A PROOF OF THE GAN-LOH-SUDAKOV CONJECTURE

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Abstract. We prove that any max-degree \( d \) graph on \( n \) vertices has at most 
\[ q\left(\binom{d+1}{3}\right) + \binom{r}{3} \] 
triangles, where \( n = q(d + 1) + r \), \( 0 \leq r \leq d \). This resolves a conjecture of Gan, Loh, and Sudakov.

1. Introduction

Fix positive integers \( d \) and \( n \) with \( d + 1 \leq n \leq 2d + 1 \). Galvin [7] conjectured that the maximum number of cliques in an \( n \)-vertex graph with maximum degree \( d \) comes from a disjoint union \( K_{d+1} \cup K_r \) of a clique on \( d + 1 \) vertices and a clique on \( r := n - d - 1 \) vertices. Cutler and Radcliffe [4] proved this conjecture. Engbers and Galvin [6] then conjectured that, for any fixed \( t \geq 3 \), the same graph \( K_{d+1} \cup K_r \) maximizes the number of cliques of size \( t \), over all \( (d + 1 + r) \)-vertex graphs with maximum degree \( d \). Engbers and Galvin [6]; Alexander, Cutler, and Mink [1]; Law and McDiarmid [11]; and Alexander and Mink [2] all made progress on this conjecture before Gan, Loh, and Sudakov [9] resolved it in the affirmative. Gan, Loh, and Sudakov then extended the conjecture to arbitrary \( n \geq 1 \) (for any \( d \)).

Conjecture (Gan-Loh-Sudakov Conjecture). Any graph on \( n \) vertices with maximum degree \( d \) has at most 
\[ q\left(\binom{d+1}{3}\right) + \binom{r}{3} \] 
triangles, where \( n = q(d + 1) + r \), \( 0 \leq r \leq d \).

They showed their conjecture implies that, for any fixed \( t \geq 4 \), any max-degree \( d \) graph on \( n = q(d + 1) + r \) vertices has at most 
\[ q\left(\binom{d+1}{t}\right) + \binom{r}{t} \] 
cliques of size \( t \), so we have restricted attention to triangles.

The Gan-Loh-Sudakov conjecture (GLS conjecture) has attracted substantial attention. Cutler and Radcliffe [5] proved the conjecture for \( d \leq 6 \) and showed that a minimal counterexample, in terms of number of vertices, must have \( q = O(d) \). Gan [8] proved the conjecture if \( d + 1 - \frac{9}{4096}d \leq r \leq d \) (there are some errors in his proof, but they can be mended). Using fourier analysis, the author [3] proved the conjecture for Cayley graphs with \( q \geq 7 \). Kirsch and Radcliffe [10] investigated a variant of the GLS conjecture in which the number of edges is fixed instead of the number of vertices (with still a maximum degree condition).

In this paper, we fully resolve the Gan-Loh-Sudakov conjecture.

Theorem 1. For any positive integers \( n, d \geq 1 \), any graph on \( n \) vertices with maximum degree \( d \) has at most 
\[ q\left(\binom{d+1}{3}\right) + \binom{r}{3} \] 
triangles, where \( n = q(d + 1) + r \), \( 0 \leq r \leq d \).
Analyzing the proof shows that \(qK_{d+1} \sqcup K_r\) is the unique extremal graph if \(r \geq 3\), and that \(qK_{d+1} \sqcup H\), for any \(H\) on \(r\) vertices, are the extremal graphs if \(0 \leq r \leq 2\).

The heart of the proof is the following Lemma, of independent interest, which says that, in any graph, we can find a closed neighborhood whose removal from the graph removes few triangles. Theorem 1 will follow from its repeated application.

**Lemma 1.** In any graph \(G\), there is a vertex \(v\) whose closed neighborhood meets at most \(\binom{d(v) + 1}{3}\) triangles.

As mentioned above, Theorem 1, together with the work of Gan, Loh, and Sudakov [9], yields the general result, for cliques of any fixed size.

**Theorem 2.** Fix \(t \geq 3\). For any positive integers \(n, d \geq 1\), any graph on \(n\) vertices with maximum degree \(d\) has at most \(q(d+1) + \binom{r}{t}\) cliques of size \(t\), where \(n = q(d + 1) + r, 0 \leq r \leq d\).

Theorem 2 gives another proof of (the generalization of) Galvin’s conjecture (to \(n \geq 2d + 2\)) that a disjoint union of cliques maximizes the total number of cliques in a graph with prescribed number of vertices and maximum degree.

Finally, the author would like to point out a connection to a related problem, that of determining the minimum number of triangles that a graph of fixed number of vertices \(n\) and prescribed minimum degree \(\delta\) can have. The connection stems from a relation, observed in [2] and [9], between the number of triangles in a graph and the number of triangles in its complement:

\[
|T(G)| + |T(G^c)| = \binom{n}{3} - \frac{1}{2} \sum_v d(v)[n - 1 - d(v)].
\]

Lo [12] resolved this “dual” problem when \(\delta \leq \frac{4n}{5}\). His results resolve the GLS conjecture for regular graphs for \(q = 2, 3\), and the GLS conjecture implies his results, up to an additive factor of \(O(\delta^2)\), for \(q = 2, 3\), and yield an extension of his results for \(q \geq 4\) — these are the optimal results asymptotically, in the natural regime of \(\frac{\delta}{n}\) fixed, and \(n \to \infty\).

2. **Notation**

Denote by \(E\) the edge set of \(G\); for two vertices \(u, v\), we write “\(uv \in E\)” if there is an edge between \(u\) and \(v\) and “\(uv \not\in E\)” otherwise — in particular, for any \(u, uu \not\in E\). For a vertex \(v\), let \(|T_{N[v]}|\) denote the number of triangles with at least one vertex in the closed neighborhood \(N[v] := \{u : uv \in E\} \cup \{v\}\), and let \(|T(G - N[v])|\) denote the number of triangles with all vertices in the graph \(G - N[v]\) (the subgraph induced by the vertices not in \(N[v]\)). Finally, \(d(v)\) denotes the degree of \(v\).
3. Proof of Theorem 1

For a graph $G$, let $W(G) = \{(x, u, v, w) : ux, vx, wx \in E, uv, uw, vw \not\in E\}$.

**Lemma 2.** For any graph $G$, $6 \sum_v |T_{N[v]}| + |W(G)| = \sum_v d(v)^3$.

**Proof.** Let $\Omega = \{(z, u, v, w) : uw, uz \in E \text{ and } [zu \in E \text{ or } zv \in E \text{ or } zw \in E]\}$, $\Sigma = \{(x, u, v, w) : ux, vx, wx \in E\}$, and $W = W(G)$, and note that $\sum_v 6|T_{N[v]}| = |\Omega|$. Any 4-tuple in $\Sigma, W$ or $\Omega$ gives rise to one of the induced subgraphs shown below, since one vertex must be adjacent to all the others. Recall that repeated vertices in the 4-tuples are allowed.

![Diagram of induced subgraphs](image)

It suffices to show that for each of the induced subgraphs above, the number of times it comes from a 4-tuple in $\Sigma$ is the sum of the number of times it comes from 4-tuples in $\Omega$ and $W$. Any fixed copy of $A$, say on vertices $u$ and $v$, comes 0 times from a 4-tuple in $\Omega$ (since it has no triangles), and 2 times from each of $W$ and $\Sigma ((u, v, v, v), (v, u, u, u))$. Any fixed copy of $B$, say on vertices $u, v, w$ with $vu, vw \in E$, comes 0 times from $\Omega$, and 6 times from each of $W$ and $\Sigma ((u, u, u, u), (v, u, w, w), (v, w, u, u), (v, w, u, w), (v, w, w, u), (v, w, u, w))$. Any fixed copy of $C$ comes 18 times from each of $\Omega$ and $\Sigma$ (3 choices for the first vertex and then 6 for the ordered triangle), and 0 times from $W$. Similarly, any fixed copy of $D$ comes 6 times from each of $W$ and $\Sigma$, and 0 times from $\Omega$; finally, $F, H, I$ come 6, 12, 24 times, respectively, from each of $\Omega$ and $\Sigma$, and 0 times from $W$. \qed

**Lemma 1.** In any graph $G$, there is a vertex $v$ whose closed neighborhood meets at most $\left(\frac{d(v)+1}{3}\right)$ triangles, i.e. $|T_{N[v]}| \leq \left(\frac{d(v)+1}{3}\right)$.

**Proof.** By Lemma 2, since $|W(G)| \geq |\{(x, u, u, u) : ux \in E\}| = \sum_x d(x)$, we have $\sum_v |T_{N[v]}| \leq \sum_v \frac{1}{6}[d(v)^3 - d(v)]$. By the pigeonhole principle, there is some $v$ with $|T_{N[v]}| \leq \frac{1}{6}[d(v)^3 - d(v)] = \left(\frac{d(v)+1}{3}\right)$.

**Lemma 3.** For any positive integers $a \geq b \geq 1$, it holds that $\frac{a}{3} + \frac{b}{3} \leq \frac{a+1}{3} + \frac{b-1}{3}$. Consequently, for any positive integers $a, b$ and any positive integer $c$ with $\max(a, b) \leq c \leq a + b$, it holds that $\frac{a}{3} + \frac{b}{3} \leq \frac{c}{3} + \frac{a+b-c}{3}$.

**Proof.** $(\frac{a+1}{3}) - (\frac{a}{3}) = (\frac{a}{2})$, and $(\frac{b}{3}) - (\frac{b-1}{3}) = (\frac{b-1}{2})$. Iterate to get the consequence. \qed

We now finish the proof of Theorem 1. With a fixed $d$, we induct on $n$. For $n = 1$, the result is obvious. Take some $n \geq 2$, and suppose the theorem holds for all
smaller values of \( n \). Let \( G \) be a max-degree \( d \) graph on \( n \) vertices. By Lemma 1, we may take \( v \) with \( |T_{N[v]}| \leq \binom{d(v)+1}{3} \). Write \( n = q(d+1) + r \) for \( 0 \leq r \leq d \). Note \( |T(G)| = |T(G - N[v])| + |T_{N[v]}| \). Since \( G - N[v] \) has maximum degree (at most) \( d \), if \( d(v) + 1 \leq r \), then induction and Lemma 3 give
\[
|T(G)| \leq q \binom{d+1}{3} + \binom{r - (d(v) + 1)}{3} + \binom{d(v) + 1}{3} \leq q \binom{d+1}{3} + \binom{r}{3},
\]
and if \( d(v) + 1 > r \), then induction and Lemma 3 give
\[
|T(G)| \leq (q-1) \binom{d+1}{3} + \binom{d+1 + r - (d(v) + 1)}{3} + \binom{d(v) + 1}{3} \leq q \binom{d+1}{3} + \binom{r}{3}.
\]
The maximum degree condition ensured \( d + 1 + r - (d(v) + 1) \geq 0 \) and \( d(v) + 1 \leq d+1 \).

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