D-branes and synthetic/$C^\infty$-algebraic symplectic/calibrated geometry,
I: Lemma on a finite algebraicness property of smooth maps
from Azumaya/matrix manifolds

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Abstract

We lay down an elementary yet fundamental lemma concerning a finite algebraicness
property of a smooth map from an Azumaya/matrix manifold with a fundamental module
to a smooth manifold. This gives us a starting point to build a synthetic (synonymously,
$C^\infty$-algebraic) symplectic geometry and calibrated geometry that are both tailored to and
guided by D-brane phenomena in string theory and along the line of our previous works
D(11.1) (arXiv:1406.0929 [math.DG]) and D(11.2) (arXiv:1412.0771 [hep-th]).

Key words: D-brane; Azumaya manifold, matrix manifold; smooth map; $C^\infty$-scheme, Weil algebra;
near-point determined; Lagrangian submanifold with nilpotent structure

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Chien-Hao Liu dedicates this note D(12.1) and D(12.2) (to be completed)
to the loving memory of
Rev. & Mrs. R. Campbell Willman (1925-2014) and Barbara M. Willman (1927-1999).

(From C.H.L.) There are memories that are too abundant to condense and too cherished and personal to reveal. A seemingly accidental encounter in my teenage years, which turned out to have profound impact on me. There is not a word with which I can express my gratitude to this family, including Ann and Lisa.
0. Introduction and outline

Lagrangian submanifolds in a symplectic manifold or special Lagrangian submanifolds in a Calabi-Yau manifolds have deep applications in string theory\(^1\). They are related to D-branes that preserve some supersymmetry (either on the open-string world-sheet or the target space(-time)). On the other hand, study of D-branes as a fundamental dynamical object in string theory leads us to the notion of differentiable maps from an Azumaya manifold (or synonymously matrix manifold) with a fundamental module to a smooth manifold. It is thus very natural to anticipate a notion of ‘Lagrangian map’ or a ‘special Lagrangian map’ from an Azumaya/matrix manifold with a fundamental module to a symplectic manifold or a Calabi-Yau manifold; cf. [L-Y3: Sec. 6.3, Sec. 7.2] (D(11.1)). However, the image of a differentiable map from an Azumaya/matrix manifold with a fundamental module in general carries a nilpotent structure with the push-forward of the fundamental module supported thereupon (cf. [L-Y3: Sec. 3, Sec. 5.2] (D(11.1)).) Such fuzzy sub-objects in a smooth manifold can be described/understood in terms of synthetic/$C^\infty$-algebraic geometry; cf., for example, [Du1], [Du2], [Joy1], [Ko], and [M-R]. (See [L-Y3: References] (D(11.1)) for more related literatures.) This suggests to relook at symplectic geometry and calibrated geometry with some input from synthetic/$C^\infty$-algebraic geometry:

In this subseries D(12) of our D-project, we explore this new direction in symplectic geometry and calibrated geometry that is motivated by D-branes in string theory and the recent progress made in [L-Y3] (D(11.1)). To begin, after some necessary background in Sec. 1 and Sec. 2, we prove in this Note D(12.1) an elementary yet fundamental lemma (Sec. 3: Lemma 3.1) concerning a finite algebraicness property of a smooth map from an Azumaya/matrix manifold with a fundamental module to a smooth manifold. It is a measure of what more to add to the traditional symplectic geometry and calibrated geometry in order to fit D-branes into them. This serves as our starting point to build a synthetic (synonymously, $C^\infty$-algebraic) symplectic geometry and calibrated geometry that are both tailored to and guided by D-brane phenomena in string theory and along the line of our previous works D(11.1) (arXiv:1406.0929 [math.DG]) and D(11.2) (arXiv:1412.0771 [hep-th]).

Convention. References for standard notations, terminology, operations and facts are (1) algebraic geometry: [Hart]; (2) symplectic geometry: [McD-S]; (3) calibrated geometry: [Harv], [H-L], [McL]; (4) synthetic geometry and $C^\infty$-algebraic geometry: [Joy1], [Ko]; and (5) D-branes: [Joh], [Po2], [Pol3].

\(^1\)This is a huge topic. Unfamiliar readers are referred to [B-B-S] and [O-O-Y] on how they are related to supersymmetric D-branes; for example, [A-B-C-D-G-K-M-S-S-W] and [H-K-K-P-T-V-V-Z] for an exposition related to mirror symmetry; and key-word search for many related themes and works.
The inclusion \( \mathbb{R} \hookrightarrow \mathbb{C} \) is referred to the field extension of \( \mathbb{R} \) to \( \mathbb{C} \) by adding \( \sqrt{-1} \), unless otherwise noted.

All manifolds are paracompact, Hausdorff, and admitting a (locally finite) partition of unity. We adopt the index convention for tensors from differential geometry. In particular, the tuple coordinate functions on an \( n \)-manifold is denoted by, for example, \((y^1, \cdots y^n)\). However, no up-low index summation convention is used.

The current Note D(12.1) spins off from the main line of D(11)-subseries in progress. It continues and focuses on themes in

[L-Y3]  
*D-branes and Azumaya/matrix noncommutative differential geometry, I: D-branes as fundamental objects in string theory and differentiable maps from Azumaya/matrix manifolds with a fundamental module to real manifolds*, arXiv:1406.0929 [math.DG]. (D(11.1))

that are related to symplectic geometry and calibrated geometry, cf. [L-Y3: Sec. 6.3, Sec. 7.2] (D(11.1)). Notations and conventions follow ibidem.

**Outline**

0. Introduction.

1. D-branes, \( \mathcal{C}^\infty \)-algebraic geometry, and differentiable maps from an Azumaya/matrix manifold with a fundamental module
   - D-branes as a fundamental dynamical object in string theory
   - Azumaya/matrix manifolds with a fundamental module and their surrogates
   - Smooth maps from \((X^A, \mathcal{E})\) to \(Y\), push-forward, associated surrogate, and graph

2. Weil algebras and determinacy of \( \mathcal{C}^\infty \)-rings
   - Weil algebras
   - Determinacy of \( \mathcal{C}^\infty \)-rings

3. Lemma on a finite algebraicness property of smooth maps \( \varphi : (X^A, \mathcal{E}) \to Y \)
   - Lemma on a finite algebraicness property of smooth maps \( \varphi : (X^A, \mathcal{E}) \to Y \)
   - A word on synthetic/\( \mathcal{C}^\infty \)-algebraic symplectic/calibrated geometry
1 D-branes, \( C^\infty \)-algebraic geometry, and differentiable maps from an Azumaya/matrix manifold with a fundamental module

To introduce the terminology and notations and to make this D(12) subseries conceptually more self-contained, we review in this section the most relevant part of [L-Y3] (D(11.1)). The setting is guided by the following three questions:

**Q1.** What is a D-brane as a fundamental dynamical object in string theory?

**Q2.** How would one understand an Azumaya/matrix manifold with a fundamental module?

**Q3.** How would one define a notion of a ‘differentiable map’ from such an “enhanced” manifold (with some open string-induced structure thereupon) to an ordinary real manifold in such a way that reflects features of D-branes?

How would one understand such maps?

We will address only the \( C^\infty \)-case in this note. Readers are referred to ibidem for omitted details, the general \( C^k \)-case, and further references. Newcomers are referred to [Liu] for a nontechnical introduction in the realm of algebraic geometry.

**D-branes as a fundamental dynamical object in string theory**

The structure of the enhanced scalar field in the massless spectrum on the world-volume of coincident D-branes that describes the deformations of the branes ([Wi] (1995), [H-W] (1996); see also [Po3: vol. I: Sec. 8.7] (1998) and [G-S] (2000)), when re-examined through Grothendieck’s setting of modern (commutative) algebraic geometry ([Hart]), leads one to the following proto-definition of D-branes (cf. [L-Y1] (D(1), 2007) and [L-L-S-Y] (D(2), 2008) in the more algebraic-geometry-oriented setting):

**Proto-Definition 1.1. [D-brane as fundamental/dynamical object].** As a fundamental/dynamical object in string theory, a *D-brane* in a space-time \( Y \) is described by a differentiable map from an Azumaya/matrix manifold with a fundamental module (and other open-string-induced structures thereupon) to the real manifold \( Y \).

**Figure 1-1 and Figure 1-3.**

Zooming into the details of this proto-definition and focusing only on the most basic structures of D-branes lead one then to *Question 2 and Question 3* above, whose answers based on \( C^\infty \)-algebraic geometry in the spirit of Grothendieck (e.g. [Joy1]) are reviewed below.

**Azumaya/matrix manifolds with a fundamental module and their surrogates**

**Definition 1.2. [Azumaya/matrix \( C^\infty \)-manifold with a fundamental module].** (cf. [L-Y3: Definition 4.0.1] (D(11.1)).) Let

- \( X \) be a smooth manifold of dimension \( m \), whose structure sheaf of smooth functions is denoted by \( \mathcal{O}_X \); denote the complexification \( \mathcal{O}_X \otimes \mathbb{R} \mathbb{C} \) of \( \mathcal{O}_X \) by \( \mathcal{O}_X^\mathbb{C} \) with the built-in inclusion \( \mathcal{O}_X \subset \mathcal{O}_X^\mathbb{C} \);

- \( \mathcal{E} \) be a locally free \( \mathcal{O}_X^\mathbb{C} \)-module of rank \( r \), (cf. the Chan-Paton sheaf on a D-brane); and
Oscillations of oriented open strings (red) that connect different D-branes (blue) give rise to an enhanced massless spectrum on D-brane world-volume when these D-branes become coincident. In particular, the enhanced massless scalar field leads to a noncommutative structure (orange) on the common D-brane world-volume of the coincident D-branes. At first, this enhanced scalar field is only (gauge) Lie-algebra-valued ([Wi], [Po2]); it can be promoted to be associative matrix-algebra-valued ([H-W]). This latter aspect is favored from Grothendieck’s viewpoint of algebraic geometry ([L-Y1] (D(1))) and at the same time leads us to a picture of a D-brane as a morphism/map from a matrix manifold with a fundamental module (i.e. the Chan-Paton bundle) (and other open-string-induced structures) to a space-time ([L-Y3] (D(11.1))), when D-brane is taken as a fundamental dynamical object in string theory. This is exactly the same as that a fundamental string is mathematically a map from a string-world-sheet to a space-time. Cf. Figure 1-3.

\[ \mathcal{O}_X^{\text{Az}} := \mathcal{End}_{\mathcal{O}_X}(\mathcal{E}) \]

be the sheaf of endomorphisms of \( \mathcal{E} \) as an \( \mathcal{O}_X^C \)-module; by construction, there is a canonical sequence of inclusions

\[ \mathcal{O}_X \subset \mathcal{O}_X^C \subset \mathcal{O}_X^{\text{Az}} \]

and \( \mathcal{O}_X^{\text{Az}} \) is a sheaf of \( \mathcal{O}_X^C \)-algebras with center the image of \( \mathcal{O}_X^C \) under the inclusion.

The smooth manifold \( X \) with the enhanced structure sheaf \( \mathcal{O}_X^{\text{Az}} := \mathcal{End}_{\mathcal{O}_X}(\mathcal{E}) \) of noncommutative function-rings from the endomorphism algebras of \( \mathcal{E} \) is called a (complex-)Azumaya (real \( m \)-dimensional) smooth manifold over \( X \); in notation,

\[ X^{\text{Az}} := (X, \mathcal{End}_{\mathcal{O}_X}(\mathcal{E})). \]

The triple

\[ (X, \mathcal{O}_X^{\text{Az}} := \mathcal{End}_{\mathcal{O}_X}(\mathcal{E}), \mathcal{E}) \]

is called an Azumaya smooth manifold with a fundamental module. With respect to a local trivialization of \( \mathcal{E} \), \( \mathcal{O}_X^{\text{Az}} \) is a sheaf of \( r \times r \)-matrix algebras with entries complexified local \( C^\infty \)-functions on \( X \). For that reason and to fit better with the terminology in quantum field theory and string theory, we shall call \( (X, \mathcal{O}_X^{\text{Az}} := \mathcal{End}_{\mathcal{O}_X}(\mathcal{E}), \mathcal{E}) \) also as a matrix \( C^\infty \)-manifold with a fundamental module, particularly in a context that is more directly related to quantum field theory and string theory.

To help having a concrete, more visualizable feeling of such a noncommutative manifold, we introduced the following intermediate commutative objects:
Definition 1.3. [(commutative) surrogate of Azumaya/matrix manifold]. (Cf. [L-Y1: Sec. 3.2] (D(1)), [L-L-S-Y: Definition 2.1.3] (D(2)), [L-Y3: Definition 5.1.4; Lemma/Definition 5.3.1.7] (D(11.1)), [L-Y4: Definition 4.1.5; Lemma/Definition 4.2.1.5] (D(11.2)).) Let $A$ be a sheaf of commutative $\mathcal{O}_X$-subalgebras of $\mathcal{O}^A_{\mathbb{X}}$ and $X_A$ be its associated $C^\infty$-scheme over $X$. Then $X_A$ is called a (commutative) surrogate of $X^A$.

Remark 1.4. [built-in diagram associated to surrogate]. Through the built-in inclusions

\[ \mathcal{O}_X \subset A \subset \mathcal{O}^A_{\mathbb{X}}, \]

one has the contravariant sequence of dominant morphisms:

\[ \xymatrix{ \mathcal{O}^A_{X} \ar[d] \ar[r] & \mathcal{A} \ar[d] \ar[r] & \mathcal{E} \ar[d] & \mathcal{E} \ar[d] \ar@{->>}[l] \ar@{->>}[rr] \ar@{->>}[ll] & & \mathbb{X} \ar[d] } \]

Here, $\mathcal{O}^E_{X}$ is $\mathcal{E}$ but as a (left) $\mathcal{O}^A_{\mathbb{X}}$-module tautologically, and similarly $\mathcal{A} \mathcal{E}$ is $\mathcal{E}$ but as an $\mathcal{A}$-module tautologically. While $\sigma_A$ is only defined conceptually by the inclusion $A \subset \mathcal{O}^A_{\mathbb{X}}$, $\pi_A$ is an honest map between $C^\infty$-schemes in the context of $C^\infty$-algebraic geometry. Furthermore, one has the following canonical isomorphisms of sheaves of modules on related ringed-spaces

\[ \sigma_{A*}(\mathcal{O}^E_X) \simeq \mathcal{A} \mathcal{E}, \quad \text{and} \quad \pi_{A*}(\mathcal{A} \mathcal{E}) \simeq \mathcal{E}. \]

Example 1.5. [surrogates of Azumaya/matrix closed string]. Examples of surrogates of the Azumaya/matrix closed string with a fundamental module $(S^1, \mathcal{A}, \mathcal{E})$, where $\mathcal{E} \simeq \mathcal{O}_{S^1}$, are illustrated in Figure 1-2.

Smooth maps from $(X^A, \mathcal{E})$ to $Y$, push-forward, associated surrogate, and graph

Continuing the notations in the previous theme. When attempting to define the notion of a ‘differentiable map’ $\varphi : (X^A, \mathcal{E}) \to Y$, there are a few subtle issues one has to face:

1. The first issue is a fundamental one. Recall that in algebraic geometry, an $\mathbb{R}$-scheme and a $\mathbb{C}$-scheme are quite different; the former contains two types of closed points, $\mathbb{R}$-points and $\mathbb{C}$-points, while the latter contains only $\mathbb{C}$-points. Thus an $\mathbb{R}$-scheme has a locus, i.e. the set of $\mathbb{R}$-points, that is familiar to differential geometers but also a generally much larger locus, i.e. the set of $\mathbb{C}$-points, which may look troublesome from the aspect of differential geometry since these points are somehow “extra”. In contrast, in differential geometry a complex manifold is simply a real manifold of even dimension with some additional structure and all the points are already there. As we are dealing with maps from a space $X$ with a complex noncommutative structure sheaf $\mathcal{O}^A_{\mathbb{X}}$ to a real manifold $Y$.

Q. Will it be that we need to add additional $\mathbb{C}$-points to $Y$ in order to make sense of a differentiable map $\varphi : (X^A, \mathcal{E}) \to Y$ in the context of $C^k$-rings and $C^k$-algebraic geometry, for $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$?
Figure 1-2. In the illustration, the Azumaya/matrix manifold $X^{Az}$ is indicated by a noncommutative cloud sitting over $X$. In-between are surrogates $X_A$ of $X^{Az} := (X, \mathcal{O}_X^{Az})$ associated to various commutative $\mathcal{O}_X$-subalgebras $\mathcal{O}_X \subset A \subset \mathcal{O}_X^{Az}$. They can form a very rich pool of patterns/varieties. From the illustrated Examples (a), (b), (c), (d), (e), (f), (g) of surrogates of Azumaya/matrix closed string $(S^1, A^{Az}, \mathcal{E})$ of rank 3, one sees, for instance,

- a long string as a connected component (cf. (a), (b)) vs. a set of short strings (cf. (c), (d), (e), (f) and $S^1$ itself),
- simple strings (cf. (a), (b), (c), (f) and $S^1$ itself) vs. strings with a fuzzy nilpotent cloud (cf. (d), (e), (g)),
- connected strings (cf. (a), (e), (g), and $S^1$ itself) vs. disconnected strings ((b), (c), (d), (f)),

Such feature will propagate to the rich pool of patterns/varieties of the image of $X^{Az}$ under smooth maps $\varphi$ therefrom; cf. Definition 1.6 and Figure 1-3, Figure 1-4, and Figure 3-1. The vertical arrows

$$X^{Az} \rightarrow X_A \quad \text{and} \quad X_A \rightarrow X$$

are the built-in maps associated to the inclusions $\mathcal{O}_X \subset A \subset \mathcal{O}_X^{Az}$. In the list of examples, $X_A$ in Example (a) – Example (f) behaves as a bundle over $X$. One should not be misled by this. For a general $A$, $X$ is stratified into pieces and $X_A$ can be of different natures over different strata; cf. Example (g).
Luckily, it turns out that the $C^k$-ring structure on the ring $C^k(Y)$ of $C^k$-functions on $Y$ enforces the eigenvalues of the matrices $\varphi^*(C^k(Y)) \subset C^k(O_X^A)$ all real ([L-Y3: Sec. 3] (D(11.1))). Here, $\varphi^* : C^k(Y) \to C^k(O_X^A)$ is a would-be ring-homomorphism defined through ‘pull back the functions via $\varphi$', if $\varphi$ is defined. Thus, $Y$ is enough:

A. There is no need to add additional $C$-points to the real manifold $Y$ in order to make sense of a “morphism” in the current context.

(2) The second issue is a technical one:

- A general ideal $I$ of the underlying ring of a $C^k$-ring $R$ may not be a good object from the aspect of $C^k$-algebraic geometry: The $C^k$-ring structure on $R$ may not descend to a $C^k$-ring structure on the quotient ring $R/I$.

This simply says that we have to look at the “right class” of ideals of a $C^k$-rings, i.e. the ideals that really reflect the nature of ideals for a $C^k$-submanifold of a $C^k$-manifold. This leads to the notion of $C^k$-normal $C^k$-ideals of a $C^k$-ring; cf. [L-Y3: Definition 2.1.3, Definition 2.1.4] (D(11.1)). For the $C^\infty$ case, things get easier:

- As a consequence of Hadamard’s Lemma, a quotient ring $R/I$ of a $C^\infty$-ring $R$ by an ideal of the undering ring is equipped with an induced $C^\infty$-ring structure such that the quotient ring-homomorphism is a $C^\infty$-ring-homomorphism.

(E.g. [Joy1: Sec. 2.2].)

(3) The third issue is a string-theoretical one:

Q. How should one “design” the notion of ‘differentiable maps’ in the current context so that it reflects fundamental features of D-branes in string theory?

For the purpose of this D-project, this is of uttermost importance. Purely mathematically, we are dealing with the notion of ‘morphisms’ between two ringed-spaces: $(X, O_X^A)$ to $(Y, O_Y)$. As already elaborated extensively in [L-Y1] (D(1)), while mathematically sensible and acceptable, it is too restrictive to define the notion of a map $(X, O_X^A) \to (Y, O_Y)$ by the standard notion of morphisms between ringed-spaces as a pair $(f, f^\#)$ where $f : X \to Y$ is an usual differentiable map between $C^k$-manifolds and $f^\# : O_Y \to f_*O_X^A$ is a map between sheaves of rings on $Y$; cf. , e.g. [Hart: Sec. II.2]. Rather,

A. To encode key features of D-branes, a map $\varphi : (X, O_X^A) \to (Y, O_Y)$ is defined contravariantly by an equivalence class $\varphi^* : O_Y \to O_X^A$ of gluing-systems of ring-homomorphisms from local function-rings of $Y$ to local function-rings of $X^A$ that satisfy some conditions.

Once these subtle issues are all passed, there is a conceptual ease here (though not necessarily a technical ease):

- A smooth manifold $M$ is affine in the context of $C^\infty$-algebraic geometry in the sense that $M$ is completely characterized by its function-ring $C^\infty(M)$. Thus, the notion of ‘morphism’ in the current context can be phrased in terms of either the structure sheaves involved $(O_X^A$ and $O_Y$) or the (global) function-rings involved $(C^\infty(O_X^A)$ and $C^\infty(Y))$.

We will use the structure-sheaf picture to match with the setting in the algebro-geometric situation in [L-Y1] (D(1)) and [L-L-S-Y] (D(2)). This is just a personal aesthetic preference.
Definition 1.6. [smooth map from Azumaya/matrix manifold]. (Cf. [L-Y1: Sec. 1] (D(1)), [L-L-S-Y: Sec. 2.1] (D(2)), [L-Y3: Sec. 5.3] (D(11.1)), and [L-Y4: Sec. 4.2.1] (D(11.2)).) A smooth map (or synonymously, infinitely differentiable map or $C^\infty$-map)

$$\varphi : (X^{A_e}, \mathcal{E}) := (X, O^A_X := \text{End}_{O^C_X}(\mathcal{E}), \mathcal{E}) \longrightarrow (Y, O_Y)$$

from a smooth Azumaya/matrix manifold with a fundamental module $(X^{A_e}, \mathcal{E})$ to a smooth manifold $Y$ is defined contravariantly by an equivalence class of gluing-systems of ring-homomorphisms (over $\mathbb{R} \subset \mathbb{C}$) from local function-rings of $Y$ to local function-rings of $X^{A_e}$ such that

1. It extends to a commutative diagram

$$
\begin{array}{ccc}
O^A_X & \xleftarrow{\varphi^\sharp} & O_Y \\
\downarrow{pr^Y_X} & & \downarrow{pr^Y_Y} \\
O^C_X & \xleftarrow{\varphi^\sharp} & O_{X \times Y}
\end{array}
$$

of equivalence classes of gluing systems of ring-homomorphisms (over $\mathbb{R}$, or $\mathbb{R} \subset \mathbb{C}$ whenever applicable). Here, $pr^Y_X$ and $pr^Y_Y$ are the pull-back maps associated to the projection maps $pr_X : X \times Y \to X$ and $pr_Y : X \times Y \to Y$ respectively.

2. In terms of the diagram in Item (1), let the $O_X$-algebra

$$O_X(\text{Im}(\varphi^\sharp)) := \text{Im}(\varphi^\sharp) \subseteq O^A_X$$

be equipped with a sheaf (over $X$) of $C^\infty$-rings structure induced from that of $O_{X \times Y}$ via the quotient map $\varphi^\sharp$. Then, it is required that, as maps to $O_X(\text{Im}(\varphi^\sharp))$, both arrows of the following subdiagram

$$
\begin{array}{ccc}
O_X(\text{Im}(\varphi^\sharp)) & \xleftarrow{\varphi^\sharp} & O_Y \\
\downarrow{O_X} & & \\
O_X & \xleftarrow{\varphi^\sharp} & O_{Y}
\end{array}
$$

of the previous 4-cornored diagram are now equivalence classes of gluing systems of $C^\infty$-ring-homomorphisms.

Since $O_X(\text{Im}(\varphi^\sharp))$ is commutative, it determines a $C^\infty$-scheme $X_{\varphi}$, with structure sheaf $O_X(\text{Im}(\varphi^\sharp))$, that fits into the following commutative diagram of morphisms between $C^\infty$-schemes:

$$
\begin{array}{ccc}
X_{\varphi} & \xleftarrow{f_\varphi} & Y \\
\downarrow{\pi_\varphi} & & \downarrow{pr_Y} \\
X & \xleftarrow{pr_X} & X \times Y.
\end{array}
$$

Cf. Figure 1-3.
From the aspect of $C^\infty$-algebraic geometry in line with Grothendieck, D-branes as a fundamental/dynamical object in string theory are given by ‘differentiable maps $\varphi : (X^{Ak}, \mathcal{E}) \to Y$ from a matrix manifold (i.e. the D-brane world-volume) with a fundamental module (and other open-string-induced structures) to the space-time’. In contrast to fundamental strings, $X^{Ak}$ carries a matrix-type “noncommutative cloud” over its underlying topology. Under a differentiable map $\varphi$ as defined in [L-Y3: Definition 5.3.1.5] (D(11.1)), cf. Definition 1-6 of the current note, the image $\varphi(X^{Ak})$ can behave in a more complicated way. In particular, it could be disconnected or carry some nilpotent fuzzy structure. See also Figure 3-1 in Sec. 3.

Remark 1.7. [in terms of function rings]. Equivalently, a $C^\infty$-map

$$\varphi : (X^{Ak}, \mathcal{E}) \to Y$$

is defined contravariantly by a ring-homomorphism

$$C^\infty(X^{Ak}) := C^\infty(O_{X}^{Ak}) := O_{X}^{Ak}(X) \xleftarrow{\varphi^\sharp} C^\infty(Y)$$

over $\mathbb{R} \subset \mathbb{C}$ such that

1’ It extends to a commutative diagram

$$\begin{array}{ccc}
C^\infty(X^{Ak}) & \xleftarrow{\varphi^\sharp} & C^\infty(Y) \\
\downarrow{\varphi^\sharp} & & \downarrow{pr_Y^\sharp} \\
C^\infty(X) & \xleftarrow{pr_X^\sharp} & C^\infty(X \times Y)
\end{array}$$

of ring-homomorphisms (over $\mathbb{R}$, or $\mathbb{R} \subset \mathbb{C}$ whenever applicable).

2’ In terms of the diagram in Item (1’), let the $C^\infty(X)$-algebra

$$C^\infty(X)\langle \text{Im}(\varphi^\sharp) \rangle := \text{Im}(\tilde{\varphi}^\sharp) \subset C^\infty(X^{Ak})$$

be equipped with the $C^\infty$-rings structure from that of $C^\infty(X \times Y)$ via the quotient map $\tilde{\varphi}^\sharp$. Then, it is required that, as maps to $C^\infty(X)\langle \text{Im}(\varphi^\sharp) \rangle$, both arrows of the following subdiagram

$$\begin{array}{ccc}
C^\infty(X)\langle \text{Im}(\varphi^\sharp) \rangle & \xleftarrow{\varphi^\sharp} & C^\infty(Y) \\
\downarrow & & \\
C^\infty(X)
\end{array}$$
of the previous 4-cornored diagram are now \( C^\infty \)-ring-homomorphisms.

**Definition 1.8. [push-forward \( \varphi_*(E) \)].** Continuing Definition 1-6. Through \( \varphi^\sharp : O_Y \rightarrow O_X^s \), the fundamental (left) \( O_X^s \)-module \( E \) becomes an \( O_Y \)-module. This defines the push-forward \( \varphi_*(E) \) of \( E \) under \( \varphi \).

**Definition 1.9. [surrogate of \( X^s \) specified by \( \varphi \)].** Continuing Definition 1-6. Through \( \varphi^\sharp : O_Y \rightarrow O_X^s \), the fundamental (left) \( O_X^s \)-module \( E \) becomes an \( O_Y \)-module. This defines the push-forward \( \varphi_*(E) \) of \( E \) under \( \varphi \).

**Definition 1.10. [graph of \( \varphi \)].** Continuing Definition 1-6. Through \( \bar{\varphi}^\sharp : O_{X \times Y} \rightarrow O_{X^s} \), the fundamental (left) \( O_{X^s} \)-module \( E \) becomes an \( O_{X \times Y} \)-module \( \bar{\varphi} \). This defines the graph of \( \varphi \).

**Example 1.11. [smooth map from Azumaya/matrix point].** ([L-Y3: Sec. 3.2] (D(11.1)).) The most elementary example of a smooth map \( \varphi : (X^s, E) := (X, O_X^s := \text{End}_{C^\infty_X}(E), E) \rightarrow Y \) from an Azumaya/matrix manifold with a fundamental module to a smooth manifold is the case when \( X \) is a point. In this case any smooth map \( (p^s, C_{\mathbb{R}^n}) \rightarrow Y \) from an Azumaya/matrix point with a fundamental module \( (p^s, C_{\mathbb{R}^n}) \) to the real manifold \( \mathbb{R}^n \) is of algebraic type, in the sense that that the corresponding \( C^\infty \)-rings homomorphism

\[ \varphi^\sharp : C^\infty(Y) \rightarrow M_{r \times r}(\mathbb{C}) \]

factors through the following commutative diagram:

\[
\begin{array}{ccc}
C^\infty(Y) & \xrightarrow{\varphi^\sharp} & M_{r \times r}(\mathbb{C}) \\
\oplus_{j=1}^s T_{q_j}^{(r-1)} & \parallel & \oplus_{j=1}^s \mathbb{R}[y_j^1, \ldots, y_j^n] \\
\end{array}
\]

for some \( q_1, \ldots, q_s \in Y \), where

- \((y_j^1, \ldots, y_j^n)\) is a local coordinate system in a neighborhood of \( q_j \in Y \) with coordinates of \( q_j \) all 0,
- \( T_{q_j}^{(r-1)} \) is the map ‘taking Taylor polynomial (of elements in \( C^\infty(Y) \)) at \( q_j \) with respect to \((y_j^1, \ldots, y_j^n)\) up to and including degree \( r - 1 \), and
- \( \varphi^\sharp \) is an (algebraic) ring-homomorphism over \( \mathbb{R} \subset \mathbb{C} \).

Thus, similar to the algebraic case in [L-Y1: Sec. 4] (D(1)), despite that \( \text{Space} M_{r \times r}(\mathbb{C}) \) may look only one-point-like, under a smooth map \( \varphi \) the Azumaya/matrix “noncommutative cloud” \( M_{r \times r}(\mathbb{C}) \) over \( \text{Space} M_{r \times r}(\mathbb{C}) \) can “split and condense” to various image 0-dimensional \( C^\infty \)-schemes with a rich geometry. The latter image \( C^\infty \)-schemes in \( Y \) can even have more than one component. These features generalize to smooth maps \( \varphi \) from general Azumaya manifolds with a fundamental module \( (X^s, E) \) to \( Y \). In the general case, one can “see” conceptually how such maps look like by “smearing”/“rolling” the above situation over \( X \); cf. [L-Y3: Sec. 5.3.3] (D(11.1)). \text{Figure 1-4}.  


Figure 1-4. (Cf. [L-Y2: Figure 2-1-1] (D(6)).) Various smooth maps $\varphi_1$, $\varphi_2$, $\varphi_2'$, $\varphi_3 : (p^A, \mathbb{C}^{\oplus r}) \rightarrow Y$ from an Azumaya/matrix point with a fundamental module to a smooth manifold $Y$ are illustrated. Observe that when the noncommutative cloud $M_{r \times r}(\mathbb{C})$ is “squeezed” into the commutative $Y$ via a smooth map $\varphi$, what’s left (i.e. $\varphi(p^A)$, which is identical to $\text{Supp}(\varphi_*(\mathbb{C}^{\oplus r}))$) is a commutative nilpotent structure in the structure sheaf of the image $C^\infty$-scheme $\varphi(p^A)$ in $Y$ if $\varphi(p^A)$ is not reduced; cf. $\varphi_2$. Observe also that, though $p^A$ is connected in any sensible aspect, for $r \geq 2$, $\varphi(p^A)$ in general can be disconnected; cf. $\varphi_2$, $\varphi_2'$, and $\varphi_3$. In the figure, a module over a $C^\infty$-scheme is indicated by a dotted arrow $\quad \cdots \cdots \infty$.
2 Weil algebras and determinacy of $C^\infty$-rings

Basic definitions and facts on Weil algebras and determinacy of ideals of smooth functions that are relevant to the current note and its follow-ups are collected in this section. They follow mainly from [M-R: Sec. I.3, Sec. I.4] of Ieke Moerdijk and Gonzalo E. Reyes. See also [B-D: Sec. 2]; [Du1], [Joy1: Example 2.9], and [Ko: Sec. I.16, Sec. III.5].

Weil algebras

Definition 2.1. [Weil algebra]. A Weil algebra $R$ is a finite-dimensional commutative $R$-algebra with a unique maximal ideal $m$ such that the residue field $R/m \cong R$.

The composition of the built-in $R$-algebra-homomorphisms $R \rightarrow R \rightarrow R$ is the identity map. It follows that $R \cong R \oplus m$ canonically as $R$-vector spaces.

Theorem 2.2. [basic properties of Weil algebras]. Some basic properties of Weil algebras are listed below:

(1) ([M-R: I.3.16 Corollary].) Let $R$ be a Weil algebra with the maximal ideal $m$. Then $m^k = 0$ for some $k$.

(2) ([M-R: I.3.18 Corollary]; [Ko: III Theorem 5.3].) There is a unique/canonical $C^\infty$-ring structure on $R$ that is compatible with the ring structure of $R$.

(3) (Cf. [M-R: I.3.19 Corollary]; [Du1].) Let $R$ be a Weil algebra and $S$ be an arbitrary $C^\infty$-ring. Then there are canonical isomorphisms (of sets)

$$\text{Hom}_{R\text{-algebra}}(R, S) \cong \text{Hom}_{C^\infty\text{-ring}}(R, S)$$

and

$$\text{Hom}_{R\text{-algebra}}(S, R) \cong \text{Hom}_{C^\infty\text{-ring}}(S, R).$$

(Cf. Item (2).)

(4) ([M-R: I.3.21 Corollary (a)].) Let $R$ and $S$ be Weil algebras. Then so is their push-out $R \otimes_\infty S$ as $C^\infty$-rings. The latter is identical to the tensor product $R \otimes_R S$ of $R$ and $S$ as $R$-algebras.

(5) ([Ko: III Theorem 5.3].) More generally, let $R$ be a Weil algebra and $S$ be a $C^\infty$-ring. Then, their push-out $R \otimes_\infty S$ as $C^\infty$-rings exists and is identical to the tensor product $R \otimes_R S$ of $R$ and $S$ as $R$-algebras.

(6) ([M-R: I.3.21 Corollary (b)].) Let $R \subset S$ be a sub-$R$-algebra. If $S$ is a Weil algebra, then so is $R$.

Remark 2.3. [finite-dimensional commutative $R$-algebra]. Note that any finite-dimensional commutative $R$-algebra $R$ that has finitely many maximal ideals with the associated residue field all isomorphic to $\mathbb{R}$ can be written uniquely, up to an isomorphism, as a (finite) direct product of Weil algebras. $R$ is thus endowed also with a unique/canonical $C^\infty$-ring structure that is compatible with the ring structure of $R$. Theorem 2.2 remains to hold with ‘Weil algebra’ replaced by ‘finite-dimensional commutative $R$-algebra that has finitely many maximal ideals with the associated residue field all isomorphic to $\mathbb{R}$’.

Caution that in general a finite-dimensional $R$-algebra can have maximal ideals with residue field $\mathbb{C}$. They may behave pathological from the aspect of $C^\infty$-algebraic geometry.
Remark 2.4. \([C^\infty\text{-ring-homomorphism from/to Weil algebra}]. \) (Cf. [M-R: I.3.19 Corollary] and [Ko: Sec. III, Theorem 5.3 and Proposition 5.12].) Let \(R\) be a Weil algebra, \(S\) be an arbitrary \(C^\infty\)-ring, and \(M\) be a smooth manifold of dimension \(n\). Then as consequences of Hadamard’s Lemma,

1. any \(C^\infty\)-ring-homomorphism \(\alpha : R \to S\) lifts to \(C^\infty\)-ring-homomorphisms

\[
\begin{array}{c}
\mathbb{R}[y_1, \ldots, y_n] \\
\downarrow \ \\
\mathbb{R}^{[y_1, \ldots, y_n]}
\end{array}
\xrightarrow{\alpha}
\begin{array}{c}
R \\
\downarrow \ \\
S
\end{array}
\xrightarrow{\hat{\alpha}}
\begin{array}{c}
\mathbb{R}[y_1, \ldots, y_n] \\
\downarrow \ \\
\mathbb{R}^{[y_1, \ldots, y_n]}
\end{array}
\]

for some \(l \in \mathbb{Z}_{\geq 0}\) and

2. any \(C^\infty\)-ring-homomorphism \(\beta : C^\infty(M) \to R\) factors through \(C^\infty\)-ring-homomorphisms

\[
\begin{array}{c}
\mathbb{R}[y_1, \ldots, y_n] \\
\downarrow \ \\
\mathbb{R}^{[y_1, \ldots, y_n]}
\end{array}
\xrightarrow{T_p^{(k)}}
\begin{array}{c}
C^\infty(M) \\
\downarrow \ \\
R
\end{array}
\xrightarrow{\beta}
\begin{array}{c}
\mathbb{R}[y_1, \ldots, y_n] \\
\downarrow \ \\
\mathbb{R}^{[y_1, \ldots, y_n]}
\end{array}
\]

for some \(p \in M\) and \(k \in \mathbb{Z}_{\geq 1}\), where \((y_1, \ldots, y_n)\) is a local coordinate system centered at \(p\) and the map \(T_p^{(k)}\) is ‘taking Taylor polynomial with respect to the local coordinate system \((y_1, \ldots, y_n)\) up to and including degree \(k\).’

Statement (2) was generalized to similar statements concerning \(C^k\)-admissible ring-homomorphisms \(C^k(Y) \to M_{r \times r}(\mathbb{C})\) when we developed the notion of ‘\(k\)-times differentiable map’ (i.e. ‘\(C^k\)-map’) from an Azumaya/matrix point \((p, M_{r \times r}(\mathbb{C}))\) to a real manifold \(Y\), \(k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}\), in [L-Y3: Sec. 3] (D(11.1)).

Determinacy of \(C^\infty\)-rings

Definition 2.5. [pointwise/algebraic/formal/local determinacy of \(C^\infty\)-ring]. ([M-R: I.4.1 Definition].) Let \(R\) be a \(C^\infty\)-ring.

(a) \(R\) is point determined if it can be embedded into a direct product \(\prod_{i \in I} \mathbb{R}\) of the base \(C^\infty\)-ring \(\mathbb{R}\).

(b) \(R\) is near-point determined if it can be embedded into a direct product of Weil algebras \(R\) can be chosen such that their respective maximal ideal \(m\) satisfies the condition that \(m^{k+1} = 0\) for a fixed \(k \in \mathbb{Z}_{\geq 0}\), then \(R\) is called order-\(k\)-near-point determined or simply \(k\)-near-point determined.

(c) \(R\) is closed if it can be embedded into a direct product of formal \(C^\infty\)-algebras (i.e. \(C^\infty\)-rings of the form \(\mathbb{R}[[x_1, \ldots, x_n]]/I\) for some \(n\) and ideal \(I\)).

(d) \(R\) is germ determined if it can be embedded into a direct product of pointed local \(C^\infty\)-rings (i.e. \(C^\infty\)-rings \(A\) that have a unique maximal ideal \(m_A\) with residue field \(A/m_A \simeq \mathbb{R}\)).

The above intrinsic determinacy property of \(C^\infty\)-rings has the following extrinsic characterization through ideals when the \(C^\infty\)-ring \(R\) is finitely generated:
Theorem 2.6. [in terms of ideals of finitely generated $C^\infty$-ring]. ([M-R: I.4.2 Theorem].) Let $R$ be a $C^\infty$-ring of the form $C^\infty(M)/I$, where $M$ is a smooth manifold, and
\[ Z(I) := \bigcap_{f \in I} \{ f^{-1}(0) \} \subset M \]
be the zero-set of $I$. For an $x \in M$ and $k \in \mathbb{Z}$, let
\[ T_x^{(k)}(I) := \{ \text{Taylor expansion } T_x^{(k)}(f) \text{ of } f \text{ at } x \text{ up to and including degree } k \mid f \in I \} \]
and
\[ T_x^{(\infty)}(I) := \{ \text{full Taylor expansion } T_x^{(\infty)}(f) \text{ of } f \text{ at } x \mid f \in I \} \]
with respect to some fixed coordinate system around $x$. For $f \in C^\infty(M)$, denote by $f(x)$ the germ of $f$ at $x \in M$ and $I(x) := \{ f(x) \mid f \in I \}$. Then
\[(a) \text{ } R \text{ is point determined if and only if for all } f \in C^\infty(M) \]
\[ \quad \text{if } f(x) = 0 \text{ for all } x \in Z(I), \text{ then } f \in I. \]
\[(b) \text{ } R \text{ is } k\text{-near-point determined, } k \in \mathbb{Z}_{\geq 0} \text{ if and only if for all } f \in C^\infty(M) \]
\[ \quad \text{if } T_x^{(k)}(f) \in T_x^{(k)}(I) \text{ for all } x \in Z(I), \text{ then } f \in I. \]
\[(c) \text{ } R \text{ is closed if and only if for all } f \in C^\infty(M) \]
\[ \quad \text{if } T_x^{(\infty)}(f) \in T_x^{(\infty)}(I) \text{ for all } x \in Z(I), \text{ then } f \in I. \]
\[(d) \text{ } R \text{ is germ determined if and only if for all } f \in C^\infty(M) \]
\[ \quad \text{if } f(x) \in I(x) \text{ for all } x \in Z(I), \text{ then } f \in I. \]

Under the setting of the Theorem, if $I$ satisfies the specified condition in Item (a) (resp. Item (b), Item (c), Item (d)), we also say that $I$ is point determined (resp. $k$-near-point determined, closed, germ determined).

Remark 2.7. [hierarchy of determinacy] ([M-R: I.4.5 Proposition].) For an ideal $I \subset C^\infty(M)$, it follows by definition that
\[ \text{point determined } \Rightarrow \text{ near-point determined } \Rightarrow \text{ closed } \Rightarrow \text{ germ determined}. \]

Remark 2.8. [Fréchet topology on $C^\infty(M)$]. ([Ma], [M-R: I.4.4 Remark].) Let $M$ be a smooth manifold. Then one can define the Fréchet topology on $C^\infty(M)$ by taking the neighborhood system around an $f \in C^\infty(M)$ to be the subsets
\[ V_{(f,\varepsilon,n,K)} := \{ g \in C^\infty(M) \mid \sup_{x \in K, \alpha, \leq n} |D^\alpha g - D^\alpha f| \} \]
of $C^\infty(M)$ for $\varepsilon > 0$, $n \in \mathbb{Z}_{\geq 1}$, and $K \subset M$ compact. Here, we fix an atlas on $M$, $D^\alpha$ is the derivative with respect to local coordinates specified by $\alpha$, $|\alpha|$ is the total degree. The topology the system generates is independent of the choices of the atlas on $M$. In terms of this topology, an ideal $I \subset C^\infty(M)$ is closed in the sense of Theorem 2.6 (c) if and only if it is closed with respect to the Fréchet topology on $C^\infty(M)$.

Readers are referred to [M-R: I.4] for more discussions.
Lemma on a finite algebraicness property of smooth maps \( \varphi : (X^{A_{k}}, \mathcal{E}) \rightarrow Y \)

With the necessary background reviewed in Sec. 1 and Sec. 2, we now state and prove the main lemma of this note.

Lemma 3.1 [pointwise finite algebraicness over \( X \)]. Let \( (X, \mathcal{O}_{X}) \) be a smooth manifold, \( \mathcal{E} \) be a locally free \( \mathcal{O}_{X}^{\underline{r}} \)-module of rank \( r \),

\[
\varphi : (X^{A_{k}}, \mathcal{E}) := (X, \mathcal{O}_{X}^{A_{k}} := \text{End}_{\mathcal{O}_{X}^{\underline{r}}} \mathcal{E}, \mathcal{E}) \rightarrow (Y, \mathcal{O}_{Y})
\]

be a smooth map from the Azumaya/matrix manifold with a fundamental module \( (X^{A_{k}}, \mathcal{E}) \) to a smooth manifold \( Y \) that is associated to a contravariant equivalence class \( \varphi^{\#} : \mathcal{O}_{Y} \rightarrow \mathcal{O}_{X}^{A_{k}} \) of gluing systems of \( C_{\infty} \)-admissible ring-homomorphisms over \( R \subset \mathbb{C} \). Let \( \tilde{\mathcal{E}}_{\varphi} \) be the graph of \( \varphi \).

Recall that \( \tilde{\mathcal{E}}_{\varphi} \) is an \( \mathcal{O}_{X \times Y}^{\underline{r}} \)-module that is flat over \( X \) of relative \( \mathbb{C} \)-length \( r \). Then

1. The \( C_{\infty} \)-scheme-theoretical support \( \text{Supp}(\tilde{\mathcal{E}}_{\varphi}) \) of \( \tilde{\mathcal{E}}_{\varphi} \) is \((r - 1)\)-near-point determined.

2. The image \( C_{\infty} \)-scheme \( \text{Im} \varphi := \varphi(X^{A_{k}}) \) of \( \varphi \) is \((r - 1)\)-near-point determined.

In particular, pointwise over \( X \), both \( \text{Supp}(\tilde{\mathcal{E}}_{\varphi}) \subset X \times Y \) and \( \text{Im} \varphi \subset Y \) are finite algebraic with respect to their respective point-determined reduced subschemes \( \text{Supp}(\tilde{\mathcal{E}}_{\varphi})_{\text{red}} \subset \text{Supp}(\tilde{\mathcal{E}}_{\varphi}) \) and \( (\text{Im} \varphi)_{\text{red}} \subset \text{Im} \varphi \).

Proof. Recall the commutative diagram of morphisms of \( C_{\infty} \)-schemes

\[
\begin{array}{ccc}
X_{\varphi} & \xrightarrow{f_{\varphi}} & Y \\
\downarrow \pi_{\varphi} & & \downarrow \text{pr}_{Y} \\
X & \xleftarrow{\text{pr}_{X}} & X \times Y
\end{array}
\]

underlying the smooth map \( \varphi \) and the built-in isomorphism

\[ \text{Supp}(\tilde{\mathcal{E}}_{\varphi}) \simeq X_{\varphi} \]

of \( C_{\infty} \)-schemes under \( \tilde{\varphi} \). We’ll show that

\[
\cdot \ C_{\infty}(X_{\varphi}) \text{ is } (r - 1)\text{-near-point determined}
\]

and hence prove Statement (1). Since \( \text{Im} \varphi = \text{Im}(f_{\varphi}) = \text{pr}_{Y}(\text{Supp}(\tilde{\mathcal{E}}_{\varphi})) \), Statement (2) follows from Statement (1).

Recall that, by construction, \( C_{\infty}(X_{\varphi}) = C_{\infty}(X) \langle \varphi^{\#}(\mathcal{O}_{Y}) \rangle \) and that there is a built-in sequence of \( \mathbb{R} \)-algebra-homomorphisms

\[ C_{\infty}(X) \subset C_{\infty}(X_{\varphi}) \subset C_{\infty}(\mathcal{O}_{X}^{A_{k}}) := C_{\infty}(\text{End}_{\mathcal{O}_{X}^{\underline{r}}} \mathcal{E}) \]

with the first inclusion a \( C_{\infty} \)-ring-monomorphism. Restrictions to all \( x \in X \) give then a sequence of \( \mathbb{R} \)-algebra-homomorphisms

\[ C_{\infty}(X_{\varphi}) \xrightarrow{\alpha} \prod_{x \in X} C_{\infty}(\pi_{\varphi}^{-1}(x)) \xrightarrow{\beta} \prod_{x \in X} \text{End}_{\mathbb{C}}(\mathcal{E}_{x}) \simeq \prod_{x \in X} M_{r \times r}(\mathbb{C}) \],
where $\mathcal{E} \simeq \mathbb{C}^r$ is the fiber of $\mathcal{E}$ at $x \in X$, $M_{r \times r}(\mathbb{C})$ is a $\mathbb{C}$-algebra of $r \times r$-matrices, and $\alpha$ is a $C^\infty$-ring-homomorphism. Again by construction, both $\beta$ and $\alpha \circ \beta$ are monomorphisms. This implies that $\alpha$ must also be a monomorphism. Since each $C^\infty(\pi^{-1}_\varphi(x))$, $x \in X$, is a finite-dimensional $\mathbb{R}$-algebra with the number of maximal ideals bounded uniformly by $r$ and all the residue field of these maximal ideals isomorphic to $\mathbb{R}$ (cf. [L-Y3: Sec. 3.2] (D(11.1))), each $C^\infty(\pi^{-1}_\varphi(x))$ is a direct product of Weil algebras. This proves that $C^\infty(X_\varphi)$ embeds into a direct product of Weil algebras and hence is near-point determined by definition. Since any nilpotent element $a \in M_{r \times r}(\mathbb{C})$ satisfies $a^r = 0$, $C^\infty(X_\varphi)$ must be then $(r - 1)$-near-point determined. This proves the lemma.

\[ \square \]

Remark 3.2. [from germs of local finite algebraic extension of $C^\infty(U)$]. Recall the local study in [L-Y3: Sec. 3, Sec. 5.1] (D(11.1)). Given a $C^\infty$-map $\varphi : (X^\mathbb{A}_r, \mathcal{E}) := (X, \mathcal{O}^\mathbb{A}_X := \text{End}_{\mathcal{O}^\mathbb{A}_X}(\mathcal{E}), \mathcal{E}) \to (Y, \mathcal{O}_Y)$ associated to a contravariant equivalence class $\varphi^\sharp : \mathcal{O}_Y \to \mathcal{O}^\mathbb{A}_X$ of gluing systems of $C^\infty$-admissible ring-homomorphisms over $\mathbb{R} \subset \mathbb{C}$ as in the Lemma, for $x \in X$ there exists a neighborhood $U$ of $x$ such that $\varphi(U)$ is contained in an open ball $V \subset Y$ diffeomorphic to $\mathbb{R}^n$ with coordinates $(y^1, \ldots, y^n)$. By shrinking $U$ if necessary, we may assume that $\mathcal{E}_U$ is trivial and trivialized by $\mathcal{E}_U \simeq \mathcal{O}_U^\mathbb{C}$. Then, locally over $U$ there is a surjection $C^\infty$-ring-homomorphism

\[
\frac{C^\infty(U \times V)}{(\det(y^1 \cdot \text{Id}_r - \varphi^\sharp(y^1)), \ldots, \det(y^n \cdot \text{Id}_r - \varphi^\sharp(y^n))} \twoheadrightarrow C^\infty(U_\varphi).
\]

Here $\text{Id}_r$ is the $r \times r$ identity matrix in $M_{r \times r}(C^\infty(U))$. Note that $\det(y^j \cdot \text{Id}_r - \varphi^\sharp(y^j)) \in C^\infty(U)[y^1, \ldots, y^n]$ for $i = 1, \ldots, n$. While the two rings

\[
\frac{C^\infty(U \times V)}{(\det(y^1 \cdot \text{Id}_r - \varphi^\sharp(y^1)), \ldots, \det(y^n \cdot \text{Id}_r - \varphi^\sharp(y^n))}
\]

versus

\[
\frac{C^\infty(U)[y^1, \ldots, y^n]}{(\det(y^1 \cdot \text{Id}_r - \varphi^\sharp(y^1)), \ldots, \det(y^n \cdot \text{Id}_r - \varphi^\sharp(y^n))}
\]

may not be isomorphic in general, it follows from the Malgrange Division Theorem ([Ma]; see also [Br]) and an induction on $n$ that they are isomorphic after passing to their respective $C^\infty$-ring of germs over $x$ (and, hence, are also isomorphic after passing to the formal neighborhood of $x \in U$). Lemma 3.1 can be proved also from this aspect.

A word on synthetic/$C^\infty$-algebraic symplectic/calibrated geometry

Remark 3.3. [synthetic/$C^\infty$-algebraic symplectic/calibrated geometry]. The lemma thus directs us to the following guiding question:

**Q. How should one enhance the current setting/notion of symplectic geometry and calibrated geometry so that near-point determined $C^\infty$-subschemes are naturally incorporated into it?** What is the notion of Fukaya(-Seidel) category in such an enhanced symplectic geometry and calibrated geometry?

(Cf. [Joy2], [Se1], [Se2].) This leads us to the new topic in synthetic/$C^\infty$-algebraic symplectic geometry and calibrated geometry. Figure 3-1; cf. [L-Y3: Sec. 7.2] (D(11.1)).
Figure 3-1. In a natural setting of the notion of a ‘special Lagrangian map’ for the current setting, a special Lagrangian map $\varphi$ from an Azumaya/matrix manifold with a fundamental module $(X^A, E)$ to a Calabi-Yau space $Y$ can have image $\text{Im} \varphi$ not only some usual special Lagrangian submanifolds (possibly with singularities) (cf. $\varphi_0$ and $\varphi_1$) but also “fuzzy” ones that carry some nilpotent structures (cf. $\varphi'_0$). Such special Lagrangian maps can deform among themselves as well (cf. $\varphi_1 \Rightarrow \varphi_0$ and $\varphi_1 \Rightarrow \varphi'_0$). Indicated in the illustration are also the corresponding $\varphi_* E$ with an associated filtration. From the target-space aspect, this suggests a notion of scheme-theoretic-like deformations of Lagrangian cycles with a generically flat sheaf/local system with singularities. This leads to the notion of a synthetic/$C^\infty$-algebraic symplectic or calibrated geometry.
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