Polar Codes: Analysis and Construction Based on Polar Spectrum

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Abstract

Polar codes are the first class of constructive channel codes achieving the symmetric capacity of the binary-input discrete memoryless channels. But the analysis and construction of polar codes involve the complex iterative-calculation. In this paper, by revisiting the error event of the polarized channel, a new concept, named polar spectrum, is introduced from the weight distribution of polar codes. Thus we establish a systematic framework in term of the polar spectrum to analyze and construct polar codes. By using polar spectrum, we derive the union bound and the union-Bhattacharyya (UB) bound of the error probability of polar codes and the upper/lower bound of the symmetric capacity of the polarized channel. The analysis based on the polar spectrum can intuitively interpret the performance of polar codes under successive cancellation (SC) decoding. Furthermore, we analyze the coding structure of polar codes and design an enumeration algorithm based on the MacWilliams identities to efficiently calculate the polar spectrum. In the end, two construction metrics named UB bound weight (UBW) and simplified UB bound weight (SUBW) respectively, are designed based on the UB bound and the polar spectrum. Not only are these two constructions simple and universal for the practical polar coding, but they can also generate polar codes with similar (in SC decoding) or superior (in SC list decoding) performance over those based on the traditional methods.

Index Terms

Polar codes, Subcode, Polar subcode, Polar spectrum, Polar weight distribution, Union bound, Union-Bhattacharyya (UB) bound.

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I. INTRODUCTION

A. Relative Research

As the first constructive capacity-achieving coding scheme, polar codes, invented by Arikan in 2009 are a great breakthrough [1] in channel coding theory. Since polar codes demonstrate advantages in error performance and other attractive application prospects, after nine years of research effort, polar codes became a coding standard for the control channel in the fifth generation (5G) wireless communication system [24] in 2018. As the most significant concept, channel polarization was introduced to interpret the behaviour of polar coding. The Bhattacharyya parameter combined with the mutual information chain rule are used to evaluate the reliability of the polarized channel and to analyze the convergence behavior of polar codes. Consequently, this theoretical framework based on the channel polarization plays a key role to design and optimize the polar codes.

According to the dependence of the original channel information, the construction method of polar codes usually falls into two categories. In the first category, by using some channel parameters of the binary-input discrete memoryless channel (B-DMC), such as, erasure probability in binary erasure channel (BEC) or signal-to-noise ratio (SNR) in binary-input additive white Gaussian noise (BI-AWGN) channel, the reliability of the polarized channel can be iteratively calculated or evaluated based on the recursive structure of polar coding. In what follows, this category is known as the channel-dependent construction. In the second category, the reliability of the polarized channel can be ordered based on some channel-independent characteristics of polar codes, which is named as the universal construction. Due to the independence of channel condition, the universal construction is more desired for the practical application.

For the channel-dependent construction, Arikan first proposed a recursive calculation of Bhattacharyya parameter [1] to evaluate the reliability of the polarized channel, whereas this method is only precise for the coding construction in a BEC. For other B-DMCs, such as binary symmetry channel (BSC) or BI-AWGN channel, exact calculation of Bhattacharyya parameter will involve the high-complexity Monte-carlo integration while approximate iterative calculation will result in some performance losses. Subsequently, Mori et al. designed the density evolution (DE) algorithm to track the distribution of logarithmic likelihood ratio (LLR) and calculate the error probability under the successive cancellation (SC) decoding [4]. However, high-precision DE
algorithm is a time-consuming process. In [5], Tal and Vardy proposed an iterative algorithm to evaluate the upper/lower bound of the error probability of the polarized channel, which can achieve preferable accuracy with the medium computational complexity. In [6], Trifonov advocated the use of Gaussian approximation (GA) to estimate the error probability in AWGN channel thereby obtain good accuracy with a low complexity. Lately, in order to further increase the accuracy of GA construction, we devised an improved GA algorithm [7] for the polar codes with the long code length. Generally, all these algorithms belong to the channel-dependent construction since the iterative calculation relies on some parameters of the original channel and the large variation of channel condition may affect the reliability order of the polarized channel.

 Commonly, from the viewpoint of system design, the coding construction should be universal and independent of channel condition so as to facilitate the implementation of the encoder and decoder. Therefore, channel-dependent construction is not convenient for the practical application of polar coding, on the contrary, the universal construction is a more desirable selection. Schürch et al. [8] found that the reliability order of a part of polarized channels is invariant thereby introduced the concept of partial order (PO). It is an efficient tool to reduce the complexity of polar coding construction. Nevertheless, we still need to calculate the reliabilities of the rest polarized channels. Furthermore, He et al. [9] exploited the index feature of polarized channels and proposed a channel-independent construction, named polarized weight (PW) algorithm. Although PW is an empirical construction, amazingly, polar codes constructed by PW can achieve almost the same performance as those constructed by GA algorithm. In fact, the polar codes in 5G standard [24] is constructed by using a fixed polar reliability sequence, which is universal for all the code configuration and obtained by computer searching [10].

B. Motivation

Traditionally, good channel codes, such as Turbo/LDPC code, can be evaluated and optimized based on the distance spectrum or weight distribution [22]. However, the high complexity involved in the weight distribution calculation of polar codes [11], [12] makes it unrealistic to use such metrics for the polar code design. Although there are plenty of research results on the weight distribution in the classic coding theory, it is lack of the theoretic interpretation based on distance spectrum/weight distribution for the channel polarization. Obviously, there is a fault between the research of the classic channel coding and that of polar coding. Thus, in this paper,
we focus on the theoretic framework based on distance spectrum to thoroughly interpret the behavior of polar codes and whereby deduce some new constructions with both the analyticity and the universality.

C. Main Contributions

In this paper, we introduce a new concept on the distance spectrum of polar codes, named polar spectrum, and establish a complete framework to analyze the performance of the polar codes under the SC decoding. Based on this framework, we obtain new universal constructions for the polar coding. The main contributions of this paper can be summarized as follows.

1) First, the theoretical framework based on the polar spectrum is established. We revisit the error event analysis of polarized channel and find that one polarized channel is associated with a subcode and its codeword subset, namely polar subcode. Consequently, we set up a one-to-one mapping between the polarized channel and the polar subcode and introduce the weight distribution of the polar subcode, named polar spectrum, which is a set of the weight enumerator with the given Hamming weight. Compared with the Bhattacharyya parameter or other performance metrics needed iterative calculation, the union bound of the error probability of the polarized channel in term of polar spectrum has an intuitive interpretation, that is, the Hamming weight affects the pairwise error probability and the weight enumerator determines the number of the corresponding error events.

By the polar spectrum, we first build the relevance between the error probability of polarized channel and the weight distribution of polar code, which is a simple analytical-metric rather than a complex iterative-calculation. Furthermore, based on the polar spectrum, the union bound and the union-Bhattacharyya (UB) bound of the block error rate (BLER) under the SC decoding is derived. Meanwhile, the upper/lower bound of the mutual information of the polarized channel is also derived by using the polar spectrum. Therefore, the framework based on polar spectrum can provide the same performance analysis as the recursive calculation based on Bhattacharyya parameter.

2) Second, thanks to the good structure of (polar) subcodes, we design an iterative enumeration algorithm for the polar spectrum. We find that a pair of specific subcodes can constitute the mutual-dual codes. By using the well-known MacWilliams identities [20], the weight distribution of these two subcodes can be easily calculated. Moreover, we prove that the
polar weight enumerator of polar subcode and the weight enumerator of subcode satisfy the recursive accumulation relation. On this basis, an iterative algorithm embedded the solution of MacWilliams identities is proposed to enumerate the polar spectrum of polar codes. Compared with the traditional searching algorithm of distance spectrum, such enumeration of polar spectrum is a low complexity and high efficiency algorithm, since its one running can generate all the polar spectra for arbitrary code configuration with a fixed code length.

3) Third, two universal and analytical construction metrics, named union-Bhattacharyya weight (UBW) and simplified UB weight (SUBW), are proposed. They are the logarithmic version of the UB bound and the latter, SUBW, only considers the minimum weight term of the UB bound. Since the polar spectrum can be calculated off-line, these two constructions have a linear complexity far below those constructions based on the iterative calculation. On the other hand, they can also be modified as universal constructions for the practical polar coding. Simulation results show that the polar codes constructed by these two metrics can achieve similar performance of those constructed by GA or PW algorithm under SC decoding. Dramatically, the former can outperform the latter under successive cancellation list (SCL) decoding.

The remainder of the paper is organized as follows. Section II presents the preliminaries of polar codes, covering polar coding and decoding, error performance analysis and polar code construction. Section III investigates the error event of polarized channel and introduces the concepts of subcode, polar subcode and polar spectrum. Then, by using polar spectrum, the union bound and union-Bhattacharyya bound of the error probability of the polarized channel are derived and analyzed. In addition, the mutual information of the polarized channel is also analyzed based on the polar spectrum. Furthermore, we explore the subcode duality of polar codes and design an iterative enumeration algorithm to calculate the polar spectrum in Section IV. Numerical analysis for the union bound and UB bound in term of polar spectrum and simulation results for comparing UBW/SUBW construction with the traditional constructions are presented in Section V. Finally, Section VI concludes the paper.
II. PRELIMINARY OF POLAR CODES

A. Notation Conventions

In this paper, calligraphy letters, such as $\mathcal{X}$ and $\mathcal{Y}$, are mainly used to denote sets, and the cardinality of $\mathcal{X}$ is defined as $|\mathcal{X}|$. The Cartesian product of $\mathcal{X}$ and $\mathcal{Y}$ is written as $\mathcal{X} \times \mathcal{Y}$ and $\mathcal{X}^n$ denotes the $n$-th Cartesian power of $\mathcal{X}$. Especially, the hollow symbol, e.g. $\mathbb{C}$, denotes the codeword set of code or subcode. Let $[a, b]$ denote the continuous integer set $\{a, a + 1, \cdots, b\}$.

We write $v_1^N$ to denote an $N$-dimensional vector $(v_1, v_2, \cdots, v_N)$ and $v_i^j$ to denote a subvector $(v_i, v_{i+1}, \cdots, v_j)$ of $v_1^N$, $1 \leq i, j \leq N$. Occasionally, we use the boldface lowercase letter, e.g. $u$, to denote a vector. Further, given an index set $A \subseteq [1, N]$ and its complement set $A^c$, we write $v_A$ and $v_{A^c}$ to denote two complementary subvectors of $v_1^N$, which consist of $v_i$s with $i \in A$ or $i \in A^c$ respectively. Then $1_A$ denotes the indicator function of a set $A$, that is, $1_A(x)$ equals 1 if $x \in A$ and 0 otherwise.

We use the boldface capital letter, such as $F_N$, to denote a matrix with dimension $N$. So the notation $F_N(a : N)$ indicates the submatrix consisting the rows from $a$ to $N$ of the matrix $F_N$.

We use $d_H(u, v)$ to denote the Hamming distance between the binary vector $u$ and $v$. Similarly, $w_H(u)$ denotes the Hamming weight of the binary vector $u$. Given $\forall a, b \in \mathbb{R}^N$, let $\|a - b\|$ denote the Euclidian distance between the vector $a$ and $b$.

Throughout this paper, $\log (\cdot)$ means “logarithm to base 2,” and $\ln (\cdot)$ stands for the natural logarithm. $(\cdot)^T$ means the transpose operation of the vector. $(a \cdot b) = ab^T$ is the inner product of two vectors $a$ and $b$. $\text{supp}(u) = \{i : u_i \neq 0\}$ is the set of indices of non-zero elements of a vector $u$. Let $\lceil x \rceil$ denote the ceiling function, that is, the least integer greater than or equal to $x$.

B. Encoding and Decoding of Polar Codes

Given a B-DMC $W : \mathcal{X} \to \mathcal{Y}$ with input alphabet $\mathcal{X} = \{0, 1\}$ and output alphabet $\mathcal{Y}$, the channel transition probabilities can be defined as $W(y|x)$, $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Then the symmetric capacity and the Bhattacharyya parameter of the B-DMC $W$ can be defined as

$$I(W) \triangleq \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \frac{1}{2} W(y|x) \log \frac{W(y|x)}{\frac{1}{2}W(y|0) + \frac{1}{2}W(y|1)}$$

(1)
and

\[ Z(W) \triangleq \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)} \tag{2} \]

respectively.

Applying channel polarization transform for \( N = 2^n \) independent uses of B-DMC \( W \), after channel combining and splitting operation \([1]\), we obtain a group of polarized channels \( W^{(i)}_N : \mathcal{X} \rightarrow \mathcal{Y} \times \mathcal{X}^{i-1}, i \in [1, N] \). By using the channel polarization, the polar coding can be described as follows.

Given the code length \( N \), the information length \( K \) and code rate \( R = K/N \), the indices set of polarized channels can be divided into two subsets: one set \( A \), named information set, to carry information bits and the other complement set \( A^c \) to assign the fixed binary sequence, named frozen bits. A message block of \( K = |A| \) bits is transmitted over the \( K \) most reliable channels \( W^{(i)}_N \) with indices \( i \in A \) and the others are used to transmit the frozen bits. So a binary source block \( u_N \) consisting of \( K \) information bits and \( N - K \) frozen bits can be encoded into a codeword \( x_1^N \) by

\[ x_1^N = u_1^N F_N, \tag{3} \]

where the matrix \( F_N \) is the \( N \)-dimension generator matrix. \([1]\) This matrix can be recursively defined as \( F_N = F_2 \otimes_n \), where “\( \otimes_n \)” denotes the \( n \)-th Kronecker product and \( F_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \) is the \( 2 \times 2 \) kernel matrix.

Such polar coding can deduce the product-form channel \( W^N \) and the synthetic channel \( W_N \). The transition probabilities of these two channels satisfy

\[ W_N \left( y_1^N \mid u_1^N \right) = W^N \left( y_1^N \mid u_1^N F_N \right) = \prod_{i=1}^N W \left( y_i \mid x_i \right). \]

So the transition probability of the \( i \)-th polarized channel \( W^{(i)}_N \) is defined as

\[ W^{(i)}_N \left( y_1^N, u_1^{i-1} \mid u_i \right) \triangleq \sum_{u_{i+1}^N \in \mathcal{X}^{N-i}} \frac{1}{2^{N-1}} W_N \left( y_1^N \mid u_1^N \right). \tag{4} \]

The Bhattacharyya parameter of \( W^{(i)}_N \) is expressed as

\[ Z \left( W^{(i)}_N \right) = \sum_{y_i^N \in \mathcal{Y}^N} \sum_{u_1^{i-1} \in \mathcal{X}^{i-1}} \sqrt{W^{(i)}_N \left( y_1^N, u_1^{i-1} \mid 0 \right) W^{(i)}_N \left( y_1^N, u_1^{i-1} \mid 1 \right)}. \tag{5} \]

1In the seminal paper \([1]\), the generator matrix is composed of the matrix \( F_N \) and the bit-reversal matrix. Since the bit-reversal operation does not affect the reliability of the polarized channel, in this paper, we will use the matrix \( F_N \) as the generator matrix. This is also the polar coding form in 5G standard \([24]\).
As proposed in [1], polar codes can be decoded by the SC decoding algorithm with a low complexity $O(N \log N)$. Furthermore, many improved SC decoding algorithms, such as successive cancellation list (SCL) [13], successive cancellation stack (SCS) [17], successive cancellation hybrid (SCH) [18], successive cancellation priority (SCP) [19], and CRC aided (CA)-SCL/SCS [13–16] decoding can be applied to improve the performance of polar codes.

C. Error Performance of Polar Codes

Based on the joint probabilities $P \left( \left\{ \left( u_1^N, y_1^N \right) \right\} \right) = 2^{-N} W_N \left( y_1^N \middle| u_1^N \right)$, we introduce the probability space $\left( \mathcal{X}^N \times \mathcal{Y}^N, P \right)$, where all $\left( u_1^N, y_1^N \right) \in \mathcal{X}^N \times \mathcal{Y}^N$. On this probability space, the random vectors $U_1^N$, $X_1^N$, $Y_1^N$, and $\hat{U}_1^N$ represent the input to the channel $W_N$, the input to the channel $W_N$, the output of $W_N$ or $W_N$, and the decisions by the decoder. Given a sample point $\left( u_1^N, y_1^N \right) \in \mathcal{X}^N \times \mathcal{Y}^N$, we have $U_1^N \left( u_1^N, y_1^N \right) = u_1^N$, $X_1^N \left( u_1^N, y_1^N \right) = u_1^N F_N$, $Y_1^N \left( u_1^N, y_1^N \right) = y_1^N$, and $\hat{U}_1^N \left( u_1^N, y_1^N \right)$ is recursively determined by using SC decoding.

Given the fixed configuration $(N, K, A)$, the block error event under SC decoding is defined as

$$E = \left\{ \left( u_1^N, y_1^N \right) \in \mathcal{X}^N \times \mathcal{Y}^N: \hat{U}_A \left( u_1^N, y_1^N \right) \neq u_A \right\}.$$  \hspace{1cm} (6)

So the block error rate can be expressed in this probability space as $P_e(N, K, A) = P(E)$. As the derivation in [1], the set of block error event can be enlarged as $E \subset \bigcup_{i \in A} E_i$, where the single-bit error event $E_i$ is defined as

$$E_i = \left\{ \left( u_1^N, y_1^N \right) \in \mathcal{X}^N \times \mathcal{Y}^N: W_N^{(i)} \left( y_1^N, u_1^i \right) \leq W_N^{(i)} \left( y_1^N, u_1^i \oplus 1 \right) \right\}. \hspace{1cm} \text{(7)}$$

Thus, the error probability of single-bit error event can be bounded as $P(E_i) \leq Z \left( W_N^{(i)} \right)$. Further, the BLER under SC decoding can be upper bounded as

$$P_e(N, K, A) = P(E) \leq \sum_{i \in A} P(E_i) \leq \sum_{i \in A} Z \left( W_N^{(i)} \right). \hspace{1cm} \text{(8)}$$

For the single-bit error event, we have the following proposition.

**Proposition 1:** Suppose the $i$-th bit is zero, that is, $u_i = 0$, then the error event $E_i$ can be equivalently written as

$$E_i = \left\{ \left( u_{i+1}^N, y_1^N \right) \in \mathcal{X}^{N-i} \times \mathcal{Y}^N: W_N^{(i)} \left( y_1^N, 0_i^{i-1} \right) \leq W_N^{(i)} \left( y_1^N, 0_i^{i-1} \right) \right\}. \hspace{1cm} \text{(9)}$$
By using the symmetry of the polarized channel $W^{(i)}_N$, this proposition can be easily concluded and the proof is omitted.

**D. Polar Code Construction**

For the construction of polar codes, the calculation of channel reliabilities and selection of good channels are the critical steps. Generally, the construction algorithms can be divided into two categories, namely, the channel-dependent construction and the universal construction.

For the former, Arikan initially proposed the construction based on the Bhattacharyya parameter [1], whereas this method is only precise for the BEC and approximate for other channels. Lately, density evolution (DE) algorithm [4] and Tal-Vardy algorithm [5] were proposed to perform high-precise construction of polar codes. However, these algorithms have a slightly high complexity. The Gaussian approximation (GA) algorithm [6] is a desirable method to construct the polar codes with a medium complexity $O(N \log N)$, especially for the AWGN channel. Further, an improved GA algorithm [7] was designed for the long code length. Nevertheless, these channel-dependent constructions are not convenient for the practical application.

On the contrary, since the universal construction is independent with the channel condition, it is more desirable for the practical design of polar code. For an example, polarized weight (PW) construction [9] is a typical method in this category. Furthermore, due to the usage of constructive property of the generator matrix, the construction based on partial order [8] is also a good method. However, these methods are still heuristic and do not fully explore the algebraic construction of polar code.

In this paper, we will re-establish the analysis framework of the error performance by investigating the distance spectrum of polar codes so as to obtain analytical construction metrics.

**III. PERFORMANCE ANALYSIS BASED ON POLAR SPECTRUM**

In this section, we will introduce a new analysis tool, namely polar spectrum, to analyze the error performance of polar codes under the SC decoding. First we investigate the structure of single-bit error event and introduce the concepts of (polar) subcode and polar spectrum. Then we derive the new upper bounds of BLER based on polar spectrum. Further, as an enlarged version of BLER bound, the union-Bhattacharyya bound is also derived. Finally, the upper/lower bound of the mutual information of the polarized channel is analyzed by using this new tool.
A. Error Event Probability Analysis

Recall that the single-bit error event $E_i$ is associated with the $i$-th polarized channel, this error event can be further decomposed into a lot of codeword error events.

**Theorem 1:** Given $\forall u_{i+1}^N, v_{i+1}^N \in X^{N-i}$ and suppose the transmission bit is zero, that is, $u_i = 0$, then the single-bit error event $E_i$ can be enlarged as

$$E_i \subset \bigcup_{u_{i+1}^N, v_{i+1}^N \in X^{N-i}} \left\{ \left( u_{i+1}^N, v_{i+1}^N, y_1^N \right) : W_N \left( y_1^N \left| 0_{i-1}^1, 0, u_{i+1}^N \right. \right) \leq W_N \left( y_1^N \left| 0_{i-1}^1, 1, v_{i+1}^N \right. \right) \right\} \quad (10)$$

**Proof:** From proposition [1], the single-bit error event $E_i$ can be written as

$$E_i = \left\{ \left( u_{i+1}^N, y_1^N \right) : W_N^{(i)} \left( y_1^N, 0_{i-1}^1 | 0 \right) \leq W_N^{(i)} \left( y_1^N, 0_{i-1}^1 | 1 \right) \right\}.$$ Extending the transition probabilities of polarized channel $W_N^{(i)}$ with Equ. [4], we have

$$E_i = \left\{ \left( u_{i+1}^N, v_{i+1}^N, y_1^N \right) : \sum_{u_{i+1}^N, \in X^{N-i}} W_N \left( y_1^N \left| 0_{i-1}^1, 0, u_{i+1}^N \right. \right) \leq \sum_{v_{i+1}^N, \in X^{N-i}} W_N \left( y_1^N \left| 0_{i-1}^1, 1, v_{i+1}^N \right. \right) \right\}. \quad \text{Based on the property of the union set, we obtain the conclusion.}$$

Concerning the derivation of Theorem [1] we find that the error event $E_i$ is associated with the codewords transmitted over the polarized channel $W_N^{(i)}$. Thus we introduce the following definitions.

**Definition 1:** Given the code length $N$, the $i$-th subcode $C_N^{(i)}$ is defined as a set of codewords, that is,

$$C_N^{(i)} \triangleq \left\{ c : c = \left( 0_{i-1}^1, u_{i+1}^N \right) F_N, \forall u_{i+1}^N \in X^{N-i+1} \right\}. \quad (11)$$

Furthermore, one subset of the subcode $C_N^{(i)}$, namely the polar subcode $D_N^{(i)}$, can be defined as

$$D_N^{(i)} \triangleq \left\{ c^{(1)} : c^{(1)} = \left( 0_{i}^{(i-1)}, 1, u_{i+1}^N \right) F_N, \forall u_{i+1}^N \in X^{N-i} \right\}. \quad (12)$$

The complement set of the polar subcode is defined as $E_N^{(i)} = C_N^{(i)} - D_N^{(i)} \triangleq \left\{ c^{(0)} \right\}$.

Obviously, subcode $C_N^{(i)}$ is a linear block code $(N, N-i+1)$ and we have $|C_N^{(i)}| = 2^{N-i+1}$ and $|D_N^{(i)}| = 2^{N-i}$. Based on Definition [1] we can further define the pairwise-codeword error event as follows.

**Definition 2:** Given the codewords $c^{(0)} \in E_N^{(i)}$ and $c^{(1)} \in D_N^{(i)}$, the pairwise-codeword error event is defined as

$$D_i (c^{(0)} \rightarrow c^{(1)}) \triangleq \left\{ \left( c^{(0)}, c^{(1)}, y_1^N \right) : W_N \left( y_1^N \left| c^{(0)} \right. \right) \leq W_N \left( y_1^N \left| c^{(1)} \right. \right) \right\}. \quad (13)$$
For the error probability of the event $\mathcal{D}_i \left( c^{(0)} \rightarrow c^{(1)} \right)$, note that
\[
P \left( \mathcal{D}_i \left( c^{(0)} \rightarrow c^{(1)} \right) \right) = \sum_{y_i^N} \frac{1}{2^{N-i}} W^N \left( y_i^N \mid c^{(0)} \right) 1_{\mathcal{D}_i \left( c^{(0)} \rightarrow c^{(1)} \right)} \left( c^{(0)}, c^{(1)}, y_i^N \right) = \frac{1}{2^{N-i}} P_N^{(i)} \left( c^{(0)} \rightarrow c^{(1)} \right),
\]
(14)
where
\[
P_N^{(i)} \left( c^{(0)} \rightarrow c^{(1)} \right) = \sum_{y_i^N} W^N \left( y_i^N \mid c^{(0)} \right) 1_{\mathcal{D}_i \left( c^{(0)} \rightarrow c^{(1)} \right)} \left( c^{(0)}, c^{(1)}, y_i^N \right)
\]
is named as the pairwise error probability (PEP). So we can derive the upper bound of the error probability of the polarized channel $W_N^{(i)}$ as follows.

**Theorem 2:** The error probability of $W_N^{(i)}$ is upper bounded by
\[
P \left( W_N^{(i)} \right) \leq \frac{1}{2^{N-i}} \sum_{c^{(0)}} \sum_{c^{(1)}} P_N^{(i)} \left( c^{(0)} \rightarrow c^{(1)} \right).
\]
(16)

**Proof:** According to Theorem 1 and Definition 2 we have
\[
P \left( W_N^{(i)} \right) = P \left( \mathcal{E}_i \right) \leq P \left( \bigcup_{c^{(0)}, c^{(1)}} \mathcal{D}_i \left( c^{(0)} \rightarrow c^{(1)} \right) \right) = \sum_{c^{(0)}} \sum_{c^{(1)}} P \left( \mathcal{D}_i \left( c^{(0)} \rightarrow c^{(1)} \right) \right).
\]
(17)
Substituting Eq. (14) into Eq. (17), we complete the proof.

From the above analysis, we establish the 1-1 mapping among the single-bit error event, the polarized channel, and the subcode, that is, $\{ \mathcal{E}_i \leftrightarrow W_N^{(i)} \leftrightarrow \mathcal{C}_N^{(i)} \}$. Next we will further analyze the block error rate based on the distance spectrum of polar subcode.

**B. Block Error Rate Bound Based on Polar Spectrum**

Let us investigate the Hamming distance between the codeword of $\mathcal{D}_N^{(i)}$ and that of $\mathcal{E}_N^{(i)}$.

**Proposition 2:** Given $c^{(1)} \in \mathcal{D}_N^{(i)}$ and $c^{(0)} \in \mathcal{E}_N^{(i)}$, the Hamming distance between these two codewords satisfies $d_H \left( c^{(1)}, c^{(0)} \right) = w_H \left( c^{(1)} \oplus c^{(0)} \right)$ and the module-2 sum of these two codewords is belong to the given polar subcode, that is, $c^{(1)} \oplus c^{(0)} \in \mathcal{D}_N^{(i)}$.

**Proof:** By Definition 1 we have $d_H \left( c^{(1)}, c^{(0)} \right) = w_H \left( \left( 0_{1}^{(i-1)}, 1, u_{i+1}^{N} \oplus v_{i+1}^{N} \right) F_N \right) = w_H \left( \left( 0_{1}^{(i-1)}, 1, u_{i+1}^{N} \oplus v_{i+1}^{N} \right) F_N \right)$. Obviously, $c^{(1)} \oplus c^{(0)} \in \mathcal{D}_N^{(i)}$.
From Proposition 2, we can simplify the PEP as below,

\[ P_N^{(i)}(c^{(0)} \rightarrow c^{(1)}) = P\left(W_N^i y_i^N | c^{(0)}\right) \leq W_N^i(y_i^N | c^{(1)}) \]

\[ = P\left(W_N^i y_i^N | 0_i^N \right) \leq W_N^i(y_i^N | c^{(0)} \oplus c^{(1)}) \]

\[ = P_N^{(i)}(d_H(c^{(0)}, c^{(1)})) \]  

(18)

Without loss of generality, hereafter we designate \(c^{(0)} = 0_i^N\). It follows that PEP \( P_N^{(i)}(c^{(0)} \rightarrow c^{(1)})\) is determined by the codeword weight of polar subcode \(D_N^{(i)}\).

**Definition 3:** The polar spectrum of the polar subcode \(D_N^{(i)}\), also named as polar weight distribution, is defined as the weight distribution set \(\{A_N^{(i)}(d)\} \), \(d \in [1, N]\), where \(d\) is the Hamming weight of non-zero codeword and the polar weight enumerator \(A_N^{(i)}(d)\) enumerates the codewords of weight \(d\) for codebook \(D_N^{(i)}\).

**Proposition 3:** The error probability of \(W_N^{(i)}\) is further upper bounded by

\[ P\left(W_N^{(i)}\right) \leq \sum_{d=1}^{N} A_N^{(i)}(d) P_N^{(i)}(d). \]  

(19)

**Proof:** Substituting Equ. (18) into Theorem 2, using Definition 3 and noting that \(|E_N^{(i)}| = 2^{N-i}\), we complete the proof.

Essentially, this bound is the union bound over the polar subcode. So we further establish the 1-1 mapping among the single-bit error event, the polarized channel, and the polar subcode, that is, \(\{E_i \leftrightarrow W_N^{(i)} \leftrightarrow D_N^{(i)}\}\). Let \(d_{min}^{(i)}\) denote the minimum Hamming distance of polar subcode \(D_N^{(i)}\). Thus, by using the PEP formula (19), we obtain a union bound of BLER under SC decoding.

**Theorem 3:** Given the fixed configuration \((N, K, A)\), the block error probability of polar code is upper bounded by

\[ P_e(N, K, A) \leq \sum_{i \in A} \sum_{d=1}^{N} A_N^{(i)}(d) P_N^{(i)}(d). \]  

(20)

Compared with the upper bound (8) proposed by Arıkan, this union bound has an analytical form, which is mainly determined by the polar spectrum of the selected polar subcodes, that is, the polar weight enumerators and the PEPs dominated by Hamming weights. So this upper bound can reveal more constructive features of polar codes than the traditional bounds. More tighter upper bounds, such as tangential bound or tangential-sphere bound (See [21] and references therein), can also be used to evaluate the error performance of polar codes under SC decoding.
Nevertheless, these improved upper bounds involve complex calculation and are not convenient for the practical application of polar coding. Therefore, in this paper, we focus on the simple upper bounds, such as union bound and union-Bhattacharyya bound. Next, we will further discuss the PEP under various B-DMC channels, such as BEC, BSC and BI-AWGN channel.

1) **PEP and BLER in the BEC:** Given the BEC \( W : \mathcal{X} \to \mathcal{Y}, \mathcal{X} = \{0, 1\} \) and \( \mathcal{Y} = \{0, e, 1\} \) (Here, \( e \) denotes an erasure), with the transition probabilities \( W(y|x) \) and the erasure probability \( \epsilon \), we have \( W(e|0) = W(e|1) = \epsilon \).

Let \( F = \{i : y_i = e\} \) denote the set of erasure-occurred indices and assume \( |F| = l \).

Assuming \( c^{(0)} = 0^N_1 \) and \( d = d_H(c^{(0)}, c^{(1)}) \), we conclude that only the case of the set of erasure-occurred indices covering the support set of \( c^{(0)} + c^{(1)} \) may result in an error, that is, \( l > d \) and \( \text{supp} \{c^{(0)} + c^{(1)}\} \subset F \). Hence, the PEP in the BEC can be expressed as

\[
P_{\text{BEC}}(c^{(0)} \to c^{(1)}) = \sum_{l=d}^{N} \binom{N}{l} (1 - \epsilon)^{N-l} \epsilon^d = \epsilon^d.
\]

Furthermore, by Proposition 3, the union bound of \( W_N^{(i)} \) is derived as

\[
P_{\text{BEC}}(W_N^{(i)}) \leq \sum_{d=d_{\text{min}}^{(i)}}^{N} A_N^{(i)}(d) \epsilon^d,
\]

where \( d_{\text{min}}^{(i)} \) is the minimum Hamming distance of polar subcode \( D_N^{(i)} \).

Correspondingly, by Theorem 3, the upper bound of BLER using SC decoding in the BEC can be written as

\[
P_{e,\text{BEC}}(N, K, A) \leq \sum_{i \in A} \sum_{d=d_{\text{min}}^{(i)}}^{N} A_N^{(i)}(d) \epsilon^d.
\]

2) **PEP and BLER in the BSC:** Suppose the BSC \( W : \mathcal{X} \to \mathcal{Y}, \mathcal{X} = \{0, 1\} \) and \( \mathcal{Y} = \{0, 1\} \), with the transition probabilities \( W(y|x) \) and the crossover error probability \( \delta \), we have \( W(1|0) = W(0|1) = \delta \).

Given the transmission codeword \( c^{(0)} = 0^N_1 \), the received vector can be written as \( y_N = c^{(0)} + e^N_1 \), where \( e^N_1 \) is the error vector. So the PEP is derived as below.

**Theorem 4:** Assuming the decision vector is \( c^{(1)}, d = d_H(c^{(0)}, c^{(1)}) \) and \( l = w_H(e^N_1) \), the PEP in the BSC can be expressed as

\[
P_{\text{BSC}}(c^{(0)} \to c^{(1)}) = \sum_{l=[d/2]}^{N} \sum_{m=[d/2]}^{d} \binom{N-d}{l-m} \binom{d}{m} \delta^l (1 - \delta)^{N-l}.
\]
Proof: When the pairwise error occurs, we have $d_H(y_1^N, c^{(0)}) > d_H(y_1^N, c^{(1)})$. This condition is equal to $w_H(y_1^N + c^{(0)}) > w_H(y_1^N + c^{(1)})$. Then we have $w_H(e_1^N) > w_H(e_1^N + c^{(0)} + c^{(1)}) = d_H(e_1^N, c^{(0)} + c^{(1)})$. It follows that an error will occur if more than half of the elements in the support set of the error vector $e_1^N$ overlap the support set of the codeword $c^{(0)} + c^{(1)}$, that is, suppose $\text{supp}(e_1^N) = G_1 \cup G_2$ and $|G_1| = m > \lceil d/2 \rceil$, we have $G_1 \subset \text{supp}(c^{(0)} + c^{(1)})$. Note that $l \geq m$ and $N - d \geq l - m$ in order to obtain the reasonable calculation results. Thus we can enumerate the number of set partition of $G_1$ and $G_2$ and complete the proof.

Furthermore, by Theorem 4, the union bound of $W_N^{(i)}$ is derived as

$$
P_{\text{BSC}}(W_N^{(i)}) \leq \sum_{d=d_{\text{min}}^{(i)}}^{N} \sum_{l=d/2}^{N} \sum_{m=d/2}^{N} A_{N}^{(i)}(d) \binom{N - d}{l - m} \binom{d}{m} \delta^l (1 - \delta)^{N - l}. \tag{25}$$

Similarly, by Theorem 3, the upper bound of BLER using SC decoding in the BSC can be written as

$$
P_{\text{e,BSC}}(N, K, A) \leq \sum_{i \in A} \sum_{d=d_{\text{min}}^{(i)}}^{N} \sum_{l=d/2}^{N} \sum_{m=d/2}^{N} A_{N}^{(i)}(d) \binom{N - d}{l - m} \binom{d}{m} \delta^l (1 - \delta)^{N - l}. \tag{26}$$

3) PEP and BLER in the AWGN channel: For a binary-input AWGN channel, the received signal is expressed as

$$
y_j = \sqrt{E_s} s_j + n_j, \tag{27}$$

where $E_s$ is the signal energy, $s_j \in \{ \pm \sqrt{E_s} \}$ is the BPSK signal, and $n_j \sim \mathcal{N}(0, N_0/2)$ is a Gaussian noise sample with the zero mean and the variance $N_0/2$. Since the BPSK modulation is used, the codeword $c^{(0)}$ is transformed into the transmitted signal vector $s^{(0)} = \sqrt{E_s} (1 - 2c^{(0)})$.

Thus, the received signal vector can be addressed as

$$
y = \sqrt{E_s} \left( 1 - 2c^{(0)} \right) + n, \tag{28}$$

where $1$ is an all-one vector and $n$ is the AWGN noise vector. Then the transition probability of the product-form channel can be written as

$$
W_N^{(i)} (y \mid c^{(0)}) = \frac{1}{(\pi N_0)^{N/2}} \exp \left\{ - \frac{||y - s^{(0)}||^2}{N_0} \right\}. \tag{29}
$$

For the PEP of SC decoding in the AWGN channel, we have the following theorem.
Theorem 5: Assuming the decision vector is $c(1)$ and $d = d_H(c(0), c(1))$, the PEP between $c(0)$ and $c(1)$ can be expressed as

$$P_{AWGN}(c(0) \rightarrow c(1)) = Q \left[ \sqrt{\frac{2E_s}{N_0} d_H(c(0), c(1))} \right],$$

(30)

where $\frac{E_s}{N_0}$ is the symbol signal-to-noise ratio (SNR) and $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$ is the tail distribution function of the standard normal distribution.

Proof: First, we assume the codeword $c(0)$ and $c(1)$ are mapped to the transmission vector $s(0)$ and the signal vector $s(1)$ respectively. If a pairwise error occurs, the Euclidian distances among the received vector and the transmission/signal vectors satisfy the inequality $\|y - s(0)\|^2 > \|y - s(1)\|^2$. Substituting (28), we have $\|n\|^2 > \|(s(0) - s(1)) + n\|^2$. Extending the inequality, we obtain the decision region $\mathcal{H} = \{ n : n (s(0) - s(1))^T < -\frac{1}{2} \|s(0) - s(1)\|^2 \}$. So the PEP can be written as

$$P(c(0) \rightarrow c(1)) = \int \cdots \int_{\mathcal{H}} W_N(y | s(0)) dy$$

$$= Q \left[ \sqrt{\frac{1}{2N_0} \|s(0) - s(1)\|^2} \right].$$

(31)

Furthermore, due to $\|s(0) - s(1)\| = \|\sqrt{E_s} [1 - 2c(0)] - \sqrt{E_s} [1 - 2c(1)]\| = 2\sqrt{E_s} \|c(1) - c(0)\|$, then $\|s(0) - s(1)\| = 2\sqrt{E_s}d_H(c(0), c(1))$. So we complete the proof.

Furthermore, by Theorem 5, the union bound of $W_N^{(i)}$ is derived as

$$P_{AWGN}(W_N^{(i)}) \leq \sum_{d=d_{(i)_{min}}}^{N} A_N^{(i)}(d)Q \left( \sqrt{\frac{2dE_s}{N_0}} \right).$$

(32)

Similarly, by Theorem 3, the upper bound of BLER using SC decoding in the BI-AWGN channel can be written as

$$P_{e,AWGN}(N, K, A) \leq \sum_{i \in A} \sum_{d=d_{(i)_{min}}}^{N} A_N^{(i)}(d)Q \left( \sqrt{\frac{2dE_s}{N_0}} \right).$$

(33)

C. Union-Bhattacharyya Bound

Union-Bhattacharyya (UB) bound provides a simple form to analyze the error performance bound of the B-DMC channel. Although the UB bound is slightly looser than the union bound, it is convenient to the theoretic analysis.
Proposition 4: Given the B-DMC $W$, the union-Bhattacharyya (UB) bound of the polarized channel $W_N^{(i)}$ is given by

$$ P \left( W_N^{(i)} \right) \leq \sum_{d=1}^{N} A_N^{(i)}(d)(Z(W))^d, \quad (34) $$

where $Z(W)$ is the Bhattacharyya parameter.

Furthermore, we can derive the UB bound of the BLER as follows.

Theorem 6: Given the fixed configuration $(N, K, A)$, the UB bound of the block error probability is written as

$$ P_e(N, K, A) \leq \sum_{i \in A} \sum_{d=1}^{N} A_N^{(i)}(d)(Z(W))^d. \quad (35) $$

Following we will discuss the detailed form of UB bound in the BEC, BSC and AWGN channels.

1) BEC: For the BEC $W$ with the erasure probability $\epsilon$, the Bhattacharyya parameter is $Z(W) = \epsilon$. By Proposition 4, the UB bound of the polarized channel $W_N^{(i)}$ is written as

$$ P \left( W_N^{(i)} \right) \leq \sum_{d=d_{\min}^{(i)}}^{N} A_N^{(i)}(d)\epsilon^d. \quad (36) $$

Indeed, this UB bound is equivalent to the union bound (22) for the BEC. Similarly, according to Theorem 6, given the fixed configuration $(N, K, A)$, the UB bound of the BLER using SC decoding in the BEC is equal to Equ. (23).

2) BSC: For the BSC $W$ with the crossover error probability $\delta$, the Bhattacharyya parameter is $Z(W) = 2\sqrt{\delta(1-\delta)}$. So the UB bound of the polarized channel $W_N^{(i)}$ is written as

$$ P \left( W_N^{(i)} \right) \leq \sum_{d=d_{\min}^{(i)}}^{N} A_N^{(i)}(d) \left(2\sqrt{\delta(1-\delta)}\right)^d. \quad (37) $$

Similarly, the UB bound of the BLER in the BSC is given by

$$ P_{e, BSC}(N, K, A) \leq \sum_{i \in A} \sum_{d=d_{\min}^{(i)}}^{N} A_N^{(i)}(d) \left(2\sqrt{\delta(1-\delta)}\right)^d. \quad (38) $$

3) AWGN channel: For the BI-AWGN channel $W$, the Bhattacharyya parameter is $Z(W) = \exp\left(-\frac{E_s}{N_0}\right)$. So the UB bounds of the polarized channel $W_N^{(i)}$ and the BLER in the AWGN channel are respectively given by

$$ P \left( W_N^{(i)} \right) \leq \sum_{d=d_{\min}^{(i)}}^{N} A_N^{(i)}(d) \exp\left(-\frac{dE_s}{N_0}\right), \quad (39) $$
and
\[ P_{e,AWGN}(N,K,A) \leq \sum_{i \in A} \sum_{d = d_{\text{min}}^{(i)}} A_N^{(i)}(d) \exp \left( -\frac{dE_s}{N_0} \right). \tag{40} \]

D. Mutual Information Analysis of Polarized Channel

Based on the polar spectrum and PEP, we further analyze the symmetric capacity of the polarized channel \( W_N^{(i)} \), that is, \( I \left( W_N^{(i)} \right) = I \left( U_i; Y_1^N \mid U_i^{i-1} \right) \). At first, we can derive the upper and lower bounds of the Bhattacharyya parameter \( Z \left( W_N^{(i)} \right) \) as below.

**Proposition 5:** Given the polarized channel \( W_N^{(i)} \), the upper and lower bounds of the corresponding Bhattacharyya parameter satisfy the following inequality,
\[ (Z(W))^{d_{\text{min}}^{(i)}} \leq Z \left( W_N^{(i)} \right) \leq \sum_{d = d_{\text{min}}^{(i)}} A_N^{(i)}(d)(Z(W))^d. \tag{41} \]

**Proof:** According to the definition in (5), Bhattacharyya parameter can be further enlarged as
\[ Z \left( W_N^{(i)} \right) \leq \sum_{y_1^N \in Y^N} \sum_{u_{i+1}^N} W_N \left( y_1^N \mid 0^i, 0, u_{i+1}^N \right) \sum_{v_{i+1}^N} W_N \left( y_1^N \mid 0^i, 1, v_{i+1}^N \right) \tag{42} \]
Without loss of generality, let \( c^{(0)} = \left( 0^i, 0, u_{i+1}^N \right) \) \( F_N = 0^N \) and \( c^{(1)} = \left( 0^i, 1, v_{i+1}^N \right) \), by using the symmetry of subcode \( C_N^{(i)} \), we have
\[ Z \left( W_N^{(i)} \right) \leq \sum_{y_1^N \in Y^N} \sum_{c^{(i)}} \sqrt{W_N \left( y_1^N \mid c^{(0)} \right) W_N \left( y_1^N \mid c^{(1)} \right)}. \tag{43} \]

Enumerating all the codewords \( c^{(1)} \) of polar subcode \( D_N^{(i)} \), we obtain the upper bound of Bhattacharyya parameter. If we only consider one codeword with the minimum Hamming weight \( d_{\text{min}}^{(i)} \) in (43), this term can be intended to serve as a lower bound to the Bhattacharyya parameter.

By using Proposition 5, we can bound the symmetric capacity of any B-DMC \( W \) as the following Theorem.
Theorem 7: Given the polarized channel \( W_N^{(i)} \) and the polar spectrum \( \{A_N^{(i)}(d)\} \) of polar subcode \( D_N^{(i)} \), the symmetric capacity \( I\left( W_N^{(i)} \right) = I\left( U_i; Y_i^N | U_i^{i-1} \right) \) can be bounded by

\[
\max \left\{ 1 - \log \left[ 1 + \sum_{d=d_{\min}^{(i)}}^{N} A_N^{(i)}(d) (Z(W))^d \right], 0 \right\} \leq I\left( W_N^{(i)} \right) \leq \sqrt{1 - (Z(W))^{2d_{\min}^{(i)}}}.
\] (44)

Proof: According to Proposition 1 in [1], the symmetric capacity \( I\left( W_N^{(i)} \right) \) can be bounded by

\[
\log \frac{2}{1 + Z\left( W_N^{(i)} \right)} \leq I\left( W_N^{(i)} \right) \leq \sqrt{1 - Z^2 \left( W_N^{(i)} \right)}. \] (45)

From Proposition 5, we complete the proof.

Especially, for the BEC, we can further derive the upper/lower bounds of the symmetric capacity of \( W_N^{(i)} \) as below.

Proposition 6: Given the BEC \( W \) with the erasure probability \( \epsilon \), then the symmetric capacity of the polarized channel \( W_N^{(i)} \) satisfies

\[
\max \left\{ 1 - \sum_{d=d_{\min}^{(i)}}^{N} A_N^{(i)}(d) \epsilon^d, 0 \right\} \leq I\left( W_N^{(i)} \right) \leq \sqrt{1 - \epsilon^{2d_{\min}^{(i)}}}. \] (46)

Proof: Noting that \( I\left( W_N^{(i)} \right) = 1 - P\left( W_N^{(i)} \right) \geq 0 \) for the BEC and using Equ. (22), we obtain the lower bound.

Fig. 1 provides the upper/lower bounds of the symmetric capacity of the polarized channels for a BEC with the erasure probability \( \epsilon = 0.5 \) and the code length \( N = 1024 \). The symmetric capacity, marked by “I”, is evaluated by the recursive relation proposed by Arikan (See Equ. (38) in [1]). The upper and lower bounds, marked by “I UB” and “I LB” respectively, are calculated by (46). Although the upper/lower bounds are the coarse approximation of the symmetric capacity, they also reveal the polarization phenomenon.

Remark 1: We establish an analytical framework to evaluate the error performance and the symmetric capacity of polar codes. The polar spectrum is the critical factor to indicate the bounds of the error probability and the mutual information of the polarized channel. So we can use the upper bound of the error probability as a metric to select the information set \( A \).
Fig. 1. The upper/lower bounds of the symmetric capacity for a BEC with the erasure probability $\epsilon = 0.5$ and the code length $N = 1024$.

IV. CALCULATION OF POLAR SPECTRUM

In this section, we will discuss the calculation of polar spectrum. First, we investigate the subcode duality of polar code. Then, by using such duality, we establish the MacWilliams identity between the subcodes. Finally, the enumeration algorithm of the polar spectrum is designed to calculate the polar weight enumerator.

A. Subcode Duality

Let $i \in \left[ \frac{N}{2} + 1, N \right]$ denote the row index of the matrix $F_N$. By the definition of subcode $C_N^{(i)}$, its generator matrix $G_{C_N^{(i)}}$ is composed of the rows (from the $i$-th to the $N$-th row) of the matrix $F_N$, that is, $G_{C_N^{(i)}} = F_N(i : N)$. Further, we introduce another subcode $C_N^{(N+2-i)}$, whose generator matrix satisfies $G_{C_N^{(N+2-i)}} = F_N(N + 2 - i : N)$. Thus, subcode $C_N^{(i)}$ is a linear code $(N, N - i + 1)$ and its code rate is $R_{C_N^{(i)}} = \frac{N-i+1}{N} = 1 - \frac{i-1}{N}$. Similarly, subcode $C_N^{(N+2-i)}$ is a linear code $(N, i - 1)$ and the code rate is $R_{C_N^{(N+2-i)}} = \frac{i-1}{N}$.

**Theorem 8:** Given $N/2 + 1 \leq i \leq N$, subcode $C_N^{(N+2-i)}$ is the dual code of subcode $C_N^{(i)}$, that is, $C_N^{(N+2-i)} = C_N^{\perp (i)}$. Especially, $C_N^{(N/2+1)}$ is a self-dual code.
Proof: We can use the mathematical induction to prove this theorem. Given the code length $N$ and the index $N/2 + 1 \leq i \leq N$, suppose the conclusion $C_{N}^{(N+2-i)} = C_{N}^{\perp(i)}$ holds. When the code length is doubled to $2N$ and the index is changed to $N + 1 \leq l \leq 2N$, we obtain two new subcodes $C_{2N}^{(l)}$ and $C_{2N}^{(2N+2-l)}$.

By the Plotkin structure $[u + v \mid v]$ of polar coding [1], due to $N + 1 \leq l \leq 2N$, the subcode $C_{2N}^{(l)}$ is consist of two identical component codes $C_{N}^{(l-N)}$. That is to say, $\forall r \in C_{2N}^{(l)}$, we have $r = (t, t), t \in C_{N}^{(l-N)}$.

On the other hand, by the Plotkin structure, due to $2 \leq 2N + 2 - l \leq N + 1$, the subcode $C_{2N}^{(2N+2-l)}$ is also regarded as a combination of two component codes: one is $C_{N}^{(2N+2-l)}$ and the other is $C_{N}^{(1)}$. Similarly, given $a \in C_{N}^{(2N+2-l)}$ and $b \in C_{N}^{(1)}$, we have $w = (a + b, b) \in C_{2N}^{(2N+2-l)}$.

Now we calculate the inner product of $r$ and $w$, that is,

$$
(r \cdot w) = (t, t) \begin{pmatrix} a^T + b^T \\ b^T \end{pmatrix} = ta^T. \quad (47)
$$

Due to $t \in C_{N}^{(l)}$ and $a \in C_{N}^{\perp(l)}$, we have $ta^T = 0$. Thus, it follows that $C_{2N}^{(2N+2-l)} = C_{2N}^{\perp(l)}$.

Let $S_{N}^{(l)}(j) (1 \leq j \leq N)$ denote the weight enumerators of the subcode $C_{N}^{(l)}$, where $j$ is the Hamming weight of non-zero codeword of codebook $C_{N}^{(l)}$. Similarly, $S_{N}^{\perp(l)}(j)$ denote the weight enumerators of the dual code $C_{N}^{\perp(l)}$.

**Proposition 7:** Given the subcode $C_{N}^{(l)}$, we have $C_{N}^{(l)} = D_{N}^{(l)} \cup C_{N}^{(l+1)}$. Thus, the weight enumerator and the polar weight enumerator satisfy $S_{N}^{(l)}(j) = A_{N}^{(l)}(j) + S_{N}^{(l+1)}(j)$.

**Proof:** Recall that the $i$-th subcode satisfies $C_{N}^{(l)} = D_{N}^{(l)} \cup E_{N}^{(l)}$. Since $E_{N}^{(l)} = C_{N}^{(l+1)}$, then we have $C_{N}^{(l)} = D_{N}^{(l)} \cup C_{N}^{(l+1)}$. It follows the relationship between the weight enumerator and the polar weight enumerator.

**Proposition 8:** The odd codeword weight of the subcode $C_{N}^{(i)}, i \in \{2, N\}$ is zero, that is, $S_{N}^{(i)}(2j + 1) = 0$. Similarly, for the polar subcode $D_{N}^{(i)}, i \in \{2, N\}$, we have $A_{N}^{(i)}(2j + 1) = 0$. For the polar subcode $D_{N}^{(1)}$, the even codeword weight is zero, that is, $A_{N}^{(1)}(2j) = 0$.

**Proof:** Due to the Plotkin structure $[u + v \mid v]$, we can easily obtain the conclusions.

**Proposition 9:** The weight distribution of the subcode $C_{N}^{(i)}$ is symmetric, that is, $S_{N}^{(i)}(j) = S_{N}^{(i)}(N - j)$. Similar result holds for the polar subcode $D_{N}^{(i)}$, that is, $A_{N}^{(i)}(j) = A_{N}^{(i)}(N - j)$.

**Proof:** Similarly, due to the Plotkin structure $[u + v \mid v]$, we can easily obtain these conclusions.
B. MacWilliams Identities of Subcodes

It is well known that the linear relations between the weight distributions of a linear code and its dual can be determined by the MacWilliams identities [20]. That means, if we know the weight distribution of one linear code, we can obtain the weight distribution of the dual code without knowing the special coding structure. These identities have become the most significant tools to investigate and calculate the weight distributions.

**Theorem 9:** Given the subcode $C_N^{(i)}$ and its dual $C_N^{⊥(i)} = C_N^{(N+2-i)}$, the weight enumerators $S_N^{(i)}(j)$ and $S_N^{⊥(i)}(j)$ satisfy the following MacWilliams identities [20]

$$
\sum_{j=0}^{N} \binom{N-j}{k} S_N^{⊥(i)}(j) = 2^{i-1-k} \sum_{j=0}^{N} \binom{N-j}{N-k} S_N^{(i)}(j),
$$

(48)

where $k \in [0, N]$.

These identities are composed of $N + 1$ linear equations. By solving these equations, we can calculate the weight distribution of one subcode and its dual.

C. Enumeration Algorithm of Polar Spectrum

Using the Plotkin structure $[u + v|v]$ and the MacWilliams identities, we design an iterative enumeration algorithm to calculate the polar spectrum. Given the weight distribution of subcodes $C_N^{(i)}$, as shown in Algorithm 1, this algorithm enumerates the weight distribution of subcodes $C_{2N}^{(i)}$ and the polar spectrum of polar subcodes $D_{2N}^{(i)}$.

Based on the iterative structure, Algorithm 1 is consist of four steps to enumerate the weight distribution and the polar spectrum. In the first step, by using Proposition 7, the polar spectrum of polar subcodes $D_N^{(i)}$ with the code length $N$ is calculated. When the code length is grown from $N$ to $2N$, due to the matrix structure $F_{2N} = \begin{bmatrix} F_N & 0 \\ F_N & F_N \end{bmatrix}$, the enumeration can be divided into two parts.

That is to say, for the second step, we enumerate the weight distribution and polar spectrum in the case of $N + 1 \leq l \leq 2N$. In this case, by using the Plotkin structure $[u + v|v]$, the subcode $C_{2N}^{(l)}$ is consist of two identical component codes $C_N^{(l-N)}$. Therefore, the subcode $C_{2N}^{(l)}$ has the same weight distribution as the subcode $C_N^{(l-N)}$. And similarly, the polar subcode $D_{2N}^{(l)}$ has the same polar spectrum as the polar subcode $D_N^{(l-N)}$. Certainly, the codeword weight of $C_{2N}^{(l)}$ or $D_{2N}^{(l)}$ is doubled.
For the third step, we enumerate the weight distribution and polar spectrum in the case of \(1 \leq l \leq N\). According to Theorem 8, the subcode \(C^{(l)}_{2N}\) is the dual of subcode \(C^{(2N+2-l)}_{2N}\). Since the weight distribution of \(C^{(2N+2-l)}_{2N}\) has been obtained in the second step, we can solve the MacWilliams identities to calculate the weight distribution of \(C^{(l)}_{2N}\).

Finally, using Proposition 7 the polar spectrum of polar subcodes \(D^{(l)}_{2N}\) with the code length \(2N\) is calculated. Note that the weight distribution of subcode \(C^{(1)}_{2N}\) obeys the binomial distribution.

The computational complexity of Algorithm 1 is mainly determined by the solution of MacWilliams identities. Due to the lower-triangle structure of the coefficient matrix in the MacWilliams identities, the weight enumerators can be recursively calculated. Hence, given the code length \(N\), the worse complexity of solving MacWilliams identities is \(\chi_M(N) = (N + 1)^2\). Furthermore, we only consider to solve \(N/2 - 1\) groups of those identities. So the computational complexity is \(\chi_E(N) = (N/2 - 1)(N + 1)^2\). If we consider the non-existence of the odd-weight codewords (Proposition 8) and the symmetry of the weight distribution (Proposition 9), the worse complexity of enumeration algorithm can be further reduced to \(\chi_E(N) = (N/2 - 1)(N/4 + 1)^2\). So the total complexity of Algorithm 1 is \(O(N^3)\).

**Example 1:** Table I shows partial results of the polar spectrum for the code length \(N = 32\). In this example, due to the symmetric distribution of polar spectrum (Proposition 9), we only provide half of the polar weight distribution (the number inside the bracket is the symmetric weight), e.g. \(A^{(3)}_{32}(2) = A^{(3)}_{32}(30) = 128\). As shown in this table, due to the duality relationship, \(A^{(2)}_{32}\) and \(A^{(32)}_{32}\), \(A^{(3)}_{32}\) and \(A^{(31)}_{32}\), etc. satisfy the MacWilliams identities. Meanwhile, the subcode \(C^{(17)}_{32}\) is a self-dual code. Furthermore, by Proposition 8 all the polar weight distributions for the index \(i \in [2, 32]\) have non-zero even codeword weight. On the contrary, for the index \(i = 1\), non-zero codeword weight is odd.

V. CONSTRUCTION OF POLAR CODES

In this section, we consider the construction of polar codes based on the UB bound. First, we derive the logarithmic version metric based on UB bound, named UB weight (UBW). Then a more simple metric, named simplified UB weight (SUBW), is designed.
A. Construction Metrics based on UB bound

By Proposition 4, we can use the UB bound of the channel error probability as a reliability metric to sort all the polarized channels. Consider the practical application, the logarithmic form of the UB bound is more convenient. Thus, we obtain the following theorem.

**Theorem 10:** Given the B-DMC $W$ and the code length $N$, the reliability of the polarized channel can be sorted by the UB Weight (UBW), that is,

$$UBW^{(i)}_N = \max_d \left\{ L^{(i)}_N (d) + d \ln(Z(W)) \right\},$$  \hspace{1cm} (50)\

where $L^{(i)}_N (d) = \ln \left[ A^{(i)}_N (d) \right]$ is the logarithmic form of the polar weight enumerator. 

**Proof:** The logarithmic form of the UB bound in Proposition 4 can be written as

$$\ln \left\{ \sum_d A^{(i)}_N (d)(Z(W))^d \right\} = \ln \left\{ \sum_d \exp \left[ \ln A^{(i)}_N (d) + d \ln(Z(W)) \right] \right\}$$
$$\approx \max_d \left\{ L^{(i)}_N (d) + d \ln(Z(W)) \right\}.$$ \hspace{1cm} (51)

Here, we use the approximation $\ln \left( \sum_k e^{a_k} \right) \approx \max_k \{ a_k \}$. 

**Example 2:** (UBW in BEC or BSC) Given the Bhattacharyya parameter of the BEC $Z(W) = \epsilon$, UBW can be written as

$$UBW^{(i)}_N = \max_d \left\{ L^{(i)}_N (d) + d \ln \epsilon \right\}. $$ \hspace{1cm} (52)\

Similarly, UBW for the BSC $\left( Z(W) = 2 \sqrt{\delta(1-\delta)} \right)$ can be written as

$$UBW^{(i)}_N = \max_d \left\{ L^{(i)}_N (d) + d \frac{2}{2} \ln[4\delta(1-\delta)] \right\}. $$ \hspace{1cm} (53)

**Example 3:** (UBW in AWGN channel) For the BI-AWGN channel $W$, the Bhattacharyya parameter is $Z(W) = \exp(-\frac{E_s}{N_0})$. Thus the corresponding UBW can be expressed as

$$UBW^{(i)}_N = \max_d \left\{ L^{(i)}_N (d) - d \frac{E_s}{N_0} \right\}. $$ \hspace{1cm} (54)\

Especially, if the symbol SNR in UBW is instead of a fixed constant $\alpha$, we obtain a universal construction metric.
B. Construction Metrics based on simplified UB bound

If we only consider the first term of the polar spectrum, that is, the minimum weight enumerator, the UBW in Theorem 10 can be further simplified. So we have the following proposition.

**Proposition 10:** Given the B-DMC $W$ and the code length $N$, the reliability of the polarized channel can be ordered by the simplified UB Weight (SUBW), that is,

$$SUBW_N^{(i)} = L_N^{(i)} \left( d_{\min}^{(i)} \right) + d_{\min}^{(i)} \ln(Z(W)).$$

(55)

Correspondingly, we can obtain the simplified union bound of the block error probability of polar codes

$$P_e(N, K, A) \lesssim \sum_{i \in A} A_N^{(i)} \left( d_{\min}^{(i)} \right) P_N^{(i)} \left( d_{\min}^{(i)} \right)$$

(56)

or the simplified UB bound

$$P_e(N, K, A) \lesssim \sum_{i \in A} A_N^{(i)} \left( d_{\min}^{(i)} \right) (Z(W))^{d_{\min}^{(i)}}.$$  

(57)

**Example 4:** (SUBW in AWGN channel) The SUBW for the BI-AWGN channel can be expressed as

$$SUBW_N^{(i)} = L_N^{(i)} \left( d_{\min}^{(i)} \right) - d_{\min}^{(i)} \frac{E_s}{N_0}. $$

(58)

Similarly, we can design a universal construction metric by selecting a suitable constant $\alpha$ for the symbol SNR in SUBW.

**Example 5:** Table II shows the reliability order of the polarized channels based on various constructions for the code length $N = 32$ and the symbol SNR $\frac{E_s}{N_0}$ is fixed as 4 dB. As shown in this table, the reliability order obtained by sorting UBW or SUBW from low to high is almost the same as that ordered by the GA algorithm except the polarized channels with indices 4 and 17. However, the reliability order obtained by sorting PW is different from that ordered by GA algorithm at the polarized channels with indices 20, 15, 8 and 25.

**Remark 2:** The UBW and SUBW are deduced from the error probability of the polarized channel, that is, union-Bhattacharyya bound analysis. They are determined by the polar spectrum and the Bhattacharyya parameter. Compared with the traditional constructions, such as, DE, GA, and Tal-Vardy algorithm etc., these metrics have explicitly analytical structure. Particularly, since the polar spectrum can be off-line calculated by using Algorithm 1, the complexity of these constructions is linear $O(N)$, which is much lower than that of the former algorithms. On the
other hand, compared with the polarized weight (PW) construction, these constructions can also be modified to the universal constructions. As an efficient SNR-independent construction, PW is an empirical construction and lack of theoretic explanation. On the contrary, UBW or SUBW is a universal construction with a good analytical property. Therefore, the construction based on UBW or SUBW has dual advantages: theoretic analyticity and practical application.

VI. Numerical Analysis and Simulation Results

In this section, we will provide the numerical and simulation results based on the polar spectrum. First, we compare the upper bounds in term of the polar spectrum with the traditional bounds based on GA or Bhattacharyya parameter under BEC, BSC and AWGN channel. Then, the BLER simulation results based on various constructions under AWGN channel are analyzed and compared.

A. Numerical analysis of upper bounds

![Graph showing BLER upper bounds for polar coding with SC decoding at code length N = 128 under different channel conditions.]

(a) BEC with the erasure probability $\epsilon = 0.5$

(b) BSC with the crossover error probability $\delta = 0.1$

Fig. 2. The BLER upper bounds for polar coding with SC decoding at code length $N = 128$ under different channel conditions.

In this part, we compare five different upper bounds of BLER which can be divided into two categories, the traditional bounds and the proposed ones based on the polar spectrum. The traditional upper bounds include the GA bound [6] and the bound provided by Arikan in [1] as
Fig. 3. The BLER upper bounds for polar coding with SC decoding over AWGN channel, where the code rate $R = 0.5$.

Presented in (8) while the proposed upper bounds contain the union bound given in (20), the UB bound expressed as (35) and the simplified UB bound (57).

Given the code length $N = 128$, Fig. 2 provides the upper bounds of BLER using SC decoding under BEC with the erasure probability $\epsilon = 0.5$ and BSC with the crossover error probability $\delta = 0.1$. The upper bound presented in (8) is named as “Arıkan’s bound” hereafter and the lower bound marked by the dash line is obtained from $\max_{i \in A} \left\{ Z \left( W^{(i)}_N \right) \right\}$ (See Fig. 7 in [1]). As shown in Fig. 2(a), all the upper bounds grow larger as the code rate increases. However, it can be observed that the UB bound (36) and the union bound (23) coincide under the BEC and they tend to diverge when the code rate is greater than 0.35, while the simplified UB bound avoids this problem and is looser than the Arıkan’s bound. When the BSC is considered, as shown in Fig. 2(b), the union bound (26) is closer to the Arıkan’s bound when the code rate is smaller than 0.3, otherwise, the simplified UB bound is closer.

The BLER upper bounds of SC decoding under AWGN channel are depicted in Fig. 3. As shown in Fig. 3(a), where the code length $N$ is set to 128 and the code rate is $R = 0.5$, all the upper bounds dramatically decrease with the increase of the symbol SNR. Due to the approximation calculation of Bhattacharyya parameter in AWGN channel, Arıkan’s bound is looser than GA bound. In addition, the union bound (33) tends to be more closer to the simulation result (marked by dash line and constructed based on GA algorithm) than the Arıkan’s bound.
when the symbol SNR exceeds 1.2 dB. It also can be observed that the UB bound (40) and the simplified UB bound (57) are gradually close to the Arikan’s bound with the increase of the symbol SNR. Similar observations can be found in Fig. 3(b), which provides the BLER upper bounds of SC decoding under AWGN channel with the configuration $N = 1024$ and $R = 0.5$.

Although the GA bound is tightly close to the BLER simulation result, it involves a complex on-line iterative-calculation. On the contrary, the proposed UB and simplified UB bounds not only have a linear complexity since the polar spectrum can be calculated off-line, but also can be used to deduce two universal and analytical construction metrics to construct good polar codes. The BLER simulation results based on these universal constructions will be presented in the next subsection.

**B. Simulation Results**

In this part, considering the AWGN channel, we compare the BLER simulation performances of polar codes generated by the proposed UBW/SUBW constructions and the traditional methods, which include GA, PW and the one based on Bhattacharyya parameter proposed by Arikan in [1]. The code length $N$ is in $\{128, 1024, 4096\}$ and the code rate $R$ is choose from $\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$.

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**Fig. 4.** The BLER performances comparison among the polar codes constructed based on GA, PW, Bhattacharyya parameter, UBW and SUBW under AWGN channel, where $N = 128$ and $R = 0.5$.

**Fig. 4** provides the BLER performances comparison among the various constructions with $N = 128$ and $R = 0.5$. For UBW/SUBW, the fixed symbol SNR in (54)/(58) is set to $\alpha = 4$. 
dB. It can be observed from Fig. 4(a) that the polar codes constructed by UBW/SUBW can achieve similar performance of those constructed by GA or PW algorithm under SC decoding. Moreover, the polar codes constructed by UBW/SUBW outperform those constructed based on the Bhattacharyya parameter in the case of $\frac{E_s}{N_0} > 0.2$ dB. When the SCL decoding with list size $L = 8$ is used, as shown in Fig. 4(b), the polar codes constructed by UBW/SUBW outperform those generated by the traditional constructions by about $0.4 \sim 0.6$ dB as the BLER is $10^{-3}$.

Fig. 5. The BLER performances comparison among the polar codes constructed based on GA, PW, Bhattacharyya parameter, UBW and SUBW under AWGN channel, where $N = 1024$ and SC decoding is used.

Given the code length $N = 1024$, Fig. 5 presents the comprehensive BLER performances comparison among the various constructions under SC decoding. The symbol SNR in (54) for UBW is respectively set to 1.5, 4 and 4.5 dB for the code rate $R = \frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{3}$, while that for SUBW in (58) is respectively fixed as 1, 3.5 and 4 dB. As shown in Fig. 5, when a low to medium code rate is considered, namely, $R = \frac{1}{3}$ or $\frac{1}{2}$, the polar codes constructed by UBW/SUBW perform slightly worse than those constructed by GA or Bhattacharyya parameter because the UB bound of the BLER under the SC decoding is looser than the GA bound or Arıkan’s bound. However, similar to the PW construction, the UBW/SUBW constructions ensure the universality
at the cost of some performance losses. In addition, when the code rate $R = \frac{2}{3}$ is considered, the polar codes constructed by UBW/SUBW can achieve nearly the same performance of those constructed by GA. Apart from these, one can also observe that the polar codes constructed by PW show error floor in the high SNR region, which can be avoided by UBW/SUBW.

![Graph](image)

Fig. 6. The BLER performances comparison among the polar codes constructed based on GA, PW, Bhattacharyya parameter, UBW and SUBW under AWGN channel, where $N = 1024$ and SCL decoding with list size 16 is used.

For the code length $N = 1024$, the BLER performances comparison among the various constructions under SCL decoding with list size $L = 16$ is shown in Fig. 6. It can be observed that the polar codes constructed by UBW/SUBW can achieve better performance than those constructed by GA/PW or the construction based on Bhattacharyya parameter. The performance gain of polar codes constructed by UBW/SUBW becomes larger with the increase of code rate. For example, given $R = \frac{1}{3}$, when the BLER is $10^{-3}$, polar codes constructed by SUBW achieve 0.3/0.5 dB gain compared to those constructed by GA/PW, it then becomes 0.4/1.3 dB under $R = \frac{2}{3}$. Actually, polar codes constructed by UBW/SUBW benefit from the adequate utilization of the polar spectrum, which is vital for the polar code construction under the SCL decoding. Furthermore, polar codes constructed by SUBW outperform those constructed by UBW when
$R = \frac{1}{3}$ because the minimum weight term of the polar spectrum plays an important role in the analysis of error probability of the polarized channels.

Fig. 7 provides the BLER performances comparison among the various constructions with $N = 4096$ and $R = \frac{2}{3}$. In this case, the symbol SNR in (54)/(58) for UBW/SUBW is set to 4.5/3.5 dB. It can be observed from Fig. 7(a) that polar codes constructed by UBW/SUBW perform worse than those constructed by GA under SC decoding but avoid the error floor in the high SNR region, which is obviously shown in the polar codes constructed by PW. When the SCL decoding with list size $L = 16$ is used, as shown in Fig. 7(b), the polar codes constructed by SUBW can achieve 0.5/1.1 dB gain compared to those constructed by Bhattacharyya parameter/PW when the BLER is $10^{-3}$.

In general, UBW and SUBW can be regarded as two good constructions, which are simple and universal for the practical polar coding as well as able to generate polar codes with superior performance under SCL decoding over those based on the traditional methods in most cases.

VII. CONCLUSIONS

In this paper, we introduce a new concept named polar spectrum from the codeword weight distribution of polar codes and then establish a systematic framework in term of the polar
spectrum to analyze and construct polar codes. On the basis of the polar spectrum, we derive the union bound and UB bound of the polarized channel and further upper bound the BLER of SC decoding. In addition, we propose an iterative algorithm embedded the solution of MacWilliams identities to enumerate the polar spectrum of polar codes, which has a low complexity and a high efficiency compared with the traditional searching algorithm. Finally, we design two universal and analytical construction metrics named UBW and SUBW, which have a linear complexity far below those constructions based on the iterative calculation since the polar spectrum can be calculated off-line. Simulation results show that the polar codes constructed by these two metrics can achieve similar performance of those constructed by GA or PW algorithm under SC decoding and even superior performance under SCL decoding.

REFERENCES

[1] E. Arıkan, “Channel polarization: a method for constructing capacity achieving codes for symmetric binary-input memoryless channels,” *IEEE Trans. Inf. Theory*, vol. 55, no. 7, pp. 3051-3073, July 2009.
[2] S. B. Korada, “Polar Codes for channel and source coding,” Dissertation of EPFL, 2009.
[3] E. Arıkan and E. Telatar, “On the rate of channel polarization,” in *Proc. IEEE Int. Symp. Inform. Theory (ISIT)*, pp. 1493-1495, July 2009.
[4] R. Mori and T. Tanaka, “Performance of polar codes with the construction using density evolution,” *IEEE Commun. Lett.*, vol. 13, no. 7, pp. 519-521, Jul. 2009.
[5] I. Tal and A. Vardy, “How to construct polar codes,” *IEEE Trans. Inf. Theory.*, vol. 59, no. 10, pp. 6562-6582, 2013.
[6] P. Trifonov, “Efficient design and decoding of polar codes,” *IEEE Trans. Commun.*, vol. 60, no. 11, pp. 3221-3227, Nov. 2012.
[7] Dai J. C., Niu K., et al., “Does Gaussian Approximation Work Well for the Long-Length Polar Code Construction?” IEEE Access, vol. 5, pp. 7950-7963, 2017.
[8] C. Schürch, “A partial order for the synthesized channels of a polar code,” *IEEE International Symposium on Information Theory (ISIT)*, pp. 220-224, July 2016.
[9] G. N. He, J. C. Belfiore , et al., “β-expansion: A Theoretical Framework for Fast and Recursive Construction of Polar Codes,” *IEEE GLOBECOM*, pp. 1-6, Dec. 2017.
[10] V. Bioglio, C. Condo, and I. Land, “Design of Polar Codes in 5G New Radio,” Arxiv 1804.04389v1, Apr. 2018.
[11] M. Valipour and S. Yousefii, “On probabilistic weight distribution of polar codes,” *IEEE Commun. Lett.*, vol. 17, no. 11, pp. 2120-2123, 2013.
[12] Z. Z. Liu, K. Chen, K. Niu, and Z. Q. He, “Distance spectrum analysis of polar codes,” in *Proc. IEEE Wireless Commun. Networking Conf. (WCNC)*, pp. 490-495, April 2014.
[13] I. Tal and A. Vardy, “List decoding of polar codes,” in *Proc. IEEE Int. Symp. Inform. Theory (ISIT)*, pp. 1-5, 2011.
[14] K. Niu and K. Chen, “CRC-aided decoding of polar codes,” *IEEE Commun. Lett.*, vol. 16, no. 10, pp. 1668-1671, Oct. 2012.
[15] B. Li, H. Shen, and D. Tse, “An adaptive successive cancellation list decoder for polar codes with cyclic redundancy check,” IEEE Commun. Lett., Vol. 16, No. 12, pp. 2044-2047, 2012.

[16] K. Niu, K. Chen, J. R. Lin, and Q. T. Zhang, “Polar codes: primary concepts and practical decoding algorithms,” IEEE Commun. Mag., pp. 192-203, July 2014.

[17] K. Niu and K. Chen, “Stack decoding of polar codes,” Electronics Letters, vol. 48, no. 12, pp. 695-697, 2012.

[18] K. Chen, K. Niu, and J. R. Lin, “Improved successive cancellation decoding of polar codes,” IEEE Trans. Commun., vol. 61, no. 8, pp. 3100-3107, 2013.

[19] D. Guan, K. Niu, C. Dong, and P. Zhang, “Successive cancellation priority decoding of polar codes,” IEEE Access, vol. 7, pp. 9575-9585, 2019.

[20] F. J. MacWilliams, “A theorem on the distribution of weights in a systematic code,” Bell System Tech., vol. 42, pp. 79-94, 1963.

[21] I. Sason and S. Shamai, “Performance analysis of linear codes under maximum-likelihood decoding: a tutorial,” Foundations and Trends in Communications and Information Theory, vol. 3, no. 1/2, pp. 1-225, 2006.

[22] S. Lin and D. J. Costello Jr., Error Control Coding: Fundamentals and Applications (2nd ed.), Pearson Education, 2004.

[23] 3rd Generation Partnership Project (3GPP) TS 36.212, “Multiplexing and channel coding,” Release 8, 2009.

[24] 3rd Generation Partnership Project (3GPP) TS 38.212, “Multiplexing and channel coding,” V15.1.0, 2018.

[25] Qualcomm, “LDPC Rate Compatible Design Overview,” 3GPP TSG R1-1610137, Lisbon, Portugal, Oct. 2016.
Algorithm 1: Iterative enumeration algorithm of polar spectrum

**Input:** The weight distribution of all the subcodes with the code length $N$,
\[ \left\{ S_N^{(i)}(j) : i = 1, 2, ..., N; j = 0, 1, ..., N \right\}; \]

**Output:** The polar spectrum of all the polar subcodes with the code length $2N$,
\[ \left\{ A_{2N}^{(l)}(j) : l = 1, 2, ..., 2N; j = 0, 1, ..., 2N \right\}; \]

1. for $i = 1 \rightarrow N - 1$
   2. for $j = 0 \rightarrow N$
      3. Calculate the polar spectrum of the polar subcodes with the code length $N$,
         \[ A_N^{(i)}(j) = S_N^{(i)}(j) - S_N^{(i+1)}(j); \]
   4. for $j = 0 \rightarrow N$
      5. Calculate the polar spectrum of the polar subcode which only contains an all-one codeword,
         \[ A_N^{(N)}(j) = 0, j = 0, 1, ..., N - 1 \text{ and } A_N^{(N)}(N) = 1; \]
   6. for $l = N + 1 \rightarrow 2N$
      7. for $j = 0 \rightarrow N$
         8. Calculate the polar spectrum of the polar subcodes with the code length $2N$,
            \[ A_{2N}^{(l)}(2j) = A_N^{(l-N)}(j); \]
      9. Calculate the weight distribution of the subcodes with the code length $2N$,
         \[ S_{2N}^{(l)}(2j) = S_N^{(l-N)}(j); \]
   10. for $l = 2 \rightarrow N$
       11. Solve the MacWilliams Identities and calculate the weight distribution $S_{2N}^{(l)}(j)$
           \[ \sum_{j=0}^{2N} \binom{2N-j}{k} S_{2N}^{(2N+2-l)}(j) = 2^{l-1-k} \sum_{j=0}^{2N} \binom{2N-j}{2N-k} S_{2N}^{(l)}(j), \quad (49) \]
           where $k = 0, 1, ..., 2N$;
   12. for $l = 2 \rightarrow N$
       13. for $j = 0 \rightarrow 2N$
          14. Calculate the polar spectrum $A_{2N}^{(l)}(j) = S_{2N}^{(l)}(j) - S_{2N}^{(l+1)}(j);$
   15. for $j = 0 \rightarrow 2N$
       16. Initialize the weight distribution $S_{2N}^{(1)}(j) = \binom{2N}{j};$
       17. Calculate the polar spectrum $A_{2N}^{(1)}(j) = S_{2N}^{(1)}(j) - S_{2N}^{(2)}(j).$
| index $i$ | weight $d$ | $A_{32}^{(i)}(d)$ | index $i$ | weight $d$ | $A_{32}^{(i)}(d)$ |
|---------|------------|-----------------|---------|------------|-----------------|
| 1       | 1(31)      | 32              | 32      | 32         | 1               |
| 1       | 3(29)      | 4960            | 31      | 16         | 2               |
| 1       | 5(27)      | 201376          | 30      | 16         | 4               |
| 1       | 7(25)      | 3365856         | 29      | 8(24)      | 4               |
| 1       | 9(23)      | 28048800        | 28      | 16         | 16              |
| 1       | 11(21)     | 129024480       | 27      | 16         | 16              |
| 1       | 13(19)     | 347373600       | 27      | 8(24)      | 8               |
| 1       | 15(17)     | 565722720       | 26      | 16         | 32              |
| 2       | 2(30)      | 256             | 26      | 8(24)      | 16              |
| 2       | 4(28)      | 17920           | 25      | 12(20)     | 56              |
| 2       | 6(26)      | 453376          | 25      | 4(28)      | 8               |
| 2       | 8(24)      | 5258240         | 24      | 16         | 256             |
| 2       | 10(22)     | 32258304        | 23      | 16         | 192             |
| 2       | 12(20)     | 112892416       | 23      | 12(20)     | 128             |
| 2       | 14(18)     | 235723520       | 23      | 8(24)      | 32              |
| 2       | 16         | 300533760       | 22      | 16         | 384             |
| 3       | 2(30)      | 128             | 22      | 12(20)     | 256             |
| 3       | 4(28)      | 8960           | 22      | 8(24)      | 64              |
| 3       | 6(26)      | 226688          | 21      | 16         | 832             |
| 3       | 8(24)      | 2629120         | 21      | 12(20)     | 496             |
| 3       | 10(22)     | 16129152        | 21      | 8(24)      | 96              |
| 3       | 12(20)     | 56446208        | 21      | 4(28)      | 16              |
| 3       | 14(18)     | 117861760       | 20      | 16         | 1536            |
| 3       | 16         | 150266880       | 20      | 12(20)     | 1024            |
| ...     | ...        | ...             | 20      | 8(24)      | 256             |
| 17      | 2(30)      | 16             | 19      | 16         | 3200            |
| 17      | 6(26)      | 560             | 19      | 12(20)     | 2016            |
| 17      | 10(22)     | 4368           | 19      | 8(24)      | 448             |
| 17      | 14(18)     | 11440           | 19      | 4(28)      | 32              |
TABLE II
RELIABILITY ORDER EXAMPLE BASED ON VARIOUS CONSTRUCTIONS FOR $N = 32$

| GA | PW | Bhattacharyya | UBW | SUBW |
|----|----|---------------|-----|------|
| 32 | 32 | 32            | 32  | 32   |
| 31 | 31 | 31            | 31  | 31   |
| 30 | 30 | 30            | 30  | 30   |
| 28 | 28 | 28            | 28  | 28   |
| 24 | 24 | 24            | 24  | 24   |
| 16 | 16 | 16            | 16  | 16   |
| 29 | 29 | 29            | 29  | 29   |
| 27 | 27 | 27            | 27  | 27   |
| 26 | 26 | 26            | 26  | 26   |
| 23 | 23 | 23            | 23  | 23   |
| 22 | 22 | 22            | 22  | 22   |
| 20 | 15 | 20            | 20  | 20   |
| 15 | 20 | 15            | 15  | 15   |
| 14 | 14 | 14            | 14  | 14   |
| 12 | 12 | 12            | 12  | 12   |
| 8  | 25 | 8             | 8   | 8    |
| 25 | 8  | 25            | 25  | 25   |
| 21 | 21 | 21            | 21  | 21   |
| 19 | 19 | 19            | 19  | 19   |
| 13 | 13 | 13            | 13  | 13   |
| 18 | 18 | 18            | 18  | 18   |
| 11 | 11 | 11            | 11  | 11   |
| 10 | 10 | 10            | 10  | 10   |
| 7  | 7  | 7             | 7   | 7    |
| 6  | 6  | 6             | 6   | 6    |
| 4  | 4  | 4             | 17  | 17   |
| 17 | 17 | 17            | 4   | 4    |
| 9  | 9  | 9             | 9   | 9    |
| 5  | 5  | 5             | 5   | 5    |
| 3  | 3  | 3             | 3   | 3    |
| 2  | 2  | 2             | 2   | 2    |
| 1  | 1  | 1             | 1   | 1    |