A KUNNETH FORMULA FOR BREDON
COHOMOLOGY OF PULLBACKS AND TWISTED
K-THEORY OF SOME 6-DIMENSIONAL ORBIFOLDS.

GERMAN COMBARIZA AND MARIO VELÁSQUEZ

Abstract. In this paper we prove a Kunneth formula for Bredon
cohomology for actions of a pullback of groups. We show how this
formula can be used to compute orbifold twisted K-theory for some
discrete twistings. Using that result, we compute orbifold K-theory
for some 6-dimensional orbifolds introduced by Vafa and Witten.
These examples also show the limitations of the method.

Introduction

One of the useful tools in singular cohomology is the Kunneth for-
mula. Let $X$ and $Y$ be topological spaces, it allows to compute $H^*(X \times Y)$ given knowledge of $H^*(X)$ and $H^*(Y)$. An interesting problem is
to extend that result to extraordinary cohomology theories. A remark-
able case is equivariant K-theory, introduced by Segal in [Seg68] an d
defined for proper actions of discrete groups in [LO01]. Let $\Gamma$ be a dis-
crete group acting properly on spaces $X$ and $Y$, one wants to compute

$$K^*_G(X \times Y)$$

in terms of $K^*_G(X)$ and $K^*_G(Y)$. In this case, to obtain an analogue to
the Kunneth formula is an open problem for actions of discrete groups
(even for finite groups). For actions of $\mathbb{Z}/2\mathbb{Z}$ a Kunneth theorem was
proved in [Ros13].

An approximation of that problem is to consider Bredon cohomology.
When we take coefficients in the Bredon module of representations
(see Def. 1.3) there is an spectral sequence converging to equivariant
K-theory whose $E^2$-term correspond Bredon cohomology.

Let $\Gamma$ be a discrete group which is obtained as a pullback diagram

(0.1)

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{p_2} & H \\
\xrightarrow{p_1} & & \xrightarrow{\pi_2} \\
G & \xrightarrow{\pi_1} & K \\
\end{array}
\]
When the maps \( \pi_1 \) and \( \pi_2 \) are clear from the context we denote this pullback by \( G \times_K H \).

If \( X \) is a proper \( G \)-CW-complex and \( Y \) is a \( H \)-CW-complex, the space \( X \times Y \) is naturally a \( \Gamma \)-CW-complex. In Theorem 3.11 we provide a Kunneth formula computing \( \Gamma \)-equivariant Bredon cohomology of \( X \times Y \) in terms of Bredon cohomology of \( X \) and \( Y \).

One of the advantages to work with pullbacks is that this includes some interesting cases. For example

- If \( K \) is the trivial group we obtain a Kunneth formula for the action of the direct product \( G \times H \) over \( X \times Y \).
- If \( K = G = H \), we obtain a Kunneth formula for the diagonal action of \( G \) over a product \( X \times Y \).

On the other hand if we take Bredon cohomology with constant coefficients we obtain a Kunneth formula for group cohomology of pullbacks.

As our main application in Corollary 4.8 and Theorem 4.12 we compute the orbifold K-theory and twisted orbifold K-theory (for an specific twisting) of some celebrated examples presented in [VW95].

These orbifolds are 6-dimensional, this makes very difficult to compute their K-theories directly. Using our formula we obtain a simple calculation of it. These examples also show the limitations of our method (Remark 4.9).

This paper continues the work started in [BJV13] where some calculations concerning pullbacks of groups were made. We clarify some questions that appear in that work about the computation of twisted K-theory for pullbacks.

1. Tensor product of Bredon modules

In this section we introduce the notion of tensor product of Bredon modules over a Green functor.

Let \( R \) be a commutative ring with unity. Let \( \text{Or}(G) \) be the orbit category of \( G \); it is a category with objects \( G/H \) for each subgroup \( H \subseteq G \) and with morphisms given by \( G \)-equivariant maps where every \( G \)-equivariant map \( G/H \to G/K \) is multiplication by an element \( g \in G \) such that \( gHg^{-1} \subseteq K \). In a similar way one can also define the orbit category relative to a family of groups \( \mathcal{F} \), which we denote by \( \text{Or}_\mathcal{F}(G) \).

For more information about \( \text{Or}_\mathcal{F}(G) \) the reader can consult [DL98].

Recall that a Bredon \( R \)-module over \( \mathcal{F} \) (or simply a Bredon module over \( \mathcal{F} \) if \( R \) is clear from the context) is a contravariant functor

\[
\text{Or}_\mathcal{F}(G) \to R \text{-MODULES}.
\]
A Mackey functor over $\text{Or}_\mathcal{F}(G)$ is a bifunctor

$$\mathcal{M} = (\mathcal{M}_*, \mathcal{M}^*) : \text{Or}_\mathcal{F}(G) \to R - \text{MODULES},$$

satisfying certain double coset formula. For details consult [Web00].

**Definition 1.1.** A Green functor over $\text{Or}_\mathcal{F}(G)$ is a Mackey functor $\mathcal{N}$ together with a natural pairing $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$ such that for every $G/P \in \text{Or}_\mathcal{F}(G)$ the map $\mathcal{N}(G/P) \times \mathcal{N}(G/P) \to \mathcal{N}(G/P)$ makes $\mathcal{N}(G/P)$ a commutative ring.

**Example 1.2.** Let $R$ be a commutative ring with unity. The trivial functor $R : \text{Or}_\mathcal{F}(G) \to R - \text{MODULES}$ such that $G/H \mapsto R$ for every subgroup $H \in \mathcal{F}$, is a Green functor.

**Example 1.3.** Let $G$ be a discrete group. Let $\mathcal{FLN}_G$ be the family of finite subgroups of $G$ (or simply $\mathcal{FLN}$ if $G$ is clear from the context). The representation functor is a Bredon module $\mathcal{R}^G : \text{Or}_{\mathcal{FLN}}(G) \to \mathcal{Z} - \text{MODULES}$ which associates to every object $G/H$ with $H \in \mathcal{FLN}$ its representation ring $R(H)$. The representation functor is a Green functor.

Now we will define a notion of tensor product of Bredon modules over a Green functor. To define it we need to recall the notion of module over a Green functor.

**Definition 1.4.** Let $\mathcal{N}$ be a Green functor over $\text{Or}_\mathcal{F}(G)$. An $\mathcal{N}$-module consists of a Bredon module $\mathcal{M}$ over $\text{Or}_\mathcal{F}(G)$ and a natural transformation $\mathcal{N} \times \mathcal{M} \to \mathcal{M}$ such that for every subgroup $P \in \mathcal{F}$, each pairing morphism $\mathcal{N}(G/P) \times \mathcal{M}(G/P) \to \mathcal{M}(G/P)$ endows to $\mathcal{M}(G/P)$ with a unitary $\mathcal{N}(G/P)$-module structure.

**Definition 1.5.** Let $\mathcal{F}$ a family of subgroups of $G$. Let $f : G \to K$ be a group homomorphism, and let $\mathcal{N}$ be a Green functor over $\text{Or}_{\mathcal{F}(K)}$ we define the Green functor $f^*\mathcal{N}$ over $\text{Or}_\mathcal{F}(G)$ as follows

$$f^*\mathcal{N} : \text{Or}_\mathcal{F}(G) \to \mathcal{Z} - \text{MODULES}$$

$$G/P \to \mathcal{N}(K/f(P)).$$

**Example 1.6.** Let $\Gamma$ be a group as in the diagram (0.1). The Bredon module $\mathcal{R}^\Gamma$, is a $p_1^*\mathcal{R}^G$-module, a $p_2^*\mathcal{R}^H$-module as well a $(\pi_1 \circ p_1)^*\mathcal{R}^K$-module.
Definition 1.7. Let $\Gamma$ be a group coming from a diagram (0.1). Let $\mathcal{F}$ be a family of subgroups of $K$, $\mathcal{F}_1$ be a family of subgroups of $G$ and $\mathcal{F}_2$ be a family of subgroups of $H$. We say that $(\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2)$ are a sequence of compatible families of subgroups for $\Gamma$ if $\pi_1(\mathcal{F}_1) \subseteq \mathcal{F}$ and $\pi_2(\mathcal{F}_2) \subseteq \mathcal{F}$.

Let $(\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2)$ be a sequence of compatible families of subgroups of $\Gamma$, we denote by $\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ to the family of subgroups of $\Gamma$ defined as

$$\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2 = \{ P \times_{\pi_1(P)} Q \mid P \in \mathcal{F}_1, \text{ and } Q \in \mathcal{F}_2 \}.$$

Definition 1.8. Consider the diagram (0.1). Let $(\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2)$ be a sequence of compatible families of subgroups for $\Gamma$. Let $N$ be a Bredon module over $\text{Or}_{\mathcal{F}_1}(K)$, let $M_1$ be a $(\pi_1^* N)$-module over $\text{Or}_{\mathcal{F}_1}(G)$ and $M_2$ be a $(\pi_2^* N)$-module over $\text{Or}_{\mathcal{F}_2}(H)$. We define the tensor product

$$(M_1 \otimes_N M_2) : \text{Or}_{\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2}(\Gamma) \to \mathbb{Z} - \text{MODULES}$$

$$\Gamma/(P \times_{\pi_1(P)} Q) \mapsto M_1(G/P) \otimes_{\text{Or}_{\mathcal{F}_1}(K/\pi_1(P))} M_2(H/Q).$$

2. Bredon cohomology associated to twisted K-theory

In this section we briefly recall the definition of Bredon cohomology with coefficients in a Bredon module, then define twisted K-theory for discrete torsion and using the coefficients of this theory we define a Bredon module associated to twisted K-theory.

2.1. Bredon cohomology.

Definition 2.1. Let $\mathcal{F}$ be a family of subgroups of $G$. A $G$-CW complex is a CW complex with a $G$-action permuting the cells and such that if a cell is sent to itself, this is done by the identity map. A $\mathcal{F}$-CW-complex is a $G$-CW-complex where all cell stabilizers are elements of $\mathcal{F}$. A $\mathcal{F}T\mathcal{L}$-CW-complex is called a proper $G$-CW-complex.

Let $X$ be a $\mathcal{F}$-CW-complex. The cellular chain complex of $X$ is defined as the $\mathbb{Z}$-graded Bredon module

$$C^G_n(X) : \text{Or}_{\mathcal{F}}(G) \to \mathbb{Z} - \text{MODULES}$$

$$C^G_n(X)(G/H) = H_n((X^n)^H, (X^{n-1})^H; \mathbb{Z})$$

together with a boundary map

$$\partial : C^G_0(X) \to C^G_{-1}(X)$$

induced from the cellular boundary map, with $\partial^2 = 0$.

Consider two Bredon modules $\mathcal{M}$ and $\mathcal{M}'$, we denote by

$$\text{Hom}_{\text{Or}_{\mathcal{F}}(G)}(\mathcal{M}, \mathcal{M}')$$
the abelian group of natural transformations from \( M \) to \( M' \). Let \( X \) be a \( \mathcal{F} \)-G-CW-complex and \( M \) a Bredon module over \( \text{Or}_{\mathcal{F}}(G) \), we define the *Bredon cochain complex with coefficients in \( M \) as*

\[
C^n_{\mathcal{F}}(X; M) = \text{Hom}_{\text{Or}_{\mathcal{F}}(G)}(\bigcup_n(X), M), \quad \text{with} \; \delta = \text{Hom}_{\text{Or}_{\mathcal{F}}(G)}(\partial, id).
\]

The homology of this complex is called the *Bredon cohomology of \( X \) with coefficients in \( M \) with respect to the family \( \mathcal{F} \), denoted by \( H^*_{\mathcal{F}}(X; M) \) or also by \( H^*_G(X; M) \) when the family \( \mathcal{F} \) is clear from the context.

2.2. **Twisted K-theory.** Equivariant twisted K-theory was defined by Atiyah and Segal in [AS04] for actions of compact Lie groups. The case of proper actions of discrete groups was treated in [BEJU14] for general twistings. On the other hand, the approach in [AR03] for orbifold twisted K-theory for discrete torsion has the advantage of a more concrete description using finite dimensional vector bundles. As is showed in [Dwy08], the last approach can be adapted to the case of proper actions of discrete groups and discrete torsion. In this section we briefly recall this construction.

Let \( V \) be a finite dimensional complex vector space, let \( \text{GL}(V) \) be the group of invertible linear transformations of \( V \). Consider the central extension

\[
0 \rightarrow \mathbb{C}^* \rightarrow \text{GL}(V) \xrightarrow{\pi} \text{GL}(V)/\mathbb{C}^* \rightarrow 0.
\]

We denote the quotient group in the above extension by \( \text{PGL}(V) \).

**Definition 2.3.** A projective representation of \( G \) is a pair \((\rho, V)\) where \( V \) is a finite dimensional complex vector space and \( \rho: G \rightarrow \text{PGL}(V) \) is a homomorphism.

Now if we choose a map (not a homomorphism)

\[ f: \text{PGL}(V) \rightarrow \text{GL}(V) \]

such that \( \pi \circ f = id \) and take the pullback of the extension \( 2.2 \), we have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathbb{C}^* & \xrightarrow{id} & \text{GL}(V) & \xrightarrow{\pi} & \text{PGL}(V) & \rightarrow & 0 \\
\downarrow{\text{id}} & & \downarrow{\bar{\rho}} & & \downarrow{\rho} & & \downarrow{\rho} & & \downarrow{\rho} \\
0 & \rightarrow & \mathbb{C}^* & \xrightarrow{\rho^* G} & \bar{G} & \xrightarrow{\pi_1} & G & \rightarrow & 0
\end{array}
\]
Consider the map \( \sigma = f \circ \rho \) and define
\[
\alpha : G \times G \to \mathbb{C} \subseteq \text{GL}(V)
\]
\[
(g, h) \mapsto \alpha(g, h) = \sigma(g)\sigma(h)\sigma(gh)^{-1}.
\]
An straightforward calculation shows that \( \alpha \in Z^2(G; \mathbb{C}^*) \).

**Definition 2.4.** Let \( V \) be a complex vector space. Let us fix \( \alpha \in Z^2(G; \mathbb{C}^*) \). An \( \alpha \)-projective representation of \( G \) is a map \( \sigma : G \to \text{GL}(V) \) satisfying
\[
(i) \quad \sigma(g)\sigma(h) = \alpha(g, h)\sigma(gh) \quad \text{for all } g, h \in G, \quad \text{and}
\]
\[
(ii) \quad \sigma(1) = \text{id}.
\]
Notice that \( \alpha \) has to be a cocycle in \( Z^2(G, \mathbb{C}^*) \), satisfying \( \alpha(g, 1) = \alpha(1, g) = 1 \).

The direct sum of two \( \alpha \)-twisted representations is again an \( \alpha \)-twisted representation. We denote by \( R_\alpha(G) \) to the Grothendieck group associated to the monoid of isomorphism classes of \( \alpha \)-twisted representations of \( G \).

The tensor product of an \( \alpha \)-twisted representation with a \( \beta \)-twisted representation is an \((\alpha + \beta)\)-twisted representation. It can be extended to Grothendieck groups obtaining a product \( R_\alpha(G) \otimes R_\beta(G) \to R_{\alpha + \beta}(G) \).

An interesting property of the groups \( R_\alpha(G) \) is the following.

**Theorem 2.5.** If \( \alpha \) is cohomologous to \( \beta \), then there is a bijective correspondence between \( \alpha \)-twisted representations of \( G \) and \( \beta \)-twisted representations of \( G \), and an isomorphism of abelian groups
\[
R_\alpha(G) \cong R_\beta(G).
\]
More details on projective representations can be found in Section 2 of [Dwy08] or in [Kar94].

Now we recall the definition of equivariant twisted K-theory for this kind of twistings. Let \( \alpha \in Z^2(G, S^1) \) be a torsion cocycle of order \( n \). One can assume that the cocycle takes values in \( \mathbb{Z}/n\mathbb{Z} \subseteq S^1 \). We call such a cocycle normalized. Then we can assign to \( \alpha \) a central extension of \( G \) by \( \mathbb{Z}/n\mathbb{Z} \) as follows.

\[
(2.6) \quad 1 \to \mathbb{Z}/n\mathbb{Z} \to G_\alpha \xrightarrow{\rho} G, \to 1
\]
where \( G_\alpha \) as a set is \( G \times \mathbb{Z}/n\mathbb{Z} \) and if we denote by \( \sigma \) a fixed generator of \( \mathbb{Z}/n\mathbb{Z} \), the product is given by
\[
(g, \sigma^j) \cdot (h, \sigma^k) = (gh, \alpha(g, h)\sigma^{j+k}).
\]
We have the following theorem. For a proof consult [Dwy08].
**Theorem 2.7.** Let $G$ be a finite group and let $\alpha$ be a 2-cocycle taking values in $\mathbb{Z}/n\mathbb{Z}$. There is a bijective correspondence between $\alpha$-twisted representations of $G$ and representations of $G_\alpha$ where $\sigma$ acts by multiplication by $e^{2\pi i/n}$.

**Definition 2.8.** Let $G$ be a discrete group and $\alpha \in Z^2(G, S^1)$ be a normalized cocycle of finite order $n$, consider the central extension \( (2.6) \) associated to $\alpha$. Let $X$ be a finite, proper $G$-CW-complex. An $\alpha$-twisted $G$-vector bundle over $X$ is a complex $G_\alpha$-vector bundle $p : E \to X$ such that

(i) The action of $G_\alpha$ on $E$ covers the action of $G$ on $X$. That means, for every $g \in G_\alpha$ and $z \in E$

\[ p(g \cdot z) = \rho(g) \cdot p(z). \]

(ii) The action of $\mathbb{Z}/n\mathbb{Z}$ on $E$ is given by multiplication by $e^{2\pi i/n}$ on the fibers.

If $E$ and $F$ are both $\alpha$-twisted $G$-vector bundles, then so is $E \oplus F$. Then one can consider the monoid of isomorphism classes of $\alpha$-twisted vector bundles with the operation of direct sum. Let $^\alpha K_G(X)$ the associated Grothendieck group. Similarly to the case of twisted representations, $^\alpha K_G(X)$ is not a ring, but we have an external product

\[ ^\alpha K_G(X) \otimes _\beta K_G(X) \to ^{\alpha+\beta} K_G(X), \]

analogous to the product in twisted representations. For details on this product see Theorem 3.4 in [Dwy08].

One crucial property of the above construction of twisted K-theory is the following. Consider an extension as \( (2.6) \), we have an inclusion

\[ ^\alpha K_G(X) \to K_{G_\alpha}(X), \]

for every finite proper $G$-CW-complex. It allows to extend many results obtained for equivariant K-theory for proper actions to the twisted case. In particular results from [LO01].

**Definition 2.9** (Def. 4.4 in [AR03]). Suppose that $X/G$ (With $G$ a finite group) is a quotient orbifold, and let $\alpha \in Z^2(G; S^1)$ be a torsion 2-cocycle, then one can define its $\alpha$-twisted orbifold K-theory just as its $\alpha$-twisted $G$-equivariant K-theory, in other words

\[ ^\alpha K^*_\text{orb}(X/G) = ^\alpha K^*_G(X). \]

**Remark 2.10.** There is a more general definition of twisted K-theory taking twistings as projective unitary stable equivariant bundles, they are classified by third degree integer cohomology classes of $X \times_G EG$ (see [AS04] and [BEJU14]). However given a torsion cocycle $\alpha \in$
it is possible to associate a projective unitary stable equivariant bundle to $\alpha$ in such way that both definition of twisted K-theory are equivalent in this case. For details see Sec. 5.4 in [BEUV13].

2.3. The Bredon module of projective representations. Fix a normalized torsion cocycle $\alpha \in Z^2(G, S^1)$. We define a Bredon module $R_\alpha$ over $\text{Or}_{FIN}(G)$. On objects its defined by $R_\alpha(G/H) = R_\alpha(H)$ and on morphism as follows, if $\phi : G/H \to G/K$ is a morphism in $\text{Or}_{FIN}(G)$, $\phi$ is determined by an element $g \in G$ with $gHg^{-1} \subseteq K$ and $\phi(g'H) = g'gK$. We define $R_\alpha(\phi) : R_\alpha(K) \to R_\alpha(H)$ as the composition

$$R_\alpha(K) \xrightarrow{\cdot g} R_\alpha(gHg^{-1}) \xrightarrow{\cdot g} R_\alpha(H),$$

where the first map is the pullback of the inclusion $i : gHg^{-1} \to K$ and the last one is the pullback of the isomorphism induced by conjugation by $g$.

**Definition 2.11.** Let $X$ be a proper $G$-CW-complex. The graded group $H^*_G(X; R_\alpha)$ is called the $\alpha$-twisted Bredon cohomology of $X$.

2.4. Spectral sequence. There is an spectral sequence constructed by Davis and Luck converging to equivariant K-theory for proper $G$-CW-complexes whose $E_2$-term is given in terms of Bredon cohomology. One can adapt this spectral sequence to be used in the case of twisted K-theory for discrete torsion. For a more general approach the reader can consult [BEUV13].

**Theorem 2.12.** Let $X$ be a finite $G$-CW-complex and let $\alpha \in Z^2(G, S^1)$ be a normalized torsion cocycle. Then there is a spectral sequence with

$$E_2^{p,q} = \begin{cases} H^p_G(X; R_\alpha) & \text{if } q \text{ is even} \\ 0 & \text{if } q \text{ is odd} \end{cases}$$

so that $E_\infty^{p,q} \Rightarrow \alpha K^{p+q}_G(X)$.

3. Pullback of groups

Let $\Gamma$ be a group as in diagram (0.1). In this section we describe the representation ring of a finite subgroup of $\Gamma$ in terms of the representation rings of finite subgroups of $G, H$ and $K$. Using that description and the algebraic Kunneth formula we obtain the main result of the paper.

If the group $\Gamma$ comes from a diagram (0.1) then it is isomorphic to a subgroup of $G \times H$, namely $\Gamma \cong \{(g, h) \in G \times H \mid \pi_1(g) = \pi_2(h)\}$. 
3.1. **The representation ring of a pullback.** Consider a pullback diagram of finite groups

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{p_2} & Q \\
\downarrow{p_1} & & \downarrow{\pi_2} \\
P & \xrightarrow{\pi_1} & S
\end{array}
\]

If we apply the representation ring functor we obtain the following diagram

\[
\begin{array}{ccc}
R(\Lambda) & \xleftarrow{p_2^*} & R(Q) \\
\downarrow{p_1^*} & & \downarrow{\pi_2^*} \\
R(P) & \xleftarrow{\pi_1^*} & R(S).
\end{array}
\]

We will prove that diagram (3.2) is a pushout. In other words we have the following theorem.

**Theorem 3.3.** Let $\Lambda$, $Q$, $P$ and $S$ be finite groups as in diagram (3.1). There is a ring isomorphism

\[m : R(P) \otimes_{R(S)} R(Q) \to R(\Lambda)\]

**Proof.** In order to avoid confusion, in this proof we denote the product on $R(\Lambda)$, $R(P)$ and $R(Q)$ by $\cdot$ and the generators of the tensor product by $\rho \otimes \gamma$.

The map $m$ is defined as

\[m : R(P) \otimes_{R(S)} R(Q) \to R(\Lambda) \]

\[\rho \otimes \gamma \mapsto p_1^*(\rho) \cdot p_2^*(\gamma).\]

First we prove that the map $m$ is well defined. Let $\xi \in R(S)$, $\rho \in R(P)$ and $\gamma \in R(Q)$, it is enough to prove that the elements

\[m(\pi_1^*(\xi) \cdot \rho \otimes \gamma) \text{ and } m(\rho \otimes \pi_2^*(\xi) \cdot \gamma)\]

in $R(\Lambda)$ have the same character. Let $(g, h) \in \Lambda$

\[\text{Ch} \left(m(\pi_1^*(\xi) \cdot \rho \otimes \gamma)\right)(g, h) = \text{Ch} \left( p_1^*(\pi_1^*(\xi)) \cdot p_1^*(\rho) \cdot p_2^*(\gamma)\right)(g, h)\]

\[= \text{Ch}(\pi_1^*(\xi) \cdot \rho)(g) \text{Ch}(\gamma)(h)\]

\[= \text{Ch}(\xi)(\pi_1(g)) \text{Ch}(\rho)(g) \text{Ch}(\gamma)(h)\]

\[= \text{Ch}(\rho)(g) \text{Ch}(\xi)(\pi_2(h)) \text{Ch}(\gamma)(h)\]

\[= \text{Ch}(m(\rho \otimes \pi_2^*(\xi) \cdot \gamma))(g, h)\]
Now we will prove that \( m \) is an isomorphism. Consider the following diagram with exact rows

\[
\begin{array}{cccccc}
0 & \rightarrow & \ker(\pi) & \rightarrow & R(P) \otimes_{\mathbb{Z}} R(Q) & \rightarrow & R(P) \otimes_{R(S)} R(Q) & \rightarrow & 0 \\
\downarrow m_2 & & \downarrow m_1 & & \downarrow m & & \\
0 & \rightarrow & \ker(i^*) & \rightarrow & R(P \times Q) & \rightarrow & R(\Lambda) & \rightarrow & 0
\end{array}
\]

where map \( \pi \) is the quotient by all relations defining the tensor product over \( R(S) \), the map \( i^* \) is the pullback of the inclusion \( i : \Lambda \rightarrow P \times Q \), the map \( m_1 \) is the natural isomorphism given by tensor product over \( \mathbb{Z} \) and the map \( m_2 \) is the restriction of \( m_1 \) to \( \ker(\pi) \). We will prove that the above diagram is commutative and that \( m_2 \) is an isomorphism.

First we need to verify that \( m_1(\ker(\pi)) \subseteq \ker(i^*) \). Let \( (g, h) \in \Lambda \),

\[
\mathcal{C}(i^*(m_2(\pi_1^*(\xi) \cdot \rho \otimes \gamma - \rho \otimes \pi_2^*(\xi) \cdot \gamma)))(g, h) = \mathcal{C}(p_1^*(\pi_1^*(\xi)) \cdot p_1^*(\rho) \cdot p_2^*(\gamma) - p_1^*(\rho) \cdot p_2^*(\pi_2^*(\xi)) \cdot p_2^*(\gamma))(g, h) = 0
\]

Now we prove that \( \ker(i^*) = m_1(\ker(\pi)) \). For this we will prove that if \( f \) is a class function in \( P \times Q \) such that \( i^*(f) \equiv 0 \) and \( f \) is orthogonal to every character in \( m_1(\ker(\pi)) \), then \( f \) has to be zero.

Suppose that for every \( \xi \in R(S) \), \( \rho \in R(P) \) and \( \gamma \in R(Q) \)

\[
\sum_{(g, h) \in P \times Q} f(g, h) \mathcal{C}(\rho)(g) \mathcal{C}(\gamma)(h)[\mathcal{C}(\xi)(\pi_2(h)) - \mathcal{C}(\xi)(\pi_1(g))]| = 0.
\]

Let us fix \( \rho \in R(P) \) and let

\[
\eta(g) = \sum_{h} f(g, h) \mathcal{C}(\gamma)(h)[\mathcal{C}(\xi)(\pi_2(h)) - \mathcal{C}(\xi)(\pi_1(g))].
\]

We observe that \( \eta \) is a class function on \( P \) that is orthogonal to every \( \rho \) in \( R(P) \), then \( \eta \equiv 0 \).

By a similar argument we conclude that for every \( (g, h) \in P \times Q \) and \( \xi \in R(S) \)

\[
(3.4) \quad f(g, h)[\mathcal{C}(\xi)(\pi_2(h)) - \mathcal{C}(\xi)(\pi_1(g))] = 0.
\]

We already know that \( f(g, h) = 0 \) if \( (g, h) \in \Lambda \), then let \( (g, h) \notin \Lambda \), we have two cases. First suppose that \( \pi_1(g) \) is conjugate to \( \pi_2(h) \) in \( S \), in this case there is \( h \in Q \) such that \((g, hh^{-1}h) \in \Lambda \) and then \( f(g, h) = f(g, hh^{-1}) = 0 \).

Suppose now that \( \pi_1(g) \) is not conjugate to \( \pi_2(h) \) in \( S \), in this case there is \( \xi \in R(S) \) such that \( \mathcal{C}(\xi)(\pi_1(g)) \neq \mathcal{C}(\xi)(\pi_2(g)) \) and equation \((3.4)\) gives us that \( f(g, h) = 0 \). Then we conclude that \( \ker(i^*) = m_1(\ker(\pi)) \). The map \( m_2 \) is an isomorphism because it is the restriction
of $m_1$ and since the diagram is commutative we conclude that $m$ is a ring isomorphism. □

An analogous result is valid considering the twisted representation group.

**Corollary 3.5.** Let $\Lambda$, $Q$, $P$ and $S$ be finite groups as in the diagram (3.1), and let $\alpha \in Z^2(P; \mathbb{Z}/n\mathbb{Z})$ be a normalized torsion cocycle. There is an isomorphism of $R(S)$-modules

$$m : R_\alpha(P) \otimes_{R(S)} R(Q) \to R_{p_1^*(\alpha)}(\Lambda).$$

### 3.2. Bredon cohomology of pullbacks.

If $X$ is a proper $G$-CW-complex and $Y$ is a proper $H$-CW-complex, the product $X \times Y$ has a natural structure of $(G \times H)$-CW-complex with each cell corresponding to a product of cells of $X$ and $Y$. Since $\Gamma$ is a subgroup of $G \times H$ by Corollary 3.4.4 on [MS06] we can suppose that $X \times Y$ has a structure of $\Gamma$-CW-complex. For this $\Gamma$-space we have the following version of the Eilenberg-Zilber theorem.

**Proposition 3.6 (Eilenberg-Zilber).** Let $X$ be a proper $G$-CW-complex and $Y$ be a proper $H$-CW-complex. There is a natural equivalence of $\text{Or}_{\mathcal{FIN}_G \times K_{\mathcal{FIN}_H}}(\Gamma)$-chain complexes

$$C^*_\Gamma(X \times Y) \cong (C_*^G(X) \otimes C_*^H(Y))|_{\Gamma}.$$

**Proof.** Let $P \times_{\pi_1(P)} Q$ be an element in $\mathcal{FIN}(G) \times_K \mathcal{FIN}(H)$. It is a well known fact that we have an isomorphism of chain complexes

$$f_{P,Q} : C_*^G(X^P) \otimes C_*^H(Y^Q) \to C_*^G(X^P) \otimes_{\mathbb{Z}} C_*^H(Y^Q),$$

$$e \otimes f \to e \otimes f.$$

We only need to verify that it is a natural transformation. Consider

$$(p, q)_* : \Gamma/P \times_{\pi_1(P)} Q \to \Gamma/P' \times_{\pi_1(P')} Q'$$

a morphism in $\text{Or}_{\mathcal{FIN}(G) \times_K \mathcal{FIN}(H)}(\Gamma)$.

We need to prove that the following diagram commutes

$$\begin{array}{ccc}
C_n((X \times Y)^P \times_{\pi_1(P)} Q^P) & \xrightarrow{(p,g)_*} & C_n((X \times Y)^{P'} \times_{\pi_1(P')} Q'^P) \\
\bigoplus_{i+j=n} C_i(X^P) \otimes_{\mathbb{Z}} C_j(Y^Q) & \xrightarrow{\oplus (p_* \otimes q_*)} & \bigoplus_{i+j=n} C_i(X^{P'}) \otimes_{\mathbb{Z}} C_j(Y^{Q'})
\end{array}$$

But this is clear because

$$(p_* \otimes q_*) \circ f_{P,Q}(e \otimes f) = (p_* \otimes q_*)(e \otimes f) = pe \otimes qf,$$
and
\[ f_{P,Q} \circ ((p, q)_s(e \times f)) = f_{P,Q}(pe \times qf) = pe \times qf. \]

Then we have a natural equivalence of Bredon modules analogous to the isomorphism in Theorem 3.3.

**Theorem 3.7.** There is a natural equivalence of Bredon modules
\[ \mathcal{R}^G \otimes_{\mathcal{R}^K} \mathcal{R}^\Gamma \xrightarrow{\tilde{m}} \mathcal{R}^\Gamma. \]

**Proof.** The map \( \tilde{m} \) is defined using the isomorphism in Theorem 3.3. Let \( \Gamma/(P \times_{\pi_1(P)} Q) \in \text{Or}_{\mathcal{F} \mathcal{F} \mathcal{N}_G \times_K \mathcal{F} \mathcal{F} \mathcal{N}_H}(\Gamma) \), we define
\[ \tilde{m}(\Gamma/(P \times_{\pi_1(P)} Q)) = m_{P,Q}, \]
where \( m_{P,Q} \) is the isomorphism in Theorem 3.3 for the group \( P \times_{\pi_1(P)} Q \).

We only need to verify that this map is natural. Consider a morphism \((g, h) : \Gamma/(P \times_{\pi_1(P)} Q) \to \Gamma/(P' \times_{\pi_1(P')} Q')\), recall that this morphism is characterized by the condition
\[ (P \times_{\pi_1(P)} Q)^{(g,h)} \subseteq (P' \times_{\pi_1(P')} Q'). \]

Following the notation in the proof of Theorem 3.3 we have
\[ (g, h)^*(m(\xi \otimes \gamma)) = (g, h)^*(p_1^*(\xi) \cdot p_2^*(\gamma)) = (p_1^*(\xi) \cdot p_2^*(\gamma)) \mid_{(P \times_{\pi_1(P)} Q)^{(g,h)}}. \]

On the other hand
\[ m((g^* \otimes h^*)(\xi \otimes \gamma)) = m(\xi \mid_{P_0} \otimes \gamma \mid_{Q_0}) = p_1^*(\xi \mid_{P_0}) \cdot p_2^*(\gamma \mid_{Q_0}). \]

And the last terms of both expressions are the same because pullbacks commute with restrictions on the representation ring. Then the map \( \tilde{m} \) is natural. \( \square \)

For a Bredon module \( \mathcal{M} \) over \( \text{Or}_\mathcal{F}(G) \) one can define the Bredon cohomology groups of \( G \) with coefficients in \( \mathcal{M} \) by
\[ H^i(G; \mathcal{M}) = \text{Ext}^i(\mathbb{Z}, \mathcal{M}), \]
where \( \text{Ext}^i \) is certain functor generalizing the Ext-functor defined for groups to the context of Bredon modules. For details consult [MV03] on pages 7-27.

In particular we have the following result

**Proposition 3.8.** There is an isomorphism
\[ H^0(G; \mathcal{M}) \cong \text{colim}_{G/K \in \text{Or}_\mathcal{F}(G)} \mathcal{M}(G/K), \]
where we are taking the inverse limit with respect to the induced morphism from the orbit category \( \text{Or}_\mathcal{F}(G) \).
If $\mathcal{M}$ is a Green functor it endows every $\mathcal{M}(G/K)$ with an $H^0(G; \mathcal{M})$-module structure.

**Theorem 3.9.** Let $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{N}$ be Bredon modules over $G$, $H$ and $K$ respectively, suppose that $\mathcal{M}$ is a $\pi_1^*\mathcal{N}$-module and $\mathcal{M}'$ is a $\pi_2^*\mathcal{N}$-module. Then there is an isomorphism of $\text{Or}_{\text{FIN}G \times \text{FIN}K}^{\text{FIN}H}(\Gamma)$-cochain complexes

$$\text{Hom}_{\text{Or}_{\text{FIN}G}(\Gamma)}( (C_*(X) \otimes C_*(Y)) |_{\Gamma}, \mathcal{M} \otimes_\mathcal{N} \mathcal{M}'),$$

where $C_*(X, \mathcal{M}) \otimes_{H^0(K; \mathcal{N})} C_*(Y, \mathcal{M}')$.

**Proof.** At degree $n$ the left hand side cochain complex is

$$\bigoplus_{e, f} \text{Hom}_Z \left( Z[e, f], \mathcal{M}(G/P) \otimes_{H^0(K; \mathcal{N})} \mathcal{M}'(H/Q) \right).$$

At degree $n$ the right hand side is

$$\left( \bigoplus_{e} \text{Hom}(Z[e], \mathcal{M}(G/P)) \right) \otimes_{H^0(K; \mathcal{N})} \left( \bigoplus_{f} \text{Hom}(Z[f], \mathcal{M}'(H/Q)) \right).$$

There is an isomorphism of cochain complexes of $H^0(K; \mathcal{N})$-modules from the right hand side to

$$\bigoplus_{\lambda, \mu} \text{Hom}_Z \left( (Z[e, \lambda] \otimes Z[f, \mu]), \mathcal{M}(G/P) \otimes_{H^0(K; \mathcal{N})} \mathcal{M}'(H/Q) \right).$$

For every $P_{\lambda} \times_{\pi_1(P_{\lambda})} Q_{\mu} \in \text{FIN}G \times_K \text{FIN}H$ we have that the $H^0(K; \mathcal{N})$-module structure of $\mathcal{M}(G/P_{\lambda})$, respectively of $\mathcal{M}'(H/Q_{\mu})$, factors through the sequences

$$H^0(K; \mathcal{N}) \to N(K/\pi_1(P_{\lambda})) \to M(G/P_{\lambda})$$

and

$$H^0(K; \mathcal{N}) \to N(K/\pi_1(P_{\lambda})) \to M'(H/Q_{\mu}).$$

Then the $H^0(K; \mathcal{N})$-module

$$\mathcal{M}(G/P_{\lambda}) \otimes_{H^0(K; \mathcal{N})} \mathcal{M}'(H/Q_{\mu})$$

is isomorphic to

$$\mathcal{M}(G/P_{\lambda}) \otimes_{N(K/\pi_1(P_{\lambda}))} \mathcal{M}'(H/Q_{\mu}).$$

Given that isomorphism is compatible with the coboundary map, then we have an isomorphism of cochain complexes.

\[ \square \]

In order to obtain a Kunneth formula for Bredon cohomology of a $\Gamma$-CW-complex $X \times Y$ we need to recall the algebraic Kunneth formula, for a proof see for example [Wei94, Thm. 3.6.3].
Lemma 3.10. Let $R$ be a commutative ring with unity and let $(C,d)$ and $(C',d')$ be $R$-cochain complexes, if $C_n$ and $d(C_n)$ are flat for each $n$ then there is a split exact sequence

$$0 \to \bigoplus_{p+q=n} H^p(C) \otimes_R H^q(C') \to H^n(C \otimes_R C') \to \bigoplus_{p+q=n+1} \text{Tor}^R(H^p(C), H^q(C')).$$

Finally we can introduce our main result.

Theorem 3.11 (Kunneth formula for Bredon cohomology of pull-backs). Let $\Gamma$ be a group coming from a diagram (0.1) and suppose that $(F,F_1,F_2)$ is a sequence of compatible families of subgroups of $\Gamma$. Let $\mathcal{N}$ be a Green functor over $\text{Or}_K(F)$, $\mathcal{M}$ be a $\pi_1^*\mathcal{N}$-module over $\text{Or}_G(F_1)$ and $\mathcal{M}'$ be a $\pi_2^*\mathcal{N}$-module over $\text{Or}_H(F_2)$. Let $X$ be a $F_1$-CW-complex and $Y$ be a $F_2$-CW-complex. Then if we consider $X \times Y$ as a $(F_1 \times_F F_2)$-CW-complex there is a split exact sequence

$$0 \to \bigoplus_{p+q=n} H^p_G(X;\mathcal{M}) \otimes_{H^0(K;\mathcal{N})} H^q_H(Y;\mathcal{M}') \to H^n_T(X \times Y; \mathcal{M} \otimes_N \mathcal{M}') \to \bigoplus_{p+q=n+1} \text{Tor}^{H^0(K;\mathcal{N})}(H^0_G(X;\mathcal{M}), H^q_H(Y;\mathcal{M}')).$$

Proof. It is a immediate consequence of Theorems 3.6, 3.9 and Lemma 3.10.

4. Examples and applications

In this section we apply Theorem 3.11 in different cases. In particular, we compute Bredon cohomology with coefficients in representations of the classifying space for proper actions for important examples studied in [VW95]. As a consequence we obtain a computation of twisted orbifold $K$-theory $\alpha K^*(T^6/(\mathbb{Z}/2\mathbb{Z})^2)$ for some discrete twisting $\alpha$.

Remark 4.1. Theorem 3.11 has the following consequences:

(i) Consider the direct product of two discrete groups $G \times H$ as a pullback over the trivial group acting over a $(G \times H)$-CW-complex. Applying Theorem 3.11 we obtain the well known Kunneth formula for direct product of groups. In this case the tensor product is taken over $\mathbb{Z}$. 


(ii) Consider the following pullback diagram over a finite group $G$

\[
\begin{array}{ccc}
G & \xrightarrow{id} & G \\
\downarrow & & \downarrow \\
G & \xrightarrow{id} & G
\end{array}
\]

If we apply Theorem 3.11 to the above pullback we obtain a Kunneth formula for Bredon cohomology of the product of two $G$-spaces with the diagonal $G$-action. In this case the tensor product is taken over $\mathcal{N}(G)$.

On the other hand if we consider the trivial family and the constant Bredon module $\mathbb{R}$ (where $\mathbb{R}$ is a commutative ring with unity) with the $G$-space $E\Gamma$ and the $H$-space $E\Delta$ we obtain the following Kunneth formula for group cohomology of pullbacks

**Theorem 4.2** (Kunneth formula for group cohomology of pullbacks). Let $R$ be a PID. There is a split exact sequence

\[
0 \to \bigoplus_{p+q=n} H^p(G; R) \otimes H^q(\mathbb{K}; R) \to H^n(\Gamma; R) \to \bigoplus_{p+q=n+1} \text{Tor}^H_{p+q}(H^p(G; R), H^q(\mathbb{K}; R)).
\]

*Proof.* Notice that the $\Gamma$-space $E\Gamma \times E\Delta$ is a model for $E\mathcal{N}$. Applying Theorem 3.11 to the $\Gamma$-CW-complex $E\Gamma \times E\Delta$ we obtain the result. \(\Box\)

Consider the Bredon modules $\mathcal{R}$ and $\mathcal{R}_\alpha$ defined on the family of finite subgroups. Applying Theorems 3.3 and 3.11 we obtain the following theorem.

**Theorem 4.3** (Kunneth formula for Bredon cohomology of proper actions with coefficients in representations). Let $X$ be a proper $G$-CW-complex, let $Y$ be a proper $H$-CW-complex. Let $\alpha \in Z^2(G; \mathbb{Z}/n\mathbb{Z})$ be a normalized torsion cocycle of $G$. There is a split exact sequence

\[
0 \to \bigoplus_{p+q=n} H^p_G(X; \mathcal{R}_\alpha) \otimes H^q(\mathbb{K}; R) \to H^n_\Gamma(X \times Y; \mathcal{R}_{p_1(\alpha)}) \to \bigoplus_{p+q=n+1} \text{Tor}^{H^0(\mathbb{K}; \mathcal{R})}_{p+q}(H^p_G(X; \mathcal{R}_\alpha), H^q(Y; \mathcal{R})).
\]

When the group $K$ is finite one can substitute $H^0(K; \mathcal{R})$ in the above formula by the representation ring $R(K)$. 
4.1. The Vafa-Witten groups. Here we apply the above results to compute the Bredon cohomology with coefficients in twisted representations for the classifying space for proper actions of the Vafa-Witten groups defined in [VW95]. Using that computation and the Atiyah Hirzebruch spectral sequence we obtain some computations of twisted and untwisted orbifold K-theory groups associated to the action of these groups on $\mathbb{R}^6$.

From now on, the cyclic group $\mathbb{Z}/4\mathbb{Z}$ will be seen as the set of 4-th roots of unity generated by $\omega = e^{\frac{2\pi i}{4}}$.

4.1.1. The group $\mathbb{Z}^6 \rtimes \mathbb{Z}/4\mathbb{Z}$. Consider the action of $\mathbb{Z}/4\mathbb{Z}$ on $\mathbb{Z}^6$ induced from the action of $\mathbb{Z}/4\mathbb{Z}$ on $\mathbb{C}^3$, given by $k(z_1, z_2, z_3) = (-z_1, iz_2, iz_3)$. We denote the $\mathbb{Z}^6$ with the above $\mathbb{Z}/4\mathbb{Z}$-action by $M$. Now we take the semidirect product

$$1 \to M \to M \rtimes \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to 1.$$ 

Note that we have the following decomposition of $\mathbb{Z}/4\mathbb{Z}$-modules

$$M = (M_1)^2 \oplus (M_2)^2,$$

where $M_1$ has rank one and the generator acts by multiplication by $-1$; $M_2$ has rank two and the generator acts by multiplication by $i$.

This decomposition induces a decomposition of the group $M \rtimes \mathbb{Z}/4\mathbb{Z}$ as the multiple pullback

$$\begin{align*}
(M_1 \rtimes \mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/4\mathbb{Z}} (M_1 \rtimes \mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/4\mathbb{Z}} (M_2 \rtimes \mathbb{Z}/4\mathbb{Z}) \times_{\mathbb{Z}/4\mathbb{Z}} (M_2 \rtimes \mathbb{Z}/4\mathbb{Z}).
\end{align*}$$

**Remark 4.5.** If we divide $\mathbb{C}^3$ by $M$, we have an action of $\mathbb{Z}/4\mathbb{Z}$ over $T^6$. Let $Y_1 = T^6/(\mathbb{Z}/4\mathbb{Z})$ the associated orbifold.

For simplicity we denote $M_1 \rtimes \mathbb{Z}/4\mathbb{Z}$ by $G$ and $M_2 \rtimes \mathbb{Z}/4\mathbb{Z}$ by $H$.

Consider $\mathbb{R}$ with the $G$-action defined as follows. Let $(n, \omega) \in G$ and $x \in \mathbb{R}$

$$(n, \omega) \cdot x = n + \omega^2 x.$$ 

A $G$-CW-complex decomposition for $\mathbb{R}$ is given by

| 0-cells |
|-------------------|
| $v_1$ $O$ $(0, \bar{1})$ $\mathbb{Z}/4\mathbb{Z}$ |
| $v_2$ $P$ $(1, \bar{1})$ $\mathbb{Z}/4\mathbb{Z}$ |
| 1-cell |
| $e_1$ $OP$ $(0, \bar{2})$ $\mathbb{Z}/2\mathbb{Z}$ |
The first column is an enumeration of equivalence classes of cells; the second lists a representative of each class; the third column gives a generating element for the stabilizer of the given representative; and the last one is the isomorphism type of the stabilizer. The inclusions $i_j : \text{stab}(e_1) \to \text{stab}(v_i)$ ($i = 1, 2$) are given by multiplication by 2.

\[
\begin{array}{c|c|c|c|c|}
O & P & OP \\
\cdot & \cdot & \cdot \\
0 & 1 & 2 & \ldots & \ldots
\end{array}
\]

Recall that $R(\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}[[\zeta]]/\langle \zeta^4 - 1 \rangle$. We use the following notations

(i) $I = \langle \zeta^2 - 1 \rangle$,
(ii) $J = \langle \zeta^2 + 1 \rangle$,
(iii) $K = \langle \zeta + 1 \rangle$,
(iv) $L = \langle \zeta - 1 \rangle$.

The Bredon cochain complex of $R$ with coefficients in $\mathcal{R}$ has the form

\[
0 \to \mathcal{R}(G/\text{stab}(v_1)) \oplus \mathcal{R}(G/\text{stab}(v_2)) \xrightarrow{\delta} \mathcal{R}(G/\text{stab}(e_1)) \to 0.
\]

The map $\delta$ is $i_1^* - i_2^*$. Note that $\delta$ is surjective therefore

$H^1_G(R; \mathcal{R}) = 0$.

On the other hand

$H^0_G(R; \mathcal{R}) = \ker(\delta)$.

Note that the Bredon cochain complex is isomorphic to

\[
0 \to R(\mathbb{Z}/4\mathbb{Z})^2 \xrightarrow{\delta} R(\mathbb{Z}/4\mathbb{Z})/I \to 0
\]

On the above complex one can see that $\ker(\delta)$ is isomorphic as a $R(\mathbb{Z}/4\mathbb{Z})$-module to $R(\mathbb{Z}/4\mathbb{Z}) \oplus I$. Note that this module is projective because $I \oplus J = 2R(\mathbb{Z}/4\mathbb{Z}) \cong R(\mathbb{Z}/4\mathbb{Z})$.

Now let us consider the group $H$. Let $(n, m, \omega) \in H$ and $z \in \mathbb{C}$, define a $H$-action on $\mathbb{C}$ as follows

$$(n, m, \omega) \cdot z = (n + im) + \omega z.$$ 

A $H$-CW-complex decomposition for $\mathbb{C}$ is given by
With similar conventions to the case of the action of $G$ over $\mathbb{R}$.

| 0-cells |          |          |          |          |
|---------|----------|----------|----------|----------|
| $a_1$   | $Q$      | $(0, 0, 1)$ | $\mathbb{Z}/4\mathbb{Z}$ |          |
| $a_2$   | $R$      | $(-1, 0, 3)$ | $\mathbb{Z}/2\mathbb{Z}$ |          |
| $a_3$   | $S$      | $(0, 1, 1)$ | $\mathbb{Z}/4\mathbb{Z}$ |          |

| 1-cells |          |          |          |          |
|---------|----------|----------|----------|----------|
| $b_1$   | $QR$     | $(0, 0, 0)$ | $0$      |          |
| $b_2$   | $RS$     | $(0, 0, 0)$ | $0$      |          |

| 2-cell   |          |          |          |          |
|----------|----------|----------|----------|----------|
| $T$      | $QRSR'$  | $(0, 0, 0)$ | $0$      |          |

In this case the boundary is given by $\partial T = 0$, $\partial b_1 = a_2 - a_1$ and $\partial b_2 = a_3 - a_2$. We denote $i_{t,m} : \text{stab}(b_m) \rightarrow \text{stab}(a_l)$ the inclusion.

The Bredon cochain complex with coefficients in representations has the form

$$0 \rightarrow \mathcal{R}(H/\text{stab}(a_1)) \oplus \mathcal{R}(H/\text{stab}(a_2)) \oplus \mathcal{R}(H/\text{stab}(a_3)) \rightarrow \mathcal{R}(H/\text{stab}(b_1)) \oplus \mathcal{R}(H/\text{stab}(b_2)) \rightarrow \mathcal{R}(H/\text{stab}(T)) \rightarrow 0.$$ 

Here, the map $\epsilon$ is defined as follows. Let $\rho_i \in \mathcal{R}(H/\text{stab}(a_i))$, then

$$\epsilon(\rho_1, \rho_2, \rho_3) = (i_{1,1}^*(\rho_1) - i_{3,1}^*(\rho_3), i_{3,2}^*(\rho_3) - i_{2,2}^*(\rho_2)).$$

The map $\epsilon$ is surjective, then $H^1_H(\mathcal{C}; \mathcal{R}) = 0$. On the other hand $H^0_H(\mathcal{C}; \mathcal{R}) = \ker\epsilon$. To determine $\ker(\delta)$ note that the map $\epsilon$ can be identified with a map

$$R(\mathbb{Z}/4\mathbb{Z})^2 \oplus R(\mathbb{Z}/4\mathbb{Z})/I \rightarrow (R(\mathbb{Z}/4\mathbb{Z})/J)^2.$$
of $R(\mathbb{Z}/4\mathbb{Z})$-modules mapping
\[
(\zeta, 0, 0) \mapsto (1, 0),
(0, \zeta, 0) \mapsto (0, -1),
(0, 0, \zeta) \mapsto (-1, 1).
\]

The kernel of the above map is isomorphic as an $R(\mathbb{Z}/4\mathbb{Z})$-module to
\[
L \oplus L/I \oplus R(\mathbb{Z}/4\mathbb{Z}).
\]
Note that the above $R(\mathbb{Z}/4\mathbb{Z})$-module is projective because we have the following isomorphisms of $R(\mathbb{Z}/4\mathbb{Z})$-modules
\[
L/I \oplus K/I \cong R(\mathbb{Z}/4\mathbb{Z})/I, \quad \text{and}
R(\mathbb{Z}/4\mathbb{Z})/I \oplus R(\mathbb{Z}/4\mathbb{Z})/J \cong R(\mathbb{Z}/4\mathbb{Z}).
\]

Then we have proved the following result.

**Theorem 4.6.** Let $H$ and $G$ be the groups defined above. There is an isomorphism of $R(\mathbb{Z}/4\mathbb{Z})/I$-modules
\[
\bullet H^n_G(\mathbb{R}; \mathcal{R}) \cong \begin{cases} 
R(\mathbb{Z}/4\mathbb{Z}) \oplus I & \text{for } n = 0 \\
0 & \text{for } n > 0.
\end{cases}
\]
\[
\bullet H^n_H(\mathbb{C}; \mathcal{R}) \cong \begin{cases} 
L \oplus (L/I) \oplus R(\mathbb{Z}/4\mathbb{Z}) & \text{for } n = 0 \\
\mathbb{Z} & \text{for } n = 2 \\
0 & \text{for } n \neq 0, 2.
\end{cases}
\]
Moreover all modules are projective.

Since all the above modules are projective, in the corresponding Kunneth formula the term with Tor is zero and then we have a complete calculation of the Bredon cohomology groups of $E\Gamma$ with coefficients in representations.

**Theorem 4.7.** Let $H$ and $G$ be the groups in Theorem 4.6. There is an isomorphism of $\mathbb{Z}$-graded, $R(\mathbb{Z}/4\mathbb{Z})$-modules
\[
H^*_M(\mathbb{R}; \mathcal{R}) \cong H^*_G(\mathbb{R}; \mathcal{R}) \otimes R(\mathbb{Z}/4\mathbb{Z}) \otimes R^*_G(\mathbb{R}; \mathcal{R}) \otimes R(\mathbb{Z}/4\mathbb{Z})
\]
\[
H^*_H(\mathbb{C}; \mathcal{R}) \otimes R(\mathbb{Z}/4\mathbb{Z}) \otimes H^*_H(\mathbb{C}; \mathcal{R}).
\]

**Proof.** Bredon cohomology of $\mathbb{R}$ and $\mathbb{C}$ are projective $R(\mathbb{Z}/4\mathbb{Z})$-modules, thus the result follows by applying Theorem 4.3. $\square$
Since the Bredon cohomology groups of $\mathbb{C}^3$ are concentrated at even degree, the Atiyah-Hirzebruch spectral sequence of Theorem 2.12 collapses at level 2 and we have also a complete calculation of $(M \rtimes \mathbb{Z}/4\mathbb{Z})$-equivariant $K$-theory of $\mathbb{C}^3$, it is the same as the orbifold $K$-theory of $Y_1$.

**Corollary 4.8.** There is an isomorphism of $\mathbb{Z}$-graded, $R(\mathbb{Z}/4\mathbb{Z})$-modules

$$K_{or}^*(T^6/(\mathbb{Z}/4\mathbb{Z})) = K_{or}^*(M \rtimes \mathbb{Z}/4\mathbb{Z})(\mathbb{C}^3) \cong H_{or}^*(M \rtimes \mathbb{Z}/4\mathbb{Z})(\mathbb{C}^3; R).$$

**Remark 4.9.** For any non trivial torsion cocycle $\beta \in Z^2(\Gamma; \mathbb{Z}/n\mathbb{Z})$ our method cannot to be used to compute $H_{or}^*(M \rtimes \mathbb{Z}/4\mathbb{Z})(\mathbb{C}^3; R_\beta)$ because from calculations contained in [AP06] we know

$$H^3(M_1 \rtimes \mathbb{Z}/4\mathbb{Z}; \mathbb{Z}) = 0 \text{ and } H^3(M_2 \rtimes \mathbb{Z}/4\mathbb{Z}; \mathbb{Z}) = 0.$$

Then it is not possible to obtain a non-trivial 2-cocycle of $M \rtimes \mathbb{Z}/4\mathbb{Z}$ coming from non trivial 2-cocycles of $G$ or $H$.

4.1.2. **The group $\mathbb{Z}^6 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$.** Consider the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{C}^3$, given on generators by

$$\sigma_1(z_1, z_2, z_3) = (-z_1, -z_2, z_3), \quad \sigma_2(z_1, z_2, z_3) = (-z_1, z_2, -z_3).$$

We denote $\mathbb{Z}^6$ with the induced action of $(\mathbb{Z}/2\mathbb{Z})^2$ by $N$. Using the above action we define the semidirect product $N \rtimes (\mathbb{Z}/2\mathbb{Z})^2$.

Note that we have the following decomposition of $(\mathbb{Z}/2\mathbb{Z})^2$-modules:

$$N = (N_1)^2 \oplus (N_2)^2 \oplus (N_3)^2,$$

where $N_1$ has rank one and $\sigma_i, i = 1, 2$, act by multiplication by $-1$; $N_2$ has rank one, $\sigma_1$ acts by multiplication by $-1$; $\sigma_2$ acts by multiplication by $1$; and $N_3$ has rank one and $\sigma_1$ acts by multiplication by $1$ and $\sigma_2$ acts by multiplication by $-1$.

That decomposition gives us a decomposition of the group $N \rtimes (\mathbb{Z}/2\mathbb{Z})^2$ as the multiple pullback of six groups over $(\mathbb{Z}/2\mathbb{Z})^2$: two copies of each $N_j \rtimes (\mathbb{Z}/2\mathbb{Z})^2, j = 1, 2, 3$.

**Remark 4.10.** If we take a quotient by $\mathbb{Z}^6$, we have an action of $(\mathbb{Z}/2\mathbb{Z})^2$ over $T^6$. Let $Y_2 = T^6/(\mathbb{Z}/2\mathbb{Z})^2$ the associated orbifold.

We will construct a nontrivial torsion cocycle

$$\beta \in Z^2(N \rtimes (\mathbb{Z}/2\mathbb{Z})^2; \mathbb{Z}/2)$$

coming from a cocycle of $N_1 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$. Consider the non-trivial central extension

$$0 \to \mathbb{Z}/2\mathbb{Z} \to D_8 \to (\mathbb{Z}/2\mathbb{Z})^2 \to 0.$$
With associated 2-cocycle $\alpha \in Z^2(G; S^1)$. To the above extension one can associate a non trivial central extension

(4.11) $0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow N_1 \rtimes D_8 \rightarrow N_1 \rtimes (\mathbb{Z}/2\mathbb{Z})^2 \rightarrow 0$.

We denote by $\beta_1 \in Z^2(N_1 \rtimes (\mathbb{Z}/2\mathbb{Z})^2; \mathbb{Z}/2\mathbb{Z})$ the cocycle associated to extension (4.11), and by $\beta \in Z^2(N \rtimes (\mathbb{Z}/2\mathbb{Z})^2; \mathbb{Z}/2\mathbb{Z})$ to the pullback of $\beta_1$ on the group $N \rtimes (\mathbb{Z}/2\mathbb{Z})^2$.

Now we compute $H^*_{N_1 \rtimes (\mathbb{Z}/2\mathbb{Z})^2}(\mathbb{R}; \mathcal{R}_{\beta_1})$.

Recall that $R((\mathbb{Z}/2\mathbb{Z})^2) = \mathbb{Z}[\gamma_1, \gamma_2]/\langle \gamma_1^2 - 1, \gamma_2^2 - 1 \rangle$.

Consider $\mathbb{R}$ with the action of $N_1 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$ defined as follows:

$(n, \sigma_i) \cdot x = n - x$ for $i = 1, 2$.

A $N_1 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$-CW-complex decomposition for $\mathbb{R}$ is given by

| 0-cells | 1-cell |
|--------|--------|
| $a_1$  | $P$    | $(0, \sigma_1), (0, \sigma_2)$ $(\mathbb{Z}/2\mathbb{Z})^2$ $b_1$ $PQ$ $(0, \sigma_1 \sigma_2)$ $\mathbb{Z}/2\mathbb{Z}$ |
| $a_2$  | $Q$    | $(-1, \sigma_1), (-1, \sigma_2)$ |

The Bredon cochain complex of $\mathbb{R}$ with coefficients in projective $\beta_1$-representations is isomorphic to (here $\beta_1 |$ denotes the restriction of $\beta_1$ to the corresponding subgroup)

$0 \rightarrow R_{\beta_1}|((\mathbb{Z}/2\mathbb{Z})^2) \oplus R_{\beta_1}|((\mathbb{Z}/2\mathbb{Z})^2) \xrightarrow{\delta} R_{\beta_1}|(\mathbb{Z}/2\mathbb{Z}) \rightarrow 0$.

The above complex can be viewed as a subcomplex of

$R(D_8) \oplus R(D_8) \xrightarrow{\delta} R(\mathbb{Z}/4\mathbb{Z}) \rightarrow 0$.

Making an easy calculation on the above complex we obtain

$$H^p_{N_1 \rtimes (\mathbb{Z}/2\mathbb{Z})^2}(\mathbb{R}; \mathcal{R}_{\beta_1}) = \begin{cases} 
\mathbb{Z} & \text{if } p = 0, 1 \\
0 & \text{if } p > 1.
\end{cases}$$

In order to obtain

$H^*_{N \rtimes (\mathbb{Z}/2\mathbb{Z})^2}(\mathbb{C}^3; \mathcal{R}_{\beta})$

we need to calculate

$H^*_{N_j \rtimes (\mathbb{Z}/2\mathbb{Z})^2}(\mathbb{R}; \mathcal{R})$ for $j = 1, 2, 3$.

First notice that the groups $N_j \rtimes (\mathbb{Z}/2\mathbb{Z})^2$ are isomorphic for $j = 1, 2, 3$ and then it suffices to calculate for $j = 1$.

In this case the Bredon cochain complex takes the form
We have an isomorphism of projective $R((\mathbb{Z}/2\mathbb{Z})^2)$-modules

\[ H^p_{N_1 \times (\mathbb{Z}/2\mathbb{Z})^2}(\mathbb{R}; R) \cong \begin{cases} R((\mathbb{Z}/2\mathbb{Z})^2) \oplus I & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases} \]

where $I = \langle (\gamma_1 - \gamma_2, 0) \rangle$. Since all above modules are projective, in the corresponding Kunneth formula the term with Tor is zero and then using Theorem 4.3 we have a complete calculation of the Bredon cohomology groups of $C^3$ with coefficients in $\beta$-representations.

**Theorem 4.12.** There is an isomorphism of $\mathbb{Z}$-graded, $R((\mathbb{Z}/2\mathbb{Z})^2)$-modules

\[ H^\alpha_{N \times (\mathbb{Z}/2\mathbb{Z})^2}(\mathbb{C}^3; R_{\beta}) \cong \bigoplus_{\sum_{i=1}^6 p_i = n} H^p_{N_1 \times (\mathbb{Z}/2\mathbb{Z})^2}(\mathbb{R}; R_{\beta_i}) \otimes \left( \bigotimes_{i=2}^6 H^p_{N_1 \times (\mathbb{Z}/2\mathbb{Z})^2}(\mathbb{R}; R) \right). \]

Finally, because as all above modules are free over $\mathbb{Z}$ by Proposition 5.8 in [LO01] we have the following result

**Corollary 4.13.** There is an isomorphism of $\mathbb{Z}$-graded, $R((\mathbb{Z}/2\mathbb{Z})^2)$-modules

\[ \alpha K^*_{\text{orb}}(T^6/(\mathbb{Z}/2\mathbb{Z})^2) = \beta K^*_{N \times (\mathbb{Z}/2\mathbb{Z})^2}(\mathbb{C}^3) \cong H^\alpha_{N \times (\mathbb{Z}/2\mathbb{Z})^2}(\mathbb{C}^3; R_{\beta}). \]

**Remark 4.14.** Although we use Theorem 4.3 to compute twisted $K$-theory for a very particular twisting, one could follow the same strategy in order to compute

\[ \alpha K^*_{\text{orb}}(T^6/(\mathbb{Z}/2\mathbb{Z})^2) \]

for every $\alpha$ coming from 2-cocycles of $G$ or $H$.

**References**

[AP06] Alejandro Adem and Jianzhong Pan. Toroidal orbifolds, Gerbes and group cohomology. *Trans. Amer. Math. Soc.*, 358(9):3969–3983, 2006.

[AR03] Alejandro Adem and Yongbin Ruan. Twisted orbifold $K$-theory. *Comm. Math. Phys.*, 237(3):533–556, 2003.

[AS04] Michael Atiyah and Graeme Segal. Twisted $K$-theory. *Ukr. Mat. Visn.*, 1(3):287–330, 2004.

[BEJU14] Noé Bárcenas, Jesús Espinoza, Michael Joachim, and Bernardo Uribe. Universal twist in equivariant $K$-theory for proper and discrete actions. *Proc. Lond. Math. Soc. (3)*, 108(5):1313–1350, 2014.
[BEUV13] Noe Barcenas, Jesus Espinoza, Bernardo Uribe, and Mario Velasquez. Segal’s spectral sequence in twisted equivariant $K$-theory for proper actions. preprint, arXiv:1307.1003 [math.AT], 2013.

[BJV13] Noe Barcenas, Daniel Juan, and Mario Velasquez. Bredon cohomology, $K$-theory and $K$-homology of pullbacks of groups. preprint, arXiv:1311.3138 [math.AT], 2013.

[DL98] James F. Davis and Wolfgang Lück. Spaces over a category and assembly maps in isomorphism conjectures in $K$- and $L$-theory. $K$-Theory, 15(3):201–252, 1998.

[Dwy08] Christopher Dwyer. Twisted equivariant $K$-theory for proper actions of discrete groups. $K$-Theory, 38(2):95–111, 2008.

[Kar94] Gregory Karpilovsky. Group representations. Vol. 3, volume 180 of North-Holland Mathematics Studies. North-Holland Publishing Co., Amsterdam, 1994.

[LO01] Wolfgang Lück and Bob Oliver. Chern characters for the equivariant $K$-theory of proper $G$-CW-complexes. In Cohomological methods in homotopy theory (Bellaterra, 1998), volume 196 of Progr. Math., pages 217–247. Birkhäuser, Basel, 2001.

[MS06] J. P. May and J. Sigurdsson. Parametrized homotopy theory, volume 132 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2006.

[MV03] Guido Mislin and Alain Valette. Proper group actions and the Baum-Connes conjecture. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2003.

[Ros13] Jonathan Rosenberg. The Künneth theorem in equivariant $K$-theory for actions of a cyclic group of order 2. Algebr. Geom. Topol., 13(2):1225–1241, 2013.

[Seg68] Graeme Segal. Equivariant $K$-theory. Inst. Hautes Études Sci. Publ. Math., (34):129–151, 1968.

[VW95] Cumrun Vafa and Edward Witten. On orbifolds with discrete torsion. J. Geom. Phys., 15(3):189–214, 1995.

[Web00] Peter Webb. A guide to Mackey functors. In Handbook of algebra, Vol. 2, pages 805–836. North-Holland, Amsterdam, 2000.

[Wei94] Charles A. Weibel. An introduction to homological algebra, volume 38 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994.

DEPARTAMENTO DE MATEMÁTICAS., PONTIFICIA UNIVERSIDAD JAVIERANA, CRA. 7 NO. 43-82 - EDIFICIO CARLOS ORTÍZ 5TO PISO, BOGOTÁ D.C, COLOMBIA

E-mail address: germancombariza@javeriana.edu.co

DEPARTAMENTO DE MATEMÁTICAS., PONTIFICIA UNIVERSIDAD JAVIERANA, CRA. 7 NO. 43-82 - EDIFICIO CARLOS ORTÍZ 5TO PISO, BOGOTÁ D.C, COLOMBIA

E-mail address: mario.velasquez@javeriana.edu.co

URL: https://sites.google.com/site/mavelasquezm/