Extreme values of the Riemann zeta function at its critical points in the critical strip

by

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In memory of Professor Andrzej Schinzel

1. Introduction. Assuming the Riemann hypothesis (RH), Littlewood proved that

\[
\limsup_{t \to \infty} \frac{|\zeta(1 + it)|}{\log \log t} \leq 2e^{C_0},
\]

where \(C_0\) denotes Euler’s constant. He also proved unconditionally that

\[
\limsup_{t \to \infty} \frac{|\zeta(1 + it)|}{\log \log t} \geq e^{C_0}.
\]

Gonek and Montgomery obtained similar results sampling zeta at the critical points of the Riemann zeta function to the right of \(\Re s = 1\). Let \(\rho' = \beta' + i\gamma'\) denote a typical critical point of the zeta function, that is, a point where \(\zeta'(\rho') = 0\). Assuming RH, they showed that for \(\beta' \geq 1\),

\[
\limsup_{\gamma' \to \infty} \frac{|\zeta(\rho')|}{\log \log \gamma'} \leq \frac{1}{2} e^{C_0}
\]

and unconditionally, for \(\beta' > 1\), that

\[
\limsup_{\gamma' \to \infty} \frac{|\zeta(\rho')|}{\log \log \gamma'} \geq \frac{1}{4} e^{C_0}.
\]

Gonek and Montgomery mention that one of their motivations was to answer a question posed by J. G. Thompson as to whether, for any large constant \(c\), there must always be infinitely many compact connected components of the level set \(|\zeta(s)| = c\). We see from (1.3) that the answer is yes, for if we take

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c to be just slightly less than $|\zeta(\rho')|$, there will be at least one compact connected component that passes close to $\rho'$. Furthermore, Montgomery and Thompson [4] have shown that the imaginary parts of points on a connected compact component of a level set $|\zeta(s)| = c$ will all lie in an interval of the form $[T - C \log T, T + C \log T]$ where $C$ is an absolute constant. Thus widely-spaced critical points $\rho'$ will give rise to disjoint compact connected components.

In this paper we obtain estimates corresponding to (1.3) and (1.4) in the right half of the critical strip.

**Theorem 1.** Assume RH. Let $\sigma_1$ and $\sigma_2$ be fixed with $1/2 < \sigma_1 < \sigma_2 < 1$. If $\rho' = \beta' + i\gamma'$ is any critical point of the Riemann zeta function with $\sigma_1 < \beta' < \sigma_2$, then there is a positive constant $A$ depending on $\sigma_1$ and $\sigma_2$ such that

$$\log |\zeta(\rho')| \leq A \frac{\log^{2-2\beta'} \gamma'}{\log \log \gamma'}.$$  \hspace{1cm} (1.5)

**Remark.** Since we do not use any specific property of the critical points, the estimate in (1.5) is in fact true with $\rho'$ replaced by any point $s = \sigma + it$ with $\sigma_1 < \sigma < \sigma_2$ (see, for example, Titchmarsh [8, equation (14.5.2)]), so it follows from it. We state the alternate version below.

Assume RH. Let $\sigma_1$ and $\sigma_2$ be fixed with $1/2 < \sigma_1 < \sigma_2 < 1$. Let $s = \sigma + it$ be such that $\sigma_1 < \sigma < \sigma_2$. Then there is a positive constant $A$ depending on $\sigma_1$ and $\sigma_2$ such that

$$\log |\zeta(s)| \leq A \frac{\log^{2-2\sigma} t}{\log \log t}. \hspace{1cm} (1.6)$$

We nevertheless give a separate proof since the method we use informs the proof of our second theorem.

**Theorem 2.** Let $\sigma_1$ be such that $1/2 < \sigma_1 < 1$. Let $\rho' = \beta' + i\gamma'$ denote a critical point of the Riemann zeta function such that $\sigma_1 < \beta' < 1$. Then for any $\epsilon, \epsilon' > 0$ and for infinitely many $\rho'$ with $\gamma' \to \infty$, we have unconditionally

$$\log |\zeta(\rho')| \geq (B(\sigma_1) - \epsilon') \frac{\log^{1-\beta'} \gamma'}{(\log \log \gamma')^{5-3\beta'+\epsilon}}, \hspace{1cm} (1.7)$$

where

$$B(\sigma_1) = \frac{(\sigma_1 - 1/2)^{1-\sigma_1} \log 2}{(1 - \sigma_1)^2 4^{1-\sigma_1}}. \hspace{1cm} (1.8)$$

One can ask about similar results concerning the small values of $\zeta(1+it)$. Assuming RH, Littlewood [3] showed that

$$\lim inf_{t \to \infty} \frac{|\zeta(1+it)| \log \log t}{\pi^2/12 e^{-C_0}}.$$

and Titchmarsh [7] showed unconditionally that
\[
\liminf_{t \to \infty} |\zeta(1 + it)| \log \log t \leq \frac{\pi^2}{6} e^{-C_0}.
\]
Gonek and Montgomery [1] obtained corresponding results at the critical points of the Riemann zeta function to the right of \( \Re s = 1 \). They showed unconditionally, for \( \beta' > 1 \), that
\[
\liminf_{\gamma' \to \infty} |\zeta(\rho')| \log \log \gamma' \geq \frac{\pi^2}{3} e^{-C_0},
\]
and assuming RH that
\[
\liminf_{\gamma' \to \infty} |\zeta(\rho')| \log \log \gamma' \leq \frac{2\pi^2}{3} e^{-C_0}.
\]
We prove the analogous results at the critical points \( \rho' \) in the right half of the critical strip.

**Theorem 3.** Assume RH. Let \( \sigma_1 \) and \( \sigma_2 \) be such that \( 1/2 < \sigma_1 < \sigma_2 < 1 \). If \( \rho' = \beta' + i\gamma' \) is any critical point of the Riemann zeta function with \( \sigma_1 < \beta' < \sigma_2 \), then for some positive constant \( C \) depending on \( \sigma_1 \) and \( \sigma_2 \) we have
\[
\log |\zeta(\rho')| \geq -C \log^2 \frac{2\beta'}{\log \log \gamma'}.
\]

**Remark.** Since we do not use any specific property of the critical points, in Theorem 3, the estimate (1.9) is true even if we replace \( \rho' \) with a generic point \( s = \sigma + it \). We state the alternative version as follows.

**Assume RH.** Let \( \sigma_1, \sigma_2 \) and \( \sigma \) be such that \( 1/2 < \sigma_1 < \sigma < \sigma_2 < 1 \). Then for some positive constant \( C \) depending on \( \sigma_1 \) and \( \sigma_2 \), we have
\[
\log |\zeta(s)| \geq \frac{-C \log^2 (2\sigma)}{\log \log t}.
\]

**Theorem 4.** Let \( \sigma_1 \) be such that \( 1/2 < \sigma_1 < 1 \). Let \( \rho' = \beta' + i\gamma' \) denote a critical point of the Riemann zeta function such that \( \sigma_1 < \beta' < 1 \). Then for any \( \epsilon, \epsilon' > 0 \) and for infinitely many \( \rho' \) with \( \gamma' \to \infty \), we have unconditionally
\[
\log |\zeta(\rho')| \leq (-B(\sigma_1) + \epsilon') \frac{\log^{1-\beta'} \gamma'}{(\log \log \gamma')^{5-3\beta'+\epsilon}},
\]
where
\[
B(\sigma_1) = \frac{(\sigma_1 - 1/2)^{1-\sigma_1} \log 2}{(1 - \sigma_1)^2 4^{1-\sigma_1}}.
\]
2. Lemmas for the proof of Theorem

**Lemma 1.** Assume RH. Let \( \sigma_1 \) and \( \sigma_2 \) be fixed with \( 1/2 < \sigma_1 < \sigma_2 < 1 \). Then for any \( \sigma \) with \( \sigma_1 \leq \sigma \leq \sigma_2 \) and any \( x \geq 2 \),

\[
\sum_{n \leq x} \frac{\Lambda(n)}{n^\sigma} = \frac{x^{1-\sigma}}{1-\sigma} + O(1),
\]

where the implied constant in the \( O \)-term depends on \( \sigma_1 \) and \( \sigma_2 \).

**Proof.** Let

\[
\psi(x) = \sum_{n \leq x} \Lambda(n),
\]

where \( \Lambda(n) \) is von Mangoldt’s function, that is, \( \Lambda(n) = \log p \) if \( n \) is a power of the prime \( p \) and \( \Lambda(n) = 0 \) otherwise. Assuming the Riemann hypothesis, we have

\[
\psi(x) = x + E(x),
\]

where \( E(x) \ll x^{1/2} \log^2 x \). Thus, using Stieltjes integration and integration by parts, we see that

\[
\sum_{n \leq x} \frac{\Lambda(n)}{n^\sigma} = \int_1^x \frac{d\psi(y)}{y^\sigma} = \int_1^x \frac{y^{-\sigma} dy}{y^\sigma} + \int_1^x \frac{dE(y)}{y^\sigma}
\]

\[
= \frac{x^{1-\sigma} - 1}{1-\sigma} + \frac{E(x)}{x^\sigma} + \sigma \int_1^x \frac{E(y)}{y^{\sigma+1}} dy.
\]

The last two terms are \( \ll x^{1/2-\sigma_1} \log x \cdot (1 + \frac{1}{\sigma_1-1/2}) \ll_1 1 \). Furthermore, \( 1/(\sigma - 1) \ll_{\sigma_2} 1 \). Hence

\[
\sum_{n \leq x} \frac{\Lambda(n)}{n^\sigma} = \frac{x^{1-\sigma}}{1-\sigma} + O(1)
\]

with the implied \( O \)-term constant depending on \( \sigma_1 \) and \( \sigma_2 \), as asserted.

In our proofs from now on, we will not keep track as explicitly as above of the dependence of our \( O \)-term constants on the parameters \( \sigma_1 \) and \( \sigma_2 \).

**Lemma 2.** With the same hypotheses as in Lemma 1, we have

\[
\sum_{2 \leq n \leq x} \frac{\Lambda(n)}{n^\sigma \log n} = \frac{x^{1-\sigma}}{(1-\sigma) \log x} + O\left( \frac{x^{1-\sigma}}{\log^2 x} \right).
\]

Here the implicit \( O \)-term constant depends on \( \sigma_1 \) and \( \sigma_2 \).

**Proof.** Define

\[
S(y) = \sum_{n \leq y} \frac{\Lambda(n)}{n^\sigma}.
\]
From Lemma 1 we find that
\[
\sum_{n \leq x} \frac{\Lambda(n)}{n^\sigma \log n} = \frac{x}{2} \frac{dS(y)}{\log y} = \frac{S(x)}{\log x} + \frac{x}{2} \frac{S(y)}{y \log^2 y} dy
\]
\[
= \frac{x^{1-\sigma}}{(1-\sigma) \log x} + O\left(\frac{1}{\log x}\right)
\]
\[+ \int \frac{x y^{1-\sigma}}{2 y \log^2 y} dy + \int \frac{x}{2} O\left(\frac{1}{y \log^2 y}\right) dy
\]
\[= \frac{x^{1-\sigma}}{(1-\sigma) \log x} + O\left(\frac{x^{1-\sigma}}{\log^2 x}\right),
\]
where the implied constant depends on \(\sigma_1\) and \(\sigma_2\).

**Lemma 3.** Let \(\sigma_1\) and \(\sigma_2\) be such that \(1/2 < \sigma_1 < \sigma_2 < 1\). Then for \(\sigma_1 < \sigma < \sigma_2\),
\[
\Re \sum_{n \leq x} \frac{\Lambda(n)}{n^s \log n} \leq \frac{x^{1-\sigma}}{(1-\sigma) \log x} + O\left(\frac{x^{1-\sigma}}{\log^2 x}\right),
\]
the implied constant depending on \(\sigma_1\) and \(\sigma_2\).

**Proof.** We have
\[
\Re \sum_{2 \leq n \leq x} \frac{\Lambda(n)}{n^s \log n} \leq \sum_{2 \leq n \leq x} \frac{\Lambda(n)}{n^\sigma \log n} = \frac{x^{1-\sigma}}{(1-\sigma) \log x} + O\left(\frac{x^{1-\sigma}}{\log^2 x}\right)
\]
by Lemma 2.

**Lemma 4.** Assume RH. Let \(\sigma_1\) be such that \(1/2 < \sigma_1 < 1\) and let \(4 \leq T \leq t \leq 2T\). Then for \(\sigma_1 \leq \sigma < 1\),
\[
\frac{-\zeta'(s)}{\zeta(s)} = \sum_{n \leq \log^2 T} \frac{\Lambda(n)}{n^s} + O(\log^{2-2\sigma} T).
\]
The implied constant depends at most on \(\sigma_1\).

**Proof.** From a theorem of Montgomery and Vaughan \[6\] (13.35), when \(s\) is not a root or a pole of a Riemann zeta function and \(x, y \geq 2\), we have
\[
-\frac{-\zeta'(s)}{\zeta(s)} = \sum_{n \leq x y} \frac{w(n) \Lambda(n)}{n^s} - \frac{(xy)^{1-s} - x^{1-s}}{(1-s)^2 \log y}
\]
\[+ \sum_{\rho} \frac{(xy)^{\rho-s} - x^{\rho-s}}{(\rho-s)^2 \log y} + \sum_{k=1}^{\infty} \frac{(xy)^{-2k-s} - x^{-2k-s}}{(2k+s)^2 \log y},
\]
where

\[ w(u) = \begin{cases} 
1 & \text{if } 1 \leq u \leq x, \\
1 - \frac{\log(u/x)}{\log y} & \text{if } x < u \leq xy, \\
0 & \text{if } u > xy.
\end{cases} \]

Substituting \( y = 2 \) and \( x = \log^2 T \) in the second term on the right-hand side of (2.3) and taking absolute values, we get

\[
\frac{(xy)^{1-s} - x^{1-s}}{(1-s)^2 \log y} = O \left( \frac{x^{1-\sigma}}{T^2} \right) = O \left( \frac{\log^{2-2\sigma} T}{T^2} \right).
\]

The last term on the right-hand side of (2.3) is absolutely bounded, thus \( O(1) \). To simplify the third term on the right-hand side of (2.3) we assume the Riemann hypothesis and that \( \sigma \geq \sigma_1 > 1/2 \). Substituting \( x = \log^2 T \) and \( y = 2 \), and then taking absolute values, we find that

\[
\sum_{\rho} \frac{(xy)^{\rho-s} - x^{\rho-s}}{(\rho - s)^2 \log y} = O \left( \sum_{\rho} \frac{x^{1/2-\sigma}}{|\rho - s|^2} \right) = O(x^{1/2-\sigma}\log T) = O(\log^{2-2\sigma} T).
\]

Here the final two \( O \)-term constants depend on \( \sigma_1 \). Now consider the first term on the right-hand side of (2.3):

\[
\sum_{n \leq x} w(n) \frac{\Lambda(n)}{n^s} = \sum_{n \leq x} \frac{\Lambda(n)}{n^s} + \sum_{x < n \leq 2x} w(n) \frac{\Lambda(n)}{n^s} + O(x^{1-\sigma})
\]

\[
= \sum_{n \leq x} \frac{\Lambda(n)}{n^s} + O(\log^{2-2\sigma} T),
\]

where the implied constant is absolute. Combining all our results, we obtain

\[
-\frac{\zeta'}{\zeta}(s) = \sum_{1 \leq n \leq \log^2 T} \frac{\Lambda(n)}{n^s} + O(\log^{2-2\sigma} T),
\]

where the implied constant depends at most on \( \sigma_1 \). This completes the proof of Lemma 4.

3. Proof of Theorem 1. By Lemma 4 we have

\[
(3.1) \quad -\frac{\zeta'}{\zeta}(s) = \sum_{n \leq \log^2 T} \frac{\Lambda(n)}{n^s} + O(\log^{2-2\sigma} T),
\]

where \( s = \sigma + it \), \( 1/2 < \sigma_1 \leq \sigma \leq \sigma_2 < 1 \), and \( 4 \leq T \leq t \leq 2T \). Integrating from \( \beta' \) to \( \infty \), where \( \rho' = \beta' + i\gamma' \) is a critical point of the zeta function and
4 \leq T \leq \gamma' \leq 2T$, we see that
\[
\log \zeta(\rho') = \int_{\beta'}^{\infty} \frac{-\zeta'(y + i\gamma')}{\zeta(y + i\gamma')} \, dy = \sum_{n \leq \log^2 T} \frac{A(n)}{n^{\rho'} \log n} + O\left(\frac{\log^{2-2\beta'} T}{\log \log T}\right).
\]
Taking the real part of both sides, we obtain
\[
\log |\zeta(\rho')| = \Re \sum_{n \leq \log^2 T} \frac{A(n)}{n^{\rho'} \log n} + O\left(\frac{\log^{2-2\beta'} T}{\log \log T}\right).
\]
From Lemma 3 we see that
\[
\Re \sum_{2 \leq n \leq \log^2 T} \frac{A(n)}{n^{\rho'} \log n} = O\left(\frac{\log^{2-2\beta'} T}{\log \log T}\right),
\]
so that
\[
\log |\zeta(\rho')| = O\left(\frac{\log^{2-2\beta'} T}{\log \log T}\right).
\]
Since $T \leq \gamma' \leq 2T$, it follows that
\[
\log |\zeta(\rho')| \ll \frac{\log^{2-2\beta'} \gamma'}{\log \log \gamma'}
\]
where the implicit constant depends on $\sigma_1$ and $\sigma_2$. This completes the proof of Theorem 1.

4. Lemmas for the proof of Theorem 2. The results in this section are all unconditional.

**Lemma 5.** Let $a(n)$ be a totally multiplicative function such that $|a(n)| \leq 1$ for all $n$. Then for all $x \geq 1$,
\[
\sum_{n \leq x} a(n) A(n) n^{-s} = \sum_{p \leq x} \frac{a(p) \log p}{p^s - a(p)} + O(x^{1/2-\sigma}),
\]
where $1/2 < \sigma < 1$.

**Proof.** We begin by observing that
\[
\sum_{p \leq x} \frac{a(p) \log p}{p^s - a(p)} = \sum_{p \leq x} \frac{a(p) \log p}{p^s} \left(\frac{1}{1 - a(p)/p^s}\right) = \sum_{p \leq x} p^{-s} a(p) \log p \cdot \sum_{k=0}^{\infty} \frac{a(p^k)}{p^{ks}}
\]
\[
= \sum_{p \leq x} \sum_{k=1}^{\infty} \frac{a(p^k) \log p}{p^{ks}}.
\]
Therefore,

\[(4.1) \quad \sum_{n \leq x} a(n) \Lambda(n) n^{-s} = \sum_{p^k \leq x} a(p^k)(\log p)p^{-ks} = \sum_{p \leq x} \frac{a(p) \log p}{p^s - a(p)} - \sum_{k=2}^{\infty} \sum_{x^{1/k} < p \leq x} \frac{a(p^k) \log p}{p^{k\sigma}}.\]

Let \( \theta(x) = \sum_{p \leq x} \log p. \) We know that \( \theta(x) \ll x. \) Thus, for any \( a > 1 \) and any \( y_1, y_2 \) such that \( 1 < y_1 < y_2, \)

\[\sum \frac{\log p}{p^a} = \int_{y_1}^{y_2} \frac{d\theta(x)}{x^a} = O(y_1^{1-a}).\]

Using this with \( x^{1/k} \geq 2, \) that is, \( 2 \leq k \leq \log x/\log 2, \) we have

\[\sum_{x^{1/k} < p \leq x} \frac{\log p}{p^{k\sigma}} = O(x^{1/k-\sigma}).\]

And when \( x^{1/k} < 2, \) that is, \( k > \log x/\log 2, \) we have

\[\sum_{2 \leq p \leq x} \frac{\log p}{p^{k\sigma}} = O(2^{1-k\sigma}).\]

Combining these estimates, we find that the second term on the far right in (4.1) equals

\[\sum_{k=2}^{\infty} \sum_{x^{1/k} < p \leq x} \frac{a(p^k) \log p}{p^{k\sigma}} \ll \frac{\log x/\log 2}{x^{1/k-\sigma}} + \sum_{k=\log x/\log 2}^{\infty} 2^{1-k\sigma} \ll x^{1/2-\sigma} + x^{1/3-\sigma} \log x + x^{-\sigma} \ll x^{1/2-\sigma}.\]

Inserting this into the right-hand side of (4.1), we obtain

\[\sum_{n \leq x} a(n) \Lambda(n) n^{-s} = \sum_{p \leq x} \frac{a(p) \log p}{p^s - a(p)} + O(x^{1/2-\sigma}).\]

Next we define some functions and parameters that will be helpful in constructing a root of \( \zeta'(s) \) arbitrarily close to the line \( \Re s = \sigma_1. \) Let \( \sigma_1 \) be fixed with \( 1/2 < \sigma_1 < 1. \) Then for \( x \geq 2, \) and \( 0 < c < 1 \) and \( \sigma_1 < \sigma < 1, \) we
define auxiliary functions $V_x(s)$ and $W_x(s)$ as

\begin{align}
V_x(s) & = \sum_{n \leq x} \Lambda(n) n^{-s}, \\
W_x(s) & = \sum_{n \leq x} b(n) \Lambda(n) n^{-s},
\end{align}

where $b(n)$ is a totally multiplicative function such that $b(p) = 1$ for all $p \leq cx$, and $b(p) = -1$ for all $p > cx$. Next we express $W_x(s)$ in terms of $V_x(s)$. By Lemma 5, if $\sigma_1 < \sigma < 1$, then

\begin{align}
W_x(s) &= 2 \sum_{p \leq cx} \frac{\log p}{p^s - 1} - \sum_{cx < p \leq x} \frac{\log p}{p^s + 1} + O(x^{1/2 - \sigma_1}) \\
&= 2 \sum_{p \leq cx} \frac{\log p}{p^s - 1} - \sum_{cx < p \leq x} \frac{\log p}{p^s - 1} + O(x^{1/2 - \sigma_1}) \\
&= 2 \sum_{p \leq cx} \frac{\log p}{p^s - 1} - \sum_{p \leq x} \frac{\log p}{p^s - 1} + O(x^{1/2 - \sigma_1}).
\end{align}

Applying Lemma 5 again to the first two sums, and separately combining the last two sums, we find that

\begin{align}
W_x(s) &= 2V_{cx}(s) - V_x(s) + \sum_{cx < p \leq x} \frac{2 \log p}{p^{2s} - 1} + O(x^{1/2 - \sigma_1}) \\
&= 2V_{cx}(s) - V_x(s) + O(x^{1/2 - \sigma_1}).
\end{align}

We now specify the value of $c$ in the definition of $W_x(s)$ as

\begin{equation}
\log c = -\frac{\log 2}{1 - \sigma_1} + \frac{\log 2}{(1 - \sigma_1)^2 \log^a x}.
\end{equation}

We use this to show that $W_x(s)$ has a root near $\sigma_1$.

**Lemma 6.** Let $1/2 < \sigma_1 < 1$, let $a > 1$ be fixed, and let $c$ in the definition of $W_x(s)$ be given by (4.5). Then for all large $x$, $W_x(s)$ has a root at

$$s = \sigma_1 + \frac{1}{\log^a x} + O\left(\frac{1}{\log^{2a} x}\right).$$

**Proof.** Writing $\psi(y) = \sum_{n \leq y} \Lambda(n)$ and applying the prime number theorem, we see that

\begin{equation}
V_x(s) = \sum_{n \leq x} \frac{\Lambda(n)}{n^s} = \frac{\int_1^x \frac{d\psi(y)}{y^s}}{1 - s} = \frac{x^{1-s}}{1-s} + O\left(x^{1-\sigma_1} \exp\left(-c_1 \sqrt{\log x}\right)\right),
\end{equation}
where \( c_1 > 0 \) is an absolute constant. Using this in (4.4), we obtain

\[
W_x(s) = \frac{2(cx)^{1-s} - x^{1-s}}{1-s} + O(x^{1-\sigma_1} \log^{-2a} x),
\]

say. Next we set \( s = \sigma_1 + z \), where \(|z| < 2/\log^a x\) and \( x \) is so large that \(|1-s| \geq |1-\sigma_1| - 2/\log^a x > 0\).

Then

\[
W_x(s) = \frac{2e^{(1-s)\log c} - 1}{1-s} + O(\log^{-2a} x).
\]

Now

\[
(1-s) \log c = (1-\sigma_1 - z) \left( -\frac{\log 2}{1-\sigma_1} + \frac{\log 2}{(1-\sigma_1)^2 \log^a x} \right)
= -\log 2 \cdot \left( 1 - \frac{z}{1-\sigma_1} - \frac{1}{(1-\sigma_1) \log^a x} + O(\log^{-2a} x) \right).
\]

Thus,

\[
\frac{W_x(s)}{x^{1-s}} = \frac{\exp \left( \log 2 \cdot \left( \frac{z}{1-\sigma_1} - \frac{1}{(1-\sigma_1) \log^a x} + O(\log^{-2a} x) \right) \right) - 1}{1-s} + O(\log^{-2a} x)
= \frac{\log 2 \cdot (z - \log^{-a} x)}{(1-\sigma_1)(1-s)} + O(\log^{-2a} x).
\]

It follows that \( W_x(s) \) has a root at

\[
s = \sigma_1 + \frac{1}{\log^a x} + O\left( \frac{1}{\log^{2a} x} \right). \quad \blacksquare
\]

If \( p^{i\tau} \) is very near to 1 for \( p \leq cx \) and \( p^{i\tau} \) is very near to \(-1\) for \( cx < p \leq x \), then \( V_x(s + i\tau) \) will be close to \( W_x(s) \). To show that such \( \tau \) exist within a reasonable height, we need a sharp form of Kronecker’s theorem concerning inhomogeneous Diophantine approximation. To this end, we follow closely the approach of Gonek and Montgomery [II].

**Lemma 7** ([II, Lemma 7]). Let \( K \) be a positive integer and suppose \( 0 < \delta \leq 1/2 \). There is a trigonometric polynomial \( f(\theta) \) of the form

\[
f(\theta) = \sum_{k=0}^{K} c_k e(-k\theta)
\]

such that \( \max_\theta |f(\theta)| = f(0) = 1 \) and \( |f(\theta)| \leq 2e^{-\pi K\delta} \) for \( \delta \leq \theta \leq 1 - \delta \).

The second moment

\[
\mu = \int_{0}^{1} |f(\theta)|^2 \, d\theta
\]
appears below. Since
\[ 1 = |f(0)|^2 = \left| \sum_{k=0}^{K} c_k \right|^2 \leq (K + 1) \sum_{k=0}^{K} |c_k|^2 \]
by Cauchy’s inequality, it follows that
\[ \frac{1}{K + 1} \leq \mu \leq 1. \]

For a given finite set \( \mathcal{P} \) of primes \( p \) and a given set of real numbers \( \beta_p \) (considered modulo 1), we want to show that there exist real numbers \( t \) in prescribed intervals such that \( \| t \log p/2\pi - \beta_p \| < \delta \), where \( \| x \| \) indicates the distance from \( x \) to the nearest integer. To accomplish this we define
\[ g(t) = \prod_{p \in \mathcal{P}} \left| f\left( \frac{t \log p}{2\pi} - \beta_p \right) \right|^2, \]
where \( f \) is as in the previous lemma.

**Lemma 8.** Let \( \mathcal{P} \) be a set of primes not exceeding \( x \). For each \( p \in \mathcal{P} \) let a number \( \beta_p \) be given. Let \( K, \mu, \) and \( g(t) \) be as in (4.11), (4.12), and (4.14). Then for \( Y \) any real number and \( T \geq 4 \),
\[ \int_{T}^{T+Y} g(t) \, dt = (Y + O(\exp(2Kx))) \mu \text{card } \mathcal{P}. \]

**Proof.** The argument is almost identical to [1, proof of Lemma 8].

From this point on, recalling that \( a > 1 \) in some of the lemmas above, we let
\[ d_1 = \sigma_1 - 1/2, \quad b > a + 1 > 2, \]
and make the following choices for the parameters \( K, x, \) and \( \delta \):
\[ x = \frac{d_1 \log T}{4(\log \log T)^b}, \quad K = \left[ \frac{1}{2} \log^b x \right], \quad \delta = \frac{1}{\log^{b-1} x}. \]

For each root \( \rho = \beta + i\gamma \) of the zeta function such that \( \beta \geq (1 + d_1)/2 \), we remove from \( [T, 2T] \) those \( \tau \) satisfying \( |\gamma - \tau| \leq T^{d_1/4} + 1 \). We let \( X \) denote the set of \( \tau \) we have removed, and let \( R \) denote the remaining set, so that
\[ [T, 2T] = R \cup X. \]
By Titchmarsh [8, Theorem 9.19(A)], the number of roots with ordinates in \([T, 2T]\) is \( \ll T^{1-d_1/2} \log^5 T \). Thus, the set \( X \) has measure \( \ll T^{1-d_1/4} \log^5 T \). Note that if \( \tau \in R \) and \( s = \sigma + it \) with \( |t| \leq 1 \), then \( \min_{\gamma} |\gamma - t - \tau| \geq T^{d_1/4} \). In particular, this holds when \( s \) is on or inside \( \mathcal{C}_1 \) (defined in (5.2)) and \( \tau \in R \).
We next prove an analogue of Lemma 8 for the integral $\int_R g(t) \, dt$.

**Lemma 9.** Under the same hypotheses as in Lemma 8 and with $R$ and $d_1$ as above, we have

$$\int_R g(t) \, dt = (T + O(T^{1-d_1/4} \log^5 T)) \mu \text{card } \mathcal{P}.$$  

**Proof.** We have

\begin{equation}
\int_R g(t) \, dt = \int_{\mathcal{T}} g(t) \, dt - \int_X g(t) \, dt.
\end{equation}

By Lemma 8

$$\int_{\mathcal{T}} g(t) \, dt = (T + O(\exp(2Kx))) \mu \text{card } \mathcal{P}.$$  

By our choice of the parameters $K$ and $x$ in (4.17) we see that $2Kx \leq d_1(\log T)/4$, so that

\begin{equation}
\exp(2Kx) \leq T^{d_1/4}.
\end{equation}

Thus

\begin{equation}
\int_{\mathcal{T}} g(t) \, dt = (T + O(T^{d_1/4})) \mu \text{card } \mathcal{P}.
\end{equation}

Next, $X$ consists of $\ll T^{1-d_1/2} \log^5 T$ intervals, each of length $\leq 2T^{d_1/4}$. Thus, by Lemma 8 and (4.20), each such interval contributes an amount $\ll (T^{d_1/4} + \exp(2Kx)) \mu \text{card } \mathcal{P} \ll T^{d_1/4} \mu \text{card } \mathcal{P}.

It follows that

\begin{equation}
\int_X g(t) \, dt \ll (T^{1-d_1/2} \log^5 T)(T^{d_1/4} \mu \text{card } \mathcal{P}) = (T^{1-d_1/4} \log^5 T) \mu \text{card } \mathcal{P}.
\end{equation}

Combining this and (4.21) in (4.19), we obtain

$$\int_R g(t) \, dt = (T + O(T^{1-d_1/4} \log^5 T)) \mu \text{card } \mathcal{P}. \quad \blacksquare$$

The function $g(t)$ is large when the numbers $\|t \log p/2\pi - \beta_p\|$ are small, but to obtain Kronecker’s theorem we need a peak function that is positive only when all of these numbers are $< \delta$. To accomplish this we define

\begin{equation}
h(t) = \prod_{p \leq x} \left| f\left(\frac{t \log p}{2\pi} - \beta_p\right)\right|^2 - \epsilon \sum_{p_1 \leq x} \prod_{p \leq x, p \neq p_1} \left| f\left(\frac{t \log p}{2\pi} - \beta_p\right)\right|^2,
\end{equation}

where

\begin{equation}
\epsilon = 4e^{-2\pi K\delta} \ll x^{-3}.
\end{equation}

It is easy to see that $h(t) > 0$ only when $\|t \log p/2\pi - \beta_p\| < \delta$ for all $p \leq x$ (cf. [1]).
Lemma 10. With \( f(\theta) \) defined as in Lemma 7, \( h(t) \) defined as above, and the choice of parameters in (4.17), for \( T \geq 4 \) and any real \( Y \) we have

\[
T + Y \int_T^T h(t) \, dt = (Y + O(T^{d_1/4}))(1 + O(x^{-1}))\mu(x).
\]

Moreover, for \( R \) as in (4.18), we have

\[
\int_R^R h(t) \, dt = (1 + O(x^{-1}))\mu(x)T.
\]

Proof. From the definition of \( h(t) \), Lemma 8, and (4.20), we have

\[
2T \int T^T h(t) \, dt = (Y + O(\exp(2Kx)))\mu(x) + O(\epsilon\pi(x)(Y + O(\exp(2Kx)\mu(x)^{-1})))
\]

\[
= (Y + O(T^{d_1/4}))\mu(x) + O(\epsilon\pi(x)(Y + O(T^{d_1/4})))\mu(x^{-1}).
\]

Now \( \pi(x) \ll x/\log x, \epsilon \ll x^{-3} \) by (4.23), and by (4.13) and our choice of \( K \) in (4.17), \( 1/\mu \ll (\log x) \). It follows that \( \epsilon\pi(x)/\mu \ll \log x/x^2 \ll 1/x \). This establishes (4.24).

Next we prove (4.25). By (4.18),

\[
\int_R^R h(t) \, dt = \int_T^T h(t) \, dt - \int_X^X h(t) \, dt.
\]

By (4.24) with \( Y = T \),

\[
2T \int T^T h(t) \, dt = (T + O(T^{d_1/4}))(1 + O(x^{-1}))\mu(x).
\]

To estimate \( \int_X^X h(t) \, dt \), recall that \( X \) consists of \( \ll T^{1-d_1/2} \log^5 T \) intervals, each of length \( \leq 2T^{d_1/4} \). Thus, by (4.24), each such interval contributes an amount

\[
\ll T^{d_1/4}(1 + O(x^{-1}))\mu(x) \ll T^{d_1/4}\mu(x).
\]

It follows that

\[
\int_X^X h(t) \, dt \ll (T^{1-d_1/2} \log^5 T)T^{d_1/4}\mu(x)
\]

\[
= (T^{1-d_1/4} \log T)\mu(x).
\]

Combining this and (4.28) with (4.27), we obtain

\[
\int_R^R h(t) \, dt = T(1 + O(x^{-1}))\mu(x),
\]

which is (4.25).
These lemmas ensure that there are $t$ for which the primes $p \leq x$ behave as we want. However, the remaining primes $p > x$ could make an unwanted contribution. The next lemma guarantees that this does not happen.

**Lemma 11.** Let $g(t)$ be as in (4.14), where $\mathcal{P}$ is the set of primes not exceeding $x$. For each $p > x$ let $b_p$ have the property that $|b_p| \leq 1/p^{\sigma_1}$. Then
\[
2T \int_{T}^{2T} g(t) \left| \sum_{x < p \leq T^{d_1/4}} \frac{b_p}{p^t} \right|^2 dt \ll T \mu^\pi(x) \frac{x^{1-2\sigma_1}}{\log x},
\]
where the implied constant depends on $\sigma_1$. The same bound holds a fortiori for the integral over $R$.

**Proof.** By (4.11) we see that
\[
\prod_{p \in \mathcal{P}} f \left( t \frac{\log p}{2\pi} - \beta_p \right) = \prod_{p \in \mathcal{P}} \left( \sum_{k=0}^{K} c_k e(k\beta_p)p^{-ikt} \right) = \sum_{n \in \mathcal{N}} a_n n^{-it},
\]
where $\mathcal{N}$ is the set of positive integers composed entirely of primes in $\mathcal{P}$, with multiplicities not exceeding $K$, and
\[
a_n = \prod_{p \in \mathcal{P}} c_k e(k\beta_p).
\]

Here the product extends over all members of $\mathcal{P}$, not just those dividing $n$. We note that a positive integer $m$ has at most one decomposition $m = np$ with $n \in \mathcal{N}$ and $p > x$. Let the numbers $C_m$ be determined by the identity
\[
\left( \sum_{n \in \mathcal{N}} a_n n^{-it} \right) \left( \sum_{x < p \leq T^{d_1/4}} b_pp^{-it} \right) = \sum_{m} C_mm^{-it}.
\]

Montgomery and Vaughan [5] have shown that if $\sum_{m} |C_m| < \infty$, then
\[
\int_{T}^{2T} \left| \sum_{m=1}^{\infty} C_mm^{-it} \right|^2 dt = \sum_{m=1}^{\infty} |C_m|^2(T + O(m)).
\]

In the main term we have
\[
\sum_{m=1}^{\infty} |C_m|^2 = \left( \sum_{n \in \mathcal{N}} |a_n|^2 \right) \left( \sum_{x < p \leq T^{d_1/4}} |b_p|^2 \right).
\]

The sum over $n$ is $\mu^\pi(x)$, and the sum over $p$ is $\ll \sum_{x < p \leq T^{d_1/4}} p^{-2\sigma_1} \ll x^{1-2\sigma_1}/\log x$. In the error term we have
\[
\sum_{m=1}^{\infty} m|C_m|^2 = \left( \sum_{n \in \mathcal{N}} n|a_n|^2 \right) \left( \sum_{x < p \leq T^{d_1/4}} p|b_p|^2 \right).
\]
For \( n \in \mathcal{N} \) we have \( n \leq \exp(2Kx) \leq T^{d_1/4} \), so the sum over \( n \) here is \( \ll \mu^n(x) T^{d_1/4} \). The sum over \( p \) is
\[
\leq \sum_{x < p \leq T^{d_1/4}} p^{1-2\sigma_1} \leq T^{d_1(1-\sigma_1)/2}/\log T.
\]
Combining our estimates, we obtain
\[
(4.29) \quad 2T \left| \sum_{x < p \leq T^{d_1/4}} \frac{b_p}{p^s} \right|^2 dt \ll T \mu^n(x) x^{1-2\sigma_1}/(\log x) + T^{d_1/4} \mu^n(x) T^{d_1(1-\sigma_1)/2}/\log T \ll T \mu^n(x) x^{1-2\sigma_1}/(\log x)
\]
by our choice of \( x \). This completes the proof of Lemma 11. 

**Lemma 12.** Let \( W_x(s) \) be as in (4.3) with \( x = (d_1 \log T)/4(\log \log T)^b \), let \( d_1 = \sigma_1 - 1/2 \), and \( T \geq 4 \). Then for \( s = \sigma + it \) with \( \sigma_1 \leq \sigma \) and \( |t| \leq 1 \), and for \( \tau \in R \), where \( R \) is defined just before (4.18), we have
\[
(4.30) \quad - \zeta'(s + i\tau) = \sum_{n \leq T^{d_1/4}} w(n) A(n)n^{-s-i\tau} + O(T^{-d_1^2/16}),
\]
where
\[
(4.31) \quad w(u) = \begin{cases} 
1 & \text{if } 1 \leq u \leq T^{d_1/8}, \\
1 - \frac{\log(u/y)}{\log y} & \text{if } T^{d_1/8} < u \leq T^{d_1/4}, \\
0 & \text{if } u > T^{d_1/4}.
\end{cases}
\]

**Proof.** By (2.3) with \( x = y = T^{d_1/8}, \ T \geq 4, \ d_1 = \sigma_1 - 1/2, \ \sigma \geq \sigma_1, \) and \( \tau \in R \), we have
\[
(4.32) \quad - \zeta'(s + i\tau) = \sum_{n \leq y^2} w(n) A(n)n^{-s+i\tau} - \frac{y^{2(1-s-i\tau)} - y^{1-s-i\tau}}{(1-s-i\tau)^2 \log y}
\]
\[
+ \sum_{\rho} \frac{y^{2(\rho-s-i\tau)} - y^{\rho-s-i\tau}}{\rho - s - i\tau)^2 \log y}
\]
\[
+ \sum_{k=1}^{\infty} \frac{y^{2(-2k-s-i\tau)} - y^{-2k-s-i\tau}}{(2k + s + i\tau)^2 \log y}.
\]
Since \( R \subseteq [T, 2T] \), the second term on the right-hand side of (4.32) becomes
\[
\frac{y^{2(1-s-i\tau)} - y^{1-s-i\tau}}{(1-s-i\tau)^2 \log y} \ll \frac{T^{d_1(1-\sigma_1)/4}}{d_1 T^2 \log T} \ll T^{d_1/8-d_1^2/4-2}.
\]
We split the third term on the right-hand side of (4.32) into two sums \( P \) and \( Q \), where \( P \) is over the zeros with \( \beta \geq (1 + d_1)/2 \), and \( Q \) is over the
zeros with $\beta < (1 + d_1)/2$. For $Q$ we have

$$Q = \sum_{\beta < (1 + d_1)/2} \frac{y^{2(\rho - s - i\tau)} - y^{\rho - s - i\tau}}{(\rho - s - i\tau)^2 \log y}$$

$$\ll \sum_{\beta < (1 + d_1)/2} \frac{y^{2(\beta - \sigma)} + y^{\beta - \sigma}}{[(\beta - \sigma)^2 + (\gamma - t - \tau)^2] \log y} \ll \frac{y^{-d_1/2} \log T}{d_1^2 \log y} \ll y^{-d_1/2} = T^{-d_1^2/16}.$$

For $P$ in turn we have

$$P = \sum_{\beta \geq (1 + d_1)/2} \frac{y^{2(\rho - s - i\tau)} - y^{\rho - s - i\tau}}{(\rho - s - i\tau)^2 \log y}$$

$$\ll \sum_{\beta \geq (1 + d_1)/2} \frac{y^{2(1 - \sigma)} + y^{1 - \sigma}}{(\gamma - t - \tau)^2 \log y} \ll \frac{y^{1 - 2d_1} \log T}{T^{d_1/4} \log y} \ll \frac{y^{1 - 2d_1}}{T^{d_1/4}} = T^{-d_1/8 - d_1^2/4}.$$

The last term on the right-hand side of (4.32) is

$$\sum_{k=1}^{\infty} \frac{y^{2(-2k - s - i\tau)} - y^{-2k - s - i\tau}}{(2k + s + i\tau)^2 \log y} \ll \frac{y^{-2 - \sigma_1}}{T \log y} \ll T^{-1 - 5d_1/16 - d_1^2/8}.$$

Combining all these estimates, we find that for $\sigma \geq \sigma_1$ and $\tau \in R$,

$$-\zeta(s + i\tau) = -\sum_{n \leq T^{d_1/4}} w(n) \frac{A(n)}{n^{s + i\tau}} + O(T^{-d_1^2/16}).$$

5. Proof of Theorem 2 Let $V_x(s)$ and $W_x(s)$ be as in (4.2) and (4.3).

Let

$$\mathcal{C}_0 = \left\{ s = \sigma_1 + \frac{1 + e^{i\theta}}{\log^a x} : 0 \leq \theta \leq 2\pi \right\},$$

$$\mathcal{C}_1 = \left\{ s = \sigma_1 + \frac{1 + e^{i\theta}}{\log^a x} : 0 \leq \theta \leq 2\pi \right\},$$

where, as previously, $a > 1$. Also, let $c$ be as in (4.5) and $\delta$ as in (4.17).

Suppose that $\tau$ is a real number such that

$$\left\| \frac{\tau \log p}{2\pi} \right\| < \delta \quad \text{for} \quad p \leq cx, \quad \left\| \frac{\tau \log p}{2\pi} + \frac{1}{2} \right\| < \delta \quad \text{for} \quad cx < p \leq x,$$

and let $\mathcal{G}$ be the set of those $\tau$ such that both inequalities hold.
By Lemma 5 (4.2), and (4.3) we see that for \( \sigma > \sigma_1 \),

\[
V_x(s) = \sum_{p \leq x} \frac{\log p}{p^s - 1} + O(x^{1/2 - \sigma_1}),
\]

(5.3)

\[
W_x(s) = \sum_{p \leq x} \frac{\log p}{b(p)p^s - 1} + O(x^{1/2 - \sigma_1}).
\]

(5.4)

Hence

\[
V_x(s + i\tau) - W_x(s) \leq \sum_{p \leq x} \log p \cdot \left( \frac{1}{p^{s+i\tau} - 1} - \frac{1}{b(p)p^s - 1} \right) + O(x^{1/2 - \sigma_1})
\]

(5.5)

\[
\ll \sum_{p \leq x} \frac{\log p}{p^{\sigma_1}} |p^{i\tau} - b(p)| + O(x^{1/2 - \sigma_1}).
\]

Now note that for real \( \theta \), if \( \|\theta\| \) is the distance between \( \theta \) and the nearest integer, then

\[
|e^{2\pi i \theta} - 1| = 2|\sin \pi \theta| \leq 2\pi \|\theta\|.
\]

Thus, taking

\[
\theta = \theta_p = \begin{cases} 
\tau \frac{\log p}{2\pi} - \frac{1}{2} & \text{if } p \leq cx, \\
\tau \frac{\log p}{2\pi} + \frac{1}{2} & \text{if } cx < p \leq x,
\end{cases}
\]

we see that if \( \tau \in \mathcal{G} \), then \( |p^{i\tau} - b(p)| \leq 2\pi \delta \) for every \( p \leq x \). Therefore, for \( \sigma \geq \sigma_1 \),

\[
|V_x(s + i\tau) - W_x(s)| \ll \delta \sum_{p \leq x} \frac{\log p}{p^{\sigma_1}} + O(x^{1/2 - \sigma_1}) \ll \frac{x^{1 - \sigma_1}}{\log^{b - 1} x},
\]

(5.6)

since \( \delta = 1/\log^{b-1} x \).

Define

\[
L_x(s) = \sum_{1 < n \leq x} \frac{b(n)\Lambda(n)}{n^s \log n}.
\]

Then we have

\[
\left| \sum_{1 < n \leq x} \frac{\Lambda(n)}{\log n} n^{-s-i\tau} - L_x(s) \right| \leq \sum_{1 < n \leq x} \frac{\Lambda(n)}{n^\sigma \log n} |n^{-i\tau} - b(n)|.
\]

If \( n = p^k \), then \( |n^{-i\tau} - b(n)| = |p^{-ik\tau} - b(p)^k| \leq k |p^{-i\tau} - b(p)| \). Thus, for \( \sigma \geq \sigma_1 \) and \( \tau \in \mathcal{G} \), the expression here is

\[
\leq 2\pi \delta \sum_{p^k \leq x} \frac{1}{p^{k\sigma_1}} \ll \delta \frac{x^{1 - \sigma_1}}{\log x} \ll \frac{x^{1 - \sigma_1}}{\log^{b} x}.
\]
by our choice of $\delta$ in (4.17). That is, for $\sigma \geq \sigma_1$ and $\tau \in \mathcal{G}$,

\begin{equation}
\left| \sum_{1 < n \leq x} \frac{\Lambda(n)}{\log n} n^{-s-i\tau} - L_x(s) \right| \ll \frac{x^{1-\sigma_1}}{\log b x}.
\end{equation}

Furthermore, when $\Re s \geq \sigma_1$,

\begin{equation}
L_x'(s) = -\sum_{n \leq x} \frac{b(n)\Lambda(n)}{n^s} \ll \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma_1}} \ll x^{1-\sigma_1}.
\end{equation}

Thus, if $s$ is on or inside $\mathcal{C}_1$, then

\begin{equation}
L_x(s) = L_x(\sigma_1) + O\left(\frac{x^{1-\sigma_1}}{\log^a x}\right).
\end{equation}

At $\sigma_1$,

\begin{equation}
L_x(\sigma_1) = \sum_{1 < n \leq x} \frac{b(n)\Lambda(n)}{n^{\sigma_1} \log n} = 2 \sum_{p \leq x} p^{-\sigma_1} - \sum_{p \leq x} p^{-\sigma_1} + O(1).
\end{equation}

By the prime number theorem, for $u \geq 2$,

\begin{equation}
\pi(u) = \frac{u}{\log u} + \frac{u}{\log^2 u} + O\left(\frac{u}{\log^3 u}\right).
\end{equation}

From this and integration by parts, we see that for $y \geq 2$,

\begin{equation}
\sum_{p \leq y} p^{-\sigma_1} = \int_{2^{-}}^{y} \frac{d\pi(u)}{u^{\sigma_1}} = \frac{y^{1-\sigma_1}}{(1-\sigma_1) \log y} + \frac{y^{1-\sigma_1}}{(1-\sigma_1)^2 \log^2 y} + O\left(\frac{y^{1-\sigma_1}}{\log^3 y}\right).
\end{equation}

Using this in (5.9), we obtain

\begin{align*}
L_x(\sigma_1) &= \frac{x^{1-\sigma_1}}{1-\sigma_1} \left( \frac{2c^{1-\sigma_1}}{\log cx} - \frac{1}{\log x} \right) \\
&+ \frac{x^{1-\sigma_1}}{(1-\sigma_1)^2} \left( \frac{2c^{1-\sigma_1}}{\log^2 cx} - \frac{1}{\log^2 x} \right) + O\left(\frac{x^{1-\sigma_1}}{\log^3 x}\right).
\end{align*}

Now, from (4.9) with $s = \sigma_1$, that is, $z = 0$, we have

\begin{equation}
2c^{1-\sigma_1} = 1 + \frac{\log 2}{(1-\sigma_1) \log^a x} + O(\log^{-2a} x) = 1 + O(\log^{-a} x).
\end{equation}

Moreover,

\begin{equation}
\frac{1}{\log cx} = \frac{1}{\log x} \left( 1 - \frac{\log c}{\log x} + O\left(\frac{1}{\log^2 x}\right) \right).
\end{equation}
Hence,
\[
\frac{2e^{1-\sigma_1}}{\log cx} - \frac{1}{\log x} = \frac{1 + O(\log^{-a} x)}{\log x} \left( 1 - \frac{\log c}{\log x} + O\left( \frac{1}{\log^2 x} \right) \right) - \frac{1}{\log x}
\]
\[
= -\frac{\log c}{\log^2 x} + O\left( \frac{1}{\log^{a+1} x} \right) + O\left( \frac{1}{\log^3 x} \right)
\]
\[
= -\frac{\log c}{\log^2 x} + O(\log^{-\min(a+1,3)} x).
\]

Similarly, one sees that
\[
\frac{2e^{1-\sigma_1}}{\log^2 cx} - \frac{1}{\log^2 x} = O(\log^{-3} x).
\]

Thus, we find that
\[
L_x(\sigma_1) = -x^{1-\sigma_1} \frac{\log c}{(1 - \sigma_1) \log^2 x} + O\left( \frac{x^{1-\sigma_1}}{\log^{\min(a+1,3)} x} \right).
\]

It now follows from (5.8) that for \( s \) on or inside \( C_1 \),
\[
L_x(s) = -x^{1-\sigma_1} \frac{\log c}{(1 - \sigma_1) \log^2 x} + O\left( \frac{x^{1-\sigma_1}}{\log^{\min(a,3)} x} \right).
\]

From this and (5.7) we see that for \( s \) on or inside \( C_1 \), and \( \tau \in \mathbb{R} \),
\[
(5.12) \quad \Re \sum_{1 < n \leq x} \frac{\nu(n)}{\log n} n^{-s-i\tau} \log n \frac{n^{-s-i\tau}}{\log n} - n^{-s} = \frac{d_1 \log T}{4(\log \log T)^{3/2}}.
\]

Let
\[
T_x(s) = \sum_{x < n \leq T^{d_1/4}} \frac{w(n) \Lambda(n)}{\log n} n^{-s}, \quad \text{where} \quad x = \frac{d_1 \log T}{4(\log \log T)^3}.
\]

Then by Lemma 12 for \( s = \sigma + it \) with \( \sigma_1 \leq \sigma \) and \( |t| \leq 1 \), and for \( \tau \in \mathbb{R} \), we have
\[
(5.13) \quad \log \zeta(s + it) = \sum_{n \leq x} \frac{\Lambda(n)}{\log n} n^{-s-i\tau} + T_x(s + it) + O(T^{-d_1^2/16}),
\]
\[
(5.14) \quad -\frac{\zeta'(s + it)}{\zeta(s + it)} = \sum_{n \leq x} \Lambda(n)n^{-s-i\tau} - T_x'(s + it) + O(T^{-d_1^2/16}).
\]
Now suppose that there is \( \tau \in R \cap \mathcal{G} \), and that
\[
T_x(s + i\tau) \ll \frac{1}{\log x}, \quad T'_x(s + i\tau) \ll 1
\]
for \( s \in \mathcal{C}_1 \). Then by (5.7) and (5.6),
\[
\Re \log \zeta(s + i\tau) = -\frac{x^{1-\sigma_1} \log c}{(1 - \sigma_1) \log^2 x} + O\left(\frac{x^{1-\sigma_1}}{\log \min(a,3) x}\right),
\]
\[
-\frac{\zeta'(s + i\tau)}{\zeta(s)} = W_x(s) + O\left(\frac{x^{1-\sigma_1}}{\log^{b-1} x}\right)
\]
for \( s \) on or inside \( \mathcal{C}_1 \).

Recall from (4.10) that if \( s = \sigma_1 + z \) and \(|z| < 2/\log^a x\), then
\[
W_x(s) = \frac{\log 2}{(1 - \sigma_1)(1 - s)} (z - \log^{-a} x) + O(\log^{-2a} x).
\]
Thus, for \( s \) on or inside \( \mathcal{C}_1 \),
\[
W_x(s) = x^{1-\sigma_1} \left(\frac{e^{i\beta} \log 2}{4(1 - \sigma_1)^2 \log^a x} + O\left(\frac{1}{\log^{2a} x}\right)\right).
\]

Since \( a > 1 \), we deduce from this that the argument of \( W_x(s) \) increases by \( 2\pi \) as \( s \) traverses \( \mathcal{C}_1 \). Thus, by (5.17), \( \zeta'(s) \) has a zero \( \rho' \) in \( \mathcal{C}_1 + i\tau \), and from (5.16) we see that
\[
\log |\zeta(\rho')| \geq \left(\frac{-\log c}{1 - \sigma_1} + o(1)\right) \frac{x^{1-\sigma_1}}{\log^2 x}.
\]

Now by (4.5),
\[
\log c = \log 2/(\sigma_1 - 1)(1 + o(1)),
\]
so that, using this and substituting again \( d_1 \log T/4(\log \log T)^b \) for \( x \), we obtain
\[
\log |\zeta(\rho')| \geq (1 + o(1)) \left(\frac{(\sigma_1 - 1/2)^{1-\sigma_1} \log 2}{(1 - \sigma_1)^2} \frac{1-\sigma_1}{4^{1-\sigma_1}}\right) \frac{\log^{1-\sigma_1} T}{(\log \log T)^{2+b(1-\sigma_1)}}.
\]
Since \( T \leq \gamma' = 3\rho' \leq 2T \), (5.21) also holds with \( T \) replaced by \( \gamma' \), which is (1.7).

Thus we get
\[
\log |\zeta(\rho')| \geq (B(\sigma_1) - \epsilon) \frac{\log^{1-\beta'} \gamma'}{(\log \log \gamma')^{2+b(1-\sigma_1)}},
\]
where
\[
B(\sigma_1) = \frac{(\sigma_1 - 1/2)^{1-\sigma_1} \log 2}{(1 - \sigma_1)^2} \cdot
\]
To complete the proof of Theorem 2 it only remains to show the existence of a \( \tau \in \mathbb{R} \cap \mathcal{G} \) satisfying (5.15). Recall that
\[
T_x(s + i\tau) = \sum_{x < p \leq T^{d_1/4}} w(p) p^{-s - i\tau},
\]
\[
T'_x(s + i\tau) = - \sum_{x < p \leq T^{d_1/4}} w(p) \log p p^{-s - i\tau}.
\]
We will prove that there is a constant \( C_1 \) that is independent of \( x \) and such that on the circle \( \mathcal{C}_0 \) we have
\[
\oint_{\mathcal{C}_0} |T_x(z + i\tau)|^2 \, dz \leq \frac{C_1}{\log^9 x}.
\]
Then, if \( s \) is on or inside \( \mathcal{C}_1 \), we see from Cauchy’s formula and the Cauchy–Schwarz inequality that
\[
T_x(s + i\tau) = \frac{1}{2\pi i} \oint_{\mathcal{C}_0} \frac{T_x(z + i\tau)}{z - s} \, dz \ll \log^{a/2} x \cdot \sqrt{\oint_{\mathcal{C}_0} |T_x(z + i\tau)|^2 \, dz} \ll \frac{1}{\log^{(9-a)/2} x},
\]
\[
T'_x(s + i\tau) = \frac{1}{2\pi i} \oint_{\mathcal{C}_0} \frac{T_x(z + i\tau)}{(z - s)^2} \, dz \ll \log^{3a/2} x \cdot \sqrt{\oint_{\mathcal{C}_0} |T_x(z + i\tau)|^2 \, dz} \ll \frac{1}{\log^{(9-3a)/2} x}.
\]
From these estimates we conclude that
\[
\sum_{x < n \leq T^{d_1/4}} \frac{w(n) A(n)}{n^{s + i\tau} \log n} = \sum_{x < p \leq T^{d_1/4}} \frac{w(p)}{p^{s + i\tau}} + O(x^{1/2 - \sigma_1})
\]
\[
= T_x(s + i\tau) + O(x^{1/2 - \sigma_1}) \ll \frac{1}{\log^{(9-a)/2} x},
\]
\[
\sum_{x < n \leq T^{d_1/4}} \frac{w(n) A(n)}{n^{s + i\tau}} = \sum_{x < p \leq T^{d_1/4}} \frac{w(p) \log p}{p^{s + i\tau}} + O(x^{1/2 - \sigma_1})
\]
\[
= -T'_x(s + i\tau) + O(x^{1/2 - \sigma_1}) \ll \frac{1}{\log^{(9-3a)/2} x}.
\]
In other words, the inequalities in (5.15) are true.

Now define
\[ h^+(t) = \max(0, h(t)) \].
Since \( h(t) \leq h^+(t) \leq g(t) \), using Lemma \[10\], we find that

\[
\int_R h^+(t) \, dt = \mu(\pi) T(1 + O(1/x)).
\]

On the other hand, we have

\[
\int_R h^+(t) \left( \int_{\mathcal{C}_0} |T_x(z + it)|^2 |dz| \right) \, dt \leq \int_R g(t) \left( \int_{\mathcal{C}_0} |T_x(z + it)|^2 |dz| \right) \, dt
\]

\[
= \int_{\mathcal{C}_0} g(t) |T_x(z + it)|^2 \, dt \, |dz| \leq \mu(\pi) T x^{1-2\sigma} \log x \leq \mu(\pi) T x^{1-2\sigma} \frac{\log x}{\log^4 x}.
\]

Thus we conclude that there is \( t \in R \cap \mathcal{G} \) such that

\[
\int_{\mathcal{C}_0} |T_x(z + it)|^2 |dz| \leq \tilde{C} \frac{x}{\log x}.
\]

Choosing \( b = 3 + \epsilon'' \), we get

\[
\log |\zeta(\rho')| \geq (B(\sigma_1) - \epsilon) \frac{\log x^{1-\sigma_1}}{\log \log T(1-\sigma) \log x} + O \left( \frac{x^{1-\sigma}}{(\log x)^{3-3\beta+\epsilon'}} \right),
\]

where \( \epsilon' = \epsilon''(1 - \sigma) \), thus completing the proof of Theorem \[2\].

6. Lemma for the proof of Theorem \[3\]. We assume RH for the proof of Theorem 3.

**Lemma 13.** Let \( \sigma_1 \) and \( \sigma_2 \) be such that \( 1/2 < \sigma_1 < \sigma_2 < 1 \). Then

\[
\Re \sum_{1 \leq n \leq x} \frac{A(n)}{n^s \log n} \geq - \frac{x^{1-\sigma} \log x}{(1-\sigma) \log x} + O \left( \frac{x^{1-\sigma}}{(\log^2 x)^{\frac{1}{2}}} \right),
\]

where \( \sigma_1 < \sigma < \sigma_2 \) and the \( O \)-term constant depends on \( \sigma_1 \) and \( \sigma_2 \).

**Proof.** We deduce the following inequality from Lemmas \[1\] and \[2\]

\[
\Re \sum_{2 \leq n \leq x} \frac{A(n)}{n^s \log n} \geq - \sum_{2 \leq n \leq x} \frac{A(n)}{n^s \log n} = - \frac{x^{1-\sigma} \log x}{(1-\sigma) \log x} + O \left( \frac{x^{1-\sigma}}{(\log^2 x)^{\frac{1}{2}}} \right).\]

7. Proof of Theorem \[3\]. By Lemma \[4\] we have

\[
-\frac{\zeta'}{\zeta}(s) = \sum_{1 \leq n \leq \log^2 T} \frac{A(n)}{n^s} + O((\log T)^{1-\sigma})
\]

for \( \sigma_1 \leq \sigma \leq \sigma_2 \); let \( 4 \leq T \leq t \leq 2T \). Integrating from \( \beta' \) to \( \infty \), where \( \rho' = \beta' + i\gamma' \) is a critical point of the zeta function and \( 4 \leq T \leq \gamma' \leq 2T \), we
see that
\[ \log \zeta(\rho') = \int_{\beta'}^{\infty} -\frac{\zeta'(y + i\gamma')}{\zeta(y + i\gamma')} \, dy = \sum_{2 \leq n \leq \log^2 T} \frac{A(n)}{n^{\rho'} \log n} + O\left(\frac{\log^{2-2\beta'} T}{\log \log T}\right). \]

Taking the real part of both sides, we obtain
\[ (7.2) \quad \log |\zeta(\rho')| = \Re \sum_{2 \leq n \leq \log^2 T} \frac{A(n)}{n^{\rho'} \log n} + O\left(\frac{\log^{2-2\beta'} T}{\log \log T}\right). \]

Because of the inequality above we can use Lemma 13. We can say that
\[ (7.3) \quad \Re \sum_{2 \leq n \leq \log^2 T} \frac{A(n)}{n^{\rho'} \log n} \geq -\frac{\log^{2-2\beta'} T}{2(1-\beta') \log \log T} + O\left(\frac{\log^{2-2\beta'} T}{(1-\beta')^2 (\log \log T)^2}\right). \]

Combining (7.2) and (7.3), we get
\[ \log |\zeta(\rho')| \geq -\frac{C \log^{2-2\beta'} T}{\log \log T} \]
for some positive constant $C$. Since $4 \leq T \leq \gamma' \leq 2T$, substituting $\gamma'$ in the above equation we get
\[ \log |\zeta(\rho')| \geq -\frac{C \log^{2-2\beta'} \gamma'}{\log \log \gamma'} \]
for some positive constant $C$ that depends on $\sigma_1$ and $\sigma_2$. This completes the proof of Theorem 3.

8. Lemmas for the proof of Theorem 4. We do not assume RH for the proof of Theorem 4.

Let $\sigma_1$ be such that $1/2 < \sigma_1 < 1$ and $\sigma_1 < \sigma < 1$. For $x \geq 1$, define
\[ Z_x(s) = \sum_{n \leq x} \frac{c(n) A(n)}{n^s}, \]
where $c(n)$ is a totally multiplicative function such that $c(p) = -1$ for $p \leq cx$ and $c(p) = 1$ for $p > cx$, and where
\[ \log c = -\frac{\log 2}{1 - \sigma_1} + \frac{\log 2}{(1 - \sigma_1)^2 \log^a x}. \]

Here we want $\sigma$ to be very close to $\sigma_1$ for large values of $x$.

**Lemma 14.** Let $\sigma_1$ be such that $1/2 < \sigma_1 < 1$, and let $a$ be fixed. Then for all large values of $x$, $Z_x(s)$ has a root at
\[ s = \sigma_1 + \frac{1}{\log^a x} + O\left(\frac{1}{\log^{2a} x}\right), \]
where the $O$-term constant depends on $\sigma_1$.
Proof. Let \( x \geq 2 \). We set \( z = s - \sigma_1 \) where \( |z| < 2/\log^a x \) and \( x \) is so large that \( |1 - s| \geq |1 - \sigma_1| - 2/\log^a x > 0 \).

By choosing \( x \) sufficiently large we make sure that \( s \) is arbitrarily close to \( \sigma_1 \). Using Lemma \( \ref{lem:approximation} \) we get

\[
Z_x(s) = - \sum_{p \leq cx} \frac{\log p}{p^s + 1} + \sum_{cx < p \leq x} \frac{\log p}{p^s - 1} + O(x^{1/2 - \sigma_1})
\]

Applying Lemma \( \ref{lem:approximation} \) again to the first two sums, and separately combining the last two sums, we find that

\[
Z_x(s) = V_x(s) - 2V_{cx} + \sum_{p \leq cx} \frac{2 \log p}{p^{2s} - 1} + O(x^{1/2 - \sigma_1})
\]

Thus we have \( Z_x(s) = V_x(s) - 2V_{cx}(s) + O(1) = -W_x(s) + O(1) \).

Using techniques similar to those in Lemma \( \ref{lem:proof_5} \) we can show that \( Z_x(s) \) has a root at

\[
s = \sigma_1 + \frac{1}{\log^a x} + O\left(\frac{1}{\log^{2a} x}\right).
\]

Lemma 15. We have the following lower bound for \( Z_x(s) \):

\[
Z_x(s) \gg \frac{x^{1 - \sigma_1}}{\log^a x}
\]
on \( \mathcal{C}_1 \), where \( \mathcal{C}_1 \) is as in Section 5.

Proof. Since the main term of \( Z_x(s) \) is just the negative of the main term of \( W_x(s) \), the proof is identical to the corresponding part of the proof of Theorem 2 (see (5.19)).

9. Proof of Theorem \( \ref{thm:main} \). Let \( \mathcal{C}_0, \mathcal{C}_1, K \) and \( \delta \) be as defined in Sections 4–5. This time let \( \tau \) be a real number such that

\[
\left| \frac{\tau \log p}{2\pi} + \frac{1}{2} \right| < \delta \quad \text{for} \ p \leq cx, \quad \left| \frac{\tau \log p}{2\pi} \right| < \delta \quad \text{for} \ cx < p \leq x,
\]

and again let \( \mathcal{G} \) be the set of those \( \tau \) such that both inequalities hold.

We have

\[
|V_x(s + i\tau) - Z_x(s)| \ll \frac{x^{1 - \sigma_1}}{\log b^{-1} x}
\]

where \( s \in \mathcal{C}_1 \) and \( \tau \in \mathcal{G} \). The proof is similar to that of (5.6).
Define
\[ L_x(s) = \sum_{1 < n \leq x} \frac{c(n)\Lambda(n)}{n^s \log n}. \]

Clearly,
\[ (9.2) \quad \left| \sum_{1 < n \leq x} \frac{A(n)n^{-s-i\tau}}{\log n} - L_x(s) \right| = \left| \sum_{1 < n \leq x} \frac{A(n)n^{-s-i\tau}}{\log n} - \sum_{1 < n \leq x} \frac{c(n)\Lambda(n)}{n^s \log n} \right| \]
\[ \leq \sum_{1 < n \leq x} \frac{A(n)}{n^\sigma \log n} |n^{-i\tau} - c(n)|. \]

If \( n = p^k \), then
\[ |n^{-i\tau} - c(n)| = |p^{-ik\tau} - c(p)^k| \leq k|p^{-i\tau} - c(p)|. \]

Thus, for \( \sigma \geq \sigma_1 \) and \( \tau \in \mathcal{G} \), the expression above is
\[ \leq 2\pi \delta \sum_{p^k \leq x} \frac{1}{p^{k\sigma_1}} \ll \delta \frac{x^{1-\sigma_1}}{\log x} \ll \frac{x^{1-\sigma_1}}{\log^b x} \]
by our choice of \( \delta \) in (4.17). That is, for \( \sigma \geq \sigma_1 \) and \( \tau \in \mathcal{G} \),
\[ (9.3) \quad \left| \sum_{1 < n \leq x} \frac{A(n)}{\log n} n^{-s-i\tau} - L_x(s) \right| \ll \frac{x^{1-\sigma_1}}{\log^b x}. \]

Also, when \( \Re s \geq \sigma_1 \),
\[ L'_x(s) = -\sum_{n \leq x} \frac{c(n)\Lambda(n)}{n^s} \ll \sum_{n \leq x} \frac{\Lambda(n)}{n^\sigma_1} \ll x^{1-\sigma_1}. \]

Thus, if \( s \) is on or inside \( \mathcal{C}_1 \), with \( \mathcal{C}_1 \) as in Section 5, then
\[ (9.4) \quad L_x(s) = L_x(\sigma_1) + O\left(\frac{x^{1-\sigma_1}}{\log^a x}\right). \]

Let us evaluate \( L_x(s) \) at \( \sigma_1 \):
\[ (9.5) \quad L_x(\sigma_1) = -2 \sum_{p \leq cx} p^{-\sigma_1} + \sum_{p \leq x} p^{-\sigma_1} + O(1) \]
\[ = \frac{x^{1-\sigma_1} \log c}{(1-\sigma_1) \log^2 x} + O\left(\frac{x^{1-\sigma_1}}{\log^{\min(3,a+1)} x}\right), \]
which we get by techniques similar to those in Sections 4–5.
Thus combining (9.2), (9.4) and (9.5), we see that for \( s \) on or inside \( C_1 \), and \( \tau \in \mathcal{G} \),

\[
\Re \sum_{1 < n \leq x} \frac{A(n)}{\log n} n^{-s-i\tau} = \frac{x^{1-\sigma_1} \log c}{(1 - \sigma_1) \log^2 x} + O\left(\frac{x^{1-\sigma_1}}{\log^{\min(a,3))} x}\right) + O\left(\frac{x^{1-\sigma_1}}{\log b x}\right),
\]

since \( b > a + 1 \) and \( a > 2 \).

We have

\[
\Re \sum_{p^{k} \leq x} \frac{p^{-k(s+i\tau)}}{k} \leq \left(\frac{\log c}{1 - \sigma_1} + o(1)\right) x^{1-\sigma_1} \log^2 x,
\]

where \( s \in C_1 \) and \( \tau \in \mathcal{G} \).

Let

\[
T_x(s) = \sum_{x < n \leq T^{d_1/4}} \frac{w(n)A(n)}{\log n} n^{-s}.
\]

Then by Lemma 12, when \( s \in C_1 \) and \( \tau \in \mathcal{G} \), for \( s = \sigma + it \) with \( \sigma_1 \leq \sigma \) and \( |t| \leq 1 \), and for \( \tau \in R \), we have

\[
\log \zeta(s + i\tau) = \sum_{1 < n \leq x} \frac{A(n)}{\log n} n^{-s-i\tau} + T_x(s + i\tau) + O(T^{-d_1^2/16}),
\]

\[
-\frac{\zeta'}{\zeta}(s + i\tau) = \sum_{n \leq x} A(n)n^{-s-i\tau} - T'_x(s + i\tau) + O(T^{-d_1^2/16}),
\]

which we get from (4.30), where \( x = d_1 \log T/4(\log \log T)^a \). Now we have to show

\[
T'_x(s + i\tau) = \sum_{x < n \leq T^{d_1/4}} w(n)A(n)n^{-s-i\tau} \ll 1,
\]

\[
T_x(s + i\tau) = \sum_{x < n \leq T^{d_1/4}} w(n)A(n)\frac{n^{-s-i\tau}}{\log n} \ll \frac{1}{\log x},
\]

for all \( s \in C_1 \) and some \( \tau \in \mathcal{G} \).

By substituting (9.9) in (9.8), we get

\[
-\frac{\zeta'}{\zeta}(s + i\tau) = Z_x(s) + O\left(\frac{x^{1-\sigma_1}}{\log^{b-1} x}\right),
\]

Since \( Z_x(s) = -W_x(s) + O(1) \), by the same argument as for (5.19) and (5.17) we can see that \( \zeta'(s) \) has a root for \( \rho' \) in \( C_1 \).
We know that
\[\log c = \log 2 / (\sigma_1 - 1)(1 + o(1)).\]
Combining the above estimates, we get
\[
(9.11) \quad \log |\zeta(\rho')| \leq (1 + o(1)) \left( - \frac{(\sigma_1 - 1/2)^{1 - \sigma_1} \log 2}{(1 - \sigma_1)^2 4^{1 - \sigma_1}} \right) \frac{\log^{1 - \sigma_1} T}{(\log \log T)^2 + b(1 - \sigma_1)}.
\]
Since \( T \leq \gamma' = \Re \rho' \leq 2T \), (9.11) also holds with \( T \) replaced by \( \gamma' \), and this is (1.11).

Thus we get
\[
\log |\zeta(\rho')| \leq (-B(\sigma_1) + \epsilon) \frac{\log^{1 - \beta'} \gamma'}{(\log \log \gamma')^{2 + b(1 - \sigma_1)}},
\]
where
\[
B(\sigma_1) = \frac{(\sigma_1 - 1/2)^{1 - \sigma_1} \log 2}{(1 - \sigma_1)^2 4^{1 - \sigma_1}}.
\]
To complete the proof of Theorem 4, it only remains to show the existence of a \( \tau \in \mathbb{R} \cap \mathcal{G} \) satisfying (9.9) and (9.10). We can apply the same idea as in Sections 4–5 and Kronecker’s formula with minor changes. The proof will proceed with just a different set of \( b_p \).

Combining all the estimates and choosing \( b = 3 + \epsilon'' \), we obtain
\[
\log |\zeta(\rho')| \leq (-B(\sigma_1) + \epsilon) \frac{\log^{1 - \beta'} \gamma'}{(\log \log \gamma')^{5 - 3\beta + \epsilon'}},
\]
where \( \epsilon' = \epsilon''(1 - \sigma) \). Thus the proof of Theorem 4 is complete.

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