Thermodynamics of the Casimir Effect
- Asymptotic Considerations -

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We study the Casimir effect with different temperatures between the plates \( T \) resp. outside of them \( T' \). If we consider the inner system as the black body radiation for a special geometry, then contrary to common belief the temperature approaches a constant value for vanishing volume during isentropic processes. This means: the reduction of the degrees of freedom can not be compensated by a concentration of the energy during an adiabatic contraction of the two-plate system. Looking at the Casimir pressure, we find one unstable equilibrium point for isothermal processes with \( T > T' \). For isentropic processes there is additionally one stable equilibrium point for larger values of the distances between the two plates.

1 Introduction

The Casimir effect is one of the fundamental effects of Quantum Field Theory. It tests the importance of the zero point energy. In principle, one considers two conducting infinitely extended parallel plates at the positions \( x_3 = 0 \) and \( x_3 = a \). These conducting plates change the vacuum energy of Quantum Electrodynamics in such a way that a measurable attractive force between both plates can be observed. This situation does not essentially change, if a non-vanishing temperature is taken into account. The thermodynamics of the Casimir effect and related problems is well investigated.

Here we shall treat the different regions separately. We assume a temperature \( T \) for the space between the plates and a temperature \( T' \) for the space outside the plates. Thereby we consider the right plate at \( x_3 = a \) as movable, so that different thermodynamic processes such as isothermal or isentropic motions, can be studied. At first we investigate the thermodynamics of the space between the two plates by setting \( T' = 0 \). This can be viewed as the black body radiation (BBR) for a special geometry. The surprising effect is, that for vanishing distance \( (a \rightarrow 0) \) in isentropic processes the temperature approaches a finite value, which is completely determined by the fixed entropy. This is in contrast to the expected behavior of the standard BBR, if the known expression derived for a large volume is extrapolated to a small volume. For large values of \( a \) the BBR takes the standard form. As a next topic we consider the Casimir pressure assuming that the two physical regions, i.e. the spaces
between and outside the two plates possess different temperatures. Depending on the choices of $T$ and $T'$ a different physical behaviour is possible. For $T' < T$ the external pressure is reduced in comparison with the standard case $T' = T$. Therefore we expect the existence of an equilibrium point, where the pure Casimir attraction ($T = 0$ effect) and the differences of the radiation pressures compensate each other. This point is unstable, so that for isothermal processes the movable plate moves either to $a \to 0$ or to $a \to \infty$. However, an isentropic motion reduces the internal radiation pressure for growing distances, so that in this case there is an additional stable equilibrium point.

2 Thermodynamic Functions

The thermodynamic functions are already determined by different methods. We recalculate them by statistical mechanics including the zero-point energy and cast it in a simpler form which can be studied in detail. For technical reasons the system is embedded in a large cube (side L). As space between the plates we consider the volume $L^2a$, the region outside is given by $L^2(L - a)$. All extensive thermodynamic functions are defined per area.

Free energy $\phi = F/L^2$:

\[
\phi_{\text{int}} = \frac{\hbar c \pi^2}{a^4} (- \frac{1}{720} + g(v)) + \frac{3hc}{\pi^2 \lambda^4}a, \quad (1)
\]
\[
\phi_{\text{ext}} = [\frac{3hc}{\pi^2 \lambda^4} - \frac{\hbar c\pi^6}{45} (\frac{v'}{a})^4](L - a). \quad (2)
\]

Energy $e = E/L^2$:

\[
e_{\text{int}} = \frac{\hbar c \pi^2}{a^4} (- \frac{1}{720} + g(v) - v \partial_v g(v)) + \frac{3hc}{\pi^2 \lambda^4}a,
\]
\[
e_{\text{ext}} = [\frac{3hc}{\pi^2 \lambda^4} + \frac{3hc\pi^6}{45} (\frac{v'}{a})^4](L - a).
\]

Pressure:

\[
p_{\text{int}} = \frac{\hbar c \pi^2}{a^4}(-\frac{1}{240} + 3g(v) - v \partial_v g(v)) - \frac{3hc}{\pi^2 \lambda^4}, \quad (3)
\]
\[
p_{\text{ext}} = [\frac{3hc}{\pi^2 \lambda^4} - \frac{\hbar c\pi^6}{45} (\frac{v'}{a})^4]. \quad (4)
\]

Entropy $\sigma = S/(kL^2)$:

\[
\sigma_{\text{int}} = -\frac{\pi}{a^3} \partial_v g(v)a; \quad \sigma_{\text{ext}} = \frac{4\pi^5}{45} (\frac{v'}{a})^3(L - a), \quad (5)
\]
λ regularizes (λ → 0) the contributions from the zero-point energy. The thermodynamics is governed by the function \(g(v)\). We list two equivalent expressions:

\[
g(v) = -v^3 \left[ \frac{1}{2} \zeta(3) + k \left( \frac{1}{v} \right) \right] = \frac{1}{720} - \frac{\pi^4}{45} v^4 - \frac{v}{4\pi^2} \left[ \frac{1}{2} \zeta(3) + k(4\pi^2 v) \right].
\]

The function \(k(x)\) is given by

\[
k(x) = (1 - x \partial_x) \sum_{n=1}^{\infty} \frac{1}{n^3} \frac{1}{\exp(nx) - 1}.
\]

It is strongly damped for large arguments. \(v\) is the known variable \(v = aTk/(\hbar\pi c)\), the variable \(v'\) contains the temperature \(T'\) instead of \(T\).

3 Black Body Radiation

As a first topic we consider the space between the two plates as a generalization of the usual black body radiation (BBR) for a special geometry \(L \times L \times a\). Contrary to the standard treatment we include here both, the internal and external the zero point energy. Thereby parameter-dependent divergent contributions compensate each other, whereas the physically irrelevant term \(L/\lambda^4\) can be omitted. If we approximate the function \(g\) for large \(v\) by

\[
g \simeq \frac{1}{720} - \frac{\pi^4}{45} v^4 - \frac{\zeta(3)}{8\pi^2} v^3,
\]

we obtain

\[
\begin{align*}
\phi_{as} & = \frac{\pi^2 \hbar c}{a^3} \left[ -\frac{\pi^4}{45} v^4 - \frac{\zeta(3)}{8\pi^2} v^3 \right], & \sigma_{as} & = \frac{\pi}{a^2} \frac{4\pi^4}{45} v^3 + \frac{\zeta(3)}{8\pi^2} v^4, \\
p_{as} & = \frac{\pi^2 \hbar c}{a^4} \left( \frac{\pi^4}{45} v^4 - \frac{\zeta(3)}{8\pi^2} v^3 \right), & e_{as} & = \frac{\pi^2 \hbar c}{a^3} \frac{3\pi^4}{45} v^4.
\end{align*}
\]

These expressions contain the large-volume contributions corresponding to the standard BBR (first term) and corrections. In the other limit of small \(v\), we have to use \(g(v) = -v^3 \zeta(3)/2\) and get

\[
\begin{align*}
\phi_o & = \frac{\pi^2 \hbar c}{a^3} \left[ -\frac{1}{720} - \frac{\zeta(3)}{2} v^3 \right], & \sigma_o & = \frac{\pi}{a^2} \frac{3\zeta(3)}{2} v^2, \\
p_o & = \frac{\pi^2 \hbar c}{a^4} \left[ -\frac{1}{2} \frac{v^3}{240} \right], & e_o & = \frac{\pi^2 \hbar c}{a^3} \left[ -\frac{1}{720} + \zeta(3)v^3 \right].
\end{align*}
\]

In this case the contributions of the zero point energy dominate. It is known that nondegenerate vacuum states do not contribute to the entropy, which indeed vanishes at \(T = 0\).
Let us now consider isentropic processes. This means that we fix the values of the entropy for the internal region during the complete process. Technically we express this fixed value according to the value of the variable either through the approximation or . Large distances and/or high temperatures lead to large values of so we have to use . Constant entropy means .

Asymptotically this is the standard relation BBR , here valid for large and . If we now consider smaller values of , then, because of eq. , also takes smaller values. It is possible to prove the inequalities , , and . This monotonic behaviour of leads to the conclusion that also the corresponding values of become smaller. Consequently, we have to apply the other representation for small and obtain

This means that for the temperature does not tend to infinity, but approaches the finite value

This is in contrast to the expectation: if we apply the standard expression of BBR, fixed entropy implies , so that the temperature tends to infinity for vanishing volume. However this standard expression for BBR, derived for a continuous frequency spectrum, is not valid for small distances. The reduction of the degrees of freedom, i.e. the transition from a continuous frequency spectrum to a discrete spectrum, is the reason for our result.

4 Equilibrium Points of the Casimir Pressure

The Casimir pressure results from the contributions of the internal and the external regions acting on the right movable plate.

\[
P(a, T, T') = P_{\text{ext}}(T') + P_{\text{int}}(a, T) = \frac{\pi^2 \hbar c}{a^4} p(v) + \frac{\pi^2 k^4}{45(hc)^3} (T^4 - T'^4),
\]

where

\[
p(v) = -\frac{1}{4\pi^2} v^4[\zeta(3) + (2 - v \partial_v)k(4\pi^2 v)] = -\frac{1}{240} + 3g(v) - v \partial_v g(v) - \frac{\pi^4}{45} v^4.
\]
Usually one considers the case $T = T'$, so that the Casimir pressure is prescribed by $p(v)$ alone. It is known, that $P(a, T, T' = T)$ is a negative but monotonically rising function from $-\infty$ (for $a \to 0$) to 0 (for $a \to \infty$). It is clear, that the addition of a positive pressure $\frac{\pi^2}{6\hbar c^2}(T^4 - T'^4)$ for $T > T'$ stops the Casimir attraction at a finite value of $v$. The question is whether this equilibrium point may be stable or not? The answer follows from the monotonically rising behaviour of the standard Casimir pressure.

$$\frac{d}{da} P(a, T, T') = \frac{d}{da} P(a, T, T' = T) > 0. \quad (16)$$

Consequently this equilibrium point is unstable (see also [4]).

Next we consider the space between the two plates not for fixed temperature but as a thermodynamically closed system with fixed entropy. In the external region we assume again a fixed temperature $T'$. To solve this problem in principle, it is sufficient to discuss our system for large $v$ (as large $v$ we mean such values of $v$ for which the asymptotic approximations (8), (9) are valid; this region starts at $v > 0.2$). Using our asymptotic formulae (8),(9) we write the Casimir pressure as

$$P(a, v, T') = \frac{\pi^2}{a^4} \left( \frac{\sigma a^2}{4\pi} - \frac{\zeta(3)}{4\pi^2} v - \frac{\pi^4}{45} v' \right), \quad (17)$$

with $v' = aT'k/(\hbar c\pi)$ where $v$ has to be determined from the condition $\sigma_{as} = \sigma$ = const. or

$$\pi v^3 = \left[ a^2 \sigma - \frac{\zeta(3)}{(8\pi^2)} \right] 45/(4\pi^4). \quad (18)$$

Then we may write

$$P(a, v, T') = \frac{\pi^2}{a^4} \left( \frac{\sigma a^2}{4\pi} - \frac{9\zeta(3)}{32\pi^2} \right) \frac{45}{4\pi^4} \left( \frac{\sigma a^2}{4\pi} - \frac{\zeta(3)}{8\pi^2} \right)^{3/2} - \frac{\pi^2}{a^4} \frac{\pi^4}{45} v'^4. \quad (19)$$

At first we consider the case $T' = 0$. We look for the possible equilibrium points $P(a, v, T' = 0) = 0$. The result is $v^3 = 45\zeta(3)/(4\pi^6)$. This corresponds to $v = 0.24$. For this value of $v$ the used approximation is not very good, but acceptable. A complete numerical estimate gives the same value. Now we express the temperature $T$ included in $v$ with the help of the equation for isentropic motions (18) and obtain $a^2 = 9\zeta(3)/(8\pi\sigma)$. The instabilty of this point can be directly seen by looking at

$$\frac{d}{da} P(a, T, T' = 0) = -4P(a, T, T' = 0) + \frac{\pi^2}{a^4} \frac{4\pi^4}{45} v'^4 - \frac{\zeta(3)}{8\pi^2} \left( \frac{dv}{da} \right)_{\sigma}.$$
It is intuitively clear that \((\frac{dv}{da})_\sigma\) is positive; an explicit proof is given in [6]. So it is clear, that this point is unstable as in the isothermal case. If we consider, in eq. (17), the variable \(v = aTk/(\hbar c\pi)\) at fixed \(T\), there is no further equilibrium point. This result for isothermal processes is, however, not valid for isentropic processes. In this case we obtain according to eq. (19) a second trivial equilibrium point at \(a \to \infty\) for vanishing external temperature \(v' = 0\). Between both zeroes we have one maximum. So we conclude: For isentropic processes there must be two equilibrium points; the left one is unstable, the right one at \(a \to \infty\) corresponds to a vanishing derivative. If we now add a not too high external pressure with the help of an external temperature \(T'\), then this second equilibrium point - present for isentropic processes - becomes stable. So, in principle we may observe oscillations at the position of the second equilibrium point.

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