Polarization squeezing and multipartite entanglement of triphoton states

G. R. Jin, S. Luo, and Y. C. Liu

Department of Physics, Beijing Jiaotong University, Beijing 100044, China

H. Jing

Department of Physics, Henan Normal University, Xinxiang, Henan 453007, China

W. M. Liu

Beijing National Laboratory for Condensed Matter Physics,
Institute of Physics, Chinese Academy of Sciences, Beijing 100080, China

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Based upon standard angular momentum theory, we develop a framework to investigate polarization squeezing and multipartite entanglement of a quantum light field. Both mean polarization and variances of the Stokes parameters are obtained analytically, with which we study recent observation of triphoton states [L. K. Shalm, et al, Nature 457, 67 (2009)]. Our results show that the appearance of maximally entangled NOON states accompanies with a flip of mean polarization and can be well understood in terms of quantum Fisher information.

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I. INTRODUCTION

Polarization squeezing and quantum entanglement of a light field have received much attention for decades not only because of fundamental physical interests, but also for potential applications in quantum metrology and quantum information [1–4]. Formally, the squeezing is defined as a reduction of polarization uncertainty below shot-noise limit (SNL), which is standard quantum limit defined as a reduction of polarization uncertainty below quantum information [1–4]. As the basis of total Hilbert space, where the photon NOON states results in super-resolving phase estimations among individual particles [5–9]. It has been shown that the squeezing, closely related to multipartite entanglement is aroused from quantum correlation effect among individual particles [5–9].

Quantum metrology based upon maximally entangled NOON states results in super-resolving phase estimations [10, 11]. However, a deterministic optical source of the entangled states is yet to be realized due to technical difficulties [12]. Using various state-projection measurements, so far there are a lot of groups have realized few-photon NOON states [1, 13–17]. In particular, Shalm et al. have succeed in preparing maximally entangled NOON state of the triphotons [17]. Counter-intuitively, they found that the NOON state does not show polarization squeezing, just like quantum uncorrelated coherent states. To explain it, in this paper we theoretically study polarization squeezing and multipartite entanglement of the triphotons. Analytical expressions of the reduced and increased variances are presented to determine the squeezing parameters and quantum Fisher information (QFI) [18–21] of the triphoton states. Our results show that the mean polarization of the triphotons changes its sign with the appearance of NOON states.

The outline of this paper is arranged as follows: in Sec. 2, we give notations and definitions of the polarization squeezing followed by a rigorous analytical approach for obtaining the reduced and the increased variances of Stokes parameters. Sec. 3 is devoted to consider the maximally entangled triphoton state. Firstly, we solve the variances and the optimal squeezing direction for the polarization squeezed state. To proceed, we investigate the relationship between the squeezing and the entanglement. Our conclusion will be presented in Sec. 4.

II. THEORETICAL BACKGROUND

In analogy with classical optics, the polarization a light field can be described by Stokes vectors \((c = \hbar = 1)\) [22]

\[
\hat{S}_0 = \frac{1}{2} (\hat{a}_{H}^{\dagger} \hat{a}_{H} + \hat{a}_{V}^{\dagger} \hat{a}_{V}), \quad \hat{S}_1 = \frac{1}{2} (\hat{a}_{H}^{\dagger} \hat{a}_{H} - \hat{a}_{V}^{\dagger} \hat{a}_{V}),
\]

\[
\hat{S}_2 = \frac{1}{2} (\hat{a}_{H}^{\dagger} \hat{a}_{V} + \hat{a}_{V}^{\dagger} \hat{a}_{H}), \quad \hat{S}_3 = \frac{1}{2} (\hat{a}_{H}^{\dagger} \hat{a}_{V} - \hat{a}_{V}^{\dagger} \hat{a}_{H})(1)
\]

where \(\hat{a}_{H,V}^{\dagger}\) and \(\hat{a}_{H,V}\) are annihilation and creation operators for the horizontal and vertical polarization modes, respectively. The photon operators satisfy bosonic commutation relations \([\hat{a}_\mu, \hat{a}_\nu^{\dagger}] = \delta_{\mu\nu}, \) with \(\mu, \nu \in \{H, V\}\).

The Stokes vectors \(\hat{S}_1, \hat{S}_2, \) and \(\hat{S}_3\) obey SU(2) algebra: \([\hat{S}_i, \hat{S}_j] = i \hat{S}_k\), with \(i, j, k \in \{1, 2, 3\}\), corresponding to horizontally, linearly at 45°, and right-circularly polarized axes \([2]\), respectively. For a fixed photon number \(N = 2s\), \(\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2 = s(s + 1)\) and \(\hat{S}_0 = s\) are invariant and commute with other three Stokes operators. Following standard theory of angular momentum, we choose eigenstates of \(\hat{S}_i, |s, n\rangle = |s + n, s - n\rangle_{H,V}\) as the basis of total Hilbert space, where the photon number states are defined as usual, \(|m, n\rangle_{H,V} = (\hat{a}_H^{\dagger})^m (\hat{a}_V^{\dagger})^n |0\rangle / \sqrt{m!n!}\). The SU(2) angular momentum...
states obey $\hat{S}_\pm(s,n) = \sqrt{(s + n)(s + n + 1)|s,n\rangle}$, with the ladder operators $\hat{S}_+ = \hat{S}_2^+ + i\hat{S}_3^+$.

Any quantum polarization state $|\Psi\rangle$ is characterized by the mean polarization $\langle \hat{S} \rangle = (\langle \hat{S}_1 \rangle, \langle \hat{S}_2 \rangle, \langle \hat{S}_3 \rangle)$ on a Poincaré sphere, where $\langle \hat{S}_i \rangle = (\Psi|\hat{S}_i|\Psi)$ for $i = 1, 2, 3$. To define the concept of polarization squeezing, it is convenient to introduce three orthogonal polarization vectors $\hat{S}_{n_i} = \hat{S} \cdot \hat{n}_i$, with

$$\hat{n}_1 = (0, -\sin \theta, \cos \phi),$$
$$\hat{n}_2 = (\sin \theta, -\cos \theta \cos \phi, -\cos \theta \sin \phi),$$
$$\hat{n}_3 = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi),$$

where the polar angle $\theta$ and the azimuth angle $\phi$ obey $\sin \theta = r/|\langle \hat{S} \rangle|$, $\cos \theta = \langle \hat{S}_3 \rangle/|\langle \hat{S} \rangle|$, $\sin \phi = \langle \hat{S}_3 \rangle/r$, $\cos \phi = \langle \hat{S}_2 \rangle/r$. Here, $|\langle \hat{S} \rangle| = (\langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2 + \langle \hat{S}_3 \rangle^2)^{1/2}$, denoting the length of the mean polarization, and $r = (\langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2)^{1/2} = |\langle \hat{S} \rangle| \sin \theta$. Note that the mean polarization $\langle \hat{S} \rangle = (\langle \hat{S}_1 \rangle, \langle \hat{S}_2 \rangle, \langle \hat{S}_3 \rangle)$ has length $|\langle \hat{S} \rangle| = |\langle \hat{S}_{n_3} \rangle|$. The two orthogonal polarization vectors normal to $\langle \hat{S} \rangle$ (i.e., $\hat{n}_3$) satisfy Heisenberg uncertainty relationship

$$\langle (\Delta \hat{S}_{n_1})^2 (\Delta \hat{S}_{n_2})^2 \rangle \geq \frac{1}{4} |\langle \hat{S}_{n_3} \rangle|^2,$$

where $(\Delta \hat{S}_{n_i})^2 \equiv |\langle \hat{S}_{n_i} \rangle|^2 - \langle \hat{S}_{n_i} \rangle^2$, denoting the variance of the Stokes operator $\hat{S}_{n_i}$ for $i = 1, 2$. It is well known that the minimal uncertainty relationship is obtained for SU(2) coherent state

$$|\theta, \phi\rangle = e^{-i\theta \hat{S}_3} |s, s\rangle = e^{i\theta (\hat{S}_2 \sin \phi - \hat{S}_3 \cos \phi)} |s, s\rangle,$$

which is also an eigenstate of $\hat{S}_{n_3}$ with eigenvalue $s$ (i.e., $\langle \hat{S}_{n_3} \rangle = |s, s\rangle = s$), and thereby $(\Delta \hat{S}_{n_1})^2 = (\Delta \hat{S}_{n_2})^2 = s/2$. Here, the value $s/2$ is termed as standard quantum limit, or the shot-noise limit (SNL). The squeezing is defined if any polarization component normal to $\langle \hat{S} \rangle$ has a reduced variance below the SNL $\frac{s}{2}$. It is obvious to choose the squeezed polarization component as

$$\hat{S}_\gamma = \hat{S} \cdot \hat{n}_\gamma = \hat{S}_n \cos \gamma + \hat{S}_2 \sin \gamma,$$

where $\gamma$ is arbitrary angle with respect to $\hat{n}_1$. Due to the relation $\langle \hat{S}_\gamma \rangle = 0$, the variance of $\hat{S}_\gamma$ takes the form

$$(\Delta \hat{S}_\gamma)^2 = [C + A \cos(2\gamma) + B \sin(2\gamma)]/2,$$

where $A = (\hat{S}_2^2 - \hat{S}_3^2)$, $B = (\hat{S}_3 \hat{S}_2^+ + \hat{S}_2 \hat{S}_3^+)$, and $C = (\hat{S}_2^+ \hat{S}_3^+ + \hat{S}_3 \hat{S}_2^+)$. Minimizing $\langle (\Delta \hat{S}_\gamma)^2 \rangle$ with respect to $\gamma$, we get $\left\{ \frac{3}{2}, 2, 3/2 \right\}$. The reduced variance $V_+ = \min_\gamma \langle (\Delta \hat{S}_\gamma)^2 \rangle$, denoting the optimal squeezing along $\hat{n}_\gamma$ with $\gamma = \gamma_{\text{opt}} \equiv [\pi + \arctan(B/A)]/2$; while the increased variance $V_- = \max_\gamma \langle (\Delta \hat{S}_\gamma)^2 \rangle$, corresponding to the anti-squeezing along $\hat{n}_\gamma$ with $\gamma = \pi/2 + \gamma_{\text{opt}}$. Remarkably, the above analysis provide us explicit form of the optimal squeezing angle $\gamma_{\text{opt}}$ and that of $V_\pm$. For the minimal uncertainty state $|\theta, \phi\rangle$, $V_+ = V_- = s/2$, so the inequality

$$\xi^2 = \frac{2(V_-)}{s} < 1$$

recognizes polarization squeezed states $[25]$. In the following, we will study polarization squeezing and entanglement of the triphotons, which has been demonstrated recently by Shalm et al. [17].

### III. THE TRIPHOTON STATES

Recently, Shalm et al. [17] have succeed in preparing the triphoton states. Due to the lack of ideal single-photon sources, they adopted type-II spontaneous parametric downconversion (SPDC) and an attenuated laser (a local oscillator, LO). A pair of orthogonally polarized photons from the SPDC and a single photon from the LO are overlapped and placed into the same mode to produce a state likes $\hat{a}_{45}^\dagger \hat{a}_{45}^\dagger \hat{a}_H^\dagger |0\rangle = (\hat{a}_H^\dagger - \hat{a}_V^\dagger) \hat{a}_H^\dagger |0\rangle$. This state is then sent to a variable partial polarizer (VPP), with which one can manipulate the polarization of light to produce [17]:

$$|\Psi\rangle_T \propto e^{-S_1 |\langle T \rangle|} (\hat{a}_H^\dagger - \hat{a}_V^\dagger) \hat{a}_H^\dagger |0\rangle,$$

where $T = TV/T_H$, denoting the transmissivity ratio of the horizontal and the vertical modes photons. In the basis of $\{|s,n\rangle\}$ with $s = 3/2$, the polarization state can be rewritten as $|\Psi\rangle_T = (3 + T^4)^{-1/2} [3^{1/2}|3/2, 3/2\rangle - T^2|3/2, -1/2\rangle]$. If the VPP is tuned to transmit only the horizontal polarized photons (i.e., $T = 0$), it becomes $|\Psi\rangle_T = |3/2, 3/2\rangle = |3, 0\rangle_{H,V}$, corresponding to a coherent state $|\theta, \phi\rangle$ with $\theta = 0$. Utilizing the VPP, it is now possible to tune the ratio $T$ from 0 up to 1.8 in the experiment [17].

After the VPP, a quarter-wave plate (QWP) is adopted to rotate the polarization state into the basis of $\hat{S}_i$. The action of the QWP can be described formally by an unitary transformation $\exp(i \frac{\pi}{2} \hat{S}_2)$, which in turn leads to a kind of triphoton states

$$|\Psi\rangle = c_2 (i|2, 1\rangle_{H,V} - |1, 2\rangle_{H,V}) + c_3 (|3, 0\rangle_{H,V} - i|0, 3\rangle_{H,V}),$$

where the $T$-dependent probability amplitudes

$$c_2 = \frac{1}{2} \sqrt{\frac{3 - T^2}{2\sqrt{3} + T^4}}, \quad c_3 = \frac{1}{2} \sqrt{\frac{3}{2\sqrt{3} + T^4}}.$$

In Fig. 1(a), we plot population distributions $|c_2|^2$ and $|c_3|^2$ as a function of $T$. It is found that (i) $c_3 = c_2/\sqrt{3} = 1/(\sqrt{2})$ at $T = 0$; (ii) $c_3 = c_2 = 1/2$ at $T = 3^{1/4}(2 - \sqrt{3})^{1/2} \approx 0.7$; (iii) $c_3 = 0$ and $c_3 = 1/\sqrt{3}$ at $T = \sqrt{3} \approx 1.7$. The first case represents a coherent state $|\theta = \pi/2, \phi = \pi/2\rangle$, obtained from $|3, 0\rangle_{H,V}$ through a rotation
of $\pi/2$ angle about $-\hat{S}_2$ axis, just a result of the QWP. The second case corresponds to a phase state with equal populations, and the latter case represents the “NOON” (i.e., Greenberger-Horne-Zeilinger, or Schrödinger “cat”) state: $|\Psi\rangle = \frac{1}{\sqrt{2}}([3,0]_{H,V} - i[0,3]_{H,V})$. Such a kind of maximally entangled triphoton state has been demonstrated by Shalm et al. [17].

![Graph](image)

**FIG. 1:** (Color online) (a) Population distribution $|c_2|^2$ (red dash) and $|c_3|^2$ (solid) as a function of the transmissivity ratio $T$. (b) Normalized mean polarization $\langle \hat{S}_x \rangle = \frac{1}{8}(\langle \hat{S}_1 \rangle, \langle \hat{S}_2 \rangle, \langle \hat{S}_3 \rangle)$. Vertical grid lines in (a) and (b) denote $T \equiv 0.7$ and $T = \sqrt{3} \equiv 1.7$, respectively.

### A. Polarization squeezing of the triphotons

Based upon Eq. (11), we begin to investigate the polarization squeezing of the triphotons. Firstly, we have to determine the mean polarization $\langle \hat{S} \rangle = (\langle \hat{S}_1 \rangle, \langle \hat{S}_2 \rangle, \langle \hat{S}_3 \rangle)$, where $\langle \hat{S}_i \rangle = \text{Re}\langle \hat{S}_i \rangle$ and $\langle \hat{S}_i \rangle = \text{Im}\langle \hat{S}_i \rangle$. Because of *imaginary* value of $\langle \hat{S}_i \rangle$, we have $\langle \hat{S}_1 \rangle = \langle \hat{S}_2 \rangle = 0$ [see Fig. 1(b)], i.e., the mean polarization being parallel along the $\hat{S}_3$ axis. In this case, Eq. (15) reduces to $\langle \hat{S}_3 \rangle = -\hat{S}_2 \cos \gamma + \hat{S}_1 \sin \gamma$, where we have set $\phi = \theta = \pi/2$ in Eq. (2). The coefficients given in Eq. (6) can be solved analytically as

$$A = \langle \hat{S}_2^2 - \hat{S}_1^2 \rangle = \frac{15}{8} - \frac{3(9c_3^2 + c_2^2)}{4} + 8\sqrt{3}c_3c_2,$$

$$C = \langle \hat{S}_2^2 + \hat{S}_1^2 \rangle = \frac{15}{8} + \frac{9c_3^2 + c_2^2 - 8\sqrt{3}c_3c_2}{4},$$  

(11)  

where probability amplitudes $c_2$ and $c_3$ are given by Eq. (10). Since $A < 0$ and $B = -\langle \hat{S}_2 \hat{S}_1 + \hat{S}_1 \hat{S}_2 \rangle = 0$, we get the optimal squeezing angle $\gamma_{\text{opt}} = \frac{\pi}{2} [\pi + \text{arctan}(B/A)] = \pi$. Without any ambiguous, our analytic results show that the optimal polarization squeezing is along $\hat{S}_2$ and the anti-squeezing along the $\hat{S}_1$ axis.

Evolution of polarization uncertainty can be illuminated schematically in terms of quasi-probability distribution. As shown in Fig. 2, we plot the Husimi Q function on Poincaré sphere, which is defined as the expectation value of the density matrix operator $\hat{\rho}$ with respect to the SU(2) coherent states $|\psi\rangle$.

$$Q(\theta, \phi) = \langle \theta, \phi | \hat{\rho} | \theta, \phi \rangle. $$  

(12)  

In comparison with Wigner function $[26, 27]$, the Q function is regular, positive definite, and especially suitable for illuminations $[28]$. From Fig. 2(a), we find that the triphoton state at $T = 0$ shows an isotropic quasi-probability distribution because of the minimum uncertainties of two orthogonal polarization components normal to $\langle \hat{S} \rangle$. When $T = 1$, the density of $Q(\theta, \phi)$ becomes an elliptic shape [Fig. 2(b)] due to the squeezing and the anti-squeezing along $\hat{S}_2$ and $\hat{S}_1$, respectively. Via a mapping $p = \cos \theta$, the Husimi Q function can be also plotted in a two-dimensional phase space $(\phi, p)$ [28, 29]. For $T = 0$ and 1 cases, maximal values of $Q(\phi, p)$ appear at $\phi = \pi/2$ and $p = \cos \theta = 0$. Inserting $\theta = \phi = \pi/2$ into the last expression of Eq. (2), one can find that the direction of the mean polarization is in fact along $\hat{S}_3$. It is shown from Fig. 2(c) that the NOON state at $T = \sqrt{3} \equiv 1.7$ shows a threefold symmetrical quasi-probability distribution [17], with its density peaked at the north and the south poles of the $\hat{S}_1$ axis $[p = \cos \theta = \pm 1]$, which indicates all the photons being either horizontally polarized or vertically polarized. To somewhat counter-intuitively, however, such a maximally entangled state shows the reduced variance $V_-$ equal to the SNL [i.e., $2V_\pm - s = 0$B, see Fig. 3(a)], just like quantum uncorrelated coherent states. How to identify the NOON state becomes a subtle but important problem.

### B. Multiparticle entanglement of the triphotons

Shalm et al [17] provide a transparent experimental results to test the relationship between spin squeezing and multiparticle entanglement. Previously, it has been proposed that a useful squeezing for quantum metrology and quantum entanglement obeys: $c_3^2 = 2s(V_\pm)/|\langle \hat{S} \rangle|^2 = (s/(\langle \hat{S} \rangle))^2 < 1$ [9]. Rather than it, Pezzé and Smerzi [20] recently proposed a more general criterion for N-particle entanglement:

$$\chi^2 = \frac{N}{F[\hat{\rho}, \hat{S}_\gamma]} < 1,$$  

(13)  

where $F[\hat{\rho}, \hat{S}_\gamma] = 4V_+$, denoting quantum Fisher information for a pure state $\hat{\rho} = |\Psi\rangle \langle \Psi|$, and thereby $\chi^2 = s/(2V_+)$. According to Ref. [20], Eq. (13) is not only a sufficient condition for multipartite entanglement, but also a sufficient and necessary condition for sub-shot-noise phase estimation. Moreover, it has been shown that the criterion $\chi^2 < 1$ can be used to distinguish and characterize quantum critical behaviors of the Lipkin-Meshkov-Glick model [21].

Substituting Eq. (11) into Eq. (6), we obtain the anti-squeezed variance $V_+ = (\Delta \hat{S}_1)^2 = \frac{1}{2}(9c_3^2 + c_2^2)$, and the squeezed variance:

$$V_- = (\Delta \hat{S}_2)^2 = \frac{1}{4} \left[ \frac{15}{2} - (9c_3^2 + c_2^2 + 8\sqrt{3}c_3c_2) \right].$$  

(14)
where the probability amplitudes $c_2$ and $c_3$ are given by Eq. \((10)\). In Fig. 3 we plot $V_\pm$, $\xi^2$, and $\chi^2$ as a function of the transmissivity ratio $T$. For the SU(2) coherent state at $T = 0$, the mean spin $\langle \hat{S}_z \rangle = s$ [see Fig. 1(b)] and the variances $V_+ = V_- = s/2$ (i.e., 0 dB), which yields $\xi^2 = \chi^2 = 1$. With the increase of $T$ up to 1, the variance $V_-$ decreases along $\hat{S}_1$ at the expense of an increased variance $V_+$ along $\hat{S}_1$. As shown in Fig. 3(a) and (b), the squeezed variance $V_-$ and also $\xi^2$ monotonically decrease to its minimum value $\xi^2_{\text{min}} = 2(V_-)_{\text{min}}/s = 1/3$ ($\sim -4.77$ dB), corresponding to a maximally squeezed state at $T = 1$. However, the squeezing parameter $\xi^2$ reaches its minimal value ($\sim 0.58$) at $T \sim 0.81$ [see blue dot-dash curve of Fig. 3(b)]. This is because of the reduced mean polarization $\langle \hat{S}_z \rangle < s$ and different evolution rates between $V_-$ and $\langle \hat{S}_z \rangle$ [4]. After $T = 1$, both $V_+$ and $V_-$ begin to increase due to the so-called oversqueezing [17]. As mentioned above, the NOON state appears at $T = \sqrt{3}$, where the squeezing parameters $\xi^2 = 1$ and $\xi^2 \rightarrow \infty$ due to $|\langle \hat{S}_z \rangle| \rightarrow 0$. From red dash line of Fig. 3(b), one can find that $\chi^2$ continuously deceases from 1 for the coherent state to the smallest value $1/3$ for the NOON state. The appearance of the NOON state accompanies with a flip of the mean polarization [see Fig. 1(h) (b)]. Such a result keeps hold for $N$-photon NOON state:

$$ |\Psi\rangle_{\text{NOON}} = \frac{1}{\sqrt{2}}(|N,0\rangle_H,V + e^{i\varphi}|0,N\rangle_H,V), \quad (15) $$

where $\varphi$ is an arbitrary phase. Note that the NOON state exhibits vanishing mean polarization $\langle \hat{S}_z \rangle = 0$ and the largest photon number fluctuation between horizontal and vertical modes $V_+ = (\Delta \hat{S}_1)^2 = s^2$ [30]. Moreover, for $|\Psi\rangle_{\text{NOON}}$, the reduced variance $V_- = (\Delta \hat{S}_2)^2 = s/2$, which is finite and equal to the SNL, so $\xi^2 \rightarrow \infty$, $\xi^2 = 1$, and $\chi^2 = 1/N$.

FIG. 2: (Color online) Quasi-probability distribution (Hu simi Q function) $Q(\theta, \phi)$ for SU(2) coherent state at $T = 0$ (a), polarization squeezed state at $T = 1$ (b), and maximally entangled NOON state at $T = \sqrt{3} \approx 1.7$ (c). The top two plots are the Q function on the Poincaré sphere, and the bottom plots are the Q function in a two-dimensional phase space $(\phi, p)$ with $0 \leq \phi < 2\pi$ and $-1 \leq p \leq 1$. The vertical coordinate $p = \cos \theta$, denoting population imbalance between the horizontally and the vertically polarized photons. Red (blue) shading represents a larger (smaller) value of the Q function.

FIG. 3: (Color online) (a) The reduced (solid) and the increased (red dash) uncertainties $V_\pm$ relative to the SNL, $s/2$. (b) The squeezing parameters $\xi^2$ (blue dot-dash), $\xi^2$ (solid), and $\chi^2$ (red dash) as a function of the transmissivity ratio $T$. The maximal squeezing with $\xi^2_{\text{min}} = 2(V_-)_{\text{min}}/s = 1/3$ (i.e., $-4.77$ dB) is obtained at $T = 1$; however, the maximal entangled NOON state appears at $T = \sqrt{3} > 1$ [see also Fig. 1(a)], which shows $\chi^2_{\text{min}} = s/(2V_+)$ [17].

IV. CONCLUSION

In summary, we investigated theoretically polarization squeezing and multipartite entanglement of the triphoton states. Analytical expressions of the reduced and increased variances $V_\pm$ of the Stokes parameters are presented by using standard angular momentum theory. As two different nonclassical effects, we find that polarization squeezing and bipartite entanglement of the triphotons can be measured respectively, by the parameters $\xi^2 = 2(V_-)/s$ and $\chi^2 = s/[2(V_+)]$. In particular, recent experimental observations of the NOON state [17] can be well understood in terms of the entanglement parameter $\chi^2$, which deceases monotonically from the shot-noise limit 1 for the coherent state, to the so-called Heisenberg limit $1/N$ (here $N = 3$) for the maximally entangled NOON state.
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