On the Resource Allocation for Political Campaigns

Sebastián Morales
Department of Industrial Engineering, University of Chile, Santiago, 8370456, Chile, sebastian.morales.a@uchile.cl

Charles Thraves*
Department of Industrial Engineering, University of Chile and Instituto Sistemas Complejos de Ingeniera (ISCI), Santiago, 8370456, Chile, cthraves@dii.uchile.cl

In an election campaign, candidates must decide how to optimally allocate their efforts/resources optimally among the regions of a country. As a result, the outcome of the election will depend on the players’ strategies and the voters’ preferences. In this work, we present a zero-sum game where two candidates decide how to invest a fixed resource in a set of regions, while considering their sizes and biases. We explore the two voting systems, the Majority System (MS) and the Electoral College (EC). We prove equilibrium existence and uniqueness under MS in a deterministic model; in addition, their closed form expressions are provided when fixing the subset of regions and relaxing the non-negative investing constraint. For the stochastic case, we use Monte Carlo simulations to compute the players’ payoffs. For the EC, given the lack of equilibrium in pure strategies, we propose an iterative algorithm to find equilibrium in mixed strategies in a subset of the simplex lattice. We illustrate numerical instances under both election systems, and contrast players’ equilibrium strategies. We show that polarization induces candidates to focus on larger regions with negative biases under MS, whereas candidates concentrate on swing states under EC. Finally, we calibrate the analyzed models with real data from the US 2020 presidential election.

Key words: electoral college; majority system; resource allocation; zero-sum game

History: Received: December 2020; Accepted: June 2021 by Dan Zhang, after 2 revisions.
*Corresponding author.
in a region will increase as the more effort she invest, while it will decrease the more effort her contenders make. Thus, the candidates’ resource allocation problem is modeled as a zero-sum game. For simplicity, we present a game theory formulation of the setting described above for two candidates, under the Majority System (MS) and the Electoral College (EC) system. The aim of this work is not only the modeling and resolution of each election system, but also analyzing the contrasts between the equilibrium strategies obtained in both. For example, how does polarization affect candidates’ resource allocations? Also, what is the impact of voter uncertainty in the two systems? Under what circumstances do swing states become attractive to candidates? We believe there are several such interesting questions that can be answered by using mathematical models able to capture the problem structure in order to analyze candidates’ actions.

Although in reality an election outcome is a result of multiple factors, we provide a simplified model that is still capable of providing insightful results able to resemble candidates’ decisions observed in reality. Some of the main challenges are: (i) create a modeling framework with the complexity that enables the representation of the agents’ actions and payoffs of the setting, while also being tractable to solve, and (ii) the model resolution under the different cases which involves for instance the estimation of complex mathematical expressions, or the computation of mixed equilibria. We analyze the case of the two following election systems:

1. **MS (Majority System):** The candidate with the most votes wins the election.
2. **EC (Electoral College):** Each region has a number of electoral votes. On each region, the candidate with the majority of votes wins all the electoral votes of the region. The candidate with more electoral votes wins the election. This is the electoral system in the United States.

It is worth to clarify that in the actual EC system used in the United States, in some states the Electors are free to vote their own choice (not necessarily matching the majority of the popular vote of the respective state). In 33 states, the electors are obligated to vote for the popular vote winner candidate.

We will focus on elections where the winner is decided entirely by the election outcome on a single round, without any phases afterwards. Therefore, systems like some parliamentary ones, where negotiations among the parties are conducted after the election, will not be considered.

For ease of exposition, we present the strategy allocation problem from an election campaign setting. However, it is important to note that there are several other applications which share a similar game theoretic framework. For example, firms that compete for a market share within a set of localities, power control games in wireless network, and resource allocations in a battlefield.

1.2. Contributions and Structure of the Paper

The main contributions of this work can be summarized as the following three:

1. **Modeling:** The development of a game theory modeling framework able to capture candidates’ resource allocation decisions, considering biases and abstention, under a MS and an EC system, showing equilibrium existence and uniqueness for some particular cases under the MS, and developing closed form solutions for certain settings.

2. **Algorithms:** The development of solution methods able to find the game equilibrium using Monte Carlo simulations to compute multidimensional integrals and its respective derivatives.

3. **Numerics:** Solve numerical experiments providing insight into what equilibria arise under different settings. In addition, contrast the impact of voters’ uncertainty as well as polarization in the candidates’ strategies and the election outcome. We also calibrate an instance representing the US Presidential Election of 2020, to show what the optimal allocation would be, and its implications.

The paper structure is described as follows: The literature review is given in Section 2. In Section 3 the electoral model under MS is presented, followed by EC in Section 4. The numerical computations are shown in Section 5. Finally, conclusions and future work are given in Section 6. All proofs are in the Appendix.

2. Literature Review

A basic model for the resource allocation problem was denoted as the “Blotto game” or “Coronel Blotto.” In this game, two players decide how to allocate a finite resource among a finite set of objects (also denoted as battlefields) where the player that allocate the most resources on an object wins it. The players’ payoff results in the number of battles won (see Borel (1921)). Since then, this problem has been studied under multiple variations; we refer the reader to Kovenock and Roberson (2012), Duffy and Matros (2017), and Thomas (2018) for more details on these variations.

An interesting setting of the Blotto game is the case with heterogeneous values (or weight) for each field.
Gross and Wagner (1950) solved the game equilibrium where players maximize the total weighted battles won. Assuming symmetric budgets and heterogeneous weights, they solved the game for three fields. The case with more than three fields and homogeneous valuations was addressed in Laslier and Picard (2002). Gross and Wagner (1950) had pointed out the directions of the results for this case without providing technical details. Recently, the result has been generalized for heterogeneous valuations, allowing for more than three regions, by Thomas (2018). The solution of the problem with asymmetric budget and homogeneous battlefields has been characterized by Roberson (2006) using $n$-copulas on the marginal distribution of the players’ strategies. This result has been extended by Schwartz et al. (2014), and Kovenock and Roberson (2021), for the heterogeneous valuations case. In all these settings, the outcome in each region is deterministic given the players’ allocations. In our paper, we address a model where the result in each region is a probability that depends on the players’ allocations, and pre-existing biases as well.

An early work that introduces uncertainty in the outcome was developed by Friedman (1958) who framed the problem as an advertising expenditure allocation. He was the first to find a closed form solution for the game equilibrium in which the players’ chances of winning each region are proportional to the players’ investments while they maximize the expected number of sales. Brams and Davis (1974) stated that under the EC system, candidates invest in states in proportion to the power of 2/3 of the state’s weight. The author assumed that both candidates maximize the expected number of electoral votes, while assuming the resources allocated to each state are the same for both candidates. Although there is literature that supports the symmetry of candidates’ allocation strategies to some degree (see Shaw (1999)), there is significant evidence for rejecting allocations to be proportional to the state’s weight (see National Popular Vote Inc. (2019)). In our work, we consider the probability of winning in the objective function, while allowing candidates’ investments to differ within the same states. Lake (1979) looked into the EC where candidates maximize the chances of winning the majority of electoral votes. They found a procedure for computing the game equilibrium in closed form expressions using the Banzhaf Power Index (see Banzhaf III (1964)). More recently, Duffy and Matros (2015) extended the results of Lake (1979) for the case of asymmetric budgets, and the results of Friedman (1958) for more than two players. Osorio (2013) generalized the closed form solution from Friedman (1958) to the case of asymmetric players’ valuations where candidates maximize the expected number of votes.

Eiselt and Marianov (2020) frames the problem studied by Friedman (1958) in a leader-follower setting. Our work differs from these in the following aspects: (i) we incorporate states’ biases, (ii) we allow for a more general representation of the stochastic voting outcome of each state (by using a Dirichlet distribution instead of a Bernoulli), and (iii) we explore equilibrium in mixed strategies.

Another study closely related to ours is Snyder (1989). The author presents a model in which two parties compete in a legislative election, and analyzes the cases where parties maximize the expected number of elected seats, and the probability of winning a majority. They consider candidates’ investments as a cost in their objective function. Similar to the work of Snyder (1989), Klumpp and Polborn (2006) studied a simultaneous and sequential equilibrium of the game, also considering the allocation cost in candidates’ objective. The main differences of our work are that we study the majority and electoral college systems with a variable number of votes/electoral votes per region, and we consider the allocation cost as a budget constraint (also known as “use it or lose it”).

Also under EC, Stromberg (2008) studied a probabilistic model in which candidates allocate resources across states to maximize the probability of winning the election. Using a limiting approximation argument of the central limit theorem, the authors characterize conditions that must satisfy an interior equilibrium of the game. Our work differs since we do not use Gaussian approximations for the probability of winning; on the contrary, we use an exact method to compute this (see Kaplan and Barnett (2003) where the authors reject the Gaussian distribution for the number of electoral votes).

Prediction of the election outcome has also captured the attention of researchers. Bayesian priors has been a technique widely used by researchers; applications of this in the EC can be found in Kaplan and Barnett (2003), Rigdon et al. (2009), and Rigdon et al. (2015). Under the same electoral system, choice models have also been used for forecasting purposes (see Wang et al. (2015)).

Another relevant area related to the subject of this study is the social choice theory, which focuses on the analysis of how to combine multiple individual preferences in order to come up with a group decision. Arrow (2012) with the Impossibility Theorem shows that there is no electoral system that remains fair with more than two candidates, simultaneously satisfying several natural properties. Recent works have analyzed the pros and cons of electoral systems in settings with more than two candidates, such as Approval Voting (AV) and Majority Judgment (MJ), see for example Balinski and Laraki (2020). The main difference between this work and ours, is that here,
we do not focus on the preferences at the individual level, but rather on how the aggregated preferences vary according to the candidates’ decisions. In addition, since we study the case with only two candidates, there is no need to explore more evolved electoral systems, such as MJ, since Majority Rule performs equivalently to AV in this case, and also, as mentioned above, the Impossibility Theorem does not hold for elections with only two candidates.

Finally, it is worth mentioning the connection of the presented problem with the market-share competition between two firms, see for example, Bell et al. (1975), Barnett (1976), and Monahan (1987). Similarly, framed as a multi-item contest problem, Robson et al. (2005) found closed form expressions for the game equilibrium using a generalized version of the Tullock functional form (see Buchanan et al. (1980)). Our work differs on treatment of the bias parameters, while we also consider the possibility of abstention. In addition, we consider a stochastic version of the electoral game, analyzing the cases where candidates maximize the expected number of votes, as well as the probability of winning.

3. Majority System

In this section, we will study the Majority System, in which the winner of the election is the candidate who obtains the largest number of votes. We provide the framework for modeling such an election under both a deterministic and a stochastic setting, and introduce algorithms that allow finding the solutions of these.

3.1. Model

Two candidates, A and B, compete on a political election campaign for president of a country. We assume throughout the study that the election is for president; however, this can be applied to any other election that shares the same settings of the model. The country is divided into a set of regions which will be denoted as \( I := \{1, \ldots, n\} \). Each region \( i \in I \) has \( v_i \) voters. Both candidates are endowed with a fixed campaign resource budget which they must allocate among the different regions. We will consider this resource to be the number of days of the campaign. Then, both candidates have a budget of \( D \) days on which they are able to run their campaign events in the different regions. Consequently, candidates must decide how much effort—how many days of campaigning—they will put into each of the regions.

Let \( x_i \geq 0 \) and \( y_i \geq 0 \) be the number of days inputted by candidates A and B, respectively, in region \( i \). For simplicity, we normalize the budget to the unit value, that is, \( D = 1 \). It can be easily seen that we can use other limited resources, besides days of campaigning, which candidates need to allocate strategically among the regions. For the sake of simplicity, the resource modeled in this work will be the days of campaigning. However, the model could easily be extended to incorporate additional resources, such as money or others, leading to a different polyhedral set as the strategy space.

The strategy space for both candidates, denoted by \( \Delta_n \), is the simplex in \( \mathbb{R}^n \), namely \( \Delta_n = \{x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = 1, x_i \geq 0\} \). Intuitively, the more days that candidate A invests in a region, the more votes she is likely to get from that particular region. Nonetheless, the more campaign her opponent (B) does in that region, the less the number of votes candidate A will receive from that particular geographical area. Therefore, the outcome of votes from each particular region will depend on the political efforts of both contenders (see Nagler and Leigheley (1992)). In addition, it is natural to think that some regions have an \textit{a-priori} bias toward one of the candidates. Put it differently, for the same level of efforts inputted by both candidates in a particular region, the outcome might favor one of the candidates over the other due to the already existing preferences of the population of the region. We assume both players play simultaneously. For each region \( i \), let \( s_i^A : \mathbb{R} \times \mathbb{R} \to [0, 1] \) be the function that maps the efforts of candidates A and B (\( x_i \) and \( y_i \), respectively) into the fraction of the votes that candidate A obtains in region \( i \). Similarly define \( s_i^B : \mathbb{R} \times \mathbb{R} \to [0, 1] \) as the fraction of votes obtained by candidate B in region \( i \). We allow the possibility for abstention to happen, therefore, \( 0 \leq s_i^A + s_i^B \leq 1 \). We will first analyze a deterministic model of the problem, and then present a stochastic version.

3.2. Deterministic Game

In this case, the vote outcome of all regions is determined by the vectors of efforts, \( x \) and \( y \), of the candidates A and B, respectively. For this setting, we will define the outcome function \( s_i^A \) and \( s_i^B \) for each region \( i \) as

\[
\begin{align*}
  s_i^A(x_i, y_i) &= \frac{x_i + \alpha_i}{x_i + \alpha_i + y_i + \beta_i + \gamma_i}, \\
  s_i^B(x_i, y_i) &= \frac{y_i + \beta_i}{x_i + \alpha_i + y_i + \beta_i + \gamma_i},
\end{align*}
\]

where \( \alpha_i, \beta_i > 0 \) are the bias parameters toward candidates A and B, respectively, and \( \gamma_i \geq 0 \) is the abstention parameter for region \( i \). Bias parameters represent the intrinsic bias of the region toward a particular candidate. If \( \alpha_i > \beta_i \), then people from region \( i \) are leaning toward candidate A since for the same levels of efforts, that is, \( x_i = y_i \), candidate A gets more votes from the region than her contender. Vice versa if \( \alpha_i < \beta_i \). Also, note that high values of the bias parameters \( \alpha_i, \beta_i \) mean that the result of region \( i \) is less sensitive with respect to the level of efforts \( x_i \) and \( y_i \), and therefore voters’ preferences are highly polarized to change their votes given the candidates’ campaigns.
On the contrary, low levels of $\alpha_i, \beta_i$ imply that the outcome of the people’s votes is more sensitive to the candidates’ efforts. The abstention parameter, $\gamma_i$ for region $i$, is such that there is no abstention if $\gamma_i = 0$. Otherwise abstention increases monotonically with the parameter. Note that the abstention can be seen as a “third” candidate option that does not campaign and has a bias parameter equal to $\gamma_i$. Then, for a given region $i$, the total number of votes candidate $A (B)$ receives is $\sum_{x \in A} s_i^A(x_i, y_i) \left( \frac{v_i s_i^B(x_i, y_i)}{v_i s_i^A(x_i, y_i)} \right)$; and the total number of votes candidate $A (B)$ receives is $\sum_{x \in A} s_i^A(x_i, y_i) \left( \sum_{y \in B} s_i^B(x_i, y_i) \right)$. All parameters are public information.

The objective of each player is to win the election. However, this can result in an infinite number of equilibria. Moreover, we can argue that some of these equilibria are more preferable than others. For instance, if candidate $A$ wins the election on one equilibrium with 51% of the votes (between both candidates), whereas on another equilibrium wins with 68% (between both candidates), there is no doubt that the second scenario is preferred by candidate $A$ (and especially the political parties behind the candidate). Then, as for the deterministic game, the objective of each candidate will be to maximize the number of votes obtained with respect to the total number of votes obtained between the two candidates. Note that an equilibrium—the formal definition of this will be given shortly—of this game is also an equilibrium of the game in which candidates aim to win the election regardless of the difference.

The optimization problem candidates $A$ and $B$ solve are written as follows:

$$\max_{x \in \Delta_a} Q^A(x, y) := \frac{\sum_{i \in A} s_i^A(x_i, y_i)}{\sum_{i \in A} s_i^A(x_i, y_i) + s_i^B(x_i, y_i)}$$  \hspace{1cm} (1)

$$\max_{y \in \Delta_b} Q^B(x, y) := \frac{\sum_{i \in B} s_i^B(x_i, y_i)}{\sum_{i \in B} s_i^A(x_i, y_i) + s_i^B(x_i, y_i)}$$  \hspace{1cm} (2)

The numerator of the candidates’ objective function in (1) and (2) has the total number of votes they get, while the denominator has the total number of votes obtained by both. This is clearly a zero-sum game since an increase in the percentage of votes of one candidate results in its loss from the opponent.

**Definition 1.** An equilibrium is a pair of effort vectors $(x^*, y^*) \in \Delta_a \times \Delta_b$ such that each vector $x^*$ and $y^*$ is the optimal solution of the respective candidate’s maximization problems given in Expressions (1) and (2).

The following theorem states the existence of equilibrium of the game presented.

**Theorem 1.** There exists an equilibrium (in pure strategies) for the deterministic game.

**Proof.** See Appendix A.

Then next theorem states the uniqueness of the equilibrium.

**Theorem 2. The equilibrium of the deterministic game is unique.**

**Proof.** See Appendix B.

Unfortunately, there is no closed form solution for the equilibrium of the game. Before showing a method to compute this, we will present a proposition that states a closed form solution for the equilibrium of an unbounded version of the game. More precisely, consider the same election game as described above except that the candidates’ efforts are allowed to take negative values (these efforts must still add up to one). Furthermore, consider that the efforts of both candidates are constrained to a particular subset of regions $I' \subseteq I$. The latter subset represents the regions on which the candidates will focus their attention, whereas regions outside this set will have null investment. The resulting game defined with the given characteristics will be called an unbounded game constrained on the set of regions $I'$. Existence and uniqueness of the equilibrium of this game can be shown using similar arguments to the ones used for the original game. The importance of the unbounded constrained game is twofold: (i) it allows us to find the equilibrium for the original game in closed form, and (ii) it enables performing sensitivity analysis on the equilibrium strategies with respect to the model parameters when using $I'$ as the set of states with positive investment in the original game equilibrium. The next proposition presents a closed form for its equilibrium.

**Proposition 1.** For any nonempty set $I' \subseteq I$, the equilibrium of the unbounded game constrained in the set of regions $I'$ is given by

$$x_i^{UB}(I') = \frac{\nu_i}{\nu_i} \left( 1 + \alpha_i \right) + \frac{Q^A(i, y_i)}{Q^A(i, y_i) + Q^B(i, y_i)} - \frac{Q^A(i, y_i)}{Q^A(i, y_i) + Q^B(i, y_i)} \gamma_i - \alpha_i$$  \hspace{1cm} (3)

$$y_i^{UB}(I') = \frac{\nu_i}{\nu_i} \left( 1 + \beta_i \right) + \frac{Q^B(i, y_i)}{Q^A(i, y_i) + Q^B(i, y_i)} - \frac{Q^B(i, y_i)}{Q^A(i, y_i) + Q^B(i, y_i)} \gamma_i - \beta_i$$  \hspace{1cm} (4)

for all $i \in I'$, where $\alpha_i := \sum_{j \in I'} x_j$ and similarly with $\beta_i$, $\gamma_i$ and $\nu_i$. Candidates’ votes can be computed as $Q^A = \frac{\nu_i \gamma_i (1+\alpha_i)}{\sum_{j \in I'} x_j + \beta_i + \gamma_i} + \nu_i \gamma_i (1+\alpha_i)$, and $Q^B = \frac{\nu_i \gamma_i (1+\beta_i)}{\sum_{j \in I'} x_j + \beta_i + \gamma_i} + \nu_i \gamma_i (1+\beta_i)$.
COROLLARY 1. If ηi = 0 for i ∈ I∗, the equilibrium of the unbounded game constraint to I∗ is such that the fraction of votes obtained by candidate A in each region in I∗ is the same. Specifically:

$$\frac{x_i^{UB(I^*)} + \alpha_i}{x_i^{UB(I^*)} + \alpha_i + y_i^{UB(I^*)} + \beta_i} = \frac{1 + \alpha_{I^*}}{2 + \alpha_{I^*} + \beta_{I^*}},$$

$$\frac{y_i^{UB(I^*)} + \beta_i}{x_i^{UB(I^*)} + \alpha_i + y_i^{UB(I^*)} + \beta_i} = \frac{1 + \beta_{I^*}}{2 + \alpha_{I^*} + \beta_{I^*}}.$$ (5)

Proof. See Appendix C.

From Proposition 1, we can obtain the equilibrium of the unbounded version of the game when fixing the set of regions where candidates can put their efforts. Note that the obtained equilibrium might have negative components, in which case it cannot be the equilibrium of the original game. Even if the unbounded equilibrium quantities are such that the unbounded version of the game can be interpreted as negative, this might not be the equilibrium of the original game. However, if I∗ matches the set of regions where candidates can put their efforts among the different regions, in which case it cannot be the equilibrium of the original game. Even if the unbounded version of the game can be interpreted as negative, this might not be the equilibrium of the original game. However, if I∗ matches the set of regions where candidates can put their efforts among the different regions, in which case it cannot be the equilibrium of the original game. However, if I∗ matches the set of regions where candidates can put their efforts among the different regions, in which case it cannot be the equilibrium of the original game.

THEOREM 3. The equilibrium of the game with non-negativity constraints, (x∗, y∗) is obtained as the limit of (x∗, y∗) when t → ∞.

Proof. See Appendix E.

Under a compulsory voting system, if I∗ matches the set of regions where the candidates’ efforts are positive in the constraint game equilibrium (i.e., the original game), the result of Corollary 1 will hold. As a result, the fraction of votes each candidate obtains in each region of the set I∗ will be the same.

In order to compute the equilibrium of the original game, with the non-negativity constraints, let us consider the parameterized zero-sum game on t > 0 with payoff Q^θ(t, x, y) := tQ^θ(x, y) + ∑i ln(x(i)) - ∑i ln(y(i)), and denote its equilibrium as (x∗, y∗). The proof of existence and uniqueness of this equilibrium is analogous to that of the original game shown in Theorems 1 and 2.

The next theorem states how to use this parameterized game to find the equilibrium of the original game.

Theorem 3. The equilibrium of the game with non-negativity constraints, (x∗, y∗) is obtained as the limit of (x∗, y∗) when t → ∞.

Proof. See Appendix F.

The idea of Theorem 3 is to exploit the zero-sum concave-convex structure by using the infeasible start Newton’s method for any given value of t > 0. Then, we iterate by increasing the value of t to determine the equilibrium of the original game. More details are provided in Appendix F.

Up to this point, we have assumed that for a given vector of (i) efforts, x and y, (ii) bias parameters, α and β, and (iii) abstention parameter, γ; the outcome of the election is perfectly known and can be computed exactly for every single region, and thus for the whole country. The latter is probably a strong assumption, since despite the amount of information we have on a particular region, we will probably not predict the result with 100% accuracy. Therefore, in the next section we introduce a stochastic model that accounts for uncertainty in the vote outcomes.

3.3. Stochastic Game

For each region i, let S^A_i, S^B_i, and S^C_i be the random variable of the fraction of votes received by candidate A, candidate B, and the abstained votes, respectively, such that

$$\text{Dir}_m(k(x_i + \alpha_i), k(y_i + \beta_i), k\gamma_i),$$ (6)

where Dir_m is an m-dimensional Dirichlet distribution (in this case, three-dimensional).
Note that the expectation of $S^A_k$ and $S^B_i$ matches with the values of the analogous parameters ($s^A_i$ and $s^B_i$) in the deterministic model. Indeed, $E(S^A_k) = \frac{(x_k + a_k)(y_i + b_i)}{x_k + a_k + y_i + b_i + r}$, $E(S^B_i) = \frac{(x_k + a_k)(y_i + b_i + r)}{x_k + a_k + y_i + b_i + r}$. The parameter $k > 0$ regulates for the noise of the vote outcomes, so that higher values of $k$ represent settings with lower variability (and vice versa). In fact, the variance for the fraction of votes for candidate $A$ is $\text{Var}(S^A_k) = \frac{(x_k + a_k)(y_i + b_i)(1+k(x_k + a_k))}{(x_k + a_k + y_i + b_i + r)^2} - \left(\frac{x_k + a_k + y_i + b_i + r}{x_k + a_k + y_i + b_i + r}\right)^2$.

Therefore, the uncertainty parameter $k$ plays a crucial role in measuring whether the model behaves more as a deterministic or as a stochastic one. Also, the combination of $k$ and the bias parameters represent the sensitivity of candidates’ campaigns. For example, consider the two game settings on a state $i$: (a) $(k, a) = (10, 1)$ and (b) $(k, a) = (1, 10)$. If $x_k = 0$, we obtain in both settings the same voting outcome, however, this happens to be more sensitive to candidate $A$’ investment when $x_k > 0$ in (a).

In this case, where the vote outcomes are stochastic, candidates will exert their efforts in order to maximize the chances of getting elected, instead of maximizing the expected number of votes. Note that if candidates were to maximize the expected number of votes, it would result in a game that is equivalent to the one introduced in the deterministic section if there was no abstention. Let $R^A$ be the number of votes obtained by candidate $A$, and similarly for $R^B$. The probability that candidate $A$ wins the election can be computed as:

$$
\mathbb{P}(R^A > R^B) = \int_{\Delta_1} \cdots \int_{\Delta_1} \prod_{i \in I} f_i(s^A_i, s^B_i) ds^A_i ds^B_i.
$$

Note that we are assuming that the results are independent among the states, $\text{corr}(S^A_i, S^A_j) = \text{corr}(S^B_i, S^B_j) = \text{corr}(S^B_i, S^A_j) = 0$, for all $i \neq j$. Then, the optimization problem of candidates $A$ and $B$ can be written as:

$$
\max_{x \in \Delta_n} \mathbb{P}(R^A > R^B),
$$

$$
\max_{y \in \Delta_n} \mathbb{P}(R^B > R^A).
$$

**Definition 2.** An equilibrium for the stochastic game is a pair of effort vectors $(x^*, y^*) \in \Delta_n \times \Delta_n$ such that each vector $x^*$ and $y^*$ is the optimal solution of the respective candidate maximization problems given in Expressions (8) and (9).

Unlike the deterministic version of the game, in the stochastic version we have no guarantee of the existence of the equilibrium in pure strategies. However, we can state the existence of equilibrium in mixed strategies.

**Theorem 4.** There exists an equilibrium in mixed strategies for the stochastic game.

**Proof.** See Appendix G.

After several numerical computations, we observed that the objective function always happens to be quasi concave, which suggests that there might always exists an equilibrium in pure strategies. Therefore, we proceed to find an equilibrium in pure strategies as described in the following subsection. Our numerical analysis suggests that there is a (unique) equilibrium in pure strategies. More details are provided below.

### 3.3.1. Computing Equilibrium

To compute the equilibrium of the stochastic game, we use a gradient descent ascent method. Namely, we move the strategies of both players simultaneously according to their payoffs at the current solution and repeat until we reach a pair of strategies $(x, y)$ such that no player has an incentive to deviate. More specifically, we take a step $\rho > 0$ in the chosen direction to update the new solution as $(x, y) \leftarrow (x + \rho d^A, y + \rho d^B)$, where $d^A$ and $d^B$ are the directions of maximum and minimum growth of $\mathbb{P}(R^A > R^B | x, y)$ within $\Delta_n$. In order to compute these directions, the following proposition is introduced:

**Proposition 2.** The direction of maximum growth of a given function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ within the simplex, $\Delta_n$, is given by the vector $d_i$ defined as $d_i = x_i \left( \frac{\partial g}{\partial x_i} - \sum_{j \neq i} \frac{\partial g}{\partial x_j} \right)$.

**Proof.** See Appendix H.

It can be seen that the values $r^A_i$ introduced in Proposition 2 actually correspond to the complementary slackness values for the Karush–Kuhn–Tucker conditions. Similarly, we define $r^B_i$. Then, these values are used in the stopping criteria.

The steps of the algorithm are described below in Algorithm 1.

The starting point is set to be the equilibrium of the deterministic version of the game. Although the game is not convex-concave, the algorithm in practice always happens to converge at a stationary point. In addition, this point is likely to be the game equilibrium in pure strategies. Indeed, we performed an empirical analysis over several random generated instances in order to check whether or not the obtained solution obtained from Algorithm 1 corresponds to a Nash equilibrium. We found that in practically all cases there were no unilateral deviations in
Algorithm 1 Equilibrium for Stochastic Game

1: Input $\alpha, \beta, \gamma, v \in \mathbb{R}_+^n$
2: Set $(x, y) = (x^d, y^d), \tau^A = \tau^B = \{\infty\}_{i=1}^n$
3: While $\max\{\|\tau^A\|, \|\tau^B\|\} > \epsilon$ do
4:   $x = x + \rho d^A, y = y - \rho d^B$
5:   Update $\tau^A, \tau^B$
6: End While
7: Return $(x, y)$

which candidates improved their winning odds with respect to the strategies obtained from Algorithm 1. More details are provided in Appendix I. We ran Algorithm 1 for different starting points, and observed that in most cases it converged to the same equilibrium, while for some starting points it converged to a non-equilibrium point. Thus, we conjecture that the game has a unique equilibrium.

The calculation of $\mathbb{P}(R^A > R^B)$ and its derivatives requires computing several composed integrals, see the expressions of these terms in Appendix J. We need to compute $3n$-dimension integrals for the terms mentioned, which is not possible in practice, not even for low values of $n$. Therefore, we use Monte Carlo simulation to approximate the value of these integrals.

### 3.3.2. Boosting

The need to compute several integrals by Monte Carlo simulation implies that at each pair of strategies $(x, y)$ we evaluate when running Algorithm 1, we need to sample multiple Dirichlet random variables; one for every region and simulation. The next proposition states a result that helps us to re-use the simulations of the Dirichlet random variables. For ease of notation, let us denote $S := \{(S^A_i, S^B_i, S^C_i)\}_{i=1}^n$.

**Proposition 3.** Let $g : (\Delta_i)^n \rightarrow \mathbb{R}$ be a scalar function. Then it holds that

$$E(g(S)|x + \Delta x, y + \Delta y) = E\left(\sum_{i \in I} \prod_{j} (S^A_j)^{\lambda x_i} (S^B_j)^{\lambda y_i}|x, y\right),$$

where $K = \prod_{i \in I} B(\lambda (x_i + \alpha_i), k(y_i + \beta_i), k_i)$ and $B(\cdot, \cdot)$ is the multivariate Beta function.

**Proof.** See Appendix K.

The intuition behind the proof of Proposition 3 relies on expressing the pdf of the Dirichlet distribution of points as the pdf of a similar distribution at a neighboring point. This allows us to reuse the simulations of the sampled Dirichlet distribution at a given point $x, y$ in other points $x + \Delta x, y + \Delta y$. In particular, we are interested in using Proposition 3 with the function $g(S) = 1\{\sum_{i \in I} S^A_i > \sum_{i \in I} S^B_i\}$, which takes value one in case the election is won by candidate $A$, and zero otherwise. As a result, we can sample the Dirichlet random variables once at a particular pair $(x, y)$, and compute an (unbiased) estimate of the probability that candidate $A$ wins at any other point $(x + \Delta x, y + \Delta y)$. However, it is worth noticing that the variance of the sample random variable in the RHS of Equation (10) might increase as we move further away. (See details in Appendix L).

### 4. Electoral College

In this section, we study the EC system. We first explore an equilibrium in pure strategies, and then provide an algorithm that allows computing an equilibrium using mixed strategies. Then, we compare some equivalence results of both electoral systems, the MS and the EC.

Under the EC, the candidates get all the electoral votes of the states where they have the majority of votes with respect to their contenders. Let $w_i \in \mathbb{Z}_+$ be the number of electoral votes of state $i \in I$. Since the electoral college is used in the United States, we prefer to denote regions as states. As in Section 3.3, we will assume that for each state $i \in I$, the fraction of votes at each state $i$ received by candidates $A$ and $B$, and the fraction of abstention votes, $(S^A_i, S^B_i, S^C_i)$, follows a Dirichlet distribution as in Equation (6). One of the consequences of using the Dirichlet distribution in the EC system is the independence between the abstention level, and the fraction of votes obtained by a candidate relative to the actually casted votes. Since, for any state, regardless of the turnout, someone is going to win all the electoral votes, the independence between
It is interesting to note that there are some equivalences between the games under the MS and EC for certain cases. The following theorems state two of these equivalences:

**THEOREM 5.** If the number of electoral votes is proportional to the number of voters, then the two following games are the same:

1. MS where candidates maximize the expected number of votes, with no abstention (\( y = 0 \)).
2. EC where candidates maximize the expected number of electoral votes.

**Proof.** See Appendix O.

The intuition behind this is that the utility of both players match in the two settings described in Theorem 5. Although it is more natural that candidates maximize the probability of winning rather than the number of electoral votes obtained in the EC; in reality, a political party with almost no odds of winning might prefer the latter objective as a damage control strategy at the expense of the few chances of winning.

**THEOREM 6.** If the number of electoral votes is proportional to the number of voters, then the two following games are equivalent in the limit where \( k \to 0 \), in the sense that players’ utilities in both cases converge in probability:

1. MS where candidates maximize the probability of winning, with no abstention (\( y = 0 \)).
2. EC where candidates maximize the probability of winning.

**Proof.** See Appendix P.

Since we are modeling \( S_i^A / (S_i^A + S_i^B) \) with a beta distribution with parameters proportional to \( k \), taking \( k \to 0 \) creates a U-shaped pdf, in which the density concentrates on the extreme cases when all the votes go to either candidate \( A \), or \( B \). In order for that to happen, voters would need to coordinate their balloting, deciding the winner before the election day. Such a scenario is obviously very unlikely to happen.

### 4.2. Computing Equilibrium in Pure Strategies

We apply a gradient descent ascent method like the one used in Section 3.3.1 but now with \( P(G) \) as the objective function of the zero-sum game. In this case, the derivative of the payoff function can be written as:

\[
\frac{\partial}{\partial x_i} P(G) = \sum_i \frac{\partial}{\partial x_i} P(G) \frac{\partial p_i}{\partial x_i} = \frac{\partial}{\partial x_i} P(G) \frac{\partial p_i}{\partial x_i}
\]

Since \( \frac{\partial p_i}{\partial x_i} \) is the derivative of the complementary cdf of the beta distribution on \( x_i \). For \( \frac{\partial}{\partial x_i} P(G) \), let \( G_i \) be the event that candidate \( A \) wins the electoral votes of state \( i \), and \( G_i^c \) be the complement of this event. Due to the Law of

\[\begin{align*}
\text{Lemma 1. If } (X, Y, Z) &\sim \text{Dir}(a, b, c), \text{ then the relative value of } X \text{ with respect to } X + Y \text{ is independent of } Z, \\
\text{Cov}(X, Z) &\sim \text{Beta}(a, b, c), \text{ which dis-}
\end{align*}\]
Total Probability, taking derivative with respect to $p_i$ results in $\frac{d}{dp_i}P(G) = P(G|G_i) - P(G|G_i')$. $P(G|G_i)$ and $P(G|G_i')$ can be computed using the same recurrence as the one introduced in Equation (34) but fixing the outcome of the $i^{th}$ state to winning ($p_i = 1$) or losing ($p_i = 0$) when the conditional event is $G_i$ or $G_i'$, respectively.

With this, we can use the same procedure described in Section 3.3.1, in particular the use of Proposition 2, and Algorithm 1. We find from numerical computations that the gradient descent ascent method converges to a point which, at least numerically, appears to be either an equilibrium, or a local Nash equilibrium. In particular, the parameter $k$ (which controls for variability) seems to have a key role in this. Low values of $k$ lead to cases with existence of equilibrium, whereas high values of $k$ tend to end up in a local Nash equilibrium. Intuitively, the latter case resembles a deterministic version of the game, where pure equilibrium does not seem plausible since in the extreme case (of $k \to \infty$), payoff functions are not even continuous.

4.3. Equilibrium in Mixed Strategies
As a result of the lack of equilibrium in pure strategies, we explore equilibrium in mixed strategies which indeed do exist.

Theorem 7. There exists an equilibrium in mixed strategies for the stochastic game under EC.

Proof. The proof follows the same arguments given in Appendix G.

Unfortunately, the search for a mixed equilibrium of the game is not a simple task. Furthermore, it might result in complicated strategies which might not be practical for the agents involved. As a result, we decide to look for mixed equilibria of the game in a finite subset of strategies. Note that since this is a zero-sum game, if the subset of strategies is finite, we can obtain a mixed equilibrium by simply solving an LP.

4.3.1. Simplex Lattice. Consider the $(n, q)$-simplex lattice (as introduced in Schefﬁe 1958) as the set of points $D^q(\Delta_n) := \{x \in \Delta_n|x_i \in \mathbb{Z}_+\}$ where $q \in \mathbb{Z}_+$. The latter set represents a discretization of the simplex, (note also that we are considering just strategies where $\sum_i x_i = 1$), in which the parameter $q$ controls for the reﬁnement of the grid so that higher values of this parameter result in a more reﬁned set, see Figure 1. Intuitively, the $(n, q)$-simplex lattice represents the players’ strategies, so that each player is endorsed with $q$ indistinguishable balls which they have to invest among the $n$ states. The number of elements in the $(n, q)$-simplex lattice is $\binom{n+q-1}{n-1}$, which unfortunately is exponential in the number of states and on the parameter $q$. However, we can think of a way to consider only a subset of strategies in $D^q(\Delta_n)$.

4.3.2. Iteration Procedure. In order to ﬁnd an equilibrium in mixed strategies, we consider the following iteration procedure:

- **Step 1:** Consider a starting strategy sets $T^A$, $T^B \subseteq D^q(\Delta_n)$ for each player. We start with a small set of strategies for both players. For example, these can be the $n$ canonical vectors $e_i$.

- **Step 2:** Let us consider an equilibrium in mixed strategies for a ﬁnite set of strategies as follows:

Definition 4. An equilibrium in mixed strategies under EC for discrete sets of strategies $T^A$ and $T^B$ is a pair of vectors $(\sigma_A^*, \sigma_B^*) \in \Delta_{|T^A|} \times \Delta_{|T^B|}$ such that $\sigma_A^* \in \arg \max_{\sigma_A} \mathbb{E}_{\sigma_A}[P(G)]$ and $\sigma_B^* \in \arg \max_{\sigma_B} \mathbb{E}_{\sigma_B}[P(G)]$.

We proceed to ﬁnd a mixed strategies equilibrium in $T^A$, $T^B$. As mentioned before, due to the zero-sum game structure, this equilibrium can be
found by solving an LP. (See details in Appendix Q).

**Step 3:** We then proceed to find each player’s best response in pure strategies in $\Delta_n$ for the given mixed strategy her opponent found in Step 2. In order to do this, we use a gradient descent method for each player considering the expectation of the objective derivative according to the contender’s mixture probabilities. Thus, candidate A considers the expectation with respect to $\sigma_B$, whereas candidate B does it with respect to $\sigma_A$.

**Step 4:** Consider $x \in \Delta_n$ as the best response of candidate A (wlog). We proceed to find the points in the simplex lattice $D^q(\Delta_n)$ that contain $x$ inside its convex hull, while at the same time, being as small as possible. The details of the algorithm, and of its properties, that performs the latter are provided in Appendix R. Let us denote by $Z(x^{BR})$ and $Z(y^{BR})$ the points obtained from the simplex lattice that contain the best response of players A and B, respectively. If all such points were already part of the sets $T^A$, $T^B$, then the algorithms finishes, and the equilibrium found is the last output of Step 2. Otherwise, we put the new points inside the respective sets, update the payoff matrix, and continue iterating again from Step 2.

This procedure will find an equilibrium in mixed strategies such that the set of possible strategies considered are constrained to those that have been explored within the lattice. Thus, finding the equilibrium over the entire simplex is not guaranteed. In summary, the complete algorithm for finding a mixed equilibrium on subsets of the simplex lattice as strategy sets is given below.

**Algorithm 2** Algorithm for mixed equilibrium in subsets of the simplex lattices

1. **Input** $\alpha, \beta \in \mathbb{R}^n_+, k \in \mathbb{R}_+, w \in \mathbb{Z}^n_+, q \in \mathbb{Z}_+$
2. **Set** $T^A, T^B = \{e_i \in \mathbb{R}^n, i \in I\}$, $P_{ij} = \mathbb{P}(G|x(i), y(j)) \forall (x(i), y(j)) \in T^A \times T^B$
3. **While** True do:
   4. $(\sigma_A, \sigma_B) = \text{solve}(P)$
   5. $x^{BR} = \arg\max_{x \in \Delta_n} \mathbb{P}(G|x, \sigma_B)$, $y^{BR} = \arg\min_{y \in \Delta_n} \mathbb{P}(G|\sigma_A, y)$
   6. **If** $Z(x^{BR}) \subseteq T^A$ and $Z(y^{BR}) \subseteq T^B$ : **Break**
   7. $T^A = T^A \cup Z(x^{BR})$, $T^B = T^A \cup Z(y^{BR})$
   8. **Update** $P$
4. **End While**
10. **Return** $(T^A, \sigma_A, T^B, \sigma_B)$

In line 2 of Algorithm 2, the strategy sets are initialized with the canonical vectors, and the payoff matrix, $P$, is computed under the pairs of these strategies. Line 4 computes the mixed equilibrium by solving Optimization problems (40) and (39) in Appendix Q. Line 5 computes each candidate best response on the continuous space $\Delta_n$. In line 6, we compute the discretized vectors in the simplex lattice for the players best responses; if these sets, $Z(x^{BR})$ and $Z(y^{BR})$, are already contained in the respective players strategy sets ($T^A$ and $T^B$), the algorithm finishes. If this is not the case, the new discretized strategies are added to the player’s strategy sets in line 7. Finally, in line 8, the payoff matrix is updated to include the payoffs for the pairs of players’ strategies that involve new strategies. Since $D^q(\Delta_n)$ is finite for every $q \in \mathbb{Z}_+$, Algorithm 2 should finish at a finite iteration. The following theorem states the convergence result of Algorithm 2.

**Theorem 8.** Algorithm 2 converges in a finite number of iterations.

**Proof.** See Appendix S.

### 5. Numerical Results

In this section, we show numerical computations of the game equilibria under the different settings introduced under the majority and electoral college voting systems. Then we analyze the effect of polarization and uncertainty for both election systems. Finally, we calibrate parameters for using our algorithms on real data from the US presidential election of 2020.
5.1. Majority System

5.1.1. Deterministic Case. We focus our analyses on cases where strategies are non-negative. Numerical results of the unbounded game (i.e., without non-negativity constraints) are shown in Appendix T. Table 1 shows the equilibrium for the deterministic game under MS in an instance composed of 10 regions. The region’s vote-share, biases, and abstention parameters are given in columns 2, 3-4, and 5, respectively. We observe that only the first three regions, which are the ones with the largest vote-share, are chosen to invest in by both candidates. Adding up the results from all regions, we see that candidate \( A \), candidate \( B \), and the abstentions are 30.0%, 28.9%, and 41.1%, respectively. Thus, candidate \( A \) wins by obtaining 50.9% of the votes in the election between \( A \) and \( B \). Among the regions that candidates invest in, it can be seen from Table 1 that they allocate most of their resources into regions in which the bias is leaning toward their contender. For example, candidate \( A \) has a less favorable bias in region 1 compared to candidate \( B \) (i.e., \( \alpha_1 < \beta_1 \)); consequently, in equilibrium candidate \( A \) ends up investing 68.3% of its resources into this region (vs. 36.4% for candidate \( B \)). As a result, the number of votes from the turnover in region 1 that candidate \( A \) ends up with is slightly more than half. (See last column of Table 1). As for the second region, the biases and candidate investments are in the opposite direction compared to the first region. Also, note that the two first regions end up getting the highest turnout nationwide. This is explained by the high number of votes there, which induce candidates’ efforts to be focused on them.

The equilibrium shown on Table 1 is computed using procedure described at the end of Section 3.2. Nonetheless, since we know \textit{ex post} that candidates focus exclusively on the first three regions, the equilibrium could have been computed using the closed form expression from Proposition 1 with \( T^* = \{1, 2, 3\} \).

5.1.2. Stochastic Case. We now proceed to solve the MS game under a stochastic model. Table 2 shows the results with \( k = 10 \). It is interesting to observe that the candidates’ efforts are similar to those in the deterministic case (see Table 1). As a result, the previously bias disadvantage effect in which candidates invest more in regions with a smaller bias parameter relative to the contender also holds. The probability that candidate \( A \) wins is 57.4%.

The equilibrium quantities given on Table 2 are obtained by running Algorithm 1. Recall that we do not have a formal proof of existence and uniqueness of the equilibrium for the stochastic game. Because of the latter, an empirical analysis is performed to test whether or not the strategies obtained are indeed an equilibrium.

Figure 2 shows the payoff ratio for different unilateral deviations for each player. Thus, if the strategies at the founded equilibrium are \((x,y)\), then the ratio for player \( A \) at a strategy \( x' \in \Delta_n \) is computed as \( \frac{P[R_1 > R'_{1}(x',y)]}{P[R_1 > R_{1}(x,y)]} \). We performed this calculation for every strategy in the simplex lattice, as introduced in Section 4.3, and similarly for player \( B \). Results are shown in Figure 2, where the y-axis corresponds to the ratios for unilateral deviations of both players, while the x-axis represents the Euclidean distance between the equilibrium point and the respective unilateral deviated strategy. Note that every unilateral deviation computed resulted in a ratio below 1. Therefore, it seems that neither player has an incentive to switch its strategies, at least from the tests unilateral deviations, which empirically suggests that the strategies are, in fact,

\begin{table}[h]
\centering
\begin{tabular}{cccccccccc}
\hline
Region & \( v \) & \( \alpha \) & \( \beta \) & \( \gamma \) & \( x \) & \( y \) & Turnout & VFT \( A \) \\
\hline
1 & 23.3 & 45 & 71 & 94 & 68.3 & 36.4 & 70.1 & 51.3 \\
2 & 18.5 & 68 & 37 & 67 & 25.8 & 52.1 & 73.2 & 51.3 \\
3 & 14.4 & 32 & 24 & 121 & 5.9 & 11.5 & 37.8 & 51.7 \\
4 & 8.9 & 43 & 39 & 89 & 0.0 & 0.0 & 48.0 & 52.4 \\
5 & 8.2 & 76 & 65 & 92 & 0.0 & 0.0 & 60.5 & 53.9 \\
6 & 8.2 & 36 & 61 & 143 & 0.0 & 0.0 & 40.4 & 37.1 \\
7 & 6.8 & 51 & 54 & 45 & 0.0 & 0.0 & 70.0 & 48.6 \\
8 & 6.2 & 42 & 41 & 79 & 0.0 & 0.0 & 51.2 & 50.6 \\
9 & 3.4 & 85 & 31 & 102 & 0.0 & 0.0 & 53.2 & 73.3 \\
10 & 2.1 & 37 & 69 & 68 & 0.0 & 0.0 & 60.9 & 34.9 \\
\hline
\end{tabular}
\end{table}
a Nash equilibrium in pure strategies for the given instance.

5.2. Electoral College
For the EC case, we consider the same instance as in the MS, with the same bias and abstention parameters, except that the states have electoral votes. With the aim of obtaining an equilibrium in fixed strategies, if there is any, we apply a gradient descent ascent method as described in Section 3.3.1 using the equations for the derivative values described in Section 4.2. Despite obtaining a pair of strategies for both candidates when doing the latter procedure, this pair of strategies is not actually an equilibrium. Figure 3 shows the payoff ratio for both candidates for unilateral deviations from the pair of strategies obtained. We can see that both candidates have an incentive to change their strategies to different ones. Nonetheless, it seems that, at least locally, there is no such incentive. Thus, the pair of strategies obtained might be a local Nash equilibrium.

Consequently, we run Algorithm 2 in order to find an equilibrium under mixed strategies. An analysis of the performance of Algorithm 2 in terms of solving time and number of iterations is provided in Appendix U. In addition, we performed a sensitivity analysis of Algorithm 2 with respect to the grid refinement $q$ (see Appendix V), which shows that as $q$ increases the equilibrium seems to converge. It is worth mentioning that Algorithm 2 uses a particular set of starting points for the strategies of both players. As a result, the equilibrium outcome might differ when using different starting strategy sets. We tested the latter on the instance shown on Table 3, as well as in other sampled instances, and conclude that the output strategies do not differ significantly when using different starting strategy sets. More details are provided in Appendix W.

Table 3 shows the instance parameters and the equilibrium obtained after running Algorithm 2. The vector efforts shown are the strategies obtained with positive probability. These probabilities are given in the last row of Table 3. It can be seen that (i) both candidates randomized their strategies, and (ii) their efforts are mostly invested in the first three states, which are the ones with more electoral votes. Candidate $A$’s equilibrium strategies are very similar to each other, focusing most efforts in the first state, and less on the second and third states. As for candidate $B$’s efforts in equilibrium, these more evenly distributed between the three first states compared to candidate $A$, with emphasis on the first two regions. Also, it can be seen that overall, the expected probability that candidate $A$ wins the election is 55.2%.

5.3. Comparing Systems: MS and the EC

5.3.1. The Consequences of Polarization. In all the examples analyzed so far, we have fixed the values of the bias parameters, and so their magnitude relative to the candidates’ budget. Nevertheless, it is not clear how big the effect of campaigning is relative to the effect of existing biases. The term polarization is used to characterize the case when voters’ position is inelastic with respect to candidates’ campaign. The latter occurs when existing biases are large compared
to candidates’ budget. In this section, we analyze the effect of polarization on candidates’ equilibrium strategies. More precisely, the same instances given on Tables 1 and 3 are solved while scaling the bias parameters $\alpha$ and $\beta$ for different factors. Let $f > 0$ denote the value of this factor so that the new bias parameters are $(f\alpha, f\beta)$. Since the candidates’ budgets remain fixed, different factors will represent different levels of power of campaigns. On the one hand, $f \rightarrow 0$ represents a low polarization case since there are virtually no biases, and therefore the voters’ decisions are triggered mostly by the candidates’ campaigns. On the other hand, in the case when $f \rightarrow \infty$, the effect of campaigning becomes negligible, except for those states in which the difference between the candidates’ bias parameters is still within the reach of what the campaign can affect. Results for the MS are shown on Table 4.

We can see that for low levels of biases ($f = 0.1$), candidates’ efforts on average are directly related to the weight of the region size (in terms of number of votes). In addition, all states get some level of investment (except for states 9 and 10). The intuition behind the latter can be easily observed in the extreme case where $f \rightarrow 0$ (i.e., there are almost no previous biases). If a candidate ignores a region, it takes the opponent just any positive effort to win most of its votes.

### Table 3

$x(i)$ and $y(i)$ with $i \in \{1, 2, 3, 4\}$ correspond to the equilibrium effort vectors obtained with positive probability after running Algorithm 1 with $q = 100$. The last row shows the probability for each strategy of being chosen.

| Region | $w$ | $\alpha$ | $\beta$ | $x(1)$ | $x(2)$ | $x(3)$ | $x(4)$ | $y(1)$ | $y(2)$ | $y(3)$ | $y(4)$ | Probability [%] |
|--------|-----|----------|----------|--------|--------|--------|--------|--------|--------|--------|--------|---------------|
| 1      | 34  | 45       | 71       | 76     | 75     | 75     | 64     | 52     |        |        |        | 11.8          |
| 2      | 27  | 68       | 37       | 8      | 9      | 8      | 47     | 48     | 61     | 61     |        | 1.1           |
| 3      | 21  | 32       | 24       | 16     | 16     | 17     | 46     | 36     | 39     | 38     |        | 82.9          |
| 4      | 13  | 43       | 39       | 7      |        |        |        |        |        |        |        | 4.2           |
| 5      | 12  | 76       | 65       |        |        |        |        |        |        |        |        | 28.4          |
| 6      | 12  | 36       | 61       |        |        |        |        |        |        |        |        | 35.9          |
| 7      | 10  | 51       | 54       |        |        |        |        |        |        |        |        | 24.0          |
| 8      | 9   | 42       | 41       |        |        |        |        |        |        |        |        | 11.7          |
| 9      | 5   | 85       | 31       |        |        |        |        |        |        |        |        |               |
| 10     | 3   | 37       | 69       |        |        |        |        |        |        |        |        |               |

Figure 3 Payoff Ratio for Unilateral Deviations for Different Strategies in $\Delta_n$ for Candidates A in the Left Panel, and B in the Right Panel [Color figure can be viewed at wileyonlinelibrary.com]
those regions with more votes. Unlike the low bias
situation, the probability of winning in such a state will be negligible.
The intuition behind the latter is that candidates invest only in state 4, as it is one of the few where the difference between its bias parameters is within the reach of their campaign budgets to offset its outcome. In summary, polarization under EC will induce the candidates to campaign in swing states. States that combine both of these elements are usually called swing states.

Table 5 shows the equilibria for EC when running Algorithm 2 for different levels of $f$.

For higher biases ($f = 5, 10, 50$), candidates focus all their efforts in a single large region, compensating the initial bias disadvantage (region 1 for candidate A; 2 for B).

All in all, candidates allocate their efforts where they have the maximum marginal return. The effect of polarization in candidate equilibrium strategies can be summarized as the interplay of the two following factors: (i) regions with a large number of votes, and (ii) the disadvantage bias.

Table 5 shows the equilibria for EC when running Algorithm 2 for different levels of $f$.

For higher biases ($f = 5, 10, 50$), candidates concentrate all their efforts in a single large region, compensating the initial bias disadvantage (region 1 for candidate A; 2 for B).

For higher biases ($f = 5, 10, 50$), candidates invest only in state 4, as it is one of the few where the difference between its bias parameters is within the reach of their campaign budgets to offset its outcome. In summary, polarization under EC will induce the candidates to campaign according to the two following factors: (i) states with similar biases and (ii) high electoral votes. States that combine both of these elements are usually called swing states.

The previous analysis helps us to understand candidates’ decisions under both election systems for different levels of polarization. Under MS, candidates put their efforts into seeking to get the larger number of votes possible. The bigger the biases are, the more they tend to invest only in those regions where there

| $f$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----|----|----|----|----|----|----|----|----|
| 0.5 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| 1   | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 |
| 5   | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50 |
| 10  | 100| 100| 100| 100| 100| 100| 100| 100|
| $\sigma_A$ [%] | 70 | 70 | 70 | 70 | 70 | 70 | 70 | 70 |

| $f$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----|----|----|----|----|----|----|----|----|
| 0.5 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| 1   | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 |
| 5   | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50 |
| 10  | 100| 100| 100| 100| 100| 100| 100| 100|
| $\sigma_B$ [%] | 70 | 70 | 70 | 70 | 70 | 70 | 70 | 70 |
are more people to convince to vote for them; large regions with relative initial disadvantages. Under EC, because of the winner-takes-all policy at the state level, some votes do not translate into its respective electoral vote. As a result, when a candidate faces a state with an initial disadvantage, such that it is virtually impossible to induce any substantial change in the probability of winning, it is simply not worth investing in, even though it might actually be the largest state. Therefore, under a highly polarized scenario, the campaign is only relevant in the undecided states; the swing states.

An interesting insight from the last result can be applied to the effect of polarization on political campaigns. In a polarized country, we would expect higher bias values, and therefore strategies should be more focused on a few states. In reality, the latter observation can have additional consequences regarding not just the candidates’ resource allocation strategies, but also on the election promises made in the different states. For example, a candidate might be more tempted to offer higher infrastructure expenditures in a swing state (under a polarized EC), despite the fact that that state might have only a small fraction of the national population, but it plays a key role in winning the election.

5.3.2. Equilibrium and States’ Uncertainty. One of the parameters that both the MS and EC required in the model presented is \( k \), which regulates the variability of the voters outcomes in each state. \( k \to 0 \) tends to the case where all the electors choose the same option, whereas \( k \to \infty \) results in a more deterministic outcome.

Figure 4 shows the probability of winning for candidate A with different levels of the uncertainty (parameter \( k \)). It can be seen that the winning odds remain barely affected under EC. By contrast, the winning probability (for candidate A) under MS has a significant increase as \( k \) increases. The intuition is that as the outcome of the voters becomes less certain, which candidate will win the election becomes more clear. This behavior does not occur under EC since the candidates mixed their strategies in equilibrium, and therefore, both candidates still ended up with almost the same odds (in the instance shown in Figure 4) as when the voters behave more deterministically. Indeed, an additional interesting observation is that the cardinality of the support of the candidates’ equilibrium strategies under EC increases with the value of \( k \). See more details of this on Table 7 in Appendix W. Despite the fact that the winning probabilities coincide for medium or high uncertainty settings, this does not imply that the players’ equilibrium strategies match between both electoral systems. We analyze the special case with homogeneous states’ weights, and observe how the games equilibria under the EC and the MS coincide for high uncertainty and non-abstention. (See details in Appendix X).

5.3.3. Variations on the Investment Effect. As stated by Equation (6), the impact of the candidates’ investments is linear in the distribution parameters. In this section we study the equilibrium of the games when adding a power term \( a > 0 \) to the candidates’ efforts. Namely, we consider that \( (S^A_i, S^B_i, S^C_i) \sim \text{Dir}_4(k(x_i^a + \alpha_i), k(y_i^a + \beta_i), k\gamma_i) \). Note that for \( x_i \in (0, 1) \) (and similarly for \( y_i \)), values of \( a \) below 1 will reward the candidates’ investments (as \( x_i^a > x_i \)). For instance, if \( a = 0.5 \) and \( x_i = 0.04 \), then \( x_i^a = 0.2 \). Contrarily, values of \( a \) above 1 will penalize investments, in the sense that \( x_i^a < x_i \). For example, if \( a = 2 \), then an effort \( x_i = 0.1 \) will translate into \( x_i^2 = 0.01 \).

Tables 6 and 7 show the equilibria obtained under MS and EC, respectively, for different values of the
Table 6 Equilibrium for Different Power Parameters $a$ Under EC. For each $a = 0.25, 0.5, 1, 2$, Candidate A Wins with Probabilities 51.65%, 51.98%, 55.05%, and 60.45%, respectively. The lattice refinement was set to $q = 100$.

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\sigma_a$ [%] |
|-----|---|---|---|---|---|---|---|---|---|----|-----------|
| 0.25 | 30 | 10 | 20 | 7 | 5 | 13 | 7 | 6 | 1 | 1 | 100.0 |
| 0.5  | 42 | 17 | 12 | 6 | 4 | 7 | 5 | 5 | 2 | 1 | 100.0 |
| 1    | 76 | 8  | 16 | 11.8 | | | | | | | |
| 2    | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |

Table 7 Equilibrium for Different Power Parameters $a$ Under MS. For each $a = 0.25, 0.5, 1, 2$, Candidate A Wins with Probabilities 54.36%, 55.30%, 57.37%, and 61.55%, respectively.

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|----|
| 0.25 | 26.6 | 18.6 | 15.2 | 9.0 | 6.3 | 6.9 | 7.5 | 6.2 | 1.9 | 1.8 |
| 0.5  | 31.6 | 22.2 | 15.5 | 8.0 | 4.3 | 5.3 | 6.7 | 4.8 | 0.7 | 0.8 |
| 1    | 68.2 | 52.9 | 5.9 | 5.9 | 5.9 | 5.9 | 5.9 | 5.9 | 5.9 | 5.9 |
| 2    | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 |

Table 8 Equilibrium for Different Power Parameters $a$ Under EC. For each $a = 0.25, 0.5, 1, 2$, Candidate A Wins with Probabilities 51.65%, 51.98%, 55.05%, and 60.45%, respectively. The lattice refinement was set to $q = 100$.

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\sigma_a$ [%] |
|-----|---|---|---|---|---|---|---|---|---|----|-----------|
| 0.25 | 22.4 | 22.3 | 16.3 | 9.3 | 6.6 | 5.6 | 7.0 | 6.1 | 2.9 | 1.3 |
| 0.5  | 24.9 | 29.5 | 16.1 | 8.2 | 4.9 | 3.8 | 6.0 | 4.7 | 1.6 | 0.4 |
| 1    | 36.3 | 52.2 | 11.4 | 11.4 | 11.4 | 11.4 | 11.4 | 11.4 | 11.4 | 11.4 |
| 2    | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 | 100.0 |

power parameter $a$ while fixing the rest of the parameters to the ones used before as was shown on Tables 2 and 3. In both election systems, we observe that for low values of $a$, the candidates’ investments are distributed among almost all regions, while regions with more votes (or electoral votes for the case of EC) are the ones that get most of the investments. Intuitively, it is convenient to invest at least some amount in every region since the low power term $a$ will amplify this investment. For high values of $a$, candidates under both electoral systems focus all their efforts on a single region, however under EC this is observed in mixed strategies. Note that the power $a$, in this case diminishes a candidates’ effort when these are in $(0, 1)$, and therefore, splitting the investment in regions is not attractive. Also, we observe that candidates’ winning odds become closer to half in equilibrium for lower values of $a$. This is reasonable since a more amplifying effect of candidates’ investments reduce the relative weight of the regions’ biases.

5.4. The 2020 US Presidential Election

We analyzed both election systems (EC and MS) with real-world data from the US Presidential Election of 2020. The idea was to compute and compare the equilibrium allocation strategies of the Democratic and Republican candidates suggested by the models presented.

More precisely, we used electoral polls from (fivethirtyeight.com 2020) for the 2020 electoral campaign, up to 90 days before election day, to estimate the states’ bias parameters of the states $(\alpha$ and $\beta$). (The idea of using data three months prior to the election is to evaluate the candidates’ allocation decisions without anticipating future information.) The abstention parameters $(\gamma)$ and the variance parameter $(k)$ are estimated from the polls data of the US presidential election of 2016 from (fivethirtyeight.com 2016). We considered $x$ and $y$ for each poll by counting the number of rallies performed in the states prior to the publication date of the respective poll. For more details on the estimation procedure see Appendix Y. Once we have estimated all the model parameters, we computed the candidates’ equilibrium strategies under: (i) EC (obtained from Algorithm 1), and (ii) MS (obtained form Algorithm 1); three months prior to the US 2020 presidential election day. For simplicity, for the EC, we considered that every state would follow the winner-takes-all policy, even though in practice some of them have divided their electoral votes in the past. Results are shown on Table 8.

Under the EC, we observe from Table 8 two strategy patterns for the Democratic candidate: (i) A risk-seeker ($x^2$), and (ii) A swing-states focus ($x^{(1)}, x^{(3)}, \text{ and } x^{(4)}$). In the first case, which happens with a probability of almost two thirds, the candidate inputs all his investment in Texas. This is because according to the estimated bias parameters, the usually Republican stronghold looks not so unlikely to shift, and the Republican candidate cannot afford to lose such a large state. In the second case, the other three strategies all basically follow the same behavior: to secure California, and to fight for the major swing-states of Florida and Pennsylvania (and to some extent, also in Georgia, Michigan, and Ohio for $x^{(1)}$). The Republicans, on the other hand, faced an initial disadvantage due to the imbalance in the bias parameters. This is what explains the structure of their strategies: $y^{(1)}, y^{(2)}$ and $y^{(4)}$ are each focused on one specific state: either
California, Florida, or Texas. Each of the three allocations has a different goal: \( y^{(1)} \) tries to turn California closer to a toss-coin state (with almost a 50-50 chance, according to the estimated bias parameters), thus, undermining the Democrat stronghold. \( y^{(2)} \) focuses on the largest swing-state, Florida, disregarding the...
risk of not appearing the remaining states. With a more conservative approach, \( y^{(4)} \), secures Texas and hopes for the best anywhere else, which is a direct response to the Democrats’ strategy \( x^{(3)} \). The remaining strategy, \( y^{(5)} \), splits its allocation among the three largest swing-states (Florida, Michigan and Pennsylvania), which were leaning slightly toward the Democrats beforehand, and turned them into 50-50 states (according to the estimated bias parameters), thus, creating a scenario with high volatility in those major battlegrounds, but at the price of putting other large states at risk. These strategies are not so different in their cores to what we see in practice: parties focus strongly on the largest swing-states (Florida, Pennsylvania, Michigan, North Carolina, etc.). Furthermore, this equilibrium suggests what might happen in the coming electoral cycles: Texas becoming a major battleground, with an increasing Democratic electoral base. The more extreme strategies (those where candidates invest their efforts basically in a single state) are probably a result of two factors the model is missing: (i) The model is not dynamic over time (in practice, once the Democrats start to invest heavily in Texas, the Republicans are very likely to react to that in order to keep their stronghold), and (ii) The only goal for the candidates in this model is to maximize the probability of winning, regardless of that meaning, if loosing, it might be by a large gap. In practice it is more likely that, in such a scenario, parties would decide to just minimize the damage, secure the key states and reorganize for the next electoral cycle, even at the cost of sacrificing most chances of winning the current election.

Under the MS, both candidates invest in one and only one state, California. This is due to the following three reasons: (i) It has the largest population, by far (roughly 25 million, compared to the 17.5 million population of Texas, which is in second place). (ii) Its bias parameters are not as large as they are in some other states. This translates into a relatively larger marginal returns, since voters are not highly polarized (medium bias parameters): for every day of campaigning, there are many more votes to win than anywhere else. (iii) The Democratic advantage is great enough so that matching the Republican strategy offsets the potential vote losses in California, ensuring winning the election.

The results obtained, especially on an hypothetical MS, should be read with some caution. First, many endogenous factors are not being considered here. Because the United States has an EC system, the bias parameters (which are somehow the result of the long term strategies of each party) are actually built in accordance with this electoral setting, which is why this result under MS seems to be so unrealistic. Also, note that the turnout was largely underestimated, probably because we considered the abstention parameter, \( \gamma \), to be the same as in 2016. It is possible that it was actually smaller this time. It is worth mentioning that polls have been biased toward the Democratic candidate in the past two elections in most of the states. This bias information could potentially be incorporated into the estimated parameters. Finally, the models introduced are calibrated only on campaign events, without considering other factors such as social and mass media.

6. Conclusions

The models and results presented show the application of a zero-sum game to analyzing the resource allocation problem of an election campaign under the MS and EC system, taking into consideration such aspects as sizes of regions, biases, abstentions, and uncertainty. In addition, the model can be calibrated with real data. Since the framework introduced is a simplification of reality, the results obtained when using the model with real data should be read with caveats.

It is interesting to note that here it is assumed that the resource being allocated (the strategy) is the time that the candidate invests in each state. This leads to a symmetrical budget constraint. However, the same model can be applied for studying the campaign resource allocation strategy in terms any other resources rather than time, such as: advertisement budgets, election promises, etc. In addition, the framework introduced can be used to address the time-dynamic version of the game in which candidates optimize their strategies simultaneously at each time period, considering the information (biases, etc.) of the current period, and future decisions in a forward-looking scheme. Myopic or bounded rationality strategies could be good points to start with, given the complexity of the problem.

Acknowledgements

The authors gratefully acknowledge financial support from CONICYT PIA/BASAL AFB180003. This research was partially supported by the supercomputing infrastructure of the NLHPC (ECM-02). In addition, we thank Jose Correa and Velibor Mišić for their useful comments that used to improve this work.

References

Arrow, K. J. 2012. Social Choice and Individual Values, vol. 12. Yale University Press, New Haven.
Balinski, M., R. Laraki. 2020. Majority judgment vs. approval voting. Operations Research INFORMS, In press. hal-02374745f.
Banzhaf III, J. F. 1964. Weighted voting doesn’t work: A mathematical analysis. Rutgers L. Rev. 19: 317.
Barnett, A. I. 1976. More on a market share theorem. *J. Mark. Res.* 13(1): 104–109.

Bell, D. E., R. L. Keeney, J. D. C. Little. 1975. A market share theorem. *J. Mark. Res.* 12(2): 136–141.

Borel, E. 1921. La théorie du jeu et les équations intégrales du noyau symétrique. *C. R. Acad. Sci.* 173(1304-1308): 58.

Brams, S. J., M. D. Davis. 1974. The 3/2’s rule in presidential campaigning. *Am. Political Sci. Rev.* 68(1): 113–134.

Buchanan, J. M., R. D. Tollison, G. Tullock. 1980. Efficient rent seeking. *Toward a Theory of the Rent Seeking Society* 97–121.

Duffy, J., A. Matros. 2015. Stochastic asymmetric blotto games: Some new results. *Econ. Lett.* 134: 4–8.

Duffy, J., A. Matros. 2017. Stochastic asymmetric blotto games: An experimental study. *J. Econ. Beh. Organ.* 139: 88–105.

Eiselt, H. A., V. Marianov. 2020. Maximizing political vote in multiple districts. *Socio-Econ. Plan. Sci.* 72: 100896.

Eiselt, H. A., V. Marianov. 2020. Maximizing political vote in multiple districts. *Socio-Econ. Plan. Sci.* 72: 100896.

Eiselt, H. A., V. Marianov. 2020. Maximizing political vote in multiple districts. *Socio-Econ. Plan. Sci.* 72: 100896.

Eiselt, H. A., V. Marianov. 2020. Maximizing political vote in multiple districts. *Socio-Econ. Plan. Sci.* 72: 100896.

Eiselt, H. A., V. Marianov. 2020. Maximizing political vote in multiple districts. *Socio-Econ. Plan. Sci.* 72: 100896.

Friedman, L. 1958. Game-theory models in the allocation of advertising expenditures. *Oper. Res.* 6(5): 699–709.

Gross, O., R. Wagner. 1950. A continuous colonel blotto game. Technical report, Rand Air Force Santa Monica Ca.

Kaplan, E. H., A. Barnett. 2003. A new approach to estimating the probability of winning the presidency. *Oper. Res.* 51(1): 32–40.

Klupp, T., M. K. Polborn. 2006. Primaries and the new Hampshire effect. *J. Public Econ.* 90(6-7): 1073–1114.

Kovenock, D., B. Roberson. 2012. Conflicts with multiple battlefields. M. Garfinkel, S. Skaperdas, eds. *The Oxford Handbook of the Economics of Peace and Conflict*. Oxford University Press, Oxford.

Kovenock, D., B. Roberson. 2021. Generalizations of the general lotto and colonel blotto games. *Econ. Theor.* 71(3): 997–1032.

Lake, M. 1979. A new campaign resource allocation model. S. J. Brams, A. Schotter, G. Schwidtiauer, eds. *Applied Game Theory*. Springer, Berlin, 118–132.

Laslier, J.-F., N. Picard. 2002. Distributive politics and electoral competition. *J. Econ. Theor.* 103(1): 106–130.

Monahan, G. E. 1987. The structure of equilibria in market share attraction models. *Management Sci.* 33(2): 228–243.

Nagler, J., J. Leighty. 1992. Presidential campaign expenditures: Evidence on allocations and effects. *Public Choice* 73(3): 319–333.

National Popular Vote Inc. 2019. Two Thirds of the Presidential Campaign is in Just 6 States. Available at https://www.nationalpopularvote.com/campaign-events-2016 (accessed date June 15, 2020).

Osorio, A. 2013. The lottery blotto game. *Econ. Lett.* 120(2): 164–166.

Our World in Data. 2019. Democracy. Available at https://ourworldindata.org/democracy (accessed date June 15, 2020).

Rigdon, S. E., J. J. Sauppe, S. H. Jacobson. 2015. Forecasting the 2012 and 2014 elections using bayesian prediction and optimization. *SAGE Open* 5(2): 2158244015579724.

Roberson, B. 2006. The colonel blotto game. *Econ. Theor.* 29(1): 1–24.

Robson, A. R. W., et al. 2005. Multi-item contests. *Working Papers in Economics and Econometrics*, W.P. No. 446.

Scheffé, H. 1958. Experiments with mixtures. *J. Roy. Stat. Soc. Ser. B* Methodol. 20(2): 344–360.

Schwartz, G., P. Loiseau, S. S. Sastry. 2014. The heterogeneous colonel blotto game. 2014 7th International Conference on NETwork Games, COntrol and OPtimization (NetGCoop). IEEE, 232–238.

Shaw, D. R. 1999. The methods behind the madness: Presidential electoral college strategies, 1988–1996. *J. Polit.* 61(4): 893–913.

Snyder, J. M. 1989. Election goals and the allocation of campaign resources. *Econometrica J. Econom. Soc.* 57: 637–660.

Stromberg, D. 2008. How the electoral college influences campaigns and policy: the probability of being florida. *Am. Econ. Rev.* 98(5): 769–807.

Thomas, C. 2018. N-dimensional blotto game with heterogeneous battlefield values. *Econ. Theor.* 65(3): 509–544.

Wang, W., D. Rothschild, S. Goel, A. Gelman. 2015. Forecasting elections with non-representative polls. *Int. J. Forecast.* 31(3): 980–991.

---

**Supporting Information**

Additional supporting information may be found online in the Supporting Information section at the end of the article.

**Data S1**: Online Appendix File.