A MAGNETIC MODULAR FORM

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Abstract. In this paper, we prove a conjecture of Broadhurst and Zudilin concerning a divisibility property of the Fourier coefficients of a meromorphic modular form using the generalization of the Shimura lift by Borcherds and Hecke operators on vector-valued modular forms developed by Bruinier and Stein. Furthermore, we construct a family of meromorphic modular forms with this property.

1. Introduction.

Let $f$ be a weakly holomorphic modular form with the Fourier expansion $\sum_{n \gg -\infty} a(n) q^n$, where $q := e(z) := e^{2\pi i z}$ for $z$ in the upper-half complex plane $\mathcal{H}$. Suppose $f$ has weight 2, level 1 and integral Fourier coefficients. Then it possesses the following divisibility property

$$n \mid a(n) \text{ for all } n \in \mathbb{N}. \quad (1.0.1)$$

Indeed, it is the image of a polynomial in the Klein $j$-invariant with integral coefficients under the differential operator $q \frac{d}{dq}$. More generally, one can apply this derivative $k - 1$ times to weakly holomorphic modular forms of weight $2 - k$ to produce forms of weight $k$ with this property (see e.g. [14]). The same phenomenon in the half-integral weight case has also been studied [10][13][24].

In [2], Ausserlechner studied how the output voltage of a Hall plate is affected by the shape of the plates and sizes of the contacts, where he encountered a double integral generalizing the classical elliptic integral used to evaluate the arithmetic geometric mean. In [4], Broadhurst and Zudilin studied this integral $I_2(f)$ in detail and showed that it satisfies

$$I_2(f) = I_2 \left( \frac{1 - f}{1 + f} \right) \text{ for } f \in [0, 1],$$

which was conjectured by Ausserlechner and also proved by Glasser and Zhou [11]. After applying a modular parametrization, they also showed that $I_2(f)$ satisfies an inhomogeneous
differential equation, whose constant term is the following modular form
\[
\phi(z) := (\eta(2z)\eta(4z))^4 \frac{1 - 96\psi(2z) + 256\psi(2z)^2}{(1 + 16\psi(2z))^2}
\]
\[
= \sum_{n \in 2\mathbb{N} - 1} a(n)q^n = q - 132q_2^3 + 5630q_5^5 - 189672q_7^7 + 5768181q_9^9 + O(q^{10}).
\]
(1.0.2)

Here,
\[
\psi(z) := \eta(z)^8 \eta(4z)^{16} \eta(2z)^{24}
\]
is a Hauptmodul on the modular curve \(X_0(4)\). The function \(\phi\) is a meromorphic modular form of weight 4 on \(\Gamma_0(8)\) and has double poles at \(z_\pm := \frac{1 \pm i}{4}\). The numbers \(a(n)\) are a priori integers. From numerical computations, Broadhurst and Zudilin made the following conjecture.

**Conjecture 1.1** (Conjecture 1 of [4]). The meromorphic modular form \(\phi(z)\) satisfies (1.0.1).

On the one hand, the conjecture above seems like a natural and expected extension of (1.0.1) for weakly holomorphic forms. On the other hand, \(p^3 \nmid a(p)\) for every odd prime \(p\) (see (1.0.9)). Therefore, unlike elements of the bases in [9], \(\phi\) does not come from applying the operator \(\left(\frac{d}{dq}\right)^3\) to a modular form of weight \(-2\), which makes this conjecture surprising.

In this note, we will prove this conjecture by first realizing \(\phi(z)\) as the regularized theta lift of a half-integral weight, vector-valued modular form. In [18], Shimura initiated the study of modular forms of half-integral weight, and showed they correspond to modular forms of integral weight. This is called the Shimura lift. Using techniques introduced by Shintani [19], Niwa expressed the Shimura lift of a holomorphic, half-integral weight modular form as its integral against a suitable theta kernel [17]. In [3], Borcherds expanded the input space to include vector-valued modular forms on the metaplectic cover \(M_{p_2}(\mathbb{Z})\) of \(SL_2(\mathbb{Z})\) with singularities at the cusps. This will be the setting that we work in.

To describe the vector-valued input, we need the following 3-dimensional representation \(\varrho : M_{p_2}(\mathbb{Z}) \to GL_3(\mathbb{C})\)
\[
\varrho(T) := \begin{pmatrix} \zeta_8 & \zeta_8^7 & \zeta_8^5 \\ \zeta_8 & \zeta_8 & \zeta_8^5 \\ \zeta_8^5 & \zeta_8 & \zeta_8 \end{pmatrix}, \quad \varrho(S) := \frac{-i}{2} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix},
\]
(1.0.4)

where \(\zeta_8 := e^{2\pi i/8}\) is an 8th root of unity, and \(T, S\) are the generators of \(M_{p_2}(\mathbb{Z})\) (see (2.1.1)). It is isomorphic to a subrepresentation of the 64 dimensional Weil representation \(\rho_L\) associated to a particular lattice \(L\) (see section 2.3). Let \(\{e_{\ell} : \ell = 0, 1, 2\}\) be the standard basis of \(\mathbb{C}^3\). Note that if we scale \(e_1\) by \(\sqrt{2}\), then the representation \(\varrho\) becomes unitary with respect to the standard inner product on \(\mathbb{C}^3\). If \(q := e(\tau)\) with \(\tau \in \mathcal{H}\), then the action of \(\varrho(T)\) implies that any modular form \(G\) in \(M'_{5/2,\varrho}\) (see section 2.1 for notation) has a Fourier expansion in
and the coefficient of $q^{m/8}$ is zero if $m \not\equiv 1, 5$ or $7$ mod 8. For any $m \in \mathbb{Z}$ congruent to 1, 7 or 5 modulo 8, let $\ell(m) \in \{0, 1, 2\}$ be the unique index satisfying $1 - m \equiv 2\ell(m) \bmod 8$. For any modular form $G \in M_{5/2,\rho}^1$, we can write out its Fourier expansion as
\begin{equation}
G(\tau) = \sum_{m \in \mathbb{Z}} c(G, m)q^m, \quad q^m := q^{m/8}e_{\ell(m)}
\end{equation}
and define the formal power series
\begin{equation}
\Phi(z, G) := \sum_{n \in 2\mathbb{N} - 1} (-1)^{(n-1)/2}q^n \sum_{r|n} r \cdot c(G, (n/r)^2),
\end{equation}
which converges absolutely for $y := \text{Im}(z)$ sufficiently large, and analytically continues to a meromorphic function in $z \in \mathcal{H}$, which we also denote by $\Phi(z, G)$. Our first result is as follows.

**Theorem 1.2.** In the notation above, the function $\Phi(z, G)$ is in $M_{4,\chi}^\text{mero}(\Gamma_0^+(8))$, where $\Gamma_0^+(8) \subset \text{SL}_2(\mathbb{R})$ is the extension of $\Gamma_0(8)$ by the matrices $R, U, W$ defined in (2.3.8) and $\chi$ is the character in (2.4.1). In particular, there exists a unique $G_1 \in M_{5/2,\rho}^1$ with $-4\phi(z) = \Phi(z, G_1)$ (see (3.1.10)).

**Remark 1.3.** Using SageMath [20], it is easy to check that $M_4(\Gamma_0(8))$ is 5-dimensional, with a $W$-eigenbasis consisting of 4 Eisenstein series and 1 cusp form. The Eisenstein series are $E_4(z) \pm 64E_4(8z)$ and $E_4(2z) \pm 4E_4(4z)$ with $E_4 \in M_4$ the Eisenstein series of weight 4. Among the eigenbasis, only $E_4(z) - 64E_4(8z)$ and $E_4(2z) - 4E_4(4z)$ have $W$-eigenvalue $-1$. From their Fourier expansions, it is clear that the only linear combination with $R$-eigenvalue $-1$ is 0. Therefore, $M_{4,\chi}(\Gamma_0^+(8))$ is trivial.

To prove the conjecture, it is necessary to study the integral structure of $M_{5/2,\rho}^1$. In section 2.4 we will see that $\rho$ can be realized as a rational subrepresentation of a Weil representation $\rho_L$ attached to the lattice $L$ in section 2.3. Therefore, the $\mathbb{Z}$-module $\mathbb{M}_{5/2,\rho}^1$ of modular forms in $M_{5/2,\rho}^1$ with integral Fourier coefficients is free and a complete lattice in $M_{5/2,\rho}^1$ by a theorem of McGraw [16] (see section 2.1). Through explicit construction in section 3.1 and the theory of Hecke operators on vector-valued modular forms developed in [7], we will prove the following result.

**Theorem 1.4.** The free $\mathbb{Z}$-module $\mathbb{M}_{5/2,\rho}^1$ has a canonical basis $\{G_d : d \in \mathbb{N}, d \equiv 1, 3, 7 \bmod 8\}$ characterized by the property
\begin{equation}
G_d(\tau) = q^{-d/8}e_{\ell(-d)} + O(q^{1/8}).
\end{equation}
Suppose $G_d$ has Fourier coefficients $c(G_d, m) \in \mathbb{Z}$ as in (1.0.5). Then
\begin{equation}
c(G_d, m^2) \in m\mathbb{Z}
\end{equation}
for all $m \in \mathbb{N}$ and square-free $d \in \mathbb{N}$.
From the definition of the map $\Phi$, it is clear that Theorems 1.2 and 1.4 together imply Conjecture 1.1, i.e., $n|a(n)$. In fact the Hecke theory allows us to study the coefficients $a(n)$ modulo $n^3$. For instance Corollary 3.6 implies that
\begin{equation}
(1.0.9)
p^3 \mid (a(p) - p)
\end{equation}
for every odd prime $p$. Furthermore, this puts $\phi_1(z) = -4\phi(z)$ into a family of modular forms $\{\phi_d(z) := \Phi(z, G_d) : d \equiv 1, 3, 7 \mod 8 \text{ square-free}\}$ in $\mathbb{M}_{3, \chi}^\text{mero}(\Gamma_0^+(8))$ that all satisfy the divisibility property (1.0.1). It is worth mentioning that the phenomenon of Zagier duality [22,24] is also present between this basis and the canonical basis of the free $\mathbb{Z}$-module $\mathbb{M}_{-1/2, \varphi^*}$, where $\varphi^*$ is the unitary dual of $\varphi$ with respect to the standard inner product on $\mathbb{C}^3$ (see Prop. 3.1).

Finally, we can apply the same idea to Example 14.4 in [3] to prove the following result.

**Theorem 1.5.** Let $\Delta(z)$ be Ramanujan’s delta function, and $E_4(z) = 1 + 240q + O(q^2)$ the Eisenstein series of weight 4. Then the meromorphic modular form $64 \frac{\Delta(z)}{E_4(z)}$ satisfies (1.0.1).

In fact numerically, also $\Delta(z)/E_4(z)^2$ satisfies (1.0.1). It is ongoing work of the authors to remove the factor 64 from the above theorem.

**Question 1.6.** Can a non-constant holomorphic modular form have the divisibility property (1.0.1)? It seems likely that even the weaker condition, that every sufficiently large prime divides the corresponding Fourier coefficient, cannot be satisfied by a holomorphic modular form.

This note is organized as follows. In section 2 we give the preliminaries concerning vector-valued modular forms as input to Borcherds' lift and prove Theorem 1.2 as a special case of Borcherds' result. In section 3 we construct the family $\{G_d\}$, prove Conjecture 1.1 and give some numerical data of the Fourier expansions of the bases.

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2. Preliminaries.

In this section we will introduce the vector-valued theta lift à la Borcherds [3].

2.1. Vector-Valued Modular Forms. Denote $\mathcal{H}^* := \mathcal{H} \cap \mathbb{P}^1(\mathbb{Q})$ the extended upper half plane, which is acted on by $\text{SL}_2(\mathbb{R})$ via linear fractional transformation. Let $\text{Mp}_2(\mathbb{R})$ be the metaplectic cover of $\text{SL}_2(\mathbb{R})$ consisting of pairs $(\gamma, \varphi)$, where $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{R})$ and $\varphi$ is a holomorphic function on $\mathcal{H}$ satisfying $\varphi(\tau)^2 = (ct + d)$. We denote the preimage of $\text{SL}_2(\mathbb{Z})$ in $\text{Mp}_2(\mathbb{R})$ under the covering map by $\text{Mp}_2(\mathbb{Z})$, which is generated by
\begin{equation}
(2.1.1)
T := \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \quad S := \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \sqrt{\tau}
\end{equation}
Here we take the principal branch of the square root.
Let $\Gamma \subset \text{Mp}_2(\mathbb{R})$ be a subgroup commensurable with $\text{Mp}_2(\mathbb{Z})$. A meromorphic function $f : \mathcal{H}^* \to \mathbb{C}$ is called a meromorphic modular form of weight $k \in \frac{1}{2} \mathbb{Z}$ with respect to a unitary representation $\rho : \Gamma \to \text{GL}(W)$ on a finite dimensional $\mathbb{C}$-vector space $W$ if it satisfies
\begin{equation}
( f |_k (\gamma, \varphi)) (\tau) := \varphi(\tau)^{-2k} f(\gamma \cdot \tau) = \rho((\gamma, \varphi)) f
\end{equation}
for all $(\gamma, \varphi) \in \Gamma$. We use $M^\text{mero}_{k, \rho}(\Gamma)$ to denote the vector space of these meromorphic modular forms. It contains the subspaces $M^1_{k, \rho}(\Gamma), M_{k, \rho}(\Gamma), S_{k, \rho}(\Gamma)$ of weakly holomorphic, holomorphic, and cuspidal modular forms.

When $k \in \mathbb{Z}$, it is necessary for $\rho$ to factor through the image of $\Gamma$ in $\text{SL}_2(\mathbb{Z})$, and we replace $\Gamma$ with its image in $\text{SL}_2(\mathbb{Z})$ in the above notation. Also, we drop the subscript $\rho$, resp. $\Gamma$, if it is 1-dimensional and trivial, resp. $\text{Mp}_2(\mathbb{Z})$. For $N \in \mathbb{N}$, we denote by $\Gamma_0^*(N)$ the extension of the congruence subgroup $\Gamma_0(N)$ by Atkin-Lehner operators $[1]$

A common type of representation comes from arithmetic. Let $L$ be an even, integral lattice with quadratic form $Q$. Let $(b^+, b^-)$ be the signature of $L_R := L \otimes_{\mathbb{Z}} \mathbb{R}$ and $L^\vee := \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ be the dual lattice. Through the bilinear form $(\cdot, \cdot)$ induced by $Q$ on $L$, we identify $L^\vee$ with a sublattice of $L_R$ containing $L$. The quotient $A_L := L^\vee / L$ is then a finite abelian group, on which $Q$ becomes a quadratic form valued in $\mathbb{Q}/\mathbb{Z}$. On the vector space $\mathbb{C}[A_L]$ with basis $\{ e_h : h \in A_L \}$, there is a hermitian inner product $\langle \cdot, \cdot \rangle$ defined by
\begin{equation}
\langle v, w \rangle := \sum_{h \in A_L} v_h \overline{w_h}
\end{equation}
for $v = \sum_{h \in A} v_h e_h$ and $w = \sum_{h \in A} w_h e_h$ in $\mathbb{C}[A_L]$, which induces the norm
\begin{equation}
\| v \| := \sqrt{\langle v, v \rangle}
\end{equation}
on $\mathbb{C}[A]$. The group $\text{Mp}_2(\mathbb{Z})$ acts through the Weil representation $\rho_L$ defined by
\begin{equation}
\rho_L(T)e_h = e(Q(h))e_h, \quad \rho_L(S)e_h = \frac{e((b^+ - b^-)/8)}{|A_L|} \sum_{\ell \in A_L} e(-\langle h, \ell \rangle)e_\ell,
\end{equation}
which is unitary with respect to the hermitian inner product in (2.1.3). By a theorem of McGraw [16], the space $M^!_{k, \rho_L}$ has a basis with Fourier coefficients in $\mathbb{Q}$. We will use $M^\text{mero}_{k, \rho_L}$ to denote the $\mathbb{Z}$-module of modular forms in $M^!_{k, \rho_L}$, with integral Fourier coefficients.

One can define the orthogonal group of $A_L$ as the following finite group
\begin{equation}
O(A_L) := \{ \sigma : A_L \to A_L \text{ group automorphism } | Q(\sigma(h)) = Q(h) \text{ for all } h \in A_L \}.
\end{equation}
Every element in $O(A_L)$ induces a $\rho$-linear automorphism on $\mathbb{C}[A_L]$, hence also acts on $M^\text{mero}_{k, \rho_L}$, which decomposes according to the irreducible representations of $O(A_L)$. For each $h \in A_L$, we have the normal subgroup $O(A_L)_h \subset O(A_L)$ consisting of the stabilizers of $h$ in $O(A_L)$. 
2.2. Symmetric Space. Let $V_\mathbb{R} := M_2(\mathbb{R})^0$ be the real vector space of 2 by 2 matrices with trace 0. It becomes a real quadratic space of signature (2, 1) with respect to the quadratic form $Q := -N \cdot \det$ for any natural number $N$. The group $GL_2(\mathbb{R})$ acts isometrically on $V_\mathbb{R}$ via conjugation, which is explicitly given by
\[
\gamma \cdot \begin{pmatrix} B & C \\ -A & -B \end{pmatrix} = \gamma \begin{pmatrix} B & C \\ -A & -B \end{pmatrix} \gamma^{-1}
\]
(2.2.1)
for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$. This identifies $SL_2(\mathbb{R})$ with Spin($V_\mathbb{R}$) and $PSL_2(\mathbb{R})$ with $SO^+(V_\mathbb{R})$, the connected component of the special orthogonal group $SO(V_\mathbb{R})$ containing the identity.

Let $D$ be the symmetric space of oriented negative lines in $V_\mathbb{R}$ and $D^0 \subset D$ the connected component containing $\mathbb{R} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. As usual, we can use $H$ to parametrize $D^0$ by defining
\[
Z(z) := \begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix}
\]
(2.2.2)
for each $z \in H$. Then $\{\text{Re}(Z(z)), \text{Im}(Z(z))\}$ always span a positive definite 2-plane in $V_\mathbb{R}$ and its orthogonal complement is an element of $D^0$. Furthermore,
\[
\gamma \cdot Z(z) = (cz + d)^{-2} Z(\gamma z)
\]
(2.2.3)
for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$.

2.3. Lattice. Let $L \subset M_2(\mathbb{Q})$ be the following lattice
\[
L := \left\{ \begin{pmatrix} B & C/2 \\ -4A & -B \end{pmatrix} : A, B, C \in \mathbb{Z} \right\},
\]
(2.3.1)
which is even integral with respect to the quadratic form $Q = -2 \cdot \det$ with dual lattice
\[
L^\vee := \left\{ \begin{pmatrix} b/4 & c/8 \\ -a & -b/4 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.
\]
(2.3.2)
The real quadratic space $L_\mathbb{R}$ has signature (2, 1). It is not hard to see that $L$ is the direct sum of the following two sublattices $L_1$ and $L_2$ of signatures (1, 1) and (1, 0):
\[
L_1 := \left\{ \begin{pmatrix} 0 & C/2 \\ -4A & 0 \end{pmatrix} \in L \right\}, \quad L_2 := \left\{ \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} \in L \right\}.
\]
(2.3.3)
Furthermore, the dual lattice $L_j^\vee \subset L_{j,\mathbb{R}} \subset L_\mathbb{R}$ is contained in $L^\vee$ for $j = 1, 2$. 
The finite quadratic module $\mathcal{A} := L^\vee / L$ is isometric to $(\mathbb{Z}/4\mathbb{Z})^3$ via the map
\[
\mathcal{A} \cong (\mathbb{Z}/4\mathbb{Z})^3
\]
\[
(b/4 \ c/8 \ -a \ -b/4) \mapsto (a, b, c),
\]
where the quadratic form on $(\mathbb{Z}/4\mathbb{Z})^3$ is $Q((a, b, c)) := \frac{b^2 - 2ac}{8} \in \mathbb{Q}/\mathbb{Z}$. The isotropic elements are
\[
\text{Iso}(\mathcal{A}) = \left\{ (0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 0, 3), (1, 0, 0), (1, 2, 2) \right\}
\]
\[
\left\{ (2, 0, 0), (2, 0, 2), (2, 2, 1), (2, 2, 3), (3, 0, 0), (3, 2, 2) \right\}.
\]
For $j = 1, 2$, let $\mathcal{A}_j$ be the finite quadratic module $L_j^\vee / L_j$ and $\rho_j$ be the Weil representations associated to $L, L_j$ respectively. Then $\mathcal{A}_j \subset \mathcal{A}$ and $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, which induces
\[
\mathbb{C}[\mathcal{A}] = \mathbb{C}[\mathcal{A}_1] \otimes \mathbb{C}[\mathcal{A}_2], \quad \rho = \rho_1 \otimes \rho_2.
\]
We also identify $\mathcal{A}_1$, resp. $\mathcal{A}_2$, with $(\mathbb{Z}/4\mathbb{Z})^2$, resp. $\mathbb{Z}/4\mathbb{Z}$, by sending $(a, 0, c)$, resp. $(0, b, 0)$, to $(a, c) \in (\mathbb{Z}/4\mathbb{Z})^2$, resp. $b \in \mathbb{Z}/4\mathbb{Z}$.

Let $SO(L) \subset SO(V)$ be the special orthogonal group stabilizing the lattice. It also fixes the dual lattice, hence induces an action on the finite quadratic module $\mathcal{A}$. Denote $\Gamma_L \subset SO(L)$ the kernel of this action. In fact, $\Gamma_L$ is contained in $SO^+(L) := SO(L) \cap SO^+(L_\mathbb{R})$ After scaling by $\sqrt{2}$, the lattice $L$ is the same as the lattice $L(8, 4)$ in Section 4 of [23], where $SO^+(L)$ and $\Gamma_L$ were given explicitly in Theorem 4.2 loc. cit. as
\[
SO^+(L) := \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0^*(2) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_L := \Gamma_0(8).
\]
Note that $\Gamma_0^*(2) \subset SL_2(\mathbb{R})$ is obtained from $\Gamma_0(2)$ by adjoining the Fricke-involution $W_2 := \begin{pmatrix} 0 & 1/\sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix}$. Consider the following elements in $SL_2(\mathbb{R})$
\[
R := \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}, \quad U := \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \quad W := \begin{pmatrix} 0 & 1/(2\sqrt{2}) \\ -2\sqrt{2} & 0 \end{pmatrix}.
\]
Then their images in $PSL_2(\mathbb{R})$, along with $\Gamma_L$, generate $SO^+(L)$, which is called the group of Atkin-Lehner operators. This is a reasonable name since if $L$ is the lattice studied in [5], then $\Gamma_L$, resp. $SO^+(L)$, is isomorphic to the image of the congruence subgroup $\Gamma_0(N) \subset SL_2(\mathbb{Z})$, resp. $\Gamma_0^*(N)$, in $PSL_2(\mathbb{R})$. We denote the preimage of $SO^+(L)$ in $SL_2(\mathbb{R})$ by $\Gamma_0^+(8)$, which contains $\Gamma_0(8)$.

2.4. Orthogonal Groups. It is important to understand the finite group $O(\mathcal{A})$, since $\mathbb{C}[\mathcal{A}]$ decomposes according to its irreducible representations. By considering $\mathcal{A}$ as a free $\mathbb{Z}/4\mathbb{Z}$-module of rank 3 through the map \((2.3.4)\), we see that the group $O(\mathcal{A})$ defined in \((2.1.6)\) has
the following form

\[(2.4.1) \quad O(A) = \{g \in \text{GL}_3(\mathbb{Z}/4\mathbb{Z}) : Q(g \cdot h) = Q(h) \text{ for all } h \in (\mathbb{Z}/4\mathbb{Z})^3 \cong A\}.
\]

The same holds for the free \(\mathbb{Z}/4\mathbb{Z}\)-modules \(A_1, A_2\), and we can then define \(\text{SO}(A)\), resp. \(\text{SO}(A_j)\), as the subgroup of \(O(A)\), resp. \(O(A_j)\), consisting of the elements with determinant 1 mod 4. Let \(\nu\) be the negative identity matrix, which is contained in all three orthogonal groups above. It is even in \(\text{SO}(A_1)\) since \(A_1\) has even rank.

There is a natural map \(\text{SO}(L)/\Gamma_L \hookrightarrow \text{SO}(A)\). By checking every element in the finite group \(\text{SO}(A)\), we know that this map is an isomorphism. We abuse the notation slightly by using \(R, U, W\) to represent the images of the elements in \(\text{(2.3.3)}\) under this map. It is now easy to check that \(R, U, W\) all have order 2 and \(U = WRW\). Therefore, the group \(\text{SO}^+(L)/\Gamma_L\) is the wreath product of \(\mathbb{Z}/2\mathbb{Z}\) by \(\mathbb{Z}/2\mathbb{Z}\), which is isomorphic to the dihedral group \(D_8\) of order 8, and \(\text{SO}(A) \cong D_8 \times \mathbb{Z}/2\mathbb{Z}\). Using \(\text{(2.2.1)}\), their actions on \(A\) are given by

\[(2.4.2) \quad R(a, b, c) = (a, b + 2a, 2a + 2b + c), \quad W(a, b, c) = (c, -b, a).
\]

To make the picture complete, we also consider the following central element in \(O(A)\)

\[(2.4.3) \quad \mu : A \to A
\]

\[(a, b, c) \mapsto (a, -b, c),
\]

which generates \(O(A) \cong D_8 \times (\mathbb{Z}/2\mathbb{Z})^2\) along with \(R, U, W\) and \(\nu\).

By acting on the basis \(\{e_h : h \in A\}\), the group \(O(A)\) naturally acts on \(\mathbb{C}[A]\), which commutes with the action of \(\text{Mp}_2(\mathbb{Z})\) under \(\rho\). Therefore, \(\text{Mp}_2(\mathbb{Z})\) acts on the \(\chi\)-isotypic subspace \(\mathbb{C}[A]^\chi \subset \mathbb{C}[A]\) as well, where \(\chi\) is a character of order 2 on \(O(A)\) defined by

\[(2.4.4) \quad \chi(R) = \chi(U) = \chi(W) = \chi(\mu) = -\chi(\nu) = -1.
\]

After composing with the map \(\text{SO}^+(L) \to \text{SO}^plus(L)/\Gamma_L \hookrightarrow O(A)\), we can also view \(\chi\) as a character of \(\text{SO}^+(L) = \Gamma_0^+(8)\). Consider the projection map \(\varpi : \mathbb{C}[A] \to \mathbb{C}[A]^\chi\) defined by

\[(2.4.5) \quad \varpi(v) := \frac{1}{|O(A)|} \sum_{s \in O(A)} \chi(s)^{-1}s(v), \quad v \in \mathbb{C}[A],
\]

which restricts to the identity map on \(\mathbb{C}[A]^\chi \subset \mathbb{C}[A]\). For any \(h \in A\), the vector \(\varpi(e_h)\) does not vanish if and only if the stabilizer \(O(A)_h\) is contained in the kernel of \(\chi\). Let \(A \subset A\) denote the subset of such elements, which is \(O(A)\)-invariant and decomposes into the following \(O(A)\)-orbits

\[(2.4.6) \quad A = A_0 \sqcup A_1 \sqcup A_2,
\]

\[A_0 := \{(0, 1, 1), (3, 3, 0), (0, 3, 3), (1, 1, 0), (0, 3, 1), (1, 3, 0), (0, 1, 3), (3, 1, 0)\},
\]

\[A_1 := \{(1, 1, 1), (1, 3, 1), (3, 3, 3), (3, 1, 3)\},
\]

\[A_2 := \{(1, 1, 2), (2, 1, 1), (3, 3, 2), (2, 3, 3), (3, 1, 2), (2, 3, 1), (1, 3, 2), (2, 1, 3)\}.
\]
Now, the identification in (2.3.6) means that the additional generator dimensions, we see that this is in fact an equality, and intertwines the representations \( \varpi \) and \( \chi \).

The product \( O(A) \) in (2.4.8) then \( \varpi \) (2.4.11) \( \chi \), where \( \varpi \) when restricted to \( C[A] \). After a straightforward calculation, we see that \( \rho \) becomes the representation \( \varrho \) in (1.0.4) when restricted to \( C[A] \).

**Lemma 2.1.** In the notations above, the map \( \varpi : C[A] \rightarrow C[A] \cong C^3 \) intertwines the representations \( \rho \) and \( \varrho \) from (1.0.4).

**Remark 2.2.** For any \( k \in \frac{1}{2} \mathbb{Z} \), we can view \( M_{k,\varrho} \) as a subspace of \( M_{k,\rho} \).

Similarly, we can analyze \( O(A_j) \). For \( j = 2 \), this is generated by \( \nu \). For \( j = 1 \), one needs the additional generator

\[
(2.4.8) \quad \sigma : A_1 \rightarrow A_1 \\
(a, c) \mapsto (c, a).
\]

The product \( O(A_1) \times O(A_2) \) canonically embeds into \( O(A) \) via (2.3.6), and we use \( O(A_1, A_2) \subset O(A) \) to denote this image. When we restrict \( \chi \) to \( O(A_1, A_2) \), it decomposes as \( \chi_1 \otimes \chi_2 \), where

\[
(2.4.9) \quad \chi_1(\sigma) = -\chi_1(\nu) = -\chi_2(\nu) = 1.
\]

Let \( \varrho_j \) denote the restriction of \( \varrho \) to the subspace \( C[A_j]^{\chi_j} \subset C[A_j] \) and \( \varpi_j : C[A_j] \rightarrow C[A_j]^{\chi_j} \) the projection defined as in (2.4.5) for \( j = 1, 2 \). The same analysis as before shows that \( C[A_j]^{\chi_j} \) has dimension 3 and 1 for \( j = 1, 2 \) respectively. Therefore, \( \varrho_2 \) is a character of \( \text{Mp}_2(\mathbb{Z}) \) given by

\[
(2.4.10) \quad \varrho_2(T) = \zeta_8, \quad \varrho_2(S) = \overline{\zeta_8}.
\]

Now, the identification in (2.3.6) means that \( C[A_1]^{\chi_1} \otimes C[A_2]^{\chi_2} \subset C[A] \). By considering the dimensions, we see that this is in fact an equality, and intertwines the representations \( \varrho_1 \otimes \varrho_2 \) and \( \varrho \). We can now identify \( C[A_1]^{\chi_1} \) and \( C[A_2]^{\chi_2} \) with \( \mathbb{C}^3 \) and \( \mathbb{C} \) respectively via the bases

\[
(2.4.11) \quad b_\ell := \sum_{s \in O(A_1)} \chi(s) s(\varepsilon_{(1,0,\ell)}) \in C[A_1]^{\chi_1}, \ell = 0, 1, 2 \quad c := \varepsilon_{(0,1,0)} - \varepsilon_{(0,3,0)} \in C[A_2]^{\chi_2}.
\]

Then \( b_\ell \otimes c = \varepsilon_\ell \) for \( \ell = 0, 1, 2 \) and the equality \( C[A_1]^{\chi_1} \otimes C[A_2]^{\chi_2} = C[A] \) becomes \( \mathbb{C}^3 \otimes \mathbb{C} = \mathbb{C}^3 \). With respect to the basis \( \{ b_\ell : \ell = 0, 1, 2 \} \), the representation \( \varrho_1 \) has the matrix representation

\[
(2.4.12) \quad \varrho_1(T) := \begin{pmatrix} 1 & -i \\ -1 & 1 \end{pmatrix}, \quad \varrho_1(S) := \frac{i}{2} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}.
\]
As in Remark 2.2 we have $M_{k,p_j} = M_{k,p_j}^\nu \subset M_{k,p_j}$ for $j = 1, 2$. We use $c^*_i, b^*_i$ and $c^*$ to represent basis vectors of the unitary dual representations $\varrho^*_i, \varrho^*_1$ and $\varrho^*_2$.

2.5. Heegner Divisor. For $m \in \mathbb{Q}$ and $h \in \mathcal{A}$, we define the $\Gamma_L$-invariant subset

$$L_{m,h} := \{ \lambda \in L + h : Q(\lambda) = m \} \subseteq L^\vee.$$  

When $m < 0$, the following analytic subset of $\mathcal{H}$

$$Z_{m,h} := \{ z \in \mathcal{H} : (Z(z), \lambda) = 0 \text{ for some } \lambda \in L_{m,h} \}$$

is $\Gamma_L$-invariant and descends to an algebraic divisor on $\Gamma_L \backslash \mathcal{H}$. Note that $Z_{m,h}$ is empty when $m > 0$. The singularities of Borcherds lifts are supported on these divisors, which are called Heegner divisors. For a function to be a regularized theta lift, it is then necessary for it to have singularity along Heegner divisors of a certain lattice. In our case, the singularity of $\phi$ appears at $z = z_\pm$. It is straightforward to check that

$$(Z(z_\pm), \lambda_\pm) = 0 \text{ for } \lambda_\pm := \left( \begin{array}{cc} \pm 1/4 & 1/8 \\ -1 & \mp 1/4 \end{array} \right) \in L_{-1/8,(1,\pm,1,1)}.$$  

We now have the following lemma.

**Lemma 2.3.** The set $\Gamma_L \backslash L_{-1/8,(1,\pm,1,1)}$ has size one.

**Proof.** We associate to $\lambda = \left( \begin{array}{cc} b/4 & c/8 \\ -a & -b/4 \end{array} \right) \in L_{-1/8,(1,\pm,1,1)}$ the binary quadratic form $[\lambda](x, y) = [8a, 4b, c](x, y) = 8ax^2 + 4bxy + cy^2$. This identifies $L_{-1/8,(1,\pm,1,1)}$ with the set

$$Q^0_{8,-16,\pm4} = \{ [8a, 4b, c] \mid b \equiv \pm 4 \text{ mod } 16, \gcd(8a, b, c) = 1 \}$$

If $[8a, b, c] \in Q^0_{8,-16,\pm4}$, we must have $a \equiv c \equiv \pm 1 \text{ mod } 8$. So $Q^0_{8,-16,\pm4}$ is the union of the images of $L_{-1/8,(1,\pm,1,1)}$ and $L_{-1/8,(1,\pm,1,1)}$ under the map $\lambda \mapsto [\lambda]$. If $\gamma \in \Gamma_L$ we have $f_{\gamma\lambda}(x, y) = f((x, y)(\gamma^{-1})^T)$. Hence $\Gamma_L \backslash (L_{-1/8,(1,\pm,1,1)} \cup L_{-1/8,(1,\pm,1,-1)})$ is in bijection with $\Gamma_L \backslash Q^0_{8,-16,\pm4}$. We also note that while, $\Gamma_L \backslash L_{-1/8,(1,\pm,1,1)}$ and $\Gamma_L \backslash L_{-1/8,(1,\pm,1,-1)}$ are disjoint, they are in bijection to each other via the map that sends $\left( \begin{array}{cc} b/4 & c/8 \\ -a & -b/4 \end{array} \right)$ to $\left( \begin{array}{cc} b/4 & c/8 \\ a & a \end{array} \right)$. In [12] §[1.1] the classes of $Q^0_{8,-16,\pm4}$ modulo $\Gamma_L = \Gamma_0(8)$ are classified. In our particular case these $\Gamma_0(8)$-classes correspond to $\text{SL}_2(\mathbb{Z})$-classes of primitive binary quadratic forms of discriminant $-16$, of which there are 2. Hence $|\Gamma_L \backslash L_{-1/8,(1,\pm,1,1)}| = |\Gamma_L \backslash Q^0_{8,-16,\pm4}|/2 = 1$. □

2.6. Additive Borcherds’ Lift. In [3], Borcherds extended the input space of theta lift from $\text{SL}_2$ to $\text{O}(p, q)$ to include weakly holomorphic, vector-valued modular forms. The outputs are then automorphic forms on orthogonal Shimura varieties with singularities along Heegner divisors. For $(p, q) = (2, 1)$, the orthogonal Shimura variety is again the modular curve, and one can obtain a generalization of the Shimura lift to include weakly holomorphic modular forms (see [8] and [15]). For the lattice $L$ in section 2.3 Borcherds’ result implies Theorem 1.2.
Proof of Theorem 1.2. Let $G \in M_{5/2,\varrho} = M_{5/2,\varrho}^I \subset M_{5/2,\varrho}$ with Fourier expansion

$$G(\tau) = \sum_{h \in A} c_h \sum_{m \in \mathbb{Z}} c_h(m) q^{m/8}$$

and $\rho = \rho_L$ with $L$ as in section 2.3. We choose the isotropic vector $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) \in L$ with $z' := (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) \in L^\vee$ in the notation of Theorem 14.3 of [3]. The lattice $K$ is then our $L_2$, $m^+ = 2$ and $c_h(0) = 0$ whenever $h$ is not in the set $A$ defined in (2.1.6). Therefore, the Fourier expansion of the regularized theta lift in part 5 of Theorem 14.3 loc. cit. becomes

$$\sum_{n>0} \sum_{b \in \mathbb{Z}, b \text{ odd}, b<0} q_z^{-(nb)/8} n \sum_{a \in \mathbb{Z}/4\mathbb{Z}} e(-na)c(a,b,0) (b^2).$$

Since $G$ is in the $\chi$-isotypic component and

$$\mu((1,1,0)) = (1,3,0), \nu((1,1,0)) = (3,3,0), (\mu \circ \nu)((1,1,0)) = (3,1,0),$$

we know that $c_{(1,1,0)}(m) = c_{(3,3,0)}(m) = -c_{(3,1,0)}(m) = -c_{(3,1,0)}(m)$ for all $m \in \mathbb{Z}$. By Lemma 2.1 we can write $G(\tau) = \sum_{m \in \mathbb{Z}} c(G,m) q^{m/8} e_{\ell(m)}$ as in (1.0.5) in the introduction with $e_{\ell}$ defined in (2.4.7). Then $c(G,m) = c_{(1,1,0)}(m)$ for all $m \in 8\mathbb{Z} + 1$ Substitute this into the expression above gives us $2i\Phi(z,G)$.

The action of the orthogonal group on the theta kernel and (2.2.3) imply that $\Phi(z,G) \mid \gamma = \Phi(z,\gamma \cdot G)$ for any $\gamma \in \text{PSL}_2(\mathbb{R})$, hence $\Phi(z,G) \in M_{4,\chi}^{\text{mero}}(\text{SO}^+(L))$ with $\text{SO}^+(L) = \Gamma_0^+(8)$. By Theorem 6.2 of [3], the function

$$\Phi(z,G) = \frac{i}{16\pi^2} \sum_{\lambda \in L^\vee, (\lambda, Z(z_0))=0} \frac{c_{\lambda}(Q(\lambda))}{(\lambda, Z(z))^2}$$

is holomorphic when $z \in \mathcal{H}$ is near $z_0 \in \mathcal{H}$. When $G = -G_1/4 = -q^{-1/8} e_1 + O(q^{1/8})$, the polar part of the expansion of $\Phi(z,G)$ near $z = z_\pm$ matches exactly with that of $\phi(z)$. Therefore, the difference $\Phi(z,G_1) - \phi(z) \in M_{4,\chi}(\Gamma_0^+(8))$ is zero by Remark 1.3. The existence follows from Serre duality [6], as there is no non-trivial cusp form in $S_{-1/2,\varrho}$. In fact, we will explicitly construct it in the following section. It is unique since $M_{5/2,\varrho} = \{0\}$ by Prop. 3.1 below.

\[\square\]

3. Bases, Dualities and Hecke Operators.

In this section, we will construct the $\mathbb{Z}$-basis $\{G_d : d \equiv 1,3 \text{ or } 7 \text{ mod } 8\}$ of $\mathbb{M}_{5/2,\varrho}$, and show that it has duality with respect to a $\mathbb{Z}$-basis of $\mathbb{M}_{5/2,\varrho}^\ast$, where $\varrho^\ast$ is the unitary dual of $\varrho$, i.e., $\varrho^\ast(T)$ and $\varrho^\ast(S)$ are the conjugate transpose of $\varrho(T)$ and $\varrho(S)$ respectively. Using Hecke operators on vector-valued modular forms [7], we will prove Conjecture 1.1.
3.1. Two Bases of Modular Forms. In the notations of the previous section, we will prove the following result.

**Proposition 3.1.** For every $d \in \mathbb{N}$ congruent to 1, 3 or 7 modulo 8, there exists a unique $G_d$ in $\mathcal{M}_{1/2,e}$ with the Fourier expansion

\begin{equation}
G_d(\tau) = q^{-d} + O(1) = \sum_{D \geq -d, \ D = 1, 5, 7 \mod 8} B(D, d)q^D, \ q^D := q^{D/8}\epsilon_{\ell(D)}.
\end{equation}

For every $D \in \mathbb{N}$ congruent to 1, 5 or 7 modulo 8, there exists a unique $F_D$ in $\mathcal{M}_{l/2, e}$ with the Fourier expansion

\begin{equation}
F_D(\tau) = \tilde{q}^{-D} + O(1) = \sum_{D \geq -D, \ D = 1, 3, 7 \mod 8} A(D, d)\tilde{q}^d, \ \tilde{q}^d := q^{d/8}\epsilon_{\ell(-d)}.
\end{equation}

Furthermore, $A(D, d) + B(D, d) = 0$ for all $D, d \in \mathbb{N}$.

To prove this proposition, we will first reduce it to the case of weight one using the presence of the unary theta function

\begin{equation}
\vartheta(\tau) := \sum_{b \in \mathbb{Z}/4\mathbb{Z}} \epsilon(b, 0) \sum_{n \in \mathbb{Z} + b} nq^n = \eta^3(\tau)\epsilon \in S_{3/2, \epsilon} \cap \mathcal{M}_{3/2, \epsilon},
\end{equation}

which only vanishes at the cusp infinity. Since tensoring with $\epsilon$ gives an isomorphism between $\mathbb{C}[A_1]^\chi$ and $\mathbb{C}[A]^\chi$ as $\text{Mp}_2(\mathbb{Z})$-modules under the representations $\varrho_1$ and $\varrho$, we obtain the following lemma.

**Lemma 3.2.** In the notations above, the following map is an isomorphism of $\mathbb{Z}$-modules

\begin{equation}
\iota_\varrho : \mathcal{M}_{1, \varrho_1} \to \mathcal{M}_{5/2, \varrho}
\end{equation}

\begin{equation}
f \mapsto f \otimes \vartheta.
\end{equation}

Furthermore, the preimage of $\mathcal{M}_{5/2, \varrho}$ under this isomorphism is the trivial subspace $\mathcal{M}_{1, \varrho_1}$ and hence $\mathcal{M}_{5/2, \varrho}$ is also trivial.

**Remark 3.3.** By taking the conjugate transpose of the representation $\varrho_1$ and dividing by $\eta^3(\tau)$, we also obtain an explicit isomorphism $\mathcal{M}_{1, \varrho_1}$ and $\mathcal{M}_{-1/2, \varrho}$. However, we will see later that $\mathcal{M}_{1, \varrho_1}$ is one dimensional.

**Proof.** The first part follows from the argument above. The inverse map is simply dividing by $\eta^3(\tau)$ on each component. For the second part, notice that $\mathcal{M}_{5/2, \varrho} \subset S_{5/2, \varrho}$ and the order of vanishing at each component of any $G \in \mathcal{M}_{5/2, \varrho}$ is at least $q^{1/8}$. Therefore, the result is still holomorphic after dividing $G$ by $\eta^3(\tau)$.

For any $f = \sum_{\ell=0,1,2} f_\ell b_\ell \in M_{1, \varrho_1}$, the function $f_1(4\tau)$ is in the 7-dimensional space $M_1(\Gamma_1(16))$ with the Fourier expansion

\begin{equation}
f_1(4\tau) = \sum_{n \geq 0} c(n)q^{4n+3}
\end{equation}
at infinity. A quick calculation with SageMath [20] shows that $f_1$ is identically zero. Then

$$f |_1 S = (f_0 | S) \varepsilon_0 + (f_2 | S) \varepsilon_2 = \varrho_1(S) \cdot f = \frac{i}{2}((f_0 + f_2)(\varepsilon_0 + \varepsilon_2) + 2(f_0 - f_2)\varepsilon_1)$$

implies that $f_0 = f_2$. Since $f_\ell(\tau + 1) = (-i)^\ell f_\ell(\tau)$, we conclude that $f_0$ and $f_2$ are both identically zero. Therefore, $M_{1,\rho_1}$ is trivial and so is $M_{1,\rho_1} \subset M_{1,\rho_1}^*$. The same procedure shows that $S_{1,\rho_1}$ is trivial and $M_{1,\rho_1}^*$ is at most one dimensional. We will construct this non-trivial element later. \hfill\Box

With the lemma above, it suffices to study the space $M_{1,\rho_1}^* = M_{1,\rho_1}^{1,\chi_1}$, which has the following property.

**Lemma 3.4.** For every $m$ in $\mathbb{N}\setminus(4\mathbb{N} - 1)$, there exists a unique $g_m \in M_{1,\rho_1}^*$ with the Fourier expansion

$$g_m(\tau) = q^{-m/4} b_{m \mod 4} + O(1).$$

For every $m$ in $\mathbb{N} \cup \{0\} \setminus (4\mathbb{N} - 3)$, there exists a unique $f_m \in M_{1,\rho_1}$ with the Fourier expansion

$$f_m(\tau) = q^{-m/4} b_{-m \mod 4} + O(q^{1/4}).$$

**Proof.** The uniqueness easily follows from the previous lemma, where we saw that $M_{1,\rho_1}$ and $S_{1,\rho_1}$ are trivial. For the existence, we will give an explicit construction, which can be implemented numerically.

Since $\rho_1$ is the representation attached to a scaled hyperbolic plane, we will use Lemma 2.6 in [3] to construct $\tilde{g}_m$ from scalar-valued modular forms in $M_1^*(\Gamma_1(4))$. For $\tilde{g} \in M_1^*(\Gamma_1(4))$ and $(a,c) \in A_1$, define

$$(3.1.5) \quad \mathcal{B}(\tilde{g})(a,c) := \sum_{d \in \mathbb{Z}/4\mathbb{Z}, (c,d)=1} i^{ad} g \left|_1 \begin{pmatrix} * & * \\ c & d \end{pmatrix} + \sum_{d \in \mathbb{Z}/4\mathbb{Z}, (a,d)=1} i^{cd} g \left|_1 \begin{pmatrix} * & * \\ a & d \end{pmatrix} \right. \right.$$

Then by the same proof of Lemma 2.6 in [3], we know that

$$(3.1.6) \quad \mathcal{B}(\tilde{g}) = \sum_{h \in A_1} \mathcal{B}(\tilde{g})_h \varepsilon_h = \mathcal{B}(\tilde{g})_{(1,0)} \varepsilon_0 + \frac{\mathcal{B}(\tilde{g})_{(1,1)}}{2} \varepsilon_1 + \mathcal{B}(\tilde{g})_{(1,0)} \varepsilon_0$$

is in $M_{1,\rho_1}^* = M_{1,\rho_1}^{1,\chi_1}$. For $(a,c) = (1,0), (1,1)$ and $(1,2)$, we can explicitly write $\mathcal{B}(\tilde{g})_{(a,c)}$ as

$$\mathcal{B}(\tilde{g})_{(1,0)} = 2i \cdot \tilde{g} + \sum_{d=0}^{3} \tilde{g} \left|_1 ST^d \right. \quad \mathcal{B}(\tilde{g})_{(1,1)} = 2 \sum_{d=0}^{3} \tilde{g} \left|_1 ST^d \right. \quad \mathcal{B}(\tilde{g})_{(1,2)} = 2i(\tilde{g} \left|_1 ST^2 S^{-1} \right. + \sum_{d=0}^{3} (-1)^d \tilde{g} \left|_1 ST^d \right. \right.$$

So the Fourier expansion of $\mathcal{B}(\tilde{g})$ is directly related to those of $\tilde{g}$ at the three cusps of $\Gamma_1(4)$.
To construct the family $g_m$, we start with the Eisenstein series
\[ \tilde{g}_0(\tau) = iE_1^{\varphi} = iL(0, \varphi) + 2i \sum_{n \geq 1} \sum_{m \mid n} \varphi(m)q^n = \frac{i}{2}(1 + 4q + O(q^2)) \in \frac{i}{2}\mathbb{Z}[q] \]
that generates $M_1(\Gamma_1(4))$. Here $\varphi$ is the Kronecker symbol modulo 4. The Fourier expansions of $E_1^{\varphi}$ at the cusps of $\Gamma_1(4)$ are given in [21, Chapter 2], from which one calculates
\begin{align*}
\tilde{g}_0|_1 S &= -\frac{i}{2} \tilde{g}_0 \left( \frac{\tau}{4} \right) = \frac{1}{4}(1 + 4q^{1/4} + O(q^{2/4})) \in \frac{1}{4}\mathbb{Z}[q^{1/4}], \\
\tilde{g}_0|_{ST^2 S^{-1}} &= 2i \sum_{n \geq 1} \sum_{m \mid n} \varphi(m)q^{n/2} = 2iq^{1/2}(1 + 2q^2 + O(q^4)) \in 2iq^{1/2}\mathbb{Z}[q].
\end{align*}
Since $M_{1, \varphi} = \{0\}$, $\tilde{g}_0$ must lift to 0 under $B$ and using the Fourier expansions we can check that this is indeed the case. Let $\psi$ be the Hauptmodul from [10,3]. Then
\[ (3.1.7) \quad \tilde{\psi}(\tau) := \psi(-1/(4\tau)) = 2^4 \eta(\tau)^{-16} \eta(2\tau)^{24} \eta(4\tau)^{-8} = 2^4(1 + O(q)) \in 2^4\mathbb{Z}[q] \]
is also a Hauptmodul of $X_1(4)$. At the other cusps, it has the expansions
\begin{align*}
(\tilde{\psi} | S)(\tau) &= \eta(\tau)^{-16} \eta(\tau/2)^{24} \eta(4\tau)^{-8} = q^{-1/4} + 8 + O(q^{1/4}) \in q^{-1/4}\mathbb{Z}[q^{1/4}], \\
(\tilde{\psi} | ST^2 S^{-1})(\tau) &= -2^8 \eta(\tau)^{-8} \eta(4\tau)^8 = -2^8 q(1 + O(q)) \in 2^8q\mathbb{Z}[q].
\end{align*}
Multiplying $\tilde{g}_0$ with monic polynomials in $\tilde{\psi}$ with integral coefficients, we can recursively construct $g_m \in M_1^0(\Gamma_1(4))$ such that $2i \cdot \tilde{g}_m(\tau), iq^{-1/2}\tilde{g}_m|_{1 ST^2 S^{-1}}(\tau) \in \mathbb{Z}[q]$ and
\[ (3.1.8) \quad (g_m \mid S)(\tau) = \frac{1}{4}(q^{-m/4} + O(q^{1/4})) \in \frac{1}{4}\mathbb{Z}[q^{1/4}] \]
for any $m \in \mathbb{N}$. For example
\[ (3.1.9) \quad \tilde{g}_1 = \tilde{g}_0(\tilde{\psi} - 12). \]
Let $a_m(n) \in \mathbb{Z}$ be the $(n/4)^{th}$ Fourier coefficient of $4\tilde{g}_m \mid S$. Then the $\epsilon_0, \epsilon_1$ and $\epsilon_2$ components of $g_m := B(\tilde{g}_m)$ are given by
\[ 2i \cdot g_m + \sum_{n \in 4\mathbb{Z}} a_m(n)q^{n/4}, \sum_{n \in 4\mathbb{Z} - 1} a_m(n)q^{n/4}, 2i(\tilde{g}_m \mid 1 ST^2 S^{-1}) + \sum_{n \in 4\mathbb{Z} + 2} a_m(n)q^{n/4} \]
respectively. Note that $g_m$ is identically zero if $m \equiv 3 \mod 4$. Therefore the $g_m$’s satisfy the condition in the lemma. The family $\{f_m : m \in \mathbb{N} \cup \{0\} \setminus (4\mathbb{N} - 3)\} \in M_{1, \varphi}^0$ can be constructed similarly. \hfill $\square$

Proof of Prop. 3.1. For $d \equiv 1, 3, 7 \mod 8$, there is a unique $G_\varphi \in M_{5/2, \varphi}$ satisfying (3.1.1). By Lemma 3.2, there exists $g \in M_{1, \varphi}^0$ such that $\iota_\varphi(g) = G_\varphi$. Since the principal part of $g$ has integral Fourier coefficients, we can express it as an integral linear combination of the $g_m$’s from Lemma 3.4. Therefore $g$ is contained in $M_{1, \varphi}^0$, and $G_\varphi = \iota_\varphi(g)$ is contained in $M_{5/2, \varphi}^0$. The same proof works for the $F_D$’s.
The last statement is a simple consequence that every weakly holomorphic modular form of weight 2 on $\text{SL}_2(\mathbb{Z})$ has vanishing constant term, as it is the derivative of a modular function. If we view $\mathbb{C}[A_L] \times \mathbb{C}[A_L]$ as an $\text{SL}_2(\mathbb{Z})$-module with respect to $\rho_L \otimes \rho_L^*$, then the hermitian inner product $\langle \cdot, \cdot \rangle : \mathbb{C}[A_L] \times \mathbb{C}[A_L] \to \mathbb{C}$ in (2.1.3) is $\text{SL}_2(\mathbb{Z})$-linear, whose action on $\mathbb{C}$ is trivial. Therefore, $\langle F_D, G_d \rangle$ is in $M^!_2$ and its constant term is given by 
\[
\sum_{D',d' \in \mathbb{Z},D'+d'=0} A(D,d') B(D',d).
\]
We are then done by (3.1.1) and (3.1.2).

Since the proof is constructive, we can explicitly give the Fourier expansions of $G_d$ for any particular $d$. For example when $d = 1$, the $c_0$-component of the modular form
\[
(3.1.10) \quad G_1 = \iota_\vartheta(g_1) = \iota_\vartheta(\mathcal{B}(\tilde{g}_1))
\]
has the Fourier expansion
\[
\eta(\tau)^3 \mathcal{B}(\tilde{g}_1)_{(1,0)} = -4(q^{1/8} + 129q^{9/8} + 1144q^{17/8} + 5625q^{25/8} + O(q^{33/8})),
\]
and one can check both numerically and from the proof of Theorem 1.2 that $\Phi(z, G_1) = -4\vartheta(z)$.

3.2. Hecke operators. In [7, Theorem 4.10] Hecke operators on vector-valued modular forms are defined. They act on $G \in M^!_{5/2,\rho}$ as follows. If
\[
G(\tau) = \sum_{h \in \mathcal{A}} \varepsilon_h \sum_{n \in \mathbb{Z}} c_h(n)q^{n/8}
\]
and $p$ is an odd prime, then multiplying by $p$ on $\mathcal{A}$ is an isometry and
\[
(G|T_p^2)(\tau) = \sum_{h \in \mathcal{A}} \varepsilon_h \sum_{n \in \mathbb{Z}} b_h(n)q^{n/8}, \quad b_h(n) := c_{ph}(p^2n) + p\left(\frac{n}{p}\right) c_h(n) + p^3 c_{p^{\nu-1}h}(n/p^2),
\]
with $\left(\frac{a}{p}\right)$ the Kronecker symbol. It is easily checked that multiplying by $p$ acts as either the identity or $\nu$ on $\mathcal{A}$. Therefore, we have $\varpi(p \cdot v) = \varpi(v)$ for any $v \in \mathbb{C}[A]$ and $T_{p^2}$ preserves the subspace $M^!_{5/2,\rho} \subset M^!_{5/2,\vartheta}$ and the lattice $M^!_{5/2,\rho} \subset M^!_{5/2,\vartheta}$. Given $\tilde{G}(\tau) = \sum_{n \in \mathbb{Z}} c(G,n)q^n$ in $M^!_{5/2,\vartheta}$, the Hecke operator $T_{p^2}$ then acts as
\[
(3.2.1) \quad (G|T_{p^2})(\tau) = \sum_{n \in \mathbb{Z}} \left(c(G,p^2n) + p\left(\frac{n}{p}\right) c(G,n) + p^3 c(G,n/p^2)\right) q^n.
\]
Using Prop. 3.1 and comparing the principal parts, we can deduce the equality
\[
(3.2.2) \quad G_d | T_{p^2} = p^3 G_{p^2d} + p\left(\frac{-d}{p}\right) G_d + G_{d/p^2}
\]
for any $G_d \in M^!_{5/2,\vartheta}$, where $G_{d/p^2}$ is zero if $p^2 \nmid d$. This leads to the following result, which implies the second half of Theorem 1.4.
Proposition 3.5. Let \( G_d(\tau) = \sum_{n \in \mathbb{Z}} B(n, d)q^n \in \mathbb{M}_{0/2, q} \) be as in Prop.
Proposition 3.1. Suppose \( d \equiv 1, 3, 7 \) mod 8 is square-free. Then
\[
(3.2.3) \quad B(p^2n, d) \equiv \left(\frac{-d}{p}\right) \left(\frac{n}{p}\right) B(n, d) \mod p^3
\]
for any \( n \in \mathbb{N} \) and odd prime \( p \). In particular, we have \( B(m^2, d) \in m\mathbb{Z} \) for all \( m \in \mathbb{N} \).

Proof. Since \( d \) is square-free, \( (3.2.2) \) becomes \( G_d \mid T_{p^2} = p^3G_{p^2d} + p\left(\frac{-d}{p}\right)G_d \). Comparing the \( n \)th Fourier coefficients of both sides then gives us the congruence \( (3.2.3) \). The last claim then follows since \( B(m, d) = 0 \) whenever \( 2 \mid m \). \( \square \)

Proposition 3.5 now leads to a proof of Conjecture 1.1. A more detailed analysis reveals higher power congruences:

Corollary 3.6. Let \( d \) be square-free and \( \Phi(z, G_d) = \sum_{n \in 2\mathbb{N} - 1} a_d(n)q^n \). Then for every odd prime \( p \) we have
\[
(3.2.4) \quad a_d(p) \equiv \left(\frac{d}{p}\right) B(1, d) \mod p^3.
\]

Proof. The coefficient \( a_d(p) \) is given by \( (1.0.6) \) and equals \((-1)^{(p-1)/2}(p \cdot B(1, d) + B(p^2, d))\). By Prop.
Proposition 3.5 we have \( B(p^2, d) \equiv \left(\frac{-d}{p}\right) - 1 \right) B(1, d) \mod p^3 \) and \( (3.2.4) \) follows from \((-1)^{(p-1)/2} = \left(\frac{-1}{p}\right) \) and the multiplicativity of the Kronecker symbol. \( \square \)

Finally, we record some examples of the bases. Two instances of Zagier duality are marked in color:

- \( G_1(\tau) = q^{-1} - 4q + 112q^5 + 19q^7 - 516q^9 + 1712q^{13} - 87q^{15} + O(q^{16}) \),
- \( G_3(\tau) = q^{-3} - 4q - 267q^5 + 1024q^7 - 3012q^9 - 19666q^{13} + 44032q^{15} + O(q^{16}) \),
- \( G_7(\tau) = q^{-7} - 7q - 3136q^5 - 20480q^7 - 102396q^9 - 1546048q^{13} - 5074944q^{15} + O(q^{16}) \),
- \( G_9(\tau) = q^{-9} - 20q + 16944q^5 - 172q^7 - 854548q^9 + 18047344q^{13} + 5031q^{15} + O(q^{16}) \),
- \( G_{11}(\tau) = q^{-11} - 12q + 21303q^5 + 216064q^7 - 1566540q^9 - 44627503q^{13} + 193840128q^{15} + O(q^{16}) \),
- \( G_{15}(\tau) = q^{-15} - 25q + 111552q^5 - 1617920q^7 - 15953955q^9 - 770664640q^{13} - 4226125824q^{15} + O(q^{16}) \).
\[ F_1(\tau) = \tilde{q}^{-1} + 4\tilde{q} + 4\tilde{q}^3 + 7\tilde{q}^7 + 20\tilde{q}^9 + 12\tilde{q}^{11} + 25\tilde{q}^{15} + O(q^{16}), \]
\[ F_5(\tau) = \tilde{q}^{-5} - 112\tilde{q} + 267\tilde{q}^3 + 3136\tilde{q}^7 - 16944\tilde{q}^9 + 21303\tilde{q}^{11} + 111552\tilde{q}^{15} + O(q^{16}), \]
\[ F_7(\tau) = \tilde{q}^{-7} - 19\tilde{q} - 1024\tilde{q}^3 + 20480\tilde{q}^7 + 172\tilde{q}^9 - 210604\tilde{q}^{11} + 1617920\tilde{q}^{15} + O(q^{16}), \]
\[ F_9(\tau) = \tilde{q}^{-9} + 516\tilde{q} + 20480\tilde{q}^3 + 102396\tilde{q}^7 + 854548\tilde{q}^9 + 1566540\tilde{q}^{11} + 15953955\tilde{q}^{15} + O(q^{16}), \]
\[ F_{13}(\tau) = \tilde{q}^{-13} - 1712\tilde{q} + 19666\tilde{q}^3 + 1546048\tilde{q}^7 - 18047344\tilde{q}^9 + 44627503\tilde{q}^{11} + 770664640\tilde{q}^{15} + O(q^{16}), \]
\[ F_{15}(\tau) = \tilde{q}^{-15} + 87\tilde{q} - 44032\tilde{q}^3 + 5074944\tilde{q}^7 - 5031\tilde{q}^9 - 193840128\tilde{q}^{11} + 4226125824\tilde{q}^{15} + O(q^{16}). \]

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