The Froissart–Martin Bound for $\pi\pi$ Scattering in QCD

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Abstract

The Froissart–Martin bound for total $\pi\pi$ scattering cross sections is reconsidered in the light of QCD properties such as spontaneous chiral symmetry breaking and the counting rules for a large number of colours $N_c$. 
I Introduction

Since the early work by Marcel Froissart [1], André Martin [2] and colleagues [3] have shown in a series of seminal papers [1] that under very general assumptions, total cross sections for $\pi\pi$, $K\bar{K}$, $\pi N$ and $\pi\Lambda$ scattering cannot grow faster than

$$\sigma^{\text{tot}}(s) \sim \frac{4\pi}{t_0} \log \frac{s}{s_0},$$

(1.1)

where $s$ is the total CM-energy squared, $t_0$ denotes the lowest mass squared singularity in the $t$–channel, which for the processes mentioned above occurs at $4m^2$, and the normalization $s_0$ in the $\log^2 s$ is arbitrary indicating where the asymptotic behaviour sets in. Although several hadronic models have been shown to saturate the FM bound [3], it is quite frustrating that the advent of QCD as the theory of the strong interactions has not added anything new, at least so far, on the Froissart–Martin (FM) bound. Some obvious questions which one would like to answer are:

1. What happens in QCD in the chiral limit where pions, the Nambu–Goldstone states of the chiral SU(2) flavour symmetry of QCD, become massless? Does the bound become irrelevant, as the presence of the pion mass in the denominator in Eq. (1.1) seems to indicate?

2. What becomes of the FM bound in the Large–$N_c$ limit of QCD? The Large–$N_c$ counting rules fix $\sigma^{\text{tot}}(s)$ in Eq. (1.1) to be of $O(1/N_c)$, while the FM–bound appears to be of $O(1)$.

3. Independently of the previous questions concerning the chiral limit and the Large–$N_c$ limit, one would also like to know: is the $\log^2 s$ behaviour of the FM bound saturated in QCD?

The purpose of this paper is to set the path to an investigation of these questions. Here we shall limit ourselves to the case of total cross sections for $\pi\pi$ scattering. In the next section we summarize well known properties of the elastic $\pi\pi$ scattering amplitudes which we shall need for our discussion. The framework of our analyses uses a Mellin–Barnes representation for the $\pi\pi$ amplitudes which we present in Section III. This will allow us to fix the discussion concerning the first question above. Section IV is dedicated to a discussion of the FM bound within the framework of the QCD Large–$N_c$ limit. Our conclusions are given in Section V.

II Elastic Pion–Pion Scattering.

Elastic $\pi\pi$ scattering in the the isospin symmetry limit is described by a single invariant Lorentz amplitude $A(s, t, u)$ [3]

$$\langle \pi^d(p_4)\pi^c(p_3) \text{ out} |\pi^a(p_1)\pi^b(p_2) \text{ in} \rangle = 1 + i(2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2) \left\{ \delta^{ab} \delta^{cd} A(s, t, u) + \delta^{ac} \delta^{bd} A(t, u, s) + \delta^{ad} \delta^{bc} A(u, s, t) \right\},$$

(2.1)

1See e.g. ref. [4] where earlier references can be found.
2See e.g. refs. [5, 6] and references therein.
3For a modern review see ref. [7].
where \( a, b, c, d \) denote the 1,2,3 components of the adjoint representation of the pion fields in \( SU(2) \) and \( s, t \) and \( u \) the usual Mandelstam variables constrained by
\[
s + t + u = 4m_\pi^2. \tag{2.2}
\]
Because of the optical theorem which relates the absorptive part of an elastic amplitude to a total cross section we shall only consider elastic scattering amplitudes with the same in and out quantum numbers, i.e.:
\[
A_{\pi^\pm \pi^0 \rightarrow \pi^\pm \pi^0}(s, t) = A(t, u, s),
\]
\[
A_{\pi^0 \pi^0 \rightarrow \pi^0 \pi^0}(s, t) = A(s, t, u) + A(t, u, s) + A(u, s, t),
\]
\[
A_{\pi^+ \pi^- \rightarrow \pi^+ \pi^-}(s, t) = A(s, t, u) + A(t, u, s),
\]
\[
A_{\pi^\pm \pi^\pm \rightarrow \pi^\pm \pi^\pm}(s, t) = A(t, u, s) + A(u, s, t). \tag{2.3}
\]

It is convenient to work with the three \( s \)–channel isospin components \( T = (T_0, T_1, T_2) \) of the amplitudes in Eq. (2.1) given by:
\[
T_0(s, t) = 3A(s, t, u) + A(t, u, s) + A(u, s, t),
\]
\[
T_1(s, t) = A(t, u, s) - A(u, s, t),
\]
\[
T_2(s, t) = A(t, u, s) + A(u, s, t). \tag{2.4}
\]

These amplitudes obey fixed-\( t \) dispersion relations, valid in the interval \(-28m_\pi^2 < t < 4m_\pi^2\). They are the so called Roy equations \[3\] which we shall consider at \( t = 0 \) and, because of our first question in the introduction concerning the chiral limit of the FM–bound, at \( m_\pi \to 0 \). The Roy equations simplify then as follows:
\[
\text{Re} \left[ \frac{T_0(s, 0)}{T_1(s, 0)} \right] = \frac{s}{t_\pi^2} \begin{pmatrix}
2 \\
1 \\
-1
\end{pmatrix}
\]
\[
+ s^2 \int_0^\infty ds' \frac{1}{s'^2} \begin{pmatrix}
1/3 & -1/3 & 5/6 \\
-1/3 & 1/2 & 5/6 \\
1/3 & 1/2 & 1/6
\end{pmatrix} \frac{1}{\pi} \text{Im} \begin{pmatrix}
T_0(s', 0) \\
T_1(s', 0) \\
T_2(s', 0)
\end{pmatrix}. \tag{2.5}
\]

The term in the r.h.s. in the first line of this equation reflects the two subtractions which have been made, as required by the Froissart bound \[4\]. In QCD, the explicit values of these subtractions are fixed by lowest order \( \chi \)PT \[5\]. We recall that in chiral \( SU(2) \) the amplitude \( A(s, t, u) \) in the limit we are considering is given by the \( \chi \)PT expansion (see ref. \[10\] and earlier references therein):

\[\text{Notice, however, that the presence of the two powers of log } s \text{ in the asymptotic behaviour of the absorptive amplitudes does not restrict any further the two subtractions which are already required for a cross section going as a constant at } s \to \infty.\]
\[ A(s, t, u) \sim 0 \quad \frac{s}{f_\pi^2} + \frac{1}{f_\pi^4} \left[ 2s^2 l_1^\mu + \left( s^2 + (t - u)^2 \right) \frac{1}{2} l_2^\mu \right] + \frac{1}{96\pi^2 f_\pi^4} \left[ 3s^2 \left( \log \frac{\mu^2}{-s} + \frac{5}{6} \right) + t(t - u) \left( \log \frac{\mu^2}{-t} + \frac{7}{6} \right) + u(u - t) \left( \log \frac{\mu^2}{-u} + \frac{7}{6} \right) \right] + \mathcal{O}(p^6), \]  

(2.6)

where \( l_{1,2}^\mu \) are renormalized coupling constants of the \( \mathcal{O}(p^4) \) effective chiral Lagrangian at the scale \( \mu \). The terms in the first line of Eq. (2.6) are leading in the QCD Large-\( N_c \) limit but so far, in this section, we are not restricting ourselves to this limit. The terms in the second line are induced by the chiral loops generated by the lowest order Lagrangian, renormalized at the scale \( \mu \). The overall contribution of \( \mathcal{O}(p^4) \) is \( \mu \)-scale independent and well defined in the chiral limit. The relation between the \( l_i^s \) constants and the more conventional \( l_i^t \) constants of the chiral \( SU(3) \) Lagrangian \([11]\) is as follows:

\[ l_1^s(\mu) = 4L_1^s(\mu) + 2L_3 - \frac{1}{96\pi^2} \frac{1}{8} \left( \log \frac{M_K^2}{\mu^2} + 1 \right), \]  

(2.7)

\[ l_2^s(\mu) = 4L_2^s(\mu) - \frac{1}{96\pi^2} \frac{1}{4} \left( \log \frac{M_K^2}{\mu^2} + 1 \right), \]  

(2.8)

where here, kaon particles have been treated as massive and integrated out, hence the dependence on their mass \( M_K \).

The linear combinations of the isospin amplitudes \( T^I(s, 0) \) which diagonalize the crossing matrix in the second line of Eq. (2.5) are:

\[ F_1(s, 0) = -\frac{1}{6} T^0(s, 0) - \frac{1}{4} T^1(s, 0) + \frac{5}{12} T^2(s, 0), \]  

\[ F_2(s, 0) = +\frac{1}{6} T^0(s, 0) + \frac{1}{4} T^1(s, 0) + \frac{7}{12} T^2(s, 0), \]  

\[ F_3(s, 0) = -\frac{1}{6} T^0(s, 0) + \frac{3}{4} T^1(s, 0) + \frac{5}{12} T^2(s, 0), \]  

(2.9)

and the physical elastic forward scattering amplitudes we are concerned with are then given by:

\[ A_{\pi^\pm \pi^0 \rightarrow \pi^\pm \pi^0}(s, 0) = \frac{1}{2} \left[ F_2(s, 0) + F_3(s, 0) \right] = \frac{1}{2} \left[ T^1(s, 0) + T^2(s, 0) \right], \]  

\[ A_{\pi^0 \pi^0 \rightarrow \pi^0 \pi^0}(s, 0) = \frac{1}{2} \left[ 3F_2(s, 0) - F_3(s, 0) \right] = \frac{1}{3} \left[ T^0(s, 0) + 2 T^2(s, 0) \right], \]  

\[ A_{\pi^+ \pi^- \rightarrow \pi^+ \pi^-}(s, 0) = -F_1(s, 0) + F_2(s, 0) = \frac{1}{3} T^0(s, 0) + \frac{1}{2} T^1(s, 0) + \frac{1}{6} T^2(s, 0), \]  

\[ A_{\pi^\pm \pi^\pm \rightarrow \pi^\pm \pi^\pm}(s, 0) = F_1(s, 0) + F_2(s, 0) = T^2(s, 0). \]  

(2.10)
The Roy equations for the $F_i(s,0)$ amplitudes are then:

$$\text{Re} \begin{pmatrix} F_1(s,0) \\ F_2(s,0) \\ F_3(s,0) \end{pmatrix} = \frac{s}{f^2} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + s^2 \int_0^\infty ds' \frac{1}{s'^2} \left[ \frac{1}{s' - s} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{s' + s} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right] \frac{1}{\pi} \text{Im} \begin{pmatrix} F_1(s',0) \\ F_2(s',0) \\ F_3(s',0) \end{pmatrix}.$$ \hspace{1cm} (2.11)

From these equations there follows that the amplitudes $F_2$ and $F_3$ obey the same dispersion relation:

$$\text{Re} F_{2,3}(s,0) = s^2 \int_0^\infty ds'^2 \frac{1}{s'^2} \frac{1}{s'^2 - s^2} \frac{1}{\pi} \text{Im} F_{2,3}(s',0),$$ \hspace{1cm} (2.12)

and are even under $s \leftrightarrow -s$, while the amplitude $F_1(s,t)$ obeys the dispersion relation:

$$\text{Re} F_1(s,0) = -\frac{s}{f^2} + 2s^3 \int_0^\infty ds'/s^2 \frac{1}{s'^2} \frac{1}{s'^2 - s^2} \frac{1}{\pi} \text{Im} F_1(s',0),$$ \hspace{1cm} (2.13)

and is odd under $s \leftrightarrow -s$. Indeed, one can check that there is no contribution of $O(s^2)$ to the $F_1(s,0)$ amplitude in $\chi$PT, while the contributions of that order from $\chi$PT to the $F_2(s,0)$ and $F_3(s,0)$ amplitudes are:

$$\text{Re} F_2(s,0) = \left. \frac{s^2}{f^2} \right|_{s \to 0} \left[ 2l_1^\pi + 3l_2^\pi + \frac{1}{12\pi^2} \left( \log \frac{\mu^2}{s} + \frac{25}{24} \right) \right] + O(s^4),$$ \hspace{1cm} (2.14)

$$\text{Re} F_3(s,0) = \left. \frac{s^2}{f^2} \right|_{s \to 0} \left[ -2l_1^\pi + l_2^\pi + \frac{1}{96\pi^2} \right] + O(s^4).$$ \hspace{1cm} (2.15)

### III Mellin–Barnes Representation for the $F_i(s,0)$ Amplitudes.

The optical theorem relates the amplitudes $\text{Im} F_i(s,0)$ to the total $\pi\pi$ cross sections as follows (massless pions):

$$\text{Im} F_i(s,0) = \frac{1}{2} \left[ s \sigma_{\pi^+\pi^+}^{\text{tot}} - s \sigma_{\pi^0\pi^0}^{\text{tot}} \right],$$

$$\text{Im} F_2(s,0) = \frac{1}{2} \left[ s \sigma_{\pi^+\pi^+}^{\text{tot}} + s \sigma_{\pi^0\pi^0}^{\text{tot}} \right] = \frac{1}{2} \left[ s \sigma_{\pi^+\pi^0}^{\text{tot}} + s \sigma_{\pi^0\pi^0}^{\text{tot}} \right],$$

$$\text{Im} F_3(s,0) = \frac{1}{2} \left[ 3s \sigma_{\pi^+\pi^0}^{\text{tot}} - s \sigma_{\pi^0\pi^0}^{\text{tot}} \right].$$ \hspace{1cm} (3.1)

Let us then consider the Mellin transforms of the $\frac{1}{\pi} \text{Im} F_i(s,0)$ amplitudes:

$$\Sigma_i(\xi) = \int_0^\infty \frac{d}{M^2} \left( \frac{s}{M^2} \right)^{\xi-1} \frac{1}{\pi} \text{Im} F_i(s,0),$$ \hspace{1cm} (3.2)
and the corresponding inverse Mellin transforms:

$$
\frac{1}{\pi} \text{Im} F_i(s, 0) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\xi \left( \frac{s}{M^2} \right)^{-\xi} \Sigma_i(\xi),
$$

(3.3)

where, for convenience, we have introduced an arbitrary mass scale $M$ (e.g. the $\rho$ mass) so as to normalize the dimensions of the $s$ variable. We then observe the following facts:

- According to Eq. (1.1), a FM–like asymptotic behaviour for the physical $\sigma_{\pi\pi}^{\text{tot}}(s)$ cross sections, implies:

$$
\begin{align*}
\sigma_{\pi^+\pi^+}^{\text{tot}}(s) &\sim_{s \to \infty} A_{\pi^+\pi^+} \frac{\pi}{M^2} \log^2 \frac{s}{M^2}, \\
\sigma_{\pi^+\pi^-}^{\text{tot}}(s) &\sim_{s \to \infty} A_{\pi^+\pi^-} \frac{\pi}{M^2} \log^2 \frac{s}{M^2}, \\
\sigma_{\pi^0\pi^0}^{\text{tot}}(s) &\sim_{s \to \infty} A_{\pi^0\pi^0} \frac{\pi}{M^2} \log^2 \frac{s}{M^2}, \\
\sigma_{\pi^0\pi^0}(s) &\sim_{s \to \infty} A_{\pi^0\pi^0} \frac{\pi}{M^2} \log^2 \frac{s}{M^2},
\end{align*}
$$

(3.4)

where the $A_{\pi\pi}$ are some appropriate constants. According to the normalization implied by Eq. (1.1) they should all be fixed to

$$
A_{\pi\pi}|_{\text{FM}} = \frac{M^2}{m_\pi^2},
$$

(3.5)

but here we consider the $A_{\pi\pi}$ constants as a priori unknown.

The inverse mapping theorem [12] requires then that, if the asymptotic behaviours in Eqs. (3.4) are satisfied, the Mellin transforms of the $\frac{1}{\pi} \text{Im} F_i(s, 0)$ amplitudes must have a triple pole at $\xi \to -1$:

$$
\Sigma_i(\xi) \sim_{\xi \to -1} \frac{-2a_i}{(\xi + 1)^3},
$$

(3.6)

where

$$
\begin{align*}
a_1 &= \frac{1}{2} [A_{\pi^+\pi^+} - A_{\pi^+\pi^-}], \\
a_2 &= \frac{1}{2} [A_{\pi^+\pi^+} + A_{\pi^+\pi^-}] = \frac{1}{2} [A_{\pi^+\pi^0} + A_{\pi^0\pi^0}], \\
a_3 &= \frac{1}{2} [3A_{\pi^+\pi^0} - A_{\pi^0\pi^0}],
\end{align*}
$$

(3.7)

The leading singularity of $\Sigma_i(\xi)$ in the Mellin plane at the right of the fundamental strip (which fixes the integration boundary $c$ in the inverse Mellin transform in Eq. (3.3)) must then be at $\xi = -1$ and it must be a triple pole. If all the $A_{\pi\pi}$ constants are equal, then $a_1 = 0$ and the corresponding pole at $\xi = -1$ becomes, at most, a double pole. We assume however, for the sake of generality, that all the $a_i \neq 0$. 

5
Let us next consider the Mellin–Barnes representation of the dispersion relations for the amplitudes \( F_i(s,0) \) in Eq. (2.11). Using the relations

\[
\frac{1}{1 + A} = \frac{1}{2\pi i} \int_{c\xi - i\infty}^{c\xi + i\infty} d\xi \ A^{-\xi} \ \Gamma(\xi)\Gamma(1 - \xi),
\]

(3.8)

\[
\frac{1}{1 - A} = \frac{1}{2\pi i} \int_{c\xi - i\infty}^{c\xi + i\infty} d\xi \ A^{-\xi} \ \Gamma(\xi)\Gamma(1 - \xi) \ \frac{\pi}{\Gamma\left(\frac{1}{2} + \xi\right)\Gamma\left(\frac{1}{2} - \xi\right)},
\]

(3.9)

and respecting the \( s \leftrightarrow -s \) symmetry properties of the \( \text{Re} F_i(s,0) \) amplitudes, one finds:

\[
\text{Re} \begin{pmatrix}
F_1(s,0) \\
F_2(s,0) \\
F_3(s,0)
\end{pmatrix}
= \frac{s}{\pi f^2} \begin{pmatrix}
-1 \\
0 \\
0
\end{pmatrix} + \frac{1}{2\pi i} \int_{c\xi - i\infty}^{c\xi + i\infty} d\xi \ \begin{pmatrix}
\frac{s}{M^2} \left(\frac{|s|}{M^2}\right)^{1-\xi} & 0 & 0 \\
0 & \left(\frac{|s|}{M^2}\right)^{2-\xi} & 0 \\
0 & 0 & \left(\frac{|s|}{M^2}\right)^{2-\xi}
\end{pmatrix}
\times \Sigma_i(\xi - 2) \ \Gamma(\xi)\Gamma(1 - \xi) \left[ \begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} + \frac{\pi}{\Gamma\left(\frac{1}{2} + \xi\right)\Gamma\left(\frac{1}{2} - \xi\right)} \begin{pmatrix}
-1 \\
1 \\
1
\end{pmatrix} \right],
\]

(3.10)

where we have used the fact that

\[
\int_0^{\infty} d\left(\frac{s'}{M^2}\right) \left(\frac{s'}{M^2}\right)^{\xi-3} \frac{1}{\pi} \text{Im} F_i(s',0) = \Sigma_i(\xi - 2),
\]

(3.11)

with \( \Sigma_i(\xi) \) the same Mellin transform as the one defined in Eq. (3.2). Notice that the fundamental strip in Eq. (3.10) is now defined by \( c\xi = \text{Re}(\xi) \in [0,1] \). Again, the low energy behaviour of the \( F_i(s,0) \) amplitudes is governed by the singularities at the left of this fundamental strip, while their high energy behaviour is governed by the singularities at the right of the same fundamental strip.

In particular, the leading low energy behaviours of the \( F_i(s,0) \) amplitudes are governed by the values of the \( \Sigma_i(\xi - 2) \) at \( \xi \to 0 \) and leads to the results:

\[
\text{Re} F_1(s,0) \quad \begin{cases}
\text{as } s \to 0 & -\frac{s}{2f^2} + O(s^3),
\end{cases}
\]

(3.12)

\[
\text{Re} F_2(s,0) \quad \begin{cases}
\text{as } s \to 0 & \frac{s^2}{M^4} \lim_{\xi \to 0} \left\{ \frac{d}{d\xi} \left[ 2\xi\Sigma_2(\xi - 2) \right] \log \frac{M^2}{s} + 2\xi\Sigma_2(\xi - 2) \right\} + O(s^4),
\end{cases}
\]

(3.13)

\[
\text{Re} F_3(s,0) \quad \begin{cases}
\text{as } s \to 0 & \frac{s^2}{M^4} 2\Sigma_3(-2) + O(s^4).
\end{cases}
\]

(3.14)

Comparison with the \( \chi \)PT expansion in Eqs. (2.13) allows us then to fix the values of \( \Sigma_{2,3} \) at \( \xi = -2 \) to:
\[
\Sigma_2(\xi) \sim_{\xi \to -2} \frac{M^4}{f_\pi^4} \left[ \frac{f_1'}{2} + \frac{5}{576\pi^2} + \frac{25}{576\pi^2} \right] + \frac{1}{24\pi^2} \frac{1}{\xi + 2}, \quad (3.15)
\]

\[
\Sigma_3(\xi) \sim_{\xi \to -2} \frac{M^4}{f_\pi^4} \left[ -\frac{f_1'}{2} + \frac{1}{192\pi^2} \right]. \quad (3.16)
\]

- On the other hand, the leading high energy behaviours of the \(F_i(s, 0)\) amplitudes are governed by the \(\Sigma_i(\xi - 2)\) at \(\xi \to 1\) which, if the FM bound is saturated for all the \(\sigma_{\pi\pi}^{\text{tot}}\) cross sections, have triple poles at the values:

\[
\Sigma_i(\xi - 2) \sim_{\xi \to 1} -\frac{2a_i}{(\xi - 2 + 1)^3}. \quad (3.17)
\]

For the amplitudes \(\text{Re} F_2(s, 0)\) and \(\text{Re} F_3(s, 0)\) the effect of this triple pole is softened by the fact that

\[
\frac{\pi}{\Gamma\left(\frac{1}{2} + \xi\right)\Gamma\left(\frac{1}{2} - \xi\right)} \sim_{\xi \to 1} -1 + \frac{\pi^2}{2}(\xi - 1)^2 + \mathcal{O}(\xi - 1)^4, \quad (3.18)
\]

and there is a cancellation between the two terms in the brackets in the second line at the r.h.s. of Eq. (3.10). Therefore, the leading asymptotic behaviour of \(F_{2,3}(s, 0)\) is then of the type

\[
\text{Re} F_{2,3}(s, 0) \sim_{s \to \infty} \mathcal{O}[a_{2,3} |s| \log |s|]. \quad (3.19)
\]

By contrast, if \(a_1 \neq 0\), there is no such a cancellation for the \(F_1(s, 0)\) amplitude and its leading high energy behaviour will then be of the type:

\[
\text{Re} F_1(s, 0) \sim_{s \to \infty} -\frac{s}{f_\pi^2} + \mathcal{O}[a_1 s \log^3 |s|], \quad (3.20)
\]

From the previous considerations we conclude that the Mellin–Barnes representation of the elastic \(\pi\pi\) forward scattering amplitudes \(F_i(s, 0)\) show explicitly how their asymptotic behaviours for \(s \to \infty\) (relevant to the FM bound), and for \(s \to 0\) (relevant to the \(\chi\)PT expansion), are governed by the Mellin transforms \(\Sigma_i(\xi)\) defined in Eq. (3.2). We find from \(\chi\)PT that the chiral limits \((m_\pi \to 0)\) of these Mellin functions exist and are perfectly well defined in QCD at the left of the corresponding \textit{fundamental strips}. We have also shown how the high energy behaviours of the \(F_i(s, 0)\) amplitudes, are governed by the same Mellin transforms and, therefore, the FM bound has direct implications on their behaviours. If, as implied by the normalization of the FM bound in Eq. (1.1), and hence Eq. (3.5), the Mellin functions \(\Sigma_i(\xi)\) at the right of their \textit{fundamental strips} are singular in the chiral limit, it means that they must have a discontinuous behaviour with respect to the pion mass in the sense that: \textit{they exist in the chiral limit at the left of their fundamental strips yet they blow up to infinity, in the same limit, at the right of their fundamental strips}. This we find a rather peculiar behaviour which, although mathematically possible, questions the presence of a pion mass factor in the denominator of the normalization of the FM bound in QCD.
IV The Froissart–Martin Bound in the QCD Large–$N_c$ Limit.

In this section we shall directly work with the $\pi\pi$ scattering amplitudes $\text{Im}T^I(s,0)$ with well defined isospin ($I = 0,1,2$). They are related to the $\text{Im}F_i(s,0)$ amplitudes which we have considered in the previous section as follows:

\begin{align*}
\text{Im}T^0(s,0) &= \frac{1}{2} \left[ -4\text{Im}F_1(s,0) + 5\text{Im}F_2(s,0) - 3\text{Im}F_3(s,0) \right], \\
\text{Im}T^1(s,0) &= -\text{Im}F_1(s,0) + \text{Im}F_3(s,0), \\
\text{Im}T^2(s,0) &= \text{Im}F_1(s,0) + \text{Im}F_2(s,0).
\end{align*}

(4.1)

In the Large–$N_c$ limit of QCD, the $\text{Im}T^I(s,0)$ amplitudes are composed of an infinite set of narrow states:

\begin{equation}
\frac{1}{\pi} \text{Im}T^I(s,0) = \sum_{n=0}^{\infty} |F_{I,n}|^2 \delta \left( s - M_{I,n}^2 \right), \quad I = 0,1,2.
\end{equation}

(4.2)

The question is then the following: is it possible to find constraints on the couplings $F_{I,n}$ and the masses $M_{I,n}$ of a possible Large–$N_c$ ansatz so as to reproduce the FM asymptotic behaviour for the $\sigma_{\pi\pi}^{\text{tot}}$ cross sections?

In order to answer this question we shall proceed as follows. The Mellin transforms of $\frac{1}{\pi} \text{Im}T^I(s,0)$ in the Large–$N_c$ limit are given by Dirichlet–like series:

\begin{equation}
\Sigma^I(\xi) = \sum_{n=0}^{\infty} |F_{I,n}|^2 \left( \frac{M^2}{M_{I,n}^2} \right)^{-\xi+1},
\end{equation}

and the corresponding inverse Mellin transforms are:

\begin{equation}
\frac{1}{\pi} \text{Im}T^I(s,0) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\xi \left( \frac{s}{M^2} \right)^{-\xi} \sum_{n=0}^{\infty} \frac{|F_{I,n}|^2}{M^2} \left( \frac{M^2}{M_{I,n}^2} \right)^{-\xi+1}.
\end{equation}

(4.4)

As discussed in the previous section, a FM–like asymptotic behaviour for the $\sigma_{\pi\pi}^{\text{tot}}$ cross sections fixes the leading singularity of the Mellin transforms of the $\frac{1}{\pi} \text{Im}F_i(s,0)$ amplitudes as given in Eqs. (3.6) and (3.7) and therefore, from Eqs. (4.1), there follows that

\begin{equation}
\Sigma^I(\xi) \sim_{\xi \to -1} \frac{-2A^I}{(\xi + 1)^3},
\end{equation}

(4.5)

where

\begin{align*}
A^0 &= \frac{1}{2} \left[ -4a_1 + 5a_2 - 3a_3 \right], \\
A^1 &= -a_1 + a_3, \\
A^2 &= a_1 + a_2.
\end{align*}

(4.6)

\footnote{For a recent discussion of QCD Large–$N_c$ properties in connexion with the asymptotic behaviours of two–point functions see ref. [19].}
In order to construct a simple Large–N\textsubscript{c} ansatz with the required properties we shall assume a Regge growth for the masses of the narrow states with $I = 1$:

$$M_{I=1,n}^2 = M_\rho^2 + n\Lambda^2,$$

(4.7)

and the absence of exotic trajectories i.e., no poles with $I = 2$. We are then assuming that the $I = 1$ channel fully dominates the physical $A[\pi^+\pi^0 \rightarrow \pi^+\pi^0]$ amplitude and focus our attention on this amplitude in the limit where

$$A_{\pi^+\pi^0 \rightarrow \pi^+\pi^0}(s,0) = \frac{1}{2} [F_2(s,0) + F_3(s,0)] \simeq \frac{1}{2} T_{I=1}^I(s,0).$$

(4.8)

Saturation of the FM bound for the corresponding total cross section $\sigma_{\pi^+\pi^0}^{\text{tot}}$ requires the couplings $|F_{I=1,n}|^2$ in Eq. (4.3) to grow like $n \log n$ as $n \rightarrow \infty$. The simplest form of a Large–N\textsubscript{c} ansatz satisfying these requirements is then

$$\frac{1}{\pi} \text{Im} T_{I=1}^I(s,0) = C \sum_{n=0}^{\infty} \left( M_\rho^2 + n\Lambda^2 \right) \log^2 \left( \frac{M_\rho^2}{\Lambda^2} + n \right) \delta(s - M_\rho^2 - n\Lambda^2),$$

(4.9)

where $C$ denotes a dimensionless constant. Here we have fixed the arbitrary scale $M = M_\rho$, and $\Lambda$ is the mass scale which as shown in Eq. (4.7) fixes the equally spaced Regge–like spectra. Quite remarkably, the Mellin transform of this Large–N\textsubscript{c} ansatz has a close analytic form:

$$\Sigma_{I=1}(\xi) = C \left( \frac{M_\rho^2}{\Lambda^2} \right)^{-\xi} \frac{d^2}{d\xi^2} \zeta \left( -\xi, \frac{M_\rho^2}{\Lambda^2} \right),$$

(4.10)

where $\zeta(-\xi, \frac{M_\rho^2}{\Lambda^2})$ is the Hurwitz function, a generalization of the Riemann zeta function, defined by the series:

$$\zeta(\xi, v) = \sum_{n=0}^{\infty} \frac{1}{(n + v)^\xi}, \quad \text{Re} \xi > 1, \quad \text{with} \quad 0 < v \leq 1,$$

(4.11)

and its analytic continuation. For $v = 1$ it reduces to the Riemann zeta function. The asymptotic behaviour of $\frac{1}{\pi} \text{Im} T_{I=1}^I(s,0)$ for $s \rightarrow \infty$ and hence of $\sigma_{\pi^+\pi^0}^{\text{tot}}$, is governed by the residue of the triple pole of $\Sigma_{I=1}(\xi)$ at $\xi = -1$ (see Eq. (4.5)). This relates the overall constant $C$ in Eq. (4.9) to the coefficient $A_{\pi^+\pi^0}$ in Eq (3.4) and hence:

$$\sigma_{\pi^+\pi^0}(s) \sim \frac{C}{2} \left( \frac{M_\rho^2}{\Lambda^2} \right)^2 \frac{\pi^2}{M_f^2} \log^2 \frac{s}{M_\rho^2}.$$

(4.12)

Let us next discuss the low–energy constraint that we can impose to the simple large–N\textsubscript{c} ansatz in Eq. (4.9) so as to fix the value of the overall constant $C$. The isospin $I = 1$ dominance assumption of the $\chi$PT expressions in Eqs. (3.15), when restricted to the Large–N\textsubscript{c} limit, fixes them to the values

$$\Sigma_2(\xi) \sim \frac{M_f^4}{f_\pi^4} \left[ l_1 + \frac{3}{2} l_2 \right] \sim \frac{1}{4} \frac{M_f^2}{f_\pi^2},$$

(4.13)

$$\Sigma_3(\xi) \sim \frac{M_f^4}{f_\pi^4} \left[ -l_1 + \frac{1}{2} l_2 \right] \sim \frac{3}{4} \frac{M_f^2}{f_\pi^2},$$

(4.14)
where in the r.h.s. we have used the $\rho$–dominance approximation for the $l_i$ constants \cite{14,15} and, as before, we have fixed $M = M_\rho$. Matching the Large–$N_c$ ansatz in Eq. (4.10) to these considerations, using Eq. (4.8), fixes the $C$ constant to

$$C \sim \frac{1}{\zeta'' \left(\frac{2}{\Lambda^2}, \frac{M^2}{\Lambda^2}\right)} \frac{\Lambda^4}{f_\pi^2 M_\rho^2},$$

(4.15)

and, therefore, the leading asymptotic growth of the total $\pi^+\pi^0$ cross section to:

$$\sigma_{\pi^+\pi^0}^{\text{tot}}(s) \sim \frac{1}{2} \zeta'' \left(\frac{2}{\Lambda^2}, \frac{M^2}{\Lambda^2}\right) \frac{\pi}{f_\pi^2} \log^2 \frac{s}{M_\rho^2} \frac{s}{M_\rho^2}.$$

(4.16)

Like the usual FM bound, it grows as $\log^2 s$, but it is finite in the chiral limit and of $\mathcal{O}(1/N_c)$ in the Large–$N_c$ counting. Numerically, for $\Lambda = M_\rho$, the r.h.s. of Eq (4.16) becomes

$$\sigma_{\pi^+\pi^0}^{\text{tot}}(s) \sim 0.25 \frac{\pi}{f_\pi^2} \log^2 \frac{s}{M_\rho^2},$$

(4.17)

but we should stress that this is only a Large–$N_c$ model estimate with many simplifications.

V Conclusions

We have shown that it is possible to construct a Large–$N_c$ QCD ansatz compatible with the Froissart–Martin bound. The bound, however, is finite in the chiral limit and it is of $\mathcal{O}(1/N_c)$ in the Large–$N_c$ counting rules. In fact, it seems very likely that these two features should be generic to full QCD because of the fact that QCD has spontaneous chiral symmetry breaking. This implies the existence of a mass gap between the Nambu–Goldstone states (the pions) and the other hadronic states. It is this property which, very likely, forces the presence of characteristic scales like $f_\pi$ in the normalization of the FM bound and which at the same time provides the correct $\mathcal{O}(1/N_c)$ Large–$N_c$ counting.

The usual derivation of the FM bound does not take into account the fact that the underlying dynamics of the strong interactions has the property of spontaneous chiral symmetry breaking. In fact, it implicitly assumes a realization of the hadronic spectrum a la Wigner–Weyl without Nambu–Goldstone particles, in which case, the normalization in Eq. (1.1) is not surprising.

Finally, we wish to emphasize that the discussion above does not answer the third question in the introduction. We have only shown that, in the Large–$N_c$ limit of QCD, it is possible to construct models which \textit{a priori} show no obstruction for the asymptotic behaviour of the total $\pi\pi$ cross sections to saturate a $\log^2 s$ like behaviour.
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