Hilbert $C^*$-bimodules over commutative $C^*$-algebras and an isomorphism condition for quantum Heisenberg manifolds.

Beatriz Abadie Ruy Exel

March 24, 2022

Abstract

Abstract: A study of Hilbert $C^*$-bimodules over commutative $C^*$-algebras is carried out and used to establish a sufficient condition for two quantum Heisenberg manifolds to be isomorphic.

Introduction. In [AEE], a theory of crossed products of $C^*$-algebras by Hilbert $C^*$-bimodules was introduced and used to describe certain deformations of Heisenberg manifolds constructed by Rieffel (see [Rf2] and [AEE, 3.3]). This deformation consists of a family of $C^*$-algebras, denoted $D_{\mu\nu}^c$, depending on two real parameters $\mu$ and $\nu$, and a positive integer $c$. In case $\mu = \nu = 0$, $D_{\mu\nu}^c$ turns out to be isomorphic to the algebra of continuous functions on the Heisenberg manifold $M^c$.

For K-theoretical reasons [Ab2], $D_{\mu\nu}^c$ and $D_{\mu'\nu'}^{c'}$ cannot be isomorphic unless $c = c'$. It is the main purpose of this work to show that the $C^*$-algebras $D_{\mu\nu}^c$ and $D_{\mu'\nu'}^{c'}$ are isomorphic when $(\mu, \nu)$ and $(\mu', \nu')$ are in the same orbit under the usual action of $GL_2(\mathbb{Z})$ on the torus $T^2$ (here the parameters are

*Partially supported by CNPq, Brazil.
viewed as running in $T^2$, since $D_{\mu,\nu}^c$ and $D_{\mu+n,\nu+m}^c$ are isomorphic for any integers $m$ and $n$).

As indicated above, the quantum Heisenberg manifold $D_{\mu,\nu}^c$ may be described as a crossed product of the commutative $C^*$-algebra $C(T^2)$ by a Hilbert $C^*$-bimodule. Motivated by this, we are led to study some special features of Hilbert $C^*$-bimodules over commutative $C^*$-algebras, which are relevant to our purposes.

In Section 1 we consider, for a commutative $C^*$-algebra $A$, two subgroups of its Picard group $\text{Pic}(A)$: the group of automorphisms of $A$ (embedded in $\text{Pic}(A)$ as in [BGR]), and the classical Picard group $C\text{Pic}(A)$ (see, for instance, [DG]) consisting of Hilbert line bundles over the spectrum of $A$. Namely, we prove that $\text{Pic}(A)$ is the semidirect product of $C\text{Pic}(A)$ by $\text{Aut}(A)$. This result carries over a slightly more general setting, and a similar statement (see Proposition 1.1) holds for Hilbert $C^*$-bimodules that are not full, partial automorphisms playing then the role of $\text{Aut}(A)$. These results provide a tool that enables us to deal with $\text{Pic}(C(T^2))$ in order to prove our isomorphism theorem for quantum Heisenberg manifolds, which is done in Section 2.

The authors would like to acknowledge the financial support from FAPESP (grant no. 95/4609-0), Brazil, and CONICYT, Uruguay.

1 The Picard group and the classical Picard group.

**Notation.** Let $A$ be $C^*$-algebra. If $M$ is a Hilbert $C^*$-bimodule over $A$ (in the sense of [EMS, 1.8]) we denote by $\langle \ , \ \rangle^L_M$ and $\langle \ , \ \rangle^R_M$, respectively, the left and right $A$-valued inner products, and drop the superscript whenever the context is clear enough. If $M$ is a left (resp. right) Hilbert $C^*$-module over $A$, we denote by $K(\mathcal{A}M)$ (resp. $K(M\mathcal{A})$) the $C^*$-algebra of compact operators on $M$. When $M$ is a Hilbert $C^*$-bimodule over $A$ we will view the elements of $\langle M, M \rangle_R$ (resp. $\langle M, M \rangle_L$) as compact operators on the left (resp. right)
module $M$, as well as elements of $A$, via the well-known identity:

$$\langle m, n \rangle_L p = m \langle n, p \rangle_R,$$

for $m, n, p \in M$.

The bimodule denoted by $\tilde{M}$ is the dual bimodule of $M$, as defined in [Rfl 6.17].

By an isomorphism of left (resp. right) Hilbert $C^*$-modules we mean an isomorphism of left (resp. right) modules that preserves the left (resp. right) inner product. An isomorphism of Hilbert $C^*$-bimodules is an isomorphism of both left and right Hilbert $C^*$-modules. We recall from [BGR 3] that Pic($A$), the Picard group of $A$, consists of isomorphism classes of full Hilbert $C^*$-bimodules over $A$ (that is, Hilbert $C^*$-bimodules $M$ such that $\langle M, M \rangle_L = \langle M, M \rangle_R = A$), equipped with the tensor product, as defined in [Rfl 5.9].

It was shown in [BGR 3.1] that there is an anti-homomorphism from Aut($A$) to Pic($A$) such that the sequence

$$1 \longrightarrow \text{Gin}(A) \longrightarrow \text{Aut}(A) \longrightarrow \text{Pic}(A)$$

is exact, where Gin($A$) is the group of generalized inner automorphisms of $A$. By this correspondence, an automorphism $\alpha$ is mapped to a bimodule that corresponds to the one we denote by $A_\alpha^{-1}$ (see below), so that $\alpha \mapsto A_\alpha$ is a group homomorphism having Gin($A$) as its kernel.

Given a partial automorphism $(I, J, \theta)$ of a $C^*$-algebra $A$, we denote by $J_\theta$ the corresponding ([AEE 3.2]) Hilbert $C^*$-bimodule over $A$. That is, $J_\theta$ consists of the vector space $J$ endowed with the $A$-actions:

$$a \cdot x = ax, \quad x \cdot a = \theta[\theta^{-1}(x)a],$$

and the inner products

$$\langle x, y \rangle_L = xy^*,$$

and

$$\langle x, y \rangle_R = \theta^{-1}(x^*y),$$

for $x, y \in J$, and $a \in A$. If $M$ is a Hilbert $C^*$-bimodule over $A$, we denote by $M_\theta$ the Hilbert $C^*$-bimodule obtained by taking the tensor product $M \otimes_A J_\theta$. 3
The map $m \otimes j \mapsto mj$, for $m \in M$, $j \in J$, identifies $M_\theta$ with the vector space $MJ$ equipped with the $A$-actions:

$$a \cdot mj = amj, \quad mj \cdot a = m\theta[\theta^{-1}(j)a],$$

and the inner products

$$\langle x, y \rangle_{L}^{M_\theta} = \langle x, y \rangle_{L}^{M},$$

and

$$\langle x, y \rangle_{R}^{M_\theta} = \theta^{-1}(\langle x, y \rangle_{R}^{M}),$$

where $m \in M$, $j \in J$, $x, y \in MJ$, and $a \in A$.

As mentioned above, when $M$ is a $C^*$-algebra $A$, equipped with its usual structure of Hilbert $C^*$-bimodule over $A$, and $\theta \in \text{Aut}(A)$ the bimodule $A_\theta$ corresponds to the element of Pic($A$) denoted by $X_\theta^{-1}$ in [3GR, 3], so we have $A_\theta \otimes A_\sigma \cong A_{\theta \sigma}$ and $\tilde{A}_\theta \cong A_{\theta^{-1}}$ for all $\theta, \sigma \in \text{Aut}(A)$.

In this section we discuss the interdependence between the left and the right structure of a Hilbert $C^*$-bimodule. Proposition 1.1 shows that the right structure is determined, up to a partial isomorphism, by the left one. By specializing this result to the case of full Hilbert $C^*$-bimodules over a commutative $C^*$-algebra, we are able to describe Pic($A$) as the semidirect product of the classical Picard group of $A$ by the group of automorphisms of $A$.

**Proposition 1.1** Let $M$ and $N$ be Hilbert $C^*$-bimodules over a $C^*$-algebra $A$. If $\Phi : M \rightarrow N$ is an isomorphism of left $A$-Hilbert $C^*$-modules, then there is a partial automorphism $(I, J, \theta)$ of $A$ such that $\Phi : M_\theta \rightarrow N$ is an isomorphism of $A - A$ Hilbert $C^*$-bimodules. Namely, $I = \langle N, N \rangle_{R}$, $J = \langle M, M \rangle_{R}$ and $\theta(\langle \Phi(m_0), \Phi(m_1) \rangle_{R}) = \langle m_0, m_1 \rangle_{R}$.

**Proof:** Let $\Phi : M \rightarrow N$ be a left $A$-Hilbert $C^*$-module isomorphism. Notice that, if $m \in M$, and $\|m\| = 1$, then, for all $m_i, m'_i \in M$, and $i = 1, ..., n$: 

4
\[ \| \sum m \langle m_i, m'_i \rangle_R \| = \| \sum \langle m, m_i \rangle_L m'_i \| \]
\[ = \| \Phi(\sum \langle m, m_i \rangle_L m'_i) \| \]
\[ = \| \sum \langle \Phi(m), \Phi(m_i) \rangle_L \Phi(m'_i) \| \]
\[ = \| \sum \langle \Phi(m), \Phi(m_i) \rangle_L \Phi(m'_i) \| \]
\[ = \| \sum \Phi(m) \langle \Phi(m_i), \Phi(m'_i) \rangle_R \| . \]

Therefore:
\[ \| \sum \langle m_i, m'_i \rangle_R \| = \sup_{\| m \| = 1} \| \sum m \langle m_i, m'_i \rangle_R \| \]
\[ = \sup_{\| m \| = 1} \| \sum \Phi(m) \langle \Phi(m_i), \Phi(m'_i) \rangle_R \| \]
\[ = \| \sum \langle \Phi(m_i), \Phi(m'_i) \rangle_R \|. \]

Set \( I = \langle N, N \rangle_R \), and \( J = \langle M, M \rangle_R \), and let \( \theta : I \rightarrow J \) be the isometry defined by
\[ \theta(\langle \Phi(m_1), \Phi(m_2) \rangle_R) = \langle m_1, m_2 \rangle_R , \]
for \( m_1, m_2 \in M \). Then,
\[ \theta(\langle \Phi(m_1), \Phi(m_2) \rangle_R^*) = \theta(\langle \Phi(m_2), \Phi(m_1) \rangle_R) \]
\[ = \langle m_2, m_1 \rangle_R \]
\[ = \langle m_1, m_2 \rangle_R^* \]
\[ = \theta(\langle \Phi(m_1), \Phi(m_2) \rangle_R)^* , \]
and
\[ \theta(\langle \Phi(m_1), \Phi(m_2) \rangle_R \langle \Phi(m'_1), \Phi(m'_2) \rangle_R) = \theta(\langle \Phi(m_1), \Phi(m_2) \langle \Phi(m'_1), \Phi(m'_2) \rangle_R \rangle_R) \]
\[ = \theta(\langle \Phi(m_1), \langle \Phi(m_2), \Phi(m'_1) \rangle_L \Phi(m'_2) \rangle_R) \]
\[ = \langle m_1, \langle \Phi(m_2), \Phi(m'_1) \rangle_L m'_2 \rangle_R \]
\[ = \langle m_1, m_2 \rangle_R \langle m'_1, m'_2 \rangle_R \]
\[ = \theta(\langle m_1, m_2 \rangle_R) \theta(\langle m'_1, m'_2 \rangle_R) , \]
which shows that \( \theta \) is an isomorphism.

Besides, \( \Phi : M_\theta \rightarrow N \) is a Hilbert \( C^* \)-bimodule isomorphism:
\[ \Phi(\langle m_1, m_2 \rangle_R \cdot a) = \Phi(m\theta^{-1}(\langle m_1, m_2 \rangle_R)a) = \Phi(m\theta(\langle \Phi(m_1), \Phi(m_2)a \rangle_R)) = \Phi(\langle m, m_1 \rangle_L \Phi^{-1}(\Phi(m_2)a)) = \langle m, m_1 \rangle_L \Phi(m_2)a = \Phi(\langle m, m_1 \rangle_L m_2)a = \Phi(m_0 \langle m_1, m_2 \rangle_R)a, \]

and
\[ \langle \Phi(m_1), \Phi(m_2) \rangle_R = \theta^{-1}(\langle m_1, m_2 \rangle_M) = \langle m_1, m_2 \rangle^{M^\theta}. \]

Finally, \( \Phi \) is onto because
\[ \Phi(M_\theta) = \Phi(M \langle M, M \rangle_R) = \Phi(M) = N. \]

Q.E.D.

**Corollary 1.2** Let \( M \) and \( N \) be Hilbert \( C^* \)-bimodules over a \( C^* \)-algebra \( A \), and let \( \Phi : M \rightarrow N \) be an isomorphism of left Hilbert \( C^* \)-modules. Then \( \Phi \) is an isomorphism of Hilbert \( C^* \)-bimodules if and only if \( \Phi \) preserves either the right inner product or the right \( A \)-action.

**Proof:** Let \( \theta \) be as in Proposition 1.1, so that \( \Phi : M_\theta \rightarrow N \) is a Hilbert \( C^* \)-bimodule isomorphism. If \( \Phi \) preserves the right inner product, then \( \theta \) is the identity map on \( \langle M, M \rangle_R \) and \( M_\theta = M \).

If \( \Phi \) preserves the right action of \( A \), then, for \( m_0, m_1, m_2 \in M \) we have:
\[ \Phi(m_0)\langle \Phi(m_1), \Phi(m_2) \rangle_R = \langle \Phi(m_0), \Phi(m_1) \rangle_L \Phi(m_2) = \langle m_0, m_1 \rangle_L \Phi(m_2) = \Phi(m_0 \langle m_1, m_2 \rangle_R) = \Phi(m_0)\langle m_1, m_2 \rangle_R, \]
so \( \Phi \) preserves the right inner product as well.

Q.E.D.
Proposition 1.3 Let $M$ and $N$ be left Hilbert $C^*$-modules over a $C^*$-algebra $A$. If $M$ and $N$ are isomorphic as left $A$-modules, and $K(M)$ is unital, then $M$ and $N$ are isomorphic as left Hilbert $C^*$-modules.

Proof: First recall that any $A$-linear map $T : M \rightarrow N$ is adjointable. For if $m_i, m'_i \in M$, $i = 1, ..., n$ are such that $\sum \langle m_i, m'_i \rangle_R = 1_{K(M)}$, then for any $m \in M$:

$$T(m) = T(\sum \langle m, m_i \rangle_L m'_i) = \sum \langle m, m_i \rangle_L T(m'_i) = (\sum \xi_{m,n}(m))(m),$$

where $\xi_{m,n} : M \rightarrow N$ is the compact operator (see, for instance, [La, 1]) defined by $\xi_{m,n}(m_0) = \langle m_0, m \rangle_L n$, for $m \in M$, and $n \in N$, which is adjointable.

Let $T : M \rightarrow N$ be an isomorphism of left modules, and set $S : M \rightarrow N$, $S = T(T^*T)^{-1/2}$. Then $S$ is an $A$-linear map, therefore adjointable. Furthermore, $S$ is a left Hilbert $C^*$-module isomorphism: if $m_0, m_1 \in M$, then

$$\langle S(m_0), S(m_1) \rangle_L = \langle T(T^*T)^{-1/2}m_0, T(T^*T)^{-1/2}m_1 \rangle_L$$

$$= \langle m_0, (T^*T)^{-1/2}T(T^*T)^{-1/2}m_1 \rangle_L$$

$$= \langle m_0, m_1 \rangle_L.$$

Q.E.D.

We next discuss the Picard group of a $C^*$-algebra $A$. Proposition 1.1 shows that the left structure of a full Hilbert $C^*$-bimodule over $A$ is determined, up to an isomorphism of $A$, by its left structure.

This suggests describing $\text{Pic}(A)$ in terms of the subgroup $\text{Aut}(A)$ together with a cross-section of the equivalence classes under left Hilbert $C^*$-modules isomorphisms. When $A$ is commutative there is a natural choice for this cross-section: the family of symmetric Hilbert $C^*$-bimodules (see Definition 1.5). That is the reason why we now concentrate on commutative $C^*$-algebras and their symmetric Hilbert $C^*$-bimodules.

Proposition 1.4 Let $A$ be a commutative $C^*$-algebra and $M$ a Hilbert $C^*$-bimodule over $A$. Then $\langle m, n \rangle_{LM} = \langle p, n \rangle_L m$ for all $m, n, p \in M$. 

7
**Proof:** We first prove the proposition for $m = n$, the statement will then follow from polarization identities.

Let $m, p \in M$, then:

\[
\langle \langle m, m \rangle_L p - \langle p, m \rangle_L m, \langle m, m \rangle_L p - \langle p, m \rangle_L m \rangle_L = 0.
\]

Now, for $m, n, p \in M$, we have:

\[
\langle m, n \rangle_L p = \frac{1}{4} \sum_{k=0}^{3} i^k \langle m + i^k n, m + i^k n \rangle_L p
\]

\[
= \frac{1}{4} \sum_{k=0}^{3} i^k \langle p, m + i^k n \rangle_L (m + i^k n)
\]

\[
= \frac{1}{4} \sum_{k=0}^{3} i^k p (m + i^k n, m + i^k n)_R
\]

\[
= p (n, m)_R
\]

\[
= \langle p, n \rangle_L m.
\]

**Definition 1.5** Let $A$ be a commutative $C^*$-algebra. A Hilbert $C^*$-bimodule $M$ over $A$ is said to be symmetric if $am = ma$ for all $m \in M$, and $a \in A$. 
If \( M \) is a Hilbert C*-bimodule over \( A \), the symmetrization of \( M \) is the symmetric Hilbert C*-bimodule \( M^* \), whose underlying vector space is \( M \) with its given left Hilbert-module structure, and right structure defined by:

\[
m \cdot a = am, \quad \langle m_0, m_1 \rangle^{M^*}_L = \langle m_1, m_0 \rangle^M_L,
\]

for \( a \in A, m, m_0, m_1 \in M^* \). The commutativity of \( A \) guarantees the compatibility of the left and right actions. As for the inner products, we have, in view of Proposition 1.4:

\[
\langle m_0, m_1 \rangle^{M^*}_L \cdot m_2 = \langle m_0, m_1 \rangle^M_L m_2
\]

\[
= \langle m_2, m_1 \rangle^{M^*}_L m_0
\]

\[
= m_0 \cdot \langle m_2, m_1 \rangle^M_L
\]

\[
= m_0 \cdot \langle m_1, m_2 \rangle^{M^*}_R
\]

for all \( m_0, m_1, m_2 \in M^* \).

Remark 1.6 By Corollary 1.2 the bimodule \( M^* \) is, up to isomorphism, the only symmetric Hilbert C*-bimodule that is isomorphic to \( M \) as a left Hilbert module.

Remark 1.7 Let \( M \) be a symmetric Hilbert C*-bimodule over a commutative C*-algebra \( A \) such that \( K(A) \) is unital. By Remark 1.6 and Proposition 1.3, a symmetric Hilbert C*-bimodule over \( A \) is isomorphic to \( M \) if and only if it is isomorphic to \( M \) as a left module.

Example 1.8 Let \( A = C(X) \) be a commutative unital C*-algebra, and let \( M \) be a Hilbert C*-bimodule over \( A \) that is, as a left Hilbert C*-module, isomorphic to \( A^p \), for some \( p \in \text{Proj}(M_n(A)) \). This implies that \( pM_n(A)p \cong K(A) \) is isomorphic to a C*-subalgebra of \( A \) and is, in particular, commutative. By viewing \( M_n(A) \) as \( C(X, M_n(C)) \) one gets that \( p(x)M_n(C)p(x) \) is a commutative C*-algebra, hence rank \( p(x) \leq 1 \) for all \( x \in X \).
Conversely, let $A = C(X)$ be as above, and let $p : X \to \text{Proj}(M_n(C))$ be a continuous map, such that $\text{rank} \, p(x) \leq 1$ for all $x \in X$. Then $A^np$ is a Hilbert $C^*$-bimodule over $A$ for its usual left structure, the right action of $A$ by pointwise multiplication, and right inner product given by:

$$\langle m, r \rangle_L = \tau(m^*r),$$

for $m, r \in A^np, a \in A$, and where $\tau$ is the usual $A$-valued trace on $M_n(A)$ (that is, $\tau([a_{ij}]) = \sum a_{ii}$).

To show the compatibility of the inner products, notice that for any $T \in M_n(A)$, and $x \in X$ we have:

$$(pTp)(x) = p(x)T(x)p(x) = [\text{trace}(p(x)T(x)p(x))](x),$$

which implies that $pTp = \tau(pTp)p$. Then, for $m, r, s \in M$:

$$\langle m, r \rangle_L s = mpr^*sp = m\tau(pr^*sp)p = m\tau(r^*s) = m \cdot \langle r, s \rangle_R.$$

Besides, $A^np$ is symmetric:

$$\langle m, r \rangle_R = \tau(m^*r) = \sum_{i=1}^n m_i^*r_i = \langle r, m \rangle_L,$$

for $m = (m_1, m_2, ..., m_n), r = (r_1, r_2, ...r_n) \in M$.

Therefore, by Remark [L], if $p, q \in \text{Proj}(M_n(A))$, the Hilbert $C^*$-bimodules $A^np$ and $A^nq$ described above are isomorphic if and only if $p$ and $q$ are Murray-von Neumann equivalent. Notice that the identity of $K(A^np)$ is $\tau(p)$, that is, the characteristic function of the set $\{x \in X : \text{rank} \, p(x) = 1\}$. Therefore $A^np$ is full as a right module if and only if $\text{rank} \, p(x) = 1$ for all $x \in X$, which happens in particular when $X$ is connected, and $p \neq 0$.

**Proposition 1.9** Let $A$ be a commutative $C^*$-algebra. For any Hilbert $C^*$-bimodule $M$ over $A$ there is a partial automorphism $(\langle M, M \rangle_R, \langle M, M \rangle_L, \theta)$ of $A$ such that the map $i : (M^*)_\theta \to M$ defined by $i(m) = m$ is an isomorphism of Hilbert $C^*$-bimodules.
Proof: The map \( i : M^s \to M \) is a left Hilbert \( C^\ast \)-modules isomorphism. The existence of \( \theta \), with \( I = \langle M, M \rangle_R \) and \( J = \langle M^s, M^s \rangle_R = \langle M, M \rangle_L \), follows from Proposition \[ \text{[1.3]} \].

Q.E.D.

We now turn to the discussion of the group \( \text{Pic}(A) \) for a commutative \( C^\ast \)-algebra \( A \). For a full Hilbert \( C^\ast \)-bimodule \( M \) over \( A \), we denote by \([M]\) its equivalence class in \( \text{Pic}(A) \). For a commutative \( C^\ast \)-algebra \( A \), the group \( \text{Gin}(A) \) is trivial, so the map \( \alpha \mapsto A_\alpha \) is one-to-one. In what follows we identify, via that map, \( \text{Aut}(A) \) with a subgroup of \( \text{Pic}(A) \).

Symmetric full Hilbert \( C^\ast \)-bimodules over a commutative \( C^\ast \)-algebra \( A = C(X) \) are known to correspond to line bundles over \( X \). The subgroup of \( \text{Pic}(A) \) consisting of isomorphism classes of symmetric Hilbert \( C^\ast \)-bimodules is usually called the classical Picard group of \( A \), and will be denoted by \( \text{CPic}(A) \). We next specialize the result above to the case of full bimodules.

Notation 1.10 For \( \alpha \in \text{Aut}(A) \), and \( M \) a Hilbert \( C^\ast \)-bimodule over \( A \), we denote by \( \alpha(M) \) the Hilbert \( C^\ast \)-bimodule \( \alpha(M) = A_\alpha \otimes M \otimes A_{\alpha^{-1}} \).

Remark 1.11 The map \( a \otimes m \otimes b \mapsto amb \) identifies \( A_\alpha \otimes M \otimes A_{\alpha^{-1}} \) with \( M \) equipped with the actions:

\[ a \cdot m = \alpha^{-1}(a)m, \quad m \cdot a = m\alpha^{-1}(a), \]

and inner products

\[ \langle m_0, m_1 \rangle_L = \alpha(\langle m_0, m_1 \rangle^M_L), \]

and

\[ \langle m_0, m_1 \rangle_R = \alpha(\langle m_0, m_1 \rangle^M_R), \]

for \( a \in A \), and \( m, m_0, m_1 \in M \).
Theorem 1.12 Let $A$ be a commutative $C^*$-algebra. Then $CPic(A)$ is a normal subgroup of $Pic(A)$ and

$$Pic(A) = CPic(A) \rtimes Aut(A),$$

where the action of $Aut(A)$ is given by conjugation, that is $\alpha \cdot M = \alpha(M)$.

Proof: Given $[M] \in Pic(A)$ write, as in Proposition 1.9, $M \cong M_\theta^s$, $\theta$ being an isomorphism from $\langle M, M \rangle_R = A$ onto $\langle M, M \rangle_L = A$.

Therefore $M \cong M^s \otimes A_\theta$, where $[M^s] \in CPic(A)$ and $\theta \in Aut(A)$. If $[S] \in CPic(A)$ and $\alpha \in Aut(A)$ are such that $M \cong S \otimes A_\alpha$, then $S$ and $M^s$ are symmetric bimodules, and they are both isomorphic to $M$ as left Hilbert $C^*$-modules. This implies, by Remark 1.11, that they are isomorphic. Thus we have:

$$M^s \otimes A_\theta \cong M^s \otimes A_\alpha \Rightarrow A_\theta \cong \overline{M^s} \otimes M^s \otimes A_\theta \cong \overline{M^s} \otimes M^s \otimes A_\alpha \cong A_\alpha,$$

which implies ([BGR, 3.1]) that $\theta \alpha^{-1} \in \text{Gin}(A) = \{id\}$, so $\alpha = \theta$, and the decomposition above is unique.

It only remains to show that $CPic(A)$ is normal in $Pic(A)$, and it suffices to prove that $[A_\alpha \otimes S \otimes A_\alpha^{-1}] \in CPic(A)$ for all $[S] \in CPic(A)$, and $\alpha \in Aut(A)$, which follows from Remark 1.11.

Q.E.D.

Notation 1.13 If $\alpha \in Aut(A)$, then for any positive integers $k, l$, we still denote by $\alpha$ the automorphism of $M_{k \times l}(A)$ defined by $\alpha[(a_{ij})] = (\alpha(a_{ij}))$.

Lemma 1.14 Let $A$ be a commutative unital $C^*$-algebra, and $p \in \text{Proj}(M_n(A))$ be such that $A^n p$ is a symmetric Hilbert $C^*$-bimodule over $A$, for the structure described in Example 1.8. If $\alpha \in Aut(A)$, then $\alpha(A^n p) \cong A^n \alpha(p)$. 

12
**Proof:** Set $J : \alpha(A^n p) \rightarrow A^n \alpha(p)$, $J(m \otimes x \otimes r) = m \alpha(xr)$, for $m \in A_\alpha$, $r \in A_{\alpha^{-1}}$, and $x \in A^n p$. Notice that

$$m \alpha(xr) = m \alpha(xpr) = m \alpha(xr) \alpha(p) \in A^n \alpha(p).$$

Besides, if $a \in A$

$$J(m \cdot a \otimes x \otimes r) = J(m \alpha(a) \otimes x \otimes r)$$
$$= m \alpha(xr)$$
$$= J(m \otimes a \cdot x \otimes r),$$

and

$$J(m \otimes x \cdot a \otimes r) = m \alpha(xar)$$
$$= J(m \otimes x \otimes a \cdot r),$$

so the definition above makes sense. We now show that $J$ is a Hilbert $C^*$-bimodule isomorphism. For $m \in A_\alpha$, $n \in A_{\alpha^{-1}}$, $x \in A^n p$, and $a \in A$, we have:

$$J(a \cdot (m \otimes x \otimes r)) = J(am \otimes x \otimes r)$$
$$= am \alpha(xr)$$
$$= a \cdot J(m \otimes x \otimes r),$$

and

$$J(m \otimes x \otimes r \cdot a) = m \alpha(xr \alpha^{-1}(a))$$
$$= m \alpha(xr) a$$
$$= J((m \otimes x \otimes r) \cdot a)$$

Finally,

$$\langle J(m \otimes x \otimes r), J(m' \otimes x' \otimes r') \rangle_L = \langle m \alpha(xr), m' \alpha(x'r') \rangle_L$$
$$= \langle m \cdot [(xr)(x'r')^*], m' \rangle_L$$
$$= \langle m \cdot \langle x \cdot (r', r'' \rangle_{A}^{A^n p}, x' \rangle_{A^n p}^{A^n p \otimes A_{\alpha^{-1}}}, m' \rangle_L$$
$$= \langle m \cdot \langle x \otimes r, x' \otimes r' \rangle_{L}^{A^n p \otimes A_{\alpha^{-1}}}, m' \rangle_L$$
$$= \langle m \otimes x \otimes r, m' \otimes x' \otimes r' \rangle_{L},$$

which shows, by Corollary 1.2, that $J$ is a Hilbert $C^*$-bimodule isomorphism.

Q.E.D.
Proposition 1.15 Let $A$ be a commutative unital $C^*$-algebra and $M$ a Hilbert $C^*$-bimodule over $A$. If $\alpha \in \text{Aut}(A)$ is homotopic to the identity, then

$$A_\alpha \otimes M \cong M \otimes A_{\gamma^{-1}\alpha \gamma},$$

where $\gamma \in \text{Aut}(A)$ is such that $M \cong (M^s)_\gamma$.

Proof: We then have that $K(AM)$ is unital so, in view of Proposition 1.3 we can assume that $M^s = A^np$ with the Hilbert $C^*$-bimodule structure described in Example 1.8, for some positive integer $n$, and $p \in \text{Proj}(M_m(A))$. Since $p$ and $\alpha(p)$ are homotopic, they are Murray-von Neumann equivalent ([Bl, 4]). Then, by Lemma 1.14 and Example 1.8, we have

$$A_\alpha \otimes M \cong A_\alpha \otimes M^s \otimes A_\gamma \cong M^s \otimes A_{\alpha \gamma} \cong M \otimes A_{\gamma^{-1}\alpha \gamma}.$$

Q.E.D.

We turn now to the discussion of crossed products by Hilbert $C^*$-bimodules, as defined in [AEE]. For a Hilbert $C^*$-bimodule $M$ over a $C^*$-algebra $A$, we denote by $A \rtimes_M \mathbb{Z}$ the crossed product $C^*$-algebra. We next establish some general results that will be used later.

Notation 1.16 In what follows, for $A - A$ Hilbert $C^*$-bimodules $M$ and $N$ we write $M \overset{cp}{\cong} N$ to denote $A \rtimes_M \mathbb{Z} \cong A \rtimes_N \mathbb{Z}$.

Proposition 1.17 Let $A$ be a $C^*$-algebra, $M$ an $A - A$ Hilbert $C^*$-bimodule and $\alpha \in \text{Aut}(A)$. Then

i) $M \overset{cp}{\cong} M$.

ii) $M \overset{cp}{\cong} \alpha(M)$.

Proof: Let $i_A$ and $i_M$ denote the standard embeddings of $A$ and $M$ in $A \rtimes_M \mathbb{Z}$, respectively.
i) Set

\[ i_M : \tilde{M} \longrightarrow A \rtimes M \mathbb{Z}, \quad i_M(\tilde{m}) = i_M(m)^*. \]

Then \((i_A, i_M)\) is covariant for \((A, M)\):

\[ i_M(a \cdot \tilde{m}) = i_M(m \alpha) = i_A(a) i_M(m)^* = i_A(a) i_M(\tilde{m}), \]

\[ i_M(\tilde{m}_1) i_M(\tilde{m}_2)^* = i_M(m_1)^* i_M(m_2) = i_A(\langle m_0, m_1 \rangle^M) = i_A(\langle m_0, m_1 \rangle^M), \]

for \(a \in A\) and \(m, m_0, m_1 \in M\). Analogous computations prove covariance on the right. By the universal property of the crossed products there is a homomorphism from \(A \rtimes \tilde{M} \mathbb{Z}\) onto \(A \rtimes M \mathbb{Z}\). Since \(\tilde{M} = M\), by reversing the construction above one gets the inverse of \(J\).

ii) Set

\[ j_A : A \longrightarrow A \rtimes M \mathbb{Z}, \quad j_{\alpha(M)} : M \longrightarrow A \rtimes M \mathbb{Z}, \]

defined by \(j_A = i_A \alpha^{-1}\), \(j_{\alpha(M)}(m) = i_M(m)\), where the sets \(M\) and \(\alpha(M)\) are identified as in Remark 1.11. Then \((j_A, j_{\alpha(M)})\) is covariant for \((A, \alpha(M))\):

\[ j_{\alpha(M)}(a \cdot m) = j_{\alpha(M)}(\alpha^{-1}(a)m) = i_A(\alpha^{-1}(a)) i_M(m) = j_A(a) i_{\alpha(M)}(m), \]

\[ j_{\alpha(M)}(m_0) j_{\alpha(M)}(m_1)^* = i_M(m_0)^* i_M(m_1)^* = i_A(\langle m_0, m_1 \rangle^M) = j_A(\langle m_0, m_1 \rangle^M), \]

for \(a \in A\), \(m, m_0, m_1 \in M\), and analogously on the right. Therefore there is a homomorphism

\[ J : A \rtimes \alpha(M) \mathbb{Z} \longrightarrow A \rtimes M \mathbb{Z}, \]

whose inverse is obtained by applying the construction above to \(\alpha^{-1}\).

Q.E.D.

2 An application: isomorphism classes for quantum Heisenberg manifolds.

For \(\mu, \nu \in \mathbb{R}\) and a positive integer \(c\), the quantum Heisenberg manifold \(D_{\mu \nu}^c \mathbb{R}^4\) is isomorphic \((\text{[AEE, Ex.3.3]})\) to the crossed product \(C(T^2) \rtimes (\mathbb{C}^*)_{\alpha_{\mu \nu}} \mathbb{Z}\),

\[ 15 \]
where \( X^c \) is the vector space of continuous functions on \( R \times T \) satisfying \( f(x + 1, y) = e^{-cy}f(x, y) \). The left and right actions of \( C(T^2) \) are defined by pointwise multiplication, the inner products by \( \langle f, g \rangle_L = \int f(x)g(x) \), and \( \langle f, g \rangle_R = \int \overline{f(y)}g(y) \), and \( \alpha_{\mu \nu} \in \text{Aut}(C(T^2)) \) is given by \( \alpha_{\mu \nu}(x, y) = (x + 2\mu, y + 2\nu) \), and, for \( t \in R \), \( e(t) = \exp(2\pi it) \).

Our purpose is to find isomorphisms in the family \( \{ D^c_{\mu \nu} : \mu, \nu \in R, c \in Z, c > 0 \} \). We concentrate in fixed values of \( c \), because \( K_0(D^c_{\mu \nu}) \cong Z^3 \oplus Z_2 \). Besides, since \( \alpha_{\mu \nu} = \alpha_{\mu+m, \nu+n} \) for all \( m, n \in Z \), we view from now on the parameters \( \mu \) and \( \nu \) as running in \( T \).

Let \( M^c \) denote the set of continuous functions on \( R \times T \) satisfying \( f(x + 1, y) = e^{-cy}f(x, y) \). Then \( M^c \) is a Hilbert \( C^* \)-bimodule over \( C(T^2) \), for pointwise action and inner products given by the same formulas as in \( X^c \).

The map \( f \mapsto \tilde{f} \), where \( \tilde{f}(x, y) = f(x, y + \nu) \), is a Hilbert \( C^* \)-bimodule isomorphism between \( (X^c_\nu)_{\alpha_{\mu \nu}} \) and \( C(T^2)_\sigma \otimes M^c \otimes C(T^2)_\rho \), where \( \sigma(x, y) = (x, y + \nu) \), and \( \rho(x, y) = (x + 2\mu, y + \nu) \). In view of Proposition 1.17 we have:

\[
D^c_{\mu \nu} \cong C(T^2) \rtimes_{\alpha_{\mu \nu}} M^c \cong C(T^2) \rtimes_{\alpha_{\mu \nu}} Z.
\]

As a left module over \( C(T^2) \), \( M^c \) corresponds to the module denoted by \( X(1, c) \) in \( [R3, 3.7] \). It is shown there that \( M^c \) represents the element \((1, c)\) of \( K_0(C(T^2)) \cong Z^2 \), where the last correspondence is given by \( [X] \mapsto (a, b) \), \( a \) being the dimension of the vector bundle corresponding to \( X \) and \(-b\) its twist. It is also proven in \( [R3] \) that any line bundle over \( C(T^2) \) corresponds to the left module \( M^c \), for exactly one value of the integer \( c \), and that \( M^c \otimes M^d \) and \( M^{c+d} \) are isomorphic as left modules. It follows now, by putting these results together, that the map \( c \mapsto [M^c] \) is a group isomorphism from \( Z \) to \( \text{CPic}(C(T^2)) \).

**Lemma 2.1**

\[
\text{Pic}(C(T^2)) \cong Z \rtimes_{\delta_\alpha} \text{Aut}(C(T^2)),
\]

where \( \delta_\alpha(c) = \det \alpha \cdot c \), for \( \alpha \in \text{Aut}(C(T^2)) \), and \( c \in Z \); \( \alpha \) being the usual automorphism of \( K_0(C(T^2)) \cong Z^2 \), viewed as an element of \( GL_2(Z) \).
Proof: By Theorem 1.12 we have:

$$\operatorname{Pic}(C(T^2)) \cong \operatorname{CPic}(C(T^2)) \rtimes \Delta \operatorname{Aut}(C(T^2)).$$

If we identify \( \operatorname{CPic}(C(T^2)) \) with \( \mathbb{Z} \) as above, it only remains to show that \( \alpha(M^c) \cong M^{\det \alpha \cdot c} \). Let us view \( \alpha_* \in GL_2(\mathbb{Z}) \) as above. Since \( \alpha_* \) preserves the dimension of a bundle, and takes \( C(T^2) \) (that is, the element \( (1, 0) \in \mathbb{Z}^2 \)) to itself, we have

$$\alpha_* = \begin{pmatrix} 1 & 0 \\ 0 & \det \alpha_* \end{pmatrix}$$

Now,

$$\alpha_*(M^c) = \alpha_*(1, c) = (1, \det \alpha_* \cdot c) = M^{\det \alpha \cdot c}.$$

Since there is cancellation in the positive semigroup of finitely generated projective modules over \( C(T^2) \) (\( \mathbb{R}_3 \)), the result above implies that \( \alpha_*(M^c) \) and \( M^{\det \alpha \cdot c} \) are isomorphic as left modules. Therefore, by Remark 1.7, they are isomorphic as Hilbert \( C^* \)-bimodules.

Q.E.D.

Theorem 2.2 If \((\mu, \nu)\) and \((\mu', \nu')\) belong to the same orbit under the usual action of \( GL(2, \mathbb{Z}) \) on \( T^2 \), then the quantum Heisenberg manifolds \( D^c_{\nu \mu} \) and \( D^{c}_{\nu' \mu'} \) are isomorphic.

Proof: If \((\mu, \nu)\) and \((\mu', \nu')\) belong to the same orbit under the action of \( GL(2, \mathbb{Z}) \), then \( \alpha_{\mu' \nu'} = \sigma \alpha_{\mu \nu} \sigma^{-1} \), for some \( \sigma \in GL(2, \mathbb{Z}) \). Therefore, by Lemma 2.1 and Proposition 1.17,

$$M^c_{\alpha_{\mu' \nu'}} \cong M^c_{\sigma \alpha_{\mu \nu} \sigma^{-1}} \cong M^c \otimes C(T^2)_{\sigma \alpha \sigma^{-1}} \cong$$

$$\cong C(T^2)_{\sigma} \otimes M^{\det \sigma^{-1} \cdot c} \otimes C(T^2)_{\alpha_{\mu \nu} \sigma^{-1}} \cong \sigma(M^{\det \sigma \cdot c}) \cong M^{\det \sigma \cdot c}_{\alpha_{\mu \nu}}.$$
In case $\det \alpha_* = -1$ we have

$$M^\text{det}_{\alpha_{\mu\nu}} \cong M_{\alpha_{\mu\nu}}^{-c} \cong \overline{M_{\alpha_{\mu\nu}}^{-c}} \cong C(T^2)_{\alpha_{\mu\nu}^{-1}} \otimes M^c \cong (M^c)_{\alpha_{\mu\nu}^{-1}},$$

since $\det \alpha_* = 1$, because $\alpha_{\mu\nu}$ is homotopic to the identity.

On the other hand, it was shown in \cite{Ab1, 0.3} that $M^c_{\alpha_{\mu\nu}^{-1}} \cong M^c_{\alpha_{\mu\nu}}$.

Thus, in any case, $M^c_{\alpha_{\mu\nu}'} \cong M^c_{\alpha_{\mu\nu}}$. Therefore

$$D^c_{\mu\nu'} \cong C(T^2) \rtimes M_{\alpha_{\mu\nu}'} \mathbb{Z} \cong C(T^2) \rtimes M_{\alpha_{\mu\nu}} \mathbb{Z} \cong D^c_{\mu\nu}.$$

Q.E.D.

References

\cite{Ab1} Abadie, B. “Vector bundles” over quantum Heisenberg manifolds. Algebraic Methods in Operator Theory, Birkhauser, pp. 307-315 (1994).

\cite{Ab2} Abadie, B. Generalized fixed-point algebras of certain actions on crossed-products. Pacific Journal of Mathematics, Vol 171, No.1, pp. 1-21 (1995).

\cite{AEE} Abadie, B.; Eilers, S.; Exel, R. Morita equivalence for crossed products by Hilbert $C^*$-bimodules. To appear in the Transactions of the AMS.

\cite{Bl} Blackadar, B. K-Theory of operator algebras. MSRI Publications, 5, Springer-Verlag, (1986).

\cite{BGR} Brown, G.; Green, P.; Rieffel, M. Stable isomorphism and strong Morita equivalence of $C^*$-algebras. Pacific Journal of Mathematics, Vol.71, Number 2, pp. 349-363 (1977).
[BMS] Brown, L.; Mingo, J. and Shen, N. *Quasi-multipliers and embeddings of Hilbert $C^*$-bimodules.* 
Canadian Journal of Mathematics, Vol. 46(6), pp. 1150-1174.

[DG] Dupré, M.J.; Gillette, R.M. *Banach bundles, Banach modules, and automorphisms of $C^*$-algebras.* 
Research Notes in Mathematics, v.92, Adv. Publ. Program, Pitman (1983).

[La] Lance, C. *Hilbert $C^*$-modules. A toolkit for operator algebraists.* 
Lecture Notes, University of Leeds (1993).

[Rf1] Rieffel, M. *Induced representations of $C^*$-algebras.* 
Advances in Mathematics, 13, No2, pp. 176-257 (1974).

[Rf2] Rieffel, M. *$C^*$-algebras associated with irrational rotations* 
Pacific Journal of Mathematics, Vol. 93, No. 2, pp. 415-429 (1981)

[Rf3] Rieffel, M. *The cancellation theorem for projective modules over irrational rotation $C^*$-algebras.* 
Proc. London Math. Soc. (3), 47 pp. 285-302 (1983).

[Rf4] Rieffel, M. *Deformation Quantization of Heisenberg manifolds.* 
Commun. Math. Phys. 122, pp. 531-562 (1989).

BA: Centro de Matemáticas, Facultad de Ciencias, Universidad de la República, 
Eduardo Acevedo 1139, C.P 11 200, Montevideo, Uruguay. E-mail address: 
abadie@@cmat.edu.uy

RE: Departamento de Matemática, Universidade de São Paulo, Cidade Universitária ”Armando de Salles Oliveira”. Rua do Matão 1010, CEP 05508-900, São Paulo, Brazil. E-mail address: exel@@ime.usp.br