Riemannian thermo-statistics geometry

L. Velazquez

Abstract. It is developed a Riemannian reformulation of classical statistical mechanics for systems in thermodynamic equilibrium, which arises as a natural extension of Ruppeiner geometry of thermodynamics. The present proposal leads to interpret entropy $S_g(I|\theta)$ and all its associated thermo-statistical quantities as purely geometric notions derived from the Riemannian structure on the manifold of macroscopic observables $M_g$ (existence of a distance $ds^2 = g_{ij}(I|\theta)dI^i dI^j$ between macroscopic configurations $I$ and $I + dI$). Moreover, the concept of statistical curvature scalar $R(I|\theta)$ arises as an invariant measure to characterize the existence of an irreducible statistical dependence among the macroscopic observables $I$ for a given value of control parameters $\theta$. This feature evidences a certain analogy with Einstein General Relativity, where the spacetime curvature $R(r,t)$ distinguishes the geometric nature of gravitation and the reducible character inertial forces with an appropriate selection of the reference frame.

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Departamento de Física, Universidad Católica del Norte, Av. Angamos 0610, Antofagasta, Chile.

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1. Introduction

Let be a classical equilibrium distribution function:

\[ dp(I|\theta) = \rho(I|\theta)dI, \]  

(1)

where the stochastic variables \( I = \{I_i\} \) denote the relevant macroscopic observables of the system under analysis, while \( \theta = \{\theta^\alpha\} \) the underlying control parameters of a given equilibrium situation. It is possible to introduce a distance notion between two close equilibrium situations \( \theta \) and \( \theta + d\theta \):

\[ ds^2_F = g_{\alpha\beta}(\theta)d\theta^\alpha d\theta^\beta \]  

(2)

in terms of the so-called Fisher’s information matrix \([1]\):

\[ g_{\alpha\beta}(\theta) = \int_{\mathcal{M}_\theta} \frac{\partial \log \rho(I|\theta)}{\partial \theta^\alpha} \frac{\partial \log \rho(I|\theta)}{\partial \theta^\beta} \rho(I|\theta) dI. \]

(3)

The existence of this type of Riemannian formulation was pioneering proposed by Rao \([2]\), which is referred to as inference geometry in the literature \([3]\). Alternately, it is also possible to introduce a distance notion between two close macroscopic configurations \( I \) and \( I + dI \) for a given value of control parameter \( \theta \):

\[ ds^2 = g_{ij}(I|\theta)dI^i dI^j \]  

(4)

starting from the same distribution function \([1]\). This latter geometric characterization appears as a suitable extension of Ruppeiner geometry of thermodynamics \([4]\) to the framework of equilibrium classical statistical mechanics. For this reason, the present proposal shall be hereafter referred to as Riemannian thermo-statistics geometry. The main interest of this work is to present a systematic development of the most relevant features of this geometric formulation starting from an axiomatic perspective. Such a procedure allows to show that fundamental concepts of statistical mechanics can be suitably rephrased in terms of geometry notions, which provides a general framework to apply powerful methods of Riemannian geometry for the analysis of properties of thermodynamical systems.
2. Riemannian thermo-statistics geometry

Let us denote by $\mathcal{M}$ the abstract manifold composed of all admissible values of macroscopic observables $I$, as well as by $\mathcal{P}$ the abstract manifold composed of all admissible values of control parameters $\theta$. Besides, let us also denote by $\mathcal{M}_\theta$ the sub-manifold of $\mathcal{M}$ composed of all values of macroscopic observables $I$ that are accessible for a given value $\theta$. In general, it is possible to consider two different types of coordinate reparametrizations: (1) the coordinate reparametrizations $\Theta(I) : \mathcal{R}_I \rightarrow \mathcal{R}_\Theta$ of the manifold of macroscopic observables $\mathcal{M}_\theta$, as well as (2) the coordinate reparametrizations $\nu(\theta) : \mathcal{R}_\theta \rightarrow \mathcal{R}_\nu$ of the manifold of control parameters $\mathcal{P}$.

2.1. Postulates of thermo-statistics geometry

**Axiom 1** The manifold of the system macroscopic observables $\mathcal{M}_\theta$ possesses a Riemannian structure, that is, it is provided of a metric tensor $g_{ij}(I|\theta)$ and a torsionless covariant differentiation $D_i$ that obeys the following constraints:

$$D_k g_{ij}(I|\theta) = 0.$$  \hspace{1cm} (5)

**Definition 1** The Riemannian structure on the manifold of macroscopic observables $\mathcal{M}_\theta$ allows to introduce the invariant volume element as follows:

$$d\mu(I|\theta) = \sqrt{\frac{g_{ij}(I|\theta)}{2\pi}} dI,$$  \hspace{1cm} (6)

where $|g_{ij}(I|\theta)|$ denotes the absolute value of the metric tensor determinant.

**Axiom 2** There exist a differentiable scalar function $S_g(I|\theta)$ defined on the manifold $\mathcal{M}_\theta$, hereafter referred to as the scalar entropy, whose knowledge determines the equilibrium distribution function $dp(I|\theta)$ of the macroscopic observables $I \in \mathcal{M}_\theta$ as follows:

$$dp(I|\theta) = \exp [S_g(I|\theta)] d\mu(I|\theta).$$  \hspace{1cm} (7)

**Definition 2** Let us consider an arbitrary curve given in parametric form $I(t) \in \mathcal{M}_\theta$ with fixed extreme points $I(t_1) = P$ and $I(t_2) = Q$. Adopting the following notation:

$$\dot{i} = \frac{dI^i(t)}{dt},$$  \hspace{1cm} (8)
the length $\Delta s$ of this curve can be expressed as:

$$\Delta s = \int_{t_1}^{t_2} \sqrt{g_{ij} [I(t)|\theta] \dot{I}^i(t) \dot{I}^j(t)} dt. \quad (9)$$

**Definition 3** The curve $I(t) \in M_\theta$ exhibits an **unitary affine parametrization** when its parameter $t$ satisfies the following constraint:

$$g_{ij}(I|\theta) \dot{I}^i(t) \dot{I}^j(t) = 1. \quad (10)$$

**Definition 4** A **geodesic** is the curve $I_g(t)$ with minimal length $\| \|_g$ between two fixed arbitrary points $(P, Q) \in M_\theta$. Moreover, the distance $D(P, Q|\theta)$ between these two points $(P, Q)$ is given by the length of its associated geodesic $I_g(t)$:

$$D(P, Q|\theta) = \int_{t_1}^{t_2} \sqrt{g_{ij} [I_g(t)|\theta] \dot{I}^i_g(t) \dot{I}^j_g(t)} dt. \quad (11)$$

**Definition 5** Considering a differentiable curve $I(t) \in M_\theta$ with an unitary affine parametrization, the **entropy production** along this curve $\Phi(t)$ is given by:

$$\Phi(t) = \frac{dS_g [I(t)|\theta]}{dt}. \quad (12)$$

**Axiom 3** The length $\Delta s$ of any interval $(t_1, t_2)$ of an arbitrary geodesic $I_g(t) \in M_\theta$ with an unitary affine parametrization is given by the variation of its entropy production $\Delta \Phi(t)$ with opposite sight:

$$\Delta s = -\Delta \Phi(t) = \Phi(t_1) - \Phi(t_2). \quad (13)$$

**Axiom 4** The probability density $\rho(I|\theta)$ associated with distribution function $\{7\}$ vanishes with its first partial derivatives for any point on the boundary $\partial M_\theta$ of the manifold $M_\theta$.

### 2.2. Interpretations and fundamental consequences

**Commentary.** **Axiom 1** allows to precise the Riemannian structure of the manifold $M_\theta$ starting from the knowledge of the metric tensor $g_{ij}(I|\theta)$, specifically, the covariant differentiation $D_i$ and the curvature tensor $R_{ijkl}(I|\theta)$. As discussed elsewhere $[7]$, Eq. (5) is an strong constraint of Riemannian geometry that determines the **affine connections** $\Gamma^k_{ij}(I|\theta)$ employed to introduce the covariant differentiation $D_i$, specifically, the so-called **Levi-Civita connection**:

$$\Gamma^k_{ij}(I|\theta) = g^{km} \frac{1}{2} \left( \frac{\partial g_{im}}{\partial I^j} + \frac{\partial g_{jm}}{\partial I^i} - \frac{\partial g_{ij}}{\partial I^m} \right). \quad (14)$$
The knowledge of the affine connections $\Gamma^k_{ij}$ allows the introduction of the curvature tensor $R^l_{ijk} = R^l_{ijk}(I|\theta)$ of the manifold $\mathcal{M}_\theta$:

$$R^l_{ijk} = \frac{\partial}{\partial I^i} \Gamma^l_{jk} - \frac{\partial}{\partial I^j} \Gamma^l_{ik} + \Gamma^l_{im} \Gamma^m_{jk} - \Gamma^l_{jm} \Gamma^m_{ik},$$

which is also determined from the knowledge of the metric tensor $g_{ij}(I|\theta)$ and its first and second partial derivatives. Using the curvature tensor $R^l_{ijk}(I|\theta)$, it is possible to obtain its fourth-rank covariant form $R_{ijkl}(I|\theta) = g_{lm}(I|\theta) R^m_{ijk}(I|\theta)$:

$$R_{ijkl} = \frac{1}{2} \left( \frac{\partial^2 g_{il}}{\partial I^j \partial I^k} + \frac{\partial^2 g_{jk}}{\partial I^i \partial I^l} - \frac{\partial^2 g_{jl}}{\partial I^i \partial I^k} - \frac{\partial^2 g_{ik}}{\partial I^j \partial I^l} \right) +$$

$$+ g_{mn} \left( \Gamma^n_{ij} \Gamma^m_{lk} - \Gamma^n_{ik} \Gamma^m_{jl} \right),$$

the Ricci curvature tensor $R_{ij}(I|\theta)$:

$$R_{ij}(I|\theta) = R_{kij}^k(I|\theta)$$

as well as the curvature scalar $R(I|\theta)$:

$$R(I|\theta) = g^{ij}(I|\theta) R_{kij}^k(I|\theta) = g^{ij}(I|\theta) g^{kl}(I|\theta) R_{kjil}(I|\theta).$$

The curvature scalar $R(I|\theta)$ has a paramount relevance in Riemannian geometry [7] because of it is the only invariant derived from the first and second partial derivatives of the metric tensor $g_{ij}(I|\theta)$.

**Commentary.** Axiom 2 is a covariant redefinition of Einstein postulate of classical fluctuation theory [5]:

$$dp_{EP}(I|\theta) = A \exp \left[ S(I|\theta) \right] dI,$$

which rephrases Boltzmann entropy $S = \log W$ to assign relative probabilities from the entropy $S(I|\theta)$. However, expression (19) has the disadvantage that the entropy $S(I|\theta)$, commonly referred to as the coarse grained entropy, does not correspond to a scalar function within a geometric theory. In fact, coarse grained entropy $S(I|\theta)$ behaves under coordinate reparametrizations $\Theta(I) : \mathcal{R}_I \rightarrow \mathcal{R}_\Theta$ as:

$$S(\Theta|\theta) = S(I|\theta) - \log \left| \frac{\partial \Theta}{\partial I} \right|.$$

The non-scalar character of the entropy $S(I|\theta)$ is inconsistent with the physical idea that thermodynamic entropy is a state function, which should not depend on the particular coordinate representation employed to describe the macroscopic properties of a thermodynamic system. One can avoid this mathematical inconsistency replacing
the usual volume element $dI$ by the invariant volume element (6). Thus, the scalar character of the entropy $S_g(I|\theta)$:

$$S_g(\Theta|\theta) = S_g(I|\theta) \quad (21)$$

and the covariance of the metric tensor $g_{ij}(I|\theta)$:

$$g_{ij}(\Theta|\theta) = \frac{\partial I^m}{\partial \Theta^i} \frac{\partial I^n}{\partial \Theta^j} g_{mn}(I|\theta) \quad (22)$$

guarantee the invariance of the equilibrium distribution function (7) under coordinate reparametrizations $\Theta(I) : \mathcal{R}_I \rightarrow \mathcal{R}_\Theta$. ■

**Remark 1 (Alternative Axiom 2)** Starting from the knowledge of probability density $\rho(I|\theta)$ of the distribution function (1) and the existence of an everywhere non-vanishing metric determinant $|g_{ij}(I|\theta)|$, it is possible to introduce the scalar entropy $S_g(I|\theta)$ as follows:

$$S_g(I|\theta) \equiv \log \frac{\rho(I|\theta)}{\sqrt{|g_{ij}(I|\theta)|/2\pi}}. \quad (23)$$

**Commentary. Axiom 3** states a direct relation between the distance notion (4) and the entropy production (83), which relates the metric tensor $g_{ij}(I|\theta)$ and the scalar entropy $S_g(I|\theta)$. This postulate can be regarded as a generalization of the Ruppeiner postulate [4]:

$$g_{ij}(\bar{I}|\theta) = -\frac{\partial^2 S_g(\bar{I}|\theta)}{\partial I^i \partial I^k} \quad (24)$$

which provides a metric tensor for thermodynamics, with $\bar{I}$ being the point with global maximum entropy. ■

**Theorem 1** The scalar entropy $S_g(I|\theta)$ is locally concave everywhere and the metric tensor $g_{ij}(I|\theta)$ is positive definite on the manifold $\mathcal{M}_\theta$. Moreover, the metric tensor $g_{ij}(I|\theta)$ can be identified with the **covariant Hessian** $\mathcal{H}_{ij}(I|\theta)$ of the scalar entropy $S_g(I|\theta)$ with opposite sign:

$$g_{ij}(I|\theta) = -\mathcal{H}_{ij}(I|\theta) = -D_i D_j S_g(I|\theta). \quad (25)$$

**Proof.** The searching of the curve with minimal length (11) between two arbitrary points $(P, Q)$ is a variational problem that leads to the following ordinary differential equations [7]:

$$\dot{I}_g^k(t) D_k \dot{I}_g^i(t) = \ddot{I}_g^i(t) + \Gamma^i_{mn}[I_g(t)|\theta] \dot{I}_g^m(t) \dot{I}_g^n(t) = 0, \quad (26)$$
which describes the geodesic $I_g(t)$ with an unitary affine parametrization. Eqs. (83) and (13) can be rephrased as follows:

$$
\Delta s = -\Delta \Phi(s) \rightarrow \frac{d\Phi(s)}{ds} = \frac{d^2 S_g}{ds^2} = -1.
$$

Taking into account the geodesic differential equations (26), constraint (27) can be rewritten as:

$$
\frac{d^2 S_g}{ds^2} = \dot{I}_k \frac{\partial S_g}{\partial I^k} + \dot{I}_i \dot{I}_j \frac{\partial^2 S_g}{\partial I^i \partial I^j} = \dot{I}_i \dot{I}_j \left\{ \frac{\partial^2 S_g}{\partial I^i \partial I^j} - \Gamma^l_{ij} \frac{\partial S_g}{\partial I^l} \right\},
$$

where it is possible to identify the covariant entropy Hessian $H_{ij}$:

$$
H_{ij} = \frac{\partial^2 S_g}{\partial I^i \partial I^j} - \Gamma^l_{ij} \frac{\partial S_g}{\partial I^l}.
$$

Eqs. (28)-(29) can be combined with constraint (10) to obtain the following expression:

$$
(g_{ij} + H_{ij}) \dot{I}_i \dot{I}_j = 0.
$$

Its covariant character leads to Eq. (25). The concave behavior of the scalar entropy $S_g(I|\theta)$ and the positive definition of the metric tensor $g_{ij}(I|\theta)$ are two direct consequences of Eq. (27).

**Corollary 1** The metric tensor $g_{ij}(I|\theta)$ can be obtained from a given scalar entropy $S_g(I|\theta)$ through the following set of covariant partial differential equations:

$$
g_{ij} = -\frac{\partial^2 S_g}{\partial I^i \partial I^j} + \frac{1}{2} g^{km} \left( \frac{\partial g_{im}}{\partial I^j} + \frac{\partial g_{jm}}{\partial I^i} - \frac{\partial g_{ij}}{\partial I^m} \right) \frac{\partial S_g}{\partial I^k}.
$$

The admissible solutions derived from the nonlinear problem (31) should be everywhere finite and differentiable, including also on boundary of the manifold $M_\theta$.

**Corollary 2** Ruppeiner geometry of thermodynamics [4] is obtained from as a particular case of thermo-statistics geometry restricting to the gaussian approximation of the distribution function [7]:

$$
dp(I|\theta) \simeq \mathcal{N} \exp \left[ -\frac{1}{2} g_{ij} \left( \bar{I}_i | \theta \right) \delta I^i \delta I^j \right] \sqrt{\frac{g_{ij} \left( \bar{I}_i | \theta \right)}{2\pi}} dI,
$$

where $\delta I = I - \bar{I}$ and $\mathcal{N} = \exp \left[ S_g \left( \bar{I}_i | \theta \right) \right]$. Moreover, the normalization of distribution function (32) leads to the following estimation for the maximum scalar entropy $S_g \left( \bar{I}_i | \theta \right)$:

$$
S_g \left( \bar{I}_i | \theta \right) \simeq 0.
$$
Proof. Eq. (25) drops to Ruppeiner definition of thermodynamic metric tensor (24) restricting to the point $\bar{I}$ with of the global maximum scalar entropy $S_g(I|\theta)$. For a sufficiently large thermodynamic short-range interacting system, it is possible to assume the following scaling dependencies with the characteristic size parameter $\Lambda$, $S_g \sim \Lambda$, $I \sim \Lambda \Rightarrow g_{ij} \sim 1/\Lambda$ and the correlation functions $\langle \delta I^i \delta I^j \rangle \sim \Lambda$, where the role of $\Lambda$ can be performed by the system volume $V$, the total mass $M$, number of constituents $N$, etc. Gaussian distribution (32) represents the lowest approximation level that accounts for the thermodynamic fluctuations of the macroscopic observables $I$ in the power expansion of the scalar entropy $S_g(I|\theta)$ and the logarithm of the factor $\sqrt{|g_{ij}(I|\theta)|}$. ■

Commentary. Axiom 4 talks about the asymptotic behavior of the equilibrium distribution function (7) for any point $I_b$ on the boundary $\partial M_\theta$:

$$\lim_{I \to I_b} \rho(I|\theta) = \lim_{I \to I_b} \frac{\partial}{\partial I^i} \rho(I|\theta) = 0. \quad (34)$$

Such conditions play a fundamental role in the character of stationary points (maxima and minima) of the scalar entropy $S_g(I|\theta)$ as well as the fluctuation theorems derived on the Riemannian structure of the manifold $M_\theta$. Due to their own importance, such fluctuation theorems shall be discussed in a forthcoming paper. ■

Remark 2 The boundary conditions (34) are independent from the admissible coordinate representation $\mathcal{R}_I$ of the manifold $M_\theta$.

Proof. This remark follows as a direct consequence of the transformation rule of the probability density:

$$\rho(\Theta|\theta) = \rho(I|\theta) \left| \frac{\partial \Theta}{\partial I} \right|^{-1} \quad (35)$$

as well as the ones associated with its derivatives:

$$\frac{\partial \rho(\Theta|\theta)}{\partial \Theta^\nu} = \frac{\partial I^i}{\partial \Theta^\nu} \left\{ \frac{\partial \rho(I|\theta)}{\partial I^i} - \rho(I|\theta) \frac{\partial}{\partial I^i} \log \left| \frac{\partial \Theta}{\partial I} \right| \right\} \left| \frac{\partial \Theta}{\partial I} \right|^{-1} \quad (36)$$

under a coordinate reparametrization $\Theta(I) : \mathcal{R}_I \to \mathcal{R}_{\Theta}$ whose Jacobian $|\partial \Theta/\partial I|$ be everywhere finite and differentiable. ■
3. Gaussian and spherical representations

3.1. Gaussian Planck potential $\mathcal{P}_g(\theta)$

**Definition 6** The covariant form of the *gradiental generalized forces* $\psi_i(I|\theta)$ are defined from the scalar entropy $S_g(I|\theta)$ as follows:

$$\psi_i(I|\theta) = -D_i S_g(I|\theta) \equiv -\partial S_g(I|\theta) / \partial I^i.$$  \hspace{1cm} (37)

Using the metric tensor $g^{ij}(I|\theta)$, it is possible to obtain its contravariant counterpart $\psi^i(I|\theta)$:

$$\psi^i(I|\theta) = g^{ij}(I|\theta) \psi_j(I|\theta),$$  \hspace{1cm} (38)

as well as its the square norm $\psi^2 = \psi^2(I|\theta)$:

$$\psi^2(I|\theta) = \psi^i(I|\theta) \psi_i(I|\theta).$$  \hspace{1cm} (39)

**Theorem 2** The scalar entropy $S_g(I|\theta)$ can be expressed in terms of the square norm of the gradiental generalized forces as follows:

$$S_g(I|\theta) = \mathcal{P}_g(\theta) - \frac{1}{2} \psi^2(I|\theta),$$  \hspace{1cm} (40)

where $\mathcal{P}_g(\theta)$ is a certain function on control parameters $\theta$ hereafter referred to as the *gaussian Planck potential*.

**Proof.** Let us introduce the scalar function $\mathcal{P}_g(I|\theta)$:

$$\mathcal{P}_g(I|\theta) = S_g(I|\theta) + \frac{1}{2} g^{ij}(I|\theta) \psi_i(I|\theta) \psi_j(I|\theta).$$  \hspace{1cm} (41)

It is easy to verify that its covariant derivatives:

$$D_k \mathcal{P}_g(I|\theta) = D_k S_g(I|\theta) + \frac{1}{2} \{ \psi_i(I|\theta) \psi_j(I|\theta) D_k g^{ij}(I|\theta) +$$

$$+ g^{ij}(I|\theta) [\psi_i(I|\theta) D_k \psi_j(I|\theta) + \psi_j(I|\theta) D_k \psi_i(I|\theta)] \}$$

vanish as direct consequences of the metric tensor properties (5) and (25), as well as the definition (37) of the gradiental generalized forces $\psi_i(I|\theta)$. Since the covariant derivatives of any scalar function are given by the usual partial derivatives:

$$D_k \mathcal{P}_g(I|\theta) = \frac{\partial}{\partial I^k} \mathcal{P}_g(I|\theta) = 0,$$  \hspace{1cm} (42)

the scalar function $\mathcal{P}_g(I|\theta)$ only depends on the control parameters:

$$\mathcal{P}_g(I|\theta) \equiv \mathcal{P}_g(\theta).$$  \hspace{1cm} (43)

This last result leads to Eq.(40). 

\[\blacksquare\]
Corollary 3 The value of scalar entropy $S_g(I|\theta)$ at all its extreme points derived from the stationary condition:
\[ \psi^2(I|\theta) = 0 \] (45)
is exactly given by the gaussian Planck potential $P_g(\theta)$.

Corollary 4 The equilibrium distribution function (7) admits the following gaussian representation:
\[ dp(I|\theta) = \frac{1}{Z_g(\theta)} \exp \left[ -\frac{1}{2} \psi^2(I|\theta) \right] d\mu(I|\theta). \] (46)

Here, the factor $Z_g(\theta)$ is related to the gaussian Planck potential as follows:
\[ P_g(\theta) = -\log Z_g(\theta), \] (47)
which shall be hereafter referred to as the gaussian partition function.

3.2. Maximum and completeness theorems

Theorem 3 The scalar entropy $S_g(I|\theta)$ exhibits a unique stationary point $\bar{I}$, which corresponds to its global maximum.

Proof. Since the metric tensor $g_{ij}(I|\theta)$ defines a positive definite metric on the manifold $M_\theta$, the vanishing of the probability density:
\[ \rho(I|\theta) = \exp [S_g(I|\theta)] \left( \sqrt{\frac{g_{ij}(I|\theta)}{2\pi}} \right) \] (48)
on the boundary $\partial M_\theta$ of the manifold $M_\theta$ for every admissible coordinate representation $R_I$ evidences the vanishing of the scalar function:
\[ \Psi(I|\theta) = \exp [S_g(I|\theta)] \] (49)
on the boundary $\partial M_\theta$. Since the function $\Psi(I|\theta)$ is nonnegative, finite and differentiable on the manifold $M_\theta$, the entropy $S_g(I|\theta)$ should exhibit at least a stationary point where takes place the stationary condition (45). Since the scalar entropy $S_g(I|\theta)$ is a concave function, its stationary points can only correspond to local maxima. Let us suppose the existence of a least two stationary points $\bar{I}_1$ and $\bar{I}_2$, which can always be connected with a certain geodesic $I_g(t)$. According to constraint (27), the entropy production $\Phi(t)$ is a monotonous function along the curve $I_g(t)$. Consequently, $\Phi(t)$ should exhibits different values at the stationary points $\bar{I}_1$ and
$I_2$, which is absurdum since the entropy production $\Phi(t)$ identically vanishes for any stationary point of the scalar entropy $S_g(I|\theta)$:
\[
\Phi(t) = -\dot{I}(t)\psi[I(t)|\theta].
\]
(50)
Consequently, there exist only one stationary point that corresponds with the global maximum of the scalar entropy $S_g(I|\theta)$. □

**Theorem 4** Any hyper-surface of constant scalar entropy $S_g(I|\theta)$ is just the boundary of a $n$-dimensional sphere $S^n(\bar{I}, r) \subset M_\theta$ centered at the point $\bar{I}$ with global maximum entropy, whose radius $r = r(I|\theta) = D(I, \bar{I}|\theta)$ at the point $I$ allows to rephrase the scalar entropy as follows:
\[
S_g(I|\theta) = \mathcal{P}_g(\theta) - \frac{1}{2} r^2(I|\theta),
\]
(51)
with $n = \dim M_\theta$ being the dimension of the manifold $M_\theta$.

**Proof.** By definition, the vector field $\nu^i(I|\theta)$:
\[
\nu^i(I|\theta) = \frac{\nu^i(I|\theta)}{\psi(I|\theta)}
\]
(52)
is the unitary normal vector of the hyper-surface with constant scalar entropy $S_g(I|\theta)$. It is easy to verify that the vector field $\nu^i(I|\theta)$ obeys the geodesic equations:
\[
\nu^k(I|\theta) D_k \nu^i(I|\theta) = \frac{\nu^k(I|\theta)}{\psi(I|\theta)} \left[ \delta^i_k - \nu^i(I|\theta) \nu_k(I|\theta) \right] = 0.
\]
(53)
Hence, $\nu^i(I|\theta)$ can be regarded as the tangent vector:
\[
\frac{dI_g(s|e)}{ds} = \nu^i[I_g(s|e)|\theta]
\]
(54)
of geodesic family $I_g(s|e)$ with unitary affine parametrization centered at the point $\bar{I}$ with maximum scalar entropy $S_g(I|\theta)$, $I_g(s = 0|e) = \bar{I}$, where the parameters $e$ distinguish geodesics with different directions at the origin. The entropy production $\Phi(s|e)$ along any of these geodesics is given by the norm of the gradiental generalized forces with opposite sign:
\[
\Phi(s|e) = -\frac{dI^i(s|e)}{ds} \psi_i[I_g(s|e)|\theta] = -\psi[I_g(s|e)|\theta].
\]
(55)
Considering Eq.(13), the norm $\psi(I|\theta)$ can be related to the length $\Delta s$ of the geodesic connecting the point $I$ with point $\bar{I}$ with maximum scalar entropy, that is, the distance $D(I, \bar{I}|\theta)$ between the points $I$ and $\bar{I}$:
\[
\psi(I|\theta) = D(I, \bar{I}|\theta).
\]
(56)
According to the gaussian decomposition (40), the hyper-surface with constant scalar entropy $S_g(I|\theta)$ is also the hyper-surface where the norm of gradiential generalized forces $\psi(I|\theta)$ is kept constant, that is, the boundary of a $n$-dimensional sphere $S^n(\bar{I}, r)$ centered at the point $\bar{I}$ with maximum entropy. ■

**Corollary 5 (Completeness)** The knowledge of the metric tensor $g_{ij}(I|\theta)$ and the point $\bar{I}$ with maximum scalar entropy $S_g(I|\theta)$ determines the equilibrium distribution function (7).

**Commentary.** The radio $r(I|\theta)$ of the $n$-dimensional sphere $S^n(\bar{I}, r)$ of the theorem 4 and the invariant volume element $d\mu(I|\theta)$ are purely geometric notions derived from the knowledge of the metric tensor $g_{ij}(I|\theta)$ and the point $\bar{I}$ with maximum scalar entropy $S_g(I|\theta)$. Therefore, the scalar entropy $S_g(I|\theta)$ and all its associated thermo-statistical quantities represents geometric notions derived from the Riemannian structure of the manifold $M_\theta$. ■

**Corollary 6** The equilibrium distribution function (7) can be expressed as follows:

$$dp(r, q|\theta) = \frac{1}{\mathcal{Z}_g(\theta)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} r^2\right) dr d\Sigma_g(q|r, \theta)$$

where $d\Sigma_g(q|r, \theta)$ is the hyper-surface element:

$$d\Sigma_g(q|r, \theta) = \sqrt{\frac{g_{\alpha\beta}(r, q|\theta)}{2\pi}} dq.$$  

obtained from the metric tensor $g_{\alpha\beta}(r, q|\theta)$ associated with the projected Riemannian structure on the hyper-surface $\partial S^n(\bar{I}, r)$ of the $n$-dimensional sphere $S^n(\bar{I}, r) \subset M_\theta$.

**Proof.** Let us consider the geodesic family $I_g(s; e)$ derived as a solution of the problem (54). The quantities $e = \{e^i\}$ represent the asymptotic values of the unitary vector field $\nu^i(I|\theta)$ at the origin:

$$e^i = \lim_{s \to 0} I^i_g(s; e),$$

which characterize the direction of the given geodesic $I_g(s; e)$ at the point $\bar{I}$. These vectors can be parameterized as $e = e(q)$ using the intersection point $q$ of the geodesic $I_g(s; e)$ with the boundary $\partial S^n(\bar{I}, r)$ of a $n$-dimensional sphere $S^n(\bar{I}, r) \subset M_\theta$ of the theorem 4 which shall be hereafter referred to as the spherical coordinates. One can employ the quantities $\rho = (r, q)$ to introduce a spherical representation
\( \mathcal{R}_\rho \) centered at the point \( \bar{I} \) with maximum scalar entropy \( \mathcal{S}_g(I|\theta) \). The coordinate reparametrization \( \rho(I) : \mathcal{R}_I \rightarrow \mathcal{R}_\rho \) is given by the geodesic family \( I = I^i_\beta[r|e(q)] \):

\[
\nu^i(I|\theta) = \frac{\partial I^i_\beta[r|e(q)]}{\partial r}, \quad \tau^i_\alpha(I|\theta) = \frac{\partial I^i_\beta[r|e(q)]}{\partial q^\alpha}.
\]

(60)

The new \( n - 1 \) vector fields \( \tau^i_\alpha(I|\theta) \) are perpendicular to the unitary vector field \( \nu^i(I|\theta) \), \( \nu^i(I|\theta)\tau^i_\alpha(I|\theta) = 0 \), since the vectors \( \nu^i(I|\theta) \) and \( \tau^i_\alpha(I|\theta) \) are respectively normal and tangential to the boundary \( \partial S^{(n)}(\bar{I}|r) \) of the n-dimensional sphere \( S^{(n)}(\bar{I}|r) \). Consequently, the components of the metric tensor in this spherical coordinate representation are given by:

\[
g_{rr}(r, q|\theta) = 1, \quad g_{r\alpha}(r, q|\theta) = g_{\alpha r}(r, q|\theta) = 0, \\
g_{\alpha\beta}(r, q|\theta) = g_{ij}(I|\theta)\tau^i_\alpha(I|\theta)\tau^j_\beta(I|\theta),
\]

(61)

whose components \( g_{\alpha\beta}(r, q|\theta) \) describe projected Riemannian structure on the boundary \( \partial S^{(n)}(\bar{I}|r) \) of the n-dimensional sphere \( S^{(n)}(\bar{I}|r) \). Eq. (57) is just the expression of the distribution function (7) in the spherical representation \( \mathcal{R}_\rho \).

4. Statistical curvature

**Definition 7** A set of stochastic variables \( I \) exhibits a **reducible statistical dependence** when there exists at least a coordinate representation \( \mathcal{R}_I \) of the manifold \( \mathcal{M}_\theta \) where the distribution function \( dp(I|\theta) \) can be factorized into independent distribution functions \( dp^{(i)}(I^{(i)}|\theta) = \rho^{(i)}(I^{(i)}|\theta) dI^{i} \) as follows:

\[
dp(I|\theta) = \prod_{i=1}^{n} dp^{(i)}(I^{(i)}|\theta)
\]

(62)

for any \( I \in \mathcal{M}_\theta \). Conversely, the set of stochastic variables \( I \) exhibits an **irreducible statistical dependence**.

**Example 1** The stochastic variables \( X \) and \( Y \) described by the distribution function:

\[
dp(X,Y) = A \exp \left[-X^2 - Y^2 - XY\right] dX dY
\]

(63)

are statistical dependent. However, this same distribution function can be decomposed into independent distributions:

\[
dp(\zeta, \eta) = \sqrt{\frac{3}{2\pi}} \exp \left[ -\frac{3}{2} \zeta^2 \right] d\zeta \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \eta^2 \right] d\eta
\]

(64)
considering the following coordinate reparametrizations:
\[ X = \frac{1}{\sqrt{2}}(\zeta + \eta), \quad Y = \frac{1}{\sqrt{2}}(\zeta - \eta). \] (65)

This fact evidences that the distribution function (63) exhibits a reducible statistical dependence. Clearly, the correlation functions:
\[ \text{cov}(X, Y) = \langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle \] (66)
cannot provide an absolute measure of statistical dependence.

**Definition 8** A manifold \( M_\theta \) is **flat** when its geometric properties are fully equivalent to the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). Otherwise, the manifold is said to be **curved**.

**Theorem 5** The flat character of the manifold \( M_\theta \) implies the existence of a reducible statistical dependence among macroscopic observables \( I \), while its curved character implies the existence of an irreducible statistical dependence.

**Proof.** According to Riemannian geometry, the flat or curved character of the manifold \( M_\theta \) is unambiguously determined by the curvature tensor \( R_{ijkl}(I|\theta) \). In fact, the manifold \( M_\theta \) is flat whenever the curvature tensor \( R^{l}_{ijk}(I|\theta) \) vanishes for any \( I \in M_\theta \) [7]. The metric tensor \( g_{ij}(I|\theta) \) of an Euclidean manifold \( M_\theta \) can be written in terms of delta Kronecker matrix \( \delta_{ij} \):
\[ g_{ij}(I|\theta) = \delta_{ij} = -\partial^2 S_g(I|\theta) / \partial I^i \partial I^j \] (67)
using an appropriate coordinate representation \( R_I \). Such a metric tensor leads to the following decomposition of the equilibrium distribution function \( dp(\Theta|\theta) \):
\[ dp(I|\theta) = \prod_i \exp \left\{ - \left[ I^i - \bar{I}^i(\theta) \right]^2 \right\} \frac{dI^i}{\sqrt{2\pi}} \] (68)
into gaussian distributions, which evidences the reducible character of the statistical dependence among macroscopic observables \( I \) for the case of a flat manifold.

To show the relationship between curved character of the manifold \( M_\theta \) and the irreducible statistical dependence among the macroscopic observables \( I \), let us consider the spherical representation (57) of the distribution function (7). The demonstration reduces to verify the irreducible coupling between the radial coordinate \( r \) and the spherical coordinates \( q \) for any coordinate representation \( R_q \) of the boundary \( \partial S^{(n)}(I, r) \). The analysis can be restricted to the asymptotic behavior.
of the distribution functions for \( r \) small, that is, within the neighborhood of the point \( \bar{I} \) with maximum scalar entropy \( S_g(I|\theta) \).

Let us consider a coordinate representation \( \mathcal{R}_x \) of the manifold \( \mathcal{M}_\theta \) where the point \( \bar{I} \) corresponds to the origin of the coordinate frame \( x = 0 \), and the first partial derivatives of the metric tensor vanish at the origin. Moreover, let us assume the notation \( \bar{A} = A(x = 0|\theta) \) to simplify the mathematical expressions. The parametric family of geodesic \( x_g(s|e) \) admits an approximate solution in terms of power-series of the unitary affine parameter \( s \):

\[
x^i_g(s|e) = e^i(q)s - \frac{1}{6}s^3 \partial_t \bar{\Gamma}^i_{jk} e^j(q)e^k(q)e^l(q) + O(s^3),
\]

where \( \partial_t \bar{\Gamma}^i_{jk} \) is the partial derivative of the affine connection at the origin:

\[
\partial_t \bar{\Gamma}^i_{jk} = \frac{1}{2} \bar{g}^{im} \partial_l \left[ \partial_j \bar{g}_{mk} + \partial_k \bar{g}_{mj} - \partial_m \bar{g}_{kl} \right].
\]

Introducing the quantities \( \xi^i_\alpha(q) \):

\[
\xi^i_\alpha(q) = \frac{\partial e^i(q)}{\partial q^\alpha},
\]

the components of the projected metric tensor \( g_{\alpha\beta}(r,q) \) on the boundary \( \partial S^{(n)}(\bar{I},r) \) are given by:

\[
g_{\alpha\beta}(r,q) = r^2 \kappa_{\alpha\beta}(q) - \frac{1}{12} r^4 \bar{R}_{ijkl} X^{ij}_\alpha(q) X^{kl}_\beta(q) + ..., \]

where \( \kappa_{\alpha\beta}(q) = \bar{g}_{ij} \xi^i_\alpha(q) \xi^j_\beta(q) \) and \( \bar{R}_{ijkl} \) is the value curvature tensor at the origin, while the quantities \( X^{ij}_\alpha(q) \) are defined as:

\[
X^{ij}_\alpha(q) = e^i(q)\xi^j_\alpha(q) - e^j(q)\xi^i_\alpha(q).
\]

The previous expressions lead to the following approximation for the spherical representation of distribution function:

\[
dp(r,q|\theta) = \frac{1}{Z_g(\theta)} d\varrho(r) \left[ 1 - \frac{1}{24} r^2 \mathcal{F}(q|\theta) + O(r^2) \right] d\Omega(q).
\]

Here, \( d\varrho(r) \) and \( d\Omega(q) \) are two independent normalized distribution functions:

\[
d\varrho(r) = \frac{1}{2^{\frac{n}{2}-1} \Gamma \left( \frac{n}{2} \right)} \exp \left( -\frac{1}{2} r^2 \right) r^{n-1} dr,
\]

\[
d\Omega(q) = 2^{\frac{n}{2}-1} \Gamma \left( \frac{n}{2} \right) \sqrt{\frac{\kappa_{\alpha\beta}(q)}{2\pi}} \frac{dq}{\sqrt{2\pi}}.
\]
while \( \mathcal{F}(q|\theta) \) is a function on the spherical coordinates \( q \) that arises as a consequence of non-vanishing curvature tensor \( \bar{R}_{ijkl} \):

\[
\mathcal{F}(q|\theta) = \bar{R}_{ijkl}\alpha\beta(q)X_{\alpha}^{ij}(q)X_{\beta}^{kl}(q).
\]  \hfill (77)

According to Eq.(74), the curved character of the manifold \( \mathcal{M}_\theta \) does not allow to decouple the radial coordinate \( r \) and the spherical coordinates \( q \). Such a coupling is irreducible since the coordinate reparametrizations of the manifold \( \mathcal{M}_\theta \) only affect spherical coordinates \( q \) in the framework of the spherical representation of the distribution function (57).

**Corollary 7** The statistical curvature tensor \( R_{ijkl}(I|\theta) \) allows to introduce some local and global invariant measures to characterize both the intrinsic curvature of the manifold \( \mathcal{M}_\theta \) as well as the existence of an irreducible statistical dependence among the macroscopic variables \( I \). They are the **curvature scalar** \( R(I|\theta) \) introduced in Eq.(18), the **spherical curvature scalar** \( \Pi(r,q|\theta) \):

\[
\Pi(r,q|\theta) = g^{ij}(r,q|\theta)R_{ijkl}(r,q|\theta)X_{\alpha}^{ij}(r,q|\theta)X_{\beta}^{kl}(r,q|\theta)
\]  \hfill (78)

with \( X_{\alpha}^{ij}(r,q|\theta) \) being:

\[
X_{\alpha}^{ij}(r,q|\theta) = \nu^{i}(r,q|\theta)\tau_{\alpha}^{j}(r,q|\theta) - \nu^{j}(r,q|\theta)\tau_{\alpha}^{i}(r,q|\theta),
\]  \hfill (79)

which arises as a local measure of the coupling between the radial \( r \) and the spherical coordinates \( q \) in the spherical representation of the distribution function (57), and finally, the **gaussian Planck potential** \( \mathcal{P}_g(\theta) \) introduced in Eq.(47), which arises as a global invariant measure of the curvature of the manifold \( \mathcal{M}_\theta \).

**Proof.** As already commented, the curvature scalar is the only invariant associated with the first and second partial derivatives of the metric tensor \( g_{ij}(I|\theta) \). The consideration of the spherical representation of the distribution function (57) allows to introduce the normal \( \nu^{i}(r,q|\theta) \) and tangential vectors \( \tau_{\alpha}^{i}(r,q|\theta) \), as well as the projected metric tensor \( g_{\alpha\beta}(r,q|\theta) = g_{ij}(r,q|\theta)\tau_{\alpha}^{i}(r,q|\theta)\tau_{\beta}^{j}(r,q|\theta) \) associated with the constant scalar entropy hyper-surface \( \partial S(\alpha)(I|r) \). This framework leads to introduce the spherical curvature scalar \( \Pi(r,q|\theta) \) as a direct generalization of the spherical function \( \mathcal{F}(q|\theta) \) of the asymptotic distribution function (74).

The role of the gaussian Planck potential \( \mathcal{P}_g(\theta) \) as a global invariant measure of the curvature of the manifold \( \mathcal{M}_\theta \) can be easily evidenced starting from the spherical
representation of the distribution function (57). Integrating over the spherical coordinates $q$, one obtains the following expression for the gaussian partition function:

$$Z_g(\theta) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \exp \left( -\frac{1}{2} r^2 \right) \Sigma_g(r|\theta) dr,$$

where $\Sigma_g(r|\theta)$ denotes the invariant area of the constant scalar entropy hyper-surface $\partial S^{(n)}(\bar{I}|r)$. For a flat manifold, the invariant area $\Sigma_{\text{flat}}(r|\theta)$ is given by:

$$\Sigma_{\text{flat}}(r|\theta) = \frac{\sqrt{\pi} r^{n-1}}{2^{2-1} \Gamma \left( \frac{n}{2} \right)},$$

which is the area of an $n$-dimensional Euclidean sphere of radius $r$ normalized by the factor $(2\pi)^{(n-1)/2}$. Eq. (80) can be rewritten as follows:

$$Z_g(\theta) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \exp \left( -\frac{1}{2} r^2 \right) \sigma(r|\theta) \Sigma_{\text{flat}}(r|\theta) dr$$

introducing the geometric rate $\sigma(r|\theta)$:

$$\sigma(r|\theta) = \Sigma_g(r|\theta)/\Sigma_{\text{flat}}(r|\theta),$$

which characterizes how much differ the hyper-surface area of the sphere $S^{(s)}(\bar{I}, r)$ of the manifold $\mathcal{M}_\theta$ due to its intrinsic curvature. Since the gaussian partition function $Z_g(\theta) = 1$ for the case of the $n$-dimensional Euclidean space $\mathbb{R}^n$, a non-vanishing gaussian Planck potential $P_g(\theta)$ is a global invariant measure of the intrinsic curvature of the manifold $\mathcal{M}_\theta$.  

**Remark 3** According to Eq. (33) and the corollary, gaussian approximation (32) employed in Ruppeiner geometry of thermodynamics [4] only accounts for the local Euclidean properties of the Riemannian manifold $\mathcal{M}_\theta$ at the point $\bar{I}$ with maximum scalar entropy $S_g(\bar{I}|\theta)$. Consequently, the intrinsic curvature of the manifold $\mathcal{M}_\theta$ can be only manifested analyzing the system fluctuating behavior beyond the Gaussian approximation (32).

**Theorem 6** For a sufficiently large short-range interacting thermodynamic system, the gaussian Planck potential $P_g(\theta)$ can be estimated as follows:

$$P_g(\theta) = \frac{1}{6} R(\bar{I}|\theta) + O(1/\Lambda),$$

where $R(\bar{I}|\theta)$ is the statistical scalar curvature (18) evaluated at the point $\bar{I}$ with maximum scalar entropy $S_g(\bar{I}|\theta)$, while $\Lambda$ is the characteristic size parameter of the system.
Proof. Assuming that scalar entropy $S_g(I|\theta)$ and macroscopic observables $I$ exhibits an extensive growth with the size parameter $\Lambda$, it is easy to check the following scaling dependencies:

$$
\begin{align*}
S_g & \sim \Lambda \\
I & \sim \Lambda \\
\end{align*}
\Rightarrow \begin{cases} 
  g_{ij} \sim 1/\Lambda, g^{ij} \sim \Lambda, \\
  \Gamma^k_{ij} \sim 1/\Lambda, R_{ijkl} \sim 1/\Lambda^3, R \sim 1/\Lambda. 
\end{cases}
$$

(85)

Thus, the curvature scalar $R(\bar{I}|\theta)$ constitutes a size effect of order $1/\Lambda$ in the fluctuating behavior associated with the distribution function (7). According to Riemannian geometry [7], the asymptotic expression of the geometric rate $\sigma(r|\theta)$ for a small sphere can be expressed in terms of the curvature scalar $R(\bar{I}|\theta)$ at the center of the n-dimensional sphere $S^{(n)}(\bar{I}, r)$ as follows:

$$
\sigma(r|\theta) = 1 - \frac{1}{6n} R(\bar{I}|\theta) r^2 + O(r^2),
$$

(86)

which leads to the desirable result (84) performing the integration of Eq.(82). It is worth to remark that the estimation formula (84) is applicable whenever the curvature scalar $R(\bar{I}|\theta)$ be sufficiently small. This observation suggests its possible applicability beyond the framework of the short-range interacting systems whenever the gaussian approximation (32) turns a licit treatment to describe the statistical behavior of the macroscopic observables $I$.

5. Final remarks

As already evidenced, Riemannian formulation of classical statistical mechanics arises as a suitable extension of Ruppeiner geometry of thermodynamics. A main consequence of this proposal is the interpretation of the scalar entropy $S_g(I|\theta)$ and its associated thermo-statistical quantities as purely geometric notions derived from the Riemannian structure of the manifold of macroscopic observables $M_\theta$. Besides, the non-vanishing of the statistical curvature tensor $R_{ijkl}(I|\theta)$ of the manifold $M_\theta$ constitutes a direct indicator about the existence of irreducible statistical dependence for the equilibrium distribution function $dp(I|\theta)$ associated with the covariant redefinition of the Einstein postulate (7).

At first glance, thermo-statistics geometry shares some analogies with Einstein gravitation theory. In particular, the notion of statistical correlations could be regarded as the statistical counterpart of the concept of interaction. Thus, the relation between statistical curvature tensor $R_{ijkl}(I|\theta)$ and the existence of irreducible
statistical dependence for the equilibrium distribution function \( (7) \) is analogous to the relation between the spacetime curvature tensor \( R_{ijkl}(t, r) \) and the irreducible character of gravitation (while the existence of inertial forces can be avoided with an appropriate selection of the reference frame, gravitational forces are irreducible because of this universal interaction is just the manifestation of the spacetime curvature).

Before to end this section, it is worth to comment some open questions that deserve a detailed analysis in forthcoming contributions. The present work inspires a geometric reformulation of classical fluctuation theory, which should lead to the derivation of a certain set of covariant fluctuation theorems and dynamical equations to describe the system relaxation. Similarly, it is worth to analyze the implications of thermo-statistics geometry on the study of phase transition phenomena as well as its possible extension to the framework of quantum distribution functions. According to the relationship between statistical curvature and the existence of irreducible statistical dependence for the distribution functions, an extension of the present proposal to quantum statistical distributions should provide an alternative characterization for the notion of quantum entanglement.

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References

[1] Fisher R A 1922 Philosophical Transactions, Royal Society of London, (A), 222, 309-368.
[2] Rao C R 1945 Bull. Calcutta Math. Soc. 37, 81-91.
[3] Amari Sh 1990 Geometrical Methods in Statistics: 22 Lecture notes in statistics, (Springer Verlag).
[4] Ruppeiner G 1979 Phys. Rev. A 20, 1608; 1995 Rev. Mod. Phys. 67, 605 and references therein.
[5] Landau L D and Lifzhitz E M 1977 Statistical Physics (Pergamon, New York).
[6] Reichl L E 1980 A modern course in Statistical Mechanics, (Univ. Texas Press, Austin).
[7] Berger A 2002 A panoramic view of Riemannian geometry, (Springer).