An Algorithmic Approach to Pick’s Theorem

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July 3, 2014

Abstract

We give an algorithmic proof of Pick’s theorem which calculates the area of a lattice-polygon in terms of the lattice-points.

Introduction

Pick’s Theorem was first discovered by Georg A. Pick in 1899 [Pic99]. Many different proofs for this elegant theorem have been published over the last 60 years. Some found a topological connection with Euler’s formula, and others, like Pick himself, proved it by geometrical means. Most of the geometric proofs prove the additivity of Pick’s formula and find a specific example for which this formula gives the area. Both Liu and Varberg [Liu79, Var85] mentioned that the most challenging part of some proofs is the fact that a primitive lattice-triangle is of area $\frac{1}{2}$. Varberg, for example, bypasses that fact in his proof. Here we do use this fact and find an explicit algorithm to find all lattice points for a lattice-polygon $P$.

Theorem 1. (Pick, [Pic99]). Let $P$ be a lattice-polygon. Then its area is $i + \frac{u}{2} - 1$, when $i$ is the number of interior lattice-points of $P$ and $u$ is the number of its boundary lattice-points.

Lemma 2. The minimal possible area of a triangle whose vertices are all lattice-points is $\frac{1}{2}$.

Proof. [Liu79] Let $A, B, C$ be lattice-points, and denote $\triangle ABC$ by $T$. We can bind $T$ with a rectangle parallel to the axes. In order to calculate the area of $T$, subtract the area outside it from the rectangle. That area consists of several right triangles and may also include a rectangle. The area of each right triangle is half the product of its legs, which are natural numbers, and thus is a multiple of $\frac{1}{2}$. The area of both the big and the small rectangles are natural numbers. Therefore, the area of $T$ is a multiple of $\frac{1}{2}$. We have found that the minimum positive area of $T$ is $\frac{1}{2}$.

Alternative Proof. [NZ67, Hon70, GKW76, MT07] Let $A, B, C$ be lattice-points. The area of $\triangle ABC$ is $\frac{1}{2} \cdot \begin{vmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \\ 1 & 1 & 1 \end{vmatrix}$, which is an integer multiple of $\frac{1}{2}$. \hfill $\Box$
Theorem 3. Let $A, B, C$ be lattice-points. The triangle $\triangle ABC$ is of minimal area, i.e., $\frac{1}{2}$, iff $\triangle ABC \cap \mathbb{Z}^2 = \{A, B, C\}$, i.e., iff there are no lattice-points on the edges of $\triangle ABC$, nor in the interior of $\triangle ABC$.

Proof. ($\Rightarrow$). We prove the contrapositive. Assume that there is another lattice-point $D$ in that intersection. If $D$ is an interior point, then we can decompose the triangle into three triangles: $\triangle ABD$, $\triangle ACD$ and $\triangle BCD$. If $D$ is on an edge of $\triangle ABC$, then we can decompose it into two triangles by drawing a line between $D$ and the opposite vertex. In any case, $\triangle ABC$ contains several disjoint sub-triangles, and thus its area is at least twice as much as the minimum. We conclude that if $\triangle ABC$ has such a point $D$, then its area is not minimal.

($\Leftarrow$). Denote $\triangle ABC$ by $T$. Move the point $C$ to the origin $(0,0)$, and denote the other points as $A = (a,c)$ and $B = (b,d)$. The area of $T$ is $\frac{ad-bc}{2}$. Since $T$ is a triangle, its area is non-zero. W.l.o.g. we assume that $ad-bc > 0$ and $c \leq d$. Denoting $n = ad-bc$, we want to prove that if $n > 1$ then there is another lattice-point $D$ in $T$. If $\gcd(a-b, c-d) = k > 1$, then the point $\frac{k}{n}A + \frac{k}{n}B$ is a new lattice-point on the edge $AB$, as desired. Thus we assume that $\gcd(a-b, c-d) = 1$. We prove that such a lattice-point exists on the segment $\frac{n}{n}AB$.

The equation of this segment is $(a-b)y-(c-d)x = n-1$. Since $a-b$ and $c-d$ are coprime, there exist $s$ and $t$ such that $(a-b)s-(c-d)t = 1$. We multiply this equation by $n-1$ and get $(a-b)(n-1)s-(c-d)(n-1)t = n-1$. Thus, we take $x = (n-1)t$ and $y = (n-1)s$, to find a lattice-point on that line. However, we need to find a lattice point not only on that line but on the segment $\frac{n}{n}AB$. Thus, we replace $x$ by $x + (a-b)i$ and $y$ by $y + (c-d)i$, to get a new lattice-point on the line. For all $r \in \mathbb{R}$, we can choose an appropriate $i$ such that $r \leq y < r - (c-d)$. We choose the appropriate $i$ for $r = \frac{n-1}{n}c$, i.e., $c - \frac{r}{n} \leq y < d - \frac{r}{n}$. Denote this point by $D = (x,y)$. We claim that this $D$ is in the segment, i.e., $\frac{n-1}{n}c \leq y \leq \frac{n-1}{n}d$, and consequently, $\frac{n-1}{n}a \leq x \leq \frac{n-1}{n}b$ as well (or $\frac{n-1}{n}a \geq x \geq \frac{n-1}{n}b$, if $a \geq b$). We need to demonstrate that $D$ does not fall past $\frac{n-1}{n}B$, i.e., that $y \notin \left(d - \frac{r}{n}, d - \frac{r}{n}\right)$. If $D$ were past $B$, then $nD$ would be a lattice-point on the line $(n-1)AB$, past the lattice-point $(n-1)B$. But the $y$-difference between these two points would be $ny - (n-1)d$. In accordance with $d - \frac{r}{n} < y < d - \frac{r}{n}$, we find that $0 < ny - (n-1)d < d - c$. This $y$-difference between lattice-points on a line with slope $\frac{c-d}{a-b}$ contradicts the fact that $a-b$ and $c-d$ are coprime. In conclusion, $D$ is in the segment $\frac{n}{n}AB$.

We have found another lattice-point $D$ in the triangle $T$ of area greater than $\frac{1}{2}$.

The above theorem is of course equivalent to [HW79, Theorem 34], which overlooked parallelograms instead of triangles, although they neglected the case in which there are two (or more) points on the diagonal $PQ$. Moreover, we want to mention this connection as evidence to the deep connection between Pick’s theorem and Farey series. We use some concepts that appear there in §3.4-3.7.
Corollary 4. For a lattice-triangle \( \triangle ABC \), with \( A = (a, c) \), \( B = (b, d) \) and \( C = (0, 0) \), if \( a - b \) and \( c - d \) are coprime, then there is one lattice point in

\[
X = \left\{ \frac{n - i}{n} A + \frac{i - 1}{n} B : i = 1, \ldots, n \right\},
\]

for \( n = |ad - bc| \).

Proof. \( X \) is a subset of the segment \( \frac{n-1}{n} AB \), and therefore \( nX \) is a subset of the segment \( (n-1)AB \). We find that \( nX \) is the set of the \( n \) lattice-points on \( (n-1)AB \). Multiplying the point \( D \) from the end of the proof of theorem \( \text{3} \) by \( n \) gives a lattice-point on the segment \( (n-1)AB \). We have shown that \( nD \in nX \), and thus \( D \in X \). \( \square \)

This proof provides an explicit way of finding the lattice-points of a lattice-polygon.

Algorithm 5. (Lattice-triangulation of a lattice-polygon \( P \)) Let \( P \) be a lattice polygon. We want to partition \( P \) into minimal triangles. We make a list of these triangles via the following steps:

1. Partition \( P \) into triangles, by drawing lines between non-adjacent vertices, without crossing any other line.
2. Choose one triangle, \( T \). Choose one vertex of \( T \), which we specify as \( C \).
3. Move \( C \) to the origin, and denote the other vertices as \( A = (a, c) \) and \( B = (b, d) \).
4. If \( a - c \) and \( b - d \) are not coprime, then take \( D = \frac{k-1}{k} A + \frac{1}{k} B \) for \( k = \gcd(a - c, b - d) \). Partition \( T \) into \( T_1 = \triangle ACD \) and \( T_2 = \triangle BCD \). Return both \( T_1 \) and \( T_2 \) to step 2.
5. If \( a - c \) and \( b - d \) are coprime and the area of \( T \) is \( \frac{n}{2} \) with \( n > 1 \), take \( D \) the one and only lattice-point in \( \text{1} \). Partition \( T \) into \( T_1 = \triangle ABD \), \( T_2 = \triangle ACD \) and \( T_3 = \triangle BCD \). Return \( T_1 \), \( T_2 \) and \( T_3 \) to step 2.
6. If the area of \( T \) is \( \frac{1}{2} \), \( T \) is minimal, and we add \( T \) to our list. If there are other triangles with area greater than \( \frac{1}{2} \), return them to step 2.
7. We have obtained a list of minimal triangles. This procedure must terminate, since the number of such triangles is twice the area of \( P \).

We conclude the proof of Pick’s Theorem (Theorem \( \text{1} \)) by proving that Pick’s formula is additive.

Proof. \( \text{[Pic99]} \) Let \( P \) be a lattice-polygon, and denote by \( i \) and \( u \) the number of its interior points and boundary points, respectively. We claim that Pick’s formula, \( i + \frac{u}{2} - 1 \), is additive under triangulation, like the total area. Thus, we can triangulate \( P \) into minimal triangles, and calculate that for a minimal triangle \( 0 + \frac{1}{2} - 1 = \frac{1}{2} \), and conclude the proof of Pick’s Theorem.
If $i \neq 0$, choose an interior point, $D$, and two boundary points $A, B$. Partition the polygon $X$ into two polygons by drawing the lines $AD$ and $BD$. Denote by $u_1$ and $u_2$ the number of boundary points in these two polygons, and by $i_1$ and $i_2$ the number of their respective interior points. Denote by $d$ the total number of lattice points on the segments $AD$ and $BD$ (count $A, B$ and $D$ only once!). Clearly, $i = i_1 + i_2 + d - 2$, since $d$ counts the points $A$ and $B$, which are not interior points. Furthermore, $u = u_1 + u_2 - 2d + 2$. We subtracted here the points on the segments from both polygons, but added the points $A$ and $B$.

Thus,

$$i + \frac{u}{2} - 1 = i_1 + i_2 + d - 2 + \frac{u_1 + u_2 - 2d + 2}{2} - 1$$

$$= \left(i_1 + \frac{u_1}{2} - 1\right) + \left(i_2 + \frac{u_2}{2} - 1\right).\quad (2)$$

We conclude that $i + \frac{u}{2} - 1$ is preserved when partitioning $P$ with respect to an interior point $D$.

If $i = 0$, but $u > 3$, choose two boundary points, $A$ and $B$, and take $D$ to be the same as $A$. Equation (2) is true in this case as well.

If $i = 0, u = 3$, we can no longer partition that minimal triangle, but by Theorem 3 we find that the area of this polygon is $\frac{1}{2} = 0 + \frac{3}{2} - 1 = i_T + \frac{u_T}{2} - 1$.

In conclusion, by decomposing $P$ by any point, interior or boundary, the total area is the sum of the area of the two parts, and Pick’s formula for $P$ is the sum of Pick’s formulas for them. Hence, we find that both these quantities are additive. If these two quantities coincide on minimal triangles, then by induction they coincide on any lattice-polygon. Indeed, by Theorem 3 the area of each minimal triangle $T$ is $\frac{1}{2} = 0 + \frac{3}{2} - 1 = i_T + \frac{u_T}{2} - 1$. From additivity of both this formula and the concept of area, $i + \frac{u}{2} - 1$ is the area of $P$. \hfill \Box

Acknowledgement. The author would like to thank Louis Rowen for his help, patience and guidance.

References

[Bla97] C. Blatter, Another Proof of Pick’s Area Theorem. Math. Mag. 70, p. 200 (1997).

[Bog] A. Bogomolny, A proof of Pick’s theorem, "Cut-the-Knot" website, http://www.cut-the-knot.org/ctk/Pick_proof.shtml.

[Cox69] H. S. M. Coxeter, Introduction to Geometry, 2nd ed., John Wiley & Sons, New York, 1969. pp. 208-210.

[DR74] D. DeTemple and J. M. Robertson, The Equivalence of Euler’s and Pick’s Theorems. Math. Teacher 67, pp. 222-226 (1974).

[DR95] R. Diaz and S. Robins, Pick’s Formula via the Weierstrass $\wp$-Function. Amer. Math. Monthly 102, pp. 431-437 (1995).
[Fun74] W. W. Funkenbusch, From Euler’s Formula to Pick’s Formula Using an Edge Theorem. *Amer. Math. Monthly* **81**, pp. 647-648 (1974).

[GKW76] R. W. Gaskell, M. S. Klamkin and P. Watson, Triangulations and Pick’s Theorem. *Math. Mag.* **49**, pp. 35-37 (1976).

[Gib76] R. A. Gibbs, Pick Iff Euler, *Math. Mag.* **49**, p. 158 (1976).

[GM10] J. E. Graver and Y. A. Monachino, A Colorful Proof of Pick’s Theorem. *Math Horizons* **18**(2), pp. 14-16 (2010).

[GS93] B. Grünbaum and G. C. Shephard, Pick’s Theorem. *Amer. Math. Monthly* **100**, pp. 150-161 (1993).

[Hai80] G. Haigh, A ’Natural’ Approach to Pick’s Theorem. *Math. Gaz.* **64** pp. 173-177 (1980).

[HW79] G. H. Hardy and E. M. Wright, *An introduction to the Theory of Numbers*, 5th ed., Clarendon Press, Oxford, 1979. pp. 25-29.

[Hon70] R. Honsberger, *Ingenuity in Mathematics*, the Mathematical Association of America, New York, 1970. pp. 27-31.

[Liu79] A. C. F. Liu, Lattice Points and Pick’s Theorem. *Math. Mag.* **52** pp. 232-235 (1979).

[MT07] M. R. Murty and M. Thain, Pick’s Theorem via Minkowski’s Theorem. *Amer. Math. Monthly* **114**, pp. 732-736 (2007).

[NZ67] I. Niven and H. S. Zuckerman, Lattice Points and Polygonal Area. *Amer. Math. Monthly* **74**, pp. 1195-1200 (1967).

[Pic99] G. A. Pick, Geometrisches zur Zahlenlehre. *Sitzensber. Lotos (Prague)* **19**, pp. 311-319 (1899).

[Rec57] J. E. Reeve, On the volume of lattice polyhedra, *Proc. London Math. Soc.* (3)7 pp. 378-395 (1957).

[Ste50] H. Steinhaus, *Mathematical Snapshots*, 2nd ed., Oxford University Press, New York, 1950. pp. 76-77.

[Tra07] J. Trainin, an Elementary Proof of Pick’s Theorem. *Math. Gaz.* **91**, pp. 536-540 (2007).

[UW04] J. Utley and J. Wolfe, Geoboard Areas: Students’ Remarkable Ideas. *Math. teacher* **97**, pp. 18-26 (2004).

[Var85] D. E. Varberg, Pick’s Theorem Revisited. *Amer. Math. Monthly* **92**, pp. 584-587 (1985).