Monogenic functions in finite-dimensional commutative associative algebras

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Let $A_{m}^{n}$ be an arbitrary $n$-dimensional commutative associative algebra over the field of complex numbers with $m$ idempotents. Let $e_{1} = 1,e_{2},\ldots,e_{k}$ with $2 \leq k \leq 2n$ be elements of $A_{m}^{n}$ which are linearly independent over the field of real numbers. We consider monogenic (i. e. continuous and differentiable in the sense of Gateaux) functions of the variable $\sum_{j=1}^{k}x_{j}e_{j}$, where $x_{1},x_{2},\ldots,x_{k}$ are real, and obtain a constructive description of all mentioned functions by means of holomorphic functions of complex variables. It follows from this description that monogenic functions have Gateaux derivatives of all orders. The present article is generalized of the author’s paper [1], where mentioned results are obtained for $k = 3$.

1 Introduction

Apparently, W. Hamilton (1843) made the first attempts to construct an algebra associated with the three-dimensional Laplace equation

$$\Delta_{3}u(x,y,z) := \left( \frac{\partial^{2}}{\partial x^{2}} + \frac{\partial^{2}}{\partial y^{2}} + \frac{\partial^{2}}{\partial z^{2}} \right) u(x,y,z) = 0$$

meaning that components of hypercomplex functions satisfy the equation (1). He constructed an algebra of noncommutative quaternions over the field of real numbers $\mathbb{R}$ and made a base for developing the hypercomplex analysis.

C. Segre [2] constructed an algebra of commutative quaternions over the field $\mathbb{R}$ that can be considered as a two-dimensional commutative semi-simple algebra of bicomplex numbers over the field of complex numbers $\mathbb{C}$. M. Futagawa [3] and J. Riley [4] obtained a constructive description of analytic function of a bicomplex variable, namely, they proved that such an analytic function can be constructed with an use of two holomorphic functions of complex variables.

F. Ringleb [5] and S. N. Volovel’skaya [6, 7] succeeded in developing a function theory for noncommutative algebras with unit over the real or complex fields, by pursuing a definition of the differential of a function on such an algebra suggested by Hausdorff in [8]. These definitions make the \textit{a priori} severe requirement that the coordinates of the function have continuous first derivatives with respect to the coordinates of the argument element. Namely, F. Ringleb [5] considered an arbitrary finite-dimensional associative (commutative or not) \textit{semi-simple} algebra over the field $\mathbb{R}$. For given class of functions which maps the mentioned algebra onto itself, he obtained a constructive description by means of real and complex analytic functions.
S. N. Volovel’skaya developed the Hausdorff’s idea defining the monogenic functions on non-semisimple associative algebras and she generalized the Ringleb’s results for such algebras. In the paper [3] was obtained a constructive description of monogenic functions in a special three-dimensional non-commutative algebra over the field \( \mathbb{R} \). The results of paper [6] were generalized in the paper [7] where Volovel’skaya obtained a constructive description of monogenic functions in non-semisimple associative algebras of the first category over \( \mathbb{R} \).

A relation between spatial potential fields and analytic functions given in commutative algebras was established by P. W. Ketchum [9] who shown that every analytic function \( \Phi(\zeta) \) of the variable \( \zeta = xe_1 + ye_2 + ze_3 \) satisfies the equation (1) in the case where the elements \( e_1, e_2, e_3 \) of a commutative algebra satisfy the condition

\[
e_1^2 + e_2^2 + e_3^2 = 0,
\]

because

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \Phi''(\zeta) (e_1^2 + e_2^2 + e_3^2) = 0,
\]

where \( \Phi'' := (\Phi')' \) and \( \Phi'(\zeta) \) is defined by the equality \( d\Phi = \Phi'(\zeta) d\zeta \).

We say that a commutative associative algebra \( A \) is harmonic (cf. [9, 10, 11]) if in \( A \) there exists a triad of linearly independent vectors \( \{e_1, e_2, e_3\} \) satisfying the equality (2) with \( e_k^2 \neq 0 \) for \( k = 1, 2, 3 \). We say also that such a triad \( \{e_1, e_2, e_3\} \) is harmonic.

P. W. Ketchum [9] considered the C. Segre algebra of quaternions [2] as an example of harmonic algebra.

Further M. N. Roşcuţelţ establishes a relation between monogenic functions in commutative algebras and partial differential equations. He defined monogenic functions \( f \) of the variable \( w \) by the equality \( df(w) = 0 \). So, in the paper [12] M. N. Roşcuţelţ proposed a procedure for constructing an infinite-dimensional topological vector space with commutative multiplication such that monogenic functions in it are the all solutions of the equation

\[
\sum_{\alpha_0+\alpha_1+\ldots+\alpha_p=N} C_{\alpha_0,\alpha_1,\ldots,\alpha_p} \frac{\partial^{N} \Phi}{\partial x_0^{\alpha_0} \partial x_1^{\alpha_1} \ldots \partial x_p^{\alpha_p}} = 0,
\]

with \( C_{\alpha_0,\alpha_1,\ldots,\alpha_p} \in \mathbb{R} \). In particular, such infinite-dimensional topological vector space are constructed for the Laplace equation (3). In the paper [13] Roşcuţelţ finds a certain connection between monogenic functions in commutative algebras and systems of partial differential equations.

I. P. Mel’nichenko proposed for describing solutions of the equation (4) to use hypercomplex functions differentiable in the sense of Gateaux, since in this case the conditions of monogenic are the least restrictive. He started to implement this approach with respect to the thee-dimensional Laplace equation (3) (see [10]). Mel’nichenko proved that there exist exactly 3 three-dimensional harmonic algebras with unit over the field \( \mathbb{C} \) (see [10, 14, 11]).

In the paper [15], the authors develop the Melnichenko’s idea for the equation (4), and considered several examples.
The investigation of partial differential equations using the hypercomplex methods is effective if hypercomplex monogenic (in any sense) functions can be constructed explicitly. On this way the following results are obtained.

Constructive descriptions of monogenic (i.e. continuous and differentiable in the sense of Gateaux) functions taking values in the mentioned three-dimensional harmonic algebras by means three corresponding holomorphic functions of the complex variable are obtained in the papers [16, 17, 18]. Such descriptions make it possible to prove the infinite differentiability in the sense of Gateaux of monogenic functions and integral theorems for these functions that are analogous to classical theorems of the complex analysis (see, e.g., [19, 20]).

Furthermore, constructive descriptions of monogenic functions taking values in special $n$-dimensional commutative algebras by means $n$ holomorphic functions of complex variables are obtained in the papers [21, 22].

In the paper [1], by author is obtained a constructive description of all monogenic functions of the variable $x_1 e_1 + x_2 e_2 + x_3 e_3$ taking values in an arbitrary $n$-dimensional commutative associative algebra with unit by means of holomorphic functions of complex variables. It follows from this description that monogenic functions have Gateaux derivatives of all orders.

In this paper we extend the results of the paper [1] to monogenic functions of the variable $\sum_{r=1}^{k} x_r e_r$, where $2 \leq k \leq 2n$.

2 The algebra $\mathbb{A}^m_n$

Let $\mathbb{N}$ be the set of natural numbers. We fix the numbers $m, n \in \mathbb{N}$ such that $m \leq n$. Let $\mathbb{A}^m_n$ be an arbitrary commutative associative algebra with unit over the field of complex number $\mathbb{C}$. E. Cartan [23, p. 33] proved that there exist a basis $\{I_r\}_{r=1}^n$ in $\mathbb{A}^m_n$ satisfying the following multiplication rules:

1. $\forall r, s \in [1, m] \cap \mathbb{N} : \quad I_r I_s = \begin{cases} 0 & \text{if } r \neq s, \\ I_r & \text{if } r = s; \end{cases}$

2. $\forall r, s \in [m+1, n] \cap \mathbb{N} : \quad I_r I_s = \sum_{p=\max\{r,s\}+1}^{n} \Upsilon_{r,p} I_p ;$

3. $\forall s \in [m+1, n] \cap \mathbb{N} \quad \exists u_s \in [1, m] \cap \mathbb{N} \quad \forall r \in [1, m] \cap \mathbb{N} : \quad I_r I_s = \begin{cases} 0 & \text{if } r \neq u_s, \\ I_s & \text{if } r = u_s. \end{cases}$ (5)

Moreover, the structure constants $\Upsilon_{r,p} \in \mathbb{C}$ satisfy the associativity conditions:

(A 1). $(I_r I_s) I_p = I_r (I_s I_p) \quad \forall r, s, p \in [m+1, n] \cap \mathbb{N};$

(A 2). $(I_u I_s) I_p = I_u (I_s I_p) \quad \forall u \in [1, m] \cap \mathbb{N} \quad \forall s, p \in [m+1, n] \cap \mathbb{N}.$

Obviously, the first $m$ basic vectors $\{I_u\}_{u=1}^m$ are idempotents and form a semi-simple subalgebra of the algebra $\mathbb{A}^m_n$. The vectors $\{I_r\}_{r=m+1}^n$ form a nilpotent
subalgebra of the algebra $A^m_n$. The element $1 = \sum_{u=1}^{m} I_u$ is the unit of $A^m_n$.

In the cases where $A^m_n$ has some specific properties, the following propositions are true.

**Proposition 1** [1]. If there exists the unique $u_0 \in [1, m] \cap \mathbb{N}$ such that $I_{u_0} I_s = I_s$ for all $s = m + 1, \ldots, n$, then the associativity condition $(A \, 1)$ is satisfied.

Thus, under the conditions of Proposition 1, the associativity condition $(A \, 1)$ is only required. It means that the nilpotent subalgebra of $A^m_n$ with the basis $\{I_r\}_{r=m+1}^{n}$ can be an arbitrary commutative associative nilpotent algebra of dimension $n - m$. We note that such nilpotent algebras are fully described for the dimensions $1, 2, 3$ in the paper [24], and some four-dimensional nilpotent algebras can be found in the papers [25], [26].

**Proposition 2** [1]. If all $u_r$ are different in the multiplication rule 3, then $I_s I_p = 0$ for all $s, p = m + 1, \ldots, n$.

Thus, under the conditions of Proposition 2, the multiplication table of the nilpotent subalgebra of $A^m_n$ with the basis $\{I_r\}_{r=m+1}^{n}$ consists only of zeros, and all associativity conditions are satisfied.

The algebra $A^m_n$ contains $m$ maximal ideals

$$I_u := \left\{ \sum_{r=1, r \neq u}^{n} \lambda_r I_r : \lambda_r \in \mathbb{C} \right\}, \quad u = 1, 2, \ldots, m,$$

and their intersection is the radical

$$R := \left\{ \sum_{r=m+1}^{n} \lambda_r I_r : \lambda_r \in \mathbb{C} \right\}.$$

Consider $m$ linear functionals $f_u : A^m_n \to \mathbb{C}$ satisfying the equalities

$$f_u(I_u) = 1, \quad f_u(\omega) = 0 \quad \forall \omega \in I_u, \quad u = 1, 2, \ldots, m.$$

Inasmuch as the kernel of functional $f_u$ is the maximal ideal $I_u$, this functional is also continuous and multiplicative (see [27, p. 147]).

### 3 Monogenic functions

Let us consider the vectors $e_1 = 1, e_2, \ldots, e_k$ in $A^m_n$, where $2 \leq k \leq 2n$, and these vectors are linearly independent over the field of real numbers $\mathbb{R}$ (see [22]). It means that the equality

$$\sum_{j=1}^{k} \alpha_j e_j = 0, \quad \alpha_j \in \mathbb{R},$$

holds if and only if $\alpha_j = 0$ for all $j = 1, 2, \ldots, k$.

Let the vectors $e_1 = 1, e_2, \ldots, e_k$ have the following decompositions with respect to the basis $\{I_r\}_{r=1}^{n}$:

$$e_1 = \sum_{r=1}^{m} I_r, \quad e_j = \sum_{r=1}^{n} a_{jr} I_r, \quad a_{jr} \in \mathbb{C}, \quad j = 2, 3, \ldots, k. \quad (6)$$

3
Let \( \zeta := \sum_{j=1}^{k} x_j e_j \), where \( x_j \in \mathbb{R} \). It is obvious that
\[
\xi_u := f_u(\zeta) = x_1 + \sum_{j=2}^{k} x_j a_{ju}, \quad u = 1, 2, \ldots, m.
\]

Let \( E_k := \{ \zeta = \sum_{j=1}^{k} x_j e_j : x_j \in \mathbb{R} \} \) be the linear span of vectors \( e_1 = 1, e_2, \ldots, e_k \) over the field \( \mathbb{R} \).

Let \( \Omega \) be a domain in \( E_k \). With a domain \( \Omega \subset E_k \) we associate the domain
\[
\Omega_{\mathbb{R}} := \{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : \zeta = \sum_{j=1}^{k} x_j e_j \in \Omega \}
\]
in \( \mathbb{R}^k \).

We say that a continuous function \( \Phi : \Omega \to \mathbb{A}^m_{\mathbb{R}} \) is monogenic in \( \Omega \) if \( \Phi \) is differentiable in the sense of Gateaux in every point of \( \Omega \), i.e. if for every \( \zeta \in \Omega \) there exists an element \( \Phi'(\zeta) \in \mathbb{A}^m_{\mathbb{R}} \) such that
\[
\lim_{\varepsilon \to 0+0} \frac{\Phi(\zeta + \varepsilon h) - \Phi(\zeta)}{\varepsilon} = h \Phi'(\zeta) \quad \forall h \in E_k.
\]
(7)

\( \Phi'(\zeta) \) is the Gateaux derivative of the function \( \Phi \) in the point \( \zeta \).

Consider the decomposition of a function \( \Phi : \Omega \to \mathbb{A}^m_{\mathbb{R}} \) with respect to the basis \( \{I_r\}_{r=1}^{n} \):
\[
\Phi(\zeta) = \sum_{r=1}^{n} U_r(x_1, x_2, \ldots, x_k) I_r.
\]
(8)

In the case where the functions \( U_r : \Omega_{\mathbb{R}} \to \mathbb{C} \) are \( \mathbb{R} \)-differentiable in \( \Omega_{\mathbb{R}} \), i.e. for every \( (x_1, x_2, \ldots, x_k) \in \Omega_{\mathbb{R}} \),
\[
U_r(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_k + \Delta x_k) - U_r(x_1, x_2, \ldots, x_k) =
= \sum_{j=1}^{k} \frac{\partial U_r}{\partial x_j} \Delta x_j + o \left( \sqrt{\sum_{j=1}^{k} (\Delta x_j)^2} \right), \quad \sum_{j=1}^{k} (\Delta x_j)^2 \to 0,
\]
the function \( \Phi \) is monogenic in the domain \( \Omega \) if and only if the following Cauchy–Riemann conditions are satisfied in \( \Omega \):
\[
\frac{\partial \Phi}{\partial x_j} = \frac{\partial \Phi}{\partial x_1} e_j \quad \text{for all} \quad j = 2, 3, \ldots, k.
\]
(9)

4 An expansion of the resolvent

Let \( b := \sum_{r=1}^{n} b_r I_r \in \mathbb{A}^m_{\mathbb{R}} \), where \( b_r \in \mathbb{C} \), and we note that \( f_u(b) = b_u, \quad u = 1, 2, \ldots, m \). It follows form the Lemmas 1, 3 of [11] that
\[
b^{-1} = \sum_{u=1}^{m} \frac{1}{b_u} I_u + \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} \frac{Q_{r,s}}{b_{u_s}'} I_s.
\]
(10)
where $\tilde{Q}_{r,s}$ are determined by the following recurrence relations:

$$
\tilde{Q}_{2,s} := b_s, \quad \tilde{Q}_{r,s} = \sum_{q=r+m-2}^{s-1} \tilde{Q}_{r-1,q} \tilde{B}_{q,s}, \quad r = 3, 4, \ldots, s - m + 1, 
$$

(11)

$$
\tilde{B}_{q,s} := \sum_{p=m+1}^{s-1} b_p \gamma_{q,s}^p, \quad p = m + 2, m + 3, \ldots, n, 
$$

(12)

and the natural numbers $u_s$ are defined in the rule 3 of the multiplication table of algebra $A_{mn}^m$.

In the next lemma we find an expansion of the resolvent $(te_1 - \zeta)^{-1}$.

Лемма 1. An expansion of the resolvent is of the form

$$
(te_1 - \zeta)^{-1} = \sum_{u=1}^{m} \frac{1}{t - \xi_u} I_u + \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} \frac{Q_{r,s}}{(t - \xi_u)} I_s 
$$

(13)

$$
\forall t \in \mathbb{C} : t \neq \xi_u, \quad u = 1, 2, \ldots, m,
$$

where the coefficients $Q_{r,s}$ are determined by the following recurrence relations:

$$
Q_{2,s} = T_s, \quad Q_{r,s} = \sum_{q=r+m-2}^{s-1} Q_{r-1,q} B_{q,s}, \quad r = 3, 4, \ldots, s - m + 1, 
$$

(14)

with

$$
T_s := \sum_{j=2}^{k} x_j a_{js}, \quad B_{q,s} := \sum_{p=m+1}^{s-1} T_p \gamma_{q,s}^p, \quad p = m + 2, m + 3, \ldots, n, 
$$

(15)

and the natural numbers $u_s$ are defined in the rule 3 of the multiplication table of algebra $A_{mn}^m$.

**Proof.** Taking into account the decomposition

$$
t e_1 - \zeta = \sum_{u=1}^{m} (t - \xi_u) I_u - \sum_{r=m+1}^{n} \sum_{j=2}^{k} x_j a_{js} I_r,
$$

we conclude that the relation (13) follows directly from the equality (10) in which instead of $b_u$, $u = 1, 2, \ldots, m$ it should be used the expansion $t - \xi_u$, and instead of $b_s$, $s = m+1, m+2, \ldots, n$ it should be used the expansion $\sum_{j=2}^{k} x_j a_{js}$. The lemma is proved.

It follows from Lemma 1 that the points $(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ corresponding to the noninvertible elements $\zeta = \sum_{j=1}^{k} x_j e_j$ form the set

$$
M^R_u : \begin{cases}
    x_1 + \sum_{j=2}^{k} x_j \text{Re} a_{ju} = 0, \\
    \sum_{j=2}^{k} x_j \text{Im} a_{ju} = 0, \quad u = 1, 2, \ldots, m
\end{cases}
$$
in the \(k\)-dimensional space \(\mathbb{R}^k\). Also we consider the set \(M_u := \{\zeta \in E_k : f_u(\zeta) = 0\}\) for \(u = 1, 2, \ldots, m\). It is obvious that the set \(M^R_u \subset \mathbb{R}^k\) is congruent with the set \(M_u \subset E_k\).

5 A constructive description of monogenic functions

We say that a domain \(\Omega \subset E_k\) is convex with respect to the set of directions \(M_u\) if \(\Omega\) contains the segment \(\{\zeta_1 + \alpha(\zeta_2 - \zeta_1) : \alpha \in [0, 1]\}\) for all \(\zeta_1, \zeta_2 \in \Omega\) such that \(\zeta_2 - \zeta_1 \in M_u\).

Denote \(f_u(E_k) := \{f_u(\zeta) : \zeta \in E_k\}\). In what follows, we make the following essential assumption: \(f_u(E_k) = \mathbb{C}\) for all \(u = 1, 2, \ldots, m\). Obviously, it holds if and only if for every fixed \(u = 1, 2, \ldots, m\) at least one of the numbers \(a_{2u}, a_{3u}, \ldots, a_{ku}\) belongs to \(\mathbb{C} \setminus \mathbb{R}\).

**Lemma 2.** Suppose that a domain \(\Omega \subset E_k\) is convex with respect to the set of directions \(M_u\) and \(f_u(E_k) = \mathbb{C}\) for all \(u = 1, 2, \ldots, m\). Suppose also that a function \(\Phi : \Omega \to \mathbb{A}^m_u\) is monogenic in the domain \(\Omega\). If points \(\zeta_1, \zeta_2 \in \Omega\) such that \(\zeta_2 - \zeta_1 \in M_u\), then

\[
\Phi(\zeta_2) - \Phi(\zeta_1) \in \mathbb{I}_u. \tag{16}
\]

**Proof.** Inasmuch as \(f_u(E_k) = \mathbb{C}\), then there exists an element \(e_2^* \in E_k\) such that \(f_u(e_2^*) = i\). Consider the lineal span \(E^* := \{\zeta = xe_1^* + ye_2^* + ze_3^* : x, y, z \in \mathbb{R}\}\) of the vectors \(e_1^* := 1, e_2^*, e_3^* := \zeta_2 - \zeta_1\) and denote \(\Omega^* := \Omega \cap E^*\).

Now, the relations \((16)\) can be proved in such a way as Lemma 2.1 \([16]\), in the proof of which one must take \(\Omega^*, f_u, \{a_\alpha e_3^* : \alpha \in \mathbb{R}\}\) instead of \(\Omega, f, L\), respectively. Lemma \(2\) is proved.

Let a domain \(\Omega \subset E_k\) be convex with respect to the set of directions \(M_u\), \(u = 1, 2, \ldots, m\). By \(D_u\) we denote that domain in \(\mathbb{C}\) onto which the domain \(\Omega\) is mapped by the functional \(f_u\).

We introduce the linear operators \(A_u, u = 1, 2, \ldots, m\), which assign holomorphic functions \(F_u : D_u \to \mathbb{C}\) to every monogenic function \(\Phi : \Omega \to \mathbb{A}^m_u\) by the formula

\[
F_u(\xi_u) = f_u(\Phi(\zeta)), \tag{17}
\]

where \(\xi_u = f_u(\zeta) \equiv x_1 + \sum_{j=2}^{k} x_j a_{j_u}\) and \(\zeta \in \Omega\). It follows from Lemma \(2\) that the value \(F_u(\xi_u)\) does not depend on a choice of a point \(\zeta\) for which \(f_u(\zeta) = \xi_u\).

Now, similar to proof of Lemma 5 \([1]\) can be proved the following statement.

**Lemma 3.** Suppose that a domain \(\Omega \subset E_k\) is convex with respect to the set of directions \(M_u\) and \(f_u(E_k) = \mathbb{C}\) for all \(u = 1, 2, \ldots, m\). Suppose also that for any fixed \(u = 1, 2, \ldots, m\), a function \(F_u : D_u \to \mathbb{C}\) is holomorphic in a domain \(D_u\) and \(\Gamma_u\) is a closed Jordan rectifiable curve in \(D_u\) which surrounds the point \(\xi_u\) and contains no points \(\xi_q, q = 1, 2, \ldots, m, q \neq u\). Then the function

\[
\Psi_u(\zeta) := I_u \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt \tag{18}
\]

is monogenic in the domain \(\Omega\).
Лемма 4. Suppose that a domain $\Omega \subset E_k$ is convex with respect to the set of directions $M_u$ and $f_u(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Suppose also that a function $V : \Omega_R \to \mathbb{C}$ satisfies the equalities

$$
\frac{\partial V}{\partial x_2} = \frac{\partial V}{\partial x_1} a_{2u}, \quad \frac{\partial V}{\partial x_3} = \frac{\partial V}{\partial x_1} a_{3u}, \quad \ldots, \quad \frac{\partial V}{\partial x_k} = \frac{\partial V}{\partial x_1} a_{ku} \tag{19}
$$

in $\Omega_R$. Then $V$ is a holomorphic function of the variable $\xi_u = f_u(\zeta) = x_1 + \sum_{j=2}^{k} x_j a_{ju}$ in the domain $D_u$.

**Proof.** We first separate the real and the imaginary part of the expression

$$
\xi_u = x_1 + \sum_{j=2}^{k} x_j \text{Re} a_{ju} + i \sum_{j=2}^{k} x_j \text{Im} a_{ju} =: \tau_u + i\eta_u \tag{20}
$$

and note that the equalities (19) yield

$$
\frac{\partial V}{\partial \eta_u} \text{Im} a_{2u} = i \frac{\partial V}{\partial \tau_u} \text{Im} a_{2u}, \quad \ldots, \quad \frac{\partial V}{\partial \eta_u} \text{Im} a_{ku} = i \frac{\partial V}{\partial \tau_u} \text{Im} a_{ku}. \tag{21}
$$

It follows from the condition $f_u(E_k) = \mathbb{C}$ that at least one of the numbers $\text{Im} a_{2u}, \text{Im} a_{3u}, \ldots, \text{Im} b_u$ is not equal to zero. Therefore, using (21), we get

$$
\frac{\partial V}{\partial \eta_u} = i \frac{\partial V}{\partial \tau_u}. \tag{22}
$$

Now we prove that $V(x_1', x_2', \ldots, x_k') = V(x_1'', x_2'', \ldots, x_k'')$ for points $(x_1', x_2', \ldots, x_k'), (x_1'', x_2'', \ldots, x_k'') \in \Omega$ such that the segment that connects these points is parallel to a straight line $L_u \subset M_u^\mathbb{R}$. To this end we use considerations with the proof of Lemma 2. Since $f_u(E_k) = \mathbb{C}$, then there exists an element $e_2^* \in E_k$ such that $f_u(e_2^*) = i$. Consider the lineal span $E^* := \{\zeta = xe_1^* + ye_2^* + ze_3^* : x, y, z \in \mathbb{R}\}$ of the vectors $e_1^* := 1, e_2^*, e_3^* := \zeta' - \zeta''$, where $\zeta' := \sum_{j=1}^{k} x_j^* e_j, \quad \zeta'' := \sum_{j=1}^{k} x''_j e_j$, and introduce the denotation $\Omega^* := \Omega \cap E^*$.

Now, the relation $V(x_1', x_2', \ldots, x_k') = V(x_1'', x_2'', \ldots, x_k'')$ can be proved in such a way as Lemma 6, in the proof of which one must take $\Omega^*, \{ae_3^* : a \in \mathbb{R}\}$ instead of $\Omega, L$, respectively. The lemma is proved.

Thus, a function $V : \Omega_R \to \mathbb{C}$ of the form $V(x_1, x_2, \ldots, x_k) := F(\xi_u)$, where $F(\xi_u)$ is an arbitrary function holomorphic in the domain $D_u$, is a general solution of the system (19). The lemma is proved.

**Теорема 1.** Suppose that a domain $\Omega \subset E_k$ is convex with respect to the set of directions $M_u$ and $f_u(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Then every monogenic function $\Phi : \Omega \to \mathbb{H}^m$ can be expressed in the form

$$
\Phi(\zeta) = \sum_{u=1}^{m} I_u \frac{1}{2\pi i} \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt + \sum_{s=m+1}^{n} I_s \frac{1}{2\pi i} \int_{\Gamma_u} G_s(t)(te_1 - \zeta)^{-1} dt, \tag{23}
$$

where $F_u$ and $G_s$ are certain holomorphic functions in the domains $D_u$ and $D_{us}$, respectively, and $\Gamma_q$ is a closed Jordan rectifiable curve in $D_q$ which surrounds the point $\xi_q$ and contains no points $\xi_\ell, \ell, q = 1, 2, \ldots, m, \ell \neq q$. 


Proof. We set
\[ F_u := A_u \Phi, \quad u = 1, 2, \ldots, m. \] (24)

Let us show that the values of monogenic function
\[ \Phi_0(\zeta) := \Phi(\zeta) - \sum_{u=1}^{m} I_u \frac{1}{2\pi i} \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt \] (25)

belong to the radical \( R \), i.e. \( \Phi_0(\zeta) \in R \) for all \( \zeta \in \Omega \). As a consequence of the equality (13), we have the equality
\[ I_u \frac{1}{2\pi i} \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt = I_u \frac{1}{2\pi i} \int_{\Gamma_u} \frac{F_u(t)}{t - \xi_u} dt + \]
\[ + \frac{1}{2\pi i} \sum_{s=m+1}^{n} \sum_{r=2}^{s} \int_{\Gamma_u} F_u(t)Q_{r,s}(t - \xi_u) dt I_s I_u, \]
from which we obtain the equality
\[ f_u \left( \sum_{u=1}^{m} I_u \frac{1}{2\pi i} \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt \right) = F_u(\xi_u). \] (26)

Operating onto the equality (25) by the functional \( f_u \) and taking into account the relations (17), (24), (26), we get the equality

\[ f_u(\Phi_0(\zeta)) = F_u(\xi_u) - F_u(\xi_u) = 0 \]

for all \( u = 1, 2, \ldots, m \), i.e. \( \Phi_0(\zeta) \in R \).

Therefore, the function \( \Phi_0 \) is of the form
\[ \Phi_0(\zeta) = \sum_{s=m+1}^{n} V_s(x_1, x_2, \ldots, x_k) I_s, \] (27)

where \( V_s : \Omega_R \to \mathbb{C} \), and the Cauchy – Riemann conditions (9) are satisfied with \( \Phi = \Phi_0 \). Substituting the expressions (6), (27) into the equality (9), we obtain

\[ \sum_{s=m+1}^{n} \frac{\partial V_s}{\partial x_2} I_s = \sum_{s=m+1}^{n} \frac{\partial V_s}{\partial x_1} I_s \sum_{r=1}^{n} a_{2r} I_r, \]

\[ \vdots \]

\[ \sum_{s=m+1}^{n} \frac{\partial V_s}{\partial x_k} I_s = \sum_{s=m+1}^{n} \frac{\partial V_s}{\partial x_1} I_s \sum_{r=1}^{n} a_{kr} I_r. \]

Equating the coefficients of \( I_{m+1} \) in these equalities, we obtain the following system of equations for determining the function \( V_{m+1}(x_1, x_2, \ldots, x_k) \):

\[ \frac{\partial V_{m+1}}{\partial x_2} = \frac{\partial V_{m+1}}{\partial x_1} a_{2w_{m+1}}, \quad \ldots, \quad \frac{\partial V_{m+1}}{\partial x_k} = \frac{\partial V_{m+1}}{\partial x_1} a_{kw_{m+1}}. \]
It follows from Lemma 4 that $V_{m+1}(x_1, x_2, \ldots, x_k) \equiv G_{m+1}(\xi_{um+1})$, where $G_{m+1}$ is a function holomorphic in the domain $D_{um+1}$. Therefore,

$$
\Phi_0(\zeta) = G_{m+1}(\xi_{um+1}) I_{m+1} + \sum_{s=m+2}^{n} V_s(x_1, x_2, \ldots, x_k) I_s. \tag{29}
$$

Due to the expansion (13), we have the representation

$$
I_{m+1} \frac{1}{2\pi i} \int_{\Gamma_{um+1}} G_{m+1}(t)(te_1 - \zeta)^{-1} dt = G_{m+1}(\xi_{um+1}) I_{m+1} + \Psi(\zeta), \tag{30}
$$

where $\Psi(\zeta)$ is a function with values in the set $\{ \sum_{s=m+2}^{n} \alpha_s I_s : \alpha_s \in \mathbb{C} \}$.

Now, consider the function

$$
\Phi_1(\zeta) := \Phi_0(\zeta) - I_{m+1} \frac{1}{2\pi i} \int_{\Gamma_{um+1}} G_{m+1}(t)(te_1 - \zeta)^{-1} dt.
$$

In view of the relations (29), (30), $\Phi_1$ can be represented in the form

$$
\Phi_1(\zeta) = \sum_{s=m+2}^{n} \tilde{V}_s(x_1, x_2, \ldots, x_k) I_s,
$$

where $\tilde{V}_s : \Omega_K \to \mathbb{C}$.

Inasmuch as $\Phi_1$ is a monogenic function in $\Omega$, the functions $\tilde{V}_{m+2}, \tilde{V}_{m+3}, \ldots, \tilde{V}_n$ satisfy the system (28), where $V_{m+1} \equiv 0$, $V_s = \tilde{V}_s$ for $s = m + 2, m + 3, \ldots, n$. Therefore, similarly to the function $V_{m+1}(x_1, x_2, \ldots, x_k) \equiv G_{m+1}(\xi_{um+1})$, the function $\tilde{V}_{m+2}$ satisfies the equations

$$
\frac{\partial \tilde{V}_{m+2}}{\partial x_2} = \frac{\partial \tilde{V}_{m+2}}{\partial x_1} a_{2um+2}, \ldots, \frac{\partial \tilde{V}_{m+2}}{\partial x_k} = \frac{\partial \tilde{V}_{m+2}}{\partial x_1} a_{kum+2}
$$

and is of the form $\tilde{V}_{m+2}(x_1, x_2, \ldots, x_k) \equiv G_{m+2}(\xi_{um+2})$, where $G_{m+2}$ is a function holomorphic in the domain $D_{um+2}$.

In such a way, step by step, considering the functions

$$
\Phi_j(\zeta) := \Phi_{j-1}(\zeta) - I_{m+j} \frac{1}{2\pi i} \int_{\Gamma_{um+j}} G_{m+j}(t)(te_1 - \zeta)^{-1} dt
$$

for $j = 2, 3, \ldots, n - m - 1$, we get the representation (23) of the function $\Phi$. The theorem is proved.

Taking into account the expansion (13), one can rewrite the equality (23) in the following equivalent form:

$$
\Phi(\zeta) = \sum_{u=1}^{m} F_u(\xi_u) I_u + \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} \frac{1}{(r-1)!} Q_{r,s} F_{as}^{(r-1)}(\xi_u) I_s + \sum_{q=m+1}^{n} G_q(\xi_{uq}) I_q + \sum_{q=m+1}^{n} \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} \frac{1}{(r-1)!} Q_{r,s} G_{aq}^{(r-1)}(\xi_u) I_q I_s. \tag{31}
$$
Thus, the equalities (23) and (31) specify methods to construct explicitly any monogenic functions \( \Phi : \Omega \rightarrow \mathbb{A}^m_n \) using \( n \) corresponding holomorphic functions of complex variables.

The following statement follows immediately from the equality (31) in which the right-hand side is a monogenic function in the domain \( \Pi := \{ \zeta \in E_k : f_u(\zeta) = D_u, u = 1, 2, \ldots, m \} \).

**Theorem 2.** Let a domain \( \Omega \subset E_k \) is convex with respect to the set of directions \( M_u \) and \( f_u(E_k) = \mathbb{C} \) for all \( u = 1, 2, \ldots, m \). Then every monogenic function \( \Phi : \Omega \rightarrow \mathbb{A}^m_n \) can be continued to a function monogenic in the domain \( \Pi \).

The next statement is a fundamental consequence of the equality (31), and it is true for an arbitrary domain \( \Omega \).

**Theorem 3.** Let \( f_u(E_k) = \mathbb{C} \) for all \( u = 1, 2, \ldots, m \). Then for every monogenic function \( \Phi : \Omega \rightarrow \mathbb{A}^m_n \) in an arbitrary domain \( \Omega \), the Gateaux \( r \)-th derivatives \( \Phi^{(r)} \) are monogenic functions in \( \Omega \) for all \( r \).

The proof is completely analogous to the proof of Theorem 4 [16].

Using the integral expression (23) of monogenic function \( \Phi : \Omega \rightarrow \mathbb{A}^m_n \) in the case where a domain \( \Omega \) is convex with respect to the set of directions \( M_u \), \( u = 1, 2, \ldots, m \), we obtain the following expression for the Gateaux \( r \)-th derivative \( \Phi^{(r)} \):

\[
\Phi^{(r)}(\zeta) = \sum_{u=1}^{m} I_u \frac{r!}{2\pi i} \int_{\Gamma_u} F_u(t) \left( (te_1 - \zeta)^{-1} \right)^{r+1} dt + \sum_{s=m+1}^{n} I_s \frac{r!}{2\pi i} \int_{\Gamma_s} G_s(t) \left( (te_1 - \zeta)^{-1} \right)^{r+1} dt \quad \forall \zeta \in \Omega.
\]

### 6 Remarks

We note that in the cases where the algebra \( \mathbb{A}^m_n \) has some specific properties (for instance, properties described in Propositions 1 and 2), it is easy to simplify the form of the equality (31).

1. In the case considered in Proposition 1, the following equalities hold:

\[
u_{m+1} = u_{m+2} = \ldots = u_n =: \eta.\]

In this case the representation (31) takes the form

\[
\Phi(\zeta) = \sum_{u=1}^{m} F_u(\xi_u) I_u + \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} \frac{1}{(r-1)!} Q_{r,s} F_{s}^{(r-1)}(\xi_\eta) I_s + \sum_{s=m+1}^{n} G_s(\xi_\eta) I_s + \sum_{q=m+1}^{n} \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} \frac{1}{(r-1)!} Q_{r,s} G_{q}^{(r-1)}(\xi_\eta) I_s I_q. \quad (32)
\]

The formula (32) generalizes representations of monogenic functions in both three-dimensional harmonic algebras (see [16, 17, 18]) and specific \( n \)-dimensional
algebras (see [21, 22]) to the case of algebras more general form and to a variable of more general form.

2. In the case considered in Proposition 2, the representation (23) takes the form

\[
\Phi(\zeta) = \sum_{u=1}^{m} F_u(\xi_u)I_u + \sum_{s=m+1}^{n} G_s(\xi_s)I_s + \sum_{s=m+1}^{n} T_s F'_u(\xi_u)I_s.
\]

(33)

The formula (33) generalizes representations of monogenic functions in both a three-dimensional harmonic algebra with one-dimensional radical (see [17]) and semi-simple algebras (see [18, 22]) to the case of algebras more general form and to a variable of more general form.

3. In the case where \( n = m \), the algebra \( \mathbb{A}_n^n \) is semi-simple and contains no nilpotent subalgebra. Then the formulae (32), (33) take the form

\[
\Phi(\zeta) = \sum_{u=1}^{n} F_u(\xi_u)I_u,
\]

because there are no vectors \( \{I_k\}_{k=m+1}^{n} \). This formula was obtained in the paper [22].

7 The relations between monogenic functions and partial differential equations

Consider the following linear partial differential equation with constant coefficients:

\[
\mathcal{L}_N U(x_1, x_2, \ldots, x_k) := \sum_{\alpha_1+\alpha_2+\ldots+\alpha_k=N} C_{\alpha_1,\alpha_2,\ldots,\alpha_k} \frac{\partial^N \Phi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_k^{\alpha_k}} = 0,
\]

(34)

If a function \( \Phi(\zeta) \) is \( N \)-times differentiable in the sense of Gateaux in every point of \( \Omega \), then

\[
\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_k} \Phi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_k^{\alpha_k}} = e_1^{\alpha_1} e_2^{\alpha_2} \ldots e_k^{\alpha_k} \Phi(\zeta) = e_1^{\alpha_2} e_3^{\alpha_3} \ldots e_k^{\alpha_k} \Phi^{(N)}(\zeta).
\]

Therefore, due to the equality

\[
\mathcal{L}_N \Phi(\zeta) = \Phi^{(N)}(\zeta) \sum_{\alpha_1+\alpha_2+\ldots+\alpha_k=N} C_{\alpha_1,\alpha_2,\ldots,\alpha_k} e_2^{\alpha_2} e_3^{\alpha_3} \ldots e_k^{\alpha_k},
\]

(35)

every \( N \)-times differentiable in the sense of Gateaux in \( \Omega \) function \( \Phi \) satisfies the equation \( \mathcal{L}_N \Phi(\zeta) = 0 \) everywhere in \( \Omega \) if and only if

\[
\sum_{\alpha_1+\alpha_2+\ldots+\alpha_k=N} C_{\alpha_1,\alpha_2,\ldots,\alpha_k} e_2^{\alpha_2} e_3^{\alpha_3} \ldots e_k^{\alpha_k} = 0.
\]

(36)

Accordingly, if the condition (36) is satisfied, then the real-valued components \( \text{Re} U_k(x_1, x_2, \ldots, x_k) \) and \( \text{Im} U_k(x_1, x_2, \ldots, x_k) \) of the decomposition (8) are solutions of the equation (34).
In the case where \( f_u(E_k) = \mathbb{C} \) for all \( u = 1, 2, \ldots, m \), it follows from Theorem 3 that the equality (35) holds for every monogenic function \( \Phi : \Omega \to \mathbb{A}^m_n \).

Thus, to construct solutions of the equation (34) in the form of components of monogenic functions, we must to find \( k \) linearly independent over the field \( \mathbb{R} \) vectors (6) satisfying the characteristic equation (36) and to verify the condition: \( f_u(E_k) = \mathbb{C} \) for all \( u = 1, 2, \ldots, m \). Then, the formula (23) gives a constructive description of all mentioned monogenic functions.

In the next theorem, we assign a special class of equations (34) for which \( f_u(E_k) = \mathbb{C} \) for all \( u = 1, 2, \ldots, m \). Let us introduce the polynomial

\[
P(b_2, b_3, \ldots, b_k) := \sum_{\alpha_1 + \alpha_2 + \ldots + \alpha_k = N} C_{\alpha_1, \alpha_2, \ldots, \alpha_k} b_2^{\alpha_2} b_3^{\alpha_3} \ldots b_k^{\alpha_k}.
\]

(37)

**Teorema 4.** Suppose that there exist linearly independent over the field \( \mathbb{R} \) vectors \( e_1, e_2, \ldots, e_k \) in \( \mathbb{A}^m_n \) of the form (6) that satisfy the equality (36). If \( P(b_2, b_3, \ldots, b_k) \neq 0 \) for all real \( b_2, b_3, \ldots, b_k \), then \( f_u(E_k) = \mathbb{C} \) for all \( u = 1, 2, \ldots, m \).

**Proof.** Using the multiplication table of \( \mathbb{A}^m_n \), we obtain the equalities

\[
e_2^{\alpha_2} = \sum_{u=1}^{m} a_{2u}^{\alpha_2} I_u + \Psi_R, \quad \ldots, \quad e_k^{\alpha_k} = \sum_{u=1}^{m} a_{ku}^{\alpha_k} I_u + \Theta_R,
\]

where \( \Psi_R, \ldots, \Theta_R \in \mathbb{R} \). Now the equality (36) takes the form

\[
\sum_{\alpha_1 + \alpha_2 + \ldots + \alpha_k = N} C_{\alpha_1, \alpha_2, \ldots, \alpha_k} \left( \sum_{u=1}^{m} a_{2u}^{\alpha_2} \ldots a_{ku}^{\alpha_k} I_u + \tilde{\Psi}_R \right) = 0,
\]

(38)

where \( \tilde{\Psi}_R \in \mathbb{R} \). Moreover, due to the assumption that the vectors \( e_1, e_2, \ldots, e_k \) of the form (6) satisfy the equality (36), there exist complex coefficients \( a_{jr} \) for \( j = 1, 2, \ldots, k, \ r = 1, 2, \ldots, n \) that satisfy the equality (38).

It follows from the equality (38) that

\[
\sum_{\alpha_1 + \alpha_2 + \ldots + \alpha_k = N} C_{\alpha_1, \alpha_2, \ldots, \alpha_k} a_{2u}^{\alpha_2} \ldots a_{ku}^{\alpha_k} = 0, \quad u = 1, 2, \ldots, m.
\]

(39)

Since \( P(b_2, b_3, \ldots, b_k) \neq 0 \) for all \( \{b_2, b_3, \ldots, b_k\} \subset \mathbb{R} \), the equalities (39) can be satisfied only if for each \( u = 1, 2, \ldots, m \) at least one of the numbers \( a_{2u}, a_{3u}, \ldots, a_{ku} \) belongs to \( \mathbb{C} \setminus \mathbb{R} \) that implies the relation \( f_u(E_k) = \mathbb{C} \) for all \( u = 1, 2, \ldots, m \). The theorem is proved.

We note that if \( P(b_2, b_3, \ldots, b_k) \neq 0 \) for all \( \{b_2, b_3, \ldots, b_k\} \subset \mathbb{R} \), then \( C_{N,0,0,\ldots,0} \neq 0 \) because otherwise \( P(b_2, b_3, \ldots, b_k) = 0 \) for \( b_2 = b_3 = \ldots = b_k = 0 \).

Since the function \( P(b_2, b_3, \ldots, b_k) \) is continuous on \( \mathbb{R}^k \), the condition \( P(b_2, b_3, \ldots, b_k) \neq 0 \) means either \( P(b_2, b_3, \ldots, b_k) > 0 \) or \( P(b_2, b_3, \ldots, b_k) < 0 \) for all real \( b_2, b_3, \ldots, b_k \). Therefore, it is obvious that for any equation (34) of elliptic type, the condition \( P(b_2, b_3, \ldots, b_k) \neq 0 \) is always satisfied for all \( \{b_2, b_3, \ldots, b_k\} \subset \mathbb{R} \). At the same time, there are equations (34) for which \( P(b_2, b_3, \ldots, b_k) > 0 \) for all \( \{b_2, b_3, \ldots, b_k\} \subset \mathbb{R} \), but which are not elliptic. For example, such is the equation

\[
\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u}{\partial x_1 \partial x_4} = 0
\]

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considered in $\mathbb{R}^4$.

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