ON THE $K$ PROPERTY FOR MAHARAM EXTENSIONS OF BERNOULLI SHIFTS AND A QUESTION OF KRENGEL

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Abstract. We show that the Maharam extension of a type III, conservative and non singular $K$ Bernoulli is a $K$-transformation. This together with the fact that the Maharam extension of a conservative transformation is conservative gives a negative answer to Krengel’s and Weiss’s questions about existence of a type $II_{\infty}$ or type $III_{\lambda}$ with $\lambda \neq 1$ Bernoulli shift. A conservative non singular $K$, in the sense of Silva and Thieullen, Bernoulli shift is either of type $II_{1}$ or of type $III_{1}$.

1. Introduction

Let $T$ be an invertible non singular transformation of the probability space $(X, \mathcal{B}, \mu)$. The Maharam extension $\tilde{T}$ of $T$ is a measure preserving transformation which is a skew product extension of $T$ with the Radon Nykodym cocycle. It is well known that the Maharam extension is ergodic if and only if $T$ is of Krieger type $III_{1}$, see below. Here we show that in the case when $T$ is a conservative non singular Bernoulli shift which satisfies the $K$-property as in [ST] with the one sided shift as the exact factor, then the Maharam extension is a $K$-transformation. Thus the Maharam extension is weak mixing in the sense that $T \times S$ is ergodic for every ergodic probability preserving transformation $S$ and it has a countable Lebesgue spectrum.

This type of non singular Bernoulli shifts was considered first in [Kre] where a shift without an absolutely continuous invariant probability was constructed. Later Hamachi in [Ham] constructed an ergodic shift without an absolutely continuous $\sigma$-finite invariant measure. Such transformations are called type III. Krengel [Kre] asked the question whether there exists a shift with an absolutely continuous invariant $\sigma$-finite measure but no such probability (these are called type $II_{\infty}$). The type III transformations can be further classified into orbit equivalence classes according to their ratio set. In [Kos] a Bernoulli shift which is of Krieger type $III_{1}$ was constructed. In a presentation of that result Benjy Weiss asked whether there are type III shifts of different Krieger types. As a corollary of the $K$ property of the Maharam extension we get a dichotomy. Namely an ergodic non singular $K$ Bernoulli shift is either of type $II_{1}$ when the measure is equivalent to a stationary product measure or of type $III_{1}$.

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The proof makes use of the fact that since the Radon Nykodym cocycle is measurable with respect to the $\sigma$-algebra $\mathcal{B}_{\{0,1\}^\mathbb{N}}$, the Maharam extension is the natural extension of a skew product $\sigma_\varphi$ of the one sided shift. Thus it is enough to show that the tail equivalence relation of the non invertible skew product is ergodic. This is done by showing that the tail equivalence relation of $\sigma_\varphi$ is the orbit equivalence relation of the Maharam extension of the odometer and proving that the odometer with the one sided measure is of type $\text{III}_1$.

One step in the proof that the corresponding odometer action is type $\text{III}_1$ is to show that for shift conservative product measures we have two subsequences $n_k \to \infty$ and $m_k \to -\infty$ for which
\[
\lim_{k \to \infty} P_{n_k} = \lim_{k \to \infty} P_{m_k}.
\]
The question arises whether for conservative shifts the limit needs to exist? We give an example of a conservative shift with
\[
\liminf_{k \to \infty} P_k(0) < \limsup_{k \to \infty} P_k(0),
\]
thus answering this question on the negative.

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2. Preliminaries

2.1. Non Singular Ergodic Theory. Let $(X, \mathcal{B}, \mu)$ be a standard measure space. Since one can always pass to an equivalent probability measure, we will always assume that $\mu$ is a probability measure. In what follows all equalities of sets are modulo the measure on the space.

A measurable transformation $T : X \to X$ is non singular if $\mu$ and $T^* \mu = \mu \circ T^{-1}$ are equivalent, meaning that they have the same collection of null sets. In the case when $T$ is invertible there exists the Radon Nykodym derivatives
\[
T^n(x) := \frac{d\mu \circ T^n}{d\mu}(x).
\]
When $T^* \mu = \mu$ we say that $T$ is $\mu$ preserving or $\mu$ is $T$ invariant. A transformation is ergodic if $T^{-1} A = A$ implies $A \in \{\emptyset, X\}$. A set $A \in \mathcal{B}$ is wandering if $\{T^{-n} A\}_{n=1}^\infty$ are disjoint. Denote by $\mathcal{D}$ the (measurable) union of all wandering sets, its complement is denoted by $\mathcal{C}$ and is called the conservative part. In the case where $\mathcal{D} = X$ we say that $T$ is dissipative. If $\mathcal{C} = X$ we say that $T$ is
conservative. By Hopf’s theorem [An Prop. 1.3.1.]

\[
\mathcal{D} = \left\{ x \in X : \sum_{n=1}^{\infty} T^n'(x) < \infty \right\}
\]

\[
\mathcal{C} = \left\{ x \in X : \sum_{n=1}^{\infty} T^n'(x) = \infty \right\}.
\]

An invertible transformation \( T \) satisfies the \( K \)-property if there exists a sub-\( \sigma \) algebra \( F \subset B \) such that \( T^{-1}F \subset F \), \( \bigcap_{n \in \mathbb{Z}} T^n F = \{ \emptyset, X \} \) and \( \bigvee_{n=1}^{\infty} T^{-n} F = B \). If \( T \) is measure preserving and \( K \) then \( T \) is either conservative or totally dissipative. This property remains true in the case of non singular Bernoulli shifts, see Lemma 5.1 or [Gre].

A measure preserving transformation \((Y, B_Y, \nu, S)\) is an extension of \((X, B_X, \mu, T)\) (equivalently \( T \) is a factor of \( X \)) if there exists a measurable map \( \pi : Y \to X \) such that \( \pi^{-1}B_X \subset B_Y \), \( \pi \circ S = T \circ \pi \) and \( \pi_* \nu = \mu \). Given a non-invertible measure preserving transformation \((X, B_X, \mu, T)\), the natural extension of \( T \) is an invertible measure preserving transformation \( \tilde{T} \) which is minimal in the sense that \( \bigvee_{n=1}^{\infty} T^n \pi^{-1}B_X = B_X \), where \( \pi : \tilde{X} \to X \) is the factor map.

2.2. Cocycles and skew product extensions. A function \( \varphi : \mathbb{N} \times X \to \mathbb{R} \) (or \( \mathbb{Z} \times X \to \mathbb{R} \) when \( T \) is invertible) is a cocycle if for every \( n, m \in \mathbb{N} \) and almost every \( x \in X \),

\[
\varphi_{n+m}(x) = \varphi_n(x) + \varphi_m(T^n x).
\]

Given a function \( \varphi : X \to \mathbb{R} \) we can define the cocycle

\[
\forall n \in \mathbb{N}, \ \varphi_n(x) = \varphi(x) + \varphi \circ T(x) + \cdots + \varphi \circ T^{n-1}(x),
\]

and the skew product extension \( \mathcal{T}_\varphi : (X \times \mathbb{R}, B_X \otimes B_{\mathbb{R}}, \mu \times e^\varphi ds) \) of \( T \) with \( \varphi \) by

\[
\mathcal{T}_\varphi(x, y) := (Tx, y + \varphi(x)).
\]

Definition. The set of essential values for \( \varphi \) is

\[
e(T, \varphi) = \left\{ t \in \mathbb{R} : \forall \epsilon > 0, \forall A \in (B_X)_+, \exists n \in \mathbb{N} \text{ s.t. } \mu (A \cap T^{-n} A \cap [||\varphi_n - a|| < \epsilon]) > 0 \right\}
\]

It follows from the cocycle equation \( \varphi_{n+m} = \varphi_n + \varphi_m(T^n x) \) that the set of essential values is a closed subset (under addition) of \( \mathbb{R} \) and therefore it is of the form \( \emptyset, \{0\}, \{0\} \cup a\mathbb{Z} (a \in \mathbb{R}) \) or \( \mathbb{R} \). The skew product \( \mathcal{T}_\varphi \) is ergodic if and only if \( T \) is ergodic and \( e(S, \varphi) = \mathbb{R} \).

We will be interested in the Maharam extension \( \tilde{T} \) which is the skew product extension of an invertible transformation \( T : (X, B, \mu) \odot \) with \( \varphi(x) = \log T'(x) \), the Radon-Nykodym cocycle. In the
case when the Maharam extension is ergodic we say that $T$ is of type III$_1$ ($e(T, \log \frac{d\nu S}{d\mu}) = \mathbb{R}$ and $T$ is ergodic). In the case where $T$ is conservative and there exists a $\mu$- equivalent $\sigma$-finite invariant measure the essential value set is $e(T, \log \frac{d\nu S}{d\mu}) = \{0\}$.

2.3. The tail and the orbital equivalence relation of a transformation. For a more detailed discussion of the contents of this subsection see [KM].

Let $(X, B_X)$ be a standard measure space. An equivalence relation on $X$ is a set $R \subset X \times X$ such that the relation $x \sim y$ if and only if $(x, y) \in R$ is an equivalence relation. It is measurable if $R \subset B_X \otimes B_X$. Given an equivalence relation $R$ and a set $A \in B_X$, the saturation of $A$ is the set

$$R(A) := \bigcup_{x \in A} R_x,$$

where $R_x := \{y \in X : (x, y) \in R\}$. Given a measure $\mu$ on $X$, we say that $R$ is $\mu$-ergodic if for each $A \in B_X$,

$$R(A) \in \{\emptyset, X\} \text{ mod } m.$$

An equivalence relation is finite (respectively countable) if for all $x \in X$, $R_x$ is a finite (countable) set. It is hyperfinite if there exists an increasing sequence of finite subequivalence relation $E_1 \subset E_2 \subset \cdots \subset R$ such that

$$R = \bigcup_{n=1}^{\infty} E_n.$$

Given a non singular non-invertible transformation $(X, B_X, \nu, S)$ we define the orbit equivalence relation on $X \times X$

$$R_S := \{(y_1, y_2) \in X \times X : \exists n, m \in \mathbb{N}, S^n y_1 = S^m y_2\}.$$

and the tail relation, which we denote by $T(S)$, by

$$T(S) = \{(y_1, y_2) : \exists n \in \mathbb{N}, S^n y_1 = S^n y_2\}.$$

A transformation is exact if for all $A \in B$,

$$T(S)_A \in \{\emptyset, Y\} \text{ mod } \nu.$$

By [WeiSIS] an equivalence relation is hyperfinite if and only if it is an orbit relation of a non singular transformation. Therefore if $T_S$ is hyperfinite, which is true in our setting since the shift is finite to one, there exists a non-singular transformation $V$ of $(Y, B_Y, \nu)$, which we call the tail action of $S$, such that

$$R_V = T_S.$$

It follows that $S$ is exact if and only if $V$ is ergodic.
A function $\hat{\varphi} : \mathcal{R} \to \mathbb{R}$ is an orbital cocycle if for every $x, y, z \in X$ in the same equivalence class of $\mathcal{R}$, 

$$\hat{\varphi}(x, y) = \hat{\varphi}(x, z) + \hat{\varphi}(z, y).$$

To every function $\varphi : X \to \mathbb{R}$ corresponds an orbital cocycle $\hat{\varphi}$ on $\mathcal{T}_S$ (notice that the sum is actually a finite sum) defined by

$$\hat{\varphi}(y_1, y_2) := \sum_{n=0}^{\infty} \{\varphi(S^n y_1) - \varphi(S^n y_2)\}, \ (y_1, y_2) \in \mathcal{T}(S).$$

and the $\mathcal{R}_V$-cocycle $\psi$ defined by

$$\psi(y) = \hat{\varphi}(y, V y).$$

The following fact shows that the skew product $S_{\varphi}$ is exact if and only if $V_{\psi}$ is ergodic where $V$ is the tail action of $S$ and $\psi$ is its corresponding cocycle.

**Fact 2.1.** [ANS] Let $(Y, \mathcal{B}_Y, \nu, S)$ be a non singular and non-invertible transformation and $(Y, \mathcal{B}_Y, \nu, V)$ its associated tail action. Let $\varphi : Y \to \mathbb{R}$ be a function and $\psi$ the corresponding $\mathcal{R}_V$ cocycle. Then

$$\mathcal{T}_{S_{\varphi}} = \mathcal{R}_{V_{\psi}}.$$

2.4. **The Zero Type property and dissipative transformations:** Given two measures on $(X, \mathcal{B})$ we can define the Hellinger Integral [Kak, Kos] by

$$\rho(\mu, \nu) = \int_X \sqrt{\frac{d\mu}{d\lambda}} \sqrt{\frac{d\nu}{d\lambda}} d\lambda$$

where $\lambda$ is any measure on $X$ such that $\nu \ll \lambda$ and $\mu \ll \lambda$.

If $T$ is a non singular transformation of $(X, \mathcal{B}, \mu)$ then since $T^n \mu \sim \mu$ we have

$$\rho(n) := \rho(\mu, T^n \mu) = \int_X \sqrt{T^n(x)} d\mu(x).$$

A transformation is Zero-Type (sometimes also called mixing) if the maximal spectral type of its Koopman operator defined by

$$\forall f \in L^2(X, \mu), \ U_T f := \sqrt{T} \cdot f \circ T$$

is a Rajchman measure. This is equivalent to the condition: For every $f, g \in L^2(X, \mu),$

$$\int_X U_T^n f \cdot g d\mu \xrightarrow{n \to \infty} 0.$$ 

Note that when $T$ is probability preserving one needs to restrict the class of functions to $L^2(X, \mu) \ominus \mathbb{C}$. The next lemma will be used to get a necessary criterion for conservativity of Bernoulli shifts.
Lemma 2.2. If \((X, \mathcal{B}, \mu, T)\) is zero type and \(\sum_{n=1}^{\infty} \rho(\mu, T^n_\ast \mu) < \infty\) then \(T\) is dissipative.

Proof. Since
\[
\mu \left( \left| T^n \right| > 1 \right) \leq \int_X \sqrt{T^n} d\mu = \rho(\mu, T^n_\ast \mu)
\]
and the right hand side is summable, it follows from the Borel Cantelli lemma that for almost every \(x \in X\) there exists \(N(x) \in \mathbb{N}\) such that for every \(n > N(x)\),
\[
T^n(x) \leq \sqrt{T^n(x)} \leq 1.
\]
In addition the summability condition on \(\rho(\mu, T^n_\ast \mu)\) ensures that
\[
\sum_{n=1}^{\infty} \sqrt{T^n} < \infty \text{ a.e. } d\mu.
\]
Therefore by comparison of sums we have that
\[
\sum_{n=1}^{\infty} T^n(x) < \infty \text{ a.e. } d\mu
\]
and so \(T\) is dissipative. \(\square\)

3. Half stationary Bernoulli Shifts

3.1. Non Singular Bernoulli Shift. Let \(X = \{0, 1\}^\mathbb{Z}, \mathcal{B} = \mathcal{B}_X, X^+ = \{0, 1\}^\mathbb{N}\) and \(\mathcal{B}^+ = \mathcal{B}_X^+\). We will write \(\sigma\) for the one-sided shift on \(X^+\) and \(T\) for the full shift on \(X\).

A product measure \(P = \prod_{k=-\infty}^{\infty} P_k \in \mathcal{P}(X)\) is **half stationary** if there exists \(p \in (0, 1)\) such that for all \(k \leq 0\),
\[
P_k(0) = 1 - P_k(1) = p.
\]

We will consider the case \(p = \frac{1}{2}\). The case of general \(p\) being similar.

Thus the general form of a half stationary product measure (with \(p = \frac{1}{2}\)) is
\[
(3.1) \quad P_k(0) = 1 - P_k(1) = \begin{cases} \frac{1-a_i}{2} & k \in \mathbb{N} \\ \frac{1}{2} & k \leq 0 \end{cases},
\]
where \(a_i \in (-1, 1)\).

Let \(P^+ = \prod_{k=1}^{\infty} P_k\) denote the measure of \(P\) restricted to \(X^+\). If \(P\) is half stationary, then the full shift \(T\) is the natural extension, in the sense of Silva and Thieullen [ST], of the one sided shift.
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\[ X^+, \mathcal{B}, P^+ = \prod_{k=1}^{\infty} P_k, \sigma \]. Since by Kolmogorov’s 0 – 1 Law the one sided shift is exact, the full shift is a $K$-transformation. Conversely every $K$-Bernoulli shift such that $T'$ is $\mathcal{B}^+$ measurable is a shift with a half stationary measure. We call such transformations non-singular $K$-shifts.

The following gives conditions on the product measures so that the shift is non-singular and ergodic.

**Theorem 3.1.** Let $P$ be of the form (3.1). Then

1. The shift $(X, \mathcal{B}, P, T)$ is non singular if and only if for all $n \in \mathbb{N}$, $|a_n| \neq 1$ and

\[
\sum_{k=0}^{\infty} \left\{ \left( \sqrt{P_k(0)} - \sqrt{P_{k+1}(0)} \right)^2 + \left( \sqrt{P_k(1)} - \sqrt{P_{k+1}(1)} \right)^2 \right\} < \infty.
\]

2. For every $n \in \mathbb{N}$,$\quad T^n'(x) = dP \circ T^n dP(x) = \prod_{k=1}^{\infty} P_{k-n}(w_k) / P_k(w_k)$.

3. If the shift is conservative then it is ergodic.

4. There is an absolutely continuous invariant probability if and only

\[
\sum_{k=1}^{\infty} a_k^2 < \infty.
\]

5. There exists constants $c, C > 0$ such that

\[
c \cdot d(P, T^n_* P) \leq -\log (\rho (P, T^n_* P)) \leq C \cdot d(P, T^n_* P)
\]

where

\[
d\left( \prod_{i \in \mathbb{Z}} P_i, \prod_{i \in \mathbb{Z}} Q_i \right) = \sum_{i \in \mathbb{Z}} \left\{ \left( \sqrt{P_i(0)} - \sqrt{Q_i(0)} \right)^2 + \left( \sqrt{P_i(1)} - \sqrt{Q_i(1)} \right)^2 \right\}.
\]

**Proof.** (1) and (2) follow from Kakutani’s Theorem, [Kak] on equivalence of product measures. Parts (3) and (4) are in [Kre]. (5) is an observation of Kakutani. $\square$

3.1.1. **The Odometer as the tail action of the shift.** We will also consider the odometer action $\tau$ on $X^+$ given by

\[
\tau \left( 1, 1, ..., 1, 0, w \right)_{n \text{-times}} = \left( 0, 0, ..., 0, 1, w \right)_{n \text{-times}}.
\]

The odometer and the one sided shift satisfy

\[ R_\tau = T_\sigma. \]
A calculation shows that

$$\tau'(x) = \frac{P_{\phi(x)}(1)}{P_{\phi(x)}(0)} \prod_{k=1}^{\phi(x)-1} \frac{P_k(0)}{P_k(1)}$$

where

$$\phi(x) := \min\{n \geq 1 : x_n = 0\}.$$

The odometer satisfies the so called *Odometer Property*, which states that for every $N \in \mathbb{N}$ and $x \in X^+$,

$$\left\{\left((\tau^k x)_1, (\tau^k x)_1, \ldots, (\tau^k x)_N\right) : k = 0, 1, \ldots, 2^N - 1\right\} = \{0, 1\}^N.$$

Using this fact one shows that for every $n \in \mathbb{N}$,

$$\tau^{(2n)'}(x) = \tau' \circ \sigma^n(x).$$

This can also be deduced from the fact that for all $n \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$,

$$\left((\tau^{2n} x)\right)_j = x_j.$$

This property plays a crucial role in calculating the essential values of the odometer action. See the proof of Lemma 3.5 below.

### 3.2. Statement of the main theorem and the Answer to Krengel’s question.

**Theorem 3.2.** For every $(X, \mathcal{B}, P, T)$ a conservative and non singular $K$-shift without an absolutely continuous invariant probability measure the Maharam extension is a $K$-transformation.

As a corollary we get a negative answer to Krengel’s question for non singular $K$-shifts.

**Corollary 3.3.** A conservative, ergodic, $K$—non singular Bernoulli shift is either of type $\text{III}_1$ or type $\text{II}_1$.

**Proof.** Assume that there exists no a.c.i.p. By Maharam’s theorem, the Maharam extension is conservative and by Theorem 3.2 it is $K$-$\sigma$-finite measure preserving transformation. Therefore by [Par] it is ergodic and so the shift is of type $\text{III}_1$. \hfill $\square$

A non singular transformation is of stable type $\text{III}_\lambda$ if for every ergodic probability preserving transformation $(Y, \mathcal{C}, \nu, S)$ the cartesian product $T \times S$ is of type $\text{III}_\lambda$. Bowen and Nevo [BN] used actions of stable type $\text{III}_\lambda$ in order to obtain ergodic theorems for measure preserving actions of countable groups. They ask which groups admit an action of stable type $\text{III}_\lambda$ with $\lambda > 0$. As a corollary of Theorem 3.2 we get the first examples of such $\mathbb{Z}$-actions.
Corollary 3.4. A conservative, ergodic, \( K \)-non singular Bernoulli shift such that
\[
\sum_{k=1}^{\infty} \left( P_k(0) - \frac{1}{2} \right)^2 = \infty
\]
is of stable type \( \text{III}_1 \).

Proof. Let \((Y,\mathcal{C},\nu,S)\) be an ergodic probability preserving transformation, \(\tilde{T}\) be the Maharam extension of the shift \(T\) and \(M_{T \times S}\) denote the Maharam extension of \(T \times S\).

Since the Maharam extension \(\tilde{T}\) is conservative then \(\tilde{T} \times S\) is conservative. It follows from [Aa Thm 2.7.6 and Corr 3.1.8] that \(\tilde{T} \times S\) is ergodic. Since \(\tilde{T} \times S = M_{T \times S}\), it follows that the Maharam extension of \(T \times S\) is ergodic and \(T \times S\) is of type \(\text{III}_1\). \(\Box\)

Remark. If \(S\) is an infinite measure preserving transformation such that \(\tilde{T} \times S\) is conservative then \(\tilde{T} \times S\) is ergodic. Thus by Dye’s Theorem \(\tilde{T} \times S\) is orbit equivalent to \(\tilde{T}\).

3.3. The proof of Theorem 3.2. By Theorem 2, the Radon-Nykodym cocycle \(\varphi(x) := \log T'(x)\) is \(\mathcal{B}^+\) measurable.

It follows that the Maharam extension of \(T\) is the natural extension of the skew product \((X^+ \times \mathbb{R}, \mathcal{B}^+ \otimes \mathcal{B}_{\mathbb{R}}, P^+ \otimes e^s ds, \sigma_{\varphi})\). Since a transformation is \(K\) if and only if it is a natural extension of an exact transformation, in order to show that the Maharam extension of the two sided shift is \(K\), we will show that the skew product extension \(\sigma_{\log T'}\) is exact.

This will be done in two steps. First we show that the odometer \((X^+, \mathcal{B}^+, P^+, \tau)\) is of type \(\text{III}_1\) and then we show that
\[
T \left( \sigma_{\log T'} \right) = R \left( \tau_{\log T'} \right),
\]
thus the tail action is ergodic.

Lemma 3.5. Let \(P\) be as in (3.1). If the shift is conservative and there exists no a.c.i.p then:

1. There exists a subsequence \(\{a_{n_k}\}\) such that \(\lim_{k \to \infty} a_{n_k} = 0\).
2. The odometer \((X^+, \mathcal{B}^+, P^+, \tau)\) is of type \(\text{III}_1\).

Proof. Denote by \(\mathfrak{A} = \left\{ a \in \mathbb{R} : \exists n_k \to \infty, a_{n_k} \xrightarrow[k \to \infty]{} a \right\}\) the set of limit points of the sequence \(\{a_n\}\).

1. Assume that \(0 \notin \mathfrak{A}\). We will show that then \(\sum_{n=1}^{\infty} \rho(P, T^n P) < \infty\) and so by Lemma 2.2 \(T\) is dissipative.

Since \(0\) is not a limit point of \(\{a_n\}\), there exists an \(\epsilon > 0\) and \(N \in \mathbb{N}\) such that for all \(i > N\),
\[
\sqrt{\frac{1 - a_i}{2} - \sqrt{\frac{1}{2}}} > \epsilon.
\]
Therefore for every $n > N$,

$$
\begin{align*}
d(P, T^n_* P) &= \sum_{i \in \mathbb{Z}} \left\{ (\sqrt{P_i(0)} - \sqrt{P_{i-n}(0)})^2 + (\sqrt{P_i(1)} - \sqrt{P_{i-n}(1)})^2 \right\} \\
&\geq \sum_{i=N}^{n} \left( \sqrt{P_i(0)} - \sqrt{P_{i-n}(0)} \right)^2 \\
&= \sum_{i=N}^{n} \left( \sqrt{\frac{1-a_i}{2}} - \sqrt{\frac{1}{2}} \right)^2.
\end{align*}
$$

The last equality follows from the fact that $P_k = \left( \frac{1}{2}, \frac{1}{2} \right)$ for $k \in \mathbb{Z} \setminus \mathbb{N}$. Therefore by (3.5) we have that

$$
d(P, T^n_* P) \geq (n - N) \epsilon^2.
$$

The conclusion follows from (3.3) and Lemma 2.2.

(2) Let $P$ be a half stationary product measure such that the shift is conservative and there is no a.c.i.p.

One can show that we can choose a subsequence such that $\lim_{n \to \infty} a_{n_k} = 0$ and $\sum_{k=1}^{\infty} a_{n_k}^2 = \infty$ and then use standard techniques.

Alternatively we can argue as follows: Since there is no a.c.i.p. then

$$
\sum_{n=1}^{\infty} a_{n_k}^2 = \infty.
$$

Therefore if $\mathcal{A} = \{0\}$ (lim $a_n = 0$) then the odometer is of type III$_1$ by [DKQ, Prop. 3.1.].

Otherwise there is $0 < \alpha < 1$ such that $\{0, \alpha\} \subset \mathcal{A}$. It follows from the non-singularity condition (3.2) that

$$[0, \alpha] \subset \mathcal{A}.$$ 

We show that $e(\tau, \log \tau') = \mathbb{R}$ by showing that for every $p \in \mathcal{A} \setminus \{-1, 1\},$

$$
\log \frac{1 + p}{1 - p} \in e(\tau, \log \tau'),
$$

so the set of essential values contains an interval. This will be done by establishing the conditions of [DKQ, Lemma 2.1].

Let $p \in \mathcal{A}$ and $a_{n_k} \xrightarrow{k \to \infty} p$.

Let

$$
C = [c]_1^n := \{ x \in X^+ : x_i = c_i \ \forall i \in [1, n] \}.
$$

be a cylinder set and write

$$
C_{n_k} = C \cap \{ x \in X^+ : x_{n_k} = 0 \}.
$$
It follows from (3.4) that for every $k \in \mathbb{N}$ such that $n_k > n$,
\[ \log \tau^{(2n_k)'} \bigg|_{C_{n_k}} = \log \frac{1 + a_{n_k}}{1 - a_{n_k}}. \]
Therefore
\[
(3.6) \quad P^+ \left( C \cap \tau^{-2n_k} C \cap \left\{ \log \tau^{(2n_k)'} = \log \frac{1 + a_{n_k}}{1 - a_{n_k}} \right\} \right) \geq P^+ (C_{n_k}) = \left( \frac{1 - a_{n_k}}{2} \right) P^+ (C).
\]
Given $\epsilon > 0$, we can choose $k$ large enough such that
\[
\left( \frac{1 - a_{n_k}}{2} \right) > \frac{1 - p}{4} := \beta > 0.
\]
and
\[
\left| \log \frac{1 + a_{n_k}}{1 - a_{n_k}} - \log \frac{1 + p}{1 - p} \right| < \epsilon.
\]
Then by (3.6) we get
\[
P^+ \left( C \cap \tau^{-2n_k} \cap \left\{ \left| \log \tau^{(2n_k)'} - \log \frac{1 + p}{1 - p} \right| < \epsilon \right\} \right) \geq \beta P^+ (C).
\]
Thus the conditions of [DKQ] Lemma 2.1 are satisfied with
\[
\gamma = \left( 0, \ldots, 0, 1, 0 \right)_{n_k - 1}
\]
and
\[
\mathcal{U} = C_{n_k}.
\]
Hence $\log \frac{1 + p}{1 - p}$ is an essential value for $\log \tau'$.

Lemma 3.6. Let $P$ be defined by (3.1), then
\[
\psi(x) = \log \tau'(x),
\]
where $\psi(x)$ is the tail-cocycle corresponding to $\varphi = \log T'$.

Proof. Since
\[
\sigma^n x = \sigma^n \tau x \iff n \geq \phi(x)
\]
it follows that
\[
\psi(x) = \sum_{k=0}^{\phi(x)-1} \left\{ \varphi \left( \sigma^k x \right) - \varphi \left( \sigma^k \tau x \right) \right\} = \varphi_{\phi(x)}(x) - \varphi_{\phi(x)}(\tau x).
\]
This together with Theorem 3.1 and the fact that
\[(\tau x)_k = \begin{cases} 1 - x_k, & k \leq \phi(x) \\ x_k & k > \phi(x) \end{cases},\]
yields
\[
\psi(x) = \log \left( \prod_{k=1}^{\phi(x)} \left[ \frac{P_{k-\phi(x)}(x_k)}{P_k(x_k)} \right] \right)
\]
\[
= \log \left( \prod_{k=1}^{\phi(x)} \left[ \frac{P_{k-\phi(x)}(x_k)}{P_k(x_k) (1 - x_k)} \right] \right).
\]
Since for all \(k < 0, P_k \equiv (1/2, 1/2),\)
\[\forall k \leq \phi(x), \quad \frac{P_{k-\phi(x)}(x_k)}{P_k(x_k) (1 - x_k)} = 1,\]
we see that
\[
\psi(x) = \log \left( \prod_{k=1}^{\phi(x)} \frac{P_k(1 - x_k)}{P_k(x_k)} \right) = \log \tau'(x).
\]
\[
□
\]

Proof of Theorem 3.2. Since the Maharam extension \(\tilde{T}\) is the natural extension of \(\sigma_\varphi\), we need to show that \(\sigma_\varphi\) is exact.

The odometer \(\tau\) is the tail action of the shift \(\sigma\). It follows from Lemma 3.6 that,
\[
\mathcal{T}(\sigma_\varphi) = \mathcal{R}(\tau_{\log \tau'}).\]
By Lemma 3.6, \(\tau_{\log \tau'}\) is ergodic (\(\tau\) is type \(\text{III}_1\)) and therefore \(\sigma_\varphi\) is exact. \(\square\)

3.4. Countable State space. By following the same arguments of the previous section, one can show that if \(X = \{1, \ldots, n\}^\mathbb{Z}\), \(T\) is the full shift and \(P\) is a half stationary measure on \(X\) which is not equivalent to a stationary product measure, then the Maharam extension is \(K\).

Consider now the full shift on a countable state space. That is \(X = \mathbb{N}^\mathbb{Z}\), \(T\) is the shift and there exists a probability measure \(p \in \mathcal{P}(\mathbb{N})\) and a sequence \(\{p_j(\cdot)\}_{j=1}^\infty\) of probability measures on \(\mathbb{N}\) so that
\[(3.7) \quad P_k(\cdot) = \begin{cases} p(\cdot), & k \leq 0 \\ p_k(\cdot), & k > 0. \end{cases}\]
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The condition for non singularity of the shift becomes now

$$d(P, T^* P) := \sum_{k=1}^{\infty} \sum_{j \in \mathbb{N}} \left( \sqrt{P_k(j)} - \sqrt{P_{k-1}(j)} \right)^2 < \infty.$$ 

and it is still true that there exists constants $M, m > 0$ so that for every $n \in \mathbb{N}$

$$m \cdot d(P, T^n P) \leq - \log \rho(P, T^n P) \leq M \cdot d(P, T^n P).$$

Therefore we can As before let $\sigma : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ be the one sided shift and $P^+ = \prod_{k \in \mathbb{N}} P_k$. The tail relation of $\sigma$ is still hyperfinite. For example if we choose

$$\mathcal{T}_N = \left\{ (x, y) : y = x \text{ or } \exists N \in \mathbb{N}, \forall n > N, x_n = y_n \text{ and } \max_{1 \leq j \leq N} (x_j, y_j) \leq N \right\},$$

then $\mathcal{T}_N$ is an increasing sequence of finite subequivalence relations with $\cup_N \mathcal{T}_N = \mathcal{T}(\sigma)$. However, unlike the finite state space case, the odometer is no longer the tail action. In this case it is easier to look at the holonomies of $\mathcal{T}(\sigma)$. An holonomy is a one to one transformations $\phi : \text{Dom}(\phi) \rightarrow \text{Ran}(\phi)$, here $\text{Dom}(\phi), \text{Ran}(\phi) \subset X = \mathbb{N}^\mathbb{N}$, where for every $x \in \text{Dom}(\phi)$,

$$(x, \phi(x)) \in \mathcal{T}(\sigma).$$

The ratio set condition for the tail action can be reformulated in the following way.

For the shift one can generalize Lemma 3.5 for the countable state case using holonomies of the form $f : [a]^n \rightarrow [b]^n$,

$$f(a_1, \ldots, a_n, x) := (b_1, b_2, \ldots, b_n, x).$$

In this way one can prove the same result. Either $P \sim \prod p$ or it’s Maharam extension is a $K$-transformation.

4. Examples

In [Kre, Ham] examples of conservative shifts were constructed without an a.c.i.p. It follows from Theorem 3.2 that the Maharam extension is $K$ and that those shifts are of type $\text{III}_1$. In these examples one has

$$\lim_{n \rightarrow \infty} P_n(0) = \frac{1}{2}. \quad (4.1)$$
We will give two more examples here. One of a dissipative half stationary shift with

$$\lim_{n \to \infty} P_n(0) = \frac{1}{2}$$

which shows that (4.1) is not sufficient for conservativity. The other is a conservative half stationary product measure with

$$\liminf_{n \to \infty} P_k(0) = \frac{1}{4}, \quad \limsup_{n \to \infty} P_k(0) = \frac{1}{2},$$

Together those examples show that Lemma 3.5.1 is all we can say about limit points of $a_n$.

Remark 4.1. Michael Grewe in his Master thesis [Gre] has constructed a different example of a dissipative shift with $P_k(0) \to \frac{1}{2}$. His method relies on the strong law of large numbers and an inductive construction. We include here a new example as the method of proof and the measure are more simple.

4.1. Dissipative example. Define a product measure by

$$P_n(0) = \begin{cases} 
\frac{1}{2} - \frac{2}{n}, & n \geq 2 \\
\frac{1}{2}, & n < 2 
\end{cases}$$

Since

$$\sum_{k=0}^{\infty} \left\{ \left( \sqrt{P_k(0)} - \sqrt{P_{k+1}(0)} \right)^2 + \left( \sqrt{P_k(1)} - \sqrt{P_{k+1}(1)} \right)^2 \right\} < \infty,$$

the shift $\left( \{0,1\}^\mathbb{Z}, P, T \right)$ is non singular. In addition

$$d(P, P \circ T^n) \geq \sum_{k=0}^{n} \left\{ \left( \sqrt{P_k(0)} - \sqrt{P_{k-n}(0)} \right)^2 + \left( \sqrt{P_k(1)} - \sqrt{P_{k-n}(1)} \right)^2 \right\} = \sum_{k=2}^{n} \left\{ 2 - \sqrt{1 - \frac{4}{k}} - \sqrt{1 + \frac{4}{k}} \right\}.$$

It follows from the Taylor expansion of $\sqrt{1 + x}$ that

$$2 - \sqrt{1 - \frac{2}{k}} - \sqrt{1 + \frac{2}{k}} = \frac{2\sqrt{2} - 1}{k} + O_{k \to \infty} \left( \frac{1}{k^2} \right).$$

Therefore there exists a constant $C \in \mathbb{R}$ such that

$$d(P, P \circ T^n) \geq \left( 2\sqrt{2} - 1 \right) \sum_{k=2}^{n} \frac{1}{k} + C.$$
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Since $\sum_{k=2}^{n} 1/2 \propto \log(n)$ and $\log(P, P \circ T^n) \propto d(P, P \circ T^n)$, it follows that

$$\sum_{n=1}^{\infty} \rho(P, P \circ T^n) < \infty.$$  

By Lemma 2,2 the shift is dissipative.

4.2. The “weird” conservative example. Given $k \in \mathbb{N}$ set

$$\lambda_n^{(k)} = \begin{cases} 1 + \frac{n}{2^k}, & n \in [0, 2^{k-1}] \\ 2 - \frac{n}{2^k}, & n \in [2^{k-1}, 2 \cdot 2^{k-1}] \\ 1, & \text{otherwise} \end{cases}$$

and let $P^{(k)}$ be the product measure on $X$ with factor measures

$$P_n^{(k)}(1) = \frac{\lambda_n^{(k)}}{1 + \lambda_n^{(k)}} = 1 - P_n^{(k)}(0).$$

Our example of a conservative product measure with

$$\lim_{k \to \infty} P_k(1) = \frac{3}{4} > \frac{1}{2} = \lim inf P_k(1)$$

consists of large intervals where $P_k(0)$ is exactly $1/2$ followed by large intervals of the form $[N, N + 2^k]$ where $P_n(1)$ equals $P_n^{(k)}(1)$ (a slow increase to $3/4$ followed by a small decrease back to $1/2$). Then this segment is followed by a larger segment where $P_k(0) = 1/2$ and so on. The main difficulty in showing that

$$\sum T_n' = \infty$$

is in showing that for some $k'$s we have $N(k)$ such that

$$T^{k'}(w) \approx \prod_{n=0}^{N(k)} \frac{P_{n-k}(w_n)}{P_n(w_n)}$$

on a set of positive measure. For that purpose we need the following lemma which states that if $k$ is large enough with respect to $m$ then the derivatives of the shift under the measure $P^{(k)}$ are bounded from below up to time $m$ on a set of large measure.

**Lemma 4.2.** Given $m$ and $t$ there exists a $k \in \mathbb{N}$ such that

$$P^{(k)} \left( \inf_{t \leq m} T_n^{(k)}(w) \geq e^{-2^{-t}} \right) \geq 1 - 2^{-t}.$$
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Proof. It follows from \(2\) and the structure of \(P^{(k)}\) that for \(l < 2^{k-1}\),

\[
\log \left( T_{(k)}^{l'}(w) \right) = \log \left( \prod_{n=0}^{2^{k-1}+l} \frac{P_{n-1}^{(k)}(w_n)}{P_{n}^{(k)}(w_n)} \right) = \log \left( \prod_{n=0}^{2^{k-1}+l} \left( \frac{\lambda_{n-l}^{(k)}}{\lambda_n^{(k)}} \right)^{w_n} \right)
\]

\[
\sum_{n=0}^{2^{k-1}+l} w_n \left( \log \lambda_{n-l}^{(k)} - \log \lambda_n^{(k)} \right).
\]

Using the fact that for every \(n < 2^{k-1}\),

\[
\lambda_n^{(k)} = \lambda_{2^{k}-n}^{(k)}
\]

and a rearrangement of the sum one has

\[\text{(4.2)} \quad \log \left( T_{(k)}^{l'}(w) \right) = \sum_{n=0}^{2^{k-1}+l} Y_{n,k,l} + f(k,l)(w),\]

where

\[Y_{n,k,l} := \left( \log \lambda_{n-l}^{(k)} - \log \lambda_n^{(k)} \right) \left( w_{n+l} - w_{2k-n} \right)\]

and

\[f(k,l)(w) = \left( \sum_{n=2^{k}-1}^{2^{k-1}+l} + \sum_{n=0}^{l} + \sum_{n=2^{k}}^{2^{k-1}+l} \right) \left[ w_n \left( \log \lambda_{n-l}^{(k)} - \log \lambda_n^{(k)} \right) \right].\]

By a trivial bound

\[\text{(4.3)} \quad |f(k,l)(w)| \leq 3l \max_{n \in \mathbb{N}} \left( \log \lambda_{n-l}^{(k)} - \log \lambda_n^{(k)} \right) \leq \frac{3l^2}{2^k}.\]

To bound the first term notice that

\[\mathbb{E}_{P^{(k)}}(Y_{n,k,l}) \propto \frac{l^2}{2^{2k}} \quad \text{and} \quad \text{Var}_{P^{(k)}}(Y_{n,k,l}) \propto \frac{l}{2^{3k}}.\]

By independence of the \(Y_{n,k,l}'s\) we have

\[\text{Var} \left( \sum_{n=0}^{2^{k-1}-l} Y_{n,k,l} \right) \propto \frac{l^2}{2^{2k}} \ll \left( \frac{l^2}{2^k} \right) \propto \mathbb{E} \left( \sum_{n=0}^{2^{k-1}-l} Y_{n,k,l} \right).\]

It follows from this equation, Equations \(\text{(4.3)}, (4.2)\) and Chebyshev’s inequality that if \(k\) is large enough relative to \(m\) and \(t\) then for every \(l < m\),

\[P^{(k)} \left( T_{(k)}^{l'}(w) \leq e^{2-t} \right) \leq \frac{e^{-t}}{m}.\]
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The Lemma follows from a union bound. □

Now we are ready to construct the product measure.

Let $P = \prod P_k$ where for $k \leq 0$,

$$P_k(0) = P_k(1) = \frac{1}{2}.$$ 

To define $P_k$ for positive $k$ we choose inductively two subsequences $\{n_t\}_{t \in \mathbb{N}}, \{m_t\}_{t=0}^{\infty}$ with

$$0 < n_t < m_t < n_{t+1}$$

and $m_0 = 0$. The factor measures will be fair coins for $j \in [n_t, m_t]$ and on the other segments we will choose them according to $P^{(k_t)}$.

**Definition of $n_t$ given $m_{t-1}$ and $P|_{[m_{t-1}, n_t]}$:**

By Lemma 4.2 there exists $k_t$ such that

$$P^{(k_t)} \left( \inf_{l \leq m_t} T_{(k_t)}^l(w) \geq e^{-2^{-t}} \right) \geq 1 - 2^{-t}. $$

Let $n_t = m_t + 2^{k_t}$. Now for $m_{t-1} \leq j < n_t$ set

$$P_j = P^{(k_t)}_{j-m_{t-1}}.$$ 

**Definition of $m_t$ given $n_t$ and $P|_{[n_t, m_t]}$:** Let

(4.4) $$m_t = n_t + 2^{n_t}.$$  

For conclusion

$$P_j(1) = 1 - P_j(0) = \begin{cases} \frac{1}{2}, & j < 0 \\ P^{(k_t)}_{j-m_{t-1}}, & m_{t-1} \leq j < n_t \\ \frac{1}{2}, & n_t \leq j < m_t \end{cases}.$$ 

The measure satisfies

$$\liminf_{k \to \infty} P_k(1) = \frac{1}{2}$$

and

$$\limsup_{k \to \infty} P_k(1) = \lim_{k \to \infty} P_{m_{t-1} + 2^{k_{t-1}}}(1) = \lim_{k \to \infty} P^{(k_t)}_{2^{k_{t-1}}}(1) = \frac{3}{4}.$$ 

**Proposition 4.3.** The shift $(X, \mathcal{B}, P, T)$ is conservative and ergodic and type $\text{III}_1$. 
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Sketch of proof: The first step will be to show that if $m < m_t$ then

$$T^{m'}(w) \geq \left( \frac{3}{2} \right)^{n_t} \prod_{u=t}^{\infty} T^{m'}_{(k_t)}(w(t))$$

where $\{w(t)\}_{t=1}^{\infty} \subset X$ are random sequences which are independent of one another and for each $t$, $w(t)$ is distributed as $P_{(k_t)}$.

Then we will use Lemma 4.2 to bound $T^{m'}$ for $m \in [n_t, m_t)$ on a set of positive measure. This will give us that $\mathcal{C} \neq \emptyset$ which by a result of Grewe, see Lemma 5.1, yields $X = \mathcal{C}$.

Lemma 4.4. For every $n_t \leq n < m_t$,

$$\frac{dP \circ T^n}{dP} = T^{m'}(w) = \left( \prod_{k=1}^{t} \prod_{u=m_{k-1}}^{n_k-1} \frac{1}{2P_u(w_u)} \right) \cdot \left( \prod_{l=t+1}^{\infty} \prod_{u=m_{l-1}}^{n_{l+1}-1} \frac{P_{u-n}(w_u)}{P(w_u)} \right).$$

Proof. This is a combination of the Theorem 3.1.2 and the fact that for every $k \notin \bigcup_{k=1}^{\infty} [m_{t-1}, n_t)$,

$$P_k(w_k) \equiv \frac{1}{2} \forall w_k \in \{0, 1\}.$$

Note that we also used the fact that for every $l > t$, and $n < m_t$

$$m_{l-1} - n > m_{t-1} - n_{t-1} > n_{t-1}$$

so the segments $[m_{t-1}, n_{t-1})$ do not overlap when we shift by $n$. □

Proof of Proposition 4.3. Set

$$A_t = \left\{ w \in X : \forall k \leq m_{t-1}, \prod_{u=m_t}^{n_{t+1}+k} \frac{P_{u-k}(w_u)}{P_u(w_u)} \geq e^{-2^{-t}} \right\}.$$

We have that $A_1, A_2, \ldots$ are independent and since

$$\prod_{u=m_t}^{n_{t+1}+n} \frac{P_{u-k}(w_u)}{P_u(w_u)} = \prod_{u=m_t}^{n_{t+1}+n} \frac{P_{u-m_{t-n}}(w_u)}{P_{u-m_t}(w_u)} = T^{m'}_{(k_{t+1})}(w|m_t, n_{t+1}+n)$$

and $P|[m_t, n_{t+1}+n] = P_{(k_t)}|[0, 2^{k_t+n}]$ we have by Lemma 4.2 and the choice of $k_t$ that

$$P(A_t) = P_{(k_{t+1})} \left( \inf_{k \leq m_t} T^{m'}_{(k_{t+1})} \geq e^{-2^{-t}} \right) \geq 1 - e^{-t}.$$

Set $A = \bigcap_t A_t$. Then

$$P(A) \geq \prod_{t=1}^{\infty} (1 - e^{-t}) > 0.$$
For every $m_{t-1} \leq n \leq m_t$, $l > t$ and $w \in A$ we have

$$\prod_{u=m_l}^{n_{l+1}+n_l} \frac{P_{u-k}(w_u)}{P_u(w_u)} \geq e^{-2^{-l}}.$$ 

Applying the last inequality together with Lemma 4.4 we see that for $w \in A$ and $n_{t-1} \leq n \leq m_t$,

$$T^{n'}(w) \geq \left( \prod_{k=1}^{t-1} \prod_{u=m_{k-1}}^{n_{k-1}} \frac{1}{2P_u(w_u)} \right) \cdot \prod_{j=l}^{\infty} e^{-2^{-j}} \geq e^{-1} \prod_{k=1}^{n_t} \frac{1}{2^{2^{k-l}}} = \frac{1}{e} \left( \frac{2}{3} \right)^{n_t}.$$ 

Therefore for every $w \in A$,

$$\sum_{n=1}^{\infty} T^{n'}(w) \geq \sum_{t=1}^{\infty} \sum_{u=n_{t-1}}^{m_t} T^{n'}(w) \geq e^{-1} \sum_{t=1}^{\infty} \left( \left( \frac{2}{3} \right)^{n_{t-1}} (m_t - n_{t-1}) \right) = \infty.$$ 

Here the last assertion follows from (4.4). Thus $A \subset \mathbb{C}$. By Lemma (5.1) the shift is conservative.

\[ \Box \]

5. Appendix

Here we give a proof of a result from [Grewe].

**Lemma 5.1.** [Grewe] Let $P$ be a product measure on $X$. Then if the factor measures are bounded away from 0 and 1 (e.g. $\exists p > 0$ s.t. $\forall k \in \mathbb{Z}, p < P_k(0) < 1 - p$) then the shift $(X, P, T)$ is either conservative or dissipative.

**Proof.** The condition on the factor measures ensures that for every $k \in \mathbb{Z}, w_1, x_1 \in \{0, 1\}$

$$c : = \min \left( \frac{p}{1-p}, \frac{1-p}{p} \right) \leq \frac{P_k(x_1)}{P_k(w_1)} \leq c^{-1}.$$ 

This means that if $x, w \in \{0, 1\}^\mathbb{Z}$ defer in only finitely many coordinates then there exists $M > 0$ s.t

$$\frac{1}{M} T^{n'}(x) \leq T^{n'}(w) = \prod_{k=1}^{\infty} \frac{P_{k-n}(w_k)}{P_k(w_k)} \leq M T^{n'}(x).$$ 

Therefore

$$\sum_{n=1}^{\infty} T^{n'}(w) = \infty \Leftrightarrow \sum_{n=1}^{\infty} T^{n'}(x) = \infty.$$
and so the conservative and the dissipative parts are in
\[ \cap \mathcal{F}_n \]
where \( \mathcal{F}_n \) is the sub sigma algebra generated by \( \{ w_k : |k| \geq n \} \). By the Zero One Law \( \mathcal{E} = X \) or \( \mathcal{D} = X \). \qed

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