A New Recursive Least-Squares Method with Multiple Forgetting Schemes

Francesco Fraccaroli, Andrea Peruffo and Mattia Zorzi

Abstract—We propose a recursive least-squares method with multiple forgetting schemes to track time-varying model parameters which change with different rates. Our approach hinges on the reformulation of the classic recursive least-squares with forgetting scheme as a regularized least-squares problem. A simulation study shows the effectiveness of the proposed method.

I. INTRODUCTION

Recursive identification methods are essential in system identification, [13], [25], [7], [9], [20], [11]. In particular, they are able to track variations of the model parameters over the time. This task is fundamental in adaptive control, [1], [12], [23].

Recursive least-squares (RLS) methods with forgetting scheme represent a natural way to cope with recursive identification. These approaches can be understood as a weighted least-squares problem wherein the old measurements are exponentially discounted through a parameter called forgetting factor. Moreover, in [3] their tracking capability has been analysed in a rigorous way.

In this paper, we deal with models having time-varying parameters which change with different rates. Many applications can be placed in this framework. An example is the automation of heavy duty vehicles, [21]. In this problem, it is required to estimate the vehicle mass and the road grade. The former is almost constant over the time, whereas the latter is time-varying. Other examples are the control of strip temperature for heating furnace, [24], and the self-tuning cruise control, [14].

In those applications the RLS with forgetting scheme provides poor performances. A refinement of this method is the RLS with directional forgetting scheme, [6], [8], [2], [4]. Roughly speaking, such approach fixes the problem that the incoming information is not uniformly distributed over all parameters. However, this nonuniformity is not equivalent to the presence of parameters with different changing rates, [21]. Indeed, it is possible to construct models with parameters having different changing rates and with incoming information uniformly distributed over all parameters. Thus, also RLS with directional forgetting scheme provides poor performances.

An ad-hoc remedy to estimate parameters with different changing rates is the RLS with vector-type forgetting (or selective forgetting) scheme, [19], [18], [15], [16]. The idea of the above method is to introduce many forgetting factors reflecting the different rates of the change of the parameters. Finally, an ad-hoc modification of the above method has been presented in [21].

In this paper, we propose a new RLS with multiple forgetting schemes. Our method is based on the reformulation of the classic RLS with forgetting scheme as a regularized least-squares problem. It turns out that the current parameters vector minimizes the current prediction error plus a penalty term. The latter is the weighted distance between the current and the previous value of the parameters vector. Moreover, the weight matrix is updated at each time step and the updating law depends on the forgetting factor. This simple observation leads us to generalize this updating to multiple forgetting factors reflecting the different changing rates of the parameters. Moreover, we provide three updating laws drawing inspiration on machine learning. For simplicity we will consider SISO models because the extension to MIMO ones is straightforward. Finally, simulation show the effectiveness of our method.

The remainder of the content in the paper is organized as follows. In Section II we present the state of the art about RLS with forgetting scheme and with vector-type forgetting scheme. The reformulation of the RLS and the three different updating laws are explained in Section III. The performance comparisons between these methods are illustrated in Section IV. Conclusions are drawn in Section V.

II. STATE OF THE ART

Consider a SISO linear, discrete time, time-varying, system

\[ A_t(z^{-1})y(t) = B_t(z^{-1})u(t) + e(t), \]  

where \( e(t) \) is additive noise with variance \( \sigma^2 \) and \( u(t) \) is a stationary Gaussian process independent of \( e(t) \).

\( A_t(z^{-1}) \) and \( B_t(z^{-1}) \) are time-varying polynomials whose degrees are \( n \) and \( m \) respectively:

\[ A_t(z^{-1}) = 1 + \sum_{i=1}^{n} a_{t,i}z^{-i}, \]  
\[ B_t(z^{-1}) = \sum_{i=1}^{m} b_{t,i}z^{-i}, \]
where $z$ is the shift operator.

Assume to collect the data

$$Z^N := \{y(1), u(1) \ldots y(N), u(N)\}. \quad (3)$$

We would estimate $A_t(z^{-1})$ and $B_t(z^{-1})$ at each time step $t$ given $Z^t$. We define

$$\theta_t = [a_{t,1} \ldots a_{t,n} b_{t,1} \ldots b_{t,m}]^T \quad (4)$$

as the vector containing the parameters of $A_t(z^{-1})$ and $B_t(z^{-1})$. Let $\Phi_t$ denote the regression matrix

$$\Phi_t = \begin{bmatrix} \varphi(t)^T \\ \vdots \\ \varphi(max(n,m) + 1)^T \end{bmatrix} \quad (5)$$

where $\varphi(t) = [y(t-1) \ldots y(t-n) u(t-1) \ldots u(t-m)]^T$.

Let $y_t$ be the vector of observations

$$y_t = [y(t) \ldots y(max(n,m) + 1)]^T \quad (6)$$

and in a similar way $e_t$ be the noise vector

$$e_t = [e(t) \ldots e(max(n,m) + 1)]^T. \quad (7)$$

A common way to solve such a problem relies on the RLS with forgetting scheme, [13], [25], where $\theta_t$ is given by

$$\hat{\theta}_t = \arg\min_{\theta_t} V(\theta_t, t), \quad (8)$$

and the loss-function is

$$V(\theta, t) = \sum_{s=1}^{t} \lambda^{t-s} (y(s) - \varphi(s)^T \theta). \quad (9)$$

Here, the forgetting factor $\lambda \in [0, 1]$ operates as an exponential weight which decreases for the more remote data.

Problem (8) admits the recursive solution

$$R_t = \lambda R_{t-1} + \varphi(t) \varphi(t)^T, \quad (10a)$$

$$\hat{\theta}_t = \hat{\theta}_{t-1} + R_{t-1}^{-1} \varphi(t)(y(t) - \varphi(t)^T \hat{\theta}_{t-1}). \quad (10b)$$

Moreover, if we define $P_t = R_t^{-1}$ we obtain the equivalent recursion

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t(y(t) - \varphi(t)^T \hat{\theta}_{t-1}), \quad (11a)$$

$$K_t = \frac{P_{t-1} \varphi(t)}{\lambda + \varphi(t)^T P_{t-1} \varphi(t)}, \quad (11b)$$

$$P_t = \frac{1}{\lambda} (I - K_t \varphi(t)^T) P_{t-1}. \quad (11c)$$

In the case that the parameters in ARX model (1) vary with a different rate it is desirable to assign different forgetting factors. The RLS with vector-type forgetting scheme, [18], [15], consists of scaling $P_t$ by a diagonal matrix $\Lambda$ of forgetting factors

$$P_t = \Lambda^{-\frac{1}{2}} (I - K_t \varphi(t)^T) P_{t-1} \Lambda^{-\frac{1}{2}} \quad (12)$$

where $\Lambda = \text{diag}(\lambda_1 \ldots \lambda_p)$ with $p = n + m$. Therefore, $\lambda_i$ is the forgetting factor reflecting the changing rate of the $i$-th parameter. Finally, an ad-hoc modification of the update law for the gain $K_t$ of the RLS has been proposed in [21]. In this case the parameters to estimate are two. Such method conceptually separates the error due to the parameters in two parts in the objective function (9), that is one part contains the error due to the parameter with faster changing rate and the second one the error due to the parameter with slower changing rate. Then two different forgetting factors have been applied for each term.

### III. RLS with Multiple Forgetting Schemes

In this Section, we introduce our RLS for model whose parameters have different changing rates. Our approach hinges on the following observation.

**Proposition 3.1:** Problem (8) is equivalent to the following problem:

$$\hat{\theta}_t = \arg\min_{\theta_t} \left( \lambda (\theta_t - \hat{\theta}_{t-1})^T R_{t-1}^{-1} (\theta_t - \hat{\theta}_{t-1}) + \gamma (\theta_t - \hat{\theta}_{t-1}) \right), \quad (13)$$

with updating law (10a).

The proof is given in Appendix A.

Proposition 3.1 shows that the RLS with forgetting scheme can be understood as regularized least squares problem. More precisely, the first term in the objective function minimizes the prediction error at time $t$, whereas the penalty term minimizes the distance between $\theta_t$ and the previous estimate $\hat{\theta}_{t-1}$ according to the weight matrix $\lambda R_{t-1}$. Moreover, the weight matrix is updated according to the law (10a).

It is then natural to allow a more general structure for the weight matrix $\lambda R_{t-1}$ and its updating law (10a). Let $F_{\lambda}(\cdot)$ be the forgetting map defined as follows

$$F_{\lambda} : S_{p}^{+} \rightarrow S_{p}^{+}, \quad (14)$$

$$R_{t-1} \mapsto F_{\lambda}(R_{t-1}), \quad (15)$$

where $S_{p}^{+}$ denotes the cone of positive definite matrices of dimension $p$ and $\lambda = [\lambda_1 \ldots \lambda_p]^T \in \mathbb{R}^p$ is the forgetting vector with $0 < \lambda_i < 1$ $i = 1 \ldots p$ forgetting factor of the $i$-th parameter.

Therefore, given $\hat{\theta}_{t-1}$, we propose the following estimation scheme for $\theta_t$.

**Proposition 3.2:** The solution to (14a) with updating law (14b) admits the recursive solution

$$\hat{\theta}_t = \hat{\theta}_{t-1} + K_t (y(t) - \varphi(t)^T \hat{\theta}_{t-1}), \quad (15a)$$

$$K_t = \frac{F_{\lambda}(P_{t-1})^{-1} \varphi(t)}{1 + \varphi(t)^T F_{\lambda}(P_{t-1})^{-1} \varphi(t)}, \quad (15b)$$

$$R_t = F_{\lambda}(R_{t-1}) + \varphi(t) \varphi(t)^T. \quad (15c)$$

Moreover, $K_t$ can be updated in the equivalent way:

$$K_t = \frac{F_{\lambda}(P_{t-1})^{-1} \varphi(t)}{1 + \varphi(t)^T F_{\lambda}(P_{t-1})^{-1} \varphi(t)}, \quad (16a)$$

$$P_t = (I - K_t \varphi(t)^T) F_{\lambda}(P_{t-1})^{-1}. \quad (16b)$$
where $P_t = R_t^{-1}$.

The proof is given in Appendix E.

To design the forgetting map $F_\lambda$, we consider the following result whose proof can be found in [17].

**Proposition 3.3:** Consider $A, B \in \mathcal{S}_p^+$. Let $C$ be a symmetric matrix of dimension $p$ such that

$$[C]_{ij} = [A]_{ij} [B]_{ij}, \quad i, j = 1 \ldots p. \quad (17)$$

Then, $C \in \mathcal{S}_p^+$. In view of the above result, a natural structure for $F_\lambda$ would be

$$F_\lambda(R_{t-1}) = |R_{t-1}| [Q_\lambda]_{ij} \quad (18)$$

where $Q_\lambda \in \mathcal{S}_p^+$. Note that, $Q_\lambda$ can be understood as a kernel matrix with hyperparameters $\lambda$ in the context of machine learning, [17], [22]. Next, we design three types of maps drawing inspiration on the diagonal kernel, the tuned/correlated kernel, [5], and the cubic spline kernel, [22].

**A. Diagonal updating**

Consider the ARX model (1) with $m = 1$ and $n = 1$, therefore we only have two parameters. Let $\theta_{1,1}$ and $\theta_{1,2}$ denote the parameter of $A_t(z^{-1})$ and $B_t(z^{-1})$, respectively. Moreover, the vector containing the two parameters is defined as $\theta_t = [\theta_{1,1} \theta_{1,2}]^T$. We assume that the changing rate of $\theta_{1,1}$ is slow over the interval $[1, N]$, whereas the changing rate of $\theta_{1,2}$ is faster. The simplest idea is to decouple the parameters in the penalty term in (14a). We associate the forgetting factor $\lambda_1$ to $\theta_{1,1}$ and $\lambda_2$ to $\theta_{1,2}$ with $\lambda_1 > \lambda_2$. Let

$$R_{t-1} = \begin{bmatrix} R_{t-1,1} & R_{t-1,12} \\ R_{t-1,2} & R_{t-1,2} \end{bmatrix}. \quad (19)$$

Then, if we define

$$F_{\lambda,DL}(R_{t-1}) = \begin{bmatrix} \lambda_1 R_{t-1,1} & 0 \\ 0 & \lambda_2 R_{t-1,2} \end{bmatrix} \quad (20)$$

the penalty term in (14a) becomes

$$\lambda_1 (\theta_{1,1} - \hat{\theta}_{1,1})^2 R_{t-1,1} + \lambda_2 (\theta_{1,2} - \hat{\theta}_{1,2})^2 R_{t-1,2} \quad (21)$$

that is the parameters of $A_t(z^{-1})$ and the ones of $B_t(z^{-1})$ have been decoupled in the penalty term. This simple example leads us to consider the diagonal updating

$$[F_{\lambda,DL}(R_{t-1})]_{i,j} = \begin{cases} 0 & \text{if } \lambda_i \neq \lambda_j \\ \{R_{t-1}\}_{i,j} \lambda_i & \text{otherwise} \end{cases}$$

Finally, it is worth noting that in the special case that $p = 2$ we obtain the method proposed in [21, formulae (22) and (23)].

**B. Tuned/Correlated updating**

We consider again the example of Section III-A. The changing rate of $R_{t-1,12}$ depends on the changing rates of $\theta_{1,1}$ and $\theta_{1,2}$. Hence, it is reasonable to forget past values of $R_{t-1,12}$ with the fastest changing rate between the one of $\theta_{1,1}$ and $\theta_{1,2}$. Therefore, we weigh $R_{t-1,12}$ with the forgetting factor $\lambda_2$.

$$F_{\lambda,TC}(R_{t-1}) = \begin{bmatrix} \lambda_1 R_{t-1,1} & \lambda_2 R_{t-1,12} \\ \lambda_2 R_{t-1,12} & \lambda_2 R_{t-1,2} \end{bmatrix}. \quad \text{Moreover, the corresponding penalty term is}$$

$$\lambda_1 (\theta_{1,1} - \hat{\theta}_{1,1})^2 R_{t-1,1} + \lambda_2 (\theta_{1,2} - \hat{\theta}_{1,2})^2 R_{t-1,2} + 2 \lambda_2 (\theta_{1,1} - \theta_{1,1}) (\theta_{1,2} - \theta_{1,2}) R_{t-1,12}. \quad (22)$$

Thus, the weight of the cross term is dominated by the smallest forgetting factor. Therefore, in the general case, a reasonable updating law is:

$$[F_{\lambda,TC}(R_{t-1})]_{i,j} = \min(\lambda_i, \lambda_j) [R_{t-1}]_{i,j}. \quad (23)$$

**C. Cubic Spline updating**

Consider the example of Section III-A. We want to construct an updating such that the weight of the cross term in the penalty term (14a) is not totally dominated by the forgetting factor $\lambda_2$. More precisely, we want that the weight of the cross term is also influenced by $\lambda_1$. We consider $Q_\lambda$ as a cubic spline like kernel matrix

$$[Q_\lambda]_{i,j} = \min \left( \frac{l_i^2}{2} \left( l_j - l_i \right), \frac{l_i^2}{2} \left( l_i - \frac{l_j}{3} \right) \right) \quad (24)$$

where $l_1, l_2 > 0, i = 1, 2$, is a function of $i$ to be determined. In our case we want that

$$[Q_\lambda]_{ii} = \frac{l_i^3}{3} \quad (25)$$

is equal to $\lambda_i$ for $i = 1, 2$. Therefore, we obtain

$$l_i = \sqrt[3]{3 \lambda_i}, \quad i = 1, 2. \quad (26)$$

In this way, we built a forgetting map whose cross term is penalized by a blend of $\lambda_1$ and $\lambda_2$.

**Remark 3.1:** One could also consider the matrix $\tilde{Q}_\lambda$ such that $[\tilde{Q}_\lambda]_{i,j} = \sqrt{\lambda_i \lambda_j}, i = 1 \ldots p$. To compare (24) and $Q_\lambda$ assume that $\lambda_1$ is fixed equal to 0.3, whereas $\lambda_2$ can vary over the interval $[0, 1]$. In Figure 1 we depict the functions $f(\lambda_2) = \frac{l_2^2}{2} \left( l_2 - \frac{l_1}{3} \right)$ and $g(\lambda_2) = \sqrt{\lambda_1 \lambda_2}$. As one can see $f(\cdot)$ takes smaller values than the ones of $g(\cdot)$, that is the influence of the smallest forgetting factor is more marked in $f(\cdot)$.

Thus, by plots evidence, (24) provides a blend of $\lambda_1$ and $\lambda_2$ in which the influence of $\lambda_2$ (forgetting factor associated to the parameter with the fastest changing rate) is more marked than the one in $Q_\lambda$.

In the general case, therefore the updating law becomes

$$[F_{\lambda,CS}(R_{t-1})]_{i,j} := [R_{t-1}]_{i,j} \times \min \left( \frac{l_i^2}{2} \left( l_j - l_i \right), \frac{l_i^2}{2} \left( l_i - \frac{l_j}{3} \right) \right) \quad (27)$$

where $l_i = \sqrt[3]{3 \lambda_i}, i = 1 \ldots p$. 

IV. SIMULATIONS RESULTS

In this section we analyse the performance of the RLS with multiple forgetting schemes that we presented in Section III. The experiment has been performed using MATLAB as the numerical platform.

A. Data generation

We consider a discrete-time, time-varying ARX model described in (1), with $n = 2$ and $m = 2$. Here, the parameters in $A_t(z^{-1})$ vary faster than the ones in $B_t(z^{-1})$. To this aim, nine stable polynomials $A_j(z^{-1})$, $j = 1 \ldots 9$, and two stable polynomials $B^{(k)}(z^{-1})$, $k = 1, 2$, have been defined. We considered the time interval $[1, N]$ with $N = 160$. The polynomial $B_t(z^{-1})$ is generated as a smooth time varying convex combination of $B^{(1)}(z^{-1})$ and $B^{(2)}(z^{-1})$. Regarding $A_t(z^{-1})$, we split the interval $[1, N]$ in eight sub-interval and at the $j$-th interval $A_t(z^{-1})$ is generated as a smooth time varying convex combination of $A^{(j)}(z^{-1})$ and $A^{(j+1)}(z^{-1})$.

Finally, the input $u(t)$ is generated as a realization of white Gaussian noise with unit variance and filtered with a $10^{th}$ order Butterworth low-pass filter. Starting from random initial condition, the output $y(t)$ is collected and corrupted by an additive white Gaussian noise with variance $\sigma^2 = 0.01$.

B. Proposed Methods

The method we consider are:

- RARX: this is the classic RARX algorithm implemented in $\text{rarx.m}$ in the MATLAB System identification Toolbox, [10];
- VF: this is the RLS with vector-type forgetting scheme described at the end of Section III-A;
- DI: this is the RLS algorithm with diagonal updating of Section III-A;
- TC: this is the RLS algorithm with tuned/correlated updating of Section III-B;
- CS: this is the RLS algorithm with cubic spline updating of Section III-C.

For each method $m = 2$ and $n = 2$, that is the estimated ARX models have the same order of the true one. Regarding VF, DI, TC and CS we set

$$\lambda = [\lambda_1 \lambda_1 \lambda_2 \lambda_2]^T$$  \hspace{1cm} (28)

that is $\lambda_1$ is the forgetting factor for the parameters in $A_t(z^{-1})$ and $\lambda_2$ is the forgetting factor for the parameters in $B_t(z^{-1})$.

C. Experiment setup

We consider a study of 500 runs. For each run, we generate the data as described in Section IV-A and we compute $\hat{\theta}_t$ with the five methods. More precisely, for each method (VF, DI, TC and CS) we compute $\hat{\theta}_t$ for twenty values of $\lambda_1$ and $\lambda_2$ uniformly sampled over the interval $[0.1, 1]$. Then, we pick $\lambda_1^* \text{ and } \lambda_2^*$ which maximize the one step ahead coefficient of determination (in percentage)

$$\text{COD} = \left(1 - \frac{1}{N} \sum_{t=1}^{N} (y(t) - \hat{y}(t))^2 \right) \times 100$$  \hspace{1cm} (29)

where $\hat{y}(t)$ is the predicted value of $y(t)$ based on the ARX model with $A_{t-1}(z^{-1})$ and $B_{t-1}(z^{-1})$, and $\overline{y}_N$ is the sample mean of the output data. It is worth noting that the performance index COD is used for time invariant models. On the other hand, it provides a rough idea whether the estimated model is good or not and it allows to choose reasonable values for $\lambda_1^*$ and $\lambda_2^*$. Then, for $\lambda_1^*$ and $\lambda_2^*$ we compute the corresponding average track fit (in percentage)

$$\text{ATF} = \left(1 - \frac{1}{N} \sum_{t=1}^{N} \frac{||\hat{\theta}_t - \theta_t||}{||\theta_t||} \right) \times 100.$$  \hspace{1cm} (30)

Regarding RARX, we use the procedure above with one forgetting factor.

D. Results

In Figure 2 are shown the values of $\lambda$. The first boxplot refers to the values chosen by the classic RARX algorithm, from the second to the fifth the values of the forgetting factor $\lambda_1$ referring to the parameters of $A_t(z^{-1})$ are represented, while the last ones refer to the forgetting factor $\lambda_2$ related to the parameters of $B_t(z^{-1})$. Since the parameters of $A_t(z^{-1})$ vary faster than the ones of $B_t(z^{-1})$, its forgetting factors are smaller than the respective others, as expected. On the other hand, the classic RARX has not the possibility to choose different forgetting factors so its best choice is to take an intermediate value among the ones picked by the proposed algorithms.

In Figure 3 are depicted the average track fit indexes. All the proposed algorithms have better performances than RARX and VF, anyway it is possible to highlight that the TC updating shows the best results. This fact suggests that the most efficient weight for the cross terms in the penalty term in (14) is the smallest forgetting factor between the eligible ones, as occurs in the TC algorithm.

Figure 4 illustrates the COD indexes. Once again the proposed algorithms outperforms the classic RARX method: if
we focus on the average value of the boxplots the difference is around 5%. In terms of outliers we can underline that RARX reaches $-100\%$ in the worst case scenario, while the proposed methods never go below $-55\%$.

![Boxplots for different algorithms](image)

Fig. 3. Average track fit of the different algorithms.

V. CONCLUSIONS

We presented a reformulation of the classic RLS algorithm, which can be split into the minimization of the current prediction error and the minimization of a quadratic function which penalizes the distance between the current and the previous value of the estimate. This reformulation is strictly connected to an updating equation which provides the weight matrix of the quadratic function: to change the updating equation given by the classic algorithm means to substitute the map that connects the present weight matrix to the past one. This permits to model multiple forgetting factors to improve the estimation of parameters with different changing rates.

In this paper we provide three different updating laws. Simulations show that these algorithms outperform the conventional ones thanks to the proposed updating law which allows the presence of several forgetting factors. Therefore, multiple forgetting factors seem to be the key to a more efficient identification. It is worth noting that the challenging step is the choice of such forgetting factors. Therefore, the next research direction will concern the estimation of such parameters from the collected data.

APPENDIX

A. Proof of Proposition 3.1

Let $Q_t = \text{diag}(1 \ldots \lambda^{-1})$. Consider
\[ \hat{\theta}_t = \arg\min_{\theta_t} \sum_{i=1}^{t} (y(i) - \varphi(i)^T \theta_t)^2 \lambda^{-i} \]

\[ = \arg\min_{\theta_t} (y(t) - \varphi(t)^T \theta_t)^2 + \lambda \sum_{i=1}^{t-1} (y(i) - \varphi(i)^T \theta_t)^2 \lambda^{-i-1} \]

\[ = \arg\min_{\theta_t} (y(t) - \varphi(t)^T \theta_t)^2 + \lambda \sum_{i=1}^{t-1} (y(i) - \varphi(i)^T \theta_t - \hat{\theta}_{t-1})^2 \lambda^{-i-1} \]

\[ = \arg\min_{\theta_t} (y(t) - \varphi(t)^T \theta_t)^2 + \lambda \sum_{i=1}^{t-1} (\varphi(i)^T (\theta_t - \hat{\theta}_{t-1}))^2 + 2(y(i) - \varphi(i)^T \hat{\theta}_{t-1}) \varphi^T (i)(\theta_t - \hat{\theta}_{t-1}) \lambda^{-i-1}, \]

where the term \((y(i) - \varphi(i)^T \hat{\theta}_{t-1})^2\) has been omitted because it does not depend on \(\theta_t\).

The last equation can be rewritten as

\[ \hat{\theta}_t = \arg\min_{\theta_t} (y(t) - \varphi(t)^T \theta_t)^2 + \lambda \| \theta_t - \hat{\theta}_{t-1} \|_{Q_{t-1}}^2 \]

\[ - 2(\theta_t - \hat{\theta}_{t-1})^T \Phi_t^T Q_{t-1} (y_{t-1} - \Phi_{t-1} \hat{\theta}_{t-1}) \].

(31)

It is not difficult to see that

\[ \hat{\theta}_{t-1} = \arg\min_{\theta_{t-1}} \| y_{t-1} - \Phi_{t-1} \theta_{t-1} \|_{Q_{t-1}}^2. \]

Therefore, it must hold the following equation (by optimality condition)

\[ \Phi_t^T Q_{t-1} (y_{t-1} - \Phi_{t-1} \hat{\theta}_{t-1}) = 0, \]

so (31) becomes

\[ \hat{\theta}_t = \arg\min_{\theta_t} (y(t) - \varphi(t)^T \theta_t)^2 + \lambda \| \theta_t - \hat{\theta}_{t-1} \|_{\Phi_t^T Q_{t-1} \Phi_t}^2. \]

Finally, it is sufficient to observe that \(R_t = \Phi_t^T Q_t \Phi_t\).

\[ \Box \]

B. Proof of Proposition 3.2

The objective function in (14a) is

\[ \mathcal{L}(\theta_t) := (y(t) - \varphi(t)^T \theta_t)^2 + \| \theta_t - \hat{\theta}_{t-1} \|^2_{F_{\lambda}(R_{t-1})}. \]

Therefore, the optimal solution takes the form

\[ \hat{\theta}_t = [\varphi(t) \varphi(t)^T + F_{\lambda}(R_{t-1})]^{-1} \varphi(t) y(t) + F_{\lambda}(R_{t-1}) \hat{\theta}_{t-1}. \]

Finally, (15a), (15b), (15c), (16a) and (16b) can be derived along similar lines used for the classic RLS with forgetting scheme.

\[ \Box \]

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