Distillation of Multipartite Entanglement by Complementary Stabilizer Measurements

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We propose a scheme of multipartite entanglement distillation driven by a complementary pair of stabilizer measurements, to distill directly a wider range of states beyond the stabilizer code states (such as the Greenberger-Horne-Zeilinger states). We make our idea explicit by constructing a recurrence protocol for the 3-qubit W state \( |001 \rangle + |010 \rangle + |100 \rangle \). Noisy W states resulting from typical decoherence can be directly purified in a few steps, if their initial fidelity is larger than a threshold. For general input mixed states, we observe distillations to hierarchical fixed points, i.e., not only to the W state but also to the 2-qubit Bell pair, depending on their initial entanglement.

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Introduction.– We recognize, through a decade of research, that entanglement is indispensable to execute quantum information processing (QIP), such as quantum computation and multi-party quantum communication. A persistent challenge is to maintain multipartite entanglement against decoherence. In this Letter, we enlarge the present applicability of a key technique, entanglement distillation, to genuine multipartite entangled states called the W states.

The W state, \( \frac{1}{\sqrt{3}} (|001 \rangle + |010 \rangle + |100 \rangle) \), i.e., the equal superposition of “single-excitation” basis vectors in \( n \) qubits is a tolerant resource against decoherence and loss of qubits. It is quite robust, because it can be compared to a symmetric web consisting of only pairwise entanglement. This state is also a Dicke state of the total spin operators \( J_z \) with eigenvalues \( \frac{1}{2} (2 + 1) \) and \( \frac{1}{2} - 1 \) respectively, due to the permutation symmetry. Thus the W state is often available more easily than the Greenberger-Horne-Zeilinger (GHZ) state. The 3-qubit W state has been created in optical systems and ion traps, and can be prepared according to several proposals in coupled quantum dots, critical spin chains, etc. Furthermore, the W state can be said to be essentially different from most multipartite entangled states known in the applications to QIP, in the following sense.

Basic software techniques to circumvent decoherence have been proposed, and implemented experimentally for prototypes. For example, entanglement distillation (or, purification) is a tool to extract high-fidelity entangled states from a larger ensemble of noisy ones. Quantum error correction codes are a way to protect entanglement from small numbers of errors. The latter can be formulated in terms of the stabilizer, i.e., as simultaneous eigenspaces of commuting “multilocal” Pauli operators. Note that, if all stabilized eigenspaces are 1-dimensional state vectors, they are often called stabilizer states or graph states (up to local unitaries). The Calderbank-Shor-Steane (CSS) code is defined by the stabilizer group which consists of only two kinds of generators: multilocal bit-flip \( X \) operators and phase-flip \( Z \) operators. In fact, beyond the “bipartite” distillation protocols for the Bell pairs, direct distillation of multipartite entanglement is so far possible just for the CSS stabilizer (or, two-colorable graph) states by the protocol in Refs. \cite{14, 15}, which extended earlier results for GHZ states \cite{16, 17}. Since the W state is not a stabilizer state, there has been no protocol to distill it directly.

Main idea.– We propose an entanglement distillation protocol that extracts directly a multipartite non-stabilizer (non-graph) state, specifically the 3-qubit W state. Our idea is to apply local measurements of the stabilizer (whose nonlocal counterpart acting at different parts stabilizes the target state), assuming that the target state belongs to a basis of equivalent entangled states. Note that such a basis, similar to the Bell basis, exists for a wider range of multipartite states than stabilizer states. We need \( n \) copies of the input state for the \( n \)-qubit case, to apply stabilizer measurements locally. In this manner, we can improve the fidelity, and attain the target state as a fixed point of the protocol. Note that if the target state is not the stabilizer state, local depolarization or twirling (over the single copy) which keeps the target state invariant seems impossible in general. Thus, we do not make the mixed states diagonal, i.e., a classical mixture of the basis states. It implies we cannot reduce the task to a “classical problem” that consists in extracting entropy from the binary strings of the stabilizer eigenvalues, as is possible by bilateral CNOT operations in all the known protocols. Nevertheless, by virtue of complementary stabilizer measurements which exchange the amplified components, our protocol works without local depolarization. The feature is favorable in efficiency, and analogous to the Oxford protocol \cite{13}.

Direct distillation of multipartite entanglement has several potential merits. In the case of CSS states such as the GHZ states, multipartite distillation was shown to be more efficient than the bipartite strategy which consists...
of distillation of Bell pairs and their connection \[14\,16\]. Under imperfect operations, the achievable fidelity can be higher \[14\,18\]. Also, the threshold for distillability may be tighter than that by indirect methods.

**W basis and its stabilizer group.** To make our idea explicit, we construct a recurrence protocol for the 3-qubit W state \(|W^{(000)}\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |010\rangle + |100\rangle)\)\(^{ABC}\), distributed over Alice, Bob, and Carol. We denote the Pauli matrices, operating on the \(j\)-th qubit at the party \(l\), as \(X^l_j, Y^l_j, Z^l_j\) along with the identity \(\mathbb{I}^l_j\). To distinguish the non-local tensor structure of the multiple Hilbert spaces controlled by different parties and the local tensor structure at a single party, we use the superscripts \(l\) as the non-local indices, and the subscripts \(j\) as the local ones. We define the Hadamard operation by \(H = \frac{1}{\sqrt{2}}(X + Z)\), the 2-qubit swap operation by \(\text{swap} : |kk\rangle \rightarrow |k'k\rangle (k, k' = 0, 1)\) in the computational basis, and a 3-qubit unitary operation \(V\), which leaves \(|000\rangle\) and \(|111\rangle\) unchanged, but exchanges the others in such a way that \(|001\rangle \leftrightarrow |110\rangle, |010\rangle \leftrightarrow |101\rangle, \text{and } |100\rangle \leftrightarrow |011\rangle\).

Let us introduce a complete orthonormal basis, called the W basis here, where each basis state \(|W^{k_1k_2k_3}\rangle\) \((k_1, k_2, k_3 = 0, 1)\) has entanglement equivalent to the W state \(|W^{(000)}\rangle\). This is because basis states transform into each other by the local unitary operations in Table I. The W basis can be obtained from the computational basis acted on by the 3-qubit unitary operation \(U^{W_{\text{basis}}} = \frac{1}{\sqrt{3}}(Z^A X^B Z^C + Z^A X^B \mathbb{I}^C + X^A \mathbb{I}^B Z^C)\), i.e., \(|W^{k_1k_2k_3}\rangle = U^{W_{\text{basis}}} |k_1k_2k_3\rangle\). It is convenient to identify the stabilizer for the W basis. To satisfy \(K_j |W^{k_1k_2k_3}\rangle = (-1)^{k_j} |W^{k_1k_2k_3}\rangle\), three generators \(K_j\) are determined as

\[
\begin{align*}
K_1^{(ABC)} &= \frac{1}{\sqrt{3}} (Z^A X^B Z^C + 2Y^A Z^B Y^C + Z^A \mathbb{I}^B \mathbb{I}^C), \\
K_2^{(ABC)} &= \frac{1}{\sqrt{3}} (2Z^A X^B Z^C + 2Y^A Z^B Y^C + \mathbb{I}^A \mathbb{I}^B Z^C), \\
K_3^{(ABC)} &= \frac{1}{\sqrt{3}} (2Z^A X^B Z^C + 2Z^A Y^B Y^C + \mathbb{I}^A \mathbb{I}^B Z^C).
\end{align*}
\]

We emphasize, by the superscript \((ABC)\), that the stabilizers are not local. Note that later we measure *locally* the stabilizers, which will be denoted, e.g., for Alice, as \(K_1^{(A)} = \frac{1}{\sqrt{3}} (2X_1^A X_2^A Z_3^A + 2Y_1^A Z_2^A Y_3^A + Z_1^A \mathbb{I}_2^A \mathbb{I}_3^A)\), and all 3 qubits specified by the subscripts belong to Alice. The stabilizer group consists of eight commuting elements, \(\{\mathbb{I}, K_1, K_2, K_3, K_1K_2, K_1K_3, K_2K_3, K_1K_2K_3\}\), where

\[
\begin{align*}
K_1K_2 &= \frac{1}{2} (2\mathbb{I}X_2X_3 + 2Y_1Y_2Z_3 - Z_1Z_2Z_3), \\
K_1K_3 &= \frac{1}{2} (2X_1X_2X_3 + 2\mathbb{I}Y_2Y_3 - Z_1Z_2Z_3), \\
K_2K_3 &= \frac{1}{2} (2X_1X_2X_3 + 2Y_1Y_2Y_3 - \mathbb{I}Z_1Z_2Z_3), \\
K_1K_2K_3 &= -Z_1Z_2Z_3.
\end{align*}
\]

**W state distillation protocol.** Our protocol consists of two subprotocols: \(\mathcal{P}\) and its dual \(\mathcal{D}\). In both, three input copies are mapped into one output copy to define a simple recurrence. Generally, any mapping to a smaller subsystem can be considered. We assume, without loss of generality, that the \(|W^{(000)}\rangle\) component of the input mixed states \(\rho\) is the largest among the diagonal elements in the W basis (otherwise we can relabel the computational basis by a local unitary operation in Table I). We define the fidelity \(F\) of \(\rho\) by \(F = |W^{(000)}\rangle \langle W^{(000)}| \rho |W^{(000)}\rangle \langle W^{(000)}|\).

**Protocol \(\mathcal{P}\):** 1) Every party \((l = A, B, C)\) applies the local measurement of two stabilizers \(K_1^{(l)}\)\(^{ABC}\) and \(K_2^{(l)}\)\(^{ABC}\) over the input state \(\gamma = (\rho_{m_0}^{ABC})\) of three copies, and obtains the 2-bit outcomes \(m^{(l)} = [m_1^{(l)}, m_2^{(l)}]\). In- 

FORMING their outcomes by two-way classical communication, parties select coincident outcomes \(m(A) = m(B) = m(C) = [0, 1]\) \((\equiv 1)\), \([1, 0]\) \((\equiv 2)\), or \([1, 1]\) \((\equiv 3)\). Otherwise they discard three copies. 3) For the coincident outcomes, each party transforms *locally* her/his state into a 1-qubit subsystem by the following “majority rule”. If \(m^{(l)} = 1\), \(P_1^{(l)} : |W_{001}\rangle |0\rangle |0\rangle \rightarrow |0\rangle, |W_{110}\rangle |1\rangle \rightarrow |1\rangle\); if \(m^{(l)} = 2\), \(P_2^{(l)} : |W_{010}\rangle |0\rangle \rightarrow |0\rangle, |W_{101}\rangle |1\rangle \rightarrow |1\rangle\); and if \(m^{(l)} = 3\), \(P_3^{(l)} : |W_{100}\rangle |0\rangle \rightarrow |0\rangle, |W_{011}\rangle |1\rangle \rightarrow |1\rangle\).

Mathematically, the stabilizer measurement \(M^{(l)(m)}\) of the party \(l\) is written by

\[
M^{(l)(m)} = \frac{1}{4} \left( \mathbb{I} + (-1)^{m_1^{(l)}} K_1^{(l)} \right) \left( \mathbb{I} + (-1)^{m_2^{(l)}} K_2^{(l)} \right),
\]

with the completeness condition \(\sum_{m^{(l)}} M^{(l)(m)+} M^{(l)(m)} = \mathbb{I}\).

Note that \(M^{(l)}\) acts on 3 qubits of the party \(l\), and it is a projector to the local W basis vectors, for example if \(m^{(l)} = 1\), \(M_1^{(l)} = |W_{001}\rangle \langle W_{001}| + |W_{110}\rangle \langle W_{110}|\). By the selection of desired coincident outcomes \(m, \mathcal{P}\) maps the input state \(\gamma = (\rho_{m_0}^{\mathcal{P}})\) to the one-copy state \(\rho'\) given by

\[
\rho' = \sum_{m = 1, 2, 3} PM^{(4)}_{m} PM^{(B)}_{m} PM^{(C)}_{\mathcal{P}} \gamma PM^{(4)}_{m} PM^{(B)}_{m} PM^{(C)}_{\mathcal{P}},
\]

with the success probability \(\text{tr}(\rho')\). We normalize the state as \(\rho_{\text{out}} = \rho' / \text{tr}(\rho')\) for the next recurrence step.

Before describing the whole protocol including \(\mathcal{D}\), we illustrate analytically how \(\mathcal{P}\) works. Suppose the perfect W state is distributed to three parties, but suffers typical decoherence as described by the local dephasing channel \(D(\rho) = \frac{1}{2}((1 + \mu)\rho + (1 - \mu)Z'\rho Z')\) with the same channel.

**TABLE I:** The 3-qubit W basis. Local unitary operations that map \(|W^{(000)}\rangle\) to \(|W^{k_1k_2k_3}\rangle\) are shown in the third column.
reliability $\mu \in [0, 1]$. Three parties initially share a noisy W state $\sigma(F) = D^A D^B D^C (|\psi_{W00}^0\rangle\langle\psi_{W00}^0|)$, which is not diagonal in the W basis, but is parametrized uniquely by $F = \frac{1}{2}(1 + 2\mu^2) \in [\frac{1}{3}, 1]$. A straightforward calculation shows that $\mathcal{P}$ maps three copies $\chi^{\otimes 3}$ to one copy $\sigma(F')$ with the higher fidelity $F'$ such that

$$F' = \frac{25}{93} F^3 + \frac{1}{9} F (1-F)^2 + \frac{13}{3} (1 - F^3)^3 F^3 / (1 - F^2)^2 + \frac{13}{16} (1 - F^3)^3 F^3.$$

Eq. (5) suggests a recurrence seen in the distillation curve of Fig. 1. Since $F$ lies in $[\frac{1}{3}, 1]$, we prove analytically that $F = 1$, corresponding to the W state, is the attractive fixed point and $F = \frac{1}{3}$ is the repulsive one. We find that any locally dephased W state except $F = \frac{1}{3}$, $\mathcal{P}$ restores it with a few steps. Indeed, this threshold coincides with a necessary condition for distillability by the partial transposition criterion $[19, 20]$. Since the mixed state $\bar{T}^l(\sigma(F))$, partially transposed for any party $l$ (i.e., bipartition), has a negative eigenvalue only for $F > \frac{1}{3}$, there is no chance to distill entanglement in $F = \frac{1}{3}$. In Fig. 1, the yield (i.e., the ratio of the number of surviving copies to that of used copies) of $\mathcal{P}$ after $F$ reaches at least 0.99 is also shown. The “stairs” of the yield come from the difference in the number of recurrence steps.

For more general noises, we need $\mathcal{P}$ which has the similar structure as $\mathcal{P}$ but employs complementary observables $\bar{K}_{ij} = \Lambda^l |K_{ij}|^l$ and $\bar{K}_{ij} \otimes |K_{ij}|^l$ in $\mathcal{P}$ and $\mathcal{P}$ are complementary (also called mutually unbiased $[21]$), i.e., $|\langle W_{ij}|K_{ij}\rangle^l| = 1 / \sqrt{3}$. Two measurement bases $|W_{ijk}\rangle$ in $\mathcal{P}$ and $|\bar{W}_{ijk}\rangle$ are complementary (also called mutually unbiased $[21]$), i.e., $|\langle W_{ijk}|\bar{W}_{ijk}\rangle| = 1 / \sqrt{3}$. $\mathcal{P}$ is a bistable system in the manner opponent to $\mathcal{P}$; $\mathcal{P}': |\bar{W}_{000}\rangle^l \mapsto |H_{111}\rangle^l \mapsto |H_{100}\rangle^l$. In brief, in $\mathcal{P}$, we replace all operators in Eqs. (5) and (6) by their “barred” dual operators. The complete distillation procedure for general mixed states consists of the sequential application of either $\mathcal{P}$ or $\mathcal{P}$, where, in every recurrence step, we select one of the subprotocols which gives the higher fidelity in the output. There seems to be no simple formula for the sequence, but we can determine the sequence if we know the initial input state $\rho$ before distillation, e.g., by state tomography. Also note that although this combination of $\mathcal{P}$ and $\mathcal{P}$ can reach numerically the region where $F > 0.999$, precisely speaking $F = 1$ is the fixed point of $\mathcal{P}$ but is not that of $\mathcal{P}$.

Hereafter, we show that, under the sequential application of $\mathcal{P}$ and $\mathcal{P}$, the W state can be distilled from arbitrary mixed states if, roughly speaking, $F$ is sufficiently large. First, consider another typical decoherence such as the local depolarizing channel (white noise) $\mathcal{E}(\rho) = \rho + \frac{1}{2\mu}(\rho + X^l\rho X^l + Y^l\rho Y^l + Z^l\rho Z^l)$, and the input state $\rho = \mathcal{E}(\rho) = \mathcal{E}(\rho) \mathcal{E}(\rho) |\psi_{W00}^0\rangle\langle\psi_{W00}^0|$ with $F = \frac{1}{3} (3 + \mu + 9\mu^2 + 11\mu^3) \in [\frac{1}{3}, 1]$. Although the locally depolarized W state does not remain in the same form under our protocol, we can still determine a threshold for distillability. As shown in Fig. 2, if initially $F \geq 0.48$, we distill the W state, and otherwise we have an undistillable mixed state $\chi = \frac{1}{2}(|\psi\rangle\langle\psi| + |\psi\rangle\langle\psi'|)$, where $|\psi\rangle = \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle - |111\rangle)$ and $|\psi'\rangle = \frac{1}{2}(-|000\rangle + |011\rangle + |101\rangle + |110\rangle)$, as another fixed point with $F = \frac{3}{8}$. This threshold is stricter than the necessary condition $F \geq 0.36$ by the partial transpose criterion $[19, 20]$. Note that the progress of the protocol is not described by a single parameter, and $F$ is not monotonic any more. A nonmonotonic behavior of $F$ was also seen in the bipartite distillation without depolarization $[13]$. However, as a long-term behavior, $F$ is increasing for the distillable cases and can be used for visualization of the progress.

Next, we consider randomly generated input mixed
states (under the Hilbert-Schmidt measure \cite{22}), and will observe numerically distillations not only to
the 3-qubit W state, but also to a 2-qubit Bell pair. This
is surprising, since it implies that we can distill a non-
stabilizer state and a stabilizer state by the same proto-
col. In Fig. 3 for 10,000 random mixed states with the
initial fidelity $F$ fixed close to 0.70 or 0.50, we display
the average fidelity and its standard deviation for each
set of samples reaching the same fixed point. When $F$ is
sufficiently large, such as $F \simeq 0.70$, the branch to the W
state is dominant. More than 99 percent of the states fol-
lowed toward each fixed point, for (in total) 10,000 randomly
generated initial mixed states with $F \in 0.70 \pm 0.01$ (left) or
0.50 $\pm$ 0.01 (right).

FIG. 3: The average fidelity and its standard deviation fol-
lowed toward each fixed point, for (in total) 10,000 randomly
generated initial mixed states with $F \in 0.70 \pm 0.01$ (left) or
0.50 $\pm$ 0.01 (right).

Conclusion.- Identifying a complementary (mutually
unbiased) pair of stabilizer measurements, which replaces
the conventional bilateral CNOT, as a key local opera-
tion for distillation, we have proposed a 3-qubit W
state distillation protocol. To our knowledge, it is the
first protocol to distill directly multipartite non-stabilizer
states. An extension to the $n$-qubit W state should be
straightforward, introducing the general W basis by
$U_{\text{Wbasis}} = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} Z_1^{l-1} \cdots Z_{l-1}^{l-1} X_1^{l-1} \cdots X_l^{l-1} \cdot 1^n$. Since our
protocol distills a non-stabilizer state and stabilizer states
on the same footing, our scheme may lead to a unified
construction of direct distillation protocols for multipar-

tite entanglement. It is still open whether a hashing pro-
tocol can be made for non-stabilizer states without lo-
cal depolarization which makes density matrices classical
mixtures of pure states. Finally, since quantum comput-
ers in which only stabilizer states are generated can be
efficiently simulated by classical computers \cite{11}, the
appearance of non-stabilizer states, such as the W state, is
necessary to exploit the power (universality) of quantum
computers. Thus, the technique to purify such states, be-
yond the “classical” parity check (exclusive or via CNOT)
for stabilizer states, might also give a new perspective on
fault-tolerant quantum computation (cf. Ref. \cite{22}).

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