Covariant actions for chiral supersymmetric bosons

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Abstract: We review a recently developed covariant Lagrangian formulation for \( p \)-forms with (anti)self–dual field–strengths and present its extension to the supersymmetric case. As explicit examples we construct covariant Lagrangians for six–dimensional models with \( N = 1 \) rigid and curved supersymmetry.

1 Introduction

Chiral bosons are described by \( p \)-form gauge potentials \( B_p \) whose curvatures \( H_{p+1} = dB_p \) satisfy, as equation of motion, a Hodge (anti)self–duality condition in a space–time with dimension \( D = 2(p + 1) \). In space–times with Minkowskian signature \( \eta_{ab} = (1, -1, \cdots, -1) \) the self–consistency of such an equation restricts \( p \) to even values and hence the relevant dimensions are \( D = 2, 6, 10, \cdots \).

Such fields populated superstring and supergravity theories, and more recently M theory, from their very beginning. Two dimensional chiral bosons (scalars) are basic ingredients in string theory, six–dimensional ones belong to the supergravity– and tensor–multiplets in \( N = 1, D = 6 \) supergravity theories and are necessary to complete the \( N = 2, D = 6 \) supermultiplet of the M-theory five–brane; finally a ten–dimensional chiral boson appears in \( IIB, D = 10 \) supergravity.

A peculiar feature of the (manifestly Lorentz covariant) self–duality equation of motion of those fields is that a manifestly Lorentz invariant lagrangian formulation for them was missing for long time. The absence of a Lorentz invariant action from which one can derive the equations of motion leads in principle to rather problematic situations e.g. the conservation of the energy–momentum tensor is not guaranteed a priori and the coupling to gravity can not be performed via the usual minimal coupling.

For previous attempts in facing this problem and for a more detailed discussion of the problematic aspects involved, see in particular \([1]\). Recently a new manifestly Lorentz–invariant lagrangian approach for chiral bosons has been proposed in \([2, 3]\). The most appealing features of this approach are the introduction of only one single scalar auxiliary field, its natural compatibility with all relevant symmetries, in particular with diffeomorphisms and with \( \kappa \)–invariance \([4]\), and its general validity in all dimensions \( D = 2(p + 1) \) with \( p \) even. Another characteristic feature of this approach is the appearance of two new local bosonic symmetries: one of them reduces the scalar auxiliary field to a non propagating ”pure gauge” field and the other one reduces the second order equation of motion for the \( p \)–form to the first order (anti)self–duality equation of motion.

A variant of this approach allowed to write manifestly duality invariant actions for Maxwell fields in four dimensions \([5]\) and to construct a covariant effective action for the M theory five–brane \([4]\). On
the other hand, the actions obtained through the non manifestly covariant approach developed in [6] can be regarded as gauge fixed versions of the actions in [4, 5] where the scalar auxiliary field has been eliminated.

The coupling of all these models with chiral bosons to gravity can be easily achieved since the new approach is manifestly covariant under Lorentz transformations; as a consequence it is obvious that the two above mentioned bosonic symmetries, which are a crucial ingredient of the new approach, are compatible with diffeomorphism invariance. To test the general validity of the approach, it remains to establish its compatibility with global and local supersymmetry. This is the aim of the present talk.

In the next section we review the covariant method, for definiteness, for chiral two–forms in six dimensions. In section three we test its compatibility with supersymmetry by writing a covariant action for the most simple cases, i.e. the rigid tensor supermultiplet and the free supergravity multiplet in six dimensions. Section four is devoted to some concluding remarks and to a brief discussion of the general case i.e. the supergravity multiplet coupled to an arbitrary number of tensor multiplets and super Yang–Mills multiplets.

The general strategy developed in this paper extends in a rather straightforward way to two and ten dimensions. Particularly interesting is the case of IIB, D = 10 supergravity whose covariant action we hope to present elsewhere. The bosonic part of this action has already been presented in [7].

For more details on the results presented here and for more detailed references, see [8].

2 Chiral bosons in six dimensions: the general method

In this section we present the method for a chiral boson in interaction with an external or dynamical gravitational field in six dimensions. To this order we introduce sechsbein one–forms $e^a = dx^m e_m^a(x)$. With $m, n = 0, \ldots, 5$ we indicate curved indices and with $a, b = 0, \ldots, 5$ we indicate tangent space indices, which are raised and lowered with the flat metric $\eta_{ab} = (1, -1, \cdots, -1)$.

To consider a slightly more general self–duality condition for interacting chiral bosons we introduce the two-form potential $B$ and its generalized curvature three–form $H$ as

$$H = dB + C \equiv \frac{1}{3!} e^a e^b e^c H_{cba},$$

where $C$ is a three-form which depends on the fields to which $B$ is coupled, such as the graviton, the gravitino and so on, but not on $B$ itself. The free (anti)self–dual boson will be recovered for $C = 0$ and $e^a_m = \delta^a_m$.

The Hodge–dual of the three–form $H$ is again a three–form $H^*$ with components $H^*_{abc} = \frac{1}{3!} \varepsilon_{abcdef} H^{def}$. The self–dual and anti self–dual parts of $H$ are defined respectively as the three–forms $H^\pm \equiv \frac{1}{2} (H \pm H^*)$.

The equations of motion for interacting chiral bosons in supersymmetric and supergravity theories, as we will see in the examples worked out in the next section, are in general of the form $H^\pm = 0$, for a suitable three–form $C$ whose explicit expression is determined by the model.

To write a covariant action which eventually gives rise to the equation $H^\pm = 0$ we introduce as new ingredient the scalar auxiliary field $a(x)$ and the one–form

$$v = \frac{1}{\sqrt{-\partial_a a^a}} da \equiv e^b v_b.$$

In particular we have $v_b = \frac{\partial_a a}{\sqrt{-\partial_a a^a}}$ and $v_b e^b = -1$. Using the vector $v^b$, to the three–forms $H, H^*$ and $H^\pm$ we can then associate two–forms $h, h^*$ and $h^\pm$ according to

$$h_{ab} = v^c H_{abc}, \quad h = \frac{1}{2} e^a e^b h_{ba},$$

and similarly for $h^*$ and $h^\pm$. 

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The action we search for can now be written equivalently in one of the following two ways

\[ S_0^\pm = \pm \int \left( v h^\pm H + \frac{1}{2} dBC \right) = \int d^6 x \sqrt{g} \left( \frac{1}{24} H_{abc} H^{abc} + \frac{1}{2} h^\pm_{abc} h^{\pm abc} \right) \pm \int \frac{1}{2} dBC. \]  

(2)

\( S_0^+ \) will describe anti self–dual bosons \((H^+ = 0)\) and \( S_0^- \) self–dual bosons \((H^- = 0)\). The last term, \( \int dBC \), is of the Wess–Zumino type and is absent for free chiral bosons.

What selects this form of the action are essentially the local symmetries it possesses. Under a general variation of the fields \( B \) and \( a \) it varies, in fact, as

\[ \delta S_0^\pm = \pm 2 \int \left( v h^\pm d\delta B + \frac{v}{\sqrt{-\partial_a \partial^a}} h^\pm h^\pm d\delta a \right). \]  

(3)

From this formula it is rather easy to see that \( \delta S_0^\pm \) vanishes for the following three bosonic transformations, with transformation parameters \( \Lambda \) and \( \psi \), which are one–forms, and \( \varphi \) which is a scalar:

\[ \begin{align*}
I) & \quad \delta B = d\Lambda, \quad \delta a = 0 \\
II) & \quad \delta B = \frac{2 h^\pm}{\sqrt{-\partial_a \partial^a}} \varphi, \quad \delta a = \varphi \\
III) & \quad \delta B = \psi da, \quad \delta a = 0.
\end{align*} \]  

(4)

For what concerns \( I \) and \( III \) invariance of the action is actually achieved also for finite transformations. This fact will be of some importance below.

The transformation \( I \) represents just the ordinary gauge invariance for abelian two–form gauge potentials. The symmetry \( II \) implies that \( a(x) \) is an auxiliary field which does, therefore, not correspond to a propagating degree of freedom. Finally, the symmetry \( III \) eliminates half of the propagating degrees of freedom carried by \( B \) and allows to reduce the second order equation of motion for this field to the desired first order equation, i.e. \( H^\pm = 0 \). To see this we note that the equations of motion for \( B \) and \( a \), which can be read from (3), are given respectively by

\[ d \left( v h^\pm \right) = 0, \quad d \left( \frac{v}{\sqrt{-\partial_a \partial^a}} h^\pm h^\pm \right) = 0. \]  

(5)

First of all it is straightforward to check that the \( a \)--equation is implied by the \( B \)--equation, as expected, while the general solution of the \( B \)--equation is given by \( v h^\pm = \frac{1}{2} d\bar{\psi} da \), for some one–form \( \bar{\psi} \). On the other hand, under a (finite) transformation \( III \) we have \( \delta (v h^\pm) = \frac{d}{2} d\bar{\psi} da \) and therefore, choosing \( \psi = -\bar{\psi} \), we can use this symmetry to reduce the \( B \)--equation to \( v h^\pm = 0 \). But \( v h^\pm = 0 \) amounts to \( h^\pm = 0 \), and this equation, in turn, can easily be seen to be equivalent to \( H^\pm = 0 \), the desired chirality condition.

This concludes the proof that the actions \( S_0^\pm \) describe indeed correctly the propagation of chiral bosons.

In a theory in which the \( B \) field is coupled to other dynamical fields, for example in supergravity theories, we can now conclude that the complete action has to be of the form

\[ S = S_0^\pm + S_6, \]

where \( S_6 \) contains the kinetic and interaction terms for the fields to which \( B \) is coupled. To maintain the symmetries \( I \)--\( III \) one has to assume that those fields are invariant under these transformations and, moreover, that \( S_6 \) is independent of \( B \) and \( a \).

For more general chirality conditions describing self–interacting chiral bosons, like e.g. those of the Born–Infeld type, see ref. \(^3\).

\footnote{Notice however that, since the action becomes singular in the limit of a vanishing or constant \( a(x) \), the gauge \( da(x) = 0 \) is not allowed.}
To conclude this section we introduce two three–form fields, $K^\pm$, which will play a central role in the next section due to their remarkable properties. They are defined as

$$K^\pm = H + 2v \mp$$

and are uniquely determined by the following peculiar properties: i) they are (anti) self–dual: $K^{\pm*} = \pm K^\pm$; ii) they reduce to $H^\pm$ respectively if $H^{\mp} = 0$; iii) they are invariant under the symmetries I and III, and under II modulo the field equations (6). These fields constitute therefore a kind of off–shell generalizations of $H^\pm$.

3 $N = 1, D = 6$ supersymmetric chiral bosons

In this section we illustrate the compatibility of the general method for chiral bosons with supersymmetry in the six–dimensional case by means of two examples: the first concerns the free tensor supermultiplet in flat space–time and the second concerns pure supergravity. The strategy developed in these examples admits natural extensions to more general cases [4, 7, 8].

1) Free tensor multiplet.

An $N = 1, D = 6$ tensor multiplet is made out of an antisymmetric tensor $B_{[ab]}$, a symplectic Majorana–Weyl spinor $\lambda_{\alpha i} \ (\alpha = 1, \ldots, 4; i = 1, 2)$ and a real scalar $\phi$. The equations of motion for this multiplet and its on–shell susy transformation rules are well known. The scalar obeys the free Klein–Gordon equation, the spinor the free Dirac equation and the $B$–field the self–duality condition $H^- = 0$, where $H = dB$, which means that in this case we have $C = 0$.

The on-shell supersymmetry transformations, with rigid transformation parameter $\xi^\alpha$, are given by

$$\delta_\xi \phi = \xi^i \lambda_i,$$

$$\delta_\xi \lambda_i = \left( \Gamma^a \partial_a \phi + \frac{1}{12} \Gamma^{abc} H^+_{abc} \right) \xi_i,$$

$$\delta_\xi B_{ab} = -\xi^i \Gamma_{ab} \lambda_i.$$  (7)

The $USp(1)$ indices $i, j$ are raised and lowered according to $K_i = \varepsilon_{ij} K^j, K^i = -\varepsilon^{ij} K_j$, where $\varepsilon_{12} = 1$ and the $\Gamma^a$ are $4 \times 4$ Weyl matrices.

Since the equations of motion are free our ansatz for the action, which depends now also on the auxiliary field $a$, is

$$S = S_0^- + S_6 = -\int vh^- H + \frac{1}{2} \int d^6x \left( \lambda^\alpha \partial_a \lambda_i + \partial_a \phi \partial^a \phi \right).$$  (8)

This action is invariant under the symmetries I–III if we assume that $\phi$ and $\lambda$ are invariant under these transformations.

For what concerns supersymmetry we choose first of all for $a$ the transformation $\delta_\xi a = 0$, which is motivated by the fact that $a$ is non propagating and does therefore not need a supersymmetric partner. Next we should find the off–shell generalizations of (6). For dimensional reasons only $\delta_\xi \lambda$ allows for such an extension. To find it we compute the susy variation of $S_0^-$, which depends only on $B$ and $a$, as

$$\delta_\xi S_0^- = -2 \int vh^- d\delta_\xi B = -\int K^+ d\delta_\xi B$$

in which the self-dual field $K^+$, defined in the previous section, appears automatically. Since $\delta_\xi S_0^-$ should be cancelled by $\delta S_6$ this suggests to define the off–shell susy transformation of $\lambda$ by making the simple replacement $H^+ \rightarrow K^+$, i.e.

$$\delta_\xi \lambda_i \rightarrow \bar{\delta}_\xi \lambda_i = \left( \Gamma^a \partial_a \phi + \frac{1}{12} \Gamma^{abc} K^+_{abc} \right) \xi_i.$$  (9)
With this modification it is now a simple exercise to show that the action (8) is indeed invariant under supersymmetry. The relative coefficients of the terms in the action are actually fixed by supersymmetry.

The general rules for writing covariant actions for supersymmetric theories with chiral bosons, which emerge from this simple example, are the following. First one has to determine the on–shell susy transformations of the fields and their equations of motion, in particular one has to determine the form of the three–form $C$. For more complicated theories this can usually be done most conveniently using superspace techniques. The off–shell extensions of the susy transformation laws are obtained by substituting in the transformations of the fermions $H^\pm \to K^\pm$. The action has then to be written as $S_0^+ + S_0$ where the relative coefficients of the various terms in $S_0$ have to be determined by susy invariance. The field $a$, finally, is required to be supersymmetry invariant.

2) Pure supergravity.

The supergravity multiplet in six dimensions contains the graviton, a gravitino and an antisymmetric tensor with anti–selfdual (generalized) field strength. The graviton is described by the vector–like vielbein $e^a = dx^m e_m^a$, the gravitino by the spinor–like one–form $e^{\alpha i} = dx^m e_m^{\alpha i}$ and the tensor by the two–form $B$.

The supersymmetry transformations of these fields and their equations of motion, obtained from the superspace approach [9], are conveniently expressed in terms of a super–covariant differential, $D = d + \omega$, with respect to a super–covariant Lorentz connection one–form $\omega^{ab} = dx^m \omega_m^{ab}$. This connection is defined by $d e^a + e^b \omega_b^a = -e^i \Gamma^a e_i$, and results in the metric connection augmented by the standard gravitino bilinears.

Among the equations of motion we recall the generalized anti–selfduality condition for $B$. This reads $H^+ = 0$, where now

$$H = dB + (e^i \Gamma_a e_i) e^a,$$

meaning that in this case the three–form $C$ is non vanishing being given by $C = (e^i \Gamma_a e_i) e^a$.

The on–shell supersymmetry transformations of $e^a$, $e^{\alpha i}$ and $B$ [9], with local transformation parameter $\xi^{\alpha i}(x)$, are given by

$$\delta \xi e^a = -2 e^i \Gamma^a e_i, \quad \delta \xi e^{\alpha i} = D \xi e^{\alpha i} - \frac{1}{8} \epsilon^{\beta \gamma} e^{\alpha i} (\Gamma^{bc})_{\beta \gamma} H_{abc}^-,$$

(9)

$$\delta \xi B = -2 (e^i \Gamma_a e_i) e^a, \quad \delta \xi a = 0.$$

(10)

According to our general rule, in the gravitino transformation we have to make the off–shell replacement $H^- \to K^-$ obtaining

$$\delta \xi e^{\alpha i} \to \tilde{\delta} \xi e^{\alpha i} = D \xi e^{\alpha i} - \frac{1}{8} \epsilon^{\beta \gamma} e^{\alpha i} (\Gamma^{bc})_{\beta \gamma} K_{abc}^-.$$

In the above relations we added the trivial transformation law for the auxiliary field $a$. As it stands, this trivial transformation law does not seem to preserve the susy algebra in that the commutator of two supersymmetries does not amount to a translation. On the other hand it is known that the supersymmetry algebra closes on the other symmetries of a theory; in the present case it is easily seen that the anticommutator of two susy transformations on the $a$ field closes on the bosonic transformations $II$).

The covariant action for pure $N = 1$, $D = 6$ supergravity can now be written as $S = S_0^+ + \int L_6$, where

$$S_0^+ = \int \left( \epsilon h^+ H + \frac{1}{2} dB C \right),$$

(11)

$$L_6 = \frac{1}{48} \epsilon_{a_1 ... a_6} e^{a_1} e^{a_2} e^{a_3} e^{a_4} R^{a_5 a_6} - \frac{1}{3} \epsilon^{a_1} e^{a_2} e^{a_3} (D e^i \Gamma_{a_1 a_2 a_3} e_i).$$

(12)

For convenience we wrote the term $S_6$ as an integral of a six–form, $L_6$. This six–form contains just the Einstein term, relative to the super–covariantized spin connection $R^{ab} = d \omega^{ab} + \omega^a \omega^b$, and the kinetic
term for the gravitino. The relative coefficients are fixed by susy invariance. In this case $S_0^+$ contains also the couplings of $B$ to the gravitino and the graviton.

This action is invariant under the symmetries (I)–(III) because $L_6$ is independent of the fields $B,a$ and we assume the graviton and the gravitino to be invariant under those transformations.

The evaluation of the supersymmetry variation of $S$ is now a merely technical point and can indeed be seen to vanish. In particular, as in the previous example, the susy variation of $S_0^+$ depends on the fields $B,a$ only through the combination $K^-$ and these contributions are cancelled by the gravitino variation, justifying again our rule for the modified susy transformation rules for the fermions.

4 Concluding remarks

The covariant lagrangians presented in this talk for six–dimensional supersymmetric chiral bosons admit several extensions. The lagrangian for $n$ tensor multiplets coupled to the supergravity multiplet, which involves $n + 1$ mixed (anti)self–duality conditions, has been worked out in [3]. The introduction of hyper multiplets, on the other hand, does not lead to any new difficulty. The coupling to Yang–Mills fields in the presence of $n$ tensor multiplets requires some caution. In this case it turns out that an action, and therefore a consistent classical theory, can be constructed only if the $n + 1$ two–forms can be arranged such that only one of them carries a Chern–Simons correction while the $n$ remaining ones have as invariant field strength $dB^{(n)}$.

In conclusion the covariant method illustrated in this talk appears compatible, at the classical level, with all relevant symmetries explored so far, in particular with supersymmetry.

Among the open problems at the quantum level one regards the existence of a covariant quantization procedure. A quantum consistency check of the covariant method for chiral bosons coupled to gravity, which has still to be performed, consists in a perturbative computation of the Lorentz anomaly and in a comparison with the result predicted by the index theorem.

Acknowledgements. It is a pleasure for me to thank G. Dall’Agata and M. Tonin for their collaboration on the results presented in this talk. I am also grateful to I. Bandos, P. Pasti and D. Sorokin for their interest in this subject and for useful discussions. This work was supported by the European Commission TMR programme ERBFMPX-CT96-0045.

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