Dancing Synchronization in Coupled Spin-Torque Nano-Oscillators

H. T. Wu, Lei Wang 王蕾, Tai Min, X. R. Wang

1 Center for Spintronics and Quantum Systems, State Key Laboratory for Mechanical Behavior of Materials, Xi’an Jiaotong University, No.28 Xianning West Road, Xi’an, Shaanxi, 710049, China
2 Department of Physics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China
3 HKUST Shenzhen Research Institute, Shenzhen, 518057, China

We are reporting a new type of synchronization, termed dancing synchronization, between two spin-torque nano-oscillators (STNOs) coupled through spin waves. Different from the known synchronizations in which two STNOs are locked with various fixed relative phases, in this new synchronized state two STNOs have the same frequency, but their relative phase varies periodically within the common period, resulting in a dynamic waving pattern. The amplitude of the oscillating relative phase depends on the coupling strength of two STNOs, as well as the driven currents. The dancing synchronization turns out to be universal, and can exist in two nonlinear Van der Pol oscillators coupled both reactively and dissipatively. Our findings open doors for new functional STNO-based devices.

I. INTRODUCTION

Synchronization is the coordination of different parts of a system working in harmony, and is an ubiquitous phenomenon that has been observed in various branches of sciences ranging from physical systems to chemical and biological systems with gain and loss [1–5]. Together with other nonlinear effects and beyond, it increases complexity of nature and organizes things at higher levels [6]. Synchronization was first discovered by Christian Huygens in 1665 [7]. He found that two pendulum clocks hanged side by side would soon swing with the same frequency and 180° out of phase regardless their initial conditions as long as their intrinsic frequencies are not too different from each other and their coupling strengths are not too weak. This completely out of phase synchronized motion is very robust against the external disturbances. Since then, our understanding of synchronization has been greatly advanced.

Two coupled nonlinear oscillators in currently known synchronizations oscillate with the same frequency, but can have different constant relative phases [1, 7–9]. They are relative simple and can be characterized by the frequency and their relative phase. For more exotic synchronizations, one needs to couple many nonlinear oscillators as a cluster or a network [10] that are commonly described by the Kuramoto model [9]. As summarized by Matheny and co-workers [11], the simplest synchronizations of many oscillators are that all oscillators have the same phase, or a few fixed relative phases. The relative phases of synchronized oscillator network can even form a complicated static pattern. Sometimes, a network can fragment into several clusters, and motions of oscillators in each cluster are synchronized with their own static phase pattern. In a word, the patterns of phase difference among oscillators in known synchronizations are static and do not change with time no matter in coupled two oscillators or in an oscillator network.

Spin torque nano-oscillators (STNOs) are important nonlinear oscillators in magnetics. STNOs [12, 13] are self-sustained oscillations driven by current generated spin-transfer torque (STT) [14, 15]. Self-sustained oscillations are a well-known nonlinear phenomenon widely existing in systems with gain and loss [1, 16, 17]. STNO is an active research topic in academia and industry because of their exotic applications in nano-technology such as microwave generation at nano-meter-scale that is crucial for microwave-assisted recording [18, 19]. Output power is an important issue in STNOs [20] because microwave power from a single STNO is of order of pico-watts due to its tiny size [21]. One promising way of increasing the output microwave power is through in-phase synchronization of many STNOs [22, 23]. Several STNOs can be coupled by static magnetic interaction [24–26]. This coupling is effective only when two STNOs are separated within a few nanometers that limits possi-
ble number of STNOs in synchronization. Coupling between STNOs through spin waves is order of magnitudes larger than that by static magnetic interaction \([20, 27–32]\). Like other nonlinear systems, various aspects of coupled STNOs have been extensively studies, such as the intrinsic mutual phase-locking \([33–36]\), STNOs due to vortex state \([37]\) and the fractional synchronization \([38]\). The temperature \([39]\) and external field \([40]\) have been used to control the frequency, linewidth of STNOs, as well as synchronization.

In this study, we report a new type of synchronization of two STNOs coupled by spin waves. In the new synchronization the relative phase of two oscillators varies periodically with time, instead of being a constant. Such an exotic synchronization is termed \textit{dancing synchronization}. Let us use the motion of two coupled clocks, shown in Fig. 1, to explain the differences between conventional synchronizations and the dancing synchronizations. The red clock (the first row) is in synchronization with both the yellow clock (the second row) and the blue clock (the third row) with the same periods, say 12 hours. The first and second rows (red and yellow clocks) illustrate various moments of two clocks in a conventional synchronization in which two clocks are in phase (always pointing to the same direction at all times). The first and third rows (red and blue clocks) schematically illustrate relative phases of the two clocks in a dancing synchronization where, within one period, the blue clock rotates slower than the red clock in the first and the third phases of the period, but faster than the red clock in the second and the last phases of their period. The distinct difference of the dancing synchronization from the known ones is that the relative phase of the red and blue clocks varies periodically with the synchronized frequency.

The paper is organized as follows. Section II includes model description of two coupled STNOs, methodology, and the demonstration of the dancing synchronization. Section III shows that dancing synchronization is universal and exists in well-known complex amplitude nonlinear oscillators and the Van der Pol oscillators when there are both reactive and dissipative couplings. Then main results are summarised.

II. DANCING SYNCHRONIZATION IN COUPLED STNOS

A. Model and Methodology

Our model, as shown in Fig. 2, consists of two nanopillar STNOs coupled through spin waves in the magnetic insulating layer physically connected with STNOs. Each STNO is made from magnetic multilayer as shown in Fig. 2(a), which consists of a polarizer of a perpendicularly magnetized layer (e.g. Pt/(Co/Pt)) to generate spin polarized current; a free layer with in-plane magnetization on the top of the polarizer separated by either a nonmagnetic metal such as Cu or nonmagnetic insulator such as MgO. Under the STT due to the spin-polarized current from the polarizer, the spins in the free layer undergo a self-sustained precession. The self-sustained precession can be detected through tunnelling magnetoresistance \([41, 42]\) of the analyzer on the top of free layer separated by another nonmagnetic layer such as a thin Cu film. The analyzer is a thick ferromagnetic film whose magnetization is pinned by an anti-ferromagnetic layer (e.g. Ir-Mn) such that self-sustained magnetization precession of free layer can generate an oscillatory voltage between the top and bottom layer of the whole nanopillar shown in the figure. Two STNOs have a nominal size of 70 nm \(\times\) 60 nm, and free-layer thickness is 3 nm. The free layer is assumed to be made of Co with saturation magnetization of \(M_{s,Co} = 886\ \text{kA/m}\), magnetic anisotropy coefficient of \(K = 4453\ \text{J/m}^3\) (parallel to the line from the center of the left STNO to the center of the right STNO), exchange stiffness constant of \(A_{Co} = 25\ \text{pJ/m}\), Gilbert damping constant of \(\alpha = 0.02\ \text{[43]}\). Our two STNOs have a slightly different spin polarization \((P)\) of \(P_1 = 0.38\) for the left STNO and \(P_2 = 0.44\) for the right one. The intrinsic oscillation frequencies of the two isolated STNOs under current density of \(1.435 \times 10^7\ \text{A/cm}^2\) are 9.87 GHz and 10.20 GHz, respectively. A Yttrium iron garnet (YIG) film of thickness 3 nm connects two STNOs as shown in Fig. 2. The material parameters of YIG are \(A_{YIG} = 4.2\ \text{pJ/m}\) and \(K_{YIG} = 754\ \text{J/m}^3\ \text{[44]}\). The interface (between the YIG film and STNOs) exchange coupling is assumed to be \(A_{eff} = 2A_{Co}A_{YIG}/(A_{Co} + A_{YIG})\ \text{[45]}\). Thus, two STNOs couple through spin waves in the YIG film generated by the STNOs \([20, 27–30]\), as well as static magnetic inter-
action [24–26].

Spin precession in STNO-free-layers will generate and modify spin waves in the YIG film such that two STNOs can interact with each other through the exchange of spin waves. This spin wave mediated coupling is much stronger [29] than the direct magnetic-dipole interactions between two STNOs when they are close to each other. The STNO separation, material parameters, and the applied electrical current can be used to control the effective coupling of STNOs. We investigate the spin dynamics of the hybrid structure consisting of free layers of STNOs and the YIG film under the injection of spin polarized currents. The current density has a non-zero value only and the YIG film under the injection of spin polarized the hybrid structure consisting of free layers of STNOs and the STNO separation, material parameters, and the applied electrical current can be used to control the effective coupling of STNOs. We investigate the spin dynamics of the hybrid structure consisting of free layers of STNOs and the YIG film under the injection of spin polarized currents. The current density has a non-zero value only and the YIG film under the injection of spin polarized currents.

FIG. 3. A snapshot of spin distribution of system in synchronization. The arrows denote the direction of the in-plane component of magnetization and the color encodes the the information of $m_z$.

FIG. 4. Time evolution of phase differences with spin-wave coupling (a) and with only dipolar coupling (b) for various distances and a fixed charge current density of $1.435 \times 10^7$ $\text{A/cm}^2$.

We first study the coupling distance of the two STNOs through the spin waves in the YIG film. We use OOMMF to simulate two identical systems described above except that one of them does not have the YIG film such that two STNOs couple with each other by dipolar field. Thus, one can attribute difference of two system to the spin wave mediated coupling. To see different behaviour of the two systems, we collect time evolution data of the average magnetization $m_i(t)$ of two STNOs, where $i = 1, 2$ label the two STNOs. The angles in-plane component of $m_i(t)$ with the $x$-axis are denoted as $\phi_i(t)$. The time dependence of phase difference $\phi_2(t) - \phi_1(t)$ can tell synchronizations from non-synchronizations. $\phi_2(t) - \phi_1(t)$ varies over $2\pi$ range in a non-synchronized motion while it is a constant in a conventional synchronization. Our OOMMF simulation results are shown in Fig. 4(a) for system with YIG film, and in Fig. 4(b) for system without YIG film. Indeed, both non-synchronisations [for $d = 70$ nm in Fig. 4(a) and $d = 10, 22$ nm in Fig. 4(b)] and conventional synchronizations [for $d = 8, 10, 44$ nm in Fig. 4(a) and $d = 2, 8$ nm in Fig. 4(b)] can be clearly identified. Interestingly, a periodically oscillating $\phi_2(t) - \phi_1(t)$ with an amplitude of $60^\circ$ appears at $d = 22$ nm in the case that two STNOs are
FIG. 5. (a) Time evolutions of $m_{1z}(t)$ (the blue curve) and $m_{2z}(t)$ (the red curve) in a dancing synchronization: $m_{1z}(t)$ and $m_{2z}(t)$ show a fast and a slow motion (at nanoseconds). The relative phase of the red and the blue curves varies with a much longer common period. (b) Time evolution of phase difference in the dancing synchronization. The common long period is about 2 ns, much longer than a tenth nanosecond oscillation. (c)-(e) Phase trajectories $\phi_2(\phi_1)$ of two STNOs on $\phi_1\phi_2$-torus ($\phi_1$ for the large circle and $\phi_2$ for the smaller one), under a current density of $1.435 \times 10^7$ A/cm$^2$; (c) is for a conventional in phase synchronization when the distance is 10 nm; (d) is for the dancing synchronization in which $\phi_2(\phi_1)$ return to its starting point after 19 turns when the distance is 22 nm and (e) is for a non-synchronized state in which $\phi_2(\phi_1)$ never closes when the distance is 70 nm. (f)-(h) $\Phi_n = \phi_2(\phi_1 = \pi)$ is the value of $\phi_2$ in the Poincare maps. (f-h) are respectively for $\Phi_n$ vs. $\Phi_{n+1}$, $\Phi_n$ vs. $\Phi_{n+18}$, and $\Phi_n$ vs. $\Phi_{n+19}$.

coupled by both dipolar field as well as by the spin waves due to YIG film. This is exactly the dancing synchronization discussed early. Without the spin waves, such a synchronization was not observed [Fig. 4(b)]. Therefore, results in Fig. 4 demonstrate not only that coupling distance between two STNOs by spin waves becomes much longer (44 nm) than that (8 nm) by dipolar field, but also it can induce a new type of synchronization never observed before. Below, we will examine this new synchronization more closely.

**C. Dancing synchronization**

For the dancing synchronization at $d = 22$ nm and under current density of $1.435 \times 10^7$ A/cm$^2$, we plot the time evolutions of $m_{1z}(t)$ (the blue curve) and $m_{2z}(t)$ (the red curve), $x$-components of average magnetization of free layer in the left and the right STNOs, respectively, in Fig. 5(a). Two curves are periodic with the same period, but have different shapes, i.e. $m_{ax}(t) = m_{ax}(t + nT)$ ($\alpha = 1, 2$), where $T$ is the period and $n$ is an arbitrary integer. For example, within one common period, both $m_{1z}(t)$ and $m_{2z}(t)$ oscillate 19 times with different amplitudes before returning to their initial values. This phenomenon is different from the conventional in phase synchronization, where time evolutions of $m_{1z}(t)$ and $m_{2z}(t)$ either overlap completely with each other or differ by a fixed lag. Fig. 5(b) plots time evolution of the phase difference $\phi_2(t) - \phi_1(t)$ of the two STNOs. Clearly, $\phi_2(t) - \phi_1(t)$ oscillates periodically with an amplitude of about $\pi/3$ and a period of 2 ns. This is different from all known synchronizations where $\phi_2(t) - \phi_1(t)$ is a constant. Because of this periodical variation of relative phase of the two STNOs that is reminiscent of two partners dancing in rhymes with different arm movements, we term this observed new synchronization of *dancing synchronization*.

To further prove the dancing synchronization of Fig. 5(a), we plot trajectory $\phi_2(\phi_1)$ on the $\phi_1\phi_2$-torus as shown in Fig. 5(c-e). In a conventional synchronization where $\phi_2(t) - \phi_1(t) = \text{const.} \phi_2(t)$ and $\phi_1(t)$ change by $2\pi$ simultaneously so that $\phi_2(\phi_1)$ is a simple one-turn closed curve as shown in Fig. 5(c). This is the case when the distance between the two STNOs is 10 nm under a current density of $1.435 \times 10^7$ A/cm$^2$. The case of $d = 22$ nm at the same current density is fundamentally different as shown in Fig. 5(b), $\phi_2(t) - \phi_1(t)$ is not a constant, but varies periodically with a longer period. The trajectory is still a closed curve as shown in Fig. 5(d) that displays data of Fig. 5(a) as $\phi_2(\phi_1)$ on the $\phi_1\phi_2$-torus. $\phi_2(\phi_1)$ returns to its starting point after 19 turns. If $\phi_1(t)$ and $\phi_2(t)$ either are neither periodic nor have a common period, the trajectory will not be a closed curve and will fill up the $\phi_1\phi_2$-torus, as shown in Fig. 5(e) that is the motion of the two STNOs for $d = 70$ nm and under a current density of $1.435 \times 10^7$ A/cm$^2$.

One can further confirm the dancing synchronization of two STNOs in Fig. 5(a) via the Poincare maps. In the map, $\Phi_n$ is defined as angle $\phi_2$ modulo $2\pi$ when $\phi_1 = (2n - 1)\pi$, i.e. $\{\Phi_n = \phi_2(\phi_1 = (2n - 1)\pi) \mod 2\pi | n = 1, 2, \ldots\}$. $\Phi_n$ can be grouped into various sets such as $\{(\Phi_n, \Phi_{n+1}) | n = 1, 2, \ldots\}$, or $\{(\Phi_n, \Phi_{n+18}) | n = 1, 2, \ldots\}$, or $\{(\Phi_n, \Phi_{n+19}) | n = 1, 2, \ldots\}$. These three sets are plotted in Fig. 5(f-h) where the x-axis is for $\Phi_n$ and the y-axis for $\Phi_{n+N}$, $n = 1, 2, 3, \ldots \{\Phi_n, \Phi_{n+N}\} | n = 1, 2, \ldots\}$ fall...
FIG. 6. Time evolution of phase differences for various distance $d$ at a fixed charge current $J = 1.435 \times 10^7 \text{ A/cm}^2$; for various charge current $J$ at a fixed distance $d = 22 \text{ nm}$ and for different conduction of the magnetic anisotropy at fixed $d = 22 \text{ nm}$ and $J = 1.435 \times 10^7 \text{ A/cm}^2$. All other unmentioned parameters are the same as those used in Fig. 5.

onto the line of $\Phi_{n+N} = \Phi_n$ if $\phi_1$ and $\phi_2$ have the common period of $N$ turns. This is exactly the case here with $N = 19$ as shown in Fig. 5(h). As a comparison, sets with $N = 1$, and 18 are off the straight line as shown in Fig. 5(f) and Fig. 5(g).

D. Robustness of the dancing synchronization

The observed dancing synchronization is very robust, and can exist in a finite region in the parameter space. For example, Fig. 6(a1~a5) shows the time evolution of $\phi_2(t) - \phi_1(t)$ for various $d$ at a fixed current density of $J = 1.435 \times 10^7 \text{ A/cm}^2$ while all other parameters keep the same as those for Fig. 5. Clearly, the dancing synchronization, featured by the periodic variation of $\phi_2(t) - \phi_1(t)$, occurs in the window of $d = 18 \sim 23 \text{ nm}$. Similarly, we observe the dancing synchronization at fixed $d = 22 \text{ nm}$ in the current density window of $J = 1.41 \sim 1.48 \times 10^7 \text{ A/cm}^2$ while all other parameters keep the same as those for Fig. 5, as shown in Fig. 6(b1~b5) where $\phi_2(t) - \phi_1(t)$ in Fig. 6(b2-b4) vary periodically. Moreover, as shown in Fig. 6(c2) at a fixed $d = 22 \text{ nm}$, $J = 1.435 \times 10^7 \text{ A/cm}^2$, the dancing synchronization occurs when the magnetic anisotropy direction as well as its magnitude vary. Interestingly, the dancing synchronization exists even in the absence of the anisotropy as shown in Fig. 6(c1).

A natural question is whether the dancing synchronization can still survive when the so-called field-like torque is included in Eq. (1). The answer is yes as shown in Fig. 7(a) for $d = 22 \text{ nm}$ and under a current density of $1.45 \times 10^7 \text{ A/cm}^2$ with 45% field-like torque. The torque modifies slightly the details of the synchronization. The dancing synchronization is still observed even when an additional external magnetic field up to $0.3 \text{ mT}$ along the $z$-axis is applied, as shown in Fig. 7(b) for $0.1 \text{ mT}$. These results demonstrate the robustness of the dancing synchronization against parameters and different types of torques.

The observed dancing synchronization is not a transient process. This can be verified by a much longer micromagnetic simulation of $300 \text{ ns}$. In this simulation, we set $d = 22 \text{ nm}$ and $K_Y = 0$ in order to show that the dancing synchronization is robust against variation of spin waves that glue two STNOs together. The rest of model parameters are the same as those in Fig. 5. As shown in Fig. 8, there is no sign that the dancing synchronization changes to another type of motion. Evolution of phase difference between $t = 290 \text{ ns}$ and $t = 300 \text{ ns}$ is the same as that between $t = 30 \text{ ns}$ and $t = 40 \text{ ns}$, and is very similar to Fig. 5(b) with $K_Y \neq 0$. 
A. Dancing synchronization in coupled complex variable oscillators

We first search dancing synchronization in the complex variable oscillation model used by Matheny and co-workers [11] who reported various fragmentation synchronizations. The nonlinear dynamical equations for $n$-complex-variables $A_j(t)$ ($j = 1, \ldots, n$) read

$$
\dot{A}_j = \lambda A_j (1 - |A_j|) + i(\omega_j A_j + \alpha |A_j|^2 A_j) + i\beta \sum_{k \neq j} (A_k - A_j) + \gamma \sum_{k \neq j} A_k (1 - |A_j|),
$$  \hspace{1cm} (2)

where $\alpha$ is the nodal nonlinearity that couples frequency to amplitude, $\beta$ measures the strength of reactive coupling among a pair of oscillators, $\gamma$ is a non-linear coupling. Each complex variable $A_j(t)$ stands for an oscillator. The real part of $A_j(t)$ represents a real variable which can be observed in the oscillation. Equation (2) is often used to introduce the concept of synchronization [1]. For STNOs, $A_j(t)$ can be $\mathbf{m}_j$. The phase of each oscillator $\phi_j(t)$ is defined as the argument of $A_j$. For $\gamma = 0$, the model has been used to describe various nonlinear systems including NEMS [11]. This model is sometimes called “a universal model for self-sustained oscillations” in comparison to the Kuramoto model [9] widely used to describe “phase synchronization” of coupled oscillators or networks. In Kuramoto model, an oscillator is represented by only one real variable.

Various nonlinear phenomena such as self-sustained oscillation and fragmentation synchronizations have been obtained from Eq. (2) with $\gamma = 0$ [11], but not the dancing synchronization. We show now that the dancing synchronization of two complex-variable oscillators can exist for certain $\gamma \neq 0$. The numerical solutions of Eq. (2) from fourth-order Runge Kutta method are plotted in Fig. 9(a) and Fig. 9(b) for $\gamma = 0.01$, $\omega_1 = 0.5 \text{ Hz}$, $\omega_2 = 0.7 \text{ Hz}$, $\alpha = 0.59126$, $\beta = 0.056$, $\lambda = 0.01$ with the initial conditions $A_1(0) = 2.51 e^{0.16i}$ and $A_2(0) = 1.62 e^{0.79i}$. Similar to the STNOs system, the amplitudes $\text{Re}(A_1)$ and $\text{Re}(A_2)$, as shown in Fig. 9(a), oscillate with a long common period of 24.75s. The phase difference $\phi_2(t) - \phi_1(t)$ as shown in Fig. 9(b) is not a constant, but varies with the same synchronized period of 24.75s with an amplitude of 0.67. Due to the fact that all nonlinear dynamical systems with gain and loss, the properties of attractors do not depend on the initial states. The dancing synchronization is also checked using phase trajectory and the Poincare map $\{\Phi_n = \phi_2(\phi_1 = (2n - 1)\pi \text{ modulo } 2\pi | n = 1, 2, \ldots \}$, and phase trajectory $\phi_2(\phi_1)$ is closed after 4
FIG. 9. Dancing synchronizations in complex amplitude model and Van der Pol model. (a), (b) and (c) are time evolution of complex amplitudes, phase difference and the Poincare map in complex amplitude model, respectively. (d), (e) and (f) are real time trace of two oscillators’ amplitudes, the time evolution of phase difference , and the Poincare map in the VdP model, respectively.

turns on \( \phi_1\phi_2 \)-torus as demonstrated by the points of \( \{ (\Phi_n, \Phi_{n+4}) | n = 1, 2, \ldots \} \) on line \( y = x \) in Fig. 9(c).

B. Dancing synchronization in two coupled Van der Pol oscillators

We have also demonstrated existence of the dancing synchronization in coupled two Van der Pol (VdP) nonlinear oscillators. The VdP equation is not only a popular model for demonstrating the self-sustained oscillation in nonlinear systems [46, 47], but also realizable by RCL-circuits with a negative differential resistor. The standard VdP equation is,

\[
\dddot{x}_i + \mu (x_i^2 - A_i) \dot{x}_i + \omega_i^2 x_i = -f_{ij},
\]

where \( i, j = 1, 2 \) label two oscillators; \( \mu > 0 \) is a parameter measuring energy gain (\( x_i^2 < A_i \)) and energy loss (\( x_i^2 > A_i \)). \( A_i > 0 \) specifies the size of energy gain region and is roughly oscillation amplitude. \( \omega_i \) and \( f_{ij} \) describe respectively the oscillatory frequency and the coupling between oscillators \( i \) and \( j \). Coupled VdP oscillators have been intensively studied before with either reactive or dissipative coupling [48, 49]. Interestingly, only conventional synchronizations were reported in all earlier studies of coupled VdP oscillators. Here we show that the dancing synchronization can appear in coupled VdP oscillators with both reactive and dissipative couplings,

\[
f_{ij} = \alpha (x_j - x_i) + (j - i)\beta \sqrt{|x_i x_j + \dot{x}_i \dot{x}_j - 1|},
\]

where the first term is a reactive coupling and the second one is dissipative. Figure 9(d) are numerical solutions of Eq. (3) from fourth-order Runge Kuta method for \( \mu = 1 \), \( A_1 = A_2 = 0.5 \), \( \omega_1 = 1 \) Hz, \( \omega_2 = 0.98 \) Hz, \( \alpha = 0.12 \), \( \beta = 0.30 \). The final self-sustained oscillations shown in those figures do not depend on the initial conditions. Two oscillators have distinguished appearances, but share a common long period of 13.19 s. To see clearly that this is a dancing synchronization, we define

\[
\phi_j(t) = \int_0^t \frac{\dot{x}_j(\tau)\dot{x}_j(\tau) - \ddot{x}_j(\tau)^2}{\dot{x}_j(\tau)^2 - \ddot{x}_j(\tau)^2} \, d\tau,
\]

which is the total winding angle of \((x(t), \dot{x})\) in \( x\dot{x} \) phase-plane. \( \phi_2 - \phi_1 \) varies periodically with an amplitude of around \( 0.2\pi \) within the common long period of 13.19 s, as plotted in Fig. 9(e). Again, the dancing synchronization is checked using phase trajectory and the Poincare map \( \{ \Phi_n = \phi_2(\phi_1 = (2n - 1)\pi) \) modulo \( 2\pi | n = 1, 2, \ldots \} \), and phase trajectory \( \phi_2(\phi_1) \) is closed after 2 turns on \( \phi_1\phi_2 \)-torus as demonstrated by the points of \( \{ (\Phi_n, \Phi_{n+2}) | n = 1, 2, \ldots \} \) on line \( y = x \) in Fig. 9(f).
ear dynamical equation of the complex amplitude model

\[ \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t) + \mathbf{G}(\mathbf{x}, t) + \mathbf{H}(\mathbf{x}, t) \]

where \( \mathbf{F}(\mathbf{x}, t) \) is the nonlinear function, \( \mathbf{G}(\mathbf{x}, t) \) is the stochastic force to the original equations, e.g., the nonlinearity between two oscillators, we set \( \gamma = 0 \) for model (2) and \( \beta = 0 \) for model (3). In both cases, we were not able to find any trace of dancing synchronization within many trials. Figure 10 are what were typically observed with only conventional synchronizations where Fig. 10(a) is the result of model (2) with \( \gamma = 0 \) and Fig. 10(b) for model (3) with \( \beta = 0 \) while all other parameters are the same as those for Fig. 9 that show dancing synchronizations. Our studies show the importance of nonlinear couplings for the dancing synchronization.

The dancing synchronization in the toy models is also robust against certain degree variation of parameters. As an example in model (2) when the intrinsic frequency of the second oscillator and \( \alpha \) change from 0.7 to 0.8 and from 0.59126 to 0.55207, respectively, dancing synchronization appears also as shown in Fig. 11(a1~a3), in which the state returns to its starting point after moving around the origin of phase plane five turns Fig. 11(a3). Similarly, if we change the intrinsic frequency of second oscillator from 0.7 to 0.9, and \( \alpha \) from 0.59126 to 0.59445, dancing synchronization is still there as shown in Fig. 11(b1~b3), in which the state returns to its starting point after six turns Fig. 11(b3).

A true natural phenomenon should be tolerable to thermal noise. To demonstrate that our dancing synchronization is insensitive to the thermal noise, we add a stochastic force to the original equations, e.g., the nonlinear dynamical equation of the complex amplitude model becomes

\[
\dot{\mathbf{A}}_j = i \mathbf{H}_j (\mathbf{A}_j - |\mathbf{A}_j|^2 \mathbf{A}_j) + i \left( \omega_j \mathbf{A}_j + \alpha |\mathbf{A}_j|^2 \mathbf{A}_j \right) + i \beta \sum_{k \neq j} (\overline{\mathbf{A}}_k - \mathbf{A}_j) + \gamma \sum_{k \neq j} \overline{\mathbf{A}}_k (1 - |\mathbf{A}_j|),
\]

and the Van der Pol model becomes

\[
f_{ij} = \alpha (\overline{x}_j - x_i) + (j - i) \beta \sqrt{|x_i \overline{x}_j + \overline{x}_i x_j - 1|},
\]

where \( \overline{\mathbf{A}}_k = A_k + \alpha S(t) \), \( \overline{x}_j = x_j + a S(t) \) and \( S(t) \) is a standard Gaussian stochastic process. In simulations, an independent Gaussian-distributed random force of standard deviation \( \sigma = 1 \) is assigned in each step (\( \Delta t = 4.7 \times 10^{-3} \)). We solved equations numerically with \( a = 1 \times 10^{-7} \) and \( a = 5 \times 10^{-7} \). The results of the complex amplitude model and the Van der Pol model are displayed in Fig. 12. The Poincaré map (collecting data from 3000 periods) is slightly dispersed for both \( a \)'s. All return points fall around the line of \( \phi_{n+4} = \phi_n \), which sustained our statement on the robustness of the dancing synchronizations. The dancing synchronization of the Van de Pol model is much resilient than that of complex amplitude model, as shown in Fig. 12(d) and Fig. 12(d) with \( a = 1 \times 10^{-7} \) and \( a = 5 \times 10^{-7} \) respectively.

IV. CONCLUSION

In summary, a new type of synchronization, termed dancing synchronization, is observed in two STNOs coupled through spin waves and the static magnetic interaction. The two STNOs oscillate with the same period and their relative phase difference varies periodically with a common long period, different from all known synchronizations in which the relative phase of two nonlinear oscillators are fixed. We further demonstrated that the dancing synchronization is a general phenomenon that can also occur in the complex variable oscillation model used by Matheny and co-workers [11], and in two coupled Van der Pol oscillators, as long as they are coupled reactively and dissipatively. The dancing synchronization exists in narrow parameter region between nonsynchronization and in phase synchronization of two nonlinear oscillators.

ACKNOWLEDGMENTS

This work is supported by the National Key Research and Development Program of China (Grant No. 2018YFB0407600 and 2016YFA0300702), the National Natural Science Foundation of China (Grant No. 12074301, 11774296, 11804266 and 11974296), the Key Research and Development Program of Shannxi (Grant No. 2020SF-24).
**ω_2=0.8Hz, α=0.55207**

FIG. 11. Dancing synchronization of model (2) with ω_2 = 0.8 Hz, α = 0.55207 and ω_2 = 0.9 Hz, α = 0.59445, respectively. Panels from the left to the right are the time evolution of the two oscillators, the corresponding phase difference and the Poincaré map.

**ω_2=0.9Hz, α=0.59445**

FIG. 12. Dancing synchronization of the complex amplitude model [model (2) (a1-a3) for a = 1 × 10^{-7} and (c1-c3) for a = 5 × 10^{-7}] and the Van der Pol model [model (3) (b1-b3) for a = 1 × 10^{-7} and (d1-d3) for a = 5 × 10^{-7}] (a1, b1, c1, d1) are the time evolution of two oscillators. (a2, b2, c2, d2) are the time evolution of phase difference. (a3, b3, c3, d3) are the Poincaré map. The periodical oscillation of phase difference and the the Poincaré map demonstrate the dancing synchronization under the noise.
[1] S. H. Strogatz, *Nonlinear Dynamics and Chaos* (Westview Press, 1994).

[2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, 2001).

[3] I. Z. Kiss, Y. Zhai, and J. L. Hudson, *Science* **296**, 1676 (2002).

[4] G. Buzsáki and A. Draguhn, *Science* **304**, 1926 (2004).

[5] B. H. Repp and Y.-H. Su, *Psychonomic Bulletin & Review* **20**, 403 (2013).

[6] P. W. Anderson, *Science* **177**, 393 (1972).

[7] H. C., *Letters to Sluse*, no. 1333 of 24 February 1665, no. 1335 of 26 February 1665, no. 1345 of 6 March 1665.

[8] F. Varela, J.-P. Lachaux, E. Rodriguez, and J. Martinerie, *Nature Reviews Neuroscience* **2**, 229 (2001).

[9] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, and R. Spigler, *Rev. Mod. Phys.* **77**, 137 (2005).

[10] L. Huang, K. Park, Y.-C. Lai, L. Yang, and K. Yang, *Phys. Rev. Lett.* **97**, 164101 (2006).

[11] M. H. Matheny, J. Emenheiser, W. Fon, A. Chapman, A. Salova, M. Rohden, J. Li, M. Tudorba de Badyn, M. Pósfai, L. Duenas-Osorio, M. Mesbah, J. P. Crutchfield, M. C. Cross, R. M. D’Souza, and M. L. Roukes, *Science* **363**, eaav7932 (2019).

[12] M. Zahedinejad, H. Mazraati, H. Fulara, J. Yue, S. Jiang, A. A. Awad, and J. Åkerman, *Applied Physics Letters* **112**, 132404 (2018).

[13] A. A. Awad and P. e. a. Dürenfeld, *Nature Physics* **13** (2016).

[14] J. Slonczewski, *Journal of Magnetism and Magnetic Materials* **159**, L1 (1996).

[15] L. Berger, *Phys. Rev. B* **54**, 9353 (1996).

[16] X. R. Wang and Q. Niu, *Phys. Rev. B* **59**, R12755 (1999).

[17] X. R. Wang, J. N. Wang, B. Q. Sun, and D. S. Jiang, *Phys. Rev. B* **61**, 7261 (2000).

[18] Z. Z. Sun and X. R. Wang, *Phys. Rev. B* **74**, 132401 (2006).

[19] Z. Z. Sun and X. R. Wang, *Phys. Rev. B* **73**, 092416 (2006).

[20] A. Slavin, *Nature Nanotechnology* **4**, 479 (2009).

[21] S. I. Kiselev, J. C. Sankey, I. N. Krivorotov, N. C. Emley, R. J. Schoelkopf, R. A. Buhrman, and D. C. Ralph, *Nature* **425**, 380 (2003).

[22] S. Kata, M. R. Pufall, W. H. Rippard, T. J. Silva, S. E. Russek, and J. A. Katine, *Nature* **437**, 389 (2005).

[23] J. Grollier, V. Cros, and A. Fert, *Phys. Rev. B* **73**, 064409 (2006).

[24] H.-H. Chen, C.-M. Lee, Z. Zhang, Y. Liu, J.-C. Wu, L. Hong, and C.-R. Chang, *Phys. Rev. B* **93**, 224410 (2016).

[25] A. D. Belanovsky, N. Locatelli, P. N. Skirdkov, F. A. Araujo, J. Grollier, K. A. Zvezdin, V. Cros, and A. K. Zvezdin, *Phys. Rev. B* **85**, 100409 (2012).

[26] H. B. Huang, X. Q. Ma, Z. H. Liu, C. P. Zhao, and L. Q. Chen, *AIP Advances* **3**, 032132 (2013).

[27] R. K. Dumas and J. Åkerman, *Nature Nanotechnology* **9**, 503 (2014).

[28] S. Sani, J. Persson, S. M. Mohseni, Y. Pogoryelov, P. K. Muduli, A. Eklund, G. Malm, M. Käll, A. Dmitriev, and J. Åkerman, *Nature Communications* **4**, 2731 (2013).

[29] A. Ruotolo, V. Cros, B. Georges, A. Dussaux, J. Grollier, C. Deranlot, R. Guillemet, K. Bouzehouane, S. Fusil, and A. Fert, *Nature Nanotechnology* **4**, 528 (2009).

[30] A. N. Slavin and V. S. Tiberkevich, *PHYSICAL REVIEW B* **74** (2006), 10.1103/PhysRevB.74.104401.

[31] M. R. Pufall, W. H. Rippard, S. E. Russek, S. Kata, and J. A. Katine, *Phys. Rev. Lett.* **97**, 087206 (2006).

[32] V. Puliafito, G. Consolo, L. Lopez-Diaz, and B. Azzerboni, *Physica B: Condensed Matter* **435**, 44 (2014), 9th International Symposium on Hysteresis Modeling and Micromagnetics (HMM 2013).

[33] S. Kata, M. R. Pufall, W. H. Rippard, T. J. Silva, S. E. Russek, and J. A. Katine, *Nature (London)* **437**, 389 (2005).

[34] V. Tiberkevich, A. Slavin, E. Bankowski, and G. Gerhardt, *Applied Physics Letters* **95**, 262505 (2009).

[35] Safin, Ansar R., Udalov, Nicolay N., and Kapranov, Mikhail V., *Eur. Phys. J. Appl. Phys.* **67**, 20601 (2014).

[36] R. Lebrun, S. Tsunegi, F. Bortolotti, H. Kubota, A. S. Jenkins, M. Romera, K. Yaskuhiji, A. Fukushima, J. Grollier, S. Yuasa, and V. Cros, *Nature Communications* **8**, 15825 (2017).

[37] R. Lehnardt, D. E. Bürgler, S. Gliga, R. Hertel, P. Grünberg, C. M. Schneider, and Z. Celinski, *Phys. Rev. B* **80**, 054412 (2009).

[38] S. Urazhdin, P. Tabor, V. Tiberkevich, and A. Slavin, *Phys. Rev. Lett.* **105**, 104401 (2010).

[39] V. S. Tiberkevich, A. N. Slavin, and J.-V. Kim, *Phys. Rev. B* **78**, 092401 (2008).

[40] K. V. Thadani, G. Finocchio, Z.-P. Li, O. Ozatay, J. C. Sankey, I. N. Krivorotov, Y.-T. Cui, R. A. Buhrman, and D. C. Ralph, *Phys. Rev. B* **78**, 024409 (2008).

[41] T. Valet and A. Fert, *Phys. Rev. B* **48**, 7099 (1993).

[42] J. S. Moodera and G. Mathon, *Journal of Magnetism and Magnetic Materials* **200**, 248 (1999).

[43] D. Houssameddine, *Nature Materials* **6** (2007).

[44] Y. Sun, H. Chang, M. Kabatek, Y.-Y. Song, Z. Wang, M. Jantz, W. Schneider, M. Wu, E. Montoya, B. Kardasz, B. Heinrich, S. G. E. te Velthuis, H. Schultheiss, and A. Hoffmann, *Phys. Rev. Lett.* **111**, 106601 (2013).

[45] M. J. Donahue and D. G. Porter, *OOMMF User’s Guide, Version 1.0*, Report Interagency Report NISTIR 6376 (National Institute of Standards and Technology, Gaithersburg, MD, 1999).

[46] J. Guckenheimer, *IEEE Transactions on Circuits and Systems* **27**, 983 (1980).

[47] Z. Z. Sun, S. Yin, X. R. Wang, J. P. Cao, Y. P. Wang, and Y. Q. Wang, *Applied Physics Letters* **87**, 182110 (2005).

[48] R. Rand and P. Holmes, *International Journal of Non-Linear Mechanics* **15**, 387 (1980).

[49] S. Wirkus and R. Rand, *Nonlinear Dynamics* **30**, 205 (2002).