Modified DJ method: Application to Boussinesq equation

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Abstract

In this paper we present a modification of DJ Method [J. Math. Anal. Appl. 316 (2006), 753-763] to solve the nonlinear equations more efficiently. It is observed that the modified DJ method is faster and hence it has accelerated convergence rate as compared to the old one. We use this new method to find the analytical solutions of Boussinesq equation. The reported results are compared with the exact solutions. Further, we compare the absolute error in our solution with those in other iterative methods. It is observed that the presented method is simple and generates more accurate solutions as compared with other methods.

Keywords: Boussinesq equation, DJ method, modified DJ method, series solution.

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1. Introduction

Boussinesq equation introduced by French mathematician Joseph Boussinesq has the form

\[ u_{tt} + pu_{xx} + q(u^2)_{xx} + ru_{xxxx} = 0, \] (1.1)

where p, q and r are constants. The Boussinesq equation have several applications in the real world. This equation play an important role in modeling various phenomena such as long waves in shallow water [1], one dimensional nonlinear lattices waves [2], vibration in a nonlinear string [3], electromagnetic waves in dielectric materials [4] and so on. Many researchers have been used analytical methods to solve Boussinesq equation such as variational iteration method [5], modified variational iteration method [6, 7], Adomian decomposition method and homotopy perturbation method [8,9]. Recently Malek et al. [10] have used potential symmetries method to solve Boussinesq equation.

The Daftardar-Gejji and Jafari Method (DJM) [11] is a simpler and more efficient technique used to solve various equations such as fractional differential equations [12], partial differential equations [13], boundary value
problems [14], evolution equations [15], system of nonlinear functional equations [16], algebraic equations [17] and so on. The method is successfully employed to solve Newell-Whitehead-Segel equation [18], Fishers equation [19], Ambartsumian equation [20], fractional-order logistic equation [21] and some nonlinear dynamical systems [22, 23]. In [24, 25] we provided the series solutions of pantograph equation in terms of new special functions. Recently DJM has been used to generate a new numerical methods [26, 27] for solving differential equations.

In this manuscript we consider the solution of Boussinesq equation by using modified DJM. We organize the paper as follows:

The DJM is described briefly in Section 2 and the modified DJM is described in section 3. A general technique used to solve Boussinesq equation using modified DJM is described in Section 4. Section 5 deals with illustrative examples and the conclusions are summarized in Section 6.

2. Daftardar-Gejji and Jafari Method

In this section we describe the Daftardar-Gejji and Jafari Method (DJM) [11], which is useful for solving the nonlinear equations of the form

\[ u = f + L(u) + N(u), \]  

where \( f \) is a source term, \( L \) and \( N \) are linear and nonlinear operators respectively. It is assumed that the DJM solution for the Eq.(2.1) has the form:

\[ u = \sum_{i=0}^{\infty} u_i. \]  

(2.2)

The convergence of series (2.2) is proved in [28].

Since \( L \) is linear

\[ L \left( \sum_{i=0}^{\infty} u_i \right) = \sum_{i=0}^{\infty} L(u_i). \]  

(2.3)

The nonlinear operator \( N \) in Eq.(2.1) is decomposed by Daftardar-Gejji and Jafari as bellow:

\[ N \left( \sum_{i=0}^{\infty} u_i \right) = N(u_0) + \sum_{i=1}^{\infty} \left\{ N \left( \sum_{j=0}^{i} u_j \right) - N \left( \sum_{j=0}^{i-1} u_j \right) \right\} \]

\[ = \sum_{i=0}^{\infty} G_i, \]  

(2.4)

where \( G_0 = N(u_0) \) and \( G_i = \left\{ N \left( \sum_{j=0}^{i} u_j \right) - N \left( \sum_{j=0}^{i-1} u_j \right) \right\}, i \geq 1. \)

Using equations (2.2), (2.3) and (2.4) in Eq.(2.1), we get

\[ \sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + \sum_{i=0}^{\infty} G_i. \]  

(2.5)
From Eq. (2.5), the DJM series terms are generated as bellow:

\[
\begin{align*}
    u_0 &= f, \\
    u_{m+1} &= L(u_m) + G_m, \quad m = 0, 1, 2, \cdots.
\end{align*}
\]  
(2.6)  

In practice, we take the approximation

\[
u = \sum_{i=0}^{k-1} u_i
\]
(2.7)

for suitable integer \( k \).

The convergence results for DJM are described in [28].

3. Modified Daftardar-Gejji and Jafari Method

In [29] Wazwaz proposed a modification in ADM to generate a rapidly converging solution series. Using same technique we modify the Daftardar-Gejji and Jafari method as follows:

We assume that

\[
f = f_1 + f_2.
\]  
(3.1)  

Then Eq. (2.1) can be written as

\[
u = f_1 + f_2 + L(u) + N(u).
\]  
(3.2)  

The modified DJM is described as:

\[
\begin{align*}
    u_0 &= f_1, \\
    u_1 &= f_2 + L(u_0) + G_0, \\
    u_{m+1} &= L(u_m) + G_m, \quad m = 1, 2, 3, \cdots.
\end{align*}
\]  
(3.3)  

It is obvious that the simpler form of \( u_0 \) in MDJM result in the reduction of computations and accelerates the convergence rate. The convergence results for MDJM are similar as that of DJM [28].

4. Applications

A bidirectional solitary wave solution of Eq. (1.1) discussed in [30] and is given by

\[
u(x, 1) = -\frac{3(\alpha^2 + p)^{\frac{1}{2}}}{2q} \text{sech}^2 \left[ \frac{1}{2} \left( \frac{\alpha^2 + p}{-\beta} \right)^{\frac{1}{2}} (x \pm \alpha t) + \beta \right],
\]  
(4.1)
where \( \alpha \) and \( \beta \) are constants and the initial conditions are

\[
\begin{align*}
    u(x, 0) &= \frac{3(\alpha^2 + p)^{\frac{1}{2}}}{2q} \operatorname{sech}^2 \left[ \frac{1}{2} \left( \frac{\alpha^2 + p}{-r} \right)^{\frac{1}{2}} x + \beta \right], \\
    u_t(x, 0) &= \mp \frac{3\alpha(\alpha^2 + p)^{\frac{1}{2}}}{2q(-r)^{\frac{1}{2}}} \operatorname{sech}^2 \left[ \frac{1}{2} \left( \frac{\alpha^2 + p}{-r} \right)^{\frac{1}{2}} x + \beta \right] \\
    &\quad \tanh \left[ \frac{1}{2} \left( \frac{\alpha^2 + p}{-r} \right)^{\frac{1}{2}} x + \beta \right] t \quad \text{.} 
\end{align*}
\]  

(4.2)

The equivalent integral equation of (1.1) is

\[
\begin{align*}
    u(x, t) &= \frac{3(\alpha^2 + p)^{\frac{1}{2}}}{2q} \operatorname{sech}^2 \left[ \frac{1}{2} \left( \frac{\alpha^2 + p}{-r} \right)^{\frac{1}{2}} x + \beta \right] \\
    &\quad \mp \frac{3\alpha(\alpha^2 + p)^{\frac{1}{2}}}{2q(-r)^{\frac{1}{2}}} \operatorname{sech}^2 \left[ \frac{1}{2} \left( \frac{\alpha^2 + p}{-r} \right)^{\frac{1}{2}} x + \beta \right] \\
    &\quad \tanh \left[ \frac{1}{2} \left( \frac{\alpha^2 + p}{-r} \right)^{\frac{1}{2}} x + \beta \right] t \quad \text{tanh} \left[ \frac{1}{2} \left( \frac{\alpha^2 + p}{-r} \right)^{\frac{1}{2}} x + \beta \right] t \\
    &\quad - \int_0^t (u_{xx} + u_{xxxx}) dx - \int_0^t (u^2)_{xx} dx. 
\end{align*}
\]  

(4.3)

This Eq.(4.4) is of the form Eq.(3.2). Using Eq.(3.3), the modified DJM series terms are generated as bellow:

\[
\begin{align*}
    u_0(x, t) &= -\frac{3(\alpha^2 + p)^{\frac{1}{2}}}{2q} \operatorname{sech}^2 \left[ \frac{1}{2} \left( \frac{\alpha^2 + p}{-r} \right)^{\frac{1}{2}} x + \beta \right], \\
    u_1(x, t) &= \mp \frac{3\alpha(\alpha^2 + p)^{\frac{1}{2}}}{2q(-r)^{\frac{1}{2}}} \operatorname{sech}^2 \left[ \frac{1}{2} \left( \frac{\alpha^2 + p}{-r} \right)^{\frac{1}{2}} (x \pm \alpha t + \beta) \right] \\
    &\quad \tanh \left[ \frac{1}{2} \left( \frac{\alpha^2 + p}{-r} \right)^{\frac{1}{2}} (x \pm \alpha t + \beta) \right] t \quad \text{tanh} \left[ \frac{1}{2} \left( \frac{\alpha^2 + p}{-r} \right)^{\frac{1}{2}} (x \pm \alpha t + \beta) \right] t \\
    &\quad - \int_0^t ((u_0(x, t))_{xx} + (u_0(x, t))_{xxxx}) dx \\
    &\quad - \int_0^t (u_0^2(x, t))_{xx} dx, \\
    u_{n+1}(x, t) &= -\int_0^t \left( \left( \sum_{i=0}^n u_i(x, t) \right)_{xx} + \left( \sum_{i=0}^n u_i(x, t) \right)_{xxxx} \right) dx \\
    &\quad - \int_0^t \left( \sum_{i=0}^n u_i(x, t) \right)^2_{xx} dx \\
    &\quad + \int_0^t \left( \sum_{i=0}^{n-1} u_i(x, t) \right)^2_{xx} dx, \quad n = 1, 2, 3, \cdots. 
\end{align*}
\]  

(4.6)
5. Illustrative examples

Besides equation Eq.(1.1), there are few more PDEs which are called Boussinesq equation. In this section, we solve such equations using MDJM.

Example 5.1. Consider the Boussinesq equation [8]

\[ u_{tt} - u_{xx} + 3(u^2)_{xx} + u_{xxxx} = 0, \]  

(5.1)

with initial condition

\[ u(x, 0) = \frac{c}{2} \text{sech}^2 \left( \frac{\sqrt{c}}{2} (x + 1) \right), \]  

(5.2)

\[ u_t(x, 0) = -\frac{c^3}{2} \text{sech}^2 \left( \frac{\sqrt{c}}{2} (x + 1) \right) \tanh \left( \frac{\sqrt{c}}{2} (x + 1) \right). \]  

(5.3)

The equivalent integral equation is

\[ u(x, t) = \frac{c}{2} \text{sech}^2 \left( \frac{\sqrt{c}}{2} (x + 1) \right) - \frac{c^3}{2} \text{sech}^2 \left( \frac{\sqrt{c}}{2} (x + 1) \right) \tanh \left( \frac{\sqrt{c}}{2} (x + 1) \right) t \]

\[ + \int_0^t (u_{xx} - u_{xxxx}) dx - 3 \int_0^t u_{xx}^2 dx. \]  

(5.4)

This Eq.(5.4) is of the form Eq.(3.2). Using Eq.(3.3), the MDJM series terms are generated as bellow:

\[ u_0(x, t) = \frac{c}{2} \text{sech}^2 \left( \frac{\sqrt{c}}{2} (x + 1) \right), \]  

(5.5)

\[ u_1(x, t) = -\frac{1}{8} c^2 t^2 \text{sech}^4 \left( \frac{1}{2} \sqrt{c} (1 + x) \right) - \frac{1}{4} \frac{c^3}{2} t^2 \text{sech}^6 \left( \frac{1}{2} \sqrt{c} (1 + x) \right) \]

\[-\frac{1}{4} \frac{c^5}{2} t \text{sech}^2 \left( \frac{1}{2} \sqrt{c} (1 + x) \right) \tanh \left( \frac{1}{2} \sqrt{c} (1 + x) \right) + \cdots, \]  

(5.6)

\[ u_2(x, t) = \frac{1}{48} c^3 t^4 \text{sech}^6 \left( \frac{1}{2} \sqrt{c} (1 + x) \right) + \frac{13}{192} \frac{c^4}{2} t^4 \text{sech}^8 \left( \frac{1}{2} \sqrt{c} (1 + x) \right) \]

\[-\frac{17}{192} \frac{c^5}{2} t^4 \text{sech}^{10} \left( \frac{1}{2} \sqrt{c} (1 + x) \right) - \cdots, \]  

(5.7)

\[ u_3(x, t) = \frac{7}{64} c^4 t^4 \text{sech}^8 \left( \frac{1}{2} \sqrt{c} (1 + x) \right) \]

\[-\frac{17 c^4}{2} t^4 \text{sech}^{10} \left( \frac{1}{2} \sqrt{c} (1 + x) \right) \]

\[+ \frac{47}{64} c^5 t^4 \text{sech}^{10} \left( \frac{1}{2} \sqrt{c} (1 + x) \right) \]

\[-\frac{77 c^5}{2} t^4 \text{sech}^{12} \left( \frac{1}{2} \sqrt{c} (1 + x) \right) + \cdots. \]  

(5.8)

and so on.
The exact solution of Eq. (5.1) is

$$u(x, t) = \frac{c}{2} \text{sech}^2 \left( \frac{\sqrt{c}}{2} x + \frac{\sqrt{c}}{2} \sqrt{1 + ct} \right)$$  \hspace{1cm} (5.9)

We compare 4-term solutions for $c = 1$ and $c = 2$ in Fig.1 and Fig.2, where MDJM solution and exact solution are shown by red and green colors respectively. From these figures, it can be observed that the modified DJM solution is well in agreement to exact solution.

Fig.1: Comparison of solutions of Eq.(5.1) for $c = 1$

Fig.2: Comparison of solutions of Eq.(5.1) for $c = 2$

The 6-term ADM and HPM solutions of (5.1) are given in [8] as
\[ u(x, t) = \frac{1}{2} \text{sech}^2 \left[ \frac{1}{2} \sqrt{c}(1 + x) \right] \]

\[ - \frac{1}{8} (-1 + c) c^2 t^2 \left( -2 + \cosh(\sqrt{c}(x + 1)) \right) \text{sech}^4 \left[ \frac{1}{2} \sqrt{c}(1 + x) \right] \]

\[ - \frac{1}{1280} ((-1 + c)^2 c^6 (-140 + 157 \cosh(\sqrt{c}(x + 1))) - 26 \cosh(2\sqrt{c}(x + 1))] \]

\[ + \cosh(3\sqrt{c}(x + 1)) \text{sech}^{10} \left[ \frac{\sqrt{c}}{2}(1 + x) \right] + \cdots \] (5.10)

and

\[ u(x, t) = \frac{1}{2} \text{sech}^2 \left[ \frac{1}{2} \sqrt{c}(1 + x) \right] \]

\[ - \frac{1}{8} (-1 + c) c^2 t^2 \left( -2 + \cosh(\sqrt{c}(x + 1)) \right) \text{sech}^4 \left[ \frac{1}{2} \sqrt{c}(1 + x) \right] \]

\[ - \frac{1}{30720} ((-1 + c)^2 c^2 t^4 (-475 - 6125c - 9480c^2 - 3360c^2 t^2 + \cdots \] (5.11)

respectively. We compare the errors in all these solutions in table 1 and 2.

Table 1: Absolute error in the 4-term MDJM solution and exact solution for \( c = 1 \).

| \( t/x \) | 20          | 25          | 30          | 35          | 40          |
|----------|-------------|-------------|-------------|-------------|-------------|
| 0.1      | 3.58376 \times 10^{-12} | 2.41472 \times 10^{-14} | 1.62702 \times 10^{-16} | 1.09628 \times 10^{-18} | 7.38668 \times 10^{-21} |
| 0.2      | 1.36005 \times 10^{-11} | 9.16392 \times 10^{-14} | 6.1746 \times 10^{-16} | 4.16041 \times 10^{-18} | 2.80326 \times 10^{-20} |
| 0.3      | 2.91257 \times 10^{-11} | 1.96248 \times 10^{-13} | 1.32231 \times 10^{-16} | 8.90963 \times 10^{-18} | 6.00326 \times 10^{-20} |
| 0.4      | 4.94206 \times 10^{-11} | 3.32993 \times 10^{-13} | 2.24369 \times 10^{-15} | 1.51179 \times 10^{-17} | 1.01863 \times 10^{-19} |
| 0.5      | 7.38844 \times 10^{-11} | 4.97829 \times 10^{-13} | 3.35435 \times 10^{-15} | 2.26014 \times 10^{-17} | 1.52287 \times 10^{-19} |
| 0.6      | 1.02022 \times 10^{-10} | 6.8742 \times 10^{-13} | 4.6318 \times 10^{-15} | 3.12088 \times 10^{-17} | 2.10283 \times 10^{-19} |
| 0.7      | 1.33421 \times 10^{-10} | 8.98981 \times 10^{-13} | 6.05728 \times 10^{-15} | 4.08137 \times 10^{-17} | 2.7500 \times 10^{-19} |
| 0.8      | 1.67731 \times 10^{-10} | 1.13017 \times 10^{-12} | 7.61499 \times 10^{-15} | 5.13094 \times 10^{-17} | 3.45720 \times 10^{-19} |
| 0.9      | 2.04658 \times 10^{-10} | 1.37898 \times 10^{-12} | 9.29146 \times 10^{-15} | 6.26054 \times 10^{-17} | 4.21832 \times 10^{-19} |
| 1        | 2.43946 \times 10^{-10} | 1.64369 \times 10^{-12} | 1.10751 \times 10^{-14} | 7.46236 \times 10^{-17} | 5.0281 \times 10^{-19} |

Table 2: Absolute error in the 4-term MDJM solution and exact solution for \( c = 2 \).
It is observed that MDJM solution has less error than other iterative methods described above. Also we have used fewer terms of MDJM series than other methods to approximate the solution.

6. Conclusions

The Boussinesq equation is an important class of PDEs arising in applied science. Various authors proposed different methods to solve these equations. In this article, we proposed a modification to DJM viz. MDJM and used it to solve few Boussinesq equations. It is observed that the MDJM is simpler iterative method than other iterative methods. Further, the solutions obtained by using this new method are good approximation to the exact solutions. This new method can be used to solve different nonlinear problems in a more efficient way.

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7. References

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