Continuity equations for entanglement

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Abstract

We introduce a complex purity density and its associated current for pure states of continuous variable systems. The scheme is constructed by analogy with the notions of probability density and probability current. Taking advantage of the formal continuity equations obtained this way we can introduce an entanglement subdynamics. We suggest the use of the dimensionality of this subdynamics as a potential measure of the complexity of entanglement. The scheme also provides insights into the relation between the global and local aspects of quantum correlations.

1 Introduction.

An elegant and intuitive description of the behavior of probability in quantum theory is provided by the continuity equation relating probability density and its associated probability current [1, 2]. This formulation has been used some times to draw formal analogies with the theory of fluids.

The reason behind a continuity equation is the conservation of a physical magnitude, in the previous case the probability. It is well-known that the entanglement of a system of freely evolving particles is also conserved. Then it is natural to consider what type of continuity equation is associated with this conservation law.

We address here the problem for continuous variable systems. In the case of pure states, the only one we shall consider in the paper, the purity is an entanglement measure. It is a measure well-suited for these systems and we shall concentrate on it. The Schmidt number, the inverse of the purity, has been frequently used in the literature [3, 4]. Concurrence [5], another popular entanglement measure, is also directly related to purity [6]. We derive the purity density, which is complex, showing that we must deal with a pair of real densities. Similarly, in our scheme we have two currents instead of one, as it was the case for probability. These densities and currents are connected by a pair of continuity equations.
The above equations are also interesting when we move to the realm of interacting particles. In this scenario the interaction potential connects the behavior of the real and imaginary parts of the densities and currents. The imaginary part of the density is contained in the source term of the real continuity equation and vice versa. We can identify the creation and destruction of entanglement with these sources terms.

The formalism of continuity equations provides a solid basis to define the underlying dynamics of entanglement or entanglement subdynamics. It is based on the temporal evolution of the density and the current. We can use the dimensionality of this subdynamics to introduce a measure of the complexity of the entanglement phenomenon. Moreover, the existence of this subdynamics suggests a picture where the entanglement phenomenon is continuous (or local) in the dynamical space, a character that is lost (it becomes global) when we move to the physical space.

We are only aware of a previous work where the existence of laws of entanglement conservation has been considered \[7\]. These authors also use the purity as measure of the entanglement degree but they consider discrete variables instead of continuous ones. Moreover, the two parties in \[7\] are continuously interacting, explicitly excluding free evolution scenarios. The analysis in \[7\] focuses on the search of invariant quantities, which can be obtained by the introduction of a third party.

The conservation of information in quantum physics has also been considered in the literature \[8\], raising some questions about conserved quantities and information flows that in some aspects resemble the discussion presented here.

2 Purity

In this section we briefly review the basic expressions for the purity as an entanglement measure. The purity, a simple tool to distinguish between pure states and mixtures, can be also used as an entanglement measure of pure states (in this paper we shall not consider mixtures). It is given by

$$\Pi = Tr_x(\hat{\rho}_x^2) = Tr_y(\hat{\rho}_y^2)$$ (1)

where we have used the reduced density matrices

$$\hat{\rho}_x = Tr_y(|\psi><\psi|) ; \hat{\rho}_y = Tr_x(|\psi><\psi|)$$ (2)

We only consider the two-particle case, denoting by \(x\) and \(y\) the spatial coordinates of the particles. In the above equations \(Tr_\mu\) denotes the trace with respect to the variable \(\mu\). On the other hand, \(|\psi>\) is the state of the system. In the position representation it can be expressed as

$$|\psi> = \int d^3x \int d^3y \psi(x,y)|x> |y>$$ (3)
with $\psi$ the wavefunction of the system. $\psi(x, y)$ determines the amplitude of the mode $|x\rangle |y\rangle$ in the state $|\psi\rangle$.

In terms of the wave functions the purity is

$$\Pi = \int d^3x \int d^3y \int d^3x' \int d^3y' \psi^*(x,y)\psi(x',y')\psi^*(x',y')$$

(4)

The Schmidt number, its inverse, is frequently used in the literature. We note that in a similar way we could also use the concurrence. It is related to the purity by the simple expression

$$C = \sqrt{2(1 - \Pi)}$$

[6]. Due to the squared root of the integral it is a little bit harder to handle than the purity, but all the conclusions obtained for $\Pi$ can be easily translated to $C$.

3 The purity density

As it is well-known entanglement is conserved during free evolution. In our case this can be verified by a simple calculation. On the other hand, a law of conservation leads to a continuity equation for the conserved variable. Let us study this equation. First of all we must introduce the density of the variable $\Pi$. From Eq. (4) we can express $\Pi$ as

$$\Pi = \int d^3x \int d^3y \int d^3X \int d^3Y \pi(x,y,X,Y)$$

(5)

with

$$\pi(x,y,X,Y) = \psi^*(x,y)\psi(X,y)\psi^*(X,Y)\psi(x,Y)$$

(6)

Clearly $\pi$ plays the role of $\Pi$-density.

One immediate consequence of its definition is that $\pi$ is a function defined in $R^{12}$. This clearly differs from the probability density for a two-particle system, which acts in $R^6$. In both cases they are densities not defined in the physical space $R^3$. This property differs from the definition of densities in classical systems as fluids, where the space of the density is the physical space.

Another relevant property of $\pi$ is that it is, in general, a complex variable. This can be easily seen from its polar decomposition. For the wave function the decomposition reads $\psi(x,y) = R(x,y)e^{i\varphi(x,y)}$, and for $\pi$:

$$\pi(x,y,X,Y) = R(x,y)R(X,y)R(X,Y)R(x,Y) \times \exp(i(-\varphi(x,y) + \varphi(X,y) - \varphi(X,Y) + \varphi(x,Y)))$$

(7)

It is clear that, except for $\varphi$ being a constant function or other special cases, $\pi$ has an imaginary part.

From a physical point of view a complex density does not make sense. It must be interpreted as a pair of real densities, $\pi_R$ and $\pi_I$, with $\pi = \pi_R + i\pi_I$. Thus, there is not one but two densities associated with $\Pi$. 

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As $\Pi$ is a real number the total contribution of $\pi_I$ must vanish
\[ \int d^3x \int d^3y \int d^3X \int d^3Y \pi_I(x, y, X, Y) = 0 \] (8)

However, as we shall see later, this property does not mean that $\pi_I$ is physically irrelevant for the problem.

We can represent $\pi_R$, $\pi_I$ and $\Pi$ as functions acting between the mathematical spaces
\[ \pi_R : R^{12} \rightarrow R ; \pi_I : R^{12} \rightarrow R \] (9)
and
\[ \Pi : C \rightarrow R \] (10)
The composite action of $\pi$ and $\Pi$ is
\[ \Pi \circ \pi : R^{12} \rightarrow C \rightarrow R \] (11)
with the symbol $\circ$ denoting the consecutive application of the two functions. This chain shows that the dynamics of purity for a two-particle system takes place in the mathematical space $R^{12}$ and implies two real densities. The dimensionality of the problem is much larger than that of the probability dynamics, which for two particles is only $R^6$.

4 Purity current and continuity equations

With the $\Pi$-density we can associate a complex $\Pi$-current obeying a continuity equation. The procedure to derive the equation is the usual one. We take the derivative $\partial\pi/\partial t$ which can be evaluated using the free two-particle Schrödinger equation
\[ i\hbar \frac{\partial \psi}{\partial t}(x, y) = -\frac{\hbar^2}{2m} \Delta_x \psi(x, y) - \frac{\hbar^2}{2M} \Delta_y \psi(x, y) \] (12)
and its complex conjugate. After simple manipulations we obtain
\[ \frac{\partial \pi}{\partial t} + \nabla \cdot \mathbf{J} = 0 \] (13)
The $\nabla$-operator and the current $\mathbf{J}$ are defined in twelve dimensions. The operator can be expressed as
\[ \nabla \equiv (\nabla_x, \nabla_y, \nabla_X, \nabla_Y) \] (14)
being $\nabla_x$ the usual three-dimensional gradient operator for the variables $x$.

Similarly, the current has four three-dimensional components
\[ \mathbf{J} \equiv (J_x, J_y, J_X, J_Y) \] (15)
With this notation we have $\nabla \cdot J = \nabla_x \cdot J_x + \nabla_y \cdot J_y + \nabla_X \cdot J_X + \nabla_Y \cdot J_Y$. The explicit expressions for the four components of the current are

$$J_x = -\frac{i\hbar}{2m} \psi(x, y)\psi^*(x, y)(\psi(x, y)\nabla_x \psi^*(x, y) - \psi^*(x, y)\nabla_x \psi(x, y))$$ (16)

$$J_y = -\frac{i\hbar}{2M} \psi(x, y)\psi^*(x, y)(\psi(x, y)\nabla_y \psi^*(x, y) - \psi^*(x, y)\nabla_y \psi(x, y))$$ (17)

$$J_X = -\frac{i\hbar}{2m} \psi^*(x, y)\psi(x, y)(\psi(x, y)\nabla_x \psi^*(X, Y) - \psi^*(X, Y)\nabla_x \psi(x, y))$$ (18)

and

$$J_Y = -\frac{i\hbar}{2M} \psi^*(x, y)\psi(x, y)(\psi(x, y)\nabla_y \psi^*(X, Y) - \psi^*(X, Y)\nabla_y \psi(x, y))$$

(19)

Clearly $J$, as $\pi$, is a complex variable. We can introduce its decomposition in real and imaginary parts, $J = J_R + iJ_I$.

The continuity-like Eq. (13) is only formal. A complex density and a complex current do not make any physical sense. They must be understood as two densities and two currents connected by two different continuity equations:

$$\frac{\partial \pi_\xi}{\partial t} + \nabla \cdot J_\xi = 0$$ (20)

with $\xi = R, I$.

5 Interacting particles

After deriving the continuity equations for free evolving particles we must analyze how they are modified when interactions are taken into account. We describe the interaction by introducing in the two-particle Schrödinger equation the term $V(x, y)\psi(x, y)$. The potential $V$ includes both inter-particle and particle-external field interactions. By the matter of simplicity we assume $V$ to be real and only to depend on the position variables of the problem.

In a straightforward way we obtain the continuity equations with interaction:

$$\frac{\partial \pi_R}{\partial t} + \nabla \cdot J_R = -\frac{\pi_I}{\hbar} U$$ (21)

and

$$\frac{\partial \pi_I}{\partial t} + \nabla \cdot J_I = \frac{\pi_R}{\hbar} U$$ (22)

with

$$U(x, y, X, Y) = V(x, y) - V(X, y) + V(X, Y) - V(x, Y)$$ (23)
The terms in the r. h. s. of the two equations represent the sources of \( \Pi \)-density. They describe as the entanglement density can be generated or destroyed. From these equations it is clear than in the absence of interaction the \( \Pi \)-density can only flow.

Another important characteristic of these equations is that they connect \( \pi_R \) and \( \pi_I \). During free evolution they are two independent magnitudes. The evolution of \( \pi_R \) does not depend at all on \( \pi_I \), and vice versa. In contrast, when there is interaction the source terms of \( \pi_R \) and \( \pi_I \) depend respectively on \( \pi_I \) and \( \pi_R \). Thus, although \( \pi_I \) does not contribute directly to \( \Pi \) (Eq. (8)), it plays an essential role in its generation.

6 Entanglement underlying dynamics

The continuity equations derived in the previous sections could be seen as a merely formal exercise. However, we shall argue that they suggest the existence of an underlying entanglement dynamics or entanglement subdynamics. The very possibility of formulating a continuity equation for a physical variable indicates that it can locally change, generating flows towards adjacent points. There is a dynamical behavior of the variable. In our case we can say that even for freely evolving two-particle systems the entanglement density is not a static phenomenon but a dynamical one.

We define the entanglement subdynamics as the dynamics of the entanglement density. We denote as the dynamical space the space where this subdynamics takes place. The subdynamics and the dynamics of entanglement must be compatible, suggesting that it is possible in principle to obtain some information about the second one from the first one. We shall discuss next two examples of such potential applications.

A fundamental magnitude to characterize any dynamics is its dimension, that is, the dimension of the space where it takes place. We take here a further step and propose to measure the complexity of the entanglement phenomenon using a parameter related to that dimension. We introduce the Dimension Comparison Parameter (DCP) as the ratio between the dimension of the subdynamics and the natural dimension. The natural dimension is that of the space where the wave functions are defined. A DCP larger that one implies that the natural space does not provide enough mathematical structure to describe the phenomenon and it is necessary to consider spaces with a larger dimension. We expect the complexity of a phenomenon to increase with the DCP.

In the case of probability the space of the probability density is \( R^6 \), whose dimension coincides with that of the natural space. We do not need to extend the natural space to describe the probability density. In contrast, for entanglement \( \pi \) is defined in \( R^{12} \), much larger than the natural space \( R^6 \). The DCP is two, indicating a large complexity for the problem. We need to double the dimensionality to accommodate the entanglement subdynamics.
This dimensionality also determines the form of the flows. In a classical fluid the flow is between points of the physical space \( R^3 \). In the case of the quantum probability the picture changes. The two-particle probability flow no longer lies on the physical space but in \( R^6 \). When we move to our problem the situation is even more involved. The flux now is a complex variable and we must consider two physical flows, one for each component of the current. Moreover, each one of the physical flows lies in \( R^{12} \). Each one of the flows connects adjacent points in the \( R^{12} \) space, which correspond to four different points of the physical space (not necessarily adjacent).

The last property suggests an interesting picture of the relation between the global and local aspects of entanglement. The possibility of formulating continuity equations for a system guarantees the existence of a local dynamics in the space where they are defined. Here, we use the term local in the sense that only adjacent points affect the behavior of a given point at a given time, not in the relativistic one. It can be understood as opposite to global dynamics (we use the word global instead of non-local to try to avoid any confusion with the use of the last term in the context of Bell-type inequalities). For two particles this local behavior takes place in the space \( R^{12} \). However, in the physical space there is not a continuity equation and the dynamics is global. The absence of a continuity equation is a consequence of the fact that the flow connects sets of four points. Our formulation provides a picture of the relation between the global aspects of entanglement (connecting separated points in the physical space) and the local ones (only connecting adjacent points in the dynamical space). It suggests an image where the entanglement is local in the dynamical space, but with a global appearance when we operate in the physical space. Extensions of this type of reasoning could be on the basis of a better understanding of the global aspects of quantum theory.

7 Discussion

In the paper we have introduced a pair of complementary continuity equations to describe the purity of entangled continuous variable systems in pure states. The continuity equations, introduced for free evolving states, remain valid in interacting scenarios after the addition of source terms. This way we can identify the physical form of the terms creating or destroying entanglement. We have used the above mathematical formalism to outline a theory of the underlying dynamics, which gives some insights on the characterization of entanglement based on its dimension and on their global aspects.

Our approach lies on the analogy with the description of the probability flow via a continuity equation. At variance with the probability description two densities and two currents are involved in the problem. This fact strongly suggests that entanglement is a more complex physical phenomenon than probability. We can make this argument more quantitative. We can use the dimension of
the space of the underlying dynamics as a measure of the complexity of the phenomenon. As discussed before, for two particles this dimension goes as the second power of that of the quantum probability space of the system. In general we can take this dimension, via the DCP, as a measure of complexity.

In [7] it has been noted that the particular dynamics of the entanglement present in that problem cannot be viewed as a flow of this quantity between the parties. The same conclusion can be easily extended to our case, where it is possible to introduce a flux or current but neither can be interpreted as a flow between parties.

The fact that the flow connects four different points in $\mathbb{R}^3$, not necessarily contiguous, is a clear manifestation of the global character of entanglement. This global feature is not exclusive from entanglement. For quantum probability the flow also connects points in $\mathbb{R}^6$, that is, pairs of points in $\mathbb{R}^3$ not necessarily contiguous. Global flows seem to be an inherent characteristic of quantum multi-particle processes.

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