Asynchronous exponential growth for a two-phase size-structured population model and comparison with the corresponding one-phase model

Meng Bai and Shihe Xu

School of Mathematics and Statistics Sciences, Zhaoqing University, Zhaoqing, 526061 Guangdong, People’s Republic of China

ABSTRACT
In this paper we study a two-phase size-structured population model with distributed delay in the birth process. This model distinguishes individuals by ‘active’ or ‘resting’ status. The individuals in the two life-stages have different growth rates. Only individuals in the ‘active’ stage give birth to the individuals in the ‘active’ stage or the ‘resting’ stage. The size of all the newborns is 0. By using the method of semigroups, we obtain that the model is globally well-posed and its solution possesses the property of asynchronous exponential growth. Moreover, we give a comparison between this two-phase model with the corresponding one-phase model and show that the asymptotic behaviours of the sum of the densities of individuals in the ‘active’ stage and the ‘resting’ stage and the solution of the corresponding one-phase model are different.

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1. Introduction

In this paper, we study a two-phase size-structured population model with distributed delay in the birth process. The individuals in this model are distinguished by two distinct life-stages: the ‘active’ stage and the ‘resting’ stage. Only individuals in the ‘active’ stage give birth to the individuals in the ‘active’ stage or the ‘resting’ stage. The individuals in the two life-stages have different growth rates. The size of all the newborns is 0. The delay in this model is given by the time lag between conception and birth or laying and hatching of the parasite eggs (see [2,17]). Moreover, unlike the non-distributed delay case, the time lag considered here can change from 0 to the maximal value, i.e. it is distributed in an interval. We denote by \( m(t,x) \) and \( n(t,x) \) the densities of individuals in the ‘active’ stage and the ‘resting’ stage, respectively, of size \( x \in [0,\tilde{a}] \) at time \( t \in [0,\infty) \), where \( \tilde{a} > 0 \) represents the finite maximal size any individual may reach in its lifetime. Then the model reads as follows:

\[
\frac{\partial m}{\partial t} + \frac{\partial (\gamma_1(x)m)}{\partial x} = -\mu_1(x)m - \rho_1(x)m + \rho_2(x)n, \quad x \in [0,\tilde{a}], \quad t \geq 0,
\]
\[
\frac{\partial n}{\partial t} + \frac{\partial (\gamma_2(x)n)}{\partial x} = -\mu_2(x)n - \rho_2(x)n + \rho_1(x)m, \quad x \in [0, \bar{a}], \quad t \geq 0,
\]
\[
\gamma_1(0)m(t, 0) = v \int_{-\tau}^{0} \int_{0}^{\bar{a}} \beta(\sigma, x)m(t + \sigma, x) \, d\sigma \, dx, \quad t \geq 0,
\]
\[
\gamma_2(0)n(t, 0) = (1 - v) \int_{-\tau}^{0} \int_{0}^{\bar{a}} \beta(\sigma, x)m(t + \sigma, x) \, d\sigma \, dx, \quad t \geq 0,
\]
\[
m(\sigma, x) = \hat{m}(\sigma, x), \quad \sigma \in [-\tau, 0], \quad x \in [0, \bar{a}],
\]
\[
n(0, x) = n_0(x), \quad x \in [0, \bar{a}]. \tag{1}
\]

Here the vital rates \(\gamma_i(x)\) and \(\mu_i(x)\) \((i = 1, 2)\) are the size-dependent growth rates and mortality rates of individuals in the ‘active’ stage and the ‘resting’ stage, respectively, the functions \(\rho_1(x)\) and \(\rho_2(x)\) are the size-dependent rates of transition between the ‘active’ stage and the ‘resting’ stage, the function \(\beta(\sigma, x)\) represents the rate that an individual of size \(x\) in the ‘active’ stage reproduces after a time lag \(-\sigma \in [0, \tau]\) starting from conception, where \(\tau\) is a constant denoting the maximal delay, and \(v\) is a constant, \(0 \leq v \leq 1\). In addition, \(\hat{m}\) and \(n_0\) are given functions defined in \([-\tau, 0] \times [0, \bar{a}]\) and \([0, \bar{a}]\), respectively. Later on we shall denote
\[
\hat{m}_0(x) = \hat{m}(0, x) \quad \text{for} \quad x \in [0, \bar{a}]. \tag{2}
\]

The one-phase model with delay in the birth process and the growth rate \(\gamma(x) = 1\) was studied in [15]. The global well-posedness and the property of so called asynchronous exponential growth (see [1, 7, 8–12, 15] for the definition) of its solution were obtained by using the method of semigroups. The author of [11] gave a completely different proof for the property of asynchronous exponential growth which can generalize to more complicated one-phase size-structured models with the growth rate \(\gamma(x) \neq 1\) by using the method of characteristics. Recently, hopf bifurcation was obtained in the similar one-phase model with two delays in [13]. The model considered here distinguishes individuals by two different stages and the two stages may have different growth rates. In this paper, we shall prove that under suitable assumptions on \(\gamma_1, \gamma_2, \mu_1, \mu_2, \rho_1, \rho_2, \beta\) and \((\hat{m}, n_0)\), the model (1) is globally well-posed and its solution possesses the property of asynchronous exponential growth. Moreover, we shall give a comparison between this two-phase model with the corresponding one-phase model by the method which is inspired by [14]. More precisely, we shall consider the special case where \(\gamma_1(x) = \gamma_2(x) = \gamma(x)\) and shows that the asymptotic behaviours of the sum of the densities of individuals in the ‘active’ stage and the ‘resting’ stage and the solution of the corresponding one-phase model are different. This implies that the research on the model which distinguishes individuals by the ‘active’ stage and the ‘resting’ stage are meaningful. In fact, the asynchronous exponential growth of solutions of size-structured population models with two stages have been investigated by many authors including [9], [12], and [3–5]. In contrast with those model, the model considered here is a non-local boundary condition problem which contains distributed delay. We use the methods of Hille–Yosida operators and isomorphic operators. The comparison between the two-phase model with the corresponding one-phase model has been also considered in [3,5]. The conjugate problem here is different because of the non-local boundary condition and the distributed delay.
Throughout this paper, \( \gamma_1(x), \gamma_2(x), \mu_1(x), \mu_2(x), \rho_1(x), \rho_2(x) \), and \( \beta(\sigma, x) \) are supposed to satisfy the following conditions:

(H.1) \( \mu_1, \mu_2, \rho_1, \) and \( \rho_2 \) are non-negative and continuous functions defined on \([0, \bar{a}]\).

(H.2) \( \gamma_1, \gamma_2 \in C^1[0, \bar{a}], \) and \( \gamma_1(x), \gamma_2(x) > 0 \) for all \( x \in [0, \bar{a}] \). \( \bar{\gamma}_1 := \max_{0 \leq x \leq \bar{a}} \{\gamma_1(x)\} \) and \( \bar{\gamma}_2 := \max_{0 \leq x \leq \bar{a}} \{\gamma_2(x)\} \).

(H.3) \( \beta \in C([-\tau, 0] \times \{0, \bar{a}\}) \) and \( \beta \geq 0 \).

In order to prove the property of asynchronous exponential growth, we make the additional assumptions:

(H.4) \( \beta(\sigma, x) > 0 \) for all \( \sigma \in [-\tau, 0] \) and \( x \in [0, \bar{a}] \).

(H.5) If \( 0 \leq v < 1, \rho_2(x) > 0 \) for all \( x \in [0, \bar{a}] \). If \( v = 1, \rho_1(x), \rho_2(x) > 0 \) for all \( x \in [0, \bar{a}] \).

We introduce the subspace \( E_1 \) of \( W^{1,1}((-\tau, 0), L^1[0,\bar{a}]) \) and the subspace \( E_2 \) of \( W^{1,1}(0,\bar{a}) \) as follows:

\[
E_1 := \{ \hat{m} \in W^{1,1}((-\tau, 0), L^1[0,\bar{a}]) : \hat{m}_0 \in W^{1,1}(0,\bar{a}) \},
\]

\[
\hat{m}_0(0) = \frac{v}{\gamma_1(0)} \int_0^\bar{a} \int_{-\tau}^0 \beta(\sigma, x) \hat{m}(\sigma, x) \, d\sigma \, dx,
\]

\[
E_2 := \{ n_0 \in W^{1,1}(0,\bar{a}) : n_0(0) = \frac{(1-v)}{\gamma_2(0)} \int_0^\bar{a} \int_{-\tau}^0 \beta(\sigma, x) \hat{m}(\sigma, x) \, d\sigma \, dx \},
\]

where \( \hat{m}_0 \) is defined in (2).

Our first main result establishes the global well-posedness of the model (1) and reads as follows:

**Theorem 1.1:** For any \( (\hat{m}, n_0) \in E_1 \times E_2 \), the model (1) has a unique solution \( (m, n) \in (C([-\tau, \infty), L^1[0,\bar{a}]) \cap C([0, \infty), W^{1,1}(0,\bar{a})) \cap C^1([0, \infty), L^1[0,\bar{a}])) \times (C([0, \infty), W^{1,1}(0,\bar{a})) \cap C^1([0, \infty), L^1[0,\bar{a}])) \), and for any \( T > 0 \), the mapping \( (\hat{m}, n_0) \to (m, n) \) from \( E_1 \times E_2 \) to \( (C([-\tau, T], L^1[0,\bar{a}]) \cap C([0, T], W^{1,1}(0,\bar{a})) \cap C^1([0, T], L^1[0,\bar{a}])) \times (C([0, T], W^{1,1}(0,\bar{a})) \cap C^1([0, T], L^1[0,\bar{a}])) \) is continuous.

The proof of this result will be given in Section 2.

From the proof of Theorem 1.1, we shall have that for any \( (\hat{m}, n_0) \in E_1 \times E_2 \), the solution of the model (1) can be expressed as \( (m(t), n(t)) = T(t)(\hat{m}, n_0) \) \( (t \geq 0) \), where \( (T(t))_{t \geq 0} \) is a strongly continuous semigroup on the space \( E := L^1([-\tau, 0], L^1[0,\bar{a}]) \times L^1[0,\bar{a}] \). Our second main result studies the asymptotic behaviour of the solution of the model (1) and reads as follows:

**Theorem 1.2:** There exist a rank projection \( \Pi \) on \( E \) and constants \( \lambda_0 \in \mathbb{R}, \varepsilon > 0, M \geq 0 \) such that

\[
\|e^{-\lambda_0 t} T(t) - \Pi\| \leq M e^{-\varepsilon t} \quad \text{for all } t \geq 0,
\]

where \( \| \cdot \| \) denotes the operator norm on \( E \).

The proof of this result will be given in Section 3. The parameter \( \lambda_0 \) is called intrinsic rate of natural increase or Malthusian parameter (see [16]). This result shows that the solution of the model (1) exhibits asynchronous exponential growth.
Next we consider the special case of the model (1), where $\gamma_1(x) = \gamma_2(x) = \gamma(x)$ and give a comparison between this two-phase model with the corresponding one-phase model. Note that the above result means that there exists a positive vector function $(\hat{u}, \hat{v}) \in \mathcal{E}$, such that for any $(\hat{m}, n_0) \in \mathcal{E}_1 \times \mathcal{E}_2$, the solution $(m(t + \cdot, \cdot), n(t, \cdot))$ of the model (1) has the following asymptotic expression:

$$
(m(t + \cdot, \cdot), n(t, \cdot)) e^{-\lambda_0 t} = c_1(\hat{u}(\cdot, \cdot), \hat{v}(\cdot)) + O(e^{-\epsilon t}) \quad \text{as } t \to \infty,
$$

(3)

where $c_1$ is a constant uniquely determined by the initial data $(\hat{m}, n_0)$. We denote

$$
\hat{U}(x) = \hat{u}(0, x) + \hat{v}(x), \quad \theta(x) = \frac{\hat{u}(0, x)}{\hat{U}(x)},
$$

(4)

$$
\mu(x) = \theta(x)\mu_1(x) + (1 - \theta(x))\mu_2(x), \quad \text{and} \quad \bar{\beta}(\sigma, x) = \theta(x)\beta(\sigma, x).
$$

(5)

We can see that $\theta(x)$ is the asymptotic proportion of the individuals in the ‘active’ stage in the population. Then we have the following problem which describes the evolution of the sum of the densities of individuals in the ‘active’ stage and the ‘resting’ stage in the asymptotic sense:

$$
\frac{\partial \bar{N}}{\partial t} + \frac{\partial (\gamma(x)\bar{N})}{\partial x} = -\mu(x)\bar{N}, \quad x \in [0, \bar{a}], \ t \geq 0,
$$

$$
\gamma(0)\bar{N}(t, 0) = \int_0^{\bar{a}} \int_{-\tau}^0 \bar{\beta}(\sigma, x)\bar{N}(t + \sigma, x) \, d\sigma \, dx, \quad t \geq 0,
$$

(6)

$$
\bar{N}(\sigma, x) = \hat{N}(\sigma, x), \quad \sigma \in [-\tau, 0], \ x \in [0, \bar{a}],
$$

where $\hat{N}(\sigma, x) = \hat{m}(\sigma, x) + n_0(x)$. By using the same method, we have that the solution $\hat{N}(t, x)$ of the model (6) has the following asymptotic expression:

$$
\hat{N}(t, x) e^{-\lambda_0 t} = c_2 \hat{U}(x) + O(e^{-\epsilon t}) \quad \text{as } t \to \infty,
$$

where $c_2$ is a constant uniquely determined by the initial data $\hat{N}(\sigma, x)$.

Let

$$
N(t, x) = m(t, x) + n(t, x) \quad \text{for } t > 0, x \in [0, \bar{a}],
$$

(7)

and

$$
N(\sigma, x) = \hat{m}(\sigma, x) + n_0(x) \quad \text{for } \sigma \in [-\tau, 0], x \in [0, \bar{a}].
$$

(8)

We can see that $N(t, x)$ is the sum of the densities of individuals in the ‘active’ stage and the ‘resting’ stage. From (3), we have the following asymptotic expression:

$$
N(t, x)e^{-\lambda_0 t} = c_1 \hat{U}(x) + O(e^{-\epsilon t}) \quad \text{as } t \to \infty,
$$

Then we want to compare $N(t, x)$ and the solution of the model (6). One might expect that $N(t, x)e^{-\lambda_0 t} - \bar{N}(t, x)e^{-\lambda_0 t} \to 0$ as $t \to \infty$. But this is actually not the case. In fact, we have the following result:
**Theorem 1.3:** Let the notation be as above. We have the following relation:

\[ N(t,x) e^{-\lambda_0 t} - \tilde{N}(t,x) e^{-\lambda_0 t} = c\tilde{U}(x) + O(e^{-\varepsilon t}) \quad \text{as } t \to \infty, \]

where \( c \) is a constant which is generally non-vanishing.

The proof of this result will be given in Section 3. This result shows that the asymptotic behaviours of the sum of the densities of individuals in the ‘active’ stage and the ‘resting’ stage and the solution of the one-phase model are different and the research on the model with two stages is meaningful.

The layout of the rest of the paper is as follows. In Section 2, we reduce the model (1) into an abstract Cauchy problem and establish the well-posedness of it by means of strongly continuous semigroups. In Section 3, we prove that the solution of the model (1) exhibits asynchronous exponential growth. In Section 4, we compare this two-phase model with the corresponding one-phase model and give the proof of Theorem 1.3.

**2. Reduction and well-posedness**

In this section we reduce the model (1) into an abstract Cauchy problem and establish the well-posedness of it. Since this model is a non-local boundary condition problem which contains distributed delay, we use the methods of Hille-Yosida operators and isomorphic operators (see [15]).

First, we introduce the following operators on the Banach spaces \( X := L^1[0,\tilde{a}] \times L^1[0,\tilde{a}] \) and \( E := L^1([-\tau,0], L^1[0,\tilde{a}]) \):

\[
A(u,v) := (-\gamma_1(\cdot)u', -\gamma_2(\cdot)v') \quad \text{with domain } D(A) := W^{1,1}(0,\tilde{a}) \times W^{1,1}(0,\tilde{a}),
\]

\[
B_1(u,v) := (-\gamma'_1(\cdot)u, -\gamma'_2(\cdot)v) \quad \text{for } (u,v) \in X,
\]

\[
B_2(u,v) := (-\mu_1(\cdot) + \rho_1(\cdot))u + \rho_2(\cdot)v, -\mu_2(\cdot) + \rho_2(\cdot))v + \rho_1(\cdot)u \quad \text{for } (u,v) \in X,
\]

\[
B := B_1 + B_2,
\]

\[
P : D(A) \to \mathbb{R}^2, P(u,v) := (u(0), v(0)),
\]

\[
C(\tilde{u}) := \begin{pmatrix} v & (1-v) \\ \gamma_1(0) & \gamma_2(0) \end{pmatrix} C_1(\tilde{u}) \quad \text{for } \tilde{u} \in E,
\]

where

\[
C_1(\tilde{u}) := \int_0^{\tilde{a}} \int_{-\tau}^0 \beta(\sigma, x)\tilde{u}(t+\sigma, x) \, d\sigma \, dx.
\]

We note that \( A \in \mathcal{L}(D(A),X), B \in \mathcal{L}(X) \) and \( C \in \mathcal{L}(E,X) \). Using these notations, we rewrite the model (1) into the following abstract initial value problem for a retarded differential equation on \( X \):

\[
\frac{d(u(t),v(t))}{dt} = (A+B)(u(t),v(t)), \quad t \geq 0
\]

\[
P(u(t),v(t)) = C(u_t),
\]

\[
(u(0),v(0)) = (\hat{m}_0, n_0),
\]

\[
u_0 = \hat{m},
\]

(9)
where \( u : [0, +\infty) \to L^1[0, \bar{a}] \) and \( v : [0, +\infty) \to L^1[0, \bar{a}] \) are defined as \( u(t) := m(t, \cdot) \) and \( v(t) := n(t, \cdot) \), respectively, and \( u_t : [-\tau, 0] \to L^1[0, \bar{a}] \) is defined as \( u_t(\sigma) := m(t + \sigma), \sigma \in [-\tau, 0] \).

Next, we introduce the following operators on \( E \):

\[
(\tilde{G} \tilde{u})(\sigma) := \frac{d}{d\sigma} \tilde{u} \quad \text{with domain} \quad D(\tilde{G}) := W^{1,1}([-\tau, 0], L^1[0, \bar{a}]),
\]

\[
Q \tilde{u} := \tilde{u}(0) \quad \text{for} \quad \tilde{u} \in D(\tilde{G}).
\]

We note that \( G \in \mathcal{L}(D(G), E) \) and \( Q \in \mathcal{L}(D(G), X) \). We now let \( \mathcal{E} := E \times X \), and introduce operator \( A \) on \( \mathcal{E} \) as follows:

\[
A \left( \begin{pmatrix} \tilde{u} \\ (u, v) \end{pmatrix} \right) := \begin{pmatrix} G & 0 \\ 0 & A + B \end{pmatrix} \begin{pmatrix} \tilde{u} \\ (u, v) \end{pmatrix}
\]

with domain \( D(A) := \left\{ \begin{pmatrix} \tilde{u} \\ (u, v) \end{pmatrix} \in D(\tilde{G}) \times D(A) : Q \tilde{u} = u, P(u, v) = C(\tilde{u}) \right\} \).

We note that \( A \in \mathcal{L}(D(A), \mathcal{E}) \). Using these notations, we see that problem (9) can be equivalently rewrite into the following abstract initial value problem of an ordinary differential equation on \( \mathcal{E} \):

\[
U'(t) = AU(t), \quad t > 0,
\]

\[
U(0) = U_0,
\]

where \( U(t) = (u_t, v(t)) \) and \( U_0 = (u_0^{(0)}, v(0)) \).

**Remark 2.1:** As usual, if the functions \( u : (-\tau, \infty) \to X \) and \( v : [0, \infty) \to X \) satisfy \( u \in C(-\tau, \infty), W^{1,1}(0, \bar{a}) \) \( \cap \) \( C(0, \infty), W^{1,1}(0, \bar{a}) \) \( \cap \) \( C^1(0, \infty), X \) and \( v \in C(0, \infty), W^{1,1}(0, \bar{a}) \) \( \cap \) \( C^1(0, \infty), X \) respectively, and problem (9), we say that functions \( (u(t), v(t)) \) is a classical solution of problem (9). It is evident that a necessary condition for problem (9) to have a classical solution is that \( \dot{m} \in W^{1,1}(-\tau, 0, X) \) and \( (\dot{m}_0, n_0) \in D(A) \).

**Remark 2.2:** As usual, we say that a function \( U : [0, \infty) \to \mathcal{E} \) is a classical solution of problem (9) if \( U \in C(0, \infty), D(A) \) \( \cap \) \( C^1(0, \infty), \mathcal{E} \) and it satisfies (10) in usual sense.

To be rigorous, we write down the following preliminary result:

**Lemma 2.1:** Let the necessary condition mentioned in Remark 2.1 is satisfied. If \( (u(t), v(t)) \) \( (u : (-\tau, \infty) \to X, v : [0, \infty) \to X) \) is a classical solution of problem (9), then \( U(t) = (u_t, v(t)) \) is a classical solution of problem (10). Conversely, if \( U \) is a classical solution of problem (10), then \( U \) has the form \( U(t) = (u_t, v(t)) \) for all \( t \geq 0 \), and by extending the first component of its second component \( u = u(t) \) to \( (-\tau, \infty) \) such that \( u(t) = u_0 \) for \( t \in (-\tau, 0) \), we have that \( (u, v) \) is a classical solution of problem (9).

**Proof:** See Lemma 2.2 of [15] and Lemma 2.1 of [6].
We consider the Banach space $X := E \times L^1[0, \bar{a}] \times X \times \mathbb{R}^2$ and the operator

$$A := \begin{pmatrix}
  G & 0 & 0 & 0 \\
  -Q & 0 & \pi_1 \text{Id} & 0 \\
  (0,0) & (0,0) & A + B & (0,0) \\
  C & (0,0) & -P & (0,0)
\end{pmatrix}$$

with domain $D(A) := D(G) \times \{0\} \times D(A) \times \{(0,0)\}$, where $\pi_1$ is the projection onto the first coordinate.

**Lemma 2.2:** The operator $(A, D(A))$ is a Hille-Yosida operator on the Banach space $X$.

**Proof:** The operator $A$ can be written as the sum of two operators on $X$ as $A = A_1 + B$, where

$$A_1 := \begin{pmatrix}
  G & 0 & 0 & 0 \\
  -Q & 0 & 0 & 0 \\
  (0,0) & (0,0) & A & (0,0) \\
  (0,0) & (0,0) & -P & (0,0)
\end{pmatrix}$$

and

$$B := \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & \pi_1 \text{Id} & 0 \\
  (0,0) & (0,0) & B & (0,0) \\
  C & (0,0) & (0,0) & (0,0)
\end{pmatrix}$$

with $D(A_1) = D(A)$ and $D(B) = X$. Since $B$ is a bounded operator on $X$, by Lemma 3.1 of [15], it suffices to prove that $(A_1, D(A_1))$ is a Hille–Yosida operator on the Banach space $X$.

The restriction $(G_0, D(G_0))$ of $G$ to the kernel of $Q$ generates the nilpotent left shift semigroup $(S_0(t))_{t \geq 0}$ on $E$, given by

$$(S_0(t)f)(\sigma, x) = \begin{cases}
  f(t + \sigma, x), & \text{if } \sigma + t \leq 0, \\
  0, & \text{if } \sigma + t > 0.
\end{cases}$$

Similarly, the restriction $(A_0, D(A_0))$ of $A$ to the kernel of $P$ generates the strongly continuous semigroup $(T_0(t))_{t \geq 0}$ on $E$, given by

$$(T_0(t)(f_1, f_2))(x) = ((T_{01}(t)f_1)(x), (T_{02}(t)f_2)(x)),$$

where

$$(T_{01}(t)f_1)(x) = \begin{cases}
  f_1(\Gamma_1^{-1}(\Gamma_1(x) - t)), & \text{if } \Gamma_1(x) \geq t, \\
  0, & \text{otherwise},
\end{cases}$$

$$(T_{02}(t)f_2)(x) = \begin{cases}
  f_2(\Gamma_2^{-1}(\Gamma_2(x) - t)), & \text{if } \Gamma_2(x) \geq t, \\
  0, & \text{otherwise},
\end{cases}$$

where $\Gamma_1(x) = \int_0^x (1/\gamma_1(s)) \, ds$ and $\Gamma_2(x) = \int_0^x (1/\gamma_2(s)) \, ds$. The resolvent of $A_1$ is

$$R(\lambda, A_1) = \begin{pmatrix}
  R(\lambda, G_0) & \varepsilon_\lambda & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  (0,0) & (0,0) & R(\lambda, A_0) & \varphi_\lambda \\
  (0,0) & (0,0) & (0,0) & (0,0)
\end{pmatrix},$$

where $(\varepsilon_\lambda)(\sigma)g := e^{\lambda \sigma}g$ for $\sigma \in [-\tau, 0]$, $g \in L^1[0, \bar{a}]$ and $(\varphi_\lambda(x))(x_1, x_2) := (e^{-\int_0^x (\lambda/\gamma_1(s)) \, ds} x_1, e^{-\int_0^x (\lambda/\gamma_2(s)) \, ds} x_2)$ for $x \in [0, \bar{a}]$, $(x_1, x_2) \in \mathbb{R}^2$. The resolvent set $\rho(A_1)$
of \(A_1\) contains \(\mathbb{R}^+\). For \((\tilde{f}, g, (f_1, f_2), (x_1, x_2)) \in X\) and \(\lambda > 0\),

\[
\|R(\lambda, A_1)(\tilde{f}, g, (f_1, f_2), (x_1, x_2))\|_X
\]

\[
= \|R(\lambda, G_0)\tilde{f} + \varepsilon_\lambda g\|_E + \|R(\lambda, A_0)(f_1, f_2) + \varphi_\lambda(x_1, x_2)\|_X
\]

\[
\leq \|R(\lambda, G_0)\tilde{f}\|_E + \|\varepsilon_\lambda g\|_E + \|R(\lambda, A_0)(f_1, f_2)\|_X + \|\varphi_\lambda(x_1, x_2)\|_X.
\]

From the proof of Proposition 3.2 of [15], we have that \(\|R(\lambda, G_0)\tilde{f}\|_E \leq (1/\lambda)\|\tilde{f}\|_E\).

Since \((A_0, D(A_0))\) generates a contraction semigroup on \(X\), by Theorem II.3.5 of [10], we have that \(\|R(\lambda, A_0)(f_1, f_2)\|_X \leq (1/\lambda)\|f_1, f_2\|_X\). A direct calculation shows that \(\|\varepsilon_\lambda g\|_E \leq (1/\lambda)\|g\|_{L^1[0, \bar{a}]}\) and \(\|\varphi_\lambda(x_1, x_2)\|_X \leq (1/\lambda)\|(x_1, x_2)\|_{\mathbb{R}^2}\), where \(\|(x_1, x_2)\|_{\mathbb{R}^2} := \max\{|x_1| + |x_2|\}\) for \((x_1, x_2) \in \mathbb{R}^2\).

Therefore, we have that \(\|\lambda R(\lambda, A_1)\| \leq 1\) and \((A_1, D(A))\) is a Hille–Yosida operator on the Banach space \(X\). This completes the proof.

By Proposition 5.9 of [15], we have the following result.

**Lemma 2.3:** The part \((A_0, D(A_0))\) of \((A, D(A))\) in the closure of its domain \(X_0 := \overline{D(A)} = E \times \{0\} \times X \times \{(0, 0)\}\) generates a strongly continuous semigroup.

**Lemma 2.4:** The operator \((A, D(A))\) is isomorphic to the part \((A_0, D(A_0))\) of the operator \((A, D(A))\) in the closure of its domain \(\overline{D(A)}\).

**Proof:** We have

\[
D(A_0) = \{(\tilde{f}, 0, (f_1, f_2), (0, 0)) : \tilde{f} \in D(G), (f_1, f_2) \in D(A), A(\tilde{f}, 0, (f_1, f_2), (0, 0))^t \in X_0\}
\
= \{(\tilde{f}, 0, (f_1, f_2), (0, 0)) : \tilde{f} \in D(G), (f_1, f_2) \in D(A), Q\tilde{f} = f_1, P(f_1, f_2) = C(\tilde{f})\}.
\]

Therefore, the operator \((A, D(A))\) is isomorphic to the part \((A_0, D(A_0))\) of the operator \((A, D(A))\) in the closure of its domain \(\overline{D(A)}\). We also refer the reader to see Theorem 3.3 of [15] for the similar proof.

By Lemma 2.3, Lemma 2.4 and Theorem II.6.7 of [10], we have the following result.

**Theorem 2.5:** The operator \(A\) generates a strongly continuous semigroup \((T(t))_{t \geq 0}\) on \(E\). For any given initial data \(U_0 = ((u_0, v_0(0))) \in D(A)\), problem (10) has a unique solution \(U \in C(0, +\infty); D(A)) \cap C^1(0, +\infty); E\), given by

\[
U(t) = T(t) \left( \begin{array}{c} u_0 \\ u_0(0, v(0)) \end{array} \right) \text{ for } t \geq 0.
\]

By Lemma 2.1 and Theorem 2.5, we see that Theorem 1.1 follows.
3. Asynchronous exponential growth

In this section we study the asymptotic behaviour of the solution of problem ([1]). We shall prove that the semigroup \((T(t))_{t \geq 0}\) has the property of asynchronous exponential growth on \(E\) by using Theorems 9.10 and 9.11 of [8]. To this end, we will prove that the semigroup \((T(t))_{t \geq 0}\) is an irreducible positive strongly continuous semigroup (see Definition II.1.7 and Theorem VI.1.2 of [10] for the definitions) satisfying the inequality \(\omega_{\text{ess}}(A) < \omega_0(A)\), where \(\omega_{\text{ess}}(A)\) and \(\omega_0(A)\) are the essential growth bound and the growth bound of the semigroup \((T(t))_{t \geq 0}\) generated by \(A\) (see Definition IV.2.1 and Definition IV.2.9 of [10] for the definitions). In contrast with the semigroups in [3–5,9,12], the generator \(A\) contains the non-local boundary condition and the distributed delay, we use the isomorphic operators in the proof of the property of asynchronous exponential growth.

We first deduce an useful expression of \(R(\lambda, A)\). For \(F \in E\), let \(U = R(\lambda, A)F\). Then \(U\) satisfies the equation

\[
(\lambda I - A)U = F. \tag{14}
\]

By writing \(U = (\tilde{u}(\sigma, x), (u(x), v(x)))\) and \(F = (\tilde{f}(\sigma, x), (f(x), g(x)))\), we see that Equation (14) can be rewritten as follows:

\[
\begin{align*}
\lambda \tilde{u}(\sigma, x) - \frac{\partial}{\partial \sigma} \tilde{u}(\sigma, x) &= \tilde{f}(\sigma, x), \quad -\tau < \sigma < 0, 0 < x < \bar{a}, \\
\lambda u(x) + \frac{d}{dx}(\gamma_1(x)u(x)) &= f(x) - \mu_1(x)u(x) - \rho_1(x)u(x) + \rho_2(x)v(x), \quad 0 < x < \bar{a}, \\
\lambda v(x) + \frac{d}{dx}(\gamma_2(x)v(x)) &= g(x) - \mu_2(x)v(x) + \rho_1(x)u(x) - \rho_2(x)v(x), \quad 0 < x < \bar{a}, \\
\tilde{u}(0, x) &= u(x), \quad 0 < x < \bar{a}, \\
\gamma_1(0)u(0) &= v \int_{0}^{\bar{a}} \int_{-\tau}^{0} \beta(\sigma, x)\tilde{u}(\sigma, x) \, d\sigma \, dx, \\
\gamma_2(0)v(0) &= (1 - v) \int_{0}^{\bar{a}} \int_{-\tau}^{0} \beta(\sigma, x)\tilde{u}(\sigma, x) \, d\sigma \, dx.
\end{align*}
\]

Then

\[
\tilde{u}(\sigma, x) = e^{\lambda \sigma} u(x) + e^{\lambda \sigma} \int_{\sigma}^{0} e^{-\lambda \xi} \tilde{f}(\xi, x) \, d\xi, \tag{15}
\]

\[
\begin{align*}
u(x) &= \frac{v}{\gamma_1(0)} E_{1\lambda}(x) \int_{0}^{\bar{a}} \int_{-\tau}^{0} \beta(\sigma, x) e^{\lambda \sigma} \left( u(x) + \int_{\sigma}^{0} e^{-\lambda \xi} \tilde{f}(\xi, x) \, d\xi \right) \, d\sigma \, dx \\
&\quad + E_{1\lambda}(x) \int_{0}^{x} f(s) + \rho_2(s)v(s) \, ds, \tag{16}
\end{align*}
\]

\[
\begin{align*}
v(x) &= \frac{(1 - v)}{\gamma_2(0)} E_{2\lambda}(x) \int_{0}^{\bar{a}} \int_{-\tau}^{0} \beta(\sigma, x) e^{\lambda \sigma} \left( u(x) + \int_{\sigma}^{0} e^{-\lambda \xi} \tilde{f}(\xi, x) \, d\xi \right) \, d\sigma \, dx \\
&\quad + E_{2\lambda}(x) \int_{0}^{x} g(s) + \rho_1(s)u(s) + \rho_1(s)u(s) \, ds. \tag{17}
\end{align*}
\]
where

\[ E_{1\lambda}(x) = \exp\left( -\int_0^x \frac{\lambda + \mu_1(s) + \rho_1(s) + \gamma_1'(s)}{\gamma_1(s)} \, ds \right) \]  

and

\[ E_{2\lambda}(x) = \exp\left( -\int_0^x \frac{\lambda + \mu_2(s) + \rho_2(s) + \gamma_2'(s)}{\gamma_2(s)} \, ds \right). \]

For each \( \lambda \in \mathbb{C} \), we define two operators \( M_\lambda \) and \( N_\lambda \) on \( E \) as follows:

\[
M_\lambda \begin{pmatrix} \tilde{f}(\sigma, x) \\ (f_1(x), f_2(x)) \end{pmatrix} = \begin{pmatrix} e^{\lambda \sigma} f_1(x) \\ (h_1(x, \lambda), h_2(x, \lambda)) \end{pmatrix},
\]

\[
N_\lambda \begin{pmatrix} \tilde{f}(\sigma, x) \\ (f_1(x), f_2(x)) \end{pmatrix} = \begin{pmatrix} e^{\lambda \sigma} \int_0^\sigma e^{-\lambda \xi \tilde{f}(\xi, x)} \, d\xi \\ (g_1(x, \lambda), g_2(x, \lambda)) \end{pmatrix},
\]

where

\[
h_1(x, \lambda) = E_{1\lambda}(x) \left( \frac{\nu}{\gamma_1(0)} \int_0^\alpha \int_{-\tau}^0 \beta(\sigma, x) e^{\lambda \sigma} f_1(x) \, d\sigma \, dx + \int_0^x \frac{\rho_2(s) f_2(s)}{E_{1\lambda}(s) \gamma_1(s)} \, ds \right),
\]

\[
h_2(x, \lambda) = E_{2\lambda}(x) \left( \frac{(1 - \nu)}{\gamma_2(0)} \int_0^\alpha \int_{-\tau}^0 \beta(\sigma, x) e^{\lambda \sigma} f_1(x) \, d\sigma \, dx + \int_0^x \frac{\rho_1(s) f_1(s)}{E_{2\lambda}(s) \gamma_2(s)} \, ds \right),
\]

\[
g_1(x, \lambda) = E_{1\lambda}(x) \left( \frac{\nu}{\gamma_1(0)} \int_0^\alpha \int_{-\tau}^0 \beta(\sigma, x) e^{\lambda \sigma} \int_0^\sigma e^{-\lambda \xi} \tilde{f}(\xi, x) \, d\xi \, d\sigma \, dx \right.
\]

\[
\left. + \int_0^x \frac{f_1(s)}{E_{1\lambda}(s) \gamma_1(s)} \, ds \right),
\]

\[
g_2(x, \lambda) = E_{2\lambda}(x) \left( \frac{(1 - \nu)}{\gamma_2(0)} \int_0^\alpha \int_{-\tau}^0 \beta(\sigma, x) e^{\lambda \sigma} \int_0^\sigma e^{-\lambda \xi} \tilde{f}(\xi, x) \, d\xi \, d\sigma \, dx \right)
\]

\[
\left. + \int_0^x \frac{f_2(s)}{E_{2\lambda}(s) \gamma_2(s)} \, ds \right).
\]

Since

\[
\left\| M_\lambda \begin{pmatrix} \tilde{f}(\sigma, x) \\ (f_1(x), f_2(x)) \end{pmatrix} \right\|_E \to 0 \quad \text{as} \quad \lambda \to +\infty,
\]

there exists \( \lambda^* > 0 \) such that \( \|M_\lambda\| < 1 \) for \( \lambda \geq \lambda^* \). This implies that the inverse \((I - M_\lambda)^{-1}\) exists and is a bounded operator for \( \lambda \geq \lambda^* \). From (15)–(21) we see that the resolvent of \( A \) is given by

\[
R(\lambda, A) = (I - M_\lambda)^{-1} N_\lambda = \sum_{n=0}^{\infty} M_\lambda^n N_\lambda \quad \text{for} \quad \lambda \geq \lambda^*.
\]

**Lemma 3.1:** The semigroup \((T(t))_{t \geq 0}\) generated by \( A \) is positive.
Proof: From Theorem VI.1.8 of [10], the desired assertion follows if we prove that the resolvent \( R(\lambda, A) \) of its generator \( A \) is positive for all sufficiently large \( \lambda \). From (20)–(22), we have that \( R(\lambda, A) > 0 \) for all sufficiently large \( \lambda \). This completes the proof.

Lemma 3.2: The semigroup \((T(t))_{t \geq 0}\) generated by \( A \) is irreducible.

Proof: By Lemma VI.1.9 of [10], if we prove that \( \langle R(\lambda, A)F, \Psi \rangle > 0 \) for some \( \lambda > 0 \) and all \( F \in \mathbb{E} \) and \( \forall \Psi \in \mathbb{E}^* \) such that \( F > 0 \) and \( \Psi > 0 \), then the desired assertion then follows. Let \( \pi_1 \) and \( \pi_2 \) be the projections onto the first and the second coordinates, respectively. We use the expression (22) to prove that \( \pi_1(R(\lambda, A)F)(\sigma, x) > 0 \) for almost all \( (\sigma, x) \in [-\tau, 0] \times [0, \bar{a}] \), \( \pi_1(\pi_2(R(\lambda, A)F))(x) > 0 \) and \( \pi_2(\pi_2(R(\lambda, A)F))(x) > 0 \) for almost all \( x \in [0, \bar{a}] \). The process of this proof is similar to Lemma 3.4 in [3] and Lemma 3.4 in [5]. We omit it here.

Lemma 3.3: \( \omega_{ess}(A) = -\infty \).

Proof: The operator \( A \) can be written as the sum of two operators on \( \mathcal{X} \) as \( A = A_2 + C \), where

\[
A_2 := \begin{pmatrix}
G & 0 & 0 & 0 \\
-Q & \pi_1 \text{Id} & 0 & 0 \\
(0,0) & (0,0) & A + B & (0,0) \\
(0,0) & (0,0) & -P & (0,0)
\end{pmatrix}
\quad \text{and} \quad
C := \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
(0,0) & (0,0) & (0,0) & (0,0) \\
C & (0,0) & (0,0) & (0,0)
\end{pmatrix}
\]

with \( D(A_2) = D(A) \) and \( D(C) = \mathcal{X} \). Since \( A_2 \) can be written as the sum of \( A_1 \) and a bounded operator \( B_1 \), where

\[
B_1 := \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \pi_1 \text{Id} & 0 \\
(0,0) & (0,0) & B & (0,0) \\
(0,0) & (0,0) & (0,0) & (0,0)
\end{pmatrix}
\]

with \( D(B_1) = \mathcal{X} \), \((A_2, D(A_2))\) is a Hille–Yosida operator on the Banach space \( \mathcal{X} \)(see Lemma 3.1 of [15]). By Proposition 4.4 of [15], we have that \((A_{20}, D(A_{20}))\) of \((A_2, D(A))\) in \( \mathcal{X}_0 \) generates a strongly continuous semigroup on \( \mathcal{X}_0 \) which is isomorphic to the semigroup \((T_0(t))_{t \geq 0}\) generated by

\[
A_0 := \begin{pmatrix}
G & 0 \\
0 & A + B
\end{pmatrix}
\]

with domain

\[
D(A_0) := \left\{ \begin{pmatrix} \tilde{u} \\ (u, v) \end{pmatrix} \in D(G) \times D(A) : Q\tilde{u} = u, P(u, v) = (0,0) \right\}.
\]

By Proposition 4.4 of [15], we have that

\[
T_0(t) := \begin{pmatrix}
S_0(t) & T_t \\
0 & T_0(t)
\end{pmatrix},
\]
where \((S_0(t))_{t \geq 0}, (T_0(t))_{t \geq 0}\) are defined in (11) and (12), and \(T_t : X \to E\) are linear operators defined as

\[
(T_t F)(\sigma) := \begin{cases} 
\pi_1(T_0(t + \sigma)F), & \text{if } \sigma + t > 0, \\
0, & \text{if } \sigma + t \leq 0,
\end{cases}
\]

where \(\pi_1\) is the projection onto the first coordinate. By Lemma 3.6 of [12], we have that \(T_0(t) = 0\) for \(t > \Gamma + \tau\), where \(\Gamma = \max\{\int_0^\alpha (d\xi / \gamma_1(\xi)), \int_0^\alpha (d\xi / \gamma_2(\xi))\}\). From the definition \(\omega_{\text{ess}}(A_0) := \lim_{t \to \infty} (1/t) \log \|T_0(t)\|_{\text{ess}}\) (see Definition IV.2.9 of [10]), \(\omega_{\text{ess}}(A_20) = \omega_{\text{ess}}(A_0) = -\infty\). Since \(C\) can be restricted to \(X_0\) and is a bounded and compact operator on \(X_0\), by Theorem 4.5 of [15] and Lemma 2.3, we have that \(\omega_{\text{ess}}(A) = \omega_{\text{ess}}(A_20) = -\infty\). This completes the proof. \(\blacksquare\)

**Lemma 3.4:** \(\omega_0(A) = s(A) > -\infty\), where \(s(A)\) is the spectral bound of \(A\), i.e. \(s(A) := \sup \{\Re \lambda : \lambda \in \sigma(A)\}\).

**Proof:** Since the semigroup \((T(t))_{t \geq 0}\) generated by \(A\) is positive, we have that \(\omega_0(A) = s(A)\) (see Theorem VI.1.15 of [10]). We split the operator \(B_2\) into the sum of two operators: \(B_2 = B_{21} + B_{22}\), where

\[
\begin{align*}
B_{21}(u, v) &:= (-\mu_1(\cdot) + \rho_1(\cdot))u + \rho_2(\cdot)v, \quad -(\mu_2(\cdot) + \rho_2(\cdot))v \quad &\text{for } (u, v) \in X, \\
B_{22}(u, v) &:= (0, \rho_1(\cdot)u) \quad &\text{for } (u, v) \in X.
\end{align*}
\]

Then we define the operator \(A_1\) as follows:

\[
A_1 \left(\begin{array}{c}
\tilde{u} \\
u
\end{array}\right) := \left(\begin{array}{c}
G \\
0
\end{array}\right) \left(\begin{array}{c}
\tilde{u} \\
u
\end{array}\right) \quad &\text{for } \left(\begin{array}{c}
\tilde{u} \\
u
\end{array}\right) \in D(A).
\]

By Corollary VI.1.11 of [10], we have that \(s(A_1) \leq s(A)\). Then the desired assertion follows if we prove that \(s(A_1) > -\infty\). To this end, we consider the eigenvalue problem

\[
(\lambda I - A_1)U_1 = 0.
\]

(23)

By writing \(U_1 = (\tilde{u}_1(\sigma, x), (u_1(x), v_1(x)))\), we see that Equation (23) can be rewritten as follows:

\[
\lambda \tilde{u}_1(\sigma, x) - \frac{\partial}{\partial \sigma} \tilde{u}_1(\sigma, x) = 0, \quad -\tau < \sigma < 0, \quad 0 < x < \tilde{a},
\]

\[
\lambda u_1(x) + \frac{d}{dx}(\gamma_1(x)u_1(x)) = -\mu_1(x)u_1(x) - \rho_1(x)u_1(x) + \rho_2(x)v_1(x), \quad 0 < x < \tilde{a},
\]

\[
\lambda v_1(x) + \frac{d}{dx}(\gamma_2(x)v_1(x)) = -\mu_2(x)v_1(x) - \rho_2(x)v_1(x), \quad 0 < x < \tilde{a},
\]

\[
\tilde{u}_1(0, x) = u_1(x), \quad 0 < x < \tilde{a},
\]

\[
\gamma_1(0)u_1(0) = v \int_0^\alpha \int_{-\tau}^0 \beta(\sigma, x)\tilde{u}_1(\sigma, x) \, d\sigma \, dx,
\]

\[
\gamma_2(0)v_1(0) = (1 - v) \int_0^\alpha \int_{-\tau}^0 \beta(\sigma, x)\tilde{u}_1(\sigma, x) \, d\sigma \, dx.
\]
we have that the eigenvalue equation (23) admits a non-trivial eigenvector if and only if
because of the non-local boundary condition and the distributed delay. The conjugate problem here is different and give the proof of Theorem 1.3 by the method which is inspired by [14]. This kind of comparison has been also considered in [3, 5].

In this section we compare the two-phase model with the corresponding one-phase model as
\[
\|e^{-s(A)t}T(t) - P\| \leq M e^{-\varepsilon t} \quad \text{for all } t > 0,
\]
where \(\| \cdot \|\) denotes the operator norm in \(E\) (see Theorem 4.1 of [15], Theorems 9.10 and 9.11 of [8], and Theorem C-IV.2.1 of [1]). This completes the proof of Theorem 1.2.

4. Relation with the one-phase model

In this section we compare the two-phase model with the corresponding one-phase model and give the proof of Theorem 1.3 by the method which is inspired by [14]. This kind of comparison has been also considered in [3, 5]. The conjugate problem here is different because of the non-local boundary condition and the distributed delay.

From (25), we have the following asymptotic expression:
\[
e^{-s(A)t}(m(t + \sigma, x), (m(t, x), n(t, x))) = c_1(\tilde{u}(\sigma, x), (\hat{u}(x), \hat{v}(x))) + O(e^{-\varepsilon_1 t})
\]
as \(t \to \infty\), where \(c_1\) is a constant uniquely determined by the initial data \((\hat{m}, n_0), s(A)\) and \((\tilde{u}(\sigma, x), (\hat{u}(x), \hat{v}(x)))\) are the dominant eigenvalue and the corresponding eigenvector of
the eigenvalue problem
\[ \lambda \tilde{u}(\sigma, x) - \frac{\partial}{\partial \sigma} \tilde{u}(\sigma, x) = 0, \quad -\tau < \sigma < 0, \quad 0 < x < \tilde{a}, \]
\[ \lambda \hat{u}(x) + \frac{d}{dx} (\gamma(x)\hat{u}(x)) = -\mu_1(x)\hat{u}(x) - \rho_1(x)\hat{u}(x) + \rho_2(x)\hat{v}(x), \quad 0 < x < \tilde{a}, \]
\[ \lambda \hat{v}(x) + \frac{d}{dx} (\gamma(x)\hat{v}(x)) = -\mu_2(x)\hat{u}(x) + \rho_1(x)\hat{u}(x) - \rho_2(x)\hat{v}(x), \quad 0 < x < \tilde{a}, \]
\[ \hat{u}(0, x) = \hat{u}(x), \quad 0 < x < \tilde{a}, \]
\[ \gamma(0)\hat{u}(0) = v \int_0^\tilde{a} \int_{-\tau}^0 \beta(\sigma, x)\tilde{u}(\sigma, x) \, d\sigma \, dx, \]
\[ \gamma(0)\hat{v}(0) = (1 - v) \int_0^\tilde{a} \int_{-\tau}^0 \beta(\sigma, x)\tilde{u}(\sigma, x) \, d\sigma \, dx. \]  

We can see that the corresponding eigenvector \((\tilde{u}(\sigma, x), (\hat{u}(x), \hat{v}(x)))\) is strictly positive, i.e. \(\tilde{u}(\sigma, x) > 0\) for all \((\sigma, x) \in [-\tau, 0] \times [0, \tilde{a}]\), \(\hat{u}(x) > 0\) and \(\hat{v}(x) > 0\) for all \(0 < x < \tilde{a}\).

Let \(\hat{N}(t, x)\) be the solution of the model (6). Similarly, we have the following asymptotic expression:
\[ e^{-\lambda_0 t} (\hat{N}(t + \sigma, x), \hat{N}(t, x)) = c_2(\hat{U}(\sigma, x), \hat{U}(x)) + O(e^{-\varepsilon t}) \quad \text{as } t \to \infty, \]  

where \(c_2\) is a constant uniquely determined by the initial data \(\hat{N}, \lambda_0\) and \((\hat{U}(\sigma, x), \hat{U}(x))\) are the dominant eigenvalue and the corresponding eigenvector of the eigenvalue problem
\[ \lambda \hat{U}(\sigma, x) - \frac{\partial}{\partial \sigma} \hat{U}(\sigma, x) = 0, \quad -\tau < \sigma < 0, \quad 0 < x < \tilde{a}, \]
\[ \lambda \hat{U}(x) + \frac{d}{dx} (\gamma(x)\hat{U}(x)) = -\mu(x)\hat{U}(x), \quad 0 < x < \tilde{a}, \]
\[ \hat{U}(0, x) = \hat{U}(x), \quad 0 < x < \tilde{a}, \]
\[ \gamma(0)\hat{U}(0) = \int_0^\tilde{a} \int_{-\tau}^0 \bar{\beta}(\sigma, x)\hat{U}(\sigma, x) \, d\sigma \, dx, \]

where \(\gamma(x), \mu(x)\) and \(\bar{\beta}(\sigma, x)\) are defined in (4) and (5). The corresponding eigenvector \((\hat{U}(\sigma, x), \hat{U}(x))\) is also strictly positive, i.e. \(\hat{U}(\sigma, x) > 0\) for all \((\sigma, x) \in [-\tau, 0] \times [0, \tilde{a}]\), \(\hat{U}(x) > 0\) for all \(0 < x < \tilde{a}\). From (4), (5), (27) and (29), we have \(\lambda_0 = s(A)\) and \(\hat{U}(x) = \hat{u}(x) + \hat{v}(x)\).

Next we want to compare \(N(t, x)\) and \(\hat{N}(t, x)\), where \(N(t, x)\) is defined in (7) and (8). The asymptotic expression (26) implies that
\[ e^{-\lambda_0 t} N(t, x) = c_1 \hat{U}(x) + O(e^{-\varepsilon_1 t}) \quad \text{as } t \to \infty, \]  

From (28) and (30), we have that
\[ e^{-\lambda_0 t} N(t, x) - e^{-\lambda_0 t} \hat{N}(t, x) = c_3 \hat{U}(x) + O(e^{-\varepsilon t}), \]
where \(c_3 = c_1 - c_2\) and \(\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}\). Then we prove that \(c_3 \neq 0\).
Let \((\phi, (\varphi, \psi))\) be the eigenvector of the conjugate problem of Equation (27), i.e.

\[
\begin{align*}
\lambda \phi(\sigma, x) + \frac{\partial}{\partial \sigma} \phi(\sigma, x) &= \nu \varphi(0) \beta(\sigma, x) + (1 - \nu) \psi(0) \beta(\sigma, x), \\
-\tau < \sigma < 0, 0 < x < \bar{a}, \\
\lambda \varphi(x) - \gamma(x) \frac{d \varphi(x)}{dx} &= -\mu_1(\sigma) \varphi(x) - \rho_1(\sigma) \varphi(x) + \rho_1(\sigma) \psi(x) + \phi(0, x), \quad 0 < x < \bar{a}, \\
\psi(x) - \gamma(x) \frac{d \psi(x)}{dx} &= -\mu_2(\sigma) \psi(x) - \rho_2(\sigma) \psi(x) + \rho_2(\sigma) \varphi(x), \quad 0 < x < \bar{a}, \\
\phi(-\tau, x) &= 0, \quad 0 < x < \bar{a}, \\
\varphi(\bar{a}) &= 0, \\
\psi(\bar{a}) &= 0. \tag{31}
\end{align*}
\]

We normalize \((\phi, (\varphi, \psi))\) such that

\[
\int_{-\tau}^{\bar{a}} \int_{0}^{0} \hat{u}(\sigma, x) \phi(\sigma, x) \, d\sigma \, dx + \int_{-\tau}^{\bar{a}} (\hat{u}(x) \varphi(x) + \hat{\psi}(x) \psi(x)) \, dx = 1. \tag{32}
\]

Due to a similar reason as that for \((\hat{u}(\sigma, x), (\hat{u}(x), \hat{\psi}(x)))\), \((\phi(\sigma, x), (\varphi(x), \psi(x)))\) is strictly positive, i.e. \(\phi(\sigma, x) > 0\) for all \((\sigma, x) \in [-\tau, 0] \times [0, \bar{a}], \varphi(x) > 0\) and \(\psi(x) > 0\) for all \(0 < x < \bar{a}\).

From (1) and (31), we easily obtain that

\[
\frac{d}{dt} \left( \int_{0}^{\bar{a}} \int_{-\tau}^{0} m(t + \sigma, x) \phi(\sigma, x) e^{-\lambda_0 t} \, d\sigma \, dx + \int_{0}^{\bar{a}} (m(t, x) \varphi(x) + n(t, x) \psi(x)) e^{-\lambda_0 t} \, dx \right) = 0.
\]

Hence

\[
\int_{0}^{\bar{a}} \int_{-\tau}^{0} m(t + \sigma, x) \phi(\sigma, x) e^{-\lambda_0 t} \, d\sigma \, dx + \int_{0}^{\bar{a}} (m(t, x) \varphi(x) + n(t, x) \psi(x)) e^{-\lambda_0 t} \, dx
\]

\[
= \int_{0}^{\bar{a}} \int_{-\tau}^{0} \hat{m}(\sigma, x) \phi(\sigma, x) \, d\sigma \, dx + \int_{0}^{\bar{a}} (\hat{m}(x) \varphi(x) + n_0(x) \psi(x)) \, dx
\]

for all \(t \geq 0\). Letting \(t \to \infty\) and using (26), we have that

\[
c_1 \left( \int_{0}^{\bar{a}} \int_{-\tau}^{0} \hat{u}(\sigma, x) \phi(\sigma, x) \, d\sigma \, dx + \int_{0}^{\bar{a}} (\hat{u}(x) \varphi(x) + \hat{\psi}(x) \psi(x)) \, dx \right)
\]

\[
= \int_{0}^{\bar{a}} \int_{-\tau}^{0} \hat{m}(\sigma, x) \phi(\sigma, x) \, d\sigma \, dx + \int_{0}^{\bar{a}} (\hat{m}(x) \varphi(x) + n_0(x) \psi(x)) \, dx.
\]
From (32), we have that
\[ c_1 = \int_0^\tilde{a} \int_{-\tau}^0 \hat{m}(\sigma, x)\phi(\sigma, x) \, d\sigma \, dx + \int_0^\tilde{a} (\hat{m}_0(x)\varphi(x) + n_0(x)\psi(x)) \, dx. \] (33)

Let \((\Phi, \Psi)\) be the eigenvector of the conjugate problem of Equation (29), i.e.
\[ \lambda \Phi(\sigma, x) + \frac{\partial}{\partial \sigma} \Phi(\sigma, x) = \Psi(0) \hat{\beta}(\sigma, x), \quad -\tau < \sigma < 0, 0 < x < \tilde{a}, \]
\[ \lambda \Psi(x) - \gamma(x) \frac{d\Psi(x)}{dx} = -\mu(x) \Psi(x) + \Phi(0, x), \quad 0 < x < \tilde{a}, \] (34)
\[ \Phi(-\tau, x) = 0, \quad 0 < x < \tilde{a}, \]
\[ \Psi(\tilde{a}) = 0. \]

We normalize \((\Phi, \Psi)\) such that
\[ \int_0^\tilde{a} \int_{-\tau}^0 \hat{U}(\sigma, x)\Phi(\sigma, x) \, d\sigma \, dx + \int_0^\tilde{a} \hat{U}(x)\Psi(x) \, dx = 1. \] (35)

\((\Phi, \Psi)\) is also strictly positive, i.e. \(\Phi(x) > 0\) for all \((\sigma, x) \in [-\tau, 0] \times [0, \tilde{a}]\), \(\Psi(x) > 0\) for all \(0 < x < \tilde{a}\). By a similar argument we have that
\[ c_2 = \int_0^\tilde{a} \int_{-\tau}^0 \hat{N}(\sigma, x)\Phi(\sigma, x) \, d\sigma \, dx + \int_0^\tilde{a} (\hat{m}_0(x) + n_0(x))\Psi(x) \, dx \]
\[ = \int_0^\tilde{a} \int_{-\tau}^0 (\hat{m}(\sigma, x) + n_0(x))\Phi(\sigma, x) \, d\sigma \, dx + \int_0^\tilde{a} (\hat{m}_0(x) + n_0(x))\Psi(x) \, dx, \]
so that generally speaking we have \(c_1 \neq c_2\) or \(c_3 \neq 0\). This proves Theorem 1.3.

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