BINOMIAL TAILS FOR COMMUNITY ANALYSIS

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ABSTRACT
An important task of community discovery in networks is assessing significance of the results and robust ranking of the generated candidate groups. Often in practice, numerous candidate communities are discovered, and focusing the analyst’s time on the most salient and promising findings is crucial. We develop simple efficient group scoring functions derived from tail probabilities using binomial models. Experiments on synthetic and numerous real-world data provides evidence that binomial scoring leads to a more robust ranking than other inexpensive scoring functions, such as conductance. Furthermore, we obtain confidence values (p-values) that can be used for filtering and labeling the discovered groups. Our analyses shed light on various properties of the approach. The binomial tail is simple and versatile, and we describe two other applications for community analysis: degree of community membership (which in turn yields group-scoring functions), and the discovery of significant edges in the community-induced graph.

Keywords Scoring Communities · Community Ranking · Community Significance · Membership Significance · Edge Significance · Binomial Tail Bounds · Community Discovery · Networks

1 Introduction
Automated discovery of candidate communities in networks finds a variety of applications in physical and social sciences (biological and biochemical networks, physical and virtual human networks) [1, 2]. Given a graph representing binary relations among nodes, informally and intuitively, a community corresponds to a subgraph, i.e. a subset of nodes, with relatively high edge density among the community members (nodes of the subgraph), and comparatively lower density of edges going outside the community. Defining communities more precisely and what overall community structure may be in various domains, and design of efficient robust algorithms for uncovering such in networks has been the subject of much research [1,3].

In our use-case, we are interested in the automated discovery and effective presentation of candidate communities comprised of computers (hosts) in an enterprise network. In particular this effort is a component of a tool that provides a user, such as a security administrator of an organization, visibility into their complex network, and importantly helps the user partition the network into groups corresponding to geographic partitions, different departments, and hosts running different applications in the organization. This partitioning and naming of the groups is a necessary step in defining and maintaining network security policies, aka network segmentation: hosts in different groups (segments) can only communicate on a few well-defined and restricted channels. Such policy enforcement severely limits penetration and spread of malware and hackers. This step of grouping hosts and assigning meaningful names/labels to the groups, with the human in the loop, is also highly useful in generating insights, for example in uncovering broad patterns of communications with applications not just for security but also for network optimization.

Given the graph of communications among hosts, we find that while many of the candidate groups generated, based on community detection, correspond well to natural groupings (such as different departments, employee groups, or applications), effective ranking and confidence assignment to groups, along with other contextual information, is crucial.
for the user to make sense of the output, as there are numerous groups of varying quality and size generated. In addition to better focusing the attention of the user, effective ranking and labeling of the groups is critical for improved user experience and bolstering user confidence in the tool. In particular, in our application, users are primarily interested in midsize and larger groups, i.e. preferably groups comprising 100s of members (hosts) and beyond.

In the published literature, we find that there exist simple efficient scores, such as conductance and triangle participation ratio, but we observed that these were prone to rank well-separated but rather small groups high. Other techniques such as stability analysis can provide more robust results, but suffer from several drawbacks: the problem of relatively small groups may still surface, the methods can be expensive as they often require multiple runs, and they depend on a choice of parameter settings and the discovery algorithm and/or the graph perturbation specifics, which can be hard to tune and explain to the user.

We explored a statistical approach based on binomial modeling, as a principled approach to incorporating a ‘natural bias’ for larger sizes, motivated by the intuition that larger groups tend to provide more evidence and may therefore attain higher significance. Two of our key findings can be summarized as:

1. The tail of the binomial is a natural fit to the task.
2. There exists an adequate efficient approximation to the tail.

The analyses of the tail formula and the bounds, together with our empirical findings show that original motivation is met well: binomial scoring favors larger groups compared to other measures such as conductance. The score is equivalent to statistical significance (p-values), which makes it convenient to (color) label the discovered groups (e.g. from ‘Very High Significance’ to ‘Weak’ and ‘No Significance’), and to filter groups when significance is too low. Furthermore, we observe that the scores are not blunt (or too coarse): ranking using the tail is quite competitive in various synthetic settings, beating other inexpensive methods. It is also competitive on a variety of real-world graphs we test on, including email graphs, co-authorship, document similarity (for text clustering), and so on (Section 5). The binomial modeling space is rich, and we explore a few variants and extensions, and discuss several properties as well as potential limitations of the approach (such as coverage, and significance vs. intensity of the community property). Community analysis is a complex multi criteria task, and we expect that binomial modeling will be a useful complement to existing measures, such as the simpler scores based on raw counts and simple ratios, and those based on stability and graph perturbation.

We emphasize that binomial tail modeling is versatile and we present two related applications in community analysis: 1) to assess the strength of a node’s membership, or node centrality with respect to a discovered community, we devise a binomial score, which in turn yields additional methods for group-scoring, and 2) we develop a binomial score for significance of edges (interactions) among communities. In all these cases, we spot a pattern: a progression from using plain ‘degree’, such as a node or group degree (a simple count), to a ratio (e.g. conductance), to binomial modeling (yields a probability and confidence). Broadly, we expect that the binomial tail is useful in other applications in domains such as information retrieval.

This paper is organized as follows. In Section 2, we describe the binomial score, the efficient approximation to the tail, and present an analysis of the growth of the score and the quality of the approximation, and explore variants of the score. We also describe and discuss the other scoring techniques we compare against, including, in addition to conductance, the triangle participation ratio, and the group-wise component of the Modularity objective (which also favors group size). Section 3 describes the evaluation methodology, and reports on our extensive experiments, both in synthetic (planted partition) settings as well as on real-world data. Section 4 presents related work, and Section 5 concludes. Appendices contain further experiments on additional synthetic settings and the different variants of the score, present a modularity score based on number of nodes, and describe the two additional applications of binomial tails for network analysis: evidence of membership and significance of edges in the community-induced graph.

2 Formulation

An undirected unweighted graph $G$ consists of a node set $V$ and edge set $E$. A group $g$ is a subset of nodes ($g \subseteq V$). An edge (unordered pair of nodes) $\{u, v\} \in E$ is referred to as an (incident) edge of the group $g$ if $u \in g$ or $v \in g$. We refer to the number of edges of a group as the degree of the group (akin to degree of a node), denoted by $d(g)$. Thus, $d(g) = |\{\{u, v\} \in E, \text{s.t. } u \in g \text{ or } v \in g\}|$. To declutter notation, we often drop the generic group $g$ and simply write $d$ as we often do for other functions of groups to be introduced. If both ends of an edge are in the group $g$, we will call it an internal edge, and otherwise (exactly one-end in $g$) it’s an outgoing edge. We refer to the number of internal

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1. Along with significant preprocessing of the graph, we use standard algorithms such as Louvain for community detection.
edges as the *internal degree*, and denote it with \( d_{in}, d_{in} \leq d \). Intuitively, a group that is dense inside and sparse outside, i.e. the observed ratio \( \frac{d_{in}}{d} \) is relatively large, may correspond to a real-world community.

Now, given is a group (generated by some algorithm or process) with degree \( d \) and internal degree \( d_{in} \), and we are interested in the probability of observing such an event by chance, which we formalize next. We assume a uniform random model of connection as the null (reference) model: for each of \( d \) edges, one end is already in the group, and the other end is picked from all vertices at random. Each such pick is a random trial, so there are \( d \) trials. We define the event of interest as: observing \( d_{in} \) or more internal edges, or \( d_{in} \) of the trials pick a node in the group, out of \( d \) trials. If the probability of such event, according to the null model, is tiny, or in other words it appears to be an extreme event, then we have good evidence that the given group is special, in terms of density inside relative to all group edges or outgoing edges (an indication of community property), i.e. this is unlikely to have happened by chance. Note that, as is common when assessing statistical significance, we are being conservative, and for example not asking for the probability of the event of observing exactly (or around) \( d_{in} \) internal edges as such point-events are often unlikely to begin with (there are many possible values for \( d_{in} \)).

Let \( p_{node} \) be the (expected) probability that the other end of a group edge is in the group (a ’success’), which according to the null model is \( p_{node} = \frac{|g|}{|V|} \) (’expected’ according to the null model). We call this model the *node-based* model (an edge-based model is described in Section 2.6). We refer to \( \frac{d_{in}}{d} \) as the *observed proportion*. The tail probability, BinomialTail\( (d, d_{in}, p_{node}) \), that we observe \( d_{in} \) or more successes out of \( d \) independent attempts, with success probability \( p = p_{node} \), is given by:

\[
\text{BinomialTail}(d, d_{in}, p) = \sum_{i \geq d_{in}} \binom{d}{i} p^i (1 - p)^{d - i}, \text{ where, for the node-based model, } p = p_{node} = \frac{|g|}{|V|} \quad (1)
\]

The tail probability in a principled manner combines the connectivity attributes of the group (degree and internal edges) and other context (group and graph size). For weighted edges, if they are integers or roundable to integers, the graph can be treated as a multigraph, and the statistics such as degree appropriately computed by adding the multiple edges (a similar approach to computing modularity on weighted graphs).

### 2.1 Evidence and Group Size

Our original motivation was a principled method for scoring with a preference for larger groups. Informally, larger groups should provide more evidence of community structure assuming other properties such as internal density remains roughly equal, since degree \( d \) tends to go up with larger groups. However, the expected proportion \( p \) \( (p_{node}) \) also goes up for larger groups, which can counteract \( d \), i.e. the larger group could get a smaller significance.

For two isolated groups (no outgoing edges, or \( d_{in} = d \)), where nodes have similar (internal) degree in both groups, it is easy to show that both situations can occur, as the following establishes. But as long as the larger group is not too big, as a fraction of the graph, indeed we obtain higher significance for the larger group:

**Proposition 1** For two groups with no outgoing edges, \( g_1 \) and \( g_2 \), \( |g_2| = 2|g_1| \), \( d(g_1) = d_{in}(g_1) = |g_1| \) (node degree of roughly 1), the larger group \( g_2 \) obtains a lower tail probability (higher significance) if and only if \( |g_1| < 1/4|V| \). More generally, with \( |g_2| = k|g_1|, k > 1 \), and \( d(g_1) = a|g_1|, a > 0 \), \( g_2 \) obtains a higher significance if \( \frac{|g_1|}{|V|} < k \frac{1}{k + a} \) (independent of \( a \)).

For the proof, first for \( k = 2 \), let \( n = |V| \) and \( |g_1| = pn \), thus \( |g_2| = 2pn \), and \( d(g_1) = d_{in}(g_1) = |g_1| = pn \), and \( d(g_2) = d_{in}(g_2) = 2pn \), and, from equation 1 we want to see when \( p^{2pn} > (2p)^{2pn} \), or \( pn \log(p) > 2p \log(2p) \Rightarrow \log(p) > \log(2p) \Rightarrow \frac{\log(p)}{\log(2p)} > p < 1/4 \). For general \( k > 1 \), replace 2 with \( k \) and insert \( a \) with \( k \alpha p^{k \alpha p} > (kp)^{akp} \Rightarrow \log(p) > k \alpha p \Rightarrow \cdots \).

Similar patterns and proof applies to dense (isolated) groups such as (near) cliques, e.g. replace number of group edges with \( 0.5(pn)^2 \), and in this case the condition on the size of the larger group to obtain higher significance becomes less strict (plausible, as \( d_{in} \) goes up quadratically). Appendix D.1.1 shows another example, to illustrate a resolution limit problem for the binomial tail, where a larger group scores higher although it may be almost half as dense (but with more total number of internal edges). In practice, it’s also the case that all or most groups discovered are a small fraction of the graph in number of nodes, and we empirically find that the tail is typically well correlated with graph size in the experiments of Section 3. The approximate bounds below also show that generally as \( d \) increases, the score increases as well.

\footnote{If we don’t allow self-edges, we could use the more precise \( p_{node} = \frac{|g| - 1}{|V| - 1} \).}
2.2 Approximation

Equation [1] is simple to compute, but it is inefficient for many generated groups in large graphs: \(O(k|E|)\) for \(k\) groups, assuming Stirling’s approximation is used for factorials in \(\binom{n}{d}\), and it can lead to numeric (floating point) problems. Indeed in our experiments when we use the exact formulation we see considerable slow down, of two orders of magnitude, even on relatively small graphs of 100s of nodes (Appendix A). Efficient computation also aids other applications of the tail such as node membership (Appendix D), and a potential use for repeated evaluation as an objective for community mining. The approximate bounds below also shed light on the properties of the tail such as on group ranking and growth, as we will see.

There exists a normal approximation to the binomial tail [10], and one could use generic bounds such as Chebychev [11], but these proved to be inadequate for our use case (based on experiments). The normal approximation works best for expected proportions \(p\) near 0.5 and when the number of trials is very large [10]. On real world datasets, we find that the expected proportion \(p\) is often relatively small (e.g. below 0.10) and the observed (connection) proportion \(q = \frac{d_{in}}{d}\) is relatively large. Fortunately, the following upper and lower bound on the tail probability is sufficiently tight (see Section 2.3) and serves us well [6][7]:

\[
\frac{1}{\sqrt{2d} U \leq \text{BinomialTail}(d, d_{in}, p) \leq U, \text{ where}}
\]

\[
U = \exp(-d \text{KL}(q||p)), \text{ where } q = \frac{d_{in}}{d} \quad (U \text{ is the upper bound on the tail probability}),
\]

where \(\text{KL}()\) is the (asymmetric) relative entropy function (or Kullback–Leibler divergence): \(\text{KL}(q||p) = q \ln \left(\frac{q}{p}\right) + (1 - q) \ln \left(\frac{1 - q}{1 - p}\right)\), where \(\ln()\) denotes the natural logarithm function. The ratio of the observed to expected proportion, \(I(g) = \frac{q}{p}\left|g\right|\), is handy in the analyses below, and we will refer to it as the intensity of the group (intensity of the community property).

Thus we simply need the readily available number of graph nodes, size of the group, and its degree and internal degree to compute the approximation to the tail, in \(O(1)\) time.

2.3 Binomial Scores

For convenience of scoring and comparisons, as the probabilities can get tiny for good groups, we will use the negative log base 10 of the above upper and lower tail probabilities, akin to Richter scale for measuring the power of earthquakes (see [2.4] below), and take the average for scoring and ranking the groups:

\[
l(g) = d \text{KL}(\frac{d_{in}}{d}||p)/\ln(10) \quad \text{lower score of group } g, \text{ using the upper bound on tail probability (3)}
\]

\[
u(g) = l(g) + 0.5 \log_{10}(2d(g)) \quad \text{upper score of group } g, \text{ via the lower bound on tail probability}
\]

In our implementation, when \(d \leq 50\) we use the exact formula of equation [1] and otherwise the approximations from above:

\[
b_{\text{node}}(g) = \begin{cases} -\log_{10}(\text{BinomialTail}(d(g), d_{in}(g), p_{\text{node}}(g))), & \text{when } d(g) < 50 \\ 0.5(u(g) + l(g)), & \text{otherwise, where } p = p_{\text{node}} = \frac{|g|}{|V|} \end{cases} \quad \text{node-based binomial score (4)}
\]

Furthermore, if the observed ratio \(q\) is not higher than the expected, \(i.e.\) when \(q = \frac{d_{in}}{d} \leq p\), we return 0 score (insignificant).

Note that the arithmetic mean of upper and lower score corresponds to (the negative log of) the geometric mean of the bounds on probabilities. Next we use the upper and lower bounds to analyze the accuracy of the approximation.

2.4 Range and Growth of Binomial Scores

Looking at the upper and lower bounds of equation [3] we see that the score can grow roughly linearly with group degree \(d\), and in the experiments we do observe such ranges for the higher scoring groups. Moreover, we also observe a positive correlation with group size as anticipated (larger groups tend to have higher \(d\)). The score also goes up with KL divergence and in particular increases in proportion with log of the intensity (so at a slower rate compared to \(d\)), as the following proposition formalizes.
Proposition 2 With $0 < p < q \leq 1$, KL divergence increases with increasing intensity $I = q/p$. The therefore the approximate binomial score (either the upper or lower or their mean) increases with either increasing $d$, when KL remains the same or also increases, or with increasing (log) intensity $I$, when $d$ remains the same.

The first term of KL divergence, $q \ln \frac{q}{p} = q \ln I$, is increasing in intensity $I$, but the 2nd term ($(1 - q) \ln \cdots$) is (slightly) decreasing. To show that KL increases with $I$, we will take the derivative and show the derivative is positive, under two cases, that either $p$ or $q$ is constant. For fixed $q$, replacing $p$ by $q/I$, \[ \frac{\partial \text{KL}(q|p)}{\partial q} = \frac{\partial}{\partial q} (q \ln (I)) + \frac{\partial}{\partial q} ((1 - q) \ln \cdots) = q + (1 - q) \frac{1 - q q^{-1} q^{-1} I^{-1}}{(1 - q)} = q \frac{I - 1 + I q^{-1}}{I} > 0 \] when $0 < p < q \leq 1$, thus $I > 1$. Similarly, for fixed $p$ (varying $q < 1$), \[ \frac{\partial \text{KL}(q|p)}{\partial q} = \frac{\partial}{\partial q} (\ln I + (1 - p) \ln \cdots) = (p \ln I + p) + \cdots > 0, \] with the remaining second term $- \ln \frac{1 - q}{1 - p} > 0$, with $p < q$.

That the binomial score should increase with $q/p$ (fixed $d$, increasing $d_{in}$) is perhaps easier to see with the exact formula of Equation 1, as the set of terms to sum over (all positive) strictly shrinks as $d_{in}$ is raised, but it’s proportional dependence on $d$ is easier to see here. As a concrete example of equal intensity but different degrees, two groups $g_1$ and $g_2$ each having 10 nodes, in a 100 node graph, thus same $p_i = 0.1, g_1$ having 10 inside and 10 outgoing edges ($q_1 = 0.5$), with $g_2$ having 20 internal and 20 outgoing edges ($q_2 = 0.5$ as well), $g_2$ obtains almost twice the score.

Since we use log base 10, the scores 1, 2, 3, \cdots correspond respectively to chance (tail) probabilities of 0.1, 0.01, 0.001, \cdots thus a score of 1 corresponds to a fairly weak confidence (0.1 p-value), and each successive increment implies 10 times more significance (akin to the Richter scale), and scores around 2 and higher indicate increasingly strong confidence. In practice, we observe many generated groups with scores well above 3, and basically the score growing proportional to group degree $d$ for many of the discovered groups.

2.5 Goodness of the Approximation

Scoring and ranking need not be so precise as we only use approximate models of reality to assess significance, and often community discovery tasks have an inherent subjective or imprecise component. However, an approximation that too far from the intended model could defeat the initial motivation for using that approach. Here we will see that the approximation quality is promising, and furthermore as the community property goes up (more precisely, when the approximate lower and upper scores increase), the error in the approximation also rapidly shrinks.

A good way to assess the quality of the approximations above is to look at the relative error, which can be achieved since we have both an upper and a lower bound: Let $s(g)$ be the actual binomial score of group $g$, and denote the relative error by $r(g), r(g) = \frac{|s(g) - b_{node}(g)|}{s(g)}$ (for $s(g) > 0$). We can readily see the following (10\% or better) guarantee on the quality of the approximation holds:

Proposition 3 Let $\tilde{r}(g) = \frac{u(g) - l(g)}{l(g)} = 0.5 \ln (2d^2 \text{KL}(g|p))$. We have $r \leq \tilde{r}$ and (1) $\tilde{r} < 0.1$ when KL($I > 0.5$ and $d > 50$, and 2) (more generally), with $0 < p < q \leq 1$, error $r()$ is reduced with increasing degree $d$, with intensity fixed or increasing, or increasing intensity $I$ (with degree fixed or increasing).

The first claim, $r(g) \leq \tilde{r}(g)$, follows from bounding the numerator of $r(g)$, i.e. noting that $|s(g) - b_{node}(g)| \leq u(g) - l(g)$ (as both $s(g)$ and $b_{node}(g)$ are in $[l(g), u(g)]$, and for the denominator, $l(g) \leq s(g)$. The second part follows from the expression for $\tilde{r}$ and that KL increases with intensity from Proposition 2.

We use the exact expression for the score when $d \leq 50$, thus we framed the above claim for $d > 50$. The higher the degree the lower the relative error, but we will focus briefly on the KL divergence component of the error to see it is above say 0.5. The divergence is above 0.5 (and easily exceeds 1) when the intensity $I \geq 2$, and $p$ is relatively high, i.e. $p \geq 0.1$, e.g. when $p = 0.1$, we need $q \geq 0.21$, for a guaranteed below 10\% relative error, when we ignore the additional help from $d$, and error goes to below 5\% for $d > 100$. However, for a constant intensity, the KL divergence goes down as $p \rightarrow 0$. For instance, for $p = 0.01$ (a group of size hundredth of the graph), KL divergence is above 0.5 only when $q \geq 0.056$ (a requirement that $I \geq 5$). In practice (on various datasets) many of the discovered candidate communities do have high intensities, and one can argue that more accurately scoring small groups with relatively low intensities (low quality groups) is not as critical (see also Appendix A).

For smaller groups, degree $d$ tends to also be small, and, looking at both the upper and lower bound on the score, we require a higher intensity to attain a comparable score. This observation is consistent with the intuition that the smaller the trials (the evidence), the smaller the community quality score of a group should be (see also Proposition 1).

\footnote{In particular, in (unweighted) sparse graphs (or $|E| = O(|V|)$), $d$ is a multiple of group size $|g|$, assuming each node has roughly the average degree.}
A tighter (smaller) bound on relative error, as we use a midpoint, is: $\max\left(\frac{b_{\text{node}}(g) - l(g)}{l(g)}, \frac{u(g) - b_{\text{node}}(g)}{u(g)}\right)$, but the above simpler expression gives an idea of the quality of the approximation. In practice, we could also use only the conservative lower score, and also report the actual worst-case error $\tilde{r}(1)$ when we use the approximate binomial score. Our ranking experiments showed little difference between using the lower or the average score, as well as, in most cases, no difference in ranking quality when using the exact vs. the approximate form of the score (Appendix A and B).

### 2.6 A Null Model Based on Edge Counts

In this null model, like above we are given a group with some number of incident edges $d$, where we are free to pick the other end of each such edge. Like above, we pick the other end of each such edge independently with some probability $p$, and $p$ is different for different groups. Here, the probability $p$ is a function of the number edges of the group $|d|$ and of the graph $|E|$ (vs. the number of nodes of the group $|g|$ and graph $|V|$), or the relative ‘chattiness’ of the group, so a rough measure is $\frac{d}{|E|}$. More precisely, we use the following small variant, which we denote by $p_{\text{edge}}$:

$$p_{\text{edge}} = \frac{d + d_{\text{in}}}{2|E|} \quad (5)$$

To motivate the small difference in the definition of proportion, imagine for each edge of the graph, we look at one end, randomly picked, to see whether or not it’s in group $g$. Then the expected number of nodes in $g$ that we observe is $d + d_{\text{in}}$ (out of $|E|$ observations). This is the same null model behind the modularity objective [9], where the expected proportion of internal edges, i.e. both ends in $g$, comes out to $p_{\text{edge}}^2$.

We can replace $p_{\text{node}}$ with $p_{\text{edge}}$ in Definition 4 to get the edge-based binomial score $b_{\text{edge}}(g)$, i.e. the main difference is the choice of the observed proportion $p$. Note that $d + d_{\text{in}}$ could also replace the number of trials $d$ in Definition 4.

Symmetrically, one could define a modularity objective based on number of nodes, and a corresponding Louvain algorithm based on node proportion vs. edge proportion. See Appendix E.

### 2.7 A Comparison of the Edge and Node Based Models

What is the consequence of using one null model vs. the other? The implications are easier seen when we look at groups that have no outgoing edges, i.e. $d = d_{\text{in}}$: the edge-based model may prefer (internally) less dense groups (as $p_{\text{edge}}$ can be smaller for less dense group, irrespective of group size), compared to the node-based model, which may prefer smaller (but possibly denser) groups (as $p_{\text{node}}$ will be smaller or the exponent can be higher). To illustrate this difference, we present an extreme example in Figure 1(a). For both groups, in either null model, following equation 1, the tail probability is $p_{\text{node}}^2(g)$ (as $d_{\text{in}} = d$). For the node-based model, $p_{\text{node}}(g_1) = p_{\text{node}}(g_2) \implies p_{\text{node}}^2(g_1) < p_{\text{node}}^2(g_2)$ (where $p_{\text{node}}(g_1) = \frac{|g_1|}{|V|} = \frac{3}{5} = p_{\text{node}}(g_2)$). For the edge-based model, $p_{\text{edge}}(g_1) = \frac{3}{5} > p_{\text{edge}}(g_2) = \frac{1}{3}$, and $\left(\frac{3}{5}\right)^6 > \left(\frac{1}{3}\right)^6$. In this example, the node-based model prefers the group that looks more like a community (both groups are isolated, but group 1 is denser).

The node set $V$ grows in Figure 1(b), but no new edges are added, and since $p_{\text{node}}(g_1)$ shrink (have $p_{\text{node}}(g_1) < p_{\text{node}}(g_2)$), we see that the significance of both groups under the node-based null model (binomial score) grows, but score grows faster for group 1, while there is no difference in the groups’ scores in the edge-based model. Finally, in Figure 1(c), as outgoing edges are added to $g_1$, the preference of $g_1$ over $g_2$ decreases, though $g_1$ may still be preferred: this depends on the ‘value’ of an internal edge vs. an outgoing edge, which in turn depends on $|V|$ relative to $|g_1|$ (or $p_{\text{node}}$). On the other hand, in the edge-based model, the preference of $g_2$ over $g_1$ only grows. We can see that the edge-based model can have an advantage now: it can prefer relatively isolated groups with good density, while the node-based model may prefer smaller dense groups that nevertheless have many outgoing edges (less community-like to the naked eye).

We have observed on several datasets that, as might be expected, the rankings by the two techniques are highly correlated (e.g. Spearman rank correlation of over 0.99 on 5000 groups from Amazon co-purchase dataset in SNAP [12]). In another experiment, on a few graphs derived from network communications, we observed that if we take the maximum of the two scores, and use 1.5 as a threshold on score (corresponding to roughly 95% confidence), we observed that we get 5% more groups deemed significant than if we use either measure in isolation.

### 2.8 A Global Bound

The above null models were conditional on an observed group, i.e. given a group with certain size $k$ and degree degree $d$ (event $A$), we sought the probability of observing $d_{\text{in}}$ or more internal edges (event $B$). The graph can be big and it may not be so surprising (so unlikely) to observe such a group in a large graph. Here, we want to add more context and
We report correlation with group size per our motivation to bias rankings towards larger groups. We also compare ranking via the binomial score with ranking using plain group size (largest first) as a baseline because in several datasets and parameter settings this can use the more conservative bound of 6 for filtering and/or labeling and ranking candidate groups, with a possible fall-back to plain binomial score for ranking the borderline cases.

Since didn’t observe a difference when we used the uniform Erdos-Renyi model for graph generation estimating reward community structure anymore, but how unlikely a group size and degree is with respect to the choice of graph generation. We account for graph size. The union bound (Boole’s inequality), i.e. the probability of a disjunction is bounded by the sum of component probabilities \( P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i) \), gives us an upper bound on the probability of the event \( C \), that we observe one or more groups (i.e. subsets) of size \( k \) with degree \( d \) and internal degree \( d_{in} \) or more:

\[
P(C) \leq \sum_{g \subseteq V, |g|=k} P(A) P(B|A), \text{ from the union bound}
\]

\[
P(C) \leq \binom{|V|}{k} P(B|A) \quad (\text{as } P(A) \leq 1)
\]

\[
= \binom{|V|}{k} \text{BinomialTail}(d, d_{in}, p)
\]

where \( P(B|A) \) is given by equation \( \text{(6)} \). We can define a corresponding score, \textit{global} binomial score, by taking the negative log. We note that \( \ln\left(\binom{|V|}{k}\right) \) can be efficiently approximated using Sterling’s formula for the factorial.

In our experiments, on synthetic datasets with relatively small groups and graphs, the global binomial score goes to \((\text{as } P(B|A) \leq 1)\) and does not observe a significant difference between the two (see Appendix \( \text{B} \)). In practice, one can use the more conservative bound of \( \text{(6)} \) for filtering and/or labeling and ranking candidate groups, with a possible fall-back to plain binomial score for ranking the borderline cases.

\[\text{2.9 Other Scoring Methods}\]

We compare ranking via the binomial score \( b_{\text{node}}(g) \) with several simple scoring methods:

1. **Modularity (group-wise)** [9, 13]: \( \text{Modularity}(g) = \frac{d_{in}(g)}{|E|} - p_c(g)^2 \), is the fraction of observed internal edges minus a expectation for such fraction, according to the null model of Section 2.6

2. **Conductance** [14]: \( \frac{d(g)-d_{in}(g)}{d(g)+d_{in}(g)} \) (the lower the better)

3. **Triangle participation ratio (TPR)** [5]: for a group \( g \): \( TPR(g) = \frac{|\{u,v,w\} |}{|\{u,v\} \cup \{v,w\} \cup \{w,u\}|} \), where \( \text{triangles}(g) = \{ \{u,v,w\} : \{u,v,w\} \subseteq g, \{\{u,v\}, \{v,w\}, \{w,u\}\} \subseteq E \} \)

4. **Descending group size (number of nodes in the group, Size)**

We report correlation with group size per our motivation to bias rankings towards larger groups. We also compare with ranking using plain group size (largest first) as a baseline because in several datasets and parameter settings this

\[\text{4To get a somewhat tighter bound, instead of replacing with 1.0, one could replace } P(A) \text{ with probability of the event of observing a size } k \text{ group with degree } \leq \text{ d. This requires a further generative assumption on the overall graph and does not purely reward community structure anymore, but how unlikely a group size and degree is with respect to the choice of graph generation. We didn’t observe a difference when we used the uniform Erdos-Renyi model for graph generation estimating } P(A) \text{.}\]

\[\text{5In the denominator, the minimum of degree and internal degree for } g \text{ and its complement (rest of the graph, } V - g \text{) is used. Since } g \text{ is substantially smaller than half of graph in our experiments, we do not see a difference in conductance value.}\]

---

**Figure 1**: Example graphs where the node-based binomial model ‘prefers’ group \( g_1 \) (i.e. scores group 1 higher), but the edge-based model prefers group \( g_2 \). In all three cases, \( P_{\text{node}}(g_1) = P_{\text{node}}(g_2) \), and for graphs in (a) and (b), for both groups and in either node- or edge-based model, the tail probability is \( P^k(g) \) (as \( d_{in} = d \)). See text for further explanations.
baseline (or its reverse) does very well and sheds light on the nature of the specific ranking problem. We also report on
the reverse (smallest first) in a few cases.

We compare against Modularity() since we expect Modularity() to correlate with group size as well, and it shares
some similarity with the Binomial score: it is also a function (the difference) of a candidate group’s internal edges
vs. an expectation of such. It can be readily available when we use the Louvain algorithm and has been used in ranking
comparisons. We note that the original modularity formulation sums per-group differences over all groups to
assess an entire partitioning while we are using the group-wise component of modularity for ranking groups.

Among a variety of simple scoring techniques tested, Yang and Leskovec found that conductance and TPR performed
among the best (one reflecting internal density, another reflecting external separation) and conductance in particular
has been used often as an objective for community discovery. Note that among these methods, only TPR requires
some nontrivial algorithm to compute.

2.9.1 Discussion

Each technique has pluses but also shortcomings. TPR is an imperfect measure of density inside the group and ignores
outgoing edges. Note also that TPR can be a perfect 1.0, and yet the internal density ($d_{in}$) of a group can have much
room for improvement. We see the issue of tied scores often for TPR in the experiments. Both TPR and conductance
are ‘normed’ measures that factor out (ignore) group size or number of connections (statistical evidence). Figure 2 show
several groups with the same (perfect) conductance of 0.0 and/or perfect TPR of 1.0 with varying sizes and internal
densities. Thus both methods can be insensitive to size and internal density in some circumstances. Under stability
and perturbation analysis tools, these structures may also obtain similar scores since they may be rediscovered under
multiple runs. But intuitively or at least in some tasks (like ours), one prefers larger sizes and higher internal densities.
We saw that the binomial score is sensitive to increasing internal edges as well as overall degree.

The binomial score provides a principled way of combining internal and outgoing edges and taking into account some
of the other contextual information (group and graph size). However, the approximations inherent in using the tail, in
equations 8 or in the union bound of equation 6 may weaken the testing (discriminative power and coverage). We
observe these possibilities in certain extreme cases in synthetic experiments (Appendix A and B). It is also possible that
real-world groups enjoy high evidence, easily passing the significance threshold, and for those passing the significance
threshold (e.g. a score above 2 or 3), the binomial score may become a blunt instrument for ranking purposes.
We conduct experiments on synthesized and real-world networks to shed light on these possibilities and some of the issues
will be explicitly raised and discussed. Note that intensity score of $I(g) = \frac{q(g)}{p(g)}$ is another measure of strength of the
(community) property. This score ignores the raw degree counts (the support or the evidence), akin to conductance and
TPR. As an example, a group A may have a high intensity of 10 (\( \frac{q}{p} = 10 \)), but a binomial score of only 2, while another
group B may have an intensity of 2, but high binomial score of 5: group B is larger than A, and has many internal edges
(providing much statistical evidence of community) but also external edges, and subjectively, to the human eye, group
A may seem more of a community. However, using intensity alone can become too biased toward small or insignificant
groups (this measure is called ratio modularity in \[13\]). In practice, likely multiple measures are needed and should be
presented to the user. For example, one possibility is to filter by a confidence threshold, and rank what remains by a
measure of intensity. We leave exploration of intensity and combinations to future work. See also \[15\] for combination
techniques.

All these measures ignore how the internal connectivity is spread among the group members. For instance, a given
group might have a small internal cut, i.e. may be separable into two good nearly-sized groups via removing only a few
edges (a dumbbell shape, imbalanced group). Or some single or few nodes may have most of the internal connections of
a group (e.g. star or core-periphery structure), while intuitively a more balanced spread of the internal edges is preferred.
The scoring techniques do rely on the group generation technique to discover a few good quality groups, possibly
mixed with lesser quality ones, and the task is to highly rank the best from among what is given, where what constitutes
‘best’ can be application specific. See also Appendix D.1.1 on scoring groups based on aggregating the strength of
membership-level scores of individual nodes, which, as explained there, may lead to more balanced groups.

3 Experiments

We conduct experiments on both synthetic and several diverse real-world datasets shown in Table 1 (social networks,
copurchases, transportation, text clustering, · · · ). On synthetic data, we follow the planted partition model and ‘plant’
the groups.

The real-world graphs come with established groupings, often according to some node attribute(s) or other external
knowledge such as explicit groupings by users. We refer to these groups as reference groups (as opposed to groundtruth).
Researchers have questioned the utility of treating such groups as groundtruth in assessing community discovery algorithms [17]. We agree that groups discovered via community analysis could be valid and highly useful and at the same time be far from the provided groups. Nevertheless, we expect that better correlation with these reference groups, on diverse datasets, can signal the robustness of a scoring method. We also critically examine and discuss the findings in some detail (e.g. reference groups can be coarse and general, while the discovered groups can be relatively fine-grained, or vice versa). For each dataset, we briefly describe the nature of the graph and the reference groups, and the reader can assess for herself the extent to which interactions that determine the graph may correlate with the provided reference groups.

In this section, we use the node-based binomial score, equation 3, from the binomial family. Appendix B reports on experiments with variants, including the edge-based approach, and a p-value expression derived by He et al. [18]. We used the Louvain algorithm for group generation [4], which we found to be fast and robust. We picked the discovered groups from the final pass, where modularity is highest. We also tried implementations of the Spectral algorithm, one from sklearn.cluster [19], but the algorithm did not converge in several scenarios, such as when noise was increased in the synthetic experiments (or the groups generated were far inferior). We report ranking comparisons only on groups of minimum size 3, consistent with [15].

3.1 Evaluating Scoring Techniques

A community discovery algorithm generates a set \( \hat{S} \) of candidate groups. We also have a set of reference groups \( S^o \) (the planted groups in synthetic experiments or the provided reference groups in the real-world datasets). Note that we are concerned with ranking the groups in the given \( \hat{S} \) set (and not on generating better such sets).

We assess the performance of a candidate group output by a community discovery algorithm by how well it overlaps the best matching group in the reference set of groups. In particular, we define the following overlap score for each candidate group:

\[
s^o(\hat{g}) = \max_{g^o \in S^o} \text{match}(\hat{g}, g^o), \quad \text{where } \text{match}(\hat{g}, g^o) = \sqrt{\text{recall}(\hat{g}, g^o) \cdot \text{precision}(\hat{g}, g^o)},
\]

(7)

Thus we use the geometric mean of ‘precision’ and ‘recall’ as the score (the ratio of intersection size to candidate group size is interpreted as the precision score, common in supervised learning), and the score of a candidate group is the highest such score among the match attempts against all the reference groups. Geometric mean is more conservative than the arithmetic mean (it is closer to the lower of the precision and recall scores). We could also use the simpler Jaccard index \([20]\) (i.e. \( \frac{|\hat{g} \cap g^o|}{|\hat{g} \cup g^o|} \)), but we had a small preference for the mean since Jaccard can be lower than both the precision and recall scores, while the mean indicates that at least one score is not lower. Note also that the matching is not exclusive: the same reference group can match multiple candidate groups.

When we sort the candidate groups based on descending order of overlap score \( s^o() \), we obtain the reference ranking. For each scoring method (binomial score, conductance, Modularity, ...), we rank the candidate groups by the scoring
method, and report the Spearman’s rank-order (SPR) correlation [21], of this candidate ranking with the reference ranking (specifically, between the two score lists), similar to the experiments of Zitnik et al. [15]. An example of evaluating is given in Section 3.1.1. We also report the overlap score \( s^o() \) of the top ranked group, averaged over the trials (in case of ties, at the top, we use the average of those tied), denoted top PR. For larger datasets, when there are many candidate groups generated, we also report the average \( s^o() \) of top 5 for each method, top 5 PR. SPR is useful to obtain an impression of the entire ranking, while in some settings we are only interested in how the top one or few groups are performing. We also report the average overlap score (average ’performance’), denoted avgPR, of the generated candidates groups, so that the overlap score of the top ranked is put in context: We expect that, for a good scoring technique, the top ranked should have higher than the average performance (otherwise, one cause could be a mismatch between the reference set and what is discovered). We only keep candidate and reference groups that are size 3 or higher, consistent with [15].

3.1.1 An Example of Evaluation

Assume the reference group has two groups, \( g^0_1, g^0_2 \), with sizes 10 and 15, while we obtained 3 candidate groups \( \hat{g}_1, \hat{g}_2, \hat{g}_3 \), with sizes respectively, 6, 14, 5, from a single run of the community detection algorithm. Say \( |\hat{g}_1 \cap g^0_1| = 5 \), and \( |\hat{g}_1 \cap g^0_2| = 1 \). Recall of \( \hat{g}_1 \), when matched to \( g^0_1 \) is \( \text{recall}(\hat{g}_1, g^0_1) = \frac{5}{10} \). The precision from this match is, \( \text{precision}(\hat{g}_1, g^0_1) = \frac{5}{6} \). Therefore, \( \text{match}(\hat{g}_1, g^0_1) = \sqrt{\frac{5}{10} \cdot \frac{5}{6}} \), and similarly \( \text{match}(\hat{g}_1, g^0_2) = \sqrt{\frac{1}{15} \cdot \frac{1}{6}} \), so overlap score of \( \hat{g}_1 \) is \( s^o(\hat{g}_1) = \text{match}(\hat{g}_1, g^0_1) \approx 0.65 \) Assume \( |\hat{g}_2 \cap g^0_2| = 14 \) and \( |\hat{g}_3 \cap g^0_3| = 5 \). Then \( s^o(\hat{g}_2) = \text{match}(\hat{g}_2, g^0_2) = \sqrt{\frac{14}{15} \cdot \frac{14}{6}} \approx 0.97 \), and \( s^o(\hat{g}_3) = \text{match}(\hat{g}_3, g^0_3) = \sqrt{\frac{5}{15} \cdot \frac{5}{6}} \approx 0.71 \) (note: \( g^0_3 \) is the best match to two of three candidate groups). Thus the sequence of overlap scores is \([0.65, 0.97, 0.71]\) \( \hat{g}_1 \) has 0.65, etc., and average overlap score of candidate groups (average ’performance’) is \( \frac{0.97+0.71+0.65}{3} \approx 0.78 \).

Any sequence of scores that yields the reference ranking, i.e. assigns its highest score to \( \hat{g}_2 \) and second highest to \( \hat{g}_3 \), such as \([0.2, 0.9, 0.3]\), gets a maximum SPR of 1.0 (and top PR of 0.97). So \( SPR([0.65, 0.97, 0.71], [1, 3, 2]) = 1 \), and similarly \( SPR([0.65, 0.97, 0.71], [-1, 10, 3, 5]) = 1 \). If a scoring technique assigns the scores \([2, 4, 5]\) \( \hat{g}_1 \) gets 2, etc., then its top PR is 0.71, since its top ranked group is \( \hat{g}_3 \) (with overlap score of 0.71), and we have \( SPR([0.65, 0.97, 0.71], [2, 4, 5]) = 0.5 \). The reverse of reference ranking yields the minimum SPR of -1, e.g. \( SPR([0.65, 0.97, 0.71], [3, 0, 1]) = -1 \). There can be ties in scores, so \( SPR([0.65, 0.97, 0.71], [3, 5, 3]) = -\frac{0.87}{3} \). However, if either scoring function (the overlap scores or a scoring technique) assigns the same score to all the generated groups, we report undefined SPR.

In each trial, we can get a different set of candidate groups and results, as Louvain is randomized and in synthetic experiments a new graph (and reference groups) is generated. We average all the results over the trials (results such as the number of groups generated, average group performance, SPR of each scoring technique, etc.).

3.2 Experiments on Synthesized Data

In these experiments, the graph is built out of 10 groups, each of size 30. The internal (edge) probability, the probability that an edge exists between a pair of nodes in the group is set at 0.6 for 5 groups and 0.2 for the other 5 groups, following [15]. We refer to this graph generation setting as Synt. Two other synthetic settings are explored in the appendix.

There are also outgoing or ’noise’ edges, connecting nodes from different groups. The edge-noise (or simply noise) probability is the probability that an edge exists between a pair of nodes from different (planted) groups. The higher the noise probability (when generating the noise edges), the harder for any discovery algorithm to fully recover the planted (intended) groups. We vary the noise probability and plot SPR and top PR (overlap score of top ranked candidate group) of various ranking methods.

For each possible value of the noise (edge-noise) probability \((0.025, 0.05, \ldots, 0.225)\), we average the results over 200 trials, where in each trial a new graph is constructed and Louvain is run to generate the groups, and each scoring method is evaluated. Note that as we increase the noise, overall graph edges increase, and the groups with lower internal density are more prone to ’corruption’ (their nodes may join other groups).

We observe from Figure 3 that for a wide range of the noise probability, the binomial score remains superior based on both Spearman rank correlation (SPR) and the performance of the top ranked group. Eventually, the noise is sufficiently high, here at 0.225, that each method’s performance nears that of the average overlap score of all the groups generated. At noise of 0.225, the binomial score, in 35 of 200 trials, assigns 0 score (all groups tied), and does not report a ranking. For these, we score 0 correlation (thus the average SPR degraded to 0.25 for Binomial at 0.225 noise). Still, the Binomial’s top group score surpasses others’, though, with a very high error-bar (standard deviation), this is no longer significant.
Binomial Tails for Community Analysis

(a) Spearman rank correlations (SPR) with reference ranking, of various ranking methods. (b) Mean performance (overlap score) of top ranked group (Top PR). Average of all candidate groups’ overlap scores in each trial (averaged over all trials) is also shown (avg).

Figure 3: Planted partition synthesized data, performance averaged over 200 trials: in each trial a graph is generated, then Louvain is applied to generate groups. A sampling of the error bars is shown too. As the noise probability is increased (x-axis), the graph becomes denser, and fewer (and larger) groups are generated (precision goes down). Average number of groups generated goes from 8 at 0.025 edge-noise, down to 6 at 0.225. The average of groups’ overlap scores (avg), shown in (b), goes down as expected with increasing edge-noise.

When noise (probability) is 0 (not shown) or very low, in most trials there is no ideal ranking in the sense that the planted groups are perfectly discovered (and all are tied in their perfect overlap score). So SPR becomes undefined. Similarly for the case of TPR, once noise-probability goes above 0.075 the graph and the few groups discovered become sufficiently dense that all generated groups reach TPR of 1.0, and no ranking is obtained from TPR. Observe also that the plot for Modularity has a similar shape to plot for Size (ranking by descending group size), although it outperforms it. We will see the high correlation of Modularity with group size on the real-world datasets as well.

Since in our implementation the approximate node-based binomial score \( b_{node}() \) (equation[4]) is used when \( d(g) >= 50 \) (equation[4]), in these experiments the approximate binomial score is always invoked, as the internal and outgoing edges of the groups exceed 50. For instance, for a planted group with the lower density of 0.2, just the expected number of internal edges, \( d_{in} \), is above 80: \( 0.2(30(30-1)) > 80 \). Indeed, the average binomial score \( b_{node}() \) of the top ranked group when noise is low at 0.025 is over 300, consistent with the discussion of Section [2.4] while at 0.225, it goes down to 2.5 (0 \( b_{node} \) in many trials).

The Syn1 set up follows that of [15], and the authors report the SPR of a few ranking techniques, in particular Conductance and Modularity rankings, in their Figure 2. However, we could not locate the noise rate(s) used for the Figure, in their publication. From their Figure 2, we note that when Modularity and Conductance are near respectively 0.3 and 0.6 SPRs (simultaneously), in Figure 3(a)) (a noise of ~0.03), Binomial ranking scores near 0.8, which is a similar performance to the performance of CRank reported in their Figure 2. We note that CRank is an aggregation technique, combining multiple rankers/scorers. It is promising that a single scoring technique appears competitive with it.

In Appendix D experiments on two other planted partition settings, equal group sizes and equal probabilities (Syn2) as well as unequal group sizes (Syn3), are reported with similar patterns of findings. There, we also observe that with a
sufficiently high noise, Binomial per our approximation implementation returns 0 scores, since the approximation to tail probability reaches or exceeds the maximum of 1.0 (thus yielding no ranking, similar to TPR), while Modularity and conductance keep some discriminatory power. Some of those groups are indeed insignificant, and using the exact formula as a special fall-back case (or the normal approximation) may help in those cases. On the real-world datasets, we find that there are ample candidate groups generated with very high significance, and this significance feature, of warning the user on certain groups with little or no significance, will be useful.

3.3 Experiments on Real-World Data

We now report on several real-world networks, where inherent complexities include the widely varying number of connections and connection patterns across nodes, and highly varied community sizes, overlapping communities, and complex community hierarchies spanning several scales. These complexities are not easily modeled in synthetic experiments, since the parameters are not known and can vary substantially from one problem or domain to another. On the other hand, some of these attributes may actually, in certain cases, simplify the task of group discovery or group ranking.

| Dataset            | nodes | edges | reference groups | reference group sizes         |
|--------------------|-------|-------|------------------|-------------------------------|
| Lazega Lawyers     | 70    | 378   | 3                | 4, 19, 48                     |
| Football           | 115   | 613   | 12               | 5, 7, ⋅⋅⋅, 12, 13             |
| Railway            | 301   | 1224  | 21               | [1, 1, 4, ⋅⋅⋅, 34, 46] median is 13 |
| Political books    | 105   | 441   | 3                | 13, 43, 49                    |
| Political blogs    | 1224  | 16716 | 2                | 732, 758                      |
| EU-core            | 986   | 16064 | 42               | [1, 1, 2, 3, ⋅⋅⋅, 92, 109] median is 14 |
| YouTube (text)     | 7k    | 30k to 1mil | 30               | roughly uniform sizes         |
| 20 newsgroups (text) | 7.5k  | 70k to 800k | 20               | roughly uniform sizes        |
| Coauthorship       | 90k   | 187k  | 24               | [29, 154, ⋅⋅⋅, 11035, 12674] median is 2058 |
| YouTube (Social)   | 90k   | 1.2mil| 5.7k             | [3, ⋅⋅⋅, 1601, 1961] median size is 5 |
| LiveJournal (Social) | 150k  | 6mil  | 98k              | [3, ⋅⋅⋅, 6k, 47k] median size is 3 |
| Amazon (Purchases) | 335k  | 926k  | 75k              | [3, ⋅⋅⋅, 29k, 44k, 53k, 54k] median is 5 |

Table 1: 12 real-world datasets, ordered roughly by size, with number of nodes, edges, and reference groups, and summary information on the distribution of the reference group sizes.

We present and discuss the results on the smaller datasets first (Table 2), then move to the two text clustering ones, and finally on the remaining big datasets. In these datasets, unlike the experiments on the synthesized graphs, the only source of variation from one trial to the next is the randomized Louvain algorithm, which is fairly stable. Consequently, we observe lower variance in the results.

3.3.1 Smaller Datasets

A description of the the smaller datasets follows, and results shown in Table 2. The reader can think for themselves, apriori, the extent to which the communities derived from the graph would be consistent with the reference groups, given the descriptions and information such as the sizes of the reference groups. The tables include the average performance (overlap scores) of the generated groups (for those with size 3 or higher).

**Lazega lawyers:** Nodes are lawyers, the network is the cowork relation: an edge exists if two lawyers have worked together on the same case. The reference groups are the 3 office locations (3 cities) [22, 17]. This is the smallest dataset where Louvain creates the fewest, 3 groups.

**Football:** Nodes are college teams, edges are games, and the reference groups are teams in the same conference (a geographic area, such as ‘Atlantic Coast’, ‘Mountain West’...). Teams play with more teams in the same conference [23].

**Political blogs:** nodes are blogs, in 2004, and edges are weblinks to other blogs (all crawled). There are 2 reference groups: conservative and liberal (some are manually labeled, some derived from a directory) [26].

**Political books:** nodes are books on US politics and edges are frequent co-purchasing of books (on Amazon). The 3 reference groups are: neutral, liberal, and conservative [25].

**Railway:** Nodes are train stations in India, edges connect if there is some train route with stops on both stations. Groups are stations in the same state [24].
Table 2: Results averaged over 200 trials. Number of groups generated by Louvain (size ≥ 3) in a trial on average and the average overlap score of these groups, avgPR, are also reported.

| Method          | Sp. Rank Corr. | Top PR | Sp. Corr. With Size |
|-----------------|----------------|--------|---------------------|
| **Lawyers:**    |                |        |                     |
| Binomial        | 0.99±0.07      | 0.96±0.04 | -0.5               |
| TPR             | 0.03           | 0.76   | -0.82               |
| Conductance     | 0.56           | 0.75   | 0.32                |
| Modularity      | 0.45           | 0.72   | 0.49                |
| (Descending) Size | 0.51          | 0.71   | 1.0                 |
| **Football:**   |                |        |                     |
| Binomial        | 0.36±0.29      | 0.84±0.15 | 0.2               |
| TPR             | -              | -      | -                   |
| Conductance     | 0.34±0.29      | 0.84±0.15 | 0.4               |
| Modularity      | -0.13          | 0.77   | 0.91                |
| Size            | -0.35          | 0.77   | 1.0                 |
| **Railway:**    |                |        |                     |
| Binomial        | 0.41±0.1       | 0.61±0.07 | 0.93             |
| TPR             | 0.34±0.09      | 0.56   | 0.27                |
| Conductance     | 0.12           | 0.48   | -0.67               |
| Modularity      | 0.32           | 0.57   | 0.98                |
| Size            | 0.29           | 0.59   | 1.0                 |
| **Political Books:** |            |        |                     |
| Binomial        | 0.82±0.11      | 0.91±0.01 | 0.84             |
| TPR             | -0.1           | 0.88   | -0.18               |
| Conductance     | 0.81±0.11      | 0.91±0.01 | 0.84             |
| Modularity      | 0.86±0.09      | 0.87   | 0.92±0.01           |
| Size            | 0.78           | 0.89   | 1.0                 |
| **Political Blogs:** |            |        |                     |
| Binomial        | 0.92±0.05      | 0.8±0.004 | 0.75             |
| TPR             | 0.82           | 0.88   | 0.82                |
| Conductance     | 0.58           | 0.88   | 0.58                |
| Modularity      | 0.97±0.03      | 0.87   | 0.99                |
| Size            | 0.93           | 0.88±0.001 | 1.0            |
| **EU-core:**    |                |        |                     |
| Binomial        | 0.32±0.15      | 0.94±0.06 | -0.25           |
| TPR             | 0.24           | 0.71   | -0.44               |
| Conductance     | 0.09           | 0.94   | 0.1                 |
| Modularity      | -0.64          | 0.34   | 0.82                |
| Size            | -0.64          | 0.31   | 1.0                 |

**Eu-core:** Nodes are users (personnel) within a European (EU) organization and edges are emails among users (an edge corresponds to at least one email between nodes u and v). Groups are departments the users belong to [27].

Looking at SPR and overlap score of highest ranked, suggest that Binomial ranking is most consistent with the reference groups. Football is the dataset where the generated candidate groups match the reference groups best (average overlap of 0.91). Interestingly, it is also the only dataset where the top ranked group score (topPR) of no method exceeds the average candidate group overlap score. On this data, the TPR score was perfect at 1.0, so no ranking was obtained by TPR (the same issue observed for many synthesized settings, see Section 3.2).

We observe that the correlation of Binomial with group size is somewhat mixed, while Modularity is almost always highly correlated with it, but at the expense of less consistency with the reference ranking.

### 3.3.2 Text Clustering

In the clustering experiments, documents are converted to tfidf vectors and cosine similarity between all pairs of document vectors is computed [31]. The pairs with similarity above a threshold are kept as edges (unweighted edges), and Louvain is run on the resulting unweighted undirected graph. We explored and show results for a few thresholds on
Table 3: Text clustering on YouTube text (tags) data. 10 trials.

| 20 Newsgroups (text) | Sp. Rank Corr. | Top PR | Top 5 PR | Sp. Corr. With Size |
|----------------------|---------------|--------|----------|---------------------|
| 20 ref. groups, edge threshold of 0.05 (772k edges), 12 groups generated on avg, avgPR =0.41 | Binomial 0.51 ±0.1 | 0.37 ±0.05 | 0.52 ±0.05 | 0.87 |
| TPR -0.18 ±0.3 | 0.39 ±0.04 | 0.41 ±0.03 | -0.02 |
| Conductance 0.48 ±0.1 | 0.29 | 0.51 | 0.90 |
| Modularity 0.48 | 0.29 | 0.51 ±0.02 | 0.95 |
| Size 0.46 ±0.1 | 0.24 | 0.50 | 1.0 |
| 20 ref. groups, edge threshold of 0.1 (66k edges), 37 groups generated on avg, avgPR =0.33 | Binomial 0.87 ±0.01 | 0.67 ±0.02 | 0.65 | 0.89 |
| TPR 0.03 | 0.07 | 0.28 | 0.13 |
| Conductance 0.1 | 0.02 | 0.24 | 0.08 |
| Modularity 0.89 ±0.01 | 0.65 ±0.04 | 0.69 ±0.02 | 0.95 |
| Size 0.86 | 0.40 | 0.61 | 1.0 |

Table 4: Text clustering on 20 newsgroups. 10 trials.

As we increase the similarity threshold starting from 0.05, the resulting graphs become sparser and more groups and smaller groups are generated by Louvain (Tables 3 and 4). We report the avgPR of top 5 groups too, since the number of groups generated is in the 10s now.

Note that the overall performance of clustering as measured by average overlap score, avgPR, of the groups generated may first increase but eventually begins to decrease, as we increase the similarity threshold. As shown in Table 3, the performances of Size and Modularity improve and may eventually surpass others. This is because the larger groups found tend to have highest recall, and their precisions are competitive with smaller generated groups. A similar trend is seen in Table 4 for newsgroups.

### 3.3.3 Coauthorship

In the Coauthorship dataset, nodes are authors, edges are coauthorship (if they coauthored a paper together), and the reference groups are the various fields of publication (subject areas/topics) [20]. Each author is assigned to the field they have most papers in, which yields the reference groups. In this dataset, unlike the remaining upcoming large datasets, we have relatively few, 24, and mostly large reference groups (100s of nodes in size), while Louvain generates
relatively many groups: nearly 5000 groups (and without a threshold of 3 on size, we’d get 10k groups!). Especially compared to the upcoming large datasets, coauthorship is sparse, and the large number of candidate groups is plausible. Many such groups are tiny and get a low recall since they are relatively small (Table 5). These small groups tend to be ranked high by conductance or TPR. Techniques that are biased towards larger sizes do well on this dataset (i.e. attain sufficient recall).

We also report the size of top group picked by each method (averaged over the trials). We see that TPR and conductance gravitate towards very small groups. We also note that over 3k and 4k groups (out of the 5k generated) tie in their perfect score respectively for TPR and conductance (avgPR is averaged over the groups tied at top).

When we increase the minimum group size threshold to 50, we obtain just over 100 such groups on average from Louvain. The size of the top group picked by the two methods also increases, and so does the average overlap score avgPR (from 0.003 to 0.05), and the performance of conductance and TPR when measured in topPR and top 5 PR (see Table 5) improve substantially on a relative scale.

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In this dataset, as with remaining large datasets, Louvain creates a small number of groups (just over 100 here), with widely varying sizes, as seen in the size of the top picked group by each method in Table 6. Binomial’s top rank is a group of size 12k (from output of Louvain) while the largest group (top ranked by Size and Modularity) is 28k. On the other hand, conductance and TPR pick sizes 4 and 6 on average. For TPR, many (30) ties at the top with a perfect TPR of 1.0 (we report their average overlap score for topPR). Similarly, 12 groups tie in conductance. No ranking method does well here in an absolute sense, as judged by topPR, as the precision or recall of all groups picked is small (avgPR is low too, at 0.1).

In a second set of experiments on this dataset, we restrict the group sizes to 50 to 500, and take these groups from first pass of Louvain (vs. the final output), where sufficiently many smaller groups are generated. This yields ∼40 groups on average. We see that, with a narrower range of group sizes, overlap performance score improve for all.

3.3.4 YouTube Social

Our YouTube social graph is a subset of the original dataset where we require a minimum degree of 10 [27], so that the Louvain algorithm (Python) finishes in 30 minutes on each trial. Here, nodes are users, and edges are friendships. Users can define groups too, and the reference groups are the connected components in the user defined groups. There are 5k such reference groups and almost all are very small (Table 1).

In this dataset, as with remaining large datasets, Louvain creates a small number of groups (just over 100 here), with widely varying sizes, as seen in the size of the top picked group by each method in Table 6. Binomial’s top rank is a group of size 12k (from output of Louvain) while the largest group (top ranked by Size and Modularity) is 28k. On the other hand, conductance and TPR pick sizes 4 and 6 on average. For TPR, many (30) ties at the top with a perfect TPR of 1.0 (we report their average overlap score for topPR). Similarly, 12 groups tie in conductance. No ranking method does well here in an absolute sense, as judged by topPR, as the precision or recall of all groups picked is small (avgPR is low too, at 0.1).

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3.3.5 LiveJournal

Our LiveJournal social graph is a subset of the original dataset where we require a minimum degree of 80 [27], so that Louvain algorithm finishes in under 2hrs in each trial. Here, nodes are users, and edges are friendships. A user can define user groups too, and the reference groups are the connected components in the user defined groups. There are 5k such reference groups and almost all are very small (Table 1).

In this dataset, as with remaining large datasets, Louvain creates a small number of groups (just over 100 here), with widely varying sizes, as seen in the size of the top picked group by each method in Table 6. Binomial’s top rank is a group of size 12k (from output of Louvain) while the largest group (top ranked by Size and Modularity) is 28k. On the other hand, conductance and TPR pick sizes 4 and 6 on average. For TPR, many (30) ties at the top with a perfect TPR of 1.0 (we report their average overlap score for topPR). Similarly, 12 groups tie in conductance. No ranking method does well here in an absolute sense, as judged by topPR, as the precision or recall of all groups picked is small (avgPR is low too, at 0.1).

In a second set of experiments on this dataset, we restrict the group sizes to 50 to 500, and take these groups from first pass of Louvain (vs. the final output), where sufficiently many smaller groups are generated. This yields ∼40 groups on average. We see that, with a narrower range of group sizes, overlap performance score improve for all.
| YouTube (Social) | Sp. Rank Corr. | Top PR | Top Size | Top 5 PR | Sp. Corr. With Size |
|------------------|---------------|--------|----------|----------|---------------------|
| Binomial         | 0.57 ± 0.02   | 0.07   | 12184    | 0.09     | 0.94                |
| TPR              | 0.87 ± 0.17   | 5.8    | 0.16     | 0.05     |
| Conductance      | -0.28         | 0.06   | 4.4      | 0.04     | -0.3                |
| Modularity       | 0.57          | 0.08   | 21586    | 0.09     | 0.98                |
| Size             | 0.57 ± 0.09   | 27016  | 0.09     | 1.0      |
| 5k ref. groups, 68 groups generated, mean avgPR =0.1 |

| LiveJournal (Social) | Sp. Rank Corr. | Top PR | Top Size | Top 5 PR | Sp. Corr. With Size |
|----------------------|---------------|--------|----------|----------|---------------------|
| Binomial             | -0.3 ± 0.02   | 0.2    | 32k      | 0.43     | 0.97                |
| TPR                  | 0.3 ± 0.006   | 0.7    | 214      | 0.68     | -0.31               |
| Conductance          | 0.4 ± 0.003   | 0.77   | 25       | 0.75 ± 0.000 | -0.25             |
| Modularity           | -0.3          | 0.2    | 32k      | 0.28     | 0.98                |
| Size                 | -0.4          | 0.14   | 38k      | 0.3      | 1.0                 |
| 98k ref. groups, 105 groups generated on avg, avgPR =0.65 |

| Amazon (CoPurchases) | Sp. Rank Corr. | Top PR | Top Size | Top 5 PR | Sp. Corr. With Size |
|----------------------|---------------|--------|----------|----------|---------------------|
| Binomial             | -0.49 ± 0.05  | 0.54   | 11894    | 0.53 ± 0.003 | 1.0                |
| TPR                  | 0.24 ± 0.04   | 0.88   | 31.4     | 0.90 ± 0.03  | -0.5               |
| Conductance          | 0.74 ± 0.02   | 0.80   | 157      | 0.83 ± 0.02  | -0.7               |
| Modularity           | -0.50 ± 0.03  | 0.54   | 11894    | 0.53 ± 0.003 | 1.0                |
| Size                 | -0.51 ± 0.03x | 0.54   | 12249.0  | 0.51 ± 0.01  | 1.0                |
| 75k ref. groups, 238±8 groups generated on avg, avgPR =0.54 |

Table 6: YouTube social graph, results averaged over 3 trials.

Table 7: Experiments on LiveJournal, 5 trials.

### 3.3.6 Amazon

In this dataset, nodes are items (products), edges are commonly co-purchased pairs of items [27]. The reference groups are defined by product category: they are connected components of items with same category. This yield cohesive high precisions groups, but rather small groups. Here, the pattern of performance is similar to the pattern on the LiveJournal graph (Table 8): there are many small reference groups, and the few in Louvain’s output that are also small match them well, so Conductance and TPR do well. Binomial does worse because it scores very large generated groups high, and such don’t find a good enough match in the reference groups.

Table 8: Experiments on Amazon. 5 trials.
As the analysis of Section 2.4 explained, the binomial scores for bigger groups can be very high. For example, for the Amazon dataset, the highest scoring generated group gets a binomial score of over 43k (observed over multiple runs), with group size over 11k items (size shown in Table 8, with \( d_{in} \) over 35k, while \( d \) is \( \approx \) 40k. Note that this group is indeed substantially significant: a group with less than 4% of nodes has nearly 90% of its many incident edges as internal. A few of this group’s most common topics, up to depth 3 in the category hierarchy, include 'Books_Subjects_Children’sBooks’, ‘Books_Subjects_Nonfiction’, and ‘FolkTales&Myths_Stories’, covering respectively 9.5k, 181, and 153 of in the group. Many of the other groups also have 'Books' as their top category, but percent covered by 'Books_Subjects_Children’sBooks’ is far below this.

The median binomial score of the 200+ groups generated is over 800, and the minimum remains over 40 (which is the score of the group with smallest size 11), thus all groups generated are highly significant in this dataset. We do see borderline or significance groups in other cases (e.g. see Appendix B), but in our experience, there exist numerous groups generated with healthy significance in real-world datasets.

### 3.4 A Comparison Summary

| Binomial vs. | Size       | Modularity | Conductance | TPR       |
|--------------|------------|------------|-------------|-----------|
| SPR          | (wins=9, losses=2, ties=1) | (6, 2, 4)   | (9, 2, 1)   | (10, 2, 0) |
| topPR        | (8, 2, 2)   | (7, 2, 3)   | (6, 3, 3)   | (6, 6, 0)   |

Table 9: Number of wins, losses and ties of Binomial vs. 4 ranking methods, based on Spearman Rank Correlation (SPR), and the overlap score of top ranked group, on the 12 datasets. In case of multiple experiments (for Newsgroups, CoAuthorship, · · · ), we use the first/default set up. A ‘win’ is whether the (average) performance is higher.

In the absence of a clear and single source of groundtruth, it is difficult to make sweeping conclusions. We aimed to see how consistent, across a variety of datasets, the rankings by various techniques were with the reference rankings, obtained via matching against the groups provided for each datasets. It is very possible that a group ranked high is informative and useful, but scores low according to the matching criterion with any of the available reference groups [17]. Table 9 gives a summary of pair-wise comparisons over all the 12 datasets (using the default or first setting for each dataset, in case of multiple experiments). We observe that, due to the nature of the ranking task and available reference groups, even a simple baseline such as Size can score wins. Overall however, given the results from synthetic experiments and the often competitive ranking performance of binomial ranking on many of the diverse datasets, we can conclude that binomial modeling with its simplicity and efficiency offers a robust alternative that complements existing scoring techniques well.

Of course, Table 9 hides the causes behind the wins or losses (as well as extent of win/loss). As explained in the previous subsections, the findings above suggest a pattern: when the reference groups are few and relatively large, or when the candidate groups are in a similar size range (e.g. within at most an order of magnitude, i.e. up to 10x, of each other), we expect Binomial ranking to do well and often better than Conductance, TPR, or Modularity. As seen above, and per our motivation, Binomial tends to score larger groups, which provide more evidence of community than other methods, even in the cases where the correlation with Size, is not as large as the case for the (group-wise) Modularity ranking, as observed in the tables. We also note that statistical significance of a property is not the same as strength of a property: given some quantification of the ‘community property’, it is possible that a group exhibits large statistically significant community property, but nevertheless the strength or intensity of the property can be relatively weak. For example, conductance can be one measure of intensity. Another is the ratio of observed to expected connection proportions, \( \mathcal{I} = \frac{q}{p} \), as explained in Section 2.9.1 We leave exploring and utilizing such distinctions to future work.

### 4 Related Work

The notion of confidence and in particular statistical confidence has a range of meanings and approaches in network analysis and community detection. A comprehensive classification and summarization of these methods appears in He et al. [18]. In particular much of the work has been concerned with the significance of the entire partitioning discovered, and not on single group (candidate community) at a time [12, 3]. Those working on significance of individual groups, have focused on developing algorithms for group discovery (community mining) and do not evaluate ranking given groups [13], with the exception of the work of Zitnik et al. [15], based on which we designed much of our evaluation protocol and experiments. Their work shows among other findings that there can be significant room for improved

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7 More accurately, the larger candidate groups in real-world datasets, provide more evidence that they are far from being generated by a random process.
We developed community analysis based on the binomial tail. Our motivation was to design a principled inexpensive modern method for what a community means, in particular under the view of what makes a good community. Thanks to the machine learning reading group at Tetration, for discussions as this research progressed, and to the members of customer adoption and services teams for providing us network data and feedback.

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5 Conclusions

We developed community analysis based on the binomial tail. Our motivation was to design a principled inexpensive scoring method that was naturally more biased towards group size compared to existing techniques such as conductance. The approach is simple, efficient, and parameter-free, and we find that it offers a robust method for confidence assignment and filtering. Extensive experiments demonstrated the utility of the binomial score for ranking. The modeling space is rich, and we explored several variants, such as node and edge-based null models, and an enhancement using the union bound. We also described how the tail is applicable to two other tasks: evidence of community assignment and filtering. Extensive experiments demonstrated the utility of the binomial score for ranking. The authors’ goal is to derive a community mining algorithm with the bound, and they show promising results. Subsequent work of the authors derives an exact bound based on the ER random graphs, with improved detection results [36], but this bound may be inefficient as it requires summing factorials up to number of edges. Traag et al. use density of all the groups found in the partitioning (called Significance) [57] and later develop the Surprise score for assessing an entire partitioning, which is based on the number of edges of all groups in the partitioning [38]. Further, they use a binomial tail (although not referred to as such) with number of trials being the lesser of the total number of possible internal edges across all groups and number of graph edges, to asymptotically approximate their measure of surprise, and present an efficient variant of Louvain algorithm for Surprise [38].

The KL-divergence for approximations also arises there. These lines of work show the promise of scoring based on a statistical significance approach using binomials, for discovering candidate communities as well. Several approaches attempt to sample from the space of random graphs (ER or configuration space), such as [39], but these becomes prohibitive for large graphs. Other approaches are somewhat indirect, and attempt to capture significance in terms of nodes that appear to least belong to a candidate group (e.g. [40, 41]).

The work of Yang and Leskovec [5] introduced a number of datasets with reference groups and assessed the performance of a variety of scoring techniques, based on several goodness criteria, using the reference groups. Later work has cautioned about treating the reference as groundtruth and basing the evaluations heavily on comparisons to them [17]. Community analysis indeed depends much on the domain and the particular goals of the task. While we used the provided reference groups, we contextualized our comparisons to shed more light on the findings.

The binomial model naturally finds numerous applications in various analyses of random graphs, in counting and deriving the probability of finding certain structures (e.g. dense subgraphs). However, we are not aware of this direct use of the binomial tail for scoring significance and ranking of candidate communities. It is a way of operationalizing what a community means, in particular under the modern view of what makes a good community [3], i.e. based on a comparison of estimates of internal (within group) vs external or overall probabilities (e.g. see Equation 2), rather than earlier definitions based on non-statistical functions of raw counts such as group degree and conductance.

There may also exist tighter bounds, but possibly limited to special cases.
References

[1] S. Fortunato. Community detection in graphs. *Physics Reports*, 486, 2010.

[2] M. E. J. Newman. *Networks: An Introduction*. Oxford University Press, 2010.

[3] S. Fortunato and D. Hric. Community detection in networks: A user guide. *Physics Reports*, 659, 2016.

[4] V. D. Blondel, J. Guillaume, R. Lambiotte, and E. Lefebvre. Fast unfolding of communities in large networks. *Journal of Statistical Mechanics: Theory and Experiment*, 2008(10), 2008.

[5] J. Yang and J. Leskovec. Defining and evaluating network communities based on ground-truth. *IEEE International Conference On Data (ICDM)*, 2012.

[6] R. Arratia and L. Gordon. Tutorial on large deviations for the binomial distribution. *Bulletin of Mathematical Biology*, 51:125–131, 1989.

[7] R. B. Ash. *Information Theory*. Dover Publications, 1990.

[8] Z. Ghalmane, M. El Hassouni, C. Cherifi, and H. Cherifi. Centrality in modular networks. *EPJ Data Science*, 2019.

[9] M. E. J. Newman. Fast algorithm for detecting community structure in networks. *Physical Review E*, 69(6), 2004.

[10] G. E. P. Box, J. S. Hunter, and W. G. Hunter. *Statistics for experimenters*. Wiley, 1978.

[11] R. Motwani and P. Raghavan. *Randomized Algorithms*. Cambridge University Press, 1995.

[12] J. Leskovec and R. Sosič. SNAP: A general-purpose network analysis and graph-mining library. *ACM Transactions on Intelligent Systems and Technology (TIST)*, 8(1):1, 2016.

[13] J. Leskovec, K. Lang, and M. Mahoney. Empirical comparison of algorithms for network community detection. In *WWW’10*, 2010.

[14] J. Shi and J. Malik. Normalized cuts and image segmentation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 2000.

[15] M. Zitnik, R. Sosič, and J. Leskovec. Prioritizing network communities. *Nature Communications*, 9, 2018.

[16] R. Andersen and K. Lang. Communities from seed sets. In *Proceedings of the 15th international conference on World Wide Web*, 2006.

[17] L. Peel, D. B. Larremore, and A. Clauset. The ground truth about metadata and community detection in networks. *Science Advances*, 3(5), 2017.

[18] Z. He, H. Liang, Z. Chen, and C. Zhao. Detecting statistically significant communities, 2018.

[19] F. Pedregosa, G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel, M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, J. Vanderplas, A. Passos, D. Cournapeau, M. Brucher, M. Perrot, and E. Duchesnay. Scikit-learn: Machine learning in Python. *Journal of Machine Learning Research*, 12:2825–2830, 2011.

[20] P. Jaccard. Étude comparative de la distribution florale dans une portion des alpes et des jura. *Bulletin de la Société vaudoise des sciences naturelles*, 1901.

[21] C. Spearman. The proof and measurement of association between two things. *American Journal of Psychology*, 1904.

[22] E. Lazega. *The collegial phenomenon: The social mechanisms of cooperation among peers in a corporate law partnership*. Oxford University Press on Demand, 2001.

[23] M. Girvan and M. E. J. Newman. Community structure in social and biological networks. In *PNAS*, 2002.

[24] S. Ghosh, A. Banerjee, N. Sharma, S. Agarwal, and N. Ganguly. Statistical analysis of the Indian railway network: a complex network approach. *Acta Physica Polonica B Proceedings Supplement*, 2011.

[25] V. Krebs. Network analysis software & services for organizations, communities, and their consultants. Accessed 2019.

[26] L. A. Adamic and N. Glance. The political blogosphere and the 2004 US election: divided they blog. In *Proc. of the 3rd Int. Workshop on Link Discovery (ACM)*, 2005.

[27] J. Leskovec and A. Krevl. SNAP Datasets: Stanford large network dataset collection. [http://snap.stanford.edu/data](http://snap.stanford.edu/data), 2014.

[28] O. Madani, M. Georg, and D. Ross. On using nearly-independent feature families for high precision and confidence. *Machine Learning Journal*, 2013.
A Experiments on Additional Synthetic Settings

Here, we look at two more distributions for the planted partition model, in particular to see whether similar patterns from Section 3.2 hold. As observed previously, we find that binomial ranking is a substantially better ranker over a considerable noise region, but we also observe limitations of coverage and accuracy of the approximation to the tail in a high noise region. These issues are unlikely to be common in practice, based on our experiments on real-world data, where there often exist ample highly significant groups. We will also briefly mention the solution of using exact or other (e.g. normal) approximations in these cases of borderline groups.

On both the datasets below, TPR always returned 1.0 on the generated groups, and thus yielded no ranking, and it’s not shown. First, we look at equal group sizes as well as equal internal edge probabilities for all the 10 groups, Syn2: each of size 30 and internal probability of 0.4, shown in Figure 4.

Similar to results of Section 3.2, initially the problem is easy and then follows a range of noise probabilities where Binomial ranking separates itself from the rest. However, as the problem gets too noisy, Binomial, as implemented based on the approximation to the tail, loses its dominance, and all the techniques perform near the average overlap score of the generated groups, which goes down to 0.3 (Figure 4(b)). Initially at 0.05, Louvain generates \( \approx 10 \) groups in every trial. In almost half the trials at 0.05 noise, there is no ideal ranking (all groups are perfectly recovered) and the results are averaged over the remaining half. With the edge noise raised eventually to 0.2, the number of generated groups goes down to near 7 on average.

Next, Figure 5 shows a case of 10 groups with unequal group sizes, Syn3, \([160, 60, 50, 40, 40, 30, 30, 30, 20]\], but equal internal probabilities, each of 0.4. Here, we see a similar pattern. However, in the high noise region of 0.2, in most trials all groups are tied with 0 \( b_{node} \) score under Binomial, while the Modularity and Conductance, as well as Size, still do well in selecting a top group (which is often the largest candidate group). The number of groups generated (in a trial) began with nearly 7 at 0.025 and went down to 4 at 0.20. At noise level of 0.25 (not shown), performances Modularity, Conductance of go down to 0.46, while avg group score goes to below 0.3.
Figure 4: Planted partition synthesized data, equal group sizes and equal group (internal) edge probabilities (all 0.4, Syn2), averaged over 200 trials. Binomial ranking does a better job until noise becomes sufficiently high.

In the implementation of binomial scoring, $b_{\text{node}}()$, if the observed ratio is not higher than the expected, i.e., $d \leq p$, we return 0 score (insignificant). Indeed, looking at the ratio of the 4 groups generated, in all trials, the observed ratio was smaller or close to the expected, at noise level of 0.2. If we drop the 0 check, we often get negative group scores for $b_{\text{node}}()$ at noise level of 0.2, and ranking is not improved either. In this region, while the group score is statistically insignificant, the ranking task may remain useful. We replaced the approximation to the binomial tail with an exact computation of the tail (Equation 1), at 0.2 noise. Each trial took a minute or so (while 200 trials would take well under a minute with the approximation), but the better binomial scores (probabilities) indeed led to superior rankings over the other techniques: For Binomial, Conductance, and Modularity, the SPRs (with the reference ranking) were respectively 0.630, 0.56, 0.57, and topPR values were 0.987, 0.929, 0.987, i.e. the largest group was often picked (averages over the 10 trials). We looked at the (exact) binomial score obtained by the top ranked group (group with the highest binomial score), and indeed it was very poor at $\sim 0.3$ on average. Note that a score of 1 or higher corresponds to a p-value of near 0.1 (when significance becomes noticeable).

We could use the exact form of (or the normal approximation to) the binomial in such exceptional scenarios (when $q \approx p$) to get better (approximate) scores, a possible improvement for future work. But the utility of this extension is perhaps very limited in practice: On the real-world datasets, in our experiments, there have been ample candidate groups generated that obtain very high scores. Regarding coverage and significance, see also Table 10. We note however that, in a community mining context, for a bottom-up search algorithm seeking to improve the binomial score in each step (as an objective), alternate scores (modularity, intensity, ...) may be useful initially when the group(s) being generated are not yet significant.

We have experimented with other variations of synthesized graphs, such as internal planted probabilities decreasing with graph size (reverse of Syn3), and we have observed similar results.

B Explorations of Variants of the Binomial

Here, we compare a few variants of binomial scoring to the default node-based. These are the lower bound score portion of the node-based scoring, the edge-based model (Section 2.6), and global scoring (Section 2.8). Table 10 and Figure 6 show SPR correlations and performances on a few datasets. We also report on a few aspects such as maximum score reported and number of groups deemed significant.

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9 Implemented with several speed ups for the factorial, including doing the complement when faster.
We observe that the different variants are very correlated, as might be expected (Table 10). In particular, the lower bound score is basically identical in its ranking to the default node-based (which is the average of high and low), but as we increase the noise, the lower bound may yield 0 scores sooner than the upper and we observe the tiny difference in ranking in the high noise region. We don’t report on it further as the differences are tiny. The global variant is strict, in the sense that a significant number of groups, even the reference groups in real-world datasets, are deemed insignificant or have weak significance, under the global model which takes the entire graph into account. Some of this is likely due to the loose upperbound (due to the union bound).

In Figure 6, we compare SPR of default node-based against global and edge-based rankings, in a couple of synthetic settings, equal size and different probabilities (from Section 3.2, Syn 1), and different sizes and probabilities (from Section A, Syn 3). We observe again that the global variant doesn’t have the range of the other two (scores become 0 with raised noise), and edge-based correlates with but appears weaker than node-based in these two synthetic settings. This underperformance may be due to how well planted partition models match either node and edge-based null models.

Table 10: Correlation (SPR) of default node-based ($b_{node}$) binomial scoring and ranking of groups vs. scoring and ranking using three variants: Use the lower score (instead of averaging low and high, see Equation 3), use edge-based binomial score (Section 2.6), and use the graph adjusted ‘global’ score (Section 2.8). The ranked groups are from: groups generated on Syn 1 scheme with 3 noise settings (200 trials each), reference groups from YouTube Social (5k), output groups from a run of Louvain (65 groups generated), and Amazon co-purchase reference groups (75k). The correlations are very high. For the global variant, only 300 groups scored above 0 on YouTube, 52 from Louvain’s output (on several runs), and 25k on Amazon, and the correlation is with respect to those subsets of groups. And on Syn 1, with high noise rates, global returns no ranking on most trials (all groups assigned 0), so no SPR is reported on such.
It is also possible that taking the average or the maximum of the node- and edge-based scores improves coverage or ranking performance. In one experiment on a graph built from hosts communicating in a datacenter, we observed that the number of groups with score above a significance threshold of 1.5 went up by \( \sim 5\% \) when using both, over using either one only. On the YouTube reference groups, 21 groups get 0 node-based scores and 31 get below 3, and median score is 11. For edge-based, the numbers are respectively 24, and over 220, and median score is just 9. Thus edge-based score seems to be smaller/weaker, but the conservative global score yields substantially lower scores than both.

![Figure 6: SPR comparisons of global and edge-based on Syn1 and Syn3 graph generation schemes.](image)

We have experimented with the global binomial variant on the real-world datasets, and we found that, with respect to SPR and overlap performance of top ranked groups, mostly tied with plain binomial, and a few mixed results but only small differences. As observed above, a bigger fraction of candidate groups are deemed insignificant when using global binomial, but the overall ranking performances didn’t change on the real-world datasets.

### B.1 \( p \)-value Based on a Configuration Model

An analytic upper bound of a \( p \)-value based on the configuration model (i.e. random graphs, same degree sequence) is developed by He et al. [18], based on the observed density, i.e. internal number of edges \( d_{\text{in}} \) out of \( d \) of a group. The formula is simple, and we replicate several of the SPR results with conductance in that paper and find that the approximate binomial bound appears better for ranking on synthetic experiments, as explained below. The authors are not focused on ranking in their work, and show that the community mining algorithm they derive based on the bound is competitive and promising against other algorithms on a range of of real-world and synthetic datasets. The \( p \)-value of a group is upper bounded by (Equation 4 [18]):

\[
p\text{-value}(g) \leq \frac{d_{\text{in}} + \Delta}{2d_{\text{in}}} \cdot \frac{|E|}{d_{\text{in}}} \tag{8}
\]

In our experiments, we use the negative log of the \( p \)-value bound as we did for the binomial score (rankings won’t change), and use Stirling’s approximation to the factorial. Table [11] shows SPR values, in particular between conductance and the \( p \)-value bound on the ranking of reference groups of a few common datasets, which is match or are consistent with the numbers in the Table 4 of [18].

We also observe very high correlation of the \( p \)-value bound with the (node-based) binomial on these groups. Given that one is based on group and graph size and the other is based on number of edges, this is remarkable. However, on the 3 synthetic regimes, the \( p \)-value bound was weaker, as shown in Figures 7 and 8. One cause for this underperformance is
Table 11: Correlation (SPR) of p-value upper bound of Equation 8 with conductance, and with binomial on the reference groups of 3 datasets. First row matches He et al. ’s Table 4 [18] (except they obtain 0.6965 for p-value with conductance). Second row shows high correlation of Binomial with p-value.

due to the limited range of the p-value bound, akin to the case for our global binomial. On Syn2 and Syn3 with noise of 0.15, respectively in around 1/4-th and more than half the trials, all the generated groups get negative scores, and if we clamp the score to 0, no ranking is returned. With lower noise rates, the bound underperforms both node-based binomial score somewhat (with edge-based, it’s more mixed), which may be due to a looser bound, and may not be a significant difference on real-world data.

Figure 7: SPR comparisons of rankings by p-value bound of Equation 8 with (node-based) binomial and edge-based binomial rankings on the three synthetic datasets (200 trials).

C Product Category Purity

Here, we explore the effect on ranking of the candidate groups having very different sizes, spanning one or more orders of magnitude, in the case of the Amazon reference groups. Furthermore, because the reference groups are determined in part based on sharing the same product categories (at lowest level of a product hierarchy), we looked at correlations with category cover size, i.e. a minimal number of (product) categories needed to cover a group, where each item in the group must belong to (be ’covered by’) at least one category in the cover. A smaller cover size may be deemed desirable as it signals category cohesiveness (’purity’). Conversely, when a group requires a very large cover, it may be deemed suspect as it signals that items that don’t belong are put together in the group. However, a natural expectation is that larger groups will require larger covers in general (and as we observe below), and furthermore the product categories may not be exhaustive and/or co-purchase patterns on which the graph is based, can also follow complimentary patterns of purchase in many cases (products from different categories that tend to complement, e.g. serve the same goal).

We do this analysis on 5000 groups picked based on having high quality on several dimensions. Not all items in a group have a category, and some may have multiple. The zero or more categories assigned appear to be at the leaf. For example, for a group of 106 items, 13 did not have any category, and a few categories that covered the most items were: ’books_Subjects_Religion&Spirituality_Hinduism_General’ (15), ’Books_Subjects_Health’ (15), ’Books_Subjects_Religion&Spirituality_NewAge_General’ (8), ···. The groups sizes range from the very small (minimum 3) to just over 500 items, and the median size is 95. The minimal covers ranged in size from 1 to 187, with median of 27.
Figure 8: Overlap score (mean of precision and recall) of top group: comparisons of rankings by $p$-value bound of Equation 8 with (node-based) binomial ranking on the three synthetic datasets (200 trials).

Note that computing the minimum cover is NP-hard, so we applied a simple randomized greedy algorithm, that repeatedly picks a category covering most remaining items in the group, until all items in the group belong to at least one category in the set (cover) picked (note: items can belong to multiple categories). Running the algorithm multiple times gave similar ranking results.

One might expect the minimal cover size of groups to correlate with the group size, i.e. the more items in a group, the more categories needed to cover all the items in the group. And this is the case. Table 12 shows the SPRs, between (minimal) cover size and a few scoring functions, on the 5000 reference groups where the conductance, modularity and CRank scores (described in 3.2) are all provided by the authors [5, 27] (and we verified that conductance and modularity matched ours). We observe that conductance and CRank highly anticorrelate with cover, while group size (as may be expected), together with modularity and binomial are highly correlated with it (column 1 on the left).

Very interestingly, however, as we narrow the range to top 2000 and top 1000 groups by size, i.e. when we compare on more 'local' rankings (limited size variations), the correlations of all converge to one another: binomial and to lesser extent modularity move in the direction of anti-correlation with cover, while this is reversed for conductance. The last columns show the SPRs as we narrow the range of sizes even further. In these experiments, in each trial a group is picked at random (from the 5000), and for a window size of say 100, 100 groups closest in size, 50 smaller and 50 larger are determined, and the SPR of the various scores with cover over the 100 groups is recorded. The SPRs are averaged over 1000 trials. We see binomial and to a lesser extent modularity become negatively correlated with cover size, similar to conductance and CRank, and binomial becomes even more anti-correlated compared to conductance. Thus when the group sizes don’t widely vary, we observe that rankings becomes more comparable.

D Further Applications of Binomial Modeling

We first describe an application of binomial modeling to computing the statistical evidence of group membership. This membership score, in turn, could also be used for scoring an entire group, with some benefits as we describe. We then present an application to discovering significant edges among groups in the group-induced graph. In all cases, we can see a pattern: a progression from using plain ‘degree’ (plain count), to a ratio (akin to conductance), to binomial modeling (yields a probability/confidence).

D.1 Strength of Group Membership

Often we want to identify node(s) that best represent the group, or that are most active or connected in the group. For example, we want to display a few such nodes to the user as a summary so the user more easily gets the essence of the group. In our application domain, the nodes are machines and often have names (hostnames), and may have other

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We also experimented with different cover requirement thresholds below 100%, e.g. 90%, and saw similar correlation results.
attributes (e.g. type of OS, ip address(es), · · · ), and by effectively identifying the more central nodes of a group, we may better summarize or name the group.

One can imagine a variety of measures that reflect different aspects of importance or belonging (e.g. see [8]). For example, one can define a counter part to the group conductance here by replacing group degree by node degree, and similarly internal degree becomes the number of edges of the node that are inside the group. However, the same drawbacks of normalization remains here for this measure. A node that has one or only a few edges, but all internal, gets perfect score according to this measure. We can also easily use the binomial counter part, which in a principled way combines \( d, d_{nn} \) and \( p \). Note that, like before, \( p \) can be determined via either using the number of nodes, of the group and the graph, or the number of edges, of the group and of the graph.

For instance, if we use the node-based null model (using number of nodes in group vs. graph), with a graph of 20 nodes and a group with 5 nodes, expected probability of connecting to a node inside the group is \( p = \frac{1}{10} \) (when self-arcs are not allowed), and if a node has \( d = 4 \) edges with \( d_{nn} = 3 \) inside, its observed connection probability is \( q = d_{nn}/d = 3/4 \) and its binomial membership level (tail score) is around 1.5 (using the approximations of \( \log_{10} \)).

Centrality of a node with respect to the whole graph has been studied [2], and recently the various centrality measures have been recently extended to communities [8]. Degree centrality (plain node degree) comes closest to the approach here, but fails to tradeoff external vs internal activity. The binomial tail score provides a simple efficient statistical alternative (a significance value) that takes the tradeoff into account.

Furthermore, one can define a new group score by taking the average or the median (or some quantile) score of the member scores, each according to a binomial membership score. Compared to binomial score at the group level (of Section 2), the median member score can better reflect the average ‘happiness’ of the members and is less prone to preferring ‘lopsided’ groups as we explain. In particular, it has an advantage with respect to the resolution-limit problem that a number of other scores don’t (including the group-level binomial score), which we explain next.

### D.1.1 Group Scores and the Resolution Problem

Consider the two (internally dense) small groups, groups 1 and 2, of Figure 9 where the two groups are connected via a single edge only (more generally, relatively few edges) and are isolated from the rest of the graph. Intuitively, the two separated groups are preferred as communities over the single union, dumbbell group, but if the remainder of the graph is sufficiently large, modularity scores the union (the dumbbell) higher than groups 1 or 2. This is the so-called resolution limit problem [42, 1]. This issue stems from the phenomenon that the presence of an edge (or a few edges) connecting the two groups becomes significant as the entire (the rest of) graph grows: the modularity scores improves when we join the two groups if the number of edges between them is higher than the expected number of edges (according to the null model). The same issue occurs for the (group-wise) binomial scores of Section 2. For instance, for the (group-wise) binomial, in the node-based modeling, letting \( p \) be the probability of connecting for group 1, then \( 2p \) is roughly the probability of connecting in the dumbbell group, since it has twice the group 1’s size. With \( d \) internal edges, the binomial tail for group 1 is at least \( p^d \), while for the dumbbell group (with at least \( 2d \) internal edges) is at most \((2p)^{2d}\). So the dumbbell group obtains a smaller tail (a higher score), as long as \( p^d \) is higher than \( (2p)^{2d} \), or \( p < 1/2^d \). For example, with \( d = 5 \) internal edges, if group 1 is smaller than roughly \( 1/2^5 \) of the entire graph. Interesting, for standard modularity, if a group is smaller than roughly \( \sqrt{|E|} \) the resolution problem may arise.

| SPRs with Cover | 5k groups | Top 2000 | Top 1000 | Window of 200 | Window of 100 | Window of 50 |
|----------------|-----------|---------|----------|---------------|---------------|---------------|
| Binomial       | 0.759     | 0.104   | -0.112   | -0.243        | -0.264        | -0.267        |
| Conductance    | -0.537    | -0.360  | -0.327   | -0.245        | -0.248        | -0.240        |
| Modularity     | 0.807     | 0.223   | -0.024   | -0.195        | -0.223        | -0.237        |
| Size           | 0.871     | 0.508   | 0.313    | 0.381         | 0.375         | 0.37          |
| CRank          | -0.723    | -0.446  | -0.358   | -0.285        | -0.281        | -0.278        |

Table 12: SPRs with a group’s product category cover size on 5000 reference groups. Cover (size) is the minimal number of categories that cover all the products in a group. It is plausible that, in general, the larger the group the more categories needed to cover it. We observed that size is most correlated with cover on all 5000 groups (left column), and we observe binomial and modularity which correlate with size also correlate with cover, while conductance which favors smaller groups, has negative SPR with cover. This is also the case with CRank scores, provided with the groups. However, as we narrow the size range of groups compared (top 2000 and 1000 groups by size, and windows of closest sizes as explained in text) we observe that the SPRs converge and furthermore binomial ranking becomes substantially anti-correlated with cover, more so than conductance.
Figure 9: Two dense groups sparsely connected to one another, and isolated from the rest of graph. As long as the rest of the graph is sufficiently large, joining the two groups (the union “dumbbell” group) yields a higher score under several scoring techniques, including modularity and group binomial score. However the median of membership scores is higher on each (separated) group.

We can also easily verify similar issues remain for a few other scores: for instance, for TPR, the scores are tied (no preference on whether to join or not), while for conductance, joining the two groups is better, as it yields a perfect conductance score for the union.

Now, considering the membership binomial score for each node, joining the two groups increases the score for the nodes that have a connection to a node in the other group (group1 to group2), while decreases it for the rest (as the group size and thus the probability of a connection increase, the tail probability under the null model increases). Thus as long as fewer than half of the group1 nodes have a connection to the other group, the median score decreases when we join the two groups.

Note that we can still join two cliques that are relatively sparsely connected to one another based on the node-based measure (as long as half the nodes have an edge to the other), but in such a situation the union.

Thus, the median membership binomial score can lead to somewhat more uniformly connected groups (avoids lopsided dumbbells), compared to the (group-wise) binomial score. Of course, directly optimizing such a score, for community discovery (community mining), is likely more challenging as, for example, small local improvements in internal density may not change the median. We leave further exploration to future work.

D.2 Highlighting Significant Group Interactions

Edges represent interactions, and once a graph is broken into candidate communities, an analyst may want to explore the most salient (significant in some sense) interactions among the discovered groups. This type of edge analysis can find applications in fields such as biological/biochemical networks (e.g. protein interaction networks) to draw scientists’ attention and effort to the more likely important interactions. Binomial modeling offer one type of edge (statistical) significance.

Once groups are discovered, one can build the group-graph where groups are the nodes and two groups $g_1$ and $g_2$ (the corresponding vertices in the group-graph) are connected if there exist any edges among members of the two groups, where the weight of an edge between groups $g_1$ and $g_2$ is the count of such node pairs. One may then wish to first focus on the significant interactions among groups, i.e., those edges with counts more than a null would typically generate. One null mode can be briefly described as follows: Given are $k > 2$ groups in the graph. A group $g_1$ has $d_1$ total count of (outgoing) edges, and it has $d_2$ edge weight to group $g_2$. We are interested in the tail probability of observing $d_2$ or more connections given the group has several ‘choices’ (groups) and the choices were picked uniformly at random among all other group members (thus success probability is $p = \frac{|g_2|}{|V| - |g_1|}$, where $V$ is the set of nodes in original graph). Note that it’s best to use the node size of the groups to account for different group sizes. There remain several avenues of exploration. For instance, as described, an edge between two groups gets different significance

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11 Averaging the scores can be influenced by extreme scores. The median of some other node-based score may also avoid the resolution problem (e.g. membership intensity defined as the ratio of observed probability $q$ of connecting to an internal node, to expected $p$).

12 The same meta-graph is built after each pass of Louvain to improve the modularity objective.

13 A ratio measure akin to conductance can be defined, to tradeoff overall degree with selectivity to certain group, but that suffers from similar aforementioned drawback.
probabilities depending on the group that we condition on, and we may want to keep this directionality, or otherwise combine the measures. For the null model, we also have a choice of whether to account for the count of internal group edges (group self-arc weights).

E  Node-Proportion Based Modularity

The modularity objective scores a given set of groups (candidate communities), and is defined as the sum, over each group \( g \), of the difference \( \delta(g) \) in observed vs. expected proportion of each group’s internal edges: \( \sum_g \delta(g) \). As in the standard applications of modularity, we assume the groups form a complete partitioning of the nodes, i.e. they are exclusive (non-overlapping) and exhaustive.

The null model in the original paper is based on what we have referred to as the ‘edge-based’ model of Section 2.6 [9], which has the desired property of adjusting for the chattiness of the group. Here we briefly develop the objective and the Louvain algorithm based on the node-based model. We are not aware of this derivation in the literature.

For each group in the partitioning (groups), using the node-based model, the expected internal edge proportion is \( \frac{d_{in}(g)}{|V|} \). That is, given a group with \( d \) incident nodes, the probability that each of the other ends also falls in the group is \( \frac{|g|}{|V|} \), thus the expected number is \( d \frac{|g|}{|V|} \), and the expected proportion is \( \frac{d_{in}(g)}{|V|} \). The observed proportion is \( \frac{d_{in}(g)}{|E|} \), and the node-based modularity, denoted \( Q_{node} \), is therefore:

\[
Q_{node} = \sum_{g \in \text{groups}} \delta_n(g) = \frac{1}{|E|} \left( \sum_g d_{in}(g) - \frac{|g|}{|V|} \right)
\]  

We have,

**Proposition 4** \( Q_{node} \in [-1, 1] \).

We present a proof that \( Q_{node} \in [-1, 1] \) (same as edge-based modularity). First \( Q_{node} \leq 1 \), as \( \frac{1}{|E|} \sum_g d_{in} \leq 1 \), and the 2nd term, \( \frac{1}{|E|} \sum_g -d \frac{|g|}{|V|} \leq 0 \). Next, to show that \( Q_{node} \geq -1 \), first note that if the partitioning consists of one group only, \( Q_{node} = 0 \) (\( \frac{|g|}{|V|} = 1 \) and \( d_{in} = d \)). With two or more groups, we can replace any internal edge of a group \( g \) with an outgoing edge that goes from it to another group, and \( Q_{node} \) strictly decreases (considering both groups affected). Now, with \( d_{in} = 0 \) for all groups, we note that \( \sum_g d(g) \frac{|g|}{|V|} \leq |E| \), as \( \sum_g d(g) = |E| \) and \( \sum_g \frac{|g|}{|V|} = 1 \), and a convex combination is no larger than the largest component: when \( a_i \in [0, 1] \), \( \sum_i a_i \leq 1 \), and letting \( i^* = \arg\max_i b_i \), we have \( \sum_i a_i b_i \leq b_{i^*} \).

For a collection of purely disassortative groups (i.e. \( d_{in} = 0 \) for all groups, such as a pure bipartite graph), we get negative modularity that approaches -1 as number of nodes grow. Symmetrically, for many purely assortative groups (\( \forall g, d_{in}(g) = d(g) \)), \( Q_{node} \) approaches 1.

Likewise, one could define a simple Louvain algorithm using the \( Q_{node} \) objective. We leave exploring the (practical) differences in the formulations and, for example, potentials for utilizing both objectives, to future work (see also Section 2.7).