Abstract

Based on the representation theory of the $q$-deformed Lorentz and Poincaré symmetries $q$-deformed relativistic wave equation are constructed. The most important cases of the Dirac-, Proca-, Rarita-Schwinger- and Maxwell- equations are treated explicitly. The $q$-deformed wave operators look structurally like the undeformed ones but they consist of the generators of a non-commutative Minkowski space. The existence of the $q$-deformed wave equations together with previous results on the representation theory of the $q$-deformed Poincaré symmetry solve the $q$-deformed relativistic one particle problem.
1 Introduction

In recent years quantum groups have been studied extensively by both mathematicians and physicists. A special point of interest is the quantum deformation of the symmetries of flat space-time, the Lorentz and Poincaré groups [1, 2, 3, 4, 5]. Following the ideas of Wigner [6], unitary irreducible representations (Irreps) of the $q$-deformed Poincaré symmetry [4] have been investigated to construct massive [7] and massless [8] representations of that symmetry. These Irreps can be interpreted as $q$-deformed relativistic one-particle states. It turns out that these states have a number of interesting properties. For example the momentum spectrum is discrete and the mass becomes quantized in terms of the deformation parameter $q$.

The construction of [7] only led to spinless massive one-particle states. To incorporate the spin, covariant- or spinor-bases for the $q$-deformed Poincaré algebra have been invented in [9] using finite dimensional corepresentations of the Quantum Lorentz Group (QLGr) [2]. As in the undeformed case these spinor bases carry by construction more spin degrees of freedom than the physical particle which they ought to describe. The additional degrees of freedom can be removed classically by covariant subsidiary conditions which turn out to be the relativistic wave equations. These subsidiary conditions can be obtained using only techniques of group representation theory [10].

The aim of this work is to show that $q$-analogues of relativistic wave equations can be constructed with the help of representation theory of quantum groups. The $q$-deformed wave equations then have the same properties as the undeformed ones. It should be noted that a $q$-analogue of the Dirac equation has already been constructed following a different approach [11].

The outline of the paper is as follows. In section 2 the classical construction of relativistic wave equations based on representation theory is briefly explained. Section 3 and 4 review the necessary material of the representation theory of the QLGr and the $q$-deformed Poincaré symmetry. In the following sections the $q$-deformed Dirac-, spin 1 Joos-Weinberg-, Maxwell-, Proca- and Rarita-Schwinger- equations are constructed. In section 9 the general case of a $q$-deformed relativistic wave equation for arbitrary spin is considered.

2 The classical situation

In order to make the construction of the $q$-deformed wave equations more transparent the procedure in the classical case is briefly outlined. Only the massive case is treated here. More details can be found in [10]. The construction is based on the representation theory of semidirect products using the method of induced representations.
When working with Mackey- or Wigner- states the unitary Irreps of the Poincaré group can be induced directly by a given unitary Irrep $D(k)$ of the stability subgroup $SU(2)$. It is possible to find a special transformation from the Mackey- or Wigner- states to the so called covariant- (spinor-) states for the Poincaré group. From the viewpoint of the inducing procedure this means that $D(k)$ is imbeded into a representation $D^{(m,n)}(g)$ of the Lorentz group whose restriction to $SU(2)$, $D(k)$, is unitary. $m$ and $n$ denote the highest weights of the Weyl representation of the Lorentz group. An important property of the spinor states is that the inner product of two states $\psi_1$ and $\psi_2$ is given by:

$$(\psi_1,\psi_2) = \int d\mu(p) \left( \psi_1(p), D^\dagger(s) D(s) \psi_2(p) \right)_D$$

(1)

The variable $p$ denotes the momenta by which the states are diagonalized, $d\mu(p)$ is some measure in momentum space and $D(s)$ is some finite dimensional representation of $s \in SL(2,\mathbb{C})$.

The problem is that $D(k)$ contains in general other representations of $SU(2)$ than the desired one $D(k)$. Hence the unrequired representations occuring in $D(k)$ have to be eliminated covariantly. This makes covariant subsidiary conditions necessary, which are the wave equations.

If the momentum eigenvalue $p$ occurring in the arguments of the states in (1) is taken to be obtained by a pure Lorentz boost from a rest system vector $p_r$: $p = D^{(1/2,1/2)}(s)p_r$, the expression (1) can be simplified to:

$$(\psi_1,\psi_2) = \int \frac{d^3p}{\omega} \psi_1^\dagger D^{-1} \left( \frac{\sigma.p}{m} \right) \psi_2(p),$$

(2)

where $m$ is the mass eigenvalue and $\sigma$ represents the ordinary Pauli matrices.

A wave operator $W^{(j)}(p)$ is a projection operator from the set of $SU(2)$ representations in $D(k)$ to the desired spin $j$ representation $D^{(j)}(k)$. It can be shown that this wave operator is in principle equivalent to the object $D^{-1} \left( \frac{\sigma.p}{m} \right)$ in (2). To establish the equivalence completely certain index symmetry properties have to be taken into account when the wave operator is applied to a wave function.

The following important theorem holds in general for induced representations:

**Theorem:** The covariant form of an induced representation is completely characterized by the set $\{D(g), W\}$ consisting of an inducing representation $D(g)$ of the full Lie group and a projection operator $W$ to the required unitary representation $D(k)$ of the stability subgroup.

In the special case of the Poincaré group this means that a massive spinor basis consisting of a direct product of a spinless unitary Irrep and a finite-dimensional representation of the Lorentz group together with a covariant wave operator which restricts the spin degrees of freedom to the physical ones in all Lorentz frames completely solves the massive relativistic one particle problem in momentum space.
The Quantum Lorentz Group (QLGr) can be defined by considering two copies of a deformed $SU(2)$ Hopf algebra corresponding to the non-equivalent fundamental representations of the quantum group $SL_q(2, \mathbb{C})$. We take the deformation parameter $q > 1$, and the abbreviation $\lambda = q - q^{-1}$ is sometimes used.

One starts by introducing a quantum matrix $M_{\alpha\beta} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The entries of $M_{\alpha\beta}$ obey commutation relations which are generated by the R-matrix \([1]\) of $SL_q(2)$:

$$\hat{R}^{\alpha\gamma}_{\delta\beta} M_{\gamma\mu} M^\delta_{\nu} = M^\alpha_{\rho} M^\beta_{\sigma} \hat{R}^{\rho\sigma}_{\mu\nu}$$ \(3\)

The Hopf algebra structure of the quantum matrix $M_{\alpha\beta}$ is given as usual by the comultiplication $\Delta(M_{\alpha\beta}) = M^\alpha_{\gamma} \otimes M^\gamma_{\beta}$, the counit $\epsilon(M_{\alpha\beta}) = \delta^{\alpha\beta}$ and the antipode which can be expressed using the deformed $\varepsilon$-tensors given explicitly in part a of the appendix: $S(M_{\alpha\beta}) = \varepsilon^{\alpha\gamma} M^\delta_{\gamma} \varepsilon_{\delta\beta}$. Together with the obvious unimodularity condition these definitions make $M$ an $SL_q(2, \mathbb{C})$-matrix.

Since Weyl representations of the QLGr shall be constructed we impose a unitarity condition using an antimultiplicative involution $\ast$: $(M_{\alpha\beta})^\ast := S(M_{\beta\alpha})$, which makes $M$ an $SU_q(2)$-matrix. We call this quantum group $A_q$. The representations of $A_q$ correspond to representations built purely from the fundamental representation of $SL_q(2, \mathbb{C})$.

To obtain the complex conjugate representations of $SL_q(2, \mathbb{C})$ an antimultiplicative algebra morphism $k : A_q \rightarrow \bar{A}_q : M_{\alpha\beta} \rightarrow \bar{M}_{\tilde{\alpha}\tilde{\beta}}$ is introduced by:

$$k \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\alpha_{\beta} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}^\tilde{\beta}_{\tilde{\alpha}}$$ \(4\)

A comment on the indices of the complex conjugate representations has to be made. In ordinary $SL(2, \mathbb{C})$ spinor calculus one works with dotted and undotted indices. To make contact with the classical case one has to identify:

$$\bar{M}_{\tilde{\alpha}\tilde{\beta}} = \bar{M}^\alpha_{\beta}$$ \(5\)

However, in most of the calculations in this paper the calculation with indices with tilde is more convenient since commutation relations can more simply be formulated. If one works with dotted indices the index-structure of the $\hat{R}$-matrices should be altered which makes the calculus more complicated.

Sometimes the dual of the complex conjugate representation is required. Classically this corresponds to $M^{\dagger -1}$. Algebraically a mapping $j : A_q \rightarrow \hat{A}_q$ can be introduced \([5]\) which maps the fundamental to the dual complex conjugate representation by:

$$j \left( M_{\alpha\beta} \right) = \bar{M}^\alpha_{\beta} := k \left( S(M^\beta_{\alpha}) \right)$$ \(6\)
It should be mentioned that \( j \) can easily be extended to arbitrary representations. Now the generating relations for \( \tilde{A}_q \) can be formulated:

\[
\hat{R}^{-1} \hat{\alpha} \hat{\beta} \tilde{M} \hat{\gamma} \tilde{\mu} \tilde{M} \hat{\delta} = \tilde{M} \hat{\alpha} \hat{\beta} \tilde{M} \hat{\gamma} \tilde{\mu} \hat{R}^{-1} \hat{\delta} \tilde{\nu} \tilde{\nu}
\]  

(7)

The comultiplication \( \Delta \) and counit \( \bar{\epsilon} \) on \( \tilde{A}_q \) are given using the mapping \( k \) by \( \Delta \circ k = (k \otimes k) \circ \Delta \) and \( - \circ \epsilon = \bar{\epsilon} \circ k \). The antipode is again defined using the \( \varepsilon \)-tensors: \( S(\tilde{M} \hat{\alpha} \hat{\beta}) = \varepsilon \hat{\alpha} \hat{\gamma} \tilde{M} \hat{\gamma} \hat{\delta} \varepsilon \hat{\delta} \hat{\mu} \). Imposing the unimodularity and the unitarity conditions analogous to the previous case shows that the isomorphism holds: \( \tilde{A}_q \simeq SU_q(2) \).

As a last step commutation relations among the generators of \( A_q \) and \( \tilde{A}_q \) have to be fixed:

\[
\hat{R} \hat{\alpha} \hat{\beta} \tilde{M} \hat{\gamma} \tilde{\mu} \tilde{M} \hat{\delta} = \tilde{M} \hat{\alpha} \hat{\rho} \tilde{M} \hat{\beta} \hat{R} \hat{\rho} \hat{\gamma} \hat{\delta}
\]  

(8)

The relations (3), (7) and (8) together with the entire Hopf structure determine the QLGr completely.

Important for this work are the left corepresentation spaces of the QLGr since they are involved in the construction of spinor bases for the \( q \)-deformed Poincaré symmetry. An irreducible right comodule of \( A_q \) belonging to the spin \( l \) representation of \( SU_q(2) \) can be defined by [12, 9]:

\[
\xi_{(l)}^\alpha = \left[ \frac{2l}{l + \alpha} \right]^\frac{1}{2} q^{l-\alpha} \bar{e}^l \xi_{(l)}^\alpha
\]  

(9)

The definition of the \( q \)-binomials can be found in appendix b. The \( \xi_{(l)}^\alpha \)'s correspond to \( q \)-deformed symmetrized undotted spinors of spin \( l \). \( \alpha \) takes values in the set \( \{-l, \ldots, l\} \). The transformation property of these undotted spinors with respect to the QLGr are given by the comultiplication of the generators:

\[
\Delta(\xi_{(l)}^\alpha) = M_{(l)}^{\alpha \beta} \otimes \xi_{(l)}^\beta
\]  

(10)

The matrix \( M_{(l)} \) is a representation of an \( SU_q(2) \)-matrix belonging to spin \( l \). They consist of a little \( q \)-Jacobi polynomial and certain powers of the quantum group generators. Details can be found in [12]. Applying the mappings \( k \) and \( j \) introduced above one obtains the other corepresentations:

\[
\tilde{\xi}_{(l)}^\alpha := k(\xi_{(l)}^\alpha), \quad \tilde{\xi}_{(l)}^\alpha := j(\xi_{(l)}^\alpha), \quad \xi_{(l)}^\alpha = k \circ j(\xi_{(l)}^\alpha)
\]  

(11)

The four corepresentation spaces (9) and (11) play the same role as their undeformed analogues do in the van der Waerden spinor calculus. Their transformation properties are obvious from (10).

A general corepresentation of the QLGr is then given by the (non-commuting) product: \( \tilde{\xi}_{(l_1)} \xi_{(l_2)} \). This is equivalent to the Weyl representation of the undeformed Lorentz group with highest weights \((l_1, l_2)\).
The last point in this section is the Iwasawa decomposition of the QLGr [2]. In the undeformed case a Lorentz transformation can be decomposed into a product of a pure space rotation and a pure boost. This is valid also in the quantum case. It holds in the fundamental representation that
\[
M = w R w B,
\]
with \((w R)_{ij}\) being a \(SU_q(2)\)-matrix. \((w B)_{\alpha \beta} := (\rho z \rho^{-1})_{\alpha \beta}\) is a \(SL_q(2, \mathbb{C})\)-matrix and its generating relations can be obtained by inserting the entries into (3), (7) and (8). This decomposition can be extended to an arbitrary representation. The Hopf structure is the same as above for an \(SL_q(2, \mathbb{C})\) matrix. It should be mentioned that because of the unitarity of \(w R\) it holds that
\[
k(M) = k(w B) S(w R).
\]

This completes the study of the QLGr for the purposes of this work. More details about the \(q\)-deformed Lorentz symmetry can be found in [2, 9, 13].

4 \(q\)-deformed one-particle states

In [7, 8] it has been shown that unitary Irreps of the \(q\)-deformed Poincaré symmetry [4] can be constructed for the massive and the massless case. We will state briefly the results here mainly for the first case.

The \(q\)-deformation of the Poincaré algebra was obtained adding an inhomogeneous part to the \(q\)-deformed Lorentz algebra [3] which consists merely of the \((1/2, 1/2)\)-corepresentation of the QLGr, i.e. a bispinor \(\tilde{\xi}^{\alpha}_{(1/2)} \xi^{\beta}_{(1/2)}\). The vector components are labeled \((A, B, C, D)\). Using the results of the previous section one obtains the commutation relations of these components:

\[
\begin{align*}
AB &= BA - q^{-1}\lambda CD + q\lambda D^2, \\
AC &= CA + q\lambda AD, \\
AD &= q^{-2}DA, \\
BC &= CB - q^{-1}\lambda BD, \\
BD &= q^2DB, \\
CD &= DC.
\end{align*}
\]

The components behave under complex conjugation \(k\) which is just denoted by a bar:
\[
\overline{A} = B, \quad \overline{B} = A, \quad \overline{C} = C, \quad \overline{D} = D.
\]

A \(q\)-analogue of a metric tensor exists which shows Minkowskian signature:

\[
g^{ij} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
q^{-2} & 0 & 0 & 0 \\
0 & 0 & -q\lambda & -1 \\
0 & 0 & -1 & 0
\end{pmatrix}_{ij}.
\]
with inverse \( g^{ij}g^{jk} = \delta^i_k \). The length of the momentum four vector

\[
P^2 = -(q^2 + 1)^{-1} P^I g_{IJ} P^J = q^{-2}CD - AB =: M^2 \tag{17}
\]
is a Casimir in the \( q \)-deformed Poincaré algebra and can be interpreted as mass. The convention \( P^A := A \) is chosen. In [7] it turned out that a unitary Irrep of the algebra is classified by the eigenvalue of \( M^2 \) and the states are labeled by the real eigenvalues of the energy- and \( z \)-component of the \( q \)-four vector: \( P^0 = q(q + q^{-1})^{-1}(C + D) \) and \( P^z = (q + q^{-1})^{-1}(qD - q^{-1}C) \) resp., the third component \( l \) of the orbital angular momentum operator \( T^3 \) and an additional parameter \( r \) which takes values 0 or 1. It should be mentioned that one can not take a basis in which all components of the four vector are diagonal as can be seen from the algebra (14). A general Hilbert space state \( |n,N,l,r,F \rangle := |P \rangle \) is labeled by the integer eigenvalues of the diagonal generators:

\[
m^2 = d_0^2 q^{2F}, \quad p^0 = d_0 q^{q^{-r}} \left( q^{2(N+1)} + q^{2(F-N+r)} \right), \quad p^z = d_0 q^{q^{-r}} \left( q^{2n} - q^{2(N+1)} q^{2(F-N+r)} q^{2+1} \right). \tag{18}
\]
d_0 is a real universal parameter whose sign characterizes the sign of the energy as in the undeformed representation theory.

An important result of this analysis is that the mass is quantized in terms of the deformation parameter, the quantum number \( F \) classifies the different Irreps. (18) shows that the spectra of energy and momentum are discretized in the deformed case. The norm of the states is obvious: \( \langle P' | P \rangle = \delta_{P' P} \). In the sense of Wigner [6] these unitary Irreps of the \( q \)-deformed Poincaré algebra can be considered as \( q \)-deformed massive relativistic one-particle states.

The construction of [7] did not allow the incorporation of spin, although the analysis showed that the stability subgroup inducing the massive representations is \( SU_q(2) \). Therefore in [8] covariant- or spinor- representations have been constructed. A general spinor state is given by:

\[
|P; l_1, \alpha; l_2, \beta \rangle := |P \rangle \otimes \tilde{\xi}^\alpha_{(l_1)} \xi^\beta_{(l_2)} \tag{19}
\]

These representations are then induced by a representation \((l_1, l_2)\) of the QLGr. For example a pure undotted spinor belonging to spin \( l_2 \) is just \( |P; 0,0; l_2, \beta \rangle \). These spinor states are of course not orthonormal by themselves as it has been shown in section 2. This problem is addressed in the next section.

Using different techniques massless representations can be obtained [9]. In this case an additional quantum number, the \( q \)-analogue of the helicity, occurs to label the unitary Irreps. From the eigenvalue of the mass in (18) one sees that in this case \( F \to -\infty \) keeping \( N \) fixed.
5   $q$-deformed Dirac equation

As has been argued in section 2 the wave operator can be extracted merely from the inner product of two spinor states. Hence there is a need for an inner product for the $q$-deformed spinor states introduced in (19).

In general a scalar product $\langle \cdot | \cdot \rangle$ for a star-Hopf algebra $(H, \Delta, \epsilon, S, *)$ and a finite dimensional left-$H$-comodule $V$ is given by the mapping [12, 9]:

$$\langle \cdot , \cdot \rangle_L : (H \otimes V) \times (H \otimes V) \rightarrow H : (a \otimes \xi) \times (b \otimes \eta) \mapsto ab^* \langle \xi | \eta \rangle_L$$

(20)

As long as the quantum group is compact and unitarily represented the inner product is given by the Haar measure and bi-invariant. The following normalization is introduced:

$$\langle \xi^\alpha_{(l_1)} | \xi^\beta_{(l_2)} \rangle_L = \delta_{l_1 l_2} \delta^{\alpha \beta}, \quad \langle \bar{\xi}^\dot{\alpha}_{(l_1)} | \bar{\xi}^\dot{\beta}_{(l_2)} \rangle_L = \delta_{l_1 l_2} \delta^{\dot{\alpha} \dot{\beta}}.$$  

(21)

Since the QLGr is not compact it can not be expected that the so constructed inner product is still invariant. This means that when transforming the expressions in (21) with a $q$-Lorentz transformation (11) in the Iwasawa decomposition of section 3 it can be seen from (20) that a term coming from the pure boost part $w_B$ survives. The pure rotation part $w_R$ vanishes since the $\xi$'s and $\bar{\xi}$'s are unitary corepresentations of $A_q$ and $\bar{A}_q$ respectively. It will be shown that this surviving part gives the wave operator in the sense of section 2. It corresponds to the expression $D(\sigma.p/m)$ occurring in (2).

Before it is shown how the procedure works in the general case a few special physically interesting cases are treated in order to make the techniques transparent.

The first example is the case of spin $1/2$. This will lead to the $q$-deformed Dirac operator. The corresponding representations of the $q$-deformed Poncaré symmetry are in this case induced by the representation $\left( (\frac{1}{2},0) \oplus (0,\frac{1}{2}) \right)$ of the QLGr. For the construction of the wave operators it is sufficient to consider only the pure Lorentz inner products. The coaction is applied to the spinors in (21):

$$\langle \Delta(\xi^\alpha_{(\frac{1}{2})}) | \Delta(\xi^\beta_{(\frac{1}{2})}) \rangle_L = (w_B)^\alpha_{\gamma} (w_B)^\beta_{\rho} \langle \xi^\gamma_{(\frac{1}{2})} | \xi^\rho_{(\frac{1}{2})} \rangle_L$$  

(22)

$$= (w_B)^\alpha_{\gamma} (w_B)^\beta_{\rho} =: (P_{\xi_{(\frac{1}{2})}})^\alpha_{\beta}$$  

(23)

An analogous procedure for the dual complex conjugate representation can be performed yielding:

$$\langle \Delta(\bar{\xi}^{\dot{\alpha}}_{(\frac{1}{2})}) | \Delta(\bar{\xi}^{\dot{\beta}}_{(\frac{1}{2})}) \rangle =: (P_{\bar{\xi}_{(\frac{1}{2})}})^{\dot{\alpha}}_{\dot{\beta}}$$  

(24)
The matrices occurring in (23) and (24) read explicitly in terms of the generators of the boost matrices $w_B$:

\[
\left( P_{\xi_{\frac{1}{2}}} \right)^{\alpha}_{\beta} = \left( \begin{array}{cc} \rho \rho^* + z^* z & z \rho \rho^* \\ \rho^{-1} z^* & \rho^{-1} \rho \rho^* \end{array} \right), \quad \left( P_{\bar{\xi}_{\frac{1}{2}}} \right)^{\bar{\alpha}}_{\bar{\beta}} = \left( \begin{array}{cc} \rho^{-1} \rho \rho^* & -q^{-1} \rho \rho^* \\ -q^{-1} z^* \rho \rho^* & q^{-2} z^* z + \rho^* \rho \end{array} \right).
\]

It is interesting that the entries of $P_{\xi_{1/2}}$ and $P_{\bar{\xi}_{1/2}}$ can be algebraically identified with the generators of the momentum part of the $q$-deformed Poincaré algebra \([14]\) normalized by the mass:

\[
M^{-1} A = -z^* \rho^{-1}, \quad M^{-1} C = q^2 \rho^* \rho + z^* z, \quad M^{-1} B = -\rho^{-1} z^*, \quad M^{-1} D = \rho^{-1} \rho^{-1} \rho^{-1}.
\]

For later use it is also mentioned that $M^{-1}(C - q\lambda D) = \rho \rho^* + zz^*$. It is easy to see that the so constructed quantities obey the algebra \([14]\) of the $q$-deformed Minkowski four vector using the relations \([13]\). One can now rewrite (23):

\[
\left( P_{\xi_{\frac{1}{2}}} \right)^{\alpha}_{\beta} = M^{-1} \left( \begin{array}{cc} C - q\lambda D & -q B \\ -q A & D \end{array} \right)^{\alpha}_{\beta}, \quad \left( P_{\bar{\xi}_{\frac{1}{2}}} \right)^{\bar{\alpha}}_{\bar{\beta}} = M^{-1} \left( \begin{array}{cc} D & q^{-1} B \\ q^{-1} A & q^{-2} C \end{array} \right)^{\bar{\alpha}}_{\bar{\beta}}.
\]

These wave operators coincide with the $q$-deformed Dirac operators constructed in \([11]\) where an approach with a $q$-deformed Clifford algebra has been used. The correspondence is:

\[
P_{\xi_{\frac{1}{2}}} \sim \frac{\sigma \cdot p}{m}, \quad P_{\bar{\xi}_{\frac{1}{2}}} \sim \frac{\bar{\sigma} \cdot p}{m}.
\]

The $\sigma$’s denote the $q$-deformed Pauli matrices in the basis of \([11]\). The $q$-deformed Dirac equation on the wave functions $\psi^\beta(P)$ and $\bar{\phi}^{\bar{\alpha}}(P)$ then is:

\[
\left( P_{\xi_{1/2}} \right)^{\bar{\alpha}}_{\bar{\beta}} \bar{\phi}^{\bar{\alpha}}(P) = \bar{\phi}^{\bar{\alpha}}(P), \quad P^2 \psi^\beta(P) = m^2 \psi^\beta(P), \quad P^2 \bar{\phi}^{\bar{\alpha}}(P) = m^2 \bar{\phi}^{\bar{\alpha}}(P).
\]

The mass shell conditions are obtained by iterating the $q$-deformed Dirac operators and are of course a consequence of the representation theory of the $q$-deformed Poincaré symmetry. For further properties of the $q$-deformed Dirac operators the reader is referred to \([11]\).

## 6 $q$-deformed spin 1 Joos-Weinberg- and Maxwell-equations

To construct a wave equation for a spin one representation one has the choice between inducing with a representation \(((1,0) \oplus (0,1))\) or \((\frac{1}{2}, \frac{1}{2})\) of the QLGr. This section deals
Using these wave operators the spin 1 wave equation can be formulated:

\[ \langle \Delta(\xi^a_{(1)}) | \Delta(\xi^b_{(1)}) \rangle_L = (w_B)^a_c (w_B)^b_c = : (P_{\xi(1)})^a_{\tilde{b}} = \]

\[ = M^{-2} \begin{pmatrix} DD & -q\omega DA & q^3 AA \\ -q\omega BD & q^2\omega BA + M^2 & -q\omega(C - q\lambda D)A \\ q^3 BB & -q^2\omega B(C - q\lambda D) & (C - q\lambda D)^2 - q^3\lambda BA \end{pmatrix}^a_{\tilde{b}} \]

The abbreviation \( \omega = \sqrt{1 + q^{-2}} \) has been used. From the analysis of \( \tilde{\xi}^a_{(1)} \) one obtains:

\[ (P_{\xi(1)})^\tilde{b}_{\tilde{a}} = M^{-2} \begin{pmatrix} q^{-4}CC + q^{-3}\lambda AB & q^{-3}\omega AC & q^{-3}AA \\ q^{-3}\omega CB & \omega^2 AB + M^2 & \omega AD \\ q^{-3}BB & \omega DB & DD \end{pmatrix}^\tilde{b}_{\tilde{a}} \]

Using these wave operators the spin 1 wave equation can be formulated:

\[ (P_{\xi(1)})^a_{\tilde{b}} \tilde{\chi}^\tilde{b}(\mathcal{P}) = \psi^a(\mathcal{P}), \quad P^2 \tilde{\chi}^\tilde{b}(\mathcal{P}) = m^2 \tilde{\chi}^\tilde{b}(\mathcal{P}), \]
\[ (P_{\xi(1)})^\tilde{b}_{\tilde{a}} \psi^a(\mathcal{P}) = \tilde{\chi}^\tilde{b}(\mathcal{P}), \quad P^2 \psi^a(\mathcal{P}) = m^2 \psi^a(\mathcal{P}). \]

It will now be shown that the system of spin 1 wave equations (32) can equivalently be written using the \( q \)-deformed Dirac operators (27). One can reexpress the fields \( \tilde{\chi} \) and \( \psi \) as follows:

\[ \tilde{\chi}^\dagger = \tilde{\phi}^{22}, \quad \tilde{\chi}^\dagger = (q\omega)^{-1}(\tilde{\phi}^{12} + q^{-1}\tilde{\phi}^{21}), \quad \tilde{\chi}^{-\dagger} = \tilde{\phi}^{11}, \]
\[ \psi^\dagger = \phi^{22}, \quad \psi^\dagger = \omega^{-1}(\phi^{12} + q^{-1}\phi^{21}), \quad \psi^{-\dagger} = \phi^{11}. \]

It is possible to show the following equivalence by direct inspection:

\[ (P_{\xi(1)})^a_{\tilde{b}} \simeq W^{\alpha\beta}_{\beta\tilde{a}} := q\delta^{-1\gamma\tilde{\sigma}}(P_{\xi(1/2)})^\alpha_{\beta} (P_{\xi(1/2)})_{\gamma \tilde{a}} \]

6.1 Spin 1 Joos-Weinberg equation

We use the comodules \( \xi^a_{(1)} \) and \( \tilde{\xi}^b_{(1)} \) of the QLGr. By construction these comodules have 3 degrees of freedom each and the indices run through the set \( \{-1, 0, +1\} \). These representations correspond to the product of two fundamental representations which have been symmetrized and from which the trace part has been eliminated.

From the inner product (21), the spin 1 representation of the \( q \)-deformed boost matrices \( w_B \) (see appendix c) and the identifications (26) of the preceding section the following can be deduced:

\[ \langle \Delta(\xi^a_{(1)}) | \Delta(\xi^b_{(1)}) \rangle_L = (w_B)^a_c (w_B)^b_c = : (P_{\xi(1)})^a_{\tilde{b}} = \]

\[ = M^{-2} \begin{pmatrix} DD & -q\omega DA & q^3 AA \\ -q\omega BD & q^2\omega BA + M^2 & -q\omega(C - q\lambda D)A \\ q^3 BB & -q^2\omega B(C - q\lambda D) & (C - q\lambda D)^2 - q^3\lambda BA \end{pmatrix}^a_{\tilde{b}} \]

The abbreviation \( \omega = \sqrt{1 + q^{-2}} \) has been used. From the analysis of \( \tilde{\xi}^a_{(1)} \) one obtains:
It is important to note that the matrix $\hat{R}^{-1}$ has the purpose of commuting the index $\tilde{\beta}$ of the first Dirac operator with the index $\gamma$ of the second one in order to have the index structure as indicated in the object $W$. This index permutation occurs every time when one expresses a $q$-deformed wave operator in terms of more than one $q$-deformed Dirac operator. One then obtains a $q$-deformed wave equation in 2-spinor indices:

$$W'^{\alpha\rho}_{\delta\tilde{\delta}} \tilde{\varphi}^{\delta\tilde{\delta}}(\mathcal{P}) = \phi'^{\alpha\rho}(\mathcal{P})$$

(36)

The same could have been done with the second pair of equations in (32).

### 6.2 $q$-deformed Maxwell equations

It will now be shown that when setting the mass equal to zero in the wave equations (32) these equations can be rewritten as first order equations. These first order equations turn out to be $q$-analogues of the Maxwell equations formulated in terms of an electromagnetic spinor cf. [16]. Performing this reduction and using (34) one obtains:

$$\begin{align*}
(C - q\lambda D) \varphi'^{1\bar{1}} - qB \varphi'^{\bar{2}\bar{1}} &= 0, \\
(C - q\lambda D) \varphi'^{1\bar{2}} - qB \varphi'^{\bar{2}\bar{2}} &= 0,
\end{align*}$$

(37)

This system can be rewritten in terms of a single $q$-deformed Dirac operator:

$$\left(P_{(1/2)}\right)^{\alpha}_{\beta} \tilde{\varphi}^{\beta\gamma}(\mathcal{P}_{m^2=0}) = 0$$

(38)

Again the same can be done for the field $\phi'^{\alpha\beta}$. When an appropriate reality condition is stated, e.g. $\bar{\varphi}^{\alpha\bar{\beta}} = j(\phi'^{\alpha\beta})$ then (37) or (38) can be considered as the $q$-deformed Maxwell equations in momentum space. It is surely possible to change the basis in such a way that the equations can be written in terms of electric- and magnetic-field.

### 7 $q$-deformed Proca equation

A more familiar description of a spin 1 field is that of an Irrep of the Poincaré group which is induced by the $(1/2, 1/2)$ representation of the Lorentz group. The wave equation belonging to this case is the Proca equation [17]. It will be shown that a $q$-deformation of this equation is possible. However, one encounters some technical problems in the deformed case.

Inducing with the $(1/2, 1/2)$ representation of the QLGr means to work with a $q$-deformed four vector $\tilde{\xi}_{(1/2)}^{\alpha}, \tilde{\bar{\xi}}_{(1/2)}^{\beta}$. In the following the representation labeling of $\xi$ and $\bar{\xi}$ will be omitted. Inserting the transformed four vector into the inner product (20) yields:

$$\Pi := \left\langle \Delta(\tilde{\xi}^{\alpha}) \Delta(\xi^{\beta}) \right| \Delta(\tilde{\xi}^{\beta}) \Delta(\xi^{\delta}) \right\rangle_L = (\bar{w}_B)^{\alpha}_{\alpha'}(w_B)^{\beta}_{\beta'}(\bar{w}_B)^{\delta}_{\delta'}(w_B)^{\gamma}_{\gamma'} \langle \xi'^{\alpha}_{\alpha'} \xi'^{\beta}_{\beta'} \xi'^{\delta}_{\delta'} \xi'^{\gamma}_{\gamma'} \rangle$$

(39)
The goal will be to permute the first matrix in (39) to the immediate left of the matrix \((w_B)_{\gamma'} \gamma\) because then the inner product can be evaluated and the remaining expression can be written in terms of \(q\)-deformed Dirac operators which is more transparent. Using the ordinary \(\hat{R}\)-matrix calculus one gets:

\[
\Pi = \hat{R}^{-1} \tilde{\alpha}^{\rho} \gamma' (w_B)_{\rho}^{\rho'} (\tilde{\omega}_B)_{\rho'}^{\rho} (w_B)_{\gamma'}^{\gamma},
\]

(40)

The commutation of the objects remaining in the inner product has not been written out explicitly only the result of the \(q\)-permutation is indicated in the bracket. Now the expression in the inner product can be evaluated:

\[
\langle \xi^{\beta'} \tilde{\xi}_{\tilde{\delta}''} \xi^{\tilde{a}'} \rangle_L \langle \xi^{\tilde{\delta}'} \tilde{\xi}_{\tilde{a}''} \rangle_L = \delta_{\tilde{a}'}^{\beta'} \delta_{\tilde{a}''}^{\tilde{\delta}'},
\]

(41)

The last term in this equation is meant only symbolically. Now \(\Pi\) can be expressed in terms of \(q\)-deformed Dirac operators:

\[
\Pi = \hat{R}^{-1} \tilde{\alpha}^{\rho} \gamma' \hat{R}^{-1} \tilde{\delta}^{\tilde{a}'} \tilde{\rho} \delta \left( P_{\xi(1/2)} \right)^{\gamma'}_{\tilde{\delta}'} \left( P_{\xi(1/2)} \right)_{\rho}^{\rho},
\]

(42)

Using the \(\varepsilon\)-tensors of the appendix the indices in the previous expression for \(\Pi\) can be raised and lowered in such a way that they can be reexpressed in a four vector index:

\[
\Pi_{IJ} = -P^I_{\tilde{\alpha}} \gamma' \delta \gamma \delta = -P^I P^J - g_{IJ},
\]

(43)

with

\[
P^I_{\tilde{\alpha}} = \left( \begin{array}{cc} C & -A \\ -B & D \end{array} \right),
\]

(44)

and the tensor \(g_{IJ}\) is the \(q\)-deformed metric (16). The capital indices run through the set \(A, B, C, D\). However, the object \(\Pi_{IJ}\) is not yet the Proca operator. This is a result of the fact that when the Dirac operators are subsequently applied to a four-vector field \(A^{\tilde{a} \beta}\) after the action of the first operator a symmetrization in the dotted indices has to be performed in order to avoid the appearance of a scalar. This could have already been implemented in the above calculation but it is now explained in a different way.

One uses the symmetrizer which comes out of the characteristic polynomial for \(\hat{R}\). It is convenient to express it as: \(S = 1 + (q + q^{-1})^{-1} \varepsilon \varepsilon\). The term with the 1 gives just \(\Pi_{IJ}\) while the second has to be applied to the first \(q\)-Dirac operator and the vector field to which it is applied. Then the second Dirac operator is multiplied from the left and the spinor indices are brought in an order comparable to that in (33). This gives an additional term \(q(q + q^{-1})^{-1} P^I P^J\) which has to be added to \(\Pi_{IJ}\).

Then the wave operator applied to a \(q\)-four vector field \(A_{J}(P)\) gives:

\[
\left( -\frac{1}{(q^2 + 1)M^2} P^I P^J - g_{IJ} \right) A_J(P) = A_I(P)
\]

(45)
The operator on the left hand side of the previous equation will be abbreviated by \( P_{f I J} \). Multiplying equation \( (45) \) from the left with \( P^K g_{K J} \) and using \( (17) \) gives the \( q \)-analogue of the Proca equation:

\[
P^I g_{I J} A^J(P) = 0 \quad (46)
\]

This equation simply means that the scalar component with respect to \( SU_q(2) \) of the \( q \)-deformed four vector has to vanish.

### 8 \( q \)-deformed Rarita-Schwinger equation

A physical field of spin \( \frac{3}{2} \) is suitably described by a representation of the Poincaré group which is induced either by a \( \left( \left( \frac{3}{2}, 0 \right) \oplus \left( 0, \frac{3}{2} \right) \right) \)- or a \( \left( \left( 1, \frac{1}{2} \right) \oplus \left( \frac{1}{2}, 1 \right) \right) \)-representation of the Lorentz group. The first possibility leads to a further example of a Joos-Weinberg equation whose \( q \)-deformation will be treated in the next section in a general setting. The second choice leads to the Rarita-Schwinger equations \[18\].

The starting point for the \( q \)-deformation of the Rarita-Schwinger equations is the product of three fundamental representations of \( SL_q(2, \mathbb{C}) \): \( \tilde{\xi}^\alpha \tilde{\xi}^\beta \xi^\gamma \). In this product an additional \((0, \frac{1}{2})\) representation of the QLGr occurs which can be eliminated by the condition 

\[
\varepsilon_{\alpha \beta \gamma} \tilde{\xi}^\alpha \xi^\beta \xi^\gamma = 0.
\]

Now the procedure of section 7 can be repeated in this more complicated case. This leads to a threefold product of \( q \)-deformed Dirac operators where the spinor indices of the operators have to be arranged in a naturally appropriate way. The Rarita-Schwinger operator \( R_1 \) which occurs can be written symbolically as:

\[
R_1 = r_{12} r_{23} r_{34} r_{45} (P_x)^{1} (P_{\tilde{x}})^{3} (P_{\tilde{x}})^{5} (P_{\tilde{x}})^{6} \quad (47)
\]

This equation requires some explanation. The \( r_{ij} \)'s are permutation operators between the \( i \)th and \( j \)th place in the sixfold product of spinor indices coming from the Dirac operators. They consist in principle of \( R \) or \( R^{-1} \) matrices. The boldface suffixes on the Dirac operators denote just the type of \( SL_q(2, \mathbb{C}) \) index entering. The explicit expression of \( (17) \) is not very transparent. It can be found in appendix d.

It can be shown that \( R_1 \) admits a decomposition into a \( q \)-deformed Proca- and a Dirac-operator:

\[
R_1 = r_{12} r_{23} r_{34} r_{45} (P_x)^{1} (P_{\tilde{x}})^{3} (P_{\tilde{x}})^{5} (P_{\tilde{x}})^{6} \quad (48)
\]

A similar procedure can be performed for the \( \left( \frac{1}{2}, 1 \right) \) product \( \tilde{\xi}^\alpha \xi^\beta \xi^\gamma \). One eliminates the unwanted \((\frac{1}{2}, 0)\) component as above by:

\[
\varepsilon_{\alpha \beta \gamma} \tilde{\xi}^\alpha \xi^\beta \xi^\gamma = 0.
\]

This leads to a second part of the Rarita-Schwinger operator:

\[
R_2 = r_{23} r_{12} r_{34} r_{23} (P_x)^{1} (P_{\tilde{x}})^{3} (P_{\tilde{x}})^{5} (P_{\tilde{x}})^{6} \quad (49)
\]
Thus one gets two equations in the parity components $\chi^{\tilde{a}\tilde{b}\gamma}(\mathcal{P})$ and $\psi^{\tilde{a}\sigma\lambda}(\mathcal{P})$ of the Rarita-Schwinger field:

\begin{align}
\mathcal{R}_1^{\tilde{a}\tilde{b}\gamma \tilde{c}\sigma\lambda} \psi^{\tilde{a}\sigma\lambda}(\mathcal{P}) &= \chi^{\tilde{a}\tilde{b}\gamma}(\mathcal{P}) \\
\mathcal{R}_2^{\tilde{a}\tilde{b}\gamma \tilde{c}\sigma\lambda} \chi^{\tilde{a}\tilde{b}\gamma}(\mathcal{P}) &= \psi^{\tilde{a}\sigma\lambda}(\mathcal{P})
\end{align}

(50, 51)

The mass shell conditions for the fields are obvious. These equations can be rewritten in terms of linear equations. The key point is to reorder the spinor indices of $\mathcal{R}_1$, $\mathcal{R}_2$ and the fields and to multiply the wave operators appropriately from the left by $q$-deformed Dirac- and Proca- operators. As in the case of the vector field in the previous section a dotted and an undotted index on the fields are combined to a four vector index. One obtains:

\begin{align}
P^I g_{IJ} \chi^J(\mathcal{P}) &= 0, & (P_{\xi(j/2)})^{\tilde{a}\tilde{b}} \chi^{\tilde{a}\tilde{b}}(\mathcal{P}) &= \psi^\alpha(\mathcal{P}) \\
(P_{\xi(j/2)})^{\tilde{a}\tilde{b}} \chi^{\tilde{a}\tilde{b}}(\mathcal{P}) &= \psi^\alpha(\mathcal{P}), & (P_{\bar{\xi}(j/2)})^{\tilde{a}\tilde{b}} \psi^{\tilde{a}\tilde{b}}(\mathcal{P}) &= \chi^\alpha(\mathcal{P}).
\end{align}

(52)

Again the result is that the $q$-deformed wave equations look structurally like the undeformed ones besides the complicated index rearrangements using $\hat{R}$-matrices. Nevertheless the key point is that the entries of the wave operators are generators of a non-commutative algebra.

9 The general case

So far $q$-deformed wave equations have been constructed for low dimensional but physically important inducing representations of the QLGr. It is possible to generalize the constructions to an arbitrary inducing representation. This is most easily done for the Joos-Weinberg equations, i.e. for an inducing representation $((j,0) \oplus (0,j))$ which is a direct generalization of sections 5 and 6.1.

One considers the corepresentations $\xi^\alpha_{(j)}$ and $\bar{\xi}^\tilde{b}_{(j)}$ introduced in (11) and (13) and studies the inner product of the transformed spinors:

\begin{align}
(P_{\xi(j)})^\alpha_{\tilde{b}} &:= \langle \Delta(\xi^\alpha_{(j)}) | \Delta(\xi^\beta_{(j)}) \rangle_L = (w_B^{(j)})^\alpha_{\beta} \left((w_B^{(j)})^\beta_{\gamma}\right)^* \\
(P_{\bar{\xi}(j)})^{\tilde{a}}_{\tilde{b}} &:= \langle \Delta(\bar{\xi}^\tilde{a}_{(j)}) | \Delta(\bar{\xi}^\tilde{b}_{(j)}) \rangle_L = (\bar{w}_B^{(j)})^{\tilde{a}}_{\tilde{b}} \left((\bar{w}_B^{(j)})^{\tilde{b}}_{\tilde{c}}\right)^*.
\end{align}

(53, 54)

The higher dimensional representations of the boost matrices $w_B$ can be taken from part c of the appendix and the entries of the $q$-deformed wave operators can be identified with the generators of the non-commutative Minkowski space using (26). Taking fields $\psi^{\beta}(\mathcal{P})$ and $\chi^{\tilde{a}}(\mathcal{P})$, the indices $\alpha$ and $\beta$ taking values in the set $\{-j, \cdots, j\}$, which are Irreps of
the \(q\)-deformed Poincaré symmetry and therefore are already on shell one can formulate
the general \(q\)-deformed Joos-Weinberg wave equations:

\[
\begin{align*}
(P_{\xi(u)})_\alpha^\beta \bar{\chi}^\beta(P) &= \psi^\alpha(P), \\
(P_{\bar{\xi}(u)})^\beta_\alpha \psi^\alpha(P) &= \bar{\chi}^\beta(P),
\end{align*}
\]

It can moreover be proven by induction from the results of sections 6, 7, 8 and especially
equation (35) that one can start equivalently with a tensor product of fundamental represen-
tations which have to be made irreducible by symmetrizing and removing the traces.
This leads to a product of \(q\)-deformed Dirac operators in which the indices have to be
arranged in a correct way. An example is shown in part d of the appendix. These expres-
sions are not transparent and therefore they are not given explicitly for the higher spins.
The \(q\)-deformed wave equations obtained in this way can be linearized. One obtains for
the symmetrized and irreducible tensor field of integer spin:

\[
P^I g_{IJ} A^{J\cdots K}(P) = 0
\]

For a state of half integral spin in a general Rarita-Schwinger basis in which the fields are
irreducible with respect to the QLGr one finds:

\[
\begin{align*}
(P_{\xi(u/2)})_\beta^\alpha \chi^{\beta\cdots\gamma}(P) &= \psi^{\beta\cdots\gamma}(P), \\
(P_{\bar{\xi}(u/2)})^{\beta\gamma} \psi^{\beta\gamma}(P) &= \bar{\chi}^{\beta\gamma}(P).
\end{align*}
\]

It should also be mentioned that the procedure outlined in section 6.2 for obtaining
massless wave equations from the massive ones applies directly to higher dimensional
systems. These equations take the form e.g.:

\[
\left(P_{\xi(u/2)}\right)^\alpha_\beta \phi^{\beta\cdots\gamma}(P_{m^2=0}) = 0
\]

10 Summary

In the sense of Wigner \[6\] elementary particles are considered as unitary Irreps of the
Poincaré group. The \(q\)-deformation of the Poincaré symmetry leads in a first step \[7\] to
spinless particles which live on a non-commutative momentum Minkowski space. This has
the effect that the spectra of energy, the measurable component of the space-momenta
and the mass admit a discretization. In a second step one goes over to spinor states \[9\] of
the \(q\)-deformed Poincaré symmetry which are not a unitray Irrep. These states have more
spin degrees of freedom than the physical particle has. The \(q\)-deformed wave equations
remove the additional degrees of freedom. This proves that the unitary Irreps of the \(q\-
deformed Poincaré symmetry actually are classified by the stability subgroup \(SU_q(2)\). One
has therefore a representation theory analogous to the undeformed case. The analysis
of this work solves the \(q\)-deformed relativistic one-particle problem in a non-commutative
momentum space from the viewpoint of representation theory.
11 Appendix

a. The $\varepsilon$ - tensors
The tensors which are used for raising and lowering $SL_q(2, \mathbb{C})$ spinor indices are [1]:

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ -q^{\frac{1}{2}} & 0 \end{pmatrix}_{\alpha\beta}, \quad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & -q^{-\frac{1}{2}} \\ q^{\frac{1}{2}} & 0 \end{pmatrix}_{\alpha\beta},$$

with normalization $\varepsilon_{\alpha\beta} \varepsilon^{\beta\gamma} = \delta^\gamma_\alpha$. The $\varepsilon$’s with tilded indices are identical to those in (59).

b. $q$-numbers
A $q$-number is defined by:

$$[n]_q = \frac{1 - q^n}{1 - q}$$

The $q$-deformed binomial coefficients are given by the definition:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^a} = \frac{[n]_{q^a}}{[k]_{q^a} [n-k]_{q^a}}$$

(c. Boost matrices in an arbitrary representation
The higher dimensional representations of the boost matrices $w_B$ introduced in section 3 are special cases of general $SL_q(2, \mathbb{C})$ matrices whose explicit form can be found in [12]. Since the $(w_B^{(l)})$’s are used in the construction of general $q$-deformed wave equations they are listed here.

Case I: $i + j \leq 0, \quad j \geq i$:

$$(w_B^{(l)})_j^i = \rho^{-i-j} z^{j-i} \left[ \begin{array}{c} l-i \\ j-i \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} l+j \\ j-i \end{array} \right]_{q^{-2}} q^{(l+i)(i-j)}$$

Case II: $i + j \geq 0, \quad j \geq i$:

$$(w_B^{(l)})_j^i = z^{j-i} \rho^{-i-j} \left[ \begin{array}{c} l-i \\ j-i \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} l+j \\ j-i \end{array} \right]_{q^{-2}} q^{(j-i)(j-i)}$$

The complex conjugate representations are obtained by $\bar{w}_B^{(l)} = k(w_B^{(l)})$ using the mapping $k$ of section 3 and $j \left( (w_B^{(l)})^{\alpha}_{\beta} \right) = (-q)^{j-i} (\bar{w}_B^{(l)})^{-\bar{\alpha}}_{-\bar{\beta}}$. This has been shown in [9].

d. Explicit form of the $q$-Rarita-Schwinger operator $\mathcal{R}_1$

$$\mathcal{R}_1^{\bar{\alpha}\bar{\beta}\gamma} \bar{\nu}_\mu \rho = \hat{R}^{\bar{\alpha}\bar{\beta}}_{\bar{\beta}_1} \hat{R}^{-1} \hat{\gamma}_{1\bar{\alpha}_1} \hat{\gamma}_{1\bar{\alpha}_2} \hat{\gamma}_1 \bar{\beta}_2 \bar{\beta}_3 \bar{\beta}_4 \bar{\beta}_5 \bar{\beta}_6 \bar{\beta}_7 \bar{\beta}_8 \bar{\beta}_9 \bar{\beta}_{10} \bar{\beta}_{11} \bar{\beta}_{12} \bar{\beta}_{13} \bar{\beta}_{14} \bar{\beta}_{15} \times$$

$$(P_\xi)^{\bar{\gamma}_2} \bar{\nu}_2 (P_\xi)^{\bar{\gamma}_3} \bar{\nu}_3 (P_\xi)^{\bar{\gamma}_4} \bar{\nu}_4 (P_\xi)^{\bar{\gamma}_5} \bar{\nu}_5 (P_\xi)^{\bar{\gamma}_6} \bar{\nu}_6 (P_\xi)^{\bar{\gamma}_7} \bar{\nu}_7 (P_\xi)^{\bar{\gamma}_8} \bar{\nu}_8 (P_\xi)^{\bar{\gamma}_9} \bar{\nu}_9 (P_\xi)^{\bar{\gamma}_{10}} \bar{\nu}_{10} (P_\xi)^{\bar{\gamma}_{11}} \bar{\nu}_{11} (P_\xi)^{\bar{\gamma}_{12}} \bar{\nu}_{12} (P_\xi)^{\bar{\gamma}_{13}} \bar{\nu}_{13} (P_\xi)^{\bar{\gamma}_{14}} \bar{\nu}_{14} (P_\xi)^{\bar{\gamma}_{15}} \bar{\nu}_{15}$$
Acknowledgement
The author would like to thank Professor J. Wess for drawing his attention to the problem of $q$-deforming relativistic wave equations and A. Schirrmacher, J. Schwenk, M. Niedermaier and W.B. Schmidke for stimulating and helpful discussions.

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