Is it possible to estimate the oscillating sum
\[
Z = e^{-N} \sum_{n=0}^{\infty} \frac{N^n}{n!} \exp\left\{ i \left( \frac{An}{\sqrt{N}} + \frac{Bn^2}{2\sqrt{N^3}} \right) \right\}
\text{for } N = 10^{23}?
\]

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Abstract

We prove that it is possible to approximate \( Z \) by an integral \( G \) with respect to the standard Gaussian measure such that the approximation error \( |G - Z| = O(N^{-1}) \) is small, and the value of the integral

\[
Z = \frac{1}{\sqrt{D}} e^{i\sqrt{N}(A+B/2) - (A+B)^2/2D} \left( 1 - i \frac{(A + B)^3}{6\sqrt{N}} \right) + O(N^{-1})
\]

with \( D = (1 - iB/\sqrt{N}) \), is finite and oscillates rapidly in \( A \) and \( B \). Straightforward summation of \( Z \) is impossible because over the huge interval \( n \in [N - \sqrt{2N\log N}, N + \sqrt{2N\log N}] \), the summands contribute significantly. We discuss approximation for some oscillating series related to \( Z \).

1 Introduction

When a mathematician says "consider the asymptotics of a state \( \psi_h(x) \) as the Plank constant \( \hbar \) goes to zero", this makes no rigorous sense in physics, because \( \hbar \) is a constant. The difference between the mathematical infinity and huge physical quantities such as the Avogadro number \( N_A \) or \( \hbar^{-1} \) is that \( \log N_a \approx 23 \), and \( \log \hbar^{-1} \approx 37 \) are rather moderate numbers, if they are used as multipliers. Therefore, for mathematicians, \( 1/\log N_A = o(1) \)

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and $1/\log h^{-1} = o(1)$ (if they keep $N_A \to \infty$ and $1/h \to \infty$ in mind), and in physics, $1/\log N_A = O(1)$, $1/\log h^{-1} = O(1)$, $\log N_A/N_A = O(N_A^{-1})$ and $h/\log h^{-1} = O(h)$, e. t. c. Fortunately, a rather general property of the nature is that the logarithms of overwhelming physical quantities are moderate numbers. On the other hand, the logarithms of large parameters occur quite rare in asymptotic expansions. But we will see that crucial estimates of the asymptotic expansion of $Z$ relay on the fact that $\log N$ is finite.

The sum $Z$ represents some quantities related to explicit solution of a problem on interferometrical detection of gravitational waves (see [2]–[4]), where $N$ is the average number of photons generated by a laser, and $A = A(t)$, $B = B(t)$ are small time-dependent functions describing interference of photons splitted into two arms of the measuring device. The main computational difficulty is that $2\sqrt{2N\log N} \approx 10^{12}$ summands centered at $n = N = 10^{23}$ give a relevant contribution to $Z$. Thus the straightforward computation of $Z$ is impossible.

In this paper we prove that the series $Z(A, B, N)$ can be approximated by the Gaussian integral, because in terms of the new variable $x_n = n - N\sqrt{N}$, $dx_n = \frac{1}{\sqrt{N}} (n = 0, 1, \ldots)$, the Stirling expansion of Gamma–function implies the Gaussian approximation of the Poisson probability distribution $P_N(n)$:

$$P_N(n) \overset{\text{def}}{=} e^{-N}\frac{N^n}{n!} = \frac{e^{-\frac{x_n^2}{2}}}{\sqrt{2\pi N}}\frac{e^{\frac{x_n^3}{6N}}}{\sqrt{1 - \frac{x_n^2}{2N} + \frac{1}{12N}}} \left(1 + O(N^{-3/2})\right) \quad (1.1)$$

if $x_n/\sqrt{N}$ is small [1] (see Appendix 3, (A.6), (A.7)).

On the other hand, by the trapezoidal approximation of the integral, it is possible to estimate the sum

$$Z = \sum_n P_N(n)e^{iS_N(n)}, \quad S_N(n) = \frac{An}{\sqrt{N}} + \frac{Bn^2}{2\sqrt{N^3}}$$

over the interval $B_N \overset{\text{def}}{=} [N - \sqrt{2N\log N}, N + \sqrt{2N\log N}] \subset \mathbb{N}$. In this case, $\max_{n \in B_N} |x_n| = \sqrt{2\log N}$. More precise, to estimate the sum $Z$, we must justify several approximations:

- estimate the error of the restriction to $B_N$ of the sum $Z$;
Figure 1: The picture shows the Poisson distribution $P_N(n)$ for $N = 10^4$, which practically coincides with the Gaussian distribution $N(10^4, 10^4)$, and the approximation error (double-well curve scaled by $10^5$), representing the difference between the Poisson and normal distributions with the Stirling corrections.

- estimate on $B_N$ the error of approximation of the Poisson distribution $P_N(b)$ by the Gaussian one as in Eq. (1.1);
- estimate the approximation error of the replacing of the sum over $B_N$ by the Gaussian integral over the set

$$D_N \overset{\text{def}}{=} [-\sqrt{2 \log N}, \sqrt{2 \log N}] \subset \mathbb{R}$$

- expand (1.1) in $1/\sqrt{N}$ on $D_N$ and estimate the approximation error;
- extend the Gaussian integral to $\mathbb{R}$ and evaluate it analytically.

First, in Section 2, we cut out the left and right tails of the probability distribution $P_N(n)$ outside of $B_N$, and then in Section 3, we approximate the sum by an integral over the interval $D_N \subset \mathbb{R}$ corresponding to $B_N$:

$$Z = \frac{1}{2\sqrt{N}} \sum_{x_n \in B_N} \left( f_N(x_n) e^{is_N(n)} + f_N(x_{n+1}) e^{is_N(n+1)} \right) + \eta_N =$$

$$= \int_{x \in D_N} dx f_N(x) e^{i\sigma_N(x + \frac{1}{2\sqrt{N}})} + \eta_N + \epsilon_N, \quad (1.2)$$
where $\epsilon_N$ and $\eta_N$ are the approximation errors, $\sigma_N(x) = S_N(x\sqrt{N} + N)$ and $f_N(x) = \sqrt{N} P_N(x\sqrt{N} + N)$. More precise, for the total approximation error $\eta_N + \epsilon_N = Z - \int dx f_N(x)e^{ix\eta_N(x)}$, we have

$$\left|\eta_N\right| + \left|\epsilon_N\right| \leq \sum_{n \in \mathbb{N}\setminus B_N} P_N(n) + \frac{1}{6} \sum_n \left\{ \max_{y \in I_n} |P_N^{(2)}(y)| + 2 \max_{y \in I_n} |P_N^{(1)}(y)S_N^{(1)}(y)| + \max_{y \in I_n} |P_N(y)S_N^{(2)}(y)| \right\},$$

(1.3)

where $I_n \stackrel{\text{def}}{=} [n, n+1]$ (see Appendix (A.1) and (A.5)).

In what follows we prove that $\left|\epsilon_N\right| + \left|\eta_N\right| = O(N^{-1})$, and Eq. (1.1) describe convenient analytical approximation of $f_N(x)$ on $D_N$ by the perturbed Gaussian distribution (see (A.6) and (A.7)).

This implies that the sum $Z$ can be approximated by the Gaussian integral

$$Z = \frac{1}{\sqrt{2\pi}} \int_{x \in D_N} dx e^{-\frac{x^2}{2}} \frac{e^{\frac{x^3}{6N} + \frac{x^4}{12N}}}{\left(1 + \frac{x}{\sqrt{N}} + \frac{1}{12N}\right)} e^{i\sigma_N(\sqrt{N}x+N)} + O(N^{-1}).$$

Eventually, the last integral can be extended to $\mathbb{R}$ and evaluated explicitly up to terms of order $O(N^{-1})$:

$$Z = \frac{e^{i(A+B)/\sqrt{N}}}{\sqrt{1-iB/\sqrt{N}}} e^{-\frac{i(A+B)^2}{2(1-iB/\sqrt{N})}} \left(1 - i \frac{(A+B)^3}{6\sqrt{N}}\right) + O(N^{-1}).$$

(1.4)

The last estimate of the error is proved by a generalization of the Komatsu inequality (3.3) proved in Appendix 4. This asymptotic expansion (1.4) is the main result of the note. It can be applied in many different ways.

In Section 4, we present similar estimates for several related oscillating series. First, it justifies the asymptotic expansion of the sum

$$Z_F = e^{-N} \sum_{n=0}^{\infty} \frac{N^n}{n!} F\left(\frac{n}{\sqrt{N}}\right) e^{i\frac{aq^2}{2N^{3/2}}},$$

where $F(x)$ is the Fourier transform of any finitely supported function $\bar{F}(p)$ such that $(\text{diam supp} \bar{F} + |B|)^2 < \log N$:

$$Z_F \approx \frac{1}{\sqrt{2\pi}} \int_R dp \bar{F}(p) e^{i(p+q/2)\sqrt{N}} \frac{e^{-\frac{(p+q)^2}{2(1-iq/\sqrt{N})}}}{\sqrt{1-iq/\sqrt{N}}} \left(1 - i \frac{(p+q)^3}{6\sqrt{N}}\right).$$
It can also be applied for the correlation and Fourier analysis of $Z(t)$ in the case, when $A = A(t)$ and $B = B(t)$.

Finally, we note that the series $Z(x, t, \hbar^{-2}) \overset{\text{def}}{=} \psi(x, t)$ represents an explicit solution of the Cauchy problem for the Schrödinger equation with a special periodic initial condition:

$$i\hbar \partial_t \psi = \frac{\hbar^2}{2} \partial_x^2 \psi, \quad \psi|_{t=0} = \exp\{-\hbar^{-2}(1 - e^{ix/\hbar})\}.$$

## 2 Estimate of the weight of tails

The total weight $T_N$ of the left and right tails of the probability distribution $P_N(n)$ on the complement of the interval $B_N$ can be estimated by use of the expansion (1.1) which is clearly valid on this set. By definition, we have

$$T_N \overset{\text{def}}{=} 1 - \sum_{n \in B_N} P_N(n) = 1 - \sum_{x_n \in D_N} \frac{1}{\sqrt{N}} f_N(x_n). \quad (2.1)$$

The trapezoidal approximation (see Appendix 1, (A.1) for the estimates of the approximation errors) implies the following integral representation of the sum:

$$\sum_{x_n \in D_N} \frac{1}{\sqrt{N}} f_N(x_n) = \int_{D_N} dx f_N(x) + \gamma_N,$$

where $|\gamma_N| \leq \frac{1}{\sqrt{N}} \sum_{n \in B_N} \max_{x \in I_n} |P_N^{(2)}(x)|$. Note that the function $P_N(x)$ has just one extremum on the set $B_N$. Hence its first and second derivatives may have no more than two and three critical points respectively. Hence the function $|P_N^{(2)}(x)|$ is monotone on all intervals $I_n$ with the exception of the three (see Fig. 2).

Therefore, we can apply the estimate (A.5):

$$|\gamma_N| \leq \sum_{n \in B_N} \max_{x \in I_n} |P_N^{(2)}(x)| \leq 3 \sum_{n \in B_N} |P_N^{(2)}(n)| = O(N^{-1}).$$

Now the expansion (1.1) ensures the equality:

$$\int_{D_N} dx f_N(x) = \frac{1}{\sqrt{2\pi}} \int_{D_N} dx e^{-\frac{x^2}{2}} \frac{e^3}{6\sqrt{N}} \int_{D_N} dx e^{-\frac{x^4}{12N}} \left(1 + \frac{x}{2\sqrt{N}} + \frac{1}{12N}\right).$$
Figure 2: The derivatives of the Poisson distribution become more flattern as $N$ increases. Each derivation add a new critical point. The picture shows $P_N^{(1)}(n)$ with two critical points, and $100 \cdot P_N^{(2)}(n)$ with three critical points, for $N = 10^4$.

$$
= \frac{1}{\sqrt{2\pi}} \int_R dx \ e^{-\frac{x^2}{2}} \left( 1 - \frac{3x - x^3}{6\sqrt{N}} \right) + O(N^{-1}) = 1 + O(N^{-1})
$$

because the functions $x$ and $x^3$ are odd. The estimate of the approximation error follows from the Komatsu inequality (see Appendix 4):

$$
4e^{-\frac{a^2}{2}} \left( \sqrt{a^2 + 4 + a} \right)^{-1} \leq \int_{|x| \geq a} dx \ e^{-\frac{x^2}{2}} \leq 4e^{-\frac{a^2}{2}} \left( \sqrt{a^2 + 2 + a} \right)^{-1}.
$$

For $a = \sqrt{2 \log N}$, we obtain

$$
\frac{1}{\sqrt{2\pi}} \int_{R \setminus D_N} dx \ e^{-\frac{x^2}{2}} \left( 1 - \frac{3x - x^3}{6\sqrt{N}} \right) = \frac{1}{\sqrt{2\pi}} \int_{R \setminus D_N} dx \ e^{-\frac{x^2}{2}} \leq \frac{4}{\sqrt{2\pi}} e^{-\frac{a^2}{2}} \left( \sqrt{a^2 + 2 + a} \right)^{-1} \bigg|_{a = \sqrt{2 \log N}} = O(N^{-1}).
$$

These estimates explain the specific choice of the sets $D_N$ and $B_N$. Therefore, the total weight $T_N$ of the tails on the complement of $B_N$ equals $O(N^{-1})$:

$$
T_N = 1 - \sum_{n \in B_N} P_N(n) = 1 - \sum_{x_n \in D_N} \frac{1}{\sqrt{N}} f_N(x_n) = \ldots
$$
= 1 - \frac{1}{\sqrt{2\pi}} \int_{D_N} dx \, e^{-\frac{x^2}{2}} \frac{e^{\frac{x^3}{12N} - \frac{x^4}{12N}}}{\left(1 + \frac{x}{2\sqrt{N}} + \frac{1}{12N}\right)} - \gamma_N \leq O(N^{-1}) + |\gamma_N| = O(N^{-1}).

(2.2)

3 Estimates of the oscillating sum

The main property of the set $B_N$ is that the Stirling formula holds true on this set. On the other hand, the probability distribution $P_N(n)$ is supported on this set, up to events with total probability of order $O(N^{-1})$. Hence the expectation of the oscillating exponents can be restricted to $B_N$ with the same approximation error:

\[ Z = \sum_{n \in B_N} P_N(n) e^{iS_N(n)} + O(N^{-1}). \quad (3.1) \]

The trapezoidal approximation method is applicable to the sum (3.1) with the approximation error (1.3) (see (A.1) for details):

\[ Z = \int_{x \in D_N} dx \, f_N(x) e^{i\sigma_N(x)} + O(N^{-1}) + \gamma_N, \]

where

\[ |\gamma_N| \leq \frac{1}{6} \sum_{n \in B_N} \left\{ \max_{y \in I_n} |P_N^{(2)}(y)| + 2 \max_{y \in I_n} |P_N^{(1)}(y)S_N^{(1)}(y)| + \max_{y \in I_n} |P_N(y)S_N^{(2)}(y)| \right\}. \]

By definition of the set $D_N$, we have the following uniform estimates of the derivatives

\[ S_N^{(1)} = \frac{A}{\sqrt{N}} + \frac{Bx}{\sqrt{N^3}} = O(N^{-1/2}), \quad S_N^{(2)} = \frac{B}{\sqrt{N^3}} = O(N^{-3/2}), \quad x \in D_N \]

and as is proved in Appendix (see (A.3) and (A.4)),

\[ P_N^{(1)}(x) = P_N(x) \log \frac{N}{x} + O(x^{-1}) = P_N(x) O(N^{-1/2}), \]

\[ P_N^{(2)}(x) = P_N(x) \left( \left( \log \frac{N}{x} \right)^2 + x^{-1} \log \left( \frac{N}{x} \right) + O(x^{-1} \log x) \right) = P_N(x) \cdot O(N^{-1}), \]
because \( N/x = 1 + O(N^{-1} \log N)^{1/2} \) uniformly on \( D_N \), and hence \( \log(N/x) = O(N^{-1/2}) \) and \( (\log(N/x))^2 = O(N^{-1}) \) uniformly in \( x \in D_N \).

Since \( \sum_n P_N(n) = 1 \), it follows that \( |\gamma_N| = O(N^{-1}) \). Thus we proved that

\[
Z = \frac{1}{\sqrt{2\pi}} \int_{x \in D_N} dx \, e^{-x^2/2} \left( 1 + \frac{x}{2 \sqrt{N}} + \frac{1}{12N} \right) e^{i S_N(\sqrt{N}x + N)} + O(N^{-1}),
\]

where \( S_N(\sqrt{N}x + N) = A(x + \sqrt{N}) + B(x + \sqrt{N})^2/(2\sqrt{N}) \). Now we expand the exponent into the series in \( x/\sqrt{N} \), and extend the of integration domain to the whole real line:

\[
Z = \frac{1}{\sqrt{2\pi}} \int_R dx \, e^{-x^2/2} \left( 1 - \frac{3x - x^3}{6 \sqrt{N}} \right) e^{i A(x + \sqrt{N}) + i B(x + \sqrt{N})^2/(2\sqrt{N})} + O(N^{-1}).
\]

(3.2)

The error \( \sigma_N \) related to the extension of the Gaussian integral from \( D_N \) to \( R \) can be estimated by an inequality of Komatsu type:

\[
\int_{|x| \geq a} dx \, x^n e^{-x^2/2} \leq 4(a + 1)^{n-1} e^{-\frac{x^2}{2}} \left( \sqrt{a^2 + 2} + a \right)^{-1}
\]

(3.3)

(see Appendix 4), which implies the required absolute estimate

\[
\frac{1}{\sqrt{2\pi}} \int_{|x| \geq a} dx \, x^n e^{-ax^2/2} \bigg|_{a=\sqrt{2\log N}} \leq (\log N)^{n/2} \cdot O(N^{-1}) = O(N^{-1}),
\]

(3.4)

n = 1, 2.

By using the standard technic, we find elementary Gaussian integral (3.2):

\[
\frac{1}{\sqrt{2\pi}} \int_R dx \, x e^{-ax^2/2 + ibx} = \frac{ib}{a^{3/2}} e^{-b^2/2a},
\]

\[
\frac{1}{\sqrt{2\pi}} \int_R dx \, x^3 e^{-ax^2/2 + ibx} = \frac{ib(3a - b^2)}{a^{7/2}} e^{-b^2/2a}.
\]

Hence, by substituting \( a = 1 - iB/\sqrt{N} \), \( b = A + B \), we obtain

**Theorem 3.1**

\[
Z = \frac{e^{i \sqrt{N}(A+B/2)}}{\sqrt{1 - iB/\sqrt{N}}} e^{-\frac{(A+B)^2}{2(1 - iB/\sqrt{N})}} \left( 1 - i \frac{(A+B)^3}{6 \sqrt{N}} \right) + O(N^{-1}).
\]

(3.4)
Figure 3: The picture shows the absolute approximation error $E$ of the sum $Z$ by integral (3.2), as a decreasing function of $A$ and $B$ for $N = 1000$.

4 Approximation of related series

In this section we consider the integral approximation of the related series

$$Z_s = e^{-N} \sum_{n=1}^{\infty} \frac{N^s}{n^s} \exp \left\{ i \frac{An}{N^{1/2}} + \frac{Bn^2}{2N^{3/2}} \right\}$$

(4.1)

for $s = 1, 2$, which are used to calculate the photo current and its dispersion (see [4], Chap. 2). Note that $Z_1$ can be rewritten as

$$Z_1 = Ne^{-N} \sum_{n=0}^{\infty} \frac{N^2}{n!} \exp \left\{ i \left( \frac{A(n+1)}{N^{1/2}} + \frac{B(n+1)^2}{2N^{3/2}} \right) \right\} =$$

$$= Ne^{A \sqrt{N} + B \sqrt{N}} \cdot Z(A + B/N, B, N) = Ne^{A \sqrt{N}} \cdot Z(A, B, N) + O(1).$$

This relation readily implies

$$Z_1 = Ne^{A \sqrt{N}} \frac{e^{i(A+B/2)\sqrt{N}}}{\sqrt{1-iB/\sqrt{N}}} e^{-\frac{(A+B)^2}{2(1-iB/\sqrt{N})}} \left( 1 - i \frac{(A+B)^3}{6\sqrt{N}} \right) + O(1).$$
Similarly,

\[
Z_2 = Z_1 + N^2 e^{-N} \sum_{n=0}^{\infty} \frac{N^n}{n!} \exp \left\{ i \left( \frac{A(n+2)}{N^{1/2}} + \frac{B(n+2)^2}{2N^{3/2}} \right) \right\} = \\
= Z_1 + N^2 e^{i \frac{A}{\sqrt{N}} (A+B/N)} \cdot Z(A+2B/N, B, N) + O(N) = \\
= N^2 e^{i \frac{2A}{\sqrt{N}}} Z(A, B, N) + O(N). \quad (4.2)
\]

For the general case, we can prove the following assertion.

**Theorem 4.1** The sum (4.1) has the following asymptotic expansion

\[
Z_s = N^s e^{i \frac{2A}{\sqrt{N}}} Z(A, B, N) + O(N^{s-1}),
\]

where $Z(A, B, N)$ is given by Eq. (3.4).

Another series which can be estimated with the help of previously developed technics is

\[
\tilde{Z} = e^{-N} \sum_{n=0}^{\infty} \frac{N^n}{n!} \exp \left\{ i \left( A\sqrt{n} + \frac{Bn}{2\sqrt{N}} \right) \right\} \quad (4.3)
\]

Indeed, for the new variable $x = (n - N)/\sqrt{N}$, we have $n/\sqrt{N} = x + \sqrt{N}$, and

\[
\sqrt{n} = (x + \sqrt{N})(1 + x/\sqrt{N})^{-1/2} = \sqrt{N} + \frac{x}{2} - \frac{3x^2}{8\sqrt{N}} + O(N^{-1}).
\]

Hence,

\[
A\sqrt{n} + \frac{Bn}{2\sqrt{N}} = A \left( \sqrt{N} + \frac{x}{2} - \frac{3x^2}{8\sqrt{N}} \right) + B(\sqrt{N} + x)/2, \quad (4.4)
\]

and by replacing the exponent in Eq. (3.2) by (4.4), we obtain the following assertion

**Theorem 4.2** The sum (4.3) has the following asymptotic expansion

\[
\tilde{Z} = e^{i \sqrt{N}(A+B/2)} \frac{e^{-\frac{(A+B)^2}{8(1+3A/4\sqrt{N})}}}{\sqrt{1 + 3iA/4\sqrt{N}}} \left( 1 - i \frac{(A+B)^3}{48\sqrt{N}} \right) + O(N^{-1}). \quad (4.5)
\]
To conclude this section, we consider an estimate of the following multiple sum:

\[ Z = e^{-2N} \sum_{m,n=0}^{\infty} \frac{N^{m+n}}{m!n!} \exp\left\{ i\left( \frac{a_1 m + a_2 n}{\sqrt{N}} + \frac{b_1 m + b_1 n + 2b_3 mn}{2\sqrt{N^3}} \right) \right\} \]  

which was derived in [3] to describe the interaction of the laser beam in some standard state with the quantum oscillator.

In variables \( x = (m - N)/\sqrt{N}, \ y = (n - N)/\sqrt{N} \), this series can be approximated by the Gaussian integral

\[ Z = \frac{e^{i\sqrt{N}\sigma(a,b)}}{2\pi} \int_{\mathbb{R}^2} dx\, dy\, e^{S_N(x,y)} \left( 1 - \frac{3(x+y) - x^3 - y^3}{6\sqrt{N}} \right) + O(N^{-1}) \]

where \( \sigma(a,b) = a_1 + a_2 + b_3 + (b_1 + b_2)/2 \) and

\[ S_N(x,y) = -(x^2 + y^2)/2 + \]

\[ + i(a_1 + b_1 + b_3)x + i(a_2 + b_2 + b_3)y + i(b_1 x^2 + b_2 y^2 + 2 b_3 x y)/(2\sqrt{N}). \]

This integral can easily be evaluated. As a result we obtain the following estimate.

**Theorem 4.3** The sum (4.6) has the following asymptotic expansion

\[ Z = \frac{e^{i\sqrt{N}\sigma(a,b) - \Theta_N(a,b)}}{\sqrt{1 - i(b_1 + b_2)/\sqrt{N}}} \left( 1 - \frac{i(a_1 + b_1 + b_3)^3 + (a_2 + b_2 + b_3)^3}{6\sqrt{N}} \right) + O(N^{-1}), \]

where

\[ \Theta_N(a, b) = \frac{(a_1 + b_1 + b_3)^2 + (a_2 + b_2 + b_3)^2}{2\sqrt{1 - i(b_1 + b_2)/\sqrt{N}}} + i\frac{b_3(a_1 + b_1 + b_3)(a_2 + b_2 + b_3)}{\sqrt{N}}. \]

**5 Appendix: Simple a priori estimates**

**5.1 Trapezoidal approximation**

Consider the estimate of the trapezoidal approximation of the integral of a smooth function \( f(x) \). The Taylor expansion implies

\[ \frac{1}{2}(f(x_n) + f(x_{n+1})) = f(x_{n+\frac{1}{2}}) + \kappa_n + \kappa_{n+\frac{1}{2}}, \]

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where the approximation error does not exceed the value 

$$ |\kappa_n| \leq \frac{1}{2} (x_n - x_{n+\frac{1}{2}})^2 \max_{y \in [x_n, x_{n+\frac{1}{2}}]} |f^{(2)}(y)|, $$

and for \( x \in [x_n, x_{n+1}] \), we have

$$ f(x) = f(x_{n+\frac{1}{2}}) + (x - x_{n+\frac{1}{2}}) f^{(1)}(x_{n+\frac{1}{2}}) + \gamma_n(x), $$

where \( |\gamma_n(x)| \leq \frac{1}{2} (x - x_{n+\frac{1}{2}})^2 \max_{y \in [x_n, x_{n+1}]} |f^{(2)}(y)| \). Hence the error of the trapezoidal approximation can be estimated as follows

$$ \int_{x_n}^{x_{n+1}} f(x) \, dx = \frac{x_{n+1} - x_n}{2} \left( f_n + f_{n+1} - \kappa_n - \kappa_{n+\frac{1}{2}} \right) - \int_{x_n}^{x_{n+1}} \gamma_n(x) \, dx = \frac{x_{n+1} - x_n}{2} (f_n + f_{n+1}) + \eta_n, \quad (A.1) $$

where \( f_n = f(x_n) \) and \( |\eta_n| \leq \frac{1}{6} (x_{n+1} - x_n)^3 \max_{y \in [x_n, x_{n+1}]} |f^{(2)}(y)| \).

### 5.2 Derivatives of the Poisson distribution

Consider the estimates of the derivatives of the distribution \( P_N(x) \). Note that from the Stirling expansion for the Gamma–function

$$ \Gamma(x + 1) = \left( \frac{x}{e} \right)^x \sqrt{2\pi x} \left( 1 + \frac{1}{12x} + \frac{1}{288x^2} + O(x^{-3}) \right) \quad (A.2) $$

we have the following estimate for its derivatives:

$$ \Gamma^{(1)}(x + 1) = \left( \log x + \frac{1}{2x} + O(x^{-2}) \right) \Gamma(x + 1) $$

and similarly,

$$ \Gamma^{(2)}(x + 1) = \left\{ \left( \log x + \frac{1}{2x} + O(x^{-2}) \right)^2 + \left( \frac{1}{x} + O(x^{-2}) \right) \right\} \Gamma(x + 1) = \left\{ \log^2 x + \frac{\log x}{x} + \frac{1}{x} + O(x^{-2} \log x) \right\} \Gamma(x + 1). $$

Therefore,

$$ \left( \frac{\Gamma^{(1)}(x + 1)}{\Gamma(x + 1)} \right)^2 - \frac{\Gamma^{(2)}(x + 1)}{\Gamma(x + 1)} = O(x^{-1} \log x), \quad (A.3) $$
and moreover,
\[
P_N^{(1)}(x) = e^{-N} \frac{d}{dx} \frac{N^x}{\Gamma(x+1)} = P_N(x) \left( \log N - \frac{\Gamma^{(1)}(x+1)}{\Gamma(x+1)} \right) = \\
= P_N(x) \left( \log \frac{N}{x} - \frac{1}{2x} + O(x^{-2}) \right).
\]

Similarly, for the second derivative we have
\[
P_N^{(2)}(x) = e^{-N} \frac{d^2}{dx^2} \frac{N^x}{\Gamma(x+1)} = \\
= P_N(x) \left\{ \left( \log \frac{N}{x} - \frac{1}{2x} + O(x^{-2}) \right)^2 - \frac{\Gamma^{(2)}(x+1)}{\Gamma(x+1)} + \left( \frac{\Gamma^{(1)}(x+1)}{\Gamma(x+1)} \right)^2 \right\} = \\
= P_N(x) \left\{ \left( \log \frac{N}{x} - \frac{1}{2x} + O(x^{-2}) \right)^2 + O(x^{-1} \log x) \right\}.
\]

Finally we obtain,
\[
\sum_{n \in B_N} P_N^{(2)}(n) = \sum_{n \in B_N} P_N(n) \left\{ \left( \log \frac{N}{n} \right)^2 - \frac{\log(Nn^{-1})}{n} + O(n^{-1} \log n) \right\},
\]
where \( \tilde{B}_N = [N e^{-\sqrt{\frac{\log N}{2n}}}, N e^{\sqrt{\frac{\log N}{2n}}}]. \) Note that for \( n \in \tilde{B}_N, \) we have \( |\log^2 \frac{N}{n}| \leq \frac{\log N}{N}, \) where \( \log N = O(1). \) By similar reasons, \( \frac{\log(Nn^{-1})}{n} + O(n^{-1} \log n) = O(N^{-1}) \) uniformly on this set; in the sense of asymptotical expansions, \( \tilde{B}_N \) is equivalent to the previously defined set \( B_N. \) Hence,
\[
\sum_{n \in \tilde{B}_N} P_N^{(2)}(n) = O(N^{-1}) \sum_{n \in B_N} P_N(n) = O(N^{-1}). \quad (A.5)
\]

### 5.3 Corrections to the Gaussian distribution

From the Stirling expansion (A.2) we have
\[
P_N(n) = \frac{1}{\sqrt{2\pi N}} e^{-N+n-n \log \frac{n}{N}} \frac{1}{\sqrt{n/N}} \left( \frac{1}{(1+12n)^{-1} + O(n^{-2})} \right).
\]

Set \( N = x \sqrt{N} + N; \) then for \( x \in D_N \) we obtain
\[
-N + n - n \log \frac{n}{N} = -N + x \sqrt{N} + N - (x \sqrt{N} + N) \log(1 + x/\sqrt{N}) = \\
= \\
= 
\]

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\[ x\sqrt{N} - (x\sqrt{N} + N) \left\{ \frac{x}{\sqrt{N}} - \frac{x^2}{2N} + \frac{x^3}{3\sqrt{N}^3} + \frac{x^4}{4N^2} + O\left( \frac{x^5}{\sqrt{N}^5} \right) \right\} = \]
\[ = -\frac{x^2}{2} + \frac{x^3}{6\sqrt{N}} - \frac{x^4}{12N} + O\left( \frac{x^5}{\sqrt{N}^5} \right). \quad (A.6) \]

This expansion contains corrections to the Gaussian exponent as in (1.1). Similarly,
\[ \sqrt{n/N} \left( 1 + (12n)^{-1} + O(n^{-2}) \right) = \sqrt{1 + \frac{x}{\sqrt{N}}} \left( 1 + \frac{1}{12N} \left( 1 - \frac{x}{\sqrt{N}} \right) + O(N^{-1}) \right) = \]
\[ = \left( 1 + \frac{x}{2\sqrt{N}} \right) \left( 1 + \frac{1}{12N} \left( 1 - \frac{x}{\sqrt{N}} \right) + O(x/N^{3/2}) + O(N^{-1}) \right) = \]
\[ = 1 + \frac{x}{2\sqrt{N}} + O(N^{-1}) + O(x/N^{3/2}). \quad (A.7) \]

Equations (A.6)–(A.7) imply expansion (1.1).

### 5.4 Komatsu inequalities

The inequalities of the Komatsu type can easily be justified by the following construction. For a positive absolutely integrable function \( f \), we consider the integral \( \int_{x}^{\infty} f(y) \, dy = F(x), \, F(\infty) = 0. \) If there exists some smooth positive decreasing function \( g(x) \) such that \( g(\infty) = 0 \) and
\[ \frac{dF(x)}{dx} = -f(x) \geq \frac{dg(x)}{dx}, \quad (A.8) \]
then \( F(x) \leq g(x) \). Indeed, the inequality (A.8) for derivatives implies that \( g \) decreases at infinity more rapidly than \( F \). Since they coincide at infinity, this proves that \( F(x) \leq g(x) \). In the Komatsu inequality and in our generalization (3.3), the right-hand sides satisfy (A.8). For the left-hand sides, the proof is similar; it uses the characteristic inequality opposite to (A.8).

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