A Tree Clock Data Structure for Causal Orderings in Concurrent Executions

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ABSTRACT

Dynamic techniques are a scalable and effective way to analyze concurrent programs. Instead of analyzing all behaviors of a program, these techniques detect errors by focusing on a single program execution. Often a crucial step in these techniques is to define a causal ordering between events in the execution, which is then computed using vector clocks, a simple data structure that stores logical times of threads. The two basic operations of vector clocks, namely join and copy, require \( \Theta(k) \) time, where \( k \) is the number of threads. Thus they are a computational bottleneck when \( k \) is large.

In this work, we introduce tree clocks, a new data structure that replaces vector clocks for computing causal orderings in program executions. Joining and copying tree clocks takes time that is roughly proportional to the number of entries being modified, and hence the two operations do not suffer the a-priori \( \Theta(k) \) cost per application. We show that when used to compute the classic happens-before (HB) partial order, tree clocks are optimal, in the sense that no other data structure can lead to smaller asymptotic running time. Moreover, we demonstrate that tree clocks can be used to compute other partial orders, such as schedulable-happens-before (SHB) and the standard Mazurkiewicz (MAZ) partial order, and thus are a versatile data structure. Our experiments show that just by replacing vector clocks with tree clocks, the computation becomes from 2.02× faster (MAZ) to 2.66× (SHB) and 2.97× (HB) on average per benchmark. These results illustrate that tree clocks have the potential to become a standard data structure with wide applications in concurrent analyses.

1 INTRODUCTION

The analysis of concurrent programs is one of the major challenges in formal methods, due to the non-determinism of inter-thread communication. The large space of communication interleavings poses a significant challenge to the programmer, as intended invariants can be broken by unexpected communication patterns. The subtlety of these patterns also makes verification a demanding task, as exposing a bug requires searching an exponentially large space [41]. Consequently, significant efforts are made towards understanding and detecting concurrency bugs efficiently [10, 19, 34, 58, 63, 68].

Dynamic analyses and partial orders. One popular approach to the scalability problem of concurrent program verification is dynamic analysis [23, 36, 39, 45]. Such techniques have the more modest goal of discovering faults by analyzing program executions instead of whole programs. Although this approach cannot prove the absence of bugs, it is far more scalable than static analysis and typically makes sound reports of errors. These advantages have rendered dynamic analyses a very effective and widely used approach to error detection in concurrent programs.

The first step in virtually all techniques that analyze concurrent executions is to establish a causal ordering between the events of
the execution. Although the notion of causality varies with the application, its transitive nature makes it naturally expressible as a partial order between these events. One prominent example is the Mazurkiewicz partial order (MAZ), which often serves as the canonical way to represent concurrent traces [7, 40] (aka Shasha-Snir traces [57]). Another vastly common partial order is Lamport’s happens-before (HB) [32], initially proposed in the context of distributed systems [55]. In the context of testing multi-threaded programs, partial orders play a crucial role in dynamic race detection techniques, and have been thoroughly exploited to explore trade-offs between soundness, completeness, and running time of the underlying analysis. Prominent examples include the widespread use of HB [18, 23, 29, 45, 56], schedulably-happensbefore (SHB) [35], causally-precedes (CP) [59], weak-causally-precedes (WCP) [30], doesn’t-commute (DC) [49], and strong/weak-dependently-precedes (SDP/WDP) [27], M2 [44] and SyncP [37]. Beyond race detection, partial orders are often employed to detect and reproduce other concurrency bugs such as atomicity violations [8, 25, 38], deadlocks [53, 61], and other concurrency vulnerabilities [66].

Vector clocks in dynamic analyses. Often, the computational task of determining the partial ordering between events of an execution is achieved using a simple data structure called vector clock. Informally, a vector clock C is an integer array indexed by the processes/threads in the execution, and succinctly encodes the knowledge of a process about the whole system. For vector clock C, associated with thread \( t_1 \), if \( C_{t_1}(t_2) = i \) then it means that the latest event of \( t_1 \) is ordered after the first \( i \) events of thread \( t_2 \) in the partial order. Vector clocks, thus seamlessly capture a partial order, with the point-wise ordering of the vector timestamps of two events capturing the ordering between the events with respect to the partial order of interest. For this reason, vector clocks are instrumental in computing the HB partial order efficiently [21, 22, 39], and are ubiquitous in the efficient implementation of analyses based on partial orders even beyond HB [23, 30, 31, 35, 38, 49, 53, 61].

The fundamental operation on vector clocks is the pointwise join \( C_{t_1} \leftarrow C_{t_1} \cup C_{t_2} \). This occurs whenever there is a causal ordering from thread \( t_2 \) to \( t_1 \). Operationally, a join is performed by updating \( C_{t_1}(t) \leftarrow \max(C_{t_1}(t), C_{t_2}(t)) \) for every thread \( t \), and captures the transitivity of causal orderings: as \( t_1 \) learns about \( t_2 \), it also learns about other threads \( t \) that \( t_2 \) knows about. Note that if \( t_1 \) is aware of a later event of \( t_2 \) this operation is vacuous. With \( k \) threads, a vector clock join takes \( \Theta(k) \) time, and can quickly become a bottleneck even in systems with moderate \( k \). This motivates the following question: is it possible to speed up join operations by proactively avoiding vacuous updates? The challenge in such a task comes from the efficiency of the join operation itself—since it only requires linear time in the size of the vector, any improvement must operate in sub-linear time, i.e., not even touch certain entries of the vector clock. We illustrate this idea on a concrete example, and present the key insight in this work.

Motivating example. Consider the example in Figure 1. It shows a partial trace from a concurrent system with 6 threads, along with the vector timestamps at each event. When event \( e_2 \) is ordered before event \( e_3 \) due to synchronization, the vector clock \( C_{t_2} \) of \( t_2 \) is joined with that of \( C_{t_1} \), i.e., the \( t_j \)-th entry of \( C_{t_1} \) is updated to the maximum of \( C_{t_1}(t_j) \) and \( C_{t_2}(t_j) \). Now assume that thread \( t_2 \) has learned of the current times of threads \( t_3, t_4, t_5 \) and \( t_6 \) via thread \( t_3 \). Since the \( t_3 \)-th component of the vector timestamp of event \( e_1 \), which is larger than the corresponding component of event \( e_2 \), \( t_1 \) cannot possibly learn any new information about threads \( t_4, t_5, t_6 \) through the join performed at event \( e_3 \). Hence the naive pointwise updates will be redundant for the indices \( j = \{3, 4, 5, 6\} \). Unfortunately, the flat structure of vector clocks is not amenable to such reasoning and cannot avoid these redundant operations.

To alleviate this problem, we introduce a new hierarchical tree-like data structure for maintaining vector times called a tree clock. The nodes of the tree encode local clocks, just like entries in a vector clock. In addition, the structure of the tree naturally captures which clocks have been learned transitively via intermediate threads. Figure 1 (right) depicts a simplified tree clock encoding the vector times of \( C_t \). The subtree rooted at thread \( t_3 \) encodes the fact that \( t_2 \) has learned about the current times of \( t_4, t_5, t_6 \), and \( t_3 \) transitively, via \( t_3 \). To perform the join operation \( C_{t_1} \leftarrow C_{t_1} \cup C_{t_2} \), we start from the root of \( C_{t_2} \), and traverse the tree as follows: Given a current node \( u \), we proceed to the children of \( u \) if and only if \( u \) represents the time of a thread that is not known to \( t_1 \). Hence, in the example, the join operation will now access only the light-gray area of the tree, and thus compute the join without accessing the whole tree, resulting in a sublinear running time of the join operation.

The above principle, which we call direct monotonicity is one of two key ideas exploited by tree clocks; the other being indirect monotonicity. The key technical challenge in developing the tree clock data structure lies in (i) using direct and indirect monotonicity to perform efficient updates, and (ii) perform these updates such that direct and indirect monotonicity are preserved for future operations. Section 3.1 illustrates the intuition behind these two principles in depth.

Contributions. Our contributions are as follows.

1. We introduce tree clock, a new data structure for maintaining logical times in concurrent executions. In contrast to the flat structure of the traditional vector clocks, the dynamic hierarchical structure of tree clocks naturally captures ad-hoc communication patterns between processes. In turn, this allows for joint and copy operations that run in sublinear time. As a data structure, tree clocks offer high versatility as they can be used in computing many different ordering relations.

2. We prove that tree clocks are an optimal data structure for computing HB, in the sense that, for every input trace, the total computation time cannot be improved (asymptotically) by replacing tree clocks with any other data structure. On the other hand, vector clocks do not enjoy this property.

3. We illustrate the versatility of tree clocks by presenting tree clock-based algorithms for the MAZ and SHB partial orders.

4. We perform a large-scale experimental evaluation of the tree clock data structure for computing the MAZ, SHB and HB partial orders, and compare its performance against the standard vector

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1 As with many presentations of dynamic analyses using vector clocks [29], we assume that the local entry of a thread’s clock increments by 1 after each event it performs. Hence, in Figure 1, the \( t_1 \)-th entry of \( C_{t_1} \) increases from 27 to 28 after \( e_1 \) is performed.
clock data structure. Our results show that just by replacing vector clocks with tree clocks, the computation becomes up to 2.97× faster on average. Given our experimental results, we believe that replacing vector clocks by tree clocks in partial order-based algorithms can lead to significant improvements on many applications.

2 PRELIMINARIES

In this section we develop relevant notation and present standard concepts regarding concurrent executions, partial orders and vector clocks.

2.1 Concurrent Model and Traces

We start with our main notation on traces. The exposition is standard and follows related work (e.g., [23, 30, 59]).

Events and traces. We consider execution traces of concurrent programs represented as a sequence of events performed by different threads. Each event is a tuple \( e = (i, t, op) \), where \( i \) is the unique event identifier of \( e \), \( t \) is the identifier of the thread that performs \( e \), and \( op \) is the operation performed by \( e \), which can be one of the following types⁵:

1. \( op = r(x) \), denoting that \( e \) reads global variable \( x \).
2. \( op = w(x) \), denoting that \( e \) writes to global variable \( x \).
3. \( op = \text{acq}(l) \), denoting that \( e \) acquires the lock \( l \).
4. \( op = \text{rel}(l) \), denoting that \( e \) releases the lock \( l \).

We write \( \text{tid}(e) \) and \( \text{op}(e) \) to denote the thread identifier and the operation of \( e \), respectively. For a read/write event \( e \), we denote by \( \text{Variable}(e) \) the (unique) variable that \( e \) accesses. We often ignore the identifier \( i \) and represent \( e \) as \((t, op)\). In addition, we are often not interested in the thread of \( e \), in which case we simply denote \( e \) by its operation, e.g., we refer to event \( r(x) \). When the variable of \( e \) is not relevant, it is also omitted (e.g., we may refer to a read event \( r \)).

A (concrete) trace is a sequence of events \( \sigma = e_1, \ldots, e_n \). The trace \( \sigma \) naturally defines a total order \( \preceq_T \) (pronounced trace order) over the set of events appearing in \( \sigma \), i.e., we have \( e \preceq_T e' \) if either \( e = e' \) or \( e \) appears before \( e' \) in \( \sigma \); when \( e \neq e' \), then we say \( e \prec_T e' \). We require that \( \sigma \) respects the semantics of locks. That

### Figure 1: (Left) Illustration of the effect of a join operation \( C_t_1 \leftarrow C_t_1 \cup C_t_2 \) on the clocks of the two threads. The \( j \)-th entry in timestamps correspond to thread \( t_j \). Red entries remain unchanged, as \( t_1 \) already knows of a later time. (Right) A tree representation of the clocks \( C_t_2 \) that encodes transitivity. Dark gray marks the threads whose clock has processed in \( C_t_2 \) compared to \( C_t_1 \) (i.e., just \( t_2 \)). Light gray marks the nodes that we need to examine when performing the join operation.

is, for every lock \( l \) and every two acquire events \( \text{acq}_l(t), \text{acq}_l(t) \) on the lock \( l \) such that \( \text{acq}_l(t) <_{\sigma} \text{acq}_l(t) \), there exists a lock release event \( \text{rel}_l(t) \) in \( \sigma \) with \( \text{tid}(\text{acq}_l(t)) = \text{tid}(\text{rel}_l(t)) \) and \( \text{acq}_l(t) <_{\sigma} \text{rel}_l(t) <_{\sigma} \text{acq}_l(t) \). Finally, we denote by \( \text{Thrds}_\sigma \) the set of thread identifiers appearing in \( \sigma \).

Thread order. Given a trace \( \sigma \), the thread order \( \preceq_{T\sigma} \) is the smallest partial order such that \( e_1 \preceq_T e_2 \) if \( i \) \( \text{Variable}(e_1) = \text{Variable}(e_2) \), \( \text{tid}(e_1) = \text{tid}(e_2) \), and \( e_1 \preceq_{\sigma} e_2 \). For an event \( e \) in a trace \( \sigma \), the local time \( \text{ITime}^\sigma(e) \) of \( e \) is the number of events that appear before \( e \) in the trace \( \sigma \) that are also performed by \( \text{tid}(e) \), i.e., \( \text{ITime}^\sigma(e) = \{e' | e' \preceq_{T\sigma} e \} \). We remark that the pair \( (\text{tid}(e), \text{ITime}^\sigma(e)) \) uniquely identifies the event \( e \) in the trace \( \sigma \).

Conflicting events. Two events of \( e_1, e_2 \) of \( \sigma \) are called conflicting, denoted by \( e_1 \parallel e_2 \), if (i) \( \text{Variable}(e_1) = \text{Variable}(e_2) \), (ii) \( \text{tid}(e_1) \neq \text{tid}(e_2) \), and (iii) at least one of \( e_1, e_2 \) is a write event. The standard approach in concurrent analyses is to detect conflicting events that are causally independent, according to some pre-defined notion of causality, and can thus be executed concurrently.

2.2 Partial Orders, Vector Times and Vector Clocks

A partial order on a set \( S \) is a reflexive, transitive and anti-symmetric binary relation on the elements of \( S \). Partial orders are the standard mathematical object for analyzing concurrent executions. The main idea behind such techniques is to define a partial order \( \preceq_{T\sigma} \) on the set of events of the trace \( \sigma \) being analyzed. The intuition is that \( \preceq_{T\sigma} \) captures causality — the relative order of two events of \( \sigma \) must be maintained if they are ordered by \( \preceq_{T\sigma} \). More importantly, when two events \( e_1 \) and \( e_2 \) are unordered by \( \preceq_{T\sigma} \) (denoted \( e_1 \parallel e_2 \)), then they can be deemed concurrent. This principle forms the backbone of all partial-order based concurrent analyses.

A naïve approach for constructing such a partial order is to explicitly represent it as an acyclic directed graph over the events of \( \sigma \), and then perform a graph search whenever needed to determine whether two events are ordered. Vector clocks, on the other hand, provide a more efficient method to represent partial orders and therefore are the key data structure in most partial order-based algorithms. The use of vector clocks enables designing streaming algorithms, which are also suitable for monitoring the system. These algorithms associate vector timestamps \([21, 22, 39]\) with events so

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⁵Fork and join events are ignored for ease of presentation. Handling such events is straightforward.
that the point-wise ordering between timestamps reflects the underlying partial order. Let us formalize these notions now.

**Vector Timestamps.** Let us fix the set of threads \( \text{Thrds} \) in the trace. A vector timestamp (or simply vector time) is a mapping \( V : \text{Thrds} \rightarrow \mathbb{N} \). It supports the following operations.

\[
V_1 \sqsubseteq V_2 \quad \text{iff} \quad \forall t : V_1(t) \leq V_2(t) \quad \text{(Comparison)}
\]

\[
V_1 \sqcup V_2 = \lambda t : \max(V_1(t), V_2(t)) \quad \text{(Join)}
\]

\[
V[t' \rightarrow +i] = \lambda t : \begin{cases} V(t) + i, & \text{if } t = t' \\ V(t), & \text{otherwise} \end{cases} \quad \text{(Increment)}
\]

We write \( V_1 = V_2 \) to denote that \( V_1 \subseteq V_2 \) and \( V_2 \subseteq V_1 \). Let us see how vector timestamps provide an efficient implicit representation of partial orders.

**Timestamping for a partial order.** Consider a partial order \( \leq^p \) defined on the set of events of \( \sigma \) such that \( \leq^p \subseteq \leq^\sigma \). In this case, we can define the \( P \)-timestamp of an event \( e \) as the following vector timestamp:

\[
C_e^\leq^p = \lambda u : \max\{ \text{ITime}^\sigma(f) | f \leq^p e, \text{tid}(f) = u \}
\]

In words, \( C_e^\leq^p \) contains the timestamps of the events that appear the latest in their respective threads such that they are ordered before \( e \) in the partial order \( \leq^p \). We remark that \( C_e^\leq^p(\text{tid}(e)) = \text{ITime}^\sigma(e) \). The following observation then shows that the timestamps defined above precisely capture the order \( \leq^p \).

**Lemma 1.** Let \( \leq^p \) be a partial order defined on the set of events of trace \( \sigma \) such that \( \leq^p \subseteq \leq^\sigma \). Then for any two events \( e_1, e_2 \) of \( \sigma \), we have, \( C_e^{\leq^p} \subseteq C_e^{\leq^\sigma} \iff e_1 \leq^p e_2. \)

In words, Lemma 1 implies that, in order to check whether two events are ordered according to \( \leq^p \), it suffices to compare their vector timestamps.

**The vector clock data structure.** When establishing a causal order over the events of a trace, the timestamps of an event is computed using timestamps of other events in the trace. Instead of explicitly storing timestamps of each event, it is often sufficient to store only the timestamps of a few events, as the algorithm is running. Typically a data-structure called vector clocks is used to store vector times. Vector clocks are implemented as a simple integer array indexed by thread identifiers, and they support all the operations on vector timestamps. A useful feature of this data-structure is the ability to perform in-place operations. In particular, there are methods such as \( \text{Join}(\cdot) \), \( \text{Copy}(\cdot) \) or \( \text{Increment}(\cdot, \cdot) \) that store the result of the corresponding vector time operation in the original instance of the data-structure. For example, for a vector clock \( C \) and a vector time \( V \), a function call \( C.\text{Join}(V) \) stores the value \( C \cup V \) back in \( C \). Each of these operations iterates over all the thread identifiers (indices of the array representation) and compares the corresponding components in \( C \) and \( V \). The running time of the join operation for the vector clock data structure is thus \( \Theta(k) \), where \( k \) is the number of threads. Similarly, copy and comparison operations take \( \Theta(k) \) time.

### 2.3 The Happens-Before Partial Order

Lamport’s Happens-Before (HB) [32] is one of the most frequently used partial orders for the analysis of concurrent executions, with wide applications in domains such as dynamic race detection. Here we use HB to illustrate the disadvantages of vector clocks and form the basis for the tree clock data structure. In later sections we show how tree clocks also apply to other partial orders, such as Schedulably-Happens-Before and the Mazurkiewicz partial order.

**Happens-before.** Given a trace \( \sigma \), the happens-before (HB) partial order \( \leq^\text{HB}_\sigma \) of \( \sigma \) is the smallest partial order over the events of \( \sigma \) that satisfies the following conditions.

1. \( \leq^\text{HB}_\sigma \subseteq \leq^p \)
2. For every release event \( \text{rel}(t) \) and acquire event \( \text{acq}(t) \) on the same lock \( t \) with \( \text{rel}(t) \leq^\text{HB} \text{acq}(t) \), we have \( \text{rel}(t) \leq^\text{HB} \text{acq}(t) \).

For two events \( e_1, e_2 \) in trace \( \sigma \), we use \( e_1 \parallel^\text{HB} e_2 \) to denote that neither \( e_1 \leq^\text{HB} e_2 \) nor \( e_2 \leq^\text{HB} e_1 \). We say \( e_1 \parallel^\text{HB} e_2 \) when \( e_1 \neq e_2 \) and \( e_1 \leq^\text{HB} e_2 \). Given a trace \( \sigma \), two events \( e_1, e_2 \) of \( \sigma \) are said to be in a happens-before (data) race if (i) \( e_1 \parallel e_2 \) and (ii) \( e_1 \parallel^\text{HB} e_2 \).

**The happens-before algorithm.** In light of Lemma 1, race detection based on \( \leq^\text{HB}_\sigma \) constructs the \( \leq^\text{HB}_\sigma \) partial order in terms of vector timestamps and detects races using these. The core algorithm for constructing \( \leq^\text{HB}_\sigma \) is shown in Algorithm 1. The algorithm maintains a vector clock \( C_t \) for every thread \( t \in \text{Thrds} \), and a similar one \( C_t \) for every lock \( L \). When processing an event \( e = (t, op) \), it performs an update \( C_t.\text{Increment}(t, L) \), which is implicit and not shown in Algorithm 1. Moreover, if \( op = \text{acq}(t) \) or \( op = \text{rel}(t) \), the algorithm executes the corresponding procedure. The HB-timestamp of \( e \) is then simply the value stored in \( C_t(\text{tid}(e)) \) right after \( e \) has been processed.

**Running time using vector clocks.** If a trace \( \sigma \) has \( n \) events and \( k \) threads, computing the HB partial order with Algorithm 1 and using vector clocks takes \( O(n \cdot k) \) time. The quadratic bound occurs because every vector clock join and copy operation iterates over all \( k \) threads.

### 3 The Tree Clock Data Structure

In this section we introduce tree clocks, a new data structure for representing logical times in concurrent and distributed systems. We first illustrate the intuition behind tree clocks, and then develop the data structure in detail.

#### 3.1 Intuition

Like vector clocks, tree clocks represent vector timestamps that record a thread’s knowledge of events in other threads. Thus, for each thread \( t \), a tree clock records the last known local time of \( t \). However, unlike a vector clock which is flat, a tree clock maintains this information hierarchically — nodes store local times of a thread,
while the tree structure records how this information has been obtained transitively through intermediate threads. In the following examples we use the operation $\text{sync}(t)$ to denote the sequence $\text{acq}(t), \text{rel}(t)$.

1. **Direct monotonicity.** Recall that a vector clock-based algorithm like Algorithm 1 maintains a vector clock $C_t$ which intuitively captures thread $t$’s knowledge about all threads. However, it does not maintain how this information was acquired. Knowledge of how such information was acquired can be exploited in join operations, as we show through an example. Consider a computation of the HB partial order for the trace $\sigma$ shown in Figure 2a. At event $e_7$, thread $t_4$ transitively learns information about events in the trace through thread $t_3$ because $e_6 <_\text{HB} e_7$ (dashed edge in Figure 2a). This is accomplished by joining with clock $C_{t_3}$ of thread $t_3$. Such a join using vector clocks will take 4 steps because we need to take the pointwise maximum of two vectors of length 4.

Suppose in addition to these timestamps, we maintain how these timestamps were updated in each clock. This would allow one to make the following observations.

1. Thread $t_3$ knows of event $e_1$ of $t_1$ transitively, through event $e_2$ of thread $t_2$.
2. Thread $t_4$ (before the join at $e_7$) knows of event $e_1$ through $e_4$ of thread $t_2$.

Before the join, since $t_4$ has a more recent view of $t_2$ when compared to $t_3$, it is aware of all the information that thread $t_3$ knows about the world via thread $t_2$. Thus, when performing the join, we need not examine the component corresponding to thread $t_1$ in the two clocks. Tree clocks, by maintaining such additional information, can avoid examining some components of a vector timestamp and yield sublinear updates.

2. **Indirect monotonicity.** We now illustrate that if in addition to information about “how a view of a thread was updated”, we also maintained “when the view of a thread was updated”, the cost of join operations can be further reduced. Consider the trace $\sigma$ of Figure 2b. At each of the events of thread $t_4$, it learns about events in the trace transitively through thread $t_3$ by performing two join operations. At the first join (event $e_5$), thread $t_4$ learns about events $e_1, e_2, e_3$ transitively through event $e_4$. At event $e_7$, thread $t_4$ finds out about new events in thread $t_3$ (namely, $e_6$). However, it does not need to update its knowledge about threads $t_1$ and $t_2$. — thread $t_3$’s information about threads $t_1$ and $t_2$ were acquired by the time of event $e_4$ about which thread $t_4$ is aware. Thus, if information about when knowledge was acquired is also kept, this form of “indirect monotonicity” can be exploited to avoid examining all components of a vector timestamp.

The flat structure of vector clocks misses the transitivity of information sharing, and thus arguments based on monotonicity are lost, resulting in vacuous operations. On the other hand, tree clocks maintain transitivity in their hierarchical structure. This enables reasoning about direct and indirect monotonicity, and thus avoid redundant operations.

### 3.2 Tree Clocks

We now present the tree clock data structure in detail.

**Tree clocks.** A tree clock TC consists of the following.

1. $T = (V, E)$ is a rooted tree of nodes of the form $(\text{tid}, \text{clk}, \text{aclk}) \in \text{Thrs} \times \mathbb{N}^2$. Every node $u$ stores its children in an ordered list $\text{Child}(u)$ of descending aclk order. We also store a pointer $\text{Prnt}(u)$ of $u$ to its parent in $T$.
2. $\text{ThrMap}$: $\text{Thrs} \rightarrow V$ is a thread map, with the property that if $\text{ThrMap}(t) = (\text{tid}, \text{clk}, \text{aclk})$, then $t = \text{tid}$.

We denote by $T_v$, root the root of $T$, and for a tree clock TC we refer by TC. $T$ and TC. $\text{ThrMap}$ to the rooted tree and thread map of TC, respectively. For a node $u = (\text{tid}, \text{clk}, \text{aclk})$ of $T$, we let $u. \text{tid} = \text{tid}$, $u. \text{clk} = \text{clk}$ and $u. \text{aclk} = \text{aclk}$, and say that $u$ points to the unique event $e$ with $\text{tid}(e) = \text{tid}$ and $\text{Time}(e) = \text{clk}$. Intuitively, if $o = \text{Prnt}(u)$, then $u$ represents the following information.

1. $T_v$ has the local time $u. \text{clk}$ for thread $u. \text{tid}$.
2. $u. \text{aclk}$ is the attachment time of $o. \text{tid}$, which is the local time of $v$ when $v$ learned about $u. \text{clk}$ of $u. \text{tid}$ (this will be the time that $v$ had when $u$ was attached to $v$).

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**Figure 2:** Illustration of the two insights behind tree clocks. An event $\text{sync}(t)$ represents two events $\text{acq}(t), \text{rel}(t)$.

**Figure 3:** The tree clock of $t_4$ after processing the event $e_7$ in the traces of Figure 2a (left) and Figure 2b (right).
Naturally, if \( u = T \cdot \text{root} \) then \( u.\text{aclock} = \bot \). See Figure 3.

**Tree clock operations.** Just like vector clocks, tree clocks provide functions for initialization, update and comparison. There are two main operations worth noting. The first is \( \text{Join} \sim TC_1.\text{Join}(TC_2) \) joins the tree clock \( TC_2 \) to \( TC_1 \). In contrast to vector clocks, this operation takes advantage of the direct and indirect monotonicity outlined in Section 3.1 to perform the join in sublinear time in the size of \( TC_1 \) and \( TC_2 \) (when possible). The second is \( \text{MonotoneCopy} \). We use \( TC_1.\text{MonotoneCopy}(TC_2) \) to copy \( TC_2 \) to \( TC_1 \) when we know that \( TC_1 \subseteq TC_2 \). The idea is that when this holds, the copy operation has the same semantics as the join, and hence the principles that make \( \text{Join} \) run in sublinear time also apply to \( \text{MonotoneCopy} \).

Algorithm 2 gives a pseudocode description of this functionality. The functions on the left column present operations that can be performed on tree clocks, while the right column lists helper routines for the more involved functions \( \text{Join} \) and \( \text{MonotoneCopy} \). In the following we give an intuitive description of each function.

1. **Init**. This function initializes a tree clock \( TC \) that belongs to thread \( t \), by creating a node \( u = \{(t, 0, \bot)\} \). Node \( u \) will always be the root of \( TC \). This initialization function is only used for tree clocks that represent the clocks of threads. Auxiliary tree clocks for storing vector times of release events do not execute this initialization.

2. **Get**. This function simply returns the time of thread \( t \) stored in \( TC \), while it returns \( 0 \) if \( t \) is not present in \( TC \).

3. **Increment**. This function increments the time of the root node of \( TC \). It is only used on tree clocks that have been initialized using \( \text{Init} \), i.e., the tree clock belongs to a thread that is always stored in the root of the tree.

4. **LessThan**. This function compares the vector time of \( TC \) to the vector time of \( TC' \), i.e., it returns \( \text{True} \) iff \( TC \subseteq TC' \).

5. **Join**. This function implements the join operation with \( TC' \), i.e., updating \( TC \leftarrow TC \sqcup TC' \). At a high level, the function performs the following steps.

1. **Routine getUpdatedNodesJoin** performs a pre-order traversal of \( TC' \), and gathers in a stack \( S \) the nodes of \( TC' \) that have progressed in \( TC' \) compared to \( TC \). The traversal may stop early due to direct or indirect monotonicity, hence, this routine generally takes sub-linear time.

2. **Routine detachNodes** detaches from \( TC \) the nodes whose \( tid \) appears in \( S \), as these will be repositioned in the tree.

3. **Routine attachNodes** updates the nodes of \( TC \) that were detached in the previous step, and repositions them in the tree. This step effectively creates a subtree of nodes of \( TC \) that is identical to the subtree of \( TC' \) that contains the progressed nodes computed by getUpdatedNodesJoin.

4. Finally, the last 4 lines of \( \text{Join} \) attach the subtree constructed in the previous step under the root \( z \) of \( TC \), at the front of the \( \text{Child}(z) \) list.

Figure 4 provides an illustration.

6. **MonotoneCopy**. This function implements the copy operation \( TC \leftarrow TC' \) assuming that \( TC \subseteq TC' \). The function is very similar to \( \text{Join} \). The key difference is that this time, the root of \( TC \) is always considered to have progressed in \( TC' \), even if the respective times are equal. This is required for changing the root of \( TC \) from the current node to one with \( tid \) equal to the root of \( TC' \). Figure 5 provides an illustration.

The crucial parts of \( \text{Join} \) and \( \text{MonotoneCopy} \) that exploit the hierarchical structure of tree clocks are in getUpdatedNodesJoin and getUpdatedNodesCopy. In each case, we proceed from a parent \( u' \) to its children \( v' \) only if \( u' \) has progressed wrt its time in \( TC \) (recall Figure 2a), capturing direct monotonicity. Moreover, we proceed from a child \( v' \) of \( u' \) to the next child \( v'' \) (in order of appearance in \( \text{Child}(u') \)) only if \( TC \) is not yet aware of the attachment time of \( v'' \) on \( u' \) (recall Figure 2b), capturing indirect monotonicity.

**Remark 1** (Constant time epoch accesses). The function \( TC.\text{Get}(t) \) returns the time of thread \( t \) stored in \( TC \) in \( O(1) \) time, just like vector clocks. This allows all epoch-related optimizations [23, 50] from vector clocks to apply to tree clocks.

### 4 TREE CLOCKS FOR HAPPENS-BEFORE

Let us see how tree clocks are employed for computing the HB partial order. We start with the following observation.

**Lemma 2** (Monotonicity of copies). Right before Algorithm 1 processes a lock-release event \( \langle t, \text{rel}(f) \rangle \), we have \( C_T \subseteq C_T' \).

**Tree clocks for HB.** Algorithm 3 shows the algorithm for computing HB using the tree clock data structure for implementing vector times. When processing a lock-acquire event, the vector-clock join operation has been replaced by a tree-clock join. Moreover, in light of Lemma 2, when processing a lock-release event, the vector-clock copy operation has been replaced by a tree-clock monotone copy.

We refer to Appendix B for an example run of Algorithm 3 on a trace \( \sigma \), showing how tree clocks grow during the execution.

**Correctness.** We now state the correctness of Algorithm 3, i.e., we show that the algorithm indeed computes the HB partial order. We start with two monotonicity invariants of tree clocks.

**Lemma 3.** Consider any tree clock \( C \) and node \( u \) of \( C. T \). For any tree clock \( C' \), the following assertions hold.

1. **Direct monotonicity:** If \( u.\text{clk} \leq C'.\text{Get}(u.\text{tid}) \) then for every descendant \( w \) of \( u \) we have that \( w.\text{clk} \leq C'.\text{Get}(w.\text{tid}) \).

2. **Indirect monotonicity:** If \( u.\text{clk} \leq C'.\text{Get}(u.\text{tid}) \) where \( v \) is a child of \( u \) then for every descendant \( w \) of \( v \) we have that \( w.\text{clk} \leq C'.\text{Get}(w.\text{tid}) \).

The following lemma follows from the above invariants and establishes that Algorithm 3 with tree clocks computes the correct timestamps on all events, i.e., the correctness of tree clocks for HB.

**Lemma 4.** When Algorithm 3 processes an event \( e \), the vector time stored in the tree clock \( C_{\text{tid}(e)} \) is \( \tau^C_e \).

**Data structure optimality.** Just like vector clocks, computing HB with tree clocks takes \( \Theta(n \cdot k) \) time in the worst case, and it is known that this quadratic bound is likely to be tight for common
Algorithm 2: The tree clock data structure.

```plaintext
// Initialize a tree clock for thread t
1 function Init(t)  
2 Let u ← (t, 0, ⊥)  
3 Make u the root of T  
4 Let ThrMap(t) ← u  
// Get the clock for thread t
5 function Get(t)  
6 if TC ⊥ then  
7 Let u ← ThrMap(t)  
8 return u, clk  
9 return 0  
// Increment the clock of the root thread
10 function Increment()  
11 Let z ← T.root  
12 z.clk ← z.clk + i  
// True iff TC'  
13 function LessThan(TC')  
14 Let z ← T.root  
15 return z.clk ≤ TC'.Get(z.tid)  
// Update with TC'  
16 function Join(TC')  
17 Let z' ← TC'.T.root  
18 if z'.clk ≤ Get(z'.tid) then  
19 return  
20 Let S ← an empty stack  
21 getUpdatedNodesJoin(S, z')  
22 detachNodes(S)  
23 attachNodes(S)  
// Place the updated subtree under the root of T  
24 Let w ← ThrMap(z'.tid)  
25 Let z ← T.root  
26 Assign w.aclk ← z.clk  
27 pushChild(w, z)  
// Monotone copy, assumes that this ⊆ TC'  
28 function MonotoneCopy(TC')  
29 Let z' ← TC'.T.root  
30 Let z ← T.root  
31 Let S ← an empty stack  
32 getUpdatedNodesCopy(S, z', z)  
33 detachNodes(S)  
34 attachNodes(S)  
// New root has the same tid as the root of TC'.T  
35 Assign T.root ← ThrMap(z'.tid)
```

Algorithm 3: HB with tree clocks.

```plaintext
1 procedure acquire(t, ℓ)  
2 Cₗ.Join(Cₗ)  
3 procedure release(t, ℓ)  
4 Cₗ.MonotoneCopy(Cₗ)  
```

applications such as dynamic race prediction [31]. However, we have seen that tree clocks can take sublinear time on join and copy operations, whereas vector clocks always require time linear in the size of the vector (i.e., Θ(k)). A natural question arises: is there a more efficient data structure than tree clocks? More generally, what is the most efficient data structure for the HB algorithm to represent vector times? To answer this question, we define vector-time work, which gives a lower bound on the number of data structure operations that HB has to perform regardless of the actual data
structure used to store vector times. Then, we show that tree clocks match this lower bound, hence achieving optimality for HB.

**Vector-time work.** Consider the general HB algorithm (Algorithm 1) and let $D = \{C_1, C_2, \ldots, C_m\}$ be the set of the vector-time data structures used. Consider the execution of the algorithm on a trace $\sigma$. Given some $1 \leq i \leq |\sigma|$, we let $C_j^i$ denote the vector time represented by $C_j$ after the algorithm has processed the $i$-th event of $\sigma$. We define the vector-time work (or vt-work, for short) on $\sigma$ as

$$VTWork(\sigma) = \sum_{1 \leq i \leq |\sigma|} \sum_{j} |\{ r \in \text{Thrds} : C_j^{i-1}(r) \neq C_j^i(r)\}|.$$

In words, for every processed event, we add the number of vector-time entries that change as a result of processing the event, and $VTWork(\sigma)$ counts the total number of entry updates in the overall course of the algorithm. Note that vt-work is independent of the data structure used to represent each $C_j$, and satisfies the inequality

$$n \leq VTWork(\sigma) \leq n \cdot k.$$

as with every event of $\sigma$ the algorithm updates one of $C_j$.

**Vector-time optimality.** Given an input trace $\sigma$, we denote by $\mathcal{T}_{DS}(\sigma)$ the time taken by the HB algorithm (Algorithm 1) to process $\sigma$ using the data structure $DS$ to store vector times. Intuitively, $VTWork(\sigma)$ captures the number of times that instances of $DS$ change state. For data structures that represent vector times explicitly, $VTWork(\sigma)$ presents a natural lower bound for $\mathcal{T}_{DS}(\sigma)$.

Hence, we say that the data structure $DS$ is **vt-optimal** if $\mathcal{T}_{DS}(\sigma) = O(VTWork(\sigma))$. It is not hard to see that vector clocks are not vt-optimal, i.e., taking $DS = VC$ to be the vector clock data structure, one can construct simple traces $\sigma$ where $VTWork(\sigma) = O(n)$ but $\mathcal{T}_{DS}(\sigma) = \Omega(n \cdot k)$, and thus the running time is $k$ times more than the vt-work that must be performed on $\sigma$. In contrast, the following theorem states that tree clocks are vt-optimal.

**Theorem 1 (Tree-clock Optimality).** For any input trace $\sigma$, we have $\mathcal{T}_{TC}(\sigma) = O(VTWork(\sigma))$.

The key observation behind Theorem 1 is that, when HB uses tree clocks, the total number of tree-clock entries that are accessed over all join and monotone copy operations (i.e., the sum of the sizes of the light-gray areas in Figure 4 and Figure 5) is $\leq 3 \cdot VTWork(\sigma)$.

**Remark 2.** Theorem 1 establishes strong optimality for tree clocks, in the sense that they are vt-optimal on every input. This is in contrast to usual notions of optimality that is guaranteed on only some inputs.

### 5.1 Schedulable-Happens-Before

SHB is a strengthening of HB, introduced recently [35] in the context of race detection. Given a trace $\sigma$ and a read event $r$, let lw$_{\sigma}(r)$
be the last write event of σ before r with Variable(w) = Variable(r). SHB is the smallest partial order that satisfies the following.

1. \( \preceq_{\text{SHB}} \preceq_{\text{MAZ}} \)
2. for every read event r, we have \( w_{\sigma}(r) \preceq_{\text{SHB}} r \).

**Algorithm for SHB.** Similarly to HB, the SHB partial order is computed by a single pass of the input trace \( \sigma \) using vector-times [35]. The SHB algorithm processes synchronization events (i.e., \( \text{acq}(t) \) and rel1(f)) similarly to HB. In addition, for each variable x, the algorithm maintains a data structure \( LW_x \) that stores the vector time of the latest write event on x. When a write event \( w(x) \) is encountered, the vector time \( C_{\text{id}}(w) \) is copied to \( LW_x \). In turn, when a read event \( r(x) \) is encountered the algorithm joins \( LW_x \) to \( C_{\text{id}}(r) \).

**SHB with tree clocks.** Tree clocks can directly be used as the data structure to store vector times in the SHB algorithm. We refer to Algorithm 4 for the pseudocode. The important new component is the function \( \text{CopyCheckMonotone} \) in Line 8 that copies the vector time of \( C_t \) to \( LW_x \). In contrast to \( \text{MonotoneCopy} \), this copy is not guaranteed to be monotone, i.e., we might have \( LW_x \not\subseteq C_t \). Note, however, that using tree clocks, this check requires only constant time. Internally, \( \text{CopyCheckMonotone} \) performs \( \text{MonotoneCopy} \) if \( LW_x \subseteq C_t \) (running in sublinear time), otherwise it performs a deep copy for the whole tree clock (running in linear time). In practice, we expect that most of the times \( \text{CopyCheckMonotone} \) results in \( \text{MonotoneCopy} \) and thus is very efficient. The key insight is that if \( \text{MonotoneCopy} \) is not used, then \( LW_x \not\subseteq C_t \) and thus we have a race (\( LW_{\sigma}(r), r \)). Hence, the number of times a deep copy is performed is bounded by the number of write-read races in \( \sigma \) between a read and its last write.

### 5.2 The Mazurkiewicz Partial Order

The Mazurkiewicz partial order (MAZ) [40] has been the canonical way to represent concurrent executions algebraically using an independence relation that defines which events can be reordered. This algebraic treatment allows to naturally lift language-inclusion problems from the verification of sequential programs to concurrent programs [7]. As such, it has been the most studied partial order in the context of concurrency, with deep applications in dynamic analyses [25, 38, 42], ensuring consistency [57] and stateless model checking [26]. In shared memory concurrency, the standard independence relation deems two events as dependent if they conflict, and independent otherwise [28]. In particular, MAZ is the smallest partial order that satisfies the following conditions.

1. \( \preceq_{\text{MAZ}} \preceq_{\text{MAZ}} \)
2. for every two events \( e_1, e_2 \) such that \( e_1 \preceq_{\sigma} e_2 \) and \( e_1 \times e_2 \), we have \( e_1 \preceq_{\sigma_{\text{MAZ}}} e_2 \).

**Algorithm 5: MAZ with tree clocks.** The algorithm for computing MAZ is similar to that for SHB. The main difference is that MAZ includes read-to-write orderings, and thus we need to store additional vector times \( R_{t,x} \) of the last event \( r(x) \) of thread \( t \). In addition, we use the set \( LRDs_x \) to store the threads that have executed a \( r(x) \) event after the latest \( w(x) \) event so far. This allows us to only spend computation time in the first read-to-write ordering, as orderings between the read event and later write events follow transitively via intermediate write-to-write orderings. Overall, this approach yields the efficient time complexity \( O(n \cdot k) \) for MAZ, similarly to HB and SHB. We refer to Algorithm 5 for the pseudocode.

### 6 EXPERIMENTS

In this section we report on an implementation and experimental evaluation of the tree clock data structure. The primary goal of these experiments is to evaluate the practical advantage of tree clocks over the vector clocks for keeping track of logical times in a concurrent program executions.

**Implementation.** Our implementation is in Java and closely follows Algorithm 2. The tree clock data structure is represented as two arrays of length \( k \) (number of threads), the first one encoding the shape of the tree and the second one encoding the integer timestamps as in a standard vector clock. For efficiency reasons, recursive routines have been made iterative.

**Benchmarks.** Our benchmark set consists of standard benchmarks found in benchmark suites and recent literature. In particular, we used the Java benchmarks from the IBM Contest suite [19], Java Grande suite [60], DaCapo [9], and SIR [16]. In addition, we used OpenMP benchmark programs, whose execution lengths and number of threads can be tuned, from DataRaceOnAccelerator [54], DataRaceBench [33], OmpSCR [17] and the NAS parallel benchmarks [6], as well as large OpenMP applications contained in the following benchmark suites: CORAL [1, 2], ECP proxy applications [3], and Mantevo project [4]. Each benchmark was instrumented and executed in order to log a single concurrent trace, using the tools RV-Predict [51] (for Java programs) and ThreadSanitizer [56] (for OpenMP programs). Overall, this process yielded a large set of 153 benchmark traces that were used in our evaluation. Table 1 presents aggregate information about the benchmark traces generated. Information on the individual traces is provided in Table 3 in the Appendix C.
Figure 6: Times for processing each benchmark trace using tree clocks (TC) and vector clocks (VC). The top row shows the time for computing the partial order, while the bottom row shows the time including the analysis component.

Table 1: Trace Statistics

|               | Min  | Max  | Mean |
|---------------|------|------|------|
| Threads       | 3    | 222  | 31   |
| Locks         | 1    | 60.5k| 688  |
| Variables     | 18   | 37.8M| 1.8M |

Table 2: Average Speedup

| Partial Order | MAZ | SHB | HB  |
|---------------|-----|-----|-----|
| Speedup       | 2.1B | 227M | 2.53 |

Running times. For each partial order, Table 2 shows the average speedup over all benchmarks, both with and without the analysis component. We see that tree clocks are very effective in reducing the running time of the computation of all 3 partial orders, with the most significant impact being on SHB where the average speedup is 2.53 times. For the cases of MAZ and SHB, this speedup also lead to a significant speedup in the overall analysis time. On the other hand, although HB with tree clocks is about 2 times faster than with vector clocks, this speedup has a smaller effect on the overall analysis time. The reason behind this observation is straightforward: SHB and MAZ are much more computationally-heavy, as they are defined using all types of events; on the other hand, HB is defined only on synchronization events (acq and rel) and on average, only \( \approx 9.5\% \) of the events are synchronization events on our benchmark traces. Since our analysis considers all events, the HB-computation component occupies a smaller fraction of the overall analysis time. We remark, however, that for programs that are more synchronization-heavy, or for analyses that are more lightweight (e.g., when checking for data races on a specific variable as opposed to all variables), the speedup of tree clocks will be larger on the whole analysis. Indeed, Figure 7 shows the obtained speedup on the total analysis time for HB as a function of synchronization events. We observe a trend for the speedup to increase as the percentage of
must perform when computing the HB partial order. We can likewise define the metrics TCWork(σ) and VCWork(σ) corresponding to the number of entries updated when processing each event when using respectively the data structures tree clocks and vector clocks. These metrics are visualized in Figure 8 for computing the HB partial order in our benchmark suite. The figure shows that the VCWork(σ)/VTWork(σ) ratio is often considerably large. In contrast, the ratio TCWork(σ)/VTWork(σ) is typically significantly smaller. The differences in running times between vector and tree clocks reflect the discrepancies between TCWork(σ) and VCWork(σ). Next, recall the intuition behind the optimality of tree clocks (Theorem 1), namely that TCWork(σ) ≤ 3·VTWork(σ). Figure 8 confirms this theoretical bound, as the TCWork(σ)/VTWork(σ) ratio stays nicely upper-bounded by 3 while the VCWork(σ)/VTWork(σ) ratio grows to nearly 100. Interestingly, for some benchmarks we have TCWork(σ) ≃ 2.99 · VTWork(σ), i.e., these benchmarks push tree clocks to their vt-work upper-bound. Going one step further, Figure 9 shows the ratio VCWork(σ)/TCWork(σ) for each partial order in our dataset. The results indicate that vector clocks perform a lot of unnecessary work compared to tree clocks, and experimentally demonstrate the source of speedup on tree clocks. Although Figure 9 indicates that the potential for speedup can be large (reaching 55x), the actual speedup in our experiments (Figure 6) is smaller, as a single tree clock operation is more computationally heavy than a single vector clock operation.

**Scalability.** To get a better insight on the scalability of tree clocks, we performed a set of controlled experiments on custom benchmarks, by controlling the number of threads and the number of locks parameters while keeping the communication patterns constant. Each trace consists of 10M events, while the number of threads varies between 10 and 360. The traces are generated in a way such that a randomly chosen thread performs two consecutive operations, acq(l) followed by a rel(l), on a randomly (when applicable) chosen lock l. We have considered four cases: (a) all threads communicate over a single common lock (single lock); (b) similar to (a), but there are 50 locks, and 20% of the threads are 5 times more likely to perform an operation compared to the rest of the threads (50 locks, skewed); (c) k = 1 client threads communicate with a single server thread via a dedicated lock per thread (star topology); (d) similar to (a), but every pair of threads communicates over a dedicated lock (pairwise communication). Figure 10 shows the obtained results. Scenario (a) shows how tree clocks have a constant-factor speedup over vector clocks in this setting. As we move to more locks in scenario (b), thread communication becomes more independent and the benefit of tree clocks may slightly diminish. As a subset of the threads is more active than the rest, timestamps are frequently exchanged through them, making tree clocks faster in this setting as well. Scenario (c) represents a case in which tree clocks thrive: while the time taken by vector clocks increases with the number of threads, it stays constant for tree clocks. This is because the star topology implies that, on average, every tree clock join and copy operation only affects a constant number of tree clock entries, despite the fact that every thread is aware of the state of every other thread. Intuitively, the star communication topology naturally affects the shape of the tree to (almost) a star, which leads to this effect. Finally, scenario (d) represents the

**Comparison with vt-work.** We also investigate the total number of entries updated using each of the data structures. Recall that the metric VTWork(σ) (Section 4) measures the minimum amount of updates that any implementation of the vector time

| PO | MAZ | SHB | HB |
|----|-----|-----|----|
| 2.02 | 2.66 | 2.97 |

Table 2: Average speedup for computing the partial order due to tree clocks.

**Figure 7:** Speedup on HB+analysis computation as a function of the percentage of synchronization events, for the traces where the total time is not too small (≥ 100ms).

**Figure 8:** Comparison of the ratios TCWork(σ)/VTWork(σ) and VCWork(σ)/VTWork(σ) across all benchmarks.

synchronization events increases in the trace. A further observation is that speedup is prominent when the number of threads are large.

Figure 6 gives a more detailed view of the tree clocks vs vector clocks times across all benchmarks. We see that tree clocks almost always outperform vector clocks on all partial orders, and in some cases by large margins. Interestingly, the speedup tends to be larger on more demanding benchmarks (i.e., on those that take more time). In the very few cases tree clocks are slower, this is only by a small factor. These are traces where the sub-linear updates of tree clocks only yield a small potential for improvement, which does not justify the overhead of maintaining the more complex tree data structure (as opposed to a vector). Nevertheless, overall tree clocks consistently deliver a generous speedup to each of MAZ, HB and SHB. Finally, we remark that all these speedups come directly from just replacing the underlying data structure, without any attempt to optimize the algorithm that computes the respective partial order, or its interaction with the data structure.
worst case for tree clocks as all pairs of threads can communicate with each other and the communication is conducted via a unique lock per thread pair. This pattern nullifies the benefit of tree clocks, while their increased complexity results in a general slowdown. However, even in this worst-case scenario, the difference between tree clocks and vector clocks remains relatively small.

7 RELATED WORK

Other partial orders and tree clocks. As we have mentioned in the introduction, besides HB and SHB, many other partial orders perform dynamic analysis using vector clocks. In such cases, tree clocks can replace vector clocks either partially or completely, sometimes requiring small extensions to the data structure as presented here. In particular, we foresee interesting applications of tree clocks for the WCP [30], DC [49] and SDP [27] partial orders.

Speeding up dynamic analyses. Vector-clock based dynamic race detection is known to be slow [52], which many prior works have aimed to mitigate. One of the most prominent performance bottlenecks is the linear dependence of the size of vector timestamps on the number of threads. Despite theoretical limits [12], prior research exploits special structures in traces [5, 13, 15, 20, 62] that enable succinct vector time representations. The Goldilocks [18] algorithm infers HB-orderings using locksets instead of vector timestamps but incurs severe slowdown [23]. The FastTrack [23] optimization uses epochs for maintaining succinct access histories and our work complements this optimization — tree clocks offer optimizations for other clocks (thread and lock clocks). Other optimizations in clock representations are catered towards dynamic thread creation [46, 47, 64]. Another major source of slowdown is program instrumentation and expensive metadata synchronization. Several approaches have attempted to minimize this slowdown, including hardware assistance [14, 69], hybrid race detection [43, 67], static analysis [24, 48], and sophisticated ownership protocols [11, 50, 65].

8 CONCLUSION

We have introduced tree clocks, a new data structure for maintaining logical times in concurrent executions. In contrast to standard vector clocks, tree clocks can dynamically capture communication patterns in their structure and perform join and copy operations in sublinear time, thereby avoiding the traditional overhead of these operations when possible. Moreover, we have shown that tree clocks are vector-time optimal for computing the HB partial order, performing at most a constant factor work compared to what is absolutely necessary, in contrast to vector clocks. Finally, our experiments show that tree clocks effectively reduce the running time for computing the MAZ, SHB and HB partial orders significantly, and thus offer a promising alternative over vector clocks.

Interesting future work includes incorporating tree clocks in an online analysis such as ThreadSanitizer [56]. Any use of additional synchronization to maintain analysis atomicity in this online setting is identical and of the same granularity to both vector clocks and tree clocks. However, the faster joins performed by tree clocks may lead to less congestion compared to vector clocks, especially for partial orders such as SHB and MAZ where synchronization occurs on all events (i.e., synchronization, as well as access events). We leave this evaluation for future work. Finally, since tree clocks are a drop-in replacement of vector clocks, most of the existing techniques that minimize the slowdown due to metadata synchronization (Section 7) are directly applicable to tree clocks.

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A PROOFS

Lemma 2 (Monotonicity of copies). Right before Algorithm 1 processes a lock-release event \((t, \text{rel}(t))\), we have \(C_t \subseteq C_i\).

Proof. Consider a trace \(\sigma\), a release event \(\text{rel}(t)\) and let \(\text{acq}(t)\) be the matching acquire event. When \(\text{acq}(t)\) is processed the algorithm performs \(C_t \leftarrow C_i \cup C_t\), and thus \(C_t \subseteq C_i\) after this operation. By lock semantics, there exists no release event \(\text{rel}'(t)\) such that \(\text{acq}(t) <_{\text{rel}} \text{rel}'(t) <_{\text{rel}} \text{rel}(t)\), and hence \(C_t\) is not modified until \(\text{rel}(t)\) is processed. Since vector clock entries are never decremented, when \(\text{rel}(t)\) is processed we have \(C_t \subseteq C_i\), as desired.

Lemma 3. Consider any tree clock \(C\) and node \(u\) of \(C\). For any tree clock \(C'\), the following assertions hold.

1. Direct monotonicity: If \(u.\text{clk} \leq C'.\text{Get}(u.\text{tid})\) then for every descendant \(w\) of \(u\) we have that \(w.\text{clk} \leq C'.\text{Get}(w.\text{tid})\).
2. Indirect monotonicity: If \(v.\text{aclk} \leq C'.\text{Get}(u.\text{tid})\) where \(v\) is a child of \(u\) then for every descendant \(w\) of \(v\) we have that \(w.\text{clk} \leq C'.\text{Get}(w.\text{tid})\).

Proof. First, note that after initialization \(u\) has no children, hence each statement is trivially true. Now assume that both statements hold when the algorithm processes an event \(e\), and we show that they both hold after the algorithm has processed \(e\). We distinguish cases based on the type of \(e\).

\(e = (t, \text{acq}(t))\). The algorithm performs the operation \(C_t.\text{Join}(\text{Load})\), hence the only tree clock modified is \(C_t\), and thus it suffices to examine the cases that \(C_t\) is \(C\) and \(C_t\) is \(C'\).

1. \(C_t\) is \(C\). First consider the case that \(u = C_t.\text{root}\). Observe that \(u.\text{clk} > C'.\text{Get}(u.\text{tid})\), and thus Item 1. holds trivially. For Item 2., we distinguish cases based on whether \(v.\text{aclk}\) has progressed by the \(\text{Join}\) operation. If yes, then we have \(v.\text{aclk} = u.\text{aclk}\), and the statement holds trivially for the same reason as in Item 1. Otherwise, we have that for every descendant \(w\) of \(v\), the clock \(w.\text{clk}\) has not progressed by the \(\text{Join}\) operation, hence the statement holds by the induction hypothesis on \(C_t\).

2. \(C_t\) is \(C'\). For Item 1., if \(u.\text{clk} \leq C'.\text{Get}(u.\text{tid})\) holds before the \(\text{Join}\) operation, then the statement holds by the induction hypothesis, since \(\text{Join}\) does not decrease the clocks of \(C_t\). Otherwise, the statement follows by the induction hypothesis on \(\text{Load}\). The analysis for Item 2. is similar. The desired result follows.

\(e = (t, \text{rel}(t))\). The algorithm performs the operation \(\text{Load}\), \(\text{MonotoneCopy}(C_t)\). The analysis is similar to the previous case, this time also using Lemma 2 to argue that no time stored in \(\text{Load}\) decreases.

Lemma 4. When Algorithm 3 processes an event \(e\), the vector time stored in the tree clock \(\text{C}_{\text{tid}}(e)\) is \(C_{e}^{\text{vt}}\).

Proof. The lemma follows directly from Lemma 3. In each case, if the corresponding operation (i.e., \(\text{Join}\) for event \((t, \text{acq}(t))\) and \(\text{MonotoneCopy}\) for \((t, \text{rel}(t))\)), if the clock of a node \(w\) of the tree clock that performs the operation does not progress, then we are guaranteed that \(w.\text{clk}\) is not smaller than the time of the thread \(w.\text{tid}\) in the tree clock that is passed as an argument to the operation.

First remote acquires. Consider a trace \(\sigma\) and a lock-release event \(e = (t, \text{rel}(t))\) of \(\sigma\), such that there exists a later acquire event \(e' = (t', \text{acq}(t))\) (\(e <_{\text{rel}} e'\)). The first remote acquire of \(e\) is the first event \(e'\) with the above property. For example, in Figure 11a, \(e_2\) is the first remote acquire of \(e_3\). When constructing the HB partial order, the algorithm makes HB orderings from lock-release events to their first remote acquires \(\text{rel}(t) <_{\text{HB}} \text{acq}(t)\). The following lemma captures the property that the edges of tree clocks are essentially the inverses of such orderings.

Lemma 5. Consider the execution of Algorithm 3 on a trace \(\sigma\). For every tree clock \(C\) and node \(u\) of \(C\), \(T\) other than the root, the following assertions hold.

1. \(u\) points to a lock-release event \(\text{rel}(t)\).
2. \(\text{rel}(t)\) has a first remote acquire \(\text{acq}(t)\) and \((v.\text{tid}, u.\text{aclk})\) points to \(\text{acq}(t)\), where \(v\) is the parent of \(u\) in \(C\), \(T\).

Proof. The lemma follows by a straightforward induction on \(\sigma\).

Lemma 5 allows us to prove the vt-optimality of tree clocks.

Theorem 1 (Tree-clock Optimality). For any input trace \(\sigma\), we have \(T_{TC}(\sigma) = O(\text{VTWork}(\sigma))\).

Proof. Consider a critical section of a thread \(t\) on lock \(l\), marked by two events \(\text{acq}(t), \text{rel}(t)\). We define the following vector times.

1. \(V^1_t\) and \(V^2_t\) are the vector times of \(C_t\) right before and right after \(\text{acq}(t)\) is processed, respectively.
2. \(V^3_l\) is the vector time of \(C_t\) right before \(\text{acq}(t)\) is processed.
3. \(V^4_l\) is the vector time of \(C_t\) right before \(\text{rel}(t)\) is processed.
4. \(V^5_l\) and \(V^6_l\) are the vector times of \(C_t\) right before and right after \(\text{rel}(t)\) is processed, respectively.

First, note that (i) \(V^1_t \subseteq V^3_t\) and (ii) due to lock semantics, we have \(V^5_l = V^1_l\). Let \(W = W_f + W_C\), where

\[W_f = |\{t' : V^4_l(t') \neq V^1_l(t')\}|\]

and

\[W_C = |\{t' : V^6_l(t') \neq V^5_l(t')\}|\]

i.e., \(W_f\) and \(W_C\) are the vt-work for handling \(\text{acq}(t)\) and \(\text{rel}(t)\), respectively. Let \(T_f\) be the time spent in \(\text{TC}\). \(\text{Join}\) due to \(\text{acq}(t)\). Similarly, let \(T_C\) be the time spent in \(\text{TC}\) \(\text{MonotoneCopy}\) due to \(\text{rel}(t)\). We will argue that \(T_f = O(W)\) and \(T_C = O(W_C)\), and
We now turn our attention to $T_J$. Note that this proves the lemma, simply by summing over all critical sections of $\sigma$.

We start with $T_J$. Observe that the time spent in this operation is proportional to the number of times the loop in Line 37 is executed, i.e., the number of nodes $v'$ that the loop iterates over. Consider the if statement in Line 38. If $\text{Get}(v'.\text{tid}) < v'.\text{clk}$, then we have $V^1_t(v'.\text{tid}) > V^1_t(v'.\text{tid})$, and thus this iteration is accounted for in $W_f$. On the other hand, if $\text{Get}(v'.\text{tid}) > v'.\text{clk}$, then we have $V^1_t(v'.\text{tid}) > V^1_t(v'.\text{tid})$. Due to (i) and (ii) above, we have $V^1_t(v'.\text{tid}) > V^1_t(v'.\text{tid})$, and thus this iteration is accounted for in $W_C$. Finally, consider the case that $\text{Get}(v'.\text{tid}) = v'.\text{clk}$, and let $v$ be the node of $C_t$ such that $v.\text{tid} = v'.\text{tid}$. There can be at most one such $v$ that is not the root of $C_t$. For every other such $v$, let $u = \text{Prnt}(v)$. Note that $v'$ is not the root of $C_t$, and let $u' = \text{Prnt}(v')$. Let $r.\text{rel}(t)$ be the lock-release event that $v$ and $v'$ point to. By Lemma 5, $r.\text{rel}(t)$ has a first remote acquire $\text{acq}(t)$ such that (i) $u.\text{tid} = u'.\text{tid} = t'$, where $t'$ is the thread of $\text{acq}(t)$, and (ii) $v.\text{aclk}$ is the local clock of $\text{acq}(t)$. Since $\text{getUpdatedNodesCopy}$ examines $v'$, we must have $u'.\text{clk} > u.\text{clk}$. In turn, we have $u.\text{clk} \geq v.\text{aclk}$, and thus $u'.\text{aclk} > v.\text{aclk}$. Hence, due to Line 39, $u'$ can have at most one child $v'$ with $v'.\text{clk} = \text{Get}(v'.\text{tid})$. Thus, we can account for the time of this case in $W_f$. Hence, $T_J = O(W)$, as desired.

We now turn our attention to $T_C$. Similarly to the previous case, the time spent in this operation is proportional to the number of times the loop in Line 63 is executed. Consider the if statement in Line 64. If $\text{Get}(v'.\text{tid}) < v'.\text{clk}$, then we have $V^1_t(v'.\text{tid}) > V^1_t(v'.\text{tid})$, and thus this iteration is accounted for in $W_C$. Note that as the copy is monotone (Lemma 2), we can’t have $\text{Get}(v'.\text{tid}) > v'.\text{clk}$. Finally, the reasoning for the case where $\text{Get}(v'.\text{tid}) = v'.\text{clk}$ is similar to the analysis of $T_J$, using Line 68 instead of Line 39. Hence, $T_C = O(W_C)$, as desired.

The desired result follows.

\[\square\]

**B EXAMPLE OF TREE CLOCKS IN HB**

Figure 11 shows an example run of Algorithm 3 on a trace $\sigma$, showing how tree clocks grow during the execution. The figure shows the tree clocks $C_t$ of the threads; the tree clocks of locks $C_L$ are just copies of the former after processing a lock-release event (shown in parentheses in the figure). Figure 12 presents a closer look of the $\text{Join}$ and $\text{MonotoneCopy}$ operations for the last events of $\sigma$. 

Figure 11: Example run of HB and the updates on the corresponding clock-trees.

(a) Details of the join operation due to $e_{15}$. $TC_6$ shows the tree clock of lock $l_6$. $TC_2$ shows the tree clock of thread $t_2$ right before performing the operation $TC_2.Join(TC_6)$. Note that $TC_2$ is not yet aware of thread $t_1$. The result is $TC_3$ shown in Figure 11b.

(b) Details of processing $e_{16}$. $TC_2$ shows the tree clock of thread $t_2$. $TC_6$ shows the tree clock of lock $l_6$ right before performing $TC_6.MonotoneCopy(TC_2)$. Note that $TC_6$ is not yet aware of thread $t_2$. The result is $TC_4$ shown in Figure 11b.

Figure 12: A closer look at the last two steps of the example of Figure 11 in the processing of events $e_{15}$ and $e_{16}$. In each figure, light gray marks the nodes of the left tree clock whose vector time is compared to the time of the tree clock on the right. Similarly, dark gray marks the nodes of the left (resp., right) tree clock that are updating (resp., being updated).
### C BENCHMARKS

| Benchmark | N | T | M | L |
|-----------|---|---|---|---|
| CoMD-omp-task-1 | 175.1M | 56 | 66.1K | 114 |
| CoMD-omp-task-2 | 175.1M | 56 | 66.1K | 114 |
| CoMD-omp-task-deps-1 | 174.1M | 63.0K | 34 |
| CoMD-omp-task-deps-2 | 174.1M | 63.0K | 34 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.badSolution-2 | 394.0M | 56 | 66.1K | 114 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.badSolution-1 | 107.0M | 16 | 101.2K | 35 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.solution1-2 | 112.6M | 16 | 101.2K | 35 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.solution1-1 | 107.0M | 16 | 101.2K | 35 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.badSolution-2 | 394.0M | 56 | 66.1K | 114 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.badSolution-1 | 107.0M | 16 | 101.2K | 35 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.solution1-2 | 112.6M | 16 | 101.2K | 35 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.solution1-1 | 107.0M | 16 | 101.2K | 35 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.badSolution-2 | 394.0M | 56 | 66.1K | 114 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.badSolution-1 | 107.0M | 16 | 101.2K | 35 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.solution1-2 | 112.6M | 16 | 101.2K | 35 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.solution1-1 | 107.0M | 16 | 101.2K | 35 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.badSolution-2 | 394.0M | 56 | 66.1K | 114 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.badSolution-1 | 107.0M | 16 | 101.2K | 35 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.solution1-2 | 112.6M | 16 | 101.2K | 35 |
| OmpSCR-v2.0-c-LoopsWithDependencies-c-loopA.solution1-1 | 107.0M | 16 | 101.2K | 35 |

Table 3: Information on Benchmark Traces. We denote by $N$, $T$, $M$ and $L$ the total number of events, number of threads, number of memory locations and number of locks, respectively.