Mathematical problems of calculation and optimization of composite structures

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Abstract. Mathematical problems of solving direct and inverse problems of mechanics of composite structures are formulated. The problems of strength calculation and optimal design of composite overwrapped pressure vessels are investigated. The solutions of the optimization problem have been received and verified by solving the direct problems with obtained design parameters using the classical and improved shell theories. Numerical solutions were obtained using the spline collocation and discrete optimization methods.

1. Introduction
Multilayer reinforced beams, plates and shells are the most important structural elements in aviation and rocket–space technology, shipbuilding and automotive industry, energy and chemical engineering, housing and industrial construction \cite{1–3}. The widespread use of composites and composite structures required the development of new models of anisotropic inhomogeneous materials which would take into account the features of their real structure, nonlinear processes of deformation and destruction. Several fundamental mathematical problems arise while researching the behavior of modern composite and hybrid structures.

The first problem is caused by the need to develop structural models of composite and hybrid materials that take into account the features of their real internal structure, nonlinear processes of deformation and fracture. To date, a large number of structural models of composites have been developed. A comparative analysis of various structural models of composite materials (CM) is given in \cite{4–7}. Carbon fiber reinforced plastics (CFRP) are the most perspective class of modern composites that combine high strength and stiffness with low specific gravity. In \cite{8, 9} a comprehensive approach to the construction of mathematical models of nonlinear-elastic deformation of CFRP under bending taking into account the effect of the multi-modulus behavior under tension and compression is presented. Mathematical model for calculating and analyzing the stress-strain state (SSS) of composite materials on the basis of an ice matrix is presented in \cite{10}.

The second problem is attributable to the need to consider refined models of the behavior of composite and hybrid structures. Currently, there are various methods for obtaining equations of theories of plates and shells. In particular, these are the methods for asymptotic integration of equations of the three-dimensional theory of elasticity using small parameters, as well as
methods for representing the characteristics of the SSS in the form of series for some systems of functions of the transverse coordinate.

The hypothesis method that includes two fundamentally different approaches has become more widespread in practice. A detailed description of the classical theory of Kirchhoff-Love and various refined theories of plates and shells can be found, in particular, in the monographs [4,11–20].

The third problem concerns the development of effective numerical methods for solving boundary value problems for nonlinear systems of partial differential equations with variable coefficients and with small parameters in the leading derivatives. Note that the transition from the classical theory of plates and shells to certain refined theories is accompanied not only by an increase in the order of systems of differential equations. It also includes qualitative changes in the structure of solutions and the emergence of new rapidly growing and rapidly decreasing solutions that have a pronounced character of boundary layers. As a result, there is a need to develop new effective numerical methods for solving problems of the mechanics of composites.

An important place among the approaches to solving of boundary value problems (BVPs) in the theory of plates and shells is taken by various methods of reducing the dimension of the problem, for example, the spline interpolation method of functions in one of the coordinate directions [21, 22], the method of separation of variables using the trigonometric representation of functions [23–26], the least-squares collocation method [27–32]. Some results of the numerical calculation of hybrid and anisogrid composite structures are presented in [33–36].

The fourth problem is related to the development of effective numerical methods for solving the problems of multicriteria optimization of composite and hybrid structures.

2. The problem statement and the mathematical models

Composite overwrapped pressure vessels (COPV) are used in the rocket and spacecraft making industry due to their high strength and lightweight. Consisting of a thin, non-structural liner wrapped with a structural fiber composite COPV are produced to hold the inner pressure of tens and hundreds atmospheres. COPV have been one of the most actual and perspective directions of research, supported especially by NASA [37,38].

Let’s consider a multilayer composite pressure vessel at a state of equilibrium under equidistributed inner pressure. We need to determine the parameters of structure and CM meeting the following requirements:

\[ V \geq V_0, \quad P \geq P_0, \quad M \leq M_0, \]

where \( V \) is the volume of the vessel, \( P \) is the inner pressure and \( M \) is the vessel’s mass; they are constrained by some preset values \( V_0, P_0, M_0 \).

We define the optimization problems in the following way: find extremum of one functional from (Eq. (1)) under other constraints.

The structures optimization problem statement includes selection of objective functional, formulation of constitutive equations and constraints on performance and design variables.

The mathematical models describing the vessel’s state are based on the following assumptions: the vessel is a multilayer thin-walled structure, the vessel’s layers can have different mechanical characteristics, the reinforced layer’s material is quasi-homogeneous, the vessel’s main loading is high inner pressure.

These assumptions allow us to reduce dimension of the corresponding mathematical problem and to build the mathematical vessel’s models based on the different theories of multilayer non-isotropic shells [4]. The Kirchhoff—Love shell theory [39] (KLST) and the improved Timoshenko [18] (TiST) and Andreev-Nemirovskii [13] (ANST) theories are used to solve the direct calculation problems of multilayer composite vessels, to analyse their behavior and to verify optimization problem solutions.
The full systems of equations were described in the paper [6]. Relations between stresses and strains are described by the structural models [7]. The main idea of these models is that CM parameters are calculated through matrix and fibers mechanical parameters, fibers volume content and winding angles. The SSS of matrix and fibers are evaluated through stresses and strains of the composite shell. A failure criterion is applied for every component of CM. Here we use the Mises criterion to determine the first stage of failure.

The objective function whose minimum is required is the minimum mass:

\[ M = 2\pi \int_{\theta_0}^{\theta_1} r R_1 h d\theta \left[ \rho_m (1 - \omega_r) + \rho_r \omega_r \right] \rightarrow \min, \]  

(2)

where \( \rho_m, \rho_r \) are the densities of matrix and reinforcing fibers, \( \omega_r \) is the volume content of reinforcement. We chose the following design functions: the curvature radius \( R_1(\theta) \) to define the generatrix, the thickness of the shell \( h(\theta) \), and the reinforcement angle \( \psi(\theta) \).

The solution has to satisfy the constraints on the shell’s inner volume:

\[ \pi \int_{\theta_0}^{\theta_1} r^2 R_1 \sin \theta d\theta = V_0 \]  

(3)

and the strength requirement:

\[ \max\{b_{sr}, b_{sm}\} \leq 1, \]  

(4)

where \( b_{sr}, b_{sm} \) are the normalized von Mises stresses in the matrix and fibers [4]. Note that a safety margin is widely used while solving engineering problems. It can be considered by correction of the right-hand member of the inequality (Eq.(4)).

We imposed the following constraints on the design functions:

\[ 0 \leq \psi \leq 90, \quad h_0^* \leq h \leq h_1^*, \quad R_0^* \leq R_1 \leq R_1^*. \]  

(5)

Estimation of the SSS of composite vessels using offered models leads to the solution of boundary value problems for rigid systems of differential equations. These problems are ill-conditioned, and their solutions have pronounced character of thin boundary layers. Numerical analysis was performed by the spline collocation and discrete orthogonalization methods, realized in the COLSYS [40] and GMDO [24] software. These computing tools have proved to be effective in numerical solving of wide range of problems of composite shell mechanics [4].

We investigated the vessel’s deformations by computing its stress-strain state based on the different shell theories. The vessel’s shape was a part of a toroid: \( R_1 = 2.46 \text{ m}, \theta_0 = 0.108^\circ, \theta_1 = 90^\circ \) (the computed half), \( r(\theta_0) = 0.04 \text{ m} \). The carbon composite parameters were: \( E_m = 3 \cdot 10^9 \text{ Pa}, \nu_m = 0.34, E_r = 300 \cdot 10^9 \text{ Pa}, \nu_r = 0.3, \omega_r = 0.55, V_0 = 350 \text{ liters} \) where \( E_m, E_r \) are the Young moduli of the matrix and fibers, \( \nu_m, \nu_r \) are their Poisson’s ratios.

Figure 1 shows the SSS characteristics of the vessel with the thickness \( h = 0.6 \text{ cm} \), reinforced in the circumferential direction (\( \psi = 90^\circ \)) under the load of 170 atm. On the left the displacements of the reference surface along the generatrix \( u(r) \) (dashed curves) and the normal displacement of these surface \( w(r) \) (solid curves) are shown. On the right are the distribution of normalized von Mises stress (nVMS) along the thickness in the matrix \( b_{sm}(r) \). The solid curves correspond to a slice at the shell edge, the dashed curves — to a slice at \( \theta = 0.1 \).

It’s easy to see that the basic kinematic characteristics coincide both qualitatively and quantitatively. Small differences are observed only for the stresses and deformations near the compressed edge. The maximum results and qualitative difference were obtained for ANST. This is due to accounting for the transverse shears by non-linear distribution in a thickness of a shell. Earlier it was shown [4] that ANST’s based results were closest to the ones of 3D elastic theory in most cases.
Figure 1. The stress-strain state characteristics of the composite vessel computed using different shell theories. The curves without symbols correspond to KLST simulations, the curves marked with △ correspond to those using TiST, and □ correspond to ANST.

Figure 2. The winding angle’s influence on the composite vessel stress-strain state. KLST’s results are drawn without marks, TiST shown with symbols △, ANST depicted with □.

The winding angle’s influence on the COPV performance was investigated using parametric analysis. Dependence of the maximum nVMS in the matrix $bs_m$ (dashed curves) and the fibers $bs_r$ (dash–dotted curves), and the maximum size of the displacement vector $||\vec{v}||$ (solid curves) are shown in Figure 2.

The calculated values are very close in the area of their minima (Figure 2 left side). The graphs of kinematic function $||\vec{v}||$ coincide qualitatively. Some noticeable quantitative difference are revealed only for KLST’s results.

The range $\psi \in (42; 45)$ corresponds to the zones of minimum values (Figure 2 right side), which practically coincide ($\min_\psi bs_m \approx 0.65$, $\min_\psi bs_r \approx 1.05$, $\min_\psi ||\vec{v}|| \approx 5 \cdot 10^{-3}$ m), as well as the angles, where these values are obtained ($\psi \approx 43.2^\circ$ for $bs_m$ and $bs_r$, $\psi \approx 43.8^\circ$ for $||\vec{v}||$).

It was revealed that the winding angles of minimum stresses values were almost insensitive to
the thickness variation. The change of \( h \) from 0.6 to 1.6 cm corresponded to the angle’s change about 0.2°.

Additionally we investigated stress-strain state of the vessel (the thickness \( h = 0.6 \) cm, the winding angles at \( \psi = \pm 43.2 \)°), when nVMS in the matrix and the fibers were near their minimum. The adopted notation is the same as in Figure 1. And again the difference is visible only in a very small region near the edge but now this difference is small enough to be neglected. Moreover the displacement values of the reference surface, the efforts and the moments completely coincide for all the theories.

All the theories (KLST, TiST, ANST) provided similar estimated characteristics of SSS. This vessel was characterized not only by essential decrease of the maximal nVMS in the matrix and the fibers, but also by their uniform distribution along the generatrix. At the same time the values of bending moments significantly reduced bringing vessel’s SSS close to momentless.

One can see that the winding angle as a design parameter gives an opportunity to increase the vessel’s strength significantly. The difference between the ”best” and ”worst” designs can reach 20±35 times comparing their nVMS in the matrix and fibers. The ”worst” designs have the winding angle close to 90°. In this case there are considerable transverse shears near the compressed edge, and the loading is redistributed to a rather weak matrix while the fibers remain unloaded.

Conclusions

- Non-constant design parameters, such as thickness, winding angles and curvature radius of composite shell give the possibility for additional reduction of COPV mass while keeping its strength.
- The solutions of the optimization problem have been received and verified by solving the direct problems with obtained design parameters using the classical and improved shell theories.
- The performed analysis showed that the optimizing problem can be solved using rather simple shell theories (Kirchhoff–Love, Timoshenko). These theories are characterized by lower computational complexity of corresponding boundary value problem if compared to Andreev–Nemirovskii theory. It takes from 10 to 20 times less resources.

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