Factorization of Polynomials in One Variable over the Tropical Semiring

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Abstract

We show factorization of polynomials in one variable over the tropical semiring is in general NP-complete, either if all coefficients are finite, or if all are either 0 or infinity (Boolean case). We give algorithms for the factorization problem which are not polynomial time in the degree, but are polynomial time for polynomials of fixed degree. For two-variable polynomials we derive an irreducibility criterion which is almost always satisfied, even for fixed degree, and is polynomial time in the degree. We prove there are unique least common multiples of tropical polynomials, but not unique greatest common divisors. We show that if two polynomials in one variable have a common tropical factor, then their eliminant matrix is singular in the tropical sense. We prove the problem of determining tropical rank is NP-hard.

1 Introduction

The tropical semiring \([15]\), earlier called the maximin algebra, is the real numbers together with infinity, and an additive operation \(\min(x, y)\) and a multiplicative operation \(x + y\). Over the past 40 years a number of applications for this structure have been found such as scheduling industrial production, hierarchical clustering, asymptotic approximations in physics, and nonstandard logics. The 2-element Boolean algebra is a subsemiring of the tropical semiring, and the nonnegative elements of the tropical semiring form the structure called an incline in \([15]\). See \([11]\), \([4]\), \([5]\), \([13]\), \([20]\), \([6]\) for additional details on history and applications of semirings with a somewhat related structure. Recently B. Sturmfels and many others \([16]\), \([17]\) have discovered and exploited a new aspect to tropical algebras, using them to study properties of algebraic varieties in terms of valuations, power series expansions, and Gröbner bases. They discovered a way to define algebraic varieties over a tropical semiring such that many properties of algebraic varieties over the complex numbers, such as Bézout’s theorem, will be valid. The paper \([17]\) lists 5 open problems in the field of tropical mathematics, and the present paper deals with the second of these, factorization of polynomials over a tropical semiring.

This paper has a general relationship to the work of S. Gao and A. Lauder \([9]\), \([10]\). The first of those papers gives criteria for irreducibility of multi-variable polynomials over the complex numbers. These will also give criteria for irreducibility of multi-variable polynomials over the tropical semiring, since any factorization over a tropical semiring can be represented by a corresponding factorization over the complex numbers. The converse does not hold, since cancellation under addition to produce zero terms does not occur in the tropical semiring. The second of their papers proves NP-completeness of the problem of decomposing polygons (in the plane) under Minkowski sum and gives an algorithm for such decompositions which is not polynomial time in the strict sense but is polynomial time in the size of the integer coordinates, for fixed numbers of points.

Here it is likewise proved that the problem of factorization over a tropical semiring is NP-complete, but our methods and the nature of the result is quite different. We consider polynomials in one variable, and coefficients
which are either 1, 2, or 3, and show that the satisfiability problem can be expressed as a problem of factorization. By factorization we do not mean factorization of polynomials as functions, which is simple in the one variable case, but factorization as formal expansions in which the coefficients are operated on by the rules of tropical algebra. We give several descriptive results on factorization and irreducibility. Our results apply to polynomials in several variables in a number of ways, such as by substituting different polynomials in 1 variable for each of several variables. We suggest an algorithm.

We pass to the case of polynomials over a Boolean semiring (coefficients 0 and $\infty$ in the tropical semiring) and show this factorization problem is likewise NP-complete, give irreducibility criteria and an algorithm also in this case. We also discuss least common multiple, greatest common divisor, and eliminants. T. Theobald [19] has recently proved other NP-completeness results in tropical geometry.

2 Polynomials over the tropical algebra and Minkowski sum

The Minkowski sum of two sets $A, B$ in $n$-space is \{a + b|a \in A, b \in B\}.

It is convenient to represent a polynomial

$$a_0 \oplus (a_1 + x) \oplus \ldots \oplus (a_n + nx)$$

over the tropical semiring geometrically, where $\oplus$ denotes the additive operation $\min(x, y)$ and the multiplicative operation is $x \otimes y = x + y$ (ordinary addition over the extended real numbers). That is, we represent the polynomial by the set of points $(n, a_n)$ in the plane, together with all line segments from $(n, a_n)$ to $(n + 1, a_{n+1})$ and all points above these line segments. When we multiply two polynomials, the sets of points at integer $x$ coordinates in this figure will combine by Minkowski sum, that is, in multiplying

$$a_0 \oplus (a_1 + x) \oplus \ldots \oplus (a_n + nx)$$

times

$$b_0 \oplus (b_1 + x) \oplus \ldots \oplus (b_m + mx)$$

at each degree $k$ the coefficient is $\min(a_i + b_{k-i})$. This gives a diagram which agrees with Minkowski sum of the diagrams of the factors at integer $x$-coordinates; we take the separate sums $a_i + b_{k-i}$ and fill in from above. Since Minkowski sum commutes with convex hull, if we take the convex hulls of these figures, they will also add under Minkowski sum. This implies the following which is familiar to those who have worked with Newton polygons.

Proposition 1. The convex hull of the diagram of a product of polynomials is obtained from the convex hulls of its factors by starting at the degree zero term, and successive drawing edges to the right, which as vectors equal the edges in the factors, taken one by one in order of increasing slope.

This follows by rational approximation and representing the tropical polynomials by polynomials over a field in the standard fashion, which
factor according to the edges. It also follows geometrically by comparing
\[\min_i (a_i + b_{k-i}), \min_i (a_i + b_{k-i+1})\]. Each is a minimum of a difference
between a sequence \(a_i\) concave upward and a sequence \(-b_{k-i}\) concave
downward. By subtracting a common linear factor \(\alpha + \beta(i)\) representing
a line separating two convex sets we can arrange that this least vertical
difference occurs at a minimum of \(a_i\) and a maximum of \(-b_{k-i}\). Now
consider what happens when we shift the latter horizontally one unit.
The least difference must occur inbetween the minimum of \(a_i\) and the
maximum of \(-b_{k-i}\). Thus in the general case the minimum occurs either
for the same \(a_i\) and the adjacent \(b_i\), or vice versa. It becomes a little more
complicated when there the minimum is not unique.

Now recursively keep track at each step of vertices for the two factors
which add to give the current vertex in the product. If we transfer any
edge as a vector from a factor to the product, its right vertex will represent
a product term entering the minimum for the product polynomial, and
for the edge of lower slope, this will be a minimum point in the product.
This will represent the next unused edge in one of the two factors.
Moreover terms which do not contribute to the convex hull in the
factors will not do so in the product.

Proposition 2. A polynomial in one variable over a tropical algebra
whose diagram lies strictly above the line segment joining its terms of
highest and lowest degree, except at those two points, is irreducible (ex-
cept for monomial factors).
This follows from proposition 1. The following result is also known.

Proposition 3. If a polynomial has a diagram which is already convex,
then it factors into linear factors, and this factorization can be done in
polynomial time.

This suggests that the case which is most challenging may be when
the convex hull of the diagram of the proposed product is spanned by 3
vertices from polynomial coefficients (with an additional infinite point),
and the diagram is above the convex hull except at those 3 vertices. This
determines the degrees and endpoints of the possible factors (by Prop.1
there can be only two factors).

Consider the subcase of this case in which the first and last coefficients
of the polynomial are 0 (corresponding to 1 for polynomials over a field),
and their degrees are equal, which we call the equal slope and degree
concave case. One question is the likelihood that a polynomial will be
irreducible, in terms of volumes over bounded sets of \(n\)-tuples of real
numbers.

Proposition 4. In the equal slope and degree concave case, the prob-
ability is 1 that a random polynomial (given the 3 vertices spanning its
convex hull) is irreducible. That is, the set of factorable polynomials of
this type has strictly lower dimension than the set of all polynomials of
this type.

Proof. Consider two factors as above
\[
\begin{align*}
a_0 &\oplus (a_1 + x) \oplus \ldots \oplus (a_n + nx) \\
b_0 &\oplus (b_1 + x) \oplus \ldots \oplus (b_m + mx)
\end{align*}
\]
where the first and last coefficients of both factors may be taken as 0. It
will suffice to show that a product polynomial with coefficients \(c\) which
are generic, that is, not lying in a finite number of sets of lower dimension to be specified in this proof, cannot be factored. The number of coefficients \(a, b\) of the factors equals the number of unknown coefficients \(c\) of the product, and if any coefficients did not enter nontrivially into formulas for \(c\), this would give a specific set of lower dimension. Hence all \(a, b\) must enter nontrivially. We may also suppose all the coefficients \(a, b\) except the highest and lowest degree coefficients are distinct, since if not this would give a set of lower dimension. By the concavity assumption, all these other coefficients are positive. Let \(b_i\) be the largest of any of these coefficients. Then \(b_i > a_i \geq c_i\) since \(c_i\) is the infimum of a collection of terms including \(a_i + 0\). But \(b_i\) must be directly involved in the minimum formula which produces some \(c_j\). Otherwise, the product does not depend on all the coefficients \(a, b\). Therefore \(c_j = b_j + a_{j-1}\). But \(c_j\) is a minimum of terms which also include either \(a_j + 0\) or \(a_{j-n} > 0\) depending on whether or not \(j > n\). Therefore some coefficient \(a\) exceeds \(b_i\) which is a contradiction.\(\blacksquare\)

Example. For two concave factors with unequal slope, the probability of being factorable is in general positive. Consider the product of

\[
0 \oplus (a_1 + x) \oplus (0 + 2x) \oplus (b_1 + x) \oplus (b_2 + 2x)
\]

where all \(a_i, b_i > 0\). For suitable inequalities the product can be

\[
0 \oplus (a_1 + x) \oplus (0 + 2x) \oplus (b_1 + 3x) \oplus (b_2 + 4x).
\]

These inequalities are

\[
a_1 < b_1, a_1 + b_2 > b_1, 2b_1 > b_2
\]

where the last ensures concavity of the second factor.

Products of polynomials over a tropical ring can be pictured in terms of minima of antidiagonals of a matrix. For instance, the product in the example is the antidiagonal minimum of

\[
\begin{pmatrix}
0 + 0 & 0 + b_1 & 0 + b_2 \\
0 + a_1 + b_1 & a_1 + b_2 \\
0 + 0 & 0 + b_1 & 0 + b_2
\end{pmatrix}
\]

3 NP-completeness

Lemma 5. Let a polynomial with \(c_0 = c_n = c_{2n} = 0\) be of the equal slope, equal degree, concave factorization type and \(c_{2n}\) the highest degree coefficient. Suppose all other \(c_i \in \{1, 2, 3\}\). Then a factorization exists only if one exists of the following form: if \(c_j = 1\) then \((a_i, b_i) = (1, 1), (1, 3)\) or \((3, 1)\) and therefore \(c_{i+n} = 1\) and conversely. If \(c_j = c_{j+n} = 2\) then \((a_j, b_j) = (2, 2)\). If \(c_j\) or \(c_{j+n} = 3\) then \((a_j, b_j) = (3, 3)\).

Proof. By Prop.1 we may suppose the factors have \(a_0 = a_n = b_0 = b_n = 0\) and other coefficients are positive. All other coefficients \(a, b\) are at least 1, or the product would have additional terms below 1. If \(c_j\) or \(c_{j+n} = 3\) then \(a_j, b_j\) must be at least \((3, 3)\) or they would reduce the
minimum in the product below 3. Without loss of generality we can reduce them to \((3, 3)\). If \(c_j = c_{j+n} = 2\) then \((a_j, b_j)\) are at least \((2, 2)\) and when \(a_j, b_j\) occur summed with other nonzero terms we have at least 3, so we can reduce them to \((2, 2)\) without loss of generality. If \(c_j = 1\) then one of \(a_j\) or \(b_j\) is 1. If the other is at least 2, then any value above 2 will not affect the products, and we can choose it as 3. If the other is strictly between 1 and 2, then the only way it can affect the product coefficients nontrivially is in another product where one of \(a_k, b_k\) is strictly between 1 and 2 and the other is 1, since \(c_k\) must be 1. But in this case the 1 + 1 will give the minimum, so the other variable can be chosen as 3. 

Lemma 6. For any positive integer \(n\), and any positive integer \(N < n^{1/8}\) we can find sequences of positive numbers \(x_{ij}, y_{ij}, z_i < \frac{n}{4}\) such that \(x_{ij} + y_{ij} = z_i\) are distinct, and all other sums \(x_{ij} + y_{rs}, x_{ij} + x_{rs}, y_{ij} + y_{rs}\) are distinct from these and from each other, and these sums are distinct from all \(x_{ij}, y_{ij}\); here \(i, j\) range from 1 to \(N\). These sequences can be found in polynomial time in terms of \(N, n\).

Proof. We first choose \(z_i\) in turn as distinct numbers between \(\frac{n}{5}, \frac{n}{4}\). Then in turn we choose the \(x_{ij}\) of size at most \(\frac{n}{8}\) which determine the \(y_{ij}\). As we choose them, we avoid any equation with a previously determined number or sum. This means avoiding at most \(15N^8\) numbers in the choice. For instance there are at most \(N^3\) sums \(x_{ij} + y_{rs}\) and \(x_{ij}\) enters at most \(N^2\) of them.

But \(n/8\) exceeds \(16N^8\) so at each stage this can be done. Choosing each variable in turn can be done in a polynomial number of steps, and there are a polynomial number of variables to choose.

The degrees involved in our problem are to be polynomial in \(n\). If we fixed the degree, then factoring reduces to a finite number of linear inequalities which can be solved in polynomial time by linear programming methods (such as homotopy methods) over the rational numbers.

Definition. The satisfiability problem is, in Boolean variables \(w_i, i = 1, \ldots, n\) and constants \(a_{ij}, b_{ij}, i = 1, \ldots, m, j = 1, \ldots, m\) to solve the Boolean equation

\[
\prod_j \sum_i (w_i a_{ij} + w_i^c b_{ij}) = 1
\]

where \(c\) superscript denotes complement. It is NP-complete, in fact almost all other NP-complete problems are ultimately proved so by comparison with it. The factors are called clauses.

Theorem 7. The problem of factoring polynomials over the tropical semiring of degree \(n\), in the equal slope, equal degree concave case, with all coefficients 0, 1, 2, 3 is NP-complete.

Proof. It suffices to show there is a polynomial time reduction of the satisfiability problem to this factoring problem, where \(m\) is of magnitude some constant positive root of \(N\). The sets of numbers given in Lemma 6 will specify the numbers \(c\) in the factoring problem. The degrees \(k = x_{ij}, y_{ij}\) are to be the degrees where \(c_k = 1\). Twice such a degree never occurs elsewhere among the sums by Lemma 6, and in such a degree \(k\) we may assume either \(a_k\) or \(b_k = 1\) but not both, setting \(c_{2k} = c_{2k+n} = 3\). We say that two such degrees, that is, degrees \(x_{ij}\) or \(y_{ij}\), have the same parity if it is the same one of \(a, b\) which is 1. In the degrees \(k = z_i\), we set
\( c_k = 2 \) and \( c_{k+n} = 3 \). The sums of two degrees \( x_{ij}, y_{rs} \) can never exceed \( n \) by Lemma 6, so \( c_{k+n} = 3 \) can be assumed without contradiction.

Whenever \( c_k = 2 \) and \( c_{k+n} = 3 \), some pair in degrees \( x_{ij}, y_{ij} \) must be added to produce \( z_k \). The two degrees from \( x_{ij}, y_{ij} \) must have opposite parity, otherwise we would be adding \( 1 + 3, 1 + 3 \) for each combination of an \( a \) and a \( b \). After some identifications the fact that some such pair adding to a \( z_i \) has opposite parity represents some variable \( w \) being 1 in the corresponding clause of the satisfiability problem. There will be one clause for each \( z_i \). By using the sums of \( x_{ij} + y_{rs}, x_{rs} + y_{ij} \), which are unique, and setting \( c_k = 2 \) and \( c_{k+n} = 3 \) in one of these degrees, we can force any two pairs of variables from the \( x_{ij}, y_{ij} \) and \( x_{rs}, y_{rs} \) to have the opposite relative parity, without affecting other pairs. By using these unique sums and \( c_k = 3 \) and \( c_{k+n} = 3 \) we can force any such pairs of variables to have the same parity. To not require any relationship between a pair from \( x_{ij}, y_{rs} \) let \( c_k = 2, c_{k+n} = 2 \). This can be solved for by setting \( a_k = 2 \) without affecting other conditions. All other coefficients \( c \) are to be 3 except \( c_0 = c_n = c_{2n} = 0 \); this imposes no constraint since \( 1 + 1 \) will never occur except when two of \( x_{ij}, y_{ij} \) are added.

This allows us to identify variables in different equations as being the same or as being complements of each other, where a single variable is associated to an \( x \) and a \( y \) which add up to be a \( z \); we can relate the \( x \) to the \( x \) and the \( y \) to the \( y \) as being either the same or complements, and make the two pairs either to have identical parity or unlike parity. By identifying pairs in the same equation we can reduce the number of variables to any desired extent. By these identifications we may produce any satisfiability problem which has at most \( N/2 \) variables (allowing for each variable to occur complemented) in each of the \( N \) clauses. All the steps going back and forth in this transformation can be done in polynomial time.

If there is a factorization, then the constructions above mean that in degrees \( z_k \) there will be some pair of variables having opposite parity, so the corresponding clause of the satisfiability problem, after we identify variables which are forced to have the same parity, and those forced to have opposite parity to be their complements, must have a 1, and the satisfiability problem is solvable. Conversely if the satisfiability problem is solvable then we may choose a consistent sequence of values of the parities of \( x_{ij} \) and \( y_{ij} \), so that whenever \( k = z_i c_k = 2, c_{n+k} = 3 \), there is some pair having opposite parity. Then for the \( x_{ij}, y_{ij} \) we choose the \( a_k, b_k \) to be \((1, 3)\) or \((3, 1)\) according to the parities, and in all the sums of pairs of these degrees \( c_k, c_{k+n} \) being \( 2, 3 \) or \( 3, 3 \) are accounted for, as well as all values \( c_k \) = 1. In other degrees choose the \( a, b \) according to Lemma 5. Sums of at least two of these other \( a, b \) will never cause a conflict. This gives a tropical factorization. \( \square \)

This construction also suggests that there are cases in which there are exponentially essentially different factorizations of tropical polynomials of degree \( n \) into irreducible factors, since we can have cases of the satisfiability problem with exponentially many solutions, and there will be no refinement into common factorizations because of the location of the two zero coefficients in the product.
4 Factorization of Boolean polynomials

Here we consider factorization of Boolean polynomials

\[ c_0 + c_1 x + \ldots + c_n x^n, \, c_i \in \{0, 1\} \]

This is the same as the question of factoring polynomials over the tropical ring whose coefficients are 0 and \(\infty\) into factors of the same type. We assume \(c_0 = c_n = 1\) (Boolean). Factoring Boolean polynomials is also the same as expressing a 1-dimensional set of numbers from 0 to \(n\) as a nontrivial Minkowski sum of two subsets of the same type.

**Proposition 8.** If the set of positive degrees \(i\) in a Boolean polynomial where \(c_i = 1\) is not a union of sets of the form \(\{d_1, d_2, d_1 + d_2\}\) then the Boolean polynomial is irreducible. In particular if the positive degrees lie in a set \(T\) of congruence classes modulo some integer \(m\), such that the sum of two congruence classes is outside \(T\), then the Boolean polynomial is irreducible. Moreover the sets of lower numbers \(\{d_1, d_2\}\) suitably ordered must form some Cartesian product.

**Proof.** This follows by consideration of products of terms \(x^{d_1}, x^{d_2}\) which produce the various positive degree terms in the polynomials. \(\Box\)

We do not know whether a random Boolean polynomial of degree \(n\), for \(n\) large, is more likely to factor or be irreducible.

**Theorem 9.** Factorization of Boolean polynomials of degree \(n\) is NP-complete.

**Proof.** We reduce the factorization problem of the previous section to a problem of factoring two Boolean polynomials. We give this factorization problem a 2-dimensional structure by choosing a modulus \(m\) less than the square root of \(n\), and restricting degrees of the product, which includes factor degrees, to those of the form \(a + bm\) where \(a, b < m/2\). That means that in products of two of these degrees, the \(a\)'s and the \(b\)'s must separately add to produce the result, and in effect we may as well consider a problem of factoring a Boolean polynomial in two variables \(x, y = x^m\). We reverse the ordering of degrees and add a constant, in passing from a tropical polynomial of the last section to a Boolean polynomial here. This means an ordered pair \((j, c_j)\) in the tropical polynomial will become a term \(x^{m(0) - j} y^{m(0) - c_j}\) in the two variable Boolean polynomial. Here \(m(0)\) is chosen large enough to avoid negatives. We fill in all degrees below these down to 0, that is, whenever a term \(x^a y^b\) occurs in a polynomial we add all terms \(x^c y^d, 0 \leq c \leq a, 0 \leq d \leq b\). This process of filling in is a multiplicative homomorphism.

Now note that if the tropical polynomial factors in terms of polynomials represented by ordered pairs \((j, a_j)\) and \((j, b_j)\) then this polynomial factors in a corresponding way, where in the factors \(m(0)/2\); if a tropical term is factored as \((a_1 + m_1 x) \otimes (a_2 + m_2 x)\) then the Boolean term factors as

\[ (x^{m(0)/2 - m_1} y^{m(0)/2 - a_1}) (x^{m(0)/2 - m_2} y^{m(0)/2 - a_2}). \]

Where previously we took minima of coefficients of products in each degree, here the maximum will be taken, since we fill in below.

Conversely if there is any factorization of the Boolean polynomial, because the constant term is 1, the factors must represent subsets of the
terms producing the product, and we can fill in all degrees from below without loss of generality. Then the top degree terms must multiply in the two factors like the products in a tropical algebra, and must produce the given product. Therefore the Boolean polynomial factors nontrivially if and only if the tropical polynomial does. □

5 Tropical eliminant and tropical rank

Given two polynomials \( f(x), g(x) \) of degrees \( r, s \), we form the eliminant matrix \( E \) of degree \( r + s \) just as over the complex numbers. Its \( r + s \) rows are the vectors of coefficients of \( f, xf, \ldots, x^{r-1}f, g, xg, \ldots, x^{s-1}g \). This produces a square matrix, and over the complex numbers if \( t \) is a common root of \( f, g \) then the matrix multiplied by the column vector \( 1, t, \ldots, t^{r+s-1} \) is zero therefore it is singular. This then gives a criterion for a common factor to exist which has many uses. In our case, missing coefficients over the tropical ring are considered infinite.

A \( n \times n \) tropical matrix \( (a_{ij}) \) is said to be nonsingular if and only if the permutation \( \pi \) minimizing \( \sum_{i=1}^{n} a_{i\pi(i)} \) is unique, otherwise singular.

This concept is isomorphic to strong regularity in max-algebra which has been studied in recent work of P. Butkovič.

Theorem 10. If \( f(x), g(x) \) are polynomials in one variable over the tropical semiring having a common factor, then their tropical eliminant is singular.

Proof. First consider the case of polynomials over the tropical rationals. By scaling we can reduce this to matrices over the nonnegative integers. Represent such polynomials as ordinary polynomials in \( t \) whose coefficients are polynomials in a parameter \( \lambda \). Assume the tropical polynomials have a common tropical factor. Then we can represent them by ordinary polynomials with a common factor, for which otherwise coefficients are generic.

Then for each value of \( \lambda \) the ordinary eliminants must be singular. Yet for \( \lambda \) sufficiently close to zero, the terms of lowest order in \( \lambda \) will be dominant, and if there is only one of given order, it cannot cancel. Therefore multiple terms of this least order must occur, and the tropical eliminant matrix must be singular.

The extension to the tropical reals follows from the case of the tropical rationals by taking sufficiently close approximations to the real numbers involved, which result from a homomorphism of \( Q \)-vector spaces \( R \) to \( Q \). For such an approximations equalities will follow from equalities in the real case, and also inequalities from inequalities in the real case. (Alternatively we could take general real powers of \( \lambda \) in the coefficients). □

Example. Two polynomials of degree \( n \) with \( a_0 = a_n = 0, b_0 = b_n = 0 \) and other coefficients positive will have singular eliminant but will not in general have a common factor, so this condition is not sufficient (except possibly for the convex case).

The following is a proposed problem in tropical mathematics not directly related to factoring polynomials, but somewhat to the above considerations.
Definition. The tropical rank of a matrix is the largest size of its nonsingular submatrices.

The row (column) space of a Boolean matrix is the space spanned by its rows (columns) under Boolean addition and multiplication by 0. It is proved by Devlin, Santos, and Sturmfels [7] that the tropical rank of a \((0, 1)\)-tropical matrix with no row (column) consisting entirely of 1 is the length of a longest chain in the row space (column space—by the next result the two are equivalent) of a complementary Boolean matrix, omitting the zero vector. For instance the rank of the tropical matrix which is 0 on the main diagonal and 1 everywhere else is its dimension. It is sufficient to consider only chains in the row space which are formed by adding one Boolean basis vector at a time to the existing sum: if a chain has \(\sum x_i > \sum y_j\) at sum step then replace this by \(\sum x_i + \sum y_j > \sum y_j\) and then add in the vectors \(x_i\) one by one when they make a difference, possibly making the chain longer. In the case of two-valued matrices with two finite values, being nonsingular is equivalent to the corresponding order-reversed Boolean \((0, 1)\)-matrix either having permanent 1 or being a direct sum of a permanent 1 matrix with a \(1 \times 1\) zero matrix: to see this consider a minimizing permutation and the two submatrices spanned by its entries with each of the higher and lower values.

Two Boolean matrices \(A, B\) are L (R) equivalent if and only if they have the same row (column) space; they are D-equivalent if and only if there is a Boolean matrix \(C\) such that \(A\) is R-equivalent to \(C\) and \(C\) is L-equivalent to \(B\). Though we will not actually use the next two results, they show the close connection between general finite lattices and Boolean matrices.

Theorem (Markowsky [14]). There is an isomorphism from the row space of a finite Boolean matrix as a lattice to the inversion of the column space. Any finite lattice is equivalent to the row space of a Boolean matrix, which is unique up to D-equivalence.

It is possible to explicitly construct the required isomorphism in the first statement. Every finite lattice can be represented in terms of its additive operation as a semilattice of sets, looking in terms of its order ideals, and this will be the row space of a Boolean matrix. A proof and reference is given in [12] that two Boolean matrices are D-equivalent if and only if their row spaces are isomorphic as lattices. This result was first proved by Zaretskii.

In connection with what lattices can arise, we mention a result of Markowsky, which is likely related to Theorem 23 [8].

Theorem (Markowsky). A finite lattice is isomorphic to the row space of an \(n \times n\) Boolean matrix if and only if it has at most \(n\) elements in its basis and at most \(n\) elements in its dual basis.

The argument goes by constructing an actual Boolean matrix by letting rows correspond to basis elements, columns to dual basis elements, and letting the \((i, j)\) entry be 1 if and only if a basis element is not less than a dual basis element. This gives an \(n \times n\) Boolean matrix whose rows and columns have the given relationship. But some other, possibly larger Boolean matrix represents the finite lattice. However the two Boolean matrices must be D-equivalent because they reflect the same relations between generators and dual generators. Dependent rows and columns will
not affect the row space lattice, or the D-class.

Work of P. Butkovič and F. Hevery [2] on strong regularity has essentially shown that determining whether a $k \times k$ tropical matrix has maximal rank can be done in polynomial time, using an algorithm for the assignment problem, and then studying the 0 and nonzero values of the final matrix to determine whether a permuted diagonal of zero entries is unique. Another case for which a polynomial time algorithm exists is the following.

Proposition 11. For $k \times n, k < n$ tropical matrices whose entries have two distinct values, such that the higher of these occurs in every column there is a polynomial time algorithm ($O(n^3)$) to decide whether they have tropical rank $k$.

Proof. As in [7] this problem is equivalent to find an increasing chain of length $k$ in the nonzero column space of the corresponding Boolean matrix. This chain must begin with a vector having one 1 entry and each time some 1 entry must be added outside the previous set of 1 entries in the sequence. This can be decided by first listing all distinct columns with a single 1, then in turn all columns having a single 1 outside the set of columns previously listed. If this process continues for $k$ stages then it produces, by sums over the initial segments of the sequence, an increasing chain of length $k$. If the process ever stops with a set $S$ of 1 entries then this chain $C$ cannot produce a solution, and also any other chain $C_1$ of that length starting with a column with a single 1 must terminate, for if not, at some point a column of $C_1$ could be inserted to add exactly one 1 outside the columns of $C$. That is because the set of 1 entries in $C$ grows by 1 each time from a single 1 entry, to all possible 1 entries. ✷

This generalizes to size $k + c$ or rank $k - c$ for each constant $c$: just consider all sets of $k$ rows and test them as above. The assumption on the lower value occurring in each column can also be removed.

By a principal triangular submatrix of size $k$ in a matrix $M$, we mean a submatrix which is conjugated by some permutation $P$ to the locations $(i,j) | i,j \leq k$ and that in these locations $PMP^T$ is 1 above the main diagonal.

Lemma 12. The problem of finding a maximal triangular principal submatrix in a given $(0,1)$-matrix is NP-complete.

Proof. In an undirected digraph this is equivalent to the NP-complete complete subgraph problem.

Theorem 13. Determining whether $n \times n$ Boolean matrices have tropical rank at least $k$ is in general NP-complete.

Proof. We consider this in terms of the longest chain in the column space of the Boolean matrix. We restrict first to a special type of Boolean matrix. Vertices (indices of the matrix) are divided into 2 sets and columns have 1 vertex in the first set and 2 vertices in the second set. This is regarded as an edge coloring of a graph where the last 2 vertices specify the edge and the first vertex specifies the color. Any element minimally greater than another element in the row space can be obtained by adding some basis element, i.e. some row, to the latter. Then the problem of finding a longest chain in the row space is to find as long a sequence of edges as possible such that each new edge either adds a new vertex or a new color. In general it is then optimal to either add only one new vertex
with existing colors or one new color with existing vertices.

The count of edges occurring in the sequence will give the size of the resulting chain in the row space.

Now consider a graph made up entirely of disjoint cycles whose edges are colored various colors. Suppose, by taking multiple edges of some colors, that in each cycle only one color is unique (occurs only on one edge of that particular cycle, though it may occur on other cycles), and these unique colors are all different from each other.

We assert that optimum chains can be made to have the following form: we add all but one edge in each cycle and then we add the remaining cycle if it has a new color. For, we can add all but one edge proceeding in sequence around the cycle, so that each edge adds a new vertex, and so the sum gives a new point in the row space. And we can then add the final edge in that cycle with an increase if and only if it has a new color. If we leave out more than one edge in a cycle, we might gain some new colors for final edges elsewhere, but the net effect will be no better than filling in all but one edge, since at most one edge is gained, one is lost in so doing.

We may as well fill in the entire cycles in the order in which we fill in their final edges, and then later do any cycles in arbitrary order, for which we do not fill in their final cycles. That is, the order of filling in non-final edges among different cycles does not matter, and we may as well have as few as possible of them at each stage. So then a cycle need not be started until the final edges prior to its final edge are put in place.

Now form a matrix to represent the possible priorities. Among the cycles which are to have some final edge, we can allow cycle $i$ to precede cycle $j$ if and only if cycle $i$ does not contain the unique color for cycle $j$ (this color will be multiple for cycle $i$, and off the main diagonal if it exists). In this case make the $(i, j)$ entry of the matrix equal to 1. This matrix can be arbitrary except for 0 on the main diagonal, by choosing the non-unique colors on each cycle.

Then we can have final edges $k_1, k_2, \ldots, k_n$ inserted if and only if all the $(k_i, k_j)$ entries of this matrix are 1, for $i < j$. This means existence of a triangular submatrix in the sense mentioned above (with some convention on the main diagonal), and by the lemma it is NP-complete. □

6 GCD and LCM

In any tropical division problem of dividing polynomials in one variable, $d$ into $s$, we can produce a least possible superquotient $q$ such that $dq$ is greater than or equal to $s$ just by the fact that the minimum of two superquotients is also a superquotient by distributivity of products over minimum.

The least superquotient in dividing $\sum_{0}^{n} a_i x^i$ into $\sum_{0}^{m} b_i x_i$ can be computed explicitly by the requirements that its coefficients $c_i$ are the least numbers which satisfy $b_i \leq \inf_k (a_{i-k} + c_k)$ so $c_k = \sup_i (b_i - a_{i+k})$.

Example. For two polynomials of the same degree, the least superquotient represents the least translate raising one above the other.
For the same reason, any two nonnegative tropical polynomials will have a unique least nonnegative tropical common multiple in each degree for which a common multiple exists, because the infimum of two common multiples is a common multiple, by the infimum of the corresponding quotients. A nonnegative tropical common multiple of finite degree is given by the product. The degree is chosen to be minimal.

In terms of degree there is a little paradox about minimal polynomials, insofar as a minimal polynomial of larger degree would be smaller, and so for this reason it may be preferable to fix the degrees in constructions involving superquotients.

We believe the following algorithm will converge rapidly to the tropical least common multiple, or else it will be possible to tell rapidly that convergence will not occur in the given degree, but we do not have a proof that it is polynomial time in the degree and sizes of the coefficients. In fact over the rationals one can note denominators are bounded, and it is likely that in projective space, the system remains within a bounded region which must become eventually periodic. In fact the maximum slope of the convex hull at each stage should be bounded by that in \( f \) and \( g \): for products this follows by Prop.1 and for superquotients by a similar argument. Given a fixed range of degrees this means modulo translation that the construction stays within a fixed rectangle.

Assume the constant terms are 0. We try different possibilities for the degree of the L.C.M. In each case we then write out the initial trial \( h \) for the L.C.M. as the tropical polynomial which has all coefficients 0 up to this degree; this will be a lower bound. Repeat the following process until it converges or can be seen to increase at a constant rate indefinitely: take the minimal superquotient of \( h \) by \( f \), and then multiply the result by \( f \). Take that as the new \( h \); the L.C.M must be at least that great by monotonicity of superquotient. Then take the minimal superquotient of the new \( h \) by \( g \), and multiply that by \( g \) take that as the new superquotient. Coefficients never decrease in this process. At each step before convergence, at least one coefficient must increase. If there is a common multiple in this degree, step by step the common multiple should remain above this.

Definition. The tropical g.c.d. of two nonnegative tropical polynomials \( f, g \) in one variable is the least superquotient of \( fg \) by their least common multiple.

The g.c.d of polynomials over a tropical semiring is not in general unique. We can take an example of nonunique factorization of some polynomial \( f \) and compare \( f \) with the product of the two factors. For example, with Boolean polynomials, both \( 1 + x^n \) and \( 1 + x^{n+1} \) divide \( f = 1 + x + x^2 + \ldots + x^{2n+1} \) but their product uniquely factors and does not divide \( f \).

### 7 Algorithms for factoring

In the Boolean case one reasonable choice of a factoring algorithm is a straightforward branch and bound algorithm based on choosing the two factors in degrees starting with 1 and increasing, and stopping any branch
when a product is not contained in the polynomial $g(x)$ to be factored. As far as we know this may have exponential time in the worst case; it would be of some interest in this respect to know what Boolean prime polynomials of given degree have the maximum number of ones. If $g(x)$ is either sparse or if the set of degrees missing from it is sparse, then this algorithm can probably be improved. In the sparse case one might compare $g(x)$ with its translates to see when the intersections are large, and this gives a likely degree for some term in a factor. In the case where the complementary set is sparse, for each sum of a pair which could produce a missing degree, choose one of the two summands to be missing from a factor, as consistently as possible. This is likely to produce a factorization, and can be made into a branch and bound algorithm with at most exponential time.

For factoring tropical polynomials, it seems from the above, that one will have to consider degrees that are not too large; perhaps degree 20 might be the most that could reasonably be done on a personal computer in the general case. If the degree is fixed, then the problem of factoring becomes a straightforward question of linear programming, which can be done in polynomial time using a homotopy algorithm over $Q^+$. Integer linear programming is NP-complete, and it is not clear whether for fixed degree, factorization using only integer coefficients can be done in polynomial time, though the linear inequalities here are comparatively simple.

Example. This example is to show that for equal degree equal slope concave factorizations, it can happen that an integer polynomial is the square of a fractional polynomial but not an integer polynomial. Consider first a case where $a_0 = a_{200} = 0, a_5 = a_{105} = a_6 = 2, a_{10} = 5, a_9 = 4$, and whenever any coefficient except $a_0, a_{10}$ is less than 24 the coefficients in degree exactly 5 higher is 4 greater than that coefficient until a value of at least 20 is reached, and all other coefficients are 24. The square of this is a degree 400 polynomial which is totally even. However in any square factorization the coefficient $c_{20} = 10$ must be produced as the square of the $a_{10}$ term; $c_{30}$ and $c_{40}$ being 24, as well as $c_j, j = 1, 2, 3, 4, 7, 8, 211, 212, 213, 214$ imply there is no other product into degree 20 which could produce a value of 10 than $a_5 + b_5$. These calculations can be checked by computer.

It seems however that if a factorization of a tropical polynomial over $Z$ exists over $Q$, not requiring the factors to be equal, then there will be a factorization of the polynomial over $Z$. This factorization is obtained simply by rounding all fractions up to the next higher integer in one factor, and rounding them down to the next lower integer in the other. Suppose that a term in the old product is given by the sum of two fractional entries, then it is the sum of the integer parts plus an integer which is a sum of two positive fractions which are less than 1, so the sum of the fractional parts is 1, which the new procedure also gives for that sum. Any other sum of one or two fractional entries which previously was at least this great will still be, because a sum of two positive fractions less than 1, is replaced by 1, and if that is a decrease, it will be the next lower integer. Suppose a term in the old product is the sum of two integer entries in the old factorization, and is not a sum of two fractional entries. The only way this rounding procedure could affect that is if that integer was less that
the sum of the fractional entries previously, but now exceeds them. But again a sum of two positive fractions less than 1 is replaced by 1; if this is a decrease it must be the next lower integer, and there can be no change.

We propose a branch and bound algorithm that in each degree $k$ of the product polynomial in turn chooses one of the possible sums $a_i + b_{k-i}$ to be $c_k$. It uses this information to deduce inequalities about the $a, b$; and uses that to either terminate a branch or restrict future choices when possible. It might be simplest to consider the concave case. Then it will be best to start simultaneously from both extreme ends, since there will be few choices, and these will help determine later choices. There will also be fewer choices for those coefficients $c$ which are closest above the convex hull of the diagram. This algorithm will be conceivably factorial time in the degree, and polynomial in the size of the coefficients given that degree.

The following result is perhaps not what it seems, but can be combined with other algorithms to improve them.

**Theorem 14.** For tropical polynomials of any given positive degree $n$ in each of $x, y$ separately, with all coefficients up to the maximal degree in each variable being present (finite) there is an algorithm which provably produces an optimal factorization with probability 1 in polynomial time, that is, the exceptional set is a finite union of algebraic sets of codimension at least 1, and a number of arithmetic operations is involved which is polynomial in $n$.

**Proof.** It suffices to give a polynomial time algorithm which will prove irreducibility except on such a codimension 1 subset. But this can be done trivially by looking at the 4 corners, degrees $(0,0), (0,2n), (2n,0), (2n,2n)$. 

A study of the convex case and convex hulls in general should produce better criteria. In particular, it seems that if one flattens out the convex hulls in particular directions, the flattened and simplified convex hulls must also factor. If one has total concavity then the 8 outer edges of the product determine the 4 edges of the factors up to choice of which is in which factor, and this should determine inner edges.

### 8 Conclusion

In the general case factorization of polynomials even in one variable over the tropical algebra is quite difficult and algorithms will be practical only for small degrees or in special cases. This is true even if coefficients are 0 and infinity. For bounded degree, factorization is quite possible, and many irreducibility conditions can be derived. Problems of both factorization and tropical rank are NP-complete.

There is a polynomial time parallel algorithm for factorization over the tropical semiring, since parallel polynomial time is equivalent to PSPACE which contains NP. (Any factorization must wind up with factors which as numbers, involve a polynomial number of digits). If, as conjectured, quantum P time is strictly a subset of NP there cannot be a quantum polynomial time algorithm for factoring tropical or Boolean polynomials, unlike for the problem of factoring integers.
We have given algorithms which in a certain sense statistically factor most tropical polynomials in 2 variables polynomial time, by proving them irreducible. We have suggested directions for branch and bound algorithms to extend these.

For Boolean two variable polynomials $h$ which are products of polynomials $f, g$ each having degree $n$ in $x, y$ separately, a simple counting argument, unlike in the 1 variable case, shows that the proportion of irreducible polynomials tends to 1. It would be of interest to find a simple criterion which in polynomial time can prove irreducibility with probability approaching 1 as $n \to \infty$.

The average polynomial $h$ will have about half of its coefficients 1 and the other half zero. For these, it will probably be the case that $f, g$ must have a much smaller proportion of their coefficients 1, something like the square roots of their degrees. It might be possible to prove some result of this kind using existing combinatorial techniques, but it seems far from trivial. Given this, knowledge of the sparsity of nonzero terms in $f, g$, a branch and bound algorithm which starts with the lowest degrees and assigns coefficients in $h$ (assumed to have constant term 1) alternately to $f$ and to $g$ might proceed somewhat rapidly because of the high likelihood of finding a product term which does not actually exist in $h$. If $B_k$ branches survive to level $k$ (which is the number of terms found so far) for $h$ of degree $n$ in $x, y$, then the number of branches which survive to the next level can be estimated as $B_k 2^{-[k/2]} (n^2/2 - k + 1)$ since we can choose the new term in about $n^2/2 - k + 1$ ways, allowing that the current term in $f$ could be repeated in $g$ and $[k/2]$ new products with it must be a 1 coefficient, and the probability of this happening could be estimated as $1/2$. This should take time about $n^{C \log n}$ to complete since as soon as $2^k$ exceeds $(n^2)^2$ the series will converge rapidly.

However this algorithm and estimate can be greatly improved by choosing the terms in order according to the maximum of the degrees in $x, y$ separately, then the minimum degree and variable, and moreover using the previous term in the factor as a lower bound, and the next unfactored term in the product as an upper bound. In this case, as soon as we have about degree $C (\log(n))^2$ in the product we would expect to have $C_1 \log(n)$ terms in the factors, and we might expect $k$ could be taken about $\log \log(n)$. This algorithm could be polynomial time in most cases. It would be however difficult to prove this behavior without making some statistical hypotheses.

Open problems:
1. Is a random Boolean polynomial of large degree in 1 variable more likely to be prime or factorable?
2. Is there a polynomial time algorithm to find a nontrivial common factor of two tropical polynomials, given that such a factor exists? Both eliminants and L.C.M. give necessary conditions.

Those readers interested in other work on tropical algebra should see the website www.arxiv.org where this paper was posted; additional open problems are listed at www.math.umn.edu/~develin/tropicalproblems.html.
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