Dilaton-Maxwell gravity with matter near two dimensions

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Abstract

Unlike Einstein gravity, dilaton-Maxwell gravity with matter is renormalizable in \(2 + \epsilon\) dimensions and has a smooth \(\epsilon \to 0\) limit. By performing a renormalization-group study of this last theory we show that the gravitational coupling constant \(G\) has a non-trivial, ultraviolet stable fixed point (asymptotic freedom) and that the dilatonic coupling functions (including the dilatonic potential) exhibit also a real, non-trivial fixed point. At such point the theory represents a standard charged string-inspired model. Stability and gauge dependence of the fixed-point solution is discussed. It is shown that all these properties remain valid in a dilatonic-Yang-Mills theory with \(n\) scalars and \(m\) spinors, that has the UF stable fixed point \(G^* = 3\epsilon(48 + 12N - m - 2n)^{-1}\). In addition, it is seen that by increasing \(N\) (number of gauge fields) the matter central charge \(C = n + m/2\) \((0 < C < 24 + 6N)\) can be increased correspondingly (in pure dilatonic gravity \(0 < C < 24\)).
1 Introduction

The recent efforts that are being invested in the study of 2-dimensional quantum gravity— in particular dilatonic gravity, with or without matter [1]-[7]— have a variety of motivations: from the elementary fact that it is much easier to study quantum gravity (QG) in two rather than in four dimensions, to the much more fundamental reason, that such kind of theories appear naturally as string-inspired models. However, first indications that some of the hard problems of 4D QG—as those associated with the Hawking radiation and black-hole evaporation [1, 2] (see [8] for a review)— might be much easier to understand in frames of 2-dimensional dilatonic gravity, appear now not to be quite correct. The impression nowadays is that 2-dimensional dilatonic gravity does not look any more as a very ‘simple’ toy model. More effort should be invested in the study of dilatonic gravity in exactly (or near) two dimensions, having always in mind its subsequent generalization to higher dimensions.

As is well accepted, 2-dimensional Einstein gravity is no more of ultimate interest, since this action represents a topological invariant in two dimensions. However, it is quite an old idea [9] to try to study Einstein gravity in $2 + \epsilon$ dimensions, what might help to remedy this shortcoming. The gravitational coupling constant in such a theory—near two dimensions— shows an asymptotically free behavior [4], what can actually be very exciting in attempts to solve the problem of non-renormalizability of 4-dimensional Einstein gravity [10]. Unfortunately, it was shown that Einstein gravity in $2 + \epsilon$ dimensions is on its turn a non-renormalizable theory [11] (in other words, it has the oversubtraction problem [12]).

Furthermore, dynamical triangulations in more than two dimensions (see, for example, [13, 14]) have clearly shown the existence of a phase transition to a strong-coupling phase of similar nature as 2-dimensional quantum gravity [15]. Because of this, of considerable interest is still to try to construct a consistent theory of QG in $2 + \epsilon$ dimensions (having in mind a subsequent continuation of $\epsilon$ to $\epsilon = 1$ or $\epsilon = 2$).

It has been suggested recently [16], that a thing to do would be to study dilatonic gravity near two dimensions (similarly, it was the idea in ref. [17] to consider Einstein gravity with a conformal scalar field near two dimensions). Notice, however, that unlike Einstein gravity, dilatonic gravity possesses a smooth $\epsilon \rightarrow 0$ limit. This difference manifests itself in the fact that dilatonic gravity—which after a proper field definition can be presented as an Einsteinian theory with a scalar field—is not equivalent to Einstein gravity in exactly two dimensions. Moreover, dilatonic gravity is renormalizable and, hence, it does not suffer from the oversubtraction problem. Even more [16], the gravitational coupling constant in this theory also has an ultraviolet stable fixed point (asymptotic freedom) for $\epsilon > 0$ and $n < 24$ ($n$ is the number of scalars or matter central charge). Thus, the matter central charge in dilatonic gravity is bounded, as it also happens in Einstein’s theory.

In the present paper we consider dilatonic-Maxwell gravity with scalar matter in $2 + \epsilon$ dimensions. This theory, which includes a dilatonic potential and a Maxwell term with an arbitrary dilatonic-vector coupling function, may be considered as a toy model for unification of gravity with matter (scalar and vector fields). We will show that in the ultraviolet stable fixed point the gravitational coupling constant has the value $G^* = 3\epsilon/[2(30 - n)]$ and, hence, owing to the contribution of the vectors, the matter central charge of our universe in such a model can be naturally increased from $0 < n < 24$ (for pure dilatonic gravity) to $0 < n < 30$ (for dilatonic-Maxwell gravity). With this, we will show explicitly that it is possible to have an even wider window for the matter central charge, provided we consider non-abelian gauge
fields. We will also present an evaluation of the corresponding, generalized beta functions, and their fixed-point solutions will be found. At the fixed point the theory can be cast under the form of a standard, string-inspired model with a dilatonic potential of Liouville type.

The paper is organized as follows. In the next section we discuss the classical action of dilatonic-Maxwell gravity with matter and we derive the corresponding one-loop effective action. Section 3 is devoted to the study of the renormalization of the model and, in particular, to the derivation of the beta functions (to first order in the gravitational coupling constant $G$). The fixed-point solutions of the renormalization group (RG) equations together with a detailed analysis of their stability are presented in section 4. The form of the action at the fixed points is discussed. In section 5 we investigate the issue of gauge dependence of the position of the fixed point for the dilatonic coupling function. Finally, in section 6 we present the conclusions of our investigation and an outlook.

## 2 Dilatonic-Maxwell gravity with matter in $2 + \epsilon$ dimensions

In this section we are going to consider a theory of dilatonic gravity interacting with scalars and vectors via dilatonic couplings in $2 + \epsilon$ dimensions. This theory can be considered as a toy model for a theory of unified gravity with matter (since fermions can be included without any problem). The action is

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \left( Z(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{\mu^\epsilon}{16\pi G} R C(\phi) + V(\phi) - \frac{1}{2} f(\phi) g^{\mu\nu} \partial_\mu \chi_i \partial_\nu \chi_i + \frac{1}{4} f_1(\phi) F_{\mu\nu}^2 \right) \right],$$

where $g_{\mu\nu}$ is the $(2 + \epsilon)$-dimensional metric, $R$ the corresponding curvature, $\chi_i$ are scalars ($i = 1, 2, \ldots, n$), $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, with $A_\mu$ a vector, and where the smooth functions $Z(\phi), C(\phi), V(\phi), f(\phi)$ and $f_1(\phi)$ describe the dilatonic interactions. Notice that $V(\phi)$ is a dimensional function and it is therefore convenient to redefine $V \rightarrow m^2 V$, where $m$ is some parameter with dimensions of mass. Notice that in the absence of an electromagnetic sector the action (1) can be easily presented under the form of a non-linear sigma model [25]. Then the $\epsilon \to 0$ limit of the theory (1) can be discussed in a string effective action manner [26].

One can easily prove that the above theory (1) is renormalizable in a generalized sense. The one-loop counterterms corresponding to (1) in two dimensions have been calculated in refs. [18, 19]. Our purpose here will be to study in some detail the renormalization structure of (1) in $2 + \epsilon$ dimensions and in particular the corresponding renormalization group equations. The very remarkable point that gives sense to this study is the fact that the theory (1) (unlike Einstein gravity [1]) has a smooth limit for $\epsilon \to 0$. This property allows for the possibility to study the behavior of (1) in $2 + \epsilon$ dimensions by simply using the counterterms calculated already in 2 dimensions — in close analogy with quantum field theory in frames of the $\epsilon$-expansion technique (for a review see [20] and [21]).

Before we start working with action (1) we will perform some simplifications. First, motivated by the form of the string effective action, we choose the dilatonic couplings in (1) as the exponents of convenient dilatonic functions. Using the rigid rescaling of scalars and vectors:

$$\chi_i \rightarrow a_1 \chi_i, \quad A_\mu \rightarrow a_2 A_\mu,$$

(2)
with $a_1$ and $a_2$ constants, one can always normalize the dilatonic couplings in such a manner that

$$f(0) = 1, \quad f_1(0) = 1.$$  

Second, as has been done in refs. [6, 7], we can also use a local Weyl transformation of the metric of the form

$$g_{\mu\nu} \longrightarrow e^{-2\alpha(\phi)} g_{\mu\nu},$$

in order to simplify the dilatonic sector of (3). In particular, one can use (4) of a specific form such that $Z(\phi) = 0$. Finally, we can perform a transformation of the dilaton field in order to simplify the function $C(\phi)$, namely to reduce it to the form $C(\phi) = e^{-2\phi}$. With all this taken into account, the action (1) can be written as follows

$$S = \int d^4x \sqrt{-g} \left[ \frac{\mu^e}{16\pi G} Re^{-2\phi} - \frac{1}{2} g^{\mu\nu} \partial_\mu \chi_i \partial_\nu \chi^i e^{-2\Phi(\phi)} + \mu^e m^2 e^{-V(\phi)} + \frac{1}{4} e^{-f_2(\phi)} F^2_{\mu\nu} \right],$$

(5)

being $\mu$ a mass parameter and where we have chosen the dilatonic couplings so that $\Phi(0) = f_2(0) = V(0) = 0$. The first two terms in (5) correspond to the dilatonic gravity action in ref. [16], which was considered in $2 + \epsilon$ dimensions. With our choice of (5) the zero modes of the dilatonic couplings are fixed with the help of reference operators in the gravitational sector [16] and with the renormalization (by constants) of the scalar, vector, and mass $m^2$ in the matter sector.

We can now start the study of the divergences of the action (5). The study of the one-loop divergences of dilatonic gravity in the covariant formalism, initiated in ref. [4], has been continued in refs. [6, 7, 18, 19, 22, 23]. It is by now well under control, and we do not consider it necessary to repeat the details of such a calculation for the case of the model (5). Let us just recall that it is based on the t’Hooft-Veltman prescription [10]. The gauge-fixing conditions that are most convenient to use are the following. The covariant gauge-fixing action is

$$S_{gf} = -\frac{\mu^e}{32\pi G} \int d^4x \sqrt{-g} g^{\mu\nu} \chi^\mu \chi^\nu e^{-2\phi},$$

(6)

where

$$\chi^\mu = \nabla^\mu \bar{h}^{\mu\nu} + 2 \nabla^\mu \varphi$$

(7)

and $\bar{h}^{\mu\nu}$ is a traceless quantum gravitational field and $\varphi$ a quantum scalar field, in the background field method. In the electromagnetic sector the gauge-fixing action is choosen as

$$S_L = \int d^4x \sqrt{-g} (\nabla_\mu Q^\mu)^2 e^{-f_2(\phi)},$$

(8)

where $Q_\mu$ is a quantum vector field. Notice that for the background fields we use the same notations as for the classical fields in (5).

The calculation of the one-loop effective action corresponding to (5) in the gauges (6) and (8) can be performed in close analogy with the one in refs. [18, 19] (which was actually carried out for a more general theory in two dimensions). Owing to the smooth behavior of (3) for $\epsilon \to 0$, the divergences of (5) can be also calculated in exactly two dimensions, what will also provide the result for $2 + \epsilon$ dimensions. After some algebra, we get

$$\Gamma_{\text{div}} = \frac{1}{4\pi\epsilon} \int d^4x \sqrt{-g} \left\{ \frac{30 - n}{6} R + m^2 e^{2\phi - V(\phi)} 16\pi G [2 + V'(\phi)] - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi [8 - n\Phi'(\phi)^2] \right\} + 4\pi G \frac{\mu^e}{\epsilon} e^{2\phi - f_2(\phi)} F^2_{\mu\nu} [f'_2(\phi) - 2],$$

(9)
where $\epsilon = d - 2$. A few remarks about the comparison of (3) in some particular cases with results that have appeared in the literature are in order. First, in the absence of scalars and vectors (pure dilatonic gravity), we obtain

$$
\Gamma_{\text{div}} = \frac{1}{4\pi \epsilon} \int d^d x \sqrt{-g} \left\{ 4R + m^2 e^{2\phi - V(\phi)} 16\pi G [2 + V'(\phi)] - 8g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\}. \tag{10}
$$

This expression coincides with the results of refs. [4, 6, 7, 18, 19, 22, 23] in the same covariant gauge. Moreover, in ref. [7], $\Gamma_{\text{div}}$ eq. (10) has been calculated in a one-parameter dependent gauge, what gives an additional check of (10) at the value of the gauge parameter corresponding to the gauge choice (3). For dilatonic gravity with a Maxwell sector, (9) coincides with the results in [18, 19, 22]. Notice that the calculation of $\Gamma_{\text{div}}$ eq. (9) for the theory (5) without vectors and for $e^{-V(\phi)} \equiv 0$ has been done in ref. [16], in the $(2 + \epsilon)$-dimensional formalism but in a slightly different gauge. When comparing the results, there is here a difference with the coefficient of the $g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ term in (9). On one hand, we get a different contribution from the scalar dilaton coupling, what may be due to the slightly different gauges. On the other, instead of our coefficient 8 there appears in [16] a coefficient 4, what might be an indication of some mistake, or maybe due to the use of the inconvenient background field method —since these authors also employ a gauge of harmonic type and do not agree with the results of [18, 6, 7, 22] either. Later we will discuss the gauge dependence of $\Gamma_{\text{div}}$.

3 Beta functions and renormalization group analysis

We turn now to the study of the RG corresponding to our model in $2 + \epsilon$ dimensions. We will follow here the approach of ref. [16], which is actually based on earlier considerations in [3]. The main idea of the whole approach is to use $G$ as the coupling constant in perturbation theory and work at some fixed power of $G$. The counterterm which follows from (3) can be written as

$$
\Gamma_{\text{count}} = -\mu^\epsilon \int d^d x \sqrt{-g} \left[ RA_1(\phi) + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi A_2(\phi) + m^2 A_3(\phi) + \frac{\mu^{-\epsilon}}{4} F^2_{\mu\nu} A_4(\phi) \right], \tag{11}
$$

where

$$
A_1(\phi) = \frac{30 - n}{24\pi \epsilon} \equiv A_1,
$$

$$
A_2(\phi) = \frac{1}{4\pi \epsilon} [n \Phi'(\phi)^2 - 8],
$$

$$
A_3(\phi) = \frac{4G}{\epsilon} [2 + V'(\phi)] e^{2\phi - V(\phi)} \equiv e^{-V(\phi)} \tilde{A}_3(\phi),
$$

$$
A_4(\phi) = \frac{4G}{\epsilon} [f'_2(\phi) - 2] e^{2\phi - f_2(\phi)} \equiv e^{-f_2(\phi)} \tilde{A}_4(\phi). \tag{12}
$$

The structure of the counterterms shows that the gravitational coupling is a really nice choice for parameter of the perturbation theory.
By looking carefully at the relations (12) we observe that our theory is one-loop finite provided the following conditions are fulfilled:

\[ n = 30, \quad \Phi(\phi) = \sqrt{\frac{8}{n}}, \quad V(\phi) = -2\phi, \quad f_2(\phi) = 2\phi. \] (13)

(Notice, however, that perturbative finiteness is a more natural property in 2-dimensional supergravity [27].)

Renormalization is now performed in the standard way

\[ S = S_{cl} + S_{count} = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G_0} Re^{-2\phi - 2f(\phi) - \epsilon A(\phi)} \right. \]

\[ + \mu^2 e^{-V(\phi)} \left. + \frac{1}{4} e^{-f_2(\phi)} g_{\mu\nu} \epsilon^\alpha g_\beta A_{\mu\nu}\right]. \] (14)

The renormalization transformations can be defined as follows

\[ \phi_0 = \phi + f(\phi), \quad g_{0\mu\nu} = g_{\mu\nu} e^{-2\Lambda(\phi)}, \quad \Phi_0(\phi_0) = \Phi(\phi) + F(\phi), \quad V_0(\phi_0) = V(\phi) + F_\nu(\phi), \]

\[ f_{02}(\phi) = f_2(\phi) + F_f(\phi), \quad \chi_0 = Z^{1/2}_\chi, \quad A_{0\mu} = Z^{1/2}_A A_{\mu}, \quad m_0^2 = Z_m m^2. \] (15)

The functions \( \Lambda, f, F, ... \) are chosen so that \( \Lambda(0), f(0), F(0), ... \) can be set equal to zero.

Substituting the renormalization transformations (13) into the renormalized action (14) (in close analogy with ref. [16]), we obtain the renormalized parameters as

\[ S_0 = \int d^4x \sqrt{-g} \left\{ \frac{1}{16\pi G_0} Re^{-2\phi - 2f(\phi) - \epsilon A(\phi)} \right. \]

\[ + \epsilon + \frac{1}{16\pi G_0} \left[ 4\Lambda'(\phi) + \epsilon \Lambda'(\phi)^2 + 4\epsilon f'(\phi) \Lambda'(\phi) \right] e^{-2\phi - 2f(\phi) - \epsilon A(\phi)} g_{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi \]

\[ - \frac{1}{2} Z_\chi g_{\mu\nu} \partial_{\mu} \chi \partial_{\nu} \chi e^{-2\phi - 2F(\phi) - \epsilon \Lambda(\phi)} + \mu^2 Z_m e^{-V(\phi) - F_\nu(\phi) - (2+\epsilon) \Lambda(\phi)} \]

\[ + \frac{Z_A}{4} e^{-f_2(\phi) - F_f(\phi) + (2+\epsilon) \Lambda(\phi)} F^2_{\mu\nu} \right\}. \] (16)

From here, one can easily obtain

\[ \frac{1}{16\pi G_0} e^{-2\phi - 2f(\phi) - \epsilon A(\phi)} = \mu^2 \left( \frac{1}{16\pi G} e^{-2\phi - A(\phi)} \right), \]

\[ \frac{1}{16\pi G_0} \left[ 4\Lambda'(\phi) + \epsilon \Lambda'(\phi)^2 + 4\epsilon f'(\phi) \Lambda'(\phi) \right] e^{-2\phi - 2f(\phi) - \epsilon A(\phi)} = -\mu^2 A_2(\phi), \]

\[ Z_\chi e^{-2\phi - 2F(\phi) - \epsilon A(\phi)} = e^{-2\phi(\phi)}, \]

\[ Z_A e^{-f_2(\phi) - F_f(\phi) + (2+\epsilon) \Lambda(\phi)} = e^{-f_2(\phi)} [1 - \bar{A}_4(\phi)], \]

\[ Z_m e^{-V(\phi) - F_\nu(\phi) - (2+\epsilon) \Lambda(\phi)} = e^{-V(\phi)} [1 - \bar{A}_3(\phi)]. \] (17)

Working to leading order in \( G \) (notice that the functions \( f, \Lambda, F, ... \), \( F_f \) are of first order in \( G \)), we may follow ref. [16] and obtain the renormalized parameters as

\[ G_0^{-1} = \mu^2 (G^{-1} - 16\pi A_1), \quad Z_\chi = 1, \quad \Lambda(\phi) = \frac{4\pi G}{\epsilon + 1} \int_0^\phi d\phi' e^{2\phi'} A_2(\phi'), \quad \epsilon \Lambda(\phi) = -2F(\phi), \]

\[ f(\phi) = 8\pi A_1 G(2\phi - 1) + \frac{2\pi \epsilon G}{\epsilon + 1} \int_0^\phi d\phi' e^{2\phi'} A_2(\phi'), \quad Z_A = 1 - A_4(0), \quad Z_m = 1 - A_3(0), \]

\[ F_f(\phi) = (2 - \epsilon) \Lambda(\phi) + \bar{A}_4(\phi) - \bar{A}_4(0), \quad F_V(\phi) = -(2 + \epsilon) \Lambda(\phi) + \bar{A}_3(\phi) - \bar{A}_3(0). \] (18)
Hence, we see that the zero modes of the functions $f$, $f_2$ and $V$ are indeed controlled by a constant renormalization of the vector, the scalar and the mass.

We can now turn to the evaluation of the beta functions. We will list below only the ones corresponding to the dilatonic couplings and gravitational constant (for the purely dilatonic case they were given in ref. [16])

\[
\beta_G = \mu \frac{\partial G}{\partial \mu} = \epsilon G - 16\pi \epsilon A_1 G^2, \\
\beta_{\Phi}(\phi_0) = \mu \frac{\partial \Phi(\phi_0)}{\partial \mu} = 8\pi \epsilon A_1 G (e^{2\phi_0} - 1) \Phi'(\phi_0) \\
\beta_{f_2}(\phi_0) = \mu \frac{\partial f_2(\phi_0)}{\partial \mu} = 8\pi \epsilon A_1 G (e^{2\phi_0} - 1) f'_2(\phi_0) \\
\beta_V(\phi_0) = \mu \frac{\partial V(\phi_0)}{\partial \mu} = 8\pi \epsilon A_1 G (e^{2\phi_0} - 1) V'(\phi_0)
\]

Similarly, one can obtain the $\gamma$-functions for the fields and mass parameter, which are however of less importance to us, due to the fact that they correspond to non-essential couplings. Notice also that the only conformal mode of the gravitational field is renormalized in $2 + \epsilon$ dimensions, what is quite well known [9, 16] (for a study of conformal factor dynamics in four dimensions, see [28]).

### 4 Fixed points

The $\beta$-function for the gravitational constant has the form [16]

\[
\beta_G = \epsilon G - \frac{2(30 - n)}{3} G^2.
\]

Hence, for $n < 30$ we obtain the infrared stable fixed point $G = 0$. There is also an ultraviolet stable fixed point as in [9]

\[
G^* = \frac{3\epsilon}{2(30 - n)}.
\]

The theory is asymptotically free in the ultraviolet limit. The inclusion of vector fields has increased the matter central charge of our universe, when compared with the cases of pure dilatonic gravity [16], Einstein gravity [9], or Einstein gravity with a conformal scalar [17]. Our theory admits more matter than any of these previous models.
We start the search for fixed-point solutions corresponding to the dilatonic couplings. To this end we choose the following Ansatz:

\[ \Phi(\phi) = \lambda \phi, \quad f_2(\phi) = \lambda f \phi, \quad V(\phi) = \lambda V \phi. \] (22)

In that case the \( \beta \)-functions are (we study the functional dependence of \( \beta \)-functions of the type (22))

\[ \beta_\Phi = G (e^{2\phi} - 1) \left[ \frac{30 - n}{3} \lambda + \frac{\epsilon}{4(1 + \epsilon)} (\lambda - 1) (n \lambda^2 - 8) \right], \]
\[ \beta_{f_2} = G (e^{2\phi} - 1) \left[ \frac{30 - n}{3} \lambda_f + \frac{n \lambda^2 - 8}{4(1 + \epsilon)} (\epsilon \lambda_f + 4 - 2 \epsilon) - 4 \lambda_f + 8 \right], \]
\[ \beta_V = G (e^{2\phi} - 1) \left[ \frac{30 - n}{3} \lambda_V + \frac{n \lambda^2 - 8}{4(1 + \epsilon)} (\epsilon \lambda_V - 4 - 2 \epsilon) - 4 \lambda_V - 8 \right]. \] (23)

Equating these \( \beta \)-functions to zero we easily get the fixed-point solutions. For the dilatonic coupling, we find

\[ \lambda^* = - \frac{6 \epsilon}{30 - n} + O(\epsilon^2). \] (24)

There are also imaginary oscillating solutions for \( \lambda \sim \epsilon^{-1/2} \), which have been mentioned in ref. [16] and which are not physical solutions.

Using (24), for the Maxwell-dilatonic coupling and dilatonic potential, we obtain

\[ \lambda^*_f = - \frac{36 \epsilon}{18 - n} + O(\epsilon^2), \quad \lambda^*_V = \frac{12 \epsilon}{18 - n} + O(\epsilon^2) \] (25)

The fixed-point solutions (25) are of the same nature as solution (24). Notice, however, that the denominator in (25) is different from the denominator for the case of a purely dilatonic sector (24). For the existence of the solutions (23) a new limitation appears: \( n \neq 18 \). Under a discontinuous transition through the point \( n = 18 \), the sign of the fixed points in (25) changes.

It is interesting to see in which way the model (5) can be rewritten at the fixed point. In particular, we will perform a Weyl rescaling (it is non-singular) in the manner suggested in ref. [16]

\[ g_{\mu\nu} \rightarrow g_{\mu\nu} \exp \left( \frac{4 \lambda^*}{\epsilon} \phi \right). \] (26)

Then, the classical action becomes

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{\mu^4}{16 \pi G^*} e^{-(1-\lambda^*)\phi} \left[ R - \frac{4(1 + \epsilon)}{\epsilon} \lambda^*(2 - \lambda^*) \right] g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \\
-\frac{1}{2} g^{\mu\nu} \partial_\mu \chi_i \partial_\nu \chi^i + \mu^2 m^2 e^{(2 \lambda^* - 4 \lambda^*/\epsilon - \lambda^*_\phi)} + \frac{1}{4} F^2 e^{(2 \lambda^* - 4 \lambda^*/\epsilon - \lambda^*_\phi)}. \right\}. \] (27)

As one can see from expression (27), without the electromagnetic sector and setting \( m^2 = 0 \), it describes the CGHS action [1] (at the limit \( \epsilon \rightarrow 0 \) is finite and scalars become non-interacting with the dilaton). As it stands, action (27) is of a similar form as dilaton-Maxwell gravity [5, 19] with the Liouville potential. Notice that such kind of dilaton-Maxwell gravity (which
can also be considered as a charged string-inspired model \cite{5} admits charged black hole solutions with multiple horizons, being in this sense analogous to four or higher-dimensional Einstein-Maxwell theories \cite{24}. Different forms of dilatonic gravity can be easily obtained too, by transforming the metric and the dilaton (see the appendix). For example, by transforming $g_{\mu \nu} \rightarrow \exp(\frac{2\epsilon}{2-\epsilon} \lambda_1^* \phi) g_{\mu \nu}$, we can present the theory at the critical point as having a free (i.e., non-interacting with the dilaton) Maxwell sector.

Notice also that, in order to fix the scale of the metric —what is certainly necessary for discussing the renormalization of the gravitational constants \cite{9}— one can use the same reference operator as in ref. \cite{16}, e.g. a combination of the trace of the metric and of the dilaton. As has been already explained in detail in \cite{16}, this choice is enough to fix the scale of $g_{\mu \nu}$ and the origin of $\phi$.

One can now study the stability of the fixed points (21), (24), (25), along the same lines as in ref. \cite{16}. Now we can perform variations along four different trajectories, but concerning the existence of fixed points the situation is pretty similar to that in ref. \cite{16}. In fact, a careful analysis of the beta-functions \cite{19} for the linear Ansatz (22) shows that the two last equations (i.e. those for $f_2$ and $V$) do not produce a new multiplicity of solutions. In other words, for each value of $G^*$ and $\lambda^*$ we just have one single value of $\lambda_1^*$ and one of $\lambda_1^* V$, that complete the four coordinates of the fixed point. For $\lambda^*$ we obtain three distinct solutions: the real one (24) and two purely imaginary ones, of order $\epsilon^{-1/2}$, namely

$$\lambda_\pm^* = \pm 2i \sqrt{\frac{30 - n}{3n\epsilon}} + O(\epsilon^0),$$  

(28)

that correspond to highly oscillating dilaton couplings and seem not to have any sensible meaning.

Expanding the beta functions near the (only real) fixed point, in the way

$$G = G^* + \delta G, \quad \Phi = \lambda^* \phi + \delta \Phi, \quad f_2 = \lambda_1^* \phi + \delta f_2, \quad V = \lambda_1^* \phi + \delta V,$$  

(29)

and assuming all the fluctuations to be small, we obtain

$$\delta \beta_G = -\epsilon \delta G,$$

$$\delta \beta_\Phi = \frac{\epsilon}{2} \left(e^{2\phi} - 1\right) \frac{d}{d\phi} \delta \Phi + O(\epsilon^2),$$

$$\delta \beta_{f_2} = \frac{\epsilon}{2} \left(\frac{18 - n}{30 - n} e^{2\phi} - 1\right) \frac{d}{d\phi} \delta f_2 + O(\epsilon^2),$$

$$\delta \beta_V = \frac{\epsilon}{2} \left(\frac{18 - n}{30 - n} e^{2\phi} - 1\right) \frac{d}{d\phi} \delta V + O(\epsilon^2).$$  

(30)

As observed in ref. \cite{16}, we may take $e^\phi$ to play the role of loop expansion parameter, and restrict ourselves to the region $e^{2\phi} \leq 1$, that is $-\infty < \phi \leq 0$. The change of variables

$$\eta_1 = \ln(e^{-2\phi} - 1), \quad \eta_2 = \ln \left[\frac{30 - n}{12} \left(e^{-2\phi} - \frac{18 - n}{30 - n}\right)\right],$$  

(31)

transform this region into the following ones

$$\phi = 0, \quad \eta_1 \rightarrow -\infty, \quad \eta_2 = 0,$$

$$\phi \rightarrow -\infty, \quad \eta_1 \rightarrow +\infty, \quad \eta_2 \rightarrow +\infty,$$  

(32)
respectively. They simplify expressions (30), which now read:

\[
\begin{align*}
\delta \beta_G &= -\epsilon \delta G, \\
\delta \beta_\Phi &= \epsilon \frac{d}{d\eta_1} \delta \Phi + \mathcal{O}(\epsilon^2), \\
\delta \beta_{f_2} &= \epsilon \frac{d}{d\eta_2} \delta f_2 + \mathcal{O}(\epsilon^2), \\
\delta \beta_V &= \epsilon \frac{d}{d\eta_2} \delta V + \mathcal{O}(\epsilon^2).
\end{align*}
\]

The eigenfunctions corresponding to the new differential operators on the right hand side are, respectively,

\[
\begin{align*}
\delta \Phi &\sim e^{\alpha_1} = (e^{-2\phi} - 1)^{\alpha_1}, \\
\delta f_2 &\sim e^{\alpha_2} = \left[\frac{30-n}{12} \left(e^{-2\phi} - \frac{18-n}{30-n}\right)\right]^{\alpha_2}, \\
\delta V &\sim e^{\alpha_3} = \left[\frac{30-n}{12} \left(e^{-2\phi} - \frac{18-n}{30-n}\right)\right]^{\alpha_3},
\end{align*}
\]

and the corresponding eigenvalues are

\[
\begin{align*}
\delta \beta_\Phi &= \epsilon \alpha_1 \delta \Phi, \\
\delta \beta_{f_2} &= \epsilon \alpha_2 \delta f_2, \\
\delta \beta_V &= \epsilon \alpha_3 \delta V.
\end{align*}
\]

By imposing the initial condition \(\delta \Phi(\phi = 0) = 0\) we see that \(\alpha_1 > 0\). Hence, we observe that the fixed point for \(G\) is not ultraviolet stable in the direction \(\delta \Phi\). At the same time, within such a picture it is clear that we can always choose \(\alpha_2\) and \(\alpha_3\) to be negative — since, for the other two directions, \(f_2\) and \(V\), there is no way to make \(\delta f_2 = 0\) nor \(\delta V = 0\) at \(\phi = 0\). (By means of an adequate choice of arbitrary constant, we can choose \(\delta f_2(\phi = 0)\) and \(\delta V(\phi = 0)\) as small as we like, but never zero — with the choice above we have set these two values equal to 1). Therefore, the fixed points for \(f_2\) and \(V\) are ultraviolet stable in the direction \(\delta \Phi\) but are always infrared unstable in this direction (since they are never zero at \(\phi = 0\)). In this sense the theory (four RG functions) possesses a saddle fixed point.

5 Gauge dependence and fixed points

It is of interest to discuss in some detail how the results of the previous sections would change if we made a different choice of the gauge condition. This question is not trivial at all, as was already mentioned in [7, 4], where it was shown that at some gauges dilatonic gravity could be rendered one-loop finite (except for the conformal anomaly term). However, explicit calculation of the divergences for dilatonic gravity with matter in a parameter-dependent gauge are extremely cumbersome. Hence, we shall here consider the simplified model of ref. [7]

\[
S = \int d^d x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + C_1 \phi R + f_3(\phi) g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \right\},
\]
where \( f_3(\phi) \) is some dilatonic coupling and \( C_1 \) some constant. The one-loop divergences of the theory (36) can be studied in the one-parameter dependent gauge

\[
S_{gf} = -\frac{C_1}{2\alpha} \int d^d x \sqrt{-g} \chi_\mu \phi_\mu, \\
\chi_\mu = \nabla_\mu h^\nu_\mu - \frac{1}{2} \nabla_\mu h - \frac{\alpha}{\phi} \partial_\mu \bar{\phi} + X(\phi) \bar{\phi} \partial_\mu \phi \\
+ h^\rho_\mu \left[ Y_1(\phi) \left( \delta_\mu^\rho \nabla^\sigma + \delta_\rho^\sigma \nabla^\mu \right) \phi + Y_2(\phi) g^{\rho \sigma} \partial_\mu \phi \right],
\]

(37)

where \( \phi \) is a background field \((\phi \rightarrow \phi + \bar{\phi}, \ g_{\mu \nu} \rightarrow g_{\mu \nu} + h_{\mu \nu})\), \( \bar{\phi} \) and \( h_{\mu \nu} \) are quantum fields, \( \alpha \) a gauge parameter, and \( X, Y_1, Y_2 \) are arbitrary dilatonic functions.

The calculation of the one-loop effective action for the theory (36) in the gauge (37) (let aside from the \( f_3(\phi) \)-dependence) has been performed in ref. [7]. The result is now

\[
\Gamma_{\text{div}} = \frac{1}{2\pi \epsilon} \int d^d x \sqrt{-g} \left\{ \frac{24 - n}{12} R + \left[ -\frac{\alpha}{\phi^2} + nV_1(\alpha, f_3(\phi)) \right] g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right\}. 
\]

(38)

Notice that in ref. [7] only the dilatonic contribution — the \((-\alpha/\phi^2)\)-term— had been found. The explicit calculation of the function \( V_1 \), which depends on \( f_3 \) and \( \alpha \), requires a huge algebra (see [7]). Fortunately, knowledge of the precise form of \( V_1 \) is not necessary for our considerations here, as we will see below.

Now, performing the background transformation in (36) (in exactly two dimensions)

\[
g_{\mu \nu} \rightarrow \exp \left( -\frac{\phi}{4C_1} \right) g_{\mu \nu}, \quad \phi = e^{-2\phi},
\]

(39)

we can rewrite (38) as follows

\[
S = \int d^d x \sqrt{-g} \left\{ \frac{1}{16\pi G} R e^{-2\phi} + f_3(\phi) g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right\},
\]

(40)

where \( C_1 \) is to be identified with \((16\pi G)^{-1} \) and \( f_3(\phi) \) with \( \exp(-2\Phi(\phi)) \). After proper transformation of the gauge-fixing Lagrangian (37), the \( \Gamma_{\text{div}} \) corresponding to the theory (36) —which is a particular case of the model (3) — in two dimensions is

\[
\Gamma_{\text{div}} = \frac{1}{2\pi \epsilon} \int d^d x \sqrt{-g} \left\{ \frac{24 - n}{12} R + 4 \left[ -\alpha + nV_1(\alpha, \phi) \right] g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right\}. 
\]

(41)

For \( \alpha = 1 \) the dilatonic gravity contribution in (41) coincides with the result given in [22, 1, 3, 4] or in Eq. (10).

Repeating the considerations of sects. 3 and 4 with \( \Gamma_{\text{div}} \) (41), we see that \( A_2(\phi) \) changes and hence \( \beta_\Phi \) changes also accordingly (but not \( \beta_G \)). Adding to the central charge in (41) the contribution from the Maxwell sector, and looking for the fixed-point solution (21) and (22), we find

\[
G^* = \frac{3\epsilon}{2(30 - n)}, \quad \lambda^* = -\frac{6\epsilon\alpha}{30 - n}. 
\]

(42)

The explicit form of \( V_1 \) (which is a combination of \( \Phi \) and its derivatives) is not necessary to find this fixed point, as it was also the case with eq. (24)). The value of the gravitational coupling constant at the ultraviolet fixed-point is gauge independent, as it should be.
We realize that for any non-zero value of the gauge parameter a fixed point $\lambda^*$ exists. (The case $\alpha = 0$, which corresponds to a gauge of Landau type, is in some sense degenerate and, hence, all the considerations in this case must be given independently, including the calculation of the explicit form of the function $V_1$). The change of gauge parameter will affect the position of the fixed point through the slope of the function $\Phi(\phi)$. However, the more physical issue of the stability (or instability) of this fixed point will not be affected.

6 Conclusions

In this paper we have studied dilatonic-Maxwell gravity with matter near two dimensions. The nice properties of this theory are: (i) its renormalizability in $2 + \epsilon$ dimensions and the asymptotically free behavior in the ultraviolet regime shown by the gravitational coupling constant. (ii) The fact that there is a non-trivial fixed-point solution for the dilatonic couplings and that, at the fixed point, the theory may be represented in the standard form of the string-inspired models that have been discussed recently. (iii) The increase of the upper limit for the matter central charge —due to the contribution of the vector field— from 24 (pure dilatonic gravity) or 25 (Einstein gravity) to 30. This gives the possibility to extend the matter content of the theory.

The investigation of the theory (3) shows the way of considering even more realistic toy models in $2 + \epsilon$ dimensions. Indeed, instead of the Maxwell term in (3) we can insert its non-abelian, Yang-Mills generalization corresponding to a gauge potential $A^a_{\mu}, a = 1, 2, \ldots, N$. For simplicity, let us consider the gauge group to be simple and compact and the structure constants antisymmetric. Let us also add to action (3) the kinetic term for $m$ fermions with a dilatonic coupling constant similar to $\Phi(\phi)$. Such theory represents the unification of dilatonic QG with matter (scalars, spinors and vectors) in $2 + \epsilon$ dimensions.

The beta-function for the gravitational coupling constant is found to be

$$\beta_G = cG - \frac{1}{3}(48 + 12N - m - 2n)G^2.$$ (43)

From this expression we see that the matter central charge, $C = n + m/2$ is limited as follows:

$$0 < C < 24 + 6N,$$ (44)

and for such a range of the central charge there is an ultraviolet stable fixed point for the gravitational coupling constant, namely

$$G^* = \frac{3\epsilon}{48 + 12N - m - 2n}, \quad \epsilon > 0.$$ (45)

Hence, we still preserve asymptotic freedom in the gravitational coupling constant. Moreover, we have now much less rigid restrictions to the central charge: by increasing the dimension $N$ of the gauge group we may increase the number of scalars and spinors in the theory. The calculation of the one-loop effective action for this theory can be carried out following ref. [18] (and the second ref. of [22]) and, at least qualitatively, the conclusions about the existence of non-trivial fixed point solutions of the form (22) remain true. In particular, we
obtain for the above model:

$$
\lambda^* = -\frac{12\epsilon}{48 + 12N - m - 2n}, \quad \lambda_f^* = -\frac{72\epsilon}{24 + 12N - m - 2n}, \quad \lambda_V^* = \frac{24\epsilon}{24 + 12N - m - 2n}.
$$

(46)

The stability of the fixed-point solutions (as well as for the corresponding fermion-dilaton coupling) can be studied similarly as in sect. 4.

It is also of interest to consider supergravity models in $2 + \epsilon$ dimensions. Some attempt in this direction has already been started in ref. [29]. Finally, an important issue is to study quantum cosmology in frames of our $(2 + \epsilon)$-dimensional model and, in particular, to check carefully the claim of ref. [17] that RG considerations in $2 + \epsilon$ dimensions may indeed help to solve the spacetime singularity problem. We plan to return to some of these questions in the near future.

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**A Appendix: Jackiw-Teitelboim model with matter in $2 + \epsilon$ dimensions**

One of the most popular models of 2D QG is the so-called Jackiw-Teitelboim (JT) model [3]

$$
S_{cl} = \int d^4x \sqrt{-g} \left[ \mu e^{-2\phi} \left( \frac{R}{16\pi G} + m^2 \right) \right].
$$

(A.1)

The classical solution of this theory is

$$
R = -16\pi Gm^2
$$

(A.2)

and shows the quite remarkable fact that there is only possibility for existence of constant curvature geometries in the model. It also gives the way to construct non-critical string theories without limitations in the matter central charge of the theory [4].

By interaction of the theory (A.1) with scalars and vectors under the form of a dilatonic coupling, as in (5), the property (A.2) is lost. Dilatonic field equations become more complicated, involving also contributions from the matter sector. By performing renormalization as it was done before, we observe that the choice of $V(\phi)$ as in (A.1) breaks down at the quantum level (25), since $\lambda_V^* \neq 0$. Hence, the JT model with matter is not a non-trivial fixed point of the RG in $2 + \epsilon$ dimensions. Of course, starting from action (5) and carrying out a non-singular gauge transformation of the metric as in (26) one can obtain in the action —at the fixed point— a term of the form (A.1). However, the price for this will be the appearance of a dilatonic kinetic term, what destroys the crucial property of the JT model (A.2) already in the sector of pure dilatonic gravity.
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