Zero-Sum Stochastic Differential Game in Finite Horizon Involving Impulse Controls

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Abstract
This paper considers the problem of two-player zero-sum stochastic differential game with both players adopting impulse controls in finite horizon under rather weak assumptions on the cost functions ($c$ and $\chi$ not decreasing in time). We use the dynamic programming principle and viscosity solutions approach to show existence and uniqueness of a solution for the Hamilton–Jacobi–Bellman–Isaacs (HJBI) partial differential equation (PDE) of the game. We prove that the upper and lower value functions coincide.

Keywords Stochastic differential game · Impulse control · Quasi-variational inequality · Viscosity solution

Mathematics Subject Classification 93E20 · 49L20 · 49L25 · 49N70

1 Introduction

The theory of differential games with Elliot–Kalton strategies in the viscosity solution framework was initiated by Evans and Souganidis [21]. Fleming and Souganidis [22] studied in a rigorous manner two-player zero-sum stochastic differential games and their work translated former results on differential games from the purely deterministic into the stochastic framework. Subsequently, Buckdahn and Li [10] generalized the framework introduced in [22].

In this paper, we consider the state process of the stochastic differential game, defined as the solution of the following stochastic equation:
\[
X_s = x + \int_t^s b(r, X_r)dr + \int_t^s \sigma(r, X_r)dW_r + \sum_{m \geq 1} \xi_m \mathbb{I}_{[\tau_m, T]}(s) \prod_{l \geq 1} \mathbb{I}_{[\tau_m \neq \rho_l]} + \sum_{l \geq 1} \eta_l \mathbb{I}_{[\rho_l, T]}(s), \quad s \geq t,
\]
for all \( s \in [t, T] \), P-a.s., with \( X_t^- = x \). Here \( W \) is a \( d \)-dimensional Wiener process, while

\[
u(s) = \sum_{m \geq 1} \xi_m \mathbb{I}_{[\tau_m, T]}(s) \quad \text{and} \quad v(s) = \sum_{l \geq 1} \eta_l \mathbb{I}_{[\rho_l, T]}(s)
\]
are the impulse controls of player I and player II, respectively. The random variables \( \xi_m \) and \( \eta_l \) take values in two convex cones \( \mathcal{U} \) and \( \mathcal{V} \) of \( \mathbb{R}^n \), respectively, called the spaces of control actions. The infinite product \( \prod_{l \geq 1} \mathbb{I}_{[\tau_m \neq \rho_l]} \) has the following meaning: When the two players act together on the system at the same time, we take into account only the action of player II. We denote by \( X_t^{l, x, u, v} = \{ X_t^{l, x, u, v}, t \leq s \leq T \} \) the state trajectory of the game with initial time \( t \), initial state \( x \), and impulse controls \( u \) and \( v \).

The gain functional for player I (resp., cost functional for player II) is given by

\[
J(t, x; u, v) = \mathbb{E} \left[ \int_t^T f(s, X_s^{l, x, u, v})ds - \sum_{m \geq 1} c(\tau_m, \xi_m) \mathbb{I}_{[\tau_m \leq T]} \prod_{l \geq 1} \mathbb{I}_{[\tau_m \neq \rho_l]} + \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{[\rho_l \leq T]} + g(X_T^{l, x, u, v}) \right],
\]
(1.1)

\( f \) is the running gain and \( g \) is the payoff. The function \( c \) is the cost function for player I and is a gain function for player II, meaning that when player I performs an action he/she has to pay a cost, resulting in a gain for player II. Analogously, \( \chi \) is the cost function for player II and is a gain function for player I. Following the framework of Buckdahn and Li [10] we define the upper and lower value functions, this framework is different from the one in the seminal paper by Fleming–Souganidis [22] since it by-passes the notion of \( r \)-strategies.

The stochastic differential games problem have recently attracted a lot of research activities, especially in connection with mathematical finance (see e.g. [7,26,29,30,32–34,36–38] and the references therein). In order to tackle those problems, authors use mainly two approaches. Either a probabilistic one [23,24] or an approach which uses the Hamilton–Jacobi–Bellman–Isaacs (HJBI) partial differential equation (PDE) [22]. Our closest related works are [2,3,14,41,42,44]. Bayraktar [3] considers a zero-sum stochastic differential controller-and-stopper game. Zhang [44] studied, in the viscosity sense, a stochastic differential game involving impulse controls in infinite horizon, but in [44] one player adopts impulse controls, while the other uses continuous controls. Azimzadeh [2] extends this result to finite horizon by using the weak dynamic programming principle. Concerning now the case when the two players adopting impulse controls: Stettner [41] studied this problem in infinite horizon by direct probabilistic method. In the finite horizon framework, Cosso [14] studied this game using viscosity solutions to the HJBI equation, where the author have proved existence and
uniqueness for the HJBI equation under stronger constraint (see also Tang and Yong [42]): the cost functions $c$ and $\chi$ are decreasing in time:

$$c(t, y) \geq c(t', y) \quad \text{and} \quad \chi(t, z) \geq \chi(t', z),$$

for every $0 \leq t \leq t' \leq T$, $y \in \mathcal{U}$ and $z \in \mathcal{V}$.

Our aim in this work lies in the fact that we investigate the solution to the zero-sum stochastic differential games without monotonicity of the costs. Therefore the main objective of our work, and this is the novelty of the paper is to characterize the value function as the only solution in viscosity sense of the associated Hamilton–Jacobi–Bellman–Isaacs (HJBI) partial differential equation (PDE) for the finite horizon problem. To derive these results, we first adapt to the case when both players use impulse controls the proof of the weak dynamic programming principle (DPP) already obtained in [2] when one player uses impulse control whereas the second uses stochastic control. Note that, in [2], the assumption of monotonicity of the costs is dropped as well. This notion of weak DPP obtained along rational stopping times is the main difference compared with [14]. Next, we use the weak DPP to prove the continuity of value functions, which in return, is used to derive the strong DPP. The last part is to show that the HJBI equation associated to the stochastic differential game, which turns out to be the same for the two value functions because the two players cannot act simultaneously on the system, is the following system:

$$\begin{cases} 
\max \left\{ \min \left[ -\frac{\partial V}{\partial t} - \mathcal{L}V - f, V - H_{sup}^c V, V - H_{inf}^\chi V \right] \right\} = 0 & (0, T) \times \mathbb{R}^n \\
V(T, x) = g(x) & \forall x \in \mathbb{R}^n.
\end{cases}$$

(1.2)

where $\mathcal{L}$ is the second-order local operator, and the nonlocal operators $H_{sup}^c$ and $H_{inf}^\chi$ are given by

$$H_{sup}^c V(t, x) = \sup_{\xi \in \mathcal{U}} \{ V(t, x + \xi) - c(t, \xi) \},$$

$$H_{inf}^\chi V(t, x) = \inf_{\eta \in \mathcal{V}} \{ V(t, x + \eta) + \chi(t, \eta) \},$$

for every $(t, x) \in [0, T] \times \mathbb{R}^n$. We prove that the value functions are viscosity solution of the HJBI equation (1.2). Uniqueness is obtained under additional assumptions.

This paper is organized as follows: in Sect. 2, we formulate the problem and we give the related definitions. In Sect. 3, we shall introduce the stochastic differential game problem and give some preliminary results of the lower and the upper value functions of stochastic differential game. Further we provide some estimate for the nearly optimal strategies. In Sect. 4, we prove the weak dynamic programming principle. Further, we show the continuity of the lower and the upper value functions and obtain as a byproduct the strong dynamic programming principle. Section 5, is devoted to the connection between the zero sum stochastic differential game problem and Hamilton–Jacobi–Bellman–Isaacs equation. In Sect. 6, we show that the solution of HJBI is
unique in the subclass of bounded continuous functions. Further, the upper and the lower value functions coincide and the game admits a value.

2 Assumptions and Formulation of the Problem

Throughout this paper $T$ (resp. $n$, $d$) is a fixed real (resp. integers) positive numbers.

Let us assume the following assumptions:

$$(H1) \quad b : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ and } \sigma : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d} \text{ be two continuous functions for which there exists a constant } C > 0 \text{ such that for any }$$

$$t \in [0, T] \text{ and } x, x' \in \mathbb{R}^n$$

$$|\sigma(t, x) - \sigma(t, x')| + |b(t, x) - b(t, x')| \leq C|x - x'|. \quad (2.1)$$

Also there exists a constant $C > 0$ such that for any $(t, x) \in [0, T] \times \mathbb{R}^n$

$$|\sigma(t, x)| + |b(t, x)| \leq C(1 + |x|). \quad (2.2)$$

$$(H2) \quad f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R} \text{ is uniformly continuous and bounded on } [0, T] \times \mathbb{R}^n. \quad g : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is uniformly continuous and bounded on } \mathbb{R}^n. \quad (H3) \text{ The cost functions } c : [0, T] \times \mathcal{U} \rightarrow \mathbb{R} \text{ and } \chi : [0, T] \times \mathcal{V} \rightarrow \mathbb{R} \text{ are measurable and uniformly continuous. Furthermore }$$

$$\inf_{[0,T] \times \mathcal{U}} c \geq k, \quad \inf_{[0,T] \times \mathcal{V}} \chi \geq k, \quad (2.3)$$

where $k > 0$.

Moreover,

$$c(t, \xi_1 + \xi_2) \leq c(t, \xi_1) + c(t, \xi_2) \quad (2.4)$$

$$\chi(t, \eta_1 + \eta_2) \leq \chi(t, \eta_1) + \chi(t, \eta_2) \quad (2.5)$$

for every $t \in [0, T]$, $\xi_1, \xi_2 \in \mathcal{U}$ and $\eta_1, \eta_2 \in \mathcal{V}$.

$$(H4) \text{ (no terminal impulse). For any } x \in \mathbb{R}^n, \eta \in \mathcal{V} \text{ and } \xi \in \mathcal{U}$$

$$\sup_{\xi \in \mathcal{U}} [g(x + \xi) - c(T, \xi)] \leq g(x) \leq \inf_{\eta \in \mathcal{V}} [g(x + \eta) + \chi(T, \eta)]. \quad (2.6)$$

$\mathcal{U}$ and $\mathcal{V}$ are two convex cones of $\mathbb{R}^n$ with $\mathcal{U} \subset \mathcal{V}$.

$\textbf{Remark 1}$ The above Assumptions (2.4) and (2.5) ensures that multiple impulses occurring at the same time are suboptimal.

We now consider the HJBI equation:

$$\begin{cases}
\max \left\{ \min \left[ -\frac{\partial V}{\partial t} - \mathcal{L}V - f, V - H^c_{\sup} V, V - H^c_{\inf} V \right] \right. , V - H^Z_{\inf} V = 0 \quad [0, T] \times \mathbb{R}^n, \\
V(T, x) = g(x) \quad \forall x \in \mathbb{R}^n.
\end{cases} \quad (2.7)$$
where $\mathcal{L}$ is the second-order local operator

$$\mathcal{L}V = \langle b, \nabla_x V \rangle + \frac{1}{2} tr [\sigma \sigma^* \nabla_x^2 V],$$

and the nonlocal operators $H^c_{sup}$ and $H^X_{inf}$ are given by

$$H^c_{sup} V(t, x) = \sup_{\xi \in \mathcal{U}} [V(t, x + \xi) - c(t, \xi)],$$

$$H^X_{inf} V(t, x) = \inf_{\eta \in \mathcal{V}} [V(t, x + \eta) + \chi(t, \eta)],$$

for every $(t, x) \in [0, T] \times \mathbb{R}^n$.

The main objective of this paper is to focus on the existence and uniqueness of the solution in viscosity sense of (2.7) whose definition is:

**Definition 1** Let $V$ be a continuous function defined on $[0, T] \times \mathbb{R}^n$ and such that $V(T, x) = g(x)$ for any $x \in \mathbb{R}^n$. The $V$ is called:

(i) A viscosity subsolution of (2.7) if for any $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ and any function $\phi \in C^{1,2}([0, T] \times \mathbb{R}^n)$, such that $(t_0, x_0)$ is a local maximum of $V - \phi$, we have:

$$\max \left\{ \min \left[ -\frac{\partial \phi}{\partial t}(t_0, x_0) - \mathcal{L}\phi(t_0, x_0) - f(t_0, x_0), V(t_0, x_0) - H^c_{sup} V(t_0, x_0) \right], V(t_0, x_0) - H^X_{inf} V(t_0, x_0) \right\} \leq 0.$$  \hspace{1cm} (2.8)

(ii) A viscosity supersolution of (2.7) if for any $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$ and any function $\phi \in C^{1,2}([0, T] \times \mathbb{R}^n)$, such that $(t_0, x_0)$ is a local minimum of $V - \phi$, we have:

$$\max \left\{ \min \left[ -\frac{\partial \phi}{\partial t}(t_0, x_0) - \mathcal{L}\phi(t_0, x_0) - f(t_0, x_0), V(t_0, x_0) - H^c_{sup} V(t_0, x_0) \right], V(t_0, x_0) - H^X_{inf} V(t_0, x_0) \right\} \geq 0.$$  \hspace{1cm} (2.9)

(iii) A viscosity solution if it is both a viscosity supersolution and subsolution. \hspace{1cm} $\square$

There is an equivalent formulation of this definition (see e.g. [15]) which we give because it will be useful later. So firstly, we define the notions of superjet and subjet of a continuous function $V$.

**Definition 2** Let $V \in C((0, T) \times \mathbb{R}^n)$, $(t, x)$ an element of $(0, T) \times \mathbb{R}^n$ and finally $S_n$ the set of $n \times n$ symmetric matrices. We denote by $J^{2,+} V(t, x)$ (resp. $J^{2,-} V(t, x)$), the superjets (resp. the subjets) of $V$ at $(t, x)$, the set of triples $(p, q, X) \in \mathbb{R} \times \mathbb{R}^n \times S_n$ such that:

$$V(s, y) \leq V(t, x) + p(s - t) + \langle q, y - x \rangle$$

$$+ \frac{1}{2} \langle X(y - x), y - x \rangle + o(|s - t| + |y - x|^2).$$
Let (resp. $V(s, y) \geq V(t, x) + p(s - t) + \langle q, y - x \rangle$
$$+
\frac{1}{2}(X(y - x), y - x) + o(|s - t| + |y - x|^2)).$$

Note that if $\phi - V$ has a local maximum (resp. minimum) at $(t, x)$, then we obviously have:
$$(D_t \phi(t, x), D_x \phi(t, x), D_{xx} \phi(t, x)) \in J^{2-} V(t, x) \ (\text{resp. } J^{2+} V(t, x)).$$

We now give an equivalent definition of a viscosity solution of HJBI equation (2.7):

**Definition 3** Let $V$ be a continuous function defined on $[0, T] \times \mathbb{R}^n$ and such that $V(T, x) = g(x)$ for any $x \in \mathbb{R}^n$. Then $V$ is a viscosity supersolution (resp., subsolution) to the HJBI equation (2.7) if and only if for every $(t, x) \in [0, T] \times \mathbb{R}^n$ and $(p, q, X) \in J^{2-} V(t, x) \ (\text{resp. } J^{2+} V(t, x))$,
$$\max \left\{ \min \left[ -p - \langle b, q \rangle - \frac{1}{2} Tr \left[ \sigma^* X \sigma \right] - f(t, x), V(t, x) - H^c_{\sup} V(t, x) \right] \right\} \geq 0 \quad (\text{resp., } \leq 0).$$

It is called a viscosity solution it is both a viscosity subsolution and supersolution. □

As pointed out previously we will show that system (2.7) has a unique solution in viscosity sense. This system is the deterministic version of the stochastic differential game problem will describe briefly in the next section.

### 3 The Stochastic Differential Game Problem

#### 3.1 Setting of the Problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space on which is defined a standard $d$-dimensional Brownian motion $W = (W_t)_{t \leq T}$, whose natural filtration is $(\mathcal{F}_t^0 := \sigma \{ W_s; s \leq t \})_{0 \leq t \leq T}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ the completed filtration of $(\mathcal{F}_t^0)_{t \leq T}$ with the $\mathbb{F}$-null sets of $\mathcal{F}$. We are given two convex cones $\mathcal{U}$ and $\mathcal{V}$ of $\mathbb{R}^n$, with $\mathcal{V} \subset \mathcal{U}$. We call $\mathcal{U}$ and $\mathcal{V}$ the spaces of control actions. We define the set $\mathcal{Q}_{[t,T]} = (\mathcal{Q} \cap [t, T]) \cup \{t, T\}$.

We begin by introducing the concept of impulse control.

**Definition 4** An impulse control $\mu = \sum_{m \geq 1} \xi_m \mathbb{1}_{[\tau_m, T]}$ for player I (resp., $\nu = \sum_{l \geq 1} \eta_l \mathbb{1}_{[\rho_l, T]}$ for player II) on $[t, T] \subset \mathbb{R}^+ = [0, +\infty)$, is such that:

(i) $(\tau_m)_m$ (resp., $(\rho_l)_l)$, the action times, is a sequence of $\mathbb{F}$-stopping times, valued in $\mathcal{Q}_{[t,T]} \cup \{+\infty\}$ such that $\mathbb{P}$-a.s. $\tau_m \leq \tau_{m+1}$ (resp., $\rho_l \leq \rho_{l+1}$).

$\mathcal{Q}$ Springer
(ii) \((\xi_m)_m\) (resp., \((\eta_l)_l\)), the actions, is a sequence of \(\mathcal{U}\)-valued (resp., \(\mathcal{V}\)-valued) random variables, where each \(\xi_m\) (resp., \(\eta_l\)) is \(\mathcal{F}_{\tau_m}\)-measurable (resp., \(\mathcal{F}_{\rho_l}\)-measurable).

Let \(t \in [0, T]\) be the initial time of the game and \(x \in \mathbb{R}^n\) the initial state. Then, given the impulse controls \(u\) and \(v\) on \([t, T]\), the state process of the stochastic differential game is defined as the solution to the following stochastic equation:

\[
X_s = x + \int_t^s b(r, X_r)dr + \int_t^s \sigma(r, X_r)dW_r + \sum_{m \geq 1} \xi_m \mathbb{1}_{[\tau_m, T]}(s) \prod_{l \geq 1} \mathbb{1}_{[\tau_m \neq \rho_l]} + \sum_{l \geq 1} \eta_l \mathbb{1}_{[\rho_l, T]}(s), \quad s \geq t, \tag{3.1}
\]

If it exists and is unique, we use \(X_t, x, u, v\) to denote a solution to (3.1).

The gain functional for player I (resp., cost functional for player II) is given by

\[
J(t, x, u, v) := \mathbb{E}\left[ \int_t^T f(s, X_s^{t, x, u, v})ds - \sum_{m \geq 1} c(\tau_m, \xi_m) \mathbb{1}_{[\tau_m \leq T]} \prod_{l \geq 1} \mathbb{1}_{[\tau_m \neq \rho_l]} + \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{1}_{[\rho_l \leq T]} + g(X_T^{t, x, u, v}) \right], \tag{3.2}
\]

\(f\) is the running gain and \(g\) is the payoff. The function \(c\) is the cost function for player I and is a gain function for player II, meaning that when player I performs an action he/she has to pay a cost, resulting in a gain for player II. Analogously, \(\chi\) is the cost function for player II and is a gain function for player I.

**Definition 5** Let \(u = \sum_{m \geq 1} \xi_m \mathbb{1}_{[\tau_m, T]}\) be an impulse control on \([t, T]\), and let \(\tau \leq \sigma\) be two \([t, T]\)-valued \(\mathbb{F}\)-stopping times. Then we define the restriction \(u_{[\tau, \sigma]}\) of the impulse control \(u\) by:

\[
u_{[\tau, \sigma]}(s) = \sum_{m \geq 1} \xi_{\mu_{t, \tau}(u)+m} \mathbb{1}_{[\tau_{\mu_{t, \tau}(u)+m} \leq s \leq \sigma]}(s), \quad \tau \leq s \leq \sigma, \tag{3.3}
\]

\(\mu_{t, \tau}\) is the number of impulses up to time \(\tau\), i.e.,

\[
\mu_{t, \tau}(u) := \sum_{m \geq 1} \mathbb{1}_{[\tau_m \leq \tau]}. \tag{3.4}
\]

**Definition 6** (Admissible impulse control). We say that the controls \(u\) and \(v\) are admissible on \([t, T] \subset \mathbb{R}^+\), if

1. A solution of (3.1) exists and is unique.
2. The average number of impulses is finite, i.e.,

\[
\mathbb{E}[\mu_{t, T}(u)] < \infty \quad \text{and} \quad \mathbb{E}[\mu_{t, T}(v)] < \infty,
\]

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The set of all admissible impulse controls for player I (resp., II) on $[t, T]$ is denoted by $\mathcal{U}_{t,T}$ (resp., $\mathcal{V}_{t,T}$).

Next, we adopt the notion of control identification.

**Definition 7** (Control identification) Let $u = \sum_{m \geq 1} \xi_m \mathbb{1}_{[\tau_m, T]}$ and $u' = \sum_{m \geq 1} \xi'_m \mathbb{1}_{[\tau'_m, T]}$ in $\mathcal{U}_{t,T}$, we write $u \equiv u'$ on $[t, T]$ if

$$\mathbb{P}(\{u = u'\ \text{a.e. on } [t, T]\}) = 1.$$  

Similarly, we interpret $v \equiv v'$ on $[t, T]$ in $\mathcal{V}_{t,T}$.

Following Cosso [14] and Buckdahn and Li [10], we define nonanticipative strategy as follows.

**Definition 8** The nonanticipative strategy set $\mathcal{A}_{t,T}$ for player I is the collection of all nonanticipative maps $\alpha$ from $\mathcal{V}_{t,T}$ to $\mathcal{U}_{t,T}$, i.e. for any $[t, T]$-valued $\mathbb{R}$-stopping times $\tau$ and any $v_1, v_2 \in \mathcal{V}_{t,T}$,

$$\text{if } v_1 \equiv v_2 \text{ on } [t, \tau], \text{ then } \alpha(v_1) \equiv \alpha(v_2) \text{ on } [t, \tau].$$

(with the notation $[t, \tau] = \{(s, \omega) \in [0, T] \times \Omega, t \leq s \leq \tau(\omega)\}$)

Analogously, the nonanticipative strategy set $\mathcal{B}_{t,T}$ for player II is the collection of all nonanticipative maps $\beta$ from $\mathcal{U}_{t,T}$ to $\mathcal{V}_{t,T}$.

We are now ready to introduce the upper and lower values of the game: For every $(t, x) \in [0, T] \times \mathbb{R}^n$ let us set

$$V^-(t, x) := \inf_{\beta \in \mathcal{B}_{t,T}} \sup_{u \in \mathcal{U}_{t,T}} J(t, x, u, \beta(u)) \quad (3.5)$$

and

$$V^+(t, x) := \sup_{\alpha \in \mathcal{A}_{t,T}} \inf_{v \in \mathcal{V}_{t,T}} J(t, x, \alpha(v), v) \quad (3.6)$$

The maps $V^-$ and $V^+$ are called the lower value and the upper value of the game, respectively. The game is said to admit a value if $V^- = V^+$.

The HJBI equation associated to the stochastic differential game, which turns out to be the same for the two value functions because the two players cannot act simultaneously on the system, is given by (2.7).

**Remark 2** The infinite product $\prod_{l \geq 1} \mathbb{1}_{\{\tau_m \neq \rho_l\}}$ in (3.1) means that when the two players act together on the system, we take into account only the action of player II. When take into account only the action of player I instead of player II. Then, using arguments analogous to those presented in the sections below, it can be proved that, with this assumption, the corresponding HJBI equation is given by

$$\mathcal{E} \text{ Springer}$$
\[
\begin{aligned}
\min \left\{ \max \left[ -\frac{\partial V}{\partial t} - LV - f, V - H_{\inf} V \right], V - H_{\sup} V \right\} = 0 \\
V(T, x) = g(x)
\end{aligned}
\]

\[0, T) \times \mathbb{R}^n \forall x \in \mathbb{R}^n.\]

(3.7)

### 3.2 Preliminary Results

In this section we present some properties of the lower and upper value functions of our differential game.

We begin by the following lemma, which is concerned with the continuous dependence of \(X^{t, x, u, v}\) with respect to \(x\).

**Lemma 1** Under assumption \((H1)\) there exists a constant \(C > 0\) such that, for every \(t \in [0, T], \ x, x' \in \mathbb{R}^n, u \in U_{t, T}\) and \(v \in V_{t, T}\) we have:

\[
\mathbb{E}\left[ |X^{t, x, u, v}_s - X^{t, x', u, v}_s| \right] \leq C|x - x'|.
\]

(3.8)

**Proof** see ([39], Appendix)

Next, in the following theorem, we prove that the two value functions are bounded.

**Theorem 1** Under the standing assumptions \((H1), (H2)\) and \((H3)\), the lower and upper value functions are bounded.

**Proof** We make the proof only for the lower value function \(V^-\), the other case being analogous.

Using the definition of lower value function, we have, for \((t, x) \in [0, T] \times \mathbb{R}^n\)

\[
V^-(t, x) = \inf_{\beta \in B_{t,T}} \sup_{u \in U_{t,T}} \mathbb{E}\left[ \int_t^T f(s, X^{t, x, u, \beta (u)}_s) + \sum_{m \geq 1} c(\tau_m, \xi_m) \mathbb{I}_{[\tau_m \leq T]} \prod_{l \geq 1} \mathbb{I}_{[\tau_m \neq \rho_l]} \right. \\
+ \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{[\rho_l \leq T]} + g(X^{t, x, u, \beta (u)}_T) \bigg] \\
\leq \sup_{u \in U_{t,T}} \mathbb{E}\left[ \int_t^T f(s, X^{t, x, u, v_0}_s) + \sum_{m \geq 1} c(\tau_m, \xi_m) \mathbb{I}_{[\tau_m \leq T]} + g(X^{t, x, u, v_0}_T) \right],
\]

where \(v_0\) is the control with no impulses.

Let \(\epsilon > 0\), then there exists a strategy \(u^\epsilon \in U_{t, T}\) such that

\[
V^-(t, x) \leq \mathbb{E}\left[ \int_t^T f(s, X^{t, x, u^\epsilon, v_0}_s) + \sum_{m \geq 1} c(\tau_m^\epsilon, \xi_m^\epsilon) \mathbb{I}_{[\tau_m^\epsilon \leq T]} + g(X^{t, x, u^\epsilon, v_0}_T) \right] + \epsilon.
\]

Since the cost \(c(\tau_m^\epsilon, \xi_m^\epsilon)\) are non negative functions, then we have

\[
V^-(t, x) \leq \mathbb{E}\left[ \int_t^T f(s, X^{t, x, u^\epsilon, v_0}_s) + g(X^{t, x, u^\epsilon, v_0}_T) \right] + \epsilon.
\]
Therefore we find, using also the boundedness of \( f \) and \( g \), that there exists a constant \( C > 0 \) such that

\[
V^-(t, x) \leq C.
\]

In a similar way we can prove that there exists a constant \( C > 0 \) such that

\[
V^-(t, x) \geq -C,
\]

from which we deduce the thesis.

\( \square \)

We are now giving some properties of nearly optimal strategies.

**Proposition 1** Let \( u \in \mathcal{U}_{t,T} \) and \( v \in \mathcal{V}_{t,T} \) be a nearly optimal strategies composed of impulses control \( (\tau, \xi) = ((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 1}) \) and \( (\rho, \eta) = ((\rho_n)_{n \geq 1}, (\eta_n)_{n \geq 1}) \). Then:

\[
\mathbb{E}\left[ \sum_{m \geq 1} c(\tau_m, \xi_m) 1_{[\tau_m \leq T]} \right] + \mathbb{E}\left[ \sum_{l \geq 1} \chi(\rho_l, \eta_l) 1_{[\rho_l \leq T]} \right] \leq C. \tag{3.9}
\]

There exists a positive constant \( C > 0 \) which does not depend on \( t \) and \( x \) such that:

\[
\forall n \geq 1, \quad \mathbb{P}[\tau_n \leq T] + \mathbb{P}[\rho_n \leq T] \leq \frac{C}{n}. \tag{3.10}
\]

We denote by \( \hat{\mathcal{U}}_{t,T} \) and \( \hat{\mathcal{V}}_{t,T} \) the set which satisfies the conditions (3.9) and (3.10), respectively. Similarly, by \( \hat{\mathcal{A}}_{t,T} \) and \( \hat{\mathcal{B}}_{t,T} \) the sets that contain all the nonanticipative strategies with values in \( \hat{\mathcal{U}}_{t,T} \) and \( \hat{\mathcal{V}}_{t,T} \), respectively.

**Proof** Let us choose a nearly optimal strategy \( u \in \mathcal{U}_{t,T} \) composed of impulse control \( ((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 1}) \) such that, for \((t, x) \in [0, T] \times \mathbb{R}^n \),

\[
\mathbb{E}\left[ \int_t^T f(s, X_s^{t,x,u,v_0})ds - \sum_{m \geq 1} c(\tau_m, \xi_m) 1_{[\tau_m \leq T]} + g(X_T^{t,x,u,v_0}) \right] \geq V^-(t, x) - 1.
\]

Since \( V^- \), \( f \) and \( g \) are bounded, then we have

\[
\mathbb{E}\left[ \sum_{m \geq 1} c(\tau_m, \xi_m) 1_{[\tau_m \leq T]} \right] \leq C.
\]

Next we show (3.10). Taking into account that \( c(\tau, \xi) \geq k > 0 \) for any \((\tau, \xi) \in [0, T] \times \mathcal{U}_{t,T} \), we obtain:

\[
\mathbb{E}\left[ \sum_{m \geq 1} k 1_{[\tau_m \leq T]} \right] \leq C.
\]
But for any \( m \leq n \), \([\tau_n \leq T] \subseteq [\tau_m \leq T]\), then:

\[
\mathbb{E}[nk\mathbb{1}_{[\tau_n \leq T]}] \leq C.
\]

Finally taking into account \( k > 0 \), we obtain the desired result.

The other case being analogous. \( \square \)

**Corollary 1** Under the standing assumptions \((H1), (H2)\) and \((H3)\), \(J(t, x, u, v)\) is bounded for every \((t, x) \in [0, T] \times \mathbb{R}^n, u \in \tilde{U}_{t,T} \) and \(v \in \tilde{V}_{t,T}\).

In the following proposition, using Lemma 1, we prove that the lower and upper value functions are continuous in the state variable, together with the gain functional.

**Proposition 2** The gain functional, lower and upper value functions are continuous in \( x \).

**Proof** It is enough to show that the conclusion holds for the gain functional \( J \).

For every \( t \in [0, T] \), \( x, x' \in \mathbb{R}^n, u \in \tilde{U}_{t,T}, v \in \tilde{V}_{t,T} \), we have

\[
|J(t, x, u, v) - J(t, x', u, v)| \leq \mathbb{E}\left[ \int_t^T \left| f(s, X^t,x,u,v) - f(s, X^t,x',u,v) \right| ds \\
+ |g(X^t,x,u,v) - g(X^t,x',u,v)| \right].
\]

From Lemma 1 and continuity of \( f \) and \( g \) in \( x \) we get the thesis. \( \square \)

### 4 The Value Functions

#### 4.1 A Weak Dynamic Programming Principle

In this section we present the weak dynamic programming principle for the stochastic differential game. We begin with the following lemma.

**Lemma 2** ([14, Lemma 3.3]). The lower and upper value functions are given by

\[
V^{-}(t, x) := \inf_{\beta \in \tilde{B}_{t,T}} \sup_{u \in \tilde{U}_{t,T}} J(t, x, u, \beta(u)) \tag{4.1}
\]

and

\[
V^{+}(t, x) := \sup_{\alpha \in \tilde{A}_{t,T}} \inf_{v \in \tilde{V}_{t,T}} J(t, x, \alpha(v), v) \tag{4.2}
\]

for every \((t, x) \in [0, T] \times \mathbb{R}^n\), where \(\tilde{U}_{t,T}\) and \(\tilde{V}_{t,T}\) contain all the impulse controls in \(\tilde{U}_{t,T}\) and \(\tilde{V}_{t,T}\), respectively, which have no impulses at time \( t \). Similarly, \(\tilde{A}_{t,T}\) and \(\tilde{B}_{t,T}\) are subsets of \(\tilde{A}_{t,T}\) and \(\tilde{B}_{t,T}\), respectively. In particular, they contain all the nonanticipative strategies with values in \(\tilde{U}_{t,T}\) and \(\tilde{V}_{t,T}\), respectively.
Theorem 2 Under assumptions (H1), (H2) and (H3), given \(0 \leq t \leq s \leq T, x \in \mathbb{R}^n\), and each family of \(\mathbb{Q}_t\)-valued \(\mathcal{F}_{t,s}\)-stopping times \(\{\tau^{u,v}, (u, v) \in \hat{U}_t \times \hat{V}_t\}\), we have

\[
V^-(t, x) \leq \inf_{\beta \in \hat{U}_t, \tau \in \hat{V}_t} \sup_{u \in \mathbb{R}^n} \mathbb{E} \left[ \int_t^{\tau_{u, \beta}} f(r, X_r^{t,x,u,\beta(u)})dr \right. \\
- \sum_{m \geq 1} c(\tau_m, \xi_m) \mathbb{I}_{[\tau_m \leq \tau_{u, \beta}]} \prod_{l \geq 1} \mathbb{I}_{[\tau_l \neq \rho_l]} \\
+ \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{[\rho_l \leq \tau_{u, \beta}]} + (V^-)^*(\tau^{u,v}, X_{\tau_{u, \beta}})
\]

(4.3)

\[
V^+(t, x) \geq \inf_{\beta \in \hat{U}_t, \tau \in \hat{V}_t} \sup_{u \in \mathbb{R}^n} \mathbb{E} \left[ \int_t^{\tau_{u, \beta}} f(r, X_r^{t,x,u,\beta(u)})dr \right. \\
- \sum_{m \geq 1} c(\tau_m, \xi_m) \mathbb{I}_{[\tau_m \leq \tau_{u, \beta}]} \prod_{l \geq 1} \mathbb{I}_{[\tau_l \neq \rho_l]} \\
+ \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{[\rho_l \leq \tau_{u, \beta}]} + (V^+)^*(\tau^{u,v}, X_{\tau_{u, \beta}})
\]

(4.4)

\[
V^+(t, x) \leq \sup_{\alpha \in \hat{A}_t} \inf_{v \in \hat{V}_t} \mathbb{E} \left[ \int_t^{\tau^{\alpha,v}} f(r, X_r^{t,x,\alpha(v),v})dr \right. \\
- \sum_{m \geq 1} c(\tau_m, \xi_m) \mathbb{I}_{[\tau_m \leq \tau^{\alpha,v}]} \prod_{l \geq 1} \mathbb{I}_{[\tau_l \neq \rho_l]} \\
+ \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{[\rho_l \leq \tau^{\alpha,v}]} + (V^+)^*(\tau^{\alpha,v}, X_{\tau^{\alpha,v}})
\]

(4.5)

\[
V^+(t, x) \geq \sup_{\alpha \in \hat{A}_t} \inf_{v \in \hat{V}_t} \mathbb{E} \left[ \int_t^{\tau^{\alpha,v}} f(r, X_r^{t,x,\alpha(v),v})dr \right. \\
- \sum_{m \geq 1} c(\tau_m, \xi_m) \mathbb{I}_{[\tau_m \leq \tau^{\alpha,v}]} \prod_{l \geq 1} \mathbb{I}_{[\tau_l \neq \rho_l]} \\
+ \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{[\rho_l \leq \tau^{\alpha,v}]} + (V^+)^*(\tau^{\alpha,v}, X_{\tau^{\alpha,v}})
\]

(4.6)

where \(V_*\) (resp., \(V^*\)) its lower (resp. upper) semicontinuous envelope defined by:

\[
V_*(t, x) := \liminf_{(s, y) \to (t, x)} V(s, y) \quad \text{and} \quad V^*(t, x) := \limsup_{(s, y) \to (t, x)} V(s, y).
\]
Proof We prove the weak dynamic programming principle only for $V^-$, the other case being analogous.

Step 1. Let $\epsilon > 0$ and consider an arbitrary function

$$\phi : [0, T] \times \mathbb{R}^n \to \mathbb{R} \quad \text{such that} \quad \phi \text{ continuous, bounded from above and } V^- \leq \phi.$$

For each $(s, y) \in [0, T] \times \mathbb{R}^n$, there exists $\beta^e \in \tilde{B}_{s, T}$ such that

$$\phi(s, y) \geq V^-(s, y) \geq J(s, y, u_{[s, T]}, \beta^e(u_{[s, T]})) - \epsilon, \quad (4.7)$$

where $u_{[s, T]}$ is as introduced in Definition 5. Let $Q_{[t, T]} := \{t_i\}_{i \geq 1}$ and fix one of the points $t_i$ in time. For each $y \in \mathbb{R}^n$, the continuity of $J$ established in Proposition 2 and that of $\phi$ imply that there exists $r^{(e, t, y)}$ such that

$$\left\{ \begin{array}{l}
\phi(t_i, y') \geq \phi(t_i, y) - \epsilon, \\
J(t_i, y', u_{[t_i, T]}, v) \leq J(t_i, y, u_{[t_i, T]}, v) + \epsilon,
\end{array} \right. \quad (4.8)$$

for all $y' \in B(y, r^{(e, t, y)})$ and $v \in \tilde{V}_{t_i, T}$, where $B(y, r) = \{y' \in \mathbb{R}^n : |y' - y| < r\}$.

Therefore, the family $\{B(y, r^{(e, t, y)}) : y \in \mathbb{R}^n, r^{(e, t, y)} > 0\}$ forms an open covering of $\mathbb{R}^n$. By the Lindelöf covering Theorem ([16, Theorem 6.3, Chapter VIII]), there exists a countable family $\{y_j\}_{j \geq 1}$ in $\mathbb{R}^n$ such that $\{B(y_j, r^{(e, t, y_j)})\}_{j \geq 1}$ is a countable subcover of $\mathbb{R}^n$. We set $\beta_{i, j} := \beta^{(e, t, y_j)} \in \tilde{B}_{t_i, T}$ and $B_{j}^{i} := B(y_j, r^{(e, t, y_j)})$.

We can now define, for $i$ still being fixed, a measurable partition $(A_j^i)_{j \geq 1}$ by

$$A_1^i := B_1^i, \quad A_{j+1}^i := B_{j+1}^i \setminus (B_j^i \cup \ldots \cup B_1^i), \quad j \geq 1.$$

Since $A_j^i \subseteq B_j^i$, the inequalities (4.7) and (4.8) yield that

$$\phi(t_i, y') \geq J(t_i, y', u_{[t_i, T]}, \beta_{i, j}(u_{[t_i, T]})) - 3\epsilon \quad \text{for all } y' \in A_j^i. \quad (4.9)$$

Now, let $(u, \beta) \in \tilde{U}_{t_i, T} \times \tilde{B}_{t_i, T}$ be arbitrary and set $\tau = \tau^{\beta(u), u}$. Fix an integer $k \geq 1$, we now focus on $(t_i)_{1 \leq i \leq k}$. We may assume that $t_1 < t_2 < \ldots < t_k$, by eliminating and relabeling some of the $t_i$. We define the $\mathcal{F}_t^i$-measurable sets

$$\Gamma_j^i := \{\tau = t_i \text{ and } \chi_{t_i}^{x, u, \beta(u)} \in A_j^i\} \in \mathcal{F}_t$$

and $\Gamma(k) := \bigcup_{1 \leq i, j \leq k} \Gamma_j^i$.

Since the $t_i$ are distinct and $A_j^i \cap A_{j'}^i = \emptyset$ for $j \neq j'$, we have $\Gamma_j^i \cap \Gamma_j^{i'} = \emptyset$ for $(i, j) \neq (i', j')$. We construct the strategy $\beta^k$ by

$$\beta^k(u) := \beta(u) \mathbb{I}_{[t, \tau]} + \mathbb{I}_{(\tau, T]} \left( \beta(u) \mathbb{I}_{(\Gamma(k))} + \sum_{1 \leq i, j \leq k} \beta_{i, j}(u_{[t_i, T]}) \mathbb{I}_{\Gamma_j^i} \right).$$
Then

\[
J(t, x, u, \beta^k(u)) = \mathbb{E} \left[ \int_t^\tau f(r, X_r^{t,x,u,\beta(u)})dr - \sum_{m \geq 1} c(\tau_m, \xi_m) \prod_{l \geq 1} \mathbb{I}_{\tau_m \neq \rho_l} + \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{\rho_l \leq \tau} \right] + \phi(\tau, X_\tau^{t,x,u,\beta(u)}).
\]

We deduce via (4.9) that

\[
\mathbb{E} \left[ \sum_{1 \leq i, j \leq k} J(\tau, X^{t,x,u,\beta^{(u)}}(u), \beta_{i,j}(u[\tau,T])) \mathbb{I}_{\Gamma_i \cap_j} \right] \leq \mathbb{E} \left[ \phi(\tau, X^{t,x,u,\beta^{(u)}}(u)) \mathbb{I}_{\Gamma(k)} \right] + 3\epsilon,
\]

(4.11) for every \( k \geq 1 \). Letting \( k \to \infty \), therefore,

\[
\mathbb{E} \left[ J(t, x, u, \beta(u)) \mathbb{I}_{(\Gamma(k))c} \right] \to 0
\]

by dominated convergence and Corollary 1. Moreover, monotone convergence yields

\[
\mathbb{E} \left[ \phi(\tau, X^{t,x,u,\beta^{(u)}}(u)) \mathbb{I}_{\Gamma(k)} \right] \to \mathbb{E} \left[ \phi(\tau, X^{t,x,u,\beta^{(u)}}(u)) \right].
\]

Therefore, we deduce the existence of an integer \( k_0 \geq 1 \) such that

\[
J(t, x, u, \beta^{k_0}(u)) \leq \mathbb{E} \left[ \int_t^\tau f(r, X_r^{t,x,u,\beta^{(u)}})dr - \sum_{m \geq 1} c(\tau_m, \xi_m) \prod_{l \geq 1} \mathbb{I}_{\tau_m \neq \rho_l} + \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{\rho_l \leq \tau} \right] + \phi(\tau, X_\tau^{t,x,u,\beta(u)}) + 4\epsilon.
\]

(4.12)

The arbitrariness of \( \beta \) and \( \epsilon \) implies that

\[
V^{-}(t, x) \leq \inf_{\beta \in \mathcal{B}_{t,T}} \sup_{u \in \mathcal{U}_{t,T}} \mathbb{E} \left[ \int_t^\tau f(r, X_r^{t,x,u,\beta(u)})dr - \sum_{m \geq 1} c(\tau_m, \xi_m) \prod_{l \geq 1} \mathbb{I}_{\tau_m \neq \rho_l} + \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{\rho_l \leq \tau} \right] + \phi(\tau, X_\tau^{t,x,u,\beta(u)}).
\]

(4.13)

**Step 2.**
Note that inequality (4.13) is valid for any bounded continuous function \( \phi \) such that \( \phi \geq V^- \). But \( (V^-)^* \) is upper semi-continuous and bounded then there exists a decreasing sequence \( (\phi_n)_{n \geq 0} \) of continuous functions such that for any \( n \geq 0 \), \( \phi_n \geq (V^-)^* \) and \( \lim_{n \to \infty} \phi_n = (V^-)^* \) (see e.g. [5],Theorem 1.3.7). As \( (V^-)^* \) is bounded then the functions \( \phi_n \) can be chosen uniformly bounded. Thus we have

\[
\phi_n \geq \phi_{n+1} \geq (V^-)^* \geq V^- , \lim_{n \to \infty} \phi_n = (V^-)^* \text{ and } |\phi_n| \leq C
\]

for some constant \( C \) independent of \( n \). Applying now inequality (4.13) with \( \phi_n \) yields:

\[
\forall n \geq 0, V^-(t,x) \leq \inf_{\beta \in \hat{B}_{t,T}} \sup_{u \in \hat{U}_{t,T}} \mathbb{E} \left[ \int_t^T f(r, X_r^{t,x,u}, \beta(u)) dr - \sum_{m \geq 1} c(\tau_m, \xi_m) \mathbb{I}_{[\tau_m \leq \tau]} \prod_{l \geq 1} \mathbb{I}_{[\tau_m \neq \rho_l]} + \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{[\rho_l \leq \tau]} + (V^-)^*(\tau, X_\tau^{t,x,u}, \beta(u)) \right].
\]

Next by the Lebesgue dominated convergence theorem one has:

\[
\mathbb{E}\left[ \phi_n(\tau, X_\tau^{t,x,u}, \beta(u)) \right] \longrightarrow_{n \to +\infty} \mathbb{E}\left[ (V^-)^*(\tau, X_\tau^{t,x,u}, \beta(u)) \right].
\]

Going back now to (4.14) and take the limit in \( n \) to obtain:

\[
V^-(t,x) \leq \inf_{\beta \in \hat{B}_{t,T}} \sup_{u \in \hat{U}_{t,T}} \mathbb{E} \left[ \int_t^T f(r, X_r^{t,x,u}, \beta(u)) dr - \sum_{m \geq 1} c(\tau_m, \xi_m) \mathbb{I}_{[\tau_m \leq \tau]} \prod_{l \geq 1} \mathbb{I}_{[\tau_m \neq \rho_l]} + \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{[\rho_l \leq \tau]} + (V^-)^*(\tau, X_\tau^{t,x,u}, \beta(u)) \right].
\]

In a similar way we can prove the reverse inequality, hence deducing the thesis.

4.2 Continuity of Value Functions in Time

In this section we prove the continuity of lower value function and upper value function in \( t \).

**Theorem 3** The lower and upper value functions are continuous in \( t \).
Define a strategy $\beta$. From this, we make the proof only for $V^-$, the other case being analogous. First let us show that $V^-$ is lower semi-continuous. Recall the characterization of weak dynamical programming principle that reads as

$$V^-(t', x) \leq \inf_{\beta \in B_{t', T}} \sup_{u \in U_{t', T}} \mathbb{E} \left[ \int_{t'}^{\rho_n \land T} f(r, X^{t', x, u, \beta(u)}_r) dr \right.$$ 

$$- \sum_{m \geq 1} c(t_m, \xi_m) \mathbb{I}_{[\tau_m \leq \rho_n \land T]} \prod_{l \geq 1} \mathbb{I}_{[\tau_m \neq \rho_l]}$$ 

$$+ \sum_{1 \leq l \leq n} X(\rho_l, \eta_l) \mathbb{I}_{[\rho_l \leq T]} + \mathbb{I}_{[\rho_n \leq T]}(V^-)(\rho_n, X^{t', x, u, \beta(u)}_{\rho_n})$$ 

$$+ \mathbb{I}_{[\rho_n = +\infty]} g(X^{t', x, u, \beta(u)}_T) \right].$$

Fix an arbitrary $\epsilon > 0$, and assume that $t < t'$. There exists $\beta^\epsilon \in B_{t, T}$ such that

$$V^-(t, x) + \epsilon$$ 

$$\geq \sup_{u \in U_{t, T}} \mathbb{E} \left[ \int_{0}^{\rho_n^\epsilon \land T} f(r, X^{t, x, u, \beta^\epsilon(u)}_r) \mathbb{I}_{[r \geq t]} dr \right.$$ 

$$- \sum_{m \geq 1} c(t \lor \tau_m, \xi^\epsilon_m) \mathbb{I}_{[\tau_m \leq \rho_n^\epsilon \land T]} \prod_{l \geq 1} \mathbb{I}_{[\tau_m \neq \rho_l^\epsilon]}$$ 

$$+ \sum_{1 \leq l \leq n} X(t \lor \rho_l^\epsilon, \eta_l^\epsilon) \mathbb{I}_{[\rho_l^\epsilon \leq T]} + \mathbb{I}_{[\rho_n^\epsilon \leq T]}(V^-)(\rho_n^\epsilon, X^{t, x, u, \beta^\epsilon(u)}_{\rho_n^\epsilon})$$ 

$$+ \mathbb{I}_{[\rho_n^\epsilon = +\infty]} g(X^{t, x, u, \beta^\epsilon(u)}_T) \right],$$

Define a strategy $\beta^*$ from $\beta^\epsilon$ by setting for each $u \in U_{t', T}$

$$\beta^*(u) := \sum_{1 \leq l \leq n} \eta_l^\epsilon \mathbb{I}_{[t' \lor \rho_l^\epsilon, T]}$$

\( \square \) Springer
Note that $\beta^* \in \hat{B}_{t', T}$. Now, pick $u^\epsilon \in \hat{U}_{t', T}$ such that

$$V^-(t', x) - \epsilon \leq \mathbb{E} \left[ \int_0^\rho_n^\epsilon \right] f(r, X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)}) \mathbb{I}_{[r \geq t']}dr \leq \sum_{m \geq 1} c(t' \lor \tau_m^\epsilon, \xi_m^\epsilon) \mathbb{I}_{[\tau_m^\epsilon \leq \rho_n^\epsilon \land T]} \prod_{l \geq 1} \mathbb{I}_{[\tau_m^\epsilon \neq \rho_l^\epsilon]} + \sum_{1 \leq l \leq n} \chi(t' \lor \rho_l^\epsilon, \eta_l^\epsilon) \mathbb{I}_{[\rho_l^\epsilon \leq T]} + \mathbb{I}_{[\rho_n^\epsilon = +\infty]} g(X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)}),$$

According to (4.15) we have

$$V^-(t, x) + \epsilon \geq \mathbb{E} \left[ \int_0^\rho_n^T \right] f(r, X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)}) \mathbb{I}_{[r \geq t']}dr \geq \sum_{m \geq 1} c(t \lor \tau_m^\epsilon, \xi_m^\epsilon) \mathbb{I}_{[\tau_m^\epsilon \leq \rho_n^\epsilon \land T]} \prod_{l \geq 1} \mathbb{I}_{[\tau_m^\epsilon \neq \rho_l^\epsilon]} + \sum_{1 \leq l \leq n} \chi(t \lor \rho_l^\epsilon, \eta_l^\epsilon) \mathbb{I}_{[\rho_l^\epsilon \leq T]} + \mathbb{I}_{[\rho_n^\epsilon = +\infty]} g(X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)}),$$

Combining the last two inequalities, we obtain

$$V^-(t, x) - V^-(t', x) \geq \mathbb{E} \left[ \int_0^{\rho_n^\epsilon \land T} f(r, X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)}) \mathbb{I}_{[r \geq t']}dr + \sum_{1 \leq l \leq n} \chi(t \lor \rho_l^\epsilon, \eta_l^\epsilon) \mathbb{I}_{[\rho_l^\epsilon \leq T]} + \mathbb{I}_{[\rho_n^\epsilon \leq T]}(V^-)^*(\rho_n^\epsilon, X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)}) + \mathbb{I}_{[\rho_n^\epsilon = +\infty]} g(X_T^{t', x, u^\epsilon, \beta^*(u^\epsilon)}) \right] - \mathbb{E} \left[ \int_0^{\rho_n^\epsilon \land T} f(r, X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)}) \mathbb{I}_{[r \geq t']}dr + \sum_{1 \leq l \leq n} \chi(t \lor \rho_l^\epsilon, \eta_l^\epsilon) \mathbb{I}_{[\rho_l^\epsilon \leq T]} + \mathbb{I}_{[\rho_n^\epsilon \leq T]}(V^-)^*(\rho_n^\epsilon, X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)}) + \mathbb{I}_{[\rho_n^\epsilon = +\infty]} g(X_T^{t', x, u^\epsilon, \beta^*(u^\epsilon)}) \right] - 2\epsilon \geq \mathbb{E} \left[ - \int_0^{\rho_n^\epsilon \land T} \mathbb{I}_{[f(r, X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)}) - f(r, X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)})] \mathbb{I}_{[r \geq t']}dr - \int_0^{\rho_n^\epsilon \land T} |f(r, X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)})| \mathbb{I}_{[r \leq t']}dr + \sum_{1 \leq l \leq n} \chi(t \lor \rho_l^\epsilon, \eta_l^\epsilon) \mathbb{I}_{[\rho_l^\epsilon \leq T]} \right] - \mathbb{E} \left[ - \int_0^{\rho_n^\epsilon \land T} \mathbb{I}_{[f(r, X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)})] \mathbb{I}_{[r \leq t']}dr + \sum_{1 \leq l \leq n} \chi(t \lor \rho_l^\epsilon, \eta_l^\epsilon) \mathbb{I}_{[\rho_l^\epsilon \leq T]} \right] - \sum_{1 \leq l \leq n} \chi(t \lor \rho_l^\epsilon, \eta_l^\epsilon) \mathbb{I}_{[\rho_l^\epsilon \leq T]} - \mathbb{I}_{[\rho_n^\epsilon \leq T]} \mathbb{I}_{[(V^-)^*(\rho_n^\epsilon, X_t^{t', x, u^\epsilon, \beta^*(u^\epsilon)})]}.$$
We note that, there exists a constant $C > 0$ such that

$$-\mathbb{E} \left[ \mathbb{1}_{[\rho^\varepsilon_n \leq T]} \left( |(V^-)^\varepsilon_n (\rho^\varepsilon_n, X_{\rho^\varepsilon_n}^{t_1, x, u^\varepsilon, \beta^\varepsilon (u^\varepsilon)})| + |(V^-)^\varepsilon_n (\rho^\varepsilon_n, X_{\rho^\varepsilon_n}^{t_1, x, u^\varepsilon, \beta^\varepsilon (u^\varepsilon)})| \right) \right] \geq - \frac{C}{n}.$$

Also, taking the limit as $t \to t'$, and using the uniform continuity of $f$, $g$ and $\chi$ to obtain:

$$\lim \inf_{t \to t'} V^- (t, x) \geq V^- (t', x) - \frac{C}{n} - 2 \varepsilon.$$

As $n$ and $\varepsilon$ are arbitrary then sending $\varepsilon \to 0$ and $n \to +\infty$, to obtain:

$$\lim \inf_{t \to t'} V^- (t, x) \geq V^- (t', x).$$

Therefore $V^-$ is lower semi-continuous.

Now we show that $V^-$ is upper semi-continuous. Fix an arbitrary $\varepsilon > 0$, and we assume that $t < t'$.

Pick $\beta^\varepsilon$ belongs to $\tilde{B}_{t', T}$ such that

$$V^- (t', x) + \varepsilon \geq \sup_{u \in \tilde{U}_{t', T}} \int_0^T f (r, X^{t', x, u^\varepsilon, \beta^\varepsilon (u)}_r) \mathbb{1}_{[r \geq t']} \, dr$$

$$\geq \sum_{1 \leq m \leq n} c (t' \vee \tau_m, \xi_m) \mathbb{1}_{[\tau_m \leq T]} \prod_{l \geq 1} \mathbb{1}_{[\tau_m \neq \rho^\varepsilon_l]}$$

$$+ \sum_{l \geq 1} \chi (t' \vee \rho^\varepsilon_l, \eta^\varepsilon_l) \mathbb{1}_{[\rho^\varepsilon_l \leq \tau_n \wedge T]} + \mathbb{1}_{[\tau_n \leq T]} (V^-)^\varepsilon_n (\tau_n, X_{\tau_n}^{t_1, x, u^\varepsilon, \beta^\varepsilon (u^\varepsilon)})$$

$$+ \mathbb{1}_{[\tau_n = +\infty]} g (X_{\tau_n}^{t_1, x, u^\varepsilon, \beta^\varepsilon (u^\varepsilon)}), \quad (4.16)$$

and we have
\[ V^-(t, x) \leq \sup_{u \in \tilde{U}_{t, T}} \mathbb{E} \left[ \int_0^{\tau_n \wedge T} f(r, X_r^{t, x, u, \beta^e(u)}) \mathbb{I}_{[r \geq t]} dr \right. \\
- \sum_{1 \leq m \leq n} c(t \vee \tau_m, \xi_m^e) \mathbb{I}_{[\tau_m \leq T]} \prod_{l \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_l^e\}} \\
+ \sum_{l \geq 1} \chi(t \vee \rho_l^e, \eta_l^e) \mathbb{I}_{\{\rho_l^e \leq \tau_n \wedge T\}} + \mathbb{I}_{[\tau_n \leq T]} (V^-)^*(\tau_n, X_{\tau_n}^{t, x, u, \beta^e(u)}) \\
+ \mathbb{I}_{[\tau_n = +\infty]} g(X_T^{t, x, u, \beta^e(u)}) \].

then there exists \( u^e = \sum_{1 \leq m \leq n} \xi_m^e \mathbb{I}_{[\tau_m, T]} \in \tilde{U}_{t, T} \) such that

\[ V^-(t, x) - \epsilon \leq \mathbb{E} \left[ \int_0^{\tau_n^e \wedge T} f(r, X_r^{t, x, u^e, \beta^e(u^e)}) \mathbb{I}_{[r \geq t]} dr \right. \\
- \sum_{1 \leq m \leq n} c(t \vee \tau_m^e, \xi_m^e) \mathbb{I}_{[\tau_m^e \leq T]} \prod_{l \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_l^e\}} \\
+ \sum_{l \geq 1} \chi(t \vee \rho_l^e, \eta_l^e) \mathbb{I}_{[\rho_l^e \leq \tau_n^e \wedge T]} + \mathbb{I}_{[\tau_n^e \leq T]} (V^-)^*(\tau_n^e, X_{\tau_n^e}^{t, x, u^e, \beta^e(u^e)}) \\
+ \mathbb{I}_{[\tau_n^e = +\infty]} g(X_T^{t, x, u^e, \beta^e(u^e)}) \].

Now, we define \( u^* \) by

\[ u^* = \sum_{1 \leq m \leq n} \xi_m^e \mathbb{I}_{[t^\vee \tau_m^e, T]} \].

Note that \( u^* \in \tilde{U}_{t, T} \). According to (4.16) we have

\[ V^-(t', x) + \epsilon \geq \mathbb{E} \left[ \int_0^{\tau_n^e \wedge T} f(r, X_r^{t, x, u^*, \beta^e(u^*)}) \mathbb{I}_{[r \geq t']} dr \right. \\
- \sum_{1 \leq m \leq n} c(t' \vee \tau_m^e, \xi_m^e) \mathbb{I}_{[\tau_m^e \leq T]} \prod_{l \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_l^e\}} \\
+ \sum_{l \geq 1} \chi(t' \vee \rho_l^e, \eta_l^e) \mathbb{I}_{[\rho_l^e \leq \tau_n^e \wedge T]} + \mathbb{I}_{[\tau_n^e \leq T]} (V^-)^*(\tau_n^e, X_{\tau_n^e}^{t', x, u^*, \beta^e(u^*)}) \\
+ \mathbb{I}_{[\tau_n^e = +\infty]} g(X_T^{t', x, u^*, \beta^e(u^*)}) \].

Therefore

\[ V^-(t, x) - V^-(t', x) \leq \mathbb{E} \left[ \int_0^{\tau_n^e \wedge T} f(r, X_r^{t, x, u^*, \beta^e(u^*)}) \mathbb{I}_{[r \geq t]} dr - \sum_{1 \leq m \leq n} c(t \vee \tau_m^e, \xi_m^e) \mathbb{I}_{[\tau_m^e \leq T]} \prod_{l \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_l^e\}} \right. \\
\]
\[
\begin{align*}
&+ \mathbb{I}_{[r^n \leq T]}(V^-\,^*(\tau_r^\epsilon, X_t^{r,x,u^*,\beta^\epsilon(u^*)}) + \mathbb{I}_{[r^n = +\infty]} g(X_T^{r,x,u^*,\beta^\epsilon(u^*)})

&- \mathbb{E} \left[ \int_0^{r^n \wedge T} f(r, X_r^{r,x,u^*,\beta^\epsilon(u^*)}) \mathbb{I}_{[r \geq r'] \, dr} - \sum_{1 \leq m \leq n} c(t' \vee \tau_m^\epsilon, \xi_m^\epsilon) \mathbb{I}_{[\tau_m^\epsilon \leq T]} \prod_{l \geq 1} \mathbb{I}_{[r_l^\epsilon \neq r'_l^\epsilon]} \right]

&+ \mathbb{I}_{[r^n \leq T]}(V^-\,^*(\tau_r^\epsilon, X_t^{r,x,u^*,\beta^\epsilon(u^*)}) + \mathbb{I}_{[r^n = +\infty]} g(X_T^{r,x,u^*,\beta^\epsilon(u^*)}) \right] + 2\epsilon

&\leq \mathbb{E} \left[ \int_0^{r^n \wedge T} \{ |f(r, X_r^{r,x,u^*,\beta^\epsilon(u^*)}) - f(r, X_r^{r,x,u^*,\beta^\epsilon(u^*)})| \mathbb{I}_{[r \geq r']} \, dr 

&+ \int_0^{r^n \wedge T} |f(r, X_r^{r,x,u^*,\beta^\epsilon(u^*)})| \mathbb{I}_{[r \leq r']} \, dr \right]

&+ \sum_{1 \leq m \leq n} c(t' \vee \tau_m^\epsilon, \xi_m^\epsilon) \mathbb{I}_{[\tau_m^\epsilon \leq T]} \prod_{l \geq 1} \mathbb{I}_{[r_l^\epsilon \neq r'_l^\epsilon]}

&+ \mathbb{I}_{[r^\epsilon = +\infty]} g(X_T^{r,x,u^*,\beta^\epsilon(u^*)}) - g(X_T^{r',x,u^*,\beta^\epsilon(u^*)})

&\leq \mathbb{E} \left[ \int_0^{r^n \wedge T} \{ |f(r, X_r^{r,x,u^*,\beta^\epsilon(u^*)}) - f(r, X_r^{r,x,u^*,\beta^\epsilon(u^*)})| \mathbb{I}_{[r \geq r']} \, dr 

&+ \int_0^{r^n \wedge T} |f(r, X_r^{r,x,u^*,\beta^\epsilon(u^*)})| \mathbb{I}_{[r \leq r']} \, dr \right]

&+ n \max_{1 \leq m \leq n} \{ \sup_{s \leq T} |c(t' \vee s, \xi_m^\epsilon) - c(t \vee s, \xi_m^\epsilon)| \}

&+ \mathbb{I}_{[r^n = +\infty]} g(X_T^{r,x,u^*,\beta^\epsilon(u^*)}) - g(X_T^{r',x,u^*,\beta^\epsilon(u^*)})

&\leq \mathbb{E} \left[ \int_0^{r^n \wedge T} \{ |(V^-)^*(\tau_r^\epsilon, X_t^{r,x,u^*,\beta^\epsilon(u^*)})| + |(V^-)^*(\tau_r^\epsilon, X_t^{r,x,u^*,\beta^\epsilon(u^*)})| \right] + 2\epsilon. \tag{4.17}
\end{align*}
\]

We note that, there exists a constant \( C > 0 \) such that

\[
\mathbb{E} \left[ \mathbb{I}_{[r^n \leq T]} \{ |(V^-)^*(\tau_r^\epsilon, X_t^{r,x,u^*,\beta^\epsilon(u^*)})| + |(V^-)^*(\tau_r^\epsilon, X_t^{r,x,u^*,\beta^\epsilon(u^*)})| \right] \leq \frac{C}{n}.
\]

Also, taking the limit as \( t \to t' \), and using the uniform continuity of \( f, g \) and \( c \) to obtain:

\[
\limsup_{t \to t'} V^-(t, x) \leq V^-(t', x) + \frac{C}{n} + 2\epsilon.
\]

As \( n \) and \( \epsilon \) are arbitrary then sending \( \epsilon \to 0 \) and \( n \to +\infty \), to obtain:

\[
\limsup_{t \to t'} V^-(t, x) \leq V^-(t', x).
\]

Therefore \( V^- \) is upper semi-continuous. We then proved that \( V^- \) is continuous in \( t \).

\( \square \)
Corollary 2 (A strong dynamic programming principle) The lower (resp., upper) value functions are continuous on \([0, T] \times \mathbb{R}^n\), therefore we have \(V^- = (V^-)_* = (V^-)^*\) (resp. \(V^+ = (V^+)_* = (V^+)^*\)). As a consequence, \(V^-\) (resp., \(V^+\)) satisfies the classical dynamic programming principle: given \(0 \leq t \leq s \leq T, x \in \mathbb{R}^n\), and each family of \(\bar{Q}_{t,s}\)-valued \(\mathcal{F}_{t,s}\)-stopping times \(\{\tau^{u,v}, (u, v) \in \mathcal{U}_{t,s} \times \mathcal{V}_{t,s}\}\), we have

\[
V^-(t, x) = \inf_{\beta \in \mathcal{B}_{t,T}} \sup_{u \in \mathcal{U}_{t,T}} \mathbb{E} \left[ \int_t^{\tau^{u,\beta}} f(r, X_r^{t,x,u,\beta(u)})dr 
- \sum_{m \geq 1} c(\tau_m, \xi_m) \mathbb{I}_{[\tau_m \leq \tau^{u,\beta}]} \prod_{l \geq 1} \mathbb{I}_{[\tau_m \neq \rho_l]}
+ \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{[\tau^{u,\beta}]} + V^- (\tau^{u,\beta}, X_{t}^{\tau^{u,\beta},\beta(u)}) \right],
\]

and

\[
V^+(t, x) = \sup_{\alpha \in \mathcal{A}_{t,T}} \inf_{v \in \mathcal{V}_{t,T}} \mathbb{E} \left[ \int_t^{\tau^{\alpha,v}} f(r, X_r^{t,x,\alpha(v),v})dr 
- \sum_{m \geq 1} c(\tau_m, \xi_m) \mathbb{I}_{[\tau_m \leq \tau^{\alpha,v}]} \prod_{l \geq 1} \mathbb{I}_{[\tau_m \neq \rho_l]}
+ \sum_{l \geq 1} \chi(\rho_l, \eta_l) \mathbb{I}_{[\tau^{\alpha,v}]} + V^+ (\tau^{\alpha,v}, X_{t}^{\tau^{\alpha,v},\alpha(v),v}) \right].
\]

5 Hamilton–Jacobi–Bellman–Isaacs Equation

In this section we prove that the two value functions are viscosity solutions of the Hamilton–Jacobi–Bellman–Isaacs equation (2.7) associated to the stochastic differential game. We begin with the following proposition.

Proposition 3 The lower and upper value functions satisfy the following properties: for all \(t \in [0, T)\) and \(x \in \mathbb{R}^n\),

\[
(i) \ H_{inf}^X V(t, x) \geq V(t, x).
(ii) If \ H_{inf}^X V(t, x) > V(t, x), \ then \ H_{sup}^c V(t, x) \leq V(t, x).
\]

Proof We make the proof only for \(V^-,\) the other case being analogous.

(i) Let \(\eta \in \mathcal{V},\) for \(\beta'(u) = \eta [t, T] + \sum_{l \geq 1} \eta_l' \mathbb{I}_{[\rho_l', T]}\) we have

\[
V^-(t, x) \leq \sup_{u \in \mathcal{U}_{t,T}} J(t, x, u, \beta'(u)).
\]
Choose \( \beta(u) = \sum_{l \geq 1} \eta'_l \mathbb{I}_{[\rho_l, T]} \). Then

\[
V^-(t, x) \leq \sup_{u \in \mathcal{U}_{t, T}} J(t, x + \eta, u, \beta(u)) + \chi(t, \eta),
\]

from which we deduce that the following inequality holds:

\[
V^-(t, x) \leq \inf_{\eta \in \mathcal{V}} \left[ V^-(t, x + \eta) + \chi(t, \eta) \right].
\]

(ii) the proof proceeds by a case distinction, for \( \tau = t \) in dynamic programming principle, and the suboptimality of multiple impulses at the same time [Assumptions (2.4) and (2.5)] we have

\[
V^-(t, x) = \inf_{\rho \in [t, +\infty], \eta \in \mathcal{V}} \sup_{\tau \in [t, +\infty], \xi \in \mathcal{U}} \left[ -c(t, \xi) \mathbb{I}_{[\tau=t]} \mathbb{I}_{[\rho=+\infty]} + \chi(t, \eta) \mathbb{I}_{[\rho=t]} + V^-(t, x + \xi \mathbb{I}_{[\tau=t]} \mathbb{I}_{[\rho=+\infty]} + \eta \mathbb{I}_{[\rho=t]}) \right].
\]

As a consequence we have

\[
V^-(t, x) = \inf_{\rho \in [t, +\infty]} \left[ \inf_{\eta \in \mathcal{V}} \left\{ \chi(t, \eta) + V^-(t, x + \eta) \right\} \right] \mathbb{I}_{[\rho=t]} + \left( \sup_{\tau \in [t, +\infty], \xi \in \mathcal{U}} \left[ -c(t, \xi) \mathbb{I}_{[\tau=t]} + V^-(t, x + \xi \mathbb{I}_{[\tau=t]}) \right] \mathbb{I}_{[\rho=+\infty]} \right).
\]

If \( H_{\inf}^X V(t, x) > V(t, x) \), then

\[
V^-(t, x) = \sup_{\tau \in [t, +\infty], \xi \in \mathcal{U}} \left\{ -c(t, \xi) \mathbb{I}_{[\tau=t]} + V^-(t, x + \xi \mathbb{I}_{[\tau=t]}) \right\}.
\]

Therefore,

\[
V^-(t, x) \geq \sup_{\xi \in \mathcal{U}} \left\{ -c(t, \xi) + V^-(t, x + \xi) \right\}.
\]

\( \square \)

Now we prove that the two value functions satisfy in the viscosity sense.

**Theorem 4** The lower and upper value functions are viscosity solutions to the Hamilton–Jacobi–Bellman–Isaacs equation (2.7).

**Proof** We give the proof for the lower value function \( V^- \), the other case being analogous. First, we prove the supersolution property. Suppose \( V^- - \phi \) achieves a local minimum in \( [t_0, t_0 + \delta) \times B(x_0, \delta) \) with \( V^-(t_0, x_0) = \phi(t_0, x_0) \). We have by Proposition 3,

\[
V^-(t_0, x_0) - H_{\inf}^X V^-(t_0, x_0) \leq 0.
\]
If

\[ V^-(t_0, x_0) - H_{\text{inf}}^X V^-(t_0, x_0) = 0, \]

then we are done. Now suppose

\[ V^-(t_0, x_0) - H_{\text{inf}}^X V^-(t_0, x_0) < -2\epsilon < 0, \]

we prove by contradiction that

\[ -\frac{\partial \phi}{\partial t}(t_0, x_0) - \mathcal{L}\phi(t_0, x_0) - f(t_0, x_0) \geq 0. \] (5.3)

Suppose otherwise, i.e., \(-\frac{\partial \phi}{\partial t}(t_0, x_0) - \mathcal{L}\phi(t_0, x_0) - f(t_0, x_0) < 0\). Then without loss of generality we can assume that \(-\frac{\partial \phi}{\partial t}(t, x) - \mathcal{L}\phi(t, x) - f(t, x) < 0\) and \(V^-(t, x) - H_{\text{inf}}^X V^-(t, x) < -\epsilon < 0\) on \([t_0, t_0 + \delta) \times B(x_0, \delta)\).

Define the stopping time \(\tau\) by

\[ \tau = \inf\{t \in \mathbb{Q}_{[t_0, T]} : (t, X_t) \notin [t_0, t_0 + \delta) \times B(x_0, \delta)\} \wedge T. \]

By Itô’s formula

\[ \mathbb{E}[\phi(\tau, X_{\tau}^0, x_0, u_0, v_0)] - \phi(t_0, x_0) = \mathbb{E}\left[ \int_{t_0}^{\tau} \left( \frac{\partial \phi}{\partial t} + \mathcal{L}\phi \right)(r, X_r^0, x_0, u_0, v_0)dr \right], \quad (5.4) \]

where \(u_0\) and \(v_0\) are the controls with no impulses.

Let \(\epsilon_1 > 0\), using the dynamic programming principle between time \(t_0\) and \(\tau \wedge \rho_1\) (\(\rho_1\) is the time of first impulse for \(\beta \in \mathcal{B}_{t_0, T}\), we deduce the existence of a strategy \(\beta^{\epsilon_1} \in \mathcal{B}_{t_0, T}\) such that

\[
V^-(t_0, x_0) \geq \mathbb{E}\left[ \int_{t_0}^{\tau \wedge \rho_1} f(r, X_r^0, x_0, u_0, \beta^{\epsilon_1}(u_0))dr + \chi(\rho_1, \eta_1)1_{[\rho_1 \leq \tau \wedge \rho_1]} \right. \\
\quad \left. + V^-(\tau \wedge \rho_1, X_{\tau \wedge \rho_1}^0, x_0, \beta^{\epsilon_1}(u_0)) \right] - \epsilon_1 \\
\geq \mathbb{E}\left[ \int_{t_0}^{\tau \wedge \rho_1} f(r, X_r^0, x_0, u_0, \beta^{\epsilon_1}(u_0))dr \\
\quad + 1_{[\rho_1 \leq \tau]} \left( \chi(\rho_1, \eta_1) + V^-(\rho_1, X_{\rho_1}^0, x_0, \beta^{\epsilon_1}(u_0)) \right) \right] \\
\quad + \mathbb{E}\left[ 1_{[\rho_1 \leq \tau]} V^-(\tau, X_{\tau}^0, x_0, \beta^{\epsilon_1}(u_0)) \right] - \epsilon_1 \\
\geq \mathbb{E}\left[ \int_{t_0}^{\tau \wedge \rho_1} f(r, X_r^0, x_0, v_0)dr + 1_{[\rho_1 \leq \tau]} H_{\text{inf}}^X V^-(\rho_1, X_{\rho_1}^0, x_0, v_0) \right].
\]
Therefore, without loss of generality, we only need to consider $\beta^{\varepsilon_1} \in \hat{B}_{0,T}$ such that $\rho_1 > \tau$. Then

$$
\phi(t_0, x_0) = V^-(t_0, x_0) \geq \mathbb{E} \left[ \int_{t_0}^{\tau} f(r, X_r^{t_0,x_0,u_0,v_0}) dr + V^-(\rho_1 \wedge \tau, X_{\rho_1 \wedge \tau}^{t_0,x_0,u_0,v_0}) \right] - \varepsilon_1
$$

Therefore, sending $\varepsilon_1 \to 0$ we get

$$
0 \leq \mathbb{E} \left[ \int_{t_0}^{\tau} \left( \frac{\partial \phi}{\partial t} + L \phi + f \right)(r, X_r^{t_0,x_0,u_0,v_0}) dr \right],
$$

which is a contradiction. Therefore, we must have $-\left( \frac{\partial \phi}{\partial t} + L \phi + f \right)(t_0, x_0) \geq 0$.

Thanks to Proposition 3, we have

$$
H_{sup}^c V^-(t_0, x_0) \leq V^-(t_0, x_0).
$$

Therefore,

$$
\max \left\{ \min \left[ -\frac{\partial \phi}{\partial t}(t_0, x_0) - L \phi(t_0, x_0) - f(t_0, x_0), V^-(t_0, x_0), V^-(t_0, x_0) - H_{sup}^c V^-(t_0, x_0) \right],
V^-(t_0, x_0) - H_{inf}^c V^-(t_0, x_0) \right\} \geq 0,
$$

which is the supersolution property. The subsolution property is proved analogously.

Now we give an equivalent of Hamilton–Jacobi–Bellman–Isaacs equation (2.7). We consider the new function $\Gamma(t, x) = \exp(t) V(t, x)$, for any $t \in [0, T]$ and $x \in \mathbb{R}^n$.

A second property is given by the:

**Proposition 4** $V$ is a viscosity solution of (2.7) if and only if $\Gamma$ is a viscosity solution to the Hamilton–Jacobi–Bellman–Isaacs equation in $[0, T) \times \mathbb{R}^n$,

$$
\max \left\{ \min \left[ \Gamma(t, x) - \frac{\partial \Gamma}{\partial t}(t, x) - L \Gamma(t, x) - \exp(t) f(t, x),
\Gamma(t, x) - \tilde{H}_{sup}^c \Gamma(t, x), \Gamma(t, x) - \tilde{H}_{inf}^c \Gamma(t, x) \right], \Gamma(t, x) - \tilde{H}_{inf}^c \Gamma(t, x) \right\} = 0,
$$

$$
\tilde{H}_{inf}^c \Gamma(t, x) = \inf_{\hat{u} \in \mathcal{U}} \Gamma(t, x). 
$$
where
\[
\tilde{H}^c_{\sup} \Gamma(t, x) = \sup_{\xi \in \mathcal{U}} [\Gamma(t, x + \xi) - \exp(t)c(t, \xi)]
\]
and
\[
\tilde{H}_{\inf}^X \Gamma(t, x) = \inf_{\eta \in \mathcal{V}} [\Gamma(t, x + \eta) + \exp(t)\chi(t, \eta)].
\]

The terminal condition for $\Gamma$ is: $\Gamma(T, x) = \exp(T)g(x)$ in $\mathbb{R}^n$.

6 Uniqueness of the Solution of Hamilton–Jacobi–Bellman–Isaacs Equation

In this section we deal with the issue of uniqueness of the solution of system (2.7) and to do so. We need to impose additional assumptions on cost functions

(H5) There exists a function $h : [0, T] \rightarrow (0, \infty)$ such that for all $t \in [0, T], \chi(t, \eta_1 + \eta_2) \leq \chi(t, \eta_1) + \chi(t, \eta_2) - h(t), (6.1)$
and
\[
c(t, \xi_1 + \eta + \xi_2) \leq c(t, \xi_1) - \chi(t, \eta) + c(t, \xi_2) - h(t) (6.2)
\]
for every $\xi_1, \xi_2 \in \mathcal{U}$, and $\eta, \eta_1, \eta_2 \in \mathcal{V}$.

To prove uniqueness viscosity solution to the HJBI equation (2.7), we begin with the following technical lemma which also appears in ([43, Lemma 4.4]), ([14, Lemma 6.1]).

**Lemma 3** Suppose $U, V : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ are uniformly continuous functions viscosity subsolution and a viscosity supersolution to the HJBI equation (2.7), respectively. Let $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$, be such that
\[
V(t_0, x_0) \geq H_{\inf}^X V(t_0, x_0) \quad (6.3)
\]
or
\[
V(t_0, x_0) < H_{\inf}^X V(t_0, x_0), \quad U(t_0, x_0) \leq H_{\sup}^C U(t_0, x_0). (6.4)
\]
Then for every $\epsilon > 0$, there exists $\bar{x} \in \mathbb{R}^n$ and $\delta > 0$ such that
\[
U(t_0, x_0) - V(t_0, x_0) \leq U(t_0, \bar{x}) - V(t_0, \bar{x}) + \epsilon
\]
and
\[
V(t, x) < H_{\inf}^X V(t, x), \quad U(t, x) > H_{\sup}^C U(t, x), (6.5)
\]
when $(t, x) \in [(t_0 - \delta) \vee 0, t_0 + \delta] \times \bar{B}(\bar{x}, \delta)$, with $t_0 + \delta < T$.

**Proof** Fix $\epsilon > 0$. We divide the proof into three steps.

**Step 1** Let (6.3) hold. Then for $\alpha_1 \in (0, 1)$, there exists $\eta_0 \in \mathcal{V}$ such that
\[
V(t_0, x_0) \geq V(t_0, x_0 + \eta_0) + \chi(t_0, \eta_0) - \alpha_1 \epsilon.
\]
Then, we observe that
\[
V(t_0, x_0 + \eta_0 + \eta) + \chi(t_0, \eta) - V(t_0, x_0 + \eta_0)
\geq V(t_0, x_0 + \eta_0 + \eta) + \chi(t_0, \eta_0) - V(t_0, x_0) - \alpha_1 \epsilon
\geq \chi(t_0, \eta) + \chi(t_0, \eta_0) - \chi(t_0, \eta_0 + \eta) - \alpha_1 \epsilon.
\]

Thus, by taking \(\alpha_1\) sufficiently small and (6.1), we get
\[
H_{\inf}^X V(t_0, x_0 + \eta_0) - V(t_0, x_0 + \eta_0) > 0.
\]

If \(U(t_0, x_0 + \eta_0) > H_{\sup}^c U(t_0, x_0 + \eta_0)\), we take \((t_0, \tilde{x}) := (t_0, x_0 + \eta_0)\). Otherwise at \((t_0, x_0 + \eta_0)\) condition (6.4) holds.

On the other hand, its easy to check that
\[
U(t_0, x_0 + \eta_0) - V(t_0, x_0 + \eta_0) \geq U(t_0, x_0 + \eta_0) + \chi(t_0, \eta_0) - V(t_0, x_0) - \alpha_1 \epsilon
\geq U(t_0, x_0) - V(t_0, x_0) - \alpha_1 \epsilon.
\]

**Step 2** Now, suppose that (6.4) holds at \((t_0, x_0)\), then for \(\alpha_2 \in (0, 1)\), there exists \(\xi_0 \in \mathcal{U}\) such that
\[
U(t_0, x_0) \leq U(t_0, x_0 + \xi_0) - c(t_0, \xi_0) + \alpha_2 \epsilon.
\]

Then, we have
\[
U(t_0, x_0 + \xi_0) - V(t_0, x_0 + \xi_0) \geq U(t_0, x_0) + c(t_0, \xi_0) - V(t_0, x_0 + \xi_0) - \alpha_2 \epsilon
\geq U(t_0, x_0) - V(t_0, x_0) - \alpha_2 \epsilon.
\]

By taking \(\alpha_2\) sufficiently small and (6.2), we can show that
\[
U(t_0, x_0 + \xi_0) > H_{\sup}^c U(t_0, x_0 + \xi_0).
\]

If \(V(t_0, x_0 + \xi_0) < H_{\inf}^X V(t_0, x_0 + \xi_0)\), we take \((t_0, \tilde{x}) := (t_0, x_0 + \xi_0)\). Otherwise we can proceed as in Step 1 and we find \(\eta_0 \in \mathcal{V}\) such that
\[
V(t_0, x_0 + \xi_0) \geq V(t_0, x_0 + \xi_0 + \eta_0) + \chi(t_0, \eta_0) - \alpha_2 \epsilon,
\]
\[
U(t_0, x_0 + \eta_0 + \xi_0) - V(t_0, x_0 + \eta_0 + \xi_0) \geq U(t_0, x_0) - V(t_0, x_0) - 2\alpha_2 \epsilon,
\]
and
\[
V(t_0, x_0 + \xi_0 + \eta_0) < H_{\inf}^X V(t_0, x_0 + \xi_0 + \eta_0).
\]

On the other hand, we have
\[
U(t_0, x_0 + \xi_0 + \eta_0 + \xi) - c(t_0, \xi) - U(t_0, x_0 + \xi_0 + \eta_0).
\]
\[ U(t_0, x_0 + \xi_0 + \eta_0 + \xi) - c(t_0, \xi) - c(t_0, \xi_0) - U(t_0, x_0 + \eta_0) \leq U(t_0, x_0 + \xi_0 + \eta_0 + \xi) - c(t_0, \xi) - c(t_0, \xi_0) + \chi(t_0, \eta_0) - U(t_0, x_0) \leq c(t_0, \xi_0 + \eta_0 + \xi) - c(t_0, \xi) - c(t_0, \xi_0) + \chi(t_0, \eta_0). \]

Thus, by (6.2), we get
\[ U(t_0, x_0 + \eta_0 + \xi_0) > H^c_{\sup} U(t_0, x_0 + \eta_0 + \xi_0). \]

Therefore we take \((t_0, \bar{x}) := (t_0, x_0 + \xi_0 + \eta_0)\).

**Step 3** We can find \(\alpha > 0\) such that
\[
\begin{align*}
U(t_0, \bar{x}) &> U(t_0, \bar{x} + \xi) - c(t_0, \xi) + \alpha \quad \forall \xi \in \mathcal{U}, \\
V(t_0, \bar{x}) &< V(t_0, \bar{x} + \eta) + \chi(t_0, \eta) - \alpha \quad \forall \eta \in \mathcal{V}.
\end{align*}
\]

Thus, by uniform continuity of \(U, V, c\), and \(\chi\) we have
\[
\begin{align*}
U(t, x) &> U(t, x + \xi) - c(t, \xi) - 2u(|x - \bar{x}| \vee |t - t_0|) - h(|t - t_0|) + \alpha, \\
V(t, x) &< V(t, x + \eta) + \chi(t, \eta) + 2v(|x - \bar{x}| \vee |t - t_0|) + w(|t - t_0|) - \alpha,
\end{align*}
\]
in which \(u, v, h\) and \(w\) are the modulus of continuity of \(U, V, c\) and \(\chi\) respectively. Therefore, there exists \(\delta > 0\) such that
\[
V(t, x) < H^\chi_{\inf} V(t, x), \quad U(t, x) > H^c_{\sup} U(t, x), \tag{6.6}
\]
when \((t, x) \in [(t_0 - \delta) \vee 0, t_0 + \delta] \times \bar{B}(\bar{x}, \delta), \) with \(t_0 + \delta < T. \)

We are going now to address the question of uniqueness of the viscosity solution of Hamilton–Jacobi–Bellman–Isaacs equation (2.7). We have the following:

**Theorem 5** The solution in viscosity sense of Hamilton–Jacobi–Bellman–Isaacs equation (2.7) is unique in the space of bounded continuous functions on \([0, T] \times \mathbb{R}^n\).

**Proof** We will show by contradiction that if \(U\) and \(V\) is a subsolution and a supersolution respectively for (5.6), then \(U \leq V\). Therefore if we have two solutions of (5.6) then they are obviously equal. Actually for some \(R > 0\) (large enough) suppose there exists \((\bar{t}, \bar{x}) \in [0, T] \times B(0, R)\) such that \(\sup_{t,x} (U(t, x) - V(t, x)) = U(\bar{t}, \bar{x}) - V(\bar{t}, \bar{x}) > 0. \)

**Step 1.** Using Lemma 3, we can find \((\tilde{t}, \tilde{x}) \in [0, T] \times \mathbb{R}^n\) and \(\delta\) such that
\[
\sup_{t \in [\bar{t} - \delta, \bar{t} + \delta]} (U(t, x) - V(t, x)) \geq U(\tilde{t}, \tilde{x}) - V(\tilde{t}, \tilde{x}) > 0, \tag{6.7}
\]
and
\[
V(t, x) < \tilde{H}^\chi_{\inf} V(t, x), \quad U(t, x) > \tilde{H}^c_{\sup} U(t, x), \tag{6.8}
\]
for all \((t, x) \in I \times \bar{B}(\bar{x}, \delta), \) where \(I := [\bar{t} - \delta, \bar{t} + \delta] \subset [0, T]. \)
Let \((t_0, x_0) \in I \times \bar{B}(\bar{x}, \delta)\) such that

\[
\sup_{I \times \bar{B}(\bar{x}, \delta)} (U(t, x) - V(t, x)) = U(t_0, x_0) - V(t_0, x_0) = \eta > 0.
\]

For a small \(\epsilon, \beta, \theta > 0\), let us define:

\[
\Phi_\epsilon(t, x, y) = U(t, x) - V(t, y) - \frac{1}{2\epsilon}|x - y|^2 - \theta(|x - x_0|^4 + |y - x_0|^4) - \beta(t - t_0)^2.
\]

By the boundedness of \(U\) and \(V\), that there exists a \((t_\epsilon, x_\epsilon, y_\epsilon) \in I \times \bar{B}(\bar{x}, \delta) \times \bar{B}(\bar{x}, \delta)\), attaining the maximum of \(\Phi_\epsilon\) on \(I \times B(\bar{x}, \delta) \times \bar{B}(\bar{x}, \delta)\).

On the other hand, from \(2\Phi_\epsilon(t_\epsilon, x_\epsilon, y_\epsilon) \geq \Phi_\epsilon(t_\epsilon, x_\epsilon, x_\epsilon) + \Phi_\epsilon(t_\epsilon, y_\epsilon, y_\epsilon)\), we have

\[
\frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2 \leq (U(t_\epsilon, x_\epsilon) - U(t_\epsilon, y_\epsilon)) + (V(t_\epsilon, x_\epsilon) - V(t_\epsilon, y_\epsilon)),
\]

and consequently \(\frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2\) is bounded, and as \(\epsilon \to 0, |x_\epsilon - y_\epsilon| \to 0\). Since \(U\) and \(V\) are uniformly continuous on \(I \times \bar{B}(\bar{x}, \delta)\), then \(\frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2 \to 0\) as \(\epsilon \to 0\).

Since \(\Phi_\epsilon(t_\epsilon, x_\epsilon, y_\epsilon) \geq \Phi_\epsilon(t_0, x_0, x_0)\), we have

\[
U(t_0, x_0) - V(t_0, x_0) \leq \Phi_\epsilon(t_\epsilon, x_\epsilon, y_\epsilon) \leq U(t_\epsilon, x_\epsilon) - V(t_\epsilon, y_\epsilon),
\]

it follow from the continuity of \(U\) and \(V\) that, up to a subsequence,

\[
(t_\epsilon, x_\epsilon, y_\epsilon) \to (t_0, x_0, x_0) \quad \theta(|x_\epsilon - x_0|^4 + |y_\epsilon - x_0|^4) \to 0 \quad U(t_\epsilon, x_\epsilon) - V(t_\epsilon, y_\epsilon) \to U(t_0, x_0) - V(t_0, x_0).
\]

Next, since \(x_0 \in \bar{B}(\bar{x}, \delta)\) then, for \(\epsilon\) small enough and at least for a subsequence which we still index by \(\epsilon\), we obtain

\[
V(t_\epsilon, y_\epsilon) < \tilde{H}^K_{\inf} V(t_\epsilon, y_\epsilon), \quad U(t_\epsilon, x_\epsilon) > \tilde{H}^c_{\sup} U(t_\epsilon, x_\epsilon).
\]

**Step 2.** We now show that \(t_\epsilon < T\). Actually if \(t_\epsilon = T\) then,

\[
\Phi_\epsilon(t_0, x_0, x_0) \leq \Phi_\epsilon(T, x_\epsilon, y_\epsilon),
\]

and,

\[
U(t_0, x_0) - V(t_0, x_0) \leq \exp(T)g(x_\epsilon) - \exp(T)g(y_\epsilon) - \beta(T - t_0)^2,
\]
since \( U(T, x_\epsilon) = \exp(T)g(x_\epsilon), \) \( V(T, y_\epsilon) = \exp(T)g(y_\epsilon) \) and \( g \) is uniformly continuous on \( \bar{B}(\bar{x}, \delta) \). Then as \( \epsilon \to 0 \), we have,

\[
\eta \leq -\beta(T - t_0)^2,
\]

which yields a contradiction and we have \( t_\epsilon \in [0, T) \).

**Step 3** To complete the proof it remains to show contradiction. Let us denote

\[
\varphi_\epsilon(t, x, y) = \frac{1}{2\epsilon}|x - y|^2 + \theta(|x - x_0|^4 + |y - x_0|^4) + \beta(t - t_0)^2. \tag{6.14}
\]

Then we have:

\[
\begin{cases}
  D_t \varphi_\epsilon(t, x, y) = 2\beta(t - t_0), \\
  D_x \varphi_\epsilon(t, x, y) = \frac{1}{\epsilon}(x - y) + 4\theta(x - x_0)|x - x_0|^2, \\
  D_y \varphi_\epsilon(t, x, y) = -\frac{1}{\epsilon}(x - y) + 4\theta(y - x_0)|y - x_0|^2, \\
  B(t, x, y) = D_{x,y}^2 \varphi_\epsilon(t, x, y) = \frac{1}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} a(x) & 0 \\ 0 & a(y) \end{pmatrix} \\
  \text{with } a(x) = 4\theta|x - x_0|^2 I + 8\theta(x - x_0)(x - x_0)^*.
\end{cases} \tag{6.15}
\]

Then applying the result by Crandall et al. (Theorem 8.3, [15]) to the function \( U(t, x) - V(t, y) - \varphi_\epsilon(t, x, y) \) at the point \((t_\epsilon, x_\epsilon, y_\epsilon)\), for any \( \epsilon_1 > 0 \), we can find \( c, d \in IR \) and \( X, Y \in S_n \), such that:

\[
\begin{cases}
  \left( c, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + 4\theta(x_\epsilon - x_0)|x_\epsilon - x_0|^2, X \right) \in J^{2,+}(U(t_\epsilon, x_\epsilon)), \\
  \left( -d, \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) - 4\theta(y_\epsilon - x_0)|y_\epsilon - x_0|^2, Y \right) \in J^{2,-}(V(t_\epsilon, y_\epsilon)), \\
  c + d = D_t \varphi_\epsilon(t_\epsilon, x_\epsilon, y_\epsilon) = 2\beta(t_\epsilon - t_0) \quad \text{and finally} \\
  -\frac{1}{\epsilon_1} + \|B(t_\epsilon, x_\epsilon, y_\epsilon)\| \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq B(t_\epsilon, x_\epsilon, y_\epsilon) + \epsilon_1 B(t_\epsilon, x_\epsilon, y_\epsilon)^2.
\end{cases} \tag{6.16}
\]

Then by definition of viscosity solution, we get:

\[
-c + U(t_\epsilon, x_\epsilon) - \left( \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) + 4\theta(x_\epsilon - x_0)|x_\epsilon - x_0|^2, \right) - \exp(t_\epsilon) f(t_\epsilon, x_\epsilon) \leq 0 \tag{6.17}
\]

and

\[
\begin{aligned}
  d + V(t_\epsilon, y_\epsilon) - \left( \frac{1}{\epsilon}(x_\epsilon - y_\epsilon) - 4\theta(y_\epsilon - x_0)|y_\epsilon - x_0|^2, \right) & - \exp(t_\epsilon) f(t_\epsilon, y_\epsilon) \geq 0, \tag{6.18}
\end{aligned}
\]
which implies that:

\[
-c - d + U(t_\epsilon, x_\epsilon) - V(t_\epsilon, y_\epsilon) \\
\leq \left( \frac{1}{\epsilon} (x_\epsilon - y_\epsilon), b(t_\epsilon, x_\epsilon) - b(t_\epsilon, y_\epsilon) \right) \\
+ (4\theta(x_\epsilon - x_0)|x_\epsilon - x_0|^2, b(t_\epsilon, x_\epsilon)) + (4\theta(y_\epsilon - x_0)|y_\epsilon - x_0|^2, b(t_\epsilon, y_\epsilon)) \\
+ \frac{1}{2} tr[\sigma^*(t_\epsilon, x_\epsilon)X\sigma(t_\epsilon, x_\epsilon) - \sigma^*(t_\epsilon, y_\epsilon)Y\sigma(t_\epsilon, y_\epsilon)] \\
+ \exp(t_\epsilon) f(t_\epsilon, x_\epsilon) - \exp(t_\epsilon) f(t_\epsilon, y_\epsilon)].
\] (6.19)

But from (6.15) there exist a constant \( C > 0 \) such that:

\[
(\|a(x_\epsilon)\| \vee \|a(y_\epsilon)\|) \leq C\theta.
\]

As

\[
B = B(t_\epsilon, x_\epsilon, y_\epsilon) = \frac{1}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} a(x_\epsilon) & 0 \\ 0 & a(y_\epsilon) \end{pmatrix}
\]

Then

\[
B \leq \frac{1}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C\theta I.
\]

It follows that:

\[
B + \epsilon_1 B^2 \leq \frac{\epsilon + \epsilon_1}{\epsilon^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C\theta I,
\] (6.20)

where \( C \) which hereafter may change from line to line. Choosing now \( \epsilon_1 = \epsilon \), yields the relation

\[
B + \epsilon_1 B^2 \leq \frac{2}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C\theta I.
\] (6.21)

Now, from (H1), (6.16) and (6.21) we get:

\[
\frac{1}{2} tr[\sigma^*(t_\epsilon, x_\epsilon)X\sigma(t_\epsilon, x_\epsilon) - \sigma^*(t_\epsilon, y_\epsilon)Y\sigma(t_\epsilon, y_\epsilon)] \\
\leq \frac{C}{\epsilon} |x_\epsilon - y_\epsilon|^2 + C\theta(1 + |x_\epsilon|^2 + |y_\epsilon|^2).
\]

Next

\[
\langle \frac{1}{\epsilon} (x_\epsilon - y_\epsilon), b(t_\epsilon, x_\epsilon) - b(t_\epsilon, y_\epsilon) \rangle \leq \frac{C}{\epsilon} |x_\epsilon - y_\epsilon|^2.
\]

And finally,

\[
(4\theta(x_\epsilon - x_0)|x_\epsilon - x_0|^2, b(t_\epsilon, x_\epsilon)) + (4\theta(y_\epsilon - x_0)|y_\epsilon - x_0|^2, b(t_\epsilon, y_\epsilon))
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\[
\leq C\theta(1 + |x_\epsilon||x_\epsilon - x_0|^3 + |y_\epsilon||y_\epsilon - x_0|^3).
\]

So that by plugging into (6.19) we obtain:

\[
-2\beta(t_\epsilon - t_0) + U(t_\epsilon, x_\epsilon) - V(t_\epsilon, y_\epsilon)
\]
\[
\leq \frac{C}{\epsilon}|x_\epsilon - y_\epsilon|^2 + C\theta(1 + |x_\epsilon|^2 + |y_\epsilon|^2) + \frac{C}{\epsilon}|x_\epsilon - y_\epsilon|^2
\]
\[
+C\theta(1 + |x_\epsilon||x_\epsilon - x_0|^3 + |y_\epsilon||y_\epsilon - x_0|^3)
\]
\[
+ \exp(t_\epsilon)f(t_\epsilon, x_\epsilon) - \exp(t_\epsilon)f(t_\epsilon, y_\epsilon).
\]

(6.22)

By sending \(\epsilon \to 0, \beta \to 0, \theta \to 0\), and taking into account of the continuity of \(f\), we obtain \(\eta \leq 0\), which is a contradiction. The proof of Theorem 5 is now complete. \(\square\)

**Corollary 3** The lower and upper value functions coincide, and the value function of the stochastic differential game is given by \(V(t, x) := V^-(t, x) = V^+(t, x)\) for every \((t, x) \in [0, T] \times \mathbb{R}^n\).

**7 Conclusions**

In this paper we have considered a general two-player zero-sum stochastic differential game involving impulse controls, whose state variable follows a diffusive dynamics driven by a multi-dimensional Brownian motion. We have proved the existence and uniqueness of the value function. Similar result had been obtained by Cosso [14] under additional assumption: the costs of the impulses to be decreasing in time.

We intend to develop this work in several directions in the future. First, we wish to take into account that the costs of impulses depend on the state variable. Second, we wish to prove that the game has a saddle-point, i.e., there exists a pair of controls \((u^*, v^*) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}\) such that for any \(u\) and \(v\), we have

\[
J(t, x, u, v^*) \leq J(t, x, u^*, v^*) \leq J(t, x, u^*, v).
\]

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