xi - (x, g(x))\|^2 \geq \frac{2}{9} (\xi_y - g(\xi_x))^2,
\]
which implies the desired lower estimate \((3.2)\). This completes the proof of Lemma \((3.2)\) \(\square\)

Let us return to the proof of Proposition \((3.1)\). Set
\[
\Gamma^\prime := \{(x, g(x)) : x \in \left[-\frac{3b}{2}, \frac{3b}{2}\right]^{d-1}\} \subset \Gamma^\prime.
\]
Since \(G \subset \Omega\) is attached to \(\Gamma\), there exists an open rectangular box \(Q \subset \mathbb{R}^d\) such that
\[
G^* = \{(x, y) : x \in (-2b, 2b)^{d-1}, \ 0 < y \leq g(x)\} = \Omega \cap Q.
\]
In particular, this implies that there exists a small constant \(\varepsilon_0 \in (0, b/4)\) depending only on \(\Omega\) and \(G\) such that \(\text{dist}(\xi, \Gamma \setminus \Gamma^\prime) \geq \varepsilon_0\) whenever \(\xi \in G\).

Let \(\varepsilon \in (0, \varepsilon_0)\) be a small constant to be specified later. Let \(\xi = (\xi_x, \xi_y), \eta = (\eta_x, \eta_y) \in G\). From Lemma \((3.2)\), it is easily seen that if \(\max(\text{dist}(\xi, \Gamma), \text{dist}(\eta, \Gamma)) \geq \varepsilon\) or \(\|\xi - \eta\| \geq \varepsilon\), then
\[
\rho_\Omega(\xi, \eta) \sim \tilde{\rho}_G(\xi, \eta) \sim \|\xi - \eta\|,
\]
where the constants of equivalence depend on \(\varepsilon\). Thus, it suffices to prove that there exists a constant \(\varepsilon \in (0, \varepsilon_0)\) depending only on \(\Omega\) and \(G\) such that
\[
\rho_\Omega(\xi, \eta) \sim \tilde{\rho}_G(\xi, \eta)
\]
if
\[
\text{dist}(\xi, \Gamma) < \varepsilon, \ \text{dist}(\eta, \Gamma) < \varepsilon \quad \text{and} \quad \|\xi - \eta\| < \varepsilon.
\]

To show this, we need to introduce a few notations. Given \(\alpha = (\alpha_x, \alpha_y) \in G\), we set \(s_\alpha := \text{dist}(\alpha, \Gamma^\prime)\), and denote by \(t_\alpha\) the point in \([-\frac{3b}{2}, \frac{3b}{2}]^{d-1}\) such that
\[
s_\alpha = \|\alpha - (t_\alpha, g(t_\alpha))\|.
\]
Since $\Gamma$ is a $C^2$-surface, a straightforward calculation shows that every $\alpha = (\alpha_x, \alpha_y) \in G$ can be represented as a function of $(t, s) = (\alpha_x, \alpha_y)$:

$$\alpha = \left(x(t, s), g(t) - s_A(t)\right),$$

where

$$x(t, s) := t + sA(t)\nabla g(t), \quad A(t) = \left(\sqrt{1 + \|\nabla g(t)\|^2}\right)^{-1},$$

$$t \in [-2b, 2b]^{d-1}, \quad s > 0.$$ 

Thus, for every $\alpha = (\alpha_x, \alpha_y) \in G$,

$$g(\alpha_x) - \alpha_y = F(t, s),$$

where $F$ is a function on the rectangular box $[-2b, 2b]^{d-1} \times [0, \varepsilon_0]$ given by

$$F(t, s) := g(x(t, s)) - g(t) + sA(t).$$

Now we turn to the proof of (3.4). Note first that (3.5) implies that

$$\text{dist}(\xi, \Gamma) = \text{dist}(\xi, \Gamma') = \text{dist}(\xi, \Gamma'') \quad \text{and} \quad \text{dist}(\eta, \Gamma) = \text{dist}(\xi, \Gamma') = \text{dist}(\eta, \Gamma'').$$

Thus, $s_\xi, s_\eta \in [0, \varepsilon]$. Furthermore, by (3.6), we have

$$\|t_\xi - \xi_x\| \leq s_\xi < \varepsilon \quad \text{and} \quad \|t_\eta - \eta_x\| \leq s_\eta < \varepsilon.$$

Since $\|\eta - \xi\| < \varepsilon$, it follows that $\|t_\xi - t_\eta\| < 2\varepsilon$.

Now using Lemma 3.2 and (3.8), we obtain

$$\|\nabla F(t, s)\| = \left\|\nabla g(x(t, s)) - \nabla g(t) + sA(t)\right\|$$

and

$$\frac{\partial F}{\partial s}(t, s) = \left[1 + (\nabla g)(x(t, s)) \cdot \nabla g(t)\right]A(t) = \sqrt{1 + \|\nabla g(t)\|^2} + O(\varepsilon),$$

and

$$\frac{\partial F}{\partial t}(t, s) = \left[1 + (\nabla g)(x(t, s)) \cdot \nabla g(t) + A(t)H_g(t)\right]$$

where $\nabla$ denotes the gradient operator $\nabla$ acting on the variable $t$, $\nabla g$ is treated as a row vector, and $H_g := (\partial_i \partial_j g)_{1 \leq i, j \leq d-1}$ denotes the Hessian matrix of $g$. Thus, using the mean value theorem, (3.7) and (3.11), we obtain

$$\frac{\partial F}{\partial s}(t, s) = \frac{F(t, s) - F(t, s)}{\sqrt{s_\xi} + \sqrt{s_\eta}} = \frac{F(t, s) - F(t, s)}{\sqrt{1 + |\nabla g(t)|^2} + O(\varepsilon)}\left|\sqrt{s_\xi} - \sqrt{s_\eta}\right| + O(\varepsilon),$$

which together with (3.9) implies that

$$\|\xi - \eta_x\| = \|x(t_\xi, s_\xi) - x(t_\eta, s_\eta)\| \leq \|t_\xi - t_\eta + s_\xi A(t_\xi)\nabla g(t_\xi) - s_\eta A(t_\eta)\nabla g(t_\eta)\|$$

provided that $\varepsilon \in (0, \varepsilon_0)$ is sufficiently small. On the other hand, using (3.7) and the mean value theorem, we have

$$\|\xi - \eta_x\| = \|x(t_\xi, s_\xi) - x(t_\eta, s_\eta)\| = \|t_\xi - t_\eta + s_\xi A(t_\xi)\nabla g(t_\xi) - s_\eta A(t_\eta)\nabla g(t_\eta)\|$$

$$= \|t_\xi - t_\eta\| + O(\varepsilon)\left|\sqrt{s_\xi} - \sqrt{s_\eta}\right| + O(\varepsilon).$$
which implies
\begin{equation}
(3.13) \quad \| \xi - \eta \| \sim \| t - t \| + O(\sqrt{\varepsilon}) \sqrt{s_{\xi} - s_{\eta}}.
\end{equation}
provided that $\varepsilon \in (0, \varepsilon_0)$ is small enough. Thus, combining (3.12) and (3.13), we deduce
\begin{align*}
\tilde{\rho}_G(\xi, \eta) &\sim \left| \sqrt{g(\xi)} - \xi_y - \sqrt{g(\eta)} - \eta_y \right| + \| \xi - \eta \|
\sim |\sqrt{s_{\xi}} - \sqrt{s_{\eta}}| + O(\sqrt{\varepsilon}) \| t - t \| + \| \xi - \eta \| \sim |\sqrt{s_{\xi}} - \sqrt{s_{\eta}}| + \| \xi - \eta \|
\sim \rho_G(\xi, \eta) = \max \left\{ |\sqrt{s_{\xi}} - \sqrt{s_{\eta}}|, \| \xi - \eta \| \right\},
\end{align*}
where the last step uses the fact that $|\xi_y - \eta_y| \leq C \| \xi - \eta \| \leq C \tilde{\rho}_G(\xi, \eta).

This completes the proof of Proposition 3.1. \qed

Recall that $U(\xi, \delta) := \{ \eta \in \Omega : \rho_G(\xi, \eta) \leq \delta \}$.

We conclude this section with the following useful corollary:

**Corollary 3.3.** (i) For each $\xi \in \Omega$ and $\delta \in (0, 1)$,
\begin{equation}
(3.14) \quad |U(\xi, \delta)| \sim \delta^d \left( \delta + \sqrt{\text{dist}(\xi, \Gamma)} \right)
\end{equation}
with the constant of equivalence depending only on $\Omega$.

(ii) There exists a constant $C > 1$ depending only on $\Omega$ such that for each $\xi \in \Omega$, $\delta > 0$ and $L > 1$, there exist $m \leq CL^d$ points $\xi_1, \ldots, \xi_m \in U(\xi, L\delta)$ such that $U(\xi, L\delta) \subset \bigcup_{j=1}^{m} U(\xi_j, \delta)$.

(iii) There exists a constant $C > 1$ depending only on $\Omega$ such that for each $\delta \in (0, 1)$, and each $(\delta, \rho_G)$-separated subset of $\Omega$, we have $\# \Lambda \leq C \delta^{-d}$.

**Proof.** (i) By Proposition 2.5 and Proposition 3.1, it suffices to prove the property for each domain $G \subset \Omega$ of special type attached to $\Gamma$, and for the metric $\tilde{\rho}_G$ given in (3.1). Without loss of generality, we may assume that $G := \left\{ (x, y) : x \in (-b, b)^{d-1}, \ g(x) - b < y \leq g(x) \right\}$, where $b > 0$ and $g$ is a $C^2$-function on $\mathbb{R}^{d-1}$ satisfying $\max_{x \in [-2a, 2a]^{d-1}} g(x) = 4b$. By Lemma 3.2, it is enough to show that for each $\xi = (\xi_x, \xi_y) \in G$ and $0 < \delta < b/2$,
\begin{equation}
(3.15) \quad \left| \left\{ (x, y) \in G^* : \| x - \xi \| \leq \delta, \ \sqrt{g(\xi_x) - \xi_y} - \sqrt{g(x) - y} \leq \delta \right\} \right| \sim \delta^d (\delta + \sqrt{g(\xi_x) - \xi_y}).
\end{equation}
Indeed, (3.15) can be easily verified by Fubini’s theorem. This proves (3.14).

Note that a similar argument also shows that for any $\delta_1, \delta_2 > 0$ and $\xi \in \Omega$,
\begin{equation}
(3.16) \quad \left| \left\{ \eta \in \Omega : \| \eta - \xi \| \leq \delta_1, \ \text{dist}(\eta, \Gamma) \leq \delta_2 \right\} \right| \leq C \delta_1^{d-1} \delta_2.
\end{equation}
Then there exists a point $\xi \in \{\xi_1, \ldots, \xi_N\}$ be a maximal $(\delta, \rho_0)$-separated subset of $U(\xi, L\delta)$. It is enough to show that $N \leq CL^d$. If $\text{dist}(\xi, \Gamma) \geq 4(L\delta)^2$, then $\text{dist}(\xi, \Gamma) \sim \text{dist}(\xi, \Gamma)$ for $1 \leq j \leq N$, which, by (3.14), implies that

$$|U(\xi, L\delta)| \sim (L\delta)^d \sqrt{\text{dist}(\xi, \Gamma)}$$

and $|U(\xi, \delta/2)| \sim \delta^d \sqrt{\text{dist}(\xi, \Gamma)}$.

Thus, a standard volume comparison argument shows that $N \leq CL^d$ if $\text{dist}(\xi, \Gamma) \geq 4(L\delta)^2$.

Now assume that $\text{dist}(\xi, \Gamma) < 4(L\delta)^2$. Then $\text{dist}(\xi, \Gamma) < 9(L\delta)^2$ for $1 \leq j \leq N$. Let $m_0 \in \mathbb{N}$ be such that $2^{m_0-1} \leq 3L < 2^{m_0}$. Then $\Lambda \subset \bigcup_{k=0}^{m_0} A_k$, where

$$A_0 = \{ \eta : \|\eta - \xi\| \leq L\delta, \quad \text{dist}(\eta, \Gamma) \leq \delta^2 \},$$

$$A_k = \{ \eta : \|\eta - \xi\| \leq L\delta, \quad (2^k-1)\delta^2 < \text{dist}(\eta, \Gamma) \leq (2^k)\delta^2 \}, \quad k \geq 1.$$

By (3.14) and (3.19), we have that for $0 \leq k \leq k_0$,

$$|A_k| \leq C(L\delta)^{d-1}4k\delta^2 \quad \text{and} \quad |U(\eta, \delta/2)| \sim 2^k\delta^{d+1}, \quad \forall \eta \in A_k.$$

Thus, a standard volume comparison argument shows that

$$N_k := \#(\Lambda \cap A_K) \leq C2^kL^{d-1}, \quad k = 0, 1, \ldots, m_0.$$

It follows that $N \leq \sum_{k=0}^{m_0} N_k \leq CL^d$.

(ii) Let $\kappa_0 > 0$ be the parameter of the $C^2$-domain $\Omega$ given in Definition 1.1. Then there exists a point $\xi \in \Omega$ such that $\text{dist}(\xi, \partial\Omega) \geq \frac{1}{2}\kappa_0$. By the definition of the metric $\rho_\Omega$ (see (1.3)), we have $\Omega = U(\xi, L\delta)$, where

$$L = \frac{(1 + \frac{4}{\kappa_0})\text{diam}(\Omega)}{\delta}.$$

Thus, every $\delta$-separated subset $\Lambda$ of $\Omega$ is contained in a maximal $\delta$-separated subset of $U(\xi, L\delta)$. By the proof in Part (ii), it then follows that

$$\#\Lambda \leq C_\Omega L^d \leq C_\Omega \delta^{-d}.$$

\[ \square \]

4. Bernstein inequality on domains of special type in $\mathbb{R}^2$

In this section, we shall prove a Bernstein type inequality on domains of special type in $\mathbb{R}^2$. Let $G \subset \mathbb{R}^2$ be an $x_2$-upward domain given by

$$(4.1) \quad G := \{(x, y) : \quad a < x \leq g(x)\},$$

where $a > 0$, $L \geq 1$ and $g$ is a $C^2$-function on $\mathbb{R}$ satisfying $\min_{x \in [-2a, 2a]} g(x) = 4La$. Following the notations in Section 2, we have that

$$G^* := \{(x, y) : x \in (-2a, 2a), \ \ 0 < y \leq g(x)\},$$

and for each $\mu > 0$,

$$G(\mu) := \{(x, y) : x \in (-\mu a, \mu a), \ \ g(x) - \mu L a < y \leq g(x)\},$$

$$\partial'G(\mu) := \{(x, g(x)) : x \in (-\mu a, \mu a)\}, \quad \partial'G = \partial'G(1), \quad \partial'G^* = \partial'G(2).$$

For $(x, y) \in G^*$, we define

$$\delta(x, y) := g(x) - y \quad \text{and} \quad \varphi_n(x, y) := \sqrt{\delta(x, y)} + \frac{1}{n}, \quad n = 1, 2, \ldots.$$
According to Lemma 3.2, we have
\[
\delta(x, y) = g(x) - y \sim \text{dist}(\xi, dG^*), \quad \forall \xi = (x, y) \in G.
\]

The Bernstein type inequality on the domain G is formulated in terms of the tangential derivatives along the essential boundary \(dG^* \) of G. To be precise, for each fixed \(x_0 \in (-2a, 2a)\) and positive integer \(\ell\), we denote by \(D^\ell\), the \(\ell\)-th order direction derivative along the tangential direction \((1, g'(x_0))\) to \(dG^*\) at the point \((x_0, g(x_0))\):
\[
D^\ell_{x_0} := \left( \partial_1 + g'(x_0) \partial_2 \right)^\ell = \sum_{i=0}^\ell \binom{\ell}{i} (g'(x_0))^i \partial_1^{\ell-i} \partial_2^i,
\]
where \(\partial_1 = \frac{\partial}{\partial x}\) and \(\partial_2 = \frac{\partial}{\partial y}\). With a slight abuse of notation, we denote by \(D^{(\ell)}\) the \(\ell\)-th order tangential differential operator given by
\[
D^{(\ell)} f(x, y) := (D^\ell_2 f)(x, y) = \sum_{i=0}^\ell \binom{\ell}{i} (g'(x))^i (\partial_1^{\ell-i} \partial_2^i f)(x, y),
\]
where \(f \in C^1(G^*)\) and \((x, y) \in G^*\). Clearly, the operator \(D^{(\ell)}\) is commutative with \(\partial_2\), but not commutative with \(\partial_1\). As a result, the operators \(D^{(\ell)}\), \(\ell = 1, 2, \ldots\) are not commutative. It is also worthwhile to point out here that \(D^{(\ell)}\) is not the \(\ell\)-th power \(D^\ell\) of the operator \(D\); namely, \(D^{(\ell+1)} \neq D^{(\ell)} D\).

Our aim in this section is to prove the following Bernstein type inequality on the domain G:

**Theorem 4.1.** Let \(G \subset \mathbb{R}^2\) be the domain given in (4.1). If \(0 < p \leq \infty, \lambda > 1\) and \(f \in \Pi_\lambda^2\), then
\[
\|\varphi_n \partial_1^j \partial_2^i f\|_{L^p(G)} \leq cn^{r+i+2j} \|f\|_{L^p(G(\lambda))}, \quad r, i, j = 0, 1, \ldots,
\]
where \(c\) is a positive constant independent of \(f\) and \(n\), and the set \(G(\lambda)\) is given in (4.2).

Theorem 4.1 allows us to establish a slightly stronger Bernstein type inequality on the domain G.

**Corollary 4.2.** Let \(\lambda > 1\) and \(\mu > 1\) be two given parameters. Let \(G \subset \mathbb{R}^2\) be the domain given in (4.1). Then for any \(0 < p \leq \infty\) and \(f \in \Pi_\mu^2\),
\[
\|\varphi_n(x, y)^j \varphi_{n(x, y)}^{\max} |D^\ell f(x, y)|\|_{L^p(G)} \leq c\mu r^{r+i+2j} \|f\|_{L^p(G(\lambda))}, \quad r, i, j = 0, 1, \ldots,
\]
where \(c > 0\) is independent of \(f\), \(n\) and \(\mu\), the first \(L^p\)-norm is computed with respect to the Lebesgue measure \(dx dy\) on the domain \(G\), and the set \(G(\lambda)\) is given in (4.2).

**Remark 4.3.** (i) It can be easily shown that the order \(n^{r+i+2j}\) in (4.4) and (4.5) is sharp as \(n \to \infty\).
(ii) Using translations and reflections, similar results can also be established on any domain \(G \subset \mathbb{R}^2\) of special type as defined in Definition 2.4 without the further technical assumptions made in this section.

The rest of this section is organized as follows. In subsection 4.1 we prove Corollary 4.2 assuming Theorem 4.1. The proof of Theorem 4.1 for \(r = 1\), which is relatively simpler, but already contains the crucial ideas required for the proof.
for $r > 1$, is given in subsection 1.2. Finally, we prove Theorem 4.1 for $r > 1$ in subsection 4.3 which is technically more involved.

4.1. **Proof of Corollary 4.2 (assuming Theorem 4.1).** Let $M > 1$ denote a constant such that

$$
\|g''\|_{L^\infty([-2a,2a])} \leq M \quad \text{and} \quad |g'(0)| \leq M.
$$

For any fixed $x_0, t \in [-a, a]$, we have

$$
D_{x_0+t} = D_{x_0} + \left(g(x_0 + t) - g(x_0)\right) \partial_2 = D_{x_0} + |t|O(x_0, t) \partial_2,
$$

where $\sup_{t \in [-a, a]} |O(x_0, t)| \leq M$. Since the operators $\partial_2$ and $D_{x_0}$ are commutative, we obtain from (4.7) that if $(x_0, y_0) \in G$ and $|t| \leq \min\{\mu \varphi_n(x_0, y_0, a)\}$, then

$$
\varphi_n(x_0, y_0)^{i+j} D_{x_0+t}^{r+i+r} f(x_0, y_0) \leq C \mu^r \max_{r_1 + r_2 = r} \varphi_n(x_0, y_0)^{i+j} \|D_{x_0}^{r+i+j} f(x_0, y_0)\|.
$$

It then follows from Theorem 4.1 that

$$
\max_{|t| \leq \min\{\mu \varphi_n(x, y, a)\}} \|D_{x_0+t}^{r+i+j} f(x, y)\|_{L^p(G; dx dy)} \leq C \mu^r \max_{r_1 + r_2 = r} \|D_{x_0}^{r+i+j} f\|_{L^p(G)} \leq C \mu^r n^{i+j} \|f\|_{L^p(G(\lambda))}.
$$

This completes the proof of Corollary 4.2.

4.2. **Proof of Theorem 4.1 for $r = 1$.** This subsection is devoted to the proof of Theorem 4.1 for $r = 1$. Since the term on the right hand side of (4.4) is increasing in $\lambda > 1$, without loss of generality, we may assume that $\lambda \in (1, 2)$. We need two lemmas, the first of which is a direct consequence of the univariate Bernstein inequality for algebraic polynomials.

**Lemma 4.4.** Let $G \subset \mathbb{R}^2$ be the domain given in 1.1. Assume that $f \in C(G)$ satisfies that $f(x, \cdot) \in \Pi^1_n$ for each fixed $x \in [-a, a]$. Then for $0 < p \leq \infty$ and $\lambda \in (1, 2)$,

$$
\|\varphi_n^{i+j} \partial_2^{i+j} f\|_{L^p(G)} \leq cn^{i+j} \|f\|_{L^p(G(\lambda))}, \quad i, j = 0, 1, \ldots,
$$

where $c > 0$ is a constant independent of $f$ and $n$.

**Proof.** If $P$ is an algebraic polynomial of one variable of degree $\leq n$, then by the univariate Bernstein inequality (p. 265), we have that for any $b > 0$ and $\alpha > 1$,

$$
\left\| (\sqrt{b+1} t + n^{-1}) P^{(i+j)}(t) \right\|_{L^p([0,b], dt)} \leq C \alpha n^{i+j} b^{-(i+j)} \|P\|_{L^p([0,\alpha b])}.
$$

Lemma 4.4 then follows by integration over vertical line segments. \hfill \Box

Our next lemma, Lemma 4.5 below, plays a crucial role in the proof of Theorem 4.1. We first explain briefly the basic idea behind this lemma. Note that every point $(x, y) \in G$ can be connected with the point $(x, g(x)) \in \partial G$ via a vertical line segment. Taking into account the tangential direction along the boundary of $G$, we shall construct a family of parabolas which touch and lie below the essential boundary $\partial^* G$, and use them to replace these vertical line segments. Lemma 4.5 below asserts that each point $(x, y) \in G$ can be connected with a point $(z, g(z))$ on $\partial^* G$ via a unique parabola which passes though $(x, y)$ and touches $\partial^* G$ at $(z, g(z))$. As a result, performing a change of variables, we may express double integrals over the
domain $G$ in terms of iterated integrals along the family of parabolas, and after that apply the univariate Bernstein inequality to integrals along these parabolas.

To be more precise, let $M > 10$ be any fixed constant satisfying (4.6). Given a parameter $A > 0$, we define

$$Q_A(z, t) := g(z) + g'(z)t - \frac{A}{2}t^2, \quad z \in [-2a, 2a], \quad t \in \mathbb{R}.$$  

By Taylor’s theorem, if $z, z + t \in [-2a, 2a]$, then

$$g(z + t) - Q_A(z, t) = \int_0^t [A + g''(u)](z + tu) du. \tag{4.9}$$

Thus, for each fixed $z \in [-2a, 2a]$ and each parameter $A > M$,

$$y = Q_A(z, x - z) = g(z) + g'(z)(x - z) - \frac{A}{2}(x - z)^2, \quad x \in [-2a, 2a]$$

is a parabola that lies below the curve $y = g(x)$, but touches it at the point $(z, g(z)) \in \partial G^2$.

Set $\lambda := 1 + \frac{1}{M}$. Let

$$E_A := \{(z, t) \in \mathbb{R}^2 : \quad z, z + t \in [-\lambda a, \lambda a], \quad |t| \leq a_0 := \sqrt{\frac{2La}{A + M}}\},$$

and define the mapping $\Phi_A : E_A \to \mathbb{R}^2$ by

$$\Phi_A(z, t) = (x, y) := (z + t, Q_A(z, t)), \quad (z, t) \in E_A.$$  

We set $E_A^+ = \{(z, t) \in E_A : \quad t \geq 0\}$, $E_A^- = \{(z, t) \in E_A : \quad t \leq 0\}$, and denote by $\Phi_A^+$ and $\Phi_A^-$ the restrictions of $\Phi_A$ on the sets $E_A^+$ and $E_A^-$ respectively. Also, we denote by $J_{\Phi_A}$ the Jacobian of the mapping $\Phi_A$.

With the above notation, we have the following crucial lemma:

**Lemma 4.5.** Let $\Lambda := (2 + \frac{16M}{a})M^2 + M$. Then the following statements hold for every parameter $A \geq \Lambda$:

(i) $\Phi_A(E_A) \subset G(\lambda)$, and moreover, both the mappings $\Phi_A^+ : E_A^+ \to G(\lambda)$ and $\Phi_A^- : E_A^- \to G(\lambda)$ are injective.

(ii) For every $(x, y) \in G$, there exists a unique $z \in [-\lambda a, \lambda a]$ such that $z \leq x \leq z + a_1$, and both $(x, y)$ and $(z, g(z))$ lie on the same parabola

$$\{(u, v) : \quad v = Q_A(z, u - z), \quad z \leq u \leq z + a_1\},$$

where

$$a_1 := \sqrt{\frac{2La}{A - M}} < a_0.$$

As a result, every $(x, y) \in G$ can be represented uniquely in the form $(x, y) = \Phi_A(z, t)$ with $(z, t) \in E_A^+$ and $0 \leq t \leq a_1$, and moreover, $G \subset \Phi_A(E_A^+)$.  

(iii) If $(z, t) \in E_A$, then

$$|\det (J_{\Phi_A}(z, t))| = (A + g''(z))|t|, \tag{4.10}$$

where we recall that $J_{\Phi_A}$ denotes the Jacobian of the mapping $\Phi_A$.  

(iv) Let \( u_A \) be the function on the set \( \Phi_A(E^+_A) \) given by
\[
(4.11) \quad u_A(x, y) := g'(z + t) - g'(z) + At =: w_A(z, t),
\]
where \((x, y) = \Phi_A(z, t) \) and \((z, t) \in E^+_A \). Then for every \((x, y) \in \Phi_A(E^+_A) \),
\[
(4.12) \quad \frac{(A - M)\sqrt{2}}{\sqrt{A + M}}\sqrt{\delta(x, y)} \leq u_A(x, y) \leq \frac{(A + M)\sqrt{2}}{\sqrt{A - M}}\sqrt{\delta(x, y)}.
\]

Proof. (i) First, we prove that \( \Phi_A(E_A) \subset G(\lambda) \). Indeed, if \((x, y) = \Phi_A(z, t) \) with \((z, t) \in E_A \), then \( x = z + t \in [-a\lambda, a\lambda] \) and by (4.9),
\[
(4.13) \quad g(x) - y = g(z + t) - Q_A(z, t) = \int^z_{z+t} (g''(u) + A)(z + t - u) \, du.
\]
Since \(|t| \leq a_0 \), this implies that \( 0 \leq g(x) - y \leq (M + A)a_0^2/2 = \lambda La \), which shows that \((x, y) \in G(\lambda) \).

Next, we show that both of the mappings \( \Phi^+_A \) and \( \Phi^-_A \) are injective. Assume that \( \Phi_A(z_1, t_1) = \Phi_A(z_2, t_2) \) for some \((z_1, t_1), (z_2, t_2) \in E_A \) with \( t_1 t_2 \geq 0 \) and \( t_2 \geq t_1 \). Then \( z_1 + t_1 = z_2 + t_2 =: \bar{x} \), \( Q_A(z_1, t_1) = Q_A(z_2, t_2) \), and hence, using (4.13), we obtain that for \( i = 1, 2 \),
\[
g(\bar{x}) - Q_A(z_i, t_i) = g(z_i + t_i) - Q_A(z_i, t_i) = \int^t_{t_i} [g''(\bar{x} - v) + A]v \, dv,
\]
which implies
\[
(4.14) \quad \int^t_{t_i} (g''(\bar{x} - v) + A)v \, dv = 0.
\]
Since \( g''(\bar{x} - v) + A \geq A - M > 0 \) and \( v \) doesn’t change sign on the interval \([t_1, t_2] \), (4.14) implies that \( t_1 = t_2 \), which in turn implies that \( z_1 = z_2 \). Thus, both \( \Phi^+_A \) and \( \Phi^-_A \) are injective.

(ii) First, a straightforward calculation shows that for \( A \geq \overline{A} \),
\[
a_1 < a_0 \quad \text{and} \quad a + a_1 < \lambda a.
\]

Next, we show that there exists \( z \in [-\lambda a, \lambda a] \) such that \( y = Q_A(z, x - z) \) and \( z \leq x \leq z + a_1 \). Define \( h(s) := g(x) - Q_A(x - s, s) \) for \( 0 \leq s \leq a_1 \). Using (4.9) with \( z = x - s \), we obtain
\[
h(s) = \int^x_{x-s} (g''(u) + A)(x - u) \, du,
\]
which implies that
\[
h(a_1) \geq (A - M)\int_0^{a_1} v \, dv = \frac{1}{2}(A - M)a_1^2 = La.
\]
Since \( h \) is a continuous function on \([0, a_1] \) and \( h(0) = 0 \), it follows by the Intermediate Value theorem that \([0, La] \subset h[0, a_1] \). Since \((x, y) \in G \), we have \(-a \leq x \leq a \), and \( g(x) - y \in [0, La] \). Thus, there exists \( t \in [0, a_1] \) such that \( h(t) = g(x) - y \). Setting \( z = x - t \), we have \( y = Q_A(z, x - z) \) and
\[
-\lambda a \leq -a - a_1 \leq z = x - t \leq x \leq a.
\]
Finally, we show the uniqueness of the number \( z \). Indeed, setting \( t = x - z \), we have
\[
(x, y) = \Phi_A(z, t) = (z + t, Q_A(z, t)), \quad (z, t) \in E_A, \quad 0 \leq t \leq a_1.
\]
The uniqueness of the number $z$ then follows from the fact that the mapping $\Phi_A^+$ is injective.

(iii) Equation (4.10) can be verified straightforwardly.

(iv) Let $(x, y) = \Phi_A(z, t)$ with $(z, t) \in E_A^+$. By (4.11) and the mean value theorem, we have

\[ (A - M)t \leq u_A(x, y) \leq (A + M)t. \]

On the other hand, however, using (4.9), we have that

\[ g(x) - y = g(z + t) - Q_A(z, t) = \int_z^{z+t} (g''(u) + A)(z + t - u) \, du, \]

which in particular implies

\[ (A - M)\frac{t^2}{2} \leq g(x) - y \leq (A + M)\frac{t^2}{2}. \]

\[ (A - M)t \leq u_A(x, y) \leq (A + M)t. \]

Finally, combining (4.15) with (4.16), we deduce the desired estimates (4.12).

\[ \Box \]

\textbf{Remark 4.6.} Lemma 4.4 with several different parameters $A$ will be required in our proof of Theorem 4.1 for $r > 1$. However, for the proof in the case of $r = 1$, it will be enough to have this lemma for one fixed parameter $A \geq \overline{A}$ only.

We are now in a position to prove Theorem 4.1 for $r = 1$.

\textbf{Proof of Theorem 4.1 for $r = 1$.} If $f \in \Pi_n^2$ and $x \in [-2a, 2a]$, then $D_x f(x, \cdot)$ is a polynomial of degree at most $n$ of a single variable. Since the operators $D$ and $\partial_2$ are commutative, it follows from Lemma 4.4 that

\[ \|\varphi_n^j D^{i+j} f\|_{L^p(G)} = \|\varphi_n^j \partial_2^{i+j} Df\|_{L^p(G)} \leq Cn^{i+j} \|Df\|_{L^p(G(\sqrt{x}))}. \]

Thus, it is sufficient to show that for any $\lambda \in (1, 2)$,

\[ \|Df\|_{L^p(G)} \leq Cn \|f\|_{L^p(G(\lambda))}, \quad \forall f \in \Pi_n^2 \]

Without loss of generality, we may assume that $M := \frac{1}{\lambda - 1}$ satisfies (4.10). Also, in our proof below, we shall always assume that $p < \infty$. The case $p = \infty$ can be treated similarly, and in fact, is simpler.

To prove (4.17), we set

\[ I := \|Df\|^p_{L^p(G)} = \int_G \left| D_x f(x, y) \right|^p \, dx \, dy, \]

and let $A \geq \overline{A}$ be a fixed parameter. Using Lemma 4.3(i) and (iii), and performing the change of variables $x = z + t$ and $y = Q_A(z, t)$, we obtain

\[ I = \int_{(\Phi_A^+)^{-1}(G)} \left| D_{z+t} f(\Phi_A(z, t)) \right|^p \left( A + g''(z) \right) t \, dz \, dt. \]

A straightforward calculation shows that for each $(z, t) \in E_A$,

\[ D_{z+t} f(\Phi_A(z, t)) = \frac{df}{dt} \left( f(\Phi_A(z, t)) \right) + w_A(z, t) \partial_2 f(\Phi_A(z, t)), \]

where $w_A(z, t) = g'(z + t) - g'(z) + At$. Thus, $I \leq 2^p (I_1 + I_2)$, where

\[ I_1 := \int_{(\Phi_A^+)^{-1}(G)} \left| \frac{df}{dt} \left( f(\Phi_A(z, t)) \right) \right|^p \left( A + g''(z) \right) t \, dz \, dt, \]

\[ I_2 := \int_{(\Phi_A^+)^{-1}(G)} |w_A(z, t)|^p |\partial_2 f(\Phi_A(z, t))|^p \left( A + g''(z) \right) t \, dz \, dt. \]
For the double integral $I_2$, performing the change of variables $(x, y) = \Phi_A^+(z, t)$, and using Lemma 4.3, we obtain

$$
I_2 = \int_G |u_A(x, y)|^p |\partial_x f(x, y)|^p \, dx \, dy \leq C \int_G \sqrt{\delta(x, y) |\partial_x f(x, y)|^p} \, dx \, dy
$$

$$
= C \int_{-a}^a \int_{g(x)-La}^{g(x)} |\sqrt{\delta(x, y) |\partial_x f(x, y)|^p} \, dy \, dx,
$$

which, using the Markov-Bernstein-type inequality (4.8), is bounded above by

$$
\leq Cn \int_{-a}^a \int_{g(x)-La}^{g(x)} |f(x, y)|^p \, dy \, dx \leq Cn^p \|f\|_{L^p(G(\lambda))}^p.
$$

Thus, it remains to prove that

$$
(4.18) \quad I_1 \leq C_n^p \|f\|_{L^p(G(\lambda))}^p.
$$

Since $A \geq \overline{A}$, we have $a_0 < \frac{n}{2M}$. Thus, using Lemma 4.3(ii), we obtain that in the $zt$-plane,

$$
(4.19) \quad (\Phi_A^+)^{-1}(G) \subset [-a - a_1, a] \times [0, a_1] \subset [-a - a_0, a] \times [-a_0, a_0] \in E_A.
$$

It follows that

$$
I_1 \leq C(A) \int_{-a-a_1}^a \left( \int_0^{a_1} \left[ \frac{d}{dt} \left| f(z + t, Q_A(z, t)) \right|^p \right] |t| \, dt \right) \, dz,
$$

which, using Bernstein’s inequality with doubling weights ([34, Theorem 7.3], [19, Theorem 3.1]) and the fact that $a_1 < a_0$, is bounded above by

$$
\leq C(M, p)n \int_{-a-a_1}^a \left[ \int_{-a_0}^{a_0} \left| f(z + t, Q_A(z, t)) \right|^p |t| \, dt \right] \, dz
$$

$$
\leq C(M, p)n \int_{E_A} \left| f(z + t, Q_A(z, t)) \right|^p |t| \, dz \, dt.
$$

Splitting this last double integral into two parts $\int_{E_A^+} + \int_{E_A^-}$, and applying the change of variables $(x, y) = \Phi_A(z, t)$ to each of them separately, we obtain that

$$
I_1 \leq C(M, p)n^p \int_{E_A^+} \left| f(\Phi_A(z, t)) \right|^p |A + g''(z)||t| \, dz \, dt
$$

$$
\leq C(M, p)n^p \int_{\Phi(A^+)} \left| f(x, y) \right|^p \, dx \, dy + \int_{\Phi(A^-)} \left| f(x, y) \right|^p \, dx \, dy
$$

$$
\leq C(M, p)n^p \|f\|_{L^p(G(\lambda))}^p,
$$

where we used Lemma 4.3(i), (iii) in the second step, and the fact that $\Phi_A(E_A) \subset G(\lambda)$ (i.e., Lemma 4.3(i)) in the third step. \hfill \qed

4.3. Proof of Theorem 4.1 for $r > 1$. Again, without loss of generality, we assume $\lambda \in (1, 2)$. First, we note that by (4.3) and the comments immediately after (4.3), the operators $D^{(r)}$, $r = 1, 2, \ldots$ are not commutative. As a result, we cannot deduce the Bernstein inequality (4.3) for $r > 1$ directly from the already proven case of $r = 1$ via iteration. The proof for $r > 1$ requires a few additional lemmas.

We start with the following lemma for computing higher order derivatives of certain composite functions:
Lemma 4.7. Let $f \in C^\infty(\mathbb{R}^2)$ and define $F(t) := f(z + t, Q(t))$ for $t \in \mathbb{R}$, where $z \in \mathbb{R}$ and $Q$ is a univariate polynomial of degree at most 2. Then for any $r \in \mathbb{N}$,

$$
\frac{d^r F}{dt^r} = \sum_{i+j+2k=r} \frac{r!}{i!j!k!2^k} (Q')^i (Q'')^j \partial_1^i \partial_2^j \partial_3^k f + \sum_{2(i+j)=r} \frac{r!}{i!j!2^r} (Q')^i \partial_1^i f,
$$

where the summations are taken over non-negative integers $i$, $j$, $k$, and the partial derivatives are evaluated at $(z + t, Q(t))$.

Proof. This is a special case of the multivariate Faa di Bruno formula from [5, p. 505]. □

Lemma 4.7 allows us to compute the higher order derivatives of the composite function $f \circ \Phi_A$:

Lemma 4.8. Let $A \geq \overline{A}$ be an arbitrarily given parameter, and let $(z_0, t_0) \in E_A^+$ be such that $(x_0, y_0) = \Phi_A(z_0, t_0) \in G$. Assume that $f \in C^\infty(\mathbb{R}^2)$ and $F(t) := f(\Phi_A(z_0, t))$ for $t \in \mathbb{R}$. Then for any $r \in \mathbb{N}$,

$$
\frac{d^r F(t_0)}{dt^r} = S_1(A, x_0, y_0) + S_2(A, x_0, y_0) + S_3(A, x_0, y_0),
$$

where

$$
S_1(A, x_0, y_0) := \sum_{j=0}^r \binom{r}{j} (-u_A(x_0, y_0))^j (D^{(r-j)} \partial_1^j f)(x_0, y_0),
$$

$$
S_2(A, x_0, y_0) := \sum_{i+j+2k=r, k \geq 1} \frac{r!}{i!j!k!2^k} (-u_A(x_0, y_0))^j (-A)^k (D^{(i)} \partial_2^i \partial_3^k f)(x_0, y_0),
$$

$$
S_3(A, x_0, y_0) := \sum_{2(i+j)=r} \frac{r!}{i!j!2^r} (-A)^i (D^{(i)} \partial_1^i f)(x_0, y_0).
$$

Proof. Define $\tilde{f}(u, v) := f(u, g'(x_0)u + v)$ for $(u, v) \in \mathbb{R}^2$. Then

$$
\partial_1^i \tilde{f}(u, v) = D^i_{x_0} f(u, g'(x_0)u + v) \quad \text{and} \quad \partial_2^i \tilde{f}(u, v) = \partial_2^i f(u, g'(x_0)u + v).
$$

We rewrite the function $F$ as

$$
F(t) = f(z_0 + t, Q_A(z_0, t)) = \tilde{f}(z_0 + t, Q(t)),
$$

where

$$
Q(t) := Q_A(z_0, t) - g'(x_0)(z_0 + t) = g(z_0) + g'(z_0)t - \frac{A}{2} t^2 - g'(z_0 + t_0)(z_0 + t).
$$

Then (4.24) implies that

$$
\partial_1^i \tilde{f}(z_0 + t_0, Q(t_0)) = D^i_{x_0} f(x_0, y_0) \quad \text{and} \quad \partial_2^i \tilde{f}(z_0 + t_0, Q(t_0)) = \partial_2^i f(x_0, y_0).
$$

To complete the proof of (4.20), we just need to apply Lemma 4.7 to the function $F(t)$, observing that

$$
Q'(t_0) = g'(z_0) - At_0 - g'(z_0 + t_0) = -w_A(z_0, t_0) = -u_A(x_0, y_0),
$$

and $Q''(t) = -A$.

□

The following lemma allows us to use Lemma 4.5 for several distinct parameters $A$. 

Lemma 4.9. Given a positive integer \( r \), there exist constants \( A_0, A_1, \ldots, A_r \) depending only on \( M \) and \( r \) such that \( A \leq A_0 < \cdots < A_r \) and for any \( (x_0, y_0) \in G \), and \( f \in C^r(\mathbb{R}^2) \),
\[
(4.25) \quad \max_{0 \leq i \leq r} \left| \frac{(\delta(x_0, y_0))^{1/2}(D^{(r-i)}\partial_2^i f)(x_0, y_0)}{(\delta(x_0, y_0))^{1/2}(D^{(r-i)}\partial_2^i f)(x_0, y_0)} \right| \leq c \max_{0 \leq i \leq r} |S_1(A_i, x_0, y_0)|,
\]
where the constant \( c \) depends only on \( M, r \) and \( a \), and \( S_1(A, x_0, y_0) \) is given in (4.21).

Proof. Note first that if \( \delta(x_0, y_0) = 0 \), then by (4.12), \( u_A(x_0, y_0) = 0 \), and hence by (4.21), \( S_1(A, x_0, y_0) = D^{(i)}f(x_0, y_0) \). Thus, (4.25) holds trivially if \( \delta(x_0, y_0) = 0 \).

For the reminder of the proof, we always assume that \( \delta(x_0, y_0) > 0 \).

Next, select a strictly increasing sequence of constants \( A_0, A_1, \ldots, A_r \geq \overline{A} \) depending only on \( M \) and \( r \) such that for \( i = 0, 1, \ldots, r - 1 \),
\[
\frac{2(A_i + M)}{\sqrt{A_i M}} < \frac{A_{i+1} - M}{\sqrt{A_{i+1} M}}
\]
which, using Lemma 4.5 (iv), implies
\[
(4.26) \quad 0 < c_{M,r} \leq \frac{u_A(x_0, y_0)}{\delta(x_0, y_0)} < \frac{u_{A_{i+1}}(x_0, y_0)}{\delta(x_0, y_0)} \leq C_{M,r}.
\]

Setting
\[
B_j = -\frac{u_{A_{j}}(x_0, y_0)}{\delta(x_0, y_0)} \quad \text{and} \quad u_j := \binom{r}{j} \delta(x_0, y_0)^{r-j} D^{(r-j)} \partial_2^j f(x_0, y_0)
\]
for \( j = 0, 1, \ldots, r \), we may apply (4.21) to obtain
\[
S_1(A_i, x_0, y_0) = \sum_{j=0}^{r} (B_j)^j u_j, \quad i = 0, 1, \ldots, r,
\]
which can be written equivalently as
\[
(4.27) \quad S = M u,
\]
where
\[
S = \begin{pmatrix} S_1(A_0, x_0, y_0), \ldots, S_1(A_r, x_0, y_0) \end{pmatrix}^t, \quad u = (u_0, u_1, \ldots, u_r)^t,
\]
and \( M \) is the \((r + 1) \times (r + 1)\) Vandermonde matrix with \((i, j)\)-entry \( M_{i,j} := (B_i)^j \).

By (4.26), we have
\[
0 < c_M \leq \min_{0 \leq i \neq j \leq r} |B_j - B_i| \leq C_M.
\]
Thus, the stated estimate (4.25) follows from (4.27) and Cramer’s rule. \( \square \)

Now we are in a position to prove Theorem 4.1 for \( r > 1 \).

Proof of Theorem 4.1. For \( r > 1 \). We use induction on \( r \in \mathbb{N} \). The case \( r = 0 \) is given in Lemma 4.4, whereas the case \( r = 1 \) has already been proven in subsection 4.2. Now suppose the Bernstein type inequality (4.1) has been established for all the derivatives \( D^{(\ell)} \partial_2^{i+j} \), with \( \ell = 0, 1, \ldots, r - 1 \) and \( i, j = 0, 1, \ldots, r \). Our goal is to prove that for any \( f \in \Pi^n_\alpha \) and \( \lambda \in (1, 2) \),
\[
\|f^t D^{(r')} \partial_2^{i+j} f\|_{L^p(G)} \leq c n^{r+i+j+2} \|f\|_{L^p(G(\lambda))}, \quad i, j = 0, 1, \ldots
\]
Since the operators $D^{(i)}$ and $\partial^j_r$ are commutative, by Lemma [4.4], it is sufficient to show that for every $f \in \mathcal{P}_n^{(p)}$ and any $\lambda \in (1, 2)$,

$$
(4.28) \quad \|D^{(r)}f\|_{L^p(G)} \leq cn^r \|f\|_{L^p(G(\lambda))}.
$$

Here and throughout the proof, the constant $c$ depends only on $p$, $a$, $M$ and $r$.

We start with the case of $p = \infty$, which is simpler. By Lemma [4.9] it is enough to show that

$$
(4.29) \quad \max_{0 \leq v \leq r} \max_{(x_0, y_0) \in G} |S_1(A_v, x_0, y_0)| \leq cn^r \|f\|_{L^\infty(G(\lambda))}.
$$

Fix temporarily $0 \leq v \leq r$ and $(x_0, y_0) \in G$. By Lemma [4.9 (ii)], there exists $(z_0, t_0) \in E^+_\lambda_n$ such that $0 \leq t_0 \leq a_1$ and $\Phi_{A_n}(z_0, t_0) = (x_0, y_0)$. Applying Lemma [4.8] to the function $F(t) := f(\Phi_{A_n}(z_0, t))$, and recalling (Lemma [4.5 (iv)])

$$
|u_{A_n}(x_0, y_0)| \leq c\sqrt{\delta(x_0, y_0)},
$$

we obtain

$$
|S_1(A_v, x_0, y_0)| \leq |F^{(r)}(t_0)| + |S_2(A_v, x_0, y_0)| + |S_3(A_v, x_0, y_0)|
$$

$$
\leq |F^{(r)}(t_0)| + C \max_{i+j+2k=r, k \geq 1} (\delta(x_0, y_0)^{j/2}|D^{(i)}\partial^j_r f(x, y)| + C \max_{2(i+j) = r} |D^{(i)}\partial^j_r f(x, y)|.
$$

It then follows by the induction hypothesis that

$$
|S_1(A_v, x_0, y_0)| \leq |F^{(r)}(t_0)| + Cn^r \|f\|_{L^\infty(G(\lambda))}.
$$

On the other hand, since the function $F(t) := f(\Phi_{A_n}(z_0, t))$ is an algebraic polynomial of degree $\leq 2n$, by the univariate Bernstein inequality, we obtain

$$
|F^{(r)}(t_0)| \leq \|F^{(r)}\|_{C([-a_1, a_1])} \leq c(a_1, a_0)n^r \|F\|_{C([-a_0, a_0])} \leq cn^r \|f\|_{L^\infty(G(\lambda))},
$$

where the last step uses Lemma [4.3] (i). Summarizing the above, we derive (4.29) and hence complete the proof of (4.28) for $p = \infty$.

Next, we prove (4.28) for $0 < p < \infty$. By Lemma [4.9] it is enough to show that for each $0 \leq v \leq r$,

$$
(4.30) \quad \int_G |S_1(A_v, x, y)|^p \, dx dy \leq cn^r \|f\|^p_{L^p(G(\lambda))},
$$

Fix $0 \leq v \leq r$ and set

$$
I_\ell := \int_G |S_\ell(A_v, x, y)|^p \, dx dy, \quad \ell = 1, 2, 3.
$$

Using (4.22), (4.23) and the induction hypothesis, we have

$$
(4.31) \quad I_2 \leq C \max_{i+j+2k=r} \int_G (\delta(x, y))^{j/2}|D^{(i)}\partial^j_r f(x, y)|^p \, dx dy \leq Cn^r \|f\|^p_{L^p(G(\lambda))},
$$

(4.32) \quad I_3 \leq C \max_{2(i+j) = r} \int_G |D^{(i)}\partial^j_r f(x, y)|^p \, dx dy \leq Cn^r \|f\|^p_{L^p(G(\lambda))}.
On the other hand, performing the change of variables \( x = z + t \) and \( y = Q_{A_v}(z,t) \) and using Lemma 4.5 (i), (iii), we obtain

\[
I_\ell := \int \Phi_{A_v}^{-1}(G) |S_{\ell}(A_v, z + t, Q_{A_v}(z,t))|^{p}(A_v + g''(z))t \, dz \, dt, \quad \ell = 1, 2, 3.
\]

Thus, setting

\[
I := \int \Phi_{A_v}^{-1}(G) \left| \frac{d^r}{dt^r} \left[ f(z + t, Q_{A_v}(z,t)) \right] \right|^p (A_v + g''(z))t \, dz \, dt,
\]

and using (4.31), (4.32) and Lemma 4.8, we obtain

\[
I_1 \leq c_p (I_2 + I_3) \leq c_p I + c_p n^r p \| f \|_{L^p(G(\lambda))}^p.
\]

Thus, for the proof (4.30), we reduce to showing that

\[
I := \int \Phi_{A_v}^{-1}(G) \left| \frac{d^r}{dt^r} \left[ f(z + t, Q_{A_v}(z,t)) \right] \right|^p (A_v + g''(z))t \, dz \, dt
\]

\[
\leq C n^r p \| f \|_{L^p(G(\lambda))}^p.
\]

This proves (4.33) and hence completes the proof of the theorem.

\[\square\]

5. Bernstein type inequality on domains of special type in \( \mathbb{R}^d \)

In this section, we shall extend the Bernstein type inequality, stated in Theorem 4.1, to higher-dimensional domains of special type. One of the main difficulties in this case comes from the fact that we have to deal with non-commutative mixed directional derivatives along different tangential directions. For convenience, we often write a point in \( \mathbb{R}^d \) and write it in the form \((x, y)\) with \( x \in \mathbb{R}^{d-1} \) and \( y \in \mathbb{R} \). Sometimes we also use a Greek letter to denote a point in \( \mathbb{R}^d \) and write it in the form \( \xi = (\xi_x, \xi_y) \) with \( \xi_x \in \mathbb{R}^{d-1} \) and \( \xi_y \in \mathbb{R} \).

Let \( d \geq 3 \). Let \( G \subset \mathbb{R}^d \) be an \( x_d \)-upward domain with base size \( a > 0 \) and parameter \( L \geq 1 \) given by

\[
G := \left\{ (x, y) \in \mathbb{R}^d : x \in (-a, a)^{d-1}, \ g(x) - La < y \leq g(x) \right\},
\]

where \( g : \mathbb{R}^{d-1} \to \mathbb{R} \) is a \( C^2 \)-function satisfying that \( \min_{x \in [-2a, 2a]} g(x) = 4La \). Following the notations in Section 2, we have

\[
G^* := \left\{ (x, y) : x \in (-2a, 2a)^{d-1}, 0 < y \leq g(x) \right\},
\]
and for each \( \mu > 0 \),

\[
G(\mu) := \{(x, y) : \ x \in (-\mu a, \mu a)^{d-1}, \ g(x) - \mu La < y \leq g(x)\},
\]

\[
\partial G(\mu) := \{(x, g(x)) : \ x \in (-\mu a, \mu a)^{d-1}\}, \quad \partial' G = \partial' G(1), \quad \partial' G^* = \partial' G(2).
\]

For \((x, y) \in G^*\), we define

\[
\delta(x, y) := g(x) - y \quad \text{and} \quad \varphi_n(x, y) := \sqrt{\delta(x, y)} + \frac{1}{n}, \ n = 1, 2, \ldots.
\]

According to Lemma 3.2, we have

\[
\delta(x, y) = g(x) - y \sim \text{dist}(\xi, \partial' G^*), \quad \forall \xi = (x, y) \in G.
\]

The Bernstein type inequality on the domain \( G \) is formulated in terms of certain tangential derivatives along the essential boundary \( \partial' G \) of \( G \), whose definition is given as follows. For \( x_0 \in [-2a, 2a]^{d-1} \), let

\[
(5.2) \quad \xi_j(x_0) := e_j + \partial_j g(x_0)e_d, \quad j = 1, \ldots, d - 1
\]

be the tangent vector to \( \partial' G \) at the point \((x_0, g(x_0))\) that is parallel to the \(x_jx_d\)-coordinate plane. We denote by \( \partial_{\xi_j(x_0)}^\ell \) the \( \ell \)-th order directional derivative along the direction of \( \xi_j(x_0) \):

\[
\partial_{\xi_j(x_0)}^\ell := (\xi_j(x_0) \cdot \nabla)^\ell = \sum_{i=0}^\ell \binom{\ell}{i} (\partial_j g(x_0))^i \partial_j^{\ell-i} \partial_d^i,
\]

where \( j = 1, 2, \ldots, d - 1 \) and \( x_0 \in [-2a, 2a]^{d-1} \). Thus, for \((x, y) \in G \) and \( f \in C^1(G) \),

\[
\partial_{\xi_j(x)}^\ell f(x, y) = \sum_{i=0}^\ell \binom{\ell}{i} (\partial_j g(x))^i (\partial_j^{\ell-i} \partial_d^i f)(x, y), \quad 1 \leq j < d.
\]

We also need to deal with certain mixed directional derivatives. Let \( \mathbb{N}_0 \) denote the set of all nonnegative integers. For \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^{d-1} \), we set \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d \), and define

\[
D_{\alpha, t_0, x_0} := \partial_{\xi_{\alpha_1}(x_0)} \partial_{\xi_{\alpha_2}(x_0)} \cdots \partial_{\xi_{\alpha_{d-1}}(x_0)}, \quad x_0 \in [-2a, 2a]^{d-1}.
\]

For \((x, y) \in G \) and \( f \in C^1(G) \), we also define

\[
D_{\alpha, t}^\alpha f(x, y) := (D_{\alpha, t_0, x}^\alpha f)(x, y).
\]

Note that each of the tangential operators \( D_{\alpha, t}^\alpha \) defined above is commutative with \( \partial_d \), but the operators \( D_{\alpha, t}^\alpha \) themselves are not commutative. Finally, let \( M > 10 \) be a constant satisfying that

\[
(5.3) \quad \max_{1 \leq i \leq j \leq d-1} \|\partial_i \partial_d g\|_{L^\infty([-2a, 2a]^{d-1})} \leq M \quad \text{and} \quad \|\nabla g(0)\| \leq M.
\]

We will keep the above notations and assumptions throughout this section.

Our aim in this section is to prove the following theorem, which is a higher dimensional extension of Theorem 4.1.

**Theorem 5.1.** Let \( G \subset \mathbb{R}^d \) be the domain given in (5.1), and let \( M > 10 \) be the constant satisfying (5.3). If \( 0 < p \leq \infty, \ \lambda > 1 \) and \( f \in \Pi_n^d \), then for any \( \alpha \in \mathbb{N}_0^{d-1}, \)

\[
\|\varphi_{\lambda, d} D_{\alpha, t}^\alpha \partial_d^i f\|_{L^p(G)} \leq c \|\alpha\|^{i+j} \|f\|_{L^p(G(\lambda))}, \quad i, j = 0, 1, \ldots,
\]

where \( c \) is a positive constant depending only on \( M, d, \lambda, \alpha, i, j, \) and \( p \).
As in the case of $d = 2$, Theorem 5.1 also allows us to deduce a slightly stronger Bernstein type inequality:

**Corollary 5.2.** Let $\lambda > 1$ and $\mu > 1$ be two given parameters. If $0 < p \leq \infty$ and $f \in \Pi^d$, then for any $\alpha \in \mathbb{N}^{d-1}$, and $i, j = 0, 1, \ldots$,

$$
\left\| \varphi_n(\xi)^i \max_{u \in \Xi_{n, \alpha, \lambda}(\xi)} \mathcal{D}^\alpha_{\tan, u} \partial_d^{i+j} f(\xi) \right\|_{L^p(G; d\xi)} \leq c\mu n^{\alpha|+j+i} \|f\|_{L^p(G(\lambda))},
$$

where

$$
\Xi_{n, \alpha, \lambda}(\xi) := \left\{ u \in [-\lambda, \lambda]^d : \|u - \xi\| \leq \mu \varphi_n(\xi) \right\}.
$$

**Remark 5.3.** One can also establish similar results on a more general domain $G \subset \mathbb{R}^d$ of special type as defined in Definition 2.4, which can be deduced directly from Theorem 5.1 through translations and reflections of the domain.

The proof of Corollary 5.2 is almost identical to that of Corollary 4.2 for $d = 2$, so we skip the details here. The rest of this section is devoted to the proof of Theorem 5.1.

In the case when $\mathcal{D}^\alpha_{\tan, x} = \partial^r_{\xi_1(x)}$ for some positive integer $r$, Theorem 5.1 can be deduced directly from Theorem 4.1 and Fubini’s theorem, as the next lemma shows.

**Lemma 5.4.** Let $\xi_1(x)$ be the vector given in (5.2). If $0 < p \leq \infty$, $\lambda > 1$ and $f \in \Pi^d$, then

$$
\left\| \mathcal{D}^r_{\xi_1(x)} f(x, y) \right\|_{L^p(G)} \leq cn^r \|f\|_{L^p(G(\lambda))}, \quad r = 0, 1, \ldots,
$$

where $c$ is a positive constant depending only on $M$, $d$, $\lambda$, $r$, and $p$, and the $L^p$-norm on the left hand side is computed with respect to the variables $x$ and $y$ and the Lebesgue measure on $G$.

**Proof.** For simplicity, we assume that $d = 3$ and $p < \infty$. (The proof below works equally well for $d > 3$ or $p = \infty$.) First, by Fubini’s theorem, we have

$$
\int\int\int_{G} \left\| \mathcal{D}^r_{\xi_1(x_1, x_2)} f(x_1, x_2, y) \right\|^p dx_1 dx_2 dy = \int_{-a}^{a} I(x_2) \, dx_2,
$$

where

$$
I(x_2) := \int_{-a}^{a} \int_{g(x_1, x_2) - La}^{g(x_1, x_2)} \left| \partial_1 + \partial_1 g(x_1, x_2) \partial_3 \right|^p f(x_1, x_2, y) \, dy dx_1.
$$

For each fixed $x_2 \in [-a, a]$, applying Theorem 4.1 to the function $g(\cdot, x_2)$ and the polynomial $f(\cdot, x_2, \cdot)$, we obtain

$$
I(x_2) \leq Cn^{rp} \int_{-a}^{a} \int_{g(x_1, x_2) - La}^{g(x_1, x_2)} |f(x_1, x_2, y)|^p \, dy dx_1.
$$

Integrating this last inequality over $x_2 \in [-a, a]$, we deduce (5.4) from (5.5). \qed

For the proof of Theorem 5.1 in the general case, we need to extend Lemma 5.4 to the case of more general directional derivatives, defined below. Given $\xi \in S^{d-2}$
and \( f \in C^\ell(G) \), define
\[
\tilde{D}_\xi^{(\ell)} f(x_0, y_0) = \partial^{(\ell)}_{(\xi, \partial_\xi g(x_0))} f(x_0, y_0) = \left[ (\xi, \partial_\xi g(x_0)) \cdot \nabla \right]^{(\ell)} f(x_0, y_0)
\]
\[
= \sum_{i=0}^{\ell} \left( \frac{\ell}{i} \right) (\partial_\xi g(x_0))^i (\partial^{\ell-i}_{\partial_\xi} \partial f)(x_0, y_0),
\]
where \((x_0, y_0) \in G\). Note here that \((\xi, \partial_\xi g(x_0))\) is the tangent vector to \(\partial'G\) at the point \((x_0, g(x_0))\) that is parallel to the plane spanned by the vectors \((\xi, 0)\) and \(e_d\).

By the definition, we have
\[
\tilde{\mathcal{D}}^{(\ell)} f(x, y) = \left( \frac{d}{dt} \right)^\ell \left[ f(x + t\xi, y + t\partial_\xi g(x)) \right]_{|t=0}, \quad (x, y) \in G, \quad f \in C^\ell(\mathbb{R}^d),
\]
and
\[
\tilde{\mathcal{D}}^{(\ell)}_{\xi_j} f(x, y) = \partial^{\ell}_{\xi_j(x)} f(x, y), \quad 1 \leq j \leq d - 1.
\]

As an extension of Lemma 5.4, we have

**Lemma 5.5.** If \(0 < p \leq \infty, \lambda > 1\) and \(f \in \Pi^d_1\), then
\[
(5.6) \quad \max_{\xi \in \mathbb{S}^{d-2}} \|\tilde{\mathcal{D}}^{(r)} f\|_{L^p(G)} \leq c n^r \|f\|_{L^p(G(\lambda))}, \quad r = 0, 1, \ldots,
\]
where \(c\) is a positive constant depending only on \(M, d, \lambda, r, \) and \(p\).

**Proof.** Without loss of generality, we may assume that \(p < \infty\) since the case \(p = \infty\) is simpler and can be treated similarly. Let \(\xi \in \mathbb{S}^{d-2}\) be a fixed direction. For \(\mu \geq 1\), we write
\[
[-\mu a, \mu a]^d = \left\{ \eta + t\xi : \eta \in E_{\mu, \xi}, \; t \in [a_{\mu, \xi}(\eta), b_{\mu, \xi}(\eta)] \right\},
\]
where
\[
E_{\mu, \xi} = \left\{ x - (x : \xi)\xi : x \in [-\mu a, \mu a]^d \right\},
\]
and
\[
[a_{\mu, \xi}(\eta), b_{\mu, \xi}(\eta)] = \left\{ t \in \mathbb{R} : \eta + t\xi \in [-\mu a, \mu a]^d \right\}, \quad \eta \in E_{\mu, \xi}.
\]

By Fubini’s theorem, we then have
\[
\int_G |\tilde{\mathcal{D}}^{(r)} f(x, y)|^p dx dy = \int_{E_{1, \xi}} \left[ \int_{a_{1, \xi}(\eta)}^{b_{1, \xi}(\eta)} \int_{g_\eta(s) - La}^{g_\eta(s)} \left| \frac{d^r}{dt^r} \left[ f_\eta(s + t, y + t g_\eta'(s)) \right] \right|_{t=0}^p dy ds \right] d\eta
\]
\[
= \int_{E_{1, \xi}} \left[ \int_{a_{1, \xi}(\eta)}^{b_{1, \xi}(\eta)} \int_{g_\eta(s) - La}^{g_\eta(s)} \left| (\partial_1 + g_\eta'(s) \partial_2)^r f_\eta(s, y) \right|^p dy ds \right] d\eta =: I,
\]
where
\[
g_\eta(s) := g(\eta + s\xi), \quad f_\eta(s, y) := f(\eta + s\xi, y), \quad y, s \in \mathbb{R}, \quad \eta \in \mathbb{R}^{d-1}.
\]

Next, let \(1 < \mu < \lambda\) be a fixed parameter. A straightforward calculation shows that for each \(\eta \in E_{1, \xi}\),
\[
b_{\mu, \xi}(\eta) - a_{\mu, \xi}(\eta) \geq 2(\mu - 1)a > 0, \quad \text{and}
\]
\[
[a_{\mu, \xi}(\eta) - (\lambda - \mu)a, b_{\mu, \xi}(\eta) + (\lambda - \mu)a] \subset [a_{\lambda, \xi}(\eta), b_{\lambda, \xi}(\eta)].
\]
It then follows from Theorem \[\text{[1]}\] that
\[
I \leq \int_{E_{1,\xi}} \int_{a_{\alpha,\xi}(\eta)}^{b_{\alpha,\xi}(\eta)} \int_{g_1(s)}^{g_2(s)} \left| \left( \partial_1 + g_1'(s) \partial_2 \right)^r f_\eta(s, y) \right|^p dy ds d\eta
\]
\[
\leq C n^p r \int_{E_{1,\xi}} \left[ \int_{a_{\alpha,\xi}(\eta)}^{b_{\alpha,\xi}(\eta)} \int_{g_1(s)}^{g_2(s)} |f_\eta(s, y)|^p dy ds \right] d\eta \leq C n^p r \int_{G(\lambda)} |f(s)|^p d\xi.
\]
This proves (5.6). \( \square \)

To deal with the mixed directional derivatives \( D_{\text{tanh}}^{(\alpha)} \), one main difficulty lies in the fact that these operators are not commutative. To overcome the difficulty, we will use a combinatorial identity on mixed directional derivatives on \( \mathbb{R}^n \). Recall that \( \partial_\xi := \xi \cdot \nabla = \sum_{j=1}^n \xi_j \partial_j \) for each \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \). As a consequence of the Kemperman lemma on mixed differences (see \[\text{[7]}, (3.7)\] or \[\text{[4]}, \text{Lemma 4.11}, \text{p.338}\]), we have

**Lemma 5.6.** Let \( \xi_1, \ldots, \xi_r \) be arbitrary vectors in \( \mathbb{R}^n \). Then

\[
(5.7) \quad \partial_{\xi_1} \partial_{\xi_2} \cdots \partial_{\xi_r} = \sum_{S \subset \{1, 2, \ldots, r\}} (-1)^{\# S} \partial_{\xi_S},
\]

where the sum is taken over all subsets \( S \) of \( \{1, 2, \ldots, r\} \), \( \# S \) is the cardinality of \( S \) and \( \xi_S = -\sum_{j \in S} j^{-1} \xi_j \).

**Remark 5.7.** In the case when all the vectors \( \xi_1, \ldots, \xi_r \) are unit, the following interesting inequality with sharp constant was proved in \[\text{[7]}\]:

\[
|\partial_{\xi_1} \cdots \partial_{\xi_r} f(x)| \leq \max_{\xi \in S^{n-1}} |\partial_\xi f(x)|, \quad x \in \mathbb{R}^n, \quad \xi_1, \ldots, \xi_r \in S^{n-1}, \quad f \in C^r(\mathbb{R}^n).
\]

Note that in Lemma 5.6 the vectors \( \xi_1, \ldots, \xi_r \) are not necessarily unit, and the sum on the right hand side of (5.7) uses at most \( 2^r \) directions. This is very important in our later applications.

**Proof.** For \( x, h \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \to \mathbb{R} \), define

\[
T_h f(x) = f(x + h), \quad \Delta_h f(x) = (T_h - I) f(x) = f(x + h) - f(x).
\]

The proof relies on the following identity on finite differences, whose proof can be found in \[\text{[4]}, \text{Lemma 4.11}, \text{p.338}\]:

\[
(5.8) \quad \Delta_h_1 \Delta_h_2 \cdots \Delta_h_r = \sum_{S \subset \{1, 2, \ldots, r\}} (-1)^{\# S} T_{h^*_S} \Delta_{h_S}, \quad \forall h_1, \ldots, h_r \in \mathbb{R}^n,
\]

where

\[
h^*_S = \sum_{j \in S} h_j \quad \text{and} \quad h_S = -\sum_{j \in S} j^{-1} h_j.
\]

For each \( f \in C^r(\mathbb{R}^n) \) and \( x, \xi_1, \ldots, \xi_r \in \mathbb{R}^n \), we have

\[
\Delta_{t \xi_1} \Delta_{t \xi_2} \cdots \Delta_{t \xi_r} f(x) = \int_{[0,t]^r} \left( \partial_{t \xi_1} \cdots \partial_{t \xi_r} f \right)(x + \sum_{j=1}^r u_j \xi_j) \, du_1 \cdots du_r, \quad \forall t > 0,
\]

which, in particular, implies

\[
\lim_{t \to 0^+} t^{-r} \left( \Delta_{t \xi_1} \cdots \Delta_{t \xi_r} f \right)(x) = \left( \partial_{t \xi_1} \cdots \partial_{t \xi_r} f \right)(x), \quad f \in C^r(\mathbb{R}^n).
\]
On the other hand, however, using \((5.8)\) with \(h_j = t \xi_j\), we obtain

\[
t^{-r} \left( \prod_{j=1}^{r} \Delta t \xi_j \right) f(x) = t^{-r} \sum_{S \subseteq \{1, 2, \ldots, r\}} (-1)^{\# S} \Delta t \xi_S f \left( x + t \sum_{j \in S} \xi_j \right).
\]

\((5.7)\) then follows by letting \(t \to 0^+\).

Now we are in a position to prove Theorem \ref{thm:5.1}.

\textbf{Proof of Theorem \ref{thm:5.1}.} Since the operators \(D^{(a)}_\text{tan} \) and \(\partial_d\) are commutative, by the univariate Bernstein inequality (or the higher-dimensional analogue of Lemma \ref{lem:4.4}), it is sufficient to prove that

\[
\|D^{(a)}_\text{tan} f\|_{L^p(G)} \leq c n^{n|a|} \|f\|_{L^p(G(\lambda))}, \quad f \in \Pi_n^d.
\]

Let \((x_0, y_0)\) be an arbitrarily fixed point in \(G\). Let \(\ell = |a|\) and let \(\{\eta_j(x_0)\}_{j=1}^\ell\) be the sequence of vectors such that

\[
\eta_{a_j+1}(x_0) = \cdots = \eta_{a_j+\alpha_j}(x_0) = \xi_j(x_0), \quad j = 1, 2, \ldots, d-1,
\]

where \(a_0 = 0\). Then using \((5.7)\), we have that

\[
\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_{d-1}}^{\alpha_{d-1}} \left( \prod_{j=1}^{\ell} \partial_{\eta_j}(x_0) = \sum_{S \subseteq \{1, 2, \ldots, \ell\}} (-1)^{\# S} \partial_{\eta_S}(x_0) \right)
\]

where \(\eta_S(x_0) = - \sum_{j \in S} j^{-1} \eta_j(x_0)\). As can be easily seen from \((5.2)\), for each \(S \subseteq \{1, 2, \ldots, \ell\}\), the vector \(\eta_S(x_0)\) can be written in the form

\[
\eta_S(x_0) = c_S \left( \xi_S, \partial_{\xi_S} g(x_0) \right), \quad c_S > 0, \quad \xi_S \in \mathbb{S}^{d-2},
\]

where \(c_S\) and \(\xi_S\) depend only on the set \(S\) and \(a\) (but independent of \(x_0\)). Thus, by \((5.7)\), it follows that

\[
\left| \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_{d-1}}^{\alpha_{d-1}} f(x_0, y_0) \right| \leq C_\alpha \sum_{S \subseteq \{1, 2, \ldots, \ell\}} |\tilde{D}^{(\ell)} f(x_0, y_0)|.
\]

Since the maximum in \((5.10)\) is taken over a set of \(2^\ell\) elements, and all the vectors \(\xi_S\) are independent of \((x_0, y_0)\), it follows by Lemma \ref{lem:5.5} that

\[
\|D^{(a)}_\text{tan} f\|_{L^p(G)} \leq C \sum_{S \subseteq \{1, \ldots, \ell\}} \|\tilde{D}^{(\ell)} f\|_{L^p(G)} \leq c n^{n|a|} \|f\|_{L^p(G(\lambda))}.
\]

This completes the proof. \qed

\section{6. Bernstein type inequality on general \(C^2\) domains}

In this section, we will use Proposition \ref{prop:2.10} to extend the Bernstein type inequality, established in the last section, to a more general \(C^2\)-domain. Our aim is to prove Theorem \ref{thm:1.3}.

We first recall some necessary notations. Let \(B_r(\xi) := \{ \eta \in \mathbb{R}^d : \|\eta - \xi\| < r \}\) denote the open ball with center \(\xi \in \mathbb{R}^d\) and radius \(r > 0\) in \(\mathbb{R}^d\). Given a ball \(B = B_r(x) \subset \mathbb{R}^d\), and a constant \(c > 0\), we denote by \(cB = B_{cr}(x)\) the dilation of \(B\) from its center by a factor \(c > 0\). Let \(\Omega \subset \mathbb{R}^d\) be a compact \(C^2\)-domain with
boundary $\Gamma = \partial \Omega$. Given $\eta \in \Gamma$, we denote by $n_\eta$ the unit outer normal vector to $\Gamma$ at $\eta$, and $S_\eta$ the set of all unit tangent vectors to $\Gamma$ at $\eta$:

$$S_\eta := \{ \tau \in S^{d-1} : \tau \cdot n_\eta = 0 \}.$$

For a parameter $\mu \geq 1$, set

$$\Gamma_{n,\mu}(\xi) := \{ \eta \in \Gamma : \| \eta - \xi \| \leq \mu \varphi_{n,\Gamma}(\xi) \},$$

where

$$\varphi_{n,\Gamma}(\xi) := \sqrt{\text{dist}(\xi, \Gamma)} + n^{-1}, \quad \xi \in \Omega, \quad n = 1, 2, \ldots.$$ Then for any two nonnegative integers $l_1, l_2$, 

$$\mathcal{D}^{l_1, l_2}_{n,\mu} f(\xi) := \max_{\eta \in \Gamma_{n,\mu}(\xi), \tau \in S_\eta} \max_{|\rho| \leq \mu} \| D^\rho f(\eta) \|, \quad \xi \in \Omega, \quad f \in C^\infty(\Omega).$$

Our goal is to show that for any $f \in \Pi^d_{n,\mu}, \mu > 1$ and $0 < p \leq \infty$, 

$$(6.1) \quad \left\| \mathcal{D}^{r+j+l}_{n,\mu} f \right\|_{L^p(\Omega)} \leq C \mu^{r+j+2l} \| f \|_{L^p(\Omega)}, \quad r, j, l = 0, 1, \ldots,$$ 

where the constant $C \mu$ is independent of $f$ and $n$.

To show (6.1), without loss of generality, we may assume that $\eta \in \Gamma_{n,\mu}(\xi)$ and hence

$$\| \mathcal{D}^{r+j+l}_{n,\mu} f \|_{L^p(\Omega_{\delta})} \leq C \delta^{r+j+2l} \| f \|_{L^p(\Omega)}, \quad r, j, l = 0, 1, \ldots.$$ 

Finally, we prove that for each domain $G \subset \Omega$ of special type attached to $\Gamma$ with base size $a \geq 2 \mu \sqrt{\delta}$, 

$$(6.3) \quad \left\| \mathcal{D}^{r+j+l}_{n,\mu} f \right\|_{L^p(G \cap \Gamma_{\delta})} \leq C n^{r+j+2l} \| f \|_{L^p(G^*)}, \quad r, j, l = 0, 1, \ldots,$$ 

which together with (6.2) will imply the desired inequality (6.1). Here we recall that the set $G^*$ is defined in (2.1).

To show (6.3), without loss of generality, we may assume that $n > \frac{2a}{\delta}$, and $G$ is an $x_a$-domain that takes the form $\{ \xi \in [-a, a]^{d-1} : g(x) = 4La \}$. 

Here we choose $\delta$ small enough so that the base size of each $G_j$ is bigger than $2\mu \sqrt{\delta}$. Then for any two nonnegative integers $r, j, l = 0, 1, \ldots$, 

$$(6.2) \quad \left\| \mathcal{D}^{r+j+l}_{n,\mu} f \right\|_{L^p(\Omega_{\delta})} \leq C \delta^{r+j+2l} \| f \|_{L^p(\Omega)}, \quad r, j, l = 0, 1, \ldots.$$ 

Here we may choose $\delta \in (0, 1)$ small enough so that the base size of each $G_j$ is bigger than $2\mu \sqrt{\delta}$. Second, we cover $\Omega_{\delta} := \Omega \setminus \Gamma_{\delta}$ by finitely many open balls $B_1, \ldots, B_m$ of radius $\frac{\delta}{2}$ such that $m_1 \leq C \delta^{-d}$ and $B_1, \ldots, B_m$ will imply the desired inequality (6.1). Here we recall that the set $G^*$ is defined in (2.1).
Since $G \subset \Omega$ is attached with $\Gamma$, we have $G_* \subset \Omega$ and $\partial' G_* \subset \Gamma$. Moreover, according to Proposition 5.1, we have

\[ \varphi_n(x, y) := \sqrt{g(x) - y + \frac{1}{n}} \sim \varphi_{n, \Gamma}(x, y), \quad \forall (x, y) \in G. \]

\[ (6.4) \]

Since $a \geq 2\mu\sqrt{\delta}$ and $n > \frac{2\mu}{a}$, it follows that $\mu \varphi_{n, \Gamma}(\xi) < a$ and $\Gamma_{n, \mu}(\xi) \subset \partial' G_*$ for every $\xi \in G \cap \Gamma_{\delta}$. Thus, by (6.4), there exists a constant $c_1 > 1$ such that

\[ \Gamma_{n, \mu}(\xi) \subset \left\{ (u, g(u)) : u \in \Xi_{n, c_1}(\xi) \right\}, \quad \forall \xi \in G \cap \Gamma_{\delta}, \]

where $\Xi_{n, \mu}(\xi) := \left\{ u \in [-2a, 2a]^{d-1} : \|u - \xi_x\| \leq \mu \varphi_n(\xi) \right\}$. Thus, using Corollary 5.2, we obtain

\[ (6.5) \]

where $\eta = (\eta_x, \eta_y), \eta_x \in \mathbb{R}^{d-1}$ and $\eta_y \in \mathbb{R}$.

Now let $\eta = (x_0, g(x_0)) \in \partial' G^*$ with $x_0 = \eta_x \in (-2a, 2a)^{d-1}$. Let $\xi_j(x_0)$, $j = 1, \ldots, d - 1$ be the vectors given in (5.2). A straightforward calculation then shows that

\[ \partial_{\tau} = \sum_{j=1}^{d-1} \tau_j \partial_{\xi_j(x_0)}, \quad \forall \tau = (\tau_1, \ldots, \tau_d) \in S_{\eta} \]

Recalling that

\[ D_{\tan, \eta_x}^{\alpha} f(x, y) = \partial_{\xi_1(x_0)}^{\alpha_1} \cdots \partial_{\xi_{d-1}(x_0)}^{\alpha_{d-1}} f(x, y), \quad \alpha \in \mathbb{N}_0^{d-1}; \]

we obtain

\[ (6.6) \]

Taking maximum over $\eta = (\eta_x, \eta_y) \in \Gamma_{n, \mu}(\xi)$ on both sides of (6.6) yields that for each $\xi \in G \cap \Gamma_{\delta}$,

\[ D_{n, \mu}^{\tau, j+1} f(\xi) := \max_{\eta \in \Gamma_{n, \mu}(\xi)} \max_{\tau \in S_{\eta}} |D_{\tan, \eta_x}^{\alpha} \partial_{\xi}^{j+1} f(\xi)| \leq C \max_{\eta \in \Gamma_{n, \mu}(\xi)} \max_{\tau \in S_{\eta}} |D_{\tan, \eta_x}^{\alpha} \partial_{\xi}^j f(\xi)|. \]

\[ (6.7) \]

Now combining (6.4), (6.5) with (6.7), we obtain

\[ \|\varphi_n, \Gamma D_{n, \mu}^{\tau, j+1} f\|_{L^p(G \cap \Gamma_{\delta})} \leq C \max_{\eta \in \Gamma_{n, \mu}(\xi)} |D_{\tan, \eta_x}^{\alpha} \partial_{\xi}^j f(\xi)| \leq C \left[ \sum_{|\alpha| = j+i} n^{j+i+2} \right] \|f\|_{L^p(G^*)}, \quad j, l = 0, 1, \ldots. \]

This proves the desired inequality (6.6), and hence completes the proof of the theorem.
7. Marcinkiewicz Type Inequality and Positive Cubature Formulas

In this section, we will prove Theorem 1.8 and Theorem 1.10. We start with a brief description of some necessary notations. Let \((X, \rho)\) be a metric space. Given \(\delta > 0\), a finite subset \(\Lambda\) of \(X\) is said to be \(\delta\)-separated with respect to the metric \(\rho\) if \(\rho(\omega, \omega') \geq \delta\) for any two distinct points \(\omega, \omega' \in \Lambda\), while a \(\delta\)-separated subset \(\Lambda \subset X\) is called maximal if \(\inf_{\omega \in \Lambda} \rho(x, \omega) < \varepsilon\) for any \(x \in X\). As usual, we assume that \(\Omega \subset \mathbb{R}^d\) is a compact \(C^2\)-domain with boundary \(\Gamma = \partial \Omega\). Let \(\rho \equiv \rho_\Omega\) denote the metric on \(\Omega\) defined by (1.3). Let \(U(\xi, t) := \{\eta \in \Omega : \rho_\Omega(\xi, \eta) \leq t\}\) for \(\xi \in \Omega\) and \(t > 0\). According to Corollary 3.3 (i), \(|U(\xi, t)| \sim t^d(t + \sqrt{\text{dist}(\xi, \Gamma)}), \xi \in \Omega, \ t \in (0, 1)\), while by Corollary 3.3 (iii), for each \(\delta \in (0, 1)\), every \(\delta\)-separated subset \(\Lambda\) of \(\Omega\) must satisfy \(\#\Lambda \leq C\delta^{-d}\). Given a bounded function \(f\) on \(\Omega\) and a subset \(I \subset \Omega\), we define \(\text{osc}(f; I) := \sup_{\xi, \eta \in I} |f(\xi) - f(\eta)|\).

Our goal in this section is to prove the following result, from which Theorem 1.8 and Theorem 1.10 will follow.

**Theorem 7.1.** Let \(\Omega \subset \mathbb{R}^d\) be a compact \(C^2\)-domain with boundary \(\Gamma = \partial \Omega\). Let \(\ell \geq 1\) be a given parameter, \(n\) a positive integer, and let \(\varepsilon \in (0, \ell]\). Assume that \(\Lambda \subset \Omega\) is \(\varepsilon_n\)-separated with respect to the metric \(\rho_\Omega\). Then for any \(1 \leq p \leq \infty\) and \(f \in \Pi^d_n\),

\[
\left( \sum_{\xi \in \Lambda} \left| U(\xi, \frac{\varepsilon}{n}) \right| \left( \text{osc} \left( f; U(\xi, \frac{\varepsilon}{n}) \right) \right)^p \right)^{\frac{1}{p}} \leq C_{\ell, \varepsilon} \|f\|_{L^p(\Omega)},
\]

where the constant \(C_{\ell, \varepsilon} > 0\) depends only on the parameter \(\ell\) and the domain \(\Omega\), and we have to replace the term on the left hand side of (7.1) with \(\max_{\xi \in \Lambda} \text{osc}(f; U(\xi, \frac{\varepsilon}{n}))\) if \(p = \infty\).

The rest of this section is organized as follows. We first take Theorem 7.1 for granted and show how it implies Theorem 1.8 and Theorem 1.10 in Section 7.1 and Section 7.2 respectively. Section 7.2 is devoted to the proof of Theorem 7.1, which is more involved. The Bernstein inequality stated in Theorem 4.1 will play a crucial role in our proof.

7.1. Proof of Theorem 1.8 (assuming Theorem 7.1). Without loss of generality, we may assume that \(p < \infty\). (The case \(p = \infty\) can be deduced by letting \(p \to \infty\) since all the implicit constants below are independent of \(p\).)

Let \(\{R_1, \ldots, R_N\}\) be a partition of \(\Omega\) with norm \(\leq \frac{\delta}{n}\). Let \(\Lambda\) be a maximal \(\frac{\delta}{n}\)-separated subset of \(\Omega\). For each \(\omega \in \Lambda\), set

\[I_\omega := \{j \in \{1, 2, \ldots, N\} : R_j \cap U(\omega, \frac{\delta}{n}) \neq \emptyset\}.\]
Then \( \{1, 2, \ldots, N\} = \bigcup_{\omega \in \Lambda} I_{\omega} \), and \( \bigcup_{j \in I_{\omega}} R_j \subset U(\omega, \frac{2\delta}{n}) \) for any \( \omega \in \Lambda \). Thus, using Theorem 7.1, we obtain that for any \( f \in \Pi_n^d \),

\[
\sum_{j=1}^{N} (\text{osc}(f; R_j))^p |R_j| \leq \sum_{\omega \in \Lambda} \left( \text{osc}(f, U(\omega, \frac{2\delta}{n})) \right)^p \sum_{j \in I_{\omega}} |R_j| \\
\leq \sum_{\omega \in \Lambda} \left( \text{osc}(f, U(\omega, \frac{2\delta}{n})) \right)^p |U(\omega, \frac{2\delta}{n})| \leq C \delta^p \|f\|^p.
\]

Since \( \xi_j \in R_j \) for each \( 1 \leq j \leq N \), it follows by the Minkowski inequality that

\[
(7.2) \quad \left| \left( \sum_{j=1}^{N} |f(\xi_j)|^p |R_j| \right)^{\frac{1}{p}} - \|f\|_p \right| \leq \left( \sum_{j=1}^{N} (\text{osc}(f; R_j))^p |R_j| \right)^{\frac{1}{p}} \leq C\delta \|f\|_p.
\]

Thus, if \( 0 < \delta \leq \delta_0 := \frac{1}{2C} \), then

\[
\frac{1}{2} \|f\|_p \leq \left( \sum_{j=1}^{N} |f(\xi_j)|^p |R_j| \right)^{\frac{1}{p}} \leq \frac{3}{2} \|f\|_p, \quad \forall f \in \Pi_n^d.
\]

\[ \text{7.2. Proof of Theorem 1.10 (assuming Theorem 7.1).} \] Let \( \{R_1, \ldots, R_N\} \) be a partition of \( \Omega \) with norm \( \leq \frac{\delta}{n} \). Here and throughout the proof, without loss of generality, we may assume that \( \delta_0 > 0 \) is a sufficiently small constant depending only on \( \Omega \). Some of the estimates below may not be true without this assumption.

It is easily seen from the proof of (7.2) that for any \( f \in \Pi_n^d \),

\[
\sum_{j=1}^{N} f(\xi_j)|R_j| - \int_{\Omega} f(\xi) \, d\xi \leq C\delta_0 \sum_{j=1}^{N} |f(\xi_j)||R_j|,
\]

where \( C > 0 \) is a constant depending only on \( \Omega \). Setting \( \delta_0 := \frac{1}{2C} \), we deduce that for any \( f \in \Pi_n^d \) satisfying \( \min_{1 \leq j \leq N} f(\xi_j) \geq 0 \),

\[
(7.3) \quad \frac{2}{3} \sum_{j=1}^{N} f(\xi_j)|R_j| \leq \int_{\Omega} f(\xi) \, d\xi \leq \frac{4}{3} \sum_{j=1}^{N} f(\xi_j)|R_j|.
\]

Next, we denote by \( \delta_\xi \) the Dirac probability measure supported at \( \xi \in \Omega \), and consider the following linear functional on \( \Pi_n^d \):

\[
(7.4) \quad Tf = \frac{4}{3} \left| \int_{\Omega} f(\xi) \, d\xi - \frac{1}{3|\Omega|} \sum_{j=1}^{N} f(\xi_j)|R_j| \right|, \quad f \in \Pi_n^d.
\]

We claim that

\[
(7.5) \quad T \in \text{conv} \left\{ \delta_{\xi_1}, \ldots, \delta_{\xi_N} \right\} \subset (\Pi_n^d)^*,
\]

which will imply that there exist constants \( \lambda_j \geq \frac{1}{4}|R_j|, \quad j = 1, \ldots, N \) such that

\[
(7.6) \quad \int_{\Omega} f(\xi) \, d\xi = \sum_{j=1}^{N} \lambda_j f(\xi_j), \quad \forall f \in \Pi_n^d.
\]

Assume to the contrary that (7.5) were not true. Then by the convex separation theorem, there exists \( P \in \Pi_n^d \) such that

\[
(7.7) \quad T(P) < \min_{1 \leq j \leq N} P(\xi_j).
\]
Indeed, for each \(1 \leq d \mu + Q\) where \(\xi\) can be constructed as \(P\). Then (7.7) implies \(T(P) < 0\). However, this is impossible since by (7.3) and (7.4),
\[
\|\Omega|T(P)| \geq \frac{5}{9} \sum_{j=1}^{N} P(\xi_j)|R_j| \geq 0.
\]
This proves the claim (7.3), and hence (7.0).

Finally, we prove that
\[
\lambda_j \leq C \left( \frac{1}{n} \right)^d \left( \frac{1}{n} + \sqrt{\text{dist}(\xi_j, \partial \Omega)} \right), \quad j = 1, 2, \ldots, N.
\]
Indeed, for each \(1 \leq j \leq N\), it suffices to find a nonnegative polynomial \(P_j \in \Pi^d_{\mu}\) on \(\Omega\) such that \(P_j(\xi_j) = 1\) and \(\int_{\Omega} P_j(\xi) d\xi \leq C n^{-d} (n^{-1} + \sqrt{\text{dist}(\xi_j, \partial \Omega)})\) as then (7.8) immediately follows from (7.6) with \(f = P_j\). The required construction is a standard technique in estimates of Christoffel function. Fix \(j, 1 \leq j \leq N\), and let \(\mu \in \partial \Omega\) be such that \(\|\xi_j - \mu\| = \text{dist}(\xi_j, \partial \Omega) =: a\). If \(n_\mu\) denotes the unit outer normal vector to \(\partial \Omega\) at \(\mu\), then \(\xi_j = \mu - an_\mu\) and the definition of \(C^2\) domains implies that there exist radii \(r_1, r_2 > 0\) depending only on \(\Omega\) such that \(\Omega \subset B_{r_2}(\mu + r_1 n_\mu) \setminus B_{r_1}(\mu + r_1 n_\mu) =: D\). Without loss of generality, we can assume \(\mu + r_1 n_\mu = 0\) and \(n_\mu = (1, 0, \ldots, 0)\). A nonnegative polynomial \(P_j\) on \(D\) satisfying
\[
P_j(\xi_j) = 1 \quad \text{and} \quad \int_D P_j(\xi) d\xi \leq C n^{-d} (n^{-1} + \sqrt{\text{dist}(\xi_j, \partial \Omega)})
\]
can be constructed as
\[
P_j(x_1, \ldots, x_d) = (Q(||x||^2)Q_2(x_2) \ldots Q_d(x_d))^2,
\]
where \(Q \text{ and } Q_i, 2 \leq i \leq d, \) are the polynomials provided by [16, Lemma 6.1] applied on the intervals \([r_1^2, r_2^2]\) and \([-r_2, r_2]\), respectively, with \(g\) chosen to guarantee \(P_j(\xi_j) = P_j(-a, 0, \ldots, 0) = 1\). We omit the details here referring an interested reader to the proof of [37, Lemma 4.3] which contains the required construction and estimates for the case \(d = 2\).

This completes the proof of Theorem 1.10.

7.3. Proof of Theorem 7.4. This subsection is devoted to the proof of Theorem 7.2. We will write the proof for the case of \(p < \infty\) only, as the case \(p = \infty\) is simpler and can be treated similarly. The proof relies on several lemmas.

Lemma 7.2. Let \(0 \leq \theta_1 < \theta_2 < \cdots < \theta_m \leq \pi\) satisfy that \(\min_{2 \leq i \leq m} (\theta_i - \theta_{i-1}) \geq \frac{1}{\ell}\) for some positive integer \(n\). Then for every \(f \in \Pi^1_k\), \(1 \leq p < \infty\) and any parameter \(\ell > 1\),
\[
\left( \sum_{i=1}^{m} \left( \frac{\sin \theta_i}{n} + \frac{1}{n^\ell} \right) \right) \max_{|\theta - \theta_i| \leq \frac{1}{\ell}} |f(\cos \theta)|^p \quad \leq \quad C_\ell \left( 1 + \frac{k}{n} \right) \left( \int_{-1}^{1} |f(x)|^p dx \right)^\frac{1}{p},
\]
where \(C_\ell > 0\) is a constant depending only on the parameter \(\ell\).

Proof. Lemma 7.2 can be proved by a slight modification of the proof of Theorem 3.1 of [34]. For completeness, we summarize its proof as follows. First, without loss of generality, we may assume \(k \geq n\), since the stated inequality for \(k \leq n\) follows directly from the case of \(k = n\).
Next, define \( T_k(\theta) := f(\cos \theta) \) for \( f \in \Pi_k^1 \). Let \( I_i := [\theta_i - \frac{L}{n}, \theta_i + \frac{L}{n}] \) for \( 1 \leq i \leq m \). Note that for each \( \theta \in I_i \),

\[
|T_k(\theta)| \leq \int_{I_i} |T_k'(s)| \, ds + \frac{1}{|I_i|} \int_{I_i} |T_k(s)| \, ds,
\]

which implies

\[
\max_{\theta \in I_i} |T_k(\theta)|^p \leq C \ell_n^{-1} \int_{I_i} |T_k'(s)|^p \, ds + C \ell_n \int_{I_i} |T_k(s)|^p \, ds.
\]

Since

\[
\sin \theta_i \leq \min_{\theta \in I_i} |\sin \theta| + C \ell_n^{-1}, \quad 1 \leq i \leq m,
\]

it follows that

\[
(\sum_{i=1}^m \left( \frac{\sin \theta_i}{n} + \frac{1}{n^2} \right) \max_{\theta \in I_i} |T_k(\theta)|^p)^{1/p} \leq C \ell (\Sigma_1 + \Sigma_2),
\]

where

\[
\Sigma_1 := n^{-1} \left( \int_{-\pi}^\pi |T_k'(\theta)|^p (|\sin \theta| + n^{-1}) \, d\theta \right)^{1/p},
\]

\[
\Sigma_2 := \left( \int_{\pi}^0 |T_k(\theta)|^p (|\sin \theta + n^{-1}) \, d\theta \right)^{1/p}.
\]

For the term \( \Sigma_1 \), we use the weighted Bernstein inequality for trigonometric polynomials (see [34, Theorem 4.1]) to obtain

\[
\Sigma_1 \leq \frac{C k}{n} \left( \int_{\pi}^0 |T_k(\theta)|^p (|\sin \theta + n^{-1}) \, d\theta \right)^{1/p} = \frac{C k}{n} \Sigma_2.
\]

Finally, using the Schur-type inequality for trigonometric polynomials (see [34, (3.3)]), we obtain

\[
\Sigma_2 \leq C \left( \int_{\pi}^0 |T_k(\theta)|^p \sin \theta \, d\theta \right)^{1/p} = C \left( \int_{-1}^1 |f(\theta)|^p \, d\theta \right)^{1/p}.
\]

Putting the above together, we deduce the desired inequality (7.9). \( \square \)

Our next lemma gives an analogue of (7.1) on domains of special type. Let \( G \subset \Omega \) be a domain of special type attached to \( \Gamma \). Without loss of generality, we may assume that \( G \) is an \( x,dy \)-upward domain with base size \( b \in (0, 1) \) and parameter \( L = b^{-1} \), given by

\[
G := \{(x, y) : \ x \in (-b, b)^{d-1}, \ g(x) - 1 < y \leq g(x)\},
\]

where \( g \) is a \( C^2 \)-function on \( \mathbb{R}^{d-1} \) satisfying that \( \min_{x \in [-2b, 2b]^{d-1}} g(x) = 4 \). Following the notation in Section [2] we then have

\[
G_* := \{(x, y) : \ x \in (-2b, 2b)^{d-1}, \ 0 < y \leq g(x)\}.
\]

Let \( \hat{\rho}_G \) be the metric on \( G \) defined by

\[
\hat{\rho}_G(\xi, \eta) := \max \left\{ \|\xi - \eta\|, \sqrt{g(\xi) - \xi_y} - \sqrt{g(\eta) - \eta_y} \right\},
\]

where \( \xi = (\xi_x, \xi_y) \in G \) and \( \eta = (\eta_x, \eta_y) \in G \). According to Proposition 8.1 we have

\[
\hat{\rho}_G(\xi, \eta) \sim \rho_{\Omega}(\xi, \eta), \quad \xi, \eta \in G.
\]
For $\xi \in G$ and $r \in (0, 1)$, we define
\[ B_G(\xi, r) := \{ \eta \in G : \delta_G(\xi, \eta) \leq r \} . \]

The following lemma will play an important role in the proof of Theorem 7.1

**Lemma 7.3.** Let $G \subset \mathbb{R}^d$ be the domain given in (7.14). Assume that $n \in \mathbb{N}$, $\mu \geq 1$ is a parameter, and $\Lambda \subset G$ is $\delta$-separated with respect to the metric $\delta_G$ for some $\delta \in (0, \frac{\pi}{2})$. Then for any $f \in \Pi_n^d$ and $1 \leq p < \infty$,
\[ (\sum_{\omega \in \Lambda} |B_G(\xi, \delta)| \left( \text{osc}(f; B_G(\omega, \mu \delta)) \right)^p)^{\frac{1}{p}} \leq C_\mu(n\delta) \|f\|_{L^p(G^*)}, \]
where the constant $C_\mu$ is independent of $\delta$, $n$ and $f$.

**Proof.** For simplicity, we assume $d = 2$. (The proof below with slight modifications works equally well for the case $d > 2$.) Then $G := \{ (x, y) : x \in (-b, b), \; g(x) - y \in [0, 1) \}$ and $G^* := \{ (x, y) : x \in (-2b, 2b), \; 0 < y \leq g(x) \}$, where $g$ is a $C^2$-function on $\mathbb{R}$ satisfying that $\min_{x \in [-2b, 2b]} g(x) = 4$.

First, we construct a partition of the domain $G$. Let $L := \max_{x \in [-2b, 2b]} |g(x)| + 10$, and let $n_1$ denote the smallest integer $> \frac{1}{\sqrt{5}}$, where $c_0 \in (0, 1)$ is a constant depending only on $\mu$ and $\Omega$. Indeed, we will choose $c_0$ sufficiently small so that $n_1 > 5L + n$. Let $\beta_j := L - L \cos \frac{j\pi}{n_1} = 2L \sin^2 \frac{j\pi}{2n_1}$, $j = 0, 1, \ldots, n_1$
be the Chebyshev partition of the interval $[0, 2L]$ of order $n_1$. Let $m$ denote the positive integer such that $\beta_m < 1 \leq \beta_{m+1}$. We then partition the interval $[0, 1]$ with nodes
\[ \alpha_j := \begin{cases} \beta_j, & j = 0, 1, \ldots, m - 1, \\ 1, & j = m. \end{cases} \]

It is easily seen that
\[ \frac{1}{5} \frac{n_1}{\sqrt{L}} \frac{n_1 \arccos(1 - \frac{1}{m})}{\pi} - 1 \leq m < \frac{n_1 \arccos(1 - \frac{1}{m+1})}{\pi} < \frac{3}{5} \frac{n_1}{\sqrt{L}}, \]
and
\[ \alpha_j - \alpha_{j-1} \sim \frac{1}{n_1} (\sqrt{\alpha_j} + \frac{1}{n_1}), \quad j = 1, 2, \ldots, m. \]

On the other hand, we partition the interval $[-b, b]$ with the uniform nodes:
\[ x_i = -b + \frac{2i}{n_1} b, \quad i = 0, 1, \ldots, n_1. \]

Thus, we may define a partition of $G$ as follows: for $1 \leq i \leq n_1$ and $1 \leq j \leq m$,
\[ I_{i,j} := \{ (x, y) \in G : x_{i-1} \leq x \leq x_i, \; \alpha_{j-1} \leq g(x) - y \leq \alpha_j \}. \]

For convenience, we also define, for each positive integer $\ell$, and $(i, j) \in \mathbb{Z}^2$,
\[ I_{i,j,\ell} := \{ (x, y) \in G : x_{i-1-\ell} \leq x \leq x_{i+\ell}, \; \alpha_{j-\ell-1} \leq g(x) - y \leq \alpha_{j+\ell} \}. \]
Here and elsewhere in the proof, we set
\[ \alpha_j := \begin{cases} 0, & j < 0 \\ \frac{1}{5}, & j > m, \end{cases} \quad \text{and} \quad x_i := \begin{cases} -b, & i < 0, \\ b, & i \geq n_1. \end{cases} \]

Next, for each \( \xi = (\xi_x, \xi_y) \in G \), we denote by \( \theta_\xi \) the angle in \([0, \pi]\) such that \( g(\xi_x) - \xi_y = L - L \cos \theta_\xi \). Then, for each \( \xi = (\xi_x, \xi_y) \in G \),
\[
0 \leq \theta_\xi = 2 \arcsin \sqrt{\frac{g(\xi_x) - \xi_y}{2L}} < \frac{\pi}{5}.
\]
As a result, for any \( \xi = (\xi_x, \xi_y), \eta = (\eta_x, \eta_y) \in G \),
\[
\left| \sqrt{g(\xi_x) - \xi_y} - \sqrt{g(\eta_x) - \eta_y} \right| = 2\sqrt{2L} \left| \frac{\theta_\xi - \theta_\eta}{4} \right| \cos \frac{\theta_\xi + \theta_\eta}{4} \sim |\theta_\xi - \theta_\eta|.
\]
It follows that
\[
\hat{\rho}_G(\xi, \eta) \sim \max \left\{ \| \xi_x - \eta_x \|, \| \theta_\xi - \theta_\eta \| \right\}, \quad \forall \xi, \eta \in G.
\]
Thus, there exist two constants \( c_1, c_2 > 0 \) such that
\[
B_G(\omega_{i,j}, c_1 n_1^{-1}) \subset I_{i,j} \subset B_G(\omega_{i,j}, c_2 n_1^{-1})
\]
for some \( \omega_{i,j} \in I_{i,j} \) and all \( 1 \leq i \leq n_1 \) and \( 1 \leq j \leq m \). Clearly, we may choose the parameter \( \ell = \ell_\mu \) large enough so that
\[
B_G(\omega_{i,j}, c_2 n_1^{-1}) \subset B_G(\omega_{i,j}, (3\mu + c_2) n_1^{-1}) \subset I_{i,j,\ell}.
\]
Note that by Fubini’s theorem,
\[
|I_{i,j}| \sim n_1^{-2}(\sqrt{\alpha_j} + n_1^{-1}), \quad 1 \leq i \leq n_1, \quad 1 \leq j \leq m.
\]
Finally, we prove that if \( c_0 < \frac{1}{c_2} \), then (7.13) holds for each \( \delta \)-separated subset \( \Lambda \) of \( G \). Indeed, since \( n_1 > \frac{1}{c_0} \), the set \( \Lambda \subset G \) is \( \frac{1}{n_1 c_0} \)-separated with respect to the metric \( \hat{\rho}_G \), and since \( 2c_2 < \frac{1}{c_0} \), (7.13) implies that every \( \omega \in \Lambda \) is contained in a unique set \( I_{i,j} \) with \( 1 \leq i \leq n_1 \) and \( 1 \leq j \leq m \). Thus, by (7.15), (7.16) and the fact that \( \frac{1}{n_1} \sim \delta \), it is enough to prove that
\[
\left( \sum_{i=1}^{n_1} \sum_{j=1}^{m} |I_{i,j}| \left( \text{osc}(f; I_{i,j,\ell}) \right)^p \right)^{\frac{1}{p}} \leq C_\ell(n\delta) \| f \|_{L^p(G^n)}, \quad f \in \Pi_n.
\]
To prove (7.18), we fix \( f \in \Pi_n \), and define \( F(x, \alpha) := f(x, g(x) - \alpha) \) for \(-2b \leq x \leq 2b \) and \( \alpha \in \mathbb{R} \). Then \( f(x, y) = F(x, g(x) - y) \), and
\[
\text{osc}(f; I_{i,j,\ell}) = \max_{x, x' \in [x_{i-1,\ell}, x_{i+\ell}], \alpha, \alpha' \in [\alpha_{j-1,\ell}, \alpha_{j+\ell}]} |F(x, \alpha) - F(x', \alpha')| \]
\[
\leq 2 \sup_{\alpha \in [\alpha_{j-1,\ell}, \alpha_{j+\ell}]} \sup_{x \in [x_{i-1,\ell}, x_{i+\ell}]} \left| F(x, \alpha) - \frac{n_1}{2b} \int_{x_{i-1}}^{x_{i+1}} F(u, \alpha) \, du \right| \]
\[
\leq 2 \left[ a_{i,j}(f) + b_{i,j}(f) \right],
\]
Thus, setting

\[
\xi
\]

Since the function \(\Sigma\)

\[
\leq
\]

we reduce to showing that

\[
\leq
\]

Thus, using (7.20) and (7.17), we obtain

\[
(7.20)
\]

To show (7.19), we first use Hölder’s inequality and (7.14) to obtain

\[
(7.21)
\]

Thus, using (7.20) and (7.17), we obtain

\[
\leq
\]

Since the function \(\partial_1 F(v, \alpha) = (\partial_1 + g'(v) \partial_2) f(v, g(v) - \alpha)\) is an algebraic polynomial of degree at most \(n\) in the variable \(\alpha\) for each fixed \(v \in [-2b, 2b]\), and since \(n_1 \geq n\), it follows from Lemma 7.2 that

\[
(7.22)
\]

Thus, setting \(\xi(v) = (1, g'(v))\) for \(v \in [-2b, 2b]\), we obtain from Theorem 4.1 that

\[
\leq
\]

which proves (7.19) for \(k = 1\).

Similarly, using (7.21) and (7.17), we have

\[
\leq
\]

\[
\leq
\]

\[
\leq
\]
which, using the univariate Markov-Bernstein-type inequality \((\text{[34 Theorem 7.3]})\),
is estimated above by
\[
Cn\delta\left(\int_{-b}^{b} \int_{0}^{2} |F(u, \alpha)|^p \, d\alpha \, du\right)^{\frac{1}{p}} \leq Cn\delta \|f\|_{L^p(G^*)}.
\]
This proves \((7.19)\) for \(k = 2\), and hence completes the proof of \((7.13)\). \(\square\)

We also need a polynomial inequality on the unit ball \(B^d := \{\xi \in \mathbb{R}^d : \|\xi\| \leq 1\}\),
which is stated in our third lemma. Let \(\rho_B\) denote the metric on \(B^d\) given by
\[
\rho_B(\xi, \eta) = \|\xi - \eta\| + \sqrt{1 - \|\xi\|^2} - \sqrt{1 - \|\eta\|^2}, \quad \xi, \eta \in B^d.
\]
For \(\xi \in B^d\) and \(r > 0\), we set \(B_r(\xi, r) := \{\eta \in B^d : \rho_B(\xi, \eta) \leq r\}\) and \(B = B_r(\xi, r)\) and a constant \(c > 0\), we denote by \(cB\) the set \(B_c(\xi, cr)\). It can be easily seen that for any \(r \in (0, 1)\) and \(\xi \in B^d\),
\[
|B_r(\xi, r)| \sim |B_r(\xi, 2r)| \sim r^d(\sqrt{1 - \|\xi\|^2} + r).
\]

**Lemma 7.4.** Let \(\{\xi_j\}_{j=1}^{m} \subset B^d\) be \(\frac{1}{n}\)-separated with respect to the metric \(\rho_B\) for some \(\delta \in (0, 1)\) and \(n \in \mathbb{N}\), and let \(E := \bigcup_{j=1}^{m} B(\xi_j, \frac{\delta}{m})\). If \(1 \leq p < \infty\) and \(0 < \delta < \delta_0\), where \(\delta_0 \in (0, 1)\) is a sufficiently small constant depending only on \(d\),
then
\[
\|f\|_{L^p(B^d)} \leq C\|f\|_{L^p(B^d \setminus E)}, \quad \forall f \in \Pi_n^d,
\]
where the constant \(C\) depends only on \(d\).

\textbf{Proof.} According to Theorem 11.6.1 of \([13\text{ p. 290}]\), there exists a constant \(\delta_0 \in (0, 1/4)\) depending only on \(d\) such that for every \(0 < \delta < \delta_0\), every maximal \(\frac{1}{n}\)-separated subset \(\Lambda\) of \(B^d\) ( with respect to the metric \(\rho_B\) ), and any \(f \in \Pi_n^d\),
\[
(7.23) \quad C_3 \left(\sum_{\xi \in \Lambda} |B_\xi| \max_{\eta \in 4B_\xi} |f(\eta)|^p\right)^{\frac{1}{p}} \leq \|f\|_{L^p(B^d \setminus E)} \leq C_2 \left(\sum_{\xi \in \Lambda} |B_\xi| \min_{\eta \in 4B_\xi} |f(\eta)|^p\right)^{\frac{1}{p}},
\]
where \(B_\xi = B(\xi, \frac{\delta}{n})\) for \(\xi \in \Lambda\), and the constants \(C_1, C_2 > 0\) depend only on \(d\). \(\square\)

Now let \(\{\xi_j\}_{j=1}^{m} \subset B^d\) be \(\frac{1}{n}\)-separated with respect to the metric \(\rho_B\) with \(0 < \delta < \delta_0\), and take a maximal \(\frac{1}{n}\)-separated subset \(\Lambda \subset \Omega\) which contains all the points \(\xi_j\), \(j = 1, \ldots, m\). Then using \((7.23)\) and \((7.22)\), we have
\[
\|f\|_{L^p(B^d \setminus E)} \leq C_2 \left(\sum_{\xi \in \Lambda} \int_{(2^{-1}B_\xi \setminus (4^{-1}B_\xi))} |f(\eta)|^p \, d\eta\right)^{\frac{1}{p}} \leq C\|f\|_{L^p(B^d \setminus E)},
\]
where the last step uses the facts that the sets \(2^{-1}B_\xi\), \(\xi \in \Lambda\) are pairwise disjoint and \(E \subset \bigcup_{\xi \in \Lambda} 4^{-1}B_\xi\). This completes the proof of Lemma 7.4 \(\square\)

Finally, we need a lemma to deal with points in \(\Omega\) that are not very close to the boundary \(\Gamma\). Given a parameter \(\delta \in (0, 1)\), we set \(\Gamma_\delta := \{\xi \in \Omega : \text{dist}(\xi, \Gamma) \leq \delta\}\) and \(\Omega_\delta = \Omega \setminus \Gamma_\delta\). Also, recall that
\[
B_r(\xi) = \{\eta \in \mathbb{R}^d : \|\eta - \xi\| < r\}, \quad r > 0, \quad \xi \in \mathbb{R}^d.
\]
Lemma 7.5. Let $\mu > 1$ and $\delta_0 \in (0, 1)$ be two given parameters. Let $\varepsilon \in (0, \frac{\delta_0}{n})$, and let $n$ be an integer $> \mu$. Assume that $\Lambda$ is a finite subset of $\Omega_{\delta_0}$ satisfying that $\|\xi - \eta\| \geq \frac{2}{n}$ for any two distinct points $\xi, \eta \in \Lambda$. Then any $f \in \Pi_n^d$ and $1 \leq p < \infty$,

$$\left(\left(\frac{\varepsilon}{n}\right)^d \sum_{\xi \in \Lambda} \left(\text{osc}(f; B_{\mu\varepsilon/n}(\xi))\right)^p\right)^\frac{1}{p} \leq C_{\mu, \delta_0} \varepsilon \|f\|_{L^p(\Omega)}, \tag{7.24}$$

where the constant $C_{\mu, \delta_0}$ is independent of $\varepsilon, n$ and $f$.

Proof. We first cover $\Omega_{\delta_0}$ with finitely many closed Euclidean balls $B_1, \ldots, B_{n_0} \subset \Omega$ of radius $\delta_0/8$ such that $6B_i \subset \Omega$ for all $1 \leq i \leq n_0$, where $n_0$ depends only on $\delta_0$ and $\Omega$. We then reduce to showing that for $i = 1, \ldots, n_0$,

$$\left(\left(\frac{\varepsilon}{n_i}\right)^d \sum_{\xi \in \Lambda \cap B_i} \left|\text{osc}(f; B_{\mu\varepsilon/n}(\xi))\right|^p\right)^\frac{1}{p} \leq C_{\mu, \delta_0} \varepsilon \|f\|_{L^p(\Omega)}, \quad \forall f \in \Pi_n^d. \tag{7.25}$$

To show (7.25), for each $\xi \in \Lambda$, set $f_\xi := \frac{1}{|B_{\mu\varepsilon/n}(\xi)|} \int_{B_{\mu\varepsilon/n}(\xi)} f(\eta) \, d\eta$, and let $\eta_\xi \in B_{\mu\varepsilon/n}[\xi]$ be such that

$$\max_{\eta \in B_{\mu\varepsilon/n}[\xi]} |f(\eta) - f_\xi| = |f(\eta_\xi) - f_\xi|.$$ 

Using Poincare’s inequality, we obtain that for any $\xi \in \Lambda \cap B_j$,

$$\text{osc}(f; B_{\mu\varepsilon/n}(\xi)) \leq 2|f(\eta_\xi) - f_\xi| \leq C_d \int_{B_{\mu\varepsilon/n}(\xi)} \|\nabla f(\eta)\| \|\eta - \eta_\xi\|^{-d+1} \, d\eta$$

$$\leq C_{\mu} \int_{2B_j} \|\nabla f(\eta)\| \|\eta - \eta_\xi\|^{-d+1} \left(1 + \frac{n}{\varepsilon} \|\eta - \eta_\xi\|\right)^{-d-1} \, d\eta,$$

where the last step uses the fact that $B_{\mu\varepsilon/n}(\xi) \subset 2B_j$ for any $\xi \in \Lambda \cap B_j$. By Lemma 7.4, this implies that for each $\xi \in \Lambda \cap B_j$,

$$\text{osc}(f; B_{\mu\varepsilon/n}(\xi)) \leq C \int_{\{\eta \in 2B_j : \|\eta - \eta_\xi\| \geq \frac{\varepsilon}{n}\}} \|\nabla f(\eta)\| |\eta - \eta_\xi|^{-d+1} \left(1 + \frac{n}{\varepsilon} \|\eta - \eta_\xi\|\right)^{-d-1} \, d\eta$$

$$\leq C \left(\frac{n}{\varepsilon}\right)^{d-1} \int_{\Omega_{\delta_0/4}} \|\nabla f(\eta)\| \left(1 + \frac{n}{\varepsilon} \|\eta - \eta_\xi\|\right)^{-2d} \, d\eta.$$ 

Thus, using Hölder’s inequality, we obtain that for any $\xi \in \Lambda \cap B_j$,

$$\left|\text{osc}(f; B_{\mu\varepsilon/n}(\xi))\right|^p \leq C \left(\frac{n}{\varepsilon}\right)^{d-p} \int_{\Omega_{\delta_0/4}} \|\nabla f(\eta)\|^p \left(1 + \frac{n}{\varepsilon} \|\eta - \eta_\xi\|\right)^{-2d} \, d\eta.$$ 

It follows that

$$\left(\frac{\varepsilon}{n}\right)^d \sum_{\xi \in \Lambda \cap B_j} \left|\text{osc}(f; B_{\mu\varepsilon/n}(\xi))\right|^p \leq C \left(\frac{n}{\varepsilon}\right)^{d-p} \int_{\Omega_{\delta_0/4}} \|\nabla f(\eta)\|^p \sum_{\xi \in \Lambda \cap B_j} \left(1 + \frac{n}{\varepsilon} \|\eta - \eta_\xi\|\right)^{-2d} \, d\eta. \tag{7.26}$$

However, since $\Lambda \subset \Omega_{\delta_0}$ is $\frac{\varepsilon}{n}$-separated in $\mathbb{R}^d$, we obtain that for any $\eta \in \mathbb{R}^d$,

$$\sum_{\xi \in \Lambda \cap B_j} \left(1 + \frac{n}{\varepsilon} \|\eta - \eta_\xi\|\right)^{-2d} \leq C \left(\frac{n}{\varepsilon}\right)^d \sum_{\xi \in \Lambda \cap B_j} \int_{B_{\mu\varepsilon/n}(\xi)} \left(1 + \frac{n}{\varepsilon} \|\eta - \zeta\|\right)^{-2d} \, d\zeta$$

$$\leq C \left(\frac{n}{\varepsilon}\right)^d \int_{\mathbb{R}^d} \left(1 + \frac{n}{\varepsilon} \|\eta - \zeta\|\right)^{-2d} \, d\zeta \leq C_d < \infty,$$
where the first step uses the fact that \( \eta_\ell \in B_{\frac{\mu}{n}}(\xi) \) for each \( \xi \in \Lambda \cap B_j \). This combined with (7.20) yields
\[
\left( \frac{\xi}{n}^d \sum_{\xi \in \Lambda \cap B_j} \left| \text{osc}(f; B_{\frac{\mu \xi}{n}}(\xi)) \right|^p \right)^{\frac{1}{p}} \leq \frac{C \varepsilon}{n} \left( \int_{\Omega_{\frac{\varepsilon}{n}/4}} \| \nabla f(\eta) \|^p d\eta \right)^{\frac{1}{p}} \leq C \varepsilon \| f \|_{L^p(\Omega)},
\]
where the last step uses the regular Bernstein inequality. This completes the proof of Lemma 7.5.

We are now in a position to prove Theorem 7.1.

**Proof of Theorem 7.1.** Without loss of generality, we may assume that \( n \geq N_\Omega \), where \( N_\Omega \) is a sufficiently large positive integer depending only on \( \Omega \). Indeed, (7.24) for \( n \leq N_\Omega \) can be deduced directly from the case \( n = N_\Omega \).

By Proposition 2.5 and Remark 2.6, we can find a constant \( \lambda_0 \in (0, 1) \) (as small as we wish) and finitely many domains \( G_1, \ldots, G_{m_0} \subset \Omega \) of special type attached to \( \Gamma \) such that
\[
\Gamma_{\delta_0} := \{ \xi \in \Omega : \text{dist}(\xi, \Gamma) \leq \delta_0 \} \subset \bigcup_{j=1}^{m_0} G_j(\lambda_0)
\]
with \( m_0 \in \mathbb{N} \) and \( \delta_0 \in (0, 1) \) depending only on \( \Omega \). Then \( \Lambda = \bigcup_{j=1}^{m_0+1} \Lambda_j \), where \( \Lambda_j = \Lambda \cap G_j(\lambda_0) \) for \( j = 1, \ldots, m_0 \), \( \Lambda_{m_0+1} = \Lambda \cap \Omega_{\delta_0} \) and \( \Omega_{\delta_0} = \Omega \setminus \Gamma_{\delta_0} \).

Thus,
\[
\sum_{\xi \in \Lambda} \left| U(\xi, \frac{\varepsilon}{n}) \right| \left( \text{osc}(f; U(\xi, \frac{\varepsilon}{n})) \right|^p \leq \sum_{j=1}^{m_0+1} S_j,
\]
where
\[
S_j := \sum_{\xi \in \Lambda_j} \left| U(\xi, \frac{\varepsilon}{n}) \right| \left( \text{osc}(f; U(\xi, \frac{\varepsilon}{n})) \right|^p, \quad 1 \leq j \leq m_0 + 1.
\]
Clearly, by (1.3) and (7.12), we may choose \( N_\Omega \) sufficiently large so that
\[
B_{G_j}(\xi, \frac{\mu_1 \varepsilon}{n}) \subset U(\xi, \frac{\varepsilon}{n}) \subset B_{G_j}(\xi, \frac{\mu_2 \varepsilon}{n}) \subset G_j, \quad \forall \xi \in \Lambda_j, \; 1 \leq j \leq m_0,
\]
and
\[
B_{\frac{\mu_1 \varepsilon}{n}}(\xi) \subset U(\xi, \frac{\varepsilon}{n}) \subset B_{\frac{\mu_2 \varepsilon}{n}}(\xi) \subset \Omega_{\delta_0/2}, \quad \forall \xi \in \Lambda_{m_0+1},
\]
where \( \mu_2 > 1 > \mu_1 > 0 \), and \( \mu_1, \mu_2 \) depend only on \( \ell \) and \( \Omega \).

For \( 1 \leq j \leq m_0 \), we use (7.12) and Lemma 7.5 to obtain
\[
S_j \leq C \sum_{\xi \in \Lambda_j} \left| B_{G_j}(\xi, \frac{\mu_2 \varepsilon}{n}) \right| \left( \text{osc}(f; B_{G_j}(\xi, \frac{\mu_2 \varepsilon}{n})) \right|^p \leq \left( C \varepsilon \| f \|_{L^p(\Omega)} \right)^p.
\]

For \( j = m_0 + 1 \), we use (1.3) and Lemma 7.5 to obtain
\[
S_{m_0+1} \leq C \sum_{\xi \in \Lambda_{m_0+1}} \left( \frac{\varepsilon}{n}^d \right) \left| \text{osc}(f; B_{\frac{\mu_2 \varepsilon}{n}}(\xi)) \right|^p \leq \left( C \varepsilon \| f \|_{L^p(\Omega)} \right)^p.
\]

Putting the above together, we prove the desired estimate (7.1), and therefore, complete the proof of Theorem 7.1. \( \square \)
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