Approaching Prosumer Social Optimum via Energy Sharing Mechanism with Proof of Convergence

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Abstract—Advent of prosumers helps to pave the way towards a more flexible and sustainable power grid, but also exerts great challenges on system management. Traditional centralized operation paradigms may become impracticable due to computational burden, privacy violation, and interest inconsistency. In this paper, an energy sharing mechanism is proposed to accommodate prosumers’ strategic decision-making on their self-production and demand considering capacity constraints. Under this setting, prosumers constitute a generalized Nash game. We prove main properties of the game: its equilibrium exists and is partially unique; no prosumer is worse off by energy sharing; a \(1 - O(1/n)\) price-of-anarchy (PoA) is achieved (which is less than 1 because net cost is negative in this paper). Specially, the PoA tends to 1 with growing number of prosumers, meaning that the resulting total cost under the proposed energy sharing approaches social optimum. We prove that the corresponding prosumers’ strategies converge to the social optimal solution as well. A bidding process is presented for implementation, which is proved to converge to the energy sharing equilibrium under mild conditions. Illustrative examples are tested to validate the results.

Index Terms—Energy sharing, generalized Nash equilibrium, prosumer, bidding algorithm, distributed mechanism

NOMENCLATURE

A. Indices, Sets, and Functions

- \(i, I\) Index and set of prosumers.
- \(S_i\) Action sets of prosumer \(i\), and \(S = \prod_{i \in I} S_i\).
- \(f_i(p_i)\) Cost function of prosumer \(i\).
- \(u_i(d_i)\) Utility function of prosumer \(i\).
- \(J_i(p_i, d_i)\) Net cost of prosumer \(i\), which equals to \(f_i(p_i) - u_i(d_i)\); and \(J(p, d) = \sum_{i \in I} J_i(p_i, d_i)\).
- \(\Gamma_i(p, d, b)\) Total net cost of prosumer \(i\) with sharing, which equals to \(f_i(p_i) - u_i(d_i) + \lambda(-aA + b_i)\).
- \(\text{PoA}(G)\) Price of anarchy of a game \(G\).
- \(\mathcal{Y}_i\) Any \((p_i, d_i) \in \mathcal{Y}_i\) satisfies the capacity constraint.

B. Parameters

- \(I\) Number of prosumers.
- \(p_i, \overline{p}_i\) Lower/upper bound of prosumer \(i\)’s production.
- \(d_i, \overline{d}_i\) Lower/upper bound of prosumer \(i\)’s demand.
- \(a\) Energy sharing market sensitivity.

C. Decision Variables

- \(p_i\) Production of prosumer \(i\).
- \(d_i\) Demand of prosumer \(i\).
- \(\lambda_m\) Dual variable of the power balancing condition.
- \(\lambda\) Energy sharing price.
- \(q_i\) Amount of energy prosumer \(i\) gets from sharing.
- \(b_i\) Bid of prosumer \(i\) in the energy sharing market.
- \(\hat{p}_i, \hat{d}_i\) Optimal strategies under centralized paradigm.
- \(p_i^*, d_i^*\) Strategy of prosumer \(i\) at sharing equilibrium.
- \(\hat{p}_i, \hat{d}_i\) Strategy of prosumer \(i\) under self-sufficiency.

I. INTRODUCTION

In recent years, distributed generation technologies and storages have developed rapidly. In the US, over 81,000 distributed wind turbines with a cumulative capacity of 1.076 MW had been deployed during 2003-2017 \([1]\). The residential solar photovoltaic (PV) panels had risen from 3,700 MW to 150,000 MW from 2004 to 2014 \([2]\). Advances in these technologies, together with decline in cost, have encouraged traditional consumers to produce and storage energy at home, via distributed energy resources (DERs), electrical vehicles, and batteries \([3]\), turning them into so-called “prosumers”. By reevaluating their energy, prosumers can play a proactive role in energy management. However, massive participants, asymmetric information, and interest conflicts also impose great challenges \([4]\). The “prosumer” era is calling for an engrained revolution in the power grid.

Typically there are three types of prosumer management paradigms as shown in Fig. \([1]\) \([4]\). The first one (on the left of Fig. \([1]\) adopts a centralized manner \([5]\). The operator of a microgrid or a virtual power plant (VPP) gathers all information and makes a central decision, aiming at minimizing the total net costs of all prosumers inside it. Then dispatch orders are sent to each prosumer to execute. Since the number of prosumers is increasing rapidly, the traditional centralized manner becomes impracticable both in computational burden and privacy requirements. The second one (in the middle of Fig. \([1]\) uses a market structure similar to the retail market. The operator announces a price, based on which every prosumer, as a price taker, decides how much energy to consume/produce/buy/sell \([6]\). It is hard to decide an effective energy price especially with massive prosumers, since private information may be needed and each prosumer’s capacity is too small to be observed. Inspired by the concept of “sharing” in other sectors, the third type of paradigm (on the right of Fig. \([1]\) has captured increasing attention in recent decades. Here, a prosumer is allowed to exchange/share energy with other prosumers, and they are turning from price-takers to price-makers. This can be done in a peer-to-peer (P2P) structure \([7]\) or with the assistance of a platform \([8]\). As revealed in \([9]\), energy sharing can be a promising direction in managing prosumers, since it can achieve a nearly social optimal in a distributed manner. Various research projects have been carried out on related issues, such as Piclo \([10]\) in the UK.
TransActive Grid [11] in the US, and Enexa [12] in South Australia. The successful operation of energy sharing relies on a well-designed mechanism, and existing research about energy sharing mechanism design can be classified into two categories:

**Cooperative game based approach.** In this category, first a profit distribution scheme is designed, and then each prosumer chooses its strategy taking into account the possible reallocation it might obtain. The key point here is to design an effective distribution scheme so that all prosumers are willing to collaborate to achieve a certain goal (usually social optimal). Profit distribution schemes for storage sharing were developed under two scenarios [13], [14]. Incentives were designed to encourage coagional operation of networked microgrids [15]. A mathematical program with equilibrium constraints (MPEC) was used for DER sharing among prosumers, and the coordination surplus was split among the aggregator and prosumers [16]. A random sampling method was proposed to estimate the Shapley value of a P2P energy sharing game [17].

**Noncooperative game based approach.** This category characterizes the interest inconsistency among prosumers, and can be further divided into bilateral contract based approach and auction based approach. Under the bilateral contract based paradigm, trading offers are posted and handshakes are made. First, each prosumer is registered as a seller or a buyer. Then, during the trading periods, both the sellers and buyers put forward several offers and try to find the best match. Once a contract is approved by the operator, the corresponding offers are removed [19]. This mechanism allows sharing to be performed in an asynchronous manner. A key feature of the bilateral contract based paradigm is scalability in terms of both the outcomes and the process to reach them. A bilateral contract network with forward and real-time markets was developed in [20]. Reference [21] presented a matching algorithm for microgrid prosumers with minimum risk of mismatch. There are relatively few analytical works based on bilateral contracts, because the matching procedure of bilateral contracts is hard to characterize [22]. Under the auction based paradigm, some or all of the prosumers first bid on energy, and then the market is cleared with the energy sharing prices determined. An evolutionary game was used to model the dynamics of buyers selecting sellers [23]. A Nash bargaining model was adopted in [24] to address the charge sharing among electric vehicles (EVs). In above works, the role of a prosumer as a buyer or a seller is predetermined and cannot change during the bidding process, which limits the flexibility of sharing. To overcome this limit, distributed peer-to-peer energy exchange was modeled as a generalized Nash game in [25], which proved that the set of variational equilibria coincides with the set of social optima. In our previous work [8], a generalized demand-bidding approach was proposed for node-level energy sharing, and several properties of the Nash equilibrium were proved. This paper extends [8] and makes three major contributions:

1) **Model:** A generalized Nash game model is proposed to depict the energy sharing among prosumers. Extending [8], in which only power balance constraint was considered with fixed energy demand, this paper incorporates capacity limits and variable demand, which is more flexible and practical. This complicates the analyses in two ways: Firstly, the energy sharing model in this paper can no longer be simplified to a standard Nash game as in [8], but indeed is a generalized Nash game whose equilibrium is hard to characterize [26]. Secondly, when analyzing each prosumer’s strategic behavior, the complementary slackness conditions associated with the inequality constraints exert great challenges.

2) **Equilibrium:** Main properties of the proposed energy sharing game are proved. The generalized Nash equilibrium (GNE) exists and is partially unique (explained later). A Pareto improvement is reached among all prosumers, which justifies the incentives for them to participate in sharing. Moreover, the energy sharing game achieves a $1 - O(1/l)$ price-of-anarchy (PoA) (which is less than 1 because net cost is negative in this paper). With more and more prosumers, the PoA approaches 1, meaning that the performance of energy sharing approaches the centralized social optimal operation. The corresponding prosumers’ strategies also converge to the centralized social optimal strategies.

3) **Algorithm:** A bidding process is developed for achieving energy sharing in a distributed manner. A range of sharing market sensitivity is given, within which the bidding process is proved to converge to a GNE, using the variational inequality technique. This can provide some guidance for designing and implementing energy sharing in the future.

**Notation.** We use $x := (x_i; i \in \mathcal{I})^T$ to denote a collection of $x_i$ in a set $\mathcal{I}$. The subscript $-i$ means all components in $\mathcal{I}$ except $i$. The Cartesian product of sets $\mathcal{S}_i$ is denoted as $\prod_{i \in \mathcal{I}} \mathcal{S}_i$. We use $f(.)$ to denote the first derivative of function $f(\cdot)$, and $\hat{f}(\cdot)$ to denote the second derivative. $(f)^{-1}(\cdot)$ denotes the inverse function of $f(\cdot)$.

## II. Mathematical Formulation

### A. Problem Description

In a standalone microgrid, there are $I$ prosumers, indexed by $i \in \mathcal{I} = \{1, 2, \ldots, I\}$. Each of them can both produce and consume. Prosumer $i$’s production is $p_i$, with cost function $f_i(p_i)$, which is strictly convex and twice differentiable; its demand is $d_i$ with the utility function $u_i(d_i)$, which is strictly...
concave and twice differentiable. It is reasonable to assume that $\tilde{f}_i$ and $-u_i$ are uniformly bounded over all $i \in \mathcal{I}$ where both the upper and lower bounds are strictly positive and independent from $I$. Traditionally, the operator manages all prosumers in a centralized manner by solving (1).

$$\min_{p_i,d_i}\sum_{i=1}^{I}[f_i(p_i) - u_i(d_i)] \quad (1a)$$

$$\text{s.t. } \sum_{i=1}^{I} p_i - \sum_{i=1}^{I} d_i = 0 ; \lambda_m \quad (1b)$$

$$p_i \leq \tilde{p}_i, \forall i \in \mathcal{I} \quad (1c)$$

$$d_i \leq \tilde{d}_i, \forall i \in \mathcal{I} \quad (1d)$$

The objective (1a) is to minimize prosumers’ total net cost (cost minus utility). Denote $J_i(p_i,d_i) := f_i(p_i) - u_i(d_i)$ and $J(p,d) = \sum_{i=\mathcal{I}} J_i(p_i,d_i)$. Constraint (1b) represents the microgrid-wide power balance with its dual variable $\lambda_m$, and capacity constraints are in (1c)-(1d). Here, $\tilde{p}_i$ and $\tilde{d}_i$ are the lower and upper bound of prosumer $i$’s production, respectively; $d_i$ and $\tilde{d}_i$ are the lower and upper bound of its demand, respectively. This centralized paradigm can achieve the lowest total net cost, and is used as a benchmark.

Throughout the paper, we assume that:

A1: \{$(p,d):$ s.t. (1b)-(1d) are satisfied.$\}$ \neq \emptyset

Condition A1 means the feasible set of problem (1) is not empty. Since $J(p,d)$ is strictly convex, (1) has a unique optimal solution, which is denoted as $(\hat{p}, \hat{d})$. At the optimal point, the value of dual variable $\hat{\lambda}_m$ is the “shadow price” in economics, showing the increment of total net cost would there be one more unit total production-demand mismatch.

**Remark:** Though problem (1) has a unique optimal solution, the corresponding $\hat{\lambda}_m$ is not necessarily unique. Specifically, if there exists at least one $i \in \mathcal{I}$ that satisfies either (1c) or (1d) strictly, $\hat{\lambda}_m$ is unique. Similar discussion about the uniqueness of locational marginal price (LMP) can be found in [27]. $\hat{\lambda}_m$ reflects the value of production-demand balance in a microgrid (the higher it is, the more important it is to balance production and demand). Later we will show that when the number of prosumer increases, the energy sharing price under the proposed mechanism converges to $\hat{\lambda}_m$.

**B. Practical Issues and Requirements**

Although centralized management of prosumers can achieve the lowest total net cost, it encounters two main difficulties in practice: 1) it would be time-consuming when there are massive prosumers; 2) some information including $f_i(\cdot), u_i(\cdot)$ is hard to obtain due to customers’ protection of their own privacy. To tackle these challenges, a distributed and scalable paradigm is desired, which needs to be:

For process: 1) Private. Prosumer privacy is preserved. 2) Distributed. Each prosumer makes its own decision according to individual rationality. 3) Convergent. The bidding process can converge within finite steps.

For result: 1) Incentive. Prosumers are willing to participate, and more participants lead to better performance. 2) Effective. The result satisfies physical constraints. 3) Meaningful. The price reflects the value of production-demand balance. 4) Flexible. Prosumer’s role as a seller or buyer is endogenously given instead of predetermined. 5) Economical. The resulting total cost should be as low as possible, and close to that under social optimum.

To meet the requirements above, in this paper, an energy sharing mechanism is proposed to facilitate energy exchange among prosumers. The basic setting of the proposed energy sharing is developed in Section III, which can be characterized as a generalized Nash game. Main properties of the sharing game equilibrium (a general Nash equilibrium) are revealed. The bidding process is described in detail in Section IV with proof of its convergence. Simulations are shown in Section V to support our results. Finally, we conclude our work in Section VI.

**III. ENERGY SHARING GAME**

**A. Basic Setting**

Instead of dispatching prosumers in a centralized manner, we allow each prosumer to make decision individually to maintain self-energy-balancing. Meanwhile, to be more flexible, these prosumers can participate in an energy sharing market to exchange energy with each other. The market clearing price is $\lambda$, and the amount of energy that prosumer $i$ gets is $q_i$. If $q_i > 0$, prosumer $i$ buys from the market and pays $\lambda q_i$; otherwise, it sells to the market and receives $-\lambda q_i$. The sharing framework is shown in Fig. 2. Each prosumer is connected to a platform via a smart meter through a bidirectional information channel. The information flow is as follows.

Fig. 2. Framework of energy sharing between prosumers and platform.

**Step 1:** (Initialization) Each prosumer $i$ enters its private parameters $f_i(\cdot), u_i(\cdot)$, $\tilde{p}_i, \tilde{d}_i, \tilde{d}_i$ to its smart meter $i$. Set $\lambda^1 = 0$, and $k = 1$. Choose tolerance $\epsilon$.

**Step 2:** Each smart meter $i$ updates its bid $b_i^{k+1}$ based on the latest $\lambda^k$, and sends it to the platform.

**Step 3:** After receiving all bids $b_i^{k+1}, \forall i \in \mathcal{I}$, the platform updates price $\lambda^{k+1}$ and sends it back to all smart meters.

**Step 4:** If $|\lambda^k - \lambda^{k+1}| \leq \epsilon$, $\lambda^* = \lambda^{k+1}$, go to Step 5; otherwise, $k = k + 1$ and go to Step 2.

**Step 5:** Each smart meter determines the optimal production $p_i^*$, demand $d_i^*$, sharing quantity $q_i^*$ based on $\lambda^*$, and sends them back to the corresponding prosumer to execute.

Details about bid and price updates will be explained in Section IV. The key of sharing mechanism design is to determine the sharing price $\lambda$ and quantity $q_i, \forall i \in \mathcal{I}$ based on prosumers’ bids $b_i, \forall i \in \mathcal{I}$. Here we use the generalized demand (or supply) function [28] to depict the relationship between $q_i$ and $\lambda$ as

$$q_i = -a \lambda + b_i, \quad \forall i \in \mathcal{I} \quad (2)$$
where \( a > 0 \) is a parameter reflecting the energy sharing market sensitivity, and \( b_i \) is prosumer \( i \)'s bid. Market clearing requires \( \sum_{i \in \mathcal{I}} q_i = 0 \). Therefore, the energy sharing price is

\[
\lambda = \frac{\sum_{i \in \mathcal{I}} b_i}{a}. \tag{3}
\]

Under this setting, the total net cost of prosumer \( i \) equals its self-production cost, minus its utility, plus the payment for buying from (or minus the revenue from selling to) the energy sharing market, denoted as \( \Gamma_i(p, d, b) := f_i(p_i) - u_i(d_i) + \lambda(b)(-a\lambda(b) + b_i) \). Its energy needs to be self-balanced, which means \( p_i - a\lambda(b) + b_i = d_i \). So each prosumer \( i \) solves:

\[
\begin{align*}
\min_{p_i, d_i, b_i} & \quad f_i(p_i) - u_i(d_i) + \lambda(b)(-a\lambda(b) + b_i) \tag{4a} \\
\text{s.t.} & \quad p_i - a\lambda(b) + b_i = d_i \tag{4b} \\
& \quad p_i \leq p_i \leq \overline{p}_i \tag{4c} \\
& \quad d_i \leq d_i \leq \overline{d}_i \tag{4d} \\
& \quad \lambda(b) = \frac{\sum_{i \in \mathcal{I}} b_i}{a} \tag{4e}
\end{align*}
\]

Due to the common constraint \((4e)\), the proposed energy sharing mechanism constitutes a generalized Nash game \([26]\) with the following elements: 1) a set of players \( \mathcal{I} \); 2) action sets \( S_i(p, d, b, i), \forall i \in \mathcal{I} \), and strategy space \( S = \prod_{i \in \mathcal{I}} S_i \); 3) cost function \( \Gamma_i(p, d, b), \forall i \in \mathcal{I} \), unique for simplicity, we denote the sharing game compactly as \( \mathcal{G} = \{I, S, \Gamma\} \). Different from the standard Nash game, the action set \( S_i \) of player \( i \) depends on the strategies of other players, making its equilibrium (defined below) difficult to compute and analyze.

**Definition 1.** A profile \( (\hat{p}, \hat{d}, \hat{b}) \in S \) is a generalized Nash equilibrium (GNE) of the sharing game \( \mathcal{G} \), if \( \forall i \in \mathcal{I} \)

\[
(\hat{p}_i, \hat{d}_i, \hat{b}_i) \in \arg\min_{p_i, d_i, b_i} \Gamma_i(p_i, d_i, b_i, \hat{p}_i, \hat{d}_i, \hat{b}_i) \tag{5}
\]

\[
\text{s.t. } (4b) - (4e)
\]

**B. Properties of the Sharing Equilibrium**

In the following, three main properties of the equilibrium of sharing game are revealed. Proposition 1 shows that an effective market equilibrium that satisfies related constraints always exists; Proposition 2 ensures that it is incentive enough to attract prosumers: Proposition 3 tells that it can achieve a near social optimal total cost, and is economical.

**Proposition 1.** (Existence and Partial Uniqueness) A GNE for the sharing game \( \mathcal{G} \) exists if and only if A1 holds. Moreover, suppose a profile \( (\hat{p}, \hat{d}, \hat{b}) \) is a GNE, then \( (\hat{p}, \hat{d}) \) is the unique optimal solution of:

\[
\begin{align*}
\min_{p_i, d_i, b_i} & \quad \sum_{i = 1}^{I} f_i(p_i) - \sum_{i = 1}^{I} u_i(d_i) + \frac{\sum_{i = 1}^{I} (d_i - p_i)^2}{2a(I-1)} \tag{6a} \\
\text{s.t.} & \quad \sum_{i = 1}^{I} p_i = \sum_{i = 1}^{I} d_i : \zeta \tag{6b} \\
& \quad p_i \leq p_i \leq \overline{p}_i : \delta_i^\pm, \forall i \in \mathcal{I} \tag{6c} \\
& \quad d_i \leq d_i \leq \overline{d}_i : \kappa_i^\pm, \forall i \in \mathcal{I} \tag{6d}
\end{align*}
\]

The proof of Proposition 1 can be found in Appendix A.

**Proposition 2.** (Pareto improvement) Suppose A2 holds, and \( (\hat{p}, \hat{d}, \hat{b}) \) is a GNE of the sharing game \( \mathcal{G} \). We have

\[
J_i(\hat{p}_i, \hat{d}_i) \geq \Gamma_i(\hat{p}_i, \hat{d}_i, \hat{b}), \forall i \tag{7}
\]

Moreover, strictly inequality holds for at least one prosumer \( i \) unless \( (\hat{p}, \hat{d}) = (\hat{p}, \hat{d}) \).

The proof of Proposition 2 can be found in Appendix B.

The proposed energy sharing game can incentivize prosumers to join since at least one prosumer can benefit from sharing while no prosumer is worse off. A rare special case is that the self-sufficient solution coincides with the energy sharing equilibrium (in which case it also coincides with the centralized social optimal for problem (1)).

Although every prosumer has the motivation to share energy, there is still a gap in terms of the total net cost between energy sharing \( \mathcal{G} \) and the centralized mechanism \( \mathcal{J} \). The next proposition characterizes this gap by calculating the price-of-anarchy defined as follows.

**Definition 2.** (Price of Anarchy, PoA \([29]\)) Consider a game \( \mathcal{G} = \{I, S, \Gamma\} \). Let a subset \( S_{eq} \subseteq S \) be the set of strategies in equilibrium, then the Price of Anarchy (PoA) of the game is

\[
\text{PoA} = \frac{\max_{\mathcal{A}} J(\mathcal{A})}{\min_{\mathcal{A} \in S_{eq}} J(\mathcal{A})}
\]
defined as the ratio of the total net cost between the worst equilibrium and the social optimal solution, i.e.

$$\text{PoA}(\mathcal{G}) := \frac{\max_{s \in \mathcal{S}} \sum_{i=1}^{I} \Gamma_i(s)}{\min_{s \in \mathcal{S}} \sum_{i=1}^{I} \Gamma_i(s)}$$

Note that the PoA measures how the overall efficiency degrades due to the strategic behavior of participants in a game. If the PoA equals to 1, it means that the game can achieve a social optimal outcome.

**Proposition 3.** (Tendency) Suppose A1–A3 hold, and $p_i$, $\overline{p}_i$, $d_i$, $\overline{d}_i$, $f_i(\cdot)$, $u_i(\cdot)$ over all $i \in \mathcal{I}$ are uniformly bounded by numbers independent from prosumer number $I$. Given $I$, let $(\hat{p}(I), \hat{d}(I), \hat{b}(I))$ be a GNE of the sharing game $\mathcal{G}$, and $(\hat{p}(I), \hat{d}(I))$ be the unique optimal solution of (1). We have

$$\text{PoA}(\mathcal{G}) = \frac{J((\hat{p}(I), \hat{d}(I)))}{J(\hat{p}(I), \hat{d}(I))} \geq 1 - \frac{C}{I - 1}$$

where $C$ is a constant. Moreover, we have

$$\lim_{I \to \infty} \hat{p}_i(I) - \hat{p}_i(I) = \lim_{I \to \infty} \hat{d}_i(I) - \hat{d}_i(I) = 0, \forall i \in \mathcal{I}.$$ (11)

The proof of Proposition 3 can be found in Appendix C. It is worth noting that, PoA is conventionally larger than 1 since a positive minimum total cost is achieved at social optimal $[29]$. However, by A3 and Proposition 2 the total net cost is consistently negative across our self-sufficiency, energy sharing, and centralized social optimal mechanisms, which renders PoA less than 1. Proposition 3 shows that the total net cost and prosumer strategies under GNE converge to those under the centralized social optimal mechanism, as more prosumers are involved.

IV. BIDDING PROCESS

Three properties of the energy sharing game are revealed above. To achieve such a desired equilibrium, a practical bidding process is presented in this section, and the range of the market sensitivity $a$ that can guarantee market convergence is given. The economic intuition behind Proposition 3 is also explained from another perspective.

**A. Procedure**

The procedure of the bidding process is shown in Algorithm 1. The key of the algorithm lies in how each prosumer updates its bid without knowing other prosumers’ information. First, as in Fig. 2 at $(k + 1)^{th}$ iteration, each prosumer $i \in \mathcal{I}$ utilizes the up-to-date price $\lambda^k$ to estimate (due to the fact that $\lambda(p, d)$ is not known exactly) its optimal solution for problem (4) which is equivalent to:

$$\min_{p_i, d_i} f_i(p_i) - u_i(d_i) + \lambda(p, d)(d_i - p_i) \quad (12a)$$

s.t. $p_i \leq p_i \leq \overline{p}_i$ \quad (12b)

$d_i \leq d_i \leq \overline{d}_i$ \quad (12c)

Denote this estimated optimal solution as $(\hat{p}_i^{k+1}, \hat{d}_i^{k+1})$, and the updated bid of prosumer $i$ is $b_i^{k+1} := d_i^{k+1} - p_i^{k+1} + a\lambda^k$ at $(k + 1)^{th}$ iteration. Denote the feasible set of problem (12b) as $\mathcal{Y}_i$. To obtain the estimated optimal solution above, instead of simply replacing the term $\lambda(p, d)(d_i - p_i)$ with $\lambda^k(d_i - p_i)$, prosumer $i$ incorporates the predicted impact of its decision $(\hat{p}_i^{k+1}, \hat{d}_i^{k+1})$ on price $\lambda(p, d)$, by taking the partial derivative of $\lambda(p, d)(d_i - p_i)$ over $p_i$ (similarly for $d_i$):

$$\frac{\partial \lambda(p, d)(d_i - p_i)}{\partial p_i} \bigg|_{\lambda = \lambda^k} = \frac{\partial \lambda(p, d)(d_i - p_i) - \lambda(p, d)}{\partial p_i} \bigg|_{\lambda = \lambda^k} = -\frac{d_i - p_i}{(I - 1)a} - \lambda^k$$

where the last equality is because of

$$\lambda(p, d) = \frac{(d_i - p_i) + \sum_{j \neq i} b_j}{(I - 1)a}$$

derived from (4b) and (4e). Thus we can get the following objective function as an estimate of (12a), where $\lambda^k$ is treated as a constant.

$$f_i(p_i) - u_i(d_i) + \frac{(d_i - p_i)^2}{2a(I - 1)} + \lambda^k(d_i - p_i)$$

Therefore, solving problem (12) is converted to solving:

$$\min_{p_i, d_i} \quad (12)$$

$$\forall (p_i, d_i) \in \mathcal{Y}_i, \forall i \in \mathcal{I}$$

After receiving all updated bids $b_i^{k+1}, \forall i \in \mathcal{I}$, the platform updates the energy sharing price as:

$$\lambda^{k+1} := \frac{\sum_{i=1}^{I} b_i^{k+1}}{al}$$
B. Convergence

Apart from the design of the bidding process, its convergence is also a major concern. Here we give a range of market sensitivity $a$ in Condition A4, and prove that Algorithm 1 converges under such a condition.

A4: For all $i$ in $I$, $a$ satisfies

$$ a \geq \frac{2I - 4}{T - 1} \sup \left\{ \frac{1}{f_i(p_i)} - \frac{1}{u_i(d_i)}, \forall (p_i, d_i) \in \mathcal{Y}_i, \forall i \in I \right\} $$

**Proposition 4.** When A1, A4 hold, Algorithm 1 converges to a GNE of the energy sharing game $G$.

Before proving convergence of the bidding process, we first give the following lemma with its proof in Appendix D. For conciseness, denote $y_i = [p_i, d_i]^T, y = [y_1, \ldots, y_T]^T$, and $\mathcal{Y} = \prod_{i \in I} \mathcal{Y}_i$. Let $h \in \mathbb{R}^{1 \times 2I}$ be a vector with $h_{2i-1} = 1$ and $h_{2i} = -1$ for all $i = 1, \ldots, I$. Let

$$ \phi(y) := \sum_{i=1}^{I} f_i(p_i) - \frac{1}{a} \sum_{i=1}^{I} (d_i - p_i)^2 + \frac{I}{2a(I - 1)} - \frac{\left( \sum_{i=1}^{I} d_i - \sum_{i=1}^{I} p_i \right)^2}{2a} $$

and $L(y, \lambda) := \phi(y) - \lambda hy$ with dom $L = \mathcal{Y} \times \mathbb{R}$.

**Lemma 1.** When $A4$ holds, $\phi(y)$ is a convex function, and $L(y, \lambda)$ has (not necessarily unique) saddle point.

With Lemma 1 in the following we prove Proposition 4 using variational inequality technique.

**Proof.** Given $\lambda^k$, substitute the update of $b_i, \forall i \in I$ into the update of $\lambda$, the $k$-th iteration is equivalent to:

$$ \begin{align*}
    y_i^{k+1} &= \arg \min \{ \phi(y_i) + \lambda^k h_i y_i, \forall y_i \in \mathcal{Y}_i \} \quad (17) \\
    \lambda^{k+1} &= \lambda^k - \frac{h_i^{k+1}}{a} \quad (18)
\end{align*} $$

Equation (17) can be further represented as

$$ y_i^{k+1} = \arg \min \{ \phi(y_i) - \lambda^k h_i y_i^T h_i^T y_i, \forall y_i \in \mathcal{Y_i} \} $$

Utilizing variational inequality and convexity of $\phi(\cdot)$, $y_i^{k+1} \in \mathcal{Y}_i$ generated by (19) satisfies

$$ \forall y \in \mathcal{Y}_i, \phi(y_i) - \phi(y_i^{k+1}) + (y_i - y_i^{k+1})^T \left( -\lambda^k h_i^T + \frac{1}{a} h_i^T (h_i^{k+1})^T \right) \geq 0 \quad (20) $$

Substitute (18) into (20), we can get

$$ \forall y \in \mathcal{Y}_i, \phi(y_i) - \phi(y_i^{k+1}) + (y_i - y_i^{k+1})^T ( -\lambda^k h_i^T + \frac{1}{a} h_i^T (h_i^{k+1})^T ) \geq 0 \quad (21) $$

Combining (21) and (18) gives the following inequality:

$$ \begin{align*}
    \left( \frac{y_i - y_i^{k+1}}{\lambda^k - \lambda^{k+1}} \right) &\left\{ \left( -\lambda^{k+1} \right)^T h_i^{k+1} + \frac{0}{a} (\lambda^{k+1} - \lambda^k) \right\} \\
    + \phi(y) - \phi(y_i^{k+1}) \geq 0, \forall (y, \lambda) \in \mathcal{Y} \times \mathbb{R}
\end{align*} $$

According to Lemma 1, let $(\hat{y}, \hat{\lambda})$ be a saddle point of $L(y, \lambda)$, then we have $\forall (y, \lambda) \in \mathcal{Y} \times \mathbb{R}$

$$ \phi(y) - \phi(\hat{y}) + \left( \frac{y - \hat{y}}{\lambda - \hat{\lambda}} \right)^T F(\hat{y}, \hat{\lambda}) \geq 0 $$

where the mapping $F(y, \lambda) := [-\lambda h, hy]^T$ and is monotone.}

Since (22) holds for all $(y, \lambda) \in \mathcal{Y} \times \mathbb{R}$, and particularly for $(\hat{y}, \hat{\lambda})$, we have

$$ \left( \lambda^{k+1} - \hat{\lambda} \right)^T \left( \lambda^k - \hat{\lambda} \right) \geq 0 $$

By monotonicity of mapping $F$, we have

$$ \left( \lambda^{k+1} - \hat{\lambda} \right) (\lambda^k - \hat{\lambda}) \geq 0 \quad (26) $$

which implies

$$ |\lambda^{k+1} - \hat{\lambda}|^2 \leq |\lambda^k - \hat{\lambda}|^2 = |\lambda^k - \hat{\lambda}|^2 $$

For every saddle point $(\hat{y}, \hat{\lambda})$ of $L(y, \lambda)$ on the price $\lambda$ corresponds to a primal-dual optimal of problem (6). Since problem (6) has a unique primal optimal $(p^*, d^*)$, we can get $p^* \rightarrow \hat{p}, d^* \rightarrow \hat{d}$, and thus $b^* \rightarrow \hat{b}$.

It is worth noting that the saddle point $(\hat{y}, \hat{\lambda})$ of $L(y, \lambda)$ corresponds to a primal-dual optimal of problem (6). Since problem (6) has a unique primal optimal $(p^*, d^*)$, we can get $p^* \rightarrow \hat{p}, d^* \rightarrow \hat{d}$; moreover, $\hat{\lambda} = \xi^*$ is a dual optimal. Therefore, $(p^*, d^*, \hat{b})$ is a GNE of the energy sharing game. }

Proposition 4 confirms that if parameter $a$ is chosen to meet Condition A4, the proposed bidding process converges to a GNE of the sharing game. It hence offers a guidance for implementing the proposed sharing mechanism. We can also know that the sequence $\{\lambda^k\}$ converges to the “shadow price” $\xi^*$ of problem (6), and as indicated in the proof of Proposition 3, $\lambda_m$ approaches $\lambda^*$ when $I \rightarrow \infty$. Therefore, the energy sharing price can reflect the value of production-demand balance and is meaningful.

**Remark:** With the bidding process presented in this section, we can give an economic explanation for Proposition 3. In each iteration, each prosumer updates its bid considering the impact of its strategy $(p_i, d_i)$ on the price $\lambda$ as in (15). When $I$ is small, all prosumers constitute a monopolistic competition market, and the impact of each prosumer’s strategy on price cannot be neglected. But when $I$ is large enough, it turns into a perfectly competitive market, in which the influence of each

\footnote{A mapping $F(\lambda, y)$ is monotone if and only if $\forall (y_1, \lambda_1), (y_2, \lambda_2) \in \mathcal{Y} \times \mathbb{R}$, we have}

$$ \left( \frac{y_1 - y_2}{\lambda_1 - \lambda_2} \right) \left( F(\lambda_1, y_1) - F(\lambda_2, y_2) \right) \geq 0 $$
prosumer is infinitesimal and thus the price \( \lambda \) can be regarded as exogenously given. In other words, now (13) becomes
\[
\frac{\partial \lambda (d_i - p_i)}{\partial p_i} \bigg|_{\lambda = \lambda^k} = -\lambda^k
\]
(28)

Following a similar procedure to the proof of Proposition 3 we can show that as \( I \to \infty \), the bidding process converges to the optimal solution of problem (1), which is the last statement of Proposition 3.

V. SIMULATION

In this section, numerical experiments are conducted to support our theoretical results. First, a simple case with three prosumers is used to verify the convergence of the bidding process and the effectiveness of the GNE.

A. Simple Example with Three Prosumers

A simple case with three prosumers is tested. The cost function is chosen as \( f_i(.) := \alpha_i^1 p_i^2 + \alpha_i^2 p_i \), and the utility function as \( u_i(.) := \beta_i^1 d_i^2 + \beta_i^2 d_i \), where \( \alpha_i^1, \alpha_i^2, \beta_i^1, \beta_i^2, \forall i \in \{1, 2, 3\} \). Market sensitivity is set at \( a = 100 \) and other parameters are provided in TABLE I. The bidding process in Section IV is used to seek GNE. The \( p_i^k, d_i^k, \lambda^k \) in each iteration are shown in Fig. 3. We observe that prosumer strategies and the energy sharing price converge after 6 iterations. At the GNE, the production and demand of a prosumer are not necessarily equal, where the gap is the amount of energy it buys from or sells to the energy sharing market.

$2.33$ to $2.59$, and prosumer 3 remains the same. This verifies Proposition 2. The centralized social optimal solution achieves the highest total net utility $10.98$. The relative gap between this social optimal and GNE is only $(10.98-10.94)/10.98=0.36\%$, verifying efficiency of the energy sharing mechanism.

B. Cases with More Prosumers

Simulation with a larger case consisting of 50 prosumers is used to show the scalability of the proposed bidding process. Parameters of prosumers are randomly chosen within given ranges: \( \alpha_i^1 \in [0.01, 0.02], \alpha_i^2 \in [0.02, 0.08], \beta_i^1 \in [-0.01, -0.005], \beta_i^2 \in [0.1], \gamma_i \in [20, 40], d_i \in [5, 10], \) and \( p_i \) is set to zero, \( \forall i \in \{1, 2, 3\} \). We change the value of \( a \), and test the case under \( a = 25, a = 50, a = 75, a = 100, \) and \( a = 125 \). With randomly chosen parameters, the change of energy sharing price under each \( a \) corresponds to a line in Fig. 4. We can find that: when \( a \) is too small (\( a = 25 \)), Condition A4 is violated and the bidding process fails to converge; but for other cases, the energy sharing price always converges after 8 iterations, showing the practicability of the proposed bidding process. It is worth noting that, even though the bidding process converges when \( a = 50, 75, 100, 125 \), Condition A4 is not always met. In other words, Condition A4 is just a sufficient condition but not a necessary condition.

Then, we test the performance of the proposed energy sharing mechanism with growing number of prosumers. Increase the number of prosumers from 2 to 50 with parameters randomly chosen within above ranges and fix \( a = 100 \). Change

---

**TABLE I**

| Prosumer | \( \alpha_1 \) ($/kWh^2$) | \( \alpha_2 \) ($/kWh^2$) | \( \beta_1 \) ($/kWh$) | \( \beta_2 \) ($/kWh$) |
|----------|-----------------|-----------------|-----------------|-----------------|
| 1        | 0.015           | 0.038           | -0.008          | 0.8             |
| 2        | 0.008           | 0.047           | -0.014          | 0.5             |
| 3        | 0.011           | 0.056           | -0.009          | 0.4             |

**TABLE II**

| Prosumer | \( p_i \) (kWh) | \( \tau_i \) (kWh) | \( d_i \) (kWh) | \( d_i \) (kWh) |
|----------|-----------------|-----------------|-----------------|-----------------|
| 1        | 0               | 20              | 5               | 15              |
| 2        | 0               | 25              | 7               | 25              |
| 3        | 0               | 30              | 10              | 25              |

**TABLE III**

| Prosumer | \( (\tau^*, d^*) \) | \( (\bar{p}, \bar{d}) \) | \( (\bar{p}, \bar{d}) \) |
|----------|-----------------|-----------------|-----------------|
| 1        | (9, 15.0)       | (8, 15.0)       | (10, 15.0)      |
| 2        | (13, 8, 4)      | (14, 6, 7, 8)   | (10, 3, 10, 3)  |
| 3        | (10, 5, 10)     | (10, 2, 10)     | (10, 0, 10)     |
| Net cost ($) | -2.59         | -0.68           | -2.33           |
| Total net cost ($) | -1.44         | -1.39           | -1.44           |

![Fig. 3. Prosumers’ strategies and sharing price in each iteration](image-url)

---

The \( (\tau^*, d^*) \) at GNE, the social optimal solution \( (\bar{p}, \bar{d}) \) for the centralized problem (1), and the self-sufficient strategy \( (\bar{p}, \bar{d}) \) that solves problem (7) are compared in TABLE III. The net costs of all prosumers are negative, meaning that they have positive net utility, so that Condition A3 is met. All prosumers can benefit (or at least retain the same net utility) by participating in energy sharing. Specifically, prosumer 1’s net utility increases from $6.25$ to $6.90$, prosumer 2 from
of the PoA given in \( \text{[10]} \) is recorded. We repeat this process for five times and get five lines shown in Fig. 5. The PoAs always converge to 1, validating Proposition 3.

To invoke prosumers’ flexibility, a distributed and scalable paradigm for energy sharing is proposed in this paper. A prosumer only needs to send a bid to the platform without revealing private information, and with its tradeoff between production and demand as well as capacity constraint fully considered. The energy sharing among prosumers is modeled as a generalized Nash game, whose equilibrium exists and is partially unique. At equilibrium, a Pareto improvement is achieved so that every prosumer has the incentive to participate in sharing. By analyzing the price-of-anarchy (PoA), we proved that the performance of energy sharing approaches the centralized social optimal solution with an increasing number of prosumers. A practicable bidding process is presented and its convergence condition is given. This paper provides insights into market mechanism design in a prosumer era. Future directions include incorporating renewable uncertainties, considering bounded rationality, and characterizing how big data may help improve the performance of energy sharing.

We further investigate how prosumer diversity would influence the outcome of energy sharing. The number of prosumers is fixed to 100. At the beginning, all prosumers have the same parameters, including the cost function, the utility function, the upper/lower bounds \( p_i, q_i, d_i, d_i, \forall i \in I \). Then, we gradually add some diversity to the testing group by increasing the number of prosumer types. Fifty (50) random scenarios are tested for each degree of diversity, and the mean and variance of the relative cost difference between self-sufficiency and energy sharing cases are plotted in Fig 6. The mean increases with growing diversity, showing that sharing exhibits growing potential for cost savings. The variance of relative cost difference becomes smaller when there are more types of prosumers, implying that the performance of sharing is more stable. Both results demonstrate that adding diversity to prosumers can improve the performance of sharing.

### VI. Conclusion

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The Hessian matrix of (A.1a) is

\[
\begin{bmatrix}
\frac{\partial^2 f_i}{\partial d_i^2} & \frac{\partial^2 f_i}{\partial d_i \partial p_i} \\
\frac{\partial^2 f_i}{\partial d_i \partial p_i} & \frac{\partial^2 f_i}{\partial p_i^2}
\end{bmatrix}
\]

Then, a profile \((\hat{\rho}, \hat{d}, \hat{b})\) is a GNE of \(\mathcal{G}\) if and only if \(\forall i \in \mathcal{I}\), there exists \(\mu_i^+, \eta_i^-\), such that \((\hat{p}_i, \hat{d}_i)\) together with \(\mu_i^+, \eta_i^-\) satisfies (A.3) (where \(\hat{b}_j\) is replaced by \(\hat{b}_j\)), and \(b\) satisfies:

\[
b_i = \hat{d}_i - \hat{p}_i + \hat{d}_i - \hat{p}_i + \sum_{j \neq i} \hat{b}_j
\]

Problems (6) is also a strictly convex optimization problem with the KKT condition as in (A.5).

\[
\hat{f}_i(p_i) - \frac{d_i - p_i}{(I-1)a} - \zeta - \delta_i^- + \delta_i^+ = 0, \forall i \in \mathcal{I}
\]

\[
-\hat{u}_i(d_i) + \frac{2(d_i - p_i) + \sum_{j \neq i} \hat{b}_j}{(I-1)a} - \eta_i^- + \mu_i^+ = 0
\]

\[
0 \leq \mu_i^- - (p_i - p_i) \geq 0
\]

\[
0 \leq \mu_i^+ - (p_i - p_i) \geq 0
\]

\[
0 \leq \eta_i^- - (d_i - d_i) \geq 0
\]

\[
0 \leq \eta_i^+ - (d_i - d_i) \geq 0
\]

Problem (6) is also a strictly convex optimization problem with the KKT condition as in (A.5).

\[
\hat{f}_i(p_i) - \frac{d_i - p_i}{(I-1)a} - \zeta - \delta_i^- + \delta_i^+ = 0, \forall i \in \mathcal{I}
\]

\[
-\hat{u}_i(d_i) + \frac{2(d_i - p_i) + \sum_{j \neq i} \hat{b}_j}{(I-1)a} - \eta_i^- + \mu_i^+ = 0
\]

\[
0 \leq \mu_i^- - (p_i - p_i) \geq 0
\]

\[
0 \leq \mu_i^+ - (p_i - p_i) \geq 0
\]

\[
0 \leq \eta_i^- - (d_i - d_i) \geq 0
\]

\[
0 \leq \eta_i^+ - (d_i - d_i) \geq 0
\]

\[
0 \leq \delta_i^- \leq 0, \forall i \in \mathcal{I}
\]

\[
0 \leq \delta_i^+ \leq 0, \forall i \in \mathcal{I}
\]

\[
0 \leq \kappa_i^- \leq 0, \forall i \in \mathcal{I}
\]

\[
0 \leq \kappa_i^+ \leq 0, \forall i \in \mathcal{I}
\]

Suppose a profile \((\hat{\rho}, \hat{d}, \hat{b})\) is a GNE of \(\mathcal{G}\), and \(\mu_i^+, \eta_i^-\), \(\forall i \in \mathcal{I}\) are the corresponding dual variables, such that (A.3), (A.4) are met. Obviously, (1C) and (1D) are satisfied. Summing up (A.4) for all \(i\) gives equation (1b). Thus, A1 holds.

Denote

\[
\zeta_i = \frac{\hat{d}_i - \hat{p}_i + \sum_{j \neq i} \hat{b}_j}{(I-1)a}, \forall i \in \mathcal{I}
\]

Condition (A.4) indicates that \(\forall i \in \mathcal{I}, \hat{d}_i - \hat{p}_i - \hat{b}_i\) are equal, and \(\zeta_i\) are equal. Let \(\zeta := \zeta_i, \forall i \in \mathcal{I}, \delta_i^+ = \delta_i^+, \kappa_i^- = \kappa_i^-, \zeta_i\). Then \((\hat{\rho}, \hat{d}, \delta, \kappa, \zeta)\) satisfy the KKT condition (A.5). Thus, \((\hat{\rho}, \hat{d}, \hat{b})\) is the optimal solution of problem (6) and is unique.

When A1 holds, problem (6) is also feasible and has a unique optimal solution \((\hat{\rho}, \hat{d}, \hat{b})\) as well as an optimal dual solution \((\delta, \kappa, \zeta)\), which together satisfy (A.5). Let \(\mu_i^+ = \delta_i^+, \eta_i^- = \kappa_i^-, \) and

\[
b_i = \hat{d}_i - \hat{p}_i + \hat{d}_i - \hat{p}_i + \sum_{j \neq i} \hat{b}_j
\]

Then \((\hat{\rho}, \hat{d}, \hat{b})\) and \((\mu_i^+, \eta_i^-)\) satisfy (A.3), (A.4), which implies \((\hat{\rho}, \hat{d}, \hat{b})\) is a GNE.

B. Proof of Proposition [2]

Note that A2 implies A1. For prosumer \(i\), given other prosumers’ strategies \((\hat{p}_i - \hat{d}_i, \hat{b}_i)\), it can choose \(p_i = \hat{p}_i, d_i = \hat{d}_i\) and \(b_i = \sum_{j \neq i} \hat{b}_j/(I-1)\), so that \(a\lambda + b_i = 0\) and \(\Gamma(p_i, \hat{d}_i, \hat{b}_i, p_i - \hat{d}_i, \hat{b}_i) = \Gamma(p_i, \hat{d})\). Since prosumer \(i\) aims at minimizing its net cost at GNE, we have

\[
J_i(p_i, \hat{d}_i) \geq \Gamma_i(\hat{p}, \hat{d}, \hat{b})
\]
Suppose (3) holds with equality for all i. Adding (4a) over all i ∈ I leads to:
\[ \sum_{i=1}^{I} \Gamma_i(\hat{p}, \hat{d}, \hat{b}) = \sum_{i=1}^{I} J_i(\hat{p}_i, \hat{d}_i) \] (B.1)
Thus, \( \sum_{i \in I} J_i(\hat{p}_i, \hat{d}_i) = \sum_{i \in I} J_i(\hat{p}_i, \hat{d}_i) \). The uniqueness of optimal solution of (6) implies (\( \hat{p}, \hat{d} \)) = (\( \hat{p}, \hat{d} \)).

C. Proof of Proposition 2

Part I: Prove (10), i.e.,
\[ \text{PoA}(\mathcal{G}) = \frac{J(\hat{p}(I), \hat{d}(I))}{J(\bar{p}(I), \bar{d}(I))} \geq 1 - \frac{C}{I-1}. \] (C.1)

For simplicity, without causing ambiguity, the I in \((\hat{p}(I), \hat{d}(I))\) and \((\bar{p}(I), \bar{d}(I))\) are omitted here. According to Proposition 1, \((\hat{p}, \hat{d})\) is the optimal solution of (6). Denote \( \Omega(p, d) := \sum_{i \in I} (d_i - p_i)^2, S := \{(p_i, d_i) | \forall i \in I : \text{s.t.} \ (6a)-(6b) \text{ are satisfied.} \} \) Note that S is also the feasible set for problem (1).

For every strategy combination \( s = (p, d, b) \in S \), there is:
\[ \sum_{i=1}^{I} \Gamma_i(p, d, b) = \sum_{i=1}^{I} J_i(p_i, d_i) = J(p, d) \] (C.2)

which in particular holds for every GNE \((\hat{p}, \hat{d}, \hat{b})\) and every \((\tilde{p}, \tilde{d}, \tilde{b})^* \in \arg \min_{s \in S} \sum_{i=1}^{I} \Gamma_i(s) \). Moreover, one can establish equivalence between the set of all subvectors \((p, d)\) in strategy space \( S \) and the feasible set \( S \) of problem (1), so there must be \((p^*, d^*) = (\hat{p}, \hat{d})\). Then PoA can be equivalently written as
\[ \text{PoA}(\mathcal{G}) = \frac{J(\hat{p}, \hat{d})}{J(\hat{p}, \hat{d})} \] (C.3)

Obviously \( |\Omega(p, d)| \leq C_1 I, \forall (p, d) \in S \), where
\[ C_1 := \sup \left\{ |p_i - \bar{p}_i|^2, |p_i - \bar{d}_i|^2, \forall i \in I \right\} \]
is independent from \( I \) by the uniform bound assumption on \( p_i, \bar{p}_i, \bar{d}_i, \bar{d}_i \) for all \( i \in I \). By definition, we have
\[ J(\hat{p}, \hat{d}) \leq J(\hat{p}, \hat{d}) \] (C.4)

and
\[ J(\hat{p}, \hat{d}) + \frac{\Omega(\hat{p}, \hat{d})}{2a(I-1)} \geq J(\hat{p}, \hat{d}) + \frac{\Omega(\hat{p}, \hat{d})}{2a(I-1)} \] (C.5)

so that
\[ J(\hat{p}, \hat{d}) \leq J(\hat{p}, \hat{d}) + \frac{\Omega(\hat{p}, \hat{d})}{2a(I-1)} - \frac{\Omega(\hat{p}, \hat{d})}{2a(I-1)} \leq J(\hat{p}, \hat{d}) + \frac{C_1 I}{a(I-1)} \]

When A2 and A3 hold, we have \( J(\hat{p}, \hat{d}) \leq J(\hat{p}, \hat{d}) \leq C_2 I < 0 \), where
\[ C_2 := \sup \left\{ f_i(\hat{p}_i) - u_i(\hat{d}_i), \forall i \in I \right\} \] (C.6)
is independent from \( I \) by the uniform bound assumption on \( f_i(\cdot), u_i(\cdot) \) for all \( i \in I \). Thus
\[ 1 - \text{PoA}(\mathcal{G}) = \frac{J(\hat{p}, \hat{d}) - J(\hat{p}, \hat{d})}{J(\hat{p}, \hat{d})} \leq \frac{C_1 I}{a(I-1)} \]

\[ = \frac{C}{I-1} \] (C.7)

where \( C := C_1/(a|C_2|) \).

Part II: Prove (11), i.e.,
\[ \lim_{i \to \infty} |\hat{p}_i(I) - \bar{p}_i(I)| = \lim_{i \to \infty} |\hat{d}_i(I) - \bar{d}_i(I)| = 0, \forall i \in I \]

Sketch of proof. We notice that the difference between KKT conditions of problems (1) and (6) only lies in the term \( \frac{d_i}{\alpha(I-1)} \) in stationarity equations (A.5a)-(A.5b). Due to the uniform bound assumption we made on \( p_i, \bar{p}_i, \bar{d}_i, \bar{d}_i \), this difference will diminish as prosumer number \( I \) increases to infinity. Based on this observation, we can bound the difference between solutions of the two sets of KKT conditions, i.e., between the optimal solutions of problems (1) and (6), and show that this difference also diminishes as \( I \) increases to infinity. Please see below for a detailed proof.

Full proof. The centralized social optimal problem (1) can be equivalently solved by its KKT condition \([\hat{f}_i, \hat{u}_i]) \)
\[ \hat{f}_i(p_i) - \lambda_m - \delta^{-} + \delta^{+} = 0, \forall i \in I \] (C.8a)
\[ -\hat{u}_i(d_i) + \lambda_m - \kappa^{-} + \kappa^{+} = 0, \forall i \in I \] (C.8b)
\[ \sum_{i=1}^{I} p_i = \sum_{i=1}^{I} d_i \] (C.8c)

0 ≤ \delta^{-} ⊥ (p_i - \bar{p}_i) ≥ 0, \forall i ∈ I \] (C.8d)

0 ≤ \delta^{+} ⊥ (\bar{p}_i - p_i) ≥ 0, \forall i ∈ I \] (C.8e)

0 ≤ \kappa^{-} ⊥ (d_i - \bar{d}_i) ≥ 0, \forall i ∈ I \] (C.8f)

0 ≤ \kappa^{+} ⊥ (\bar{d}_i - d_i) ≥ 0, \forall i ∈ I \] (C.8g)

All the equations in (C.8) except (C.8c) define the optimal production \( p_i \) and consumption \( d_i \) in response to a given dual variable \( \lambda_m \) as the following functions, for all \( i ∈ I \):
\[ p_i = \hat{f}_i(\lambda_m) := \begin{cases} \hat{f}_i(p_i), & \text{if } \lambda_m \leq \hat{f}_i(p_i) \\ \hat{p}_i, & \text{if } \lambda_m ≥ \hat{f}_i(\bar{p}_i) \end{cases} \] (C.8c)
\[ d_i = \hat{u}_i(\lambda_m) := \begin{cases} \hat{u}_i(d_i), & \text{if } \lambda_m ≤ \hat{u}_i(d_i) \\ \bar{d}_i, & \text{if } \lambda_m ≥ \hat{u}_i(\bar{d}_i) \end{cases} \] (C.8d)

By our assumptions on \( f_i(\cdot), u_i(\cdot) \), for all \( i ∈ I \), functions \( \hat{f}_i(\cdot) \) and \( -\hat{u}_i(\cdot) \) are well defined and monotonically increasing on \( \lambda_m ∈ \mathbb{R} \). By (C.8c), \( \lambda_m \) is a dual optimal solution of problem (1) if and only if it solves the following equation:
\[ \sum_{i=1}^{I} \left( \hat{f}_i(\lambda_m) - \hat{u}_i(\lambda_m) \right) = 0. \]

We next look at KKT condition (A.5) which equivalently characterizes primal-dual optimal solutions of problem (6).
Specifically, all the equations in (A.5) except (A.5c) define the optimal \( p_i \) and \( d_i \) in response to a given dual variable \( \zeta \) as functions \( f_i^*(\zeta) \) and \( u_i^*(\zeta) \), respectively, for all \( i \in I \). Although closed-form expressions of \( f_i^*(\cdot) \) and \( u_i^*(\cdot) \) are hard to derive, we can establish their relationships with \( f_i(\cdot) \) and \( u_i(\cdot) \), for all \( i \in I \):

\[
\begin{align*}
p_i = f_i^*(\zeta) &= \tilde{f}_i \left( \zeta + \frac{d_i - p_i}{a(l-1)} \right), \\
d_i = u_i^*(\zeta) &= \tilde{u}_i \left( \zeta + \frac{d_i - p_i}{a(l-1)} \right)
\end{align*}
\]

Besides, when \( f_i^*(\zeta) \in (p_i, \tilde{p}_i) \) and \( u_i^*(\zeta) \in (d_i, \tilde{d}_i) \) are both satisfied, the following equation holds for all \( i \in I \):

\[
\tilde{f}_i \left( f_i^*(\zeta) \right) = \zeta + \frac{u_i^*(\zeta) - f_i^*(\zeta)}{a(l-1)} = \tilde{u}_i (u_i^*(\zeta))
\]

Taking its derivative over \( \zeta \), and combining the cases where capacity constraints are binding, we get for all \( i \in I \):

\[
\begin{align*}
f_i^*(\zeta) &= \begin{cases} 
\tilde{f}_i(p_i) \left[ 1 + \frac{1}{a(l-1)} \left( \frac{1}{f_i(p_i) - u_i(d_i)} \right) \right] & \text{if } p_i < p_i < \tilde{p}_i \\
0 & \text{otherwise}
\end{cases}, \\
u_i^*(\zeta) &= \begin{cases} 
\tilde{u}_i(d_i) \left[ 1 + \frac{1}{a(l-1)} \left( \frac{1}{u_i(d_i) - \tilde{u}_i(p_i)} \right) \right] & \text{if } d_i < d_i < \tilde{d}_i \\
0 & \text{otherwise}
\end{cases}
\]

where \( p_i = f_i^*(\zeta) \) and \( d_i = u_i^*(\zeta) \). By our assumptions on \( f_i(\cdot), u_i(\cdot) \), for all \( i \in I \), functions \( f_i^*(\cdot) \) and \( -u_i^*(\cdot) \) are well defined and monotonically increasing on \( \zeta \in \mathbb{R} \). By the power balance constraint (A.5c), \( \hat{\zeta} \) is a dual optimal solution of problem (4) if and only if it solves the following equation:

\[
\sum_{i=1}^{I} \left( f_i^*(\hat{\zeta}) - u_i^*(\hat{\zeta}) \right) = 0.
\]

Let \( \lambda_m \) be any dual optimal solution of problem (1), so that \( \tilde{p}_i = \tilde{f}_i(\lambda_m), \tilde{d}_i = \tilde{u}_i(\lambda_m) \) for all \( i \in I \) constitute the (unique) primal optimal solution of problem (1). We next show that there must be a dual optimal solution \( \xi \) of problem (6) which lies near \( \lambda_m \). For that purpose, we denote

\[
\begin{align*}
p &:= \sup \{ p_i, \forall i \in I \}, \\
\bar{p} &:= \inf \{ p_i, \forall i \in I \} \\
d &:= \sup \{ d_i, \forall i \in I \}, \\
\bar{d} &:= \inf \{ d_i, \forall i \in I \}
\end{align*}
\]

which all exist and are independent from \( I \) by our uniform bound assumption. Define two numbers:

\[
\xi^+ := \lambda_m + \frac{\bar{p} - \bar{d}}{a(l-1)} \geq \xi^- := \lambda_m + \frac{p - d}{a(l-1)}.
\]

Indeed, there must be

\[
f_i^*(\xi^+) - u_i^*(\xi^+) \geq \tilde{f}_i(\lambda_m) - \tilde{u}_i(\lambda_m), \forall i \in I
\]

which can be verified by assuming \( f_i^*(\xi^+) - u_i^*(\xi^+) < \tilde{f}_i(\lambda_m) - \tilde{u}_i(\lambda_m) \) and deducing a contradiction:

\[
f_i^*(\xi^+) = \tilde{f}_i \left( \xi^+ + \frac{\tilde{d}_i - \tilde{p}_i}{a(l-1)} \right) \geq \tilde{f}_i \left( \xi^+ + \frac{d_i - p_i}{a(l-1)} \right).
\]

where \( p_i = f_i^*(\xi^+), d_i = u_i^*(\xi^+) \), for all \( i \in I \). Both inequalities above stem from monotonicity of \( f_i(\cdot) \). Similarly, \(-u_i^*(\xi^+) \geq -\tilde{u}_i(\lambda_m) \) for all \( i \in I \), and therefore \( f_i^*(\xi^+) - u_i^*(\xi^+) \geq \tilde{f}_i(\lambda_m) - \tilde{u}_i(\lambda_m) \), which contradicts our assumption.

We hence further have

\[
\sum_{i=1}^{I} \left( f_i^*(\xi^+) - u_i^*(\xi^+) \right) \geq \sum_{i=1}^{I} \left( \tilde{f}_i(\lambda_m) - \tilde{u}_i(\lambda_m) \right) = 0.
\]

Following the same procedure, we can also show

\[
\sum_{i=1}^{I} \left( f_i^*(\xi^-) - u_i^*(\xi^-) \right) \leq 0.
\]

Due to monotonicity of function \( \Sigma_{i=1}^{I} (f_i^*(\cdot) - u_i^*(\cdot)) \), there must be \( \xi \in [\xi^-, \xi^+] \), such that \( \Sigma_{i=1}^{I} \left( f_i^*(\xi) - u_i^*(\xi) \right) = 0 \), i.e., \( \xi \) is a dual optimal solution of problem (6). Further, \( \tilde{p}_i = f_i^*(\xi), \tilde{d}_i = u_i^*(\xi) \) for all \( i \in I \) constitute the (unique) primal optimal solution of problem (6), which is also the production and consumption profile at GNE.

To prepare for the final step of our proof, we point out Lipschitz continuity of functions \( f_i(\cdot), u_i(\cdot) \) for all \( i \in I \). Specifically, due to the uniform bound assumption we made on \( \tilde{f}_i(\cdot), \tilde{u}_i(\cdot) \), for all \( i \in I \), there exists a positive constant \( \gamma \) which is independent from \( I \), such that

\[
|f_i(x) - f_i(y)| \leq \gamma |x - y|, \forall i \in I, \forall x, y \in \mathbb{R}
\]

\[
|u_i(x) - u_i(y)| \leq \gamma |x - y|, \forall i \in I, \forall x, y \in \mathbb{R}.
\]

Denote \( \sigma := \max \{|p - \bar{d}|, |\bar{p} - p|, \gamma \} / a \). For any prosumer number \( I \), for all \( i \in I \), we have:

\[
|\tilde{p}_i - p_i| \leq \gamma |x - y|, \forall i \in I, \forall x, y \in \mathbb{R}
\]

\[
|\tilde{u}_i - u_i| \leq \gamma |x - y|, \forall i \in I, \forall x, y \in \mathbb{R}.
\]

To finish the proof, we apply the standard definition of convergence. For arbitrary \( \varepsilon > 0 \), we can identify integer \( I_\varepsilon \geq \frac{2\gamma \sigma}{\varepsilon} + 1 \), such that for all \( I \geq I_\varepsilon \), we can make \( |\tilde{p}_i - p_i| \leq \frac{2\gamma \sigma}{I_\varepsilon} \leq \varepsilon \). This proves \( \lim_{I \to \infty} |\tilde{p}_i - p_i| = 0 \) for all \( i \in I \). A similar argument can prove \( \lim_{I \to \infty} |\tilde{u}_i - u_i| = 0 \) for all \( i \in I \).

D. Proof of Lemma 7

The Hessian matrix of \( \phi(y) \) is \( \nabla^2 \phi = H_1 + H_2 + H_3 \), where

\[
H_1 = \begin{pmatrix}
\tilde{f}_i & -\tilde{u}_1 & \cdots & -\tilde{u}_i \\
\vdots & \ddots & \ddots & \vdots \\
-\tilde{u}_1 & \cdots & \tilde{f}_i & -\tilde{u}_i \\
-\tilde{u}_i & \cdots & -\tilde{u}_i & \tilde{f}_i
\end{pmatrix}
\]
corresponding to orthonormal eigenvectors $e_{i}$, eigenvalue of which means  

$$x = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

\[
\frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \quad \cdots \quad \frac{\sqrt{2}}{2} \quad 0 \cdots 0, \forall i = 1 \cdots I.
\]

The only non-zero eigenvalue of $H_2$ is $\frac{2}{a(I-1)}$, corresponding to orthonormal eigenvectors $e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, $\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ \cdots \\ \sqrt{2} \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$, $\forall i = 1 \cdots I$. The only non-zero eigenvalue of $H_3$ is $-\frac{2}{a}$, corresponding to unit eigenvector $e = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \cdots \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. When $A4$ holds, for any vector $x = [x_{11} x_{12} \cdots x_{1I} x_{21} \cdots x_{2I}]^T \in \mathbb{R}^{2I \times 1}$, we have

$$x^T \Xi(\phi)x$$

$$= x^T H_1 x + x^T H_2 x + x^T H_3 x$$

$$= \sum_{i=1}^{I} (f_i x_{1i} - \bar{u}_i x_{2i})^2 + \frac{2}{a(I-1)} \left( \frac{\sqrt{2}}{2} \right)^2 \sum_{i=1}^{I} (x_{1i} - x_{2i})^2$$

$$- \frac{2}{a(I-1)} \sum_{i=1}^{I} (x_{1i} - x_{2i})^2$$

$$\geq \sum_{i=1}^{I} (f_i x_{1i} - \bar{u}_i x_{2i})^2 + \left( \frac{1}{a(I-1)} - \frac{1}{a} \right) \sum_{i=1}^{I} (x_{1i} - x_{2i})^2$$

$$= \sum_{i=1}^{I} \left( f_i x_{1i} - \bar{u}_i x_{2i} - \frac{I - 2}{a(I-1)} (x_{1i} - x_{2i})^2 \right)$$

$$\geq \sum_{i=1}^{I} \left( f_i x_{1i} - \bar{u}_i x_{2i} - \frac{2I - 4}{a(I-1)} (x_{1i}^2 + x_{2i}^2) \right) \geq 0 \quad (D.1)$$

Therefore, $\Xi(\phi)$ is a positive semidefinite matrix, implying $\phi(y)$ is a convex function.

Suppose $(\hat{\phi}, \hat{d}, \hat{b})$ is an GNE of the game $G$, and \( \hat{\lambda} := \sum_{i=1}^{I} \hat{b}_i / (aI) \). According to the KKT condition (A.3) and the convexity of $\phi(y)$, it is easy to check that $(\hat{y}, \hat{\lambda})$ satisfies

$$L_{\hat{\lambda} \in \mathbb{R}}(\hat{y}, \hat{\lambda}) \leq L(\hat{y}, \hat{\lambda}) \leq L_{\lambda \in \mathbb{R}}(y, \hat{\lambda}) \quad (D.2)$$

which means $(\hat{y}, \hat{\lambda})$ is a saddle point of $L(y, \lambda)$. 

$$H_2 = \begin{pmatrix} \frac{1}{a(I-1)} & \frac{-1}{a(I-1)} \\ \frac{-1}{a(I-1)} & \frac{1}{a(I-1)} \\ \vdots \\ \frac{-1}{a(I-1)} & \frac{1}{a(I-1)} \\ \frac{1}{a(I-1)} & \frac{-1}{a(I-1)} \end{pmatrix}$$

$$H_3 = \begin{pmatrix} \frac{-1}{a} & \frac{1}{a} & \cdots & \frac{-1}{a} \\ \frac{1}{a} & \frac{-1}{a} & \cdots & \frac{1}{a} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{a} & \frac{1}{a} & \cdots & \frac{1}{a} \\ \frac{1}{a} & \frac{-1}{a} & \cdots & \frac{-1}{a} \end{pmatrix}$$