CONVERGENCE OF FOURIER TRUNCATIONS
FOR COMPACT QUANTUM GROUPS
AND FINITELY GENERATED GROUPS

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Abstract. We generalize the Fejér-Riesz operator systems defined for the
circle group by Connes and van Suijlekom to the setting of compact matrix
quantum groups and their ergodic actions on C*-algebras. These truncations
form filtrations of the containing C*-algebra. We show that when they and the
containing C*-algebra are equipped with suitable quantum metrics, then under
suitable conditions they converge to the containing C*-algebra for quantum
Gromov-Hausdorff distance. Among other examples, our results are applicable
to the quantum groups $SU_q(2)$ and their homogeneous spaces $S^2_q$.

1. Introduction

Over the past decade or so there has been slowly increasing interest in the
use of operator systems in non-commutative geometry [30, 31, 52, 22, 23,
15, 14, 3, 11, 29, 21, 33, 20, 39], spurred on quite recently by the paper
[13] of Connes and van Suijlekom concerning spectral truncations. Even
more recently, the use of quantum Gromov-Hausdorff distance for spectral
truncations in the setting of operator systems was initiated in [57]. (But see
the earlier paper [16] which used the setting of order-unit spaces.) These two
papers of Connes and van Suijlekom concentrate on revealing the remarkably
rich structure that one obtains already in the case of the circle. (See also
[24, 32].)

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For a spectral triple \((A, \mathcal{H}, D)\) the spectral truncations are the compressions \(PAP\), where \(P\) is a finite-dimensional spectral projection of \(D\), viewed as an operator system of operators on \(PH\). In [57] it was shown that, for the case in which \(A\) is the algebra \(C(S^1)\) of continuous functions on the circle, the spectral truncations converge to \(C(S^1)\) for quantum Gromov-Hausdorff distance. (Very recently the corresponding question for higher-dimensional tori has been explored in [33].) It is extremely interesting to ask to what extent this continues to be true for other spectral triples.

In [13, 57] the operator systems dual to the spectral truncations play an important role. While for the circle the spectral truncations are shown to be the vector spaces of Toeplitz matrices of various sizes, their operator system duals are shown to be the vector spaces of functions whose Fourier coefficients are 0 outside given intervals of \(\mathbb{Z}\) symmetric about 0. In [57] it is shown that these dual operator systems too converge to \(C(S^1)\) for a suitable quantum Gromov-Hausdorff distance. It is interesting to ask to what extent this continues to be true in other situations.

The purpose of the present paper is to give an answer to this latter question in a quite broad setting, namely that of compact matrix quantum groups, which includes the case of finitely generated discrete groups. Our setting also includes ergodic actions of compact matrix quantum groups on unital C*-algebras. It seems to me that the feature underlying the proof in [57] of the convergence of the dual operator systems is the fact that \(S^1\) is a compact group.

We now give an imprecise statement of our main theorem. The various technical terms will be defined as needed in the text of this paper.

**Theorem 1.1.** (Imprecise statement of Theorem 6.1) Let \((A, \Delta)\) be a compact matrix quantum group, and let \(\alpha\) be an ergodic action of \((A, \Delta)\) on a unital C*-algebra \(B\). Let \(A_1\) be the operator subsystem of \(A\) generated by the coordinate elements of a fixed faithful finite dimensional corepresentation of \(A\), and for each \(n \in \mathbb{Z}\) let \(A_n\) be the operator subsystem of \(A\) generated by all products of \(n\) elements of \(A_1\), so that the \(A_n\)'s form a filtration of \(A\). Let \(\{B_n\}\) be the corresponding filtration of \(B\). Assume that \((A, \Delta)\) is coamenable, and let \(L^B\) be a regular \(\alpha\)-invariant Lip-norm on \(B\). For each \(n\) let \(L^B_n\) be the restriction of \(L^B\) to \(B_n\). Then the compact quantum metric spaces \((B_n, L^B_n)\) converge to \((B, L^B)\) for quantum Gromov-Hausdorff distance.

We will call the \(B_n\)'s “Fourier truncations” of \(B\), for reasons that will be evident later. They correspond to the Fejér-Riesz operator systems of [13, 57]. So we are dealing with “truncations of the Fourier modes of the (bosonic) elements” as mentioned very briefly in section 6 of [13].

Of course there remains the big challenge of formulating and proving corresponding results for the spectral truncations of [13, 57].

In Section 7 we provide a number of classes of examples to which our results are applicable. These include ordinary compact Lie groups and their ergodic actions on unital C*-algebras, the group C*-algebras of amenable
finitely generated discrete groups, groups and quantum groups of rapid decay, and the quantum groups $SU_q(2)$ and their homogeneous space $S^2_q$.

The proof of our main theorem makes crucial use of parts of Hanfeng Li’s exploration of metric aspects of ergodic actions of compact quantum groups in section 8 of [36].

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2. Preliminaries about compact quantum groups

In this section we gather well-known facts about compact quantum groups that we will need. We give few proofs, and instead refer the reader to [59, 17, 56, 18] and the references they include.

A compact quantum group is a pair $(A, \Delta)$, where $A$ is a unital C*-algebra and $\Delta : A \to A \otimes A$ is a unital $\ast$-homomorphism, called the comultiplication, such that

1. $(\Delta \otimes I_A)\Delta = (I_A \otimes \Delta)\Delta$ as homomorphisms $A \to A \otimes A \otimes A$ (coassociativity)
2. The spaces $(A \otimes 1_A)\Delta(A) := \operatorname{span}\{(a \otimes 1_A)\Delta(b) : a, b \in A\}$ and $(1_A \otimes A)\Delta(A)$ are dense in $A \otimes A$ (the cancellation property).

Here $I_A$ is the identity operator on $A$, and the tensor product is the minimal tensor product for C*-algebras.

A fundamental fact about compact quantum groups is that one can prove that they have a generalization of the normalized Haar measure on ordinary compact groups. That is, there is a unique state, the “Haar state” $h : A \to \mathbb{C}$ that satisfies the left and right invariance conditions

$$(I_A \otimes h) \circ \Delta = h1_A = (h \otimes I_A) \circ \Delta.$$

In general the Haar state may not be a trace, and it may not be faithful on $A$.

The following class of examples is of importance for this paper.

**Example 2.1.** Let $\Gamma$ be a discrete group. Denote by $C_u^*(\Gamma)$ the full group C*-algebra of $\Gamma$. Because $\Gamma$ is discrete, this is a unital C*-algebra. Every $x \in \Gamma$ determines a unitary element, $\delta_x$, of $C_u^*(\Gamma)$, and the linear span, $C_f(\Gamma)$, of these unitary elements forms a dense $\ast$-subalgebra of $C_u^*(\Gamma)$. This subalgebra can be viewed as consisting of the finitely supported functions on $\Gamma$. The mapping $\delta_x \mapsto \delta_x \otimes \delta_x$ determines a unital homomorphism, $\Delta$, from $C_u^*(\Gamma)$ into $C_u^*(\Gamma) \otimes C_u^*(\Gamma)$ that makes $C_u^*(\Gamma)$ into a compact quantum group. (Note that in this situation one might expect to use the maximal C*-tensor product, but for quantum groups it is usual to use the minimal C*-tensor product, for which the construction of this example remains valid.) For this class of examples the Haar state is just the usual tracial state coming from evaluating functions in $C_f(\Gamma)$ at the identity element of $\Gamma$. (For the details of this construction see [7]; for even locally compact $\Gamma$ see the earlier
In the case that \( \Gamma \) is not amenable, the Haar state on \( C^*_u(G) \) is not faithful, almost by one definition of amenability. (But the Haar state is always faithful on \( C_f(\Gamma) \).) The reduced C*-algebra of \( \Gamma \), defined by the GNS construction for the Haar state, is a compact quantum group in essentially the same way. If \( \Gamma = F_2 \), the free group on two generators, then the Haar state is faithful on its reduced C*-algebra, but \( F_2 \) is not amenable.

For a compact quantum group \((A, \Delta)\) the C*-algebra \( A \) gives the “space”, and has its own \(*\)-representations on Hilbert spaces. The generalization of finite-dimensional unitary representations of groups to the quantum group context is called finite-dimensional unitary “corepresentations”. Let \( H \) be a finite dimensional Hilbert space (with inner product linear in the second variable). Consider the right Hilbert \( A \)-module \( H \otimes A \), with \( A \)-valued inner product \( \langle \xi \otimes a, \eta \otimes b \rangle_A = \langle \xi, \eta \rangle a^* b \) for \( a, b \in A \) and \( \xi, \eta \in H \). A unitary corepresentation of \((A, \Delta)\) on \( H \) is a linear map \( u : H \to H \otimes A \) that satisfies

- \( (u \otimes I_A) \circ u = (I_H \otimes \Delta) \circ u \)
- \( \langle u(\xi), u(\eta) \rangle_A = \langle \xi, \eta \rangle C_1 A \) for \( \xi, \eta \in H \)
- \( u(H)A = H \otimes A \).

(See [17], section 3 of [41] and definition 2.8 of [52].) Given \( \xi, \eta \in H \), we obtain an element, \( u_{\xi \eta} \) of \( A \), by

\[
u_{\xi \eta} = (\langle \xi, \cdot | \otimes I_A)(u(\eta))
\]

(where \( \langle \xi, \cdot | \) denotes the linear functional on \( H \) determined by \( \xi \)). These elements are called “coordinate elements” of the corepresentation \( u \). If \( \xi_1, \ldots, \xi_n \) is an orthonormal basis for \( H \), and if we set \( u_{jk} = u_{\xi_j \xi_k} \), then the matrix \( U = \{u_{jk}\} \) is a unitary element of \( M_n(A) \) that satisfies

\[
\Delta(u_{jk}) = \sum_{\ell=1}^n u_{j\ell} \otimes u_{\ell k}.
\]

Such a matrix provides the more common definition of a finite-dimensional unitary corepresentation of a compact quantum group. Given such a unitary matrix \( U \) that satisfies this relation, one can define the corresponding linear map \( u : H \to H \otimes A \) by \( u(\xi) = U(\xi \otimes 1_A) \), suitably interpreted. See lemma 1.7 of [17].

We will let \( \hat{\Delta} \) denote the set of unitary equivalence classes of (finite-dimensional) irreducible unitary corepresentations of \((A, \Delta)\). It can be considered to be the set of Fourier modes in a generalized sense. In the usual way, our notation will not distinguish between such corepresentations and their equivalence classes. For each \( \gamma \in \hat{\Delta} \) we let \( A^\gamma \) denote the corresponding isotypic component of \( A \). It is a finite-dimensional subspace of \( A \), and \( \Delta(A^\gamma) \subseteq A^\gamma \otimes A^\gamma \). Each unitary corepresentation has a conjugate unitary corepresentation. We denote the conjugate of \( \gamma \) by \( \gamma^c \). Then \( (A^\gamma)^* = A^{\gamma^c} \). The isotypic components are mutually orthogonal with respect to the inner product defined by the Haar state.
Thus a faithful on representations, a element for the unital Hopf *-algebra is well-defined on A and there is a well-defined coinverse (i.e. antipode) on A. Furthermore, the one-dimensional trivial corepresentation is well-defined on A, and serves as a coidentity element for A, and there is a well-defined converse (i.e. antipode) on A. Thus A is a unital Hopf *-algebra [17]. In addition, the Haar state on A is faithful on A.

Let A be the completion of A for the universal C*-norm on A. All of the unital Hopf *-algebra structure of A extends to A, except that the coinverse may not be continuous for the universal C*-norm on A. If it is continuous, then A is said to be of “Kac type”. The Haar state need not be faithful on A, as seen in Example 2.1.

Instead, let A be the completion of A for the C*-norm from the GNS representation on the Hilbert space L^2(A, h) obtained from using the Haar state. All of the unital Hopf *-algebra structure of A extends to A, except that the coinverse and the coidentity may not be continuous for the reduced norm. We call A the “reduced” compact quantum group for A. The Haar state is faithful on A. The coidentity is continuous for the reduced norm if and only if A and A coincide, in which case A is said to be “coamenable”. For a discrete group Γ the corresponding compact quantum group is coamenable exactly if Γ is amenable. In this paper, we will be concerned mostly with compact quantum groups that are coamenable.

Let A be the Banach space of all continuous linear functionals on A. For μ, ν ∈ A set μ * ν = (μ ⊗ ν) ◦ Δ. This is an associative product, “convolution”, on A for which A is a Banach algebra. The Haar state h is an element of A, and h * μ = μ(1_A)h = μ * h for all μ ∈ A. If the coidentity element is continuous on A so that it is an element of A, then it serves as an identity element of the algebra A, and we will then denote it by ε.

For any β, γ ∈ A let β ⊗ γ denote the subset of A consisting of all the elements of A that appear in the decomposition of the tensor product of β with γ into irreducible representations. For any finite subset S of A let A^S denote the direct sum of the A^γ’s with γ ∈ S. Then for any β, γ ∈ A one has A^β A^γ ⊆ A^{β ⊗ γ} (where A^β A^γ means “finite sums of products”).

The following result will be crucial for our purposes.

**Proposition 2.2.** Let a ∈ A, viewed as an element of L^2(A, h), and satisfying ||a||_2 = 1 for the norm on L^2(A, h). Let μ_a be the vector state on A determined by a. Then there is a finite subset, F_a, of A such that if γ ∈ A but γ is not in F_a, then μ_a(A^γ) = 0.

**Proof.** Since a ∈ A, it is the finite sum of non-zero elements of various A^γ’s. Let F be the set of the corresponding γ’s. Let δ ∈ A and let b ∈ A^δ.
Suppose that \( \mu_a(b) \neq 0 \), that is, \( h(a^*ba) \neq 0 \). Then there must be \( \gamma, \beta \in F \), and \( a_\gamma \in \mathcal{A}^\gamma \) and \( a_\beta \in \mathcal{A}^\beta \), such that \( h(a_\gamma^*ba_\beta) \neq 0 \).

Now \( h \) may not be a trace. But in proposition 3.12 of [59] Van Daele shows that there is an automorphism, \( \sigma \) of \( \mathcal{A}_c \), with the property that \( h(ab) = h(b\sigma(a)) \) for all \( a, b \in \mathcal{A}_c \), and also that \( h(\sigma(a)) = h(a) \). Van Daele calls this a “weak KMS property”. (To see the relationship with KMS automorphisms see theorem 1.4 of [59] or section 2.1 of [55].) However, \( \sigma \) is in general not a \(*\)-automorphism. But \( \sigma \) does carry isotypic components of \( \mathcal{A}_c \) into themselves. To see this, let \( \gamma \in \hat{\Delta} \) and let \( d \in \mathcal{A}^\gamma \) be given. Then for any \( \beta \in \hat{\Delta} \) with \( \beta \neq \gamma \), and any \( c \in \mathcal{A}^\beta \) we have \( 0 = \langle c^*, d^* \rangle = h(dc^*) = h(c^*\sigma(d)) = \langle \sigma(d), c \rangle \). This implies that \( \sigma(d) \in \mathcal{A}^\gamma \).

Returning to our original \( \gamma \) and \( \beta \), we see that \( h(a_\gamma^*ba_\beta) = h(ba_\beta\sigma(a_\gamma^*)) \), and that \( \sigma(a_\gamma^*) \) is in \( \mathcal{A}^{\gamma^*} \). Since \( 0 \neq h(ba_\beta\sigma(a_\gamma^*)) = \langle a_\beta\sigma(a_\gamma^*), b^* \rangle \) and \( a_\beta\sigma(a_\gamma^*) \) is in \( \mathcal{A}^{\beta\gamma^*} \) it follows that \( \delta \) is in \( \gamma \otimes \beta^* \).

For \( F \) as defined above, let \( F^c \) denote the set of all conjugates of elements of \( F \). Then let \( F_a \) be the set of all elements of \( \hat{\Delta} \) that are contained in the tensor product of some element of \( F \) with some element of \( F^c \), in that order. We see that if \( \gamma \in \hat{\Delta} \) and if \( \gamma \) is not in \( F_a \), then \( \mu_a(\mathcal{A}^\gamma) = 0 \). Note that the set \( F_a \) is finite.

Essentially by definition, a compact Lie group (not necessarily connected) has a faithful finite-dimensional unitary representation. By taking the direct sum of this representation with its conjugate if necessary, we can assume that this representation is self-conjugate. By then taking the direct sum with the trivial one-dimensional representation if necessary, we can assume that this representation is self-conjugate and contains the trivial representation. There are many choices of such a faithful representation. Let \( G \) be a compact Lie group, and let \( (\mathcal{H}, U) \) be a faithful finite-dimensional unitary representation that is self-conjugate and contains the trivial representation. Let \( C(G; U) \) be the linear span of the coordinate functions of \( U \). Then \( C(G; U) \) is a finite-dimensional subspace of \( C(G) \) that is closed under complex conjugation and contains the constant functions. Thus the set of \( \mathbb{R} \)-valued functions that it contains is an order-unit space, and when \( C(G) \) is viewed as a commutative \( \mathbb{C}^* \)-algebra, we see that \( C(G; U) \) is an operator system. Because the representation \( (\mathcal{H}, U) \) is faithful, the functions in \( C(G; U) \) separate the points of \( G \). Thus the (algebraic) subalgebra of \( C(G) \) generated by \( C(G; U) \) is dense in \( C(G) \), according to the Stone-Weierstrass theorem. (Because \( C(G; U) \) is finite-dimensional, we say that \( C(G) \) is “finitely generated”, with \( C(G; U) \) as generating set.) Let \( \mathcal{A} = C(G) \), and for \( n \in \mathbb{N} \) let \( \mathcal{A}_n \) be the linear span of products of \( n \) elements of \( C(G; U) \) (with \( \mathcal{A}_0 = C_1 \mathcal{A} \)). Then each \( \mathcal{A}_n \) is a finite-dimensional operator system in \( \mathcal{A} \), and for any \( m, n \in \mathbb{N} \) we have \( \mathcal{A}_m \mathcal{A}_n \subseteq \mathcal{A}_{m+n} \). Thus \( \{ \mathcal{A}_n \} \) is a filtration of the \( \mathbb{C}^* \)-algebra \( \mathcal{A} \). We view the \( \mathcal{A}_n \)'s as Fourier truncations of \( \mathcal{A} \). We now generalize this structure to the setting of compact quantum groups.
Let \((A, \Delta)\) be a compact quantum group and let \((\mathcal{H}, u)\) be a finite-dimensional corepresentation of \((A, \Delta)\). Much as above, we can arrange that this corepresentation is self-conjugate and contains the trivial corepresentation \((1_A \otimes 1)\). Let \(A_1\) be the linear span of the coordinate elements of \(u\), as defined above. It is a finite-dimensional operator system in \(A\). For each \(n \in \mathbb{N}\) let \(A_n\) be the linear span of products of \(n\) elements of \(A_1\) (with \(A_0 = \mathbb{C}1_A\)). Then each \(A_n\) is a finite-dimensional operator system in \(A\), and for any \(m, n \in \mathbb{N}\) we have \(A_mA_n \subseteq A_{m+n}\). But in general the union of the \(A_n\)'s is not dense in \(A\).

**Definition 2.3.** With notation as just above, we say that a finite-dimensional corepresentation \((\mathcal{H}, u)\) is **faithful** if the union of all the \(A_n\)'s is dense in \(A\) (so \(A\) is finitely generated and the \(A_n\)'s form a filtration of \(A\)). A faithful corepresentation is often called a “fundamental” corepresentation. In the context of [13, 57] we view the \(A_n\)'s as Fourier truncations of \(A\).

A compact quantum group that has a faithful finite-dimensional corepresentation is called a “compact matrix quantum group”. These are exactly the quantum generalization of compact Lie groups.

**Example 2.4.** Let \(\Gamma\) be a finitely generated group. We can choose a finite generating set \(S\) that is closed under taking inverses and contains the identity element of \(\Gamma\). The irreducible corepresentations of \(A = C^*(\Gamma)\) (all versions) are all one-dimensional, and correspond to the elements of \(\Gamma\). Given \(x \in \Gamma\), for the corresponding corepresentation \(u^x\) we can choose \(\mathcal{H} = \mathbb{C}\), and define \(u^x\) by \(u^x(z) = z\delta_x \in \mathcal{H} \otimes A = A\) for \(z \in \mathbb{C}\). Let \(u^S\) be the direct sum of the \(u^x\)'s as \(x\) ranges over \(S\). The range of \(u^S\) can be naturally identified with the subspace of functions supported on \(S\). It forms a finite dimensional operator system in \(A\) that generates \(A\). Thus it determines a familiar filtration of \(A = C^*(\Gamma)\) (used e.g. in [15, 40, 10]), whose elements we view as Fourier truncations of \(C^*\) of \(\Gamma\). In the case that \(\Gamma\) is \(\mathbb{Z}\) the elements of this filtration are exactly the Fejér-Riesz operator systems of [13, 57].

### 3. Preliminaries about actions

Just as actions of compact groups on compact spaces are of much importance, so too, actions of compact quantum groups on unital \(C^*\)-algebras are of much importance. We recall here the facts that we will need. For details, see for example [9, 42, 7, 17, 54]. Let \((A, \Delta)\) be a compact quantum group, and let \(B\) be a unital \(C^*\)-algebra. By a (left) action of \((A, \Delta)\) on \(B\) we mean a unital injective \(*\)-homomorphism, \(\alpha\), from \(B\) into \(B \otimes A\) such that

\[
(\alpha \otimes 1^A) \circ \alpha = (1^B \otimes \Delta) \circ \alpha
\]

and the linear span of \(\alpha(B)(1_B \otimes A)\) is dense in \(B \otimes A\). (Often \(\alpha\) is called a “coaction”. Also, some authors call this a “right” action. But Li in [36] calls it a “left” action, and since we make important use of his results there we will follow his usage.) Notice that \(\Delta\) can be viewed as giving an action
of $A$ on itself that is appropriately viewed as the “left regular action” of this quantum group.

For each $\gamma \in \hat{\Delta}$ let $U^\gamma = \{u^\gamma_{jk}\}$ be a unitary matrix in $M_n(A)$ that represents $\gamma$, and thus satisfies

$$\Delta(u^\gamma_{jk}) = \sum_{\ell=1}^n u^\gamma_{j\ell} \otimes u^\gamma_{\ell k}.$$ 

Then, as described in section 2 of [54] (and a number of other places beginning with [42]), there are elements $\phi^\gamma_{ij}$ of $A'$ such that

$$\phi^\beta_{ij}(u^\gamma_{kl}) = \delta^\beta\gamma \delta^\beta_{ik} \delta^\beta_{jl}$$

for all $\beta, \gamma$ and all appropriate $i, j, k, l$. For each $\gamma \in \hat{\Delta}$ define an operator, $E^\gamma$, from $B$ to $B \otimes \mathbb{C} = B$ by

$$E^\gamma = \sum_{k=1}^{n_\gamma} (I^B \otimes \phi^\gamma_{kk}) \circ \alpha,$$

where $n_\gamma$ is the dimension of the corepresentation $\gamma$. (Notice the hint of a trace in this formula.) Then $E^\gamma$ is the projection onto the $\gamma$-isotypic component, $B^\gamma$, of $B$ for the action $\alpha$. The product of any two of these projections for different $\gamma$’s is 0. Also, $\alpha(B^\gamma) \subseteq B^\gamma \otimes A^\gamma$.

We denote the isotypic component of the trivial corepresentation by $B^\alpha$. It consists exactly of the elements $b$ of $B$ that are $\alpha$-invariant, that is, $B^\alpha = \{b \in B : \alpha(b) = b \otimes 1_A\}$. In lemma 4 of [9] Boca shows that there is a canonical conditional expectation, $E$, from $B$ onto $B^\alpha$, given by $E(b) = (I^B \otimes h)\alpha(b)$.

For our purposes we want the action $\alpha$ to be ergodic, that is, the isotypic component of the trivial representation, $B^\alpha$ should be exactly $C_1B$. In this case, the conditional expectation $E$ is of the form $E(b) = \omega(b)1_B$ where $\omega$ is the unique $\alpha$-invariant state on $B$, that is, $\omega \in S(B)$ is said to be $\alpha$-invariant if $(\omega \otimes \mu)\alpha(b) = \mu(1_A)\omega(b)$ for all $\mu \in A'$ and $b \in B$. (Ergodic actions are sometimes called “quantum homogeneous spaces”, e.g. [25].)

Boca proved [9] that if $\alpha$ is ergodic then all of the isotypic components $B^\gamma$ are finite dimensional. Let $\mathcal{B}_c$ be the algebraic direct sum of all the isotypic components. It is a dense $\ast$-subalgebra of $B$ that is an analog of the coordinate subalgebra of $A$, and it is sometimes called the subalgebra of “regular” elements of $B$.

For a finite subset $S$ of $\hat{\Delta}$ let us set $B^S = \bigoplus_{\gamma \in S} B^\gamma$. Suppose that $S$ is closed under taking conjugate corepresentations and contains the trivial corepresentation. Then $B^S$ is an operator system in $B$. For each $n \in \mathbb{N}$ let $S^n$ denote the collection of all irreducible corepresentations that are contained in the $n$-fold tensor products of elements of $S$. Let $S^0$ consist of just the trivial corepresentation. Then the $S^n$’s form an increasing sequence of finite subsets of $\hat{\Delta}$, each of which is closed under taking conjugate corepresentations, and contains the trivial corepresentation. Thus the $B^S$’s form an increasing sequence of operator systems in $B$. Suppose further that
(A, ∆) is a compact matrix quantum group, and that the direct sum of the irreducible corepresentations in S is a faithful corepresentation of (A, ∆). Then the B^S_n's form a filtration of B. These are the filtrations in which we are interested. In the context of [13, 57] we can view the B^S_n's as Fourier truncations of B.

We summarize the above discussion with:

**Proposition 3.1.** Let (A, ∆) be a compact matrix quantum group, and let α be an ergodic action of (A, ∆) on a unital C*-algebra B. Let S be a finite subset of ˆΔ such that the direct sum of the elements of S is a faithful unitary corepresentation of (A, ∆), and that S is closed under taking conjugate corepresentations, and contains the trivial corepresentation. Then, with notation as above, the Fourier truncations B^S_n's form a filtration of B.

4. **Quantum Metrics**

The definition of quantum metrics was given in [44] in the setting of order-unit spaces. Here we only need the definition for the case of operator systems (notably those of the filtrations discussed above), so we now recall the definition for that case [30, 31, 36, 20]. We will need the fact that an operator system C has a well-defined state space, which we will denote by S(C). It is compact for the weak-* topology. Let C be an operator system. A quantum metric on C is a seminorm, L, on C that plays the role of assigning the Lipschitz constant to functions on a compact metric space. As such, it can take the value +∞.

**Definition 4.1.** Let C be an operator system, and let L be a seminorm on C that may take the value +∞. We say that L is a Lip-norm if it satisfies the following properties:

1. For any c ∈ C we have L(c*) = L(c), and L(c) = 0 if and only if c ∈ C^1_C.
2. Dom(L) := {c : L(c) < +∞} is dense in C (and so is a dense subspace).
3. Define a metric, d_L, on S(C) by

\[ d_L(µ, ν) = \sup\{ |µ(c) - ν(c)| : L(c) \leq 1 \}. \]

(A priori this can take the value +∞. Also, an argument given just before definition 2.1 of [46] shows that it suffices to take the supremum only over self-adjoint c's.) We require that the topology on S(C) determined by this metric agrees with the weak-* topology.

4. L is lower semi-continuous with respect to the operator norm.

We will call a pair (C, L) with C an operator system and L a Lip-norm on C a “metrized operator system”.

The third condition is the one that is often difficult to verify for specific examples, but it is crucial for our purposes. If the fourth condition is not satisfied, then the closure of L as defined in [44] will satisfy it, with no change
in the metric \(d^L\). Since a compact space that is metrizable is separable (i.e. has a countable dense subset), any operator system on which a Lip-norm is defined must be separable.

In proposition 1.1 of [45] it is shown that if \(E\) is a countable subset of a separable order-unit space \(C\) then there are many Lip-norms on \(C\) that are finite on \(E\). Let \((\mathcal{A}, \Delta)\) be a compact quantum group such that \(\mathcal{A}\) is separable, and let \(\mathcal{A}_c\) be the corresponding dense coordinate subalgebra. It follows that there are many Lip-norms on \(\mathcal{A}\) that are finite on \(\mathcal{A}_c\). In [36] Hanfeng Li calls such Lip-norms “regular”.

It will be important for us that the Lip-norms that we use are suitably invariant. Here are the definitions given by Li in [36]:

**Definition 4.2.** A regular Lip-norm \(L\) on a compact quantum group \((\mathcal{A}, \Delta)\) is said to be “left-invariant” (or “right-invariant”) if for all \(a \in \mathcal{A}\) with \(a^* = a\) and all \(\mu \in S(\mathcal{A})\) we have
\[
L(a \ast \mu) \leq L(a) \quad \text{(or} \quad L(\mu \ast a) \leq L(a))
\]
where \(a \ast \mu = (\mu \otimes I^\mathcal{A})\Delta(a)\) and similarly for \(\mu \ast a\). Then \(L\) is said to be “bi-invariant” if it is both left and right invariant.

Li shows in proposition 8.9 of [36] that if \(L\) is any regular Lip-norm on \((\mathcal{A}, \Delta)\), and if \((\mathcal{A}, \Delta)\) is coamenable, then one obtains a left-invariant regular Lip-norm \(L'\) by setting
\[
L'(a) = \sup_{\mu \in S(\mathcal{A})} L(a \ast \mu).
\]
There is a similar result producing right-invariant regular Lip-norms. Using these constructions, Li shows that one can obtain bi-invariant regular Lip-norms.

We summarize the above results of Li that we need with:

**Proposition 4.3.** Let \((\mathcal{A}, \Delta)\) be a separable coamenable compact quantum group. Then there exist (probably many) bi-invariant regular Lip-norms on \(\mathcal{A}\).

We emphasize that the Lip-norms on the algebras discussed above may well not satisfy the Leibniz inequality
\[
(4.1) \quad L(aa') \leq L(a)\|a'\| + \|a\|L(a'),
\]
which they would satisfy if they came from a first-order differential calculus or a spectral triple. Thus they may not relate particularly well to the products on the algebras. (Of course, for operator systems that are not algebras this Leibniz condition has no meaning.)

**Question 4.4.** How does one characterize the separable compact quantum groups that admit a bi-invariant regular Lip-norm that satisfies the Leibniz inequality (or, even better, the strong Leibniz condition defined in [49] and studied in [1], or better yet, comes from a spectral triple)?
Examples of ones which do have Lip-norms coming from spectral triples can be found in [15, 10, 10, 29]. The compact quantum groups implicit in the first three of these papers are those corresponding to certain classes of finitely generated groups. But many related questions remain for other classes of finitely generated groups.

In the next section we will need the results we give below, which are mostly due to Li. Much as in section 2 of [44], we define the radius of a metrized operator system \((C, L)\) to be half of the diameter of the compact metric space \((S(C), d^L)\). (For a thorough discussion of the nuances concerning Lip-norms on operator systems as opposed to order-unit spaces see section 2 of [29].) Equivalently, according to proposition 2.2 of [44], \(r_C\) is the smallest constant, \(r\), such that for any \(c \in C\) with \(c^* = c\) we have \(\|c\|' \leq r L(c)\), where \(\| \cdot \|'\) is the quotient norm on \(C/C1_C\) (so that there is some \(t \in \mathbb{R}\) such that \(\|c - t1_C\| \leq r L(c)\)).

**Lemma 4.5.** Let \((C, L)\) be a metrized operator system. For any \(c \in C\) with \(c^* = c\) and any \(\mu \in S(C)\) we have \(\|c - \mu(c)1_C\| \leq 2r_C L(c)\).

**Proof.** Given \(c \in C\) with \(c^* = c\), choose \(t \in \mathbb{R}\) such that \(\|c - t1_C\| \leq r_C L(c)\). Then for any \(\mu \in S(C)\) we have

\[
\|c - \mu(c)1_C\| \leq \|c - t1_C\| + \|t1_C - \mu(c)1_C\|
\leq r_C L(c) + \|\mu(t1_C - c)1_C\| \leq 2r_C L(c).
\]

\[\square\]

**Corollary 4.6.** (lemma 8.4 of [36]) Let \((A, \Delta)\) be a compact quantum group with Haar state \(h\), and let \(L\) be a regular Lip-norm on \((A, \Delta)\). Then for any \(a \in A\) with \(a^* = a\) we have

\[\|a - h(a)1_A\| \leq 2r_A L(a)\]

The following key proposition is a general version of much of lemma 8.6 of [36].

**Proposition 4.7.** Let \(A\) be a unital \(C^*\)-algebra, and let \(L\) be a Lip-norm on \(A\). Let \((H, \pi)\) be a faithful \(*\)-representation of \(A\), and let \(K\) be a dense subspace of \(H\). Let \(S_K(A)\) be the set of finite convex combinations of vector states determined by unit vectors in \(K\). Then for every \(\mu \in S(A)\) and every \(\epsilon > 0\) there exists a \(\nu \in S_K(A)\) such that

\[|\mu(a) - \nu(a)| \leq \epsilon L(a)\]

for all \(a \in A\).

**Proof.** Let \(r_A\) be the radius of \((S(A), d^L)\), as defined before Lemma 4.5 and let

\[B_L = \{a \in A : a^* = a, \ L(a) \leq 1, \ \|a\| \leq r_A\}\]

Then if \(a^* = a\) and \(L(a) \leq 1\), there is a \(t \in \mathbb{R}\) such that \(a - t1_A\) is in \(B_L\). By the basic criterion for verifying property 3 in the definition of a Lip-norm (see theorem 4.5 of [46]) \(B_L\) is a totally bounded subset of \(A\).
Let $\mu$ and $\epsilon > 0$ be given. Let $a_1, \ldots, a_m$ be elements of $B_L$ such that the balls around them of radius $\epsilon/4$ cover $B_L$. It is easily seen that $S_K(\mathcal{A})$ is dense in $S(\mathcal{A})$ for the weak-* topology. (Argue as for corollary T.5.10 of [58].) Accordingly, we can find a $\nu \in S_K(\mathcal{A})$ such that $\|\mu(a_j) - \nu(a_j)\| \leq \epsilon/2$ for $j = 1, \ldots, m$.

Let $a \in \mathcal{A}$. We only need to consider $a$'s for which $L(a) < \infty$. By scaling we can assume that $L(a) = 1$. For such an $a$ with $a^* = a$, there is a $t \in \mathbb{R}$ such that $a - t1_\mathcal{A}$ is in $B_L$. Thus there is a $j_0$ such that $\|a - t1_\mathcal{A} - a_{j_0}\| \leq \epsilon/4$. Then

$$|\mu(a) - \nu(a)| = |(\mu - \nu)(a - t1_\mathcal{A})|$$

$$\leq |(\mu - \nu)(a - t1_\mathcal{A} - a_{j_0})| + |(\mu - \nu)(a_{j_0})|$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon = \epsilon L(a).$$

For general $a \in \mathcal{A}$ we can replace $\epsilon$ with $\epsilon/2$ in the above argument, and apply the result to the real and imaginary parts of $a$.

**Proposition 4.8.** (Lemma 8.6 of [36]) Let $(\mathcal{A}, \Delta)$ be a compact quantum group and let $L$ be a regular Lip-norm on $\mathcal{A}$. Assume that $(\mathcal{A}, \Delta)$ is in reduced form, that is, its Haar state is faithful. Then for any $\mu \in S(\mathcal{A})$ and any $\epsilon > 0$ there is a $\nu \in S(\mathcal{A})$ and a finite subset $F \subseteq \hat{\Delta}$ such that

$$|\mu(a) - \nu(a)| \leq \epsilon L(a)$$

for all $a \in \mathcal{A}$, but $\nu(\mathcal{A}^\gamma) = 0$ whenever $\gamma \in \hat{\Delta}$ but $\gamma$ is not in $F$.

**Proof.** Let $(\mathcal{H}, \pi)$ be the GNS representation of $\mathcal{A}$ for the Haar state. Since $(\mathcal{A}, \Delta)$ is in reduced form, this representation is faithful. Let $\mathcal{K}$ be the image in $\mathcal{H}$ of the dense subalgebra $\mathcal{A}_c$ of $\mathcal{A}$. Then $\mathcal{K}$ is a dense subspace of $\mathcal{H}$. Let $\mu \in S(\mathcal{A})$ and $\epsilon > 0$ be given. Then according to Proposition 4.7 there is a finite convex combination, $\nu$, of vector states using vectors from $\mathcal{K}$, such that

$$|\mu(a) - \nu(a)| \leq \epsilon L(a)$$

for all $a \in \mathcal{A}$. Let $a_1, \ldots, a_m$ be the elements of $\mathcal{A}_c$ of length-one in $\mathcal{K}$ determining the vector states whose convex combination is $\nu$, and for each $j : 1 \leq j \leq m$ let $F_{a_j}$ be defined as in Proposition 2.2. Set $F = \bigcup F_{a_j}$. From Proposition 2.2 it follows quickly that $F$ has the desired properties. \hfill \Box

5. **Induced Lip-norms for actions**

Suppose now that $\alpha$ is an action of a compact quantum group $(\mathcal{A}, \Delta)$ on a unital C*-algebra $\mathcal{B}$. If $\mathcal{A}$ is separable then $\hat{\Delta}$ is countable. If also $\alpha$ is ergodic, then $\mathcal{B}$ is separable. In theorem 1.4 and section 8 of [36] Li shows that if $(\mathcal{A}, \Delta)$ also is coamenable (and $\alpha$ is ergodic), then any regular Lip-norm $L^A$ on $\mathcal{A}$ induces a corresponding Lip-norm $L^B$ on $\mathcal{B}$, defined by

$$(5.1) \quad L^B(b) = \sup_{\phi \in S(\mathcal{B})} L^A(b * \phi)$$
where \( b \ast \phi = (\phi \otimes I^A)\alpha(b) \), and that \( L^B \) is regular on \( B \) in the sense that it is finite on \( B \). As Li remarks, this is a generalization to the setting of compact quantum groups of the corresponding fact for actions of ordinary compact groups given in theorem 2.3 of [43].

Let \( L^B \) be any regular Lip-norm on \( B \). Then Li calls \( L^B \) "\( \alpha \)-invariant" if

\[
L^B(\mu \ast b) \leq L^B(b)
\]

for all \( b \in B \) with \( b^* = b \) and all \( \mu \in S(A) \), where \( \mu \ast b = (I^B \otimes \mu)\alpha(b) \). Li shows that if \( L^B \) is induced as above from a right-invariant regular Lip-norm on \( A \) then \( L^B \) is \( \alpha \)-invariant. (Note that \( (\mu, b) \mapsto \mu \ast b \) gives in action of the algebra \( A' \), with its convolution product, on \( B \).)

We will now present an important step in Li’s proof of these results, since this step is crucial for our purposes.

**Lemma 5.1.** Let \((A, \Delta)\) be a compact quantum group, and let \( \alpha \) be an action of \((A, \Delta)\) on a unital \( C^* \)-algebra \( B \). Assume that \((A, \Delta)\) is coamenable, and let \( \varepsilon \) be the coidentity element (which is a \(*\)-homomorphism from \( A \) into \( \mathbb{C} \), so an element of \( S(A) \)). Then \( \varepsilon * b = b \) for all \( b \in B \), that is, \((I^B \otimes \varepsilon) \alpha = I^B \).

**Proof.** (a fragment from the proof of lemma 2.2 of [53]) Set \( \alpha_0 = (I^B \otimes \varepsilon) \alpha \). On multiplying equation (3.1) on the left by \( I^B \otimes \varepsilon \otimes I^A \) and simplifying, we obtain

\[
(\alpha_0 \otimes I^A) \circ \alpha = \alpha.
\]

Then for all \( a \in A \) and \( b \in B \) we have

\[
(\alpha_0 \otimes I^A)((\alpha(b))(1_B \otimes a)) = ((\alpha_0 \otimes I^A)\alpha(b))(1_B \otimes a) = \alpha(b)(1_B \otimes a).
\]

Thus \( \alpha_0 \otimes I^A \) coincides with \( I^B \otimes I^A \) on \( \alpha(1_B \otimes A) \). But the latter spans a dense subspace of \( B \otimes A \) according to the non-degeneracy hypothesis in the definition of an action. It follows that \( \alpha_0 \otimes I^A = I^B \otimes I^A \), and so \( \alpha_0 = I^B \) as desired. \( \square \)

**Proposition 5.2.** (Related to lemma 8.7 of [36]) Let \((A, \Delta)\) be a compact quantum group, and let \( \alpha \) be an ergodic action of \((A, \Delta)\) on a unital \( C^* \)-algebra \( B \). Let \( L^A \) be a regular right-invariant Lip-norm on \( A \). Assume that \((A, \Delta)\) is coamenable, and let \( L^B \) be defined by equation (5.1), so that \( L^B \) is an \( \alpha \)-invariant regular Lip-norm on \( B \). Let \( \varepsilon \) be the coidentity of \( A \), viewed as an element of \( S(A) \). Let \( \varepsilon > 0 \) be given, and let \( \varepsilon \) be used as \( \mu \) in Proposition 4.8 to produce the state \( \nu \) and the finite set \( F \) with the properties described in that proposition. Let \( P_\nu \) be the operator on \( B \) defined by

\[
P_\nu(b) = \nu * b = (I^B \otimes \nu)\alpha(b)
\]

for \( b \in B \). Then the range of \( P_\nu \) is contained in \( B^F \) (the direct sum of the isotypic components \( B^\gamma \) for \( \gamma \in F \)), and

\[
L^B(P_\nu(b)) \leq L^B(b) \quad \text{and} \quad \|b - P_\nu(b)\| \leq \varepsilon L^B(b).
\]

for all \( b \in B \) with \( b^* = b \).
Proof. The fact that $L^B(P_\nu(b)) \leq L^B(b)$ follows directly from the definition of $\alpha$-invariance (5.2) and the definition of $P_\nu(b)$. Because $\alpha(B^\gamma) \subseteq B^\gamma \otimes A^\gamma$ for each $\gamma \in \hat{\Delta}$, we see from the properties of $\nu$ that the range of $P_\nu$ is contained in $B^F$. Finally, let $b \in B$ be given with $b^* = b$. Notice that for any $\phi \in S(B)$, any $\mu \in S(A)$, and any $c \in B$ we have $\phi(\mu * c) = (\phi \otimes \mu)\alpha(c) = \mu(c^* \phi)$. Then

$$
\|b - P_\nu(b)\| = \|\varepsilon \ast (b - P_\nu(b))\| = \sup_{\phi \in S(B)} |\phi(\varepsilon \ast (b - \nu \ast b))|
$$

$$
= \sup_{\phi \in S(B)} |\varepsilon(b \ast \phi) - \nu(b \ast \phi)| \leq \sup_{\phi \in S(B)} \epsilon L^A(b \ast \phi)
$$

$$
= \epsilon L^B(b),
$$

where we have used Lemma [5.1] for the first equality, the self-adjointness of $b$ for the second equality, Proposition [4.8] and the choice of $\nu$ for the inequality, and equation (5.1) for the final equality. \hfill \Box

We remark that the $P_\nu$ above can be viewed as a generalization of the $P_n$ used in the proof of theorem 8.2 of [46], and that the above Proposition 5.2 can be viewed as a generalization of lemma 8.3 in the proof of that theorem. Also, $P_\nu$ and $P_n$ can be viewed as analogues of the classical Fejer kernels of harmonic analysis.

6. THE MAIN THEOREM: CONVERGENCE OF TRUNCATIONS

In discussing the convergence of truncations we will use the quantum Gromov-Hausdorff distance that was first introduced in [46]. Its setting is order-unit spaces. But the spaces of operators which we will use below are operator systems, which are order-unit spaces with important extra structure. Quite soon after the appearance of [46] David Kerr introduced a stronger version of quantum Gromov-Hausdorff distance [30] that was especially tailored to the setting of operator systems. It uses spaces of unital completely positive maps into matrix algebras (as generalizations of the state space), and Kerr has referred to it as “complete Gromov-Hausdorff distance”. At about the same time Hanfeng Li developed a fairly different strategy for defining quantum Gromov-Hausdorff-type distances, and provided a version tailored for C*-algebras [33], and a version tailored for order-unit spaces [35]. Subsequently Kerr and Li wrote a paper [31] in which they showed that when Li’s strategy is applied to operator systems, it leads to exactly the same quantum Gromov-Hausdorff-type distance as Kerr’s complete Gromov-Hausdorff distance. They then call this “operator Gromov-Hausdorff distance”. (For C*-algebras the best current quantum Gromov-Hausdorff-type distance is Latréomolière’s dual propinquity [32], but since it explicitly uses the Leibniz inequality, it can not be applied to operator systems.)

Since most of the spaces of operators used below are operator systems, it would be appropriate to use here Kerr and Li’s operator Gromov-Hausdorff
distance. But I have chosen not to do this since it would considerably complicate the notation, and so somewhat obscure the ideas. But I fully expect that with small adjustments the arguments given below would work well for operator Gromov-Hausdorff distance, though I have not checked thoroughly that this is the case.

For the readers’ convenience we begin by recalling definition 4.2 of [46], which is the definition of quantum Gromov-Hausdorff distance, adapted here for operator systems much as in [29]. Let \((C, L^C)\) and \((D, L^D)\) be metrized operator systems. Let \(M(L^C, L^D)\) be the set of all Lip-norms \(L\) on \(C \oplus D\) such that \(\text{Dom}(L) = \text{Dom}(L^C) \oplus \text{Dom}(L^D)\) and such that the quotient of \(L\) on \(C\) coincides with \(L^C\) and similarly for \(D\). For this condition the inclusion of \(S(C)\) into \(S(C \oplus D)\) is an isometry for the metric \(d\) on \(S(C)\) and the metric \(d\) on \(S(C \oplus D)\), and similarly for \(D\), so we can view \(S(C)\) and \(S(D)\) as subspaces of \(S(C \oplus D)\) with the induced metric from \(d\). Then the quantum Gromov-Hausdorff distance, \(\text{dist}_q(C, D)\), between \(C\) and \(D\) is defined by

\[
\text{dist}_q(C, D) = \inf \{ \text{dist}_H(S(C), S(D)) : L \in M(L^C, L^D) \},
\]

where \(\text{dist}_H\) denotes ordinary Hausdorff distance.

Thus for any particular example, the challenge is to construct elements \(L\) of \(M(L^C, L^D)\) that bring \(S(C)\) and \(S(D)\) appropriately close together. A convenient way to approach this (section 5 of [46]) is to look for \(L\)'s of the form

\[
L(c, d) = L^C(c) \lor L^D(d) \lor N(c, d)
\]

for \(c \in C\) and \(d \in D\) (and \(\lor\) means “max”). Here \(N\) should be a norm-continuous seminorm on \(C \oplus D\) such that \(N(1^C, 1^D) = 0\) but \(N(1^C, 0^D) \neq 0\), and such that for any \(c \in C\) and \(\epsilon > 0\) there is a \(d \in D\) such that

\[
L^D(d) \lor N(c, d) \leq L^C(c) + \epsilon,
\]

and similarly for \(C\) and \(D\) interchanged. In [46] the term \(N\) is called a “bridge”.

We now assume that \((A, \Delta)\) is a compact matrix quantum group, and we let \((\mathcal{H}, u)\) be a fundamental unitary corepresentation of \((A, \Delta)\), as defined in Definition 2.3. As discussed just before that definition, arrange that \((\mathcal{H}, u)\) is self-conjugate and contains the trivial corepresentation. Let \(S\) be the set of irreducible unitary corepresentations that appear in the decomposition of \((\mathcal{H}, u)\). For each \(n \in \mathbb{N}\) we define \(S^n\) as done just before Proposition 3.1, so the \(S^n\)'s form a “filtration” of \(\hat{\Delta}\).

We also assume that \(\alpha\) is an ergodic action of \((A, \Delta)\) on a unital C*-algebra \(B\), and we let the \(B^{S^n}\)'s be defined just as before Proposition 3.1, so that they form a filtration of \(B\).

The following theorem, which is the main theorem of this paper, can be viewed as a generalization of theorem 8.2 of [46] (which is the case in which our compact matrix quantum group is an ordinary compact Lie group).
**Theorem 6.1.** Let \((A, \Delta)\) be a coamenable compact matrix quantum group, and let \(\alpha\) be an ergodic action of \((A, \Delta)\) on a unital C*-algebra \(B\). Let notation be as above. Let \(L^A\) be a regular Lip-norm on \(A\) which is right invariant, and let \(L^B\) be the seminorm on \(B\) defined by Equation 5.1 (which is a regular \(\alpha\)-invariant Lip-norm). For each \(n \in \mathbb{N}\) let \(L^n\) be the restriction of \(L^B\) to the operator system \(B^{S^n}\), so that \((B^{S^n}, L^n)\) is a metrized operator system. Then

\[
\text{dist}_q(B^{S^n}, B) \to 0 \quad \text{as} \quad n \to \infty.
\]

**Proof.** Let \(\epsilon > 0\) be given. According to Proposition 5.2 we can find a state \(\nu\) of \(A\) and a finite subset \(F\) of \(\hat{\Delta}\) such that the range of \(P_\nu\) is contained in \(B_F\) and

\[
L^B(P_\nu(b)) \leq L^B(b) \quad \text{and} \quad \|b - P_\nu(b)\| \leq \epsilon L^B(b)
\]

for all \(b \in B\) with \(b^* = b\). Since \(F\) is finite, we can find an \(N \in \mathbb{N}\) such that \(F \subseteq S^N\). Then for all \(n \geq N\) we have \(F \subseteq S^n\) so that the range of \(P_\nu\) is contained in \(B^{S^n}\). We can then immediately apply proposition 8.5 of [46] to conclude that \(\text{dist}_q(B^{S^n}, B) < \epsilon\) for all \(n \geq N\). The bridge for this situation is simply \(\epsilon^{-1}\|b - a\|\).

For the convenience of the reader, we now state proposition 8.5 of [46], for the case of metrized operator systems. We let \(A^{sa}\) denote the set of self-adjoint elements of \(A\), and similarly for \(B\).

**Proposition 6.2.** (proposition 8.5 of [46]) Let \((A, L^A)\) be a metrized operator system, and let \(B\) be an operator subsystem of \(A\). Let \(L^B\) denote the restriction of \(L^A\) to \(B\), so that \((B, L^B)\) is a metrized operator system. Let \(P\) be a function (not necessarily even linear or continuous) from \(A^{sa}\) to \(B^{sa}\) for which there is an \(\epsilon > 0\) such that

\[
L^B(P(a)) \leq L^A(a) \quad \text{and} \quad \|a - P(a)\| \leq \epsilon L^A(a)
\]

for all \(a \in A^{sa}\). Then \(\text{dist}_q(B, A) < \epsilon\).

7. Examples

We will now give a number of examples to which our results above apply.

**Example 7.1.** Let \(G\) be a compact Lie group, and let \(A = C(G)\). Choose an \(Ad\)-invariant inner product on the Lie algebra of \(G\), and let \(D\) be the corresponding Dirac operator on the Hilbert space \(S\) of spinor fields, as described in many places, for example in [51, 48]. Then \((A, S, D)\) is a spectral triple, and so one can define a seminorm, \(L^D\), (with value \(+\infty\) allowed) on \(A\) by

\[
L^D(a) = \|[D, a]\|
\]

for any \(a \in A\). Then \(L^D\) is a C*-metric, and in particular \(L^D\) is a Lip-norm that satisfies the Leibniz inequality 4.1. See proposition 6.5 of [51], as well as its antecedent theorem 4.2 of [43]. Our results in the preceding sections...
apply to this class of examples, including to the many ergodic actions of compact Lie groups on unital C*-algebras \cite{51}.

The case in which $G$ is the circle group is the example treated in section 3.2 of \cite{57} concerning Fejér-Riesz operator systems.

If we only have a continuous length function on $G$, it too can be used to define a Lip-norm satisfying the Leibniz inequality on any unital C*-algebra on which $G$ has an ergodic action. When our results of previous sections are applied, one obtains theorem 8.2 of \cite{46}.

**Example 7.2.** Let $\Gamma$ be a finitely generated group, as in Example 2.1. Both its full and its reduced C*-algebras are compact quantum groups. We can view them as (co)acting on themselves on the left using $\Delta$. They both acquire a filtration consisting of operator systems from any given finite set $S$ of generators of $\Gamma$ closed under taking inverses and containing the identity element of $G$. Let $\mathcal{A} = C^*_r(\Gamma)$, the reduced C*-algebra, with its faithful representation on $\mathcal{H} = l^2(\Gamma)$, and let $\{\mathcal{A}_n\}$ be the corresponding filtration by operator systems using $S$. For each $n$ let $\mathcal{H}_n$ be the image of $\mathcal{A}_n$ in $\mathcal{H}$, so that the $\mathcal{H}_n$'s form an increasing family of finite dimensional subspaces whose union is dense, with $\mathcal{H}_0$ the span of $1_{\mathcal{A}}$. Set $\mathcal{K}_0 = \mathcal{H}_0$, and for each integer $n \geq 1$ set $\mathcal{K}_n = \mathcal{H}_n \ominus \mathcal{H}_{n-1}$. Let $D$ be the unbounded operator on $\mathcal{H}$ whose domain is the algebraic sum $\oplus \mathcal{K}_n$, and which multiplies all elements of $\mathcal{K}_n$ by $n$, for all $n \in \mathbb{N}$. Then $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple. This is the main class of examples discussed in Connes' first paper \cite{12} on the metric aspects of non-commutative geometry. One can again define a seminorm, $L$, on $\mathcal{A}$ by

$$L(a) = \|[D, a]\|$$

for any $a \in \mathcal{A}$. It is easily seen that $L$ satisfies properties 1, 2 and 4 of the definition \ref{def:lip-norm} of a Lip-norm, as well as the Leibniz inequality \ref{def:lip-norm}. But property 3 of definition \ref{def:lip-norm} is only known to hold in the case of finitely generated groups of polynomial growth \cite{45,10} (so virtually nilpotent), and the case of hyperbolic groups and some related free product groups \cite{40}. No counterexamples are known for other groups. It is a very interesting open question to determine for which other groups property 3 is satisfied. (The corresponding spectral triples for group algebras twisted by a 2-cocycle are studied in \cite{38}, but that is somewhat outside the the scope of the present paper.)

Of course one can always use Li's results discussed above to choose a regular Lip-norm $L$ on $\mathcal{A}$ that is invariant for the left (co)action of $\mathcal{A}$ on itself (but which may well not satisfy the Leibniz inequality). But it is only if $\mathcal{A}$ is coamenable, that is, if $\Gamma$ is amenable, that we can apply the results in the earlier sections to conclude that the operator systems $\mathcal{A}_n$, equipped with the restrictions of $L$ to them, converge to $(\mathcal{A}, L)$ for quantum Gromov-Hausdorff distance.

**Example 7.3.** Let $G$ be a group of rapid decay. (See \cite{5} for the definition.) In \cite{5} it is shown how to use a proper length function on $G$ and suitable
Sobolev-type norms to define in a natural way Lip-norms on $C^*(G)$. Then when $G$ is amenable it is easily seen that our results in earlier sections apply. But these Lip-norms will seldom satisfy the Leibniz inequality. (The corresponding Lip-norms for the group algebra of $G$ twisted by a 2-cocycle are studied in [37], but that is again somewhat outside the the scope of the present paper.)

**Example 7.4.** In [27] M. Junge and T. Mei use the theory of one-parameter semigroups of completely positive operators to show how to use a conditionally negative functions on a group $G$ of rapid decay to produce a Lip-norm on the reduced $C^*$-algebra of $G$ that is Leibniz, and even strongly Leibniz in the sense defined in definition 1.1 of [49] and studied in [1]. They show that this applies, for example, to cocompact lattices in certain semisimple Lie groups. There is no suggestion that there is a Dirac-type operator associated to this situation. But in [28] several Dirac-type operators are examined that are associated to this situation (starting two paragraphs before proposition C.4). In chapter 5 of the book [6], especially in section 5.8, there is further examination of such Dirac-type operators. (I thank Cédric Arhancet, co-author of this book, for bringing this book and its chapter 5, and thus also reference [28], to my attention.)

**Example 7.5.** Let $(\mathcal{A},\Delta)$ be any coamenable compact matrix quantum group. So it is finitely generated, and any finite set of generators will yield a filtration of it. View it as (co)acting on itself on the left using $\Delta$. Because it is coamenable, our results above apply to it. Thus we can use Li’s results discussed above to choose a regular Lip-norm $L$ on $\mathcal{A}$ that is invariant for the left (co)action of $\mathcal{A}$ on itself. Then when the operator systems of the filtration are equipped with the restriction of $L$ to them, they converge to $(\mathcal{A},L)$ for quantum Gromov-Hausdorff distance.

**Example 7.6.** In [8] the authors give a definition of what it means for a discrete quantum group to have rapid decay (by modifying a definition given earlier by Vergnioux for the unimodular case). They then generalize the results from [5] described in Example 7.3 by showing that for any compact quantum matrix group $(\mathcal{A},\Delta)$ whose dual discrete quantum group has rapid decay one can again use suitable Sobolev-type norms to define in a natural way Lip-norms on $(\mathcal{A},\Delta)$. When $(\mathcal{A},\Delta)$ is coamenable it is easily seen that our results in earlier sections apply. But again, these Lip-norms will seldom satisfy the Leibniz inequality.

**Example 7.7.** Let $\mathcal{A}$ be the compact quantum group $SU_q(2)$. For its definition and properties see [29] and the many references contained therein. It is a compact matrix quantum group, and is coamenable.

Even better, in [29] the authors construct for each $q$ (for $0 < q \leq 1$) a 1-parameter family of Dirac-type operators $D_{t,q}$ on $SU_q(2)$, each of which they prove determines a regular Lip-norm on $SU_q(2)$. These Lip-norms satisfy a twisted Leibniz inequality (lemma 4.8 of [29]). And from our results
described above, for each of these Lip-norms the operator systems of the filtration coming from any faithful finite-dimensional unitary corepresentation of \( SU_q(2) \) (modified as discussed above so as to determine an operator subsystem of \( SU_q(2) \)) will converge to \( SU_q(2) \) for quantum Gromov-Hausdorff distance.

**Example 7.8.** Let \( \mathcal{B} \) be the standard Podleś sphere, \( C(S^2_q) \). For its definition and properties see [2, 3, 29] and the many references contained therein. We can view \( C(S^2_q) \) as a subalgebra of \( SU_q(2) \), and as such, as an embedded homogeneous space in \( SU_q(2) \). Here, for a compact quantum group \( (\mathcal{A}, \Delta) \) we say [17] that a unital C*-subalgebra \( \mathcal{B} \) of \( \mathcal{A} \) is an embedded homogeneous space of \( (\mathcal{A}, \Delta) \) if \( \Delta(\mathcal{B}) \subseteq \mathcal{B} \otimes \mathcal{A} \) so that the restriction of \( \Delta \) to \( \mathcal{B} \) is an action of \( \mathcal{A} \) on \( \mathcal{B} \).

There have been many proposals for Dirac operators on \( C(S^2_q) \). Let us denote one of these proposals, that of Dabrowski and Sitarz [19], by \( D_q \). It is shown in [2] that the corresponding seminorm, \( L^{D_q} \), is in fact a Lip-norm, and that \( L^{D_q} \) is \( \alpha \)-invariant, where here the action \( \alpha \) is just the restriction of \( \Delta \) to the subalgebra \( C(S^2_q) \). For the Dirac operators \( D_{t,q} \) on \( SU_q(2) \) of the previous example, let \( L^{D_{t,q}} \) be the corresponding Lip-norms. In proposition 5.2 of [29] it is shown that for each \( t \) the restriction of \( L^{D_{t,q}} \) to \( C(S^2_q) \) is \( L^{D_q} \).

To put this in the context of Li’s framework, we use the following simple result.

**Proposition 7.9.** Let \( (\mathcal{A}, \Delta) \) be a coamenable compact quantum group, and let \( \mathcal{B} \) be an embedded homogeneous space in \( \mathcal{A} \). Let \( L^A \) be a regular Lip-norm on \( \mathcal{A} \). Let \( L^B \) be Li’s corresponding induced Lip-norm on \( \mathcal{B} \) as defined in equation (5.1). Then \( L^B \) coincides with the restrictions of \( L^A \) to \( \mathcal{B} \).

**Proof.** Let \( \mu \in S(\mathcal{A}) \), and let \( \phi \) be its restriction to \( \mathcal{B} \), so \( \phi \in S(\mathcal{B}) \). Since for any \( b \in \mathcal{B} \) we have \( \Delta(b) \in \mathcal{B} \otimes \mathcal{A} \), we see that

\[
b \ast \phi = (\phi \otimes I^A)\Delta(b) = (\mu \otimes I^A)\Delta(b) = b \ast \mu.
\]

But any \( \phi \in S(\mathcal{B}) \) can be extended (perhaps in many ways) to be an element \( \mu \) of \( S(\mathcal{A}) \). It follows easily that \( L^B(b) \leq L^A(b) \). But if we let \( \mu = \varepsilon \) we see that in fact \( L^B(b) = L^A(b) \).

Consequently, the filtration of \( C(S^2_q) \) constructed as in Proposition 3.1 from any faithful finite-dimensional unitary corepresentation of \( SU_q(2) \) (modified as discussed above so as to determine an operator subsystem of \( SU_q(2) \)) will converge to \( C(S^2_q) \) for quantum Gromov-Hausdorff distance.

**Example 7.10.** The situation involving “matrix algebras converge to the sphere” which is discussed in [47, 49, 50] can be viewed as being a closely related situation in which extra structure is present that equips the finite-dimensional operator systems with a C*-algebra product making them full matrix algebras. But when this is generalized to the Podleś spheres [52, 3] one gets only operator systems.
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