Two–Component Dust in Spherically Symmetric Motion

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Abstract

Two components of spherically symmetric, inhomogeneous dust penetrating each other are introduced as a generalization of the well–known Tolman–Bondi dust solution. The field equations of this model are formulated and general properties are discussed. Special solutions with additional symmetries — an extra Killing– or homothetic vector — and their matching to the corresponding Tolman–Bondi solution are investigated.

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1 Introduction

The Tolman–Bondi solution (TBS) for spherically symmetric has been found by Lemaitre [1] 1933 and Tolman [2] 1934 as one of the first inhomogeneous cosmological models. Its physical properties has been widely investigated, e.g. Bondi 1947 [3]. It has been used as a model to study the gravitational collapse and the formation of a black hole. In recent years spherically symmetric dust was studied to simulate such phenomena like shell–crossing [4] or voids [5]. The TBS has also been taken as background metric for thin shells of dust to model voids and thin shells [6].

Spherically symmetric dust has been generalized in various directions. Mixtures of dust and radiation [8] or dust and a perfect fluid [9] have been investigated. Coley and Tupper [7] introduced a two–fluid model for general viscous fluids and examined various special cases. In this paper we consider dust shells of finite size moving with different four velocities. So two shells of dust can move towards each other to form a common region where two dust components exist simultaneously. To model voids spherically symmetric dust with different values for the energy densities
has been matched by an intermediate region. Here comoving boundary surfaces have been considered. By introducing two dust components the boundaries need no longer to be comoving. Furthermore one can examine a central core of two dust components with a mass flux through the boundary that can be matched to an exterior one–component dust.  

A solution for the two–component dust (TCD) has to solve a coupled nonlinear system of partial differential equations (PDEs) in two independent variables for two functions. To find an explicit solutions further assumptions as additional symmetries are made. These special solutions still allow us to study different physically and geometrically interesting space times and lead to different topologies.

The paper is organized as follows. In section 2 the Tolman–Bondi solution is given. The model of two–component dust is introduced in section 3. The field equations are formulated in an invariant way and general properties of the metric are discussed. Section four deals with the matching procedure. In the sections 5 and 6 some special solution with an extra Killing– or homothetic vector in the two–component dust region are examined in detail.

2 One component of spherically symmetric dust

For dust, i.e. a perfect fluid without pressure, the Einstein field equations read

\[ R_{ab} - \frac{1}{2} R g_{ab} = T_{ab}, \]

with the energy momentum tensor

\[ T_{ab} = \rho u_a u_b, \quad u_a u^a = -1, \quad u_{a;b} u^b = 0, \]

(2.1)

where \( \rho \) is the energy density and \( u_a \) a time-like geodesic vector field. In a comoving spherically symmetric coordinate system the metric can be written in the form

\[ ds^2 = Y^2(r,t)d\Omega^2 + R^2(r,t)dr^2 - dt^2 \]

(2.2)

where \( d\Omega^2 \) is the usual line element on the unit sphere. Here the solution of the field equations can be given explicitly, see e.g. [10].

For a non–comoving frame of reference, which we will use later for the matching procedure, the line element is best given by

\[ ds^2 = Y^2d\Omega^2 + \frac{1}{Y^2 + Y_a Y^a} \left[ (Y_a + \dot{Y} u_a)dx^a(Y_b + \dot{Y} u_b)dx^b \right] - u_a u_b dx^a dx^b. \]

(2.3)

Here \( Y_a \) is the gradient of the sphere radius \( Y \) and

\[ \dot{Y} = Y_a u^a = L_a Y \]
the derivative of $Y$ along $u_a$. The free functions $M$ and $f$ can be expressed by

\begin{align}
M &= Y(1 - Y_a Y^a), \quad (2.4) \\
f &= -1 + \dot{Y}^2 + Y_a Y^a. \quad (2.5)
\end{align}

Their specific coordinate dependence is defined by

$$
\mathcal{L}_u M = \dot{M} = 0, \quad \mathcal{L}_u f = \dot{f} = 0, \quad (2.6)
$$

as $f$ and $M$ are constant along the fluid’s world lines. With the expressions \((2.4), (2.5)\) and the metric \((2.3)\) the field equations for one–component dust are fulfilled identically.

The sign of the free function $f$, sometimes called the ”energy function”, leads to three different types of solutions. The hyperbolic case, $f > 0$, represents an open unbound model in contrast to the elliptic case, $-1 < f < 0$, which is a bound one. The parabolic case, $f = 0$, is marginally bound. The function $M$ gives twice the mass inside a sphere with radius $Y$ so it is usually called the ”mass function”.

### 3 The model

The energy momentum tensor for two–component dust in radial motion is

$$
T_{ab} = \mu v_a v_b + \epsilon u_a u_b. \quad (3.1)
$$

We assume that both dust components are only coupled by their gravitational attraction and the mass density is conserved for each component, so they have to move on geodesics:

\begin{align}
v_a v^a &= -1 \quad v_{a;b} v^b = 0 \quad \text{and} \quad u_a u^a = -1 \quad u_{a;b} u^b = 0. \quad (3.2)
\end{align}

For this model one can again introduce a mass function $M$ given by \((2.4)\) which was first considered by Lemaitre \cite{1} and rediscovered by Misner \cite{11} for perfect fluids. $M$ represents the active gravitational mass and two field equations can be reformulated with the aid of $M$. The rate of change for the mass function along each fluid line is given by

\begin{align}
M_a v^a &= \epsilon Y^2 Y_a \left( v^a + u_b v^b u_a \right) \quad (3.3) \\
M_a u^a &= \mu Y^2 Y_a \left( u^a + u_b v^b u^a \right). \quad (3.4)
\end{align}

$M$ is only conserved for one component if the other component vanishes. The remaining field equations not containing the energy densities can be formulated in an invariant way, too. The first one requires that the tangential pressure vanishes, $G_{ab} = 0$, and the second one reads

$$
-2Y Y_{a;b} u^a v^b + \frac{M}{Y} u_a v^a = 0 \quad (3.5)
$$
which is a consequence that \( u^a \) and \( v^a \) are both geodesic.

The line element can be written as

\[
ds^2 = Y^2 d\Omega^2 + \frac{1}{A^2 - 1} \left[ dx^a dx^b (u_a u_b + v_a v_b - A(u_a v_b + v_a u_b)) \right], \tag{3.6}
\]

with

\[
u_a = (0, 0, u_3, u_4) \quad v_a = (0, 0, u_3, u_4),
\]

where \(-A = u_a v^a\) is the scalar product of the four velocities (\(|A| > 1\)).

To ensure that both four velocities are geodesic they must be gradients,

\[
u_a = W_{,a} \quad v_a = V_{,a}. \tag{3.7}
\]

The scalars \( V \) and \( W \) are arbitrary functions of \( x^3 \) and \( x^4 \).

For explicit calculations two coordinate systems were quite useful. In the first one both velocities have only one nontrivial covariant component. With

\[
u_a = (0, 0, 0, -1) \quad v_a = (0, 0, 1, 0) \tag{3.8}
\]

we get the line element \[12\]

\[
ds^2 = Y^2 d\Omega^2 + \frac{1}{A^2 - 1} \left( dT^2 + dt^2 + 2AdTdT \right). \tag{3.9}
\]

In the coordinate chart \(3.9\) the energy momentum tensor is diagonal, so the field equations, \( G_{11} = 0 \) and \( G_{34} = 0 \), not involving the energy densities, are given explicitly by a a coupled nonlinear system of PDEs of second order:

\[
(A_t^2 + A_T^2)Y(-2A^2 - 1) + A_t A_T YA(A^2 + 5)
+ (A_t Y_t + A_T Y_T) A(A^2 - 1) + (Y_t A_T + A_t Y_T)(1 - A^2)
+ (A_{tt} + A_{TT}) Y A(A^2 - 1) + A_{tt} Y (1 - A^4)
+ (Y_{tt} + Y_{TT})(-(A^2 - 1)^2) + Y_{TT} 2A(A^2 - 1)^2 = 0 \tag{3.10}
A^2(2Y_t Y_t + 2YY_{TT})
- A(Y_t^2 + Y_T^2 + 2YY_{tt} + 2YY_{TT} + 1) + 2YY_{tt} = 0. \tag{3.11}
\]

The other non vanishing two components of the Einstein tensor determine the energy densities:

\[
G_{33} = \mu, \quad G_{44} = \epsilon. \tag{3.12}
\]

Another line element is of the Tolman–Bondi form, comoving to \( u_a \) \((W = -\dot{t})\):

\[
ds^2 = Y^2 d\Omega^2 + \frac{V_r^2}{V_t^2 - 1} dr^2 - dt^2, \quad u_a = (0, 0, 0, -1) \quad \text{and} \quad v_a = (0, 0, V_r, V_t) \tag{3.13}
\]

The line elements \(3.9\) and \(3.13\) are related by the transformation

\[
T = V(r, t) \quad t = \bar{t}. \tag{3.14}
\]
In general both four velocities have got non zero shear and expansion. But further restrictions can be obtained by using the Raychaudhuri equation (see [10] and references therein) for each velocity and considering the field equations. There are no solutions with positive energy densities where the expansion is constant along the corresponding vector fields $u^a$ or $v^a$ respectively. Additionally, after a lengthy calculation one can show that vanishing shear of both $u_a$ and $v_a$ implies that the space time is flat [12, 13].

It is hopeless to find the general solution of the PDEs (3.10) and (3.11). One way to find explicit solutions is to look for additional symmetries in order to reduce the number of variables. So the specific shape for the free functions and the radius $Y$ of the corresponding TBS is determined by the matching procedure. But general properties and features of quite different models can be examined. Before discussing this in detail in the sections 5 and 6, we will consider the junction conditions between two- and one-component dust regions.

4 Matching procedure

Here we discuss the matching between solutions with the energy momentum tensors

$$T^{(2)} = \mu v_a v_b + \epsilon u_a u_b \quad \text{and} \quad T^{(1)} = \rho u_a u_b$$

across a time like hyper surface $\Sigma$. The matching procedure is based on the Darmois junction conditions outlined in the work of Israel [14] and especially for dust by Bonnor et.al. [15]. They require the continuity of the first and second fundamental form. $\Sigma$ has to be tangent to one of the four velocities because there has to be no mass flux of the second dust component through the boundary surface. As a consequence of the junction conditions the sphere radius $Y$ and the free functions $M$ (2.4) and $f$ (2.5) of the Tolman–Bondi solution "produced" by the mass flux through the boundary are continuous on $\Sigma$. Hence $M$ and $f$ for one-component dust can be calculated completely in the two-component dust metric on $\Sigma$ because $M$ and $f$ are constant along the world lines of the corresponding dust component. Vice versa a TBS for one-component dust gives the initial values for the TCD on $\Sigma$.

The matching of two-component dust to the vacuum is only possible if both velocities are parallel, i.e. in fact only for one-component dust. A continuous matching of the energy momentum tensor on $\Sigma$, that means one of the two energy densities vanishes, is included in the general scheme as a special case, too.
5 Solutions with an additional Killing vector

Special solutions with additional symmetries, i.e. Lie symmetries of the PDEs (3.10) and (3.11), lead to space times which admit an additional Killing vector (or a homothetic vector, see section 6) with only radial–time components. The Killing vector \( \xi^a = (0, 0, \xi^3, \xi^4) \) reflects a translational invariance in the radial–time coordinates which do not appear explicitly in the field equations. \( \xi^a \) can be space like or time like but not null because \( \xi^a \xi_a = 0 \) implies \( \epsilon \mu < 0 \). Before discussing space like and time like Killing vectors separately in the next two subsections, we will examine common properties of the TCD admitting an extra Killing vector.

In the coordinate chart (3.9) the Killing vector \( \xi^a \) is

\[
\xi^a = \left(0, 0, 1, -\frac{1}{k}\right) \quad \xi^a \xi_a = \frac{k^2 + 1 - 2kA}{k^2(A^2 - 1)},
\]

where the metric coefficients depend only on the reduction variable \( z = T + kt \) with a constant \( k \neq 0 \):

\[
A = A(T + kt) \\
Y = Y(T + kt).
\]

The field equations \( G_{11} = 0 \) (3.10) and \( G_{34} = 0 \) (3.11) now read

\[
A^2 2k(YY'' + (Y')^2) + A(-1 - (Y')^2(1 + k^2) - 2YY''(1 + k^2)) + 2kYY'' = 0 \tag{5.3}
\]

\[
Y(A')^2((1 + k^2)(-1 - 2A^2) + kA(5 + A^2)) + AY'(A^2 - 1)(-2k + A + kA) + YA''((A^2 - 1)(A + k^2A) - k(A^4 - 1)) + Y''((A^2 - 1)^2(-1 - k^2 + 2kA)) = 0. \tag{5.4}
\]

They have reduced to an autonomous system of ODEs where the prime denotes differentiation w.r.t. \( z \). The energy densities obtained from (3.12) become

\[
\mu = \frac{E(z)}{(-1 + kA)Y^2(A^2 - 1)} \tag{5.5}
\]

\[
\epsilon = \frac{kE(z)}{(-k + A)Y^2(A^2 - 1)}, \tag{5.6}
\]

when eliminating the second derivatives with the aid of the field equations (5.3) and (5.4). Because of the common factor \( E(z) \) in the two energy densities (5.5) and (5.6) it is not possible that only one vanishes, i.e. the Tolman–Bondi solution cannot admit such a Killing vector except in the vacuum case. As a consequence the geometry and topology of the corresponding TBS differs from the two–dust–component.
When the Killing vector is orthogonal to one of the four velocities, i.e. $k = 0$ or $k = \infty$, one gets one–component dust which belongs to the class of Kantowski–Sachs because the Killing vector is space like [10].

In the coordinate system (3.13), comoving to $u^a$, the line element can be transformed in a form similar to (5.2):

$$ds^2 = F_1^2(\tilde{z})d\Omega^2 + \frac{F_2^2(\tilde{z})}{(kF_2(\tilde{z}) + c_1)^2 - 1}dr^2 - dt^2 \quad \tilde{z} = r + \tilde{k}t$$

(5.7)

with

$$u_a = (0, 0, 0, 1), \quad v_a = (0, 0, F_2(\tilde{z}), kF_2(\tilde{z}) + c_1) \quad \text{and} \quad \xi^a = (0, 0, 1, -\frac{1}{k}).$$

(5.8)

Because of the existence of an additional Killing vector the free functions $M$ and $f$ and the sphere radius $Y$ of the corresponding solution for one dust component are determined completely by the Darmois junction conditions. So one knows the properties of the corresponding TBS by calculating the metric coefficients of the TCD.

### 5.1 Solution with a space like Killing vector

Solutions of two–component dust with a space like Killing vector, i.e a group $G_4$ acting on a $S_3$ (notation see [10]), are generalizations of the Kantowski–Sachs class for one–component dust. Here an Euclidean "center" of symmetry is lacking. A radial coordinate can extend from $-\infty$ to $\infty$. Such models for dust and perfect fluid (where $Y$ depends only on the comoving time) are also called T–models; see Ruban [16] and references therein for a deeper discussion.

Possible singularities in the energy densities might occur for $Y = 0$, which could be interpreted as a "big bang" or gravitational collapse "big crunch". As an approximation of first order for a singularity at $z = 0$ for the reduction variable one gets with the aid of the field equations

$$|A| \approx 1 + \frac{9}{8ck^2}z^{2/3}, \quad Y \approx cz^{2/3} \quad \text{and} \quad \epsilon \sim \frac{1}{z^{4/3}}, \quad \mu \sim \frac{1}{z^{4/3}}.$$

So $|A| = 1$ and $Y = 0$ occur simultaneously, i.e. in the neighbourhood of the big bang singularity both four velocities are nearly parallel. Here the asymptotic is the same as for one dust component with Kantowski–Sachs geometry. The expansion is stopped when the TCD reaches its own Schwarzschild sphere because the derivative $Y'$ vanishes for $Y = M$ when substituting the ansatz (5.2) in the formula (2.4) for the mass function $M$. Then the re–collapse begins and ends in the final big crunch singularity (see Figure 1). The T–models for dust and perfect fluid (especially the "uniform T–models") discussed in [16] are completely analogous; so that solution for itself can be interpreted as a generalization of the uniform T–models for two–component dust and represents a closed universe. To get
a concrete solution for the TCD and the corresponding TBS the ODEs (5.3) and (5.4) have to be solved numerically. The initial values determine the maximal expansion, the relative velocity and the values of the energy densities for each dust component in a nonlinear way.

Because of the range of $Y$ this model can be interpreted as central core consisting of two–component dust related to one–component dust outside due to a mass flux. The mass function $M$ is not a monotonic function (see Figure 2). The decreasing part indicating a mass loss for the TCD region is separated from the increasing part (ingoing mass flux) by the ”turning point” $Y = M$. The ingoing and outgoing mass flux is not balanced because there is a sign change for the ”energy function” $f$ (see Figure 2). So a part of the exterior TBS is connected to an open model while otherwise $f$ lead always to a closed model ($f < 0$). The properties for $f$ and $M$ can be compared to the ”neck–and–two–sheets” topology with the same qualitative features for the TBS discussed by Hellaby [17].

5.2 Solution with a time like Killing vector

For a time like Killing vector the metric is static. There exists no corresponding non–vacuum solution for one–component dust. The type of shell–crossing singularities can occur when $\xi^a$ becomes parallel to $u^a$ or $v^a$. Here the corresponding energy density is diverging but the metric itself and the other energy density remain finite for a non–comoving coordinate system. Because the condition $Y \geq M$ is fulfilled for time like Killing vectors the sphere radius $Y$ has its maximal and minimal values when the shell–crossing singularities occur. To get a quantitative solution the ODEs (5.3) and (5.4) have to be solved numerically. Numerical solutions show that the other energy density also reaches its extremal values at the shell–crossing singularities (see figure 3), e.g.

$$Y_1 < Y < Y_2 \quad \text{and} \quad \mu_1 < \mu < \mu_2.$$ 

The constants $Y_1$, $Y_2$, $\mu_1$ and $\mu_2$ are determined by the initial values.

Because of the finite values of $Y$ this model can be interpreted as a thick shell consisting of two dust components. The shell is ”produced” by two shells of one-component dust where the thickness of the shells is restricted by shell–crossing singularities. First the shells are separated then they crash together and penetrate through each other. So the interior of the ”mixing zone” is static but the boundaries are varying in time. The two inner boundaries of the shells meet each other when the mass function $M$ (figure 4) vanishes, let us say for $Y = \bar{Y}$, because the junction conditions for the flat space has to be fulfilled for a moment. Before this moment the normal vector of the surface is orthogonal to e.g. $u^a$ then orthogonal to $v^a$. Because the mass function is negative a TCD shell cannot be matched for $Y_1 < Y < \bar{Y}$. The shells have to re–separate because the dust components never become comoving. The energy function $f$ for $v_a$ (Figure 4) changes
from positive to negative values. An ever expanding sphere is slowed down due to the shell crossing. That can lead to a bound state and end in an gravitational collapse. The other dust shell belongs to a closed model.

6 Solutions with an additional homothetic vector

A homothetic vector $\eta^a$ represents a self–similar behaviour in the radial–time coordinates of the metric. Eardley [18] examined self–similar space times in general. Cahill and Taub [19] considered spherically symmetric self–similar space times for a perfect fluid, especially they investigated self–similar dust solutions. In recent years self–similar space times were extensively studied, for dust see e.g. [20]. The self–similar dust solution is included here as a particular solution. The main difference to the solutions of the previous section is that $\eta^a \eta^a$ can change sign for the same solution, i.e. $\eta^a$ can be space like or time like in different regions.

In the coordinate system (3.9) the explicit expression for the homothetic vector is

$$\eta^a = (0, 0, T, t), \quad \eta_a \eta^a = \frac{t^2}{G^2 - 1} \left(1 + z^2 + 2zG\right)$$

with the line element

$$ds^2 = t^2 F^2(z) d\Omega^2 + \frac{1}{G^2(z) - 1} \left(dT^2 + dT^2 + 2G(z) dT dt\right) \quad z = \frac{T}{t}.$$ (6.2)

The field equations (3.10) and (3.11) are thus reduced to a non–autonomous system of ODEs

$$GG'F(z + G)(G^2 - 1)$$
$$+ F(G')^2 (-1 - z^2 - 5zG - 2G - 2z^2 G^2 - zG^3)$$
$$+ G'F'(2z + G + z^2 G)(G^2 - 1)$$
$$+ G''F((G + z^2 G)(G^2 - 1) + z(G^4 - 1))$$
$$+ F''(G^2 - 1)^2 (-1 - z^2 - 2zG) = 0$$
$$G + GF^2 - 2zGFF' - 2G^2 F' = 0$$
$$+ G(F')^2 + z^2 G(F')^2 + 2zG^2 (F')^2$$
$$+ 2zFF'' + 2GFF'' + 2z^2 GFF'' + 2zG^2 F' = 0$$ (6.3)

which cannot be solved analytically. The energy densities can be expressed in the form

$$\epsilon = \frac{1}{t^2(G^2(z) - 1)F^2(z)} \epsilon_1(z)$$
$$\mu = \frac{1}{t^2(G^2(z) - 1)F^2(z)} \mu_1(z)$$ (6.5)
for some functions $\mu_1$ and $\epsilon_1$ depending on the reduction variable $z$ alone. The energy densities do not vanish except for the particular TBS. In the comoving coordinate system (3.8) the line element can be written in a form analogous to (3.2).

The qualitative behaviour for the solutions of the field equations (6.3) and (6.4) can be compared to the solutions discussed in the previous section. A big bang singularity with space like $\eta^a$ as well as a shell-crossing singularity with time like $\eta^a$ can occur both. The coordinate dependence of the singularities show the same asymptotic. But in contrast to the space like Killing vector with Kantowski–Sachs geometry a sphere can expand through its own Schwarzschild sphere for the closed model with big bang and big crunch singularity. So the mass function $M$ is a strict monotonic function indicating that there always is a mass loss through the boundary generating the TBS outside. The phenomena with two shells of dust penetrating each other and re-separating is possible, too. Here the homothetic vector is time like for the TCD region.

In addition to the already known models space times admitting a homothetic vector contain another type of solutions, presenting a compact ball of one–component dust inside surrounded by a shell of dust with finite size outside. The dust shell is crashing on the dust ball and ends in a gravitational collapse. Here the space time is restricted by a big bang (big crunch) and a shell–crossing singularity where the maximal radius $Y$ is reached (see Figure 5). The mass function $M$ of the corresponding one–component dust is a positive monotonic function, i.e mass flux in one direction. The energy function $f$ leads to an elliptic (closed) TBS w.r.t. to both $u^a$ or $v^a$.

\section{Conclusions}

As a generalization of the well–known Tolman–Bondi solution for spherically symmetric dust the model of two–component dust in radial motion was introduced. The corresponding field equations were formulated in an invariant way. The connection of the two–component dust to the Tolman–Bondi solution was discussed and the matching procedure outlined. This model can describe different physical models and topologies. Two–component dust with a time like Killing vector can describe the crossing of thick shells of dust where the thickness of the shells is restricted by shell–crossing singularities. A space like Killing vector leads to a central core of two dust components with Kantowski–Sachs geometry and ingoing and outgoing mass flux. The topology of the corresponding Tolman–Bondi solution outside can be compared with the ”neck–and–two–sheets” topology discussed by Hellaby. In the case with the homothetic vector the mass flux can go only in one direction. Additionally big bang and shell–crossing singularities can occur simultaneously leading to the model with a ball of dust surrounded by a dust shell.
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Figure 1: Metric coefficients and energy densities for a solution of two-component dust with a space like Killing vector $\xi^a = (0, 0, 1, -1/2)$ and the initial values $Y(2) = 1$, $Y'(2) = -1$, $A(2) = 2$ and $A'(2) = -1$. In the neighborhood of the big bang (big crunch) singularity both dust components are nearly comoving indicated by $A = 1$. 
Figure 2: The mass function $M$ of the exterior one-component dust is not monotonic because of ingoing and outgoing mass flux. The mass flux is not balanced because the sign change of the energy function $f$ indicates that a part belongs to an open model. The energy function $f$ belonging to the other four velocity is only multiplied by the factor $k$. 
Figure 3: Metric coefficients and energy densities for two–component dust with a time like Killing vector $\xi^a = (0, 0, 1, -2)$ in the coordinate chart \((5.7)\) comoving to $u^a$: with the parameter $c_1 = -2$ and the initial values $F_1(2) = 2, F_1'(2) = 1/5, F_2(2) = -1/2$ and $F_2'(2) = 3/2$. In this coordinate chart the shell–crossing singularity with diverging $\epsilon$ occurs at $R = 0$ where $Y$ and the other energy density $\mu$ has got its extremal valuesa.
Figure 4: Free functions for the corresponding dust component: $M = 0$ belongs to the moment when the inner spheres of the shells meet each other. While $f_u$, denoting the energy function w.r.t. $u_a$, belongs to a closed model $f_v$ changes sign, i.e. an ever expanding dust shell inside is slowed down due to the shell crossing and can re-collapse afterwards.
Figure 5: Metric coefficients and energy densities with the initial values $F(3) = 1$, $F'(3) = 1/4$, $G(3) = -2$ and $G'(3) = -6$: The scaling factor $t$ has been set to unity. A big bang (big crunch) singularity occurs at $Y = 0$. The energy density $\epsilon$ diverges because of a shell–crossing singularity when the homothetic vector becomes parallel to the four velocity $u^a$. 