Nonperturbative QCD corrections to electroweak observables

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focusing on work with X. Feng, M. Petschlies and K. Jansen (ETMC)

also showing work from:

- Kerrane, Boyle, Del Debbio, Zanotti [LAT11]
- Jäger, Della Morte, Jüttner, Wittig [LAT11]
- Aubin and Blum [PRD (2007)]
Main point

- precision measurements are becoming sensitive to QCD corrections

- a prominent example is the muon $g-2$, which shows a $3\sigma$ discrepancy

- but, there are many such opportunities for lattice calculations

- these calculations may be more feasible than previously thought
Outline

- start with the muon $g - 2$ as a concrete example
  - discuss the relevant phenomenology
  - explain our modified method
  - compare the results from current lattice calculations
- continue to illustrate our modified method with calculations of
  - $g - 2$ for the electron and tau, quite distinct from the muon
  - $\Delta\alpha(Q^2)$, the QCD corrections to the running QED coupling
  - higher-order QCD corrections, using $g_\mu - 2$ as an example
- ask me about: $D(Q^2)$, $\Lambda_{QCD}$, and $\Delta E_{\text{Lamb}}$ for muonic hydrogen
Muon $g-2$

- **anomalous magnetic moment due solely to radiative corrections**

\[ a_\mu \equiv \frac{g_\mu - 2}{2} = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2) \]

- **experimental measurement at BNL** [Muon G-2, PRD 2006]

\[ a_{\mu}^{ex} = 1.16592080(63) \times 10^{-3} \ [0.54 \text{ ppm}] \]

- **Standard Model estimate** [Jegerlehner, Nyffeler Phys. Rept. 2009]

\[ a_{\mu}^{th} = 1.16591790(65) \times 10^{-3} \ [0.56 \text{ ppm}] \]

- **a 3.2\sigma discrepancy might indicate physics beyond the Standard Model**

\[ a_{\mu}^{ex} - a_{\mu}^{th} = 2.90(91) \times 10^{-9} \]
Future muon $g-2$ experiments

- proposed experiments at Fermilab (2016/17) and J-PARC

\[ \sigma^{\text{ex}} = 6.3 \times 10^{-10} \rightarrow 1.6 \times 10^{-10} \, [\text{FNAL}] \approx 10^{-10} \]

- comparison would then be dominated by theory errors alone

\[ a_{\mu}^{\text{ex}} - a_{\mu}^{\text{th}} = 3.2 \, \sigma \rightarrow (3.4 - 5.3) \, \sigma \]

- Standard Model error is dominated by hadronic physics

| Contribution    | $\sigma^{\text{th}} \, [10^{-10}]$ |
|-----------------|------------------------------------|
| QCD-LO $[\alpha^2]$ | 5.3                      |
| QCD-NLO $[\alpha^3]$ | 3.9                      |
| QED/EW          | 0.2                         |
| **Total**       | **6.6**                     |

- improvement from the theory side is highly desirable
Standard Model estimate of muon $g-2$

- QED corrections are known to 4 loops and EW to 2 loops

\[ a_\mu \equiv \frac{g_\mu - 2}{2} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{QCD}} \]

- QCD corrections first occur at $O(\alpha^2)$, only smaller than QED piece

- leading-order hadronic contribution (hlo) is in fact measured

\[ a_\mu^{\text{hlo}} = \alpha^2 \int_{4m_\pi^2}^{\infty} \frac{ds}{s} K^{\text{lo}}(s/m_\mu^2) R(s) \quad R(s) = \frac{\sigma(\gamma^* \rightarrow \text{hadrons})}{\sigma(\gamma^* \rightarrow e^+e^-)} \]

- thus the "theory" calculation requires significant experimental input
Phenomenology of $R(s)$

- lowest vector-mesons, $\rho$, $\omega$ and $\phi$, account for 80% of $a_{\mu}^{\text{hlo}}$

![Graph showing $R(E)$ vs $E$ in GeV with peaks at $\pi^+\pi^-$, $\rho$, $\omega$, $\phi$, and $J/\psi$.]

- pheno. analysis uses $R_{N_f}(s)$ to extract $N_f = 2$ and 3 contributions

$$R_{N_f}(s) = R(s)\left(\sum_{N_f} Q_f^2\right)/\left(\sum_{N} Q_f^2\right) \quad 4m_N^2 \leq s \leq 4m_{N+1}^2$$

- improvement in $\sigma(e^+e^- \rightarrow \text{hadrons})$ coming from many experiments

[$R(E)$ given by F. Jegerlehner’s compilation of $\sigma(e^+e^- \rightarrow \text{hadrons})$]
Lattice calculation of $a_{\mu}^{\text{hlo}}$

- $a_{\mu}^{\text{hlo}}$ can also be calculated directly in Euclidean space (Tom Blum)

- vacuum polarization tensor is a simple two-point function

$$\pi_{\mu\nu}(Q^2) = \int d^4X e^{iQ\cdot(X-Y)} \langle J_\mu(X) J_\nu(Y) \rangle = (Q_\mu Q_\nu - Q^2 \delta_{\mu\nu}) \pi(Q^2)$$

- leading-order QCD contribution [Blum, PRL 2003]

$$a_{\mu}^{\text{hlo}} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w^{\text{lo}}(Q^2/m_\mu^2) \pi_R(Q^2)$$

- $\pi_R(Q^2) = \pi(Q^2) - \pi(0)$ is finite with $R(s) \propto \text{Im} \pi(-s + i\epsilon)$
Problem with external scales

- $a_{\mu}^{\text{hlo}}$ is made dimensionless at the expense of introducing $m_{\mu}$

$$a_{\mu}^{\text{hlo}} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w^{\text{lo}}(Q^2/m_{\mu}^2) \pi_R(Q^2)$$

- introduces dependence on lattice spacing in dimensionless quantity

$$\frac{Q^2}{m_{\mu}^2} = \frac{1}{a^2} \frac{a^2 Q^2}{m_{\mu}^2} = \frac{1}{a^2} \frac{[Q^2]}{[m_{\mu}^2]}_{\text{latt}}$$

- creates strong $m_{PS}$ dep., as seen in leading vector-meson contribution

$$a_{\mu,V} \propto g_V^2 \frac{m_{\mu}^2}{m_V^2}$$

- effective dimension captures dimensionality of QCD scales only

$$d_{\text{eff}}[X] = -\frac{a}{X} \left. \frac{\partial X}{\partial a} \right|_{g_0=\text{fixed}} \quad d_{\text{eff}}[M^n] = n \quad d_{\text{eff}}[a_{\mu}] = -1.887 \ (5)$$
Eliminating external scales

- this understanding leads to a class of modified observables

\[ a_{hlo}^{\mu} = \alpha^2 \int_0^\infty dQ^2 \frac{Q^2}{Q^2} w^{lo} \left( \frac{Q^2}{H^2} \cdot \frac{H_{phys}^2}{m_\mu^2} \right) \pi_R(Q^2) \]

- \( H \) is any hadronic scale and \( H(m_{PS} \to m_\pi) = H_{phys} \), so

\[ \lim_{m_{PS} \to m_\pi} a_{hlo}^{\mu} = a_{\mu}^{hlo} \]

- each \( a_{hlo}^{\mu} \) behaves like a proper dimensionless QCD quantity

\[ d_{eff}[a_{hlo}^{\mu}] = 0 \]

- each \( a_{hlo}^{\mu} \) is composed of hadronic scales only
Modified method

- bottom to top: $H = 1$ (std. method), $H = f_V$ and $H = m_V$

- standard method matched to VMD model for illustration only

- VMD model predicts linear behavior for $a_{\mu}^{\text{hlo}}$ with correct slope
Two-flavor lattice results

- both groups have two lattice spacings, multiple physical volumes

![Graph showing lattice results](image)

- low $Q^2$ treatment, twisting versus extrapolation, seems consistent
- Mainz results rise consistently with ETMC vector-meson contribution
Three-flavor lattice results

- trend is clear but agreement is less compelling for $N_f = 3$

- very recent extrapolated result from Edinburgh with a 7% error

- also, Edinburgh finds similar extrapolation with our modified method
Electron and tau $g - 2$

- High precision measurement of $g_e$ [Harvard, PRL 100:120801 (2008)]
  
  $$g_e/2 = 1.00115965218073(28) \ [0.28 \text{ ppt}]$$

- Extraction of $\alpha$ from $g_e$ just becoming sensitive to QCD corrections
  
  $$\alpha^{-1} = 137.035999084(51) \ [0.37 \text{ ppb}]$$

- $g_e$ provides an very different probe of the QCD vacuum polarization
  
  $$a_e^{\text{hlo}} \approx \frac{4}{3} \alpha^2 m_e \frac{d\pi_R}{dQ^2} \bigg|_{Q^2=0} \quad d_{\text{eff}}[a_e] = -1.999984 (1)$$

- $g_\tau$ is sensitive to larger $Q^2$ and provides another test of our calculation
  
  $$d_{\text{eff}}[a_\tau] = -0.936 (13)$$

- $g_\tau$ is much more difficult to measure directly but $a_\tau^{\text{hlo}}$ is not
Calculation of all three charged leptons

- using modified method with $H = m_V$ for all leptons $l \in \{e, \mu, \tau\}$

- no QCD perturbation, complete non-perturbative calculation of $a_l^{\text{hlo}}$

- $a_e^{\text{hlo}}$ and $a_{\mu}^{\text{hlo}}$ are accurate to < 3% and $a_{\tau}^{\text{hlo}}$ is accurate to 2%
QCD corrections to the QED coupling

- an effective QED coupling is normally defined by

\[
\alpha(Q^2) = \frac{\alpha}{1 - \Delta \alpha(Q^2)}
\]

- the hadronic piece is again related to \(\pi_R(Q^2)\)

\[
\Delta \alpha_{\text{had}}(Q^2) = 4\pi \alpha \pi_R(Q^2)
\]

- precision of \(\alpha\) is eroded by QCD corrections

\[
\frac{\sigma \alpha}{\alpha} \approx 4 \cdot 10^{-10} \quad \rightarrow \quad \frac{\sigma \alpha(M_Z^2)}{\alpha(M_Z^2)} \approx 3 \cdot 10^{-4}
\]

- this impacts many SM predictions, for example the Gfitter fit for \(m_H\)

\[
m_H = 44^{+62}_{-43} \text{ GeV} \quad \text{without } \Delta \alpha(M_Z^2)
\]

\[
m_H = 96^{+31}_{-24} \text{ GeV} \quad \text{with } \Delta \alpha(M_Z^2)
\]
Modified definition of $\Delta \alpha_{\text{had}}(Q^2)$

- treat $Q^2$ as an external scale and similarly define a new observable
  \[
  \Delta \bar{\alpha}_{\text{had}}(Q^2) = 4\pi \alpha \pi R \left( \frac{Q^2}{H_{\text{phys}}^2} \cdot H^2 \right)
  \]

- $M_0 = 2.5$ GeV is a common matching point in pheno. work

\[\Delta \alpha(M_0^2) = 5.72 (12) \cdot 10^{-3} \quad \text{LQCD}\]
\[\Delta \alpha(M_0^2) = 5.60 (06) \cdot 10^{-3} \quad \text{PHENO}\]
• lattice artifacts only show up slowly for $Q^2 \gtrsim 7$ GeV$^2$

\begin{align*}
\Delta \alpha_{\text{had}}(Q^2) \times 10^{-3} & \quad \text{Pheno, } N_f=2 \\
\Delta \alpha_{\text{had}}(Q^2) \times 10^{-3} & \quad \text{Lattice, } N_f=2
\end{align*}

• $\alpha_s$ from $\pi(Q^2)$ used to determine $\Delta \alpha(M_Z^2) - \Delta \alpha(M_0^2)$ at 5 loops

$$
\Delta \alpha(M_Z^2) = \Delta \alpha(M_0^2) + \Delta \alpha(M_Z^2) - \Delta \alpha(M_0^2) = 0.01715 (42)
$$
NLO QCD correction to $g_\mu - 2$

- calculated all three classes of 17 NLO diagrams involving $\pi_R(Q^2)$

- complete non-pert. NLO ($\alpha^3$) correction, excluding light-by-light

\[
\begin{align*}
    a_{\mu}^{nlo,hvp} &= -8.41 (22) \cdot 10^{-10} \quad \text{Lattice, } N_f = 2 \\
    a_{\mu}^{nlo,hvp} &= -8.33 (17) \cdot 10^{-10} \quad \text{Pheno, } N_f = 2
\end{align*}
\]

- light-by-light corrections require a different technology

- ongoing work by Blum et. al, QCDSF, JLQCD

\[
\begin{align*}
    a_{\mu}^{nlo, lbl} &= 8 (4) \cdot 10^{-10} \leftrightarrow 12 (4) \cdot 10^{-10} \quad \text{Pheno}
\end{align*}
\]
**Summary and outlook**

- Current SM error on $a_{\mu}^{\text{hlo}}$ is $6 \cdot 10^{-10}$, Fermilab target requires $2 \cdot 10^{-10}$

| $N_f$ | $a_{\mu}^{\text{hlo}}$ [Pheno] | $a_{\mu}^{\text{hlo}}$ [Lattice] |
|-------|---------------------------------|----------------------------------|
| 5     | $6.93 (06) \cdot 10^{-8}$      | $-$                              |
| 4     | $6.93 (06) \cdot 10^{-8}$      | $-$                              |
| 3     | $6.81 (05) \cdot 10^{-8}$      | $6.41 (46)$                      |
| 2     | $5.67 (05) \cdot 10^{-8}$      | $5.72 (16)$                      |

- $N_f = 4$ with improvement by factor of 3 will match current estimates
- An additional factor of 3 eventually needed by 2016/2017
Conclusions

- QCD corrections are becoming important for precision measurements
- Lattice calculations may be more feasible than previously thought
- Currently, we have the LO + NLO vac. pol. corrections < 3%
- Demanding light-by-light needed to complete the NLO calculation
- Near term: $N_f = 4$ with a factor of 2 – 3 improvement needed
- Longer term: an additional factor of 4 improvement required
Extra slides
• no complicated resonance or threshold structure for Euclidean $Q^2$

• we parameterize the entire range of $Q^2$ with a smooth curve
Definition of $a_{\mu}^{\text{hlo}}$ for $a > 0$

- the large $Q^2$ behavior is parameterized by fitting to

$$\pi_R(Q^2) = c + \ln Q^2 \cdot \sum_n a_n Q^{2n}$$

- to be precise, we fix the definition at non-zero lattice spacing with

$$\int_0^\infty dQ^2 \rightarrow \int_0^{Q_{uv}^2} dQ^2 \quad Q_{uv}^2 = 16/a^2$$

- the integral is convergent, so this is just a choice of cut-off effects

- this choice does not require QCD perturbation theory

- this definition does not force us to introduce a lattice spacing

- this last point is important given that $d_{\text{eff}}[a_{\mu}] \approx -2$
Definition of $a_{\mu}^{\text{hlo}}$ for $L < \infty$

- define $\pi_R$ for low $Q^2$ by including the lowest meson and fitting the $a_n$

$$\pi_R(Q^2) = \frac{5}{9} g_V^2 \frac{Q^2}{Q^2 + m_V^2} + \sum_n a_n Q^{2n}$$

- fit ensures that $\pi_R(Q^2)$ matches lattice calculation for accessible $Q^2$
- extrapolation provides a well-defined finite-volume definition
- explicit vector-meson term is systematically reabsorbed as $L$ increases

$$\frac{5}{9} g_V^2 \frac{Q^2}{Q^2 + m_V^2} = \sum_n b_n Q^2 \quad \text{for} \quad Q^2 < m_V^2$$

- this is not a systematic error but a proper finite-volume definition
- a practical matter of explicitly verifying controlled finite-size effects
Effective dimension

- $d_{\text{eff}}$ attempts to capture the dimensionality of only the QCD scales
  \[ d_{\text{eff}}[X] = -\frac{a}{X} \frac{\partial X}{\partial a} \bigg|_{g_0=\text{fixed}} \]

- for a standard mass scale $M$, definition is the usual mass dimension
  \[ d_{\text{eff}}[M^n] = -\frac{a}{M^n} \frac{\partial}{\partial a} \left( \frac{1}{a^n} \tilde{M}^n(g_0) \right) = -\frac{a}{M^n} \tilde{M}^n(g_0) \frac{\partial}{\partial a} \left( \frac{1}{a^n} \right) = n \]

- however, it differs for a composite observable
  \[ d_{\text{eff}} \left[ \frac{m_{\mu}^2}{m_V^2} \right] = d_{\text{eff}} \left[ \frac{1}{m_V^2} \right] = -2 \]

- for $a_\mu$, we have an expression that must be evaluated on the lattice
  \[ d_{\text{eff}}[a_\mu] = -2 \left( \int \frac{dQ^2}{Q^2} w(Q^2/m_{\mu}^2) Q^2 d\pi_R \right) / \left( \int \frac{dQ^2}{Q^2} w(Q^2/m_{\mu}^2) \pi_R \right) < 0 \]

- you can easily prove that $d_{\text{eff}}[a_\mu] \to -2$ (0) for $m_\mu \to 0$ ($\infty$)
Renormalization of QCD + QED

- introduce a variable muon mass $m_{\mu}$ and quark mass $m_{\bar{q}}$

- both paths, with $m_{\mu}$ or $m_{\mu}/m_{\bar{q}}$ fixed, define valid physical limits

- but $m_{\mu} = (m_{\mu}/m_{\bar{q}}) m_{\bar{q}}$ follows a contour of $a_{\mu}^{hvp}$ in pQCD
- introduce variable muon mass $m_{\mu}$ and pseudo-scalar mass $m_{PS}$

- curve $m_{\mu} = (m_{\mu}/m_{\rho}) m_V$ is implicitly defined so that $m_{\mu} \rightarrow m_{\mu}$

- contours from VMD model (ask me) matched to the lattice calc.
Vector meson contribution to $a_\mu$

- the vector-mesons dominate the hadronic contribution to $a_\mu$

\[ a_{\mu,V} \approx c \frac{m_\mu^2}{m_V^2} \]

- on general grounds we expect any model to give

\[ a_{\mu,V} = \alpha^2 g_V^2 f(m_\mu^2/m_V^2) = \frac{2}{3} \alpha^2 g_V^2 \frac{m_\mu^2}{m_V^2} + O(m_\mu^4/m_V^4) \]

- tree-level chiral perturbation theory gives

- this allows us to model the vector meson contribution to $a_\mu^{\text{hvp}}$
Electromagnetic coupling of vector-meson

- dimensionless quantities are typically better calculated

\[ g_V \]

result for \( g_V \) represents quantitative success for our calculation
• dimensionful quantities are sensitive to the overall scale setting

• phenomenological fit includes the PDG value of $m_\rho$
Phenomenological description of $a_\mu^{\text{hvp}}$

- can combine model expectations with our calc. of $g_V$ and $m_V$

- apparently strong $m_{PS}$ dependence of $m_V$ is reflected in $a_\mu^{\text{hvp}}$
Modified definition of $\Delta \alpha(Q^2)$

- a change of variables gives $a^{\text{hvp}}_{\mu}$ as

$$a^{\text{hvp}}_{\mu} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} w(Q^2/m^2_{\mu}) \pi R \left( \frac{Q^2}{H_{\text{phys}}^2} \cdot H^2 \right)$$

- this suggests treating $Q^2$ as an external scale like $m^2_{\mu}$ and defining

$$\Delta \overline{\alpha}_{\text{had}}(Q^2) = 4\pi \alpha \pi R \left( \frac{Q^2}{H_{\text{phys}}^2} \cdot H^2 \right)$$

- this choice for $\pi_R(Q^2)$ then defines all other observables consistently
Running of $\alpha$

- includes only the QCD corrections, remember full $\alpha^{-1}(M_Z) \approx 129$

- a step-scaling strategy is being pursued for large $Q^2$ running
the Adler function eliminates the UV divergence by a derivative

\[ D(Q^2) = 12\pi^2 Q^2 \frac{d\pi R}{dQ^2} \rightarrow \overline{D}(Q^2) = D(Q^2/H_{\text{phys}}^2 \cdot H^2) \]

this makes \( D(Q^2) \) much more sensitive to cut-off effects

\[ \alpha_s^{(2)}(2 \text{ GeV}^2) = 0.263 (16) \]

\[ \Lambda^{(2)} = 222 (27) \text{ MeV} \]

can determine \( \alpha_s \) and \( \Lambda \) at each \( Q^2 \) (2 GeV\(^2\) used) without OPE
Muonic hydrogen

- the LO QCD corrections to the 2P/2S splitting in $\mu^- p$

$$\Delta E_{\text{hfs}}^{\text{hlo}} = 2\pi\alpha^5 \mu^3 \frac{d\pi R}{dQ^2}\bigg|_{Q^2=0}$$

- this is closely related to $a_e$ and similarly tests the low $Q^2$ region

![Graph showing a plot of $\Delta E_{\text{hfs}}$ vs. $m_{\text{PS}}^2$ with markers and error bars. The graph includes markers for different $a$ and $L$ values with linear and quadratic fits.]

Lattice, $N_f = 2$

$$\Delta E_{\text{hfs}}^{\text{hlo}} = 9.06 (29) \, \mu\text{eV}$$

Pheno, $N_f = 2$

$$\Delta E_{\text{hfs}}^{\text{hlo}} = 9.17 (07) \, \mu\text{eV}$$

- current $5 \sigma$ discrepancy corresponds to 0.3.., only rough checks needed