ON THE LOG CANONICAL RING OF PROJECTIVE PLT PAIRS WITH THE KODAIRA DIMENSION TWO

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Abstract. The log canonical ring of a projective plt pair with the Kodaira dimension two is finitely generated.

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1. INTRODUCTION

One of the most important open problems in the theory of minimal models for higher-dimensional algebraic varieties is the finite generation of log canonical rings for lc pairs.

Conjecture 1.1. Let $(X, \Delta)$ be a projective lc pair defined over $\mathbb{C}$ such that $\Delta$ is a $\mathbb{Q}$-divisor on $X$. Then the log canonical ring

$$R(X, \Delta) := \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

In [FG2], Yoshinori Gongyo and the first author showed that Conjecture 1.1 is closely related to the abundance conjecture and is essentially equivalent to the existence problem of good minimal models for lower-dimensional varieties. Therefore, Conjecture 1.1 is thought to be a very difficult open problem.

When $(X, \Delta)$ is klt, Shigefumi Mori and the first author showed that it is sufficient to prove Conjecture 1.1 under the extra assumption that $K_X + \Delta$ is big in [FM]. Then Birkar–Cascini–Hacon–McKernan completely solved Conjecture 1.1 for projective klt pairs in [BCHM]. More generally, in [Fu4], the first author slightly generalized a canonical bundle formula in [FM] and showed that Conjecture 1.1 holds true even when $X$ is in Fujiki’s class $\mathcal{C}$ and $(X, \Delta)$ is klt. We note that Conjecture 1.1 is not necessarily true when $X$ is not in Fujiki’s class $\mathcal{C}$ (see [Fu4] for the details). Anyway, we have already established the finite generation of log canonical rings for klt pairs. So we are mainly interested in Conjecture 1.1 for $(X, \Delta)$ which is lc but is not klt.

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If \((X, \Delta)\) is lc, then we have already known that Conjecture \([\text{I.1}]\) holds true when \(\dim X \leq 4\) (see \([\text{Fu1}]\)). If \((X, \Delta)\) is lc and \(\dim X = 5\), then Kenta Hashizume showed that Conjecture \([\text{I.1}]\) holds true when \(\kappa(X, K_X + \Delta) < 5\) in \([\text{H}]\). We note that

\[
\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(|mD|))
\]

is always a finitely generated \(\mathbb{C}\)-algebra when \(X\) is a normal projective variety and \(D\) is a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor on \(X\) with \(\kappa(X, D) \leq 1\). Therefore, the following theorem is the first nontrivial step towards the complete solution of Conjecture \([\text{I.1}]\) for higher-dimensional algebraic varieties.

**Theorem 1.2 (Main Theorem).** Let \((X, \Delta)\) be a projective plt pair such that \(\Delta\) is a \(\mathbb{Q}\)-divisor. Assume that \(\kappa(X, K_X + \Delta) = 2\). Then the log canonical ring

\[
R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(|m(K_X + \Delta)|))
\]

is a finitely generated \(\mathbb{C}\)-algebra.

In this paper, we will describe the proof of Theorem 1.2.

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We will work over \(\mathbb{C}\), the complex number field, throughout this paper. We will freely use the basic notation of the minimal model program as in \([\text{Fu2}]\) and \([\text{Fu5}]\). In this paper, we do not use \(\mathbb{R}\)-divisors. We only use \(\mathbb{Q}\)-divisors.

2. \(\mathbb{Q}\)-divisors

Let \(D\) be a \(\mathbb{Q}\)-divisor on a normal variety \(X\), that is, \(D\) is a finite formal sum \(\sum_i d_i D_i\) where \(d_i\) is a rational number and \(D_i\) is a prime divisor on \(X\) for every \(i\) such that \(D_i \neq D_j\) for \(i \neq j\). We put

\[
D^{< 1} = \sum_{d_i < 1} d_i D_i, \quad D^{\leq 1} = \sum_{d_i \leq 1} d_i D_i, \quad \text{and} \quad D^{= 1} = \sum_{d_i = 1} D_i.
\]

We also put

\[
[D] = \sum_i [d_i] D_i, \quad [D] = -[-D], \quad \text{and} \quad \{D\} = D - [D],
\]

where \([d_i]\) is the integer defined by \(d_i \leq [d_i] < d_i + 1\). A \(\mathbb{Q}\)-divisor \(D\) on a normal variety \(X\) is called a boundary \(\mathbb{Q}\)-divisor if \(D\) is effective and \(D = D^{\leq 1}\) holds.

Let \(B_1\) and \(B_2\) be two \(\mathbb{Q}\)-divisors on a normal variety \(X\). Then we write \(B_1 \sim_\mathbb{Q} B_2\) if there exists a positive integer \(m\) such that \(mB_1 \sim mB_2\), that is, \(mB_1\) is linearly equivalent to \(mB_2\).

Let \(f : X \to Y\) be a proper surjective morphism between normal varieties and let \(D\) be a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor on \(X\). Then we write \(D \sim_\mathbb{Q} f^* B\) if there exists a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(B\) on \(Y\) such that \(D \sim_\mathbb{Q} f^* B\).

Let \(D\) be a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor on a normal projective variety \(X\). Let \(m_0\) be a positive integer such that \(m_0 D\) is a Cartier divisor. Let

\[
\Phi_{[m_0 D]} : X \longrightarrow \mathbb{P}^{\dim [m_0 D]}
\]
be the rational map given by the complete linear system \(|mm_0D|\) for a positive integer \(m\). We put
\[
\kappa(X, D) := \max_m \dim \Phi_{|mm_0D|}(X)
\]
if \(|mm_0D| \neq \emptyset\) for some \(m\) and \(\kappa(X, D) = -\infty\) otherwise. We call \(\kappa(X, D)\) the Iitaka dimension of \(D\). Note that \(\Phi_{|mm_0D|}(X)\) denotes the closure of the image of the rational map \(\Phi_{|mm_0D|}\).

Let \(D\) be a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor on a normal projective variety \(X\). If \(D \cdot C \geq 0\) for every curve \(C\) on \(X\), then we say that \(D\) is nef. If \(\kappa(X, D) = \dim X\) holds, then we say that \(D\) is big.

In this paper, we will repeatedly use the following well-known easy lemma.

**Lemma 2.1.** Let \(\varphi : X \to X'\) be a birational morphism between normal projective surfaces and let \(M\) be a nef \(\mathbb{Q}\)-divisor on \(X\). Assume that \(M' := \varphi_*M\) is \(\mathbb{Q}\)-Cartier. Then \(M'\) is nef.

**Proof.** By the negativity lemma, we can write \(\varphi_*M' = M + E\) for some effective \(\varphi\)-exceptional \(\mathbb{Q}\)-divisor \(E\) on \(X\). We can easily see that \((M + E) \cdot C \geq 0\) for every curve \(C\) on \(X\). Therefore, \(M'\) is a nef \(\mathbb{Q}\)-divisor on \(X'\). \(\square\)

## 3. Singularities of pairs

Let us quickly recall the notion of singularities of pairs. For the details, we recommend the reader to see [Fu2] and [Fu5].

A pair \((X, \Delta)\) consists of a normal variety \(X\) and a \(\mathbb{Q}\)-divisor \(\Delta\) on \(X\) such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier. Let \(f : Y \to X\) be a projective birational morphism from a normal variety \(Y\). Then we can write
\[
K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta)E
\]
with
\[
f_* \left(\sum_E a(E, X, \Delta)E\right) = -\Delta,
\]
where \(E\) runs over prime divisors on \(Y\). We call \(a(E, X, \Delta)\) the discrepancy of \(E\) with respect to \((X, \Delta)\). Note that we can define the discrepancy \(a(E, X, \Delta)\) for any prime divisor \(E\) over \(X\) by taking a suitable resolution of singularities of \(X\). If \(a(E, X, \Delta) \geq -1\) (resp. > \(-1\)) for every prime divisor \(E\) over \(X\), then \((X, \Delta)\) is called sub lc (resp. sub klt). If \(a(E, X, \Delta) > -1\) holds for every exceptional divisor \(E\) over \(X\), then \((X, \Delta)\) is called sub plt. It is well known that \((X, \Delta)\) is sub lc if it is sub plt.

Let \((X, \Delta)\) be a sub lc pair. If there exist a projective birational morphism \(f : Y \to X\) from a normal variety \(Y\) and a prime divisor \(E\) on \(Y\) with \(a(E, X, \Delta) = -1\), then \(f(E)\) is called an lc center of \((X, \Delta)\). We say that \(W\) is an lc stratum of \((X, \Delta)\) when \(W\) is an lc center of \((X, \Delta)\) or \(W = X\).

We assume that \(\Delta\) is effective. Then \((X, \Delta)\) is called lc, plt, and klt if it is sub lc, sub plt, and sub klt, respectively. In this paper, we call \(\kappa(X, K_X + \Delta)\) the Kodaira dimension of \((X, \Delta)\) when \((X, \Delta)\) is a projective lc pair.
4. Preliminary lemmas

In this section, we prepare two useful lemmas. One of them is a kind of connectedness lemma and will play a crucial role in this paper. Another one is a well-known generalization of the Kawamata–Shokurov basepoint-free theorem, which is essentially due to Yujiro Kawamata. We state it explicitly for the reader’s convenience.

The following lemma is a key observation. As we mentioned above, it is a kind of connectedness lemma and will play a crucial role in this paper.

**Lemma 4.1.** Let \( f : V \to W \) be a surjective morphism from a smooth projective variety \( V \) onto a normal projective variety \( W \). Let \( B_V \) be a \( \mathbb{Q} \)-divisor on \( V \) such that \( K_V + B_V \sim_{\mathbb{Q}, f} 0 \), \((V, B_V)\) is sub\-plt, and \( \text{Supp} B_V \) is a simple normal crossing divisor. Assume that the natural map

\[
\mathcal{O}_W \to f_* \mathcal{O}_V([-B_V^{\leq 1}])
\]

is an isomorphism. Let \( S_i \) be an irreducible component of \( B_V^{=1} \) such that \( f(S_i) \nsubseteq W \) for \( i = 1, 2 \). We assume that \( f(S_1) \cap f(S_2) \neq \emptyset \). Then \( S_1 = S_2 \) holds. In particular, we have \( f(S_1) = f(S_2) \).

**Proof.** We note that \( B_V^{=1} \) is a disjoint union of smooth prime divisors since \((V, B_V)\) is sub\-plt and \( \text{Supp} B_V \) is a simple normal crossing divisor. We put \( C_i = f(S_i) \) for \( i = 1, 2 \). Then we put \( Z = C_1 \cup C_2 \) with the reduced scheme structure. By taking some suitable birational modification of \( V \) and replacing \( S_i \) with its strict transform for \( i = 1, 2 \), we may further assume that \( f^{-1}(Z) \) is a divisor and that \( f^{-1}(Z) \cup \text{Supp} B_V \) is contained in a simple normal crossing divisor. Let \( T \) be the union of the irreducible components of \( B_V^{=1} \) that are mapped into \( Z \) by \( f \). Let us consider the following short exact sequence

\[
0 \to \mathcal{O}_V(A - T) \to \mathcal{O}_V(A) \to \mathcal{O}_T(A|_T) \to 0
\]

with \( A = [-B_V^{\leq 1}] \). Then we obtain the long exact sequence

\[
0 \to f_* \mathcal{O}_V(A - T) \to f_* \mathcal{O}_V(A) \to f_* \mathcal{O}_T(A|_T) \xrightarrow{\delta} R^1 f_* \mathcal{O}_V(A - T) \to \cdots
\]

Note that

\[
A - T - (K_V + \{B_V\} + B_V^{=1} - T) = -(K_V + B_V) \sim_{\mathbb{Q}, f} 0.
\]

Therefore, by [Fu2, Theorem 6.3 (i)], every nonzero local section of \( R^1 f_* \mathcal{O}_V(A - T) \) contains in its support the \( f \)-image of some lc stratum of \((V, \{B_V\} + B_V^{=1} - T) \). On the other hand, the support of \( f_* \mathcal{O}_T(A|_T) \) is contained in \( Z = f(T) \). We note that no lc strata of \((V, \{B_V\} + B_V^{=1} - T) \) are mapped into \( Z \) by \( f \) by construction. Therefore, the connecting homomorphism \( \delta \) is a zero map. Thus we get a short exact sequence

\[
0 \to f_* \mathcal{O}_V(A - T) \to \mathcal{O}_W \to f_* \mathcal{O}_T(A|_T) \to 0.
\]

Since \( f_* \mathcal{O}_V(A - T) \) is contained in \( \mathcal{O}_W \) and \( f(T) = Z \), we have \( f_* \mathcal{O}_V(A - T) = \mathcal{I}_Z \), where \( \mathcal{I}_Z \) is the defining ideal sheaf of \( Z \) on \( W \). Thus, by the above short exact sequence, we obtain that the natural map \( \mathcal{O}_Z \to f_* \mathcal{O}_V(A|_T) \) is an isomorphism. Hence we obtain

\[
\mathcal{O}_Z \sim f_* \mathcal{O}_T \sim f_* \mathcal{O}_T(A|_T).
\]

In particular, \( f : T \to Z \) has connected fibers. Therefore, \( f^{-1}(P) \cap T \) is connected for every \( P \in C_1 \cap C_2 \). Note that \( T \) is a disjoint union of smooth prime divisors since \( T \leq B_V^{=1} \). Thus we get \( T = S_1 = S_2 \) since \( S_1, S_2 \leq T \).

As a corollary of Lemma 4.1, we have:
Corollary 4.2. Let \( f : V \to W \) be a surjective morphism from a smooth projective variety \( V \) onto a normal projective variety \( W \). Let \( B_V \) be a \( \mathbb{Q} \)-divisor on \( V \) such that \( K_V + B_V \sim_{\mathbb{Q}, f} 0 \), \((V, B_V)\) is sub plt, and \( \text{Supp} B_V \) is a simple normal crossing divisor. Assume that the natural map

\[
\mathcal{O}_W \to f_* \mathcal{O}_V([-B_V^{\leq 1}])
\]

is an isomorphism. Let \( S \) be an irreducible component of \( B_V^{\leq 1} \) such that \( Z := f(S) \subsetneq W \). We put \( K_S + B_S = (K_V + B_V)|_S \) by adjunction. Then \((S, B_S)\) is sub klt and the natural map

\[
\mathcal{O}_Z \to g_* \mathcal{O}_S([-B_S])
\]

is an isomorphism, where \( g := f|_S \). In particular, \( Z \) is normal.

Proof. We can easily check that \((S, B_S)\) is sub klt by adjunction. We consider the following short exact sequence

\[
0 \to \mathcal{O}_V([-B_V^{\leq 1}] - S) \to \mathcal{O}_V([-B_V^{\leq 1}]) \to \mathcal{O}_S([-B_S]) \to 0.
\]

Note that \( B_V^{\leq 1}|_S = B_S^{\leq 1} = B_S \) holds. By Lemma 4.1, we know that no lc strata of \((V, \{B_V\} + B_V^{\leq 1} - S)\) are mapped into \( Z \) by \( f \). By the same argument as in the proof of Lemma 4.1 we obtain that the natural map

\[
\mathcal{O}_Z \to g_* \mathcal{O}_S([-B_S])
\]

is an isomorphism. Therefore, the natural map \( \mathcal{O}_Z \to g_* \mathcal{O}_S \) is an isomorphism. This implies that \( Z \) is normal. \( \Box \)

Lemma 4.3 is well known to the experts. It is a slight refinement of the Kawamata–Shokurov basepoint-free theorem and is essentially due to Yujiro Kawamata (see [K, Lemma 3]).

Lemma 4.3. Let \((V, B_V)\) be a projective plt pair and let \( D \) be a nef Cartier divisor on \( V \). Assume that \( aD - (K_V + B_V) \) is nef and big for some \( a > 0 \) and that \( \mathcal{O}_V(D)|_{B_V} \) is semi-ample. Then \( D \) is semi-ample.

Proof. By replacing \( D \) with a multiple, we may assume that \( |\mathcal{O}_V(mD)|_{B_V} \) is free for every nonnegative integer \( m \). Since \((V, B_V)\) is plt, the non-klt locus of \((V, B_V)\) is \([B_V]\). Therefore, by [Fu5, Corollary 4.5.6], \( D \) is semi-ample. \( \Box \)

5. ON LC-TRIVIAL FIBRATIONS

In this section, we recall some results on klt-trivial fibrations in [A] and lc-trivial fibrations in [FG1] for the reader’s convenience. We give only the definition which will be necessary to our purposes.

Let \( f : V \to W \) be a surjective morphism from a smooth projective variety \( V \) onto a normal projective variety \( W \). Let \( B_V \) be a \( \mathbb{Q} \)-divisor on \( V \) such that \((V, B_V)\) is sub lc and \( \text{Supp} B_V \) is a simple normal crossing divisor on \( V \). Let \( P \) be a prime divisor on \( W \). By shrinking \( W \) around the generic point of \( P \), we assume that \( P \) is Cartier. We set

\[
b_P := \max \{ t \in \mathbb{Q} \mid (V, B_V + tf^*P) \text{ is sub lc over the generic point of } P \}.
\]

Then we put

\[
B_W := \sum_P (1 - b_P)P,
\]

where \( P \) runs over prime divisors on \( W \). It is easy to see that \( B_W \) is a well-defined \( \mathbb{Q} \)-divisor on \( W \) (see the proof of Lemma 5.1 below). We call \( B_W \) the discriminant \( \mathbb{Q} \)-divisor of \( f : (V, B_V) \to W \). We assume that the natural map

\[
\mathcal{O}_W \to f_* \mathcal{O}_V([-B_V^{\leq 1}])
\]
is an isomorphism. In this situation, we have:

**Lemma 5.1.** $B_W$ is a boundary $\mathbb{Q}$-divisor on $W$.

We give a detailed proof of Lemma 5.1 for the reader’s convenience.

**Proof of Lemma 5.1.** We can easily see that there exists a nonempty Zariski open set $U$ of $W$ such that $b_P = 1$ holds for every prime divisor $P$ on $W$ with $P \cap U \neq \emptyset$. Therefore, $B_W$ is a well-defined $\mathbb{Q}$-divisor on $W$. Since $(V, B_V)$ is sub lc, $b_P \geq 0$ holds for every prime divisor $P$ on $W$. Thus, we have $B_W = B_{W}^{\leq 1}$ by definition. If $b_P > 1$ holds for some prime divisor $P$ on $W$, then we see that the natural map $\mathcal{O}_W \to f_*\mathcal{O}_V([-B_{W}^{\leq 1}])$ factors through $\mathcal{O}_W(P)$ in a neighborhood of the generic point of $P$. This is a contradiction. Therefore, $b_P \leq 1$ always holds for every prime divisor $P$ on $W$. This means that $B_W$ is effective. Hence we see that $B_W$ is a boundary $\mathbb{Q}$-divisor on $W$.

We further assume that $K_V + B_V \sim_{\mathbb{Q}} f^*D$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ on $W$. We set

$$M_W := D - K_W - B_W,$$

where $K_W$ is the canonical divisor of $W$. We call $M_W$ the moduli $\mathbb{Q}$-divisor of $K_V + B_V \sim_{\mathbb{Q}} f^*D$. Then we have:

**Theorem 5.2.** There exist a birational morphism $p : W' \to W$ from a smooth projective variety $W'$ and a nef $\mathbb{Q}$-divisor $M_{W'}$ on $W'$ such that $p_*M_{W'} = M_W$.

Theorem 5.2 is a special case of [FG1] Theorem 3.6, which is a generalization of [A Theorem 2.7]. When $W$ is a curve, we have:

**Theorem 5.3 ([A Theorem 0.1]).** If $\dim W = 1$ and $(V, B_V)$ is sub klt, then $M_W$ is semi-ample.

As an easy consequence of Theorem 5.3 we have:

**Corollary 5.4.** If $\dim W = 1$, $(V, B_V)$ is sub klt, and $D$ is nef, then $D$ is semi-ample.

**Proof.** If $\deg D > 0$, then $D$ is semi-ample. In particular, $D$ is semi-ample. From now on, we assume that $D$ is numerically trivial. By definition, $D = K_W + B_W + M_W$. Since $B_W$ is effective by Lemma 5.1 and $M_W$ is nef by Theorem 5.2, $W$ is $\mathbb{P}^1$ or an elliptic curve. If $W = \mathbb{P}^1$, then $D \sim_{\mathbb{Q}} 0$. Of course, $D$ is semi-ample. If $W$ is an elliptic curve, then $D \sim M_W$, that is, $D$ is linearly equivalent to $M_W$. In this case, $D$ is semi-ample by Theorem 5.3. Anyway, $D$ is always semi-ample.

Corollary 5.5 is a key ingredient of the proof of the main theorem: Theorem 1.2.

**Corollary 5.5.** If $\dim W = 2$, $(V, B_V)$ is sub plt, $(W, B_W)$ is plt, and $D$ is nef and big, then $D$ is semi-ample.

**Proof.** Let $Z$ be an irreducible component of $[B_W]$. Then, by the definition of $B_W$ and Lemma 4.1, there exists an irreducible component $S$ of $B_{W}^{=1}$ such that $f(S) = Z$. Therefore, by Corollary 4.2, the natural map $\mathcal{O}_Z \to g_*\mathcal{O}_S([-B_S])$ is an isomorphism, where $K_S + B_S = (K_V + B_V)|_S$ and $g = f|_S$. Note that $(S, B_S)$ is sub klt and that $K_S + B_S \sim_{\mathbb{Q}} g^*(D|_Z)$. Thus, $D|_Z$ is semi-ample by Corollary 5.4. On the other hand, by Theorem 5.2, $M_W$ is nef since $M_W = D - (K_W + B_W)$ is $\mathbb{Q}$-Cartier and $W$ is a normal projective surface (see Lemma 2.11). Therefore, $2D - (K_W + B_W) = D + M_W$ is nef and big. Thus we obtain that $D$ is semi-ample by Lemma 4.3.

We close this section with a short remark on recent preprints [Fu6] and [FFL].
Remark 5.6. In [Fu6], the first author introduced the notion of basic slc-trivial fibrations, which is a generalization of that of lc-trivial fibrations, and got a much more general result than Theorem 5.2 (see [Fu6, Theorem 1.2]). In [FFL], we established the semi-ampleness of $M_W$ for basic slc-trivial fibrations when the base space $W$ is a curve (see [FFL, Corollary 1.4]). We strongly recommend the interested reader to see [Fu6] and [FFL].

6. Minimal model program for surfaces

In this section, we quickly see a special case of the minimal model program for projective plt surfaces.

We can easily check the following lemma by the usual minimal model program for surfaces. We recommend the interested reader to see [Fu3] for the general theory of log surfaces.

Lemma 6.1. Let $(X, B)$ be a projective plt surface such that $B$ is a $\mathbb{Q}$-divisor and let $M$ be a nef $\mathbb{Q}$-divisor on $X$. Assume that $K_X + B + M$ is big. Then we can run the minimal model program with respect to $K_X + B + M$ and get a sequence of extremal contraction morphisms

$$(X, B + M) =: (X_0, B_0 + M_0) \xrightarrow{\varphi_0} \cdots \xrightarrow{\varphi_{k-1}} (X_k, B_k + M_k) =: (X^*, B^* + M^*)$$

with the following properties:

(i) each $\varphi_i$ is a $(K_{X_i} + B_i + M_i)$-negative extremal birational contraction morphism,
(ii) $K_{X_{i+1}} = \varphi_i K_{X_i}, B_{i+1} = \varphi_i B_i$, and $M_{i+1} = \varphi_i M_i$ for every $i$,
(iii) $M_i$ is nef for every $i$, and
(iv) $K_{X^*} + B^* + M^*$ is nef and big.

Proof. If $K_{X_i} + B_i + M_i$ is not nef, then we can take an ample $\mathbb{Q}$-divisor $H_i$ and an effective $\mathbb{Q}$-divisor $\Delta_i$ on $X_i$ such that $K_{X_i} + B_i + M_i + H_i \sim q_i * K_{X_i} + \Delta_i$, $(X_i, \Delta_i)$ is plt, and $K_{X_i} + \Delta_i$ is not nef. By the cone and contraction theorem, we can construct a $(K_{X_i} + \Delta_i)$-negative extremal birational contraction morphism $\varphi_i : X_i \to X_{i+1}$. Since $H_i$ is ample, $\varphi_i$ is a $(K_{X_i} + B_i + M_i)$-negative extremal contraction morphism. Moreover, since $M_i$ is nef, $\varphi_i$ is $(K_{X_i} + B_i)$-negative. Therefore, $(X_{i+1}, B_{i+1})$ is plt by the negativity lemma. In particular, $X_{i+1}$ is $\mathbb{Q}$-factorial. By Lemma 5.1, we obtain that $M_{i+1} = \varphi_i M_i$ is nef. Since $K_X + B + M$ is big by assumption, this minimal model program terminates. Then we finally get a model $(X^*, B^* + M^*)$ such that $K_{X^*} + B^* + M^*$ is nef and big.

If we put $\varphi := \varphi_{k-1} \circ \cdots \circ \varphi_0 : X \to X^*$, then we have

$$K_X + B + M = \varphi^*(K_{X^*} + B^* + M^*) + E$$

for some effective $\varphi$-exceptional $\mathbb{Q}$-divisor $E$ on $X$ by the negativity lemma.

We will use Lemma 6.1 in the proof of the main theorem: Theorem 1.2.

7. Proof of the main theorem: Theorem 1.2

In this section, let us prove the main theorem: Theorem 1.2.

Let $(X, \Delta)$ be a projective plt (resp. lc) pair such that $\Delta$ is a $\mathbb{Q}$-divisor. Assume that $0 < \kappa(X, K_X + \Delta) < \dim X$. Then we consider the Iitaka fibration

$$f := \Phi_{\mid m_0(K_X + \Delta)} : X \dashrightarrow Y$$
where \( m_0 \) is a sufficiently large and divisible positive integer. By taking a suitable birational modification of \( f : X \rightarrow Y \), we get a commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X' \\
\downarrow{f} & & \downarrow{f'} \\
Y & \xrightarrow{h} & Y'
\end{array}
\]

which satisfies the following properties:

(i) \( X' \) and \( Y' \) are smooth projective varieties,
(ii) \( g \) and \( h \) are birational morphisms, and
(iii) \( K_{X'} + \Delta' = g^*(K_X + \Delta) \) such that \( \text{Supp} \Delta' \) is a simple normal crossing divisor on \( X' \).

We note that \((X', (\Delta')^{>0})\) is plt (resp. lc) and that

\[
H^0(X, \mathcal{O}_X([m(K_X + \Delta)])) \simeq H^0(X', \mathcal{O}_{X'}([m(K_{X'} + (\Delta')^{>0})]))
\]

holds for every nonnegative integer \( m \). Therefore, for the proof of the finite generation of the log canonical ring \( R(X, \Delta) \), we may replace \((X, \Delta)\) with \((X', (\Delta')^{>0})\) and assume that the Itaka fibration \( f : X \rightarrow Y \) with respect to \( K_X + \Delta \) is a morphism between smooth projective varieties. By construction, \( \dim Y = \kappa(X, K_X + \Delta) \) and \( \kappa(X_{\pi}, K_{X_{\pi}} + \Delta|_{X_{\pi}}) = 0 \), where \( X_{\pi} \) is the geometric generic fiber of \( f : X \rightarrow Y \).

By [AK] Theorem 2.1, Proposition 4.4, and Remark 4.5], we can construct a commutative diagram

\[
\begin{array}{ccc}
U_{X'} & \xrightarrow{g} & X \\
\downarrow{f'} & & \downarrow{f} \\
U_{Y'} & \xrightarrow{h} & Y
\end{array}
\]

such that \( g \) and \( h \) are projective birational morphisms, \( X' \) and \( Y' \) are normal projective varieties, the inclusions \( U_{X'} \subset X' \) and \( U_{Y'} \subset Y' \) are toroidal embeddings without self-intersection satisfying the following conditions:

(a) \( f' \) is toroidal with respect to \((U_{X'}, \subset X')\) and \((U_{Y'}, \subset Y')\),
(b) \( f' \) is equidimensional,
(c) \( Y' \) is smooth,
(d) \( X' \) has only quotient singularities, and
(e) there exists a dense Zariski open set \( U \) of \( X \) such that \( g \) is an isomorphism over \( U \),

\[
U_{X'} = g^{-1}(U), \quad \text{and} \quad U \cap \Delta = \emptyset.
\]

Although it is not treated explicitly in [AK], we can make \( g : X' \rightarrow X \) satisfy condition (e) (see Remark 7.1).

Remark 7.1. In this remark, we will freely use the notation in [AK]. For condition (e), it is sufficient to prove that there exists a Zariski open set \( U \) of \( X \) such that \( U_{X'} = m^{-1}_{X'}(U) \) and that \( m_X \) is an isomorphism over \( U_{X'} \) in [AK] Theorem 2.1. Precisely speaking, we enlarge \( Z \) and may assume that \( X \setminus Z \) is a Zariski open set of the original \( X \) in [AK] 2.3, and enlarge \( \Delta \) suitably in [AK] 2.5. Then we can construct \( m_X : X' \rightarrow X \) such that \( U \) is a Zariski open set of \( X \), \( U_{X'} = m^{-1}_{X'}(U) \), and \( m_X : U_{X'} \rightarrow U \) is an isomorphism.

By condition (e), we have \( \text{Supp} \Delta' \subset X' \setminus U_{X'} \), where \( \Delta' \) is a \( \mathbb{Q} \)-divisor defined by \( K_{X'} + \Delta' = g^*(K_X + \Delta) \). We note that \((X', (\Delta')^{>0})\) is plt (resp. lc) and that

\[
H^0(X, \mathcal{O}_X([m(K_X + \Delta)])) \simeq H^0(X', \mathcal{O}_{X'}([m(K_{X'} + (\Delta')^{>0})]))
\]

holds for every nonnegative integer \( m \). Therefore, by replacing \( f : X \rightarrow Y \) and \((X, \Delta)\) with \( f' : X' \rightarrow Y' \) and \((X', (\Delta')^{>0})\), respectively, we may assume that \( f : X \rightarrow Y \) satisfies
conditions (a), (b), (c), (d), and $\text{Supp} \Delta \subset X \setminus U_X$, where $(U_X \subset X)$ is the toroidal structure in (a).

Since $\kappa(X, K_X + \Delta) > 0$, we can take a divisible positive integer $a$ such that
\[ H^0(X, \mathcal{O}_X(a(K_X + \Delta))) \neq 0. \]

Therefore, there exists an effective Cartier divisor $L$ on $X$ such that
\[ a(K_X + \Delta) \sim L. \]

We put
\[ G := \max \{ N | N \text{ is an effective } \mathbb{Q} \text{-divisor on } Y \text{ such that } L \geq f^*N \}. \]

Then we set
\[ D := \frac{1}{a}G \text{ and } F := \frac{1}{a}(L - f^*G). \]

By definition, we have
\[ K_X + \Delta \sim Q f^*D + F. \]

**Lemma 7.2.** For every nonnegative integer $i$, the natural map
\[ \mathcal{O}_Y \to f_*\mathcal{O}_X([iF]) \]

is an isomorphism.

**Proof.** By definition, $F$ is effective. Therefore, we have a natural nontrivial map
\[ \mathcal{O}_Y \to f_*\mathcal{O}_X([iF]) \]

for every nonnegative integer $i$. By $\kappa(X_\pi, K_{X_\pi} + \Delta|_{X_\pi}) = 0$, we have $\kappa(X_\pi, F|_{X_\pi}) = 0$.

Thus, we see that $f_*\mathcal{O}_X([iF])$ is a reflexive sheaf of rank one since $f$ is equidimensional.

Moreover, since $Y$ is smooth, $f_*\mathcal{O}_X([iF])$ is invertible. Let $P$ be any prime divisor on $Y$.

By construction, $\text{Supp} F$ does not contain the whole fiber of $f$ over the generic point of $P$.

Therefore, we obtain that $\mathcal{O}_Y \to f_*\mathcal{O}_X([iF])$ is an isomorphism in codimension one. This implies that the natural map
\[ \mathcal{O}_Y \to f_*\mathcal{O}_X([iF]) \]

is an isomorphism for every nonnegative integer $i$. \qed

By construction and Lemma 7.2, there exists a divisible positive integer $r$ such that
\[ r(K_X + \Delta) \text{ and } rD \text{ are Cartier and that } \]
\[ H^0(X, \mathcal{O}_X(mr(K_X + \Delta))) \simeq H^0(Y, \mathcal{O}_Y(mrD)) \]

holds for every nonnegative integer $m$. In particular, $D$ is a big $\mathbb{Q}$-divisor on $Y$. We put
\[ B := \Delta - F \text{ and consider } K_X + B \sim Q f^*D. \]

Let $p : V \to X$ be a birational morphism from a smooth projective variety $V$ such that $K_V + B_V = p^*(K_X + B)$ and that $\text{Supp} B_V$ is a simple normal crossing divisor.

\[ \xymatrix{ V \ar[r]^{p} \ar[dr]^\pi & X \ar[d]^f & \\ & Y } \]

It is obvious that $K_V + B_V \sim Q \pi^*D$ holds. Since $p_*\mathcal{O}_V([-(B_V^{\leq 1})]) \subset \mathcal{O}_X(kF)$ for some divisible positive integer $k$, the natural map $\mathcal{O}_Y \to \pi_*\mathcal{O}_V([-(B_V^{\leq 1})])$ is an isomorphism. For any prime divisor $P$ on $Y$, we put
\[ b_P := \max \{ t \in \mathbb{Q} | (X, B + tf^*P) \text{ is sub } \text{lc over the generic point of } P \}. \]

Then we set
\[ B_Y := \sum_P (1 - b_P)P \]
as in Section 3. Since $K_V + B_V = p^*(K_X + B)$ and the natural map $O_Y \to \pi_*O_V([- (B_V^{<1})])$ is an isomorphism, $B_Y$ is the discriminant $\mathbb{Q}$-divisor of $\pi : (V_B) \to Y$ and is a boundary $\mathbb{Q}$-divisor on $Y$ (see Lemma 5.1). By construction, we have $b_p = 1$ if $P \cap U_Y \neq \emptyset$, where $(U_Y \subset Y)$ is the toroidal structure in (a). Therefore, $\text{Supp} B_Y \subset Y \setminus U_Y$.

From now on, we assume that $(X, \Delta)$ is plt and $\kappa(X, K_X + \Delta) = 2$. Then $(V, B_V)$ is sub plt and $Y$ is a smooth projective surface. As in Section 5 we write

$$D = K_V + B_Y + M_Y,$$

where $M_Y$ is the moduli $\mathbb{Q}$-divisor of $K_V + B_Y \sim_{\mathbb{Q}} \pi^*D$. As we saw above, $\text{Supp} B_Y \subset Y \setminus U_Y$, where $(U_Y \subset Y)$ is the toroidal structure in (a). In particular, this means that $\text{Supp} B_Y$ is a simple normal crossing divisor on $Y$ because $Y$ is smooth. By Lemma 4.1 $[B_Y]$ is a disjoint union of some smooth prime divisors. Therefore, $(Y, B_Y)$ is plt. By Lemma 6.1 there exists a projective birational contraction morphism $\varphi : Y \to Y'$ onto a normal projective surface $Y'$ such that $D' = K_{Y'} + B_{Y'} + M_{Y'}$ is nef and big and that $D = \varphi^*D' + E$ for some effective $\varphi$-exceptional $\mathbb{Q}$-divisor $E$ on $Y$. Of course, $D'$, $K_{Y'}$, $B_{Y'}$, and $M_{Y'}$ are the pushforwards of $D$, $K_Y$, $B_Y$, and $M_Y$ by $\varphi$, respectively.

$$\begin{array}{ccc}
V & \xrightarrow{\pi'} & Y' \\
\pi & \searrow & \\
Y & \swarrow & \varphi
\end{array}$$

By replacing $V$ birationally, we may further assume that the union of $\text{Supp} B_Y$ and $\text{Supp} \pi^*E$ is a simple normal crossing divisor on $V$. We consider

$$K_V + B_V - \pi^*E \sim_{\mathbb{Q}} \pi'^*D'.$$

We note that the natural map

$$O_{Y'} \to \pi'_*O_V([- (B_V - \pi^*E)^{<1}])$$

is an isomorphism since $\pi_*O_V([- (B_V - \pi^*E)^{<1}]) \subset O_Y(kE)$ for some divisible positive integer $k$ and $O_{Y'} \sim \varphi_*O_Y(kE)$. By construction, $(Y', B_{Y'})$ is plt (see Lemma 6.1) and $B_{Y'}$ is the discriminant $\mathbb{Q}$-divisor of $\pi' : (V, B_V - \pi^*E) \to Y'$. Therefore, by Corollary 6.5 $D'$ is semi-ample. Thus, we obtain that

$$\bigoplus_{m \geq 0} H^0(Y, O_Y([mD])) \simeq \bigoplus_{m \geq 0} H^0(Y', O_{Y'}([mD']))$$

is a finitely generated $\mathbb{C}$-algebra. This implies that the log canonical ring $R(X, \Delta)$ of $(X, \Delta)$ is also a finitely generated $\mathbb{C}$-algebra.

Hence we have finished the proof of Theorem 1.2

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