Inverse boundary value problem for the Schrödinger equation in a cylindrical domain by partial boundary data

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Abstract

Let $\Omega_1 \subset \mathbb{R}^2$ be a bounded domain with $\partial \Omega_1 \in C^\infty$ and $L$ be a positive number. For a three-dimensional cylindrical domain $Q = \Omega_1 \times (0, L)$, we obtain some uniqueness result in determining a complex-valued potential for the Schrödinger equation from partial Cauchy data when Dirichlet data vanish on a sub-boundary $(\partial \Omega_1 \setminus \tilde{\Gamma}) \times [0, L]$ and the corresponding Neumann data are observed on $\tilde{\Gamma} \times [0, L]$, where $\tilde{\Gamma}$ is an arbitrary fixed open set of $\partial \Omega_1$.

This paper is concerned with the inverse boundary value problem of determination of a complex-valued potential for the Schrödinger equation in a cylindrical domain from partial boundary data. More precisely the problem is as follows. Let $Q = \Omega_1 \times (0, L)$, where $\Omega_1 \subset \mathbb{R}^2$ is a bounded domain with $\partial \Omega_1 \in C^\infty$. Let $\Gamma_0 = \text{Int}(\partial \Omega_1 \setminus \tilde{\Gamma})$, $\Sigma = \tilde{\Gamma} \times [0, L]$ and $\Sigma_0 = \Gamma_0 \times [0, L]$.

In $Q$, we consider the Schrödinger equation with some complex-valued potential $q$:

$$L_q(x, D)u = (\Delta + q)u = 0 \quad \text{in} \quad Q, \quad (1)$$

Consider the following Dirichlet-to-Neumann map $\Lambda_{q, \Sigma_0}$:

$$\Lambda_{q, \Sigma_0} f = \frac{\partial u}{\partial \nu}|_{\partial Q \setminus \Sigma_0}, \quad \text{where} \quad L_q(x, D)u = 0 \quad \text{in} \quad Q, \quad u|_{\Sigma_0} = 0, \quad u|_{\partial Q \setminus \Sigma_0} = f \quad (2)$$

with the domain

$$D(\Lambda_{q, \Sigma_0}) = \{f \in H^\frac{1}{2}(\partial Q)| \text{supp} f \subset \partial Q \setminus \Sigma_0, \quad (f, g)_{L^2(\partial Q)} = 0, \quad \forall \ g \in \mathcal{N}\}$$

and

$$\mathcal{N} = \left\{ \frac{\partial u}{\partial \nu}|_{\partial Q} | L_q(x, D)u = 0 \quad \text{in} \quad Q, \quad u|_{\partial Q} = 0, \quad u \in H^1(Q) \right\}.$$

Problem (1) and (2) is the generalization of the inverse boundary value problem of recovery of the conductivity, which is also known as Calderón’s problem (see [3]). It is related to...
many practical applications, for example, detecting oil or minerals by applying voltage and measuring the fluxes on the surface of the earth. See also Cheney et al [4] for applications to medical imaging of EIT.

In the case when \( Q \) is a general domain in \( \mathbb{R}^n \) with \( n \geq 2 \), \( \Sigma_0 = \emptyset \) (i.e. the case of full Dirichlet-to-Neumann map), the unique recovery of the conductivity was established in [14] and [17] in two- and three-dimensional cases respectively. For reconstruction of the conductivity, see [15]. The assumption \( \Sigma_0 = \emptyset \) means that one has to set up voltages and measure the fluxes on the whole boundary. In practice, this assumption does not often hold, for example because the domain \( Q \) is extremely large or we cannot have access to some part of \( \partial Q \), e.g., the domain has cavities located inside. For the inverse boundary value problem with such partial Dirichlet-to-Neumann map, we refer to the following works. In [2], Bukhgeim and Uhlmann show that if voltages are applied on the boundary \( \partial Q \) and the corresponding fluxes are measured on some part of \( \partial Q \), then the potential can be uniquely determined. This result and a recent improved result [11] still require access to the whole boundary \( \partial Q \). In [10], Isakov solves the case where voltages are applied and the currents measured on the same set \( \partial Q \) provided that the sub-boundary \( \partial Q \setminus \partial Q^- \) is a part of some sphere or some plane. All the above-mentioned papers treat the case where the dimension of the domain \( \Omega \) is more than or equal to 3. As for related works in slabs, see Ikehata [6], Krupchyk et al [12], and Li and Uhlmann [13].

For general two-dimensional domains, [7] proved the unique recovery of a potential for the Schrödinger equation in the case when voltages are applied and the fluxes measured both on an arbitrary open set of \( \partial Q \). Thus, [7] established the best possible uniqueness in the two-dimensional case with data \( \Lambda_{q, \Sigma_0} \) defined by (2). Also see [8], which deals with the same inverse problem for more general second-order elliptic equations in the two-dimensional case and [9] improves the result of [7] in terms of regularity assumption of the potential for the Schrödinger equation. See Novikov [16] for conditional stability results for Calderón’s problem.

The purpose of this paper is to establish the uniqueness with weak constraints on such a sub-boundary in the case of three-dimensional cylindrical domain \( Q \).

By our method we can obtain the uniqueness for potentials in more general domains (not only the cylindrical one) but we do not discuss the details here.

We introduce the subset \( \mathcal{O} \) of domain \( \Omega \)
\[
\mathcal{O} = \Omega \setminus Ch(\Gamma_0), \quad Ch(\Gamma_0) = \{x| x = \lambda x^1 + (1 - \lambda)x^2, x^1, x^2 \in \Gamma_0, \lambda \in (0, 1)\}
\]

We have

**Theorem 1.** Let \( q_1, q_2 \) be continuous functions on \( \overline{\Omega} \). If \( \Lambda_{q_1, \Sigma_0} = \Lambda_{q_2, \Sigma_0} \) and \( D(\Lambda_{q_1, \Sigma_0}) \subset D(\Lambda_{q_2, \Sigma_0}) \), then \( q_1 = q_2 \) in \( \mathcal{O} \times [0, L] \).

From theorem 1 we immediately obtain

**Corollary 2.** Let \( \Omega \) be concave near \( \Gamma_0 \) and potentials \( q_1, q_2 \) be continuous functions on \( \overline{\Omega} \) such that \( \Lambda_{q_1, \Sigma_0} = \Lambda_{q_2, \Sigma_0} \) and \( D(\Lambda_{q_1, \Sigma_0}) \subset D(\Lambda_{q_2, \Sigma_0}) \). Then \( q_1 = q_2 \) in \( Q \).

First we formulate the following Carleman estimate with the linear weight function \( \varphi = x_3 \) for the Schrödinger operator (1). Denote \( \| \cdot \|_{H^{1,1}(Q)} = \| \cdot \|_{H^1(Q)} + |\tau| \| \cdot \|_{L^2(Q)} \). In [2] (see lemma 2.1) the following is proved.

**Theorem 3.** Let \( q \in L^\infty(Q) \). There exist constants \( \tau_0 \) and \( C \) independent of \( \tau \) such that for all \( \tau \geq \tau_0 \)
\[
\|ae^{\tau x_3}\|_{H^{1,1}(Q)} \leq C(\| (L_q(x, D)u) e^{\tau x_3}\|_{L^2(Q)} + \sqrt{\tau} \left( \frac{\partial u}{\partial x_3} e^{\tau x_3} \right) (\cdot, L)\|_{L^2(\Omega_1)}) \quad \forall u \in H^1_0(Q).
\]
Next we formulate some known result on the Radon transform
\[
(Rf)(\omega, p) = \int_{(x,x')=p} f(x') \, ds, \quad (\omega, p) \in S^l \times \mathbb{R}.
\]
The following result is proved in theorem 2.1 in [1] (see also [5]).

**Theorem 4.** Assume that \((\omega_0, p_0) \in S^l \times \mathbb{R}^l \) and \( f \in L^1(\mathbb{R}^3) \) has a compact support. Let \( V \) be an open neighborhood of \( \omega_0 \). Assume that \((Rf)(\omega, p) = 0 \) for \( p > p_0 \) and \( \omega \in V \). Then \( f = 0 \) on the half space \((x', \omega_0) > p_0\).

**Proof of theorem 1.** Let a point \( \hat{\omega} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \) exist, have the following form:
\[
\text{where } \vec{e}_1 = \bigg(1 \quad 0\bigg), \quad \vec{e}_2 = \bigg(0 \quad 1\bigg), \quad \alpha(x_0) = m'(x_0), \quad \beta(x_0) = \sqrt{1 - \alpha^2(x_0)}.
\]
Without loss of generality we may assume that \( \Omega \subset \mathbb{R}^l \times \mathbb{R}^l \). Let \( m \) be a smooth function defined on \( \mathbb{R}^l \times \{0\} \) such that \(|m'| < 1\). Denote \( \nabla' = (\partial_{x_1}, \partial_{x_2}) \) and \( x' = (x_1, x_2) \). Consider the eikonal equation
\[
|\nabla' \Psi| = 1 \quad \text{in } \Omega, \quad \Psi|_{x_2=0} = m.
\]
This equation can be integrated by the method of characteristics. The solutions, as long as they exist, have the following form:
\[
\Psi(x_0 + t\alpha(x_0)e_1 + t\beta(x_0)e_2) = m(x_0) + t \quad \forall x_0 \in [x'|x_2 = 0], \quad t > 0,
\]
Next construct the function \( \Psi \) more explicitly using the implicit function theorem. Consider the following mapping:
\[
F(y) = y_2\alpha(y_1)e_1 + y_2\beta(y_1)e_2 + \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \quad y = (y_1, y_2).
\]
Assume that
\[
\alpha(0) = m'(0) = 0.
\]
Then
\[
F(0, t) = (0, t)
\]
and
\[
F'(y) = (y_2\alpha'(y_1)e_1 + y_2\beta'(y_1)e_2) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \alpha(y_1)e_1 + \beta(y_1)e_2 = \begin{pmatrix} y_2\alpha' + \alpha \\ y_2\beta' \end{pmatrix}.
\]
In particular,
\[
F'(0, y_2) = \begin{pmatrix} 1 + y_2\alpha'(0) \\ 0 \end{pmatrix}.
\]
As long as the function \( 1 + y_2\alpha'(0) \) is positive, there exists the inverse matrix
\[
(F')^{-1}(0, y_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
Let $K$ be a positive number such that
\[ \Omega \cap \{x' | x_1 = 0\} \subset \{x' | x_1 = 0, 0 < x_2 < K\}. \]
By (8) there exists $\epsilon(K)$ such that for any $a \in \mathcal{X} = \{ \phi \in C^3_0(-1, 1) | \phi(0) = 0 \}$ satisfying $\|a\|_{C[-1, 1]} \leq \epsilon(K)$, there exists $\delta > 0$ such that on the set $[-\delta, \delta] \times [0, 2K]$ the matrix $(F')^{-1}$ is correctly defined:
\[ (F')^{-1}(y) = \frac{1}{\det F'(y)} \times \begin{pmatrix} -\alpha & -\beta \\ y_2 & y_2 \alpha' + 1 \end{pmatrix}. \]
Then by (7) and the implicit function theorem there exists $\tilde{\delta} > 0$ such that the mapping $x' \mapsto y(x')$ is correctly defined on $\mathcal{D} = [-\delta, \delta] \times [0, K]$ and the derivative of this mapping is given by the formula
\[ \frac{\partial y}{\partial x'} = (F')^{-1}(y(x')). \]
Differentiating the first columns on both sides of the matrix equation (9) with respect to $x_1$ and using (6) we have
\[ \begin{pmatrix} \frac{\partial^2 y_1}{\partial x_1^2} \\ \frac{\partial^2 y_1}{\partial x_1 \partial x_2} \\ \frac{\partial^2 y_2}{\partial x_2^2} \end{pmatrix}(0, x_2) = -\frac{2y_2 \alpha''(0)}{(1 + y_2 \alpha'(0))^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{(1 + y_2 \alpha'(0))^2} \begin{pmatrix} 0 \\ y_2 \alpha'(0) \end{pmatrix}. \]
Then the function $\Psi$ can be determined by the formula
\[ \Psi(x') = m(y_1(x')) + y_2(x'). \]
Observe that by (5) the function $\Psi(0, x_2)$ is a linear function and therefore $\partial^2_{x_2} \Psi(0, x_2) = 0$. Then short computations and (8), (10) imply
\[ (\Delta \Psi)(0, x_2) = \frac{y_2 \alpha'(0)}{(1 + y_2 \alpha'(0))^2} \]
\[ + \frac{y_2 \alpha''(0)}{(1 + y_2 \alpha'(0))^2} = \frac{\alpha'(0)}{(1 + y_2 \alpha'(0))^2} + \frac{y_2 \alpha''(0)}{(1 + y_2 \alpha'(0))^2} = \frac{\alpha'(0)}{(1 + y_2 \alpha'(0))}. \]
Under the assumption (6), setting $\gamma_0 = (0, 0)$ in (5), we obtain that
\[ \Psi(0, x_2) = m(0) + x_2. \]
Let $a_0(x')$ be a function such that
\[ 2(\nabla \Psi, \nabla a_0) + \Delta \Psi a_0 = 0 \quad \text{in} \quad \mathcal{D}, \]
and $a(x')$ be a smooth function such that
\[ (\nabla \Psi, \nabla a) = 0 \quad \text{in} \quad \mathcal{D}. \]
Next we construct the functions $a_0$ and $a$. In order to construct the function $a(x')$ we take a smooth function $r$
\[ r \in C^\infty_0(-\epsilon, \epsilon), \]
where $\epsilon$ is a small parameter and we set
\[ a(x') = r(x_0) \quad \text{on the line} \left\{ x' \in \mathbb{R}^2 | x' = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} + t \alpha(x_0) \tilde{e}_1 + t \beta(x_0) \tilde{e}_2, t > 0 \right\}. \]
We claim that for the function \( a \) defined by these formulae we have \( (14) \). Set \( \vec{v}_1 = \alpha(x_0)\vec{e}_1 + \beta(x_0)\vec{e}_2, \vec{v}_2 = \beta(x_0)\vec{e}_1 - \alpha(x_0)\vec{e}_2 \). Then by \( (5) \) \( |\partial_{\vec{v}_1}\Psi| = 1 \). Since \( |\nabla\Psi| = 1 \), we have that the vector \( \vec{v}_1 \) is parallel to \( \nabla\Psi \). The vectors \( \vec{v}_j \) are orthogonal. Hence

\[
\partial_{\vec{v}_j}\Psi = 0.
\]

Therefore,

\[
\partial_{\nabla\Psi} a = |\nabla\Psi|\partial_{\vec{v}_j} a = |\nabla\Psi|\partial_{\vec{v}_j} a = 0.
\]

Hence, the formula \( (14) \) is proved.

We integrate equation \( (13) \) by the method of characteristics. In particular, using \( (11) \) we set

\[
\tau = a_0(0, x_2) = e^{-\frac{1}{2} \int_{x_0}^{x_2} \frac{\partial D^0}{\partial \tau} \, dt} = e^{-\frac{1}{2} \ln(1+x_2\alpha'(0))} = \frac{1}{\sqrt{1+x_2\alpha'(0)}}. \tag{17}
\]

**Step 2: construction of complex geometric optics solutions.** Next we construct the complex geometric optics solution \( u_1(x, \tau) \) for the Schrödinger operator with the potential \( q_1 \). For the principal term of complex geometric optics solution we set

\[
U = e^{(\tau + i\gamma)(x_0 + i\psi(\tau'))} a_0,
\]

where \( \gamma \in \mathbb{C} \) is a parameter. The set \( Ch(\Gamma_0) \) is closed and the axis \( x_2 \) does not intersect this set. Hence there exists a neighborhood of the set \( \{x' | x_2 \in [0, K] \} \) such that it does not intersect \( Ch(\Gamma_0) \). Thanks to \( (15) \) and \( (16) \), choosing a positive parameter \( \epsilon \) sufficiently small, we obtain

\[
U|_{\Sigma_0} = 0. \tag{19}
\]

The simple computations imply

\[
L_{q_1}(x, D)U = (\tau + i\gamma)(\nabla(x_2 + i\nabla\psi), \nabla(x_2 + i\nabla\psi))U \\
+ (\tau + i\gamma)(2(\nabla\nabla\psi, \nabla a_0) + \Delta x\nabla\psi a_0) a e^{(\tau + i\gamma)(x_2 + i\nabla\psi(\tau'))} \\
+ e^{(\tau + i\gamma)(x_2 + i\nabla\psi(\tau'))} \Delta x (a_0 a) + q_1 U = e^{(\tau + i\gamma)(x_2 + i\nabla\psi(\tau'))} \Delta x (a_0 a) + q_1 U. \tag{20}
\]

Observe that the functions \( e^{(\tau + i\gamma)(x_2 + i\nabla\psi(\tau'))} \Delta x (a_0 a) + q_1 U \) are uniformly bounded in \( \tau \) in the norm of the space \( L^2(Q) \). Consequently, using lemma 3.1 of [2], we construct the last term in complex geometric optics solution—the function \( u_{\text{cor}}(\cdot, \tau) \)—such that

\[
L_{q_1}(x, D)(e^{\tau x_2} u_{\text{cor}}) = -L_{q_1}(x, D)U \quad \text{in} \quad Q, \quad u_{\text{cor}}|_{\Sigma_0} = 0 \tag{21}
\]

and

\[
\|u_{\text{cor}}\|_{L^2(Q)} = O\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty. \tag{22}
\]

Hence, by \( (19), (21) \) and \( (22) \) for the function \( u_1 = U + u_{\text{cor}} e^{\tau x_2} \), we have the representation

\[
u_1 = U + e^{\tau x_2} O_{L^2(Q)} \left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty, \quad u_1|_{\Sigma_0} = 0. \tag{23}
\]

By \( O_{L^2(Q)} \), we mean any function \( f(\cdot, \tau) \) such that \( \|f(\cdot, \tau)\|_{L^2(Q)} = O(1) \) as \( \tau \to +\infty. \) Similarly we set

\[
V = e^{-\tau (x_3 + i\psi(\tau'))} a_0, \quad V|_{\Sigma_0} = 0. \tag{24}
\]
We multiply any smooth function $a_0$ satisfying (13) with a solution of equation (14) which is supported around the ray $\{x | x_2 > 0, x_1 = 0\}$ and is equal to 1 on this ray. Hence we can assume that the support of function $a_0$ is concentrated around this ray and (17) holds true.

**Step 3: proof of uniqueness.** Since the Dirichlet-to-Neumann maps of the Schrödinger equations with potentials $q_1, q_2$ are the same, there exists a solution to the following boundary value problem:

$$L_{q_2}(x, D)u_2 = 0 \text{ in } Q, \quad u_1 = u_2 \text{ on } \partial Q, \quad \text{and } \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \text{ on } \partial Q \setminus \Sigma_0.$$  \hfill (25)

Setting $u = u_1 - u_2$ and using (25), we have

$$L_{q_2}(x, D)u = (q_1 - q_2)u_1 \text{ in } Q, \quad u|_{\partial Q} = \frac{\partial u}{\partial \nu}|_{\partial Q | \Sigma_2} = 0.$$  \hfill (26)

Applying to equation (26) the Carleman estimate (3), we have that there exist constants $C$ and $\tau_0$ independent of $\tau$ such that

$$\|ue^{-x_1}\|_{H^1(Q)} \leq C \quad \forall \tau \geq \tau_0.$$  \hfill (27)

Then taking the scalar product in $L^2(Q)$ with $V$, and using (27), (26), (24), (23) we have

$$\int_{Q} (q_1 - q_2)u_1 V \, dx = \int_{\Omega} uL_{q_2}(x, D)V \, dx = O\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty.$$  

This equality and (22) imply

$$\int_{Q} (q_1 - q_2)e^{(x_1 + i\beta(x_1'))}a_0^2 \, dx = 0.$$  \hfill (28)

Setting $p_z(x') = \int_{0}^{t} (q_1 - q_2)e^{x_2} dx_2$ and using (17) we obtain from (28)

$$\int_{0}^{k} p_z e^{x_2} dx_2 = 0 \quad \forall \tau \in \mathbb{C}^l.$$  \hfill (29)

Indeed, let $r(s) = r_{h, \delta}(s) \in C^\infty_c(\mathbb{R}^1)$ be a function such that it is equal to 1/2$\delta$ on the segment $[-h, h]$ and zero for $|s| \geq h + \delta$. On the segment $[-h - \delta, -h]$ the function $r_{h, \delta}$ is monotone increasing and on the segment $[h, h + \delta]$ the function $r_{h, \delta}$ is monotone decreasing. Denote the solution to equation (14) given by (16) with the initial condition $r_{h, \delta}$ as $a(h, \delta)$. By (28) we have

$$\int_{Q} (q_1 - q_2)e^{(x_1 + i\beta(x_1'))}a(h, \delta)a_0^2 \, dx = 0.$$  \hfill (30)

Passing to the limit as $\delta$ approaches to $+0$ in (30) we obtain

$$\int_{Q} (q_1 - q_2)e^{(x_1 + i\beta(x_1'))}a(h) a_0^2 \, dx = 0.$$  \hfill (31)

The function $a(h)$ is given by the formula

$$a(h) = \left\{ \begin{array}{ll} 1 & x' \in \Pi_h, \\ \frac{1}{2h} & x' \notin \Pi_h, \\ \end{array} \right.$$

where $\Pi_h = \{x' \in \mathbb{R}^1 \times \mathbb{R}^1 \mid x_2 \in [0, K], -h + \frac{a(h)}{2h} x_2 \leq x_1 \leq h + \frac{a(h)}{2h} x_2\}$. Therefore, for any fixed $x_2$ from the segment $[0, K]$ the function $a(h)$ equals $\frac{1}{2h}$ on the segment
Therefore, by (32) we obtain
\[
0 = \int_0^L (q_1 - q_2) e^{i\alpha(x, x')} a(h) a_0^2 \, dx = \frac{1}{2h} \int_{\mathbb{R}_0} p_e e^{i\phi(x')} a_0^2 \, dx'.
\]
\[
= \int_0^K \int_{-h + \overline{(x_0 + i)h}}^{h + \overline{(x_0 - i)h}} \frac{p_e e^{i\phi(x')} a_0^2}{2h} \, dx_1 \, dx_2.
\]
By (33) there exist constants 0 such that
\[
\exists \forall \beta(x) \in \mathbb{C}^1
\]
for all \( p \in S^1 \times \mathbb{R}_1 \) such that \( \{x'\} \cap Ch(\Gamma_0) = \emptyset \).

This proves (29).

By (29) we have
\[
\mathcal{P}(z, \omega, p) = \int_{(\omega, x') = p} e^{i\alpha(x', x)} \, ds = 0 \quad \alpha^\perp = (\omega_2, -\omega_1),
\]
for all \( (\omega, p) \in S^1 \times \mathbb{R}_1 \) such that \( \omega\perp \cap Ch(\Gamma_0) = \emptyset \).

For any fixed \( (\omega, p) \in S^1 \times \mathbb{R}_1 \) the function \( \mathcal{P}(\omega, p) \) is holomorphic in the variable \( z \).

Therefore, by (32) we obtain
\[
\frac{\partial^\ell \mathcal{P}}{\partial z^\ell}(0, \omega, p) = \int_{(\omega, x') = p}^L (q_1 - q_2) (x_3 + i\omega', x') d_3 \, ds = 0 \quad \forall (\ell, z) \in \mathbb{N}_+ \times \mathbb{C}^1
\]
and for all \( (\omega, p) \in S^1 \times \mathbb{R}_1 \) such that \( \{x'\} \cap Ch(\Gamma_0) = \emptyset \).

We can claim that
\[
p_{0,k}(x') = \int_0^L (q_1 - q_2) (x) x_3^k \, dx_3 = 0 \quad \forall k \in \mathbb{N}_+ \quad \text{and} \quad \forall x' \in \mathbb{R}^2 \setminus Ch(\Gamma_0).
\]

By (33) there exist constants \( C_{k, \ell} \) such that
\[
\int_{(\omega, x') = p} p_{0,k} \, ds = \sum_{\ell = 0}^{k-1} C_{k, \ell} \int_{(\omega, x') = p} (\omega^\perp, x')^{k-\ell} \, p_{0,\ell} \, ds
\]
for all \( (\omega, p) \in S^1 \times \mathbb{R}_1 \) such that \( \{x'\} \cap Ch(\Gamma_0) = \emptyset \).

The function \( \mathcal{P}(0, \omega, p) \) is the Radon transform of the function \( p_0 \). Applying theorem 4 for the Radon transform, we have
\[
p_0(x') = p_{0,0}(x') = 0 \quad \forall x' \in \mathbb{R}^2 \setminus Ch(\Gamma_0).
\]
Suppose that the equalities (34) are already proved for all \( \ell \) less than \( k \). Then equality (33) immediately implies (34) for \( \ell = k \).

The function \( p_{\ell} \) is holomorphic in the variable \( z \) for any fixed \( x' \). Equality (34) implies that the derivatives of any orders with respect to \( z \) of this function are equal to zero. Therefore,
\[
p_{\ell}(x') = 0 \quad \forall z \in \mathbb{C}^1 \quad \text{and} \quad \forall x' \in \mathbb{R}^2 \setminus Ch(\Gamma_0).
\]

Setting in equation (37) \( z = iy \), for any fixed \( x' \in \mathcal{O} \) and any \( y \in \mathbb{R}_1 \) we have
\[
\int_0^L (q_1 - q_2)(x', x_3) e^{iyx_3} \, dx_3 = 0.
\]
This equality immediately implies that the function \( x_3 \rightarrow (q_1 - q_2)(x', x_3) \) on the segment \( [0, L] \) is orthogonal to any polynomials. Therefore \( (q_1 - q_2)(x)|_{\mathcal{O} \times [0, L]} = 0 \). The proof of the theorem is complete. \( \square \)
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