NON-MEAGER FREE SETS FOR MEAGER RELATIONS
ON POLISH SPACES

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Abstract. We prove that for each meager relation $E \subset X \times X$ on a Polish space $X$ there is a nowhere meager subspace $F \subset X$ which is $E$-free in the sense that $(x, y) \notin E$ for any distinct points $x, y \in F$.

1. Introduction

This paper is devoted to the problem of finding non-meager free subsets for meager relations on Polish spaces. For a relation $E \subset X \times X$, a subset $F \subset X$ is called $E$-free if $(x, y) \notin E$ for any distinct points $x, y \in F$. This is equivalent to saying that $F^2 \cap E \subset \Delta_X$ where $\Delta_X = \{(x, y) \in X^2 : x = y\}$ is the diagonal of $X^2$.

The problem of finding “large” free sets for certain “small” relations was considered by many authors; see [10], [11], [9], [6], [7]. Observe that the classical Mycielski-Kuratowski Theorem [8, 18.1] implies that for each meager relation $E \subset X^2$ on a perfect Polish space $X$ there is an $E$-free perfect subset $F \subset X$. We recall that a non-empty subset of a Polish space is perfect if it is closed and has no isolated points. Nonetheless, the following result seems to be new.

Theorem 1. For each meager relation $E \subset X^2$ on a Polish space $X$ there is an $E$-free nowhere meager subspace $F \subset X$. Moreover, if the set of isolated points is not dense in $X$, then $F$ may be chosen of any cardinality $\kappa \in [\text{cof}(\mathcal{M}), \mathfrak{c}]$.

Let us recall that a subspace $A$ of a topological space $X$

- is meager in $X$, if $A$ can be written as a countable union $A = \bigcup_{n \in \omega} A_n$ of nowhere dense subsets of $X$;
- is nowhere meager in $X$, if for any non-empty open set $U \subset X$ the intersection $U \cap A$ is not meager in $X$.

It is clear that a subset $A \subset X$ of a Polish space $X$ is nowhere meager if and only if $A$ is dense in $X$ and contains no open meager subspace. By definition, $\text{cof}(\mathcal{M})$ is the minimal cardinality of a collection $\mathcal{X}$ of meager subsets of the Baire space $\omega^\omega$ such that for every meager $A \subset \omega^\omega$ there exists $X \in \mathcal{X}$ containing $A$. It is known [5] that $\text{cof}(\mathcal{M}) = \mathfrak{c}$ under Martin’s Axiom, and $\text{cof}(\mathcal{M}) < \mathfrak{c}$ in some models of ZFC; see [4].

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Theorem 1 will be proved in Section 3. One of its applications is the existence of a first-countable uniform Eberlein compact space which is not supercompact (see [10, 5.2]), which was our initial motivation for considering free non-meager sets for meager relations. The following simple example shows that the nowhere meager set $F$ in Theorem 1 cannot have the Baire property. We recall that a subset $A$ of a topological space $X$ has the Baire property in $X$ if for some open set $U \subset X$ the symmetric difference $A \triangle U = (A \setminus U) \cup (U \setminus A)$ is meager in $X$.

**Example 2.** For the nowhere dense relation

$$E = \bigcup_{n \in \omega} \{(x, y) \in \mathbb{R}^2 : |x - y| = 2^{-n}\} \subset \mathbb{R} \times \mathbb{R}$$

on the real line $\mathbb{R}$, each $E$-free subset $F \subset \mathbb{R}$ with the Baire property is meager.

**Proof.** Assuming that $F$ is not meager, and using the Baire property of $F$, find a non-empty open subset $U \subset \mathbb{R}$ such that $U \setminus F$ is meager and hence lies in some meager $F_\sigma$-set $M \subset \mathbb{R}$. Then $G = U \setminus M \subset F$ is a dense $G_\delta$-set in $U$. By the Steinhaus-Pettis Theorem [8, 9.9], the difference $G - G = \{x - y : x, y \in G\}$ is a neighborhood of zero in $\mathbb{R}$ and hence $2^{-n} \in G - G$ for some $n \in \omega$. Then any points $x, y \in G \cap F$ with $|x - y| = 2^{-n}$ witness that the set $F \ni x, y$ is not $E$-free. □

**Remark 3.** By a classical result of Solovay [11], there are models of ZF in which all subsets of the real line have the Baire property. In such models each $E$-free subset for the relation $E = \bigcup_{n \in \omega} \{(x, y) \in \mathbb{R}^2 : |x - y| = 2^{-n}\}$ is meager. This means that the proof of Theorem 1 must essentially use the Axiom of Choice.

2. **Some auxiliary results**

We recall [2] that a family $\mathcal{F}$ of infinite subsets of a countable set $X$ is called a semifilter, if $A \in \mathcal{F}$ provided $F \subset^* A \subset X$ for some set $F \in \mathcal{F}$. Here $F \subset^* A$ means that $F \setminus A$ is finite. Each semifilter on $X$ is contained in the semifilter $[X]^\omega$ of all infinite subsets of $X$. The semifilter $[X]^\omega$ is a subset of the power set $\mathcal{P}(X)$ which can be identified with the Tychonoff product $2^X$ via characteristic functions. So, we can speak about topological properties of semifilters as subspaces of the compact Hausdorff space $\mathcal{P}(X)$. According to Talagrand’s characterization [13] of meager semifilters on $\omega$, a semifilter $\mathcal{F}$ on a countable set $X$ is meager (as a subset of $\mathcal{P}(X)$) if and only if $\mathcal{F}$ can be enlarged to a $\sigma$-compact semifilter $\overline{\mathcal{F}} \subset [X]^\omega$. This characterization implies the following:

**Corollary 4.** For any finite-to-one map $\phi : X \to Y$ between countable sets, a semifilter $\mathcal{F} \subset \mathcal{P}(X)$ is meager if and only if the semifilter $\phi[\mathcal{F}] = \{E \subset Y : \phi^{-1}(E) \in \mathcal{F}\} \subset \mathcal{P}(Y)$ is meager.

We recall that a map $f : X \to Y$ between two sets is called finite-to-one if for each $y \in Y$ the preimage $\psi^{-1}(y)$ is finite and non-empty. In particular, each monotone surjection $\psi : \omega \to \omega$ is finite-to-one.

A key ingredient of the proof of Theorem 1 is the following proposition.

**Proposition 5.** For any meager relation $E \subset 2^\omega \times 2^\omega$ on the Cantor cube $2^\omega$ there is a family $(G_\alpha)_{\alpha < \lambda}$ of nowhere meager subsets in $2^\omega$ such that $(G_\alpha \times G_\beta) \cap E = \emptyset$ for any distinct ordinals $\alpha, \beta < \lambda$.

**Proof.** An indexed family $(A_\alpha)_{\alpha < \lambda}$ of infinite subsets of $\omega$ is called almost disjoint if $A_\alpha \cap A_\beta$ is finite for any distinct ordinals $\alpha, \beta < \lambda$. 

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Lemma 6. There are a finite-to-one map \( \varphi : \omega \to \omega \) and an almost disjoint family \((A_\alpha)_{\alpha < \kappa} \) of infinite subsets of \( \omega \) such that \( \varphi(A_\alpha) = \omega \) for all \( \alpha < \kappa \).

Proof. Let \( \xi : \omega \to 2^{<\omega} \) be any bijection of \( \omega \) onto the binary tree \( 2^{<\omega} = \bigcup_{n \in \omega} 2^n \).
Let \( \ell : 2^{<\omega} \to \omega \) be the level map assigning to each finite sequence \( s \in 2^{<\omega} \) the unique number \( n \in \omega \) such that \( s \in 2^n \). It is clear that the map \( \ell : 2^{<\omega} \to \omega \) is finite-to-one and so is the composition \( \varphi = \ell \circ \xi : \omega \to \omega \).

The family \( B \) of all branches (i.e., maximal linearly ordered subsets) of the tree \( 2^{<\omega} \) is almost disjoint and has cardinality \( \kappa \). So, it admits an (injective) enumeration \( B = \{ B_\alpha \}_{\alpha < \kappa} \) such that the indexed family \( (B_\alpha)_{\alpha < \kappa} \) is almost disjoint.

For every \( \alpha < \kappa \) put \( A_\alpha = \xi^{-1}(B_\alpha) \) and observe that \( (A_\alpha)_{\alpha < \kappa} \) is a required almost disjoint family of infinite subsets of \( \omega \) with \( \varphi(A_\alpha) = \ell(B_\alpha) = \omega \) for all \( \alpha < \kappa \). \( \square \)

Using Lemma 6 fix a finite-to-one surjection \( \varphi : \omega \to \omega \) and an almost disjoint family \( (A_\alpha)_{\alpha < \kappa} \) of infinite subsets of \( \omega \) such that \( \varphi(A_\alpha) = \omega \) for all \( \alpha < \kappa \).

Next, fix any free ultrafilter \( U \) on \( \omega \) and for every \( \alpha < \kappa \) choose an ultrafilter \( U_\alpha \) on \( \omega \) extending the family \( \{ A_\alpha \cap \varphi^{-1}[U] : U \in U \} \). The almost disjoint property of the family \( (A_\alpha)_{\alpha < \kappa} \) guarantees that \( \omega \setminus A_\alpha \in U_\xi \) for any distinct ordinals \( \alpha, \xi < \kappa \).

Lemma 7. For every \( \alpha < \kappa \), the filter
\[
F_\alpha = \mathcal{P}(\omega \setminus A_\alpha) \cap \bigcap_{\alpha \neq \xi < \kappa} U_\xi
\]
is non-meager in \( \mathcal{P}(\omega \setminus A_\alpha) \).

Proof. By Corollary 4, the filter \( F_\alpha \) is not meager in \( \mathcal{P}(\omega \setminus A_\alpha) \) as its image \( \varphi[F_\alpha] = \{ E \in \omega : \varphi^{-1}[E] \in F_\alpha \} \) coincides with the ultrafilter \( U \) and hence is not meager in \( \mathcal{P}(\omega) \). \( \square \)

Let \( E \subset 2^\omega \times 2^\omega \) be a meager relation on \( 2^\omega \). By [3, Theorem 2.2.4], there exist a monotone surjection \( \phi : \omega \to \omega \) and functions \( f_0, f_1 : \omega \to 2 \) such that
\[
E \subset \{(g, g') \in 2^\omega \times 2^\omega : \forall n \in \omega \, (g \upharpoonright \phi^{-1}(n) \neq f_0 \upharpoonright \phi^{-1}(n)) \}
\]

For every ordinal \( \alpha < \kappa \) consider the subset
\[
G_\alpha = \{ g \in 2^\omega : \exists X_0, X_1 \in U_\alpha \setminus \bigcup_{\alpha \neq \xi < \kappa} U_\xi \}
\]
\[
\left( X_0 \subset X_1 \right) \wedge (g \upharpoonright \phi^{-1}[X_0] = f_0 \upharpoonright \phi^{-1}[X_0])
\]
\[
\wedge (g \upharpoonright \phi^{-1}[\omega \setminus X_1] = f_1 \upharpoonright \phi^{-1}[\omega \setminus X_1]) \}
\]
in the Cantor cube \( 2^\omega \).

Lemma 8. For every ordinal \( \alpha < \kappa \) the set \( G_\alpha \) is nowhere meager in \( 2^\omega \).

Proof. Since \( G_\alpha \) is closed under finite modifications of its elements, it is enough to show that \( G_\alpha \) is non-meager in \( 2^\omega \). Observe that \( G_\alpha \) contains the set
\[
G'_\alpha = \{ g \in 2^\omega : \exists Y_0 \in U_\alpha \cap \mathcal{P}(A_\alpha) \, \exists Y_1 \in \mathcal{P}(\omega \setminus A_\alpha) \setminus \bigcup_{\alpha \neq \xi < \kappa} U_\xi \}
\]
\[
\left( g \upharpoonright \phi^{-1}[Y_0] = f_0 \upharpoonright \phi^{-1}[Y_0]) \right) \wedge (g \upharpoonright \phi^{-1}[\omega \setminus (A_\alpha \cup Y_1])
\]
\[
= f_1 \upharpoonright \phi^{-1}[\omega \setminus (A_\alpha \cup Y_1)] \}
\]
Indeed, if \( g \in G'_\alpha \) is witnessed by \( Y_0, Y_1 \), then \( X_0 = Y_0 \) and \( X_1 = A_\alpha \cup Y_1 \) are witnessing that \( g \in G_\alpha \). Now \( G'_\alpha \) may be written as the product \( R_\alpha \times H_\alpha \), where

\[
R_\alpha = \{ g \in 2^{\phi^{-1}[A_\alpha]} : \exists Y_0 \in U_\alpha \cap \mathcal{P}(A_\alpha) \ (g \upharpoonright \phi^{-1}[Y_0] = \emptyset \upharpoonright \phi^{-1}[Y_0]) \}
\]

and

\[
H_\alpha = \{ g \in 2^{\phi^{-1}[\omega \setminus A_\alpha]} : \exists Y_1 \in \mathcal{P}(\omega \setminus A_\alpha) \setminus \bigcup_{\alpha \neq \xi < \zeta} U_\xi \ (g \upharpoonright \phi^{-1}[\omega \setminus (A_\alpha \cup Y_1)] = \emptyset \upharpoonright \phi^{-1}[\omega \setminus (A_\alpha \cup Y_1)]) \}.
\]

Thus it suffices to show that both \( R_\alpha \) and \( H_\alpha \) are non-meager. By the homogeneity of \( 2^\omega \) there is no loss of generality to assume that \( \emptyset \upharpoonright \phi^{-1}[A_\alpha] \equiv 1 \) and \( \emptyset \upharpoonright \phi^{-1}[\omega \setminus A_\alpha] \equiv 1 \).

With \( f_1 \) as above we see that \( H_\alpha \) is simply the set of characteristic functions of elements of the semifilter

\[
\mathcal{H}_\alpha = \{ Z \subset \phi^{-1}[\omega \setminus A_\alpha] : \exists Y_1 \in \mathcal{P}(\omega \setminus A_\alpha) \setminus \bigcup_{\alpha \neq \xi < \zeta} U_\xi \ (\phi^{-1}[\omega \setminus (A_\alpha \cup Y_1)] \subset Z) \}
\]
on \( \phi^{-1}[\omega \setminus A_\alpha] \). Therefore

\[
\phi[\mathcal{H}_\alpha] = \{ T \subset \omega \setminus A_\alpha : \exists Y_1 \in \mathcal{P}(\omega \setminus A_\alpha) \setminus \bigcup_{\alpha \neq \xi < \zeta} U_\xi \ (\omega \setminus (A_\alpha \cup Y_1) \subset T) \}.
\]

Observe that \( Y_1 \in \mathcal{P}(\omega \setminus A_\alpha) \setminus \bigcup_{\alpha \neq \xi < \zeta} U_\xi \) iff \( \omega \setminus (A_\alpha \cup Y_1) \in \bigcap_{\alpha \neq \xi < \zeta} U_\xi \), and hence \( \phi[\mathcal{H}_\alpha] \) is equal to the filter \( \mathcal{P}(\omega \setminus A_\alpha) \cap \bigcap_{\alpha \neq \xi < \zeta} U_\xi \) which is non-meager in \( \mathcal{P}(\omega \setminus A_\alpha) \) by Lemma \( \Box \) and consequently the filter \( \mathcal{H}_\alpha \) is non-meager in \( \mathcal{P}(\phi^{-1}[\omega \setminus A_\alpha]) \) by Corollary \( \Box \).

In other words, \( H_\alpha \) is a non-meager subset of \( 2^{\phi^{-1}[\omega \setminus A_\alpha]} \).

The proof of the fact that \( R_\alpha \) is non-meager is analogous. However, we present it for the sake of completeness. With \( f_0 \) as above we see that \( R_\alpha \) is simply the set of characteristic functions of elements of the semifilter

\[
\mathcal{R}_\alpha = \{ Z \subset \phi^{-1}[A_\alpha] : \exists Y_0 \in \mathcal{P}(A_\alpha) \cap \mathcal{U}_\alpha \ (\phi^{-1}[Y_0] \subset Z) \}
\]
on \( \phi^{-1}[A_\alpha] \). It follows that

\[
\phi[\mathcal{R}_\alpha] = \{ T \subset A_\alpha : \exists Y_0 \in \mathcal{P}(A_\alpha) \cap \mathcal{U}_\alpha \ (Y_0 \subset T) \} = \mathcal{P}(A_\alpha) \cap \mathcal{U}_\alpha
\]
is a non-meager ultrafilter on \( A_\alpha \), and hence \( \mathcal{R}_\alpha \) is a non-meager semifilter on \( \phi^{-1}[A_\alpha] \) according to Corollary \( \Box \). Consequently, \( R_\alpha \) is a non-meager subset of \( 2^{\phi^{-1}[A_\alpha]} \). \( \Box \)

**Lemma 9.** For any distinct ordinals \( \alpha, \beta < \epsilon \) we get \( (G_\alpha \times G_\beta) \cap E = \emptyset \).

**Proof.** Fix any \((g_\alpha, g_\beta) \in G_\alpha \times G_\beta\) and find some pair \((g_\alpha, g_\beta)\). Fix sets \( X_0^\alpha, X_1^\alpha \) and \( X_0^\beta, X_1^\beta \) witnessing that \( g_\alpha \in G_\alpha \) and \( g_\beta \in G_\beta \), respectively. The intersection \( X_0^\alpha \cap (\omega \setminus X_1^\beta) \) is infinite: otherwise \( X_0^\alpha \subset^* X_1^\beta \) and \( X_1^\beta \in \mathcal{U}_\alpha \), which contradicts the definition of \( G_\beta \). Thus the set \( X_0^\alpha \setminus X_1^\beta \) is infinite and for every \( n \in X_0^\alpha \setminus X_1^\beta \) we get \( g_\alpha \upharpoonright \phi^{-1}(n) = f_0 \upharpoonright \phi^{-1}(n) \) and \( g_\beta \upharpoonright \phi^{-1}(n) = f_1 \upharpoonright \phi^{-1}(n) \), which implies \((g_\alpha, g_\beta) \notin E\). \( \Box \)

This completes the proof of Proposition \( \Box \)

Using the well-known fact that each perfect Polish space \( X \) contains a dense \( G_\delta \)-subset homeomorphic to the space of irrationals \( \omega^\omega \), we can generalize Proposition \( \Box \) as follows.

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Proposition 10. For any meager relation $E \subset X \times X$ on a perfect Polish space $X$ there is a family $(G_\alpha)_{\alpha < \kappa}$ of nowhere meager subsets in $X$ such that $(G_\alpha \times G_\beta) \cap E = \emptyset$ for any distinct ordinals $\alpha, \beta < \kappa$.

3. Proof of Theorem

Let $E \subset X \times X$ be a meager relation on a Polish space $X$. If the set $D$ of isolated points is dense in $X$, then $F = D$ is a required nowhere meager $E$-free subset of $X$. So, we assume that the set $D$ is not dense in $X$. Then the open subspace $Y = X \setminus \overline{D}$ of $X$ is not empty and has no isolated points. Let $\kappa \in [\text{cof}(\mathcal{M}), \mathfrak{c}]$ be any cardinal. By Proposition 10 there is a family $(G_\alpha)_{\alpha < \kappa}$ of nowhere meager subsets in $Y$ such that $(G_\alpha \times G_\beta) \cap E = \emptyset$ for any distinct ordinals $\alpha, \beta < \kappa$.

Let $\mathcal{U}$ be a countable base of the topology of $Y$ and $\mathcal{X}$ be a cofinal with respect to inclusion family of meager subsets in $Y$ of size $\kappa$. It is clear that the set $\mathcal{U} \times \mathcal{X}$ has cardinality $\kappa$ and hence can be enumerated as $\mathcal{U} \times \mathcal{X} = \{ (U_\alpha, X_\alpha) : \alpha < \kappa \}$. Since the set $D$ is at most countable and $E$ is meager in $X \times X$, the set $E_0 = \{ y \in Y : \exists x \in D \ (x, y) \in E \text{ or } (y, x) \in E \}$ is meager in $Y$. For every ordinal $\alpha < \kappa$ the set $G_\alpha$ is nowhere meager in $Y$, which allows us to find a point $y_\alpha \in U_\alpha \cap G_\alpha \setminus (X_\alpha \cup E_0)$. Then $F = D \cup \{ y_\alpha \}_{\alpha < \kappa}$ is a nowhere meager $E$-free set in $X$.

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