On a Spectral Sequence for Twisted Cohomologies*

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Abstract Let \((\Omega^*(M), d)\) be the de Rham cochain complex for a smooth compact closed manifold \(M\) of dimension \(n\). For an odd-degree closed form \(H\), there is a twisted de Rham cochain complex \((\Omega^*(M), d + H)\) and its associated twisted de Rham cohomology \(H^*(M, H)\). The authors show that there exists a spectral sequence \(\{E_{r}^{p,q}, d_r\}\) derived from the filtration \(F_p(\Omega^*(M)) = \bigoplus_{i \geq p} \Omega^i(M)\) of \(\Omega^*(M)\), which converges to the twisted de Rham cohomology \(H^*(M, H)\). It is also shown that the differentials in the spectral sequence can be given in terms of cup products and specific elements of Massey products as well, which generalizes a result of Atiyah and Segal. Some results about the indeterminacy of differentials are also given in this paper.

Keywords Spectral sequence, Twisted de Rham cohomology, Massey product, Differential

2000 MR Subject Classification 58J52, 55T99, 81T30

1 Introduction

Let \(M\) be a smooth compact closed manifold of dimension \(n\), and \(\Omega^*(M)\) be the space of smooth differential forms over \(\mathbb{R}\) on \(M\). We have the de Rham cochain complex \((\Omega^*(M), d)\), where \(d : \Omega^p(M) \to \Omega^{p+1}(M)\) is the exterior differentiation, and its cohomology \(H^*(M)\) (the de Rham cohomology). The de Rham cohomology with coefficients in a flat vector bundle is an extension of the de Rham cohomology.

The twisted de Rham cohomology was first studied by Rohm and Witten [13] for the antisymmetric field in superstring theory. By analyzing the massless fermion states in the string sector, Rohm and Witten obtained the twisted de Rham cochain complex \((\Omega^*(M), d + H_3)\) for a closed 3-form \(H_3\), and mentioned the possible generalization to a sum of odd closed forms. A key feature in the twisted de Rham cohomology is that the theory is not integer-graded but (like \(K\)-theory) is filtered with the grading mod 2. This has a close relation with the twisted \(K\)-theory and the Atiyah-Hirzebruch spectral sequence (see [1]).

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*This work was supported by the National Natural Science Foundation of China (No. 11171161) and the Scientific Research Foundation for the Returned Overseas Chinese Scholars of the State Education Ministry (No. 2012940).
Let $H$ be $\sum_{i=1}^{n-1} H_{2i+1}$, where $H_{2i+1}$ is a closed $(2i+1)$-form. Then one can define a new operator $D = d + H$ on $\Omega^*(M)$, where $H$ is understood as an operator acting by exterior multiplication (for any differential form $\omega$, $H(\omega) = H \wedge \omega$). As in [1, 13], there is a filtration on $(\Omega^*(M), D)$ as follows:

$$K_p = F_p(\Omega^*(M)) = \bigoplus_{i \geq p} \Omega^i(M).$$

This filtration gives rise to a spectral sequence

$$\{E^{r,q}_2, d_r\}$$

converging to the twisted de Rham cohomology $H^*(M, H)$ with

$$E^{p,q}_2 = \begin{cases} H^p(M), & q \text{ is even,} \\ 0, & q \text{ is odd.} \end{cases}$$

For convenience, we first fix some notations in this paper. The notation $[r]$ denotes the greatest integer part of $r \in \mathbb{R}$. In the spectral sequence (1.2), for any $[y_p]_k \in E^{p,q}_k$, $[y_p]_{k+t}$ represents its class to which $[y_p]_k$ survives in $E^{p,q}_{k+t}$. In particular, as in Proposition 3.2, for $x_p \in E_1^{p,q}$, $[x_p]_2 \in [x_p]_3 \in E_2^{p,q} = E_3^{p,q}$ represents the de Rham cohomology class $[x_p]$. $d_r[x_p]$ represents a class in $E^{p+r,q-r+1}_2$, which survives to $d_r[x_p] \in E^{p+r,q-r+1}_2$.

In [13, Appendix I], Rohm and Witten first gave a description of the differentials $d_3$ and $d_5$ for the case $D = d + H_3$. Atiyah and Segal [1] showed a method about how to construct the differentials in terms of Massey products, and gave a generalization of Rohm and Witten’s result: The iterated Massey products with $H_3$ give (up to sign) all the higher differentials of the spectral sequence for the twisted cohomology (see [1, Proposition 6.1]). Mathai and Wu [9, p. 5] considered the general case of $H = \sum_{i=1}^{n-1} H_{2i+1}$ and claimed, without proof, that $d_2 = d_4 = \cdots = 0$, while $d_3, d_5, \cdots$ are given by the cup products with $H_3, H_5, \cdots$ and the higher Massey products with them. Motivated by the method in [1], we give an explicit description of the differentials in the spectral sequence (1.2) in terms of Massey products.

We now describe our main results. Let $A$ denote a defining system for the $n$-fold Massey product $\langle x_1, x_2, \ldots, x_n \rangle$, and $c(A)$ denote its related cocycle (see Definition 5.1). Then

$$\langle x_1, x_2, \ldots, x_n \rangle = \{c(A) \mid A \text{ is a defining system for } \langle x_1, x_2, \ldots, x_n \rangle \}$$

by Definition 5.2. To obtain our desired theorems by specific elements of Massey products, we restrict the allowable choices of defining systems for Massey products (see [14]). By Theorems 4.1–4.2 in this paper, there are defining systems for the two Massey products that we need (see Lemma 5.1). The notation $\langle H_3, \ldots, H_3, x_p \rangle_A$ in Theorem 1.1 below denotes a cohomology class in $H^*(M)$ represented by $c(A)$, where $A$ is a defining system obtained by Theorem 4.1 (see Definition 5.3). Similarly, the notation $\langle H_{12s+1}, \ldots, H_{12s+1}, x_p \rangle_A$ in Theorem 1.2 below denotes a cohomology class in $H^*(M)$ represented by $c(A)$, where $A$ is a defining system obtained by Theorem 4.2 (see Definition 5.3).
Theorem 1.1 For $H = \sum_{i=1}^{n-1} H_{2i+1}$ and $[x_p]_{2t+3} \in E_{2t+3}^{p,q}$ ($t \geq 1$), the differential of the spectral sequence (1.2), i.e., $d_{2t+3} : E_{2t+3}^{p,q} \rightarrow E_{2t+3}^{p+2t+3, q-2t-2}$, is given by

$$d_{2t+3}[x_p]_{2t+3} = (-1)^t[(H_3, \cdots, H_{t+1}, x_p)_{A}]_{2t+3},$$

and $[(H_3, \cdots, H_{t+1}, x_p)_{A}]_{2t+3}$ is independent of the choice of the defining system $A$ obtained from Theorem 4.1.

Specializing Theorem 1.1 to the case $H = H_{2s+1}$ ($s \geq 2$), we obtain

$$d_{2t+3}[x_p]_{2t+3} = (-1)^t[(0, \cdots, 0, x_p)_{A}]_{2t+3}. \quad (1.5)$$

Obviously, much information has been concealed in the above expression. In particular, we give a more explicit expression of differentials for this special case, which is compatible with Theorem 1.1 (see Remark 5.6).

Theorem 1.2 For $H = H_{2s+1}$ ($s \geq 1$) only and $[x_p]_{2t+3} \in E_{2t+3}^{p,q}$ ($t \geq 1$), the differential of the spectral sequence (1.2), i.e., $d_{2t+3} : E_{2t+3}^{p,q} \rightarrow E_{2t+3}^{p+2t+3, q-2t-2}$, is given by

$$d_{2t+3}[x_p]_{2t+3} = \begin{cases} [H_{2s+1} \wedge x_p]_{2t+3}, & t = s - 1, \\ (-1)^{t-l}[(H_{2s+1}, \cdots, H_{2s+1}, x_p)_{B}]_{2t+3}, & t = ls - 1 \ (l \geq 2), \\ 0, & \text{otherwise}, \end{cases}$$

and $[(H_{2s+1}, \cdots, H_{2s+1}, x_p)_{B}]_{2t+3}$ is independent of the choice of the defining system $B$ obtained from Theorem 4.2.

Atiyah and Segal [1] gave the differential expression in terms of Massey products when $H = H_3$ (see [1, Proposition 6.1]). Obviously, the result of Atiyah and Segal is a special case of Theorem 1.2.

Some of the results above are known to experts in this field, but there is a lack of mathematical proof in the literature.

This paper is organized as follows. In Section 2, we recall some backgrounds about the twisted de Rham cohomology. In Section 3, we consider the structure of the spectral sequence converging to the twisted de Rham cohomology, and give the differentials $d_i$ ($1 \leq i \leq 3$) and $d_2k$ ($k \geq 1$). With the formulas of the differentials in $E_{2t+3}^{p,q}$ in Section 4, Theorems 1.1 and 1.2 are proved in Section 5. In Section 6, we discuss the indeterminacy of differentials of the spectral sequence (1.2).

2 Twisted de Rham Cohomology

For completeness, in this section, we recall some knowledge about the twisted de Rham cohomology. Let $M$ be a smooth compact closed manifold of dimension $n$, and $\Omega^*(M)$ be the
space of smooth differential forms on $M$. We have the de Rham cochain complex $(\Omega^*(M), d)$ with the exterior differentiation $d : \Omega^p(M) \to \Omega^{p+1}(M)$, and its cohomology $H^*(M)$ (the de Rham cohomology).

Let $H$ denote $\sum_{i=1}^{[\frac{n+1}{2}]} H_{2i+1}$, where $H_{2i+1}$ is a closed $(2i+1)$-form. Define a new operator $D = d + H$ on $\Omega^*(M)$, where $H$ is understood as an operator acting by exterior multiplication (for any differential form $w$, $H(w) = H \wedge w$, also denoted by $H \wedge$). It is easy to show that

$$D^2 = (d + H)^2 = d^2 + dH + Hd + H^2 = 0.$$  

However, $D$ is not homogeneous on the space of smooth differential forms $\Omega^*(M) = \bigoplus_{i \geq 0} \Omega^i(M)$.

Define $\Omega^*(M)$ to be a new (mod 2) grading as follows:

$$\Omega^*(M) = \Omega^o(M) \oplus \Omega^e(M),$$

(2.1)

where

$$\Omega^o(M) = \bigoplus_{i \equiv 1 \text{ (mod 2)}} \Omega^i(M), \quad \Omega^e(M) = \bigoplus_{i \equiv 0 \text{ (mod 2)}} \Omega^i(M).$$

(2.2)

Then $D$ is homogeneous for this new (mod 2) grading,

$$\Omega^o(M) \xrightarrow{D} \Omega^o(M) \xrightarrow{D} \Omega^o(M).$$

Define the twisted de Rham cohomology groups of $M$ as follows:

$$H^o(M, H) = \frac{\ker[D : \Omega^o(M) \to \Omega^o(M)]}{\text{im}[D : \Omega^o(M) \to \Omega^o(M)]},$$

(2.3)

$$H^e(M, H) = \frac{\ker[D : \Omega^e(M) \to \Omega^e(M)]}{\text{im}[D : \Omega^e(M) \to \Omega^e(M)]}. $$

(2.4)

Remark 2.1 (i) The twisted de Rham cohomology groups $H^*(M, H)$ ($* = o, e$) depend on the closed form $H$ but not just on its cohomology class. If $H$ and $H'$ are cohomologous, then $H^*(M, H) \cong H^*(M, H')$ (see [1, Section 6]).

(ii) The twisted de Rham cohomology is also an important homotopy invariant (see [9, Section 1.4]).

Let $E$ be a flat vector bundle over $M$, and $\Omega^i(M, E)$ be the space of smooth differential $i$-forms on $M$ with values in $E$. A flat connection on $E$ gives a linear map

$$\nabla^E : \Omega^i(M, E) \to \Omega^{i+1}(M, E),$$

such that for any smooth function $f$ on $M$ and any $\omega \in \Omega^i(M, E)$,

$$\nabla^E(f \omega) = df \wedge \omega + f \cdot \nabla^E \omega, \quad \nabla^E \circ \nabla^E = 0.$$  

Similarly, define $\Omega^*(M, E)$ to be a new (mod 2) grading as follows:

$$\Omega^*(M, E) = \Omega^o(M, E) \oplus \Omega^e(M, E).$$

(2.5)
Then $D^E = \nabla^E + H_\wedge$ is homogenous for the new (mod 2) grading,

$$
\Omega^*(M, E) = \bigoplus_{i \geq 0} \Omega^i(M, E), \quad \Omega^p(M, E) = \bigoplus_{i \geq 0} \Omega^i(M, E). \quad (2.6)
$$

Then $D^E = \nabla^E + H_\wedge$ is homogenous for the new (mod 2) grading,

$$
\Omega^*(M, E) \xrightarrow{D^E} \Omega^p(M, E) \xrightarrow{D^E} \Omega^*(M, E).
$$

Define the twisted de Rham cohomology groups of $E$ as follows:

$$
H^p(M, E, H) = \frac{\ker[D^E : \Omega^p(M, E) \to \Omega^p(M, E)]}{\text{im}[D^E : \Omega^p(M, E) \to \Omega^p(M, E)]}, \quad (2.7)
$$

$$
H^*(M, E, H) = \frac{\ker[D^E : \Omega^*(M, E) \to \Omega^*(M, E)]}{\text{im}[D^E : \Omega^*(M, E) \to \Omega^*(M, E)]}. \quad (2.8)
$$

Results proved in this paper are also true for the twisted de Rham cohomology groups $H^*(M, E, H)$ ($* = o, e$) with twisted coefficients in $E$ without any change.

### 3 A Spectral Sequence for Twisted de Rham Cohomology and Its Differentials $d_i$ (1 ≤ $i$ ≤ 3), $d_{2k}$ ($k \geq 1$)

Recall $D = d + H$ and $H = \bigoplus_{i=1}^{[\frac{m}{2}]} H_{2i+1}$, where $H_{2i+1}$ is a closed $(2i+1)$-form. Define the usual filtration on the graded vector space $\Omega^*(M)$ to be

$$
K_p = F_p(\Omega^*(M)) = \bigoplus_{i \geq p} \Omega^i(M),
$$

and $K = K_0 = \Omega^*(M)$. The filtration is bounded and complete,

$$
K \equiv K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_n \supset K_{n+1} = \{0\}. \quad (3.1)
$$

We have $D(K_p) \subset K_p$ and $D(K_p) \subset K_{p+1}$. The differential $D(= d + H)$ does not preserve the grading of the de Rham complex. However, it does preserve the filtration $\{K_p\}_{p \geq 0}$.

The filtration $\{K_p\}_{p \geq 0}$ gives an exact couple (with bidegree) (see [12]). For each $p$, $K_p$ is a graded vector space with

$$
K_p = (K_p \cap \Omega^p(M)) \oplus (K_p \cap \Omega^p(M)) = K^o_p \oplus K^e_p,
$$

where $K^o_p = K_p \cap \Omega^p(M)$ and $K^e_p = K_p \cap \Omega^p(M)$. The cochain complex $(K_p, D)$ is induced by $D : \Omega^*(M) \longrightarrow \Omega^*(M)$. In a way similar to (2.4), there are two well-defined cohomology groups $H^*_D(K_p)$ and $H^*_D(K_p)$. Note that a cochain complex with grading

$$
K_p/K_{p+1} = (K^o_p/K^o_{p+1}) \oplus (K^e_p/K^e_{p+1})
$$

derives cohomology groups $H^*_D(K_p/K_{p+1})$ and $H^*_D(K_p/K_{p+1})$. Since $D(K_p) \subset K_{p+1}$, we have $D = 0$ in the cochain complex $(K_p/K_{p+1}, D)$.
Lemma 3.1 For the cochain complex \((K_p/K_{p+1}, D)\), we have

\[
H^p_D(K_p/K_{p+1}) \cong \begin{cases} 
\Omega^p(M), & p \text{ is odd,} \\
0, & p \text{ is even,}
\end{cases}
\]

\[
H^p_D(K_p/K_{p+1}) \cong \begin{cases} 
\Omega^p(M), & p \text{ is even,} \\
0, & p \text{ is odd.}
\end{cases}
\]

Proof If \(p\) is odd, then

\[
K_p \cap \Omega^p(M) = K_{p+1} \cap \Omega^p(M) \text{ and } (K_p \cap \Omega^p(M))/(K_{p+1} \cap \Omega^p(M)) = 0.
\]

We have

\[
(K_p \cap \Omega^p(M))/(K_{p+1} \cap \Omega^p(M)) = K_p^e/K_{p+1}^e \cong \Omega^p(M),
\]

\[
H^p_D(K_p/K_{p+1}) \cong \Omega^p(M), \quad H^p_D(K_p/K_{p+1}) = 0.
\]

Similarly, for even \(p\), we have

\[
H^p_D(K_p/K_{p+1}) \cong \Omega^p(M), \quad H^p_D(K_p/K_{p+1}) = 0.
\]

By the filtration (3.1), we obtain a short exact sequence of cochain complexes

\[
0 \rightarrow K_{p+1} \xrightarrow{i} K_p \xrightarrow{j} K_p/K_{p+1} \rightarrow 0,
\]

(3.2)

which gives rise to a long exact sequence of cohomology groups

\[
\cdots \rightarrow H^{p+q}_D(K_{p+1}) \xrightarrow{i^*} H^{p+q}_D(K_p) \xrightarrow{j^*} H^{p+q}_D(K_p/K_{p+1})
\]

\[
\xrightarrow{\delta} H^{p+q+1}_D(K_{p+1}) \xrightarrow{i^*} H^{p+q+1}_D(K_p) \xrightarrow{j^*} \cdots.
\]

(3.3)

Note that in the exact sequence above,

\[
H^i_D(K_p) = \begin{cases} 
H^i_D(K_p), & i \text{ is even,} \\
H^i_D(K_{p+1}), & i \text{ is odd,}
\end{cases}
\]

\[
H^i_D(K_p/K_{p+1}) = \begin{cases} 
H^i_D(K_p/K_{p+1}), & i \text{ is even,} \\
H^i_D(K_p/K_{p+1}), & i \text{ is odd.}
\end{cases}
\]

Let

\[
E_1^{p,q} = H^{p+q}_D(K_p/K_{p+1}), \quad D_1^{p,q} = H^{p+q}_D(K_p), \quad i_1 = i^*, \quad j_1 = j^*, \quad k_1 = \delta.
\]

(3.4)

We get an exact couple from the long exact sequence (3.3)

\[
\begin{array}{ccc}
D_1^{*,*} & \xrightarrow{i_1} & D_1^{*,*} \\
& k_1 \downarrow & \downarrow j_1 \\
E_1^{*,*}
\end{array}
\]

(3.5)
Then we have that \( d \) and \( \{ \text{sequence} \) by the following:

\[
D_2^{*,*} \xrightarrow{i_2} D_2^{*,*} \xrightarrow{k_2} E_2^{*,*}
\]

by the following:

1. \( D_2^{*,*} = i_1 D_1^{*,*}, E_2^{*,*} = H_{d_1}(E_1^{*,*}) \).
2. \( i_2 = i_1 |_{D_2^{*,*}} \), also denoted by \( i_1 \).
3. If \( a_2 = i_1 a_1 \in D_2^{*,*} \), define \( j_2(a_2) = [j_1 a_1]_{d_1} \), where \([ \cdot ]_{d_1} \) denotes the cohomology class in \( H_{d_1}(E_1^{*,*}) \).
4. For \( [b]_{d_1} \in E_2^{*,*} = H_{d_1}(E_1^{*,*}) \), define \( k_2([b]_{d_1}) = k_1 b \in D_2^{*,*} \).

The derived couple (3.6) is also an exact couple, and \( j_2 \) and \( k_2 \) are well defined (see [6, 12]).

**Proposition 3.1**

(i) There exists a spectral sequence \( (E_p^{*,q}, d_r) \) derived from the filtration \( \{K_n\}_{n \geq 0} \), where \( E_1^{*,q} = H^+_D(K_p/K_{p+1}) \), \( d_1 = j_1 k_1 \), and \( E_2^{*,q} = H_{d_1}(E_1^{*,q}) \), \( d_2 = j_2 k_2 \). The bidegree of \( d_r \) is \( (r, 1-r) \).

(ii) The spectral sequence \( \{E_p^{*,q}, d_r\} \) converges to the twisted de Rham cohomology

\[
\bigoplus_{p+q=1} E_p^{*,q} \cong H^*(M, H), \quad \bigoplus_{p+q=0} E_p^{*,q} \cong H^*(M, H).
\]

**Proof** Since the filtration is bounded and complete, the proof follows from the standard algebraic topology method (see [12]).

**Remark 3.1**

1. Note that

\[
H^*_D(K_p) = \begin{cases} H^*_D(K_p), & i \text{ is even}, \\ H^*_D(K_p), & i \text{ is odd}, \end{cases}
\]

\[
H^*_D(K_p/K_{p+1}) = \begin{cases} H^*_D(K_p/K_{p+1}), & i \text{ is even}, \\ H^*_D(K_p/K_{p+1}), & i \text{ is odd}. \end{cases}
\]

Then we have that \( H^*_D(K_p) \) and \( H^*_D(K_p/K_{p+1}) \) are 2-periodic on \( i \). Consequently, the spectral sequence \( \{E_p^{*,q}, d_r\} \) is 2-periodic on \( q \).

2. There is also a spectral sequence converging to the twisted cohomology \( H^*(M, E, H) \) for a flat vector bundle \( E \) over \( M \).

**Proposition 3.2** For the spectral sequence in Proposition 3.1,

(i) The \( E_1^{*,*} \)-term is given by

\[
E_1^{*,q} = H^{p+q}_D(K_p/K_{p+1}) \cong \begin{cases} \Omega^p(M), & q \text{ is even}, \\ 0, & q \text{ is odd}, \end{cases}
\]

and \( d_1 x_p = dx_p \) for any \( x_p \in E_1^{0,q} \).
(ii) The $E_2^{p,q}$-term is given by

$$E_2^{p,q} = H_{d_1}(E_1^{p,q}) \cong \begin{cases} H^p(M), & q \text{ is even}, \\ 0, & q \text{ is odd}, \end{cases}$$

and $d_2 = 0$.

(iii) $E_3^{p,q} = E_2^{p,q}$ and $d_3[x_p] = [H_3 \wedge x_p]$ for $[x_p]_3 \in E_3^{p,q}$.

**Proof**

(i) By Lemma 3.1, we have the $E_1^{p,q}$-term as desired, and by definition, we obtain $d_1 = j_1k_1 : E_1^{p,q} \to E_1^{p+1,q}$. We only need to consider the case when $q$ is even, otherwise $d_1 = 0$. By (3.2) for odd $p$ (the case, when $p$ is even, is similar), we have a large commutative diagram

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
0 & K_{p+1}^e & K_p^e \\
& D & D & D \\
0 & K_{p+1}^o & K_p^o \\
& D & D & D \\
0 & K_{p+1}^e & K_p^e \\
& D & D & D \\
\vdots & \vdots & \vdots \\
\end{array}
\]

(3.8)

where the rows are exact and the columns are cochain complexes.

Let $x_p \in \Omega^p(M) \cong H_D^{p+q}(K_p/K_{p+1}) \cong E_1^{p,q}$, and

$$x = \sum_{i=0}^{\lfloor \frac{n-p}{2} \rfloor} x_{p+2i}$$

(3.9)

be an (inhomogeneous) form, where $x_{p+2i}$ is a $(p+2i)$-form $0 \leq i \leq \lfloor \frac{n-p}{2} \rfloor$. Then $x \in K_p^o$, $jx = x_p$ and $Dx \in K_p^e$. Also $Dx \in K_{p+1}^e$. By the definition of the homomorphism $\delta$ in (3.3), we have

$$k_1x_p = [Dx]_D.$$  

(3.10)

where $[\ ]_D$ is the cohomology class in $H_D(K_{p+1})$. The class $[Dx]_D$ is well defined and independent of the choices of $x_{p+2i}$ ($1 \leq i \leq \lfloor \frac{n-p}{2} \rfloor$) (see [3, p. 116]).

Choose $x_{p+2i} = 0$ ($1 \leq i \leq \lfloor \frac{n-p}{2} \rfloor$). Then we have

$$k_1x_p = [Dx]_D = [dx_p + H \wedge x_p]_D = \left[ dx_p + \sum_{t=1}^{\lfloor \frac{n-1}{2} \rfloor} H_{2t+1} \wedge x_p \right]_D \in H_D^{p+q+1}(K_{p+1}).$$
Thus, one obtains
\[ d_1 x_p = (j_1 k_1) x_p = j_1 (k_1 (x_p)) = j_1 \left[ dx_p + \sum_{l=1}^{(n-1)/2} H_{2l+1} \land x_p \right]_D = dx_p. \]

(ii) By the definition of the spectral sequence and (i), one obtains that \( E_2^{p,q} \cong H^p(M) \) when \( q \) is even, and \( E_2^{p,q} = 0 \) when \( q \) is odd. Note \( d_2 : E_2^{p,q} \rightarrow E_2^{p+2, q-1} \). It follows that \( d_2 = 0 \) by degree reasons.

(iii) Note that \( [x_p]_3 \in E_3^{p,q} \) implies \( dx_p = 0 \). Choosing \( x_{p+2i} = 0 \) for \( 1 \leq i \leq \left[ \frac{n-2}{2} \right] \), we get
\[ [Dx]_D = [H \land x_p]_D = \left[ \sum_{l=1}^{(n-1)/2} H_{2l+1} \land x_p \right]_D \in H_2^{p+q+1}(K_{p+1}), \]
where \( x \) is given in the proof of (i). Note
\[ H_2^{p+q+1}(K_{p+1}) \xrightarrow{j_1} H_2^{p+q+1}(K_{p+3}) \xrightarrow{j_1} H_2^{p+q+1}(K_{p+3}/K_{p+4}) \]
which implies that \( d_3[k]_3 = (j_3)_3[k]_3 = j_3[k_3 x_p]_3 = j_3[k_1 x_p] = j_3[Dx]_D = [j_1((i_1^{-1})^2[Dx]_D)]_3 = [H_3x_p]_3 \),
\[ d_3[x_p] = [H_3x_p], \]
where the first, second and fourth identities follow from the definitions of \( d_3, k_3 \) and \( j_3 \), respectively, and the third and last identities follow from (3.10) and (3.11), respectively. By (ii), \( d_2 = 0 \), so \( E_3^{p,q} = E_2^{p,q} \). Then we have
\[ d_3[x_p] = [H_3x_p]. \]

**Corollary 3.1** \( d_{2k} = 0 \) for \( k \geq 1 \). Therefore, for \( k \geq 1 \),
\[ E_4^{p,q} = E_2^{p,q}. \]

**Proof** Note \( d_{2k} : E_2^{p,q} \rightarrow E_2^{p+2k, q+1-2k} \). By Proposition 3.2(ii), if \( q \) is odd, then \( E_2^{p,q} = 0 \), which implies that \( E_2^{p,q} = 0 \). By degree reasons, we have \( d_{2k} = 0 \) and \( E_2^{p,q} = E_2^{p,q} \) for \( k \geq 1 \).

The differential \( d_3 \) for the case \( H = H_3 \) is shown in [1, Section 6], and the \( E_2^{p,q} \)-term is also known.

4 **Differentials** \( d_{2t+3} (t \geq 1) \) in Terms of Cup Products

In this section, we will show that the differentials \( d_{2t+3} (t \geq 1) \) can be given in terms of cup products.
We first consider the general case of \( H = \sum_{i=1}^{n-1} H_{2i+1} \). For \([x_p]_{2t+3} \in E^{p,q}_{2t+3}\), we let \( x = \sum_{j=0}^{\lfloor n/2 \rfloor} x_{p+2j} \in F_p(\Omega^*(M))\). Then we have

\[
Dx = \left( d + \sum_{i=1}^{\lfloor n/2 \rfloor} H_{2i+1} \right) \left( \sum_{j=0}^{\lfloor n/2 \rfloor} x_{p+2j} \right)
\]

\[
= dx_p + \sum_{j=0}^{\lfloor n/2 \rfloor - 1} \left( dx_{p+2j+2} + \sum_{i=1}^{j+1} H_{2i+1} \wedge x_{p+2(j-i)+2} \right). \tag{4.1}
\]

Denote \( y = Dx = \sum_{j=0}^{\lfloor n/2 \rfloor} y_{p+2j+1} \), where

\[
\begin{align*}
y_{p+1} &= dx_p, \\
y_{p+2j+3} &= dx_{p+2j+2} + \sum_{i=1}^{j+1} H_{2i+1} \wedge x_{p+2(j-i)+2}, \quad 0 \leq j \leq \left[ \frac{n-p}{2} \right] - 1. \tag{4.2}
\end{align*}
\]

**Theorem 4.1** For \([x_p]_{2t+3} \in E^{p,q}_{2t+3}\) \((t \geq 1)\), there exist \( x_{p+2i} = x_{p+2i}^{(t)} \) \((1 \leq i \leq t)\), such that \( y_{p+2j+1} = 0 \) \((0 \leq j \leq t)\) and

\[
d_{2t+3}[x_p]_{2t+3} = \left[ \sum_{i=1}^{t} H_{2i+1} \wedge x_{p+2(t-i)+2}^{(t)} + H_{2t+3} \wedge x_p \right]_{2t+3},
\]

where the \((p+2i)\)-form \( x_{p+2i}^{(t)} \) depends on \( t \).

**Proof** The theorem is shown by mathematical induction on \( t \).

When \( t = 1 \), \([x_p]_{2t+3} = [x_p]_5\). \([x_p]_5 \in E^{5,5}_5\) implies that \( dx_p = 0 \) and \( d_3[x_p] = [H_3 \wedge x_p] = 0 \) by Proposition 3.2. Thus there exists a \((p+2)\)-form \( v_1 \), such that \( H_3 \wedge x_p = d(-v_1) \). We can choose \( x_{p+2}^{(1)} = v_1 \) to get \( y_{p+3} = dx_{p+2}^{(1)} + H_3 \wedge x_p = dv_1 + H_3 \wedge x_p = 0 \) from (4.2). Noting

\[
H^{p+q+1}_D(K_{p+1}) \xrightarrow{i_1^1} H^{p+q+1}_D(K_{p+5}) \xrightarrow{j_1} H^{p+q+1}_D(K_{p+5}/K_{p+6})
\]

\[
[Dx]_D \xrightarrow{(i_1^{-1})^4} [Dx]_D \xrightarrow{j_1} y_{p+5}, \tag{4.3}
\]

we obtain

\[
d_5[x_p]_5 = j_5[k_5[x_p]_5] = j_5[k_1 x_p] = j_5[Dx]_D = [j_1(i_1^{-1})^4[Dx]_D]_5 = [y_{p+5}]_5. \tag{4.4}
\]

The reasons for the identities in (4.4) are similar to those of (3.12). Thus, we have

\[
d_5[x_p]_5 = [dx_{p+4} + H_3 \wedge x_{p+2}^{(1)} + H_5 \wedge x_p]_5 = [H_3 \wedge x_{p+2}^{(1)} + H_5 \wedge x_p]_5,
\]

where the first identity follows from (4.4) and the definition of \( y_{p+5} \) in (4.2), and the second one follows from the fact that \( dx_{p+4} \) vanishes in \( E^{5,*}_5 \). Hence the result holds for \( t = 1 \).
Suppose that the result holds for $t \leq m - 1$. Now we show that the theorem also holds for $t = m$.

From $[x_p]_{2m+1} \in E_{2m+1}^{p,q}$, we have $[x_p]_{2m+1} \in E_{2m+1}^{p,q}$ and $d_{2m+1}[x_p]_{2m+1} = 0$. By induction, there exist $x_{p+2i}^{(m-1)}$ ($1 \leq i \leq m - 1$), such that

$$
\begin{align*}
\begin{cases} 
  y_{p+1}^{(m-1)}(x_p) = dx_p = 0, \\
  y_{p+3}^{(m-1)}(x_p) = dx_{p+2}^{(m-1)} + H_3 \land x_p = 0, \\
  y_{p+2i+1}^{(m-1)}(x_p) = dx_{p+2i}^{(m-1)} + \sum_{j=1}^{i-1} H_{2j+1} \land x_{p+2(1-j)}^{(m-1)} + H_{2i+1} \land x_p \\
    = 0 (2 \leq i \leq m - 1), \\
  d_{2m+1}[x_p]_{2m+1} = \left[ \sum_{i=1}^{m-1} H_{2i+1} \land x_{p+2(i-1)}^{(m-1)} + H_{2m+1} \land x_p \right]_{2m-1} = 0.
\end{cases}
\end{align*}
$$

(4.5)

By $d_{2m} = 0$ and the last equation in (4.5), there exists a $(p + 2)$-form $w_{p+2}$, such that

$$
\left[ \sum_{i=1}^{m-1} H_{2i+1} \land x_{p+2(i-1)}^{(m-1)} + H_{2m+1} \land x_p \right]_{2m-1} = d_{2m-1}[w_{p+2}]_{2m-1}.
$$

(4.6)

By induction and $[w_{p+2}]_{2m-1} \in E_{2m-1}^{p+2,q-2}$, there exist $w_{p+2(i+1)}^{(m-2)}$ ($1 \leq i \leq m - 2$), such that

$$
\begin{align*}
\begin{cases} 
  y_{p+3}^{(m-2)}(w_{p+2}) = dw_{p+2} = 0, \\
  y_{p+5}^{(m-2)}(w_{p+2}) = dw_{p+4}^{(m-2)} + H_3 \land w_{p+2} = 0, \\
  y_{p+2i+3}^{(m-2)}(w_{p+2}) = dw_{p+2(i+1)}^{(m-2)} + \sum_{j=1}^{i-1} H_{2j+1} \land w_{p+2(i-j+1)}^{(m-2)} + H_{2i+1} \land w_{p+2} \\
    = 0 (2 \leq i \leq m - 2), \\
  d_{2m-1}[w_{p+2}]_{2m-1} = \left[ \sum_{i=1}^{m-2} H_{2i+1} \land w_{p+2(i-1)}^{(m-2)} + H_{2m-1} \land w_{p+2} \right]_{2m-1}.
\end{cases}
\end{align*}
$$

(4.7)

By (4.6) and the last equation in (4.7), we obtain

$$
\left[ \sum_{i=1}^{m-2} H_{2i+1} \land (x_{p+2(i-1)}^{(m-1)} - w_{p+2(i-1)}^{(m-2)}) + H_{2m-1} \land (x_{p+2}^{(m-1)} - w_{p+2}) + H_{2m+1} \land x_p \right]_{2m-1} = 0.
$$

Note that $d_{2m-2} = 0$, and it follows that there exists a $(p + 4)$-form $w_{p+4}$, such that

$$
\left[ \sum_{i=1}^{m-2} H_{2i+1} \land (x_{p+2(i-1)}^{(m-1)} - w_{p+2(i-1)}^{(m-2)}) + H_{2m-1} \land (x_{p+2}^{(m-1)} - w_{p+2}) + H_{2m+1} \land x_p \right]_{2m-3} = d_{2m-3}[w_{p+4}]_{2m-3}.
$$

Keeping the same iteration process as mentioned above, we have

$$
\left[ \sum_{i=1}^{2} H_{2i+1} \land (x_{p+2(i-1)}^{(m-1)} - w_{p+2(i-1)}^{(m-2)}) + \sum_{j=1}^{m-3} w_{p+2(i-1)}^{(m-2-j)} \right].
$$
From (4.2), by a direct computation, we have
\[ y_{p+2(m-1)} = y_{p+2} - w_{p+2} \]
Comparing (4.10) with (4.2), we choose at this time
\[ x_{p+2i} = x_{p+2i} - \sum_{j=1}^{i-1} w_{p+2i-j} - w_{p+2i} \quad (2 \leq i \leq m - 1), \]

From (4.2), by a direct computation, we have
\[
\begin{align*}
    y_{p+1} &= y_{p+1} (x_p) = 0, \\
    y_{p+2i-1} &= y_{p+2i-1} (x_p) - \sum_{j=1}^{i-1} y_{p+2i-1} (w_{p+j}) = 0 \quad (2 \leq i \leq m), \\
    y_{p+2m+1} &= 0.
\end{align*}
\]
Note

\[
H_{p+q+1}^p(K_{p+1}) \xrightarrow{(i_1)^{2(m+1)}} H_{p+q+1}^p(K_{p+2m+3}) \xrightarrow{j_1} H_{p+q+1}^p(K_{p+2m+3}/K_{p+2m+4})
\]  
(4.13)

By the similar reasons as in (3.12), the following identities hold:

\[
d_{2m+3}[x_p]_{2m+3} = j_{2m+3}k_{2m+3}[x_p]_{2m+3}
= j_{2m+3}(k_1x_p)
= j_{2m+3}[Dx]_D
= [j_1(i_1^{-1})^{2(m+1)}][Dx]_D|_{2m+3}
= [y_{p+2m+3}]_{2m+3}.
\]  
(4.14)

So we have

\[
d_{2m+3}[x_p]_{2m+3} = [y_{p+2m+3}]_{2m+3}
= \left[ dx_{p+2m+2} + \sum_{i=1}^{m} H_{2i+1} \wedge x_p^{(m)}_{p+2(m-i)+1} + H_{2m+3} \wedge x_p \right]_{2m+3} \quad \text{(by (4.2))}
= \left[ \sum_{i=1}^{m} H_{2i+1} \wedge x_p^{(m)}_{p+2(m-i)+1} + H_{2m+3} \wedge x_p \right]_{2m+3},
\]

showing that the result also holds for \( t = m \).

The proof of the theorem is completed.

**Remark 4.1** Note that \( x_{p+2i}^{(t_i)} (1 \leq i \leq t) \) depend on \( t \), and that \( x_{p+2i}^{(t_1)} \not= x_{p+2i}^{(t_2)} \) depend on the condition \( t_1 \not= t_2 \) generally. \( x_{p+2i}^{(t_i)} (1 \leq i \leq t) \) are related to \( x_{p+2j}^{(t-1)} (1 \leq j \leq t-1, j \leq i) \).

Now we consider the special case in which \( H = H_{2s+1} \) (\( s \geq 1 \)) only. For this special case, we will give a more explicit result which is stronger than Theorem 4.1.

For \( x = \sum_{j=0}^{n-p} x_{p+2j} \), we have

\[
Dx = (d + H_{2s+1}) \left( \sum_{j=0}^{n-p} x_{p+2j} \right) = \sum_{j=0}^{s-1} dx_{p+2j} + \sum_{j=s}^{n-p} (dx_{p+2j} + H_{2s+1} \wedge x_{p+2(j-s)}).
\]

Denote

\[
\begin{align*}
    y_{p+2j+1} &= dx_{p+2j}, & 0 \leq j \leq s-1, \\
    y_{p+2j+3} &= dx_{p+2j+2} + H_{2s+1} \wedge x_{p+2(j-s)+2}, & s-1 \leq j \leq \left\lfloor \frac{n-p}{2} \right\rfloor - 1.
\end{align*}
\]  
(4.15)

Then \( Dx = \sum_{j=0}^{n-p} y_{p+2j+1} \).
Theorem 4.2 For $H = H_{2s+1}$ $(s \geq 1)$ only and $[x_p]_{2t+3} \in E_{2t+3}^{p,q}$ $(t \geq 1)$, there exist $x_{p+2i} = x_{p+2i}^{(l)}$, $x_{p+2(i-1)+s+2j} = 0$ and $x_{p+2(s+2k)} = 0$ for $1 \leq i \leq \left[\frac{s}{l}\right]$, $1 \leq j \leq s-1$ and $1 \leq k \leq t - \left[\frac{t}{s}\right]$, such that $y_{p+2u+1} = 0$ $(0 \leq u \leq t)$ and

$$d_{2t+3}[x_p]_{2t+3} = \begin{cases} [H_{2s+1} \wedge x_p]_{2s+1}, & t = s - 1, \\ [H_{2s+1} \wedge x_p^{(l-1)}]_{2t+3}, & t = ls - 1 (l \geq 2), \\ 0, & \text{otherwise}, \end{cases}$$

where the $(p+2i)$-form $x_{p+2i}^{(l)}$ depend on $\left[\frac{s}{l}\right]$.

Proof We prove the theorem by mathematical induction on $s$.

When $s = 1$, the result follows from Theorem 4.1.

When $s \geq 2$, we prove the result by mathematical induction on $t$. We first show that the result holds for $t = 1$. Note that $[x_p]_5 \in E_5^{p,q}$ implies $y_{p+1} = dx_p = 0$. Choose $x_{p+2} = 0$ and make $y_{p+3} = 0$.

(i) When $s = 2$, by (4.4), we have

$$d_5[x_p]_5 = [y_{p+5}]_5 = [dx_{p+4} + H_5 \wedge x_p]_5 = [H_5 \wedge x_p]_5.$$ 

(ii) When $s \geq 3$, by (4.4), we have

$$d_5[x_p]_5 = [y_{p+5}]_5 = [dx_{p+4}]_5 = 0.$$

Combining (i) and (ii), we have that the theorem holds for $t = 1$.

Suppose that the theorem holds for $t \leq m - 1$. Now we show that the theorem also holds for $t = m$.

Case 1 $2 \leq m \leq s - 1$.

By induction, the theorem holds for $1 \leq t \leq m - 1$. Choosing $x_{p+2i} = 0$ $(1 \leq i \leq m)$, from (4.15), we easily get that $y_{p+2j+1} = 0$ $(0 \leq j \leq m)$. By (4.14)–(4.15), we have

$$d_{2m+3}[x_p]_{2m+3} = [y_{p+2m+3}]_{2m+3} = \begin{cases} [dx_{p+2(m+1)}]_{2m+3}, & 2 \leq m \leq s - 2, \\ [dx_{p+2(m+1)} + H_{2s+1} \wedge x_p]_{2m+3}, & m = s - 1, \\ 0, & 2 \leq m \leq s - 2, \\ [H_{2s+1} \wedge x_p]_{2s+1}, & m = s - 1. \end{cases}$$

Case 2 $m = ls - 1 (l \geq 2)$.

By induction, the theorem holds for $t = m - 1 = ls - 2$. Thus, there exist $x_{p+2i} = x_{p+2i}^{(l)}$, $x_{p+2(i-1)+s+2j} = 0$ and $x_{p+2(l-1)+s+2k} = 0$ for $1 \leq i \leq l - 1$, $1 \leq j \leq s-1$ and $1 \leq k \leq s - 2$, such that $y_{p+2u+1} = 0$ $(0 \leq u \leq ls - 2)$. Choosing $x_{p+2(l-1)} = 0$, by (4.15), we get

$$y_{p+2(l-1)+1} = dx_{p+2(l-1)} + H_{2s+1} \wedge x_{p+2(l-1)s-2} = 0 + H_{2s+1} \wedge 0 = 0.$$
Then we have
\[ d_{2(l-1)+3}[x_p]_{2(l-1)+3} = [y_p+2ls+1]_{2(l-1)+3} \quad \text{(by (4.14))} \]
\[ = [dx_p+2ls + H_{2s+1} \wedge x_p^{(l-1)}]_{2(l-1)+3} \quad \text{(by (4.15))} \]
\[ = [H_{2s+1} \wedge x_p^{(l-1)}]_{2(l-1)+3}. \]

**Case 3** \( m = ls \) \((l \geq 1)\).

By induction, there exist \( x_{p+2is} = x_{p+2is}^{(l)} \), \( x_{p+2(l-1)s+2j} = 0 \) and \( x_{p+2(l-1)s+2k} = 0 \) for \( 1 \leq i \leq l-1, 1 \leq j \leq s-1 \) and \( 1 \leq k \leq s-1 \), such that \( y_{p+2u+1} = 0 \) \((0 \leq u \leq ls-1)\). By the same method as in Theorem 4.1, one has that there exist \( x_{p+2is} = x_{p+2is}^{(l)} \), \( x_{p+2(l-1)s+2j} = 0 \) and \( x_{p+2(l-1)s+2k} = 0 \) for \( 1 \leq i \leq l, 1 \leq j \leq s-1 \) and \( 1 \leq k \leq s-1 \), such that \( y_{p+2u+1} = 0 \) \((0 \leq u \leq ls)\). By (4.14)–(4.15) and \( x_{p+2ls-2s+2} = 0 \), we have
\[ d_{2ls+3}[x_p]_{2ls+3} = [y_{p+2ls+3}]_{2ls+3} \]
\[ = [dx_{p+2ls+2} + H_{2s+1} \wedge x_{p+2ls-2s+2}]_{2ls+3} \]
\[ = 0. \]

**Case 4** \( ls < m < (l+1)s - 1 \) \((l \geq 1)\).

By induction, there exist \( x_{p+2is} = x_{p+2is}^{(l)} \), \( x_{p+2(l-1)s+2j} = 0 \) and \( x_{p+2ls+2k} = 0 \) for \( 1 \leq i \leq l, 1 \leq j \leq s-1 \) and \( 1 \leq k \leq m - ls - 1 \), such that \( y_{p+2u+1} = 0 \) \((0 \leq u \leq m - 1)\). Choose \( x_{p+2m} = 0 \) and make \( y_{p+2m+1} = 0 \). By (4.14)–(4.15) and \( x_{p+2m-2s+2} = 0 \), we have
\[ d_{2m+3}[x_p]_{2m+3} = [y_{p+2m+3}]_{2m+3} \]
\[ = [dx_{p+2m+2} + H_{2s+1} \wedge x_{p+2m-2s+2}]_{2m+3} \]
\[ = 0. \]

Combining Cases 1–4, we have that the result holds for \( t = m \), and the proof is completed.

**Remark 4.2** (1) Theorems 4.1–4.2 show that the differentials in the spectral sequence (1.2) can be computed in terms of cup products with \( H_{2s+1} \)'s. The existence of \( x_{p+2is}^{(l)} \)'s and \( x_{p+2is}^{(l)} \)'s in Theorems 4.1–4.2 plays an essential role in proving Theorems 1.1–1.2, respectively. Theorems 4.1–4.2 give a description of the differentials at the level of \( E_{2t+3}^{p,q} \) for the spectral sequence (1.2), which was ignored in the previous studies of the twisted de Rham cohomology in \([1, 9]\).

(2) Note that Theorem 4.2 is not a corollary of Theorem 4.1, and it cannot be obtained from Theorem 4.1 directly.

5 Differentials \( d_{2l+3} \) \((t \geq 1)\) in Terms of Massey Products

The Massey product is a cohomology operation of higher order introduced in \([8]\), which generalizes the cup product. May \([10]\) showed that the differentials in the Eilenberg-Moore spectral sequence associated with the path-loop fibration of a path connected, simply connected space are completely determined by higher order Massey products. Kraines and Schochet \([5]\) also described the differentials in Eilenberg-Moore spectral sequence by Massey products. In
order to describe the differentials $d_{2t+3}$ ($t \geq 1$) in terms of Massey products, we first recall briefly the definition of Massey products (see [4, 10–12]). Then the main theorems in this paper will be shown.

Because of different conventions in the literature used to define Massey products, we present the following definitions. If $x \in \Omega^p(M)$, the symbol $\mathbf{v}$ will denote $(-1)^{1+\deg v}v = (-1)^{1+p}x$. We first define the Massey triple product.

Let $x_1$, $x_2$, $x_3$ be closed differential forms on $M$ of degrees $r_1$, $r_2$, $r_3$ with $[x_1][x_2] = 0$ and $[x_2][x_3] = 0$, where $[\ ]$ denotes the de Rham cohomology class. Thus, there are differential forms $v_1$ of degree $r_1 + r_2 - 1$ and $v_2$ of degree $r_2 + r_3 - 1$, such that $dv_1 = \mathbf{v}_1 \wedge x_2$ and $dv_2 = \mathbf{v}_2 \wedge x_3$. Define the $(r_1 + r_2 + r_3 - 1)$-form

$$\omega = \mathbf{v}_1 \wedge x_3 + \mathbf{v}_2 \wedge x_2.$$  (5.1)

Then $\omega$ satisfies

$$d(\omega) = (-1)^{r_1+r_2}dv_1 \wedge x_3 + (-1)^{r_1}x_1 \wedge dv_2$$

$$= (-1)^{r_1+r_2}x_1 \wedge x_2 \wedge x_3 + (-1)^{r_1+r_2+1}x_1 \wedge x_2 \wedge x_3$$

$$= 0.$$

Hence a set of all the cohomology classes $[\omega]$ obtained by the above procedure is defined to be the Massey triple product $\langle x_1, x_2, x_3 \rangle$ of $x_1$, $x_2$ and $x_3$. Due to the ambiguity of $v_i$, $i = 1, 2$, the Massey triple product $\langle x_1, x_2, x_3 \rangle$ is a representative of the quotient group

$$H^{r_1+r_2+r_3-1}(M)/([x_1]H^{r_2+r_3-1}(M) + H^{r_1+r_2-1}(M)[x_3]).$$

**Definition 5.1** Let $(\Omega^*(M), d)$ be de Rham complex, and $x_1$, $x_2$, $\cdots$, $x_n$ be closed differential forms on $M$ with $[x_i] \in H^r(M)$. A collection of forms, $A = (a_{i,j})$ for $1 \leq i \leq j \leq k$ and $(i, j) \neq (1, n)$, is said to be a defining system for the $n$-fold Massey product $(x_1, x_2, \cdots, x_n)$ if

1. $a_{i,j} \in \Omega^{r_1+r_2+\cdots+r_j-j+i}(M)$,
2. $a_{i,i} = x_i$ for $i = 1, 2, \cdots, k$,
3. $d(a_{i,j}) = \sum_{r=1}^{j-1} \pi_{i,r} \wedge a_{r+1,j}$.

The $(r_1 + \cdots + r_n - n + 2)$-dimensional cocycle, $c(A)$, defined by

$$c(A) = \sum_{r=1}^{n-1} \pi_{i,r} \wedge a_{r+1,n} \in \Omega^{r_1+\cdots+r_n-n+2}(M)$$  (5.2)

is called the related cocycle of the defining system $A$.

**Remark 5.1** There is a unique matrix associated to each defining system $A$ as follows:

$$\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\
a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} \\
a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\
a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-2} & a_{n,n-1} & a_{n,n}
\end{pmatrix}_{n \times n}.$$
Definition 5.2 The $n$-fold Massey product $\langle x_1, x_2, \cdots, x_n \rangle$ is said to be defined, if there is a defining system for it. If it is defined, then $\langle x_1, x_2, \cdots, x_n \rangle$ consists of all classes $w \in H^{r_1+r_2+\cdots+r_n-n+2}(M)$ for which there exists a defining system $A$, such that $c(A)$ represents $w$.

Remark 5.2 There is an inherent ambiguity in the definition of the Massey product arising from the choices of defining systems. In general, the $n$-fold Massey product may or may not be a coset of a subgroup, but its indeterminacy is a subset of a matrix Massey product (see [10, Section 2]).

Based on Theorems 4.1–4.2, we have the following lemma on defining systems for the two Massey products we consider in this paper.

Lemma 5.1 (1) For $[x_p]_{2t+3} \in E^p_{2t+3}$ ($t \geq 1$), there are defining systems for $\langle H_3, \cdots, H_{t+1}, x_p \rangle$ obtained from Theorem 4.1.

(2) For $[x_p]_{2t+3} \in E^p_{2t+3}$, when $t = ls - 1$ ($l \geq 2$), there are defining systems for $\langle H_{2s+1}, \cdots, H_{2s+1}, x_p \rangle$ obtained from Theorem 4.2.

Proof (1) From Theorem 4.1, there exist $x_{p+2j}^{(t)}$ ($1 \leq j \leq t$), such that $y_{p+2i+1} = 0$ ($0 \leq i \leq t$) and $d_{2t+3}[x_p]_{2t+3} = \left[ \sum_{i=1}^t H_{2t+1} \wedge x_{p+2(t-i+1)}^{(t)} + H_{2t+3} \wedge x_p \right]_{2t+3}$. By Theorem 4.1 and (4.2), there exists a defining system $A = (a_{i,j})$ for $\langle H_3, \cdots, H_{t+1}, x_p \rangle$ as follows:

$$
\begin{align*}
& a_{i+2,t+2} = x_p, \\
& a_{i,t+k} = (-1)^k H_{2k+3}, \\
& a_{i,t+2} = (-1)^{t+1} x_p^{(t)},
\end{align*}
$$

(5.3)

for $1 \leq i \leq t - 1$, $0 \leq k < t$, to which the matrix associated is given by

$$
\begin{pmatrix}
H_3 & -H_5 & H_7 & \cdots & (-1)^{t-1} H_{2t+1} & (-1)^{t} H_{2t+3} \\
H_3 & -H_5 & \cdots & (-1)^{t-2} H_{2t-1} & (-1)^{t-1} H_{2t+1} & (-1)^{t} x_p^{(t)} \\
H_3 & \cdots & (-1)^{t-3} H_{2t-3} & (-1)^{t-2} H_{2t-1} & (-1)^{t-1} x_p^{(t)} & (-1)^{t} x_p^{(t)} \\
& \vdots & \vdots & \vdots & \vdots & \vdots \\
& H_3 & -H_5 & \cdots & (-1)^{t} x_p^{(t)} & (-1)^{t+1} x_p^{(t)} \\
& H_3 & \cdots & (-1)^{t} x_p^{(t)} & (-1)^{t+1} x_p^{(t)} & (-1)^{t+2} x_p^{(t)} \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
$$

(5.4)

The desired result follows.

(2) By Theorem 4.2, there exist $x_{p+2is} = x_{p+2is}^{(t-1)}$, $x_{p+2(i-1)s+2j} = 0$ and $x_{p+2(l-1)s+2k} = 0$ for $1 \leq i \leq l - 1$, $1 \leq j \leq s - 1$ and $1 \leq k \leq s - 1$, such that $y_{p+2i+1} = 0$ ($0 \leq i \leq t$) and $d_{2t+3}[x_p]_{2t+3} = [H_{2s+1} \wedge x_{p+2(l-1)s}]_{2t+3}$. By Theorem 4.2 and (4.15), there also exists a defining
system $A = (a_{i,j})$ for $(H_{2s+1}, \cdots, H_{2s+1}, x_p)$ as follows:

\[
\begin{cases}
    a_{i,j} = 0, & 1 \leq i < j \leq l, \\
    a_{i,i} = H_{2s+1}, & 1 \leq i \leq l, \\
    a_{i,t+1} = x_p, & 1 \leq i \leq l, \\
    a_{i,t+1} = (-1)^{l+1-i} x_p^{1/t+2(t+1-i)s}, & 2 \leq i \leq l,
\end{cases}
\]

(5.5)

to which the matrix associated is given by

\[
\begin{pmatrix}
    H_{2s+1} & 0 & 0 & \cdots & 0 & 0 \\
    H_{2s+1} & 0 & \cdots & 0 & 0 & (-1)^{l-1} x_p^{11/2} \times \text{A}_{s+1} \\
    H_{2s+1} & \cdots & 0 & 0 & (-1)^{l-2} x_p^{11/2} \times \text{B}_{s+1} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    H_{2s+1} & 0 & \cdots & (-1)^{l-1} x_p^{11/2} & (-1)^{l} x_p^{11/2} \\
    H_{2s+1} & \cdots & \cdots & \cdots & \cdots & \cdots \\
    H_{2s+1} & 0 & \cdots & (-1)^{l} x_p^{11/2} & (-1)^{l+1} x_p^{11/2} & (-1)^{l+2} x_p^{11/2} \\
\end{pmatrix}
\]

(5.6)

\[(l+1) \times (l+1)\]

The desired result follows.

To obtain our desired theorems by specific elements of Massey products, we restrict the allowable choices of defining systems for the two Massey products in Lemma 5.1 (see [14]). By Lemma 5.1, we give the following definitions.

**Definition 5.3**

(1) Given a class $[x_p]_{2t+3} \in E_{2t+3}^{p,q} \; (t \geq 1)$, a specific element of $(t + 2)$-fold Massey product $\langle H_3, \cdots, H_{3}, x_p \rangle_A$, denoted by $\langle H_3, \cdots, H_{3}, x_p \rangle_A$, is a class in $H^{p+2t+3}(M)$ represented by $c(A)$, where $A$ is a defining system obtained from Theorem 4.1. We define the $(t+2)$-fold allowable Massey product $\langle H_3, \cdots, H_{3}, x_p \rangle_A$, to be the set of all the cohomology classes $w \in H^{p+2t+3}(M)$ for which there exists a defining system $A$ obtained from Theorem 4.1, such that $c(A)$ represents $w$.

(2) Similarly, given a class $[x_p]_{2t+3} \in E_{2t+3}^{p,q} \; (t \geq 1)$, when $t = ls - 1 \; (l \geq 2)$, we define the specific element of $(l + 1)$-fold Massey product $\langle H_{2s+1}, \cdots, H_{2s+1}, x_p \rangle$ and the $(l + 1)$-fold allowable Massey product $\langle H_{2s+1}, \cdots, H_{2s+1}, x_p \rangle$, by replacing Theorem 4.1 by Theorem 4.2 in (1).

**Remark 5.3**

(1) From Definition 5.3, we can get the following:

$\langle H_3, \cdots, H_{3}, x_p \rangle_A \subseteq \langle H_3, \cdots, H_{3}, x_p \rangle_A$

(2) The allowable Massey product $\langle H_3, \cdots, H_{3}, x_p \rangle_A$, is less ambiguous than the general Massey product $\langle H_3, \cdots, H_{3}, x_p \rangle_A$. Take $\langle H_3, H_3, x_p \rangle_A$ in Definition 5.3 for example. Suppose
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\[ H = \sum_{i=1}^{n-1} H_{2i+1}. \]

By Theorem 4.1 and (4.2), there exist \( x^{(1)}_{p+2j} \), such that \( y_{p+2i+1} = 0 \) (0 \( \leq i \leq 1 \)) and \( d_5[x_p] = [H_3 \wedge x^{(1)}_{p+2} + H_5 \wedge x_p]_5 \). By Lemma 5.1, we get a defining system \( A \) for \( (H_3, H_3, x_p) \) and its related cocycle \( c(A) = -H_3 \wedge x^{(1)}_{p+2} - H_5 \wedge x_p \). Thus, we have

\[ \langle H_3, H_3, x_p \rangle_A = \langle \begin{array}{l} H_3 \\ H_3 \\ x_p \end{array} \rangle_A = \langle -H_3 \wedge x^{(1)}_{p+2} - H_5 \wedge x_p \rangle. \] (5.7)

Obviously, the indeterminacy of the allowable Massey product \( \langle H_3, H_3, x_p \rangle_* \) is \([H_3]H^{p+2}(M)\). However, in the general case, the indeterminacy of the Massey product \( \langle H_3, H_3, x_p \rangle_3 \) is \([H_3]H^{p+2}(M) + H^3(M)[x_p] \).

Similarly, the allowable Massey product \( \langle H_{2s+1}, \ldots, H_{2s+1}, x_p \rangle_* \) is less ambiguous than the general Massey product \( \langle H_{2s+1}, \ldots, H_{2s+1}, x_p \rangle \).

Now we begin to prove our main theorems.

**Proof of Theorem 1.1** By Lemma 5.1(1), there exist defining systems for \( \langle H_3, \ldots, H_3, x_p \rangle \)

given by Theorem 4.1. For any defining system \( A = (a_{i,j}) \) given by Theorem 4.1, by (5.4), we have

\[ c(A) = (-1)^t \left( \sum_{i=1}^{t} H_{2i+1} \wedge x^{(t)}_{p+2i+1} + H_2 x_p \right). \]

By Definition 5.3, we have

\[ \langle H_3, \ldots, H_3, x_p \rangle_A = [c(A)]. \] (5.8)

Then by Theorem 4.1, we have

\[ d_{2t+3}[x_p]_{2t+3} = \sum_{i=1}^{t} H_{2i+1} \wedge x^{(t)}_{p+2(t-i+1)} + H_2 x_p \]

Thus, we have \( d_{2t+3}[x_p]_{2t+3} = (-1)^t \langle H_3, \ldots, H_3, x_p \rangle_A \).

By the arbitrariness of \( A \), we have that \( \langle H_3, \ldots, H_3, x_p \rangle_A \) is independent of the choice of the defining system \( A \) obtained by Theorem 4.1.

**Example 5.1** For formal manifolds, which are manifolds with vanishing Massey products, it is easy to get

\[ E_4^{p,q} \cong E_{\infty}^{p,q} \]

by Theorem 1.1. Note that simply connected compact Kähler manifolds are an important class of formal manifolds (see [2]).
Remark 5.4  (1) From the proof of Theorem 1.1, we have that the specific element
\[
\langle H_3, \ldots, H_3, x_p \rangle_{A}^{t+1}
\]
represents a class in \(E_{2t+3}^{*,*}\). For two different defining systems \(A_1\) and \(A_2\) given by Theorem 4.1, we have
\[
\langle H_3, \ldots, H_3, x_p \rangle_{A_1}^{t+1} \neq \langle H_3, \ldots, H_3, x_p \rangle_{A_2}^{t+1}
\]
generally. However, in the spectral sequence (1.2), we have
\[
[(\langle H_3, \ldots, H_3, x_p \rangle_{A_1})]_{2t+3} = [(\langle H_3, \ldots, H_3, x_p \rangle_{A_2})_{2t+3}].
\]

(2) Since the indeterminacy of \(\langle H_3, \ldots, H_3, x_p \rangle_{*}\) does not affect our results, we will not
analyze the indeterminacy of Massey products in this paper.

(3) By Theorem 1.1, \(d_{2t+3}[x_p]_{2t+3} = (-1)^t[(\langle H_3, \ldots, H_3, x_p \rangle_{A})_{2t+3}] \) for \(t \geq 1\), which is ex-
pressed only by \(H_3\) and \(x_p\). From the proof of Theorem 1.1, we know that the above expression
conceals some information, because the other \(H_{2i+1}\)'s affect the result implicitly.

We have the following corollary (see [1, Proposition 6.1]).

Corollary 5.1  For \(H = H_3\) only and \([x_p]_{2t+3} \in E_{2t+3}^{p,q} (t \geq 1)\), we have that in the spectral
sequence (1.2),
\[
d_{2t+3}[x_p]_{2t+3} = (-1)^t[(\langle H_3, \ldots, H_3, x_p \rangle_{A})_{2t+3}],
\]
and \([(\langle H_3, \ldots, H_3, x_p \rangle_{A})]_{2t+3}\) is independent of the choice of the defining system \(A\) obtained from
Theorem 4.1.

Remark 5.5  (1) Because the definition of Massey products is different from the definition
in [1], the expression of differentials in Corollary 5.1 differs from the one in [1, Proposition 6.1].

(2) The two specific elements of \((\langle H_3, \ldots, H_3, x_p \rangle)^{t+1}\) in Theorem 1.1 and Corollary 5.1 are
completely different, and equal \([c(A_1)]\) and \([c(A_2)]\), respectively, where \(c(A_i) (i = 1, 2)\) are
related cocycles of the defining systems \(A_i (i = 1, 2)\) obtained from Theorem 4.1. The matrices
associated to the two defining systems are given by
\[
\begin{pmatrix}
H_3 & -H_5 & H_7 & \cdots & (-1)^{t-1}H_{2t+1} & (-1)^tH_{2t+3} \\
H_3 & -H_5 & \cdots & (-1)^{t-2}H_{2t-1} & (-1)^{t-1}H_{2t+1} & (-1)^{t-1}x_{p+2t}^{(t)} \\
H_3 & \cdots & (-1)^tH_{2t-3} & (-1)^{t-2}H_{2t-1} & (-1)^{t-1}x_{p+2t-2}^{(t)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
H_3 & -H_5 & \cdots & (-1)^tH_{2t} & (-1)^t2x_{p+4}^{(t)} \\
H_3 & \cdots & (-1)^tH_{2t+1} & (-1)^t2x_{p+2}^{(t)} \\
x_p & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}_{(t+2) \times (t+2)}
\]
and

\[
\begin{pmatrix}
H_3 & 0 & 0 & \cdots & 0 & 0 \\
H_3 & 0 & 0 & \cdots & 0 & 0 \\
H_3 & \cdots & 0 & 0 & (-1)^t x_{p+2t}^{(t)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
H_3 & 0 & \cdots & 0 & (-1)^{t-1} x_{p+2t-2}^{(t)} \\
\end{pmatrix}
\]

respectively. Here \(x_{p+2i}^{(t)} (1 \leq i \leq t)\) in the first matrix are different from those in the second one.

For \(H = H_{2s+1} (s \geq 2)\) only (i.e., in the case \(H_i = 0, i \neq 2s + 1\) and \([x_p]_{2t+3} \in E^{p,q}_{2t+3} (t \geq 1)\), we make use of Theorem 1.1 to get

\[
d_{2t+3}[x_p]_{2t+3} = (-1)^t \left( \sum_{i=1}^{t+1} A_i \right)_{2t+3}.
\]

Obviously, some information has been concealed in the expression above. Another description of the differentials for this special case is shown in Theorem 1.2.

**Proof of Theorem 1.2** When \(t = s - 1\), the result follows from Theorem 4.2.

When \(t = ts - 1 (l \geq 2)\), from Lemma 5.1(2), we know that there exist defining systems for \(\langle H_{2s+1}, \cdots, H_{2s+1}, x_p \rangle\) obtained from Theorem 4.2. For any defining system \(B\) given by Theorem 4.2, by (5.6), we get

\[
\langle H_{2s+1}, \cdots, H_{2s+1}, x_p \rangle_B = [c(B)].
\]

(5.10)

Then by Theorem 4.2, we have

\[
d_{2t+3}[x_p]_{2t+3} = \left[ H_{2s+1} \wedge x_{p+2(l-1)}^{(l-1)} \right]_{2t+3}
\]

\[
= (-1)^{l-1} \left[ \langle H_{2s+1}, \cdots, H_{2s+1}, x_p \rangle_B \right]_{2t+3}.
\]

Thus

\[
d_{2t+3}[x_p]_{2t+3} = (-1)^{l-1} \left[ \langle H_{2s+1}, \cdots, H_{2s+1}, x_p \rangle_B \right]_{2t+3}.
\]

By the arbitrariness of \(B\), we have that \([\langle H_{2s+1}, \cdots, H_{2s+1}, x_p \rangle_B \]_{2t+3} is independent of the choice of the defining system \(B\) obtained from Theorem 4.2.

For the rest cases of \(t\), the results follow from Theorem 4.2.

The proof of this theorem is completed.
Remark 5.6 We now use the special case $H = H_5$ and $d_9[x_p]_9$ to illustrate the compatibility between Theorems 1.1 and 1.2 for $s = 2$ and $t = 3$.

Note that in this case, we have $H_3 = 0$ and $H_i = 0$ for $i > 5$. By Theorem 1.1, we get the corresponding matrix associated to the defining system $A$ for $⟨0, 0, 0, 0, x_p⟩_A$ is

$$
\begin{pmatrix}
0 & -H_5 & 0 & 0 & -x_p^{(3)} \\
0 & -H_5 & 0 & -x_p^{(3)} & -x_p^{(3)} + 6 \\
0 & -H_5 & 0 & -x_p^{(3)} & -x_p^{(3)} + 4 \\
0 & -H_5 & 0 & -x_p^{(3)} & -x_p^{(3)} + 2 \\
0 & -H_5 & 0 & 0 & x_p
\end{pmatrix}_{5 \times 5}
$$

and

$$
d_9[x_p]_9 = -[0, 0, 0, 0, x_p]_A. \quad (5.12)
$$

By Theorem 1.2, in this case, the matrix associated to the defining system $B$ for $⟨H_5, H_5, x_p⟩_B$ is

$$
\begin{pmatrix}
H_5 & 0 & -x_p^{(1)} \\
H_5 & -x_p^{(1)} & -x_p^{(1)} + 4 \\
0 & -x_p^{(1)} & -x_p^{(1)} + 2 \\
0 & 0 & x_p
\end{pmatrix}_{3 \times 3}
$$

and

$$
d_9[x_p]_9 = -[⟨H_5, H_5, x_p⟩_B]. \quad (5.14)
$$

We claim that $⟨H_5, H_5, x_p⟩_* = (0, 0, 0, 0, x_p)_*$. For any defining system $B$ above, there is a defining system $\tilde{B}$

$$
\begin{pmatrix}
0 & -H_5 & 0 & 0 & 0 \\
0 & -H_5 & 0 & 0 & 0 \\
0 & -H_5 & 0 & -x_p^{(1)} & -x_p^{(1)} + 4 \\
0 & -H_5 & 0 & -x_p^{(1)} & -x_p^{(1)} + 2 \\
0 & 0 & 0 & 0 & x_p
\end{pmatrix}_{5 \times 5}
$$

for $⟨0, 0, 0, 0, x_p⟩$, which can be obtained from Theorem 4.1, such that

$$
(0, 0, 0, 0, x_p)_B = (H_5, H_5, x_p)_B.
$$

Hence $⟨H_5, H_5, x_p⟩_* \subseteq (0, 0, 0, 0, [x_p])_*$. On the other hand, for any defining system $A$ above, there also exists a defining system $\tilde{A}$

$$
\begin{pmatrix}
H_5 & 0 & -x_p^{(3)} \\
H_5 & -x_p^{(3)} & -x_p^{(3)} + 6 \\
0 & -x_p^{(3)} & -x_p^{(3)} + 4 \\
0 & -x_p^{(3)} & -x_p^{(3)} + 2 \\
0 & 0 & x_p
\end{pmatrix}_{3 \times 3}
$$

for $⟨H_5, H_5, x_p⟩$, which can be obtained from Theorem 4.2, such that

$$
⟨H_5, H_5, x_p⟩_\tilde{A} = (0, 0, 0, 0, x_p)_A.
$$
Therefore \( \langle 0, 0, 0, x_p \rangle_* \subseteq \langle H_5, H_5, x_p \rangle_* \), and thus the claim follows.

By Theorem 1.1 and Remark 5.3, we have
\[
\tilde{d}_5[y_p]_5 = -(0, 0, y_p)_4 = -[H_5 \wedge y_p]_5.
\]
By Theorem 1.2, \( \tilde{d}_5[y_p]_5 = [H_5 \wedge y_p]_5 \). By Proposition 3.4, \( \tilde{d}_4 = d_1 = d \) and \( \tilde{d}_3 = d_3 = 0 \). It follows that \( \tilde{d}_5 = d_5 \).

By Theorems 1.1 and 4.1, we have
\[
\tilde{d}_7[z_p]_7 = [(0, 0, 0, z_p)_4]_7 = [-H_5 \wedge z_{p+2}^{(2)}]_7,
\]
where \( z_{p+2}^{(2)} \) is an arbitrary \((p+2)\)-form satisfying \( d(z_{p+2}^{(2)}) = 0 \wedge z_p \). By Remark 5.4(2), we take \( z_{p+2}^{(2)} = 0 \). Then we have \( \tilde{d}_7[z_p]_7 = 0 \), i.e., \( \tilde{d}_7 = 0 \). At the same time, we also have \( \tilde{d}_7 = 0 \) from Theorem 1.2. Thus \( \tilde{d}_7 = d_7 = 0 \).

By \( E_1^{p,q} = K_1^{p,q} \), \( \tilde{d}_i = d_i \) for \( 1 \leq i \leq 7 \) and \( \langle H_5, H_5, x_p \rangle_* \equiv \langle 0, 0, 0, x_p \rangle_* \), we can conclude that \( \tilde{d}_9 = d_9 \) from (5.12) and (5.14).

### 6 The Indeterminacy of Differentials in the Spectral Sequence (1.2)

Let \([x_p], \in E_r^{p,q} \). The indeterminacy of \([x_p] \) is a normal subgroup \( G \) of \( H^*(M) \), which means that if there is another element \([y_p] \in H^p(M)\), which also represents the class \([x_p], \in E_r^{p,q} \), then \([y_p] - [x_p] \in G \).

In this section, we will show that for \( H = \sum_{i=1}^{2t+1} H_{2i+1} \) and \([x_p]_{2t+3} \), the indeterminacy of the differential \( d_{2t+3}[x_p] \in E_2^{p+2t+3,q-2t-2} \) is a normal subgroup of \( H^*(M) \).

From the long exact sequence (3.3), we have a commutative diagram
\[
\begin{array}{ccccccc}
\vdots & & & & & & \\
& \delta & \rightarrow & H_D^{p+q}(K_{p+1}) \\
\rightarrow & \delta & \rightarrow & H_D^{p+q}(K_{p+1}/K_{p+2}) & \delta & \rightarrow & H_D^{p+q+1}(K_{p+2}) & \rightarrow & \cdots \\
& i^* & & & & i^* & & \\
\vdots & & & & & & \\
& \delta & \rightarrow & H_D^{p+q}(K_{p}) \\
\rightarrow & \delta & \rightarrow & H_D^{p+q}(K_{p}/K_{p+1}) & \delta & \rightarrow & H_D^{p+q+1}(K_{p+1}) & \rightarrow & \cdots \\
& i^* & & & & i^* & & \\
\vdots & & & & & & \\
& \delta & \rightarrow & H_D^{p+q}(K_{p-1}) \\
\rightarrow & \delta & \rightarrow & H_D^{p+q}(K_{p-1}/K_{p}) & \delta & \rightarrow & H_D^{p+q+1}(K_{p}) & \rightarrow & \cdots \\
& i^* & & & & i^* & & \\
\vdots & & & & & & \\
\end{array}
\]

in which any sequence consisting of a vertical map \( i^* \) followed by two horizontal maps \( j^* \) and \( \delta \) and then a vertical map \( i^* \) followed again by \( j^*, \delta, \) and iteration of this is exact. From this diagram, there is a spectral sequence, in which \( E_1^{p,q} = H_D^{p+q}(K_{p}/K_{p+1}) \) and for \( r \geq 2, E_r^{p,q} \) is
defined to be the quotient $Z^{p,q}_r/B^{p,q}_r$, where
\begin{align*}
Z^{p,q}_r &= \delta^{-1}(i^{*r-1}H_D^{p+q+1}(K_{p+r})) , \\
B^{p,q}_r &= j^*(\ker[i^{*r-1} : H_D^{p+q}(K_p) \to H_D^{p+q}(K_{p-r+1})]).
\end{align*}

(6.2)

We also have a sequence of inclusions
\[ B^{p,q}_2 \subset \cdots \subset B^{p,q}_r \subset B^{p,q}_{r+1} \subset \cdots \subset Z^{p,q}_r \subset Z^{p,q}_{r+1} \subset \cdots \subset Z^{p,q}_2. \quad (6.3) \]

By [6–7], the $E_r^{*,*}$-term defined above is the same as the one in the spectral sequence (1.2). A similar argument about a homology spectral sequence is given in [15, p. 472–473].

\textbf{Theorem 6.1} Let $H = \bigoplus_{i=1}^{\infty} H_{2i+1}$ and $[x_p] \in E^{p,q}_2 (r \geq 3)$. Then the indeterminacy of $[x_p] \in E^{p,q}_2 \cong H^p(M)$ is the following normal subgroup of $H^p(M)$:
\[ \text{im}(\delta) : H_D^{p+q+1}(K_{p-r+1}/K_p) \to H_D^{p+q}(K_{p}/K_{p+1}) \]
where $\delta$ is the connecting homomorphism of the long exact sequence induced by the short exact sequence of cochain complexes
\[ 0 \longrightarrow K_p/K_{p+1} \xrightarrow{\bar{7}} K_{p-r+1}/K_{p+1} \xrightarrow{\bar{7}} K_{p-r+1}/K_p \longrightarrow 0. \]

\textbf{Proof} From the above tower (6.3), we get a tower of subgroups of $E^{p,q}_2$
\[ B^{p,q}_2 / B^{p,q}_2 \subset \cdots \subset B^{p,q}_r / B^{p,q}_{r+1} \subset \cdots \subset Z^{p,q}_r / B^{p,q}_2 \subset Z^{p,q}_{r+1} / B^{p,q}_{r+1} = E^{p,q}_2. \]

Note
\[ E^{p,q}_2 \cong (Z^{p,q}_2 / B^{p,q}_2) / (B^{p,q}_2 / B^{p,q}_{2+1}). \]

It follows that the indeterminacy of $[x_p]$ is the normal subgroup $B^{p,q}_2 / B^{p,q}_2$ of $H^p(M)$.

From the short exact sequences of cochain complexes
\[ 0 \longrightarrow K_p \xrightarrow{i^*} K_{p-r+1} \xrightarrow{j^*} K_{p-r+1}/K_p \longrightarrow 0, \]
\[ 0 \longrightarrow K_p/K_{p+1} \xrightarrow{\bar{7}} K_{p-r+1}/K_{p+1} \xrightarrow{\bar{7}} K_{p-r+1}/K_p \longrightarrow 0, \]
we can get the following long exact sequence of cohomology groups:
\begin{align*}
\cdots & \xrightarrow{\delta'} H_D^p(K_p) \xrightarrow{i^*} H_D^p(K_{p-r+1}) \xrightarrow{j^*} H_D^p(K_{p-r+1}/K_p) \xrightarrow{\delta'} \cdots , \\
\cdots & \xrightarrow{\bar{7}} H_D^p(K_p/K_{p+1}) \xrightarrow{\bar{7}} H_D^p(K_{p-r+1}/K_{p+1}) \xrightarrow{\bar{7}} H_D^p(K_{p-r+1}/K_p) \xrightarrow{\bar{7}} \cdots ,
\end{align*}

(6.4)

where $\delta'$ and $\bar{\delta}$ are the connecting homomorphisms.
Combining (3.3) and (6.4), we have the following commutative diagram of long exact sequences:

\[
\begin{array}{cccc}
H_{D}^{p+q-1}(K_{p-r+1}/K_{p}) & \xrightarrow{\delta} & H_{D}^{p+q}(K_{p}) & \xrightarrow{i^*} & H_{D}^{p+q}(K_{p-r+1}) \\
\downarrow{} & & \downarrow{} & & \uparrow{} \\
H_{D}^{p+q}(K_{p}/K_{p+1}) & \xrightarrow{\delta} & H_{D}^{p+q+1}(K_{p+1}) \\
\end{array}
\]

(6.5)

Using the above commutative diagram and the fact that \(i^{*r-1} = i^*\), we have

\[
B_{r}^{p,q} = j^*(\ker[i^{*r-1} : H_{D}^{p+q}(K_{p}) \rightarrow H_{D}^{p+q}(K_{p-r+1})])
\]
\[
= j^*(\ker[i^* : H_{D}^{p+q}(K_{p}) \rightarrow H_{D}^{p+q}(K_{p-r+1})])
\]
\[
\cong j^*(\text{im}[\delta' : H_{D}^{p+q-1}(K_{p-r+1}/K_{p}) \rightarrow H_{D}^{p+q}(K_{p})])
\]
\[
\cong \text{im}[\delta : H_{D}^{p+q-1}(K_{p-r+1}/K_{p}) \rightarrow H_{D}^{p+q}(K_{p}/K_{p+1})].
\]

When \(r = 2\), from (6.5), we have

\[
\overline{\delta} = \delta' j^* : H_{D}^{p+q-1}(K_{p-1}/K_{p}) \rightarrow H_{D}^{p+q}(K_{p}/K_{p+1}).
\]

From (3.4), it follows that \(\overline{\delta} = d_1\). By Proposition 3.2, \(\overline{\delta} = d\). Thus, we have

\[
B_{2}^{p,q} \cong \text{im}[\delta : H_{D}^{p+q-1}(K_{p-1}/K_{p}) \rightarrow H_{D}^{p+q}(K_{p}/K_{p+1})]
\]
\[
\cong \text{im}[d : \Omega^{p+1}(M) \rightarrow \Omega^p(M)].
\]

The desired result follows.

By Theorem 6.1, we obtain the following corollary.

**Corollary 6.1** In Theorem 1.1, for \(d_{2t+3,3}[x_p]_{2t+3} \in E_{2t+3}^{p+2t+3,q-2t-2}\), we have that the indeterminacy of \(d_{2t+3,3}[x_p]\) is a normal subgroup of \(H_{D}^{p+2t+3}(M)\)

\[
\frac{\text{im}[\overline{\delta} : H_{D}^{p+q}(K_{p+1}/K_{p+2t+3}) \rightarrow H_{D}^{p+q+1}(K_{p+2t+3}/K_{p+2t+4})]}{\text{im}[d : \Omega^{p+2t+2}(M) \rightarrow \Omega^{p+2t+3}(M)]},
\]

where \(d\) is just the exterior differentiation, and \(\overline{\delta}\) is the connecting homomorphism of the long exact sequence induced by the short exact sequence of cochain complexes

\[
0 \rightarrow K_{p+2t+3}/K_{p+2t+4} \xrightarrow{\overline{\delta}} K_{p+1}/K_{p+2t+4} \xrightarrow{\overline{\delta}} K_{p+1}/K_{p+2t+3} \rightarrow 0.
\]

**Proof** In Theorem 6.1, \(r, p\) and \(q\) are replaced by \(2t+3, p+2t+3\) and \(q-2t-2\), respectively. Then the desired result follows.

**Acknowledgements** The authors would like to express gratitude to Professor Jim Stasheff for his helpful comments, and thank the referees for their suggestions.
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