Effects of Janus Oscillators in the Kuramoto Model with Positive and Negative Couplings

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We study the effects of Janus oscillators in a system of phase oscillators in which the coupling constants take both positive and negative values. Janus oscillators may also form a cluster when the other ones are ordered and we calculate numerically the traveling speed of three clusters emerging in the system and average separations between them as well as the order parameters for three groups of oscillators, as the coupling constants and the fractions of positive and Janus oscillators are varied. An expression explaining the dependence of the traveling speed on these parameters is obtained and observed to fit well the numerical data. With the help of this, we describe how Janus oscillators affect the traveling of the clusters in the system.

I. INTRODUCTION

A system of coupled phase oscillators is a typical model exhibiting collective synchronization behavior, which is revealed as a cluster formation on a phase circle. The model was first introduced by Winfree [1], and later refined by Kuramoto [2, 3]. After these works, have appeared many variations of the Kuramoto model [4]. One interesting extension of the model in view of cluster formation is incorporating both repulsive and attractive couplings [5, 6].

Each oscillator is identified either as a negative (repulsive) or as a positive (attractive) one, and there may form two clusters accordingly, which are separated in a phase circle. When this phase separation is less than $\pi$ radians, two clusters travel on the phase circle, maintaining the average separation angle. Recently, dynamics of the traveling state in this model as well as in a variant has been studied in terms of the phase speed of clusters [7, 8].

Systems suggested as a possible application of this model include the sociophysical models of opinion formation [9], for which the positive/negative coupling corresponds to conformists/contrarians [5]. In this context, it is thus of interest to incorporate an intermediate group in the model. One possibility of this is a group of constituents with two-faced character, which interacts the positive/negative oscillators with positive/negative couplings and hence the name of “Janus” oscillators [10]. In this work, we study numerically an oscillator model in which Janus oscillators are also present. It is observed that a cluster of Janus oscillators may form and travel, in addition to the two clusters of positive and negative oscillators. We calculate the traveling speed of the three clusters emerging in the system and average separations between them as well as the order parameters for three groups of oscillators, as the coupling constants and the composition of three types of oscillators are varied. An expression explaining the dependence of the traveling speed on these parameters is obtained and observed to fit well the numerical data. With the help of this, we describe how Janus oscillators affect the traveling speed of the clusters in the system.

This paper consists of four sections: In Section II, the oscillator model and its dynamics are described. Section III presents numerical results together with phenomenological interpretations of the effects of Janus oscillators. Finally, a brief summary is given in Section IV.

II. MODEL AND NUMERICAL CALCULATION

We consider a system of $N$ oscillators, the $i$th of which has intrinsic frequency $\omega_i$. The oscillator is described by its phase and is coupled globally to other oscillators. The dynamics of such a coupled oscillator system is governed by the set of equations of motion for the phase $\phi_i$ of the $i$th oscillator ($i = 1, ..., N$):

$$\dot{\phi}_i = \omega_i - \frac{1}{N} \sum_{j=1}^{N} K_i \sin(\phi_i - \phi_j),$$

(1)

where the intrinsic frequencies are assumed to be symmetrically distributed according to the Lorentzian distribution $g(\omega) = (\gamma/\pi)(\omega^2 + \gamma^2)^{-1}$. The other term on the right-hand side represents sinusoidal interactions with other oscillators, where the coupling constant $K_i$
takes a positive or negative value. A new feature in this work is that a fraction $q$ of oscillators are of Janus character, which means that the coupling constant for such a Janus oscillator takes a positive/negative value when it interacts with a positive/negative oscillator. Thus, for this oscillator, the coupling constant $K_j$ takes a value $K_{jp}/K_{jn}$ when it couples with a positive/negative oscillator. Further, we take $K_j = 0$ between Janus oscillators which means Janus oscillators do not interact with each another. Specifically, the coupling is taken from the distribution $\Gamma(K) = (1-q)p\delta(K-K_p) + (1-q)(1-p)\delta(K-K_n) + q\delta(K-K_j)$, where $K_{p/n}$ is the positive/negative coupling constant ($K_p > 0$ and $K_n < 0$) and $p$ is the fraction of normal oscillators having a positive coupling constant.

In order to measure the synchronization of the system, we introduce the complex order parameter

$$\Psi = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j} = \Delta e^{i\theta},$$

which characterizes the synchronization of the oscillators, with the magnitude $\Delta$ and the average phase $\theta$. The order parameter defined in Eq. (2) allows us to reduce Eq. (1) to a single decoupled equation:

$$\dot{\phi}_i = \omega_i - K_i \Delta \sin(\phi_i - \theta).$$

To investigate the behavior of the system governed by Eq. (3), we resort mainly to numerical methods. Using the second-order Runge-Kutta algorithm, we integrate Eq. (3) with the time step $\Delta t = 0.01$ for the system size $N = 2000$. Initially ($t=0$), $\phi_i$’s are randomly distributed between 0 and $2\pi$ for all $i$. We fix the positive coupling constant to be $K_p = 1$, and set the ratio $K_{jn}/K_{jp}$ equal to $K_n/K_p$ such that $K_{jn} = K_{jp}K_n$ throughout this work.

After the initial transient behavior, the system reaches stationarity, and we obtain the time series of the order parameter information on the time evolution of the phase distribution. In order to understand the clustering behavior, we define the order parameter of the positive/negative oscillators $\Psi_{p/n} = \Delta_{p/n} e^{i\theta_{p/n}}$ as well as that of the Janus oscillators $\Psi_j = \Delta_j e^{i\theta_j}$ similarly to Eq. (2). These parameters and the phase separations $\delta_{p/n}, \delta_{jp}$ and $\delta_{jn}$ between phases of the three kinds of oscillators are also calculated in each run. Namely, for example, $\delta_{jp} \equiv \theta_{jp} - \theta_{p}$ stands for the angular distance between the two phases $\theta_{jp}$ and $\theta_{p}$ for Janus and positive oscillators, respectively, and likewise $\delta_{pn}$ and $\delta_{jn}$.

The average values of these, as well as the traveling speed $w$, are obtained in the following way: At each time, we calculate these parameters and the average value of the phase velocities $w_i = \dot{\phi}_i$ over the oscillators. Then, we get the time average over $10^4$ time steps and take the absolute value to obtain the traveling speed. The parameters $\Delta$’s and $\theta$’s do not vary much in time, and we usually take the values after the time evolution. Finally, we take the averages over 30 initial configurations to obtain the (average) speed $w$. We also examine populations in given regions of the phase space. Specifically, we divide one cycle of the phase angle into 72 different ranges, the $k$th of which is defined by the phase interval $(k\pi/36 - \pi/72, k\pi/36 + \pi/72)$ (modulo $2\pi$), and we obtain the number $n_k$ of oscillators belonging to the $k$th range ($k = 0, ..., 71$).

The traveling speed $w$ obtained numerically can be explained essentially in the same way as in Refs. [7, 8]. Suppose that $Np(1-q)\Delta_p$ oscillators with the phase $\theta_p$ are in the positive cluster, $N(1-p)(1-q)\Delta_n$ oscillators with the phase $\theta_n$ are in the negative cluster and $Nq\Delta_j$ oscillators with the phase $\theta_j$ are in the third (Janus) cluster. Assuming that the remaining oscillators are desynchronized and have no net effects on $w$, one can derive straightforwardly, from Eq. (1),

$$w = \Delta_n \Delta_p (K_p - K_n) p(1-p)(1-q)^2 \sin \delta_{pn} + \Delta_j \Delta_p (K_{jp} - K_{jn}) p(1-q) \sin \delta_{jp} + \Delta_j \Delta_n (K_{jn} - K_n)(1-p)q(1-q) \sin \delta_{jn}. \quad (4)$$

Note that only the first term in Eq. (4) remains when $K_p = K_{jp} = K_{jn}$, the condition under which the data in last two figures of this work have been calculated (as well as for a system without Janus oscillators). This equation, when the width $\gamma_\omega$ of the frequency distribution is small, has been found to give a good description of the data for the traveling speed in this work. If $\gamma_\omega$ becomes too large, the system becomes disordered, and no clusters are formed.

III. RESULTS AND DISCUSSION

Before explaining the traveling of the three clusters, we describe how the traveling occurs in a system of pos-
itive and negative oscillators without Janus ones. When the positive coupling is sufficiently large and the positive cluster is formed, negative oscillators move away from the positive cluster. This may result in the formation of the negative cluster if it can get over the repulsive interactions between negative oscillators. If this repelling between two clusters is too strong, the separation angle becomes \( \pi \) radians and no traveling occurs, which is easily understood in view of Eq. (1) with \( q = 0 \) or Eq. (4). Thus we find that the clusters do travel only if the separation is less than \( \pi \).

When Janus oscillators are present together with positive and negative clusters, they move away from the negative oscillators farther than positive ones because a Janus oscillator does not attract negative ones like positive oscillator does. Consider a system in which the repulsive interaction as well as the Janus coupling is strong enough so that a Janus cluster is formed at the farthest angle \( \pi \) from the negative cluster. Then the positive cluster will be located at an angle between phases of the negative and the Janus ones; otherwise the system would become unstable. The distance between the Janus and the positive clusters \( \delta_{jp} \) depends on the strength of the overall repulsive interaction due to the negative cluster. Note that this overall interaction depends on the coupling constants and phase separations as well as the populations of the clusters. The separation \( \delta_{pn} \) is important in determining the traveling speed of this system. To see this, we may examine Eq. (4): Since \( \delta_{jn} \) is close to, only slightly less than \( \pi \), the third term contributing to \( w \) is nearly zero. Moreover, we have \( \sin \delta_{pn} \approx \sin \delta_{jp} \) because \( \delta_{jp} \) is approximately equal to \( \pi - \delta_{pn} \). Thus the traveling speed is roughly proportional to \( \sin \delta_{pn} \).

Of course, we should resort to the full form of Eq. (4) for the other systems in which \( \delta_{jn} \) is somewhat less than \( \pi \) radians and/or \( \delta_{jp} \) is larger than, say, one radian. Two cases are met: either the sum of three separation angles is 2\( \pi \) radians or \( \delta_{jn} = \delta_{pn} + \delta_{jp}(< \pi) \). We find that the former corresponds to a system in which the overall repulsive interaction due to negative oscillators is very weak and the population of Janus cluster is very small.

With these ideas in mind, we now present our numerical results. Figure 1, which has been obtained from a system having \( \delta_{jn} = \pi \), shows the number \( n_k \) of oscillators belonging to domain \( k \) in a system of \( N=2000 \) oscillators with \( q = 0.04 \), \( K_{jp} = 0.5 \), \( K_n = -0.5 \), \( p = 0.6 \) and \( \gamma_\omega = 0.01 \). Three clusters are clearly observed: negative, positive and Janus clusters from left. The value of the separation \( \delta_{jp} \) is about 0.9 in this figure. Here, one may note that clustering of mutually noninteracting members is less surprising than that of mutually repelling members.

We then show how the system behaves as the Janus coupling strength is varied. Figure 2 displays (a) the average traveling speed \( w \), (b) the phase separation \( \delta_{pn} \) between positive and negative clusters, (c) the phase separation \( \delta_{jp} \) between positive and Janus clusters, and (d) the order parameter \( \Delta_n \), plotted versus the Janus coupling constant \( K_{jp} \) in a system of oscillators with \( p = 0.6 \), \( K_n = -0.5 \), and \( \gamma_\omega = 0.01 \), for three values of the fraction \( q \) of Janus oscillators as shown in the legend. The lines in (a) are plots of Eq. (4) whereas the lines in (b) to (d) are merely guides for the eye. Error bars represent standard deviations.

**FIG. 2:** (color online) (a) Average traveling speed \( w \), phase separations (b) \( \delta_{pn} \) between positive and negative clusters and (c) \( \delta_{jp} \) between positive and Janus clusters, and (d) order parameter \( \Delta_n \), versus coupling constant \( K_{jp} \) in a system of \( N=2000 \) oscillators with \( p = 0.6 \), \( K_n = -0.5 \) and \( \gamma_\omega = 0.01 \), for three values of the fraction \( q \) of Janus oscillators as shown in the legend. The lines in (a) are plots of Eq. (4) whereas the lines in (b) to (d) are merely guides for the eye. Error bars represent standard deviations.
run away has increased noticeably. The results are the increased order parameter $\Delta_n = 1$ as is shown in (d) and the increased separation $\delta_{pn}$ or the $\pi$ state. Here, the presence itself of Janus oscillators causes this behavior, with the coupling between them being negligible.

As the Janus coupling constant $K_{jp}$ is increased, dramatic changes arise in the data for $q = 0.08$; the positive cluster moves away from the Janus cluster and the traveling speed increases as a result. This motion is not easy to explain thoroughly because it results from intriguing attractive and repulsive interactions among three clusters. Roughly speaking, the configuration such that the positive cluster is separated from the Janus one is usually more favorable in the presence of Janus coupling, except when the overall repulsion due to the negative cluster is very strong. Otherwise, due to the Janus coupling, this oscillator may move to both directions at the $\pi$ position if the negative oscillators are not almost completely ordered. Thus the positive oscillators move slightly even for the small value of $K_{jp}$. The slight increase of $w$ with $K_{jp}$ in the data for $q = 0.12$ can be explained similarly: As the coupling is increased, the separation $\delta_{pn}$ decreases, but not as much as the data for $q = 0.08$ because $\Delta_n \approx 1$ and the overall effect of repulsive interaction is so strong.

Now we move to an issue about the domain of the parameter $p$ in which traveling occurs. Figure 3 shows the average traveling speed $w$ versus the fraction $p$ in a system of $N=2000$ oscillators with $\gamma_w = 0.01$, $K_n = -0.7$, and $K_{jp} = 1$ for four values of $q$ as shown in the legend. It is found that the domain of the traveling state moves towards the origin of the $p$ axis and broadens as $q$ is increased, although the peak speed is decreased. Argument similar to that in the last paragraph can be put forward on how a $\pi$-state becomes a traveling state in the presence of Janus oscillators at smaller values of $p$ and vice versa at larger values of $p$.

In doing this, it is helpful to inspect Fig. 4 which presents (a) the average traveling speed $w$, (b) the phase separation $\delta_{pn}$ between positive and negative clusters, and (c) the order parameter $\Delta_n$, versus $q$ in a system of $N=2000$ oscillators with $K_n = -0.7$, $K_{jp} = 1$, and $\gamma_w = 0.01$, for three values of $p$ as shown in the legend. The lines in (a) plot Eq. (4) whereas the lines in (b) and (c) are merely guides for the eye. Error bars represent standard deviations.

![FIG. 3: (color online) Average traveling speed $w$ versus the fraction $p$ in a system of $N=2000$ oscillators with $\gamma_w = 0.01$, $K_n = -0.7$ and $K_{jp} = 1$ for four values of $q$ as shown in the legend. Lines are plots of Eq. (4) whereas symbols represent the data points obtained numerically.](image1)

![FIG. 4: (color online) (a) Average traveling speed $w$, (b) phase separation $\delta_{pn}$ between positive and negative clusters, and (c) order parameter $\Delta_n$, versus $q$ in a system of $N=2000$ oscillators with $K_n = -0.7$, $K_{jp} = 1$, and $\gamma_w = 0.01$, for three values of $p$ as shown in the legend. The lines in (a) plot Eq. (4) whereas the lines in (b) and (c) are merely guides for the eye. Error bars represent standard deviations.](image2)
than 0.9, which in turn makes the separation angle $\delta_{pn}$ increase, resulting in a $\pi$-state.

IV. SUMMARY

We have considered a system of phase oscillators in which the coupling constants take both positive and negative values, and probed the effects of Janus oscillators. It has been found that Janus oscillators also form a cluster when positive and negative oscillators are ordered. The resulting three clusters are traveling when the phase separation between positive and negative clusters is less than $\pi$ radians. We have obtained an expression explaining the dependence of the traveling speed on these parameters and observed that it fit well the numerical data. With the help of this, we have described how Janus oscillators affect the traveling of the clusters in the system. Depending on the parameters of the system, the Janus oscillators may facilitate or hinder the traveling.

Acknowledgments

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