In Ref. [1], a Josephson junction shunted by an ohmic transmission line is studied. The authors present a phase diagram with features not anticipated in the established literature [2]. We show that their Numerical Renormalization Group (NRG) calculation suffers from several flaws, and cannot be trusted to substantiate their claims. In Ref. [1], a Josephson junction shunted by an ohmic transmission line is studied. The authors present a phase diagram with features not anticipated in the established literature [2]. We show that their Numerical Renormalization Group (NRG) calculation suffers from several flaws, and cannot be trusted to substantiate their claims. NRG captures low energy physics by building recursive Hamiltonians, $H_{N+1} = H_N + \Delta H_N$, that are iteratively diagonalized. Scale separation is required for NRG to work, i.e. $\Delta H_N$ should decrease exponentially with $N$ [3]. For the NRG scheme in Ref. [1], $\Delta H_N$ is of the same order as $H_0$ [See Eqs. (S51) and (S52) in the supplementary material to [1].] This is a known problem that can only be cured by introducing an infrared cutoff [4]. As a result, the NRG fails to flow to the correct infrared fixed point. To demonstrate this, we considered large conductance $\alpha$ and large $E_J/E_C$, where the system studied in [1] is nearly harmonic, allowing us to expand $-E_J \cos(\Xi) \simeq E_J(\Xi^2/2 - 1)$. We compared low energy spectra obtained with the NRG scheme of [1] for the cosine and quadratic potentials, to the exact spectrum obtained for the latter. As the top panel of Fig. 1 shows, the NRG results diverge from the exact spectrum after the seventh RG step. Thus the NRG scheme proposed in [1] is unreliable and cannot be trusted to predict the phase diagram. (See Appendix A for discussion of the RG flow of mobility $\mu_{10}$.)

The phase diagram in [1] is flawed in another way. Even if one trusted the employed NRG scheme, the re-entrant superconductivity seen at small $\alpha$ and small $E_J/E_C$ is a numerical artefact. The blue dots in the bottom panel of Fig. 1 reproduce the result for $\langle \cos(\varphi) \rangle$ v. $\alpha$ at $E_J/E_C = 0.15$ in the upper panel of Fig. 4 of [1], obtained with the truncation parameter $n_B = 15$ in each mode for $N > 0$. For this result to be correct, it must not change when $n_B$ is increased. Instead we see that the region where $\langle \cos(\varphi) \rangle$ vanishes, grows to include the interval $\alpha \in [0, 0.2]$ when $n_B$ is increased. Thus, the apparent re-entrant superconductivity in the phase diagram in [1] stems from un converged data. In [1] it is argued that superconductivity makes common sense when the junction is shunted by a sufficiently large impedance. We stress that taking the thermodynamic limit $N \rightarrow \infty$ before $\alpha \rightarrow 0$, couples the junction to divergent $\varphi$-fluctuations that render the junction’s zero-frequency response non-trivial. The object Letter also contains a brief functional Renormalization Group (fRG) argument in support of superconductivity at $\alpha < 1$ and large $E_J/E_C$. The approximations involved are not controlled by any obvious small parameter. It is still not known whether fRG can reproduce infrared Luttinger exponents for $1 < \alpha < 2$ [4], where phase-slips affect results non-trivially. Until this is settled, fRG’s validity in the more challenging $\alpha < 1$ regime remains unclear.
Appendix A: Additional Information

Here we present further information that length restrictions prevented us from presenting in the published comment. It concerns the contribution $\mu_{10}$ to the phase mobility, that is employed as an order parameter in the Object Letter.

![Graph showing mobility as a function of site index N]

**FIG. 2.** Top panel: Mobility $\mu_{10}$ as a function of site index $N$ in the NRG discretization, for $E_J = 10E_C$, $\alpha = 10$, $N_{\text{kept}} = 50$, $n_B = 14$ for $n > 0$. Other parameters as in the Comment. Bottom panel: Same quantity, replotted on a logarithmic vertical axis, showing the breakdown in the NRG computation, both for the cosine (red dots) and quadratic (green triangles) potentials. In contrast to the vanishing mobility at large $N$, as expected in the superconducting phase, the NRG leads to a finite mobility at the end of the flow (namely $N \gg 1$).

We have calculated $\mu_{10}$, which sheds further light on the convergence issues pointed out in the Comment, and investigated is dependence with $N$, the number of sites in the NRG discretization. This observable shows a crossover between ultraviolet behaviour (small $N$) and infrared behavior (large $N$). At small $N$, one has:

$$\mu_{10} \simeq \alpha \xi_N^2.$$  \hspace{1cm} (A1)

(See the Supplemental Material to the Object Letter for details on the notation). In the harmonic limit, where the cosine potential can be replaced by one that is quadratic, the asymptotic behavior at large $N$ is

$$\mu_{10} \simeq \left(\frac{\Lambda}{2}\right)^4 \frac{\gamma_0^2}{8E_J^2\xi_N^4}.$$  \hspace{1cm} (A2)

The top panel of Fig. 2 shows behavior very similar to the non-monotone flow presented in Fig. 3 of the Object Letter. It shows the flow of the mobility, which at first sight seems to indicate that the NRG results are reliable and lead to a vanishing mobility in the ground state. However, the correct value of the mobility $\mu_{10}$ should be read after complete iteration of the NRG scheme (namely for large $N$ values in the plot), corresponding to the final stage of the renormalization flow. In the bottom panel of Fig. 2, we therefore show exactly the same mobility as in the top panel, but we have extended the horizontal axis to larger $N$ and displayed the data in a better way using a logarithmic scale. What can be seen here is again a complete breakdown of this NRG after few iterations: the mobility does not vanish (either for the cosine or quadratic potentials), contrary to the claim of Masuki et al. Rather, the mobility saturates to a finite value in the NRG simulation, which is physically incorrect. In contrast, the exact solution does display a vanishing mobility at large $N$, as expected in the superconducting phase of the model. The same issues are found for all values of the parameters of the considered model, and thus the results cannot be trusted to establish a phase diagram. Again, we stress that these problems are fully expected since the NRG of Masuki et al. does not converge. A devil’s advocate could perhaps argue that a finite but small mobility could be used as an approximate way to describe the superconducting phase, although there would be no qualitative difference with the insulating regime, so that a careful scaling analysis would be required to establish a proper phase diagram. However, this argument cannot be made, because all the NRG calculations of Masuki et al. are plagued by convergence problems. A clear example for this serious issue is given for the parameters $E_J = 10E_C$, $\alpha = 2$, which indisputably fall inside the superconducting phase. If we increase the truncation parameters $n_B$ and $N_{\text{kept}}$, from respectively 15 and 50 (their values in the original Letter by Masuki et al.) to respectively 29 and 100, which should make the result more accurate, the mobility $\mu_{10}$ switches from superconducting-like to strongly insulating for the cosine potential, see Fig. 3 below.

A further point, of importance to anyone wishing to reproduce our results, or those of the object Letter, is the following: To achieve agreement with the results in the Object Letter, we had to reverse engineer a mistake in the numerics, whose presence is revealed by the spectra of Fig.S2 in the Supplemental Material to the Object Letter. At low $N$, the spectrum should be $n\gamma_0$, $n = 1, 2, \ldots$. According to Eq. (S50), one should have $\gamma_0 = 4.33$ in Fig.S2. Instead, one sees $\gamma_0 = 0.5$. There is in fact a typo in Eq. (S50): $(1 + \Lambda + \Lambda^{-2})/3$ should be replaced with $(1 + \Lambda^{-1} + \Lambda^{-2})/3$. Correcting the typo however...
still does not give $\gamma_0 = 0.5$ as in Fig. S2. We therefore tried various plausible mistakes, and we found one that reproduced exactly the NRG results in the Object Letter. Apparently, the numerics in the Object Letter were performed using $(1+\Lambda^{-3})/3$ instead of $(1+\Lambda+\Lambda^{-2})/3$ in Eq. (S50) that defines $\gamma_n$. When the correct expression for $\gamma_n$ is used, the position of the phase boundary in the phase diagram changes, making the apparent agreement to fRG predictions rather fortuitous.

![Convergence issues in the NRG scheme of Masuki et al. comparing two choices of truncation parameters. Top panel: $N_{\text{kept}} = 50$, $n_B = 14$ for $n > 0$, leading to a finite but small mobility. Bottom panel: $E_J = N_{\text{kept}} = 100$, $n_B = 29$ for $n > 0$, leading to a finite but large mobility. The mobility should in any case vanish in the limit $N \gg 1$, since parameters $E_J = 10E_C$, $\alpha = 2$ correspond to the superconducting phase.](image)

FIG. 3. Convergence issues in the NRG scheme of Masuki et al., comparing two choices of truncation parameters. Top panel: $N_{\text{kept}} = 50$, $n_B = 14$ for $n > 0$, leading to a finite but small mobility. Bottom panel: $E_J = N_{\text{kept}} = 100$, $n_B = 29$ for $n > 0$, leading to a finite but large mobility. The mobility should in any case vanish in the limit $N \gg 1$, since parameters $E_J = 10E_C$, $\alpha = 2$ correspond to the superconducting phase.

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