SUFFICIENT CONDITIONS FOR HOLOMORPHIC LINEARISATION

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Abstract. Let $G$ be a reductive complex Lie group acting holomorphically on Stein manifolds $X$ and $Y$. When is there a $G$-equivariant biholomorphism of $X$ and $Y$? The categorical quotients $Q_X$ and $Q_Y$ have canonical Luna stratifications, where the strata correspond to the slice representations of closed orbits in $X$ and $Y$. Then a necessary condition is that $Q_X$ and $Q_Y$ are biholomorphic and that the biholomorphism sends the Luna stratum of $Q_X$ corresponding to any given slice representation isomorphically onto the Luna stratum of $Q_Y$ corresponding to the same slice representation. Suppose that $Y = V$ is a $G$-module. Then we show that, for most $V$, the necessary condition is sufficient. We also show that, for any $V$, we get the desired result if the biholomorphism of $Q_X$ and $Q_V$ locally lifts to $G$-biholomorphisms.

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1. Introduction

We first recall some of our earlier work on the linearisation problem appearing in [KLSb]. Throughout the paper, $G$ will be a reductive complex Lie group. Let $X$ and $Y$ be Stein manifolds (always taken to be connected) on which $G$ acts holomorphically. The categorical quotients $Q_X$ and $Q_Y$ are normal Stein spaces. Assume that there is a biholomorphism $\tau : Q_X \to Q_Y$ that locally lifts to $G$-equivariant biholomorphisms between $G$-saturated open subsets of $X$ and $Y$. We use $\tau$ to identify the quotients.

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and call the common quotient $Q$ with quotient maps $p : X \to Q$ and $r : Y \to Q$. Our assumption, then, is that there is an open cover $(U_i)_{i \in I}$ of $Q$ and $G$-equivariant biholomorphisms $\varphi_i : p^{-1}(U_i) \to r^{-1}(U_i)$ over $U_i$ (meaning that $\varphi_i$ descends to the identity map of $U_i$). We express the assumption by saying that $X$ and $Y$ are locally $G$-biholomorphic over a common quotient.

We need one more assumption. We assume that the set of closed orbits with trivial isotropy group is open in $X$ and that the complement, a closed subvariety of $X$, has complex codimension at least two. We say that $X$ is generic (equivalently, $Y$ is generic).

For justification of the term “generic” see [KLSb, Remark 5] or Remark 2.1 below. Now assume that $Y = V$ is a $G$-module. From [KLSb, Corollary 14] we have:

**Theorem 1.1.** Suppose that $X$ and $V$ satisfy the following conditions.

1. $X$ and $V$ are locally $G$-biholomorphic over a common quotient.
2. $X$ and $V$ are generic.

Then $X$ and $V$ are $G$-biholomorphic.

**Corollary 1.2.** Suppose that $X = \mathbb{C}^n$ and that (1) and (2) hold for some $V$. Then the $G$-action on $\mathbb{C}^n$ is linearisable, that is, there is a holomorphic automorphism $\Phi$ of $\mathbb{C}^n$ such that $\Phi \circ g \circ \Phi^{-1}$ is linear for every $g \in G$.

The problem of linearising actions of reductive groups on $\mathbb{C}^n$ has attracted much attention both in the algebraic and holomorphic settings ([Huc90], [Kra96]). The first counterexamples for the algebraic linearisation problem were constructed by Schwarz [Sch89] for $n \geq 4$. His examples are holomorphically linearisable. Derksen and Kutzschebauch [DK98] show that for $G$ nontrivial, there is $N_G \in \mathbb{N}$ such that there are nonlinearisable actions of $G$ on $\mathbb{C}^n$, for every $n \geq N_G$. Their method was to construct actions whose stratified quotients cannot be isomorphic to the stratified quotient of a linear action. We will show that this is essentially the only way to get a counterexample to linearisation.

**Remark 1.3.** Assume that $X$ and $V$ are locally $G$-biholomorphic over a common quotient $Q$. Then there is an open cover $\{U_i\}$ of $Q$ and $G$-biholomorphisms $\psi_i : p^{-1}(U_i) \to r^{-1}(U_i)$ inducing the identity on $U_i$. The maps $\psi_j \circ \psi_i^{-1}$ give us an element of $H^1(Q, \mathcal{F})$, where for $U$ open in $Q$, $\mathcal{F}(U)$ is the group of $G$-biholomorphisms of $r^{-1}(U)$ which induce the identity on $U$. Theorem 1.1 says that the cohomology class associated to $X$ is trivial. On the other hand, given any element of $H^1(Q, \mathcal{F})$ one constructs a corresponding $X$ (which is not a priori Stein). Our proof does not use that $X$ is Stein, hence our theorem is equivalent to the statement that $H^1(Q, \mathcal{F})$ is trivial.

Our goal in this paper is to present two strengthenings of Theorem 1.1.

**Theorem 1.4.** Suppose that $X$ is a Stein $G$-manifold, $V$ is a $G$-module and $X$ and $V$ are locally $G$-biholomorphic over a common quotient. Then $X$ and $V$ are $G$-biholomorphic.

We have removed the assumption of genericity in Theorem 1.1. The proof constructs a vector field on $X$ which is analogous to the Euler vector field on $V$. The flows of these vector fields are used to reduce Theorem 1.4 to [KLSa, Theorem 1.1], which we now
state. Let $Y$ be a Stein $G$-manifold as above and let $\Phi: X \to Y$ be a $G$-diffeomorphism inducing the identity on the common quotient $Q$. We say that $\Phi$ is strict if its restriction to the reduced fibers of the quotient maps is biholomorphic. Then we have [KLSa, Theorem 1.1]:

**Theorem 1.5.** Let $X$ and $Y$ be Stein $G$-manifolds with common quotient $Q$. Suppose that there is a strict $G$-diffeomorphism $\Phi$ of $X$ and $Y$. Then $\Phi$ is homotopic, through strict $G$-diffeomorphisms, to a $G$-biholomorphism of $X$ and $Y$.

Now we find a condition that implies that $X$ and $V$ are locally $G$-biholomorphic over a common quotient. Let $X_{(n)}$ denote the subset of $X$ with isotropy group of dimension $n$. We say that $X$ is 2-large if it is generic and $\text{codim} \ X_{(n)} \geq n + 2$ for $n \geq 1$. For other conditions equivalent to 2-largeness see [Sch95, Section 9]. We say that a biholomorphism $\tau: Q_X \to Q_V$ is strata preserving or preserves the Luna stratification if it sends the Luna stratum of $Q_X$ corresponding to any slice representation onto the Luna stratum of $Q_V$ corresponding to the same slice representation (see Section 2).

**Theorem 1.6.** Suppose that $X$ is a Stein $G$-manifold and $V$ is a $G$-module satisfying the following conditions.

1. There is a biholomorphism $\tau$ from $Q_X$ to $Q_V$ which preserves the Luna stratifications.
2. $V$ is 2-large.

Then, by perhaps changing $\tau$, one can arrange that $X$ and $V$ are locally $G$-biholomorphic over $Q_X \simeq Q_V$, hence $X$ and $V$ are $G$-biholomorphic.

Our proof of the theorem again uses the flows of the Euler vector field on $V$ and an analogous vector field on $X$ to show that $X$ and $V$ admit local $G$-biholomorphisms covering $\tau: Q_X \to Q_V$. Then we can apply Theorem 1.1. Our proofs of Theorems 1.4 and 1.6 essentially use the fact that we have a smooth deformation retraction of $Q_X \simeq Q_V$ to a point which is covered by $G$-equivariant retractions of $X$ and $V$ to fixed points.

Let $X$ and $Y$ be Stein $G$-manifolds with quotients $Q_X$ and $Q_Y$. In [KLSa, Theorems 1.1 and 1.2] we give sufficient conditions for a strata preserving biholomorphism $\tau: Q_X \to Q_Y$ to lift to a $G$-biholomorphism. The nature of the sufficient conditions is to assume the existence of a strict $G$-diffeomorphism or a special kind of $G$-homeomorphism lifting $\tau$. What is nice about Theorem 1.6 is that we do not have to assume the existence of any kind of liftings of $\tau$.

One can ask if assumption (2) of Theorem 1.6 can be removed. In Section 3 we show that one has no problem if $\dim Q \leq 1$ or $G = \text{SL}(2, \mathbb{C})$. We would be surprised if there is a counterexample to Theorem 1.6 with (2) omitted.

2. Background

We start with some background. For more information, see [Lun73] and [Sno82, Section 6]. Let $X$ be a normal Stein space with a holomorphic action of a reductive complex Lie group $G$. The categorical quotient $Q_X = X//G$ of $X$ by the action of $G$ is the set of closed orbits in $X$ with a reduced Stein structure that makes the quotient
map $p: X \to Q_X$ the universal $G$-invariant holomorphic map from $X$ to a Stein space. When $X$ is understood, we drop the subscript $X$ in $Q_X$. Since $X$ is normal, $Q = Q_X$ is normal. If $U$ is an open subset of $Q$, then $\mathcal{O}_X(p^{-1}(U))^G \cong \mathcal{O}_Q(U)$. We say that a subset of $X$ is $G$-saturated if it is a union of fibres of $p$. If $X$ is a $G$-module, then $Q$ is just the complex space corresponding to the affine algebraic variety with coordinate ring $\mathfrak{o}_{\text{alg}}(X)^G$.

If $Gx$ is a closed orbit, then the stabiliser (or isotropy group) $G_x$ is reductive. We say that closed orbits $Gx$ and $Gy$ have the same isotropy type if $G_x$ is $G$-conjugate to $G_y$. Thus we get the isotropy type stratification of $Q$ with strata whose labels are conjugacy classes of reductive subgroups of $G$.

Assume that $X$ is smooth and let $Gx$ be a closed orbit. Then we can consider the slice representation which is the action of $G_x$ on $T_xX/T_x(Gx)$. We say that closed orbits $Gx$ and $Gy$ have the same slice type if they have the same isotropy type and after arranging that $G_x = G_y$, the slice representations are isomorphic representations of $G_x$. The stratification by slice type (the Luna stratification) is finer than the isotropy type stratification, but the Luna strata are unions of irreducible components of the isotropy type strata [Sch80, Proposition 1.2]. Hence if the isotropy strata are irreducible, the Luna strata and isotropy type strata are the same. This occurs for the case of a $G$-module [Sch80, Lemma 5.5], hence in Theorem 1.6 one could replace “Luna stratification” by “isotropy type stratification.” Alternatively, one can show directly that in a $G$-module, the isotropy group of a closed orbit determines the slice representation (see [Sch80, proof of Proposition 1.2]).

There is a unique open stratum $Q_{\text{pr}} \subset Q$, corresponding to the closed orbits with minimal stabiliser. We call this the principal stratum and the closed orbits above $Q_{\text{pr}}$ are called principal orbits. The isotropy groups of principal orbits are called principal isotropy groups. Then $X$ is generic when the principal isotropy groups are trivial and the closed subvariety $p^{-1}(Q \setminus Q_{\text{pr}})$ has codimension at least 2 in $X$. Recall that $X$ is 2-large if it is generic and codim $X(n) \geq n + 2$ for $n \geq 1$.

**Remark 2.1.** If $G$ is simple, then, up to isomorphism, all but finitely many $G$-modules $V$ with $V^G = 0$ are 2-large [Sch95, Corollary 11.6 (1)]. The same result holds for semisimple groups but one needs to assume that every irreducible component of $V$ is a faithful module for the Lie algebra of $G$ [Sch95, Corollary 11.6 (2)]. A “random” $\mathbb{C}^*$-module is 2-large, although infinite families of counterexamples exist. More precisely, a faithful $n$-dimensional $\mathbb{C}^*$-module without zero weights is 2-large if and only if it has at least two positive weights and at least two negative weights and any $n - 1$ weights are coprime. Finally, $X$ is 2-large if and only if every slice representation is 2-large and the property of being 2-large only depends upon the Luna stratification of $Q$.

3. **Proof of Theorem 1.4**

We are assuming that the Stein $G$-manifold $X$ and the $G$-module $V$ are locally $G$-biholomorphic over a common quotient $Q$.

The scalar action of $\mathbb{C}^*$ on $V$ descends to a $\mathbb{C}^*$-action on $Q$ (see below), hence we have an action of $\mathbb{R}^{++} = \{ u \in \mathbb{R} \mid u > 0 \}$ on $Q$. The idea is to lift the $\mathbb{R}^{++}$-action to
Let $\mathcal{A}_Q$ denote the sheaf of holomorphic vector fields on $Q$ (derivations of $\mathcal{O}_Q$) and let $\mathcal{A}(Q)$ denote the global sections. Similarly we have the sheaf of holomorphic vector fields $\mathcal{A}_X$ on $X$. Let $U$ be open in $Q$ and let $\mathcal{A}_X^G(U)$ denote $\mathcal{A}_X(p^{-1}(U))^G$. Then $\mathcal{A}_X^G$ is a coherent sheaf of $\mathcal{O}_Q$-modules as is $\mathcal{A}_Q$. We have $p_*: \mathcal{A}_X(p^{-1}(U))^G \to \mathcal{A}_Q(U)$ where $p_*(A)(f) = A(p^*(f))$ for $f \in \mathcal{O}_Q(U) \simeq \mathcal{O}_X(p^{-1}(U))^G$. Then $p_*: \mathcal{A}_X^G \to \mathcal{A}_Q$ is a morphism of coherent sheaves of $\mathcal{O}_Q$-modules. Hence the kernel $\mathcal{M}$ of $p_*$ is coherent.

Let $r_1, \ldots, r_m$ be homogeneous generators of $\mathcal{O}_{alg}(V)^G$, where $r_i$ has degree $d_i$. Then $(r_1, \ldots, r_m): V \to \mathbb{C}^m$ induces a map $f: Q \to \mathbb{C}^m$ which is an algebraic isomorphism of $Q$ onto the image of $f$. Hence we can think of the quotient map $r: V \to \mathbb{C}^m$ as the polynomial map with entries $r_i$. Note that we have an induced $\mathbb{C}^*$-action on $Q$ where $t \in \mathbb{C}^*$ sends $(q_1, \ldots, q_m) \in Q$ to $(t^{d_1}q_1, \ldots, t^{d_m}q_m)$. We have the Euler vector field $E = \sum x_i\partial/\partial x_i$ on $V$, where the $x_i$ are the coordinate functions on $V$. Let $y_1, \ldots, y_m$ be the usual coordinate functions on $\mathbb{C}^m$. Then $r_*(E) = \sum d_iy_i\partial/\partial y_i \in \mathcal{A}(Q)$.

**Definition 3.1.** Let $B$ be a holomorphic vector field on $Q$. We say that a $G$-equivariant holomorphic vector field $A$ on $X$ is a lift of $B$ if $A(p^*f) = p^*B(f)$ for every $f \in \mathcal{O}(Q)$.

**Lemma 3.2.** Let $B = r_*E$. Then $B$ lifts to a $G$-invariant holomorphic vector field $A$ on $X$.

**Proof.** Let $U$ be open and $G$-saturated in $V$ and let $\Phi: U \to \Phi(U) \subset X$ be a $G$-biholomorphism inducing the identity on $r(U)$. Let $A_U$ denote the image of $E|_U$ in $\mathcal{A}_X(\Phi(U))^G$ under the action of $\Phi$. Then $A_U$ is a lift of $B|_{r(U)}$. The various $A_U$ differ by elements in the kernel $\mathcal{M}$ of $p_*$, hence a global lift of $B$ is obstructed by an element of $H^1(Q, \mathcal{M})$, which vanishes by Cartan’s Theorem B. Hence $A$ exists.

Choose a lift $A$ of $B$ and let $\psi_t$ denote the flow of $A$ on $X$. From [Sch14, proof of Theorem 3.4] we have:

**Lemma 3.3.** The flow $\psi_t$ exists for all $t \in \mathbb{R}$.

**Remark 3.4.** Since everything in sight is real analytic, $\psi_t$ is real analytic in $t$ and extends to be holomorphic in a neighborhood of $\mathbb{R}$ in $\mathbb{C}$ near any $x \in X$. One can show that $\psi_t$ exists for all complex $t$ but we do not need this.

We need to find retractions of $Q$, $X$ and $V$. Choose positive integers $c_i$ such that $d_ie_i = d$ is independent of $i = 1, \ldots, m$. For $q = (q_1, \ldots, q_m) \in Q$ let $\rho(q) = \sum_i |q_i|^{c_i}$. Choose $u \in \mathbb{R}^+$ and set $Q_u = \{q \in Q \mid \rho(q) < u\}$. Let $h: [0, \infty) \to [0, u)$ be a diffeomorphism which is the identity in a neighborhood of 0. Set

$$a(q) = \left(\frac{h(\rho(q))}{\rho(q)}\right)^{1/d} \quad \text{and} \quad \alpha(q) = a(q) \cdot q, \quad q \in Q.$$

Here $a(q) \cdot q$ denotes the $\mathbb{C}^*$-action. Now $\alpha$ is a diffeomorphism of $Q$ with $Q_u$ with inverse

$$\beta(q) = b(q) \cdot q \quad \text{where} \quad b(q) = \left(\frac{h^{-1}(\rho(q))}{\rho(q)}\right)^{1/d}, \quad q \in Q_u.$$

**Proof of Theorem 1.4.** For $u > 0$, let $X_u$ denote $p^{-1}(Q_u)$ and let $V_u$ denote $r^{-1}(Q_u)$. Choose $u > 0$ so that we have a local $G$-biholomorphism $\Phi: X_u \to V_u$ inducing the
identity on $Q_u$. Let $\rho_t$ be the flow of the Euler vector field on $V$. Then we have a $G$-diffeomorphism $\sigma$ of $V$ with $V_u$ which sends $v \in V$ to $\rho_{\ln a(r(v))}(v)$. Using the flow $\psi_t$ of the vector field $A$ that we constructed on $X$, we have a $G$-diffeomorphism $\tau$ of $X$ with $X_u$ which sends $x \in X$ to $\psi_{\ln a(p(x))}(x)$. By construction, $\sigma$ and $\tau$ map fibers $G$-biholomorphically to fibers and $\sigma^{-1} \circ \Phi \circ \tau$ is a strict $G$-diffeomorphism of $X$ and $V$. By Theorem 1.5, $X$ and $V$ are $G$-biholomorphic. □

Remark 3.5. We used the fact that $X$ and $V$ are locally $G$-biholomorphic over $Q$ to construct our special vector field $A$. But given any $A$ lifting $B = r_s E$, we can construct our strict $G$-diffeomorphism, as long as we have a $G$-biholomorphism of neighborhoods of $p^{-1}(r(0))$ and $r^{-1}(r(0))$ inducing the identity on $Q$.

4. Proof of Theorem 1.6

Assume for now that we have a biholomorphism $\varphi: Q_X \to Q_V$ which preserves the Luna stratifications. Note that $X^G$ is smooth and closed in $X$. We may identify $X^G$ with its image in $Q_X$ and similarly for $V^G$. Then $\varphi$ induces a biholomorphism (which we also call $\varphi$) from $X^G$ to $V^G$. We have $V = V^G \oplus V'$ where $V'$ is a $G$-module. Since $X^G \cong V^G$ is contractible, the normal bundle $\mathcal{N}(X^G) = (TX|_{X^G})/T(X^G)$ is trivial $G$-vector bundle [HK95] and we have an isomorphism $\Phi: \mathcal{N}(X^G) \to V^G \times V'$ (viewing the latter as the $G$-vector bundle $V^G \times V' \to V^G$). Since $TX|_{X^G}$ is also $G$-trivial, we may think of $\mathcal{N}(X^G)$ as a subbundle of $TX|_{X^G}$. Note that $\Phi$ restricts to $\varphi$ on the zero section.

Proposition 4.1. Let $\varphi: X^G \to V^G$ and $\Phi$ be as above. Then there is a $G$-saturated neighborhood $U$ of $X^G$ in $X$ and a $G$-saturated neighborhood $U'$ of $V^G$ in $V$ and a $G$-biholomorphism $\Psi: U \to U'$ whose differential induces $\Phi$ on $\mathcal{N}(X^G)$.

Proof. Let $v_1, \ldots, v_k$ be a basis of $V'$ and let $A_1, \ldots, A_k$ denote the corresponding constant vector fields on $V$. Let $X_1, \ldots, X_k$ denote their inverse images under $\Phi$. Then the $X_i$ are holomorphic vector fields defined on $X^G$ and they extend to global vector fields on $X$, which we also denote as $X_i$. Let $\rho_i^{(p)}$ denote the complex flow of $X_i$, $i = 1, \ldots, k$. For $(v, v') \in V^G \oplus V'$, $v' = \sum a_i v_i$, let

$$F(v, v') = \rho_{a_1}^{(1)} \rho_{a_2}^{(2)} \cdots \rho_{a_k}^{(k)}(\varphi^{-1}(v)).$$

Then $F$ is defined and biholomorphic on a neighborhood of $V^G \times \{0\}$ and the derivative of $F$ along $V^G \times \{0\}$ is $\Phi^{-1}$. The inverse of $F$ gives us a biholomorphism $\Psi: U \to U'$ with the following properties:

1. $U$ is a neighborhood of $X^G$ in $X$ and $U'$ is a neighborhood of $V^G$ in $V$.
2. $\Psi$ restricts to $\varphi$ on $X^G$.
3. $d\Psi$ restricted to $\mathcal{N}(X^G)$ gives $\Phi$.

Let $K$ be a maximal compact subgroup of $G$. Averaging $\Psi$ over $K$ gives us a new holomorphic map (also called $\Psi$) which still satisfies the conditions above, perhaps with respect to smaller neighborhoods $U_0$ and $U'_0$. Shrinking we may assume that $U_0$ and $U'_0$ are $K$-stable. It follows from [HK95, Section 5, Lemma 1, Proposition 1 and Corollary 1] that shrinking further we may achieve the following:
(4) The restriction of $\Psi$ to $U_0$ extends to a $G$-equivariant map on $U = G \cdot U_0$.
(5) The restriction of $\Psi^{-1}$ to $U'_0$ extends to a $G$-equivariant map $\Theta$ on $U' = G \cdot U'_0$.

We can reduce to the case that $X^G$ is connected and then reduce to the case that $U$ and $U'$ are connected. Then it is clear that $\Psi \circ \Theta$ and $\Theta \circ \Psi$ are identity maps. Removing $p^{-1}(p(X \setminus U))$ from $U$ we can arrange that it is $G$-saturated, and similarly for $U'$.

Note that our $\Psi$ only induces $\varphi$ on $X^G$. Let $\psi$ denote the biholomorphism of $U/\!/G$ and $U'/\!/G$ induced by $\Psi$ and let $\tau$ denote $\varphi \circ \psi^{-1}$. Then $\tau$ is a strata preserving biholomorphism of a neighborhood of $V^G$ in $Q_\ell$ which is the identity on $V^G$. We now show that $\tau$ has a local $G$-biholomorphic lift to $V$ if we modify $\varphi$ and $\Psi$ (hence $\psi$).

Let $s_1, \ldots, s_n$ denote homogeneous invariant polynomials generating $\mathcal{O}_{\text{alg}}(V')^G$, where $\delta_i$ is the degree of $s_i$, $i = 1, \ldots, n$. Let $s = (s_1, \ldots, s_n): V' \to \mathbb{C}^n$. We can identify $Q' = V'/\!/G$ with the image of $s$ and we can identify $Q_\ell$ with $V^G \times Q'$. We have an action of $\mathbb{C}^*$ on $Q'$ where $t \in \mathbb{C}^*$ sends $(q_1, \ldots, q_n) \in Q'$ to $(t^{\delta_1}q_1, \ldots, t^{\delta_n}q_n)$.

Let $\text{Aut}_{q}(Q')$ denote the quasilinear automorphisms of $Q'$, that is, the automorphisms which commute with the $\mathbb{C}^*$-action. An element of $\text{Aut}_{q}(Q')$ is determined by its (linear) action on the invariant polynomials of degrees $\delta_i$, $i = 1, \ldots, n$. Hence $\text{Aut}_{q}(Q')$ is a linear algebraic group. Let $\sigma$ be a germ of a strata preserving automorphism of $Q'$ at $s(0)$. Then $\sigma(s(0)) = s(0)$. Let $\sigma_t$ denote the automorphism of $Q'$ which sends $y$ to $t^{-1}\cdot \sigma(t\cdot y)$, $t \in \mathbb{C}^*$, $y \in Q'$. It is not automatic that the limit of $\sigma_t$ exists as $t \to 0$. One needs to have the vanishing of certain terms of the Taylor series of $\sigma$ (see [Sch14, Section 2]). But this occurs in the case that $V$ is 2-large (equivalently $V'$ is 2-large) [Sch14 Theorem 2.2], which we now assume. Let us denote the limit of the $\sigma_t$ as $\sigma_0$.

Consider our automorphism $\tau$ of a neighborhood of $V^G$ in $Q_\ell \simeq V^G \times Q'$. Write $\tau = (\tau_1, \tau_2)$, where $\tau_1$ takes values in $V^G$ and $\tau_2$ in $Q'$. Then $\tau_1(v, s(0)) = v$ and $\tau_2(v, s(0)) = s(0)$ for all $v \in V^G$. It follows from the inverse function theorem that $\tau_2(v, \cdot)$ is a germ of a strata preserving automorphism of $Q'$. Thus we have an isotopy $\tau_2(v, \cdot)_t$ with $\tau_2(v, \cdot)_0 = \text{Aut}_{q}(Q')$. Set $\tau_1(v, q)_t = \tau_1(v, t\cdot q)$. Then

$$\tau_t(v, q) = (\tau_1(v, q)_t, \tau_2(v, q)_t)$$

is an isotopy connecting $\tau$ with $\tau_0$ where $\tau_0(v, q) = (v, \tau_2(v, q))$. The isotopy is holomorphic in $v \in V^G$. The connected component of $\text{Aut}_{q}(Q')$ containing $\tau_2$ is independent of $v$. Let $\rho \in \text{Aut}_{q}(Q')$ denote any of the $\tau_2$, say $\tau_{2,0}$, which we can consider as an automorphism of $Q_\ell$. Change our original $\varphi$ by $\rho^{-1}$, which we are allowed to do, and we find ourselves in the case where $\tau_2$ lies in the identity component of $\text{Aut}_{q}(Q')$ for every $v \in V^G$. Now the identity component of $\text{Aut}_{q}(Q')$ is the image of $\text{GL}(V')^G$ [Sch14 Proposition 2.8]. Thus there is a $G$-saturated neighborhood of $0 \in V$ on which we have lifts of the $\tau_2$ to elements of $\text{GL}(V')^G$. Hence shrinking $U$ and $U'$ and changing our map $\Psi$ we can arrange that $\tau_0$ is the identity of $U'/\!/G$. Thus $\tau_t$ is an isotopy. It is obtained by integrating a time-dependent vector field on $Q$. Since $V$ is 2-large, the time dependent vector field lifts to a $G$-equivariant time dependent vector field on $U$ ([Sch95 Theorem 0.4 ] and [Sch13 Remark 2.4]). As in Lemma 3.3 we can integrate to get an isotopy whose value $\Theta$ at time 1 is a $G$-automorphism of $U'$ which covers $\tau$. Then $\Theta \circ \Psi$
is a $G$-equivariant biholomorphism inducing $\varphi$ sending a $G$-saturated neighborhood $U_X$ of $r^{-1}(r(0))$ onto a $G$-saturated neighborhood $U_V$ of $0 \in V$.

**Proof of Theorem 1.6.** We use ideas of the proof of Theorem 1.4. Let $E$ denote the Euler vector field on $V$. Then by [Sch95, Theorem 0.4], [Sch13, Remark 2.4] we can lift the vector field $r_*E$ to a $G$-invariant vector field $A$ on $X$. Recall the $G$-equivariant flows $\rho_t$ of $E$ on $V$ and $\psi_t$ of $A$ on $X$. Let $X_u$ and $V_u$ be as before, $u > 0$. Let $\Phi: U_X \rightarrow U_V$ be a $G$-biholomorphism inducing $\varphi$ as above. Then there is $t$ such that $\psi_t(X_u) \subset U_X$ and $\rho_t(V_u) \subset U_V$. The composition $\rho_{-t} \circ \psi_t$ is a $G$-biholomorphism of $X_u$ with $V_u$ which induces $\varphi$. Hence $X$ and $V$ are locally $G$-biholomorphic over a common quotient and we can apply Theorem 1.4. \hfill \Box

## 5. Small representations

Suppose that we have a strata preserving biholomorphism $\tau: Q_X \rightarrow Q_V$ as in Theorem 1.6. We know that $X$ and $V$ are $G$-equivariantly biholomorphic if $V$ is 2-large. In this section we investigate “small” $G$-modules $V$ which are not 2-large and see if we can still prove that $X$ and $V$ are $G$-equivariantly biholomorphic. The proof of Theorem 1.6 goes through if we can establish the following two statements where $Q = V//G$ and $V^G = (0)$:

1. Let $\varphi: Q \rightarrow Q$ be a strata preserving automorphism and let $\varphi_t = t^{-1} \circ \varphi \circ t$.

Then $\lim_{t \rightarrow 0} \varphi_t$ exists.

2. Let $B$ be a holomorphic vector field on $Q$ which preserves the strata, that is, $B(s) \in T_s(S)$ for every $s \in S$, where $S$ is any stratum of $Q$. Then $B$ lifts to a $G$-invariant holomorphic vector field on $V$.

**Remark 5.1.** Suppose that the minimal homogeneous generators of $\mathcal{O}_{\text{alg}}(V)^G$ have the same degree. Then $\varphi_0 = \varphi'(0)$ exists.

The following theorem is one of the results in [Jia92].

**Theorem 5.2.** Suppose that $\dim Q \leq 1$. Then $X$ and $V$ are $G$-biholomorphic.

**Proof.** The case $\dim Q = 0$ is an immediate consequence of Luna’s slice theorem, so let us assume that $\dim Q = 1$. Then $\mathcal{O}_{\text{alg}}(V)^G$ is normal of dimension one, hence regular, and it is graded. Thus $\mathcal{O}_{\text{alg}}(V)^G = \mathbb{C}[f]$, where $f$ is homogeneous and $Q \simeq \mathbb{C}$. First suppose that $Q$ has one stratum. Then the closed orbits in $V$ are the fixed points and Proposition 1.1 gives the required biholomorphism. The remaining case is where the strata of $Q$ are $\mathbb{C} \setminus \{0\}$ and $\{0\}$. Then (1) follows from Remark 5.1. As for (2), our vector field is of the form $h(z)z\partial / \partial z$ where $h(z)$ is holomorphic. The vector field lifts to an invariant holomorphic function times the Euler vector field on $V$. \hfill \Box

**Theorem 5.3.** Suppose that $G = \text{SL}_2(\mathbb{C})$. Then $X$ and $V$ are $G$-biholomorphic.

**Proof.** We may assume that $V^G = (0)$. Let $R_d$ denote the representation of $G$ on $S^d \mathbb{C}^2$. Then the representations which are not 2-large are [Sch95, Theorem 11.9]

1. $kR_1$, $1 \leq k \leq 3$.
2. $R_2$, $2R_2$, $R_2 \oplus R_1$.
3. $R_3$, $R_4$. 

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In all cases the quotient is $\mathbb{C}^k$ for some $k \leq 3$. The cases $R_1, 2R_1, R_2, R_3$ have quotient of dimension at most 1, hence they present no problem. Suppose that $V = 3R_1$. Then the generating invariants are determinants of degree 2, so we have (*). Let $z_{ij}$ be the variable on $Q = \mathbb{C}^3$ corresponding to the $i$th and $j$th copy of $\mathbb{C}^2$. Then the strata preserving vector fields are generated by the $z_{ij}\partial/\partial z_{kl}$. Thus we have 9 generators. But we have a canonical action of $GL(3, \mathbb{C})$ on $V$ commuting with the action of $G$ and the image of $\mathfrak{gl}(3, \mathbb{C})$ is the span of the 9 generators. Hence we have (**). For the case of $2R_2$ we have (*) because the generators are polynomials of degree 2 and we have (**) because $2R_2$ is an orthogonal representation [Sch80, Theorems 3.7 and 6.7].

Suppose that $V = R_4$. Then $V$ is orthogonal, so (** holds. The quotient $Q$ is isomorphic to the quotient of $\mathbb{C}^2$ by $S_3$, and it is known that strata preserving automorphisms have local lifts [Lya83], [KLM03, Theorem 5.4], hence we certainly have (*). Finally, there is the case $V = R_2 \oplus R_1$. Then there are generating invariants homogeneous of degrees 2 and 3 and the zeroes of the degree three invariant define the closure of the codimension one stratum. Thus we may think of $Q$ as $\mathbb{C}^2$ with coordinate functions $z_2$ and $z_3$ where $z_i$ has weight $i$ for the action of $\mathbb{C}^*$. A strata preserving $\varphi$ has to send $z_3$ to a multiple of $z_3$ (and fix the origin), so that $\varphi = (\varphi_2, \varphi_3)$ where $\varphi_3(z_2, z_3) = \alpha(z_2, z_3)z_3$. It follows easily that $\varphi_0$ exists and we have (*). The strata preserving vector fields must all vanish at the origin and preserve the ideal of $z_3$, so they are generated by $z_3\partial/\partial z_3, z_2\partial/\partial z_2$ and $z_3\partial/\partial z_2$. Since $V$ is self dual, we can change the differentials of the generators $f_2$ and $f_3$ into invariant vector fields $A_2$ and $A_3$, and one can see that our three strata preserving vector fields below are in the span of the images of $A_2, A_3$ and the Euler vector field. Hence we have (**).

□

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