Computable real function $\mathcal{F}$
such that $\mathcal{F} \in \text{FEXPTIME}_{C[0,1]}$

but $\mathcal{F} \notin \text{FPTIME}_{C[0,1]}$

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Abstract
In the present paper, a computable real function $\mathcal{F}$ on $[0,1]$ is constructed such that there exists an exponential time algorithm for the evaluation of the function on $[0,1]$ on Turing machine but there does not exist any polynomial time algorithm for the evaluation of the function on $[0,1]$ on Turing machine (moreover, it holds for any rational point on $(0,1)$).

Keywords: Computable real numbers and functions, Cauchy function representation, modulus function of uniform continuity, polynomial time computable real functions, exponential time computable real functions.

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1 Introduction
In the present paper, we consider computable real numbers and functions that are represented by Cauchy functions computable on Turing machines [1] (main results regarding computational complexity of computations on Turing machines can be found in [2]).

Book [1] studies mainly polynomial time computable real numbers and functions. It is a common point of view that polynomial time algorithms can be considered as efficient algorithms, so such algorithms are very important for theoretical and practical computer science.
In book [1] a general theory of polynomial time computable real numbers and functions is considered. But it is interesting to consider specific examples of computable real numbers and functions with specific lower and upper bounds of time and space complexity of them.

In the present paper, a computable real function $F$ on $[0, 1]$ is constructed such that the function is exponential time computable on $[0, 1]$ but it is not polynomial time computable on $[0, 1]$ (moreover, it holds for any rational point on $(0, 1)$).

To show that function $F$ is not polynomial time computable on $[0, 1]$, it is proved that there does not exist any polynomial time modulus function of uniform continuity of function $F$ on $[0, 1]$ (moreover, there does not exist any polynomial time modulus function of continuity of function $F$ at any rational point on $(0, 1)$). So, a function such that there is no efficient algorithm to evaluate it is given.

The construction of function $F$ is very similar to the constructions of nowhere-differentiable real functions like Weierstrass function (many of such functions could be found in [3]) in the sense that function $F$ is an infinite sum of simple functions with certain properties. A similar construction can also be found in [1] (in the examples regarding complexity of derivatives of polynomial time computable real functions).

1.1 $CF$ computable real numbers and functions

Cauchy functions in the model defined in [1] are functions binary converging to real numbers. A function $\phi : \mathbb{N} \rightarrow \mathbb{D}$ (here $\mathbb{D}$ is the set of dyadic rational numbers) is said to binary converge to real number $x$ if

$$|\phi(n) - x| \leq 2^{-n}$$

for all $n \in \mathbb{N}$; $CF_x$ denotes the set of all functions binary converging to $x$.

Real number $x$ is said to be a $CF$ computable real number if $CF_x$ contains a computable function $\phi$.

Real function $f : [a, b] \rightarrow \mathbb{R}$ is said to be a $CF$ computable function on $[a, b]$ if there exists a function-oracle Turing machine $M$ such that for all $x \in [a, b]$ and for all $\phi \in CF_x$ function $\psi$ computed by $M$ with oracle $\phi$ is in $CF_{f(x)}$.

When the time complexity of a computable function $f$ is considered, the whole process of querying for oracle $\phi$ costs only one time unit.

**Definition 1.** [1] Function $f : [a, b] \rightarrow \mathbb{R}$ is said to be $\text{FPTIME}$ ($\text{FEXPTIME}$) computable real function on $[a, b]$ if for all computable $x \in [a, b]$ function $\psi \in CF_{f(x)}$ ($\psi$ is from the definition of $CF$ computable real function) is $\text{FPTIME}$ ($\text{FEXPTIME}$) computable.

Function $f$ computable at point $x \in [a, b]$ is defined in a similar way. Set of polynomial time (exponential time) computable real functions on $[a, b]$ is denoted by $\text{FPTIME}_{C[a,b]}$ ($\text{FEXPTIME}_{C[a,b]}$).

1.2 Modulus function of uniform continuity

**Definition 2.** [1] Let $f : [a, b] \rightarrow \mathbb{R}$ be a uniformly continuous function on $[a, b]$. Function $\omega : \mathbb{N} \rightarrow \mathbb{N}$ is said to be a modulus function of uniform continuity of function $f$ on $[a, b]$ if

$$|x - y| \leq 2^{-\omega(n)} \text{ implies } |f(x) - f(y)| \leq 2^{-n}$$

for all $n \in \mathbb{N}$ and for all $x, y \in [a, b]$.

In the following theorem it is said that $\omega$ is a computable function for computable real functions (this theorem holds if a model of computable real functions is considered such that real functions on $[a, b]$ are defined for all real numbers in $[a, b]$).
Theorem 1. If function $f : [a, b] \to \mathbb{R}$ is a $CF$ computable real function on $[a, b]$ then there exists a modulus function of uniform continuity of function $f$ on $[a, b]$.

It is said that there is a polynomial modulus function of uniform continuity of function $f$ on $[a, b]$ if there exists a polynomial $\omega : \mathbb{N} \to \mathbb{N}$ such that

$$|x - y| \leq 2^{-\omega(n)} \text{ implies } |f(x) - f(y)| \leq 2^{-n}$$

for all $n \in \mathbb{N}$ and for all $x, y \in [a, b]$.

Theorem 2. If $f$ is a $FPTIME$ computable real function then there exists a polynomial modulus function of uniform continuity of function $f$ on $[a, b]$; it means there exists a polynomial $\omega : \mathbb{N} \to \mathbb{N}$ such that

$$|x - y| \leq 2^{-\omega(n)} \text{ implies } |f(x) - f(y)| \leq 2^{-n}$$

for all $n \in \mathbb{N}$ and for all $x, y \in [a, b]$.

The same results hold if we consider the evaluation of function $f$ at a point $x \in [a, b]$ and consider the notion of modulus function of continuity at this point.

2 Construction of computable real function $F$

Let’s construct computable real function $F$ on $[0, 1]$ such that

1) there does not exist polynomial modulus function of uniform continuity of function $F$ on $[0, 1]$, and therefore function $F$ is not a polynomial time computable function on $[0, 1]$ (moreover, there does not exist any polynomial time modulus function of continuity of function $F$ at any rational point on $(0, 1)$, so function $F$ is not a polynomial time computable function at any rational point on $(0, 1)$),

2) there exists an exponential time algorithm for the evaluation of function $F$ on $[0, 1]$.

The construction of function $F$ on $[0, 1]$ is as follows:

1) let’s define real function $\beta_{p,q}$ on $[0, 1]$: $\forall p, q$ such that $q \geq 1$ and $p \in [0..(q - 1)]$;

$$\beta_{p,q}(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{p}{q}, \\ 2^q \left( x - \frac{p}{q} \right) & \text{if } \frac{p}{q} \leq x \leq \frac{p + \frac{1}{q}}{q}, \\ \frac{1}{q} & \text{if } \frac{p + \frac{1}{q}}{q} \leq x \leq 1, \end{cases}$$

for natural numbers $p$ and $q$ such that $q \geq 1$ and $p \in [0..(q - 1)]$;

2) let’s define real function $\alpha_q$ on $[0, 1]$:

$$\alpha_q(x) = \sum_{p=0}^{q-1} \beta_{p,q}(x);$$

3) $F(x) = \sum_{q=1}^{\infty} \frac{1}{q^2} \alpha_q(x).$ (1)
2.1 Function $F$ is not polynomial time computable

Let’s prove that function $F$ has the following properties:

1) function $F$ is a monotonically increasing function on $[0, 1]$;
2) function $F$ is a uniformly continuous function on $[0, 1]$;
3) $\omega(n) \geq 2^{C \cdot n}$, wherein $\omega$ is the function from definition 2 should hold if we evaluate function $F$ in precision $2^{-n}$.

Functions $\beta_{p,q}$ are monotonically increasing functions, therefore $\alpha_q$ are monotonically increasing functions as sums of monotonically increasing functions $\beta_{p,q}$, and function $F$ is monotonically increasing function as a sum of monotonically increasing functions $\alpha_q$. So, point 1) holds.

Because $|\alpha_q(x + \delta) - \alpha_q(x)| \leq 1$, the following holds for all real $x \in [0, 1]$ and real $\delta$, wherein $\delta \leq 1$:

$$|F(x + \delta) - F(x)| \leq \sum_{q: \delta \leq \frac{1}{q}} \frac{1}{q^2} + \sum_{q: \delta > \frac{1}{q}} \frac{1}{q^2} \leq \frac{1}{\delta} + \delta.$$

So, point 2) holds (taking into account point 1)).

Further, we have

a) functions $\beta_{p,q}$ and function $F$ are monotonically increasing functions, and

b) if $\delta \leq \frac{1}{2n}$, $n \leq q \leq n^2$, and $x = \frac{p}{q}$ then

$$x = \frac{p}{q} < \frac{p + \frac{1}{2}}{q} < x + \delta$$ and $$\left| \frac{1}{q} \right| \geq \left| \frac{1}{\log(\delta)^2} \right|.$$

Therefore, for all real $\delta$, wherein $\delta \leq 1$, there exist $x$, namely $x = \frac{p}{q}$, such that the following holds:

$$|F(x + \delta) - F(x)| > \frac{1}{q^2} |\beta_{p,q}(x + \delta) - \beta_{p,q}(x)| = \frac{1}{q^2} \cdot \frac{1}{q} = (\log_2(\delta))^{-6}.$$

Hence,

$$(\log(\delta))^{-6} \leq 2^{-n}$$

should hold if we require that $|F(x + \delta) - F(x)| \leq 2^{-n}$ holds; it means $\omega(n) \geq 2^{C \cdot n}$ should hold for $\delta = 2^{-\omega(n)}$. So, point 3) holds.

It follows from point 3) that proposition 1 holds.

**Proposition 1.** There does not exist polynomial modulus function of uniform continuity of function $F$ on $[0, 1]$.

Let $x$ be rational $\frac{p}{q} \in (0, 1)$, real $\delta$ be $\frac{1}{2n}$; in that case point b) holds for $q = (q')^k$, such that $n \leq (q')^k \leq n^2$ for some natural number $k$, and for $x = \frac{p}{(q')^k}$; therefore, $\omega(n) \geq 2^{C \cdot n}$ should hold at $x$. So, proposition 2 holds.

**Proposition 2.** For each rational number $x \in (0, 1)$ there does not exist polynomial modulus function of continuity of function $F$ at point $x$.

So, the following theorems hold.

**Theorem 3.** Real function $F$ is not a polynomial time computable real function on $[0, 1]$ of real numbers.

**Theorem 4.** For each rational number $x \in (0, 1)$ real function $F$ is not a polynomial time computable real function at point $x$.

Let’s note that function $F$ is a nowhere-differentiable function on $[0, 1]$. 


2.2 Evaluation of function $F$ in exponential time

Because

$$|\alpha_q(x)| \leq 1,$$

for the remainder of series (1) the following holds:

$$|R_t(x)| \leq \left| \sum_{q=t+1}^{\infty} \frac{1}{q^2} \right| < C_1 \cdot 2^{-C_2 \log(q)}.$$ 

Therefore, if we sum $2^{C \cdot n}$ terms of the series and evaluate each of that terms (which are functions $\alpha_q$) in precision $2^{-C \cdot n}$ then we evaluate series (1) in precision $2^{-n}$.

Because $\omega(n)$, wherein $\omega$ is the function from definition 2, is exponential in $n$, the time complexity of functions $\alpha_q$ is exponential in $n$; therefore, the time complexity of the evaluations of $F(x)$ in precision $2^{-n}$ is exponential in $n$.

To evaluate functions $\beta_{p,q}$ at points $p$ and $p+1$ respectively, let’s use the following result from [1]. Let

1) $f$ be a computable real function on $[a,b]$,

2) $g$ be a computable real function on $[b,c]$, and

3) $f(b) = g(b)$.

In that case, function $h$ defined by equation

$$h(x) = f(\max(x,z)) + g(\max(x,z)) - f(z)$$

is a computable real function and has the following property:

$$h(x) = \begin{cases} 
 f(x) & \text{if } x \in [a,b], \\
 g(x) & \text{if } x \in [b,c].
\end{cases}$$

**Theorem 5.** Real function $F$ is an exponential time computable real function on $[0,1]$ of real numbers.

3 Conclusion

One of the open questions here is as follows: if function $F$ is polynomial time computable at irrational numbers $x \in (0,1)$ ? If not, how a function that is not polynomial time computable at every real $x \in [0,1]$ could be constructed ?

It could as well be interesting to construct computable real functions that are computable in a time complexity class, but not computable in a space complexity class, or that are computable in a space complexity class, but not computable in a time complexity class, and so on.

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