POLYNOMIAL GENERATORS OF MSU$^*[1/2]$ RELATED TO CLASSIFYING MAPS OF CERTAIN FORMAL GROUP LAWS

MALKHAZ BAKURADZE

Abstract. This paper presents a commutative complex oriented cohomology theory that realizes the Buchstaber formal group law $F_B$ localized away from 2. It is shown that the restriction of the classifying map of $F_B$ on special unitary cobordism ring localized away from 2 defines a four parameter genus, studied by Hoehn and Totaro.

1. Introduction

The ring of complex cobordism $\text{MU}_*$ and the ring $\text{MSU}_*$ of special unitary cobordism has been studied by many authors. We refer the reader to [20], [17] for details. In particular the ring $\text{MSU}_*[1/2]$, localized away from 2, is torsion free

$$\text{MSU}_*[1/2] = \mathbb{Z}[1/2][x_2, x_3, \cdots], \quad |x_i| = 2i$$

and $SU$-structure forgetful homomorphism is the inclusion in complex cobordism ring

$$\text{MSU}_*[1/2] \subset \text{MU}_*[1/2] = \mathbb{Z}[1/2][x_1, x_2, x_3, \cdots].$$

In this paper we construct a commutative complex oriented cohomology theory (Theorem 5.1) such that the coefficient ring is the scalar ring of the Buchstaber formal group law $F_B$ with inverted 2, and show (Proposition 5.1) that after restricted to $\text{MSU}_*[1/2]$, the classifying map of $F_B$ can become a genus

$$(1.1) \quad \text{MSU}_*[1/2] \to \mathbb{Z}[1/2][x_2, x_3, x_4], \quad |x_i| = 2i,$$

studied by Hoehn [13] and Totaro [21].

Since $\text{MSU}^*$ is not complex oriented, it is difficult to compute the genus (1.1) on specific explicit elements. Using the polynomial generators of the spherical cobordism ring $W^*[1/2]$ given by G. Chernykh and T. Panov [12], we derive certain polynomial generators of $\text{MSU}^*[1/2]$ in terms of the universal formal group law. This gives a new understanding of the genus (1.1).

In particular, the classifying map $f_B$ of $F_B$ is a surjection on some infinitely generated ring $\Lambda_B$, with kernel generated by some explicit elements (Proposition 2.1). After it is tensored with rationals it is identical (Proposition 2.2) to the complex elliptic genus

$$(1.2) \quad \text{MU}_* \otimes \mathbb{Q} \to \mathbb{Q}[x_1, x_2, x_3, x_4],$$

2010 Mathematics Subject Classification. 55N22; 55N35.
Key words and phrases. Complex bordism, $SU$-bordism, Formal group law, Complex elliptic genus.

The author was supported by Shota Rustaveli NSF grant FR-21-4713.
defined in Hoehn’s thesis [13](Section 2.5). Here $x_1$ is the image of complex projective plane $\mathbb{CP}^1$ and $x_2, x_3, x_4$ are the images of any first three generators of the polynomial ring $\text{MSU}_* \otimes \mathbb{Q} = \mathbb{Q}[x_2, x_3, x_4, \ldots]$.

By Hoehn [13], for $X$ an $SU$-manifold of complex dimension $n$, the exponential characteristic class $\phi(X)$ is in fact a Jacobi form of weight $n$. Jacobi forms are generalizations of modular forms. See details in [21]. Hoehn showed that the Jacobi forms $x_2, x_3, x_4$ arise as the elliptic genera of certain explicit $SU$-manifolds, of complex dimensions 2, 3, 4, so that the homomorphism

$$\text{MSU}_* \rightarrow (\text{Jacobi forms over } \mathbb{Z})$$

becomes surjective after it is tensored with $\mathbb{Z}[1/2]$.

In [21] (Theorem 4.1) Totaro proved that the Krachever-Hoehn complex elliptic genus on complex cobordism viewed as a homomorphism (1.2) is surjective and the kernel is equal to the ideal of complex flops.

Then Totaro proved (Theorem 6.1) that the kernel of the complex elliptic genus on $\text{MSU}_* \otimes \mathbb{Z}[1/2]$ is equal to the ideal $I$ of $SU$-flops. Also, the quotient ring is a polynomial ring:

$$(1.3) \quad \text{MSU}_*[1/2]/I = \mathbb{Z}[1/2][x_2, x_3, x_4].$$

Unfortunately $\text{MSU}_*$ is not complex oriented. It would be nice to develop a method for calculating (1.3) explicitly in terms of the universal formal group law using some generators of $\text{MSU}_*[1/2]$ treated as explicit elements in $\text{MU}_*[1/2]$.

This goal can be achieved as follows: in Section 5 we replace the ideal of complex flops with a more explicit ideal by considering the integral Buchstaber genus which is identical to the Krachever-Hoehn complex elliptic genus over $\text{MU}_* \otimes \mathbb{Q}$.

In Section 6 we use the polynomial generators of the spherical cobordism ring $W_*[1/2]$ constructed in [12] and define certain polynomial generators of $\text{MSU}_*[1/2]$. In particular, we use fact that $W_*[1/2]$ is generated by the coefficients of the corresponding formal group law. Given Novikov’s criteria and that $W_*[1/2]$ is a free $\text{MSU}_*[1/2]$ module generated by 1 and $\mathbb{CP}^1$, we define some generators in $\text{MSU}_*[1/2]$ (Proposition 6.3) in terms of the universal formal group law. Finally the explicit quotient map of the Buchstaber formal group law (Proposition 2.1) gives the decompositions of constructed generators $\text{MSU}_*$ of dimensions $> 10$ in polynomial ring $\mathbb{Z}[1/2][x_2, x_3, x_4]$. It is a way to calculate the genus (1.3) in terms of the universal formal group law.

In Section 7 we consider the restriction of classifying map of the universal abelian formal group law on $\text{MSU}_*[1/2]$ to define a genus with one parameter.

2. Preliminaries

The theory $W_*$ of $c_1$-spherical bordism is defined geometrically in [20] (Chapter VIII). The closed manifolds $M$ with a $c_1$-spherical structure, consist of

– a stably complex structure on the tangent bundle $TM$;
– a $\mathbb{CP}^1$-reduction of the determinant bundle, that is, a map $f : M \rightarrow \mathbb{CP}^1$ and an equivalence $f^*(\eta) \simeq \text{det } TM$, where $\eta$ is the tautological bundle over $\mathbb{CP}^1$.

This is a natural generalization of an $SU$-structure, which can be thought of as a trivialization of the determinant bundle. The corresponding bordism theory is called $c_1$-spherical bordism and is denoted $W_*$. The unitary and special unitary
bordism rings are denoted by $\text{MU}_*$ and $\text{MSU}_*$ respectively. We refer to [17] and [12] for details on $\text{MSU}_*$ and $W_*$ that will be used throughout the paper.

Motivated by string theory in [14], [13], [21] the universal Krichever-Hoehn complex elliptic genus $\phi_{KH}$ is defined as the ring homomorphism
\begin{equation}
\phi_{KH} : \text{MU}^* \to \mathbb{Q}[q_1, q_2, q_3, q_4]
\end{equation}
associated to the Hirzebruch characteristic power series $Q(x) = \frac{x}{f(x)}$, where
\[ h(x) = \frac{f'(x)}{f(x)} \]
is the solution of the differential equation in $\text{MU}^* \otimes \mathbb{Q}$
\begin{equation}
(h')^2 = S(h),
\end{equation}
where
\[ S(x) = x^4 + q_1 x^3 + q_2 x^2 + q_3 x + q_4, \]
for some formal parameters $|q_i| = 2i$.

One consequence of Krichever-Hoehn’s rigidity theorem [14], [13], [21], is that (Kor 2.2.3) if $F \to E \to B$ is a fiber bundle of closed connected weakly complex manifolds, with structure group a compact connected Lie group $G$, and if $F$ is an $SU$-manifold, then the elliptic genus $\phi_{KH}$ satisfies $\phi_{KH}(E) = \phi_{KH}(F)\phi_{KH}(B)$. In fact, the elliptic genus is the universal genus with the above multiplicative property.

In (21, Theorem 6.1) Totaro gave a geometric description of the kernel ideal $I$ of the complex elliptic genus restricted to $\text{MSU}[1/2]$, the ideal of $SU$-flops. This kernel is equal to the ideal in $\text{MSU}_*[1/2]$ generated by twisted projective bundles $\mathbb{CP}(A \oplus B)$ over weakly complex manifolds $Z$ such that the complex vector bundles $A$ and $B$ over $Z$ have rank 2 and $c_1 Z + c_1 A + c_1 B = 0$; in this case, the total space is an $SU$-manifold. Then Totaro’s result says that the $I \in \text{MU}_*[1/2]$ contains a polynomial generator of $\text{MSU}_*[1/2]$ in real dimension $2n$ for all $n \geq 5$ and
\begin{equation}
\text{MSU}_*[1/2]/I \simeq \mathbb{Z}[1/2][x_2, x_3, x_4].
\end{equation}

In [13] by using of characteristic classes, Hoehn constructed a base sequence $W_1, W_2, W_3, \cdots$ of the rational cobordism ring $\text{MU}^* \otimes \mathbb{Q}$ on which $\phi_{KH}$ has the values $A, B, C, D,$ and 0 for $W_i$ with $i > 4$.

As another generalization of Oshanin’s elliptic genus Schreieder in [19] studied a genus $\psi$ with logarithmic series
\[ \log_\psi(x) = \int_0^x \frac{dt}{R(t)}, \quad R(t) = \sqrt{1 + q_1 t + q_2 t^2 + q_3 t^3 + q_4 t^4}. \]

The genus $\psi$ is easily calculable on cobordism classes of complex projective spaces $\mathbb{CP}_i$, the generators of the domain
\[ \text{MU}^* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}_1, \mathbb{CP}_2, \cdots]. \]
This is because of the equation $(\log_\psi(x'))^2 = 1/R^2(x) = \sum_{i \geq 1} \psi(\mathbb{CP}_i) x^i$ we need only the Taylor expansion of $(1 + y)^{-1/2}$.

It is natural to ask whether one can calculate $\phi_{KH}$ in an elementary manner, different from that relying on the formulas in [13] and [8].

Viewed as a classifying map $\psi$ is strongly isomorphic to genus $\phi$ by the series $\mu(x) = \sum_{i \geq 0} \mathbb{CP}_i x^{i+1}$ [4]. This gives a method for explicit calculation of $\phi_{KH}$.
In [2, 3] we introduced the formal power series
\begin{equation}
A(x, y) = \sum A_{ij} x^i y^j = F(x, y)(x\omega(y) - y\omega(x)) \in \text{MU}^*[[x, y]],
\end{equation}
where
\[
F = F(x, y) = \sum \alpha_{ij} x^i y^j
\]
is the universal formal group law over complex cobordism ring $\text{MU}^*$ and
\[
\omega(x) = \frac{\partial F(x, y)}{\partial y}(x, 0) = 1 + \sum_{i \geq 1} w_i x^i
\]
is the invariant differential of $F$.

The series $A(x, y)$ has proven to be interesting for the following reasons.

**Proposition 2.1.** [3]. i) The obvious quotient map
\[
f_B : \text{MU}^* \to \text{MU}^*/(A_{ij}, i, j \geq 3)
\]
classifies a formal group law which is identical to the universal Buchstaber formal group law $F_B$, the universal formal group law of the form
\[
\frac{x^2 A(y) - y^2 A(x)}{xB(y) - yB(x)},
\]
where $A(0) = B(0) = 1$.

(ii) If $A'(0) = B'(0)$ then $B(x)$ is identical to the image of $\omega(x)$ under the classifying map $f_B$.

**Proposition 2.2.** [4]. After it is tensored with rationals the classifying map $f_B$ of the Buchstaber formal group law is identical to the Krichever-Hoehn complex elliptic genus
\[
\phi : \text{MU}^* \otimes \mathbb{Q} \to \Lambda_B \otimes \mathbb{Q} = \mathbb{Q}[\text{CP}_1, \text{CP}_2, \text{CP}_3, \text{CP}_4],
\]
where $\text{CP}_i$ are complex projective spaces.

For explicit calculation of the Krichever-Hoehn genus the following observation is helpful.

**Proposition 2.3.** [4]. Over the ring $\text{MU}^* \otimes \mathbb{Q}$ the series $\frac{\int x}{\omega(x)}$ is the strong isomorphism from the formal group law with logarithm series
\[
\int_0^x \frac{dt}{\sqrt{1 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4}}
\]
in [19] to the formal group law classified by $f$. 

---

4  MALKHAZ BAKURADZE
3. SOME AUXILIARY COMBINATORIAL DEFINITIONS

By Euclid’s algorithm for the natural numbers $m_1, m_2, \cdots, m_k$ one can find integers $\lambda_1, \lambda_2, \cdots, \lambda_k$ such that

$$\lambda_1 m_1 + \lambda_2 m_2 + \cdots + \lambda_k m_k = \gcd(m_1, m_2, \cdots, m_k).$$

Let

$$d(m) = \gcd\left\{ \binom{m+1}{1}, \binom{m+1}{2}, \cdots, \binom{m+1}{m-1} \mid m \geq 1 \right\}.$$

By [15] one has

$$d(m) = \begin{cases} p, & \text{if } m+1 = p^s \text{ for some prime } p, \\ 1, & \text{otherwise}. \end{cases}$$

For the coefficients of universal formal group law $F(x, y) = \sum \alpha_{ij} x^i y^j$ the elements

$$e_m = \lambda_1 \alpha_{1m} + \lambda_2 \alpha_{2m-1} + \cdots + \lambda_m \alpha_{m1}$$

are multiplicative generators in $\text{MU}_*$.

By [9], Theorem 9.9, or [22]

$$\frac{D(m)}{d(m)} = \begin{cases} d(m-1) & \text{if } m \neq 2^k - 2, \\ 2 & \text{if } m = 2^k - 2. \end{cases}$$

where

$$D(m) = \gcd\left\{ \binom{m+1}{i} - \binom{m+1}{i-1} \mid 2 < i \leq m - 1, m \geq 5 \right\}.$$

Let $m \geq 4$ and let $\lambda_2, \cdots, \lambda_{m-2}$ are such integers that

$$d_2(m) := \sum_{i=2}^{m-2} \lambda_i \binom{m+1}{i} = \gcd\left\{ \binom{m+1}{2}, \cdots, \binom{m+1}{m-2} \right\}.$$

Then by [9] Lemma 9.7 one has for $m \geq 3$

$$d_2(m) = d(m)d(m-1).$$

Note $d(n)$ are the Chern numbers of the generators in complex cobordism of dimension $2n$.

For the generators

$$A_{ij}, \ i, j \geq 3, \ i + j - 2 = m$$

of the quotient ideal corresponding to $\Lambda_B$, the scalar ring of the universal Buchstaber formal group law in Proposition [21] and the integers $\lambda_3, \cdots, \lambda_{m-1}$ corresponding to (3.5) consider the linear combinations

$$T_m = \lambda_3 A_{3m-1} + \lambda_4 A_{4m-2} + \cdots + \lambda_{m-1} A_{m-13}.$$

The elements $T_p$, where $p$ is a prime number, and $e_i$ in (3.3) for $i \neq p^s$ will play a major role in Section 4.
4. Realization of the universal Buchstaber formal group law localized away from 2.

Let \( e_i \) and \( T_i \) be as in \((3.3)\) and \((3.8)\) respectively.

Let \( J_B \) be the ideal of \( \text{MU}_* = \mathbb{Z}[e_1, e_2, \ldots] \)
\[ J_B = \{ A_{ij}, i, j \geq 3 \}, \]
the quotient ideal of the universal Buchstaber formal group law \( F_B \) classified by
\[ f_B : \text{MU}_* \to \text{MU}_*/J_B = \Lambda_B. \]

The ideal \( J_B \) is not prime as the quotient ring \( \Lambda_B \) is not an integral domain: it has 2-torsion element of degree 12 \[3\]. Here we use the results of \[9\], that \( \Lambda_B \) is generated by \( f_B(e_j), j = 1, 2, 3, 4 \) and \( j = p^r, r \geq 1, p \) is prime, and \( j = 2^k - 2, k \geq 3 \). Then the ideal \( \text{Tor}(\Lambda_B) \) is generated by the elements of order 2, namely \( f_B(e_j), j = 2^k - 2, k \geq 3 \).

The ideal \( \text{Tor}(\Lambda_B) \) is prime as \( \Lambda_B \) in
\[ f_B : \text{MU}_* \to \Lambda_B \to \Lambda_B/\text{Tor}(\Lambda_B) = \Lambda_B. \]
is an integral domain and so is \( J_B = f_B^{-1}(\text{Tor}(\Lambda_B)) \), the preimage ideal in \( \text{MU}_* \).

Then the ideal
\[ J_B = J_B + (e_{2^k-2}), k \geq 3 \]
is the kernel of the composition \((4.1)\). Denote by \( F_B \) the formal group law classified by \( f_B \).

Let
\[ J = (T_l, l \geq 5), \quad \text{where} \quad T_l = \begin{cases} T_l, & n = p^r, p \text{ is a prime} \\ e_l, & \text{otherwise.} \end{cases} \]

Let \( J_B(l) \subset J_B \), generated by those elements whose degree is greater or equal \(-2l\), and let \( J(l) \subset J \) be generated by \( T_5, \ldots, T_l \).

Remark 4.1. We note that those polynomials in \( \mathbb{Z}[e_1, e_2, \ldots, e_l] \) that are in the kernel of \( f_B \) can be viewed as the elements of \( J_B(l) \).

**Proposition 4.2.** \( J(l) = J_B(l) \) for any natural \( l \geq 5 \).

Proof. It is clear that \( J(l) \subset J_B(l) \).

Let us prove \( J_B(l) \subset J(l) \) by induction on \( l \). It is obvious for \( l = 5 \) as \( T_5 = A_{34} \).

To prove \( J_B(l) = J_B(l-1) + T_l \) note
\[ s_{i+j-2}(A_{ij}) = \binom{i + j - 1}{j - 1} - \binom{i + j - 1}{j}. \]

Indeed, modulo decomposable elements
\[ A_{ij} = \alpha_{i-1j} - \alpha_{ij-1} \]
and \( s_{i+j-1}(\alpha_{ij}) = -(i+j) \). Now apply Euclid’s algorithm for \( m_i = s_l(A_{i,l+2-i}) \), fix the integers \( \lambda_i \) and consider the elements \( T_l \) in \((3.8)\).

The combinatorial identities in Section 3 implies that
\[ s_l(T_l) = D(l) \]
is the greatest common divisor of the integers \( s_l(A_{ij}) \) for \( A_{ij} \in J_B \). \( i + j - 2 = l \).
It follows that
\[ A_{ij} = \frac{s_n(A_{ij})}{D(l)} T_l + P(e_1, e_2, \cdots, e_{l-1}), \]
for some polynomial $P$, i.e.,
\[ A_{ij} \equiv P(e_1, e_2, \cdots, e_{l-1}) \mod T_l. \]
Therefore $P(e_1, e_2, \cdots, e_{l-1})$ is in the kernel of $f_B$, i.e., is in $J_B(l - 1) = J(l - 1)$ by above Remark 4.1.

Let $A_l = \mathbb{Z}[e_1, e_2, \cdots, e_l]$ and $A^{l+1} = \mathbb{Z}[e_{l+1}, e_{l+2}, \cdots]$ i.e., $MU_\ast = A_l \otimes A^{l+1}$. Let $J(l)$ as above be generated by $\mathcal{T}_1, \cdots, \mathcal{T}_l$. The preimage of $J(l)$ by obvious inclusion defines the ideal of $A_l$ denoted by same symbol so that
\[ MU_\ast / J(l) = A_l / J(l) \otimes A^{l+1}. \]

**Proposition 4.3.** i) The ideal $J$ in (4.3) is regular;
ii) $A_l / J(l)$ and $MU_\ast / J(l)$ are integral domains, or equivalently $J(l)$ is prime.

It is clear that ii) implies i): If $MU_\ast / J(l)$ is integral domain for any $l \geq 5$, i.e., it has no zero divisors, then multiplication by $\mathcal{T}_{l+1}$ is morphism. Therefore the sequence $\mathcal{T}_5, \mathcal{T}_6, \cdots$ of generators of $J$ is regular.

We will see that ii) follows from the proof of Proposition 6.5 in [9] and the following

**Lemma 4.4.** For $p^r \leq l < p^{r+1}$ the ring $A_l / J(l) \otimes \mathbb{F}_p$ is additively generated by the following monomials

For $p = 2$,
\[ \alpha_1^{m_1} \beta_2^{m_2} \alpha_3^{m_3} \beta_4 \cdots \beta_{2r-1}^{m_{2r-1}} \beta_{2r}^m, \quad k_1, k_2, k_{r-1} = 0, 1; \]

For $p = 3$,
\[ \alpha_1^{m_1} \beta_2^{m_2} \alpha_3^{m_3} \beta_4 \cdots \beta_{3r-1}^{m_{3r-1}} \beta_{3r}^m, \quad k_1, k_2, k_{r-1} = 0, 1, 2; \]

For prime $p > 3$,
\[ \alpha_1^{m_1} \beta_2^{m_2} \alpha_3^{m_3} \beta_4 \cdots \beta_{pr-1}^{m_{pr-1}} \beta_p^m, \quad 0 \leq k_1, k_2, k_{r-1} \leq p - 1, \]

not divisible by $\alpha_1^{j_1} \beta_2^{j_2} \alpha_3^{j_3} \beta_4^{j_4}$, where $(j_1, j_2, j_3, j_4)$ corresponds to leading lexicographical monomial ordering for which $\lambda_{j_1, j_2, j_3, j_4} \neq 0 \mod p$ in
\[ p \beta_p = \sum \lambda_{j_1, j_2, j_3, j_4} \alpha_1^{j_1} \beta_2^{j_2} \alpha_3^{j_3} \beta_4^{j_4}. \]

Proof. We follow the proof of Proposition 6.5 in [9]. To get the generating monomials in Lemma 4.4 we need only to modify the generating monomials of $\Lambda_\mathbb{F} \otimes \mathbb{F}_p$. In particular, in (6.17), (6.18) and (6.20) there are extra factors for $A_l / J(l)$, $p^r \leq l < p^{r+1}$, namely
\[ \beta_{p^{r+1}}^{k_1} \beta_{p^{r+2}}^{k_2} \cdots \beta_{p^r}^{k_{r-1}} \cdots , \quad k_1, k_2, \cdots \leq p - 1. \]

We have to replace these factors by
\[ \beta_p^{p k_1 + p^r k_2 + \cdots + p^r k_r}. \]

This is because of $\beta_{p^{r+1}} \notin A_l$. In this way for each $m$ we keep the total number of generating monomials of $\Lambda^{2m} \otimes \mathbb{F}_p$ since there is no relation (6.2) of [9] in our ring $A_l / J(l) \otimes \mathbb{F}_p$. \qed
Denote by $[T_i]$ the cobordism class representing the generator $T_i$ of the ideal $J_i$. Consider the sequence

$$
\Sigma = ([T_5], [T_6], \cdots).
$$

The Sullivan-Baas construction \[1\] of cobordism with singularities $\Sigma$ gives a cohomology theory $\mathbf{MU}_\Sigma^*(\_)$ which by regularity of the ideal $J$ has a scalar ring

$$
\mathbf{MU}_\Sigma^*(pt) = \mathbf{MU}^*/J = \Lambda.
$$

By Mironov \[16\] (Theorem 4.3 and Theorem 4.5) $\mathbf{MU}_\Sigma^*(\_)$ admits an associate multiplication and all obstructions to commutativity are in $\Lambda \otimes \mathbb{F}_2$. Therefore after localization away from 2 all obstructions vanish and we get a commutative cohomology

$$
h^*_B := \mathbf{MU}^*_\Sigma[1/2](\_).
$$

Here we recall that $\Lambda[1/2]$ is the ring of coefficients of the universal Buchstaber formal group law localized away from 2.

Thus we can state

**Theorem 4.5.** There exist a commutative complex oriented cohomology $h^*_B(\_)$ with scalar ring isomorphic to $\Lambda[1/2]$, the ring of coefficients of the universal Buchstaber formal group law localized away from 2.

This result without a complete proof, is announced in short communications of MMS \[5\].

5. The restriction of the Buchstaber genus on $\text{MSU}_*[1/2]$.

Taking into account \[13\] that after localized away from 2, the forgetful map from the special unitary cobordism $\text{MSU}_*[1/2] = \mathbb{Z}[1/2][x_2, x_3, \cdots]$ to complex cobordism $\mathbf{MU}_*[1/2]$ is an injection, define the following ideal extensions in $\mathbf{MU}_*[1/2]$:

- $J^e_{SU}$, generated by any polynomial generators $x_n$ of $\text{MSU}_*[1/2]$, $n \geq 5$ viewed as elements in $\mathbf{MU}_*[1/2]$ by forgetful injection map;
- $J^e_T$, generated by $SU$-flips \[21\] of dimension $\geq 10$ again viewed as elements in $\mathbf{MU}_*[1/2]$;
- $J^e_B$, the contraction ideal by the obvious inclusion $\mathbf{MU}_* \to \mathbf{MU}_*[1/2]$ of the ideal $J_B$ of $\mathbf{MU}_*$ generated by the elements $\{A_{ij}, i, j \geq 3\}$, defined in Section 3.

**Proposition 5.1.** i) $J^e_B = J^e_T = J^e_{SU}$; ii) When restricted on $\text{MSU}_*[1/2]$ the classifying map of the Buchstaber formal group law localized away from 2, gives a genus with the scalar ring $\mathbb{Z}[1/2][x_2, x_3, x_4]$, $|x_i| = 2i$.

One motivation is the restricted Krichever-Hoehn complex elliptic genus below, studied in \[13\] and \[21\]. Another construction with the scalar ring $\mathbb{Z}(2)[a, b]$, $|a| = 2$, $|b| = 6$ see in \[6\].

Proof. $J^e_T = J^e_{SU} \subseteq J^e_B$: The homomorphism

$$
\mathbf{MU}_* \overset{J^e_B}{\to} \Lambda_B \otimes \mathbb{Q} = \mathbb{Q}[x_1, x_2, x_3, x_4]
$$

is a specialization of the complex elliptic genus

$$
\mathbf{MU}_* \overset{J^e_B}{\to} \mathbf{MU}_* \otimes \mathbb{Q} \to \mathbb{Q}[x_1, x_2, x_3, x_4].
$$
by [2], therefore vanishes on the kernel of the complex elliptic genus which is the ideal $I$ of complex flops by [21] (Theorem 4.1). On the other hand the ring $\Lambda_B[1/2]$ is torsion free and injected in $\mathbb{Q}[x_1, x_2, x_3, x_4]$ by [9]. So the ring homomorphism $\text{MU}_*[1/2] \xrightarrow{f_B} \Lambda_B[1/2]$ vanishes on $I$, therefore it vanishes on $SU$-flops. Moreover by [21] the ideal of $SU$-flops $J_T$ in $\text{MSU}_*[1/2]$ contains the polynomial generators $x_n, \ n \geq 5$ constructed by using Euclid's algorithm and $SU$-flops. Therefore $J_T = (x_5, \cdots)$ and $J_T^q = J_{SU}^q \subseteq J_B$.

To prove $J_B^q \subseteq J_{SU}^q$, note by (4.3) $T_n$ satisfies the criteria for the membership of the set of polynomial generators in $\text{MSU}_*[1/2] = \mathbb{Z}[1/2][x_2, x_3, \cdots], \ |x_i| = 2i$, described by Novikov in [13]. In particular, an $SU$-manifold $M$ of real dimension $2n, \ n \geq 2$ is a polynomial generator if and only if $s_n(M)$, the main Chern characteristic number is as follows

$$s_n(M) = \begin{cases} \pm 2^k p & \text{if } n = p^l, \ p \text{ is odd prime, } \\ \pm 2^{k+2} & \text{if } n + 1 = p^l, \ p \text{ is odd prime, } \\ \pm 2^k & \text{otherwise.} \end{cases}$$

(5.1)

It follows that the generators $A_{ij}$ of $J_B$, with $i + j = n + 2$ and the generators $x_n$ of $J_{SU}^q \subset J_B^q$ are related as follows $A_{ij} \equiv \frac{s_n(A_{ij})}{D(n)} T_n, \ T_n \equiv \pm 2^{k(n)} x_n, \ \text{mod decomposables. Therefore we can proceed as in the proof of Proposition 4.2.}$

Recall also that $\Lambda_B[1/2] = \Lambda_B[1/2]$ is an integral domain. Note that Proposition 4.3 implies

**Proposition 5.2.** The sequence $\{x_n\}, \ n \geq 5$ of any polynomial generators in $\text{MSU}_*[1/2]$ viewed as elements in $\text{MU}_*[1/2]$ by forgetful map is regular.

**Corollary 5.3.** After restriction on $\text{MSU}_*[1/2]$ the Buchstuber genus $f_B$ gives a cohomology theory $\text{MSU}_*[1/2]$ with singularities $\Sigma = (x_5, \cdots)$, with the scalar ring $\mathbb{Z}[1/2][x_2, x_3, x_4], \ |x_i| = 2i$.

6. $\text{MSU}_*[1/2]$

Let $F_U(x, y) = \sum \alpha_{ij} u^i v^j$ be the universal formal group law. Recall the idempotent in [7, 12]

$$\pi_0 : \text{MU}_* \rightarrow \text{MU}_* : \ \pi_0 = 1 + \sum_{k \geq 2} \alpha_{1k} \partial_k$$

and the projection $\pi_0 : \text{MU}_* \rightarrow W_* = \text{Im} \pi_0$.

Then $W_*$ is a ring with multiplication $*$ and $\pi_0(a \cdot b) = a * b$. By [12] Proposition 2.15 the multiplication $*$ is given by

$$a * b = ab + 2[V] \partial a \partial b, \ \text{where } [V] = \alpha_{12} \in \text{MU}_4 \text{ is the cobordism class } \text{CP}_1^2 - \text{CP}_2.$$

$W_*$ is complex oriented and by [12] Proposition 3.12 the ring $W_*[1/2]$ is generated by the coefficients of the formal group law

$$F_W = F_W(x, y) = \sum w_{ij} x^i y^j.$$
Following \cite{7,12} one can calculate \(w_{ij}\) in terms of \(\alpha_{kl}\) as follows. Consider the multiplicative cohomology theory \(\Gamma\) with
\[\pi_*(\Gamma) = \text{MU}_*[t]/(t^2 - a_{11}t - 2a_{21}),\]
the free \(\text{MU}_*\) module generated by 1 and \(t\).

There is a natural multiplicative transformation \(\phi: W \to \Gamma\) given by
\[\phi_*(x) = x + t\partial x\]
for \(x \in W_*\). The restriction of \(\phi_*[1/2]\) on \(\text{MSU}_*[1/2]\), the subring in \(W_*[1/2]\) of cycles of \(\partial\), is the natural inclusion in \(\text{MU}_*[1/2]\).

Then
\[\phi_* F_W = u + v + \sum_{i,j \geq 1} (w_{ij} + t\partial w_{ij})u^i v^j\]
is strongly isomorphic to \(F_U\) (considered as a formal group low over \(\pi_*(\Gamma)\) via the natural inclusion) by a series \(\gamma^{-1}\), i.e.,
\[(6.1) \quad u + v + \sum_{i,j \geq 1} (w_{ij} + t\partial w_{ij})u^i v^j = \gamma F_U(\gamma^{-1}(u), \gamma^{-1}(v)).\]

Finally, we need to apply for \(\gamma\) in \(6.1\) Lemma 3 in \cite{7}, which says that any orientation \(w \in W_2\) gives the following identity in \(\pi_*(\Gamma)\)
\[(6.2) \quad \gamma(u) = \phi_*(w) = \pi_0(u) + t\partial u = u + twu + \sum_{i \geq 2} \alpha_{1i}wu^i.\]

By \cite{12} one can specify an orientation of \(W\) such that
\[\text{gcd}(w_{ij}, i + j - 1 = k) = d(k)d(k - 1)\]
modulo a power of 2. This allows to construct the generators of \(W_*[1/2]\).

In particular, one can calculate main Chern numbers \(s_k(w_{ij})\) in terms of main Chern numbers of \(\alpha_{ij}\) as follows. By \cite{12} Lemma 3.5 any orientation \(w \in W_2\) gives \(w_i \in W_{2n}\) and \(\lambda \in \text{MU}_2 = W_2\) such that one has in \(\pi_*(\Gamma)\)
\[\gamma(u) = u - (\lambda + (2l + 1)t)u^2 + \sum_{i \geq 2} \gamma_{i+1}u^{i+1}\mod J^2 + tj,\]
where \(2l = \partial \lambda, l \in \mathbb{Z}, \gamma_{i+1} = (-1)^i\alpha_{1i} + w_i, J\) is the ideal in \(\text{MU}_*\) of elements of positive degree.

**Lemma 6.1.** Let \(k = i + j - 1 \geq 3\) is not of the form \(k = 2^l = p^s - 1\) for some odd prime \(p\). There is a choice of complex orientation \(w\) for the theory \(W\) such that
\[s_k(w_{ij}) = \begin{cases} p\text{s}_k(\alpha_{ij}) & \text{if } k = p^s, \ p \text{ is any prime, } s > 0, \\ s_k(\alpha_{ij}) & \text{otherwise.} \end{cases}\]

**Proof.** By using \(6.2\) it is proved in \cite{12} (Lemma 3.9) that for \(k \geq 3\) one has modulo decomposable elements
\[(6.3) \quad \phi_*(w_{1k}) = \alpha_{1k} + (k + 1)(-1)^k\alpha_{1k} + w_k;\]
\[(6.4) \quad \phi_*(w_{ij}) = \alpha_{ij} + (-1)^k\binom{k + 1}{i}\alpha_{1k} + \binom{k + 1}{i}w_k;\]
Choose a complex orientation \( w \) for the theory \( W \) such that elements \( w_k \) satisfy the following conditions

\[
1 + (-1)^k (k+1) - s_k(w_k) = d(k-1)
\]

The it is easily checked that \( w_k \) indeed belongs to \( W \), that is \( s_k(w_k) \) is divisible by \( d(k-1)d(k) \).

Then by (6.3) and (6.5) we have for \( k = p^s \)

\[
w_{1k} = -k\alpha_{1k} + (k+1)w_k = -k\alpha_{1k} + \alpha_{1k}(p^s + p) = p\alpha_{1k};
\]

\[
w_{ij} = \alpha_{ij} - \binom{k+1}{i} \alpha_{1k} + \binom{k+1}{i} w_k
\]

\[
= \alpha_{ij} - \alpha_{ij}(k+1) + \alpha_{ij}(k+p) = p\alpha_{ij}.
\]

For \( k = 2^l \) one has

\[
w_{1k} = (2+k)\alpha_{1k} + (k+1)w_k = (2+k)\alpha_{1k} - \alpha_{1k}k = 2\alpha_{1k};
\]

\[
w_{ij} = \alpha_{ij} + \binom{k+1}{i} \alpha_{1k} - \binom{k+1}{i} w_k = \alpha_{ij} + \alpha_{ij}(k+1) - k\alpha_{ij} = 2\alpha_{ij}.
\]

Similarly for other cases.

If \( k = 2^l = p^s - 1 \geq 3 \) for some odd prime \( p \), then by [12] Lemma 3.15 one has \( k = 8 \) or \( k = 2^{2^n} \). We have to replace \( d(k-1) = 2 \) in (6.5) by \( 4 \), if \( k = 8 \) to get \( \phi_s(w_{i9-1}) = 4\alpha_{i9-1} \) and by \(-2^{2^n} \), if \( k = 2^{2^n} \) to get \( \phi_s(w_{i,2^{2n+1}-1}) = -2^{2^n}\alpha_{i,2^{2n+1}-1} \).

Together with Lemma 6.1 this implies

**Corollary 6.2.** Let \( k \geq 3 \). By (5.1) and (5.2) let \( \lambda_i \) be such that the linear combination \( a_k = \sum_{i=1}^{k} \lambda_i \alpha_{ik+1-i} \) is a polynomial generator in \( \text{MU}_2k \). Then

\[
b_k = \sum_{i=1}^{k} \lambda_i w_{ik+1-i}.
\]

is a polynomial generator in \( W_{2k}[1/2] \).

Then \( \text{MSU}_{*}[1/2] \) is a subring of cycles of the boundary operation \( \partial \) in \( W_{*}[1/2] \) with multiplication *

\[
\partial(a \ast b) = a \ast \partial b + \partial a \ast b - \text{CP}_1 \partial a \ast \partial b.
\]

One has \( \partial \text{CP}_1 = 2 \) and \( a \ast b = a \cdot b \) whenever \( a \in 1m\partial \) or \( b \in 1m\partial \). Therefore

\[
\partial(\text{CP}_1 \ast \alpha) = 2\alpha - \text{CP}_1 \cdot \partial \alpha, \forall \alpha \in W.
\]

As mentioned in [17] this implies that

\[
\alpha = 1/2\partial(\text{CP}_1 \ast \alpha) + 1/2\text{CP}_1 \cdot \partial \alpha,
\]

and therefore \( W[1/2] \) is generated by \( 1 \) and \( \text{CP}_1 \) as a \( \text{MSU}_{*}[1/2] \) module. It is easily checked that this module is free.
Proposition 6.3. Let $b_k$ be as in Corollary 6.2. Then

$$\text{MSU}_*\{1/2\} = \mathbb{Z}\{1/2\}[x_2, x_k : k \geq 3],$$

where $x_2 = \text{CP}_2 - 9/8\text{CP}_1^2$ and $x_k = \partial(\text{CP}_1 \ast b_k)$.

Proof. One has for the values of the Chern numbers

$$c_1c_1[\text{CP}_1^2] = 8, \quad c_1c_1[\text{CP}_2] = 9, \quad c_2[\text{CP}_1^2] = 4, \quad c_2[\text{CP}_2] = 3.$$

This imply that $c_1c_1[b_2] = 0$. There are no more Chern numbers having $c_1$ as a factor and $s_2[b_2] = s_2[\text{CP}_2^2] = 3$. Therefore $b_2$ forms a generator of $\text{MSU}_4\{1/2\}$.

Apply (6.6). The main Chern number vanishes on the second (decomposable) component of $b_k = 1/2\partial(\text{CP}_1 \ast b_k) + 1/2\text{CP}_1 \partial b_k$, i.e., the first component $x_k$ has the main Chern number $2s_k(b_k)$. \qed

7. The restriction of the classifying map of $F_{Ab}$ on $\text{MSU}^*\{1/2\}$.

As above let $F_U = \sum \alpha_{ij}x^iy^j$ be the universal formal group law. By definition the coefficient ring of the universal abelian formal group law $F_{Ab}$ is the quotient

$$(7.1) \quad \Lambda_{Ab} = \text{MU}_*/I_{Ab}, \quad \text{where } I_{Ab} = (\alpha_{ij}, i, j > 1).$$

Let us apply Euclid’s algorithm for the Chern numbers $s_{m-1}(\alpha_{i,m-i})$ in (3.6) Let

$$z_k = \sum_{i=2}^{k-1} \lambda_i\alpha_{i,k+1-i}, \quad k \geq 3.$$

By [10], [11] one has $I_{Ab} = I_{AB} = (z_k, \ k \geq 3)$.

Consider the composition

$$(7.2) \quad r_{Ab} : \text{MSU}_*\{1/2\} \xrightarrow{\subset} \text{MU}_*\{1/2\} \xrightarrow{I_{Ab}} \Lambda_{Ab}[1/2],$$

where $\subset$ is forgetful map.

Proposition 7.1. One has the following polynomial generators in $\text{MSU}_*\{1/2\}$ viewed as the elements in $\text{MU}_*\{1/2\}$

$$x_2 = \text{CP}_2 - 9/8\text{CP}_1^2, \quad x_3 = -\alpha_{22}, \quad x_4 = -\alpha_{23} - 3/2x_3\text{CP}_1.$$

To prove this we have to check that all Chern numbers of $x_3$ having factor $c_1$ are zero. Then we have to check the main Chern number $s_i(x_i)$ for Novikov’s criteria.

We already did this for $x_2$ in the proof of Proposition 6.3. Then by definition $x_3$ is the coefficient $-\alpha_{22} \in I_{AB}$ of the universal formal group law. In $\text{MU}_*$ one has

$$2\alpha_{22} = -3\text{CP}_3 + 8\text{CP}_1\text{CP}_2 - 5\text{CP}_1^2,$$

$$\alpha_{23} = 2\text{CP}_1^4 - 7\text{CP}_1^2 \ast \text{CP}_2 + 3\text{CP}_2^2 + 4\text{CP}_1\text{CP}_3 - 2\text{CP}_4.$$

Let us compute the Chern numbers of $x_3 = -\alpha_{22}$. One has
It follows all Chern numbers of $\alpha_{22}$ having factor $c_1$ are zero. Then $\alpha_{22}$ forms a generator in $\text{MSU}_6[1/2]$ as $s_3[-2\alpha_{22}] = 4 \cdot 3$.

Similarly for $x_4$: the main Chern number $s_4(x_4) = 2 \cdot 5$ fits for Novikov’s criteria and one has

| $X$    | $c_3(X)$ | $c_1c_2(X)$ | $c_1c_1c_1(X)$ |
|--------|----------|-------------|-----------------|
| $\text{CP}_3$ | 4        | 24          | 64              |
| $\text{CP}_1\text{CP}_2$ | 6        | 24          | 54              |
| $\text{CP}_1^3$ | 8        | 24          | 48              |

Note $F_{Ab}$ is a specialization of the Buchstaber formal group law $F_B$. In particular one can put $A(x) = B(x)^2$ in Proposition (2.1) to specify $F_B$ to $F_{Ab}$ over torsion free ring $\Lambda_{Ab}[1/2]$. Then Proposition 5.2 implies

**Proposition 7.2.** After restriction on $\text{MSU}_*[1/2]$ the classifying map of the universal abelian formal group law becomes the one-parameter genus

$$r_{Ab} : \text{MSU}_*[1/2] \to \mathbb{Z}[1/2][x_2].$$

**References**

[1] N. Baas, *On bordism theory of manifolds with singularities*, Math. Scand., 33(1973), 279-302.
[2] M. Bakuradze, *On the Buchstaber formal group law and some related genera*, Proc. Steklov Math. Inst., 286(2014), 7-21.
[3] M. Bakuradze, *Formal group laws by Buchstaber, Krichever and Nadiradze coincide*, Russian Math. Surveys, 68:3 (2013), 571–573.
[4] M. Bakuradze, *Comparing the Krichever genus*, J. Homotopy Relat. Struct, 9, 1(2014), 85-93.
[5] M. Bakuradze, *Cohomological realization of the Buchstaber formal group law*, Uspekhi Mat. Nauk, 77:5(467) (2022), 189-190.
[6] V. M. Buchstaber, E. Yu. Netay, $CP(2)$-multiplicative Hirzebruch genera and elliptic cohomology, Russian Math. Surveys, 69:4 (2014), 757-759.
[7] V. M. Buchstaber, *Projectors in unitary cobordisms that are related to SU-theory*, Uspekhi Mat. Nauk, 27:6(168)(1972), 231-232.
[8] V. Buchstaber, T. Panov, N. Ray, *Toric genera*, Intern. Math. Res. Notices, 16(2010), 3207–3262.
[9] V. Buchstaber, K. Ustinov, *Coefficient rings of Buchstaber formal group laws*, Math. Notices, 206:11(2015), 19-60.
[10] Bukhshhtaber V.M., Kholodov A.N., *Formal groups, functional equations, and generalized cohomology theories, (English. Russian original)*, Math. USSR Sbornik 69:1 (1991), 77–97.
[11] Ph. Busato, *Realization of Abel’s universal formal group law*, Math. Z., 239(2002), 527–561.
[12] G. Chernykh, T. Panov, SU-linear operations in complex cobordism and the $c_1$-spherical bordism theory, Izvestiya Ross. Akad. Nauk Ser. Mat. 87 (2023), no. 4.
[13] G. Höhn, Komplexe elliptische Geschlechter und $S^1$-äquivariante Kobordismustheorie, 2004, https://arxiv.org/abs/math/0405232
[14] I. Krichever, Generalized elliptic genera and Baker-Akhiezer functions, Math. Notes, 47(1990), 132–142.
[15] E. E. Kummer, Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen, J. Reine Angew. Math., 44 (1852), 93–146.
[16] O. K. Mironov, Multiplications in cobordism theories with singularities, and Steenrod – tom Dieck operations, Math. USSR Izvestija 13(1979), 89–106;
[17] I. Yu. Limonchenko, T. E. Panov, and G. S. Chernykh SU-bordism: structure results and geometric representatives, Russian Math. Surv., 74:3(2019), 461–524.
[18] S. P. Novikov, Homotopy properties of Thom complexes (Russian), Mat. Sb. 57(1962), 407–442.
[19] S. Schreieder, Dualization invariance and a new complex elliptic genus, 2012, arXiv:1109.6394 [math.AT]
[20] R. E. Stong, Notes On Cobordism Theory, Princeton University Press and University of Tokyo Press, (1968)
[21] B. Totaro, Chern numbers for singular varieties and elliptic homology, Annals of Mathematics, 151(2000), 757–792.
[22] Zhi Lu, T. Panov, On toric generators in the unitary and special unitary bordism rings, Algebraic and Geometric Topology 16 no. 5 (2016), 2865–2893; arXiv:1412.5684

Faculty of exact and natural sciences, A. Razmadze Math. Institute, IV. Javakhishvili Tbilisi State University, Georgia
Email address: malkhaz.bakuradze@tsu.ge