Quantum Entanglement on Boundaries

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Abstract

Quantum entanglement in 3 spatial dimensions is studied in systems with physical boundaries when an entangling surface intersects the boundary. We show that there are universal logarithmic boundary terms in the entanglement Rényi entropy and derive them for different conformal field theories and geometrical configurations. The paper covers such topics as spectral geometry on manifolds with conical singularities crossing the boundaries, the dependence of the entanglement entropy on mutual position of the boundary and the entangling surface, effects of acceleration and rotation of the boundary, relations of coefficients in the trace anomaly to coefficients in the boundary logarithmic terms in the entropy. The computations are done for scalar, spinor and gauge fields.
1 Introduction

Studying entanglement of degrees of freedom of a quantum system on its physical boundary is an interesting and motivated problem. It may tell us about features which distinguish boundary correlations from those in the bulk, explain how the structure of the boundary, its composition, shape, roughness, and etc affect the strength of the correlations. One of the possibilities to probe this phenomenon is to consider entanglement across a spatial 'entangling' surface (denoted further as $B$) and to assume that $B$ ends on the boundary of the system. Some of the effects at the intersection of the two surfaces may be rather complicated since they depend on the material the boundary is made of. As studies of the Casimir effect in real media show, taking into account these properties may be a challenging problem. In this paper we follow a common approach and assume idealized boundaries with conditions of the Dirichlet type. We consider such an approach as a first step toward understanding more physical conditions.

We work with quantum field theories in three spatial dimensions (a four dimensional spacetime). To quantify the entanglement we use entanglement entropy and entanglement Rényi entropies (ERE). In a free quantum field theory with a spatial boundary $\partial M$ the leading terms in the entanglement entropy have been studied in simple configurations: in [1] for flat rectangular boundaries, and in [2] for a system in a waveguide geometry. In both cases $B$ was assumed to be orthogonal to the boundary. These results can be extended to Rényi entropies which have a similar geometrical structure. By taking into account results of [1]-[3] one expects the following asymptotic behaviour of ERE in a four dimensional spacetime:

$$S^{(n)}(B) \simeq \frac{1}{2} \lambda^2 s_2^{(n)} + \lambda s_3^{(n)} + s_4^{(n)} \ln(\Lambda \mu) + \ldots , \quad (1.1)$$

where $n$ is an order of a Rényi entropy (see the definition in sec. [2]), $\Lambda$ is an ultraviolet cutoff, $\mu$ is a typical scale of the theory. The canonical mass dimensions of $\Lambda$ and $\mu$ are +1 and −1, respectively. The entanglement entropy follows from (1.1) in the limit $n \to 1$. As a result of idealized conditions microscopical parameters such as, for example, the atomic spacing of the boundary material, do not appear in (1.1).

In (1.1) the leading term $s_2^{(n)}$ is proportional to the area of $B$. Intersection of $B$ with $\partial M$ is a curve $C$. We call $C$ the 'entangling' curve. Computations show [1] that the next term $s_3^{(n)}$ is proportional to the length of $C$.

The focus of this paper is on boundary effects in the logarithmic term $s_4^{(n)}$ where one expects a combination of invariants on $B$ and $C$. A primary motivation is that $s_4^{(n)}$ is related to the trace anomaly of the stress tensor. Another motivation is that in lower dimensions the boundary contributions to analogous logarithmic terms carry an important physical information. As was shown in [4] in two-dimensional conformal theories with a boundary [5] the logarithmic term depends on the so-called boundary entropy or $g$-function introduced in [6]. The $g$-function decreases under the renormalization, from a critical point to a critical point [7]. One may expect that boundary terms in four dimensions may have similar features.

In four-dimensional conformal theories there is a universal part of $s_4^{(n)}$ depending on geometrical properties on entangling curve $C$. More precisely,

$$s_4^{(n)} = a(n)F_a + c(n)F_c + b(n)F_b + d(n)F_d + e(n)F_e + z(n) , \quad (1.2)$$
where functionals $F_a, F_b, F_c$ are set on $B$ ($F_a$ being a topological invariant of $B$) while $F_d, F_e$ are defined entirely on $C$. The numbers $z(n)$ are related to zero modes. All these 5 functionals are scale invariant and independent. Coefficients $a(n), b(n)$ and $F_a, F_b, F_c$ have been determined in [3] for the case without boundaries. The coefficients are 3d order polynomials of $\gamma_n = 1/n$. (Note that $a(\gamma_n), b(\gamma_n), c(\gamma_n)$ in [3] correspond to $a(n), b(n), c(n)$ in (1.2).) It is the identification of $F_d, F_e$, and calculation of one of the coefficients, $d(n)$, is the main technical problem solved in the present paper for the case of conformal theories.

The paper is organized as follows. In Sec. 2 we define the Rényi entropy, introduce necessary elements of spectral geometry, and describe a relation of $s^{(n)}_p$ in (1.1) to heat kernel coefficients. Special attention is paid here to boundary conditions for scalar, spinor and gauge fields which preserve the conformal invariance. Section 3 is devoted to a heat kernel coefficient $A_4$ of a Laplace operator on a 4-dimensional manifold with conical singularities and boundaries. $A_4$ allows one to determine $s^{(n)}_4$. Conformal properties of the Laplace operator and conformal invariance of boundary conditions are assumed. When a co-dimension 2 surface where conical singularities are located (an entangling surface $B$) crosses the boundary, $A_4$ acquires boundary terms located on a curve $C = B \cap \partial M$. If the boundary is smooth we prove that only two independent terms, $F_d$ and $F_e$, on $C$ can exist in $A_4$ and we fix their structure by the conformal invariance. We then compute a coefficient function $d(n)$ at $F_d$. The section ends with examples of $F_d$ and $F_e$ for some geometrical configurations in a flat spacetime. In Sec. 4 we discuss a number of physical applications where the boundary entanglement plays a key role. We consider a difference of two entanglement entropies $\Delta S$ for a fixed model and the same boundary conditions. We assume that entangling surfaces and curves for the two systems have, respectively, equal areas and lengths. The two entropies then differ by the logarithmic terms presented in (1.2). In a flat spacetime $\Delta S$ may depend only on the entanglement on boundaries. The results of this section include: a study of scaling properties of $\Delta S$ for different geometrical configurations and for boundaries which have non-zero local rotation and acceleration. The acceleration and rotation effects follow from the structure of $F_d$. In a simplest case $F_d$ may be proportional to an integral over $C$ of the acceleration in the direction orthogonal to $\partial M$. A discussion of the above results with an emphasis on open problems is given in sec. 5. In particular, we analyse possible relations of coefficients in the trace anomaly to a coefficient $d$ at $F_d$ in the entropy and discuss its relevance in studying renormalization group flow in the boundary terms. Appendix A collects all geometrical notations. Unavoidable technicalities are left for Appendix B (calculations of the heat coefficients for different spins to fix $d(n)$) and Appendix C (a 3+1 decomposition of the boundary extrinsic curvature tensor in a Killing frame of reference).

## 2 Definitions

### 2.1 Entropy

We consider a quantum field theory on a static four-dimensional spacetime with constant time sections $\Sigma$. The state of the system is specified by a density matrix $\hat{\rho}$ and the entanglement between spatially separated parts, $A$ and $B$, with a common boundary $B$ is determined by a reduced density matrix. Say, for the region $A$, the reduced matrix is
defined as

\[ \hat{\rho}_A = \text{Tr}_B \hat{\rho} , \]  

(2.1)

by taking trace over the states located in the region B. The entanglement Rényi entropy of an order \( n \) is defined as

\[ S^{(n)}_A = \frac{\ln \text{Tr}_A \hat{\rho}_A^n}{1 - n} , \]  

(2.2)

where \( n \) is a non-negative parameter, \( n \neq 1 \). A related notion, the corresponding entanglement entropy

\[ S_A = -\text{Tr}_A \hat{\rho}_A \ln \hat{\rho}_A , \]  

(2.3)

follows from (2.2) in the limit \( n \rightarrow 1 \). In the rest part of this paper we consider integer values \( n = 2, 3, ... \). In local terms \( s_p^{(n)} \) the limit \( n \rightarrow 1 \) does not pose a problem.

Analogously, one can define the Rényi entropy \( S^{(n)}_B \) for a density matrix obtained by integrating over states in region A. There is an important symmetry property, \( S^{(n)}_B = S^{(n)}_A \), when the system is in a pure state. In what follows we do not write explicitly the indexes \( A \) or \( B \) in the entropy. Properties of the entropy we discuss do not depend on the choice of the reduction procedure even when the state is not pure.

For technical reasons it is convenient to choose thermal density matrix, \( \hat{\rho} = e^{-\hat{H}/T}/Z(T) \), where \( T \) is the temperature, \( \hat{H} \) is a Hamiltonian, \( Z(T) = \text{Tr} \exp(-\hat{H}/T) \) is a partition function. One recovers the vacuum state in the limit \( T \rightarrow 0 \). The finite-temperature theory corresponds to a Euclidean four-dimensional manifold \( \mathcal{M} \) with constant time sections \( \Sigma \). The orbits of the Killing vector field generating translations in Euclidean time are the circles \( S^1 \) with the length equal \( 1/T \).

The entropy can be written as

\[ S^{(n)}(T) = \frac{1}{1 - n} \left( \ln Z(n, T) - n \ln Z(T) \right) , \]  

(2.4)
Here $Z(n, T)$ is an 'entanglement partition function', $Z(1, T) = Z(T)$.

In a quantum field theory the partition function $Z(T)$ is represented as a functional integral over field configurations which live on $\mathcal{M}$. Analogously, $Z(n, T)$ can be written in terms of a path integral where field configurations are set on a 'replicated' manifold $\mathcal{M}_n$ which is glued from $n$ copies (replicas) of $\mathcal{M}$ along some cuts which meet on $\mathcal{B}$, see [1]. $\mathcal{M}_n$ are locally identical to $\mathcal{M}$ but have conical singularities on $\mathcal{B}$ with the length of a small unit circle around each point on $\mathcal{B}$ equals $2\pi n$. By the definition $\mathcal{M}_1 = \mathcal{M}$.

For a free QFT the partition function $Z(n, T)$ is defined in terms of a regularized determinant $(\det \Delta)^{\mp 1/2}$ of a Laplace operator $\Delta$. The base manifold for the Laplace operators is $\mathcal{M}_n$. The details of these constructions are described, e.g. in [3].

\subsection{2.2 Heat kernels}

Let $A_p(\Delta)$ be heat coefficients for the asymptotic expansion of the heat kernel of a Laplacian $\Delta$,

$$K(\Delta; t) = \text{Tr} e^{-t\Delta} \approx \sum_{p=0} A_p(\Delta) t^{(p-4)/2}, \quad t \to 0.$$  

(2.6)

The number of spacetime dimensions in (2.6) is 4. By taking into account (2.4), (2.5) one can show [3] that entropies in (1.1) are expressed as

$$s^{(n)}_p = \eta \frac{nA_p(1) - A_p(n)}{n - 1}, \quad p \neq 4,$$  

(2.7)

$$s^{(n)}_4 = \eta \frac{nA_4(1) - A_4(n)}{n - 1} + z(n),$$  

(2.8)

where $A_p(n)$ are heat coefficients $A_p(\Delta)$ for $\Delta$ on $\mathcal{M}_n$, $\eta = +1$ for Bosons and $\eta = -1$ for Fermions. If $\Delta$ on $\mathcal{M}_n$ has a non-vanishing number of zero modes $N_{zm}(n)$, the zero modes have to be excluded from $\det \Delta$. This subtraction affects only the coefficient $A_4(\Delta)$ and it yields the last term in the r.h.s. of $s^{(n)}_4$ in (2.8). One can show that

$$z(n) = -\eta \frac{nN_{zm}(1) - N_{zm}(n)}{n - 1}.$$  

(2.9)

For given boundary conditions the number of zero modes and $z(n)$ are determined only by topologies of the background manifolds. Contribution of $z(n)$ in (2.8) can be important. For example, in a pure 2D gauge theory without this contribution the entanglement entropy computed by the method of conical singularities would be non-trivial, see [8].

We consider manifolds whose constant time sections $\Sigma$ have boundaries $\partial \Sigma$. Boundary $\partial \mathcal{M}$ of the 4-dimensional manifold is $\partial \mathcal{M} \sim S^1 \times \partial \Sigma$.

The definitions related to the geometry of entangling surface $\mathcal{B}$ are as follows (see Appendix [A]). We assume that $\mathcal{B}$ crosses $\partial \mathcal{M}$. Since $\mathcal{B}$ lies in a constant time section $\Sigma$ it also crosses $\partial \Sigma$. The intersection of $\mathcal{B}$ and $\partial \Sigma$ (or $\partial \mathcal{M}$) is denoted by $\mathcal{C}$ and is called the entangling curve, see Fig. [1]. We also introduce different unit vectors: $v$ is a tangent vector to $\mathcal{C}$, $a = \nabla_v v$ is an acceleration of $\mathcal{C}$, $N$ is an outward pointing normal vector to $\partial \mathcal{M}$, $p_i$ is an ortho-normalized pair of normal vectors to $\mathcal{B}$, and $m_a$ is an ortho-normalized pair of vectors at $\mathcal{C}$ which are orthogonal to $v$ and $N$. We denote $R$, $R_{\mu\nu}$, $R_{\mu\nu\lambda\rho}$ the scalar curvature, the Ricci tensor and the Riemann tensor of the regular part of $\mathcal{M}_n$, respectively.
2.3 Operators and boundary conditions

In this paper we consider only models with massless conformal scalar and spinor fields, as well as gauge models. We choose boundary conditions which are invariant under conformal transformations defined in Sec. 3. The scalar Laplacian is taken as $\Delta^{(0)} = -\nabla^2 + \frac{1}{6} R$, and the boundary condition is the Dirichlet condition

$$\varphi |_{\partial M} = 0 \quad ,$$

which is manifestly conformally invariant.

Quantization of an Abelian gauge field $V_\mu$ is considered in the Lorentz gauge $\nabla V = 0$. The corresponding vector Laplacian is $(\Delta^{(1)})_\mu = -\nabla^2 \delta_\mu^\nu + R_\mu^\nu$ and the Laplacian for ghosts is $\Delta^{(gh)} = -\nabla^2$. We use the following boundary condition:

$$N^\mu F_{\mu\nu} |_{\partial M} = 0 \quad ,$$

where $F_{\mu\nu} = \nabla_\mu V_\nu - \nabla_\nu V_\mu$. This condition is manifestly gauge and conformally invariant. It requires that components of an electric field which normal to $\partial M$ and components of the magnetic field which tangential to $\partial M$ vanish on the boundary. The condition like this is physically motivated when the boundary is a perfect conductor. In the Lorentz gauge we use the so called absolute boundary conditions [9]

$$V_N |_{\partial M} = 0 \quad , \quad (N^\mu \nabla_\mu V_\nu^\| + K_\mu^\nu V_\mu^\|) |_{\partial M} = 0 \quad (2.12)$$

where $K_{\mu\nu}$ is an extrinsic curvature tensor of $\partial M$, $V_N = N^\mu V_\mu$ and $V_\|\|\|\|$ are, respectively, normal and tangential components of the vector field to $\partial M$. The corresponding boundary condition for a ghost field $c$ is

$$\partial_N c |_{\partial M} = 0 \quad .$$

Physical condition (2.11) follows from (2.12), (2.13).

In case of a massless Dirac field $\psi$ the operator is $\Delta^{(1/2)} = (i\gamma^\mu \nabla_\mu)^2$ and we require that

$$\Pi_- \psi |_{\partial M} = 0 \quad ,$$

where $\Pi_- = \frac{1}{2}(1 \pm i \gamma_s N^\mu \gamma_\mu)$, and $\gamma_s$ is a chirality gamma matrix. The physical meaning of (2.14) is that the normal component of the spinor current vanishes on the boundary. Condition (2.14) does not break conformal invariance.

If a 4D classical theory is conformally invariant one can show that the heat coefficient $A_4(\Delta)$ for the corresponding Laplacian $\Delta$ remains invariant under local conformal transformations, see e.g. [10], [11]. In general, gauge fixing procedure breaks the conformal invariance in gauge models (also on the level of boundary conditions (2.12),(2.13)). For the gauge field in the Lorentz gauge we always consider a “total” heat coefficient

$$A_4^{(gauge)} = A_4(\Delta^{(1)}) - 2A_4(\Delta^{(gh)})$$

since this is a gauge invariant combination which determines the one-loop divergences in the effective action (in the dimensional regularization, for example). A proof that these divergences are conformally invariant can be found in [12]. An explicit demonstration of conformal invariance of the part of $A_4^{(gauge)}$ which appears due to conical singularities in the absence of boundaries is presented in [3].
3 Spectral geometry

3.1 Heat kernels and conical singularities located on boundaries

Equations (2.7), (2.8) show that leading terms in the entanglement entropies are related to contributions from conical singularities to corresponding heat kernel coefficients. In this section, therefore, we study the spectral geometry on manifolds which, like $M_n$, have conical singularities located on a co-dimension 2 hypersurface $B$ which crosses the boundary. Such manifolds close to the intersection of $B$ and $\partial M_n$ have structure $C_n \times C$. A conical angle of the conical space $C_n$ is $2\pi n$. Boundary conical singularities produce extra contributions to heat coefficients in a form of local invariant functionals given on $C$. Our aim is to fix the structure of these functionals.

The dimensionality $A_p(\Delta)$ is $L^{d-p}$, where $d$ is the number of spacetime dimensions and $L$ is a length parameter. Therefore, functionals on $C$ must be integrals of curvature invariants which have the dimensionality $L^{d-p-(d-3)} = L^{3-p}$. Since dimensionality of curvature invariants should be non-positive we expect that relevant boundary terms appear in $A_p(\Delta)$ only if $p \geq 3$. This is exactly what one can learn from particular geometries [1],[2]. The analysis of [1] shows that boundary conical singularities yield a contribution to $A_3(\Delta)$ (proportional to volume of $C$) which is not universal and depends on boundary conditions. According to (2.7) the contribution $s^{(n)}_3$ is not universal as well.

In the rest of this paper we focus only on boundary terms in $A_4(\Delta)$. One may write $A_4(\Delta) = A_4(n) = nA_4(n = 1) + \bar{A}_4(n)$ where $nA_4(n = 1)$ is a part of the heat coefficient determined on a regular domain of $M_n$ in a standard way. The part of the heat coefficient which depends on conical singularities can be written as

$$\bar{A}_4(n) = \bar{a}(n)F_a + \bar{c}(n)F_c + \bar{b}(n)F_b + \bar{d}(n)F_d + \bar{e}(n)F_e \ , \quad (3.1)$$

and if one takes into account (1.2) and (2.8),

$$\bar{A}_4(n) = \eta(1 - n)(s^{(n)}_4 - z(n)) \quad (3.2)$$

$$\bar{a}(n) = \eta(1 - n)a(n) \ , \ \bar{b}(n) = \eta(1 - n)b(n) \ , \ \bar{c}(n) = \eta(1 - n)c(n) \ , \ \bar{d}(n) = \eta(1 - n)d(n) \ , \ \bar{e}(n) = \eta(1 - n)e(n) \ . \quad (3.3)$$

Since we require that theory is conformally invariant, quantities $F_a - F_e$ are defined as a number of independent conformal invariants. By the definition, $F_d, F_e$ are present only for boundary conical singularities. We call them boundary terms. In the absence of boundary conical singularities invariant functionals, $F_a, F_b, F_c$, have been determined in [3] on the base of previous results, see references therein.

The quantity $F_a$ has been expressed in [3] in terms of the Euler characteristic $\chi_2$ of $B$. It is convenient to keep this relation also in case of boundary conical singularities and define

$$F_a = -2\chi_2[B] = -\frac{1}{2\pi} \left[ \int_B \sqrt{\sigma} d^2x \ R(B) + 2 \int_C k_B ds \right] \quad (3.4)$$

by adding a standard boundary term. Here $R(B)$ is the scalar curvature of $B$ and $k_B$ is an extrinsic curvature of $C$ in $B$, $ds$ is the arclength of $C$. The functional $F_c$ is defined as

$$F_c = \frac{1}{2\pi} \int_B \sqrt{\sigma} d^2x \ C_{ijij} \ , \quad (3.5)$$
\[ C_{ijij} = C_{\mu\nu\lambda\rho} p_i^\mu p_j^\nu p_i^\lambda p_j^\rho \quad . \] (3.6)
p_i, i = 1, 2, are two unit mutually orthogonal normal vectors to \( \mathcal{B} \). Summation over repeated indexes is implied. The Weyl tensor in four dimensions is

\[ C_{\mu\nu\lambda\rho} = R_{\mu\nu\lambda\rho} + \frac{1}{2} (g_{\mu\rho} R_{\nu\lambda} + g_{\nu\lambda} R_{\mu\rho} - g_{\mu\lambda} R_{\nu\rho} - g_{\nu\rho} R_{\mu\lambda}) \]

\[ + \frac{1}{6} R (g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}) \quad . \] (3.7)

Finally,

\[ F_b = \frac{1}{2\pi} \int_\mathcal{B} \sqrt{\sigma} d^2x \left( \frac{1}{2} k_i^2 - \text{Tr}(k_i^2) \right) = - \frac{1}{2\pi} \int_\mathcal{B} \sqrt{\sigma} d^2x \text{Tr}(\hat{k}_i^2) \quad , \] (3.8)

where \((k_i)_{\mu\nu} = p^\lambda p^\rho (p_i)_{\lambda\rho}\) are extrinsic curvatures of \( \mathcal{B} \), \( p^\lambda = \delta^\lambda - (p_i)_{\mu}(p_i)^\lambda \) is a projector on directions tangent to \( \mathcal{B} \), \( k_i = g_{\mu\nu}(k_i)_{\mu\nu} \), \( \text{Tr}(k_i^2) = (k_i)_{\mu\nu}(k_i)^{\mu\nu} \). The traceless part \((\hat{k}_i)_{\mu\nu} = (k_i)_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} k_i \) changes homogeneously under scaling transformations.

Let us emphasize that \( F_a, F_b, \) and \( F_c \) are universal functionals which do not depend on boundary conditions. Explicit expressions of coefficients \( \bar{a}(n), \bar{c}(n) \) for different CFT’s can be found in [3].

### 3.2 Conformal invariants on entangling curve

Let us discuss now boundary functionals in (3.1). They have a form of integrals over \( \mathcal{C} \) of some curvature invariants. Since \( A_4(\Delta) \) is dimensionless in \( d = 4 \) the curvature invariants should have dimensionality \( L^{-1} \). The only appropriate material the invariants can be made of are different extrinsic curvatures. There may be three sorts of terms related to \( \partial M, \mathcal{B}, \) and \( \mathcal{C} \).

**Curvatures of the spacetime boundary.** We define the extrinsic curvature tensor of \( \partial M \) as \( K_{\mu\nu} = H^\lambda_{\mu} H^{\nu}_{\lambda} N_{\lambda\rho} \), where \( H^\nu_\mu = \delta^\nu_\mu - N_{\mu} N^\nu \). According to definitions set in Sec. 2 one has the following invariant quantities on \( \mathcal{C} \): \( \hat{K}_{ab} = K_{\mu\nu} m^a_\mu m^b_\nu, \hat{K}_{va} = K_{\mu\nu} v^\mu m^a_\nu, \hat{K}_{vv} = K_{\mu\nu} v^\mu v^\nu \), where \( v \) is a tangent vector to \( \mathcal{C} \) and \( m_a \) is an orthonormalized pair of vectors at \( \mathcal{C} \) which are orthogonal to \( v \) and \( N \). Since the boundary terms should not depend on the choice of the basis they must obey an additional \( O(2) \) symmetry related to a rotation of the basis \( m_1, m_2 \). This requirement leaves only two possible terms: \( \hat{K}_{ab} \delta^{ab} \) and \( \hat{K}_{vv} \), or, as an equivalent option, \( \hat{K} \) and \( \hat{K}_{vv} \), where \( \hat{K} = \hat{K}^\mu_\mu \). This leaves us with the following functional:

\[ F_d = - \frac{1}{2\pi} \int_\mathcal{C} ds \left( K - 3 \hat{K}_{vv} \right) = \frac{3}{2\pi} \int_\mathcal{C} ds \hat{K}_{vv} \quad , \] (3.9)

\[ \hat{K}_{\mu\nu} = K_{\mu\nu} - \frac{1}{3} H_{\mu\nu} K \quad . \] (3.10)

where numerical coefficients are chosen just for further convenience. Functional (3.9) is invariant under conformal transformations

\[ \bar{g}_{\mu\nu}(x) = e^{-2\omega(x)} g_{\mu\nu}(x) \quad , \] (3.11)

\[ \bar{K}_{\mu\nu} = e^{-\omega} \left[ K_{\mu\nu} - H_{\mu\nu} \omega, N \right] \quad , \quad \bar{N}^\mu = e^{\omega} N^\mu \quad , \quad \bar{v}^\mu = e^{\omega} v^\mu \quad , \] (3.12)
where $\omega_N = N^\lambda \omega_\lambda$. The traceless part \(3.10\) transforms homogeneously.

**Extrinsic curvatures of entangling surface on the boundary.** Consider the extrinsic curvature tensors \((k_i)_{\mu\nu}\) of \(\mathcal{B}\). Since they are 2 by 2 symmetric matrices we can use only three sorts of coordinate invariants on \(\mathcal{C}\): \((k_i)_{vv} = (k_i)_{\mu\nu} v^\mu v^\nu\), \((k_i)_{vl} = (k_i)_{\mu\nu} v^\mu l^\nu\), where \(l\) is some unit vector which is in a tangent space to \(\mathcal{B}\) and is orthogonal to \(\mathcal{C}\). We choose \(l\) as outward directed vector. The term \((k_i)_{vl}\) should be excluded since it depends on the direction of tangent vector \(v\). Instead of two remaining quantities it is convenient to choose \(k_i\) and \((k_i)_{vv}\). Next one must ensure independence on the choice of the pair \(p_i\) and require the corresponding \(O(2)\) symmetry. This can be done by taking \(O(2)\) invariant combinations \((N \cdot p_i)k_i\) and \((N \cdot p_i)(k_i)_{vv}\). One cannot use prefactors such as \((m_a \cdot p_i)\) since they would depend on the choice of another pair of normal vectors \(m_a\). Other vectors associated to orientation of \(\mathcal{C}\) are orthogonal to \(p_i\). Thus, we come to the following functional:

$$F_e = \frac{1}{\pi} \int_{\mathcal{C}} ds (N \cdot p_i)(\hat{k}_i)_{vv} \ ,$$

where summation over the index \(i\) is implied. Conformal invariance of \(3.13\) results from \(3.11\) and transformations

$$\hat{k}_i = e^{-\omega} \left[ (k_i)_{\mu\nu} - h_{\mu\nu} \omega_{,\lambda} \right] \ , \quad \hat{p}_i^\mu = e^\omega p_i^\mu \ ,$$

where $\omega_{,\lambda} = p_i^\lambda \omega_{,\lambda}$. The traceless part \((\hat{k}_i)_{\mu\nu} = (k_i)_{\mu\nu} - 1/2 g_{\mu\nu} k_i\) changes homogeneously under \(3.14\).

Since \(F_e\) depends on the traceless part \((\hat{k}_i)_{\mu\nu}\) of extrinsic curvatures of \(\mathcal{B}\) it is a complete boundary analogue of the bulk functional \(F_b\), see \(3.8\).

**Invariants related to the entangling curve.** The curve \(\mathcal{C}\) is characterized by an acceleration vector $a^\mu = \nabla_v v^\mu$. The norm of this vector is called the curvature of \(\mathcal{C}\). Scalar products of the acceleration vector with vectors orthogonal to \(v\) yield other invariant structures on \(\mathcal{C}\). One can use the definition of \(a\) and orthogonality of \(N, p_i, l\) to \(v\) to see that all these quantities are reduced to

$$\langle a \cdot N \rangle = -K_{vv} \ , \quad \langle a \cdot p_i \rangle = -(k_i)_{vv} \ , \quad \langle a \cdot l \rangle = -k_B \ .$$

Here \(k_B\) is an extrinsic curvature of \(\mathcal{C}\) in \(\mathcal{B}\), see \(3.4\). \(k_B\) is not an independent quantity since \(l\) is a linear combination of \(n\) and \(p_i\). Vectors \(N\) and \(p_i\) are linear independent in general, and one cannot construct a conformal invariant by using quantities \(3.15\) alone. This brings us back to invariants considered before.

We proved, therefore, that the boundary terms due to conical singularities in heat coefficient \(A_4(\Delta)\) in conformally invariant theories are given by two invariant functionals \(3.9, 3.13\).

### 3.3 Fixing coefficients

We now fix coefficient \(d(n)\) by studying particular cases. Consider a flat spacetime and suppose that a quantum system is in a domain \(\Sigma\) with a cylinder-like boundary \(\partial \Sigma\) stretched along an axis parametrized, say, by a \(z\) coordinate. We assume also a translational invariance of \(\partial \Sigma\) along the \(z\) coordinate. As is shown in Appendix \(3\) the heat
Table 1: Coefficient functions. The used notations are \( \gamma \equiv 1/n, d = d(1) \).

| field      | \( \bar{a}(n) \) | \( \bar{g}(n) \) | \( \bar{d}(n) \) | \( d(n) \) | \( d \) |
|------------|------------------|------------------|------------------|-----------|-------|
| real scalar| \( \frac{\gamma^4 - 1}{1440} \) | \( \frac{\gamma^2 - 1}{144} \) | \( \frac{\gamma^4 + 10\gamma^2 - 11}{1440} \) | \( \frac{\gamma^3 + \gamma^2 + 11\gamma + 11}{1440} \) | \( \frac{1}{60} \) |
| Dirac spinor| \( -7\frac{\gamma^4 + 30\gamma^2 - 37}{2880} \) | \( \frac{\gamma^2 - 1}{144} \) | \( -\frac{7\gamma^4 + 10\gamma^2 - 17}{2880} \) | \( \frac{7\gamma^3 + 7\gamma^2 + 17\gamma + 17}{2880} \) | \( \frac{1}{60} \) |
| gauge Boson | \( \frac{\gamma^2 + 30\gamma^2 + 60\gamma - 91}{720\gamma} \) | \( -\frac{\gamma^2 + 37\gamma - 4}{36\gamma} \) | \( \frac{\gamma^2 + 10\gamma^2 - 11}{720\gamma} \) | \( \frac{\gamma^3 + \gamma^2 + 11\gamma + 11}{720} \) | \( \frac{1}{30} \) |

Coefficients on \( M_n \) can be computed for this problem if the entangling surface \( B \) is flat and orthogonal to \( \partial \Sigma \). In fact, the only contribution from conical singularities in \( A_4 \) comes from the boundary,

\[
\bar{A}_4(n) = \frac{\bar{g}(n)}{\pi} \int_C k_B ds . \tag{3.16}
\]

The coefficient functions \( \bar{g}(n) \) for different spins are given in Table 1 for boundary conditions (2.10), (2.12)-(2.14). This result can be compared with a general formula (3.1). Definitions (3.4), (3.5), (3.8), (3.9), (3.13) yield

\[
F_a = -F_d = -\frac{1}{\pi} \int_C k_B ds , \quad F_b = F_c = F_e = 0 . \tag{3.17}
\]

Here we took into account that \( N = l \), and, therefore, \( K = K_{vv} = k_B \) in (3.9), see (3.15). By comparing (3.16) with (3.1) one finds that

\[
\bar{d}(n) = \bar{a}(n) + \bar{g}(n) . \tag{3.18}
\]

Since \( \bar{d}(n) \) are non-trivial and \( \bar{d}(n) \) and \( \bar{g}(n) \) are independent functions the boundary functionals \( F_d \) must appear in \( A_4 \). Results for \( \bar{a}(n) \), \( \bar{d}(n) \) are presented in Table 1. This table also includes the values of coefficient functions \( d(n) \) which appear in the Rényi entropy, Eq. (1.2). \( d(n) \) are derived from \( \bar{d}(n) \) with the help of (3.3). Note that for all spins \( d(n) \) allow analytical continuation to arguments \( n = 1 \). Values \( d = d(1) \) are coefficients which are present in corresponding logarithmic terms in entanglement entropies (2.3).

We do not suggest here a method how to fix coefficient \( e(n) \). It remains as unknown as the analogous bulk coefficient \( \bar{b}(n) \).

### 3.4 Examples

Let us discuss now some simple but not trivial examples of boundary terms in a flat spacetime. In a flat spacetime Eqs. (3.9), (3.13) are simplified as

\[
F_d = \frac{1}{2\pi} \int_C ds \left( 3K_{vv}^{(3)} - K^{(3)} \right) , \tag{3.19}
\]

\[
F_e = \frac{1}{2\pi} \int_C ds \left( N \cdot p \right) \left( 2k_{vv} - k \right) . \tag{3.20}
\]
Figure 2: The figure shows different examples of the entangling surface $\mathcal{B}$ crossing the boundary at $\mathcal{C}$. The boundary is a cylinder (a), sphere (b) and a cone (c).

In (3.19) the extrinsic curvature $K_{\mu\nu}$ of $\partial \mathcal{M}$ is reduced to the extrinsic curvature $K_{(3)}^{\mu\nu}$ of boundary $\partial \Sigma$ of $\Sigma$. In (3.20) also we took into account that one of extrinsic curvatures of $\mathcal{B}$, the one associated with a time-like normal vector, is identically zero. We define $k_{\mu\nu}$ as an extrinsic curvature of $\mathcal{B}$ for a space-like normal (i.e. a vector lying in a constant section $\Sigma$ and orthogonal to $\mathcal{B}$). It is clear that $k_{\mu\nu}$ is just an extrinsic curvature of $\mathcal{B}$ in $\Sigma$.

Let us start with the invariant functional $F_d$ and consider three examples shown on Fig. 2. The boundary $\partial \Sigma$ can be taken as an infinite cylinder (a), as a sphere (b) of the radius $R$, and as a cone (c) with a conical angle $\varphi$. In all three cases the surface $\mathcal{B}$ is disc of some radius $R_B$.

For the spherical and conical cases the singular surface is tilted to the boundary. After some simple algebra one finds with the help of (3.19)

$$F_d = 2 , \text{ cylindrical boundary} ,$$

$$F_d = \frac{R_B}{R} , \text{ spherical boundary} ,$$

$$F_d = 2 \sqrt{1 - \left(\frac{\varphi}{2\pi}\right)^2} , \text{ conical boundary} .$$

For the cylinder $K^{(3)} = K_{vv}^{(3)} = 1/R_B$. For the sphere $K^{(3)} = 2K_{vv}^{(3)} = 1/R$, where $R$ is the radius of the sphere. In the case of the cone $K^{(3)} = K_{vv}^{(3)} = \cos \alpha/R_B$, where the angle $\alpha$ is related to the conical angle $\varphi$ as $\varphi = 2\pi \sin \alpha$.

The boundary functional $F_e$ shares a common property with the bulk functional $F_b$: the both invariants vanish when $(k_i)_{\mu\nu} = \frac{1}{2} h_{\mu\nu} \text{Tr} k_i$. In particular the functionals vanish
when \( \mathcal{B} \) is a segment of \( S^2 \). \( F_e \) is non-zero only when \( \mathcal{B} \) is tilted to the boundary. This property differs \( F_e \) from \( F_d \).

A simple example of non-zero \( F_e \) is the case of a planar boundary \( \partial \Sigma \) and a cylindrical surface \( \mathcal{B} \). Let \( \mathcal{B} \) be a cylinder of the radius \( R \) and \( \alpha \) be an angle between axis of the cylinder and normal vector \( N \) to \( \partial \Sigma \). One finds for the integrand in (3.20)

\[
(N \cdot p) (2k_{vv} - k) = \frac{(N \cdot p) \cos 2\alpha + (N \cdot p)^2}{1 - (N \cdot p)^2},
\]

where \( p \) is a normal to \( \mathcal{B} \) at \( \mathcal{C} \).

### 4 Some manifestations of boundary entanglement

#### 4.1 The anomalous scaling

The quantum entanglement in the presence of physical boundaries depends on a material of the boundary and on a cutoff parameter. It is possible however to identify some properties of the boundary entanglement which are cutoff independent.

There is a simple way to get rid of the leading terms in entanglement entropy, Eq. (1.1), by taking the difference

\[
\Delta S^{(n)} \equiv S_{a}^{(n)} - S_{b}^{(n)} \simeq \Delta s_4^{(n)} \ln(\Lambda \mu) + \Delta s_{\text{fin}}^{(n)}. \tag{4.25}
\]

Here \( S_{a}^{(n)} \) and \( S_{b}^{(n)} \) are entanglement entropies for a fixed field model but for different geometrical configurations. In (4.25) the leading terms with a power dependence on the cutoff cancel out when areas of entangling surfaces as well as lengths of the entangling curves for the two configurations coincide. The remaining logarithmic term depends on the difference \( \Delta s_4^{(n)} = s_4^{(n)}(a) - s_4^{(n)}(b) \).

The last term in the r.h.s. of (4.25) is \( \Delta s_{\text{fin}}^{(n)} = S_{\text{fin},a}^{(n)} - S_{\text{fin},b}^{(n)} \). It is related to parts of the entropies which are finite when the cutoff \( \Lambda \) is sent to infinity. One may call \( S_{\text{fin}}^{(n)} \) a renormalized entropy.

In some simple configurations (4.25) depends only on the boundary entanglement. Consider, as an example, configurations discussed in Sec. 3.4, see Fig. 2. This is the case of a system in a flat spacetime with a boundary and a flat entangling surface \( \mathcal{B} \). We choose \( \mathcal{B} \) to be a disc of a radius \( R \). The bulk terms \( F_b, F_e \) in \( s_4^{(n)} \) equal to zero, \( F_a = -2 \) and cancel out as well. For the given configurations the boundary invariant \( F_e \) is absent as well.

Suppose the system is in a ground state. Then \( R \) is the single dimensional parameter, and one can write (4.25) as

\[
\Delta S^{(n)}(R) = [d(n)(F_{d,1} - F_{d,2}) + \Delta z(n)] \ln(\Lambda R) + \Delta S_{\text{fin}}^{(n)}. \tag{4.26}
\]

When the cutoff is removed the finite part \( \Delta S_{\text{fin}}^{(n)} \) depends only on dimensionless parameters of the system but not on \( R \).

Therefore, in the ground state one can consider the following cutoff independent quantity:

\[
\frac{\partial}{\partial \ln R} \Delta S^{(n)}(R) = \Delta \frac{\partial S^{(n)}(R)}{\partial \ln R} = d(n)(F_{d,1} - F_{d,2}) + \Delta z(n), \tag{4.27}
\]
where the derivative is taken under fixed dimensionless parameters and $\Delta z(n) = z_1(n) - z_2(n)$. The coefficient $d(n)$ depends on the model. For example, for a conformal theory in a cylinder and in a cone, if zero modes are ignored, one gets from (3.21), (3.23)

$$\frac{\partial}{\partial \ln R} (S_{cyl}^{(n)}(R) - S_{cone}^{(n)}(R)) = 2d(n) \left( 1 - \sqrt{1 - \left( \frac{\varphi}{2\pi} \right)^2} \right). \quad (4.28)$$

This equation compares the evolution of the two entropies for a disc entangling surface under a change of the disc radius. The property that the entropies behave differently is entirely due to the boundary effects and different orientations of the disc to boundary surfaces.

4.2 Acceleration effects

Systems which are at rest in non-inertial frames of references represent other examples where effects of the boundary entanglement in a flat spacetime are significant. Let us show how a non-vanishing acceleration or rotation of boundaries change the entanglement.

We start with a general case. Consider a stationary spacetime with a time-like Killing vector field $\xi^\mu = \partial_\tau$. Suppose a system is at rest in a Killing frame of reference (a frame whose velocity four-vector $u^\mu$ is directed along $\xi^\mu$). Physical relations in the given frame are determined by a standard 3+1 decomposition of the metric

$$ds^2 = B(d\tau + a_i dx^i)^2 + h_{ij} dx^i dx^j, \quad (4.29)$$

where $B = g_{\tau\tau} = u_0^2$, $a_i = u_i/u_0$. The 3-dimensional metric $h_{ij}$ allows one to measure physical distances between nearby observers while a 3-vector $a_i$ can be used to synchronize their clocks.
In 4-dimensional notations one can write \( g_{\mu\nu} = h_{\mu\nu} + u_{\mu} u_{\nu} \). Since we use a Euclidean formulation of the theory, \( u^{\mu} u_{\mu} = 1 \). For stationary solutions of the Einstein equations metric \( (4.29) \) is obtained from the Lorentzian metric via a Wick rotation. For stationary but non-static metrics this procedure also implies the Wick rotation of certain parameters of the solution.

The 3+1 decomposition applied to invariant \( (3.9) \) yields the following result:

\[
F_d = F_d^{(3)} + F_d^{(acc)},
\]

where \( F_d^{(3)} \) is a pure 3-dimensional part, and \( F_d^{(acc)} \) is a part related to acceleration and rotation of the boundary. The first term in the r.h.s. of \( (4.30) \) is analogous to \( (3.19) \) and is defined as

\[
F_d^{(3)} = \frac{1}{2\pi} \int_C ds \left( 3K_{vv}^{(3)} - K^{(3)} \right),
\]

where \( K_{vv}^{(3)} = h^\lambda_v h^\rho_v K_{\lambda\rho} \) and \( h^\mu_v = \delta^\mu_v - u^{\mu} u_{\nu} \). As is shown in Appendix C, \( K_{vv}^{(3)} \) coincides with an extrinsic curvature tensor of a boundary of a 3D space with metric \( h_{ij} \), see Eq. \( (4.29) \).

Acceleration and local rotation of the boundary result in a non-trivial term

\[
F_d^{(acc)} = \frac{1}{2\pi} \int_C ds \left[ (w \cdot N)(1 - 3(v \cdot u)^2) - 6(v \cdot u)\Omega_{\perp} \right],
\]

where \( w^\mu = \nabla_\mu u^\mu \) is the physical acceleration of an observer in the given frame, and \( \Omega_{\perp} \) is a projection of the local angular velocity 3-vector \( \Omega^i = -\frac{1}{2}\sqrt{B}e^{ijk}a_{j,k} \) on a direction orthogonal to vectors \( N \) and \( v \). The local angular velocity describes a rotation with respect to a local inertial frame. Derivation of \( (4.31) \) and \( (4.32) \) is presented in Appendix C.

Eqs. \( (4.31) \), \( (4.32) \) can be used to study entanglement entropies in different non-inertial frames in a flat spacetime. Consider, as an example, a system which accelerates along a \( z \) coordinate, see Fig. 3.a. Suppose that it is at rest in the so called Rindler coordinates defined by metric \( (4.29) \) with \( a_t = 0, B = \rho^2, h_{ij} dx^i dx^j = d\rho^2 + dx^2 + dy^2 \), where \( \rho > 0 \). Acceleration of a point with a coordinate \( \rho \) is \( w = \sqrt{w^\mu w_{\mu}} = 1/\rho \).

Let us require that the geometry of the system in the coordinates \( \rho, x, y \) is the same as in the Minkowsky spacetime. This can be required since \( h_{ij} \) is a flat metric. We also suppose that entangling surfaces \( B \) in the Rindler and in the Minkowsky coordinates are identical. Since the spacetime is flat \( F_c = 0 \). The invariants \( F_b \) and \( F_c \) depend on extrinsic curvatures of \( B \), see \( (3.8), (3.13) \), and may be non-trivial. However, they coincide with corresponding invariants for a system in Minkowsky coordinates. The reason is very simple: \( B \) is defined as a surface in a constant time slice. Such slices in the Rindler coordinates are parts of constant time sections in inertial frames. Finally, since topology is fixed, \( F_a \) also does not depend on the acceleration.

Therefore, the difference of entanglement entropies in the Rindler and in the Minkowsky frames is determined by the \( F_d \) invariants and it is

\[
\Delta S^{(n)} = S^{(n)}_{\text{Rind}} - S^{(n)}_{\text{Mink}} = \frac{d(n)}{2\pi} \int_C ds \left( w \cdot N \right) \ln(\Lambda\mu) + \Delta S_{\text{fin}}^{(n)}. \tag{4.33}
\]

In the accelerated frame the \( F_d^{(3)} \) part coincides with the Minkowsky invariant \( F_{d,\text{Mink}} \). This means that \( F_{d,\text{Rind}} - F_{d,\text{Mink}} = F_d^{(acc)} \). Eq. \( (4.33) \) follows if one uses \( (4.32) \) with \( (v \cdot u) = 0 \).
According to (4.33) an acceleration in the direction orthogonal to the boundary $\partial \Sigma$ results in a logarithmic term in the entropy. Acceleration along the boundary does not affect the entropy. Computations for particular cases can be easily done, taking into account that the only non-zero component of $w^\mu$ is $w^\rho = 1/\rho$. If the acceleration vector is strictly orthogonal to $\partial \Sigma$, $(w \cdot N) = -1$, one finds from (4.33)

$$\Delta S^{(n)} = -\frac{d(n)L}{2\pi \rho} \ln(\Lambda \mu) + \Delta S^{(n)}_{\text{fin}},$$

(4.34)

where $L$ is the length of $\mathcal{C}$, $\rho$ is a coordinate of the boundary.

Another interesting case is a system rotating with a constant angular velocity $\Omega$ in the Minkowsky spacetime. Suppose that the system has an axial symmetry and a symmetry axis coincides with an axis of the rotation. Let the entangling surface $\mathcal{B}$ lie in a plane which goes through the symmetry axis, as is shown on Fig. 3.b. Since $\mathcal{B}$ is flat $F_b = F_c = 0$. Also $F_c = 0$, while $F_a$ does not depend on the rotation. A computation shows that the entanglement entropy of a rotating system differs from the entropy of the same system in an inertial frame as

$$\Delta S^{(n)} = S^{(n)}_{\text{Rot}} - S^{(n)}_{\text{Mink}} = d(n)F_d^{(\text{acc})} \ln(\Lambda \mu) + \Delta S^{(n)}_{\text{fin}},$$

(4.35)

$$F_d^{(\text{acc})} = \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 - \Omega^2 R^2}}\right),$$

(4.36)

where $R$ is the radius of the boundary. To get this result we used (4.32) and took into account that the velocity vector $u$ is orthogonal to $\mathcal{B}$ and, hence, $(u \cdot v) = 0$. The acceleration vector has a single non-zero component $w^\rho = \frac{1}{2} \partial_r \ln(1 - \Omega^2 r^2)$, where $r$ is a distance from a point to the rotation axis.

5 Discussion and open problems

The aim of this work was to study boundary effects in entanglement, and, in particular, boundary terms in anomalous scaling of entanglement entropies for different conformal field theories with three spatial dimensions. Our results can describe the boundary entanglement of electromagnetic field excitations, as a most important physical application.

We identified 2 independent conformal structures in the leading boundary terms and succeeded in calculating one of the coefficient functions $d(n)$ at one of them. As was shown for simple examples, the anomalous boundary entanglement is sensitive to a tilt angle between an entangling surface and the boundary. As well it is sensitive to a curvature of the entangling curve. Acceleration and rotation of the boundary also affect the boundary entanglement. These leading boundary terms do not depend on the temperature of the system.

From the analysis of Sec. 4 we conclude that boundary effects in 3 spatial dimensions not only affect the entanglement entropies, but may define the entropies through the scaling properties, for example, for systems with a single dimensional parameter in a ground state in Minkowsky spacetime. Equation (4.27) allows one to quantify entanglement of vacuum electromagnetic fluctuations. Consider the difference of entropies $\Delta S$ for cylindrical and spherical boundaries of a radius $R$ when the entangling surfaces are flat and orthogonal to these boundaries. For the sphere it means that the plane lies in the
equator. The length of the cylinder is assumed to be infinite. A variation $\delta R$ results in
the following change of the entanglement of vacuum electromagnetic fluctuations:

$$\delta \Delta S = \frac{1}{30} \frac{\delta R}{R}, \quad (5.1)$$

where we used (3.21), (3.22), (4.27), results from Table 1 and ignored zero modes. Re-
markably, this change of entropies is finite, cutoff independent, and it is entirely due to
boundary effects.

We could not fix by our method the coefficient function $e(n)$. This problem is left for a
future research. We have not studied non-smooth boundaries, although boundary defects
yield extra contributions to the entropy [1], [2]. There are also other directions where a
study of boundary effects in 3 dimensions can be continued.

One can consider conformal theories with a more general class of invariant boundary
conditions. This can be done by including background fields in the boundary conditions
in some appropriate way. For example, for a scalar field one can impose a Robin condition
of a special form

$$\left( \partial_N + \frac{1}{3} K \right) \varphi |_{\partial \mathcal{M}} = 0, \quad (5.2)$$

where the role of a background field is played by the extrinsic curvature $K$ of $\partial \mathcal{M}$. Condition (5.2) is invariant under conformal transformations (3.11), (3.12) accompanied by
the corresponding transformation of the scalar field $\hat{\phi} = e^\sigma \phi$. The function $g(n)$ which
 corresponds to (5.2) differs from the scalar function for the Dirichlet condition by the
sign. Thus, (5.2) changes also the coefficient function $d(n)$ in the boundary entanglement.
One can also extend these results to theories without conformal symmetry, for example,
massive scalar and spinor fields with Dirichlet conditions, or models where boundary
conditions break conformal symmetry.

The second interesting problem is to understand if the coefficients $d = d(1)$, $e = e(1)$
in the entanglement entropy are new parameters of the theory or they are related to the
bulk coefficients $a$ and $c$ in the trace anomaly. The anomaly has the following form [13]:

$$\langle T^\mu_\mu \rangle = -2a E - c I - \frac{c}{24\pi^2} \nabla^2 R, \quad (5.3)$$

where $E$ is the volume density of the Euler characteristics, see Sec. A and

$$I = -\frac{1}{16\pi^2} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \quad (5.4)$$

with $C_{\mu\nu\lambda\rho}$ defined in (3.7). In case of boundaries a more appropriate quantity is the
integral of the trace anomaly with boundary terms included. The general structure of
the boundary terms has been established in [14]. The integral of $\langle T^\mu_\mu \rangle$ is just the heat
coefficient which can be written in manifestly scale invariant form

$$\eta A_4 = -a \chi_4 - c i_4 - c' j'_4 - c'' j''_4, \quad (5.5)$$

where, as earlier, $\eta = +1$ for Bosons and $\eta = -1$ for Fermions. The Euler invariant $\chi_4$
is defined in Sec. A by Eqs. (A.1)-(A.4), $i_4$ is the integral of $I$, and

$$j'_4 = \frac{1}{16\pi^2} \int_{\partial \mathcal{M}} \sqrt{H} d^3 x \, \text{Tr}(\hat{K}^3), \quad j''_4 = \frac{1}{16\pi^2} \int_{\partial \mathcal{M}} \sqrt{H} d^3 x \, C_{\mu\nu\lambda\rho} n^\nu n^\rho \hat{K}^{\mu\lambda}. \quad (5.6)$$
$\hat{K}_{\mu \nu}$ is defined in (3.10). The boundary terms $j'_4$ and $j''_4$ are conformal invariants since $\hat{K}_{\mu \nu}$ transforms homogeneously under transformations (3.12). Coefficients $c'$, $c''$ are given in [13] for a conformal scalar field with the Dirichlet condition, $c' = -2/35$, $c'' = 1/15$.

We note that $a$ and $c$ in (5.3), (5.5) are limiting values at $n \to 1$ of the coefficient functions $a(n)$ and $c(n)$ introduced in (1.2). Eq. (3.18) implies that $d$ is connected with the bulk coefficient $a$ as

$$d = a + g \quad (5.7)$$

Here $g = g(1)$, $g(n) = \eta \bar{g}(n)/(1 - n)$, and $\bar{g}(n)$ is defined in (3.16). In certain models $d$ and $a$ may coincide, for example, in a model which consists of two conformal scalar fields, one with Dirichlet boundary condition (2.10) and another with condition (5.2). Constants $g$ for these fields differ by the sign. Thus, for this model $d = a = 2/360 = 1/180$.

We mentioned in Sec. 1 that the boundary $g$-function in 2D CFT’s decreases under the RG flow [7]. One may expect that some boundary terms in four dimensions have similar features. A possible relation between $d$ and $a$ is interesting since, as was suggested in [15], $a$ might be a four dimensional analogue of the 2D Zamolodchikov $C$-function [16]. There are arguments that this ‘$a$-function’ monotonically decreases under a renormalization group (RG) flow from IR to UV fixed points, see also discussion in [17]. Similar arguments that the $a$-theorem holds for the entanglement entropy are given in [18]. An incomplete list of works where aspects of RG behaviour of entanglement entropies in different dimensions have been studied includes [19]-[24]. It would be interesting to analyse the boundary function $d$ along the lines of [18].

In general, since $d$ in (5.7) is a sum of the bulk charge $a$ and the seemingly independent constant $g$ which is determined by boundary conditions, one should not exclude that $d$ is a new parameter of the theory. It is also worth pointing out a possible relation of $d$ to the bulk coefficient $c$ at the ‘Weyl part’ of the trace anomaly (5.3). For massless Dirac spinors one has the following expression [3] for function $\bar{c}(n)$ defined in (3.1):

$$\bar{c}(n) = -\frac{7\gamma^4 + 10\gamma^2 - 17}{960\gamma}, \quad (5.8)$$

with $\gamma = 1/n$. (The r.h.s of (5.8) is twice the result for the Weyl spinors reported in [3].) For gauge fields [3]

$$\bar{c}(n) = \frac{\gamma^4 + 10\gamma^2 - 11}{240\gamma}, \quad (5.9)$$

By comparing (5.8), (5.9) with results collected in Table 1 we find a set of ‘magic’ identities between the boundary and bulk coefficients

$$\bar{d}(n) = \frac{\bar{c}(n)}{3}, \quad d(n) = \frac{c(n)}{3}, \quad (5.10)$$

which hold for all arguments $n$ for the gauge field and spin 1/2 massless Dirac field.

Formulas (5.10) are not universal and do not apply in case of a conformal scalar field where $\bar{d}(n)$ cannot be represented as a linear combination of the corresponding scalar functions $\bar{a}(n)$ and $\bar{c}(n)$. One may point out, however, another ‘magic’ relation

$$\bar{d}_{\text{scalar}}(n) = \frac{1}{2} \bar{d}_{\text{gauge}}(n), \quad (5.11)$$
which holds between scalar and gauge functions, see Table 1. Interpretation of (5.11) is that each degree of freedom of electromagnetic field has the same entanglement as a scalar quantum.

Finally, another research direction is in studying a holographic representation of our results for the boundary entanglement. The four dimensional conformal theory which admits a dual description in terms of the AdS gravity one dimension higher is the $SU(N)$ supersymmetric Yang-Mills theory. Our results applied to a multiplet of fields in this model in the limit of a weak coupling and large $N$ yield the value $d = N^2/10$ for the total boundary charge (for the Weyl spinors we just take half of the coefficient of Dirac spinors). It is interesting to see if Eqs. (1.1), (1.2) with this coefficient can be reproduced by applying the holographic formula of [25]. We hope this can be done by taking into account a recent progress [26] in understanding the holographic formulation of CFT’s with boundaries.

Acknowledgement

The author is grateful to I.L. Buchbinder, G. Esposito, A. Yu. Kamenshchik, D.V. Vassilevich for helpful discussions. This work was supported by RFBR grant 13-02-00950.
A Main notations and definitions

For a convenience we collect here main geometrical notations used in the paper.

\( \mathcal{M} \): a four-dimensional manifold (with the Euclidean signature);
\( \partial \mathcal{M} \): a boundary of \( \mathcal{M} \);
\( N \): a unit outward pointing normal vector to \( \partial \mathcal{M} \);
\( \nabla_N = N^\mu \nabla_\mu \): a normal derivative operator;
\( K_{\mu \nu} = H^\lambda_\mu H^\rho_\nu N_\lambda N_\rho \): an extrinsic curvature tensor of \( \partial \mathcal{M} \), \( K = g^{\mu \nu} K_{\mu \nu} \);
\( H_\mu = \delta_\mu - N_\mu N^\nu \): a projector on a tangent space of \( \partial \mathcal{M} \);
\( \tau \): a time coordinate of \( \mathcal{M} \);
\( \xi_\mu \partial_\mu = \partial_\tau \): a Killing vector field of \( \mathcal{M} \), \( \xi_\mu \) is time-like in the Lorentzian signature;
\( u^\mu \): a unit velocity four-vector of the Killing frame, \( u^\mu \) is directed along \( \xi_\mu \);
\( w_\mu = \nabla_\mu u^\nu u_\nu \) is an acceleration of the Killing frame;
\( K^{(3)}_{\mu \nu} = h^\lambda_\mu h^\rho_\nu K_{\lambda \rho} \), where \( h^\mu_\nu = \delta^\mu_\nu - u^\mu u_\nu \), is a 3-dimensional extrinsic curvature in the Killing frame;
\( \Sigma \): a constant \( \tau \) hypersurface;
\( \partial \Sigma \): a boundary of \( \Sigma \);
\( B \): a co-dimension 2 entangling surface which lies in \( \Sigma \);
\( p_i \): a pair of ortho-normalized normal vectors to \( B \);
\( (k_i)_{\mu \nu} = P^\lambda_\mu P^\rho_\nu (p_i)_\lambda (p_i)_\rho \): extrinsic curvatures of \( B \), \( k_i = g^{\mu \nu} (k_i)_{\mu \nu} \);
\( P^\lambda_\mu = \delta^\lambda_\mu - (p_i)_\mu (p_i)_\lambda \): a projector on a tangent space of \( B \);
\( C = B \cap \partial \Sigma \): an entangling curve;
\( l \): a unit outward directed vector in a tangent space to \( B \), \( l \) is orthogonal to \( C \);
\( v \): a unit tangent vector to \( C \);
\( a = \nabla_v v \): an acceleration of \( C \);
\( m_a \): an ortho-normalized pair of vectors at \( C \) which are orthogonal to \( v \) and \( N \);
\( K_{\nu \nu} = K_{\mu \nu v^\mu v^\nu} \), \( (k_i)_{\nu \nu} = (k_i)_{\mu \nu v^\mu v^\nu} \): quantities defined on \( C \).

The definition of the Euler characteristic \( \chi_4 \) on a four-dimensional manifold \( \mathcal{M} \) with the boundary \( \partial \mathcal{M} \) is [14]

\[
\chi_4 = B_4[\mathcal{M}] + S_4[\partial \mathcal{M}] \quad , \tag{A.1}
\]

\[
B_4[\mathcal{M}] = \int_{\mathcal{M}} \sqrt{g} d^4x \ E = \frac{1}{32\pi^2} \int_{\mathcal{M}} \sqrt{g} d^4x \left[ R^2 - 4R_{\mu \nu} R^{\mu \nu} + R_{\mu \nu \lambda \rho} R^{\mu \nu \lambda \rho} \right] \quad , \tag{A.2}
\]

\[
S_4[\partial \mathcal{M}] = -\frac{1}{4\pi^2} \int_{\partial \mathcal{M}} \sqrt{H} d^3x \left[ \det K_{\mu \nu} + \hat{G}^{\mu \nu} K_{\mu \nu} \right] \quad , \tag{A.3}
\]

\[
\hat{G}^{\mu \nu} = \hat{R}^{\mu \nu} - \frac{1}{2} H^{\mu \nu} \hat{R} \quad , \tag{A.4}
\]

where \( \hat{G}^{\mu \nu} \) is the Einstein tensor on \( \partial \mathcal{M} \). The use of \( H_{\mu \nu} \) in (A.4) is equivalent to use of \( \hat{G}^{\mu \nu} \) to induced metric tensor.

B Heat coefficients for entangling surfaces orthogonal to boundaries

Let \( \mathcal{M} \) be a domain with a boundary \( \partial \mathcal{M} \) in \( R^4 \). In a constant time section \( \Sigma \) of \( \mathcal{M} \) we choose a \( z \) coordinate along one of \( R^1 \) directions and assume that \( \Sigma = B \times R^1 \), where \( B \)
is a constant \( z \) section of \( \Sigma \). Thus, \( \mathcal{B} \) is flat. The boundary of \( \Sigma \) is \( \partial \Sigma = \mathcal{C} \times \mathbb{R}^3 \), where \( \mathcal{C} = \partial \mathcal{B} \). If \( \mathcal{C} \) is closed \( \Sigma \) has a topology of a cylinder. Hence, \( \mathcal{M} = \mathcal{B} \times \mathbb{R}^2 \) and the spacetime boundary is \( \partial \mathcal{M} = \mathcal{C} \times \mathbb{R}^2 \).

Construction of replicated versions \( \mathcal{M}_n \) of \( \mathcal{M} \) leads to manifolds which have the structure \( \mathcal{M}_n = \mathcal{B} \times \mathcal{M}^{(2)}_n \). Here \( \mathcal{M}^{(2)}_n \) is a 2D manifold with conical singularities which has been explicitly described in [1]. We calculate now contributions to the heat coefficient \( A_4 \) on \( \mathcal{M}_n \) for operators and boundary conditions discussed in Sec. 2.3.

**Scalar Laplacian.** The heat kernel \( K_n(\Delta; t) \) on \( \mathcal{M}_n = \mathcal{B} \times \mathcal{M}^{(2)}_n \) can be written as

\[
K_n(\Delta; t) = K_n(\Delta_{(2)}, t) K(\Delta_{\mathcal{B}}; t) ,
\]

where \( K_n(\Delta_{(2)}, t) \) and \( K(\Delta_{\mathcal{B}}; t) \) are heat kernels on \( \mathcal{M}^{(2)}_n \) and \( \mathcal{B} \), respectively. Consider asymptotics

\[
K_n(\Delta_{(2)}; t) \simeq \sum_{p=0}^{\infty} A_{p}(\Delta_{(2)}, n) t^{(p-2)/2} , \quad t \to 0 ,
\]

\[
K(\Delta_{\mathcal{B}}; t) \simeq \sum_{p=0}^{\infty} A_{p}(\Delta_{\mathcal{B}}) t^{(p-2)/2} , \quad t \to 0 .
\]

We are interested in the singular part \( \bar{A}_4(n) \) of \( A_4 \), see (3.1), which comes from conical singularities. It follows from (B.1), (B.2) that

\[
\bar{A}_4(n) = \bar{A}_2(n) A_2(\Delta_{\mathcal{B}}) ,
\]

where \( \bar{A}_2(n) \) is the singular part of \( A_2(\Delta_{(2)}, n) \). It can be shown that

\[
\bar{A}_2(n) = \frac{1}{12} \left( \gamma^2 - 1 \right) ,
\]

where \( \gamma = 1/n \), and for the Dirichlet boundary condition

\[
A_2(\Delta_{\mathcal{B}}) = \frac{1}{12\pi} \int_{\mathcal{C}} k_{B} ds .
\]

(Eq. (B.6) also holds for the Neumann condition, which we use for the ghost operator.) Thus, \( \bar{A}_4(n) \) is a purely boundary term. Substitution of (B.5), (B.6) in (B.4) yields formula (3.16), where the factor \( \bar{g}(n) \) for the scalar Laplacian is given in Table 1.

**Spinor Laplacian.** We take the spinor Laplacian and use the boundary conditions discussed in Sec. 2.3. Eqs. (B.1), (B.4) hold for the spinor operators as well but expressions (B.5), (B.6) for the heat coefficients of 2D spinor operators are modified.

\[
\bar{A}_2(n) = - \frac{1}{12\gamma} \left( \gamma^2 - 1 \right) ,
\]

\[
A_2(\Delta_{\mathcal{B}}) = - \frac{1}{12\pi} \int_{\mathcal{C}} k_{B} ds .
\]

The above spinor coefficients differ from scalar coefficients (B.5), (B.6) only by the sign. Functions \( g(n) \), \( \bar{d}(n) \) and \( d(n) \) for the spinor coefficient \( A_4(\Delta_n) \), which are listed in Table 1, follow from (B.2), (B.3).
Derivation of (B.7) can be found, for example, in [8]. Several comments are in order concerning (B.8). A self-consistent formulation of boundary problem (2.14) for the spinor Laplacian requires additional condition

\[(\nabla_N - S)\Pi_+ \psi |_{\partial \mathcal{M}} = 0 \quad , \tag{B.9}\]

where \(\Pi_+ = 1 - \Pi_-\), \(S = -\frac{1}{2} K \Pi_+\), see details in [9]. (For the considered case \(K = k_B\).) Thus, spinor boundary conditions are the mixed ones. For the two dimensional spin 1/2

Laplacian \(\Delta_B\) acting on 2D spinors \(\psi_2\) the corresponding boundary condition is

\[(\nabla_N - S_2)(\Pi_+)_2 \psi_2 |_{\partial \mathcal{M}} = 0 \quad , \tag{B.10}\]

with \(S_2 = -\frac{1}{2} k_B (\Pi_+)_2\) and a 2D projector \((\Pi_+)_2\). The boundary part of \(A_2\) in this case is

\[A_2(\Delta_B) = \frac{1}{12\pi} \int_C ds \, \text{Tr}_2(k_B + 6S_2) \quad . \tag{B.11}\]

To get (B.8) from (B.11) one should take into account that \(\text{Tr}_2 I = 2, \text{Tr}_2 (\Pi_+)_2 = 1\).

**Coefficients for gauge fields.** We need to calculate contributions from boundary conical singularities to 'total' heat coefficient (2.15) which is

\[\hat{A}_4^{(\text{gauge})} = \hat{A}_4(\Delta) - 2\hat{A}_4(\Delta^{(gh)}) \quad , \tag{B.12}\]

where \(\Delta = \Delta^{(1)}\) is the vector Laplacian defined in Sec. 2.3. The ghost operator \(\Delta^{(gh)}\) is just the scalar Laplacian with minimal coupling and Neumann boundary condition (2.13). As a result, \(\hat{A}_4(\Delta^{(gh)})\) coincides with the scalar coefficient.

Calculation of the vector coefficient is more tricky. Let us denote coordinates on \(\mathcal{B}\) and \(\mathcal{M}_n^{(2)}\) as \(x\) and \(y\), respectively. The vector field on \(\mathcal{M}_n = \mathcal{B} \times \mathcal{M}_n^{(2)}\) is decomposed as

\[V^\mu(x, y) = b^\mu(x) \varphi(y) + d^\mu(y) \chi(x) \quad , \tag{B.13}\]

where vector \(b^\mu(x)\) is tangent to \(\mathcal{B}\) and \(d^\mu(y)\) is tangent to \(\mathcal{M}_n^{(2)}\). Vectors \(b^\mu(x)\), \(d^\mu(y)\) have two non-vanishing components each, \(\varphi(y)\), \(\chi(x)\) are scalars.

Instead of (B.1) for the vector Laplacian \(\Delta^{(1)}\) one has

\[K_n(\Delta; t) = K_n(\Delta(2), t) K_{\parallel}(\Delta_B; t) + K_{n,\parallel}(\Delta(2), t) K(\Delta_B; t) \quad . \tag{B.14}\]

The first term in the r.h.s. of (B.14) corresponds to the component tangent to \(\mathcal{B}\). Here \(K_{\parallel}(\Delta_B; t)\) is the heat kernel of 2D vector Laplacian for \(b^\mu(x)\), \(K_n(\Delta(2), t)\) is the scalar heat kernel for \(\varphi(y)\). The second term in the r.h.s. of (B.14) is for the component tangent to \(\mathcal{M}_n^{(2)}\). Therefore, \(K_{n,\parallel}(\Delta(2), t)\) is the heat kernel of 2D vector Laplacian for \(d^\mu(y)\) on \(\mathcal{M}_n^{(2)}\), \(K(\Delta_B; t)\) is the scalar heat kernel for \(\chi(x)\).

Eq. (B.4) for the heat coefficient in case of \(\Delta^{(1)}\) is replaced with

\[\hat{A}_4(n) = \hat{A}_2(n) A_{2,\parallel}(\Delta_B) + \hat{A}_2(\parallel) A_2(\Delta_B) \quad , \tag{B.15}\]

where \(\hat{A}_2(n), A_{2,\parallel}(\Delta_B), \hat{A}_2(\parallel), A_2(\Delta_B)\) correspond to \(K_n(\Delta(2), t), K(\Delta_B; t), K_{n,\parallel}(\Delta(2), t), K(\Delta_B; t)\), respectively.
\( \bar{A}_2(n) \) and \( \bar{A}_2(\parallel)(n) \) are singular parts of coefficients of scalar and vector Laplacians on 2D manifolds with conical singularities \((d^\mu(y) \text{ and } \varphi(y) \text{ fields})\). \( \bar{A}_2(n) \) is given by \( (B.5) \), and

\[
\bar{A}_2(\parallel)(n) = \frac{1}{6\gamma} \left( \gamma^2 - 1 \right) + \frac{1}{\gamma} - 1 = \frac{1}{6\gamma} \left( \gamma^2 - 6\gamma + 5 \right) ,
\]

(B.16)

see, for example, \[8\].

To calculate \( A_2(\Delta_B), A_2(\parallel)(\Delta_B) \) one has to use boundary conditions \( (2.12) \). The normal vector \( N \) to \( \partial M \) is tangent to \( B, K_{\mu\nu} \) does not have components orthogonal to \( B \). This yields the following conditions:

\[
\partial_N \chi \bigg|_C = 0 \quad , \quad (b \cdot N) \bigg|_C = 0 \quad , \quad (\partial_N b^i - S^i_j b^j) \bigg|_C = 0 , \quad (B.17)
\]

where \( S^i_j = - (k_B)^i_j \), and \( b^i \) are the components of \( b^\mu \) tangent to \( C \). \( \bar{A}_2(\Delta_B) \) is given by \( (B.6) \). Boundary conditions for the vector fields are the mixed conditions. One has

\[
A_2(\parallel)(\Delta_B) = \frac{1}{12\pi} \int_C ds \ Tr(\parallel)(k_B + 6S) = - \frac{1}{3\pi} \int_C ds \ k_B , \quad (B.18)
\]

where \( Tr(\parallel) = 2, Tr(\parallel)S = -k_B \). By collecting all results above and using \( (B.12) \) one gets

\[
\bar{A}_4(\Delta) = - \frac{1}{72\pi\gamma} \left( \gamma^2 + 6\gamma - 7 \right) \int_C ds \ k_B , \quad (B.19)
\]

\[
\bar{A}_4^{(\text{gauge})} = - \frac{1}{36\pi\gamma} \left( \gamma^2 + 3\gamma - 4 \right) \int_C ds \ k_B . \quad (B.20)
\]

Eq. \( (B.20) \) yields necessary functions \( g(n), \bar{d}(n) \) and \( d(n) \) listed in Table 1.

### C Boundary effects in the Killing frame

Here we give a more detailed description of results in the Killing frame discussed in Sec. 4.2. The first covariant derivative of the velocity 4-vector allows the following presentation:

\[
u_{\mu;\nu} = w_\mu u_\nu + \Omega_{\mu\nu} , \quad (C.1)
\]

\[
\Omega_{\mu\nu} = \frac{1}{2} h^\lambda_\mu h^\rho_\nu (u_{\lambda;\rho} - u_{\rho;\lambda}) . \quad (C.2)
\]

Quantity \( (C.2) \) is related to an antisymmetric part of \( u_{\mu;\nu} \) and is called a rotation tensor. A projection of the symmetric part of \( u_{\mu;\nu} \) (called a deformation tensor) vanishes in the Killing frame.

We are interested in 3+1 decomposition of boundary invariants. One can easily see that the normal vector to \( \partial M \) is orthogonal to 4-velocity vector, \((N \cdot u) = 0\). The tangent vector \( v \) to the entangling curve may be not orthogonal to \( u \) in stationary but non-static spacetimes (when \( u \) is not orthogonal to constant time sections). Therefore, we should keep \((v \cdot u)\) non-vanishing in all expressions.

We define a 3-dimensional projection of the boundary extrinsic curvature tensor onto directions orthogonal to the velocity 4-vector

\[
K^{(3)}_{\mu\nu} = h^\lambda_\mu h^\rho_\nu K_{\lambda\rho} = h^\lambda_\mu H^\alpha_\lambda h^\rho_\nu H^\beta_\rho N_\alpha;\beta , \quad (C.3)
\]

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where \( H_\lambda^\alpha = \delta_\lambda^\alpha - N_\lambda N^\alpha \). Since \((N \cdot u) = 0\) one can write (C.3) also as
\[
K^{(3)}_{\mu\nu} = H_\mu^\lambda h_\nu^\rho N_{\lambda\rho} , \quad N_{\lambda\rho} = h_\lambda^\alpha h_\rho^\beta N_{\alpha\beta} .
\] (C.4)

This shows that \( K^{(3)}_{\mu\nu} \) is an extrinsic curvature of the boundary on 3D space with metric \( h_{ij} \). The trace of \( K^{(3)}_{\mu\nu} \) can be written as
\[
K = K^{(3)} + K_{uu} = K^{(3)} - (w \cdot N) ,
\] (C.5)
where \( K^{(3)} \) is the trace of \( K^{(3)}_{\mu\nu} \), and \( K_{uu} = u^\mu u^\nu K_{\mu\nu} = u^\mu u^\nu N_{\mu\nu} = -(w \cdot N) \).

The tangent vector \( v \) to \( C \) can be decomposed as \( v = v_\perp + (v \cdot u) u \), where \( v_\perp \) is a component of \( v \) orthogonal to 4-velocity vector, \((v_\perp \cdot u) = 0\). Therefore, on \( C \)
\[
K_{vv} = K^{(3)}_{vv} + (v \cdot u)^2 K_{uu} + 2(v \cdot u) u^\mu v_\perp^\nu K_{\mu\nu} ,
\] (C.6)
where \( K^{(3)}_{vv} = v^\mu v_\perp^\nu K_{\mu\nu} \) is a component of 3D extrinsic curvature along \( v \). We can use now (C.2) to make further transformations
\[
u^\mu v_\perp^\nu K_{\mu\nu} = u^\mu v_\perp^\nu N_{\mu\nu} = -N^\mu v_\perp^\nu u_{\mu\nu} = -\Omega_\perp ,
\] (C.7)
where \( \Omega_\perp = N^\mu v_\perp^\nu \Omega_{\mu\nu} = N^i v_\perp^i \Omega_{ij} \). One can show that \( \Omega_{ij} = -\frac{1}{2} \sqrt{B} (a_{i,j} - a_{j,i}) \), where we used (4.29). In 3D notations \( \Omega_\perp = (\overrightarrow{\Omega} \cdot [\overrightarrow{N} \times \overrightarrow{v}]) \), where \( (\overrightarrow{\Omega})_i = \frac{1}{2} \epsilon_{ijk} \Omega^{jk} \). Eq. (C.6) takes the form
\[
K_{vv} = K^{(3)}_{vv} - (v \cdot u)^2 (w \cdot N) - 2(v \cdot u) \Omega_\perp .
\] (C.8)
As a result we get
\[
3K_{vv} - K = (3K^{(3)}_{vv} - K^{(3)}) + (w \cdot N)(1 - 3(v \cdot u)^2) - 6(v \cdot u) \Omega_\perp .
\] (C.9)
This yields relations (4.30)-(4.32).
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