Characterization of the Matrix Class \((\ell_\alpha, \ell_\beta), 0 < \alpha \leq \beta \leq 1\)

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Abstract. Throughout the present paper, entries of sequences, infinite series and infinite matrices are real or complex numbers. In this paper, we characterize the matrix class \((\ell_\alpha, \ell_\beta), 0 < \alpha \leq \beta \leq 1\).

1. Introduction and Preliminaries

Throughout the present paper, entries of sequences, infinite series and infinite matrices are real or complex numbers; \(\alpha, \beta\) are real numbers satisfying \(0 < \alpha \leq \beta \leq 1\).

We need the following sequence space in the sequel.

\[ \ell_\alpha = \left\{ x = \{x_k\} : \sum_{k=0}^{\infty} |x_k|^\alpha < \infty \right\}, \alpha > 0. \]

If \(A = (a_{nk}), n, k = 0, 1, 2, \ldots\) is an infinite matrix, we write

\[ A \in (\ell_\alpha, \ell_\beta), \alpha, \beta > 0, \]

if

\[ (Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \]

is defined, \(n = 0, 1, 2, \ldots\) and the sequence \(A(x) = (Ax)_n \in \ell_\beta,\) whenever \(x = \{x_k\} \in \ell_\alpha.\) \(A(x)\) is called the \(A\)-transform of \(x.\)

We now present a short summary of the research done so far regarding the characterization of the matrix class \((\ell_\alpha, \ell_\beta).\) A complete characterization of the matrix class \((\ell_\alpha, \ell_\beta), \alpha, \beta \geq 2,\) does not seem to be available in the literature. The latest result in this direction \([3]\) characterizes only non-negative matrices in \((\ell_\alpha, \ell_\beta), \alpha \geq \beta > 1.\) A known simple sufficient condition \([4], p. 174, Theorem 9) for \(A = (a_{nk}) \in (\ell_\alpha, \ell_\beta)\) is

\[ A \in (\ell_\infty, \ell_\infty) \cap (\ell_1, \ell_1). \]

Sufficient conditions or necessary conditions for \(A \in (\ell_\alpha, \ell_\beta)\) are available in the literature (for instance, see \([7]\)). Necessary and sufficient conditions for \(A \in (\ell_1, \ell_1)\) are due to Mears \([5]\) (for alternative proofs, see Knopp and Lorentz \([2]\), Fridy \([1]\)). In \([6],\) Natarajan characterized the matrix class \((\ell_\alpha, \ell_\alpha), 0 < \alpha \leq 1.\)

In the context of the above survey, the main result of the present paper is interesting.
2. Main Result

In this section, we need the following lemma.

Lemma 2.1. ([4], p. 22)
(i) \[ |a^\alpha - b^\alpha| \leq |a + b|^\alpha \leq |a|^\alpha + |b|^\alpha, \quad 0 < \alpha \leq 1; \]

(ii) \[ \sum_{k=0}^{\infty} |a_k + b_k|^\alpha \leq \sum_{k=0}^{\infty} |a_k|^\alpha + \sum_{k=0}^{\infty} |b_k|^\alpha, \quad 0 < \alpha \leq 1. \]

We now take up the main result of the paper.

Theorem 2.2. \( A = (a_{nk}) \in (\ell_\alpha, \ell_\beta), \quad 0 < \alpha \leq \beta \leq 1, \) if and only if
\[
\sup_{k \geq 0} \sum_{n=0}^{\infty} |a_{nk}|^\beta < \infty. \tag{3}
\]

Proof. Sufficiency. Let (3) hold. We first claim that \( 0 < \alpha \leq \beta \leq 1 \) implies that \( \ell_\alpha \subseteq \ell_\beta \subseteq \ell_1. \) Let \( x = \{x_k\} \in \ell_\alpha, \)

i.e., \( \sum_{k=0}^{\infty} |x_k|^\alpha < \infty. \) So \( x_k \to 0, \ k \to \infty. \) We can find a positive integer \( N \) such that
\[ |x_k| < 1, \ k \geq N. \]

Since \( \frac{\beta}{\alpha} \geq 1, \)
\[ |x_k|^\beta \leq |x_k|, \]
i.e., \( |x_k|^\beta \leq |x_k|^\alpha, k \geq N. \)

Thus,
\[ \sum_{k=N}^{\infty} |x_k|^\beta \leq \sum_{k=N}^{\infty} |x_k|^\alpha < \infty \]

and so \( \sum_{k=0}^{\infty} |x_k|^\beta < \infty, \) i.e., \( x = \{x_k\} \in \ell_\beta. \) Hence \( \ell_\alpha \subseteq \ell_\beta. \) Similarly, \( \beta \leq 1 \) implies that \( \ell_\beta \subseteq \ell_1. \) Consequently,
\[ \ell_\alpha \subseteq \ell_\beta \subseteq \ell_1. \] Now, let \( x = \{x_k\} \in \ell_\alpha. \) So \( \{x_k\} \in \ell_1, \) i.e., \( \sum_{k=0}^{\infty} |x_k| < \infty. \) Using (3), \( \sup_{n,k} |a_{nk}| < \infty. \) Hence
\[ \sum_{k=0}^{\infty} |a_{nk}x_k| \leq \left( \sup_{n,k} |a_{nk}| \right) \left( \sum_{k=0}^{\infty} |x_k| \right) \]
\[ < \infty, \]

from which it follows that \( \sum_{k=0}^{\infty} a_{nk}x_k \) converges, \( n = 0, 1, 2, \ldots. \)

So,
\[ (Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k \]
is defined, \( n = 0, 1, 2, \ldots \). Since \( \ell_\alpha \subseteq \ell_\beta \), \( \sum_{k=0}^{\infty} |x_k|^\beta < \infty \).

Now, using Lemma 2.1 and condition (3), we get

\[
\sum_{n=0}^{\infty} |(Ax)_n|^\beta = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{nk}x_k \right)^\beta \\
\leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_{nk}|^\beta |x_k|^\beta \\
= \sum_{k=0}^{\infty} |x_k|^\beta \sum_{n=0}^{\infty} |a_{nk}|^\beta \\
\leq \left( \sup_{k \geq 0} \sum_{n=0}^{\infty} |a_{nk}|^\beta \right) \left( \sum_{k=0}^{\infty} |x_k|^\beta \right)
\]

\[
< \infty.
\]

Hence \( \{(Ax)_n\} \in \ell_\beta \), i.e., \( A \in (\ell_\alpha, \ell_\beta) \).

Necessity. Let \( A \in (\ell_\alpha, \ell_\beta) \). First, we note that

\[
B_n = \sup_{k \geq 0} |a_{nk}|^\beta < \infty, \ n = 0, 1, 2, \ldots. \tag{4}
\]

Suppose not. Then, for some positive integer \( m \),

\[
B_m = \sup_{k \geq 0} |a_{mk}|^\beta = \infty.
\]

We can now choose a strictly increasing sequence \( \{k(i)\} \) of positive integers such that

\[
|a_{m,k(i)}|^\beta > i^2, i = 1, 2, \ldots.
\]

Define the sequence \( x = \{x_k\} \) by

\[
x_k = \begin{cases} \frac{1}{a_{m,k(i)}}, & \text{if } k = k(i); \\
0, & \text{if } k \neq k(i), \ i = 1, 2, \ldots.
\end{cases}
\]

\( x = \{x_k\} \in \ell_\alpha \), for,

\[
\sum_{k=0}^{\infty} |x_k|^\alpha = \sum_{i=1}^{\infty} |x_{k(i)}|^\alpha = \sum_{i=1}^{\infty} \frac{1}{|a_{m,k(i)}|^\beta} \\
< \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.
\]

On the other hand,

\[
d_{m,k(i)}x_{k(i)} = 1 \not\to 0, \ i \to \infty,
\]

which implies that

\[
(Ax)_m = \sum_{k=0}^{\infty} a_{mk}x_k
\]
is not defined, a contradiction, proving (4). For $k = 0, 1, 2, \ldots$, the sequence $x = \{x_k\} = \{0, 0, \ldots, 0, 1, 0, \ldots\}$, 1 occurring in the $k$th place, is in $\ell$, for which $(Ax)_n = a_{nk}$. \((Ax)_n = |a_{nk}|_{n=0}^\infty \in \ell\) implies that

$$\mu_k = \sum_{n=0}^\infty |a_{nk}|^\beta < \infty, k = 0, 1, 2, \ldots.$$  

We now claim that $\{\mu_k\}$ is bounded. Suppose not, i.e., $\{\mu_k\}$ is unbounded. Choose a positive integer $k(1)$ such that

$$\mu_k(1) > 3.$$  

We now choose a positive integer $n(1)$ such that

$$\sum_{n=n(1)+1}^\infty |a_{n,k(1)}|^\beta < 1,$$  

so that

$$\mu_k(1) = \sum_{n=0}^{n(1)} |a_{n,k(1)}|^\beta + \sum_{n=n(1)+1}^\infty |a_{n,k(1)}|^\beta,$$  

i.e.,

$$\sum_{n=0}^{n(1)} |a_{n,k(1)}|^\beta = \mu_k(1) - \sum_{n=n(1)+1}^\infty |a_{n,k(1)}|^\beta > 3 - 1 = 2.$$  

More generally, having chosen the positive integers $k(j), n(j), j \leq m - 1$, choose the positive integers $k(m), n(m)$ such that $k(m) > k(m - 1), n(m) > n(m - 1), n(m-1)$,

$$\sum_{n=n(m-1)+1}^{n(m)} \sum_{k=k(m)} B_n^\beta/k^2 < 1,$$  

$$\mu_k(m) > 2 \sum_{n=0}^{n(m-1)} B_n + \rho^{-a} m^2 \left\{ 2 + \sum_{i=1}^{m-1} i^2 \mu_k(i) \right\}$$  

and

$$\sum_{n=n(m)+1}^\infty |a_{n,k(m)}|^\beta \leq \sum_{n=0}^\infty B_n,$$  

where, $0 < \rho < 1$. Now, using (6) and (7), we get

$$\sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\beta = \mu_k(m) - \sum_{n=0}^{n(m-1)} |a_{n,k(m)}|^\beta - \sum_{n=n(m)+1}^\infty |a_{n,k(m)}|^\beta$$  

$$> 2 \sum_{n=0}^{n(m-1)} B_n + \rho^{-a} m^2 \left\{ 2 + \sum_{i=1}^{m-1} i^2 \mu_k(i) \right\} - \sum_{n=0}^{n(m-1)} B_n - \sum_{n=0}^{n(m-1)} B_n$$  

$$= \rho^{-a} m^2 \left\{ 2 + \sum_{i=1}^{m-1} i^2 \mu_k(i) \right\}.$$  

$$\square$$
Now, for every $i = 1, 2, 3, \ldots$, we can choose a non-negative integer $\lambda(i)$ such that
\[ \rho^{\lambda(i)+1} \leq i^{\alpha} < \rho^{\lambda(i)}. \] (9)

Define the sequence $x = \{x_k\}$, where
\[ x_k = \begin{cases} 
\rho^{\lambda(i)} + 1, & \text{if } k = k(i); \\
0, & \text{if } k \neq k(i), i = 1, 2, \ldots.
\end{cases} \]

We note that $x = \{x_k\} \in \ell_\alpha$, since
\[
\sum_{k=0}^{\infty} |x_k|^\alpha = \sum_{i=1}^{\infty} |x_k(i)|^\alpha \\
= \sum_{i=1}^{\infty} \rho^{\lambda(i)+1} \alpha \\
\leq \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}}, \text{ using (9)} \\
< \infty.
\]

In view of Lemma 2.1, we have
\[
\sum_{n=n(m-1)+1}^{n(m)} |(Ax)_n|^\beta \geq \Sigma_1 - \Sigma_2 - \Sigma_3,
\]
where
\[ \Sigma_1 = \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\beta |x_{k(m)}|^\beta, \]
\[ \Sigma_2 = \sum_{n=n(m-1)+1}^{n(m)} \sum_{i=1}^{m-1} |a_{n,k(i)}|^\beta |x_{k(i)}|^\beta, \]
and
\[ \Sigma_3 = \sum_{n=n(m-1)+1}^{n(m)} \sum_{n=m+1}^{\infty} |a_{n,k(i)}|^\beta |x_{k(i)}|^\beta. \]

Now,
\[ \Sigma_1 = \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\beta \rho^{\lambda(m)+1} \alpha, \]
\[ > \rho^{\alpha} \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\beta m^{-2}, \text{ using (9)} \]
\[ = \rho^{\alpha} m^{-2} \sum_{n=n(m-1)+1}^{n(m)} |a_{n,k(m)}|^\beta \]
\[ > 2 + \sum_{i=1}^{m-1} i^{-2} \mu(k(i)), \text{ using (8)}; \]
(10)
\[ \Sigma_2 = \sum_{n=n(m)+1}^m \sum_{i=1}^{m-1} |a_{n,k(i)}|^\beta \rho^{(i(0)+1)\beta} \]

\[ \leq \sum_{n=n(m)+1}^m \sum_{i=1}^{m-1} |a_{n,k(i)}|^\beta \rho^{(i(0)+1)\alpha}, \]

since \( 0 < \rho < 1 \) and \( \alpha \leq \beta \)

\[ \leq \sum_{n=n(m)+1}^m \sum_{i=1}^{m-1} |a_{n,k(i)}|^\beta i^{-2}, \quad \text{using (9)} \]

\[ = \sum_{i=1}^{m-1} i^{-2} \sum_{n=n(m)+1}^m |a_{n,k(i)}|^\beta \]

\[ \leq \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)} \]

\[ = \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)} \quad (11) \]

and

\[ \Sigma_3 = \sum_{n=n(m)+1}^m \sum_{i=m+1}^\infty |a_{n,k(i)}|^\beta \rho^{(i(0)+1)\beta} \]

\[ < \sum_{n=n(m)+1}^m \sum_{i=m+1}^\infty |a_{n,k(i)}|^\beta \rho^{(i(0)+1)\alpha} \]

since \( 0 < \rho < 1 \) and \( \alpha \leq \beta \)

\[ \leq \sum_{n=n(m)+1}^m \sum_{i=m+1}^\infty B_{n,i}^{\beta/\alpha} i^{-2}, \quad \text{using (4)} \]

\[ < 1, \quad \text{using (5)}. \quad (12) \]

Now, using (10), (11) and (12), we get

\[ \sum_{n=n(m)+1}^m |(Ax)_n|^\beta > 2 + \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)} - \sum_{i=1}^{m-1} i^{-2} \mu_{k(i)} - 1 \]

\[ = 1, \quad m = 2, 3, \ldots, \]

from which it follows that \( \{Ax\}_n \notin \ell_\beta \), while, \( x = \{x_k\} \in \ell_\alpha \), which is a contradiction. Thus (3) is necessary, completing the proof of the theorem. \( \square \)

**Corollary 2.3.** If we put \( \beta = \alpha \), we get a characterization of the matrix class \( (\ell_\alpha, \ell_\alpha) \), \( 0 < \alpha \leq 1 \), which was obtained by the author in [6].

**Corollary 2.4.** \( A = (a_{nk}) \in (\ell_\alpha, \ell_\beta), \ 0 < \alpha < \beta \leq 1 \) if and only if

\[ A \in (\ell_\beta, \ell_\beta). \]
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