SL(2) spin chain and spinning strings on $AdS_5 \times S^5$

S. Bellucci, P.-Y. Casteill, J.F. Morales†, C. Sochichiu∗

INFN – Laboratori Nazionali di Frascati,
Via E. Fermi 40, 00044 Frascati, Italy

† Dipartimento di Fisica Teorica, Universita di Torino.

August 4, 2018

Abstract

We derive the coherent state representation of the integrable spin chain Hamiltonian with symmetry group SL(2, $\mathbb{R}$). By passing to the continuum limit, we find a spin chain sigma model describing a string moving on the hyperboloid SL(2, $\mathbb{R}$)/U(1). The same sigma model is found by considering strings rotating with large angular momentum in $AdS_5 \times S^5$. The spinning strings are identified with semiclassical coherent states built out of SL(2, $\mathbb{R}$) spin chain states.

Contents

1 Introduction 2

2 sl(2) coherent states 3

2.1 sl(2) and its representations ........................................ 3

2.2 Coherent states .................................................. 4

3 Hamiltonian in the coherent state basis 6

4 Spinning strings on $AdS_5 \times S^5$ 9

5 Classical solutions 10

6 Conclusions 12

A Coherent states 14

B Hamiltonian in the CS representation 14

* On leave from: Bogoliubov Lab. Theor. Phys., JINR, 141980 Dubna, Moscow Reg., RUSSIA and Institutul de Fizică Aplicată AŞ, str. Academiei, nr. 5, Chişinău, MD2028 MOLDOVA.
1 Introduction

AdS/CFT correspondence [1–3] relates string theories on AdS spaces and gauge theories living on the AdS boundaries. The typical example is the correspondence between $\mathcal{N} = 4$ SYM and IIB string on $AdS_5 \times S^5$. The duality maps bulk gravity and string fields to excitations of the gauge theory living in the holographic screen and puts in correspondence their dynamics. The two dual descriptions are complementary in the sense that, when spacetime curvatures are large and a low energy supergravity description is reliable, the gauge coupling constant is large and vice versa, when the gauge coupling is small, the supergravity description breaks down and one has to consider the full string theory on AdS. Unfortunately, string theory on AdS, like gauge theories beyond the perturbative regime, is hard to deal with making most tests of the correspondence out of reach. In supersymmetric models which are dealt with, however, one can test the correspondence using BPS states, i.e. states that preserve part of supersymmetry and whose properties are protected by it against quantum corrections, even strong ones (see [4] for a review and a complete list of references). More recently, strings on AdS have been tested against free SYM [5–11], but how to bring interactions into the game is far from clear.

In [12], the authors considered the holographic correspondence near null geodesics of $AdS_5 \times S^5$, where the geometry looks like a pp-wave [13]. On the gauge theory side, this corresponds to consider SYM operators with large $R$-symmetry charge $J$. String theory on a pp-wave is known to be integrable [14, 15], allowing for quantitative tests of the correspondence beyond the supergravity level. This appears possible since corrections to the string theory for states not far from BPS can be interpreted as “semiclassical” [16,17]. Multispin solutions were further found and analyzed in [18–24]. In this approach, energies of classical string solutions are compared to anomalous dimensions of SYM operators with large $J$ charges (see [25] for a review and a complete list of references).

On the other hand there has been an enormous progress in the understanding of $\mathcal{N} = 4$ SYM dynamics. In a series of nice papers [26–28] the planar limit of the dilatation operator for $\mathcal{N} = 4$ SYM was identified with the Hamiltonian of integrable spin chains, while non planar contributions were realized in [29–31] in terms of a joining-splitting spin chain operator mimicking string interactions.

There are two representative sectors of the SYM theory: the su(2) sector associated to scalar impurities (excitations along $S^5$) and the sl(2) sector associated to vector impurities (excitations along $AdS_5$). In [32] the su(2) sector was explored using a spin chain sigma model description, and classical string spinning on $S^5$ were identified with semiclassical coherent states built out of su(2) eigenstates in the spin chain system. This was inspired by the observation that in the thermodynamic limit the spin chain Hamiltonian and higher conserved charges fit those coming from semiclassical string sigma model on $AdS_5 \times S^5$ [33, 34]. An extensive check of the correspondence beyond one loop [35] and the generalization to the compact su(3), so(6) sectors was worked out in [36, 37, 39].

The aim of this paper is to test the non-compact sector of the theory and consider the sl(2) case\(^1\). It corresponds to SYM operators made out of a single scalar and its derivatives along a fixed direction. States in this sector carry charges associated to both the $R$-symmetry SO(6) and conformal SO(4,2) groups. Accordingly sl(2) spin states corresponds to strings spinning in both $S^5$ and $AdS_5$. The preliminary task will be, as in [32], to derive

\(^1\)In this paper sl(2) will always refers to the real form sl(2, $\mathbb{R}$).
a sigma model representation of the spin chain dynamics. The \( \text{su}(2) \) case was well studied, in part because of important applications to condensed matter \cite{40}. Unlike the \( \text{su}(2) \) case, spin chain and, respectively, sigma models with non-compact symmetry groups are less familiar (as far as we know) to the condensed matter literature. Here we combine the coherent state representation of the \( \text{sl}(2) \) algebra and the spin chain Hamiltonian in order to derive a path integral formulation of the gauge dynamics. In the continuum limit this Hamiltonian results into a sigma model describing a string moving on an hyperboloid. \( \text{sl}(2) \) spin chain/string sigma models were considered earlier in \cite{38}.

The plan of the paper is as follows. In Section 2 we review the representations and the construction of coherent states of the \( \text{sl}(2) \) algebra. In Section 3 we derive a coherent state representation of the \( \text{sl}(2) \) spin chain Hamiltonian and a path integral formulation of its dynamics. By passing to the continuum limit we found a two-dimensional sigma model on an hyperboloid. In Section 4 we describe the string duals and match the spin chain sigma model to that following from a string spinning fast on \( S^1_\phi \times S^1_\varphi \) with \( S^1_\phi \in \text{AdS}_5 \) and \( S^1_\varphi \in S^5 \). In Section 5 we draw some conclusions.

\section{sl(2) coherent states}

In this section we collect the main properties of \( \text{sl}(2) \) coherent states. We refer the reader to \cite{41} for details.

\subsection{sl(2) and its representations}

We start by describing the \( \text{sl}(2) \) algebra and its representations. The algebra \( \text{sl}(2) \sim \text{su}(1,1) \sim \text{so}(2,1) \sim \text{sp}(2,\mathbb{R}) \) is non-compact and it is defined by the commutation relations

\begin{align}
\begin{bmatrix}
[J_-, J_+] &= 2J_0, \\
[J_0, J_\pm] &= \pm J_\pm.
\end{bmatrix}
\end{align}

The operators \( J_\pm \) are related to generators \( J_{1,2}, i = 1, 2 \) through

\begin{align}
J_\pm &= J_2 \mp iJ_1,
\end{align}

with the following commutation relations:

\begin{align}
[J_0, J_1] &= iJ_2, \quad [J_1, J_2] = -iJ_0, \quad [J_2, J_0] = iJ_1.
\end{align}

It is not difficult to verify that the quadratic operator

\begin{align}
\hat{C}_2 &= J_0^2 - J_1^2 - J_2^2 = J_0^2 - \frac{1}{2}(J_+J_- + J_-J_+)
\end{align}

commutes with all generators, i.e. it is a Casimir of the algebra. Therefore, it is proportional to unit matrix

\begin{align}
\hat{C}_2 &= j(j - 1)\mathbb{I}.
\end{align}

The number \( j \) labels the irreducible representations. The algebra \( SU(1,1) \) is noncompact, so, unlike the case of \( SU(2) \), all its unitary irreducible representations are infinite-dimensional. The irreducible representations fall into three types of series: principal,
discrete and supplementary. For the discrete series of representations of \( \text{sl}(2) \), the spin \( j \) takes the values \( j = 1, 3/2, 2, 5/2, \ldots \). For the universal covering group \( \widetilde{\text{sl}}(2) \) this is extended to arbitrary positive numbers \( 0 < j < \infty \). The expressions in this section will always refer to the latter type of representations. For applications to SYM gauge theories we will later specify to the case \( j \rightarrow \frac{1}{2} \).

The states in a spin \( j \) representation of \( SU(1,1) \) will be denoted by \( |j,j+m\rangle \) with \( m = 0, 1, 2, \ldots \). The unitarity of the representation implies that one can choose an orthonormal basis for the spin states:

\[
\langle j,m|j,m'\rangle = \delta_{m,m'}.
\]

The representations for the discrete series of \( \text{sl}(2) \) are essentially the analytic continuation of those of \( \text{su}(2) \). The action of the generators on the states of spin \( m \) of representation \( j \) is given by\(^2\)

\[
J_+ |j,j+m\rangle = \sqrt{(m+1)(m+2j)} |j,j+m+1\rangle, \quad (2.5)
\]

\[
J_- |j,j+m\rangle = \sqrt{m(m+2j-1)} |j,j+m-1\rangle, \quad (2.6)
\]

\[
J_0 |j,j+m\rangle = (m+j) |j,j+m\rangle. \quad (2.7)
\]

In contrast to \( \text{su}(2) \) representations the action of \( J_+ \) never ends and therefore the representation is infinite dimensional. We will mainly concern ourselves with the spin \( j = 1/2 \) case

\[
J_+ |m\rangle = (m+1) |m+1\rangle, \quad (2.8)
\]

\[
J_- |m\rangle = m |m-1\rangle, \quad (2.9)
\]

\[
J_0 |m\rangle = (m+\frac{1}{2}) |m\rangle. \quad (2.10)
\]

with \( |m\rangle \equiv |\frac{1}{2}, \frac{1}{2} + m\rangle \) from now on.

## 2.2 Coherent states

(Generalized) coherent states in the Perelomov’s sense are given by the action of a finite group transformation on a particular state \( |\psi_0\rangle \). It is suitable to take this state to be the vacuum, i.e. the lowest weight state \( |0\rangle = |j,j\rangle \) of a spin \( j \) representation of \( \text{sl}(2) \). The coherent state will be parameterized by a point

\[
\bar{n} = (\cosh \rho, \sinh \rho \sin \phi, \sinh \rho \cos \phi)
\]

on the upper sheet of two-sheet hyperboloid:

\[
\bar{n}^2 = n_0^2 - n_1^2 - n_2^2 = 1, \quad n_0 > 0.
\]

We choose the following definition for the coherent state:

\[
|\bar{n}\rangle = D(\bar{n}) |0\rangle = e^{\xi J_+ - \xi^* J_-} |0\rangle \quad \xi = \frac{1}{2} \rho e^{i\phi}. \quad (2.11)
\]

\(^2\)Alternatively the algebra can be represented in the space of functions \( f(z) \) holomorphic within the unit circle (generalized Fock–Bargmann representation) by \( J_+ = z^2 \partial_z + 2jz, J_- = \partial_z, J_0 = z \partial_z + j \).
The element $D(\vec{n})$ is the matrix of the hyperbolic rotation which maps the “north pole” $\vec{n}_0 = (1, 0, 0)$ into the point $\vec{n}$ on the hyperboloid (see Figure \ref{fig:n0n} below). The operators $D(\vec{n})$ do not form a group, but the multiplication law is $D(\vec{n}_1)D(\vec{n}_2) = D(\vec{n}_3)\exp(\Phi(\vec{n}_1, \vec{n}_2)J_0)$ where $\Phi(\vec{n}_1, \vec{n}_2)$ is the area of the hyperbolic triangle with vertices $\vec{n}_1, \vec{n}_2$ and $\vec{n}_0 = (1, 0, 0)$.

Next we list the main properties of the coherent states $|\vec{n}\rangle$.

1) Coherent state content:
Using (2.5), coherent states can be expanded in the spin state basis $|j, j + m\rangle$ building the $\mathfrak{sl}(2)$ representation. The expansion coefficients are given in [41]:

$$|\vec{n}\rangle = \sum_{m=0}^{\infty} c_{j,m} |j, j + m\rangle,$$

$$c_{j,m} = \frac{1}{\cosh(\frac{\rho}{2})^{2j}} \frac{\Gamma(m + 2j)}{m! \Gamma(2j)} \frac{1}{2} e^{im\phi} \tanh(\frac{\rho}{2})^m. \quad (2.12)$$

The expansion is such that

$$\langle \vec{n} | \vec{n} \rangle = 1.$$ 

We are mostly interested in the particular case $j = \frac{1}{2}$. The expansion coefficients drastically simplifies in this case

$$|\vec{n}\rangle = \frac{1}{\cosh(\frac{\rho}{2})} \sum_{m=0}^{\infty} e^{im\phi} \tanh(\frac{\rho}{2})^m |m\rangle, \quad (2.13)$$

where as before $|m\rangle \equiv |\frac{1}{2}, \frac{1}{2} + m\rangle$.

2) Coherent states are over-complete.
Using (2.12) it is not difficult to show that the unity operator $\mathbb{I}$ can be written in terms of the coherent state $|\vec{n}\rangle$ as

$$\mathbb{I} = \frac{2j - 1}{4\pi} \int \sinh \rho \, d\rho \, d\phi \, |\vec{n}\rangle \langle \vec{n} | \equiv \int d^2n \, |\vec{n}\rangle \langle \vec{n} |. \quad (2.14)$$

Notice that in this formula the point $j = \frac{1}{2}$ is singular but the limit $j \to \frac{1}{2}$ is well defined (see appendix \ref{app:details} for details). In the rest this limit is always assumed.

3) Coherent states are not orthonormal

$$\langle \vec{n}_1 | \vec{n}_2 \rangle = \left( \cosh(\frac{\rho_1}{2}) \cosh(\frac{\rho_2}{2}) - \sinh(\frac{\rho_1}{2}) \sinh(\frac{\rho_2}{2}) e^{i(\phi_2 - \phi_1)} \right)^{2j}. \quad (2.15)$$

4) To each coherent state $|\vec{n}\rangle$ we can associate a point $\vec{n}$ on the hyperboloid via the important property

$$\langle \vec{n} | \vec{J} | \vec{n} \rangle = j \vec{n}.$$ 

This property is demonstrated in appendix \ref{app:details} In particular, we notice that

$$\langle \vec{n} | J_0 | \vec{n} \rangle = j \cosh \rho.$$ 

This means that $\cosh \rho$ measures the “average spin” of the coherent state.
3 Hamiltonian in the coherent state basis

In this section we derive a coherent state representation of the sl(2) spin chain Hamiltonian and a path integral formulation of the spin chain dynamics. We follow closely [40] where the derivation of the su(2) ferromagnetic sigma model is presented in full details. Unlike the more familiar su(2) case, sl(2) is a non-compact algebra and its representations are infinite-dimensional. This will result into a non-polynomial Hamiltonian.

Here we consider the spin chain describing one-loop planar anomalous dimensions for SYM operators in the sl(2) sector. The Hamiltonian is given by [28]

$$ H = \sum_{k=1}^{L} H_{k,k+1} $$

with

$$ H_{k,k+1} |m_{k1}m_{k2}\rangle = [h(m_{k1}) + h(m_{k2})] |m_{k1}m_{k2}\rangle 
- \sum_{l=1}^{m_{k1}} \frac{1}{2} |m_{k1} - l, m_{k2} + l\rangle 
- \sum_{l=1}^{m_{k2}} \frac{1}{2} |m_{k1} + l, m_{k2} - l\rangle \quad (3.1) $$

and

$$ |m_{k1}m_{k2}\rangle \equiv |m_{k1}\rangle \otimes |m_{k1}\rangle , \quad h(m) = \sum_{i=1}^{m} \frac{1}{i} . $$

The underscripts $k_i = 1, 2, \ldots, L$ specify the spin chain site where the corresponding state $|m_{k_i}\rangle$ leaves.

The Hamiltonian (3.1) has a nice representation in the coherent state basis $|\vec{n}\rangle$. Indeed, using (2.13) one can compute the average of $H_{k,k+1}$ over two-site coherent states $|\vec{n}_k\vec{n}_{k+1}\rangle \equiv |\vec{n}_k\rangle \otimes |\vec{n}_{k+1}\rangle$. This was first done in [38] (see appendix B for an independent derivation). The result reads:

$$ \langle \vec{n}_k\vec{n}_{k+1} | H_{k,k+1} | \vec{n}_k\vec{n}_{k+1}\rangle = \log \left( \frac{1 + \vec{n}_k\vec{n}_{k+1}}{2} \right) = \log \left( 1 - \frac{(\vec{n}_k - \vec{n}_{k+1})^2}{4} \right) . \quad (3.2) $$

Here and below the dot product is defined via

$$ \vec{n} \cdot \vec{m} = n_0m_0 - n_1m_1 - n_2m_2 . $$

In order to pass to path integral representation, one considers the Hamiltonian as the generator of time translations in the spin chain system and discretizes the time interval $\Delta T$. At a given time $t_s$ we introduce the “spin field vector”

$$ |\vec{n}_{s}\rangle \equiv |\vec{n}_{1,s}\rangle \otimes |\vec{n}_{2,s}\rangle \otimes \ldots \otimes |\vec{n}_{L,s}\rangle \quad (3.3) $$

with $\vec{n}_{k,s} \equiv \vec{n}_k(t_s)$ specifying the spin chain configuration at the $k$th site and time $t_s$. Each $\vec{n}_k$ describes therefore a trajectory on the hyperboloid. Alternatively, the “L-plet” \{\vec{n}_k(t_s)\} specifies the spin chain profile at a fixed time $t_s$. 

6
Figure 1: The vector $\vec{n}_k$ moving on the hyperboloid.

From (3.2) one then finds

$$H_{kk+1}(t_s) \equiv \langle \vec{n}_{s-1} | H_{kk+1} | \vec{n}_s \rangle = \log \left( 1 - \frac{(\vec{n}_{k,s} - \vec{n}_{k+1,s})^2}{4} \right).$$  (3.4)

The evolution operator is then represented as a product over small intervals of time $\delta t = \Delta T/M$. Finally we insert the expression of unity (2.14) and then take the limit $\delta t \to 0$

$$Z = \text{Tr} e^{-i\Delta T \tilde{\lambda} H} = \text{Tr} \int \prod_{s=1}^M d^2 n_s \langle \vec{n}_{s-1} | e^{-i\delta t \tilde{\lambda} H} | \vec{n}_s \rangle$$

$$= \int [dn] e^{i\mathcal{S}(\vec{n})}, \quad [dn] \equiv \prod_s d^2 n_s.$$  (3.5)

To the leading order in $\delta t$, one finds

$$\langle \vec{n}_{s-1} | e^{-i\tilde{\lambda} \delta t H} | \vec{n}_s \rangle = \langle \vec{n}_{s-1} | \vec{n}_s \rangle (1 - i \tilde{\lambda} \delta t \langle \vec{n}_{s-1} | H | \vec{n}_s \rangle + \ldots)$$

$$= 1 + i \delta t \sum_k \left[ \frac{1}{2} (\cosh \rho_{s,k} - 1) \dot{\phi}_{s,k} - \tilde{\lambda} \langle \vec{n}_s | H_{kk+1} | \vec{n}_s \rangle \right] + \ldots$$

$$\approx \exp \left( i \delta t \sum_k \left[ \frac{1}{2} (\cosh \rho_k - 1) \dot{\phi}_k - \tilde{\lambda} H_{kk+1} \right] (t_s) \right),$$  (3.6)

where we have used (2.15) to rewrite

$$\langle \vec{n}_{s-1} | \vec{n}_s \rangle \approx 1 + \frac{i \delta t}{2} \sum_k (\cosh \rho_k - 1) \dot{\phi}_k$$

with $\vec{n}_s = \vec{n}_{s-1} + \delta t \vec{n}_s + \ldots$.

Taking the limit $\delta t \to 0$, one finds for the action

$$S = \int dt \sum_{k=1}^L \left[ \frac{1}{2} \dot{\phi}_k (\cosh \rho_k - 1) - \tilde{\lambda} H_{kk+1} \right] (t).$$  (3.7)
As in the su(2) case, the unconventional first-derivative term in (3.7) is a Wess-Zumino like term. Indeed, it can be written as a boundary term in one higher dimension \( y \in [0,1] \). Introducing a dependence of the coherent state \( \vec{n}_k = \vec{n}_k(t,y) \) on the extra dimension \( y \) and using

\[
\vec{n} \cdot (\partial_y \vec{n} \times \partial_t \vec{n}) = \sinh \rho (\partial_y \phi \partial_t \rho - \partial_t \phi \partial_y \rho) = \partial_t [(\cosh \rho - 1)\partial_y \phi] - \partial_y [(\cosh \rho - 1)\partial_t \phi] \tag{3.8}
\]

one has for the Wess-Zumino term\(^3\):

\[
S_{WZ} \equiv \frac{1}{2} \int dt \sum_{k=1}^L \dot{\phi}_k (\cosh \rho_k - 1) = -\frac{1}{2} \sum_k \int dt \int_0^1 dy \vec{n}_k \cdot (\partial_y \vec{n}_k \times \partial_t \vec{n}_k). \tag{3.9}
\]

The two sides in eq. (3.9) agree for the boundary condition choice \( \vec{n}_k(t,0) \equiv \vec{n}_k(t), \vec{n}_k(t,1) \equiv \vec{n}_0, \partial_y \phi|_{t=\Delta T} = \partial_y \phi|_{t=0} = 0 \).

For each site, the double integral (3.9) is nothing more than the spin \( j = \frac{1}{2} \) times the area \( A \) between the trajectory of \( \vec{n} \) and the “north pole” \( \vec{n}_0 = (1,0,0) \) (see Figure (1)).

In addition, \( S_{WZ} \) generates the right Poisson brackets

\[
\{n_\alpha, n_\beta\} = \epsilon_{\alpha\beta\gamma} n_\gamma
\]

which reproduce the sl(2) commutation relations upon quantization.

The action (3.7) with the two-site Hamiltonian defined as in (3.2) looks like a discretization of a hyperboloid sigma model. The correspondence can be made precise by going to the continuum limit \( a = \frac{1}{L} \to 0 \)

\[
\vec{n}_k \to \vec{n}(\sigma)|_{\sigma=ka} \quad \sum_{k=1}^L \to \frac{1}{a} \int_0^1 d\sigma.
\]

One finds

\[
\tilde{\lambda} \sum_k H_{kk+1}(t) \to -\frac{\tilde{\lambda}}{4a} \int d\sigma a^2 (\partial_\sigma \vec{n})^2 = \frac{\tilde{\lambda}}{4L} \int d\sigma \left[ (\partial_\sigma \rho)^2 + \sinh^2 \rho (\partial_\sigma \phi)^2 \right] \tag{3.10}
\]

leading to

\[
S = \frac{L}{2} \int d\sigma d\tau \left( (\cosh \rho - 1) \dot{\phi} - \frac{\tilde{\lambda}}{2L^2} \left[ (\partial_\sigma \rho)^2 + \sinh^2 \rho (\partial_\sigma \phi)^2 \right] \right). \tag{3.11}
\]

The same result will be found below by considering semiclassical strings spinning on \( \text{AdS}_5 \times S^5 \). A coherent spin chain state will be specified by its spin

\[
S_z = \sum_{k=1}^L \langle \vec{n}_k | J_0 | \vec{n}_k \rangle \to \frac{L}{2} \int d\sigma \cosh \rho. \tag{3.12}
\]

\(^3\)Here and below \( \vec{a} \cdot (\vec{b} \times \vec{c}) = \epsilon_{\alpha\beta\gamma} a^\alpha b^\beta c^\gamma, \ n \cdot m = n^\alpha m_\alpha \) with indices raising and lowering with the metric \( \eta_{\alpha\beta} = (+ - -) \) .
4 Spinning strings on $AdS_5 \times S^5$

Here we describe the string duals of excitations in the sl(2) spin chain system. We follow closely [32] where analogous results were found for strings on $S^5$ dual to su(2) spin states. Here sl(2) coherent states will be associated to classical solutions of the string equations with non-trivial angular velocities both in $S^5$ and $AdS_5$. We refer the reader to [38] for an independent study of these solutions and [25] for a review and a list of references on spinning strings on $AdS_5 \times S^5$.

The bosonic part of Polyakov action describing a string moving on $AdS_5 \times S^5$ can be written as

$$S = \frac{R^2}{4\pi \alpha'} \int g_{MN} (\partial^X \partial^X - \partial^\sigma \partial^\sigma)$$

(4.1)

with

$$ds^2 = g_{MN} dX^M dX^N = ds^2_{AdS_5} + ds^2_{S^5}$$

and

$$ds^2_{AdS_5} = d\rho^2 - \cosh^2 \rho d\rho_0^2$$
$$ds^2_{S^5} = d\gamma^2 + \cos^2 \gamma d\varphi^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\varphi_1^2 + \sin^2 \psi d\varphi_2^2)$$

(4.2)

String states are classified by charges with respect to the Cartan generators $S_{1,2,3}$ and $J_{1,2,3}$ of the SO(4,2) and SO(6) isometry groups of $AdS_5$ and $S^5$ respectively. In the spacetime theory they correspond to shifts of $\phi_{1,2,3}$ and $\varphi_{1,2,3}$ coordinates respectively

$$S_{1,2} = \partial_{\phi_{1,2}}, \quad E = \partial_t, \quad J_{1,2,3} = \partial_{\varphi_{1,2,3}}.$$  

(4.3)

We are interested in string duals of sl(2) SYM operators made out of a single scalar $\phi$ and its derivatives. To the direction $\phi$ we associate a circle on $S^5$ parametrized by $\varphi_3$ while derivatives correspond to excitations along a circle in $AdS_5$ parametrized by $\phi_1$. The remaining excitations will be turned off. The other choices of the sl(2) sector are related to this one via SO(4,2) $\times$ SO(6) rotations.

More precisely we look for solutions of the string equations following from (4.1) with

$$\gamma = \theta = 0 \quad \rho = \rho(\sigma, \tau) \quad \phi_1 = \phi_1(\sigma, \tau) \quad \varphi_3 = \varphi_3(\sigma, \tau) \quad t = \kappa \tau.$$  

(4.4)

In addition, the solutions must satisfy the Virasoro constraints

$$g_{MN} \partial^X \partial^X = 0$$
$$g_{MN} (\partial^X \partial^X + \partial^\sigma \partial^\sigma) = 0.$$  

(4.5)

As in [32] we consider the limit $\kappa \to \infty$. To this end it is convenient to rewrite the metric in coordinates where $g_{tt} = 0$. This can be done by the following change of variables:

$$\phi_1 = \phi + t \quad \varphi_3 = \varphi + t \quad \rho_0 = \frac{1}{2} \rho.$$  

(4.6)

The metric in the new variables reads (at $\gamma = \theta = 0$)

$$ds^2 = \frac{1}{4} d\rho^2 + \frac{1}{2} (\cosh \rho - 1) d\phi^2 + d\varphi^2 + dt \left[ 2 d\varphi + (\cosh \rho - 1) d\phi \right].$$  

(4.7)
Then we consider the limit

$$\kappa \to \infty \quad \text{with} \quad \kappa \partial_\tau X^M \neq t \quad \text{fixed} \quad X^M = \phi, \varphi, \rho.$$  

(4.8)

Notice that in the original variables the limit corresponds to $\partial_\tau \varphi_3 \approx \partial_\tau \phi_1 \approx k \gg 1$, i.e. the string spins fast on $S^1_{\varphi_3} \subset S^5$ and $S^1_\phi \subset \text{AdS}_5$. This is expected from holography since states in the sl(2) sector carry charges $J_3 \in SO(6)$ and $E, S_1 \in SO(4,2)$.

To the leading order in $\kappa$, the first of the Virasoro constraints in (4.5) reads

$$\kappa [2 \partial_\sigma \varphi + (\cosh \rho - 1) \partial_\sigma \phi] = 0,$$

(4.9)

which can be used to solve $\partial_\sigma \varphi$ in favor of $\partial_\sigma \phi$.

Evaluating (4.1) and using (4.7,4.9), one finds (to the leading order in $\kappa$)

$$S = \frac{R^2}{4\pi \alpha'} \int d\sigma d\tau \left( \kappa [(\cosh \rho - 1) \partial_\tau \phi + 2 \partial_\tau \varphi] - \frac{1}{4} [(\partial_\sigma \rho)^2 + \sinh^2 \rho (\partial_\sigma \phi)^2] \right).$$

(4.10)

Finally identifying

$$L = \frac{R^2 \kappa}{2\pi \alpha'} \quad \tilde{\lambda} = \frac{R^4}{8\pi^2 \alpha'^2}$$

and changing to variable $t = k \tau$ one finds

$$S = \frac{L}{2} \int d\sigma dt \left( (\cosh \rho - 1) \partial_t \phi + 2 \partial_t \varphi - \frac{\tilde{\lambda}}{2L^2} [(\partial_\sigma \rho)^2 + \sinh^2 \rho (\partial_\sigma \phi)^2] \right).$$

(4.11)

Besides the $\varphi$-dependent total derivative term, the result (4.11) is in perfect agreement with the spin chain sigma model result (3.11). It can be compared with the result in [32] corresponding to strings spinning on $S^5$. The two results differ (up to total derivative terms) for the replacing of trigonometric with hyperbolic functions.

5 Classical solutions

The classical string solutions following from the string/spin chain action (4.11) can be found via analytic continuation from those in [32]. In this section we sketch the results and refer the reader to [32] for details. In addition we will work out in detail classical solutions associated to string sitting at the bottom end of the hyperboloid where BMN frequencies and energies are reproduced.

The classical equations of motions are

$$\sinh \rho \partial_\tau \rho - \frac{\tilde{\lambda}}{L^2} \partial_\sigma (\partial_\sigma \phi \sinh^2 \rho) = 0$$

$$\sinh \rho \partial_\tau \phi + \frac{\tilde{\lambda}}{L^2} \left[ \partial_\sigma^2 \rho - \frac{1}{2} \sinh(2\rho)(\partial_\sigma \phi)^2 \right] = 0.$$  

(5.1)

The simplest solutions of eqs. (5.1) are found by taking $\partial_\sigma \phi = 0$. The two equations then imply $\partial_\tau \rho = 0$ and $\partial_\tau^2 \phi = 0$ i.e. $\rho = \rho(\sigma)$ and $\partial_\tau \phi = \omega$ respectively. The solution

4Notice that both integrands go like $\frac{1}{\kappa^2}$ in the limit (4.8) and therefore the action is finite.
then represents a string with profile $\rho(\sigma)$, rotating at constant velocity $\omega$ on a circle inside $AdS_5$. The profile $\rho(\sigma)$ is determined by the second equation in (5.1)

$$\omega \sinh \rho + \frac{\tilde{\lambda}}{L^2} \rho^2 = 0$$

solved by

$$\partial_\sigma \rho = \pm \sqrt{a - b \cosh \rho} \quad a = \text{const} \quad b = \frac{2 L^2}{\tilde{\lambda}} \omega$$

with $a$ denoting an integration constant. Solutions exist only for $a > b$. In this case one can take for $\rho(\sigma)$ an oscillating solution between $\pm \rho_{\text{max}}$ with $\rho_{\text{max}} = \text{Arccosh} \frac{a}{b}$.  

The energy $E$ and spin $S_z$ of the string/spin chain state are given by (3.10,3.12)

$$S_z = \frac{L}{2} \int_0^1 \cosh \rho \, d\sigma = 2 L \int_0^{\rho_{\text{max}}} \frac{\cosh \rho}{\sqrt{a - b \cosh \rho}} \, d\rho$$

$$E = \frac{\tilde{\lambda}}{4L} \int_0^1 (\partial_\sigma \rho)^2 = \frac{\tilde{\lambda}}{L} \int_0^{\rho_{\text{max}}} \sqrt{a - b \cosh \rho} \, d\rho$$

$$= -2 \sqrt{2} b \frac{\tilde{\lambda}}{L} [E(x) - (1 - x)K(x)] ,$$

in terms of the elliptic integrals

$$K(x) = \sqrt{\frac{b}{2}} \int_0^{\rho_{\text{max}}} \frac{1}{\sqrt{a - b \cosh \rho}} \, d\rho \quad x = \frac{b - a}{2 b}$$

$$E(x) = \frac{1}{2} \sqrt{\frac{b}{2}} \int_0^{\rho_{\text{max}}} \frac{1 + \cosh \rho}{\sqrt{a - b \cosh \rho}} \, d\rho .$$

Eqs. (5.4) has to be evaluated on $x$, determined by

$$1 = \int_0^1 d\sigma = 4 \int_0^{\rho_{\text{max}}} \frac{1}{\sqrt{a - b \cosh \rho}} \, d\rho = 4 \sqrt{\frac{2}{b}} K(x) .$$

Eqs. (5.4,5.6) describe a string/spin chain solution of length $L$, total spin $S_z$ and anomalous dimension/energy $E$.

It is instructive to consider solutions localized near the bottom end of the hyperboloid with $\rho_{\text{max}} \ll 1$. In this limit one can consider the linearized version of (5.1)

$$\rho \partial_\rho \rho - \frac{\tilde{\lambda}}{L^2} \rho^2 = 0$$

$$\rho \partial_\phi \rho + \frac{\tilde{\lambda}}{L^2} [\partial_\rho^2 \rho - \rho (\partial_\phi)^2] = 0 .$$

Notice that in coordinates where $\phi \in [0,2\pi]$ the points $(\rho, \phi)$ and $(-\rho, \phi + \pi)$ are identified.
It is interesting to note that similar equations appear also in the case of $su(2)$ sector with $\rho \to \theta, \phi \to \varphi$. This is the clear case because at low scales $\rho_{\text{max}}$ the string does not distinguish the hyperboloid from the sphere.

As before, Eq. (5.7) can be easily solved by taking $\partial_\sigma \phi = 0$. The system reduces to the harmonic oscillator equation

$$\partial_\sigma^2 \rho + \nu^2 \rho = 0, \quad \nu^2 \equiv \frac{\omega L^2}{\lambda},$$

$$\partial_\sigma \phi = \omega \quad \partial_\rho = 0. \quad (5.8)$$

For $\omega > 0$ this leads to the harmonic solution

$$\rho = \rho_{\text{max}} \cos \nu (\sigma - \sigma_0).$$

Requirement of periodicity in $\sigma$: $\rho(\sigma + 1) = \rho(\sigma)$ leads to quantization condition for $\nu$

$$\nu_n = 2\pi n, \quad n \in \mathbb{Z}. \quad (5.9)$$

For the above solution is not difficult to compute the integrals of motion of interest:

$$S_z = \frac{L}{2} \int_0^1 d\sigma \left(1 + \frac{1}{2} \rho^2 + \ldots\right) = \frac{L}{2} + \frac{L \rho_{\text{max}}^2}{4} + \ldots \quad (5.10)$$

$$E = \frac{\tilde{\lambda}}{4L} \int_0^1 d\sigma (\partial_\sigma \rho)^2 = \frac{\tilde{\lambda} \rho_{\text{max}}^2 \nu_n^2}{8L} = \frac{\rho_{\text{max}}^2 \omega_n L}{8L}. \quad (5.11)$$

From (5.10) one can express $\rho_{\text{max}}$ in terms of the spin $s = S_z - L/2^6$. Then the energy/anomalous dimension of a string/spin chain state at level $n$ and spin $s$ is given by

$$E_{s,n} = 4\pi^2 n^2 s \frac{\tilde{\lambda}}{L^2}. \quad (5.12)$$

Formula (5.12) is in perfect agreement with the expectations coming from BMN analysis. It would be interesting to compare this equation to the lowest energy levels in $sl(2)$ spin chain produced by Bethe ansatz. Let us note also that the same equation for energy is valid for the $su(2)$ sector as well.

### 6 Conclusions

In this paper we derived a coherent state representation for the Hamiltonian of a spin chain with symmetry group $sl(2)$. Coherent states for $sl(2)$ are in one-to-one correspondence with points on a hyperboloid and carry a natural coset structure $sl(2)/u(1)$. The result for the $sl(2)$ Hamiltonian can be compared with its $su(2)$ analog

$$H_{su(2)} = \frac{1}{2} \sum_k (\vec{n}_k - \vec{n}_{k+1})^2$$

$$H_{sl(2)} = \sum_k \log \left(1 - \frac{(\vec{n}_k - \vec{n}_{k+1})^2}{4}\right) \quad (6.1)$$

---

$^6$Notice that in our conventions vacuum corresponds to the state with minimal total spin $L/2$. 

12
with \( \vec{n} \) living on \( S^2 \) for su(2) and on a two-dimensional hyperboloid for sl(2). The two results are similar\(^7\) but there are important differences. First, by being sl(2) non-compact, representations are infinite-dimensional and therefore coherent states involve a linear combination of an infinite number of states. Second, in contrast with the su(2) case, the sl(2) Hamiltonian has a non-polynomial character.

In the continuum limit (where the number of chain sites gets large), both Hamiltonians reduce to that of string sigma models on the sphere/hyperboloid spaces. We derive a path integral formulation of the sl(2) spin chain dynamics. As in the su(2) the resulting Lagrangian contains a Wess–Zumino term, which ensures the right commutation relations between the hyperboloid coordinates upon quantization.

Spin chain coherent states are identified with strings spinning fast on a torus \( S^1_\phi \times S^1_\psi \) with \( S^1_\phi \in AdS_5 \) and \( S^1_\psi \in S^5 \). The spin chain and string sigma models are shown to be in perfect agreement in consistency with the results of [38]. In addition we consider classical solutions of the string/spin chain sigma model actions.

As expected, the results are often related to those in su(2) via an analytic continuation.

There are several interesting directions to go on. One can extend the analysis here either for higher number of loops, or by considering the first non-planar corrections where the hamiltonian (although no longer integrable) is known. In addition, it would nice to generalize our results to the case of the psu(2,2|4) spin chain describing \( \mathcal{N} = 4 \) SYM. At this stage the generalization looks conceptually straightforward although technically involved.

The sl(2) symmetry is common to any (asymptotically) free gauge theory since it is part of the conformal group in arbitrary dimensions, for instance see [42, 43] for studies in the framework of QCD and [44] for a review of recent development and a complete list of references. We believe that our results here can be helpful in further studies along these lines. It would be nice to understand whether the results here generalize to gauge theories in \( d \neq 4 \) and their string dual on warped AdS spaces [45–48].

**Acknowledgements**

We thank A. Tseytlin for pointing out our attention to the recent paper [38] where part of the results here were independently found. We thank E. Orazi and H. Samtleben for discussions and G. Arutyunov and A. Belitsky for correspondence. This work was partially supported by NATO Collaborative Linkage Grant PST.CLG. 97938, INTAS-00-00254 grant, INTAS-00-00262, RF Presidential grants MD-252.2003.02, NS-1252.2003.2, INTAS grant 03-51-6346, RFBR-DFG grant 436 RYS 113/669/0-2, RFBR grant 03-02-16193 and the European Community’s Human Potential Programme under contract HPRN-CT-2000-00131.

\(^7\)As expected they are related to each other via an analytic continuation.
A Coherent states

We prove here important relations concerning the coherent states defined in Section 2. These are defined with the infinite series

$$|\vec{n}(\rho, \phi)\rangle = \sum_{m=0}^{\infty} c_{j,m}(\rho, \phi) \left| j, j + m \right\rangle,$$

$$c_{j,m}(\rho, \phi) = \frac{1}{\cosh(\frac{\rho}{2})^{2j}} \left( \frac{\Gamma(m + 2j)}{m! \Gamma(2j)} \right)^{\frac{1}{2}} e^{i\phi} \tanh(\frac{\rho}{2})^m.$$  

Using

$$\sum_{m=0}^{\infty} \frac{\Gamma(m + 2j)}{m! \Gamma(2j)} x^m = (1 - x)^{-2j},$$

it is easy to prove that

$$\langle \vec{n} | \vec{n} \rangle = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} c_{j,m}^* c_{j,m} = \cosh(\frac{\rho}{2})^{-2j} (1 - \tanh^2(\frac{\rho}{2}))^{-2j} = 1$$

$$\langle \vec{n} | J_0 | \vec{n} \rangle = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} c_{j,m}^* c_{j,m} (m + j) = j \cosh(\frac{\rho}{2})^{-2j} \frac{1 + \tanh^2(\frac{\rho}{2})}{(1 - \tanh^2(\frac{\rho}{2}))^{2j+1}} = j \cosh \rho$$

$$\langle \vec{n} | J_+ | \vec{n} \rangle = \sum_{m=1}^{\infty} \sum_{j=0}^{\infty} c_{j,m}^* c_{j,m-1} \sqrt{m(m - 1 + 2j)} = j \frac{e^{-i\phi}}{\cosh(\frac{\rho}{2})^{2j}} \frac{2 \tanh(\frac{\rho}{2})}{(1 - \tanh(\frac{\rho}{2}))^{2j+1}} = j e^{-i\phi} \sinh \rho$$

$$\langle \vec{n} | J_- | \vec{n} \rangle = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} c_{j,m}^* c_{j,m+1} (m + 1)(m + 2j) = j \frac{e^{i\phi}}{\cosh(\frac{\rho}{2})^{2j}} \frac{2 \tanh(\frac{\rho}{2})}{(1 - \tanh(\frac{\rho}{2}))^{2j+1}} = j e^{i\phi} \sinh \rho$$

In order to prove the resolution of unity (2.14), we act with its l.h.s on an arbitrary state $|j, m\rangle$:

$$\int d\rho d\phi \sinh \rho \langle \vec{n} | \left| j, j + m \right\rangle = \sum_{k=0}^{\infty} \int d\rho d\phi \sinh \rho \frac{\tanh(\frac{\rho}{2})^{k+m}}{\cosh(\frac{\rho}{2})^{2k}} \sqrt{\frac{\Gamma(m + 2j) \Gamma(k + 2j)}{k! m! \Gamma(2j)^2}} e^{i(k-m)\phi} |j, j + k\rangle$$

$$= 2 \frac{\Gamma(m + 2j)}{m! \Gamma(2j)} \int_0^\infty d\rho \sinh \rho \frac{\tanh(\frac{\rho}{2})^{2m}}{\cosh(\frac{\rho}{2})^{4j}} |j, j + m\rangle$$

$$= 8 \frac{\Gamma(m + 2j)}{m! \Gamma(2j)} \int_0^\infty d\tau \frac{(\sinh \tau)^{2m+1}}{(\cosh \tau)^{4j+2m-1}} |j, j + m\rangle .$$

As

$$\int_0^\infty d\rho \frac{(\sinh \rho)^A}{(\cosh \rho)^B} = \frac{\Gamma(\frac{A+1}{2})\Gamma(\frac{B-A}{2})}{2 \Gamma(\frac{B+1}{2})}$$

for $A < B$, one ends with

$$\int d\rho d\phi \sinh \rho \langle \vec{n} | \left| j, j + m \right\rangle = (2j - 1) \frac{\Gamma(2j - 1)}{\Gamma(2j)} |j, j + m\rangle = |j, j + m\rangle$$

for any $j > 1/2$. Notice that also the limit $j \to 1/2$ is well defined. This proves the identity (2.14).

B Hamiltonian in the CS representation

We restrict ourselves here to the $j = \frac{1}{2}$ case. The two-site Hamiltonian can then be rewritten as

$$H_{k_1 k_2} |m_1 m_2\rangle = (h(m_1) + h(m_2)) |m_1 m_2\rangle - \sum_{l=1}^{m_1} \frac{1}{l} |m_1 - l, m_2 + l\rangle - \sum_{l=1}^{m_2} \frac{1}{l} |m_1 + l, m_2 - l\rangle .$$
Acting on coherent states, it gives

$$H_{k_1 k_2} |\vec{n}_1, \vec{n}_2\rangle = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} c_{m_1}(\rho_1, \phi_1) c_{m_2}(\rho_2, \phi_2) \left[ (h(m_1) + h(m_2)) |m_1, m_2\rangle 
- \sum_{l=1}^{m_2} \frac{1}{l} |m_1 + l, m_2 - l\rangle - \sum_{l=1}^{m_1} \frac{1}{l} |m_1 - l, m_2 + l\rangle \right].$$

The sums over $l$ can be extended to infinity using the relation

$$\sum_{m=1}^{\infty} \sum_{l=1}^{m} = \sum_{M=m-l=0}^{\infty} \sum_{l=1}^{\infty}.$$

We get:

$$\langle \vec{n}_1 \vec{n}_2 | H_{12} | \vec{n}_1 \vec{n}_2\rangle = \frac{1}{\cosh(\frac{\rho_1}{2})^2 \cosh(\frac{\rho_2}{2})^2} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \tanh(\frac{\rho_1}{2})^{2m_1} \tanh(\frac{\rho_2}{2})^{2m_2} (h(m_1) + h(m_2))$$

$$- \frac{1}{\cosh(\frac{\rho_1}{2})^2 \cosh(\frac{\rho_2}{2})^2} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{l=1}^{\infty} \sum_{l=1}^{\infty} e^{-i l (\phi_1 - \phi_2)} \tanh(\frac{\rho_1}{2})^{2m_1+l} \tanh(\frac{\rho_2}{2})^{2M_2+l}$$

$$- \frac{1}{\cosh(\frac{\rho_1}{2})^2 \cosh(\frac{\rho_2}{2})^2} \sum_{M_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{l=1}^{\infty} \sum_{l=1}^{\infty} e^{i l (\phi_1 - \phi_2)} \tanh(\frac{\rho_1}{2})^{2M_1+l} \tanh(\frac{\rho_2}{2})^{2m_2+l}.$$

The first term sums to

$$\log \left( \cosh(\frac{\rho_1}{2})^2 \cosh(\frac{\rho_2}{2})^2 \right),$$

the second and third terms to

$$\log \left( 1 - e^{\mp i (\phi_1 - \phi_2)} \tanh(\frac{\rho_1}{2}) \tanh(\frac{\rho_2}{2}) \right).$$

Adding the three logarithms yields finally

$$\langle \vec{n}_1, \vec{n}_2 | H_{12} | \vec{n}_1, \vec{n}_2\rangle = \log \frac{1}{2} (1 + \cosh \rho_1 \cosh \rho_2 - \cos(\phi_1 - \phi_2) \sinh \rho_1 \sinh \rho_2)$$

$$= \log \frac{1}{2} (1 + \vec{n}_1.\vec{n}_2)$$

$$= \log \left( 1 - \frac{(\vec{n}_1 - \vec{n}_2)^2}{4} \right)$$

where as before the product is defined via

$$\vec{n} \cdot \vec{m} = n_0 m_0 - n_1 m_1 - n_2 m_2.$$

References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* 2 (1998) 231–252. **[hep-th/9711200]**
[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109](https://arxiv.org/abs/hep-th/9802109)

[3] E. Witten, “Anti-de sitter space and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150](https://arxiv.org/abs/hep-th/9802150)

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, “Large N field theories, string theory and gravity,” *Phys. Rept.* **323** (2000) 183–386, [hep-th/9905111](https://arxiv.org/abs/hep-th/9905111)

[5] M. Bianchi, J. F. Morales, and H. Samtleben, “On stringy $AdS_5 \times S^5$ and higher spin holography,” [hep-th/0305052](https://arxiv.org/abs/hep-th/0305052)

[6] N. Beisert, M. Bianchi, J. F. Morales, and H. Samtleben, “On the spectrum of AdS/CFT beyond supergravity,” *JHEP* **02** (2004) 001, [hep-th/0310292](https://arxiv.org/abs/hep-th/0310292)

[7] R. Gopakumar, “From free fields to AdS,” *Phys. Rev.* **D70** (2004) 025009, [hep-th/0308184](https://arxiv.org/abs/hep-th/0308184)

[8] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, “The hagedorn / deconfinement phase transition in weakly coupled large n gauge theories,” [hep-th/0310285](https://arxiv.org/abs/hep-th/0310285)

[9] R. Gopakumar, “From free fields to AdS. II,” *Phys. Rev.* **D70** (2004) 025010, [hep-th/0402063](https://arxiv.org/abs/hep-th/0402063)

[10] N. Beisert, M. Bianchi, J. F. Morales, and H. Samtleben, “Higher spin symmetry and $\mathcal{N} = 4$ SYM,” *JHEP* **07** (2004) 058, [hep-th/0405057](https://arxiv.org/abs/hep-th/0405057)

[11] G. Bonelli, “On the boundary gauge dual of closed tensionless free strings in AdS,” [hep-th/0407144](https://arxiv.org/abs/hep-th/0407144)

[12] D. Berenstein, J. M. Maldacena, and H. Nastase, “Strings in flat space and pp waves from N=4 super Yang Mills,” *JHEP* **04** (2002) 013, [hep-th/0202021](https://arxiv.org/abs/hep-th/0202021)

[13] M. Blau, J. Figueroa-O’Farrill, C. Hull, and G. Papadopoulos, “A new maximally supersymmetric background of IIB superstring theory,” *JHEP* **01** (2002) 047, [hep-th/0110242](https://arxiv.org/abs/hep-th/0110242)

[14] R. R. Metsaev, “Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background,” *Nucl. Phys.* **B625** (2002) 70–96, [hep-th/0112044](https://arxiv.org/abs/hep-th/0112044)

[15] R. R. Metsaev and A. A. Tseytlin, “Exactly solvable model of superstring in plane wave Ramond–Ramond background,” *Phys. Rev.* **D65** (2002) 126004, [hep-th/0202109](https://arxiv.org/abs/hep-th/0202109)

[16] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” *Nucl. Phys.* **B636** (2002) 99–114, [hep-th/0204051](https://arxiv.org/abs/hep-th/0204051)

[17] S. Frolov and A. A. Tseytlin, “Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$,” *JHEP* **06** (2002) 007, [hep-th/0204226](https://arxiv.org/abs/hep-th/0204226)
[18] A. A. Tseytlin, “On semiclassical approximation and spinning string vertex operators in AdS5 × S5,” *Nucl. Phys.* B664 (2003) 247–275, hep-th/0304139.

[19] S. Frolov and A. A. Tseytlin, “Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors,” *Phys. Lett.* B570 (2003) 96–104, hep-th/0306143.

[20] S. Frolov and A. A. Tseytlin, “Multi-spin string solutions in AdS5 × S5,” *Nucl. Phys.* B668 (2003) 77–110, hep-th/0304255.

[21] S. Frolov and A. A. Tseytlin, “Quantizing three-spin string solution in AdS5 × S5,” *JHEP* 07 (2003) 016, hep-th/0306130.

[22] G. Arutyunov, S. Frolov, J. Russo, and A. A. Tseytlin, “Spinning strings in AdS5 × S5 and integrable systems,” *Nucl. Phys.* B671 (2003) 3–50, hep-th/0307191.

[23] N. Beisert, S. Frolov, M. Staudacher, and A. A. Tseytlin, “Precision spectroscopy of AdS/CFT,” *JHEP* 10 (2003) 037, hep-th/0308117.

[24] N. Beisert, J. A. Minahan, M. Staudacher, and K. Zarembo, “Stringing spins and spinning strings,” *JHEP* 09 (2003) 010, hep-th/0306139.

[25] A. A. Tseytlin, “Spinning strings and AdS/CFT duality,” hep-th/0311139.

[26] J. A. Minahan and K. Zarembo, “The Bethe-ansatz for N = 4 super Yang–Mills,” *JHEP* 03 (2003) 013, hep-th/0212208.

[27] N. Beisert and M. Staudacher, “The N = 4 SYM integrable super spin chain,” hep-th/0307042.

[28] N. Beisert, “The complete one-loop dilatation operator of N = 4 super yang-mills theory,” hep-th/0307015.

[29] N. Beisert, C. Kristjansen, J. Plefka, and M. Staudacher, “BMN gauge theory as a quantum mechanical system,” *Phys. Lett.* B558 (2003) 229–237, hep-th/0212269.

[30] S. Bellucci, P. Y. Casteill, J. F. Morales, and C. Sochichiu, “Spin bit models from non-planar N = 4 SYM,” to appear in Nucl. Phys. B, hep-th/0404066.

[31] S. Bellucci, P. Y. Casteill, J. F. Morales, and C. Sochichiu, “Chaining spins from (super)Yang–Mills,” hep-th/0408102.

[32] M. Kruczenski, “Spin chains and string theory,” hep-th/0311203.

[33] G. Arutyunov and M. Staudacher, *JHEP* 0403 (2004) 004 arXiv:hep-th/0310182.

[34] G. Arutyunov and M. Staudacher, arXiv:hep-th/0403077.

[35] M. Kruczenski, A. V. Ryzhov, and A. A. Tseytlin, “Large spin limit of AdS5 × S5 string theory and low energy expansion of ferromagnetic spin chains,” *Nucl. Phys.* B692 (2004) 3–49, hep-th/0403120.
[36] R. Hernandez and E. Lopez, “The SU(3) spin chain sigma model and string theory,” JHEP 04 (2004) 052. hep-th/0403139

[37] C. Kristjansen and T. Mansson, “The circular, elliptic three-spin string from the SU(3) spin chain,” Phys. Lett. B596 (2004) 265–276, hep-th/0406176

[38] B. J. Stefanski and A. A. Tseytlin, JHEP 0405 (2004) 042 arXiv:hep-th/0404133.

[39] M. Kruczenski and A. A. Tseytlin, “Semiclassical relativistic strings in $S^5$ and long coherent operators in $\mathcal{N} = 4$ SYM theory,” hep-th/0406189.

[40] E. Fradkin, “Field Theories of Condensed Matter Systems”. Addison-Wesley Publishing Company, Redwood City, CA, 1991.

[41] A. Perelomov, “Generalized Coherent States and their Applications”. Springer-Verlag, Berlin, 1986.

[42] L. N. Lipatov, “High-energy asymptotics of multicolor QCD and exactly solvable lattice models,” JETP Lett. 59 (1994) 596–599, hep-th/9311037

[43] L. D. Faddeev and G. P. Korchemsky, “High-energy QCD as a completely integrable model,” Phys. Lett. B342 (1995) 311–322, hep-th/9404173

[44] A. V. Belitsky, V. M. Braun, A. S. Gorsky and G. P. Korchemsky, arXiv:hep-th/0407232

[45] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, “Supergravity and the large $N$ limit of theories with sixteen supercharges,” Phys. Rev. D58 (1998) 046004, hep-th/9802042.

[46] T. Gherghetta and Y. Oz, “Supergravity, non-conformal field theories and braneworlds,” Phys. Rev. D65 (2002) 046001, hep-th/0106255.

[47] J. F. Morales and H. Samtleben, “Supergravity duals of matrix string theory,” JHEP 08 (2002) 042, hep-th/0206247.

[48] J. F. Morales and H. Samtleben, “AdS duals of matrix strings,” Class. Quant. Grav. 20 (2003) S553–S558, hep-th/0211278.

18