Analytic theory of finite asymptotic expansions
in the real domain.
Part II:
the factorizational theory for Chebyshev asymptotic scales.

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Abstract. This paper contains a general theory for asymptotic expansions of type
\[ f(x) = a_1\phi_1(x) + \cdots + a_n\phi_n(x) + o(\phi_n(x)), \quad x \to x_0, \quad n \geq 3, \]
where the asymptotic scale
\[ \phi_1(x) \gg \phi_2(x) \gg \cdots \gg \phi_n(x), \quad x \to x_0. \]
is assumed to be an extended complete Chebyshev system on a one-sided neighborhood of \( x_0 \). "Factorizational theory" refers to proofs being based on various types of factorizations of a differential operator associated to \( (\phi_1, \cdots, \phi_n) \), hence we preliminarily collect various results scattered (and not all of them available) in the literature concerning Chebyshev systems, associated disconjugate operators and canonical factorizations. Another guiding thread of our theory is the property of formal differentiation and we aim at characterizing some \( n \)-tuples of asymptotic expansions formed by (*) and \( n - 1 \) expansions obtained by formal applications of suitable linear differential operators of orders 1, 2, \ldots, \( n - 1 \). Whereas for \( n = 2 \) there are only two such operators “naturally” suggested by the structure of the scale and the theory is comparatively simple, for \( n \geq 3 \) a result by Levin on the hierarchies of the Wronskians highlights a large class of operators which preserve the hierarchy of the \( \phi_i \)'s and, as such, are a-priori candidates to be formally applicable to (*). Our second preliminary step will be that of a closer investigation discovering that the restricted class of the operators naturally associated to “canonical” factorizations seems to be the most meaningful to be used in a context of formal differentiation. This gives rise to conjectures whose proofs build an analytic theory of finite asymptotic expansions in the real domain which, though not elementary, parallels the familiar results about Taylor’s formula. One of the results states that to each scale of the type under consideration it remains associated an important class of functions (namely that of generalized convex functions) enjoying the property that the expansion (*), if valid, is automatically formally differentiable \( n - 1 \) times in two special senses.

Keywords. Asymptotic expansions, formal differentiation of asymptotic expansions, factorizations of ordinary differential operators, Chebyshev asymptotic scales.

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1. Introduction

In this paper we develop a general analytic theory of asymptotic expansions of type

\[ f(x) = a_1 \phi_1(x) + \cdots + a_n \phi_n(x) + o(\phi_n(x)), \quad x \to x_0; \ n \geq 3, \]

where

\[ \phi_1(x) \gg \phi_2(x) \gg \cdots \gg \phi_n(x), \quad x \to x_0. \]

Though asymptotic expansions are since long a very useful tool in pure and applied mathematics, as far as asymptotic expansions in the real domain are concerned the general theory lacks basic results as are: (i) the classical Taylor’s formula for polynomial expansions at a point \( x_0 \in \mathbb{R} \); (ii) the theory of polynomial expansions at \( \infty \) systematized in \([5]\); (iii) the (not-too-trivial) case \( n = 2 \) thoroughly investigated in \([8]\). Here we have in mind characterizations of (1.1) via integro-differential conditions useful for applications unlike the trivial characterization of (1.1) by means of the existence (as finite numbers) of the following \( n \) limits defining the coefficients \( a_i \):

\[ a_1 := \lim_{x \to x_0} \frac{f(x)}{\phi_1(x)}, \quad a_i := \lim_{x \to x_0} \frac{[f(x) - a_1 \phi_1(x) - \cdots - a_i-1 \phi_i-1(x)]}{\phi_i(x)}, \quad 2 \leq i \leq n, \]

the \( \phi_i \)'s being supposed non-vanishing on a deleted neighborhood of \( x_0 \). All the mentioned cases show that a proper approach to a satisfying theory consists in studying (1.1) not by itself but matched to other expansions obtained by formal application of certain differential operators. For this we need a nontrivial background about the given asymptotic scale and some related differential operators. This preliminary material may be considered of an algebraic character: a first part concerning algebra of linear ordinary differential operators, a second part concerning those operators which act on the vector space “span (\( \phi_1, \ldots, \phi_n \))” preserving asymptotic scales. As for algebraic properties of differential operators we shall take advantage of certain Pólya-Mammana factorizations of an associated operator \( L_{\phi_1, \ldots, \phi_n} \) such that

\[ \ker L_{\phi_1, \ldots, \phi_n} = \text{span} (\phi_1, \ldots, \phi_n). \]

The scale of comparison functions (\( \phi_1, \ldots, \phi_n \)) is assumed to form an extended Chebyshev system on some left deleted neighborhood \( I \) of \( x_0 \); this is equivalent to the property of disconjugacy of \( L_{\phi_1, \ldots, \phi_n} \) on \( I \) which in turn is equivalent to the existence of Pólya-Mammana factorizations on \( I \) paying due attention to whether or not the left endpoint
is included. Concerning the regularity of the \( \phi_i \)'s and \( f \) they will be assumed of class \( AC^{n-1} \) i.e. with \((n-1)\)th-order derivatives absolutely continuous on the chosen neighborhood of \( x_0 \); \( AC^0 \equiv AC \). We shall exploit two special types of such factorizations, called canonical factorizations (C.F.'s for short) of type (I) or type (II) at \( x_0 \) making the most of both Pólya’s expressions for the coefficients of a factorization and a deep result by Levin on hierarchies of Wronskians. All this material (not all available in the literature) is systematized in §2.

Besides this, our theory revolves around the idea of formal differentiation of an asymptotic expansion; now, in general, applying an arbitrary differential operator to an asymptotic expansion yields a meaningless result; so it is necessary to have some a-priori information on the differential operators which are most likely to be formally applicable to (1.1) in the sense of generating a new asymptotic expansion. A possible approach to obtain such an information consists in investigating the case of an asymptotic expansion with an identically-zero remainder and this is done in §3. For this case Levin’s result highlights certain differential operators, defined by means of Wronskians involving the \( \phi_i \)'s, which preserve the hierarchy of the \( \phi_i \)'s and, as such, are a-priori candidates to be formally applicable to (1.1). These operators may be too many (for \( n \geq 3 \)) to be included in a useful theory whereas canonical factorizations automatically define two \((n-1)\)-tuples of differential operators, of orders \( 1, 2, \ldots, n-1 \), which are practically more meaningful than a generic Levin’s Wronskian. Their investigation gives rise to certain “natural” conjectures whose proofs are the core of our theory which we call “the factorizational theory” developed in §§4,5,6. All proofs are collected in §7. The main features of this theory are:

(i) It yields applicable analytic characterizations of an expansion (1.1) matched to other asymptotic relations obtained by formal differentiations in suitable senses.

(ii) For each asymptotic scale, which is also an extended complete Chebyshev system, there are at least two well-defined \((n-1)\)-tuples of differential operators \((L_1, \ldots, L_{n-1})\) and \((M_1, \ldots, M_{n-1})\), of orders \( 1, 2, \ldots, n-1 \) respectively, which can be formally applied to (1.1) under suitable integrability conditions. In one of the two circumstances useful representations of the remainders are also available.

(iii) A special family of functions is associated to each Chebyshev asymptotic scale, namely that of generalized convex functions, for which the validity of the sole relation (1.1) automatically implies its formal differentiability \((n-1)\) times in the two senses involving the above-mentioned operators \((L_1, \ldots, L_{n-1})\) and \((M_1, \ldots, M_{n-1})\).

In §8, as an appendix, we exhibit two algorithms to construct both types of canonical factorizations which are simple, intuitive and meaningful in our asymptotic context; they show once again, besides the considerations in §3, how “natural” is the factorizational theory.

As a last remark we point out that the same theory can be approached through different considerations. The limits in (1.3), being recursively defined, indicate the ”dynamic” nature of a relation (1.1) in the sense that, theoretically, (1.1) originates from a step-by-step procedure showing in sequence the validity of the \( n \) asymptotic expansions

\[
(1.5) \quad f(x) = a_1 \phi_1(x) + \cdots + a_i \phi_i(x) + o(\phi_i(x)), \quad x \to x_0, \quad 1 \leq i \leq n.
\]

This procedure may work well in elementary cases but is useless for general results. Now if the scale \((\phi_1, \ldots, \phi_n)\) is fixed a priori we can look at (1.1) as an approximation of \( f(x) \), in the specified asymptotic sense, by means of a linear combination of \( \phi_1, \ldots, \phi_n \).
This "static" viewpoint changes the situation and allows one to obtain results on the
validity of \((1.1)\) using conditions involving the operator \(L_{\phi_1,\ldots,\phi_n}\).

The introductions in [5] and [8] contain other comments but the general theory to be
developed in this paper is independent of any previous results in these references: only
the line of thought is the same.

Occasionally an asymptotic expansion \((1.5)\) with \(i < n\) will be called "incomplete"
— with respect to the given scale \((\phi_1,\ldots,\phi_n)\), of course — whereas \((1.1)\) will be called
"complete", and these locutions refer to the specified growth-order of the remainder and
not to the terms effectively present in the expansion i.e. those with non-zero coefficients.

The reader is urged to read just now (before going to §2) the remark at the end of §5.

Notations

— \(f \in AC^0(I) \equiv AC(I) \iff f\) is absolutely continuous on each compact subinterval
of \(I\); \(f \in AC^k(I) \iff f^{(k)} \in AC(I);\)

— For \(f \in AC^k(I)\) we write \(\lim_{x \to x_0} f^{(k+1)}(x)\) meaning that \(x\) runs through the points
wherein \(f^{(k+1)}\) exists as a finite number. Applying L'Hospital's rule in such a context
means using Ostrowski's version [21] valid for absolutely continuous functions.

— \(\mathbb{R} := \mathbb{R} \cup \{\pm\infty\}\) denotes the extended real line.

— If no ambiguity arises we use the following shorthand notations or similar ones:

\[
\begin{align*}
\int_T^x f_1 \int_T^t f_2 \cdots \int_T^{t_{n-2}} f_{n-1} \int_T^{t_{n-1}} f_n(t) \, dt := \\
\quad = \int_T^x f_1(t_1) dt_1 \int_T^{t_1} f_2(t_2) dt_2 \cdots \int_T^{t_{n-2}} f_{n-1}(t_{n-1}) dt_{n-1} \int_T^{t_{n-1}} f_n(t_n) dt_n;
\end{align*}
\]

\[
\begin{align*}
\int_x^{x_0} f_1 \int_{t_1}^{x_0} f_2 \cdots \int_{t_{n-2}}^{x_0} f_{n-1} \int_{t_{n-1}}^{x_0} f_n(t) \, dt := \\
\quad = \int_x^{x_0} f_1(t_1) dt_1 \int_{t_1}^{x_0} f_2(t_2) dt_2 \cdots \int_{t_{n-2}}^{x_0} f_{n-1}(t_{n-1}) dt_{n-1} \int_{t_{n-1}}^{x_0} f_n(t_n) dt_n;
\end{align*}
\]

wherein each integral \(\int_x^{x_0} f \) is to be understood as an improper integral.

— The main acronyms we systematically use are:

\(ECT\)-system := extended complete Chebyshev system: Lemma 2.1;

\(C.F.\) := canonical factorization: Proposition 2.2-(iv) and (v).

— Propositions are numbered consecutively in each section irrespective of their labelling as lemma, theorem and so on. For later references we report here a classic fundamental formula for Wronskians:

\[(1.6)\quad W(\phi(x)\phi_1(x),\ldots,\phi(x)\phi_n(x)) = (\phi(x))^n \cdot W(\phi_1(x),\ldots,\phi_n(x)),\]

valid under the required order of differentiability and no restriction on the sign of \(\phi\).
2. Chebyshev asymptotic scales, disconjugate operators and canonical factorizations

Our theory is built upon certain integral representations stemming from a special structure of the n-tuple \((\phi_1, \ldots, \phi_n)\).

**Definition 2.1** (Basic assumptions on the asymptotic scale). Let \(T\) be an half-open interval, \(I := [T, x_0[, \ T \in \mathbb{R}, \ x_0 \leq +\infty, \) and let \((\phi_1, \ldots, \phi_n)\) be an ordered n-tuple of real-valued functions defined on \(I\). We call \((\phi_1, \ldots, \phi_n)\) an "extended complete Chebyshev asymptotic scale" on the half-open interval \(I\) if the following properties are satisfied:

\[
\begin{align*}
(2.1) & \quad \phi_i \in C^{n-1}(I), \ 1 \leq i \leq n; \ n \geq 3; \\
(2.2) & \quad \phi_1(x) \gg \phi_2(x) \gg \cdots \gg \phi_n(x), \ x \to x_0^-; \\
(2.3) & \quad \phi_i(x) \neq 0 \quad \forall \ x \in I, \ 1 \leq i \leq n; \\
(2.4) & \quad W(\phi_1(x), \ldots, \phi_i(x)) > 0 \quad \forall \ x \in I, \ 1 \leq i \leq n;
\end{align*}
\]

where \(W(\phi_1(x), \ldots, \phi_i(x)) \equiv W(\phi_1, \ldots, \phi_i; x)\) denotes the Wronskian determinant of the functions \(\phi_1, \ldots, \phi_i ; W(\phi(x)) := \phi(x)\). In the factorizational theory condition (2.1) will be strengthened to \(\phi_i \in AC^{n-1}(I), 1 \leq i \leq n\). This additional regularity will be clearly stated whenever used.

In this definition the interval must be half-open and the point \(x_0\) appearing in the asymptotic scale is characterized as the endpoint not belonging to the interval, possibly \(x_0 = +\infty\). An analogous definition can be formulated for an interval \([x_0, T]\). Definitions and concepts in this section hold true for \(n = 2\) as well and the pertinent results are much simpler to state, but we have assumed \(n \geq 3\) because when mentioning assumption (2.1) we mean that \(n \geq 3\), the case \(n = 2\) having been treated separately in [8].

Such \(\phi_i's\) obviously are linearly independent and strictly one-signed on \(I\) and \(\phi_1 > 0\) on \(I\). It is shown in Proposition 2.6-(ii) below that (2.3) is implied by the other three conditions but we have written it down explicitly to make clear the basic hypotheses. The real import of (2.4) is that each involved determinant has one strict sign on \(I\). If, instead of (2.4), one assumes

\[
(2.4)_{\text{bis}} \quad W(\phi_1(x), \ldots, \phi_i(x)) \neq 0 \quad \forall \ x \in [T, x_0[, \ 1 \leq i \leq n,
\]

then suitably changing the signs of the \(\phi_i's\) would yield positive Wronskians. As these signs have no import in our asymptotic investigations we may freely assume (2.4) and we do so for definiteness and easier references to classical results. Proposition 2.6-(v) also points out that, for an n-tuple satisfying (2.1)-(2.2) on an half-open interval, condition (2.4) or equivalently (2.4)_{\text{bis}} implies the non-vanishingness on \([T, x_0]\) of the Wronskians \(W(\phi_1, \ldots, \phi_n)\) whatever the choice of the distinct functions \(\phi_i;\) in particular

\[
(2.4)_{\text{ter}} \quad W(\phi_n(x), \phi_{n-1}(x), \ldots, \phi_1(x)) \neq 0 \quad \text{on} \ [T, x_0[, \ 1 \leq i \leq n.
\]
But in general \((2.4)_{\text{ter}}\) does not imply \((2.4)_{\text{bis}}\). Hence the order of the \(\phi_i\)'s is not immaterial. However condition \((2.4)_{\text{ter}}\) implies that the inequalities in \((2.4)_{\text{bis}}\) hold true on some left neighborhood of \(x_0\) (Proposition 2.5), hence, from a practical viewpoint, we are merely fixing a neighborhood of \(x_0\) of type \([T, x_0]\) wherein the inequalities in \((2.4)_{\text{bis}}\) hold true and this condition has many important consequences that we are going to elucidate.

As concrete examples of such asymptotic scales on \([T, +\infty)\) the reader may think of scales whose non-identically zero and infinitely-differentiable functions are represented by linear combinations, products, ratios and compositions of a finite number of powers, exponentials and logarithms. As a rule such functions and their Wronskians have a principal part at \(+\infty\) which can be expressed by products of similar functions, hence they do not vanish on a neighborhood of \(+\infty\).

The locution used in the above definition explicitly refers to the concept of "extended complete Chebyshev system" which plays a basic role in approximation and interpolation theory, in ordinary differential equations and in the framework of Computer-Aided Geometric Design. As standard books on such topics we here mention Karlin and Studden [14], Karlin [13], Schumaker [23], and from these sources we freely draw terminology and results. Some recent papers by Mazure also contain very clear summaries of the classical theory; we only mention [16; 17; 18] referring the reader to them for gaining a more complete view of the theory. We do not report here the general definition of an "extended complete Chebyshev system" on an interval via certain determinants but point out that our Definition 2.1 is motivated by the following well-known characterization.

**Lemma 2.1** ([14; Th.4.3, p. 24, and Th.1.1, p. 376] or [23; Th.2.33, p. 34, and Th.9.1, p. 363]). For an ordered \(n\)-tuple of functions \(\phi_i \in C^{n-1}(J), 1 \leq i \leq n\), and \(J\) a generic interval of \(\mathbb{R}\), the following are equivalent properties:

(i) \(W(\phi_1(x), \ldots, \phi_i(x)) > 0 \quad \forall x \in J, 1 \leq i \leq n\);

(ii) \((\phi_1, \ldots, \phi_n)\) is an extended complete Chebyshev system (≡ ECT-system) on \(J\).

Moreover, the signs of the \(\phi_i\)'s apart, we may add the property:

(iii) Every nontrivial linear combination

\[
\phi(x) := c_1\phi_1(x) + \cdots + c_i\phi_i(x), \quad 1 \leq i \leq n, 
\]

has at most \((i - 1)\) zeros on \(J\) counting multiplicities.

The equivalence ",(i) ⇔ (iii)" is meant in the sense that ",(iii) ⇒ (i)" except possibly for the signs of the \(\phi_i\)'s. The equivalence ",(i) ⇔ (ii)" is stated in [14; Thm. 1.1, p. 376] for a compact interval but the argument does not depend on the type of \(J\): see Mazure [18; Prop. 2.6]. Under the stronger regularity assumption

\[
\phi_i \in AC^{n-1}(J), \quad 1 \leq i \leq n, \quad \text{and} \quad J \text{ is any interval},
\]

an ECT-system on \(J\) may be thought of as a fundamental system of solutions to \(L_nu = 0\) where \(L_n\) is a linear ordinary differential operator of type

\[
L_nu := u^{(n)} + \alpha_{n-1}(x)u^{(n-1)} + \cdots + \alpha_0(x)u \quad \forall u \in AC^{n-1}(J),
\]

\[
\alpha_i \in L^1_{\text{loc}}(J), \quad 0 \leq i \leq n - 1,
\]
i.e. "\(\alpha_i\) Lebesgue-summable on every compact subinterval of \(J\)". Hence if \(J\) contains one of its endpoints, call it \(a\), (2.7) implies that the \(\alpha_i\)'s are summable on a neighborhood of \(a\). To specify the value of \(L_nu\) at a point \(x\) we always write \(L_n[u(x)]\).

We now illustrate the relationships between the concepts of ECT-systems, disconjugacy, factorizations and canonical factorizations of linear ordinary differential operators. In addition to the three above-mentioned monographs we mention the book by Coppel [1] and the papers by Levin [15], Trench [24] and the author [3; 4].

**Definition 2.2** (Disconjugacy via the number of zeros). Both \(L_n\) and equation \(L_nu = 0\) are termed "disconjugate" on an interval \(J\), open or half-open, if any of the following equivalent properties holds true:

(i) Every nontrivial solution to \(L_nu = 0\) has at most \((n - 1)\) zeros on \(J\) counting multiplicities.

(ii) Any nontrivial solution to \(L_nu = 0\) has at most \((n - 1)\) distinct zeros on \([a, b]\).

Here the zeros and their multiplicities at an endpoint where not all the coefficients \(\alpha_i\) are continuous are defined in a suitable generalized sense: Levin [15; pp. 59-60 and Lemma 2.3, p. 61] or [3; Definition 3.4, p. 163]; and in this generalized context Levin shows that disconjugacy on \([a, b]\) is equivalent to disconjugacy on \([a, b]\) and to disconjugacy on \([a, b]\).

See more comments at the outset of §7.

In the next proposition specific terms are defined in the statement itself.

**Proposition 2.2** (Disconjugacy on an open interval via factorizations). For an operator \(L_n\) of type (2.7)\(_{1,2}\), \(n \geq 2\), on an open interval \([a, b]\), bounded or not, the following properties are equivalent:

(i) \(L_n\) is disconjugate on \([a, b]\).

(ii) \(L_nu = 0\) has a fundamental system of solutions on \([a, b]\), \((u_1, \ldots, u_n)\), satisfying Pólya’s W-property:

\[
W(u_1(x), \ldots, u_i(x)) > 0 \quad \forall x \in [a, b], \quad 1 \leq i \leq n;
\]

i.e. \((u_1, \ldots, u_n)\) is an ECT-system on \([a, b]\). Equivalently \(L_nu = 0\) has \((n - 1)\) solutions \(u_1, \ldots, u_{n-1}\) satisfying (2.8) for \(1 \leq i \leq n - 1\).

(iii) \(L_n\) has a Pólya-Mammana factorization on \([a, b]\) i.e.

\[
L_nu \equiv r_n[r_{n-1}(\ldots(r_1(r_0)')')']' \quad \forall u \in AC^{n-1}[a, b],
\]

where the \(r_i\)'s are suitable functions such that

\[
\begin{align*}
\text{(2.10)} & \quad \left\{ \begin{array}{l}
\quad r_i(x) > 0 \quad \forall x \in [a, b]; \quad r_i \in AC^{n-1-i}[a, b], \quad 0 \leq i \leq n - 1; \\
\quad r_n \in AC^0[a, b].
\end{array} \right.
\end{align*}
\]

(iv) \(L_n\) has a "canonical factorization (C.F. for short) of type (1) at the endpoint \(a\)", i.e. a factorization of type (2.9)-(2.10) with the additional conditions

\[
\text{(2.11)}_a \quad \int_{a}^{1/r_i} = +\infty, \quad 1 \leq i \leq n - 1,
\]
and a similar "C.F. of type (I) at the endpoint b", i.e. with the \( r_i \)'s satisfying

\[
\int_{a}^{b} \frac{1}{r_i} = +\infty, \quad 1 \leq i \leq n - 1.
\]

\( (v) \) For each \( c, a < c < b \), \( L_n \) has a "C.F. on the interval \([a, c]\) which is of type (II) at the endpoint \( a \)"; i.e. a factorization (2.9)-(2.10) valid on the interval \([a, c]\) and with the \( r_i \)'s satisfying

\[
\int_{a}^{c} \frac{1}{r_i} < +\infty, \quad 1 \leq i \leq n - 1.
\]

And \( L_n \) has a "C.F. on the interval \([c, b]\) which is of type (II) at the endpoint \( b \)"; i.e. a factorization (2.9)-(2.10) valid on the interval \([c, b]\) and with the \( r_i \)'s satisfying

\[
\int_{c}^{b} \frac{1}{r_i} < +\infty, \quad 1 \leq i \leq n - 1.
\]

**Remarks.** 1. In the definition of a C.F. conditions (2.11) or (2.12) are required to hold for the index \( i \) running from 1 to \((n - 1)\): there are no conditions on \( r_0 \) and \( r_n \). Factorizations in properties (iii)-(iv) are global i.e. valid on the whole given interval \([a, b]\), whereas property (v) claims the existence of local C.F.'s of type (II). The existence of a global C.F. of type (II) at \( a \) or at \( b \) is a special circumstance [3; Thm. 3.11, p. 163].

2. A global C.F. of type (I) at a specified endpoint does always exist for a conjugate operator on an open interval and is "essentially" unique in the sense that the functions \( r_i \) are determined up to multiplicative constants with product 1: Trench [24]. The situation is quite different for C.F.'s of type (II). For example the operator \( L_n \equiv u^{(n)} \) has no global C.F. on \((\infty, +\infty)\) of type (II) at any of the endpoints for it has only "one" (up to constant factors) Pólya-Mammana factorization on \((\infty, +\infty)\), namely

\[
u^{(n)} \equiv (\ldots (u')' \ldots)', \]

which is a special contingency characterized in [3; Thm. 3.3] and in [4; Thm. 7.1]. But the operator \( u^{(n)} \) thought of as acting on the space \( AC^{n-1}[0, +\infty) \), or even on the space \( C^\infty[0, +\infty) \), has infinitely many "essentially" different C.F.'s of type (II), for instance the following ones

\[
\begin{align*}
u^{(n)} & \equiv \frac{1}{(x-c)^n-1} \left[ (x-c)^2 \left( \ldots \left( (x-c)^2 \left( \frac{u}{(x-c)^{n-1}} \right)' \ldots \right)' \right) \right]' ,
\end{align*}
\]

which are C.F.'s of type (II) at both the endpoints "0" and "+\infty" whatever the choice of the constant \( c < 0 \). For \( c = 0 \) we get a factorization on \([0, +\infty)\) which is a C.F. of type (I) at "0" and of type (II) at "+\infty"; for \( c > 0 \) we have nonglobal factorizations which are of type (II) at +\infty.

C.F.'s are naturally linked to asymptotic scales of solutions to \( L_n u = 0 \) and the pertinent formulas are written down in Proposition 2.6 in the context of our basic scale. The reader, either acquainted with the subject or in a first reading, may jump directly to Proposition 2.6.
For an half-open interval, as required in our study, some points in Proposition 2.2 must be cleared up and some further characterizations may be added.

First the disconjugacy of \( L_n \) on \([a, b]\) is not equivalent to property (ii) relatively to \([a, b]\), i.e. with the inequalities (10.11) also valid for \( x = a \), as shown by the example of “\( u'' + u'' \)” on \([0, \pi]\). Property (ii) relatively to \([a, b]\) is a stronger property than disconjugacy on \([a, b]\) and is related to the possibility of extending the operator on a larger interval preserving disconjugacy. We report in the next proposition some characterizations of this fact not to be found in the literature in the present complete form for operators with locally-summable coefficients. The special case wherein all coefficients are continuous at an endpoint may be traced back to Hartman [9; 10; 11; 12]; see also Coppel [1; Lemma 6, p.93, and Thm. 3, p. 94] and Fink [2]. We need a preliminary definition.

**Definition 2.3.** If \( L_n \) is of type (2.7)\( _{1,2} \) on an open interval \([a, b]\) then the endpoint \( a \) is called ”nonsingular” for \( L_n \) if: (i) \( a \in \mathbb{R} \) (and \( b \leq +\infty \)); (ii) all the coefficients \( \alpha_i \) in (2.7) are also summable on an interval \((a, a + \epsilon)\). Analogous definition for the endpoint \( b \).

**Proposition 2.3** (Disconjugacy on an interval with a nonsingular endpoint). For an operator \( L_n \) of type (2.7)\( _{1,2} \) on an open interval \([a, b], a \in \mathbb{R} \) and \( b \leq +\infty \), the following properties are equivalent:

(i) \( L_n u = 0 \) has a fundamental system of solutions \((u_1, \ldots, u_n)\) such that

\[
\begin{align*}
\begin{cases}
\alpha_i \in AC^{n-1}[a, b], \quad 1 \leq i \leq n, \\
W(u_1(x), \ldots, u_i(x)) > 0 \quad \forall \ x \in [a, b], \quad 1 \leq i \leq n.
\end{cases}
\end{align*}
\]

(This implies that \( a \) is a nonsingular endpoint of \( L_n \) hence \( L_n \) may be thought of as acting on the space \( AC^{n-1}[a, b]\).)

(ii) ”The” global C.F. of \( L_n \) of type (I) at \( b \), say

\[
L_n u = r_n[r_{n-1}(\ldots (r_0 u)')']',
\]

is a C.F. of type (II) at \( a \) and the functions \( r_i \) satisfy the regularity conditions (2.10) on \([a, b]\) i.e.

\[
\begin{align*}
\begin{cases}
r_i(x) > 0 \quad \forall \ x \in [a, b]; \quad r_i \in AC^{n-1-i}[a, b], \quad 0 \leq i \leq n - 1; \\
r_n \in AC^0[a, b].
\end{cases}
\end{align*}
\]

(iii) \( L_n \) admits of a C.F. on \([a, b]\) of type (II) at both endpoints, say

\[
L_n u = \tilde{r}_n[\tilde{r}_{n-1}(\ldots (\tilde{r}_0 u)')']',
\]

such that the functions \( \tilde{r}_i \) satisfy the regularity conditions (2.10) on \([a, b]\) like the functions \( r_i \) in (2.17).

(iv) The endpoint \( a \) is a nonsingular endpoint for \( L_n \) and \( L_n \) can be extended as a disconjugate operator on a larger interval \([a - \epsilon, b]\).

(v) The endpoint \( a \) is a nonsingular endpoint for \( L_n \) and \( L_n \) is disconjugate on \([a, b]\) in the generalized sense mentioned above.
(vi) $L_n u = 0$ has a fundamental system of solutions $(\tilde{u}_1, \ldots, \tilde{u}_n)$ satisfying, in addition to (2.15), the stronger requirements for the Wronskians

\begin{equation}
W(\tilde{u}_i(x), \tilde{u}_{i+1}(x), \ldots, \tilde{u}_j(x)) > 0 \quad \forall \, x \in [a, b] \quad \text{for all}
\end{equation}

and even the still stronger requirements

\begin{equation}
W(\tilde{u}_{i_1}(x), \ldots, \tilde{u}_{i_k}(x)) > 0 \quad \forall \, x \in [a, b] \quad \text{for all}
\end{equation}

Remarks. 1. For several characterizations of property (ii) in the context of disconjugacy on open intervals the reader is referred to [3; Thms. 3.1 and 3.2, pp. 163-164]. In a subsequent restriction that at least one endpoint be singular for $L_n$. Proposition 2.3 completes the result in this last reference by adding the regularity at the nonsingular endpoint of the coefficients of certain C.F.'s and the existence of a system of solutions forming a Chebyshev, indeed a Cartesian, system on $[a, b]$; see the proof in §7 for correcting two misprints in the last reference.

2. Notice that there may exist infinitely many ”essentially” different C.F.'s of type (II) at both endpoints, as shown by (2.14) on $[0, +\infty)$ with $c < 0$, and that property (iii) does not claim that each factorization of $L_n$ on $[a, b]$ of type (II) at both endpoints has coefficients with the same regularity of the $r_i$'s in (2.17). The special factorizations (2.16) and (2.17) are indeed factorizations on the half-open interval $[a, b]$.

3. Not any fundamental system satisfying (2.15) automatically satisfies (2.19)-(2.20) as shown by the operator $d^n/dx^n$ on $[0, +\infty)$ and the system $(1, x, \ldots, x^{n-1})$, $n \geq 2$. Neither the existence of a system satisfying (2.19) on an open interval $[a, b]$ is granted for any disconjugate operator as shown by the same operator $d^n/dx^n$ on $\mathbb{R}$, $n \geq 2$; for any such operator it is known, Coppel [1; Lemma 6, p. 93], that for each chosen $t_0 \in [a, b]$ there exists a fundamental system satisfying (2.19)-(2.20) on $[t_0, b]$. Our proposition exclusively refers to half-open intervals. An ordered $n$-tuple satisfying (2.19) or, equivalently, (2.20) is variously labelled in the literature, an admissible locution being that of ”extended positive Cartesian (≡ Descartes) system”.

Having highlighted properties of $n$-tuples satisfying (2.1) and (2.4) we are going to say something about scales (2.2)-(2.3) and then to link together all conditions (2.1) to (2.4). The fundamental properties in the following two propositions are due to Levin [15; §2], and concern the Wronskians constructed with an asymptotic scale, namely their signs and their hierarchies inherited from (2.2).

**Proposition 2.4** (Asymptotic scales in linear spaces).

(i) If $S_n$ is an $n$-dimensional linear space of continuous real-valued functions on an open interval $[T, x_0]$ and if each function $\tilde{\phi} \in S_n \setminus \{0\}$ does not vanish on a left neighborhood (depending on $\tilde{\phi}$) of $x_0$ then there exists some basis $(\tilde{\phi}_1, \ldots, \tilde{\phi}_n)$ in $S_n$ such that

\begin{equation}
\begin{cases}
\tilde{\phi}_i > 0 \text{ on some interval } [x_0 - r, x_0], & 1 \leq i \leq n, \\
\tilde{\phi}_1(x) \ll \tilde{\phi}_2(x) \ll \cdots \ll \tilde{\phi}_n(x), & x \to x_0.
\end{cases}
\end{equation}
(ii) If $L_n$ is an operator of type $(2.7)_{1,2}$ disconjugate on an open interval $]T, x_0[$ then its kernel has some basis $(\tilde{\phi}_1, \ldots, \tilde{\phi}_n)$ satisfying (2.21) and for each such basis
\begin{equation}
W(\tilde{\phi}_1(x), \ldots, \tilde{\phi}_i(x)) > 0 \text{ on the whole interval } ]T, x_0[ , \quad 1 \leq i \leq n.
\end{equation}

In this statement we have preserved the numbering of the $\tilde{\phi}_i$'s used by Levin which is essential for the inequalities in (2.22). The situation changes at the left endpoint. In our context we must adapt Proposition 2.4 to our notations.

By (2.22) our Chebyshev asymptotic scale would satisfy the inequalities in (2.4)$_{\text{ter}}$ on $]T, x_0[$ which are not equivalent to those in (2.4)$_{\text{bis}}$ on $]T, x_0[$ generally speaking, but our basic assumptions refer to an half-open interval with a nonsingular endpoint (Proposition 2.3 and Proposition 2.6 below).

**Proposition 2.5** (The hierarchies of the Wronskians). If the $n$-tuple $(\phi_1, \ldots, \phi_n)$ satisfies conditions (2.1) to (2.4) with respect to the open interval $]T, x_0[$ then:

(i) For any strictly increasing set of indexes $\{i_1, \ldots, i_k\}$, i.e. such that
\begin{equation}
1 \leq i_1 < i_2 < \cdots < i_k \leq n, \quad 1 \leq k \leq n - 1,
\end{equation}
we have
\begin{equation}
W(\phi_{i_1}(x), \ldots, \phi_{i_k}(x)) \neq 0 \text{ on a left neighborhood of } x_0.
\end{equation}

(ii) For each $k$, $1 \leq k \leq n - 1$, and for any two distinct and strictly increasing sets of indexes $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ such that $i_h \leq j_h, 1 \leq h \leq k$, we have
\begin{equation}
W(\phi_{i_1}(x), \ldots, \phi_{i_k}(x)) \gg W(\phi_{j_1}(x), \ldots, \phi_{j_k}(x)), \quad x \to x_0^-.
\end{equation}

To visualize the second part of the proposition we list a few asymptotic scales at $x_0^-$ constructed with the Wronskians:
\begin{equation}
\begin{cases}
W(\phi_1, \phi_2) \gg W(\phi_1, \phi_3) \gg \cdots \gg W(\phi_1, \phi_n) \\
W(\phi_2, \phi_3) \gg W(\phi_2, \phi_4) \gg \cdots \gg W(\phi_2, \phi_n) \\
\ldots \\
W(\phi_{n-2}, \phi_{n-1}) \gg W(\phi_{n-2}, \phi_n)
\end{cases}, \quad x \to x_0^-,
\end{equation}
\begin{equation}
W(\phi_1, \phi_2, \phi_3) \gg W(\phi_1, \phi_2, \phi_4) \gg \cdots \gg W(\phi_1, \phi_2, \phi_n), \quad x \to x_0^-.
\end{equation}

We are now in a position to write down explicitly all the properties and formulas related to the differential operator implicitly defined by our Chebyshev asymptotic scale with emphasis on the two C.F.'s and on extending the validity of (2.24) to the whole interval.

**Definition 2.4.** For a fixed asymptotic scale $(\phi_1, \ldots, \phi_n)$ satisfying the basic assumptions (2.1) to (2.4) and (2.6) we denote by $L_{\phi_1, \ldots, \phi_n}$ the $n$th-order linear ordinary differential operator of type $(2.7)_{1,2}$ acting on the space $AC^{n-1}[T, x_0]$ and such that
\begin{equation}
\ker L_{\phi_1, \ldots, \phi_n} = \text{span}(\phi_1, \ldots, \phi_n).
\end{equation}
Proposition 2.6. (Properties of the basic differential operator). Under conditions (2.1) to (2.4) and (2.6) the above-defined operator \( L_{\phi_1, \ldots, \phi_n} \) is disconjugate on the open interval \([T, x_0]\) and, as such, enjoys the properties in Proposition 2.2. Moreover, as an operator acting on \( AC^{n-1}[T, x_0] \) it has the following further properties partly inferred from Proposition 2.3:

(i) Define the following \((n+1)\) functions on \([T, x_0]\)

\[
\begin{align*}
q_0 &:= 1/\phi_1; \quad q_1 := (\phi_1)^2/W(\phi_1, \phi_2); \\
q_i &:= [W(\phi_1, \ldots, \phi_i)]^{2}/[W(\phi_1, \ldots, \phi_{i-1}) \cdot W(\phi_1, \ldots, \phi_{i+1})], \quad 2 \leq i \leq n - 1; \\
q_n &:= [q_0 q_1 \ldots q_{n-1}]^{-1} \equiv W(\phi_1, \ldots, \phi_n)/W(\phi_1, \ldots, \phi_{n-1}).
\end{align*}
\]

Then the \(q_i\)'s satisfy the following regularity conditions:

\[
\begin{align*}
q_i(x) > 0 & \quad \forall \, x \in [T, x_0]; \quad q_i \in AC^{n-1-i}[T, x_0], \quad 0 \leq i \leq n - 1; \\
q_n &\in AC^0[T, x_0].
\end{align*}
\]

Their reciprocals, left apart \(q_0\) and \(q_n\), may be expressed as derivatives of certain ratios

\[
\begin{align*}
1/q_1(x) &= (\phi_2(x)/\phi_1(x))', \\
1/q_i(x) &= \left[\frac{W(\phi_1(x), \ldots, \phi_{i-1}(x), \phi_{i+1}(x))}{W(\phi_1(x), \ldots, \phi_{i-1}(x), \phi_i(x))}\right]', \quad 2 \leq i \leq n - 1,
\end{align*}
\]

on the interval \([T, x_0]\), and

\[
\int_T^{x_0} (1/q_i) < +\infty, \quad 1 \leq i \leq n - 1.
\]

Moreover \(L_{\phi_1, \ldots, \phi_n}\) admits of the following factorization on \([T, x_0]\)

\[
L_{\phi_1, \ldots, \phi_n} u \equiv q_n [(q_{n-1} \cdots (q_0 u)' \cdots)']',
\]

which, by (2.32), is a global C.F. of type (II) at both endpoints \(T\) and \(x_0\).

(ii) Our Chebyshev asymptotic scale \((\phi_1, \ldots, \phi_n)\) admits of the following integral representation in terms of the \(q_i\)'s defined in (2.29):

\[
\begin{align*}
\phi_1(x) &= \frac{1}{q_0(x)}; \quad \phi_2(x) = -\frac{1}{q_0(x)} \int_x^{x_0} \frac{1}{q_1}; \\
\phi_i(x) &= \frac{(-1)^{i-1}}{q_0(x)} \int_x^{x_0} \frac{1}{q_1} \cdots \int_{t_{i-2}}^{x_0} \frac{1}{q_{i-1}}, \quad 2 \leq i \leq n.
\end{align*}
\]

Hence the \(\phi_i\)'s have alternating signs on \([T, x_0]\), namely

\[
(-1)^{i+1} \phi_i(x) > 0 \quad \forall \, x \in [T, x_0], \quad 1 \leq i \leq n,
\]

and they have the same order of growth at \(T\), namely

\[
\lim_{x \to T^+} \phi_i(x)/\phi_j(x) = c_{ij} \in \mathbb{R} \setminus \{0\} \quad \forall \, i \neq j.
\]
(iii) Denoting "the" global C.F. of \( L_{\phi_1, \ldots, \phi_n} \) of type (1) at \( x_0 \) by

\[
L_{\phi_1, \ldots, \phi_n} u \equiv p_n [p_{n-1} \cdots (p_0 u)']',
\]
the \( p_i \)'s satisfy the same regularity conditions on the half-open interval \([T, x_0[\) as the \( q_i \)'s in (2.30). Moreover:

\[
\int_{x_0}^{x_0} \frac{1}{p_i} = +\infty, \quad 1 \leq i \leq n - 1,
\]

\[
\int_T \frac{1}{p_i} < +\infty, \quad 1 \leq i \leq n - 1.
\]

The \( p_i \)'s (constant factors apart) may be recovered from the \( q_i \)'s by means of the following formulas which are the analogue of (2.29) with respect to the inverted \( n \)-tuple \((|\phi_n|, \ldots, |\phi_1|)\):

\[
p_0 := 1/|\phi_n|; \quad p_1 := (\phi_n)^2/W(|\phi_n|, |\phi_{n-1}|);
\]

\[
p_i := \left[ W(\phi_n, \phi_{n-1}, \ldots, \phi_{n-i+1}) \right]^2 \times \\
\times \left[ W(|\phi_n|, |\phi_{n-1}|, \ldots, |\phi_{n-i+2}|) \cdot W(|\phi_n|, |\phi_{n-1}|, \ldots, |\phi_{n-i}|) \right]^{-1}, \quad 2 \leq i \leq n - 1;
\]

\[
p_n := [p_0 p_1 \cdots p_{n-1}]^{-1} \equiv W(|\phi_n|, |\phi_{n-1}|, \ldots, |\phi_1|)/W(|\phi_n|, |\phi_{n-1}|, \ldots, |\phi_2|).
\]

The reciprocals of the functions \( p_1, \ldots, p_{n-1} \) defined in (2.40) may be expressed as derivatives of the following ratios analogous to (2.31):

\[
\begin{align*}
1/p_1(x) &= (|\phi_{n-1}(x)/\phi_n(x)|)', \\
1/p_i(x) &= \left[ \frac{W(\phi_n(x), \ldots, \phi_{n-i+2}(x), \phi_{n-i}(x))}{W(\phi_n(x), \ldots, \phi_{n-i+2}(x), \phi_{n-i+1}(x))} \right]’, \quad 2 \leq i \leq n - 1;
\end{align*}
\]

on the interval \([T, x_0[\). The simplified use of the absolute values in (2.41), written outside the Wronskians rather than applied to each of their arguments \( \phi_j \), is allowed by property (v) below.

(iv) The special fundamental system of solutions to \( L_{\phi_1, \ldots, \phi_n} u = 0 \) defined by

\[
\begin{align*}
P_0(x) &:= \frac{1}{p_0(x)}; \quad P_1(x) := \frac{1}{p_0(x)} \int_T^x \frac{1}{p_1}; \\
P_i(x) &:= \frac{1}{p_0(x)} \int_T^x \frac{1}{p_1} \cdots \int_T^{t_{i-1}} \frac{1}{p_i}, \quad 1 \leq i \leq n - 1,
\end{align*}
\]

satisfies the asymptotic relations:

\[
\begin{align*}
P_0(x) &\gg P_1(x) \gg \cdots \gg P_{n-2}(x) \gg P_{n-1}(x), \quad x \to T^+, \\
P_{n-1}(x) &\gg P_{n-2}(x) \gg \cdots \gg P_1(x) \gg P_0(x), \quad x \to x_0^-.
\end{align*}
\]
Relations (2.43) uniquely determine the fundamental system \( (P_0, \ldots, P_{n-1}) \) up to multiplicative constants. (In the terminology used by the author \([3; 4]\) the \( n \)-tuple \( (P_0, \ldots, P_{n-1}) \) is a "mixed hierarchical system" on \( [T, x_0] \) whereas Levin \([15; \text{p. 80}] \) would call it a "doubly hierarchical system" because he uses different arrangements for asymptotic scales at the left or right endpoints \([15; \text{p. 59}]. \))

(v) Both the ordered \( n \)-tuples \( (P_0, \ldots, P_{n-1}) \) and \( (|\phi_0|, \ldots, |\phi_1|) \) are extended positive Cartesian systems on \( [T, x_0] \); see (2.19)-(2.20). The Wronskians of the \( P_i \)'s and \( \phi_i \)'s can be expressed in terms of \( p_i \)'s and \( q_i \)'s respectively through the formulas:

\[
\begin{align*}
W(P_0(x), \ldots, P_i(x)) &= [(p_0(x))^i \cdot (p_1(x))^i \cdots (p_{i-1}(x))^2 \cdot p_i(x)]^{-1}, \\
& \quad x \in [T, x_0], \ 0 \leq i \leq n - 1; \\
W(\phi_1(x), \ldots, \phi_i(x)) &= [(q_0(x))^i \cdot (q_1(x))^{i-1} \cdots (q_{i-2}(x))^2 \cdot q_{i-1}(x)]^{-1}, \\
& \quad x \in [T, x_0], \ 1 \leq i \leq n.
\end{align*}
\]

**Remarks.** It follows from (v) that, under conditions (2.1)-(2.2), the inequalities in (2.4) hold implies

\[
W(\phi_1, \ldots, \phi_k) \neq 0 \text{ on } [T, x_0] \text{ for any distinct functions } \phi_i,
\]

and in particular condition (2.4) hold. That (2.4) hold do not in general imply (2.4) on the whole given interval is shown by the following two scales:

\[
1 \gg cx + x^2 \gg x^2, \ x \to 0^-, \ (c > 0), \ \text{on } (-\infty, 0];
\]

and

\[
x^2 \gg c - x \gg 1, \ x \to +\infty, \ (c > 0), \ \text{on } ]0, +\infty).
\]

With our notations the scale (2.47) satisfies (2.4) hold on \((-\infty, 0] \) whereas:

\[
\begin{align*}
\phi_1 \text{ and } W(\phi_1, \phi_2, \phi_3) &\neq 0 \text{ on } (-\infty, 0]; \\
W(\phi_1, \phi_2) &\equiv W(1, cx + x^2) = c + 2x \neq 0 \text{ on } ] - c/2, 0[ \text{ but not on } (-\infty, 0].
\end{align*}
\]

Analogously the scale (2.48) satisfies (2.4) hold on \( \mathbb{R} \) whereas:

\[
\phi_1 > 0 \text{ on } [0, +\infty), \ W(\phi_1, \phi_2) = x(x - 2c) > 0 \text{ on } ]2c, +\infty).
\]

Notice that the functions \( p_i \)'s in (2.37) are unique, constant factors apart, by a mentioned result by Trench but they may be recovered from many different asymptotic scales and not just from one! The main feature of the above proposition is that we can express all the properties of our basic operator (at least those needed in our theory) in terms of the a-priori given Chebyshev asymptotic scale.

A quick proof of the existence of C.F.’s of type (I). The global existence of C.F.’s of type (I) was for the first time proved by Trench \([24]\) by an original procedure which was subsequently adapted by the author \([3]\) to show the local existence of C.F.’s of type (II).
Trench’s result played a historical role as it had a great impact on the asymptotic theory of ordinary differential equations. In compiling the present résumé of the theory of C.F.’s we noticed that Trench’s result concerning global existence can be also obtained as an easy consequence of Levin’s results. The synthetic proof is reported in §7. An analogous brief proof is not available for C.F.’s of type (II) in order to show their existence on each interval specified in property (v) of Proposition 2.2. However we must point out that Trench’s procedure, independent of properties of Wronskians, applies to a larger class of operators [24, §1].

3. Applying differential operators to asymptotic scales

In the elementary case of Taylor’s formula the simple condition

\[ \exists f^{(n)}(x_0) \]

is not a mere sufficient condition for the validity of the asymptotic expansion

\[ f(x) = \sum_{i=0}^{n} a_i(x - x_0)^i + o((x - x_0)^n) \equiv T_n(x) + o((x - x_0)^n), \ x \to x_0; \]

it in fact characterizes the set of the \( n \) asymptotic expansions

\[ f^{(k)}(x) = \sum_{i=0}^{n-k} T_n^{(k)}(x) + o((x - x_0)^{n-k}), \ x \to x_0, \ 0 \leq k \leq n - 1, \]

which is formed by (3.2) together with the relations obtained by formal differentiation \( 1, 2, \ldots n - 1 \) times. In this case we have the known formulas for the coefficients:

\[ a_i = f^{(i)}(x_0)/i!, \ 0 \leq i \leq n. \]

If we strengthen condition (3.1) by assuming

\[ f \in AC^n(I_{x_0}), \ I_{x_0} : a neighborhood of x_0, \]

we have representation

\[ f^{(n)}(x) = f^{(n)}(x_0) + \int_{x_0}^{x} f^{(n+1)}(t)dt, \]

which, besides implying the validity of (3.3) for \( k = n \) as well, gives rise to the integral representation formulas of all the remainders in (3.3).

A similar situation occurs in the factorizational theory of polynomial asymptotic expansions at \(+\infty, [5]\), where the standard operators of differentiation \( D^k := d^k/dx^k \) happen to be formally applicable \( n \) times to the expansion

\[ f(x) = a_n x^n + \ldots + a_1 x + a_0 + o(1), \ x \to +\infty, \]

in two quite different senses and under suitable integral conditions. But in the analogous theory for expansions in arbitrary real powers

\[ f(x) = a_1 x^{\alpha_1} + \ldots + a_n x^{\alpha_n} + o(x^{\alpha_n}), \ x \to +\infty, \ (\alpha_1 > \ldots > \alpha_n), \]
developed in [7], it turns out that the most natural operators on which to build a simple
type are those linked to the C.F.’s of the differential operator in Definition 2.4 with
φ_i(x) := x^{\alpha_i} and not the operators D^k though, in this special instance, the set of
the formally-differentiated expansions may be equivalently expressed by expansions involving
the standard derivatives. In our present general context it may be opportune to preliminarily
investigate which differential operators are likely to be formally applicable to an expansion
(1.1). A possible approach consists in investigating the case of an asymptotic expansion
with a zero remainder i.e. a relation of type
\begin{equation}
(3.9) \quad f(x) = a_1 \phi_1(x) + \cdots + a_n \phi_n(x).
\end{equation}

A first very general answer comes from the hierarchies of the Wronskians; a second, less
general but practically more meaningful, answer comes from the use of C.F.’s; third, a C.F.
of type (II) turns out to play a special role in computing the coefficients of an asymptotic
expansion. We have used locutions such as “formal application of an operator” in an
intuitive way but it is good to give a precise definition to avoid possible incongruences
arising from identically-zero terms.

**Definition 3.1** (Asymptotically-admissible operators). Let L be a linear operator act-
ing between two linear spaces of real- or complex-valued functions of one real variable.
First, if L[\phi_i(x)] \equiv 0 on some neighborhood of x_0 \forall i, in particular if \phi_i \in \ker L \forall i, then
the concept in question is not defined. If this is not the case we put:
\begin{equation}
(3.10) \quad m := \max\{i \in \{1, \ldots, n\} : L[\phi_i(x)] \neq 0 \text{ on a neighborhood of } x_0\},
\end{equation}
and say that L is asymptotically admissible with respect to a given asymptotic expansion
\begin{equation}
(3.11) \quad f(x) = a_1 \phi_1(x) + \cdots + a_n \phi_n(x) + o(\phi_n(x)), \ x \to x_0,
\end{equation}
if its formal application to both sides of (3.11) yields
\begin{equation}
(3.12)_1 \quad L[f(x)] = a_1 L[\phi_1(x)] + \cdots + a_m L[\phi_m(x)] + o(L[\phi_m(x)]), \ x \to x_0,
\end{equation}
wherein
\begin{equation}
(3.12)_2 \quad \begin{cases} L[\phi_1(x)] \gg \cdots \gg L[\phi_n(x)], \ x \to x_0, \text{ after suppression} \\
\text{of all the terms } \equiv 0 \text{ on some neighborhood of } x_0. \end{cases}
\end{equation}

An alternative location for an asymptotically-admissible L is “L is formally applicable
to the asymptotic expansion (3.11)”.

We exhibit two simple examples clarifying the above definition; in each of them the
standard operator d/dx is asymptotically admissible according to Definition 3.1 and in-
consistencies would occur without suppression of the identically-zero terms:
\begin{equation}
(3.13) \quad \begin{cases} f_1(x) := x^2 + \log x + 1 + x^{-1} + e^{-x}, \ x > 0, \\
\quad f_1(x) = x^2 + \log x + 1 + x^{-1} + o(x^{-1}), \ x \to +\infty, \\
f_1'(x) = 2x + x^{-1} - x^{-2} + o(x^{-2}), \ x \to +\infty. \end{cases}
\end{equation}
\[
\begin{aligned}
&f_2(x) := \log x + 1 + \sqrt{x} + x^2, \ x > 0, \\
&f_2(x) = \log x + 1 + \sqrt{x} + o(x), \ x \to 0^+, \\
&f_2'(x) = x^{-1} + \frac{1}{2}x^{-1/2} + o(1), \ x \to 0^+.
\end{aligned}
\]

(3.14)

3-A. The approach through the Wronskians, based on Levin’s theorem on hierarchies

A mere rereading of Proposition 2.5-(ii) gives

**Proposition 3.1.** Referring to a scale satisfying (2.1)-(2.4) and (2.6) consider the operators

\[
\mathcal{L}_{\phi_1, \ldots, \phi_k} u := W(\phi_{i_1}, \ldots, \phi_{i_k}, u), \ 1 \leq i_1 < i_2 < \cdots < i_k \leq n; \ 1 \leq k \leq n - 1,
\]

which are kth-order linear differential operators whose leading coefficients never vanish on \([T, x_0]\) (as follows from Proposition 2.6-(v)). Then all these operators are asymptotically admissible with respect to relation (3.9) viewed as an asymptotic expansion with zero remainder; and this means that each relation

\[
\mathcal{L}_{\phi_1, \ldots, \phi_k} f = \sum_{i=1}^{n} a_i W(\phi_{i_1}, \ldots, \phi_{i_k}, \phi_i)
\]

is again an asymptotic expansion at \(x_0\) with zero remainder. For instance we have the identities

\[
\mathcal{L}_{\phi_k} f = a_1 W(\phi_k, \phi_1) + \cdots + a_{k-1} W(\phi_k, \phi_{k-1}) + \\
+ a_{k+1} W(\phi_k, \phi_{k+1}) + \cdots + a_n W(\phi_k, \phi_n),
\]

wherein

\[
W(\phi_k, \phi_1) \gg W(\phi_k, \phi_2) \gg \cdots \gg W(\phi_k, \phi_{k-1}) \gg \\
\gg W(\phi_k, \phi_{k+1}) \gg \cdots \gg W(\phi_k, \phi_n), \ x \to x_0^-,
\]

(3.18)

for each fixed \(k, 1 \leq k \leq n - 1, n \geq 3\). (For \(n = 2\) the chain (3.18) has only one term). And we also have the identities

\[
\mathcal{L}_{\phi_h, \phi_k} f = \sum_{i=1}^{n} a_i W(\phi_h, \phi_k, \phi_i)
\]

wherein

\[
\begin{aligned}
&W(\phi_h, \phi_k, \phi_1) \gg W(\phi_h, \phi_k, \phi_2) \gg \cdots \gg \\
&\gg W(\phi_h, \phi_k, \phi_{h-1}) \gg W(\phi_h, \phi_k, \phi_{h+1}) \gg \cdots \gg \\
&\gg W(\phi_h, \phi_k, \phi_{k-1}) \gg W(\phi_h, \phi_k, \phi_{k+1}) \gg \cdots \gg W(\phi_h, \phi_k, \phi_n), \ x \to x_0^-,
\end{aligned}
\]

(3.20)

for fixed \(h, k : 1 \leq h < k \leq n, n \geq 4\). (For \(n = 3\) the chain (3.20) has only one term).
Proposition 3.1 gives rise to a conjecture:

**Conjecture A.** Referring to the asymptotic expansion (1.1), or (1.5) with $i < n$, there are many linear differential operators, namely (3.15), which are likely to be formally applicable under reasonable hypotheses.

3-B. The special operators associated to canonical factorizations

In the Wronskians in (3.15) a permutation of $(\phi_1, \ldots, \phi_n)$ is immaterial a sign apart, hence there are exactly $(2^n - 2)$ essentially different operators of type (11.5) i.e. the number of the distinct subsets of $(\phi_1, \ldots, \phi_n)$ with cardinality $k : 1 \leq k \leq n - 1$. Now for $n \geq 3$ the object of our study, in a general formulation, involves a sequence of "nested" operators:

\[
\mathcal{L}_{\phi_1}; \mathcal{L}_{\phi_1, \phi_2}; \ldots; \mathcal{L}_{\phi_1, \phi_2, \ldots, \phi_k};
\]

where "nested" refers to the inclusions of their kernels and the problem consists in finding sufficient, and possibly necessary, conditions for the validity of the set of asymptotic relations

\[
\begin{align*}
\mathcal{L}_{\phi_1}[f(x)] &= \sum_{i=1}^{n} a_i W(\phi_i, \phi_i; x) + o(\psi_1(x)), \\
\mathcal{L}_{\phi_1, \phi_2}[f(x)] &= \sum_{i \neq j}^{1, \ldots, n} a_i W(\phi_i, \phi_i; x) + o(\psi_1(x)), \\
\mathcal{L}_{\phi_1, \phi_2, \ldots, \phi_k}[f(x)] &= \sum_{i \neq j}^{1, \ldots, n} a_i W(\phi_i, \ldots, \phi_i, \phi_i; x) + o(\psi_1(x)),
\end{align*}
\]

with proper choices of the $\psi_i$’s. Once a subset $(\phi_1, \ldots, \phi_k)$ has been fixed there is no a-priori reason to prefer one permutation of the $\phi_i$’s to another but it turns out that each ordered $k$-tuple $(\phi_1, \ldots, \phi_k)$ is linked to a special factorization of $\mathcal{L}_{\phi_1, \ldots, \phi_k}$, possibly valid on a neighborhood of $x_0$ smaller than $[T, x_0]$ and calculations can be successfully carried out only under proper integrability assumptions on the coefficients of the factorization. Now a generic factorization of $\mathcal{L}_{\phi_1, \ldots, \phi_k}$, say (2.9), assumed valid on $[T, x_0]$, involves the differential operators

\[
r_0(x)u; \ r_1(x)(r_0(x)u)'; \ r_2(x)[r_1(x)(r_0(x)u)]'' \ldots
\]

which we label as "weighted derivatives of orders 0, 1, 2 etc with respect to the weights $(r_0, r_1, \ldots, r_n)$" in preference to the (some-times used) generic locutions of "quasi-derivatives or generalized derivatives" with no reference to the $n$-tuples of weights. For convenience we include the operator of order zero. Operators (3.23) are not always linked to operators (3.15) nor they preserve the hierarchy (2.2) but the two $C.F.$’s highlighted in Proposition 2.6 yield two sequences of differential operators of orders 0, 1, 2, $\ldots, n - 1$ which are strictly related to operators (3.15) and preserve the hierarchy (2.2); these operators were the core of the asymptotic theory in the case of real-power expansions [6; 7] hence they deserve a special attention and, as a matter of fact, the most meaningful results of our theory are based on them.
Referring to the factorization of type (I) in (2.37), with the $p_i$’s in (2.40), we define the differential operators

$$
\begin{align*}
L_0u &:= p_0(x)u; \quad L_ku \equiv pk[p_k-1(\ldots(p_0u)\ldots)]', \quad 1 \leq k \leq n; \\
M_nu &\equiv L_{\phi_1,\ldots,\phi_n}u;
\end{align*}
$$

which satisfy the recursive formula

$$L_ku := p_k(x)(L_{k-1}u)', \quad 1 \leq k \leq n.
$$

And referring to the factorization of type (II) (2.33), with the $q_i$’s in (2.29), we define the differential operators

$$
\begin{align*}
M_0u &:= q_0(x)u; \quad M_ku := q_k(x)[q_{k-1}(x)(\ldots(q_0(x)u)\ldots)]', \quad 1 \leq k \leq n; \\
M_nu &\equiv L_{\phi_1,\ldots,\phi_n}u;
\end{align*}
$$

which satisfy the recursive formula

$$M_ku := q_k(x)(M_{k-1}u)', \quad 1 \leq k \leq n.
$$

It is easily shown that the operators $L_k$ and $M_k$, $1 \leq k \leq n - 1$, are linked to (3.15) through the formulas

$$L_ku = c_kW(\phi_n, \phi_{n-1}, \ldots, \phi_{n-k+1}, u)W(\phi_n, \phi_{n-1}, \ldots, \phi_{n-k}), \quad 1 \leq k \leq n - 1, c_k = constant \neq 0;
$$

$$M_ku = \frac{W(\phi_1, \ldots, \phi_k, u)}{W(\phi_1, \ldots, \phi_k, \phi_{k+1})}, \quad 1 \leq k \leq n - 1;
$$

hence they preserve the hierarchy (2.2), namely we have the following asymptotic scales

$$L_k[\phi_1(x)] \gg L_k[\phi_2(x)] \gg \cdots \gg L_k[\phi_{n-k}(x)], x \to x_0^-;
$$

$$M_k[\phi_{k+1}(x)] \gg M_k[\phi_{k+2}(x)] \gg \cdots \gg M_k[\phi_n(x)], x \to x_0^-;
$$

for each fixed $k, 0 \leq k \leq n - 2$. For $k = 0$ they respectively reduce to

$$p_0(x)\phi_1(x) \gg p_0(x)\phi_2(x) \gg \cdots \gg p_0(x)\phi_n(x), x \to x_0^-;
$$

$$q_0(x)\phi_1(x) \gg q_0(x)\phi_2(x) \gg \cdots \gg q_0(x)\phi_n(x), x \to x_0^-;
$$

both equivalent to (2.2). It follows that applying each $n$-tuple of operators $L_k$ and $M_k$, $0 \leq k \leq n - 1$, to (3.9) yields again an asymptotic expansion with zero remainders and in this sense we may say that "the asymptotic expansion (3.9) is formally differentiable $(n - 1)$ times with respect to the $n$-tuples of weights $(p_0, \ldots, p_{n-1})$ and $(q_0, \ldots, q_{n-1})$" neglecting the $n$th-order weighted derivative which is $\equiv 0$. The above discussion leads to the following
Conjecture B (Particularization of Conjecture A). Chosen a factorization of $L_{\phi_1, \ldots, \phi_n}$

\[(3.34) \quad L_{\phi_1, \ldots, \phi_n} u \equiv r_n [r_{n-1} \ldots (r_0 u') \ldots']' \quad \forall u \in AC^{n-1}[T, x_0],\]

which be a C.F. of type either (I) or (II) at $x_0$, there exists a linear subspace $\mathcal{D} \in AC^{n-1}[T, x_0]$, such that

(i) $\mathcal{D} \supset \not= \text{span} (\phi_1, \ldots, \phi_n),$

(ii) each $f \in \mathcal{D}$ has an asymptotic expansion of type (1.1) which is formally differentiable $(n-1)$ times with respect to the n-tuples of weights $(r_0, r_1, \ldots, r_{n-1})$.

The problem consists in finding out analytic conditions characterizing the elements of $\mathcal{D}$ for a C.F. of type (I) or (II) separately. The foregoing approach suggests a smallness condition involving the quantity $L_{\phi_1, \ldots, \phi_n}[f(x)]$ which is $\equiv 0$ whenever the remainder in the expansion is.

3-C. The coefficients of an asymptotic expansion with zero remainder

A third fact we wish to investigate is the possible expressions of the coefficients of an asymptotic expansion alternatively to the recurrent formulas (1.3), so generalizing (3.4). It is clear from the study of polynomial expansions in [5] that the C.F. of type (I) is of no use to this end whereas the right approach is via a C.F. of type (II) by establishing a link between the coefficients of (3.9) and the limits of the weighted derivatives.

**Proposition 3.2** (The coefficients of an asymptotic expansion with zero remainder). Referring to the Chebyshev asymptotic scale (2.1)-(2.4) and (2.6) and to the special factorization (2.33) the following facts hold true for the differential operators $M_k$ in (3.26).

(I) Besides formula (3.29) the $M_k$’s satisfy the following relations:

\[(3.35) \quad M_0[\phi_h(x)] \equiv (-1)^{h+1}, \quad 1 \leq h \leq n;\]

\[(3.36) \quad \ker M_k = \text{span} (\phi_1, \ldots, \phi_k), \quad 1 \leq k \leq n;\]

\[(3.37) \quad M_k[\phi_{k+1}(x)] \equiv 1, \quad 1 \leq k \leq n-1;\]

\[(3.38) \quad \begin{cases} 
M_k[\phi_h(x)] = (-1)^{h+k+1} \int_{x_0}^{x} \frac{1}{q_{k+1}} \cdots \int_{x_0}^{x_0} \frac{1}{q_{h-1}} = \\
\quad = o(1), \quad x \to x_0^+; \quad 1 \leq k \leq h-2, \quad h \leq n.
\end{cases}\]

(II) For a fixed $k$, $1 \leq k \leq n$, we have the logical equivalence:

\[(3.39) \quad M_{k-1}[f(x)] \equiv a_k = \text{constant on some interval } J\]

iff

\[(3.40) \quad f(x) = a_1 \phi_1(x) + \cdots + a_k \phi_k(x) \text{ on } J \text{ for some constants } a_i,\]
\( a_k \) being the same as in (3.39).

If (3.39)-(3.40) hold true on a left neighborhood of \( x_0 \) then the following limits exist as finite numbers and

\[
(3.41) \quad a_h = \lim_{x \to x_0^-} M_{h-1}[f(x)], \; 1 \leq h \leq k.
\]

For \( h=k \) (3.41) is the identity (3.39).

We stress that the equivalence "(3.39) \Leftrightarrow (3.40)" is an algebraic fact based on (3.36)-(3.37) whereas the inference "(3.39)-(3.40) \Rightarrow (3.41)" is an asymptotic property whose validity requires that \( (\phi_1, \ldots, \phi_k) \) be an asymptotic scale at \( x_0 \) and that the operators \( M_k \) be defined as specified. Proposition 3.2 suggests the following

**Conjecture C.** If all the limits in (3.41) exist as finite numbers for some function \( f \) sufficiently regular on a left deleted neighborhood of \( x_0 \) then an asymptotic expansion

\[
(3.42) \quad f(x) = a_1 \phi_1(x) + \cdots + a_k \phi_k(x) + o(\phi_k(x)), \; x \to x_0^-,
\]

holds true matched to other expansions obtained by formal applications of the operators \( M_k \). Moreover it is worth investigating if the validity of the sole last relation in (3.41), i.e. for \( h = k \), implies the validity of the other relations.

We shall give complete answers to Conjectures B and C in §§4.5.

**3-D. An heuristic approach via L'Hôpital's rule.**

There is another way to arrive at Conjecture C by the elementary use of L'Hôpital's rule. The following calculations on the limits in (1.3), if legitimate, would yield:

\[
(3.43) \quad \begin{cases} 
  a_1 = \lim_{x \to x_0^-} f/\phi_1; \\
  a_2 = \lim_{x \to x_0^-} \frac{f - a_1 \phi_1}{\phi_2} = \lim_{x \to x_0^-} \frac{(f/\phi_1) - a_1 (= o(1))}{\phi_2/\phi_1} = H \lim_{x \to x_0^-} \frac{(f/\phi_1)' - a_1}{\phi_2/\phi_1}' = H \lim_{x \to x_0^-} M_1[f(x)]; \\
  a_3 = \lim_{x \to x_0^-} \frac{f - a_1 \phi_1 - a_2 \phi_2}{\phi_3} = \lim_{x \to x_0^-} \frac{(f/\phi_1) - a_1 - a_2(\phi_2/\phi_1)}{\phi_3/\phi_1} = H \lim_{x \to x_0^-} \frac{(f/\phi_1)' - a_1 - a_2(\phi_2/\phi_1)'}{\phi_3/\phi_1}' = H \lim_{x \to x_0^-} M_2[f(x)]; \\
  \vdots \\
  H \lim_{x \to x_0^-} \frac{(f/\phi_1)'(\phi_2/\phi_1)'}{(\phi_3/\phi_1)'(\phi_2/\phi_1)'} = H \lim_{x \to x_0^-} M_2[f(x)]; \text{ and so on.}
\end{cases}
\]

Now such kind of manipulations may seem artificial and awkward from an elementary viewpoint and it is by no means obvious that iterating the procedure yields the relations in (3.41) for \( h \geq 4 \) as well. In one of the two algorithms presented in §8 (Proposition 8.1) it will be shown that the procedure is quite natural in the context of formal differentiation of an asymptotic expansion and that it actually leads to (3.41) for all values of \( h \).
4. The first factorizational approach

We start from the "unique" $C.F.$ of our operator $L_{\phi_1,\ldots,\phi_n}$ on the interval $[T,x_0]$ of type (I) at $x_0$, i.e. identity (2.37) with conditions (2.38)-(2.39) and the $p_i$'s satisfying the same conditions as the $q_i$'s in (2.30). We consider the fundamental system (2.42). The ordered $n$-tuple $(P_{n-1},\ldots,P_0)$ is an asymptotic scale at $x_0$, by (2.43), but it cannot coincide (constant factors apart) with the given scale $(\phi_1,\ldots,\phi_n)$ as (2.36) and (2.43) are incompatible. However (2.2) and (2.43) imply that the two scales are linked by the following relations

\begin{equation}
\phi_i(x) \sim b_i P_{n-i}(x), \ x \to x_0^-, \ 1 \leq i \leq n,
\end{equation}

with suitable nonzero constants $b_i$, hence

\begin{equation}
\begin{cases}
\phi_i(x) = b_i P_{n-i}(x) + \sum_{j=i+1}^{n} \beta_{i,j} P_{n-j}(x), & 1 \leq i \leq n-1, \\
\phi_n(x) = b_n P_0(x),
\end{cases}
\end{equation}

and vice versa

\begin{equation}
P_0(x) = \frac{1}{b_n} \phi_n(x), \ P_i(x) = \frac{1}{b_{n-i}} \phi_{n-i}(x) + \sum_{j=n-i+1}^{n} \tilde{\beta}_{i,j} \phi_j(x), \ 1 \leq i \leq n-1,
\end{equation}

with suitable constants $\beta_{i,j}, \tilde{\beta}_{i,j}$.

In this approach the appropriate differential operators to be used are the $L_k$'s defined in (3.24) and here are some elementary properties of these operators.

**Lemma 4.1.** The following relations are checked at once

\begin{equation}
\ker L_k = \begin{cases}
\text{span} (P_0, P_1, \ldots, P_{k-1}) \\
\text{span} (\phi_n, \phi_{n-1}, \ldots, \phi_{n-k+1}), & 1 \leq k \leq n;
\end{cases}
\end{equation}

\begin{equation}
L_k[P_k(x)] \equiv 1, \ 0 \leq k \leq n-1;
\end{equation}

\begin{equation}
L_k[P_i(x)] \equiv \int_{T}^{x} \frac{dt_{k+1}}{p_{k+1}(t_{k+1})} \cdots \int_{T}^{t_{i-1}} \frac{dt_i}{p_i(t_i)}, \ 0 \leq k < i \leq n-1;
\end{equation}

\begin{equation}
L_k[P_i(x)] \ll L_k[P_{i+1}(x)], \ x \to x_0^-, \ 0 \leq k \leq i \leq n-2.
\end{equation}

Hence we have the following chains of asymptotic relations:

\begin{equation}
\begin{cases}
L_0[P_0(x)] \ll L_0[P_1(x)] \ll \cdots \ll L_0[P_{n-1}(x)], \\
L_1[P_1(x)] \ll L_1[P_2(x)] \ll \cdots \ll L_1[P_{n-1}(x)], \\
L_2[P_2(x)] \ll L_2[P_3(x)] \ll \cdots \ll L_2[P_{n-1}(x)], \ x \to x_0^-, \\
\cdots \cdots \\
L_{n-2}[P_{n-2}(x)] \ll L_{n-2}[P_{n-1}(x)].
\end{cases}
\end{equation}
The first chain in (4.8) coincides with the second chain in (2.43) apart from the ordering and the multiplicative factor \( p_0(x) \). As the first term in each chain is the constant ”1” all the other terms diverge to \(+\infty\).

**Lemma 4.2.** If a solution \( \phi \) of \( L_{\phi_1,\ldots,\phi_n}u = 0 \) satisfies the asymptotic relation
\[
\phi(x) \sim cP_i(x), \quad x \to x_0^-,
\]
for some \( i \in \{0, 1, \ldots, n-1\} \) and some nonzero constant \( c \) then the following relations hold true
\[
L_k[\phi(x)] \sim cL_k[P_i(x)], \quad x \to x_0^-, \quad 0 \leq i \leq n-1;
\]
(4.10)
\[
L_k[\phi(x)] \equiv 0, \quad i + 1 \leq k \leq n.
\]
Moreover
(4.12)
\[
L_k[\phi_{n-i}(x)] = \begin{cases} b_{n-k}, & 0 \leq i = k \leq n-1, \\ 0, & 0 \leq i < k, \end{cases}
\]
with the \( b_i \)'s defined in (4.1). It follows from (4.1) and (4.10) that all relations in (4.8) hold true after replacing \( P_i \) by \( \phi_{n-i} \) hence we have the asymptotic scales:

\[
\begin{align*}
L_0[\phi_1(x)] & \gg L_0[\phi_2(x)] \gg \cdots \gg L_0[\phi_n(x)], \\
L_1[\phi_1(x)] & \gg L_1[\phi_2(x)] \gg \cdots \gg L_1[\phi_{n-1}(x)], \\
L_2[\phi_1(x)] & \gg L_2[\phi_2(x)] \gg \cdots \gg L_2[\phi_{n-2}(x)], \quad x \to x_0^-, \\
\cdots \cdots \\
L_{n-2}[\phi_1(x)] & \gg L_{n-2}[\phi_2(x)].
\end{align*}
\]

Last, with the \( b_i \)'s defined in (4.1), formula (3.28) becomes
\[
L_k u = b_{n-k} \frac{W(\phi_n, \phi_{n-1}, \ldots, \phi_{n-k+1}, u)}{W(\phi_n, \phi_{n-1}, \ldots, \phi_{n-k})}, \quad 1 \leq k \leq n-1.
\]

**Lemma 4.3.** Any function \( f \in AC^{n-1}[T, x_0] \) admits of a representation of type
\[
f(x) = c_1\phi_1(x) + \cdots + c_n\phi_n(x) +
\]
(4.15)
\[
+ \frac{1}{p_0(x)} \int_T^x \cdots \int_T^{t_{n-2}} \frac{1}{p_{n-1}} \int_T^{t_{n-1}} \frac{L_{\phi_1,\ldots,\phi_n}[f(t)]}{p_n(t)} dt, \quad x \in [T, x_0],
\]
with suitable constants \( c_i \). From (4.6), (4.12) and (4.15) we infer the following representations of the weighted derivatives of \( f \) with respect to the weight functions \((p_0, \ldots, p_n)\):
\[
L_k[f(x)] = c_1L_k[\phi_1(x)] + \cdots + c_{n-k}L_k[\phi_{n-k}(x)] +
\]
(4.16)
\[
+ \int_T^x \frac{dt_{k+1}}{p_{k+1}(t_{k+1})} \cdots \int_T^{t_{n-2}} \frac{dt_{n-1}}{p_{n-1}(t_{n-1})} \int_T^{t_{n-1}} \frac{L_{\phi_1,\ldots,\phi_n}[f(t)]}{p_n(t)} dt,
\]
for \( x \in [T, x_0]; \quad 0 \leq k \leq n-2; \)
which is the explicit form of the relation in (4).

Let of type (I) ).

We shall now characterize various situations wherein relations (4.16)-(4.17) become asymptotic expansions. In the following two theorems we state separately three cases of a single claim lest a unified statement be obscure. The reader is referred to the first remark after next theorem to grasp the meaning of the differentiated asymptotic expansions which exhibit a special non-common phenomenon.

**Theorem 4.4** (Asymptotic expansions formally differentiable according to the C.F. of type (I)). Let \( f \in AC^{n-1}[T,x_0][T,x_0]. \)

(I) The following are equivalent properties for a suitable constant \( a_1 \):

(i) The set of asymptotic relations

\[
L_k[f(x)] = a_1 L_k[\phi_1(x)] + o(L_k[\phi_1(x)]), \ x \to x_0^-, \ 0 \leq k \leq n - 1.
\]

(ii) The single asymptotic relation

\[
L_{n-1}[f(x)] = a_1 b_1 + o(1), \ x \to x_0^-, \text{ with } b_1 \text{ defined in (4.1)},
\]

which is the explicit form of the relation in (4.18) for \( k = n - 1 \).

(iii) The improper integral

\[
\int_T^{x_0} L_{\phi_1, \ldots, \phi_n}[f(t)] \frac{dt}{p_n(t)} \text{ converges.}
\]

Under condition (4.20) we have the representation formula:

\[
L_{n-1}[f(x)] = a_1 b_1 - \int_T^{x_0} L_{\phi_1, \ldots, \phi_n}[f(t)] \frac{dt}{p_n(t)}, \ x \in [T,x_0].
\]

(II) For a fixed \( i \in \{2, \ldots, n\} \) the following are equivalent properties for suitable constants \( a_i \) (the same in each set of conditions):

(iv) The set of asymptotic expansions as \( x \to x_0^+ \):

\[
L_k[f(x)] = a_1 L_k[\phi_1(x)] + \cdots + a_i L_k[\phi_i(x)] + o(L_k[\phi_i(x)]), \ 0 \leq k \leq n - i;
\]

\[
L_{n-i+h}[f(x)] = a_1 L_{n-i+h}[\phi_1(x)] + \cdots + a_{i-h} L_{n-i+h}[\phi_{i-h}(x)] + o(1); \ 0 \leq h \leq i - 1.
\]

(v) The second group of asymptotic expansions in (4.22), i.e.

\[
L_{n-i+h}[f(x)] = a_1 L_{n-i+h}[\phi_1(x)] + \cdots + a_{i-h} L_{n-i+h}[\phi_{i-h}(x)] + o(1), \ x \to x_0^+, \ 0 \leq h \leq i - 1,
\]

By (4.13) the linear combination \( \sum_{i=1}^{n-k} c_i L_k[\phi_i(x)] \) in the right-hand side of (4.16) is in itself an asymptotic expansion at \( x_0^+ \) for each fixed \( k \).
where we point out that the last meaningful term in the right-hand side is a constant.

(vi) The following improper integral, involving “i” iterated integrations,

$$\int_T^{x_0} \frac{1}{p_{n-i+1}} \int_{t_{n-i+1}}^{x_0} \frac{1}{p_{n-i+2}} \cdots \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} L_{\phi_1, \ldots, \phi_n}[f(t)] \, dt$$ converges.

Under condition (4.24) we have the representation formula:

$$L_{n-i}[f(x)] = a_1 L_{n-i-1}[\phi_1(x)] + \cdots + a_i L_{n-i}[\phi_i(x)] + \int_x^{x_0} \frac{1}{p_{n-i+1}} \int_{t_{n-i+1}}^{x_0} \frac{1}{p_{n-i+2}} \cdots \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} L_{\phi_1, \ldots, \phi_n}[f(t)] \, dt,$$

for $x \in [T, x_0]$, as well as the corresponding formulas for the functions $L_{n-i+h}[f(x)]$ with $0 \leq h \leq i - 1$, obtained by suitable differentiations of (4.25): see remark 3 below.

Remarks. 1. Relations in (4.22) may be read as follows. The first relation, involving $L_0$, is equivalent to the asymptotic expansion

$$f(x) = a_1 \phi_1(x) + \cdots + a_i \phi_i(x) + o(\phi_i(x)), \ x \to x_0^-,$$

and the relations involving $L_k$, with $1 \leq k \leq n - i$, state that (4.26) can be formally differentiated $(n - i)$ times in the sense of formally applying the operators $L_k$ to the remainder in (4.26). In so doing one arrives at the expansion

$$L_{n-i}[f(x)] = a_1 L_{n-i-1}[\phi_1(x)] + \cdots + a_i L_{n-i}[\phi_i(x)] + o(1), \ x \to x_0^-,$$

where $L_{n-i}[\phi_i(x)] \equiv$ constant. The process of formal differentiation, from the order $(n-i+1)$ up to $(n-1)$, goes on according to the following rule: in (4.27) and in each expansion in (4.23) the last term is constant and is lost after one further weighted differentiation while the remainder preserves its simple growth-order estimate of ”$O(1)$”. So the first $(n-i+1)$ expansions, i.e. those involving $L_0, L_1, \ldots, L_{n-i}$, have the same number of meaningful terms whereas each of the other $(i-1)$ expansions is deprived of the last meaningful term at each successive differentiation. We rewrite more explicitly the expansions in (4.22) to better highlight the dynamics of this process:

$$\begin{align*}
\begin{cases}
 f(x) = a_1 \phi_1(x) + \cdots + a_i \phi_i(x) + o(\phi_i(x)), \\
 L_1[f(x)] = a_1 L_1[\phi_1(x)] + \cdots + a_i L_1[\phi_i(x)] + o(L_1[\phi_i(x)]), \\
 \cdots \cdots \\
 L_{n-i}[f(x)] = a_1 L_{n-i-1}[\phi_1(x)] + \cdots + a_i L_{n-i}[\phi_i(x)] + o(1), \\
 L_{n-i+1}[f(x)] = a_1 L_{n-i+1}[\phi_1(x)] + \cdots + a_i L_{n-i+1}[\phi_i(x)] + o(1), \\
 \cdots \cdots \\
 L_{n-2}[f(x)] = a_1 L_{n-2}[\phi_1(x)] + a_2 L_{n-2}[\phi_2(x)] + o(1), \\
 L_{n-1}[f(x)] = a_1 L_{n-1}[\phi_1(x)] + o(1).
\end{cases}
\end{align*}$$
The loss of the last meaningful term, where it occurs, is caused by formula (4.12) for \( i = k - 1 \) which, after renaming the indexes, reads

\[
L_{n-i} = a_1L_{n-i}[\phi_1(x)] + \cdots + a_iL_{n-i}[\phi_i(x)] + o(1), \ x \to x_0^-.
\]

Notice that in the second group of expansions in (4.28) the meaningful terms disappear one after one in reversed order if compared with Taylor's formula.

2. It is shown in §7, after the proof of Theorem 4.4, that the set (4.23) is not equivalent in general to the single relation

\[
L_{n-i}[f(x)] = a_1L_{n-i}[\phi_1(x)] + \cdots + a_iL_{n-i}[\phi_i(x)] + o(1), \ x \to x_0^-,
\]
as in part (I) of the theorem (case \( i = 1 \)).

3. Suitable weighted differentiations of (4.25) yield integral representations of the remainders in the differentiated expansions of orders greater than \((n-i)\). The improper integral

\[
\int_T^x \frac{1}{p_1} \cdots \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} \frac{L_{\phi_1,\ldots,\phi_n}[f(t)]}{p_n(t)} \, dt
\]

converges.

(iii) There exist \( n \) real numbers \( a_1, \ldots, a_n \) and a function \( \Phi_n \) Lebesgue-summable on \([T, x_0] \) such that

\[
f(x) = a_1\phi_1(x) + \cdots + a_n\phi_n(x) + \left( -\frac{1}{p_0(x)} \right) \int_x^{x_0} \frac{1}{p_1} \int_{t_1}^{x_0} \frac{1}{p_2} \cdots \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} \Phi_n(t) \, dt, \ x \in [T, x_0].
\]

If this is the case \( \Phi_n \) is determined up to a set of measure zero and

\[
\Phi_n(x) = \frac{1}{p_n(x)} L_{\phi_1,\ldots,\phi_n}[f(x)] \ a.e. \ on \ [T, x_0].
\]
Notice that in (4.33) it is \((-1)^n/p_0(x) = -\phi_n(x)\), by (2.40) and (2.35).

The phenomenon described in the above theorems is intrinsic in the theory; it occurs even in the seemingly elementary case of real-power expansions, [7; Thm. 4.2-(ii), p. 181, and formula (7.2), p. 195], where the asymptotic scale enjoys the most favourable algebraic properties. This type of formal differentiation of an asymptotic expansion does not frequently occur in the literature though the results in this section show that it is one of the possible natural situations. An instance (not inserted in a general theory) is to be found in a paper by Schoenberg [22; Thm. 3, p. 258] and refers to the asymptotic expansion

\[(4.35) \quad f(x) = a_1x^{-1} + a_2x^{-2} + \cdots + a_nx^{-n} + O(x^{-n-1}), \ x \to +\infty.\]

5. The second factorizational approach and estimates of the remainder

Now we face our problem starting from a C.F. of type (II) at \(x_0\). Referring to Proposition 2.6 the most natural choice is the special C.F. of \(L_{\phi_1,\ldots,\phi_n}\) defined in (2.33), with the \(q_i\)'s in (2.29) and satisfying conditions (2.32). According to Conjectures B and C in §3 we shall characterize a set of asymptotic expansions, involving the operators \(M_k\) defined in (3.26), wherein each coefficient of the first expansion may be found by an independent limiting process instead of the recursive formulas (1.3), and the existence of the sole last coefficient implies the existence of all the preceding coefficients.

In this new context a representation of the following type is appropriate for any function \(f \in AC^{n-1}[T, x_0]\):

\[
(5.1) \quad f(x) = c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \frac{1}{q_0(x)} \int_T^x \frac{1}{q_1} \cdots \int_T^{t_{n-2}} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L_{\phi_1,\ldots,\phi_n}[f(t)]}{q_n(t)} dt, \ x \in [T, x_0[,
\]

with suitable constants \(c_i\). Applying the operators \(M_k\) to (5.1) we get the following representations of the weighted derivatives of \(f\) with respect to the weight functions \((q_0, \ldots, q_n)\):

\[
(5.2) \quad M_k[f(x)] = c_{k+1}M_k[\phi_{k+1}(x)] + \cdots + c_nM_k[\phi_n(x)] + \int_T^x \frac{dt_{k+1}}{q_{k+1}(t_{k+1})} \cdots \int_T^{t_{n-2}} \frac{dt_{n-1}}{q_{n-1}(t_{n-1})} \int_T^{t_{n-1}} \frac{L_{\phi_1,\ldots,\phi_n}[f(t)]}{q_n(t)} dt, \ 0 \leq k \leq n-1.
\]

By (3.37) the last relation in (5.2) explicitly is

\[
(5.3) \quad M_{n-1}[f(x)] = c_n + \int_T^x \frac{L_{\phi_1,\ldots,\phi_n}[f(t)]}{q_n(t)} dt, \ x \in [T, x_0[.
\]

By (3.31) the ordered linear combination in (5.2),

\[
(5.4) \quad \sum_{i=k+1}^n c_iM_k[\phi_i(x)],
\]
is an asymptotic expansion at \( x_0 \) for each fixed \( k, 0 \leq k \leq n - 1 \).

We state separately the result concerning a complete asymptotic expansion, i.e. of type (1.1), because it is the most expressive result in this paper and characterizes the simple circumstance that \( M_{n-1}[f(x)] = a_n + o(1) \) via a set of \( n \) asymptotic expansions. Always refer to Proposition 3.2 for properties of the \( M_k \)’s.

Theorem 5.1 (Complete asymptotic expansions formally differentiable according to a C.F. of type (II)). Let \( f \in AC^{n-1}[T, x_0] \).

(I) The following are equivalent properties:

(i) There exist \( n \) real numbers \( a_1, \ldots, a_n \) such that

\[
(5.5) \quad f(x) = a_1 \phi_1(x) + \cdots + a_n \phi_n(x) + o(\phi_n(x)), \quad x \to x_0;
\]

(ii) All the following limits exist as finite numbers:

\[
(5.6) \quad \begin{cases} 
M_k[f(x)] = a_{k+1} M_k[\phi_{k+1}(x)] + \cdots + a_n M_k[\phi_n(x)] + \\
+ o(M_k[\phi_n(x)]), \quad x \to x_0; & 1 \leq k \leq n - 1.
\end{cases}
\]

where the first term in each expansion is lost in the successive expansion.

Notice that the relation that would be obtained in (5.6) for \( k = 0 \) differs from relation in (5.5) by the common factor \( q_0(x) \).

(ii) All the following limits exist as finite numbers:

\[
(5.7) \quad \lim_{x \to x_0} M_k[f(x)] \equiv a_{k+1}, \quad 0 \leq k \leq n - 1,
\]

where the \( a_k \)’s coincide with those in (5.5).

(iii) The single last limit in (5.7) exists as a finite number, i.e.

\[
(5.8) \quad \lim_{x \to x_0} M_{n-1}[f(x)] \equiv a_n,
\]

and (5.8) is nothing but the relation in (5.6) for \( k = n - 1 \) which reads \( M_{n-1}[f(x)] = a_n + o(1) \), \( x \to x_0 \).

(iv) The improper integral

\[
(5.9) \quad \int_T^{x_0} \frac{L_{\phi_1, \ldots, \phi_n}[f(t)]}{q_n(t)} dt \quad \text{converges},
\]

and automatically also the iterated improper integral

\[
(5.10) \quad \int_T^{x_0} \frac{1}{q_1} \int_T^{x_0} \frac{1}{q_2} \cdots \int_T^{x_0} \frac{1}{q_{n-1}} \int_T^{x_0} \frac{L_{\phi_1, \ldots, \phi_n}[f(t)]}{q_n(t)} dt \quad \text{converges}.
\]

(v) There exist \( n \) real numbers \( a_1, \ldots, a_n \) and a function \( \Psi_n \) Lebesgue-summable on \([T, x_0]\) such that

\[
(5.11) \quad f(x) = \sum_{i=1}^{n} a_i \phi_i(x) + \frac{(-1)^n}{q_0(x)} \int_T^{x_0} \frac{1}{q_1} \cdots \int_T^{x_0} \frac{1}{q_{n-1}} \int_T^{x_0} \Psi_n(t) dt, \quad x \in [T, x_0),
\]
where we remind that, by (2.29), 1/q₀(x) = φ₁(x). In this case Ψₙ is determined up to a set of measure zero and

\[
Ψₙ(x) = \frac{1}{qₙ(x)}L_{φ₁,...,φₙ}[f(x)] \text{ a.e. on } [T, x₀].
\]

(II) Whenever properties in part (I) hold true we have integral representation formulas for the remainders

\[
\begin{aligned}
R₀(x) &:= f(x) - \sum_{i=1}^{n} a_i φ_i(x) \\
R_k(x) &:= M_k[f(x)] - \sum_{i=1}^{n-k} a_{k+i} M_k[φ_{k+i}(x)], \ 1 ≤ k ≤ n - 1,
\end{aligned}
\]

namely

\[
R₀(x) = (-1)^n \frac{1}{q₀(x)} \int^{x₀} \frac{1}{q₁} \cdots \int^{x₀} \frac{1}{qₙ₋₁} \int^{x₀} \frac{L_{φ₁,...,φₙ}[f(t)]}{qₙ(t)} dt,
\]

\[
R_k(x) = (-1)^{n+k} \int^{x₀} \frac{1}{q_{k+1}} \cdots \int^{x₀} \frac{1}{qₙ₋₁} \int^{x₀} \frac{L_{φ₁,...,φₙ}[f(t)]}{qₙ(t)} dt,
\]

for x ∈ [T, x₀], 1 ≤ k ≤ n - 1. From (5.14) we get the following estimate of R₀ wherein the order of smallness with respect to φₙ is made more explicit than in Theorem 4.5 (formula (2.34) for i = n is used):

\[
|R₀(x)| ≤ |φₙ(x)| \cdot \sup_{t ≥ x} \left| \int^{x₀} \frac{L_{φ₁,...,φₙ}[f(τ)]}{qₙ(τ)} dτ \right|, \ x ∈ [T, x₀].
\]

Under the stronger hypothesis of absolute convergence for the improper integral we get

\[
|R₀(x)| ≤ |φₙ(x)| \cdot \int^{x₀} \frac{L_{φ₁,...,φₙ}[f(t)]}{qₙ(t)} dt, \ x ∈ [T, x₀].
\]

Similar estimates can be obtained for the R_k’s.

Remarks. 1. As noticed in [7; Remark 1 after Thm. 4.1, pp. 179-180] the remarkable inference ”(iii) ⇒ (ii)” is true for the special operator Mₙ₋₁ stemming out from a C.F. of type (II) at x₀ but not for any (n - 1)th-order differential operator originating from an arbitrary factorization of L_{φ₁,...,φₙ}.

2. Condition (5.9) involves the sole coefficient qₙ which admits of the explicit expression (2.29) in terms of φ₁,...,φₙ:

\[
qₙ = W(φ₁,...,φₙ)/W(φ₁,...,φₙ₋₁);
\]

hence (2.9) can be rewritten as

\[
\int^{x₀} \frac{W(φ₁,...,φₙ₋₁; t)}{W(φ₁,...,φₙ; t)} L_{φ₁,...,φₙ}[f(t)] dt \text{ converges.}
\]
For \( n = 2 \) the ratio inside the integral equals \( \phi_1 / W(\phi_1, \phi_2) \) and we reobtain the result in [8; condition (5.15), p. 265].

3. In Theorem 4.5, generally speaking, no such estimates as in (5.16)-(5.17) can be obtained due to the divergence of all the improper integrals in (4.33) if the innermost integral is factored out.

4. Referring to the elementary characterizations in (1.3) of the coefficients \( a_k \), Theorem 5.1 changes the perspective: in (1.3) the \( a_k \)’s are defined recursively whereas in (5.7) each \( a_k \) has its own independent expression and, moreover, the existence of \( a_n \), as the limit in (5.8), implies the existence of \( a_1, \ldots, a_{n-1} \).

In the following result about incomplete expansions formal differentiation is in general legitimate a number of times less than the "length" of the expansion (see Remark 2 after the statement).

**Theorem 5.2** (A result on incomplete asymptotic expansions). Let \( f \in AC^{n-1}[T, x_0] \).

(I) For a fixed \( i \in \{2, \ldots, n-1\} \) the following are equivalent properties:

(i) There exist \( i \) real numbers \( a_1, \ldots, a_i \) such that

\[
(5.20) \quad f(x) = a_1 \phi_1(x) + \cdots + a_i \phi_i(x) + o(\phi_i(x)), \ x \to x_0^{-};
\]

\[
(5.21) \quad \begin{cases} 
M_k[f(x)] = a_{k+1}M_k[\phi_{k+1}(x)] + \cdots + a_i M_k[\phi_i(x)] + \\
\quad + o(M_k[\phi_i(x)]), \ x \to x_0^{-}; \ 1 \leq k \leq i - 1.
\end{cases}
\]

(ii) All the following limits exist as finite numbers

\[
(5.22) \quad \lim_{x \to x_0^{-}} M_k[f(x)] \equiv a_{k+1}, \ 0 \leq k \leq i - 1,
\]

where the \( a_k \)’s coincide with those in (5.20)-(5.21).

(iii) The single last limit in (5.22) exists as a finite number, i.e.

\[
(5.23) \quad \lim_{x \to x_0^{-}} M_{i-1}[f(x)] \equiv a_i,
\]

and (5.23) coincides with the relation in (5.21) for \( k = i - 1 \).

(iv) The improper integral

\[
(5.24) \quad \int_{x_0}^{\to x_0} dt_i \frac{dt_i}{q_i(t_i)} \int_T^{t_i} dt_{i+1} \frac{dt_{i+1}}{q_{i+1}(t_{i+1})} \cdots \int_T^{t_{n-1}} L_{\phi_1, \ldots, \phi_n} [f(t)] \frac{dt}{q_n(t)} \quad \text{converges},
\]

and automatically also the iterated improper integral

\[
(5.25) \quad \int_T^{x_0} \frac{1}{q_i} \cdots \int_{l_{i-1}}^{x_0} \frac{1}{q_i} \int_T^{t_i} \frac{1}{q_{i+1}} \cdots \int_T^{t_{n-1}} L_{\phi_1, \ldots, \phi_n} [f(t)] \frac{dt}{q_n(t)} \quad \text{converges}.
\]

(II) For \( i = 1 \) the theorem simply asserts that the asymptotic relation

\[
(5.26) \quad f(x) = a_1 \phi_1(x) + o(\phi_1(x)), \ x \to x_0^{-},
\]
holds true for some real number $a_1$ iff the improper integral

\[(5.27) \quad \int_T^{x_0} \frac{1}{q_1} \int_T^{t_1} \frac{1}{q_2} \ldots \int_T^{t_{n-1}} L_{\phi_1, \ldots, \phi_n}[f(t)] \frac{dt}{q_n(t)} \text{ converges.} \]

Remarks. 1. We shall see in the proof of Theorem 5.2, formula (7.37), that the representations of the quantities $M_k[f(x)], 0 \leq k \leq i - 1,$ contain some unspecified constants not determinable through the sole condition (5.24) which, for this reason, grants neither explicit representations nor numerical estimates of the remainders of the expansions in (5.20)-(5.21).

2. As concerns estimates of the quantities $M_k[f(x)]$ for $i \leq k \leq n - 1,$ the situation is as follows. Formula (7.36), given in the proof, reads:

\[(5.28) \quad M_i[f(x)] = \int_T^x \frac{dt_{i+1}}{q_{i+1}(t_{i+1})} \ldots \int_T^{t_{n-1}} L_{\phi_1, \ldots, \phi_n}[f(t)] \frac{dt}{q_n(t)} + c + o(1), \]

for some constant $c.$ If, as $x \to x_0^-$, $I(x)$ converges to a real number then we may apply Theorem 5.2 with $i$ replaced by $i + 1$; but if it is unbounded and oscillatory no asymptotic relation more expressive than (5.28) can be obtained generally speaking. On the contrary a favourable situation occurs when it is known a priori that $I(x)$ either converges or diverges to $\pm \infty$ and the corresponding estimates are reported in Theorem 6.3.

**Theorem 5.3** (The analogue of Theorems 5.1-5.2 with "$O$"-estimates). Let $f \in AC^{n-1}[T, x_0]$ and let $i \in \{2, \ldots, n\}$ be fixed. The following are equivalent properties:

(i) There exist $(i - 1)$ real numbers $a_1, \ldots, a_{i-1}$ such that

\[(5.29) \quad f(x) = a_1 \phi_1(x) + \ldots + a_{i-1} \phi_{i-1}(x) + O(\phi_i(x)), \quad x \to x_0^-; \]

\[(5.30) \quad \begin{cases} M_k[f(x)] = a_{k+1}M_k[\phi_{k+1}(x)] + \ldots + a_{i-1}M_k[\phi_{i-1}(x)] + O(M_k[\phi_i(x)]), \quad x \to x_0^-; 1 \leq k \leq i - 1. \end{cases} \]

(ii) All the following relations hold true:

\[(5.31) \quad \begin{cases} \lim_{x \to x_0^-} M_k[f(x)] = a_{k+1}, 0 \leq k \leq i - 2; \\ M_{i-1}[f(x)] = O(1), \quad x \to x_0^-; \end{cases} \]

where the $a_k$’s coincide with those in (5.29)-(5.30).

(iii) It holds true the single last relation in (5.31), i.e.

\[(5.32) \quad M_{i-1}[f(x)] = O(1), \quad x \to x_0^-; \]

(iv) We have the following estimate instead of condition (5.24):

\[(5.33) \quad \int_T^x \frac{dt_i}{q_i(t_i)} \int_T^{t_i} \frac{dt_{i+1}}{q_{i+1}(t_{i+1})} \ldots \int_T^{t_{n-1}} L_{\phi_1, \ldots, \phi_n}[f(t)] \frac{dt}{q_n(t)} = O(1), \quad x \to x_0^-; \]
For $i = n$ condition (5.32) reads

$$
(5.34) \quad \int_T^x \frac{L_{\phi_1, \ldots, \phi_n}[f(t)]}{q_n(t)} dt = O(1), \quad x \to x_0^-,
$$

and representation (5.11)-(5.12) must be replaced by

$$
(5.35) \quad f(x) = \sum_{i=1}^{n-1} a_i \phi_i(x) + \frac{(-1)^{n-1}}{q_0(x)} \int_x^{x_0} \frac{1}{q_1} \int_{t_1}^{x_0} \frac{1}{q_2} \cdots \int_{t_{n-2}}^{x_0} \frac{1}{q_{n-1}} \int_T^{x_{n-1}} \frac{L_{\phi_1, \ldots, \phi_n}[f(t)]}{q_n(t)} dt.
$$

For $i = 1$ the theorem simply asserts that the asymptotic relation

$$
(5.36) \quad f(x) = O(\phi_1(x)), \quad x \to x_0^-,
$$

holds true iff

$$
(5.37) \quad \int_T^x \frac{1}{q_1} \int_{t_1}^{x_0} \frac{1}{q_2} \cdots \int_T^{x_{n-1}} \frac{L_{\phi_1, \ldots, \phi_n}[f(t)]}{q_n(t)} dt = O(1), \quad x \to x_0^-.
$$

An outlook on the theory developed so far. We suggest a way of visualizing what our theory is all about. Referring, say, to the situations characterized in Theorem 5.2 we have an asymptotic scale of the type:

$$
(5.38) \quad \frac{1}{q_0(x)} \gg \frac{1}{q_0(x)} \int_x^{x_0} \frac{1}{q_1} \gg \cdots \gg \frac{1}{q_0(x)} \int_x^{x_0} \frac{1}{q_1} \int_{t_1}^{x_0} \frac{1}{q_2} \gg \frac{1}{q_0(x)} \int_x^{x_0} \frac{1}{q_1} \int_{t_1}^{x_0} \frac{1}{q_2} \int_{t_{i-2}}^{x_0} \frac{1}{q_{i-1}}, \quad x \to x_0^-,
$$

where the $q_i$’s are continuous and everywhere-nonzero functions on some interval $[T, x_0]$, and are interested in the validity of an asymptotic expansion of the type:

$$
(5.39) \quad \begin{cases} 
\frac{f(x)}{q_0(x)} = a_1 \frac{1}{q_0(x)} + a_2 \int_x^{x_0} \frac{1}{q_1} + a_3 \int_x^{x_0} \frac{1}{q_1} \int_{t_1}^{x_0} \frac{1}{q_2} + \cdots \\
+ a_i \frac{1}{q_0(x)} \left[ \int_x^{x_0} \frac{1}{q_1} \int_{t_1}^{x_0} \frac{1}{q_2} \int_{t_{i-2}}^{x_0} \frac{1}{q_{i-1}} \right] \cdot [a_i + o(1)], \quad x \to x_0^-.
\end{cases}
$$

Theorem 5.2 gives characterizations of the set formed by (5.39) and the following expansions obtained in a quite natural way:

$$
(5.40) \quad \begin{cases} 
M_1[f(x)] = q_1(x)(q_0(x)f(x))' = -a_2 \frac{1}{q_1} a_3 \int_x^{x_0} \frac{1}{q_2} + \cdots + \\
- a_i \frac{1}{q_1(x)} \left[ \int_x^{x_0} \frac{1}{q_2} \int_{t_1}^{x_0} \frac{1}{q_2} \int_{t_{i-2}}^{x_0} \frac{1}{q_{i-1}} \right] \cdot [a_i + o(1)], \quad x \to x_0^-, \\
\cdots \cdots \\
M_{i-1}[f(x)] = q_i(x)(q_{i-1}(x)(q_0(x)f(x))' \cdots)' = (-1)^{i-1} a_i + o(1), \quad x \to x_0^-.
\end{cases}
$$
So far the continuity of the $\phi_i$’s would be enough but the stronger regularity assumptions required by our theory make the whole matter richer and with useful representations of the remainders. The formal derivations of (5.40) from (5.39) may even seem a triviality but it is not an automatic fact and we have tied up our theory with the concepts of Chebyshev systems and canonical factorizations, useful in other contexts. Moreover in some applications the asymptotic scale is explicitly given whereas the expressions of the coefficients $q_i$ or $p_i$ of the canonical factorizations, as given by formulas (2.29) or (2.40), are unmanageable even for small values of $n$ and only some properties of them can be detected and used. In other applications, e.g. when a function $f$ is defined as a solution of a functional equation, it may happen that the asymptotic scale is implicitly defined and only the principal parts of the $\phi_i$’s are known; in such cases there is no searching out the expressions of the $q_i$’s and $p_i$’s, but $q_n$, as the ratio in (2.29), might be indirectly known and this would let us decide whether or not Theorem 5.1 applies.

6. Absolute convergence and solutions of differential inequalities

The foregoing theory becomes particularly simple when the involved improper integrals are absolutely convergent and still more expressive for a function $f$ satisfying the $n$th-order differential inequality

\[(6.1) \quad L_{\phi_1,\ldots,\phi_n}[f(x)] \geq 0 \quad a.e. \quad [T,x_0].\]

This is a subclass of the so-called "generalized convex functions with respect to the system $(\phi_1,\ldots,\phi_n)$". The nice result stated in the next theorem claims that: if such a function admits of an asymptotic expansion (1.1) then this expansion is automatically differentiable $(n-1)$ times in the senses of both relations (4.31) and (5.6).

**Theorem 6.1** (Complete asymptotic expansions). If $f \in AC^{n-1}[T,x_0]$ satisfies (6.1) then the following are equivalent properties:

(i) There exist $(n-1)$ real numbers $a_1,\ldots,a_{n-1}$ such that

\[(6.2) \quad f(x) = a_1\phi_1(x) + \cdots + a_{n-1}\phi_{n-1}(x) + O(\phi_n(x)), \quad x \to x_0^-.

(ii) There exist $n$ real numbers $a_1,\ldots,a_n$ such that

\[(6.3) \quad f(x) = a_1\phi_1(x) + \cdots + a_{n-1}\phi_{n-1}(x) + a_n\phi_n(x) + o(\phi_n(x)), \quad x \to x_0^+.

(iii) The following set of asymptotic expansions holds true:

\[(6.4) \quad \begin{cases} L_k[f(x)] = a_1L_k[\phi_1(x)] + \cdots + a_{n-k}L_k[\phi_{n-k}(x)] + \\
\quad \quad \quad \quad \quad + o(1), \quad x \to x_0^-, \quad 0 \leq k \leq n-1; \quad \text{see (4.31)}.
\end{cases}

(iv) The following set of asymptotic expansions holds true:

\[(6.5) \quad \begin{cases} M_k[f(x)] = a_{k+1}M_k[\phi_{k+1}(x)] + \cdots + a_nM_k[\phi_n(x)] + \\
\quad \quad \quad \quad \quad + o(M_k[\phi_n(x)]), \quad x \to x_0^+; \quad 0 \leq k \leq n-1; \quad \text{see (5.5)-(5.6)}.
\end{cases}

\]
(v) The following integral condition is satisfied:

\[
\int_T^{x_0} \frac{1}{p_1} \cdots \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} \frac{1}{p_n(t)} L_{\phi_1, \ldots, \phi_n}[f(t)] dt < +\infty; \text{ see (4.32).}
\]

(vi) The following integral condition is satisfied

\[
\int_T^{x_0} \frac{1}{q_n(t)} L_{\phi_1, \ldots, \phi_n}[f(t)] dt < +\infty, \text{ see (5.9) and (5.19).}
\]

To this list we may obviously add the other properties in Theorem 5.1.

If this is the case the remainder \( R_0(x) \) of the expansion in (6.3) admits of both representations:

\[
\begin{aligned}
R_0(x) &= \frac{(-1)^n}{p_0(x)} \int_x^{x_0} \frac{1}{p_1} \cdots \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} \frac{L_{\phi_1, \ldots, \phi_n}[f(t)]}{p_n(t)} dt = \\
&= \frac{(-1)^n}{q_0(x)} \int_x^{x_0} \frac{1}{q_1} \cdots \int_{t_{n-2}}^{x_0} \frac{1}{q_{n-1}} \int_{t_{n-1}}^{x_0} \frac{L_{\phi_1, \ldots, \phi_n}[f(t)]}{q_n(t)} dt,
\end{aligned}
\]

whence it follows that

\[
(-1)^n R_0(x) \geq 0 \quad \forall x \in [T, x_0],
\]

and that both the following functions

\[
(-1)^n R_0(x)p_0(x) \equiv -R_0(x)\phi_n(x) \quad \text{and} \quad (-1)^n R_0(x)q_0(x) \equiv (-1)^n R_0(x)\phi_1(x)
\]

are decreasing on \([T, x_0]\). (The second expressions of the above functions follow from (2.40) and (2.35) for the first one and from (2.29) for the second one.)

Moreover \((-1)^n R_0(x)\) is decreasing on \([T, x_0]\) under any of the following four conditions

\[(6.11)\quad \text{either} \quad (1/p_0(x))' \leq 0 \quad \text{or} \quad (1/q_0(x))' \leq 0 \quad \text{on} \quad [T, x_0],
\]

\[(6.12)\quad \text{or} \quad \left(\frac{1}{p_0(x)} \cdot \int_x^{x_0} \frac{1}{p_1} \right)' \leq 0 \quad \text{or} \quad \left(\frac{1}{q_0(x)} \cdot \int_x^{x_0} \frac{1}{q_1} \right)' \leq 0 \quad \text{on} \quad [T, x_0].
\]

In addition to the equivalence “\((iii) \leftrightarrow (iv)\)” stated in Theorem 6.1, there is another remarkable circumstance wherein the two types of formal differentiations are simultaneously admissible namely when the convergence of the pertinent improper integrals is absolute.

**Theorem 6.2.** For \( f \in AC^{n-1}[T, x_0]\), the following integral conditions are equivalent:

\[
\int_T^{x_0} \frac{1}{p_1} \cdots \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} \frac{1}{p_n(t)} |L_{\phi_1, \ldots, \phi_n}[f(t)]| dt < +\infty;
\]
\[
(6.14) \quad \left\{ \begin{array}{l}
\int_T^{x_0} P(t) |L_{\phi_1, \ldots, \phi_n}[f(t)]| \, dt < +\infty, \quad \text{where} \\

P(t) := \frac{1}{p_n(t)} \int_T^t \frac{dt_{n-1}}{p_{n-1}(t_{n-1})} \cdots \int_T^{t_3} \frac{dt_2}{p_2(t_2)} \int_T^{t_1} \frac{dt_1}{p_1(t_1)} ;
\end{array} \right.
\]

\[
(6.15) \quad \int_T^{x_0} \frac{|L_{\phi_1, \ldots, \phi_n}[f(t)]|}{q_n(t)} \, dt \equiv \int_T^{x_0} \frac{W(\phi_1, \ldots, \phi_{n-1}; t)}{W(\phi_1, \ldots, \phi_n; t)} |L_{\phi_1, \ldots, \phi_n}[f(t)]| \, dt < +\infty.
\]

Hence each of these three conditions implies both sets of asymptotic expansions (4.31) and (5.5)-(5.6).

**Open problem 1.** In §7 we give an indirect proof of the equivalence “(6.14) ⇔ (6.15)” based on Theorem 6.1; a more refined statement would be:

\[
(6.16) \quad P(x) \sim c \frac{W(\phi_1, \ldots, \phi_{n-1}; x)}{W(\phi_1, \ldots, \phi_n; x)}, \quad x \to x_0^-,
\]

for some constant \( c \neq 0 \). For \( n = 2 \) this is quite elementary and we also found a proof for \( n = 3 \); but for the time being we leave this minor question as an open problem.

Using Theorems 4.4 and 5.2 we can also get the analogues of Theorems 6.1.6.2 for incomplete asymptotic expansions left apart the integral representations of the remainders but with meaningful estimates for weighted derivatives of orders \( \geq i \). We give here a simplified statement wherein all asymptotic relations refer to \( x \to x_0^- \) of course.

**Theorem 6.3** (Incomplete asymptotic expansions). Let \( f \in AC^{n-1}[T, x_0] \) satisfy (6.1) and let \( i \in \{1, \ldots, n-1\} \) be fixed. Then the following are equivalent properties:

\[
(6.17) \quad f(x) = a_1 \phi_1(x) + \cdots + a_i \phi_{i-1}(x) + O(\phi_i(x));
\]

\[
(6.18) \quad f(x) = a_1 \phi_1(x) + \cdots + a_{i-1} \phi_{i-1}(x) + a_i \phi_i(x) + o(\phi_i(x));
\]

\[
(6.19) \quad \left\{ \begin{array}{l}
L_k[f(x)] = a_1 L_k[\phi_1(x)] + \cdots + a_i L_k[\phi_i(x)] + \nonumber \\
\quad \quad \quad + o(L_k[\phi_i(x)]), \quad 0 \leq k \leq n-i;
\end{array} \right.
\]

\[
L_{n-i+h}[f(x)] = a_1 L_{n-i+h}[\phi_1(x)] + \cdots + \nonumber \\
\quad \quad \quad + a_i L_{n-i+h}[\phi_i(h(x)) + o(1)], \quad 0 \leq h \leq i-1,
\]

(which last relations are written in (4.28) in an expanded form):

\[
(6.20) \quad \left\{ \begin{array}{l}
M_k[f(x)] = a_{k+1} M_k[\phi_{k+1}(x)] + \cdots + a_i M_k[\phi_i(x)] + \nonumber \\
\quad \quad \quad + o(M_k[\phi_i(x)]), \quad 0 \leq k \leq i-1;
\end{array} \right.
\]

\[
M_k[f(x)] = O\left( \int_T^x \frac{1}{q_{k+1}} \cdots \int_T^{t_{n-i}} \frac{L_{\phi_1, \ldots, \phi_n}[f(t)]}{q_n(t)} \, dt \right), \quad i \leq k \leq n-2;
\]

\[
M_{n-1}[f(x)] = O\left( \int_T^x \frac{L_{\phi_1, \ldots, \phi_n}[f(t)]}{q_n(t)} \, dt \right);
\]
\[
\int_{T}^{x_0} \frac{1}{p_{n+i+1}} \int_{t_{n-i+1}}^{x_0} \frac{1}{p_{n-i+2}} \cdots \int_{t_{n-1}}^{x_0} \frac{L_{\phi_1,\ldots,\phi_n}[f(t)]}{p_n(t)} dt < +\infty;
\]

(6.21)

\[
\left\{ \int_{T}^{x_0} P(t) L_{\phi_1,\ldots,\phi_n}[f(t)] dt < +\infty, \quad \text{where} \right. \\
P(t) := \frac{1}{p_n(t)} \int_{T}^{t_i} \frac{dt_{n-1}}{p_{n-1}} \cdots \int_{T}^{t_{n-i+2}} \frac{dt_{n-1+i}}{p_{n-1+i}} \quad \text{if } i \geq 2;
\]

(6.22)

\[
\int_{T}^{x_0} \frac{dt_i}{q_i} \int_{T}^{t_i} \frac{dt_{i+1}}{q_{i+1}} \cdots \int_{T}^{t_{n-1}} \frac{L_{\phi_1,\ldots,\phi_n}[f(t)]}{q_n(t)} dt < +\infty.
\]

(6.23)

To the foregoing list we may obviously add property (ii) or property (v) in Theorem 4.4 and properties (ii)-(iii) in Theorem 5.1. For \( i = 1 \) the first group of expansions in (6.20) reduces to relation in (5.26).

As pointed out in Remark 1 after Theorem 5.2 the "O"-estimates in (6.20) are meaningful whenever all the involved integrals diverge as \( x \to x_0 \) i.e. whenever the asymptotic expansion in (6.18) cannot be improved by adding more meaningful terms of the form \( a_{i+j} \phi_{i+j}(x) \). As soon as one of these integrals converges to a real number as \( x \to x_0 \) then we may apply the theorem with a greater value of \( i \).

Remark. In Theorem 6.1 the two types of formal differentiability 1, 2, \ldots, \( n-1 \) times are equivalent facts. It is not so for a generic \( f \) such that \( L_{\phi_1,\ldots,\phi_n}[f(x)] \) changes sign on each deleted left neighborhood of \( x_0 \). This has been proved for polynomial expansions [5] and for real-power expansions [7] in an indirect way by expressing the two sets of differentiated expansions as suitable sets of expansions involving the standard operators \( d^k/dx^k \); these new sets of expansions make evident that what we called "weak formal differentiability" indeed is a weaker property than what we called "strong formal differentiability". The same circumstance occurs for a general two-term expansion [8; Remarks, p. 261] but is not a self-evident fact. In each of these three cases direct proofs could be also provided working on the corresponding integral conditions. Hence in these cases the locations of "weak or strong formal differentiation" are legitimate. But in the general theory for \( n \geq 3 \) we face a nontrivial situation and state

**Open problem 2.** For \( n \geq 3 \) let us consider the two types of formal differentiability characterized in Theorems 4.5 and 5.1. Investigate whether or not property in Theorem 5.1 always implies that in Theorem 4.5.

**7. Proofs**

**Definition 2.2.** The nontrivial inference "(ii)⇒(i)" is valid for either open or half-open but not compact intervals if \( n \geq 3 \) and a proof may be found in Coppel [1; Prop. 3, p. 82, and Proposition 12, p. 102] for operators with continuous coefficients. The proof for open intervals is still valid for our more general operators as the argument only involves derivatives up to order \( n-1 \), whereas the proof for half-open intervals rests on the theory of conjugate points which has been extended to our operators by Levin [15; pp. 70-72]. □
Proof of Proposition 2.2. "(i)⇔(ii)" is proved in Levin [15; Thm. 2.1, p. 66] where the interval $I$ is explicitly stated to be open not in the statement of the cited theorem but at the outset of §2 on p. 58; "(ii)⇔(iii)" is the classical result by Pólya [20]; "(i)⇔(iv)" is the fundamental result by Trench [24]; "(i)⇒(v)" is to be found in [3; Thm. 2.2, p. 162] whereas the converse rests on the trivial fact that disconjugacy on $[a, b]$ is equivalent to disconjugacy on every compact subinterval of $[a, b]$.

Proof of Proposition 2.3. Postponing to Proposition 2.6 the construction of a double C.F. of type (II) at both endpoints we choose the following pattern (among several possible ones) for the proof:

(i)⇒(iv)⇔(v); (iv)⇒(ii)⇒(v); (iv)⇒(iii)⇒(v); (i)⇔(vi).

(i)⇒(iv). First, conditions (2.15) imply that the coefficients $\alpha_i$ in the standard representation (2.7) of $L_n$ are of class $L_{loc}^1(a, b)$. Second, in whatever manner we extend the given functions $(u_1, \ldots, u_n)$ to a larger interval $[a_1, b_1]$, $a_1 < a$, preserving properties (2.15) with respect to the new interval $[a_1, b_1]$, the corresponding Wronskians are extended as functions continuous on $[a_1, b_1]$ and strictly positive on some interval $]a - \epsilon, b[$. Hence $L_n$ may be considered as the restriction to $[a, b]$ of an operator of the same type, say $\tilde{L}_n$, disconjugate (by Proposition 2.2-(ii)) on a larger interval $]a - \epsilon, b[$.

(iv)⇒(i). If $\tilde{L}_n$ is the assumed extension of $L_n$ to $]a - \epsilon, b[$ then Proposition 2.2 applied to $\tilde{L}_n u = 0$ yields an $n$-tuple of solutions satisfying properties (2.15) with respect to the interval $]a - \epsilon, b[$ and, a fortiori, properties (2.15) themselves.

(iv)⇒(v). The result by Levin mentioned after Definition 2.2 states the equivalence between disconjugacy on any of the three intervals $]\alpha, \beta[, ]\alpha, \beta[, ]\alpha, \beta[$ in the generalized sense wherein either $\alpha$ or $\beta$ may be singular endpoints. Hence $L_n$, the assumed extension of $L_n$, is also disconjugate on $]a - \epsilon, b[$ and, a fortiori, on $[a, b]$.

(v)⇒(iv). This is contained in Lemma 8.2 in [4; p. 1169], more precisely in the version of this lemma wherein $a$ is supposed to be a nonsingular endpoint. By the way we point out two misprints in the statement of Lemma 8.2 in this reference: in the first line of the statement the symbol $D_n(a, b)$ must be replaced by $D_n[a, b]$ which denotes the class of $n$th-order operators disconjugate on $[a, b]$ in the generalized sense; in the last-but-one line of the statement the open interval $(a, a + \epsilon)$ must be replaced by the open interval $(a, b + \epsilon)$.

(iv)⇒(ii). Let $\tilde{L}_n$ be the extension of $L_n$ disconjugate on $]a - \epsilon, b[$; the "unique" C.F. of $\tilde{L}_n$ of type (I) at $b$ and valid on $]a - \epsilon, b[$ necessarily coincides on $[a, b]$ with the "unique" C.F. of $L_n$ of type (I) at $b$; hence the functions $r_i$ in (2.16) satisfy the stated regularity properties as they are restrictions of functions $\tilde{r}_i$ strictly positive and regular on a larger interval.

(ii)⇒(v). The regularity properties of the $r_i$'s in (2.16) imply that $a$ is a nonsingular endpoint for $L_n$; moreover the hypothesis that the C.F. of type (I) at $b$ is a C.F. of type (II) at $a$ is equivalent to disconjugacy on $[a, b]$; [3; Theorems. 3.1 and 3.2, pp. 163-164].

(iv)⇒(iii). As the extension $\tilde{L}_n$ is disconjugate on $]a - \epsilon, b[$ it is disconjugate on $]a - \epsilon, b[$ as well and, a fortiori, on $]a - \frac{\epsilon}{2}, b[$. This implies, by [4; Theorem. 8.3, p. 1170], that $\tilde{L}_n$ admits of a C.F. on $]a - \frac{\epsilon}{2}, b[$ of type (II) at both endpoints; its restriction to $[a, b]$ gives the sought-for C.F.
(iii)⇒(v). The regularity of the $\tau_i$'s in (2.18) implies that $a$ is a nonsingular endpoint; under this condition the existence of a double C.F. of type (II) on $[a, b]$ is equivalent to disconjugacy on $[a, b]$: [4; Theorem. 8.3, p. 1170].

It remains to prove "(i)⇒(vi)" the converse being obvious. This inference depends on the existence of a C.F. on $[a, b]$ whose coefficients are as regular at the endpoint $a$ as those in (2.16) or (2.18). In fact starting, e.g., from (2.16) we can construct an $n$-tuple of independent solutions to $L_n u = 0$ by the formulas

$$
\tilde{u}_1(x) := \frac{1}{r_0(x)}; \quad \tilde{u}_k(x) := \frac{1}{r_0(x)} \int_a^x \frac{1}{r_1} \cdots \int_a^{t_{k-2}} \frac{1}{r_{k-1}}, \quad 2 \leq k \leq n.
$$

It is known that the Wronskians of the $\tilde{u}_i(x)$ can be expressed by the formulas

$$
W(\tilde{u}_1(x), \ldots, \tilde{u}_k(x)) = \left( \frac{1}{r_0(x)} \right)^k \left( \frac{1}{r_1(x)} \right)^{k-1} \cdots \left( \frac{1}{r_{k-2}(x)} \right)^{2} \left( \frac{1}{r_{k-1}(x)} \right), \quad x \in [a, b], \quad 1 \leq k \leq n;
$$

see, e.g., [14; Chp. XI, p. 380]. This easily implies conditions (2.19) which in turn imply (2.20): [1; Proposition 4, p. 88].

Proposition 2.4 is contained in Levin [15; Lemma 2.1, p. 58, and Theorem 2.1-part a), p. 66].

**Proof of Proposition 2.5.** This follows from Levin [15; Theorem. 2.1, p. 66] by simple remarks. In Levin’s theorem the operator $L$, whose kernel is spanned by $(\phi_1, \ldots, \phi_n)$, is explicitly assumed to be disconjugate on an open interval, in our case $[T, x_0]$, and this is granted by (2.4) and Proposition 2.2. By (2.3) the $n$-tuple $(|\phi_n|, \ldots, |\phi_1|)$ is a fundamental system of solutions of $L u = 0$ and is hierarchical at $x_0^-$ in the sense that

$$
|\phi_n(x)| \ll |\phi_{n-1}(x)| \ll \cdots \ll |\phi_1(x)|, \quad x \to x_0^-.
$$

Taking account of the reverse numbering of the $\phi_i$’s Levin’s theorem asserts that (2.23) implies:

$$
W(|\phi_{i_k}(x)|, \ldots, |\phi_{i_1}(x)|) > 0 \quad \text{near } x_0^-,
$$

and

$$
W(|\phi_{j_k}(x)|, \ldots, |\phi_{j_1}(x)|) \ll W(|\phi_{i_k}(x)|, \ldots, |\phi_{i_1}(x)|), \quad x \to x_0^-,
$$

if the indexes $i$ and $j$ are chosen as in (ii) of our proposition. Changing the orders of the $\phi_i$’s and dropping the absolute values may only alter the signs and our proof is over. □

**Proof of Proposition 2.6.** Concerning (i) just notice that factorization (2.33), with the $q_i$’s defined by (2.29), is the classical factorization discovered for the first time by Pólya [20]; a simpler proof has been given almost simultaneously by Coppel [1; Theorem. 2, pp. 91-92] and Ristroph [21]. Any proof relies on the identities (2.31) which are a standard fact [1; Lemma 4, p. 87]. To prove (2.32) notice that the continuity of the $q_i$’s at the
endpoint $T$ implies $\int_T^1 \alpha(x) \, dx < +\infty$ whereas $\int_T^{x_0} \alpha(x) \, dx < +\infty$ follows from (2.31) and the following relation inferred from (2.25):

\begin{equation}
W(\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}) \ll W(\phi_1, \ldots, \phi_{i-1}, \phi_i), \ x \to x_0^-.
\end{equation}

(ii). Representations (2.34) are proved by a word-for-word repetition of the proof in Karlin and Studden [14; Theorem. 1.2, pp. 379-380] concerning the analogous representations when $x_0$ is replaced by the "regular" endpoint $T$. In our present context we have that: the assumption (1.7) on p. 379 in this reference is replaced by our conditions (2.1)-(2.2); moreover the reduced system, defined in formula (1.8) on p. 379, is

\begin{equation}
\psi_i(x) = (\phi_{i+1}(x)/\phi_i(x))', \ 1 \leq i \leq n-1,
\end{equation}

and conditions (1.9) on p. 379 are replaced by the asymptotic relations

\begin{equation}
\psi_1(x) \gg \cdots \gg \psi_{n-1}(x), \ x \to x_0^-,
\end{equation}

inferred from the above equality (7.7) and Levin’s relations reported in (2.25).

(iii) and (iv). Both the regularity of the $\rho_i$’s and conditions (2.38)-(2.39) follow from (ii) of Proposition 2.3; the functions $P_i$ defined by (2.42), obviously satisfy (2.43). Now we prove (v) coming back afterwards to the remaining part of (iii).

(v). A simple inductive argument based on (1.6), see e.g. [14; p. 380], shows the validity of formulas (2.44) starting from representations (2.42). Similar formulas then show the positivity on $[T, x_0]$ of the Wronskians $W(P_{i_1}(x), \ldots, P_{i_k}(x))$ with indexes satisfying (2.23). To prove the assertion concerning the $n$-tuple ($\phi_n, \ldots, \phi_1$) we may use an indirect argument. Notice that representations (2.34) and (2.42) have the same structures hence the formulas expressing the Wronskians $W(\phi_{i_1}(x), \ldots, \phi_{i_k}(x))$ via the $q_i$’s are the same as the corresponding ones for $W(P_{i_1}(x), \ldots, P_{i_k}(x))$. Hence the following Wronskians

\begin{equation}
W(|\phi_{i_1}(x)|, \ldots, |\phi_{i_k}(x)|)
\end{equation}

do not vanish on $[T, x_0]$ and the inequalities in (7.4), with indexes satisfying (2.23), hold true on $[T, x_0]$.

To complete our proof we must prove (2.40) whereas (2.41) is a standard consequence of (2.40) as remarked above for the functions $q_i$. Now, if for the time being we denote by $\tilde{p}_i$ the functions $\rho_i$ defined by (2.40), the situation is just the same as for the functions $q_i$ defined by (2.29) after replacing ($\phi_1, \ldots, \phi_n$) by ($|\phi_n|, \ldots, |\phi_1|$); hence, by (i), our operator admits of the factorization

\begin{equation}
L_{\phi_1, \ldots, \phi_n} u = \tilde{p}_n[\tilde{p}_{n-1}(\ldots (\tilde{p}_0 u)' \cdots )]',
\end{equation}
on $[T, x_0]$. Moreover (2.41) implies

\begin{equation}
\int_T^x \frac{1}{\tilde{p}_i} = \frac{W(\phi_n(x), \ldots, \phi_{n-i+2}(x), \phi_{n-i}(x))}{W(\phi_n(x), \ldots, \phi_{n-i+2}(x), \phi_{n-i+1}(x))} + \text{constant}, \ 2 \leq i \leq n-1,
\end{equation}

which, by Proposition 2.5, diverges as $x \to x_0^-$. Hence (7.10) is a C.F. on $[T, x_0]$ of type (I) at $x_0$ and, by uniqueness, the functions $\tilde{p}_i$ must coincide (constant factors apart) with the functions $p_i$ in (2.37). \qed
**Proof of Proposition 3.2.** Relations (3.35) to (3.38) can be directly checked using representations (2.34). If (3.39) holds true for some sufficiently regular \( f \) then (3.40) follows from repeated integrations of 

\[ q_{k-1}(x) (\ldots (q_0(x) u)')' = a_k \]

taking account of the integrability properties in (2.32) and formulas (2.34). The converse trivially follows from (3.36)-(3.37). Now suppose (3.39)-(3.40) to be true on the left of \( x_0 \); relation (3.41) for \( h = 1 \) is nothing but the obvious relation \( a_1 = \lim_{x \to x_0^-} f(x)/\phi_1(x) \).

For \( h \geq 2 \) we use all relations (3.36),(3.37),(3.38) and get from (3.41):

\[
M_{h-1}[f(x)] = \sum_{i=0}^{k-h} a_{h+i} M_{h-1}[\phi_{h+i}(x)] = a_h + \sum_{i=1}^{k-h} a_{h+i} M_{h-1}[\phi_{h+i}(x)] = a_h + o(1),
\]

where the remainder "\( o(1) \)" is \( \equiv 0 \) for \( h = k \).

Formulas (3.29) are simply proved noticing that both sides in (3.29) define linear differential operators of order \( k \) whose kernels are spanned by \( (\phi_1, \ldots, \phi_k) \), hence they only differ by a factor which is a never-vanishing function. Relation (3.37) completes the proof. The proof of (3.28) is completely analogous and the constant \( c_k \), is specified in Lemma 4.2. \( \square \)

**A quick proof of the existence of C.F.'s of type (I).** If \( L_n \) is an operator of type (2.7), disconjugate on an open interval \( [T, x_0] \) then its kernel has some basis \( (\bar{\phi}_1, \ldots, \bar{\phi}_n) \) satisfying (2.21)-(2.22). Let us now consider the global factorization of \( L_n \) constructed by Pólya’s procedure i.e.

\[
L_n u \equiv \bar{p}_n [\bar{p}_{n-1}(\ldots (\bar{p}_0 u)' \ldots )]',
\]

where the \( \bar{p}_i \)'s have the same expressions of the \( q_i \)'s in (2.29) with \( \phi_i \) replaced by \( \bar{\phi}_i \); formulas (3.31) become

\[
\frac{1}{\bar{p}_i(x)} = \left( \frac{\bar{\phi}_2(x)}{\bar{\phi}_1(x)} \right)' \quad \frac{1}{\bar{p}_i(x)} = \left[ \frac{W(\bar{\phi}_1(x), \ldots, \bar{\phi}_{i-1}(x), \bar{\phi}_{i+1}(x))}{W(\bar{\phi}_1(x), \ldots, \bar{\phi}_i(x))} \right]', \quad 2 \leq i \leq n-1.
\]

Now for a fixed \( T, T < T < x_0 \), and as \( x \to x_0^- \) we have

\[
\int_T^x \frac{1}{\bar{p}_i} \, dt = \text{constant} + \frac{W(\bar{\phi}_1(x), \ldots, \bar{\phi}_{i-1}(x), \bar{\phi}_{i+1}(x))}{W(\bar{\phi}_1(x), \ldots, \bar{\phi}_i(x))} \xrightarrow{\text{by (2.25)}} +\infty,
\]

hence \( \int^{+\infty} 1/\bar{p}_i = +\infty \), \( 1 \leq i \leq n-1 \). \( \square \)

Using the Wronskians \( W(\bar{\phi}_n(x), \ldots, \bar{\phi}_{n-i}(x)) \) and (2.24) an analogous short proof would give the existence of a C.F. of type (II) at \( x_0 \) valid on that left neighborhood of \( x_0 \) whereon the involved Wronskians do not vanish. \( \square \)

**Proof of Lemma 4.2.** From the chain \( P_{n-1}(x) \gg \ldots \gg P_0(x) \), \( x \to x_0 \), we get

\[
\phi(x) = cP_1(x) + \alpha_{i-1}P_{i-1}(x) + \ldots + \alpha_0 P_0(x)
\]

for suitable constants \( \alpha_k \), hence

\[
L_k[\phi(x)] = cL_k[P_1(x)] + \alpha_{i-1}L_k[P_{i-1}(x)] + \ldots + \alpha_0 L_k[P_0(x)];
\]
now (4.10) follows from (4.7), and (4.11) follows from (4.4). If in (7.15) we replace \( \phi \) by \( \phi_{n-i} \) we have \( c = b_{n-i} \) and the identities in (4.12) follow from (4.4) and (4.5). The identity in (4.12) for \( i = k \), i.e. \( L_k[\phi_{n-k}(x)] = b_{n-k} \), implies (4.14). \( \square \)

**Proof of Theorem 4.4.** Part (I). From (4.12) and (4.17), with \( c_1 = a_1 \), we infer at once the equivalence "(ii) \( \Leftrightarrow \) (iii)" as well as representation in (4.21). The inference "(i) \( \Rightarrow \) (ii)" being obvious let us prove the converse simply denoting by \( L \) our operator \( L_{\phi_1,...,\phi_n} \). We shall repeatedly use the recursive formulas (3.25) in the form

\[
L_{k-1}u = \int_T^x \frac{1}{p_k(t)} L_k[u(t)] dt + \text{constant, } 1 \leq k \leq n.
\]

If (4.19) holds true we have (4.21), and representations in (4.16) can be rewritten as

\[
L_k[f(x)] = \{c_1 L_k[\phi_1(x)] + \cdots + c_{n-k} L_k[\phi_{n-k}(x)]\} +
\]

\[
\left[ \int_T^{x_0} \frac{L[f(t)]}{p_n(t)} dt \right] \cdot \int_T^x \frac{dt_{k+1}}{p_{k+1}(t_{k+1})} \cdots \int_T^{t_{n-2}} \frac{dt_{n-1}}{p_{n-1}(t_{n-1})} +
\]

\[
- \int_T^{x_0} \frac{dt_{k+1}}{p_{k+1}(t_{k+1})} \cdots \int_T^{t_{n-2}} \frac{dt_{n-1}}{p_{n-1}(t_{n-1})} \int_T^{t_{n-1}} \frac{L[f(t)]}{p_n(t)} dt; \quad 0 \leq k \leq n-2.
\]

Now we have

\[
\int_T^x \frac{dt_{k+1}}{p_{k+1}(t_{k+1})} \cdots \int_T^{t_{n-2}} \frac{dt_{n-1}}{p_{n-1}(t_{n-1})} \quad \overset{\text{by (4.6)}}{=} \quad L_k[P_{n-1}(x)] = \cdots
\]

by (4.1) and (4.10)

\[
\cdots = \frac{1}{b_1} L_k[\phi_1(x)] + o(L_k[\phi_1(x)]);
\]

and after substituting into (7.18) we get

\[
\left\{ \begin{aligned}
L_k[f(x)] &= c_1 L_k[\phi_1(x)] + o(L_k[\phi_1(x)]) + \bar{c} L_k[P_{n-1}(x)] + o(L_k[P_{n-1}(x)]) = \\
&= \left(c_1 + \frac{\bar{c}}{b_1}\right) L_k[\phi_1(x)] + o(L_k[\phi_1(x)]), \quad 0 \leq k \leq n-2,
\end{aligned} \right.
\]

where \( \bar{c} := \int_T^{x_0} L[f(t)]/p_n(t) \) dt, and the coefficient

\[
c := c_1 + (\bar{c}/b_1)
\]

is independent of \( k \). To show that \( c \) coincides with the \( a_1 \) appearing in (4.19) we may suitably integrate (4.19) to obtain, by (3.24),

\[
\left\{ \begin{aligned}
L_{n-2}[f(x)] &= \int_T^x \frac{L_{n-1}[f(t)]}{p_{n-1}(t)} dt + \text{constant} \quad \overset{\text{by (4.19)}}{=} \\
&= a_1 b_1 \int_T^x \frac{dt}{p_{n-1}(t)} + o\left(\int_T^x \frac{dt}{p_{n-1}(t)}\right) \quad \overset{\text{by (4.6)}}{=} \quad a_1 b_1 L_{n-2}[P_{n-1}(x)] + o(L_{n-2}[P_{n-1}(x)]) = \cdots
\end{aligned} \right.
\]
by (4.1) and (4.10)
\[ \cdots = a_1 L_{n-2}[\phi_1(x)] + o(L_{n-2}[\phi_1(x)]). \]

Part (II). Case \( i = 2 \). We must prove the equivalence of the following three contingencies:

\[
\begin{align*}
    f(x) &= a_1 \phi_1(x) + a_2 \phi_2(x) + o(\phi_2(x)), \\
    L_k[f(x)] &= a_1 L_k[\phi_1(x)] + a_2 L_k[\phi_2(x)] + o(L_k[\phi_2(x)]), \quad 1 \leq k \leq n - 2, \\
    L_{n-1}[f(x)] &= a_1 L_{n-1}[\phi_1(x)] + o(L_{n-1}[\phi_1(x)]);
\end{align*}
\]

(7.23)

\[
\begin{align*}
    L_{n-2}[f(x)] &= a_1 L_{n-2}[\phi_1(x)] + a_2 L_{n-2}[\phi_2(x)] + o(L_{n-2}[\phi_2(x)]), \\
    L_{n-1}[f(x)] &= a_1 L_{n-1}[\phi_1(x)] + o(L_{n-1}[\phi_1(x)]);
\end{align*}
\]

(7.24)

\[
\int_T^{x_0} \frac{dt}{p_{n-1}(t)} \int_T^{x_0} \frac{L[f(\tau)]}{p_n(\tau)} d\tau \text{ convergent.}
\]

(7.25)

First we prove ”(7.24)\( \Leftrightarrow (7.25)\)” if (7.25) holds true then , by part (I) of our theorem, we have all relations in (4.18) and in particular the second relation in (7.24). Moreover we can rewrite representation in (4.16) for \( k = n - 2 \) in the form

\[
L_{n-2}[f(x)] = a_1 L_{n-2}[\phi_1(x)] + a_2 L_{n-2}[\phi_2(x)] + \int_T^{x_0} \frac{dt}{p_{n-1}(t)} \int_T^{x_0} \frac{L[f(\tau)]}{p_n(\tau)} d\tau,
\]

where \( a_1 \) is just the same as in the second relation in (7.24) and \( a_2 \) is a suitable constant. This yields the first relation in (7.24) because \( L_{n-2}[\phi_2(x)] \) is a nonzero constant by (4.12). Viceversa if relations in (7.24) hold true then, by part (I), we have representation in (4.21) by which we replace the quantity \( L_{n-1}[f(t)] \) in the first equality in (7.26). Denoting by \( c_{n-2}, \bar{c}_{n-2} \) suitable constants we get

\[
\begin{align*}
    L_{n-2}[f(x)] &= c_{n-2} + \int_T^x \frac{L_{n-1}[f(t)]}{p_{n-1}(t)} dt = \\
    &= c_{n-2} + \int_T^x \frac{1}{p_{n-1}(t)} \left[ a_1 L_{n-1}[\phi_1(t)] - \int_T^{x_0} \frac{L[f(\tau)]}{p_n(\tau)} d\tau \right] dt = \\
    &= \bar{c}_{n-2} + a_1 L_{n-2}[\phi_1(x)] - \int_T^x \frac{dt}{p_{n-1}(t)} \int_T^{x_0} \frac{L[f(\tau)]}{p_n(\tau)} d\tau.
\end{align*}
\]

(7.27)

By comparison with the first relation in (7.24) we get (7.25) because \( L_{n-2}[\phi_2(x)] \) is a constant. As the inference ”(7.23)\( \Rightarrow (7.24)\)” is obvious it remains to prove the converse. Using (7.17) and integrating the first relation in (7.24) we get (with suitable constants \( c_{n-3}, \bar{c}_{n-3} \)):

\[
\begin{align*}
    L_{n-3}[f(x)] &= c_{n-3} + \int_T^x \frac{L_{n-2}[f(t)]}{p_{n-2}(t)} dt = c_{n-3} + \\
    &+ a_1 \int_T^x \frac{L_{n-2}[\phi_1(t)]}{p_{n-2}(t)} dt + a_2 \int_T^x \frac{L_{n-2}[\phi_2(t)]}{p_{n-2}(t)} dt + \int_T^x o\left( \frac{L_{n-2}[\phi_2(t)]}{p_{n-2}(t)} \right) dt = \ldots
\end{align*}
\]

(7.28)
as $L_{n-2}[\phi_2(x)]$ is a nonzero constant and $\int^{x_0}_x 1/p_{n-2}$ diverges
\[ \cdots = \epsilon_{n-3} + a_1 L_{n-3}[\phi_1(x)] + a_2 L_{n-3}[\phi_2(x)] + o(L_{n-3}[\phi_2(x)]). \]

Here the constant $\epsilon_{n-3}$ is meaningless as the comparison functions are divergent as $x \to x_0^0$. Iterating the procedure we get all relations in (7.23). By induction on $i$ and the same kind of reasonings our theorem is proved for each $i \leq n$. \hfill \Box

Proof of Theorem 5.1. (i) $\Rightarrow$ (ii). Relation (5.5) implies the existence of $a_1 \equiv \lim_{x \to x_0^0} f(x)/\phi_1(x) \equiv \lim_{x \to x_0^0} M_0[f(x)]$, and each relation in (5.6) implies the relation in (5.7) with the same value of $k$ because of (3.27)-(3.28). (ii) $\Rightarrow$ (iii) : obvious. (iii) $\Rightarrow$ (iv). It follows from (5.3) that the limit in (5.8) exists in $\mathbb{R}$ iff (5.9) holds true and, in this case, (5.3) can be written as

\[ M_{n-1}[f(x)] = a_n - \int_{x}^{x_0} \frac{L[f(t)]}{q_n(t)} dt, \] (7.29)

where, as above, $L \equiv L_{\phi_1, \ldots, \phi_n}$.

(iv) $\Rightarrow$ (i). We have already proved (7.29) which is (5.6) for $k = n - 1$ together with an integral representation of the remainder. For $k = n - 2$ the recursive formulas (3.27) give

\[ (M_{n-2}[f(x)])' = \frac{1}{q_{n-1}(x)} M_{n-1}[f(x)], \] (7.30)

whence, by (7.29) and (2.32), we get

\[ M_{n-2}[f(x)] = a_{n-1} - a_n \int_{x}^{x_0} \frac{1}{q_{n-1}} + \int_{x}^{x_0} \frac{d\tau}{q_{n-1}(\tau)} \int_{\tau}^{x_0} \frac{L[f(t)]}{q_n(t)} dt \] (7.31)

for a suitable constant $a_{n-1}$. By (3.37)-(3.38) this is nothing but

\[ M_{n-2}[f(x)] = a_{n-1} M_{n-2}[\phi_{n-1}(x)] + a_n M_{n-2}[\phi_n(x)] + \int_{x}^{x_0} \frac{d\tau}{q_{n-1}(\tau)} \int_{\tau}^{x_0} \frac{L[f(t)]}{q_n(t)} dt, \] (7.32)

which is the relation in (5.6) for $k = n - 2$ with a representation of the remainder. In a similar way for $k = n - 3$ we start from

\[ (M_{n-3}[f(x)])' = \frac{1}{q_{n-2}(x)} M_{n-2}[f(x)], \] (7.33)

and integrate (7.31) after dividing by $1/q_{n-2}$, so getting

\[
\begin{cases}
M_{n-3}[f(x)] = a_{n-2} - a_{n-1} \int_{x}^{x_0} \frac{1}{q_{n-2}} + a_n \int_{x}^{x_0} \frac{dt_{n-2}}{q_{n-2}} \int_{t_{n-2}}^{x_0} \frac{dt_{n-2}}{q_{n-1}} + \\
- \int_{x}^{x_0} \frac{dt_{n-2}}{q_{n-2}} \int_{t_{n-2}}^{x_0} \frac{dt_{n-1}}{q_{n-1}} \int_{t_{n-1}}^{x_0} \frac{L[f(t)]}{q_n(t)} dt
\end{cases}
\] (7.34)
for a suitable constant \( a_{n-2} \). By (3.37)-(3.38) this can be rewritten as

\[
M_{n-3}[f(x)] = a_{n-2}M_{n-3}[\phi_{n-2}(x)] + a_{n-1}M_{n-3}[\phi_{n-1}(x)] + \int_{x}^{x_0} \frac{1}{q_{n-2}} \int_{x}^{x_0} \frac{1}{q_{n-1}} \int_{x}^{x_0} L[f(t)] \frac{dt}{q_n(t)} dt,
\]

with a suitable constant \( a_{n-2} \). An iteration of the procedure gives all relations in (5.6) together with the representation formulas (5.11)-(5.12) for \( R_0(x) \) and (5.15) for \( R_k(x), k \geq 1 \). And "(i) \( \Rightarrow \) (v)" has been proved. The last inference "(v) \( \Rightarrow \) (i)" and (5.12) are trivially proved by applying the operators \( M_k \) to (5.11).

\[\text{Proof of Theorem 5.2. } \text{"(i) } \Rightarrow \text{(ii)" follows from (3.37)-(3.38). "(ii) } \Rightarrow \text{(iii)" is obvious. (iii) } \Leftrightarrow \text{(iv). Again by (3.37)-(3.38) the representation in (5.2) for } k = i - 1 \text{ has the form}
\]

\[
M_{i-1}[f(x)] = c_i + \int_{T}^{x} \frac{1}{q_{i}} \int_{T}^{t_i} \frac{1}{q_{i+1}} \ldots \int_{T}^{t_{n-2}} \frac{1}{q_{n-1}} \int_{T}^{t_{n-1}} L[f(t)] \frac{dt}{q_n(t)} dt + o(1),
\]

whence our equivalence follows at once. If this is the case then (5.2) can be rewritten as

\[
M_{i-1}[f(x)] = a_i - \int_{x}^{x_0} \frac{1}{q_{i}} \int_{x}^{t_i} \frac{1}{q_{i+1}} \ldots \int_{x}^{t_{n-2}} \frac{1}{q_{n-1}} \int_{x}^{t_{n-1}} L[f(t)] \frac{dt}{q_n(t)} dt +
\]

\[
+ c_{i+1} M_{i-1}[\phi_{i+1}(x)] + \ldots + c_n M_{i-1}[\phi_n(x)], \ x \in [T, x_0],
\]

where \( a_i \) is uniquely determined by (5.23) but \( c_{i+1}, \ldots, c_n \) are non-better specified constants not determinable by the sole condition (5.23).

(iv) \( \Rightarrow \) (i). This is proved like the corresponding inference in Theorem 5.1 by successive integrations of (7.37) starting from

\[
(M_{i-2}[f(x)])' = \frac{1}{q_{i-1}(x)} M_{i-1}[f(x)],
\]

whence, by (2.37) and (3.38), we get

\[
M_{i-2}[f(x)] = a_{i-1} - a_i \int_{x}^{x_0} \frac{1}{q_{i-1}} + \int_{x}^{x_0} \frac{1}{q_{i-1}} \int_{T}^{t_i} \frac{1}{q_{i+1}} \ldots \int_{T}^{t_{n-1}} L[f(t)] \frac{dt}{q_n(t)} dt +
\]

\[
+ c_{i+1} M_{i-2}[\phi_{i+1}(x)] + \ldots + c_n M_{i-2}[\phi_n(x)] \quad \text{by (11.24)}
\]

\[
= a_{i-1} + a_i M_{i-2}[\phi_i(x)] + o(M_{i-2}[\phi_i(x)]),
\]

where the constant \( a_{i-1} \), which includes all the constants from integration of the various terms, is uniquely determined by (5.22). By iteration of the procedure we get all relations in (5.20)-(15.21). Relation (5.28) easily follows from (7.37) by (3.37)-(3.38).

\[\text{Proof of Theorem 5.3. } \text{This is almost a word-for word repetition of the proofs of Theorems 5.1-5.2. "(i) } \Rightarrow \text{(ii)". For } 0 \leq k \leq i - 2 \text{ this is included in the same inference in}\]
Theorems 5.1-5.2: whereas the relation in (5.30) for \( k = i - 1 \) just reads \( M_{i-1}[f(x)] = O(1) \). "(ii) \( \Rightarrow \) (iii)" is obvious. "(iii) \( \Leftrightarrow \) (iv)" follows from (7.36). "(iv) \( \Rightarrow \) (i)". We now use (7.38) and the representation in (5.2) for \( k = i - 1 \) instead of (7.37) as in the proof of Theorem 5.2. Due to the convergence of \( \int x_0^{x_0} 1/q_{i-1} \) we may still apply the operator \( \int_x^{x_0} \) so getting, instead of (7.39),

\[
M_{i-2}[f(x)] = a_{i-1} - \int_x^{x_0} \frac{1}{q_{i-1}} \int_T^{t_{i-1}} 1 \int_T^{t_i} 1 \int_T^{t_{i+1}} \cdots \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt + \]

\[
- \int_x^{x_0} \frac{1}{q_{i-1}} \left[ \sum_{j=i+1}^{n} c_j M_{i-2}[\phi_j(x)] \right] = a_{i-1} + O \left( \int_x^{x_0} \frac{1}{q_{i-1}} \right) = a_{i-1} + O(M_{i-2}[\phi_i(x)]) .
\]

By iteration we get all relations in (5.28)-(5.29).

**Proof of Theorem 6.1.** The only thing to be proved is the inference "(i) \( \Rightarrow \) (v) \( \land \) (vi)"; the other properties being included in Theorems 4.5 and 5.1. We use a procedure already used in [5; p. 193] and in [7; p. 213]. From representation in (4.15) we get (using the simplified notation \( L \equiv L_{\phi_1, \ldots, \phi_n} \)):

\[
(7.40) \quad \frac{f(x)}{\phi_1(x)} - c_1 + O(1) = \frac{1/p_0(x)}{\phi_1(x)} \int_T^{x} \frac{1}{p_1} \int_T^{t_{n-2}} \frac{1}{p_{n-1}} \int_T^{t_{n-1}} \frac{L[f(t)]}{p_n(t)} dt , \quad x \in [T, x_0].
\]

By the assumption (6.2) the left-hand side has a finite limit as \( x \to x_0 \), and for the limit of the right-hand side we have:

\[
(7.41) \quad \lim_{x \to x_0} \frac{1}{p_0(x)} \int_T^{x} \frac{1}{p_1} \int_T^{t_{n-2}} \frac{1}{p_{n-1}} \int_T^{t_{n-1}} \frac{L[f(t)]}{p_n(t)} dt \quad \text{by (2.42) and (2.43)} = \]

\[
= \frac{1}{b_1} \lim_{x \to x_0} \int_T^{x} \frac{1}{p_1} \int_T^{t_{n-2}} \frac{1}{p_{n-1}} \int_T^{t_{n-1}} L[f(t)]/p_n(t) dt = \]

\[
= \frac{1}{b_1} \lim_{x \to x_0} \int_T^{x} \frac{1}{p_1} \int_T^{t_{n-2}} \frac{1}{p_{n-1}} \int_T^{t_{n-1}} L[f(t)]/p_n(t) dt = \cdots = \frac{1}{b_1} \lim_{x \to x_0} \int_T^{x} L/p_n, \]

after applying L'Hospital's rule \( (n - 1) \) times (which is legitimate as all the denominators diverge to \( +\infty \)). By the positivity of the integrand this last limit exists in \( \mathbb{R} \) and coincides with the limit of the left-hand side in (7.40) hence it must be a real number and (4.15) can take the form:

\[
(7.42) \quad \begin{cases} f(x) = a_1 \phi_1(x) + c_2 \phi_2(x) + \cdots + c_n \phi_n(x) + \\ - \frac{1}{p_0(x)} \int_T^{x} \frac{1}{p_1} \int_T^{t_{n-2}} \frac{1}{p_{n-1}} \int_T^{t_{n-1}} L[f(t)]/p_n(t) dt , \quad x \in [T, x_0]. \end{cases}
\]
with suitable constants \(c_2, \ldots, c_n\). From this we get:

\[
\begin{align*}
\frac{f(x) - a_1\phi_1(x)}{\phi_2(x)} - c_2 + o(1) &= \\
&= -\frac{1/p_0(x)}{\phi_2(x)} \int_T^x \frac{1}{p_1} \cdots \frac{1}{p_{n-1}} \int_{t_{n-2}}^{x_0} L[f(t)] \frac{1}{p_n(t)} dt,
\end{align*}
\]

(7.43)

Here again the left-hand side has a finite limit as \(x \to x_0\) whereas the limit of the right-hand side, by (4.1), equals:

\[
\begin{align*}
&= -\frac{1}{b_2} \lim_{x \to x_0^-} \int_T^x \frac{1}{p_1} \cdots \frac{1}{p_{n-1}} \int_{t_{n-2}}^{x_0} L[f(t)]/p_n(t) dt \\
&= -\frac{1}{b_2} \lim_{x \to x_0^-} \int_T^x \frac{1}{p_1} \cdots \frac{1}{p_{n-2}} \int_{t_{n-3}}^{x_0} L/f(t)/p_n(t) dt \\
&= \cdots = -\frac{1}{b_2} \lim_{x \to x_0^-} \int_T^x \frac{1}{p_1} \int_{t_{n-1}}^{x_0} L/p_n(t),
\end{align*}
\]

after applying L’Hospital’s rule \((n - 2)\) times. Hence this last limit, which exists in \(\mathbb{R}\), must be a real number and (7.42) can be rewritten as:

\[
\begin{align*}
&f(x) = a_1\phi_1(x) + a_2\phi_2(x) + c_3\phi_3(x) + \cdots + c_n\phi_n(x) + \\
&\quad + \frac{1}{p_0(x)} \int_T^x \frac{1}{p_1} \cdots \frac{1}{p_{n-1}} \int_{t_{n-2}}^{x_0} L[f(t)]/p_n(t) dt,
\end{align*}
\]

(7.44)

with suitable constants \(c_3, \ldots, c_n\). It is now clear how this procedure works and by induction one can prove the validity of representation:

\[
\begin{align*}
&f(x) = a_1\phi_1(x) + \cdots + a_{n-1}\phi_{n-1}(x) + c_n\phi_n(x) + \\
&\quad + \frac{(-1)^{n-1}}{p_0(x)} \int_T^x \frac{1}{p_1} \int_{x_0}^{x_0} \frac{1}{p_2} \cdots \frac{1}{p_{n-2}} \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} L[f(t)]/p_n(t) dt,
\end{align*}
\]

(7.45)

with a suitable constant \(c_n\). As a last step we observe that (6.2) implies:

\[
[f(x) - a_1\phi_1(x) - \cdots - a_{n-1}\phi_{n-1}(x)]/\phi_n(x) = O(1), \; x \to x_0^-,
\]

(7.46)

and (7.45) in turn implies:

\[
\begin{align*}
&\frac{1/p_0(x)}{\phi_n(x)} \int_T^x \frac{1}{p_1} \int_{x_0}^{x_0} \frac{1}{p_2} \cdots \frac{1}{p_{n-2}} \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} L[f(t)]/p_n(t) dt \overset{by (4.2)}{=} \\
&\quad \int_T^x \frac{1}{p_1} \int_{x_0}^{x_0} \frac{1}{p_2} \cdots \frac{1}{p_{n-2}} \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} L[f(t)]/p_n(t) dt = O(1), \; x \to x_0^-.
\end{align*}
\]

(7.47)
By the positivity of the integrand this last relation implies (6.6) and the first representation in (6.8) for $R_0(x)$. To prove (6.7) we apply the same ideas starting from representation (5.1) and dividing by $\phi_1$; recalling that $\phi_1 = 1/q_0$ we get:

\begin{equation}
(7.48) \quad \frac{f(x)}{\phi_1(x)} - c_1 + o(1) = \int_T^{x_0} \frac{1}{q_1} \cdots \int_T^{t_{n-2}} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt, \quad x \in [T, x_0, T].
\end{equation}

This implies:

\begin{equation}
(7.49) \quad \int_T^{x_0} \frac{1}{q_1} \int_T^{t_1} \frac{1}{q_2} \cdots \int_T^{t_{n-2}} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt < +\infty,
\end{equation}

and (5.1) can be rewritten as:

\begin{equation}
(7.50) \quad \begin{cases}
    f(x) = a_1 \phi_1(x) + c_2 \phi_2(x) + \cdots + c_n \phi_n(x) + \\
    -\frac{1}{q_0(x)} \int_T^{x_0} \frac{1}{q_1} \int_T^{t_1} \frac{1}{q_2} \cdots \int_T^{t_{n-2}} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt, \quad x \in [T, x_0, T],
\end{cases}
\end{equation}

with suitable constants $c_2, \ldots, c_n$. From this we get:

\begin{equation}
(7.51) \quad \frac{f(x) - a_1 \phi_1(x)}{\phi_2(x)} - c_2 + o(1) = \frac{\phi_1(x)}{\phi_2(x)} \int_T^{x_0} \frac{1}{q_1} \int_T^{t_1} \frac{1}{q_2} \cdots \int_T^{t_{n-2}} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt.
\end{equation}

Evaluating the limit of the right-hand side by L'Hospital's rule and using formula in (2.31), $1/q_1 = (\phi_2/\phi_1)'$, we get:

\begin{equation}
\lim_{x \to x_0} \frac{-\int_T^{x_0} \frac{1}{q_1} \int_T^{t_1} \frac{1}{q_2} \cdots \int_T^{t_{n-2}} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt}{\phi_2(x)/\phi_1(x)} = \lim_{x \to x_0} \int_T^{x_0} \frac{1}{q_1} \int_T^{t_1} \frac{1}{q_2} \cdots \int_T^{t_{n-2}} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt,
\end{equation}

and this last limit, which exists in $\mathbb{R}$, must be a real number. This means that

\begin{equation}
(7.52) \quad \int_T^{x_0} \frac{1}{q_1} \int_T^{t_1} \frac{1}{q_2} \cdots \int_T^{t_{n-2}} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt < +\infty,
\end{equation}

and (7.50) can be rewritten as:

\begin{equation}
(7.53) \quad \begin{cases}
    f(x) = a_1 \phi_1(x) + a_2 \phi_2(x) + c_3 \phi_3(x) + \cdots + c_n \phi_n(x) + \\
    + \frac{1}{q_0(x)} \int_T^{x_0} \frac{1}{q_1} \int_T^{t_1} \frac{1}{q_2} \cdots \int_T^{t_{n-2}} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt,
\end{cases}
\end{equation}

with suitable constants $c_3, \ldots, c_n$. For the clarity's sake we make explicit the steps of this second part of our proof. Assume by induction that the following two conditions hold true:

\begin{equation}
(7.54) \quad \int_T^{x_0} \frac{1}{q_1} \int_T^{t_1} \frac{1}{q_{i+1}} \cdots \int_T^{t_{n-2}} \frac{1}{q_{n-1}} \int_T^{t_{n-1}} \frac{L[f(t)]}{q_n(t)} dt < +\infty;
\end{equation}
\[
\begin{align*}
\begin{cases}
f(x) = a_1 \phi_1(x) + \cdots + a_i \phi_i(x) + c_{i+1} \phi_{i+1}(x) + \cdots + c_n \phi_n(x) + \\
+ \frac{(-1)^i}{q_0(x)} \int_x^{x_0} \frac{1}{q_1} \cdots \int_{t_{i-1}}^{x_0} \frac{1}{q_i} \int_{q_{i+1}}^{t_i} \frac{1}{q_{i+1}} \cdots \int_T^{t_{n-1}} \frac{1}{q_n(t)} \int_T^{t_{n-1}} L[f(t)] \, dt,
\end{cases}
\end{align*}
\]

for some \( i, 1 \leq i \leq n - 2 \), and suitable constants \( c_{i+1}, \ldots, c_n \). Dividing both sides of (7.55) by \( \phi_{i+1} \) and taking account of (6.2) we infer that the limit of the quantity

\[
\int_T^{t_{n-1}} L[f(t)] \, dt
\]

exists in \( \mathbb{R} \). Applying L'Hospital's rule \( i \) times to evaluate this limit we get the new limit

\[
\lim_{x \to x_0} \int_T^{t_{n-1}} L[f(t)] \, dt
\]

which, by the positivity of the integrand, exists in \( \mathbb{R} \) hence it must be a real number. We infer that condition (7.54) holds true with \( i \) replaced by \( i + 1 \) and this implies representation (7.55) with \( i \) replaced by \( i + 1 \) and suitable constants \( c_{i+2}, \ldots, c_n \). By this inductive procedure we arrive at representation:

\[
\begin{align*}
\begin{cases}
f(x) = a_1 \phi_1(x) + \cdots + a_{n-1} \phi_{n-1}(x) + c_n \phi_n(x) + \\
+ \frac{(-1)^n}{q_0(x)} \int_x^{x_0} \frac{1}{q_1} \cdots \int_{t_{n-2}}^{x_0} \frac{1}{q_{n-1}} \int_{t_{n-1}}^{t_{n-1}} \frac{1}{q_n(t)} \int_T^{t_{n-1}} L[f(t)] \, dt,
\end{cases}
\end{align*}
\]

with some constant \( c_n \). Dividing by \( \phi_n \) and using (6.2) we may now conclude that

\[
\int_T^{t_{n-1}} L[f(t)] \, dt = O(1),
\]

and if we try to evaluate the limit of the ratio on the left applying L'Hospital's rule \( (n - 1) \) times we get the limit \( \lim_{x \to x_0} \int_T^{t_{n-1}} L[f(t)] / q_n(t) \, dt \), which exists in \( \mathbb{R} \) and must be a finite number. This is condition (6.7) which allows the second representation in (6.8) for the remainder in (6.3).

Properties in (6.9),(6.10) follow immediately from (6.8). For the last assertion about the monotonicity of \( R_0 \) just evaluate

\[
\begin{align*}
\begin{cases}
(-1)^n R_0(x) = \left( \frac{1}{p_0(x)} \right)' \int_x^{x_0} \frac{1}{p_1} \cdots \int_{t_{n-2}}^{x_0} \frac{1}{p_{n-1}} \int_{t_{n-1}}^{x_0} L[f(t)] \, dt + \\
- \frac{1}{p_0(x)p_1(x)} \int_x^{x_0} \frac{1}{p_2} \cdots \int_{t_{n-3}}^{x_0} \frac{1}{p_{n-2}} \int_{t_{n-1}}^{x_0} L[f(t)] \, dt \leq \ldots
\end{cases}
\end{align*}
\]
\[
\begin{aligned}
(7.60) \quad & \left\{ \begin{array}{l}
\leq 0 \text{ under the first inequality (6.11),} \\
\leq \left( \int_{x}^{x_0} \frac{1}{p_2} \cdots \int_{x_{n-1}}^{x_0} \frac{L[f(t)]}{p_{n}(t)} dt \right) \left[ \left( \frac{1}{p_0(x)} \right) \int_{x}^{x_0} \frac{1}{p_1} \frac{1}{p_0(x)p_1(x)} dt \right] \leq 0
\end{array} \right.
\end{aligned}
\]

under the first inequality (6.12).

Identical calculations for the inequalities involving the \( q_i \)'s. The proof is over. \( \square \)

**Proof of Theorem 6.2.** The equivalence between (6.13) and (6.14) easily follows from Fubini’s theorem by interchanging the order of integrations in (6.13) whereas the equivalence between (6.14) and (6.15) is by no means an obvious fact. We give a concise proof based on Theorem 6.1. Putting

\[
F(x) := \frac{1}{p_0(x)} \int_{x}^{x_0} 1 \cdots \int_{x_{n-1}}^{x_0} \frac{L[f(t)]}{p_{n}(t)} dt \int_{x}^{x_0} \frac{1}{p_1} \frac{1}{p_0(x)p_1(x)} dt, \quad x \in [T,x_0],
\]

we have

\[
F \in \mathcal{AC}^{n-1}[T,x_0]; \quad L_{\phi_1,...,\phi_n}[F(x)] = |L_{\phi_1,...,\phi_n}[f(x)]| \quad \text{a.e. on } [T,x_0];
\]

hence \( F \) satisfies \( L_{\phi_1,...,\phi_n}[F(x)] \geq 0 \) a.e. on \( [T,x_0] \) and Theorem 6.1 implies the equivalence between (6.14) and (6.15). \( \square \)

**Proof of Theorem 6.3.** The only thing to prove is the \( O \)-estimates in (6.20). From representation (7.37) we get:

\[
(7.63) \quad M_i[f(x)] = \int_{T}^{x} \frac{1}{q_{i+1}} \cdots \int_{T}^{x_{n-1}} \frac{L[f(t)]}{q_{n}(t)} dt + c_{i+1} M_i[\phi_{i+1}(x)] + \cdots + c_n M_i[\phi_n(x)] = \cdots
\]

by (3.37) and (3.38)

\[
\cdots = \int_{T}^{x} \frac{1}{q_{i+1}} \cdots \int_{T}^{x_{n-1}} \frac{L[f(t)]}{q_{n}(t)} dt + c + o(1)
\]

for some constant \( c \) whence it follows the estimate for \( M_i[f(x)] \). And so on for the other estimates. \( \square \)

8. Appendix: algorithms for constructing canonical factorizations

The original procedure used by Trench [24] to construct a C.F. of type (I) for a disconjugate operator is not an intuitive one. Here we exhibit two easier algorithms to construct both types of C.F.’s starting from an explicit fundamental system of solutions which is also an asymptotic scale at one endpoint. The so-obtained factorizations will be proved to coincide with those obtainable by Pólya’s procedure when applied either to the asymptotic scale \( (\phi_1,\ldots,\phi_n) \) or to the inverted \( n \)-tuple \( (\phi_n,\ldots,\phi_1) \). As each step in the algorithms has an asymptotic meaning they provide asymptotic interpretations of Pólya’s procedure and they may sometimes be quicker to apply than Pólya’s procedure, especially for small values of \( n \), avoiding the explicit use of Wronskians. The algorithm for a C.F. of type (II)
is particularly meaningful as it highlights how the operators \( M_k \) naturally arise from an asymptotic expansion with an identically-zero remainder when one attempts to find out expressions of the coefficients: see §3-C and §3-D.

Let us consider a generic element \( u \in \text{span} \ (\phi_1, \ldots, \phi_n) \) of the type
\[
(8.1) \quad u = a_1 \phi_1(x) + \cdots + a_n \phi_n(x), \quad a_i \neq 0 \ \forall \ i,
\]
which we interpret as an asymptotic expansion at \( x \) with the largest growth-order at \( x \).

We shall first present the algorithm for a C.F. of type (II) as it is more simple to describe.

**Proposition 8.1** (The algorithm for a special C.F. of type (II)). Under conditions (2.1)-(2.4) and (2.6) the following algorithm yields the special global C.F. of \( L_{\phi_1, \ldots, \phi_n} \) of type (II) at \( x_0 \) in (2.33) together with \( (n-1) \) asymptotic expansions which, after dividing by the first meaningful term on the right, coincide with the expansions obtained by applying to (8.1) the operators \( M_k \) defined in (3.27). Formulas for the coefficients \( a_k \) in (3.41) are reobtained.

(A) **Verbal description of the algorithm.**

1\textsuperscript{st} step. Divide both sides of (8.1) by the first term on the right, which is the term with the largest growth-order at \( x_0 \), and then take derivatives so obtaining
\[
(8.2) \quad \left( \frac{u(x)}{\phi_1(x)} \right)' = a_2 \left( \frac{\phi_2(x)}{\phi_1(x)} \right)' + \cdots + a_n \left( \frac{\phi_n(x)}{\phi_1(x)} \right)'.
\]

Notice that division of both sides by the first term on the right just yields the expansion obtained by applying to (8.1) the operators \( M_1 \); a similar remark applies to each of the subsequent expansions both in this and in the next proposition.

2\textsuperscript{nd} step. Divide both sides of (8.2) by the first term on the right and take derivatives so obtaining
\[
(8.3) \quad \left[ \frac{1}{(\phi_2(x)/\phi_1(x))'} \left( \frac{u(x)}{\phi_1(x)} \right)' \right]' = a_3 \left( \frac{\phi_3(x)/\phi_1(x)}{\phi_2(x)/\phi_1(x)} \right)' + \cdots + a_n \left( \frac{\phi_n(x)/\phi_1(x)}{\phi_2(x)/\phi_1(x)} \right)'.
\]

3\textsuperscript{rd} step. Repeat the procedure on (8.3) dividing by the first term on the right and then taking derivatives so getting
\[
(8.4) \quad \left[ \frac{1}{(\phi_3/\phi_1)'} \left( \frac{1}{(\phi_2/\phi_1)'} \left( \frac{u}{\phi_1} \right)' \right)' \right]' =
\]
\[
= a_4 \left( \frac{\phi_4/\phi_1}{\phi_2/\phi_1} \right)' / \left( \frac{\phi_3/\phi_1}{\phi_2/\phi_1} \right)' + \cdots + a_n \left( \frac{\phi_n/\phi_1}{\phi_2/\phi_1} \right)' / \left( \frac{\phi_3/\phi_1}{\phi_2/\phi_1} \right)'.
\]

Iterating the procedure each of the obtained relation is an identity on \([T, x_0]\) and is an asymptotic expansion at \( x_0 \), hence at each step we are dividing by the term on the right with the largest growth-order at \( x_0 \). Notice that at each step the asymptotic expansion loses
its first meaningful term and this is the same phenomenon occurring in differentiation of Taylor’s
formula. After \( n \) steps we arrive at an identity:

\[
[q_{n-1}(\ldots(q_0 u)')\ldots]' = 0 \text{ on } [T, x_0],
\]

where the \( q_i \)’s coincide with those in (2.29).

\[\text{(B) Schematic description of the algorithm.}\]

**Step \( "1" \):**

\[
u = a_1 \phi_1 + \cdots + a_n \phi_n
\]

\[
\left. \begin{array}{c}
\uparrow \\
\downarrow d & \& d
\end{array} \right\} \text{ \&} \left. \begin{array}{c}
\downarrow \\
\uparrow \downarrow d & \& d
\end{array} \right\}
\]

**Step \( "2" \):**

\[
(u/\phi_1)' = a_2 (\phi_2/\phi_1)' + \cdots + a_n (\phi_n/\phi_1)'
\]

\[
\left. \begin{array}{c}
\uparrow \\
\downarrow d & \& d
\end{array} \right\} \text{ \&} \left. \begin{array}{c}
\downarrow \\
\uparrow \downarrow d & \& d
\end{array} \right\}
\]

**Step \( "3" \):**

\[
\left( \frac{u/\phi_1}{\phi_2/\phi_1} \right)' = a_3 \left( \frac{(\phi_3/\phi_1)'}{(\phi_2/\phi_1)'} \right) + \cdots + a_n \left( \frac{(\phi_n/\phi_1)'}{(\phi_2/\phi_1)'} \right)'
\]

\[
\left. \begin{array}{c}
\uparrow \\
\downarrow d & \& d
\end{array} \right\} \text{ \&} \left. \begin{array}{c}
\downarrow \\
\uparrow \downarrow d & \& d
\end{array} \right\}
\]

**Step \( "4" \):**

\[
\left[ \frac{1}{(\phi_3/\phi_1)'} \left( \frac{1}{(\phi_2/\phi_1)'} \left( \frac{u}{\phi_1} \right)' \right) \right]'
\]

\[
\left. \begin{array}{c}
\uparrow \\
\downarrow d & \& d
\end{array} \right\} \text{ \&} \left. \begin{array}{c}
\downarrow \\
\uparrow \downarrow d & \& d
\end{array} \right\}
\]

\[
= a_4 \left( \frac{(\phi_4/\phi_1)'}{(\phi_2/\phi_1)'} \right) / \left( \frac{(\phi_3/\phi_1)'}{(\phi_2/\phi_1)'} \right) + \cdots + a_n \left( \frac{(\phi_n/\phi_1)'}{(\phi_2/\phi_1)'} \right) / \left( \frac{(\phi_3/\phi_1)'}{(\phi_2/\phi_1)'} \right),
\]

and so on, where the symbol "\( d \) \& \( d \)" stands for the two operations "divide" both sides by the underbraced term on the right and then "differentiate" both sides" (the equation in each step being the result of the preceding step).

**Proposition 8.2** (The algorithm for the C.F. of type (I)). Under conditions (2.1)-(2.4) and (2.6) the following algorithm yields "the" global C.F. of \( L_{\phi_1, \ldots, \phi_n} \) of type (I) at \( x_0 \) and \((n-1)\) asymptotic expansions which, after dividing by the last meaningful term on the right, coincide (apart from the signs of the coefficients) with the expansions obtained by applying to (8.1) the operators \( L_k \) defined in (3.24).

**A** **Verbal description of the algorithm.**
1\textsuperscript{st} step. Divide both sides of (8.1) by the last term on the right, which is the term with the smallest growth-order at \(x_0\), and then take derivatives so obtaining

\[
(8.6) \quad \frac{u(x)}{\phi_n(x)}' = a_1 \left( \frac{\phi_1(x)}{\phi_n(x)} \right)' + \cdots + a_{n-1} \left( \frac{\phi_{n-1}(x)}{\phi_n(x)} \right)'.
\]

2\textsuperscript{nd} step. Divide both sides of (8.6) by the last term on the right and then take derivatives so obtaining

\[
(8.7) \quad \left[ \frac{1}{(\phi_{n-1}(x)/\phi_n(x))'} \right] \left( \frac{u(x)}{\phi_n(x)} \right)' = a_1 \left( \frac{\phi_1(x)/\phi_n(x)}{\phi_{n-1}(x)/\phi_n(x)} \right)' + \cdots + a_{n-2} \left( \frac{\phi_{n-2}(x)/\phi_n(x)}{\phi_{n-1}(x)/\phi_n(x)} \right)'.
\]

3\textsuperscript{rd} step. Repeat the procedure on (8.7) dividing by the last term on the right and then taking derivatives so getting

\[
(8.8) \quad \left[ \frac{1}{(\phi_{n-2}/\phi_{n-1})'} \right] \left( \frac{1}{(\phi_{n-1}/\phi_n)'} \left( \frac{u}{\phi_n} \right) \right)' = a_1 \left( \frac{\phi_1/\phi_n}{(\phi_{n-1}/\phi_n)'} \right)' + \cdots + a_{n-3} \left( \frac{\phi_{n-3}/\phi_n}{(\phi_{n-2}/\phi_n)'} \right)' + \cdots + a_{n-2} \left( \frac{\phi_{n-2}/\phi_n}{(\phi_{n-1}/\phi_n)'} \right)'.
\]

Iterating the procedure each of the obtained relation is an identity on \([T,x_0]\) and is an asymptotic expansion at \(x_0\), hence at each step we are dividing by the term on the right with the smallest growth-order at \(x_0\). Also notice that at each step the asymptotic expansion loses its last meaningful term and this is a phenomenon different from that occurring in differentiation of Taylor’s formula (see the foregoing proposition). After \(n\) steps we arrive at an identity:

\[
(8.9) \quad [p_{n-1}(\ldots(p_0u')\ldots)]' \equiv 0 \text{ on } [T,x_0],
\]

where the \(p_i\)’s coincide, signs apart, with those in (2.40).

(B) \textbf{Schematic description of the algorithm.}

\begin{itemize}
  \item \textbf{Step } "1":
  \[
  u = a_1\phi_1 + \cdots + a_n\phi_n
  \]

\end{itemize}

\begin{itemize}
  \item \textbf{Step } "2":
  \[
  (u/\phi_n)' = a_1(\phi_1/\phi_n)' + \cdots + a_{n-1}(\phi_{n-1}/\phi_n)'
  \]

\end{itemize}

\begin{itemize}
  \item \textbf{Step } "3":
  \[
  \left( \frac{u(\phi_n)'}{(\phi_{n-1}/\phi_n)'} \right)' = a_1 \left( \frac{(\phi_1/\phi_n)'}{(\phi_{n-1}/\phi_n)'} \right)' + \cdots + a_{n-2} \left( \frac{(\phi_{n-2}/\phi_n)'}{(\phi_{n-1}/\phi_n)'} \right)'.
  \]

\end{itemize}
Step ",4\):

\[
\begin{bmatrix}
\frac{1}{(\phi_{n-2}/\phi_n)'} & 1 & (u/\phi_n)' \\
(\phi_{n-1}/\phi_n)' & (\phi_{n-1}/\phi_n)' & (\phi_{n-1}/\phi_n)'
\end{bmatrix}' =
\]

\[\downarrow \quad \text{d} & \text{d} \quad \downarrow \]

\[
= a_1 \left( \frac{(\phi_1/\phi_n)'}{(\phi_{n-1}/\phi_n)'} \right)' + \cdots + a_{n-3} \left( \frac{(\phi_{n-3}/\phi_n)'}{(\phi_{n-1}/\phi_n)'} \right)' + \left( \frac{(\phi_{n-2}/\phi_n)'}{(\phi_{n-1}/\phi_n)'} \right)',
\]

\[\downarrow \]

and so on with the symbol ",d & d\" reminding of the two operations ",divide\" both sides by the underbraced term on the right and then ",differentiate\" both sides\" (the equation in each step being the result of the preceding step).

Remarks. 1. In order to obtain any C.F. by the above procedures one may simply put 
\[a_1 = \cdots = a_n = 1.\]

2. If some operator \(L_n\) of type (2.7) is known to be disconjugate on a left neighborhood of \(x_0, I_{x_0}\), and if

\[
(8.10) \left\{ \begin{array}{l}
\phi_1, \ldots, \phi_n \in \ker L_n; \\
\phi_1(x) \gg \cdots \gg \phi_n(x), \quad x \to x_0^-;
\end{array} \right.
\]

then the algorithm in Proposition 8.1 yields the C.F. of type (I) at \(x_0\), valid on the whole given interval \(I_{x_0}\) by Proposition 2.4-(ii): pay attention to the different numbering of the \(n\)-tuple! But the algorithm in Proposition 8.1 yields a C.F. of type (II) at \(x_0\) valid on some left neighborhood of \(x_0\), the largest of them being characterized by the non-vanishingness of all the Wronskians \(W(\phi_1, \ldots, \phi_i), 1 \leq i \leq n - 1\), which does not automatically follow from the non-vanishingness of \(W(\phi_n, \phi_{n-1}, \ldots, \phi_i), 1 \leq i \leq n\); a fact amply commented on in §2.

3. By applying the above algorithms to (8.1) with a random choice of the term to be factored out at each step one may well obtain, after \(n\) steps, a factorization valid on a certain subinterval of the given interval but, in general for \(n \geq 3\), it will not be a C.F. at an endpoint.

4. In practical applications of the algorithms there is a fatal pitfall to avoid, namely the temptation at each step of suppressing brackets, cancelling possible opposite terms and rearranging in an aesthetically-nicer asymptotic scale. This in general gives rise to a factorization of an operator quite different from \(L_{\phi_1, \ldots, \phi_n}\). Hence it is essential that all the terms coming from a single term in the preceding step be kept grouped together as a single term to the end of the procedure: see examples at the end of this section.

Proof of Proposition 8.1, that of Proposition 8.2 being exactly the same after replacing \((\phi_1, \ldots, \phi_n)\) by \((\phi_n, \ldots, \phi_1)\). We have to prove that the \(q_i\)'s in (8.5) coincide with those in (2.29) and this does not seem to be an obvious fact though it is made explicit in the algorithm that the first three coefficients \(q_0, q_1, q_2\) coincide with Pólya’s coefficients in
Karlin [13, p. 60], which we report here in the version needed in our proof:

\[ 1/q_{i+1} = \left[ q_i \times \text{(the expression for } 1/q_i \text{ with the one change: } \phi_{i+1} \text{ replaced by } \phi_{i+2}) \right]' \]

Hence it is enough to show that Pólya’s expression for \( 1/q_{i+1} \) is obtained by the same rule. We present two different proofs, the first being based on the equivalent representations (2.29) and (2.31). We have:

\[ [q_i \times \text{(expression in (2.29) for } 1/q_i \text{ with } \phi_{i+1} \text{ replaced by } \phi_{i+2})]'] = \]

\[ = \left[ \frac{\left[ W(\phi_1, \ldots, \phi_i) \right]^2}{W(\phi_1, \ldots, \phi_i)W(\phi_1, \ldots, \phi_i, \phi_{i+1})} \cdot \frac{W(\phi_1, \ldots, \phi_i)W(\phi_1, \ldots, \phi_i, \phi_{i+2})}{\left[ W(\phi_1, \ldots, \phi_i) \right]^2} \right]' = \]

\[ = \left[ \frac{W(\phi_1, \ldots, \phi_i, \phi_{i+2})}{W(\phi_1, \ldots, \phi_i, \phi_{i+1})} \right]' \text{ by (2.31)} = \frac{1}{q_{i+1}}. \]

It is also clear that the various identities obtained are nothing but those obtained by applying to (8.1) the operators \( M_k \) defined in (3.26) which, by (3.29), differ from \( L_{\phi_1, \ldots, \phi_k} \) by a factor which is a non-vanishing function.

The second proof is based on a nontrivial identity involving Wronskians of Wronskians, Karlin [13, p. 60], which we report here in the version needed in our proof:

\[ W(g_1, \ldots, g_n, f_1, f_2) \cdot W(g_1, \ldots, g_n) \equiv W(W(g_1, \ldots, g_n, f_1), W(g_1, \ldots, g_n, f_2)). \]

Comparing the expressions in (2.31) and those given by our algorithm we see that the two procedures coincide if the following identity holds true:

\[ \left[ \frac{W(\phi_1, \ldots, \phi_i, \phi_{i+2})}{W(\phi_1, \ldots, \phi_i, \phi_{i+1})} \right]' = \left\{ \left[ \frac{W(\phi_1, \ldots, \phi_{i-1}, \phi_{i+2})}{W(\phi_1, \ldots, \phi_{i-1}, \phi_i)} \right]' \right\}'. \]

We shall show the validity of this identity even if the outer derivatives are suppressed. Using the elementary formula \( (g_2/g_1)' = W(g_1, g_2) \cdot g_1^{-2} \) we have:

\[ \left\{ \frac{W(\phi_1, \ldots, \phi_{i-1}, \phi_{i+2})}{W(\phi_1, \ldots, \phi_{i-1}, \phi_i)} \right\}' = \]

\[ = \frac{W(W(\phi_1, \ldots, \phi_i), \phi_{i-1}, \phi_{i+2}))(W(\phi_1, \ldots, \phi_i))^{-2}}{W(W(\phi_1, \ldots, \phi_i), \phi_{i-1}, \phi_{i+1}))(W(\phi_1, \ldots, \phi_i))^{-2}} = \ldots \]

by (8.13) with \( g_1, \ldots, g_n \) replaced by \( \phi_1, \ldots, \phi_i \cdot \phi_{i-1} \)

\[ \ldots = \frac{W(\phi_1, \ldots, \phi_{i-1}, \phi_i, \phi_{i+2})}{W(\phi_1, \ldots, \phi_{i-1}, \phi_i, \phi_{i+1})} \cdot \frac{W(\phi_1, \ldots, \phi_i)}{W(\phi_1, \ldots, \phi_{i-1})} = \frac{W(\phi_1, \ldots, \phi_i, \phi_{i+2})}{W(\phi_1, \ldots, \phi_{i-1})}. \]

\[ \square \]

An example illustrating the two algorithms. Consider the fourth-order operator \( L \) of type (2.7), such that

\[ \ker L = \text{span} \ (e^x, x, \log x, 1), \]
acting on $AC^3[0, +\infty)$ or even on $C^\infty[0, +\infty)$. Starting from the asymptotic scale
\begin{equation}
(8.17) \quad e^x \gg x \gg \log x \gg 1, \ x \to +\infty,
\end{equation}
the algorithm in Proposition 8.2 yields in sequence:

\[
\begin{align*}
u &= e^x + x + \log x + 1; \\
u' &= e^x + 1 + x^{-1}; \\
(xu')' &= [(x+1)e^x] + 1; \\
(xu')'' &= [(x+2)e^x]; \\
\frac{d}{dx}[(x+2)^{-1}e^{-x}(xu')'''] &= 0.
\end{align*}
\]

Hence
\begin{equation}
(8.18) \quad Lu \equiv x^{-1}(x+2)e^x [(x+2)^{-1}e^{-x}(xu')''']',
\end{equation}
where
\begin{equation}
(8.19) \quad p_1(x) = x; \ p_2(x) = 1; \ p_3(x) = (x+2)^{-1}e^{-x};
\end{equation}
and (8.18) is "the " global C.F. of $L$ of type (I) at $+\infty$.

On the other hand the algorithm in Proposition 8.1 yields in sequence:

\[
\begin{align*}
u &= e^x + x + \log x + 1; \\
(e^{-x}u)' &= [(1-x)e^x] + [(x^{-1} - \log x)e^{-x}] - e^{-x}; \\
[(1-x)^{-1}e^x(e^{-x}u)']' &= [(1-x)^{-2}(-\log x + 1 + x^{-1} - x^{-2})] - (1-x)^{-2}; \\
\frac{d}{dx}[(1-x)^2(-\log x + 1 + x^{-1} - x^{-2})^{-1}[(1-x)^{-1}e^x(e^{-x}u)']]' &= \\
= (-\log x + 1 + x^{-1} - x^{-2})^{-2}x^{-3}(-x^2 - x + 2); \\
\frac{d}{dx}\left[(-\log x + 1 + x^{-1} - x^{-2})^2x^3(-x^2 - x + 2)^{-1}\times (1-x)^2(-\log x + 1 + x^{-1} - x^{-2})^{-1}[(1-x)^{-1}e^x(e^{-x}u)']\right]' &= 0.
\end{align*}
\]

(The underbraced terms on the right are those by which one must divide and then differentiate.) Hence:
\begin{equation}
(8.20) \quad Lu \equiv (1-x)^{-1}x^{-3}(-x^2 - x + 2)(-\log x + 1 + x^{-1} - x^{-2})^{-1} \times
\end{equation}
For instance the first procedure involves quite simple terms and only at the last-but-one step we may split the remaining term on the right by writing
\[ \times \left[ (-\log x + 1 + x^{-1} - x^2)^2 x^3 (-x^2 - x + 2)^{-1} \right] \times \left[ (1 - x)^2 (-\log x + 1 + x^{-1} - x^{-2})^{-1} (1 - x)^{-1} e^x (e^{-x} u)' \right]' \]

where
\[
\begin{align*}
\mathcal{L}_0 (x) & := e^{-x}; \quad \mathcal{L}_1 (x) := (1 - x)^{-1} e^x; \\
\mathcal{L}_2 (x) & := (1 - x^2) (-\log x + 1 + x^{-1} - x^{-2})^{-1} \sim x^2 / \log x, \ \ x \to +\infty; \\
\mathcal{L}_3 (x) & := x^3 (-x^2 - x + 2)^{-1} (-\log x + 1 + x^{-1} - x^{-2})^2 \sim \\
& \sim -x (\log x)^2, \ \ x \to +\infty; \quad \int^{+\infty} 1 / |\mathcal{L}_i| < +\infty, \ i = 1, 2, 3.
\end{align*}
\]

(8.21)

Hence (8.20) is a C.F. of L of type (II) at +∞ valid on the largest neighborhood of +∞ whereon
\[ 1 - x \neq 0; \quad 1 - x^2 \neq 0; \quad (\log x - 1 - x^{-1} + x^{-2}) \neq 0; \quad x^2 + x - 2 \neq 0, \]
which is easily seen to be the interval \([1, +\infty)\). In conclusion: changing the signs of the \(\mathcal{L}_i\)'s, if necessary, we get a Pólya-Mammana factorization of L on \([1, +\infty)\) which is a C.F. of type (II) at +∞. The standard non-factorized form of L is
\[ Lu \equiv u^{(4)} + x^{-1} (6 - 2x) (x + 2)^{-1} u^{(3)} - 2x^{-1} (x + 2) \log x. \]

(8.22)

In the various steps of the above procedures one must carefully avoid the temptation of rearranging the terms in the right-hand side in (supposedly) nicer asymptotic scales. For instance the first procedure involves quite simple terms and only at the last-but-one step we may split the remaining term on the right by writing
\[
(x u')'' = xe^x + 2e^x
\]
and taking \(e^x\) as the term with the smallest growth-order. The procedure then yields
\[ (e^{-x} (x u')')' = 1 \quad \text{and} \quad (e^{-x} (x u')'')'' \equiv 0. \]

This gives a fifth-order operator
\[ \tilde{L}u := x^{-1} e^x (e^{-x} (x u')'')'', \]
distinct from the given fourth-order operator.

On the contrary the second procedure offers a great number of temptations! For instance if one rewrites the result of the first step as
\[ (e^{-x} u)' = -xe^{-x} - \log x \cdot e^{-x} + x^{-1} e^{-x}, \]
and then goes on applying the second algorithm to (8.23) as if the right-hand side would be an asymptotic expansion with three meaningful terms, one gets:
\[ (x^{-1} e^x (e^{-x} u')')' = \frac{(x^{-2} \log x - x^{-2}) - 2x^{-3}}{x^{-2} \log x - x^{-2} - 2x^{-3}}, \]

(8.24)
the only difference between the two expressions on the right being the term-grouping. From the upper relation in (8.24), considered as an asymptotic expansion at $+\infty$ with two meaningful terms, one gets:

$$\left[ x^2 (\log x - 1)^{-1} (x^{-1} e^x (e^{-x} u)' \prime) \prime \right] \prime = (-2x^{-1} (\log x - 1)^{-1}) \prime = 2x^{-2} (\log x - 1)^{-1} + 2x^{-2} (\log x - 1)^{-2} = 2x^{-2} (\log x - 1)^{-2} \log x,$$

and then

$$\left\{ x^2 (\log x)^{-1} (\log x - 1)^2 \left[ x^2 (\log x - 1)^{-1} (x^{-1} e^x (e^{-x} u)' \prime) \prime \right] \prime \right\} \prime \equiv 0,$$

whose left-hand side is a fourth-order operator distinct from our operator.

If, instead, one starts from the lower relation in (8.24), considered as an asymptotic expansion at $+\infty$ with three meaningful terms, one gets:

$$(x^2 (\log x)^{-1} (x^{-1} e^x (e^{-x} u)' \prime) \prime) \prime = x^{-1} (\log x)^{-2} + 2x^{-2} (\log x)^{-1} + 2x^{-2} (\log x)^{-2}$$

and so forth in an endless process leading nowhere!!

An example showing that application of any procedure in §11-D regardless of the relative growth-orders of the terms may yield a non-canonical factorization. Let us consider $Lu := u'''$ acting on $AC^2[0, +\infty)$ or even on $C^\infty[0, +\infty)$ and the tern $(1, x, x^2)$ which satisfies

$$1 \gg x \gg x^2, \ x \to 0^+; \quad x^2 \gg x \gg 1, x \to +\infty,$$

and is such that all the possible Wronskians constructed with these three functions do not vanish on $]0, +\infty)$. We now apply the following two procedures:

$$u = x^2 + x + 1$$

and obtain the two factorizations:

$$u''' \equiv x^{-2} (x^3 (x^{-1} u)' \prime) \prime; \quad u''' \equiv (x^{-1} (x^2 (x^{-1} u)' \prime) \prime) \prime \equiv 0,$$

both valid on $]0, +\infty)$ but none of which is a C.F. at any of the endpoints.

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