BOUNDEDNESS OF TRACE FIELDS OF RANK TWO LOCAL SYSTEMS

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Abstract. Let \( p \) be a fixed prime number, and \( q \) a power of \( p \). For any curve over \( \mathbb{F}_q \) and any local system on it, we have a number field generated by the traces of Frobenii at closed points, known as the trace field. We show that as we range over all pointed curves of type \((g, n)\) in characteristic \( p \) and rank two local systems with infinite monodromy at infinity, the set of trace fields which are unramified at \( p \) and of bounded degree is finite. This proves observations of Kontsevich obtained via numerical computations, which are in turn closely related to the analogue of Maeda’s conjecture over function fields. We also prove a similar finiteness result across all primes \( p \). The key ingredients of the proofs are Chin’s theorem on independence of \( \ell \) of monodromy groups, and the boundedness of abelian schemes of GL\(_2\)-type over curves in positive characteristics, obtained using partial Hasse invariants; the latter is an analogue of Faltings’ Arakelov theorem for abelian varieties in our setting.

1. Introduction

Let \( p \) be a prime number, and \( q \) some power of \( p \). Let \( \bar{C}/\mathbb{F}_q \) be a smooth, projective curve of genus \( g \), and \( Z \) a non-empty subset of \( n \) points of \( \bar{C} \); we write \( C := \bar{C} - Z \) for the open curve. We will refer to \((\bar{C}, Z)\) as a pointed curve of type \((g, n)\), and to the points in \( Z \) as the cusps of \( C \).

Definition 1.1. For \((\bar{C}, Z)/\mathbb{F}_q\) as above, let \( \mathcal{L}(\bar{C}, Z) \) be the set of isomorphism classes of rank two local systems \( \mathbb{L} \) on \( C \) with infinite monodromy around each point in \( Z \), and such that \( \det \mathbb{L} \cong \mathbb{Q}_\ell(-1) \).

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For such a local system $\mathbb{L}$, the work of Drinfeld, which was later generalized to arbitrary rank by Lafforgue, implies that there is a unique number field $F$ and an embedding $\sigma: F \to \overline{\mathbb{Q}}_p$ such that $\sigma(F)$ is the field generated by the traces of Frobenii at closed points of $C$. We refer to $F$ as the trace field of $L$, and denote by $\mathfrak{F}(L(C, Z))$ the set of trace fields of local systems in $\mathcal{L}(C, Z)$. It is natural to wonder about the distribution of such trace fields; our first main result is the following boundedness statement.

**Theorem 1.2.** Fix a pair $(g, n)$. Let $\mathfrak{F}_{g,n} := \bigcup_{\bar{C}, Z} \mathfrak{F}(L(\bar{C}, Z))$ be the set of trace fields of local systems in $L(C, Z)$, as $(C, Z)$ and $q$ vary over all pointed curves of type $(g, n)$ and powers of $p$, respectively. Then, for any integer $d$, there are only finitely many fields in $\mathfrak{F}_{g,n}$ with degree $\leq d$ and which are unramified at $p$.

**Remark 1.3.** It is also straightforward to see that, if we fix the trace field $F$ and the pointed curve $(\bar{C}, Z)/\mathbb{F}_q$ as in Theorem 1.2, then only finitely many elements of $\bigcup_q \mathcal{L}(\bar{C}_{F_q}, Z_{F_q})$ have trace field $F$, where $(\bar{C}_{F_q}, Z_{F_q})$ denotes the basechange to Spec$(\mathbb{F}_q)$. On the other hand, it is possible to have a positive dimensional family of curves, all of which admit rank two local systems with the same trace field; this can be ruled out in certain special cases, as is done in Lam22.

We now state a result where the characteristic $p$ is allowed to vary. For this purpose, for a pointed curve $(\bar{C}, Z)/\mathbb{F}_q$ in characteristic $p$, we write $L_p(\bar{C}, Z)$ for $L(C, Z)$ to emphasize the characteristic of the base field.

**Theorem 1.4.** There are only finitely many fields in $\bigcup_{p,C,Z} \mathfrak{F}(L_p(\bar{C}, Z))$ of degree $d$ and completely split at $p$.

The heuristic in the next section leads us to expect that Theorem 1.2 should hold without the unramified-at-$p$ assumption, and Theorem 1.4 should hold without the splitness condition, though we are not able to prove these: indeed, our method to bound the degree of the Hodge bundle uses partial Hasse invariants, and the obtained bounds depend on the prime $p$ if the field is not totally split at $p$. In another direction, one can try to remove the infinite monodromy condition in the definition of $L(C, Z)$: the main obstruction here is that such local systems are not known to come from abelian varieties, or some other family of varieties with a “uniform” description: instead, they are known to arise in the cohomology of moduli of shtukas, whose geometry depends heavily on $C$ and $q$.

1.5. **Context.** Our work is motivated by Maeda’s conjecture in the function field setting. For example, for each curve $(\bar{C}, Z)/\mathbb{F}_q$ of type $(g, n) = (0, 4)$, computations of Kontsevich [Kon09, § 0.1] shows that, in almost all cases, there are four trace fields, each of degree roughly $(q+1)/4$. This contributes to the Maeda philosophy that, generically, trace fields should be as large as

\[ \text{the splitting into four fields comes from the Atkin-Lehner operators} \]
possible. We refer the reader to [Lam22] for more on the analogue of Maeda’s conjecture over function fields, as well as references for the number field case.

We were also heavily influenced by the results of Falting [Fal83] on Arakelov’s theorem for abelian varieties, as well as Deligne’s finiteness theorem [Del87]. More precisely, we were motivated by the possibility of uniformity in Deligne’s finiteness theorem: the latter says that on a fixed complex curve $C$, only finitely many rank $N \mathbb{Q}$-local systems can come from algebraic geometry, and by uniformity we mean whether this finite number depends on the underlying curve in its moduli space. As a first step towards this question, for a fixed $N$, one may ask about the contribution to rank $N \mathbb{Q}$-local systems coming from abelian varieties: in this case uniformity is known and is a corollary of Faltings’ theorem [Fal83, Theorem 1]. Note that this uniformity from Faltings’ proof seems, at least to the author, to be stronger than the subsequent re-proofs of the same result due to Deligne [Del87], as well as Jost and Yau [JY93], although these works are more general in that they apply to local systems not coming from abelian schemes.

In any case, for fixed $(g, n)$, one sees that there are only finitely many $F$’s of fixed degree such that a curve of type $(g, n)$ carries a non-trivial family of abelian varieties of $\text{GL}_2(F)$-type. Theorem 1.2 is the analogue of this in positive characteristic. It would be interesting to investigate the analogue of the full strength version (i.e. beyond abelian varieties of $\text{GL}_2$-type) of Faltings’ theorem in positive characteristic. For example, is it true that, in characteristic $p$, any abelian scheme of dimension $D$ on a curve $(\bar{C}, \mathcal{Z})$ of type $g, n$ is isogenous to one whose Hodge bundle has degree bounded by $g, n, p, D$? We should also mention a related result of Litt [Lit21] which is an analogue of Deligne’s result in positive characteristic, but for $\ell$-adic coefficients.

1.6. Sketch of proof of Theorem 1.2. For this sketch, we will focus on the case of local systems with unipotent monodromy around each cusp, which is the key part. By work of Drinfeld, we know that the rank two local $\mathbb{Q}_\ell$-local systems in question come from abelian schemes of $\text{GL}_2(F)$-type, with $F$ being the trace field of Frobenii; suppose that there are infinitely many such $F$’s of degree $d$. Using Zarhin’s trick, we obtain principally polarized abelian schemes of dimension $N := 8d$ over $C$, and therefore infinitely many maps $C \to \mathcal{A}_N$, which extend to maps $\bar{C} \to \mathcal{A}_N^\ast$, where the latter denotes the minimal compactification. The first key idea is that, using partial Hasse invariants and a Frobenius untwisting result, we can pass to an isogenous abelian scheme and bound the degree (with respect to the Hodge bundle of $\mathcal{A}_N^\ast$) of such maps in terms of just $(g, n, p, d)$. This step may be seen as an enhancement of the recent beautiful work [KY22] of Krishnamoorthy, Yang, and Zuo by the use of partial Hasse invariants; it is also where the unramified at $p$ condition is used.

Now consider the moduli space $\mathcal{M}$ of curves $C$ of type $(g, n)$, along with maps $\bar{C} \to \mathcal{A}_N^\ast$ of some fixed degree. Using the previous step, we have
ininitely many points $s_i \in M$, corresponding to infinitely many fields $F_i$'s. Since $M$ is of finite type, taking the Zariski closure of the $s_i$'s give a positive dimensional family of curves $C \to S$, and an abelian scheme $A \to C$, such that the fiber of $A$ at $s_i \in S$ is a (power of an) abelian scheme of $\text{GL}_2(F_i)$-type. If we were in characteristic zero, this would already give a contradiction since by isomonodromy all the monodromy representations of $\pi_1(C_{s_i})$ must be the same. In our situation, the $\mathbb{Q}$-structure on Betti cohomology is of course not available, and we crucially make use of Chin’s theorem on $\ell$-independence of monodromy groups and some tricks, such as the finiteness of number fields of fixed degree and bounded ramification, to conclude.

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1.7. Notation. Throughout, $p > 0$ will denote a prime, $q$ a power of $p$. For a scheme $Z$, $\pi_1(Z)$ will always denote the étale fundamental group.

2. Drinfeld’s work on function field Langlands

We first recall the results of Drinfeld from his work on function field Langlands for $\text{GL}_2$. Throughout this section, we use notation as follows. As in the introduction, let $\overline{C}/\mathbb{F}_q$ be a smooth, projective curve of genus $g$, and $Z$ an effective Cartier divisor on $\overline{C}/\mathbb{F}_q$ of degree $n$; we write $C := \overline{C} - Z$ for the open curve. We will refer to $(\overline{C}, Z)$ as a pointed curve of type $(g, n)$.

Let $\ell$ be a prime distinct from $p$, and $L$ be a rank two $\mathbb{Q}_\ell$-local system on $C/\mathbb{F}_q$, such that $\det L \cong \mathbb{Q}_\ell(1)$. Let $F \subset \overline{\mathbb{Q}}_\ell$ be the field generated by Frobenius traces of $L$; we write $[F : \mathbb{Q}] = d$, and sometimes refer to $F$ simply as the Frobenius trace field, or simply trace field, of $L$.

Theorem 2.1. Suppose $L$ has infinite local monodromy around some $z \in Z$. Then there exists an abelian scheme $\pi_{\text{Drinf}} : B_{\text{Drinf}} \to C$ of relative dimension $d$, such that $\text{End}_{\mathcal{C}}(B_{\text{Drinf}}) \otimes \mathbb{Q} = F$, and $L$ appears as a direct summand of $R^1\pi_{\text{Drinf}*}\mathbb{Q}_\ell$. Moreover, if we write $D := R^1\pi_{\text{Drinf}*}\overline{\mathbb{Q}}_p \in F-\text{Isoc}(C)_{\overline{\mathbb{Q}}_p}$, then we have a decomposition

$$D = \bigoplus \mathbb{D}_\tau,$$

where the above sum is over embeddings $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$, each $\mathbb{D}_\tau$ is two dimensional, and the induced action of $F$ on $\mathbb{D}_\tau$ is through $\tau$.

We refer the reader to [KYZ22 Theorem 2.2] as well as to the remark in loc. cit. that follows for how to deduce the above theorem from the works of Drinfeld. Our next goal is to refine the abelian scheme $B_{\text{Drinf}}$ as follows, under further assumptions.
**Definition 2.2.** For any scheme $Z/\mathbb{F}_q$ and number field $F$, we say that an abelian scheme $\pi : B \to Z$ is of $\text{GL}_2(F)$-type if it satisfies the properties in Theorem 2.1: that is

- $\text{End}(B) \otimes \mathbb{Q} = F$,
- for $\ell \neq p$, there is a decomposition

$$R^1 \pi_* \overline{\mathbb{Q}}_\ell = \bigoplus \mathbb{L}_\tau,$$

where each $\mathbb{L}_\tau$ has rank two, the induced action of $F$ on $\mathbb{L}_\tau$ is via the embedding $\tau$, and $\det \mathbb{L}_\tau \cong \overline{\mathbb{Q}}_\ell(-1)$. Note that the field of Frobenius traces of each $\mathbb{L}_\tau$ is $F$.

We recall the definition of companions.

**Definition 2.3.** Let $Y/\mathbb{F}_q$ be a smooth scheme, and $L$ a $\mathbb{Q}_\ell$-local system on $Y$. Let $\ell' \neq p$ be a prime number (possibly equal to $\ell$). Let $\iota : \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_{\ell'}$ be a (possibly non-continuous) field isomorphism; abusing notation, let $\iota$ also denote the induced isomorphism $\overline{\mathbb{Q}}_\ell[t] \to \overline{\mathbb{Q}}_{\ell'}[t]$. A $\iota$-companion to $L$ is a lisse $\mathbb{Q}_{\ell'}$-Weil sheaf $L'$ on $Y$, such that for all closed points $y$, we have:

$$\iota(P_y(L,t)) = P_y(L', t) \in \overline{\mathbb{Q}}_{\ell'}[t].$$

**Proposition 2.4.** For an abelian scheme $\pi : B \to Z$ of $\text{GL}_2(F)$-type, and let $L_\tau$'s be as in Definition 2.2. For any $\tau, \tau'$, the local systems $\mathbb{L}_\tau, \mathbb{L}_{\tau'}$ are companions. Moreover, every $\overline{\mathbb{Q}}_\ell$-companion of $\mathbb{L}_\tau$ is isomorphic to $\mathbb{L}_{\tau'}$ for some $\tau'$.

**Proof.** This is recorded in [KP22, Remark 2.8].

Given a local system $\mathbb{L}$ on $C$, and $z \in Z$, we can restrict it to the punctured neighborhood of $z \in \overset{\circ}{C}$: upon picking a local coordinate $t$ at $z$, we obtain a representation of $\text{Gal}(\mathbb{F}_q((t)))$. We say that $\mathbb{L}$ has unipotent local monodromy at $z$ if the inertia subgroup $I \subset \text{Gal}(\mathbb{F}_q((t)))$ acts unipotently on this representation. We say that the monodromy is infinite at $z$ if the representation of $I$ does not have finite image.

**Proposition 2.5.** Suppose that $\mathbb{L}$ has infinite unipotent monodromy around each $z \in Z$, and that its trace field $F$ is unramified at $p$. Then there exists an abelian scheme $\pi : B \to C$ of relative dimension $d$, with Néron model $\overset{\text{N}}{B} \to C$ such that

1. $B$ has semi-stable and totally degenerate reduction around each $z \in Z$,
2. $\text{End}_C(B) \otimes \mathbb{Q} = F$,
3. $\mathbb{L}$ appears as a direct summand of $R^1 \pi_* \overline{\mathbb{Q}}_\ell$,
4. the abelian scheme $(B \times B^t)^4$ admits a principal polarization, inducing a map $C \to \mathcal{A}_{8h}$. The latter extends to a map

$$\tilde{f} : \overset{\text{N}}{C} \to \mathcal{A}_{8d}^\text{sh}.$$
such that the line bundle $\bar{f}^*\omega$ has degree bounded above by a function of $g, n, d,$ and $p$ (crucially, the dependence is on $p$ and not $q$). Here, $\mathcal{M}_N^p$ denotes the minimal compactification of the moduli stack of principally polarized abelian varieties of dimension $N$, with $\omega$ the Hodge line bundle on it.

The proof of this will appear at the end of §4.

Remark 2.6. In the case where $F$ is completely split above $p$, the above follows from [KYZ22, Lemma 2.7]; in fact, in this case the degree of $\bar{f}^*\omega$ is boundedly above by a function of only $g, n, d$.

3. Partial Hasse invariants

For any abelian scheme $A$ over $C/\mathbb{F}_q$ with semi-stable reduction along $Z$, we let $\bar{A} \to \bar{C}$ denote its Néron model. Let $\omega_A$ denote its Hodge vector bundle, which is a vector bundle of dimension $\text{dim} A$ on $\bar{C}$; similarly we denote by $\omega_{\bar{A}}$ the Hodge vector bundle of $\bar{A}$, which is a vector bundle over $\bar{C}$, whose restriction to $C$ is given by $\omega_A$.

Set-up 3.1. For the rest of this section we suppose $B \to C$ is an abelian scheme with semi-stable and totally degenerate reduction along $Z$, and moreover $\text{End}(B) \otimes \mathbb{Q} = F$, which is unramified at $p$ and such that $\text{dim} B = [F : \mathbb{Q}] = d$. It is straightforward to check that, passing to an isogenous abelian scheme if necessary, we can (and do) assume that the ring of integers $\mathcal{O}_F \subset F$ acts on $B$; we refer to such abelian schemes as $\text{GL}_2(\mathcal{O}_F)$-type.

Notation 3.2. Suppose $\mathbb{F}_q$ contains all the residue fields $k_p$ for primes $p$ of $\mathcal{O}_F$ lying above $p$; for each $\tau : k_p \hookrightarrow \mathbb{F}_q$, let $\omega_{B, \tau}$ denote the summand of $\omega_B$ on which the $\mathcal{O}_F$-action is through $\mathcal{O}_F \to k_p \hookrightarrow \mathbb{F}_q$. We define $\omega_{\bar{B}, \tau}$ similarly. When the context is clear, we sometimes omit the subscripts $B, \bar{B}$ and simply denote the Hodge bundles by $\omega_\tau$. Similarly, whenever there is a module with an $\mathcal{O}_F$-action, we use the subscript $\tau$ to denote the component where the action is through $\tau$.

Definition/Proposition 3.3. (1) Let $(M, F, V)$ denote the (covariant) logarithmic Dieudonné crystal on $(\bar{C}, Z)$ associated to $\bar{B}$, as constructed by Kato and Trihan [KT03, §4]. We denote by $M_{(\bar{C}, Z)}$ the evaluation of the crystal on the trivial thickening $(\bar{C}, Z)$; this is a vector bundle on $\bar{C}$ with integrable connection with logarithmic poles along $Z$, and is moreover equipped with Frobenius and Verschiebung maps $F$ and $V$.

(2) For each prime $p$ of $F$ and each embedding $\tau : k_p \hookrightarrow \mathbb{F}_q$, let $H^{dR}_{1, \tau} := M_{(\bar{C}, Z), \tau}$ denote the summand on which the $\mathcal{O}_F$-action is through $\tau$.

\footnote{see also [KP22, Appendix A] for a very nice summary}

\footnote{the notation here is to reflect that, when restricted to $C$, the bundle is given by the $\tau$-component of the relative de Rham homology of $B$}
\[ \mathcal{O}_F \to k_p \hookrightarrow \mathbb{F}_q. \] The latter is equipped with a Hodge filtration
\[ 0 \to \omega_{B,\tau} \to H^1_{dR} \to \omega^\ast_{B,\tau} \to 0. \]

The Hodge filtration is defined in [KT03, § 5.1], and proved to be locally free in Lemma 5.2 of loc.cit.. The proof that the sub and quotient are isomorphic to \( \omega_{B,\tau}, \omega^\ast_{B,\tau} \), respectively, is given in [KT03 Example 5.4 (b)]. We sometimes also denote the sub-bundle \( \omega_{B,\tau} \) by \( \mathrm{Fil}^1 \). The Frobenius action on \( H^1_{dR}(B) \) annihilates each of \( \mathrm{Fil}^1 \).

**Proposition 3.4.** We have \( \dim H^1_{dR} = 2, \dim \mathrm{Fil}^1 = 1 \) for all embeddings \( \tau \).

**Proof.** For each point \( c : \text{Spec}(\mathbb{F}_p) \to C \), let \( H^1_{\text{cris}}(B_c) \) denote the crystalline homology of the fiber \( B_c \). Since \( B \) is of \( \text{GL}_2(\mathcal{O}_F) \)-type, we have \( \dim H^1_{\text{cris}}(B_c) = 2 \), where \( H^1_{\text{cris}}(B_c) \) denotes the summand on which the \( \mathcal{O}_F \)-action is through the embedding \( \mathcal{O}_F \to W(\mathbb{F}_p) \) lifting \( \tau \). Therefore \( \dim H^1_{dR} = 2 \) for each \( \tau \), and it remains to show that \( \dim \mathrm{Fil}^1 = 1 \).

It suffices to show this when restricted to a neighborhood of a cusp \( z \in Z \). Let \( \mathcal{G} \) denote the \( p \)-divisible group of \( B \), and let \( k[[t]] \) denote the local ring around \( z \). The totally degenerate assumption implies \( \mathcal{G} |_{\text{Spec}(k((t)))} \) has a sub \( p \)-divisible group \( \mathcal{H} \) of multiplicative type and dimension \( d \). Let \( \kappa := k((t^{1/p^\infty})) \) denote the perfection of \( k((t)) \), and consider the inclusion of Dieudonné modules \( D(\mathcal{H}) \subset D(\mathcal{G} |_{\text{Spec}(\kappa)}) \). We denote by \( H^1_{dR}(\mathcal{H}) \) the reduction mod \( p \) of \( D(\mathcal{H}) \), which we may take as the definition of the de Rham homology of \( \mathcal{H} \).

Recall that Frobenius annihilates \( \mathrm{Fil}^1 \subset H^1_{dR}(B |_{\text{Spec}(\kappa)}) \). Since \( \mathcal{H} \) is of multiplicative type, the Frobenius action is bijective (as we are using covariant Dieudonné theory) on \( H^1_{dR}(\mathcal{H}) \), the induced map
\[ H^1_{dR}(\mathcal{H}) \to H^1_{dR}(B |_{\text{Spec}(\kappa)})/\mathrm{Fil}^1 = \omega^\ast_{B |_{\text{Spec}(\kappa)}} \]
is injective, and hence bijective for dimension reasons. Now \( \mathcal{H} \) is acted on by \( \mathcal{O}_F \otimes \mathbb{Z}_p = \mathcal{O}_{p_1} \times \cdots \times \mathcal{O}_{p_k} \), where \( p_i \) are the primes of \( F \) lying over \( p \), and \( \mathcal{O}_{p_i} \) denotes the completion of \( \mathcal{O}_F \) at \( p_i \), which by our assumption is unramified over \( \mathbb{Z}_p \); hence we have a corresponding decomposition of \( p \)-divisible groups
\[ \mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_k, \]
where \( \mathcal{H}_i \) is the image of the idempotent corresponding to the \( \mathcal{O}_{p_i} \) factor. By construction, \( \mathcal{H}_i \) is a \( p \)-divisible group of \( \text{GL}_2(\mathcal{O}_{p_i}) \)-type. Recall that each \( \tau \) specifies one of the primes \( p_i \), and an embedding of the residue field \( k_{p_i} \hookrightarrow \mathbb{F}_q \). Of course, fixing such an embedding \( \tau : k_{p_i} \hookrightarrow \mathbb{F}_q \), every other embedding is of the form \( \sigma^m \tau \), the composition of \( \tau \) with the \( m \)-th power of absolute Frobenius on \( \mathbb{F}_q \).

Each of \( \mathcal{H}_i \) is necessarily of multiplicative type, and hence the Frobenius
\[ F_i : H^1_{dR}(\mathcal{H}_i)(\nu) \to H^1_{dR}(\mathcal{H}_i) \]
is again bijective. Let $H^{dR}_i(\mathcal{H}_\tau) = \bigoplus_\tau H^{dR}_{1,\tau}(\mathcal{H}_i)$ denote the decomposition into $\tau$-components, as $\tau$ varies over embeddings $k_p \hookrightarrow \mathbb{F}_q$. Now, for each $\tau$, $F_i$ sends $H^{dR}_{1,\tau}(\mathcal{H}_i)$ to $H^{dR}_{1,\sigma_\tau}(\mathcal{H}_i)$; as $\mathcal{H}_i$ is of multiplicative type, this is an isomorphism. In other words, for a fixed $i$, the dimension of $H^{dR}_{1,\tau}(\mathcal{H}_i)$ is independent of $\tau$, and hence at least one. Since $H^{dR}_i(\mathcal{H}_i)$ has dimension $d$, we deduce that $\dim H^{dR}_{1,\tau}(\mathcal{H}_i) = 1$ for each choice of $i$ and $\tau$. This implies that $\omega_{B,\tau}^*$ has rank one for each $\tau$, and hence the same is true of $\text{Fil}^1$, as required.

We now recall the notion of partial Hasse invariants, following [TX16, §4.4]. For any abelian scheme $A$ over a characteristic $p$ scheme $S$, we denote by $A_{\mathbb{F}_q}$ the pull-back of $A$ along absolute Frobenius on $S$; similarly for any coherent sheaf $M$ on $S$ we have the Frobenius pull-back $M_{\mathbb{F}_q}$. Let $\sigma$ be absolute Frobenius on $\mathbb{F}_q$, and denote by $\sigma_\tau$ the composition $k_p \hookrightarrow \mathbb{F}_q \xrightarrow{\sigma} \mathbb{F}_q$; let $\sigma^{-1}\tau$ be the embedding such that $\sigma(\sigma^{-1}\tau) = \tau$.

**Definition 3.5.** Notation as in 3.1. For each $\tau$, we have the map $\omega_{B^1,\tau} \to \omega_{B^1,\tau} \otimes \omega_{B^1,\tau}^{-1}$ induced by Verschiebung. We view this as a section $h_\tau \in \Gamma(\omega_{B^1,\tau} \otimes \omega_{B^1,\tau}^{-1})$, and refer to it as a partial Hasse invariant.

The following is a simple by-product of the proof of Proposition 3.4, and we therefore omit the proof.

**Proposition 3.6.** Each of the $h_\tau$’s is not identically zero.

### 4. Frobenius untwisting

**Definition 4.1.** Let $K/\mathbb{Q}_p$ be a local field with ring of integers $\mathcal{O}_K$. We say that a $p$-divisible group $G$ is of $GL_2(\mathcal{O}_K)$-type if $\mathcal{O}_K \hookrightarrow \text{End}(G)$, and moreover $[K: \mathbb{Q}_p] = \dim G$.

We prove the following result, which generalizes a result of Xia’s [Xia13, Theorem 6.1]. As before, suppose $C/\mathbb{F}_q$ is a smooth affine curve. For the definition and notation surrounding the Kodaira–Spencer map of a family of $p$-divisible groups, we refer the reader to [Xia13, §1.1], as well as Theorem 3.11 of loc.cit.. For any sheaf $\mathcal{F}$ on $C$ (e.g. a vector bundle or a $p$-divisible group), we write $\mathcal{F}^{(p)}$ for the Frobenius twist, i.e. the pullback under absolute Frobenius on $C$.

**Lemma 4.2.** Let $G$ be any $p$-divisible group over $C$, such that the Kodaira-Spencer map of $G$ is identically zero. Then there exists a $p$-divisible group $\mathcal{H}$ isogenous to $G$ and such that $\mathcal{H}^{(p)} \cong G$. Moreover, if $G$ is of $GL_2(\mathcal{O}_K)$-type, then so is $\mathcal{H}$.

**Proof.** Suppose $G$ has dimension $d$ and height $h$. Let $\mathcal{V}$ be the Dieudonné crystal of $G$, and let $\mathcal{V}_C$ denote its evaluation on $C$, which is a vector bundle
on $C$ of dimension $h$; moreover, it has a sub-bundle $\Fil^1$ given by the Hodge filtration, as well as the usual Frobenius and Verschiebung maps

$$F_C : \mathcal{V}_C^{(p)} \to \mathcal{V}_C, \ V_C : \mathcal{V}_C \to \mathcal{V}_C^{(p)}.$$  

Fix any lifting $\tilde{C}$ of $C$ over $W(F_q)$, along with a lift $\sigma$ of absolute Frobenius. Evaluating the crystal on $\tilde{C}$ we obtain a vector bundle on $\tilde{C}$ with integrable connection, which we denote by $(\mathcal{V}_{\tilde{C}}, \nabla)$; this has its own Frobenius and Verschiebung, which we denote by $F_{\tilde{C}}, V_{\tilde{C}}$ respectively; moreover we have a mod $p$ reduction map $\pi : \mathcal{V}_{\tilde{C}} \to \mathcal{V}_C$.

We now construct the Dieudonné crystal for the hypothetical “Frobenius untwisting” of $\mathcal{G}$. Let $V'_{\tilde{C}} \subset \mathcal{V}_{\tilde{C}}$ denote the sub-module $\pi^{-1}(\Fil^1)$, which contains $pV_{\tilde{C}}$; from the conditions $(\Fil^1)^{(p)} = \ker F_{\tilde{C}} = \Image V_{\tilde{C}}$ it is straightforward to see that $V'_{\tilde{C}}$ is stable under both $F_{\tilde{C}}$ and $V_{\tilde{C}}$, as well as the $\mathcal{O}_K$-action if $G$ is of $GL_2(\mathcal{O}_K)$-type. Moreover, since we are assuming that Kodaira-Spencer map is identically zero, $V'_{\tilde{C}}$ is stable under $\nabla$, and therefore $V'_{\tilde{C}}$ is the Dieudonné module for a $p$-divisible group $H$, which is of $GL_2(\mathcal{O}_K)$-type if $G$ is.

We claim that

$$(4.2.1) \quad F_{\tilde{C}}(V_{\tilde{C}}^{(p)}) = pV_{\tilde{C}} :$$

indeed, by the construction of $V'$, we have the inclusions

$$F_{\tilde{C}}(V_{\tilde{C}}^{(p)}) \subseteq pV_{\tilde{C}} \subseteq V'_{\tilde{C}} \subseteq V_{\tilde{C}}.$$ 

It remains to check that the left most inclusion is an equality. We have $\dim V'/F_{\tilde{C}}(V_{\tilde{C}}^{(p)}) = h - d$, and also $\dim V'/pV_{\tilde{C}} = h - d$, where dimension refers to the dimension as vector bundles on $\tilde{C}$; these, together with the chain of inclusions $F_{\tilde{C}}(V_{\tilde{C}}^{(p)}) \subseteq pV_{\tilde{C}} \subseteq V'_{\tilde{C}}$, prove the claim. But $(4.2.1)$ says precisely that $\mathcal{H}^{(p)} \cong \mathcal{G}$; finally, the construction gives a natural isogeny $\mathcal{H} \to \mathcal{G}$, as required.

**Proof of Proposition 2.5.** Let $B := B_{Drinf} \to C$ be the abelian scheme given by Theorem 2.1, which is of $GL_2(F)$-type, and we assume that $\mathcal{O}_F$ acts on $B_{Drinf}$ as in 3.1. Let $\mathcal{G}$ denote the $p$-divisible group of $B$; for a prime $\mathfrak{p}$ of $F$ lying above $p$, let $\mathcal{G}_\mathfrak{p}$ denote the factor of $\mathcal{G}$ such that the $\mathcal{O}_F$-action is through the embeddings corresponding to $\mathfrak{p}$ (exactly as in the proof of Proposition 3.4). By applying Lemma 4.2 repeatedly, which must terminate by the same argument as in [KYZ22, Proof of Proposition 2.4], we may assume that the Kodaira-Spencer map for $\mathcal{G}_\mathfrak{p}$ is non-vanishing. Let $\tau_0$ be an embedding for the prime $\mathfrak{p}$ for which the Kodaira-Spencer map

$$\Fil^1_{\tau_0} \to H^d_{1,\tau_0}/\Fil^1_{\tau_0} \otimes \Omega^1_{C}(Z)$$



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4 another proof of this termination is given in [KYZ22, Appendix A], which also applies essentially without change in our set-up.
is non-zero: note that this is a map of line bundles, by Proposition 3.4. Therefore \( \deg \frac{H^1_{dR} / \Fil^1_0}{\Fil^1_0} = - \deg \Fil^1_0 \), from which we may conclude that \( \deg \Fil^1_0 \leq (2g - 2 + n)/2 \).

By Proposition 3.6, for any other embedding \( \tau' = \sigma^j \tau_0 \) for the prime \( p \) for some \( j \geq 0 \), we have a non-zero map \( \omega_{\sigma^j \tau_0} \to \omega_{\tau_0}^{\otimes p^j} \) by iterating Verschiebung, and hence \( \deg \omega_{\sigma^j \tau_0} \leq p^j \deg \omega_{\tau_0} \). This implies that, for each \( \tau \), \( \deg \omega_{\tau} \) is bounded above purely in terms of \( g, n, p, d \). Finally, recall that \( \Fil^1_0 \cong \omega_{B, \tau} \) and \( H^1_{dR} / \Fil^1_0 \cong \omega_{B, \tau}^* \), and therefore the Hodge bundle of the abelian scheme \((B \times B)^4\) has degree bounded above purely in terms of \( p, g, n, d \), as required.

### 5. Mapping spaces

For any \( N \geq 1 \), let \( \mathcal{A}_N \) denote the moduli stack of principally polarized abelian varieties of dimension \( N \), with minimal, i.e. Baily-Borel-Satake, compactification \( \mathcal{A}^*_N \). Let \( \omega \) denote the Hodge bundle on \( \mathcal{A}^*_N \).

**Definition 5.1.** Let \( \mathcal{M}_{g,n}(\mathcal{A}^*_N, h) \) denote the moduli stack\(^5\) of \( n \)-pointed genus \( g \) curves \((\bar{C}, p_1, \ldots, p_n)\), along with a map \( \bar{f} : \bar{C} \to \mathcal{A}^*_N \) such that

1. \( \bar{f}^{-1}(\partial \mathcal{A}^*_N) \subset \{p_1, \ldots, p_n\} \), and
2. \( \bar{f}^* \omega \) has degree \( h \).

Let \( S \) be the set of trace fields defined in the introduction, and let \( S_{\text{unip}} \subset S \) denote the subset of trace fields coming from local systems with unipotent local monodromy around each cusp of \( C \).

**Lemma 5.2.** Suppose there are infinitely many fields \( F_1, F_2, \ldots \) in \( S_{\text{unip}} \). Then there exists a smooth scheme of finite type \( S/\mathbb{F}_q \), a family of smooth curves \( C \to S \), an abelian scheme \( A \to C \), and a Zariski dense set of points \( s_i \in S \) such that the family \( \mathcal{A}_{s_i} \to C_{s_i} \) is isomorphic to \((B_{s_i} \times B_{s_i}^*)^4\), where \( B_{s_i} \to C_{s_i} \) is of \( \text{GL}_2(F_i) \)-type.

**Proof.** By Proposition 2.5, for each \( i \), there is a pointed curve \((\bar{C}_i, Z_i)\) of type \((g, n)\), a map 

\[ \bar{f}_i : \bar{C}_i \to \mathcal{A}^*_N \]

with \( \bar{f}_i^{-1}(\partial \mathcal{A}^*_N) \subset Z_i \), and such that the degree \( \bar{f}_i \) is bounded in terms of \( g, n, p, d \). Therefore, for some \( h \), we have infinitely many points \( s_i \in \mathcal{M}_{g,n}(\mathcal{A}^*_N, h) \). Since the latter is of finite type, taking \( S \) to be the smooth part of the Zariski closure of the \( s_i \) gives the desired family of curves \( C \to S \) and abelian scheme \( A \).

\(^5\)similar mapping stacks have been studied by Faltings [Fal83, § 3]
6. Proofs of main results

**Proposition 6.1.** Suppose $k$ is an algebraically closed field of characteristic $p > 0$, $Z/k$ a smooth scheme of finite type, and $Z \subset \overline{Z}/k$ is a compactification, i.e. $\overline{Z}$ is normal, projective, and $D := \overline{Z} \setminus Z$ is a simple normal crossings divisor. If $\pi_A : A \to \overline{Z}$ is an abelian scheme such that the action of $\pi_1(Z)$ on the $\ell$-torsion of $A$ is trivial, for $\ell > 2$ a prime number, then $R^1\pi_A\overline{\mathbb{Q}}_\ell$ has unipotent (geometric) monodromy along each component of the boundary $D$. Moreover, the image of the representation of $\pi_1(Z)$ has pro prime-to-$p$ image.

**Proof.** The claim about unipotent monodromy is given on [KP22, p. 879] in the third paragraph. For the prime-to-$p$ claim, note that, for $\ell > 2$, the subgroup $\Gamma(\ell) \subset \text{GL}_n(\mathbb{Z}_\ell)$, consisting of matrices which are the identity mod $\ell$, is pro-$\ell$, as long as $\ell > 2$. □

**Lemma 6.2.** Let $C/F_\ell$ be a smooth curve, and $\pi : B \to C$ an abelian scheme of $\text{GL}_2(F)$-type, for a number field $F$. Let $W := R^1\pi_*\mathbb{Q}_\ell$ be the relative Tate module of $B$, so that we have a decomposition

$$W \otimes \overline{\mathbb{Q}}_\ell \cong \bigoplus_{\sigma:F\to \overline{\mathbb{Q}}_\ell} L_\sigma$$

with each $L_\sigma$ being of rank two.

1. For any finite étale cover $C' \to C$, and any choice of $\sigma$, the trace field of $L_\sigma|_{C'}$ is still $F$. For $\sigma \neq \tau$, the local systems $L_\sigma$, $L_\tau$ are not isomorphic.

2. Moreover, for any finite index subgroup $H \leq \pi_1(C_{F_\ell})$,

$$\text{End}(W \otimes \overline{\mathbb{Q}}_\ell)^H = F \otimes \overline{\mathbb{Q}}_\ell = \prod_{\sigma:F\to \overline{\mathbb{Q}}_\ell} \overline{\mathbb{Q}}_\ell.$$

**Proof.** We first prove point (1). Note that the second assertion of point (1) follows immediately from the first: indeed, if $\sigma \neq \tau$ and $L_\sigma|_{C'} \cong L_\tau|_{C'}$, then $\sigma$ and $\tau$ must agree on the trace field of $L_\sigma|_{C'}$, which means this latter trace field is strictly smaller than $F$.

We now prove the first assertion of (1). Suppose the contrary, i.e. that the trace field of $L_\sigma|_{C'}$ is a strict sub-field $E \subset F$, for some $\sigma$. Taking a different embedding $\tau : F \to \overline{\mathbb{Q}}_\ell$ which agrees with $\sigma$ on $E$, we get

$$L_\sigma|_{C'} \cong L_\tau|_{C'}.$$  

We may assume without loss of generality that $C'$ is Galois over $C$, so that $\pi_1(C)/\pi_1(C')$ is a finite group $\Gamma$. Since $L_\sigma|_{C'}$, $L_\tau|_{C'}$ are irreducible, (6.2.1) implies that there exists a character $\chi : \Gamma \to \overline{\mathbb{Q}}_\ell^\times$ such that $L_\sigma \cong L_\tau \otimes \chi$. Since $\det L_\sigma \cong \overline{\mathbb{Q}}_\ell(-1) \cong \det L_\tau$, we deduce that $\chi$ factors as

$$\chi : \Gamma \to \{\pm 1\} \to \overline{\mathbb{Q}}_\ell^\times.$$
Replacing $C'$ by the cover of $C$ defined by $\chi$, we may therefore assume that $C'$ has degree two over $C$. Suppose that the trace field of $L_\sigma|_{C'}$ is a number field $E \subset F$.

Applying Drinfeld’s result (Theorem 2.1) to $L_\sigma|_{C'}$, we see that there is some abelian scheme $B' \to C'$ of dimension $[E : \mathbb{Q}]$, and of $\text{GL}_2(E)$-type, so that

$$\mathbb{W}|_{C'} \cong \mathbb{W}(B')^\text{sm},$$

where $\mathbb{W}(B')$ denotes the $\mathbb{Q}_\ell$-Tate module of $B'$, and $m := [F : E]$.

Consider the abelian scheme $\text{Res}_{C'}C_B'$ over $C$. By the definition of Weil-restriction and the Tate conjecture for homomorphisms of abelian varieties over function fields (proven by Zarhin), we have

$$\text{dim}_\mathbb{Q}\text{Hom}_{C'}(B, \text{Res}_{C'}C_B') \otimes \mathbb{Q} = m,$$

which is strictly greater than one by our assumption.

On the other hand, $B$ is simple over $C$, and since $C'$ is a degree two cover of $C$, $\text{dim} \text{Res}_{C'}C_B' = 2[E : \mathbb{Q}] \leq m[E : \mathbb{Q}] = \text{dim} B$. This contradicts Equation 6.2.2.

We now deduce the second part. For $H$ a finite index subgroup of $\pi_1(\mathbb{C}_{\overline{\mathbb{F}}_p})$, let

$$\mathbb{C}_{\overline{\mathbb{F}}_p} \to C_{\overline{\mathbb{F}}_p}$$

be the corresponding finite étale cover of $C_{\overline{\mathbb{F}}_p}$. Note that there exists a finite extension $\mathbb{F}_{q'}$ over which (6.2.3) is defined over: i.e. a curve $\mathbb{C}_{\overline{\mathbb{F}}_{q'}}$, along with a map $\mathbb{C} \to C_{\overline{\mathbb{F}}_{q'}}$ whose basechange along $\mathbb{F}_{q'} \to \overline{\mathbb{F}}_p$ is the map (6.2.3).

We have the natural embedding $\mathbb{F} \otimes \overline{\mathbb{Q}}_{\ell} \hookrightarrow \text{End}(\mathbb{W} \otimes \overline{\mathbb{Q}}_{\ell})^H$, and suppose for the sake of contradiction that this is a strict inclusion.

This implies that two of the $L_\sigma$’s become isomorphic when viewed as representations of $H$: more precisely, there are distinct embeddings $\sigma, \tau : F \to \overline{\mathbb{Q}}_{\ell}$, such that $L_\sigma|_{\mathbb{C}_{\overline{\mathbb{F}}_p}} \cong L_\tau|_{\mathbb{C}_{\overline{\mathbb{F}}_p}}$. This in turn implies that there is a character $\chi : \pi_1(\text{Spec}(\mathbb{F}_{q'})) \to \overline{\mathbb{Q}}_{\ell}^\times$ such that

$$L_\sigma|_{\mathbb{C}} \cong L_\tau|_{\mathbb{C}} \otimes \chi.$$

As before, comparing determinants, we deduce that $\chi$ factors through $\{\pm 1\} \subset \overline{\mathbb{Q}}_{\ell}^\times$; therefore $\chi$ determines a degree two étale cover $\mathbb{C}' \to \mathbb{C}$ over which $L_\sigma$ and $L_\tau$ are isomorphic. This contradicts the first part of the lemma. □

**Proposition 6.3.** Let $(\mathcal{C}, \mathcal{D}) \to S$ be a family of smooth pointed curves: that is, there exists a smooth proper $S$-curve $\overline{\mathcal{C}}$, $\mathcal{D}$ is a relatively étale Cartier divisor on $\overline{\mathcal{C}}/S$, and $\mathcal{C} = \overline{\mathcal{C}} \setminus \mathcal{D}$. Suppose $S$ is geometrically connected. Suppose there is an abelian scheme $\mathcal{A} \to \mathcal{C}$, such that the action of $\pi_1(\mathcal{C})$ on the $\ell$-torsion is trivial. Then

- for any closed point $s \in S$ with geometric point $\overline{s}$ lying over it, the representation of $\pi_1(\mathcal{C}_s)$ on the $\ell$-adic Tate module of $\mathcal{A}|_{\mathcal{C}_s}$ factors through the prime-to-$p$ quotient $\pi_1(\mathcal{C}_s)(\mathcal{C}_s)$, and
for any other closed point $t \in S$, with geometric points $\bar{t}$ lying over it, the representations of $\pi_1^{(p')}(\mathcal{C}_s)$ and $\pi_1^{(p')}(\mathcal{C}_\bar{t})$ (on the $\ell$-adic Tate modules of $\mathcal{A}|_{\mathcal{C}_s}$ and $\mathcal{A}|_{\mathcal{C}_\bar{t}}$, respectively) are isomorphic: that is, there exists a continuous isomorphism $\pi_1^{(p')}(\mathcal{C}_s) \cong \pi_1^{(p')}(\mathcal{C}_\bar{t})$ such that these representations are isomorphic.

Proof. The first part follows from Proposition 6.1. We now prove the second part. Pick a point $s_{\text{gen}}$ of $S$ specializing to both $s$ and $t$, and let $\bar{s}_{\text{gen}}$ be a geometric point lying over $s_{\text{gen}}$.

We have specialization isomorphisms of prime-to-$p$ fundamental groups (see [SGA60, Exposée XIII, Corollaire 2.12] for this precise statement; we also refer the reader to [Ota18 §3] for a nice exposition):

$$\text{sp}_s : \pi_1^{(p')}(\mathcal{C}_{s_{\text{gen}}}) \cong \pi_1^{(p')}(\mathcal{C}_s), \; \text{sp}_t : \pi_1^{(p')}(\mathcal{C}_{s_{\text{gen}}}) \cong \pi_1^{(p')}(\mathcal{C}_t).$$

By the construction of the specialization map and proper base-change, the representation of $\pi_1^{(p')}(\mathcal{C}_s)$ on the $\ell$-adic Tate module of $\mathcal{A}|_{\mathcal{C}_s}$ is obtained by taking the corresponding representation of $\mathcal{C}_{s_{\text{gen}}}$ and composing with the isomorphism $\text{sp}_s^{-1}$. The same goes for the representation of $\pi_1^{(p')}(\mathcal{C}_t)$, which implies the claim. \qed

The following is the key lemma to show finiteness of trace fields.

**Lemma 6.4.** Let $(\mathcal{C}, \mathcal{D}) \to S$ be a family of smooth pointed curves. Suppose there is an abelian scheme $\mathcal{A} \to \mathcal{C}$, and an infinite set of points $s_i \in S$ such that, for each $i$, the abelian scheme $\mathcal{A}_{s_i} \to \mathcal{C}_{s_i}$ is isogenous to $B_{k_i}^\ell$ for an abelian scheme $B_i \to \mathcal{C}_{s_i}$ which is of $\text{GL}_2(F_i)$-type, for some number field $F_i$, and $k_i \geq 1$ (the latter being independent of $i$). Then the set $\{F_i\}$ of trace fields is finite.

**Proof.** Since passing to finite étale covers of $\mathcal{C}_{s_i}$ does not change the trace field by Lemma 6.2, we may as well assume that the action of $\pi_1(\mathcal{C})$ on the $\ell$-torsion of $\mathcal{A}$ is trivial. Under this assumption, by Proposition 6.3 for each $s_i$, the representation $\mathcal{V}_{s_i}^{\ell}$ of $\pi_1(\mathcal{C}_{s_i})$ on the $\ell$-adic Tate module of $\mathcal{A}|_{\mathcal{C}_{s_i}}$ factors through the prime-to-$p$ quotient $\pi_1^{(p')}(\mathcal{C}_{s_i})$, and moreover the resulting representations of $\pi_1^{(p')}(\mathcal{C}_{s_i})$ are isomorphic for all $i$, in the sense of the statement of Proposition 6.3.

Fix $s = s_1$, and let $\mathcal{V}_\ell$ denote the $\ell$-adic Tate module of $\mathcal{A}_s$. As $\ell$ varies over primes different from $p$, the $\mathcal{V}_\ell$’s form a $\mathbb{Q}$-compatible system.\footnote{see the introduction of [Chi04] for the definition of compatible systems} Let $G_{\text{geom}, \ell}$ denote the Zariski closure of the image of the geometric fundamental group $\pi_1(\mathcal{C}_s)$ under this representation; we view $\mathcal{V}_\ell$ as a representation of $G_{\text{geom}, \ell}$. Let $G^0_{\text{geom}, \ell}$ denote the neutral component of $G_{\text{geom}, \ell}$. By the previous paragraph, the isomorphism class of the triple $(G_{\text{geom}, \ell}, G^0_{\text{geom}, \ell}, \mathcal{V}_\ell)$ is independent of $i$, and we therefore chose to omit $i$ from the notation.
By Chin’s independence of $\ell$ theorem [Chi04, Theorem 1.4], there exists a number field $E$, a split algebraic group $G_0/E$, and an $E$-rational representation $U$ such that, for any place $\lambda$ of $E$ above $\ell$, $G_{0,\text{geom}} \otimes E_\lambda \simeq G_{E_\lambda}$, and $V_\ell \otimes Q E_\lambda \simeq U \otimes E_\lambda$ as representations of these groups. Define $Z := Z(\text{End}(U)^G)$, which is a finitely generated $E$-algebra.

**Claim 6.5.** We have $F_i \otimes Q_\ell \hookrightarrow Z \otimes E_\lambda$ for each $i$ and $\lambda$.

The claim implies the finiteness of the $F_i$’s: indeed, $Z$ is a product of number fields, so is unramified at $\ell$ for $\ell$ large enough. Therefore the $F_i$’s belong to the finite set [Neu13, Ch. II Theorem 2.13] of number fields with bounded degree and unramified outside a finite set of primes.

**Proof of Claim.** By assumption, $A_{\kappa_i}$ is isogenous to $B_i^k$, where $B_i$ is of $\text{GL}_2(F_i)$-type and $k \geq 1$. Letting $F_i$ act on $B_i^k$ diagonally, we have

$$F_i \otimes Q_\ell \hookrightarrow \text{End}(V_\ell)^{G_{0,\text{geom},\ell}} \hookrightarrow \text{End}(V_\ell \otimes Q_\ell E_\lambda)^{G_{0,\text{geom},\ell}} \simeq \text{End}(U \otimes E_\lambda)^{G_{E_\lambda}} = \text{End}(U)^{G_0 \otimes E_\lambda}.$$

Note that we have $V_\ell \simeq W \otimes k$ as $\pi_1(\kappa_{\kappa})$-representations. To show the image of the above map lies in the center of $\text{End}(V_\ell)^{G_{0,\text{geom},\ell}}$, it suffices to show that $\text{End}(W \otimes \overline{Q_\ell})^H = F_i \otimes \overline{Q_\ell}$, where $W$ denotes the rational $\ell$-adic Tate module of $B_i$, and $H$ ranges over all finite index subgroups of $\pi_1(\kappa_{\kappa})$: indeed, this implies that $\text{End}(V_\ell)^{G_{0,\text{geom},\ell},\overline{Q_\ell}}$ is precisely $M_k(F_i \otimes \overline{Q_\ell})$, the ring of $k \times k$-matrices with entries in $F_i \otimes \overline{Q_\ell}$, and the embedding of $F_i \otimes Q_\ell$ into $M_k(F_i \otimes \overline{Q_\ell})$ is precisely the diagonal one.

But the equality $\text{End}(W \otimes \overline{Q_\ell})^H = F_i \otimes \overline{Q_\ell}$ follows immediately from Lemma 6.2 as required.

We now assemble all the ingredients to deduce our main result.

**Proof of Theorem 1.2.** As in the statement of the theorem, we fix a pair $(g, n)$. Suppose, for the sake of contradiction, that we have infinitely many distinct trace fields $F_1$’s in $\mathfrak{F}_{g,n} = \bigcup_{C, Z, g} \mathfrak{F}(\mathcal{L}(C, Z))$, corresponding to $Q_\ell$-local systems $L_i$ on pointed curves $(C_i, Z_i)$ of type $(g, n)$, so that, by Theorem 2.1, $L_i$ comes from an abelian scheme $B_i \to C_i$ of relative dimension $d_i$, which is moreover of $\text{GL}_2(F_i)$-type. We first treat the case when each $L_i$ has unipotent monodromy around each cusp: in this case, by Lemma 5.2

$\dagger$note that the Tate conjecture does not apply here since we are considering representations of the geometric fundamental group $\pi_1(\kappa_{\kappa})$.
there exists $A \to C \to S$ as in the statement of Lemma 6.4, at which point
the lemma allows us to conclude in this case.

We now handle the general case. By the same argument as in \cite[p. 879]{KP22}, by passing to the étale cover $\varphi_i : C'_i \to C_i$ trivializing the $\ell$-torsion of $A_i$, we have that the pullbacks $\varphi_i^* A_i$ has unipotent monodromy or good reduction around each of the cusps of $C'_i$. By the Riemann–Hurwitz formula and the pigeonhole principle we may assume that the $C'_i$ have the same topological type. By Lemma 6.2, passing from $C_i$ to $C'_i$ does not change the trace field, and hence we are done by the unipotent case.

\[\square\]

6.6. Proof sketch of Theorem 1.4 Since the proof of Theorem 1.4 is very similar to that of Theorem 1.2 we only provide a sketch here.

Proof sketch of Theorem 1.4 The proof is essentially the same as that of Theorem 1.2 Suppose again that there are infinitely many $F_i \in \bigcup_{p \subset C, Z} \mathfrak{H}(\mathcal{Z}_{p}(\bar{C}, Z))$, which are of degree $d$ and are furthermore completely split at $p$. Suppose $F_i$ is attached to a local system $L_i$, which in turn arises from an abelian scheme $B_i \to C_i$ of $\text{GL}_2(F_i)$-type. By taking the cover $\varphi_i : C'_i \to C_i$ trivializing the $\ell$-torsion of $A_i$, we have that $\varphi_i^* A_i$ has unipotent monodromy around each cusp; let $\bar{C}'_i$ be a smooth compactification of $C'_i$. By Remark 2.6 we may assume that the induced map $\bar{C}'_i \to A^* N$ has degree bounded by $g, n, d$, so as in the proof of Theorem 1.2 we can find a family of curves $C \to S$, an abelian scheme $A \to C$, and points $s_i$ such that the fiber $A_{s_i}$ is isogenous to $B^8_i$-type. If $S$ is purely in characteristic $p$ for some $p$, then we conclude by Lemma 6.4 it therefore remains to treat the case when $S$ is in characteristic zero.

We may therefore pick a complex point $s_C : \text{Spec}(\mathbb{C}) \to S$. Let $C_{sc}$ be the fiber of $C$ over $s_C$, and $\pi_C A_{sc} \to C_{sc}$ the abelian scheme gotten by basechanging $A \to S$.

By specialization of prime-to-$p$ fundamental groups, we deduce that $\nabla_{sc} := R^1 \pi_{C} \mathbb{Q}$ decomposes into rank two pieces, and has infinite local monodromy at the cusps. The infinite local monodromy condition implies that $A_{sc}$ is in fact isogenous to $B^8$ where $B \to C_{sc}$ is an abelian scheme of $\text{GL}_2(F)$-type, for some number field $F$ of degree $d$.

We may then spread out the elements of $\text{End}(A_{sc})$ to an open dense subset $U \subset S$. Since the $s_i$’s are Zariski dense, we may assume that $s_i \in U$ for all $i$ (by throwing out the $s_i$’s which lie outside $U$). Since $\text{End}(A_{s_i}) = M_8(F_i)$, we deduce that there is an injection $M_8(F) \to M_8(F_i)$, and hence $F = F_i$ for all $i$, which is a contradiction.

\[\square\]

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