LOCALISATION AND COLOCALISATION OF TRIANGULATED CATEGORIES AT THICK SUBCATEGORIES

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Abstract. Given a thick subcategory of a triangulated category, we define a co-localisation and a natural long exact sequence that involves the original category and its localisation and co-localisation at the subcategory. Similarly, we construct a natural long exact sequence containing the canonical map between a homological functor and its total derived functor with respect to a thick subcategory.

1. Introduction

Throughout this article, $\mathcal{T}$ is a triangulated category and $\mathcal{E}$ is a thick subcategory. The localisation of $\mathcal{T}$ at $\mathcal{E}$ is a triangulated category $\mathcal{T}/\mathcal{E}$ with a triangulated functor $L: \mathcal{T} \to \mathcal{T}/\mathcal{E}$ such that $L|_\mathcal{E} \cong 0$, and such that any other triangulated functor with this property factors uniquely through $L$. This universal property determines the localisation uniquely up to isomorphism if it exists. The localisation is constructed in [4], disregarding set-theoretic issues. These also appear in our context. We must assume that certain colimits exist, which follows, for instance, if the triangulated categories in question are essentially small.

Let $A[n]$ denote the $n$-fold suspension of $A$ and let $T_n(A, B) := T(A, B[n]) \cong T(A[-n], B)$, and similarly for $\mathcal{T}/\mathcal{E}$. We are going to embed the localisation functor $L: \mathcal{T} \to \mathcal{T}/\mathcal{E}$ into a natural long exact sequence

$$
(1.1) \quad \cdots \to \mathcal{T}/\mathcal{E}_1^\perp(A, B) \to T_1(A, B) \xrightarrow{L} \mathcal{T}/\mathcal{E}_1(LA, LB) \to \to \mathcal{T}/\mathcal{E}_0^\perp(A, B) \to T_0(A, B) \xrightarrow{L} \mathcal{T}/\mathcal{E}_0(LA, LB) \to \cdots,
$$

with $\mathcal{T}/\mathcal{E}_n^\perp(A, B) := \mathcal{T}/\mathcal{E}_n^\perp(A, B[n]) \cong \mathcal{T}/\mathcal{E}_n^\perp(A[-n], B)$. We call $\mathcal{T}/\mathcal{E}^\perp$ the colocalisation of $\mathcal{T}$ at $\mathcal{E}$. The naturality of (1.1) means that $\mathcal{T}/\mathcal{E}^\perp(A, B)$ is a bifunctor on $\mathcal{T}$ contravariant in the first and covariant in the second variable, and that the maps in the above exact sequence are natural transformations. Using the natural transformation $\mathcal{T}/\mathcal{E}^\perp \to \mathcal{T}$ in (1.1) and the $\mathcal{T}$-bimodule structure, $\mathcal{T}/\mathcal{E}^\perp$ is equipped with an associative composition product. It is not a category because it lacks identity morphisms.

Given an Abelian category $\mathcal{C}$ and a homological functor $F: \mathcal{T} \to \mathcal{C}$, its right localisation at $\mathcal{E}$ is a homological functor $\mathbb{R}F: \mathcal{T} \to \mathcal{C}$ together with a natural transformation $\mathbb{R}F \Rightarrow \mathbb{R}F$, such that $\mathbb{R}F$ vanishes on $\mathcal{E}$ and such that any other

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natural transformation $F \Rightarrow G$ for a homological functor $G$ with $G|_E = 0$ factors uniquely through $F \Rightarrow RF$. Left localisations are defined similarly, but they will not be treated here.

If $F$ is a homological functor to the category of Abelian groups, then our construction of the colocalisation functor also provides a long exact sequence

\[
\cdots \to \mathbb{R}^1 F_1(A) \to F_1(A) \to \mathbb{R}F_1(A) \to \mathbb{R}^2 F_0(A) \to F_0(A) \to \mathbb{R}F_0(A) \to \cdots
\]

which is functorial in $A$ and $F$; that is, a natural transformation of homological functors $F \Rightarrow F'$ and an arrow $A \to A'$ induce a chain map from (1.2) for $F_*(A)$ to (1.2) for $F'_*(A')$ in a functorial manner.

We call a homological functor $F : T \to \mathfrak{A}b$ local if the natural map $F \Rightarrow RF$ is invertible, and colocal if the map $\mathbb{R}^1 F \Rightarrow F$ is invertible. As we shall explain, local homological functors correspond to homological functors $T/E \to \mathfrak{A}b$, while colocal homological functors correspond to homological functors $E \to \mathfrak{A}b$. Thus (1.2) decomposes homological functors on $T$ into homological functors on $T/E$ and $E$.

Dual statements apply to cohomological functors $T \to \mathfrak{A}b$ because we may view them as homological functors $T^\text{op} \to \mathfrak{A}b$ on the opposite category $T^\text{op}$.

The exact sequence (1.1) is easy to construct in the special situation of a complementary pair of thick subcategories in the notation of [3]. (In [4], this is called a Bousfield localisation.) Recall that $(E^\perp, E)$ is a complementary pair of thick subcategories if $T(E^\perp, E) = 0$ and for any object $B$ of $T$ there is an exact triangle

\[
L^1 B \to B \to LB \to L^1 B[1]
\]

with $LB \in E^\perp$, $L^1 B \in E$. Here we write $\in$ for objects of categories, as opposed to $\in$ for morphisms of categories. The triangle (1.3) is unique up to canonical isomorphism and depends functorially on $B$. Since

\[
T/E(A, B) \cong T(A, LB) \cong T(LA, LB), \quad RF(B) \cong F(LB),
\]

we may define

\[
T/E^\perp(A, B) := T(A, L^1 B), \quad \mathbb{R}^1 F(B) := F(L^1 B).
\]

Then the exact sequences (1.1) and (1.2) follow by applying the homological functors $T\bigwedge B$ and $F$ to the exact triangle (1.3).

Without $E^\perp$, we cannot single out a unique exact triangle as in (1.3). Instead, we construct the localisation and the colocalisation as colimits of $T(A, D)$ and $T(A, C)$, where $C$ and $D$ are indexed by the category of exact triangles $C \to B \to D \to C[1]$ with $C \in E$. The main issue is to prove that this category of exact triangles is filtered, so that the colimit preserves exact sequences.

A typical example where the general construction of the colocalisation is useful is considered in [2]. There $T(A, B) = KK(A, B)$ is Kasparov’s bivariant K-theory for $C^*$-algebras $A$ and $B$, and $T/E(A, B) = KK(A, B) \otimes \mathbb{Q}$. The colocalisation provides a torsion variant of KK-theory. It can be shown that $T/E$ is the localisation of KK at a suitable thick subcategory. It is unclear, however, whether this is part of a complementary pair of thick subcategories.

This article is organised as follows. Section [2] recalls the well-known construction of the localisation and its basic properties. Similar techniques are used in Section [3] to define and study colocalisations. Section [4] establishes the exact sequences (1.1) and (1.2). Each section uses a particular filtered category.
2. Localisation

We fix a triangulated category $\mathcal{T}$ and a thick subcategory $\mathcal{E}$ throughout. The cone of a morphism $f: A \to B$ in $\mathcal{T}$ is the object $C$ in an exact triangle

$$A \xrightarrow{f} B \to C \to A[1]$$

A morphism $f$ in $\mathcal{T}$ is an $\mathcal{E}$-weak equivalence if its cone belongs to $\mathcal{E}$. Let $\text{we}_\mathcal{E}(A, B)$ be the set of $\mathcal{E}$-weak equivalences from $A$ to $B$.

Recall that an object of $\mathcal{T}$ becomes zero in $\mathcal{T}/\mathcal{E}$ if and only if it belongs to $\mathcal{E}$, and an arrow in $\mathcal{T}$ becomes invertible in $\mathcal{T}/\mathcal{E}$ if and only if it is an $\mathcal{E}$-weak equivalence (see [1]). Thus $\text{we}_\mathcal{E}$ is closed under composition and contains all isomorphisms in $\mathcal{T}$. Even more, $\text{we}_\mathcal{E}$ has the two-out-of-three and even the two-out-of-six property:

**Lemma 2.1.** Let $f$, $g$ and $h$ be composable morphisms in $\mathcal{T}$. If $fg$ and $gh$ are $\mathcal{E}$-weak equivalences, so are $f$, $g$, $h$, and $fgh$.

If two of $f$, $g$, and $fg$ are $\mathcal{E}$-weak equivalences, so is the third.

**Proof.** The assumption means that $L(fg)$ and $L(gh)$ are invertible in $\mathcal{T}/\mathcal{E}$. This implies that $L(g)$ has both a left and a right inverse, so that $L(g)$ is invertible. Then $L(f)$ and $L(h)$ must also be invertible, and finally $L(fgh)$ is invertible. The second statement follows from the first by considering the special cases where one of $f$, $g$ or $h$ is an identity (see also [1] Lemma 1.5.6) for a more direct proof of the two-out-of-three property. □

The construction of the localisation $\mathcal{T}/\mathcal{E}$ in [1] shows that its underlying category is isomorphic to the localisation of $\mathcal{T}$ at $\text{we}_\mathcal{E}$ in the category theoretic sense of [1]. To prepare for this, it is shown that $\text{we}_\mathcal{E}$ allows a calculus of left fractions in the sense of [1, §2.2], that is, it satisfies the following conditions (LF1)–(LF3):

**Lemma 2.2.** Let $\mathcal{E}$ be a thick subcategory of a triangulated category $\mathcal{T}$.

(LF1) Identities belong to $\text{we}_\mathcal{E}$, and $\text{we}_\mathcal{E}$ is closed under composition.

(LF2) For each pair of morphisms $B \xleftarrow{s} A \xrightarrow{f} C$ with $s \in \text{we}_\mathcal{E}$ there are $g \in \mathcal{T}$ and $t \in \text{we}_\mathcal{E}$ with $gs = tf$, that is, there is a commuting square

$$\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow{s} & \sim & \downarrow{t} \\
B & \xrightarrow{g} & D.
\end{array}$$

(LF3) If $f, g: A \xrightarrow{\sim} B$ are parallel morphisms in $\mathcal{T}$ and $s: A' \to A$ is in $\text{we}_\mathcal{E}$ and satisfies $fs = gs$, then there is $t: B \to B'$ in $\text{we}_\mathcal{E}$ with $tf = tg$.

**Proof.** (LF1) is obvious, and (LF2) follows from [1] Lemma 1.5.8 and the Octahedral Axiom. We only have to prove (LF3). We construct the following commuting diagram:

$$\begin{array}{ccc}
A' & \xrightarrow{s} & A & \xrightarrow{f} & C \\
\downarrow{0} & & \downarrow{t-g} & & \downarrow{t} \\
B & \sim & B & \sim & B
\end{array}$$
First we embed $s : A' \to A$ in an exact triangle $A' \xrightarrow{s} A \to C \to A'[1]$. Then $C \in \mathcal{E}$ because $s \in \text{we}_\mathcal{E}$. Since $(f - g) \circ s = 0$ and $T(\mathcal{A}, B)$ is cohomological, we may factor $f - g$ through the map $A \to C$. We embed the resulting map $C \to B$ into an exact triangle $C \to B \xrightarrow{t} B' \to C[1]$. Since $C \in \mathcal{E}$, $t \in \text{we}_\mathcal{E}$. By construction $t \circ (f - g) = 0$.

Dually, there is a calculus of right fractions for the same reasons. It is useful for us to rewrite $T/\mathcal{E}(A, B)$ as a filtered colimit. For a fixed object $B \in \mathcal{T}$, we consider the category $B \downarrow \text{we}_\mathcal{E}$ whose objects are arrows $B \to C$ in $\text{we}_\mathcal{E}$ and whose arrows are commuting triangles:

$$
\begin{array}{ccc}
C & \xrightarrow{r} & C' \\
\downarrow{s} & & \downarrow{t} \\
B & & \\
\end{array}
$$

where $s$ and $t$ are morphisms in $\text{we}_\mathcal{E}$ and $r$ is a morphism in $\mathcal{T}$. By the two-out-of-three property, we also get $r \in \text{we}_\mathcal{E}$.

Recall that a category is called filtered if it satisfies the following conditions:

- (F1) for two objects $E$ and $D$ there are an object $H$ and morphisms $f : E \to H$ and $g : D \to H$;
- (F2) for a pair of parallel morphisms $h, j : A \Rightarrow B$ there is a morphism $l : B \to F$ with $lh = lj$.

Since $\text{we}_\mathcal{E}$ allows a calculus of left fractions by Lemma 2.2, the category $B \downarrow \text{we}_\mathcal{E}$ is filtered for each $B$. (The proof that (LF1)–(LF3) imply (F1) and (F2) is an easy exercise.)

The map $s \mapsto T(A, C)$ for $(s : B \to C) \in B \downarrow \text{we}_\mathcal{E}$ defines a covariant functor from $B \downarrow \text{we}_\mathcal{E}$ to the category of sets. The colimit of this diagram yields the space of arrows $T/\mathcal{E}(A, B)$ in the localisation of $\mathcal{T}$ at $\text{we}_\mathcal{E}$:

$$
(2.1) \quad T/\mathcal{E}(A, B) := \lim_{(s : B \to C) \in (B \downarrow \text{we}_\mathcal{E})} T(A, C).
$$

The existence of these colimits for all $A$ and $B$ ensures that the localisation exists. Recall that any small diagram of Abelian groups has a colimit. If $\mathcal{T}$ is (essentially) small, then so is $B \downarrow \text{we}_\mathcal{E}$, and hence the colimit in (2.1) exists. There are many examples where $\mathcal{T}$ is large but the colimits in (2.1) exist for different reasons.

For instance, if $\mathcal{E}$ is part of a complementary pair of thick subcategories $(\mathcal{E}^\perp, \mathcal{E})$ in the notation of [3], then the arrow $B \to LB$ in (1.3) is a final object of $B \downarrow \text{we}_\mathcal{E}$. Thus the colimit above exists and reduces to $T/\mathcal{E}(A, B) \cong T(A, LB)$. Conversely, if $B \downarrow \text{we}_\mathcal{E}$ has a final object for each $B$, then $(\mathcal{E}^\perp, \mathcal{E})$ is a complementary pair of thick subcategories.

**Lemma 2.3.** The inductive system $(C)_s : B \to C \in B \downarrow \text{we}_\mathcal{E}$ depends functorially on $B$.

**Proof.** Let $f : B \to B'$ be a morphism in $\mathcal{T}$. By (LF1), we associate to an element $s : B \to C$ an element $t : B' \to C'$ of $B' \downarrow \text{we}_\mathcal{E}$ and a map $g : C \to C'$ with $gs = tf$. It follows from the calculus of left fractions that this defines a morphism of inductive systems that does not depend on our auxiliary choices of $t$ and $g$.

Alternatively, morphisms of inductive systems $(C_s) \to (C'_s)$ correspond bijectively to natural transformations $T(X, (C_s)) \to T(X, (C'_s))$. Since $T(X, (C_s)) =$
Our statement is equivalent to the existence of a natural composition product
\[ T/E(X, B) \times T(B, B') \to T/E(X, B'). \]
This exists because \( T/E \) is a category and the maps \( T \to T/E \) are a functor.

**Definition 2.4.** Let \( F: T \to \text{Ab} \) be a homological functor to the category of Abelian groups. Its right derived functor or right localisation \( RF: T \to \text{Ab} \) is defined by
\[
RF(B) := \lim_{(s: B \to C) \in B \setminus \text{weg}} F(C).
\]
This is a functor by Lemma 2.3. The maps \( s: B \to C \) provide a natural transformation \( F \Rightarrow RF \).

**Theorem 2.5.** Let \( T \) be a triangulated category, \( E \) a thick subcategory, and \( F: T \to \text{Ab} \) a homological functor. Then the right localisation \( RF: T \to \text{Ab} \) is homological.

**Proof.** Let \( B' \xrightarrow{f} B \xrightarrow{g} B'' \to B'[1] \) be an exact triangle. We must show that
\[
RF(B') \xrightarrow{RF(f)} RF(B) \xrightarrow{RF(g)} RF(B'')
\]
is an exact sequence of groups. Functoriality of \( RF \) already implies \( \text{im } RF(f) \subseteq \ker RF(g) \). It remains to prove \( \ker RF(f) \subseteq \ker RF(g) \).

Let \( s \in \text{weg}(B, C) \) and \( x \in F(C) \) represent an element in \( \ker RF(g) \). By the definition of the functoriality of \( RF \), this means that there is a commuting diagram
\[
\begin{array}{ccc}
C & \xrightarrow{\hat{g}} & C'' \\
\downarrow{s} & & \downarrow{\hat{g}''} \\
B & \xrightarrow{g} & B''
\end{array}
\]
with \( s'' \in \text{weg} \) and \( F(\hat{g})(x) = 0 \). By the properties of triangulated categories, we may complete this to a morphism of exact triangles
\[
\begin{array}{ccc}
C' & \xrightarrow{f} & C & \xrightarrow{\hat{g}} & C'' & \xrightarrow{C'[1]} \\
\downarrow{s'} & & \downarrow{s} & & \downarrow{\hat{g}''} & & \downarrow{C'[1]} \\
B' & \xrightarrow{f} & B & \xrightarrow{g} & B'' & \xrightarrow{B'[1]}
\end{array}
\]
Since \( s, s'' \in \text{weg} \), we also get \( s' \in \text{weg} \) from the Octahedral Axiom. Since \( F \) is homological, there is an element \( x' \in F(C') \) with \( F(f)(x') = x \). Therefore, the class of the pair \( (s', x') \) in \( RF(B') \) maps to the class of \( (s, x) \). \( \square \)

Recall that the category \( T^{\text{op}} \) is again triangulated, with new suspension automorphism \( A \to A[-1] \) and the same exact triangles: \( A[1] \leftarrow C \leftarrow B \leftarrow A \) is an exact triangle in \( T^{\text{op}} \) if \( A \to B \to C \to A[1] \) is an exact triangle in \( T \). Thus we may view a cohomological functor \( T \to \text{Ab} \) as a homological functor \( T^{\text{op}} \to \text{Ab} \).

**Definition 2.4.** applied to a homological functor \( T^{\text{op}} \to \text{Ab} \) yields a right localisation for the original cohomological functor \( T \to \text{Ab} \).

Theorem 2.5 may be extended to the case where \( F \) is a homological functor with values in an Abelian category \( C \) with exact filtered colimits. However, some assumption on \( C \) seems necessary. In particular, it is unclear how to treat homological
functors $T \to \mathbb{Ab}^{\text{op}}$ because filtered colimits in $\mathbb{Ab}^{\text{op}}$ are filtered limits in $\mathbb{Ab}$, and these need not be exact.

By definition, the functor $B \mapsto T/E(A, B)$ is the right localisation of $B \mapsto T(A, B)$. Therefore, this functor is homological by Theorem 2.5. Since filtered colimits are exact, the functor $A \mapsto T/E(A, B)$ is cohomological by (2.1). Of course, this also follows from the stronger statement that $T/E$ is a triangulated category.

We may also apply the construction above to get the localisation of $T^{\text{op}}$ at $E^{\text{op}}$. The opposite category of this localisation provides another model for $T/E$ that is based on a calculus of right fractions instead of left fractions. Both constructions agree because both localisations share the same universal property.

**Proposition 2.6.** Let $F: T \to \mathbb{Ab}$ be a homological functor. The following assertions are equivalent:

1. the natural transformation $F \Rightarrow RF$ is invertible;
2. $F(E) \cong 0$ for all $E \in E$;
3. $F(s)$ is invertible for all $s \in \text{we}E$;
4. $F$ factors through a homological functor $T/E \to \mathbb{Ab}$.

Furthermore, $RF$ always satisfies these equivalent conditions.

**Proof.** If $E \in E$, then the zero map $E \to 0$ is an $E$-weak equivalence. It is a final object in $E \downarrow \text{we}E$, so that $RF(E) = F(0) = 0$. Thus $RF(E) = 0$ always satisfies (b), and hence (a) implies (b). If $s \in \text{we}E$, then $s$ is part of an exact triangle $A \xrightarrow{s} B \to E \to A[1]$ with $E \in E$. The long exact sequence for $F$ applied to this exact triangle shows that $F_s(s)$ is invertible if and only if $F_s(E) \cong 0$. Since $\text{we}E$ and $E$ are closed under suspensions, this yields (b) $\iff$ (c). Since $T/E$ is the localisation of $T$ at $\text{we}E$ in the sense of category theory, a functor on $T$ factors through $T/E$ if and only if it maps all arrows in $\text{we}E$ to invertible arrows. Furthermore, the functor on $T/E$ induced in this way is again homological if $F$ was (see [4]). Thus (c) $\iff$ (d). Finally, (c) implies that the maps $F(B) \to F(C)$ are invertible for all $s: B \to C$ in $B \downarrow \text{we}E$. Thus (c) implies (a). \hfill $\square$

There is an analogous result for cohomological functors.

A (co)homological functor with the equivalent properties in Proposition 2.6 is called *local*. Condition (d) means that local (co)homological functors $T \to \mathbb{Ab}$ are equivalent to local (co)homological functors $T/E \to \mathbb{Ab}$.

**Proposition 2.7.** The localisation $RF$ is the universal local homological functor on $T$ equipped with a natural transformation $F \Rightarrow RF$: if $G$ is any local homological functor on $T$, then there is a natural bijection between natural transformations $F \Rightarrow G$ and natural transformations $RF \Rightarrow G$.

This universal property characterises $RF$ uniquely up to natural isomorphism.

**Proof.** If $G$ is local, then a natural transformation $\Phi: F \Rightarrow G$ induces a natural transformation $\Phi': RF \Rightarrow RG \cong G$. The product of $\Phi'$ with the natural transformation $\Psi: F \Rightarrow RF$ is again $\Phi$, and $\Phi'$ is the only natural transformation $RF \Rightarrow G$ with $\Phi' \circ \Psi = \Phi$. \hfill $\square$
3. Colocalisation

Let $\mathcal{E} \downarrow B$ be the category, whose objects are arrows $f: E \to B$ with $E \in \mathcal{E}$ and whose arrows are commuting triangles

$$
\begin{array}{ccc}
E & \xrightarrow{f} & B \\
\downarrow{r} & & \downarrow{f'} \\
E' & \xrightarrow{f'} & B
\end{array}
$$

**Lemma 3.1.** The category $\mathcal{E} \downarrow B$ is filtered for all $B \in T$.

**Proof.** The first axiom (F1) of a filtered category follows easily because $\mathcal{E}$ is additive: any pair of maps $E_1 \to B$, $E_2 \to B$ is dominated by $E_1 \oplus E_2 \to B$. To verify (F2), we must equalise a diagram of the form

$$
\begin{array}{ccc}
E_1 & \xrightarrow{f_1} & B \\
\downarrow{r} & & \downarrow{r'} \\
E_2 & \xrightarrow{f_2} & B
\end{array}
$$

Embed $r-r'$ in an exact triangle $E_1 \xrightarrow{r-r'} E_2 \xrightarrow{f_2} E_3 \xrightarrow{f_3} E_1$ [1]. Since $f_2 \circ (r-r') = 0$ and $T(A, B)$ is cohomological, we may factor $f_2$ through the map $E_2 \to E_3$. This yields a commuting diagram

$$
\begin{array}{ccc}
E_1 & \xrightarrow{f_1} & B \\
\downarrow{r} & & \downarrow{f} \\
E_2 & \xrightarrow{f_2} & B, \\
\downarrow{r'} & & \downarrow{f_3} \\
E_3 & &
\end{array}
$$

which equalises the diagram (3.1) in $\mathcal{E} \downarrow B$. □

**Definition 3.2.** The colocalisation of $T$ at $\mathcal{E}$ is defined by

$$
\frac{T}{\mathcal{E}^\perp}(A, B) := \lim_{\to} T(A, C).
$$

As in the discussion after 2.1, this colimit exists if $T$ is small or if $\mathcal{E} \downarrow B$ has a final element. The latter happens for all $B$ if and only if $\mathcal{E}$ is part of a complementary pair of thick subcategories $(\mathcal{E}^\perp, \mathcal{E})$. In this case, the map $L^\perp B \to B$ in (1.3) is a final object in $\mathcal{E} \downarrow B$, so that $\frac{T}{\mathcal{E}^\perp}(A, B) \cong T(A, L^\perp B)$.

We will see below that Definition 3.2 leads to an exact sequence as in (1.1).

The naturality of the inductive system $(C)_{s \in \mathcal{E} \downarrow B}$ is trivial: an arrow $f: B \to B'$ in $T$ induces a morphism of inductive systems

$$
(C)_{(s: C \to B) \in \mathcal{E} \downarrow B} \to (C')_{(s': C' \to B') \in \mathcal{E} \downarrow B'},
$$

which maps $s: C \to B$ to $f \circ s: C \to B'$ and acts identically on $C$. As a result, $B \mapsto \frac{T}{\mathcal{E}^\perp}(A, B)$ is a bifunctor that is contravariant in $A$ and covariant in $B$.

The maps $s: C \to B$ provide a natural natural transformation

$$
\frac{T}{\mathcal{E}^\perp}(A, B) \to T(A, B).
$$

We may describe elements of $\frac{T}{\mathcal{E}^\perp}(A, B)$ as diagrams $A \xrightarrow{f} \hat{B} \xrightarrow{s} B$ with $\hat{B} \in \mathcal{E}$. The natural map to $T(A, B)$ maps this diagram to $sf: A \to B$. 
The naturality $\mathcal{T}(B, C) \times \mathcal{T}/\mathcal{E} \to \mathcal{T}/\mathcal{E}$ and the natural map $\mathcal{T}/\mathcal{E} \to \mathcal{T}(B, C)$ provide a multiplication
\[
\mathcal{T}/\mathcal{E}(B, C) \times \mathcal{T}/\mathcal{E}(A, B) \to \mathcal{T}/\mathcal{E}(A, C).
\]
The product of $B \xrightarrow{f_1} \tilde{C} \xrightarrow{s_1} C$ and $A \xrightarrow{f_2} \tilde{B} \xrightarrow{s_2} B$ is
\[
A \xrightarrow{f_1 s_2 f_2} \tilde{C} \xrightarrow{s_1} C \sim A \xrightarrow{f_2} \tilde{B} \xrightarrow{s_1 s_2} C,
\]
where $\sim$ denotes equivalence in the inductive limit $\mathcal{T}/\mathcal{E}(A, C)$. That is, we get the same multiplication on $\mathcal{T}/\mathcal{E}$ if we use the right multiplication map
\[
\mathcal{T}/\mathcal{E}(B, C) \times \mathcal{T}(A, B) \to \mathcal{T}/\mathcal{E}(A, C)
\]
and the natural map $\mathcal{T}/\mathcal{E}(A, B) \to \mathcal{T}(A, B)$. It is also straightforward to see that the multiplication on $\mathcal{T}/\mathcal{E}$ is associative. However, $\mathcal{T}/\mathcal{E}$ has no identity maps, so that it is not a category.

**Definition 3.3.** Given a homological functor $F: \mathcal{T} \to \mathfrak{Ab}$ to the category of Abelian groups, we define its *right colocalisation* at $\mathcal{E}$ by
\[
\mathbb{R}^+ F(B) := \lim_{(s: C \to B) \in \mathcal{E}} F(C).
\]

The naturality of the inductive system $(C)_{s \in \mathcal{E}}$ implies that $\mathbb{R}^+ F$ is a functor on $\mathcal{T}$. The maps $s: C \to B$ induce a natural map $\mathbb{R}^+ F(B) \to F(B)$. A natural transformation $\mathcal{F} \Rightarrow \mathcal{F}'$ clearly induces a natural transformation $\mathbb{R}^+ \mathcal{F} \Rightarrow \mathbb{R}^+ \mathcal{F}'$.

**Theorem 3.4.** If $F$ is a homological functor on $\mathcal{T}$, then so is $\mathbb{R}^+ F$.

**Proof.** Let $B' \xrightarrow{\tilde{f}} B \xrightarrow{s_2} B'' \to B'[1]$ be an exact triangle. We must show that
\[
\mathbb{R}^+ F(B') \xrightarrow{\mathbb{R}^+ F(f)} \mathbb{R}^+ F(B) \xrightarrow{\mathbb{R}^+ F(g)} \mathbb{R}^+ F(B'')
\]
is an exact sequence. Functoriality already implies $\ker \mathbb{R}^+ F(g) \subseteq \text{im} \mathbb{R}^+ F(f)$. It remains to show $\ker \mathbb{R}^+ F(g) \subseteq \text{im} \mathbb{R}^+ F(f)$.

Represent an element of $\mathbb{R}^+ F(B)$ by a pair $(s, x)$ with $E \in \mathcal{E}$, $s: E \to B$ and $x \in F(E)$. If $\mathbb{R}^+ F(g)(s, x) = 0$, then there is a commuting diagram
\[
\begin{array}{ccc}
E & \xrightarrow{\tilde{g}} & E'' \\
\downarrow{s} & & \downarrow{s''} \\
B & \xrightarrow{g} & B''
\end{array}
\]
with $E'' \in \mathcal{E}$ such that $F(\tilde{g})(x) = 0$. We may embed the above commuting square into a morphism of exact triangles
\[
\begin{array}{ccc}
E' & \xrightarrow{\tilde{f}} & E \xrightarrow{\tilde{g}} E'' \xrightarrow{E''[1]} \\
\downarrow{s'} & & \downarrow{s} & & \downarrow{s''} \\
B' & \xrightarrow{f} & B \xrightarrow{g} B'' \xrightarrow{B''[1]}
\end{array}
\]
Since $F$ is homological and $F(\tilde{g})(x) = 0$, there is $x' \in F(E')$ with $x = F(\tilde{f})(x')$. Thus the class of $(s', x')$ in $\mathbb{R}^+ F(B')$ is a pre-image for the class of $(s, x)$. \qed
Theorem 3.5. The map $(A, B) \mapsto T/E \bot (A, B)$ defines a bifunctor that is cohomological in the first and homological in the second variable.

Proof. Since $T/E \bot (A, \_)$ is the right colocalisation of $T(A, \_)$, Theorem 3.4 shows that $T/E \bot (A, B)$ is homological in the second variable. It is cohomological in the first variable because filtered colimits are exact. □

Definition 3.6. We call a (co)homological functor $F: T \to Ab$ colocal if the natural transformation $R \bot F \Rightarrow F$ is invertible.

Proposition 3.7. Let $F: E \to Ab$ be a homological functor. Then there is a unique colocal homological functor $\bar{F}: T \to Ab$ that extends $F$. Thus colocal homological functors $T \to Ab$ are essentially equivalent to homological functors $E \to Ab$.

Furthermore, $R \bot G$ is colocal for any homological functor $G: T \to Ab$. The natural transformation $R \bot G \Rightarrow G$ is universal among natural transformations from colocal functors to $G$.

Proof. We extend $F: E \to Ab$ by $\bar{F}(B) := \lim_{(s: C \to B) \in E \downarrow B} F(C)$.

The proof of Theorem 3.4 only needs that $F|_E$ is a homological functor. Hence it shows that $\bar{F}$ is a homological functor.

Let $G: T \to Ab$ be any homological functor. If $B \in E$, then $\text{id}_B: B \to B$ is a final object in $E \downarrow B$, so that we get $R\bot G(B) \cong G(B)$. In particular, this shows that $\bar{F}|_E = F$. Hence $R \bot (R\bot G) \cong R\bot G$ and $R\bot G$ is colocal. The definition of the colocalisation $R\bot G$ shows that two colocal functors that agree on $E$ already agree on all of $T$.

If $H$ is any colocal homological functor with a natural transformation $H \Rightarrow G$, then we get an induced natural transformation $H \cong R\bot H \Rightarrow R\bot G$. It is straightforward to see that this provides a bijection between natural transformations $H \Rightarrow G$ and $H \Rightarrow R\bot G$. □

4. The localisation–colocalisation exact sequence

To relate the localisation and colocalisation of $T$ at $E$, we introduce a third filtered category $\Delta E B$ that combines $B \downarrow \text{we} E$ and $E \downarrow B$ for an object $B$ of $T$. Objects of $\Delta E B$ are exact triangles of the form

$E \to B \rightarrowtail C \to E[1]$

with $E \in E$ or, equivalently, $s \in \text{we} E$; arrows in $\Delta E B$ are morphisms of triangles of the form

$E \to B \rightarrowtail C \to E[1]$

$E' \to B \rightarrowtail C' \to E'[1]$.

There are obvious forgetful functors from $\Delta E B$ to $B \downarrow \text{we} E$ and $E \downarrow B$ that extract the map $B \to C$ or the map $E \to B$, respectively. Since $s \in \text{we} E$ if and only if $E \in E$, any object of $B \downarrow \text{we} E$ or $E \downarrow B$ is in the range of this forgetful functor. The axiom (TR3) for triangulated categories implies that these two forgetful functors are surjective on arrows as well.
Proposition 4.1. The category $\Delta_\mathcal{E}B$ is filtered.

Proof. First we check (F1). Consider two exact triangles

$$\Delta := (E \to B \xrightarrow{\delta} C \to E[1]), \quad \Delta' := (E' \to B \xrightarrow{\delta'} C' \to E'[1])$$

with $E, E' \in \mathcal{E}$ and $s, s' \in \text{we}_\mathcal{E}$. (LF2) yields a commuting diagram

$$\begin{array}{ccc}
E & \xrightarrow{s} & C' \\
\downarrow & & \downarrow \\
B & \xrightarrow{t} & C''
\end{array}$$

with $t \in \text{we}_\mathcal{E}$. Then $ts' \in \text{we}_\mathcal{E}$ as well. We embed $ts'$ in an exact triangle

$$\Delta'' := (E'' \to B \xrightarrow{ts'} C'' \to E''[1]),$$

this yields an object of $\Delta_\mathcal{E}B$. The axioms of a triangulated category yield morphisms of triangles

$$\begin{array}{ccc}
E & \xrightarrow{g} & B \\
\downarrow & & \downarrow \\
E'' & \xrightarrow{B} & C' \\
\downarrow & & \downarrow \\
E''[1] & \xrightarrow{E'[1]} & E \xrightarrow{s} C \\
\downarrow & & \downarrow \\
E''[1] & \xrightarrow{E''[1]} & E'' \xrightarrow{s'} C' \\
\downarrow & & \downarrow \\
E''[1] & \xrightarrow{E''[1]} & E''[1].
\end{array}$$

Thus $\Delta''$ dominates both $\Delta$ and $\Delta'$ in $\Delta_\mathcal{E}B$, verifying (F1).

Next we construct equalisers for parallel arrows in $\Delta_\mathcal{E}B$:

$$\begin{array}{ccc}
E & \xrightarrow{g} & B \\
\downarrow & & \downarrow \\
E' & \xrightarrow{g'} & B \\
\downarrow & & \downarrow \\
E'[1] & \xrightarrow{E'[1]} & E \xrightarrow{s} C \\
\downarrow & & \downarrow \\
E'[1] & \xrightarrow{E'[1]} & E' \xrightarrow{s'} C' \\
\downarrow & & \downarrow \\
E'[1] & \xrightarrow{E'[1]} & E'[1].
\end{array}$$

Since the category $B \downarrow \text{we}_\mathcal{E}$ is filtered, it is easy to equalise $\gamma_1$ and $\gamma_2$ by a morphism in $B \downarrow \text{we}_\mathcal{E}$. This lifts to a morphism in $\Delta_\mathcal{E}B$ that equals $\gamma_1$ and $\gamma_2$. Since the category $\mathcal{E} \downarrow B$ is filtered as well by Proposition 3.1, we may equalise $\varepsilon_1$ and $\varepsilon_2$ by a morphism in $\mathcal{E} \downarrow B$, which once again lifts to a morphism in $\Delta_\mathcal{E}B$ that still equals $\varepsilon_1$ and $\varepsilon_2$. By (F1), we may equalise both at the same time, that is, we get a commuting diagram

$$\begin{array}{ccc}
E & \xrightarrow{g} & B \\
\downarrow & & \downarrow \\
E' & \xrightarrow{g'} & B \\
\downarrow & & \downarrow \\
E'[1] & \xrightarrow{E'[1]} & E \xrightarrow{s} C \\
\downarrow & & \downarrow \\
E'[1] & \xrightarrow{E'[1]} & E' \xrightarrow{s'} C' \\
\downarrow & & \downarrow \\
E'[1] & \xrightarrow{E'[1]} & E'[1].
\end{array}$$

with $\gamma_3 \gamma_1 = \gamma_3 \gamma_2$ and $\varepsilon_3 \varepsilon_1 = \varepsilon_3 \varepsilon_2$. Thus $\Delta_\mathcal{E}B$ satisfies (F2). \qed

Since the forgetful functors from $\Delta_\mathcal{E}B$ to $B \downarrow \text{we}_\mathcal{E}$ and $\mathcal{E} \downarrow B$ are surjective both on objects and arrows, they preserve colimits. That is, we may rewrite localisations and colocalisations as colimits over $\Delta_\mathcal{E}B$. 
Theorem 4.2. Let $T$ be a triangulated category and $E$ a thick subcategory. Let $F: T \to \mathbb{Ab}$ be a homological functor to the category of Abelian groups. Then there is a natural exact sequence

$$\cdots \to R^1F_1(B) \to F_1(B) \to RF_1(B) \to R^1F_0(B) \to F_0(B) \to RF_0(B) \to \cdots$$

This is the exact sequence (1.2) promised in the introduction.

Proof. For each object $E \to B \to C \to E[1]$ of $\triangle E B$, we get a long exact sequence (4.2)

$$\cdots \to F_1(E) \to F_1(B) \to F_1(C) \to F_0(E) \to F_0(B) \to F_0(C) \to \cdots$$

This is a functor from $\triangle E B$ to the category of exact chain complexes in $\mathbb{Ab}$. Since $\triangle E B$ is filtered, the colimit of this diagram of chain complexes is again an exact chain complex.

Since the forgetful functors from $\triangle E B$ to $B \downarrow \mathcal{E}$ and $\mathcal{E} \downarrow B$ are surjective on objects and arrows, we have

$$RF(B) \cong \lim_{\triangle E B} F(C), \quad R^1F(B) \cong \lim_{\triangle E B} F(E).$$

Since $\triangle E B$ is filtered, the colimit of the constant diagram $F(B)$ on $\triangle E B$ is $F(B)$. Hence the colimit of the exact sequences (4.2) is the desired exact sequence (1.2). □

Corollary 4.3. A homological or cohomological functor $F$ is local if and only if $R^1F \cong 0$, and $F$ is colocal if and only if $RF \cong 0$.

Corollary 4.4. A natural transformation $\Phi: F \Rightarrow F'$ between two homological or cohomological functors is invertible if and only if both its localisation $RF \Rightarrow RF'$ and colocalisation $R^1\Phi: R^1F \Rightarrow R^1F'$ are invertible.

Proof. It is clear that invertibility of $\Phi$ implies invertibility of $R\Phi$ and $R^1\Phi$. The converse follows from the Five Lemma and Theorem 4.2. □

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