A second-order stability analysis for the continuous model of indirect reciprocity

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Abstract
Reputation is one of key mechanisms to maintain human cooperation, but its analysis gets complicated if we consider the possibility that reputation does not reach consensus because of erroneous assessment. The difficulty is alleviated if we assume that reputation and cooperation do not take binary values but have continuous spectra so that disagreement over reputation can be analysed in a perturbative way. In this work, we carry out the analysis by expanding the dynamics of reputation to the second order of perturbation under the assumption that everyone initially cooperates with good reputation. The second-order theory clarifies the difference between Image Scoring and Simple Standing in that punishment for defection against a well-reputed player should be regarded as good for maintaining cooperation. Moreover, comparison among the leading eight shows that the stabilizing effect of justified punishment weakens if cooperation between two ill-reputed players is regarded as bad. Our analysis thus explains how Simple Standing achieves a high level of stability by permitting justified punishment and also by disregarding irrelevant information in assessing cooperation. This observation suggests which factors affect the stability of a social norm when reputation can be perturbed by noise.

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Keywords: Indirect reciprocity, Prisoner’s dilemma, Evolution of cooperation, Perturbation

1. Introduction

The power and the instability of reputation have attracted interest among researchers in the field of social evolution (Alexander, 1987). Reputation strongly affects our behaviour from early childhood (Silver and Shaw, 2018), but it can turn to a capricious tyrant: Sometimes a small mistake ruins it without deserving, and it may be unrecoverable nowadays when digital footprints last forever. The question is how to make a stable reputation system that recovers from erroneous assessment.

The dynamics of reputation was first analysed in mathematical terms by considering a norm called Image Scoring (Nowak and Sigmund, 1998). It is a first-order norm in the sense that it assigns reputation to an individual depending on what she did to her co-player (Nowak and Sigmund, 2005). It also implies that Image Scoring ignores the co-player’s reputation. This is problematic because conditional cooperation crucially hinges on the ability to base a decision on the co-player’s reputation. By referring to the co-player’s reputation, one must be allowed to refuse to cooperate toward an ill-reputed co-player without risking her own reputation: Otherwise, well-intentioned punishment will not be distinguished from malicious defection, and conditional cooperators cannot thrive in such an environment. Based on this argument, some authors have advocated ‘Standing’ strategy, according to which a player loses good reputation only by defecting against a well-reputed player (Sugden, 1986; Leimar and Hammerstein, 2001). ‘Simple Standing’ and its variant called ‘Consistent Standing’ actually belong to the leading eight, the set of cooperative norms that are evolutionarily stable against every behavioural mutant (Ohtsuki and Iwasa, 2004, 2006), whereas Image Scoring does not (Table 1). The lesson of ‘Standing’ (Sugden, 1986; Leimar and Hammerstein, 2001) holds true for every member norm of the leading eight (Ohtsuki and Iwasa, 2004, 2006): A well-reputed player’s defection against an ill-reputed co-player should be regarded as good so as to secure cooperation at the societal level. We believe that this property should generally be true even beyond the binary-reputation system (Murase et al., 2022).

How to ensure stable cooperation in the presence of error and private reputation rules is still under active investigation (Uchida, 2010; Uchida and Sasaki, 2020).
Table 1: Leading eight and Image Scoring (IS). We denote cooperation and defection as $C$ and $D$, respectively, and a player’s reputation as either good (1) or bad (0). By $\alpha_{uXv}$, therefore, we mean the reputation assigned to a player who had reputation $u$ and did $X \in \{C, D\}$ to another player with reputation $v$. The behavioural rule $\beta_{uv}$ prescribes an action between $C$ and $D$ when the focal player has reputation $u$ and the co-player has reputation $v$.

|    | $\alpha_{1C1}$ | $\alpha_{1D1}$ | $\alpha_{1C0}$ | $\alpha_{1D0}$ | $\alpha_{0C1}$ | $\alpha_{0D1}$ | $\alpha_{0C0}$ | $\alpha_{0D0}$ | $\beta_{11}$ | $\beta_{10}$ | $\beta_{01}$ | $\beta_{00}$ |
|----|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|--------------|--------------|--------------|--------------|
| L1 | 1              | 0              | 1              | 1              | 0              | 1              | 0              | 1              | C            | D            | C            | 0            |
| L2 | 1              | 0              | 0              | 1              | 0              | 1              | 0              | 1              | C            | D            | C            | 0            |
| L3 | 1              | 0              | 1              | 1              | 0              | 1              | 1              | 0              | C            | D            | C            | 0            |
| L4 | 1              | 0              | 1              | 1              | 1              | 0              | 0              | 1              | C            | D            | C            | 0            |
| L5 | 1              | 0              | 0              | 1              | 1              | 0              | 1              | 1              | C            | D            | C            | 0            |
| L6 | 1              | 0              | 0              | 1              | 1              | 0              | 0              | 1              | C            | D            | C            | 0            |
| L7 | 1              | 0              | 1              | 1              | 1              | 0              | 0              | 0              | C            | D            | C            | 0            |
| L8 | 1              | 0              | 0              | 1              | 1              | 0              | 0              | 0              | C            | D            | C            | 0            |
| IS | 1              | 0              | 1              | 0              | 1              | 0              | 1              | 0              | C            | D            | C            | 0            |

The problem can be illustrated in the following way (Hilbe et al., 2018): Suppose that Alice and Bob hold different views on Charlie because of perception error or their own private assessment rules. Their different views may also lead to different opinions about what to do to Charlie, especially if they have adopted strict social norms. Therefore, when David chooses to (or not to) help Charlie, Alice and Bob will also judge David’s action differently. We can imagine that the disagreement spreads further as time goes by, and the process may end up with complete segregation of the society (Oishi et al., 2021). The crisis can be mitigated if we have an institutional observer who assesses each member of the society and broadcasts it to all others (Okada et al., 2018; Radzvilavicius et al., 2021) or if players have tendency to conform to others’ average opinions (Krellner et al., 2021). Another way suggested to suppress the chain reaction is to introduce insensitivity deliberately, e.g., by keeping the existing views about David unchanged regardless of his action as well as Charlie’s reputation (Quan et al., 2019; Okada, 2020; Quan et al., 2022).

Our goal is to understand where the problem arises in general terms. In our previous work (Lee et al., 2021), we proposed to regard reputation and cooperation as continuous variables to calculate the effects of different assessments in a perturbative way. This continuum approach offered a simple
understanding of the reasons why some of the leading eight are vulnerable to error in reputation. In addition, by assuming small difference between the resident and mutant norms, we derived a threshold for the benefit-to-cost ratio of cooperation to suppress mutants (Lee et al., 2021), replacing an earlier prediction relating the threshold to the probability of observation (Nowak and Sigmund, 1998, 2005; Nowak, 2006). However, by taking into account only linear-order perturbation, our previous work failed to address the difference between first- and second-order norms because the simultaneous action of defection and bad reputation appears as a second-order effect if everyone initially cooperates with good reputation.

In this work, we wish to fill this gap by extending the stability analysis to the second order. Our analysis identifies second-order effects such as justified punishment on the stability of the reputation system when perturbation is caused by erroneous assessment. The analysis in this work differs from exhaustive invasion analysis (Perret et al., 2021) in that we focus on the stability of a pure system where everyone uses the same norm, rather than evolutionary stability against mutant norms.

2. Analysis

The basic dynamical process goes as follows: At every time step, we pick up a random pair of players, say, $i$ and $j$, as a donor and a recipient, respectively, from a large population of size $N \gg 1$. The conventional setting is that they play the one-shot donation game so that the donor makes a binary decision on whether to donate $b$ to the recipient by paying $c$, where $b$ and $c$ represent the benefit and the cost of cooperation, respectively, with $b > c > 0$. In our continuous version of the donation game, the donor $i$ chooses the level of donation, $\beta_i \in [0, 1]$, so that $b\beta_i$ is donated to $j$ at the cost of $c\beta_i$. Cooperation and defection thus correspond to $\beta_i = 1$ and 0, respectively.

Players in the population observe the interaction with probability $q$. Let $k$ be one of the observers. We introduce $m_{ki}$ as a continuous variable between 0 (bad) and 1 (good), for describing $i$’s reputation according to $k$’s assessment rule $\alpha_k = \alpha_k(m_{ki}, \beta_i, m_{kj})$. Likewise, the donation level $\beta_i$ depends on how the donor assesses herself as well as the recipient, i.e., $\beta_i = \beta_i(m_{ii}, m_{ij})$. As time goes by, the new assessment replaces the older one, so the averaged dynamics of $m_{ki}$ in the continuous-time limit can be written as fol-
lows (Hilbe et al., 2018; Lee et al., 2021):

\[
\frac{d}{dt} m_{ki} = -qm_{ki} + \frac{q}{N-1} \sum_{j \neq i} \alpha_k [m_{ki}, \beta_i(m_{ii}, m_{ij}), m_{kj}],
\]

\[(1)\]

where \( q \) is the probability of observation. We assume that implementation error and perception error occur at a low rate, so that error plays the role of perturbation to the initial condition without affecting the governing equation itself.

Let us assume that the society has adopted a common social norm \((\alpha, \beta)\), one of whose stationary fixed points is a fully cooperative initial state with \( m_{ij} = 1 \) for every pair of \( i \) and \( j \). In other words, we consider norms that satisfy

\[
\alpha(1, 1, 1) = \beta(1, 1) = 1,
\]

\[(2)\]

which implies that the norm under consideration maintains a fully cooperative state unless error occurs. Error perturbs the players’ reputations from this fixed point, and we are interested in how small perturbation \( \epsilon_{ki} \equiv 1 - m_{ki} \) grows over time. By expanding Eq. (1) to the first order in \( \epsilon_{ki} \)'s, we obtain the following equation:

\[
\frac{d}{dt} \epsilon_{ki} \approx -q(1 - A_x) \epsilon_{ki} + qA_y B_x \epsilon_{ii} + \frac{q}{N-1} \sum_{j \neq i} [A_y B_y \epsilon_{ij} + A_z \epsilon_{kj}],
\]

\[(3)\]

where

\[
A_x \equiv \partial_x \alpha(x, y, z)|_{(1,1,1)} \quad (4a)
\]

\[
A_y \equiv \partial_y \alpha(x, y, z)|_{(1,1,1)} \quad (4b)
\]

\[
A_z \equiv \partial_z \alpha(x, y, z)|_{(1,1,1)} \quad (4c)
\]

\[
B_x \equiv \partial_x \beta(x, y)|_{(1,1)} \quad (4d)
\]

\[
B_y \equiv \partial_y \beta(x, y)|_{(1,1)} \quad (4e)
\]

Each partial derivative can be interpreted as a sensitivity measure: For example, \( A_z \) means how much the judgment about a well-reputed donor’s cooperation toward a recipient is affected when the recipient has low reputation. Likewise, \( B_x \) means how much a donor should decrease the level of cooperation toward a well-reputed recipient when the donor’s own reputation is low.
To make the story more concrete, we can construct a continuous version of Simple Standing (Sugden, 1986), denoted by L3 in Table 1, by using the bilinear and trilinear interpolation methods as follows:

\[
\alpha_{SS}(x, y, z) = yz - z + 1 \tag{5a}
\]
\[
\beta_{SS}(x, y) = y. \tag{5b}
\]

Table 2 shows the full list of interpolated expressions for the leading eight and Image Scoring. If we look at Table 3, all the leading eight have \( A_x = B_x = 0 \) and \( A_y = B_y = 1 \) in this linear description. The only difference among the leading eight lies in \( A_z \): That is, L1, L3, L4, and L7 have \( A_z = 0 \), whereas L2, L5, L6, and L8 have \( A_z = 1 \). As for Image Scoring, the expression is even simpler:

\[
\alpha_{IS} = \beta_{IS} = y. \tag{6}
\]

Note that the linear-order description for Simple Standing and Image Scoring is given as

\[
(A_x, A_y, A_z, B_x, B_y) = (0, 1, 0, 0, 1) \tag{7}
\]

in common. The above values of partial derivatives are interpreted as follows: Because \( A_y = 1 \), an observer must be sensitive to the reduction of cooperation toward a well-reputed recipient when the donor is also well-reputed. A well-reputed donor must reduce the level of cooperation when he or she meets an ill-reputed recipient (\( B_y = 1 \)). At the same time, the norms with Eq. (7) are indifferent to the donor’s reputation (\( A_x = B_x = 0 \)) as well as to the recipient’s in assessing the donor’s cooperation (\( A_z = 0 \)). In plain language, one can make an apology because of \( B_x = 0 \), according to which the donor has to help a well-reputed player even if his or her own reputation is not good. Then, forgiveness due to \( A_x = 0 \) comes into play because helping a well-reputed recipient is still regarded as good, even if the donor has once lost reputation.

The difference between Simple Standing and Image Scoring is manifested in the second-order description because \( A_{yz} = 1 \) for Simple Standing, whereas \( A_{yz} = 0 \) for Image Scoring as shown in Table 4, where \( A_{\mu\nu} \equiv \partial^2 \alpha / \partial \mu \partial \nu \rvert_{(1,1,1)} \) and \( B_{\mu\nu} \equiv \partial^2 \beta / \partial \mu \partial \nu \rvert_{(1,1,1)} \). The second derivatives can be interpreted in the same way as above: This time, two variables can change simultaneously from the initial state, so \( A_{yz} \) would be related to how much an observer changes the assessment of a well-reputed donor when the donor reduces the level of cooperation toward a relatively ill-reputed recipient (see Appendix A for more details).
Table 2: Continuous versions of the leading eight and Image Scoring. The assessment rule \( \alpha(x, y, z) \) is obtained by applying the trilinear interpolation to \( \alpha_{xyz} \)'s in Table 1 where \( C \) and \( D \) correspond to 1 and 0, respectively. Likewise, the behavioural rule \( \beta(x, y) \) results from the bilinear interpolation applied to \( \beta_{xy} \)'s. We note that L1 has been nicknamed Contrite Tit-for-Tat in the context of direct reciprocity (Sugden, 1986; Brandt et al., 2007).

\[
\begin{array}{|c|c|c|}
\hline
\text{Norm} & \alpha(x, y, z) & \beta(x, y) \\
\hline
L1 & x + y - xy - xz + xyz & -x + xy + 1 \\
L2 (Consistent Standing) & x + y - 2xy - xz + 2xyz & -x + xy + 1 \\
L3 (Simple Standing) & yz - z + 1 & y \\
L4 & -y - z + xy + 2yz - xz + 1 & y \\
L5 & -z - xy + yz + xyz + 1 & y \\
L6 (Stern Judging) & -y - z + 2yz + 1 & y \\
L7 (Staying) & x - xz + yz & y \\
L8 (Judging) & x - xy - xz + yz + xyz & y \\
\hline
\end{array}
\]

Table 3: First-order derivatives of the continuous leading eight and Image Scoring at \( (x, y, z) = (1, 1, 1) \). Note that their differences lie only in \( A_z \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Norm} & A_x & A_y & A_z & B_x & B_y \\
\hline
L1 & 0 & 1 & 0 & 0 & 1 \\
L2 & 0 & 1 & 1 & 0 & 1 \\
L3 & 0 & 1 & 0 & 0 & 1 \\
L4 & 0 & 1 & 0 & 0 & 1 \\
L5 & 0 & 1 & 1 & 0 & 1 \\
L6 & 0 & 1 & 1 & 0 & 1 \\
L7 & 0 & 1 & 0 & 0 & 1 \\
L8 & 0 & 1 & 1 & 0 & 1 \\
\hline
\text{Image Scoring} & 0 & 1 & 0 & 0 & 1 \\
\hline
\end{array}
\]
Table 4: Second-order derivatives of the continuous leading eight and Image Scoring at 
$(x, y, z) = (1, 1, 1)$. Note that their differences lie only in $A_{zx}$, $A_{yz}$, and $B_{xy}$.

| Norm    | $A_{xx}$ | $A_{xy}$ | $A_{xz}$ | $A_{yy}$ | $A_{yz}$ | $B_{xx}$ | $B_{xy}$ | $B_{yy}$ |
|---------|----------|----------|----------|----------|----------|----------|----------|----------|
| L1      | 0        | 0        | 0        | 0        | 0        | 0        | 1        | 0        |
| L2      | 0        | 0        | 1        | 0        | 2        | 0        | 1        | 0        |
| L3      | 0        | 0        | 0        | 0        | 1        | 0        | 0        | 0        |
| L4      | 0        | 0        | -1       | 0        | 1        | 0        | 0        | 0        |
| L5      | 0        | 0        | 1        | 0        | 2        | 0        | 0        | 0        |
| L6      | 0        | 0        | 0        | 0        | 2        | 0        | 0        | 0        |
| L7      | 0        | 0        | -1       | 0        | 1        | 0        | 0        | 0        |
| L8      | 0        | 0        | 0        | 0        | 2        | 0        | 0        | 0        |
| Image Scoring | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Equation (3) can be expressed as a linear-algebraic equation for an $N^2$-dimensional vector $\vec{V} = (\epsilon_{11}, \ldots, \epsilon_{NN})$. The $N^2 \times N^2$ matrix acted on $\vec{V}$ has the largest eigenvalue in the following form (Lee et al., 2021):

$$\Lambda_1 = q \left[-1 + A_x + A_z + A_y(B_x + B_y)\right],$$

and the corresponding eigenvector is

$$\vec{V}_1 = (1, 1, \ldots, 1)$$

up to a proportionality constant. If $\Lambda_1 > 0$, the fixed point in Eq. (2) is unstable, which is the case of L2, L5, L6, and L8 because they have $A_z = 1$. For the others norms of the leading eight as well as for Image Scoring, this linear-order analysis leaves stability indeterminate by having $\Lambda_1 = 0$. From our viewpoint, the important point is that Simple Standing is not distinguished from Image Scoring because they have exactly the same first derivatives [Eq. (7)].

To proceed, we take into account second-order terms to write down the
following equation (see Appendix B):

\[
\frac{d\epsilon_{ki}}{dt} = -q\epsilon_{ki} - \frac{q}{N - 1} \sum_{j \neq i} \left[ -A_x\epsilon_{ki} - A_y(B_x\epsilon_{ii} + B_y\epsilon_{ij}) - A_z\epsilon_{kj} \right]
\] (10)

\[
-\frac{q}{N - 1} \sum_{j \neq i} \left[ A_y \left( \frac{1}{2}B_{xx}\epsilon_{ii}^2 + B_{xy}\epsilon_{ii}\epsilon_{ij} + \frac{1}{2}B_{yy}\epsilon_{ij}^2 \right) + \frac{1}{2}A_{xx}\epsilon_{ki}^2 + \frac{1}{2}A_{yy}(B_x\epsilon_{ii} + B_y\epsilon_{ij})^2 + \frac{1}{2}A_{zz}\epsilon_{kj}^2 
\right]
\]

\[
+ A_{xy}\epsilon_{ki}(B_x\epsilon_{ii} + B_y\epsilon_{ij}) + A_{yz}(B_x\epsilon_{ii} + B_y\epsilon_{ij})\epsilon_{kj} + A_{xx}\epsilon_{ki}\epsilon_{kj}\] + . . . .

We wish to reduce the \( N^2 \)-dimensional dynamics into a one-dimensional one along the principal eigenvector \( \vec{V}_1 = (\epsilon, \epsilon, \ldots, \epsilon) \) [Eq. (9)]. The shape of the eigenvector reflects the symmetry among players, according to which everyone should deviate from the initial state by an equal amount, although marginal differences may exist. To argue the validity of this approximation, let us write \( \epsilon_{ki} = \epsilon + c_{ki} \), where the difference from the approximation is assumed to be small, i.e., \( |c_{ki}| \ll \epsilon \). We then decompose Eq. (10) into two parts, i.e., one for \( \epsilon \) and the other for \( c_{ki} \). The former is written as

\[
\frac{d\epsilon}{dt} \approx \Lambda_1 \epsilon - q\left[ \frac{1}{2}A_{xx} + A_{zz} + (A_{xy} + A_{yz})(B_x + B_y) + \frac{1}{2}A_{yy}(B_x + B_y)^2 + A_y \left( \frac{1}{2}B_{xx} + B_{xy} + \frac{1}{2}B_{yy} \right) \right] \epsilon^2,
\] (11)

and the latter obeys the following dynamics:

\[
\frac{dc_{ki}}{dt} \approx q\{( -1 + A_x)c_{ki} + A_yB_xc_{ii} \} + \frac{q}{N - 1} \sum_{j \neq i} (A_yB_yc_{ij} + A_zc_{kj})
\]

\[
- \frac{q\epsilon}{N - 1} \sum_{j \neq i} \left[ A_y\{B_{xx}c_{ii} + B_{xy}(c_{ii} + c_{ij}) + B_{yy}c_{ij} \}
\right]
\]

\[
+ A_{xx}c_{ki} + A_{yy}(B_xc_{ii} + B_yc_{ij})(B_x + B_y) + A_{zz}c_{kj}
\]

\[
+ (A_{xy} + A_{yz})(B_xc_{ii} + B_yc_{ij}) + (A_{xy}c_{ki} + A_{yz}c_{kj})(B_x + B_y)
\]

\[
+ A_{xx}(c_{ki} + c_{kj}) \],
\] (12)

where \( \epsilon \) is regarded as a constant. For the norms with \( \Lambda_1 = 0 \), i.e., L1, L3, L4, L7, and Image Scoring, the largest eigenvalue of Eq. (12) is either negative or zero (see Appendix C). Specifically, it is \(-2q\epsilon\) for L1 and L3.
Table 5: Stability of the initial cooperative state with $\epsilon = 0$ under each of the leading eight and Image Scoring, according to our second-order analysis of the continuous model.

| Norm                  | Dynamics                                      | Stability of $\epsilon = 0$ |
|-----------------------|-----------------------------------------------|------------------------------|
| L1                    | $de/dt = -2q\epsilon^2 + \ldots$             | Stable                       |
| L2 (Consistent Standing) | $de/dt = q\epsilon + \ldots$                  | Unstable                     |
| L3 (Simple Standing)  | $de/dt = -q\epsilon^2 + \ldots$              | Stable                       |
| L4                    | $de/dt = 0 + \ldots$                          | Neutral                      |
| L5                    | $de/dt = q\epsilon + \ldots$                  | Unstable                     |
| L6 (Stern Judging)    | $de/dt = q\epsilon + \ldots$                  | Unstable                     |
| L7 (Staying)          | $de/dt = 0 + \ldots$                          | Neutral                      |
| L8 (Judging)          | $de/dt = q\epsilon + \ldots$                  | Unstable                     |
| Image Scoring         | $de/dt = 0 + \ldots$                          | Neutral                      |

(Simple Standing), and zero for L4, L7, and Image Scoring. The point is that $c_{ki}$ does not grow over time, so that the one-dimensional dynamics in Eq. (11) remains valid when we consider the leading eight and Image Scoring. If we plug the partial derivatives in Tables 3 and 4 into Eq. (11), the dynamics of $\epsilon$ reduces to

$$\frac{d\epsilon}{dt} \approx \Lambda_1 \epsilon - q(A_{zx} + A_{yz} + B_{xy})\epsilon^2.$$  \hspace{1cm} (13)

For Simple Standing, we arrive at

$$\frac{d\epsilon}{dt} \approx -q\epsilon^2,$$  \hspace{1cm} (14)

which admits a solution in the form of $\epsilon \sim 1/t$. Its diverging time scale is self-consistent with our assumption that $\epsilon$ can be approximated as a constant in Eq. (12). As for Image Scoring, on the other hand, $d\epsilon/dt$ is zero up to the second order of perturbation, meaning that the restoring force toward the initial cooperation is still absent. Among the nine norms under consideration, L1 is predicted to show the fastest recovery with the aid of $B_{xy} = 1$. Its nonvanishing $B_{xy}$ originates from $\beta_{00} = C$ in Table 1. By prescribing cooperation between two ill-reputed players, it promotes recovery more efficiently than Simple Standing, although it again has a diverging time scale. Table 5 shows the results of recovery analysis when applied to each of the leading eight and Image Scoring. These results are consistent with the numerical simulation as shown in the next section.
Figure 1: Recovery from disagreement under the leading eight and Image Scoring (abbreviated as IS). Consistently with Table 5, the average of $\epsilon_{ki}$ converges to zero only under L1 and L3, whereas it remains finite under L7 and IS. Differently from the prediction in Table 5, L4 shows an extremely slow decay of $\epsilon_{avg}$, which may be attributed to its high-order terms. The other four norms worsen a small decline in reputation as seen from the gradual increase of $\epsilon_{avg}$, consistently with $\Lambda_1 > 0$. The black dotted line shows the decay inversely proportional to time for comparison. The population size is $N = 50$, and the curves are average results over $10^2$ samples. The error bars are smaller than the symbol size.

3. Results

To check the recovery process from error, we conduct numerical simulation. The code is identical to the one used in our previous work (Lee et al., 2021): Let us assume that every player uses the same $\alpha$ and $\beta$. We consider a population of size $N \gg 1$ by using an $N \times N$ image matrix, $\{m_{ij}\}$. The matrix elements are random numbers uniformly drawn from $[0.9, 1.0]$. At each time step, we pick up a random pair of players $i$ and $j$, the former as the donor and the latter as the recipient. The donor chooses the donation level $\beta_i(m_{ii}, m_{ij})$. This interaction is observed by each of the other members in the society with probability $q$. Each observer, say, $k$, updates $m_{ki}$ according to $\alpha_k \left[ m_{kii}, \beta_i(m_{ii}, m_{ij}), m_{kj} \right]$. We repeat the above procedure $M$ times, so $M$ can be regarded as a time index.

Our Monte Carlo results in Fig. 1 corroborates the predictions in Table 5.
The instability of L2, L5, L6, and L8 is already clear from the fact that \( \Lambda_1 > 0 \). The stability of L1 and L3 is, however, correctly predicted only when we go through the second-order stability analysis [Eq. (11)]. The behaviour of \( \epsilon \sim 1/t \) is also confirmed by this simulation. Still, the analysis leaves the stability of L4 and L7 undetermined, and Fig. 2 suggests that the average of \( \epsilon_{ij} \)'s will converge to a finite value for L7, whereas it decays extremely slowly in the case of L4. The latter behaviour is not captured by our present study, and a higher-order analysis will be required to understand it. Under Image Scoring, the initial state certainly has neutral stability. To sum up, our second-order stability analysis successfully explains the difference between Simple Standing and Image Scoring: The former recovers from erroneous disagreement, albeit slowly, whereas the latter does not.

The governing equation [Eq. (1)] suggests that the observation probability \( q \) will only change the overall time scale. That is, if we rescale time by defining \( \tau \equiv qt \), the derivative with respect to \( \tau \) will become independent of \( q \). Figure 2 indeed shows that the average deviations from the initial state, \( \epsilon_{avg} \), as functions of \( \tau \) behave similarly regardless of \( q \) in the case of Simple Standing. Another point to mention is that the cost-benefit ratio is irrelevant.
to this stability analysis because it does not enter the dynamics [Eq. (1)]. The
ratio becomes crucial when the symmetry among players breaks down, e.g.,
when some players adopt a different norm from the existing one. In such
a case, the cost-benefit ratio will determine whether the mutant norm can
invade the population (Lee et al. [2021]).

4. Discussion and Summary

The continuous dynamics of reputation and behaviour opens up the possi-
bility to apply powerful analytic tools to the study of indirect reciprocity.
For this reason, the continuum framework is one of the most convenient
ways to study general conditions for norms to be successful when variations
in $\alpha$ and $\beta$ can be treated in a perturbative way. This idea is especially
relevant when norms resist abrupt changes, as suggested by empirical ob-
servations (Mackie et al. 2014; Amato et al. 2018). A comprehensive and
systematic investigation of this framework would thus greatly enhance our
understanding of cooperation through indirect reciprocity. However, our pre-
vious linear-order solution provided an inconclusive or incorrect answer when
applied to the well-known first-order norm called Image Scoring because the
solution did not consider the second-order effects such as justified punish-
ment. In addition, the difference among the leading eight in their responses
to error, despite the ability of justified punishment shared by all of them,
remained unanswered there.

In this work, we have shown how one can go beyond the linear-order
analysis and identified where the difference arises, which will be an impor-
tant piece of information in analysing a social norm. We have expanded the
governing equation [Eq. (1)] to the second order of $\epsilon_k$'s and argued the reason
that the effective dynamics can reduce to a one-dimensional one. Such re-
duced dynamics agrees well with numerical simulation for each of the leading
eight as well as for Image Scoring. The success of this one-dimensional reduc-
tion must be related to the mean-field nature of Eq. (1), although it involves
intricate three-body interaction among a donor, a recipient, and an observer
at each time step. The mean-field assumption can readily be justified in a
well-mixed population, where everyone participates in the interaction with
equal probability at a symmetric position. The effect of heterogeneity among
players on the process of rebuilding a consensus remains as a future work.

We also mention the following limitations of our approach: First, we have
examined only the vicinity of a fixed point at which everyone cooperates
Table 6: Characteristics of the leading eight for successful recovery from error. The left two columns show mathematical representations in the continuous and binary models, respectively. The third column means which of the leading eight satisfy the condition, and the last column explains how to interpret the characteristics.

| Continuous | Binary | Norms | Interpretation |
|------------|--------|-------|----------------|
| $\{ \alpha(1, 1, 1) = 1 \} \beta(1, 1) = 1$ | $\{ \alpha_{1C1} = 1 \} \beta_{11} = C$ | L1–L8 | Maintenance of cooperation |
| $A_x = 0$ | $\alpha_{0C1} = 1$ | L1–L8 | Forgiveness |
| $B_x = 0$ | $\beta_{01} = C$ | L1–L8 | Apology |
| $A_y = 1$ | $\alpha_{1D1} = 0$ | L1–L8 | Identification of defectors |
| $B_y = 1$ | $\beta_{10} = D$ | L1–L8 | Punishment |
| $A_{yz} > 0$ | $\alpha_{1D0} = 1$ | L1–L8 | Justification of punishment |
| $A_z = 0$ | $\alpha_{1C0} = 1$ | L1,L3,L4,L7 | Approval for cooperation |
| $A_{xz} = 0$ | $\alpha_{0C0} = 1$ | L1,L3 | to the ill-reputed |
| $B_{xy} = 1$ | $\beta_{00} = C$ | L1 | Cooperation between the ill-reputed |

with a good reputation. If the initial state is arbitrarily far away from the fixed point, the analysis loses validity in principle. In addition, the second-order stability analysis may fail to address the difference among high-order norms, as can be seen from the fact that the behaviour of L4 in Fig. 1 is not described by our analysis. Another limitation is that our approach considers the presence of error only to escape from the initial state, leaving the subsequent dynamics unchanged [Eq. (1)] if error occurs continuously in time with a small rate of $\zeta$, the dynamics of $\epsilon$ for Simple Standing [Eq. (14)] will be rewritten as

$$\frac{d\epsilon}{dt} \approx -q\epsilon^2 + \zeta.$$  \hspace{1cm} (15)

In a stationary state, we would thus expect $\epsilon_{st} \propto \sqrt{\zeta}$, which is much greater than the magnitude of $\zeta$ itself. Such amplification of error is due to the lack of linear stability at $\epsilon = 0$. In other words, the recovery process is so slow that any finite error rate can easily keep the system away from the state of $\epsilon = 0$. In this sense, the initial cooperative state is only weakly protected from erroneous assessment even under the action of L1 or L3.

Let us mention a few points on the leading eight from the viewpoint of our second-order stability analysis (see Table 6). In our continuum framework,
all of the leading eight have

\[ A_y = B_y = 1 \]
\[ A_x = B_x = 0 \]  (16)

and these slope values are related to the basic properties for being nice, retaliatory, apologetic, and forgiving (Ohtsuki and Iwasa, 2006). If we look further into their second derivatives, we find another common feature that they all have

\[ A_{yz} > 0 \]  (17)

so as to justify punishment on an ill-reputed player (Appendix A). The linear-order stability analysis shows that L2, L5, L6, and L8 are nevertheless unstable because \( \Lambda_1 = qA_z > 0 \) [see Eq. (8)]. This observation imposes an additional condition that

\[ A_z = 0, \]  (18)

which means that a well-reputed player’s cooperation to an ill-reputed player should be regarded as good. Among the leading eight, L1, L3, L4, and L7 share this property, and we note that they actually show the quickest recovery from error in the original discrete version of the model (Hilbe et al., 2018). Now when it comes to L4 and L7, which lack restoring force in spite of all the above three conditions [Eqs. (16) to (18)], it turns out that the effect of justified punishment is exactly cancelled out by \( A_{zx} = -1 \) [see Eq. (13)]. This second derivative is related with how to judge cooperation between two ill-reputed players (see \( \alpha_0C_0 \) in Table I). L4 and L7 regard such cooperation as bad (\( \alpha_0C_0 = 0 \)) and fail to recover from error. When it is regarded as good (\( \alpha_0C_0 = 1 \)), the recovery process succeeds, albeit slowly, as we see from L1 and L3. Our continuum approach thus suggests that the following property stabilizes L1 and L3 in the second-order analysis:

\[ A_{zx} = 0. \]  (19)

Together with Eq. (18), this last condition implies that cooperation to an ill-reputed player should be regarded as good, irrespective of the donor’s reputation. All these features are shared by L1 and L3 in common, and their sole difference in assessment lies in \( \alpha_{0D0} \) (Table I). If we focus on L3 (Simple Standing), when you encounter an ill-reputed player, it is always good for your reputation whether you choose to cooperate or punish the
co-player. Only defection against a well-reputed player is regarded as bad ($\alpha_{1D1} = \alpha_{0D1} = 0$). By disregarding irrelevant information on reputation, Simple Standing achieves a high level of stability in a noisy environment. Note that invasion analysis also shows that those who defect against ill-reputed individuals do not have to be regarded as bad for a social norm to have evolutionary stability [Perret et al., 2021]. Finally, L1 shows faster recovery than L3 because it prescribes cooperation between two ill-reputed players by having $\beta_{00} = C$.

One of the open issues that remain untouched in this paper is the evolutionary stability against mutants. Our previous paper analytically obtained the critical benefit-to-cost ratio above which close mutants are driven out, and we may improve the theory by incorporating the second-order effects properly as is done in this paper. This is a promising direction to deepen our understanding of the mechanism to sustain cooperation.

Acknowledgements

S.K.B. acknowledges support by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2020R1I1A2071670). Y.M. acknowledges support from Japan Society for the Promotion of Science (JSPS) (JSPS KAKENHI; Grant no. 18H03621 and Grant no. 21K03362). We appreciate the APCTP for its hospitality during the completion of this work.

Appendix A. Second derivatives

The second derivative of $\alpha$ with respect to $y$ and $z$ at the fixed point $(x, y, z) = (1, 1, 1)$ can be approximated as

$$A_{yz} \approx \frac{\alpha(1, 1, 1) - \alpha(1, 1 - h, 1) - \alpha(1, 1, 1 - h) + \alpha(1, 1 - h, 1 - h)}{h^2}$$

(Appendix A.1)

with a small parameter $h$. To see the meaning of $A_{yz} > 0$ clearly, let us consider a special case where $\alpha$ has no $z$-dependence when $x = y = 1$. Then, the positivity of $A_{yz}$ is equivalent to

$$\alpha(1, 1 - h, 1 - h) > \alpha(1, 1 - h, 1).$$

(Appendix A.2)

In other words, when one reduces the level of cooperation ($y = 1 - h$), it is regarded as good if the co-player has bad reputation ($z = 1 - h$). We
can generally consider the case with \( z \)-dependence, and the point is that one earns better reputation by punishing an ill-reputed player than not.

Likewise, we can approximate \( A_{xx} \) as

\[
A_{xx} \approx \frac{\alpha(1, 1, 1) - \alpha(1 - h, 1, 1) - \alpha(1, 1, 1 - h) + \alpha(1 - h, 1, 1 - h)}{h^2}.
\]

(Appendix A.3)

Again, let us assume that \( \alpha \) has no \( x \)-dependence when \( y = z = 1 \) for convenience of explanation. This assumption is especially relevant to the leading eight because they all have \( A_x = 0 \). Then, the negativity of \( A_{xx} \) means the following:

\[
\alpha(1 - h, 1, 1 - h) < \alpha(1, 1, 1 - h).
\]

(Appendix A.4)

Note that the right-hand side is effectively the same as \( \alpha(1, 1, 1) = 1 \) for \( L_1, L_3, L_4, \) and \( L_7 \) because they have \( A_z = 0 \). The inequality in Eq. (Appendix A.4) implies that cooperation \((y = 1)\) between two ill-reputed players \((x = z = 1 - h)\) is regarded as bad by \( L_4 \) and \( L_7 \), for which \( A_{xx} = -1 \).

By the same token, we can say that \( B_{xy} > 0 \) of \( L_1 \) under the condition that \( B_x = 0 \) implies

\[
\beta(1 - h, 1 - h) > \beta(1, 1 - h),
\]

(Appendix A.5)

which corresponds to \( \beta_{00} = C \) and \( \beta_{10} = D \) in Table II. The latter prescription is required to punish defectors, but the former is nontrivial: If both the donor and the recipient are ill-reputed, \( L_1 \) tells the donor to cooperate, which speeds up the recovery of the reputation.

**Appendix B. Second-order perturbation**

When \( \epsilon_{ij} \)’s are small parameters, the second-order perturbation for \( \beta \) can be written as follows:

\[
\beta(m_{11}, m_{1j}) = \beta(1 - \epsilon_{11}, 1 - \epsilon_{1j}) \quad \text{(Appendix B.1)}
\]

\[
\approx 1 - B_x \epsilon_{11} - B_y \epsilon_{1j} + \frac{1}{2} B_{xx} \epsilon_{11}^2 + B_{xy} \epsilon_{11} \epsilon_{1j} + \frac{1}{2} \beta_{10}^2 \quad \text{(Appendix B.2)}
\]

\[
\equiv 1 - \kappa. \quad \text{(Appendix B.3)}
\]
Here, we write $\kappa \equiv \kappa^{(1)} + \kappa^{(2)}$, where $\kappa^{(1)} \equiv B_x \epsilon_{1i} + B_y \epsilon_{1j}$ and $\kappa^{(2)} \equiv -(\frac{1}{2} B_{xx} \epsilon_{11}^2 + B_{xy} \epsilon_{11} \epsilon_{1j} + \frac{1}{2} B_{yy} \epsilon_{1j}^2)$ are first- and second-order corrections, respectively. The second-order perturbation for $\alpha$ is also straightforward:

$$\alpha[m_{i1}, \beta_i(m_{ii}, m_{ij}), m_{1j}] \approx \alpha(1 - \epsilon_{1i}, 1 - \kappa, 1 - \epsilon_{1j})(\text{Appendix B.4})$$

$$\approx 1 - A_x \epsilon_{1i} - A_y \kappa - A_z \epsilon_{1j} + \frac{1}{2} A_{xx} \epsilon_{1i}^2 + \frac{1}{2} A_{yy} (\kappa^{(1)})^2 + \frac{1}{2} A_{zz} \epsilon_{1j}^2$$

$$+ A_{xy} \epsilon_{1i} \kappa^{(1)} + A_{yz} \kappa^{(1)} \epsilon_{1j} + A_{zx} \epsilon_{1i} \epsilon_{1j}.$$  (Appendix B.5)

### Appendix C. Eigenvalues of the system of $c_{ki}$’s

For $N = 2$, the full eigenvalue structure of Eq. (12) is obtained as follows:

$$\lambda_1 = q[-1 + A_x - A_z$$

$$- \{A_{xx} - A_{zz} + (A_{xy} - A_{yz})(B_x + B_y)\} \epsilon]$$

$$\lambda_2 = q[-1 + A_x - A_z + A_y B_x - A_y B_y$$

$$- \{A_{xx} - A_{zz} + 2 A_{xy} B_x - 2 A_{yz} B_y + A_{yy}(B_x - B_y)(B_x + B_y)$$

$$+ A_y (B_{xx} - B_{yy})\} \epsilon]$$

$$\lambda_3 = q[-1 + A_x + A_z$$

$$- \{A_{xx} + 2 A_{xz} + A_{zz} + (A_{xy} + A_{yz})(B_x + B_y)\} \epsilon]$$

$$\lambda_4 = q[-1 + A_x + A_z + A_y B_x + A_y B_y$$

$$- \{A_{xx} + 2 A_{xz} + A_{zz} + 2 (A_{xy} + A_{yz})(B_x + B_y)$$

$$+ A_{yy}(B_x + B_y)^2 + A_y (B_{xx} + 2 B_{xy} + B_{yy})\} \epsilon].$$  (Appendix C.1)

If we consider the leading eight, we can readily obtain eigenvalues for $N = 2, \ldots, 5$ and generalize the patterns: For example, L1 has the following structure:

$$\lambda_1^{(N^2-2N+1)} = \frac{1}{N-1} \{- (N - 1) + \epsilon\} q$$

$$\lambda_2^{(N-1)} = - \frac{1}{N-1} \{ N + (N - 4) \epsilon\} q$$

$$\lambda_3^{(N-1)} = -(1 + \epsilon) q$$

$$\lambda_4^{(1)} = - 4 \epsilon q,$$  (Appendix C.2)
where the superscript on each eigenvalue indicates its multiplicity. For L3, we find a similar result:

$$
\lambda_1^{(N^2-2N+1)} = \frac{1}{N-1}\{- (N - 1) + \epsilon\} q
$$

$$
\lambda_2^{(N-1)} = -\frac{1}{N-1}(N - 2\epsilon) q
$$

$$
\lambda_3^{(N-1)} = -(1 + \epsilon) q
$$

$$
\lambda_4^{(1)} = -2\epsilon q. \quad \text{(Appendix C.3)}
$$

Finally, the structure becomes even simpler for L4 and L7:

$$
\lambda_1^{(N^2-N)} = -(1 - \epsilon) q
$$

$$
\lambda_2^{(N-1)} = -\frac{N}{N-1}(1 - \epsilon) q
$$

$$
\lambda_3^{(1)} = 0. \quad \text{(Appendix C.4)}
$$

In every case, the largest eigenvalue is the last one, which is either negative or zero.

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