The variational principle and effective action for a spherical dust shell

Valentin D. Gladush

Department of Physics, Dnepropetrovsk National University,
per. Nauchniy 13, Dnepropetrovsk 49050, Ukraine
E-mail: gladush@ff.dsu.dp.ua

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Abstract

The variational principle for a spherical configuration consisting of a thin spherical dust shell in gravitational field is constructed. The principle is consistent with the boundary-value problem of the corresponding Euler-Lagrange equations, and leads to “natural boundary conditions”. These conditions and the field equations following from the variational principle are used for performing of the reduction of this system. The equations of motion for the shell follow from the obtained reduced action. The transformation of the variational formula for the reduced action leads to two natural variants of the effective action. One of them describes the shell from a stationary interior observer’s point of view, another from the exterior one. The conditions of isometry of the exterior and interior faces of the shell lead to the momentum and Hamiltonian constraints.

I Introduction

A spherically-symmetric dust shell is among the simplest popular models of collapsing gravitating configurations. The equations of motion for these objects were obtained in Refs. [1] and [2]. The construction of a variational principle for such systems was discussed in Refs. [3]–[5]. There are a number of problems here, basic of which is a choice of the evolution parameter (internal, external, proper). The choice of time coordinate, in turn, affects the choice of a particular quantization scheme, leading, in general, to quantum theories which are not unitarily equivalent.

In most of these papers the variational principle for shells is usually constructed in a comoving frame of reference, or in one of variants of freely falling frames of reference. However, using of such frames of reference frequently leads to effects unrelated to the object under consideration. In the approach related to proper time of the shell reduction of the system leads to complicated Lagrangians and Hamiltonians which creates difficulties on quantization. In particular it leads to theories with higher derivatives or to finite difference equations.

The essential physics involves a picture of a gravitational collapse from the point of view of an infinitely remote stationary observer. In quantum theory this point of view enables us
theory correctly. On the other hand, to treat primordial black holes in the theory of self-gravitating shells it is convenient to take the viewpoint of a central stationary observer. In our opinion, the choice of exterior or interior stationary observers is most natural and corresponds to real physics. The natural Hamiltonian formulation of a self-gravitating shell was considered in Refs. [6], [7]. However, this formulation was not obtained by a variational procedure from some initial action containing the standard Einstein-Hilbert term. The general Lagrange approach to the theory of dust shells in General Relativity was developed in Ref. [8]. In present paper the natural Lagrange and Hamiltonian formulation of the spherical self-gravitating dust shell is constructed, which has some specific features in comparison with general approach. The system under consideration is regarded as a compound spherical configuration consisting of two vacuum spherical regions $D_-$ and $D_+$ with spherical boundary surface $\Sigma$ formed by the shell. The initial action is taken as the sum of actions of the York type $I_Y = I_{EH} + I_{\partial D}$ [9] for each regions and the action for the dust matter on the singular hypersurface $\Sigma$. Here $I_{EH}$ is the the Einstein–Hilbert action, and $I_{\partial D}$ is the boundary term. The constructed variational principle is compatible with boundary-value problems of the corresponding Euler–Lagrange equations for each region of the configuration, and, when we vary with respect to metric, leads to the “natural boundary conditions” on the shell. The obtained conditions together with gravitation field equations, are used for performing the reduction of the system and eliminating of the gravitational degrees of freedom. The equation of motion for the shell is obtained from the reduced action by considering normal variations of the shell. Transforming of the variational formula and applying of the surface equations leads to two variants of effective action. One of them describes the shell from an interior stationary observer’s point of view, and the other from the exterior one. Going over to the Hamiltonian description and using the isometry conditions of the exterior and interior faces of the shell generates momentum and Hamiltonian constraints.

Here $c$ is the velocity of light, $k$ is the gravitational constant, $\chi = 8\pi k/c^2$. The metric tensor $g_{\mu\nu} (\mu, \nu = 0, 1, 2, 3)$ has signature (+ - - -).

II Total action for the configuration, bulk and surface equations

Consider a set of the regions $D = D_- \cup \Sigma \cup D_+ \subset V^{(4)}$ in spherically symmetric space-time $V^{(4)}$. Here $D_-$ and $D_+$ are the interior and exterior regions, respectively, which are separated by the spherically-symmetric infinitely thin dust shell $\Sigma$ with the surface dust density $\sigma$. Choose in $D_\pm$ the general angle coordinates $x^i : \{x^2 = \theta, \ x^3 = \alpha\}$ ($i, k = 2, 3$) and individual space-time coordinates $x^a_\pm (a, b = 0, 1)$ for $D_\pm$, respectively. Then the gravitational fields in the regions $D_\pm$ are described by the metrics

\begin{align}
(4) ds_\pm^2 &= (2) ds^2_{\pm} - r^2 d\sigma^2, \\
(2) ds^2_\pm &= \gamma_{ab}^\pm dx^a_\pm dx^b_\pm, \quad d\sigma^2 = h_{ij}dx^i dx^j = d\theta^2 + \sin^2 \theta d\alpha^2,
\end{align}

where the two-dimensional metrics $\gamma_{ab}^\pm$ and the function $r$ depend on the coordinates $x^a_\pm$.

Einstein’s equations and the curvature scalar for each region $D_\pm$ can be represented in the form

\begin{align}
(4) G_{\mu\nu} &= \frac{8\pi G}{c^4} (\kappa \sigma - \rho), \\
(2) G_{ab} &= \frac{\kappa}{\hat{r}} \left( e^a - e^{a_0} \frac{e^{a_0}}{r} \right) \left( e^b - e^{b_0} \frac{e^{b_0}}{r} \right) - \frac{1}{r} \left( \left( e^a - e^{a_0} \frac{e^{a_0}}{r} \right) \frac{\partial}{\partial x^a} \left( e^b - e^{b_0} \frac{e^{b_0}}{r} \right) \right.
\end{align}
\[ (5) G_{\alpha j} \equiv 0, \]  
\[ (4) G_{ij} = \frac{r}{2} \left( r \frac{(2) R - 2\Delta r}{r} \right) h_{ij} = 0, \]  
\[ (4) R = \frac{(2) R - 4}{r} \Delta r - \frac{2}{r^2} (\nabla r)^2 - \frac{2}{r^2}, \]  

where \( \Delta r = \nabla^a \nabla_a r = r ; a, \) \( (\nabla r)^2 = \gamma^{ab} \nabla_a r \nabla_b r = r ; a \cdot r, \) \( \nabla_a \equiv ; a \) is the covariant derivative with respect to \( x^a \) in metric \( \gamma_{ab}, \) and \( (2) R \) is the curvature scalar of two-dimensional space with metric \( \gamma_{ab}, r, a, \equiv \partial r/\partial x^a. \) Here, for simplicity, we temporarily omit the signs “±”.

Now we introduce a general coordinate map \( x^a \in D, \) and metrics \( \gamma_{ab} \) such that \( \gamma_{ab|\Sigma} = \gamma_{ab|\Sigma}^\pm = \gamma_{ab}. \) Then \( (2) ds_{+|\Sigma} = (2) ds_{-|\Sigma} = (2) ds, \) and the world line \( \gamma \) of the shell in this map are given by equation \( x^a = x^a(s). \) Let

\[ \{ \vec{u} = u^a \partial_a, \vec{n} = n^a \partial_a \}, \quad \{ \omega = u_a dx^a, \eta = n_a dx^a \} \]  

be the general orthonormal vector and covector bases in the regions \( D_\pm. \) Here \( \partial_a = \partial/\partial x^a \) is the partial derivative with respect to \( x^a. \) The components of vectors \( \{ \vec{u}, \vec{n} \} \) and covectors \( \{ \omega, \eta \} \) satisfy the conditions \( u_a u^a = -n_a n^a = 1 \) and \( u_a n^a = 0. \) Hence, accurate to a general factor \( \epsilon = \pm 1, \) we obtain

\[ n_0 = \sqrt{-\gamma} u^1, \quad n_1 = \sqrt{-\gamma} u^0, \]  
\[ u_0 = \sqrt{-\gamma} n^1, \quad u_1 = \sqrt{-\gamma} n^0. \]  

where \( \gamma = \det |\gamma_{ab}|. \) With respect to the bases \( \{ \vec{u}, \vec{n} \} \) and \( \{ \omega, \eta \} \) we have

\[ \gamma_{ab} = u_a u_b - n_a n_b, \quad \gamma^{ab} = u^a u^b - n^a n^b, \quad \delta^a_b = u^a u_b - n^a n_b. \]  

Further, we shall suppose that the vector field \( \vec{u} \) in points \( p \in \Sigma \) is tangential to a world line of a shell \( \gamma \) so, that \( u_a^2| \Sigma = dx^a/(2)ds. \) The vector field \( \vec{n} \) in points \( p \in \Sigma \) is normal to \( \Sigma \) and is directed from \( D_- \) in \( D_+. \) Inside regions \( D_\pm \) the field of an orthonormal dyad \( \{ u^a, n^a \} \) is arbitrary.

Define the one-forms \( d\Sigma_a \) as

\[ dx^a \wedge d\Sigma_b = \delta^a_b dx^0 \wedge dx^1 = \delta^a_b d^2 x, \]  

where the symbol “\( \wedge \)” denotes the exterior product. It is also useful to define the one-forms

\[ d\Sigma_u = u^a d\Sigma_a, \quad d\Sigma_n = -n^a d\Sigma_a, \]  

which are dual to the one-forms \( \omega, \eta, \) so that

\[ \sqrt{-\gamma} \omega \wedge d\Sigma_u = \sqrt{-\gamma} \eta \wedge d\Sigma_n = \sqrt{-\gamma} d^2 x = \omega \wedge \eta, \]  
\[ \omega = -\sqrt{-\gamma} d\Sigma_n, \quad \eta = \sqrt{-\gamma} d\Sigma_u. \]

In addition we have \( g = \det |g_{\mu \nu}| = \gamma r^4 \sin^2 \theta. \)

Now we introduce the tensors of extrinsic curvature

\[ K_{\mu \nu} = -n_{\mu ; \rho} (n^\rho n_\nu + \delta^\rho_\nu), \quad K = g^{\mu \nu} K_{\mu \nu} = -n^\mu_{\mu}, \]
of local subspaces $\Sigma_n$ and $\Sigma_u$, which are orthogonal to the vectors $n^\mu = \{n^a, 0, 0\}$ and $u^\mu = \{u^a, 0, 0\}$, respectively. Here "\(\cdot\)\" is the covariant derivative with respect to metric $g_{\mu\nu}$, $K$ and $D$ are the trace of the tensors $K_{\mu\nu}$ and $D_{\mu\nu}$, respectively. On the shell the tensor $K_{\mu\nu}$ is the tensor of extrinsic curvature hypersurface $\Sigma$.

From the definitions (2.15) and (2.16) we can obtain

\[
K_{ik} = r(\vec{n}r)h_{ik}, \quad K_{ai} = K_{ia} = 0, \quad K_{ab} = K_{uu}u_au_b, \quad (2.17)
\]

\[
K_{uu} = K_{ab}u^au^b = -n^a_{;a}, \quad K = -\frac{1}{r^2} \left( r^2 n^a_{;a} \right) = K_{uu} - \frac{2(\vec{n}r)}{r}, \quad (2.18)
\]

\[
D_{ik} = r(\vec{u}r)h_{ik}, \quad D_{ai} = D_{ia} = 0, \quad D_{ab} = D_{nn}n_an_b, \quad (2.19)
\]

\[
D_{nn} = D_{ab}n_an_b = u^a_{;a}, \quad D = -\frac{1}{r^2} \left( r^2 u^a_{;a} \right) = -D_{nn} - \frac{2(\vec{u}r)}{r}, \quad (2.20)
\]

where $\vec{n}r = n^a r_a$, $\vec{u}r = u^a r_a$.

We take the total action for the spherically symmetric compound configuration under consideration in the form

\[
I_{\text{tot}} = I_{EH} + I_m + I_\Sigma + I_{\partial D} + I_0, \quad (2.21)
\]

where

\[
I_{EH} = -\frac{c}{2\chi} \int_{D_- \cup D_+} \sqrt{-g} (4) R \ d^2x \wedge d\theta \wedge d\alpha \quad (2.22)
\]

is the sum of Einstein-Hilbert actions for the regions $D_{\pm}$.

The dust on the singular shell $\Sigma$ is described by the action

\[
I_m = c \int_{\Sigma} \sigma \sqrt{-g} \ d\Sigma_a \wedge d\theta \wedge d\alpha. \quad (2.23)
\]

The third term in the right-hand side (2.21) is the matching term

\[
I_\Sigma = -\frac{c}{\chi} \int_{\Sigma} \sqrt{-g} [K] \ d\Sigma_a \wedge d\theta \wedge d\alpha, \quad (2.24)
\]

where the symbol $[A] = A_+ - A_-$ denotes the jump of the quantity $A$ on the shell $\Sigma$. The signs "\(\cdot\)\" indicate that the marked quantities are calculated as the limit values when we approach to $\Sigma$ from inside and outside, respectively.

The fourth term in the right-hand side (2.21)

\[
I_{\partial D} = \frac{c}{\chi} \int_{\partial D} \sqrt{-g} (Du^a - Kn^a) d\Sigma_a \wedge d\theta \wedge d\alpha \quad (2.25)
\]

contains the surface terms similar to Gibbons-Hawking surface term, which are introduced to fix the metric on the boundary $\partial D$ of the region $D$. Note, that the boundary $\partial D$ consists of the pieces of timelike as well as spacelike hypersurfaces. The last term $I_0$ in (2.21) contains the boundary terms, necessary for normalization of the action. It is needed when exterior boundary $\partial D_+$ of the region $D_+$ is situated on the timelike infinitely remote hypersurfaces.

Thus the total action $I_{\text{tot}}$ is the functional of the metrics $\{\gamma_{ab}, \gamma_{ab}^\perp\}$, of the shell radius $r$ and of the hypersurface $\Sigma$.

\[
I_{\text{tot}} \equiv I_{\text{tot}}[\gamma_{ab}, \gamma_{ab}^\perp, r; \Sigma].
\]
The first and the fourth terms in (2.21) form the action of the York type $I_Y = I_{EH} + I_{\partial D}$. It is used in variational problems with the fixed metric on the boundary $\partial D$ of the configuration $D$. This action can also be used in variational problems with the general relativistic version of the “natural boundary conditions” for “free edge” [10], when the metric on the boundary is arbitrary and the corresponding momenta vanishes. Together with $I_0$ it forms the York–Gibbons–Hawking action $I_{YGH} = I_Y + I_0$ for a free gravitational field.

In our case of the compound configuration we also fix the metric on the boundary $\partial D$. However, in addition, we have the boundary surface $\Sigma$ inside the system, with the singular distribution of matter on it. We can treat this configuration as two vacuum regions $D_{\pm}$ with common “loaded edge” (or “massive edge”) $\Sigma$. The sum of the actions of type $I_Y$ for these regions, and of the action for matter $I_m$, and normalizing term $I_0$ do leads to the action $I_{tot}$.

If there is no dust, $\sigma = 0$, the common boundary is not “loaded”. Then, the requirement $\delta I_{tot} = 0$, at arbitrary, everywhere continuous variations of the metric, gives generalization of the above “natural boundary conditions” for free hypersurface $\Sigma$. They coincide with the continuity conditions for the extrinsic curvature on $\Sigma$, i.e. with the standard matching conditions. If the matched edges “are loaded” by some surface matter distribution, then we obtain the surface equations or the boundary conditions for $D_{\pm}$. They are the analog of the generalized “natural boundary conditions” for “loaded edges”. The initial action $I_{tot}$ was so chosen, that the surface equations on $\Sigma$ which follow from requirement $\delta I_{tot} = 0$ coincide with the matching conditions on singular hypersurfaces [1]. In that case the variational principle for the action $I_{tot}$ will be compatible with the boundary-value problem of the corresponding Euler–Lagrange equations [11], [12].

After integrating with respect to angles and taking into account the relations (2.6) and (2.14), the actions (2.22) and (2.23) can be written in the form

$$I_{EH} = -\frac{c^3}{4k} \int_{D_{(2)}} \sqrt{-\gamma} \left( (2) R r^2 - 4r \Delta r - 2(\nabla r)^2 - 2 \right) d^2 x , \quad (2.26)$$

$$I_m = mc \int_{\Sigma^{(1)}} \sqrt{-\gamma} d\Sigma_m = -mc \int_{\Sigma} \omega , \quad (2.27)$$

where $m = 4\pi \sigma r^2 = \text{const}$ is the shell mass.

The matching (2.24) and the boundary surface (2.25) terms can be written as

$$I_{\Sigma} = \frac{c^3}{2k} \int_{\gamma} r^2 [K] \omega = \frac{c^3}{2k} \int_{\gamma} r [r K_{uu} - 2(\bar{m}r)] \omega , \quad (2.28)$$

$$I_{\partial D} = \frac{c^3}{2k} \int_{\partial D} r^2 \sqrt{-\gamma} (Du^a - Kn^a) d\Sigma_a . \quad (2.29)$$

In order to simplify the total action $I_{tot}$ we reduce the action (2.26) to the form including only the first-order derivatives. To this end, we use the fact that, in the two-dimensional space, a curvature scalar can be reduced (locally!) to the divergence of a vector (see Appendix A)

$$(2) R = 2V_{,a}^a , \quad (2.30)$$
Then, using the formulae
\[ \sqrt{-\gamma} r^2 R = 2\sqrt{-\gamma} r^2 V^a = 2(\sqrt{-\gamma} r^2 V^a)_{,a} - 4rr_{,a}V^a, \] (2.32)
\[ r\sqrt{-\gamma} \Delta r = (\sqrt{-\gamma} rr_{,a})_{,a} - \sqrt{-\gamma} (\nabla r)^2, \] (2.33)
the Einstein-Hilbert actions (2.26) can be rewritten as
\[ I_{EH} = I_g - I_\theta, \] (2.34)
where
\[ I_g = \int_{D^{(2)} \cup D^+_{(2)}} L_g d^2x. \] (2.35)

is the gravitational action for the gravitational field with the Lagrangian, which includes only the first order derivatives
\[ L_g = \frac{c^3}{2k} \sqrt{-\gamma} (2rr_{,a}V^a - r_{,a}r_{,a} + 1). \] (2.36)

Here \( r_{,a} = \gamma^{ab}r_{,a}, \) \( r_{,a} = \partial r/\partial x^a = (\bar{u}r)u_a - (\bar{n}r)n_a, \) \( r_{,a}r_{,a} = (\bar{u}r)^2 - (\bar{n}r)^2. \)

The second term in (2.34) is the sum of two surface terms
\[ I_\theta = \frac{c^3}{2k} \int_{\partial D^{(2)}} r\sqrt{-\gamma} W^a d\Sigma_a + \frac{c^3}{2k} \int_{\partial D^+_{(2)}} r\sqrt{-\gamma} W^a d\Sigma_a, \] (2.37)

where
\[ W^a = rV^a - 2r_{,a}. \] (2.38)

The term (2.37) includes the integration over total boundaries \( \partial D_+ \) and \( \partial D_- \) of the regions \( D_- \) and \( D_+ \). Further, take into account (2.31), (2.18) and (2.20), we find
\[ W^a = (rD_{nn} - 2(\bar{u}r)) u^a - (rK_{uu} - 2(\bar{u}r)) n^a = r(D_{uu}^a - Kn^a). \] (2.39)

Now the term (2.37) can be rewritten as the sum of two addends
\[ I_\theta = \tilde{I}_\Sigma + \tilde{I}_{\partial D}. \] (2.40)

The addend \( \tilde{I}_{\partial D} \) includes the integration only over that part of boundaries \( \partial D_+ \) and \( \partial D_- \) of region \( D_- \) and \( D_+ \) which coincides with the boundary \( \partial D \) of configuration \( D = D_- \cup \Sigma \cup D_+ \).

In the addend \( \tilde{I}_\Sigma \) we integrate over the remaining parts of the boundaries \( \partial D_\pm \), which means the integration over exterior and interior sides of common boundary \( \Sigma \) of the regions \( D_+ \) and \( D_- \), i.e. over the exterior and interior faces of the dust shell. Taking into account (2.37), (2.39), (2.28) and (2.29), it is easy to see, that \( \tilde{I}_{\partial D} = I_{\partial D} \) and \( \tilde{I}_\Sigma = I_\Sigma. \) After substitution (2.31) and (2.40) into (2.21), the surface terms are reduced and complete action acquires the ordinary and natural form.
\[ I_{tot} = I_g + I_m + I_0, \] (2.41)

where the action \( I_g \) contains the Lagrangian (2.36) with first-order derivatives only.

The forms (2.21) and (2.41) of the action \( I_{tot} \) are equivalent. Applying the action (2.21), we can evaluate the value of \( I_{tot} \) on the extremals, whereas we use formula (2.41) for finding

Now find the variation $\delta I_{tot}$ generated by varying $r$ and $\gamma^{ab}$. Using relations (2.10) and
\[ \delta\sqrt{-\gamma} = -\frac{1}{2}\sqrt{-\gamma}\gamma_{ab}\delta\gamma^{ab} = \sqrt{-\gamma}(n_a\delta n^a - u_a\delta u^a), \]
we can easily show that the equations
\[ \delta(r, a r^a) = 2(\dot{r} r, a \delta u^a - r^r_a \delta n^a - r^r_i a \delta r) + 2(r^r_i a \delta r)_i, \]
are specific to the two-dimensional case. It can easily be shown that the equations
\[ \delta\gamma_{ab} = n_a \delta n^a - u_a \delta u^a, \]
are satisfied by the variations $\delta r^a$ and $\delta n^a$ in the final formulas.

In order to calculate the variation $\delta I_g$, we use formulae
\[ \delta I_g = \frac{1}{2} \epsilon_{abcd} \sqrt{g} \frac{\partial \sqrt{g}}{\partial x^d} \delta I_m, \]
and also the metric on it and the normal to be fixed. Therefore
\[ \delta I_m = -mc \delta \int_\gamma \omega = -mc \int_\gamma \delta \omega_{\gamma}. \]
The sign “$|$” denotes the restriction of the one-form $\omega$ on the shell world line $\gamma$:
\[ \omega_{\gamma} = (u_a dx^a)_\gamma = (u_a dx^a/(2)ds) (2)ds = u_a u^a (2)ds = (2)ds, \]
such that
\[ \delta\omega_{\gamma} = \delta (2)ds = \frac{1}{2} u^a u_b (2)ds \delta\gamma_{ab} = -\frac{1}{2} u_a u_{b|\gamma} \delta\gamma_{ab} = -u_a \omega_{\gamma} \delta u^a. \]
The requirement of stationarity $\delta I_{tot} = 0$ with respect to arbitrary variations $\delta u^a$, $\delta n^a$ satisfying the above-mentioned conditions leads to
\[ \dot{r}' - r' u^b_{,b} = 0, \quad \dot{r}^r_i - \dot{r} n^b_{,b} = 0, \]
\[ 2r \dot{r} - 2rr' n^b_{,b} + r^2 - r'^2 + 1 = 0, \]
\[ 2r \dot{r}' - 2r \dot{r} u^b_{,b} - \dot{r}^2 + r'^2 - 1 = 0. \]
In deriving formulas (2.44) - (2.49) we used equations
\[ u^a_{,b} u^b = -n^a n_c u^c_{,b} u^b = n^a n^b_{,b}, \quad u^a_{,b} n^b = n^a u^b_{,b}, \]
\[ n^a_{,b} n^b = u^a u_c n^c_{,b} n^b = u^a n^b_{,b}, \quad n^a_{,b} u^b = u^a n^b_{,b}, \]
which are specific to the two-dimensional case. It can easily be shown that the equations
\[ u^a_{,b} u^b = -n^a n_c u^c_{,b} u^b = n^a n^b_{,b}, \quad u^a_{,b} n^b = n^a u^b_{,b}, \]
\[ n^a_{,b} n^b = u^a u_c n^c_{,b} n^b = u^a n^b_{,b}, \quad n^a_{,b} u^b = u^a n^b_{,b}, \]
\[ u^a_{,b} u^b = -2n^a n_c u^c_{,b} u^b = n^a n^b_{,b}, \quad u^a_{,b} n^b = u^a n^b_{,b}, \]
\[ n^a_{,b} n^b = 2u^a u_c n^c_{,b} n^b = u^a n^b_{,b}, \quad n^a_{,b} u^b = 2u^a n^b_{,b}, \]
\[ u^a_{,b} u^b = n^a n_c u^c_{,b} u^b = n^a n^b_{,b}, \quad u^a_{,b} n^b = n^a u^b_{,b}, \]
\[ n^a_{,b} n^b = u^a u_c n^c_{,b} n^b = u^a n^b_{,b}, \quad n^a_{,b} u^b = u^a n^b_{,b}, \]
\[ u^a_{,b} u^b = -2n^a n_c u^c_{,b} u^b = n^a n^b_{,b}, \quad u^a_{,b} n^b = u^a n^b_{,b}, \]
\[ n^a_{,b} n^b = 2u^a u_c n^c_{,b} n^b = u^a n^b_{,b}, \quad n^a_{,b} u^b = 2u^a n^b_{,b}, \]
\[ u^a_{,b} u^b = n^a n_c u^c_{,b} u^b = n^a n^b_{,b}, \quad u^a_{,b} n^b = n^a u^b_{,b}, \]
\[ n^a_{,b} n^b = u^a u_c n^c_{,b} n^b = u^a n^b_{,b}, \quad n^a_{,b} u^b = u^a n^b_{,b}, \]
which are specific to the two-dimensional case. It can easily be shown that the equations
\[ u^a_{,b} u^b = -n^a n_c u^c_{,b} u^b = n^a n^b_{,b}, \quad u^a_{,b} n^b = n^a u^b_{,b}, \]
\[ n^a_{,b} n^b = u^a u_c n^c_{,b} n^b = u^a n^b_{,b}, \quad n^a_{,b} u^b = u^a n^b_{,b}, \]
"
The variations of $I_{tot}$ with respect to $r$ lead to equation
\[ rV^a_a - \Delta r = 0 , \] (2.51)
which, in view of (2.30), is equivalent to the rest of the Einstein equations (2.5). Besides equations (2.47)-(2.51) we also obtain the surface equation for jumps
\[ [\tilde{n} \cdot r] - r[n_a V^a] = 0 , \] (2.52)
\[ c^2 r[\tilde{n} \cdot r] + km = 0 . \] (2.53)

Note that by virtue of (2.31) and (2.50) there exist formulae
\[ n_a V^a = -n^a_a = K_{uu} = n_a u^a_b b = n_a f^a \equiv f , \] (2.54)
where $f^a = u^a_b b = -f n^a$ is the acceleration vector of the shell. Therefore, formula (2.52) can be written as
\[ [\tilde{n} \cdot r] = r[K_{uu}] = r[n_a f^a] . \] (2.55)

In order to obtain the equations of motion for the dust spherical shell we shall consider the normal variations of the hypersurface $\Sigma$. Let each point $p$ be displaced at a coordinate distance $\delta x^a(p) = n^a \delta \lambda(p)$ in the direction of the normal. As a result of the displacement, we obtain a new hypersurface $\tilde{\Sigma}$. The initial and final positions of the shell are fixed, therefore we have $\delta \lambda(p) = 0$, $\forall p \in \Sigma \cap \partial D = \tilde{\Sigma} \cap \partial D$. In addition, we fix the metric $\gamma_{ab}$, and also all quantities on $\Sigma$, so that $\delta \lambda_m = 0$.

As a result of displacement of the hypersurface $\Sigma$, the original regions $D_+$ and $D_-$ are transformed into new regions $\tilde{D}_+$ and $\tilde{D}_-$, such that $\tilde{D}_- \cup \tilde{\Sigma} \cup \tilde{D}_+ = D_- \cup \Sigma \cup D_+ = D$. Then, for example, the variation of the region $D_-$ can be represented as $\delta D_- = \tilde{D}_- \setminus D_- = D_+ \setminus \tilde{D}_+$. The change of the action (2.55) induced by the displacement $\Sigma$, under the above conditions, is given by
\[ \delta I_{tot} = \delta I_g = \int_{\tilde{D}_- \cup \tilde{D}_+} L_g \ d^2 x - \int_{D_- \cup D_+} L_g \ d^2 x \cong - \int_{\tilde{D}_-} \left( L_g^+ - L_g^- \right) d^2 x . \] (2.56)
Here $L_g^+$ and $L_g^-$ are the Lagrangians determined by the relation (2.30) and calculated to the right and to the left of $\Sigma$, respectively. Under the infinitesimal normal displacement of the hypersurface $\Sigma$, the variation of the total action takes the form
\[ \delta I_{tot} = - \int_\Sigma (L_g^+ - L_g^-) \delta x^a d\Sigma_a = \int_\Sigma [L_g] \delta \lambda d\Sigma . \] (2.57)

Hence, owing to the arbitrariness of $\delta \lambda$ and the requirement $\delta I_{tot} = 0$, we find
\[ [L_g] = (L_g^+ - L_g^-) \big|_\Sigma = -\frac{c^2}{2\gamma} \sqrt{-\gamma} \left[ 2r(\tilde{n} \cdot r)K_{uu} - (\tilde{n} \cdot r)^2 \right] = 0 . \] (2.58)

Using formulas such as $[AB] = \tilde{A}[B] + \tilde{B}[A]$, where $\tilde{A} = (A_+ + A_-)/2$, we obtain the equations of motion in the form
\[ r(\tilde{n} \cdot r)[K_{uu}] + r[\tilde{n} \cdot r]K_{uu} - (\tilde{n} \cdot r)^2 \tilde{n} \cdot r = 0 . \] (2.59)

After substitution of expressions for $[\tilde{n} \cdot r]$ from (2.55) into (2.59), the equations of motion for the dust spherical shell can be written as
The relations (2.53), (2.55) and (2.60) form the necessary complete set of the boundary algebraic conditions imposed on normal derivatives of the shell radius \((\vec{n} r)_{|+}\), \((\vec{n} r)_{|-}\) and of the shell acceleration \(f_{|+}\) and \(f_{|-}\) (or component \(K_{uu|+}\) and \(K_{uu|-}\) of the extrinsic curvature tensor on the shell) with respect to the internal or external coordinates, respectively. In particular, this equations imply

\[
K_{uu} = n^a u^{;a}_{\pm} = n^a \frac{Du_a}{ds} \bigg|_\pm = \pm \frac{km}{2c^2r^2},
\]

or

\[
u^{a;\pm}_b u^b \bigg|_\pm = \frac{Du_a}{ds} \bigg|_\pm = \pm \frac{km}{2R^2} n^a.
\]

where \(Du_a = u^{a;\pm}_b dx^b\) is the covariant differential. These relations give us the equations of motions for the spherical dust shell with respect to coordinates \(x^a_+\) or \(x^a_-\) of the regions \(D_+\) or \(D_-\), respectively.

From the equations (2.61) it follows the two-dimensional spherically symmetric analog of the well-known Israel equations [1]

\[
n^a \frac{Du_a}{ds} \bigg|_+ + n^a \frac{Du_a}{ds} \bigg|_- = 0,
\]

\[
n^a \frac{Du_a}{ds} \bigg|_+ - n^a \frac{Du_a}{ds} \bigg|_- = -\frac{km}{c^2r^2} = -\frac{\chi \sigma}{2}.
\]

### III The reduced and effective actions for the dust spherical shell

Now we can realize a reduction of system and construct the reduced action for the shell. For this purpose we shall calculate action \(I_{tot}\) on the solutions of the vacuum Einstein equations (2.3)-(2.5) (or the equations (2.47)-(2.51)). In this case it is convenient to take action \(I_{tot}\) in the form (2.21). In addition we shall take into account the surface equations (2.53), (2.55) and (2.60). Note, that on this stage we explicitly use the following consequences of these equations

\[(4) \quad R = 0, \quad [\vec{n} \vec{r}] = r[K_{uu}], \quad r^2[K_{uu}] = -\frac{km}{c^2} .\]

Substituting these relations into (2.21) and taking into account (2.27), one finds

\[I_{tot\{Eqs. (3.1)\}} = J_{sh} + I_{0},\]

where

\[J_{sh} = I_{\Sigma\{Eqs. (3.1)\}} - m c \int_\gamma \omega,
\]

\[-\frac{1}{2} \int mc \omega\]

is the reduced action for the dust shell. This action must be considered together with the boundary conditions (2.53), (2.55) and (2.60). The action \(J_{sh}\) is quite certain if in the
Einstein equations \((2.3) - (2.5)\) which satisfy the boundary conditions \((2.63), (2.65)\) and \((2.61)\). The boundary term \(I_{\partial D} = I_{\partial D | (Eqs. \ 3.1)}\) in \((3.2)\) now has a fixed value and not essentially for further.

Note, that one usually comes to the action for the shell in the other form. In our approach this form of the action can be found by partial reduction of the initial action \(I_{\text{tot}}\) when the last boundary condition in \((3.1)\) is not taken into account. As a result, we come to the action similar to the expression in braces in \((3.3)\) or to some of its modification. Hence one can obtain the Lagrangian of the shell in the frame of reference of the comoving observer. However, the quantity \(K_{uu} = n_a f^a = n_a u^a u^b\) contains second derivatives of coordinates \(x^a\) with respect to the proper time of the shell. When these derivatives are eliminated by integrating by parts, we obtain rather complicated Lagrangians and Hamiltonians.

Now we introduce independent coordinates \(x^a_{\pm}\) in each of the regions \(D_{\pm}\). Then, the reduced action is the functional of embedding functions \(x^a_{\pm}(s)\) of the shell: \(J_{sh} \equiv J_{sh}[x^a_{\pm}(s), x^a_{\pm}(s)]\). Consider the variation of integrand in \(J_{sh}\) with respect to these functions. We have

\[
\delta \omega^a_{\pm} = \delta(2) ds_{\pm} = \delta \left( \sqrt{\gamma} a^a dx^a dx^b \right)_{\pm} = -(u_{a;b} u^b \delta x^a)(2) ds_{\pm} + d(u_a \delta x^a)_{\pm}. \tag{3.4}
\]

Hence, applying formulas \(\delta^a_b = u^a u_b - n^a n_b\) and \(u^a u_{a;b} = 0\), we obtain

\[
\left( \delta(2) ds - u^a u_{\pm} \delta x^a(2) ds \right)_{\pm} = d(u_a \delta x^a)_{\pm}, \tag{3.5}
\]

or, considering the boundary conditions \((2.61)\),

\[
\left( \delta(2) ds \pm \frac{km}{2c^2 r^2} n_a \delta x^a(2) ds \right)_{\pm} = d(u_a \delta x^a)_{\pm}. \tag{3.6}
\]

Further, using the conditions \((2.8)\) and \((2.9)\), we have \(^1\)

\[
(n_a \delta x^a ds)_{\pm} = \{ \sqrt{-\gamma} (u^1 \delta x^0 - u^0 \delta x^1) ds \}_{\pm} = \{ \sqrt{-\gamma} (dx^1 \delta x^0 - dx^0 \delta x^1) \}_{\pm}. \tag{3.7}
\]

Therefore the variational formula \((3.5)\) takes the form

\[
\left\{ \delta(2) ds \pm \frac{km}{2c^2 r^2} \sqrt{-\gamma} (dx^1 \delta x^0 - dx^0 \delta x^1) \right\}_{\pm} = d(u_a \delta x^a)_{\pm}. \tag{3.8}
\]

Now introduce the vector potential \(B_a = B_a(x^0, x^1)\) with the help of equation

\[
d \wedge (B_a dx^a) \equiv G_{01} dx^0 \wedge dx^1 = -\frac{km}{2c^2 r^2} \sqrt{-\gamma} dx^0 \wedge dx^1, \tag{3.9}
\]

where \(G_{ab} \equiv B_{b,a} - B_{a,b}\). Note that, in two-dimensional space, the integrability condition for the just introduced relation holds identically. With this definition in mind and owing to the fact that

\[
\delta(B_a dx^a) - d(B_a \delta x^a) = G_{10} (dx^0 \delta x^1 - dx^1 \delta x^0), \tag{3.10}
\]

the variational formula \((3.8)\) can be written in the form

\[
\delta \left\{ \delta(2) ds \pm B_a dx^a \right\}_{\pm} = d \{ (u_a \pm B_a) \delta x^a \}_{\pm}. \tag{3.11}
\]

\(^1\)In the paper \([8]\) the factor \(\sqrt{-\gamma}\) was lost in these formulas. However, in the particular case of the Schwarzschild solution it is of no importance since

\[
\sqrt{1 - \frac{2M}{r}} \approx 1 - \frac{2M}{r}.
\]
Thus, if we introduce the actions in the form

\[ I_{sh}^\pm = -mc \int \left( (2) ds \mp B_a dx^a \right) \bigg|_{\pm}, \tag{3.12} \]

then, owing to the variational formula (3.11), we shall obtain the stationarity condition \( \delta I_{sh}^\pm = 0 \) for the fixed initial and final positions of the shell. The just obtained actions are the natural modification of the action (3.3) which is compatible with the boundary conditions. The stationarity condition \( \delta I_{sh}^\pm = 0 \) for arbitrary variation of coordinates \( x^a_{\pm} \) yield the equations of motion for the shell with respect to external or internal coordinates. Therefore, formula (3.12) is the general form of the effective actions for the dust spherical shell in general relativity, where the vector potential \( B_a \) is found from equation (3.9).

\section*{IV The effective action for the spherical dust shell}

Now let us construct the effective actions for the spherical dust shell in the Schwarzschild gravitational fields. Using curvature coordinates, we choose common spatial spherical coordinates \( \{ r, \theta, \alpha \} \) in \( D_\pm \), and individual time coordinates \( t_\pm \) in \( D_\pm \), respectively. Then the world sheet of the shell \( \Sigma \), in interior and exterior coordinates, is given by equations \( r = R_-(t_-) \) and \( r = R_+(t_+) \), respectively.

The gravitational fields in the regions \( D_\pm \) are described by the metrics

\[ (4) ds^2_\pm = f_\pm c^2 dt^2_\pm - f^{-1}_\pm dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\alpha^2), \tag{4.1} \]

where

\[ f_\pm = 1 - \frac{2kM_\pm}{c^2 r}, \tag{4.2} \]

and \( M_+ \) and \( M_- \) are the Schwarzschild masses (\( M_+ > M_- \)).

In this case we have

\[ B_a dx^a = c\varphi(t_\pm, R) dt_\pm + U_R(t_\pm, R) dR. \tag{4.3} \]

Using the gauge condition \( U_R(t_\pm, R) = 0 \), the action (3.12) can be written as

\[ I_{sh}^\pm = -mc \int \left( (2) ds \mp c \varphi dt \right) \bigg|_{\pm}. \tag{4.4} \]

Further, formula (3.9) implies

\[ \frac{\partial \varphi}{\partial R} = \frac{km}{2c^2 R^2}. \tag{4.5} \]

From here, up to an additive constant, we find

\[ \varphi = -\frac{km}{2c^2 R}. \tag{4.6} \]

Eventually, the effective action for the shell can be represented as

\[ I_{sh}^\pm = \int L_{sh}^\pm dt_\pm = - \int \left( mc (2) ds \mp \frac{km^2}{2} dt \right) \bigg|_{\pm}. \tag{4.7} \]
where
\[ L_{\pm}^{{sh}} = -mc^2 \sqrt{f_{\pm} - f_{\pm}^{-1}R_{t_{\pm}}^2/c^2} \pm U \quad (4.8) \]
are the Lagrangians of the dust shell in the frames of reference of stationary observers in the regions \( D_{\pm}, \ (R_{t_{\pm}} = dR/dt_{\pm}) \), respectively, and
\[ U = -\frac{km^2}{2R} \quad (4.9) \]
is the effective potential energy of the shell gravitational self-action.

It is easy to check that from the actions (4.7) the equations of motion (2.61) of the dust spherical shell follow.

The actions \( I_{\pm}^{{sh}} \) transform each into other under the discrete gauge transformation
\[ M_{\pm} \rightarrow M_{\mp}, \quad (f_{\pm} \rightarrow f_{\mp}), \quad U \rightarrow -U, \quad t_{\pm} \rightarrow t_{\mp}. \]
They describe the transition from the interior observer to the exterior one and vice versa.

Note that the actions \( I_{\pm}^{{sh}} \) can be considered quite independently. The regions \( D_{\pm} \) together with the gravitational fields (4.1) can also be regarded separately and independently as manifolds with edges \( \Sigma_{\pm} \). These edges acquire the physical sense of the different faces of the dust shell with the world sheet \( \Sigma \) if the regions \( D_{\pm} \) are joined along these boundaries. The last can be realized only if the conditions of isometry for the edges \( \Sigma_{\pm} \) are fulfilled (or if the curves \( \gamma_{\pm} \) representing the world lines of the shell in the coordinates \( \{R, t_{\pm}\} \) coincide), where \( \tau \) is the proper time of the shell.

Consider some consequences following from the conditions of isometry for the edges. First of all we have the relationships between the velocities
\[ f_{\pm}^2 d\tau^2 - f_{\mp}^2 dR^2 = f_{\pm} c^2 d\tau^2 - f_{\mp}^{-1} c^2 dR^2 = c^2 d\tau^2, \quad (4.10) \]
for the edges \( \Sigma_{\pm} \) are fulfilled (or if the curves \( \gamma_{\pm} \) representing the world lines of the shell in the coordinates \( \{R, t_{\pm}\} \) coincide), where \( \tau \) is the proper time of the shell.

Further, from the Lagrangians \( L_{\pm}^{{sh}} \) we find the momenta and Hamiltonians for the shell
\[ P_{\pm} = \frac{\partial L_{\pm}^{{sh}}}{\partial R_{t_{\pm}}} = mR_{t_{\pm}} \left( f_{\pm} \sqrt{f_{\pm} - f_{\pm}^{-1}R_{t_{\pm}}^2/c^2} \right) = \frac{m}{f_{\pm}} R_{\tau}, \quad (4.13) \]
\[ H_{\pm}^{{sh}} = mc^2 \frac{f_{\pm}}{\sqrt{f_{\pm} - f_{\pm}^{-1}R_{t_{\pm}}^2/c^2}} \mp U = mc^2 \frac{dt_{\pm}}{d\tau} \mp U \quad (4.14) \]
or
\[ H_{\pm}^{{sh}} = c\sqrt{f_{\pm}(m^2c^2 + f_{\pm}P_{\pm}^2)} \mp U = mc^2 \sqrt{R_{t_{\pm}}^2/c^2} \mp U = E_{\pm}, \quad (4.15) \]
where \( E_{\pm} \) are the shell energies which are conjugated to the coordinate times \( t_{\pm} \) and
or exterior one). Eliminating the velocity $R_t$ from (4.13) and (4.15), the conditions of isometry for edges can be written as

$$f_+ P_+ = f_- P_- , \quad (E_+ + U)^2 - m^2 c^4 f_+ = (E_- - U)^2 - m^2 c^4 f_- . \quad (4.16)$$

The last equation can be rewritten as

$$(E_+ + E_-)(E_+ - E_- + 2U) = m^2 (F_+ - F_-) . \quad (4.17)$$

Substituting the expressions for $f_\pm$ and $U$ taken from (4.2), (4.9) in this equation, we obtain

$$(E_+ + E_-)(E_+ - E_-) = \frac{k m^2}{R} (E_+ + E_- - 2(M_+ + M_-)) . \quad (4.18)$$

Hence we find the relation between the Hamiltonians $H^\pm_{sh}$ and the Schwarzschild masses $M_\pm$

$$H^+_\text{sh} = H^-_{\text{sh}} = (M_+ - M_-) c^2 = E . \quad (4.20)$$

Here $E = E_+ = E_-$ denotes the total energy of the shell, which is conjugated both of the coordinate time $t_+$ and of that $t_-$ and whose value is independent of the stationary observer’s position (inside or outside of the shell). From now on we shall treat the relationships (4.16) and (4.20), which appear in the above independent description of the shell faces, as momentum and Hamiltonian constraints. Thus, the dynamic systems with Lagrangians $L^{\pm}_{sh}$ are not independent. They satisfy momentum and Hamiltonian constraints (4.16), (4.20) which ensure of isometry of the shell faces.

The Lagrangians $L^\pm_{sh}$ (4.8), as well as the relations (4.11) - (4.20), are valid only in a limited domain, since the used curvature coordinates are valid outside the event horizon only. Therefore, $L^-_{sh}$ can be used when $R > 2kM_- / c^2$, and $L^+_{sh}$ for $R > 2kM_+ / c^2$ ($M_+ > M_-$).

As is known, the complete description of the shells can be performed, for example, in the Kruskal-Szekeres coordinates. With respect to these coordinates the full Schwarzschild geometry consists of the four regions $R^+$, $T^-$, $R^-$, $T^+$, detached by the event horizons. Our above consideration concerned with the $R^+$ region only.

Formally, assuming $r$ to be the time coordinate, we can also use the action for the shell in the form (4.7) under the horizon, i.e. in the regions $T^-$ and $T^+$. In order to use the simplicity and convenience of the curvature coordinates and to conserve the information about the shells in the region $R^-$, it is sufficient to introduce an auxiliary discrete variable $\varepsilon = \pm 1$ and to make a change $\gamma (2) ds_\pm \rightarrow \varepsilon_\pm (2) ds_\pm$ in $I^\pm_{sh}$ (4.4) [5]. Here, $\varepsilon_\pm = 1$ corresponds to the shell into the $R^+$-region, and $\varepsilon_\pm = -1$, to the shell into the $R^-$-region. Further, introduce the quantities $\mu_\pm = \varepsilon_\pm m$. Then the extended action takes the form

$$I^\pm_{\text{sh}}(\mu_\pm) = \frac{1}{2} \int_{t_+}^{t_-} L^\pm_{\text{sh}}(\mu_\pm) dt_\pm = - \frac{1}{2} \int_{t_+}^{t_-} (\mu \gamma c (2) ds \mp U dt)_\pm , \quad (4.21)$$

where

$$L^\pm_{\text{sh}}(\mu_\pm) = - \mu_\pm c^2 \sqrt{f_\pm - f_\pm^{-1} R^2_{\pm} / c^2} \pm U \quad (4.22)$$

are the generalized Lagrangians describing the shell inside any of the $R^\pm$-regions with respect to the curvature coordinates of the interior $\{t_-, R\}$ or exterior $\{t_+, R\}$ regions. The event horizons $R^\pm_\gamma = 2kM_\pm / c^2$, as before, are the singular points of the dynamical
For the extended system (4.21) the Hamiltonians has the form

\[ H_{\pm}^\pm(\mu_\pm) = c\varepsilon_\pm \sqrt{f_\pm(m^2c^2 + f_\pm P_\pm^2)} \mp U = \mu_\pm c^2 \sqrt{f_\pm + R_\pm^2/c^2} \mp U. \] (4.23)

Hence, taking into account the Hamiltonian constraints (4.20), we find the standard relationships of the theory of the spherical dust shell [1] and rewrite them using new notation

\[ \mu_-\sqrt{f_- + R_-^2/c^2} - \mu_+\sqrt{f_+ + R_+^2/c^2} = \frac{k\mu^2}{Rc^2}, \] (4.24)
\[ \mu_-\sqrt{f_- + R_-^2/c^2} + \mu_+\sqrt{f_+ + R_+^2/c^2} = 2(M_+ - M_-). \] (4.25)

Now consider briefly the self-gravitating shell when \( M_- = 0 \). Denote \( M_+ = M \) and consider the shell moving in the \( R^+ \)-region. In the exterior coordinates, the Lagrangian and Hamiltonian of the shell are of the form

\[ L_{sh}^+ = -mc^2 \sqrt{1 - \frac{2\gamma M}{c^2 R}} - \left(1 - \frac{2\gamma M}{c^2 R}\right)^{-1} \frac{R_{\pm}^2}{c^2} - \frac{\gamma m^2}{2R}, \] (4.26)
\[ H_{sh}^+ = c \sqrt{m^2c^2 + \left(1 - \frac{2\gamma M}{c^2 R}\right)P_\pm^2 + \frac{\gamma m^2}{2R}}. \] (4.27)

In the interior coordinates, the same shell is described by the Lagrangian and Hamiltonian

\[ L_{sh}^- = -mc^2 \sqrt{1 - \frac{R_-^2}{c^2}} + \frac{\gamma m^2}{2R}, \] (4.28)
\[ H_{sh}^- = c \sqrt{m^2c^2 + P_-^2 - \frac{\gamma m^2}{2R}}. \] (4.29)

The dynamical systems with \( L_{sh}^\pm \) obey the momentum and Hamiltonian constraints \( P_- = f_+P_+ \), \( H_{sh}^+ = H_{sh}^- = Mc^2 \), and are canonically equivalent in the extended phase space [8]. However they are not canonically equivalent dynamic system, which is obtained at a choice of proper time as evolutional parameter.

### V Conclusions

In the paper, on the basis of the standard Einstein–Hilbert bulk action and surface action for the dust the variational formalism for a spherical dust shell is constructed. The total action also includes the surface matching and boundary terms. The variational principle is compatible with the boundary-value problem of the corresponding Euler–Lagrange equations. From the total action by variational procedure the bulk equations and complete set of boundary conditions are found. These equations are used for performing the reduction of the system. As a result, we come to the reduced action (3.3) for the spherical dust shell which must be considered together with surface equations. From reduced action we obtain the equations of motion for the dust shell.

Further, by transforming the variational formula (3.4) for the reduced action and taking into account the surface equations (2.61), we obtain the effective action \( I_{sh}^\pm \) for the dust shell which leads to correct equations of motion. The above procedure is carried out in the curvature coordinates for the interior and exterior regions \( D_\pm \) of the configuration.
the gravitational collapse from the point of view of the interior stationary observer and exterior remote stationary one.

The regions \( D_{\pm} \subset D \) together with the corresponding gravitational fields \( \Sigma_{\pm} \) can be treated as independent submanifolds with “loaded edges” \( \Sigma_{\pm} \) which can be described by the actions \( I_{sh}^{\pm} \). These edges acquire the physical sense of different faces of the dust shell with the world sheet \( \Sigma \) if the regions \( D_{\pm} \) are matched along these boundaries. From the conditions of isometric equivalence of the edges \( \Sigma_{\pm} \) we obtain the momentum and Hamiltonian constraints (4.16), (4.20).

The effective Hamiltonian \( H_{sh}^{-} \) was virtually postulated in [6] and was used for finding the energy spectrum of quantum states of the dust shell with the bare mass \( m \) that was less than the Planck mass \( m_{pl} \). In Ref. [13] the Hamiltonians \( H_{sh}^{\pm} \) was used for constructing the quasi-classical model of collapsing spherical configuration, for describing the tunneling spherical dust shell, and also for the model of the pair creation and annihilation of the shells. The method of constructing the effective action is easily generalized to the case of more complex spherical configurations with the space and surface distribution of fields and matter. The present approach (see also [8]) can be readily generalized to the case of higher dimensions and can be used for constructing the effective Lagrangians describing the cosmological scenarios with branes. In that case, by using the variational procedure, we can also find the complete set of boundary conditions on the singular hypersurfaces, which are necessary both in the theory of brane worlds and in the shell theory (see, for example, the papers [14] and [15] and references therein). In conclusion, it should be stressed that, in contrast to [8], the approach taken in this paper is specially adapted for the configurations which, after dimension reduction, are reduced to 2D-models. This allows us to use the equations specific to the two-dimensional case only, simplifies the variational technique and makes clearer the procedure of constructing the effective action.

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**Appendix A: The representation of the curvature scalar in the two-dimensional space**

In the two-dimensional space there are specific relationships, which can exist only in the spaces with dimensionality that equals two. Some of them have been already written (see (2.50) and (2.54)). Here we show that in the two-dimensional space the curvature scalar is expressed (locally!) in terms of the divergence of the vector constructed with the help of the vectors of two-dimensional orthogonal basis \( \{u^{a}, n^{a}\} \).

By definition we have

\[
\begin{align*}
    u_{a;b;c} - u_{a;c;b} &= R_{abc}^{d} u_{d}, \quad (A.1) \\
    n_{a;b;c} - n_{a;c;b} &= R_{abc}^{d} n_{d}. \quad (A.2)
\end{align*}
\]

Multiplying equation (A.1) by \( u^{f} \), and equation (A.2) by \( n^{f} \) and applying formula \( u_{d} u^{f} - n_{d} n^{f} = \delta_{d}^{f} \) gives
From here we find

\[ R_{ac} = u^b (u_{a;b;c} - u_{a;c;b}) - n^b (n_{a;b;c} - n_{a:c;b}) , \]  
\[ R = u^b (u_{b;a} - u_{a;b}) - n^b (n_{b;a} - n_{a;b}) = (u^b u^a - u^a u^b - n^a n^b + n^b n^a) :a , \]  
\[ (A.4) \]
\[ (A.5) \]

With the help of equations (2.50), this formula can be written in the form

\[ R = 2 (n^b n^a - u^b u^a) :a = 2 V^a , \]  
\[ (A.6) \]

which gives the required equation (2.30) expressing the curvature scalar in terms of the divergence of vector \( V^a \).

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