Streaming instability in the quasi-global protoplanetary discs

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ABSTRACT

We investigate streaming instability using two-fluid approximation (neutral gas and dust) in a quasi-global, unstratified protoplanetary disc, with the help of PIERNIK code. We compare amplification rate of the eigenmode in numerical simulations, with the corresponding growth resulting from the linear stability analysis of full system of Euler’s equation including aerodynamic drag. Following Youdin & Goodman (2005), we show that (1) rapid dust clumping occurs due to the difference in azimuthal velocities of gas and dust, coupled by the drag force, (2) initial density perturbations are amplified by several orders of magnitude. We demonstrate that the multifluid extension of the simple and efficient Relaxing TVD scheme, implemented in PIERNIK, leads to results, which are compatible with those obtained with other methods.

Key words: hydrodynamics – instabilities – planets and satellites: formation.

1 INTRODUCTION

The formation of planets is a complex process that requires sub-μm dust grains to grow over a dozen orders of magnitude in size. The smallest particles, although they are highly coupled to gas, can slowly drift in both radial and vertical direction, and collide with each other. These physical collisions with low relative velocities lead to the creation of larger agglomerates (Blum & Wurm 2008). On the other side of the scale, 10−10 m bodies are large enough to be almost completely decoupled from gas dynamics and be mostly influenced by gravitational interactions (Kokubo, Kominami & Ida 2006). The most problematic is the intermediate step of growing dust from cm grains to km-sized bodies, due to the existence of several processes that counteract the possible growth or pose significant time constraints. The latter is mainly caused by a fast radial drift of the bodies that are marginally coupled to the gas, i.e. characteristic scale of their aerodynamic drag is of the order of their orbital time (Weidenschilling 1977). Moreover, the characteristic relative velocities for the bodies of sizes ranging from 1 cm to 1 m are of the order of 1−10 m s−1 which, during collisions, result in fragmentation and bouncing rather than sticking (Zsom et al. 2010).

One of the possible scenarios is a fast enhancement of dust density via sedimentation to the mid-plane due to vertical component of host star gravity. Then, further fragmentation and collapse of dense dust layer are caused by a self-gravity (Goldreich & Ward 1973). Though the sedimentation itself leads to the rise of Kelvin–Helmholz instability (Johansen, Henning & Klahr 2006) (KHI) which prevents forming an infinitesimally thin layer in the mid-plane; recent studies show that it is not the case for massive discs with higher than solar metallicities (Lee et al. 2010). However, a significant flaw behind presented reasoning is a complete negligence of disc’s global turbulence, which in turn is the only mechanism that would allow us to explain observed accretion rates assuming α-disc model (Shakura & Sunyaev 1973). Even though, the most plausible mechanism responsible for such turbulence, i.e. magnetorotational instability (Balbus & Hawley 1998), allows for formation of local, low-level of turbulence areas due to insufficient ionization of gas, there are still other instabilities that stir the fluids (Lesur & Papaloizou 2010).

Despite of this unfavourable circumstances there is a process that commences to dominate dust evolution when ratio of concentration of dust particles to gas density approaches unity. This mechanism was first presented by Youdin & Goodman (2005) (hereafter YG05) and named the streaming instability. It appears that combination of dust trapping in gas pressure maxima and mutual interaction leading to dust dragging the gas, therefore enhancing the maxima even further, results in significant dust pile-up (Jacquet, Balbus & Latter 2011). Even without the presence of self-gravity, dust concentration may be risen up to the three orders of magnitude (Johansen & Youdin 2007) (hereafter JY07) which could possibly lead to gravitationally bound objects (Johansen et al. 2007). Recent numerical studies of streaming instability concentrate on various physical aspects that may influence its evolution; that includes: the influence of wide range of dust species (Bai & Stone 2010b), effects of global pressure gradients (Bai & Stone 2010a) and stratification of discs (Takeuchi et al. 2012). However, all known to authors’ works limit themselves to local disc approximation.
The main goal of this paper is to investigate the streaming instability in a more realistic circumstances of radially extended discs. The essential points of interest are: (1) resolution studies of the streaming instability in a 2D radially extended domain, to see what is the minimum grid resolution necessary to ensure growth rates of simulated instability modes consistent with the results of linear stability analysis, (2) investigation of the streaming instability in a 3D radially extended domain, in an optimal grid resolution, deduced from resolution studies in 2D.

The plan of the paper is as follows: In Section 2, we set up the basic assumptions and equations relevant for this paper. In Section 3, we perform linear analysis of streaming instability in protoplanetary discs. In Section 4, we describe specific numerical algorithm that were used in the performed simulations along with the initial conditions and the basic simulation parameters. Section 5 shows the obtained results and their detailed comparison with linear stability analysis. Finally in Section 6, we discuss the outcome of our experiments and give short outlook of the future work.

2 BASIC ASSUMPTIONS AND EQUATIONS

We investigate global dynamics of two interacting fluids – gas and dust – in the protoplanetary disc. We assume that neutral gas obeys the isothermal equation of state, whereas dust is treated as a pressureless fluid.

\[ \partial_t \rho_g + \nabla \cdot (\rho_g \mathbf{u}) = 0, \]
\[ \partial_t \rho_d + \nabla \cdot (\rho_d \mathbf{w}) = 0, \]
\[ \partial_t (\rho_g \mathbf{u}) + \nabla \cdot (\rho_g \mathbf{u} \otimes (\rho_g \mathbf{u})) + P = -\rho_g \left( \nabla \Phi + \frac{\rho_d}{\tau_f \rho_g} (\mathbf{u} - \mathbf{w}) \right), \]
\[ \partial_t (\rho_d \mathbf{w}) + \nabla \cdot (\rho_d \mathbf{w} \otimes (\rho_d \mathbf{w})) = -\rho_d \left( \nabla \Phi + \frac{1}{\tau_f} (\mathbf{w} - \mathbf{u}) \right), \]

where \( \rho_g, \rho_d \) are densities of gas and dust, respectively, \( \mathbf{u}, \mathbf{w} \) corresponding velocities, \( P \) gas pressure, \( \tau_f \) is a friction time and \( \Phi \) is gravitational potential. Our aim is to isolate the streaming instability from the other possible processes acting in a radially extended protoplanetary disc. In order to obtain a clear diagnostics of the excited instability, we neglect the vertical component of gravity arising from the point mass source in the centre of the reference frame, which would lead to dust sedimentation and eventually to the onset of KHI (Johansen et al. 2006).

We shall simulate disc starting from the relatively large radius (2 au). We assume, following Cuzzi, Dobrovolskis & Champness (1993) that transition to Stokes regime occurs for particle radius \( a = 9/4\lambda_g \), where \( \lambda_g = 4.2 \times 10^6 \text{ cm} (10^{-14} \text{ g cm}^{-3}/\rho_0)^{1/2} (R/1 \text{ au})^{1/2} \text{ cm} \) is the mean free path of the gas molecules (Weidenschilling 1977; Balsara et al. 2009), and \( R \) is the radial distance to the disc centre. Under these assumptions, Epstein’s regime applies in the dominating part of our domain, even for the largest simulated grain sizes. Therefore, to compute the friction time, we use Epstein’s law

\[ \tau_f = \frac{\rho_* a}{\rho_0 c_s^2 + |\mathbf{u} - \mathbf{w}|^2}, \]

where \( c_s \) is the gas sound speed, and \( \rho_* = 1.6 \text{ g cm}^{-3} \) is the density of the solid material. While the term \( |\mathbf{u} - \mathbf{w}|^2 \) does not play any significant role during the evolution of Streaming Instability (SI), i.e. its value never exceeds 1 per cent of \( c_s^2 \), we have decided not to neglect it in our simulations for the completeness sake.

3 LINEAR STABILITY ANALYSIS

One of the aims of this paper is to validate numerical results related to the growth of streaming instability unstable modes via the comparison to the growth rate and mode geometry resulting from linear stability analysis. Therefore, following the previous authors (YG05), in this section, we review the main steps of the linear analysis of the streaming instability in protoplanetary discs.

The linear stability analysis relies on three essential elements: (1) finding an equilibrium state of the system, (2) perturbing the system with small amplitude (linear) perturbations and (3) deriving the eigenmodes and their growth rates. Unfortunately, there is no strictly stationary solution to global Keplerian disc consisting of two fluids coupled via the drag force. The reason is the dust migration, implying variations of the radial disc profile. We note however, that the characteristic time-scale of radial migration of metre-size dust agglomerates is of the order of 100 years or longer, therefore we consider the variations of the disc profile as a slow process, as compared to typical growth times of streaming instability.

Another difficulty results from the fact that the wide range of the radial coordinate in our numerical models implies radial variations of the equilibrium state of the physical parameters. In these circumstances, it would be appropriate to perform global stability analysis by means of solving the two-point boundary value problem (see e.g. Perucho et al. 2004; Kosiński & Hanasz 2006); however, the latter approach is significantly more complex than the local stability analysis. Therefore, we consider the local stability analysis as a first approximation to the full analysis of streaming instability in protoplanetary discs. The modes derived in the framework of local linear stability analysis will serve us as a reference solution for validation of the modes existed in the global numerical simulations.

The most suitable local treatment of streaming instability is provided by the ‘shearing-sheet’ approximation (Hawley, Gammie & Balbus 1995), which is achieved by placing Cartesian coordinate frame on an orbit of radius \( R \), that corotates with Keplerian frequency \( \Omega \). We assume that x-axis points radially outward, and y corresponds to the azimuthal direction, whereas z is a vertical axis. Following Youdin & Johansen (2007), we write down the continuity and Euler equations for both the gas and dust components

\[ \partial_t \rho_g + \mathbf{u} \cdot \nabla \rho_g - \frac{3}{2} \Omega x \partial_y \rho_g = -\rho_g \nabla \cdot \mathbf{u}, \]
\[ \partial_t \rho_d + \mathbf{w} \cdot \nabla \rho_d - \frac{3}{2} \Omega x \partial_y \rho_d = -\rho_d \nabla \cdot \mathbf{w}, \]
\[ \partial_t (\mathbf{u} \cdot \mathbf{v}) - \frac{3}{2} \Omega x \partial_y \mathbf{u} = 2 \Omega u_x \hat{x} - \frac{1}{2} \Omega u_y \hat{y}, \]
\[ \partial_t (\mathbf{w} \cdot \mathbf{v}) - \frac{3}{2} \Omega x \partial_y \mathbf{w} = 2 \Omega w_x \hat{x} - \frac{1}{2} \Omega w_y \hat{y} - \frac{1}{\tau_f} (\mathbf{w} - \mathbf{u}), \]

where transport terms, such as \((3/2)\Omega x\) on the left-hand side of equations arise due to fact that we measure all velocities relative to linear, Keplerian shear flow in the rotating frame \( v_0 = -(3/2)\Omega x \hat{y} \). It is worth noting that term \(-(1/2)\Omega (\mathbf{u}, \mathbf{w})\), \( \hat{y} \) on the right-hand
side of the equations of motion (8)–(9) is a sum of two components: \((-2\Omega [u, u_y] + (3/2)\Omega [u, v_z])\dot{y},\) where the first one is the component of Coriolis force and the second results from aforementioned subtraction of the mean flow. The main difference between equations (8) and (9) is the radial pressure gradient term influencing the dynamics of gas only. As noted by YG05, it is possible, within the local approximation, to include consistently the disc’s global pressure gradient parametrized by the dimensionless measure of sub-Keplerian rotation

\[
\eta = \frac{\partial \xi P}{2 \rho g R^2} \sim \frac{c_s^2}{\tau_f^2},
\]  

(10)

The set of equations (6)–(9) has a known equilibrium solution (Nakagawa, Sekiya & Hayashi 1986)

\[
\dot{w} = \left[ -2 \tau_i \xi - \frac{1}{1 + \epsilon}, 0 \right] \eta v_K,
\]  

(11)

\[
\dot{u} = \left[ 2 \tau_i \xi - \frac{1 + \epsilon}{1 + \epsilon}, 0 \right] \eta v_K,
\]  

(12)

where \(\tau_i = \Omega \tau_f\) is dimensionless stopping time and \(\xi = ((1 + \epsilon)x + \frac{1}{2}z)^{-1}\). We linearize equations (6)–(9), decomposing variables into a steady part and a perturbation \(q = \hat{q} + q',\) where \(q = \rho, v_x, v_y, v_z, u_x, u_y, u_z\). We assume subsequently that the perturbations are axisymmetric (independent on \(y\)-coordinate in the present analysis) and can be expressed as plane waves

\[
q'(x, z, t) = \hat{q} \exp[i (k_x x + k_z z - \omega t)].
\]  

(13)

After the substitution of the linear perturbations the equations read

\[ -i(\omega - k_x \dot{\bar{w}})\hat{\bar{p}}_d = -i\hat{\bar{p}}_d (k_x \hat{\bar{w}} + k_z \hat{\bar{w}}), \]  

(14)

\[ -i(\omega - k_z \hat{\bar{u}})\hat{\bar{p}}_g = -i\hat{\bar{p}}_g (k_z \hat{\bar{u}} + k_x \hat{\bar{u}}), \]  

(15)

\[ -i(\omega - k_x \hat{\bar{u}})\hat{\bar{u}} = 2\Omega \hat{\bar{u}} \hat{\bar{u}} - \frac{1}{2} \Omega \hat{\bar{u}} \hat{\bar{y}} - \frac{\epsilon}{\tau_f} (\hat{\bar{u}} - \hat{\bar{w}}) \]  

\[ - \frac{\hat{\bar{p}}_d}{\rho \tau_f} (\hat{\bar{u}} - \hat{\bar{w}}) = \frac{c_s^2}{\rho \tau_f} (\hat{\bar{u}} - \hat{\bar{w}}), \]  

(16)

\[ -i(\omega - k_z \hat{\bar{w}})\hat{\bar{w}} = 2\Omega \hat{\bar{w}} \hat{\bar{w}} - \frac{1}{2} \Omega \hat{\bar{w}} \hat{\bar{y}} - \frac{\epsilon}{\tau_f} (\hat{\bar{w}} - \hat{\bar{u}}), \]  

(17)

where \(\epsilon = \hat{\bar{p}}_d/\hat{\bar{p}}_g\). The set of equations (14)–(17) can be expressed as

\[ A(k_x, k_z, \omega)\hat{q} = 0. \]  

(18)

Non-trivial solutions of the linear system (18) exist if

\[ \det[A(k_x, k_z, \omega)] = 0. \]  

(19)

We solve the dispersion relation (19) for given values of \((k_x, k_z),\) with respect to the complex frequency \(\omega,\) using the Durand–Kerner method (Durand 1960; Kerner 1966), and subsequently obtain relation between constant amplitudes of components of the vector \(\hat{q}.\)

The growth rate of the instability is defined as imaginary part of the complex frequency \(s = \text{Im}(\omega)\).

### 4 SIMULATION SETUP

#### 4.1 Algorithms

We conduct numerical simulations with the aid of a parallel magnetohydrodynamic (MHD) code PIERNIK using the cylindrical coordinate system. Following Mignone et al. (2007) and Skinner & Ostriker (2010) we use ‘angular momentum-conserving form’ of the \(\phi\)-momentum equation, which with respect to Cartesian geometry introduces only one additional source term to equations (3)–(4):

\[
((\rho u_\phi + P)R)\dot{R} \quad \text{and} \quad (\rho u_\phi u_\phi / R)\dot{R},
\]

respectively.

Both components, i.e. gas and dust are treated as fluids (Hanasz et al. 2010) which are dynamically coupled via the friction force. In order to prevent large timestep constraint from drag force acceleration, we used the semi-implicit scheme by Tilley et al. (2009). We allow the system to relax numerically (during the period of initial 10 yr of the simulation) before we ‘turn on’ the aerodynamic drag force and seed dust velocities with low-amplitude random noise. The feedback of the linear drag force scales with the density ratio of dust to gas \(\epsilon = \rho_d/\rho_g.\)

#### 4.2 Initial disc configuration

The initial density profile for gas roughly follows prescription of Minimal Mass Solar Nebula (Hayashi 1981)

\[ \Sigma(R) = 1700 \left( \frac{R}{1 \text{ au}} \right)^{-3/2} \text{ g cm}^{-2}. \]  

(20)

We assume isothermal equation of state and a constant temperature \(T_0 = 170 \text{ K}\) across the whole disc. We assume gravitational field from a point mass \(M = 1 \text{ M}_\odot\), and neglect the vertical component of gravitational acceleration towards the central mass, implying no vertical stratification of the disc. Yet, we refer to the vertical scaleheight \(H\) to estimate volume density of gas at given radius, relying on the hydrostatic equilibrium density distribution in the presence of the vertical gravity of a point mass

\[ \rho(R, z) = \rho(R, 0) \exp \left(-\frac{z^2}{2H(R)^2}\right), \]  

(21)

where \(\rho(R, 0)\) is gas density at the disc mid-plane and \(H^2 = 2c_s^2 R^3 / GM.\) The surface gas density is given by

\[ \Sigma(R) = \int_{-\infty}^{\infty} \rho(R, z) \, dz. \]  

(22)

The corresponding value of mid-plane gas density is

\[ \rho(R, 0) = \int_{-\infty}^{\infty} \frac{\Sigma(R)}{H(R)^2} \, dz. \]  

(23)

For the chosen disc temperature \(T_0,\) the integral on the right-hand side of (23) varies from 0.4 to 2.0 au over the range of radii \(R \in [2, 6] \text{ au}\). We assume for simplicity its value equal to 1 au.

We assume the vertical extent of the computational domain \(L_z = 0.3 \text{ au},\) and impose periodic boundary conditions at upper and lower \(z\)-boundaries. The initial condition relies on a radial force balance for the gas and dust components independently. The gas component remains in a hydrostatic equilibrium resulting from the radial balance of gravity, centrifugal and pressure forces, while the pressure gradient term is absent in the equation of motion for the dust component. Reflecting boundary conditions are set on the inner and outer boundaries of the computational grid to prevent mass escape from the computational domain.
To minimize unphysical wave reflections, we implement wave killing zones close to the inner and outer radial boundaries. The inner wave killing zones cover 0.5 au near the inner and outer edges of the computational domain. In these zones, we add an additional damping term, in the evolution equations of each fluid variable

\[ \frac{dX}{dt} = -\frac{X - X_0}{T_d} f(R), \]

(24)

together with

\[ f(R) = 1 - \tanh \left( (R - R_{in} + 1)^\epsilon \right) + \max \left\{ \tanh \left( (R - R_{out} + 1)^\epsilon \right), 0 \right\}, \]

(25)

where \( X_0 \) is the initial value of \( X \) and \( T_d \) is the damping time-scale. We chose \( T_d \) of the order of orbital period. The exponents \( f_{in} = f_{out} = 10 \) control the width of the transition layer between the unmodified to the damped zones. Within the damped zones, the effects of undesired wave reflection are essentially minimized.

### 4.3 Simulation parameters

We have performed a parameter sweep for the ratio of dust to gas density \( \eta \) and the size of particles, at two different resolutions of the computational grid in 2D, and additionally we have realized one of the models in 3D. Our choice of these parameters follows closely that of JY07, to enable detailed comparison of the results.

The full list of simulations together with their main parameters is presented in Table 1. In all simulations the domain has a height of 0.293 au, for 2D it covers radii from 2 to 7 au, whereas BB3D extends from 2 to 12 au and spans \( \varphi \in [0, \pi/6] \). Following JY07, we vary \( \epsilon \) from 0.2 to 2.0 in order to exhibit morphologically different outcomes of non-linear phase of streaming instability. We choose particle radii to be greater than 10 cm and less than 50 cm so that (1) particles fall into Epstein regime in all simulations, (2) their size could be plausibly explained by the growth processes e.g. collisional agglomeration.

### 5 RESULTS

The initial stage of streaming instability evolution is similar for all cases of \( \epsilon \) or \( \alpha \) and is governed by the dominant linear modes. Detailed analysis of linear phase is covered in Section 5.4. The following two sections describe different non-linear evolution of streaming instability in quasi-global setup with reference to similar case shown by JY07.

#### 5.1 Marginally coupled boulders (\( \alpha = 50 \text{ cm}, \tau_s \approx 1.2 \))

As noted by JY07, streaming instability is the most prominent for \( \tau_s = 1 \) and exhibits fast linear growth and heavy dust clumping. Fig. 1 shows temporal evolution of streaming instability for different initial dust to gas ratios \( \epsilon \). In all cases (BA, BB, BC) elongated clumps are formed. When the saturated state is reached, due to the high vertical speed of dust overdensities, the filaments undergo occasional fragmentation and collisions with each other, though still following ‘v’-shaped trajectories that emerged during linear phase. The most prominent differences between BA, BB and BC are: (1) the characteristic length of local clumps is decreasing with the increase of initial \( \epsilon \) and (2) their tendency to lean in radial direction: in BB structures are almost purely diagonal, whereas in BA and BC are much more elongated in the vertical direction. Similarly to JY07, in all cases (BA, BB, BC) the density peaks of the dust component in the non-linear regime settle at levels about 2 order of magnitude higher than initial density.

#### 5.2 Tightly coupled boulders (\( \alpha = 10 \text{ cm}, \tau_s \approx 0.24 \))

Since the fluid approach, contrary to the particle description, is much less susceptible to effects of Poisson noise we were able to follow linear phase for run AB and ABh for much longer than JY07, though in the end we also observe sudden increase in growth rate of the instability due to spontaneous cavitation (initial phase is caught in the middle sub-panel of bottom-right panel of Fig. 2). The bubbles of void start to spawn at outer radii and greatly increase the dust density and velocity on their edges (see Fig. 3). After just few orbital periods the accelerated instability expands to whole domain and dominate the non-linear evolution. Afterwards the quasi-stationary state of vivid turbulence is achieved. In both cases (AB, AC) the non-linear evolution leads to the enhancement of the peak densities of dust component over one order of magnitude, which is in close agreement with results of JY07 (see Fig. 8 in their work).

In the dust-dominated case (AC) our results closely follow those of JY07 up to the point of saturation, where we achieve highly turbulent flow (lower sub-panel of upper-right panel of Fig. 2) and about 20 fold growth in dust density. However, we note that secular evolution of the physical system leads to the formation of elongated dust overdensities and further growth up to 2 orders of magnitude as seen in runs with other parameters (see lower sub-panel of bottom-right panel of Fig. 2).

In the gas-dominated case (AA), we observe initial overdensity growth over two orders of magnitude in a characteristic grid-like pattern. However, at time 1200, 440, 325 for runs AA, AAh, AAu, respectively, in the course of non-linear evolution the dense dust filaments are abruptly smoothed out resulting in oscillatory motion of mild overdensities that are only one order of magnitude denser than initial dust distribution (see Fig. 4).

#### 5.3 3D run

The course of the evolution of our single 3D (BB3D) closely follows the corresponding 2D run BB (compare Fig. 5 and Fig. 1). During the linear phase of growth-elongated clumps or rather sheets of dense dust are formed, as the deviation from axis axisymmetry is almost negligible. In order to estimate whether we should expect
Figure 1. Dust density snapshots for runs with 50 cm grain sizes at 100, 200, 450 and 700 yr for upper-left, upper-right, bottom-left and bottom-right panel, respectively. Each of the panels is divided into three sub-panels for initial $\epsilon = 0.2, 1, 2.0$ on upper (BAh), middle (BB) and lower (BC) sub-panel, respectively.

Figure 2. Dust density snapshots for runs with 10 cm grain sizes at 100, 200, 800 and 1200 yr for upper-left, upper-right, bottom-left and bottom-right panel, respectively. Each of the panels is divided into three sub-panels for initial $\epsilon = 0.2, 1, 2.0$ on upper (AA), middle (AB) and lower (AC) sub-panel, respectively.

Figure 3. Dust density snapshot for run ABh showing two ‘zoom-in’ regions where cavities emerge from overdensities created during linear phase of evolution (left-hand panel) and region totally overwhelmed by bursting void bubbles and vivid dust turbulence (right-hand panel).

Figure 4. Maximum ratio of dust to gas density for three runs with the same initial conditions, but different resolution. Streaming instability follows regular pattern of evolution: (1) fast linear growth that locally increases $\epsilon$ over 2 orders of magnitude; (2) after reaching certain level of $\epsilon \approx 10$, overdensities are abruptly smoothed out; (3) instability reaches out saturated, non-linear phase where large, smooth clumps of dust are only 10 times denser than initial condition. Varying resolution only influences the availability of shorter and faster growing modes, i.e. shortens phase (1).

effects of self-gravity in the non-linear evolution of streaming instability, we have used YT (Turk et al. 2011) clump finding facility to identify all the largest disconnected contour spanning for minimum value of 50 computational cells in BB3D run. Then, we
velocity \( \bar{\Omega} \) is taken for the centre of the patch. Dimensionless measure of sub-Keplerian gas rotation is calculated by means of the formula (see YG05 equation 16 or JY07 equation 1)

\[
\bar{\eta} = -\frac{c_s^2 (\partial_x (\rho g)_i)_R}{2 \bar{\rho}_g \bar{\Omega}^2 R}.
\]  

In formula (27), we average gas density in the vertical direction, then we calculate the mean radial derivative of \( (\partial_x \rho g) \). The expression for mean stopping time is derived from (5)

\[
\bar{t}_f = \rho c / \left( \bar{\rho}_g \sqrt{c_s^2 + (|u - v|^2)} \right).
\]  

For the sake of consistency gas and dust velocities \( \bar{u}, \bar{w} \) are also approximated by their mean values \( (u), (w) \) instead of calculating them with the aid of (11)–(12). However, we note that the mean values that are naturally achieved during the simulation, do not diverge from the equilibrium solution by more than 10 per cent.

To determine linear growth rates of the excited instability modes, we calculate Fourier transforms of density and velocity distributions of the dust component. We analyse subsequently time variation of amplitudes of individual modes. Intermediate results are shown in Fig. 7.

We identify time \( T_e \) at which the dominating modes emerge from fluctuations and start their linear growth phase. Similarly, we identify the end of the linear growth phase \( T_s \) when modes have grown by few orders of magnitude, and start to saturate their growth. We fit exponential function

\[
f(t) = A \exp(-st)
\]  

to the measured density amplitudes in the period \( \Delta T = T_e - T_s \).

That procedure allows us to determine the growth rate \( s(k, \kappa) \) of each individual unstable mode, during the linear phase of the instability growth in the numerical experiments. We identify the most rapidly growing modes by selecting those with the greatest amplitude at fixed point at the reference time \( T_{ref} \) before the saturation. We compare the growth rates \( s(k, \kappa) \) obtained for each mode with

5.4 Comparison to the results of linear stability analysis

In order to analyse the growth of streaming instability, we extract small square patches from various locations across the computational domain. The patches have sizes of 0.15\(^2\) \( \text{au} \); thus are small enough so that mean values of physical quantities do not vary significantly across them. In the case of 3D run the patches are chosen on the \( r-z \) plane at \( \psi = \psi_{max}/2 \).

We note that in the place of the fixed parameters describing the unperturbed equilibrium in (14)–(15), we use patch-averaged values of corresponding dependent variables in equations (6)–(9). We calculate the spatially averaged densities of gas and dust component \( \bar{\rho}_g = (\rho_g), \bar{\rho}_d = (\rho_d) \) and their mutual ratio \( \bar{\epsilon} = (\rho_g/\rho_d) \). Mean angular
the solution \( s_0(k_x, k_z) \) resulting from the linear analysis for the local mean flow parameters.

In Fig. 8, we show temporal evolution of dust density perturbation amplitudes for three dominating instability modes in three experiments BB3d, BB and BBh, together with lines fitted to the phase of linear growth and lines representing growth of the amplitudes predicted by the linear analysis of the streaming instability. In the mid-resolution run BB the growth rates are 10\%–30\% smaller than those predicted by linear analysis, what indicates that still higher resolution is required to resolve these modes. In the high-resolution run BBh the results of the numerical experiment tend to converge to the results of linear stability analysis. The final saturation amplitudes seem to be slightly smaller in high-resolution runs. One should note, however, that we have chosen the modes of the highest growth rate for each run, which are not the same modes in terms of wavenumbers.

To extend our analysis, we show in Fig. 9 a contour plot (following YG05 and YJ07) of the growth rate \( s(k_x, k_z) \), resulting from solutions of the dispersion relation (19). The shape of isocontours indicates that the most unstable modes of the streaming instability, predicted by linear analysis, form a ridge extending towards large values of \( k_z \) and \( k_x \). The plots are constructed for the same set of the physical parameters and the mean state parameters derived for the domain patches at \( T = T_{\text{ref}} \). We then place \( (k_x, k_z) \) of nine dominating modes extracted at \( T = T_{\text{ref}} \) for simulations performed at different resolutions.

Our procedure allows us to confirm that wavenumbers of the modes, emerging in presented numerical models, align with the ridge of linear solutions at the growth rate map. It is apparent that dominating modes are limited by available numerical resolution. At lower numerical resolutions (run BB3d), the dominant modes locate below the contour labelled as ‘\(-1.000\)’. At the mid-resolution of run BB, some of the modes locate above the contour ‘\(-1.000\)’, and at the high resolution most of the modes locate above this contour. Similar tendency can be observed for the pair of runs AB and ABh. We anticipate that the reason for the absence of the very short wavenumber modes is the numerical diffusivity of the presently used Relaxing TVD algorithm. Our estimations show that our code requires at least 32 computational cells per wavelength of the unstable mode, to accurately represent the linear growth rate, which is much more than is required by higher order schemes Youdin & Johansen (2007); Balsara et al. (2009).

We varied the resolution of our simulations to check how our numerical scheme affects the obtained solutions. As the streaming instability in general ‘prefers’ shorter wavelengths for the optimal growth, increasing resolution always leads to more, smaller overdensities emerging during the course of evolution (see Fig. 10). However, we note that our results follow the fastest growing, linear modes (see Fig. 9) and that most phenomena described in previous sections, i.e. cavitation (see Fig. 3) or sudden growth dumping of the tightly coupled boulders in the gas-dominated regime (see Fig. 4) are independent of the size of the smallest computational cell.

6 DISCUSSION AND CONCLUSIONS

We performed a set of 2D simulations and 3D simulation of two mutually interacting fluids, i.e. gas and dust in protoplanetary disc, while neglecting vertical component of gravity. For the spectrum of grain sizes ranging from 10–50 cm we observe rapid growth of unstable modes in dust component, which are consistent with modes obtained for streaming instability within the framework of local linear analysis.

Our model enhances previous work of other authors by taking into account the full dynamics of protoplanetary disc, e.g. radial migration, that lead to significant variation in physical quantities, such as gas pressure gradient, and were previously treated as constant. Moreover, instead of using a general dimensionless stopping time parameter to couple both fluid components, we implement specific drag law (equation 5) which is best suited to the physical properties of simulated fluids.

In the present quasi-global approach, we relieved the restriction of the fixed radial pressure gradient, adopted for numerical modelling of the streaming instability within the shearing box approximation. It remains unknown how much this term affects the non-linear evolution of the system, since the additional term acts as an infinite energy reservoir driving the difference in azimuthal velocities of both fluids. In the global approach, even around fixed orbit, the parameter \( \eta \), measuring the radial pressure gradient, is function of time. While locality of shearing box fully justifies using a constant dimensionless stopping time to calculate mutual linear drag force, that approximation is no longer valid in the global disc.

The wave numbers of the fastest growing modes, extracted from numerical models, and their positions at the map of growth rate \( s \) versus \( k_x \) and \( k_z \) coincide with the predictions of linear analysis of streaming instability. We reproduce reasonably well the linear growth rates, getting rapid convergence near \(~32\) cells per wavelength. It is apparent, however, that due to diffusive nature of the Relaxing TVD scheme, PIERNIK needs high resolution to capture linear growth phase accurately.

The instability saturates when dust overdensities synchronize their velocity with the gas component. We must note that despite the steady outward migration of the gas, dust blobs never really gain positive radial velocity and never cease to migrate inwards (this was also noted by JY07). Though in certain cases, their radial velocity is significantly slower than what is expected from the standard estimation of the migration rate. The lack of additional force that would keep the blobs bounded also leads to high dispersion rate of the overdensities.
In our simulations, we have neglected the role of stratification of the disc, self-gravity of both fluids and also Magnetorotational Instability (MRI)-induced turbulence. As each and every of the aforementioned physical processes plays significant and integral role in the protoplanetary disc evolution, we plan to gradually expand our setup increasing its complexity in order to create the more complete global model.

We have found that non-linear evolution of the streaming instability forms conditions for the formation of gravitationally bound dust blobs in the examined 3D semi-global discs configuration. This indicates, in accordance with the predictions by Johansen et al. (2007), the possibility for planetesimals formation in accreting circumstellar discs, and therefore we plan to incorporate self-gravity in our future work.

We note that some of our runs may be underresolved, especially the gas-dominated setups. However the non-linear outcomes of all setups converge to common solutions. Moreover, our initial simulation of full 3D setup, which took 0.36 MCPUh (Million Computer Processing Unit hours), shows that we are able to perform global simulation of streaming instability in a convincing manner. However, due to the severe Courant–Friedrichs–Lewy (CFL) constraint on timestep imposed by the high azimuthal velocity in 3D simulations, we consider implementing FARGO-like algorithm Masset (2000) for our future work. This is an important step towards complete model of planetesimals formation due to combined action of various fluid instabilities postulated by Johansen et al. (2007).

**NULLIUS IN VERBA**

We would like to stress that everything that is required to reproduce results presented in this paper is publicly available: the code (http://piernik.astri.umk.pl), initial conditions (part of PIERNIK’s example problem set – streaming global), analysis and visualization routines (aforementioned github repository and YT package). Runs in the basic resolution took approximately 400 CPUh and can be conveniently reproduced on small clusters or even modern workstations. KK is willing to assist inquisitive readers with performing said simulations along with providing access to data obtained in much more computationally demanding runs.

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2 http://yt-project.org/
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