MASSIVE SIGMA MODELS WITH (p,q) SUPERSYMMETRY

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ABSTRACT

We determine the general scalar potential consistent with (p,q) supersymmetry in two-dimensional non-linear sigma models with torsion, generalizing previous results for special cases. We thereby find many new supersymmetric sigma models with potentials, including new (2,2) and (4,4) models.
1. Introduction

The general \((p,q)\) supersymmetry algebra for two-dimensional \((d=2)\) Minkowski space field theories is spanned by \(p\) Hermitian spinorial charges \(\{Q^I_+; I = 1, \ldots, p\}\) of one chirality, \(q\) charges \(\{Q^I_-; I' = 1, \ldots, q\}\) of the other chirality, and the self-dual and anti-self-dual components, \(P_+\), \(P_-\), of the 2-momentum (we use a Lorentz charge notation for \(d=2\) spinors and vectors). Excluding spinorial or tensorial central charges, the re remains only the possibility of an additional \(pq\) scalar central charges, \(Z^{II'}\). The non-zero (anti)commutation relations of this algebra are

\[
\{Q^I_+, Q^J_+\} = 2\delta^{IJ}P_+ \quad \{Q^I_-, Q^J_-\} = 2\delta^{I'J'}P_- \quad \{Q^I_+, Q^I_-\} = Z^{II'} .
\] (1.1)

All \(d=2\) supersymmetric field theories have at least \((1,0)\) or \((0,1)\) supersymmetry and, since \((0,1)\) is the parity reflection of \((1,0)\), we may assume without loss of generality that all \(d=2\) supersymmetric field theories are \((1,0)\) supersymmetric. If we restrict our attention to scalar and spinor fields only then all such theories can be written in terms of the \((1,0)\) scalar superfields \(\{\phi^i(x, \theta^+); i = 1, \ldots, D\}\), which define a map \(\phi\) from \((1,0)\) superspace, \(\Sigma^{(1,0)}\), into a \(D\)-dimensional Riemannian target manifold \(\mathcal{M}\), and the spinor superfields \(\{\psi^a(x, \theta^+); a = 1, \ldots, n\}\), which define a section \(\psi\) of the vector bundle \(S_- \otimes \phi^*\xi\) where \(\xi\) is a vector bundle over \(\mathcal{M}\) of rank \(n\) and \(S_-\) is the spin bundle over \(\Sigma^{(1,0)}\). Subject to the further restriction that all scalar field equations be second order, the action for the general \((p,q)\)-supersymmetric model, generally called a ‘sigma-model’, can be written as the \((1,0)\)-superspace integral [1]

\[
S = \int d^2x d\theta^+ \left\{ D_+ \phi^i \partial_+ \phi^j (g_{ij} + b_{ij}) + i \psi^a \nabla_+ \psi^b h_{ab} + ims_a \psi^a \right\} ,
\] (1.2)

where \(m\) is a constant with dimensions of mass, \(D_+ = i\partial/\partial \theta^+ + \theta^+ \partial_+\) is the (real) supercovariant derivative satisfying \(D^2_+ = i \partial_+\), and

\[
\nabla_+ \psi_-^b \equiv (D_+ \psi_-^b + D_+ \phi^i \Omega_i^b \psi_-^c) .
\] (1.3)

Thus, in this formulation, the general \((p,q)\)-supersymmetric model is characterised by (i) a metric \(g\) on \(\mathcal{M}\), which we require to be positive definite in order that the Hamiltonian be positive semi-definite (ii) a two-form \(b\) on \(\mathcal{M}\); this need be defined only locally since it is \(H = db\) that occurs in the field equations (as a torsion tensor), (iii) a metric \(h\) and connection \(\Omega\) on the fibre \(\xi\); without loss of generality we can choose \(h\) to be covariantly constant, i.e.

\[
\nabla_i h_{ab} = 0 ,
\] (1.4)

and (iv), when \(m \neq 0\), a section \(s\) of \(\xi\).

When \(m = 0\) this action is classically conformally invariant. It will be shown below that when \(m \neq 0\) the component action contains the potential term

\[
V(\phi) = \frac{1}{4} m^2 h^{ab} s_a(\phi) s_b(\phi) .
\] (1.5)
In many models of interest the section $s$, and hence the potential $V$, will have isolated zeros. Linearisation about an isolated zero of the potential yields a massive supersymmetric field theory with mass proportional to $m$. We shall therefore refer to the general model with $m \neq 0$ as a ‘massive sigma-model’. Note that when $m \neq 0$, the fibre metric $h$ (as well as the target space metric $g$) must be positive definite for the Hamiltonian to be positive semi-definite. We henceforth assume that $h$ is positive definite, in which case the structure group of the bundle $\xi$ is a subgroup of $O(n)$.

As written, the action (1.2) has manifest $(1,0)$ supersymmetry. If certain conditions on the couplings $g$, $b$, $h$, $\Omega$, and $s$ are satisfied it will have further, although non-manifest, supersymmetries. The conditions for $(p,q)$ supersymmetry in the massless case, $m = 0$, have been thoroughly investigated [2,3,4] in past years because of the applications of these sigma-models to string theory and conformal field theory. More recently attention has focused on certain massive models. In particular it has been shown that some massive $(2,2)$ supersymmetric sigma models are integrable quantum field theories (see e.g. [5]). A feature of these models is that they admit solitons which interpolate between distinct zeros of the potential and carry a complex topological charge which appears in the (on-shell) supersymmetry algebra as a central charge. More recently, it has been shown that there exist $(4,4)$-supersymmetric models with solitons that carry a quaternionic charge which again appears in the supersymmetry algebra as a central charge [6]. Also, certain $(2,0)$ massive models have acquired importance in the context of the Landau-Ginsburg approach to integrable models [7].

The principal purpose of this paper is to provide a complete analysis of the conditions required for $(p,q)$ supersymmetry in massive sigma-models. A study of the conditions for $(p,p)$ supersymmetry for models without torsion was undertaken some ten years ago [8]. Very recently, results for massive models with torsion, i.e. with $H \neq 0$, were presented for off-shell $(p,0)$ and $(1,1)$ models [1] using $(1,0)$ and $(p,0)$ superfield methods. Here we use the $(1,0)$ superfield methods to determine the conditions required for on-shell $(p,0)$ supersymmetry. All remaining models can be considered as special cases of the general $(1,1)$ model. In this case, the bundle $\xi$ is isomorphic to the tangent bundle of $\mathcal{M}$ and the general form of the section $s$ is $s_i = (u - X)_i$, where $X$ is any Killing vector field of the target space $\mathcal{M}$ (should one exist) and $u$ is a one-form on $\mathcal{M}$ defined by $\iota_X H = du$. The potential $V$ is therefore [1]

$$V = \frac{m^2}{4} g^{ij} (u - X)_i (u - X)_j.$$  

The Noether charge corresponding to the symmetry generated by $X$ appears in the $(1,1)$ supersymmetry algebra as a central charge [8]. For $(p,q)$ supersymmetric models with $p > 1$, the supersymmetry algebra of the massless model has an $SO(p) \times SO(q)$ automorphism group which acts in the obvious way on the central charge matrix $Z^{II'}$. Since $X$ is associated with a particular central charge, the potential (1.6) appears to break the $SO(p) \times SO(q)$ symmetry. However, as we shall see, $(p,q)$ supersymmetry implies that the potential $V$ can be written in many different but equivalent ways. A consequence of this fact is that $V$ is actually $SO(p) \times SO(q)$ invariant (up to a constant). There are also further restrictions on $X$ and $u$ imposed by $(p,q)$ supersymmetry which we state and analyse.

The $(2,2)$ and $(4,4)$ models are of particular current interest. For these cases one can use the $SO(p) \times SO(q)$ invariance to diagonalize $Z^{II'}$. The $(p,p)$ supersymmetry then implies that all the diagonal elements are simply related to the Killing vector field $X$. For the $(2,2)$
models with zero torsion we find agreement with previous results [8] but we believe that the (2,2) models with non-zero torsion and non-zero central charge are new. It would be of interest to determine their soliton solutions and whether any of them are integrable quantum field theories. For the (4,4) case it was known that zero torsion massive models exist with \( u = 0 \) and \( X \) tri-holomorphic. Here we prove that \( u = 0 \) and \( X \) is tri-holomorphic for all (4,4) models with zero torsion, but for non-zero torsion we find new (off-shell) models for which \( X = 0 \) but \( u \neq 0 \).

The results presented here will have implications for the renormalization properties of the quantum theory of massive supersymmetric sigma models. For example, previous results concerning the ultraviolet finiteness of massless off-shell supersymmetric (4,q) models [4] can be extended to massive (4,q) models for which the supersymmetry algebra closes off-shell without central charges, so some of the models constructed here will be ultraviolet finite.

Finally, we remark that an alternative method of constructing (off-shell) massive (1,1)-supersymmetric sigma models based on gauged massless sigma models was also presented in [1]. This method allows an alternative construction of massive off-shell (p,q)-supersymmetric models [9].

2. (1,0) Supersymmetry

We begin with a discussion of some general features of the action (1.2). To determine the component version of this action we define the component fields contained in \((\phi^i, \psi^a)\) by

\[ \phi^i = \phi^i \Big| \lambda^i_+ = D_+ \phi^i \big| \quad \psi^a_- = \psi^a \big| \quad F^a = \nabla_+ \psi^a_- \]  

(2.1)

where the vertical bar indicates the \( \theta^+ = 0 \) component of a superfield. Then (1.2) becomes

\[ S = \int d^2 x \left\{ \partial_+ \phi^i \partial_- \phi^j (g_{ij} + b_{ij}) + ig_{ij} \lambda^i_+ \nabla^{(+)i}_- \lambda^j_- - i\psi^a_- \nabla_+ \psi^b_- h_{ab} - \frac{1}{2} \psi^a_- \psi^b_- \lambda^i_+ \lambda^j_- F_{ijab} + F^a_- h_{ab} + m \nabla_i s_a \lambda^i_- \psi^a_- + m s_a F^a \right\}, \]

(2.2)

where \( \nabla^{(\pm)}_\pm \) is the covariantization of \( \partial_\pm \) with the pull-back of the connection with torsion

\[ \Gamma_{ij}^{(\pm)k} = \{ k_{ij} \} \pm H_{ij}^k, \]

(2.3)

so that \( \nabla^{(\pm)}_\pm \lambda^k_+ = \partial_\pm \lambda^k_+ + \Gamma^{(\pm)k}_i \partial_\pm \phi^i \lambda^j_k \). The torsion tensor \( H_{ijk} \) is given by

\[ H_{ijk} = \frac{1}{2} (\partial_i b_{jk} + \partial_k b_{ij} + \partial_j b_{ki}) \equiv \frac{3}{2} \partial_i [b_{jk}], \]

(2.4)

and \( F_{ijab} = h_{ac} F_{ij}^{\ c} b \), where

\[ F_{ij}^{\ a} b = \partial_i \Omega^a_j b - \partial_j \Omega^a_i b + \Omega^a_i c \Omega^c_j b - \Omega^a_j c \Omega^c_i b. \]

(2.5)

Elimination of \( F^a \) from (2.2) by means of its algebraic field equation yields the scalar potential \( V \) of (1.5).
Returning now to the (1,0)-superspace action (1.2), we note that its variation with respect to the arbitrary variations $\delta \phi^i$ and $\delta \psi^a_-$ of $\phi^i$ and $\psi^a_-$ is (up to a surface term)

$$\delta S = \int d^2 x d\theta^+ \left\{ \delta \phi^i \mathcal{S}_{-i} + \Delta \psi^a_- \mathcal{S}_a \right\} ,$$

(2.6)

where

$$\Delta \psi^a_- \equiv \delta \psi^a_- + \delta \phi^i \psi^b_- \Omega^a_{i b}$$

(2.7)

is the covariantization of $\delta \psi^a_-$, and

$$\mathcal{S}_{-i} \equiv -2g_{ij} \nabla^\perp_{+} \partial_\pm \phi^j - i\psi^a_- \psi^b_- \nabla^i \phi^j + im \nabla_i s_a \psi^a_- + i m s^a .$$

(2.8)

Using this result the action (1.2) is readily verified to be invariant under the transformations

$$\delta \epsilon \phi^i = -\frac{i}{2} D_+ \epsilon_- \phi^i + \epsilon_- \partial_\perp \phi^i$$

$$\Delta \epsilon \psi^a_- = -\frac{i}{2} D_+ \epsilon_- \nabla^\perp \psi^a_- + \epsilon_- \nabla_\perp \psi^a_-$$

(2.9)

for $x$-independent (but $\theta$-dependent) superfield parameter $\epsilon_- \equiv |+ i \theta^+ \epsilon_-$. The $\epsilon_-$ part of these transformations can be rewritten as

$$\delta \phi^i = -\frac{1}{2} \epsilon_- Q_+ \phi^i \quad \delta \psi^a_- = -\frac{1}{2} \epsilon_- Q_+ \psi^a_-$$

(2.10)

where

$$Q_+ = -i D_+ + 2i \theta^+ \partial_+ \phi^i \quad \partial_+ = \partial_+ + i \theta^+ \partial_ \perp$$

(2.11)

is the (hermitian) differential operator that generates (1,0) supersymmetry transformations (satisfying $\{Q_+, D_+\} = 0$). The $\epsilon_-$ part of the transformations (2.9) are therefore those of the manifest (1,0) supersymmetry. The remaining $\epsilon_- |$ part is the transformation generated by the $P_\perp$ component of the 2-momentum. By combining the two transformations they become expressible in terms of (1,0) superfields. The symmetry transformations generated by the $P_\pm$ component of the 2-momentum may similarly be expressed in (1,0) superfield form as

$$\delta \phi^i = \epsilon_\pm \partial_\pm \phi^i \quad \Delta \psi^a_- = \epsilon_\pm \nabla_\pm \psi^a_-$$

(2.12)

where the parameter $\epsilon_\pm$ is a constant (independent of both $x$ and $\theta^+$).
3. (p,0) Supersymmetry

Any additional supersymmetries of (1,2) of the same chirality must have Noether charges that anticommute with the first one. This implies that the additional supersymmetry transformations can be expressed in terms of (1,0) superfields and a set of constant, anticommuting, parameter(s) \( \{ \eta^r_-; r = 1, \ldots, p - 1 \} \). The form of these transformations for \( m = 0 \) is fixed by dimensional analysis; when \( m \neq 0 \) we must allow for an additional variation of \( \psi \) proportional to \( m \). We are thus led to consider

\[
\delta_\eta \phi^i = i\eta^-_r I^i_r(\phi)D_+ \phi^j \\
\Delta_\eta \psi^a_- = \frac{1}{2} \eta^r_- \tilde{I}^a_r(\phi)S^b + \frac{im}{2} \eta^r_- t^a_r(\phi)
\]

where \( I_r \) are tensors on \( \mathcal{M} \), and \( t_r \) and \( \tilde{I}_r \) are sections of \( \xi^* \) (the dual of \( \xi \)) and \( \xi \otimes \xi^* \), respectively. We shall now investigate the conditions for on-shell closure of the (p,0) supersymmetry transformations (i.e. using \( S_i^- = 0 \) and \( S_a = 0 \)). We then determine the conditions for invariance of the action. The constraints arising from these two requirements are necessary and sufficient for the existence of conserved charges \( \{ Q^I_+; I = 1, \ldots, p \} \) obeying the (p,0) supersymmetry algebra. Off-shell closure of (p,0) supersymmetry algebra requires stronger conditions, which were investigated in [1].

The conditions for closure of the algebra on \( \phi^i \) are

\[
I_r I_s = -\delta_{rs} + f^t_{rs} I_t
\]

(in matrix notation) and

\[
N(I_r, I_s)^i_{jk} = 0
\]

where \( f^t_{rs} \) is zero for \( p=2 \) and equal to the quaternion structure constants \( \epsilon_{rst} \) for \( p=4 \), and \( N \) is the generalised Nijenhuis tensor defined by

\[
N(I_r, I_s)^i_{jk} \equiv 2\left[ \partial_i I^i_r[k I^j_s l] - I^i_r l \partial[j I^l_s k] + (r \leftrightarrow s) \right].
\]

On-shell closure on \( \psi^a_- \) implies

\[
F^a_i j b I^i_r [k I^j_s l] = F^a_{kl} b \delta_{rs}
\]

for \( m = 0 \). In addition, if \( m \neq 0 \) we find the condition

\[
[I^i_r j \nabla_i G^a_s + (r \leftrightarrow s)] + 2\delta_{rs} \nabla_i s^a = 0 .
\]

There is no condition on \( \tilde{I} \) from on-shell closure since \( \delta_\eta S^a = -i\eta^r_- \nabla_+ (\tilde{I}^a_r b S^b) \) (when the above conditions are satisfied) and this vanishes on-shell.

If \( m = 0 \) the action (1.2) is invariant under the transformations (3.1) provided that

\[
I^k_r (i g_j) k = 0 \quad \nabla^{(+)}_i I^i_r j k = 0
\]
\[
\hat{I}_{(ab)} \equiv h_{c(a} \hat{I}^c_{b)} = 0 
\] (3.8)

(with no condition on \( \hat{I}_{[ab]} \)). In addition, if \( m \neq 0 \) we find that

\[
\partial_i (t^a s_a) = 0 
\] (3.9)

and

\[
\nabla_i t^a_r = I_{r j i} \nabla^a_j. 
\] (3.10)

We shall begin the analysis of these conditions by considering the (2,0) case. The conditions arising from both closure of the supersymmetry algebra and invariance of the action can be summarised as follows, for \( m = 0 \) [3]: \( \mathcal{M} \) is a complex manifold with complex structure \( I; g \) is an Hermitian metric with respect to \( I \), and the holonomy of the connection \( \Gamma^{(+)} \) is a subgroup of \( U(D/2) \). Furthermore (3.5) implies that the vector bundle \( \xi \otimes \mathbb{C} \) is holomorphic*. The additional conditions that arise for \( m \neq 0 \) are just (3.9) and (3.10) since (3.6) is implied by (3.2) and (3.10). To discuss these conditions it is convenient to choose complex coordinates \( \{ \phi^i \} \to \{ \phi^\mu, \overline{\phi}^\bar{\mu} \equiv (\phi^\mu)^* \} \) adapted to the complex structure \( I \). Condition (3.10) then reduces to

\[
\nabla_{\bar{\mu}} (s + it)^a = 0; 
\] (3.11)

i.e. \( s \) is the real part of a holomorphic section of the bundle \( \xi \otimes \mathbb{C} \). The integrability condition of (3.11) is precisely (3.5). Combining (3.11) with (3.9) we deduce that

\[
s_a s^a = t_a t^a + \text{const.} 
\] (3.12)

We turn now to \( p = 4 \). The conditions for \( m = 0 \) can be summarised as follows: \( \mathcal{M} \) admits a quaternionic structure, i.e the three (integrable) complex structures obey the algebra of imaginary unit quaternions, the metric \( g \) is tri-Hermitian and the holonomy of the connection \( \Gamma^{(+)} \) is a subgroup of \( Sp(D/4) \). Furthermore, (3.5) implies that the bundle \( \xi \otimes \mathbb{C} \) is holomorphic with respect to all three complex structures. The additional conditions arising for \( m \neq 0 \) are (3.9) and (3.10) since (3.6) is again implied by (3.2) and (3.10). The integrability conditions of (3.10) are eqs. (3.4) and (3.5). It is again convenient to discuss the conditions (3.9) and (3.10) by choosing complex coordinates adapted to any one of the three complex structures (it is not possible, in general, to find a coordinate system such that all complex structures are simultaneously constant). Let us choose coordinates adapted to the complex structure \( I_1 \). From (3.10) we may then derive the two conditions

\[
\nabla_{\bar{\mu}} (s + it_1)^a = 0, \quad \nabla_{\bar{\mu}} (t_2 - it_3)^a = 0, 
\] (3.13)

i.e. \( s + it_1 \) and \( t_2 - it_3 \) are holomorphic sections (with respect to \( I_1 \)) of the bundle \( \xi \otimes \mathbb{C} \). The conditions (3.9) combined with (3.13) become

\[
s_a s^a = h_{ab} t^a_1 t^b_1 + \text{const.} 
\] (3.14)

* Note that, in contrast to off-shell supersymmetry, on-shell supersymmetry does not require \( \hat{I} \) to be a complex structure so the rank of \( \xi \) is not necessarily even.
By adopting coordinates adapted to each of the other two complex structures one can similarly deduce the cyclic permutations of the above relations.

4. (p,1) Supersymmetry

Since the (p,1) models are all special cases of (1,1) we shall make use here of the (1,1) results given in [1]. The vector bundle $\xi$ is now isomorphic to the tangent bundle, which allows us to convert all bundle indices to tangent bundle indices. In addition, the connection $\Omega$ of the bundle $\xi$ becomes the $\Gamma(-)$ connection of the tangent bundle. To allow for a central charge in the (1,1) supersymmetry algebra, one supposes that the manifold $\mathcal{M}$ has an isometry generated by a Killing vector field $X$, i.e.

$$\nabla(X_i X_j) = 0 . \quad (4.1)$$

Then, in the presence of torsion, invariance of the action requires that $L_X H = 0$ which implies that the two-form $\iota_X H$ is closed. Thus,

$$X^k H_{ijk} = \partial_i u_j \quad (4.2)$$

for some locally defined vector $u_i$. In fact, $u$ is globally defined on $\mathcal{M}$ since the section $s$ of $\xi$ that determines the potential is now given by

$$s_i = u_i - X_i . \quad (4.3)$$

The action can now be rewritten as

$$S = \int d^2 x d\theta^+ \left\{ D_+ \phi^i \partial_- \phi^j (g_{ij} + b_{ij}) + i \psi_i^- \nabla_+ (\phi^-) \psi_j^+ g_{ij} + im s_i \psi_i^- \right\} . \quad (4.4)$$

In addition

$$\partial_i (X^j u_j) = 0 . \quad (4.5)$$

The potential $V$ for the (1,1) model is

$$V(\phi) = \frac{1}{4} m^2 g^{ij} (u - X)_i (u - X)_j + \text{const.} . \quad (4.6)$$

Using (4.5) this can be rewritten as

$$V = \frac{1}{4} m^2 (g^{ij} u_i u_j + g^{ij} X_i X_j) + \text{const.} . \quad (4.7)$$

The action (4.4) is invariant under the (1,0) and (0,1) transformations,

$$\delta_\epsilon \phi^i = -\frac{i}{2} D_+ \epsilon \phi^i + \epsilon \partial_\pm \phi^i$$

$$\delta_\epsilon \psi_+^i = -\frac{i}{2} D_+ \epsilon \psi_+^i + \epsilon \partial_\pm \psi_-^i \quad (4.8)$$
\[ \delta_\zeta \phi^i = D_+ \zeta \psi^i_+ + m \zeta X^i \]
\[ \delta_\zeta \psi^i_- = -iD_+ \zeta \partial_\zeta \phi^i + m \zeta \partial_j X^i \psi^j_+ , \quad (4.9) \]
respectively \[1\].

The extended \((p,0)\) transformations take the form
\[ \delta_\eta \phi^i = i \eta^j \Gamma_{jkl} \phi^k \]
\[ \Delta_\eta \psi^i_+ = \frac{1}{2} \eta^j \tilde{I}_{rj} \phi^i S^j + \frac{im}{2} \eta^j t^i_r \phi \]
(4.10)
where \(S^i = 0\) is the \(\psi^-\) field equation (\(S^i_+ = 0\) being the field equation for \(\phi^i\) and \(\psi^i_\) field equation.

For simplicity we will use freely all the conditions above and those derived in the previous section. The action is therefore invariant and the only commutator that needs to be checked to ensure \((p,1)\) supersymmetry is the \(\zeta-\eta\) one. A calculation yields the following result
\[ [\delta_\eta, \delta_\zeta] \phi^i = -im(\zeta \eta^r) D_+ \phi^k (L_x I^j_r k) + imD_+(\zeta \eta^r) Z^j_r - \frac{1}{2} (\eta^r D_+ \zeta) (I_r - I_r) j S^j \]
\[ [\delta_\eta, \delta_\zeta] \psi^i_- = -[\delta_\eta, \delta_\zeta] \phi^j \Gamma_{jk} \psi^k_+ + imD_+(\zeta \eta^r) V^j_r k \psi^k_+ - \frac{m^2}{2} (\zeta \eta^r) L_x t^i_r \]
\[ + \frac{1}{2} (D_+ \zeta \eta^r) \psi^k_- \nabla^{(+)}_k \tilde{I}^i_r - \frac{m}{2} (\zeta \eta^r) L_x \tilde{I}^i_r \]
\[ - \frac{1}{2} (\eta^r D_+ \zeta) (I_r - I_r) j S^j_- \] (4.11)
where
\[ Z^i_r \equiv \frac{1}{2} (t^i_r + I^j_r (s^j + 2X^j)) \]
\[ V^i_{rj} \equiv -\nabla_i [t_{rj}] . \] (4.12)
The right-hand side of the above commutators is necessarily a symmetry of the action. The terms which vanish with the field equations leave the action invariant, so the remaining terms must do so too. Of these terms, note that the those proportional to the parameter \((\zeta \eta^r)\) have the same form as the \((p,0)\) transformations of (4.10). Here, however, the parameter is not \(\theta\)-independent so we must impose the conditions
\[ L_x I^i_r = 0 \quad L_x t^i_r = 0 . \] (4.13)

Using (3.7) and (3.10) one can re-express \(V_r\) as
\[ V^i_{rj} = \nabla^{(+)}_j Z^i_r . \] (4.14)
The transformations appearing on the right-hand-side of (4.11) that survive on-shell may now be rewritten as
\[ \delta_\lambda \phi^i = m \lambda^r Z^i_r \]
\[ \delta_\lambda \psi^i_+ = m \lambda^r \partial_j Z^i_r \psi^j_+ \] (4.15)
where \(\lambda^r = i D_+ (\zeta) \eta^r\). Observe that this takes the same form as the transformation with parameter \(\zeta\), generated by the Killing vector \(X\), in (4.9). As just remarked, these transformations are necessarily symmetries of the action and this implies that \(Z_r\) are Killing vector
fields (which is also easily seen from the calculation leading to (4.11)) and leave invariant the sigma-model couplings $H_{ijk}$ and $s$.

We now turn to the calculation of the commutators of the transformations (4.15) among themselves (for $p = 4$) and with the supersymmetry transformations. The commutators of (4.15) with the $(1, 0)$ supersymmetry transformations vanish. The commutators of (4.15) with the extended $(p, 0)$ supersymmetry transformations are

$$[\delta_\lambda, \delta_\eta] \phi^i = i m \eta^r \lambda^s D_+ \phi^j \left( L_{Z_i} I^i_j \right)$$

$$[\delta_\lambda, \delta_\eta] \psi_+^i = \frac{1}{2} m \eta^r \lambda^s \left( L_{Z_i} \hat{I}^i_j \right) S^j + i \frac{1}{2} m^2 \eta^r \lambda^s [Z_s, t_r]^i - [\delta_\lambda, \delta_\eta] \phi^j \Gamma^{(-)}_{jk} \psi_-^k .$$

(4.16)

The commutators of (4.15) with the $(0, 1)$ supersymmetry transformations vanish as a consequence of (4.13). Finally, the commutators of the transformations (4.15) among themselves are

$$[\delta_\lambda, \delta_\lambda'] \phi^i = m^2 \lambda^r \lambda^s [Z_s, Z_r]^i$$

$$[\delta_\lambda, \delta_\lambda'] \psi_+^i = m^2 \lambda^r \lambda^s \partial_j [Z_s, Z_r]^i \psi_-^j .$$

(4.17)

The transformations appearing on the right-hand side of these commutators are necessarily symmetries of the action. They have the general structure of $(p, 0)$ supersymmetry and (4.15) transformations, respectively. Because of this the weakest condition we can impose is that these new symmetries be linear combinations of the existing supersymmetry and (4.15) transformations, respectively, which will be the case if

$$L_{Z_s} I^i_j = A_{st} I^i_j \quad [Z_s, t_r]^i = A_{sr} t_r^i$$

$$[Z_s, Z_r]^i = B_{sr} Z_r^i .$$

(4.18)

where $A$ and $B$ are structure constants, which are restricted by Jacobi identities.

If $A$ or $B$ is non-zero the supersymmetry algebra is not of the form assumed in the introduction because the additional scalar charges associated with the invariance under (4.15) are then not central. In principle one might wish to consider scalar charges that are not central but we leave the investigation of this point to the future. Hence we shall require, for on-shell closure, that $A = B = 0$, i.e. that

$$L_{Z_s} I^i_j = 0 \quad [Z_s, t_r]^i = 0 \quad [Z_s, Z_r]^i = 0 .$$

(4.19)

We remark that the result stated above for the commutators of $(0, 1)$ supersymmetry with (4.15) implies that the commutators of the transformations generated by $X$ commute with those generated by $Z_r$.

We now have all the conditions required for on-shell closure of $(p, 1)$ transformations, and invariance of the action. Not all of these conditions are independent. For example, (4.13) and (4.19) are easily seen to be consequences of the other conditions if $t_r$ is expressed in terms of the central charge generator $Z_r$ by the first of eqs. (4.12). Furthermore, substituting the result for $t_r$ into (3.10) we deduce that

$$2\nabla_i^{(-)} Z^k_r - (\nabla_i^{(-)} I^k_r) (s^i + 2X^j) - I^k_r (\nabla_i^{(-)} s^j + 2\nabla_i^{(-)} X^j) - I^j_r \nabla_j^{(-)} s^k = 0 .$$

(4.20)
Using freely all the conditions derived previously, one finds after some computation that this equation is equivalent to
\[
2\nabla_{[i}^{(-)} Z_{r]j} - 2(u + X)^{l}H_{ijk}I_{r}^{k}l + I_{r}^{k}j \nabla_{k}^{(-)} u_{i} - I_{r}^{k}i \nabla_{k}^{(-)} u_{j} = 0 . \tag{4.21}
\]
This is in turn equivalent to
\[
(Z_{r} + v_{r})_{i} + I_{r}^{k}i(X + u)_{k} = 0 \tag{4.22}
\]
where \( v_{r} \) is defined locally by
\[
(Z_{r})^{k}H_{kij} = \partial_{[i}(v_{r})_{j]} . \tag{4.23}
\]
Actually, (4.21) implies only that the one-form defined by the left-hand side of (4.22) is closed, so that it can be written locally as the derivative of a scalar. One then arrives at (4.22) by absorbing this scalar into the definition of \( v_{r} \).

Note that, by using (4.22) to eliminate \( Z_{r} \) in the first of eqs. (4.12), \( t_{r} \) can be expressed as
\[
t_{r} = (Z - v)_{r} . \tag{4.24}
\]
Using (3.12) (or its cyclic permutations) we now see that the potential \( V \) can be expressed in terms of \( Z_{r} \) and \( v_{r} \) (for each value of \( r \)) in the same way as it was expressed in terms of \( X \) and \( u \) in (4.7), i.e.
\[
V = \frac{m^{2}}{4}g^{ij}(v_{r} - Z_{r})_{i}(v_{r} - Z_{r})_{j} \quad (r = 1, \ldots, p - 1) . \tag{4.25}
\]
Furthermore, the condition (3.9) can now be re-expressed as
\[
u \cdot v_{r} + X \cdot Z_{r} = \text{const.} \tag{4.26}
\]
but this can be shown to be a consequence of the other conditions including, in particular, (4.22). We shall not present the details of the calculation here since a very similar calculation will be described in the next section for the (1,q) models. Analogous calculations lead also to the relations
\[
v_{r} \cdot v_{s} + Z_{r} \cdot Z_{s} = \text{const.} \quad (r \neq s) \tag{4.27}
\]
A consequence of the relations (4.26) and (4.27) is that the potential \( V \) can be rewritten as
\[
V = \frac{m^{2}}{4|c|^{2}}g^{ij}(c \cdot [X - u])_{i}(c \cdot [X - u])_{j} \tag{4.28}
\]
where \([X - u]\) is a p-vector in the space of central charges with components \((X - u, Z_{r} - v_{r})\) and \( c \) is any constant p-vector in this space. Despite the explicit appearance in the potential of \( c \), the potential is \( O(p) \) invariant because it is actually independent of the choice of \( c \) (up to a constant).

This result agrees with that obtained in [1] for (1,1) supersymmetry on the assumption of off-shell closure of the supersymmetry algebra. Off-shell closure requires \( \hat{I}_{r} = I_{r} \), but here,
in contrast to the (p,0) case, this places no conditions on the action that were not already required for on-shell closure, so we may choose \( I_r = I_r \) without loss of generality.

5. (1,q) supersymmetry

We now investigate the conditions under which the action (4.4) has (1,q) supersymmetry. The results are of course equivalent to those just obtained for (p,1), again assuming that any scalar charges appearing in the supersymmetry commutators are central. By subsequently combining the (1,q) with the (p,1) results we shall be able to discuss the remaining (p,q) models.

On dimensional grounds the extended (0,q) supersymmetry transformations can be written, in terms of (1,0) superfields, as

\[
\delta \kappa \phi^i = D\kappa J^i_{\ j} \psi^j_\perp + m\kappa Y^i_r(\phi) \\
\Delta \kappa \psi^i_\perp = -iD\kappa J^i_{\ j} \partial_\perp \phi^j + D\kappa L^i_{\ jk} \phi^j \psi^k_\perp + m\kappa W^i_{\ j} \psi^j_\perp
\]

where \( J, \dot{J}, L, Y \) and \( W \) are tensors on \( \mathcal{M} \), and \( \kappa^r(\theta) \) are \( \theta \)-dependent but \( x \)-independent (1,0) superfield parameters.

Using freely the conditions obtained previously from (1,1) supersymmetry, the following conditions result from requiring closure of the algebra of the (0,q) extended supersymmetry transformations (5.1) when \( m = 0 \):

\[
\dot{J}_r = -J_r ,
\]

\[
L^i_{\ rjk} = \nabla^{(-)}_{\ jk} J^i_{\ rj} + H_{jk} m J^i_{\ rm} + 2H_{[i} J^l_{\ jk]} \]

\[
J_r J_s = -\delta_{rs} + f_{rst} J^t ,
\]

and the Nijenhuis conditions

\[
N(J_r, J_s) = 0 .
\]

For \( m \neq 0 \) we find the following additional conditions:

\[
W^i_{\ rj} = \nabla^{(+)}_{\ j} Y^i_r \quad [X, Y_r] = 0 \quad [Y_r, Y_s] = 0
\]

and

\[
\mathcal{L}_X J_r = 0 \quad \mathcal{L}_Y J_r = 0 .
\]

We have still to check closure of the algebra of the (0,q) with the (1,0) supersymmetry transformations, but this we leave for the moment.

The invariance of the action under the (0,q) supersymmetry transformations imposes various new conditions. Simplifying these with the aid of those derived above, we find (after some computation) the following independent additional conditions. Firstly, for \( m = 0 \),

\[
J_r (ij) = 0 \quad \nabla^{(-)}_i J^j_{\ r} = 0 .
\]

For \( m \neq 0 \) the new independent conditions are

\[
\nabla_{(i} Y_{r\ j)} = 0 \quad W_{rijd} = J^k_{\ r} J^l_{\ rj} W_{rkl}
\]
i.e. that the vector fields $Y_r$ are Killing and the tensor $W_r$ is (1,1) with respect to the complex structure $J_r$. We further find that the Lie derivative of the torsion with respect to $Y_r$ must vanish, a condition that is (locally) equivalent to

$$Y_r^i H_{ijk} = \partial_j[w_r k], \quad (5.9)$$

which defines $w_r$ up to the gradient of a locally defined function. Also, the section $s$ satisfies

$$J_{rs}^{ij} s^j = (w_r - Y_r)^i \quad (5.10)$$

and (no summation over the index $r$)

$$\partial_i(Y_r^j w_r j) = 0 \quad (r = 1, \ldots, p - 1). \quad (5.11)$$

We next compute the commutator of the $(0,q)$ with the $(1,0)$ supersymmetry transformations. Taking into account that the parameters are superfields and hence $\theta$-dependent, we find that

$$[\delta_\kappa, \delta_\epsilon] \phi^i = -i 2m D_+ \epsilon = D_+ \kappa^i Y_r^i$$

$$[\delta_\kappa, \delta_\epsilon] \psi^i_- = -i 2m D_+ \epsilon = D_+ \kappa^i W_r^{ij} \psi^j_- - [\delta_\kappa, \delta_\epsilon] \phi^i k j \psi^j_- \quad (5.12)$$

i.e. that when $m \neq 0$ there are $p - 1$ possible additional central charges associated with the Killing vector fields $Y_r$. Clearly the $Y_r$ are the analogues of the Killing vector fields $Z_r$ found in the previous section. Finally, it can be verified that the transformations generated by the vector fields $Y_r$ are indeed those of central charges, without the need for imposing any further conditions.

We turn now to the form of the potential $V$. Following roughly the same steps as for the $(p,1)$ case one can show that the potential $V$ can be expressed in terms of $Y_r$ and $w_r$ (for each value of $r$) in the same way as it was expressed in terms of $X$ and $u$ in (4.7), i.e.

$$V = \frac{m^2}{4} g^{ij}(w_r - Y_r)^i(w_r - Y_r)^j \quad (r = 1, \ldots, p - 1). \quad (5.13)$$

Note now that by contracting (5.10) with the vector $(u - X)$ one has

$$(u \cdot w_r + X \cdot Y_r) = (u \cdot Y_r + X \cdot w_r). \quad (5.14)$$

But the right-hand-side of (5.14) is constant. This can be seen as follows:

$$d(u \cdot Y_r + X \cdot w_r) = d\iota_{Y_r} u + d\iota_X w_r$$

$$= - (\iota_{Y_r} du + \iota_X dw_r)$$

$$= - (\iota_{Y_r} du + \iota_X dw_r) \equiv 0. \quad (5.15)$$

One can show by a similar argument that

$$w_r \cdot w_s + Y_r \cdot Y_s = \text{const.} \quad (r \neq s) \quad (5.16)$$
Following the same reasoning as in the (p,1) case the potential $V$ can be expressed in the form

$$V = \frac{m^2}{4|c|^2} g^{ij} (c \cdot [X - u])_i (c \cdot [X - u])_j$$  \hspace{1cm} (5.17)$$

where $[X - u]$ is a now a q-vector in the space of central charges with components $(X - u, (Y_r - w_r))$ and $c$ is again any constant p-vector in this space. As expected the final results for (1,q) are equivalent to those found previously for (p,1) supersymmetry.

6. (p,q) Supersymmetry

We are now in a position to investigate the conditions under which the action (4.4) has (p,q) supersymmetry for p and q both greater than 1. The conditions for invariance of the action are just those obtained previously for either (p,1) or (1,q) supersymmetry. The only additional requirement for (p,q) supersymmetry is the closure of the algebra of the extended (p,0) with the extended (0,q) transformations. Since these correspond to supersymmetry charges of opposite chirality we expect additional central charges. Using freely conditions previously derived, a calculation of the commutator on $\phi$ leads to the following result:

$$[\delta_\eta, \delta_\kappa] \phi^i = \{mD_+ (\kappa_s \eta^-) Z_{sr} - i m \kappa_s \eta^- D_+ \phi^k (\mathcal{L}_{Y_s} I_r^i) + \frac{1}{2} D_+ (\kappa_s \eta^-)(J_s \hat{I}_r - I_r J_s)^i_j S^j \}$$  \hspace{1cm} (6.1)$$

where

$$Z_{sr} \equiv \frac{1}{2} (J_s i j I_r^i + I_r j^i (2Y_s^j + J_s^i k s^k))$$  \hspace{1cm} (6.2)$$

A similar calculation for $\psi$ leads to

$$[\delta_\eta, \delta_\kappa] \psi_+^i = i mD_+ (\kappa_s \eta^-) V_{sr}^i j \psi_+^j - [\delta_\eta, \delta_\kappa] \phi^k \Gamma^{(+)}_{k j} \psi_+^j$$

$$+ \frac{1}{2} D_+ (\kappa_s \eta^-) (J_s I_r - \hat{I}_r J_s)^i_j S^j - \frac{1}{2} m \kappa \eta^- (\mathcal{L}_{Y_s} \hat{I}_r^i j) S^j$$

$$+ D_+ (\kappa_s \eta^-) \psi_+^i \{ \frac{1}{2} J_s k m \nabla^k \nabla^m \hat{I}_r^i + 2 \hat{I}_r^i j L_s m k + 2 \hat{I}_r^i j H^l q k J_s q m \} S^l$$

$$- \frac{i}{2} m \eta^- \kappa \eta [Y_s, t_r]^i$$  \hspace{1cm} (6.3)$$

where

$$V_{sr}^i j \equiv - [L_s i j k t_r^k + \frac{1}{2} (J_s I_r)^{ik} \nabla^m (-)_{m s j} - \frac{1}{2} J_s ^m j \nabla^m \hat{I}_r^i] .$$  \hspace{1cm} (6.4)$$

The terms that vanish with the field equations are (trivially) symmetries of the action so the remaining terms must also be symmetries. Of these, the terms with coefficient $\kappa_s \eta^- \eta$ have the form of an $\eta$-supersymmetry, but the $\theta$-dependence of this coefficient precludes this identification and we must set

$$\mathcal{L}_{Y_s} I_r = 0 \quad [Y_s, t_r] = 0 .$$  \hspace{1cm} (6.5)$$

The remaining (on-shell) transformations may now be identified as those of new central charge symmetries. In particular, it follows that the vector fields $Z_{sr}$ are Killing, that the
torsion and the section $s$ are invariant with respect to these symmetries, and that $V_{sr}^i j = \nabla_j^{(+)} Z_{sr}^i$, exactly as for the previous central charges. The new central charge transformations may be simplified to
\[
\delta \lambda \phi^i = m \lambda^{sr} Z_{sr}^i,
\]
\[
\delta \lambda \psi_\pm^i = m \lambda^{sr} \partial_j Z_{sr}^i \psi_\pm^j.
\]
(6.6)
where $\lambda_{sr}$ is a constant parameter. Finally, the commutators of the new central charge transformations with themselves and with all other central charge and supersymmetry transformations will vanish, as required, provided that all commutators of the Killing vector fields $Z_{sr}$ with themselves and with the other Killing vector fields vanish, and provided that the Lie derivatives of the complex structures $I_r$ and $J_s$ with respect to $Z_{sr}$ vanish.

As in the $(p,1)$ case we may now use (6.2) and (3.10) to eliminate $t_r$. Following the same steps as those described in previous sections, one finds that
\[
(Z_{sr} + v_{sr})_i + I_r^k i (Y_s + w_s)_k = 0,
\]
(6.7)
where $v_{sr}$ is locally defined by $\iota_{Z_{sr}} H = dv_{sr}$. Using this result in (6.2) we now find that
\[
J_s^i j t_r^j = -(v_{sr} - Z_{sr})^i.
\]
(6.8)
This implies that $V$ can be written in $pq$ different ways as
\[
V = \frac{m^2}{4} g^{ij} (v_{sr} - Z_{sr})_i (v_{sr} - Z_{sr})_j \left( r = 1, \ldots, p - 1 \right) \left( s = 1, \ldots, q - 1 \right).
\]
(6.9)
In analogy with (5.17), the potential can now be rewritten, up to a constant, as
\[
V = \frac{m^2}{4 |c_s|^2} g^{ij} \left( c_s \cdot [Y_s - w_s] \right)_i \left( c_s \cdot [Y_s - w_s] \right)_j \left( s = 1, \ldots, q - 1 \right).
\]
(6.10)
where $[Y_s - w_s]$ is a $p$-vector (for each value of $s$) with components $(Y_s - w_s, Z_{sr} - v_{sr})$, and $c_s$ is a $p$-vector (for each value of $s$) with constant components. To establish this result one notices that the cross terms sum to a constant as a result of (6.7) while the diagonal terms are all equal. The details follow the same lines as in the previous two sections.

Similarly, from (4.24) and (6.8) one can deduce that the potential can also be written as
\[
V = \frac{m^2}{4 |c_r|^2} g^{ij} \left( c_r \cdot [Z_r - v_r] \right)_i \left( c_r \cdot [Z_r - v_r] \right)_j \left( r = 1, \ldots, p - 1 \right)
\]
(6.11)
where $[Z_r - v_r]$ is a $q$-vector (for each value of $r$) with components $(Z_r - v_r, Z_{sr} - v_{sr})$, and $c_r$ is a $q$-vector (for each value of $r$) with constant components.

This concludes our discussion of the conditions required for on-shell $(p,q)$ supersymmetry. In subsequent sections we shall consider their implications for certain interesting special cases. We turn now to the conditions required for off-shell closure of the supersymmetry algebra. From our discussion of the conditions for off-shell $(p,1)$ supersymmetry in section 4, we know
that this requires us to set \( \hat{I}_r = I_r \). From (6.1) we deduce that the additional requirement for off-shell \((p,q)\) supersymmetry is that, in matrix notation,

\[
[I_s, J_r] = 0 \tag{6.12}
\]

In fact, this condition is sufficient to show that all field-equation terms in the commutator (6.3), on \( \psi \), vanish. Thus (6.12) is the only additional condition for off-shell closure of the extended \((p,0)\) and \((0,q)\) supersymmetries.

7. Potentials for \((2,2)\) and \((4,4)\) models

The supersymmetry algebra of a massive sigma-model with \((p,q)\) supersymmetry has a possible \(pq\) central charges, \(Z_{II}'\) in the notation of the introduction. These correspond to the Killing vector fields \(X, Y_r, Z_s,\) and \(Z_{sr}\). We have seen previously that the potential \(V\) is expressed in terms of these Killing vector fields and the associated one-forms \(u, w_r, v_s\) and \(v_{sr}\), respectively. We have seen previously (eqs. (4.22), (5.12), (6.7) and (6.8)) that these quantities are constrained by the relations

\[
(Z_r + v_r)_i + I_r^k i (X + u)_k = 0 \\
(Y_s - w_s)_i + J_s^k i (X - u)_k = 0 \\
(Z_{sr} + v_{sr})_i + I_r^k i (Y_s + w_s)_k = 0 \\
(Z_{sr} + v_{sr})_i + J_s^k i (Z_r - v_r)_k = 0 \tag{7.1}
\]

where \(r = (1, \ldots, p-1)\) and \(s = (1, \ldots, q-1)\). To complete the determination of the general form of the potential we must therefore solve these relations for, e.g., \(X\) and \(u\). This task is greatly simplified by the observation that the massless model has an \(SO(p) \times SO(q)\) symmetry which translates into an \(SO(p) \times SO(q)\) isometry group of the supersymmetry algebra. By means of such an \(SO(p) \times SO(q)\) transformation the number of non-zero central charges of the massive model can be reduced. For example, if \(p = q\) a basis of the supersymmetry charges may always be found for which \(Z_{II}'\) is diagonal, i.e. the only non-zero central charges are those generated by \(X\) and \(T_r \equiv Z_{rr}\). This observation would not be so useful if the potential \(V\) were not also invariant under \(SO(p) \times SO(q)\) because the form of the potential in a special basis would then be a special form and our intention is to find the general form. Fortunately, we showed in the previous section that the potential \(V\) is also \(SO(p) \times SO(q)\) invariant. Hence no generality is lost if, for the \(p=q\) models with torsion, we set

\[
Y_r = Z_r = 0 \quad Z_{rs} = 0 \quad (r \neq s) \tag{7.2}
\]

so that the only non-zero Killing vector fields are \(X\) and \(T_r \equiv Z_{rr}\). In this case,

\[
w_r = dc_r \quad v_s = db_s \quad v_{sr} = de_{sr} \quad (r \neq s) \tag{7.3}
\]

for locally-defined scalar functions \(c_r, b_s\) and \(e_{sr}\). Substituting (7.2) and (7.3) into (7.1) we obtain the new relations

\[
\partial_i b_r + I_r^k i (X + u)_k = 0 \\
\partial_i c_s - J_s^k i (X - u)_k = 0 \tag{7.4}
\]
\[(T_r + n_r)_i + I_{r}^{k}i \partial_k c_r = 0 \]
\[(T_s - n_s)_i - J_{s}^{k}i \partial_k b_s = 0 \]  \hspace{1cm} (7.5)

where \( n_r \equiv v_{rr} \), and

\[
\partial_i e_{sr} + I_{r}^{k}i \partial_k e_s = 0 \ (r \neq s) \\
\partial_i e_{sr} + J_{s}^{k}i \partial_k b_r = 0 \ (r \neq s) .
\]  \hspace{1cm} (7.6)

Using (7.4), the eqs. (7.5) can be solved for \( T_r \) and \( n_r \) in terms of \( X \) and \( u \) as follows

\[
(T_r)_i = \frac{1}{2}\{I_r, J_r\}^k_i X_k + \frac{1}{2}[I_r, J_r]^k_i u_k
\]
\[
(n_r)_i = -\frac{1}{2}\{I_r, J_r\}^k_i u_k - \frac{1}{2}[I_r, J_r]^k_i X_k .
\]  \hspace{1cm} (7.7)

Eliminating the functions \( e_{sr} \) from (7.6) and then using (7.4) we find the following constraint on \( X \) and \( u \):

\[
(u + X)_i = (J_s I_r J_s)^k_i (u - X)_k \ (r \neq s) .
\]  \hspace{1cm} (7.8)

Since the potential \( V \) can be expressed entirely in terms of \( X \) and \( u \), the relevant relations are those of (7.4) and (7.8). We now turn to a discussion of the consequences of these relations for the (2,2) and (4,4) models.

For (2,2) models (7.8) does not apply since, necessarily, \( r = s = 1 \) and (7.4) is equivalent to

\[
(X + u)_i = I_{r}^{k}i \partial_k b \\
(X - u)_i = -J_{r}^{k}i \partial_k c .
\]  \hspace{1cm} (7.9)

These equations generalize the expression in [8] for a holomorphic Killing vector field in terms of a Killing potential to the case of non-zero torsion. Note that \( V \) is expressed in terms of the \((X - u)\) so the potential is the square of the derivative of \( c \).

Consider now the special case of zero torsion and \( I = J \) discussed in [8]. Solving (7.4) for \( u \) and \( X \) we get

\[
X_i = -I_{r}^{k}i \partial_k \left(\frac{c - b}{2}\right) \quad u_i = I_{r}^{k}i \partial_k \left(\frac{c + b}{2}\right) .
\]  \hspace{1cm} (7.10)

From the first of these equations we identify \((c - b)/2\) as the Killing potential of the holomorphic Killing vector field \( X \). Moreover, since the torsion vanishes, \( u = da \) for some locally defined scalar function \( a \). Thus the second of the equations (7.9) implies that \( a \) is the real part of a locally defined holomorphic function (which is the superpotential in the superspace formulation). Note further that in this special case \( T = -X \). These results agree with those of [8] but the potential given there was expressed in terms of two commuting Killing vector fields. However, as we have seen, the general scalar potential can be written (up to a constant) in terms of a single holomorphic Killing vector field (which must be a linear combination of those considered in [8]).

We now turn to some special classes of (4,4) models. For (4,4) models eq. (7.8) is applicable and has important consequences for the potential \( V \). First consider the zero torsion case, for which \( u = da \), and \( I_r = J_r \). In this case, (7.8) implies that \( da = 0 \), and the
relation (7.4) allows the identification of the functions \((c_r - b_r)/2\) with the Killing potentials of the tri-holomorphic Killing vector field \(X\). Note also that \(T_r = -X\). We conclude that the general scalar potential \(V\) for (4,4) models with zero torsion and \(I_r = J_r\) is given by the length of a tri-holomorphic Killing vector field. This is consistent with the results of [6] where such a model was constructed\(^*\).

Another class of (4,4) models are those for which the supersymmetry algebra closes off-shell, i.e. \([I_r, J_s] = 0\). Note that this is not possible when \(I_r = J_r\). In this case (7.8) and (7.5) imply that \(X = 0\) and \(T_r = 0\). Because \(X = 0\), \(u = da\). The (4,4) analogue of (7.9) now implies that \(a\) can be written in three different ways as the real part of a holomorphic function with respect to each of the complex structures. Because there are no central charges in this case, the (4,4)-superfield formalism of [4] applies, so there exists an off-shell (4,4) superfield action.

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\(^*\) We have not understand the results of the authors of [8] on (4,4) models well enough to know whether their results agree with ours.