CROSSED PRODUCTS BY TWISTED PARTIAL ACTIONS AND GRADED ALGEBRAS

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Abstract. For a twisted partial action \( \Theta \) of a group \( G \) on an (associative non-necessarily unital) algebra \( A \) over a commutative unital ring \( k \), the crossed product \( A \rtimes_{\Theta} G \) is proved to be associative. Given a \( G \)-graded \( k \)-algebra \( B = \bigoplus_{g \in G} B_g \) with the mild restriction of homogeneous non-degeneracy, a criteria is established for \( B \) to be isomorphic to the crossed product \( B_1 \rtimes_{\Theta} G \) for some twisted partial action of \( G \) on \( B_1 \). The equality \( B_g B_g^{-1} B_g = B_g \) (\( \forall g \in G \)) is one of the ingredients of the criteria, and if it holds and, moreover, \( B \) has enough local units, then it is shown that \( B \) is stably isomorphic to a crossed product by a twisted partial action of \( G \).

1. INTRODUCTION

In the Theory of Operator Algebras partial actions of groups appeared as a general approach to study \( C^* \)-algebras generated by partial isometries, several relevant classes of which turned out to be crossed products by partial actions. Crossed products classically are in the center of the rich interplay between dynamical systems and operator algebras, and the efforts in generalizing them produce structural knowledge on algebras underlied by the new constructions. Thus the notion of a \( C^* \)-crossed product by a partial automorphism, given in [19], permitted to characterize the approximately finite \( C^* \)-algebras, the Bunce-Deddence and the Bunce-Deddence-Toeplitz algebras as such crossed products [20], [21]. Further generalizations and steps were made in [34],
In [27] a machinery was developed, based on the interaction between partial actions and partial representations, enabling to study representations and the ideal structure of partial crossed products, and including into consideration prominent examples such as the Toeplitz \( C^* \)-algebras of quasi-lattice ordered groups, as well as the Cuntz-Krieger algebras. The technique was also used in [26] to define and study Cuntz-Krieger algebras for arbitrary infinite matrices, and recently these ideas were also applied to Hecke algebras defined by subnormal subgroups of length two [25], as well as to \( C^* \)-algebras generated by crystals and quasi-crystals [6].

The general notion of a (continuous) twisted partial action of a locally compact group on a \( C^* \)-algebra (a twisted partial \( C^* \)-dynamical system) and the corresponding crossed products were introduced in [22], where the associativity of crossed products was proved by means of approximate identities. The construction permitted to show in particular that any second countable \( C^* \)-algebraic bundle\(^1\) with stable unit fibre is a crossed product by a twisted partial action [22].

Algebraic counterparts for some of the above mentioned notions were introduced and studied in [14] and [13], stimulating further investigations in [5], [8], [9], [10], [11], [15], [16], [17], [18], [28], [29] and [33]. Interesting results have been obtained also in [36] and [38] (with respect to the topological part of [36] see also [1]). In [36] the authors point out several old and more recent results in various areas of mathematics underlied by partial group actions, and, more generally, the importance of the partial symmetry is discussed in [37], while developing the theory of inverse semigroups as a natural approach to such symmetries.

The goal of the present article is to pursue the algebraic part of the programm by introducing the twisted partial actions of groups on abstract algebras over commutative rings and the corresponding crossed products, and comparing them with general graded algebras. The reader will note a change in our treatment compared to previous ones: partial actions on rings, as well as on other structures, were defined so far in terms of intersections of the involved domains, and in the present paper we prefer to do it in terms of products. Moreover, the domains are supposed to be idempotent and pairwise commuting ideals. Observe firstly that this does not mean a change with respect to \( C^* \)-algebras: in that context the domains are closed two-sided ideals, and a product of such two ideals coincides with their intersection, in particular, they commute and are idempotent. Secondly, in the majority of the algebraic situations considered so far, the domains are unital (or more generally \( s \)-unital) rings and, as a consequence, again the products coincide with the intersections. Thirdly, the “product definition” serves our treatment of graded rings, whereas it is not even clear to us what would be the “nicest” definition in terms of intersections.

The definition of a twisted partial action incorporates a “partial 2-cocycle equality” which is definitely an associativity ingredient. The above mentioned conditions on the

\(^1\)A \( C^* \)-algebraic bundle is roughly a “\( C^* \)”-algebra graded by a locally compact group.
domains, i.e. to be idempotent and pairwise commuting, also serve the associativity purpose. In Section 2.4, after giving our main definitions, we prove the associativity of the crossed product in Theorem 2.4. This is done by means of a commuting property of left and right multipliers (2), introduced in [13] to serve the non-twisted case.

Our next question is to give a criteria for a graded algebra to be a twisted partial crossed product. It is an easy exercise that for a group $G$, a $G$-graded unital algebra $B = \bigoplus_{g \in G} B_g$ is a global crossed product $B \rtimes G$ exactly when each $B_g$ contains an element which is invertible in $B$, i.e.

$$\forall g \in G \exists u_g \in B_g, v_g \in B_g^{-1} \text{ such that } u_g v_g = v_g u_g = 1_B.$$  

The corresponding criteria with twisted partial actions is more involved: using some analogue of the $u_g$’s and $v_g$’s one needs to construct isomorphisms $D_g \rightarrow D_g^{-1}$ formed by certain multipliers and everything should be combined into a twisted partial action $\Theta$, so that the given $G$-graded algebra $B = \bigoplus_{g \in G} B_g$ will be $B \rtimes G$. The easy part is to determine the $D_g$’s: simply looking at the partial crossed product, one readily comes to define: $D_g = B_g B_g^{-1} = \{ \sum a_i b_i : a_i \in B_g^{-1}, b_i \in B_g \}$. Next a crucial point comes: each $B_g$ is a $(D_g, D_g^{-1})$-bimodule, and one immediately forms the surjective Morita context $(D_g, D_g^{-1}, B_g, B_g^{-1})$ in which the bimodule maps are determined by the product in $B$. The rings in the context are non-necessarily unital and if we start with an arbitrary $G$-graded $B$, they may be non-idempotent and even degenerate (in the sense of Section 3). So a restriction which guarantees that the $D_g$’s are idempotent is needed and there is also a reasonable assumption designed to avoid degeneracy. The latter is provided by the homogeneous non-degeneracy condition (16) (see Section 3) which we impose on our graded algebra\(^2\). As to the first one, $D_g^2 = D_g$ is a direct consequence of the equality

$$(1) \quad B_g B_g^{-1} B_g = B_g \quad (\forall g \in G),$$

which is immediately seen to be true for twisted partial crossed products (see Section 3). The latter equality also has several other useful consequences given in Lemma 5.3, and it enters as one of the ingredients of our criteria. In particular, it implies that the $D_g$’s pairwise commute. The other ingredient deals with the $u_g$’s and $v_g$’s we are looking for. To find them one takes the linking algebra (also called Morita context ring)

$$C_g = \left( \begin{array}{cc} D_g & B_g \\ B_g^{-1} & D_g^{-1} \end{array} \right),$$

and it turns out that the elements $u_g$ and $v_g$ are multipliers of $C_g$ with

$$u_g v_g = e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad v_g u_g = e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  

The matrices $e_{11}$ and $e_{22}$ are not elements of $C_g$ in general, but they can be easily interpreted as multipliers (see Section 3). That crossed products by twisted partial

\(^2\)Graded algebras with (16) are often called non-degenerate in the literature, however, we use the latter term in a different sense (see Section 3).
The consequences of the existence of such multipliers in the linking algebra of a given Morita context is analyzed in Section 4. An isomorphism between the rings in the context comes out as expected, as well as a couple of maps with a list of associativity properties (see Proposition 4.1) being used in further considerations. The next task is to get out from the constraints of a single Morita context in order to construct the multipliers which will form the partial twisting. This is done in Section 5. Then the criteria, Theorem 6.1, is proved in Section 6 with the restriction of homogeneous non-degeneracy of $B$. The condition of the existence of the $u_g$'s and $v_g$'s can be replaced by a more straightforward condition, which in practice is easier to check, if we assume that each $D_g$ is an $s$-unital ring (see Theorem 6.5).

The main result of [22] (Theorem 7.3) says that a second countable $C^*$-algebraic bundle $B$, which satisfies a certain regularity condition, can be obtained as a twisted partial $C^*$-crossed product. The regularity condition is based on the existence of a partial isometry $u$ in the multiplier algebra of a certain linking algebra such that $uu^* = e_{11}$ and $u^*u = e_{22}$. If the unit fibre $B_1$ is stable, i.e. $B_1$ is isometrically isomorphic to $B_1 \otimes \mathcal{K}$, where $\mathcal{K}$ is the $C^*$-algebra of the compact operators on an infinite dimensional separable Hilbert space, then using [7], the regularity condition is guaranteed (see [22, Propositions 5.2, 7.1]). In [7] the authors prove the interesting fact that two strongly Morita equivalent $C^*$-algebras $A$ and $A'$ with countable approximate units are necessarily stably isomorphic, i.e. $A \otimes \mathcal{K} \cong A' \otimes \mathcal{K}$. It is also shown that this fails in the absence of countable approximate units. The condition of $B$ to be second countable serves to ensure the existence of countable approximate unities (or equivalently the existence of strictly positive elements) which is needed when using [7].

It is our next main goal to establish an algebraic version of Theorem 7.3 from [22]. Any $C^*$-algebra has approximate unities, which are “units in limit”, and it is an analytic property. Our algebraic analogue of this are the local units (see Section 7 for the definitions). The property of having countable approximate unities then corresponds to that of having a countable set of local units. It is easily seen that such a set of idempotents can be orthogonalized (see Section 7), and we shall work in the more general situation of rings with orthogonal local units. One also should note that any closed ideal in a $C^*$-algebra is again a $C^*$-algebra, so it possesses approximate units. For an abstract ring with local units the analogue is not true, a two-sided ideal in such a ring may not have local units. So we deal with a $G$-graded $B$ with enough local units, which means that each $D_g$ has orthogonal local units. Further, $\mathcal{K}$ can be viewed as the $C^*$-direct (also called inductive) limit of the matrix algebras $M_n(\mathbb{C})$, in which $M_n(\mathbb{C})$ is embedded into the left-upper corner of $M_{n+1}(\mathbb{C})$. The algebraic direct limit of the $M_n(\mathbb{C})$'s is the algebra $\text{FMat}_\omega(\mathbb{C})$ of all $\omega \times \omega$-matrices over $\mathbb{C}$ which have only finitely many non-zero entries, where $\omega$ stands for the first infinite ordinal. Since we do not impose restrictions on the cardinalities of the orthogonal units, our analogue of $\mathcal{K}$ will be $\text{FMat}_X(k)$, where $X$ is an appropriate infinite set of indices and $k$ is the
Our aim is to prove Theorem 8.5 which says that if \( B \) is a \( G \)-graded \( k \)-algebra with enough local units which satisfies (1), then \( \text{FMat}_X(B) \), endowed with the grading directly extended from \( B \), is the crossed product by a twisted partial action of \( G \) on \( \text{FMat}_X(B_1) \). Our criteria permits us to work in a single Morita context \( (R, R', R M R', R' M' R) \). The main tool is the concept of a finitely determined map, defined in Section 5. In the same section we build up several finitely determined maps in order to use them in the Eilenberg trick which results into a finitely determined isomorphism \( \psi : R(X) \rightarrow M(X) \) (see Proposition 7.6), where \( X \) is an appropriate infinite index set. Then in Section 8 the main step is done in Theorem 8.2, in the proof of which the map \( \psi \) is interpreted as what we call a row and column summable \( X \times X \)-matrix over \( \text{Hom}_R(R, M) \), which can be used to define the multiplier \( u \) (the analogue of the above \( u_g \)) of the linking algebra of the Morita context \( (\text{FMat}_X(R), \text{FMat}_X(R'), \text{FMat}_X(M), \text{FMat}_X(M')) \). The multiplier \( v \) can be defined by using the inverse of \( \psi \) which is also finitely determined. In practice we need only to give two certain maps and Proposition 6.2 guarantees the existence of \( u \) and \( v \).

Theorem 8.5 is then a quick consequence of Theorem 8.2 thanks to the criteria. An algebraic version of the above mentioned stable isomorphism result from [7] is given in Corollary 8.4, which is an immediate by product of Theorem 8.2 (see also [4]).

In Section 9 we show by means of examples with uncountable local units that Theorem 8.2 and Theorem 8.5 fail if we abandon the orthogonality condition on local units.

2. Associativity of crossed products by twisted partial actions

In all what follows \( k \) will be a commutative associative unital ring, which will be the base ring for our algebras. The latter will be assumed to be associative and non-necessarily unital. Let \( A \) be such an algebra. We remind that the multiplier algebra \( M(A) \) of \( A \) is the set

\[
M(A) = \{(R, L) \in \text{End}(A A) \times \text{End}(A A) : (aR)b = a(Lb) \text{ for all } a, b \in A\}
\]

with component-wise addition and multiplication (see [13] or [30, 3.12.2] for more details). Here we use the right hand side notation for homomorphisms of left \( A \)-modules, while for homomorphisms of right modules the usual notation shall be used. In particular, we write \( a \mapsto aR \) and \( a \mapsto La \) for \( R : A \rightarrow A, L : A \rightarrow A \) with \( a \in A \). For a multiplier \( w = (R, L) \in M(A) \) and \( a \in A \) we set \( aw = aR \) and \( wa = La \). Thus one always has \( (aw)b = a(wb) \) \((a, b \in A)\). The first (resp. second) components of the elements of \( M(A) \) are called right (resp. left) multipliers of \( A \).

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\Theta = \{\{D_g\}_{g \in G}, \{\theta_g\}_{g \in G}, \{w_{g, h}\}_{(g, h) \in G \times G}\},
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where for each \( g \in G, D_g \) is a two-sided ideal in \( A \), \( \theta_g \) is an isomorphism of \( k \)-algebras \( D_{g^{-1}} \rightarrow D_g \), and for each \( (g, h) \in G \times G, w_{g, h} \) is an invertible element from \( M(D_g D_{gh}) \),

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satisfying the following postulates, for all \( g, h \) and \( t \) in \( G \):

(i) \( D_{g}^{2} = D_{g}, \ D_{g} \cdot D_{h} = D_{h} \cdot D_{g} \);

(ii) \( D_{1} = A \) and \( \theta_{1} \) is the identity map of \( A \);

(iii) \( \theta_{g}(D_{g^{-1}} \cdot D_{h}) = D_{g} \cdot D_{gh} \);

(iv) \( \theta_{g} \circ \theta_{h}(a) = w_{g,h} \theta_{gh}(a) w_{g,h}^{-1}, \ \forall a \in D_{h^{-1}} \cdot D_{h^{-1}g^{-1}} \);

(v) \( w_{1,g} = w_{g,1} = 1 \);

(vi) \( \theta_{g}(aw_{h,t})w_{g,ht} = \theta_{g}(a)w_{g,h}w_{gh,t}, \ \forall a \in D_{g^{-1}} \cdot D_{h} \cdot D_{ht} \).

Some comments are needed concerning the above definition. It obviously follows from (i) that a finite product of ideals \( D_{g} \cdot D_{h} \cdot \ldots \) is idempotent, and

\[
\theta_{g}(D_{g^{-1}} \cdot D_{h} \cdot D_{f}) = D_{g} \cdot D_{gh} \cdot D_{gf}
\]

for all \( g, h, f \in G \), by (iii). Thus all multipliers in (vi) are applicable. Given an (associative) algebra \( \mathcal{I} \) with \( \mathcal{I}^{2} = \mathcal{I} \), by [13, Prop. 2.5] one has

\[
(wx)w' = w(xw')
\]

for any \( w, w' \in \mathcal{M}(\mathcal{I}), \ x \in \mathcal{I} \). This explains the absence of brackets in the right hand side of (iv).

Observe also that (iii) implies

\[
\theta_{g}^{-1}(D_{g} \cdot D_{h}) = D_{g^{-1}} \cdot D_{g^{-1}h},
\]

for all \( g, h \in G \).

**Definition 2.2.** Given a twisted partial action \( \Theta \) of \( G \) on \( A \), the crossed product \( A \rtimes_{\Theta} G \) is the direct sum:

\[
\bigoplus_{g \in G} D_{g} \delta_{g},
\]

in which the \( \delta_{g} \)'s are symbols. The multiplication is defined by the rule:

\[
(a_{g} \delta_{g}) \cdot (b_{h} \delta_{h}) = \theta_{g}(\theta_{g}^{-1}(a_{g})b_{h})w_{g,h} \delta_{gh}.
\]

Here \( w_{g,h} \) acts as a right multiplier on \( \theta_{g}(\theta_{g}^{-1}(a_{g})b_{h}) \in \theta_{g}(D_{g^{-1}} \cdot D_{h}) = D_{g} \cdot D_{gh} \).

An element \( a \) in an algebra \( A \) obviously determines the multiplier \( (R_{a}, L_{a}) \in \mathcal{M}(A) \) where \( xR_{a} = xa \) and \( L_{a}x = ax (x \in A) \). If \( \mathcal{I} \) is a two-sided ideal in \( A \) then this multiplier evidently restricts to one of \( \mathcal{I} \) which shall be denoted by the same pair of symbols \( (R_{a}, L_{a}) \). Given an isomorphism \( \alpha : \mathcal{I} \rightarrow \mathcal{J} \) of algebras and a multiplier \( u = (R, L) \) of \( \mathcal{I} \), we have that \( u^{\alpha} = (\alpha^{-1}R\alpha, \alpha L\alpha^{-1}) \) is a multiplier of \( \mathcal{J} \). Observe that the effect
of $\alpha^{-1}Ra$ on $x \in \mathcal{I}$ is $x \cdot (\alpha^{-1}Ra) = \alpha(\alpha^{-1}(x) \cdot R)$, whereas $(\alpha L \alpha^{-1}) \cdot x = \alpha(L \cdot \alpha^{-1}(x))$.

Before proving the associativity of $\mathcal{A} \times_\Theta G$ we separate two technical equalities.

**Lemma 2.3.** We have the following two properties:

(i) 

$$a \theta_h(\theta_h^{-1}(b)c) = \theta_h(\theta_h^{-1}(ab)c),$$

for any $a, c \in \mathcal{A}$, $b \in \mathcal{D}_h$ and $h \in G$;

(ii) 

$$[\theta^{-1}_{gh}(w_{g,h}\theta_{gh}(x))]c = \theta^{-1}_{gh}(w_{g,h}\theta_{gh}(x)c),$$

for any $x \in \mathcal{D}_{h^{-1}} \cdot \mathcal{D}_{h^{-1}}g^{-1}$, $g, h \in G$ and $c \in \mathcal{A}$.

**Proof.** (i) Since $\theta_h : \mathcal{D}_{h^{-1}} \rightarrow \mathcal{D}_h$ is an isomorphism, $(\theta_h^{-1}R_c\theta_h, \theta_hL_c\theta_h^{-1})$ is a multiplier of $\mathcal{D}_h$, and applying (2) for the multipliers $(\theta_h^{-1}R_c\theta_h, \theta_hL_c\theta_h^{-1})$ and $(R_a, L_a)$ we see that $L_a \cdot (b \cdot (\theta_h^{-1}R_c\theta_h)) = (L_a \cdot b) \cdot (\theta_h^{-1}R_c\theta_h)$. But this is precisely what (i) does say.

(ii) By (iii) of Definition 2.1, $\theta_{gh}$, restricted to $\mathcal{D}_{h^{-1}} \cdot \mathcal{D}_{h^{-1}}g^{-1}$, gives an isomorphism $\mathcal{D}_{h^{-1}} \cdot \mathcal{D}_{h^{-1}}g^{-1} \rightarrow \mathcal{D}_g \cdot \mathcal{D}_{gh}$. Hence $w_{g,h}^{\theta_{gh}^{-1}}$ is a multiplier of $\mathcal{D}_{h^{-1}} \cdot \mathcal{D}_{h^{-1}}g^{-1}$, and combining it in (2) with the multiplier $(R_c, L_c)$ we obtain

$$[(\theta_h^{-1}w_{g,h}\theta_{gh}) \cdot x] \cdot R_c = (\theta_{gh}^{-1}w_{g,h}\theta_{gh}) \cdot [x \cdot R_c],$$

which is exactly (ii). \hfill \Box

**Theorem 2.4.** The crossed product $\mathcal{A} \times_\Theta G$ is associative.

**Proof.** Obviously, $\mathcal{A} \times_\Theta G$ is associative if and only if

$$\theta_{g,h}^{-1}(a)\theta_{h}^{-1}(b)c\delta_t = \theta_{g,h}^{-1}(ab)c\delta_t$$

for arbitrary $g, h, t \in G$ and $a \in \mathcal{D}_g, b \in \mathcal{D}_h, c \in \mathcal{D}_t$. Computing the left hand side of the above equality, we have

$$\theta_{g,h}^{-1}(a)w_{g,h}\theta_{h}^{-1}(b)c\delta_t = \theta_{g,h}^{-1}[(\theta_{g,h}^{-1}(a)b)w_{g,h}]c\delta_t.$$ 

On the other hand,

$$a\delta_g(b\delta_h)\delta_t = a\delta_g\theta_h(\theta_h^{-1}(b)c)w_{h,t}\delta_{ht} = \theta_g[\theta_g^{-1}(a)\theta_h(\theta_h^{-1}(b)c)w_{h,t}]w_{g,h}\delta_{ght} = \theta_g[\theta_g^{-1}(a)\theta_h(\theta_h^{-1}(b)c)]w_{g,h}w_{g,h}w_{h,t}\delta_{ght},$$

by the co-cycle equality (vi) of Definition 2.1, taking into account that

$$\theta_g^{-1}(a)\theta_h(\theta_h^{-1}(b)c) \in \mathcal{D}_{g^{-1}} \cdot \theta_h(\mathcal{D}_{h^{-1}} \cdot \mathcal{D}_t) = \mathcal{D}_{g^{-1}} \cdot \mathcal{D}_h \cdot \mathcal{D}_t.$$
The last equality is obtained by using (iii) of Definition 2.1. Comparing the two sides of (3) we may cancel $w_{gh,t}$ as it is invertible. Observing also that $\theta_g^{-1}(a)$ runs over $\mathcal{D}_{g^{-1}}$ when $a$ runs over $\mathcal{D}_g$, we have that (3) holds if and only if

$$\theta_{gh}\{\theta_g^{-1}[\theta_g(ab)w_{g,h}]c\} = \theta_g[\theta_h(\theta_h^{-1}(b)c)]w_{g,h}$$

is verified for any $g, h \in G$ and $a \in \mathcal{D}_{g^{-1}}, b \in \mathcal{D}_h, c \in \mathcal{A}$ (take $t = 1$ to see that $c$ may be arbitrary in $A$). Applying (i) of Lemma 2.3 to the right hand side of (4), we have

$$\theta_g[a\theta_h(\theta_h^{-1}(b)c)]w_{g,h} = \theta_g[\theta_h(\theta_h^{-1}(ab)c)]w_{g,h}.$$  

Now $y = ab$ lies in $\mathcal{D}_{g^{-1}} \mathcal{D}_h$, and therefore it is enough to show that

$$\theta_{gh}\{\theta_g^{-1}[\theta_g(y)w_{g,h}]c\} = \theta_g[\theta_h(\theta_h^{-1}(y)c)]w_{g,h}$$

is satisfied for arbitrary $g, h \in G, y \in \mathcal{D}_{g^{-1}} \cdot \mathcal{D}_h, c \in \mathcal{A}$. Write $x = \theta_h^{-1}(y) \in \theta_h^{-1}(\mathcal{D}_{g^{-1}} \cdot \mathcal{D}_h) = \mathcal{D}_{h^{-1}} \cdot \mathcal{D}_{h^{-1}g^{-1}}$. Then by (iv) of Definition 2.1 the left hand side of (5) becomes

$$\theta_{gh}\{\theta_g^{-1}[\theta_g \circ \theta_h(x)w_{g,h}]c\} = \theta_{gh}\{\theta_g^{-1}[w_{g,h}\theta_{gh}(x)c]\}.$$  

Applying (ii) of Lemma 2.3 we see that the latter equals to $\theta_{gh}\{\theta_g^{-1}[w_{g,h}\theta_{gh}(xc)]\} = w_{g,h}\theta_{gh}(xc)$. Thus taking $z = xc$, (5) becomes

$$w_{g,h}\theta_{gh}(z) = \theta_g[\theta_h(z)]w_{g,h}$$

with arbitrary $g, h \in G, z \in \mathcal{D}_{h^{-1}} \cdot \mathcal{D}_{h^{-1}g^{-1}}$, but this is (iv) of Definition 2.1. 

3. Some multipliers related to crossed products

We first derive some consequences from the definition of a twisted partial action $\Theta$. Taking $h = g^{-1}, t = g$ in (vi) of Definition 2.1 we easily obtain that

$$\theta_{g}(aw_{g^{-1},g}) = \theta_g(a)w_{g,g^{-1}}$$

for any $g \in G, a \in \mathcal{D}_{g^{-1}}$. Applying the multiplier $w_{g,g^{-1}}^{-1}$ to the both sides of the above equality and replacing $a$ by $aw_{g^{-1},g}^{-1}$, one has

$$\theta_g(aw_{g^{-1},g}) = \theta_g(a)w_{g,g^{-1}}^{-1}$$

with any $g \in G, a \in \mathcal{D}_{g^{-1}}$. From (6) and (7) we also obtain

$$\theta_{g}^{-1}(aw_{g,g^{-1}}) = \theta_{g}^{-1}(a)w_{g,g^{-1}}^{-1} \forall g \in G, a \in \mathcal{D}_g;$$

and

$$\theta_{g}^{-1}(aw_{g^{-1},g}^{-1}) = \theta_{g}^{-1}(a)w_{g^{-1},g}^{-1} \forall g \in G, a \in \mathcal{D}_g.$$  

Item (iv) of Definition 2.1 implies

$$\theta_{g} \circ \theta_{g^{-1}}(x) = w_{g,g^{-1}}xw_{g,g^{-1}}^{-1}$$
with \( x \in D_g \). Taking \( x = \theta_g(a), a \in D_{g^{-1}} \) and using (7), one comes to
\[
\theta_g(w_{g^{-1},g}a) = w_{g,g^{-1}}\theta_g(a) \quad \forall g \in G, a \in D_{g^{-1}}.
\]
(10)

Similarly as above one has the following equalities:
\[
\theta_g(w_{g^{-1},g}^{-1}a) = w_{g,g^{-1}}^{-1}\theta_g(a) \quad \forall g \in G, a \in D_{g^{-1}};
\]
(11)
\[
\theta_{g^{-1}}(w_{g,g^{-1}}a) = w_{g^{-1},g}\theta_{g^{-1}}(a) \quad \forall g \in G, a \in D_g;
\]
(12)
and
\[
\theta_{g^{-1}}(w_{g^{-1},g}^{-1}a) = w_{g^{-1},g}^{-1}\theta_{g^{-1}}(a) \quad \forall g \in G, a \in D_g.
\]
(13)

Using the equality \( \theta_{g^{-1}} \circ \theta_g(x) = w_{g^{-1},g}xw_{g^{-1},g}^{-1} \) \( (x \in D_{g^{-1}}) \), we have
\[
\theta_{g^{-1}}(a) = w_{g^{-1},g}\theta_{g^{-1}}(a)w_{g^{-1},g}^{-1} \quad \forall g \in G, a \in D_g.
\]
(14)

Given a \( G \)-graded algebra \( B = \bigoplus_{g \in G} B_g \), the set
\[
C_g = \begin{pmatrix}
B_g \bigoplus B_{g^{-1}} & B_g \\
B_{g^{-1}} & B_{g^{-1}} \bigoplus B_g
\end{pmatrix}
\]
is evidently a \( k \)-algebra with respect to the usual matrix operations. Here \( B_g \bigoplus B_{g^{-1}} \)
stands for the \( k \)-span of the products \( ab \) with \( a \in B_g, b \in B_{g^{-1}} \), and similarly \( B_{g^{-1}} \bigoplus B_g \)
is defined. Obviously \( C_g \) is contained in the algebra of all \( 2 \times 2 \) matrices over \( B \), and
therefore, if \( B \) is unital, the elementary matrices \( e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \)
can be seen as multipliers for \( C_g \). However even if \( B \) is not a unital algebra, one can define
the multipliers \( e_{11} \) and \( e_{22} \) of \( C_g \) in the natural way:
\[
\begin{pmatrix} x & y \\ y' & x' \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ y' & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ y' & x' \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix},
\]
\[
\begin{pmatrix} x & y \\ y' & x' \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & y \\ y' & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ y' & x' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ y' & x' \end{pmatrix},
\]
where \( x \in B_g, x' \in B_{g^{-1}}, y \in B_g, y' \in B_{g^{-1}} \).

In the case of a crossed product \( B = A \rtimes \Theta G \) the algebra \( C_g \) is obviously of the form
\[
\begin{pmatrix}
D_g \delta_1 & D_g \delta_g \\
D_{g^{-1}} \delta_{g^{-1}} & D_{g^{-1}} \delta_{g^{-1}}
\end{pmatrix}
\]
If we assume that each algebra \( D_g \) is unital with the unity element denoted by \( 1_g \), then
$C_g$ is also unital and, consequently, $C_g$ can be identified with its multiplier algebra. Then taking the elements

$$u_g = \begin{pmatrix} 0 & 1_g \delta_g \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad v_g = \begin{pmatrix} 0 & 0 \\ w_{g^{-1},g}^{-1} \delta_g & 0 \end{pmatrix},$$

we see that $C_g$ possesses multipliers $u_g$ and $v_g$ which satisfy the equalities $u_g v_g = e_{11}$ and $v_g u_g = e_{22}$. This however holds for general crossed products:

**Proposition 3.1.** Given a crossed product $\mathcal{A} \rtimes_{\Theta} G$ by a twisted partial action $\Theta$, for each $g \in G$ there exist multipliers $u_g, v_g \in \mathcal{M}(C_g)$ such that $u_g v_g = e_{11}$ and $v_g u_g = e_{22}$.

**Proof.** Define the multipliers $u_g$ and $v_g$ as follows.

$$u_g = \begin{pmatrix} a\delta_1 & b\delta_g \\ c\delta_{g^{-1}} & d\delta_1 \end{pmatrix}, \quad v_g = \begin{pmatrix} a\delta_1 & b\delta_g \\ c\delta_{g^{-1}} & d\delta_1 \end{pmatrix},$$

where

$$u_g \begin{pmatrix} a\delta_1 & b\delta_g \\ c\delta_{g^{-1}} & d\delta_1 \end{pmatrix} = \begin{pmatrix} \theta_g(c)w_{g,g^{-1}}\delta_1 & \theta_g(d)\delta_g \\ 0 & 0 \end{pmatrix},$$

$$v_g \begin{pmatrix} a\delta_1 & b\delta_g \\ c\delta_{g^{-1}} & d\delta_1 \end{pmatrix} = \begin{pmatrix} 0 & \theta_g^{-1}(aw_{g,g^{-1}})\delta_g \\ \theta_g^{-1}(aw_{g,g^{-1}})\delta_g & 0 \end{pmatrix}.$$

Check first that $u_g$ is a multiplier. Write

$$x = \begin{pmatrix} a\delta_1 & b\delta_g \\ c\delta_{g^{-1}} & d\delta_1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} a'\delta_1 & b'\delta_g \\ c'\delta_{g^{-1}} & d'\delta_1 \end{pmatrix}.$$

By (6) we easily have

$$(xy)u_g = \begin{pmatrix} 0 & (aa' + b\theta_g(c')w_{g,g^{-1}})\delta_g \\ 0 & (c\theta_g(a') + d\delta c')w_{g,g^{-1}}\delta_1 \end{pmatrix} = x(yu_g).$$

Furthermore, for the $(1,1)$-entry of $u_g(xy)$, by (iv) of Definition 2.1, one has

$$\theta_g(c) \theta_g \circ \theta_g^{-1}(a')w_{g,g^{-1}} + \theta_g(d\delta c')w_{g,g^{-1}} =$$

$$\theta_g(c)w_{g,g^{-1}}a'w_{g,g^{-1}}w_{g,g^{-1}} + \theta_g(d)\theta_g(c')w_{g,g^{-1}},$$

which is the $(1,1)$-entry of $(u_g x) y$. On the other hand, by (6) the $(1,2)$-entry of $u_g(xy)$ equals

$$(\theta_g(c) \theta_g \circ \theta_g^{-1}(b')w_{g,g^{-1}} + \theta_g(dd'))\delta_g =$$

$$(\theta_g(c)w_{g,g^{-1}}b' + \theta_g(d)\theta_g(d'))\delta_g,$$

using again (iv) of Definition 2.1. It is easily seen that this is the $(1,2)$-entry of $(u_g x) y$, implying $u_g(xy) = (u_g x)y$. One also has

$$x(u_g y) = \begin{pmatrix} a\theta_g(c')w_{g,g^{-1}}\delta_1 & a\theta_g(d')\delta_g \\ c\theta_g^{-1}(\theta_g(c')w_{g,g^{-1}})\delta_g & c\theta_g^{-1}(\theta_g(d'))w_{g,g^{-1}}\delta_1 \end{pmatrix} = (xu_g)y.$$
applying once more (6) and (iv) of Definition 2.1. This shows that \( u_g \) is a multiplier of \( \mathcal{C}_g \).

Similarly, using (7) one has \((xy)v_g = x(yv_g)\), and by (9) and (14) it is easily seen that \((xv_g)y = x(v_gy)\). As to the equality \( v_g(xy) = (v_gx)y \), the (2,1)-entry of \( v_g(xy) \)

\[
\theta_g^{-1}(a'a + b\theta_g(c')w_{g,g^{-1}})w_{g^{-1},g}^{-1} = \theta_g^{-1}(a\theta_g^{-1}(a')w_{g^{-1},g}^{-1}g + \theta_g^{-1}(b)c')\delta_{g^{-1}},
\]

by (9). Applying (14) this equals to the (2,1)-entry of \( (v_gx)y \). Using (9) and (14) we also see that the the (2,2)-entries coincide, which shows that \( v_g(xy) = (v_gx)y \). Thus \( v_g \in \mathcal{M}(\mathcal{C}_g) \). By easy calculations we have \( u_gv_g = e_{11} \) and \( v_gu_g = e_{22} \).

We need one more property of a crossed product \( A \rtimes_{\Theta} G \) by a twisted partial action, which is in fact an immediate consequence of its definition. More precisely, we easily see that \( \mathcal{D}_g\delta_g \cdot \mathcal{D}_{g^{-1}}\delta_{g^{-1}} \cdot \mathcal{D}_g\delta_g = (\mathcal{D}_g^2)w_{g,g^{-1}}\cdot\delta_e \cdot \mathcal{D}_g\delta_g = \mathcal{D}_g\delta_g \cdot \mathcal{D}_g\delta_g = \mathcal{D}_g\delta_g \), as \( \mathcal{D}_g \) is idempotent and \( w_{g,g^{-1}} \) is an invertible multiplier of \( \mathcal{D}_g \). Thus the equality

\[
\mathcal{D}_g\delta_g \cdot \mathcal{D}_{g^{-1}}\delta_{g^{-1}} \cdot \mathcal{D}_g\delta_g = \mathcal{D}_g\delta_g
\]

holds for any \( g \in G \).

We shall work with a mild restriction on \( \Theta \), supposing that each domain \( \mathcal{D}_g \) is left and right non-degenerate. Following [13] and [32] we say that an algebra \( \mathcal{I} \) is right non-degenerate if for any \( 0 \neq x \in \mathcal{I} \) one has \( x\mathcal{I} \neq \{0\} \), and similarly is defined the concept of a left non-degenerate algebra. If each \( \mathcal{D}_g \) is left and right non-degenerate then we evidently have

\[
x \neq 0 \implies x \cdot \mathcal{D}_{g^{-1}}\delta_{g^{-1}} \neq 0 \quad \text{and} \quad \mathcal{D}_{g^{-1}}\delta_{g^{-1}} \cdot x \neq 0 \quad \forall x \in \mathcal{D}_g\delta_g.
\]

For a general \( G \)-graded algebra \( \mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g \) this corresponds to the property

\[
x \neq 0 \implies x \cdot \mathcal{B}_{g^{-1}} \neq 0 \quad \text{and} \quad \mathcal{B}_{g^{-1}} \cdot x \neq 0 \quad \forall x \in \mathcal{B}_g.
\]

We shall say that a \( G \)-graded algebra \( \mathcal{B} \) is homogeneously non-degenerate if \( \mathcal{B} \) satisfies (16). It turns out that among the homogeneously non-degenerate graded algebras the crossed products by twisted partial actions are distinguished by the property (15) and the existence of the multipliers like in Proposition 3.1. The precise statement will be given in Theorem 6.1.

Given a graded algebra \( \mathcal{B} \), write \( \mathcal{D}_g = \mathcal{B}_g\mathcal{B}_{g^{-1}} \). Observe that \( \mathcal{B}_g \) is a \( (\mathcal{D}_g, \mathcal{D}_{g^{-1}}) \)-bimodule, whereas \( \mathcal{B}_{g^{-1}} \) is a \( (\mathcal{D}_{g^{-1}}, \mathcal{D}_g) \)-bimodule, so that \( (\mathcal{D}_g, \mathcal{D}_{g^{-1}}, \mathcal{B}_g, \mathcal{B}_{g^{-1}}) \) together with the surjective maps \( \mathcal{B}_g \otimes \mathcal{B}_{g^{-1}} \to \mathcal{D}_g \) and \( \mathcal{B}_{g^{-1}} \otimes \mathcal{B}_g \to \mathcal{D}_{g^{-1}} \), determined by the multiplication in \( \mathcal{B} \), form a Morita context. In general we know nothing about the existence of an identity element in \( \mathcal{D}_g \), however in the situation in which we are interested in, the \( \mathcal{D}_g \)'s are idempotent rings and the bimodules \( \mathcal{B}_g, \mathcal{B}_{g^{-1}} \) are unital,
and so by a result of [32], \( D_{g^{-1}} \) and \( D_g \) are Morita equivalent (see Section 7 for the precise definitions). The idempotence of the \( D_g \)'s, as well as the property of the \( B_g \)'s being unital modules, is an immediate consequence of the analogue of (15) for graded algebras:

\[
(17) \quad B_g \cdot B_{g^{-1}} \cdot B_g = B_g \quad \forall g \in G.
\]

The existence of multipliers of \( B \), like in Proposition 3.1, turns out to be a powerful tool: we shall have that \( D_{g^{-1}} \) and \( D_g \) are isomorphic and, the obtained isomorphisms will fit into a twisted partial action \( \Theta \) of \( G \) on \( B_1 \) so that \( B \cong B_1 \rtimes_G G \). In order to carry out this idea we need some technical tools, which we develop in the next sections.

4. Morita context and multipliers

Let \( (\mathcal{R}, \mathcal{R}', M, M', \tau, \tau') \) be a Morita context in which \( \mathcal{R} \) and \( \mathcal{R}' \) are some (non-necessarily unital) algebras. We recall that this means that \( M \) is an \( \mathcal{R} \)-\( \mathcal{R}' \)-bimodule, \( M' \) is an \( \mathcal{R}' \)-\( \mathcal{R} \)-bimodule, \( \tau : M \otimes_{\mathcal{R}} M' \to \mathcal{R} \) is an \( \mathcal{R} \)-bimodule map, \( \tau' : M' \otimes_{\mathcal{R}} M \to \mathcal{R}' \) is an \( \mathcal{R}' \)-bimodule map, such that

\[
\tau(m_1 \otimes m') m_2 = m_1 \tau'(m' \otimes m_2), \quad \forall m_1, m_2 \in M, \ m' \in M',
\]

and

\[
\tau'(m'_1 \otimes m) m'_2 = m'_1 \tau(m \otimes m'_2), \quad \forall m'_1, m'_2 \in M', \ m \in M.
\]

One can construct the linking algebra of the Morita context, which is the set

\[
C = \left( \begin{array}{cc}
\mathcal{R} & M \\
M' & \mathcal{R}'
\end{array} \right),
\]

with the obvious addition of matrices and multiplication by scalars, and the matrix multiplication determined by the bimodule structures on \( M \) and \( M' \) and the maps \( \tau \) and \( \tau' \), so that \( m \cdot m' = \tau(m \otimes m') \) and \( m' \cdot m = \tau'(m' \otimes m) \).

If we have

\[
(18) \quad \mathcal{R}^2 = \mathcal{R}, (\mathcal{R}')^2 = \mathcal{R}', \mathcal{R}M + M\mathcal{R}' = M \quad \text{and} \quad \mathcal{R}'M' + M'\mathcal{R} = M',
\]

it is easily verified that \( C^2 = C \). Thus by [13, Prop. 2.5]

\[
(19) \quad (wx)w' = w(xw') \quad \forall w, w' \in \mathcal{M}(C), x \in C.
\]

In the case of a graded algebra \( B \) with \( \mathcal{R} = D_g, \mathcal{R}' = D_{g^{-1}}, M = B_g \) and \( M' = B_{g^{-1}} \), the linking algebra \( C \) evidently coincides with the algebra \( C_g \) considered in the previous section. If the graded algebra \( B \) satisfies (17) then obviously (18) is also satisfied.

Similarly, as it was done in the previous section for \( C_g \), one defines the multipliers \( e_{11} \) and \( e_{22} \) of \( C \) in the natural way.
We are going to examine the situation in which the linking algebra $C$ of a Morita context contains multipliers with properties announced in Proposition 3.1. More precisely, suppose that we have a Morita context $(R, R', M, M', \tau, \tau')$ such that the linking algebra $C$ is idempotent. Suppose furthermore that there exists $u, v \in \mathcal{M}(C)$ such that $uv = e_{11}$ and $vu = e_{22}$. One may assume that $e_{11}u = ue_{22} = u$ and $e_{22}v = ve_{11} = v$. For it is enough to replace $u$ by $e_{11}ue_{22}$ and $v$ by $e_{22}ve_{11}$, as $(e_{11}ue_{22})(e_{22}ve_{11}) = (uv)^4 = e_{11}$ and similarly $(e_{22}ve_{11})(e_{11}ue_{22}) = e_{22}$. Let $x = \begin{pmatrix} r & m \\ m' & r' \end{pmatrix}$ be arbitrary in $C$. Then

$$ux = e_{11}u(e_{22}x) = e_{11}u \begin{pmatrix} 0 & 0 \\ m' & r' \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}.$$ 

Moreover, by (19),

$$u \begin{pmatrix} 0 & 0 \\ m' & 0 \end{pmatrix} = u(xe_{11}) = (ux)e_{11} = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix},$$

and the $(1,1)$-entry of the latter matrix depends only on $m'$. Denote it by $u \cdot m'$. Thus we have a map of $k$-modules $M' \ni m' \mapsto u \cdot m' \in R$ which we denote by $L_u$. Similarly,

$$u \begin{pmatrix} 0 & 0 \\ 0 & r' \end{pmatrix} = u(xe_{22}) = (ux)e_{22} = \begin{pmatrix} 0 & u \cdot r' \\ 0 & 0 \end{pmatrix},$$

and we have a $k$-linear map $R' \ni r' \mapsto u \cdot r' \in M$, which we denote by the same symbol $L_u$. Thus we have

$$u \begin{pmatrix} 0 & 0 \\ m' & r' \end{pmatrix} = u(xm) = \begin{pmatrix} um' & ur' \\ 0 & 0 \end{pmatrix}.$$ 

Analogously, $u$, as a right multiplier, determines a pair of maps

$$R_u : R \ni r \mapsto r \cdot u \in M \quad \text{and} \quad R_u : M' \ni m' \mapsto m' \cdot u \in R'$$

so that

$$(\begin{pmatrix} r & 0 \\ m' & 0 \end{pmatrix} u = \begin{pmatrix} 0 & ru \\ 0 & m'u \end{pmatrix}.$$ 

In a similar way the multiplier $v$ gives rise to the pairs of $k$-linear maps:

$$L_v : R \ni r \mapsto v \cdot r \in M' \quad \text{and} \quad L_v : M' \ni m' \mapsto v \cdot m' \in R'$$

and

$$R_v : M \ni m \mapsto m \cdot v \in R' \quad \text{and} \quad R_v : R' \ni r' \mapsto r' \cdot v \in M'.$$

so that

$$v \begin{pmatrix} r & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ vr & vm \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & m \\ 0 & r' \end{pmatrix} v = \begin{pmatrix} mv & 0 \\ 0 & vr \end{pmatrix}.$$
Lemma 4.1. Let \((\mathcal{R}, \mathcal{R}', M, M', \tau, \tau')\) be a Morita context such that the linking algebra \(C\) is idempotent. Suppose that the multiplier algebra \(M(C)\) contains elements \(u\) and \(v\) with \(uv = e_{11}\) and \(vu = e_{22}\). Then the multipliers \(u\) and \(v\) can be chosen such that \(u\) determines the maps

\[ L_u : M' \ni m' \mapsto u \cdot m' \in \mathcal{R}, \quad \mathcal{R}' \ni r' \mapsto u \cdot r' \in M, \]

\[ R_u : \mathcal{R} \ni r \mapsto r \cdot u \in M, \quad M' \ni m' \mapsto m' \cdot u \in \mathcal{R}', \]

which are isomorphisms of \(k\)-modules whose inverses are respectively the maps

\[ L_v : \mathcal{R} \ni r \mapsto v \cdot r \in M', \quad M \ni m \mapsto v \cdot m \in \mathcal{R}', \]

\[ R_v : M \ni m \mapsto m \cdot v \in \mathcal{R}, \quad \mathcal{R}' \ni r' \mapsto r' \cdot v \in M', \]

determined by \(v\). Moreover, the following properties hold with arbitrary \(r, r_1, r_2 \in \mathcal{R}, r', r'_1, r'_2 \in \mathcal{R}', m, m_1, m_2 \in M, m', m'_1, m'_2 \in M' :\)

\[ u(m'r) = (um')r, \quad (r_1 r_2)u = r_1(r_2u), \quad (ru)m' = r(u m'), \]

\[ u(m'm) = (um'm), \quad (mm')u = m(m'u), \quad (ru)r' = r(ur'), \]

\[ u(r'm') = (ur')m', \quad (r'm')u = r'(m'u), \quad (m'_1 u)m'_2 = m'_1(u m'_2), \]

\[ u(r'_1 r'_2) = (ur'_1')r'_2, \quad (m'r)u = m'(ru), \quad (m'u)r' = m'(ur'), \]

and the corresponding properties involving \(v\):

\[ v(r_1 r_2) = (vr_1)r_2, \quad (rm)v = r(mv), \quad (mv)r = m(vr), \]

\[ v(mm') = (vm)m', \quad (mr')v = m(r'v), \quad (m_1 v)m_2 = m_1(v m_2), \]

\[ v(rm) = (vr)m, \quad (m'm)v = m'(mv), \quad (r'v)m = r'(vm), \]

\[ v(m'r) = (vm)r', \quad (r'_1 r'_2)v = r'_1(r'_2 v), \quad (r'v)r = r'(vr). \]

**Proof.** The pairs of maps \(L_u, R_u, L_v, R_v\) are already defined. It follows immediately from the equalities \(uv = e_{11}, vu = e_{22}\) that all these maps are isomorphisms of \(k\)-modules with inverses as announced in the lemma.

Using the fact that \(u\) is a left multiplier, we have

\[ u \left( \begin{array}{cc} 0 & 0 \\ m' & 0 \end{array} \right) \left( \begin{array}{cc} r & m \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} (um')r & (um')m \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} u(m'r) & u(mm') \\ 0 & 0 \end{array} \right), \]

which gives the first two properties enounced in the lemma. The next two properties of the first column come from

\[ u \left( \begin{array}{cc} 0 & 0 \\ 0 & r'_1 \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ m' & r'_2 \end{array} \right) = \left( \begin{array}{cc} (ur'_1)m' & (ur'_1)r'_2 \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} u(r'm') & u(r'_1 r'_2) \\ 0 & 0 \end{array} \right). \]

The second column of properties involving \(u\) is obtained from the fact that \(u\) is a right multiplier, whereas the last column comes from the equality \((xu)y = x(uy)\) with appropriate \(x, y \in \mathcal{C}\). The analogous properties involving \(v\) are obtained similarly. \(\Box\)
Remark 4.2. The list of equalities in the lemma above say that whenever we form a product of three elements one of which is \( u \) or \( v \) and the other two belong to \( M \cup M' \cup \mathcal{R} \cup \mathcal{R}' \), it is associative, i.e. we may put the brackets arbitrarily.

Remark 4.3. It is directly seen that two pairs of some \( k \)-linear maps
\[
L_u : M' \ni m' \mapsto u \cdot m' \in \mathcal{R}, \quad \mathcal{R}' \ni r' \mapsto u \cdot r' \in M,
\]
\[
R_u : \mathcal{R} \ni r \mapsto r \cdot u \in M, \quad M' \ni m' \mapsto m' \cdot u \in \mathcal{R}'
\]
determine a multiplier \( u \) of \( \mathcal{M}(\mathcal{C}) \) exactly when the equalities of the three columns involving \( u \) in Lemma 4.1 hold (and similarly for \( v \)).

Remark 4.4. One can directly check that in Lemma 4.1 the assumption that \( \mathcal{C} \) is idempotent can be replaced by that of \( (\mathcal{R}')^2 = \mathcal{R}', \mathcal{R}'M' = M'\mathcal{R} = M' \) or by the assumption \( \mathcal{R}^2 = \mathcal{R}, \mathcal{R}M = M\mathcal{R}' = M \).

The pairs of maps \( L_u \) and \( R_u \) given in Lemma 4.1 permit to establish an isomorphism between \( \mathcal{R}' \) and \( \mathcal{R} \):

**Proposition 4.5.** Let \( (\mathcal{R}, \mathcal{R}', M, M', \tau, \tau') \) be a Morita context with idempotent linking algebra \( \mathcal{C} \) such that \( \mathcal{M}(\mathcal{C}) \) contains elements \( u \) and \( v \) with \( uv = e_{11} \) and \( vu = e_{22} \). Then \( u \) and \( v \) can be chosen such that the map
\[
\theta : \mathcal{R}' \ni r' \mapsto ur'v \in \mathcal{R}
\]
is an isomorphism \( k \)-algebras, whose inverse is
\[
\mathcal{R} \ni r \mapsto vr' \in \mathcal{R}'.
\]

**Proof.** Observe first that by (19) we have
\[
\left( u \begin{pmatrix} 0 & 0 \\ 0 & r' \end{pmatrix} \right) v = u \left( \begin{pmatrix} 0 & 0 \\ 0 & r' \end{pmatrix} v \right) \quad \text{and} \quad \left( v \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \right) u = v \left( \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} u \right),
\]
which implies \( (ur'v)v = u(r'v)v \) and \( (vr)u = v(ru) \) so that it makes sense to omit brackets in \( ur'v \) and \( vr' \).

Clearly, \( (u(\mathcal{R}'))v = (M)v = \mathcal{R} \) so \( \theta \) maps \( \mathcal{R}' \) onto \( \mathcal{R} \). Moreover, because \( L_u \) and \( R_u \) are \( k \)-isomorphisms, it follows that \( \theta \) is also a \( k \)-isomorphism. Clearly \( \mathcal{R} \ni x \mapsto vxu \in \mathcal{R}' \) is the inverse of \( \theta \). Thus it remains to show that \( \theta \) preserves multiplication. This follows from the next calculations:
\[
\begin{pmatrix} \theta(r_1' r_2') & 0 \\ 0 & 0 \end{pmatrix} u \begin{pmatrix} 0 & 0 \\ 0 & r_1' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & r_2' \end{pmatrix} v = \begin{pmatrix} u \begin{pmatrix} 0 & 0 \\ 0 & r_1' \end{pmatrix} e_{22} \begin{pmatrix} 0 & 0 \\ 0 & r_2' \end{pmatrix} \end{pmatrix} v =
\begin{pmatrix} u \begin{pmatrix} 0 & 0 \\ 0 & r_1' \end{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & r_2' \end{pmatrix} v = \begin{pmatrix} \theta(r_1') & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \theta(r_2') & 0 \\ 0 & 0 \end{pmatrix} v.
\]

\( \square \)

Given a \( G \)-graded algebra \( \mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g \) which satisfies (17) and such that for any \( g \in G \) there exist \( u_g, v_g \in \mathcal{M}(\mathcal{C}_g) \) with \( u_g v_g = e_{11} \) and \( v_g u_g = e_{22} \), applying Lemma 4.1 to the Morita context \( (\mathcal{D}_g, \mathcal{D}_g^{-1}, \mathcal{B}_g, \mathcal{B}_g^{-1}, \cdot, \cdot) \), we have the isomorphisms of \( k \)-modules
\[
L_{u_g} : \mathcal{B}_{g^{-1}} \rightarrow \mathcal{D}_g, \quad R_{u_g} : \mathcal{D}_g \rightarrow \mathcal{B}_{g^{-1}} \rightarrow \mathcal{D}_{g^{-1}}
\]
with inverses
\[ L_{v_g} : \mathcal{D}_g \to \mathcal{B}_{g^{-1}}, \mathcal{B}_g \to \mathcal{D}_{g^{-1}}, R_{v_g} : \mathcal{B}_g \to \mathcal{D}_g, \mathcal{D}_{g^{-1}} \to \mathcal{B}_{g^{-1}}, \]
respectively. Moreover, by Proposition 4.5
\[ (20) \quad \theta_g : \mathcal{D}_{g^{-1}} \ni x \mapsto u_g x v_g \in \mathcal{D}_g \]
is an isomorphism of \( k \)-algebras.

5. Constructing \( w_{g,h} \)

**Lemma 5.1.** Let \( \mathcal{B} \) be a homogeneously non-degenerate \( G \)-graded algebra satisfying (17) and such that for any \( g \in G \) the multiplier algebra \( \mathcal{M}(\mathcal{C}_g) \) contains elements \( u_g \) and \( v_g \) with \( u_g v_g = e_{11} \) and \( v_g u_g = e_{22} \). Then there are isomorphisms of right \( \mathcal{B} \)-modules
\[ \tilde{L}_{u_g} : \mathcal{B}_{g^{-1}} \mathcal{B} \ni xy \mapsto (u_g x) y \in \mathcal{B}_g \mathcal{B} \quad (x \in \mathcal{B}_{g^{-1}}, y \in \mathcal{B}), \]
\[ \tilde{L}_{v_g} : \mathcal{B}_g \mathcal{B} \ni xy \mapsto (v_g x) y \in \mathcal{B}_{g^{-1}} \mathcal{B} \quad (x \in \mathcal{B}, y \in \mathcal{B}) \]
and isomorphisms of left \( \mathcal{B} \)-modules
\[ \tilde{R}_{u_g} : \mathcal{B} \mathcal{B}_{g^{-1}} \ni xy \mapsto x(y u_g) \in \mathcal{B} \mathcal{B}_g \quad (x \in \mathcal{B}, y \in \mathcal{B}_{g^{-1}}), \]
\[ \tilde{R}_{v_g} : \mathcal{B} \mathcal{B} \ni xy \mapsto x(y v_g) \in \mathcal{B} \mathcal{B}_{g^{-1}} \quad (x \in \mathcal{B}, y \in \mathcal{B}_g), \]
such that \( \tilde{L}_{u_g} \) extends the pair of maps
\[ L_{u_g} : \mathcal{B}_{g^{-1}} \to \mathcal{D}_g, \mathcal{D}_{g^{-1}} \to \mathcal{B}_g, \]
\[ \tilde{R}_{u_g} \text{ extends} \]
\[ R_{u_g} : \mathcal{D}_g \to \mathcal{B}_g, \mathcal{B}_{g^{-1}} \to \mathcal{D}_{g^{-1}}, \]
and similarly for \( \tilde{L}_{v_g} \) and \( \tilde{R}_{v_g} \). Moreover, \( \tilde{L}_{v_g} = \tilde{L}_{u_g}^{-1} \) and \( \tilde{R}_{v_g} = \tilde{R}_{u_g}^{-1} \).

**Proof.** Observe first that
\[ (21) \quad u_g(xy) = u_g(x)y \quad \forall x \in \mathcal{B}_{g^{-1}}, y \in \mathcal{B}_g. \]
Indeed, by (17), \( \mathcal{B}_{g^{-1}} \mathcal{D}_g = \mathcal{B}_g \mathcal{B} \mathcal{B} \mathcal{B} = \mathcal{B}_{g^{-1}} \), and thus we can write \( x = \sum x_i' x_i'' \) with \( x_i' \in \mathcal{B}_{g^{-1}}, x_i'' \in \mathcal{D}_g \). Then using the first equality in Lemma 4.1, and keeping in mind that \( x_i'' y \in \mathcal{D}_g \mathcal{B} \subseteq \mathcal{D}_g \), we have
\[ u_g(xy) = \sum u_g(x_i' x_i'' y) = \sum u_g(x_i' x_i'' y) = \sum u_g(x_i' x_i'' y) = u_g(xy), \]
as desired.

We define \( \tilde{L}_{u_g} \) first on \( \mathcal{B}_{g^{-1}} \mathcal{B}_h \) with arbitrary \( h \in G \) by
\[ \tilde{L}_{u_g}(xy) = (u_g x) y \quad \forall x \in \mathcal{B}_{g^{-1}}, y \in \mathcal{B}_h, \]
and extending it additively to arbitrary elements of \( \mathcal{B}_{g^{-1}} \mathcal{B}_h \). Then
\[ \tilde{L}_{u_g} : \mathcal{B}_{g^{-1}} \mathcal{B}_h \to \mathcal{D}_g \mathcal{B}_h. \]
is a well-defined map. For suppose \( \sum x_i y_i = 0 \) for some \( x_i \in B_{g^{-1}}, y_i \in B_h \). Then for any \( z \in B_{h^{-1}} \) applying (21) we have
\[
(\tilde{L}_{u_g}(\sum x_i y_i)) z = \sum u_g(x_i) y_i z = \sum u_g(x_i y_i z) = u_g(\sum x_i y_i z) = 0,
\]
as \( y_i z \in B_e \). Since \( z \in B_{h^{-1}} \) is arbitrary, it follows that \( (\tilde{L}_{u_g}(\sum x_i y_i)) B_{h^{-1}} = 0 \), which implies \( \tilde{L}_{u_g}(\sum x_i y_i) = 0 \), as \( B \) is homogeneously non-degenerate.

Since \( B = \bigoplus_{h \in G} B_h \) we can extend \( \tilde{L}_{u_g} \) additively to a well-defined map
\[
\tilde{L}_{u_g} : B_{g^{-1}} B \to D_g B,
\]
which sends \( xy \mapsto (u_g x)y \) with \( x \in B_{g^{-1}}, y \in B \). Observe now that
\[
B_g B_h = B_g B_{g^{-1}} B_g B_h \subseteq D_g B,
\]
in view of (17). Consequently, \( B_g B \subseteq D_g B \), and since obviously \( D_g B = B_g B \), one has \( D_g B = B_g B \).

Thus we obtain a map
\[
\tilde{L}_{u_g} : B_{g^{-1}} B \to B_g B,
\]
which is a homomorphism of right \( B \)-modules by its construction. Its restriction to \( B_{g^{-1}} \) clearly coincides with \( L_{u_g} \). Now let \( z \in D_{g^{-1}} = B_{g^{-1}} B_g \) be arbitrary and write \( z = \sum x_i y_i, \ x_i \in B_{g^{-1}}, y_i \in B_g \). Then using the second equality of the first column in Lemma 4.1, we have
\[
\tilde{L}_{u_g}(z) = \sum (u_g x_i) y_i = \sum u_g(x_i y_i) = L_{u_g}(z),
\]
so that \( \tilde{L}_{u_g} \), when restricted to \( D_{g^{-1}} \), coincides with \( L_{u_g} \).

We proceed similarly with \( R_{u_g} \). Using the equality \( B_{g^{-1}} = B_{g^{-1}} B_g B_{g^{-1}} \), we readily obtain
\[
(xy)u_g = x(yu_g) \quad \forall x \in B_e, y \in B_{g^{-1}}.
\]
Then for an arbitrary fixed \( h \in G \) set
\[
\tilde{R}_{u_g}(xy) = x(yu_g) \quad \forall x \in B_h, y \in B_{g^{-1}},
\]
which gives a well-defined map
\[
\tilde{R}_{u_g} : B_h B_{g^{-1}} \to B_h D_{g^{-1}}.
\]
Indeed, if \( \sum x_i y_i = 0 \) with \( x_i \in B_h, y_i \in B_{g^{-1}} \) then \( z x_i \in B_1 \) for any \( z \in B_{h^{-1}} \) and using (22) we have
\[
z \sum x_i(y_i u_g) = \sum (zx_i y_i) u_g = 0.
\]
As above this yields \( \tilde{R}_{u_g}(\sum x_i y_i) = 0 \), as \( B \) is homogeneously non-degenerate.

Extending additively we come to a well-defined homomorphism of left \( B \)-modules
\[
\tilde{R}_{u_g} : BB_{g^{-1}} \to BD_{g^{-1}} = BB_g.
\]

The maps \( \tilde{L}_{u_g} \) and \( \tilde{R}_{u_g} \) are obtained by analogous arguments. Finally, for \( x \in B_{g^{-1}}, y \in B \) we obviously have
\[
\tilde{L}_{u_g} \circ \tilde{L}_{u_g}(xy) = \tilde{L}_{u_g}((u_g x)y) = (v_g u_g x)y = xy,
\]
as \( L_{ug} = L_{ug}^{-1} \). It is also evident that \( \tilde{L}_{ug} \circ \tilde{L}_{vg} \) is identity on \( B_gB \). Hence \( \tilde{L}_{vg} = \tilde{L}_{vg}^{-1} \) and similarly \( \tilde{R}_{vg} = \tilde{R}_{vg}^{-1} \).

We shall write \( u_gx = \tilde{L}_{ug}(x) \), \( xu_g = \tilde{R}_{ug}(x) \), \( v_gx = \tilde{L}_{vg}(x) \) and \( xv_g = \tilde{R}_{vg}(x) \), provided that \( x \) belongs to the domain of the considered map.

**Remark 5.2.** Suppose that \( B \) is as in the above lemma. It will be convenient to observe the following equalities which easily follow from Lemma 5.1:

\[
\begin{align*}
    u_g(xy) &= (u_gx)y \quad \forall x \in D_{g^{-1}}, y \in B, \\
    (xy)u_g &= x(yu_g) \quad \forall x \in B, y \in D_g, \\
    v_g(xy) &= (v_gx)y \quad \forall x \in D_g, y \in B, \\
    (xy)v_g &= x(yv_g) \quad \forall x \in B, y \in D_{g^{-1}}.
\end{align*}
\]

We shall use the maps obtained in Lemma 5.1 to construct the multipliers \( w_{g,h} \). We list first some easy consequences of the property (17).

**Lemma 5.3.** Suppose that \( B = \bigoplus_{g \in G} B_g \) is a graded algebra satisfying (17). Then for all \( g, h \in G \) we have:

\[
\begin{align*}
    (i) \quad & B_{g^{-1}}B_gB_h = B_{g^{-1}}B_{gh}, \quad B_gB_hB_{h^{-1}} = B_{gh}B_{h^{-1}}, \\
    (ii) \quad & D_gD_h = D_hD_g, \\
    (iii) \quad & D_gB_g = B_g, \quad B_gD_g^{-1} = B_g.
\end{align*}
\]

**Proof.** Clearly,

\[
B_{g^{-1}}B_gB_h \subseteq B_{g^{-1}}B_{gh} = B_{g^{-1}}B_gB_{g^{-1}}B_{gh} \subseteq B_{g^{-1}}B_{gh}B_h,
\]

which gives the first equality of (i). The second one follows similarly. Using (i) we have

\[
\begin{align*}
    D_gD_h &= B_gB_{g^{-1}}B_hB_{h^{-1}} = B_gB_{g^{-1}}B_{gh}B_{h^{-1}} = B_gB_{g^{-1}}B_{gh}B_{g^{-1}h}B_{h^{-1}} = B_{gh}B_{g^{-1}}B_{gh}B_{h^{-1}} = B_{gh}B_{g^{-1}}B_{gh}B_{g^{-1}},
\end{align*}
\]

and (ii) follows. Finally, item (iii) is immediate.

**Proposition 5.4.** Suppose that \( B \) is a \( G \)-graded algebra as in Lemma 5.1. Then for any \( g, h \in G \) the equality

\[
w_{g,h} = u_gu_hv_{gh}
\]

defines an invertible multiplier of \( D_gD_{gh} \) with

\[
w_{g,h}^{-1} = u_{gh}v_{h}v_{g}.
\]
Proof. Take arbitrary \( x, y \in D_g D_{gh} = D_{gh} D_g \) and write \( x = \sum x' x'' \) with \( x' \in D_{gh} \) and \( x'' \in D_g \) (we omit the indices for simplicity). Then by Remark 5.2

\[
v_{gh}(xy) = \sum v_{gh}(x' x'' y) = \sum v_{gh}(x') x'' y = \sum v_{gh}(x' x'') y = (v_{gh} x)y.
\]

We also have

\[
v_{gh}(x) \in v_{gh}(D_{gh} D_g) = v_{gh}(D_{gh}) D_g = B_{h^{-1} g^{-1}} D_g = B_{h^{-1} B_{g^{-1}} D_g} = B_{h^{-1} B_{g}} D_g,
\]

by Remark 5.2 and (17). So we can write \( v_{gh}(x) = \sum x' x'' \) with \( x' \in B_{h^{-1}}, x'' \in B_{g^{-1}} \).

Then by Lemma 5.1

\[
u_h v_{gh}(xy) = \sum u_h(x' x'' y) = \sum u_h(x') x'' y = \sum u_h(x' x'') y = (u_h v_{gh}(x)) y.
\]

We see that

\[
u_h v_{gh}(x) \in u_h(B_{h^{-1} B_{g^{-1}}}) = u_h(B_{h^{-1} B} B_{g^{-1}}) = D_{h} D_{g} = D_{g} D_{h} = D_{h^{-1} g^{-1}} D_{g^{-1}} D_{h^{-1}} D_{h} D_{g^{-1}},
\]

and we can write \( u_h v_{gh}(x) = \sum x' x'' \) with \( x' \in D_{gh} \) and \( x'' \in B \). Then

\[
u_g((u_h v_{gh} x) y) = \sum u_g(x' x'' y) = \sum u_g(x') x'' y = \sum u_g(x' x'') y = (u_g u_h v_{gh}(x) y).
\]

This shows that

\[
u_g u_h v_{gh}(x) y = u_g u_h v_{gh}(x) y \quad \forall x, y \in D_g D_{gh}.
\]

We also have

\[
u_g u_h v_{gh}(D_{gh} D_g) = u_g(D_{gh}^{-1} D_{h} B_{g^{-1}}) = u_g(D_{g^{-1}}) D_{h} B_{g^{-1}} = B_{g} D_{h} B_{g^{-1}}
\]

\[
= B_{g} D_{h} B_{g^{-1} D_{g}} = B_{g} B_{g} B_{h^{-1} B_{g^{-1}}} D_{g} = B_{g} B_{g^{-1}} D_{g} = D_{g} D_{g}
\]

Using similar calculations one also shows that

\[
(x y) u_g u_h v_{gh} = x (y u_g u_h v_{gh}) \quad \forall x, y \in D_g D_{gh}
\]

and

\[
(D_{gh} D_g) u_g u_h v_{gh} = D_{gh} D_g.
\]

On order to prove that \( w_{g,h} = u_g u_h v_{gh} \) is a multiplier of \( D_g D_{gh} \) it remains to show that

\[
(x w_{g,h}) y = x (w_{g,h} y) \quad \forall x, y \in D_g D_{gh}.
\]

However, the above property is a consequence of the next two equalities which can be easily verified by Lemma 4.1:

\[
(x u_g) y = x (u_g y) \quad \forall x \in B B_{g^{-1}}, y \in B_{g^{-1} B},
\]

\[
(x v_g) y = x (v_g y) \quad \forall x \in B B_{g}, y \in B_{g} B.
\]

Thus

\[
w_{g,h} \in \mathcal{M}(D_g D_{gh}).
\]

Finally, because all involved maps are invertible, \( w_{g,h} \) is also invertible with \( w_{g,h}^{-1} = u_{gh} v_h v_g \).

\[\Box\]
6. The criteria

Now we are ready to formulate the next result.

**Theorem 6.1.** A homogeneously non-degenerate graded algebra $B = \bigoplus_{g \in G} B_g$ is isomorphic as a graded algebra to a crossed product by a twisted partial action of $G$ if and only if the following two conditions are satisfied:

(i) $B_g \cdot B_{g^{-1}} \cdot B_g = B_g \ \forall g \in G$;

(ii) for any $g \in G$ there exist $u_g, v_g \in M(C_g)$ such that $u_g v_g = e_{11}$ and $v_g u_g = e_{22}$.

Moreover, if a homogeneously non-degenerate graded algebra $B$ satisfies (i) and (ii), then with the notation introduced in the previous sections

$$\Theta = \{\{D_g\}_{g \in G}, \{\theta_g\}_{g \in G}, \{w_{g,h}\}_{(g,h) \in G \times G}\}$$

form a twisted partial action of $G$ on $A = B_1$ such that

$$\varphi : \bigoplus_{g \in G} B_g \ni \sum a_g \mapsto \sum a_g v_g \delta_g \in A \rtimes_{\Theta} G$$

is an isomorphism of graded algebras, whose inverse is

$$\varphi^{-1} : A \rtimes_{\Theta} G \ni \sum a_g \delta_g \mapsto \sum a_g u_g \in \bigoplus_{g \in G} B_g.$$

**Proof.** The “only if” part is proved already in Section 3 so suppose that $B$ is homogeneously non-degenerate satisfying (i) and (ii). Evidently, $D_g = B_g B_{g^{-1}}$ ($g \in G$) are idempotent two-sided ideals in $A$ which commute by (ii) of Lemma 5.3. Thus we have (i) of Definition 2.1. We know already by (20) that

$$\theta_g : D_g^{-1} \ni x \mapsto u_g x v_g \in D_g$$

are isomorphisms of $k$-algebras, and by Proposition 5.4, $w_{g,h}$ are invertible multipliers of $D_g D_{gh}$.

Obviously, $D_1 = (A)^2 = A$, by (i). Then evidently $C_1$ is the algebra of $2 \times 2$-matrices over $A$ and one can define the multipliers $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ by setting

$$\begin{pmatrix} x & y \\ x' & y' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & x' \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \begin{pmatrix} x' & y' \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} x & y \\ x' & y' \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} y & 0 \\ y' & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix},$$

where $x, y, x', y' \in A$. The multipliers $u_e$ and $v_e$ can be chosen to be equal to $e_{12}$ and $e_{21}$ respectively. Then $L_{u_1}, L_{v_1} : A \rightarrow A, R_{u_1}, R_{v_1} : A \rightarrow A, L_{u_1}, L_{v_1} : AB \rightarrow AB$ and $R_{u_1}, R_{v_1} : BA \rightarrow BA$ are all identity maps. Consequently, $\theta_1 : A \rightarrow A$ is the identity.
isomorphism, and \( w_{1,g} = u_g v_g = w_{g,1} \) is the identity multiplier of \( \mathcal{D}_g^2 = \mathcal{D}_g \). This gives (ii) and (v) of Definition 2.1.

Next observe that

\[
\theta_g(\mathcal{D}_{g^{-1}} \mathcal{D}_h) = (u_g \mathcal{D}_{g^{-1}} \mathcal{D}_h) v_g = ((u_g \mathcal{D}_{g^{-1}}) \mathcal{D}_h) v_g = (B_g \mathcal{D}_h) v_g = (B_g \mathcal{D}_{g^{-1}} \mathcal{D}_h) v_g = (B_g \mathcal{D}_h \mathcal{D}_{g^{-1}}) v_g = B_g \mathcal{D}_h \mathcal{B}_{g^{-1}} \mathcal{D}_g = B_g \mathcal{B}_h \mathcal{B}_{g^{-1}} \mathcal{B}_g \mathcal{D}_g = B_g \mathcal{B}_h \mathcal{B}_{g^{-1}} \mathcal{D}_g = \mathcal{D}_g \mathcal{D}_g,
\]

and (iii) of Definition 2.1 is also satisfied.

For arbitrary \( x \in \mathcal{D}_{h^{-1}} \mathcal{D}_{h^{-1}g} \) we have

\[
(u_{g,\theta_{gh}}(x)) w_{g,1}^{-1} = (u_g u_h (v_{gh} \circ u_{gh}(x v_{gh}))) u_{gh} v_g v_g = (u_g u_h (x v_{gh})) u_{gh} v_g v_g,
\]

as \( v_{gh} \circ u_{gh} \) is identity. By a direct inspection we see that

\[
u_h(x v_{gh}) \in \mathcal{B}_h \mathcal{B}_{h^{-1}g^{-1}} = \mathcal{B}_h \mathcal{B}_{h^{-1}g^{-1}} \mathcal{D}_g = \mathcal{B}_g \mathcal{D}_g = \mathcal{D}_g \mathcal{B}_g \subseteq \mathcal{D}_g^{-1} \mathcal{B} \mathcal{D}_g.
\]

We also have that

\[
(u_g y u_t = u_g (y u_t)) \quad \forall y \in \mathcal{D}_g^{-1} \mathcal{BD}_t.
\]

Indeed, write \( y = \sum z z' z'' \) with \( z \in \mathcal{D}_g, z' \in \mathcal{B}, z'' \in \mathcal{D}_t \). Then

\[
(u_g(z z' z'')) u_t = (u_g(z) z' z'') u_t = u_g(z) z'(z'' u_t) = (u_g z)((z' z'') u_t) = u_g(z((z' z'') u_t)),
\]

which implies the claimed equality.

Applying (23) with \( t = gh \) it follows that

\[
(u_g(u_h(x v_{gh}))) u_{gh} = u_g((u_h(x v_{gh})) u_{gh}).
\]

Now it is readily seen that

\[
x v_{gh} \in \mathcal{D}_{h^{-1}} \mathcal{B}_{h^{-1}g^{-1}} = \mathcal{D}_{h^{-1}} \mathcal{B}_{h^{-1}g^{-1}} \mathcal{D}_g \subseteq \mathcal{D}_{h^{-1}} \mathcal{B} \mathcal{D}_g,
\]

and using (23) again we have

\[
(u_h(x v_{gh})) u_{gh} = u_h(x v_{gh} \circ u_{gh}) = u_h x,
\]

as \( v_{gh} \circ u_{gh} \) is identity. Thus we obtain that

\[
(w_{g,\theta_{gh}}(x)) w_{g,1}^{-1} = (u_g u_h x) v_g v_g.
\]

Observe next that

\[
u_h x \in \mathcal{B}_h \mathcal{D}_{h^{-1}g^{-1}} = \mathcal{B}_{g^{-1}} \mathcal{B}_g = \mathcal{D}_{g^{-1}} \mathcal{B}_h = \mathcal{D}_{g^{-1}} \mathcal{B}_h \mathcal{D}_{h^{-1}} \subseteq \mathcal{D}_{g^{-1}} \mathcal{B} \mathcal{D}_{h^{-1}}
\]

and one can easily check that

\[
(u_g y v_h = u_g (y v_h)) \quad \forall y \in \mathcal{D}_g^{-1} \mathcal{BD}_h^{-1}.
\]

This yields that

\[
(w_{g,\theta_{gh}}(x)) w_{g,1}^{-1} = u_g(u_h x v_h) v_g = \theta_g \circ \theta_h(x),
\]

proving (iv) of Definition 2.1.
In order to prove that \( \Theta \) is a twisted partial action it remains to check the co-cycle equality (vi) of Definition 2.1. For any \( x \in D_{g^{-1}Dht} \) we have

\[
\theta_g(xu_h t)w_{g,h} = (u_g(xu_h u_t v_{ht}))v_g u_g u_{ht} v_{ght} = (u_g(xu_h u_t))u_{ht} v_{ght},
\]

as \( v_g u_g \) is identity. We see that

\[
xu_h \in D_{g^{-1}Dht}B_h = D_{g^{-1}B_h}B_{t^{-1}},
\]

and

\[
xu_h u_t \in D_{g^{-1}Bht}D_{t^{-1}} = D_{g^{-1}Bht}D_{t^{-1}h^{-1}} \subseteq D_{g^{-1}BD_{t^{-1}h^{-1}}}.
\]

Then by (23) and (24)

\[
u_g(xu_h u_t v_{ht}) = (u_g(xu_h u_t))v_{ht} = ((u_g(xu_h))u_t)v_{ht},
\]

and thus

\[
\theta_g(xu_h t)w_{g,h} = (u_g(xu_h))u_t v_{ht} u_{ht} v_{ght} = (u_g(xu_h))u_t v_{ght}.
\]

Furthermore, since clearly \( x \in D_{g^{-1}BD_h} \), applying once more (23) we obtain that

\[
\theta_g(xu_h t)w_{g,h} = (u_g x)u_h u_t v_{ght}.
\]

On the other hand,

\[
\theta_g(x)w_{g,h} w_{gh,t} = (u_g x)v_g u_g u_h v_{gh} u_{ht} u_t v_{ght} = (u_g x)u_h u_t v_{ght},
\]

completing the proof of the co-cycle equality.

It remains to show that \( \varphi \) is an isomorphism of algebras. Clearly \( a_g v_g \in D_g \) for \( a_g \in B_g \) so that \( a_g v_g \delta_g \) is an element of \( A \otimes \Theta G \). It is also evident that \( \varphi \) is \( k \)-linear, as \( v_g \) is so. Moreover, because each \( v_g : B_g \to D_g \) is bijective with inverse \( u_g \), it follows that \( \varphi \) is bijective with \( \varphi^{-1} : \sum a_g \delta_g \mapsto \sum a_g u_g \). Thus it remains to check that \( \varphi \) preserves multiplication. Taking \( x \in B_g \) and \( y \in B_h \) we have

\[
\varphi(x) \varphi(y) = (x v_g \delta_g) \cdot (y v_h \delta_h) = \theta_g(\theta_g^{-1}(x v_g)(y v_h)) w_{g,h} \delta_{gh}.
\]

By Remark 5.2,

\[
\theta_g(\theta_g^{-1}(x v_g)(y v_h)) w_{g,h} = (u_g[\theta_g^{-1}(x v_g) y v_h]) u_{gh} v_{ght} = (u_g[\theta_g^{-1}(x v_g)] y v_h) u_{gh} v_{ght} = (u_g[\theta_g^{-1}(x v_g)] y v_h u_h) v_{gh} = (u_g[\theta_g^{-1}(x v_g)] y v_{gh}),
\]

as \( v_g u_g \) and \( v_h u_h \) are identity. Furthermore,

\[
(u_g[\theta_g^{-1}(x v_g)] y v_{gh}) v_{gh} = (u_g[\theta_g^{-1}(x v_g)] y v_{gh}) v_{gh} = (u_g[\theta_g^{-1}(x v_g)] y v_{gh}) v_{gh} = (x y) v_{gh}.
\]

On the other hand,

\[
\varphi(xy) = (xy) v_{gh} \delta_{gh}
\]

which completes the proof of the equality \( \varphi(xy) = \varphi(x) \varphi(y) \).

The above criteria can be modified under the assumption that each \( D_g \) is an \( s \)-unital ring. We remind that a ring \( \mathcal{R} \) is called left \( s \)-unital if for any \( r \in \mathcal{R} \) there exists an element \( y \in \mathcal{R} \) with \( yr = r \). Equivalently, for any finite subset \( F \) of \( \mathcal{R} \) there exists
$y \in R$ such that $yr = r$ for all $r \in F$ (see [17, Lemma 2.4]). A left $s$-unital ring is clearly idempotent. A ring $R$ is said to be $s$-unital if it is both left $s$-unital and right $s$-unital. We proceed with the next:

**Proposition 6.2.** Let $(R, R', M, M', \tau, \tau')$ be a Morita context with $s$-unital algebras $R$ and $R'$, and surjective $\tau$ and $\tau'$, such that $M'R = R'M = M'$ and $M'R' = RM = M$. Then the following are equivalent:

(i) There exist $u, v$ in $M(C)$ such that $uv = e_{11}$ and $vu = e_{22}$. 

(ii) There are maps

$$\psi : R \to M, \quad \text{and} \quad \psi' : R' \to M,$$

such that $\psi$ is an isomorphism of left $R$-modules, $\psi'$ is an isomorphism of right $R'$-modules, and

$$(r\psi)r' = r(\psi'r')$$

for all $r \in R$ and $r' \in R'$.

**Proof.** The algebra $C$ is easily seen to be idempotent, so (i) $\Rightarrow$ (ii) follows by Lemma 4.1.

For the converse, observe first that one has the isomorphism $M' \otimes_R R \cong M'$ given by $m' \otimes r \mapsto m'r$. Indeed, the map is clearly surjective in view of $M'R = M'$. Suppose that $\sum m'_i \otimes r_i \mapsto 0$, i.e. $\sum m'_i r_i = 0$. Then there exists $r \in R$ with $r_i r = r_i$ for all $i$, and $\sum m'_i \otimes r_i = \sum m'_i \otimes r'_r = \sum m'_i r_i \otimes r = 0$, as claimed. One similarly has the isomorphism $R' \otimes_{R'} M' \cong M'$ defined by $r' \otimes m' \mapsto r'm'$. Moreover, the maps $M \otimes R'M \to R$, and $M' \otimes R'M \to R'$ given by the context are also isomorphisms. For the maps $\tau, \tau'$ are surjective by the assumption. Suppose that $\tau(\sum m_i \otimes m'_i) = 0$. In view of $M'R = M'$ there exists $r \in R$ with $m'_i r = m'_i$ for all $i$. Write $r = \tau(\sum_j x_j \otimes x'_j)$ with $x_j \in M, x'_j \in M'$. Then

$$\sum_i m_i \otimes m'_i = \sum_i m_i \otimes m'_i r = \sum_i m_i \otimes m'_i \tau(x_j \otimes x'_j) =$$

$$\sum_i \sum_j m_i \otimes \tau(m'_i \otimes x_j) x'_j = \sum_i \sum_j m_i \tau'(m'_i \otimes x_j) \otimes x'_j =$$

$$\sum_j \sum_i \tau(m_i \otimes m'_i) x_j \otimes x'_j = 0,$$

which shows that $\tau$ is injective, and thus it is an isomorphism. The verification for $\tau'$ is similar.

We shall use notation consistent with that of Lemma 4.1, so write

$L_u : R' \ni r' \mapsto \psi'r' \in M, \quad \text{and} \quad R_u : R \ni r \mapsto r\psi \in M$.

The composition

$$M' = R' \otimes_{R'} M' \xrightarrow{\psi' \otimes 1} M \otimes_{R'} M' \to R$$

(25)
clearly possesses an inverse and thus it is an isomorphism of right $\mathcal{R}$-modules which we denote by $L_u$ (so that that as in Lemma 4.1 $L_u$ stands for the pair $M' \to \mathcal{R}, \mathcal{R}' \to M$). Use $R_u$ to denote the following composed map:

$$M' = M' \otimes_\mathcal{R} \mathcal{R} \xrightarrow{1 \otimes \psi} M' \otimes_\mathcal{R} M \to \mathcal{R}'$$

which is an isomorphism of left $\mathcal{R}'$-modules (thus $R_u$ stands now for the pair $\mathcal{R} \to M, M' \to M'$). Denote the inverses of the above maps by $L_v : \mathcal{R} \to M', \ M \to \mathcal{R}'$ and $R_v : M \to \mathcal{R}, \ \mathcal{R}' \to M'$. Next define the multiplier $u$ of $\mathcal{C}$ by setting

$$u \begin{pmatrix} r \\ m' \\ m' \\ r' \end{pmatrix} = \begin{pmatrix} um' \\ ur' \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad u \begin{pmatrix} r \\ m' \\ m' \\ r' \end{pmatrix} u = \begin{pmatrix} 0 \\ ru \\ 0 \\ m'u \end{pmatrix},$$

where $um' := L_u(m'), \ ur' := L_u(r'), \ ru := (r)R_u, \ m'u := (m')R_u$. One needs to make sure that this in fact defines a multiplier. By Remark 4.3 one has to check the equalities in the three columns involving $u$. By Remark 4.4 one directly verifies that the above proposition follows.

**Remark 6.3.** In view of Remark 4.4 one directly verifies that the above proposition remains valid if we remove from the assumptions one of the pairs of equalities $\mathcal{R}M = M \mathcal{R}' = M$ or $\mathcal{R}'M = M' \mathcal{R} = M'$.

A left module $M$ over a (non-necessarily unital) ring $\mathcal{R}$ is called torsion-free (or non-degenerate) if

$$x \in M, \ \mathcal{R}x = 0 \Rightarrow x = 0.$$ 

On easily obtains the following:

**Lemma 6.4.** Let $\mathcal{R}$ be a left $s$-unital ring and let $M$ be a left $\mathcal{R}$-module such that $M = \mathcal{R}M$. If $x \in M$ then $x \in \mathcal{R}x$. In particular, $M$ is torsion-free.

Now we can state the next:

**Theorem 6.5.** Suppose that $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ is a $G$-graded algebra such that $\mathcal{D}_g$ is an $s$-unital ring for each $g \in G$. Then $\mathcal{B}$ is isomorphic as a graded algebra to a crossed product by a twisted partial action of $G$ on $\mathcal{B}_1$ exactly when the following two conditions

...
are satisfied:

(i) \[ B_g \cdot B_{g^{-1}} \cdot B_g = B_g \quad \forall g \in G; \]

(ii) for any \( g \in G \) there exist maps

\[ \psi_g : D_g \rightarrow B_g \quad \text{and} \quad \psi'_g : D_{g^{-1}} \rightarrow B_g, \]

such that \( \psi_g \) is an isomorphism of left \( D_g \)-modules, \( \psi'_g \) is an isomorphism of right \( D_{g^{-1}} \)-modules, and

\[ (d \psi_g) d' = d(\psi'_{g^{-1}} d') \quad \forall d \in D_g, d' \in D_{g^{-1}}. \]

**Proof.** By (i) we have

\[ D_g B_g = B_g = B_g D_{g^{-1}}, \quad D_{g^{-1}} B_g = B_{g^{-1}} = B_{g^{-1}} D_g \]

for each \( g \in G \). By Lemma 6.4, the left \( D_g \)-module \( B_g \) is torsion-free. Hence if \( x \in B_g \) and \( B_{g^{-1}} x = 0 \) then \( D_g x = 0 \), and consequently \( x = 0 \). Similarly, \( x \in B_{g^{-1}}, x B_g = 0 \) implies \( x = 0 \), and thus \( B \) is homogeneously non-degenerate. Our result then follows from Theorem 6.1 and Proposition 6.2.

**Remark 6.6.** If \( B = A \times_\Theta G \) is a crossed product by a twisted partial action, then \( \psi_g \) is the map \( D_g \ni a \mapsto a \delta_g \in B_g \) and \( \psi'_g \) is \( D_{g^{-1}} \ni a \mapsto \theta_g(a) \delta_g \in B_g \). If \( \Theta \) is global and \( A \) has \( 1_A \) then each \( D_g = A \) and identifying \( A \) with \( A \delta_1 \subseteq B \) we see that \( \psi_g \) is the multiplication by \( 1_A \delta_g \) from the right whereas \( \psi' \) is multiplication by the same element from the left.

### 7. Constructing some finitely determined maps

Our next aim is to obtain an algebraic version of Theorem 7.3 from [22] for which we do some preparatory work in this section. As it is mentioned in the Introduction, we view the \( C^* \)-property of having a countable approximate identity algebraically as that of possessing a countable set of local units. We say that an algebra \( R \) has local units if for any \( r \in R \) there exists an idempotent \( e \in R \) such that \( r \in e Re \). Equivalently, any finite subset of \( R \) is contained in a subring of the form \( eRe \) for some \( e = e^2 \in R \) (see Remark 7.1 below). The algebra \( R \) possesses a countable set of local units exactly when there exists an increasing sequence \( e_1 \leq e_2 \leq \ldots \) of idempotents in \( R \) with \( R = \bigcup_{i=1}^{\infty} e_i Re_i \). Here \( e_1 \leq e_2 \) means that \( e_1 e_2 = e_2 e_1 = e_1 \). The elements \( e'_1 = e_1 \) and \( e'_i = e_i - e_{i-1} \) \((i \geq 2)\) are pairwise orthogonal idempotents and

\[ R = \bigoplus_{i \in \mathbb{N}} R e'_i = \bigoplus_{i \in \mathbb{N}} e'_i R. \]

More generally, we say that \( R \) possesses orthogonal local units if there exists a set of (non-necessarily central) pairwise orthogonal idempotents \( E \in R \) such that

\[ R = \bigoplus_{e \in E} R e = \bigoplus_{e \in E} e R. \]
Rings $\mathcal{R}$ with (27) are also called rings with enough idempotents (see [31]). As it is shown by P. N. Ánh and L. Márki [3], every Morita equivalence class of rings with local units contains rings with orthogonal local units.

**Remark 7.1.** We observe that if $\mathcal{R}$ is a ring such that for any $r \in \mathcal{R}$ there exists an idempotent $e \in \mathcal{R}$ with $r \in e\mathcal{R}e$, then for any finite subset $\{r_1, r_2, \ldots, r_m\} \subseteq \mathcal{R}$ there exists an idempotent $e \in \mathcal{R}$ such that $r_i \in e\mathcal{R}e$ for all $i = 1, \ldots, m$. Indeed, our ring $\mathcal{R}$ is right $s$-unital and thus there exists $y \in \mathcal{R}$ such that $r_iy = r_i$ for all $i = 1, \ldots, m$. Take an idempotent $f \in \mathcal{R}$ with $yf = y$. Then $r_i = r_iyf = r_if, i = 1, \ldots, m$. Analogously, since $\mathcal{R}$ is left $s$-unital, there exists $x \in \mathcal{R}$ with $xr_1 = r_1, \ldots, xr_m = r_m$ and $xf = f$. Take an idempotent $e \in \mathcal{R}$ such that $ex = x$. Then $r_i = er_i, i = 1, \ldots, m$ and $f = ef$. Setting now $e' = e + f - fe$ it is directly verified that $e'$ is an idempotent with $r_1, \ldots, r_m \in e'\mathcal{R}e'$.

We shall need some information on the Morita Theory for idempotent rings. The reader is referred to [32] for the details. Given an idempotent ring $\mathcal{R}$, a left $\mathcal{R}$-module is said to be **unital** if $\mathcal{R}M = M$. The category of left (respectively, right) unital torsion-free modules is denoted by $\text{mod-}\mathcal{R}$ (respectively, $\text{mod-}\mathcal{R}'$). Two idempotent rings $\mathcal{R}$ and $\mathcal{R}'$ are called Morita equivalent if the categories $\text{mod-}\mathcal{R}$ and $\text{mod-}\mathcal{R}'$ are equivalent. This happens exactly when there is a Morita context $(\mathcal{R}, \mathcal{R}', M, M', \tau, \tau')$ with unital $\mathcal{R}M, M_{\mathcal{R}'}, M', M'_{\mathcal{R}}$ and surjective $\tau : M \otimes_{\mathcal{R}'} M' \to \mathcal{R}, \tau' : M' \otimes_{\mathcal{R}} M \to \mathcal{R}'$. Then the categories $\text{mod-}\mathcal{R}$ and $\text{mod-}\mathcal{R}'$ are also equivalent and vice versa.

When dealing with modules over rings without identity, the concept of a finitely generated module is refined by using the categorical definition with respect to a family of generators (see [35, p. 72]). For rings with orthogonal local units, however, we will be satisfied with the “usual” definition, so that we say that a left $\mathcal{R}$-module $N$ is **finitely generated** if there exist finitely many $y_i \in N$ such that $N = \sum_i \mathcal{R}y_i$. Note that $\mathcal{R}$ in general does not need to be finitely generated as a module over itself. Note also that a finitely generated $\mathcal{R}$-module is necessarily unital, as $\mathcal{R}$ is idempotent.

We shall use the next easy fact which also explains that our definition of finitely generated modules is equivalent to the categorical one with the most naturally chosen family of generators.

**Lemma 7.2.** Let $\mathcal{R}$ be a ring with a set of orthogonal local units $E$ and set $\mathcal{E} = \{\mathcal{R}e : e \in E\}$. A left $\mathcal{R}$-module $M$ is $\mathcal{E}$-finitely generated if and only if there exists a finite number of elements $y_1, \ldots, y_t$ of $M$, such that $M = \sum_i \mathcal{R}y_i$.

**Proof.** For the reader’s convenience we remind that according to [35, p. 72], a left $\mathcal{R}$-module $M$ is said to be finitely generated with respect to the generating set $\mathcal{E}$ if there exists a finite subset $E_0 \subseteq E$ such that $M$ is an epimorphic image of a finite direct sum of modules each of which is isomorphic to some $\mathcal{R}e$ with $e \in E_0$. Now our assertion easily follows by observing that for any $x \in M$, there exists a finite subset $E' \subseteq E$ such that $x \in \sum_{e \in E'} \mathcal{R}ex$. \qed
Let $P$ be a left $\mathcal{R}$-module with given fixed decomposition
\[ P = \bigoplus P_i \]
into a (non-necessarily finite) direct sum of finitely generated submodules. By an \textit{admissible summand} of $P$ we shall mean a direct summand of $P$ which is a finite sum of some of the $P_i$’s. The $P_i$’s shall be called \textit{admissible components} of $P$. Let $Q$ be also a left $\mathcal{R}$-module with the fixed decomposition
\[ Q = \bigoplus Q_j, \]
with finitely generated $Q_j$’s which are taken to be the admissible components of $Q$.

\textbf{Definition 7.3.} An $\mathcal{R}$-module homomorphism $\psi : P \to Q$ shall be called finitely determined if for any admissible summand $L_1$ of $Q$ there exists an admissible summand $L_2$ of $P$ such that
\[ (x)\psi \circ \lambda_1 = (x)\lambda_2 \circ \psi|_{L_2} \circ \lambda_1 \]
for all $x \in P$, where $\lambda_1 : Q \to L_1$ and $\lambda_2 : P \to L_2$ are the projections and $\psi|_{L_2}$ is the restriction.

Evidently the composition of finitely determined maps is finitely determined, and when checking that $\psi$ is finitely determined is enough to take $L_1 = Q_j$ with arbitrary $Q_j$.

For the module $P \bigoplus Q$ the admissible components will be those of $P$ and $Q$, and the same will be used for infinite direct sums of modules with fixed admissible components.

In order to construct some finitely determined maps we shall fix the following notation. The rings $\mathcal{R}$ and $\mathcal{R}'$ both will have orthogonal local units, respectively denoted by $E$ and $F$. Thus $\mathcal{R}$ satisfies (27) and similarly for $\mathcal{R}'$ we have
\[ \mathcal{R}' = \bigoplus_{f \in F} \mathcal{R}'f = \bigoplus_{f \in F} f\mathcal{R}'. \]

The cardinalities of $E$ and $F$ are arbitrary. Suppose further that $\mathcal{R}$ and $\mathcal{R}'$ are Morita equivalent. This means that there exists a Morita context $(\mathcal{R}, \mathcal{R}', M, M', \tau, \tau')$ with surjective maps (in fact, by [32, Proposition 2.4] the maps $\tau, \tau'$ are bijective). Moreover, by Lemma 6.4, the modules $\mathcal{R}M, M_{\mathcal{R}'}, M', M'_{\mathcal{R}}$ are all torsion-free and by [32, Theorem 2.7], the functors $M' \otimes_{\mathcal{R}} - : \mathcal{R}\text{-mod} \to \mathcal{R}'\text{-mod}$ and $M \otimes_{\mathcal{R}}' - : \mathcal{R}'\text{-mod} \to \mathcal{R}\text{-mod}$ are inverse category equivalences, which are unique up to natural isomorphisms. Note that our category equivalences preserve short exact sequences. Write $x \cdot x' = \tau(x \otimes x')$ and $x' \cdot x = \tau'(x' \otimes x)$ where $x \in M, x' \in M'$. Since $M = \mathcal{R}M = M\mathcal{R}'$ and $M' = \mathcal{R}'M' = M'\mathcal{R}$, it follows by (27) and (28) that
\[ M = \bigoplus_{e \in E} eM = \bigoplus_{f \in F} fMf \quad \text{and} \quad M' = \bigoplus_{f \in F} fM'f = \bigoplus_{e \in E} eM'e. \]

For any $f \in F$ write
\[ f = \sum_{i=1}^{n_f} x_i' \cdot x_i, \]
where \( x_i = x_i^{(f)} \in M \), and \( x'_i = x_i^{(f)}' \in M' \). One evidently may suppose that \( x_i = x_i f \), and \( f x'_i = x'_i \) for all \( i \).

Observe next the following easy facts.

**Lemma 7.4.** With the above notation we have:

(i) For each \( e \in E \) and \( f \in F \) the modules \( R(Mf) \), and \( R'(M'e) \) are finitely generated.

(ii) Finitely generated modules are preserved under the equivalences \( M' \otimes_{R'} M \) and \( M \otimes_{R} M \).

**Proof.** (i) For any \( y \in M \) by (30) one has
\[
yf = \sum_i (yx'_i)x_i.
\]
Since \( yx'_i \in R \), it follows that \( Mf = Rx_1 + \ldots + Rx_{n_f} \), so that the left \( R \)-module \( Mf \) is finitely generated.

Similarly, so too is \( R'(M'e) \).

(ii) Follows easily by using (i) and Lemma 7.2. \( \square \)

We proceed by constructing some homomorphisms. For each \( f \in F \), by (30) we may define the map
\[
\pi_f : R^n_f \ni (r_1, r_2, \ldots, r_{n_f}) \mapsto \sum_i r_i x_i \in Mf,
\]
where \( x_i = x_i^{(f)} \) and \( x'_i = x_i^{(f)}' \). This is an epimorphism of left \( R \)-modules, as \( yf = \pi_f(yx'_1, yx'_2, \ldots, yx'_{n_f}) \) for any \( y \in M \). The map
\[
\rho_f : Mf \ni yf \mapsto (yfx'_1, yfx'_2, \ldots, yfx'_{n_f}) \in R^n_f,
\]
obviously splits \( \pi_f \), i.e \( \rho_f \circ \pi_f = \text{id} \). Then we have the exact sequence of left \( R \)-modules and homomorphisms
\[
0 \longrightarrow K_f \xrightarrow{\mu_f} R^n_f \xrightarrow{\pi_f} Mf \longrightarrow 0,
\]
in which \( K_f \) is the kernel of \( \pi_f \) and \( \mu_f \) is its imbedding, and also the exact sequence of the splitting maps
\[
0 \leftarrow K_f \xrightarrow{\tau_f} R^n_f \xrightarrow{\rho_f} Mf \leftarrow 0 \quad (\mu_f \circ \tau_f = \text{id}).
\]

Note that since \( x'_i = f x'_i \), one can consider \( \rho_f \) to be defined on \( M \) and \( x \rho_f = (x f) \rho_f \) for all \( x \in M \).

Let \( X \) be an infinite set of indices whose cardinality is bigger or equal to those of \( E \) and \( F \). Completing \( E \) or \( F \) with zeros, we may assume that \( |X| = |E| = |F| \). Let further \( R^{(X)} \) be the direct sum of copies of \( R \) indexed by the elements of \( X \) and write \( R^{(X)} = \bigoplus_{f \in F} R^n_f \). Here we assume \( R^{(X)} = 0 \) if \( f = 0 \). Define the map
\[
\pi : \bigoplus_{f \in F} R^n_f \ni (\bar{r}_f) \mapsto \sum_f (\bar{r}_f) \pi_f \in M,
\]
here \( \bar{\tau}_f \in \mathcal{R}^{n_f} \). By (29) \( \pi \) is an epimorphism of left \( \mathcal{R} \)-modules. Since \( x = \sum_{f \in F} x_f \), we see that \( \pi \) is split by the homomorphism
\[
\rho : M \ni x \mapsto (x \rho_f) \in \bigoplus_{f \in F} \mathcal{R}^{n_f},
\]
i.e. \( \rho \circ \pi = id \). Similarly, the maps \( \mu_f, \tau_f \ (f \in F) \) determine the left \( \mathcal{R} \)-module homomorphisms \( \mu : K \to \mathcal{R}(X) \) and \( \tau : \mathcal{R}(X) \to K \), where \( K = \bigoplus_{f \in F} K_f \), resulting in the exact sequences
\[
0 \to K \xrightarrow{\mu} \mathcal{R}(X) \xrightarrow{\pi} M \to 0 \quad \text{and} \quad 0 \leftarrow K \xleftarrow{\tau} \mathcal{R}(X) \xleftarrow{\rho} M \leftarrow 0,
\]
where \( \mu \circ \tau = id \).

Taking \( \mathcal{R}' \) instead of \( \mathcal{R} \) and \( M' \) instead of \( M \) in (31), we obtain that there are exact sequences of left \( \mathcal{R}' \)-modules and homomorphisms
\[
0 \to K' \xrightarrow{\mu'} \mathcal{R}'(X) \xrightarrow{\tau'} M' \to 0 \quad \text{and} \quad 0 \leftarrow K' \xleftarrow{\tau'} \mathcal{R}'(X) \xleftarrow{\rho'} M' \leftarrow 0,
\]
with \( \rho' \circ \tau' = id, \mu' \circ \tau' = id \). Applying the equivalence functor \( \mathcal{M} \otimes _{\mathcal{R}'} - \) we come to the exact sequences of left \( \mathcal{R} \)-modules and homomorphisms:
\[
0 \to \tilde{K} \xrightarrow{\tilde{\mu}} M(\mathcal{X}) \xrightarrow{\tilde{\pi}} M \to 0 \quad \text{and} \quad 0 \leftarrow \tilde{K} \xleftarrow{\tilde{\tau}} M(\mathcal{X}) \xleftarrow{\tilde{\rho}} \mathcal{R} \leftarrow 0,
\]
with \( \tilde{\rho} \circ \tilde{\pi} = id, \tilde{\mu} \circ \tilde{\tau} = id \), where \( \tilde{K} \) is the submodule of \( M(\mathcal{X}) \) obtained from \( M \otimes _{\mathcal{R}'} \mathcal{R}' \) by using \( M \otimes _{\mathcal{R}'} \mathcal{R}' \cong _{\mathcal{R}} M \), and \( \tilde{\mu} \) is the embedding of \( \tilde{K} \).

We shall fix for the rest of this section the decomposition for the modules \( \mathcal{R}M, \mathcal{R}M' \), \( \mathcal{R} \), \( \mathcal{R}' \), \( \mathcal{R}K \), \( \mathcal{R}'K \), and \( \mathcal{R} \). For \( \mathcal{R} \) and \( \mathcal{R}' \) take as fixed the decompositions
\[
\mathcal{R} = \bigoplus_{e \in E} \mathcal{R}e \quad \text{and} \quad \mathcal{R}' = \bigoplus_{f \in F} \mathcal{R}'f,
\]
so that the admissible components of \( \mathcal{R} \) are the \( \mathcal{R}e \)'s and those of \( \mathcal{R}' \) are the \( \mathcal{R}'f \)'s.

We fix now the decompositions for \( \mathcal{R}M \) and \( \mathcal{R}M' \) :
\[
M = \bigoplus_{f \in F} M_f \quad \text{and} \quad M' = \bigoplus_{e \in E} M'e.
\]

As to \( \mathcal{R}K \), since \( M = \bigoplus_{e \in E} eM \), there exists an idempotent \( e^{(f)} \in \mathcal{R} \) such that \( e^{(f)} = e^{(f)}_j \eta_j^{(f)} \) for all \( j = 1, \ldots n_f \), which is a (finite) sum of some \( e \)'s from \( E \). Clearly
\[
K^{(f)} = \bigoplus_{e \in E, e^{(f)} = 0} (\mathcal{R}e)^{n_f} \subseteq K_f,
\]
where \( \mathcal{R}e \) is the submodule of \( \mathcal{R} \) obtained from \( \mathcal{R} \) by using \( \mathcal{R} \otimes _{\mathcal{R}'} \mathcal{R} \cong _{\mathcal{R}'} \mathcal{R} \), and \( \mathcal{R}e \) is the embedding of \( \mathcal{R} \).
and we have $K_f = K^{(f)} \oplus L^{(f)}$ with $L^{(f)} = K_f \cap (\mathcal{R}e^{(f)})^{n_{f}}$. Since $K_f$ is a direct summand of $\mathcal{R}^{n_{f}}$, the left $\mathcal{R}$-module $L^{(f)}$ is a direct summand of $(\mathcal{R}e^{(f)})^{n_{f}}$. Because $(\mathcal{R}e^{(f)})^{n_{f}}$ is finitely generated, so too is $L^{(f)}$. Then the sum

$$(34) \quad K = \bigoplus_{f \in F} (L^{(f)} \oplus (\mathcal{R}e)^{n_{f}})$$

is a decomposition of $K$ into finitely generated direct summands, and we fix the modules $L^{(f)}$ and $\mathcal{R}e$ with $f \in F$ and $ee^{(f)} = 0, e \in E$, as the admissible components of $K$.

Working similarly with $\mathcal{R}', M'$ and $K'$, we define for each $e \in E$ the idempotent $f^{(e)}$ and the module $L'^{(e)}$, so that

$$(35) \quad K' = \bigoplus_{e \in E} (L'^{(e)} \oplus (\mathcal{R}'e)^{n_{e}})$$

is a decomposition of $K'$ as a direct sum of finitely generated submodules, and the modules $L'^{(e)}$ and $\mathcal{R}'f$ with $e \in E$ and $ff^{(e)} = 0, f \in F$, are defined to be the admissible components of $K'$.

Applying the functor $M \otimes_{\mathcal{R}'} -$ to the decomposition of $K'$ we obtain the decomposition:

$$(36) \quad \tilde{K} = \bigoplus_{e \in E} (\tilde{L}^{(e)} \oplus (Mf)^{n_{e}})$$

of $\tilde{K}$ into a direct sum of submodules, where $\tilde{L}^{(e)}$ is the submodule of $M^{(X)}$ obtained from $M \otimes_{\mathcal{R}'} L'^{(e)}$ by using $M \otimes_{\mathcal{R}'} \mathcal{R}' \cong M$. We know already that each $Mf$ is finitely generated and, moreover, by (ii) of Lemma 7.4, so too is each $\tilde{L}^{(e)}$. Thus the modules $Mf$ and $\tilde{L}^{(e)}$ with $e \in E$ and $ff^{(e)} = 0, f \in F$, can be defined as the admissible components of $\tilde{K}$, so that the functor $M \otimes_{\mathcal{R}'} -$ preserves the admissible components when passing from (35) to (36).

Now we are ready to state the next:

**Lemma 7.5.** All homomorphisms in (31) and (33) are finitely determined.

**Proof.** Taking $L_1 = Mf$ with arbitrary $f \in F$ and $L_2 = (\mathcal{R}e^{(f)})^{n_{f}}$, it is readily seen that $\pi$ is finitely determined. Next, any admissible summand $L_1$ of $\mathcal{R}^{(X)}$ is contained in some $\mathcal{R}^{n_{f}}$, so that taking $L_2 = Mf$ we immediately have that $\rho$ is finitely determined. On the other hand, it follows from (34) that each admissible summand of $K$ is either equal to an admissible one of $\mathcal{R}^{(X)}$ or contained in a finite direct sum of them. It follows that both $\mu$ and $\tau$ are finitely determined.

Analogously all maps in (32) are finitely determined, and since the admissible summands of (36) are defined so that they are preserved by $M \otimes_{\mathcal{R}'} -$, all maps in (33) are also finitely determined. \qed
We apply the above lemma to give the next:

**Proposition 7.6.** There is a finitely determined isomorphism of left \( \mathcal{R} \)-modules \( \psi : \mathcal{R}^{(X)} \rightarrow M^{(X)} \) whose inverse is also finitely determined.

**Proof.** Observe first that by Lemma 7.5 the following are finitely determined isomorphisms of left \( \mathcal{R} \)-modules:

\[
\pi \oplus \tau : \mathcal{R}^{(X)} \rightarrow M \oplus K, \quad \tilde{\pi} \oplus \tilde{\tau} : M^{(X)} \rightarrow \mathcal{R} \oplus \tilde{K},
\]

whose inverses are also finitely determined. Then using these maps and rearrangements of indices we apply the Eilenberg’s trick:

\[
\mathcal{R}^{(X)} \cong (\mathcal{R}^{(X)})^{(X)} \cong (M \oplus K)^{(X)} \cong M^{(X)} \oplus M^{(X)} \oplus K^{(X)}
\]

\[
M^{(X)} \oplus (M \oplus K)^{(X)} \cong M^{(X)} \oplus (\mathcal{R}^{(X)})^{(X)} \cong M^{(X)} \oplus \mathcal{R}^{(X)},
\]

and similarly

\[
M^{(X)} \cong (M^{(X)})^{(X)} \cong (\mathcal{R} \oplus \tilde{K})^{(X)} \cong \ldots \cong \mathcal{R}^{(X)} \oplus M^{(X)}.
\]

It is directly seen that each step is a finitely determined isomorphism and thus so too is their composition \( \psi : \mathcal{R}^{(X)} \rightarrow M^{(X)} \), as well as \( \psi^{-1} \).

It seems to be interesting to give here some easy examples of non-finitely determined isomorphisms.

**Example 7.7.** Let \( \phi : \mathbb{R}^{(N)} \rightarrow \mathbb{R}^{(N)} \) be defined by \( e_1 \mapsto e_1, e_i \mapsto e_1 + e_i, i \geq 2 \), where the \( e_i \)'s are the canonical basis of \( \mathbb{R}^{(N)} \). Then \( \phi \) is an \( \mathbb{R} \)-isomorphism whose inverse is determined by \( e_1 \mapsto e_1, e_i \mapsto e_i - e_1, i \geq 2 \). Taking the canonical copies of \( \mathbb{R} \) as the admissible components, it is immediately seen that \( \phi \) is not finitely determined, as well as its inverse.

**Example 7.8.** Using notation from the above example define \( \phi : \mathbb{R}^{(N)} \rightarrow \mathbb{R}^{(N)} \) by \( e_1 \mapsto e_1, e_i \mapsto e_{i-1} + e_i, i \geq 2 \). Then \( \phi \) is an \( \mathbb{R} \)-isomorphism whose inverse is given by \( e_i \mapsto \sum_{j=0}^{i-1}(-1)^j e_{i-j} \). It is readily seen that \( \phi \) is finitely determined, while \( \phi^{-1} \) is not.

8. **The stable characterization of graded algebras**

Recall that given an algebra \( \mathcal{R} \) and a (non-necessarily finite) set of indices \( X \), we denote by \( \text{RFMat}_X(\mathcal{R}) \) the algebra of all \( X \times X \)-matrices over \( \mathcal{R} \) which are row-finite and by \( \text{FMat}_X(\mathcal{R}) \) the algebra of all \( X \times X \)-matrices over \( \mathcal{R} \) with only a finite number of non-zero entries.

**Lemma 8.1.** Let \( \mathcal{R} \) be a ring with a set \( E \) of orthogonal local units and \( P \) and \( Q \) left unital \( \mathcal{R} \)-modules with decompositions \( P = \bigoplus_{i \in X} P_i \) and \( Q = \bigoplus_{j \in Y} Q_j \), where \( P_i \) and \( Q_j \) are finitely generated. Then

(i) \( \text{Hom}_\mathcal{R}(P, Q) \cong \text{RFMat}_{X \times Y}(\text{Hom}_\mathcal{R}(P_i, Q_j)) \).

(ii) \( P \cong \text{FMat}_{(e,i) \in E \times X}(eP_i) \).
Proof. (i) Since each \( P_i \) is a finitely generated module then \( \text{Hom}_R(P_i, \oplus_{j \in Y} Q_j) \cong \oplus_{j \in Y} \text{Hom}_R(P_i, Q_j) \). Thus \( \text{Hom}_R(\oplus_{i \in X} P_i, \oplus_{j \in Y} Q_j) \cong \prod_{i \in X} \oplus_{j \in Y} \text{Hom}_R(P_i, Q_j) \cong \text{RFMat}_{X \times Y}(\text{Hom}_R(P_i, Q_j)) \).

(ii) Clearly, \( \text{Hom}_R(\mathcal{R}e, P_i) \cong eP_i \). Then, by (i), \( \text{Hom}_R(\mathcal{R}, P) \cong \text{RFMat}_{E \times X}(eP_i) \).

Now \( P \) embeds in \( \text{Hom}_R(\mathcal{R}, P) \) as right multiplication; so that, for each \( p \in P \), with \( p = \sum p_i \), we have \( p \mapsto (ep_i) \in \text{RFMat}_{E \times X}(eP_i) \). Conversely, one may check that \( a \in \text{RFMat}_{E \times X}(eP_i) \), comes from the right multiplication by \( p = \sum a(e, i) \).

The following is the main fact that we need about Morita equivalent rings with orthogonal local units.

**Theorem 8.2.** Let \( \mathcal{R} \) and \( \mathcal{R}' \) be rings with orthogonal local units \( E \) and \( F \) respectively and \( X \) be an infinite set of indices whose cardinality is bigger than or equal to those of \( E \) and \( F \). Let further \( (\mathcal{R}, \mathcal{R}', M, M', \tau, \tau') \) be a Morita context with unital (and thus, by Lemma 6.4, torsion-free) \( \mathcal{R}M, M_{\mathcal{R}, \mathcal{R}'} M', M'_{\mathcal{R}} \) and surjective \( \tau : M \otimes_{\mathcal{R}'} M' \rightarrow \mathcal{R}, \tau' : M' \otimes_{\mathcal{R}} M \rightarrow \mathcal{R}' \), and let \( C \) be the corresponding linking algebra. There exist maps

\[
\Psi : \text{FMat}_X(\mathcal{R}) \rightarrow \text{FMat}_X(M), \quad \Psi' : \text{FMat}_X(\mathcal{R}') \rightarrow \text{FMat}_X(M),
\]

such that \( \Psi \) is an isomorphism of left \( \text{FMat}_X(\mathcal{R}) \)-modules, \( \Psi' \) is an isomorphism of right \( \text{FMat}_X(\mathcal{R}') \)-modules, and

\[
(\Psi A') A' = A(\Psi' A')
\]

for all \( A \in \text{FMat}_X(\mathcal{R}) \) and \( A' \in \text{FMat}_X(\mathcal{R}') \).

Proof. A homomorphism \( \phi \) of left \( \mathcal{R} \)-modules \( \mathcal{R}^X \rightarrow M^X \) can be viewed as an \( X \times X \)-matrix with entries in \( \text{Hom}_R(\mathcal{R}, M) \). By (i) of Lemma 8.1 \( \phi \) becomes an element \([\phi]\) of the set \( \text{Mat}_X(\text{RFMat}_{(e,f) \in E \times F}(eMf)) \), i.e. \([\phi]\) has block structure the block-rows and block-columns being indexed by the elements of \( X \), and each block is a matrix from \( \text{RFMat}_{(e,f) \in E \times F}(eMf) \).

Fixing \( i \in X \) and \( e \in E \) we have the (“thin”) \((i,e)\)-row, which has to be finite, as the sum of its elements is the image under \( \phi \) of \( e \) from the \( i \)-copy of \( \mathcal{R} \) in \( \mathcal{R}^X \). Now taking the finitely determined isomorphism \( \psi \) from Proposition 7.6, we have that for any \( i \in X \) and \( f \in F \) the (“thin”) \((i,f)\)-column of \([\psi]\) is also finite. It follows that the sum of each block-row as well as that of each block-column of \([\psi]\) makes sense, and one may call such matrices as row and column summable \( X \times X \)-matrices over \( \text{Hom}_R(\mathcal{R}, M) \) and denote the set of them by \( \text{RCSumMat}_X(\text{Hom}_R(\mathcal{R}, M)) \). It is evident that each block of \([\psi]\) is a row and column finite matrix, so that denoting the set of all row and column finite matrices from \( \text{Mat}_{(e,f) \in E \times F}(eMf) \) by \( \text{RCFMat}_{(e,f) \in E \times F}(eMf) \), one can write that

\([\psi] \in \text{RCSumMat}_X(\text{RCFMat}_{(e,f) \in E \times F}(eMf))\).

It may be interesting to observe that the sum of each block-row of \([\psi]\) is a row-finite matrix whereas that of each block-column is column-finite.

Now by (ii) of Lemma 8.1 we have the following ring isomorphisms

\[
\varepsilon : \mathcal{R} \cong \text{FMat}_{e,e' \in E}(eR e') \quad \text{and} \quad \varepsilon' : \mathcal{R}' \cong \text{FMat}_{f,f' \in E}(fR f')
\]
and a $k$-linear isomorphisms

$$\zeta : M \rightarrow \text{FMat}_{e \in E, f \in F}(eMf)$$

and

$$\zeta' : M' \rightarrow \text{FMat}_{f \in F, e \in E}(fM'e).$$

Moreover, using matrix multiplication, one can check that $\text{FMat}_{e \in E, f \in F}(eMf)$ has a structure of a left $\text{FMat}_{f' \in F}(f'Rf')$-right $\text{FMat}_{e \in E}(eRe')$-bimodule, and similarly $\text{FMat}_{f \in F, e \in E}(fM'e)$ also has its corresponding bimodule structure, so that the maps $\zeta$ and $\zeta'$ are both semilinear, i.e.

$$\zeta(rY) = \varepsilon(r)\zeta(y), \zeta(yr') = \zeta(y)\varepsilon'(r'), \zeta'(r'r') = \varepsilon'(r'')\zeta'(y'), \zeta'(y'y) = \zeta'(y')\varepsilon(r),$$

for any $r \in R, r' \in R', y \in M, y' \in M'$. Extending $\varepsilon, \zeta, \varepsilon', \zeta'$ to finite matrices, the semi-linearity is obviously preserved. Moreover, the maps $\tau$ and $\tau'$ from the Morita context yield the equalities

$$\varepsilon(YY') = \zeta(Y)\zeta'(Y'), \ \varepsilon'(Y'Y) = \zeta'(Y)\zeta(Y).$$

for all $Y \in \text{FMat}_X(M), Y' \in \text{FMat}_X(M')$.

An element from $\text{FMat}_X(\text{FMat}_{e,e' \in E}(eRe'))$ can be multiplied from the right by $[\psi]$ and

$$\text{FMat}_X(\text{FMat}_{e,e' \in E}(eRe')) \cdot [\psi] \subseteq \text{FMat}_X(\text{FMat}_{e \in E}, F(eMf)).$$

Then setting $A \cdot [\psi] = \zeta^{-1}(\varepsilon(A)[\psi])$ with $A \in \text{FMat}_X(R)$, the map

$$\Psi : \text{FMat}_X(R) \ni A \mapsto A \cdot [\psi] \in \text{FMat}_X(M)$$

is clearly $k$-linear and

$$(AB) \cdot [\psi] = A(B \cdot [\psi])$$

for all $A, B \in \text{FMat}_X(R)$.

One also defines the $k$-linear map

$$\Psi' : \text{FMat}_X(R') \ni A' \mapsto [\psi] \cdot A' \in \text{FMat}_X(M),$$

such that

$$[\psi] \cdot (A_1'A_2') = ([\psi] \cdot A_1')A_2'$$

and

$$(A \cdot [\psi])A' = A([\psi] \cdot A')$$

for any $A \in \text{FMat}_X(R), A_1, A_2, A_1', A_2' \in \text{FMat}_X(R')$.

Now as it was done for $\psi$, the isomorphism $\psi^{-1} : M^{(X)} \rightarrow R^{(X)}$, may be viewed as a matrix in $\text{Mat}_X(\text{RFMat}_{(f,e) \in F \times E}(\text{Hom}_R(Mf, Re)))$, by Lemma 8.1. The equivalence functor $M' \otimes_R -$ transforms $\text{Hom}_R(Mf, Re)$ into $fM'e$; so that $\text{Hom}_R(M, R) \cong \text{RFMat}_{(f,e) \in F \times E}(fM'e)$. As in the case of $\psi$, it is directly verified that

$$[\psi^{-1}] \in \text{RCSumMat}_X(\text{RCFMat}_{(f,e) \in F \times E}(fM'e)).$$

Also, as it was done for $[\psi]$ we define the map

$$\text{FMat}_X(M) \ni Y \mapsto Y \cdot [\psi^{-1}] \in \text{FMat}_X(R),$$

where $[\psi^{-1}]$ is the inverse of $[\psi]$.
which is $\Psi^{-1}$, and the map

$$\operatorname{FMat}_X(M) \ni Y \mapsto [\psi^{-1}] \cdot Y \in \operatorname{FMat}_X(R'),$$

which is the inverse of $\Psi'$. \hfill \square

**Remark 8.3.** Under the assumptions in the above result, Proposition 6.2 guarantees that there exist multipliers $u, v$ of $\operatorname{FMat}_X(C) \cong \left( \begin{array}{cc} \operatorname{FMat}_X(R) & \operatorname{FMat}_X(M) \\ \operatorname{FMat}_X(M') & \operatorname{FMat}_X(R') \end{array} \right)$ such that $uv = e_{11}$ and $vu = e_{22}$.

In view of Proposition 4.5, we immediately have the next:

**Corollary 8.4.** The algebras $\operatorname{FMat}_X(R)$ and $\operatorname{FMat}_X(R')$ are isomorphic.

The above corollary is one of the algebraic versions of the Brown-Green-Rieffel Theorem [7, Theorem 1.2], and in the case of countable local units also follows from [4, Theorem 2.1].

We shall say that a $G$-graded algebra $B$ has *enough local units* if for any $g \in G$ the ring $D_g$ has orthogonal local units. Now it is an easy matter to derive the next:

**Theorem 8.5.** Let $G$ be an arbitrary group and $B = \bigoplus_{g \in G} B_g$ be $G$-graded algebra such that

$$B_g \cdot B_g^{-1} \cdot B_g = B_g \ \forall g \in G. \ (39)$$

If $B$ has enough local units then there exists an infinite cardinal $|X|$ and a twisted partial action $\Theta$ of $G$ on $A = \operatorname{FMat}_X(B_1)$ such that

$$\operatorname{FMat}_X(B) \cong A \rtimes \Theta G$$

as graded algebras, the $g$-homogeneous component of $\operatorname{FMat}_X(B)$ being $\operatorname{FMat}_X(B_g)$, $g \in G$.

**Proof.** By (39), each $B_g$ is a unital left module over the ring with orthogonal idempotents $D_g$, and thus by Lemma 6.4, it is torsion-free. Then as in the proof of Theorem 6.5 it follows that $B$ is homogeneously non-degenerate.

Let $X$ be an infinite set whose cardinality is bigger than or equal to $|E_g|$ for each $g \in G$, where $E_g$ is the set of orthogonal local units of $D_g$. Add zeros to $E_g$ so that one may assume that $|E_g| = |X|$ for all $g \in G$. Write $B' = \operatorname{FMat}_X(B)$ and $D'_g = B'_g B'_{g^{-1}}$ $(g \in G)$. Since $B$ is homogeneously non-degenerate and satisfies (39), it is easy to see that both of these properties are also satisfied by $B'$. Moreover, for any $g \in G$, $D'_g = \operatorname{FMat}_X(D_g)$ and $D'_g$ has orthogonal local units, as so does $D_g$. Thus our result directly follows from Theorem 8.2 and Theorem 6.5 (or Theorem 6.1 in view of Remark 8.3). \hfill \square
Remark 8.6. As it is observed in the beginning of this section, a countable set of local units in a ring can be orthogonalized. Thus the above theorem can be applied for a \(G\)-graded algebra \(B\) which satisfies (39) and such that each \(D_g\) has at most countable set of local units.

9. Examples with Uncountable Local Units

For the concept of a projective module over a non-necessarily unital ring \(\mathcal{R}\) we use the next definition: \(\pi\mathcal{P}\) is said to be projective if \(\text{Hom}(\pi\mathcal{P}, -)\) is an exact functor. Evidently, a direct summand of a projective module is again projective.

We need the following remark, the details can be found in [2]. We keep denoting by \(\mathcal{R}\)-mod the category of all unital and non-degenerate \(\mathcal{R}\)-modules.

Remark 9.1. If \(\mathcal{R}\) is a ring with local units then for an arbitrary cardinal \(\gamma\) the rings \(\mathcal{R}\) and \(\text{FMat}_\gamma(\mathcal{R})\) are Morita equivalent, and since \(\mathcal{R}^{(\gamma)}\) and \(\text{FMat}_\gamma(\mathcal{R})\) are corresponding objects, we have that \(\mathcal{R}\) is projective in \(\mathcal{R}\)mod if and only if \(\text{FMat}_\gamma(\mathcal{R})\) is projective in \(\text{FMat}_\gamma(\mathcal{R})\)-mod.

We give the next:

Example 9.2. There exist rings \(\mathcal{R}\) and \(\mathcal{R}'\) with local units such that \(\mathcal{R}\) and \(\mathcal{R}'\) are Morita equivalent rings; however, \(\text{FMat}_\gamma(\mathcal{R})\) and \(\text{FMat}_\gamma(\mathcal{R}')\) are not isomorphic for any cardinal \(\gamma\).

Proof. Let \(K\) be a field, and denote by \(\omega\) the first infinite ordinal, and by \(\alpha\) the least ordinal whose cardinality is next to that of \(\omega\), i.e. \(|\alpha| = \aleph_0^\omega\).

Let \(a \in \text{RFMat}_\alpha(K)\), we denote by \(a(i, j)\) its \((i, j)\)-entry. As usual, \(e_{ij}\) stands for the matrix in \(\text{RFMat}_\alpha(K)\) such that \(e_{ij}(i, j) = 1\) and 0 otherwise. We write \(e_i = e_{ii}\). Moreover, if \(X \subseteq \alpha\) is a subset, then the “sum” \(\sum_{i \in X} e_i\) will be denoted by \(e_X\).

Let \(\mathcal{E}\) be the subring of \(\text{RFMat}_\alpha(K)\) formed by the matrices each column of which has at most countably many nonzero entries (\(\mathcal{E} = \mathcal{E}_{\alpha, \alpha}\) in the notation of [12]).

Set \(e = \sum_{i \in \mathbb{N}} e_i = e_\omega\), \(\mathcal{R} = e\mathcal{E}e\) and \(\mathcal{R}' = e\mathcal{E}\mathcal{E}\). It is known that \(\mathcal{R}\) and \(\mathcal{R}'\) are Morita equivalent rings [32]. Moreover,

\[
(40) \quad (e\mathcal{E}e, e\mathcal{E}e, e\mathcal{E}, e\mathcal{E})
\]

form a Morita context with surjective bimodule maps given by the multiplication. In particular, the Morita correspondent of \(e\mathcal{E}e\) is \(e\mathcal{E}e\) and thus it is a (projective) generator of \(e\mathcal{E}e\)-mod.

It is easily observed that \(\mathcal{R}'\) consists of all \(\alpha \times \alpha\)-matrices with at most countably many non-zero entries. Clearly \(\mathcal{R}\) has local units as it is a ring with identity, and we check that \(\mathcal{R}'\) is also a ring with local units. Let \(N\) be the set of all subsets \(X \subseteq \alpha\) with \(|X| \leq \aleph_0\). It is easily seen that \(e\mathcal{E}e = \sum_{X \in N} e_X\mathcal{E}e\). Now take \(a \in e\mathcal{E}e\). Then there exist \(X, Y \in N\) such that \(a = ae_X = e_Y a\). Setting \(Z = X \cup Y\) we have that \(a = e_Z ae_Z\); so that \(\{e_X\}_{X \in N}\) is a set of local units for \(e\mathcal{E}e\).

If \(e\mathcal{E}e\) was a finitely generated object in \(e\mathcal{E}e\)-mod (with respect to the generator \(e\mathcal{E}e\)), then \(e\mathcal{E}e = \sum e_i\mathcal{E}e x_i\) for some finitely many \(x_i \in e\mathcal{E}e\) which would imply \(e\mathcal{E}e \not= e\mathcal{E}e\) for some \(X \in N\). Since this is not the case, \(e\mathcal{E}e\) is not finitely generated \(e\mathcal{E}e\)-module.
We shall show that \( \mathcal{E}e\mathcal{E} \) is not a projective object in \( \mathcal{E}e\mathcal{E}\)-mod, hence, by Remark 9.1 we will be done, as \( \mathcal{E}e\mathcal{E} \) is a unital ring and consequently is projective in \( e\mathcal{E}e \)-mod.

Observe first that \( \text{Hom}_{\mathcal{E}e\mathcal{E}}(e\mathcal{E},e\mathcal{E}) \cong \text{RFMat}_{\alpha}(K) \cdot e \) (notice that it is not \( e\mathcal{E}e \) as one may suspect). Indeed, let \( \varphi : e\mathcal{E} \to e\mathcal{E}e \) be an arbitrary homomorphism of left \( e\mathcal{E}e \)-modules. Evidently \( e_{1t} \in \mathcal{E} \) and \( e_{1t} = e_{1t} \) for any \( t \in \alpha \). Set \( y_t = \varphi(e_{1t}) (t \in \alpha \) and \( y = \sum_{t \in \alpha} e_{1t} y_t \in \text{RFMat}_{\alpha}(K)e \). For any \( x \in \mathcal{E} \) and \( i \in \omega \) write \( e_ix = \sum_{j \in F_i} e_i xe_j \), where \( F_i \subset \alpha \) is a finite subset. Then we have

\[
e_ix \sum_{j \in F_i} e_j y_j = e_i xy.
\]

Since this holds for any \( i \in \omega \), it follows that \( \varphi(x) = xy \) for any \( x \in e\mathcal{E} \). Because any matrix from \( \text{RFMat}_{\alpha}(K) \cdot e \) obviously determines a homomorphism \( e\mathcal{E} \to e\mathcal{E}e \), the claimed isomorphism follows.

Suppose by contradiction that \( \mathcal{E}e\mathcal{E} \) is a projective object in \( \mathcal{E}e\mathcal{E}\)-mod. Then its Morita correspondent \( e\mathcal{E}e\mathcal{E} = e\mathcal{E} \) is a projective (and infinitely generated) object in \( e\mathcal{E}e\mathcal{E}\)-mod, and then \( \mathcal{E} \) has a (infinite) dual basis \( \{f_i, x_i\} \), where \( f_i \in \text{Hom}_{\mathcal{E}e\mathcal{E}}(e\mathcal{E},e\mathcal{E}e) \cong \text{RFMat}_{\alpha}(K) \cdot e \). Hence \( f_i = y_i \), the right multiplication by \( y_i \in \text{RFMat}_{\alpha}(K) \cdot e \).

We proceed by showing that there is an infinite sequence \( y_{k1}, \ldots, y_{k_n}, \ldots \in \{y_i\} \) and an element \( a \in e\mathcal{E} \), such that \( ay_{kn} \neq 0 \) for all \( n \in \mathbb{N} \). For take first \( e_{11} \). Clearly \( e_1 \in e\mathcal{E} \). There is at least one \( y_i \), such that \( e_1 y_i \neq 0 \). Choose one of them, and denote it by \( y_{k1} \).

Suppose we have chosen \( y_{k1}, \ldots, y_{kn} \), and \( e_{11}, \ldots, e_{nn} \), such that

\[
(1) \quad e_{ri}, y_r \neq 0.
(2) \quad e_{ri}, y_s = 0, \text{ for all } 1 \leq r < s \leq n.
\]

By the definition of a dual basis, the set \( \mathcal{U} = \{y_i : e_{ri}, y_i = 0, 1 \leq r \leq n\} \) can not be empty (in fact, it is infinite). Pick \( y_{kn+1} \in \mathcal{U} \), and choose \( i_{n+1} \) such that \( e_{i_{n+1}}, y_{kn+1} \neq 0 \). Clearly, all elements in \( \{y_{kn}\} \) are distinct.

Now set \( a = \sum_{n \in \mathbb{N}} e_{nn} \). Since \( e_n a = e_{nn} \), we have that \( ay_{kn} \neq 0 \) for all \( n \in \mathbb{N} \), which contradicts the existence of a dual basis.

This means that \( e\mathcal{E} \) is not projective and hence \( \mathcal{E}e\mathcal{E} \) can not be projective.

\[\square\]

**Example 9.3.** Given a group \( G \), which has an element \( g \) with \( g^2 \neq 1 \), there is a homogeneously non-degenerate \( G \)-graded algebra \( \mathcal{B} \) which satisfies (39) such that \( \text{FMat}_{\gamma}(\mathcal{B}) \) (with the \( h \)-homogeneous component of \( \text{FMat}_{\gamma}(\mathcal{B}) \) being defined to be \( \text{FMat}_{\gamma}(\mathcal{B}_h) \), \( h \in G \)) is not isomorphic as a graded algebra to a crossed product by a twisted partial action of \( G \) on \( \text{FMat}_{\gamma}(\mathcal{B}_1) \) for any cardinal \( \gamma \).

**Proof.** Let \( \mathcal{B} \) be the linking algebra of the Morita context (40). Define a \( G \)-grading on \( \mathcal{B} \) by setting

\[
\mathcal{B}_1 = \begin{pmatrix} e\mathcal{E}e & 0 \\ 0 & e\mathcal{E} \end{pmatrix}, \quad \mathcal{B}_g = \begin{pmatrix} 0 & e\mathcal{E} \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B}_g^{-1} = \begin{pmatrix} 0 & 0 \\ e\mathcal{E} & 0 \end{pmatrix}, \quad \mathcal{B}_h = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

for all \( h \in G \) with \( h \neq 1, g, g^{-1} \). We have that
and \( D_g = \begin{pmatrix} e & e \\ 0 & 0 \end{pmatrix}, \quad D_g^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \)

and \( \text{FMat}_\gamma(B_h)\text{FMat}_\gamma(B_{h^{-1}}) = \text{FMat}_\gamma(D_h) \), for any \( h \in G \). Moreover, \( B \) is homogeneously non-degenerate, satisfies (39) and, consequently both of these properties are verified by \( \text{FMat}_\gamma(B) \). It is directly checked that \( \text{FMat}_\gamma(D_g) \cong \text{FMat}_\gamma(e\mathbf{E}e) \) and \( \text{FMat}_\gamma(D_g^{-1}) \cong \text{FMat}_\gamma(\mathbf{EeE}) \). Hence by the previous example \( \text{FMat}_\gamma(D_g) \) is not isomorphic to \( \text{FMat}_\gamma(D_g^{-1}) \) and thus \( \text{FMat}_\gamma(B) \) does not satisfy (ii) of Theorem 6.1. \( \Box \)

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