Dynamic Oligopoly with Incomplete Information

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Motivation

- New markets: firms often know their costs better than rivals ⇒ opportunities for **signaling** and **learning**

- How does “jockeying for a position” play out?
  - Strategies; learning; prices, profits and welfare

- Challenge: handling beliefs ⇒ existing works typically assume:
  - two periods or static:
    - one-sided info: Milgrom and Roberts 82, Fudenberg and Tirole 86, Mailath 89
    - two-sided info: Bernhardt and Taub 15 (supply functions)
  - patient firms (e.g., Athey & Bagwell 08), or
  - non-standard solution concepts (e.g., Fershtman & Pakes, 12; Doraszelski, Lewis, Pakes, 14)

- This paper: a tractable analysis of **dynamic oligopolistic competition** with incomplete information
Model

“Repeated Cournot w/ pvte. costs and imperfect monitoring”

- Time is continuous $t \in [0, T]$
- $n$ firms; compete in quantities $Q^i_t \in \mathbb{R}$, $i = 1, \ldots, n$
- **Incomplete information:** costs are private; $C^i \overset{iid}{\sim} \mathcal{N}(\pi_0, g_0)$
- **Imperfect monitoring:** firms only observe cumulative price process:

$$dY_t = \left( \bar{p} - \sum_i Q^i_t \right) dt + \sigma dZ_t$$

where $(Z_t)_{t \geq 0}$ is a Brownian motion

- Firm $i$’s flow payoff: $Q^i_t(dY_t - C^i dt)$; Total expected payoff:

$$\mathbb{E} \left[ \int_0^T e^{-rt} \left( \bar{p} - \sum_j Q^j_t - C^i \right) Q^i_t dt \right]$$
Markov strategies: condition only on beliefs about costs and calendar time

Representation of symmetric linear Markov strategies as

\[ Q^i_t = \alpha_t C^i_t + \beta_t \Pi_t + \delta_t, \]

where \( \Pi_t := \mathbb{E} \left[ \frac{1}{n} \sum_j C^j \mid Y^t \right] \) “public belief”

Characterization of symmetric linear Markov equilibria: system of nonlinear ODEs

Economic implications
1. Limit behavior as $T \to \infty$:
   - Play converges to static complete info Nash; firm only learns avg. cost

2. Incentives: signal jamming, learning and signaling
   - Non-monotone $\alpha$ and $\beta$ decreasing

3. Implications for prices, quantities:
   - Prices have an “upward trend”; difference $Q^i - Q^j$ is non-monotone

4. Welfare effects of dynamic competition:
   - Consumer surplus higher than static case
   - Expected profits may be highest in the medium run
Strategies and Equilibrium

- Info comes from \((C, Y)\) only; \(Y\) satisfies full-support assumption
  - “Nash eq. is outcome-equivalent to sequential eq.” \(\Rightarrow\) focus on Nash

- A profile \((Q_t^1, ..., Q_t^n)_{t \in [0, T]}\) is admissible if
  1. \((Q_t^i)_{t \in [0, T]}\) is \(\mathcal{F}_t^{C_i, Y}\)-measurable
  2. \(\mathbb{E}[\int_0^T (Q_t^i)^2 dt] < \infty\)

Definition (Pure-Strategy Nash Equilibrium)

Profile \(Q\) such that no firm can improve time-zero expected profits by deviating to an admissible \(\tilde{Q}^i\)
Linear Strategies

- Tractable **learning** dynamics ⇒ linearity in $C$

- A strategy is said to be **linear** if

$$Q_t^i = \alpha_t C^i + \int_0^t f^t_s dY_s + \delta_t$$

where $\alpha, \delta : [0, T] \to \mathbb{R}$ and $f^t : [0, t] \to \mathbb{R}$, $t \in [0, T]$.

- Focus on **symmetric linear** equilibria $(\alpha, f, \delta)$
Beliefs and Linear Markov Strategies
Private Learning

- Fix strategy profile $Q^{-i} = (\alpha, f, \delta)$. Firm $i$ always observes

$$dY_t^i := dY_t - \left[ \bar{p} - (n - 1) \left( \int_0^t f_s^t dY_s + \delta_t \right) \right] dt - Q_t^i dt$$

\[ \text{public component of rival's strategies} \]

$$\Rightarrow dY_t^i = -\alpha \sum_{j \neq i} C_j^i dt + \sigma dZ_t : \text{pvte. signal of rivals' costs}$$

- By symmetry and normality, firm $i$’s belief is summarized by

$$M_t^i = \frac{1}{n-1} \mathbb{E}_t^i \left[ \sum_{j \neq i} C_j^i \right] \quad \text{and} \quad \gamma_t^M = \frac{(n-1)g_0}{1 + (n-1)g_0 \int_0^t (\frac{\alpha_s}{\sigma})^2 ds}.$$

with $M_t^i$ is private and $\gamma_t^M$ is deterministic (given $\alpha$)

- Conditioning actions of $M_t^i \Rightarrow$ Forecasting the forecasts of others
Representation of $M^i$ Through Public Belief

- Fix strategy profile $(\alpha, f, \delta)$. Define the public belief $\sum_j C^j \mid Y$

\[
\Pi_t := \frac{1}{n} \mathbb{E} \left[ \sum_j C^j \right] \quad \text{and} \quad \gamma_t := \frac{ng_0}{1 + ng_0 \int_0^t \left( \frac{\alpha_s}{\sigma} \right)^2 ds}
\]

- Belief of an outsider (cf. market maker in finance models)

**Lemma (Adaptation of Foster and Viswanathan (JF, 1996))**

*Under a symmetric linear strategy profile, for each firm $i,*

\[
M^i_t = z_t \Pi_t - (z_t - 1) C^i, \quad t \geq 0, \quad \text{where}
\]

\[
z_t = \frac{n \gamma_t^M}{(n - 1) \gamma_t} = \frac{n^2 g_0}{n(n - 1) g_0 + \gamma_t} \in \left[ 1, \frac{n}{n - 1} \right]
\]

- Along the path of play, payoff-relevant info summarized by $(C, \Pi, \gamma)$
Linear Markov Strategies

- A strategy is **Markov** if each $Q^i$ depends on history only through the firm $i$’s beliefs about $(C^1, ..., C^n)$ and calendar time
  - $(C, \Pi, \gamma, t)$ is sufficient
- Gaussian learning: $\Pi_t = \Pi_t[Y]$ is linear and $\gamma$ deterministic

**Lemma**

*A symmetric linear strategy profile is **Markov** if and only if there exist coefficients with $\alpha, \beta, \delta : [0, T] \rightarrow \mathbb{R}$ such that*

$$Q_t = \alpha_t C + \beta_t \Pi_t + \delta_t$$

- Focus on **symmetric linear Markov** equilibrium
- Performing eq. analysis requires evaluating hypothetical deviations
Beliefs Off the Equilibrium Path

- Deviations ⇒ $\Pi_t$ is biased and Lemma no longer holds

- But $M_t^i = M_t^i[Y^i]$ is uniquely determined by $(Y_s^i : 0 \leq s \leq t)$

- Construct **counterfactual** public belief: $\hat{\Pi}_t = \hat{\Pi}_t[Y^i]$

**Lemma**

$M_t^i = z_t \hat{\Pi}_t - (z_t - 1) C^i$ a.s., $t \in [0, T]$. If all firms play symmetrically, then $\hat{\Pi} = \Pi$

- If other firms play a symmetric Markov profile, then firm $i$’s payoff-relevant *state* is $(C^i, \Pi_t, \hat{\Pi}_t^i, t)$
Dynamic Programming
Best-Response Problem

- Fixing strategy profile $Q^{-i} = (\alpha, \beta, \delta)$, firm $i$ solves:

$$
\max_{(Q^i_s)_{0 \leq s \leq t}} \mathbb{E} \int_0^T e^{-rs} \left[ \bar{p} - Q^i_s - (n - 1)(\alpha_s M^i_s + \beta_s \Pi_s + \delta_s) - c \right] Q^i_s ds \sum_{j \neq i} \mathbb{E}[Q^j_s | F^i_s]
$$

s.t. $d\Pi_s = \lambda_s \left[ (\alpha_s + \beta_s) \Pi_s + \delta_s - Q^i_s \right] + (n - 1) \alpha_s \left[ \Pi_s - M^i_s \right] ds + \lambda_s \sigma dZ^i_s,$

$$d\hat{\Pi}^i_s = \lambda_s \left[ \alpha_s (\hat{\Pi}^i_s - c) + (n - 1) \alpha_s (\hat{\Pi}^i_s - M^i_s) \right] ds + \lambda_s \sigma dZ^i_s,$$

$$M^i_s = z_s \hat{\Pi}^i_s + (1 - z_s) c, \quad \lambda_t := -\frac{\alpha_t \gamma_t}{n \sigma^2}, \quad \Pi_t = \pi, \quad \hat{\Pi}^i_t = \hat{\pi},$$

where $(Q^i_s)_{0 \leq s \leq T}$ is p.m. w.r.t. $\mathcal{F}^C_i, \Pi, Z^i_t$, $t \geq 0$

- B-R problem is a L-Q control problem: quadratic value and linear BR
Finding Markov Equilibria

- Suppose that \((\alpha, \beta, \delta)\) are such that there exists \(V(c, \pi, \hat{\pi}, t)\) solving

\[
    rV(c, \pi, \hat{\pi}, t) = \sup_{q \in \mathbb{R}} \left\{ \left[ \bar{p} - q - \left( n - 1 \right) \left( \alpha_t (z_t \hat{\pi} + (1 - z_t)c) + \beta_t \pi + \delta_t \right) - c \right] q 
\right. 
\]

\[
\left. + \mathcal{L}^q V(c, \pi, \hat{\pi}, t) \right\} 
\]

s.t. \(\arg \max \text{ RHS}(c, \pi, \hat{\pi}, t) |_{\pi = \hat{\pi}} = \alpha_t c + \beta_t \pi + \delta_t \)

with \(\mathcal{L}^q V = \mu_t(q) \frac{\partial V}{\partial \pi} + \hat{\mu}_t \frac{\partial V}{\partial \hat{\pi}} + \frac{\partial V}{\partial t} + \frac{\lambda_t^2 \sigma^2}{2} \left( \frac{\partial^2 V}{\partial \pi^2} + 2 \frac{\partial^2 V}{\partial \pi \partial \hat{\pi}} + \frac{\partial^2 V}{\partial \hat{\pi}^2} \right) \)

\(\Rightarrow (\alpha, \beta, \delta)\) is a Markov eq., and \(V(c, \pi, \pi, t)\) the on-path payoff

- Quadratic solution in \((c, \pi, \hat{\pi})\) with time-dependent coefficients
  - ODEs are non-standard: game is not a L-Q game; \((\gamma)_{t \in [0, T]}\) is non-linear
Characterization: Boundary Value Problem

Let $\alpha^m(x)$, $\beta^m(x)$ and $\xi^m(x)$: myopic best-response given variance $x > 0$

Consider the BVP

\[
\dot{\alpha}_t = r\alpha_t \frac{\alpha_t - \alpha^m(\gamma_t)}{\alpha^m(\gamma_t)} - \frac{\alpha_t^2 \beta_t \gamma_t [(n - 1)n\alpha_t (z_t - 1) + 1]}{n\sigma^2} \tag{1}
\]

\[
\dot{\beta}_t = r\alpha_t \frac{(n + 1)(\beta^m(\gamma_t) - \beta_t)}{(n + 1)\beta^m(\gamma_t) + 1} + \frac{\alpha_t \beta_t \gamma_t [n\alpha_t (n + 1 - (n - 1)z_t - (n^2 - 1)\beta_t (z_t - 1)) + (n - 1)\beta_t]}{n(n + 1)\sigma^2}, \tag{2}
\]

\[
\dot{\gamma}_t = -\frac{\alpha_t^2 \gamma_t^2}{\sigma^2}, \tag{3}
\]

\[
\dot{\xi}_t = r\alpha_t \frac{(n + 1)(\xi^m(\gamma_t) - \xi_t)}{2\xi^m(\gamma_t) + 1} + \frac{\alpha_t \gamma_t \xi_t}{n\sigma^2} [\xi_t - (n\alpha_t((n - 1)\beta_t(z_t - 1) - 1) + \beta_t)] - \frac{(n - 1)\alpha_t^2 \beta_t \gamma_t z_t}{2\sigma^2} \tag{4}
\]

with terminal conditions $\alpha_T = \alpha^m(\gamma_T)$, $\beta_T = \beta^m(\gamma_T)$ and $\xi_T = \xi^m(\gamma_T)$, and

initial condition $\gamma_0 = n\gamma_0$, respectively.
Existence and Characterization of Linear Markov Eq.

Theorem

A symmetric linear Markov equilibrium with differentiable coefficients \((\alpha, \beta, \delta)\) exists if and only if

(i) \(\delta = -\bar{p}(\alpha + \beta)\)

(ii) \(\exists \xi : [0, T] \rightarrow \mathbb{R}\) such that \((\alpha, \beta, \gamma, \xi)\) solve the BVP (1)-(4).

In any such an equilibrium (on- and off-path) behavior is given by

\[
Q^*(c, \pi, \hat{\pi}, t) = \alpha_t c + \beta_t \pi + \delta_t + \xi_t (\hat{\pi} - \pi)
\]

A sufficient condition for a solution to the BVP to exist is

\[
\frac{ng_0}{\sigma^2} < \max \left\{ \frac{4r}{27}, \frac{1}{3T} \right\}.
\]
**Equilibrium Coefficients:** \[ Q_t^i = \alpha_t C_t^i + \beta_t \Pi_t + \delta_t \]

**Proposition**

1. \( \alpha \) is ↘ and \( \beta \) is ↗ at zero; for \( T \) large, \( \alpha \) is ↗ and \( \beta \) ↘ at \( T \)

2. \( \delta \) is eventually decreasing

3. \( r = 0 \): \( \alpha \) is quasi-convex; \( \beta \) is quasi-concave; \( \delta \) is strictly decreasing.
Economic Implications

1. Beliefs and behavior as \( T \to \infty \)
2. Incentives: signal jamming, learning and signaling
3. Paths of prices and quantities
4. Expected flow profits and consumer surplus
Preliminary Result: Dynamic vs. Myopic Coefficients

Proposition

\[ (-\alpha_t, \beta_t, \delta_t) \geq (-\alpha^m(\gamma_t), \beta^m(\gamma_t), \delta^m(\gamma_t)) \]
1) Learning and Limit Behavior

\[ \gamma_t \propto (1 + n g_0 \int_0^T \frac{\alpha_s^2}{\sigma^2} ds)^{-1}; \quad q^N(\vec{C}) : \text{Nash eq. of static complete info} \]

**Corollary**

\[ \gamma_t^T \to 0 \text{ as } t \text{ grows, uniformly along any sequence of games. Thus, for all } \epsilon > 0, \text{ there exists } t_\epsilon < \infty \text{ s.t. in every symmetric linear Markov eq.} \]

\[ \mathbb{P}[\| Q_t - q^N(\vec{C}) \| \leq \epsilon] > 1 - \epsilon \text{ for } t \in [t_\epsilon, T] \]
2) Incentives: Signal-Jamming $\delta$

\[ \delta, \delta^m \]

Corollary (Overproduction)

\[ \mathbb{E} \left[ \sum_i Q^i_t \right] = n\delta_t \left( 1 - \frac{\pi_0}{\bar{p}} \right) > \frac{n}{n+1} (\bar{p} - \pi_0) \text{ if } \bar{p} - \pi_0 > 0. \]
2) Incentives: Learning and Signaling

- Goal: understand forces behind non-monotonicities
- Learning ⇒ move from \((C^i, \Pi)\) to \((C^i, M^i)\) space:

\[
Q^i_t = \alpha_t C^i + \beta_t \Pi_t + \delta_t = \left( \alpha_t - \frac{1 - z_t}{z_t} \beta_t \right) C^i + \frac{\beta}{z_t} M^i_t + \delta_t
\]

- Learning: \((\hat{\alpha}_t^{BR}, \hat{\beta}_t^{BR}, \hat{\delta}_t^{BR})\)
  - Adjusting output as information accumulates
- Signaling: \((\hat{\alpha}_t - \hat{\alpha}_t^{BR}, \hat{\beta}_t - \hat{\beta}_t^{BR}, \hat{\delta}_t - \hat{\delta}_t^{BR})\)
  - Deviations from myopic production: affecting continuation values
2) Incentives: Learning and Signaling

Proposition

\[ \hat{\alpha}_t - \hat{\alpha}_t^{m, BR} < 0, \hat{\beta}_t - \hat{\beta}_t^{m, BR} > 0 \text{ and } \hat{\delta}_t - \hat{\delta}_t^{m, BR} > 0. \text{ If } r = 0, \]

\[ |\hat{\alpha}_t - \hat{\alpha}_t^{m, BR}|, |\hat{\beta}_t - \hat{\beta}_t^{m, BR}| \text{ and } |\hat{\delta}_t - \hat{\delta}_t^{m, BR}| \text{ are decreasing.} \]

Figure: \((\hat{\alpha}_t, \hat{\beta}_t); (\hat{\alpha}_t^{m, BR}, \hat{\beta}_t^{m, BR}); (\hat{\alpha}_t - \hat{\alpha}_t^{m, BR}, \hat{\beta}_t - \hat{\beta}_t^{m, BR})\)
Incentives: Summary

• Fix $t$. Behavior across different types:

$$Q_t^i - Q_t^i,m,BR = \delta_t - \delta_t^m,BR$$

( all firms have the incentive to overproduce )

$$+ (\hat{\alpha}_t - \hat{\alpha}_t^m,BR) C^i$$

( more efficient firms scale back less )

$$+ (\hat{\beta}_t - \hat{\beta}_t^m,BR) M^i_t$$

( optimistic firms produce more )

• Fix type. Using $\delta = -\bar{p}(\alpha + \beta)$, behavior across time:

$$Q_t^i - Q_t^i,m,BR = -(\hat{\alpha}_t - \hat{\alpha}_t^m,BR)[\bar{p} - C^i] - (\hat{\beta}_t - \hat{\beta}_t^m,BR)[\bar{p} - M^i_t]$$

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3) Prices and Quantities

![Graph showing prices and quantities over time.](image-url)
3) Prices and Quantities

- Over-production $\Rightarrow$ in expectation prices everywhere depressed
  - Expected prices eventually increasing in $t$

- If $r = 0$, expected total quantity decreases (price increases) in $t$

- Output differences across firms are deterministic (given $C$):
  \[ Q^i_t - Q^j_t = \alpha_t (C^i_t - C^j_t), \]
  where $\alpha$ is non-monotone (for large $T$) and “U-shaped” (if $r = 0$)
4) Welfare Effects

- Expected flow consumer surplus > complete-information, for all $t$
- Expected flow profits < complete-information at $t \in \{0, T\}$
- Non-monotone $\alpha \Rightarrow$ implications for allocative efficiency:
  - Low-cost firm has largest advantage in the medium run
  - “Taking turns” from ex-ante perspective
  - Variance increases expected profits (esp. if avg. margin is low)
  - Highest industry profits at interior $t$
Expected Flow Profits

Figure: Comparison of expected flow profits under complete information (dashed) and incomplete information (blue) for parameter values $\pi_0 = 0$ (left), $\pi_0 = \bar{p}$ (right), and $(r, \sigma, n, \bar{p}, T, g_0) = (0.2, 1, 2, 5, 15.44, 2)$. 
Infinite Horizon

- Sequence of symmetric linear Markov equilibria indexed by $T^n \uparrow \infty$
- Let $g_0/\sigma^2 < 4r/(27n)$.

Proposition

Any sequence of symmetric linear Markov equilibria contains a subsequence that converges uniformly to a symmetric linear Markov equilibrium $(\alpha^*, \beta^*, \delta^*, \xi^*, \gamma^*)$ of the infinite-horizon game.

Furthermore, $\delta^* = -\bar{p}(\alpha^* + \beta^*)$ and $(\alpha^*, \beta^*, \xi^*, \gamma^*)$ is a solution to the BVP on $[0, \infty)$ with $\lim_{t \to \infty} \alpha_t^* = \alpha^m(0)$, $\lim_{t \to \infty} \beta_t^* = \beta^m(0)$, $\lim_{t \to \infty} \xi_t^* = \xi^m(0)$, and $\gamma_0^* = ng_0$. 
Literature

- Industrial Organization
  - Milgrom and Roberts 82, Fudenberg and Tirole 86, Mailath 89, etc.
  - Raith 96, Vives 08 book, Bernhardt and Taub 15
  - Fershtman and Pakes 12; Doraszelski, Lewis and Pakes 14

- Finance
  - Kyle 85; Foster and Viswanathan 96; Back, Cao and Willard 00

- Repeated Bayesian Games
  - Athey and Bagwell 08
  - Aumann and Maschler 95, Peski 14
  - Horner and Lovo 09
  - Escobar and Toikka 13; Horner, Takahashi and Vieille 15

- Continuous-time Games and Private Beliefs
  - Cisternas 15
Conclusions

- A tractable dynamic oligopoly game with independent private values and imperfect monitoring
  - Symmetric linear Markov equilibrium in private and public info
  - Private information is gradually revealed over time; convergence to full info play
  - Overproduction $\Rightarrow$ depressed prices
  - Non-monotone incentives
    - Non-monotone differences in quantities: $Q_t^i - Q_t^j = \alpha_t(C_j^i - C_j^j)$
    - Non-monotone ex-ante flow profits

- Extensions
  - Correlation
  - Interdependent values
  - Asymmetric case
  - Changing costs
Myopic Coefficients

Given $\gamma_t = x \in \mathbb{R}_+$, the unique equilibrium for myopic firms is given by

\[
\alpha^m(x) = -\frac{(n-1)ng_0 + x}{(n-1)ng_0 + (n+1)x} < -\frac{1}{2},
\]

\[
\beta^m(x) = \frac{(n-1)n^2g_0}{(n+1)[(n-1)ng_0 + (n+1)x]} > \frac{n-1}{2(n+1)},
\]

\[
\delta^m(x) = -\bar{p}(\alpha^m(x) + \beta^m(x)) = \frac{-\bar{p}}{n+1}
\]

\[
\xi^m(x) = \frac{(n-1)n^2g_0}{2[(n-1)ng_0 + (n+1)x]} > \frac{n-1}{2}.
\]