PIERI ALGEBRAS FOR THE ORTHOGONAL AND SYMPLECTIC GROUPS

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Abstract. We study the structure of a family of algebras which encodes an iterated version of the Pieri Rule for the complex orthogonal group. In particular, we show that each of these algebras has a standard monomial basis and has a flat deformation to a Hibi algebra. There is also a parallel theory for the complex symplectic group.

1. Introduction

The standard monomial theory of Hodge, describing the homogeneous coordinate ring of the Grassmannian and its natural extension to the case of the flag manifold for $GL_n = GL_n(\mathbb{C})$ (aka, the flag algebra), is a landmark at the border between algebraic geometry and representation theory. Hodge’s results inspired repeated attempts both to generalize them and to find an abstract viewpoint from which they could be understood (Mu, La and the references therein). Important progress toward the second goal was made by Gonciulea and Lakshmibai (GL), who showed that the $GL_n$ flag manifold could be described as a flat deformation of a toric variety. Such a result has since been established for the flag algebras of all reductive groups, and even for all multiplicity-free actions (AB, Ca).

However, simply saying that a flag manifold is a flat deformation of a toric variety does not capture all the structure inherent in Hodge’s original description, which also identified a finite family of polynomial subrings of the $GL_n$ flag algebra, and showed that the flag algebra was almost a direct sum of these subrings.

An affine toric variety has a semigroup ring as its coordinate ring. The semigroup in question has finite index in a lattice cone - the intersection of $\mathbb{Z}^n$ with a rational polyhedral convex cone in $\mathbb{R}^n$. In this context, it was remarked in GL, Ki1, KM that the extra structure of Hodge’s theory could be expressed by describing the relevant cone (known as the Gelfand-Tsetlin cone, or equivalently Gelfand-Tsetlin polytope) as a lattice cone attached to a Hibi ring: the cone of non-negative integer-valued, order-preserving functions on a certain partially ordered set (dubbed the Gelfand-Tsetlin poset (Ho2, Ki1)).

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Hibi rings constitute an attractive class of algebras. Polynomial rings are Hibi rings, and all Hibi rings share the concrete and visible nature of polynomial rings. In particular, all Hibi rings can be explicitly described in terms of generators and relations, and they all have an explicit “abstract standard monomial theory”, describing them as almost direct sums of certain polynomial subrings ([Ho2]). However, instead of a unique prototype in each dimension, Hibi rings constitute a rich class with many examples in each dimension.

It is therefore natural to ask, whether other rings besides the $GL_n$ flag algebra occurring in representation theory can be described as Hibi rings. A positive answer for the $Sp_{2n}(\mathbb{C})$ flag algebra was already given in [Ki1]. For the $SO_n(\mathbb{C})$ flag algebra, a stable range case was studied in [Ki2].

Standard monomial theory for $GL_n$ is closely linked with branching - the restriction of representations of $GL_n$ to $GL_{n-1}$, then to $GL_{n-2}$, and so forth. A reciprocity law links branching from $GL_n$ to $GL_{n-1}$ to decomposing a tensor product $\rho \otimes S^m$, where $\rho$ indicates any irreducible representation of $GL_n$, and $S^m = S^m(\mathbb{C}^n)$ is a symmetric power of the standard representation on $\mathbb{C}^n$. The decomposition of such tensor products turns out to be the same as the case of the Schubert calculus named after Mario Pieri, so the result is known as the Pieri rule.

Successive branchings correspond to decomposition of multiple tensor products

$$\rho \otimes S^{m_1} \otimes S^{m_2} \otimes S^{m_3} \otimes \cdots \otimes S^{m_k}. \quad (1.1)$$

Because of the connection with the Pieri rule, we call the algebra that describes these tensor products an iterated Pieri algebra, or simply a Pieri algebra. The same techniques that show that the flag algebra of $GL_n$ is a flat deformation of a Hibi ring also establish that a Pieri algebra is a flat deformation of a Hibi ring.

In this paper, we consider the problem of describing certain families of tensor products for the orthogonal and symplectic groups, analogous to the products (1.1) for $GL_n$. In the manner of [HL1, HTW2], we construct an algebra that describes these tensor products, which we again call a Pieri algebra. Our main result is that each of these algebras has a standard monomial basis and has a flat deformation to a Hibi ring.

Here is a slightly more detailed overview of the paper. The irreducible rational representations of $O_n$ are labeled by Young diagrams such that the sum of the lengths of the first two columns is at most $n$ ([Wy, GW, Ho1]). For such a Young diagram $D$, let $\sigma_n^D$ be the corresponding irreducible representation of $O_n$. For positive integers $k$ and $\ell$ such that $2(k+\ell) < n$ (we call this the stable range condition), we construct an algebra $\mathfrak{A}_{n,k,\ell}$ with the following properties: $\mathfrak{A}_{n,k,\ell}$ carries a multigrading, and each of its homogeneous components can be identified with the space of $SO_n$ highest weight vectors of a certain weight in a tensor product of the form

$$\sigma_n^D \otimes \sigma_n^{(p_1)} \otimes \sigma_n^{(p_2)} \otimes \cdots \otimes \sigma_n^{(p_\ell)} \quad (1.2)$$

where $D$ is a Young diagram with at most $k$ rows and $p_1, \ldots, p_\ell \geq 0$. Thus, the multiplicity of an irreducible representation of $O_n$ in this tensor product coincides
with the dimension of the corresponding homogeneous component. In this sense, the algebra encodes an iterated version of the Pieri rule for $O_n$, so we call the algebra $A_{n,k,\ell}$ a Pieri algebra for $O_n$.

The structure of $A_{n,k,\ell}$ is closely related to a finite poset $\tilde{\Gamma} = \tilde{\Gamma}(k, \ell)$ which arises from a consideration of the multiplicities in tensor products of the form \((1,2)\). Let $\Omega = \Omega(k, \ell)$ be the set of order preserving functions on $\tilde{\Gamma}$ with nonnegative integral values. Then $\Omega$ can be identified with a lattice cone. In particular, $\Omega$ is a semigroup with respect to the usual addition of functions and it has a distinguished finite set $\mathcal{G} = \mathcal{G}(k, \ell)$ of generators. Using $\mathcal{G}$, we identify a set $S = S(k, \ell)$ of generators for the algebra $A_{n,k,\ell}$. The generating set $\mathcal{G}$ of $\Omega$ has a structure of a distributive lattice, so it gives rise to a Hibi algebra (\cite{Hi, Ho2}). This Hibi algebra is isomorphic to the semigroup algebra $\mathbb{C}[\Omega]$ on $\Omega$. The partial ordering on $\mathcal{G}$ also induces a partial ordering on $S$.

**MAIN THEOREM.** Let $n$, $k$ and $\ell$ be positive integers such that $2(k + \ell) < n$.

(a) The algebra $A_{n,k,\ell}$ has a standard monomial theory for $S$, that is, the monomials in elements of the maximal chains in $S$ form a basis for $A_{n,k,\ell}$.

(b) There is a flat family of complex algebras with general fibre $A_{n,k,\ell}$ and special fibre $\mathbb{C}[\Omega]$.

We have also obtained the following multiplicity formula from the structure of the algebras $A_{n,k,\ell}$:

**The Iterated Pieri Rule for $O_n$.** Let $k, \ell$ and $n$ be positive integers such that $2(k + \ell) < n$, $D$ a Young diagram with at most $k$ rows, and $P = (p_1, \ldots, p_\ell) \in \mathbb{Z}_{\geq 0}^\ell$. For any Young diagram $F$ which labels a representation of $O_n$, the multiplicity $m(F, D, P)$ of $\sigma_n^F$ in the tensor product

$$\sigma_n^D \otimes \sigma_n^{(p_1)} \otimes \sigma_n^{(p_2)} \otimes \cdots \otimes \sigma_n^{(p_\ell)}$$

is given by

$$m(F, D, P) = \sum_{E,A,B,C} K_{F/E,A} K_{D/E,B}$$

where $K_{F/E,A}$ and $K_{D/E,B}$ are skew Kostka numbers (\cite{Sta}) and the sum is taken over all Young diagrams $E$ with at most $k$ rows, $A = (a_i), B = (b_j) \in \mathbb{Z}_{\geq 0}^\ell$ and $C = (c_{s,t})_{1 \leq s < t \leq \ell} \in \mathbb{Z}_{\geq 0}^{(\ell-1)/2}$ such that

$$p_i = a_i + b_i + c_{1i} + \cdots + c_{i-1,i} + c_{i,i+1} + \cdots + c_{i,\ell} \quad (1 \leq i \leq \ell).$$

Alternatively, the multiplicity $m(F, D, P)$ can also be described as the number of integral points in some polytope in an Euclidean space. See Lemma 5.1.

As we will see explicitly in Section 2.4, the combinatorial yoga that has been constructed to describe representations of $\text{GL}_n$ uses Young diagrams and semistandard tableau. This formalism has also been extended to deal with the classical isometry...
groups. This yoga can be used to describe the Pieri rules for isometry groups, but tensor products for the isometry groups do not sit as comfortably in this formalism as does branching. Our results show that (at least under some restrictions, which we refer to as the stable range condition) there is a uniform type of structure, that of Hibi rings, that can describe tensor products as well as branching for all classical groups. Moreover, the posets for the Hibi rings of iterated Pieri rules have a family resemblance to the Hibi rings of Hodge’s standard monomial theory. This raises the possibility that there is a uniform theory that encompasses all iterated branching algebras and all iterated Pieri algebras for all classical groups. We hope to study this issue in a future publication.

This paper is arranged as follows: In Section 2, we introduce notation for the representations of GLn, On and Sp2n, and review the Pieri rule for GLn. We define the Pieri algebra A_{n,k,ℓ} in Section 3, where 2(k + ℓ) < n. In Section 4, we compute the dimension of the homogeneous components of A_{n,k,ℓ}. We study the structure of A_{n,k,ℓ} in Section 5. Finally, we briefly explain the parallel theory for Sp2n in Section 6.

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2. Preliminaries

In this section, we introduce notation for the representations of GLn, On and Sp2n, and review the Pieri rule for GLn. We will use the following notation throughout this paper: If G is a group and V a G-module, then V^G will denote the space of all vectors in V fixed by G.

2.1. Representations of GLn. Let GLn = GLn(C) denote the general linear group consisting of all n × n invertible complex matrices, and let Bn = AnUn be the standard Borel subgroup of upper triangular matrices in GLn, where An is the diagonal torus in GLn and Un is the maximal unipotent subgroup consisting of all the upper triangular matrices with 1’s on the diagonal.

Recall that a Young diagram D is an array of square boxes arranged in left-justified horizontal rows, with each row no longer than the one above it ([Fu]). If D has at most m rows, then we shall write it as

D = (λ1, ..., λm)

where for each i, λi is the number of boxes in the i-th row of D. We shall denote the number of rows in D by r(D), and |D| = λ1 + · · · + λm.

For a Young diagram D = (λ1, ..., λn) with at most n rows, let ψ^D \colon A_n \to \mathbb{C}^× be the character given by

ψ^D[\text{diag}(a_1, ..., a_n)] = a_1^{λ1} \cdots a_n^{λn}. \tag{2.1}

Let \hat{A}_n^+ be the semigroup of dominant weights for GLn with respect to the Borel subgroup Bn. Then ψ^D \in \hat{A}_n^+ ([GW]), and we shall denote the irreducible representation
of $GL_n$ with highest weight $\psi^D_n$ by $\rho^D_n$. By the theory of highest weight (GW), the space $(\rho^D_n)^{U_n}$ of $U_n$-invariants in $\rho^D_n$ is one-dimensional, and $A_n$ acts on $(\rho^D_n)^{U_n}$ by the character $\psi^D_n$, i.e. the nonzero elements in $(\rho^D_n)^{U_n}$ are the $GL_n$ highest weight vectors of weight $\psi^D_n$.

2.2. **Representations of $O_n$.** Let $O_n = O_n(\mathbb{C})$ be the subgroup of $GL_n$ which preserves the symmetric bilinear form

$$
\langle \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \rangle = \sum_{j=1}^n u_jv_{n-j+1}
$$

(2.2)
on $\mathbb{C}^n$. The irreducible finite dimensional representations of $O_n$ are parameterized by Young diagrams $D$ such that the sum of the lengths of the first two columns of $D$ does not exceed $n$ ([Wy] GW Ho1). For such a Young diagram $D$, we shall denote the $O_n$ representation associated with $D$ by $\sigma_n^D$. Specifically, $\sigma_n^D$ is the irreducible representation of $O_n$ generated by the $GL_n$ highest weight vector in $\rho^D_n$. See Section 3.6 of [Ho1] for more details.

Let $SO_n$ denote the subgroup of $O_n$ containing elements of $O_n$ with determinant 1, and let

$$A_{SO_n} = A_n \cap SO_n, \quad U_{SO_n} = U_n \cap SO_n.$$ Explicitly,

$$A_{SO_n} = \begin{cases} 
\{ \text{diag}(a_1, \ldots, a_m, a_m^{-1}, \ldots, a_1^{-1}) : a_1, \ldots, a_m \in \mathbb{C}^\times \} & n = 2m \\
\{ \text{diag}(a_1, \ldots, a_m, 1, a_m^{-1}, \ldots, a_1^{-1}) : a_1, \ldots, a_m \in \mathbb{C}^\times \} & n = 2m + 1.
\end{cases}$$

If $D$ is a Young diagram with $r(D) \neq n/2$, then the restriction of $\sigma_n^D$ to $SO_n$ is irreducible. If in addition, $r(D) < n/2$ and $\phi_n^D : A_{SO_n} \to \mathbb{C}^\times$ is the restriction of the character $\psi_n^D$ to $A_{SO_n}$, then as a $SO_n$ module, $\sigma_n^D$ has highest weight $\phi_n^D$. In this case, we also have $(\sigma_n^D)^{U_{SO_n}} = (\rho_n^D)^{U_n}$.

2.3. **Representations of $Sp_{2n}$.** Let $Sp_{2n} = Sp_{2n}(\mathbb{C})$ be the subgroup of $GL_{2n}$ which preserves the symplectic form $(\cdot, \cdot)$ on $\mathbb{C}^{2n}$ given by

$$
\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_n \end{pmatrix}, \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \\ y'_1 \\ \vdots \\ y'_n \end{pmatrix} = \sum_{j=1}^n (x_jy'_j - y_jx'_j).
$$

(2.3)
The diagonal torus $A_{Sp_{2n}}$ of $Sp_{2n}$ is isomorphic to $(\mathbb{C}^\times)^n$. The highest weights and hence the irreducible finite dimensional representations of $Sp_{2n}$ are parametrized by Young diagrams with at most $n$ rows. If $D$ is such a Young diagram, we shall denote the corresponding highest weight and representation by $\chi^D_{2n}$ and $\tau^D_{2n}$ respectively. See Section 3.8 of [Ho1] for more details.
2.4. Pieri rule for GL

and Kostka coefficients. The Pieri rule for GL

describes the decomposition of a tensor product of a general irreducible representation with a representation corresponding to a Young diagram with only one row. In this section, we shall review this rule and one of its generalizations.

Definition. For two Young diagrams \( A = (a_1, a_2, \ldots) \) and \( B = (b_1, b_2, \ldots) \), we denote \( A \supseteq B \) (or \( B \subseteq A \)) and say \( A \) interlaces \( B \) if

\[
a_j \geq b_j \geq a_{j+1} \quad \text{for all } j.
\]

The following result is well known (\cite{GW, Ho1}):

Pieri rule for GL

Let \( D \) be a Young diagram with at most \( n \) rows and let \( p \) be a nonnegative integer. Then

\[
\rho_D^n \otimes \rho_n^p = \sum_F \rho_F^n
\]

where the sum is taken over all Young diagrams \( F \) with at most \( n \) rows such that \( D \subseteq F \) and \( |F| - |D| = p \).

By applying the Pieri rule repeatedly, we can give a description of the decomposition of the tensor product

\[
\rho_D^n \otimes \rho_n^{(p_1)} \otimes \cdots \otimes \rho_n^{(p_\ell)}.
\]

In fact, a representation \( \rho_F^n \) occurs in this tensor product if and only if there is a sequence of Young diagrams \( (F_0, F_1, ..., F_\ell) \) such that

\[
D = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{\ell-1} \subseteq F_\ell = F,
\]

and

\[
|F_j| - |F_{j-1}| = p_j, \quad 1 \leq j \leq \ell.
\]

Moreover, the number of sequences \( (F_0, F_1, ..., F_\ell) \) satisfying (2.5) and (2.6) gives the multiplicity of \( \rho_F^n \) in the tensor product in (2.4).

We shall give another description of this multiplicity. If a Young diagram \( D \) sits inside another Young diagram \( F \), then we write \( D \subset F \). In this case, by removing all boxes belonging to \( D \), we obtain the skew diagram \( F/D \). If we put a positive number in each box of \( F/D \), then it becomes a skew tableau and we say that the shape of this skew tableau is \( F/D \). If the entries of this skew tableau are taken from \( \{1, 2, ..., m\} \), and \( \mu_j \) of them are \( j \) for \( 1 \leq j \leq m \), then we say the content of this skew tableau is \( E = (\mu_1, ..., \mu_m) \). A skew tableau \( T \) is called semistandard if the numbers in each row of \( T \) weakly increase from left-to-right, and the numbers in each column of \( T \) strictly increase from top-to-bottom. The number of semistandard tableaux of shape \( F/D \) and with content \( E \) is denoted by \( K_{F/D, E} \) and it is called a skew Kostka number (\cite{Sta}).

We now fix two Young diagrams \( D \subset F \), and consider a sequence \( (F_1, ..., F_\ell) \) which satisfies conditions (2.5) and (2.6). We regard \( F/D \) as a union of the skew diagrams \( F_i/F_{i-1} \), \( 1 \leq i \leq \ell \). By filling the boxes in \( F_i/F_{i-1} \) with \( i \) for each \( 1 \leq i \leq \ell \), we obtain a semistandard tableau \( T \) of shape \( F/D \) and content \( P = (p_1, ..., p_\ell) \). This sets up a bijection between sequences of Young diagrams \( (F_1, ..., F_\ell) \) which satisfy conditions
and (2.6), and semistandard tableaux of shape $F/D$ and content $P$. It follows that $K_{F/D, P}$ is the multiplicity of $\rho_n^F$ in the tensor product (2.4). This proves:

**Iterated Pieri rule for $\text{GL}_n$.** Let $D$ be a Young diagram with at most $n$ rows and $P = (p_1, \ldots, p_\ell)$ be a sequence of nonnegative integers. Then

$$
\rho_n^D \otimes \rho_n^{(p_1)} \otimes \cdots \otimes \rho_n^{(p_\ell)} = \sum_F K_{F/D, P} \rho_n^F.
$$

(2.7)

### 3. The construction of a Pieri algebra for $\text{O}_n$

Consider a tensor product of $\text{O}_n$ representations of the form $\sigma_n^D \otimes \sigma_n^E$ where the Young diagram $E$ consists of only one row, that is, $E = (p)$ for some nonnegative integer $p$. Under the diagonal action of $\text{O}_n$, $\sigma_n^D \otimes \sigma_n^E$ is decomposed as a direct sum

$$
\sigma_n^D \otimes \sigma_n^E = \bigoplus_F m_F \sigma_n^F
$$

(3.1)

where for each Young diagram $F$ appearing in the sum, $m_F$ is the multiplicity of $\sigma_n^F$ in $\sigma_n^D \otimes \sigma_n^E$. A description of the multiplicity $m_F$ is called the **Pieri Rule** for $\text{O}_n$.

It is natural to generalize (3.1) to the tensor product of $\sigma_n^D$ with any number of representations indexed by one-row Young diagrams:

$$
\sigma_n^D \otimes \sigma_n^{(p_1)} \otimes \sigma_n^{(p_2)} \otimes \cdots \otimes \sigma_n^{(p_\ell)}.
$$

We call a description of the multiplicities in this tensor product the **iterated Pieri rule** for $\text{O}_n$. In this section, we shall construct an algebra, called a **Pieri algebra** of $\text{O}_n$, whose algebra structure encodes information on the iterated Pieri rule for the case when $r(D) \leq k$ and $2(k + \ell) < n$. We call the condition $2(k + \ell) < n$ the **stable range condition**.

#### 3.1. A realization of irreducible representations of $\text{O}_n$

Let $n$ and $k$ be positive integers such that $2k < n$, and let $M_{nk} = M_{nk}(\mathbb{C})$ denote the space of all $n \times k$ complex matrices. Let $\mathcal{P}(M_{nk})$ be the algebra of polynomial functions on $M_{nk}$, that is, each $p \in \mathcal{P}(M_{nk})$ is of the form

$$
p(x) = \sum_\alpha a_\alpha x^\alpha,
$$

where the sum is finite, $x = (x_{ij}) \in M_{nk}$, each $\alpha = (\alpha_{ij})$ appearing in the sum is an $n \times k$ matrix of nonnegative integers, $a_\alpha \in \mathbb{C}$ and

$$
x^\alpha = \prod_{i,j} x_{ij}^{\alpha_{ij}}.
$$

We define an action of $\text{O}_n \times \text{GL}_k$ on $\mathcal{P}(M_{nk})$ as follows: For $(g, h) \in \text{O}_n \times \text{GL}_k$, $f \in \mathcal{P}(M_{nk})$ and $X \in M_{nk}$, let

$$
((g, h) \cdot f)(X) = f(g^{-1}X h).
$$

(3.2)

For $1 \leq i, j \leq k$, let

$$
\mathcal{r}_{i,j}(X) = \langle X_i, X_j \rangle,
$$
where $\langle \cdot, \cdot \rangle$ is the symmetric bilinear form defined in \eqref{2.2} and $X_i$ and $X_j$ are the $i$-th and the $j$-th columns of $X$ respectively. These polynomials generate the algebra of $O_n$ invariants in $P(M_{nk})$. Let $I_{nk}$ be the ideal of $P(M_{nk})$ generated by \{ $r_{i,j}: 1 \leq i,j \leq k$ \}. It is stable under the action by $O_n \times GL_k$. So the action \eqref{3.2} induces an action by $O_n \times GL_k$ on the quotient algebra $P(M_{nk})/I_{nk}$. Under this action, $P(M_{nk})/I_{nk}$ has a decomposition given by \eqref{3.3}

$$P(M_{nk})/I_{nk} \cong \sum_{r(D) \leq k} \sigma_n^D \otimes \rho_k^D. \quad \text{(3.3)}$$

Let $(P(M_{nk})/I_{nk})^{U_k}$ be the subalgebra of $P(M_{nk})/I_{nk}$ consisting of all the elements fixed by the maximal unipotent subgroup $U_k$ of $GL_k$. Then by extracting $U_k$ invariants in \eqref{3.3}, we obtain

$$(P(M_{nk})/I_{nk})^{U_k} \cong \sum_{r(D) \leq k} \sigma_n^D \otimes (\rho_k^D)^{U_k}.$$ 

By the theory of highest weight, $\dim (\rho_k^D)^{U_k} = 1$ and the diagonal torus $A_k$ of $GL_k$ acts on $(\rho_k^D)^{U_k}$ by the character $\psi_k^D$ given in \eqref{2.1}. It follows that $(P(M_{nk})/I_{nk})^{U_k}$ is a module for $O_n \times A_k$. For each Young diagram $D$ with $r(D) \leq k$,

$$\sigma_n^D \cong \sigma_n^D \otimes (\rho_k^D)^{U_k}.$$

Thus we can realize the irreducible representation $\sigma_n^D$ as the $\psi_k^D$-eigenspace of $A_k$ in $(P(M_{nk})/I_{nk})^{U_k}$.

We also need to realize those $O_n$ representations indexed by one-row Young diagrams. To do this, we simply repeat the above construction with $k = 1$. For each $1 \leq j \leq \ell$, let $C^n_j$ be a copy of $C^n$. We shall denote a typical vector in $C^n_j$ as

$$Y_j = \begin{pmatrix} y_{1j} \\
 y_{2j} \\
 \vdots \\
 y_{nj} \end{pmatrix},$$

so the algebra $P(C^n_j)$ of polynomial functions on $C^n_j$ can be regarded as the polynomial algebras on the variables $y_{1j}, y_{2j}, \ldots, y_{nj}$. Let

$$t_j(Y_j) = \langle Y_j, Y_j \rangle,$$

and let $I_{n1}^{(j)}$ be the ideal of $P(C^n_j)$ generated by $t_j$. We form the quotient algebra $P(C^n_j)/I_{n1}^{(j)}$. Then $P(C^n_j)/I_{n1}^{(j)}$ is a module for $O_n \times GL_1$ and it can be decomposed as

$$P(C^n_j)/I_{n1}^{(j)} \cong \sum_{\rho_j \geq 0} \sigma_n^{(\rho_j)} \otimes \rho_1^{(\rho_j)}.$$ 

As before, the $O_n$ representation $\sigma_n^{(\rho_j)}$ can now be identified with the $\psi_1^{(\rho_j)}$-eigenspace of $GL_1$ in $P(C^n_j)/I_{n1}^{(j)}$. 
3.2. The algebra $\mathfrak{A}_{n,k,\ell}$. We shall assume $2(k + \ell) < n$ from now on. To realize the tensor product $\sigma_n^D \otimes \sigma_n^{(p_1)} \otimes \cdots \otimes \sigma_n^{(p_r)}$ of $O_n$ representations, we form the tensor product of the algebras

$$\mathfrak{T}_{n,k,\ell} := (\mathcal{P}(M_{nk})/I_{nk})^U_k \otimes \left( \mathcal{P}^n(C)/I_{n1}^{(1)} \right) \otimes \cdots \otimes \left( \mathcal{P}^n(C)/I_{n1}^{(\ell)} \right).$$

It is a module for $O_n \times A_k \times GL_\ell$, and it can be decomposed as

$$\mathfrak{T}_{n,k,\ell} \cong \prod_{r(D) \leq k, p_1, \ldots, p_r \geq 0} \left( \sum_{r(D) \leq k} \sigma_n^D \otimes \sigma_n^{(p_1)} \otimes \cdots \otimes \sigma_n^{(p_r)} \otimes (\rho_k^D)_{U_k} \otimes \rho_1^{(p_1)} \otimes \cdots \otimes \rho_1^{(p_r)} \right).$$

We can identify $\sigma_n^D \otimes \sigma_n^{(p_1)} \otimes \cdots \otimes \sigma_n^{(p_r)}$ with the $\psi_k^D \times \psi_1^{(p_1)} \times \cdots \times \psi_1^{(p_r)}$-eigenspace of $A_k \times GL_\ell$ in $\mathfrak{T}_{n,k,\ell}$. To decompose this tensor product, we extract the $U_{SO_n}$ invariants from this space. Let

$$\mathfrak{A}_{n,k,\ell} := (\mathfrak{T}_{n,k,\ell})^{U_{SO_n}}, \quad (3.4)$$

that is, $\mathfrak{A}_{n,k,\ell}$ be the subalgebra of $\mathfrak{T}_{n,k,\ell}$ consisting of all elements of $\mathfrak{T}_{n,k,\ell}$ fixed by $U_{SO_n}$. It is a module for $A_{SO_n} \times A_k \times GL_\ell$ and can be decomposed as

$$\mathfrak{A}_{n,k,\ell} \cong \sum_{r(D) \leq k, p_1, \ldots, p_r \geq 0} (\sigma_n^D \otimes \sigma_n^{(p_1)} \otimes \cdots \otimes \sigma_n^{(p_r)})_{U_{SO_n}} \otimes (\rho_k^D)_{U_k} \otimes \rho_1^{(p_1)} \otimes \cdots \otimes \rho_1^{(p_r)}.$$

The space $(\sigma_n^D \otimes \sigma_n^{(p_1)} \otimes \cdots \otimes \sigma_n^{(p_r)})_{U_{SO_n}}$ can be further decomposed as a sum of eigenspaces of $A_{SO_n}$, so we can write

$$\mathfrak{A}_{n,k,\ell} = \sum_{F,D,P} \mathcal{E}_{F,D,P}, \quad (3.5)$$

where the sum is taken over all Young diagrams $D$ and $F$ with $r(D) \leq k, r(F) \leq k + \ell$ and all $P = (p_1, \ldots, p_r) \in \mathbb{Z}_{\geq 0}^r$, and $\mathcal{E}_{F,D,P}$ is the $\phi_n^F \times \psi_k^D \times \psi_1^{(p_1)} \times \cdots \times \psi_1^{(p_r)}$-eigenspace of $A_{SO_n} \times A_k \times GL_\ell$ (recall that $\phi_n^F$ is the restriction of $\psi_n^F$ to $A_{SO_n}$). Since $A_{SO_n} \times A_k \times GL_\ell$ acts on $\mathfrak{A}_{n,k,\ell}$ by algebra automorphisms, the direct sum decomposition (3.5) defines a multi-grading on $\mathfrak{A}_{n,k,\ell}$.

Observe that the homogeneous component $\mathcal{E}_{F,D,P}$ can be identified with the space of all $SO_n$ highest weight vectors of weight $\phi_n^F$ in $\sigma_n^D \otimes \sigma_n^{(p_1)} \otimes \cdots \otimes \sigma_n^{(p_r)}$. Now the Young diagrams which label the irreducible $O_n$ representations occurring in the tensor product $\sigma_n^D \otimes \sigma_n^{(p_1)} \otimes \cdots \otimes \sigma_n^{(p_r)}$ have at most $k + \ell$ rows. Since $k + \ell < n/2$, these $O_n$ representations remain irreducible under the action by $SO_n$ and they are determined by the $SO_n$ highest weight vectors they contain. Consequently, the dimension of $\mathcal{E}_{F,D,P}$ coincides with the multiplicity of $\sigma_n^F$ in $\sigma_n^D \otimes \sigma_n^{(p_1)} \otimes \cdots \otimes \sigma_n^{(p_r)}$. Thus the algebra structure of $\mathfrak{A}_{n,k,\ell}$ encodes the iterated Pieri rule in the case $2(k + \ell) < n$. In
view of this property, we will call $\mathfrak{A}_{n,k,\ell}$ an $O_n$ Pieri algebra. The remaining paper will be devoted to determining the structure of this algebra.

4. The generalized Pieri rule for $O_n$

In this section, we show that the $O_n$ Pieri algebra $\mathfrak{A}_{n,k,\ell}$ is isomorphic to a subalgebra of a polynomial algebra. Using this isomorphism, we determine the dimension of the homogeneous components $E_{r,d,p}$ of $\mathfrak{A}_{n,k,\ell}$.

As before, we assume that $2(k+\ell) < n$. The group $O_n \times GL_{k+\ell}$ acts on the algebra $\mathcal{P}(M_{n,k+\ell})$ of polynomial functions on $M_{n,k+\ell}$ in a similar way as in Section 3.1 (with $k$ replaced by $k+\ell$), and we restrict this action to the subgroup $O_n \times (GL_k \times GL^+_1)$ of $O_n \times GL_{k+\ell}$. We write a typical matrix in $M_{n,k+\ell}$ as $X = (x_{ij})$, and the $j$th column of $X$ as $X_j$. Let

\[ r_{ij} = \langle X_i, X_j \rangle \quad (1 \leq i, j \leq k+\ell), \]

\[ R = \{ r_{ij} : 1 \leq i, j \leq k \} \cup \{ r_{k+j,k+j}^2 : 1 \leq j \leq \ell \}, \]

and let $J$ be the ideal of $\mathcal{P}(M_{nk})$ generated by $R$. Then $J$ is stable under the action by $O_n \times (GL_k \times GL^+_1)$, so the action by $O_n \times (GL_k \times GL^+_1)$ on $\mathcal{P}(M_{n,k+\ell})$ induces an action on the quotient algebra $\mathcal{P}(M_{n,k+\ell})/J$. There is an obvious isomorphism of algebras $O_n \times (GL_k \times GL^+_1)$ modules:

\[ (\mathcal{P}(M_{nk})/I_{nk}) \otimes \left( \mathcal{P}(\mathbb{C}^n)/I^{(1)}_{n1} \right) \otimes \cdots \otimes \left( \mathcal{P}(\mathbb{C}^n)/I^{(\ell)}_{n1} \right) \cong \mathcal{P}(M_{n,k+\ell})/J. \]

Here, we identify $X_{k+j} = Y_j$, i.e. $x_{i,k+j} = y_{i,j}$, $1 \leq i \leq n$ and $1 \leq j \leq \ell$, so that $r_{k+j,k+j} = t_j$ (see Section 3 for notation). It now follows from this and the definition of $\mathfrak{A}_{n,k,\ell}$ given in equation (3.1) that

\[ \mathfrak{A}_{n,k,\ell} \cong (\mathcal{P}(M_{n,k+\ell})/J)^{U_{SO_n} \times U_k} \]  \hspace{1cm} (4.1)

as $A_{SO_n} \times A_k \times GL^+_1$ modules and algebras.

**Proposition 4.1.** The $O_n$ Pieri algebra $\mathfrak{A}_{n,k,\ell}$ is isomorphic to a subalgebra of a polynomial algebra.

**Proof.** For $1 \leq i, j \leq k + \ell$, let

\[ \Delta_{ij} = \sum_{a=1}^{n} \frac{\partial^2}{\partial x_{a,i} \partial x_{n-a+1,j}}, \]

and let

\[ \mathcal{H}(M_{n,k+\ell}) = \{ f \in \mathcal{P}(M_{n,k+\ell}) : \Delta_{ij}(f) = 0 \ \forall 1 \leq i \leq j \leq k + \ell \} \]

be the space of $O_n$ harmonic polynomials in $\mathcal{P}(M_{n,k+\ell})$. Let $\mathcal{P}(M_{n,k+\ell})^{O_n}$ be the subalgebra of $O_n$ invariant polynomials on $M_{n,k+\ell}$, which is a polynomial algebra on $r_{ij}, 1 \leq i \leq j \leq k$. Both $\mathcal{H}(M_{n,k+\ell})$ and $\mathcal{P}(M_{n,k+\ell})^{O_n}$ are stable under the action by $O_n \times GL_{k+\ell}$, so the tensor product $\mathcal{H}(M_{n,k+\ell}) \otimes \mathcal{P}(M_{n,k+\ell})^{O_n}$ is an $O_n \times GL_{k+\ell}$ module. We consider the multiplication map

\[ m : \mathcal{H}(M_{n,k+\ell}) \otimes \mathcal{P}(M_{n,k+\ell})^{O_n} \to \mathcal{P}(M_{n,k+\ell}), \]
Since \(2(k + \ell) < n\), \(m\) is an \(O_n \times GL_{k+\ell}\) module isomorphism (\([Ho1]\)).

We now label the various copies of \(GL_1\) in \(GL_k \times GL_{\ell}\) as \(GL_1^{(j)}, j = 1, 2, \ldots, \ell\), and note that for each \(1 \leq j \leq \ell\), the subalgebra of \(\mathcal{P}(M_{n,k+\ell}^{(j)})\) generated by \(\{r_{1,k+j}, \ldots, r_{k,k+j}\}\) is stable under the action by \((GL_k \times GL_1^{(j)})\) and is isomorphic to \(\mathcal{P}(C^k)\). Similarly, for \(1 \leq s < t \leq \ell\), the subalgebra of \(\mathcal{P}(M_{n,k+\ell}^{(s,t)})\) generated by \(r_{k+s,k+t}\) is stable under the action by \(GL_1^{(s)} \times GL_1^{(t)}\), and is isomorphic to \(\mathcal{P}(C)\). So we shall denote this subalgebra by \(\mathcal{P}(C_{s,t})\), where \(C_{s,t}\) denote a copy of \(C\). The map \(m\) now induces an isomorphism of \(O_n \times (GL_k \times GL_{\ell})\) modules:

\[
\mathcal{P}(M_{n,k+\ell})/J \cong \mathcal{H}(M_{n,k+\ell}) \otimes \left( \bigotimes_{j=1}^{\ell} \mathcal{P}(C_j^k) \right) \otimes \left( \bigotimes_{1 \leq s < t \leq \ell} \mathcal{P}(C_{s,t}) \right). 
\]  

Using this and (4.4), we obtain an \(A_{SO_n} \times (A_k \times GL_{\ell})\) module isomorphisms:

\[
\mathfrak{A}_{n,k,\ell} \cong \left\{ \mathcal{H}(M_{n,k+\ell}) \otimes \left( \bigotimes_{j=1}^{\ell} \mathcal{P}(C_j^k) \right) \otimes \left( \bigotimes_{1 \leq s < t \leq \ell} \mathcal{P}(C_{s,t}) \right) \right\}_{U_{SO_n \times U_k}} 
\]

\[
= \left\{ \mathcal{H}(M_{n,k+\ell})^{U_{SO_n}} \otimes \left( \bigotimes_{j=1}^{\ell} \mathcal{P}(C_j^k) \right) \otimes \left( \bigotimes_{1 \leq s < t \leq \ell} \mathcal{P}(C_{s,t}) \right) \right\}_{U_k} 
\]

\[
= \left\{ \mathcal{P}(M_{n,k+\ell})^{U_n} \otimes \left( \bigotimes_{j=1}^{\ell} \mathcal{P}(C_j^k) \right) \otimes \left( \bigotimes_{1 \leq s < t \leq \ell} \mathcal{P}(C_{s,t}) \right) \right\}_{U_{n \times U_k}} 
\]

where \(\mathcal{P}_{n,k,\ell}\) is the polynomial algebra

\[
\mathcal{P}(M_{nk}) \otimes \left( \bigotimes_{i=1}^{\ell} \mathcal{P}(C_i^n) \right) \otimes \left( \bigotimes_{j=1}^{\ell} \mathcal{P}(C_j^k) \right) \otimes \left( \bigotimes_{1 \leq s < t \leq \ell} \mathcal{P}(C_{s,t}) \right),
\]

and for each \(1 \leq i \leq \ell\), \(C_i^n\) is a copy of \(C^n\). In establishing the isomorphisms above, we have used the fact that \(\mathcal{H}(M_{n,k+\ell})^{U_{SO_n}} = \mathcal{P}(M_{n,k+\ell})^{U_n}\). Since this module isomorphism is induced by the multiplication map (4.2), the isomorphism for \(\mathfrak{A}_{n,k,\ell} \cong (\mathcal{P}_{n,k,\ell})^{U_{n \times U_k}}\) is actually an algebra isomorphism. \(\Box\)

In view of proposition 4.1 we shall identify \(\mathfrak{A}_{n,k,\ell}\) with \((\mathcal{P}_{n,k,\ell})^{U_{n \times U_k}}\) from now on.

**Notation.** For \(A = (a_i), B = (b_j) \in \mathbb{R}^\ell\) and \(C = (c_{st})\) \(1 \leq s < t \leq \ell \in \mathbb{R}^{\ell(\ell-1)}/2\), let \(S(A, B, C) = (p_i) \in \mathbb{R}^\ell\) be defined by

\[
p_i = a_i + b_i + c_{1i} + \cdots + c_{i-1,i} + c_{i,i+1} + \cdots + c_{i,\ell} \quad (1 \leq i \leq \ell).
\]  

(4.4)
Proposition 4.2. The dimension of the homogeneous component \( \mathcal{E}_{F,D,P} \) is given by

\[
\dim \mathcal{E}_{F,D,P} = \sum_{r(E) \leq k, S(A,B,C) = P} K_{F/E,A} K_{D/E,B}.
\]

Proof. By \((\text{GL}_n, \text{GL}_m)\) duality \([\text{Ho}1]\), we have

\[
\mathcal{P}(M_{nk}) \cong \sum_{r(E) \leq k} \rho_n^E \otimes \rho_k^E, \quad \mathcal{P}(C_{s,t}) \cong \sum_{c_{s,t} \geq 0} \rho_{1,s}^{(c_{s,t})} \otimes \rho_{1,t}^{(c_{s,t})} \quad (1 \leq s < t \leq \ell),
\]

\[
\mathcal{P}(C^n_i) \cong \sum_{a_i \geq 0} \rho_n^{(a_i)} \otimes \rho_{1,i}^{(a_i)}, \quad \mathcal{P}(C^k_j) \cong \sum_{b_j \geq 0} \rho_k^{(b_j)} \otimes \rho_{1,j}^{(b_j)} \quad (1 \leq i, j \leq \ell).
\]

Here, to indicate that a particular copy \(\text{GL}_1^{(j)}\) of \(\text{GL}_1\) acts on the representation \(\rho_1^{(m)}\), we have written \(\rho_1^{(m)}\) as \(\rho_1^{(m)}\). Then

\[
\mathcal{P}_{n,k,\ell} = \mathcal{P}(M_{nk}) \otimes \left( \bigotimes_{i=1}^\ell \mathcal{P}(C^n_i) \right) \otimes \left( \bigotimes_{j=1}^\ell \mathcal{P}(C^k_j) \right) \otimes \left( \bigotimes_{1 \leq s < t \leq \ell} \mathcal{P}(C_{s,t}) \right)
\]

\[
\cong \sum_{r(E) \leq k, A,B,C} \left\{ \left( \rho_n^E \otimes \left( \bigotimes_{i=1}^\ell \rho_n^{(a_i)} \right) \right) \otimes \left( \rho_k^E \otimes \left( \bigotimes_{j=1}^\ell \rho_k^{(b_j)} \right) \right) \right. \\
\left. \otimes \left( \bigotimes_{1 \leq s < t \leq \ell} \rho_{1,s}^{(c_{s,t})} \otimes \rho_{1,t}^{(c_{s,t})} \right) \otimes \left( \bigotimes_{i=1}^\ell \rho_{1,i}^{(a_i)} \right) \otimes \left( \bigotimes_{j=1}^\ell \rho_{1,j}^{(b_j)} \right) \right\}
\]

\[
\cong \sum_{r(E) \leq k, A,B,C} \left\{ \left( \sum_F K_{F/E,A} \rho_n^F \right) \otimes \left( \sum_D K_{D/E,B} \rho_k^D \right) \otimes \left( \bigotimes_{1 \leq i \leq \ell} \rho_{1,i}^{(p_i)} \right) \right\}
\]

\[
\cong \sum_{F,D,P} \left( \sum_{r(E) \leq k, S(A,B,C) = P} K_{F/E,A} K_{D/E,B} \right) \rho_n^F \otimes \rho_k^D \otimes \left( \bigotimes_{1 \leq i \leq \ell} \rho_{1,i}^{(p_i)} \right)
\]

where \(A = (a_1, \ldots, a_\ell), B = (b_1, \ldots, b_\ell), C = (c_{s,t})\) and \(P = S(A,B,C) = (p_1, \ldots, p_\ell)\) in the sums. The proposition now follows from extracting the \(U_n \times U_k\) invariants from \(\mathcal{P}_{n,k,\ell}\). \(\square\)

Corollary 4.3. Let \(k, \ell\) and \(n\) be positive integers such that \(2(k+\ell) < n\), \(D\) a Young diagram such that \(r(D) \leq k\) and \(P = (p_1, \ldots, p_\ell) \in \mathbb{Z}_{\geq 0}^\ell\). Then for any Young diagram \(F\) which labels a representation of \(O_n\), the multiplicity \(m(F, D, P)\) of \(\sigma_n^F\) in the tensor product

\[
\sigma_n^D \otimes \sigma_n^{(p_1)} \otimes \sigma_n^{(p_2)} \otimes \cdots \otimes \sigma_n^{(p_\ell)}
\]

is given by

\[
m(F, D, P) = \sum_{r(E) \leq k, S(A,B,C) = P} K_{F/E,A} K_{D/E,B}.
\]
5. The structure of $\mathfrak{A}_{n,k,\ell}$

In this section, we shall determine the structure of the algebra $\mathfrak{A}_{n,k,\ell}$. We first define a poset $\tilde{\Gamma}(k, \ell)$ which arises from a consideration of the dimension formula given in Proposition 4.2. The poset $\tilde{\Gamma}(k, \ell)$ gives rise to a finite distributive lattice $G(k, \ell)$ as well as a generating set $S(k, \ell)$ for the algebra $\mathfrak{A}_{n,k,\ell}$. By defining an appropriate partial ordering on $S(k, \ell)$, standard monomials on $S(k, \ell)$ form a basis for $\mathfrak{A}_{n,k,\ell}$.

This standard monomial basis allows us to show that $\mathfrak{A}_{n,k,\ell}$ has a flat deformation to the Hibi algebra associated with the distributive lattice $G(k, \ell)$ ([Hi, Ho2]).

5.1. The poset $\Gamma(k, \ell)$. Suppose that the homogeneous component $E_{F,D,P}$ of $\mathfrak{A}_{n,k,\ell}$ is nonzero. Then by Proposition 4.2,

$$\dim E_{F,D,P} = \sum_{\substack{r(F) \leq k \\ S(A,B,C)=P}} K_{F/E,A}K_{D/E,B}.$$  

Fix $A = (a_1, ..., a_\ell), B = (b_1, ..., b_\ell) \in \mathbb{Z}_k^\ell$ and $C = (c_{st})_{1 \leq s < t \leq \ell} \in \mathbb{Z}_{\geq 0}^{{(\ell-1)/2}}$ in the sum. The Kostka number $K_{F/E,A}$ counts the number of sequences of Young diagrams $(F_0, F_1, ..., F_\ell)$ such that

$$E = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{\ell-1} \subseteq F_\ell = F, \text{ and } |F_j| - |F_{j-1}| = a_j \quad (1 \leq j \leq \ell). \quad (5.1)$$

This sequence of Young diagrams can be identified with the “truncated” Gelfand-Tsetlin pattern

$$\begin{array}{cccccccc}
\alpha_{\ell,1} & \alpha_{\ell,2} & \alpha_{\ell-1,1} & \alpha_{\ell-1,2} & \alpha_{\ell-1,3} & \alpha_{\ell-1,4} & \cdots & \alpha_{\ell,k+\ell-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_{0,1} & \alpha_{0,2} & \cdots & \alpha_{0,k} \\
\end{array}$$

where $F_j = (\alpha_{j,1}, \alpha_{j,2}, ..., \alpha_{j,k+j})$ for $0 \leq j \leq \ell$, which is obtained from a Gelfand-Tsetlin pattern with $\ell + k$ rows by removing its lowest $k - 1$ rows. Similarly, $K_{D/E,B}$ counts the number of sequences of Young diagrams $(D_0, D_1, ..., D_\ell)$ such that

$$E = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_{\ell-1} \subseteq D_\ell = D, \text{ and } |D_j| - |D_{j-1}| = b_j \quad (1 \leq j \leq \ell). \quad (5.2)$$

which can be viewed as another “truncated” Gelfand-Tsetlin pattern:

$$\begin{array}{cccccccc}
\beta_{\ell,1} & \beta_{\ell,2} & \beta_{\ell-1,1} & \beta_{\ell-1,2} & \beta_{\ell-1,3} & \beta_{\ell-1,4} & \cdots & \beta_{\ell,k} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\beta_{0,1} & \beta_{0,2} & \beta_{0,3} & \cdots & \beta_{0,k} \\
\end{array}$$

where $D_j = (\beta_{j,1}, \beta_{j,2}, ..., \beta_{j,k})$ for $0 \leq j \leq \ell$. Note that these two patterns share the same lowest row, so we can invert the first pattern and stack it below the second to
obtain a pattern of the form
\[
\begin{array}{cccccc}
\beta_{\ell,1} & \beta_{\ell,2} & \cdots & \beta_{\ell,k} \\
\vdots & \ddots & & \vdots \\
\beta_{1,1} & \beta_{1,2} & \cdots & \beta_{1,k} \\
\alpha_{0,1} & \alpha_{0,2} & \cdots & \alpha_{0,k} \\
\alpha_{\ell-1,1} & \alpha_{\ell-1,2} & \cdots & \alpha_{\ell-1,k+\ell-1} & \alpha_{\ell,k+\ell}
\end{array}
\] (5.3)

Thus the product \( K_{F/E,A}K_{D/E,B} \) counts certain patterns of this form. Following [Ki1], we can view the pattern (5.3) as an order preserving function \( f \) on a poset \( \Gamma(k, \ell) \), which is defined in the following way. Each entry in (5.3) should correspond to a unique element in \( \Gamma(k, \ell) \), and the partial ordering on \( \Gamma(k, \ell) \) is induced by the partial ordering on the entries of the pattern. Thus we let
\[
\Gamma(k, \ell) = \{ \gamma^{(j)}_i : 1 \leq i \leq \max(k, k+j), -\ell \leq j \leq \ell \},
\]

and the partial ordering on \( \Gamma(k, \ell) \) is given as follows: for \( 1 \leq i \leq \ell \),
\[
\gamma_a^{(-i)} \geq \gamma_a^{(-i+1)} \geq \gamma_a^{(-i+2)} \quad \text{for} \quad 1 \leq a \leq k-1, \quad \gamma_k^{(-i)} \geq \gamma_k^{(-i+1)}, \quad \text{and}
\]
\[
\gamma_b^{(i)} \geq \gamma_b^{(i-1)} \geq \gamma_b^{(i-2)} \quad \text{for} \quad 1 \leq b \leq k+i-1.
\]

For example, if we display the elements of \( \Gamma(k, \ell) \) as the entries in (5.3), then \( \Gamma(k, 1) \) and \( \Gamma(k, 2) \) are given by
\[
\Gamma(k, 1) = \begin{array}{ccccccc}
\gamma_1^{(-1)} & \gamma_2^{(-1)} & \gamma_3^{(-1)} & \cdots & \gamma_k^{(-1)} \\
\gamma_1^{(0)} & \gamma_2^{(0)} & \gamma_3^{(0)} & \cdots & \gamma_k^{(0)} \\
\gamma_1^{(1)} & \gamma_2^{(1)} & \gamma_3^{(1)} & \cdots & \gamma_k^{(1)} \\
\gamma_1^{(2)} & \gamma_2^{(2)} & \gamma_3^{(2)} & \cdots & \gamma_k^{(2)} \end{array} \quad (5.4)
\]

\[
\Gamma(k, 2) = \begin{array}{ccccccc}
\gamma_1^{(-2)} & \gamma_2^{(-2)} & \gamma_3^{(-2)} & \cdots & \gamma_k^{(-2)} \\
\gamma_1^{(-1)} & \gamma_2^{(-1)} & \gamma_3^{(-1)} & \cdots & \gamma_k^{(-1)} \\
\gamma_1^{(0)} & \gamma_2^{(0)} & \gamma_3^{(0)} & \cdots & \gamma_k^{(0)} \\
\gamma_1^{(1)} & \gamma_2^{(1)} & \gamma_3^{(1)} & \cdots & \gamma_k^{(1)} \end{array} \quad (5.5)
\]

The poset \( \Gamma(k, \ell) \) is still insufficient for describing \( \dim E_{F,D,P} \); we also need to encode the information in the matrix \( C = (c_{s,t})_{1 \leq s \leq t \leq \ell} \). We can view \( C \) as a function on another poset with \( (\ell - 1)/2 \) elements. Recall that for posets \( P \) and \( Q \) on disjoint sets, the disjoint union \( P + Q \) is the poset on \( P \cup Q \) such that \( x \leq y \) in \( P + Q \) if either \( x, y \in P \) and \( x \leq y \) in \( P \), or \( x, y \in Q \) and \( x \leq y \) in \( Q \). Let \( F_\ell \) be the disjoint union of the singleton posets \{\( \varepsilon_{s,t} \)\}, \( 1 \leq s < t \leq \ell \). Finally, we let
\[
\tilde{\Gamma}(k, \ell) = \Gamma(k, \ell) + F_\ell.
\]

The poset \( \tilde{\Gamma}(k, \ell) \) will play a pivotal role in the structure of \( \mathfrak{A}_{n,k,\ell} \).
5.2. **The lattice cone** \( \Omega(k, \ell) \). By a lattice cone, we mean the intersection of a convex polyhedral cone in \( \mathbb{R}^n \) for some \( n \) with \( \mathbb{Z}^n \). It has a structure of an affine semigroup with respect to vector addition, i.e. it is a finitely generated subsemigroup of \( \mathbb{Z}^n \) containing 0. In [Ho2], Howe describes how to construct a lattice cone from a finite poset, and gives the generators and relations for this lattice cone. In this subsection, we shall use results in [Ho2] to describe the lattice cone \( \Omega(k, \ell) \) corresponding to \( \tilde{\Gamma}(k, \ell) \).

We call a real-valued function \( f \) on \( \tilde{\Gamma} = \tilde{\Gamma}(k, \ell) \) order preserving if

\[
x, y \in \tilde{\Gamma}, \ x \geq y \implies f(x) \geq f(y).
\]

Let

\[
C = C(k, \ell) = \{ f : \tilde{\Gamma} \to \mathbb{R}_{\geq 0} \mid f \text{ is order preserving} \},
\]

and let \( \mathbb{R}^{\tilde{\Gamma}} = \mathbb{R}^\tilde{\Gamma}(k, \ell) \) be the space of all functions \( f : \tilde{\Gamma} \to \mathbb{R} \). By listing the elements of \( \tilde{\Gamma} \) in a specific order, say \( u_1, u_2, \ldots, u_N \) where

\[
N = (2\ell + 1)k + \ell^2,
\]

we can identify each function \( f \in \mathbb{R}^{\tilde{\Gamma}} \) with the point

\[
(f(u_1), f(u_2), \ldots, f(u_N)) \in \mathbb{R}^N.
\]

With this identification, it is easy to see that the subset of \( \mathbb{R}^N \) corresponding to \( C \) is a convex polyhedral cone. Let

\[
\Omega = \Omega(k, \ell) = C \cap \mathbb{Z}^{\tilde{\Gamma}},
\]

where \( \mathbb{Z}^{\tilde{\Gamma}} \) is the space of all integer-valued functions on \( \tilde{\Gamma} \) (which can be identified with the lattice \( \mathbb{Z}^N \) in \( \mathbb{R}^N \)). So \( \Omega \) is a lattice cone. In particular, it forms an affine semigroup with the usual addition of functions.

Each element of \( \Omega \) gives rise to a pattern of the form \([5.3]\) and a matrix \( C \in \mathbb{Z}^{(\ell-1)\times \ell} \). To describe this more precisely, we need some notation. If \( f \in C \), we write

\[
\begin{align*}
    f_{i,j} &= f\left(\gamma_{j}^{(i)}\right), & -\ell \leq i \leq \ell, & 1 \leq j \leq k + \max(0, i), \\
    f^{(s,t)} &= f(\varepsilon_{s,t}), & 1 \leq s < t \leq \ell, \\
    f_i &= (f_{i,1}, f_{i,2}, \ldots, f_{i,k+\max(0,i)}), & -\ell \leq i \leq \ell, \\
    C(f) &= (f^{(s,t)})_{1 \leq s < t \leq \ell}.
\end{align*}
\]

When \( f \in \Omega \), then \( f \) corresponds to the pattern formed by the rows \( f_{-\ell}, f_{-\ell+1}, \ldots, f_{\ell} \) and to the matrix \( C = C(f) \in \mathbb{Z}_{\geq 0}^{(\ell-1)/2} \). Next, we define certain linear functionals on
\[ A_j(f) = \sum_{a=1}^{k+j} f_{j,a} - \sum_{b=1}^{k+j-1} f_{j-1,b}, \]
\[ B_j(f) = \sum_{a=1}^{k} (f_{-j,a} - f_{-j+1,a}), \]
\[ P_j(f) = A_j(f) + B_j(f) + \sum_{a<j} f(a,j) + \sum_{b>j} f(j,b). \]

We also write
\[ A(f) = (A_1(f), \ldots, A_\ell(f)) \]
\[ B(f) = (B_1(f), \ldots, B_\ell(f)) \]
\[ P(f) = (P_1(f), \ldots, P_\ell(f)). \]

For a finite set \( X \), \(|X|\) shall denotes its cardinality.

Lemma 5.1. For Young diagrams \( D \) and \( F \) such that \( r(D) \leq k \) and \( r(F) \leq k + \ell \), and for \( P \in \mathbb{Z}_{\geq 0}^\ell \geq 0 \), let
\[ \mathcal{C}_{F,D,P} = \{ f \in \mathcal{C} : f_{-\ell} = D, f_\ell = F, P(f) = P \} \]
and
\[ \Omega_{F,D,P} = \mathcal{C}_{F,D,P} \cap \mathbb{Z}^{\mathbb{R}}. \]

Then \( \mathcal{C}_{F,D,P} \) is a polytope and
\[ |\Omega_{F,D,P}| = \dim \mathcal{E}_{F,D,P}. \]

Proof. For a Young diagram \( E \) with \( r(E) \leq k \), \( A, B \in \mathbb{Z}_{\geq 0}^\ell \geq 0 \) and \( C \in \mathbb{Z}_{\geq 0}^{\ell(\ell-1)/2} \) such that \( S(A, B, C) = P \) (see (4.4)), let \( \Omega^{(E,A,B,C)}_{F,D,P} \) be the subset of \( \Omega_{F,D,P} \) containing all elements \( f \) with the properties
\[ f_0 = E, \ A(f) = A, \ B(f) = B, \ \text{and} \ C(f) = C. \]

Then by identifying elements of \( \Omega^{(E,A,B,C)}_{F,D,P} \) with patterns of the form (5.3), we see that
\[ |\Omega^{(E,A,B,C)}_{F,D,P}| = K_{F/E,A} K_{D/E,B}. \]

Now \( \Omega_{F,D,P} \) is the disjoint union of the subsets \( \Omega^{(E,A,B,C)}_{F,D,P} \). So the cardinality of \( \Omega_{F,D,P} \) is given by the sum of \( K_{F/E,A} K_{D/E,B} \) over all possible \( E, A, B \) and \( C \). The number obtained is the dimension of \( \mathcal{E}_{F,D,P} \) given in Proposition 4.2. \( \square \)

Lemma 5.1 suggests that \( \Omega_{F,D,P} \) may be used to index a basis for \( \mathcal{E}_{F,D,P} \). Since
\[ \Omega = \bigcup_{F,D,P} \Omega_{F,D,P}, \]
\( \Omega \) may be used to index a basis for the algebra \( \mathfrak{A}_{n,k,\ell} \). We also observe that if \( f \in \Omega_{F,D,P} \) and \( g \in \Omega_{F',D',P'} \), then \( f + g \in \Omega_{F+D',F+F',P+P'} \). We express this property as
\[ \Omega_{F,D,P} + \Omega_{F',D',P'} \subseteq \Omega_{F+F',D+D',P+P'}. \]
5.3. **The generators of the lattice cone** $\Omega$. In this subsection, we shall use the results in \cite{Ho2} to specify a set of generators for $\Omega$ and the relations between the generators.

Recall the poset $\tilde{\Gamma} = \Gamma + F_\ell$ defined in \eqref{eq:5.3} where $\Gamma = \Gamma(k, \ell)$. A subset $A$ of a poset $P$ is said to be increasing if for any $a \in P$,

$$x \in P \text{ and } x \geq a \implies x \in A.$$  

Let $J^*(\tilde{\Gamma})$ be the collection of increasing subsets of $\tilde{\Gamma}$. For each $A \in J^*(\tilde{\Gamma})$, let $\chi_A : \tilde{\Gamma} \to \mathbb{Z}_{\geq 0}$ be the characteristic function of $A$, that is,

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

(If $A = \emptyset$, then $\chi_A(x) = 0$ for all $x \in A$.) Then $\chi_A \in \Omega$. By Theorem 3.3 of \cite{Ho2}, $\Omega$ is generated by

$$\mathcal{G} = \mathcal{G}(k, \ell) = \{ \chi_A : A \in J^*(\tilde{\Gamma}) \},$$

and a defining set of relations is given by

$$\chi_A + \chi_B = \chi_{A \cup B} + \chi_{A \cap B}, \quad A, B \in J^*(\tilde{\Gamma}).$$

We now specify the increasing sets in $\tilde{\Gamma}$. Since $\tilde{\Gamma} = \Gamma + F_\ell$, every increasing subset $A$ of $\tilde{\Gamma}$ is of the form $A_1 \cup A_2$ where $A_1$ and $A_2$ are increasing subsets of $\Gamma$ and $F_\ell$, respectively. On the other hand, every subset of $F_\ell$ is increasing. Thus we only need to determine the increasing subsets of $\Gamma$. For $0 \leq c \leq k$, $I = \{ i_1, ..., i_u \}$, $J = \{ j_1, ..., j_v \} \subseteq \{ 1, ..., \ell \}$ with $|I| = u \leq k - c$, we define a sequence $(a_{-\ell}, ..., a_0, ..., a_\ell)$ as follows:

(i) $a_0 = c$.

(ii) For $0 \leq s \leq \ell - 1$,

$$a_{s+1} = \begin{cases} a_s + 1 & s + 1 \in \{ i_1, ..., i_u \} \\ a_s & s + 1 \notin \{ i_1, ..., i_u \} \end{cases}$$

and

$$a_{s+1} = \begin{cases} a_s + 1 & s + 1 \in \{ j_1, ..., j_v \} \\ a_s & s + 1 \notin \{ j_1, ..., j_v \} \end{cases}.$$ (5.13)

Let

$$A_{(c, I, J)} = \{ \gamma^{(I)}_j : 1 \leq j \leq a_i, -\ell \leq i \leq \ell \}.$$ (So $A_{0,\emptyset,\emptyset} = \emptyset$.) Then one verifies that $A_{(c, I, J)}$ is an increasing subset of $\Gamma$. In fact, the sets $A_{(c, I, J)}$ exhaust all the increasing subsets of $\Gamma$.

**Lemma 5.2.** Every increasing subset of $\Gamma$ is of the form $A_{(c, I, J)}$. Consequently,

$$J^*(\tilde{\Gamma}) = \{ A_{(c, I, J)} \cup Z : 0 \leq c \leq k, \ I, J \subseteq \{ 1, ..., \ell \}, |I| \leq c - k, Z \subseteq F_\ell \}.$$
Proof. Let $A$ be an increasing subset of $\Gamma$. Then for $-\ell \leq j \leq \ell$, 
\[ A \cap \{ \gamma^{(j)}_i : 1 \leq i \leq k + \max(0,j) \} = \{ \gamma^{(j)}_i : 1 \leq i \leq a_j \} \]
for some $a_j \geq 0$. ($a_j = 0$ means the intersection is empty.)

Let $0 \leq s \leq \ell - 1$. Then $\gamma^{(s)}_a \in A$ and $\gamma^{(s)}_{a+1} \not\in A$. Since $A$ is increasing and $\gamma^{(s+1)}_{a+1} \geq \gamma^{(s)}_{a+1}$, $\gamma^{(s+1)}_a \in A$. Similarly, since $\gamma^{(s)}_{a+2} \geq \gamma^{(s+1)}_{a+2}$, we must have $\gamma^{(s+1)}_{a+2} \not\in A$. So $a_{s+1} = a_s$ or $a_{s+1} = a_s + 1$. Similarly, $a_{s-1} = a_s$ or $a_{s-1} = a_s + 1$. We now let $c = a_0$ and define $I$ and $J$ by equations (5.12) and (5.13). Then $A = A_{(c,l,l)}$. □

5.4. Standard monomial basis for $\mathfrak{A}_{n,k,l}$. Recall that $\mathfrak{A}_{n,k,l} = (\mathcal{P}_{n,k,l})^{U_n \times U_k}$ where

\[ \mathcal{P}_{n,k,l} = \mathcal{P}(M_{nk}) \otimes \left( \bigotimes_{i=1}^{\ell} \mathcal{P}(C_i^n) \otimes \left( \bigotimes_{j=1}^{\ell} \mathcal{P}(C_j^k) \otimes \left( \bigotimes_{1 \leq s \leq t \leq \ell} \mathcal{P}(C_{s,t}) \right) \right) \right). \]

We denote the standard coordinates on $M_{nk}$, $C_i^n$, $C_j^k$ and $C_{s,t}$ as follows:

(i) $(x_{ij})_{1 \leq i \leq n, 1 \leq j \leq k} \in M_{nk}$.

(ii) $(y_{ij})_{1 \leq i \leq n} \in C_{i}^n$, $1 \leq j \leq \ell$.

(iii) $(r_{i,j})_{1 \leq i \leq k} \in C_{j}^k$, $1 \leq j \leq \ell$.

(iv) $(r_{k+s,k+t}) \in C_{s,t}$, $1 \leq s < t \leq \ell$.

So $\mathcal{P}_{n,k,l}$ can be regarded as a polynomial algebra on these variables.

For later use, we define a monomial ordering on $\mathcal{P}_{n,k,l}$ as follows: it is the graded lexicographic order (CLO) with respect to

\[ x_{11} > x_{21} > \cdots > x_{1n} > x_{12} > x_{22} > \cdots > x_{nk} \]
\[ > y_{11} > y_{21} > \cdots > y_{1n} > y_{21} > \cdots > y_{nk} \]
\[ > r_{1,k+1} > r_{2,k+1} > \cdots > r_{k,k+1} > r_{1,k+2} > \cdots > r_{k,k+\ell} \]
\[ > r_{k+1,k+2} > \cdots > r_{k+1,k+3} > \cdots > r_{k+\ell-1,k+\ell}. \]

For each polynomial $f$ in $\mathcal{P}_{n,k,l}$, we write the leading monomial of $f$ with respect to the above monomial ordering as $\text{LM}(f)$.

**Definition.** For $0 \leq c \leq k$, $I = \{ i_1 < i_2 < \cdots < i_u \}$, $J = \{ j_1 < j_2 < \cdots < j_v \} \subseteq \{1, \ldots, \ell\}$ with $|I| = u \leq k - c$, let

\[ \eta_{(c,l,l)} = \begin{vmatrix}
    x_{11} & x_{12} & \cdots & x_{1,c+u} & y_{1,j_1} & \cdots & y_{1,j_v} \\
    x_{21} & x_{22} & \cdots & x_{2,c+u} & y_{2,j_1} & \cdots & y_{2,j_v} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    x_{c+v,1} & x_{c+v,2} & \cdots & x_{c+v,c+u} & y_{c+v,j_1} & \cdots & y_{c+v,j_v} \\
    r_{1,k+i_1} & r_{2,k+i_1} & \cdots & r_{c+u,k+i_1} & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    r_{1,k+i_u} & r_{2,k+i_u} & \cdots & r_{c+u,k+i_u} & 0 & \cdots & 0
\end{vmatrix}. \]

(So $\eta_{(0,0,0)} = 1$.)

For each $g \in \mathcal{G}$, we define the polynomial $\eta_g \in \mathfrak{A}_{n,k,l}$ as follows:

(i) If $g = \chi_{A_{(c,l,l)}}$, then $\eta_g = \eta_{(c,l,l)}$. 

(ii) If $g = \chi_{\{s,t\}}$, then $\eta_g = r_{k+s,k+t}$.

(iii) If $g = \chi_B$ where $B = A_{(c,I,J)} \cup Z$ and $Z \subseteq \Gamma_\ell$, then

$$\eta_g = \eta_{(c,I,J)} \cdot \prod_{\varepsilon_{s,t} \in Z} r_{k+s,k+t}.$$ 

More generally, we can extend the map $g \to \eta_g$ on $G$ to $\Omega$. This is done in the following way. Recall from [Ho2] and [Hi] that each element $g$ has a unique standard expression as a sum

$$g = \sum_{j=1}^{N} c_j \chi_{A_j} \quad (5.14)$$

where $N$ is given in equation (5.6), $c_j \geq 0$ and $\emptyset \subset A_1 \subset A_2 \subset \cdots \subset A_N = \tilde{\Gamma}$ is a maximal chain in $J^*(\tilde{\Gamma})$. Then we define

$$\eta_g = \prod_{j=1}^{N} \eta_{\chi_{A_j}} \cdot \prod_{\varepsilon_{s,t} \in Z} r_{k+s,k+t} \quad (5.15)$$

Let

$$B = \{ \eta_g : g \in \Omega \}.$$ 

Note that $B$ can be characterized as follows. Let

$$S = S(k, \ell) = \{ \eta_g : g \in G \}.$$ 

Then the partial ordering on $J^*(\tilde{\Gamma})$ induces the following partial ordering on $S$: if $g_1, g_2 \in S$, then $g_1 = \chi_{A_1}$ and $g_2 = \chi_{A_2}$ for some $A_1, A_2 \in J^*(\tilde{\Gamma})$. We define

$$g_1 \leq g_2 \iff A_1 \subseteq A_2.$$ 

Since $J^*(\tilde{\Gamma})$ is a distributive lattice, so is $S$. We now observe that each $\eta_g \in B$ is a monomial in elements of a maximal chains in $S$, that is, it is a standard monomial. We shall show that $B$ is a basis for $\mathfrak{A}_{n,k,\ell}$, so that the algebra $\mathfrak{A}_{n,k,\ell}$ has a standard monomial theory for $S$. In the appendix, we will give the Hasse diagram of the poset $J^*(\tilde{\Gamma})$ for $\ell = 1$.

**Lemma 5.3.** The leading monomial of $\eta_{(c,I,J)}$ is given by

$$\text{LM}(\eta_{(c,I,J)}) = \left( \prod_{a=1}^{c} x_{aa} \right) \left( \prod_{a=1}^{v} g_{c+a,j_a} \right) \left( \prod_{a=1}^{u} r_{c+a,k+i_a} \right).$$ 

**Proof.** By observation. $\square$

**Proposition 5.4.** For $g \in \Omega$, with notation given in (5.7),

$$\text{LM}(\eta_g) = \left( \prod_{u=1}^{k} x_{a_u,u}^{g_{0,i}} \right) \left( \prod_{1 \leq a \leq k+b} y_{a,b}^{g_{0,a} - g_{0,b-1,a}} \right) \left( \prod_{1 \leq i \leq k} r_{i,k+i}^{g_{-i,j,i} - g_{-j,i} + 1} \right) \left( \prod_{1 \leq s < t \leq \ell} r_{k+s,k+t}^{(s,t)} \right),$$

(5.16)
Proof. For each \( g \in \Omega \), let \( m(g) \) be the monomial given in the right hand side of (5.16). This defines a map \( m \) from \( \Omega \) to the semigroup of all monomials in \( \mathcal{P}_{n,k,\ell} \). It is easy to check that \( m \) is a semigroup homomorphism, and \( m(g) = \text{LM}(\eta_g) \) for all \( g \in \mathcal{G} \). In fact, \( m \) is a semigroup isomorphism onto its image.

We now let \( g \in \Omega \) and assume that its standard form is given in equation (5.14). Then \( \eta_g \) is given in equation (5.15). Since \( m \) is a semigroup homomorphism,

\[
\text{LM}(\eta_g) = \prod_{j=1}^{N} \text{LM}(\eta_{\chi_{A_j}})^{c_j} = \prod_{j=1}^{N} m\left(\chi_{A_j}\right)^{c_j} = m\left(\sum_{j=1}^{N} c_j \chi_{A_j}\right) = m(g).
\]

Corollary 5.5. The set

\[
\text{LM(\mathcal{B})} = \{\text{LM}(\eta_g) : g \in \Omega\}
\]

of monomials forms an affine semigroup isomorphic to \( \Omega \).

Proof. This is because the map \( m(g) = \text{LM}(\eta_g) \) defined in the proof Proposition 5.4 is an semigroup isomorphism from \( \Omega \) onto \( \text{LM(\mathcal{B})} \).

Corollary 5.6. The set \( \mathcal{B} \) is linearly independent.

Proof. This is because the elements in \( \mathcal{B} \) have distinct leading monomials.

Proposition 5.7. Let

\[
\mathcal{B}_{F,D,P} = \{\eta_g : g \in \Omega_{F,D,P}\}.
\]

Then \( \mathcal{B}_{F,D,P} \) is a basis for the homogeneous component \( \mathcal{E}_{F,D,P} \) of \( \mathfrak{A}_{n,k,\ell} \).

Proof. We claim that if \( g \in \Omega_{F,D,P} \), then \( \eta_g \in \mathcal{E}_{F,D,P} \). This is true if \( g \in \mathcal{G} \). The general case follows from this and (5.10). It follows that \( \mathcal{B}_{F,D,P} \subseteq \mathcal{E}_{F,D,P} \). Moreover, \( \mathcal{B}_{F,D,P} \) is linearly independent and the number of vectors it contains coincides with the dimension of \( \mathcal{E}_{F,D,P} \). So \( \mathcal{B}_{F,D,P} \) is a basis for \( \mathcal{E}_{F,D,P} \).

Theorem 5.8. The algebra \( \mathfrak{A}_{n,k,\ell} \) has a standard monomial theory for \( S \), that is, the monomials in elements of the maximal chains in \( S \) form a basis for \( \mathfrak{A}_{n,k,\ell} \).

Proof. The set \( \mathcal{B} \) is precisely the set of all monomials in elements of the maximal chains in \( S \). It forms a basis for \( \mathfrak{A}_{n,k,\ell} \) follows from \( \mathcal{B} = \bigcup_{F,D,P} \mathcal{B}_{F,D,P} \), (3.2) and Proposition 5.7.
5.5. Toric degeneration of $\mathfrak{A}_{n,k,\ell}$. We first recall the notion of a SAGBI basis (RS Stu). Let $R = k[x_1,...,x_n]$ be a polynomial algebra over a field $k$ and $L$ a subalgebra of $R$. Assume that $R$ is given a monomial ordering. For each $a \in R$, let $\text{LM}(a)$ be the leading monomial of $a$ with respect to this monomial ordering. The initial algebra of $L$, denoted by $\text{LM}(L)$, is the subalgebra of $R$ generated by the set $\{\text{LM}(a) : a \in L\}$ of leading monomials. A subset $F$ of $L$ is called a SAGBI basis for $L$ if the set $\{\text{LM}(a) : a \in F\}$ generates the initial algebra $\text{LM}(L)$. We will need the following result.

**Proposition 5.9.** [CHV] Let $L$ be a subalgebra of a polynomial algebra $R = k[x_1,...,x_n]$ over a field $k$. If the initial algebra $\text{LM}(L)$ of $L$ with respect to some monomial ordering is finitely generated, then there exists a flat 1-parameter family of $k$-algebras with general fibre $L$ and special fibre $\text{LM}(L)$.

Let $\mathbb{C}[[\Omega]]$ be the semigroup algebra ([BH]) on $\Omega$, that is, $\mathbb{C}[[\Omega]]$ is a complex algebra with basis $\{X^f : f \in \Omega\}$ and it satisfies the multiplication law

$$X^{f_1}X^{f_2} = X^{f_1+f_2}, \quad f_1, f_2 \in \Omega.$$ 

On the other hand, the distributive lattice $J^*(\tilde{\Gamma})$ gives rise to the Hibi algebra ([Hi Ho2]) generated by $J^*(\tilde{\Gamma})$ and with relations

$$A_1 \cdot A_2 = (A_1 \cup A_2) \cdot (A_1 \cap A_2).$$

It is easy to see that this algebra and the semigroup algebra $\mathbb{C}[[\Omega]]$ are isomorphic ([Ho2]).

**Lemma 5.10.** The initial algebra $\text{LM}(\mathfrak{A}_{n,k,\ell})$ of $\mathfrak{A}_{n,k,\ell}$ is isomorphic to the semigroup algebra $\mathbb{C}[[\Omega]]$.

**Proof.** Let $f \in \mathfrak{A}_{n,k,\ell}$. By Theorem 5.8

$$f = c_1\eta_{g_1} + c_2\eta_{g_2} + \cdots + c_r\eta_{g_r}$$

for some $c_1, ..., c_r \in \mathbb{C}$ and $g_1, ..., g_r \in \Omega$. Since the leading monomials $\text{LM}(\eta_{g_1}), ..., \text{LM}(\eta_{g_r})$ are distinct, $\text{LM}(f) = \text{LM}(\eta_{g_j})$ for some $1 \leq j \leq r$. This shows that $\text{LM}(f) \in \text{LM}(\mathfrak{B})$. It follows that the initial algebra $\text{LM}(\mathfrak{A}_{n,k,\ell})$ is generated by $\text{LM}(\mathfrak{B})$. Now by Lemma 5.10 the semigroups $\text{LM}(\mathfrak{B})$ and $\Omega$ are isomorphic. So the algebras $\text{LM}(\mathfrak{A}_{n,k,\ell})$ and $\mathbb{C}[[\Omega]]$ are also isomorphic. □

**Theorem 5.11.** There exists a flat one-parameter family of complex algebras with general fibre $\mathfrak{A}_{n,k,\ell}$ and special fibre $\mathbb{C}[[\Omega]]$.

**Proof.** Since $G$ generates the semigroup $\Omega$, the set $\{\text{LM}(\eta_f) : \eta_f \in S\}$ generates the initial algebra $\text{LM}(\mathfrak{A}_{n,k,\ell})$. So $S$ is a finite SAGBI basis for $\mathfrak{A}_{n,k,\ell}$. The theorem now follows from Proposition 5.9 and Lemma 5.10 □
6. Pieri Algebras for Symplectic Groups

There is a parallel construction of Pieri algebras for the complex symplectic group $\text{Sp}_{2n}$ which we will briefly explain in this section. This algebra encodes information on how a tensor product of $\text{Sp}_{2n}$ representations of the form

$$\tau_{2n}^D \otimes \tau_{2n}^{(p_1)} \otimes \tau_{2n}^{(p_2)} \otimes \cdots \otimes \tau_{2n}^{(p_r)}$$

decomposes, where $r(D) \leq k$, $p_1,\ldots,p_r \geq 0$ and $k + \ell \leq n$ (see Section 2.3 for notation).

Let the groups $\text{Sp}_{2n}$ and $\text{GL}_k$ act on the algebra $\mathcal{P}(M_{2n,k})$ of polynomial functions on $M_{2n,k}$ by

$$[(g,h).f](T) = f(g^T h), \quad g \in \text{Sp}_{2n}, \ h \in \text{GL}_k, \ f \in \mathcal{P}(M_{2n,k}), \ T \in M_{2n,k}. \quad (6.1)$$

Let $I_{2n,k}$ be the ideal of $\mathcal{P}(M_{2n,k})$ generated by all $\text{Sp}_{2n}$ invariants of positive degree, and form the quotient algebra $\mathcal{P}(M_{2n,k})/I_{2n,k}$. Then the action (6.1) induces an action of $\text{Sp}_{2n} \times \text{GL}_k$ on $\mathcal{P}(M_{2n,k})/I_{2n,k}$, and

$$\mathcal{P}(M_{2n,k})/I_{2n,k} \cong \sum_{r(D) \leq k} \tau_{2n}^D \otimes \rho_k^D.$$

Let $\mathcal{P}(M_{2n,k})/I_{2n,k})^{U_k}$ be the subalgebra of $U_k$ invariants in $\mathcal{P}(M_{2n,k})/I_{2n,k}$. It is a module for $\text{Sp}_{2n} \times A_k$.

Next, for each $1 \leq j \leq \ell$, let $C_{2n}^j$ (resp. $\text{GL}_1^{(j)}$) be a copy of $C_{2n}$ (resp. $\text{GL}_1$) and consider the algebra $\mathcal{P}(C_{2n}^j)$. Since $C_{2n}^j \cong M_{2n,1}$, by the $(\text{GL}_{2n}, \text{GL}_1^{(j)})$-duality,

$$\mathcal{P}(C_{2n}^j) \cong \sum_{p_j \geq 0} \rho_{2n}^{(p_j)} \otimes \rho_1^{(p_j)}$$

as a $\text{GL}_{2n} \times \text{GL}_1^{(j)}$ module. We now restrict the action of $\text{GL}_{2n}$ to $\text{Sp}_{2n}$. But for each $p_j \geq 0$, $\rho_{2n}^{(p_j)}$ remains irreducible under the action by $\text{Sp}_{2n}$, so that $\rho_{2n}^{(p_j)} = \tau_{2n}^{(p_j)}$. Thus

$$\mathcal{P}(C_{2n}^j) \cong \sum_{p_j \geq 0} \tau_{2n}^{(p_j)} \otimes \rho_{1,j}^{(p_j)}.$$

as a $\text{Sp}_{2n} \times \text{GL}_1^{(j)}$ module.

We now let

$$A_{n,k,\ell} := \left\{ \mathcal{P}(M_{2n,k})/I_{2n,k})^{U_k} \otimes \mathcal{P}(C_{2n}^1) \times \cdots \times \mathcal{P}(C_{2n}^\ell) \right\}^{U_{\text{Sp}_{2n}}}$$

and

$$A_{n,k,\ell} := \left\{ \mathcal{P}(M_{2n,k})/I_{2n,k}) \otimes \mathcal{P}(C_{2n}^1) \times \cdots \times \mathcal{P}(C_{2n}^\ell) \right\}^{U_{\text{Sp}_{2n}} \times U_k}.$$

This algebra is a module for $A_{\text{Sp}_{2n}} \times A_k \times \text{GL}_1^{(1)} \times \cdots \times \text{GL}_1^{(\ell)}$ and can be decomposed as

$$A_{n,k,\ell} \cong \sum_{r(D) \leq k} \left( \tau_{2n}^D \otimes \tau_{n}^{(p_1)} \otimes \cdots \otimes \tau_{n}^{(p_r)} \right)^{U_{\text{Sp}_{2n}}} \otimes \left( \rho_k^D \otimes \rho_1^{(p_1)} \otimes \cdots \otimes \rho_1^{(p_r)} \right).$$
For Young diagrams $D$ and $F$ with $r(D) \leq k$ and $r(F) \leq k + \ell$, and $P = (p_1, ..., p_\ell) \in \mathbb{Z}_{\geq 0}^\ell$, the $\chi^F \times \psi^D \times \psi_1^{(p_1)} \times \cdots \times \psi_1^{(p_\ell)}$-eigenspace in this algebra can be identified with the space of all Sp$_{2n}$ highest weight vectors of weight $\chi^F$ in $\tau^D \otimes \tau_2^{(p_1)} \otimes \cdots \otimes \tau_2^{(p_\ell)}$. Thus the multiplicity of $\tau^F_{2n}$ in this tensor product coincides with the dimension of the eigenspace. In view of this property, we will call $A_{n,k,\ell}$ an Sp$_{2n}$ Pieri algebra.

Using similar arguments as in Proposition 4.1, we can show that the algebras $A_{n,k,\ell}$ and $A_{2n,k,\ell}$ are isomorphic. Consequently, one can deduce the structure of $A_{n,k,\ell}$ from the results of Section 5. In particular, $A_{n,k,\ell}$ has a standard monomial basis and has a flat deformation to a Hibi algebra.

7. Appendix

We show the poset structure of the generating set $S(k, \ell)$ of the algebra $A_{n,k,\ell}$ for the case $\ell = 1$. Note that if $\ell = 1$, then we have $\tilde{\Gamma}(k, \ell) = \Gamma(k, \ell)$. Therefore, from Lemma 5.2, the poset structure of $S(k, \ell)$ can be obtained by considering order increasing subsets of $\tilde{\Gamma}(k, 1)$ drawn in (5.3).

Following the notation used in (5.12) and (5.13), we let $(a_{-1}, a_0, a_1)$ denote the order preserving characteristic function $\chi$ over $\tilde{\Gamma}(k, 1)$ such that the integers $a_{-1}$, $a_0$, and $a_1$ are equal to the cardinalities of the intersections between $\chi^{-1}(1)$ and the top row, the middle row, and the bottom row of $\tilde{\Gamma}(k, 1)$ respectively.
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