Modified teleparallel gravity: Inflation without an inflaton

Rafael Ferraro\textsuperscript{1,2,*} and Franco Fiorini\textsuperscript{1,†}

\textsuperscript{1}Instituto de Astronomía y Física del Espacio, Casilla de Correo 67, Sucursal 28, 1428 Buenos Aires, Argentina
\textsuperscript{2}Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina

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The Born-Infeld strategy to smooth theories having divergent solutions is applied to the teleparallel equivalent of general relativity. Differing from other theories of modified gravity, modified teleparallelism leads to second order equations, since the teleparallel Lagrangian only contains first derivatives of the vierbein. We show that the Born-Infeld-modified teleparallelism solves the particle horizon problem in a spatially flat Friedmann-Robertson-Walker (FRW) universe by providing an initial exponential expansion without resorting to an inflaton field.

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I. MODIFIED GRAVITY: A BORN-INFELD APPROACH

In 1934 Born and Infeld (BI) [1] proposed the following scheme for modifying a field theory governed by a Lagrangian density $\mathcal{L} = \sqrt{-g} L$:

$$\mathcal{L} \rightarrow \mathcal{L}_{\text{BI}} = \sqrt{-g} \lambda \left[ \left(1 + \frac{2L}{\lambda} \right) - 1 \right]. \quad (1)$$

The basic idea was to introduce a new scale $\lambda$ with the aim of smoothing singularities. The scheme (1) is essentially the way for going from the classical free particle Lagrangian to the relativistic one; in such case, the scale is $\lambda = mc^2$, which smooths the particle velocity by preventing its unlimited growth. Besides, Born and Infeld subtracted the “rest energy” to get that $\mathcal{L}_{\text{BI}}$ vanishes when $\mathcal{L}$ is zero. We can then expect that Born-Infeld dynamics will differ from the original dynamics for those configurations where $L$ is large. In fact, Born and Infeld looked for a reformulation of Maxwell’s electrodynamics in order to smooth the divergence of the pointlike charge electric field, and they have succeeded in obtaining a finite self-energy for this configuration. On the other hand, the original Lagrangian is recovered if $L \ll \lambda$; hence the solutions of both theories do not appreciably differ in these regions. Nowadays Born-Infeld Lagrangians have reappeared in developments of string theories at low energies [2–7]; they have also been used in quintessence theories for modeling matter fluids able to drive both inflation and the present accelerated expansion [8]. However, although the subject has received some attention [9–11], no gravitational BI analogue leading to second order equations was yet proposed in four dimensions.

A wide variety of modified gravity theories have been considered in the last decades. For instance, the Lovelock Lagrangian is a polynomial in Riemann curvature which leads to second order equations for the metric tensor [12]. Nevertheless, the Lovelock Lagrangian only differs from the Einstein-Hilbert (EH) Lagrangian, $\mathcal{L}_{\text{EH}}[g_{\mu\nu}(x)] = -(16\pi G)^{-1}\sqrt{-g} R (R$ being the scalar curvature), for a dimension larger than 4. On the other hand “$f(R)$” theories are being currently studied, mostly connected with the attempts to explain the cosmic acceleration without resorting to quintessence models [13,14]. For instance, a $f(R)$ theory could be obtained by using the Born-Infeld scheme:

$$\mathcal{L} = -\frac{1}{16\pi G} \sqrt{-g} \lambda \left[ \left(1 + \frac{2R}{\lambda} \right) - 1 \right]. \quad (2)$$

However we find this strategy unsatisfactory because: (1) fourth order dynamical equations will result, since $R$ contains second derivatives of the metric (a feature that is common to $f(R)$ theories); (2) this strategy is unable to smooth black holes, since they have $R = 0$ (then the scale $\lambda$ could not play any role).

Concerning the first objection, it is well known that the second derivatives of the metric in the Einstein-Hilbert Lagrangian do not lead to fourth order equations because they only give rise to surface terms in the action. This characteristic only remains valid in Lovelock Lagrangians but is lost in $f(R)$ theories.

II. TELEPARALLEL EQUIVALENT OF GENERAL RELATIVITY

In order to build a modified gravity leading to second order equations in four dimensions, we will start not from the Einstein-Hilbert Lagrangian but from the teleparallel equivalent of general relativity (TEGR). While general relativity uses the Levi-Civita connection (curvature but no torsion), teleparallelism uses the Weitzenböck connection [15] (torsion but no curvature). In this sense teleparallelism [16] is a sector of Einstein-Cartan theories [17,18], which describe gravity by means of a connection having both torsion and curvature. In teleparallelism the dynami-
The Weitzenböck connection,\[ \Gamma^\mu_{\nu\rho} = -h^\mu_{\nu} \partial_{\rho} h^\rho_{\nu} = h^\mu_{\rho} \partial_{\nu} h^\rho_{\mu}, \] is such that the Weitzenböck covariant derivative of a vector \( V^\mu h_{\nu} \) becomes\[ \nabla^w_{\nu} V^\mu = \partial^w_{\nu} V^\mu + \Gamma^w_{\nu\mu} V^\lambda = \partial^w_{\nu} V^\mu + \Gamma^w_{\nu\mu} V^\lambda = h^\mu_{\nu} \partial^w_{\nu} V^\lambda. \] Hence a vector \( V^\mu \) will be autoparallel if its components \( V^i = h^i_{\nu} V^\mu \) are constant.

The Weitzenböck connection has zero Riemann curvature and non-null torsion:
\[ T^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \Gamma^\mu_{\rho\nu} = h^\mu_{\rho} \partial_{\nu} h^\rho_{\mu} - \partial_{\nu} h_{\mu}^\rho, \] i.e., \( h^\mu_{\nu} T^\lambda_{\mu\nu} \) are the components of the 2-form \( dh^\mu \), where \( \{h^i\} \) is the dual basis (whose elements have components \( h^i_{\mu} \)). The TEGR Lagrangian is [19,20]
\[ L_T[h^i_{\mu}(x)] = \frac{1}{16\pi G} h S^\rho_{\mu\nu} T^\rho_{\mu\nu}, \] where \( h = \det(h^i_{\mu}) \) and \( S^\rho_{\mu\nu} \) is given by
\[ S^\rho_{\mu\nu} = \frac{1}{2} \left[ K^\rho_{\mu\nu} + \delta^\rho_{\mu} T^\nu_{\mu\nu} - \delta^\nu_{\mu} T^\rho_{\mu\nu} \right]. \] In this last equation, the contorsion tensor is
\[ K^\rho_{\mu\nu} = -\frac{1}{2} (T^\rho_{\mu\nu} - T^\nu_{\rho\mu} - T^\mu_{\rho\nu}). \] In (8) and (9), indexes have been raised and lowered with the metric
\[ g^\rho_{\mu\nu}(x) = \eta_{ij} h^i_{\mu}(x) h^j_{\nu}(x), \quad g^\mu_{\nu}(x) = \eta_{ij} h^i_{\mu}(x) h^j_{\nu}(x) \] \[ (\eta_{ij} = \text{diag}(1, -1, -1, -1)), \text{ so it is } h = (-\det g^\rho_{\mu\nu})^{1/2}. \] Note that the vierbein is orthonormal in this metric:
\[ g^\rho_{\nu\mu}(x) h^i_{\mu}(x) h^j_{\nu}(x) = \eta_{ij}. \] Moreover, the Weitzenböck connection proves to be metric compatible. It is easy to show that the contorsion equals the difference between the Levi-Civita connection associated with the metric (10) and the Weitzenböck connection:
\[ \Gamma^w_{\nu\mu} = \Gamma^l_{\nu\mu} + K^\lambda_{\mu\nu}. \] Taking into account the Weitzenböck connection definition (4), this means that
\[ \Gamma^w_{\nu\mu} = \Gamma^l_{\nu\mu} + \frac{1}{2} (T^\nu_{\rho\mu} - T^\mu_{\rho\nu} - T^\rho_{\mu\nu}). \]
vectorial, axial, and traceless-symmetric. The coefficients of the vectorial and traceless-symmetric pieces are strongly constrained by physics in the solar system. The axial part is dynamically linked to the antisymmetric part of the energy-momentum tensor (associated with the intrinsic spin of matter). This ingredient renders the theory invariant only under global Lorentz transformations of the vierbein. However the use of the axial term has been questioned (see Ref. [21]).

In spite of the above mentioned defect, the structure of the teleparallel Lagrangian (7) is very appealing because it resembles the structure for a gauge field: it is quadratic in the torsion $h^i_a dh^i$. In particular, it has only first derivatives of the vierbein field. This feature can be exploited to build a modified teleparallel gravity leading to second order dynamical equations. Remarkably, modified teleparallel gravity will be invariant only under global Lorentz transformations of the vierbein. Concretely, we are going to use a teleparallel gravity leading to second order dynamical equations. The Born-Infeld equations in regions where $\lambda$ is small,

$$\mathcal{L}_{BI} = \frac{\lambda}{16\pi G} h \left[ 1 + \frac{2S_{\mu}^{\nu} T_{\mu}^{\nu}}{\lambda} - 1 \right].$$

which results from varying with respect to $h^i_a$ ($H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter), and a second order one,

$$\left( \frac{12H^2}{N^2\lambda} \right)^{3/2} - 1 = \frac{16\pi G}{\lambda} N^2 \rho, \quad (21)$$

$$\left( \frac{16H^2}{N^2\lambda} + \frac{4H^2}{N^2\lambda} q - 1 \right) \left( 1 - \frac{12H^2}{N^2\lambda} \right)^{3/2} + 1 = \frac{16\pi G}{\lambda} p. \quad (22)$$

which results from varying with respect to $h^i_a$ with $i = \sigma = 1, 2, 3$ or $3$ ($q = -\dot{a}/a^2$ is the deceleration parameter). Actually Eqs. (21) and (22) could also be obtained by replacing the proposed solution (19) in the Lagrangian (18) and then varying with respect to $N(t)$ and $a(t)$; this is a typical feature of high symmetry solutions. Note that $S^{\mu\nu} T^{\mu\nu} = -6H(t)^2/N(t)^2$, so $\lambda$ in (18) will prevent the Hubble parameter from becoming infinite. As it was expected, Eq. (21) is not a dynamical equation for $N(t)$ but a constraint for $a(t)$ (“initial value equation”), as a consequence of the fact that $N(t)$ is not a genuine degree of freedom: $N(t)$ can be absorbed by redefining the $t$ coordinate, so we will choose $N(t) = 1$.

By differentiating Eq. (21) with respect to $t$ and combining it with Eq. (22), the fluid energy-momentum conservation is obtained:

$$\frac{d}{dt}(pa^2) = -p \frac{d}{dt} a^3. \quad (23)$$

If the fluid is described by the state equation $p = \omega \rho$ then one obtains

$$a^{3(1+\omega)} \rho = \text{constant} = a_0^{3(1+\omega)} \rho_0, \quad (24)$$

where $a_o$ and $\rho_o$ indicate the presentday values.

Combining Eqs. (21) and (22) it results

$$1 + q = \frac{3}{2} \left( 1 + \frac{\omega}{\frac{2}{3}} \right) \left( 1 + \frac{2}{3} \right) \rho.$$ \hspace{1cm} (25)

In general relativity ($\lambda \rightarrow \infty$) an accelerated expansion ($q < 0$) is only possible if $\omega = -1/3$ (negative pressure). Instead, in Born-Infeld modified teleparallelism an accelerated expansion can be handled without resorting to negative pressure; a large density $\rho$ is sufficient:

$$\frac{32\pi G}{\lambda} \rho > -3 + \sqrt{13 + 12\omega}. \quad (26)$$

Actually, for $\rho \rightarrow \infty$ in (25), it is $q \rightarrow -1$ and the expansion becomes exponential.

In a context where the cosmological model had spatial curvature, Eq. (21) would define the critical density $\rho_c$ making the universe spatially flat. Therefore, it is useful to measure the contributions to the density coming from different constituents as fractions $\Omega_i = \rho_i/\rho_c$. In this way, by combining Eqs. (21) and (24) we obtain
\[
\left(1 - \frac{12\alpha^2}{\lambda a^2}\right)^{-1/2} - 1 = \frac{16\pi G}{\lambda} \sum_{i} \rho_{ci}(a/a_0)^{-3(1+\omega_i)}, \tag{27}
\]
which can be rewritten in the form
\[
\dot{x}^2 + \mathcal{V}(x) = 0, \quad x = \frac{a}{a_0}, \tag{28}
\]
\[
\mathcal{V}(x) \text{ being an effective potential given by}
\]
\[
\mathcal{V}(x) = \frac{\lambda}{12} x^2 \left[\left(1 + \beta_o \sum_i \Omega_{ci} x^{-3(1+\omega_i)}\right)^2 - 1\right], \tag{29}
\]
where \(\beta_o \equiv (1 - 12H_0^2/\lambda)^{-1/2} - 1\), is a constant. The potential is always negative and vanishes with null derivative when \(a \to 0\), for any value of \(\omega\). Moreover, if \(\omega > -1/3\) the potential will asymptotically approach zero when \(x\) goes to infinity. Instead if \(\omega < -1/3\) then \(\mathcal{V}\) will be a decreasing function. More relevant is the fact that \(|\mathcal{V}|\) is proportional to \(x^2\) when \(x\) goes to zero, so giving an exponential expansion for the early universe, as was anticipated. If \(\omega > -1\) then the initial behavior is \(a(t) \propto \exp[(\lambda/12)^{1/2} t]\). Therefore, the Hubble parameter is equal to the maximum value \(H_{\text{max}} = (\lambda/12)^{1/2}\) at the early stage. Equation (28) also says that
\[
H(z)^2 = H_{\text{max}}^2 \left[1 - \left(1 + \beta_o \sum_i \Omega_{ci} (1 + z)^{3(1+\omega_i)}\right)^{-2}\right], \tag{30}
\]
where \(z = a_0/a(t) - 1\) is the redshift. Equation (28) for only one constituent (\(\Omega = 1\)) can be easily integrated to obtain the evolution in an implicit way:
\[
\ln[2(1 + \nu) + 2\sqrt{\nu(2 + \nu)}] - \sqrt{\nu^{-1} (2 + \nu)} = \mathcal{T}, \tag{31}
\]
being \(\nu = \beta_o (a/a_o)^{-3(1+\omega)}\) and \(\mathcal{T} = -3(1 + \omega)H_{\text{max}} t\).

IV. CONCLUDING COMMENTS

Figure 1 shows the dimensionless scale factor \(a(t)/a_0\) as a function of \(H_0 t\) for several values of \(\alpha = H_{\text{max}} / H_o\), as implied by Eq. (31) with \(\omega = 1/3\). The standard \((a/a_0 = (2H_0 t)^{1/2})\) behavior is plotted as a reference (dashed curve). Remarkably, modified teleparallelism smooths the singularity because the scale factor goes to zero asymptotically.

The main feature of the scale factor behavior is its asymptotic exponential character for any value of \(\omega\). This means that \(H(z)\) becomes a constant when \(z\) goes to infinity. This feature implies that the particle horizon radius \(\sigma = a_0 \int_0^{\sigma_0} (a\dot{a})^{-1} da\) diverges. Hence the whole spacetime ends up being causally connected, in agreement with the isotropy of the cosmic microwave background radiation. This fact appears as an essential property of modified teleparallelism which does not require any special assumption about the sources of the gravitational field.

The standard big bang model successfully explains the relative abundances of light elements. Therefore, a modified gravity theory cannot noticeably change the standard evolution of the universe from the epoch of nucleosynthesis. This means that \(H(z)\) at \(z_{\text{max}} \sim 10^9 - 10^{10}\) should not appreciably differ from its standard value. Figure 2 shows how the Hubble parameter moves away from the GR behavior, represented by the dashed line, to approach the value \(H_{\text{max}}\) as the redshift increases. The redshift \(z_{\text{c}}\) characterizing the transition between both behaviors can be defined as the value of \(z\) at which the asymptotic lines intersect. Since the GR behavior for only one constituent is

FIG. 1. Scale factor as a function of the cosmological time for \(\omega = 1/3\) and different values of \(\alpha = H_{\text{max}} / H_o\). The dashed line represents the GR behavior.

FIG. 2. Hubble parameter as a function of the redshift for \(\omega = 1/3\) and different values of \(\alpha = H_{\text{max}} / H_o\). The dashed line represents the GR behavior.
log(H/H_0) = (3/2)(1 + \omega)(1 + z), one obtains

(1 + z_t)^{3(1+\omega)/2} = H_{max}/H_0. \hspace{1cm} (32)

The condition z_t \gg z_{nuc} implies a lower bound for H_{max}. For a radiation dominated universe (\omega = 1/3) one obtains that H_{max}/H_0 \gg 10^{18}.

Although inflation without inflaton was already obtained in the framework of Einstein-Cartan theories (see for instance Refs. [23,24]), those solutions rely on the existence of spinning matter (the antisymmetric part of the energy-momentum tensor does not vanish). On the contrary, an inflationary phase exists in modified teleparallel gravity for a symmetric energy-momentum tensor. In this case the inflation is ruled by the parameter \lambda entering the Born-Infeld Lagragian. \lambda^{-1/2} has dimensions of time, and behaves as a scale governing the transition from the inflationary phase to the standard GR regime. Besides giving the value of H_{max} = (\lambda/12)^{1/2}, \lambda controls the redshift at the transition between both regimes.

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