The Odd Log-Logistic Poisson Inverse Rayleigh Distribution: Statistical Properties and Applications

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Abstract
In this work, a new extension of the Inverse Rayleigh model is proposed and studied. We derive some of its fundamental properties. We assess the performance of the maximum likelihood estimators via a simulation study. The importance of the new model is shown via two applications to real data sets. The new model is better fit than other important competitive models based on two real data sets.

Key Words: Odd Log-Logistic-G Family; Inverse Rayleigh Distribution; Simulation; Generating Function, Maximum Likelihood.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction and interpretation
A random variable (rv) $X$ is said to have the Inverse Rayleigh (IR) distribution if its probability density function (PDF), cumulative distribution function (CDF) are given by

$$g_\eta(x) = 2\eta^2 x^{-3} \exp\left[-\eta^2 x^{-2}\right],$$

$$F_\eta(x) = \exp\left[-\eta^2 x^{-2}\right],$$

and

where $\alpha, \beta > 0$ and $\Phi$ is the vector of parameter for baseline CDF. The corresponding PDF is given by

$$F_{a,\beta,\phi}(x) = \frac{\left[\exp[\beta G_\phi(x)] - 1\right]^{\alpha}}{\tau(\beta)} + \left[1 - \frac{\exp[\beta G_\phi(x)] - 1}{\tau(\beta)}\right]^{\alpha},$$

$$\times \frac{\exp[\beta G_\phi(x)] - 1^{\alpha-1}}{\exp(\beta) - \exp[\beta G_\phi(x)]^{2} - \exp(\beta) - \exp[\beta G_\phi(x)]^{2}}.$$  

and

$$F_{a,\beta,\phi}(x) = \left(\frac{\exp[\beta \exp[-\eta^2 x^{-2}]] - 1}{\tau(\beta)}\right)^{\alpha} + \left[1 - \frac{\exp[\beta \exp[-\eta^2 x^{-2}]] - 1}{\tau(\beta)}\right]^{\alpha}.$$  

The corresponding PDF is given by
In this section, mixture representations for Equations (5) and (6) are obtained, firstly we have

\[
f_{a,\beta,\eta}(x) = \frac{2a\beta\eta^{-2}(\tau_\beta)^{-3}x^{-3} \exp[-\eta^2x^{-2}] \left[\exp(\beta \exp[-\eta^2x^{-2}]) - 1\right]^{a-1}}{\exp(-\beta \exp[-\eta^2x^{-2}]) \left[\exp(\beta - \exp(\beta \exp[-\eta^2x^{-2}]))\right]^{-a+1}} \times \left\{\frac{1}{\tau_\beta}(\exp(\beta \exp[-\eta^2x^{-2}]) - 1)\right\}^a + \left[1 - \frac{1}{\tau_\beta}(\exp(\beta \exp[-\eta^2x^{-2}]) - 1)\right]^{a-1},
\]

where \(\tau_\beta = \exp(\beta) - 1\). The survival function can be derived as

\[
S_{a,\beta,\eta}(x) = 1 - \left[\frac{1}{\tau_\beta}(\exp(\beta \exp[-\eta^2x^{-2}]) - 1)\right]^a + \left[1 - \frac{1}{\tau_\beta}(\exp(\beta \exp[-\eta^2x^{-2}]) - 1)\right]^{a-1}.
\]

The hazard rate functions (HRF) is defined as \(f_{a,\beta,\eta}(x)/[1 - F_{a,\beta,\eta}(x)]\) can be easily derived. When \(\beta \to 0\), we have the odd log-logistic Inverse Rayleigh (OLL-IR) model and when \(a = 1\), we have the Poisson Inverse Rayleigh (P-IR) model.

Let \(T\) be a rv describing a stochastic system by the CDF, \(G_{\beta,\eta}(x) = \frac{\exp(\beta \exp[-\eta^2x^{-2}])^{-1}}{\tau_\beta}\), which is the CDF of the compound Poisson Inverse Rayleigh model. If the rv \(X\) represents the odds ratio, the risk that the system following the lifetime \(T\) will not be working at time \(x\) is given by

\[
\vartheta_{\beta,\eta}(x) = \frac{\frac{1}{\tau_\beta}(\exp(\beta \exp[-\eta^2x^{-2}]) - 1)}{1 - \frac{1}{\tau_\beta}(\exp(\beta \exp[-\eta^2x^{-2}]) - 1)}.
\]

If we are interested in modeling the randomness of the odds ratio, \(\vartheta_{\beta,\eta}(x)\), by the exponentiated half-logistic CDF

\[
R(t) = \frac{t^a}{1 + t^a} |_{t>0},
\]

the CDF of \(X\) is given by

\[
Pr(X \leq x) = R \left(\frac{\frac{1}{\tau_\beta}(\exp(\beta \exp[-\eta^2x^{-2}]) - 1)}{1 - \frac{1}{\tau_\beta}(\exp(\beta \exp[-\eta^2x^{-2}]) - 1)}\right) = \frac{(\exp(\beta \exp[-\eta^2x^{-2}]) - 1 - 1)^a}{(\exp(\beta \exp[-\eta^2x^{-2}]) - 1 - 1)^a + (\exp(\beta - \exp(\beta \exp[-\eta^2x^{-2}])))^a}.
\]

Furthermore, the basic motivations for using the OLLP-IR model in practice are the following: to make the kurtosis more flexible compared to the baseline Fr model; to produce a skewness for symmetrical distributions; to construct heavy-tailed distributions that are not longer-tailed for modeling real data; to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped; to define special models with flexible types of the HRF; to provide consistently better fits than other generated models under the same baseline distribution. Although, we have stated that \(\beta \in (0, \infty)\), equation (5) is still a CDF if \(\beta < 0\). Hence, we can consider the OLLP-IR model defined for any \(\beta \in R - [0]\). Many extensions of the IR model can be cited as: beta IR (BIR) by Barreto-Souza et al. (2011), Marshall-Olkin IR (MOIR) by Krishna et al. (2013), Kumaraswamy transmutated Marshall-Olkin IR (KTMOIR) by Yousof et al. (2016), Odd Lindley IR (OLIR) by Korkmaz et al. (2017), odd log-logistic IR (OLLIR) by Yousof et al. (2018a), Transmuted Topp Leone IR (TTLIR) by Yousof et al. (2018b), among others. Many other extensions can be found in Brito et al. (2017), Chakraborty et al. (2018), Cordeiro et al. (2018), Korkmaz et al. (2018) and Korkmaz et al. (2019).

2. Linear representations

In this section, mixture representations for Equations (5) and (6) are obtained, firstly we have

\[
\left[\frac{1}{\tau_\beta}(\exp(\beta \exp[-\eta^2x^{-2}]) - 1)\right]^a = \sum_{j=0}^{\infty} a_j \left[\frac{1}{\tau_\beta}(\exp(\beta \exp[-\eta^2x^{-2}]) - 1)\right]^i,
\]

where \(a_j = \sum_{i=j}^{\infty}(-1)^{i+j} \binom{a}{i} \binom{i}{j}\) and

\[
\binom{a}{i} = \frac{a(a-1)(a-2)...(a-i+1)}{i!},
\]

\[
\binom{i}{j} = \frac{i!}{j!(i-j)!}.
\]
\[
\left[ \frac{1}{\tau(\beta)} \left( \exp[\beta \exp[-\eta^2 x^2]] - 1 \right) \right]^\alpha + \left[ 1 - \frac{1}{\tau(\beta)} \left( \exp[\beta \exp[-\eta^2 x^2]] - 1 \right) \right] = \sum_{j=0}^{\infty} b_j \left( \frac{\exp[\beta \exp[-\eta^2 x^2]] - 1}{\tau(\beta)} \right)^j,
\]
where \( b_j = a_j + (-1)^j \left( \frac{\alpha}{j} \right) \). Then, the OLLP CDF in (5) can be written as
\[
F(x) = \frac{\sum_{j=0}^{\infty} a_j \left( \frac{\exp[\beta \exp[-\eta^2 x^2]] - 1}{\tau(\beta)} \right)^j}{\sum_{j=0}^{\infty} c_j \left( \frac{\exp[\beta \exp[-\eta^2 x^2]] - 1}{\tau(\beta)} \right)^j},
\]
where \( c_0 = \frac{a_0}{b_0} \) and for \( j \geq 1 \), we have \( c_j = \frac{1}{b_0} \left( a_j + \frac{1}{b_0} \sum_{r=1}^{\infty} b_r c_{j-r} \right) \)
\[
F(x) = \sum_{k=0}^{\infty} \xi_k H_{k+1}(x), \tag{7}
\]
where \( \xi_k = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{(1+j)j!+1}{(1+k)!} \frac{(1+k)!}{j!} \left( \frac{\alpha}{j} \right)^j \) and \( H_\delta(x) = \exp[-(k+1)\eta^2 x^2] \) is CDF of the IR distribution with scale parameter \( \eta(k+1)^2 \). The corresponding OLLP-IR density function is obtained by differentiating (7) as
\[
f(x) = h_{k+1}(x), \tag{8}
\]
where \( h_{k+1}(x) = 2(k+1)^2 \eta^2 x^3 \exp[-(k+1)\eta^2 x^2] \) is PDF of the IR distribution with scale parameter \( \eta(k+1)^2 \).

Equation (7) and (8) reveal that PDF of OLLP-IR is a linear combination of IR densities. Thereby, some properties of the proposed family such as moments and generating function can be determined by means of IR distribution.

3. Mathematical properties
Asymptotics
Let \( \varepsilon = \inf \{ x \mid G(x) > 0 \} \), the asymptotics of equations (3), (4) and (5) as \( x \to d \) are given by
\[
F(x) \sim \left( \beta \exp[-\eta^2 x^2] \right)^\alpha \left( \frac{\alpha}{\tau(\beta)} \right)^{1/\alpha} \exp[-2\eta^2 x^2],
\]
and
\[
\tau(x) \sim \frac{2\beta^\alpha}{\tau(\beta)} \eta^2 x^{-(2+1)} \exp\left[-\alpha \left( \frac{\alpha}{x} \right)^2 \right].
\]
The asymptotics of equations (3), (4) and (5) as \( x \to \infty \) are given by
\[
1 - F(x) \sim \left( \frac{\beta(1 - \exp[-\eta^2 x^2])}{\tau(\beta)} \right)^\alpha \left( \frac{\alpha}{\tau(\beta)} \right)^{1/\alpha} \frac{2\eta^2 \beta^\alpha [1 - \exp(-\eta^2 x^2)]}{x^3 \exp[\eta^2 x^2]},
\]
and
\[
\tau(x) \sim \frac{2\alpha^2 \eta^2 x^{-(2+1)} \exp[-\eta^2 x^2]}{1 - \exp[-\eta^2 x^2]},
\]
These equations show the effect of parameters on tails of OLLP-IR distribution.

Moments, incomplete moments and generating function
The \( r \)th ordinary moment of \( X \) is given by \( \mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx \). Then we obtain
\[
\mu'_r = \sum_{k=0}^{\infty} \xi_k \eta^r (k+1)^{2r} \Gamma \left( 1 - \frac{r}{2} \right), \tag{9}
\]
where \( \Gamma(1+\delta) \mid_{\delta \in R^+} = \delta! = \prod_{k=1}^{\delta} (\delta - k) = \int_{0}^{\infty} t^\delta \exp(-t) dt \) is the complete gamma function. Setting \( r = 1 \) in (9), we have the mean of \( X \). The last integration can be computed numerically for most parent distributions. The \( r \)th incomplete moment, say \( I_r(t) \), of \( X \) can be expressed from (7) as
\[
I_r(t) = \int_{-\infty}^{t} x^r f(x) dx = \sum_{k=0}^{\infty} \xi_k \eta^r (k+1)^{2r} \left( \frac{r}{2} , (k+1)\eta^2 x^2 \right) \tag{10}
\]
where \( \gamma(\delta, \vartheta) \) is the incomplete gamma function.

\[
\gamma(\delta, \vartheta) = \int_0^\vartheta t^{\delta-1} \exp(-t) \, dt = \frac{\vartheta^\delta}{\delta} \{1F_1[\delta; \delta + 1; -\vartheta]\} = \sum_{\zeta=0}^{\infty} \frac{(-1)^\zeta}{\zeta! (\delta + \zeta)} \vartheta^{\delta + \zeta},
\]

and \( 1F_1[\cdot; \cdot; \cdot] \) is a confluent hypergeometric function. The first incomplete moment \( l_1(t) \) given by (9) with \( r = 1 \). The moment generating function (mgf) \( M_X(t) = E(e^{tx}) \) of \( X \) can be derived from equation (8) as

\[
M_X(t) = \sum_{k,r=0}^{\infty} \frac{t^r}{r!} \xi_k^n (k + 1) 2^r \left(1 - \frac{r}{2}\right)(r < 2),
\]

Skewness and kurtosis for the OLLP-IR model can be calculated from the ordinary moments by using well-known relationships. The mean, variance, skewness and kurtosis of the OLLP-IR distribution are computed numerically using the R software and reported in Table 1. Table 1 indicate that the skewness of the OLLP-IR distribution is almost always positive. The kurtosis is always more than 3.

| \( \alpha \) | \( \beta \) | \( \eta \) | mean | variance | skewness | kurtosis |
|---|---|---|---|---|---|---|
| 2 | 0.5 | 1.5 | 0.3099428 | 9.819942 | 426.3933 | 6811390 |
| 1.5 | 0.5 | 14.35429 | 346.1332 | 457474.1 |
| 1 | 1.2393480 | 299.4811 | 348174.4 |
| 1.5 | 0.1 | 1.5 | 0.0009350 | 0.21149 | 6959.573 | 29912471 |
| 0.5 | 0.5 | 14.35429 | 346.1332 | 457474.1 |
| 0.75 | 4.1792060 | 91.85036 | 159.0526 | 83479.36 |
| 1 | 19.855450 | 373.4145 | 153301.3 |
| 1.5 | 0.5 | 0.5 | 0.1973479 | 1.825138 | 849.4490 | 3145095 |
| 0.75 | 0.2960061 | 607.9861 | 1537535 |
| 1 | 0.3946538 | 480.6036 | 927976.9 |
| 2.5 | 0.9863203 | 230.2226 | 189177.9 |
| 5 | 1.9715930 | 133.9198 | 58031.79 |
| 10 | 3.9389940 | 79.10100 | 18205.84 |
| 25 | 9.8160510 | 40.55250 | 4110.307 |
| 50 | 19.527330 | 25.08110 | 1390.652 |
| 200 | 75.597090 | 10.38714 | 185.8402 |
| 500 | 176.51360 | 6.240796 | 58.02135 |
| 2000 | 477.21200 | 2.994811 | 16.25156 |

Quantile function
The OLLP-G family can easily be simulated by inverting (5) as follows: if \( U \sim U(0,1) \), then the random variable \( X_U \) can be obtained from the baseline qf, say \( Q_G(u) = G^{-1}(u) \). In fact, the random variable

\[
X_U = \sqrt{-\eta^2 \left( \ln \left( \beta^{-1} \ln \left[ 1 + \frac{\tau(1) \sqrt{u}}{\sqrt{u} + \sqrt{1-u}} \right] \right) \right)^{-1}},
\]

has CDF (5). The effects of the shape parameters on the skewness and kurtosis can be based on quantile measures.

Moments of residual and reversed life
The \( n \)th moment of the residual life say \( z_n(t) = E[(X - t)^n \mid X > t] \), \( n = 1, 2, \ldots \) uniquely determines \( F(x) \). The \( n \)th moment of the residual life of \( X \) is given by \( z_n(t) = \int_0^\infty (x-t)^n dF(x) \). Therefore

\[
z_n(t) = \frac{1}{1 - F(t)} \sum_{k=0}^{\infty} \sum_{r=0}^{n} \xi_k (1-t)^n \int_r^\infty x^r h_{k+1}(x) \, dx
\]

\[
= \frac{1}{1 - F(t)} \sum_{k=0}^{\infty} \sum_{r=0}^{n} \xi_k (1-t)^n \eta^k (k+1) 2^r \left(1 - \frac{r}{2}, (k + 1) \eta^2 x^{-2}\right) \left| (n < 2) \right.
\]
where $\Gamma(\delta, \theta)|_{\delta > 0} = \int_0^\infty t^{\delta-1} e^{-t} dt$ and $\Gamma(\delta, \theta) = \Gamma(\omega) - \gamma(\delta, \theta)$. The $n$th moment of the reversed residual life say, $Z_n(t) = E[(t - X)^n]$ \( (X \leq t, t > 0 \text{ and } n = 1, 2, ...) \) uniquely determines $F(x)$. We obtain $Z_n(t) = \int_0^t (t-x)^n dF(x)$. Then, the $n$th moment of the reversed residual life (RRL) of $X$ becomes
\[
Z_n(t) = \frac{1}{F(t)} \sum_{k=0}^\infty \sum_{r=0}^n \xi_k (-1)^r \binom{n}{r} t^{n-r} \int_0^t x^r h_{k+1}(x) dx
\]
\[
= \frac{1}{F(t)} \sum_{k=0}^\infty \sum_{r=0}^n \xi_k (-1)^r \binom{n}{r} t^{n-r} \eta^n (k + 1) \eta^r \left( 1 - \frac{n}{2}, (k + 1)\eta^2 x^2 \right) (n < 2).
\]

The mean residual life (MRL) function or the life expectation at age $t$ defined by $Z_1(t) = E[(X - t) | X > t]$ is given, which represents the expected additional life length for a unit which is alive at age $t$. The MRL of $X$ can be obtained when $n = 1$ in $Z_n(t)$ equation. The mean inactivity time (MIT) or mean waiting time also called the mean reversed residual life function (MRR) is given by $Z_1(t) = E[(t - X) | X \leq t]$ and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$. The MIT of the OLLP-IR distribution can be obtained easily when $n = 1$ in $Z_n(t)$ equation.

Order statistics
Suppose $X_1, ..., X_n$ is a random sample from any OLLP-G model, let $X_{i:n}$ denote the $i^{th}$ order statistic. The PDF of $X_{i:n}$ can be expressed as $f_{i:n}(x) = \binom{n}{i-n} \sum_{j=0}^{n-i} (-1)^j \binom{n-j}{j} F(x)^j (1 - F(x))^{n-j-i}$. Then, we can write the density function of $X_{i:n}$ as
\[
f_{i:n}(x) = \sum_{k=0}^\infty d_{i,k} h_{i+k+1}(x), \tag{12}
\]
where $d_{i,k} = \frac{n!(i+1)(i-1)!d_{i+1}}{(i+k+1)(n-i)!l_{i+1}} \sum_{j=0}^{n-i} (-1)^j \binom{n-j}{j} l_{j+i-1,k}$ and the quantities $\zeta_{j+i-1,k}$ can be determined with $\zeta_{j+i-1,0} = d_{0}^{j+i-1}$ and recursively for $k \geq 1$, $\zeta_{j+i-1,k} = (k d_{0})^{-1} \sum_{m=1}^{k} [m(j + i) - k] \zeta_{j+i-1,k-m}$. Equation (12) is the main result of this section. It reveals that the PDF of the OLLP-IR order statistics is a linear combination of IR density functions. So, several mathematical quantities of the OLLP-IR order statistics such as ordinary, incomplete and factorial moments, mean deviations and several others can be determined from those quantities of the IR distribution. Then
\[
E(X_{i:n}^q) = \sum_{l,k=0}^\infty d_{i,k} \eta^q (k + 1) \eta^r \left( 1 - \frac{q}{2} \right) (q < 2),
\]

4. Maximum likelihood estimation
Here, we consider estimation of the unknown parameters of the OLLP-IR distribution by the maximum likelihood method. Let $x_1, ..., x_n$ be a random sample from the OLLP-G distribution with a $4 \times 1$ parameter vector. The log-likelihood function for $\Psi$ is given by
\[
\ell(\Psi) = n \log \alpha + n \log \beta + n \log \eta + n \log 2 - n \log (\tau(\beta)) (2 + 1) \sum_{i=1}^n \log(x_i) + \sum_{i=1}^n \log \left[ \zeta_{i}^{(q)} \right] + (\alpha - 1) \sum_{i=1}^n \log \left( \exp(\beta) - \exp \left[ \beta \zeta_{i}^{(q)} \right] \right) + \beta \sum_{i=1}^n \zeta_{i}^{(q)} + (\alpha - 1) \sum_{i=1}^n \log \left( \exp \left[ \beta \zeta_{i}^{(q)} \right] - 1 \right) - 2 \sum_{i=1}^n \log \left( \begin{array}{c} \exp \left[ \beta \zeta_{i}^{(q)} \right] - 1 \\ \tau(\beta) \end{array} \right)^a + \left( 1 - \frac{\exp \left[ \beta \zeta_{i}^{(q)} \right]}{\tau(\beta)} \right)^a
\]
where
\[
\zeta_{i}^{(q)} = \exp[-\eta^2 x_i^{-2}].
\]

The components of the score vector
\[
U(\Psi) = \frac{\partial \ell(\Psi)}{\partial \Psi} = \left( U_{\alpha}, U_{\beta}, U_{\eta}, U_{\delta(\beta)} \right),
\]
can be easily derived. Procedures of the are available in literature so we can ignore this for avoiding the redundant details.
5 Simulation studies

Upon (14), we simulate the OLLP-IR model by taking \( n = 20, 50, 200, 500 \) and 1000. For each sample size, we evaluate the ML estimations (MLEs) of the parameters using the optim function of the R software (see the R code in the Appendix). Then, we repeat this process 1000 times and compute the averages of the estimates (AEs), biases (Bias) and mean squared errors (MSEs). Table 1 gives all simulation results. The values in Table 2 indicate that the MSEs decrease toward zero when \( n \) increases for all settings of \( \alpha, \beta \) and \( \eta \), as expected under first-order asymptotic theory. The AEs of the parameters tend to be closer to the true parameter values (I: \( \alpha = 1.5, \beta = 1.5 \) and \( \eta = 2.5 \) and II: \( \alpha = 2.5, \beta = 0.5 \) and \( \eta = 2.5 \)) when \( n \) increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. Table 2 gives the AEs and MSEs based on 1000 simulations of the OLLP-IR distribution for some values of \( a \) and \( b \) when by taking \( n = 20, 50, 150, 500 \) and 1000.

Table 2: The AEs, biases and MSEs based on 1000 simulations.

| n     | AE  | MSE   | AE   | MSE   |
|-------|-----|-------|------|-------|
| 20    | (I) | \( \alpha \) | 1.793350 | 0.861535 | (II) | \( \alpha \) | 2.767703 | 0.990319 |
|       | \( \beta \) | 1.845321 | 0.314960 | \( \beta \) | 0.680413 | 0.390565 |
|       | \( \eta \) | 2.679219 | 0.523719 | \( \eta \) | 2.683424 | 0.518902 |
| 50    | \( \alpha \) | 1.594410 | 0.695429 | \( \alpha \) | 2.573102 | 0.637859 |
|       | \( \beta \) | 1.697538 | 0.305324 | \( \beta \) | 0.606314 | 0.193455 |
|       | \( \eta \) | 2.559800 | 0.359734 | \( \eta \) | 2.609100 | 0.396730 |
| 200   | \( \alpha \) | 1.525549 | 0.494399 | \( \alpha \) | 2.547899 | 0.415617 |
|       | \( \beta \) | 1.530891 | 0.225603 | \( \beta \) | 0.556720 | 0.099810 |
|       | \( \eta \) | 2.515467 | 0.125894 | \( \eta \) | 2.534517 | 0.275609 |
| 500   | \( \alpha \) | 1.505480 | 0.191170 | \( \alpha \) | 2.503258 | 0.225699 |
|       | \( \beta \) | 1.508474 | 0.008005 | \( \beta \) | 0.513240 | 0.067683 |
|       | \( \eta \) | 2.501775 | 0.016781 | \( \eta \) | 2.505690 | 0.038110 |
| 1000  | \( \alpha \) | 1.500711 | 0.002130 | \( \alpha \) | 2.500219 | 0.001103 |
|       | \( \beta \) | 1.500843 | 0.000215 | \( \beta \) | 0.500205 | 0.006102 |
|       | \( \eta \) | 2.500111 | 0.000405 | \( \eta \) | 2.500136 | 0.000224 |

6. Real data modeling

This section presents two applications of the new distribution using real data sets. We shall compare the fit of the new distribution with the Weibull Inverse Weibull (W-IW), exponentiated IW (E-IW), Kumaraswamy IW(Kum-IW), beta IW (B-IW) transmuted IW (T-IW), gamma extended IW (GE-IW), Marshall-Olkin IW (MO-IW), MOKum-IW, generalized MO-IW(GMO-IW), KumMO-IW and IW distributions. The PDFs of the competitive model are available in statistical literature. The unknown parameters of the above PDFs are all positive with the exception of the first setting of the parameters. The AEs and MSEs of the parameters tend to be closer to the true parameter values (I: \( \alpha = 1.5, \beta = 1.5 \) and \( \eta = 2.5 \) and II: \( \alpha = 2.5, \beta = 0.5 \) and \( \eta = 2.5 \)) when \( n \) increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. Table 2 gives the AEs and MSEs based on 1000 simulations of the OLLP-IR distribution for some values of \( a \) and \( b \) when by taking \( n = 20, 50, 150, 500 \) and 1000.
### Table 3: The statistics AIC, BIC, HQIC and CAIC values for breaking stress data.

| Model  | Measures | AIC | BIC | HQIC | CAIC |
|--------|----------|-----|-----|------|------|
| OLLP-IR |          | 144.76 | 152.58 | 147.92 | 146.01 |
| OLLP-IE |          | 381.85 | 390.19 | 382.06 | 385.24 |
| W-IW    |          | 294.5 | 304.9 | 298.7 | 294.9 |
| E-IW    |          | 295.7 | 303.5 | 298.9 | 296.0 |
| Kum-IW  |          | 297.1 | 307.5 | 301.3 | 297.5 |
| B-IW    |          | 311.1 | 321.6 | 315.4 | 311.6 |
| GE-IW   |          | 312.0 | 332.4 | 316.2 | 312.4 |
| IW      |          | 348.3 | 353.5 | 350.4 | 348.4 |
| T-IW    |          | 350.5 | 358.3 | 353.6 | 350.7 |
| MO-IW   |          | 351.3 | 359.1 | 354.5 | 351.6 |

### Table 4: MLEs and their standard errors (in parentheses) for breaking stress of carbon fiber data.

| Model  | Estimates | α,β,η | α,β,a,b | α,β,a | α,β |
|--------|-----------|-------|---------|-------|-----|
| OLLP-IR(α,β,η) | 0.87510 | 1.9025 | 1.3623 | (0.000) | (4.3222) |
| OLLP-IE(α,β,η) | 0.0670 | 18.5425 | 16.4366 | (0.08291) | (20.377) |
| W-IW(α,β,a,b) | 2.2231 | 0.355 | 6.9721 | (11.409) | (113.811) |
| Kum-IW(α,β,a,b) | 2.0556 | 0.4654 | 6.2815 | (0.071) | (0.00701) |
| B-IW(α,β,a,b) | 1.6097 | 0.4046 | 22.0143 | (2.498) | (21.432) |
| GE-IW(α,β,a,b) | 1.3692 | 0.4776 | 27.6452 | (2.017) | (14.136) |
| E-IW(α,β,a) | 69.1489 | 0.5019 | 145.3275 | (57.349) | (122.924) |
| T-IW(α,β,a) | 1.9315 | 1.7435 | 0.0819 | (0.097) | (0.076) |
| MO-IW(α,β,a) | 2.3066 | 1.5796 | 0.5988 | (0.498) | (0.16) |
| IW(α,β) | 1.8705 | 1.7766 | 1.8705 | (0.112) | (0.113) |

### Table 5: The statistics AIC, BIC, HQIC and CAIC values for glass fiber data.

| Model  | Measures | AIC | BIC | HQIC | CAIC |
|--------|----------|-----|-----|------|------|
| OLLP-IR |          | 66.72 | 73.15 | 69.25 | 67.12 |
| OLLP-IE |          | 101.3 | 109.1 | 104.4 | 101.5 |
| B-IW    |          | 68.60 | 77.20 | 72.00 | 69.30 |
| GE-IW   |          | 69.60 | 78.10 | 72.90 | 70.30 |
| IW      |          | 97.70 | 102.0 | 99.40 | 97.90 |
| T-IW    |          | 100.1 | 106.5 | 102.6 | 100.5 |
| MO-IW   |          | 101.7 | 108.2 | 104.2 | 102.1 |
Table 5: The statistics AIC, BIC, HQIC and CAIC values for glass fiber data.

| Model                  | Estimates                  |
|------------------------|----------------------------|
| OLLP-IR(η,θ,α)         | 1.0125                     |
|                        | (4.772)                    |
| OLLP-IE(η,θ,α)         | 0.8419                     |
|                        | (0.000)                    |
| B-IW(α,β,a,b)          | 2.0518                     |
|                        | (0.986)                    |
| GE-IW(α,β,a,b)         | 1.6625                     |
|                        | (0.952)                    |
| T-IW(α,β,a)            | 1.3068                     |
|                        | (0.034)                    |
| MO-IW(α,β,a)           | 1.5441                     |
|                        | (0.226)                    |
| IW(α,β)                | 1.264                      |
|                        | (0.059)                    |

Figure 1: Box plots.
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Figure 2: Q-Q plots.

Figure 3: TTT plots.
Figure 4: P-P plots.

Figure 5: Estimated PDFs.
Figure 6: Estimated HRFs.

Figure 7: Estimated CDFs.
7. Conclusions
A new extension of the Inverse Rayleigh model is proposed and studied. Some of its fundamental properties are derived. We assessed the performance of the maximum likelihood estimators via a simulation study. The mean, variance, skewness and kurtosis of the new distribution are computed numerically using the R software. The skewness of the new distribution is always positive, the kurtosis is always more than 3. The importance of the new model is shown via two applications to real data sets. The new model is better fit than other important competitive models based on two real data sets.

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