The short time (large energy) behavior of the Sachdev-Ye-Kitaev model (SYK) is one of the main motivations to the growing interest garnered by this model. True chaotic behavior sets in at the Thouless time, which can be extracted from the energy spectrum. In order to do so, it is necessary to unfold the spectrum, i.e., to filter out global tendencies. Using a simple ensemble average for unfolding results in a parametrically low estimation of the Thouless energy. By examining the behavior of the spectrum as the distribution of the matrix elements is changed into a log-normal distribution it is shown that the sample to sample level spacing variance determines this estimation of the Thouless energy. Using the singular value decomposition method, SVD, which filters out these sample to sample fluctuations, the Thouless energy becomes parametrically much larger, essentially of order of the bandwidth. It is shown that the SYK model in non-self-averaging even in the thermodynamic limit which must be taken into account in considering its short time properties.

### I. INTRODUCTION

The interplay between disorder and interactions in quantum systems has been a central theme in condensed-matter physics for the last half century. Recently the Sachdev-Ye-Kitaev (SYK) model \[1,2\], has garnered much interest in the fields of quantum gravity and quantum field theory \[3,4\]. A key feature of the model is that it follows random matrix theory (RMT) behavior, as is manifested in the chaotic behavior of its energy spectrum.

The SYK model first appeared in the context of spin liquids \[1\] and then in string theory \[5\] and quantum gravity \[4\]. The SYK model is known to be chaotic \[6-8\], showing a Wigner-like behavior of the energy spectra. Once the SYK model is perturbed by a single-body random term which mimics diagonal disorder in the Anderson model \[9-13\], or several SYK dots are coupled by single-body random terms \[14,20\], a transition from metallic (chaotic) to insulating behavior occurs which leads to a Wigner to Poisson crossover of the statistical properties of the spectra.

While studying nuclear and condensed-matter systems it became clear that many physical systems exhibits universal behavior at long times (short energy scales) \[21-26\]. Nevertheless, universality may break at shorter times (large energy scales) for which a particle had no time to sample the entire phase space of the system and its behavior depends on local non-universal features \[27\]. Thus, in the context of disordered metals the scale for which the metallic spectrum deviates from the universal behavior is known as the Thouless energy, \(E_{Th} = \hbar D/L^2 = g \Delta\) (\(D\) is the diffusion constant, \(L\) is the linear dimension, \(g\) is the dimensionless conductance, \(\Delta\) the average level spacing) and the Thouless time \(t_{Th} = \hbar/E_{Th} = L^2/D\).

The question whether an analogue of the Thouless energy manifests itself in the SYK model has surfaced in Ref. \[6\] where Garcia-Garcia and Verbaarschot have studied (among other things) the variance of the number of levels as function of the size of an energy window \(E\), denoted by \(\langle \delta^2 n(E) \rangle\) (where \(\langle \ldots \rangle\) represents an ensemble average and \(n(E)\) is the number of levels within \(E\)). A departure from the RMT behavior is apparent above a certain value of \(E\), which quite naturally was identified with the Thouless energy. Moreover, at larger energy windows, the number variance adopts a quadratic form \(\langle \delta^2 n(E) \rangle = a + b(n(E))^2\). Evidence for the Thouless energy has also been seen for other measures such as off diagonal expectation values \[28\]. In Ref. \[19\] the origin of the Thouless energy for the SYK was identified as the relaxation of modes prevailing at shorter scale. For disordered metals these modes are known as the diffusion modes. For SYK it was suggested that similar modes can be constructed, where the number of such modes is connected to the number of independent interaction terms in the SYK model which is much smaller than the size of the Hilbert space. This leads to an energy scale which determines the Thouless energy. Thus, the energy scale for the departure of the SYK model level number variance from the RMT prediction is determined by the scale of the sample to sample fluctuations of the ensemble.

For a single disordered realization of the SYK model all the terms have just a single variance scale, which is well behaved (box or Gaussian) and therefore the origin of the additional energy scale determining the Thouless energy is not obvious. As noted recently by Jia and Verbaarschot \[30\] this deviation from RMT behavior has its root in large scale sample to sample fluctuations of the spectrum. They attribute these fluctuations to the relatively small number of independent random variables contributing to the SYK Hamiltonian. When these small number of long-wavelength sample to sample fluctuations are parameterized by terms of Q-Hermite orthogonal polynomials and removed from the spectrum of a particular realization a pure RMT behavior is retained up to a very large energy scale.

The influence could be quantified \[30\] by estimating the energy scale for which the small number of independent random variables will change the number variance. Using the notation for the complex SYK (CSYK) half-filled model defined in the next section, where \(L\) is the number of sites, the size of the Hilbert space is \(\langle L/2 \rangle \sim 2^L\).
number of independent variables is \( \frac{2L}{n} \sim L^4 \) leading to a variance of \( L^{-2} \) in any observable. Thus one expects the number of levels \( n \) to deviate significantly \( O(1) \) from RMT predictions on a scale of \( n \sim L^2 \). A similar result emerges from the calculation in Ref. [39] where the coefficient \( b \) in the number variance was estimated as \( b \sim L^{-4} \) thus becoming significant at \( n \sim L^2 \).

Although the deviation from RMT is shifted to larger \( n \) as the system size \( L \) increases, the proportion of levels following RMT predictions out of the total number of states goes to zero as \( L^2/2^L \). On the other hand, one would expect that after filtering out long wavelength sample to sample fluctuations the RMT behavior will be followed for a finite portion of the spectrum. Thus, one expects the Thouless energy to crucially depend on whether one simply averages over an ensemble or takes into account the sample specific fluctuations. This is a hallmark of a non-self-averaging system [31]. The number variance for the ensemble average is different than the number variance adjusted to a particular sample.

Hence, the energy scale for which the spectrum departs from the RMT predictions crucially depends on the the unfolding, i.e., the method by which the averaged over the density of states is performed. Estimating the local density of states by a simple ensemble average will give a different value than an unfolding method that is able to take into account sample specific global behavior of the spectrum. In recent studies it has been shown that [32,33], these sample specific long ranged features of the spectrum can be identified by the singular value decomposition (SVD) procedure. As detailed below, similar in a sense to Fourier transform, SVD actually reconstructs the energy spectrum of each realization by a sum over a series of SVD amplitudes multiplied by the corresponding SVD mode. Unlike the Fourier transform, the amplitudes of the SVD are identical for all realizations while the modes are realization specific. Generally, plotting the amplitudes from large to small (known as a Scree plot) shows that the largest amplitudes (usually \( O(1) \) modes) are orders of magnitude larger than the rest, while the following modes amplitudes obey a power law. The largest amplitudes modes depict the very long wave length behavior of the spectrum, while the modes whose amplitudes follow a power law capture the shorter range properties. Thus SVD is a very natural method to examine the SYK spectrum behavior, and uncover the realization specific universal properties of the spectrum.

One may conclude that there are two possible energy scales for the deviation of the spectrum of the SYK model from RMT predictions. The first is the energy scale corresponding to the deviation from the ensemble average, which in self averaging systems such as disordered metals is the Thouless energy. The second energy scale corresponds to the deviation from RMT when the sample to sample long wavelength fluctuations are removed from the spectrum. This sample specific parameterization of these long range behavior can be done using Q-Hermite orthogonal polynomials [30] or by SVD [32,33], or probably by other method. The energy scale of these fluctuations is another relevant energy scale which for self-averaging systems such as disordered metals is equivalent to the Thouless energy obtained from the ensemble average [39]. A similar result emerges from the calculation in Ref. [19] where the Thouless energy, \( E_{Th} \), for the case where the spectrum is unfolded by a local average over all realizations, while the realization adjusted Thouless energy \( E_{Th,*} \) is obtained using a realization adapted unfolding method.

Here we will show that one can tweak the behavior of the CSYK model to a more non-self-averaging behavior by changing the distribution of the off-diagonal to a log-normal distribution. Thus it is possible to enhance the sample to sample fluctuations and study its influence on \( E_{Th} \) and \( E_{Th,*} \). Such a wide distribution was not previously considered for the SYK model and should help to clarify the divergence of \( E_{Th,*} \) from \( E_{Th} \).

The paper is arranged as follows: CSYK is defined in Sec. [II]. Corroborating the expected behavior for short range energy scales is performed in Sec. [III]. The universal statistics 4-fold symmetry as function of the system size is observed. In Sec. [IV] long range energy scales are probed by the number variance. The spectrum is unfolded by using the local ensemble averaged level spacing, RMT predictions hold only up to \( E_{Th} \), which becomes smaller as the log-normal distribution acquires a thicker tail towards larger values. Above \( E_{Th} \) the variance increases quadratically as function of the number of levels in the energy window. In Sec. [V] we switch to the SVD analysis. This analysis reveals that each realization has a distinct long range correlated level spacing structure. Thus, one should adapt the unfolding for each realization. After a realization adapted unfolding the number variation follows the RMT prediction for much larger energy scales, i.e., \( E_{Th,*} > E_{Th} \). Actually, \( E_{Th} \) could be estimated from the sample to sample variance in the level spacing. In Sec. [VI] these results are discussed in the limit of large CSYK systems, showing the in the thermodynamic limit the SYK is non-self-averaging.

### II. COMPLEX SYK MODEL

Here we use the the complex spinless fermions version of the SYK model given by the following Hamiltonian:

\[
\hat{H} = \sum_{i>j>k>l} L V_{i,j,k,l} \hat{c}_i \hat{c}_j \hat{c}_k \hat{c}_l, \tag{1}
\]

the couplings \( V_{i,j,k,l} \) are complex numbers, where the real and imaginary components are independently drawn from an identical distribution. We study here two different distribution: The first is a box distribution between \(-L^{-3/2}/2 \ldots L^{-3/2}/2 \), where \( L \) is the number of sites.
The second distribution is the log-normal distribution

\[ P(V) = (A/V_c)e^{-\frac{\ln^2(V/V_{typ})}{2\sigma^2(V/V_{typ})}}, \tag{2} \]

with \( V_{typ} \sim K^{-\gamma/2} \), \( K \) is the size of the Hilbert space, \( A \) a normalization and \( \gamma, \sigma \) are parameters. For simplicity here we shall set \( \sigma = 0.5 \), while \( \gamma \) is varied \([39]\). Thus, the log-normal distribution becomes wider and more skewed as \( \gamma \) increases. The number of fermions is conserved and we considered the \( N = L/2 \) sector for even \( L \) and \( N = (L + 1)/2 \) sector for odd \( L \). Resulting in a Hilbert space size of \( K = \left( \frac{N}{L} \right)^{L/2} \), and a matrix size of \( K \times K \).

### III. SHORT ENERGY SCALES

As a first step we would like to probe the nearest neighbor level spacing statistics of the CSYK box distributed in order to establish the extended regime of this model. One expects that in the extended regime short energy scales (long times) follow the Wigner statistics. The short energy scale statistics is revealed by the ratio statistics, which depends on \( \gamma \). For the CSYK model, the spectrum of Eq. (1) shows statistics which depends on \( L \) through the number of fermions is conserved and the number of realizations of disorder and half of the eigenvalues around the middle of the band. For the Wigner distribution \( r_s \approx 0.5307 \) for the GOE symmetry, \( r_s \approx 0.5996 \) for GUE and \( r_s \approx 0.6744 \) for GSE. \([40]\).

An interesting behavior emerges for the CSYK. It is known that as a consequence of the symmetries of the SYK model, the spectrum of Eq. (1) shows statistics which depends on \( L \) \([38]\). For \( L \mod 4 = 0 \) the statistics are GOE, for \( L \mod 4 = 2 \) the statistics are GSE, and for \( L \mod 4 = 1, 3 \) the statistics are GUE. This is indeed seen in Fig. 1 where \( r_s \) averaged over the middle half of the energy spectrum, for different sizes \( L = 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 \) and number of fermions \( N = 4, 5, 6, 7, 8, 9, 10 \) is plotted. In all cases we exactly diagonalize the corresponding \( K \times K \) matrix and average over 3000 realizations of disorder (except for the largest size for which only 100 realization were computed). It can be seen that the expected 4-fold symmetry is followed.

Concentrating on the \( L = 16 \) with \( N = 8 \) systems, we investigate the role played by changing the distribution from the box distribution to a log-normal. Setting \( p = 0.5 \) and increasing \( \gamma \) we sweep through the values \( \gamma = 1, 1.5, 2, 2.5, 3, 4, 5 \). For all these values \( r_s \approx 0.5307 \pm 0.001 \). Thus, GOE universal behavior on short energy scales is perfectly followed.

![FIG. 1. The nearest neighbor level spacing statistics as manifested in the behavior of the ratio statistics \( r_s \) for different system sizes \( L \) of the CSYK with a box distribution are indicated by the black circles. The number of fermions is \( N = L/2 \) for even \( L \) and \( N = (L + 1)/2 \) for odd \( L \). The \( r_s \) values expected for GOE (dashed red) GUE (dashed green) and GSE (dashed blue) are marked. The expected 4-fold symmetry is seen.](image)

### IV. LOCAL ENSEMBLE UNFOLDING

As discussed, the practice of determining the Thouless energy is fraught with difficulties. In order to compare any spectrum to RMT predictions, one must recast the spectrum such that it will exhibit an averaged constant density of states, i.e., a constant level spacing throughout the region examined. What is the averaging procedure? Usually, one averages the level spacing over an ensemble of disordered realizations in a given region and then reconstructs a particular spectrum such as the level spacing is on the average equal to 1 everywhere. Specifically, the averaged level spacing for the \( i \)-th level is \( \Delta_i = \langle E_{i+m} - E_{i-m} \rangle / 2m \) (where \( m = 0 \) here chosen as \( m = 5 \)), and the unfolded spectrum for the \( j \)-th realization is \( \tilde{\epsilon}_j = \tilde{\epsilon}_{j-1} + (E_j - E_{j-1}) / \Delta_j \). For brevity we shall call this unfolding procedure local ensemble unfolding.

Implementation of this local unfolding procedure for the level number variance of CSYK with a box distribution and for a log-normal distribution \( \gamma = 1, 1.5 \) results in the behavior depicted in the inset of Fig 2, which is in agreement with the behavior observed in Refs. \([30, 31]\). Here the number variance begins by following the GOE predictions and grows quadratically for larger \( \langle n \rangle \). The Thouless energy corresponds to the point where the number variance deviates from GOE predictions. As an estimate of the Thouless energy we chose the point for which the variance deviates by an arbitrary amount (set as 0.2), resulting in \( E_{Th} \sim 95\Delta \), for the box distribution,
b = 1

ble unfolding, fitted for larger values by a = 1, blue

⟨ the GOE prediction = 1, and

γ

box distribution,

constant of 0

sert: zoom into smaller values of E

intersection between the variance and the dashed cyan line to

are are random, SYK models have just

like typical RMT models for which all non-diagonal terms

of

γ

Δ = (⟨

i

j

⟩

3

5

2

δ

2

n

Box

γ=1

γ=1.5

GOE

a+bn

2

0 20 40 60 80 100

<n>

0

0.5

1

1.5

2

<δ

2

n>

E

V. SINGULAR VALUE DECOMPOSITION

SVD can be used to characterize the features of the spectrum, for example on what scale does it follow RMT predictions [32][35]. For the analysis, one tabulates P

eigenvalues around the center of the band of M realizations of disorder as a matrix X of size M × P where X

is the p level of the n-th realization. The matrix X is de-

composed to X = UΣVT, where U and V are M × M and

P × P matrices correspondingly, and Σ is a diagonal ma-

trix of size M × P and rank r = min(M, P). The r diagonal

elements of Σ are the singular values amplitudes (SV)

σk of X. All σk are positive and therefore may be ordered by

their size σ1 ≥ σ2 ≥ . . . σr. The Hilbert-Schmidt norm of the matrix ||X||HS = √TrXTX = ∑ λk (where

λk = σk2). The matrix X could be written as a series

composed of matrices X(k), where Xij(k) = UikVjk and

Xij = ∑k σkXij(k). This series in an approximation of

matrix X, where the sum of the first m modes gives a matrix

X = ∑k=1m σkX(k), for which ||X||HS = ∑ X||HS is minimal.

The idea is that low modes capture the long-

wavelength fluctuations of the spectrum, while higher

modes sample the shorter wavelength fluctuations. If a

distinct pattern of behavior of the amplitude as func-
tion of the mode number can be seen for a particular
range of k, it is meaningful to discuss different behaviors
at different energy scales. Indeed, examining the scree
plot of the singular values λk for box distribution and

log-normal distribution with different values of γ and a

fixed p = 0.5, for L = 16, N = 8, with M = 4096 re-
alizations and P = 4096 eigenvalues around the middle
of the band one sees two distinct regions. The lowest
modes (k = 1, 2 for box and γ = 1; k = 1, 2, 3, 4 for

1.5 ≤ γ ≤ 2.5; and k = 1, 2, 3, 4, 5, 6 for 3 ≤ γ ≤ 5) am-

plitudes are much larger than all the other modes, and
determine the very long-wave behavior of the spectrum.
The bulk of the modes for larger k follow a power-law be-

havior (λk ~ κγ) with α = 1, as expected for Wigner-

Dyson statistics [32][34].

In order to illustrate the difference between the local
ensemble unfolding and unfolding using the lower modes of
the SVD it is useful to examine the difference in the
behavior of level spacing of the i-th level, Δi, obtained
by each method. While for the local ensemble unfold-
ing the level spacing Δi is averaged over all realizations
and therefore is not realization dependent, for the SVD
unfolding the i-th level spacing of the j-th realization

Δij = (Δij,m − Δij,m)/2m, where Δij,m = ∑k=1,2,...,m σkUikVjkT,
is realization dependent. As can be seen in Fig. 3 there is a noticeable difference between the realization specific
level spacing Δij and the ensemble average Δi. This dif-
derence becomes larger as γ increases, i.e., as the width
of the log-normal distribution increases. Moreover, the
difference, Δij − Δi, is long-range correlated for a given
realization across thousands of levels. Thus, it makes
sense to define the typical spacing difference between re-
alizations δΔ = (⟨(Δij − Δi)2⟩)1/2. For L = 16, N = 8
with M = 4096 realizations and P = 4096 levels around
the center of the band one gets δΔ = 4.44 × 10−3Δ for

the box distribution δΔ = 3.52 × 10−2Δ for γ = 1 and

δΔ = 6.61 × 10−2Δ for γ = 1.5.
The long range correlations within a sample are very significant, and different log-normal distributions with \( \gamma = 1, 1.5, 2, 2.5, 3, 4, 5 \) for \( L = 16, N = 8 \), with \( M = 4096 \) realizations and \( P = 4096 \) eigenvalues around the middle of the band. The square amplitude of the singular vale \( \lambda_k \) indicated by the symbols. The cyan line corresponds to 

\[ \lambda_k \sim k^{-\alpha}, \quad \text{with } \alpha = 1, \text{ as expected for a spectrum which follows Wigner-Dyson statistics.} \]

The SVD unfolded level spacing, \( \Delta_i \), is realization dependent. Five individual realizations for each distribution are shown (red, green, blue, yellow, brown curves). As \( \gamma \) increases the sample to sample fluctuations increases. It is clear that the long range correlations within a sample are very significant.

\[ \Delta_i \sim \langle \Delta \rangle \sim (n-i)^2, \quad \text{taking into account that after unfolding } \langle n \rangle = n \text{ and for a typical realization } n \sim n(1 + \delta\Delta/\Delta). \]

Thus, \( \langle \delta^2 n \rangle \sim \langle n \delta\Delta/\Delta \rangle \), and after plugging in the above mentioned values of \( \delta\Delta \) one obtains 

\[ \langle \delta^2 n \rangle \sim 1.97 \times 10^{-5} n^2 \quad \text{for the box distribution,} \]

\[ \langle \delta^2 n \rangle \sim 1.24 \times 10^{-3} n^2 \quad \text{for } \gamma = 1 \text{ and } \langle \delta^2 n \rangle \sim 4.36 \times 10^{-5} n^2 \quad \text{for } \gamma = 1.5 \text{ in good agreement with the numerical value quoted above.} \]

Moreover, using our previous definition of the Thouless energy as the energy for which the deviation from RMT results becomes larger than some threshold, \( b(E_{TH}/\Delta)^2 = 0.2 \), and using \( b = (\delta\Delta/\Delta)^2 \), one gets

\[ E_{TH} = 0.44\Delta^2/\delta\Delta \text{ resulting in } E_{TH} = 100\Delta, 18.5\Delta, 6.7\Delta \text{ for box and log-normal distributions with } \gamma = 1, 1.5 \text{ correspondingly, in good agreement with the results in Fig. 3.} \]

One concludes that the main contribution to the local ensemble average Thouless energy, \( E_{TH} \), comes from the sample to sample long range fluctuations which can be quantified by \( \delta\Delta \), the typical level spacing difference between the different realizations in the ensemble.

Thus, using the sample specific level spacing from the SVD (i.e., \( \Delta_i^1 \)) for unfolding will eliminate the sample

\[ \langle \delta^2 n(E) \rangle = (2/\pi^2) \ln(\langle n(E) \rangle) + 0.44. \]
The level number variance, $\langle \delta^2 n(E) \rangle$, as function of the energy window $E$ for box distribution using local ensemble unfolding (black symbols) and SVD unfolding (red symbols) for $L = 15$, $N = 8$, $P = 3000$ and $M = 3000$. The cyan dashed lines line is the GUE prediction $\langle \delta^2 n(E) \rangle = (1/\pi^2) \ln(\langle n(E) \rangle) + 0.35$ for finite size SYK systems. The local ensemble unfolding results in $E_{Th} \sim 90\Delta$, and a fit to a quadratic behavior of the form $a + bn^2$ leads to $b = 2.48 \times 10^{-5}$, while for the SVD unfolding $E_{Th} \sim 800\Delta$. Comparing with the typical spacing difference between realizations for the box distribution which for $L = 15$ equals to $\delta \Delta = 5.17 \times 10^{-3}$, one would estimates $E_{Th} = 0.44\Delta^2/\delta \Delta \sim 85\Delta$ and $b = (\delta \Delta/\Delta)^2 \sim 2.67 \times 10^{-3}$, both in good agreement with the numerical results. As explained above for $E_{Th} = r\Delta/2k_{Th}$, and here $r = 3000$ and $k_{Th} \sim 2$, resulting in $E_{Th} \sim 750\Delta$, again in good agreement with the results.

VI. DISCUSSION

Much of the fascination with the SYK is connected to its chaotic behavior. Since short time scales are associated with large energy scales, deviation from GOE behavior of the spectra on large energy scales indicate non-chaotic behavior on short time scales. Estimating the time for which the chaotic behavior emerges is relevant to the estimation of the scrambling time of the SYK models, which motivates its application to studies of quantum gravity in black holes [41]. Ensemble averaging also plays an important role in the duality between SYK and classical general relativity.

For self-averaging systems there is no difference between averaging over the ensemble or averaging over a large single realization. As discussed, for finite size SYK samples, there is a huge difference between the Thouless time for a realization adjusted unfolding compared to the ensemble averaged unfolding, a difference which is larger as the distribution of the interaction elements is more skewed.

Nevertheless, it is relevant to extrapolate to infinite systems ($L \to \infty$) in order to see whether this difference persists. SYK has three time scales [22]: (i) band structure time $t_B$ associated with the band width, $B$. (ii) Thouless time, $t_{Th}$ or $t_{Th}^*$, on which much of the paper concentrated and (iii) Heisenberg time, $t_H$. Since the band width depends linearly on $L$, $t_B = h/B \sim h/L$. Thus, the Heisenberg time $t_H = h/\Delta$ which $\Delta \sim B^{2/3}$, is $t_H = h^{2/3}/L$. The Thouless time for the ensemble average is $t_{Th} = h/E_{Th}$, where $E_{Th} \sim \Delta L^2$ and resulting in $t_{Th} = h^{2/3}/L$. For the realization adjusted unfolding, $E_{Th} \sim 2\Delta B/k_{Th}$, with $k_{Th} \sim O(1)$, thus, $t_{Th} \sim k_{Th}/h/B \sim k_{Th}/L$. Hence, $t_{Th} > t_B > t_{Th} \gg t_H$, and while the realization adjusted Thouless time $lim_{L \to \infty} t_{Th} \to 0$, the ensemble averaged Thouless time $lim_{L \to \infty} t_{Th} \to \infty$. The difference between $t_{Th}$ and $t_{Th}^*$.
increases as the distribution is more skewed (Figs. 2 and 3).

This leads to the conclusion that one can not determine the behavior of the SYK model by ensemble average for times shorter than $t_{TH}$ since shorter times are non-self-averaging. Moreover, these shorter times $(t_{TH} < t < t_{RN})$ correspond to a parametrically large span of times. Therefore, when one wishes to study the transition from chaotic to localized behavior in modified SYK models [9][20] for which the number of independent random variables remain low, sample to sample fluctuations are expected to remain important and non-self-averaging at short times should be considered. In principal, although the transition occurs at long times nevertheless these effects could obscure the transition point and influence the perceived nature of the metallic regime. This will be considered in future studies.

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