Constants of motion in the dynamics of a 2N-junction SQUID

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Abstract

We show that a 2N junction SQUID (Superconducting QUantum Interference Device) made of 2N overdamped, shunted, identical junctions may be described as a system having only 6 degrees of freedom for any \( N \geq 3 \). This is achieved by means of the reduction introduced by Watanabe and Strogatz (Physica D, 74, (1994) 197) for series biased arrays. In our case 6 rather than 3 degrees of freedom are necessary to describe the system, due to the requirement of phase quantization along the superconducting loop constituting the device. Generalization to multijunction parallel arrays is straightforward.

1 Introduction

Two linear arrays, each containing \( N \) Josephson junctions, closed into a superconductor loop constitute a 2N-junction SQUID (Superconducting QUantum Interference Device), (see Fig.1). Such multijunction structures are of interest in designing high gain SQUIDs[1, 2, 3, 4]. The case with a single junction for each branch (\( N=1 \)) is the well studied case of a dc-SQUID[5, 6, 7]. Structures containing \( N=2 \), and \( N=3 \) junctions
for each branch, have also been fabricated and investigated in some detail\cite{1, 3}. Moreover, theoretical analysis based on numerical simulations of the 2N-junction SQUID has appeared too\cite{4}. The aim of this letter is to point out that, as for series arrays of overdamped Josephson junctions shunted by passive elements, a 2N-junction SQUID, made of overdamped (vanishing capacitance) identical junctions, exhibits a highly degenerate dynamics. In particular we will show that the transformation introduced by Watanabe and Strogatz (WS)\cite{8} may be fruitfully applied to the case considered here, with few modifications, to demonstrate the existence of $2N - 6$ constants of motion, i.e., that any trajectory in the phase space of a 2N SQUID is confined to a 6-dimensional subspace.

2 The 2N-junction SQUID model

The system we are interested in is sketched in Fig. 1 as a superconducting loop in which each of the two branches include N identical Josephson junctions. The behavior of the circuit is described by 2N dynamical variables, $\{\phi_k^{(a)}\}, \{\phi_k^{(b)}\} (k = 1, ...N)$ representing the gauge invariant phase difference across each junction for the branch (a) and (b), respectively. In the case of overdamped junctions, the governing equations are \cite{5, 8, 7}

$$\frac{\hbar}{2eR} \frac{1}{dt} d\phi_k^{(a)} + I_0 \sin \phi_k^{(a)} - \frac{I}{2} = J \quad (1)$$

$$\frac{\hbar}{2eR} \frac{1}{dt} d\phi_k^{(b)} + I_0 \sin \phi_k^{(b)} - \frac{I}{2} = -J \quad (2)$$

$R$ is the normal resistance, $I_0$ is the critical current, $I$ is the total bias current. The coupling terms $J$ and $-J$ coincides with the screening current, which is determined through the fluxoid quantization:

$$J = \frac{\Phi_0}{4\pi L} \left\{ \sum_{i=1}^{N} \phi_i^{(a)} - \sum_{i=1}^{N} \phi_i^{(b)} \right\} - \frac{\Phi_a}{2L}. \quad (3)$$

Here $\Phi_0$ is the flux quantum, $2L$ is the inductance of the loop, and $\Phi_a$ is the externally applied magnetic flux. The key observation is that the phases $\phi_k^{(a)}$ and $\phi_k^{(b)}$ are globally coupled through Eq. (3) and that for each subsystem, (a) and (b), this coupling is independent of the index $k$. This generalizes the case of globally coupled junction series arrays.
studied by WS [8]. Indeed, although our system is not globally coupled (there is a change of sign of the coupling term when passing from a branch to another), it can be split into two globally coupled sub-systems. Each subsystem contains N series connected junctions and satisfies the conditions for the applicability of the reduction introduced by WS. If we introduce the dimensionless variables

\[ t' = \frac{2\pi I_0 R}{\Phi_0} i \]
\[ i = \frac{I}{I_0} \]
\[ j = \frac{J}{I_0} \]
\[ \beta = \frac{2LI_0}{\Phi_0} \]
\[ \Phi_a' = \frac{\Phi_a}{\Phi_0} \]

Eq.s (1-3) take the form (dots denote derivative with respect to the normalized time):

\[ \dot{\phi}_k^{(a)} = \frac{i}{2} - j - \sin \phi_k^{(a)} \quad ; \quad k = 1, \ldots, N \] (4)
\[ \dot{\phi}_k^{(b)} = \frac{i}{2} + j - \sin \phi_k^{(b)} \quad ; \quad k = 1, \ldots, N \] (5)
\[ j = \frac{\sum_{k=1}^{N} \dot{\phi}_k^{(a)} - \sum_{k=1}^{N} \dot{\phi}_k^{(b)} - 2\pi \Phi_a'}{\pi \beta} \] (6)

Therefore we introduce a change of coordinates separately for each branch, as [8]:

\[ \tan \left[ \frac{1}{2} \left( \phi_k^{(a)}(t') - \Theta^{(a)}(t') \right) \right] = \left[ \frac{1 + \gamma^{(a)}(t')}{1 - \gamma^{(a)}(t')} \right] \tan \left[ \frac{1}{2} \left( \psi_k^{(a)} - \Psi^{(a)}(t') \right) \right] \quad k = 1, \ldots N \] (7)
\[ \tan \left[ \frac{1}{2} \left( \phi_k^{(b)}(t') - \Theta^{(b)}(t') \right) \right] = \left[ \frac{1 + \gamma^{(b)}(t')}{1 - \gamma^{(b)}(t')} \right] \tan \left[ \frac{1}{2} \left( \psi_k^{(b)} - \Psi^{(b)}(t') \right) \right] \quad k = 1, \ldots N. \] (8)

Here \( \Theta^{(a)}, \gamma^{(a)}, \Psi^{(a)}, \Theta^{(b)}, \gamma^{(b)}, \) and \( \Psi^{(b)}, \) are unknown functions of time \( (0 \leq \gamma^{(a)}, \gamma^{(b)} < 1), \) \( \psi_k^{(a)} \) and \( \psi_k^{(b)} \) are \( 2N \) constants. It is a straightforward exercise to prove that the transformations (7,8) reduce the 2N Eq.s (4-6), for every \( N \geq 3, \) to the following six equations:

\[ \dot{\gamma}^{(a)} = -(1 - \gamma^{(a)^2}) \cos \Theta^{(a)}, \] (9)
\[
\gamma^{(a)} \dot{\Psi}^{(a)} = \sqrt{1 - \gamma^{(a)^2}} \sin \Theta^{(a)}, \quad (10)
\]
\[
\gamma^{(a)} \dot{\Theta}^{(a)} = \gamma^{(a)} \left( \frac{i}{2} - j \right) + \sin \Theta^{(a)}, \quad (11)
\]
\[
\dot{\gamma}^{(b)} = -(1 - \gamma^{(b)^2}) \cos \Theta^{(b)}, \quad (12)
\]
\[
\gamma^{(b)} \dot{\Psi}^{(b)} = \sqrt{1 - \gamma^{(b)^2}} \sin \Theta^{(b)}, \quad (13)
\]
\[
\gamma^{(b)} \dot{\Theta}^{(b)} = \gamma^{(b)} \left( \frac{i}{2} + j \right) + \sin \Theta^{(b)}. \quad (14)
\]

Eqs (9-14) form a closed system for the six unknown functions, since \( j \) can be expressed in terms of the new variables inserting the change of coordinates (7,8) in Eq. (6). The 'frozen phases' (adopting the WS’s language) appear in the system merely as parameters. Some comments are in order. Firstly, the above system becomes singular if either \( \gamma^{(a)} \), or \( \gamma^{(b)} \) vanish. Moreover negative values for \( \gamma^{(a)} \), or \( \gamma^{(b)} \) seem not excluded, contrary to their definition. WS have shown that these troubles are not essential, but just an artifact of the chosen coordinate system. As a matter of fact they can be avoided by the additional change of coordinates:

\[
x^{(a)} = \gamma^{(a)} \cos \Theta^{(a)}, \quad (15)
\]
\[
y^{(a)} = \gamma^{(a)} \sin \Theta^{(a)}, \quad (16)
\]
\[
z^{(a)} = \Theta^{(a)} - \Psi^{(a)}, \quad (17)
\]
\[
x^{(b)} = \gamma^{(b)} \cos \Theta^{(b)}, \quad (18)
\]
\[
y^{(b)} = \gamma^{(b)} \sin \Theta^{(b)}, \quad (19)
\]
\[
z^{(b)} = \Theta^{(b)} - \Psi^{(b)} \quad (20)
\]

Thus the system, in terms of the new variables, becomes :

\[
\dot{x}^{(a)} = -1 + x^{(a)^2} - y^{(a)} \left( \frac{i}{2} - j \right), \quad (21)
\]
\[
\dot{y}^{(a)} = x^{(a)} y^{(a)} + x^{(a)} \left( \frac{i}{2} - j \right), \quad (22)
\]
\[
\dot{z}^{(a)} = \left( \frac{i}{2} - j \right) + \frac{1 - \sqrt{1 - x^{(a)^2} - y^{(a)^2}}}{x^{(a)^2} + y^{(a)^2}} y^{(a)}, \quad (23)
\]
\[
\dot{x}^{(b)} = -1 + x^{(b)^2} - y^{(b)} \left( \frac{i}{2} + j \right), \quad (24)
\]
\[ y^{(b)} = x^{(b)} y^{(b)} + x^{(b)} \left( \frac{i}{2} + j \right), \quad (25) \]
\[ z^{(b)} = \left( \frac{i}{2} + j \right) + \frac{1 - \sqrt{1 - x^{(b)2} - y^{(b)2}}}{x^{(b)2} + y^{(b)2}} y^{(b)} \quad (26) \]

which is well-behaved even at \( \gamma^{(a)} = 0 \), or \( \gamma^{(b)} = 0 \). The second point is about the initial conditions. Given \( 2N \) arbitrary initial junction phase values, the values for the \( 2N \) ‘frozen phases’ \( \psi^{(a), (b)}_k \) and the initial values for the six ‘reduced variables’ \( x^{(a), (b)}, y^{(a), (b)}, z^{(a), (b)} \) should be obtained from Eqs. (7, 8). This of course can be done in many ways: a simple and natural choice is the ‘identity transformation’:

\[ x^{(a)}(0) = y^{(a)}(0) = z^{(a)}(0) = x^{(b)}(0) = y^{(b)}(0) = z^{(b)}(0) = 0 \quad (27) \]

which straightforwardly gives:

\[ \psi^{(a), (b)}_k = \phi^{(a), (b)}_k(0) \quad (28) \]

The investigation of the meaning of such multiplicity of choice for the initial values is an important issue for the understanding of the dynamics of the system, but goes outside the scope of this letter. We just comment that, as (27) and (28) show, there is at least one way to effectively achieve the reduction.

We have numerically verified, for few values of the parameters, that the reduced system of six equations yields the same results of the originary system, Eqs. (4-6). This was checked with increasing number of junctions up to \( N = 20 \) junctions per branch. Care must be taken in reconstructing the phase values from the six unknown functions to avoid spurious \( 2\pi \) jumps in the solution.

### 3 Parallel arrays

The above property may be exploited, more generally, to a multijunction parallel array of \( M \) branches each branch containing \( N \) identical junctions (Fig. 2). Then Eqs. (1-2) can be generalized as follows:

\[ \frac{\hbar}{2eR} \frac{1}{dt} \phi^{(m)}_k + I_0 \sin \phi^{(m)}_k = I^{(m)} \quad (m = 1, 2, \ldots, M; \ k = 1, 2, \ldots, N) \quad (29) \]
where $I^{(m)}$ is the current in the $m-th$ branch, and $\phi_k^{(m)}$ is the phase of the $k$-th junction in the $m$-th branch. The $I^{(m)}$ sum, of course, to the total bias current $I$:

$$\sum_{l=1}^{M} I^{(l)} = I$$

(30)

Moreover the condition for the fluxoid quantization in each loop writes:

$$\sum_{i=1}^{N} \phi_i^{(j)} - \sum_{i=1}^{N} \phi_i^{(j+1)} = 2\pi \frac{\Phi_a}{\Phi_0} + 2\pi \sum_{l=1}^{M} c_l^{(j)} I^{(l)} \quad j = 1, \ldots, M - 1$$

(31)

Here $c_l^{(j)}$ is the mutual inductance coefficient accounting for the contribution to the magnetic flux at the loop $(j, j+1)$ due to the current flowing in the $l$-th branch. These coefficients are determined by the specific geometry of the circuit. Eqs. (30) and (31) are a set of linear equations for the unknown $I^{(m)}$, and allow a unique determination of their value as function of all the phase variables. In the simple case in which only the nearest neighbours branch currents contribute to the magnetic flux in each loop [9, 10], $c_l^{(j)}$ assume the simple form:

$$c_l^{(j)} = L (\delta_l^{(j+1)} - \delta_l^{(j)})$$

(32)

where $L$ is the inductance of each branch and $\delta_l^{(j)}$ is the Kronecker’s symbol. Under these simplifying hypotheses it is easy to show that:

$$I^{(m)} = \frac{I}{M} - \frac{\Phi_a}{L} \left( \frac{M+1}{2} - m \right) - \frac{\Phi_0}{2\pi L} \sum_{i=1}^{N} \phi_i^{(m)} + \frac{\Phi_0}{2\pi ML} \sum_{k=1}^{M} \sum_{i=1}^{N} \phi_i^{(k)} \quad m = 1, \ldots, M$$

(33)

The $N \times M$ Eqs. (29) and (33) describe completely a multi-junction parallel array, and, as in the case of the 2Nj-SQUID, it can be viewed as M sub-systems each one globally coupled to the rest of the system. The voltage $V$ developed across the circuit may be obtained by time derivating Eq.(33):

$$V \equiv \frac{\hbar}{2e} \sum_{i=1}^{N} d\phi_i^{(m)} \frac{dt}{dt} + L \frac{dI^{(m)}}{dt} = \frac{\hbar}{2eM} \sum_{k=1}^{M} \sum_{i=1}^{N} d\phi_i^{(k)}$$

(34)

It should be pointed out that the simplification introduced here (to neglect off diagonal terms of the mutual inductance matrix) is not essential to obtain a global coupling, i.e. also the more general case will lead qualitatively to the same kind of (global) coupling. As in
the previous discussion of the 2N-junction SQUID, it is therefore possible to introduce $M$ transformations analogous to (7,8) (one for each superconducting branch) and to conclude that only the following $3M$ equations ($m = 1...M$)

\begin{align}
\dot{x}^{(m)} &= -1 + x^{(m)2} - y^{(m)} \left( \frac{I^{(m)}}{I_o} \right), \\
\dot{y}^{(m)} &= x^{(m)} y^{(m)} + x^{(m)} \left( \frac{I^{(m)}}{I_o} \right), \\
\dot{z}^{(m)} &= \frac{I^{(m)}}{I_o} + \frac{1 - \sqrt{1 - x^{(m)2} - y^{(m)2}}}{x^{(m)2} + y^{(m)2}} y^{(m)}
\end{align}

should actually be solved for the $3M$ variables \{$x^{(m)}$, $y^{(m)}$, $z^{(m)}$\}, rather than $N \times M$. It should finally remarked that the all identical junction assumption may be partially relaxed. Indeed no particular troubles are met in the previous derivation if one assumes the junctions in a branch (or even their number) to be different from those of another branch. That is to say, the requirement of identical parameters concerns only the junctions in the same branch.

4 Conclusion

The possibility to reduce an $N \times M$ system to $3 \times M$ one is in itself of some advantage to simplify numerical simulations. Still we believe that the major achievement that we have reached is that the low dimensional motion for 'globally coupled' systems with sine nonlinearity can be extended also to sets of equations that are not 'globally coupled' in a strict sense, but can be divided in subsystems exhibiting global coupling (at the price to increase the number of dynamical variables necessary to describe the system). So we have explicitly proved that phase quantization in superconducting loops has no other effect that to introduce 3 more variables for each loop. Beside the extension of the theory there is also a consequence of practical importance: neutral stability is a drawback for practical applications (as local oscillators) because of the consequent deterioration of the linewidth of the emitted microwave. Our findings can be interpreted as a criterion for the maximum
number of junctions that can be inserted in a loop: above 3 the system undergoes a reduction similar to that illustrated in this letter. It should be noted that this criterion is not a positive one, i.e., there might be configuration that are neutrally stable but do not contain more than 3 junctions per loop [11, 12]. At this stage it is not known to which extent the formal analogy between the system studied in Ref. [8] may be carried on. Indeed, there are differences in the actual form of the coupling term that will lead to substantial differences: for instance Eq.s (4-6) do not exhibit the important property of reversibility (changing \(\phi_{k}^{(m)}\) in \(-\phi_{k}^{(m)}\) and \(t'\) in \(-t'\) does not leave unchanged the equations). This will lead to profound effects on the dynamics of the system.

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Figure Captions

Fig.1 Schematic circuit model for a 2N-junction SQUID;

Fig.2 Schematic circuit model for a multi-junction parallel array;