A VLASOV-POISSON PLASMA OF INFINITE MASS WITH A
POINT CHARGE

GANG LI AND XIANWEN ZHANG∗

School of Mathematics and Statistics
Huazhong University of Science and Technology
Wuhan 430074, China

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ABSTRACT. We study the time evolution of the three dimensional Vlasov-
Poisson plasma interacting with a positive point charge in the case of infi-
nite mass. We prove the existence and uniqueness of the classical solution to
the system by assuming that the initial density slightly decays in space, but
not integrable. This result extends a previous theorem for Yukawa potential
obtained in [10] to the case of Coulomb interaction.

1. Introduction. In this paper, we study the time evolution of the Vlasov-
Poison plasma interacting with a positive point charge in the case of infinite mass in $\mathbb{R}^3$.
We denote by $f(t, x, v)$ the distribution of the particles in the plasma at time $t \geq 0$
and position $x \in \mathbb{R}^3$, moving with velocity $v \in \mathbb{R}^3$, then the spatial density is defined
by $\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v) dv$ and the time evolution of the plasma-charge system is
governed by the following Vlasov-Poisson type system [9]:

\[
\begin{cases}
\partial_t f + v \cdot \nabla_x f + (E + F) \cdot \nabla_v f = 0, \\
E(t, x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho(t, y) dy, \\
F(t, x) = \frac{x-\xi(t)}{|x-\xi(t)|^3}, \\
\dot{\xi}(t) = \eta(t), \quad \dot{\eta}(t) = E(t, \xi(t)), \\
f(0, x, v) = f_0(x, v), \quad (\xi(0), \eta(0)) = (\xi_0, \eta_0).
\end{cases}
\]

Here, $E(t, x)$ indicates the self-consistent electrostatic field, which is induced by the
positively charged particles of the plasma. The position and velocity at time $t$ of the
point charge is denoted by $\xi(t)$ and $\eta(t)$ respectively. $F(t, x)$ is the Coulomb’s force
field, which models the repulsive interactions between the plasma and the positive
point charge.

When there is no point charge, the system (1) becomes the classical Vlasov-
Poison system which has been extensively studied (see, e.g.: [1, 2, 15, 16, 18, 20,
21, 23, 25, 26, 27, 28, 32, 33, 34]). Global solution for two dimensional problem was
constructed in [23, 32]. Local existence and uniqueness of classical solutions were
established in [1, 20]. Almost simultaneously, two different approaches to prove
global existence for general data were independently given in [21] and [25]. A full

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∗ Corresponding author: Xianwen Zhang.
discussion of the literature about the classical Vlasov-Poisson system before 2007 was presented in [18, 27].

When considering the Vlasov-Poisson system with infinite mass, we can refer to [3, 4, 5, 6, 7, 8, 12, 19, 24, 29, 30, 31]. In [3], by ignoring the singularities in the interaction potential, which is assumed to be positive, bounded and short-range in $\mathbb{R}^3$, the authors obtained the existence and uniqueness of the solution to the system. In [12], the Yukawa potential was considered to deal with the well-posed problem with infinite charge in $\mathbb{R}^2$. Moreover, the problem that the plasma of infinite mass is confined by an external magnetic field has been studied in [4, 5] for the Yukawa and Coulomb potentials respectively. When considering the plasma with charged particles mutually interacting via the Coulomb force in $\mathbb{R}^3$, the authors of [6] proved the existence and uniqueness of the solution to the system under the assumption that the initial density is slowly decaying at infinity. Recently, the Vlasov-Poisson plasma with infinite mass and velocities confined in a cylinder and in the whole space respectively has been studied in [7, 8].

The study of the system (1) is relatively few. For attractive case, the authors of [11] found a convenient condition on support of the initial density $f_0$ of the plasma, under which they succeed in establishing a global-in-time existence of solutions in two dimensional case. Global existence of weak solutions in three space variables was obtained in [14] by the theory of Diperna-Lions flow and compactness argument. For repulsive case, the first global existence and uniqueness result for compactly supported classical solution was established in [9] in two dimensional case, which was then extended to the case of three space variables [22]. Recently, higher order velocity moments were shown to be propagated by weak solutions [17]. This result is important and extends partially the Lions-Perthame theory [21] for the classical Vlasov-Poisson system to the case of the Vlasov-Poisson plasma interacting with a positive point charge (see also [13] for a better upper bound).

We stress that the initial density is assumed to belong to $L^\infty \cap L^1$ in the papers above. In the present paper, we consider the case that the initial density is not in $L^1$. In fact, in [10] the assumption of the initial density having spatial compact support is removed and it is only assumed that the initial density is bounded. However, it is only investigated that the mutual interaction of the charged particles of the plasma is Yukawa type in $\mathbb{R}^2$. There is no result about the three dimensional Vlasov equation with the Coulomb mutual interaction for the case that the plasma of the infinite charge distribution interacts with a positive point charge.

In this paper, we extend the result of [10] to the case that the charged particles of the plasma mutually interact by the Coulomb force in $\mathbb{R}^3$. To overcome the difficulty induced by infinite mass, we use two fundamental tools: one is the local energy introduced in [10], and the other is a new energy function $h(t, x, v)$, which consists of the pointwise energy for a plasma particle and the kinetic energy of the point charge.

We denote the position and velocity of the plasma particles beginning from $(x, v)$ at time $t = 0$ by $(X(t, 0, x, v), V(t, 0, x, v))$, the characteristic equations corresponding to the system (1) is given by

$$
\begin{align*}
\dot{X}(t, 0, x, v) &= V(t, 0, x, v), \\
\dot{V}(t, 0, x, v) &= E(t, X(t, 0, x, v)) + F(t, X(t, 0, x, v)), \\
(X(0, 0, x, v), V(0, 0, x, v)) &= (x, v),
\end{align*}
$$

and the evolution of the point charge is governed by
we have
\[
\begin{align*}
\dot{\xi}(t) &= \eta(t), \\
\dot{\eta}(t) &= E(t, \xi(t)), \\
(\xi(0), \eta(0)) &= (\xi_0, \eta_0).
\end{align*}
\] (3)
Along the characteristics, we have
\[
f(t, X(t, 0, x, v), V(t, 0, x, v)) = f_0(x, v),
\]
which implies that
\[
||f(t)||_{L^\infty} = ||f_0||_{L^\infty} = C_0,
\]
where $C_0$ is a constant.
We introduce the set
\[
S = \{(x, v) | |x - \xi_0| \geq \delta_0, |v| \leq V_0\},
\]
where $\delta_0$ and $V_0$ are given positive constants.
In the following, positive constants denoted by $C$, which may change from line to line, depend only on the initial data and the time $T$. Some constants indexed by the number 1, 2, 3 will be quoted in the sequel. Then, we give the main result of the paper.

**Theorem 1.1.** Let $f_0(x, v) \in C^1_b(\mathbb{R}^3 \times \mathbb{R}^3)$ be nonnegative and supported on $S$, $(\xi_0, \eta_0) \in \mathbb{R}^3 \times \mathbb{R}^3$. Assume that
\[
\int_{|\xi - \eta| \leq \epsilon} \rho_0(x)dx \leq \frac{C_1}{|\xi|^{2+\tau}}, \forall i \in \mathbb{Z}^3 : |i| \geq 1
\] (4)
for some fixed $\epsilon > \frac{1}{8}$ and $C_1 > 0$. Then, for arbitrary time $T > 0$, there exists a unique solution $(f(t, x, v), \xi(t), \eta(t))$ to (1) on $[0, T]$ such that:
\[
\sup\{|V(t, 0, x, v)| + \frac{1}{|X(t, 0, x, v) - \xi(t)|} | t \in [0, T], (x, v) \in \text{supp}f_0\} \leq C_2,
\] (5)
and
\[
\int_{|\xi - \eta| \leq \epsilon} \rho(t, x)dx \leq \frac{C_3}{|\xi|^{2+\tau}}, \forall i \in \mathbb{Z}^3 : |i| \geq 1, \ t \in [0, T],
\] (6)
where $C_2$ and $C_3$ are positive constants depending only upon $T$ and the initial data.
As pointed out above, we introduce the energy function
\[
h(t, x, v) = \frac{|v - \eta(t)|^2}{2} + \frac{1}{|x - \xi(t)|} + \frac{\eta(t)^2}{2} + k,
\]
where $k > 1$ is a suitably large positive constant which will be given in (24). Then, we have
\[
|v| \leq 2\sqrt{h}.
\]
Along the characteristic of the plasma particles, we obtain that
\[
\left|\frac{d}{dt} \sqrt{h}(t, X(t, 0, x, v), V(t, 0, x, v))\right|
\]
\[= \frac{1}{2\sqrt{h}} (V(t, 0, x, v) - \eta(t))(E(t, X(t, 0, x, v)) - E(t, \xi(t)) + 2\eta(t)\dot{\eta}(t))
\] (7)
\[\leq 2(|E(t, X(t, 0, x, v))| + |E(t, \xi(t))|)
\]
The maximal velocity and the maximal displacement of the plasma particles are respectively defined by
\[
V(t) = \max\{k, \sup_{s \in [0, t]} \sup_{(x, v) \in S} |V(s, 0, x, v)|, \sup_{s \in [0, t]} |\eta(s)|\}
\]
and 

\[ R(t) = 1 + \int_0^t V(s) \, ds. \]

2. The estimate of the local energy. To prove Theorem 1.1, we consider the following initial condition:

\[ f_0^N(x,v) = f_0(x,v) \chi_{\{|x| \leq N\}}(x), \]

where \( \chi_A(x) \) is the characteristic function of the set \( A \). And we introduce the cut-off system:

\[
\begin{aligned}
& \partial_t f_N + v \cdot \nabla_x f_N + (E_N + F_N) \cdot \nabla_v f_N = 0, \\
& E_N(t,x) = \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho(t,y) \, dy, \\
& F_N(t,x) = \frac{x-\xi_N(t)}{|x-\xi_N(t)|^3}, \\
& \rho_N(t,x) = \int_{\mathbb{R}^3} f_N(t,x,v) \, dv, \\
& \xi_N(t) = \eta_N(t), \quad \eta_N(t) = E_N(t,\xi_N(t)), \\
& f_N(0, x, v) = f_0^N(x,v), \quad (\xi_N(0), \eta_N(0)) = (\xi_0, \eta_0).
\end{aligned}
\]  

(8)

Then, the system (8) has a unique global solution \((f_N, \xi_N, \eta_N)\) for any fixed \( N \), which has been proven in [22].

The characteristic equations corresponding to the system (8) is given by

\[
\begin{aligned}
& \dot{X}_N(t,0,x,v) = V_N(t,0,x,v), \\
& \dot{V}_N(t,0,x,v) = E_N(t,X_N(t,0,x,v)) + F_N(t,X_N(t,0,x,v)), \\
& (X_N(0,0,x,v), V_N(0,0,x,v)) = (x,v).
\end{aligned}
\]  

(9)

and the evolution of the point charge is governed by

\[
\begin{aligned}
& \dot{\xi}_N(t) = \eta_N(t), \\
& \dot{\eta}_N(t) = E_N(t,\xi_N(t)), \\
& (\xi_N(0), \eta_N(0)) = (\xi_0, \eta_0).
\end{aligned}
\]  

(10)

Along the characteristics, we have

\[ f_N(t, X_N(t,0,x,v), V_N(t,0,x,v)) = f_0^N(x,v). \]

We set

\[ P_N(t) = \sup \{ \sqrt{h(t,X_N(t,0,x,v),V_N(t,0,x,v))} | t \in [0,T], (x,v) \in \text{supp} f_0^N \}. \]

For any given time interval \([0,T]\), the main purpose of the paper is to prove that \( P_N(t) \) is bounded uniformly in \( N \).

Now, we introduce the local energy, which has been used to deal with the unboundedness of the charged particles in \([3, 4, 5, 6, 7, 8, 10, 12]\). For any given \( \mu \in \mathbb{R}^3 \) and \( R > 0 \), we define the function

\[ \varphi^{\mu,R}(x) = \varphi \left( \frac{|x-\mu|}{R} \right), \]

where \( \varphi \in C^\infty(\mathbb{R}) \) such that

\[
\begin{aligned}
\varphi(r) &= 1 \text{ if } r \in [0,1], \\
\varphi(r) &= 0 \text{ if } r \in [2,\infty], \\
-2 &\leq \varphi'(r) \leq 0.
\end{aligned}
\]
In this paper, similar to [10], the local energy is defined as follows:

\[
W^N(\mu, R, t) = \frac{1}{2} \int \int \phi^{\mu,R}(x)|v|^2 f^N(t, x, v) dx dv \\
+ \frac{1}{2} \int \int \phi^{\mu,R}(x) \rho^N(t, x) dx \int \rho^N(t, y) G(|x - y|) dy \\
+ \int \phi^{\mu,R}(x) \rho^N(t, x) G(|x - \xi^N(t)|) dx \\
+ \frac{1}{2} \rho^{\mu,R}(\xi^N(t)) |\eta^N(t)|^2,
\]

where

\[
G(|x|) = \frac{1}{|x|}.
\]

Let us define

\[
Q^N(R, t) = \max\{1, \sup_{\mu \in \mathbb{R}^3} W^N(\mu, R, t)\}.
\]

Furthermore, for any fixed \(N\), the maximal velocity and the maximal displacement corresponding to the cut-off system (8) are respectively defined by

\[
V^N(t) = \max\{k, \sup_{s \in [0, t]} \sup_{(x,v) \in S} |V^N(s, 0, x, v)|, \sup_{s \in [0, t]} |\eta^N(s)|\}
\]

and

\[
R^N(t) = 1 + \int_0^t V^N(s) ds.
\]

For the local energy, we have the following fundamental estimate,

**Proposition 1.** In the hypotheses of Theorem 1.1, there exists a positive constant \(C\), independent of \(N\), such that

\[
Q^N(R^N(t), t) \leq CR^N(t)^{1-\varepsilon}.
\]

The proof of Proposition 1 can be deduced by Lemma 2.1 and Lemma 2.2.

**Lemma 2.1.** There exists a positive constant \(C\), independent of \(N\), such that

\[
Q^N(R^N(t), t) \leq CQ^N(2R^N(t), 0).
\]

**Proof.** Firstly, we set

\[
R^N(t, s) = R^N(t) + \int_s^t V^N(\tau) d\tau.
\]

Then, by the definition of the \(R^N(t)\), we get

\[
R^N(t, 0) = R^N(t) + \int_0^t V^N(\tau) d\tau \leq 2R^N(t).
\]

According to the definition of the local energy, we have

\[
W^N(\mu, R^N(t, s), s) = \frac{1}{2} \int \int \phi^{\mu,R^N(t,s)}(x)|v|^2 f^N(s, x, v) dx dv \\
+ \frac{1}{2} \int \int \phi^{\mu,R^N(t,s)}(x) \rho^N(s, x) dx \int \rho^N(s, y) G(|x - y|) dy \\
+ \int \phi^{\mu,R^N(t,s)}(x) \rho^N(s, x) G(|x - \xi^N(s)|) dx \\
+ \frac{1}{2} \rho^{\mu,R^N(t,s)}(\xi^N(s)) |\eta^N(s)|^2.
\]
We set
\[(X^N(s), V^N(s)) = (X^N(s, 0, x, v), V^N(s, 0, x, v)),\]
and
\[(Y^N(s), W^N(s)) = (Y^N(s, 0, y, w), W^N(s, 0, y, w)).\]

By the change of variables \((x, v) \rightarrow (X^N(s), V^N(s))\) and \((y, w) \rightarrow (Y^N(s), W^N(s))\), we obtain that
\[
W^N(\mu, R^N(t, s), s) = \frac{1}{2} \int \int \varphi^{\mu, R^N(t, s)}(X^N(s)) f_0^N(x, v) dx dv \int \int f_0^N(y, w) G(|X^N| - Y^N(s)) dy dw
+ \frac{1}{2} \int \int \varphi^{\mu, R^N(t, s)}(X^N(s)) |V^N(s)|^2 f_0^N(x, v) dx dv
+ \int \int \varphi^{\mu, R^N(t, s)}(X^N(s)) f_0^N(x, v) G(|X^N - \xi^N(s)|) dx dv
+ \frac{1}{2} \varphi^{\mu, R^N(t, s)}(\xi^N(s)) |\eta^N(s)|^2.
\]

The time derivative of \(W^N(\mu, R^N(t, s), s)\) can be divided into three parts [10]:
\[
\partial_s W^N(\mu, R^N(t, s), s) = A + B + C,
\]
where
\[
A = \int \int \varphi^{\mu, R^N(t, s)}(X^N(s)) f_0^N(x, v) \left[ \nabla V^N(s) \cdot \nabla (E^N(s, X^N(s))
+ \frac{1}{2} \int \int f_0^N(y, w) \nabla G(|X^N(s) - Y^N(s)|) \cdot (V^N(s) - W^N(s)) dy dw \right] dx dv,
\]
\[
B = \int \int \varphi^{\mu, R^N(t, s)}(X^N(s)) f_0^N(x, v) \left[ \nabla V^N(s) \cdot \nabla (F(s, X^N(s))
+ \nabla G(|X^N(s) - \xi^N(s)|) \cdot (V^N(s) - \eta^N(s)) \right] dx dv
+ \varphi^{\mu, R^N(t, s)}(\xi^N(s)) \eta^N(s) \cdot \nabla E^N(s, \xi^N(s)),
\]
and
\[
C = \int \int \partial_s \varphi^{\mu, R^N(t, s)}(X^N(s)) f_0^N(x, v) \left[ \frac{|V^N(s)|^2}{2} + G(|X^N(s) - \xi^N(s)|)
+ \frac{1}{2} \int \int f_0^N(y, w) G(|X^N(s) - Y^N(s)|) dy dw \right] dx dv
+ \frac{1}{2} \partial_s \varphi^{\mu, R^N(t, s)}(\xi^N(s)) |\eta^N(s)|^2.
\]

For the term \(C\), similar to [6, 10], we deduce that
\[
C \leq 0.
\]
For the term $\mathcal{A}$, the estimate has been established in [6], we repeat it here for the completeness of the proof, according to the system (8), we have

\[
\mathcal{A} = -\frac{1}{2} \int \int \varphi_{\mu,R^N(t,s)}(X^N(s)) f_0^N(x,v)dx dv \int \int f_0^N(y,w) \nabla G(|X^N(s) - Y^N(s)|) \cdot (V^N(s) + W^N(s)) dy dw
\]

\[= -\frac{1}{2} \int \int f_0^N(x,v)dx dv \int \int [\varphi_{\mu,R^N(t,s)}(X^N(s)) - \varphi_{\mu,R^N(t,s)}(Y^N(s))] f_0^N(y,w) \nabla G(|X^N(s) - \xi^N(s)|) \cdot V^N(s) dy dw,
\]

where we have used the change of the variables $(y,w) \to (x,v)$. By the mean value theorem and the definition of $\varphi_{\mu,R}$, we get

\[|\varphi_{\mu,R^N(t,s)}(X^N(s)) - \varphi_{\mu,R^N(t,s)}(Y^N(s))| \leq 2 \frac{|X^N(s) - Y^N(s)|}{R^N(t,s)}.
\]

Hence,

\[|\mathcal{A}| \leq \frac{V^N(s)}{R^N(t,s)} \int \int f_0^N(x,v)dx dv \int \int f_0^N(y,w) \left| \frac{\chi_{\{\mu - X^N(s) \leq 2R^N(t,s)\}}(x)}{X^N(s) - Y^N(s)} \right| dy dw
\]

\[+ \chi_{\{\mu - Y^N(s) \leq 2R^N(t,s)\}}(y) \right| dy dw = 2 \frac{V^N(s)}{R^N(t,s)} \int \int f_0^N(x,v)dx dv \int \int f_0^N(y,w) \left| \frac{\chi_{\{\mu - X^N(s) \leq 2R^N(t,s)\}}(x)}{X^N(s) - Y^N(s)} \right| dy dw.
\]

By the change of variables $(X^N(s), V^N(s)) \to (x, v)$ and $(Y^N(s), W^N(s)) \to (y, w)$, we have

\[|\mathcal{A}| \leq C \frac{V^N(s)}{R^N(t,s)} \int_{|\mu - x| \leq 2R^N(t,s)} \rho^N(s,x)dx \int \frac{\rho^N(s,y)}{|x - y|} dy.
\]

Denote $\mu_i = \mu + iR^N(t,s)$, we observe that

\[\{x : |\mu - x| \leq 2R^N(t,s)\} \subset \bigcup_{i \in \mathbb{Z}^3 : |i| \leq 2} \{x : |\mu_i - x| \leq R^N(t,s)\}.
\]

Then, by (11) and the definition of the local energy, we get

\[\int_{|\mu - x| \leq 2R^N(t,s)} \rho^N(s,x)dx \int \frac{\rho^N(s,y)}{|x - y|} dy
\]

\[\leq \sum_{i \in \mathbb{Z}^3 : |i| \leq 2} \int_{|\mu - x| \leq 2R^N(t,s)} \int \varphi_{\mu_i,R^N(t,s)}(x) \rho^N(s,x) \frac{\rho^N(s,y)}{|x - y|} dx dy
\]

\[\leq \sum_{i \in \mathbb{Z}^3 : |i| \leq 2} W^N(\mu_i, R^N(t,s), s) \leq CQ^N(R^N(t,s), s).
\]

Hence,

\[|\mathcal{A}| \leq C \frac{V^N(s)}{R^N(t,s)} Q^N(R^N(t,s), s).
\]
For the term $B$, according to the system (8), we have

\[
B = \int \int \varphi^{\mu,R}(t,s)(X^N(s)) f_0(x,v) \left[ V^N(s) \cdot F^N(s,X^N(s)) + \nabla G([X^N(s) - \xi^N(s)] \cdot (V^N(s) - \eta^N(s))] dx dv \\
+ \varphi^{\mu,R}(t,s)(\xi^N(s)) \eta^N(s) \cdot E^N(s,\xi^N(s)) \right] dx dv \\
= \int \int \left[ \varphi^{\mu,R}(t,s)(X^N(s)) - \varphi^{\mu,R}(t,s)(\xi^N(s)) \right] \eta^N(s) dx dv \\
\cdot \frac{X^N(s) - \xi^N(s)}{|X^N(s) - \xi^N(s)|} f_0(x,v) dx dv.
\]

Since

\[
|\varphi^{\mu,R}(t,s)(X^N(s)) - \varphi^{\mu,R}(t,s)(\xi^N(s))| \leq \frac{2}{R^N(t,s)} |X^N(s) - \xi^N(s)|,
\]

we get

\[
|B| \leq C \frac{V^N(s)}{R^N(t,s)} \int \int f_0(x,v) \frac{1}{|X^N(s) - \xi^N(s)|} \chi_{\{|\mu - X^N(s)| \leq 2R^N(t,s)\}}(x) dx dv \\
+ \frac{1}{|X^N(s) - \xi^N(s)|} \chi_{\{|\mu - \xi^N(s)| \leq 2R^N(t,s)\}}(x) dx dv.
\]

By the change of variables $(X^N(s), V^N(s)) \to (x, v)$, we obtain that

\[
|B| \leq C \frac{V^N(s)}{R^N(t,s)} \int \int \frac{\rho^N(s,x)}{|x - \xi^N(s)|} \chi_{\{|\mu - x| \leq 3R^N(t,s)\}}(x) dx \\
+ \int \frac{\rho^N(s,x)}{|x - \xi^N(s)|} \chi_{\{|\mu - x| \leq 5R^N(t,s) \cap |\mu - \xi^N(s)| \leq 2R^N(t,s)\}}(x) dx \\
+ \int \frac{\rho^N(s,x)}{|x - \xi^N(s)|} \chi_{\{|\mu - x| > 5R^N(t,s) \cap |\mu - \xi^N(s)| \leq 2R^N(t,s)\}}(x) dx.
\]

Due to

\[
|\xi^N(s) - x| \geq |\mu - x| - |\mu - \xi^N(s)| > 3R^N(t,s),
\]

we have

\[
|B| \leq C \frac{V^N(s)}{R^N(t,s)} \int \int \frac{\rho^N(s,x)}{|x - \xi^N(s)|} \chi_{\{|\mu - x| \leq 3R^N(t,s)\}}(x) dx \\
+ \int \frac{\rho^N(s,x)}{|x - \xi^N(s)|} \chi_{\{|\mu - x| \leq 5R^N(t,s)\}}(x) dx \\
+ \int \frac{\rho^N(s,x)}{|x - \xi^N(s)|} \chi_{\{|\xi^N(s) - x| > 3R^N(t,s)\}}(x) dx \\
\leq C \frac{V^N(s)}{R^N(t,s)} \int \int \frac{\rho^N(s,x)}{|x - \xi^N(s)|} \chi_{\{|\mu - x| \leq 5R^N(t,s)\}}(x) dx \\
+ \int \frac{\rho^N(s,x)}{|x - \xi^N(s)|} \chi_{\{|\xi^N(s) - x| > 3R^N(t,s)\}}(x) dx.
\]

Now, we estimate each term on the right hand side of the above inequality separately. Firstly, we have

\[
\int \frac{\rho^N(s,x)}{|x - \xi^N(s)|} \chi_{\{|\xi^N(s) - x| > 3R^N(t,s)\}}(x) dx \leq C.
\]
Indeed, by the change of variables \((x, v) \rightarrow (X^N(s), V^N(s))\), we obtain that
\[
\int \frac{\rho^N(s, x)}{|x - \xi^N(s)|} \chi(|\xi^N(s) - x| > 3R^N(t)) \left| \frac{f_0^N(x)}{\xi^N(s) - X^N(s)} \right| dx
\leq C \int \frac{f_0^N(x)}{|\xi^N(s) - X^N(s)|} dxdv
\leq C \int \frac{\rho_0^N(x)}{|\xi^N(s) - X^N(s)|} dx,
\]
where we have used the fact that
\[
|\xi^N(s) - X^N(s)| > |\xi^N(s) - x| - R^N(t) > \frac{|\xi^N(s) - x|}{2}.
\]
Then, we obtain that
\[
\int \frac{\rho^N(s, x)}{|x - \xi^N(s)|} \chi(|\xi^N(s) - x| > 3R^N(t,s)) \left| \frac{\rho_0^N(x)}{|\xi^N(s) - x|} \right| dx
\leq C \sum_{i \in \mathbb{Z}^3 : |i| \geq 1} \int_{|x - i| \leq 1} \frac{\rho_0^N(x)}{|\xi^N(s) - i|} dx,
\]
where we have used the fact that
\[
|\xi^N(s) - x| \geq |\xi^N(s) - i| - |i - x| \geq |\xi^N(s) - i| - R^N(t) \geq \frac{|\xi^N(s) - i|}{2}.
\]
By the assumption of (4),
\[
\int \frac{\rho^N(s, x)}{|x - \xi^N(s)|} \chi(|\xi^N(s) - x| > 3R^N(t,s)) \left| \frac{1}{|\xi^N(s) - i|^{2+\varepsilon}} \right| dx
\leq C \sum_{i \in \mathbb{Z}^3 : |i| \geq 1} \frac{1}{|\xi^N(s) - i|^{2+\varepsilon}}.
\]
If \(\{ i : |i| \leq |\xi^N(s) - i| \}\), then,
\[
\sum_{i \in \mathbb{Z}^3 : |i| \geq 1} \frac{1}{|\xi^N(s) - i|^{2+\varepsilon}} \leq \sum_{i \in \mathbb{Z}^3 : |i| \geq 1} \frac{1}{|i|^{3+\varepsilon}} \leq C.
\]
If \(\{ i : |i| > |\xi^N(s) - i| \}\), then,
\[
\sum_{i \in \mathbb{Z}^3 : |i| \geq 1} \frac{1}{|\xi^N(s) - i|^{2+\varepsilon}} \leq \sum_{i \in \mathbb{Z}^3 : |\xi^N(s) - i| \geq 1} \frac{1}{|\xi^N(s) - i|^{3+\varepsilon}} \leq C.
\]
Hence, (15) is proved. Secondly, similar to the proof of (2), we have
\[
\int \frac{\rho^N(s, x)}{|x - \xi^N(s)|} \chi(|x - s| \leq 5R^N(t,s)) \left( \frac{1}{|x - s|^2} \right) dx \leq CQ^N(R^N(t,s), s).
\]
Then, by (15), (16) and the fact that $Q^N(R^N(t, s), s) \geq 1$, we deduce that

$$|B| \leq C \frac{\mathcal{V}^N(s)}{R^N(t, s)} Q^N(R^N(t, s), s).$$

(17)

By (13), (14), (17), we obtain that

$$\partial_s W^N(\mu, R^N(t, s), s) \leq C \frac{\mathcal{V}^N(s)}{R^N(t, s)} Q^N(R^N(t, s), s).$$

Hence, we get

$$Q^N(R^N(t, s), s) \leq Q^N(R^N(t, 0), 0) + C \int_0^s \frac{\mathcal{V}^N(\tau)}{R^N(t, \tau)} Q^N(R^N(t, \tau), \tau)d\tau.$$ 

By the Gronwall lemma, we get

$$Q^N(R^N(t, s), s) \leq C Q^N(R^N(t, 0), 0) \leq C Q^N(2R^N(t), 0),$$

where we have used the fact that

$$\int_t^s \mathcal{V}^N(s) \frac{R^N(t, s)}{R^N(t, s)} = - \int_0^t \partial_s R^N(t, s) \leq \log 2,$$

which is implied by the estimate (12) and the proof is completed.

Lemma 2.2. In the hypotheses of Theorem 1.1, we have

$$Q^N(2R^N(t), 0) \leq CR^N(t)^{1-\epsilon}.$$ 

Proof. Due to assumptions on the initial data and the definition of $\varphi$, we have

$$W^N(\mu, 2R^N(t), 0) \leq \frac{V^N_0}{2} \int \int \varphi^{\mu, 2R^N(t)}(x) f^N_0(x, v) dx dv + \frac{1}{2} \eta^2_0$$ 

$$+ \frac{1}{2} \int \varphi^{\mu, 2R^N(t)}(x) \rho^N_0(x) dx \int \rho^N_0(y) G(|x-y|) dy$$ 

$$+ \frac{1}{\delta_0} \int \varphi^{\mu, 2R^N(t)}(x) \rho^N_0(x) dx$$ 

$$\leq C \left( 1 + \int \varphi^{\mu, 2R^N(t)}(x) \rho^N_0(x) \left[ 1 + \int \rho^N_0(y) G(|x-y|) dy \right] dx \right),$$

where $C$ is a constant only depending on $V_0, \delta_0$ and the initial data.

Firstly, by the hypotheses of Theorem 1.1, we get

$$\int \rho^N_0(y) G(|x-y|) dy \leq \int_{|x-y| \leq 2} \frac{\rho^N_0(y)}{|x-y|} dy + \int_{|x-y| > 2} \frac{\rho^N_0(y)}{|x-y|} dy$$ 

$$\leq C + \sum_{i \in \mathbb{Z}, |i| \geq 1} \int_{|x-y| \leq 1} \frac{\rho^N_0(y)}{|x-y|} dy.$$ 

Due to

$$|x-y| \geq |x-i| - |i-y| \geq \frac{|x-i|}{2},$$

we have

$$\int \rho^N_0(y) G(|x-y|) dy \leq \sum_{i \in \mathbb{Z}, |i| \geq 1} \int_{|x-y| \leq 1} \frac{\rho^N_0(y)}{|x-y|} dy.$$ 

Hence

$$\int \rho^N_0(y) G(|x-y|) dy \leq C + \sum_{i \in \mathbb{Z}, |i| \geq 1} \int_{|x-y| \leq 1} \frac{\rho^N_0(y)}{|x-y|} dy.$$ 

This completes the proof.
by the assumption of (4), we deduce that
\[
\int \rho_0^N(y)G(|x - y|)dy \leq C + C \sum_{i \in \mathbb{Z}^3, |i| \geq 1} \frac{1}{|x - i|} \int_{|x - y| \leq 1} \rho_0^N(y)dy
\]
\[
\leq C \left[ 1 + \sum_{i \in \mathbb{Z}^3, |i| \geq 1} \frac{C_1}{|i|^{2+\epsilon}} \frac{1}{|x - i|} \right]
\]
\[
\leq C \left[ 1 + \sum_{i \in \mathbb{Z}^3, |i| \geq 1} \frac{1}{|i|^{2+\epsilon}} \frac{1}{|x - i|} \right].
\]
If \( \{ i : |i| \leq |x - i| \} \), then,
\[
\sum_{i \in \mathbb{Z}^3, |i| \geq 1} \frac{1}{|i|^{2+\epsilon}} \frac{1}{|x - i|} \leq \sum_{i \in \mathbb{Z}^3, |i| \geq 1} \frac{1}{|i|^{3+\epsilon}} \leq C.
\]
If \( \{ i : |i| > |x - i| \} \), then,
\[
\sum_{i \in \mathbb{Z}^3, |i| \geq 1} \frac{1}{|i|^{2+\epsilon}} \frac{1}{|x - i|} \leq \sum_{i \in \mathbb{Z}^3, |x - i| \geq 1} \frac{1}{|x - i|^{3+\epsilon}} \leq C.
\]
Hence,
\[
\sum_{i \in \mathbb{Z}^3, |i| \geq 1} \frac{1}{|i|^{2+\epsilon}} \frac{1}{|x - i|} \leq C,
\]
namely,
\[
\int \rho_0^N(y)G(|x - y|)dy \leq C. \tag{18}
\]
Secondly, by the hypotheses of Theorem 1.1, if \( |\mu| \leq 5R^N(t) \), we get
\[
\int \varphi^{\mu, 5R^N(t)}(x) \rho_0^N(x)dx \leq \int_{|\mu - x| \leq 4R^N(t)} \rho_0^N(x)dx
\]
\[
\leq \int_{|x| \leq 9R^N(t)} \rho_0^N(x)dx
\]
\[
= \int_{|x| \leq 1} \rho_0^N(x)dx + \int_{1 \leq |x| \leq 9R^N(t)} \rho_0^N(x)dx
\]
\[
\leq C + \sum_{i \in \mathbb{Z}^3, 1 \leq |i| \leq 9R^N(t)} \int_{|x - i| \leq 1} \rho_0^N(x)dx
\]
\[
\leq C + \sum_{i \in \mathbb{Z}^3, 1 \leq |i| \leq 9R^N(t)} \frac{C_1}{|i|^{2+\epsilon}} \leq CR^N(t)^{1-\epsilon}. \tag{19}
\]
If \( |\mu| > 5R^N(t) \), by the assumption of (4), we deduce that
\[
\int \varphi^{\mu, 5R^N(t)}(x) \rho_0^N(x)dx \leq \int_{|\mu - x| \leq 4R^N(t)} \rho_0^N(x)dx
\]
\[ \leq \int_{|x-u| \leq 1} \rho_0^N(x) dx + \sum_{1 \leq |i| \leq 4R^N(t)} \int_{|\mu+i-x| \leq 1} \rho_0^N(x) dx \]
\[ \leq C + \sum_{1 \leq |i| \leq 4R^N(t)} \frac{C_1}{|\mu+i|^{2+\epsilon}} \]
\[ \leq C + \sum_{1 \leq |i| \leq 4R^N(t)} \frac{C}{|i|^{2+\epsilon}} \leq CR^N(t)^{1-\epsilon}, \] (20)

where we have used the fact that
\[ |\mu + i| \geq |\mu| - |i| \geq \frac{|i|}{4}. \]

combining (18), (19) and (20), we get
\[ W^N(\mu, 2R^N(t), 0) \leq C(1 + R^N(t)^{1-\epsilon}), \]
by (11) and the definition of \( R^N(t) \), we have
\[ Q^N(2R^N(t), 0) \leq CR^N(t)^{1-\epsilon}. \]

This is the desired result. \( \square \)

3. The estimate of the electric field.

**Proposition 2.** There exists a positive number \( C \) such that
\[ |E^N(t, x)| + |E^N(t, \xi^N(t))| \leq CV^N(t)^{\frac{1}{2}} Q^N(R^N(t), t)^{\frac{1}{2}}. \]

**Proof.** For any given \( \mu \in \mathbb{R}^3 \) and \( 0 < \nu \leq 3R^N(t) \), we have
\[ |E^N(t, \mu)| \leq \mathcal{I}_1(t, \mu) + \mathcal{I}_2(t, \mu) + \mathcal{I}_3(t, \mu), \]
where
\[ \mathcal{I}_1(t, \mu) = \int_{0 < |\mu-y| \leq \nu} \frac{\rho^N(t, y)}{|\mu-y|^2} dy, \] (21)
\[ \mathcal{I}_2(t, \mu) = \int_{\nu \leq |\mu-y| \leq 3R^N(t)} \frac{\rho^N(t, y)}{|\mu-y|^2} dy, \] (22)
\[ \mathcal{I}_3(t, \mu) = \int_{3R^N(t) \leq |\mu-y|} \frac{\rho^N(t, y)}{|\mu-y|^2} dy. \] (23)

Because
\[ \rho^N(t, x) = \int_{|v| \leq V^N(t)} f^N(t, x, v) dv \leq C\|f_0\|_{L^\infty} V^N(t)^3 \leq CV^N(t)^3, \]
for the term (21), we deduce that
\[ \mathcal{I}_1(t, \mu) \leq C\nu \|\rho^N(t)\|_{L^\infty} \leq C\nu V^N(t)^3 \]
For the term (22), we choose \( \kappa = \left( \int |v|^2 f^N(t, x, v) dv \right)^{\frac{1}{2}} \), then we get
\[
\rho^N(t, x) = \int f^N(t, x, v) dv \\
\leq \int_{|v| \leq \kappa} f^N(t, x, v) dv + \frac{1}{\kappa^2} \int_{|v| > \kappa} v^2 f^N(t, x, v) dv \\
\leq C \kappa^3 + \frac{1}{\kappa^2} \int |v|^2 f^N(t, x, v) dv \\
\leq C \left( \int |v|^2 f^N(t, x, v) dv \right)^{\frac{3}{2}}.
\]
Denote \( \mu_i = \mu + i R^N(t, s) \), we observe that
\[
\{ x : |x - \mu| \leq 3 R^N(t) \} \subset \bigcup_{i \in \mathbb{Z}^3: |i| \leq 3} \{ x : |x - \mu_i| \leq R^N(t) \}.
\]
By (11) and the definition of the local energy, we deduce that
\[
\int_{|x - \mu| \leq 3 R^N(t)} \rho^N(t, x)^{\frac{1}{2}} dx \leq C \int_{|x - \mu| \leq 3 R^N(t)} dx \int |v|^2 f^N(t, x, v) dv \\
\leq C \sum_{i \in \mathbb{Z}^3: |i| \leq 3} \int_{|x - \mu| \leq R^N(t)} dx \int |v|^2 f^N(t, x, v) dv \\
\leq C \sum_{i \in \mathbb{Z}^3: |i| \leq 3} W^N(\mu_i, R^N(t), t) \leq C Q^N(R^N(t), t).
\]
Then, by Hölder inequality, we have
\[
I_2(t, \mu) \leq C \left( \int_{|y - \mu| \leq 3 R^N(t)} \rho^N(y, t)^{\frac{1}{2}} dy \right)^{\frac{3}{2}} \left( \int_{|y - \mu| \geq \kappa} \frac{1}{|\mu - y|^5} dy \right)^{\frac{1}{2}} \\
\leq C \left( Q^N(R^N(t), t) \right)^{\frac{3}{2}} \kappa^{-\frac{5}{2}}.
\]
By the definition of \( V^N(t) \), we have
\[
V^N(t) \geq k,
\]
by Proposition 1, we can choose \( k \) to be suitably large such that
\[
\nu = V^N(t)^{-\frac{5}{2}} Q^N(R^N(t), t)^{\frac{1}{2}} \leq C k^{-\frac{5}{2}} R^N(t)^{\frac{1}{2}} \leq 3 R^N(t).
\]
Then, we deduce that
\[
I_1(t, \mu) + I_2(t, \mu) \leq C V^N(t)^{\frac{3}{2}} Q^N(R^N(t), t)^{\frac{1}{2}}. \tag{24}
\]
For the term (23), by the change of variables \((\bar{y}, \bar{w}) \mapsto (Y^N(t, 0, y, w), W^N(t, 0, y, w))\), we obtain that
\[
I_3(t, \mu) = \int_{|y - \mu| \geq 3 R^N(t)} \frac{\rho^N(t, \bar{y})}{|\mu - \bar{y}|^2} d\bar{y} \\
\leq \int \int_{|y - \mu| \geq 2 R^N(t)} \frac{f^N_0(y)}{|\mu - Y(t, 0, y, w)|^2} dy dw \\
\leq C \int_{|y - \mu| \geq 2 R^N(t)} \rho^N_0(y) |\mu - y|^2 dy,
\]
where we have used the fact that
\[
|\mu - Y^N(t, 0, y, w)| \geq |\mu - y| - |Y^N(t, 0, y, w) - y| \geq |\mu - y| - R^N(t) \geq \frac{|\mu - y|}{2}.
\]
Then, we have
\[
\mathcal{I}_3(t, \mu) \leq C \sum_{i \in \mathbb{Z}^3: |i| \geq 1 \atop |\mu - i| \geq 2R^N(t)} \int_{|y - i| \leq 1} \frac{\rho^N_0(y)}{|\mu - y|^2} \, dy
\]
\[
\leq C \sum_{i \in \mathbb{Z}^3: |i| \geq 1 \atop |\mu - i| \geq 2R^N(t)} \frac{1}{(|\mu - i| - R^N(t))^2} \int_{|y - i| \leq 1} \rho^N_0(y) \, dy
\]
\[
\leq C \sum_{i \in \mathbb{Z}^3: |i| \geq 1 \atop |\mu - i| \geq 2R^N(t)} \frac{C_1}{(|\mu - i| - R^N(t))^2 |i|^{2+\epsilon}}.
\]
where we have used the fact that \(|\mu - y| \geq |\mu - i| - |i - y| \geq |\mu - i| - R^N(t)\) and (4).
If \( \{ i: |i| \leq |\mu - i| \} \), then,
\[
\sum_{i \in \mathbb{Z}^3: |i| \geq 1 \atop |\mu - i| \geq 2R^N(t)} \frac{1}{(|\mu - i| - R^N(t))^2 |i|^{2+\epsilon}} \leq \sum_{i \in \mathbb{Z}^3: |i| \geq 1} \frac{1}{|i|^{4+\epsilon}} \leq C.
\]
If \( \{ i: |i| > |\mu - i| \} \), then,
\[
\sum_{i \in \mathbb{Z}^3: |i| \geq 1 \atop |\mu - i| \geq 2R^N(t)} \frac{1}{(|\mu - i| - R^N(t))^2 |i|^{2+\epsilon}} \leq \sum_{i \in \mathbb{Z}^3: |\mu - i| \geq 2R^N(t)} \frac{1}{(|\mu - i| - R^N(t))^{4+\epsilon}} \leq C.
\]
Hence, we obtain that
\[
\mathcal{I}_3(t, \mu) \leq C. \quad (26)
\]
By (25) and (26), we have
\[
|E^N(t, x)| + |E^N(t, \xi(t))| \leq C V^N(t) \frac{1}{2} Q^N(R^N(t), t)^{\frac{1}{2}}.
\]
This is the desired result. \(\square\)

To proceed further, we choose
\[
\Delta = \frac{1}{4CP^{\frac{1}{2} - \gamma} Q^N},
\]
where
\[
P = P^N(T), \quad Q = \sup_{t \in [0, T]} Q^N(R^N(t), t), \quad 0 < \gamma < \frac{2 + \epsilon}{4}.
\]
We partition the interval \([0, T]\) into \(n\) subintervals \([t_j, t_{j+1}], j = 0, 1, \cdots, n - 1\), we have
\[
\int_0^T (|E^N(s, X^N(s, 0, x, v))| + |E^N(s, \xi^N(s))|) \, ds = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (|E^N(s, X^N(s, 0, x, v))| + |E^N(s, \xi^N(s))|) \, ds,
\]
where
\[ t_0 = 0, \quad t_n = T, \quad \frac{1}{2} \Delta \leq t_{j+1} - t_j \leq \Delta. \]

Then, we have

**Lemma 3.1.** Let \( t' \in [0, T] \). If \( \sqrt{h}(x, v, t') \leq P^\gamma \), then,
\[
\sup_{t \in [t', t' + \Delta]} \sqrt{h}(x, v, t) \leq 2P^\gamma.
\]

If \( \sqrt{h}(x, v, t') \geq P^\gamma \), then,
\[
\sup_{t \in [t', t' + \Delta]} \sqrt{h}(x, v, t) \geq \frac{1}{2}P^\gamma.
\]

**Proof.** By the estimate of (7), we get
\[
\left| \sqrt{h}(x, v, t) - \sqrt{h}(x, v, t') \right| \leq 2 \int_{t'}^t (|E(s, x(s))| + |E(s, \xi(s))|) ds
\]
for \( t \in [t', t' + \Delta] \). If \( \sqrt{h}(x, v, t') \leq P^\gamma \), by the definition of \( \Delta \) and Proposition 2, we have
\[
\sqrt{h}(x, v, t) \leq \sqrt{h}(x, v, t') + 2CV(t)^\frac{1}{2} Q^\frac{1}{3} \Delta
\leq P^\gamma + 2CV(t)^\frac{1}{2} Q^\frac{1}{3} \Delta \leq 2P^\gamma.
\]
Similarly, if \( \sqrt{h}(x, v, t') \geq P^\gamma \), we have
\[
\sqrt{h}(x, v, t) \geq P^\gamma - 2CV(t)^\frac{1}{2} Q^\frac{1}{3} \Delta \geq \frac{1}{2} P^\gamma.
\]

The proof is completed. \( \Box \)

The following proposition, whose proof is similar to that of [6, 8], is important in the proof of Theorem 1.1. For the sake of completeness of this paper we give the detailed proof of it.

**Proposition 3.** In the hypotheses of Theorem 1.1, there exist positive constants \( C \) and \( \beta < 1 \) such that
\[
\int_0^T [ |E^N(s, X^N(s, 0, x, v))| + |E^N(s, \xi^N(s))|] ds \leq CP^N(T)^\beta. \tag{27}
\]

**Proof.** We consider the system (9) on \([t_j, t_{j+1}]\) for a fixed \( j \). By the change of variables \((\bar{y}, \bar{w}) \to (Y^N(t, t_j, \bar{y}, \bar{w}), W^N(t, t_j, \bar{y}, \bar{w}))\), we obtain that
\[
|E(t, X^N(t, 0, x, v))| + |E^N(t, \xi^N(t))| \leq \int \frac{\rho^N(t, \bar{y})}{|X^N(t, 0, x, v) - \bar{y}|^2} d\bar{y} + \int \frac{\rho^N(t, \bar{y})}{|\xi^N(t) - \bar{y}|^2} d\bar{y}
= \int \int \frac{f^N(t_j, \bar{y}, \bar{w})}{|X^N(t, 0, x, v) - Y^N(t, t_j, \bar{y}, \bar{w})|^2} d\bar{y} d\bar{w}
+ \int \int \frac{f^N(t_j, \bar{y}, \bar{w})}{|\xi^N(t) - Y^N(t, t_j, \bar{y}, \bar{w})|^2} d\bar{y} d\bar{w}.
\]
Firstly, we give the estimate of $|E^N(t, \xi^N(t))|$, setting

$$ S = \{(\hat{y}, \hat{w}) : |\xi^N(t_j) - \hat{y}| \leq 2R^N(T)\}, $$

$$ S_1 = \{(\hat{y}, \hat{w}) : \sqrt{h}(t_j, \hat{y}, \hat{w}) \leq P\}, $$

$$ S_2 = \{(\hat{y}, \hat{w}) : \sqrt{h}(t_j, \hat{y}, \hat{w}) > P\}, $$

then

$$ |E^N(t, \xi^N(t))| \leq \int \int_{S \cup S_1 \cup S_2} \frac{f^N(t_j, \hat{y}, \hat{w})}{|\xi^N(t) - Y^N(t, t_j, \hat{y}, \hat{w})|^2} d\hat{y} d\hat{w} $$

$$ \leq \int \int_{S \cap S_1} \frac{f^N(t_j, \hat{y}, \hat{w})}{|\xi^N(t) - Y^N(t, t_j, \hat{y}, \hat{w})|^2} d\hat{y} d\hat{w} $$

$$ + \int \int_{S \cap S_2} \frac{f^N(t_j, \hat{y}, \hat{w})}{|\xi^N(t) - Y^N(t, t_j, \hat{y}, \hat{w})|^2} d\hat{y} d\hat{w} $$

$$ + \int \int_{S_1} \frac{f^N(t_j, \hat{y}, \hat{w})}{|\xi^N(t) - Y^N(t, t_j, \hat{y}, \hat{w})|^2} d\hat{y} d\hat{w}. $$

By Lemma 3.1 and by the change of variables $(Y^N(t, t_j, \hat{y}, \hat{w}), W^N(t, t_j, \hat{y}, \hat{w})) \to (\hat{y}, \hat{w})$, we have

$$ |E^N(t, \xi^N(t))| \leq \int \int_{S' \cap S'_1} \frac{f^N(t_j, \hat{y}, \hat{w})}{|\xi^N(t) - Y^N(t, t_j, \hat{y}, \hat{w})|^2} d\hat{y} d\hat{w} $$

$$ + \int \int_{S' \cap S'_2} \frac{f^N(t_j, \hat{y}, \hat{w})}{|\xi^N(t) - Y^N(t, t_j, \hat{y}, \hat{w})|^2} d\hat{y} d\hat{w} $$

$$ + \int \int_{S'_1} \frac{f^N(t_j, \hat{y}, \hat{w})}{|\xi^N(t) - Y^N(t, t_j, \hat{y}, \hat{w})|^2} d\hat{y} d\hat{w} $$

$$ + \int \int_{S'_2} \frac{f^N(t_j, \hat{y}, \hat{w})}{|\xi^N(t) - Y^N(t, t_j, \hat{y}, \hat{w})|^2} d\hat{y} d\hat{w}, $$

where

$$ S' = \{(\hat{y}, \hat{w}) : |\xi^N(t_j) - \hat{y}| \leq 3R^N(T)\}, $$

$$ S'_1 = \{(\hat{y}, \hat{w}) : \sqrt{h}(t_j, \hat{y}, \hat{w}) \leq 2P\}, $$

$$ S'_2 = \{(\hat{y}, \hat{w}) : \sqrt{h}(t_j, \hat{y}, \hat{w}) > \frac{P}{2}\}. $$

Then, we deduce that

$$ |E^N(t, \xi^N(t))| \leq 2 \sum_{i=1}^{3} \Xi_i, $$

where

$$ \Xi_i = \int \int_{S' \cap S'_i} \frac{f^N(t, \hat{y}, \hat{w})}{|\xi^N(t) - \hat{y}|^2} d\hat{y} d\hat{w} $$

for $i = 1, 2$, and

$$ \Xi_3 = \int \int_{S''} \frac{f^N(t, \hat{y}, \hat{w})}{|\xi^N(t) - \hat{y}|^2} d\hat{y} d\hat{w}, \quad (28) $$
with 

\[ S' = \{(\bar{\xi}, \bar{\omega}) : |\xi^{N}(t_{j}) - \bar{\xi}| > 3R^{N}(T)\} \]

For the term \( \Xi_{1} \), similar to the proof of (25), we can deduce that 

\[
\Xi_{1} = \int \int_{S' \cap S'_{1}} \frac{f^{N}(t, \bar{\xi}, \bar{\omega})}{|\xi^{N}(t) - \bar{\xi}|^{2}} d\bar{\xi} d\bar{\omega} \\
\leq \int_{|\xi^{N}(t) - \bar{\xi}| \leq 4R(t,s)} \int_{|\bar{\omega}| \leq 2P^{2}} \frac{f^{N}(t, \bar{\xi}, \bar{\omega})}{|\xi^{N}(t) - \bar{\xi}|^{2}} d\bar{\xi} d\bar{\omega} \\
\leq CP^{2} \gamma Q^{3/4}.
\]  

(29)

For the term \( \Xi_{2} \), we define

\[ A = \{(t, \bar{\xi}, \bar{\omega}) : \frac{P^{\gamma}}{2} \leq \sqrt{\bar{\xi}(t, \bar{\xi}, \bar{\omega})} \leq P, |\xi^{N}(t) - \bar{\xi}| \leq l\}, \]

\[ B = \{(t, \bar{\xi}, \bar{\omega}) : \frac{P^{\gamma}}{2} \leq \sqrt{\bar{\xi}(t, \bar{\xi}, \bar{\omega})} \leq P, |\xi^{N}(t) - \bar{\xi}| > l\}, \]

where 

\[ l = \frac{Q^{3/4}}{P^{\frac{1}{4} + \eta}}, \eta > 1 - \frac{\epsilon}{3}. \]

Then, we have

\[ \Xi_{2} \leq \Xi_{2}' + \Xi_{2}'' \]

where

\[
\Xi_{2}' = \int \int_{S' \cap A} \frac{f^{N}(t, \bar{\xi}, \bar{\omega})}{|\xi^{N}(t) - \bar{\xi}|^{2}} d\bar{\xi} d\bar{\omega}, \]

(30)

\[
\Xi_{2}'' = \int \int_{S' \cap B} \frac{f^{N}(t, \bar{\xi}, \bar{\omega})}{|\xi^{N}(t) - \bar{\xi}|^{2}} d\bar{\xi} d\bar{\omega}. \]

(31)

For the term (30), by the assumption of Theorem 1.1, we obtain that 

\[
\Xi_{2}' \leq \|f_{0}\|_{L^{\infty}} \int_{|\bar{\xi}| \leq 2P} d\bar{\omega} \int_{|\xi^{N}(t) - \bar{\xi}| \leq l} \frac{d\bar{\xi}}{|\xi^{N}(t) - \bar{\xi}|^{2}} \\
\leq CC_{0}dP^{3} \leq CP^{2} \gamma Q^{3/4}.
\]

(32)

For the term (31), along the characteristics, we obtain that 

\[
\Xi_{2}'' \leq \int_{|\xi^{N}(t_{j}) - \bar{\xi}| \leq 3R(T)} \int_{|\xi^{N}(t_{j}) - \bar{\xi}| > l} \frac{f^{N}(t, \bar{\xi}, \bar{\omega})}{|\xi^{N}(t) - \bar{\xi}|^{2}} d\bar{\xi} d\bar{\omega} \\
\leq l^{2} \int_{|\xi^{N}(t_{j}) - \bar{\xi}| \leq 3R(T)} \int f^{N}(t, \bar{\xi}, \bar{\omega}) d\bar{\xi} d\bar{\omega} \\
\leq l^{2} \int_{|\xi^{N}(t_{j}) - \bar{\xi}| \leq 4R(T)} \int f^{N}(y, \omega) dy dw,
\]

where we have used the fact that

\[ |\xi^{N}(t_{j}) - \bar{\xi}| \leq |\xi^{N}(t_{j}) - \bar{\xi}| + |\bar{\xi} - \bar{\xi}| \leq 4R(T). \]
If \(|\xi_N(t_j)| \leq 5R_N(T)|, by the assumption of Theorem 1.1 and the definition of \(R_N(t)|, we obtain that
\[
\int_{|\xi_N(t_j) - y| \leq 4R_N(T)} \rho_0^N(y)dy \leq \int_{|y| \leq 9R_N(T)} \rho_0^N(y)dy \\
\leq \int_{|y| \leq 1} \rho_0^N(y)dy + \sum_{1 < |i| \leq 9R_N(T)} \int_{|i-y| \leq 1} \rho_0^N(y)dy \\
\leq C + \sum_{1 < |i| \leq 9R_N(T)} \frac{C_1}{|i|^{2+\epsilon}} \\
\leq C + CR_N(T)^{1-\epsilon} \leq CP^{1-\epsilon}.
\] (33)

If \(|\xi_N(t_j)| > 5R_N(T)|, we have
\[
\int_{|\xi_N(t_j) - y| \leq 4R_N(T)} \rho_0^N(y)dy = \int_{|\xi_N(t_j) - y| \leq 1} \rho_0^N(y)dy + \int_{1 < |\xi_N(t_j) - y| \leq 4R_N(T)} \rho_0^N(y)dy \\
\leq \int_{|\xi_N(t_j) - y| \leq 1} \rho_0^N(y)dy \\
\quad + \sum_{1 < |i| \leq 4R_N(T)} \int_{|\xi_N(t_j) + i - y| \leq 1} \rho_0^N(y)dy \\
\leq C + \sum_{1 < |i| \leq 4R_N(T)} \frac{C_1}{|\xi_N(t_j) + i|^2 + \epsilon} \\
\leq C + \sum_{1 < |i| \leq 4R_N(T)} \frac{C}{|i|^{2+\epsilon}} \\
\leq C + CR_N(T)^{1-\epsilon} \leq CP^{1-\epsilon},
\] (34)

where we have used the fact that \(|\xi_N(t_j) + i| \geq |\xi_N(t_j)| - |i| \geq \frac{|\xi_N(t_j)|}{4}, and the assumption of (4). Hence, by (33) and (34), we obtain that
\[
\Xi'' \leq CT^2 P^{1-\epsilon} \leq C \frac{Q^2}{P^{2+\epsilon+2\gamma}}.
\] (35)

For the term (28), if \(|\xi_N(t_j) - \bar{y}| \geq 3R_N(T)|, we deduce that
\[
|\xi_N(t) - \bar{y}| \geq |\xi_N(t_j) - \bar{y}| - |\xi_N(t_j) - \xi_N(t)| \\
\geq |\xi_N(t_j) - \bar{y} - R_N(T)| \geq \frac{2}{3}|\xi_N(t_j) - \bar{y}|
\]

By the change of variables \((\bar{y}, \bar{w}) \rightarrow (Y_N(t, 0, y, w), W_N(t, 0, y, w))\), we obtain that
\[
\Xi_3 = \int \int_{|\xi_N(t_j) - y| \geq 3R_N(T)} \frac{f_N(t, \bar{y}, \bar{w})}{|\xi_N(t) - \bar{y}|^2} d\bar{y}d\bar{w} \\
\leq C \int \int_{|\xi_N(t_j) - y| \geq 3R_N(T)} \frac{f_N(t, \bar{y}, \bar{w})}{|\xi_N(t_j) - \bar{y}|^2} d\bar{y}d\bar{w} \\
\leq C \int \int_{|\xi_N(t_j) - y| \geq 2R_N(T)} \frac{f_0^N(y, w)}{|\xi_N(t_j) - Y_N(t, 0, y, w)|^2} dydw.
\]
where we have used the fact that
\[ |\xi^N(t_j) - y| \geq |\xi^N(t_j) - Y^N(t, 0, y, w)| - |Y^N(t, 0, y, w) - y| \geq 2R^N(T). \]
Due to
\[ |\xi^N(t_j) - Y^N(t, 0, y, w)| \geq |\xi^N(t_j) - y| - |Y^N(t, 0, y, w) - y| \geq |\xi^N(t_j) - y| - R^N(T) \geq \frac{|\xi^N(t_j) - y|}{2}, \]
we obtain that
\[
\Xi_3 \leq C \int \int_{|\xi^N(t_j) - y| \geq 2R^N(T)} f_0^N(y, w) |\xi^N(t_j) - y|^2 dy dw \\
\leq C \sum_{i \in \mathbb{Z}^3: |i| \geq 1} \int_{|i-y| \leq 1} \frac{\rho_0^N(y)}{|\xi^N(t_j) - y|^2} dy.
\]
Then, we deduce that
\[
\sum_{i \in \mathbb{Z}^3: |i| \geq 1} \int_{|i-y| \leq 1} \frac{\rho_0^N(y)}{|\xi^N(t_j) - y|^2} dy \\
\leq C \sum_{i \in \mathbb{Z}^3: |i| \geq 1} \frac{1}{|\xi^N(t_j) - i|^2} \int_{|i-y| \leq 1} \rho_0^N(y) dy \\
\leq C \sum_{i \in \mathbb{Z}^3: |i| \geq 1} \frac{C_1}{|\xi^N(t_j) - i|^2 |i|^{2+\gamma}} \\
\leq C \sum_{i \in \mathbb{Z}^3: |i| \geq 1} \frac{1}{|\xi^N(t_j) - i|^2 |i|^{2+\gamma}},
\]
where we have used the fact that
\[ |\xi^N(t_j) - y| \geq |\xi^N(t_j) - i| - |i - y| \geq \frac{|\xi^N(t_j) - i|}{2}. \]
If \( |\xi^N(t_j) - i| \leq |i| \), we have
\[ \sum_{i \in \mathbb{Z}^3: |i| \geq 1} \frac{1}{|\xi^N(t_j) - i|^2 |i|^{2+\gamma}} \leq \sum_{i \in \mathbb{Z}^3: |i| \geq 1} \frac{1}{|\xi^N(t_j) - i|^{4+\gamma}} \leq C. \]
If \( |\xi^N(t_j) - i| \geq |i| \), we have
\[ \sum_{i \in \mathbb{Z}^3: |i| \geq 1} \frac{1}{|\xi^N(t_j) - i|^2 |i|^{2+\gamma}} \leq \sum_{i \in \mathbb{Z}^3: |i| \geq 1} \frac{1}{|i|^{4+\gamma}} \leq C. \]
Hence,
\[
\Xi_3 = \int \int_{|\xi^N(t_j) - y| \geq 3R^N(T)} f^N(t, \bar{y}, \bar{w}) |\xi^N(t) - y|^2 dy d\bar{w} \leq C. \tag{36}
\]
By (29), (32), (35) and (36), we have
\[
|E^N(t, \xi^N(t))| \leq CP^4 \gamma Q^4 + CP^4 \gamma Q^4 + C \frac{Q^4}{p^{2+\gamma+2\eta}}, \tag{37}
\]
Secondly, analogous to the way above, we give the estimate of $|E^N(t, X^N(t, 0, x, v))|$. Setting
\[
M = \{(\hat{y}, \hat{w}) : |X^N(t_j, 0, x, v) - \hat{y}| \leq 2R^N(T)\},
\]
\[
M_1 = \{(\hat{y}, \hat{w}) : \sqrt{h}(t_j, \hat{y}, \hat{w}) \leq P^\gamma\},
\]
\[
M_2 = \{(\hat{y}, \hat{w}) : \sqrt{h}(t_j, \hat{y}, \hat{w}) > P^\gamma\}.
\]
Similar to the estimate of $|E^N(t, \xi^N(t))|$, we have
\[
|E(X^N(t, 0, x, v))| \leq 2 \sum_{j=1}^{3} A_j,
\]
where
\[
A_j = \int \int_{M_j} \frac{f^N(t, \hat{y}, \hat{w})}{|X^N(t, 0, x, v) - \hat{y}|^2} d\hat{y} d\hat{w}
\]
for $j = 1, 2,$ and
\[
A_3 = \int \int_{M_3} \frac{f^N(t, \hat{y}, \hat{w})}{|X^N(t, 0, x, v) - \hat{y}|^2} d\hat{y} d\hat{w}, \tag{38}
\]
with
\[
M' = \{(\hat{y}, \hat{w}) : |X^N(t_j, 0, x, v) - \hat{y}| \leq 3R^N(T)\},
\]
\[
M'' = \{(\hat{y}, \hat{w}) : |X^N(t_j, 0, x, v) - \hat{y}| > 3R^N(T)\},
\]
\[
M_1' = \{(\hat{y}, \hat{w}) : \sqrt{h}(t_j, \hat{y}, \hat{w}) \leq 2P^\gamma\},
\]
\[
M_2' = \{(\hat{y}, \hat{w}) : \sqrt{h}(t_j, \hat{y}, \hat{w}) > \frac{P^\gamma}{2}\}.
\]
For the term $A_1$, analogous to the proof of (25), we deduce that
\[
A_1 \leq CP^{\frac{3}{2}}Q^{\frac{1}{2}}. \tag{39}
\]
For the term $A_2$, we also define the sets as follows:
\[
A_1 = \{(t, \hat{y}, \hat{w}) : \frac{P^\gamma}{2} \leq \sqrt{h}(t, \hat{y}, \hat{w}) \leq P, |X^N(t, 0, x, v) - \hat{y}| \leq l\},
\]
\[
B_1 = \{(t, \hat{y}, \hat{w}) : \frac{P^\gamma}{2} \leq \sqrt{h}(t, \hat{y}, \hat{w}) \leq P, |X^N(t, 0, x, v) - \hat{y}| > l\},
\]
where
\[
l = \frac{Q^{\frac{1}{2}}}{P^{\frac{1}{2} + \eta}}, \quad \eta > 1 - \frac{\epsilon}{3}.
\]
Then, we have
\[
A_2 = A_2' + A_2'',
\]
where
\[
A_2' = \int \int_{M' \cap A} \frac{f^N(t, \hat{y}, \hat{w})}{|X^N(t, 0, x, v) - \hat{y}|^2} d\hat{y} d\hat{w}, \tag{40}
\]
\[
A_2'' = \int \int_{M' \cap B} \frac{f^N(t, \hat{y}, \hat{w})}{|X^N(t, 0, x, v) - \hat{y}|^2} d\hat{y} d\hat{w}. \tag{41}
\]
For the term \((40)\), by the assumption of Theorem 1.1, we obtain that

\[
A_2' \leq \|f_0\|_{L^\infty} \int_{|\tilde{y}| \leq P} d\tilde{y} \int_{X^N(t,0,x,v)-\tilde{y}) \leq t} \frac{d\tilde{y}}{|X^N(t,0,x,v) - \tilde{y}|^2}
\]

\[
\leq C P^3 \leq C P^4 \eta Q^4.
\] (42)

For the term \((41)\), analogous to the proof of \((31)\), we get

\[
A_2' \leq \frac{Q^2}{P^{\frac{4}{3} + \frac{2}{3}}}.
\] (43)

Now, we give the estimate \((38)\). If \(|X^N(t_j,0,x,v) - \bar{y}| \geq 3R(T)\), we deduce that

\[
|X^N(t,0,x,v) - \bar{y}| \geq |X^N(t_j,0,x,v) - \bar{y}| - |X^N(t,0,x,v) - X^N(t_j,0,x,v)|
\]

\[
\geq |X^N(t_j,0,x,v) - \bar{y}| - R(T)
\]

\[
\geq \frac{2}{3}|X^N(t_j,0,x,v) - \bar{y}|.
\]

By the change of variables \((\bar{y}, \bar{w}) \to (Y^N(t,0,y,w), W^N(t,0,y,w))\), we obtain that

\[
A_3 = \int \int_{|X^N(t_j,0,x,v) - \bar{y}| \geq 3R(T)} f^N(t, \bar{y}, \bar{w}) |X^N(t,0,x,v) - \bar{y}|^2 d\bar{y} d\bar{w}
\]

\[
\leq C \int \int_{|X^N(t_j,0,x,v) - \bar{y}| \geq 3R(T)} f^N(t, \bar{y}, \bar{w}) |X^N(t_j,0,x,v) - \bar{y}|^2 d\bar{y} d\bar{w}
\]

\[
\leq C \int \int_{|X^N(t_j,0,x,v) - \bar{y}| \geq 2R(T)} f^N_0(y,w) |X^N(t_j,0,x,v) - Y^N(t,0,y,w)|^2 dy dw,
\]

where we have used the fact that

\[
|X^N(t_j,0,x,v) - y| \geq |X^N(t_j,0,x,v) - Y^N(t,0,y,w)| - |Y^N(t,0,y,w) - y|
\]

\[
\geq 2R(T).
\]

Due to

\[
|X^N(t_j,0,x,v) - Y^N(t,0,y,w)|
\]

\[
\geq |X^N(t_j,0,x,v) - y| - |Y^N(t,0,y,w) - y|
\]

\[
\geq |X^N(t_j,0,x,v) - y| - R(T)
\]

\[
\geq \frac{|X^N(t_j,0,x,v) - y|}{2},
\]

we obtain that

\[
A_3 \leq C \int \int_{|X^N(t_j,0,x,v) - y| \geq 2R(T)} f^N_0(y,w) |X^N(t_j,0,x,v) - y|^2 dy dw
\]

\[
\leq C \sum_{i \in \mathbb{Z}, |i| \geq 1} \int_{|X^N(t_j,0,x,v) - i| \geq 2R(T)} \rho_i^N(y) |X^N(t_j,0,x,v) - y|^2 dy,
\]

since

\[
|X^N(t_j,0,x,v) - y| \geq |X^N(t_j,0,x,v) - i| - |y - i| \geq \frac{|X^N(t_j,0,x,v) - i|}{2},
\]
by the assumption of (4), we deduce that
\[
\sum_{i \in \mathbb{Z}^3, |i| \geq 1} \frac{C}{|X^N(t_j, 0, x, v) - i|^2} \int_{|y| \leq 1} \rho^N_0(y) dy \\
\leq \sum_{i \in \mathbb{Z}^3, |i| \geq 1} \frac{CC_1}{|X^N(t_j, 0, x, v) - i|^2 |i|^{2+\epsilon}} \\
\leq \sum_{i \in \mathbb{Z}^3, |X^N(t_j, 0, x, v) - i| \geq 2R(t)} \frac{C}{|X^N(t_j, 0, x, v) - i|^4} \\
\leq \sum_{i \in \mathbb{Z}^3, |X^N(t_j, 0, x, v) - i| \geq 2R(t)} \frac{1}{|X^N(t_j, 0, x, v) - i|^4 + \epsilon} \leq C.
\]

If $|X^N(t_j, 0, x, v) - i| \leq |i|$, we have
\[
\sum_{i \in \mathbb{Z}^3, |i| \geq 1} \frac{1}{|X^N(t_j, 0, x, v) - i|^2 |i|^{2+\epsilon}} \\
\leq \sum_{i \in \mathbb{Z}^3, |X^N(t_j, 0, x, v) - i| \geq 2R(t)} \frac{1}{|X^N(t_j, 0, x, v) - i|^4} + \epsilon \leq C.
\]

Hence, we obtain that
\[
\mathcal{A}_3 = \int \int_{|X^N(t_j, 0, x, v) - \tilde{y}| \geq 3R(t)} \frac{f_N(t, \tilde{y}, \tilde{w})}{|X^N(t, 0, x, v) - \tilde{y}|^2} d\tilde{y} d\tilde{w} \leq C. \quad (44)
\]

By (39), (42), (43), (44), we have
\[
|E^N(s, X^N(s, 0, x, v))| \leq CP^\frac{4}{n}Q_1 + CP^\frac{2}{n}Q_2 + C \frac{Q_2^2}{P^{2+\epsilon+2\eta}}. \quad (45)
\]

Finally, by (37), (45), we obtain that
\[
\int_0^T (|E^N(s, X^N(s, 0, x, v))| + |E^N(s, \xi^N(s))|) ds \\
= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (|E^N(s, X^N(s, 0, x, v))| + |E^N(s, \xi^N(s))|) ds \\
\leq CT(P^\frac{4}{n}Q_1^2 + P^\frac{2}{n}Q_2^2 + \frac{Q_2^2}{P^{2+\epsilon+2\eta}}) \leq CP^\beta,
\]

where
\[
\beta = \max \{ \frac{1 + 4\gamma - \epsilon}{3}, 2 - \eta - \frac{\epsilon}{3} \} < 1.
\]

The proof of Proposition 3 is completed. \qed
4. The proof of Theorem 1.1. By the estimate (7) and (27), we obtain that
\[ \sqrt{h(T, X^N(T, 0, x, v), V^N(T, 0, x, v))} \]
\[ \leq \sqrt{h(0, x, v)} + C \int_0^T \left( |E^N(s, X^N(s, 0, x, v))| + |E^N(s, \xi^N(s))| \right) ds \]
\[ \leq C + CP^N(T)^\beta, \]
by the definition of \( P^N(t) \), we get
\[ P^N(T) \leq C + CP^N(T)^\beta. \]
Hence, we obtain that
\[ P^N(T) \leq C_2, \] (46)
where \( C_2 \) is a positive constant depending only on the initial data and the time \( T \).

Since the estimates of the fields and (46) are uniform in \( N \), then the solutions of system (9) and (10) converge to the solutions of system (2) and (3) in the limit \( N \to \infty \).

Indeed, for any positive integer \( K \), we define that \((X^N(t), V^N(t), \xi^N(t), \eta^N(t))\) and \((X^{N+K}(t), V^{N+K}(t), \xi^{N+K}(t), \eta^{N+K}(t))\) are the two solutions to the system (9) and (10), with the initial distributions \( f^N_0 \) and \( f^{N+K}_0 \) respectively, where \((X^N(t), V^N(t), 0, x, v)) = (X^N(t, 0, x, v), V^N(t, 0, x, v))\) and \((X^{N+K}(t), V^{N+K}(t)) = (X^{N+K}(t, 0, x, v), V^{N+K}(t, 0, x, v))\). We set
\[
A^{N,K}(t) = \sup_{(x,v) \in S, \, |x| \leq N} |X^N(t) - X^{N+K}(t)| + |\xi^N(t) - \xi^{N+K}(t)|,
\]
\[
B^{N,K}(t) = \sup_{(x,v) \in S, \, |x| \leq N} |V^N(t) - V^{N+K}(t)| + |\eta^N(t) - \eta^{N+K}(t)|,
\]
\[
C^{N,K}(t) = A^{N,K}(t) + B^{N,K}(t).
\]
Then, we have the following estimate:

**Proposition 4.** For any \( t \in [0, T] \), we have
\[ C^{N,K}(t) \leq \left( C \int \int |f^N_0(y,w) - f^{N+K}_0(y,w)| dy dw \right)^{\exp(Ct)} \] (47)

The proof of Proposition 4 will be given in the last section. Now, we give the proof of the main result.

**Proof of Theorem 1.1.** By the assumption of (4) and the Proposition 4, we deduce that
\[ C^{N,K}(t) \leq \left( C \int_{N \leq |y| \leq N+K} \rho_0(y) dy \right)^{\exp(Ct)} \]
\[ \leq \left( C \sum_{N \leq |i| \leq N+K} \int_{|y| \leq 1} \rho_0(y) dy \right)^{\exp(Ct)} \]
\[
\left( C \sum_{N \leq |i| \leq N + K} \frac{1}{|i|^{2+\epsilon}} \right)^{\exp(C_1)} \leq \left( C \sum_{N \leq |i| \leq N + K} \frac{1}{|i|^{2+\epsilon}} \right)^{\exp(C_1)} \leq \left( \sum_{N \leq |i| \leq N + K} \frac{C}{|i|^{2+\epsilon}} \right)^{\exp(C_1)}
\]

then, for \( \epsilon > \frac{1}{3} \), \( C_{N,K}(t) \) converges to zero when \( N \to \infty \) uniformly on \([0, T] \). Hence, for any fixed \((x, v)\), \( \{X^N(t)\} \) and \( \{V^N(t)\} \) (respectively \( \{\xi^N(t)\} \) and \( \{\eta^N(t)\} \)) are Cauchy sequences in space \( C([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3) \) (respectively in space \( C([0, T]) \)). Consequently, \( \{f^N(t, x, v)\} \) is also a Cauchy sequence in space \( C^3_0([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3) \).

Then, there exist some limit functions \( X(t), \ V(t), \ \xi(t), \ \eta(t), \ f(t, x, v) \) such that

\[
\xi^N(t) \to \xi(t), \ \eta^N(t) \to \eta(t)
\]

uniformly on \([0, T]\) as \( N \to \infty \), and

\[
X^N(t) \to X(t), \ V^N(t) \to V(t), \ f^N(t, x, v) \to f(t, x, v)
\]

uniformly on \([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3\) as \( N \to \infty \).

Moreover, by Proposition 1, Proposition 2 and the estimate (46), we know that the fields \( E^N \) and \( F^N \) are uniformly bounded in \( N \). Analogous to the estimate of (57), we get

\[
|E^N(t, x) - E(t, x)| \leq C \left( \int \int |f^N_0(y, w) - f_0(y, w)|dydw \right)^{\frac{1}{2}} + C|X^N(s) - X(s)|(1 + |\log|X^N(s) - X(s)||).
\]

Thus, we have

\[
E^N(t, x) \to E(t, x), \ \text{in} \ C([0, T] \times \mathbb{R}^3), \ N \to \infty.
\]

By the estimate of (46), we get

\[
\begin{align*}
F^N(t, x) & \to F(t, x), \ \text{in} \ C([0, T] \times \mathbb{R}^3), \ N \to \infty. \\
E^N(t, \xi^N(t)) & \to E(t, \xi(t)), \ \text{in} \ C([0, T] \times \mathbb{R}^3), \ N \to \infty.
\end{align*}
\]

Hence, \( (f(t, x, v), \xi(t), \eta(t)) \) satisfy system (1) on \([0, T]\).

Now, we give the proof of (5) and (6). Firstly, by the estimate (46), we deduce that

\[
\sup\{|V(t, 0, x, v)| + \frac{1}{|X(t, 0, x, v) - \xi(t)|} | t \in [0, T], (x, v) \in \text{supp}f_0\} \leq C_2.
\]

Secondly, by the change of variables \((x, v) \to (X(t, 0, x, v), V(t, 0, x, v))\), similar to [6], we obtain that

\[
\int_{|x| \leq 1} f(t, x, v)dxdv \leq \sum_{\mu \leq 3, |\mu| \leq R(t)} \int_{|x| \leq 1 + R(t)} f_0(x, v)dxdv \leq \sum_{\mu \leq 3, |\mu| \leq R(t)} \rho_0(x)dx.
\]

where \( i \in \mathbb{Z}^3 : |i| \geq 1 \). If \( |i| \leq 2R(t) \), we get

\[
\sum_{\mu \leq 3, |\mu| \leq R(t)} \int_{|x| \leq 3R(t)} \rho_0(x)dx \leq C \sum_{\mu \leq 3, |\mu| \leq R(t)} R(t)^3.
\]
If $|i| > 2R(t)$, we get
\[
\sum_{\mu \in \mathbb{R}^d : |\mu| \leq R(t)} \frac{\rho_0(x)dx}{|i + \mu - z| \leq 1} \leq \sum_{\mu \in \mathbb{R}^d : |\mu| \leq R(t)} \frac{C_1}{|i + \mu|^2 + \epsilon} \leq \sum_{\mu \in \mathbb{R}^d : |\mu| \leq R(t)} \frac{C}{|i|^2 + \epsilon},
\]
where we have used the fact that
\[
|\mu + i| \geq |i| - |\mu| \geq \frac{|i|}{2}.
\]
By the definition of $R(t)$, we have
\[
R(t) \leq CV(T) < C.
\]
Hence, we obtain that
\[
\int_{|i - z| \leq 1} f(t, x, v)dx dv \leq \sum_{\mu \in \mathbb{R}^d : |\mu| \leq R(t)} \left[ CR(t)^3 + \frac{C}{|i|^2 + \epsilon} \right] \leq \frac{C_3}{|i|^2 + \epsilon}.
\]
Furthermore, the uniqueness of the solution to the system (2) and (3) can be deduced by putting two different solutions in place of $f^N$ and $f^{N+K}$. The proof of Theorem 1.1 is completed.

5. **The proof of Proposition 4.** We need to prove the almost-Lipschitz property of the field $E^N$ at first.

**Lemma 5.1.** In the hypotheses of Theorem 1.1, we have
\[
|E^N(t, x) - E^N(t, y)| \leq C|x - y|(1 + \log |x - y||).
\]

**Proof.** The proof is similar to [6]. We define $|x - y| = D$. In case $D \geq 1$, by Proposition 1, Proposition 2 and the estimate (46), we deduce that
\[
|E^N(t, x) - E^N(t, y)| \leq |E^N(t, x)| + |E^N(t, y)| \\
\leq CV^N(N(t))Q^N(R^N(t), t)^\frac{1}{2} \\
\leq CV^N(t)^{\frac{1}{2}} \leq C \leq CD,
\]
where we have used the fact that
\[
R^N(t) \leq CV^N(t).
\]
In case $D < 1$, setting $\lambda = \frac{x + y}{2}$, we have
\[
|E^N(t, x) - E^N(t, y)| \leq \sum_{n=1}^{3} T_n,
\]
where
\[
T_1 = \int_{|\mu - \lambda| \leq 2D} \frac{1}{|x - \mu|^2} - \frac{1}{|y - \mu|^2} \rho^N(t, \mu) d\mu,
\]
\[
T_2 = \int_{2D < |\mu - \lambda| \leq \frac{3}{2}D} \frac{1}{|x - \mu|^2} - \frac{1}{|y - \mu|^2} \rho^N(t, \mu) d\mu,
\]
\[
T_3 = \int_{|\mu - \lambda| \geq \frac{3}{2}D} \frac{1}{|x - \mu|^2} - \frac{1}{|y - \mu|^2} \rho^N(t, \mu) d\mu.
\]
For the term $T_1$, if $|\mu - \lambda| \leq 2D$, we deduce that
\[
T_1 \leq 2\|\rho^N(t)\|_{L^\infty} \int_{|\mu - x| \leq 3D} \frac{1}{|x - \mu|^2} dz \leq CD,
\]
(48)
where we have used the fact that $|\mu - x| \leq 3D$ and $|\mu - y| \leq 3D$.

For the term $I'_2$, by the mean value theorem, we have

$$T'_2 \leq CD \int_{2D < |\mu - \lambda| \leq \frac{3}{2}} \frac{1}{|\mu - \zeta|^3} d\mu,$$

where

$$\zeta = \theta x + (1 - \theta) y, \ \theta \in [0, 1].$$

Due to

$$|\mu - \zeta| \geq |\mu - \lambda| - |\lambda - \zeta| > \frac{|\mu - \lambda|}{2},$$

we have

$$T'_2 \leq CD \int_{2D < |\mu - \lambda| \leq \frac{3}{2}} \frac{8}{|\mu - \lambda|^3} d\mu \leq CD(1 + |\log D|).$$

(49)

For the term $I'_3$, we have

$$T'_3 \leq CD \int_{|\mu - \lambda| \geq \frac{3}{2}} \frac{1}{|\mu - \zeta|^3} d\mu$$

$$\leq CD \int_{|\mu - \lambda| \geq \frac{3}{2}} \frac{8}{|\mu - \lambda|^3} d\mu$$

$$\leq CD \int_{|\mu - \lambda| > 1} \frac{1}{|\mu - \lambda|^2} d\mu,$$

where we have used the fact that

$$|\mu - \zeta| \geq |\mu - \lambda| - |\lambda - \zeta| > \frac{|\mu - \lambda|}{2},$$

similar to the estimate (22) and (23), we get

$$T'_3 \leq CD.$$

(50)

Hence, by (48), (49) and (50), we obtain that

$$|E^N(t, x) - E^N(t, y)| \leq CD(1 + |\log D|).$$

The proof of Lemma 5.1 is completed.

Lemma 5.2. For any $t \in [0, T]$, we have

$$|\eta^N(t) - \eta^{N+K}(t)| \leq C \left( \int \int |f^N_0(y, w) - f^{N+K}_0(y, w)| dy dw \right)^{\frac{3}{2}} + C \int_0^t \psi(A^N(s)) ds.$$

Proof. According to the system (10), we have

$$|\eta^N(t) - \eta^{N+K}(t)| = \left| \int_0^t (E^N(s, \xi^N(s)) - E^{N+K}(s, \xi^{N+K}(s))) ds \right|$$

$$\leq \int_0^t |(E^N(s, \xi^N(s)) - E^{N+K}(s, \xi^{N+K}(s)))| ds.$$

We set

$$|E^N(s, \xi^N(s)) - E^{N+K}(s, \xi^{N+K}(s))| \leq \mathcal{E}_1 + \mathcal{E}_2,$$

where

$$\mathcal{E}_1 = |E^N(s, \xi^N(s)) - E^N(s, \xi^{N+K}(s))|,$$

$$\mathcal{E}_2 = |E^N(s, \xi^{N+K}(s)) - E^{N+K}(s, \xi^{N+K}(s))|.$$
For the term $\mathcal{E}_1$, by Lemma 5.1, we have
\[
\mathcal{E}_1 \leq C|\xi^{N+K}(s) - \xi^N(s)| \left(1 + \log|\xi^{N+K}(s) - \xi^N(s)|\right).
\] (51)

For the term $\mathcal{E}_2$, we have
\[
|E^N(s, \xi^{N+K}(s)) - E^{N+K}(s, \xi^{N+K}(s))|
= \left| \int \frac{\xi^{N+K}(s) - y}{|\xi^{N+K}(s) - y|^3} \rho^N(s, y)dy - \int \frac{\xi^{N+K}(s) - y}{|\xi^{N+K}(s) - y|^3} \rho^{N+K}(s, y)dy \right|.
\]
by the change of variables $(y, w) \to (Y^N(s), W^N(s))$ and $(y, w) \to (Y^{N+K}(s), W^{N+K}(s))$, we obtain that
\[
|E^N(s, \xi^{N+K}(s)) - E^{N+K}(s, \xi^{N+K}(s))|
= \left| \int \int \frac{\xi^{N+K}(s) - y}{|\xi^{N+K}(s) - y|^3} f^N_0(y, w)dydw - \int \int \frac{\xi^{N+K}(s) - y}{|\xi^{N+K}(s) - y|^3} f^{N+K}_0(y, w)dydw \right|
\]
where
\[
\mathcal{E}_2' = \left| \int \int \left( \frac{\xi^{N+K}(s) - Y^N(s)}{|\xi^{N+K}(s) - Y^N(s)|^3} - \frac{\xi^{N+K}(s) - Y^{N+K}(s)}{|\xi^{N+K}(s) - Y^{N+K}(s)|^3} \right) f^N_0(y, w)dydw \right|
\]
and
\[
\mathcal{E}_2'' = \left| \int \int \frac{\xi^{N+K}(s) - Y^{N+K}(s)}{|\xi^{N+K}(s) - Y^{N+K}(s)|^3} f^{N+K}_0(y, w)dydw \right|.
\]
For the term $\mathcal{E}_2'$, setting,
\[
a^{N,K}(t) = \sup_{(x,v) \in S, |x| \leq N} |X^N(t) - X^{N+K}(t)|,
\]
we have
\[
\mathcal{E}_2' \leq F_1 + F_2,
\]
where
\[
F_1 = \int \int_{|\xi^{N+K}(s) - Y^N(s)| \leq 2a^{N,K}(s)} \frac{1}{|\xi^{N+K}(s) - Y^N(s)|^2} \left| \int \frac{\xi^{N+K}(s) - y}{|\xi^{N+K}(s) - y|^3} \rho^N(s, y)dy \right| \left| \int \frac{\xi^{N+K}(s) - y}{|\xi^{N+K}(s) - y|^3} \rho^{N+K}(s, y)dy \right|
\]
and
\[
F_2 = \int \int_{|\xi^{N+K}(s) - Y^N(s)| > 2a^{N,K}(s)} \frac{1}{|\xi^{N+K}(s) - Y^N(s)|^2} \left| \int \frac{\xi^{N+K}(s) - y}{|\xi^{N+K}(s) - y|^3} \rho^N(s, y)dy \right| \left| \int \frac{\xi^{N+K}(s) - y}{|\xi^{N+K}(s) - y|^3} \rho^{N+K}(s, y)dy \right|.
\]
For the term $F_1$, we get
\[
F_1 \leq \int \int_{|\xi^{N+K}(s)-Y^N(s)| \leq 2\alpha^{N,K}(s)} \frac{f_0^N(y,w)}{\|\xi^{N+K}(s)-Y^N(s)\|^2} dydw \\
+ \int \int_{|\xi^{N+K}(s)-Y^N(s)| \leq 2\alpha^{N,K}(s)} \frac{f_0^N(y,w)}{\|\xi^{N+K}(s)-Y^N(s)\|^2} dydw,
\]

For the first term on the right hand side of the above inequality, due to
\[
|\xi^{N+K}(s) - Y^{N+K}(s)| \\
\leq |\xi^{N+K}(s) - Y^N(s)| + |Y^N(s) - Y^{N+K}(s)| \leq 3\alpha^{N,K}(s),
\]
we obtain that
\[
\int \int_{|\xi^{N+K}(s)-Y^N(s)| \leq 2\alpha^{N,K}(s)} \frac{f_0^N(y,w)}{\|\xi^{N+K}(s)-Y^N(s)\|^2} dydw \\
\leq \|f_0\|_{L^\infty} \int \int_{3\alpha^{N,K}(s)} \frac{1}{|\xi^{N+K}(s)-Y^N(s)|^2} dydw \\
\leq C\alpha^{N,K}(s)|Y^{N+K}(s)|^3 \leq C\alpha^{N,K}(s),
\]

by the change of variables $(Y^{N+K}(s), W^{N+K}(s)) \rightarrow (y, w)$, we get
\[
\int \int_{|\xi^{N+K}(s)-Y^N(s)| \leq 2\alpha^{N,K}(s)} \frac{f_0^N(y,w)}{\|\xi^{N+K}(s)-Y^N(s)\|^2} dydw \\
\leq \|f_0\|_{L^\infty} \int \int_{|w| \leq V^{N+K}(s)} \frac{1}{|\xi^{N+K}(s)-y|^2} dydw \\
\leq C\alpha^{N,K}(s)|V^{N+K}(s)|^3 \leq C\alpha^{N,K}(s),
\]

where we have used the estimate (46). Similar to the estimate above, we have
\[
\int \int_{|\xi^{N+K}(s)-Y^N(s)| \leq 2\alpha^{N,K}(s)} \frac{f_0^N(y,w)}{\|\xi^{N+K}(s)-Y^N(s)\|^2} dydw \leq C\alpha^{N,K}(s).
\]

Thus, we get
\[
F_1 \leq C\alpha^{N,K}(s).
\]

For the term $F_2$, by the mean value theorem, we deduce that
\[
F_2 \leq 2 \int \int_{|\xi^{N+K}(s)-Y^N(s)| > 2\alpha^{N,K}(s)} \frac{f_0^N(y,w)}{|\xi^{N+K}(s)-\zeta|^3} |Y^N(s) - Y^{N+K}(s)| dydw,
\]
where
\[
\zeta = \theta Y^N(s) + (1-\theta)Y^{N+K}(s), \quad \theta \in [0, 1].
\]

Due to
\[
a^{N,K}(t) = \sup_{(x,v)\in S} |X^N(t) - X^{N+K}(t)|,
\]
we obtain that
\[
|\xi^{N+K}(s) - \zeta^N(s)| \geq |\xi^{N+K}(s) - Y^N(s)| - |Y^N(s) - \zeta^N(s)| \\
\geq |\xi^{N+K}(s) - Y^N(s)| - (1-\theta)|Y^N(s) - Y^{N+K}(s)| \\
\geq |\xi^{N+K}(s) - Y^N(s)| - |Y^N(s) - Y^{N+K}(s)| \\
\geq \frac{|\xi^{N+K}(s) - Y^N(s)|}{2}.
\]
Then, we obtain that
\[
F_2 \leq C a^{N,K}(s) \int \int_{|\xi^{N+K}(s) - Y^{N}(s)| > 2a^{N,K}(s)} \frac{f_N^N(y, w)}{|\xi^{N+K}(s) - Y^{N}(s)|^3} dy dw.
\]
By the change of variables \((Y^{N}(s), W^{N}(s)) \rightarrow (y, w)\), we get
\[
F_2 = C a^{N,K}(s) \int \int_{|\xi^{N+K}(s) - y| > 2a^{N,K}(s)} \frac{f_N^N(s, y, w)}{|\xi^{N+K}(s) - y|^3} dy dw.
\]
If \(2a^{N,K}(s) > 1\), we have
\[
F_2 \leq C a^{N,K}(s) \int \int_{|\xi^{N+K}(s) - y| > 1} \frac{f_N^N(s, y, w)}{|\xi^{N+K}(s) - y|^3} dy dw
\]
\[
= C a^{N,K}(s) \int \int_{1 < |\xi^{N+K}(s) - y| \leq 3R^N(t)} \frac{f_N^N(s, y, w)}{|\xi^{N+K}(s) - y|^3} dy dw
\]
\[
+ \int \int_{|\xi^{N+K}(s) - y| > 3R^N(t)} \frac{f_N^N(s, y, w)}{|\xi^{N+K}(s) - y|^3} dy dw
\]
\[
\leq C a^{N,K}(s) \int \int_{1 < |\xi^{N+K}(s) - y| \leq 3R^N(t)} \frac{f_N^N(s, y, w)}{|\xi^{N+K}(s) - y|^2} dy dw
\]
\[
+ \int \int_{|\xi^{N+K}(s) - y| > 3R^N(t)} \frac{f_N^N(s, y, w)}{|\xi^{N+K}(s) - y|^2} dy dw,
\]
similar to the estimate of (22) and (23), we have
\[
F_2 \leq C a^{N,K}(s)(C Q^N(R^N(s), s)^{\frac{7}{2}} + C),
\]
by Proposition 1 and the estimate of (46), we obtain that
\[
F_2 \leq C a^{N,K}(s)\nu^N(s)^{\frac{1}{2}} \leq C a^{N,K}(s).
\]
If \(2a^{N,K}(s) \leq 1\), we have
\[
F_2 \leq C a^{N,K}(s) \int \int_{2a^{N,K}(s) < |\xi^{N+K}(s) - y| \leq 1} \frac{f_N^N(s, y, w)}{|\xi^{N+K}(s) - y|^3} dy dw
\]
\[
+ C a^{N,K}(s) \int \int_{|\xi^{N+K}(s) - y| > 1} \frac{f_N^N(s, y, w)}{|\xi^{N+K}(s) - y|^3} dy dw.
\]
For the first term on the right hand side of the above inequality, we have
\[
C a^{N,K}(s) \int \int_{2a^{N,K}(s) < |\xi^{N+K}(s) - y| \leq 1} \frac{f_N^N(s, y, w)}{|\xi^{N+K}(s) - y|^3} dy dw
\]
\[
\leq C \|f_0\|_{L^\infty} a^{N,K}(s) \int_{|y| \leq \nu^N(s)} dw \int_{|\xi^{N+K}(s) - y| \leq 1} \frac{1}{|\xi^{N+K}(s) - y|^3} dy
\]
\[
\leq C \|f_0\|_{L^\infty} a^{N,K}(s) \nu^N(s)^3 \log a^{N,K}(s).
\]
By the estimate (46), we have
\[
C a^{N,K}(s) \int \int_{2a^{N,K}(s) < |\xi^{N+K}(s) - y| \leq 1} \frac{f_N^N(s, y, w)}{|\xi^{N+K}(s) - y|^3} dy dw
\]
\[
\leq C a^{N,K}(s) \log a^{N,K}(s).
\]
Thus, we deduce that
\[
F_2 \leq C a^{N,K}(s)(1 + |\log a^{N,K}(s)|).
\]
(53)
By (52) and (53), we obtain that
\[ E'_2 \leq F_1 + F_2 \leq C a^{N,K}(s)(1 + | \log a^{N,K}(s) |). \]  

(54)

For the term \( E''_2 \), we have
\[ E''_2 \leq \int \int_{|\xi^{N,K} - Y^{N,K(s)}| \leq R} \frac{|f^N_0(y,w) - f^{N+K}_0(y,w)|}{|\xi^{N,K}(s) - Y^{N,K(s)}|^2} dydw 
+ \int \int_{|\xi^{N,K} - Y^{N,K(s)}| > R} \frac{|f^N_0(y,w) - f^{N+K}_0(y,w)|}{|\xi^{N,K}(s) - Y^{N,K(s)}|^2} dydw, \]

(55)

where we choose
\[ R = \left( \int \int |f^N_0(y,w) - f^{N+K}_0(y,w)| dydw \right)^{\frac{1}{2}}. \]

Firstly, for the first term on the right hand side of the inequality (55), by the change of variables \((Y^{N,K(s)}, W^{N,K(s)}) \rightarrow (y, w)\), we have
\[ \int \int_{|\xi^{N,K} - Y^{N,K(s)}| \leq R} \frac{|f^N_0(y,w) - f^{N+K}_0(y,w)|}{|\xi^{N,K}(s) - Y^{N,K(s)}|^2} dydw \leq 2 \|f_0\|_{L^\infty} \int \int_{|\xi^{N,K} - Y^{N,K(s)}| \leq R} \frac{1}{|\xi^{N,K}(s) - Y^{N,K(s)}|^2} dydw \]
\[ = 2 \|f_0\|_{L^\infty} \int_{|w| \leq Y^{N,K(s)}} \int_{|\xi^{N,K(s)} - y| \leq R} \frac{1}{|\xi^{N,K}(s) - y|^2} dydw \leq CR \nu^{N,K(s)} \leq CR, \]

where we have used the estimate (46).

Secondly, for the second term on the right hand side of the inequality (55), by Hölder inequality, we deduce that
\[ \int \int_{|\xi^{N,K} - Y^{N,K(s)}| > R} \frac{|f^N_0(y,w) - f^{N+K}_0(y,w)|}{|\xi^{N,K}(s) - Y^{N,K(s)}|^2} dydw \leq \sqrt{2} \|f_0\|_{L^\frac{1}{2}} \left( \int \int |f^N_0(y,w) - f^{N+K}_0(y,w)| dydw \right)^{\frac{1}{2}} 
\cdot \left( \int \int_{|\xi^{N,K} - Y^{N,K(s)}| > R} \frac{1}{|\xi^{N,K}(s) - Y^{N,K(s)}|^4} dydw \right)^{\frac{1}{2}} \leq CR^{-\frac{1}{2}} \nu^{N,K(s)} \left( \int \int |f^N_0(y,w) - f^{N+K}_0(y,w)| dydw \right)^{\frac{1}{2}} 
\leq CR^{-\frac{1}{2}} \left( \int \int |f^N_0(y,w) - f^{N+K}_0(y,w)| dydw \right)^{\frac{1}{2}}, \]

where we have used the estimate (46).
Hence, we obtain that
\[ \mathcal{E}_2'' \leq CR + CR^{-\frac{1}{2}} \left( \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dy dw \right)^{\frac{1}{2}} \]
\[ \leq C \left( \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dy dw \right)^{\frac{1}{2}}. \] (56)

By (54) and (56), we get
\[ E_2 = |E^N(s, \xi^{N+K}(s)) - E^{N+K}(s, \xi^{N+K}(s))| \]
\[ \leq \mathcal{E}_2' + \mathcal{E}_2'' \]
\[ \leq C \left( \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dy dw \right)^{\frac{1}{2}} + Ca^{N,K}(s)(1 + |\log a^{N,K}(s)|). \] (57)

By (51) and (57), we obtain that
\[ |E^N(s, \xi^{N}(s)) - E^{N+K}(s, \xi^{N+K}(s))| \]
\[ \leq C \psi(a^{N,K}(s)) + C \psi(|\xi^{N+K}(s) - \xi^{N}(s)|) \]
\[ + C \left( \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dy dw \right)^{\frac{1}{2}}, \]

where we set
\[ \psi(r) = r(1 + |\log r|), \quad r > 0. \]

Then, we observe that
\[ \psi'(r) \geq 0, \]

hence, we deduce that
\[ |E^N(s, \xi^{N}(s)) - E^{N+K}(s, \xi^{N+K}(s))| \]
\[ \leq C \psi(A^{N,K}(s)) + C \left( \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dy dw \right)^{\frac{1}{2}}. \]

Finally, we have
\[ |\eta^N(t) - \eta^{N+K}(t)| \]
\[ \leq C \left( \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dy dw \right)^{\frac{1}{2}} + C \int_t^0 \psi(A^{N,K}(s)) ds. \]

The proof of Lemma 5.2 is completed. \qed

Lemma 5.3. For any \( t \in [0, T] \), we have
\[ |V^N(t) - V^{N+K}(t)| \]
\[ \leq C \int_0^t (\psi(A^{N,K}(s)) + A^{N,K}(s)) ds + C \left( \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dy dw \right)^{\frac{1}{2}}. \]

Proof. According to the system (9), we have
\[ |V^N(t) - V^{N+K}(t)| \]
\[ \leq \int_0^t |E^N(s, X^N(s)) - E^{N+K}(s, X^{N+K}(s))| + |F(s, X^N(s)) - F(s, X^{N+K}(s))| ds. \]
Firstly, for the term \( |F_N(s, X_N(s)) - F_{N+K}(s, X_{N+K}(s))| \), we have
\[
|F_N(s, X_N(s)) - F_{N+K}(s, X_{N+K}(s))| \\
= \left| \frac{1}{|X_N(s) - \xi_N(s)|^2} - \frac{1}{|X_{N+K}(s) - \xi_{N+K}(s)|^2} \right| \\
\leq \left( \frac{|X_N(s) - \xi_N(s)| + |X_{N+K}(s) - \xi_{N+K}(s)|}{|X_{N+K}(s) - \xi_{N+K}(s)|^2 |X_N(s) - \xi_N(s)|^2} \right) |X_N(s) - \xi_N(s)| - |X_{N+K}(s) - \xi_{N+K}(s)| \\
\leq C \left( |X_N(s) - X_{N+K}(s)| + |\xi_N(s) - \xi_{N+K}(s)| \right),
\]
where we have used the fact that \( |X_N(s) - \xi_N(s)| > C \) and \( |X_{N+K}(s) - \xi_{N+K}(s)| > C \).

Secondly, the estimate of the term \( |E_N(s, X_N(s)) - E_{N+K}(s, X_{N+K}(s))| \) is similar to the estimate of \( |E_N(s, \xi_N(s)) - E_{N+K}(s, \xi_{N+K}(s))| \), hence, we have
\[
|E_N(s, X_N(s)) - E_{N+K}(s, X_{N+K}(s))| \\
\leq C\psi(A^{N,K}(s)) + C \left( \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dydw \right)^{\frac{1}{2}}.
\]

Finally, we obtain that
\[
|V_N(t) - V_{N+K}(t)| \\
\leq C \int_0^t (\psi(A^{N,K}(s)) + A^{N,K}(s)) ds + C \left( \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dydw \right)^{\frac{1}{2}}.
\]

The proof of Lemma 5.3 is completed. \( \square \)

**Proof of Proposition 4.** By Lemma 5.2 and Lemma 5.3, we obtain that
\[
B^{N,K}(t) \leq C \left[ \left( \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dydw \right)^{\frac{1}{2}} \right. \\
\left. + \int_0^t \psi(A^{N,K}(s)) ds + \int_0^t A^{N,K}(s) ds \right].
\]

Moreover, we have the fact that
\[
A^{N,K}(t) \leq \int_0^t B^{N,K}(s) ds.
\]

Hence, we deduce that
\[
C^{N,K}(t) \leq C \left[ \left( \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dydw \right)^{\frac{1}{2}} \right. \\
\left. + \int_0^t \psi(C^{N,K}(s)) ds + \int_0^t C^{N,K}(s) ds \right] \\
\leq C \left[ \left( \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dydw \right)^{\frac{1}{2}} + \int_0^t \psi(C^{N,K}(s)) ds \right],
\]
so that, we have
\[
C^{N,K}(t) \leq \left( C \int \int |f_0^N(y, w) - f_0^{N+K}(y, w)| dydw \right)^{\frac{\exp(Ct)}{3}}.
\]
The proof of Proposition 4 is completed.

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E-mail address: gang_li1989@163.com
E-mail address: xwzhang@hust.edu.cn