ON SCHUR PARAMETERS IN STEKLOV’S PROBLEM

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Abstract. We study the recursion (aka Schur) parameters for monic polynomials orthogonal on the unit circle with respect to a weight which provides negative answer to the conjecture of Steklov.

1. Introduction

Given a probability measure $d\sigma$ on the unit circle $\mathbb{T} = \partial \mathbb{D}, \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with infinitely many growth points, we define the monic polynomials $\{\Phi_n\}$ orthogonal in $L^2(d\sigma)$ by beginning with the set $\{1, z, z^2, \ldots\}$ and applying the Gram-Schmidt orthogonalization procedure, i.e.,

$$\Phi_n(z, \sigma) = z^n + a_{n-1}z^{n-1} + \ldots + a_{0,n}, \quad \int_{\mathbb{T}} \Phi_n(e^{i\theta}, \sigma)e^{-ij\theta}d\sigma(\theta) = 0, \quad j = 0, \ldots, n-1.$$ 

Consider the so-called $*$-operation (of order $n$) defined as

$$P_n^*(z) = z^n \overline{P_n(z)}, \quad z \in \mathbb{T},$$

for any polynomial $P_n$ of degree at most $n$. Given $\{\Phi_n\}$, one can define the orthonormal polynomials by the formula

$$\phi_n(z, \sigma) = \frac{\Phi_n(z, \sigma)}{\|\Phi_n\|_{2,\sigma}}, \quad \|\Phi_n\|_{2,\sigma} = \left(\int_{\mathbb{T}} |\Phi_n|^2d\sigma\right)^{1/2}. \quad (1)$$

The polynomials of the second kind will be denoted by $\{\Psi_n\}$ (monic) and $\Psi_n^*(z) = z^n \overline{\Psi_n(z^{-1})}$, $z \in \mathbb{C}$, $\{\psi_n\}$ (orthonormal), and $\{\psi_n^*\}$. The pair $(\Phi_n, \Phi_n^*)$ satisfies the Szegő recurrence:

$$\begin{cases}
\Phi_{n+1}(z, \sigma) = z\Phi_n(z, \sigma) - \gamma_n \overline{\Phi_n(z, \sigma)} \\
\Phi_n^*_{n+1}(z, \sigma) = \Phi_n^*(z, \sigma) - \gamma_n z \Phi_n(z, \sigma)
\end{cases} \quad (2)$$

The pair $(\Psi_j, \Psi_j^*)$ satisfies the same recurrence except that the parameters are $\{-\gamma_n\}$. We refer the reader to [19][20] for the basic theory (our $\gamma_j = \alpha_j$ from [20]). The recursion parameters $\{\gamma_n\} \subseteq \mathbb{D}^\infty$ are sometimes called the Schur parameters due to their relationship with Schur functions and the Schur algorithm. Two of the key identities in the theory are

$$\|\Phi_n\|_{2,\sigma} = \left(\prod_{j=0}^{n-1} (1 - |\gamma_j|^2)\right)^{1/2},$$

$$\exp\left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \ln(2\pi\sigma'(\theta))d\theta\right) = \prod_{j \geq 0} \rho_j, \quad \rho_j = (1 - |\gamma_j|^2)^{1/2}. \quad (3)$$

and the measure $\sigma$ is said to satisfy the Szegő condition if the left hand side in the last formula is positive (which is equivalent to $\{\gamma_j\} \in \ell^2(\mathbb{Z}^+)$. Due to $\|\cdot\|$, we have

$$|\phi_n(z)| \sim |\Phi_n(z)| \quad (4)$$

uniformly in $n$ and $z$ provided that $\sigma$ is Szegő measure.

The condition on the measure to be probability controls only the size of the orthonormal polynomials. Indeed,

$$\Phi_j(z, \alpha \sigma) = \Phi_j(z, \sigma), \quad \phi_j(z, \alpha \sigma) = \alpha^{-1/2} \phi_j(z, \sigma). \quad (5)$$
In 1921, Steklov conjectured in [21] that, for a positive weight \( p \) given on the interval \([a, b]\) of the real line \( \mathbb{R} \), the corresponding sequence of orthonormal polynomials \( \{P_n(x, p)\} \) is bounded in \( n \) for every fixed \( x \in (a, b) \). This conjecture attracted considerable interest (see, e.g., [10-14]) and was answered negatively by Rakhmanov in the series of papers [17, 18]. The proofs first dealt with polynomials orthogonal on the unit circle and then the obtained results were recast to handle \( \{P_n\} \) by making use of a formula due to Geronimus. In this paper, we will only be focusing on the case of orthogonality on the unit circle. We define the Steklov class of measures as

\[
S_{\delta} = \left\{ \sigma : \int d\sigma = 1, \quad \sigma' \geq \delta/(2\pi), \quad \text{a.e.} \quad \theta \in [0, 2\pi) \right\},
\]

where \( \delta \in (0, 1) \). The following variational problem was considered in [17, 18]

\[
M_{n, \delta} = \sup_{\sigma \in S_{\delta}} \|\phi_n(z, \sigma)\|_{L^\infty(\mathbb{T})}.
\]

In [18], the estimates

\[
\left( \frac{n}{\ln n} \right)^{1/2} \leq M_{n, \delta} \leq n^{1/2}
\]

were established (see formula (3) below for the definition of \( <_{\delta} \)) and [2] improved it to

\[
M_{n, \delta} \sim_{\delta} n^{1/2}.
\]

The paper [2] contained a method that presents both the measure \( \sigma^{(n)} \in S_{\delta} \) and the polynomial \( \phi_n(z, \sigma^{(n)}) \), which satisfies \( \|\phi_n(z, \sigma^{(n)})\|_{\infty} \sim_{\delta} n^{1/2} \), explicitly. However, the parameters \( \{\gamma_j^{(n)}\} \) that correspond to this construction were defined only implicitly. Neither did methods of Rakhmanov provide much information about the behavior of \( \{\gamma_j^{(n)}\} \). In the current paper, we study the recursion parameters that give a negative answer to the conjecture of Steklov. That opens new, difference-equation perspective to the problem and answers partially an open question raised in [2]. The main result of the current paper is

**Theorem 1.1.** Fix \( \epsilon \in (0, \epsilon_0) \) where \( \epsilon_0 \) is sufficiently small. Then, there is \( n_0(\epsilon) \) such that for every \( n > n_0(\epsilon) \), there is a weight \( \tilde{w}^{(n)} \) such that

\[
\tilde{w}^{(n)} \text{ satisfies the uniform Steklov condition : } \left\| \frac{1}{\tilde{w}^{(n)}} \right\|_{L^\infty(\mathbb{T})} \lesssim 1,
\]

and the asymptotics for \( \gamma_j^{(n)} \) is given by

\[
\gamma_j^{(n)} = \begin{cases} 
\frac{j^{j+1}}{j+1} \sum_{\sigma=\pm 1} \sigma^{j+1}(2(j+1))^{\frac{\lambda j + \sigma}{\lambda}} \frac{\Gamma(1-\frac{i\sigma j}{\pi})}{\Gamma\left(\frac{\mu + \sigma}{\pi}\right)} + r_{j, \epsilon}, & 0 \leq j \leq n-1 \\
\frac{j^{j+1}}{j+1} \sum_{\sigma=\pm 1} \sigma^{j+1}(2(j'+1))^{\frac{\lambda j' + \sigma}{\lambda}} \frac{\Gamma(1-\frac{i\sigma j'}{\pi})}{\Gamma\left(\frac{\mu + \sigma}{\pi}\right)} + r_{j', \epsilon}, & n \leq j \leq 2n-1 \\
0, & j = 2n \\
\frac{(-1)^{j-2n}}{j-2n} \sum_{\sigma=\pm 1} \sigma^{j-2n}(2(j-2n))^{\frac{\lambda j - 2n + \sigma}{\lambda}} \frac{\Gamma(1-\frac{i\sigma j - 2n}{\pi})}{\Gamma\left(\frac{\mu + \sigma}{\pi}\right)} + r_{j-2n, \epsilon}, & 2n+1 \leq j \leq 3n \\
0, & j > 3n
\end{cases}
\]

where \( j' = 2n - 1 - j \) and \( |r_{j, \epsilon}| < C_\epsilon(j+1)^{-2} \).

**Remark.** The existence of the measure satisfying these conditions is not a new result [1] and the logarithmic growth is not optimal [6]. However, the construction and the asymptotics of the Schur parameters are new and might be interesting. The polynomial constructed by our method has a structure similar to the one from [17, 7].

**Remark.** The careful analysis of (3) shows that the main terms are real-valued and converge to 0 as \( \epsilon \to 0 \) for every fixed \( j \). The results in [4] allow one to control the dependence of \( C_\epsilon \) on \( \epsilon \) when it converges to zero and we conjecture that \( \lim_{\epsilon \to 0} C_\epsilon = 0 \).
We will use the following notation. If \( f_1(x) \) are two positive functions for which \( f_1 < C f_2 \) with some absolute constant \( C \), uniformly in the argument, we will write \( f_1 \lesssim f_2 \). If \( f_1 \lesssim f_2 \) and \( f_2 \lesssim f_1 \), then \( f_1 \sim f_2 \).

If
\[
\sup_z \frac{f_1}{f_2} < C(\epsilon),
\]
where \( \epsilon \) is a parameter, then we will write \( f_1 \sim_\epsilon f_2 \). Relations \( f_1 \sim f_2, f_1 \sim_\epsilon f_2 \) are defined similarly.

The symbol \( \Gamma \) denotes the Gamma function. We will call a function \( F \) to be Carathéodory if it is analytic in \( \mathbb{D} \) and its real part is positive. Given a measure \( \sigma \), we will denote the Carathéodory function given by the Schwarz transform of \( \sigma \) as
\[
F(z) = S(\sigma) = \int_{-\pi}^{\pi} e^{i\eta} + z e^{i\eta} - z d\sigma(\eta).
\]

If \( \sigma \) is in Szegő class, \( \Pi \) will denote the outer function in \( \mathbb{D} \) that satisfies
\[
|\Pi(z)|^{-2} = 2\pi |\sigma|, \quad \text{a.e. } z \in \mathbb{T}, \quad \Pi(0) > 0.
\]

The structure of the paper is as follows: the second section contains the proofs of auxiliary Lemmas and the main Theorem. The third section is an Appendix with the proof of a Lemma in the main text.

2. Proof of Theorem

The measures considered below will be symmetric with respect to \( \mathbb{R} \) and the related Schur parameters will be real. We will need the following simple Lemma.

Lemma 2.1. Suppose \( \Phi_k, \Psi_k, \Phi_k^*, \Psi_k^* \) are the polynomials that correspond to real Schur parameters \( \{\alpha_j\}_{j=0}^{k-1} \). Then, the polynomials associated to the sequence of Schur parameters \( \{\gamma_j\} \) given by
\[
\gamma_j = \begin{cases} 
\alpha_j & : 0 \leq j \leq k - 1 \\
-\alpha_{2k-1-j} & : k \leq j \leq 2k - 1
\end{cases}
\]
satisfy
\[
2\Phi_{2k} = \Phi_k^2 + \Phi_k \Psi_k - z^{-1} (\Phi_k^*)^2 + z^{-1} \Phi_k^* \Psi_k^*,
\]
\[
2\Phi_{2k}^* = (\Phi_k^*)^2 + \Phi_k^* \Psi_k^* - z \Phi_k^2 + z \Psi_k \Phi_k.
\]

Proof. It is known that the pair \( (\Psi_j, \Psi_j^*) \) satisfies the Szegő recurrence with parameters \( \{\alpha_j\} \). If
\[
A = \prod_{j=k-1}^0 \begin{bmatrix} z & -\gamma_j \\ -z^{-1} \gamma_j & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},
\]
then
\[
\begin{bmatrix} \Phi_k & \Phi_k^* \\ \Phi_k^* & -\Psi_k \end{bmatrix} = A \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.
\]

Thus,
\[
a = \frac{\Phi_k + \Psi_k}{2}, \quad b = \frac{\Phi_k - \Psi_k}{2}, \quad c = \frac{\Phi_k^* - \Psi_k^*}{2}, \quad d = \frac{\Phi_k^* + \Psi_k^*}{2}.
\]

First we reverse the dynamics. We are interested in
\[
\prod_{j=0}^{k-1} \begin{bmatrix} z & -\gamma_j \\ -z^{-1} \gamma_j & 1 \end{bmatrix}.
\]

We see that
\[
A^T = \prod_{j=0}^{k-1} \begin{bmatrix} z & -\gamma_j \\ -z^{-1} \gamma_j & 1 \end{bmatrix} = \prod_{j=0}^{k-1} \left( \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} z & -\gamma_j \\ -z^{-1} \gamma_j & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \prod_{j=0}^{k-1} \left( \begin{bmatrix} z & -\gamma_j \\ -z^{-1} \gamma_j & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}.
\]

Therefore,
\[
\prod_{j=0}^{k-1} \begin{bmatrix} z & -\gamma_j \\ -z^{-1} \gamma_j & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} A^T \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}.
\]
We have
\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
z & -\gamma_j \\
-z\gamma_j & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
= \begin{bmatrix}
z & \gamma_j \\
z\gamma_j & 1
\end{bmatrix}.
\]
Therefore, \(\text{(10)}\) implies
\[
\prod_{j=k-1}^{0} \begin{bmatrix}
z & \gamma_j \\
z\gamma_j & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix} A \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}.
\]
Combining the above results, we get
\[
\prod_{j=0}^{k-1} \begin{bmatrix}
z & \gamma_j \\
z\gamma_j & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix} A^T \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix},
\]
and so
\[
\begin{bmatrix}
\Phi_{2k} & \Psi_{2k} \\
\Phi_{2k}^* & -\Psi_{2k}^*
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & -z
\end{bmatrix} \begin{bmatrix}
a & c \\
b & d
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & -z^{-1}
\end{bmatrix} \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix}
= \begin{bmatrix}
a(a+b) - z^{-1}c(c+d) & a(a-b) + z^{-1}c(d-c) \\
d(c+d) - zb(a+b) & d(c-d) + zb(b-a)
\end{bmatrix}.
\]
Thus,
\[
2\Phi_{2k} = \Phi_k^2 + \Phi_k \Psi_k - z^{-1}(\Phi_k^*)^2 + z^{-1}\Phi_k^* \Psi_k^*
\]
and
\[
2\Phi_{2k}^* = (\Phi_k^*)^2 + \Phi_k^* \Psi_k^* - z\Phi_k^2 + z\Phi_k \Psi_k.
\]

The following result is well-known.

**Lemma 2.2.** If \(\{\alpha_j\}\) are the Schur parameters for the measure \(\sigma(\theta)\), then parameters \(\{\alpha_j^{(\beta)}\}\) for the translated measure \(\sigma^{(\beta)} = \sigma(\theta - \beta)\) are given by
\[
\alpha_j^{(\beta)} = e^{-i(j+1)\beta} \alpha_j, \quad j = 0, 1, \ldots
\]

**Proof.** The proof follows from making the following two observations:
\[
\Phi_n(z, \sigma^{(\beta)}) = e^{i\beta} \Phi_n(ze^{-i\beta})
\]
(which follows from the definition) and, take \(z = 0\) in \(\text{(11)}\), \(\Phi_{j+1}(0, \sigma^{(\beta)}) = -\alpha_j^{(\beta)}\).

We will need the following decoupling Lemma. Its proof is contained in \([2, 7]\). However, following \([7]\), we include the sketch for the reader’s convenience.

**Lemma 2.3.** Suppose we are given a polynomial \(\phi_n\) of degree \(n\) and Carathéodory function \(\tilde{F}\) which satisfy the following properties
1. \(\phi_n^*(z)\) has no roots in \(\overline{\mathbb{D}}\).
2. Normalization on the size and “rotation”
\[
\int_T |\phi_n^*(z)|^{-2} d\theta = 2\pi, \quad \phi_n^*(0) > 0.
\]
3. \(\tilde{F} \in C^\infty(\mathbb{T}), \text{Re} \tilde{F} > 0\) on \(\mathbb{T}\), and
\[
\frac{1}{2\pi} \int_T \text{Re} \tilde{F}(e^{i\theta}) d\theta = 1.
\]

Denote the Schur parameters given by the probability measures \(\mu_n\) and \(\tilde{\sigma}\)
\[
d\mu_n = \frac{d\theta}{2\pi|\phi_n^*(e^{i\theta})|^2}, \quad d\tilde{\sigma} = \tilde{\sigma}' d\theta = \frac{\text{Re} \tilde{F}(e^{i\theta})}{2\pi} d\theta,
\]
as \(\{\gamma_j\}\) and \(\{\tilde{\gamma}_j\}\), respectively. Then, the probability measure \(\sigma\), corresponding to Schur coefficients
\[
\gamma_0, \ldots, \gamma_{n-1}, \tilde{\gamma}_0, \tilde{\gamma}_1, \ldots
\]
The polynomial \( \phi_n \) is the \( n \)-th orthonormal polynomial for \( \sigma \).

**Proof.** First, notice that \( \{ \tilde{\gamma}_j \} \in \ell^1 \) by Baxter’s Theorem (see, e.g., [20], Vol.1, Chapter 5). Therefore, \( \sigma \) is purely absolutely continuous by the same Baxter’s criterion. Define the orthonormal polynomials of the first/second kind corresponding to measure \( \tilde{\sigma} \) by \( \{ \tilde{\phi}_j \}, \{ \tilde{\psi}_j \} \). Similarly, let \( \{ \phi_j \}, \{ \psi_j \} \) be orthonormal polynomials for \( \sigma \). Since, by construction, \( \mu_n \) and \( \sigma \) have identical first \( n \) Schur parameters, \( \phi_n \) is \( n \)-th orthonormal polynomial for \( \sigma \).

Let us compute the polynomials \( \phi_j \) and \( \psi_j \), orthonormal with respect to \( \sigma \), for the indexes \( j > n \). The recursion can be rewritten in the following matrix form

\[
\begin{pmatrix}
\phi_{n+m} \\
\phi^*_{n+m} \\
\psi_{n+m} \\
\psi^*_{n+m}
\end{pmatrix} = \begin{pmatrix}
A_n & B_n \\
C_n & D_n
\end{pmatrix} \begin{pmatrix}
\phi_n \\
\phi^*_n \\
\psi_n \\
\psi^*_n
\end{pmatrix},
\]

where \( A_n, B_n, C_n, D_n \) satisfy

\[
\begin{pmatrix}
A_n & B_n \\
C_n & D_n
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

and thus depend only on \( \tilde{\gamma}_0, \ldots, \tilde{\gamma}_{m-1} \). Moreover, we have

\[
\begin{pmatrix}
\tilde{\phi}_m \\
\tilde{\phi}^*_m \\
\tilde{\psi}_m \\
\tilde{\psi}^*_m
\end{pmatrix} = \begin{pmatrix}
A_m & B_m \\
C_m & D_m
\end{pmatrix} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix},
\]

Thus, \( A_m = (\tilde{\phi}_m + \tilde{\psi}_m)/2 \), \( B_m = (\tilde{\phi}_m - \tilde{\psi}_m)/2 \), \( C_m = (\tilde{\phi}^*_m - \tilde{\psi}^*_m)/2 \), \( D_m = (\tilde{\phi}^*_m + \tilde{\psi}^*_m)/2 \) and their substitution into (14) yields

\[
2\phi^*_{n+m} = \phi_n(\tilde{\phi}_m - \tilde{\psi}_m) + \phi^*_n(\tilde{\phi}^*_m + \tilde{\psi}^*_m) = \tilde{\phi}_m = \phi_n + \phi^*_n + \tilde{F}_m(\phi^*_n - \phi_n)
\]

where

\[
\tilde{F}_m(z) = \frac{\tilde{\psi}_m(z)}{\tilde{\phi}_m(z)}.
\]

Since \( \{ \tilde{\gamma}_n \} \in \ell^1 \) and \( \{ \gamma_n \} \in \ell^1 \), we have ([20], p. 225)

\[
\tilde{F}_m \rightarrow \tilde{F} \text{ as } m \rightarrow \infty \quad \text{and} \quad \phi^*_j \rightarrow \Pi, \quad \tilde{\phi}^*_j \rightarrow \tilde{\Pi} \text{ as } j \rightarrow \infty.
\]

uniformly on \( \overline{D} \). The functions \( \Pi \) and \( \tilde{\Pi} \) are related to \( \sigma \) and \( \tilde{\sigma} \) as follows: they are the outer functions in \( D \) that satisfy

\[
|\Pi|^2 = 2\pi\sigma', \quad |\tilde{\Pi}|^2 = 2\pi\tilde{\sigma}', \quad \Pi(0) > 0, \quad \tilde{\Pi}(0) > 0
\]

on \( T \). In (15), send \( m \rightarrow \infty \) to get

\[
2\Pi = \tilde{\Pi} \left( \phi_n + \phi^*_n + \tilde{F}(\phi^*_n - \phi_n) \right)
\]

and we have (13) after taking the square of absolute values and using (16). 

The next result has to do with the recent analysis of the asymptotics of the polynomials orthogonal with the respect to the so-called Fisher-Hartwig weights. We will use [3] as the main reference.

Consider the weight on \( T \) given by

\[
f(z) = \epsilon^x g_{\nu_0}(z)g_{-\nu_0}(z),
\]

where \( z = e^{i\theta}, \theta \in [0, 2\pi) \) and \( \epsilon \in (0, \epsilon_0] \) with \( \epsilon_0 \) to be chosen sufficiently small. For \( z_j = e^{i\theta_j} \),

\[
g_{z_j, \beta_j} = \begin{cases} 
  e^{i\pi\beta_j} & : 0 \leq \arg z < \theta_j \\
  e^{-i\pi\beta_j} & : \theta_j \leq \arg z < 2\pi 
\end{cases}
\]
That is, \( f \) is a weight with two jumps, one from \( e^\epsilon \) to \( e^{-\epsilon} \) around \( i \) (in counterclockwise direction), and one from \( e^{-\epsilon} \) back to \( e^\epsilon \) around \( -i \) (again in counterclockwise direction). It does not define a probability measure but this will not influence the polynomials much due to (4),(5). Notice that \( f \) is symmetric with respect to \( \mathbb{R} \) so all its recursion parameters are real.

We will need the following two Lemmas the proofs of which (essentially contained in [4]) will be discussed in the Appendix.

**Lemma 2.4.** The Schur parameters \( \{\gamma_j\} \) associated to the \( f \) above satisfy:

\[
\gamma_j = -(j+1)^{-1}i^{j+1} \left( (2(j+1))^{\frac{2\nu}{\pi}} \frac{\Gamma(1 - \frac{2\nu}{\pi})}{\Gamma(\frac{2\nu}{\pi})} + (-1)^{j+1}(2(j+1))^\epsilon \frac{\Gamma(1 + \frac{2\nu}{\pi})}{\Gamma(-\frac{2\nu}{\pi})} \right) + r_{j,\epsilon}, \quad |r_{j,\epsilon}| < C_\epsilon(j+1)^{-2}.
\]

The next Lemma will be needed in the proof of Theorem 1.1.

**Lemma 2.5.** Let \( \epsilon \in (0,\epsilon_0] \) and \( n > n_0(\epsilon) \). Then, for the weight \( f \) given by (18), the \( n \)-th associated monic polynomials of the first and second kinds \( \Phi_n \) and \( \Psi_n \) respectively, and their \( * \)-polynomials \( \Psi_n^* \) and \( \Phi_n^* \) satisfy the following estimates:

\[
|\Phi_n^*(z)| \sim 1, \quad z \in \mathbb{T}, \quad |\Psi_n(z)| \sim 1, \quad z \in \mathbb{T},
\]

\[
\|\Phi_n^*\Psi_n^* + z\Phi_n\Psi_n\|_{L_\infty(\mathbb{T})} \sim \epsilon \ln n, \quad z \in \mathbb{T},
\]

\[
|\frac{\Psi_n^*(z)}{\Phi_n^*(z)} + \frac{\Psi_n^*(-z)}{\Phi_n^*(-z)}| \lesssim 1, \quad z \in \mathbb{T}.
\]

**Remark.** The following general identity is immediate from Szegő recursion by taking the determinant

\[
\det \begin{bmatrix} \Psi_n^* & \Psi_n \\ \Phi_n^* & -\Phi_n \end{bmatrix} = -2z^n \prod_{j=0}^{n-1} (1 - |\gamma_j|^2)
\]

so

\[
\Phi_n^*\Psi_n^* + \Phi_n^*\Psi_n = 2 \prod_{j=0}^{n-1} (1 - |\gamma_j|^2).
\]

Then, (19) gives

\[
\left| \frac{\Psi_n^*}{\Phi_n^*} + \frac{\Psi_n^*}{\Phi_n^*} \right| \lesssim 1
\]

uniformly over \( \mathbb{T} \) if the polynomials correspond to the weight (18).

Now we are in position to give a proof of Theorem 1.1.

**Proof. (of Theorem 1.1).** Given \( \epsilon > 0 \) and large \( n \), we consider \( f \) defined by (18). If the corresponding coefficients are denoted by \( \{\alpha_j\} \), we consider the weight \( w^{(n)} \) given by Schur parameters \( \{\gamma_j^{(n)}\} \) defined as

\[
\gamma_j^{(n)} = \begin{cases} 
\alpha_j, & j \leq n - 1 \\
-\alpha_{j'}, & j' = 2n - 1 - j, \quad n - j \leq 2n - 1 \\
0, & j = 2n \\
(-1)^{j-2n}\alpha_{j'-2n-1}, & 2n + 1 \leq j \leq 3n
\end{cases}
\]

Now, (22) follows immediately from Lemma 2.4 and we need to show (6) and (7).

Let us prove (7). Notice first that \( |\phi_j| \sim |\Phi_j| \) by (11) and (13) so it is sufficient to consider \( \Phi_{2n+1} \). We apply Lemma 2.1 to say

\[
2\Phi_{2n} = \Phi_{2n}^2 + \Phi_n\Psi_n - z^{-1}\Phi_n^2 + z\Phi_n^*\Psi_n^* = 2\Phi_{2n} = \Phi_{2n}^2 + \Phi_n^*\Psi_n^* - z\Phi_n^2 + z\Phi_n\Psi_n.
\]

Since the \( \gamma_{2n}^{(n)} = 0 \), we get \( \Phi_{2n+1} = z\Phi_{2n}^2 + \Phi_{2n+1} = z\Psi_{2n}^* \Phi_{2n+1} = \Phi_{2n+1}^2 = \Psi_{2n+1}^* \Phi_{2n+1} = \Phi_{2n+1}^* \Phi_{2n+1} = \Psi_{2n+1}^* \Psi_{2n+1} = \Psi_{2n+1}^* \)

\[
2\Phi_{2n+1} = z\Phi_{2n}^2 + z\Phi_n\Psi_n - \Phi_n^2 + \Phi_n^*\Psi_n^*, \quad 2\Phi_{2n+1} = \Phi_{2n}^2 + \Phi_n^*\Psi_n^* - z\Phi_n^2 + z\Phi_n\Psi_n.
\]
We have 
\[ 2\|\Phi_{2n+1}\|_{\infty} \geq \|\Phi_n^* \Phi_n^* + z\Phi_n \Psi_n\|_{\infty} - 2\|\Phi_n\|_{\infty}^2 > \epsilon \ln n \]
by Lemma 2.26.

To show (23), we will use Lemma 2.3. We choose Carathéodory function for the decoupled problem as
\[ \bar{F}(z) = \frac{\psi_n^*(z)}{\Phi_n^*(z)} = \frac{\Psi_n^*(z)}{\Phi_n^*(z)} = -\frac{\Psi_n^*(z)}{\Phi_n^*(z)} + O(1) \quad \text{by (21)}. \]
We have (see [20], Theorem 3.2.4)
\[ \Re \bar{F}(e^{i\theta}) = |\phi_n^*(e^{i(\theta + \pi)})|^{-2} \]
and \( \bar{F} \) is Carathéodory function of the Bernstein-Szegő weight \( (2\pi)^{-1}|\phi_n^*(e^{i(\theta + \pi)})|^{-2} \) having the Schur parameters
\[ \left\{ \begin{array}{ll}
(-1)^{j+1} \alpha_j, & j < n \\
0, & j \geq n
\end{array} \right. \]
as follows from Lemma 2.2. Since our Schur parameters \( \gamma_j^{(n)} = 0, j > 3n \), we have
\[ |\Pi(-z)| = |\phi_n^*(-z,f)| \sim 1, \quad \text{by (19)}. \]
So, by the identity (17), we only need to show
\[ |\Phi_{2n+1} + \Phi_{2n+1}^* + \bar{F}(\Phi_{2n+1}^* - \Phi_{2n+1})| \lesssim 1 \]
uniformly on \( T \). Recall that \( \bar{F} = -\frac{\Psi_n^*}{\Phi_n} + O(1) \) by (21). We introduce auxiliary
\[ D = \frac{\Phi_n^* \Psi_n^*}{2}, \quad A = -\frac{\Phi_n^*^2}{2}. \]
Notice that
\[ |A| \sim 1, \quad \frac{D}{A} = -\frac{\Psi_n^*}{\Phi_n^*} = -\left( \frac{D}{A} \right) + O(1) \quad \text{(25)} \]
by (19) and (22). We can now rewrite (23) as
\[ \Phi_{2n+1} = -A^* + D^* + A + D, \quad \Phi_{2n+1}^* = -A + D + A^* + D^*. \]
(Where the \((*)\)-operations in these identities are of order \( 2^j, j > 2n + 1 \)).

Then, by (24),
\[ \Phi_{2n+1} + \Phi_{2n+1}^* + \bar{F}(\Phi_{2n+1}^* - \Phi_{2n+1}) = 2 \left( D^* + D \right) + \left( D + D^* \right) \left( A^* - A \right) + O(1) = 2 \left( D^* + \frac{D}{A} A^* \right) + O(1) = 2 \left( \frac{2z_n^2}{A} + \frac{D}{A} \right) = O(1) \]
by (25).

\[ \square \]

3. Appendix: properties of polynomials and Schur parameters, proofs of Lemma 2.26 and Lemma 2.4

3.1. Setup. We wish to analyze the asymptotics of the orthogonal polynomials with respect to the weight
\[ f(z) = e^{i\theta} g_{z,j}(z) g_{z,j}(z), \]
where \( z = |z| e^{i\theta}, \theta \in [0, 2\pi) \). For \( z = e^{i\theta} \),
\[ g_{z,j, \beta_j} = \left\{ \begin{array}{ll}
e^{i\pi \beta_j} & : 0 \leq \arg z < \theta_j \\
e^{-i\pi \beta_j} & : \theta_j \leq \arg z < 2\pi
\end{array} \right. \]
as defined in the main text.

This \( f \) belongs to the class of Fisher-Hartwig weights considered in [4] (see also [8] for the weight on the real line), so much of this Appendix consists of examining the results of [4], which performs asymptotic
analysis of a Riemann-Hilbert problem involving the orthogonal polynomials. Recall the three properties of these polynomials we need:

\begin{align*}
(1) & \quad |\Phi_n^*(z)| \sim 1, \\
(2) & \quad \|\Phi_n^*\Psi_n^* + z\Phi_n \Psi_n\|_{L^\infty(T)} \gg \ln n, \\
(3) & \quad \frac{\Psi_n^*(z)}{\Phi_n(z)} + \frac{\Psi_n^*(-z)}{\Phi_n^*(-z)} = O(1),
\end{align*}

for \( z \in T, n > n_0(\epsilon) \).

As was noted in \cite{3} and \cite{9}, the orthogonal polynomials of the first and second kinds satisfy a particular Riemann-Hilbert problem with contour \( C = T \). If \( Y \) is defined by

\begin{equation}
Y(z) = \begin{pmatrix}
\Phi_n(z) & \int_T \Phi_n(\xi) f(\xi) d\xi \\
-\Phi_{n-1}(z) & -\int_T \Phi_{n-1}(\xi) f(\xi) d\xi
\end{pmatrix}, \tag{26}
\end{equation}

then it satisfies the following Riemann-Hilbert problem:

- \( Y(z) \) is analytic in \( \mathbb{C} \setminus T \).
- For \( z \in T \setminus \{i, -i\} \), \( Y \) has continuous boundary values \( Y_+(z) \) as \( z \) approaches \( T \) from the inside, and \( Y_-(z) \) from the outside, related by the jump condition
  \[ Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n} f(z) \\ 0 & 1 \end{pmatrix}. \]
- \( Y(z) \) has the following asymptotic behavior at infinity:
  \[ Y(z) = \left( I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 1 & z^{-n} \end{pmatrix}. \]
- As \( z \to \pm i, z \in \mathbb{C} \setminus T \),
  \[ Y(z) = \begin{pmatrix} O(1) & O(\ln|z \pm i|) \\ O(1) & O(\ln|z \pm i|) \end{pmatrix}. \]

In what follows, we consider \( n \) to be a sufficiently large but fixed parameter.

The analysis of Riemann-Hilbert problems of this type proceeds by: enclosing the singularity points \( z_j \) by small but fixed disks \( U_{z_j} \); enclosing the curve \( C = T \) by lenses meeting in these disks; solving the Riemann-Hilbert problems induced in these regions; finally stitching them back together. For Fisher-Hartwig singularities this was performed in \cite{4}. Since two of our estimates must be uniform over \( R \) and \( C_0 \), we will be concerned with the formulas that result in the regions enclosed by \( U_{z_j} \) as well as the the regions outside \( U_{z_j} \), but inside the lenses.

Before we begin the analysis of Riemann-Hilbert problem, we list a few useful identities and notations. Following \cite{4}, our complex logarithm will be cut at the negative real axis unless otherwise noted.

The measure \( d\mu = \frac{f}{2\pi \cosh \epsilon} d\theta \) is a probability one. Therefore, one has (see \cite{20}, equation (3.2.53)):

\begin{align*}
S(\mu)\Phi_{n-1}(z) - \Psi_{n-1}^*(z) &= z^{n-1} \int_T \frac{e^{i\theta} + z}{e^{i\theta} - z} \Phi_{n-1}(e^{i\theta}) d\mu \\
&= z^{n-1} \int_T \frac{e^{i\theta} - z + 2z \Phi_{n-1}(e^{i\theta})}{e^{i\theta} - z} d\mu = 0 + 2z^n \int_T \frac{\Phi_{n-1}(\xi)}{\xi - z} f(\xi) d\xi \\
&= -Y_{22}(z) = \int_T \frac{\Phi_{n-1}(\xi)}{\xi - z} f(\xi) d\xi = \frac{1}{2\pi n} \left( F(z) \Phi_n^*(z) - (\cosh \epsilon) \Psi_n^*(z) \right), \tag{27}
\end{align*}

where \( F \) is the Carathéodory function associated to \( f(\theta) \frac{d\theta}{2\pi} \) (i.e. \( F = S(f/2\pi) \)), and \( \Psi_n \) is the second kind polynomial associated to \( \Phi_n \).
The exact expression for $F$ is easy to compute:

$$F(z) = \int_\mathbb{T} \frac{e^{i\theta} + z}{e^{i\theta} - z} f(\theta) \frac{d\theta}{2\pi} = -\frac{i(e^z - e^{-z})}{\pi} \ln \left( \frac{i - z}{i + z} \right) + \frac{1}{2}(e^{-z} + e^z).$$ (28)

We will use the following notation from [4]:

$$\beta_1 = -i\varepsilon/\pi, \quad \alpha_1 = 0, \quad z_1 = i$$
$$\beta_2 = i\varepsilon/\pi, \quad \alpha_2 = 0, \quad z_2 = -i$$

Before we discuss the results obtained in [4], we need to introduce confluent hypergeometric function $\psi(a, b, \zeta)$ which plays the key role in the analysis of the Riemann-Hilbert problem. We will have to use many facts about $\psi$. For this purpose we refer the reader to the National Institute of Standards and Technology’s Digital Library of Mathematical Functions [23] and the appendix of [15].

The function $\psi(a, b, \zeta)$ is the confluent hypergeometric function of the second kind, often written as $U(a, b, \zeta)$. It is defined as the unique solution to Kummer’s equation

$$\zeta \frac{d^2w}{dz^2} + (b - \zeta) \frac{dw}{d\zeta} - aw = 0,$$ satisfying $w(a, b, \zeta) \sim \zeta^{-a}$ as $\zeta \to \infty$. We will be interested in the following choices of the parameters: $b = 1$ and $a = \{ \beta_j, 1 + \beta_j \}, j = 1, 2$. The function $\psi$ is analytic in $\zeta$ on the universal cover of $\mathbb{C}\setminus 0$ and can be represented by the series (formula 13.2.9 in [23]):

$$\psi(a, 1, \zeta) = -\frac{1}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(a)_k}{k!^2} \zeta^k \left( \ln \zeta + \frac{\Gamma'(a+k)}{\Gamma(a+k)} - \frac{2\Gamma'(k+1)}{\Gamma(k+1)} \right),$$ (29)

where $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$ is the Pochhammer symbol. This allows us to write $\psi$ as

$$\psi(\zeta) = g(\zeta) \ln \zeta + h(\zeta),$$ (30)

where $g$ and $h$ are entire and single-valued. In particular, we have

$$\psi(a, 1, \zeta) = -\frac{1}{\Gamma(a)} \left( \ln \zeta + \frac{\Gamma'(a)}{\Gamma(a)} - \frac{2\Gamma'(1)}{\Gamma(1)} \right) + O(\ln \zeta)$$ (31)

for $\zeta : |\zeta| < 1$. The precise asymptotics of $\psi$ as $\zeta \in \mathbb{C} \to \infty$, $-3\pi/2 < \arg \zeta < 3\pi/2$ for fixed $a$ is (formula (7.2) in [15])

$$\psi(a, 1, \zeta) = \zeta^{-a} \left[ 1 - a^2 \zeta^{-1} + O(\zeta^{-2}) \right]$$ (32)

This asymptotics is a consequence of the following integral representation of $\psi$ (formula (7.3), [15]):

$$\psi(a, 1, \zeta) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{-\zeta a}} t^{a-1}(1+t)^{-a} e^{-zt} dt, \quad -\pi < \alpha < \pi, -\pi/2 + \alpha < \arg \zeta < \pi/2 + \alpha.$$ (33)

That representation, in particular, implies that

$$\sup_{u \in i\mathbb{R}, |u| > 1} |\psi(\pm i\varepsilon, 1, u)| \to 1, \quad \varepsilon \to 0$$ (34)

and

$$\sup_{u \in i\mathbb{R}, |u| > 1} |\psi(1 \pm i\varepsilon, 1, u) - \psi(1, 1, u)| \to 0, \quad \varepsilon \to 0.$$ (35)

The crucial property of $\psi$ which makes it indispensable for the Riemann-Hilbert analysis is the following transformation formula (formula (7.30) in [15])

$$\psi(a, c, e^{-2\pi i} \zeta) = e^{2\pi i a} \psi(a, c, \zeta) - \frac{2\pi i}{\Gamma(a) \Gamma(c-a+1)} e^{\pi i a} e^{\zeta} \psi(c - a, c, e^{-i\pi} \zeta).$$ (36)

Following [4], we will use the convention that, unless otherwise mentioned, $\zeta$ always satisfies $0 \leq \arg \zeta < 2\pi$. 
Concerning the logarithmic derivative of the Gamma function (the digamma function) which appears above, we will have occasion to use its reflection formula (equation 5.15.6, [23])
\[
\frac{\Gamma'(1-z)}{\Gamma(1-z)} - \frac{\Gamma'(z)}{\Gamma(z)} = \pi \cot(\pi z).
\]  
(35)

Now, we need to discuss another function which will be important below. Consider
\[
\mathcal{D}(z) = \exp\left(\frac{1}{2\pi i} \int_T \frac{\ln f(s)}{s-z} ds\right).
\]
This function is analytic away from $T$ with a.e. boundary values satisfying $\mathcal{D}_+(z) = \mathcal{D}_-(z)f(z), |\mathcal{D}_+(z)|^2 = f(z)$ (compare with $\Pi$ defined in [16]). Notice this definition a priori differs from that in [20], section 2.4. However due to the normalization $\int_T \ln(f(\theta))d\theta = 0$, in fact these two agree. We may compute $\mathcal{D}$ explicitly:
\[
\mathcal{D}(z) = \exp\left(\frac{1}{2\pi i} \int_T \frac{\ln f(s)}{s-z} ds\right) = \exp\left(\frac{\epsilon}{\pi i} \ln \left(\frac{i-z}{i+z}\right)\right).
\]  
(36)

3.2. Asymptotic formulas for solution of the Riemann-Hilbert problem. In the next two subsections, we recall how asymptotics of $Y$ on $T$ is obtained through solving Riemann-Hilbert problem. Then we will apply this asymptotics to prove Lemma 2.5.

In [4], the Riemann-Hilbert problem undergoes various transformations until it is in a form for which explicit solutions can be written. The singularity points $i$ and $-i$, along with the artificially introduced point $z = 1$ (though the analysis reduces to triviality here and we drop this case) are all enclosed by the small disks $U_i, U_{-i}$ of fixed radius $\delta > 0$. The remainder of the unit circle is enclosed in “lenses” (see Figure 1).

We trace through the various transformations of the RH problem for $z$ away from the points of singularity $i$ and $-i$. These reductions are:
\[
Y \to T \to S \to R.
\]  
(37)

We will explain each of these transformations below. However, our analysis will be limited to the case when $\epsilon \in (0,\epsilon_0), n > n_0(\epsilon), |z| \leq 1$ and $z$ belongs to one of the lenses. This choice is motivated by our goal to control $Y$ only on the unit circle $T$ itself so we will only need to take $|z| = 1$ later on. In these domains, some of the transformations in [3] are trivial, e.g., $T = Y$ (formula (4.1) in [4]).
Then, (formula (4.3), [4]), \( S \) is related to \( T \) by

\[
S(z) = T(z) \begin{pmatrix} 1 & 0 \\ -f(z)^{-1} z^n & 1 \end{pmatrix}.
\]

Now, the original Riemann-Hilbert problem for \( Y \) can be written in terms of \( S \) and its solution proceeds by first choosing various parametrices (approximate solutions) in each of the domains. The parametrices outside of \( \cup_j U_j \) and inside of each \( U_j \) will be denoted by \( N \) and \( P_z \), respectively. Our final transformation is to \( R \), which satisfies the Riemann-Hilbert problem of very special form. In fact, the correct choices of parametrices \( N \) and \( P_z \) makes it possible to say that each jump in the Riemann-Hilbert problem for \( R \) is of the form \( I + O(n^{-1}) \) when \( n \to \infty \) on each of the contours involved (see (4.57)–(4.59)) and an asymptotics of \( R \) at infinity is \( I + O(1/z) \).

Then, the standard argument (see, e.g., [5]) implies that

\[
R = I + O(n^{-1})
\]

uniformly over \( z \in \mathbb{C} \). It is clear now that the main asymptotics of \( Y \) is captured by \( N \) and \( P_z \). Below we will discuss these parametrices in detail.

3.2.1. Case 1. Parametrix \( N \), \( z \) outside of \( U_z \). For \( z : |z| < 1 \), we write (formula (4.7), [4])

\[
N(z) = \begin{pmatrix} \mathcal{D}(z) & 0 \\ 0 & \mathcal{D}(z)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

\[
R(z) = S(z) N^{-1}(z)
\]

and

\[
R(z) = I + O \left( \frac{1}{n} \right)
\]

by equations (4.61) and (4.65-71) in [4]. Since we are away from the singularities of the weight, all terms are uniformly bounded. Collecting \( O \left( \frac{1}{n} \right) \) errors, we have

\[
S(z) = N(z) + O \left( \frac{1}{n} \right).
\]

Reversing these transformations

\[
Y(z) = T(z) = S(z) \begin{pmatrix} 1 & 0 \\ -f(z)^{-1} z^n & 1 \end{pmatrix}^{-1} = \left( N(z) + O \left( \frac{1}{n} \right) \right) \begin{pmatrix} 1 & 0 \\ f(z)^{-1} z^n & 1 \end{pmatrix}
\]

\[
= \left( \begin{pmatrix} \mathcal{D}(z) & 0 \\ 0 & \mathcal{D}(z)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + O \left( \frac{1}{n} \right) \right) \begin{pmatrix} 1 & 0 \\ f(z)^{-1} z^n & 1 \end{pmatrix}.
\]

So,

\[
Y(z) = \begin{pmatrix} z^n f^{-1} \mathcal{D}(z) & \mathcal{D}(z) \\ -\mathcal{D}(z)^{-1} & 0 \end{pmatrix} + O \left( \frac{1}{n} \right),
\]

when \( |z| < 1 \) and \( z \notin U_z \). Now, to get asymptotical behavior of \( Y(z) \) (and thus of \( \Phi^*_n(z) \) and \( \Psi^*_n(z) \) by formula (27) on \( T \) but outside \( U_z \)), we need to take \( |z| \to 1 \) in (39) and this gives

\[
Y(z) = \begin{pmatrix} z^n f^{-1} \mathcal{D}_+(z) & \mathcal{D}_+(z) \\ -\mathcal{D}_+(z)^{-1} & 0 \end{pmatrix} + O \left( \frac{1}{n} \right)
\]

uniformly over \( z \in T, z \notin U_z \).
3.2.2. Case 2: Parametrix $P_{z_j}$, $z \in U_{z_j}$. The nature of the singularities of the weight $f$ at points $z_1$ and $z_2$ is the same so we will discuss only asymptotics in $U_{z_1}$ in detail. By formula (4.23) in [4], we write

$$P_{z_j}(z) = E(z)\Psi^{(j)}(z) \left( \begin{array}{cc} z^{n/2} & 0 \\ 0 & z^{-n/2} \end{array} \right),$$

where $E(z)$ is chosen so that $P_{z_j}$ and $N$ approximately match across $\partial U_{z_j}$. For example, we choose region I on Figure 1 (we can also take the similar domain around $z_2$). This corresponds to $|z| < 1, 0 < \arg \frac{z}{z_j} < \pi/2$. By introducing $\zeta = n \ln \frac{z}{z_j}$ we get $\arg \zeta \in (\pi/2, \pi)$ and the choice of $E$ and $\Psi^{(j)}$ are made by (formula (4.32), [4])

$$\Psi^{(j)}_j(\zeta) = \left( \begin{array}{cc} \psi(\beta_j, 1, \zeta)e^{2i\pi\beta_j} \left( \frac{z}{z_j} \right)^{-n/2} & -e^{i\pi\beta_j}\psi(1 - \beta_j, 1, e^{-i\pi\zeta}) \left( \frac{z}{z_j} \right)^{n/2} \frac{\Gamma(1-\beta_j)}{\Gamma(z_j)} \\ -e^{i\pi\beta_j}\psi(1 + \beta_j, 1, \zeta) \left( \frac{z}{z_j} \right)^{-n/2} & \psi(-\beta_j, 1, e^{-i\pi\zeta}) \left( \frac{z}{z_j} \right)^{n/2} \end{array} \right)$$

$$(1) \left( \begin{array}{cc} \zeta \psi_1^{(j)} & 0 \\ 0 & -\zeta^{-1} \psi_1^{(j)} \end{array} \right) \left( \begin{array}{cc} \zeta \psi_2^{(j)} & 0 \\ 0 & -\zeta^{-1} \psi_2^{(j)} \end{array} \right) \left( \begin{array}{cc} 0 & e^{-2i\pi\beta_j} \\ 0 & e^{i\pi\beta_j} \end{array} \right).$$

Our matrices, we find

$$P_{z_j}(z) = \left( \begin{array}{cc} 0 & D(z) \\ -D(z)^{-1} & 0 \end{array} \right) \left( \begin{array}{cc} \psi_1^{(j)} & 0 \\ 0 & \psi_1^{(j)} \end{array} \right) \left( \begin{array}{cc} \zeta \psi_1^{(j)} & 0 \\ 0 & -\zeta^{-1} \psi_1^{(j)} \end{array} \right) \left( \begin{array}{cc} 0 & e^{-2i\pi\beta_j} \\ 0 & e^{i\pi\beta_j} \end{array} \right),$$

Restricting our attention to $j = 1$, and using the notation $Y^{(1,I)}$ to identify this as the solution $Y$ in region I around point $z_1 = i$

$$Y^{(1,I)} = T = RP_{z_1} \left( \begin{array}{cc} 1 & 0 \\ f(z)^{-1}z^n & 1 \end{array} \right).$$

We have again

$$R(z) = S(z)P_{z_1}^{-1}(z)$$

and

$$R(z) = I + O\left( \frac{1}{n} \right)$$

as follows from [8]. However, since we have singularities in our expressions, we will leave $R$ in this form for now. This yields

$$Y^{(1,I)}(z) = \left( I + O\left( \frac{1}{n} \right) \right) P_{z_1}(z) \left( \begin{array}{cc} f(z)^{-1}z^n & 0 \\ 1 & 1 \end{array} \right).$$

(41)

$$= \left( I + O\left( \frac{1}{n} \right) \right) \left( \begin{array}{cc} 0 & D(z) \\ -D(z)^{-1} & 0 \end{array} \right) \left( \begin{array}{cc} \zeta^{-i\pi/\psi_1^{(1)}} & 0 \\ -e^{2i\pi\beta_1} \zeta^{i\pi/\psi_1^{(1)}} & \zeta^{i\pi/\psi_1^{(1)}} \end{array} \right) \left( \begin{array}{cc} 0 & e^{-2i\pi\beta_1} \\ 0 & e^{i\pi\beta_1} \end{array} \right)$$

$$= \left( I + O\left( \frac{1}{n} \right) \right) \left( \begin{array}{cc} -D(z)^{-1} & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} \zeta^{-i\pi/\psi_1^{(1)}} & 0 \\ -e^{2i\pi\beta_1} \zeta^{i\pi/\psi_1^{(1)}} & \zeta^{i\pi/\psi_1^{(1)}} \end{array} \right) \left( \begin{array}{cc} 0 & e^{-2i\pi\beta_1} \\ 0 & e^{i\pi\beta_1} \end{array} \right)$$

$$= \left( I + O\left( \frac{1}{n} \right) \right) \left( \begin{array}{cc} \frac{2}{i\pi} \zeta^{i\pi/\psi_1^{(1)}} & -e^{2i\pi\beta_1} \zeta^{-i\pi/\psi_1^{(1)}} \\ -e^{2i\pi\beta_1} \zeta^{i\pi/\psi_1^{(1)}} & \zeta^{-i\pi/\psi_1^{(1)}} \end{array} \right) \left( \begin{array}{cc} 0 & e^{-2i\pi\beta_1} \\ 0 & e^{i\pi\beta_1} \end{array} \right).$$
In $U_{z_2}$, the calculations are exactly the same, but with $z_1 = i$ replaced by $z_2 = -i$ and $\beta_1$ replaced by $\beta_2 = -\beta_1$. Therefore, we have

$$Y^{(2,1)}(z) = \left( I + O\left( \frac{1}{n} \right) \right) \left( \begin{array}{c} \mathcal{D}(z)(-c_1^{(2)}(-i))n\zeta^{ie/\pi}\psi_1^{(2)} + e^{-\epsilon}f_{-1}^{(2)}n\zeta^{ie/\pi}\psi_2^{(2)} \\ \mathcal{D}(z)(-1)(\zeta^{ie/\pi}\psi_1^{(2)}) - e^{-\epsilon}f_{-1}^{(2)}n\zeta^{ie/\pi}(-i) - n\psi_3^{(2)} \end{array} \right).$$

Recalling that

$$f(z) = \left\{ \begin{array}{ll} e^{\epsilon i}, & -\pi/2 \leq \arg z < \pi/2 \\ e^{-\epsilon}, & \text{otherwise} \end{array} \right.,$$

we see that $f = e^{-\epsilon}$ in $U_{z_1}$, region I, and $f = e^{\epsilon}$ in region I of $U_{z_2}$. Thus, we get

$$Y^{(1,1)}(z) = \left( I + O\left( \frac{1}{n} \right) \right) \left( \begin{array}{c} e^{2\epsilon}\mathcal{D}(z)(-c_1^{(1)}\zeta^{ie/\pi}\psi_1^{(1)} + z_n\zeta^{ie/\pi}\psi_1^{(1)}) \\ \mathcal{D}(z)(-1)(\zeta^{ie/\pi}\psi_1^{(1)}) - z_n\zeta^{ie/\pi}(-i) - n\psi_3^{(1)} \end{array} \right) \quad \text{and}$$

$$Y^{(2,1)}(z) = \left( I + O\left( \frac{1}{n} \right) \right) \left( \begin{array}{c} e^{-2\epsilon}\mathcal{D}(z)(-c_2^{(2)}\zeta^{ie/\pi}(-i)\psi_1^{(2)} + z_n\zeta^{ie/\pi}\psi_2^{(2)}) \\ \mathcal{D}(z)(-1)(\zeta^{ie/\pi}\psi_1^{(2)}) - z_n\zeta^{ie/\pi}(-i) - n\psi_3^{(2)} \end{array} \right).$$

Because of the singularities involved, we must take care in performing this last multiplication. Denote by $Y_{\delta, j}$ the right-hand matrix in the above equation. That is

$$\tilde{Y}_{i,j}(z) := \left( \begin{array}{c} e^{2\epsilon}\mathcal{D}(z)(-c_1^{(1)}\zeta^{ie/\pi}\psi_1^{(1)} + z_n\zeta^{ie/\pi}\psi_2^{(2)}) \\ \mathcal{D}(z)(-1)(\zeta^{ie/\pi}\psi_1^{(1)}) - z_n\zeta^{ie/\pi}(-i) - n\psi_3^{(1)} \end{array} \right),$$

$$\tilde{Y}_{i,1,j}(z) := \left( \begin{array}{c} e^{-2\epsilon}\mathcal{D}(z)(-c_2^{(2)}\zeta^{ie/\pi}(-i)\psi_1^{(2)} + z_n\zeta^{ie/\pi}\psi_2^{(2)}) \\ \mathcal{D}(z)(-1)(\zeta^{ie/\pi}\psi_1^{(2)}) - z_n\zeta^{ie/\pi}(-i) - n\psi_3^{(2)} \end{array} \right).$$

We start with proving Lemma 2.4.

**Proof.** (of Lemma 2.4) We only need to use formula (1.23) from [1]. This equation shows, in the notations introduced above,

$$-\tau_{k-1} = \Phi_k(0) = k^{-2\beta_1-1}z^{2\beta_2-1}2^{\gamma}2^{\beta_2-1}2^{\gamma}(\frac{1 + \beta_2}{\Gamma(-\beta_2)} + k^{-2\beta_1-1}2^{2\beta_2-1}2^{2\beta_2}2^{\beta_2-1}z^{\gamma}2^{\gamma}\Gamma(1 + \beta_2)\Gamma(-\beta_1) + r_{k, \epsilon})$$

and

$$|r_{k, \epsilon}| < C_{\epsilon}(k + 1)^{-2}. \quad (45)$$

**Remark.** The estimates (44), (45), and (33) imply that the recursion coefficients for the weight $f$ satisfy

$$||\gamma_k||_{L^2} \lesssim \sqrt{c}, \quad |\gamma_k| \lesssim (k + 1)^{-1}. \quad (46)$$

Recall that $\Psi_n$ and $\Psi_n^*$ satisfy recursion

$$\begin{cases} \Psi_{n+1} = z\Psi_n + \tau_n \Psi_n^* \\ \Psi_{n+1} = \Psi_n + \gamma_n z\Psi_n \end{cases}$$

and we have

$$|\Psi_{n+1}| \leq |\Psi_n|(|1 + |\gamma_n||, \quad |\Psi_0| = 1$$

Iterating this formula and using (46) we get a rough upper bound

$$||\Psi_n||_{L^\infty(T)} \lesssim e^{nC_\epsilon}. \quad (48)$$

This estimate can be substantially improved by Riemann-Hilbert analysis but [48] will be good enough for our purposes.

Now we are ready to verify Lemma 2.6.

**Proof.** (of Lemma 2.6).
3.3. \(|\Phi_n^*(z)| \sim 1, z \in \mathbb{T}\) for \(\epsilon \in (0, \epsilon_0]\) and \(n > n_0(\epsilon)\).

We consider two cases.

3.3.1. \(z\) outside \(U_2\). By equation (29), \(\Phi_n^*(z) \to \mathbb{D}_+(z)^{-1}\) uniformly in this region. Since \(|\mathbb{D}_+| = f^{1/2}\) a.e. on \(\mathbb{T}\), this trivially implies our desired estimate for \(z\) outside of \(U_2\).

3.3.2. \(z\) inside \(U_2\). We consider the boundary values as \(|z| \to 1\) in the asymptotics for \(Y\). Notice that, since \(|\mathbb{D}_+|^2 = f\), we have \(|\mathbb{D}_+| = e^{\epsilon/2}\) in region I around \(z = i\). We will focus on region I where \(\zeta = iu, u > 0\).

In the other regions, analysis is the same. Recall that \(\zeta = n \ln \frac{\psi}{\Gamma(1)}\). So, if \(z = e^{i(\pi/2 + \tau)}, \tau > 0\), we have
\[
\zeta^{-i\epsilon/\pi} = \left(n \ln \frac{z}{i}\right)^{-i\epsilon/\pi} = (n i \tau)^{-i\epsilon/\pi} = e^{\epsilon/2} e^{-i \epsilon \frac{\pi}{n \tau} \ln(n \pi)}.
\]

Therefore, in region I, in which \(\arg \frac{\psi}{\Gamma(1)} > 0\), one has
\[
\frac{1}{|\mathbb{D}(z)\zeta^{i\epsilon/\pi}|} = \epsilon^\epsilon.
\]

Similarly, in region IV, the other side of \(i\), \(|\mathbb{D}(z)| = e^{\epsilon/2}\) and \(|\zeta^{-i\epsilon/\pi}| = |e^{-i \epsilon \pi / n \tau} \ln(n \pi)| = e^{3 \epsilon/2}\) since \(\arg \zeta = 3 \pi / 2\). We again obtain
\[
\frac{1}{|\mathbb{D}(z)\zeta^{i\epsilon/\pi}|} = \epsilon^\epsilon.
\]

Consider the expressions involving the \(\psi\) in the first column of \(\tilde{Y}_{i,21}(z)\). We focus on \(\tilde{Y}_{i,21}\), the bottom left corner of the \(Y\) matrix. Due to (10) and definition of \(\zeta\),
\[
|\tilde{Y}_{i,21}(z)| = \epsilon^\epsilon \left|\psi_1^{(1)} \left(\frac{z}{i}\right)^n c_2^{(1)}\psi_3^{(1)}\right| = \epsilon^\epsilon \left|\psi\left(-\frac{i \epsilon}{\pi}, 1, \zeta\right) - e^{\zeta^{-i \epsilon / \pi}} \frac{\Gamma(1 + \frac{i \epsilon}{\pi})}{\Gamma(-\frac{i \epsilon}{\pi})} \psi\left(1 + \frac{i \epsilon}{\pi}, 1, e^{-i \pi} \zeta\right)\right|.
\]

Consider
\[
\Omega(\zeta, \epsilon) = \psi\left(-\frac{i \epsilon}{\pi}, 1, \zeta\right) - e^{\zeta^{-i \epsilon / \pi}} \frac{\Gamma(1 + \frac{i \epsilon}{\pi})}{\Gamma(-\frac{i \epsilon}{\pi})} \psi\left(1 + \frac{i \epsilon}{\pi}, 1, e^{-i \pi} \zeta\right).
\]

It is the analysis of this function which concerns us. We want to show that
\[
\max_{\zeta = iu, u \in \mathbb{R}} |\Omega(\zeta, \epsilon)| - 1 \to 0, \quad \epsilon \to 0.
\]

We do this in two steps. The estimates (32), (33), and (35) imply that
\[
\max_{\zeta = iu, |u| > 1} |\Omega(\zeta, \epsilon)| - 1 \to 0, \quad \epsilon \to 0.
\]

For \(|\zeta| < 1\), we will use series (29) for \(\psi\). We want to show
\[
\max_{\zeta = iu, |u| < 1} |\Omega(\zeta, \epsilon)| - 1 \to 0, \quad \epsilon \to 0.
\]

Recall that \(\Gamma(\zeta)\) has a pole at 0 so \(\lim_{\epsilon \to 0} \Gamma^{-1}(\pm i \epsilon) = 0\). From (30), we get
\[
\Omega(\zeta, \epsilon) = (\ln \zeta) g(-i \epsilon / \pi, \zeta) + h(-i \epsilon / \pi, \zeta) - e^{\zeta^{-i \epsilon / \pi}} \frac{\Gamma(1 + i \epsilon / \pi)}{\Gamma(-i \epsilon / \pi)} \left(|\ln(e^{-i \pi} \zeta)| g(1 + i \epsilon / \pi, e^{-i \pi} \zeta) + h(1 + i \epsilon / \pi, e^{-i \pi} \zeta)\right),
\]

where
\[
g(a, \zeta) = -\frac{1}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \zeta^k,
\]
\[
h(a, \zeta) = -\frac{1}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(a)_k}{k!} \zeta^k \left(\frac{\Gamma'(a + k)}{\Gamma(a + k)} - \frac{2 \Gamma''(k + 1)}{\Gamma(k + 1)}\right).
\]

These expansions converge uniformly in \(\zeta: |\zeta| < 1\) and the coefficients depend on \(\epsilon\) explicitly. In \(\Omega\), the logarithmic singularities cancel each other as follows (recall that \(\zeta = iu, u > 0\))
\[
-\frac{\ln \zeta}{\Gamma(-i \epsilon / \pi)} + \frac{\ln(e^{-i \pi} \zeta)}{\Gamma(-i \epsilon / \pi)} = -\frac{i \pi}{\Gamma(-i \epsilon / \pi)} \to 0, \quad \epsilon \to 0,
\]
Performing the multiplication and noting the error, (42) gives 3.4. 

\[ \text{Therefore, the required asymptotics of } \Omega \text{ will follow from} \]

\[ \lim_{\epsilon \to 0} \frac{\Gamma'(-i\epsilon/\pi)}{\Gamma^2(-i\epsilon/\pi)} = 1. \]

Now, recall that (26), (43), (41) yield

\[ -\Phi_{n-1}^*(z) = O(n^{-1})\tilde{Y}_{i,11}^{(1)}(z) + (1 + O(n^{-1})\tilde{Y}_{i,21}^{(1)}(z). \]

This also allows us to easily perform the final multiplication in (41), since all elements in the \( \tilde{Y} \) matrix are \( O(1) \) when \( |z| > 1 \). Recall equation (27):

\[ 2z^nY_{22}(z) = (\cosh \epsilon)\Psi_{n-1}^*(z) - F(z)\Phi_{n-1}^*(z). \]

Performing the multiplication and noting the error, (42) gives

\[ Y_{22}(z) = \mathcal{D}(z)^{-1}e^{-c_2^{(1)}z - i\epsilon/\pi i - n\psi_3^{(1)}} + O\left(\frac{1}{n}\right). \]

By (42), \( \mathcal{D}(z)^{-1}e^{-c_2^{(1)}z - i\epsilon/\pi i - n\psi_3^{(1)}} = O(n^{-1/2}) \) and \( Y_{22}(z) = O(n^{-1/2}) \). Similarly, by equation (42),

\[ \Phi_n^* = \mathcal{D}^{-1} + O\left(\frac{1}{\sqrt{n}}\right) \]

for \( \zeta = iu, u \sim \sqrt{n} \). Therefore, we finally have

\[ |\Psi_n^*(z)| \sim \epsilon \ln n \]

due to (26). Recall that \( z^0\overline{\Phi_n} = \Phi_n \). So, we can write

\[ \Phi_n^*\Psi_n + z\Phi_n\Psi_n = \Phi_n^*\Psi_n + z^{2n+1}\overline{\Phi_n^*\Psi_n} = \Phi_n^*\Psi_n \left(1 + z^{2n+1}\frac{\Phi_n^*\Psi_n}{\overline{\Phi_n^*\Psi_n}}\right). \]

Inserting (52), (53), and using \( |\mathcal{D}| \sim 1 \), we have

\[ \frac{\overline{\Phi_n^*\Psi_n}}{\Phi_n^*\Psi_n} = \frac{F(z)(\mathcal{D}(z)^{-2} + O(1/V_n^2))}{F(z)(\mathcal{D}(z)^{-2} + O(1/V_{\tilde{n}}^2))} = \overline{F} \cdot \left(\frac{\overline{\mathcal{D}}}{\mathcal{D}}\right)^{-2} (1 + O(n^{-0.5})). \]

From (3.1) and (3.0) we can compute

\[ \overline{F} = -1 + o(1), \quad n \to \infty \]

and

\[ \frac{\overline{\mathcal{D}}}{\mathcal{D}} = \exp\left(\frac{2i\epsilon}{\pi}\ln(\theta/2 - \pi/4)\right) (1 + o(1)), \quad n \to \infty \]

in our range of \( z = e^{i\theta} \). Therefore, substitution into (55) shows that there is some \( \theta_0 : \theta_0 \in (\pi/2 + n^{-0.5}, \pi/2 + 2n^{-0.5}) \) for which

\[ \left|1 + z_0^{2n+1}\frac{\Phi_n^*(z_0)\Psi_n^*(z_0)}{\overline{\Phi_n^*(z_0)\Psi_n^*(z_0)}}\right| \sim 1 \]

where \( z_0 = e^{i\theta_0} \).

Since, by (54), we have \( |\Psi_n^*(z_0)| = O(\ln n) \), and \( |\Phi_n^*| \sim 1 \), we get

\[ \|\Phi_n^*\Psi_n + z\Phi_n\Psi_n\|_{L^\infty(\mathbb{T})} > \epsilon \ln n \]
as desired.

3.5. \( \Psi_n^*(z) + \Psi_n^*(-z) = O(1), z \in \mathbb{T}, n > n_0(\epsilon) \). Outside of \( U_{z_1(2)} \), this statement is trivial by (39), so we only consider \( z \) inside \( U_{z_1(2)} \). Further, since the calculations are exactly similar in \( U_i \) and \( U_{-i} \), we let \( z \in U_i \).

Before we proceed with the analysis of Riemann-Hilbert problem, let us make two remarks. Firstly, since \( |\Phi_n^*| \sim 1, n > n_0(\epsilon) \), we only need to show that \( U_{2n} \), defined by \( U_{2n}(z) = \Psi_n^*(z)\Phi_n^*(z) + \Psi_n^*(-z)\Phi_n^*(-z) \), satisfies

\[ \|U_{2n}\|_{L^\infty(\mathbb{T})} \lesssim 1, \]

Secondly, \( U_{2n} \) is a polynomial of degree at most \( 2n \) and

\[ \|U_{2n}\|_{L^\infty(\mathbb{T})} < \epsilon n^C, \]

by (41). The Bernstein inequality gives us

\[ \|U_{2n}\|_{L^\infty(\mathbb{T})} < \epsilon n^{1+C}. \]

Thus, to prove \( \|U_{2n}\|_{L^\infty(\mathbb{T})} = O(1) \), we only need

\[ |U_{2n}(e^{i\theta})| \lesssim 1 \]

for \( \theta : \theta \in (\pi/2 + e^{-\sqrt{n}}, \pi/2 + \delta_1) \) and the parameter \( \delta_1 \) here is of the same size as the radius of \( U_i \).

Recall that \( Y^{(1),(2),I} \) denotes the \( Y \) matrix in the region \( I \) that corresponds to point \( z_{1(2)} \), respectively. By (27),

\[ 2^n n Y_{21}^{(1)}(z) + 2(-z)^n n Y_{21}^{(2)}(-z) = \left( F(z) + F(-z) \right) - (\cosh \epsilon) \left( \frac{\Psi_{n-1}^*(z)}{\Phi_{n-1}^*(z)} + \frac{\Psi_{n-1}^*(-z)}{\Phi_{n-1}^*(-z)} \right), \]

so we want to show

\[ \left( F(z) + F(-z) \right) - 2 \left( z^n n Y_{21}^{(1)}(z) + (-z)^n n Y_{21}^{(2)}(-z) \right) = O(1) \]

uniformly on \( \mathbb{T} \) provided that \( n \) is large enough.

The formula (3.1) implies

\[ F(z) + F(-z) = -\frac{i(e^\epsilon - e^{-\epsilon})}{\pi} \left( \ln \left( \frac{i - z}{i + z} \right) + \ln \left( \frac{i + z}{i - z} \right) \right) + e^{-\epsilon + \epsilon} = e^\epsilon + e^{-\epsilon}. \]

Due to this cancelation, we may focus on

\[ z^n n Y_{21}^{(1)}(z) + (-z)^n n Y_{21}^{(2)}(-z) \]

where \( z = e^{i\theta}, \theta \in (\pi/2, \pi/2 + \delta_1) \). In fact, since we know that \( |\Phi_n^*| \sim 1 \) uniformly on \( \mathbb{T} \), we may multiply out the denominators and only examine

\[ z^n n Y_{21}^{(1)}(z) Y_{21}^{(2)}(-z) + (-z)^n n Y_{21}^{(2)}(-z) Y_{21}^{(1)}(z). \]

We must take care in performing the final multiplication in the Riemann-Hilbert problem. We have previously seen that \( Y_{21}(z) = \tilde{Y}_{21}(z) + O\left( \frac{1}{n} \right) \sim O(1) \) uniformly \( z \in \mathbb{T} \). Therefore,

\[ z^n n Y_{21}^{(1)}(z) Y_{21}^{(2)}(-z) + (-z)^n n Y_{21}^{(2)}(-z) Y_{21}^{(1)}(z) = \]

\[ z^n \tilde{Y}_{21}^{(1)}(z) \tilde{Y}_{21}^{(2)}(-z) + (-z)^n \tilde{Y}_{21}^{(2)}(-z) \tilde{Y}_{21}^{(1)}(z) + O \left( \max_{j=1,2} \frac{|\tilde{Y}_{12}(z)| + |\tilde{Y}_{12}(z)|}{n} \right). \]

Since the final term has a logarithmic singularity at \( z = i \), we will handle it away from this point. In the range \( z = e^{i\theta}, \pi/2 + e^{-\sqrt{n}} < \theta < \pi/2 + \delta_1 \), we have (by (42) and estimates on \( \psi \))

\[ \max_{j=1,2} \frac{|\tilde{Y}_{12}(z)| + |\tilde{Y}_{12}(z)|}{n} = O \left( \frac{\ln \zeta}{n} \right) = O(\sqrt{n}/n) = O(n^{-1/2}). \]
So, it suffices to show
\[ z^n \bar{Y}^{(1)}_{22}(z) \bar{Y}^{(2)}_{21}(-z) + (-z)^n \bar{Y}^{(2)}_{22}(-z) \bar{Y}^{(1)}_{21}(z) = O(1). \]  
(58)

By (42) and (43), we have
\[ z^n \bar{Y}^{(1)}_{22}(z) \bar{Y}^{(2)}_{21}(z) + (-z)^n \bar{Y}^{(2)}_{22}(-z) \bar{Y}^{(1)}_{21}(z) = \frac{1}{e^{\pi \text{D}(z) \zeta e^{i\pi}}} \left( \frac{z}{i} \right)^n c_{21}^{(1)} \psi^{(1)}_{3} \left( \frac{-i e^{i\pi}}{\text{D}(z)} \left( \psi^{(2)}_{1} - \left( \frac{z}{i} \right)^n c_{22}^{(2)} \psi^{(2)}_{3} \right) \right) + e^\epsilon e^{i e^{i\pi}/\text{D}(z)} \left( \frac{-z}{i} \right)^n c_{22}^{(2)} \psi^{(2)}_{3} \left( \frac{-1}{\text{D}(z) e^{i\pi}} \left( \psi^{(1)}_{1} - \left( \frac{z}{i} \right)^n c_{22}^{(2)} \psi^{(2)}_{3} \right) \right). \]

Notice it is not ambiguous to leave the arguments of the \( \psi \) functions unidentified, as \( \zeta = n \ln \frac{z}{z_j} \) and \( \frac{z}{i} = \frac{-z}{i} \). Taking absolute values, we have
\[ |z^n \bar{Y}^{(1)}_{22}(z) \bar{Y}^{(2)}_{21}(z) + (-z)^n \bar{Y}^{(2)}_{22}(-z) \bar{Y}^{(1)}_{21}(z)| = \frac{C}{|\text{D}(z)\text{D}(z)|^{-1}} \left| e^{-\epsilon} \left( c_{21}^{(1)} \psi^{(1)}_{3} - \left( \frac{z}{i} \right)^n c_{22}^{(2)} \psi^{(2)}_{3} \right) + e^\epsilon \left( c_{22}^{(2)} \psi^{(1)}_{3} - \left( \frac{z}{i} \right)^n c_{22}^{(2)} \psi^{(2)}_{3} \right) \right|. \]

By (32), (33), (34), these expressions are uniformly bounded in \( \zeta, |\zeta| > 1, \epsilon < \epsilon_0, n > n_0(\epsilon) \). Thus, we only need to consider the case \( \zeta : |\zeta| < 1 \). On that interval, \((z/i)^n = e^\epsilon = 1 + O(\zeta)\). We are concerned with the logarithmic singularities and the constant terms in the series expansions for \( \psi \). We isolate these terms and denote their sum \( c_0 \). Using the notation \( d(z) = \frac{\Gamma'(z)}{\Gamma(z)} \) for the digamma function, we get
\[
c_0 = e^{-\epsilon} e^\epsilon \frac{1}{\Gamma(\frac{i\pi}{\pi})\Gamma(\frac{-i\pi}{\pi})} \ln(e^{-i\pi} + d \left( 1 + \frac{i\pi}{\pi} \right) - 2d(1)) \ln(\zeta + d \left( \frac{i\pi}{\pi} \right) - 2d(1))
\]
\[
- (e^\epsilon + e^{-\epsilon}) \frac{1}{\Gamma(\frac{-i\pi}{\pi})\Gamma(\frac{i\pi}{\pi})} \ln(1 + e^{-i\pi}) \ln(1 + i\pi) - 2d(1)) \ln(\zeta + d \left( \frac{i\pi}{\pi} \right) - 2d(1))
\]
\[
+ e^\epsilon \frac{1}{\Gamma(\frac{i\pi}{\pi})\Gamma(\frac{-i\pi}{\pi})} \ln(1 - e^{-i\pi}) \ln(1 - i\pi) - 2d(1)) \ln(\zeta + d \left( \frac{i\pi}{\pi} \right) - 2d(1)).
\]

Performing these multiplications, writing \( \ln(iu) = \ln u + i\pi/2 \), \( \ln(e^{-i\pi} iu) = \ln u - i\pi/2 \), and pulling out the common factor yields
\[
c_0 = \ln u \left( e^{-\epsilon} \left( d \left( \frac{i\pi}{\pi} \right) - d \left( 1 - \frac{i\pi}{i\pi} \right) + i\pi \right) + e^\epsilon \left( d \left( -\frac{i\pi}{\pi} \right) + d \left( 1 + \frac{i\pi}{i\pi} \right) + i\pi \right) \right) + O(1),
\]
where \( \zeta = iu \). Using the reflection formula (35) gives
\[
\ln u \left( e^\epsilon \left( -\pi \cot(-i\pi) + e^i\pi + e^{-\epsilon}(-\pi \cot(i\pi)) + e^{-\epsilon}i\pi \right)
\]
\[
- \pi \ln u \left( i(e^\epsilon + e^{-\epsilon}) - e^{-\epsilon} \cot(-i\pi) - e^\epsilon \cot(i\pi) \right) = 0,
\]
because \( \cot z = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}} \). Therefore,
\[
z^n \bar{Y}^{(1)}_{22}(z) \bar{Y}^{(2)}_{21}(-z) + (-z)^n \bar{Y}^{(2)}_{22}(-z) \bar{Y}^{(1)}_{21}(z) = O(1)
\]
and
\[
\frac{\Psi_n^\epsilon(z)}{\Phi_n^\epsilon(z)} + \frac{\Psi_n^\epsilon(-z)}{\Phi_n^\epsilon(-z)} = O(1)
\]
uniformly in \( z \in \mathbb{T} \) for \( n \) large enough. This finishes the proof of Lemma 2.5. \( \square \)
Acknowledgement

The work of SD done in the first part of the paper was supported by RSF-14-21-00025 and his research on the rest of the paper was supported by the grant NSF-DMS-1464479. The research of KR was supported by the RTG grant NSF-DMS-1147523.

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