Interrelation of nonclassicality features in higher dimensional systems through logical operators

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Abstract. Interrelation of different nonclassical correlations with quantum coherence has been shown in [S. Asthana. New Journal of Physics 24.5 (2022): 053026] for multiqubit systems through logical qubits. In this work, we generalise that work to higher dimensional systems. For this, we assume different forms of logical qudits and logical continuous-variable (cv) systems in terms of their physical constituent qudits and physical cv systems. Thereafter, we show how conditions for coherence in logical qudits and logical cv systems themselves give rise to conditions for nonlocality and entanglement in their physical constituent qudits and cv systems. As we increase the number of parties in a logical qudit, conditions for coherence map to those for entanglement and then, for nonlocality. We illustrate it with the examples of SLK inequality and a condition for entanglement in continuous-variable systems. Furthermore, with nonclassicality of logical cv systems, we show that a recently introduced correlation can also be understood. Finally, we show how a condition for imaginarity maps to partial-positive transpose criterion in two-qubit systems. This shows that a single nonclassicality condition detects different types of nonclassicalities in different physical systems. Thereby, it reflects interrelations of different nonclassical features of states belonging to Hilbert spaces of nonidentical dimensions.

Keywords: logical qudits, nonlocality, entanglement, quantum coherence

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1. Introduction

In recent times, there is an unprecedented surge of interest in the studies of quantum foundations \[1\] - \[10\]. This is driven mainly by two reasons: (i) different nonclassical features act as resources in various quantum communication protocols, quantum computation and quantum search algorithms \[11\] - \[14\], (ii) study of quantum foundations has unravelled many features that are at variance with their classical counterparts or they do not have any classical counterpart. Their prime examples are quantum nonlocality (NL) and quantum entanglement which have been noticed in the seminal papers by Einstein, Podolsky and Rosen \[15\] and by Schrödinger \[16\]. Since NL acts as a resource in various protocols (for example, \[11\]), obtaining criteria for NL from various approaches has attracted a lot of attention. Many NL inequalities have been proposed– for multiqubit as well as multiqudit systems (see, for example, \[17\] and references therein). Their derivations are based on different local-hidden-variable (LHV) models \[14, 16, 18, 19\], violations of classical probability rules \[20\], and group-theoretic approaches (for example, stabiliser groups) \[21 – 23\]. Similarly, resource theory of entanglement has also been developed (see, for example, \[24\] and references therein). Sufficiency conditions for entanglement, known as entanglement witnesses, have been derived through various approaches (see, for example, \[25\]).

In parallel, resource theory of quantum coherence has gained a lot of significance \[26\]. The interrelation of quantum coherence with different nonclassical correlations, viz., entanglement, nonlocality and quantum discord has been studied from different approaches \[27 – 30\]. In all these works, interrelations of nonclassical features have been studied in the quantum systems belonging to Hilbert spaces of identical dimensions. This prompts us to ask a question: how are the nonclassical features of a monoparty lower-dimensional system related to that of a multi-party higher-dimensional system and vice-versa?

To answer this, we have recently employed a framework based on nonclassicality in logical qubits. Logical bits have been used in classical error-correcting codes (for example, in parity codes) \[31\]. In contrast, quantum mechanics allows for many different forms of logical qubits in terms of physical qubits, thanks to the principle of superposition \[32\]. Hence, all the criteria for quantum coherence of logical qubits should map to those for nonclassical correlations (be it nonlocality or entanglement or quantum discord) of its physical constituent qubits and vice-versa. Employing the equivalence between logical qubits and different multiqubit physical systems, we have shown that this is indeed the case \[33\].

In this paper, we generalise the framework proposed in \[33\] to multiqudit systems and to infinite-dimensional systems. We first employ nonclassicality of logical qudits (or logical infinite-dimensional systems) to trace back the coherence witnesses underlying nonlocality and entanglement inequalities derived for multiqudit systems. Recently, a quantum correlation beyond entanglement and quantum discord has been shown in \[34\]. We show in this paper that the nonclassical correlation described in \[34\]
Interrelation of nonclassicality features in higher dimensional systems through logical operators

can be understood through nonclassicality in logical infinite-dimensional systems and conditions for coherence. Logical qudits and stabilisers have already been used extensively in the context of quantum error-correcting codes [35]. To the best of our knowledge, they have not yet been employed to study interrelations of different nonclassical features in different dimensions.

The plan of the paper is as follows. In section (3), we lay down the framework to be used throughout the paper. Sections (4, 6, 7, 8, 9) contain the central results of the paper. Section (4) presents how a condition for coherence in a single qutrit maps to a condition for entanglement in a two–qutrit system, and subsequently to three-qutrit systems. In section (5), we recapitulate GHZ nonlocality for the purpose of pedagogy. In section (6), we show how a condition for coherence leads to a condition for generalised GHZ nonlocality. Section (7) shows how conditions for coherence in a single continuous-variable system map to condition for entanglement in two and three-mode states. In section (8), we show how a correlation beyond discord and entanglement can also be understood through nonclassicality in logical quantum system in continuous-variable systems. Finally, we show in section (9) how the condition for imaginarity in a single qubit maps to the celebrated PPT criterion for entanglement in two-qubit systems. Section (10) concludes the paper with closing remarks.

2. Notation

In this section, we set up the notations to be used throughout the paper.

(i) The subscript ‘L’ is used for representing logical states or logical operators acting on the logical Hilbert space. For example, the symbols $|\psi\rangle_L$ and $A_L$ represent a logical state and a logical operator respectively.

(ii) The numeral subscript of an observable represents the party on which this observable acts. For example, the observables $A_1, \cdots, A_N$ act over the Hilbert spaces of the first, second, $\cdots$, $N^{th}$ qudit.

(iii) For continuous-level systems, we employ the symbols $X_i(\eta)$ and $P_j(\epsilon)$ to represent the phase space translation operators generated by position and momentum operators ($\hat{x}_i$ and $\hat{p}_j$) acting on the $i^{th}$ and $j^{th}$ modes respectively. The symbols $\eta$ and $\epsilon$ denote the magnitudes of respective translations.

(iv) For qudit systems, the symbol $X_i$ represents the cyclic translation operator by one unit acting on the $i^{th}$ qudit. The symbol $Y_i$ represents the translation operator followed by a phase shift acting on the $i^{th}$ qudit. The dimension in which these operators are defined will be clear from the context.

3. Framework

In this section, we present the framework to obtain conditions for nonclassical correlations, given a condition for quantum coherence.
We start with some observations to elucidate the interrelation between quantum coherence of logical quantum systems and underlying quantum correlations in its constituent physical subsystems:

**Observation 1:** A monoparty system shows no nonclassical feature with respect to the observables which are diagonal in its eigenbasis. It is because this scenario can be simulated by classical distribution functions and classical observables. So, nonzero values of off-diagonal elements, i.e., coherence is essential for a density matrix to exhibit nonclassical features with respect to a set of operators which are not diagonal in its eigenbasis.

**Observation 2:** If we club all the qudits of a nonlocal (entangled) multiparty state and map it to an effective monoparty system with logical qudits, it exhibits coherence in the effective computational basis (obtained by the tensor product of the computational bases elements of the constituents)‡.

**Example:** The three-qudit entangled GHZ state, $|\psi_{\text{GHZ}}\rangle \equiv \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |iii\rangle$, is equivalent to a maximally coherent logical state, $|\psi_{\text{GHZ}}\rangle_L \equiv \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_L$, if $|iii\rangle \equiv |i\rangle_L$.

This observation is valid for mixed states as well. It can be seen by invoking the convexity argument. Suppose that there is a mixed bipartite entangled state, $\rho = \sum_i p_i |\psi_i\rangle\langle \psi_i|$, in its eigenbasis. Since $\rho$ is entangled, there exists at least one value of $i$ (say, $i = 1$) for which the state $|\psi_1\rangle$ has a Schmidt rank $r > 1$, i.e., $|\psi_1\rangle \equiv \sum_{j=1}^{r} \sqrt{\lambda_j} |jj\rangle$. Under the mapping $|jj\rangle \equiv |j\rangle_L$, $|\psi_1\rangle \equiv \sum_{j=1}^{r} \sqrt{\lambda_j} |j\rangle_L$. So, the state $\rho$ will exhibit coherence in the effective logical basis. This argument can be straightforwardly extended to multipartite states.

**Observation 3:** Suppose that the logical basis states are assumed to be separable physical states (e.g., $|i\rangle_L \equiv |i\rangle^\otimes N$). Let a monoparty logical pure state be incoherent across any bipartition (that is to say, the logical pure state does not involve any superposition of its constituent physical systems). Under the mapping of logical states to physical states, the monoparty logical state will map to a separable state across that bipartition.

**Example:** Consider the logical state,

$$|\xi\rangle_L \equiv |\xi_1\rangle_L |\xi_2\rangle_L \equiv \left( \sum_i \sqrt{\lambda_i} |i\rangle_L \right) \left( \sum_j \sqrt{\mu_j} |j\rangle_L \right), \quad 0 \leq \lambda_i, \mu_j \leq 1, \quad \sum_i \lambda_i = \sum_j \mu_j = 1.$$

(1)

There is no coherent superposition between $|\xi_1\rangle_L$ and $|\xi_2\rangle_L$. The logical state $|\xi\rangle_L$ is equivalent to a biseparable state, $\left( \sum_i \sqrt{\lambda_i} |i\rangle^\otimes N \right) \left( \sum_j \sqrt{\mu_j} |j\rangle^\otimes M \right)$ if $|i\rangle_L \equiv |i\rangle^\otimes N$ and $|j\rangle_L \equiv |j\rangle^\otimes M$.

‡ At this juncture, we wish to stress that this clubbing is not just a mathematical artifice, its physical instances are manifest in atomic physics through coupling of angular momenta, i.e., $L - S$ coupling and $jj$ coupling.
Interrelation of nonclassicality features in higher dimensional systems through logical operators

It can be generalised straightforwardly to any number of bipartitions.

Observation 4: Suppose that the logical basis states involve superpositions of those multiparty physical states, which are not orthogonal in all the parties. These logical basis states lead to different classes of entangled states.

Example: Let $|0\rangle_L \equiv |001\rangle$, $|1\rangle_L \equiv |010\rangle, |2\rangle_L \equiv |100\rangle$. The three logical states are mutually orthogonal, but they are not orthogonal in all the constituent qubits. That is to say, the first constituent qubits of the logical states $|0\rangle_L$ and $|1\rangle_L$ are not orthogonal. Under this choice of logical qutrits, the state, $\frac{1}{\sqrt{3}}(|0\rangle_L + |1\rangle_L + |2\rangle_L)$ is equivalent to a three-qubit W state. On the other hand, if we assume $|0\rangle_L \equiv |000\rangle, |1\rangle_L \equiv |111\rangle, |2\rangle_L \equiv |222\rangle$, the same logical state, $|\psi\rangle_L \equiv \frac{1}{\sqrt{3}}(|0\rangle_L + |1\rangle_L + |2\rangle_L)$, is equivalent to the three-qutrit GHZ state, $\frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$. The difference between the three-qubit W-state and the three-qutrit GHZ state is that the former leads to a quantum-correlated state when one of its qubits is traced over. The latter, however, leads to a fully separable state if any one of the qutrits is traced over.

Observation 5: Suppose that the logical basis states are themselves equivalent to entangled physical qudits. For this choice of logical states, a separable (incoherent) parent logical state may map to an entangled daughter state.

Example: Let $|0\rangle_L \equiv \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. For this choice of logical states, an incoherent separable state (viz., $|00\rangle_L$) is equivalent to a two-copy entangled two-qubit Bell state, $\left(\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\right)^{\otimes 2}$.

Hence, in this language, all the conditions for nonlocality and entanglement (in general, for any nonclassical correlation) in multiqutrit physical systems (and multimode continuous-variable states) should reduce to those for coherence in logical qudits (resp., continuous-variable single-mode logical states). We note that nonlocality is a nonclassical feature not restricted to quantum mechanics, whereas quantum coherence (in probability amplitudes) is a nonclassical feature restricted to quantum mechanics. A condition for quantum coherence provides us with appropriate observables. The bound on their combinations is set after ensuring that the combination is obeyed by all the local hidden variable models. This provides us with a tool to study interrelation between different kinds of nonclassical correlations and quantum coherence.

With this preface, we lay down the procedure for obtaining conditions for nonclassical correlations from that for quantum coherence.

3.1. Procedure

3.1.1. For finite dimensional systems Suppose that the set, $\{|i\rangle_L\}$, represents the basis of an effective logical Hilbert space. Each logical qudit $|i\rangle_L$ is composed of $N$ qudits, i.e.,

$$|i\rangle_L \equiv |i\rangle^{\otimes N}; \quad 0 \leq i \leq (d - 1).$$  (2)
Interrelation of nonclassicality features in higher dimensional systems through logical operators

Let there be an operator,

$$A_L = \sum_{i,j=0}^{d-1} a_{ij} |i\rangle_L \langle j| = \sum_{i,j=0}^{d-1} a_{ij} ([i] \langle j|)^{\otimes N}, \quad (3)$$

acting over an effective logical Hilbert space.

Assume that a condition for quantum coherence in the effective logical basis, \( \{|i\rangle_L\} \), is given by violation of the inequality,

$$C_L : \langle A_L \rangle \leq c, \quad (4)$$

where \( c \) is a positive real number. The state that maximally violates this condition is assumed to be,

$$|\psi\rangle_L \equiv \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_L. \quad (5)$$

The steps to obtain conditions for quantum correlations as descendants of the condition for quantum coherence, \( C_L \), are as follows:

(i) First, we rewrite the logical operator \( A_L \) in terms of tensor product basis of all the subsystems, that is, we use \( |i\rangle_L \equiv |i\rangle^{\otimes N} \).

(ii) Thereafter, we find \( N \) operators \( A_1, \ldots, A_N \) acting locally over the Hilbert spaces of the first, second, \( \ldots, N^{th} \) qudits, such that the action of \( A_1 \otimes A_2 \otimes \cdots \otimes A_N \) on the states \( |i\rangle_L \) is identical to that of \( A_L \). Crucially, a single operator \( A_L \) maps to more than one operator, \( A_1, \ldots, A_N \). The matrix elements of these operators are fixed as follows.

(a) If \( ij^{th} \) entry of the operator \( A_L \), \( a_{ij}^{(L)} \), is equal to zero, we fix \( ij^{th} \) entries of all the operators \( A_1, \ldots, A_N \) to be equal to zero. That is,

$$a_{ij}^{(L)} = 0 \implies a_{ij}^{(\alpha)} = 0, \forall \alpha \in \{1, \ldots, N\}. \quad (6)$$

(b) On the other hand, suppose that the \( ij^{th} \) entry of the operator \( A_L \), represented by \( a_{ij}^{(L)} \), is nonzero. Then, the corresponding \( ij^{th} \) entries, \( a_{ij}^{(\alpha)} \), of the operators \( A_\alpha \ (1 \leq \alpha \leq N) \) are chosen such that they obey the following equation,

$$\prod_{\alpha=1}^{N} a_{ij}^{(\alpha)} = a_{ij}^{(L)}. \quad (7)$$

These entries, \( a_{ij}^{(\alpha)} \), may be real or complex with their moduli bounded by 1. For these choices of \( A_1, \ldots, A_N \), the following equality holds,

$$\langle \psi_L | A_L | \psi_L \rangle = \langle \psi_L | A_1 A_2 \cdots A_N | \psi_L \rangle. \quad (8)$$

The state \( |\psi_L\rangle \) is written on the right-hand side of equation (8) in the tensor product basis of all the \( N \) qudits (i.e., \( |\psi_L\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle^{\otimes N} \)). At this juncture, it is worth
mentioning that due to this procedure, there is a freedom in choosing the matrix elements. Thanks to this freedom, a single logical operator acting over the logical space maps to a set of operators of cardinality greater than one. Each operator in this set is a tensor product of different local operators. Moreover, increasing the number of parties will increase the cardinality of this set. Its underlying reason is that the logical qudit \( |i\rangle \) belongs to a Hilbert space of dimension \( d^N \). The operator \( A_L \) has a support only in a \( d \)-dimensional subspace. On the other hand, the operator, \( A_1 \otimes A_2 \otimes \cdots \otimes A_N \), in general, may have support in the full \( d^N \) dimensional Hilbert space.

(iii) The final step involves ensuring that the bound satisfied by the combination of such observables is respected by all the local hidden variable models or separable states or nondiscordant states, depending upon the nonclassical correlation under consideration.

We wish to stress that we have taken but a simple choice of logical qudits \( |i\rangle_L \equiv |i\rangle^\otimes N \). The procedure, however, is amenable to other choices of logical states discussed at the beginning of section (3). Obviously, any other choice of logical state will change equations (7) and (8), thereby leading to a new set of observables.

We now give an example to show how a single logical operator acting on a single logical qutrit maps to three different operators acting on a two-qutrit system (equivalent to the logical qutrit).

Example: Consider a maximally coherent logical qutrit, \( |\phi\rangle_L \equiv \frac{1}{\sqrt{3}}(|0\rangle_L + |1\rangle_L + |2\rangle_L) \). It is an eigenstate of the operator \( X_L \equiv |1\rangle_L \langle 0| + |2\rangle_L \langle 1| + |0\rangle_L \langle 2| \). The matrix form of the operator \( X_L \) in the logical basis, \( \{|0\rangle_L, |1\rangle_L, |2\rangle_L\} \), is as follows:

\[
X_L = \begin{pmatrix}
|0\rangle_L & |1\rangle_L & |2\rangle_L \\
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]

We now assume that each logical qutrit is composed of two qutrits (i.e., \( |i\rangle_L \equiv |ii\rangle \)). Let,

\[
X \equiv |1\rangle \langle 0| + |2\rangle \langle 1| + |0\rangle \langle 2|, \quad Z \equiv |0\rangle \langle 0| + \omega |1\rangle \langle 1| + \omega^2 |2\rangle \langle 2|,
\]

where \( \omega \) is a cube root of identity. Employing \( |i\rangle_L \equiv |ii\rangle \), the state \( |\phi\rangle_L \) acquires the form, \( \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \). Making use of the points ii(a) and ii(b), the logical operator
Interrelation of nonclassicality features in higher dimensional systems through logical operators

\(X_L\) maps to the following two operators,

\[
X_L = \begin{pmatrix}
|0\rangle_L & |1\rangle_L & |2\rangle_L \\
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
|0\rangle & |1\rangle & |2\rangle \\
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \otimes \begin{pmatrix}
|0\rangle & |1\rangle & |2\rangle \\
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} = X_1X_2
\]

Similarly, it can be seen that the operator \(X_L\) also maps to \((X_1Z_1^2)(X_2Z_2^2)\). The state, \(\frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)\), is an eigenstate of all the three operators \(X_1X_2, (X_1Z_1^2)(X_2Z_2^2), (X_1Z_1)(X_2Z_2^2)\).

3.1.2. For continuous-variable systems While transiting from finite-dimensional systems to infinite-dimensional systems, we recall the mapping, given in [30], from generators of the special unitary group to those of Heisenberg group.

Suppose that \(L_x, L_y, L_z\) represent generators of the SU(2) group in a dimension \(\ell\). The symbols \(\hat{x}, \hat{p}\) and \(I\) represent infinite-dimensional position operator, momentum operator and identity operator respectively. Then, under the mapping,

\[
\lim_{\ell \to \infty} \frac{L_x}{\sqrt{\ell}} \rightarrow \hat{x}, \quad \lim_{\ell \to \infty} \frac{L_y}{\sqrt{\ell}} \rightarrow \hat{p}, \quad \lim_{\ell \to \infty} \frac{L_z}{\ell} \rightarrow I,
\]

the canonical commutation relations, \([L_i, L_j] = iL_k\) \((i, j, k\) vary cyclically over the symbols \(x, y, z\)) and \([\hat{x}, \hat{p}] = iI\) are preserved (we have assumed \(\hbar = 1\)).

This mapping provides us with a choice of observables in an infinite-dimensional space, contingent on a given choice of observables in a finite dimensional Hilbert space. To illustrate it, consider a translationally invariant \(d\)-dimensional state, \(|\psi_d\rangle \equiv \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle\). Its continuous-variable analogue is the translationally invariant zero-momentum state, \(|\psi_c\rangle \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx |x\rangle\). Thanks to the mapping given in equation (10), the unitary operator which leaves the state, \(|\psi_d\rangle\) invariant, maps to a unitary operator which is generated by \(\hat{x}\) and \(\hat{p}\). In particular, the following three mappings follow immediately:

(i) Finite-dimensional translation operators \([e.g., \sum_{i=0}^{d-1} |i\rangle \langle i + 1|]\) map to translation operators, \(\exp(-i\hat{p}\epsilon)\), generated by the momentum operator. That is,

\[
\sum_{i=0}^{d-1} |i\rangle \langle i + 1| \mapsto \exp(-i\hat{p}\epsilon); \epsilon \neq 0,
\]

where \(\epsilon\) has the dimensions of inverse momentum.
(ii) Suppose that $\omega$ is a $d$th root of identity. Finite-dimensional relative phase shift operators [e.g., $\text{diag}(1, \omega, \cdots, \omega^{d-1})$] map to an operator, $\exp[-i\hat{x}\eta(x)]$, where $\eta(x)$ is a function of the coordinate, $x$. The operator, $\exp[-i\hat{x}\eta(x)]$, introduces relative phase shifts between different position kets. That is, $\text{diag}(1, \omega, \cdots, \omega^{d-1}) \mapsto \exp[-i\hat{x}\eta(x)]$, where $\eta(x)$ has the dimensions of inverse length.

Thereafter, the procedure for mapping the observables for a single-mode state to a two-mode state is exactly the same as for finite-dimensional systems.

We now give an example. Suppose that there is a logical single mode state, $|\psi_c\rangle_L \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx|L||x\rangle_L$. By employing the above procedure, the logical operator, $\exp(-i\hat{p}_L \epsilon)$, acting on the logical state, $|\psi_c\rangle_L$, maps to a tensor product of operators, viz., $\exp(-i\hat{p}_1 \epsilon)\exp(-i\hat{p}_2 \epsilon)$. It is straightforward to see that, $\exp(-i\hat{p}_L \epsilon)|\psi_c\rangle_L = |\psi_c\rangle_L$.

4. Coherence in a single qutrit $\rightarrow$ Entanglement in a two-qutrit system

We now turn our attention to mapping the condition for quantum coherence in a single logical qutrit to that for quantum entanglement in a two–qutrit system. In what follows, we show that coherence witness for a single qutrit maps to an entanglement witness for a two–qutrit system. Consider the translation operator, $X_L$, by one unit acting on a three-dimensional logical vector space spanned by $|0\rangle_L, |1\rangle_L, |2\rangle_L$. The action of the translation operator $X_L$ is given by, $X_L|i\rangle_L = |i + 1\rangle_L, i \in \{0, 1, 2\}$. The summation is assumed to be modulo 3. The 1-D invariant symmetric subspace is given as, $|\phi\rangle_L = \frac{1}{\sqrt{3}}(|0\rangle_L + |1\rangle_L + |2\rangle_L)$. So, a condition for coherence in the basis spanned by $\{|0\rangle_L, |1\rangle_L, |2\rangle_L\}$ can be written as, $|\langle X_L|\rangle| \leq \frac{1}{2}$.
Interrelation of nonclassicality features in higher dimensional systems through logical operators

Since, \( \langle i_L|X_L|i_L \rangle = 0, i \in \{0, 1, 2\} \), their incoherent superpositions of the form,

\[
\rho_{\text{inc}} \equiv \sum_{i=0}^{2} p_i \langle i | L \rangle \langle L | i \rangle, \quad 0 \leq p_i \leq 1; \sum_i p_i = 1, \quad (18)
\]

would satisfy the condition given in equation (17). The state \( |\phi \rangle_L = \frac{1}{\sqrt{3}}(|0 \rangle_L + |1 \rangle_L + |2 \rangle_L) \) violates this condition, as \( \langle \phi_L|X_L|\phi_L \rangle = 1 \). For this reason, violation of the condition given in (17) is sufficient for detecting coherence in the basis \( \{ |0 \rangle_L, |1 \rangle_L, |2 \rangle_L \} \). The bound of \( \frac{1}{2} \) has been chosen so as to ensure that the condition (17) maps to entanglement witnesses for two- and three-qutrit systems. We now move on to show how the condition for coherence in a logical system, given in (17), gives rise to conditions for entanglement under suitable choices of logical qutrits.

4.1. Logical qutrit consisting of a pair of qutrit: \( |i \rangle_L \equiv |ii \rangle \)

If we assume each qutrit to be composed of a pair of qutrits, i.e, \( |0 \rangle_L \equiv |00 \rangle, |1 \rangle_L \equiv |11 \rangle, |2 \rangle_L \equiv |22 \rangle \), the state \( |\phi \rangle_L \) assumes the following form,

\[
|\phi \rangle_L \equiv \frac{1}{\sqrt{3}}(|00 \rangle + |11 \rangle + |22 \rangle). \quad (19)
\]

The logical operator, \( X_L \equiv |1 \rangle L \langle 0 | + |2 \rangle L \langle 1 | + |0 \rangle L \langle 2 | \), maps to many operators that act locally on the two qutrits. These operators can be identified by employing the procedure described in section (3.1) and are given as,

\[
X_L \rightarrow \{X_1 X_2, (X_1 Z_1)(X_2 Z_2^2), (X_1 Z_2^2)(X_2 Z_1)\}, \quad (20)
\]

where the subscript of an operator denotes the party over which it acts. The operator \( Z \) is defined to be, \( Z \equiv \text{diag}(1, \omega, \omega^2) \), where \( \omega \) is a cube root of identity. Obviously, \( \omega^3 = 1 \) and \( [X_1, Z_1] \neq 0, [X_2, Z_2^2] \neq 0 \).

If we choose a pair of operators from this set, \( \{X_1 X_2, X_1 Z_1 X_2 Z_2^2\} \), the condition for coherence given in (17) yields the following condition for entanglement:

\[
|\langle X_1 X_2 + X_1 Z_1 X_2 Z_2^2 \rangle| \leq 1. \quad (21)
\]

All the separable states satisfy this condition, whereas the state \( |\phi \rangle_L \) violates it. So, the operator, \( X_1 X_2 + X_1 Z_1 X_2 Z_2^2 \), provides a condition for entanglement in a two-qutrit system. Similarly, entanglement witnesses with a larger number of operators can be constructed.

4.2. Logical qutrit consisting of a triplet of qutrit: \( |i \rangle_L \equiv |iii \rangle \)

If we consider each logical qutrit to be comprised of three qutrits, the logical state \( |\phi \rangle_L \) assumes the following form,

\[
|\phi \rangle_L \equiv \frac{1}{\sqrt{3}}(|000 \rangle + |111 \rangle + |222 \rangle). \quad (22)
\]
The state $|\phi\rangle_L$ is stabilised by the following observables (which are identified by employing the procedure given in section (3.1)),

$$X_L \rightarrow \left\{ X_1X_2X_3, (X_1Z_1)(X_2Z_2)(X_3Z_3), (X_1Z_1)(X_2Z_2^2)X_3, \\
(X_1Z_1^2)(X_2Z_2)X_3, (X_1Z_1)X_2(X_3Z_3^2), (X_1Z_1^2)X_2(X_3Z_3), \\
X_1(X_2Z_2)(X_3Z_3^2), X_1(X_2Z_2^2)(X_3Z_3) \right\}. \quad (23)$$

We choose a pair of operators from this set, \{ $X_1X_2X_3, (X_1Z_1)(X_2Z_2)(X_3Z_3)$ \}. Under this choice of operators, the condition (17) yields the following condition for three-qutrit entanglement:

$$|\langle X_1X_2X_3 + X_1Z_1X_2Z_2X_3Z_3 \rangle| \leq 1. \quad (24)$$

It is satisfied by all the separable states and violated by the state (22). Similarly, other entanglement inequalities can be constructed by involving other observables in the equation (23).

As another illustration, we now apply this procedure to show how observables employed in SLK inequality [19] can be straightforwardly identified by using condition for quantum coherence in a single logical system.

5. GHZ Nonlocality in arbitrary even dimensions

This is a brief recapitulation of GHZ nonlocality in higher dimensions for pedagogic purposes. For details, see, for example, [37].

Consider a tripartite system, with each subsystem being $d$– dimensional. The composite system is in the following state:

$$|\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} |n, n, n\rangle, \quad (25)$$

where the set, \{ $|n\rangle : 0 \leq n \leq (d - 1)$ \}, represents an orthonormal basis.

Suppose that the three parties choose one of two variables, $X$, and $Y$. Each variable takes its value in the set, \{ $1, \omega, \cdots, \omega^{d-1}$ \}. The elements of this set are the $d$– th roots of unity. The observable $X$ can be written in the basis set, \{ $|n\rangle$ \}, as,

$$X = \sum_{n=0}^{d-1} |n + 1\rangle \langle n|, \quad (26)$$

where the addition is modulo $d$, i.e., $|n\rangle \equiv |n \mod d\rangle$. That is, the operator $X$ is a translation operator by one unit on a circle with $d$ discrete points. Obviously, the $d^{th}$ point is identified with the first point. The generalised GHZ state $|\psi_{\text{GHZ}}\rangle$ is the eigenstate of the observable $X_1X_2X_3$ with eigenvalue +1 as,

$$X_1X_2X_3|\psi_{\text{GHZ}}\rangle = |\psi_{\text{GHZ}}\rangle. \quad (27)$$
At this juncture, we note that the state $|\psi_{\text{GHZ}}\rangle$ is a translationally invariant state. But, thanks to the presence of three parties, it also allows for other symmetry transformations, through the incorporation of phases in the following manner.

By using the symmetry operations (viz., translational and permutational invariance) for the generalised GHZ state $|\psi_{\text{GHZ}}\rangle$, other observables can be constructed. One such operator is given by $\omega X_1 Y_2 Y_3$ ($\omega$ is the $d^{th}$ root of identity). The operator $Y$ can be written as,

$$Y = \omega^{-1/2} \left( \sum_{n=0}^{d-2} |n+1\rangle\langle n| - |0\rangle\langle d-1| \right).$$

(28)

In a similar manner, the other two observables, $\omega Y_1 X_2 Y_3$ and $\omega Y_1 Y_2 X_3$ can be obtained. The obtained observables respectively satisfy,

$$X_1 Y_2 Y_3 |\psi_{\text{GHZ}}\rangle = \omega^{-1} |\psi_{\text{GHZ}}\rangle$$

$$Y_1 X_2 Y_3 |\psi_{\text{GHZ}}\rangle = \omega^{-1} |\psi_{\text{GHZ}}\rangle$$

$$Y_1 Y_2 X_3 |\psi_{\text{GHZ}}\rangle = \omega^{-1} |\psi_{\text{GHZ}}\rangle.$$  

(29)

We assume the existence of an underlying LHV model and employ the forms $X_\alpha = \omega^{x_\alpha}$ and $Y_\alpha = \omega^{y_\alpha}$ for the outcomes of $X$ and $Y$. $x_\alpha$ and $y_\alpha$ are integers. The constraint of the values assumed by the variables $X_\alpha$ and $Y_\alpha$ (outcomes of the corresponding observables) for each qudit $\alpha$ to be consistent with an underlying LHV model gets converted to the following constraints:

$$x_1 + y_2 + y_3 \equiv -1 \text{ mod } d$$

$$y_1 + x_2 + y_3 \equiv -1 \text{ mod } d$$

$$y_1 + y_2 + x_3 \equiv -1 \text{ mod } d.$$  

(30)

Adding these equations yields the following condition,

$$x_1 + x_2 + x_3 \equiv -2(y_1 + y_2 + y_3) - 3 \text{ mod } d.$$  

(31)

Since the outcomes of $X_\alpha$ are $\omega^{x_\alpha}$, equating the powers of $\omega$ both sides in equation (27) under the assumption of an underlying LHV model leads to the following equation,

$$x_1 + x_2 + x_3 \equiv 0 \text{ mod } d.$$  

(32)

For an even integer $d$, the RHS of Equation (31) is always an odd integer modulo $d$ for arbitrary $y_\alpha$. In other words, for even $d$, no integer can satisfy the equation, $2y + 3 \equiv 0 \text{ mod } d$ where $y = y_1 + y_2 + y_3$. This contradicts the condition (32) emergent from eq. (27). That is to say, equations (31) and (32) can not be solved simultaneously for any integer value of $y$.

This gives rise to a Hardy-type condition for nonlocality without inequality [38] for an arbitrary even-dimensional tripartite system. In any local hidden variable model, the
probability for joint outcomes of observables is equal to convex sum of the product of probabilities for individual outcomes of local observables conditioned on a local hidden variable. That is to say, the joint probability $P(a_i b_j c_k | ABC)$ for the outcomes $a_i, b_j, c_k$ of the observables $A, B, C$ is of the form,

$$P(a_i b_j c_k | ABC) = \int d\lambda \rho(\lambda) P(a_i | A, \lambda) P(b_j | B, \lambda) P(c_k | C, \lambda),$$

(33)

where $\lambda$ is a hidden variable with a probability distribution represented by $\rho(\lambda)$. The symbols $P(a_i | A, \lambda), P(b_j | B, \lambda), P(c_k | C, \lambda)$ represent the probabilities for outcomes $a_i, b_j, c_k$ of the observables $A, B, C$ conditioned over the hidden-variable $\lambda$.

Since the conditions (27) and (29) can not be obeyed by an arbitrary local hidden variable model simultaneously, they can be used to construct a Bell inequality. The inequality is given by,

$$|\langle X_1 X_2 X_3 + \omega X_1 Y_2 Y_3 + \omega Y_1 X_2 Y_3 + \omega Y_1 Y_2 Y_3 \rangle| \leq 3.$$

(34)

Since under a LHV model, only three conditions out of four (given in equations (27) and (29)) can be satisfied. So, the LHV bound is 3.

This can be straightaway generalised to $N$–party systems, with each subsystem being $d$–dimensional, where $N$ is odd and $d$ is an even integer. In fact, the $N$-partite SLK inequality for even dimensions has been given in [19].

Equations similar to (27, 29) for higher powers of these observables have been shown in Appendix A for $d = 4$ for the purpose of illustration.

6. Coherence witness → GHZ nonlocality for qudits in even dimensions

In this section, we show that the operators that are employed in GHZ nonlocality map simply to a shift operator acting on a single logical qudit system if we assume the $N$ qudits to be a single logical qudit. Let,

$$|0\rangle^{\otimes N} \equiv |0\rangle_L, \cdots, |d-1\rangle^{\otimes N} \equiv |d-1\rangle_L.$$

(35)

So, the $N$–partite generalised GHZ state can be written as an effective monoparty logical state,

$$|\psi_{\text{GHZ}}\rangle_L \equiv \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} |n\rangle_L.$$

(36)

Since the state (36) is a permutationally invariant state, it is invariant under the operators $X_L, X_L^2, \cdots, X_L^{d-1}$, where

$$X_L^i = \sum_{n=0}^{d-1} |n + i\rangle_L \langle n|, 1 \leq i \leq (d - 1).$$

(37)
The set \( \{1, X_L, \cdots, X_{d-1}^L\} \) forms a cyclic group. The addition is assumed to be modulo \( d \). The operator \( X_L^d \) enacts a translation of logical basis qudits by \( i \) steps, i.e., under its action, \( |0\rangle_L \mapsto |i\rangle_L \).

The crucial point is that since the logical qudit itself is composed of three qudits, the operator \( X_L^d \) will map to products of locally noncommuting operators that give rise to GHZ nonlocality in the multipartite system. That is, if we employ the equality \( |n\rangle_L \equiv |n, n, n\rangle, \ n \in \{0, \cdots, d-1\} \), the operators \( X_L^k \) may be reexpressed as,

\[
X_L^k \equiv \sum_{i=0}^{d-1} |i + k, i + k, i + k\rangle \langle i + k|, \ \ 1 \leq k \leq d - 1. \tag{38}
\]

The sets of tensor products of local operators to which these operators map to are as follows (by applying the procedure laid down in section \( \text{3.1} \)):

\[
X_L \mapsto \{X_1X_2X_3, \ \omega X_1Y_2Y_3, \ \omega Y_1X_2Y_3, \ \omega Y_1Y_2X_3\} \equiv S_1
\]

\[
X_L^2 \mapsto \{X_1^2X_2^2X_3, \ \omega^2 X_1^2Y_2^2Y_3, \ \omega^2 Y_1^2X_2^2Y_3, \ \omega^2 Y_1^2Y_2^2X_3\} \equiv S_2
\]

\[
X_L^3 \mapsto \{X_1^3X_2^3X_3, \ \omega^3 X_1^3Y_2^3Y_3, \ \omega^3 Y_1^3X_2^3Y_3, \ \omega^3 Y_1^3Y_2^3X_3\} \equiv S_3,
\]

\[
\vdots
\]

\[
X_L^{d-1} \mapsto \{X_1^{d-1}X_2^{d-1}X_3^{d-1}, \ \omega^{d-1}X_1^{d-1}Y_2^{d-1}Y_3^{d-1}, \ \omega^{d-1}Y_1^{d-1}X_2^{d-1}Y_3^{d-1}, \ \omega^{d-1}Y_1^{d-1}Y_2^{d-1}X_3^{d-1}\} \equiv S_{d-1}, \tag{39}
\]

where \( \omega \) is a \( d^{th} \) root of identity and the operators \( X \) and \( Y \) are defined in equations \( \text{26} \) and \( \text{28} \) respectively.

We now employ this mapping in the reverse direction. That is to say, we replace each operators in the set \( S_1 \) by \( X_L \) in the equation \( \text{27} \text{29} \). In this way, the conditions of GHZ nonlocality \( \text{27} \text{29} \) reduce to the following single equation,

\[
X_L|\psi_{\text{GHZ}}\rangle_L = |\psi_{\text{GHZ}}\rangle_L, \tag{40}
\]

which is a reflection of fact that \( |\psi_{\text{GHZ}}\rangle_L \) is a maximally coherent state in the computational basis \( \{|0\rangle_L, \cdots, |d-1\rangle_L\} \). The nonlocality inequality given in equation \( \text{34} \) maps to the following condition for quantum coherence in a single logical system,

\[
|\langle X_L\rangle| \leq \frac{3}{4}. \tag{41}
\]

In a similar manner, consider the nonlocality conditions for the three-qudit GHZ state emergent from \( k^{th} \) power of observables (i.e., \( X^k, Y^k \) (given in \( \text{19} \))

\[
X_1^kX_2^kX_3^k|\psi_{\text{GHZ}}\rangle = |\psi_{\text{GHZ}}\rangle, \ \omega^kX_1^kY_2^kY_3^k|\psi_{\text{GHZ}}\rangle = |\psi_{\text{GHZ}}\rangle,
\omega^kY_1^kX_2^kX_3^k|\psi_{\text{GHZ}}\rangle = |\psi_{\text{GHZ}}\rangle, \ \omega^kY_1^kY_2^kX_3^k|\psi_{\text{GHZ}}\rangle = |\psi_{\text{GHZ}}\rangle; \ (1 \leq k \leq (d - 1)). \tag{42}
\]

Under the mapping \( |iii\rangle \equiv |i\rangle_L \), these conditions reduce to the following conditions for a logical system,

\[
X_L^k|\psi_{\text{GHZ}}\rangle_L = |\psi_{\text{GHZ}}\rangle_L. \tag{43}
\]
Interrelation of nonclassicality features in higher dimensional systems through logical operators

So, the conditions for nonlocality in a multiparty system reduce to the following conditions for coherence,

$$|\langle X^k \rangle| = 1; \ 1 \leq k \leq (d - 1),$$

in the effective monoparty logical system. It is a condition for coherence because it is satisfied by the state $\frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_L$ and it is violated by all the incoherent states in the basis $\{|i\rangle_L\}$.

In fact, with the mappings given in equation (39), the nonlocality inequality derived in [19],

$$\frac{1}{4} \sum_{n=1}^{d-1} (\langle X^n_1 X^n_2 X^n_3 + \omega^n X^n_1 Y^n_2 Y^n_3 + \omega^n Y^n_1 X^n_2 Y^n_3 + \omega^n Y^n_1 Y^n_2 Y^n_3 \rangle) + \text{c.c.} \leq \frac{3d}{4} - 1, \quad d \text{ even},$$

reduces to the following condition for quantum coherence in a single logical qudit system,

$$\sum_{n=1}^{d-1} \langle X^n_L \rangle + \text{c.c.} \leq \frac{3d}{4} - 1.$$  (46)

This condition gets maximally violated by the logical state, $\frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_L$.

This procedure admits a straightforward generalisation to $N$ qudits and for arbitrary $d$.

In a similar manner, the coherence witness underlying CGLMP inequality [20] can be obtained by employing the form of CGLMP inequality given in [39].

7. Generalisation to Continuous variable systems

We now turn our attention to infinite-dimensional systems to show how a condition for coherence in a single infinite-dimensional system maps to conditions for entanglement for a bipartite infinite-dimensional system.

7.1. Condition for coherence in a single mode system $\rightarrow$ Condition for entanglement in a two-mode system

A maximally coherent logical state in the position basis may be written as,

$$|\phi\rangle_L = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx_L |x\rangle_L.$$  (47)

It is invariant under translation generated by the logical momentum operator $\hat{p}_L$ and the logical identity operator, i.e.,

$$\exp(-i\hat{p}_L \epsilon)|\phi\rangle_L = |\phi\rangle_L, \quad 1\rangle_L |\phi\rangle_L = |\phi\rangle_L.$$  (48)
The operators $e^{-i\hat{p}_L \epsilon}$ and $\mathbb{1}_L$ can be represented as $\int dx_L |x + \epsilon\rangle_{LL} \langle x|$ and $\int dx_L |x\rangle_{LL} \langle x|$ respectively. Now, we assume that $|x\rangle_L \equiv |x, x\rangle$ (i.e., a two-mode state). Under this assumption, the state $|\phi\rangle_L$ assumes the following form:

$$|\phi\rangle_L \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx |x, x\rangle. \quad (49)$$

This state is physically realised when two oppositely squeezed vacuum states,

$$\psi_1(x_1) \equiv e^{\zeta/2 \pi^{-1/4}} \exp\left(-\frac{1}{2} e^{2\zeta} x_1^2\right),$$

$$\psi_2(x_2) \equiv e^{-\zeta/2 \pi^{-1/4}} \exp\left(-\frac{1}{2} e^{-2\zeta} x_2^2\right), \quad (50)$$

interfere at 50:50 beam splitter to produce two-mode squeezed vacuum,

$$\psi'(x_1, x_2) \equiv \pi^{-1/2} \exp\left(-\frac{1}{4} e^{2\zeta}(x_1 + x_2)^2 - \frac{1}{4} e^{-2\zeta}(x_1 - x_2)^2\right). \quad (51)$$

This state provides a physical realisation of EPR like state in the limit of infinite squeezing ($\zeta \to \infty$) [40].

Suppose that the translation operators are represented by,

$$\mathcal{P}_i(\epsilon) \equiv \exp(-i\hat{p}_i \epsilon), \quad \mathcal{X}_j(\eta) \equiv \exp(-i\hat{x}_j \eta), \quad i, j \in \{1, 2\}. \quad (52)$$

The representation of the operator $e^{-i\hat{p}_L \epsilon}$ in the logical basis $\{|x\rangle_L\}$ is given by $\int dx_L |x + \epsilon\rangle_{LL} \langle x|$. If $|x_L\rangle \equiv |x, x\rangle$, the action of $e^{-i\hat{p}_L \epsilon}$ is identical to that of the product of operators $e^{-i\hat{p}_1 \epsilon} e^{-i\hat{p}_2 \epsilon}$ on the state $\frac{1}{\sqrt{2\pi}} \int dx |x, x\rangle$. Similarly, the action of the identity operator $\mathbb{1}_L$ on the state $|\psi\rangle_L$ is identical to the action of the product of the operators $e^{-i\hat{x}_1 \eta} e^{i\hat{x}_2 \eta}$ on the state $|\psi\rangle_L$. Employing the procedure laid down in section (3.1), the operators $e^{-i\hat{p}_L \epsilon}$ and $\mathbb{1}_L$ map to the operators $\mathcal{P}_1(\epsilon) \mathcal{P}_2(\epsilon)$ and $\mathcal{X}_1(\eta) \mathcal{X}_2(-\eta)$ respectively.

Of relevance to this work is the fact that the state $|\phi\rangle_L$ is stabilised by the following operators for all values of $\epsilon$ and $\eta$,

$$\mathcal{P}_1(\epsilon) \mathcal{P}_2(\epsilon)|\phi\rangle_L = |\phi\rangle_L, \quad \mathcal{X}_1(\eta) \mathcal{X}_2(-\eta)|\phi\rangle_L = |\phi\rangle_L. \quad (53)$$

The operators $\mathcal{P}_1(\epsilon) \mathcal{P}_2(\epsilon)$ and $\mathcal{X}_1(\eta) \mathcal{X}_2(-\eta)$ are locally noncommuting (in fact, the operators acting on the first party, viz., $\mathcal{X}_1(\eta), \mathcal{P}_1(\epsilon)$ or those acting only on the second party, viz., $\mathcal{X}_2(\eta), \mathcal{P}_2(\epsilon)$ yield the unitary version of canonical commutation relation), and they are globally commuting. It is because $[\hat{x}_1 - \hat{x}_2, \hat{p}_1 + \hat{p}_2] = 0$. So, we employ the relation,

$$\langle \phi_L | e^{-i\hat{p}_L \epsilon} | \phi_L \rangle = \langle \phi_L | e^{-i\hat{p}_1 \epsilon} e^{-i\hat{p}_2 \epsilon} | \phi_L \rangle, \quad \langle \phi_L | \mathbb{1}_L | \phi_L \rangle = \langle \phi_L | e^{-i\hat{x}_1 \eta} e^{i\hat{x}_2 \eta} | \phi_L \rangle. \quad (54)$$

We take a derivative with respect to $\epsilon, \eta$ followed by taking the limit $\epsilon, \eta \to 0$ to obtain
Interrelation of nonclassicality features in higher dimensional systems through logical operators

the following relation,

\[ \lim_{\varepsilon \to 0} \frac{\partial}{\partial \varepsilon} \langle \phi_L | e^{-i\hat{p}_L \varepsilon} | \phi_L \rangle = \lim_{\varepsilon \to 0} \frac{\partial}{\partial \varepsilon} \langle \phi_L | e^{-i\hat{p}_1 \varepsilon - i\hat{p}_2 \varepsilon} | \phi_L \rangle \]

\[ \implies \langle \phi_L | \hat{p}_L | \phi_L \rangle = \langle \phi_L | (\hat{p}_1 + \hat{p}_2) | \phi_L \rangle; \]

\[ \lim_{\eta \to 0} \frac{\partial}{\partial \eta} \langle \phi_L | \hat{1}_L | \phi_L \rangle = \lim_{\eta \to 0} \frac{\partial}{\partial \eta} \langle \phi_L | e^{-i\hat{x}_1 \eta e^{i\hat{x}_2 \eta}} | \phi_L \rangle \]

\[ \implies \langle \phi_L | (\hat{x}_1 - \hat{x}_2) | \phi_L \rangle = 0. \] (55)

So, the operators \((\hat{p}_1 + \hat{p}_2)\) and \((\hat{x}_1 - \hat{x}_2)\) can be employed for constructing entanglement inequalities. In fact, they have been employed in [41] to obtain the following condition for entanglement,

\[ \text{Var}(\hat{x}_1 - \hat{x}_2) + \text{Var}(\hat{p}_1 + \hat{p}_2) \geq 2, \] (56)

where \(\text{Var}(A)\) represents variance of the observable \(A\). The condition in (56) corresponds to the condition for coherence,

\[ \text{Var} \hat{p}_L \geq 2, \] (57)

in a logical continuous-variable system. It is because we substitute the operator, \((\hat{p}_1 + \hat{p}_2)\) by the logical operator, \(\hat{p}_L\) and the operator \((\hat{x}_1 - \hat{x}_2)\) by the null operator, \(0_L\) in accordance with equation (55).

7.2. Condition for coherence in a single-mode state \(\rightarrow\) Condition for entanglement in a three-mode state

The aforementioned procedure admits a straightforward generalisation to multiparty continuous variable systems as well. To show this, we consider an analog of GHZ state in the continuous-variable system,

\[ |\psi_{\text{GHZ}}^c\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx |x, x, x\rangle. \] (58)

This three-mode continuous variable GHZ state has been experimentally realised by sending a momentum squeezed vacuum state \(|p = 0\rangle_1\) and two position squeezed vacuum states, \(|x = 0\rangle_2\) and \(|x = 0\rangle_3\) into a “tritter”. It consists of two beam splitters with transmittance/ reflectivity of 1/2 and 1/1 [42]. The state \(|\psi_{\text{GHZ}}^c\rangle\) is invariant under the following operations due to translation invariance of the state for all values of \(\varepsilon\) and \(\eta\),

\[
\mathcal{P}_1(\varepsilon) \mathcal{P}_2(\varepsilon) \mathcal{P}_3(\varepsilon) |\psi_{\text{GHZ}}^c\rangle = |\psi_{\text{GHZ}}^c\rangle, \\
\mathcal{P}_1(\varepsilon) \{ \mathcal{X}_2(\eta) \mathcal{P}_2(\varepsilon) \} \{ \mathcal{X}_3(-\eta) \mathcal{P}_3(\varepsilon) \} |\psi_{\text{GHZ}}^c\rangle = |\psi_{\text{GHZ}}^c\rangle, \\
\{ \mathcal{X}_1(\eta) \mathcal{P}_1(\varepsilon) \} \{ \mathcal{X}_2(-\eta) \mathcal{P}_2(\varepsilon) \} \{ \mathcal{X}_3(-\eta) \mathcal{P}_3(\varepsilon) \} |\psi_{\text{GHZ}}^c\rangle = |\psi_{\text{GHZ}}^c\rangle, \\
\{ \mathcal{X}_1(\eta) \mathcal{P}_1(\varepsilon) \} \{ \mathcal{X}_2(-\eta) \mathcal{P}_2(\varepsilon) \} \mathcal{P}_3(\varepsilon) |\psi_{\text{GHZ}}^c\rangle = |\psi_{\text{GHZ}}^c\rangle. \] (59)
The three-mode GHZ state, $|\psi^c_{\text{GHZ}}\rangle$, is also an eigenstate of the operator, $X_1(\eta)X_2(-\eta)$, i.e.,

$$X_1(\eta)X_2(-\eta)|\psi^c_{\text{GHZ}}\rangle = |\psi^c_{\text{GHZ}}\rangle.$$ \hspace{1cm} (60)

Here $X_i, P_i$ represent the operators acting over the $i^{th}$ subsystem.

As before, all the operators in equation (59) map to a single logical translation operator, $e^{-i\hat{p}_L\epsilon}$ if we assume $|x, x, x\rangle \equiv |x\rangle_L$. That is to say,

$$\langle \psi^c_{\text{GHZ}}|P_1(\epsilon)P_2(\epsilon)P_3(\epsilon)|\psi^c_{\text{GHZ}}\rangle = \langle \psi^c_{\text{GHZ}}|e^{-i\hat{p}_L\epsilon}|\psi^c_{\text{GHZ}}\rangle.$$ \hspace{1cm} (61)

As before, taking a derivative with respect to $\epsilon$ and taking a limit $\epsilon \rightarrow 0$, we obtain the following relation,

$$\lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial \epsilon} \langle \psi^c_{\text{GHZ}}|P_1(\epsilon)P_2(\epsilon)P_3(\epsilon)|\psi^c_{\text{GHZ}}\rangle = \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial \epsilon} \langle \psi^c_{\text{GHZ}}|e^{-i\hat{p}_L\epsilon}|\psi^c_{\text{GHZ}}\rangle,$$

$$\Rightarrow \langle \psi^c_{\text{GHZ}}|\hat{p}_1 + \hat{p}_2 + \hat{p}_3|\psi^c_{\text{GHZ}}\rangle = \langle \psi^c_{\text{GHZ}}|\hat{p}_L|\psi^c_{\text{GHZ}}\rangle.$$ \hspace{1cm} (62)

Similarly, under the assumption $|x\rangle_L \equiv |x, x, x\rangle$, taking a derivative with respect to $\eta$, equation (61) yields the following equation,

$$\lim_{\eta \rightarrow 0} \frac{\partial}{\partial \eta} \langle \psi^c_{\text{GHZ}}|X_1(\eta)X_2(-\eta)|\psi^c_{\text{GHZ}}\rangle = 0,$$

$$\Rightarrow \langle \psi^c_{\text{GHZ}}|\hat{x}_1 - \hat{x}_2|\psi^c_{\text{GHZ}}\rangle = 0.$$ \hspace{1cm} (63)

The observables, $(\hat{p}_1 + \hat{p}_2 + \hat{p}_3)$ and $(\hat{x}_1 - \hat{x}_2)$ have been employed in [33] to obtain the following condition for separability in a three-mode continuous-variable system,

$$\text{Var}(\hat{x}_1 - \hat{x}_2) + \text{Var}(\hat{p}_1 + \hat{p}_2 + \hat{p}_3) \geq 1.$$ \hspace{1cm} (64)

It reduces to the condition for coherence,

$$\text{Var} \hat{p}_L \geq 1,$$ \hspace{1cm} (65)

in a single logical system. It is because the operator $(\hat{x}_1 - \hat{x}_2)$ and $(\hat{p}_1 + \hat{p}_2 + \hat{p}_3)$ map to the null logical operator and the operator $\hat{p}_L$ respectively in accordance with equations (63) and (62) respectively. In a similar manner, the coherence witness underlying entanglement witnesses for an $N$-mode state may be identified.

8. Explanation of correlation beyond entanglement and discord

Recently, a new quantum correlation beyond entanglement and discord has been shown in [34]. We show that the quantum correlation shown in [34] can also be understood by a suitable choice of logical quantum systems. First, we briefly recapitulate the quantum correlation beyond entanglement and discord in the state,

$$\rho_{12} = \sum_{n=1}^{\infty} (1-p)p^n|n\rangle_1\langle n| \otimes |n\rangle_2\langle n|.$$ \hspace{1cm} (66)
Interrelation of nonclassicality features in higher dimensional systems through logical operators

given in [34]. The subscripts 1 and 2 label the two modes. Quite clearly, this state has zero entanglement and zero quantum discord. However, the $P$-function of this state cannot be interpreted as a classical probability and hence, it is inferred that this state has a nonclassical correlation. It is because both the subsystems have zero coherence and hence the nonclassicality can only be attributable to correlations. That correlation has been termed as 'beyond the previously considered notions'.

It has been shown that the correlation in the state (66) can be employed to activate entanglement. For this, both the modes of the state (66) are made incident on a two beam–splitters. The action of a beam splitter is represented by the transformation,

$$|\alpha\rangle \otimes |\alpha'\rangle \rightarrow \frac{1}{\sqrt{2}} \left( |\alpha + \alpha'\rangle \otimes |\alpha - \alpha'\rangle \right),$$  

where $|\alpha\rangle (|\alpha'\rangle)$ are coherent states. With respect to this basis, the action of a beam–splitter is classical. It will be represented by a diagonal matrix. However, with respect to the basis, $|n\rangle \otimes |0\rangle$, the action of a beam-splitter is given by,

$$|n\rangle \otimes |0\rangle \rightarrow \frac{1}{2^{n/2}} \sum_{j=0}^{n} \binom{n}{j} \frac{1}{2} (-1)^{n-j} |j\rangle \otimes |n - j\rangle \equiv |\Psi_n\rangle,$$

where $\{|n\rangle\}$ is the number basis and $|0\rangle$ is the vacuum state. With respect to this basis, the action of a beam–splitter is nonclassical. It will be represented by a matrix having nonzero off-diagonal elements. So, under the action of two beam-splitters, the state (66) transforms as follows,

$$\rho_{12} \otimes |0\rangle_{33} \langle 0| \otimes |0\rangle_{44} \langle 4| \rightarrow \sum_{n=1}^{\infty} (1 - p)^n |\Psi_n\rangle_{13} \langle \Psi_n| \otimes |\Psi_n\rangle_{24} \langle \Psi_n|.$$

The final state is entangled in the modes (1, 2) and (3, 4), and separable across the bipartition (1, 3) and (2, 4) (i.e., when the modes (1, 3) and (2, 4) are clubbed together). Hence, the state (66) is claimed to have nonclassical correlations, because it can be employed to activate entanglement.

### 8.1. Relation with nonclassicality of logical quantum systems

We now move on to show that the nonclassical correlation identified in [34] can also be mapped to coherence, provided a logical continuous-variable system is chosen suitably. At this juncture, it is pertinent to give the notion for classicality in our approach:

“All the observables and density matrices which are diagonal with respect to a fixed basis are to be treated as classical (with respect to that basis). If there is a nonvanishing off-diagonal term, it is a signature of coherence. Suppose that the system represented by this density matrix is a logical one, which is composed of many physical subsystems. By a suitable correspondence between single party logical basis states and multiparty physical states, coherence in the logical system maps to entanglement (in general, nonclassical correlation) in the physical multiparty system.”
We now proceed to show how nonclassicality of state given in equation (66) can be understood through logical systems.

First choice of logical states:

In equation (69), we first treat the states corresponding to the modes (1, 3) and (2, 4) as two logical states. That is, we make the following mapping:

\[
|\Psi_n\rangle_{13} \equiv |\Psi_{nL}\rangle_{(13)}; \quad |\Psi_n\rangle_{24} \equiv |\Psi_{nL}\rangle_{(24)}.
\]

The resulting logical state which is counterpart of equation (69) will be incoherent,

\[
\rho_L \equiv \sum_{n=1}^{\infty} (1 - p)p^n|\Psi_{nL}\rangle_{(13)} \langle \Psi_{nL}| \otimes |\Psi_{nL}\rangle_{(24)} \langle \Psi_{nL}|.
\]

Clearly, the transformation given in (68) does not generate any coherence for the choice of logical states given in equation (70).

Second choice of logical states:

In a multiparty system, we have freedom in choosing the logical systems. So, we can treat the states corresponding to the modes 1, 2 and 3, 4 as two logical states. That is to say, we choose the following logical states,

\[
|j,k\rangle_{12} \equiv |j,k\rangle_{(12),L}; \quad |n - j, n - k\rangle_{34} \equiv |n - j, n - k\rangle_{(34),L}.
\]

For this choice of logical states, the state \(|\Psi_n\rangle_{13}|\Psi_n\rangle_{24}\) assumes the following form,

\[
|\Psi_n\rangle_{13}|\Psi_n\rangle_{24} \equiv \frac{1}{2^n} \sum_{j,k=0}^{n} \binom{n}{j}^{1/2} \binom{n}{k}^{1/2} (-1)^{2n-j-k}|j,k\rangle_{(12),L}\langle n - j, n - k|_{(34),L}.
\]

For this choice of logical states, the counterpart of equation (69) is given by,

\[
\rho_L \equiv \frac{1}{2^{2n}} \sum_{n=1}^{\infty} \sum_{j,k,l,m=0}^{n} (1 - p)p^n \left\{ \binom{n}{j} \binom{n}{k} \binom{n}{l} \binom{n}{m} \right\}^{1/2} (-1)^{2n-j-k}|j,k\rangle_{(12),L}\langle l,m| \otimes |n - j, n - k\rangle_{(34),L}\langle n - j, n - k|,
\]

which is coherent across the bipartition (1, 2) and (3, 4). It implies that logical counterpart of the transformation given in equation (68) generates coherence for the choice of logical states given in (72). So, the physical state does not show any nonclassical correlation for that partition across which the logical state is incoherent. On the other hand, the physical state shows nonclassical correlation for that partition across which the logical state is coherent.

The nonclassicality of the state \(\rho_{12}\) given in the equation (66) can be understood through logical quantum systems in the following manner. Since the state \(\rho_{12} \otimes |00\rangle_{34}\langle 00|\)
is a four-mode state, it allows for more than one choice of logical quantum systems. The choices that lead to incoherent initial and final logical states (e.g., the one given in equation (70)) do not yield any non-classical correlation in the physical multimode system. On the other hand, the choices that lead to coherence in the final state (e.g., the one given in equation (72)) would yield nonclassical correlation in the physical multimode state.

This feature can be generalised to any $2^N$-mode state having zero entanglement and quantum discord.

8.1.1. Implication An implication of this observation is that not only negative P function captures nonclassical correlations in the state $\rho$, they are also be understood through nonclassicality in logical systems.

9. Partial-positive transpose and its analog in a single qubit system

So far, we have studied interrelations of conditions for coherence to those for different nonclassical correlations through logical quantum systems. Put differently, we have considered the invariance of a given state under different transformations. Now, we consider the invariance of a set of states under a group of transformations. In particular, we turn our attention to PPT criterion for entanglement [7] and show how it reduces to condition for imaginarity of density matrix, whose resource theory has been recently developed [44].

In this section, we show how, in a fixed basis, the transposition for a single qubit maps to PPT criterion for two-qubit systems. Let there be two groups,

- $G_1 \equiv \{1, T\}$. Here 1 and $T$ represent identity transformation and transposition respectively. It is isomorphic to the group $\mathbb{Z}_2 \equiv \{0, 1\}$ addition modulo 2.
- $G_2 \equiv G_1 \otimes G_1$ which is a Kronecker product of $G_1$ with itself.

There is a homomorphic map from $G_2 \equiv G_1 \otimes G_1$ to $G_1$, which is given below,

\[
\begin{align*}
\{1, T \otimes T\} &\mapsto 1 \\
\{1 \otimes T, T \otimes 1\} &\mapsto T.
\end{align*}
\]

(75)

The kernel of homomorphism is $\{1, T \otimes T\}$. Employing representation theory, we may make the groups $G_1$ and $G_2$ act as superoperators in the space of a single system, $\rho_1$ and that of a composite system, $\rho_{12}$. Suppose that a single qubit density matrix $\rho_1$ and a two-qubit density matrix $\rho_{12}$ are given as:

\[
\begin{align*}
\rho_1 &\equiv \sum_{ij} c_{ij} |i\rangle \langle j|, \\
\rho_{12} &\equiv \sum_{i\mu, j\nu} c_{i\mu,j\nu} |i\mu\rangle \langle j\nu|.
\end{align*}
\]

(76)
Interrelation of nonclassicality features in higher dimensional systems through logical operators

Note that $\rho_1$ is not the reduced density matrix of $\rho_{12}$. The transpositions will act as superoperators as follows:

\[
T|i\rangle\langle j| = |j\rangle\langle i|,
(T \otimes 1)|i\mu\rangle\langle j\nu| = |j\mu\rangle\langle i\nu|,
(1 \otimes T)|i\mu\rangle\langle j\nu| = |i\nu\rangle\langle j\mu|,
(T \otimes T)|i\mu\rangle\langle j\nu| = |j\nu\rangle\langle i\mu|.
\]

(77)

The coset $\{1 \otimes T, T \otimes 1\}$ provides us with a positive but not completely-positive map that distinguishes separable and entangled states. One more point—transposition in a single qudit is a physical map and its image provides us with a test that distinguishes entangled and separable states.

9.1. Classical–quantum correspondence

At this juncture, it is pertinent to ask: how does classical-quantum correspondence manifest itself in this case? For example, in the earlier scenarios, a diagonal density matrix with all the diagonal observables (in a fixed basis) constitutes a classical scenario. Here, we may see the correspondence as follows:

(i) A quantum mechanical density matrix is a quantum counterpart of a classical probability distribution function or a probability mass function.

(ii) An eigenvalue-preserving superoperator (that acts on a density matrix) is a quantum counterpart of a classical doubly stochastic matrix (that acts on a probability mass function).

A doubly stochastic matrix maps to a doubly stochastic matrix under transposition (this follows from the definition of a doubly stochastic matrix. An $n \times n$ matrix, $A = [a_{ij}]$, is said to be doubly stochastic if $a_{ij} \geq 0$ and $\sum_i a_{ij} = 1$ for all $i$ and $\sum_j a_{ij} = 1$ for all $j$). Even the tensor product of two doubly stochastic matrices maps to a doubly stochastic matrix under partial transposition. On the other hand, the tensor product of an eigenvalue preserving superoperator with identity need not map to another eigenvalue preserving superoperator (e.g., eigenvalues are preserved under transposition but not under partial transposition). Hence, the interpretation of the aforementioned observation is as follows:

We start with a group $\{1, T\}$ (identity along with transposition). Under the action of transposition, a single qubit transforms as follows:

\[
\rho \equiv \frac{1}{2}(1 + \vec{\sigma} \cdot \vec{p}) \xrightarrow{T} \tilde{\rho} \equiv \frac{1}{2}(1 + \sigma_x p_x - \sigma_y p_y + \sigma_z p_z)
\]

(78)

Thus, transposition reverses the sign of $\langle \sigma_y \rangle$ and so, it is sensitive to the expectation value of $\sigma_y$. Since transposition maps to $1 \otimes T$ or $T \otimes 1$ for a two-qubit system, we infer that a detector of the value of $\langle \sigma_y \rangle$ maps to a criterion of entanglement in two-qubit systems. At this point, we mention that $\langle \sigma_y \rangle$ has been recently identified as a detector
of nonclassicality in the resource theory of imaginarity [41]. What we have shown is that it is this nonclassicality that maps to a condition for entanglement in two-qubit systems.

This observation admits a straightforward generalisation to a higher number of parties.

10. Conclusion

In summary, we have laid down a formalism to obtain condition for entanglement and nonlocality, given a condition for quantum coherence. As an application, we have shown how generalised GHZ nonlocality may be looked upon as a condition for coherence if the three qudits are looked as a single logical qudit. We have also applied the formalism to continuous-variable systems and shown that the observables detecting entanglement in continuous-variable systems are easily identified by this procedure. Finally, we have shown that a new form of quantum correlation identified in [34] may be understood through coherence of logical quantum systems. This work shows how different nonclassical features are related to each other through logical qudits. Therefore, we believe that all the observables that are employed for higher dimensional quantum-error correcting codes [45] may also be used for detection of entanglement in the corresponding states.

In this work, we have studied interconnections of nonclassicality features in quantum states belonging to Hilbert spaces of different dimensions. It is because the space of operators forms a vector space and superoperators can be defined acting over this space of operators. That constitutes an interesting study that will be taken up elsewhere.

Furthermore, the interrelations of different monogamy relations of different quantifiers of a resource, e.g., coherence with those of another resource, e.g., entanglement, nonlocality, etc. can also be studied. In fact, what this work seems to suggest is that mathematically the same nonclassicality conditions detect different forms of nonclassicalities in different physical systems. Finally, if we employ the simplest decimal-to-binary mapping, the resulting mappings will naturally lead to another set of hierarchical relations of nonclassicality conditions in multiqubit systems and multiqudit systems. These conditions will be significant in resource theory of irreducible dimensions [46,47].

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Appendix A. SLK nonlocality for higher powers of observables

In this section, we show how higher powers of observables \(X_i, Y_i\) (given by equations (26) and (28) respectively) also give rise to nonlocality conditions (for a detailed discussion, we refer the reader to [48]). The analysis in this appendix is restricted to \(d = 4\) only. We note that for the observables, \(X_i^2, Y_i^2\), the following set of eigenvalue equations holds:

\[
\begin{align*}
X_i^2X_j^2X_k^2|\psi\rangle &= |\psi\rangle \\
\omega^2X_i^2Y_j^2Y_k^2|\psi\rangle &= |\psi\rangle \\
\omega^2Y_i^2X_j^2Y_k^2|\psi\rangle &= |\psi\rangle \\
\omega^2Y_i^2Y_j^2X_k^2|\psi\rangle &= |\psi\rangle,
\end{align*}
\]

(A.1)

where \(|\psi\rangle\) is the three-ququart GHZ state, \(\frac{1}{2}\sum_{i=0}^3|iii\rangle\). As in section (5), assuming a LHV model and denoting the outcomes of \(X_i, Y_i\) by \(\omega^{x_i}, \omega^{y_i}\) and equating the powers of \(\omega\) both sides, the following set of equations results:

\[
\begin{align*}
2(x_1 + x_2 + x_3) &\equiv 0 \mod 4 \\
2(1 + x_1 + y_2 + y_3) &\equiv 0 \mod 4 \\
2(1 + y_1 + x_2 + y_3) &\equiv 0 \mod 4 \\
2(1 + y_1 + y_2 + x_3) &\equiv 0 \mod 4.
\end{align*}
\]

(A.2)

Adding all the three equations except the first one, the following equation results:

\[
2\left\{3 + (x_1 + x_2 + x_3) + 2(y_1 + y_2 + y_3)\right\} \equiv 0 \mod 4
\]

(A.3)

We incorporate the first equation, \(2(x_1 + x_2 + x_3) \equiv 0\), in this equation to get

\[
6 + 4(y_1 + y_2 + y_3) \equiv 0 \mod 4,
\]

(A.4)

which cannot be satisfied for any integer values of \(y_1, y_2, y_3\).

Similarly, for \(X_i^3, Y_i^3\), the following set of eigenvalue equations holds:

\[
\begin{align*}
X_i^3X_j^3X_k^3|\psi\rangle &\equiv |\psi\rangle \\
\omega^3X_i^3Y_j^3Y_k^3|\psi\rangle &\equiv |\psi\rangle \\
\omega^3Y_i^3X_j^3Y_k^3|\psi\rangle &\equiv |\psi\rangle \\
\omega^3Y_i^3Y_j^3X_k^3|\psi\rangle &\equiv |\psi|.
\end{align*}
\]

(A.5)

As before, assuming a LHV model and denoting the outcomes of \(X_i, Y_i\) by \(\omega^{x_i}, \omega^{y_i}\) and equating the powers of \(\omega\) both sides, the following set of equations results:

\[
\begin{align*}
3(x_1 + x_2 + x_3) &\equiv 0 \mod 4 \\
3(1 + x_1 + y_2 + y_3) &\equiv 0 \mod 4 \\
3(1 + y_1 + x_2 + y_3) &\equiv 0 \mod 4 \\
3(1 + y_1 + y_2 + x_3) &\equiv 0 \mod 4.
\end{align*}
\]

(A.6)
Adding all but the first equation, the following equation results:

$$3\{3 + (x_1 + x_2 + x_3) + 2(y_1 + y_2 + y_3)\} = 0$$  \hspace{1cm} (A.7)

We incorporate the first equation, $3(x_1 + x_2 + x_3) = 0$, in this equation to get

$$9 + 6(y_1 + y_2 + y_3) \equiv 0 \mod 4,$$  \hspace{1cm} (A.8)

which cannot be satisfied for any integer values of $y_1, y_2, y_3$. Similarly, it can be extended to higher values of $d$ by a suitable choice of $Y_i$.

**Bibliography**

[1] John S Bell. On the einstein podolsky rosen paradox. *Physics Physique Fizika*, 1(3):195, 1964.
[2] Simon Kochen and Ernst P Specker. The problem of hidden variables in quantum mechanics. In *The logico-algebraic approach to quantum mechanics*, pages 293–328. Springer, 1975.
[3] Arthur Fine. Hidden variables, joint probability, and the bell inequalities. *Physical Review Letters*, 48(5):291, 1982.
[4] G Svetlichny. Quantum nonlocality as an axiom. *Phys. Rev. D*, 35:3066, 1987.
[5] Reinhard F Werner. Quantum states with einstein-podolsky-rosen correlations admitting a hidden-variable model. *Physical Review A*, 40(8):4277, 1989.
[6] N David Mermin. Extreme quantum entanglement in a superposition of macroscopically distinct states. *Physical Review Letters*, 65(15):1838, 1990.
[7] Asher Peres. Separability criterion for density matrices. *Physical Review Letters*, 77(8):1413, 1996.
[8] Harold Ollivier and Wojciech H Zurek. Quantum discord: a measure of the quantumness of correlations. *Physical review letters*, 88(1):017901, 2001.
[9] Howard M Wiseman, Steve James Jones, and Andrew C Doherty. Steering, entanglement, nonlocality, and the einstein-podolsky-rosen paradox. *Physical review letters*, 98(14):140402, 2007.
[10] Alexander Streltsov, Gerardo Adesso, and Martin B Plenio. Colloquium: Quantum coherence as a resource. *Reviews of Modern Physics*, 89(4):041003, 2017.
[11] Artur K Ekert. Quantum cryptography based on bell’s theorem. *Physical review letters*, 67(6):661, 1991.
[12] Charles H Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K Wootters. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Physical review letters*, 70(13):1895, 1993.
[13] Lov K Grover. Quantum mechanics helps in searching for a needle in a haystack. *Physical review letters*, 79(2):325, 1997.
[14] Mark Hillery. Coherence as a resource in decision problems: The deutsch-jozsa algorithm and a variation. *Physical Review A*, 93(1):012111, 2016.
[15] Albert Einstein, Boris Podolsky, and Nathan Rosen. Can quantum-mechanical description of physical reality be considered complete? *Physical review*, 47(10):777, 1935.
[16] Erwin Schrödinger. Discussion of probability relations between separated systems. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 31, pages 555–563. Cambridge University Press, 1935.
[17] Nicolas Brunner, Daniel Cavalcanti, Stefano Pironio, Valerio Scarani, and Stephanie Wehner. Bell nonlocality. *Reviews of Modern Physics*, 86(2):419, 2014.
[18] Michael Seevinck and George Svetlichny. Bell-type inequalities for partial separability in n-particle systems and quantum mechanical violations. *Physical review letters*, 89(6):060401, 2002.
Interrelation of nonclassicality features in higher dimensional systems through logical operators

[19] W. Son, Jinhyoung Lee, and M. S. Kim. Generic bell inequalities for multipartite arbitrary dimensional systems. *Phys. Rev. Lett.*, 96:060406, Feb 2006.

[20] Daniel Collins, Nicolas Gisin, Noah Linden, Serge Massar, and Sandu Popescu. Bell inequalities for arbitrarily high-dimensional systems. *Physical review letters*, 88(4):040404, 2002.

[21] V Ugur Guney and Mark Hillery. Bell inequalities from group actions of single-generator groups. *Physical Review A*, 90(6):062121, 2014.

[22] V Ugur Guney and Mark Hillery. Bell inequalities from group actions: Three parties and non-abelian groups. *Physical Review A*, 91(5):052110, 2015.

[23] Mariami Gachechiladze, Costantino Budroni, and Otfried Gühne. Extreme violation of local realism in quantum hypergraph states. *Physical review letters*, 116(7):070401, 2016.

[24] Ryszard Horodecki, Paweł Horodecki, Michal Horodecki, and Karol Horodecki. Quantum entanglement. *Reviews of modern physics*, 81(2):865, 2009.

[25] Otfried Gühne and Géza Tóth. Entanglement detection. *Physics Reports*, 474(1-6):1–75, 2009.

[26] Andreas Winter and Dong Yang. Operational resource theory of coherence. *Physical review letters*, 116(12):120404, 2016.

[27] Chang-shui Yu, Yang Zhang, and Haiqing Zhao. Quantum correlation via quantum coherence. *Quantum Information Processing*, 13(6):1437–1456, 2014.

[28] Eric Chitambar and Min-Hsiu Hsieh. Relating the resource theories of entanglement and quantum coherence. *Phys. Rev. Lett.*, 117:020402, Jul 2016.

[29] Yuan Sun, Yuanyuan Mao, and Shunlong Luo. From quantum coherence to quantum correlations. *EPL (Europhysics Letters)*, 118(6):60007, 2017.

[30] Lu-Ming Duan, G. Giedke, J. I. Cirac, and P. Zoller. Inseparability criterion for continuous variable systems. *Phys. Rev. Lett.*, 84:2722–2725, Mar 2000.
Interrelation of nonclassicality features in higher dimensional systems through logical operators

[44] Kang-Da Wu, Tulja Varun Kondra, Swapan Rana, Carlo Maria Scandolo, Guo-Yong Xiang, Chuan-Feng Li, Guang-Can Guo, and Alexander Streltsov. Operational resource theory of imaginarity. *Phys. Rev. Lett.*, 126:090401, Mar 2021.

[45] Priya J Nadkarni and Shayan Srinivasa Garani. Quantum error correction architecture for qudit stabilizer codes. *Physical Review A*, 103(4):042420, 2021.

[46] Wan Cong, Yu Cai, Jean-Daniel Bancal, and Valerio Scarani. Witnessing irreducible dimension. *Physical review letters*, 119(8):080401, 2017.

[47] Tristan Kraft, Christina Ritz, Nicolas Brunner, Marcus Huber, and Otfried Gühne. Characterizing genuine multilevel entanglement. *Phys. Rev. Lett.*, 120:060502, Feb 2018.

[48] Jinhyoung Lee, Seung-Woo Lee, and Myungshik S Kim. Greenberger-horne-zeilinger nonlocality in arbitrary even dimensions. *Physical Review A*, 73(3):032316, 2006.