The BCS-Bose Crossover Theory

S.K. Adhikari, a M. de Llano, b,c,e F.J. Sevilla, d,e M.A. Solís d & J.J. Valencia e,f

a Instituto de Física Teórica, UNESP-São Paulo State University, 01405-900 São Paulo, SP, Brazil
b Texas Center for Superconductivity, University of Houston, Houston, TX 77204, USA
c Instituto de Investigaciones en Materiales, Universidad Nacional Autónoma de México 04510 México, DF, Mexico
d Instituto de Física, UNAM, 01000 México, DF, Mexico
e Consortium of the Americas for Interdisciplinary Science, University of New Mexico Albuquerque, NM 87131, USA
f Universidad de la Ciudad de México, San Lorenzo Tezonco, 09940 México, DF, Mexico

October 28, 2021

Abstract

We contrast four distinct versions of the BCS-Bose statistical crossover theory according to the form assumed for the electron-number equation that accompanies the BCS gap equation. The four versions correspond to explicitly accounting for two-hole-(2h) as well as two-electron-(2e) Cooper pairs (CPs), or both in equal proportions, or only either kind. This follows from a recent generalization of the Bose-Einstein condensation (GBEC) statistical theory that includes not boson-boson interactions but rather 2e- and also (without loss of generality) 2h-CPs interacting with unpaired electrons and holes in a single-band model that is easily converted into a two-band model. The GBEC theory is essentially an extension of the Friedberg-T.D. Lee 1989 BEC theory of superconductors that excludes 2h-CPs. It can thus recover, when the numbers of 2h- and 2e-CPs in both BE-condensed and noncondensed states are separately equal, the BCS gap equation for all temperatures and couplings as well as the zero-temperature BCS (rigorous-upper-bound) condensation energy for all couplings. But ignoring either 2h- or 2e-CPs it can do neither. In particular, only half the BCS condensation energy is obtained in the two crossover versions ignoring either kind of CPs. We show how critical temperatures $T_c$ from the original BCS-Bose crossover theory in 2D require unphysically large couplings for the Cooper/BCS model interaction to differ significantly from the $T_c$s of ordinary BCS theory (where the number equation is substituted by the assumption that the chemical potential equals the Fermi energy). PACS # 74.20.-z; 74.20.Mn; 05.30.Fk; 05.30.Jp

Key words: Bose-Einstein condensation statistical model; BCS-Bose crossover theory.

1 Introduction

Boson-fermion (BF) statistical models of superconductivity (SC) as a Bose-Einstein condensation (BEC) [1, 2] began to be seriously studied in the mid-1950’s [3-6], pre-dating even the BCS-Bogoliubov statistical theory [7-9]. Although BCS theory only contemplates the presence of “Cooper correlations” of single-particle states, BF models [3-6,10-18] posit the existence of actual bosonic CPs. A drawback of early BF models is the notorious absence of an electronic gap $\Delta(T)$, with $T$ the absolute temperature. Perhaps the first BF model with a gap was introduced in Ref. [19]. Somewhat later, the remarkable relation $\Delta(T) \propto \sqrt{n_0(T)}$, with $n_0(T)$ the BEC condensate number density of electron pairs, first seems to have appeared [20]. It resurfaced a year later in the BEC BF model in 3D of Friedberg and T.D. Lee [21, 22] applied to cuprate superconductors. With just one adjustable parameter (the ratio of perpendicular to CuO$_2$-plane boson masses) this theory fitted [22] quasi-2D cuprate $T_c/T_F$ empirical values [23] rather well. The ratio turned out to be 66,650—just under the $10^5$ anisotropy ratio reported for $B_{2+\pm\delta}Sr_{2-\delta}CuO_{2\pm\delta}$ [24] almost contemporaneously.
An extension of the work in Refs. [21, 22] is a generalized BEC (GBEC) statistical single-band theory whereby a superconducting BCS condensate was recently suggested [10], and subsequently confirmed [11] (but only to lowest order in the BF coupling) to be precisely a Bose-Einstein condensate (BEC) of equal numbers of bosonic two-electron (2e) and two-hole (2h) Cooper pairs (CPs), at least inasmuch as the GBEC reproduces the same BCS gap equation for all temperature and coupling as well as the same $T = 0$ condensation energy found from BCS theory. The holes that make up the 2h-CPs originate precisely the Fermi sea associated with the model studied here. One advantage of the single-band model is that it allows recovering, among other theories, the BCS theory as a special case. The distinction (Ref. [25] pp. 70-72) between single particles and holes is, in a sense, trivial. Not so for particle-pairs and hole-pairs, as will be seen shortly.

The BF coupling assumed appears in an interaction many-body Hamiltonian $H_{int}$ which defines the BEC theory. Added to $H_{int}$ is an unperturbed Hamiltonian $H_0$ describing a free ternary gas of unpaired electrons plus 2e-CPs plus 2h-CPs. The noninteracting ternary gas represents the normal state of the original, strongly-correlated many-electron system under study, and is a viable candidate for a so-called “non-Fermi-liquid.” The new GBEC theory embodied in

$$H = H_0 + H_{int}$$

is in essence a complete BF (statistical) single-band model that, however, admits departure from the perfect 2e-/2h-CP symmetry that constrains BCS theory by construction. It can be diagonalized via a Bogoliubov canonical transformation exactly if one neglects nonzero center-of-mass-momentum (CMM) CPs in $H_{int}$ as is done in BCS theory but not in $H_0$ which in BCS theory represents a pure electron gas. The GBEC theory is complete only in the sense that 2h-CPs are not ignored. It reduces to all the known statistical theories of superconductors (SCs), including the BCS-Bose “crossover” picture in the four versions to be distinguished below. Its practical impact is that it yields [12, 13] robustly higher $T_c$’s than BCS theory without abandoning electron-phonon dynamics when one departs from the perfect 50/50 symmetry of 2e-/2h-CPs in the condensate.

In the literature, electron-phonon dynamics have been widely mimicked by the s-wave BCS/Cooper model interaction $V_{k, k'}$ [7, 26]. It is a nonzero negative constant $-V$, and if only if single-particle energies $\epsilon_k, \epsilon_{k'}$ lie within the energy interval $[\max\{0, \mu - h\omega_D\}, \mu + h\omega_D]$ where $\mu$ is the electron chemical potential and $\omega_D$ the Debye frequency. We employ this model interaction here. Other pairing symmetries beyond pure s-wave can also be accommodated. Although it sheds considerable light on different possible crossover schemes, our single-band picture where single-electron and single-holes are assumed to have the same effective mass is not as realistic in describing real materials as a multiband (say, a valence-like band for holes and a conduction-like one for electrons) theory where these two masses can differ, as will be discussed later.

A fundamental drawback of early [3]-[6] BF models, which took 2e CPs as analogous to diatomic molecules in a classical atom-molecule binary gas mixture, is the cumbersome introduction of an electron energy gap $\Delta(T)$. “Gapless” models, however, are useful [15, 16] in locating transition-temperature singularities if approached from above, i.e., from the normal state where $T > T_c$.

The “ordinary” CP problem [26] for two distinct interference interactions (the $\delta$-potential well [17, 18] or the Cooper/BCS model [7, 26] interactions) neglects the effect of 2h-CPs treated on an equal footing with 2e [or, in general, two-particle (2p)] CPs. On the other hand, Green’s functions [25] can naturally deal with hole propagation and thus accommodate both 2e- and 2h-CPs via, e.g., the Bethe-Salpeter equation [27, 28]. In addition to the generalized CP problem, a crucial result [10, 11] as already mentioned is that the BCS condensate consists of equal numbers of 2e- and 2h-CPs. This was implicitly already suggested from the perfect symmetry about electron energy $\epsilon = \mu$ of the well-known Bogoliubov [29] $v^2(\epsilon)$ and $u^2(\epsilon)$ coefficients, with the tail of $v^2(\epsilon)$ above $\epsilon = \mu$ representing 2e correlations and that of $u^2(\epsilon)$ below $\epsilon = \mu$ refers to 2h correlations.

In this paper we show how: a) four versions of the BCS-Bose statistical crossover theory can be obtained by ignoring either 2h- or 2e-CPs or by including both; c) for only two of the four versions can the precise BCS gap equation for all temperatures $T$ be derived; c) crossover-picture $T_c$’s, defined self-consistently by both the gap and fermion-number equations, requires unphysically large couplings (at least for the Cooper/BCS model interaction in 2D SCs) to differ significantly from the $T_c$ of ordinary BCS theory defined without the number equation since here the chemical potential is assumed equal to the Fermi energy; and d) the full $T = 0$ BCS condensation energy follows from one crossover version but only half of it from the two versions ignoring either kind of CPs. The condensation energy is simply related to the ground-state energy of the many-fermion system, which in the case of BCS is a rigorous upper bound to the exact many-body value for the given Hamiltonian as BCS theory starts from a variational wave function for the superconductor ground state. These results, with the exception of (c) which does not apply, are expected to hold
also for neutral-fermion superfluids (SFs)—such as liquid $^3$He [30–31], neutron matter and trapped ultra-cold fermion atomic gases [32–33]—where the pair-forming two-fermion interaction, of course, differs from the Cooper/BCS one for SCs.

2 Generalized BEC Theory (GBEC)

The GBEC theory is described in detail in Refs. [10–13]; here we summarize its main equations. It applies in $d$ dimensions and is defined by a Hamiltonian of the form $H = H_0 + H_{\text{int}}$. The unperturbed Hamiltonian $H_0$ should ideally be, to quote Leggett [44] “an appropriate ‘zerohth-oder’ starting point” accounting for “pairs of electronic excitations with charge 2e that all have the same ground-state wavefunction.” Thus, our $H_0$ corresponds to a non-Fermi-liquid “normal” state which, besides just fermions, is an ideal (i.e., noninteracting) ternary gas mixture of unpaired fermions and both types of CPs namely, 2e and 2h, the latter introduced without loss of generality. Specifically

$$H_0 = \sum_{k_1,s_1} \epsilon_{k_1} a_{k_1,s_1}^+ a_{k_1,s_1} + \sum_{K} E_+(K)b_K^+ b_K - \sum_{K} E_-(K)c_K^+ c_K$$

(1)

where $K \equiv k_1 + k_2$ is the CMM wavevector of the pair, while $\epsilon_{k_1} \equiv h^2 k_1^2 / 2m$ are the single-electron, and $E_\pm(K)$ the 2e-/2h-CP phenomenological, energies. Here $a_{k_1,s_1}^+ (a_{k_1,s_1})$ are creation (annihilation) operators for fermions and similarly $b_K^+ (b_K)$ and $c_K^+ (c_K)$ for 2e- and 2h-CP bosons, respectively. These $b$ and $c$ operators depend only on $K$ and so are distinct from the BCS operators depending on both $K$ and the relative $k = k_1 - k_2$ discussed in Ref. [7].

Eqs. (2.9) to (2.13) for the particular case of $K = 0$ and shown there not to satisfy the ordinary Bose commutation relations. But because two pairs cannot exactly overlap in real space without violating the Pauli Principle, they are often considered “hard-core bosons,” albeit of hard-core radii $0^+$. For this reason, one can probably not expect to be able to construct the $b$ and $c$ operators directly from the $a$ operators in order to establish that $b$ and $c$ obey Bose commutation relations precisely. Nonetheless, these pairs stand for objects that can easily be seen to obey Bose-Einstein statistics as, in the thermodynamic limit, an indefinitely large number of $K$ values correspond to a given $K$ value defining an energy level $E_+(K)$ or $E_-(K)$. This is all that is needed to ensure the BEC (or macroscopic occupation of a given state that appears below a certain fixed $T = T_c$) found [10–13] numerically a posteriori in the GBEC theory. Furthermore, being non-interacting (except for the Pauli Principle restriction mentioned), CPs satisfy the Ehrenfest-Oppenheimer [45] criteria for two clusters of charges to conserve a specific kind of statistics, either Bose or Fermi. These assumed properties are justified a posteriori when in the GBEC theory: a) the BCS gap equation is recovered for equal numbers of both kinds of pairs, both in the $K = 0$ state and in all $K \neq 0$ states taken collectively, and in weak coupling, regardless of CP overlaps; and b) the precise familiar BEC $T_c$ formula emerges [10] when i) 2h-CPs are ignored, the Friedberg-T.D. Lee model [21, 22] equations are recovered and ii) one switches off the BF interaction. The only difference in the recovered BEC $T_c$ formula is that the boson number density now depends on $T_c$, as expected in a boson-fermion mixture where populations are $T$-dependent. Finally, we note that fermion scattering terms [46] are not included in [11] as they are not expected to be substantial, say, in the BCS limit of high electron density where they would be the most effective, which in turn is included in the GBEC model as a special case.

Two-hole CPs in [11] are postulated to be distinct and kinematically independent from both the 2e-CPs and the unpaired electrons, i.e., $a$, $b$ and $c$ are assumed to commute with each other. This postulate is grounded on magnetic-flux-quantization measurements establishing the presence of pair charge carriers in both conventional [47, 48] as well as cuprate [49] superconductors, and on the fact that no experiment has yet been done, to our knowledge [50], that distinguishes between electron and hole CPs. The latter uncertainty further motivates a Hamiltonian such as [11] with both kinds of CPs.

The interaction Hamiltonian $H_{\text{int}}$ in the expression $H = H_0 + H_{\text{int}}$ describes the formation and disintegration of CPs, respectively, from and into unpaired electrons and holes. It is further simplified by dropping all $K \neq 0$ terms. This is also done in BCS theory in its full Hamiltonian $H = H_0 + H_{\text{int}}$, but kept in the GBEC theory in its unperturbed $H_0$ portion [11]. The GBEC $H_{\text{int}}$ is made up of four distinct BF interaction vertices each with two-fermion/one-boson creation and/or annihilation operators. These vertices depict how unpaired electrons (subindex +) [or holes (subindex −)] are involved in the formation and disintegration of the 2e- (and 2h-) $K = 0$ CPs in the $d$-dimensional system of
The BCS and recover BCS theory, in Refs. [10]-[13] the corresponding energy form factors were picked as analogous to (2) but without hole terms. In contrast, the full BCS Hamiltonian superconductivity, they must then be allowed to interact via the Fröhlich electron-phonon interaction [51] of a form replaced by, say, \( n \equiv \frac{1}{2}(k_1 - k_2) \) is again the relative wavevector of a CP. The interaction vertex form factors \( f_\pm(k) \) in (2) are essentially the Fourier transforms of the 2e- and 2h-CP intrinsic wavefunctions, respectively, in the relative coordinate of the two fermions. The GBEC theory is thus reminiscent of the Sommerfeld theory of the electron gas combined with the Debye picture of the phonon gas which together give a binary mixture of noninteracting electrons and phonons, a picture which describes low-T specific heats in metals and insulators. But to explain either resistance and superconductivity, they must then be allowed to interact via the Fröhlich electron-phonon interaction [51] of a form analogous to (2) but without hole terms. In contrast, the full BCS Hamiltonian \( H_{BCS} + H_{int} \) consists of only the first (electron) term on the rhs of (1), namely

\[
H_{int} = L^{-d/2} \sum_k f_+(k) \{ a_{k \uparrow}^+ a_{-k \downarrow}^+ b_0^+ + a_{-k \downarrow}^+ a_{k \uparrow}^+ b_0 \} + L^{-d/2} \sum_k f_-(k) \{ a_{k \uparrow}^+ a_{-k \downarrow}^+ c_0^+ + a_{-k \downarrow}^+ a_{k \uparrow}^+ c_0 \}
\]

(2)

where \( k = \frac{1}{2}(k_1 - k_2) \) is again the relative wavevector of a CP. The interaction vertex form factors \( f_\pm(k) \) in (2) are essentially the Fourier transforms of the 2e- and 2h-CP intrinsic wavefunctions, respectively, in the relative coordinate of the two fermions. The GBEC theory is thus reminiscent of the Sommerfeld theory of the electron gas combined with the Debye picture of the phonon gas which together give a binary mixture of noninteracting electrons and phonons, a picture which describes low-T specific heats in metals and insulators. But to explain either resistance and superconductivity, they must then be allowed to interact via the Fröhlich electron-phonon interaction [51] of a form analogous to (2) but without hole terms. In contrast, the full BCS Hamiltonian \( H_{BCS} + H_{int} \) consists of only the first (electron) term on the rhs of (1), namely

\[
H_{0BCS}^{\text{BCS}} = \sum_{k_1,s_1} \epsilon_{k_1} a_{k_1,s_1}^+ a_{k_1,s_1}
\]

(3)

and

\[
H_{int}^{\text{BCS}} = \sum_{k_1,k_2} v_{k_1,k_2} a_{k_1,s_1}^+ a_{-k_2,s_2} - a_{-k_2,s_2}^+ a_{k_1,s_1}.
\]

(4)

The BCS \( H_{0BCS}^{\text{BCS}} \) thus represents a Fermi liquid normal state.

The interaction vertex form factors \( f_\pm(k) \) in (2) are essentially the Fourier transforms of the 2e- and 2h-CP intrinsic wavefunctions, respectively, in the relative coordinate between the paired fermions of the CP. In order to eventually recover BCS theory, in Refs. [10]-[13] the corresponding energy form factors were picked as

\[
f_+(\epsilon) = \begin{cases} 
  f & \text{for } E_f < \epsilon < E_f + \delta \varepsilon \\
  0 & \text{otherwise,}
\end{cases}
\]

(5)

\[
f_-(\epsilon) = \begin{cases} 
  f & \text{for } E_f - \delta \varepsilon < \epsilon < E_f \\
  0 & \text{otherwise.}
\end{cases}
\]

(6)

This is after one introduces the quantities \( E_f \) and \( \delta \varepsilon \) as new phenomenological dynamical energy parameters (in addition to the positive BF vertex coupling parameter \( f \)) that replace the previous phenomenological CP energy parameters \( E_\pm(0) \), through the definitions

\[
E_f \equiv \frac{1}{2}[E_+(0) + E_- (0)] \quad \text{and} \quad \delta \varepsilon \equiv \frac{1}{2}[E_+(0) - E_- (0)] \geq 0
\]

(7)

where \( E_+(0) \) and \( E_- (0) \) are the (empirically unknown) zero-CMM energies of the 2e- and 2h-CPs, respectively. Note that \( 2E_f \) lies midway between \( E_+(0) \) and \( E_- (0) \). Alternately, instead of (7) one can write the two relations

\[
E_\pm(0) = 2E_f \pm \delta \varepsilon.
\]

(8)

The quantity \( E_f \) serves as a convenient energy scale; it is not to be confused with the Fermi energy \( E_F = \frac{1}{2}mv_F^2 = k_BT_F \) where \( T_F \) is the Fermi temperature. The Fermi energy \( E_F \) equals \( \pi \hbar^2 n/m \) in 2D and \( (\hbar^2/2m)(3\pi^2n)^{2/3} \) in 3D, with \( n \equiv N/L^d \) the total number-density of charge-carrier mobile electrons, while \( E_f \) is of the same form but with \( n \) replaced by, say, \( n_f \), which in turn serves as convenient electron-density scale. The quantities \( E_f \) and \( E_F \) coincide only when perfect 2e/2h-CP symmetry holds, i.e., when \( n = n_f \).

The grand potential \( \Omega \) for the full Hamiltonian \( H = H_0 + H_{int} \) given by (11) and (2) is then constructed via (Ref. [25] Eq. 4.14) the definition

\[
\Omega(T, L^d, \mu, N_0, M_0) = -k_BT \ln \left[ \text{Tr} e^{-\beta(H-\mu N)} \right]
\]

(9)

where “Tr” stands for “trace” and \( \beta \equiv 1/k_BT \) with \( T \) the absolute temperature. It is related to the system pressure \( P \), internal energy \( E \) and entropy \( S \) by \( \Omega = -PL^d = F - \mu N = E - TS - \mu N \), where \( F \) is the Helmholtz free energy. Following the Bogoliubov prescription [52], one sets \( b_0, b_0^\dagger \) equal to \( \sqrt{N_0} \) and \( c_0^\dagger, c_0 \) equal to \( \sqrt{M_0} \) in (2), where \( N_0 \) is
the $T$-dependent number of zero-CMM 2e-CPs and $M_0$ likewise for 2h-CPs. This allows exact diagonalization for any coupling, through a Bogoliubov transformation of the $a^+, a$ fermion operators, giving \[\text{Eq} (5)\] after some algebra

$$
\Omega(T, L^d, \mu, N_0, M_0)/L^d = \int_0^\infty d\epsilon N_0(\epsilon) [\epsilon - \mu - E(\epsilon)] - 2k_B T \int_0^\infty d\epsilon N_0(\epsilon) \ln[1 - \exp(-\beta E(\epsilon))] + [E+(0) - 2\mu] n_0 + k_B T \int_0^\infty d\epsilon M(\epsilon) \ln[1 - \exp(-\beta E+(0) + \epsilon - 2\mu)] + [2\mu - E-(0)] m_0 + k_B T \int_0^\infty d\epsilon M(\epsilon) \ln[1 - \exp(-\beta(2\mu - E-(0) + \epsilon))].$$

(10)

Here $N(\epsilon)$ and $M(\epsilon)$ are respectively the electronic and bosonic density of states, while

$$E(\epsilon) \equiv \sqrt{(\epsilon - \mu)^2 + \Delta^2(\epsilon)} \equiv \sqrt{(\epsilon - \mu)^2 + n_0 f^2_+(\epsilon) + m_0 f^2_-(\epsilon)}$$

(11)

since $\Delta(\epsilon) \equiv \sqrt{n_0 f^2_+(\epsilon) + \sqrt{m_0 f^2_-(\epsilon)}}$ and $f_+(\epsilon) = 0$ from \[\text{Eq} (5)\] and \[\text{Eq} (6)\], with $n_0(T) \equiv N_0(T)/L^d$ and $m_0(T) \equiv M_0(T)/L^d$ being the 2e-CP and 2h-CP number densities, respectively, of BE-condensed (i.e., with $K = 0$) bosons.

Minimizing $F$ with respect to $N_0$ and $M_0$, while simultaneously fixing the total number $N$ of electrons by introducing the electron chemical potential $\mu$ in the usual way, namely

$$\frac{\partial F}{\partial N_0} = 0, \quad \frac{\partial F}{\partial M_0} = 0, \quad \text{and} \quad \frac{\partial \Omega}{\partial \mu} = -N$$

(12)

ensures an equilibrium thermodynamic state of the system with volume $L^d$ at temperature $T$ and chemical potential $\mu$. Evidently, $N$ includes both paired and unpaired CP electrons. Some algebra then leads \[\text{to the three coupled integral Eqs. (7)-(9) of Ref. \[\text{10}\]}\] which, since from \[\text{Eq} (5)\] and \[\text{Eq} (6)\] one has that $f_+(\epsilon) f_-(\epsilon) \equiv 0$, can be simplified to the two “gap-like equations”

$$[2E_f + \delta \epsilon - 2\mu(T)] = \frac{1}{2} \int_{E_f}^{E_f + \delta \epsilon} d\epsilon N(\epsilon) \frac{\tanh \frac{1}{2} \beta \sqrt{\epsilon - \mu(T)} + f^2\left(\epsilon - \mu(T)\right)}{\sqrt{\epsilon - \mu(T)}^2 + f^2\left(\epsilon - \mu(T)\right)}$$

(13)

$$[2\mu(T) - 2E_f + \delta \epsilon] = \frac{1}{2} \int_{E_f - \delta \epsilon}^{E_f} d\epsilon N(\epsilon) \frac{\tanh \frac{1}{2} \beta \sqrt{\epsilon - \mu(T)} + f^2\left(\epsilon - \mu(T)\right)}{\sqrt{\epsilon - \mu(T)}^2 + f^2\left(\epsilon - \mu(T)\right)}$$

(14)

and a single “number equation” (that guarantees charge conservation)

$$2n_B(T) - 2m_B(T) + n_f(T) = n.$$ 

(15)

where

$$n_f(T) \equiv \int_0^\infty d\epsilon N(\epsilon) [1 - \frac{\epsilon - \mu}{E(\epsilon)}] \frac{1}{\tanh \frac{1}{2} \beta E(\epsilon)}$$

(16)

is clearly the number of unpaired electrons. This is identical with $2 \sum_k v_n^2(T)$ with $v_n^2(T)$ the well-known $T$-dependent Bogoliubov $v^2$-coefficient. In \[\text{Eq} (15)\] $n \equiv N/L^d$ is the number density of electrons while $n_B(T)$ and $m_B(T)$ are, respectively, the number densities of 2e- and 2h-CPs in all bosonic states (both $K = 0$ as well as $K > 0$). The “complete” number equation \[\text{Eq} (15)\] can be rewritten more explicitly as

$$2n_0(T) + 2n_{B+}(T) - 2m_0(T) - 2m_{B+}(T) + n_f(T) = n \quad \text{crossover version A}$$

(17)

where $n_B(T)$ is \[\text{Eq} (13)\]

$$n_B(T) \equiv n_0(T) + n_{B+}(T); \quad n_{B+}(T) \equiv \int_0^\infty d\epsilon M(\epsilon) \exp \beta \{E_+(0) + \epsilon - 2\mu\} - 1 \Bigg]^{-1}$$

(18)
and similarly for $m_B(T)$ which is

$$m_B(T) \equiv m_0(T) + m_{B+}(T); \quad m_{B+}(T) \equiv \int_{0^+}^{\infty} d\varepsilon M(\varepsilon) \exp \beta \{2\mu - E_-(0) + \varepsilon\} - 1 \}^{-1}. \quad (19)$$

Clearly, $m_{B+}(T)$ are precisely the number of “pre-formed” $K > 0$ 2h-CPs, and $n_{B+}(T)$ that of 2e-CPs. These CPs are non-condensed in contrast with the $K = 0$ CPs which are BE condensed. Evaluating the integrals requires knowing the bosonic density-of-states $M(\varepsilon)$ of CPs of energy $\varepsilon$, which in turn requires knowing the dispersion relation $\varepsilon$ vs. $K$, e.g., as has been determined via the Bethe-Salpeter equation in the ladder approximation in 3D [27] and in 2D [28].

Self-consistent (at worst, numerical) solution of the three coupled integral equations (13), (14) and (17) then yields the three thermodynamic variables of the GBEC theory

$$n_0(T, n, \mu), \quad m_0(T, n, \mu), \quad \text{and} \quad \mu(T, n). \quad (20)$$

Figure 1 displays the three BE condensed phases—labeled $s^+, s^-$ and $ss$—along with the normal phase $n$, that emerge [11] from the GBEC theory. Phase $s^+$ stands for a pure 2e-CP BE condensate, $s^-$ for a pure 2h-CP such condensate and $ss$ denotes a mixed phase. Only the two pure phases were found [11] to display $T_c$ values higher than the corresponding BCS value, while the mixed phase occurs below this value.

For the two pure phases one can, in principle, shift from the single-band model implied so far to a two-band model by allowing the particle (e) masses to differ from hole (h) masses; this can be done by introducing two different Fermi energies $E^e_F$ and $E^h_F$ that differ precisely by these two masses.

The GBEC theory contains [13] the key equations of all five distinct statistical theories as special cases. These range from ordinary BCS to ordinary BEC theories, which are thereby completely unified by the GBEC theory.
Perfect $2e/2h$ CP symmetry signifies equal numbers of $2e$- and $2h$-CPs, more specifically, $n_B(T) = m_B(T)$ as well as $n_0(T) = m_0(T)$. This implies that $n_{B+}(T) = m_{B+}(T)$ for all $T$, meaning that the exponents in (18) and (19) coincide so that with (8) this makes $E_f = \mu$. The GBEC theory then reduces to the gap and number equations [viz., in 2D for $T = T_c$ both (28) and (29) below] of the original [55] BCS-Base crossover picture with the Cooper/BCS model interaction—if its parameters $V$ and $\hbar \omega_D$ are identified with the BF interaction GBEC Hamiltonian $H_{\text{int}}$ parameters $f^2/2\delta \epsilon$ and $\delta \epsilon$, respectively. This one-to-one correspondence between $H_{\text{int}}$ and $H^{\text{BCS}}_{\text{int}}$ defined in (2) and (4) justifies the particular choice of form factors (5) and (6) for the BF interaction. The original crossover picture for unknowns $\Delta(T)$ and $\mu(T)$ is now supplemented by the central relation

$$\Delta(T) = f \sqrt{n_0(T)} = f \sqrt{m_0(T)}.$$  \hspace{1cm} (21)

All three functions $\Delta(T), n_0(T)$ and $m_0(T)$ have the familiar “half-bell-shaped” forms. Namely, they are zero above a certain critical temperature $T_c$, and rise monotonically upon cooling (lowering $T$) to maximum values $\Delta(0)$, $n_0(0)$ and $m_0(0)$ at $T = 0$. The energy gap $\Delta(T)$ is the order parameter describing the superconducting (or superfluid) condensed state, while $n_0(T)$ and $m_0(T)$ are the BEC order parameters depicting the macroscopic occupation that occurs below $T_c$ in a BE condensate. This $\Delta(T)$ is precisely the BCS energy gap if the boson-fermion coupling $f$ is made to correspond to $\sqrt{2V\hbar \omega_D}$ within the GBEC formalism. Evidently, the BCS and BEC $T_c$s are the same. Writing (21) for $T = 0$ and dividing this into (21) gives the much simpler $f$-independent relation involving order parameters normalized to unity in the interval $[0, 1]$

$$\Delta(T)/\Delta(0) = \sqrt{n_0(T)/n_0(0)} = \sqrt{m_0(T)/m_0(0)} \xrightarrow{T \rightarrow 0} 1 \xrightarrow{T \geq T_c} 0.$$ \hspace{1cm} (22)

The first equality, apparently first obtained in Ref. [20], connects in a simple way the two heretofore unrelated “half-bell-shaped” order parameters of the BCS and the BEC theories. The second equality [10, 11] implies that a BCS condensate is precisely a BE condensate of equal numbers of $2e$- and $2h$-CPs. Since (22) is independent of the particular two-fermion dynamics of the problem, it can be expected to hold for either SCs and SFs.

### 3 Gap equation

The standard procedure in all SC and SF theories of many-fermion systems is to ignore dealing explicitly with $2h$-CPs altogether. Neglecting in (10) all terms containing $m_0(T), E_-(0)$ and $f_-(\epsilon)$ leaves an $\Omega(T, L^d, \mu, N_0)$ defining a binary, instead of ternary, BF model. Minimizing the associated Heilmholtz free energy $F(T, L^d, \mu, N_0) = \Omega(T, L^d, \mu, N_0) + \mu N$ over $N_0$ (for fixed total electron number $N$) requires that $\partial F/\partial N_0 = 0 = \partial F/\partial n_0$, which becomes

$$\int_0^\infty d\epsilon N(\epsilon) \left[-1 + \frac{2 \exp\{-\beta E(\epsilon)\}}{1 + \exp\{-\beta E(\epsilon)\}}\right] \frac{dE(\epsilon)}{dn_0} + \left[E_+(0) - 2\mu\right] = 0$$

or

$$2[2E_f + \delta \epsilon - 2\mu] = f^2 \int_{E_f}^{E_f + \delta \epsilon} d\epsilon N(\epsilon) \frac{1}{E(\epsilon)} \tanh \frac{1}{2} \beta E(\epsilon).$$ \hspace{1cm} (23)

Using (8) yields precisely the BCS gap equation for all $T$, Eq. (3.27) of Ref. [7], provided one picks $E_f = \mu$, namely

$$1 = \lambda \int_0^{\hbar \omega_D} d\xi \frac{1}{\sqrt{\xi^2 + \Delta^2(T)}} \tanh \frac{1}{2} \beta \sqrt{\xi^2 + \Delta^2(T)}$$ \hspace{1cm} (24)

where $\xi = \epsilon - \mu$, since $\lambda \equiv N(E_f)V = f^2 N(E_f)/2\delta \epsilon$ while $\delta \epsilon = \hbar \omega_D$ [see relation between $V$ and $f$ stated just above (21)], and provided $N(\epsilon)$ can be taken outside the integral sign in (23). This last operation is exact in 2D when $N(\epsilon)$ is independent of $\epsilon$ and is otherwise a good approximation if $\hbar \omega_D \ll \mu$.

However, the choice $E_f = \mu$ cannot be justified, to our knowledge, without assuming within the GBEC that $n_B(T) = m_B(T)$ as well as $n_0(T) = m_0(T)$, i.e., by explicitly recognizing the existence of $2h$-CPs along with $2e$-CPs and taking them in equal or 50-50 proportions.
4 Number equation

Besides the normal phase consisting of the ideal BF ternary gas described by \( H_0 \), three different stable BEC phases emerge \cite{11} when solving all three equations \( \text{(13)} \) to \( \text{(14)} \) or \( \text{(17)} \): two pure phases, a pure 2e-CP BEC and a pure 2h-CP BEC, as well as a mixed phase consisting of both types of BECs in varying proportions. For a half-and-half mixed phase, i.e., \( n_0(T) = m_0(T) \) and \( n_{B+}(T) = m_{B+}(T) \), all the boson number-density terms in \( \text{(17)} \) cancel and the BCS number equation

\[
n = n_f(T) \quad \text{crossover version B (special case of A)} \quad \text{(25)}
\]

is recovered, with \( n_f(T) \) defined by \( \text{(10)} \). Crossover version B does not explicitly neglect either kind of CP, nor does it draw a distinction between them. and is the version applied below in 2D to obtain \( \text{(28)} \) to \( \text{(30)} \) and the results of Fig. 2. This is the original crossover version first presented in 1967 in Ref. \cite{55}.

If 2h-CPs are ignored altogether, the companion number equation follows from the last equation of \( \text{(12)} \) as

\[
n = n_f(T) + 2n_B(T) \quad \text{crossover version C} \quad \text{(26)}
\]

where \( n_f(T) \) is interpreted as the number density of unpaired but BCS-correlated electrons and is given by \( \text{(10)} \). In perhaps the first attempt to discuss \( \text{(19)} \) BEC in 1969 within the BCS-Bose crossover picture, Eagles \cite{54} imposed instead of \( \text{(26)} \) but with the same

\[
\text{noting that one set } 
\int_0^{\hbar \omega_D/2k_BT_c} dx \frac{\tanh x}{x} \quad \text{if } \mu > \hbar \omega_D; \quad 1 = \lambda \int_{-\mu(T_c)/2k_BT_c}^{\hbar \omega_D/2k_BT_c} dx \frac{\tanh x}{2x} \quad \text{if } \mu < \hbar \omega_D \quad \text{(28)}
\]

\[
\int_0^\infty \frac{dx}{\exp[(\epsilon - \mu(T_c))/2k_BT_c] + 1} = E_F. \quad \text{(29)}
\]

The last integral can be done analytically and leaves the explicit expression

\[
\mu(T_c) = k_BT_c \ln(e^{E_F/k_BT_c} - 1) \quad \text{(30)}
\]

which is then eliminated symbolically from \( \text{(25)} \) to render \( T_c \) as an implicit function of \( \lambda \) alone. Using \( \hbar \omega_D/E_F = 0.05 \) as a typical value for cuprates, increasing \( \lambda \) makes \( \mu(T_c) \) decrease from its weak-coupling (where \( T_c \rightarrow 0 \) value of \( E_F \) down to \( \hbar \omega_D \) when \( \lambda \simeq 56 \), an unphysically large value as it well exceeds the Migdal ionic-lattice stability upper limit \cite{59} of 1/2, although \cite{60} “there is no universally accepted, simple, and quantitative stability criterion.”

Fig. 2 displays \( T_c \) (in units of \( T_F \)) as function of \( \lambda \). Note that room-temperature SCs (RTSC) are predicted by BCS-Bose crossover theory but only for \( \lambda \) values definitely larger than about 10 that are still too unphysical.

5 BCS-Bose crossover \( T_c \) compared with BCS \( T_c \) in 2D

The original crossover theory \cite{55} is defined by two simultaneous coupled equations, the BCS gap and number equations, without the BCS assumption that the chemical potential \( \mu \) equals the Fermi energy \( E_F \). For subsequent extensions of the original version, see reviews in Refs. \cite{59, 67}. The critical temperature \( T_c \) is defined by \( \Delta(T_c) = 0 \), and is to be determined self-consistently with \( \mu(T_c) \) by solving both gap and number equations. Because of its interest in quasi-2D cuprate superconductors \cite{58}, in this section we concentrate on 2D only. For the Cooper/BCS model interaction, if \( \lambda \equiv N(E_F)V \) where \( N(E_F) = m/(2\pi \hbar^2) \), the two crossover equations to be solved self-consistently reduce to

\[
1 = \lambda \int \frac{\hbar \omega_D/2k_BT_c}{2k_BT_c} dx \frac{\tanh x}{x} \quad (\text{if } \mu > \hbar \omega_D); \quad 1 = \lambda \int -\mu(T_c)/2k_BT_c \hbar \omega_D/2k_BT_c dx \frac{\tanh x}{2x} \quad (\text{if } \mu < \hbar \omega_D) \quad \text{(28)}
\]

\[
\int_0^\infty \frac{dx}{\exp[(\epsilon - \mu(T_c))/2k_BT_c] + 1} = E_F. \quad \text{(29)}
\]

The last integral can be done analytically and leaves the explicit expression

\[
\mu(T_c) = k_BT_c \ln(e^{E_F/k_BT_c} - 1) \quad \text{(30)}
\]

which is then eliminated symbolically from \( \text{(25)} \) to render \( T_c \) as an implicit function of \( \lambda \) alone. Using \( \hbar \omega_D/E_F = 0.05 \) as a typical value for cuprates, increasing \( \lambda \) makes \( \mu(T_c) \) decrease from its weak-coupling (where \( T_c \rightarrow 0 \) value of \( E_F \) down to \( \hbar \omega_D \) when \( \lambda \simeq 56 \), an unphysically large value as it well exceeds the Migdal ionic-lattice stability upper limit \cite{59} of 1/2, although \cite{60} “there is no universally accepted, simple, and quantitative stability criterion.”

Fig. 2 displays \( T_c \) (in units of \( T_F \)) as function of \( \lambda \). Note that room-temperature SCs (RTSC) are predicted by BCS-Bose crossover theory but only for \( \lambda \) values definitely larger than about 10 that are still too unphysical.
Figure 2: Critical SC temperatures $T_c$ in units of $T_F$ for the BCS-Bose crossover theory (full curve) in 2D compared with the BCS value from the exact implicit $T_c$ equation (see, e.g., Ref. [25], p. 447) $1 = \lambda \int_0^{\hbar \omega_D/2k_B T_c} dx x^{-1} \tanh x$ (upper dashed curve) valid for any $d > 0$ in any coupling $\lambda$, and its weak-coupling explicit solution $T_c \approx 1.134 \hbar \omega_D \exp(-1/\lambda)$ (lower dashed curve). The dot-dashed “appendage” signals a breakdown in the BCS/Cooper interaction model when $\mu(T_c)$ turns negative, as the Fermi surface at $\mu$ then washes out and the interaction model becomes meaningless. The value of $\lambda = 1/2$ marked is the maximum possible value allowed just short of lattice instability in 3D for this interaction model, at least by one criterion [59]. Unshaded region refers to room-temperature superconductivity (RTSC) for SCs with $T_F \lesssim 10^3$ K.

6 Condensation energy

The $T = 0$ condensation energy per unit volume according to the GBEC theory, given (10), is

$$\frac{E_s - E_n}{L^d} = \frac{\Omega_s(T = 0) - \Omega_n(T = 0)}{L^d}$$

(31)

since for any $T$ the Helmholtz free energy $F \equiv E - TS = \Omega + \mu N$, with $S$ the entropy, and $\mu$ is the same for either superconducting $s$ or normal $n$ phases with internal energies $E_s$ and $E_n$, respectively. In the normal phase $n_0(T) = 0$, $m_0(T) = 0$ so that $\Delta(T) = 0$ for all $T \geq 0$, so that (10) reduces to

$$\frac{\Omega_n(T = 0)}{L^d} = \int_0^\infty d\epsilon N(\epsilon)(\epsilon - \mu - |\epsilon - \mu|) = 2 \int_0^\mu d\epsilon N(\epsilon)(\epsilon - \mu) = 2 \int_{-\mu}^0 d\xi N(\xi)\xi.$$  

(32)

For the superconducting phase, and when $n_0(T) = m_0(T)$ and $n_B(T) = m_B(T)$ hold, i.e., crossover scenario B, one deduces from (8) and (10) that $\mu = E_f$. Putting $\Delta(T = 0) \equiv \Delta$ in (10) as well as $\delta \epsilon \equiv \hbar \omega_D$, while using (8), gives

$$\frac{\Omega_s(T = 0)}{L^d} = 2\hbar \omega_D n_0(0) + \int_{-\mu}^\infty d\xi N(\xi) \left(\xi - \sqrt{\xi^2 + \Delta^2}\right)$$

9
The first factor of 2 in the last line comes precisely from the condition \( m_0(T) = m_0(T) \) while the last two factors of 2 arise from the condition that according to (5) and (6) the magnitudes of \( f_+(\epsilon) \) and \( f_-(\epsilon) \) are the same and equal \( f \). Subtracting (32) from (37) and putting \( N(\xi) \equiv N(0) \), the density of electronic states at the Fermi surface [designated before as \( N(E_F) \)] yields

\[
\frac{E_s - E_n}{L^d} = \frac{2\hbar \omega_D n_0(0) + 2N(0)}{L^d} \int_{-\mu}^{\hbar \omega_D} \frac{d\xi N(\xi)}{\xi - \sqrt{\xi^2 + \Delta^2}}. \tag{33}
\]

exactly, by standard integrations. Using the expression that follows from (24) for \( T = 0 \) gives Eq. (2.40) of Ref. [7], namely

\[
\Delta = \frac{\hbar \omega_D}{\sinh(1/\lambda)} \tag{35}
\]

where \( \lambda \) is related to GBEC BF interaction parameter \( f \) through

\[
\lambda \equiv V N(0) = f^2 N(0)/2\hbar \omega_D.
\]

This makes the first term on the rhs of (34) exactly equal to \( \Delta^2 N(0)/\lambda \) which in turn can be shown to cancel exactly against the log term if one recalls the hyperbolic-function identity \( \sinh^2 x + 1 = \cosh^2 x \). Thus, the GBEC theory condensation energy (34) is identical for any coupling to that of BCS theory, Eq. (2.42) of Ref. [7], namely

\[
\frac{E_s - E_n}{L^d} = \frac{N(0)(\hbar \omega_D)^2}{2} \left[ 1 - \frac{1}{\sqrt{1 + (\Delta/\hbar \omega_D)^2}} \right], \quad \text{(BCS)} \tag{36}
\]

which in turn can be shown to cancel exactly against the log term if one recalls the hyperbolic-function identity \( \sinh^2 x + 1 = \cosh^2 x \). Thus, the GBEC theory condensation energy (34) is identical for any coupling to that of BCS theory, Eq. (2.42) of Ref. [7], namely

\[
\Delta = \frac{\hbar \omega_D}{\sinh(1/\lambda)} \tag{35}
\]

where \( \lambda \) is related to GBEC BF interaction parameter \( f \) through

\[
\lambda \equiv V N(0) = f^2 N(0)/2\hbar \omega_D.
\]

This makes the first term on the rhs of (34) exactly equal to \( \Delta^2 N(0)/\lambda \) which in turn can be shown to cancel exactly against the log term if one recalls the hyperbolic-function identity \( \sinh^2 x + 1 = \cosh^2 x \). Thus, the GBEC theory condensation energy (34) is identical for any coupling to that of BCS theory, Eq. (2.42) of Ref. [7], namely

\[
\frac{E_s - E_n}{L^d} = \frac{N(0)(\hbar \omega_D)^2}{2} \left[ 1 - \frac{1}{\sqrt{1 + (\Delta/\hbar \omega_D)^2}} \right], \quad \text{(BCS)} \tag{36}
\]

where \( \lambda \) is related to GBEC BF interaction parameter \( f \) through

\[
\lambda \equiv V N(0) = f^2 N(0)/2\hbar \omega_D.
\]
which again is just one half the full GBEC theory condensation energy \(\frac{1}{3}\) that was found to be identical to the full BCS condensation energy \(\frac{1}{2}\). Though not too surprising as the function \(E(\epsilon) = \sqrt{(\epsilon - \mu)^2 + \Delta^2(\epsilon)}\), where \(\Delta(\epsilon) = \sqrt{n_0 f_+} + \sqrt{m_0 f_-}\), becomes “half-gapless” in either crossover versions C or D, this one-half difference occurs precisely because either \(n_0\) or \(m_0\), and \(f_+\) or \(f_-\), have been deleted. Including both 2e- and 2h-CPs gave similarly striking conclusions on generalizing via the Bethe-Salpeter equation the ordinary \[26\] CP problem from unrealistic infinite-lifetime pairs to the physically expected finite-lifetime ones of Refs. \[27, 28\].

7 Conclusions

The recent generalized BEC (GBEC) statistical single-band theory was employed to distinguish four different versions of the BCS-Bose crossover picture. One of these is the original BCS-Bose crossover theory with number equation \(\frac{1}{2}\), crossover version B. For the Cooper/BCS model interaction predicts in 2D virtually the same \(T_c\) to well beyond physically unreasonable values of coupling, as the (allegedly less general) BCS statistical theory where the number equation becomes trivial on assuming that the electron chemical potential \(\mu = E_F\), the Fermi energy. However, \(T_c\) much higher than those of the BCS-Bose crossover theory have been obtained \[63\] via the GBEC number equation \(\frac{1}{2}\), designated here as crossover version A, that includes both electron- or hole-pair bosons explicitly but in different proportions.

The GBEC statistical theory also reveals that the BCS gap equation for all temperatures follows rigorously only when neither hole- nor electron-pairs are ignored and occur in equal proportions, separately for zero- and nonzero-CMM pairs, and that the resulting GBEC \(T = 0\) condensation energy equals the entire (rigorous-upper-bound) BCS value for any coupling. But that it is only half as large when either kind of pair is ignored. Hence, if a BEC theory that reduces properly to BCS theory is at all relevant in SCs and SFs taken as many-fermion systems where pairing into bosonic CPs can occur, two-hole CPs must play an unambiguously crucial role.

Acknowledgements MdeLl thanks J.F. Annett, D.M. Eagles, M. Fortes, J. Javanainen, T.A. Mamedov, J.P. Rodríguez, O. Rojo, V.V. Tolmachev, J.A. Wilson and G.M. Zhao for discussions. MdeLll and MAS acknowledge UNAM-DGAPA-PAPIIT (Mexico) for grants IN106401 & IN111405, and CONACyT (Mexico) grants 41302F & 43234F, for partial support. MdeLl also thanks the Texas Center for Superconductivity, University of Houston, Houston, TX 77204 for partial support, as well as the Royal Society (UK) and the Academia Mexicana de Ciencias (Mexico) for a joint fellowship.

References

[1] R.A. Ogg, Jr., Phys. Rev. 69, 243 (1946).
[2] V.L. Ginzburg, Usp. Fiz. Nauk. 48, 25 (1952); Fortschr. Phys. 1, 101 (1953).
[3] M.R. Schafroth, Phys. Rev. 96, 1442 (1954).
[4] M.R. Schafroth, S.T. Butler, & J.M. Blatt, Helv. Phys. Acta 30, 93 (1957).
[5] M.R. Schafroth, Sol. State Phys. 10, 293 (1960).
[6] J.M. Blatt, Theory of Superconductivity (Academic, New York, 1964).
[7] J. Bardeen, L.N. Cooper & J.R. Schrieffer, Phys. Rev. 108, 1175 (1957).
[8] N.N. Bogoliubov, JETP 34, 41 (1958).
[9] N.N. Bogoliubov, V.V. Tolmachev & D.V. Shirkov, Fortschr. Phys. 6, 605 (1958); and also in A New Method in the Theory of Superconductivity (Consultants Bureau, NY, 1959).
[10] V.V. Tolmachev, Phys. Lett. A 266, 400 (2000).
[11] M. de Llano & V.V. Tolmachev, Physica A 317, 546 (2003).
[12] J. Batle, M. Casas, M. Fortes, M. de Llano, F.J. Sevilla, M.A. Solís, & V.V. Tolmachev, Cond. Matter Theories 18, 111 (2003).
[13] M. de Llano, in Frontiers in Superconductivity Research (ed. B.P. Martins) (Nova Science Publishers, NY, 2004).
[14] M. Casas, S. Fujita, M. de Llano, A. Puente, A. Rigo, & M.A. Solís, Phys. Lett. A 245, 5 (1998).
[15] M. Casas, M. de Llano, A. Puente, A. Rigo, & M.A. Solís, Physica A 295, 146 (2001).
[16] M. Casas, M. de Llano, A. Puente, A. Rigo & M.A. Solís, Sol. State Comm. 123, 101 (2002).
[17] S.K. Adhikari, M. Casas, A. Puente, A. Rigo, M. Fortes, M.A. Solís, M. de Llano, A.A. Valladares & O. Rojo, Phys. Rev. B 62, 8671 (2000).
[18] S.K. Adhikari, M. Casas, A. Puente, A. Rigo, M. Fortes, M.A. Solís, M. de Llano, A.A. Valladares & O. Rojo, Physica C 351, 341 (2001).
[19] D.M. Eagles, Phys. Rev. 186, 456 (1969).
[20] J. Ranninger, R. Micnas & S. Robaszkiewicz, Ann. Phys. Fr. 13, 455 (1988).
[21] R. Friedberg & T.D. Lee, Phys. Rev. B 40, 6745 (1989).
[22] R. Friedberg, T.D. Lee, & H.-C. Ren, Phys. Lett. A 152, 417 and 423 (1991).
[23] Y.J. Uemura et al., Phys. Rev. Lett. 62 (1989) 2317.
[24] A.T. Fiory, S. Martin, R.M. Fleming, L.F. Schneemeyer, J.V. Waszczak, A.F. Hebard, and S.A. Sunshine, Physica C 162-164, 1195 (1989).
[25] A.L. Fetter and J.D. Walecka, Quantum Theory of Many-Particle Systems (McGraw-Hill, New York, 1971).
[26] L.N. Cooper, Phys. Rev. 104, 1189 (1956).
[27] M. Fortes, M.A. Solís, M. de Llano, & V.V. Tolmachev, Physica C 364, 95 (2001).
[28] V.C. Aguilera-Navarro, M. Fortes, & M. de Llano, Sol. St. Comm. 129, 577 (2004).
[29] N.N. Bogoliubov, N. Cim. 7, 794 (1958).
[30] D. Vollkardt & P. Wölfe, The Superfluid Phases of Helium 3 (Taylor & Francis, London, 1990).
[31] E.R. Dobbs, Helium Three (Oxford University Press, Oxford, UK, 2000).
[32] M.J. Holland, B. DeMarco & D.S. Jin, Phys. Rev. A 61, 053610 (2000).
[33] M. Holland, S.J.J.M.F. Kokkelmans, M.L. Chiofalo & R. Walser, Phys. Rev. Lett. 87, 120406 (2001).
[34] E. Timmermans, K. Furuya, P.W. Milonni & A.K. Kerman, Phys. Lett. A 285, 228 (2001).
[35] K.M. O’Hara, S.L. Hammer, M.E. Ghem, S.R. Granade & J.E. Thomas, Science 298, 2179 (2002).
[36] M.L. Chiofalo, S.J.J.M.F. Kokkelmans, J.N. Milstein & M.J. Holland, Phys. Rev. Lett. 88, 090402 (2002).
[37] Y. Ohashi & A. Griffin, Phys. Rev. Lett. 89, 130402 (2002).
[38] L. Pitaevskii & S. Stringari, Science 298, 2144 (2002).
[39] M. Greiner, C.A. Regal & D.S. Jin, Nature 426, 537 (2003).
[40] K.E. Strecker, G.B. Partridge & R.G. Hulet, Phys. Rev. Lett. 91, 080406 (2003).
[41] M.W. Zwierlein, C.A. Stan, C.H. Schunck, S.M. F. Raupach, S. Gupta, Z. Hadzibabic & W. Ketterle, Phys. Rev. Lett. 91, 250401 (2003).
[42] S. Jochim, M. Bartenstein, A. Altmeyer, G. Hendl, S. Riedl, C. Chin, J. Hecker Denschlag & R. Grimm, Science 302, 2101 (2003).
[43] C.A. Regal, M. Greiner & D.S. Jin, Phys. Rev. Lett. 92, 040403 (2004).
[44] A.J. Leggett, Nature Physics 2 (2006) 134.
[45] P. Ehrenfest & J.R. Oppenheimer, Phys. Rev. 37, 333 (1931).
[46] P. Nozières and S. Schmitt-Rink, J. Low. Temp. Phys. 59, 195 (1985).
[47] B.S. Deaver, Jr. and W.M. Fairbank, Phys. Rev. Lett. 7, 43 (1961).
[48] R. Doll and M. Nährer, Phys. Rev. Lett. 7, 51 (1961).
[49] C.E. Gough, M.S. Colclough, E.M. Forgan, R.G. Jordan, M. Keene, C.M. Muirhead, I.M. Rae, N. Thomas, J.S. Abell, and S. Sutton, Nature 326, 855 (1987).
[50] C.E. Gough, priv. comm.
[51] H. Fröhlich, Proc. Roy. Soc. (London) 215 (1952) 291.
[52] N.N. Bogoliubov, J. Phys. (USSR) 11, 23 (1947).
[53] M. Casas, M. de Llano & V.V. Tolmachev, to be published.
[54] D.M. Eagles, priv. comm.
[55] J. Labbé, S. Barisic & J. Friedel, Phys. Rev. Lett. 19, 1039 (1967).
[56] M. Randeria, in Bose-Einstein Condensation, ed. A. Griffin, D.W. Snoke, & S. Stringari (Cambridge University, Cambridge, 1995) p. 355.
[57] Q. Chen, J. Stajic, S. Tan, K. Levin, Phys. Repts. 412 (2005) 1.
[58] B.D. Brandow, Phys. Repts. 296, 1 (1998).
[59] A.B. Migdal, JETP 7, 996 (1958).
[60] J.P. Carbotte, Revs. Mod. Phys. 62 (1990) 1027.
[61] A. Jeffrey, Handbook of Mathematical Formulas and Integrals (Academic Press, Newcastle, UK, 1995).
[62] J.F. Annett, Superconductivity, Superfluids and Condensates (Oxford University Press, UK, 2004) p. 70.
[63] M. de Llano and M. Grether, AIP Conf. Proc. 846, 221 (2006) Lectures on the Physics of Highly-Correlated Electron Systems X.