A set of the Viète-like recurrence relations for the unity constant

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Abstract

Using a simple Viète-like formula for $\pi$ based on the nested radicals $a_k = \sqrt{2 + a_{k-1}}$ and $a_1 = \sqrt{2}$, we derive a set of the recurrence relations for the constant 1. Computational test shows that application of this set of the Viète-like recurrence relations results in a rapid convergence to unity.

Keywords: arctangent function, constant $\pi$, constant 1

1 Description and implementation

1.1 Derivation

Several centuries ago the French mathematician François Viète derived a remarkable formula for $\pi$

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2 + \sqrt{2}}}{2} \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots \tag{1}$$

Nowadays this well-known equation is commonly regarded as the Viète’s formula for $\pi$ [1, 2, 3, 4]. The uniqueness of this formula is due to nested

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radicals consisting of square roots of twos only. Defining these nested radicals as

\[ a_1 = \sqrt{2}, \]
\[ a_2 = \sqrt{2 + \sqrt{2}}, \]
\[ a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \]
\[ \vdots \]
\[ a_k = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} \]

the Viète’s formula \([1]\) for \( \pi \) can be rewritten in a compact form as follows

\[ \frac{2}{\pi} = \lim_{k \to \infty} K \prod_{k=1}^{K} \frac{a_k}{2}. \]

There is a simple Viète-like formula for \( \pi \) that can be represented in form \([5]\)

\[ \frac{\pi}{2k+1} = \arctan \left( \frac{\sqrt{2} - a_{k-1}}{a_k} \right), \quad k \geq 2, \quad (2) \]

From this formula it follows that

\[ \frac{\pi}{2^3} + \frac{\pi}{2^4} + \frac{\pi}{2^5} + \cdots = \arctan \left( \frac{\sqrt{2} - a_1}{a_2} \right) + \arctan \left( \frac{\sqrt{2} - a_2}{a_3} \right) + \arctan \left( \frac{\sqrt{2} - a_3}{a_4} \right) + \cdots \quad (3) \]

and because of the decreasing geometric series

\[ \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots = \frac{1}{4} \]

the equation \([3]\) can be expressed in a more simplified form

\[ \frac{\pi}{4} = \lim_{K \to \infty} \sum_{k=1}^{K} \arctan \left( \frac{\sqrt{2} - a_k}{a_{k+1}} \right). \quad (4) \]
It is more convenient for our purpose to represent the equation (4) as

\[
\frac{\pi}{4} = \arctan \left( \frac{\sqrt{2} - \sqrt{2}}{\sqrt{2} + \sqrt{2}} \right) + \arctan \left( \frac{\sqrt{2} + \sqrt{2}}{\sqrt{2} + \sqrt{2}} \right) + \arctan \left( \frac{\sqrt{2} - \sqrt{2} + \sqrt{2}}{\sqrt{2} + \sqrt{2} + \sqrt{2}} \right) + \cdots
\]

or

\[
\frac{\pi}{4} = \arctan (b_1) + \arctan (b_2) + \arctan (b_3) \cdots
\]

\[
= \lim_{{K \to \infty}} \sum_{{k=1}}^{K} \arctan (b_k),
\]

where the arguments of the arctangent functions can be found by using the recurrence relations

\[
b_k = \frac{\sqrt{2} - a_k}{a_{k+1}}
\]

and

\[
a_k = \sqrt{2} + a_{k-1}, \quad a_1 = \sqrt{2}.
\]

Since

\[
\arctan (1) = \frac{\pi}{4}
\]

we can also write

\[
\arctan (1) = \lim_{{K \to \infty}} \sum_{{k=1}}^{K} \arctan (b_k). \quad (5)
\]

The right side of the equation (5) consists of the infinite summation terms of the arctangent functions. We may attempt to exclude the infinite sum using the identity

\[
arctan (x) + \arctan (y) = \arctan \left( \frac{x + y}{1 - xy} \right) \quad (6)
\]

repeatedly. Specifically, we employ the following recurrence relations that just reflects the successive application of the identity (6) above

\[
c_k = \frac{c_{k-1} + b_k}{1 - c_{k-1} b_k}, \quad c_1 = b_1.
\]
This enables us to rewrite the equation (5) as

\[
\arctan (1) = \arctan (c_k) + \lim_{L \to \infty} \sum_{\ell=k+1}^{L} \arctan (b_\ell).
\] (7)

According to the Maclaurin expansion series

\[
\arctan (b_\ell) = b_\ell - \frac{b_\ell^3}{3} + \frac{b_\ell^5}{5} - \frac{b_\ell^7}{7} + \cdots = b_\ell + O (b_\ell^3).
\]

Since at \( \ell \to \infty \) the variable \( b_\ell \to 0 \) and, therefore, due to negligible \( O (b_\ell^3) \) we can simply replace it by \( \arctan (b_\ell) \) and then use the equation (2) in order to find a ratio of the limit

\[
\lim_{\ell \to \infty} \frac{b_{\ell+1}}{b_\ell} = \lim_{\ell \to \infty} \frac{\arctan (b_{\ell+1})}{\arctan (b_\ell)} = \lim_{\ell \to \infty} \frac{\pi/2^{\ell+2}}{\pi/2^{\ell+1}} = \frac{1}{2}.
\] (8)

Consider the following infinite sequence

\[
\{ b_1, b_2, b_3, \ldots, b_\ell, \ldots \}.
\] (9)

According to the limit (8) the ratio \( b_{\ell+1}/b_\ell \) tends to \( \frac{1}{2} \) with increasing index \( \ell \). Consequently, it is not difficult to see now that

\[
\frac{b_2}{b_1} < \frac{b_3}{b_2} < \frac{b_4}{b_3} < \cdots < \frac{b_{\ell+1}}{b_\ell} < \cdots < \frac{1}{2}.
\]

In fact, the tendency of the ratio \( b_{\ell+1}/b_\ell \) towards \( 1/2 \) with increasing index \( \ell \) is very fast. In particular, when the index \( \ell \) is large enough, say at \( \ell > 10 \), the sequence (9) behaves almost like a decreasing geometric progression where a common ratio is \( 1/2 \).

Since the index \( k \) in the equation (7) can be taken arbitrarily large, we can rewrite it in form

\[
\arctan (1) = \lim_{k \to \infty} \left[ \arctan (c_k) + \lim_{L \to \infty} \sum_{\ell=k+1}^{L} b_\ell \right].
\] (10)

Taking into account that the ratio \( b_{\ell+1}/b_\ell \) tends to but never exceeds \( 1/2 \), we can conclude that the damping rate in the sequence (9) is faster than that of in a decreasing geometric progression

\[
\{ b_1, \frac{b_1}{2}, \frac{b_1}{2^2}, \frac{b_1}{2^3}, \ldots, \frac{b_1}{2^\ell}, \ldots \}
\]
with fixed common ratio \(1/2\). This signifies that
\[
\sum_{\ell=k+1}^{L} b_\ell < \sum_{\ell=k+1}^{L} \frac{b_1}{2^{\ell-1}}, \quad L > k > 0,
\]
and since the limit of the decreasing geometric series
\[
\lim_{L \to \infty} \sum_{\ell=k+1}^{L} \frac{b_1}{2^{\ell-1}} \to 0, \quad k \to \infty,
\]
we prove that
\[
\lim_{L \to \infty} \sum_{\ell=k+1}^{L} b_\ell \to 0, \quad k \to \infty.
\]
As a consequence, the equation (10) can be further simplified as
\[
\arctan(1) = \lim_{k \to \infty} \arctan \left( \frac{c_k}{k} \right) \Leftrightarrow 1 = \lim_{k \to \infty} c_k.
\]
Thus, we can infer that the constant 1 can be approached successively by increment of the index \(k\) in a set of the Viète-like recurrence relations
\[
\begin{align*}
a_1 &= \sqrt{2}, \\
a_k &= \sqrt{2 + a_{k-1}}, \\
b_k &= \frac{\sqrt{2 - a_k}}{a_{k+1}}, \\
c_1 &= b_1, \\
c_k &= \frac{c_{k-1} + b_k}{1 - c_{k-1}b_k},
\end{align*}
\]
such that \(c_{k \to \infty} \to 1\).

1.2 Computation
Consider the first three elements from the sequence (9)
\[
b_1 = \frac{\sqrt{2 - a_1}}{a_2} = \frac{\sqrt{2 - \sqrt{2}}}{\sqrt{2} + \sqrt{2}},
\]
\[ b_2 = \frac{\sqrt{2} - a_2}{a_3} = \frac{\sqrt{2} - \sqrt{2 + \sqrt{2}}}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}} \]

and

\[ b_3 = \frac{\sqrt{2} - a_3}{a_4} = \frac{\sqrt{2} - \sqrt{2 + \sqrt{2 + \sqrt{2}}} + \sqrt{2} - \sqrt{2 + \sqrt{2 + \sqrt{2}} + \sqrt{2}}}{\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}. \]

Consequently, the corresponding first three values of the variable \( c_k \) are

\[ c_1 = b_1 = \frac{\sqrt{2} - \sqrt{2}}{\sqrt{2} + \sqrt{2}} = 0.41421356237309504880\ldots, \]

\[ c_2 = \frac{c_1 + b_2}{1 - c_1 b_2} = \frac{\sqrt{2} - \sqrt{2} + \sqrt{2 - \sqrt{2} + \sqrt{2}}}{\sqrt{2 + \sqrt{2} + \sqrt{2}}} = 0.66817863791929891999\ldots \]

and

\[ c_3 = \frac{c_2 + b_3}{1 - c_2 b_3} = \frac{\sqrt{2} - \sqrt{2} + \sqrt{2 - \sqrt{2} + \sqrt{2}}}{\sqrt{2 + \sqrt{2} + \sqrt{2}}} = 0.82067879082866033097\ldots, \]

respectively.

From these examples one can see that the set (11) of the Viète-like recurrence relations gradually builds the continued fractions in the numerator and denominator of the variable \( c_k \) at each successive step in increment of the index \( k \). It is also interesting to note that each value of the variable \( c_k \) is based on nested radicals consisting of square roots of twos only.

Figure 1 shows the dependence of the variables \( a_k, b_k \) and \( c_k \) as a function of the index \( k \) by blue, green and red colors, respectively. We can observe
Fig. 1. Evolution of the variables $a_k$ (blue), $b_k$ (green) and $c_k$ (red).

how the variable $c_k$ tends to 1 while the variables $a_k$ and $b_k$ tend to 2 and 0, respectively.

Table 1 shows the values of variable $c_k$ and error term $\varepsilon_k = 1 - c_k$ with corresponding index $k$ ranging from 4 to 15. As we can see from this table, the variable $c_k$ quite rapidly tends to unity with increasing index $k$. In particular, the error term $\varepsilon_k$ decreases by factor of about 2 at each increment of the index $k$ by one.

Table 1. The variable $c_k$ and error term $\varepsilon_k$ at index $k$ ranging from 4 to 15.

| $k$ | $c_k$ | $\varepsilon_k$ |
|-----|-------|-----------------|
| 4   | 0.90634716901914715794... | 0.09365283098085284205... |
| 5   | 0.95207914670092534858... | 0.04792085329907465141... |
| 6   | 0.9755264993237653232... | 0.02424735006762346767... |
| 7   | 0.98780284145152917070... | 0.01219715854847082929... |
| 8   | 0.99388282491415211156... | 0.00611717508584788843... |
| 9   | 0.99603673501114949604... | 0.00306326498885050395... |
| 10  | 0.99846719455859369106... | 0.00153280544140630893... |
| 11  | 0.99923303559286120490... | 0.0007669640713879509... |
| 12  | 0.99961657831851611515... | 0.00038342168148388484... |
| 13  | 0.9998082707827333526... | 0.0001917292172666473... |
| 14  | 0.99990413079635610519... | 0.00009586920364389480... |
| 15  | 0.99995206424931502866... | 0.00004793575068497133... |
2 New formula for pi

As the error term $\varepsilon_k$ decreases successively by factor of about 2 (see third column in the Table 1), we may expect that $2^k \varepsilon_k$ is convergent and tends to some constant when the index $k$ tends to infinity. The computational test shows that the value $2^k \varepsilon_k$ approaches to $\pi/2$ as the index $k$ increases. Therefore, we assume that

$$\lim_{k \to \infty} 2^k \varepsilon_k = \frac{\pi}{2}$$

or

$$\pi = \lim_{k \to \infty} 2^{k+1} (1 - c_k).$$

Furthermore, relying on numerical results we also suggest a generalization to the power $m$ as given by

$$m \pi = \lim_{k \to \infty} 2^{k+1} (1 - c_k^m). \quad (12)$$

Since the variable $c_k$ is determined within the set (11) of the Viète-like recurrence relations, the new equation (12) can also be regarded as the Viète-like formula for pi.

3 Conclusion

We show a set (11) of the Viète-like recurrence relations for the constant 1 derived by using the Viète-like formula (2) for pi. Sample computations reveal that the variable $c_k$ quite rapidly tends to unity as the index $k$ increases.

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