Homogeneous variational complexes and bicomplexes

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Abstract

We present a family of complexes playing the same rôle, for homogeneous variational problems, that the horizontal parts of the variational bicomplex play for variational problems on a fibred manifold. We show that, modulo certain pullbacks, each of these complexes (apart from the first one) is globally exact. All the complexes may be embedded in bicomplexes, and we show that, again modulo pullbacks, the latter are locally exact. The edge sequence is an important part of such a bicomplex, and may be used for the study of homogeneous variational problems.

Keywords: variational complex, variational bicomplex

MSC2000 Classification: 58E99

1 Introduction

The variational bicomplex, introduced around 30 years ago, plays an important rôle in the calculus of variations on a fibration $\pi : E \to M$. The spaces in this bicomplex are spaces of differential forms on the infinite jet bundle of the fibration. Each form can be decomposed into a number of components; this decomposition is generated by the decomposition of a 1-form into horizontal and contact components. The exterior derivative $d$ may thus be written as the sum of two anticommuting parts, $d_h$ and $d_v$, and both operators are locally exact. The local exactness of $d_v$ essentially mirrors that of $d$ on the fibres. However, the local exactness of $d_h$ is significantly harder to demonstrate, although several proofs are available: see, for example, [1, 7, 8, 9, 10].
By its nature, the variational bicomplex contains no information about the order of the forms involved, although it is also possible to define a version of the bicomplex on finite-order jet bundles (see, for example, [11]).

The finite-order variational complexes are defined in a somewhat different way. Here, attention is restricted to a particular $k$-jet manifold. The first observation is that every contact 1-form on this manifold is horizontal over the $(k-1)$-jets. An important result (the $C\Omega$ hypothesis) is that every contact form may be expressed in terms of these contact 1-forms and their exterior derivatives. Thus a complex may be obtained from the spaces of forms on the $k$-jet manifold by taking quotients with respect to the contact forms and their exterior derivatives. The exterior derivative $d$ passes to this quotient, and is locally exact [5].

The finite-order variational complex can also be defined for the manifold of $k$-th order contact elements of a manifold $E$. However the variational bicomplex makes essential use of the fibration over the base manifold $M$.

In this paper, we consider the possibility of defining similar complexes for homogeneous variational problems. These are problems defined on $m$-frame bundles, that is, bundles of regular $m$-velocities (see, for example, [2, 3]). The prototype for such problems is Finsler geometry, where the problem is defined on the slit tangent bundle of $E$ (in other words, the first-order 1-frame bundle). Any variational problem on a jet bundle of a fibration induces a homogeneous variational problem on (an open submanifold of) a frame bundle and studying the problem in this context can sometimes give important insights.

Our definition is given for finite-order $m$-frame bundles. The terms in each complex contain vector-valued forms rather than scalar forms, and we show that, modulo certain pull-backs, each complex is exact. We also embed these complexes in suitable bicomplexes, where the vertical differential $d_v$ is replaced by the ordinary exterior derivative $d$. The edge sequence of this bicomplex has some similarities with finite-order variational complex on a jet manifold.

We start, therefore, with some preliminary remarks about the frame bundles and their properties, and then introduce the spaces of each complex and the coboundary operator $d_T$. The main result of the paper is the proof that, for all except the first complex, this operator is globally exact (modulo pullbacks) and, indeed, that there is a canonically-defined ‘homotopy operator’ — the quotation marks indicate that it takes its values in a pull-back of the domain of $d_T$, so that the term pseudo-homotopy operator might be more appropriate. The following section introduces the bicomplexes and demonstrates (local) exactness of the first complex, again modulo pullbacks, while the final section shows how certain terms of the edge sequence containing equivalence classes of vector-valued forms may be mapped globally to spaces of representative forms, and considers the relationship of this sequence to problems in the calculus of variations. A subsequent paper [6] will give a preliminary report on a project to apply this theory to find, for a homogeneous Lagrangian, a scalar form which is closed precisely when the Lagrangian is
null: this corresponds to the ‘fundamental Lepage equivalent’ known only for first-order Lagrangians in the case of fibred manifolds.

2 Preliminary remarks

The manifolds studied in the context of homogeneous variational problems are the (higher-order) \( m \)-frame bundles (see, for example, [3]). Given a smooth manifold \( E \) with \( \dim E = n \), consider the \( k \)-th order \( m \)-velocities in \( E \). These are the \( k \)-jets at \( 0 \in \mathbb{R}^m \) of maps \( \gamma \) from a neighbourhood of \( 0 \) to \( E \). If a map \( \gamma \) is an immersion then its \( k \)-jet \( j^k_0 \gamma \) is called a \( k \)-th order \( m \)-frame. The set of all such \( j^k_0 \gamma \) is denoted \( \mathcal{F}^k_{(m)} E \). It is a smooth manifold, and a bundle over \( E \) with projection \( \tau^k_{(m)} : \mathcal{F}^k_{(m)} E \to E \). There are also projections \( \tau^{k,l}_{(m)} : \mathcal{F}^k_{(m)} E \to \mathcal{F}^l_{(m)} E \) where \( l < k \). If \( (u^\alpha) \) are local coordinates on \( U \subset E \), where \( 1 \leq \alpha \leq n \), then

\[
(u^\alpha, u_i^\alpha, u_{(ij)}^\alpha, \ldots, u_{(i_1 i_2 \ldots i_k)}^\alpha)
\]

are local coordinates on \( (\tau^k_{(m)})^{-1}(U) \subset \mathcal{F}^k_{(m)} E \), where \( 1 \leq i, j, \ldots \leq m \) and the parentheses \((ij)\) indicate symmetrization. In multi-index notation, these coordinates may be written as \( (u^\gamma_I) \) where \( I \in \mathbb{N}^m \) and where \( 0 \leq |I| \leq k \). Note that the two notations often give rise to different constant multiples.

A basic example of this construction arises when \( k = 1 \). In this case the bundle of first-order \( m \)-velocities may be identified with the Whitney sum bundle \( \oplus^m TE \to E \), and the bundle of first-order \( m \)-frames may be identified with the set of \( m \)-tuples \( (\xi_1, \xi_2, \ldots, \xi_m) \) where the vectors \( \xi_i \) are linearly independent. For this example, there is another interpretation of the bundle of \( m \)-velocities as a tensor product bundle \( TE \otimes \mathbb{R}^{m*} \). If \( (\xi_1, \xi_2, \ldots, \xi_m) \) is a first-order \( m \)-velocity (where \( \xi_i \in T_p E \) then the corresponding tensor is \( \xi_i \otimes t^i \), where \( (e_i) \) is the standard basis of \( \mathbb{R}^m \) and \( (t^i) \) is the dual basis of \( \mathbb{R}^{m*} \).

Frame bundles are closely related to bundles of contact elements, and to jet bundles. The bundle \( J^k(E, m) \) of \( m \)-dimensional \( k \)-contact elements on \( E \), also known as the bundle of \( k \)-jets of immersed \( m \)-dimensional submanifolds, or the bundle of \( k \)-th order Grassmannians, is a quotient of the frame bundle \( \mathcal{F}^k_{(m)} E \). The group of diffeomorphisms of \( \mathbb{R}^m \) preserving the origin acts on elements of \( \mathcal{F}^k_{(m)} E \) by composition with the immersion defining the frame, and this action factors through to an action of the \( k \)-th order jet group; if \( k = 1 \) then this group is \( \text{GL}(m) \). The quotient of \( \mathcal{F}^k_{(m)} E \) by this action of the jet group is \( J^k(E, m) \). There is also an oriented version of this quotient construction, given by taking the subgroup of orientation-preserving diffeomorphisms.

Now suppose that there is a fibration \( \pi : E \to M \) over some \( m \)-dimensional manifold \( M \). The image of any local section \( \phi \) of \( \pi \) is an \( m \)-dimensional submanifold of \( E \), and so any jet \( j^k_p \phi \in J^k \pi \) is an element of \( J^k(E, m) \); in this way \( J^k \pi \) is an open submanifold of
$J^k(E, m)$. It is not the whole of $J^k(E, m)$, because submanifolds that are not transverse to the fibration do not have jets in $J^k\pi$.

In the special case where $M = \mathbb{R}^m$, each local section of $\pi$ gives rise by translation to a map from a neighbourhood of the origin to $E$, and so the jet $j^k_{0}\phi$ gives rise to an $m$-velocity; it is in fact an $m$-frame because a local section has rank $m$. But only those maps $\mathbb{R}^m \to E$ that are (local) sections of $\pi$ have velocities corresponding to jets in $J^k\pi$, and so in this particular special case $J^k\pi$ becomes a closed submanifold of $\mathcal{F}^k(\pi) E$.

A choice of adapted coordinates $(x^i, y^a)$ on $E$ gives a local identification of $M$ with $\mathbb{R}^m$, and so locally $J^k\pi$ may be considered as a closed submanifold of $\mathcal{F}^k(\pi) E$ (but not in an invariant way: the identification depends on the chart). Put $u^i = x^i$ for $i = 1, \ldots, m$ and $u^{m+a} = y^a$ for $a = 1, \ldots, n - m$; then $J^k\pi$ is the submanifold of $\mathcal{F}^k(\pi) E$ given by $u^i_1 = \delta^i_j$, $u^j_1 = 0$ for $|I| > 1$.

There are a number of objects associated with frame bundles, and corresponding formulæ relating them. We shall need to consider, in particular, the total derivatives and the vertical endomorphisms.

First, the total derivatives. These arise because a prolonged map $j^k\gamma : \mathbb{R}^m \to \mathcal{F}^k(\pi) E$ defines both a point $j^k_0\gamma$ in the $(k + 1)$-th frame bundle $\mathcal{F}^{k+1}(\pi) E$, and a first-order $m$-velocity $j^k_1(j^k\gamma) = j^k_0\gamma \in \mathcal{F}^k(\pi) E$. This relationship gives an embedding $\mathbf{T} : \mathcal{F}^{k+1}(\pi) E \to T(\mathcal{F}(\pi) E) \otimes \mathbb{R}^{mx}$ called the $(k + 1)$-th order total derivative for $m$-frames. The $i$-th component of $\mathbf{T}$ is the contraction $\mathbf{T}_i = \langle \mathbf{T}, e_i \rangle$, and is a vector field along the projection $\mathcal{T}^{k+1}_k : \mathcal{F}^{k+1}(\pi) E \to \mathcal{F}^k(\pi) E$; in coordinates it is

$$\mathbf{T}_i = \sum_{|I| = 0}^k u^{a}_{I+1} \frac{\partial}{\partial u^a_I}.$$ 

We shall write $d_i$ for the action of $\mathbf{T}_i$ on the functions on $\mathcal{F}^k(\pi) E$, giving functions on $\mathcal{F}^{k+1}(\pi) E$, and of course this may be extended to a map of $r$-forms $d_i : \Omega^r\mathcal{F}^k(\pi) E \to \Omega^{r-1}\mathcal{F}^{k+1}(\pi) E$. We also write $i_i : \Omega^r\mathcal{F}^k(\pi) E \to \Omega^{r-1}\mathcal{F}^{k+1}(\pi) E$ for the corresponding contraction.

The ‘vertical endomorphisms’ may be defined in the following way. Suppose that $\chi : \mathbb{R}^m \times \mathbb{R}^m \to E$ is a map, and put $\chi_y(x) = \chi(x, y)$; suppose that each $k$-th order $m$-velocity $j^k\chi_y$ is a $k$-th order $m$-frame, and so an element of $\mathcal{F}^k(\pi) E$. The 1-jet at zero of the map $\mathbb{R}^m \to \mathcal{F}^k(\pi) E$, $y \to j^k\chi_y$, is a 1st-order velocity on $\mathcal{F}^k(\pi) E$ at $j^k\chi_0$, and so is an element of $T\mathcal{F}^k(\pi) E \otimes \mathbb{R}^{mx}$; any such velocity may be represented by a map $\chi$ in this way.

Given $\chi$, define a new map $\overline{\chi} : \mathbb{R}^m \times \mathbb{R} \to E$ by $\overline{\chi}(x, t) = \chi(x, tx)$, and write $\overline{\chi}_t(x) = \overline{\chi}(x, t)$. The map $\mathbf{R} \to \mathcal{F}^{k+1}(\pi) E, t \to j^{k+1}_0\overline{\chi}_t$ is a curve in $\mathcal{F}^{k+1}(\pi) E$ defining a tangent vector
in $\mathcal{F}_{(m)}^{k+1}E$.

Given a 1st-order velocity in $T\mathcal{F}_{(m)}^k E \otimes \mathbb{R}^{m*}$ at $a \in \mathcal{F}_{(m)}^k E$ and a point $b \in \mathcal{F}_{(m)}^{k+1}E$ projecting to $a$, we may represent the 1-velocity by a map $\chi$ as described above, choosing $\chi$ so that $j^{k+1} \chi_0 = b$. We may then construct the corresponding map $\varphi$, noting that the curve $t \mapsto j^{k+1} \varphi_t$ passes through $b$ at zero. We thus obtain a tangent vector at $b$, and it may be shown that this is independent of the choice of the representative map $\chi$. This tangent vector is the vertical lift of the velocity to the point $b$.

We may now map a tangent vector $\xi$ at $b \in \mathcal{F}_{(m)}^{k+1}E$ to another tangent vector at the same point. First we project $\xi$ to $a \in \mathcal{F}_{(m)}^k E$; then we construct the velocity $(0, \ldots, i^k(\xi), \ldots, 0)$ with the projected tangent vector in the $i$-th place; and finally we take the vertical lift to $b$. We denote the resulting tangent vector by $S^i(\xi)$. The operations are all linear, and so $S^i$ is a tensor field of type $(1, 1)$ on $\mathcal{F}_{(m)}^{k+1}E$. These are the vertical endomorphisms of the frame bundles. In coordinates, they are

$$S^i = \sum_{|I| = 0}^k (|I| + 1) \frac{\partial}{\partial u^r_{I+1}} \otimes du^q_I.$$  

3 Homogeneous variational complexes

The variational bicomplex is constructed from spaces of (scalar) differential forms, and the finite-order variational sequences are constructed from spaces of equivalence class of scalar forms. By contrast, the homogeneous variational complexes that we shall describe below are constructed from spaces of vector-valued forms: to be precise, we consider the vector spaces $\Lambda^r \mathbb{R}^{m*}$ of multilinear forms on $\mathbb{R}^m$, and study the differential forms on $\mathcal{F}_{(m)}^k E$ taking their values in these vector spaces. We shall denote the set of such $r$-forms by $\Omega^r_{k} = \Omega^r \mathcal{F}_{(m)}^k E$ is the space of scalar $r$-forms on $\mathcal{F}_{(m)}^k E$. In coordinates, an element of $\Omega^r_{k}$ would be written

$$\Phi = \Phi^{I_1 \ldots I_r} \left( du^{a_1}_{I_1} \wedge \ldots \wedge du^{a_r}_{I_r} \right) \otimes \left( dt^{i_1} \wedge \ldots \wedge dt^{i_s} \right),$$

where $t^i$ are the standard linear coordinate functions on $\mathbb{R}^m$ and we write $dt^i$ rather than $t^i$, using the identification between the constant 1-form $dt^i$ and the element $t^i \in \mathbb{R}^{m*}$ of the dual space.

The wedge products on the two components of these vector-valued forms induce a wedge product on the direct sum $\bigoplus_{r,s} \Omega^r_{k} \otimes \Omega^s_{k}$: for decomposable elements,

$$(\theta \otimes w) \wedge (\theta' \otimes w') = (\theta \wedge \theta') \otimes (w \wedge w')$$

where $\theta \in \Omega^r_{k}$, $\theta' \in \Omega^r_{k}$, $w \in \Lambda^s \mathbb{R}^{m*}$, $w' \in \Lambda^s \mathbb{R}^{m*}$. The fact that, for $\lambda \in \mathbb{R}$, $\lambda \neq 0$,

$$(\lambda \theta) \otimes (\lambda^{-1} w) = \theta \otimes w$$
causes no ambiguity in this definition.

The exterior derivative on the first component induces a derivation $d : \Omega^r,s_k \rightarrow \Omega^{r+1,s}_k$ for decomposable elements

$$d(\theta \otimes w) = d\theta \otimes w,$$

and this is extended by linearity. Again there is no ambiguity in this definition.

Finally, the vertical endomorphisms $S^i$ may be combined in this context to produce a map $S : \Omega^r,s_k \rightarrow \Omega^{r,s-1}_k$ by writing

$$S(\theta \otimes w) = S^i\theta \otimes w_i,$$

where $w_i$ denotes the contraction of $\partial/\partial t^i$ with $w$.

We now introduce two new operations on these vector-valued forms, both derived from the total derivative operators on scalar forms. These are the maps $i_T : \Omega^r,s_k \rightarrow \Omega^{r-1,s+1}_{k+1}$ and $d_T : \Omega^{r,s}_k \rightarrow \Omega^{r,s+1}_{k+1}$, defined for decomposable elements by

$$i_T(\theta \otimes w) = i_i\theta \otimes (dt^i \wedge w)$$
$$d_T(\theta \otimes w) = d_j\theta \otimes (dt^i \wedge w)$$

and extended by linearity. As the $i_i$ and $d_j$ are derivations, it follows that $i_T$ and $d_T$ are derivations. Note that, in general, both $i_T$ and $d_T$ increase the order of a vector-valued form by one. We call $d_T$ the total exterior derivative.

We can now build the homogenous variational complexes from the spaces of vector-valued forms and the maps $d_T$. It is immediate that $d_T^2 = 0$, because

$$d_T^2(\theta \otimes w) = d_i d_j \theta \otimes (dt^i \wedge dt^j \wedge w) = 0$$

as $d_i d_j$ is symmetric in $i, j$ whereas $dt^i \wedge dt^j$ is skew-symmetric. Thus for each $k$ and each $r$ satisfying $1 \leq r \leq \dim \mathcal{F}^k_{(m)} E$ we have a complex

$$0 \rightarrow \Omega^{r,0}_k \rightarrow \Omega^{r,1}_{k+1} \rightarrow \ldots \Omega^{r,s}_{k+s} \rightarrow \Omega^{r,s+1}_{k+s+1} \rightarrow \ldots \rightarrow \Omega^{r,m}_{k+m} \rightarrow \Omega^{r,m+1}_{k+m}/d_T(\Omega^{r,m-1}_{k+m-1}) \rightarrow 0,$$

and for each $k$ when $r = 0$ a complex

$$0 \rightarrow \mathbb{R} \rightarrow \Omega^{0,0}_k \rightarrow \Omega^{0,1}_{k+1} \rightarrow \ldots \Omega^{0,s}_{k+s} \rightarrow \Omega^{0,s+1}_{k+s+1} \rightarrow \ldots \rightarrow \Omega^{0,m}_{k+m} \rightarrow \Omega^{0,m+1}_{k+m}/d_T(\Omega^{0,m-1}_{k+m-1}) \rightarrow 0.$$

We shall, in fact, replace the latter complex by

$$0 \rightarrow \Omega^{0,0}_k \rightarrow \Omega^{0,1}_{k+1} \rightarrow \ldots \Omega^{0,s}_{k+s} \rightarrow \Omega^{0,s+1}_{k+s+1} \rightarrow \ldots \rightarrow \Omega^{0,m}_{k+m} \rightarrow \Omega^{0,m+1}_{k+m}/d_T(\Omega^{0,m-1}_{k+m-1}) \rightarrow 0$$
where $\Omega_{k+s}^{0,s} = \Omega_{k+s}^{0,s} \wedge R^m$ taking quotients by the constant vector-valued functions. The reason for this will be given when we consider exactness.

There are also some shorter complexes arising because we are considering finite-order $m$-frame bundles. We have

$$0 \to \Omega_0^{r,s} \to \Omega_1^{r,s+1} \to \ldots \to \Omega_{m-s}^{r,m} \to d_T(\Omega_{m-s-1}^{r,m-1}) \to 0$$

for $r \geq 1$ and

$$0 \to \Omega_0^{0,s} \to \Omega_1^{0,s+1} \to \ldots \to \Omega_{m-s}^{0,m} \to d_T(\Omega_{m-s-1}^{0,m-1}) \to 0$$

arising when we consider zeroth-order forms taking their values in $\wedge^s R^m$ for $s \geq 1$, and

$$0 \to \Omega_k^{r,s} \to \Omega_{k+1}^{r,s+1} \to \ldots \to \Omega_{k+m-s}^{r,m} \to d_T(\Omega_{k+m-s-1}^{r,m-1}) \to 0$$

arising when $r > \dim F_{k-1} (m)$ (so that $\Omega_{k-1}^{r,s-1} = 0$). We shall call each of the complexes described above a homogeneous variational complex.

### 4 Exactness

The main result of this paper concerns the exactness of the homogeneous variational complexes described above. A first remark is that it is, in general, not possible for all of these to be exact in the usual sense, even locally. To see why, let $E = \mathbb{R}^n$ and let $u^\alpha$ be the standard (global) coordinate functions on $E$. Let $\Phi$ be the non-zero element of $\Omega_1^{2,2}$ given in these coordinates by

$$\Phi = \delta_{\alpha\beta} d\xi^\alpha_i \wedge d\xi^\beta_j \otimes dt^i \wedge dt^j,$$

so that

$$d_T \Phi = \delta_{\alpha\beta} \left( d\xi^\alpha_i \wedge d\xi^\beta_j + d\xi^\alpha_i \wedge d\xi^\beta_j \right) \otimes dt^k \wedge dt^i \wedge dt^j = 0.$$

But for any $\Psi \in \Omega_0^{2,1}$ we must have

$$\Psi = \Psi_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta \otimes dt^j$$

with $\Psi_{\alpha\beta} + \Psi_{\beta\alpha} = 0$, so that

$$d_T \Psi = \left( (d_i \Psi_{\alpha\beta}) d\xi^\alpha \wedge d\xi^\beta + 2\Psi_{\alpha\beta} d\xi^\alpha_i \wedge d\xi^\beta \right) \otimes dt^i \wedge dt^j \neq \Phi.$$

It is clear that this apparent setback arises because total derivatives increase the order of forms; indeed, exactly the same phenomenon arises in the affine case, and this is why the variational bicomplex is usually defined on the infinite jet bundle. We could adopt a similar procedure here, but again we would lose all information about the order of the form. Instead, we shall look at particular finite pullbacks, and we shall see that, modulo
these pull-backs, the homogeneous variational complexes are globally exact. Indeed, the result is even stronger than this: we are able to construct a canonical 'pseudo-homotopy operator', so that if \( d_T \Phi = 0 \) then we can make a canonical choice of \( \Psi \) such that 
\[
d_T \Psi = (T^{(r+1)(k+1)}_{(m)})^* \Phi.
\]

**Definition** If \( r \geq 1 \), the operator \( P : \Omega^{r,s}_k \rightarrow \Omega^{r,s+1}_{(r+1)k-1} \) is defined by

\[
P(\Phi) = P^{i}(s) (\phi_{i_1...i_s}) \otimes \left\{ \frac{\partial}{\partial t^j} J \left( dt^{i_1} \wedge ... \wedge dt^{i_s} \right) \right\} = s \ P^{i}(s) (\phi_{ij_2...i_s}) \otimes \left( dt^{i_2} \wedge ... \wedge dt^{i_s} \right)
\]

where \( \Phi = \phi_{i_1...i_s} \otimes (dt^{i_1} \wedge ... \wedge dt^{i_s}) \), the \( \phi_{i_1...i_s} \) are scalar \( r \)-forms, completely skew-symmetric in the indices \( i_1, ..., i_s \), and \( P^{i}(s) \) is the differential operator on scalar \( r \)-forms defined by

\[
P^{i}(s) = \sum_{|J| = 0}^{r-1} \frac{(-1)^{|J|} (m-s)! |J|!}{r^{||J||+1} (m-s+|J|+1)! |J|!} d_J S^{j+1} \).
\]

Note that the scalar forms \( \phi_{i_1...i_s} \) are globally well-defined. Indeed, \( \Phi \) takes its values in the vector space \( \bigwedge^s \mathbb{R}^m^* \), and \( \phi_{i_1...i_s} \) is obtained by contracting the image of \( \Phi \) with the \( s \)-vector \( e_{i_1} \wedge ... \wedge e_{i_s} \) where \( \{e_i\} \) is the standard basis of \( \mathbb{R}^m \).

**Theorem** The operator \( P \) is a pseudo-homotopy operator for \( d_T \) for the case \( r \geq 1 \), in that

\[
d_T P + P d_T = id
\]

modulo pullbacks; thus the homogeneous variational complexes are globally exact, again modulo pullbacks. This formula is also valid for the case \( s = 0 \) where, by default, \( P^{i}(0) = 0 \), and for the case \( s > 0, k = 0 \) where \( P^{i}(s) = 0 \) explicitly.

A remark is needed about the spaces on which the homotopy formula is defined. In principle, if \( \Phi \in \Omega^{r,s}_k \) then \( d_T P \Phi \in \Omega^{r,s}_{(r+1)k} \) and \( P d_T \Phi \in \Omega^{r,s}_{(r+1)(k+1)-1} \), so that the formula would be an equation in \( \Omega^{r,s}_{(r+1)(k+1)-1} \); to be precise it would be

\[
(T^{(r+1)(k+1)-1}_m)^* d_T P + P d_T = (T^{(r+1)(k+1)-1}_m)^* \).
\]

But in fact it turns out that \( P d_T \Phi \) is always projectable to \( \Omega^{r,s}_{(r+1)k} \), and indeed this is essential for the success of the proof; in the particular case of interest, where \( d_T \Phi = 0 \), both sides of the formula are projectable further to \( \Omega^{r,s}_k \). Note however that, for the case \( r > \dim X^{k-1}_m E \) where the complex starts \( \Omega^{r,s}_k \rightarrow \Omega^{r,s+1}_{k+1} \rightarrow ... \), the first map is not injective, and indeed the codomain of \( P : \Omega^{r,s}_k \rightarrow \Omega^{r,s-1}_{2k-1} \) is not trivial.

We also mention that, for certain classes of forms, \( P \Psi \) may be projectable to a lower-order frame bundle; this turns out to be important for variational problems (see [6]).
In the proofs below we shall normally omit the pullback maps from the formulæ where there is no chance of confusion.

The proof of the theorem is based upon the relationship between the total derivative operators $d_j$ and the vertical endomorphisms $S^i$. The fundamental lemma is this one.

**Lemma 1** If $\theta$ is a scalar 1-form on $\mathcal{F}_{(m)}^k E$ then

$$S^i d_j \theta - d_j S^i \theta = \delta^i_j (\tau^{k+1,k}_{(m)})* \theta .$$

**Proof** This result is mentioned in [2]; we include a proof for completeness. We use local coordinates. Let

$$\theta = \sum_{|J|=0}^k \theta^J_{\alpha} du^\alpha_J ,$$

and note that we may rewrite $S^i$ as

$$S^i = \sum_{|I|=0}^{k-1} (I(i) + 1)du^\alpha_I \otimes \frac{\partial}{\partial u^\alpha_{I+1}} = \sum_{|J|=1}^k J(i)du^\alpha_{J-1_i} \otimes \frac{\partial}{\partial u^\alpha_J}$$

(the multi-index $J - 1_i$ does not make sense when $J(i) = 0$, but then the coefficient of the term vanishes so that the formula is still valid.) Thus

$$d_j \theta = \sum_{|J|=0}^k (d_j \theta^J_{\alpha} du^\alpha_J + \theta^J_{\alpha} du^\alpha_{J+1})$$

so that

$$S^i d_j \theta = \sum_{|J|=0}^k (J(i)(d_j \theta^J_{\alpha} du^\alpha_J + \theta^J_{\alpha} du^\alpha_{J+1}) + (J + 1)(i)) \theta^J_{\alpha} du^\alpha_{J-1_i} ;$$

also

$$S^i \theta = \sum_{|J|=0}^k J(i) \theta^J_{\alpha} du^\alpha_{J-1_i}$$

so that

$$d_j S^i \theta = \sum_{|J|=0}^k J(i) \left( (d_j \theta^J_{\alpha} du^\alpha_{J-1_i} + \theta^J_{\alpha} du^\alpha_{J-1_i+1}) \right) .$$

Subtracting,

$$S^i d_j \theta - d_j S^i \theta = \sum_{|J|=0}^k [(J + 1_j)(i) - J(i)] \theta^J_{\alpha} du^\alpha_{J+1_j-1_i} = \sum_{|J|=0}^k \delta^J_{i} \theta^J_{\alpha} du^\alpha_{J+1_j-1_i} = \delta^i_j \theta .$$
Lemma 2 If $\phi$ is a scalar $r$-form then

$$S^i d_j \phi - d_j S^i \phi = r \delta^i_j \phi.$$  

Proof By induction on $r$, using the fact that the commutator of two derivations is again a derivation. 

We shall now assume that all scalar forms are $r$-forms, and write

$$S^i d_j - d_j S^i = r \delta^i_j$$  

without further comment.

Lemma 3 For any natural number $p$,

$$(d_{i_1} \ldots d_{i_p} S^{i_1} \ldots S^{i_p}) d_j = d_j (d_{i_1} \ldots d_{i_p} S^{i_1} \ldots S^{i_p}) + rp d_j (d_{i_1} \ldots d_{i_{p-1}} S^{i_1} \ldots S^{i_{p-1}}).$$

Proof By induction on $p$:

$$(d_{i_1} \ldots d_{i_p} S^{i_1} \ldots S^{i_p}) d_j = (d_{i_1} \ldots d_{i_p} S^{i_1} \ldots S^{i_{p-1}}) (d_j S^{i_p} + r \delta^{i_p}_j)

= [d_j (d_{i_1} \ldots d_{i_p} S^{i_1} \ldots S^{i_{p-1}})

+ r(p-1)d_j d_{i_p} (d_{i_1} \ldots d_{i_{p-2}} S^{i_1} \ldots S^{i_{p-2}}) S^{i_p}]

+ r d_j (d_{i_1} \ldots d_{i_{p-1}} S^{i_1} \ldots S^{i_{p-1}})

= d_j (d_{i_1} \ldots d_{i_p} S^{i_1} \ldots S^{i_p})

+ rp d_j (d_{i_1} \ldots d_{i_{p-1}} S^{i_1} \ldots S^{i_{p-1}}),$$

using Lemma 2, the induction hypothesis, and the commutativity of total derivatives.

We shall need to rewrite Lemma 3 using multi-index notation.

Lemma 4 For any natural number $p$,

$$\sum_{|J|=p} \frac{|I|!}{I!} d_I S^I d_j = d_j \left( \sum_{|I|=p} \frac{|I|!}{I!} d_I S^I + rp \sum_{|J|=p-1} \frac{|J|!}{J!} d_J S^J \right).$$

Proof For any multi-index $I$ with

$$I = i_1 + \ldots + i_p,$$

the number of distinct rearrangements of the indices $i_1, \ldots, i_p$ (giving the same multi-index $I$) is $|I|!/I!$; this is called the weight of $I$.
We now give a proof of the main theorem, using Lemma 4. As before, we take \( \Phi \in \Omega^{r,s}_k \) with \( \Phi = \phi_{i_1 \ldots i_s} \otimes (dt^1 \wedge \ldots \wedge dt^s) \); then

\[
d_T P(\Phi) = d_T \left\{ s \ P^j_{(s)}(\phi_{j_2 \ldots i_s}) \otimes (dt^{i_2} \wedge \ldots \wedge dt^{i_s}) \right\} \\
= s \left( d_{i_1} P^j_{(s)}(\phi_{j_2 \ldots i_s}) \right) \otimes (dt^{i_1} \wedge \ldots \wedge dt^{i_s}) ,
\]

whereas

\[
P(d_T \Phi) = P \left\{ (d_j \phi_{i_1 i_2 \ldots i_s}) \otimes (dt^j \wedge dt^{i_1} \wedge \ldots \wedge dt^{i_s}) \right\} \\
= P^q_{(s+1)}(d_j \phi_{i_1 i_2 \ldots i_s}) \otimes \left\{ \frac{\partial}{\partial t^j} J \ (dt^j \wedge dt^{i_1} \wedge \ldots \wedge dt^{i_s}) \right\} \\
= P^q_{(s+1)}(d_j \phi_{i_1 i_2 \ldots i_s}) \otimes (dt^j \wedge dt^{i_1} \wedge \ldots \wedge dt^{i_s}) \\
- s P^q_{(s+1)}(d_j \phi_{i_1 i_2 \ldots i_s}) \otimes (dt^j \wedge dt^{i_2} \wedge \ldots \wedge dt^{i_s}) \\
= \left( P^q_{(s+1)}(d_j \phi_{i_1 i_2 \ldots i_s}) \right) - s P^q_{(s+1)}(d_{i_1} \phi_{j_2 \ldots i_s}) \otimes (dt^{i_1} \wedge \ldots \wedge dt^{i_s}) ,
\]

using the definitions of \( d_T \) and \( P \). So the task is to show that, as scalar \( r \)-forms,

\[
s d_{i_1} P^j_{(s)}(\phi_{j_2 \ldots i_s}) + P^q_{(s+1)}(d_j \phi_{i_1 i_2 \ldots i_s}) - s P^q_{(s+1)}(d_{i_1} \phi_{j_2 \ldots i_s}) = \phi_{i_1 i_2 \ldots i_s} ,
\]

and we carry out this task by examining the operator

\[
s d_{i_1} P^j_{(s)} + \delta^j_{i_1} P^q_{(s+1)} d_q - s P^j_{(s+1)} d_{i_1}
\]

acting on \( \phi_{j_2 \ldots i_s} \). We expand each of the three terms, using Lemma 2 and Lemma 4 to ensure that all the total derivatives \( d_j \) are moved to the left of all the vertical endomorphisms \( S^i \).

First,

\[
s d_{i_1} P^j_{(s)} = s \sum_{|J|=0}^{r-1} \frac{(-1)^{|J|} (m-s)! |J|!}{r^{|J|+1} (m-s+|J|+1)! J!} d_{J+1} S^{J+1} ;
\]

next,

\[
\delta^j_{i_1} P^q_{(s+1)} d_q = \sum_{|J|=0}^{r(k+1)-1} \frac{(-1)^{|J|} (m-s-1)! |J|!}{r^{|J|+1} (m-s+|J|)! J!} d_J S^{J+1} d_q
\]

\[
= \sum_{|J|=0}^{r(k+1)-1} \frac{(-1)^{|J|} (m-s-1)! |J|!}{r^{|J|+1} (m-s+|J|)! J!} d_J S^J (d_Q S^q + m r)
\]

\[
= \sum_{|J|=0}^{r(k+1)-1} \frac{(-1)^p (m-s-1)!}{r^{p+1} (m-s+p)!} \left\{ \sum_{|J|=p}^{|J|} \frac{|J|!}{J!} d_{J+1} S^{J+1} + m r \right\}
\]

\[
+ r p \sum_{|J|=p-1}^{|J|} \frac{|J|!}{J!} d_{J+1} S^{J+1} + m r \sum_{|J|=p}^{|J|} \frac{|J|!}{J!} d_J S^J \right\}
\]
and finally, using similar manipulations,

\[-s P_{(s+1)}^j d_{i_1} = -s \sum_{|J|=0}^{r(k+1)-1} \frac{(-1)^{|J|}(m-s-1)!|J|!}{r^{|J|+1}(m-s+|J|)!J!} d_{J+1} S^{J+1} d_{i_1} \]

\[-s \sum_{|J|=0}^{r(k+1)-2} \frac{(-1)^{|J|+1}(m-s-1)!(|J|+1)!}{r^{|J|+1}(m-s+|J|+1)!J!} d_{J+1} S^{J+1} \]

\[+ m \sum_{|J|=0}^{r(k+1)-1} \frac{(-1)^{|J|}(m-s-1)!|J|!}{r^{|J|}(m-s+|J|)!J!} d_{J} S^{J} \]

and altogether there are seven sums to consider:

\[sd_{i_1} P_{(s)}^j + \delta_{i_1} P_{(s+1)}^q d_q - s P_{(s+1)}^j d_{i_1} \]

\[= \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|}(m-s)!|J|!}{r^{|J|+1}(m-s+|J|+1)!J!} d_{J+1} S^{J+1} \]

\[+ \delta_{i_1} \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|}(m-s-1)!|J|!}{r^{|J|+1}(m-s+|J|)!J!} d_{J+1} S^{J+1} \]

\[+ \delta_{i_1} \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|+1}(m-s-1)!(|J|+1)!}{r^{|J|+1}(m-s+|J|+1)!J!} d_{J+1} S^{J+1} \]

\[+ m \delta_{i_1} \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|}(m-s-1)!|J|!}{r^{|J|}(m-s+|J|)!J!} d_{J} S^{J} \]

\[+ \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|}(m-s-1)!|J|!}{r^{|J|+1}(m-s+|J|)!J!} d_{J+1} S^{J+1} \]

\[+ \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|+1}(m-s-1)!(|J|+1)!}{r^{|J|+1}(m-s+|J|+1)!J!} d_{J+1} S^{J+1} \]

\[+ \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|}(m-s-1)!|J|!}{r^{|J|}(m-s+|J|)!J!} d_{J} S^{J} \]

\[+ \sum_{|J|=0}^{rk-1} \frac{(-1)^{|J|+1}(m-s-1)!(|J|+1)!}{r^{|J|+1}(m-s+|J|+1)!J!} d_{J+1} S^{J+1} \]
\[ -s \delta^j_{i_1} \sum_{|J|=0}^{r_k} \frac{(-1)^{|J|(m-s-1)!|J|!}}{r^{|J|(m-s+|J|)!}|J|!} d_J S^J, \tag{7} \]

where we have reduced the maximum summation in sums (2), (3), (5) and (6) to \(|J| = rk - 1\), and in sums (4) and (7) to \(|J| = rk\): this is because the scalar forms \(\phi_{j_2...i_s}\) in the domain of this operator are defined on \(F^k_{(m)}E\), so that \(S^J \phi_{j_2...i_s} = 0\) whenever \(|J| > rk\).

We now show that the seven sums collapse to give \(\delta^j_{i_1}\).

First, we see that sums (1), (5) and (6) cancel when taken together. The range of summation is the same in all three cases, and for any given multi-index \(J\) in that range we may extract a common factor

\[ s \frac{(-1)^{|J|(m-s-1)!|J|!}}{r^{|J|(m-s+|J|)!}|J|!} d_{J+1_{i_1}} S^{J+1_{i_1}} \]

from the terms in the three sums to give

\[(m-s) - (m-s + |J| + 1) + (|J| + 1) = 0.\]

Next, we take the remaining expression and combine sums (2) and (3), and also combine sums (4) and (7), to give

\[ s d_{i_1} P^j_{(s)} + \delta^j_{i_1} P^k_{(s+1)} d_k - s P^j_{(s+1)} d_{i_1} \]

\[ = \delta^j_{i_1} \sum_{|J|=0}^{r_k-1} \frac{(-1)^{|J|(m-s)!|J|!}}{r^{|J|(m-s+|J|+1)!}|J|!} d_{J+1_{i_1}} S^{J+1_{i_1}} \]

\[ + \delta^j_{i_1} \sum_{|J|=0}^{r_k} \frac{(-1)^{|J|(m-s)!|J|!}}{r^{|J|(m-s+|J|)!}|J|!} d_J S^J \]

\[ = \delta^j_{i_1} \sum_{|J|=0}^{r_k-1} \frac{(-1)^{|J|(m-s)!|J|!}}{r^{|J|(m-s+|J|+1)!}|J|!} d_{J+1_{i_1}} S^{J+1_{i_1}} \]

\[ + \delta^j_{i_1} \sum_{|J|=1}^{r_k} \frac{(-1)^{|J|(m-s)!|J|!}}{r^{|J|(m-s+|J|)!}|J|!} d_J S^J \]

\[ + \delta^j_{i_1}, \]

where we have separated out the term \(|J| = 0\) from the second sum to give the required \(\delta^j_{i_1}\). But we can rewrite this modified second sum as
\[ \delta^j_{i_1} \sum_{|J|=1}^{r^k} \frac{(-1)^{|J|} (m-s)!! |J|!}{r^{|J|} (m-s + |J|)! J!!} d_J S^J \]

\[ = \delta^j_{i_1} \sum_{|J|=0}^{r^k-1} \frac{(-1)^{|J|+1} (m-s)!(|J|+1)!}{r^{|J|+1} (m-s + |J|+1)! (|J|+1)!!} d_{J+1_q} S^{J+1_q} \times \frac{J(q) + 1}{|J| + 1} \]

\[ = \delta^j_{i_1} \sum_{|J|=0}^{r^k-1} \frac{(-1)^{|J|+1} (m-s)!|J|!}{r^{|J|+1} (m-s + |J|+1)! |J|!!} d_{J+1_q} S^{J+1_q} \]

where the weight \((J+1_q)!!/(|J|+1)!!\) of the multi-index \(J+1_q\) has been replaced by the weight \(J!/|J|!!\) of the multi-index \(J\) to take account of the change of notation; thus the modified second sum cancels with the first sum, and the proof is complete. \(\blacksquare\)

5 The homogeneous variational bicomplexes

In order to obtain make use of the homogeneous variational complexes, we embed them in bicomplexes where the commuting map is the usual exterior derivative \(d : \Omega^r_{k,s} \rightarrow \Omega^{r+1}_{k,s}\).

In the diagram below, we give an example of such a bicomplex where the scalar forms are of order \(k \geq 0\); it should be understood that one or more upper rows of the diagram will have to be omitted if \(-m < k < 0\). We also write

\[ \Xi^r_{k+m} = \Omega^r_{k+m}/d_T \Omega^{r,m-1}_{k+m-1} \]

\[ \Xi^0_{k+m} = \Omega^0_{k+m}/d_T \Omega^{0,m-1}_{k+m-1} \]

for the final quotient spaces in each column.
\[
\begin{align*}
0 & \\
& \downarrow \\
0 & \longrightarrow \overline{\Omega}_{k}^{0,0} \longrightarrow \Omega_{k}^{1,0} \longrightarrow \Omega_{k}^{2,0} \longrightarrow \cdots \\
& \downarrow d_{T} \downarrow d_{T} \downarrow d_{T} \\
0 & \longrightarrow \overline{\Omega}_{k+1}^{0,1} \longrightarrow \Omega_{k+1}^{1,1} \longrightarrow \Omega_{k+1}^{2,1} \longrightarrow \cdots \\
& \vdots \\
0 & \longrightarrow \overline{\Omega}_{k+m}^{0, m-1} \longrightarrow \Omega_{k+m}^{1, m-1} \longrightarrow \Omega_{k+m}^{2, m-1} \longrightarrow \cdots \\
& \downarrow d_{T} \downarrow d_{T} \downarrow d_{T} \\
0 & \longrightarrow \overline{\Omega}_{k+m}^{0, m} \longrightarrow \Omega_{k+m}^{1, m} \longrightarrow \Omega_{k+m}^{2, m} \longrightarrow \cdots \\
& \downarrow p_{0} \downarrow p_{1} \downarrow p_{2} \\
0 & \longrightarrow \overline{\Xi}_{k+m}^{0} \longrightarrow \overline{\Xi}_{k+m}^{1} \longrightarrow \overline{\Xi}_{k+m}^{2} \longrightarrow \cdots \\
& \downarrow \\
0 & 0 0 0
\end{align*}
\]
On the right-hand side of the diagram, the numbers $N_a$ represent the dimensions of the manifolds $\mathcal{F}_{(m)}^{k+s} E$. Note that $d$ and $d_T$ commute rather than anti-commute.
We have proved that the columns of this diagram (apart from the first) are exact modulo pullbacks, and we know that the rows (apart from the last) are locally exact by the standard property of the exterior derivative: recall that we have replaced the usual zeroth term of the de Rham sequence with its quotient by the constants. Local exactness elsewhere (again, possibly modulo pull-backs) comes from diagram chasing. The following arguments are based on those in [8], although given there in a slightly different context. The assertions hold only locally, so a symbol such as $\Omega^{r,s}$ should be interpreted here as a sheaf of germs of vector-valued forms, rather than as a space of globally-defined forms. (We shall not be specific about the orders of the forms here, and we shall again omit the pull-back maps, as the formulæ for the orders become increasingly complex and obscure the main thrust of the argument.) Some modifications will be needed in the case of zeroth-order forms taking their values in $\bigwedge^s \mathbb{R}^{m*}$ for $s \geq 1$, but the nature of these modifications should be clear. The gradual shortening of the columns on the right-hand side of the bicomplex has no effect on our argument, as we always increase the order when moving upwards.

**Lemma 5** If $1 \leq r \leq N_0 - 1$ and $1 \leq s \leq m - 1$ then

$$\ker \left( d_T d : \Omega^{r,0} \to \Omega^{r+1,1} \right) \subset \text{im} \left( d : \Omega^{r-1,0} \to \Omega^{r,0} \right)$$

and

$$\ker \left( d_T d : \Omega^{r,s} \to \Omega^{r+1,s+1} \right) \subset \text{im} \left( d : \Omega^{r-1,s} \to \Omega^{r,s} \right) + \text{im} \left( d_T : \Omega^{r,s-1} \to \Omega^{r,s} \right).$$

**Proof** Suppose first that

$$\Phi \in \ker \left( d_T d : \Omega^{r,0} \to \Omega^{r+1,1} \right),$$

so that $d_T d \phi = 0$. Then $d \phi = 0$ because $d_T : \Omega^{r,0} \to \Omega^{r-1}$ is injective; thus

$$\Phi \in \text{im} \left( d : \Omega^{r-1,0} \to \Omega^{r,0} \right).$$

The second assertion follows from the first by induction on $s$. Suppose that

$$\Phi \in \ker \left( d_T d : \Omega^{r,s} \to \Omega^{r+1,s+1} \right),$$

so that

$$d \Phi \in \ker \left( d_T : \Omega^{r+1,s} \to \Omega^{r+1,s+1} \right).$$

Put $\Phi_0 = P d \Phi \in \Omega^{r+1,s-1}$, so that $d_T \Phi_0 = d \Phi$. Then

$$d_T d \Phi_0 = dd_T \Phi_0 = d^2 \Phi = 0,$$

so that

$$\Phi_0 \in \ker \left( d_T d : \Omega^{r+1,s-1} \to \Omega^{r+2,s} \right).$$
Thus by the induction hypothesis we may write
\[ \Phi_0 = d\Phi_1 + d_T\Phi_2 \]
where
\[ \Phi_1 \in \Omega^{r,s-1}, \quad \Phi_2 \in \Omega^{r+1,s-2}. \]
Now consider \( \Phi - d_T\Phi_1 \). We have
\[ d(\Phi - d_T\Phi_1) = d_T\Phi_0 - d_T(\Phi_0 - d_T\Phi_2) = 0, \]
so that
\[ \Phi - d_T\Phi_1 = d\Psi \]
for some \( \Psi \in \Omega^{r-1,s} \) and the result follows. \( \blacksquare \)

**Lemma 6** If \( 1 \leq s \leq m - 1 \) then
\[ \ker(d_T \delta^0_r : \Omega^0_r \to \Omega^{1,s+1}) = 0 \]
and
\[ \ker(d_T \delta^0_r : \Omega^0_0 \to \Omega^{1,s+1}) \subset \im(d_T : \Omega^{0,s-1} \to \Omega^{0,s}). \]

**Proof** Take the results of Lemma 5 with \( r = 1 \) and apply the same method of proof once again. \( \blacksquare \)

**Corollary** The left-hand column of each homogeneous variational bicomplex is locally exact. \( \blacksquare \)

It should now be clear why we have chosen to use the spaces \( \Omega^{0,s}_{k+s} \) rather than \( \Omega^{0,s}_k; \) the constant vector-valued functions are certainly \( d_T \)-closed, but are not locally \( d_T \)-exact, even to within pullback. This is rather different from the behaviour of the left-hand column of the variational bicomplex on jet bundles, which is locally \( d_h \)-exact on forms rather than classes of forms.

**Lemma 7** If \( 1 \leq r \leq N_{m-1} \) then
\[ \ker(\delta p_r : \Omega^{r,m} \to \Xi^{r+1}) \subset \im(d : \Omega^{r-1,m} \to \Omega^{r,m}) + \im(d_T : \Omega^{r,m-1} \to \Omega^{r,m}) \]
and in addition
\[ \ker(\delta p_0 : \Omega^{0,m} \to \Xi^{1}) \subset \im(d_T : \Omega^{0,m-1} \to \Xi^{1,m}). \]

**Proof** Take the results of Lemma 5 with \( s = m \) and apply the same method of proof once again. \( \blacksquare \)

**Corollary** The bottom row of each homogeneous variational bicomplex is locally exact. \( \blacksquare \)
6 Homogeneous Lagrangians and the edge sequence

The edge sequence of a homogeneous variational bicomplex, as given above, is

\[
0 \to \Omega^{0,0}_k \to \Omega^{1,1}_{k+1} \to \ldots \to \Omega^{s,s}_{k+s} \to \Omega^{s+1,s+1}_{k+s+1} \to \ldots \\
\to \Omega^{0,m}_{k+m} \to \Omega^{1}_{k+m} \to \ldots \to \Omega^{N,m-1}_{k+m} \to \Omega^{N,m-1+1,m}_{k+m} \to \ldots \to \Omega^{N,m,m}_{k+m} \to 0,
\]

and we have seen that this is locally exact, modulo pull-backs. This sequence is particularly significant for homogeneous variational problems. In this final section, it is convenient to relabel the order of the various spaces so that we have \(\Omega^{0,m}_k\) rather than \(\Omega^{0,m}_k\); if \(k < 0\) we shall, of course, need to omit the first \(-k\) non-zero terms of the edge sequence.

A homogeneous \(k\)-th order Lagrangian may be considered as a function \(L\) on an \(m\)-frame bundle \(F^k_{(m)}E\) satisfying the conditions

\[
\Delta^j_i(L) = \delta^j_i L, \quad \Delta^I_j(L) = 0 \quad \text{for} \quad \vert I \vert > 1.
\]

where \(\Delta^I_j = S^I(T_j)\) (note that, although \(T_j\) is a vector field along, in this case, the projection \(\tau_{(m)}^{k,k-1}\), the contraction is unambiguous). The Hilbert forms \(\vartheta^i\) of the Lagrangian are defined by

\[
\vartheta^i = P^i_1 dL,
\]

and the Euler-Lagrange form \(\varepsilon\) by

\[
\varepsilon = dL - d_i \vartheta^i
\]

so that, in coordinates,

\[
\varepsilon = \sum_{\vert I \vert = 0}^{k} (-1)^{\vert I \vert} d_I \left( \frac{\partial L}{\partial u^\alpha_I} \right)
\]

(see, for example, [3] for further details of these constructions).

Now the function \(L\) may be considered as the vector valued function \(\Lambda = L d^m t\), where \(d^m t = dt^1 \wedge \ldots \wedge dt^m\) is the canonical volume element on \(\mathbb{R}^m\), and hence we may determine whether an element \(\Lambda \in \Omega^{0,m}_k E\) is homogeneous. An element of the quotient space \(\Omega^{0,m}_k\) will be said to be homogeneous if it has a homogeneous representative; such a representative must be unique as the only homogeneous constant function is zero. The individual Hilbert forms may be combined into a single vector-valued form

\[
\Theta_1 = \vartheta^i \otimes \left( \frac{\partial}{\partial t_i} \lrcorner d^m t \right) = P dL,
\]

and then the formula for the vector-valued version of the Euler-Lagrange form becomes

\[
\varepsilon \otimes d^m t = d\Lambda - d_T \Theta_1 \in \Omega^{1,m}_{2k}.
\]
Indeed, the pseudo-homotopy formula for $d_T$ tells us that the map $\Xi^r_k \to \Omega^{r,m}_k$ given by

$$p_r(\Phi) \mapsto \Phi - d_T P\Phi$$

is globally well-defined for $r \geq 1$, and gives a canonical global representative for each class in $\Xi^r_k$; the Euler-Lagrange form $\varepsilon \otimes d^mt$ is the canonical representative of the class $p_1(dL)$. In a similar way, if $\Phi \in \Omega^{1,m}_{2k}$ then the map $\mathcal{H} : \Omega^{1,m}_{2k} \to \Omega^{2,m}_{4k}$ given by

$$\mathcal{H}(\Phi) = d\Phi - d_T P\Phi$$

is called the Helmholtz-Sonin map, and if $\mathcal{H}(\Phi) = 0$ then local exactness of the bicomplex shows that $\Phi$ must locally be an Euler-Lagrange form.

There are also questions that may be answered by studying parts of the bicomplex away from the edge sequence. One such question involves the existence of a scalar $m$-form $\Theta_m$ corresponding to a Lagrangian $\Lambda$, having the property that $\Theta_m$ is closed precisely when $\Lambda$ is a null Lagrangian (that is, when $\varepsilon \otimes d^mt = 0$). For a first-order Lagrangian, such an $m$-form is given by

$$\Theta_m = S^1 dS^2 d \ldots S^m dL$$

(see [4]); using the language of vector-valued forms described above, we may write this as

$$\Theta_m = (Pd)^m \Lambda \in \Omega^{m,0}_1$$

where the formula $(Pd)^m \Lambda$ gives in principle a form of higher order, but the result is projectable to the first-order frame bundle. One might ask whether a similar result is true for $k$-th order Lagrangians with $k > 1$. Some preliminary work [6] shows that this is indeed the case for second-order Lagrangians in two independent variables, where $\Theta_2$ is projectable to the fourth-order frame bundle, and investigation of the more general problem continues.

In summary, therefore, we conclude that the homogeneous variational bicomplexes may be used to analyse homogeneous variational problems on $m$-frame bundles in the same way that the variational bicomplex and finite-order variational sequence are used for variational problems on jet bundles.

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