Lorentz Ricci Solitons of Four-Dimensional Non-Abelian Nilpotent Lie Groups

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Abstract. The goal of this paper is to investigate which one of the non-isometric left-invariant Lorentz metrics \( g \) on four-dimensional nilpotent Lie groups \( H_3 \times \mathbb{R} \) and \( G_4 \) satisfies the Ricci soliton equation \( 2\text{Ric}[g] + \mathcal{L}_X g + \alpha g = 0 \), here \( X \) is a vector field and \( \alpha \) is a constant. Among the left-invariant Lorentzian metrics on \( H_3 \times \mathbb{R} \), \( g^- \) is a shrinking while \( g^+ \) and \( g_\mu \) are expanding and also \( g_0^1, g_0^2, g_0^3 \) have Ricci solitons. We exhibit among the non-isometric left-invariant Lorentz metric on the group \( G_4 \) only \( g_\lambda^1, g_\lambda^2 \) have Lorentz Ricci solitons and \( g_\lambda^2 \) is a shrinking.

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1. Introduction

Let \((N,g)\) be a nilpotent Lie group with left-invariant metric \( g \) and let \( \mathfrak{n} \) be its corresponding Lie algebra with the induced inner product that we denote by \( \langle ., . \rangle \). If \( g \) satisfies

\[
2\text{Ric}[g] + \mathcal{L}_X g + \alpha g = 0,
\]

where \( X \) is some vector field and \( \alpha \) is some constant, then \((M^n, g, X, \alpha)\) is called a Ricci soliton structure and \( g \) is called the Ricci soliton. Moreover we say that the Ricci soliton \( g \) is a gradient Ricci soliton if the vector field \( X \) satisfies \( X = \nabla f \), where \( f \) is some function, and the Ricci soliton \( g \) is a non-gradient Ricci soliton if the vector field \( X \) satisfies \( X \neq \nabla f \) for any function \( f \). If a constant \( \alpha \) is negative, zero, or positive, then \( g \) is called a shrinking, steady, or expanding Ricci soliton, respectively. Recall that the Ricci solitons have a relation with the Ricci flow. A pseudo-Riemannian metric \( g \) is a Ricci soliton if and only if \( g \) is a solution of the Ricci flow equation,

\[
\frac{\partial g}{\partial t} = -2\text{Ric}[g(t)],
\]
with initial condition \( g(0) = g \) where \( g(t) = c(t)(\varphi_t)^*g \), and here \( c(t) \) is a scaling parameter, and \( \varphi_t \) is a diffeomorphism \([3,4]\).

Rahmani \([7]\) showed the Heisenberg group \( H_3 \) has three non-isometric left-invariant Lorentzian metrics \( g_1, g_2, \) and \( g_3 \) and in \([8]\), he and his collaborator presented that the associated Lie algebras of infinitesimal isometries of \( (H_3, g_1) \) and \( (H_3, g_2) \) are four-dimensional and solvable but not nilpotent, and the associated Lie algebra of infinitesimal isometries of \( (H_3, g_3) \) is six-dimensional and the left-invariant Lorentzian metric \( g_2 \) has negative constant curvature \(-\frac{1}{4}\), \( g_3 \) is flat, and \( g_1 \) is not Einstein. In \([6]\), Onda characterized the left-invariant Lorentzian metric \( g_1 \) as a Lorentz Ricci soliton. Moreover, he proved that the group rigid motions of Euclidean 2-space, \( E(2) \), and the group rigid motions of Minkowski 2-space, \( E(1,1) \), have Lorentz Ricci solitons.

In \([11]\), Wears carried out a complete investigation of the left-invariant Lorentzian metrics on a five-dimensional, connected, simply-connected, two-step nilpotent Lie group and its left-invariant Ricci soliton and algebraic Ricci soliton metrics. One of his student in her dissertation \([9]\) tried to the present classification of Lorentzian scalar products of some four dimensional Lie algebras as Ricci solitons.

Yong Wang classified affine Ricci solitons associated with canonical connections and Kobayashi–Nomizu connections and perturbed canonical connections and perturbed Kobayashi–Nomizu connections on three-dimensional Lorentzian Lie groups with some product structure \([10]\).

Recently, Calvaruso \([2]\) studied the semi-direct extensions \( G_S = H \rtimes \exp(RS) \) of the three-dimensional Heisenberg group \( H \), equipped with a one-parameter family of left-invariant metrics \( g_a, a^2 \neq 1 \) and he calculated several curvature properties of \( (G_S, g_a) \) and presented a complete classification of its algebraic Ricci solitons with constructing some new examples of non-algebraic Lorentzian Ricci solitons.

Bokan et al. \([1]\) classified left-invariant Lorentz metrics on four-dimensional nilpotent Lie groups \( H_3 \times \mathbb{R} \) and \( G_4 \). Magnin \([5]\) proved that, up to isomorphism, \( h_3 \oplus \mathbb{R} \) and \( g_4 \) with corresponding Lie groups \( H_3 \times \mathbb{R} \) and \( G_4 \) are only two non-Abelian nilpotent Lie algebras of dimension 4.

### 2. The Ricci Soliton on \( H_3 \times \mathbb{R} \)

Algebra \( h_3 \oplus \mathbb{R} \) is spanned by basis \( \{x_1, x_2, x_3, x_4\} \) with nonzero commutator

\[
[x_1, x_2] = x_3.
\]  

(3)

The algebra \( h_3 \oplus \mathbb{R} \) is 2-step nilpotent with two dimensional center \( Z(h_3 \oplus \mathbb{R}) = \mathcal{L}(x_4, x_3) \) and one dimensional commutator subalgebra spanned by \( x_3 \). The group action on \( H_3 \times \mathbb{R} \) for coordinates \( (x, y, z, w) \) is given by

\[
(x, y, z, w) \cdot (x', y', z', w') = (x + x', y + y' + xz', z + z', w + w').
\]  

(4)

The Lie algebra \( h_3 \oplus \mathbb{R} \) of \( H_3 \times \mathbb{R} \) has a basis with frame

\[
X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z}, \quad X_4 = \frac{\partial}{\partial w}.
\]  

(5)
The coframe dual to the left-invariant frame (5) is given by the basis of one-forms
\[ \omega_1 = dx, \quad \omega_2 = dy, \quad \omega_3 = dz - xdy, \quad \omega_4 = dw. \] (6)
Therefore
\[ d\omega_1 = 0, \quad d\omega_2 = 0, \quad d\omega_3 = -\omega_1 \wedge \omega_2, \quad d\omega_4 = 0. \] (7)

**Theorem 1.** Each left-invariant Lorentz metric on the group \( H_3 \times \mathbb{R} \), up to an automorphism of \( H_3 \times \mathbb{R} \), is isometric to one of the following
\[ g_\mu = dx^2 - dy^2 + \mu(xdy - dz)^2 + dw^2, \]
\[ g_\lambda^\pm = dx^2 + dy^2 \pm \lambda(xdy - dz)^2 \mp dw^2, \]
\[ g_0^1 = dx^2 + dy^2 - 2x\,dxdy \,dw + 2dz \,dw, \]
\[ g_0^2 = dx^2 - 2x \,dy^2 + dw^2 + 2dy \,dz, \]
\[ g_0^3 = dx^2 + 2dy \,dw + (xdy - dz)^2, \]
where \( \lambda, \mu > 0. \)

**Theorem 2.** Non-isometric left-invariant Lorentz metric on the group \( H_3 \times \mathbb{R} \) are given by
\[ g_\mu = \omega_1 \otimes \omega_1 - \omega_2 \otimes \omega_2 + \mu \omega_3 \otimes \omega_3 + \omega_4 \otimes \omega_4, \]
\[ g_\lambda^\pm = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 \pm \lambda \omega_3 \otimes \omega_3 \mp \omega_4 \otimes \omega_4, \]
\[ g_0^1 = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + 2\omega_3 \otimes \omega_3, \]
\[ g_0^2 = \omega_1 \otimes \omega_1 + 2\omega_2 \otimes \omega_3 + \omega_4 \otimes \omega_4, \]
\[ g_0^3 = \omega_1 \otimes \omega_1 + 2\omega_2 \otimes \omega_4 + \omega_3 \otimes \omega_3. \]

We can write \( g_\mu \) and \( g_\lambda^\pm \) metrics in general form
\[ g = \omega_1 \otimes \omega_1 + a_1 \omega_2 \otimes \omega_2 + a_2 \omega_3 \otimes \omega_3 + a_3 \omega_4 \otimes \omega_4, \] (8)
where substituting \( a_1 = -a_3 = -1 \), \( a_2 = \mu \) in the (8) we find \( g_\mu \) and substituting \( a_1 = 1, a_2 = \pm \lambda, a_3 = \mp 1 \) in (8), leads to \( g_\lambda^\pm \).

**Theorem 3.** The left-invariant Lorentzian metrics \( g_\mu \) and \( g_\lambda^\pm \) with general form (8) satisfy a Ricci soliton equation
\[ 2\text{Ric}[g] + \mathcal{L}_X g - \frac{3a_2}{a_1} g = 0, \] (9)
where the vector field \( X \) is defined by
\[ X = \left( -\frac{a_2}{a_1} x + C_2 y + C_3 \right) X_1 - \left( \frac{1}{a_1} (C_2 x + a_2 y) + C_4 \right) X_2 \]
\[ + \left( \frac{C_2}{2a_1} (x^2 - a_1 y^2) + C_3 y + C_4 x - \frac{a_2}{a_1} (xy - 2z) + C_5 \right) X_3 \]
\[ + \left( a_2 \frac{y - 3a_2}{2a_1} w + C_1 \right) X_4. \] (10)
Here for $g_\mu$ we have $a_1 = -a_3 = -1$, $a_2 = \mu$ and for $g_\lambda^\pm$ we have $a_1 = 1$, $a_2 = \pm \lambda$, $a_3 = \mp 1$ and $C_i$ for $i = 1, \ldots, 5$ are constants. Therefore the left-invariant Lorentzian metric $g_\pm^\lambda$ is a shrinking Lorentz Ricci soliton while $g_\pm^\mu$ and $g_\mu$ are expanding.

**Proof.** For (8), there is a unique solution $g(t)$ to the Ricci flow satisfied in (2), with the following form

$$g(t) = f_1(t) \omega_1 \otimes \omega_1 + f_2(t) \omega_2 \otimes \omega_2$$

$$+ f_3(t) \omega_3 \otimes \omega_3 + f_4(t) \omega_4 \otimes \omega_4,$$

(11)
satisfying the initial condition $g(0) = g_0$. The matrix of connection one-forms of a metric $g$ as in (8) is

$$\omega = (\omega^i_j) = \frac{1}{2} \begin{pmatrix}
0 & a_2 \omega^3 & c \omega^2 & 0 \\
-a_2 \omega^3 & 0 & -a_2 \omega^1 & 0 \\
-a_1 \omega^3 & a_1^2 \omega^2 & a_1^2 \omega^1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},$$

and the matrix of curvature two-forms is given by

$$\Omega = (\Omega^i_j) = \frac{1}{4} \begin{pmatrix}
3a_2 & 0 & -a_2 \omega^1 \wedge \omega^2 & a_2 \omega^1 \wedge \omega^3 & 0 \\
a_2^3 & 0 & -a_a \omega^2 \wedge \omega^3 & 0 \\
-a_2^3 & a_2^3 \omega^2 \wedge \omega^3 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$

Therefore the Ricci curvature is

$$\text{Ric} \ [g] = -\frac{a_2}{2a_1} \omega_1 \otimes \omega_1 - \frac{a_2}{2} \omega_2 \otimes \omega_2 + \frac{a_2^2}{2a_1} \omega_3 \otimes \omega_3.$$  

(12)

The scalar curvature is

$$S [g] = -\frac{a_2}{2a_1},$$

(13)

and the corresponding Ricci operator is

$$\text{Rc} \ [g] = \text{diag} \left( -\frac{3a_2}{8a_1}, -\frac{3a_2}{8}, \frac{5a_2^2}{8a_1}, \frac{a_2a_3}{8a_1} \right).$$

(14)

Now assume $X$ be an arbitrary vector field on $H_3 \times \mathbb{R}$ by

$$X = \sum_{i=1}^{4} P^i(x, y, z, w) \ X_i,$$

(15)

where the $P^i$'s are smooth functions on $H_3 \times \mathbb{R}$. The component functions of the Lie derivative $\mathcal{L}_X g$ are recorded in the following symmetric matrix

$$(\mathcal{L}_X g)_{ij} = \begin{pmatrix}
2P^1_{x^2} & P^1_{x^2+y^2} & P^1_{x^2+y^2} - xP^2_{y^2} & P^1_{x^2+y^2} - xP^2_{y^2} \\
2P^1_{y^2} & P^1_{x^2+y^2} & P^1_{x^2+y^2} - yP^2_{x^2} & P^1_{x^2+y^2} - yP^2_{x^2} \\
2P^1_{z^2} & P^1_{x^2+y^2} & P^1_{x^2+y^2} - zP^2_{x^2} & P^1_{x^2+y^2} - zP^2_{x^2} \\
2P^1_{w^2} & P^1_{x^2+y^2} & P^1_{x^2+y^2} - wP^2_{x^2} & P^1_{x^2+y^2} - wP^2_{x^2}
\end{pmatrix}.$$
The corresponding Ricci soliton equation $2\text{Ric}[g] + \mathcal{L}_X g + \alpha g = 0$, concludes the following system of partial differential equations

$$\begin{cases}
-\frac{a_2^2}{a_1} + 2P_1^1 + \alpha = 0, \\
\frac{a_2^2}{a_1} x^2 - a_2 + 2xP_1^1 - 2xP_3^3 + 2(1 + x^2)P_3^2 + a_1 \alpha + a_2 \alpha x^2 = 0, \\
\frac{a_2^2}{a_1} + 2P_3^3 - 2xP_3^2 + a_2 \alpha = 0, \\
2P_4^4 + a_3 \alpha = 0, \\
-\frac{a_2^2}{a_1} x^2 + P_3^3 + (1 + x^2)P_3^2 - x(P_3^3 + P_3^2) - P_1^1 - a_2 \alpha x = 0, \\
P_1^1 + (1 + x^2)P_3^2 - xp_3^3 = 0, \\
P_3^3 + P_1^1 - xp_3^2 = 0, \\
P_4^4 - xp_3^3 + (1 + x^2)P_3^2 = 0, \\
P_4^4 + P_3^3 - xp_3^2 = 0
\end{cases}$$

We can obtain the coefficient functions $P_i$ of the vector field (10), by solving the above system of PDE’s and additionally it concludes $\alpha = -\frac{3a_2}{a_1}$. Putting $a_1 = 1$, $a_2 = \lambda$ in metric $g^\lambda_0$ we have $\alpha = -3\lambda$. Then it is a shrinking Lorentz Ricci soliton while in $g^-\lambda$ and $g^\mu$ substituting $a_1 = 1$, $a_2 = -\lambda$ and $a_1 = -1$, $a_2 = \mu$ respectively, we have $\alpha = 3\lambda$ and $\alpha = 3\mu$ respectively that shows they are expanding.

**Theorem 4.** The left-invariant Lorentzian metrics

$$g^1_0 = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + 2\omega_3 \otimes \omega_4,$$

satisfies a Ricci soliton equation

$$2\text{Ric}[g^1_0] + \mathcal{L}_X g^1_0 + \alpha g^1_0 = 0,$$

where the vector field $X$ is defined by

$$X = (C_1 y - \frac{\alpha}{2} x + C_5 \cos w + C_6 \sin w + C_2) \ X_1$$

$$- \left( C_1 x + \frac{\alpha}{2} y - C_5 \sin w + C_6 \cos w + C_3 \right) \ X_2$$

$$+ \left( \frac{C_1}{2} (x^2 + y^2) + (C_3 + \frac{\alpha y}{2})x + C_2 y - \alpha z - \frac{w}{2} + C_4 \right) \ X_3 + C_7 \ X_4,$$

where $C_i$ for $i = 1, \ldots, 7$ are arbitrary constants and $\alpha \in \mathbb{R}$.

**Proof.** The matrix of connection one-forms of the metric $g^1_0$ is

$$\omega = (\omega^i_j) = \frac{1}{2} \begin{pmatrix}
0 & \omega^4 & 0 & \omega^2 \\
-\omega^4 & 0 & 0 & -\omega^1 \\
-\omega^2 & \omega^1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$
and the matrix of curvature two-forms is represented by
\[
\Omega = (\Omega^i_j) = \frac{1}{4} \begin{pmatrix}
0 & 0 & 0 & \omega^1 \wedge \omega^4 \\
0 & 0 & 0 & \omega^2 \wedge \omega^4 \\
-\omega^1 \wedge \omega^4 & -\omega^2 \wedge \omega^4 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Thus the Ricci curvature is
\[
\text{Ric} \left[ g^1_0 \right] = \frac{1}{2} \omega_4 \otimes \omega_4, \tag{20}
\]
and the corresponding scalar curvature is equal to zero. Therefore the Ricci operator is
\[
\text{Rc} \left[ g^1_0 \right] = \text{diag} \left( 0, 0, 0, \frac{1}{2} \right). \tag{21}
\]
Let \( X \) be an arbitrary vector field \( X = \sum_{i=1}^{4} P^i(x, y, z, w) X_i \) on \( H_3 \times \mathbb{R} \). The Lie derivative \( \mathcal{L}_X g^1_0 \) as a symmetric matrix is given by
\[
(\mathcal{L}_X g^1_0)_{ij} = \begin{pmatrix}
2P^1_x & P^1_y + P^2_x - xP^4_x & P^1_z + P^4_x & P^1_w + P^2_y + P^3_x - xP^4_w \\
2P^2_y - 2xP^4_y & P^2_x + P^4_y - xP^4_z & P^2_z + P^4_w & P^3_z + P^4_w \\
2P^3_z & 2P^3_w & P^3_w & P^3_w \\
2P^3_w & 2P^3_w & 2P^3_w & 2P^3_w
\end{pmatrix}.
\]
Putting the arrays of \( (\mathcal{L}_X g^1_0)_{ij} \) and Ricci curvature (20) in the Ricci soliton equation (18), leads to
\[
\begin{cases}
(\mathcal{L}_X g)_{11} + \alpha = 0, \\
(\mathcal{L}_X g)_{22} + \alpha = 0, \\
1 + (\mathcal{L}_X g)_{44} = 0, \\
(\mathcal{L}_X g)_{34} + \alpha = 0, \\
(\mathcal{L}_X g)_{ij} = 0, \text{ other cases.}
\end{cases} \tag{22}
\]
Solving the system (22), the coefficient functions \( P^i \)'s of the vector field (19) are obtained.

**Theorem 5.** The left-invariant Lorentzian metric
\[
g^2_0 = \omega_1 \otimes \omega_1 + 2\omega_2 \otimes \omega_3 + \omega_4 \otimes \omega_4, \tag{23}
\]

satisfies a Ricci soliton equation
\[
2\text{Ric} \left[ g^2_0 \right] + \mathcal{L}_X g^2_0 + \alpha g^2_0 = 0, \tag{24}
\]
where the vector field $X$ is defined by

$$X = \left(-\frac{\alpha}{2} x - C_1 y w - C_2 w + C_5 z \right. $$

$$+ \left(\frac{C_5}{3} y^3 + \frac{C_6}{2} y^2 + C_7 y + C_8\right) X_1 + \left(C_1 w - C_5 \left(\frac{y^2}{2} + x\right) \right. $$

$$- \left(\frac{\alpha + 2 C_6}{4} y + C_9\right) X_2 + \left(\frac{C_6}{2} - \frac{3 \alpha}{4}\right) z $$

$$+ \left(\frac{C_5}{12} y^2 + \frac{C_6}{6} y + \frac{C_1}{2} w - \frac{C_5}{2} x + C_7 \right) y^2 + \left(\frac{\alpha}{4} - \frac{C_6}{2}\right) x y $$

$$+ C_8 y + C_5 y z - C_2 y w - (C_7 + C_9) x - C_3 w + C_{10}\right) X_3 $$

$$+ \left(-\frac{\alpha}{2} w + C_1 (x y - z + \frac{y^3}{6}) \right. $$

$$+ C_2 (x + \frac{y^2}{2}) + C_3 y + C_4\right) X_4. $$

(25)

Here $C_i$ for $i = 1, \ldots, 10$ and $\alpha$ are arbitrary constants.

**Proof.** Since the matrix of connection one-forms of the metric $g_0^2$ is

$$\omega = (\omega^i_j) = \frac{1}{2} \begin{pmatrix}
0 & \omega^2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\omega^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},$$

then the matrix of curvature two-forms equals to zero matrix and the Ricci curvature is

$$\text{Ric} \left[g_0^2\right] = 0. $$

(26)

Suppose that $X$ is an arbitrary vector field $X = \sum_{i=1}^{4} P^i(x, y, z, w) X_i$ on $H_3 \times \mathbb{R}$. The Lie derivative $\mathcal{L}_X g_0^2$ as a symmetric matrix is given by

$$\left(\mathcal{L}_X g_0^2\right)_{ij} = \begin{pmatrix}
2 P^1_x P^2 + P^1_y - x P^2_y + P^3_y \\
-2 P^1_y + 2 P^2_y + 2 P^3_y \\
P^2_x - x P^2_z + P^3_z \\
2 P^2_z - x P^2_w + P^3_w + P^4_w
\end{pmatrix}. $$

Substituting the arrays of above matrix and (26) in the Ricci soliton equation (24), concludes to

$$\begin{cases}
(\mathcal{L}_X g)_{11} + \alpha = 0, \\
(\mathcal{L}_X g)_{44} + \alpha = 0, \\
(\mathcal{L}_X g)_{23} + \alpha = 0, \\
(\mathcal{L}_X g)_{ij} = 0, \text{ other cases}
\end{cases} $$

(27)

Finally by solving the system (27), one finds immediately the coefficient functions $P^i$’s of the vector field (25).

**Theorem 6.** The left-invariant Lorentzian metrics

$$g_0^3 = \omega_1 \otimes \omega_1 + 2 \omega_2 \otimes \omega_4 + \omega_3 \otimes \omega_3. $$

(28)

satisfies a Ricci soliton equation

$$2\text{Ric}[g_0^3] + \mathcal{L}_X g_0^3 + \alpha g_0^3 = 0, $$

(29)
where the vector field $X$ is defined by

$$X = \left(-\frac{\alpha}{2}x + \frac{C_1}{2}y^2 + C_2y + C_3\right)X_1 + C_4X_2 + \left(-\frac{\alpha}{2}z + \frac{C_1}{6}(y^3 - 6y) + \frac{C_2}{2}y^2 + C_3y - C_4x + C_5\right)X_3 + \left(\frac{y}{2} - \alpha w + \frac{C_1}{2}(2z - xy) - C_2x + C_6\right)X_4,$$

(30)

$C_i$ for $i = 1, \ldots, 6$ are arbitrary constants and $\alpha \in \mathbb{R}$.

Proof. The matrix of connection one-forms of the metric $g^3_0$ is

$$\omega = (\omega^i_j) = \frac{1}{2} \begin{pmatrix} 0 & \omega^3 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 \\ -\omega^2 & \omega^1 & 0 & 0 \\ -\omega^3 & 0 & -\omega^1 & 0 \end{pmatrix},$$

and the matrix of curvature two-forms is represented by

$$\Omega = (\Omega^i_j) = \frac{1}{4} \begin{pmatrix} 0 & -3\omega^1 \wedge \omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\omega^2 \wedge \omega^3 & 0 & 0 \\ 3\omega^1 \wedge \omega^2 & 0 & \omega^2 \wedge \omega^3 & 0 \end{pmatrix}.$$

Thus the Ricci curvature tensor is

$$\text{Ric} [g^1_0] = -\frac{1}{2} \omega_2 \otimes \omega_2$$

(31)

and the corresponding scalar curvature equals to zero. Therefore the Ricci operator is

$$\text{Rc} [g^1_0] = \text{diag} \left(0, -\frac{1}{2}, 0, 0\right).$$

(32)

The Lie derivative $L_Xg^3_0$ where $X = \sum_{i=1}^{4} P^i(x, y, z, w)X_i$ is an arbitrary vector field on $H_3 \times \mathbb{R}$, as a symmetric matrix is given by

$$(L_Xg^3_0)_{ij} = \begin{pmatrix} 2P^1_x & P^1_y - P^2_x \frac{y^2}{2} - P^3_x + P^4_x & P^1_y + P^2 + P^3_x & P^1_y + P^2_x \\ 2x(P^1_y - P^3_y) + 2P^4_y & -P^1_y + P^3_x - P^3_z + P^4_z & P^1_y + P^2 + P^3_x & P^1_y + P^2_x \\ 2P^3_x & P^3_y - P^3_y - xP^3_x - P^4_x & P^3_y - xP^3_x + P^4_x & P^3_y + P^3_x \\ 2P^3_x & P^3_y - xP^3_x + P^4_x & P^3_y + P^3_x & P^3_y - xP^3_x + P^4_x \end{pmatrix}.$$

When we put the above matrix and (31) in the Ricci soliton equation (29), we have

$$\begin{cases} (L_Xg)_{11} + \alpha = 0, \\
-1 + (L_Xg)_{22} = 0, \\
(L_Xg)_{33} + \alpha = 0, \\
(L_Xg)_{24} + \alpha = 0, \\
(L_Xg)_{ij} = 0, \text{ other cases.} \end{cases}$$

(33)

The coefficient functions $P^i$s of the vector field (30) are readily seen to be a solution to the system (33).
3. The Ricci Soliton on $G_4$

The algebra $g_4$ is spanned by basis \{$x_1, x_2, x_3, x_4$\} with nonzero commutators

$$[x_1, x_2] = x_3, \quad [x_1, x_3] = x_4. \quad (34)$$

If $X_1, X_2, X_3, X_4$ are left-invariant vector fields on $G_4$ defined by $x_1, x_2, x_3, x_4 \in g_4$, for global coordinates $(x, y, z, w)$ on $G_4$ we have the relations

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} + \frac{x^2}{2} \frac{\partial}{\partial w}, \quad X_3 = \frac{\partial}{\partial z} + x \frac{\partial}{\partial w}, \quad X_4 = \frac{\partial}{\partial w}. \quad (35)$$

The coframe dual to the left-invariant frame (35) is obtained by the basis of one-forms

$$\omega_1 = dx, \quad \omega_2 = dy, \quad \omega_3 = dz - x dy, \quad \omega_4 = \frac{x^2}{2} dy - x dz + dw. \quad (36)$$

therefore

$$d\omega_1 = 0, \quad d\omega_2 = 0, \quad d\omega_3 = -\omega_1 \wedge \omega_2, \quad d\omega_4 = -\omega_1 \wedge \omega_3. \quad (37)$$

**Theorem 7.** [1] Each left-invariant Lorentz metric on the group $G_4$, up to an automorphism of $G_4$, is isometric to one of the following

$$g_A^\pm = \pm dx^2 \mp dy^2 + a(xdy - dz)^2 - b(xdy - dz)(2dw - 2xdz + x^2dy) + \frac{c}{4}(2dw + x(xdy - 2dz))^2,$$

$$g_A = dx^2 + dy^2 + a(xdy - dz)^2 - b(xdy - dz)(2dw - 2xdz + x^2dy) + \frac{c}{4}(2dw + x(xdy - 2dz))^2,$$

$$g_1^\lambda = dx^2 + 2dwdy + xdy(xdy - 2dz) + \lambda(xdy - dz)^2,$$

$$g_2^\lambda = 2dwdx + dy^2 + xdx(xdy - 2dz) + \lambda(xdy - dz)^2,$$

$$g_3^\lambda = dy^2 - 2dx(xdy - dz) + \frac{\lambda}{4}(2dw + x(xdy - 2dz))^2,$$

$$g_4^\lambda = dx^2 - 2dy(xdy - dz) + \frac{\lambda}{4}(2dw + x(xdy - 2dz))^2,$$

where $\lambda > 0$.

**Theorem 8.** By a calculation in Theorem 7, we can rewrite

$$g_A^\pm = \pm \omega_1 \otimes \omega_1 \mp \omega_2 \otimes \omega_2 + a \omega_3 \otimes \omega_3 + 2b \omega_3 \otimes \omega_4 + c \omega_4 \otimes \omega_4,$$

$$g_A = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + a \omega_3 \otimes \omega_3 + 2b \omega_3 \otimes \omega_4 + c \omega_4 \otimes \omega_4,$$

$$g_1^\lambda = \omega_1 \otimes \omega_1 + 2 \omega_2 \otimes \omega_4 + \lambda \omega_3 \otimes \omega_3,$$

$$g_2^\lambda = \omega_2 \otimes \omega_2 + 2 \omega_1 \otimes \omega_4 + \lambda \omega_3 \otimes \omega_3,$$

$$g_3^\lambda = \omega_2 \otimes \omega_2 + 2 \omega_1 \otimes \omega_3 + \lambda \omega_4 \otimes \omega_4,$$

$$g_4^\lambda = \omega_1 \otimes \omega_1 + 2 \omega_2 \otimes \omega_3 + \lambda \omega_4 \otimes \omega_4.$$
Theorem 9. The left-invariant Lorentzian metric $g_1^\lambda$ satisfies a Ricci soliton equation

$$2\text{Ric}[g_1^\lambda] + \mathcal{L}X g_1^\lambda + \alpha g_1^\lambda = 0,$$

where the vector field $X$ is defined by

$$X = -\frac{\alpha}{2} x X_1 + C_1 X_2 - \left(\frac{\alpha}{2} z + C_1 x + C_2\right) X_3 + \frac{1}{2} \left(\lambda y + \alpha(2w) + C_1 x^2 - 2C_2 x + 2C_3\right) X_4,$$

where $C_i$ for $i = 1, \ldots, 3$ are arbitrary constants and $\alpha \in \mathbb{R}$.

Proof. The matrix of connection one-forms of a metric $g_1^\lambda$ is

$$\omega = (\omega^i_j) = \frac{1}{2} \begin{pmatrix}
0 & (\lambda + 1) \omega^3 & (\lambda + 1) \omega^2 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{1 + \lambda}{\lambda} \omega^2 & \frac{\lambda - 1}{\lambda} \omega^1 & 0 & 0 \\
-(1 + \lambda) \omega^3 & 0 & (1 - \lambda) \omega^1 & 0
\end{pmatrix},$$

and the matrix of curvature two-forms is given by

$$\Omega = (\Omega^i_j) = \frac{1}{4} \begin{pmatrix}
0 & \frac{-3\lambda^2 + 2\lambda - 1}{\lambda} \omega^1 \wedge \omega^2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \frac{(1 + \lambda)^2}{\lambda} \omega^2 \wedge \omega^3 & 0 & 0 \\
\frac{3\lambda^2 + 2\lambda - 1}{\lambda} \omega^1 \wedge \omega^2 & 0 & (1 + \lambda)^2 \omega^2 \wedge \omega^3 & 0
\end{pmatrix}.$$

Therefore the Ricci curvature is

$$\text{Ric} [g_1^\lambda] = \frac{1 - \lambda^2}{2\lambda} \omega_2 \otimes \omega_2,$$

and the scalar curvature equals to zero. The corresponding Ricci operator is

$$\text{Rc} [g_1^\lambda] = \text{diag} \left(0, \frac{1 - \lambda^2}{2\lambda}, 0, 0\right).$$

Assume $X = \sum_{i=1}^4 P^i(x, y, z, w) X_i$ as an arbitrary vector field on $G_4$ where the $P^i$ are smooth functions on $G_4$. The component functions of the
Lie derivative $\mathcal{L}_X g^\lambda_1$ are listed by
\[
\begin{align*}
(\mathcal{L}_X g^\lambda_1)_{11} &= 2P_x^1, \\
(\mathcal{L}_X g^\lambda_1)_{12} &= P_y^1 - \frac{3}{2} x^2 P_x^2 + x(P^2 + P_x^3)(1 - \lambda) + P^3 + P_x^4, \\
(\mathcal{L}_X g^\lambda_1)_{13} &= P_z^1 + \lambda P^3 - 2x P_x^2 + \lambda P_x^3, \\
(\mathcal{L}_X g^\lambda_1)_{14} &= P_w^1 + P_x^2, \\
(\mathcal{L}_X g^\lambda_1)_{22} &= 2(1 + \lambda)x P^1 + x^2 P_y^2 - 2\lambda x P_y^3 + 2P_y^4, \\
(\mathcal{L}_X g^\lambda_1)_{23} &= -(2 + \lambda)P^1 - 2x P_x^2 + \lambda P_y^3 - \lambda x P_x^3 - \frac{x^2}{2} P_y^2 - P^3 + P_x^4, \\
(\mathcal{L}_X g^\lambda_1)_{24} &= P_y^2 - \frac{x^2}{2} P_w^2 - (1 + \lambda)x P_y^3 + P, \\
(\mathcal{L}_X g^\lambda_1)_{33} &= 2\lambda P^3_z - 2x P_x^2, \\
(\mathcal{L}_X g^\lambda_1)_{34} &= P_z^2 + \lambda P^3_w, \\
(\mathcal{L}_X g^\lambda_1)_{44} &= 2P_w^2.
\end{align*}
\]

The corresponding Ricci soliton equation $2\text{Ric}[g^\lambda_1] + \mathcal{L}_X g^\lambda_1 + \alpha g^\lambda_1 = 0$, deduces the following system of PDE’s
\[
\begin{align*}
\begin{cases}
(\mathcal{L}_X g^\lambda_1)_{11} + \alpha &= 0, \\
\frac{1 - \lambda^2}{\lambda} + (\mathcal{L}_X g^\lambda_1)_{22} &= 0, \\
(\mathcal{L}_X g^\lambda_1)_{33} + \lambda\alpha &= 0, \\
(\mathcal{L}_X g)_{24} - a_2 \alpha x &= 0, \\
(\mathcal{L}_X g)_{ij} &= 0, \quad \text{other cases.}
\end{cases}
\end{align*}
\tag{42}
\]

When we solve the system (42), the coefficient functions $P^i$ of the vector field (39) are found.

**Theorem 10.** The left-invariant Lorentzian metric $g^\lambda_2$ satisfies a Ricci soliton equation
\[2\text{Ric}[g^\lambda_2] + \mathcal{L}_X g^\lambda_2 = 0,\tag{43}\]
where the vector field $X$ is defined by
\[
X = -\left(\frac{C_1}{2} x^2 + \frac{2}{\lambda}\right) X_2 + \left(\frac{C_1}{6} x^3 + \frac{C_2}{2} x^2 + C_3 x + C_4\right) X_3
+ \left(-\frac{C_1}{24} x^4 - \frac{C_2}{2} x^3 - \frac{C_3}{2} x^2 + \left(\frac{\lambda}{2} - C_4\right) x + C_2 y + C_1 z + C_5\right) X_4,\tag{44}
\]
where $C_i$ for $i = 1, \ldots, 5$ are arbitrary constants and then $g^\lambda_2$ is a shrinking.

**Proof.** The matrix of connection one-forms of a metric $g^\lambda_2$ is
\[
\omega = (\omega^i_j) = \frac{1}{2}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
-\lambda \omega^3 & 0 & -\lambda \omega^1 & 0 \\
-\frac{1}{\lambda} \omega^1 - \omega^2 & -\omega^2 & 0 & 0 \\
0 & \lambda \omega^3 & \omega^1 + \lambda \omega^2 & 0
\end{pmatrix}
\]
and the matrix of curvature two-forms is given by
\[
\Omega = (\Omega^i_j) = \frac{1}{4} \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
3\lambda \omega^1 \wedge \omega^2 & 0 & 0 & 0 \\
-\lambda \omega^1 \wedge \omega^3 & 0 & 0 & 0 \\
0 & -3\lambda \omega^1 \wedge \omega^2 & \lambda^2 \omega^1 \wedge \omega^3 & 0
\end{array} \right).
\]

Therefore the Ricci curvature is
\[
\text{Ric} [g^\lambda_2] = -\frac{\lambda}{2} \omega_1 \otimes \omega_1, \tag{45}
\]
and the scalar curvature is equal to zero. The corresponding Ricci operator is
\[
\text{Rc} [g^\lambda_2] = \text{diag} \left( -\frac{\lambda}{2}, 0, 0, 0 \right). \tag{46}
\]

Let \( X = \sum_{i=1}^{4} P_i(x, y, z, w) X_i \) is an arbitrary vector field on \( G_4 \), that means the \( P_i \) are smooth functions on \( G_4 \). The component functions of the Lie derivative symmetric matrix \( \mathcal{L}_X g^\lambda_2 \) are listed by
\[
(\mathcal{L}_X g^\lambda_2)_{11} = 2P_x^4 + 2P^3_x,
(\mathcal{L}_X g^\lambda_2)_{12} = P_y^4 - xP^3_y - \frac{x^2}{2}P^2_y - \lambda xP^3_x + x^2P^1_x - \lambda xP^2 + P^2_x + 2xP^1_x,
(\mathcal{L}_X g^\lambda_2)_{13} = P_y^4 - xP^3_z - \frac{x^2}{2}P^2_z + \lambda P^3_x - 2xP^1_x + \lambda P^2 - 2P^1_x,
(\mathcal{L}_X g^\lambda_2)_{14} = P_y^4 - xP^3_w - \frac{x^2}{2}P^2_w + P^1_x,
(\mathcal{L}_X g^\lambda_2)_{22} = -2\lambda xP^3_y + x^2P^1_y + 2\lambda xP^1_y + 2P^2_y,
(\mathcal{L}_X g^\lambda_2)_{23} = -\lambda xP^3_y + \lambda P^3_y - 2xP^1_y - \lambda P^1_y + P^2_y,
(\mathcal{L}_X g^\lambda_2)_{24} = P_y^1 + P^2_w - \lambda xP^3_w,
(\mathcal{L}_X g^\lambda_2)_{33} = -2xP^1_z + 2\lambda P^3_z,
(\mathcal{L}_X g^\lambda_2)_{34} = P^1_z + \lambda P^3_z,
(\mathcal{L}_X g^\lambda_2)_{44} = 2P^1_w.
\]

The corresponding Ricci soliton equation with \( \alpha = 0 \), is \( 2\text{Ric}[g^\lambda_2] + \mathcal{L}_X g^\lambda_2 = 0 \), and it leads to the following system
\[
\begin{cases}
-\lambda + (\mathcal{L}_X g^\lambda_2)_{44} = 0, \\
(\mathcal{L}_X g^\lambda_2)_{ij} = 0, \quad \text{other cases}.
\end{cases} \tag{47}
\]
Solving the above system (47), we can find the vector field (44).

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