BASES OF THE EQUIVARIANT COHOMOLOGIES OF
REGULAR SEMISIMPLE HESSENBERG VARIETIES

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Abstract. We consider bases for the cohomology space of regular semisimple Hessenberg varieties, consisting of the classes that naturally arise from the Bialynicki-Birula decomposition of the Hessenberg varieties. We give an explicit combinatorial description of the support of each class, which enables us to compute the symmetric group actions on the classes in our bases. We then successfully apply the results to the permutohedral varieties to explicitly write down each class and to construct permutation submodules that constitute summands of a decomposition of cohomology space of each degree. This resolves the problem posed by Stembridge on the geometric construction of permutation module decomposition and also the conjecture posed by Chow on the construction of bases for the equivariant cohomology spaces of permutohedral varieties.

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1. INTRODUCTION

Hessenberg varieties were introduced and investigated by De Mari, Shayman and Procesi [14, 13, 12] around 1990 as a generalization of subvarieties of the full flag variety $Fl(C^n)$ that arise in the study of Hessenberg matrices. Important varieties such as Springer fibers, Peterson varieties, toric varieties associated with the Weyl chambers as well as the flag varieties appear as examples of Hessenberg varieties, and there have been many interesting researches on the structure of the Hessenberg varieties. We refer the reader to the survey article [2] and references therein for more details on the known results of Hessenberg varieties.

Tymoczko applied GKM theory to regular semisimple Hessenberg varieties in [41, 40] to obtain combinatorial description of the (equivariant) cohomology spaces and successfully define Weyl group actions.
actions on them. Tymoczko’s Weyl group actions, called dot actions, provide a way to understand the structure of the cohomology of regular semisimple Hessenberg varieties.

Our main concern is on the structure of the equivariant cohomology of regular semisimple Hessenberg varieties $Hess(S,h)$ of type $A$ described as follows:

$$Hess(S,h) = \{ \{0\} \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n \mid SV_i \subset V_{h(i)} \text{ for all } 1 \leq i \leq n \} \subset \mathcal{F}(\mathbb{C}^n),$$

where $S$ is a given regular semisimple matrix and $h: \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a nondecreasing function satisfying $h(i) \geq i$ for all $i$, called a Hessenberg function. The toric variety $H_n$ associated with the Weyl chambers of type $A_{n-1}$, also known as the permutohedral variety, is $Hess(S,h)$ with the Hessenberg function $h$ given by $h(i) = i + 1$ for $i < n$. The symmetric group $\mathfrak{S}_n$, that is the Weyl group of type $A_{n-1}$, acts on the Weyl chambers of type $A_{n-1}$ and it naturally defines an action on $H^*(H_n)$. Throughout the whole paper, we will use the $\mathbb{C}$-coefficients for cohomology rings unless otherwise specified. It is known that this action coincides with Tymoczko’s dot action.

The $\mathfrak{S}_n$-module structure of $H^*(H_n)$ was first considered by Procesi in [30], then by Stanley in [33] in the language of symmetric functions. Following the work of Stanley, Stembridge constructed a graded $\mathfrak{S}_n$-module carrying permutation representation, which is isomorphic to $H^*(H_n)$ by defining combinatorial objects called codes in [36]. In the same paper, Stembridge posed the problem to give a geometric explanation of the fact that $H^*(H_n)$ is decomposed as direct sum of permutation representations. There has been no known solution to this problem.

Chromatic symmetric functions were introduced by Stanley in [34] as a generalization of the chromatic polynomials of a graph, and they were refined to chromatic quasi-symmetric functions by Shareshian and Wachs in [32]. The Stanley–Stembridge conjecture on chromatic symmetric functions states that the chromatic symmetric function of the incomparability graph of a $(3+1)$-free poset expands positively as a sum of elementary symmetric functions, that is $e$-positive, and it is refined by Shareshian and Wachs to a conjecture on chromatic quasisymmetric functions. A work by Guay-Paquet [20] shows that for the $e$-positivity conjecture it is enough to consider the incomparability graphs of natural unit interval orders which can be identified with a Hessenberg function.

In [32], Shareshian and Wachs made a conjecture that the regular semisimple Hessenberg varieties of type $A$ and the chromatic quasisymmetric functions are related in the following way, which was independently proved by Brosnan and Chow [6], and Guay-Paquet [21].

**Theorem 1.1.** For a Hessenberg function $h$,

$$\sum_k \text{ch}H^{2k}(Hess(S,h)) t^k = \omega X_{G(h)}(x, t),$$

where $\text{ch}$ is the Frobenius characteristic map and $\omega$ is the involution on the symmetric function algebra sending elementary symmetric functions to homogeneous symmetric functions; $\omega(e_i) = h_i$.

Theorem 1.1 plays a crucial role in connecting geometry, combinatorics and representation theory. The $e$-positivity conjecture on chromatic (quasi)symmetric functions translates into the statement that $\text{ch}H^*(Hess(S,h))$ is positively expanded as a sum of homogeneous symmetric functions:
**Conjecture 1.2** (Stanley–Stembridge [33, 34], Shareshian–Wachs [32]). Under the dot action of the symmetric group, the $(2k)$th cohomology space $H^{2k}(\text{Hess}(S, h))$ is decomposed as a direct sum of permutation modules $M^\lambda$ for each degree $2k$.

Conjecture 1.2 is proved to be true for some special cases; see [34, 8, 16, 11, 9, 24, 25]. Even in those cases, the geometric construction of permutation module decomposition for $H^*(\text{Hess}(S, h))$ has not been provided except for trivial cases. Our work in this paper is motivated by Conjecture 1.2, especially on the permutation module decomposition of $H^*(\text{Hess}(S, h))$. For the construction of permutation submodules inside $H^*(\text{Hess}(S, h))$, it is essential to choose a good basis that behaves well under the symmetric group action. This is not a trivial work and there are only some sets of cohomology classes conjectured to form good bases of $H^*(\text{Hess}(S, h))$ for certain $h$ or for some permutation modules of certain type: Erasing marks conjecture due to Chow (personal communication, cf. [10]) is for toric varieties $\mathcal{H}_n$ for example. We first consider a basis of $H^*_T(\text{Hess}(S, h))$ that naturally arises from a geometric structure of Hess$(S, h)$ and examine the properties of the basis elements. We then compute the symmetric group actions using our bases and finally apply the results to construct a permutation module decomposition of the cohomology spaces of the permutohedral variety. In the rest of the current section we summarize the work we have done.

Any regular semisimple Hessenberg variety Hess$(S, h)$ admits an affine paving, which is a Bialynicki–Birula decomposition (see [12]). Considering the closure $\Omega_{w, h}$ of each minus cell $\Omega^c_{w, h}$, we obtain an equivariant cohomology class $\sigma_{w, h}$ for each $w \in \mathcal{S}_n$ (see Definition 2.9). The set $\{\sigma_{w, h} \mid w \in \mathcal{S}_n\}$ forms a basis of the equivariant cohomology ring $H^*_T(\text{Hess}(S, h))$. Our basis satisfies nice properties which are already considered in other known results. Indeed, when Hess$(S, h) = F^l(\mathbb{C}^n)$, our basis coincides with the basis constructed by Tymoczko [41]. When Hess$(S, h)$ is a permutohedral variety, which is a smooth projective toric variety, then our basis is the ‘canonical basis’ provided by Pabiniak and Sabatini [28]. Moreover, the basis $\sigma_{w, h}$ is a ‘flow-up basis’ studied by Teff [37, 38] (see Proposition 2.11).

To study the action of $\mathcal{S}_n$ on each class $\sigma_{w, h}$, we have to specify the support of $\sigma_{w, h}$, that is, we need to describe elements $v \in \mathcal{S}_n$ such that $\sigma_{w, h}(v) \neq 0$. We note that the support $\sigma_{w, h}$ is the same as the fixed point set $\Omega^T_{w, h}$ (see Proposition 2.11). One of the primary goals of this manuscript is to present an explicit description of the support of $\sigma_{w, h}$ in terms of $w$ and $h$. In order to introduce our result, we prepare some terminologies (see Section 3 for precise definitions). For a positive integer $n$, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. Let $h$ be a Hessenberg function and $w \in \mathcal{S}_n$. We define a directed graph $G_{w, h}$ with the vertex set $[n]$ such that for each pair of indices $1 \leq j < i \leq n$, there is an edge $j \to i$ in the graph $G_{w, h}$ if and only if $j < i \leq h(j)$ and $w(j) < w(i)$. For example, when $n = 5$, the graph $G_{15342,(3,3,4,5,5)}$ is given as follows.

$$
\begin{array}{cccccc}
1 & \to & 2 & \to & 4 & \to & 5 \\
3 & \to & 4 & & & \\
\end{array}
$$

We say a vertex $i$ is reachable from $j$ if there exists a directed path from $j$ to $i$. For subsets $A = \{1 \leq a_1 < \cdots < a_k \leq n\}$ and $B = \{1 \leq b_1 < \cdots < b_k \leq n\}$ of $[n]$, we say that $A$ is reachable from $B$ if there exists a permutation $\sigma \in \mathcal{S}_k$ such that $a_{\sigma(d)}$ is reachable from $b_d$ for all $d = 1, \ldots, k$. We then compute the symmetric group actions using our bases and finally apply the results to construct a permutation module decomposition of the cohomology spaces of the permutohedral variety. In the rest of the current section we summarize the work we have done.

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in the graph $G_{w,h}$. For example, $\{3,4\}$ is reachable from $\{1,3\}$ while $\{5\}$ is not reachable from $\{3\}$ in the graph $G_{15342,(3,3,4,5,5)}$. Using these expressions, we state our first main theorem.

**Theorem A** (Theorem 3.13). Let $h$ be a Hessenberg function and $w \in \mathcal{S}_n$. An element $u \in \mathcal{S}_n$ is in $\text{supp}(\sigma_{w,h})$ if and only if \{w$^{-1}$(1), ..., w$^{-1}$(j)\} is reachable from $[j]$ in the graph $G_{w,h}$ for all $j = 1, \ldots, n$.

The GKM graphs of regular semisimple Hessenberg varieties are subgraphs of the Bruhat graphs of the symmetric group and the vertices are permutations $w \in \mathcal{S}_n$. For two permutations $v, w \in \mathcal{S}_n$ such that $v = ws_{jk}$ and $\ell(w) > \ell(v)$, we use $w \rightarrow v (w \rightarrow v$, respectively) if $v$ and $w$ are connected (not connected, respectively) by an edge in the GKM graph.

We follow the same line of the computation of the $\mathcal{S}_n$ action on the cohomology space $H^*_T(G/B)$ of the full flag variety done by Brion [5] to compute the Weyl group action on $H^*_T(\text{Hess}(S, h))$ for any Hessenberg variety. To deal with the minus cells $\Omega_w^\circ, h \in \mathcal{G}$, we have to do delicate analyses and the main results are stated as follows. For $u \in \Omega_u^T, w \in \mathcal{S}_{s_iw,h}$, we define $T_u$ to be the closure of $T_u := \Omega_u^c \cap \Omega_{s_iw,h}$ and define $\tau_u$ to be the class given by $T_u$.

**Theorem B** (Propositions 4.6 and 4.8). Let $h : [n] \rightarrow [n]$ be a Hessenberg function. Let $w \in \mathcal{S}_n$ be a permutation and $s_i \in \mathcal{S}_n$ be a simple reflection.

1. If $w \rightarrow s_iw$ or $s_iw \rightarrow w$, then $s_i\sigma_{w,h} = \sigma_{s_iw,h}$.

2. Define $A$ to be the set of all $u \in \Omega^T_{s_iw,h} \cap \Omega^T_w$ such that $\dim T_u = \dim \Omega_{w,h}, u \rightarrow s_iu$ and $s_iu \notin \Omega^T_w$.

   (a) If $s_iw \rightarrow w$, then $s_i\sigma_{w,h} - \sigma_{w,h} = 0$.

   (b) If $w \rightarrow s_iw$, then $s_i(\sigma_{w,h} + \sum_{v \in A} \tau_v) - (\sigma_{w,h} + \sum_{v \in A} \tau_v) = (t_{i+1} - t_i)\sigma_{s_iw,h}$.

As we already mentioned, Conjecture [1.2] is true for the permutohedral varieties $\mathcal{H}_n$ but no geometric explanation (construction) was known before. We apply Theorem A and Theorem B to $H^*_T(\mathcal{H}_n)$ to construct explicit permutation submodule decomposition of each $H^*_{T_k}(\mathcal{H}_n)$ resolving a question by Stembridge posed in [30]. We present a nice set of permutations of order $\dim(H^*_T(\mathcal{H}_n))^\mathcal{S}_n$; that is the number of permutation modules appearing in the decomposition of $H^*_T(\mathcal{H}_n)$. We then construct one permutation submodule for each permutation we choose, where we use the ‘erasing marks’ due to Chow to make a submodule of the right degree.

It is well known that the number of descents $\text{des}(w) := \# \{ i \mid w(i) > w(i+1) \}$ of a permutation $w \in \mathcal{S}_n$ determines the degree of the corresponding cohomology class of $w$; $\sigma_w \in H^*_{T_k}(\mathcal{H}_n)$ if $\text{des}(w) = k$, where we use $\sigma_w$ instead of $\sigma_{w,h}$ since we fix $h$. Symmetric group actions in Theorem [3] are computed in more explicit ways for permutohedral varieties; see Proposition [7.8] and Proposition [5.11].

The final step in a construction of permutation module decomposition of $H^*(\mathcal{H}_n)$ is to choose a right module generator of each permutation submodule. The cohomology classes of the permutations with descent number $k$ are reasonable candidates for the generators, but the isotropy subgroups of them do not make the right dimension. We ‘erase marks (descents)’ of each candidates by the symmetrization, where ‘erasing marks’ is defined by Chow in his conjecture on bases for $H^*(\mathcal{H}_n)$ (see [10]). Let $G_k$ be the set of $w \in \mathcal{S}_n$ with $\text{des}(w) = k$ such that $\text{supp}(\sigma_w)$ contains $w_0$, the longest element in $\mathcal{S}_n$. The erasing descents from a permutation $w$ erases the descent 1 if 1 is a
descent of \( w \), and descent \( d \) if both \((d - 1)\) and \( d \) are descents of \( w \). We then symmetrize \( \sigma_w \) to have \( \hat{\sigma}_w = \sum_{u \in \bar{\mathcal{S}}_w} u \sigma_w \) where \( \bar{\mathcal{S}}_w \) is the subgroup of permutations that permute the elements in each newly made block by erasing: For example, if \( n = 52341 \) and \( w = 45312 \) are permutations in \( \mathcal{G}_2 \) then erasing descents from \( v, w \) result in \( 52341 \) and \( 45312 \) respectively, and \( \hat{\sigma}_v = \sum_{u \in \bar{\mathcal{S}}(15,2,3,4)} u \sigma_v \) and \( \hat{\sigma}_w = \sum_{u \in \bar{\mathcal{S}}(3,1,2)} u \sigma_w \). Then the isotropy subgroups of \( \hat{\sigma}_v \) and \( \hat{\sigma}_w \) are \( \bar{\mathcal{S}}(1) \times \bar{\mathcal{S}}(5,2,3,4) \) and \( \bar{\mathcal{S}}(3,1,2) \times \bar{\mathcal{S}}(4,5) \) respectively, so that \( \hat{\sigma}_v \) and \( \hat{\sigma}_w \) generate the permutation modules \( M^{(1,4)} \) and \( M^{(3,2)} \) respectively. As a \( \mathbb{C}S_n \)-module, \( H^{2k}(\mathcal{H}_n) \) is decomposed as permutation submodules:

**Theorem C** (Theorem 5.23). For each \( k \),

\[
H^{2k}(\mathcal{H}_n) = \bigoplus_{w \in \mathcal{G}_k} \mathbb{C}S_n(\hat{\sigma}_w).
\]

This paper is organized as follows. In Section 2 we recall Białynicki-Birula decompositions on regular semisimple Hessenberg varieties and the basis of the equivariant cohomology of a regular semisimple Hessenberg variety constructed by them. In Section 3 we provide an explicit description of the support of the basis we constructed in terms of the reachability of a certain acyclic directed graph. Symmetric group actions on the basis elements are calculated in Section 4. Sections 5 and 6 are devoted to the construction of permutation decomposition of the cohomology space of the permutohedral variety.

### 2. Preliminaries

#### 2.1. Hessenberg varieties and Białynicki-Birula decompositions

In this subsection, we first review some properties of the flag variety and then we present the definition of Hessenberg varieties and Białynicki-Birula decompositions on them.

Let \( G \) be a general linear group \( \text{GL}_n(\mathbb{C}) \) and let \( B \) be the set of upper triangular matrices in \( G \). Let \( T \) be set of diagonal matrices in \( G \), i.e., \( T \cong (\mathbb{C}^*)^n \). We denote by \( B^- \) the set of lower triangular matrices in \( G \). The flag variety \( F\ell(\mathbb{C}^n) \) is isomorphic to the quotient space \( G/B \):

\[
F\ell(\mathbb{C}^n) = \{ V_\bullet = \{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \mid \dim \mathbb{C} V_i = i \quad \text{for all } 1 \leq i \leq n \} \cong G/B.
\]

For \( g \in G \), the corresponding element in \( F\ell(\mathbb{C}^n) \) to \( gB \in G/B \) is given by the column vectors \( v_1, \ldots, v_n \) of the matrix \( g \):

\[
gB = (\{0\} \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \ldots, v_i \rangle \subset \cdots \subset \mathbb{C}^n).
\]

The left multiplication of \( T \) on \( G \) induces an action of \( T \) on the flag variety \( F\ell(\mathbb{C}^n) \) which is indeed the same as that induced by coordinate-wise multiplication of \( T \) on \( \mathbb{C}^n \). The set of \( T \)-fixed points in \( F\ell(\mathbb{C}^n) \) is identified with the set \( \mathcal{S}_n \) of permutations on \( [n] := \{1, \ldots, n\} \) which is the Weyl group of \( G \) (cf. [15] Lemma 2 in §10.1). More precisely, for a permutation \( w = w(1)w(2)\cdots w(n) \in \mathcal{S}_n \), the corresponding element in \( F\ell(\mathbb{C}^n) \) is a coordinate flag \( \hat{w}B \)

\[
(2.1) \quad \hat{w}B = (\{0\} \subset \langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \cdots \subset \langle e_{w(1)}, \ldots, e_{w(i)} \rangle \subset \cdots \subset \mathbb{C}^n)
\]

Here, \( \hat{w} \) is the column permutation matrix of \( w \), i.e., it has 1 on \( (w(i), i) \)-entry and the others are all zero. We denote by \( e_1, \ldots, e_n \) the standard basis vectors in \( \mathbb{C}^n \).
The flag variety $G/B$ admits a cell decomposition called the *Bruhat decomposition* 

\[(2.2) \quad G/B = \bigsqcup_{w \in \mathfrak{S}_n} BwB/B = \bigsqcup_{w \in \mathfrak{S}_n} B^{-w}B/B.\]

For each $w \in \mathfrak{S}_n$, we denote by 

\[(2.3) \quad X_w = BwB/B, \quad \Omega_w = B^{-w}B/B\]

and call them *Schubert cell* and *opposite Schubert cell*, respectively. Both are indeed affine cells of dimension $\ell(w)$ and codimension $\ell(w)$, respectively. Here, $\ell(w)$ is the *length* of $w$ defined as follows. The symmetric group $\mathfrak{S}_n$ is generated by simple reflections $s_i$, which are the adjacent transpositions exchanging $i$ and $i+1$ for $i = 1, \ldots, n-1$. Any element $w \in \mathfrak{S}_n$ can be written as a product of generators $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ for $i_1, \ldots, i_k \in [n-1]$. If $k$ is minimal among all such expressions for $w$, then $k$ is called the *length* of $w$. Note that the length $\ell(w)$ is the same as the number of inversions of $w$.

\[(2.4) \quad \ell(w) = \# \{ (j, i) \mid 1 \leq j < i \leq n, w(j) > w(i) \}.\]

To define Hessenberg varieties, we first present the definition of *Hessenberg functions*. A function $h: [n] \to [n]$ is called a *Hessenberg function* if

- it is weakly increasing: $1 \leq h(1) \leq h(2) \leq \cdots \leq h(n) \leq n$, and
- $h(i) \geq i$ for all $1 \leq i \leq n$.

**Definition 2.1.** Let $h$ be a Hessenberg function and let $X$ be a linear operator. Then the *Hessenberg variety* $\text{Hess}(X, h)$ is a subvariety of the flag variety defined by 

$$\text{Hess}(X, h) = \{ V_\bullet \in F\ell(\mathbb{C}^n) \mid XV_i \subset V_{h(i)} \text{ for all } 1 \leq i \leq n \}.$$

When the linear operator $X$ is regular (respectively, semisimple or nilpotent) then we call $\text{Hess}(X, h)$ regular (respectively, semisimple or nilpotent).

In this article, we concentrate on *regular semisimple Hessenberg varieties*, i.e., the Jordan canonical form of the linear operator $X$ is

\[(2.5) \quad P^{-1}XP = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n \end{bmatrix} =: S\]

with mutually distinct numbers $c_1, c_2, \ldots, c_n$. There is an isomorphism

$$\text{Hess}(X, h) \cong \text{Hess}(gXg^{-1}, h)$$

for any $g \in G$ by sending $V_\bullet$ to $gV_\bullet$. Accordingly, we may assume that the linear operator $X$ is of the form \[2.5\]. To emphasize we are considering regular semisimple Hessenberg varieties, we denote our linear operator by $S$.

**Example 2.2.**

1. Let $h = (n, n, \ldots, n)$. Then $\text{Hess}(S, h)$ is the flag variety $F\ell(\mathbb{C}^n)$. 

(2) Let \( h = (2, 3, 4, \ldots, n, n) \). Then Hess\((S, h)\) is a toric variety called \textit{permutohedral variety} with the fan consisting of Weyl chambers of the root system of type \( A_{n-1} \) (see [12, Theorem 11]).

The torus action on the flag variety preserves a regular semisimple Hessenberg variety. Using this torus action, De Mari, Procesi and Shayman [12] computed the Poincaré polynomial of Hess\((S, h)\):

\textbf{Proposition 2.3} ([12]). Let Hess\((S, h)\) be a regular semisimple Hessenberg variety. Then Hess\((S, h)\) is smooth of \( \mathbb{C} \)-dimension \( \sum_{i=1}^{n}(h(i) - i) \). The Poincaré polynomial \( \text{Poin}(\text{Hess}(S, h), q) \) is given by

\[
\text{Poin}(\text{Hess}(S, h), q) := \sum_{k \geq 0} \dim_{\mathbb{Q}} H^{k}(\text{Hess}(S, h); \mathbb{Q}) q^{k} = \sum_{w \in S_n} q^{2\ell_{h}(w)}.
\]

Here, the number \( \ell_{h}(w) \) is defined by

\[
(2.6) \quad \ell_{h}(w) := \# \{(j, i) \mid 1 \leq j < i \leq h(j), w(j) > w(i)\}.
\]

We notice that the number \( \ell_{h}(w) \) is the same as \( \ell(w) \) if \( h = (n, \ldots, n) \) (see (2.4)). To prove the previous proposition, they applied the theory of Białynicki-Birula [3] to Hessenberg varieties. We briefly review their proof. The Hessenberg variety is \( T \)-stable and the fixed point set is again identified with \( S_n \) like as before (see (2.1)). Fix a regular dominant weight \( \lambda : \mathbb{C}^{*} \rightarrow T, t \mapsto \text{diag}(t^{\lambda_{1}}, t^{\lambda_{2}}, \ldots, t^{\lambda_{n}}) \) determined by integers \( \lambda_{1} > \lambda_{2} > \cdots > \lambda_{n} \).

For each \( w \in S_n \), consider the plus cells and minus cells.

\[
(2.7) \quad X_{w, h}^{\circ} := \left\{ gB \in \text{Hess}(S, h) \mid \lim_{t \to 0} \lambda(t) gB = \dot{w}B \right\},
\]

\[
(2.8) \quad \Omega_{w, h}^{\circ} := \left\{ gB \in \text{Hess}(S, h) \mid \lim_{t \to \infty} \lambda(t) gB = \dot{w}B \right\}.
\]

Then, both \( X_{w, h}^{\circ} \) and \( \Omega_{w, h}^{\circ} \) are isomorphic to affine spaces and they define the plus and minus decompositions. Note that when we take \( h = (n, \ldots, n) \) then Hess\((S, h)\) is the flag variety, and moreover, we have \( X_{w, h}^{\circ} = X_{w}^{\circ} \) and \( \Omega_{w, h}^{\circ} = \Omega_{w}^{\circ} \) (cf. (2.3)). Counting the numbers of positive and negative weights in the tangent space of Hess\((S, h)\) at \( \dot{w}B \), we have that

\[
(2.9) \quad \dim_{\mathbb{C}} X_{w, h}^{\circ} = \ell_{h}(w) (= \dim_{\mathbb{C}}(\text{Hess}(S, h)) - \dim_{\mathbb{C}} \Omega_{w, h}^{\circ})
\]

which implies the above proposition.

\textbf{Example 2.4.} Suppose that \( n = 3 \) and \( h = (2, 3, 3) \). For a permutation \( w = 231 \), there are two inversions on locations \((1, 3)\) and \((2, 3)\) because \( w(1) > w(3) \) and \( w(2) > w(3) \). However, because of the condition \( j < h(i) \), we only count \((2, 3)\) to compute \( \ell_{h}(w) \). Therefore, we obtain \( \ell_{h}(w) = \# \{(2, 3)\} = 1 \). By a similar computation, we present \( \ell_{h}(w) \).

| \( w \) | 123 | 132 | 213 | 231 | 312 | 321 |
|---|---|---|---|---|---|---|
| \( \ell_{h}(w) \) | 0 | 1 | 1 | 1 | 1 | 2 |

\textbf{Remark 2.5.} It is known that the closure of a Schubert cell is a union of other Schubert cells:

\[
X_{w} := \overline{X_{w}^{\circ}} = \bigcup_{v \leq w} X_{v}^{\circ}, \quad \Omega_{w} := \overline{\Omega_{w}^{\circ}} = \bigcup_{v \geq w} \Omega_{v}^{\circ}.
\]
See, for example, [15, §10.2]. However, a plus or minus cell decomposition is not cellular in general, indeed, the boundary of a cell of dimension $k$ is not always contained in a union of cells of dimension $\leq (k - 1)$. See [13, Remark in Section 5].

**Remark 2.6.** The Hessenberg varieties are defined for any reductive linear algebraic group and any linear operator. Especially, it is proved that Hessenberg varieties corresponding to a reductive linear group and a regular element are paved by affines, i.e., each Hessenberg variety is a finite, disjoint union of affine spaces (see [29, 39] and references therein).

2.2. **Basis of the equivariant cohomology.** In this subsection, we consider the torus action on regular semisimple Hessenberg varieties, and then define a basis $\{\sigma_{w,h} \mid w \in \mathfrak{S}_n\}$ of the equivariant cohomology ring in Definition 2.9. Moreover, we study their combinatorial properties in Proposition 2.11.

We first briefly recall the GKM theory from [19, 40]. Let $T$ be a complex torus $\mathbb{C}^r$. Let $S = \mathbb{Z}[t_1, \ldots, t_n]$ be the symmetric algebra over $\mathbb{Z}$ of the character group $M := \text{Hom}(T, \mathbb{C}^r)$. Let $X$ be a complex projective variety with an action of $T$. Then $X$ is called a $GKM$ space if it satisfies the following three conditions.

1. $X$ has finitely many $T$-fixed points,
2. $X$ has finitely many (complex) one-dimensional $T$-orbits $X^{(1)}$, and
3. $H^{\text{odd}}(X)$ vanishes.

Toric varieties are GKM spaces, and the flag variety, Grassmannians, and Schubert varieties are also GKM. Since there are finitely many fixed points, the closure $\overline{O}$ of a one-dimensional orbit $O$ is homeomorphic to $\mathbb{C}P^1$.

The $GKM$ graph $\Gamma = (V, E, \alpha)$ of a GKM space is an oriented graph with label $\alpha$ on edges defined as follows.

1. $V = X^T$.
2. $(v \to w) \in E$ if and only if there exists a one-dimensional orbit $O_{v,w}$ whose closure $\overline{O_{v,w}}$ contains $v$ and $w$.
3. We label each edge $v \to w$ with the weight $\alpha(v \to w) \in S$ of the $T$-action corresponding to the closure $\overline{O_{v,w}}$ at the fixed point $v$.

From the second condition, we have that $(v \to w) \in E$ if and only if $(w \to v) \in E$. In particular, $\alpha(v \to w) = -\alpha(w \to v)$.

**Example 2.7.** Let $h : [n] \to [n]$ be a Hessenberg function. The regular semisimple Hessenberg variety $\text{Hess}(S, h)$ is a GKM manifold (cf. [12, Section III] or [40, Proposition 5.4(1)]). The GKM graph $\Gamma_h = (V, E, \alpha)$ of $\text{Hess}(S, h)$ is given as follows.

1. $V = \mathfrak{S}_n$.
2. $(w \to ws_{j,i}) \in E$ if and only if $j < i \leq h(j)$.
3. $\alpha(w \to ws_{j,i}) = t_{w(i)} - t_{w(j)}$.

Here, $s_{j,i}$ is the transposition in $\mathfrak{S}_n$ exchanging $j$ and $i$. Because of the equality $s_{j,i}w = ws_{w^{-1}(j), w^{-1}(i)}$, the second condition is equivalent to the following.
\( \bullet (w \to s_{j,i}w) \in E \) if and only if \( w^{-1}(j) < w^{-1}(i) \leq h(w^{-1}(j)) \).

With this description, \( \alpha(w \to s_{j,i}w) = t_i - t_j \).

For example, suppose that \( n = 3 \) and \( h = (2, 3, 3) \). For \( w = 132 \), there are two edges starting at \( w \).

\[
(w \to ws_1) = (132 \to 312), \quad (w \to ws_2) = (132 \to 123).
\]

For each edge, the labeling is given by \( \alpha(132 \to 312) = t_3 - t_1 \) and \( \alpha(132 \to 123) = t_2 - t_3 \).

In Figure 1, we present GKM graphs of regular semisimple Hessenberg varieties for \( n = 3 \) and \( h = (2, 3, 3), (3, 3, 3) \).

Goresky–Kotwitz–MacPherson [19] provides a description of the equivariant cohomology ring of a GKM space.

**Theorem 2.8 ([19]).** Let \( T = (\mathbb{C}^*)^n \) and let \( (X, T) \) be a GKM space. Let \( \Gamma = (V, E, \alpha) \) be the GKM graph. Then,

\[
H^*_T(X; \mathbb{C}) \cong \left\{ (p(v)) \in \bigoplus_{v \in V} \mathbb{C}[t_1, \ldots, t_n] : \alpha(v \to w) \mid p(v) - p(w) \quad \text{for all } (v \to w) \in E \right\}.
\]

Now we concentrate on the equivariant cohomology ring of a regular semisimple Hessenberg variety. One may wonder whether there exist nice classes which form a \( H^*(BT) \)-basis of the equivariant cohomology ring. We take a basis using the minus cell decomposition. We have the minus cell decomposition of regular semisimple Hessenberg variety (cf. (2.8)):

\[
(2.10) \quad \text{Hess}(S, h) = \bigsqcup_{w \in \mathcal{S}_n} \Omega^w_{w, h}.
\]

When \( h = (n, \ldots, n) \), each cell agrees with the opposite Schubert cell \( \Omega^\circ_w \).

The closure \( \Omega_{w,h} := \overline{\Omega^w_{w,h}} \) of a minus cell \( \Omega^w_{w,h} \) might not be smooth even though any regular semisimple Hessenberg variety \( \text{Hess}(S, h) \) is smooth. The closure \( \Omega_{w,h} \) defines a class \([\Omega_{w,h}]\) in the equivariant Chow ring \( A^*_T(\text{Hess}(S, h)) \) which is graded by codimension. (See [5] for more details on equivariant Chow rings.) On the other hand, since the \( T \)-fixed points are isolated and \( \text{Hess}(S, h) \) is smooth, the cycle map is an isomorphism by [5, Corollary 2 in Section 3.2]:

\[
(2.11) \quad \text{cl}_T^H(\text{Hess}(S, h)) : A^*_T(\text{Hess}(S, h))_\mathbb{Q} \cong H^*_{\mathbb{A}}(\text{Hess}(S, h); \mathbb{Q}).
\]

Using the cycle map, we provide the following definition.
Figure 2. $\Gamma_h^0$ for $h = (2,3,3)$ and $h = (3,3,3)$.

**Definition 2.9.** Let $\text{Hess}(S,h)$ be a regular semisimple Hessenberg variety. For $w \in \mathcal{S}_n$, we define an equivariant class $\sigma_{w,h} \in H^{2\ell_h(w)}(\text{Hess}(S,h); \mathbb{C})$ to be the image of the class $[\Omega_{w,h}] \in A^T_{N - \ell_h(w)}(\text{Hess}(S,h))_{\mathbb{Q}} = A^T_{T}(\text{Hess}(S,h))_{\mathbb{Q}}$ under the cycle map $\mathbf{(2.11)}$.

Since the Hessenberg variety is decomposed by the minus cells as in $\mathbf{(2.10)}$, the equivariant classes $\sigma_{w,h}$ form a basis of the equivariant cohomology.

**Proposition 2.10.** Let $\text{Hess}(S,h)$ be a regular semisimple Hessenberg variety. Then the classes $\{\sigma_{w,h} \mid w \in \mathcal{S}_n\}$ form a basis of the equivariant cohomology ring $H^*_T(\text{Hess}(S,h); \mathbb{C})$.

Let $\Gamma_h = (V, E, \alpha)$ be the GKM graph of a regular semisimple Hessenberg variety $\text{Hess}(S,h)$. We consider the edges $E^o \subset E$ such that $(v \to w) \in E^o$ if $(v \to w) \in E$ and $\ell(v) > \ell(w)$. Indeed, we choose one edge from a pair of edges $v \to w$ and $w \to v$ comparing the lengths of elements. With abuse of notation, we denote by $(v \to w) \in \Gamma_h^0$ if $(v \to w)$ is an edge of the graph $\Gamma_h$. We present $\Gamma_h^0$ for $h = (2,3,3)$ and $h = (3,3,3)$ in Figure 2. For an equivariant cohomology class $p = (p(v))_{v \in X^T}$, the support $\text{supp}(p)$ of $p$ is defined to be

$$\text{(2.12)} \quad \text{supp}(p) := \{v \in X^T \mid p(v) \neq 0\}.$$ 

From the definition of the equivariant class $\sigma_{w,h}$, we obtain subsequent properties.

**Proposition 2.11.**

1. For each $v \in \Omega^T_{w,h}$, there exists a descending chain $v \to \cdots \to w$ in the graph $\Gamma_h^0$.
2. The support $\text{supp}(\sigma_{w,h})$ of $\sigma_{w,h}$ is $\Omega^T_{w,h}$, and $\sigma_{w,h}(v)$ is homogeneous of degree $\ell_h(w)$ for each $v \in \Omega^T_{w,h}$.
3. $\sigma_{w,h}(w) = \prod_{(w \to v) \in \Gamma_h^0} \alpha(w \to v)$.

In the foregoing section, we will provide a concrete description of the support $\text{supp}(\sigma_{w,h})$ in Theorem 3.5. To present a proof of Proposition 2.11, we use the following result of Brion [5].

**Lemma 2.12** ([5] Theorems 4.2 and 4.5, Proposition 4.4]). Let $X$ be a variety with an action of $T$, let $x \in X$ be an isolated fixed point. Let $\chi_1, \ldots, \chi_m$ be the weights of $T_x X$, where $m = \dim_{\mathbb{C}}(X)$.

1. There exists a unique $S$-linear map

$$e_x : A^*_T(X) \to \frac{1}{\chi_1 \cdots \chi_m} S$$

$\square$
such that $e_x[x] = 1$ and that $e_x[Y] = 0$ for any $T$-invariant subvariety $Y \subset X$ which does not contain $x$.

(2) If $x$ is an attractive point in $X$, that is, all weights in the tangent space $T_xX$ are contained in some open half-space of $M \otimes_{\mathbb{Z}} \mathbb{R}$, then $e_x[X] \neq 0$.

(3) For any $T$-invariant subvariety $Y \subset X$, the rational function $e_x[Y]$ is homogeneous of degree $(-\dim_C(Y))$.

(4) The point $x$ is nonsingular in $X$ if and only if

$$e_x[X] = \frac{1}{\chi_1 \cdots \chi_m}.$$ 

Moreover, in this case, for any $T$-invariant subvariety $Y \subset X$, we have that $[Y]_x = e_x[Y] \chi_1 \cdots \chi_m$

where $[Y]_x$ denotes the pull-back of $[Y]$ by the inclusion of $x$ into $X$.

Proof of Proposition 2.11. (1) The first statement follows from the result of Carrell and Sommese [7, Lemma 1 in Section IV].

(2) It is enough to show that the claim holds for $[\Omega_{w,h}]$, that is, $\text{supp}([\Omega_{w,h}]) = \Omega^T_w,h$, and $[\Omega_{w,h}]_v$ is homogeneous of degree $\ell_h(w)$. If $v \notin \Omega^T_w,h$, then

$$[\Omega_{w,h}]_v \overset{(4)}{=} e_v[\Omega_{w,h}] \cdot \chi_1 \cdots \chi_m \overset{(1)}{=} 0 \cdot \chi_1 \cdots \chi_m = 0$$

using (1) and (4) in Lemma 2.12 where $\chi_1, \ldots, \chi_m$ are weights in the tangent space $T_v \text{Hess}(S,h)$. Moreover, since all $v \in \Omega^T_w,h$ are attractive in $\Omega_{w,h}$ (cf. [5, §6.5]), we obtain $e_v[\Omega_{w,h}] \neq 0$ by Lemma 2.12(2). Then by Lemma 2.12(3) and (4), we have that $[\Omega_{w,h}]_v \neq 0$ and is homogeneous of degree $\dim_C(\text{Hess}(S,h)) - \dim_C \Omega_{w,h} = \ell_h(w)$. Therefore, we obtain the second statement.

(3) Note that the point $wB$ is smooth in $\Omega_{w,h}$. By Lemma 2.12 we have that

$$[\Omega_{w,h}]_w = e_w[\Omega_{w,h}] \chi_1 \cdots \chi_m = \prod_{(w \rightarrow v) \in \Gamma_h} \alpha(w \rightarrow v) \prod_{(w \rightarrow v) \notin \Gamma_h^0} \alpha(w \rightarrow v) \overset{(w \rightarrow v) \in \Gamma_h^0}{} \alpha(w \rightarrow v).$$

Here, the second equality is deduced from the fact the $\Omega_{w,h}$ is the closure of the minus cell and the weights of $T_w \Omega_{w,h}$ are given by $\alpha(w \rightarrow v)$ for $(w \rightarrow v) \notin \Gamma_h^0$. This completes the proof of the third statement. \qed

Remark 2.13. There are several attempts to find bases of the equivariant cohomology of a complex variety with an action of $T$. Guillemin and Zara [22, 23] introduced ‘equivariant Thom classes’ which can be considered as the equivariant Poincaré duals of the closures of the minus cells when the closures are smooth. Goldin and Tolman [18] considered a similar problem for Hamiltonian $(S^1)^k$-manifolds. However, their bases are not uniquely defined in general. Recently, Pabiniak and Sabatini [28] defined ‘canonical bases’ for symplectic toric manifolds which are uniquely defined. For regular semisimple Hessenberg varieties, Tymoczko [31] concentrated on the case of $\text{Hess}(S,h) = F\ell(\mathbb{C}^n)$ and studied a basis called ‘Knutson–Tao classes’ which are uniquely determined. Moreover, Teff [37, 38] studied bases of the equivariant cohomology rings of arbitrary regular semisimple Hessenberg varieties called ‘flow-up bases’. 
Our basis \( \sigma_{w,h} \) satisfies nice properties which are already considered in the known results. Indeed, when \( \text{Hess}(S,h) = F\ell(C^n) \), our basis coincides with the basis constructed by Tymoczko. When \( \text{Hess}(S,h) \) is toric, then our basis is the ‘canonical basis’ provided by Pabiniak and Sabatini. Moreover, by Proposition 2.11, the basis \( \sigma_{w,h} \) is a ‘flow-up basis’ studied by Teff.

We enclose this subsection with a property of an equivariant cohomology class which will be used later. The following is directly obtained by combining the equation (2.11) and Lemma 2.12(4).

**Proposition 2.14.** Let \( (X,T) \) be a nonsingular GKM space and let \( \Gamma = (V,E,\alpha) \) be the GKM graph. Let \( Y \subset X \) be a \( T \)-invariant nonsingular subvariety. For \( v \in \text{supp}(\mu(Y)) \), if \( v \) is smooth at \( Y \), then we have that

\[
[Y]_v = \prod_{(v \to w) \in \Gamma, \ w \notin \text{supp}(\mu(Y))} \alpha(v \to w).
\]

### 3. Fixed points in the closure of a minus cell

In this section, we provide a concrete formula on the support \( \sigma_{w,h} \) in Theorem 3.5 using the reachability of a certain graph. In order to prove the theorem, we study an explicit description of the minus cell \( \Omega^0_{w,h} \) and its properties in Proposition 3.12 and Theorem 3.13. Furthermore, we present the defining equations of \( \Omega_{w,h} \) in \( \text{Hess}(S,h) \) (see Corollary 3.17).

#### 3.1. Support of \( \sigma_{w,h} \)

**Definition 3.1.** Let \( h \) be a Hessenberg function and \( w \in S_n \). We define a directed graph \( G_{w,h} \) with the vertex set \([n]\) such that for each pair of indices \( 1 \leq j < i \leq n \), there is an edge \( j \rightarrow i \) in \( G_{w,h} \) if and only if

\[
 (3.1) \quad j < i \leq h(j), \quad w(j) < w(i).
\]

**Example 3.2.** Suppose that \( n = 5 \) and \( h = (3,3,4,5,5) \). We present the graphs \( G_{24135,h}, G_{15342,h}, \) and \( G_{12345,h} \) in Figure 3.

By the definitions of the graph \( G_{w,h} \) and the length \( \ell_h(w) \) (see Definition 3.1 and (2.6)), and by the dimension formula in (2.9), we obtain

\[
 \#E(G_{w,h}) = \dim_C \Omega^0_{w,h}.
\]

We note that the graph \( G_{e,h} \) agrees with the incomparability graph of the natural unit interval order determined by the Hessenberg function \( h \), where \( e \) is the identity element in \( S_n \); see [32].

We say a vertex \( i \) is reachable from \( j \) if there exists a sequence of vertices \( v_0, v_1, \ldots, v_k \) such that \( j = v_0 \to v_1 \to \cdots \to v_{k-1} \to v_k = i \). We allow the length of a sequence to be 0, that is, \( j \) is reachable from \( j \). Let \( A = \{ 1 \leq a_1 < a_2 < \cdots < a_k \leq n \} \) and \( B = \{ 1 \leq b_1 < b_2 < \cdots < b_k \leq n \} \) be

\[
(1) \quad G_{24135,h}, \quad (2) \quad G_{15342,h}, \quad (3) \quad G_{12345,h}.
\]
subsets of \([n]\). We say that \(A\) is reachable from \(B\) if there exists a permutation \(\sigma \in \mathfrak{S}_k\) such that \(a_{\sigma(d)}\) is reachable from \(b_d\) for all \(d = 1, \ldots, k\). For two vertices \(j\) and \(i\) in the graph \(G_{w,h}\), the distance \(d(j, i)\) is defined to be the length of the shortest sequences \(j = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{k-1} \rightarrow v_k = i\). We set \(d(j, j) = 0\) and if \(i\) is not reachable from \(j\), then \(d(j, i) = \infty\).

For \(1 \leq j \leq n\), we define the following set of ordered tuples.

\[
I_{j,n} := \{(i_1, \ldots, i_j) \in \mathbb{Z}^j \mid 1 \leq i_1 < \cdots < i_j \leq n\}.
\]

For \(u \in \mathfrak{S}_n\), and \((i_1, \ldots, i_j) \in I_{j,n}\), we set

\[
u(i_1, \ldots, i_j) = \{u(i_1), \ldots, u(i_j)\} = \{u(i_1), \ldots, u(i_j)\}^+ \in I_{j,n} \quad \text{for } 1 \leq j \leq n.
\]

Using the reachability of the graph \(G_{w,h}\), we define \(J_{w,h,j} \subset I_{j,n}\) as follows.

(3.2) \[
J_{w,h,j} := \{(i_1, \ldots, i_j) \in I_{j,n} \mid \{i_1, \ldots, i_j\} \text{ is reachable from } [j]\}.
\]

**Example 3.3.** We continue Example 3.2. Using graphs in Figure 3, we obtain the subsets \(J_{w,h,j}\) as follows.

1. When \(w = 24135\), we have

\[
J_{w,h,1} = \{(1), (2)\}, J_{w,h,2} = \{(1, 2)\}, J_{w,h,3} = \{(1, 2, 3), (1, 2, 4), (1, 2, 5)\}, J_{w,h,4} = \{(1, 2, 3, 4), (1, 2, 3, 5), (1, 2, 4, 5)\}, J_{w,h,5} = \{(1, 2, 3, 4, 5)\}.
\]

2. When \(w = 15342\), we have

\[
J_{w,h,1} = \{(1), (2), (3), (4)\}, J_{w,h,2} = \{(1, 2), (2, 3), (2, 4)\}, J_{w,h,3} = \{(1, 2, 3), (1, 2, 4), (2, 3, 4)\}, J_{w,h,4} = \{(1, 2, 3, 4)\}, J_{w,h,5} = \{(1, 2, 3, 4, 5)\}.
\]

3. When \(w = 12345\), we have

\[
J_{w,h,j} = I_{j,n} \quad \text{for } 1 \leq j \leq 5.
\]

Using these terminologies, we define a subset of \(\mathfrak{S}_n\) which is determined by a Hessenberg function \(h\) and a permutation \(w \in \mathfrak{S}_n\).

**Definition 3.4.** Let \(h\) be a Hessenberg function on \([n]\) and \(w \in \mathfrak{S}_n\). We define a subset \(A_{w,h} \subset \mathfrak{S}_n\) to be

\[
A_{w,h} := \{u \in \mathfrak{S}_n \mid u(i_j) \in \{w \cdot (i_1, \ldots, i_j) \mid (i_1, \ldots, i_j) \in J_{w,h,j}\} \quad \text{for all } 1 \leq j \leq n\}.
\]

The subsequent theorem is the main result of this section whose proof will be given in Subsection 3.4.

**Theorem 3.5.** Let \(h\) be a Hessenberg function and \(w \in \mathfrak{S}_n\). Then the \(T\)-fixed points in the closure of the minus cell is \(A_{w,h}\), that is, \(A_{w,h} = (\Omega_{w,h})^T = \text{supp}(\sigma_{w,h})\).

**Example 3.6.** We continue Example 3.3. Let \(h = (3, 3, 4, 5, 5)\).
(1) Suppose that \( w = 24135 \). Then the subset \( A_{w,h} \) is given by
\[
A_{w,h} = \{24135, 24153, 24351, 24315, 24513, 42135, 42153, 42351, 42315, 42513, 42531, 42513\}.
\]

(2) Suppose that \( w = 15342 \). Then the subset \( A_{w,h} \) is given by
\[
A_{w,h} = \{15342, 15432, 35142, 35412, 45312, 51342, 51432, 53142, 53412, 54132, 54312\}.
\]

(3) Suppose that \( w = 12345 \). Then the subset \( A_{w,h} \) is the same as the set \( \mathfrak{S}_n \) of permutations.

**Example 3.7.** Suppose that \( h = (n, n, \ldots, n) \). Then we have \( \text{Hess}(S, h) = F \ell(C^n) \). In this case, for \( 1 \leq j < i \leq n \), the graph \( G_{w,h} \) has an edge \( j \to i \) if and only if \( w(j) < w(i) \). Accordingly, for \( \hat{i} = (i_1, \ldots, i_j) \in I_{j,n} \), the set \{\( i_1, \ldots, i_j \)\} is reachable from \([j]\) if and only if \( w \cdot \hat{i} \geq w^{(j)} \). Here, we use an order on \( I_{j,n} \) defined as follows: for \((a_1, \ldots, a_j), (b_1, \ldots, b_j) \in I_{j,n} \),
\[
(a_1, \ldots, a_j) \geq (b_1, \ldots, b_j) \iff a_t \geq b_t \quad \text{for all } 1 \leq t \leq j.
\]

Therefore, we have that \( J_{w,h,j} = \{\hat{i} \mid w \cdot \hat{i} \geq w^{(j)}\} \), and moreover,
\[
A_{w,h} = \{u \in \mathfrak{S}_n \mid u^{(j)} \geq w^{(j)} \text{ for all } j\}.
\]

This proves that \( A_{w,h} = \{u \in \mathfrak{S}_n \mid u \geq w\} \) in this case because \( u \geq w \) if and only if \( u^{(j)} \geq w^{(j)} \) for all \( j \) (see, for example, [4, \S 3.2]).

3.2. **Description of minus cells.** By the definition of the minus cell \( \Omega_{w,h}^c \), we have that
\[
\Omega_{w,h}^c = \text{Hess}(S, h) \cap \Omega_{w}^c.
\]

Note that elements in the opposite Schubert cell \( \Omega_{w}^c \) are described by
\[
\Omega_{w}^c = \{g = (g_{i,j}) \in \text{GL}_n(C) \mid g_{w(j),j} = 1, \quad g_{i,j} = 0 \text{ if } i < w(j) \text{ or } w^{-1}(i) < j\} B \subset G/B.
\]

We recall the result [13] which describes the minus cell \( \Omega_{w,h}^c \) explicitly. They consider the case only when a Hessenberg function has the form \( h(i) = i + p \) while their results can be extended to all Hessenberg functions. Recall that \( n \) distinct numbers \( c_1, \ldots, c_n \) are the diagonal entries of our regular semisimple matrix \( S \).

**Proposition 3.8.** [13, \S 5] Let \( x = (x_{i,j}) \) be a lower triangular matrix having 1 on its diagonal. Then \( \hat{x} \in \Omega_{w,h}^c \) if and only if

1. \( x_{i,j} = 0 \) if \( w(i) < w(j) \);
2. if \( w(i) > w(j) \) and \( i > h(j) \), we have
\[
x_{i,j} = \frac{-1}{c_{w(i)} - c_{w(j)}} \left[ \sum_{\substack{i > \gamma_1 > \gamma_2 > \cdots > \gamma_{i-j} > j}} \sum_{t=1}^{i-j-1} (-1)^t (c_{w(\gamma_t)} - c_{w(j)}) x_{\gamma_1, \gamma_2} x_{\gamma_3, \gamma_4} \cdots x_{\gamma_{i-j}, \gamma_j} \right].
\]

From the description of the minus cell, the entries \( x_{i,j} \) for \( w(i) > w(j) \) and \( i \leq h(j) \) are free, i.e., there is no restriction on these entries. The number of such entries agrees with the dimension of the minus cell. Using Proposition 3.8, we obtain the following corollary.

**Corollary 3.9.** Let \( x = (x_{i,j}) \) be a lower triangular matrix having 1 on its diagonal. Then \( \hat{x} \in \Omega_{w,h}^c \) if and only if
(1) \( x_{i,j} = 0 \) if \( i \) is not a reachable vertex from \( j \);
(2) if \( i \) is reachable from \( j \) with \( d(j, i) > 1 \), then

\[
x_{i,j} = \frac{-1}{c_{w(i)} - c_{w(j)}} \left[ \sum_{l=1}^{i-1} \sum_{\gamma_0=i>\gamma_1>\gamma_2>\cdots>\gamma_l=j=\gamma_{l+1},}^{-1}(c_{w(\gamma_l)} - c_{w(\gamma_j)})x_{i,\gamma_1,\gamma_2,\ldots,\gamma_{l-1},j} \right].
\]

**Example 3.10.** We continue Example 3.2. Let \( h = (3, 4, 5, 5) \) and \( x = (x_{i,j}) \) be a lower triangular matrix having 1 on its diagonal.

(1) Suppose that \( w = 24135 \). By considering the reachability of \( G_{w,h} \) (see Figure 3(1)), the entries \( x_{3,1}, x_{4,1}, x_{5,1}, x_{3,2}, x_{4,2}, x_{5,2} \) vanish. Moreover, three edges \((1 \rightarrow 2), (3 \rightarrow 4), \) and \((4 \rightarrow 5)\) define free variables \( x_{2,1}, x_{4,3}, \) and \( x_{5,4}. \) The variable \( x_{5,3} \) is given by the following equation.

\[
x_{5,3} = \frac{-1}{c_5 - c_1} \left[ (-1)^{1}(c_3 - c_1)x_{5,4}x_{4,3} \right] = \frac{(c_3 - c_1)x_{5,4}x_{4,3}}{c_5 - c_1}.
\]

(2) Suppose that \( w = 15342 \). By considering the reachability of \( G_{w,h} \) (see Figure 3(2)), the entries \( x_{5,1}, x_{3,2}, x_{4,2}, x_{5,2}, x_{5,3}, x_{5,4} \) vanish, and there are three free variables \( x_{2,1}, x_{3,1}, x_{4,3}. \) The variable \( x_{4,1} \) is given by the following equation.

\[
x_{4,1} = \frac{-1}{c_4 - c_1} \left[ (-1)^{1}(c_3 - c_1)x_{4,3}x_{3,1} \right] = \frac{(c_3 - c_1)x_{4,3}x_{3,1}}{c_4 - c_1}.
\]

(3) Suppose that \( w = 12345 \). Then there are five free variables \( x_{2,1}, x_{3,1}, x_{3,2}, x_{4,3}, x_{5,4}, \) and the other variables are given as follows.

\[
x_{4,1} = \frac{-1}{c_4 - c_1} \left[ (-1)^{1}(c_3 - c_1)x_{4,3}x_{3,1} + (-1)^{1}(c_2 - c_1)x_{4,2}x_{2,1} + (-1)^{2}(c_2 - c_1)x_{4,3}x_{3,2}x_{2,1} \right],
\]

\[
x_{5,1} = \frac{-1}{c_5 - c_1} \left[ (-1)^{2}(c_3 - c_1)x_{5,4}x_{4,3}x_{3,1} + (-1)^{1}(c_3 - c_1)x_{5,3}x_{3,1} + (-1)^{1}(c_4 - c_1)x_{5,4}x_{4,1} \right.
\]
\[
\left. + (-1)^{3}(c_2 - c_1)x_{5,4}x_{4,3}x_{3,2}x_{2,1} + (-1)^{2}(c_2 - c_1)x_{5,4}x_{4,2}x_{2,1} \right],
\]

\[
x_{4,2} = \frac{-1}{c_4 - c_2} \left[ (-1)^{1}(c_3 - c_2)x_{4,3}x_{3,2} \right] = \frac{(c_3 - c_2)x_{4,3}x_{3,2}}{c_4 - c_2},
\]

\[
x_{5,2} = \frac{-1}{c_5 - c_2} \left[ (-1)^{2}(c_3 - c_2)x_{5,4}x_{4,3}x_{3,2} + (-1)^{1}(c_4 - c_2)x_{5,4}x_{4,2} + (-1)^{1}(c_3 - c_2)x_{5,3}x_{3,2} \right],
\]

\[
x_{5,3} = \frac{-1}{c_5 - c_3} \left[ (-1)^{1}(c_4 - c_3)x_{5,4}x_{4,3} \right] = \frac{(c_4 - c_3)x_{5,4}x_{4,3}}{c_5 - c_3}.
\]

### 3.3. Minors and reachability.

In this subsection, we study the expression of \( x_{i,j} \) in terms of free variables. Applying Corollary 3.3 iteratively, one may get this expression of \( x_{i,j} \) in terms of free variables. Moreover, each path \( j = \gamma_{l+1} \rightarrow \gamma_l \rightarrow \cdots \rightarrow \gamma_1 \rightarrow i = \gamma_0 \) in the graph \( G_{w,h} \) associates a monomial in the expression of \( x_{i,j}. \) Indeed, we have that

\[
x_{i,j} = \sum_{j=\gamma_{l+1} \rightarrow \gamma_l \rightarrow \cdots \rightarrow \gamma_1 \rightarrow i=\gamma_0} \text{a path in } G_{w,h} \ C(i, \gamma_1, \ldots, \gamma_l, j)x_{i,\gamma_1,\gamma_2,\ldots,\gamma_{l-1},j}
\]

Here, \( C(i, \gamma_1, \ldots, \gamma_l, j) \) is the coefficient of the monomial \( x_{\gamma_1,\gamma_2,\ldots,\gamma_{l-1},j}. \) Note that each coefficient \( C(i, \gamma_1, \ldots, \gamma_l, j) \) is a Laurent polynomial in variables \( c_1, \ldots, c_n. \)
For a given path $j = \gamma_{t+1} \rightarrow \gamma_{t} \rightarrow \cdots \rightarrow \gamma_{1} \rightarrow i = \gamma_{0}$ in the graph $G_{w,h}$, we call it minimal if there are no edges $\gamma_{\ell} \rightarrow \gamma_{\ell'}$ for $\ell - \ell' > 1$. For example, the path $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ in the graph $G_{12345,(3,3,4,5,5)}$ is not minimal because there is an edge $1 \rightarrow 3$. See Figure 3(3). We note that there is a minimal path from $j$ to $i$ if $i$ is reachable from $j$. We will see in Proposition 3.12 that if $i$ is reachable from $j$, then $x_{i,j}$ is not identically zero, indeed, one of the coefficients in the expression (3.4) is not identically zero as a Laurent polynomial in variables $c_{1}, \ldots, c_{n}$. Before presenting the statement, we provide an example.

**Example 3.11.** We continue Example 3.10(3). Substituting the expression of $x_{4,2}$ in $x_{4,1}$, we have the following.

$$x_{4,1} = \frac{-1}{c_{4} - c_{1}} \left[ (-1)^{1} (c_{3} - c_{1}) x_{4,3} x_{3,1} + (-1)^{1} (c_{2} - c_{1}) x_{4,2} x_{2,1} + (-1)^{2} (c_{2} - c_{1}) x_{4,3} x_{3,2} x_{2,1} \right]$$

$$= \frac{-1}{c_{4} - c_{1}} \left[ (-1)^{1} (c_{3} - c_{1}) x_{4,3} x_{3,1} + (-1)^{1} (c_{2} - c_{1}) \cdot \frac{(c_{3} - c_{2}) x_{4,3} x_{3,2}}{c_{4} - c_{2}} x_{2,1} + (-1)^{2} (c_{2} - c_{1}) x_{4,3} x_{3,2} x_{2,1} \right]$$

$$= \frac{c_{3} - c_{1}}{c_{4} - c_{1}} x_{4,3} x_{3,1} - \frac{(c_{2} - c_{1}) (c_{4} - c_{3})}{(c_{4} - c_{1}) (c_{4} - c_{2})} x_{4,3} x_{3,2} x_{2,1}.$$
Substituting $x_{4,2}$ and $x_{5,3}$ to $x_{5,2}$, we have
\[
x_{5,2} = \frac{-1}{c_5 - c_2} \left[ \left( -1 \right)^2 (c_3 - c_2) x_{5,4} x_{4,3} x_{3,2} + \left( -1 \right)^1 (c_4 - c_2) x_{5,4} x_{4,2} + \left( -1 \right)^1 (c_3 - c_2) x_{5,3} x_{3,2} \right]
\]
\[
= \frac{-1}{c_5 - c_2} \left[ \left( -1 \right)^2 (c_3 - c_2) x_{5,4} x_{4,3} x_{3,2} + \left( -1 \right)^1 (c_4 - c_2) x_{5,4} \frac{c_3 - c_2}{c_4 - c_2} x_{4,3} x_{3,2} \right]
\]
\[
+ \left( -1 \right)^1 (c_3 - c_2) \frac{c_4 - c_3}{c_5 - c_3} x_{5,4} x_{4,3} x_{3,2}
\]
\[
= \frac{(c_4 - c_3)(c_3 - c_2)}{(c_5 - c_2)(c_5 - c_3)} x_{5,4} x_{4,3} x_{3,2}.
\]

There is one path $2 \to 3 \to 4 \to 5$ connecting 2 and 5 in the graph $G_{w,h}$, and the corresponding coefficient $C(2,3,4,5)$ is given by
\[
C(2,3,4,5) = \frac{(c_4 - c_3)(c_3 - c_2)}{(c_5 - c_2)(c_5 - c_3)}.
\]

**Proposition 3.12.** Let $x = (x_{i,j})$ be a lower triangular matrix having 1 on its diagonal. Suppose that $wix \in \Omega_{w,h}^*$ and $i$ is reachable from $j$. For a minimal path $\gamma_{t+1} = j \to \gamma_t \to \cdots \to \gamma_1 \to i = \gamma_0$, the coefficient $C(i,\gamma_1,\ldots,\gamma_t,j)$ of the monomial $x_{i,\gamma_1} x_{\gamma_1,\gamma_2} \cdots x_{\gamma_t,j}$ in the expression \eqref{eq:3.3} is given as follows:
\[
\frac{(c_{w(\gamma_1)} - c_{w(\gamma_2)}) (c_{w(\gamma_2)} - c_{w(\gamma_3)}) \cdots (c_{w(\gamma_t)} - c_{w(\gamma_{t+1})})}{(c_{w(\gamma_0)} - c_{w(\gamma_1)}) (c_{w(\gamma_0)} - c_{w(\gamma_1)}) \cdots (c_{w(\gamma_0)} - c_{w(\gamma_2)})}.
\]

Thus, $x_{i,j}$ is not identically zero if $i$ is reachable from $j$.

**Proof.** To get the coefficient of a certain monomial, we substitute $x_{k,l} = 0$ for variables which do not appear in the monomial. Indeed, for a given minimal path $j \to \gamma_t \to \cdots \to \gamma_1 \to i$, we set $I = \{x_{i,\gamma_1}, x_{\gamma_1,\gamma_2}, \ldots, x_{\gamma_t,j}\}$. Then, we have that
\[
x_{i,j}\{x_{k,l}=0|x_{k,l}\#I\} = C(i,\gamma_1,\ldots,\gamma_t,j) x_{i,\gamma_1} x_{\gamma_1,\gamma_2} \cdots x_{\gamma_t,j}.
\]

We use an induction on $t$, which is one less than the length of the minimal path. Suppose that $t = 0$, i.e., there is an edge $j \to i$ in the graph $G_{w,h}$. In this case, $x_{i,j}$ itself is a free variable, and the result follows. Suppose that $t = 1$. Then, by Corollary \ref{cor:3.9}(2), we have that
\[
x_{i,j}\{x_{k,l}=0|x_{k,l}\#I\} = \frac{-1}{c_{w(i)} - c_{w(j)}} \left( -1 \right)^1 (c_{w(\gamma)} - c_{w(j)}) x_{i,\gamma} x_{\gamma,j} = \frac{c_{w(\gamma)} - c_{w(j)}}{c_{w(i)} - c_{w(j)}} x_{i,\gamma} x_{\gamma,j}.
\]

Therefore, the proposition holds.

Now, for $t > 1$, we assume that the above proposition holds for any path having the length less than $t$. We claim the following. Suppose that we have a minimal path $j \to \gamma_t \to \cdots \to \gamma_1 \to i$ in $G_{w,h}$. Then, for every nonempty subset $S = \{s_1,\ldots,s_k\} \subset \{t-1\}$, we have that
\[
\left( c_{w(\gamma_t)} - c_{w(j)} \right) x_{i,\gamma_{s_1}} x_{\gamma_{s_1},\gamma_{s_2}} \cdots x_{\gamma_{s_k},\gamma_t} x_{\gamma_t,j} - \left( c_{w(\gamma_{s_k})} - c_{w(j)} \right) x_{i,\gamma_{s_1}} x_{\gamma_{s_1},\gamma_{s_2}} \cdots x_{\gamma_{s_k},j} \right)\{x_{k,l}=0|x_{k,l}\#I\} = 0.
\]
By the induction assumption, using the minimal path \( j \to \gamma_t \to \cdots \to \gamma_{s_k+1} \to \gamma_{s_k} \), we have that
\[
\begin{align*}
\frac{\prod_{i=1}^{k} (c_{w(i)} - c_{w(j)})}{c_{w(\gamma_t)} - c_{w(j)}} \cdot x_{\gamma_{s_k},\gamma_t} y_{\gamma_{s_k+1} \cdots y_{\gamma_{s_k}}} &= \frac{\prod_{i=1}^{k} (c_{w(i)} - c_{w(j)})}{c_{w(\gamma_t)} - c_{w(j)}} \cdot x_{\gamma_{s_k},\gamma_t} y_{\gamma_{s_k+1} \cdots y_{\gamma_{s_k}}}. 
\end{align*}
\]
Here, the second equality comes from applying the induction hypothesis to the path \( \gamma_t \to \cdots \to \gamma_{s_k+1} \to \gamma_{s_k} \). Using the equality (3.6), we prove the claim (3.5).

We present a proof using the equality (3.5). Suppose that \( j \to \gamma_t \to \cdots \to \gamma_{s_k} \to i \) is a minimal path. Then the terms in Corollary 3.9 contribute to the monomial \( x_{i,\gamma_{s_k}} x_{\gamma_{s_k},\gamma_{s_k+1}} \cdots x_{\gamma_{s_k},\gamma_t} \). Let \( A, B \) be subsets of \([n]\). Then there exists a Hessenberg function and
\[
\begin{align*}
\det(x_{i,j}) &:= \det(x_{a,b})_{a \in A, b \in B}.
\end{align*}
\]

**Theorem 3.13.** Let \( h : [n] \to [n] \) be a Hessenberg function and \( w \in S_n \). Let \( A = \{a_1 < a_2 < \cdots < a_k \leq n\} \) and \( B = \{b_1 < b_2 < \cdots < b_k \leq n\} \) be subsets of \([n]\). Then there exists \( n \) distinct numbers \( c_1, \ldots, c_n \) satisfying the following: If we set a regular semisimple matrix \( S \) to be the diagonal matrix \( \text{diag}(c_1, \ldots, c_n) \), then for a lower triangular matrix \( x = (x_{i,j}) \) having 1 on its diagonal and satisfying \( w x \in \Omega_{w,h}^n \), we have that \( p_{A,B}(x) \) is not identically zero as a polynomial with the free variables \( \{x_{i,j} \mid (j \to i) \in G_{w,h}\} \) if and only if \( A \) is reachable from \( B \).
Proof. We first note that by Corollary 3.9 (2) and Proposition 3.12, we have that $x_{i,j} = 0$ if and only if $i$ is not reachable from $j$. This proves the theorem for $k = 1$.

Now we consider the ’only if’ part of the statement: if $A$ is not reachable from $B$, then $p_{A,B}(x)$ is identically zero. By the definition of reachability, for every $\sigma \in S_k$, there exists an index $d_{\sigma} \in [k]$ such that $a_{\sigma(d_{\sigma})}$ is not reachable from $b_{d_{\sigma}}$. Accordingly, we have $x_{a_{\sigma(d_{\sigma})},b_{d_{\sigma}}} = 0$. Therefore, we have that

$$p_{A,B}(x) = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{d=1}^{k} x_{a_{\sigma(d)},b_{d}} = \sum_{\sigma \in S_k} 0 = 0.$$ 

Here, the second equality comes from $x_{a_{\sigma(d_{\sigma})},b_{d_{\sigma}}} = 0$. This proves the only if part of the statement.

To prove the ‘if’ part of the statement, we use an induction argument on $|A| = |B| = k$. We already have seen that the statement holds when $k = 1$. Suppose that $k > 1$ and assume that the claim holds for subsets $A$ and $B$ satisfying $|A| = |B| < k$. Because of the description in Corollary 3.9 (2), we have that

$$\text{the entry } x_{i,j} \text{ is expressed by free variables } x_{\gamma,\gamma'} \text{ such that } j \leq \gamma' < \gamma \leq i.$$ 

Let $a_p$ be the maximal element in $A$ which is reachable from $b_1$ and such that $A \setminus \{a_p\}$ is reachable from $B \setminus \{b_1\}$, i.e., $x_{a_i,b_1} = 0$ for $i > p$ and $p_{A \setminus \{a_p\},B \setminus \{b_1\}}(x)$ is not identically zero. There are two possibilities: $(b_1 \rightarrow a_p) \in G_{w,h}$ or $(b_1 \rightarrow a_p) \notin G_{w,h}$. We consider the case that $b_1 \rightarrow a_p$ is an edge in $G_{w,h}$ first. Considering the first column and the respective minors, we have that

$$p_{A,B}(x) = \sum_{i=1}^{k} (-1)^{i+1} x_{a_i,b_1} \cdot p_{A \setminus \{a_i\},B \setminus \{b_1\}}(x) = \sum_{i=1}^{p} (-1)^{i+1} x_{a_i,b_1} \cdot p_{A \setminus \{a_i\},B \setminus \{b_1\}}(x).$$

Then, the term $x_{a_p,b_1} \cdot p_{A \setminus \{a_p\},B \setminus \{b_1\}}(x)$ is not identically zero by the induction argument and it is the only term having $x_{a_p,b_1}$ because of (3.8). Therefore, $p_{A,B}(x)$ is not identically zero.

Before considering the remaining case, we observe the following. Suppose that $\sigma \in S_k$ satisfies that $x_{a_{\sigma(d)}}$ is reachable from $b_d$ for all $d \in [k]$. Applying Proposition 3.12 for the variables $x_{a_{\sigma(d)},b_d}$, we have that the product $\prod_{d=1}^{k} x_{a_{\sigma(d)},b_d}$ has a monomial of the following form:

$$\frac{F_{\sigma}(c)}{(c_{w(a_{\sigma(1)})} - c_{w(b_{1})})G_{\sigma}(c)} X_1 \cdots X_k,$$

where $X_d, 1 \leq d \leq k$, is a monomial of free variables given by a certain minimal path $b_d \rightarrow \cdots \rightarrow a_{\sigma(d)}$, and moreover, $F_{\sigma}(c)$ and $G_{\sigma}(c)$ are not identically zero polynomials in variables $c_1, \ldots, c_n$. Here, we notice that $F_{\sigma}(c)$ is not divisible by $(c_{w(a_{\sigma(1)})} - c_{w(b_{1})})$, and moreover, in the polynomial $G_{\sigma}(c)$, the variable $c_{w(b_1)}$ cannot appear because $G_{\sigma}(c)$ consists of variables appearing in paths $b_d \rightarrow \cdots \rightarrow a_{\sigma(d)}$ and $b_1$ is the smallest value in $B$. Therefore, we have that

$$(c_{w(a)} - c_{w(b_1)}) \nmid G_{\sigma}(c) \quad \text{for } a \in A.$$ 

Using the above observation, we consider the remaining case: $(b_1 \rightarrow a_p) \notin G_{w,h}$. In the expression (3.9), we have a monomial of the form

$$\frac{(-1)^{p+1}F(c)}{(c_{w(a_p)} - c_{w(b_1)})G(c)} X.$$
coming from \((-1)^{p+1} x_{a_p, b_1} \cdot p_{A \setminus \{a_p\}, B \setminus \{b_1\}}(x)\). If the summation \(\sum_{i < p} (-1)^{i+1} x_{a_i, b_1} \cdot p_{A \setminus \{a_i\}, B \setminus \{b_1\}}(x)\) does not have the monomial \(X\), then this monomial is not identically zero since \(F(c)\) is not identically zero as a polynomial in variables \(c_1, \ldots, c_n\).

Suppose that the summation \(\sum_{i < p} (-1)^{i+1} x_{a_i, b_1} \cdot p_{A \setminus \{a_i\}, B \setminus \{b_1\}}(x)\) has the monomial \(X\). Then the coefficient of \(X\) in \(p_{A, B}(x)\) is written by
\[
\frac{(-1)^{p+1} F(c)}{(c_{w(a_p)} - c_{w(b_1)}) G(c)} + \frac{J(c)}{K(c)} = \frac{(-1)^{p+1} F(c) K(c) + (c_{w(a_p)} - c_{w(b_1)}) G(c) J(c)}{(c_{w(a_p)} - c_{w(b_1)}) G(c) K(c)}
\]
for appropriate not identically zero polynomials \(J(c)\) and \(K(c)\). Note that
\[
K(c) = (c_{w(a_{\sigma_1(1)})} - c_{w(b_1)}) \cdots (c_{w(a_{\sigma_{\ell}(1)})} - c_{w(b_1)}) G_{\sigma_1}(c) \cdots G_{\sigma_{\ell}}(c)
\]
for some permutations \(\sigma_1, \ldots, \sigma_{\ell}\) satisfying \(\sigma_1(1), \ldots, \sigma_{\ell}(1) < p\). Accordingly, we have that \((c_{w(a_p)} - c_{w(b_1)}) \nmid K(c)\) because of (3.10). Since \((c_{w(a_p)} - c_{w(b_1)}) \nmid F(c)\), the polynomial \((-1)^{p+1} F(c) K(c) + (c_{w(a_p)} - c_{w(b_1)}) G(c) J(c)\) is not identically zero as a polynomial in variables \(c_1, \ldots, c_n\). Hence the result follows.

\[\square\]

**Remark 3.14.** Following the proof of Theorem 3.13 one may see that the set of distinct numbers \(c_1, \ldots, c_n\) satisfying the theorem is a dense subset. We believe that this subset is the whole set while we couldn’t find a proof of this.

### 3.4. Proof of Theorem 3.15

We first notice that the equivariant cohomology of Hessenberg variety \(\text{Hess}(S, h)\) does not depend on the choice of \(S\) as we have seen in the description of the GKM graph in Example 2.7. Indeed, for any regular semisimple elements \(S\) and \(S’\), we have \(H^*_T(\text{Hess}(S, h)) \cong H^*_T(\text{Hess}(S’, h))\). Accordingly, it is enough to consider a certain regular semisimple element \(S\) to provide a proof. From now on, we suppose that \(S\) satisfies Theorem 3.13 that is, for a given lower triangular matrix \(x\) satisfying \(wx \in \Omega_w, h\), a minor \(p_{A, B}(x)\) is not identically zero if and only if \(A\) is reachable from \(B\).

To give a proof of Theorem 3.15, we recall the Plücker embedding of \(G/B\). For an element \(g = (g_{i,j}) \in G = \text{GL}_n(\mathbb{C})\) and \(i = (i_1, \ldots, i_j) \in I_{j,n}\), the \(i\)th Plücker coordinate \(p_{\bar{i}}(g)\) of \(g\) is given by the \(j \times j\) minor of \(g\) with row indices \(i_1, \ldots, i_j\) and the column indices \(1, \ldots, j\). That is, with the notation (3.7), we have that
\[
(3.11) \quad p_{\bar{i}}(g) = p_{\bar{i}[j]}(g).
\]

The Plücker embedding \(\psi\) is defined to be
\[
\psi: G/B \to \prod_{j=1}^{n-1} \mathbb{CP}^{(j)}_n, \quad gB \mapsto \prod_{j=1}^{n-1} (p_{\bar{i}}(g))_{\bar{i} \in I_{j,n}}.
\]

The Plücker embedding \(\psi\) is well-defined, and it is \(T\)-equivariant with respect to the action of \(T\) on \(\prod_{j=1}^{n-1} \mathbb{CP}^{(j)}_n\) given by
\[
(t_1, \ldots, t_n) \cdot (p_{\bar{i}})_{\bar{i} \in I_{j,n}} := (t_{i_1} \cdots t_{i_j}) \cdot (p_{\bar{i}})_{\bar{i} \in I_{j,n}}
\]
for \((t_1, \ldots, t_n) \in T\) and \(\bar{i} = (i_1, \ldots, i_j)\).
Example 3.15. Suppose that \( G = \text{GL}_3(\mathbb{C}) \). For an element \( g = (g_{i,j}) \in G \), the image \( \psi(gB) \) is given by
\[
\left( [p(1)_w(g), p(2)_w(g), p(3)_w(g)], [p(1,2)_w(g), p(1,3)_w(g), p(2,3)_w(g)] \right) \\
= \left( [g_{1,1}, g_{2,1}, g_{3,1}], [g_{1,1}g_{2,2} - g_{2,1}g_{1,2}, g_{1,1}g_{3,2} - g_{3,1}g_{1,2}, g_{2,1}g_{3,2} - g_{3,1}g_{2,2}] \right).
\]

Before presenting a proof of Theorem 3.3 we recall the following result of Gelfand and Serganova [17].

Lemma 3.16 ([17 Proposition 5.2.1]). Let \( gB \in G/B \), and let
\[
L_g = \left\{ \hat{i} \in \bigcup_{j=1}^{n} I_{j,n} \mid p_{\hat{i}}(g) \neq 0 \right\}.
\]
Then, for \( u \in S_n \), a point \( \hat{u}B \) is contained in \( T \cdot gB \) if and only if \( u^{(j)} \in L_g \) for all \( 1 \leq j \leq n - 1 \).

Proof of Theorem 3.3 We first recall Theorem 3.13. Let \( x = (x_{i,j}) \) be a lower triangular matrix having 1 on its diagonal. Suppose that \( \hat{w}x \in \Omega_{w,h} \). Then, for \( \hat{i} = (i_1, \ldots, i_j) \in I_{j,n} \), \( p_{\hat{i}}(x) \) is not identically zero if and only if \( \{i_1, \ldots, i_j\} \) is reachable from \( [j] \). Moreover, \( p_{\hat{i}}(\hat{w}x) \) is not identically zero if and only if \( \hat{k} = w \cdot (i_1, \ldots, i_j) \) for some \( \{i_1, \ldots, i_j\} \) which is reachable from \( [j] \). Therefore, there exists a point \( \hat{w}x \in \Omega_{w,h} \) such that
\[
L_{\hat{w}x} = \bigcup_{j=1}^{n} \{ w \cdot (i_1, \ldots, i_j) \mid (i_1, \ldots, i_j) \in J_{w,h,j} \}.
\]
By Lemma 3.16, a point \( \hat{u}B \) is contained in the closure \( \overline{T \cdot (\hat{w}x)B} \) if and only if \( u^{(j)} \in \{ w \cdot (i_1, \ldots, i_j) \mid (i_1, \ldots, i_j) \in J_{w,h,j} \} \) for all \( 1 \leq j \leq n - 1 \). This proves the theorem. \( \square \)

We close this section considering the description of \( \Omega_{w,h} \). As a direct corollary of Theorem 3.13 the closure \( \Omega_{w,h} \) in \( \text{Hess}(S, h) \) is given as follows.

Corollary 3.17. Let \( h: [n] \to [n] \) be a Hessenberg function and \( w \in S_n \). The closure \( \Omega_{w,h} \) of \( \Omega_{w,h} \) in \( \text{Hess}(S, h) \) is given by
\[
\Omega_{w,h} = \bigcap_{j=1}^{n-1} \{ gB \in \text{Hess}(S, h) \mid p_{w^{(j)}}(gB) = 0 \text{ for } \hat{i} \in I_{j,n} \setminus J_{w,h,j} \}.
\]

Example 3.18. (1) Suppose that \( h = (n, n, n, \ldots, n) \). Recall from Example 3.7 that \( J_{w,h,j} = \{ \hat{i} \mid w \cdot \hat{i} \succeq w^{(j)} \} \). This shows that the description in Corollary 3.17 generalizes that in [26 §10] (also, see [41 Theorem 3.2.10]).

(2) Let \( h = (3, 3, 4, 5, 5) \) and \( w = 15342 \). From Example 3.3 we have that
\[
\Omega_{w,h} = \{ gB \in \text{Hess}(S, h) \mid p(2)(g) = 0, p(1,3)(g) = p(1,4)(g) = p(1,2)(g) = p(2,5)(g) = 0, \\
p(3,4)(g) = p(2,3)(g) = p(2,4)(g) = 0, \\
p(1,2,5)(g) = p(1,3,4)(g) = p(1,2,3)(g) = 0, \\
p(1,2,4)(g) = p(2,3,5)(g) = p(2,4,5)(g) = p(2,3,4)(g) = 0, \\
p(1,2,3,5)(g) = p(1,2,4,5)(g) = p(1,2,3,4)(g) = p(2,3,4,5)(g) = 0 \}.
\]
4. Symmetric group action

4.1. $H^*_T(G/B)$. In [11] Tymoczko defines an action of the symmetric group $\mathfrak{S}_n$ on the equivariant cohomology $H^*_T(G/B)$ as follows. For $u \in \mathfrak{S}_n$ and $\sigma = (\sigma(v))_{v \in \mathfrak{S}_n} \in H^*_T(G/B) \subset \bigoplus_{v \in \mathfrak{S}_n} \mathbb{C}[t_1, \ldots, t_n]$, 

$$(u \cdot \sigma)(v) := (\sigma(u^{-1} v))(t_{u(1)}, \ldots, t_{u(n)}).$$

Geometrically, this action corresponds to the action of the symmetric group $\mathfrak{S}_n$ on $G/B$ by $u \cdot [g] = [u^{-1} g]$ for $u \in \mathfrak{S}_n$ and $[g] \in G/B$ (Corollary 2.10 of [11]).

The edge set of the GKM graph of the full flag variety $G/B$ (= the Bruhat graph of $\mathfrak{S}_n$) consists of $(w \to v)$ satisfying that $v = ws_{j,k}$ and $\ell(w) > \ell(v)$. The symbol $w \to v$ will mean that the pair $(w, v)$ satisfies these conditions.

**Proposition 4.1** (Proposition 6.2 of [5], Proposition 3.5 of [11]). Assume that $\text{Hess}(S, h)$ is the full flag variety $G/B$ of type $A$.

1. If $s_i w \to w$, then $s_i[X_w] - [X_w] = 0$.
2. If $w \to s_i w$, then $s_i[X_w] - [X_w] = (t_{i+1} - t_i)[X_{s_i w}]$.

We recall how to prove Proposition 4.1 following the proof in [5]. Let $X$ be a variety with an action of a complex torus $T$. Let $M$ be the character group of $T$ and $S$ be the character ring of $T$. Then $S$ acts on the equivariant Chow group $A^*_T(X)$ of $X$. For details, see Section 2.1 of [5].

**Theorem 4.2** (Theorem 2.1 of [5]). Let $X$ be a variety with an action of a complex torus $T$. Then the $S$-module $A^*_T(X)$ is generated by the classes $[Y]$ of closed $T$-invariant subvarieties $Y$ of $X$ with relations

$$\text{div}_Y(f) - \chi[Y]$$

where $f$ is a non-constant rational function on $Y$ which is a $T$-eigenvector of weight $\chi$.

**Lemma 4.3** (cf. Section 6.2 of [5]). Assume that $\text{Hess}(S, h)$ is the full flag variety. Let $X_w = B\bar{w}B/B$ and let $P_\alpha := B \cup B\bar{s}_\alpha B$ be the minimal parabolic subgroup associated to a simple root $\alpha$.

1. If $w \to s_\alpha w$, then $P_\alpha X_w = X_w$.
2. If $s_\alpha w \to w$, then $\varphi: P_\alpha \times_B X_w \to P_\alpha X_w$ is a birational morphism.

**Sketch of the proof of Proposition 4.1**. Consider

$$Z := P_\alpha \times_B X_w \xrightarrow{\varphi} P_\alpha X_w$$

where $\varphi([p, x]) = px$ for $p \in P_\alpha$ and $x \in X_w$ and $\pi([p, x]) = pB/B \in P_\alpha/B$.

Applying Theorem 4.2 to $\pi: Z \to \mathbb{P}^1$, we get

$$-\alpha[Z] = -[\ell(X_w)] + s_\alpha[\ell(X_w)],$$
where $\nu: X_w \to Z$ is the inclusion map. Then Lemma \ref{lem:cohomology-action} implies that

$$s_\alpha[X_w] - [X_w] = \begin{cases} 0 & \text{if } w \to s_\alpha w, \\ -\alpha [P_\alpha X_w] & \text{if } s_\alpha w \to w. \end{cases}$$

\hfill $\square$

4.2. $H^*_T(\text{Hess}(S, h))$. As in the case when $\text{Hess}(S, h)$ is the full flag variety $G/B$ of type $A$, the symmetric group $S_n$ acts on $H^*_T(\text{Hess}(S, h))$ for any Hessenberg function $h$: For $u \in S_n$ and $\sigma \in \bigoplus_{e \in S_n} \mathbb{C}[[t_1, \ldots, t_n]]$, if $\sigma \in H^*_T(\text{Hess}(S, h))$ then $u \cdot \sigma \in H^*_T(\text{Hess}(S, h))$ (cf. \cite{Deodhar} §4.2). We remark that the symmetric group $S_n$ does not stabilize $\text{Hess}(S, h)$, so that it does not act on the family of $T$-invariant subvarieties of $\text{Hess}(S, h)$. In other words, the translate of a $T$-invariant subvariety of $\text{Hess}(S, h)$ by an element of $S_n$ is not necessarily contained in $\text{Hess}(S, h)$. We will get a geometric description of the action of $S_n$ on $H^*_T(\text{Hess}(S, h))$ similar to that on the equivariant cohomology of the full flag variety $G/B$, by considering the action on the $T$-varieties $\Omega_{w, h}$, where $w \in S_n$. This is enough because their classes form a basis of $H^*_T(\text{Hess}(S, h))$.

Let $w, v \in S_n$ be such that $v = ws_{j,k}$ and $\ell(w) > \ell(v)$. Then $(w \to v)$ is contained in the edge set of the GKM graph of the full flag variety $G/B$ (= the Bruhat graph of $S_n$). By $w \to v$ we mean that $(w \to v)$ is contained in the edge set of the GKM graph of $\text{Hess}(S, h)$, and by $w \to v$ we mean that $(w \to v)$ is not contained in the edge set of the GKM graph of $\text{Hess}(S, h)$. In the following we will describe the action of a simple reflection $s_i$ on the class $\sigma_{w, h}$ (Proposition \ref{prop:action-simple-reflection}) and Proposition \ref{prop:action-simple-reflection}). For this, we will use some combinatorial properties which can be read from the GKM graph of $\text{Hess}(S, h)$ and the graphs $G_{w, h}$ and $G_{s_{i,w}, h}$ (Lemma \ref{lem:action-on-gkm}) and Lemma \ref{lem:action-on-gkm}.

**Lemma 4.4.** Let $w \in S_n$ and let $s_i$ be a simple reflection.

1. If $w \to s_i w$, then $E(G_{w, h}) = E(G_{s_{i,w}, h})$ and $\ell_h(s_i w) = \ell_h(w)$ (see Figure \ref{fig:1}).

2. If $w \to s_i w$, then $E(G_{s_{i,w}, h}) = E(G_{w, h}) \cup \{w^{-1}(i + 1) \to w^{-1}(i)\}$ and $\ell_h(w) = \ell_h(s_i w) + 1$ (see Figure \ref{fig:1}).

**Proof.** Assume that $\ell(w) > \ell(s_i w)$. Then $j := w^{-1}(i + 1) > w^{-1}(i) =: k$. Furthermore, $(w \to s_i w)$ is contained in the edge set of the GKM graph of $\text{Hess}(S, h)$ if and only if $k \leq h(j)$, or equivalently, there is an edge $j \to k$ in $G_{s_{i,w}, h}$. This completes the proof. $\square$
Lemma 4.5. Assume that $\ell(w) > \ell(s_iw)$, or equivalently, $w^{-1}(i + 1) < w^{-1}(i)$. Recall that we have a coordinate chart $X = (x_{a,b}) \in L^+ \rightarrow wX \in \{wX \in G \mid X = (x_{a,b}) \in L^+\}/B$ of $G/B$ around $w$, where $L^+$ is the unipotent part of $B^-$. Here, for the coordinate chart $(x_{a,b})$, the index $(a,b)$ satisfies the inequality $a > b$. In the following we will assume that for the pairing $(a,b)$, it holds that $a > b$. Then we have

$$w^{-1}\Omega_w^o = \{(x_{a,b}) \in L^+ \mid x_{a,b} = 0 \text{ for all } (a,b) \text{ with } w(a) < w(b)\}.$$

Similarly, we have

$$(s_iw)^{-1}\Omega_{s_iw}^o = \{(x_{a,b}) \in L^+ \mid x_{a,b} = 0 \text{ for all } (a,b) \text{ with } s_iw(a) < s_iw(b)\}.$$ 

By the condition $w^{-1}(i + 1) < w^{-1}(i)$, $w^{-1}\Omega_w^o$ is contained in $(s_iw)^{-1}\Omega_{s_iw}^o$ and

$$\{(a,b) \mid w(a) > w(b)\} \cup \{(w^{-1}(i), w^{-1}(i + 1))\} = \{(a,b) \mid s_iw(a) > s_iw(b)\}.$$ 

Furthermore, by Proposition 3.8,

$$w^{-1}\Omega_{w,h}^o = \{(x_{a,b}) \in w^{-1}\Omega_w^o \mid f_{a,b}^w = 0 \text{ for all } (a,b) \text{ with } a > h(b)\},$$

where

$$f_{a,b}^w = (c_{w(a)} - c_{w(b)})x_{a,b} + \sum_{l=1}^{a-1} \sum_{\gamma_1 > \gamma_2 > \cdots > \gamma_l > b} (-1)^l (c_{w(\gamma_l)} - c_{w(b)})x_{a,\gamma_1,\gamma_2,\cdots,\gamma_l,b}.$$ 

Thus $w^{-1}\Omega_{w,h}^o$ can be considered as the graph $\{x_{a,b} = g_{a,b}^w(x_{c,d}) : w(a) > w(b) \text{ and } a > h(b)\}$ of a function $g_{a,b}^w$ of free variables $x_{c,d}$, those variables with $(c,d)$ satisfying $w(c) > w(d)$ and $c \leq h(d)$.

Similarly,

$$(s_iw)^{-1}\Omega_{s_iw,h}^o = \{(x_{a,b}) \in (s_iw)^{-1}\Omega_{s_iw}^o \mid f_{a,b}^{s_iw} = 0 \text{ for all } (a,b) \text{ with } a > h(b)\},$$

where

$$f_{a,b}^{s_iw} = (c_{s_iw(a)} - c_{s_iw(b)})x_{a,b} + \sum_{l=1}^{a-1} \sum_{\gamma_1 > \gamma_2 > \cdots > \gamma_l > b} (-1)^l (c_{s_iw(\gamma_l)} - c_{s_iw(b)})x_{a,\gamma_1,\gamma_2,\cdots,\gamma_l,b},$$

and thus $(s_iw)^{-1}\Omega_{s_iw,h}^o$ can be considered as the graph $\{x_{a,b} = g_{a,b}^{s_iw}(x_{c,d}) \mid s_iw(a) > s_iw(b) \text{ and } a > h(b)\}$ of a function $g_{a,b}^{s_iw}$ of free variables $x_{c,d}$, those variables with $(c,d)$ satisfying $s_iw(c) > s_iw(d)$ and $c \leq h(d)$.

Lemma 4.5. Assume that $\ell(w) > \ell(s_iw)$.

1. If $w \rightarrow s_iw$, i.e., $w^{-1}(i) > h(w^{-1}(i + 1))$, then

$$\{(a,b) \mid w(a) > w(b) \text{ and } a \leq h(b)\} = \{(a,b) \mid s_iw(a) > s_iw(b) \text{ and } a \leq h(b)\}.$$ 

Thus $\Omega_{w,h}^o$ and $\Omega_{s_iw,h}^o$ have the same number of free variables.

2. If $w \rightarrow s_iw$, i.e., $w^{-1}(i) \leq h(w^{-1}(i + 1))$, then

$$\{(a,b) \mid w(a) > w(b) \text{ and } a \leq h(b)\} \cup \{(w^{-1}(i), w^{-1}(i + 1))\} = \{(a,b) \mid s_iw(a) > s_iw(b) \text{ and } a \leq h(b)\}.$$ 

Thus $\Omega_{s_iw,h}^o$ has one more free variable $x_{w^{-1}(i),w^{-1}(i+1)}$ than $\Omega_{w,h}^o$. 

Furthermore, their defining equations \( f^{w}_{a,b} \) and \( f^{s_{i}w}_{a,b} \) differ only by the coefficients of monomials, which are again determined by the choice of a regular semisimple element \( S = \text{diag}(c_{1}, \ldots, c_{n}) \). Let \( S' \) denote the diagonal matrix \( \text{diag}(c'_{1}, \ldots, c'_{n}) \), where \( c'_{a} = c_{s_{i}(a)} \) for \( a \in [n] \). Then

\[
f^{s_{i}w}_{a,b} = (c'_{w(a)} - c'_{w(b)})x_{a,b} + \sum_{a > \gamma_{1} > \gamma_{2} > \cdots > \gamma_{b}} (-1)^{t}(c'_{w(\gamma_{1})} - c'_{w(\gamma_{b})})x_{a,\gamma_{1},\gamma_{1},\gamma_{2} \cdots \gamma_{b}}.
\]

**Proposition 4.6.** Let \( w \in \mathfrak{S}_{n} \) and \( s_{i} \) be a simple reflection. If \( w \rightarrow s_{i}w \) or \( s_{i}w \rightarrow w \), then \( s_{i}\sigma_{w,h} = \sigma_{s_{i}w,h} \).

**Proof.** We may assume that \( w \rightarrow s_{i}w \). By Lemma 4.5 (1), \( \Omega_{w,h}^{\circ} \) and \( \Omega_{s_{i}w,h}^{\circ} \) have the same number of free variables, and their defining equations \( f^{w}_{a,b} \) and \( f^{s_{i}w}_{a,b} \) are homotopic. Therefore, they define homotopically equivalent subsets, \( w^{-1}\Omega_{w,h}^{\circ} \) and \( (s_{i}w)^{-1}\Omega_{s_{i}w,h}^{\circ} \). This proves the desired equality. \( \square \)

**Remark 4.7.** As the proof of Proposition 4.6 shows, the subvariety \( \Omega_{w,h} \subset \text{Hess}(S,h) \) may be moved out by the action of \( s_{i} \). In this case, \( s_{i}\Omega_{w,h} \) is no longer contained in \( \text{Hess}(S,h) \), but it is contained in \( \text{Hess}(S',h) \) for another choice of \( S' \), so that it defines a cohomology class in \( H^{*}(\text{Hess}(S',h)) \) and in turn defines one in \( H^{*}(\text{Hess}(S,h)) \) by the isomorphism \( \text{Hess}(S',h) \cong \text{Hess}(S,h) \) given by the action of \( g \in G \) such that \( S' = gSg^{-1} \).

Let \( w \in \mathfrak{S}_{n} \) and \( s_{i} \) be a simple reflection such that \( w \rightarrow s_{i}w \). Then the Bialynicki-Birula decomposition of \( \Omega_{s_{i}w,h} \) is given by

\[
\Omega_{s_{i}w,h} = \bigsqcup_{u \in \Omega_{s_{i}w,h}^{T}} (\Omega_{u}^{\circ} \cap \Omega_{s_{i}w,h}).
\]

For \( u \in \Omega_{s_{i}w,h}^{T} \), define \( T_{u} \) by the closure of \( T_{u}^{\circ} := \Omega_{u}^{\circ} \cap \Omega_{s_{i}w,h} \). Then \( T_{u} \) is irreducible because it is the closure of an affine cell. Let \( \tau_{u} \) denote the equivariant class in \( H_{T}^{*}(\text{Hess}(S,h)) \) induced by \( T_{u} \).

**Proposition 4.8.** Let \( w \in \mathfrak{S}_{n} \) and let \( s_{i} \) be a simple reflection.

1. If \( s_{i}w \rightarrow w \), then \( s_{i}\sigma_{w,h} - \sigma_{w,h} = 0 \).
2. If \( w \rightarrow s_{i}w \), then \( s_{i}(\sigma_{w,h} + \sum_{v \in \mathcal{A}} \tau_{v}) - (\sigma_{w,h} + \sum_{v \in \mathcal{A}} \tau_{v}) = (t_{i+1} - t_{i})\sigma_{s_{i}w,h} \), where \( \mathcal{A} \) is the set of all \( u \in \Omega_{s_{i}w,h}^{T} \cap \Omega_{w}^{T} \) such that \( \dim T_{u} = \dim \Omega_{w}, u \rightarrow s_{i}u \) and \( s_{i}u \notin \Omega_{w}^{T} \).

Note that \( \Omega_{w}^{T} \) consists of \( u \in \mathfrak{S}_{n} \) such that there is a descending chain \( u \rightarrow \cdots \rightarrow w \) in the Bruhat graph and a combinatorial description of \( \Omega_{w}^{T} \) is given in Theorem 3.5. Since \( \Omega_{u}^{\circ} \cap \Omega_{s_{i}w,h} \subset \Omega_{u,h} \), if \( \dim(\Omega_{u}^{\circ} \cap \Omega_{s_{i}w,h}) = \dim \Omega_{w,h} \), then \( \dim \Omega_{u,h} \geq \dim \Omega_{w,h} \). Let \( \tilde{\mathcal{A}} \) be the set of all \( u \in \Omega_{s_{i}w,h}^{T} \cap \Omega_{w}^{T} \) such that \( u \rightarrow s_{i}u \) and \( s_{i}u \notin \Omega_{w}^{T} \) and \( \dim \Omega_{u,h} \geq \dim \Omega_{w,h} \). Then \( \mathcal{A} = \{ u \in \tilde{\mathcal{A}} : \dim(\Omega_{u,h}^{\circ} \cap \Omega_{s_{i}w,h}) = \dim \Omega_{w,h} \} \).

**Example 4.9.** We illustrate the \( s_{i} \)-action on \( \sigma_{w,h} \) when \( w \rightarrow s_{i}w \). (See Figure 3)

1. Let \( h = (2, 4, 4, 4) \) and \( w = 2143 \) and \( i = 1 \). Then we have

\[
(t_{2} - t_{1})\sigma_{1243,h} = s_{1}(\sigma_{2143,h} + \sigma_{2413,h} + \sigma_{2341,h}) - (\sigma_{2143,h} + \sigma_{2413,h} + \sigma_{2341,h}).
\]

In this case, \( \Omega_{s_{i}w,h}^{T} \cap \Omega_{w}^{T} = \{ 2143, 2413, 2341, 3142, 4213, 3412 \} \) and \( \tilde{\mathcal{A}} = \{ 2413, 2341 \} = \mathcal{A} \).
(2) Let \( h = (2, 4, 4, 4) \) and \( w = 1423 \) and \( i = 3 \). Then we have
\[
(t_4 - t_3)\sigma_{1324,h} = s_3(\sigma_{1423,h} + s_1\sigma_{4213,h}) - (\sigma_{1423,h} + s_1\sigma_{4213,h})
\]
and \( s_1\sigma_{4213,h} = \sigma_{4213,h} + (t_2 - t_1)\sigma_{4123,h} \).

In this case, \( \Omega^T_{s,w,h} \cap \Omega^T_w = \{1423, 1432, 4132, 4312, 4321, 3412, 3421\} \) and
\[
\tilde{A} = \{4132, 4123\} \supset A = \{4123\}.
\]

For \( u = 4123 \in A, T_u = s_1\Omega_{4213,h} \).

(3) Let \( h = (2, 3, 4, 4) \) and \( w = 2143 \) and \( i = 1 \). Then we have
\[
(t_2 - t_1)\sigma_{1243,h} = s_1(\sigma_{2143,h} + s_1\sigma_{2431,h}) - (\sigma_{2143,h} + s_1\sigma_{2431,h})
\]
and \( s_1\sigma_{2431,h} = \sigma_{2431,h} + (t_3 - t_1)\sigma_{2413,h} \).

In this case, \( \Omega^T_{s,w,h} \cap \Omega^T_w = \{2143, 2413, 4213\} \) and
\[
\tilde{A} = \{2413\} = A.
\]

For \( u = 2413 \in A, T_u = s_1\Omega_{2431,h} \).

(4) Let \( h = (2, 3, 4, 4) \) and \( w = 1324 \) and \( i = 2 \). Then we have
\[
(t_3 - t_2)\sigma_{1234,h} = s_2(\sigma_{1324,h} + s_1\sigma_{1342,h} + s_3\sigma_{3124,h} + s_3\sigma_{4312,h}) - (\sigma_{1324,h} + s_1\sigma_{1342,h} + s_3\sigma_{3124,h} + s_3\sigma_{4312,h}).
\]

In this case, \( \Omega^T_{s,w,h} \cap \Omega^T_w = \{1324, 1342, 3124, 3142, 3412, 3412, 4312, 3421\} \) and
\[
\tilde{A} = \{1342, 3124, 4312\} = A.
\]

For \( u \in A, T_u = \Omega_{u,h} \).
We will give a more detailed description of the action of simple reflections when $h = (2, 3, \ldots, n, n)$ in Proposition 5.11.

4.3. **Proof of Proposition 4.8.** Let $\alpha$ be a simple root. The minimal parabolic subgroup $P^-_{\alpha}$ associated to $\alpha$ containing $B^-$ is defined by $P^-_{\alpha} = B^- \cup B^- \hat{s}_\alpha B^-$, where $s_\alpha$ is the simple reflection associated to $\alpha$. Then $P^-_{\alpha}/B^-$ is isomorphic to $\mathbb{C} P^1$ and is decomposed as $B^-/B^- \sqcup U_{-\alpha} \hat{s}_\alpha B^-/B^- \simeq \{ \text{a point} \} \sqcup \mathbb{C}$, where $U_{-\alpha}$ is the root group of root $-\alpha$. Denote by $0$ the point $B^-/B^-$ and by $\infty$ the point $\hat{s}_\alpha B^-/B^-$. As in the previous section, consider $Z := P^-_{\alpha} \times_{B^-} \Omega_w$ and two maps $\varphi$ and $\pi$:

$$Z = P^-_{\alpha} \times_{B^-} \Omega_w \xrightarrow{\varphi} P^-_{\alpha} \Omega_w$$

$$\xrightarrow{\pi} P^-_{\alpha}/B^-$$

where $\varphi([p, x]) = px$ for $p \in P^-_{\alpha}$ and $x \in \Omega_w$ and $\pi([p, x]) = pB^-/B^- \in P^-_{\alpha}/B^-$. Then $P^-_{\alpha} \times_{B^-} \Omega_w$ is the disjoint union

$$((B^- \hat{s}_\alpha B^-) \times_{B^-} \Omega^\circ_w) \sqcup (B^- \times_{B^-} \Omega^\circ_w).$$

Under the map $\varphi$, the image of $(B^- \hat{s}_\alpha B^-) \times_{B^-} \Omega^\circ_w$ is $U_{-\alpha} \hat{s}_\alpha \Omega^\circ_w$ and the image of $B^- \times_{B^-} \Omega^\circ_w$ is $\Omega_w$.

(-) If $\ell(s_\alpha w) = \ell(w) - 1$, then $U_{-\alpha} \hat{s}_\alpha \Omega^\circ_w$ is $\Omega^\circ_{s_\alpha w}$. In this case, $\varphi$ maps $(B^- \hat{s}_\alpha B^-) \times_{B^-} \Omega^\circ_w$ onto $\Omega^\circ_{s_\alpha w}$ of the same dimension.

(+) If $\ell(s_\alpha w) = \ell(w) + 1$, then $\Omega^\circ_w$ is $U_{-\alpha} \hat{s}_\alpha \Omega^\circ_{s_\alpha w}$, and thus $U_{-\alpha} \hat{s}_\alpha \Omega^\circ_w$ is $U_{-\alpha} \hat{s}_\alpha U_{-\alpha} \hat{s}_\alpha \Omega^\circ_{s_\alpha w}$.

From

$$U_{-\alpha} \hat{s}_\alpha U_{-\alpha} \hat{s}_\alpha \Omega^\circ_{s_\alpha w} = (U_{-\alpha} \sqcup U_{-\alpha} \hat{s}_\alpha U_{-\alpha}) \Omega^\circ_{s_\alpha w} = \Omega^\circ_{s_\alpha w} \sqcup \Omega^\circ_w$$

it follows that $U_{-\alpha} \hat{s}_\alpha \Omega^\circ_w$ is $\Omega^\circ_{s_\alpha w} \sqcup \Omega^\circ_w$. In this case, $\varphi$ maps $(B^- \hat{s}_\alpha B^-) \times_{B^-} \Omega^\circ_w$ onto $\Omega^\circ_{s_\alpha w} \sqcup \Omega^\circ_w$ of dimension less by one.

The action of $T$ on $Z = P^-_{\alpha} \times_{B^-} \Omega_w$ is given as follows: For $t \in T$, $b \in P^-_{\alpha}$ and $x \in \Omega_w$, $t \cdot [b, x] = [tb, x] = [t b t^{-1}, t x]$.

Thus the map $\varphi$ is $T$-equivariant. The fixed point set $Z^T$ is

$$\{ [s_\alpha, v] \mid v \in \Omega^T_w \} \cup \{ [e, v] \mid v \in \Omega^T_w \}$$

and $Z$ can be written as the union

$$\left( \bigsqcup_{v \in \Omega^T_w} (B^- \hat{s}_\alpha B^-) \times_{B^-} \Omega^\circ_v \right) \sqcup \left( \bigsqcup_{v \in \Omega^T_w} B^- \times_{B^-} \Omega^\circ_v \right),$$

which is the Bialynicki-Birula decomposition of $Z$ for the choice of a one-parameter subgroup given in Subsection 2.1.

Furthermore, (+) and (−) in the above hold also for $v \in \Omega^T_w$.

(+) If $\ell(s_\alpha v) = \ell(v) + 1$, then $\varphi$ maps $(B^- \hat{s}_\alpha B^-) \times_{B^-} \Omega^\circ_v$ onto $\Omega^\circ_{s_\alpha v} \cup \Omega^\circ_v \subset \Omega_v$. 


where \( A \) is a subsystem of \( \pi \). Applying Theorem 4.2 to \( Z \) and \( \pi \), it remains to describe the divisors of \( \varphi \) and \( \pi \) components of \( Z \).

Proof of Proposition 4.8. We will use the notation \( s_i \) for the simple reflection \( s_\alpha \) and \( P^-_i \) for the corresponding minimal parabolic subgroup \( P^-_\alpha \).

(1) Assume that \( s_1 w \rightarrow w \), that is, the edge \( (s_1 w \rightarrow w) \) is contained in the edge set of the GKM graph of \( \text{Hess}(S, h) \). Then \( \ell_h(w) = \ell_h(s_1 w) + 1 \) by Lemma 4.4 (2), that is, \( \dim \Omega_{s_1 w, h} = \dim \Omega_{w, h} + 1 \). Let \( Z' \) be the closure of \( \varphi^{-1}(\Omega^c_w \cap \text{Hess}(S, h)) \) in \( Z \) and let \( \varphi' \) and \( \pi' \) denote the restriction of \( \varphi \) and \( \pi \) to \( Z' \):

\[
Z' := \varphi^{-1}(\Omega^c_{s_1 w, h}) \xrightarrow{\varphi'} \Omega_{s_1 w, h}
\]

Applying Theorem 4.2 to \( Z' \) and \( \varphi' \) as in the previous section, we have

\[
\varphi'_s[\pi'^{-1}(x)] - \varphi'_s[\pi'^{-1}(0)] = 0.
\]

Furthermore, the divisors \( \varphi'_s[\pi'^{-1}(x)] \) and \( \varphi'_s[\pi'^{-1}(0)] \) are given by \([s_1 \Omega_{w, h}] \) and \([\Omega_{w, h}] \). Therefore, we have

\[
s_1 \sigma_{w, h} - \sigma_{w, h} = 0.
\]

This completes the proof of Proposition 4.8 (1).

(2) Assume that \( w \rightarrow s_1 w \), that is, the edge \( (w \rightarrow s_1 w) \) is contained in the edge set of the GKM graph of \( \text{Hess}(S, h) \). Then \( \ell_h(w) = \ell_h(s_1 w) - 1 \) by Lemma 4.4 (2), that is, \( \dim \Omega_{s_1 w, h} = \dim \Omega_{w, h} - 1 \). Let \( Z' \) be the closure of \( \varphi^{-1}(\Omega^c_{s_1 w} \cap \text{Hess}(S, h)) \) in \( Z \) and let \( \varphi' \) and \( \pi' \) denote the restriction of \( \varphi \) and \( \pi \) to \( Z' \):

\[
Z' := \varphi^{-1}(\Omega^c_{s_1 w, h}) \xrightarrow{\varphi'} \Omega_{s_1 w, h}
\]

Applying Theorem 4.2 to \( Z' \) and \( \varphi' \) as in the previous section, we have

\[
\varphi'_s[\pi'^{-1}(x)] - \varphi'_s[\pi'^{-1}(0)] = -\alpha \sigma_{s_1 w, h}.
\]

It remains to describe the divisors \( \varphi'_s[\pi'^{-1}(x)] \) and \( \varphi'_s[\pi'^{-1}(0)] \) in \( \Omega_{w, h} \), that is, irreducible components of \( \pi'^{-1}(x) \) of dimension \( \dim \Omega_{w, h} \) whose images by \( \varphi \) have the same dimension, and irreducible components of \( \pi'^{-1}(0) \) of dimension \( \dim \Omega_{w, h} \).

Since \( Z' \) is invariant under the action of \( T \), the Białynicki-Birula decomposition of \( Z' \) is given by

\[
Z' = \bigcup_{\ell_i \in \mathcal{A}_\infty} \left( (B^-s_i B^-) \times_{B^-} \Omega^c_{i, v} \right) \cap Z' \cup \bigcup_{\ell_i \in \mathcal{A}_0} \left( (B^- \times_{B^-} \Omega^c_{i, v}) \cap Z' \right),
\]

where \( \mathcal{A}_\infty = \{ u \in \Omega^T_w \mid s_1 u \in \Omega^T_{s_1 w, h} \} \) and \( \mathcal{A}_0 = \{ u \in \Omega^T_w \mid u \in \Omega^T_{s_1 w, h} \} \). For \( v \in \mathcal{A}_\infty \) and \( u \in \mathcal{A}_0 \), put

\[
\tilde{T}^c_{\infty, v} := ((B^-s_i B^-) \times_{B^-} \Omega^c_{i, v}) \cap Z' \text{ and } \tilde{T}^c_{0, u} := (B^- \times_{B^-} \Omega^c_{i, v}) \cap Z'.
\]
and put $\tilde{T}_{x,v}$ and $\tilde{T}_{0,u}$ to be their closures in $Z'$. Then, $\tilde{T}_{x,v}$ is the unique cell in $Z'$ of maximal dimension $\dim \Omega_{s_iw,h}$ and other cells $\tilde{T}_{0,u}$ for all $u \in A_0$ and $\tilde{T}_{0,v}$ for all $v \in A_\infty \setminus \{w\}$ have dimension less than or equal to $\dim \Omega_{w,h}$. Note that $\tilde{T}_{x,v} \cap \pi^{-1}(x) = \{[s_a, x] \mid x \in \Omega_{w,h}\}$ is an irreducible component of $\pi^{-1}(x)$ and $\tilde{T}_{0,w} = \{[e, x] \mid x \in \Omega_{w,h}\}$ is an irreducible component of $\pi^{-1}(0)$. Furthermore, any irreducible component of $\pi^{-1}(0)$ other than $\tilde{T}_{0,w}$ is of the form $\tilde{T}_{0,u}$, where $u \in A_0$ satisfies $\dim \tilde{T}_{0,u} = \dim \Omega_{w,h}$ and any irreducible component of $\pi^{-1}(x)$ other than $\tilde{T}_{x,v}$ is of the form $\tilde{T}_{x,v} \cap \pi^{-1}(x)$, where $v \in A_\infty$ satisfies $\dim (\tilde{T}_{x,v} \cap \pi^{-1}(x)) = \dim \Omega_{w,h}$. Consequently,

$$\left[\pi^{-1}(0)\right] = \sum_{u \in A_0} [\tilde{T}_{0,u}]$$

where $A'_0 := \{u \in A_0 \mid \dim \tilde{T}_{0,u} = \dim \Omega_{w,h}\}$, and

$$\left[\pi^{-1}(x)\right] = \sum_{v \in A_\infty} [\tilde{T}_{x,v} \cap \pi^{-1}(x)]$$

where $A'_\infty := \{v \in A_\infty \mid \dim (\tilde{T}_{x,v} \cap \pi^{-1}(x)) = \dim \Omega_{w,h}\}$.

**Lemma 4.10.** Assume that $w \to s_iw$.

1. For $u \in \Omega^T_w$, we have

$$u \in \Omega^T_{s_iw,h} \iff s_iu \in \Omega^T_{s_iw,h}.$$  

2. For $u \in \Omega^T_w \cap \Omega^T_{s_iw,h}$ with $u \to s_iu$, we have

$$\dim \tilde{T}_{0,u} = \dim \Omega_{w,h} \iff \dim (\tilde{T}_{x,u} \cap \pi^{-1}(x)) = \dim \Omega_{w,h}.$$  

If $u$ is an element of $A_0$ with $\dim \tilde{T}_{0,u} = \dim \Omega_{w,h}$, then, by Lemma 4.10, $u$ is an element of $A_\infty$ and $\tilde{T}_{x,u} \cap \pi^{-1}(x) = \{[s_a, x] \mid x \in T_u\}$ is an irreducible component of $\pi^{-1}(x)$. Conversely, if $u$ is an element of $A_\infty$ with $\dim (\tilde{T}_{x,u} \cap \pi^{-1}(x)) = \dim \Omega_{w,h}$, then by Lemma 4.10, $u$ is an element of $A_0$. There is a subvariety $T$ of $\Omega_u$ such that $\tilde{T}_{x,u} = \{[s_i, x] \mid x \in T\}$. Then $\{[e, x] \mid x \in T\}$ is an irreducible component of $\pi^{-1}(0)$. Therefore, we have $A'_0 = A'_\infty$.

We postpone the proof of Lemma 4.10 until the end of this section and proceed with the proof of Proposition 4.8 (2).

Let $u \in A'_0$.

1. If $\ell(s_iu) > \ell(u)$, then $s_iu \in \Omega^T_w$, and thus $s_iu \in A_\infty$. By Lemma 4.10, $s_iu$ belongs to $A_0$.

(a) If $s_iu \to u$, then we have $\varphi'_s[\tilde{T}_{x,u} \cap \pi^{-1}(x)] - \varphi'_s[\tilde{T}_{0,u}] = 0$.

(b) If $s_iu \to u$, then $\dim \tilde{T}_{0,s_iu} = \dim \tilde{T}_{0,u}$, and thus $\dim \tilde{T}_{0,s_iu} = \dim \Omega_{w,h}$. Consequently, both $u$ and $s_iu$ are contained in $A'_0$. Since $s_iu \to u$, $\pi'$ maps $\tilde{T}_{x,u}$ and $\tilde{T}_{x,s_iu}$ to $\{x\}$, that is, $\tilde{T}_{x,u}$ and $\tilde{T}_{x,s_iu}$ are contained in $\pi^{-1}(x)$. It follows that $\varphi'_s[\tilde{T}_{x,u} \cap \pi^{-1}(x)] = [T_{s_iu}]$ and $\varphi'_s[\tilde{T}_{x,s_iu} \cap \pi^{-1}(x)] = [T_{s_iu}] = [T_u]$, and that

$$[\varphi'_s[\tilde{T}_{x,u} \cap \pi^{-1}(x)] + \varphi'_s[\tilde{T}_{x,s_iu} \cap \pi^{-1}(x)] - \varphi'_s[\tilde{T}_{0,u}] + \varphi'_s[\tilde{T}_{0,s_iu}]] = ([T_{s_iu}] + [T_u]) - ([T_u] + [T_{s_iu}]) = 0.$$
Therefore, the pair \((u, s_i u)\) does not contribute to \(\varphi'_s[\pi'^{-1}(x)] - \varphi'_s[\pi'^{-1}(0)]\).

2. If \(\ell(s_i u) < \ell(u)\), then \(\varphi\) maps \(\tilde{\mathcal{T}}^r_{x,u}\) biholomorphically onto \(\mathcal{T}_{s_i u}\).

(a) If \(u \rightarrow s_i u\), then \(\pi'\) maps \(\tilde{\mathcal{T}}^r_{x,u}\) onto \(P^r_{\alpha}/B^-\), and thus \(\dim(\tilde{\mathcal{T}}^r_{x,u} \cap \pi'^{-1}(x)) \leq \dim \Omega_{w,h} - 1\), contradicting to the fact that \(\tilde{\mathcal{T}}^r_{x,u} \cap \pi'^{-1}(x)\) an irreducible component of \(\pi'^{-1}(x)\).

(b) If \(u \rightarrow s_i u\), then \(\pi'\) maps \(\tilde{\mathcal{T}}^r_{x,u}\) onto \(\{x\} = \{s_\alpha B^-/B^-\}\), and thus \(\tilde{\mathcal{T}}^r_{x,u} \cap \pi'^{-1}(x) = \tilde{\mathcal{T}}^r_{x,u}\). In this case, \(\varphi'_s[\tilde{T}_{0,u}] = [T_u]\) and \(\varphi'_s[\tilde{T}_{x,u} \cap \pi'^{-1}(x)] = [T_{s_i u}] = s_i[T_u]\).

i. If \(s_i u \notin \Omega^T_w\), then these terms will contribute nontrivially to \(\varphi'_s[\pi'^{-1}(x)] - \varphi'_s[\pi'^{-1}(0)]\).

ii. If \(s_i u \in \Omega^T_w\), then \(v := s_i u\) is an element of \(\mathcal{A}_\infty\) and thus is an element of \(\mathcal{A}_0\) by Lemma 4.11. As in the case 1-(b), we see that the pair \((v, s_i v)\) does not contribute to \(\varphi'_s[\pi'^{-1}(x)] - \varphi'_s[\pi'^{-1}(0)]\).

Therefore,

\[
\varphi'_s[\pi'^{-1}(x)] - \varphi'_s[\pi'^{-1}(0)] = \sum_{u \in \mathcal{A}_c} \varphi'_s[\tilde{T}_{x,u} \cap \pi'^{-1}(x)] - \sum_{u \in \mathcal{A}_0} \varphi'_s[\tilde{T}_{0,u}] = \left( s_i \sigma_{w,h} + \sum_{v \in \mathcal{A}} s_i \tau_v \right) - \left( \sigma_{w,h} + \sum_{v \in \mathcal{A}} \tau_v \right),
\]

where \(\mathcal{A}\) is the set of all \(u \in \Omega^T_{s_i w,h} \cap \Omega^T_w\) such that \(\dim T_u = \dim \Omega_{w,h}, u \rightarrow s_i u\) and \(s_i u \notin \Omega^T_w\).

This completes the proof of Proposition 4.8(2).

In the remaining part of this section, we will present a proof of Lemma 4.11. Recall that \(\Omega^T_{v,h}\) consists of \(u \in \mathcal{S}_n\) having the property that

for each \(j \in [n]\) there is \(\hat{i} = (i_1, \ldots, i_j)\) which is reachable from \([j]\) with \(\{u(1), \ldots, u(j)\} \uparrow = v \cdot \hat{i}\) (Theorem 3.5) and \(\Omega_{v,h}\) is defined by the equations

\(p_{v,\hat{i}} = 0\) for all \(\hat{i} = (i_1, \ldots, i_j)\) which is not reachable from \([j]\) and for all \(j \in [n - 1]\) (Corollary 3.11). From these properties we may notice that Lemma 4.11 will follow from the symmetricity of the reachability. We will prove this first in the following Lemma.

**Lemma 4.11.** Assume that two vertices \(j\) and \(k\) \((j < k)\) in \(G_{v,h}\) are connected by an edge in \(G_{v,h}\) and \(v(j) = i < v(k) = i + 1\). Let \(\{i_1, \ldots, i_{d-1}\}\) be a set consisting of \(d - 1\) vertices in \(G_{v,h}\) distinct from \(j\) or \(k\). Then \(\{i_1, \ldots, i_{d-1}, j\}\) is reachable from \([d]\) if and only if \(\{i_1, \ldots, i_{d-1}, k\}\) is reachable from \([d]\).

**Proof.** Let \(\{i_1, \ldots, i_{d-1}\}\) be a set consisting of \(d - 1\) vertices in \(G_{v,h}\) distinct from \(j\) or \(k\).

\[
i \quad i + 1
\
\begin{array}{c}
\hline
j \quad \rightarrow \quad k
\end{array}
\]

\(G_{v,h}\).

\((\Leftarrow\Rightarrow)\) If \(\{i_1, \ldots, i_{d-1}, j\}\) is reachable from \([d]\), then the vertex \(j\) is reachable from \(a\) for some \(a \leq d\). Thus there is a sequence \(a = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{l-1} \rightarrow v_l = j\) of vertices in \(G_{v,h}\).
Noting that there is an edge \((j \rightarrow k)\) in \(G_{w,h}\), extend the above sequence by adding \(v_l = j \rightarrow v_{l+1} = k\). Thus \(k\) is reachable from \(a\).

By assigning \(k\) to \(a\), we get that \(\{i_1, \ldots, i_{d-1}, j\}\) is reachable from \([d]\).

\((\Rightarrow)\) Assume that \(\{i_1, \ldots, i_{d-1}, k\}\) is reachable from \([d]\). Then the vertex \(k\) is reachable from \(a\) for some \(a \leq j'\). Thus there is a sequence \(a = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{l-1} \rightarrow v_l = k\) of vertices in \(G_{v,h}\).

- If \(a = j\), then \(j\) is reachable from \(a\), and thus we get that \(\{i_1, \ldots, i_{d-1}, j\}\) is reachable from \([d]\).

- If \(a < j\), then \(v_m < j \leq v_{m+1}\) for some \(m\). Then there is an edge \((v_m \rightarrow v_{m+1})\) in \(G_{v,h}\), and we have \(v(v_m) < i\) because \(v(v_m) < i + 1\) and \(v(v_m) = i\).

Thus there is an edge \((v_m \rightarrow j)\) in \(G_{s,w,h}\) and hence \(j\) is reachable from \(a\). Replacing the sequence \(a = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{l-1} \rightarrow v_l = k\) with \(a = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_m \rightarrow j\), we get \(\{i_1, \ldots, i_{d-1}, j\}\) which is reachable from \([d]\).

- If \(a > j\), then \(j < a \leq d\). By the assumption that \(\{i_1, \ldots, i_{d-1}, k\}\) is reachable from \([d]\), there is \(\sigma \in \mathcal{S}_d\) such that \(\sigma(c)\) is reachable from \(c \in [d] - \{a\}\). Note that \(k\) is reachable from \(a \in [d]\). Since \(j \notin \{i_1, \ldots, i_{d-1}\}\), we have \(\sigma(j) \neq j\) and thus \(\sigma(j) > j\). Then \(\sigma(i_{\sigma(j)}) > \sigma(j)\), too. Repeating this process, we get a sequence \(b_0 := j < b_1 := i_{\sigma(j)} < \cdots < b_d < b_{d+1} := i_{\sigma(b_d)} < \cdots\). Then \(b_d < a \leq b_{d+1}\) for some \(d\).
Since there is a sequence of edges \((b_{d} \rightarrow \cdots \rightarrow b_{d+1})\), there is a sequence of edges \((a \rightarrow \cdots \rightarrow b_{d+1})\). By replacing the sequences of vertices
\[
\begin{align*}
  j &\rightarrow \cdots \rightarrow b_{1}, \\
  b_{1} &\rightarrow \cdots \rightarrow b_{2}, \\
  \cdots \\
  b_{d} &\rightarrow \cdots \rightarrow b_{d+1}, \\
  a &\rightarrow \cdots \rightarrow k,
\end{align*}
\]
with
\[
\begin{align*}
  j, \\
  b_{1}, \\
  \cdots \\
  b_{d}, \\
  a &\rightarrow \cdots \rightarrow b_{d+1},
\end{align*}
\]
we get that \(\{i_{1}, \ldots, i_{d-1}, j\}\) is reachable from \([d]\). 

\[\Box\]

*Proof of Lemma 4.10* Assume that \(w \rightarrow s_{i}w\). Then \(j := w^{-1}(i + 1) < w^{-1}(i) =: k\), so that \(w(j) = i + 1 > w(k) = i\) and \(s_{i}w(j) = i < s_{i}w(k) = i + 1\). By Lemma 4.4 (2), \(G_{s_{i}w,h}\) has \((j \rightarrow k)\) as an edge.

\[
\begin{array}{ccc}
  \circ & \circ & \circ \\
  j & k & j \\
\end{array}
\]

\[G_{w,h} \quad G_{s_{i}w,h}\]

(1) Let \(u \in \Omega_{w}^{T}\).

(\(\Leftarrow\)) Assume that \(s_{i}u \in \Omega_{s_{i}w,h}^{T}\). Then \(u > s_{i}u\). Thus \(j' := u^{-1}(i + 1) < u^{-1}(i) =: k'\), so that \(u(j') = i + 1 > u(k') = i\) and \(s_{i}u(j') = i < s_{i}u(k') = i + 1\). Since \(s_{i}u \in \Omega_{s_{i}w,h}^{T}\), by Theorem 3.5 for all \(j \in [n - 1]\), there is \(\{i_{1}, \ldots, i_{j}\}\) which is reachable from \([j]\) in \(G_{s_{i}w,h}\) and such that \(\{s_{i}u(1), \ldots, s_{i}u(j)\} = \{s_{i}w(i_{1}), \ldots, s_{i}w(i_{j})\}\). We will show that \(\{u(1), \ldots, u(j)\}\) can be obtained in the same way. Note that \(u(a) = s_{i}u(a)\) for all \(a\) other than \(j', k'\). Hence it suffices to show that \(\{u(1), \ldots, u(d) = i\}\) is of the form \(\{s_{i}w(i_{1}), \ldots, s_{i}w(i_{d})\}\) for some \(\{i_{1}, \ldots, i_{d}\} \in J_{s_{i}w,h,d}\).

Let \(\{i_{1}, \ldots, i_{d}\}\) be a set such that
- \(i_{c}\) is reachable from \(c\) for all \(c \in [d]\); and
- \(\{s_{i}u(1), \ldots, s_{i}u(d) = i\} = \{s_{i}w(i_{1}), \ldots, s_{i}w(i_{d})\}\).

Here, we do not assume the monotonicity of \(\{i_{c}\}_{c}\), i.e., we do not assume that \(i_{1} < \cdots < i_{d}\). By the fact that \(s_{i}w(k) = i\), the vertex \(j\) is reachable from \(a\) for some \(a \leq d\). By Lemma 4.11 \(k\) is reachable from \(a\). By replacing the sequence \(a \rightarrow \cdots \rightarrow j\) with \(a \rightarrow \cdots \rightarrow k\), we get \(\{i_{1}, \ldots, i_{d}\}\)
which is reachable from $[d]$ and such that \( \{u(1), \ldots, u(d)\} = \{s_iw(i_1), \ldots, s_iw(i_d)\} \). Therefore, \( u \) is contained in \( \Omega^T_{s_iw,h} \).

\((\Longrightarrow)\) Assume that \( u \in \Omega^T_{s_iw,h} \). If \( s_iu > u \) then applying the same arguments as in the above to \( v = s_iu \), we get that \( s_iu \) is contained in \( \Omega^T_{s_iw,h} \). If \( u > s_iu \), then \( u' = u^{-1}(i + 1) < u^{-1}(i) = k' \), so that \( u(j') = i + 1 > u(k') = i \) and \( s_iu(j') = i < s_iu(k') = i + 1 \).

Since \( u \in \Omega^T_{s_iw,h} \), by Theorem 3.5 for all \( j \in [n - 1] \), there is \( \{i_1, \ldots, i_j\} \) which is reachable from \( [j] \) in \( G_{s_iw,h} \) and such that \( \{u(1), \ldots, u(j)\} = \{s_iw(i_1), \ldots, s_iw(i_j)\} \). We will show that \( \{s_iu(1), \ldots, s_iu(j)\} \) can be obtained in the same way. Note that \( u(a) = s_iu(a) \) for all \( a \) other than \( j', k' \). Hence it suffices to show that \( \{s_iu(1), \ldots, s_iu(d) = i\} \) is of the form \( \{s_iw(i_1), \ldots, s_iw(i_d)\} \) for some \( \{i_1, \ldots, i_d\} \in J_{s_iw,h,d} \).

Let \( \{i_1, \ldots, i_d\} \) be a set such that

- \( i_1 \) is reachable from \( c \) for all \( c \in [d] \); and
- \( \{u(1), \ldots, u(d) = i + 1\} = \{s_iw(i_1), \ldots, s_iw(i_d)\} \).

Here, we do not assume the monotonicity of \( \{i_c\} \), i.e., we do not assume that \( i_1 < \cdots < i_d \). By the fact that \( s_iw(k) = i + 1 \), the vertex \( k \) is \( i_a \) for some \( a \leq d \). By Lemma 4.11 \( (\{i_1, \ldots, i_d\}\setminus\{k\}) \cup \{j\} \) is reachable from \( [d] \). Furthermore, \( \{i_1', \ldots, i_d'\} = ((\{i_1, \ldots, i_d\}\setminus\{k\}) \cup \{j\}) \) satisfies that \( \{u(1), \ldots, u(d) = i\} = \{s_iw(i_1'), \ldots, s_iw(i_d')\} \). Therefore, \( u \) is contained in \( \Omega^T_{s_iw,h} \).

\((\Longleftarrow)\) Let \( u \in \Omega^T_{w} \cap \Omega^T_{s_iw,h} \) with \( u \leftrightarrow s_iu \). Then \( \dim \tilde{T}_{0,u} = \dim(\Omega_{u,h} \cap \Omega_{s_iw,h}) \) and \( \dim(\tilde{T}_{\infty,u} \cap \pi^{-1}(\infty)) = \dim(\Omega_{s_iu,h} \cap \Omega_{s_iw,h}) \). By Proposition 4.6 \( \Omega_{s_iu,h} = s_i\Omega_{u,h} \). Thus \( \dim(\tilde{T}_{\infty,u} \cap \pi^{-1}(\infty)) = \dim(s_i\Omega_{u,h} \cap \Omega_{s_iw,h}) \).

By Lemma 4.11 \( p_{s_iw \cdot \{i_1, \ldots, i_d\}} = 0 \) on \( \Omega_{s_iw,h} \) if and only if \( p_{s_iw \cdot \{i_1, \ldots, i_d\}} = 0 \) on \( \Omega_{s_iw,h} \), that is, the set of \( \mathcal{L} \) such that \( p_{\mathcal{L}} = 0 \) on \( \Omega_{s_iw,h} \) is stable under the action of the simple reflection \( s_i \).

Here, the action of \( s_i \) on \( p_{\{i_1, \ldots, i_d\}} \) is given by \( s_i \cdot p_{\{i_1, \ldots, i_d\}} = p_{\{s_i(i_1), \ldots, s_i(i_d)\}} \). Therefore,

\[ \dim \tilde{T}_{0,u} = \dim \Omega_{w,h} \iff \dim(\tilde{T}_{\infty,u} \cap \pi^{-1}(\infty)) = \dim \Omega_{w,h}. \]

\[ \square \]

5. Permutohedral varieties

In this section we consider the Hessenberg variety associated with the function \( h \) defined as \( h(i) = i + 1 \) for \( i = 1, \ldots, n - 1 \) and \( h(n) = n \). This variety, denoted by \( H_n \), is called the permutohedral variety and is known to be the toric variety corresponding to the fans determined by the chambers of the root system of type \( A_{n-1} \). We give explicit descriptions of the classes \( \sigma_{w,h} \) and also the symmetric group actions for each class. It is known from [32] that \( H^2_k(H_n) \) is a direct sum of permutation modules of \( S_n \) for each \( k \). The explicit descriptions of basis classes and the symmetric group action enable us to construct permutation modules that constitute the \( S_n \)-module \( H^2_k(H_n) \) for all \( k \).

In this section we only consider the permutohedral varieties, and we drop \( h \) in \( A_{w,h}, \sigma_{w,h}, J_{w,h,j}, G_{w,h} \) and so forth for convenience.
5.1. **Bases of** $H_{T}^{2k}(\mathcal{H}_{n})$. For a permutation $w \in \mathfrak{S}_{n}$, we say that $i \in [n - 1]$ is a *descent* of $w$ if $w(i) > w(i + 1)$. Let $D(w)$ be the set of descents of $w$, and let $\text{des}(w) = |D(w)|$ be the number of descents of $w$. Then the following lemma follows from the definition of the GKM graphs.

**Lemma 5.1.** Let $w \in \mathfrak{S}_{n}$ be a vertex in the GKM graph of $\mathcal{H}_{n}$.

1. The number of all edges adjacent to $w$ is $n - 1$.
2. The number of outgoing(downward) edges from $w$ is the number of descent of $w$; that is, $\ell_{h}(w) = \text{des}(w)$.

For a set partition $I_{1}, \ldots, I_{t}$ of $[n]$, the *Young subgroup* $\mathfrak{S}_{I_{1}} \times \cdots \times \mathfrak{S}_{I_{t}}$ is defined to be the stabilizer of $I_{1}, \ldots, I_{t}$ in the symmetric group $\mathfrak{S}_{n}$. For a nonempty subset $I$ of $[n]$, we denote by $\mathfrak{S}_{I} \subseteq \mathfrak{S}_{n}$ the Young subgroup $\mathfrak{S}_{I} \times \mathfrak{S}_{[n] \setminus I}$ for notational simplicity.

We will give explicit descriptions of the support $A_{w}$ of $\sigma_{w}$ and the value $\sigma_{w}(v)$ for each $v \in A_{w}$.

**Proposition 5.2.** For a permutation $w \in \mathfrak{S}_{n}$, let $D(w) = \{d_{1} < \cdots < d_{k}\} \subseteq [n - 1]$ be the set of descents of $w$, and let $d_{0} = 0$, $d_{k+1} = n$. If we let $J_{s} = \{w(d_{s} + 1), \ldots, w(d_{s})\}$ for $s = 1, 2, \ldots, k + 1$, then

$$A_{w} = (\mathfrak{S}_{J_{1}} \times \cdots \times \mathfrak{S}_{J_{k+1}}) w = \{uw \mid u \in \mathfrak{S}_{J_{1}} \times \cdots \times \mathfrak{S}_{J_{k+1}}\}.$$ 

**Proof.** Because we are considering the Hessenberg function $h(i) = i + 1$ for all $i$, the graph $G_{w}$ is a disjoint union of path graphs and it is enough to consider pairs $(j, j + 1)$ when we find the graph $G_{w}$. Indeed, an edge $j \to (j + 1)$ is in the graph $G_{w}$ if and only if $w(j) < w(j + 1)$. Therefore, we have that $(j \to j + 1)$ is an edge in $G_{w}$ if and only if $j \notin D(w) = \{d_{1} < \cdots < d_{k}\}$ and we obtain

$$G_{w} = \bigcup_{s \in \{0, 1, \ldots, k\}} \mathcal{P}(d_{s} + 1, \ldots, d_{s+1})$$

where $\mathcal{P}(v_{1}, \ldots, v_{m})$ denote the path graph with vertices $v_{1}, \ldots, v_{m}$ and edges $(v_{p}, v_{p+1})$ for $p \in [m - 1]$. For an index $j$ satisfying $d_{s} < j \leq d_{s+1}$, the set of reachable vertices from $j$ is $\{d_{s} + 1, \ldots, d_{s+1}\}$. For any index $j$ satisfying $d_{s} < j \leq d_{s+1}$, we get

$$L_{w,j} = \{w(i) \mid i \in \{d_{s} + 1, \ldots, d_{s+1}\}\} \cup w(j) = \{w(d_{s} + 1), \ldots, w(d_{s+1})\} = J_{s+1}.$$ 

By the definition of $J_{w,j}$ and $A_{w}$, we obtain $A_{w} = (\mathfrak{S}_{J_{1}} \times \cdots \times \mathfrak{S}_{J_{k+1}}) w$. \hfill \Box

**Remark 5.3.** It is well known that the GKM graph of $\mathcal{H}_{n}$ is the one skeleton of the permutohedron of type $A_{n-1}$, that is the convex hull of the permutation vectors $\{(w(1), \ldots, w(n)) \mid w \in \mathfrak{S}_{n}\}$ in $\mathbb{R}^{n}$. Proposition 5.2 can be proved using the arguments on the face structure of the permutohedron; as it is shown in [28].

We now give an explicit description of each class $\sigma_{w}$ for $w \in \mathfrak{S}_{n}$.

**Theorem 5.4.** For a permutation $w \in \mathfrak{S}_{n}$, let $D(w) = \{d_{1} < \cdots < d_{k}\} \subseteq [n - 1]$ be the set of descents of $w$ and $v \in A_{w}$. Then

$$\sigma_{w}(v) = \prod_{s=1}^{k} (t_{v(d_{s}+1)} - t_{v(d_{s})}).$$
Proof. The induced subgraph $\Gamma_w$ of the GKM graph $\Gamma$ of $H_n$, with the vertex set $A_w$, is $(n-1-k)$-regular. Hence, by Proposition 2.14, $\sigma_w$ is uniquely determined by the condition that the value $\sigma_w(v)$ is the product of $\alpha(v \rightarrow u)$ for $v \rightarrow u$ is an edge in $\Gamma$ but not in $\Gamma_w$. □

Example 5.5. Let $w = 25347168$ be a permutation in $S_8$. Then $D(w) = \{2, 5\}$ and

$$A_w = \{ v \in S_8 \mid \{v(1), v(2)\} = \{2, 5\}, \{v(3), v(4), v(5)\} = \{3, 4, 7\}, \{v(6), v(7), v(8)\} = \{1, 6, 8\} \}$$

has $2 \times 6 \times 6 = 72$ elements. For example, $v = 52437681$ is an element of $A_w$. Moreover, due to Theorem 5.4 we have $\sigma_w(w) = (t_3 - t_5)(t_1 - t_7)$ and $\sigma_w(v) = (t_4 - t_2)(t_6 - t_7)$.

From Lemma 5.1 and Theorem 5.4 we obtain a nice basis for the $(2k)$th cohomology space of the permutohedral variety $H_n$.

Proposition 5.6. For each $k = 0, 1, \ldots, n-1$, $B_k := \{ \sigma_w \mid w \in S_n, \text{des}(w) = k \}$ forms a basis of the $(2k)$th cohomology space $H^{2k}(H_n)$; hence the $(2k)$th Betti number of $H_n$ is the number of permutations in $S_n$ with $k$ descents.

5.2. Symmetric group action on $H^*(H_n)$. In this subsection, we will consider the symmetric group action on the equivariant cohomology ring $H^*_T(H_n)$, and then study that on the singular cohomology ring $H^*(H_n)$.

We denote by

$$B_k := \{ \sigma_w \mid w \in S_n, \text{des}(w) = k \}$$

a basis of the cohomology space $H^{2k}_T(H_n)$. An $l$-composition of $n$ is a sequence of positive integers $a = (a_1, \ldots, a_l)$ satisfying $\sum_i a_i = n$. For a composition $a = (a_1, \ldots, a_l)$ of $n$, we let $S(a) = \{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_{l-1}\}$ be a subset of $[n-1]$ and let

$$S_n(a) = \{ w \in S_n \mid D(w) = S(a) \}.$$ 

Then

$$B_k = \bigcup \limits_{a} B_k(a)$$

where the union is over the $(k+1)$-compositions of $n$ and $B_k(a) := \{ \sigma_w \mid w \in S_n(a) \}$.

Lemma 5.7. For a $(k+1)$-composition $a = (a_1, \ldots, a_{k+1})$ of $n$, any pair of two distinct permutations in $S_n(a)$ is not connected in the GKM graph of $H_n$.

Proof. Let $w \rightarrow w(i, j), i < j$, be an edge in the GKM graph of $H_n$ for $w \in S_n(a)$, then $j$ must be $i + 1$. However, $ws_i$ is not in $S_n(a)$ since exchanging the $i$th and the $(i+1)$th element of $w$ will change the descent set; $D(ws_i) = D(w) \cup \{i\}$ if $i \notin D(w)$ and $D(ws_i) = D(w) \setminus \{i\}$ otherwise. □

Lemma 5.7 together with Proposition 4.6 proves the following proposition.

Proposition 5.8. For a permutation $w \in S_n$ and a simple transposition $s_i \in S_n$, if $D(s_iw) = D(w)$, then $s_i\sigma_w = \sigma_{s_iw}$ as elements in $H^*_T(H_n)$.

Example 5.9. When $n = 4$ and $k = 1$, there are three $(k+1)$-compositions of $n$; $a = (1, 3), b = (2, 2)$ and $c = (3, 1)$. The corresponding subsets of $S_4$ are

$$S_4(a) = \{4|123, 3|124, 2|134\},$$

$$S_4(b) = \{4|13, 3|2, 2|14\},$$

$$S_4(c) = \{4|1, 3|2, 2|13\}.$$
\[ \mathcal{S}_4(b) = \{ w_{34} := 34|12, w_{24} := 24|13, w_{23} := 23|14, w_{14} := 14|23, w_{13} := 13|24 \}, \]
\[ \mathcal{S}_4(c) = \{ 234|1, 134|2, 124|3 \}. \]

We consider the permutations in \( \mathcal{S}_4(b) \) in more detail: Since \( s_2 w_{34} = w_{24}, s_1 w_{24} = w_{14}, s_3 w_{24} = w_{23}, s_3 w_{14} = w_{13}, \) and \( s_1 w_{23} = w_{13}, \) by Proposition 5.8 we can obtain all \( \sigma_w \) for \( w \in \mathcal{S}_4(b) \) by applying sequences of \( s_i \)'s to \( \sigma_w \). Note however that \( s_2 w_{13} \notin \mathcal{S}_4(b) \), that is \( w_{13} \) and \( s_2 w_{13} \) have different descent sets, and we cannot apply Proposition 5.8 in this case.

**Proposition 5.10.** For a permutation \( w \in \mathcal{S}_n \), let \( D(w) = \{ d_1 < \cdots < d_k \} \subseteq [n-1] \) be the set of descents of \( w \), and let \( d_0 = 0, d_{k+1} = n \). The stabilizer subgroup of \( \mathcal{S}_n \) for the class \( \sigma_w \in H_1^+ \mathcal{H}_n \) is \( \mathcal{S}_{J_1} \times \cdots \times \mathcal{S}_{J_{k+1}} \), where \( J_s = \{ w(d_{s-1} + 1), \ldots, w(d_s) \} \) for \( s = 1, 2, \ldots, k+1 \).

**Proof.** By Proposition 5.2, the support \( A_w \) of \( \sigma_w \) is given by
\[ A_w = (\mathcal{S}_{J_1} \times \cdots \times \mathcal{S}_{J_{k+1}})w = \{ uw \mid u \in \mathcal{S}_{J_1} \times \cdots \times \mathcal{S}_{J_{k+1}} \}. \]
This implies that for an element \( v \notin \mathcal{S}_{J_1} \times \cdots \times \mathcal{S}_{J_{k+1}} \), the support is not invariant, that is, \( v \cdot A_w \neq A_w \). Accordingly, the stabilizer subgroup for the class \( \sigma_w \) is contained in \( \mathcal{S}_{J_1} \times \cdots \times \mathcal{S}_{J_{k+1}} \).

On the other hand, for elements \( u \) and \( v \) in \( \mathcal{S}_{J_1} \times \cdots \times \mathcal{S}_{J_{k+1}} \), we have the following
\[ (u \cdot \sigma_w)(vw) = u(\sigma_w^{-1}uvw) = u \left( \prod_{s=1}^k (t_{uvw(d_s+1)} - t_{uvw(d_s)}) \right) = u \left( \prod_{s=1}^k (t_{v(d_s+1)} - t_{v(d_s)}) \right) = \sigma_w(vw). \]

Here, the second and the last equalities come from Theorem 5.4. This proves the proposition. \( \square \)

We now consider the \( s_i \)-action on the element \( \sigma_w \) when \( w^{-1}(i+1) + 1 = w^{-1}(i) \); that is there is a descent in between \( i + 1 \) and \( i \) in the one-line notation of \( w \). Let \( D(w) = \{ d_1, \ldots, d_k \} \). We denote by \( d_\ell \) the location where \( i + 1 \) appears in \( w \); that is \( w(d_\ell) = i + 1 \). Consider numbers between descents \( d_\ell-1 \) and \( d_\ell+1 \) as follows.
\[ \tilde{P} := \{ w(j) \mid d_\ell-1 < j < d_\ell \}, \quad \tilde{Q} := \{ w(j) \mid d_\ell+1 < j \leq d_{\ell+1} \}. \]
Choose subsets \( P = \{ p_1 < p_2 < \cdots < p_x \} \subset \tilde{P} \), and \( Q = \{ q_1 < q_2 < \cdots < q_y \} \subset \tilde{Q} \). We define \( \tilde{w}_{P,Q}^{(i)} \in \mathcal{S}_n \) to be
\[ \tilde{w}_{P,Q}^{(i)} = w(1) \ldots w(d_{\ell-1}) p_1 p_2 \ldots p_x i+1 q_1 q_2 \ldots q_y \{ [n] \setminus (P \cup Q \cup \{ i+1 \}) \} \uparrow w(d_{\ell+1} + 1) \ldots w(n). \]

Now we compare the descents sets \( D(w) \) and \( D(\tilde{w}_{P,Q}^{(i)}) \). First of all, because we are taking the same numbers as \( w \) before \( d_\ell-1 \) and after \( d_\ell+1 + 1 \), we have that
\[ \{ d_a \mid a \in [k] \setminus \{ \ell-1, \ell, \ell+1 \} \} \subset D(\tilde{w}_{P,Q}^{(i)}). \]
Since \( P \subset [i-1] \) and \( Q \subset [i+2, n] \), we have that
\[ p_1 < p_2 < \cdots < p_x < i+1 < q_1 < q_2 < \cdots < q_y. \]
Moreover, \( \min([n]\setminus (P \cup Q \cup \{i + 1\})) \leq i \). Therefore, there is a descent \( x + y + 1 + d_{\ell-1} \) in \( \tilde{w}_{P,Q}^{(i)} \).

Depending on whether \( d_{\ell-1}, d_{\ell+1} \) are descents in \( \tilde{w}_{P,Q}^{(i)} \) or not, we modify \( \tilde{w}_{P,Q}^{(i)} \) as follows.

\[
\tilde{w}_{P,Q}^{(i)} := \begin{cases} 
\tilde{w}_{P,Q}^{(i)} & \text{if } d_{\ell-1}, d_{\ell+1} \in D(\tilde{w}_{P,Q}^{(i)}), \\
\tilde{s}_{\tilde{w}_{P,Q}^{(i)}(d_{\ell-1}),\tilde{w}_{P,Q}^{(i)}(d_{\ell-1}+1)}^{(i)} & \text{if } d_{\ell-1} \notin D(\tilde{w}_{P,Q}^{(i)}) \text{ and } d_{\ell+1} \in D(\tilde{w}_{P,Q}^{(i)}), \\
\tilde{s}_{\tilde{w}_{P,Q}^{(i)}(d_{\ell+1}),\tilde{w}_{P,Q}^{(i)}(d_{\ell+1}+1)}^{(i)} & \text{if } d_{\ell-1} \in D(\tilde{w}_{P,Q}^{(i)}) \text{ and } d_{\ell+1} \notin D(\tilde{w}_{P,Q}^{(i)}), \\
\tilde{s}_{\tilde{w}_{P,Q}^{(i)}(d_{\ell-1}),\tilde{w}_{P,Q}^{(i)}(d_{\ell+1}+1)}^{(i)} & \text{if } d_{\ell-1} \notin D(\tilde{w}_{P,Q}^{(i)}) \text{ and } d_{\ell+1} \notin D(\tilde{w}_{P,Q}^{(i)}).
\end{cases}
\]  

(5.2)

Because of the construction, we have that

\[
D(w_{P,Q}^{(i)}) = \{a \mid a \in [k]\setminus \{\ell\} \} \cup \{x + y + 1 + d_{\ell-1}\}.
\]  

Moreover, we define the class \( \tilde{\sigma}_{P,Q}^{(i)} \) by

\[
\tilde{\sigma}_{P,Q}^{(i)} := \begin{cases} 
\sigma_{\tilde{w}_{P,Q}^{(i)}} & \text{if } d_{\ell-1}, d_{\ell+1} \in D(\tilde{w}_{P,Q}^{(i)}), \\
\tilde{s}_{\tilde{w}_{P,Q}^{(i)}(d_{\ell-1}),\tilde{w}_{P,Q}^{(i)}(d_{\ell-1}+1)}^{(i)} & \text{if } d_{\ell-1} \notin D(\tilde{w}_{P,Q}^{(i)}) \text{ and } d_{\ell+1} \in D(\tilde{w}_{P,Q}^{(i)}), \\
\tilde{s}_{\tilde{w}_{P,Q}^{(i)}(d_{\ell+1}),\tilde{w}_{P,Q}^{(i)}(d_{\ell+1}+1)}^{(i)} & \text{if } d_{\ell-1} \in D(\tilde{w}_{P,Q}^{(i)}) \text{ and } d_{\ell+1} \notin D(\tilde{w}_{P,Q}^{(i)}), \\
\tilde{s}_{\tilde{w}_{P,Q}^{(i)}(d_{\ell-1}),\tilde{w}_{P,Q}^{(i)}(d_{\ell+1}+1)}^{(i)} & \text{if } d_{\ell-1} \notin D(\tilde{w}_{P,Q}^{(i)}) \text{ and } d_{\ell+1} \notin D(\tilde{w}_{P,Q}^{(i)}).
\end{cases}
\]  

With these terminologies, we describe the \( s_i \)-action as follows.

**Proposition 5.11.** Let \( \sigma_{w}^{(i)} \) be an element defined by

\[
\sigma_{w}^{(i)} := \sum_{P \subset [k] \setminus \{i\}} \tilde{\sigma}_{P,Q}^{(i)} \in H_T^*(\mathcal{H}_n).
\]

Then, we have that

\[
(t_{i+1} - t_i)\sigma_{s_i w} = s_i \cdot \sigma_{w}^{(i)} - \sigma_{w}^{(i)}.
\]

**Example 5.12.** Let \( n = 5 \) and consider \( s_2 = 13245 \). Then for \( P \subset [1] = \{1\} \) and \( Q \subset [4,5] = \{4,5\} \), permutations \( w_{P,Q}^{(i)} \) are given as follows.

| \( P \) | \( Q \) | \{4,5\} | \{4\} | \{5\} | \( \emptyset \) |
|---|---|---|---|---|---|
| \{1\} | 1345| 13452| 134521| 134524| 1345245 |
| \( \emptyset \) | 34512| 345123| 3451235| 34512354| 345123545 |

We put \( | \) on descents. Therefore, we have that

\[
\sigma_{13245}^{(2)} = \sigma_{13452} + \sigma_{13452} + \sigma_{13524} + \sigma_{13245} + \sigma_{34512} + \sigma_{34512} + \sigma_{35124} + \sigma_{31245}.
\]

Accordingly, we get

\[
s_2 \sigma_{13245} = (t_3 - t_2)\sigma_{12345} + \sigma_{13245}
\]

\[
+ (\sigma_{13452} + \sigma_{13452} + \sigma_{13524} + \sigma_{34512} + \sigma_{34512} + \sigma_{31245} + \sigma_{35124})
\]

\[
- (\sigma_{12453} + \sigma_{12453} + \sigma_{12534} + \sigma_{24513} + \sigma_{24135} + \sigma_{25134} + \sigma_{21345}).
\]
Example 5.13. Let \( n = 5 \) and consider \( w = 21435 \). Then \( \tilde{P} = \{1\} \) and \( \tilde{Q} = \{5\} \). For \( P \subset \tilde{P} \) and \( Q \subset \tilde{Q} \), permutations \( \tilde{w}^{(i)}_{P,Q} \) and \( w^{(i)}_{P,Q} \) are given as follows.

\[
\begin{array}{c|ccc}
  P & Q & \{5\} & \emptyset \\
  \hline
  \{1\} & 2 & 1453 & 21435 \\
  \emptyset & 24513 & 24135 \\
\end{array}
\begin{array}{c|ccc}
  P & Q & \{5\} & \emptyset \\
  \hline
  \{1\} & 2 & 1453 & 21435 \\
  \emptyset & 42513 & 42135 \\
\end{array}
\]

Therefore, we have that
\[
\sigma_{21435}^{(3)} = \sigma_{21453} + \sigma_{21435} + s_{2.4}\sigma_{42513} + s_{2.4}\sigma_{42135}.
\]

Since \( s_{2.4} = s_{2.3}s_{2} \), we have that
\[
\begin{align*}
s_{2.4}\sigma_{42513} &= s_{2.3}s_{2}\sigma_{42513} = s_{2.3}\sigma_{43512} \\
&= s_2((t_4 - t_3)\sigma_{34512} + \sigma_{34512} + \sigma_{45312} - \sigma_{35412}) \\
&= (t_4 - t_2)\sigma_{24513} + \sigma_{42513} + \sigma_{45213} - \sigma_{25413},
\end{align*}
\]
\[
\begin{align*}
s_{2.4}\sigma_{42135} &= s_{2.3}s_{2}\sigma_{42135} = s_{2.3}\sigma_{43125} \\
&= s_2((t_4 - t_3)\sigma_{34125} + \sigma_{43125}) \\
&= (t_4 - t_2)\sigma_{24135} + \sigma_{42135}.
\end{align*}
\]

Accordingly, we get
\[
\begin{align*}
s_{3}\sigma_{21435} &= (t_4 - t_3)\sigma_{21345} + \sigma_{21435} \\
+ \sigma_{21453} + (t_4 - t_2)\sigma_{24513} + \sigma_{42513} + \sigma_{45213} - \sigma_{25413} + (t_4 - t_2)\sigma_{24135} + \sigma_{42135} \\
- (\sigma_{21354} + (t_3 - t_2)\sigma_{23514} + \sigma_{32514} - \sigma_{25314} + (t_3 - t_2)\sigma_{23145} + \sigma_{32145}).
\end{align*}
\]

Proof of Proposition 5.11. Let \( P = \{p_1 < p_2 < \cdots < p_x\} \subset \tilde{P} \) and \( Q = \{q_1 < q_2 < \cdots < q_y\} \subset \tilde{Q} \). Then at any point \( u \in A_{w^{(i)}_{P,Q}} \), we have that
\[
(5.4) \quad \tilde{\sigma}_{P,Q}^{(i)}(u) = (t_u(x+y+2+d_{\ell-1}) - t_u(x+y+1+d_{\ell-1})) \prod_{a \in [k] \setminus \{\ell\}} (t_u(d_{a+1}) - t_u(d_a))
\]
since \( D(w^{(i)}_{P,Q}) = \{d_a \mid a \in [k] \setminus \{\ell\} \} \cup \{x + y + d_{\ell-1}\} \) (see (5.3)) and by Theorem 5.4. Moreover, we also note that by the description of the support of \( \sigma_w \) in Proposition 5.2 for a permutation \( u \in \mathfrak{S}_n \), we have
\[
u \in \bigcup_{P \subset \tilde{P}, \quad Q \subset \tilde{Q}} \text{supp}(\tilde{\sigma}_{P,Q}^{(i)})
\]

\[\iff \quad u^{-1}(i + 1) < u^{-1}(i), \quad \{u(d_{a-1} + 1), \ldots, u(d_a)\} = \{w(d_{a-1} + 1), \ldots, w(d_a)\} \quad \text{for all} \ a \in [k] \setminus \{\ell, \ell + 1\}.
\]

We claim that for \( u \in \mathfrak{S}_n \) in the union of the supports of \( \tilde{\sigma}_{P,Q}^{(i)} \), we have
\[
(5.5) \quad \sigma_w^{(i)}(u) = (t_i - t_{i+1}) \prod_{a \in [k] \setminus \{\ell\}} (t_u(d_{a+1}) - t_u(d_a)).
\]
We denote by $b = u^{-1}(i + 1)$ and $b' = u^{-1}(i)$. By the assumption, we have that $d_{\ell - 1} < b < b'$. For given $P \subset \tilde{P}$ and $Q \subset \tilde{Q}$, the permutation $u$ is in the support $\text{supp}(\sigma_{P,Q}^{(i)})$ if and only if \( \{u(d_{\ell - 1} + 1), \ldots, u(\beta)\} = P \cup Q \cup \{i + 1\} \) for some $b \leq \beta < b'$. Therefore, we obtain that

\[
\sigma_w^{(i)}(u) = \sum_{P \subset P, \atop Q \subset \tilde{Q}} \tilde{\sigma}_{P,Q}^{(i)}(u) = \sum_{b \leq \beta < b'} \tilde{\sigma}_{P,\beta,Q}^{(i)}(u)
\]

(5.6)

where $P_\beta$ and $Q_\beta$ are subsets satisfying $P_\beta \cup Q_\beta \cup \{i + 1\} = \{u(d_{\ell - 1} + 1), \ldots, u(\beta)\}$ for $b \leq \beta < b'$. Moreover, we have that

\[
\bigcup_{b \leq \beta < b'} D(w_{P_\beta Q_\beta}^{(i)}) = \{\beta \mid b \leq \beta < b'\} \cup \{d_a \mid a \in [k]\{\ell}\}\bigcup_{b \leq \beta < b'}.
\]

By applying (5.4), we obtain

\[
\sum_{b \leq \beta < b'} \tilde{\sigma}_{P_\beta Q_\beta}^{(i)}(u) = \left( \sum_{b \leq \beta < b'} t_{u(\beta + 1)} - t_{u(\beta)} \right) \prod_{a \in [k]\{\ell}\} (t_{u(d_a + 1)} - t_{u(d_a)})
\]

(5.7)

\[
= (t_{u(b')} - t_{u(b)}) \prod_{a \in [k]\{\ell}\} (t_{u(d_a + 1)} - t_{u(d_a)})
\]

\[
= (t_i - t_{i + 1}) \prod_{a \in [k]\{\ell}\} (t_{u(d_a + 1)} - t_{u(d_a)}).
\]

Combining equations (5.6) and (5.7), we prove the claim (5.5).

Now we give a proof of the proposition. For $u \in \mathcal{S}_n$, suppose that $u \notin \bigcup_{P \subset \tilde{P}, \atop Q \subset \tilde{Q}} \text{supp}(\tilde{\sigma}_{P,Q}^{(i)})$ but in $\text{supp}(\sigma_{s_i w})$, that is, $u^{-1}(i + 1) > u^{-1}(i)$ and $\{u(d_{a - 1} + 1), \ldots, u(d_a)\} = \{w(d_{a - 1} + 1), \ldots, w(d_a)\}$ for all $a \in [k]\{\ell, \ell + 1\}$. For this case, $(s_i \cdot u)^{-1}(i + 1) < (s_i \cdot u)^{-1}(i)$. Accordingly, $s_i \cdot u \in \bigcup_{P \subset \tilde{P}, \atop Q \subset \tilde{Q}} \text{supp}(\tilde{\sigma}_{P,Q}^{(i)})$, and we have that

\[
(s_i \cdot \sigma_w^{(i)} - \sigma_w^{(i)}(u)) = s_i \cdot \sigma_w^{(i)}(s_i \cdot u) - \sigma_w^{(i)}(u)
\]

\[
= s_i \left[ (t_i - t_{i + 1}) \prod_{a \in [k]\{\ell\}} (t_{s_i \cdot u(d_a + 1)} - t_{s_i \cdot u(d_a)}) \right] - 0
\]

\[
= (t_{i + 1} - t_i) \cdot s_i \left[ \prod_{a \in [k]\{\ell\}} (t_{s_i \cdot u(d_a + 1)} - t_{s_i \cdot u(d_a)}) \right]
\]

\[
= (t_{i + 1} - t_i) \prod_{a \in [k]\{\ell\}} (t_{u(d_a + 1)} - t_{u(d_a)}).
\]

Here, the second equality comes from (5.5). On the other hand, we have $D(s_i w) = \{d_a \mid a \in [k]\{\ell\}\}$, and

\[
\sigma_{s_i w}(u) = \prod_{a \in [k]\{\ell\}} (t_{u(d_a + 1)} - t_{u(d_a)})
\]
by Theorem 5.4. Therefore, this proves the proposition for \( u \notin \bigcup_{P \subseteq \tilde{P}} \text{supp}(\tilde{\sigma}_{P,Q}^{(i)}) \). Suppose that 
\[ u \in \bigcup_{P \subseteq \tilde{P}} \text{supp}(\tilde{\sigma}_{P,Q}^{(i)}) \] 
Then we have that \( s_i \cdot u \notin \bigcup_{P \subseteq \tilde{P}} \text{supp}(\tilde{\sigma}_{P,Q}^{(i)}) \). Moreover, we obtain 
\[ (s_i \cdot \sigma_w^{(i)} - \sigma_w^{(i)})(u) = s_i \cdot \sigma_w^{(i)}(s_i \cdot u) - \sigma_w^{(i)}(u) \]
\[ = 0 - (t_{i} - t_{i+1}) \prod_{a \in [k] \setminus \ell} (t_{u(d_a+1)} - t_{u(d_a)}) \]
\[ = (t_{i+1} - t_{i}) \prod_{a \in [k] \setminus \ell} (t_{u(d_a+1)} - t_{u(d_a)}) \].

This completes the proof. \( \square \)

We enclose this section by considering a corollary of Proposition 5.11 which will be used later.

**Corollary 5.14.** For a permutation \( w \in S_n \), let \( D(w) \) be the set of descents of \( w \). If \( w^{-1}(i+1) - 1, w^{-1}(i+1), w^{-1}(i) \in D(w) \cup \{0\} \) and \( w^{-1}(i+1) = w^{-1}(i) - 1 \), then the class \( \sigma_w \) is invariant under \( s_i \) as an element in \( H^*(\mathcal{H}_n) \), that is, \( s_i \sigma_w = \sigma_w \) in \( H^*(\mathcal{H}_n) \).

**Proof.** The condition \( w^{-1}(i+1) - 1, w^{-1}(i+1), w^{-1}(i) \in D(w) \) and \( w^{-1}(i+1) = w^{-1}(i) - 1 \) implies that we have the following one-line notation of \( w \):
\[ w = \ldots |i+1|i| \ldots \]
In this case, the sets \( \tilde{P} \) and \( \tilde{Q} \) are both empty. Therefore, the class \( \sigma_w^{(i)} \) is given by \( \sigma_w^{(i)} = \sigma_w \), so we have
\[ (t_{i+1} - t_{i})\sigma_{s_i \cdot w} = s_i \cdot \sigma_w - \sigma_w. \]
Accordingly, as an element in \( H^*(\mathcal{H}_n) \), the class \( \sigma_w \) is invariant under the action of \( s_i \). \( \square \)

### 5.3. Permutation module decompositions.

The \( e \)-positivity conjecture on the chromatic quasi-symmetric functions of Shareshian and Wachs [32] is shown independently by Brosnan–Chow and Guy-Paquet, to be equivalent to the conjecture that \( H^{2k}(\text{Hess}(X,h)) \) is isomorphic to a direct sum of permutation modules of \( S_n \) for each \( k \). Permutation modules of the symmetric group \( S_n \) are \( M^\lambda = 1_{\text{S}_{\lambda_1} \times \cdots \times \text{S}_{\lambda_k+1}} \mathbb{C}[\text{S}_n(t^\lambda)] \) for partitions \( \lambda = (\lambda_1, \ldots, \lambda_{k+1}) \) of \( n \), where \( J_s := \{\lambda_1 + \cdots + \lambda_{s-1} + 1, \ldots, \lambda_1 + \cdots + \lambda_s\} \) for \( s = 1, \ldots, k+1 \) and \( t^\lambda = (J_1, \ldots, J_{k+1}) \). A natural basis of \( M^\lambda \) is the set \( \{(J_1, \ldots, J_{k+1}) : |J_s| = \lambda_s, \bigcup_s J_s = [n]\} \). See [31] for the representation theory of the symmetric groups. We remark that \( M^a \), for a composition \( a \) of \( n \) can be defined in the same way as \( M^\lambda \) is defined, while \( M^a \) is isomorphic to \( M^\lambda(a) \) for the partition \( \lambda(a) \) obtained from \( a \) by arranging the parts of \( a \) in nonincreasing order.

**Proposition 5.15** ([6], [21]). For a Hessenberg function \( h \),
\[ \sum_{k} \text{ch} H^{2k}(\text{Hess}(S,h)) t^k = \omega X_{G(h)}(x,t), \]
where \( \text{ch} \) is the Frobenius characteristic map and \( \omega \) is the involution on the ring \( \Lambda \) of symmetric functions sending elementary symmetric functions to homogeneous symmetric functions.
The e-positivity conjecture is proved to be true for permutohedral varieties by Shareshian and Wachs in their seminal paper [32], where they gave a closed form formula for the expansion of the corresponding chromatic quasisymmetric functions as sums of elementary symmetric functions (Theorem C.4 or Table 1 of [32]). Transforming this formula in \( \Lambda[t] \) into a formula in \( R[t] \) via the isomorphism of the ring \( \Lambda \) of symmetric functions with the ring of the nilpotent Hessenberg variety \( \text{Hess}_p \) (Remark 5.18).

We will construct a basis \( \bigcup_{k=0}^{n-1} \{ \mathcal{G}_w : w \in \mathcal{G}_k \} \) of \( \sum_{k=0}^{n-1} H^{2k}(\mathcal{H}_n) \) generating permutation modules in the right hand side of (5.8).

Recall that for a permutation \( w \in \mathfrak{S}_n \) with \( D(w) = \{ d_1 < \cdots < d_k \} \), the support \( A_w \) of \( \sigma_w \) is \( (\mathfrak{S}_{J_1} \times \cdots \times \mathfrak{S}_{J_{k+1}})w \), where \( J_s = \{ w(d_{s-1}+1), \ldots, w(d_s) \} \) for \( s = 1, 2, \ldots, k+1 \). Here, we set \( d_0 = 0 \) and \( d_{k+1} = n \) (Proposition 5.2).

Let \( \mathcal{G}_k \) be the set of \( w \in \mathfrak{S}_n \) with \( \text{des}(w) = k \) such that \( A_w \) contains \( w_0 \), the longest element in \( \mathfrak{S}_n \), and let \( \mathcal{G} \) be the union \( \bigcup_{k=0}^{n-1} \mathcal{G}_k \).

**Lemma 5.17.** There is a bijective correspondence between the set of compositions and \( \mathcal{G} := \bigcup_{k=0}^{n-1} \mathcal{G}_k \). Indeed, a composition \( \mathbf{a} = (a_1, \ldots, a_{k+1}) \) corresponds to an element in \( \mathcal{G}_k \).

**Proof.** Let \( \mathbf{a} = (a_1, \ldots, a_{k+1}) \) be a composition of \( n \). Recall that \( S(\mathbf{a}) \) is the ordered set \( \{ d_1 < \cdots < d_k \} \), where \( d_s = \sum_{i=1}^{s} a_i \) for \( s = 1, \ldots, k \). Define \( w(\mathbf{a}) \in \mathfrak{S}_n \) by

\[
    n - d_1 + 1 \cdots n|n - d_2 + 1 \cdots n - d_1| \cdots |1 \cdots n - d_k.
\]

Then, \( A_w(\mathbf{a}) \) contains \( w_0 \), which shows that \( w(\mathbf{a}) \in \mathcal{G}_k \).

Conversely, suppose that \( A_w \) contains \( w_0 \). Let \( \mathbf{a} \) be the composition with \( S(\mathbf{a}) = D(w) = \{ d_1 < \cdots < d_k \} \). Then \( w \) is of the form

\[
    n - d_1 + 1 \cdots n|n - d_2 + 1 \cdots n - d_1| \cdots |1 \cdots n - d_k,
\]

and thus we have \( w(\mathbf{a}) = w \). \( \square \)

**Remark 5.18.** In general, when \( h : [n] \to [n] \) is a Hessenberg function, the intersection of \( X_w^h \) with the nilpotent Hessenberg variety \( \text{Hess}(N, h) \) is nonempty if and only if

\[
    w^{-1}(w(j) - 1) \leq h(j) \text{ for all } j \in [n]
\]
(Lemma 2.3 of \text{[1]}). When \( h = (2, 3, \ldots, n, n) \), our \( \mathcal{G} = \bigcup_k \mathcal{G}_k \) is the image of the set of these \( w \)'s by the involution \( \iota : \mathcal{S}_n \to \mathcal{S}_n \) given by \( \iota(w)(i) = n - w(i) + 1 \) for \( i \in [n] \). We expect that for general \( h \) the latter set will play the same role as \( \mathcal{G} \) does.

\textbf{Definition 5.19 (cf. Chow \text{[10]}).} \hspace{1em} (1) For a set \( D = \{d_1 < \cdots < d_k\} \subset [n] \) of marks, define the \textit{erasure} \( e(D) \) of \( D \) by

\[ e(D) = \{d \in D : d \neq 1 \text{ and } d - 1 \notin D\}. \]

(2) For a composition \( a = (a_1, \ldots, a_{k+1}) \) of \( n \), define the \textit{erasure} \( \hat{a} \) of \( a \) by the composition with \( S(\hat{a}) = e(S(a)) \).

\textbf{Definition 5.20.} For \( w \in \mathcal{G} \) with \( D(w) = \{d_1 < \cdots < d_k\} \), let \( e(D) = \{\epsilon_1 < \cdots < \epsilon_l\} \subset D \) be the erasure of \( D \). For \( s = 1, 2, \ldots, k+1 \), we let \( J_s = \{w(d_{s-1} + 1), \ldots, w(d_s)\} \) and for \( t = 1, 2, \ldots, l+1 \), we let \( \tilde{J}_t = \{w(\epsilon_{t-1} + 1), \ldots, w(\epsilon_t)\} \), where \( d_0 = \epsilon_0 = 0 \) and \( d_{s+1} = \epsilon_{t+1} = n \).

1. Define a subgroup \( \mathcal{S}_w \) of \( \mathcal{S}_n \) as

\[ \mathcal{S}_w := \mathcal{S}_{\tilde{J}_1} \times \cdots \times \mathcal{S}_{\tilde{J}_{k+1}}, \]

that contains \( \mathcal{S}_{J_1} \times \cdots \times \mathcal{S}_{J_{k+1}} \) as a subgroup.

2. Let

\[ \mathcal{S}_w^\circ := \mathcal{S}_w/(\mathcal{S}_{J_1} \times \cdots \times \mathcal{S}_{J_{k+1}}). \]

3. Define an element in \( H^2k_1(\mathcal{H}_n) \);

\[ \hat{\sigma}_w := \sum_{v \in \mathcal{S}_w^\circ} v\sigma_w. \]

4. Define \( a(w) \) by the composition such that \( S(a(w)) = D(w) \). Put \( \hat{a}(w) := \hat{a}(w) \).

We note that each \( \mathcal{S}_{J_s}, s = 1, \ldots, k+1, \) stabilizes \( \sigma_w \) in Definition \text{[5.20]} so the element \( \hat{\sigma}_w \) in \text{[3]} is well defined.

Before we continue on the construction of permutation module decomposition of \( H^*(\mathcal{H}_n) \), we state a conjecture on bases of the equivariant cohomology space of \( \mathcal{H}_n \) due to Chow, whose proof follows from our main theorem in the current section; see Corollary \text{[5.24]}. The description of the suggested basis elements by Chow is different from what we have in the following lemma; but they can be shown to coincide using the explicit formula for the classes \( \sigma_w \):

\textbf{Lemma 5.21.} For \( w \in \mathcal{G} \) with \( D(w) = \{d_1 < \cdots < d_k\} \) and \( e(D(w)) = \{\epsilon_1 < \cdots < \epsilon_l\} \), let \( \hat{a} = \hat{a}(w) \). Then

(1) \( \text{supp}\hat{\sigma}_w = A_w(\hat{a}) \), and

(2) \( \text{for } u \in \text{supp}\hat{\sigma}_w, \hat{\sigma}_w(u) = \prod_{s=1}^k (t_{u(d_{s+1})} - t_{u(d_s)}). \)

\textbf{Proof.} By Proposition \text{[5.2]} the support of \( \sigma_w = A_w \) is given by

\[ A_w = (\mathcal{S}_{J_1} \times \cdots \times \mathcal{S}_{J_{k+1}})w. \]

Because of the definition of \( \mathcal{S}_w^\circ \), the support of the class \( \hat{\sigma}_w := \sum_{v \in \mathcal{S}_w^\circ} v\sigma_w \) is

\[ \text{supp}(\hat{\sigma}_w) = (\mathcal{S}_w/(\mathcal{S}_{J_1} \times \cdots \times \mathcal{S}_{J_{k+1}})) \times A_w = \mathcal{S}_w \cdot w. \]
On the other hand, we have that $D(w(\tilde{a})) = e(D(w))$. This proves the first statement. The second statement follows from Theorem 5.4 and the definition of the action (see Subsection 4.1).

**Conjecture 5.22** (Erasing marks conjecture (Chow [10])).

\[ \bigcup_{k=0}^{n-1} \bigcup_{w \in \mathcal{G}_k} \{ v\hat{\sigma}_w : v \in \mathcal{S}_n/\mathcal{S}_w \} \]

forms a basis of the equivariant cohomology space $H^*_T(\mathcal{H}_n)$.

We state the main theorem of this section; whose proof will be done in Section 6. We will use the same notation for the image of $\hat{\sigma}_w$ in the singular cohomology space $H^*(\mathcal{H}_n)$.

**Theorem 5.23.** For $w \in \mathcal{G}_k$, let $M(w) := \mathbb{C}\mathcal{S}_n(\hat{\sigma}_w)$ be the $\mathcal{S}_n$-module generated by $\hat{\sigma}_w$.

1. For $w \in \mathcal{G}_k$, the $\mathcal{S}_n$-module $M(w)$ is isomorphic to the permutation module $M^{\tilde{a}(w)}$.
2. $H^{2k}(\mathcal{H}_n) = \bigoplus_{w \in \mathcal{G}_k} M(w)$.

Theorem 5.23 proves the erasing marks conjecture of Chow.

**Corollary 5.24** (Erasing marks conjecture is true). For each $k = 0, \ldots, n-1$,

\[ \bigcup_{w \in \mathcal{G}_k} \{ v\hat{\sigma}_w \in H^{2k}_T(\mathcal{H}_n) : v \in \mathcal{S}_n/\mathcal{S}_w \} \]

forms a basis of the $(2k)\text{th}$ equivariant cohomology module $H^{2k}_T(\mathcal{H}_n)$ over $\mathbb{C}[t_1, \ldots, t_n]$.

**Proof.** For a fixed $k$ and $w \in \mathcal{G}_k$, we have $|\mathcal{S}_n/\mathcal{S}_w| = \dim(M^{\tilde{a}(w)})$. Hence, the second part of Theorem 5.23 proves that $\bigcup_{w \in \mathcal{G}_k} \{ v\hat{\sigma}_w \in H^{2k}_T(\mathcal{H}_n) : v \in \mathcal{S}_n/\mathcal{S}_w \}$ is a $\mathbb{C}$-basis of $H^{2k}(\mathcal{H}_n)$ for each $k$.

We now recall from [27] Lemma 2.1 that for a manifold $M$ with a smooth action of a compact torus $T = (S_1)^n$ such that the fixed point set is finite and non-empty, then $H^*_T(M)$ is free as an $\mathbb{C}[t_1, \ldots, t_n]$-module if and only if $H^{\text{odd}}(M) = 0$. In this case,

\[ H^*_T(M) \cong H^*(M) \otimes \mathbb{C}[t_1, \ldots, t_n] \]

as $\mathbb{C}[t_1, \ldots, t_n]$-modules. Since the odd degree cohomology of the permutohedral variety vanishes, we obtain that

\[ H^*_T(\mathcal{H}_n) \cong H^*(\mathcal{H}_n) \otimes \mathbb{C}[t_1, \ldots, t_n] \]

as $\mathbb{C}[t_1, \ldots, t_n]$-modules. This shows that $\bigcup_{w \in \mathcal{G}_k} \{ v\hat{\sigma}_w \in H^{2k}_T(\mathcal{H}_n) : v \in \mathcal{S}_n/\mathcal{S}_w \}$ is a $\mathbb{C}[t_1, \ldots, t_n]$-module basis of $H^{2k}_T(\mathcal{H}_n)$.

**Example 5.25.** If $n = 5$ and $k = 2$, $\mathcal{G}_k = \{54123, 53412, 52341, 45312, 45231, 34521\}$. 
We have that
\[
\sum_{k=0}^{n-1} \sum_{w \in \mathcal{G}_k} M(\hat{a}(w)) t^k = \left[ \sum_{m=1}^{n+1} \sum_{k_1, \ldots, k_m \geq 2, \sum k_i = n+1} M(k_1, \ldots, k_m) t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t \right].
\]

In other words, for each \( (k_1 - 1, k_2, \ldots, k_m) \) with \( k_1, \ldots, k_m \geq 2 \) and \( \sum k_i = n + 1 \),

\[
|\{w \in \mathcal{G}_k : \hat{a}(w) = (k_1 - 1, k_2, \ldots, k_m)\}|
\]
is equal to the coefficient of \( t^k \) in the polynomial \( t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t \).

In particular, \( \sum_{k=0}^{n-1} \sum_{w \in \mathcal{G}_k} \dim M(\hat{a}(w)) = \dim H^*(\mathcal{H}_n) = n! \).
In the remaining part of this section we will explain the idea of proof by proving the theorem for the case when \( n = 4 \) and \( k = 2 \).

**Example 5.27.** \( n = 4 \) and \( k = 2 \). We will show that \( \bigoplus_{w \in \mathcal{G}_k} M(w) \) generates \( H^{2k}(\mathcal{H}_n) \) linearly. Note that \( H^{2k}(\mathcal{H}_n) \) is linearly generated by \( \sigma_w \) for \( w \in \mathcal{S}(2, 1, 1) \cup \mathcal{S}(1, 2, 1) \cup \mathcal{S}(1, 1, 2) \).

\[
\begin{array}{c|c|c|c}
 w & a(w) & \hat{a}(w) & \mathcal{G}_w \\
 \hline
 4|3|2 & (1, 1, 2) & (4) & \mathcal{G}_{4,3,1,2}/\mathcal{G}_{1,2} \\
 4|23|1 & (1, 2, 1) & (3, 1) & \mathcal{G}_{4,2,3}/\mathcal{G}_{1,2} \\
 34|2|1 & (2, 1, 1) & (2, 2) & \mathcal{G}_{2,1} \\
\end{array}
\]

- \( \mathcal{G}(2, 1, 1) = \{3421, 2431, 1432\} \) and \( \hat{\sigma}_{3421} = \sigma_{3421} + \sigma_{3421} = 2\sigma_{3421} \).

\[
\sigma_{3421} \xrightarrow{s_2} \sigma_{2431} \xrightarrow{s_1} \sigma_{1432} \\
\sigma_{2431} \xrightarrow{s_1} \sigma_{1432} \\
\sigma_{1432} \xrightarrow{s_2} \sigma_{1432}
\]

\[+ (\sigma_{4231} - \sigma_{3241})\]

\[
\left(\sigma_{4231} - \sigma_{3241}, (\sigma_{4132} - \sigma_{3142}), (\sigma_{4132} + (\sigma_{4132} - \sigma_{4213})) - \sigma_{2143}\right)
\]

We remark that

\[s_3 (\sigma_{4132} + (\sigma_{4132} - \sigma_{4213})) - s_3 \sigma_{2143} = (\sigma_{3142} + (\sigma_{4132} - \sigma_{3214})) - (\sigma_{2143} + (\sigma_{4213} - \sigma_{3214}))\]

is also contained in \( M(3421) \) because \( M(3421) \) is an \( \mathcal{S}_n \)-module.

- \( \mathcal{G}(1, 2, 1) = \{4231, 3241, 4132, 3412, 2143\}, \) \( \hat{\sigma}_{4231} = \sigma_{4231} + \sigma_{3421} + (\sigma_{3421} + (\sigma_{4213} - \sigma_{2431})) \).

We have already shown that \( (\sigma_{3421} - \sigma_{2431}) \) is contained in \( M(3421) \).

\[
\sigma_{4231} + \sigma_{3421} \xrightarrow{s_1} \sigma_{4132} + \sigma_{3142} \xrightarrow{s_2} \sigma_{4132} + \sigma_{2143} \\
\sigma_{4132} + \sigma_{2143} \xrightarrow{s_3} \sigma_{3142} + \sigma_{2143}
\]

\[+ (\sigma_{4312} - \sigma_{4213})\]

\[
\left(\sigma_{4231} + \sigma_{3241}, (\sigma_{4132} + \sigma_{3142}), (\sigma_{4132} + (\sigma_{4132} - \sigma_{4213}) + \sigma_{2143},
(\sigma_{3142} + (\sigma_{4312} - \sigma_{3214}) + (\sigma_{2143} + (\sigma_{4213} - \sigma_{3214}))
\right)
\]

and thus

\[
\sigma_{4231}, \sigma_{3241}, \sigma_{4132}, \sigma_{3142}, (\sigma_{4312} - \sigma_{4213}), \sigma_{2143},
\sigma_{3142} + (\sigma_{4312} - \sigma_{3214}), \sigma_{2143} + (\sigma_{4213} - \sigma_{3214})
\]
are contained in the $S_n$-module generated by $M(3421)$ and $M(4231)$ because of (5.10) and (5.11). Therefore,

$$\sigma_{4312}, \sigma_{3421}, \sigma_{4132}, \sigma_{3142}, \sigma_{4132}, \sigma_{2143}$$

are contained in the $S_n$-module generated by $M(3421)$ and $M(4231)$. We remark that the module $M(3421) + M(4231)$ also contains

$$\sigma_{4312} - \sigma_{4213}, \sigma_{4312} - \sigma_{3214}, (\sigma_{4213} - \sigma_{3214}).$$

- $S(1, 1, 2) = \{4312, 4213, 3241\}$, to compute the class $\hat{\sigma}_{4312}$, we consider the following diagram.

Accordingly, we obtain that

$$\hat{\sigma}_{4312} = 2\sigma_{4312} + 4\sigma_{4213} + 6\sigma_{3241} + 2(\sigma_{4231} - \sigma_{4132}) + 4\sigma_{3241} - 2\sigma_{3142} - 2\sigma_{2143} + 2(\sigma_{4321} - \sigma_{4231}).$$

We have already shown that $(\sigma_{4231} - \sigma_{4132})$ is contained in the $S_n$-module generated by $M(3421)$ and $M(4231)$, and it suffices to show that

$$\sigma_{4312}, \sigma_{4213}, \sigma_{3214}$$

is contained in the $S_n$-module generated by $M(3421)$ and $M(4231)$ and $M(4312)$. This is obvious because the module $M(3421) + M(4231) + M(4312)$ contains

$$(\sigma_{4312} - \sigma_{4213}), (\sigma_{4312} - \sigma_{3214}), 2\sigma_{4312} + 4\sigma_{4213} + 6\sigma_{3241}.$$

See (5.12).

6. Proof of Theorem 5.23

In this section, we will present a proof of Theorem 5.23. For an element $w \in S_n$, we say $a$ is the composition of $w$ if $D(w) = S(a)$ and we denote it by $a(w)$. We first consider the case when the erasing occurs consecutively, that is, the composition $a$ of $w$ is given by

$$a_1, a_2, \ldots, a_{\ell-1}, 1, 1, \ldots, 1, a_{\ell+m}, \ldots, a_k$$
and $a_i \neq 1$ for $i \notin [\ell, \ell + m - 1]$. For this case, by Lemma 5.17, the corresponding element $w(a)$ in $G_k$ is given by

$$w(a) = \ldots |x|x - 1|x - 2| \ldots |x - m + 1|y y + 1 \ldots y + a_{\ell+m} - 1| \ldots .$$

Here, we decorate the places where the descents appear. Because of the construction of $w(a)$, we have that $y + a_{\ell+m} - 1 = x - m$.

Now we describe the element $\hat{\sigma}_{w(a)}$ using a certain (edge) labeled graph $G(a) = (V, E)$ embedded in the Euclidean space $\mathbb{R}^m$ (see Proposition 6.5).

- The set $V = V(G(a))$ of vertices is given by

$$V = \{(z_1, \ldots, z_m) \in \mathbb{Z}^m \mid a_{\ell+m} - 1 \geq z_1 \geq z_2 \geq \cdots \geq z_m \geq 0\}.$$

- Two vertices $(z_1, \ldots, z_m)$ and $(z'_1, \ldots, z'_m)$ are connected by an edge if and only if $|z_j - z'_j| = 1$ for some $j \in [m]$ and all the other coordinates are the same. We label the value $x - m + (j - 1) - z_j$ for such an edge with $z_j < z'_j$.

For the case when the erasing does not occur consecutively, we may break the composition $a$ into smaller pieces $a_1, \ldots, a_3$ so that the erasing appears consecutively in each piece, and moreover, the erasure $\hat{a}$ is the concatenation of $\hat{a}_1, \ldots, \hat{a}_3$. For instance, for a composition $a = (2, 1, 1, 3, 4, 1, 1, 1, 5, 1, 2)$, we break it into

$$a_1 = (2, 1, 1, 3, 4), a_2 = (1, 1, 5), a_3 = (1, 2).$$

In this case, the erasure $\hat{a}$ is $(2, 5, 4, 8, 3)$, and we have $\hat{a}_1 = (2, 5, 4), \hat{a}_2 = (8), \hat{a}_3 = (3)$. See the following diagram:

Here, we draw dotted vertical line for erasing descents. Without loss of generality, we may break the composition $a$ so that in each piece has at least one erasing occur. We define the graph $G(a)$ by the product $G(a_1) \times \cdots \times G(a_3)$ of graphs. Moreover, for an edge $(v_1, \ldots, v_{k-1}, v_k, v_{k+1}, \ldots, v_\beta) \to (v_1, \ldots, v_{k-1}, v'_k, v_{k+1}, \ldots, v_\beta)$ in the graph $G(a)$, we put the same label as that on the edge $v_k \to v'_k$ in $G(a_k)$ (see Example 6.8).

**Example 6.1.**

1. Suppose that $a = (1, 1, 4)$. The corresponding element $w(a)$ is $6|5|1234$. Then we need to erase two consecutive descents, so the graph $G(1, 1, 4)$ is drawn in $\mathbb{R}^2$. The set $V$ of vertices of the corresponding graph $G(a)$ is given by

$$V = \{(z_1, z_2) \in \mathbb{Z}^2 \mid 3 \geq z_1 \geq z_2 \geq 0\}$$

and the graph $G(1, 1, 4)$ is presented in Figure 7(1). For example, the edge connecting $(1, 1)$ and $(2, 1)$ is decorated by $6 - 2 + (1 - 1) - 1 = 3$. Here, we have $x = 6$, $m = 2$, $j = 1$, and $z_j = 1$.

2. Suppose that $a = (1, 1, 1, 3)$. The corresponding element $w(a)$ is $6|5|4|123$. Then we need to erase three consecutive descents, and the vertex set of the graph $G(1, 1, 1, 3)$ is given by

$$V = \{(z_1, z_2, z_3) \in \mathbb{Z}^3 \mid 2 \geq z_1 \geq z_2 \geq z_3 \geq 0\}.$$
The graph \(G(1,1,3)\) is given in Figure \(7(2)\).

For each vertex \(z\) in the graph \(G(\mathbf{a})\), we associate an element \(w_z \in \mathfrak{S}_n\) such that \(\mathbf{a}(w_z) = \mathbf{a}\). (They will be used to describe \(\mathbf{\bar{\sigma}}_w\) in Proposition \(6.5\).) For a vertex \(z = (z_1, \ldots, z_m)\), consider a shortest path from the origin to the vertex \(z\). If the path is given by a sequence of edges labeled by \((i_1, \ldots, i_k)\), then we associate the permutation

\[
(6.3) \quad w_z := s_{i_k} \cdots s_{i_1} \cdot w(\mathbf{a})
\]

to the vertex \(z\).

We notice that if there are two different paths connecting vertices

\[(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, z_{j+2}, \ldots, z_m)\]

and

\[(z_1, \ldots, z_{j-1}, z_j + 1, z_{j+1} + 1, z_{j+2}, \ldots, z_m)\]

then permutations obtained from these two paths are the same. More precisely, suppose that there are four vertices

\[
v_1 = (z_1, \ldots, z_{j-1}, z_j, z_{j+1}, z_{j+2}, \ldots, z_m),
\]

\[
v_2 = (z_1, \ldots, z_{j-1}, z_j + 1, z_{j+1}, z_{j+2}, \ldots, z_m),
\]

\[
v_3 = (z_1, \ldots, z_{j-1}, z_j, z_{j+1} + 1, z_{j+2}, \ldots, z_m),
\]

\[
v_4 = (z_1, \ldots, z_{j-1}, z_j + 1, z_{j+1} + 1, z_{j+2}, \ldots, z_m)
\]

in the graph \(G(\mathbf{a})\). By the definition of the graph \(G(\mathbf{a})\), we have that \(z_j \geq z_{j+1}\). Otherwise, one cannot have the vertex \(v_3\). There are four edges whose labels are given as follows.

\[
(v_1, v_2) \text{ and } (v_3, v_4) : x - m + (j - 1) - z_j,
\]

\[
(v_1, v_3) \text{ and } (v_2, v_4) : x - m + j - z_{j+1}.
\]

Since \(z_j > z_{j+1}\), we have that

\[
|(x - m + j - z_{j+1}) - (x - m + (j - 1) - z_j)| = |1 + z_j - z_{j+1}| > 1.
\]
This implies that $s_{x-m+(j-1)-z_j}s_{x-m+j-z_{j+1}} = s_{x-m+j-z_{j+1}}s_{x-m+(j-1)-z_j}$ from the commutativity of two simple reflections $s_i$ and $s_j$ satisfying $|i - j| > 1$. Therefore, the permutation $w_z$ in (6.3) is well-defined.

By the construction of $w_z$, we have the following lemma which is directly obtained:

**Lemma 6.2.** Let $a$ be a composition of $n$ having parts $k+1$. We denote by $d_1 < d_2 < \cdots < d_k$ the elements of $S(a)$. Then for elements $z = (z_1, \ldots, z_m)$ and $z' = (z'_1, \ldots, z'_m)$ connected by an edge satisfying $|z_j - z'_j| = 1$, we have that

$$w_z(i) = w_{z'}(i) \quad \text{for } i \neq d_{\ell+1-m}, d_{\ell-m+1} + 1, \ldots, d_{\ell+m}.$$  

**Example 6.3.**

(1) Continuing Example 6.1(1), we have the elements $w_z$ as follows.

$$
\begin{array}{cccc}
6 | 5 | 1234 & \overset{s_4}{\rightarrow} & 6 | 4 | 1235 & \overset{s_3}{\rightarrow} & 6 | 3 | 1245 & \overset{s_2}{\rightarrow} & 6 | 2 | 1345 \\
5 | 4 | 1236 & \overset{s_5}{\downarrow} & 5 | 3 | 1246 & \overset{s_5}{\downarrow} & 5 | 2 | 1346 & \overset{s_5}{\downarrow} \\
4 | 3 | 1256 & \overset{s_4}{\downarrow} & 4 | 2 | 1356 & \overset{s_4}{\downarrow} \\
3 | 2 | 1456 & \overset{s_3}{\downarrow} \\
\end{array}
$$

Here, one may see that along the edge on the $z_1$-direction, the first value on the permutation does not change. For example, for $z = (z_1, 0)$ (which are on the first row in the diagram), $z_3 = 0$ we have that $w_z(1) = 6$. On the other hand, for $z = (3, z_2)$ (which are on the fourth column in the diagram), we have that $w_z(2) = 2$.

(2) Continuing Example 6.1(2), we have the elements $w_z$ as follows.

$$
\begin{array}{cccc}
6 | 5 | 123 & \overset{s_3}{\rightarrow} & 6 | 5 | 3 | 124 & \overset{s_2}{\rightarrow} & 6 | 5 | 2 | 134 \\
6 | 4 | 3 | 125 & \overset{s_4}{\downarrow} & 6 | 4 | 2 | 135 & \overset{s_4}{\downarrow} \\
5 | 4 | 3 | 126 & \overset{s_5}{\downarrow} & 5 | 4 | 2 | 136 & \overset{s_5}{\downarrow} \\
5 | 3 | 2 | 145 & \overset{s_5}{\downarrow} & 5 | 3 | 2 | 146 & \overset{s_5}{\downarrow} \\
4 | 3 | 2 | 156 & \overset{s_3}{\downarrow} \\
\end{array}
$$

$z_3 = 1$

$z_3 = 2$
Example 6.4. We consider compositions $(1, 1, 2)$, $(1, 2, 1)$ and $(2, 1, 1)$ in Example 5.27. For this case, the lexicographic order defines a total order:

$$(1, 1, 2) < (1, 2, 1) < (2, 1, 1).$$

The corresponding graphs $G(a)$ are given in Figure 8. For each composition $a$, we compute the elements $w_z$ for $z \in V(G(a))$. Moreover, we present $\hat{\sigma}_{w(a)}$ such that $w \in G_k$.

- $a = (1, 1, 2)$.

\[
\begin{align*}
4|3|12 & \xrightarrow{s_2} 4|2|13 \\
& \xrightarrow{s_3} 3|2|14
\end{align*}
\]

$$\hat{\sigma}_{4312} = 2\sigma_{4312} + 4\sigma_{4213} + 6\sigma_{3214} + 2(\sigma_{4321} - \sigma_{4132}) + 4\sigma_{3241} - 2\sigma_{3142} - 2\sigma_{2143} + 2(\sigma_{3421} - \sigma_{2431}).$$

- $a = (1, 2, 1)$.

\[
\begin{align*}
4|23|1 & \xrightarrow{s_3} 3|24|1
\end{align*}
\]

$$\hat{\sigma}_{4231} = \sigma_{4231} + 2\sigma_{3241} + (\sigma_{3421} - \sigma_{2431}).$$

- $a = (2, 1, 1)$.

$$\hat{\sigma}_{3421} = 2\sigma_{3421}.$$
Proposition 6.5. Let \( w \) be an element in \( G_k \) such that \( a(w) = a \). Then, the element \( \hat{\sigma}_w \) is written by
\[
\hat{\sigma}_w = \sum_{z \in V(G(a))} c_z \sigma_{w_z} + \sum_{a(v) > a} k_v \sigma_v \quad \text{for} \ c_z > 0.
\]
Here, we consider the lexicographic order on the set of compositions.

Before providing a proof of Theorem 5.23, we prepare one lemma.

Lemma 6.6. Let \( a \) be a composition of \( n \) consisting of \( k + 1 \) parts. Then, for each \( z \in V(G(a)) \), the element \( \sigma_{w_z} \) is contained in \( \sum_{u \in G_k} M(u) \).

Proof. Recall that we define the total order on the set of compositions lexicographically. We prove the statement using the induction with respect to this total order. For simplicity, we denote by \( M \) the sum \( \sum_{u \in G_k} M(u) \). For a given \( k \), the maximal element \( a \) among the compositions having \( k + 1 \) parts is
\[
a = (n - k, 1, \ldots, 1).
\]
In this case, the graph \( G(a) \) has one vertex. Therefore, by Proposition 6.5 we obtain that
\[
\hat{\sigma}_{w(a)} = c \sigma_{w(a)}
\]
for some positive integer \( c \). This proves the claim for this case.

Now we suppose that we prove the claim for any composition which is greater than \( a \). Recall the expression of \( \hat{\sigma}_w \) in Proposition 6.5
\[
\hat{\sigma}_w = \sum_{z \in V(G(a))} c_z \sigma_{w_z} + \sum_{a(v) > a} k_v \sigma_v, \quad c_z > 0.
\]
By the induction argument, we have \( \sigma_v \) satisfying \( a(v) > a \) is in the sum \( M \). Therefore, to prove the lemma, it is enough to show that for all \( z \in V(G(a)) \setminus \{0\} \), there exists a linear combination
\[
\sigma_{w_{z'}} - \sigma_{w_z}
\]
in the sum \( M \) for some \( z' \in V(G(a)) \) such that the distance between \( z' \) and the origin is shorter than that of \( z \). Indeed, since the coefficients \( c_z \) are all positive the existence of linear combinations of the form (6.4) implies that
\[
\text{span}_C \left( \sum_{z \in V(G(a))} c_z \sigma_{w_z} \right) \cup \{ \sigma_{w_{z'}} - \sigma_{w_z} \mid z \in V(G(a)) \setminus \{0\} \} = \text{span}_C \{ \sigma_{w_z} \mid z \in V(G(a)) \}.
\]
We first provide the linear combination (6.4) when the erasing occurs consecutively, i.e., \( a \) has the form (6.1). First we consider elements \( z \in V(G(a)) \) such that \( z' := z - (1, 0, \ldots, 0) \) is also contained in \( V(G(a)) \), that is, \( z \in \{(z_1, \ldots, z_m) \in V(G(a)) \mid z_1 \geq z_2 \} \). Then they are connected by an edge, and we denote by \( i \) the label on the edge. Because of the construction of \( w_z \), we have that
\[
w_{z'} = s_{i} \cdot w_z
\]
and moreover, \( w_{z'}^{-1}(i + 1) < w_z^{-1}(i) \). Note that such elements \( z \) and \( z' \) have different values only between descents \( d_{\ell+m-2} \) and \( d_{\ell+m} \) by Lemma 6.2. Furthermore, \( w_z(d_{\ell+m-1}) = i + 1 \). We denote \( j = w_z^{-1}(i) \). Since the descents sets of \( w_{z'} \) and \( w_z \) are the same, \( d_{\ell+m-1} + 1 < j \). Now we define a
permutation $u_z$ by

$$u_z = w_z s_{d_{t+m-1}} s_{d_{t+m-1}+1} \ldots s_{j-2}.$$ 

In words, we move the descent $d_{t+m-1}$ to the backward until its location is just before $i$, i.e., two numbers $i+1$ and $i$ are consecutive in the one-line notation of $u_z$. Therefore, by applying Proposition [5.11] we obtain that

$$s_i \sigma_{u_z} = \sigma_{w_{z'}} + (t_{i+1} - t_i) \sigma_{s_i u_z} + \sum_{a(v) > a} q_v \sigma_v.$$ 

We notice that $a(u_z) \geq a$. Accordingly, by the induction argument, both $\sigma_{u_z}$ and $\sum_{a(v) > a} q_v \sigma_v$ are in the sum $M$. Because $M$ is invariant under the action of $\mathfrak{S}_n$, we prove that the linear combination $\sigma_{w_{z'}} - \sigma_{w_z}$ is contained in $M$.

Now we consider elements $z = (z_1, \ldots, z_m) \in V(G(a))$ such that $z_1 = z_2 \geq z_3$. For this case, we notice that two elements $z' := z - (1, 1, 0, \ldots, 0)$ and $\bar{z} := z - (0, 1, 0, \ldots, 0)$ are in $V(G(a))$. By the construction of the graph $G(a)$, elements $z$ and $\bar{z}$ are connected by an edge, and we denote by $i$ the label on the edge, that is, $x - m + (2 - 1) - z_2 = i$. Since $z_1 = z_2$, by the assumption, we have $x - m + (1 - 1) - z_1 = i - 1$, that is, the label on the edge connecting $z'$ and $\bar{z}$ is $i - 1$. See the left hand side of Figure 9. On the other hand, $z_1 = z_2$ implies that $z' - (0, 1, \ldots, 0)$ is not in $G(a)$, i.e., $D(s_i z') \neq D(z')$. Moreover, we have $D(z') = D(s_{i-1} z') = D(s_i s_{i-1} z')$. Therefore, the permutations $w_z, w_{z'}, w_{\bar{z}}$ have the form in Figure 9.

By the previous observation, we have a linear combination

$$\sigma_{w_{z'}} - \sigma_{w_z} \in M.$$ 

Then, by applying $s_i$ on the relation, we obtain the desired linear combination

$$\sigma_{w_{z'}} - \sigma_{w_z} \in M.$$ 

Here, $s_i \cdot \sigma_{w_{z'}} = \sigma_{w_{z'}}$ since $\{w_{z'}^{-1}(i + 1) - 1, w_{z'}^{-1}(i + 1), w_{z'}^{-1}(i)\} \subset D(w_{z'}) \cup \{0\}$ and by Corollary [5.14]. Continuing this process, we obtain the desired linear combinations when the erasing occurs consecutively.

Now we suppose that the graph $G(a)$ is the product $G(a_1) \times \cdots \times G(a_\beta)$ of graphs. We need to show that for any vertex $v = (v_1, \ldots, v_\beta) \in G(a)$, there exists $v' = (v'_1, \ldots, v'_\beta) \in G(a)$ such that the distance between $v'$ and the origin is shorter than that of $v$ and

$$\sigma_{w_{v'}} - \sigma_{w_v} \in M.$$ 

Among indices $1, \ldots, \beta$, choose the minimal index $x$ such that $v_x \neq 0$. Moreover, choose the maximal index $y$ such that $v_y \neq 0$. Consider the vertex $v_x$ in the graph $G(a_x)$. Then by the
previous argument, we have a vertex \( v'_x \) in the graph \( G(a) \) such that
\[
\sigma_{w(0, \ldots, 0, v'_x, 0, \ldots, 0)} - \sigma_{w(0, \ldots, 0, v_x, 0, \ldots, 0)} \in M.
\]
We apply the permutation \( u \) on this relation obtained from the path connecting the vertex \((0, \ldots, 0, v_x, 0, \ldots, 0)\) and \( v = (0, \ldots, 0, v_x, v_{x+1}, \ldots, v_y, 0, \ldots, 0) \):
\[
u(\sigma_{w(0, \ldots, 0, v'_x, 0, \ldots, 0)}) = \sigma_{uw(0, \ldots, 0, v'_x, 0, \ldots, 0)} - \sigma_{uw(0, \ldots, 0, v_x, 0, \ldots, 0)} = \sigma_{w(0, \ldots, 0, v'_x, v_{x+1}, \ldots, v_y, 0, \ldots, 0)} - \sigma_{vw} \in M.
\]
Since \((0, \ldots, 0, v'_x, v_{x+1}, \ldots, v_y, 0, \ldots, 0) \in G(a)\), we obtain a desired linear relation. This completes the proof.

**Example 6.7.** We consider compositions \((1, 1, 2), (1, 2, 1)\) and \((2, 1, 1)\) in Examples [5.27] and [6.4]. We already have seen that
\[
\bullet \ a = (1, 1, 2): \quad \hat{\sigma}_{4312} = 2\sigma_{4312} + 4\sigma_{4213} + 6\sigma_{3214} + 2(\sigma_{4231} - \sigma_{4132}) + 4\sigma_{3241} - 2\sigma_{3142} - 2\sigma_{3143} + 2(\sigma_{3421} - \sigma_{3431}) \cdot
\]
\[
\bullet \ a = (1, 2, 1): \quad \hat{\sigma}_{4231} = \sigma_{4231} + 2\sigma_{3241} + (\sigma_{3421} - \sigma_{2431}) \cdot
\]
\[
\bullet \ a = (2, 1, 1): \quad \hat{\sigma}_{3421} = 2\sigma_{3421}.
\]
The maximal element is \( a = (2, 1, 1) \), and we obtain the element \( \sigma_{3421} \) in \( M := \bigoplus_{u \in G_k} M(u) \). For \( a = (1, 2, 1) \), consider
\[
s_3\sigma_{2431} = \sigma_{2431} + \sigma_{2431} - \sigma_{3241} \in M.
\]
Here, since \( \sigma_{2431} = s_2 \cdot \sigma_{4231} \), both \( \sigma_{2431} \) and \( s_3\sigma_{2431} \) are contained in \( M \). Therefore, we have \( \sigma_{4231} - \sigma_{3241} \in M \). Since the matrix \(
\begin{bmatrix}
1 & 2 \\
1 & -1
\end{bmatrix}
\)
is nonsingular, both elements \( \sigma_{4231} \) and \( \sigma_{3241} \) are contained in \( M \). Finally, consider \( a = (1, 1, 2) \). Since \( s_1\sigma_{4231} = \sigma_{4132} \) and \( \sigma_{4231} \in M \), we have \( \sigma_{4132} \in M \). Accordingly,
\[
s_2\sigma_{4132} = \sigma_{4132} + \sigma_{4312} - \sigma_{4213} \in M
\]
and this implies that \( \sigma_{4312} - \sigma_{4213} \in M \). Moreover,
\[
s_3(\sigma_{4312} - \sigma_{4213}) = \sigma_{4312} - \sigma_{3214} \in M.
\]
Considering the following matrix
\[
\begin{bmatrix}
\sigma_{4312} & \sigma_{4213} & \sigma_{3214} \\
1 & 2 & 2 \\
1 & -1 & 0 \\
1 & 0 & -1
\end{bmatrix}
\]
is nonsingular, we obtain that \( \sigma_{4312}, \sigma_{4213}, \sigma_{3214} \in M \).
Example 6.8. Suppose that $a = (1,2,1,3)$. To draw the graph $G(a)$, we need to consider $a_1 = (1,2)$ and $a_2 = (1,3)$. Accordingly, the graph $G(a)$ and the elements $w_z$ are given as follows.

We first find the following linear relations obtained from $G(a_1) \times \{0\}$ and $\{0\} \times G(a_2)$.

For the vertex $(1,1)$, we need to consider the $s_3$-action on the linear relation related to the vertex $(1,0)$:

$$s_3(\sigma_7564123 - \sigma_6574123) = \sigma_7563124 - \sigma_6573124$$

which is a desired relation related to the vertex $w_{(1,1)}$.

Proof of Theorem 5.23. We denote by $M$ the sum $\sum_{u \in \mathcal{G}_k} M(u)$ of the modules. Then $M$ is contained in the whole vector space $H^{2k}(\mathcal{H}_n)$. We first claim that

$$M = H^{2k}(\mathcal{H}_n).$$

To prove the claim, it is enough to show that $M$ contains $H^{2k}(\mathcal{H}_n)$. We will show that for any composition $a$, the element $\sigma_v$ satisfying $a(v) = a$ are all contained in $\sum_{u \in \mathcal{G}_k} M(u)$, which proves the claim. Let $a$ be a composition. For any element $u$ satisfying $a(u) = a$, there exists a sequence of simple reflections $s_i, \ldots, s_i$ such that

- $u = s_{i_1} \cdots s_{i_l} w(a)$; and
- $a(w(a)) = a(s_{i_1} w(a)) = a(s_{i_2} s_{i_1} w(a)) = \cdots = a(s_{i_l} \cdots s_{i_1} w(a)) = a$.

Then, by Proposition 5.8 we obtain that

$$s_{i_j} \cdots s_{i_1} w(a) = \sigma_{s_{i_j} \cdots s_{i_1} w(a)}$$

for $1 \leq j \leq l$.

On the other hand, we have already shown in Lemma 5.17 that $v \sigma_{w_z}$ is also in $\sum_{u \in \mathcal{G}_k} M(u)$ for any $v \in \mathcal{G}_n/\prod \mathcal{S}_{I_j}$ and $z \in V(G(a))$. Therefore, any element $\sigma_v$ satisfying $a(v) = a$ is contained in $\sum_{u \in \mathcal{G}_k} M(u)$. This proves the claim (6.5).

For $u \in \mathcal{G}_k$, the stabilizer subgroup of $\sigma_u$ in $H^*_T(\mathcal{H}_n)$ is the same as $\mathcal{G}_u$ by Lemma 5.21. Therefore the stabilizer subgroup $\text{Stab}(\sigma_u)$ of $\sigma_u$ in $H^*(\mathcal{H}_n)$ contains $\mathcal{G}_u$. Because of the construction of $M(u)$ and the orbit-stabilizer theorem, we have that

$$\dim M(u) \leq \frac{|\mathcal{G}_n|}{|\text{Stab}(\sigma_u)|} \leq \frac{|\mathcal{G}_n|}{|\mathcal{G}_u|}.$$
Accordingly, we obtain

\[(6.7) \dim H^{2k}(\mathcal{H}_n) \leq \sum_{u \in G_k} \dim M(u) \leq \sum_{u \in G_k} \frac{|\mathcal{G}_u|}{|\mathcal{G}_u|} \dim H^{2k}(\mathcal{H}_n).\]

Here, the first inequality comes from (6.5), and the second inequality comes from (6.6). The last equality is provided by Proposition 5.26. From the inequalities (6.7), we obtain that

\[\text{Stab}(\mathcal{G}_u) = \mathcal{G}_u \quad \text{and} \quad \bigoplus_{u \in G_k} M(u) = H^{2k}(\mathcal{H}_n),\]

which proves the theorem. \(\square\)

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