The large $N$ limit of the topological susceptibility of Yang-Mills gauge theory

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The $U(1)_A$ problem

Chiral symmetry breaking:
$SU(3)_L \times SU(3)_R \rightarrow SU(3)_V \rightarrow 8$ Goldstone bosons $\pi, K, \eta$

What happens with $U(1)_A$?

Could the $\eta'$ be the Goldstone boson associated to this symmetry?

$m_{NG} < \sqrt{3}m_\pi$, but $m_{\eta'} \approx 958$ MeV  

[Weinberg (1975)]

The symmetry is explicitly broken by an anomaly: $\partial_\mu J_{\mu 5} = -i \frac{N_f g^2}{16\pi^2} F^a_{\mu \nu} \tilde{F}^a_{\mu \nu}$

[Adler, Bell (1969)]

**Witten - Veneziano, 1979** (Based on the $N \rightarrow \infty$, $g^2 N$ fixed limit)

At leading order in the $1/N$ expansion:

$\chi \neq 0$ for pure YM theory, but $\chi = 0$ when massless fermions are added (?)

Solution: $m_{\eta'}^2 \propto 1/N$, $\partial_\mu J_{\mu 5} \propto 1/N \rightarrow \eta'$ is a Goldstone boson at large $N$
Objective

Our goal: Compute the large $N$ limit of $\chi_{YM}$

Previous work:

- Cooling methods [Lucini et al. (2001), Del Debbio et al. (2002), Lucini et al. (2005)].
- Definition of $\chi$ using the index of the Dirac operator [Cundy et al. (2002)] → expensive.
- Periodic boundary conditions (PBC) → large autocorrelations when approaching the continuum and large $N$ limits [Del Debbio et al. (2002)].

This work:

- We use the theoretically clean definition of $\chi$ based on the Yang-Mills gradient flow [Narayanan, Neuberger (2006), Lüscher (2010)].
- We use open boundary conditions (OBC) to avoid the freezing of topology near the continuum [Lüscher, Schaefer (2010)].
Observables

The topological susceptibility $\chi^t$ at flow time $t$ is defined as the two point function of the topological charge density $q^t(x)$

$$\chi^t = \int d^4x \langle q^t(x) q^t(0) \rangle$$

Provides a correct field theoretical definition of $\chi$ in the continuum [Cè et al. (2015)].

- Topological charge density

$$q^t(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr} G_{\mu\nu}(x) G_{\rho\sigma}(x)$$

- Yang-Mills Energy density

$$e^t(x) = \frac{1}{2} \text{Tr} G_{\mu\nu}(x) G_{\rho\sigma}(x)$$

We use the clover definition of $G_{\mu\nu}$ on the lattice.
Definition of $t_0$

We want to compute the dimensionless quantity $t_0^2 \chi_{YM}$.

In $SU(3)$, the reference flow time $t_0$ is defined implicitly by the equation:

$$t^2 \langle e^t \rangle_{t=t_0} = 0.3$$

For general gauge group $SU(N)$:

$$t^2 \langle e^t \rangle = \frac{3(N^2 - 1)}{128\pi^2 N} \lambda_t(q) \left[ 1 + c_1 \lambda_t(q) + O(\lambda_t(q)^2) \right]$$

where $\lambda_t(q) = g^2(q)N$ at the scale $q = (8t)^{-1/2}$.

We define the scale $t_0$ as:

$$t^2 \langle e^t \rangle_{t=t_0} = 0.1125 \frac{(N^2 - 1)}{N}$$
Computation of the topological susceptibility with OBC

Why use OBC? → The freezing of the topology is worse at larger $N$
[Del Debbio et al. (2002), Amato et al. (2015)]

OBC have been shown to reduce $\tau_{\text{int}}$ for the slowly decaying topological modes
[Lüscher (2011,2013), Amato et al. (2015)]

With PBC:
\[ \chi = \langle Q^2 \rangle / V \]
→ Not possible with OBC as translation invariance in broken in the time direction.

With OBC:
\[ \bar{q}^t(x_0) = \sum_{\vec{x}} q^t(\vec{x}, x_0) \]
[Bruno et al. (2014)]

\[ \bar{C}^t(\Delta) = \frac{1}{(T - 2d - \Delta)L^3} \sum_{x_0=d}^{T-1-d-\Delta} \langle \bar{q}^t(x_0)\bar{q}^t(x_0 + \Delta) \rangle \]

\[ \chi_{\text{YM}}^t(r) = \bar{C}^t(0) + 2 \sum_{\Delta=a}^{r} \bar{C}^t(\Delta) \]
Ensembles

| #run    | N | T/a | L/a | a[fm] | #meas. | #it. |
|---------|---|-----|-----|-------|--------|------|
| A(4)_1  | 4 | 64  | 16  | 0.096 | 22k    | 40   |
| A(4)_2  | 4 | 80  | 20  | 0.078 | 41k    | 80   |
| A(4)_3  | 4 | 96  | 24  | 0.065 | 21k    | 160  |
| A(5)_1  | 5 | 64  | 16  | 0.095 | 15k    | 120  |
| A(5)_2  | 5 | 80  | 20  | 0.077 | 27k    | 240  |
| A(5)_3  | 5 | 96  | 24  | 0.064 | 14k    | 480  |
| A(6)_1  | 6 | 64  | 16  | 0.095 | 30k    | 250  |
| A(6)_2  | 6 | 80  | 20  | 0.076 | 17k    | 500  |
| A(6)_3  | 6 | 96  | 24  | 0.063 | 16k    | 450  |

**Table:** The approximate lattice spacing using $\sqrt{t_0} = 0.166$ fm.

- 1 it. correspond to $n_{ov} \propto a^{-1}$ overrelaxation sweeps followed by one heatbath sweep.

- The updates are done using the Cabibbo-Marinari strategy updating all the $N(N - 1)/2$ SU(2) subgroups of SU($N$).
Open boundary effects

We fit the data to a one excited state contribution from the boundary:

\[ f(x_0) = A + Be^{-mx_0} \]

Plateau region:

\[ |f(d) - A| < 0.25\sigma \]

Sufficiently far away from the boundaries, observables assume their vacuum expectation values up to small exponential corrections.

\[ d_e = 9.5\sqrt{t_0} \]
\[ d_\chi = 7.5\sqrt{t_0} \]

For both \( e \) and \( \chi \), the plateau region is larger or equal than \( T/2a \).
Systematics from our definition of $\chi$

Is it reasonable to compute $\sum_{\Delta=0}^{r} \langle \bar{q}^t(0)\bar{q}^t(\Delta) \rangle$ up $r = T - 2d$?

[Bazavov et al. (2010), Bruno et al. (2014)]
Systematics from our definition of $\chi$

Is it reasonable to compute $\sum_{\Delta=0}^{r} \langle \bar{q}^t(0) \bar{q}^t(\Delta) \rangle$ up $r = T - 2d$?

[Bazavov et al. (2010), Bruno et al. (2014)]

$SU(3)$, $\beta = 6.11$, $t_0 = 4.5776(15)$

Using multilevel algorithms [MGV, Schaefer (2016)] \( \rightarrow r = 7.0 \sqrt{t_0} \)
Autocorrelations

Simulations at fine lattice spacings are only possible due to the use of OBC.
Large-$N$ and continuum limits

\[ t_0^2 \chi_{YM} \left( \frac{1}{N}, a \right) = \begin{cases} 
\chi_{YM} (0, 0) + c_1 \frac{1}{N^2} + c_2 \frac{a^2}{t_0} & \text{if} \quad N > 3 \\
c_3 + c_2 \frac{a^2}{t_0} & \text{if} \quad N = 3 
\end{cases} \]

\[ \chi^2 / \text{dof} = 0.94 \]
Large-$N$ and continuum limits

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Fits group by group
Large-$N$ and continuum limits

Different fit strategies give compatible results.

\[ t_0^2 \chi_{YM}(0, 0) = 7.03(13) \times 10^{-4} \]
Results

$t$ dependent discretization effects

\[ \chi_t / \chi_{t_0} = a^2 / t_0 \]

- $t = t_0 / 4$
- $t = t_0 / 2$
- $t = 3t_0 / 4$

SU(6)  
SU(5)  
SU(4)
Conclusions

- We have computed the large $N$ limit of $\chi_{YM}$ with an unprecedented accuracy thanks to a solid definition through the YM Gradient flow and the use of open boundary conditions (OBC).
- By using OBC we were able to go to finer lattice spacings and keep the autocorrelations under control.
- Through a careful study of all the systematic effects we quote a result in the large $N$ and continuum limit with a percent level accuracy.
- The value computed for $t_0^2\chi_{YM} = 7.03(13) \times 10^{-4}$ is a new verification of the Witten-Veneziano relation that gives mass to the $\eta'$ meson.
- We find the large $N$ effects to be small at our level of accuracy.

Thank you very much for your attention!