NORMAL FORMS AND TENSOR RANKS OF PURE
STATES OF FOUR QUBITS

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2000 Mathematics Subject Classification 81P68, 15A69

Abstract. We examine the SLOCC classification of the non-
normalized pure states of four qubits obtained by F. Verstraete et al. in [31]. The rigorous proofs of their basic results are provided and necessary corrections implemented. We use Invariant Theory to solve the problem of equivalence of non-normalized pure states under SLOCC transformations of determinant 1 and qubit permutations. As a byproduct, we produce a new set of generators for the invariants of the Weyl group of type $F_4$. We complete the determination of the tensor ranks of four-qubit pure states initiated by J.-L. Brylinski [3]. As a result we obtain a simple algorithm for computing these ranks. We obtain also a very simple classification of states of rank $\leq 3$.

1. Introduction

We use the methods of Linear Algebra and Invariant Theory to study the problem of classification of pure quantum states of four qubits. Although we use the terminology common to Quantum Physics, we do not assume the reader is familiar with it, and we shall provide necessary definitions or references. We do not need a precise definition of qubits. It suffices to say that a qubit is a mathematical model for the quantum analog of an ordinary computer bit. A basic ingredient of this model is a 2–dimensional complex Hilbert space (see [26, 27]).

We shall work with four qubits. The Hilbert space of the $k$-th qubit will be denoted by $\mathcal{H}_k = \mathbb{C}^2$ with an orthonormal basis $\{e_0, e_1\}$. The Hilbert space for the quantum system consisting of four qubits is the tensor product

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4.$$ 

Occasionally we shall use Dirac’s bra-ket notation in abbreviated form, e.g.,

$$|ijkl\rangle = e_i \otimes e_j \otimes e_k \otimes e_l.$$ 

The first author was supported by an NSERC Undergraduate Student Research Award, and the second by NSERC Grant A-5285.
If $A_k$ is an invertible linear operator on $\mathcal{H}_k$, then we refer to $A_1 \otimes A_2 \otimes A_3 \otimes A_4$ as an invertible SLOCC operation (reversible stochastic local quantum operations assisted by classical communication).

A normalized pure state is a unit vector $\psi \in \mathcal{H}$ up to a phase factor. However we shall work mostly with non-normalized pure states, i.e., mostly with nonzero vectors of $\mathcal{H}$ and we refer to them simply as pure states.

The classification of pure states of three qubits is now well-known for both the group of SLOCC operations [10] and the group of local unitary transformations [3]. The SLOCC classification of the pure states of four qubits was obtained by Verstraete et al. in [31]. However their list has an error which has not been noticed so far: the family $L_{ab3}$ is equivalent to a subfamily of $L_{abc2}$. See Remark 3.5 for more detailed comments. The need to redo this classification on a more rigorous basis is also shared by some physicists [18, 22].

The study of the tensor ranks of four-qubit pure states was initiated in a recent paper of J.-L. Brylinski [3], where he proposed that these ranks can be used as an algebraic measure of entanglement. We recall that the tensor rank of a pure state $\psi \in \mathcal{H}$ is defined as the least number $r$ of product states whose sum is $\psi$. By a product state we mean a (non-normalized) pure state of the form $v_1 \otimes v_2 \otimes v_3 \otimes v_4$. It is worthy of mention that the problem of calculating tensor ranks may be potentially very challenging. The case of tensor products of three vector spaces, of arbitrary dimensions, is relevant to the theory of algebraic complexity in Computer Science and we refer the reader to [4] for the exploration of this topic.

Our objective in this paper is threefold. First of all we shall reprove Theorems 1 and 2 of [31] and at the same time improve and correct their formulations. Second, we give a simple method to test whether two semisimple states (see section 3 for the definition) are equivalent under SLOCC operations of determinant one and qubit permutations. The case of arbitrary pure states is also discussed. Third, we shall present an algorithm for computing tensor ranks for arbitrary pure states of four qubits. Along the way we shall give a different and simple classification of tensors of rank $\leq 3$, we shall determine the Zariski closure of the tensors of rank $\leq 2$ (a question left open in [3]), and we shall construct a nice set of generators for the algebra of polynomial invariants of the Weyl group of type $F_4$ (see Appendix B).

There are several groups operating on $\mathcal{H}$ that are important for this paper. The most important ones are $\text{SL}_{\text{loc}} = \text{SL}_2 \times \text{SL}_2 \times \text{SL}_2 \times \text{SL}_2$ and $\text{SL}_{\text{loc}}^* = \text{SL}_{\text{loc}} \cdot \text{Sym}_4$, where $\text{SL}_2 = \text{SL}_2(\mathbb{C})$ is the special linear group in two dimensions and $\text{Sym}_4$ is the symmetric group on four symbols.
which acts by permuting the four qubits. Similarly define GL$_{\text{loc}}$ and GL$_{\text{loc}}^*$ by replacing the group $\text{SL}_2$ with $\text{GL}_2(\mathbb{C})$. When we refer to $M_n$ or $M_{m,n}$ as the spaces of $n \times n$ resp. $m \times n$ matrices it is assumed they are over $\mathbb{C}$, otherwise we will explicitly write $M_n(\mathbb{R})$, etc.

Since we are interested in classifying the orbits of $\text{SL}_{\text{loc}}^*$, we shall need some basic facts about the polynomial invariants of this group. We denote by $A$ resp. $A^*$ the algebra of complex analytic polynomial functions on $H$ which are invariant under the action of $\text{SL}_{\text{loc}}$ resp. $\text{SL}_{\text{loc}}^*$. It is well known that $A$ is a polynomial algebra in four variables. Its generators are algebraically independent homogeneous polynomials $H$, $L$, $M$ and $D$ of degree 2, 4, 4 and 6 respectively. See the paper [19] where these generators are explicitly constructed. The quadratic invariant (first constructed by Cayley, see [19]) has the following simple expression

$$H(\psi) = \sum_{i,j,k=0}^1 (-1)^{i+j+k} \psi_{i,j,k,0,0} \psi_{1-i,1-j,1-k,1}.$$ 

The quartic invariants $L$ and $M$ are defined in section 3 and the sextic invariant $D$ in section 4.

In section 2 we prove the basic result, Theorem 2.8, which gives the classification of indecomposable orthogonal representations of a very simple quiver $Q$ (see Figure 1). The matrix version of this result is stated in Theorem 2.10. One can view this theorem as a complex analog of the real SVD decomposition. The paper [31] contains such a theorem, for the case of square matrices only, but the uniqueness assertion for their proposed normal form is not valid.

In section 3 we recall some basic facts about $A$. We also recall some important facts from the theory of infinitesimal complex semisimple symmetric spaces and introduce the notion of semisimple and nilpotent states $\psi \in H$. The main results are stated in Theorems 3.4 and 3.6. They provide the classification of $O_4 \times O_4$-orbits in $M_4$ and $\text{SL}_{\text{loc}}^*$-orbits in $H$, respectively. The representatives of the $O_4 \times O_4$-orbits, organized in 17 families, are given in Table 1. Nine of the families are selected to obtain a set of representatives of the $\text{SL}_{\text{loc}}^*$-orbits in $H$. The explicit expressions for these families are listed in Table 7 in appendix A.

In section 4 we show that $A^*$ is also a polynomial algebra in four variables and exhibit its generators $H$, $\Gamma$, $\Sigma$, $\Pi$. These generators have degrees 2, 6, 8, and 12 and appear in a paper of Schläfli in 1852. We then show that these generators can be used to separate semisimple orbits.
The tensors $\psi \in \mathcal{H}$ of rank $\leq 3$ can be described by elementary means. This is accomplished in section 5. The results obtained here are used in an essential way in the development of our rank algorithm.

In section 6 we examine the nine families giving representatives of $\text{SL}^*_\text{loc}$-orbits in $\mathcal{H}$. For each of them we compute the tensor ranks of all states $\psi$ in the family. Finally, in section 7 we present a simple algorithm which computes the tensor rank of arbitrary pure state $\psi \in \mathcal{H}$.

In section 8 we summarize our results and make some comments on the problem of equivalence of two states under the group of local unitary operations. From the $\text{SL}^*_\text{loc}$-classification of the pure states of four qubits it is easy to derive the $\text{GL}^*_\text{loc}$-classification. For a different approach to the $\text{GL}_\text{loc}$ and $\text{GL}^*_\text{loc}$-classifications see the recent posting [18] on the arXiv.

We thank A. Osterloh for his comments regarding the examples in section 4.

2. A complex analog of the real Singular Value Decomposition

Our classification of $\text{SL}^*_\text{loc}$-classes of pure states of four qubits is based on a result of Linear Algebra which we discuss in this section.

Consider $M_{m,n}(\mathbb{R})$ where we assume $m \leq n$. By $A^T$ we denote the transpose of a matrix $A$. If we consider the usual action of the real orthogonal groups $\text{O}(m)$ and $\text{O}(n)$ on $M_{m,n}(\mathbb{R})$, i.e,

$$(S_1, S_2) \cdot A = S_1 A S_2^{-1}$$

where $S_1 \in \text{O}(m)$, $S_2 \in \text{O}(n)$ and $A \in M_{m,n}(\mathbb{R})$, then the orbits for this action are classified by diagonal matrices $\Sigma \in M_{m,n}(\mathbb{R})$ with non-increasing and nonnegative diagonal entries (the singular value decomposition theorem).

There is a complex version of this theorem where $M_{m,n}(\mathbb{R})$ is replaced by $M_{m,n}$, the complex $m \times n$ matrices, and $\text{O}(m)$ and $\text{O}(n)$ replaced by the unitary groups $\text{U}(m)$ and $\text{U}(n)$. We are interested in another complex version of the above theorem where instead of $\text{U}(m)$ and $\text{U}(n)$ we use the complex orthogonal groups $\text{O}_m = \text{O}_m(\mathbb{C})$ and $\text{O}_n = \text{O}_n(\mathbb{C})$. We recall that $\text{O}_n$ is the subgroup of $\text{GL}_n(\mathbb{C})$ consisting of all complex matrices $X$ such that $X^T X = I_n$. In the case $m = n$, such a theorem appears in the recent paper [31]. As the proof presented there is not completely clear, and the statement of the theorem we feel can be improved upon, we shall offer a different approach in this section.

It is convenient to use the language of quivers. We just need one very simple quiver $\mathcal{Q}$ which has two vertices, say 1 and 2, and a single
directed edge from 1 to 2, as in Figure 1.

We are interested only in orthogonal representations of this quiver. For simplicity we shall refer to them simply as representations.

**Definition 2.1.** A representation of the above quiver $Q$ is a 5-tuple $(V_1, V_2, \phi_1, \phi_2, A)$ where $V_1$ and $V_2$ are finite dimensional complex vector spaces equipped with nondegenerate symmetric bilinear forms $\phi_1$ and $\phi_2$ respectively, and $A$ is a linear map $V_1 \to V_2$. We think of $V_1$ and $V_2$ as being attached to vertices 1 and 2 respectively while $A$ is attached to the directed edge.

**Example 2.2.** The most basic example of such a representation is given by $V_1 = \mathbb{C}^n$, $V_2 = \mathbb{C}^m$, where $\phi_1$ and $\phi_2$ are the usual dot products on $\mathbb{C}^n$ and $\mathbb{C}^m$, and the linear transformation $A : \mathbb{C}^n \to \mathbb{C}^m$ is identified with an $m \times n$ complex matrix. Henceforth whenever we refer to $\mathbb{C}^k$ as a vector space we will assume that the bilinear form is the usual dot product.

Let us recall some basic facts about representations of quivers in our setting. We start with the definition of a homomorphism of orthogonal representations.

**Definition 2.3.** A homomorphism

\[(2.2) \quad S : (V_1, V_2, \phi_1, \phi_2, A) \to (V_1', V_2', \phi_1', \phi_2', A')\]

is a pair of linear maps $S_1 : V_1 \to V_1'$ and $S_2 : V_2 \to V_2'$ such that

\[(2.3) \quad \phi_1'(S_1v_1, S_1w_1) = \phi_1(v_1, w_1), \forall v_1, w_1 \in V_1\]

\[(2.4) \quad \phi_2'(S_2v_2, S_2w_2) = \phi_2(v_2, w_2), \forall v_2, w_2 \in V_2\]

and $S_2 \circ A = A' \circ S_1$.

If $S_1$ and $S_2$ are isomorphisms we say that $S$ is an isomorphism. If there exists an isomorphism between two representations, then we say that the representations are *isomorphic* or *equivalent*. For instance two representations

\[(2.5) \quad A : \mathbb{C}^n \to \mathbb{C}^m; \quad B : \mathbb{C}^n \to \mathbb{C}^m\]
are isomorphic if and only if there exist \( S_1 \in O_n \) and \( S_2 \in O_m \) such that \( S_2 A = BS_1 \) i.e., \( B = S_2 A S_1^{-1} \). This means that two matrices \( A, B \in M_{m,n} \) belong to the same orbit of \( O_m \times O_n \), acting on \( M_{m,n} \) in the usual way, if and only if the two representations above are isomorphic. Clearly, every representation of \( Q \) is isomorphic to one of the type given in example 2.2.

One defines the direct sum of representations in the obvious way. A representation \( (V_1, V_2, \phi_1, \phi_2, A) \) is nonzero if \( V_1 \) or \( V_2 \) is a nonzero space. A nonzero representation is said to be indecomposable if it is not a direct sum of two nonzero representations. The Krull–Schmidt theorem is valid, i.e., every representation decomposes as a direct sum of indecomposable ones and these indecomposable summands are unique up to permutation and isomorphism.

Hence the classification of all representations of our quiver \( Q \) reduces to the description of its indecomposable representations (up to isomorphism).

The three simplest nonzero representations are the following.

**Example 2.4.** The representation \( A : 0 \to \mathbb{C}^1 \) is indecomposable. The matrix of the linear transformation \( A \) is the unique \( 1 \times 0 \) matrix.

**Example 2.5.** The representation \( A : \mathbb{C}^1 \to 0 \) is indecomposable. Its matrix is the unique \( 0 \times 1 \) matrix.

**Example 2.6.** Every representation \( A : \mathbb{C}^1 \to \mathbb{C}^1 \) is given by multiplication by a fixed \( \alpha \in \mathbb{C} \). Its matrix is the \( 1 \times 1 \) matrix \([\alpha]\). If \( \alpha = 0 \) this representation is isomorphic to the direct sum of the representations from examples 2.4 and 2.5. If \( \alpha \neq 0 \), the representation is indecomposable.

The \( n \times n \) symmetrized Jordan block \( J_n^\times (\alpha) \) is the sum of the band matrix having \( \alpha \)'s on the diagonal and \( 1 \)'s on the sub and super-diagonal and the matrix with \( i \)'s on the opposite super-diagonal and \(-i \)'s on the opposite sub-diagonal. For example

\[
J_5^\times (\alpha) = \begin{bmatrix}
\alpha & 1 & 0 & i & 0 \\
1 & \alpha & 1 + i & 0 & -i \\
0 & 1 + i & \alpha & 1 - i & 0 \\
i & 0 & 1 - i & \alpha & 1 \\
0 & -i & 0 & 1 & \alpha
\end{bmatrix}.
\]

We point out that if \( n \) is odd, then the representation of \( Q \) given by \( J_n^\times (0) \) is decomposable. We have already seen this in Example 2.6.
when \( n = 1 \). As a less trivial example consider

\[
J_3^x(0) = \begin{bmatrix}
0 & 1+i & 0 \\
1+i & 0 & 1-i \\
0 & 1-i & 0
\end{bmatrix}.
\]

By permuting rows and columns, we obtain

\[
\begin{bmatrix}
1+i & 0 & 0 \\
1-i & 0 & 0 \\
0 & 1+i & 1-i
\end{bmatrix}.
\]

Hence this representation of \( Q \) is the direct sum of the representations given by the two nonzero blocks, which are indecomposable.

**Remark 2.7.** These blocks can be replaced with \([1 \ i]^T\) and \([1 \ i]\) respectively. Indeed, we have \([1 + i \ 1 - i] = [1 \ i] P\) where

\[
P = \frac{1}{4} \begin{bmatrix} 3 + i & 3 - i \\ 3 - i & -3 - i \end{bmatrix}
\]

is orthogonal. We may simplify the symmetrized Jordan blocks as well. Note that for example \( J_3^x(\alpha) \) is similar to

\[
\begin{bmatrix} \alpha & 1 & 0 \\ 1 & \alpha & i \\ 0 & i & \alpha \end{bmatrix}
\]

and since they are symmetric they are orthogonally similar. Hence, this matrix is in the same \( O_3 \times O_3 \)–orbit as \( J_3^x(\alpha) \).

The blocks \( J_n^x(\alpha) \) can be replaced by another kind of symmetrized Jordan blocks which consist of 3–diagonal symmetric matrices. For the description of these blocks see the recent paper [9].

We shall use a single matrix, as described in Example 2.2, to denote a representation of the quiver \( Q \). With this in mind we can now state the following important classification theorem.

**Theorem 2.8.** The representatives of the isomorphism classes of indecomposable (orthogonal) representations of the quiver \( Q \) are given by the following matrices:

1. \( J_n^x(\alpha) \) for \( n \geq 1 \) where \( \alpha \neq 0 \) if \( n \) is odd. The two values \( \pm \alpha \) give the same isomorphism class.
2. The \((m + 1) \times m\) matrix, \( m \geq 0 \), formed by using even index columns and odd index rows of \( J_{2m+1}^x(0) \).
3. The transpose of the previous indecomposable.
Proof. Let us first show that the representations \( A \) given in (1-3) are indeed indecomposable. Note that if the representations \( A \) and \( B \) of \( Q \) are isomorphic, then the matrices \( A^T A \) and \( B^T B \) are similar. Consequently, the number of indecomposable direct summands of \( A \) is at most equal to the number of Jordan blocks of \( A^T A \).

In case (1), \( A = J_n^\times(\alpha) \). If \( \alpha \neq 0 \) then \( A^T A = A^2 \) has just one Jordan block and so the representation given by \( A \) must be indecomposable. If \( \alpha = 0 \) then \( n = 2m \) is even and \( A^2 \) is similar to \( J_m(0) \oplus J_m(0) \). We leave to the reader to show that \( A \) must be indecomposable.

The cases (2) and (3) can be handled together. Let \( A = J_n^\times(0) \) with \( n = 2m + 1 \) odd. Then \( A^T A = A^2 \) is similar to \( J_m(0) \oplus J_m+1(0) \) and so the representation \( A \) has at most two indecomposable direct summands. But we have seen in the discussion preceding this theorem that it indeed is a direct sum of two representations. Hence these summands must be indecomposable.

We show next that every indecomposable representation of \( Q \) is isomorphic to one of the representations (1-3).

Let \( A : V \to W \) be an indecomposable representation where \( V = \mathbb{C}^n \) and \( W = \mathbb{C}^m \). We have that \( A^T A \) and \( AA^T \) are linear operators on \( V \) and \( W \) respectively. Let us apply the Fitting decomposition

\[
V = V_0 \oplus V_1, \quad W = W_0 \oplus W_1,
\]

where \( V_0 \) and \( V_1 \) are \( A^T A \)-invariant subspaces, \( A^T A \) is nilpotent on \( V_0 \) and invertible on \( V_1 \), and similar properties hold for \( W_0, W_1 \) and \( AA^T \). Then it is easy to show that \( V_0 \perp V_1, W_0 \perp W_1 \), and \( A(V_i) \subseteq W_i \) and \( A^T(W_i) \subseteq V_i \) for \( i = 0, 1 \).

This means that the representation \( A : V \to W \) is the direct sum of the representations \( A_i : V_i \to W_i \) where \( A_i \) is the restriction of \( A \) for \( i = 0, 1 \). As our representation is assumed to be indecomposable, we have \( V_0 = W_0 = 0 \) or \( V_1 = W_1 = 0 \).

Case 1: \( V_0 = W_0 = 0 \). Then \( m = n \) and \( A \) and \( A^T \) are isomorphisms. By [16, 5] \( A \) is a product of an orthogonal matrix and a symmetric one. A symmetric matrix is orthogonally similar to the direct sum of symmetrized Jordan blocks [11]. Consequently, we can write \( A = PBQ \) where \( P, Q \in O_n \) and \( B \) is the direct sum of symmetrized Jordan blocks. There is only one block, i.e., \( B = J_n^\times(\alpha) \) by the indecomposability assumption. As \( B \) is invertible, we have \( \alpha \neq 0 \). It is also required to show that two symmetrized Jordan blocks \( J_n^\times(\alpha) \) and \( J_n^\times(\beta) \) are in the same isomorphism class if and only if \( \alpha = \pm\beta \). This can be seen as follows. If \( J_n^\times(\alpha) \) and \( J_n^\times(\beta) \) give isomorphic representations then

\[
J_n^\times(\beta) = PJ_n^\times(\alpha)Q
\]
for some $P, Q \in O_n$. Then

$$J_n^x(\beta)^2 = J_n^x(\beta)^T J_n^x(\beta) = Q^T J_n^x(\alpha)^2 Q = Q^{-1} J_n^x(\alpha)^2 Q$$

so $J_n^x(\alpha)^2$ and $J_n^x(\beta)^2$ are similar and thus they have the same eigenvalues hence $\beta^2 = \alpha^2$. Conversely, note that $J_n^x(0)$ and $-J_n^x(0)$ are similar and symmetric so they are orthogonally similar. So there exists $P \in O_n$ such that

$$PJ_n^x(0)P^{-1} = -J_n^x(0).$$

Adding $\alpha I_n$ to both sides we have

$$PJ_n^x(\alpha)P^{-1} = \alpha I_n - J_n^x(0) = -J_n^x(-\alpha).$$

As $-I_n \in O_n$ we see that $J_n^x(\alpha)$ and $J_n^x(-\alpha)$ are in the same $O_n \times O_n$–orbit, and so they give equivalent orthogonal representations of $Q$.

**Case 2:** $V_1 = W_1 = 0$. In this case the matrices $A^T A$ and $AA^T$ are nilpotent. This case occurs naturally in the theory of (infinitesimal) semisimple complex symmetric spaces. We omit the proof and refer the reader to [25, 8].

**Remark 2.9.** The classification of indecomposables in the above theorem can be deduced from the general results on representations of symmetric quivers. We refer the reader to the recent paper of Derksen and Weyman [7], where this new type of quiver is introduced and their representations (including the orthogonal and symplectic ones) are studied. In order to apply their results, our quiver has to be modified by adding an additional directed edge from the second to the first vertex. The involution $\sigma$, required by the definition of symmetric quivers, fixes the vertices and interchanges the two arrows.

We can reformulate Theorem 2.8 in terms of matrices.

**Theorem 2.10.** Let $O_m \times O_n$ act on $M_{m,n}$ in the usual way. Then the block-diagonal matrices

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_k \in M_{m,n},$$

where each $A_i$ is one of the matrices listed in Theorem 2.8 are representatives of $O_m \times O_n$–orbits. These representatives are unique up to permutation of the $A_i$’s and sign changes mentioned in that theorem.

**Remark 2.11.** This theorem should be compared with [31, Theorem 1]. The authors consider only the square case $m = n$. Contrary to their claim, the canonical forms given there are not unique up to permutation of the diagonal blocks because some of their blocks have the shape

$$\begin{bmatrix} 0 & R_1 \\ R_2 & 0 \end{bmatrix}$$
and are made up of two of our rectangular blocks, one of size $(p+1) \times p$ and the other $q \times (q+1)$. In the formulation given in Theorem 2.10 there are more possible ways of combining such blocks, which leads to non-uniqueness. They also failed to mention that $J_n^\alpha$ and $J_n^{-\alpha}$ belong to the same $O_n \times O_n$–orbit.

3. Classification of $O_4 \times O_4$–orbits in $M_4$ and $SL_{lo}^*$–orbits in $H$

Our main objective here is to apply Theorem 2.10 to the problem of $SL_{lo}^*$–classification of pure states of $H$. We start with some preliminary results, mostly well known. We have mentioned that a product state $\psi$ is of the form $v_1 \otimes v_2 \otimes v_3 \otimes v_4$. We say a state $\psi$ is factorizable if, after a permutation of qubits, it can be written as the product of two tensors $\psi = \psi_1 \otimes \psi_2$. As in [31], for a tensor

$$\psi = \sum_{i,j,k,l=0}^1 \psi_{ijkl} |ijkl\rangle$$

we define the $4 \times 4$ matrix $\tilde{\psi}$ by using the pairs $ij$ as the row index and the pairs $kl$ as the column index (we order these pairs as 00, 01, 10, 11). By permuting cyclically the indices $jkl$ we obtain two more such matrices $\tilde{\psi}'$ and $\tilde{\psi}''$. As in [19], we denote their determinants by $L = \Delta_{1234}, M = \Delta_{1342}, N = \Delta_{1423}$, respectively. It is easy to verify that $L + M + N = 0$.

Let $S_k$ be the set of tensors with rank less than or equal to $k$. Surprisingly, it may happen that $S_k$ is not Zariski closed. We shall denote its Zariski closure by $\overline{S}_k$. We need the following result proved by Brylinski [3].

**Proposition 3.1.** The maximum rank of a tensor $\psi \in H$ is 4. The affine variety $\overline{S}_3$ is irreducible and is defined by the equations $L = M = N = 0$. Hence $\overline{S}_3$ has dimension 14.

We prove the analogous result for $\overline{S}_2$, which was alluded to in [3].

**Proposition 3.2.** The affine variety $\overline{S}_2 \subseteq H$ is irreducible of dimension 10. Its ideal is exactly the ideal generated by the forty-eight $3 \times 3$ minors of the matrices $\tilde{\psi}, \tilde{\psi}'$ and $\tilde{\psi}''$.

**Proof.** Let $I$ be the ideal generated by the forty-eight minors and $W \subseteq H$ its zero set. We used Singular [13] to verify that $I$ is a prime ideal and that $\dim W = 10$. On the other hand a simple computation (using Maple [23]) shows that the $GL_{lo}^*$–orbit $O$ of $|0000\rangle + |1111\rangle$ also has...
dimension 10. Since $\mathcal{O} \subseteq S_2 \subseteq W$, we have $\mathcal{O} \subseteq S_2 \subseteq W$. As $\mathcal{O}$ is dense in $W$, the proposition is proved. □

Consider the action of $SL_4 \times SL_4$ on $M_4$ given by:

$$((P, Q), R) \rightarrow PRQT^T.$$  

The image of $SL_4 \times SL_4$ under this representation is contained in $SL(M_4)$, the special linear group of the space $M_4$. This image is usually written as $SL_4 \otimes SL_4$, which means that we have two copies of $SL_4$ with their centers identified (glued together).

The action of $SL_{\text{loc}}$ on $H$ gives rise to an action on $M_4$ via the map $\psi \rightarrow \tilde{\psi}$. Explicitly, this action is given by

$$((A_1, A_2, A_3, A_4), \psi) \rightarrow (A_1 \otimes A_2)\tilde{\psi}(A_3 \otimes A_4)^T,$$

where $A_1 \otimes A_2$ is the usual tensor product of matrices.

The image of the first two factors $SL_2$ of $SL_{\text{loc}}$ under the action on $M_4$ is contained in the first factor of $SL_4 \otimes SL_4$. It is well known that this image is isomorphic to $SL_2 \otimes SL_2 \cong SO_4$ but is different from $SO_4$. We need to conjugate this image to obtain $SO_4$. Clearly, the matrix which performs this conjugation is not unique. For that purpose we use the unitary matrix

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \end{bmatrix},$$

which we borrow from [31]. A slightly different such matrix $Q$ is given in Makhlin’s paper [21]. Now define $R = R_\psi = T\tilde{\psi}T^\dagger$, where the superscript $\dagger$ indicates the hermitian transpose. It is assumed that the particular $\psi$ is obvious from the context when it is not written in the subscript. Finally we set

$$\tilde{R} = \tilde{R}_\psi = \begin{bmatrix} 0 & R & R^T \\ -R^T & 0 & 0 \end{bmatrix}.$$  

If $A_k \in SL_2$ then $|\phi\rangle = A_1 \otimes A_2 \otimes A_3 \otimes A_4|\psi\rangle$ corresponds to $R_\phi = P_1R_\psi P_2$ where $P_1, P_2 \in SO_4$ are given by $P_1 = T(A_1 \otimes A_2)T^\dagger$ and $P_2 = T(A_3 \otimes A_4)^T T^\dagger$. Hence there is a 1-to-1 correspondence between the $SL_{\text{loc}}$-orbits in $H$ and $SO_4 \times SO_4$-orbits in $M_4$.

For the following facts the reader can consult chapter 38 of [29], and in particular Proposition 38.6.8. The $SL_{\text{loc}}$-orbit of $\psi$ is closed (in the Zariski topology) iff the $SO_4 \times SO_4$-orbit of $R_\psi$ is closed. It is well known that this is the case iff the matrix $\tilde{R}_\psi$ is semisimple (i.e. diagonalizable). In this case we shall also say that $\psi$ is semisimple.
The Zariski closure of the \( SL_{\text{loc}} \)-orbit of \( \psi \) contains the zero vector iff the same is true for the \( SO_4 \times SO_4 \)-orbit of \( R_\psi \). Furthermore, this is the case iff the matrix \( \tilde{R}_\psi \) is nilpotent. In that case we shall also say that \( \psi \) is nilpotent. A nilpotent \( SL_{\text{loc}} \)-orbit, say \( \mathcal{O} \), is conical, i.e., if \( \psi \in \mathcal{O} \) then also \( \lambda \psi \in \mathcal{O} \) for all nonzero scalars \( \lambda \in \mathbb{C} \). Hence \( \mathcal{O} \) is also a \( GL_{\text{loc}} \)-orbit.

For any \( \psi \in \mathcal{H} \), the characteristic polynomial of \( \tilde{R}_\psi \) is given by

\[
t^8 + 2Ht^6 + (H^2 + 2L + 4M)t^4 + 2(HL + 2D)t^2 + L^2,
\]

where it is understood that \( H \) is short for \( H(\psi) \) etc. We will define the invariant \( D \) in the next section. If \( \psi \in \bar{S}_3 \) then \( L = M = 0 \) and we obtain

\[
t^2(t^6 + 2Ht^4 + H^2t^2 + 4D).
\]

The discriminant of the cubic

\[
s^3 + 2Hs^2 + H^2s + 4D \]

is equal to

\[
16D(H^3 - 27D).
\]

The conjugation by the diagonal matrix \( I_4 \oplus (-I_4) \) induces an involutory automorphism \( \theta \) of \( O_8 \) and its Lie algebra \( \mathfrak{g} = so_8(\mathbb{C}) \) consisting of the skew-symmetric matrices in \( M_8 \). Let \( \mathfrak{k} \) and \( \mathfrak{p} \) be the eigenspaces of \( \theta \) in \( \mathfrak{g} \) with eigenvalues \(+1\) and \(-1\), respectively. These eigenspaces consist of the matrices having the form

\[
\begin{bmatrix}
* & 0 \\
0 & *
\end{bmatrix}
\]
resp.

\[
\begin{bmatrix}
0 & * \\
* & 0
\end{bmatrix},
\]

all blocks being of size 4. The space \( \mathfrak{k} \) is in fact a subalgebra of \( \mathfrak{g} \), the Lie algebra of the subgroup \( K = O_4 \times O_4 \) of \( O_8 \). The space \( \mathfrak{p} \) is a \( K \)-module with the action

\[
\begin{pmatrix}
P_1 & 0 \\
0 & P_2
\end{pmatrix}, \begin{pmatrix}
0 & \tilde{R} \\
-R^T & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
0 & P_1RP_2^{-1} \\
-P_2R^TP_1^{-1} & 0
\end{pmatrix}.
\]

This is an example of an (infinitesimal) semisimple complex symmetric space. The following theorem is a special case of general results about such spaces \cite[Lemma 38.7.14]{30}:

**Theorem 3.3.** Let \( \phi, \psi \in \mathcal{H} \) be semisimple states. The invariants \( H, L, M, D \) take the same values at \( \phi \) and \( \psi \) iff \( R_\phi \) and \( R_\psi \) belong to the same \( SO_4 \times SO_4 \)-orbit, i.e., \( \phi \) and \( \psi \) belong to the same \( SL_{\text{loc}} \)-orbit.

The special case \( m = n = 4 \) of Theorem \cite[10]{2} plays an important role in the sequel. We now state this special case in more detail.

**Theorem 3.4.** The 17 families of matrices \( R \), listed in Table \cite{7}, classify the \( O_4 \times O_4 \) orbits on \( M_4 \) up to permutation of diagonal blocks of the
same size and replacing the parameters $a, b, c, d$ by $\pm a$, $\pm b$, $\pm c$, $\pm d$ respectively.

Table 1. $R$-matrix representatives of $O_4 \times O_4$-orbits

|   |   |   |   |
|---|---|---|---|
| 1 | $a$ | $b$ | $c$ |
|   | $b$ | $c$ | $d$ |
| 2 | $a$ | $b$ | $c+i$ | $1$ |
|   | $b$ | $c-i$ | $1$ |
| 3 | $a$ | $b$ | $1+i$ | $1$ |
|   | $1+i$ | $1$ |
| 4 | $a$ | $b$ | $1+i$ | $1$ |
|   | $1+i$ | $1$ |
| 5 | $a+i$ | $1$ | $b+i$ | $1$ |
|   | $b+i$ | $1$ |
| 6 | $a$ | $b$ | $1$ | $0$ |
|   | $1$ | $b+i$ |
| 7 | $a$ | $1$ | $i$ | $1+i$ |
|   | $1+i$ | $1-i$ |
| 8 | $a$ | $1$ | $i+1$ | $1$ |
|   | $i$ | $1-i$ |
| 9 | $a$ | $1$ | $i$ | $0$ |
|   | $1$ | $a+i$ | $1$ |
| 10 | $a+i$ | $1$ | $a-i$ | $1$ |
|   | $1$ | $i$ |
| 11 | $a+i$ | $1$ | $a-i$ | $1$ |
|   | $1$ | $i$ |
| 12 | $1$ | $0$ | $i$ | $1+i$ |
|   | $i$ | $1-i$ |
| 13 | $1$ | $1$ | $i$ | $-i$ |
|   | $0$ | $1+i$ | $1-i$ |
| 14 | $1$ | $i$ | $1+i$ | $1$ |
|   | $-i$ | $1$ |
It should be noted that the representatives given in Table 1 may contain blocks different from those listed in Theorem 2.8; some of them have been simplified using Remark 2.7. Table 2 describes the Jordan structure of the $\tilde{R}_\psi$ matrices. The $1 \times 1$ Jordan blocks are given by listing their eigenvalues. The symbol $J_2(\pm ic)$ indicates two $2 \times 2$ Jordan blocks with eigenvalues $ic$ and $-ic$ respectively, etc. The family 1 consists of semisimple elements. On the other hand none of the other families contains a semisimple element. The nilpotent $O_4 \times O_4$–orbits are easy to identify by using Table 2. Just set all parameters (if any) to 0 in each of the 17 families.

Table 3 gives a correspondence between the families of orbits found in [31] and those that we have identified. More precisely, for each of the 9 families given in [31] by explicit expressions we have determined the corresponding matrices $\tilde{R}_\psi$ and their Jordan structure as well as the corresponding $O_4 \times O_4$–family in our notation (see Table 1). The appearance of the imaginary units $i$ in the expressions for eigenvalues of $\tilde{R}_\psi$ is due to the fact that this matrix is skew-symmetric while the matrix $P$ used in [31] is symmetric.

**Remark 3.5.** Verstraete et al. [31] state that they found only 12 $O_4 \times O_4$–families, while we found 17. This is probably due to the fact that their Theorem 1 is not correct as stated. Their family $L_{abs}$ is equivalent to the subfamily of $L_{abc2}$ obtained by setting $c = a$. We believe that there are two misprints in the formula for $L_{abs}$: the two $+$ signs, in the last line of this formula, should be replaced by $-$ signs. After this change, the family $L_{abs}$ is equivalent to our family 6 and we have a perfect correspondence between their nine $SL_{loc}^*$–families and ours.

Some groups of families become one family once we examine how they behave under permutations of qubits. That is, an $SL_{loc}$–orbit from one family may be taken to an orbit in another family by permuting qubits. After this consideration there are nine different groups of families as
found in [31]. They are \{1\}, \{2\}, \{3, 4, 5\}, \{6\}, \{7, 8, 9\}, \{10, 11\}, \{12, 13\}, 
\{14, 15\} and \{16, 17\}.

Table 2. Jordan structure of $\tilde{R}_\psi$

|   |          |                  |
|---|----------|------------------|
| 1 | $\pm ia$, $\pm ib$, $\pm ic$, $\pm id$ | 2. $\pm ia$, $\pm ib$, $J_2(\pm ic)$ |
| 3 | $\pm ia$, $\pm ib$, 0, $J_3(0)$ | 6. $\pm ia$, $J_3(\pm ib)$ |
| 4 | Same as 3 |                  |
| 5 | $J_2(\pm ia)$, $J_2(\pm ib)$ |                  |
| 7 | $\pm ia$, 0, $J_5(0)$ | 10. $J_2(\pm ia)$, 0, $J_3(0)$ |
| 8 | Same as 7 | 11. Same as 10 |
| 9 | $J_4(\pm ia)$ |                  |
| 12 | 0, $J_7(0)$ | 14. $J_3(0)$, $J_5(0)$ |
| 13 | Same as 12 | 15. Same as 14 |
| 16 | 0, 0, $J_3(0)$, $J_3(0)$ |                  |
| 17 | Same as 16 |                  |

Theorem 3.6. The orbits of $\text{SL}^*_\text{loc}$ on $\mathcal{H}$ are classified by the nine families $1, 2, 3, 6, 9, 10, 12, 14$ and $16$ listed in appendix A (Table 7). Their $R$–matrices are given in Table 2. States belonging to two different families (from this list of nine) are not equivalent under $\text{SL}^*_\text{loc}$–operations. However, within the same family, different values of the parameters may give states belonging to the same $\text{SL}^*_\text{loc}$–orbit.

Proof. Denote by $R_i$ the $R$–matrix of the $i$–th family as given in Table 2. Assume that $k \in \{3, 7, 10, 12, 14, 16\}$. One can easily compute the new $R$–matrix, $R'_k$, which results by applying the permutation $(1, 4)(2, 3)$ of the four qubits. Then it is easy to see that after multiplying the first row and the first column of $R'_k$ by $-1$, we obtain exactly the transpose of $R_k$ (if $k = 3$ or 7 this step is redundant). By inspection of Table 2 we see that $R'_k = R_{k+1}$. This means that the $k$-th and $(k + 1)$-st family of $O_4 \times O_4$–orbits fuse into a single family of $\text{SL}^*_\text{loc}$–orbits.

We leave to the reader to verify that the family 5 resp. 9 fuses with the family 3 resp. 7 into a single $\text{SL}^*_\text{loc}$–family.

Let us point out that the $O_4 \times O_4$–orbits may be disconnected and that different connected components may behave differently under qubit
permutations. We shall illustrate this in the case of the families 14 and 15. These families are in fact single $O_4 \times O_4$–orbits which we denote as $\mathcal{O}_{14}$ and $\mathcal{O}_{15}$, respectively. Each of them has two connected components:

$$\mathcal{O}_{14} = \mathcal{O}_{14}^I \cup \mathcal{O}_{14}^{II}, \quad \mathcal{O}_{15} = \mathcal{O}_{15}^I \cup \mathcal{O}_{15}^{II}.$$ 

These facts and some others that we will use are explained in [8]. To be precise, we assume that the Roman superscripts I and II are chosen so that the representative of $\mathcal{O}_{14}$ resp. $\mathcal{O}_{15}$ given in Table 1 belongs to $\mathcal{O}_{14}^I$ resp. $\mathcal{O}_{15}^I$. The left resp. right multiplication of $R$ by an orthogonal matrix with determinant $-1$ has the effect of switching the first two resp. last two qubits. The former leaves $\mathcal{O}_{14}^I$ and $\mathcal{O}_{14}^{II}$ invariant and switches $\mathcal{O}_{15}^I$ and $\mathcal{O}_{15}^{II}$, the latter switches $\mathcal{O}_{14}^I$ and $\mathcal{O}_{14}^{II}$ and leaves $\mathcal{O}_{15}^I$ and $\mathcal{O}_{15}^{II}$ invariant. Switching qubits 2 and 3 is a new feature, not discussed in [8]. We claim that in this case it has the following effect: The components $\mathcal{O}_{14}^I$ and $\mathcal{O}_{15}^{II}$ get interchanged while the components $\mathcal{O}_{14}^{II}$ and $\mathcal{O}_{15}^I$ remain invariant. To carry out this verification, one cannot rely on the Jordan structure of the matrices $\tilde{R}$ as they are the same for both $\mathcal{O}_{14}$ and $\mathcal{O}_{15}$. However these orbits have different $ab$–diagrams which makes it possible to verify the claim. For the definition of $ab$–diagrams see [17, 25, 8]. Since the transpositions $(1, 2), (2, 3)$ and $(3, 4)$ generate $\text{Sym}_4$, it follows that $\mathcal{O}_{14} \cup \mathcal{O}_{15}$ is indeed a single $\text{SL}_{4\text{c}}^*$–orbit.

The redundancies mentioned in Theorem 3.6 will be addressed in the next section.

| Family in [31] | Jordan blocks of $\tilde{R}_\psi$ | Our family |
|----------------|----------------------------------|------------|
| $G_{abcd}$     | $\pm ia, \pm ib, \pm ic, \pm id$ | 1          |
| $L_{abc2}$     | $\pm ia, \pm ib, J_2(\pm ic)$   | 2          |
| $L_{a2b}$      | $J_2(\pm ia), J_2(\pm ib)$      | 5          |
| $L_{ab3}$      | $\pm ia, \pm ib, J_2(\pm ia)$   | ?          |
| $L_{a4}$       | $J_4(\pm ia)$                    | 9          |
| $L_{a203}$     | $J_2(\pm ia), 0, J_3(0)$         | 10         |
| $L_{053}$      | $J_3(0), J_5(0)$                 | 15         |
| $L_{072}$      | $0, J_7(0)$                      | 13         |
| $L_{03+1,03+1}$| $0, 0, J_3(0), J_3(0)$           | 16         |
4. Criterion for $SL^{*}_{\text{loc}}$–equivalence

In this section we give a criterion for testing the equivalence of two states $\phi, \psi \in \mathcal{H}$ under $SL^{*}_{\text{loc}}$–operations. In the case when $\phi$ and $\psi$ are semisimple the criterion is very easy to use: one just has to verify whether the four invariants $H, \Gamma, \Sigma, \Pi$ (the latter three to be defined below) take the same values on $\phi$ and $\psi$.

As mentioned in the introduction, the algebra $A$ of complex analytic polynomial functions $f : \mathcal{H} \to \mathbb{C}$ which are $SL^{*}_{\text{loc}}$–invariant, i.e., satisfy
\[ f(g \cdot \psi) = f(\psi), \quad \forall g \in SL_{\text{loc}}, \forall \psi \in \mathcal{H}, \]
is isomorphic to a polynomial algebra in four generators. Explicit generators, as constructed in [19], are $H, L, M$ and another polynomial $D$. The definition of $D$ is somewhat involved.

For $j, k \in \{0, 1\}$ let
\[ \Psi_{jk} = \sum_{i,l=0}^{1} \psi_{ijkl} x^i y^l \]
where $x_0, x_1, y_0, y_1$ are independent commuting indeterminates. The determinant
\[ \begin{vmatrix} \Psi_{00} & \Psi_{01} \\ \Psi_{10} & \Psi_{11} \end{vmatrix} \]
is a biquadratic form in the two sets of variables $\{x_0, x_1\}$ and $\{y_0, y_1\}$. There is a unique $3 \times 3$ matrix $B$ such that this form can be written as
\[ \begin{bmatrix} x_0^2 & x_0 x_1 & x_1^2 \end{bmatrix} B \begin{bmatrix} y_0^2 \\ y_0 y_1 \\ y_1^2 \end{bmatrix}. \]
Then $D(\psi) = \det B \in A$ and it is homogeneous of degree 6. By permuting cyclically the last three indices of $\psi$, we obtain two more such invariants which we denote by $E$ and $F$.

One can easily verify that the four homogeneous polynomials
\[ H, \quad \Gamma = D + E + F, \quad \Sigma = L^2 + M^2 + N^2, \]
\[ \Pi = (L - M)(M - N)(N - L), \]
are algebraically independent and invariant under the action of $SL^{*}_{\text{loc}}$. The degrees of these polynomials are 2, 6, 8 and 12, respectively. These polynomials appear in a work of Schläfli in 1852, who also noticed their invariance property under permutations of indices [29].

Let us now examine the mentioned redundancies of Theorem 3.6. The most interesting case is that of family 1, the family of all semisimple orbits. The question we raise is the following: When are two states
ψ_{abcd} and ψ_{a'b'c'd'} in the same SL_{loc}^∗-orbit? (By ψ_{abcd} we denote the state whose R-matrix is the first matrix in Table 1.)

Let a be the subspace of ℋ consisting of tensors ψ with R_{ψ} a diagonal matrix. If we identify ℋ with p using the map ψ → ˜\hat{R}_{ψ} then a is a maximal abelian subspace of p consisting of semisimple elements. Such a subspace is known as a Cartan subspace of p. We mention that all Cartan subspaces of p are conjugate by SO_4 × SO_4, the identity component of the subgroup K = O_4 × O_4 of O_8.

Let N_a resp. Z_a be the subgroup of SL_{loc} which leaves a globally resp. pointwise invariant. Define similarly the subgroups N^*_a and Z^*_a of SL_{loc}^∗. The quotient groups W_a = N_a/Z_a and W^*_a = N^*_a/Z^*_a act on a effectively.

Let us use the diagonal entries of R_ψ as coordinates in a. It is easy to see that W_a can permute arbitrarily the coordinates a, b, c, d and also replace them with ±a, ±b, ±c, ±d provided the number of ”−” signs is even. We conclude that W_a has order at least 192. On the other hand, a is a Cartan subalgebra of so_8 and W_a is a subgroup of the Weyl group of the pair (so_8, a). Since so_8 has Cartan type D_4, this Weyl group has exactly order 192. We conclude that W_a coincides with this Weyl group.

It is easy to check that Sym_4 ⊂ N^*_a, i.e., all qubit permutations map a into itself. It is also easy to check that the permutation (1, 2)(3, 4) acts trivially on a. It follows that the Klein four-group V ⊳ Sym_4 also acts trivially. The transposition (2, 3) sends the point (a, b, c, d) to the point
\[
\frac{1}{2}(a + b + c + d, a + b - c - d, a - b + c - d, a - b - c + d),
\]
i.e., it acts as the reflection in the hyperplane a = b + c + d. By using GAP [12], one can easily check that W_a and this reflection generate a group of order 1152. Clearly this is the Weyl group of type F_4.

**Lemma 4.1.** W_a^* is the Weyl group of type F_4, of order 1152 = 2^7 · 3^2.

**Proof.** We have
\[
[N^*_a : Z_a] = [N^*_a : N_a][N_a : Z_a] = [N^*_a : Z^*_a][Z^*_a : Z_a].
\]
We have seen above that [N^*_a : Z^*_a] = |W^*_a| ≥ 1152. Since the Klein four-group V ⊲ Sym_4 acts trivially on a, we have [Z^*_a : Z_a] ≥ 4. On the other hand, since N_a = N^*_a ∩ SL_{loc}, we have [N^*_a : N_a] ≤ |SL_{loc}^∗ : SL_{loc}| = 24. Recall also that [N_a : Z_a] = 192. The equality implies now that all these inequalities are in fact equalities. In particular,
\[
|W^*_a| = [N^*_a : Z^*_a] = 1152.\]
Since $W_\mathfrak{a}$ is a finite irreducible reflection group of rank 4, it must be the Weyl group of type $F_4$. □

**Theorem 4.2.** The restriction homomorphism $\rho : \mathcal{A}^* \to \mathcal{B}$ from the algebra $\mathcal{A}^*$ to the algebra $\mathcal{B}$ of polynomial $W_\mathfrak{a}^*$ invariants on $\mathfrak{a}$ is an isomorphism of graded algebras. The algebra $\mathcal{A}^*$ is generated by the four homogenous algebraically independent polynomials $H$, $\Gamma$, $\Sigma$, and $\Pi$ of degree 2, 6, 8 and 12, respectively.

**Proof.** If $f \in \ker \rho$, i.e., $f \in \mathcal{A}^*$ and $f$ vanishes on $\mathfrak{a}$, then $f$ vanishes on all semisimple elements of $\mathcal{H}$. But the semisimple elements are dense in $\mathcal{H}$, and so $f \equiv 0$. This shows that $\rho$ is injective.

In view of Lemma 4.1, we can apply to $W_\mathfrak{a}^*$ some well known facts about finite reflection groups, see for example [14, Section 3.7]. The algebra $\mathcal{B}$ is isomorphic to a polynomial algebra in four variables and it is generated by four homogeneous polynomials of degree 2, 6, 8 and 12. Moreover any set of four homogeneous generators of $\mathcal{B}$ must have these degrees. Now recall that the $\text{SL}_{\text{loc}}^*$-invariants $H$, $\Gamma$, $\Sigma$, $\Pi$ have exactly these degrees. Since they are algebraically independent, and $\rho$ is injective, their restrictions to $\mathfrak{a}$ are also algebraically independent. As their degrees are 2, 6, 8 and 12, they must generate $\mathcal{B}$. Hence $\rho$ is also surjective. We can now prove the following analog of Theorem 3.3. □

**Theorem 4.3.** Two semisimples states $\phi, \psi \in \mathcal{H}$ are $\text{SL}_{\text{loc}}^*$-equivalent iff the invariants $H, \Gamma, \Sigma$ and $\Pi$ take the same values at $\phi$ and $\psi$. For arbitrary states $\phi, \psi \in \mathcal{H}$, if at least one of the invariants $H$, $\Gamma$, $\Sigma$, $\Pi$ takes different values on $\phi$ and $\psi$, then the $\text{SL}_{\text{loc}}^*$-orbits of $\phi$ and $\psi$ are different.

**Proof.** The second assertion is obvious. For the first assertion, we need only prove that its condition is sufficient. Assume that the condition is satisfied. Let $\mathfrak{a} \subset \mathcal{H}$ be the Cartan subspace introduced above. Since every semisimple element $\psi \in \mathcal{H}$ is $\text{SL}_{\text{loc}}^*$-equivalent to an element of $\mathfrak{a}$, we may assume that $\phi, \psi \in \mathfrak{a}$. By Theorem 4.2 and our hypothesis, all invariants of $W_\mathfrak{a}^*$ take the same values on $\phi$ and $\psi$. Since $W_\mathfrak{a}^*$ is a finite reflection group, we conclude that $\phi$ and $\psi$ are $W_\mathfrak{a}^*$-equivalent. Since $W_\mathfrak{a}^* = N_\mathfrak{a}^*/Z_\mathfrak{a}^*$, it follows that $\phi$ and $\psi$ are $N_\mathfrak{a}^*$-equivalent. In particular, they are $\text{SL}_{\text{loc}}^*$-equivalent. □

We can now use this theorem to show in a straightforward manner that certain tensors are not $\text{SL}_{\text{loc}}^*$-equivalent. In some situations one may use certain Bell inequalities as in [33] to show that two states
Table 4. $SL_{\text{loc}}^*$-invariants of some pure 4-qubit states

| State  | $H$ | $\Gamma$ | $\Sigma$ | $\Pi$ |
|--------|-----|----------|----------|------|
| GHZ    | $\frac{1}{2}$ | 0 | 0 | 0 |
| $W$    | 0 | 0 | 0 | 0 |
| $|\phi\rangle$ | 0 | 0 | $\frac{1}{128}$ | $\frac{1}{2048}$ |
| $|\phi'\rangle$ | 0 | 0 | $\frac{1}{128}$ | $\frac{1}{2048}$ |
| $|\chi\rangle$ | 0 | 0 | $\frac{1}{128}$ | $-\frac{1}{2048}$ |

are not equivalent. However we feel that simply calculating the $SL_{\text{loc}}^*$-invariants is a more straightforward approach and the above theorem will be enough for the majority of situations.

Let us look at some examples. The generalized GHZ and $W$ states in four qubits are

$\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$

and

(4.2) $\frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$

respectively. Two important states in quantum teleportation [33, Eq. (22) and Eq. (2)] are the cluster state

$|\phi\rangle = \frac{1}{2}(|\beta^+0\beta^+0\rangle + |\beta^+0\beta^-1\rangle + |\beta^-1\beta^-0\rangle + |\beta^-1\beta^+1\rangle)$

where $|\beta^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ and the state $|\chi\rangle$ given by

$|\chi\rangle = \frac{1}{2\sqrt{2}}(|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle).$

There is also another cluster state mentioned in [2, Eq. (4)]:

$|\phi'\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$.

In Table 4 we tabulate the values of $H$, $\Gamma$, $\Sigma$ and $\Pi$ on these five states. It is clear from this table that all five states belong to different $SL_{\text{loc}}^*$-orbits except possibly for the pair $|\phi\rangle$ and $|\phi'\rangle$. These two states
share the same invariants and, as both are semisimple, they indeed belong to the same $SL^*_\text{loc}$--orbit.

We remark that the states $e^{i\pi/4}|\phi\rangle$ and $|\chi\rangle$ belong to the same $SL^*_\text{loc}$--orbit because they are both semisimple and share the same invariants. As A. Osterloh has pointed to us, from the quantum physics point of view, the unit vectors $|\phi\rangle$ and $e^{i\pi/4}|\phi\rangle$ represent the same pure state because they differ only by a phase factor. On the other hand, they belong to different $SL^*_\text{loc}$--orbits. The apparent discrepancy is explained by the fact that we are classifying the nonzero vectors in $\mathcal{H}$ rather than the genuine pure states (see the Introduction).

We can now sketch our procedure that one can use to decide whether two arbitrary pure states $\phi, \psi \in \mathcal{H}$ are $SL^*_\text{loc}$--equivalent. It is understood that equivalence will mean $SL^*_\text{loc}$--equivalence for the rest of this section. Clearly if $\phi$ and $\psi$ are equivalent and $\phi$ is semisimple or nilpotent then $\psi$ must have the same property.

Step 1. We compute the values of $H$, $\Gamma$, $\Sigma$ and $\Pi$ at $\phi$ and $\psi$. If they do not agree then $\phi$ and $\psi$ are not equivalent. From now on we assume that they do agree. If $\phi$ and $\psi$ are semisimple, they must be equivalent. We shall now assume they are not semisimple.

Step 2. Assume $\phi$ and $\psi$ are nilpotent and compute the Jordan structures of $\tilde{R}_\phi$ and $\tilde{R}_\psi$. By inspecting Table 2, with all eigenvalues set to 0, we see that apart from one case the Jordan structure of the $\tilde{R}$--matrix determines uniquely the $SL^*_\text{loc}$--orbit. The exceptional case is when the Jordan blocks are of size 1, 1, 3, 3. Then there are two orbits. They can be distinguished by using $ab$--diagrams. One of these orbits is that of the generalized $W$ state and the other is in family 16 (or 17). From now on we assume that $\phi$ and $\psi$ are not nilpotent.

Step 3. The families 5 and 9 can be distinguished from the families 3, 4 and 7, 8 respectively by the sizes of Jordan blocks of $\tilde{R}$--matrices. By permuting qubits in both $\phi$ and $\psi$ we can assume that both states belong to one of the families 2, 5, 6, 9, 10 or 11. After this reduction $\phi$ and $\psi$ are equivalent iff they have the same Jordan structure.

5. Classification of tensors of rank at most three

Here we provide some normal forms for tensors of rank 1,2 and 3 under the action of $SL^*_\text{loc}$ and investigate some of their properties. A rank 1 tensor is just a product state and so it is in the same orbit as $|0000\rangle$.

For the rank 2 case we have a few more situations to consider. Let

$$
\psi = a_1 \otimes a_2 \otimes a_3 \otimes a_4
+ b_1 \otimes b_2 \otimes b_3 \otimes b_4
$$
be a rank 2 tensor. We may consider where linear dependencies occur amongst the sets \( \{a_i, b_i\} \). Since we also consider the action of \( \text{Sym}_4 \) there are really only 3 cases as in Figure 2. A line connecting two points in the \( i \)-th column means those corresponding two vectors are scalar multiples of each other. In case (a) let \( g_i \in \text{GL}_2 \) be such that \( g_i(a_i) = \nu_i e_0 \) and \( g_i(b_i) = \nu_i e_1 \) where \( \nu_i \) is chosen so that \( g_i \in \text{SL}_2 \), for \( i = 1, 2, 3, 4 \). Hence the tensor reduces to \( \alpha (|0000\rangle + |1111\rangle) \), where \( \alpha = \nu_1 \nu_2 \nu_3 \nu_4 \). If \( \{a_i, b_i\} \) is linearly dependent then we may set \( \nu_i = 1 \) since we are free to choose how the transformation acts on some other vector that creates a basis. Let us also describe how to handle (b). Since \( \{a_1, b_1\} \) is linearly dependent, we may assume \( b_1 = a_1 \). Hence the tensor is in the same orbit as \( \alpha |00\rangle \otimes (|000\rangle + |111\rangle) \). Similarly (c) is in the same orbit as \( \alpha |00\rangle \otimes (|00\rangle + |11\rangle) \). Note that conversely any tensor in one of these forms is of rank 2.

**Proposition 5.1.** A rank 2 tensor \( \psi \in \mathcal{H} \) is \( \text{SL}^*_{\text{loc}} \)-equivalent to one of the following:

(a) \( \alpha (|0000\rangle + |1111\rangle) \), \( \alpha \neq 0 \),

(b) \( |0\rangle \otimes (|000\rangle + |111\rangle) \),

(c) \( |00\rangle \otimes (|00\rangle + |11\rangle) \).

In case (a) \( \psi \) is semisimple and non-factorizable, while in cases (b) and (c) it is nilpotent and factorizable.

**Proof.** The first assertion follows from the above discussion. We can assume that \( \alpha = 1 \) in cases (b) and (c) since they are nilpotent orbits and so \( \text{SL}^*_{\text{loc}} \) and \( \text{GL}^*_{\text{loc}} \)-orbits coincide. The second assertion is easy to verify. \( \square \)

The case of a rank 3 tensor is not as easy to breakdown. The complications arise because now we have 3 vectors \( a_i, b_i, c_i \) being mapped under a \( \text{SL}_2 \)-transformation but can only control where 2 of them are mapped to in most cases. Let

\[
\psi = a_1 \otimes a_2 \otimes a_3 \otimes a_4 \\
+ b_1 \otimes b_2 \otimes b_3 \otimes b_4 \\
+ c_1 \otimes c_2 \otimes c_3 \otimes c_4
\]

be a rank 3 tensor.
We can bring $\psi$ into a reduced form by using $\text{SL}_2$ transformation on each qubit. The first three qubits are handled a bit differently than the last one as we shall see. We outline how to construct the $g_i \in \text{SL}_2$ that will act on each of the first three qubits.

Case 1: The set \{a_i, b_i, c_i\} spans $H_i$. Assume \{a_i, c_i\} is linearly independent and $b_i = \lambda_i a_i + \mu_i c_i$. With $\lambda_i \mu_i \neq 0$ we can choose $g_i \in \text{SL}_2$ such that $g_i(\lambda_i a_i) = \nu_i e_0$ and $g_i(\mu_i c_i) = \nu_i e_0$ where $\nu_i^2 = \det [\lambda_i a_i \mid \mu_i c_i]$. In this situation $g_i(b_i) = \nu_i (e_0 + e_1)$. If $\lambda_i = 0$ then we can choose $g_i \in \text{SL}_2$ such that $g_i(a_i) = \nu_i e_0$ and $g_i(\mu_i c_i) = \nu_i e_1$. A similar argument holds when $\mu_i = 0$.

Case 2: The set \{a_i, b_i, c_i\} does not span $H_i$. We can assume that $a_i = b_i = c_i$ and choose $g_i \in \text{SL}_2$ such that $g_i(a_i) = g_i(b_i) = g_i(c_i) = e_0$.

Now when $i = 4$ the only difference is that if say \{a_4, b_4\} is linearly dependent then we cannot simply assume that $a_4 = b_4$, but must take into account a scalar factor.

Figure 3 contains the different ways that the sets \{a_i, b_i, c_i\} can contain the same vector multiple. It is not surprising that it is more complicated than Figure 2 and it is indeed slightly more difficult to show that it captures all possibilities. Nevertheless we have:

**Proposition 5.2.** Any rank 3 tensor $\psi \in H$ is $\text{SL}_2^*\text{–equivalent to a tensor having one of the patterns (a–g) in Table 3. This pattern is uniquely determined by } \psi.$
Proof. Let $\psi = a + b + c$ where 
\begin{align*}
a &= a_1 \otimes a_2 \otimes a_3 \otimes a_4, \\
b &= b_1 \otimes b_2 \otimes b_3 \otimes b_4, \\
c &= c_1 \otimes c_2 \otimes c_3 \otimes c_4.
\end{align*}

If $\psi$ is factorizable, then since it has rank 3, by the classification in [3] it must be $\text{SL}^{\ast}_{\text{loc}}$-equivalent to a tensor of the form $(g)$. Assume $\psi$ is not factorizable. If for each $i$ there are no linear dependencies between any two of the factors $a_i, b_i, c_i$ then $\psi$ is clearly in the form $(a)$. If there is one linear dependency, by permuting qubits, we may assume $\psi$ is of the form $(b)$. Assume there are two linear dependencies. If they are both between the factors of the same two summands, say $a$ and $b$, then we may assume that $b = a_1 \otimes a_2 \otimes b_3 \otimes b_4$. Now we can use the $\text{SL}^{\ast}_{\text{loc}}$-operations as described above to get 
\begin{align*}
\nu^{-1}\psi' &= e_0 \otimes e_0 \otimes e_0 \otimes e_0 \\
&+ e_0 \otimes e_0 \otimes (e_0 + e_1) \otimes (e_0 + e_1) \\
&+ e_1 \otimes e_1 \otimes e_1 \otimes e_1 \\
&= e_0 \otimes e_0 \otimes e_0 \otimes (2e_0 + e_1) \\
&+ e_0 \otimes e_0 \otimes e_1 \otimes (e_0 + e_1) \\
&+ e_1 \otimes e_1 \otimes e_1 \otimes e_1
\end{align*}

which has diagram

Now we can repeat this process one more time to get a tensor in the form $3(e)$. If there is a dependency between a factor of $a$ and $b$, and another dependency between a factor of $b$ and $c$, then $\psi$ is in the form $(c)$. Assume there are 3 dependencies. If they are between the same two summands then $\psi$ is no longer a rank 3 tensor. If two are between $a$ and $b$, and one is between $b$ and $c$, then as before we can bring this into the form $(e)$. If we have a dependency between factors of $a$ and $b$, and $b$ and $c$, and between $a$ and $c$ then $\psi$ is of the form $(d)$. Assume that there are 4 dependencies. Clearly if at least 3 are between two summands then $\psi$ is not rank 3. Otherwise it is straightforward to see that the only possibility is a tensor of the form $(f)$.

The uniqueness assertion follows by inspection of Table 5. □

With this in mind, the normal forms follow.

Remark 5.3. In one of the cases, the proof below depends on the following important fact, a special case of [30, Theorem 38.6.1]. Let us identify $\mathcal{H}$ with the subspace $\mathfrak{p}$ of $\mathfrak{g} = \mathfrak{so}_8$. Fix $\psi \in \mathfrak{p}$ and let $\mathcal{O} \subset \mathfrak{g}$ be its $\text{SO}_8$-orbit under the adjoint action. Then each irreducible component
It is easy to verify that all matrices $\mathcal{O} \cap p$ of $\mathbb{SO}_4 \times \mathbb{SO}_4$–orbit. Moreover all these components have the same dimension.

**Proposition 5.4.** A rank 3 tensor $\psi \in \mathcal{H}$ of the given pattern (see Figure 3) can be reduced using $\text{SL}^\ast_{\text{loc}}$–operations to the form:

(a) $\alpha|0000\rangle + \beta(|0\rangle + |1\rangle)^{\otimes 4} + \gamma|1111\rangle$, $\alpha \beta \gamma \neq 0$,

(b) $\alpha(|0000\rangle + |1111\rangle) + |0\rangle \otimes (|0\rangle + |1\rangle)^{\otimes 3}$, $\alpha \neq 0$,

(c) $\alpha(|0000\rangle + |01\rangle \otimes (|0\rangle + |1\rangle)^{\otimes 2} + |1111\rangle)$, $\alpha \neq 0$,

(d) $|0000\rangle + |011\rangle \otimes (|0\rangle + |1\rangle) + |1101\rangle$,

(e) $\alpha(|0000\rangle + |0011\rangle + |1111\rangle)$, $\alpha \neq 0$,

(f) $|0000\rangle + |0011\rangle + |1110\rangle$,

(g) $|0000\rangle + |0110\rangle + |1100\rangle$.

**Proof.** Let us outline the procedure in cases (b) and (d). The rest of the cases follow from much the same reasoning, although case (b) is uniquely non-trivial. Using Figure 3 we see that in the case (b) we may assume that

$$\psi = \alpha|0000\rangle + \beta|0\rangle \otimes (|0\rangle + |1\rangle)^{\otimes 3} + \gamma|1111\rangle$$

where $\alpha \beta \gamma \neq 0$. We apply the $\text{SL}_2$–transformation $\alpha e_0 \to ne_0$ and $\gamma e_1 \to ne_1$ to the first qubit to get

$$\alpha'|0000\rangle + |1111\rangle) + \beta'|0\rangle \otimes (|0\rangle + |1\rangle)^{\otimes 3}$$

where $\alpha' = \nu$ and $\beta' = \nu \beta \alpha^{-1}$.

Let

$$\psi = \alpha(|0000\rangle + |1111\rangle) + \beta|0\rangle \otimes (|0\rangle + |1\rangle)^{\otimes 3}$$

with $\alpha \beta \neq 0$ (we rename $\alpha'$ to $\alpha$ and $\beta'$ to $\beta$ for convenience). Choose $\gamma \in \mathbb{C}$ such that $\gamma^2 + \gamma = \alpha(\alpha + \beta)$. If $\beta = -\alpha$ we assume that $\gamma = -1$.

We claim that $\psi$ is $\text{SL}^\ast_{\text{loc}}$–equivalent to

$$\phi = \gamma(|0000\rangle + |1111\rangle) + |0\rangle \otimes (|0\rangle + |1\rangle)^{\otimes 3}$$

If $\beta = -\alpha$ then $\psi = -\alpha \phi$. Since $\phi$ is nilpotent, our claim holds.

Now assume that $\alpha + \beta \neq 0$. Choose a continuous function $f : [0, 1] \to \mathbb{C} \setminus \{0\}$ such that $f(0) = \alpha$, $f(1) = \gamma$ and $f(t)^2 \neq \alpha(\alpha + \beta)$ for all $t \in [0, 1]$. Consider the one-parameter tensor family

$$\chi(t) = f(t)(|0000\rangle + |1111\rangle) + \frac{\alpha^2 - f(t)^2 + \alpha \beta}{f(t)}|0\rangle \otimes (|0\rangle + |1\rangle)^{\otimes 3}.$$

It is easy to verify that all matrices $R_{\chi(t)}$ have the same Jordan structure:

$$0, J_2(\pm i \sqrt{\alpha^2 + \alpha \beta}), J_3(0).$$
Hence they all belong to a single $GL_8$–orbit (a similarity class) $\mathcal{O} \subset M_8$. Since $\mathfrak{g} \subset M_8$ is the space of skew-symmetric matrices, $\mathcal{O} \cap \mathfrak{g}$ is a single $O_8$–orbit, and so it is the union of at most two $SO_8$–orbits. By Remark 5.3, each irreducible component of $\mathcal{O} \cap p$ is a single $SO_4 \times SO_4$–orbit and all these components have the same dimension. Since $\{ R_\chi(0) \}$ is contained in a single irreducible component of $\mathcal{O} \cap p$, and $\chi(0) = \psi$ and $\chi(1) = \phi$, we conclude that $R_\psi$ and $R_\phi$ belong to the same $SO_4 \times SO_4$–orbit, and so $\psi$ and $\phi$ belong to the same $SL_{loc}$–orbit. This concludes the proof of our claim.

A tensor $\psi$ of the form $(d)$ can be reduced to

$$\alpha|0000\rangle + \beta|011\rangle \otimes (|0\rangle + |1\rangle) + \gamma|1101\rangle.$$ 

By mapping $\alpha e_0 \rightarrow \nu e_0$ and $\gamma e_1 \rightarrow \nu e_1$ in the third qubit we attain

$$\alpha'|0000\rangle + \beta'|011\rangle \otimes (|0\rangle + |1\rangle) + \alpha'|1101\rangle$$

and we can check that $\alpha' = \nu$ and $\beta' = \nu\beta\alpha^{-1}$. Now by applying the $SL_2$–transformation which sends $\alpha'e_0 \rightarrow \nu'e_0$ and $\beta'e_1 \rightarrow \nu'e_1$ in the third qubit we obtain

$$\nu'(|0000\rangle + |011\rangle \otimes (|0\rangle + |1\rangle) + |1101\rangle).$$

Since $\psi$ is nilpotent, the $SL_{loc}^*$ and $GL_{loc}^*$–orbits coincide and we can replace $\nu'$ with 1. □

We shall say that the tensors listed in Proposition 5.4 are of type $3(a−g)$, respectively. Note that the invariants $L$ and $M$ vanish on each of these tensors.

For the computation of tensor ranks it is important to know the Jordan structure of the matrices $R_\psi$ for all types of tensors of ranks $\leq 3$. We shall investigate the tensor $\psi$ of type $3(a)$ in detail. The other cases are easy to analyze and we omit their discussion. By using (3.1) we find that the characteristic polynomial of $R_\psi$ is

$$t^2(t^6 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)t^4 + (\alpha\beta + \alpha\gamma + \beta\gamma)^2t^2 + 4(\alpha\beta\gamma)^2).$$

If we let $s = t^2$ then it is $sg(s)$ where

$$g(s) = s^3 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)s^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)^2s + 4(\alpha\beta\gamma)^2.$$ 

The discriminant of $g(s)$ (see (3.2)) is

$$16(\alpha\beta\gamma)^2((\alpha\beta + \alpha\gamma + \beta\gamma)^3 - 27(\alpha\beta\gamma)^2).$$

If the discriminant does not vanish then $R_\psi$ is semisimple and belongs in family 1 from Table 1. Now if it vanishes then we must have

$$(\alpha\beta + \alpha\gamma + \beta\gamma)^3 - 27(\alpha\beta\gamma)^2 = 0$$

(5.1)
and the roots of \( g(s) \) are

\[
\lambda^2 = -\frac{4}{3}(\alpha\beta + \alpha\gamma + \beta\gamma), \quad \mu^2 = -\frac{1}{3}(\alpha\beta + \alpha\gamma + \beta\gamma),
\]

where the latter is a double root.

Now we can determine the Jordan structure of \( \tilde{R}_\psi \). Since the eigenvalue 0 of \( \tilde{R}_\psi \) has multiplicity two, and the matrix rank of \( \tilde{R}_\psi \) is 6 (there are two rows of zeros) we conclude that there are two \( 1 \times 1 \) Jordan blocks of 0. Consider the matrix \( \tilde{R}_\psi^2 \). By permuting rows and columns, we see that it is similar to a matrix of the form \( P_1 \oplus 0 \oplus P_2 \) where the second summand is a \( 2 \times 2 \) zero matrix and the other two summands are \( 3 \times 3 \) matrices. It is straightforward to see that \( P_1 \) and \( P_2 \) are similar. Now assume that \( \tilde{R}_\psi \) is semisimple, then so is \( \tilde{R}_\psi^2 \). In particular the matrices \( P_1 - \lambda^2 I_3 \) and \( P_1 - \mu^2 I_3 \) have ranks 1 and 2 respectively. But then by evaluating the \( 2 \times 2 \) minors we can conclude that \( \alpha = \beta = \gamma \). Conversely one can check that \( \alpha = \beta = \gamma \) implies that \( \tilde{R}_\psi \) is semisimple. We find that \( \tilde{R}_\psi \) then has the Jordan structure: \( 0,0,0,0, J_2(\pm\mu) \) and it is in family 2, unless \( \alpha = \beta = \gamma \) in which case \( \tilde{R}_\psi \) is semisimple and is again in family 1.

To summarize, we find that if a tensor of rank 3 is semisimple, then it must be of type \( 3(a) \). Furthermore, a rank 3 tensor of type \( (a) \) is semisimple unless \( \eqref{5.1} \) holds and \( \alpha, \beta, \gamma \) are not all equal.

Table 5 presents the Jordan structures for the different type of tensors of rank \( \leq 3 \). Note that one has to permute the four qubits in order to obtain all possible Jordan structures for a given type.

### Table 5. Jordan structure of \( \tilde{R}_\psi \) for tensors of rank \( \leq 3 \)

| Type | Structure |
|------|-----------|
| 1(a) | \( 0,0,0,0, J_2(0), J_2(0) \). |
| 2(a) | \( 0,0,0,0, \pm i\sqrt{\alpha}, \pm i\sqrt{\alpha} \). |
| 2(b) | \( 0,0, J_3(0), J_3(0) \). |
| 2(c) | \( 0,0,0,0, J_3(0) \) and \( J_2(0), J_2(0), J_2(0), J_2(0) \). |
| 3(a) | Discussed above: Semisimple or \( 0,0,0,0, \pm i\lambda, J_2(\pm i\mu) \). |
| 3(b) | If \( \alpha = -1 \) then 0, \( J_7(0) \) else 0, \( J_2(\pm i\sqrt{\alpha^2 + \alpha}), J_3(0) \). |
| 3(c) | \( \pm i\alpha, \pm i\alpha, 0, J_3(0) \) and \( J_2(0), J_2(0), J_2(\pm i\alpha) \). |
| 3(d) | \( J_3(0), J_5(0) \). |
| 3(e) | \( \pm i\alpha, \pm i\alpha, J_2(0), J_2(0) \) and \( 0,0,0,0, J_2(\pm i\alpha) \). |
| 3(f) | 0, 0, 0, \( J_5(0) \) and \( J_4(0), J_4(0) \). |
| 3(g) | 0, \( J_2(0), J_2(0), J_3(0) \). |
6. Determining the Tensor Ranks

Here we will compute the ranks of tensors \( \psi \) in each of the nine families listed in Theorem 3.6. For any subsequence \( \{ a_i \} \) of 1, 2, 3, 4 we set \( H_{a_1 \cdots a_k} = \bigotimes_{i=1}^{k} H_{a_i} \). For \( \psi = e_0 \otimes t_0 + e_1 \otimes t_1 \), where \( \{ t_0, t_1 \} \) is linearly independent, we form the linear transformation \( T_{\psi} : \mathbb{C}^2 \to \mathcal{H}_{234} \) sending \( (x, y) \to xt_0 + yt_1 \). The image of \( \mathbb{C}^2 \setminus \{ 0 \} \) under \( T_{\psi} \) is a projective line \( l_{\psi} \) in the projective space \( \mathbb{P}(\mathcal{H}_{234}) \). For a pure state \( \phi \in \mathcal{H}_{234} \) we will denote by \( \det \phi \) the Cayley hyperdeterminant as described in [3]. Explicitly we have
\[
\det \phi = (\text{tr} A \text{tr} B - \text{tr} AB)^2 - 4 \det A \det B
\]
where
\[
A = \begin{bmatrix}
\phi_{000} & \phi_{001} \\
\phi_{010} & \phi_{011}
\end{bmatrix}, \quad
B = \begin{bmatrix}
\phi_{100} & \phi_{101} \\
\phi_{110} & \phi_{111}
\end{bmatrix}.
\]
Note that \( 4 | \det \phi | \) is the residual entanglement (also known as the 3–tangle) of the pure state \( \phi \) described in [6].

**Lemma 6.1.** If \( (x_1, y_1) \) and \( (x_2, y_2) \) are linearly independent then we have that \( \text{rank} \, \psi \leq \text{rank} \, T_{\psi}(x_1, y_1) + \text{rank} \, T_{\psi}(x_2, y_2) \).

**Proof.** Since \( \{ (x_1, y_1), (x_2, y_2) \} \) is linearly independent, so is \( \{ x_1 e_0 + x_2 e_1, y_1 e_0 + y_2 e_1 \} \). The tensor \( \psi' = (x_1 e_0 + x_2 e_1) \otimes t_0 + (y_1 e_0 + y_2 e_1) \otimes t_1 \) is in the same \( \text{GL}_\text{loc} \)-orbit as \( \psi \) so \( \text{rank} \psi' = \text{rank} \psi \). But \( \psi' = e_0 \otimes (x_1 t_0 + y_1 t_1) + e_1 \otimes (x_2 t_0 + y_2 t_1) \) so \( \text{rank} \psi' \leq \text{rank} \, T_{\psi}(x_1, y_1) + \text{rank} \, T_{\psi}(x_2, y_2) \) and the result follows. \( \square \)

Table [5] lists, in order, the invariants \( H, L, M, N, D \) and \( \Gamma \) for each of the 17 families from Table [1]. It will be helpful to refer to the list of invariants of the families as we proceed. We will often use the fact that if either \( L \) or \( M \) does not vanish on \( \psi \) then \( \psi \notin \bar{S}_3 \) and so \( \psi \) must have rank 4 (see Proposition 3.1). It is easy to determine whether a state \( \psi \) is factorizable. For instance, we have \( \psi = \phi \otimes \chi \) with \( \phi \in \mathcal{H}_{12} \) and \( \chi \in \mathcal{H}_{34} \) iff \( \text{rank} \psi = 1 \). Similarly we have \( \psi = \phi \otimes \chi \) with \( \phi \in \mathcal{H}_1 \) and \( \chi \in \mathcal{H}_{234} \) iff the 2 \( \times \) 8 matrix \( [\psi_{i,j,k,l}] \) has rank 1. In general, one has first to permute the qubits.

We shall now consider separately each of the nine families mentioned in Theorem 3.6.

**Family 1.** We may permute the diagonal entries \( a, b, c, d \) of \( R \) (see Table [1] and replace them by \( \pm a, \pm b, \pm c, \pm d \) without changing the \( \text{SL}_\text{loc} \)-orbit of \( \psi \). If \( L \neq 0 \) or \( M \neq 0 \) then \( \text{rank} \psi = 4 \). From now on we may assume that \( a + b + c = d = 0 \) (see the expressions for \( L \) and \( M \) in Table [6].
Table 6. The SL_{loc}-invariants $H, L, M, N, D, \Gamma$

1. $\frac{1}{2}(a^2 + b^2 + c^2 + d^2)$; $\frac{1}{32}(a^2 - b^2 - c^2 + d^2)^2$; $\frac{1}{10}(4(ac + bd)^2 - (a^2 - b^2 + c^2 - d^2)^2)$; $\frac{1}{10}(4(ac + bd)^2 - (a^2 - b^2 + c^2 - d^2)^2) - \frac{1}{4}(ad - bc)(ab - cd)(ac - bd)$
2. $\frac{1}{2}(a^2 + b^2 + 2c^2)$; $\frac{1}{2}(a^2 + b^2 + 2c^2)$
3. $\frac{1}{2}(a^2 + b^2)$; $\frac{1}{16}(a^2 - b^2)^2$; $\frac{1}{16}(a^2 - b^2)^2$; $\frac{1}{32}(a^2 - b^2)^2$; $\frac{1}{32}(a^2 - b^2)^2$
4. $\frac{1}{2}(a^2 + b^2)$; $\frac{1}{16}(a^2 - b^2)^2$; $\frac{1}{16}(a^2 - b^2)^2$; $\frac{1}{32}(a^2 - b^2)(a^4 - b^4)$
5. $\frac{1}{2}(a^2 + b^2)$; $\frac{1}{2}(a^2 + b^2)$; $\frac{1}{2}(a^2 + b^2)$; $\frac{1}{2}(a^2 + b^2)$
6. $\frac{1}{2}(a^2 + b^2)$; $\frac{1}{2}(a^2 + b^2)$; $\frac{1}{2}(a^2 + b^2)$; $\frac{1}{2}(a^2 + b^2)$
7. $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$
8. $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$
9. $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$
10. $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$
11. $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$; $\frac{1}{2}(a^2)$

 Families 12 - 17 are nilpotent so all invariants are 0.

Then we have

$$4\psi = a(e_0 + e_1) \otimes (e_0 + e_1) \otimes (e_0 - e_1) \otimes (e_0 - e_1) + a(e_0 - e_1) \otimes (e_0 + e_1) \otimes (e_0 + e_1) \otimes (e_0 + e_1) - 4c(e_0 \otimes e_1 \otimes e_0 \otimes e_0 \otimes e_0 \otimes e_1).$$

If $abc = 0$, say $c = 0$, then rank $\psi = 2$. Note that after normalization, this $\psi$ represents the generalized GHZ state. Now we may assume that $abc \neq 0$. Let $\phi$ be the tensor of type $3(a)$ (see Proposition 5.4). We shall choose the scalars $\alpha, \beta, \gamma$ to satisfy the two equations

$$(6.1) \quad 2(\alpha \beta + \beta \gamma + \gamma \alpha) = a^2 + b^2 + c^2, \quad 4(\alpha \beta \gamma)^2 = a^2 b^2 c^2.$$

and to ensure that $\phi$ is semisimple. If $a, b, c$ are not distinct, say $a = b$, we can take $\alpha = \beta = \gamma = a$. For semisimplicity of $\phi$ in this case see the end of the previous section.

From now on we assume that $a, b, c$ are distinct. If $a^2 + b^2 + c^2 = 0$ we take $\beta = \alpha \zeta$ and $\gamma = \alpha \zeta^2$; where $\zeta = e^{2\pi i/3}$, and we choose $\alpha$ such that $-2\alpha^3 = abc$. Otherwise we take $\beta = -\alpha, \gamma = \frac{abc}{2\alpha^2}$ and choose $\alpha$ such that $-2\alpha^2 = a^2 + b^2 + c^2$. The equations (6.1) imply that $\bar{R}_\phi$ and
have the same characteristic polynomial. Since the nonzero eigenvalues of $\tilde{R}_\phi$, i.e. $\pm ia, \pm ib, \pm ic$, are distinct, one can verify easily that $\tilde{R}_\phi$ is semisimple. As $L$ and $M$ vanish on $\psi$ and $\phi$ and the equations (6.1) show that $H$ and $D$ also agree on $\psi$ and $\phi$, Theorem 3.3 implies that $\psi$ and $\phi$ are in the same $\text{SL}_{\text{loc}}$-orbit. Hence rank $\psi = \text{rank} \phi = 3$.

**Family 2.** If $\psi \not\in \tilde{S}_3$ then rank $\psi = 4$. Otherwise we must have $L = M = 0$, i.e., $abc = (a - b)((a + b)^2 - 4c^2) = 0$. As we may switch $a$ and $b$ and multiply $a, b, c$ by $\pm 1$, there are only three cases to consider:

(i) $a = b, c = 0$;
(ii) $a = b = 0, c \neq 0$;
(iii) $b = 0, a = -2c \neq 0$.

In case (i) we have

$$4\psi = (e_0 - e_1) \otimes (e_0 + e_1) \otimes (e_0 + e_1) \otimes ((a - 2i)e_0 + (a + 2i)e_1)$$
$$+ a(e_0 - e_1) \otimes (e_0 - e_1) \otimes (e_0 - e_1) \otimes (e_0 - e_1)$$
$$+ 2ae_1 \otimes (e_0 + e_1) \otimes (e_0 + e_1) \otimes (e_0 + e_1),$$

and in case (ii)

$$4\psi = c(e_0 + e_1) \otimes (e_0 - e_1) \otimes (e_0 + e_1) \otimes (e_0 - e_1)$$
$$+ c(e_0 - e_1) \otimes (e_0 + e_1) \otimes (e_0 - e_1) \otimes (e_0 + e_1)$$
$$- 2i(e_0 - e_1) \otimes (e_0 + e_1) \otimes (e_0 + e_1) \otimes (e_0 - e_1).$$

Clearly if $a = 0$ in case (i) then rank $\psi = 1$. Otherwise in cases (i) and (ii) it is easy to verify that $\psi \not\in \tilde{S}_2$ and so rank $\psi = 3$. Now we consider case (iii). Let $\phi \in \mathcal{H}$ be of type $3(a)$ with $\alpha = \frac{\xi}{3}, \beta = ic\sqrt{3}$ and $\gamma = -\beta$. Then the matrices $R_\psi$ and $PR_\phi$, where $P$ is the diagonal matrix with $(-1, 1, 1, 1)$ diagonal entries, are symmetric and have the same Jordan structure: $0, -2c, J_2(c)$. Hence they are orthogonally similar. This shows that $R_\psi$ and $R_\phi$ are in the same $O_4 \times O_4$-orbit. Hence $\psi$ and $\phi$ are in the same $\text{SL}_{\text{loc}}$-orbit and so rank $\psi = \text{rank} \phi = 3$.

**Family 3.** If $a^2 \neq b^2$ then $\psi \not\in \tilde{S}_3$ and rank $\psi = 4$. Since we can interchange $a$ and $b$ and replace them by $\pm a$ and $\pm b$, we may assume that $a = b$. If $a = 0$ then

$$2\psi = (e_0 + e_1) \otimes (e_0 - e_1) \otimes (e_1 \otimes e_0 - e_0 \otimes e_1)$$
and rank $\psi = 2$. If $a \neq 0$ then $\psi \not\in \bar{S}_2$ and

$$4a\psi = (e_0 + e_1) \otimes ((a^2 + 1)e_0 + (a^2 - 1)e_1) \otimes (e_0 + e_1) \otimes (e_0 + e_1)$$

$$- (e_0 + e_1) \otimes (e_0 - e_1) \otimes ((1 + a)e_0 + (1 - a)e_1)$$

$$\otimes ((1 - a)e_0 + (1 + a)e_1)$$

$$- 2a^2e_1 \otimes (e_0 - e_1) \otimes (e_0 - e_1) \otimes (e_0 - e_1),$$

and so rank $\psi = 3$.

**Family 6.** If $\psi \not\in S_3$ then rank $\psi = 4$. Otherwise $L = M = 0$ and by using Table 6 we have $a = b = 0$. Then $\psi$ is nilpotent and it is easy to verify that $\psi \in S_2$. Since $\psi$ is not factorizable, it cannot have rank 1 or 2 (see Proposition 5.1). The matrix $\tilde{R}_\psi$ has Jordan structure $0, 0, J_3(0), J_3(0)$ (see Table 1). Since this is absent from the rank 3 section of Table 5, we infer that rank $\psi \neq 3$. Thus rank $\psi = 4$.

**Family 9.** If $a \neq 0$ then $\psi \not\in \bar{S}_3$ and so rank $\psi = 4$. If $a = 0$ then

$$\psi = 2i(|1110\rangle - |0010\rangle + |1001\rangle)$$

and $\psi \not\in \bar{S}_2$ so rank $\psi = 3$.

**Family 10.** Permute qubits 1 and 2. (The effect on the matrix $R_\psi$ is just to change its $(3, 3)$–entry from 1 to $-1$.) If $a \neq 0$ then rank $T(1, 1) = 1$ and rank $T(1, 0) = 2$. Since $\psi \not\in \bar{S}_2$ we have that rank $\psi = 3$. If $a = 0$ then $\psi = (e_1 - e_0) \otimes \phi$. Since det $\phi = 0$ and $\phi$ is not factorizable, we have rank $\psi = \text{rank} \phi = 3$.

**Family 12.** We check that $\psi \not\in \bar{S}_2$ since the 1,1 minor of $\tilde{\psi}$ is nonzero. We have that rank $T(1, 0) = 1$ and rank $T(1, 1) = 2$. Hence rank $\psi = 3$. A computation gives

$$(1 - i)\sqrt{2}\psi = (1 - i)\sqrt{2}(e_0 - e_1) \otimes e_1 \otimes (e_0 - ie_1) \otimes (-ie_0 + e_1)$$

$$- e_1 \otimes (e_0 - ie_1) \otimes (e_0 + \beta e_1) \otimes (e_0 + \alpha e_1)$$

$$- e_1 \otimes (e_0 + ie_1) \otimes (-e_0 + \alpha e_1) \otimes (e_0 - \beta e_1),$$

where $\alpha = \sqrt{2} + 1$ and $\beta = \sqrt{2} - 1$.

**Family 14.** It is again straightforward to verify that $\psi \not\in \bar{S}_2$. Then using that rank $T(i, 1) = 1$ and rank $T(1, 0) = 2$, we obtain that rank $\psi = 3$.

**Family 16.** In this case $\psi = (e_0 + e_1) \otimes \phi$. Since det $\phi \neq 0$ we have rank $\psi = \text{rank} \phi = 2$. 
After all these computations it is worthwhile observing that $S_3$ contains only one $\text{SL}_{10}^* – \text{orbit of rank 4 tensors}$. This exceptional orbit is the unique nilpotent orbit of family 6 (with $a = b = 0$). It is the orbit of the generalized $W$ state given by (4.2) and it is contained in $\bar{S}_2$. In particular we have $\bar{S}_2 \neq S_2$ and $\bar{S}_3 \neq S_3$.

7. Tensor rank algorithm

By using the results of the previous section, we can now construct a simple algorithm for computing the tensor rank. We have explained in the previous section how to test a state $\psi$ for factorization. If one of the factors is from a single $H_k$, we may use density matrices. For a state $\psi \in H$ let $\rho = \langle \psi | \psi \rangle$ be its density matrix. Denote by $\rho_k$ its reduced density matrix obtained by tracing out all qubits but the $k$-th one (for the definition of the density matrices and partial trace see e.g. [27]). Then $\psi$ factorizes, with one of the factors in $H_k$, iff the matrix $\rho_k$ has rank 1, so we let $r_k$ be the matrix rank of $\rho_k$. With an abuse of notation, we let $r_k$ be the rank of the corresponding $\rho_k$ for 3-qubit tensors as well.

We now give our algorithm for computing the tensor rank of an arbitrary state $\psi \in H$. The algorithm uses another procedure which computes the tensor rank of 3–qubit states, which can be deduced from [3]. It should be understood that the algorithms halt as soon as the rank is returned. Recall the definition of the hyperdeterminant $\text{det} \psi$ for 3–qubit pure states $\psi$ given in the previous section.

3–Qubit Tensor Rank Algorithm.

Input: A nonzero tensor $\psi \in H_{123}$
Output: The tensor rank of $\psi$

1 If $\text{det} \psi$ is nonzero then return 2.
2 Compute the ranks $r_k$ of $\rho_k$ for $k \in \{1, 2, 3\}$.
3 If $r_k = 1$ for at least two different $k$ then return 1.
4 If some $r_k = 1$ then return 2.
5 Return 3.

With this we may compute ranks of 4–qubit tensors.

4-Qubit Tensor Rank Algorithm.

Input: A nonzero tensor $\psi \in H$
Output: The tensor rank of $\psi$

1 If $L(\psi)$ or $M(\psi)$ is nonzero then return 4.
2 If at least one of the forty-eight $3 \times 3$ minors of the matrices $\tilde{\psi}$, $\tilde{\psi}'$, $\tilde{\psi}''$ is nonzero then return 3.

3 Compute the ranks $r_k$ of $\rho_k$.

4 If say $r_1 = 1$, then $\psi = v_1 \otimes \phi$ with $v_1 \in \mathcal{H}_1$ and $\phi \in \mathcal{H}_{234}$, and return rank $\phi$.

5 Now all $r_k = 2$. If $\psi$ is nilpotent, i.e., $\tilde{R}_\psi$ is nilpotent, then return 4.

6 Return 2.

Let us show that the algorithm is correct. It may be helpful to look at Figure 4 where some of the sets we use below are exhibited. Step 1 is clear. In order to justify step 2, it suffices to verify that $$\psi \in \bar{S}_3 \setminus \bar{S}_2 \Rightarrow \text{rank } \psi = 3.$$ This follows from the case-by-case analysis of the previous section.

After reaching Step 3, we have $\psi \in \bar{S}_2$. Consequently, the families 7, 8, 9, 12, 13, 14 and 15 are ruled out, i.e., $\psi$ does not belong to any of them. Indeed it is easy to verify that none of these families meets $\bar{S}_2$.

Steps 3 and 4 are also clear.

After reaching step 5 our $\psi$ is non-factorizable and we can rule out the families 2, 3, 4, 5, 10, 11, 16 and 17. Indeed the families 16 and 17 are factorizable and the orbits in the families 2, 3, 4, 5, 10 and 11 which are contained in $\bar{S}_2$ are also factorizable. Hence $\psi$ belongs to the family 1 or 6. If it is nilpotent, it is in family 6 and has rank 4. Otherwise it is in family 1 and the detailed analysis of this case in the previous section shows that the rank of $\psi$ is 2.

Figure 4 describes the structure of $\mathcal{H}$ with respect to tensor ranks. Each vertex represents a Zariski closed set and it is ordered by inclusion as one progresses to the top vertex. All sets on or below the horizontal line satisfy the equation $H = 0$ and consist of nilpotent orbits. However there exist nilpotent orbits not contained in $\bar{S}_2$. The numbers between pairs of adjacent vertices indicate the rank of the tensors that are in the set theoretic difference between the higher and lower vertices. The numbers on the far left indicate the dimension of the corresponding affine variety.

8. Conclusion

In this paper we have investigated the SLOCC classification of pure states of four qubits first described by Verstraete et al. in [31]. The families of representatives provided in that paper were accurate except for some possible misprints in the family $L_{ab_3}$. However their claim of
uniqueness in Theorem 1 is not true and the subsequent proof was not easy to follow.

We have provided a more general version of that theorem in our Theorem 2.8. We presented this theorem within the framework of orthogonal representations of a certain quiver $Q$. We felt that this approach lead to a simpler proof of Theorem 2.8. We also observed that this theorem can be deduced (with some additional work) from the theory of symmetric quivers as presented in a recent paper of Derksen and Weyman [7].

We found it beneficial to embed the 4–qubit Hilbert space $\mathcal{H}$ into the Lie algebra $\mathfrak{g}$ of the complex orthogonal group $O_8$. This naturally lead to the notion of semisimple and nilpotent states.
states are dense in $\mathcal{H}$ while the nilpotent ones comprise only finitely many $\text{SL}_{\text{loc}}^*$-orbits. The subgroup $O_4 \times O_4 \subset O_8$ acts naturally on $\mathcal{H}$.

We have also provided a more complete description of the behaviour of the $O_4 \times O_4$-orbits under permutations of qubits. This was addressed in [31] as well, but an important characteristic was not stressed. Namely that the action of a permutation of qubits on a family of orbits may map some orbits into a different family while at the same time map others back into itself. So in general, a permutation of qubits does not induce a permutation of families of orbits as naivete would have one think.

The problem of showing that two states $\phi, \psi \in \mathcal{H}$ are not $\text{SL}_{\text{loc}}^*$-equivalent appears in a number of recent papers [2, 33]. We derived polynomial invariants for $\text{SL}_{\text{loc}}^*$ and show that two semisimple states are $\text{SL}_{\text{loc}}^*$-equivalent iff they agree on the invariants. In 1852, these same invariants were considered by Schl"afli who noted their invariance under permutations of indices. The general case is somewhat more complicated as it requires computing the Jordan structure of associated matrices $\tilde{R}$, and possibly the use of $ab$-diagrams.

The other focus of this paper was to ultimately develop an algorithm that would calculate the tensor rank for a nonzero tensor in $\mathcal{H}$. This was accomplished by a thorough examination of each of the families in the $\text{SL}_{\text{loc}}^*$ classification. To carry out this analysis we used the results of Brylinski in [3]. In particular it was essential to know that the maximum rank of a tensor in $\mathcal{H}$ is 4 and that the polynomial $\text{SL}_{\text{loc}}$ invariants $L$ and $M$ define the Zariski closure of the tensors of rank $\leq 3$. We found that another set of 48 equations define the Zariski closure of the tensors of rank $\leq 2$ based on the speculation by Brylinski in [3]. It was also fortunate that the tensors of rank $\leq 3$ admitted a simple classification which allowed us to deduce the ranks of some other tensors in certain cases. We then were able to construct the algorithm which is pleasantly simple compared to the machinery involved in the analysis mentioned above.

The authors of [31] claim to have solved the problem of equivalence of two states under the group $U_{\text{loc}}$ of local unitary operations. For that purpose they propose a normal form via a two step procedure. Apparently they failed to observe that the second step of their procedure may undo the beneficial effect of the first step. It is unclear how their proposed normal form is actually defined. Since $U_{\text{loc}}$ is a compact group, this equivalence problem can be solved by computing the algebra of real polynomial invariants $\mathcal{H} \rightarrow \mathbb{R}$ for $U_{\text{loc}}$. Indeed it is known in general that these invariants separate $U_{\text{loc}}$-orbits. However, a set of generators
for this algebra of invariants has not been computed so far although its Poincaré series has been computed independently in [32] and [20]. The problem of local unitary equivalence is also considered in [22] but the results are far from conclusive. Hence this problem remains open.

Appendix A

The following table gives the representatives $\psi$ of the nine families of $\text{SL}_{\text{loc}}^*$–orbits of non-normalized pure states of four qubits. They are derived from the corresponding $R$–matrices given in Table 1 by the transformation $R_\psi \rightarrow \tilde{\psi} = T^{-1}R_\psi T$. We point out that our $R$–matrices for these families are chosen to be as simple as possible, consequently the expressions for the corresponding states $\psi$ are not. The representations given in [31] are in some cases shorter than our representations, e.g. for family 16 their representative is

$$|0\rangle \otimes (|000\rangle + |111\rangle).$$

We recall that $\text{SL}_{\text{loc}}^*$–orbits originating from different families are necessarily distinct, and that two states in the same family may be in the same $\text{SL}_{\text{loc}}^*$–orbit.
Table 7. Representatives $\psi$ of $\text{SL}_{\text{loc}}^*$--orbits

1. \[
\frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{3}(|0011\rangle + |1100\rangle) \\
+ \frac{b+c+i}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle)
\]

2. \[
\frac{a+c-i}{2}(|0000\rangle + |1111\rangle) + \frac{a-c+i}{2}(|0011\rangle + |1100\rangle) \\
+ \frac{b+c+i}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle) \\
+ \frac{1}{2}(|0001\rangle + |0111\rangle + |1000\rangle + |1110\rangle) \\
- |0010\rangle - |0100\rangle - |1011\rangle - |1101\rangle)
\]

3. \[
\frac{a}{2}(|0000\rangle + |1111\rangle + |0011\rangle + |1100\rangle) + \frac{b+1}{2}(|0101\rangle + |1010\rangle) \\
+ \frac{b-1}{2}(|0110\rangle + |1001\rangle) + \frac{1}{2}(|1101\rangle + |0010\rangle - |0001\rangle - |1110\rangle)
\]

6. \[
\frac{a+b}{2}(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) + i(|1001\rangle - |0110\rangle) \\
+ \frac{a-b}{2}(|0011\rangle + |1100\rangle) + \frac{1}{2}(|0001\rangle + |0110\rangle + |1001\rangle + |1101\rangle) \\
- |0001\rangle - |0111\rangle - |1000\rangle - |1110\rangle)
\]

9. \[
a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) \\
- 2i(|0100\rangle - |1001\rangle - |1110\rangle)
\]

10. \[
\frac{a+i}{2}(|0000\rangle + |1111\rangle + |0011\rangle + |1100\rangle) + \frac{a-i+1}{2}(|0101\rangle + |1010\rangle) \\
+ \frac{a-i-1}{2}(|0110\rangle + |1001\rangle) + \frac{i+1}{2}(|1101\rangle + |0010\rangle) \\
+ \frac{1}{2}(|0001\rangle + |1110\rangle) - \frac{1}{2}(|0100\rangle + |0111\rangle + |1000\rangle + |1111\rangle)
\]

12. \[
(|0101\rangle - |0110\rangle + |1100\rangle + |1111\rangle) + (i + 1)(|1001\rangle + |1010\rangle) \\
- i(|0100\rangle + |0111\rangle + |1101\rangle - |1110\rangle)
\]

14. \[
\frac{i+1}{2}(|0000\rangle + |1111\rangle - |0010\rangle - |1101\rangle) \\
+ \frac{i-1}{2}(|0001\rangle + |1110\rangle - |0011\rangle - |1100\rangle) \\
+ \frac{1}{2}(|0100\rangle + |1001\rangle + |1010\rangle + |0111\rangle) \\
+ \frac{1-2i}{2}(|1000\rangle + |0101\rangle + |0110\rangle + |1011\rangle)
\]

16. \[
\frac{1}{2}(|0\rangle + |1\rangle) \otimes (|000\rangle + |011\rangle + |100\rangle + |111\rangle) \\
+ i(|001\rangle + |010\rangle - |101\rangle - |110\rangle)
\]

Appendix B

To simplify the notation, we shall use the same symbols to denote the polynomials in $\mathcal{A}$ or $\mathcal{A}^*$ and their restrictions to the Cartan subspace $\mathfrak{a}$. As a byproduct of our construction of generators of
the algebra $A^*$, we have obtained a nice set of generators of the algebra of polynomial invariants of the Weyl group of type $F_4$. Their degrees are, of course, 2, 6, 8 and 12. The invariant of degree 12 has the factorization $Π = (L - M)(M - N)(N - L)$ and the one of degree 8 is a sum of three squares $Σ = L^2 + M^2 + N^2$ where

$L = abcd, \\
M = -\frac{1}{16}(4(ad - bc)^2 - (a^2 - b^2 - c^2 + d^2)^2) \\
N = -\frac{1}{16}(4(ac + bd)^2 - (a^2 - b^2 + c^2 - d^2)^2) \\
= \frac{1}{16}((a + b)^2 - (c - d)^2)((a - b)^2 - (c + d)^2)$.

The known sets of generators which we could find in the literature [15, 24, 28] do not share these special features.

Table 8. Generators $H$, $Γ$, $Σ$, $Π$ of invariants of the Weyl group of type $F_4$

$2H = a^2 + b^2 + c^2 + d^2$

$2^5Γ = 2(a^6 + b^6 + c^6 + d^6) - (a^2 + b^2 + c^2 + d^2)(a^4 + b^4 + c^4 + d^4) + 18(a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2)$

$2^7Σ = 2^8a^2b^2c^2d^2 + (4(ad - bc)^2 - (a^2 - b^2 - c^2 + d^2)^2) \\
(4(ac + bd)^2 - (a^2 - b^2 + c^2 - d^2)^2)$

$2^{12}Π = (16abcd - 4(ad - bc)^2 + (a^2 - b^2 - c^2 + d^2)^2) \\
(4(ad - bc)^2 - (a^2 - b^2 - c^2 + d^2)^2) \\
+4(ac + bd)^2 - (a^2 - b^2 + c^2 - d^2)^2) \\
(-4(ac + bd)^2 + (a^2 - b^2 + c^2 - d^2)^2 - 16abcd)$

The generators $I_2$, $I_6$, $I_8$ and $I_{12}$ found in [28] relate to our generators as follows:

$I_2 = 12H, \\
I_6 = 72H^3 - 96Γ, \\
I_8 = 264H^4 - 832ΓH + 320Σ, \\
I_{12} = 4104H^6 - 24096H^3Γ + 17440H^2Σ + 3904Γ^2 - 3840Π.$
REFERENCES

[1] E. Briand, J.-G. Luque and J.-Y. Thibon, A complete set of covariants of the four qubit system, *J. Phys. A.: Math. Gen.* **38** (2003), 9915-9927.

[2] H.J. Briegel and R. Raussendorf, Persistent entanglement in arrays of interacting particles, *Phys. Rev. Lett.* **86**, 910 (2000).

[3] J.-L. Brylinski, Algebraic measures of entanglement, *Chapter I in Mathematics of Quantum Computation*, Chapman and Hall/CRC, 2002, pp. 3-23.

[4] P. Bürgisser, M. Clausen and M. A. Shokrollahi, Algebraic Complexity Theory, Springer, 1997.

[5] D. Choudhury and R. A. Horn, A complex orthogonal-symmetric analog of the polar decomposition, *SIAM J. Algebraic Discrete Methods* **8** (1987), 219-225.

[6] V. Coffman, J. Kundu and W. K. Wootters, Distributed entanglement, *Phys. Rev. A* **67**, 052306 (2000).

[7] H. Derksen and J. Weyman, Generalized quivers associated to reductive groups, *Colloquium Mathematicum* **94** (2002), 151-173.

[8] D.Ž. Doković, N. Lemire and J. Sekiguchi, The closure ordering of adjoint nilpotent orbits in $so(p,q)$, *Tohoku Math. J.* **53** (2001), 395-442.

[9] D.Ž. Đoković and K. Zhao, Tridiagonal normal forms for orthogonal similarity classes of symmetric matrices, *Linear Algebra and Its Applications* **384** (2004) 77–84.

[10] W. Dürr, G. Vidal and J.I. Cirac, Three qubits can be entangled in two ways, *Phys. Rev. A* **62**, 062314 (2000).

[11] F. R. Gantmacher, *Theory of Matrices*, Vol. 2, Chelsea Publishing Company, 1959.

[12] [GAP] The GAP Group, *GAP — Groups, Algorithms, and Programming*, Version 4.4.9: 2006 [http://www.gap-system.org](http://www.gap-system.org).

[13] [GPS03] G.-M. Greuel, G. Pfister, and H. Schönenmann. *SINGULAR 2.05*. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern (2003). [http://www.singular.uni-kl.de](http://www.singular.uni-kl.de).

[14] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, 1990.

[15] V.F. Ignatenko, Some questions in the geometric theory of invariants of groups generated by orthogonal and oblique reflections, *J. Soviet Math.* **33** (1986), 933-953.

[16] I. Kaplansky, Algebraic polar decomposition, *SIAM Journal on Matrix Analysis and Applications* **11** (1990), 213-217.

[17] H. Kraft and C. Procesi, Closures of conjugacy classes of matrices are normal, *Inventiones Math.* **53** (1979), 227-247.

[18] L. Lamata, J. León, D. Salgado and E. Solano, Inductive entanglement classification of four qubits under SLOCC, *arXiv: quant-ph/0602333 v.1*, 27 Oct 2006.

[19] J.-G. Luque and J.-Y. Thibon, Polynomial invariants of four qubits, *Phys. Rev. A* **67**, 042303 (2003).

[20] J.-G. Luque, J.-Y. Thibon and F. Toumazet, Unitary invariants of qubit systems, *arXiv: quant-ph/0604202 v1* 27 Apr 2006.
[21] Y. Makhlin, Nonlocal properties of two-qubit gates and mixed states, and optimization of quantum computation, *Quantum Information Processing* 1 (2002), 243-252.

[22] A. Mandilara, V.M. Akulin, A.V. Smilga and L. Viola, Quantum entanglement via nilpotent polynomials, *arXiv: quant-ph/0508232* v2 15 Jan 2006.

[23] Maplesoft, Waterloo Maple Inc., Maple 9.01, 2003.

[24] M. L. Mehta, Basic sets of invariant polynomials for finite reflection groups, *Comm. Algebra* 16 (1988), 1083-1098.

[25] T. Ohta, The closures of nilpotent orbits in the classical symmetric pairs and their singularities, *Tohoku Math. J.* 43 (1991), 161-211.

[26] A.O. Pittenger, *An Introduction to Quantum Computing Algorithms*, Birkhäuser, Boston, 2001.

[27] J. Preskill, Lectures Notes for Physics 229: Quantum Information and Computation, California Institute of Technology, 1998.

[28] K. Suito, T. Yano and J. Sekiguchi, On a certain generator system of the ring of invariants of a finite reflection group, *Comm. Algebra* 8 (1980), 373-408.

[29] L. Schl"afli, "Uber die Resultante eines Systemes mehrerer algebraischer Gleichungen. Ein Beitrag zur Theorie der Elimination. Denkschriften der Kaiserlichen Akademie Wissenschaften, mathematisch-naturwissenschaftliche Klasse* 4* (1852), Wien. Verzeichnis Grad, Nr. 24. Reprinted in Gesammelte mathematische Abhandlungen*, vol. 2, Birkhäuser, Basel, 1950-1956.

[30] P. Tauvel and R.W.T. Yu, *Lie Algebras and Algebraic Groups*, Springer, 2005.

[31] F. Verstraete, J. Dehaene, B. De Moor and H. Verschelde, Four qubits can be entangled in nine different ways, *Phys. Rev. A* 65, 052112 (2002).

[32] N.R. Wallach, The Hilbert series of measures of entanglement for 4 qubits, *Acta Applicandae Mathematicae* 86 (2005), 203-220.

[33] C. Wu, Y. Yeo, L.C. Kwek and C.H. Oh, Quantum nonlocality of four-qubit entangled states, *arXiv: quant-ph/0611172* v1 16 Nov 2006.

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