Implications of the Super-K atmospheric data for the mixing angles $\theta_{13}$ and $\theta_{23}$

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A three-neutrino analysis of oscillation data is performed using the recent, more finely binned Super-K oscillation data, together with the CHOOZ, K2K, and MINOS data. The solar parameters, $\Delta_{21}$ and $\theta_{12}$, are fixed from a recent analysis, and $\Delta_{32}$, $\theta_{13}$, and $\theta_{23}$ are varied. We utilize the full three-neutrino oscillation probability and an exact treatment of the Earth’s MSW effect with a castel-wall density. By including terms linear in $m_k^2$ and two independent mass-squared differences $\Delta_{kj} := m_k^2 - m_j^2$, the separation between two of the mass-squared differences is sufficiently large so that the data from a given experiment, which may span some range of baselines and neutrino energies, can be approximately understood within the context of an effective two-flavor theory. Experiments detecting solar neutrinos $^1$ and MINOS $^{17,18}$ and the long baseline (LBL) reactor experiment KamLAND $^{11,12}$ are particularly sensitive to the mixing angle $\theta_{12}$ and the mass-squared difference $\Delta_{21}$ assuming the standard representation of the neutrino mixing matrix $^{22}$. A three neutrino analysis $^{23}$ gives a value for the mixing angle $\sin^2 \theta_{12} = 0.304^{+0.046}_{-0.034}$ ($2 \sigma$ error), with a precision of 8% at 3$\sigma$. The solar mass-squared difference is determined predominantly by the SNO data $^{10}$ and is found to be $\Delta_{21} = 7.65^{+0.34}_{-0.40} \times 10^{-5}$ eV$^2$. Atmospheric and accelerator beam stop neutrinos provide experiment- als with a good source with which to measure $\theta_{23}$ and $\Delta_{32}$. MINOS $^{17,18}$ predominantly determines $\Delta_{32}$ while the mixing angle $\theta_{23}$ is determined mainly by the Super-K atmospheric data $^{13,14,15,16}$. Present values for these parameters $^{23}$ are $\Delta_{32} = 2.40^{+0.24}_{-0.22} \times 10^{-3}$ eV$^2$ and $\sin^2 \theta_{23} = 0.50^{+0.13}_{-0.11}$. The remaining mixing angle, $\theta_{13}$, mixes the two scales. This same analysis gives $\sin^2 \theta_{13} \leq 0.040$; recent analyses hint at a value of $\theta_{13}$ differing from zero $^{24,25,26}$. Recent review articles can be found at Refs. $^{27,28}$.

As we enter the era of precision measurements, global analyzes of neutrino data must employ a full three-neutrino framework in order to correctly assess the neutrino mixing parameters. This will become evident herein as we consider various experiments’ impact upon the small parameters $\theta_{13}$ and $\varepsilon := \theta_{23} - \pi/4$, the deviation of $\theta_{23}$ from maximal mixing. The quantitative knowledge of $\theta_{13}$ is a particularly important part of neutrino oscillation phenomenology because it sets the magnitude of possible CP violating effects as well as the size of effects that might be used to determine the neutrino mass hierarchy. There are presently three new reactor experiments planned or under construction which are designed to measure $\theta_{13}$, Daya Bay $^{29}$, Double CHOOZ $^{30}$, and RENO $^{31}$; an LBL experiment is also planned, T2K $^{32}$. The subsequent generation of experiments, which will be designed to ascertain the level of CP violation, cannot proceed until the current generation better determines the value of $\theta_{13}$. In addition, a more quantitative knowledge of the mixing angles, and particularly of $\theta_{13}$, can help discern between models and symmetries of the physics that underlies neutrino mixing $^{33}$. The deviation of $\theta_{23}$ from maximal mixing is also important in model building as it might indicate the presence of a broken symmetry. At short baselines, the oscillation probabilities which might probe the mixing angle $\theta_{13}$ are quadratic in this small parameter; however, we have previously shown there are terms in the oscillation probability which are linear and appreciable at very long baselines (VLBL) $^{34,35,36}$ and arise from interference between the oscillations driven by the two mass-squared differences. This is also a region of the parameter space where one can look $^{37,38}$ for CP violating effects. The sub-GeV data set of the Super-K atmospheric experiments is potentially sensitive to such effects. Furthermore, it was shown in Ref. $^{39}$ that there is a non-trivial relation between $\varepsilon$ and $\theta_{13}$ for sub-GeV neutrinos at VLBLs. As such, the extraction of these parameters from the at-
mospheric data requires a full three neutrino treatment since approximations overly simplify the correlations of the parameters.

We here investigate atmospheric neutrino oscillations with the full three neutrino oscillation probabilities. Because we do not use truncated expansions, all terms linear in \( \theta_{13} \) and \( \varepsilon \) will be considered as well as higher order contributions. We do not expect a large change in the extracted parameters as only a limited number of the Super-K data bins lie in the region where linear terms will be significant. On the other hand, in the context of atmospheric data, \( \theta_{13} \) is itself a small effect as is the octant of \( \theta_{23} \). Small effects can sometimes have a proportionally larger impact on something that is inherently small. In keeping with the use of the full three neutrino oscillation probabilities, we also utilize the method proposed in Refs. [39, 40] to treat the MSW effect [41, 42]. With a crustal density profile of the earth, this treatment of the MSW effect is exact so that approximate expressions for the oscillation probabilities are not needed. We also include a model for the multi-ring events, a data subset often neglected.

**II. ANALYSIS**

In vacuo, the probability that a neutrino of flavor \( \alpha \) and energy \( E_\nu \) will be detected as a neutrino of flavor \( \beta \) after traveling a distance \( L \) is given by

\[
P_{\alpha\beta}(L/E_\nu) = \delta_{\alpha\beta} - 4 \sum_{k,j=1}^{3} (U_{\alpha j} U_{\alpha k} U_{\beta j} U_{\beta k}) \sin^2 \varphi_{jk},
\]

where \( \varphi_{jk} := 1.27 \Delta_{jk} L/E_\nu \), where \( L \) is measured in kilometers, \( E_\nu \) in GeV, and the mass eigenvalues \( m_i \) in eV. The matrix \( U_{\alpha j} \) is the unitary matrix that relates the mass basis \( i \) to the flavor basis \( \alpha \). We assume CP conservation so that the \( U_{\alpha j} \) are real. Neutrinos which propagate long distances through matter of sufficient densities can incur significant interactions which are diagonal in flavor. For matter of constant density, the upshot of these interactions is a modification of the effective mixing angles and mass-squared differences so that an oscillation formula similar to Eq. (1) holds. The density of the earth may be approximated as piecewise constant \([43]\). In addition, for certain energies and densities, the neutrinos can undergo parametric resonances in regions of varying densities \([44]\). To account for these interactions, we employ a simple model of the earth: a mantle of density 4.5 gm/cm\(^3\) and a core of density 11.5 gm/cm\(^3\) with radius 3486 km. Using the methods in Refs. [39, 40], we are able to fully incorporate an exact three neutrino model of the neutrino-matter interactions which automatically incorporates any possible parametric resonances.

Our interest is to study and extract the following parameters from the experimental data: \( \theta_{13} \), \( \theta_{23} \), and \( \Delta_{32} \). As such, we fix the solar mixing parameters from a recent analysis \([27]\); \( \theta_{12} = 0.58 \) and \( \Delta_{21} = 8.0 \times 10^{-5} \) eV\(^2\). Given that there is no evidence to indicate CP violation, we assume CP conservation and take \( \delta = 0 \). We include the details of our analyses of the relevant experiments in the Appendix. We comment on them briefly here. The Super-K atmospheric data is statistically the most significant data set, and it covers a range of over four orders of magnitude in \( L/E \). Our analysis employs the most recent results from the full atmospheric data, the most recent MINOS results \([18]\), the K2K results \([20]\), and the CHOOZ results \([21]\).

In order to ascertain the importance of the linear and higher-order terms in \( \theta_{13} \) (and also \( \varepsilon \)), we compare our results with those generated by the often used sub-dominant approximation which arises from an expansion in the ratio of the mass-squared differences, \( \alpha = \Delta_{12}/\Delta_{32} \). In this approximation, the leading order oscillation probabilities are given by

\[
\begin{align*}
\mathcal{P}_{ee} &= 1 - \sin^2 2\theta_{13} \sin^2 (\varphi_{32}) \\
\mathcal{P}_{e\mu} &= \sin^2 \theta_{23} \sin^2 2\theta_{13} \sin^2 (\varphi_{32}) \\
\mathcal{P}_{\mu\mu} &= 1 - 4 \cos^2 \theta_{13} \sin^2 \theta_{23} (1 - \cos^2 \theta_{13} \sin^2 \theta_{23}) \times \sin^2 (\varphi_{32})
\end{align*}
\]

Additional correction terms \([27, 28, 45]\) can then be added. The results for the sub-dominant approximation when compared to the results for the full three-neutrino oscillation probabilities will inform us of the size of the correction terms.

We begin by examining the mass-squared difference \( \Delta_{32} \). We plot \( \Delta \chi^2 \) versus \( \Delta_{32} \) using both the sub-dominant approximation, Fig. 1a, and the full three-neutrino calculation, Fig. 1b, with \( \theta_{13} \) and \( \theta_{23} \) as varied parameters. The [black] solid curves are obtained from the Super-K atmospheric data alone. The [red] dash-dot curves employ the K2K, MINOS, and CHOOZ data, omitting the Super-K atmospheric data. These curves are largely determined by the recent MINOS data which constrain the mass-squared difference more so than Super-K, as is well known. The analysis utilizing all of the data sets (atmospheric, K2K, MINOS, and CHOOZ) is depicted by the [blue] dashed curves. Notice that although the Super-K atmospheric data is not as constraining as MINOS, it combines with MINOS to produce a reduced bound, particularly from above. In comparing the approximation, Fig. 1a, with the full calculation, Fig. 1b, we see that the sub-dominant approximation is useful for determining the mass-squared difference \( \Delta_{32} \). A very careful inspection will reveal that the full three-neutrino analysis produces a slightly larger bound than does the sub-dominant approximation. Our results are \( \Delta_{32} = 0.25^{+0.02}_{-0.03} \) eV\(^2\) at the 90\% confidence level. (The errors quoted for our calculations will be for \( \Delta \chi^2 = 6.25 \), the 90\% confidence level for a three parameter fit.)

We next present \( \Delta \chi^2 \) versus \( \varepsilon = \theta_{23} - \pi/4 \) using the sub-dominant approximation, Fig. 2a, and the full three-neutrino calculation, Fig. 2b, with \( \theta_{13} \) and \( \Delta_{32} \) as var-
FIG. 1: [color online] $\Delta \chi^2$ versus mass-squared difference $\Delta_{32}$ for the a.) sub-dominant approximation and b.) full three-neutrino calculation. The [black] solid curves utilize only atmospheric data; the [red] dot-dash curves utilize K2K, MINOS, and CHOOZ data; the dashed [blue] curves utilize all the data sets: atmospheric, K2K, MINOS, and CHOOZ.

FIG. 2: [color online] $\Delta \chi^2$ versus $\varepsilon$ for the a.) sub-dominant approximation and b.) full three-neutrino calculation. The [black] solid curve utilizes only atmospheric data; the dashed [blue] curve utilizes all the data sets: atmospheric, K2K, MINOS, and CHOOZ.

FIG. 3: [color online] The same as Fig. 1 except $\Delta \chi^2$ versus $\theta_{13}$ is presented.

ied parameters. We express our result in terms of $\theta_{23}$, rather than $\sin^2 2\theta_{23}$ or $\sin^2 \theta_{23}$, because the oscillation probabilities truly are a function of $\theta_{23}$. The [black] solid curve again represents the Super-K atmospheric data alone. The [blue] dashed curve represents the results from all data sets: Super-K atmospheric, K2K, MINOS, and CHOOZ. Adding K2K, MINOS, and CHOOZ hardly alters the Super-K result. We do not present the results for K2K, MINOS, and CHOOZ alone because this data does not yield a reasonable constraint on $\theta_{23}$ when treated as a linear variable with a varied $\theta_{13}$ included in the analysis. Only in a two neutrino analysis does K2K and MINOS restrict the appropriate variable $\sin^2 2\theta_{23}$. Comparing the sub-dominant approximation with the full calculation, we see that the full three-neutrino probabilities produce an allowed region which is much more asymmetric about $\varepsilon = 0$. In fact, we find a statistically insignificant indication that $\theta_{23}$ is greater than $\pi/4$, maximal mixing. The ability of atmospheric data to determine the octant of $\theta_{23}$ has also been investigated in [46]. We find the value at 90% CL is $\varepsilon = 0.03^{+0.09}_{-0.15}$.

In Fig. 3a, we present $\Delta \chi^2$ versus $\theta_{13}$ calculated in the sub-dominant approximation and full three-neutrino formulation with $\theta_{23}$ and $\Delta_{32}$ as varied parameters. Previously, it has been shown [16] that in the sub-dominant approximation the atmospheric data alone restrict $\theta_{13}$. Focusing upon our sub-dominant calculation, Fig. 3a, the [black] solid curve depicts the corresponding result from our analysis. As noted in the Appendix, our analysis quantitatively reproduces the results in Ref. [16]. For $\Delta \chi^2 < 4.6$, the 90% confidence level for a two-neutrino analysis, we both find $\sin^2 \theta_{13} < 0.14$ (or $|\theta_{13}| < 0.38$). This is a very important calibration of our analysis tool.
The effect of $\theta_{13}$ on atmospheric oscillations is small, and obtaining the same result implies we are reproducing small effects, not just the global features of the analysis. The dash-dot [red] curve in Fig. 3a is the result of analyzing the K2K, MINOS, and CHOOZ data, neglecting the Super-K atmospheric data. This curve is mainly determined by the CHOOZ data. We see that CHOOZ is more constraining on $\theta_{13}$ than is the Super-K atmospheric data. However, the dashed [blue] curve presents the results utilizing all of the data sets and shows that the Super-K data does somewhat reduce the error on $\theta_{13}$; this is due to the indirect effect arising from Super-K further constraining the mass-squared difference $\Delta m_{23}^{2}$.

To obtain the constraints on $\theta_{13}$ implied by the Super-K atmospheric data, it is important to use the data from Ref. [16] which is more finely binned than earlier Super-K work. Note that the curves in the sub-dominant approximation are symmetric about $\theta_{13} = 0$ as is manifest from the approximate oscillation formulae, Eq. (2). In Refs. [24, 25, 26], it has been observed that recent data imply a statistically insignificant non-zero value for $\theta_{13}$; our results are likewise consistent.

Turning to the full three-neutrino calculation, Fig. 3b, we find $\Delta \chi^2$ to be very asymmetric with a strong preference for negative $\theta_{13}$ when using only Super-K data, the [black] solid curve. The [red] dash-dot curve employs only K2K, MINOS, and CHOOZ data; it is symmetric about the origin so that the asymmetry present when all data is included, the [blue] dashed curve, is due to the Super-K data. What is more, we see the novel result [17] that $\theta_{13}$ is constrained from above by the Super-K atmospheric data, not by CHOOZ, while it is constrained from below primarily by CHOOZ.

This conclusion is further reinforced by looking at the allowed region for the parameters $\theta_{13}$ and $\theta_{23}$ as depicted in Fig. 4. We plot the 90% confidence level of $\Delta \chi^2$ for a two parameter analysis as we fix the third parameter $\Delta m_{32}^{2}$ in calculating these curves. The dash-dot [green] curve depicts the results for Super-K atmospheric data alone in the sub-dominant approximation. We compare this with the dashed [red] curve which also utilizes only the Super-K atmospheric data alone but incorporates the full three-neutrino probabilities. Again, we see the significant change brought about by incorporating the linear, and higher-order, terms in $\theta_{13}$. The allowed region grows, favoring negative $\theta_{13}$. The dash-dot [blue] curve utilizes all the data in the sub-dominant approximation. It is similar to the dash-dot [green] curve because the mass-squared difference $\Delta m_{32}^{2}$ is fixed when calculating the curves; the main effect of the MINOS experiment is to restrict $\Delta m_{32}^{2}$. Finally, the solid [black] curve utilizes all the data and the full three-neutrino oscillation probabilities. Note that the upper bound on $\theta_{13}$ is similar to that from the dashed [red] curve, i.e. the curve also utilizing the full three-neutrino oscillation probabilities but only the Super-K data. For the lower bound on $\theta_{13}$, however, we find similarities to the dash-dot-dot [blue] curve where the restriction on $\theta_{13}$ originates primarily from the CHOOZ experiment. Thus we again see that the upper bound on $\theta_{13}$ no longer arises from the CHOOZ experiment but is determined by the Super-K atmospheric experiment, while the lower bound continues to come from the CHOOZ experiment. Our final result for this mixing angle is $\theta_{13} = 0.07^{+0.18}_{-0.11}$.

The principle effect of utilizing the full three neutrino oscillation probabilities is the alteration of the shape of the allowed region for $\theta_{13}$, particularly the introduction of the asymmetry about zero. The absolute minimum for $\chi^2$ is lowered by only 1.3 [17]. This is because the minima are very close to $\theta_{13} = 0$ where the linear and higher order terms contribute little.

III. DISCUSSION

The most striking differences between the sub-dominant approximation and the full three-neutrino probabilities were seen in the determination of the mixing angle $\theta_{13}$, Fig. 3. Additionally, the deviation of $\theta_{23}$ from maximal mixing also produced noticeable features, though less striking, Fig. 2. Clearly, the two features are non-trivially linked as demonstrated in the allowed regions depicted in Fig. 4. In fact, we see from Fig. 2 that the Super-K data is the source of the asymmetry about $\theta_{13} = 0$ in the full three-neutrino model. To flesh out
we examine the relevant oscillation probabilities in the
core suppresses the oscillations and thus this angular bin
through the earth’s higher density core. We find that the
typical neutrino which produces leptons in this bin passes
this
L/E
ment that the angular bin,
be able to take data at energies below 100 MeV. We com-

FIG. 5: [color online] The two lowest eigenvalues of the effect-
which subset of the Super-K data results in these asym-
metries, we examine the various contributions of the data
to \( \chi^2 \) for a fixed positive and negative value of the mixing
angle \( \theta_{13} \) taken to be \( \pm 0.15 \). The total difference in \( \Delta \chi^2 \)
for \( \theta_{13} = +0.15 \) and \( \theta_{13} = -0.15 \) is \( \sim 7.0 \). Focusing on
the fully contained events, we find that two thirds of this
change in \( \Delta \chi^2 \) between the positive and negative values
of the mixing angle comes from the sub-GeV electron-like
events. Half of the total change in \( \Delta \chi^2 \) (3.5) arises from
a single angular bin within this subset of data, namely
the bin for e-like events in which the detected charged
lepton has zenith angle \( \vartheta \) satisfying \( -0.8 < \cos \vartheta < -0.6 \)
and momentum less than 250 MeV. The detected leptons
in this bin are produced by neutrinos which travel along
a very long baseline upward through the earth. Such
neutrinos fall into the region of \( L/E \) where we have pre-
viously shown effects linear in \( \theta_{13} \) to be significant in the oscillation probabilities \( P_{ee} \) and \( P_{\mu\mu} \) \cite{34,35,36}; such effects occur in a region where the sub-dominant approxi-
mation is the leading term in an expansion which is not
convergent. Terms linear in \( \theta_{13} \) can be even more signif-
icant \cite{18} should an atmospheric oscillation experiment
be able to take data at energies below 100 MeV. We com-
ment that the angular bin, \( -1.0 < \cos \vartheta < -0.8 \), is also in this
\( L/E \) region for the low-energy neutrinos; however, a
typical neutrino which produces leptons in this bin passes
through the earth’s higher density core. We find that the
core suppresses the oscillations and thus this angular bin
is not as sensitive to the effects linear in \( \theta_{13} \).

The preference of the data for negative \( \theta_{13} \) can be
linked to the excess of e-like events in the sub-GeV data
set \cite{37}. This excess is not present in the \( \mu \)-like data or in
the multi-GeV data so that an overall renormalization of
the atmospheric flux cannot account for the excess. To
understand the role of the data in extracting \( \theta_{13} \) and \( \theta_{23} \),
we examine the relevant oscillation probabilities in the
limit of a constant density mantle and sub-GeV neutrino
energies, keeping only terms linear in \( \theta_{13} \) and \( \varepsilon \) and av-
eraging over the \( \Delta_{21} \) oscillations; these approximations
have been discussed previously \cite{35,36}. As detailed in
the Appendix, the electron-like events at the Super-K detec-
tor are related to the \( \nu_e \) survival probability and the
\( \nu_\mu \) conversion probability via \( R_e = P_{ee} + r P_{e\mu} \) where \( r \)
is the ratio of the \( \nu_\mu \) to \( \nu_e \) flux at the source. This yields
the approximate expression

\[
R_e \approx 1 + r \sin^2 2\theta_{12} \left[ \frac{1}{2} - \frac{1}{r} \cos(2\theta_{12}) \theta_{13} - \varepsilon \right] \sin^2 \varphi_{21} \cos 2\vartheta_{0}. \tag{3}
\]

Here, \( \theta_{12} \) is the effective mixing angle in matter; addition-
ally, the phase \( \varphi_{21} \) employs the effective mass-squared
difference in matter corresponding to \( \Delta_{21} \). In this approxi-
mation, we can understand how to effect an excess
of electron-like events for sub-GeV neutrinos over a long
baseline,

\[
\frac{1}{2} - \frac{1}{r} \cos(2\theta_{12}) \theta_{13} - \varepsilon > 0. \tag{4}
\]

Using the same approximations, we can simply express
the MSW resonant energy

\[
E_R = \frac{\Delta_{21} \cos 2\theta_{12}}{2V \cos^2 \theta_{13}}. \tag{5}
\]

with \( V \approx 1.7 \times 10^{-13} \) eV in the mantle; this yields \( E_R \)
on the order of 100 MeV. This resonance is apparent when
we plot the eigenvalues of the effective mass-squared ma-
trix in the mantle, Fig. 5; the “resonance” is indicated
by the slight bowing in the curves toward each other and
is located at the point where the effective mass-squared
difference is minimal.

At the resonant energy, one has \( \theta_{12}^2 = \pi/4 \); for neutrino
energies above the resonance, the effective mixing angle
in matter increases up to \( \pi/2 \). As a consequence, for neu-
trino energies above 100 MeV in the mantle, the function
\( \cot(2\theta_{12}) \) is negative; to reiterate, the coefficient of the
\( \theta_{13} \) term in the inequality, Eq. [4], is negative. We note
that for these low energy atmospheric neutrinos \( r \approx 2 \) so that
the first two terms of the inequality approximately
sum to zero. If \( \theta_{13} \) is restricted to positive values, then
the mixing angle \( \theta_{23} \) must lie in the first octant \( \varepsilon < 0 \)
in order to account for the excess in \( R_e \). However, if we
allow \( \theta_{13} \) to run the full range of allowed parameter space
in a CP conserving theory, then a negative value of this
mixing angle can easily accommodate the excess in \( R_e \),
even permitting \( \theta_{23} \) to lie in the second octant as is the
case in our analysis.

To demonstrate the point regarding the effect of terms
linear in \( \theta_{13} \), we plot \( R_e \) in Fig. 5 for sub-GeV neutrinos
in angular bin \( -0.8 < \cos \vartheta < -0.6 \). The solid [black]
curve employs our best fit parameters. To show the effect
of \( \theta_{13} \), we also plot the e-like events for \( \theta_{13} = \pm 0.15 \) with
\( \Delta_{23} \) and \( \theta_{23} \) unchanged. The dash-dot [red] curve has
negative \( \theta_{13} \), and the dashed [blue] curve has positive
θ_{13}. It is clear that a negative value of this mixing angle permits an excess of e-like events for sub-GeV neutrinos.

Returning to Fig. 4, we see how the full three-neutrino oscillation probabilities jointly affect the allowed region for the two mixing angles θ_{23} and θ_{13}. Using the subdominant approximation, the dash-dot-dot [blue] curve represents the allowed region when all the data is employed. As expected, the region is symmetric about θ_{13} = 0, and we note that the actual value of θ_{13} has little impact upon the allowed values of θ_{23}, save the neighborhood immediately around θ_{13} = 0. Inclusion of the higher order terms in the oscillation probability dramatically alters this picture. As discussed above, the data now favor negative θ_{13}, with the atmospheric Super-K data shrinking the 90% CL contour, the solid [black] curve, for positive θ_{13}. No longer is the contour symmetric about a particular value of θ_{13}; hence, the true value of this mixing angle will impact the allowed region for the θ_{23} mixing angle. In particular, the allowed region for θ_{23} shrinks as θ_{13} approaches positive values. In the future, should a reactor neutrino experiment confirm a nonzero value for |θ_{13}|, it will have interesting consequences for the allowed value of θ_{23}. With such a measurement, we would perhaps see two true local minima in the Δχ^2 versus θ_{13} plot in Fig. 3. The impact upon Fig. 4 would be to separate the jointly allowed regions into two disconnected curves with the limit on θ_{23} more tightly constrained for positive values of θ_{13}.

IV. CONCLUSIONS

As we enter into the era of precision neutrino experiments, small effects, such as those arising from θ_{13} or the octant of θ_{23}, require a careful treatment in the analysis. Future reactor experiments [29, 30, 31] are sensitive to θ_{13}^2 and thus can determine the magnitude of θ_{13}, but not its sign. Long-baseline experiments, e.g., [32], will contain small effects that are linear in θ_{13}, while an upgraded Super-K will produce additional data in the region where we have found significant effects linear in θ_{13}. How these different data interplay with each other in determining θ_{13}, including its sign, and the octant of θ_{23} will be most interesting.

In Ref. [49], it was shown that the constancy of R_ee imposes an upper bound on |θ_{13}| as well as constrains θ_{23} to be near maximal mixing. We find that present atmospheric data restrict the value of θ_{13} from above, while the limit from below remains as determined by CHOOZ. We find θ_{13} = −0.07±0.18, assuming no CP violation. We have investigated which data points lead to the asymmetry in θ_{13} and find that it is the atmospheric data in the very long baseline region previously noted [34, 55, 36] to have significant terms linear in θ_{13}. We further found that the Earth’s MSW effect plays an important role as it increases the effective value of θ_{13} in matter such that the atmospheric data provides a strict upper bound on θ_{13}. Further, the data producing the preference for a negative θ_{13} is data with an excess of e-like events, R_ee > 1. Allowing θ_{13} to be negative supports this excess and permits θ_{23} to be in the second octant. The parameters θ_{13} and θ_{23} are found to be correlated; the statistically insignificant negative value for the minimum of θ_{13} relates to the minimum for θ_{23} being statistically insignificantly in the second octant, and the error in θ_{23} is dependent on the value of θ_{13}. Future measurements of θ_{13} will impact the allowed value for θ_{23}.

For Δ_{32} and θ_{23} we find Δ_{32} = 0.25^{+0.07}_{−0.05} eV^2 and θ_{23} = π/4 ± 0.01, where the use of the full three-neutrino oscillation probabilities leads to the asymmetry in the errors. We find that a quantitative analysis requires utilizing the more finely binned atmospheric data of Ref. [16], the use of the full three-neutrino oscillation probabilities, and the inclusion of the full three-neutrino MSW effect.

APPENDIX: EXPERIMENTAL SIMULATION

In this appendix, we present the computational tools we use to analyze the Super-K atmospheric, CHOOZ, K2K, and MINOS experiments. The analysis tool for the Super-K atmospheric data is similar to that being used by others [23, 27, 28]; however, it is distinct in that we employ a full three-neutrino oscillation probability rather than an approximate expansion, use a full three-neutrino treatment of the Earth’s MSW effect, and include a model of the multi-ring data. The analysis of CHOOZ, K2K, and MINOS data is standard. Additional details can be found in Ref. [50]. Also, in this appendix, we demonstrate the efficacy of our analysis tools by comparing our results with others’ when appropriate.

The appendix is organized as follows. We first discuss
the Super-K atmospheric experiment, beginning with the contained events followed by the upgoing muon events. Then we discuss our statistical treatment of this experiment. Finally, we include a similar discussion for the CHOOZ, K2K, and MINOS experiments.

1. Super-K contained events

In order to observe atmospheric neutrinos at Super-Kamiokande, the neutrinos must interact with matter in either the detector or the surrounding environment to produce charged particles. The direction and energy of these charged particles can be deduced from the Cherenkov light they emit while traveling through the water-filled detector; from this data, one can infer on average the direction and energy of the initial neutrino. The Super-K experiment classifies the various detections in terms of the production point of the charged lepton, the number of charged particles produced, and their subsequent motion through the detector.

Contained events refer to events in which the charged lepton is produced by the neutrino within the detector. These events are subdivided into fully contained and partially contained events. If an event is fully contained, then the charged lepton(s) produced within the detector do not escape the detector. An event is partially contained if the charged lepton(s) exit the detector. Finally, these two data sets are further separated into single-ring and multi-ring events according to the number of charged particles produced by the neutrino; if only one charged lepton is observed in the detector, it is termed a single-ring event. We first discuss the fully contained single-ring events and then extend this analysis to the other classes of data. The fully contained single-ring events are statistically the most significant subset of the data and the cleanest to analyze. Preliminary discussions of the analysis technique utilized for the fully contained events can be found in Refs. 51, 52.

The Super-K detector distinguishes between electrons and muons by the fuzziness of the Cherenkov ring generated by the charged lepton; however, the detector cannot differentiate an electron $e^-$ from a positron $e^+$ or a $\mu^-$ from a $\mu^+$ and charge conservation, the detector can only determine if an event is $e$–like, originating from either a $\nu_e$ or $\bar{\nu}_e$ interaction, or $\mu$–like, originating from either a $\nu_\mu$ or $\bar{\nu}_\mu$ interaction. Thus the detector counts charged leptons of flavor $\alpha$ in energy bin $m$ and zenith angular bin $n$ over the run time $T$

$$N^\alpha_{mn} = \sum_{\nu, \tau} \left( \frac{dN^\nu_{mn}}{dt} + \frac{dN^\tau_{mn}}{dt} \right) T, \quad \text{(A.1)}$$

where the quantity $dN^\nu_{mn}/dt$ represents the rate at which a neutrino, created in the atmosphere with flavor $\beta$, will be detected as an $\alpha$–like event in the appropriate energy and angular bins within the detector.

This rate depends upon the atmospheric neutrino flux, the neutrino oscillation probability from source to detector, the kinematics of the charged lepton production, and the detector efficiencies. We may write it as

\[
\frac{dN^\alpha_{\alpha-\beta}}{dt} = N \int_{E_{vis}^{m,\text{min}}}^{E_{vis}^{m,\text{max}}} dE_{vis} \int d\cos \theta_\nu \int dE_\nu \int d\cos \theta_s \int d\phi_s \\
\times \varepsilon(E_{vis}) \frac{d^2 \Phi_\alpha(E_\nu, \cos \theta_\nu)}{dE_\nu \cos \theta_\nu} \hat{\sigma}_{\alpha,\beta}(E_\nu, \cos \theta_\nu) \frac{d^3 \sigma_\beta(E_\nu, E_\ell, \cos \theta_\ell)}{dE_\ell d\cos \theta_\ell d\phi_\ell} \\
\times \Theta(\cos \vartheta^{m,\text{max}} - \cos \vartheta) \Theta(\cos \vartheta - \cos \vartheta^{m,\text{min}}). \quad \text{(A.2)}
\]

We define the variables in Eq. (A.2). $N$ represents the number of target protons. $E_{vis}$ is the energy measured by the detector (this quantity is defined differently depending on the data sample); $E_{vis}^{m,\text{max}}$ ($E_{vis}^{m,\text{min}}$) is the maximum (minimum) value of $E_{vis}$ for bin $m$. (For single-ring fully contained events, $E_{vis}$ is simply the energy of the created lepton, $E_{vis} = E_\ell$.) $\vartheta$ is the zenith angle of the detected charged lepton with $\cos \vartheta = 1$ indicating the vertically downward direction. The relative angle between the incident neutrino and the produced charged lepton are described by the $\theta_s$ scattering angle and the $\phi_s$ azimuthal angle. The energy of the incident neutrino is $E_\nu$ with zenith angle $\theta_\nu$. The azimuthally averaged atmospheric neutrino flux for a neutrino of flavor $\alpha$ is $d^2 \Phi_\alpha(E_\nu, \cos \theta_\nu)/dE_\nu d\cos \theta_\nu$ which we take from Ref. 53. $\varepsilon(E_{vis})$ corresponds to the detection efficiency. $d^3 \sigma_\beta(E_\nu, E_\ell, \cos \theta_\ell)/dE_\ell d\cos \theta_\ell d\phi_\ell$ is the differential cross section for a neutrino of energy $E_\nu$ and flavor $\beta$ to produce a charged lepton of flavor $\beta$ with energy $E_\ell$ through a scattering angle $\theta_\ell$. Although the differential cross section which occurs in Eq. (A.2) does not depend on the azimuthal angle $\phi_s$, the geometry that determines in which angular bin an event lies does depend on $\phi_s$. This is because the zenith angle $\vartheta$ of the charged lepton is given in terms of the neutrino zenith angle $\theta_\nu$ and the scattering angles $\theta_s$ and $\phi_s$ by

$$\cos \vartheta = \cos \theta_\nu \cos \theta_s - \sin \theta_s \sin \theta_\nu \cos \phi_s. \quad \text{(A.3)}$$
The energy range for atmospheric neutrinos as measured at Super-K requires the use of several cross sections. At low energies, below 1 GeV, the dominant process is charged-current quasi-elastic scattering from the proton and the nucleons in the oxygen nucleus in H$_2$O, e.g., $\nu_e + p^+ \rightarrow e^+ + n$ and $\nu_e + n \rightarrow e^- + p^+$. At intermediate energies, peaking around 1.5 GeV, the dominant process is single-pion resonance production, i.e. $\nu_e + N \rightarrow \ell_e + N^*$ followed by $N^* \rightarrow N^+ + \pi$. At higher energies, starting at 1 GeV and dominating above 10 GeV, deeply inelastic scattering occurs, $\nu_e + N \rightarrow \ell_e + X$.

We utilize the same set of cross sections as was used with $R$, e.g., $\nu_e + p^+ \rightarrow e^+ + n$ and $\nu_e + n \rightarrow e^- + p^+$. The energy range for atmospheric neutrinos as measured at Super-K requires the use of several cross sections. At low energies, below 1 GeV, the dominant process is charged-current quasi-elastic scattering from the proton and the nucleons in the oxygen nucleus in H$_2$O, e.g., $\nu_e + p^+ \rightarrow e^+ + n$ and $\nu_e + n \rightarrow e^- + p^+$. At intermediate energies, peaking around 1.5 GeV, the dominant process is single-pion resonance production, i.e. $\nu_e + N \rightarrow \ell_e + N^*$ followed by $N^* \rightarrow N^+ + \pi$. At higher energies, starting at 1 GeV and dominating above 10 GeV, deeply inelastic scattering occurs, $\nu_e + N \rightarrow \ell_e + X$.

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The neutrino oscillation probability in vacuum $\mathcal{P}_{\alpha\beta}(E_\nu, \cos \theta_\nu)$ averaged over this production height in the atmosphere

$$\mathcal{P}_{\alpha\beta}(E_\nu, \cos \theta_\nu) = \int_0^L dh \, P_\alpha(h, E_\nu) \, \mathcal{P}_{\alpha\beta}(L(h, \cos \theta_\nu)/E_\nu),$$

(A.4)

where $L$ is related to $h$ and $\cos \theta_\nu$ by

$$L = \sqrt{R^2 \cos^2 \theta_\nu + h(2R + h) - R \cos \theta_\nu},$$

(A.5)

with $R$ the radius of the Earth. $P_\alpha(h, E_\nu)$ is the normalized probability for a neutrino of flavor $\alpha$ to be created at a height $h$, a quantity we take from Ref. [54].

The neutrino oscillation probability in vacuum $\mathcal{P}_{\alpha\beta}(L/E_\nu)$ is given in Eq. (1); however, the coherent forward scattering of neutrinos on matter alters the probability for those neutrinos which pass through the Earth [44]. Neutral current interactions between the neutrinos and matter are not flavor dependent leaving the oscillation probabilities unaffected; however, charged current interactions will introduce into the Hamiltonian a flavor dependent potential. In the flavor basis, we may write [54] the neutrino evolution equation as

$$i\partial_t \nu_f = \left[\frac{1}{2E_\nu} U M U^\dagger + V\right] \nu_f$$

(A.6)
Finally, we must determine the detector efficiency $\varepsilon(E_{\text{vis}})$, which has not been furnished by the experimentalists. We can, however, extract it from information provided. For no oscillations, Eq. (A.2) becomes, utilizing our assumptions,
We have assumed, appropriate for these high energies, that the scattering is forward. This allows us to replace the charged lepton angle \( \vartheta \) with the neutrino angle \( \theta_\nu \); we may also perform the integration over the scattering angles \( \theta_\nu \) and \( \phi_\nu \) in the cross section. \( N_A \) is Avogadro’s number. The function \( R(E_\mu, E_{\text{th}}) \) is the average distance that a muon of energy \( E_\mu \) will travel until its energy reaches the value \( E_{\text{th}} \), the amount of energy needed to traverse the detector; this quantity is expressed in the natural units for range, distance times the Earth’s density. \( A_T(E_{\text{th}}, \cos \theta_\nu) \) is the area projected onto a plane perpendicular to the muon direction such that a muon of energy \( E_{\text{th}} \) or greater can pass through this part of the detector. The details for calculating \( A_T(E_{\text{th}}, \cos \theta_\nu) \) can be found in Ref. [60]. For the stopping muons, \( A_S(E_{\text{th}}, \cos \theta_\nu) = A(\ell_{\text{min}}, \cos \theta_\nu) - A_T(E_{\text{th}}, \cos \theta_\nu) \), where \( A(\ell_{\text{min}}, \cos \theta_\nu) \) is the projected area of the detector with a path length greater than \( \ell_{\text{min}} \) taken to be 7 m by the experimentalists. Note that there is only muon data as electrons/positrons produced in the rock are unable to travel to the detector. The data covers the angular region from \( \cos \theta_\nu = 0 \) to \( \cos \theta_\nu = -1 \), directions where muon production from the rock exceeds the cosmic ray background. Since the neutrinos can originate as either electron or muon neutrinos, we sum over the two neutrino flavors \( \alpha \) as in Eq. (A.10). The upgoing muon data is binned in ten angular bins and not binned in energy, resulting in a total of 20 data points.

Analysis of these events is not as computationally intensive as the calculation of the contained events because the forward scattering allows the integration over the scattering angles and the muon energy \( E_\mu \) to be performed outside the fitting program. The parameters being fit are contained in \( \mathcal{P}_{\alpha\mu}(E_\nu, \cos \theta_\nu) \) which is independent of the muon energy and scattering angle. We have found it efficient to change the integration over these variables to the Feynman scaling variables \( x \) and \( y \), as is natural for the deeply inelastic region.

3. Super-K Statistical Analysis

We use the most recent experimental data from Ref. [16] which includes 180 data points for fully contained single-ring events, 90 for fully contained multi-ring events, 80 for partially-contained events, 10 for upward through-going muons, and 10 for upward stopping muons; in all, this constitutes 370 data points. In order to determine the neutrino oscillation parameters, we construct a \( \chi^2 \) based upon a Poisson distribution, following the same procedure used in Ref. [16]. We incorporate systematic errors by utilizing the “pull” approach as described in Ref. [?], which allows one to incorporate systematic errors in the analysis without adding adjustable parameters. The approach is based upon allowing linear corrections to the theoretical predictions for each systematic error. Our \( \chi^2 \) function is

\[
\chi^2 = \sum_{n=1}^{370} \left[ 2 \left( N_{\text{the}}(n) - N_{\text{obs}}(n) \right) + 2N_{\text{obs}}(n) \ln \left( \frac{N_{\text{obs}}(n)}{N_{\text{the}}(n)} \right) \right] + \sum_{i=1}^{43} \left( \frac{\xi_i}{\sigma_i} \right)^2.
\]

\( N_{\text{obs}}(n) \) is the number of observed events in the bin \( n \); \( N_{\text{the}}(n) \) is the theoretical prediction of the number of events in that bin; \( \xi_i \) is the systematic error pull for the systematic error \( i \); and \( \sigma_i \) is the one-sigma value for the systematic error \( i \). \( N_{\text{the}}(n) \) represents a modified prediction of the expected number of events due to the inclusion of systematic errors; the systematic errors adjust this quantity through an assumed linear dependence on the pulls \( \xi_i \). Here we use 45 systematic errors arising from different inputs into the data analysis as described in Tables VII–X taken from Ref. [15]. For these 45 errors, all of them contributed to the \( \chi^2 \) except the overall

\[
\frac{dN_{\mu,S,T}}{dt} = N_A \int_0^\infty dE_\nu \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} d\cos \theta_\nu \int_0^{E_{\mu}} dE_\mu \\
\times \frac{d^2 \Phi_\alpha(E_\nu, \cos \theta_\nu)}{dE_\nu \, d\cos \theta_\nu} \mathcal{P}_{\alpha\mu}(E_\nu, \cos \theta_\nu) \frac{d\sigma_{\mu}(E_\nu; E_\mu)}{dE_\mu} R(E_\mu, E_{\text{th}}) A_{S,T}(E_{\text{th}}, \cos \theta_\nu).
\]
We compare our analysis with that performed by the experimentalists in Ref. [16]. To do this, we utilize the sub-dominant approximation in Eq. (2) and minimize the above $\chi^2$. Our best fit oscillation parameters are $(\Delta_{32}, \sin^2 \theta_{23}, \theta_{13}) = (2.5 \times 10^{-3} \text{ eV}^2, 0.51, 0.01)$ with an overall $\chi^2$ of 416 for the 370 data points. In Fig. 1a, the [black] solid curve represents $\Delta \chi^2$ versus $\Delta_{32}$ for the sub-dominant approximation, using only atmospheric data. In Fig. 7 we present the allowed region for $\Delta_{32}$ and $\sin^2 2\theta_{23}$ corresponding to $\Delta \chi^2 = 4.61$, the 90% confidence level for a two parameter fit. We also present $\Delta \chi^2$ versus $\sin^2 2\theta_{23}$ in Fig. 8.

At the 90% confidence level for a two parameter fit, we find the allowed parameter values $2.1 \times 10^{-3} \text{ eV}^2 < \Delta_{32} < 5.1 \times 10^{-3} \text{ eV}^2$ and $0.938 < \sin^2 2\theta_{23}$. Additionally, we extract the allowed value for $\theta_{13}$ from the [black] solid curve in Fig. 3a. $-0.38 < \theta_{13} < 0.38$. This is exactly the result of Ref. [16], and our other results are in excellent agreement with that analysis. As noted previously, our reproduction of the allowed region for $\theta_{13}$, which has a nonzero but small effect on the atmospheric data, is a very strong test of our analysis.

Finally, in Figs. 9 and 10, we compare the predicted number of neutrino events corresponding to our best fit parameters with the experimental data as a function of the zenith angle. We also present the Monte Carlo predictions for the expected number of events in the absence of neutrino oscillations. Each of the different Super-K atmospheric data sets is depicted. The results are a good fit to the data, comparable to that found in Ref. [16].

4. CHOOZ experiment

For the CHOOZ reactor experiment, we follow a standard procedure as described in Ref. [21]. In our analysis, we use experimental data that consists of seven positron energy bins for each of the two reactors, giving a total of 14 bins. We include a $14 \times 14$ covariance matrix, $V_{ij}^{-1}$, to account for the correlation between the energy bins, and we include the systematic error from the overall normalization and energy calibration. We write the expected positron yield for the $k$th reactor and the $j$th energy spectrum bin as

$$\overline{X}(E_j, L_k, \theta, \Delta_{32}) = \overline{X}(E_j) \overline{P}(E_j, L_k, \theta, \Delta_{32}),$$

$$j = 1, \ldots, 7, \ k = 1, 2, \quad (A.13)$$
where \( \tilde{X}(E_i) \) is the distance-independent positron yield in the absence of neutrino oscillations, \( L_k \) is the reactor-detector distance, and \( \mathcal{P}(E_j, L_k, \theta, \Delta_{32}) \) is the oscillation probability averaged over the energy bin and the detector and reactor core sizes. In our fitting routine, we minimize the following \( \chi^2 \) function with respect to the neutrino oscillation parameters

\[
\chi^2(\theta, \Delta_{32}, \alpha, g) = \sum_{i=1}^{14} \sum_{j=1}^{14} \left( X_i - \alpha \tilde{X}(g E_i, L_i, \theta, \Delta_{32}) \right) V_{ij}^{-1} \left( X_j - \alpha \tilde{X}(g E_j, L_j, \theta, \Delta_{32}) \right) + \left( \frac{\alpha - 1}{\sigma_\alpha} \right)^2 + \left( \frac{g - 1}{\sigma_g} \right)^2,
\]

(A.14)

with the absolute normalization constant \( \alpha \) and the energy-scale calibration factor \( g \).

\[
\begin{align*}
\text{PC stopping} & \quad \text{PC through-going} \\
\text{Upward stopping} & \quad \text{Upward through-going}
\end{align*}
\]

FIG. 10: The same as Fig. 9 except the data sets are now the partially contained stopping events (upper left), the partially contained through-going events (upper right), the upward stopping muon events (lower left), and the upward through-going muon events (lower right).

5. K2K experiment

For the K2K experiment \cite{20}, we employ the method developed in Ref. \cite{61} to estimate the expected no-oscillation neutrino spectrum, \( S(E_\nu) \), in the relevant energy range of \( \sim 0.2 \) to \( \sim 3.0 \) GeV. The expected number of neutrino events for oscillating neutrinos is then

\[
N_{n}^{\text{theo}} = \int_{E_{\text{min}}(n)}^{E_{\text{max}}(n)} S(E_\nu) P_{\mu\mu}(L/E_\nu),
\]

(A.15)

where \( P_{\mu\mu}(L/E_\nu) \) is the muon neutrino survival probability and \( E_{\text{max}}(n) \) (\( E_{\text{min}}(n) \)) are the maximum (minimum) energy values for the energy bin \( n \). For the statistical analysis, we follow the procedure described in Ref. \cite{62}. We only use the single-ring sub-sample, which consist of 58 neutrino events. The signature for neutrino oscillations from \( \nu_\mu \) to \( \nu_\tau \) in a two neutrino analysis are both a reduction in the total number of observed neutrino events and a distortion in the neutrino energy spectrum. The \( \chi^2 \) function is divided into two terms: the observed to-theoretical number of events detected at the Super-K detector, \( \chi^2_{\text{norm}} \), and the shape of the spectrum included in \( \chi^2_{\text{shape}} \). We use the “pull” method \cite{21} to account for 31 systematic uncertainties by adding a third term \( \chi^2_{\text{syst}} \).

\[
\chi^2_{\text{K2K}} = \chi^2_{\text{norm}} + \chi^2_{\text{shape}} + \chi^2_{\text{syst}}.
\]

(A.16)

The best fit oscillation parameters, \( \Delta_{32} \) and \( \theta_{23} \), are obtained by minimizing \( \chi^2_{\text{K2K}} \).

The systematic parameters included in \( \chi^2_{\text{syst}} \) arise from the neutrino energy spectrum at the near detector site, the flux ratio, the neutrino-nucleus cross-section, the efficiency and the energy scale of the Super-K detector, and the overall normalization. The \( k \)th systematic error is represented by the coefficient \( C_{k}^{\text{syst}} \) and modifies the expected number of neutrino events, Eq. (A.15), in a linear

\[
\Delta \chi^2 = \sum_{i=1}^{14} \sum_{j=1}^{14} \left( X_i - \alpha \tilde{X}(g E_i, L_i, \theta, \Delta_{32}) \right) V_{ij}^{-1} \left( X_j - \alpha \tilde{X}(g E_j, L_j, \theta, \Delta_{32}) \right) + \left( \frac{\alpha - 1}{\sigma_\alpha} \right)^2 + \left( \frac{g - 1}{\sigma_g} \right)^2,
\]

(A.14)

where 58 neutrino events. The signature for neutrino oscillations from \( \nu_\mu \) to \( \nu_\tau \) in a two neutrino analysis are both a reduction in the total number of observed neutrino events and a distortion in the neutrino energy spectrum. The \( \chi^2 \) function is divided into two terms: the observed to-theoretical number of events detected at the Super-K detector, \( \chi^2_{\text{norm}} \), and the shape of the spectrum included in \( \chi^2_{\text{shape}} \). We use the “pull” method \cite{21} to account for 31 systematic uncertainties by adding a third term \( \chi^2_{\text{syst}} \).

\[
\chi^2_{\text{K2K}} = \chi^2_{\text{norm}} + \chi^2_{\text{shape}} + \chi^2_{\text{syst}}.
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The best fit oscillation parameters, \( \Delta_{32} \) and \( \theta_{23} \), are obtained by minimizing \( \chi^2_{\text{K2K}} \).

The systematic parameters included in \( \chi^2_{\text{syst}} \) arise from the neutrino energy spectrum at the near detector site, the flux ratio, the neutrino-nucleus cross-section, the efficiency and the energy scale of the Super-K detector, and the overall normalization. The \( k \)th systematic error is represented by the coefficient \( C_{k}^{\text{syst}} \) and modifies the expected number of neutrino events, Eq. (A.15), in a linear

\[
\Delta \chi^2 = \sum_{i=1}^{14} \sum_{j=1}^{14} \left( X_i - \alpha \tilde{X}(g E_i, L_i, \theta, \Delta_{32}) \right) V_{ij}^{-1} \left( X_j - \alpha \tilde{X}(g E_j, L_j, \theta, \Delta_{32}) \right) + \left( \frac{\alpha - 1}{\sigma_\alpha} \right)^2 + \left( \frac{g - 1}{\sigma_g} \right)^2,
\]

(A.14)
manner according to the “pull” method

\[ \tilde{N}_n^{\text{theo}} = N_n^{\text{theo}}(n) + \sum_{k=1}^{31} C_n^k \xi_k, \]

\[ \tilde{N}_n^{\text{total}} = \sum_{n=1}^{8} \tilde{N}_n^{\text{theo}}, \]  

(A.17)

with \( \xi_k \) the pull corresponding to systematic error \( k \).

Due to the low statistics, we employ a Poisson distribution; hence, the expressions for \( \chi^2_{\text{norm}} \) and \( \chi^2_{\text{shape}} \) are given by

\[ \chi^2_{\text{norm}} = 2 \left( \tilde{N}_n^{\text{theo}} - N_n^{\text{data}} - N_n^{\text{data}} \ln \left( \frac{\tilde{N}_n^{\text{theo}}}{N_n^{\text{data}}} \right) \right), \]

\[ \chi^2_{\text{shape}} = 2 \sum_{n=1}^{8} \left( \tilde{N}_n^{\text{theo}} - N_n^{\text{data}} - N_n^{\text{data}} \ln \left( \frac{\tilde{N}_n^{\text{theo}}}{N_n^{\text{data}}} \right) \right)^2, \]

(A.18)

where \( N_n^{\text{data}} \) is the experimental data provided by the K2K collaboration [20] and the superscript “total” implies a sum over \( n \). The contribution to \( \chi^2 \) from the systematic errors is

\[ \chi^2_{\text{syst}} = \sum_{j,k=1}^{31} \xi_k M_{kj}^{-1} \xi_j, \]

(A.19)

where we use an error matrix \( M_{kj} \) constructed from Tables 8.1 and 8.2 provided in Ref. [62].

In Fig. 11 we depict \( \Delta \chi^2 \) versus \( \Delta_{32} \) for an analysis that utilizes only the K2K data in the subdominant approximation; also, Fig. 12 shows the 90% CL allowed region in the [red] dashed contour for \( \Delta_{32} \) and \( \sin^2 2\theta_{23} \) in the subdominant approximation. The absolute minimum in our fit is \((\Delta_{32}, \sin^2 2\theta_{23}) = (2.78 \times 10^{-3} \text{ eV}^2, 0.998)\). At 90% CL, we find \( 2.2 \times 10^{-3} \text{ eV}^2 \leq \Delta_{32} < 3.2 \times 10^{-3} \text{ eV}^2 \). The total number of observed events 58 is in agreement with the 56 events found from the model. All of these results are consistent with the analysis performed by the experimentalists in Ref. [20].

6. MINOS experiment

The MINOS experiment is quite similar to the K2K experiment and thus we apply a similar analysis technique. For the no-oscillation spectrum, we use the Monte Carlo simulation provided by the MINOS collaboration [18]. We normalized this spectrum to measurements made at the near detector. A total of 1065 events were expected in the absence of neutrino oscillations. With the no-oscillation spectrum and Eq. (A.15), we calculate the expected number of neutrino events in the presence of neutrino oscillations. We use the MINOS data [18] corresponding to two years of beam operation in which 884 \( \nu_\mu \) neutrino events are observed. The data consists of 15 energy bins along with three systematic errors: the relative normalization between the far and near detectors with a 4% uncertainty; the absolute hadronic energy scale with a 11% uncertainty; and a 50% uncertainty in the neutral-current background rate. For the definition of \( \chi^2 \), we use the Poisson distribution function

\[ \chi^2_{\text{MINOS}} = \frac{1}{2} \sum_{n=1}^{15} \left( \tilde{N}_n^{\text{theo}} - N_n^{\text{data}} - N_n^{\text{data}} \ln \left( \frac{\tilde{N}_n^{\text{theo}}}{N_n^{\text{data}}} \right) \right)^2 + \sum_{j=1}^{3} \left( \frac{\xi_j}{\sigma_j} \right)^2, \]

(A.20)

where the symbols are analogously defined to those in the K2K section.

In Fig. 13 we plot \( \Delta \chi^2 \) versus \( \Delta_{32} \) for the K2K data in the subdominant approximation; likewise, in Fig. 12
we present the allowed region at 90% CL, the [blue] solid curve, for $\Delta_{22}$ and $\sin^22\theta_{23}$. The minimum of $\chi^2$ is located at $\Delta_{22} = 2.41 \times 10^{-3}$ eV$^2$ and $\sin^22\theta_{23} = 0.9990$. The allowed intervals of these parameters at 90 % CL are \( 2.25 \times 10^{-3} \text{ eV}^2 < \Delta_{22} < 2.8 \times 10^{-3} \text{ eV}^2 \) and $0.86 < \sin^22\theta_{23}$ for $\Delta\chi^2 = 4.6$. All of these results are consistent with the analysis performed by the experimentalists in Ref. 13.

Results which combine CHOOZ, K2K, and MINOS are presented in the main body of this work. This Appendix provides details of the analysis tools we use throughout this work; additional details may be found in Ref. 51.

APPENDIX: ACKNOWLEDGMENTS

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