Topological Shiba Metals

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Shiba bands formed by overlapping Yu-Shiba-Rusinov subgap states in magnetic impurities on a superconductor play an important role in topological superconductors. Here, we theoretically demonstrate the existence of a new type of Shiba bands (dubbed topological Shiba metal) on a magnetically doped s-wave superconducting surface with Rashba spin-orbit coupling in the presence of a weak in-plane magnetic field. Such topological gapless Shiba bands develop from gapped Shiba bands through Lifshitz phase transitions accompanied by second-order quantum phase transitions for the intrinsic thermal Hall conductance. We also find a new mechanism in Shiba lattices that protects the first-order quantum phase transitions for the intrinsic thermal Hall conductance. Due to the long-range hopping in Shiba lattices, the topological Shiba metal exhibits intrinsic thermal Hall conductance with large nonquantized values. As a consequence, there emerge a large number of second-order quantum phase transitions.

Topological superconductors have attracted a great amount of attention during the last decade due to their potential applications in topological quantum computation [1–10]. In the context, Shiba lattices play an important role, since they provide a versatile tool to realize highly controllable topological superconducting phases [11–18]. Shiba lattices are formed by overlapping Yu-Shiba-Rusinov (YSR) subgap states, bound states occurring in magnetic impurities when placed on a superconducting surface [19–27]. Such lattices can be utilized to realize topological superconductivity with high Chern numbers due to the long-range hopping between two YSR subgap states [14]. Remarkably, a very recent experiment reports on an observation of topological Shiba bands in a magnet-superconductor hybrid system [28]. Topological phases can not only exist in gapped systems but also in gapless systems [29, 30]. However, for Shiba lattices, previous studies either focused on gapped superconductors in regular Shiba lattices [14, 15] or gapless superconductors (but Anderson localized) in Shiba glasses [18]. Although it has been theoretically predicted that topological metals can emerge in fermionic superfluids in cold atom systems [31–35], it is unclear whether topological metals can arise from the subgap band formed by the YSR states.

Motivated by the experimental progress in topological Shiba bands, we here study the topological phases in a two-dimensional (2D) lattice formed by ferromagnetic impurities on a 2D s-wave superconducting surface with Rashba spin-orbit coupling. We theoretically predict the emergence of topological metallic phases in Shiba bands (dubbed topological Shiba metals) in the presence of a weak in-plane magnetic field, which drives the direction of the magnetic moment away from a surface normal vector. Starting from a gapped topological superconducting phase, one can obtain the metallic phase through a Lifshitz phase transition by varying a system parameter, such as the Fermi wavevector or the spin-orbit coupling strength. The transition also manifests in a second-order quantum phase transition for the intrinsic thermal Hall conductance. In addition, it has been shown that owing to particle-hole symmetry, the intrinsic thermal Hall conductance always exhibits a first-order quantum phase transition if an energy gap closes at a high-symmetry momentum [35]. When the energy gap closing points deviate from high-symmetry momenta, the first-order quantum phase transition is not protected in a metallic phase since these points are usually not pinned at zero energy. Remarkably, we find abundant first-order quantum phase transitions in Shiba metals arising from the energy gap closing at non-high-symmetry momenta. We demonstrate a new mechanism (called reciprocal lattice reflection symmetry) in Shiba lattices that fixes the band touching point at zero energy and thus protects the first-order phase transition. Moreover, we illustrate that the topological metals exhibit intrinsic thermal Hall conductance with large nonquantized values due to the long-range hopping supported by the YSR states, leading to many continuous quantum phase transitions.

![FIG. 1. (Color online) Ferromagnetic impurities are arranged into a square lattice on an s-wave superconducting surface with Rashba spin-orbit coupling, resulting in topological metallic phases in Shiba bands. The direction of the magnetization (specified by the polar angle $\theta$ and azimuthal angle $\varphi$) deviates from the surface normal direction (z direction) due to the presence of an in-plane magnetic field $B$.](attachment://image.png)
properties of $H_S$ (S5). We remarkably find the existence of a ferromagnetic phase with tilted magnetization, as shown in Fig. 2(a) by a white regime. The tilted angle $\theta$ in the ferromagnetic regime can be enlarged by increasing the Zeeman field $M_{S}S$ [see Fig. 2(b)]. All these parameters are in reasonable scales, implying an experimentally accessible ferromagnetic Shiba lattice with tilted magnetization. We note that the impurities can be arranged on the superconducting surface by a scanning tunneling microscope with a resolution about 0.3 nm [28], which is sufficient to form a ferromagnetic lattice in Fig. 2.

With a ferromagnetic phase for impurities, we now turn to study the properties of the YSR states. Since any spin flip of an individual impurity atom is suppressed by its ferromagnetic neighbors via RKKY interactions, we assume that the classical spin model is valid in such a ferromagnetic regime [14, 38]. In this condition, the electronic Hamiltonian can be written as

$$H_e = \zeta \tau_z + \alpha_R (\sigma \times k)_z \tau_z + \Delta \tau_x + m\parallel \sigma || - J \sum_i (S_i \cdot \sigma) \delta(r - r_i),$$

(2)

where $\zeta = \frac{k^2}{2m} - E_F$ denotes the kinetic energy of free electrons measured relative to the Fermi energy $E_F$ with $m$ being the effective mass, and $\Delta$ denotes the superconducting order parameter. The Pauli matrices $\sigma$ and $\tau$ are defined on the spin and particle-hole subspaces, respectively. An impurity $i$ is treated as a classical spin $S_i$ localized at $r_i$, coupled to the bulk electrons with the exchange coupling strength $J$. Such a spin binds a YSR subgap state with eigenenergy $\Delta(1 - \alpha^2)/(1 + \alpha^2)$ determined by a dimensionless coefficient $\alpha = mJS/2$.

A deep-in-gap YSR state occurs when $\alpha \approx 1$. In the presence of multiple impurities, the corresponding YSR states couple with each other and constitute a Shiba lattice. The second term in Eq. (2) describes the Rashba spin-orbit coupling, an essential term to create nontrivial topology in Shiba lattices. The in-plane magnetic field leads to a Zeeman splitting term $m\parallel \sigma ||$ for electrons, which seems to disturb the YSR state. To suppress such an effect, the magnetic field is limited to a relatively weak scale. For instance, consider the Zeeman splitting $M_{S}S$ of a magnetic atom below 0.3 meV [see Fig. 2]. If a magnetic moment of an impurity atom is five times larger than that of a free electron, then the Zeeman splitting $m\parallel \sigma ||$ can be limited to $0 \sim 0.06$ meV, which is much smaller than the superconducting gap $\Delta \sim 1$ meV. In the Supplementary Material, we show that such a weak Zeeman term is negligible.

For an isolated impurity, there are two YSR states described by $|+\uparrow\rangle$ and $|\pm \downarrow\rangle$ in the Nambu representation [11], where $|\tau\rangle$ ($\tau = \pm$) and $|\sigma\rangle$ ($\sigma = \uparrow, \downarrow$) are the eigenstates of $\tau_z$ and $\sigma_i = (S_i/S) \cdot \sigma$, respectively. We derive a $2 \times 2$ tight-binding Hamiltonian to describe the low energy behavior of Shiba lattices by projecting $H_e$ on these YSR states (see the Supplementary Material for
Here $H(r)$ represents the hopping matrix between two impurities with a displacement vector $r = (r, \psi_r)$ in polar coordinates, and for $r \neq 0$,
\begin{align}
    d_0(r) &= \Delta \text{Im} A(r) \sin \theta \sin(\varphi - \psi_r)/2 \\
    d_x(r) &= \Delta \text{Re} A(r) \cos \theta \sin(\varphi - \psi_r)/2 \\
    d_y(r) &= \Delta \text{Re} A(r) \cos(\psi_r - \varphi)/2 \\
    d_z(r) &= -\Delta \text{Re} S(r)/2,
\end{align}
where $S(r)$ and $A(r)$ are special functions composed of Bessel functions. At $r = 0$, $d_z(0) = \Delta(1 - \alpha^2)/(1 + \alpha^2)$ and $d_{0,x,y}(0) = 0$. In the case with the magnetization aligning along $z$ (i.e., $\theta = 0$), Eq. (3) reduces to the traditional case, which has been widely explored [14, 15, 18]. In momentum space, the Hamiltonian reads
\begin{equation}
    H(k) = d_0(k) + d(k) \cdot \sigma,
\end{equation}
where $d_{0,x,y}(k)$ and $d_z(k)$ are odd and even functions with respect to $k$, respectively, due to the particle-hole symmetry, i.e., $P^{-1}H(k)P = -H(-k)$ with $P = \sigma_x \kappa$ and $\kappa$ being the complex conjugate operator. One can clearly see that a metallic phase cannot appear due to the vanishing of $d_0$ when $\theta = 0$.

One may ask whether the quasiparticle excitation spectrum can exhibit a metallic phase. The answer is affirmative. For simplicity, we first consider the nearest-neighbor hopping terms, which dominate, and neglect other long-range hopping ones. In this case, considering that an energy gap closes at $k = 0$, one can easily find that the eigenenergies of $H(k)$ near the band touching point can be approximated by $E(k) \approx k_0 \Delta |\text{Im} A(a)\sin(\theta)\sin(\psi_k - \varphi)|$, where $k_0 \Delta \text{Im} A(a)\sin(\theta)$ is an odd function with respect to $k$, $k_0 \Delta \text{Im} A(a)\sin(\theta)$ is an odd function with respect to $k$, and $k_0 \Delta \text{Im} A(a)\sin(\theta)$ is an odd function with respect to $k$. For example, when $\theta = \pi/2$, it requires that $\text{tan}|\psi_k - \varphi| > |\text{Re} A(a)/|\text{Im} A(a)|$, which can always be satisfied if $|\text{Im} A(a)| \neq 0$. In a realistic case, a very small $\theta$ is able to render the energy spectrum gapless, which will be discussed in the following.

The topological features of the metallic phase can be characterized by the intrinsic thermal Hall conductance,
\begin{equation}
    \sigma_H = \frac{g_0}{2\pi} \sum_n \int_{\mathcal{BZ}} d^2 k [E_n(k)] \Omega_n(k),
\end{equation}
where $\Omega_n$ denotes the Berry curvature of the $n$th band ($n = 1, 2$ refer to the valence and conduction bands, respectively), $\mathcal{BZ}$ stands for the first Brillouin zone, $f(E)$ is the Fermi-Dirac distribution function, and $g_0 = \pi^2 k_B^2 T/(6h)$ is the thermal conductance quantum with $T$ being the temperature and $k_B$ being the Boltzmann constant [39]. For a gapped system with temperatures much lower than the band gap, this thermal Hall conductance is equal to the Chern number multiplied by $g_0$ due to the fully occupied valence band. However, in the metallic regime, the intrinsic thermal Hall conductance is no longer quantized to an integer multiple of $g_0$ since both bands are partially occupied [see Fig. 2(c3)].

**Reciprocal lattice reflection symmetry protected topological quantum phase transitions.**—A topological phase transition occurs when the energy gap between the valence and conduction bands closes. In the traditional gapped case without external magnetic fields, $d_0$ vanishes so that the energy gap can only close at zero energy, leading to a quantized jump in $\sigma_H$ across the phase transition point. However, with nonzero $d_0$, the energy gap can close at nonzero energy, in which case such a first-order quantum phase transition does not happen. Fortunately, the particle-hole symmetry guarantees that $d_0$ is an odd function with respect to $k$ so...
that \(d_0\) has to vanish at high-symmetry momenta such as \((k_xa, k_ya) = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\). As a result, the first-order phase transition will take place if there is an energy gap closing at these high-symmetry points [35].

In Fig. 3(a), we indeed observe the appearance of sharp changes in the thermal Hall conductance \(\sigma_H\) as we vary \(k_F\), revealing the first-order topological quantum phase transitions. For example, when \(\varphi = \pi/4\), \(\sigma_H\) experiences a sharp decline at \(k_{F1}\). However, the energy spectra at \(k_{F1}\) do not exhibit a gap closing at high-symmetry momenta [see Fig. 3(b)]. Instead, two gap closings occur at momenta along \(k_x = k_y\) at zero energy. We show that the gap closings are protected by a reflection symmetry of the reciprocal lattices about the direction of the magnetic field. Such a symmetry ensures that \(\mathcal{M}K = K\) where \(K\) is a set consisting of all reciprocal lattice vectors. \(\mathcal{M}\) is a reflection operator that acts on a reciprocal lattice vector \(\mathbf{K} = K_\| e_\| + K_\perp e_\perp\) resulting in \(\mathcal{M}K = K_\| e_\| - K_\perp e_\perp\) with \(e_\| = B/B\) and \(e_\perp\) being vertical to \(e_\|\). With this symmetry, \(d_0(k)\) has to vanish at momenta on a symmetry line \(k = ke_\|\) so that if the band touching happens at these momenta, then the first-order topological phase transition arises (see the proof in the Supplemental Material).

Specifically, for a square lattice geometry as we consider, there exists a reciprocal lattice reflection symmetry when \(\varphi = n\pi/4\) with \(n\) being an integer. At \(k_F = k_{F1}\), the jump in \(\sigma_H\) is associated with gap closings at momenta on the symmetry line, as shown in Fig. 3. Because of the symmetry, the energy at the crossing points must vanish, giving rise to the first-order topological phase transition. Although the energy gap remains closed when we vary \(\varphi\) [see Fig. 3(c4)], for other \(\varphi\), such as \(\varphi = 0\), \(d_0\) is not enforced to vanish at momenta along \(k_x = k_y\) so that the first-order phase transition does not occur [see the blue line in Fig. 3(a)]. However, for \(\varphi = 0\), we see the occurrence of a first-order phase transition at \(k_F = k_{F2}\). There, the band touching occurs at the outer four valleys on the \(k_x\) and \(k_y\) axes; the band touching on the \(k_x\) axis is protected to occur at zero energy, resulting in the first-order topological quantum phase transition. In the Supplemental Material, we have also demonstrated the universality of the reciprocal lattice reflection symmetry protected topological phase transitions in Shiba metals.

When \(\varphi = 0\), at \(k_F = k_{F1}\), the first-order phase transition disappears because the gap closing points are not pinned at zero energy [see Fig. 3(c4)]. Interestingly, there appear two second-order quantum phase transitions around this point revealed by discontinuous changes in \(\partial\sigma_H/\partial k_F\). Such a phase transition arises from the Lifshitz transition where the topology of the Fermi surface changes. Specifically, as we increase \(k_F\), the conduction band declines and the valence band rises, so that these bands approach and then cross the zero energy, generating an electron pocket in the conduction band and a hole pocket in the valence band [see Fig. 3(c3)] corresponding to a sharp change in the Fermi surface. Once the pockets appear, the integral of the Berry curvature around the electron (hole) pocket is approximated by \(\Omega_2(k_0)dS\) \([-\Omega_1(k_0)\delta S = -\Omega_2(k_0)\delta S\], where \(k_0\) and \(\delta S\) denote the momentum and the area of the electron pocket, respectively. The derivative of the intrinsic thermal Hall conductance contributed by the two pockets is proportional to \(2\Omega_2(k_0)dS/dk_F\). Clearly, this derivative develops a discontinuous change from zero to a nonzero value as the pockets appear, leading to a second-order quantum phase transition manifesting in the singularity of the intrinsic thermal Hall conductance. In fact, such second-order topological quantum phase transitions are widespread in a Shiba metal [see Fig. 4] due to the ubiquitous existence of pocket structures in the energy bands, which is attributed to the long-range hopping. Another manifestation of the long-range hopping is the high thermal Hall conductance. In fact, it can be much higher, but it is harder to identify the phase transitions numerically.

In summary, we have theoretically predicted the existence of topological Shiba metals in a magnet-superconductor hybrid system subject to a very weak in-plane magnetic field. The topological Shiba metallic phase arises due to the formation of tilted magnetization of magnetic impurities. Such a metallic phase exhibits intrinsic thermal Hall conductance with large nonquantized values and undergoes many second-order quantum phase transitions for the intrinsic thermal Hall conductance. We also find a new mechanism (reciprocal lattice reflection symmetry) that protects the first-order topological quantum phase transitions for the intrinsic thermal Hall conductance. Our work thus open the door to studying topological metallic phases in Shiba lattices.

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In the Supplemental Material, we will present the details on the formation of the ferromagnetic order in Section S-1, derive the model Hamiltonian for the Shiba lattice with a comment on the disturbance caused by the magnetic field in Section S-2, and finally demonstrate how a reciprocal lattice reflection symmetry protects a topological quantum phase transition in Section S-3.

S-1. MAGNETIC ORDER OF AN IMPURITY LATTICE

In this section, we will fill in the details about the formation of the ferromagnetic order. The full Hamiltonian of the Shiba metal, including magnetic impurities and the underlying superconducting substrate, is presented as

\[ H = H_{e0} + H_m + H_{s-d} \]

\[ = \zeta \tau_z + \alpha_R (\sigma \times k)_z \tau_z + \Delta \tau_x + m_{||} \sigma || + \sum_i \left[ -\frac{D}{2} S_{i,z}^2 + M_{||} S_{i,||} - J(S_i \cdot \sigma) \delta(r - r_i) \right]. \]  

Here the first term,

\[ H_{e0} = \zeta \tau_z + \alpha_R (\sigma \times k)_z \tau_z + \Delta \tau_x + m_{||} \sigma ||, \]

is the Hamiltonian of electrons in \( k \)-space, where \( \zeta = \frac{k^2}{2m} - E_F \) is the kinetic energy of free electrons with \( m \) (\( E_F \)) denoting the effective mass (Fermi energy), \( \alpha_R \) is the Rashba coefficient, \( \Delta \) denotes the \( s \)-wave superconducting order
parameter, $m_{||}$ denotes the Zeeman splitting of electrons, $\sigma_{||} = \sigma \cdot e_{||}$ with $e_{||}$ being the unit vector along the direction of the magnetic field, and the Pauli matrices $\sigma$ and $\tau$ are defined on the spin and particle-hole subspaces, respectively. The second term

$$H_m = \sum_i \left( -\frac{D}{2} S_{i,z}^2 + M_{||} S_{i,||} \right),$$  \hspace{1cm} (S3)

which is the energy of the magnetic impurities, consists of the crystal field anisotropy term $-DS_{i,z}^2/2$ and the Zeeman splitting term $M_{||} S_{i,||}$ with $S_i$ being the classical spin of the $i$th impurity atom. Finally, the magnetic impurities and electrons are coupled via the s-d exchange interaction, which is given by

$$H_{s-d} = -J \sum_i (S_i \cdot \sigma) \delta(r - r_i).$$ \hspace{1cm} (S4)

When only the impurities are concerned, we neglect the electron Hamiltonian $H_{e0}$ and arrive at an effective Hamiltonian describing the magnetic impurities,

$$H_S = \sum_i \left( -\frac{D}{2} S_{i,z}^2 + M_{||} S_{i,||} \right) + H_{\text{RKKY}},$$ \hspace{1cm} (S5)

where the RKKY interaction $H_{\text{RKKY}}$ can be obtained by the second order perturbation theory [S1]. In the presence of Rashba spin-orbit coupling (SOC), the RKKY interaction takes the form of [S1, S2]

$$H_{\text{RKKY}} = -m \left( \frac{Jk_F}{\pi} \right)^2 \sum_{ij} \frac{\sin(2k_F r_{ij})}{(2k_F r_{ij})^2} \left( \cos(2nR r_{ij}) S_i \cdot S_j \right.$$

$$+ [1 - \cos(2nR r_{ij})](S_i \cdot e_{i,j})(S_j \cdot e_{i,j}) + \sin(2nR r_{ij})(S_i \times S_j) \cdot e_{i,j} \left. \right) \right),$$ \hspace{1cm} (S6)

where $J$ denotes the strength of the s-d exchange coupling, $e_{i,j} = e_z \times e_{ij}$ and $e_{ij}$ represents the unit vector from site $i$ to $j$. Using $E_F = k_F^2/2m$ and $\alpha = mJS^2/2 = \sqrt{1 + \xi^2}$. Eq. (S6) can be rewritten as

$$H_{\text{RKKY}} = -\frac{8E_F}{\pi^2} \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right) \sum_{ij} \frac{\sin(2k_F r_{ij})}{(2k_F r_{ij})^2} \left( \cos(2nR r_{ij}) S_i \cdot S_j \right.$$

$$+ [1 - \cos(2nR r_{ij})](S_i \cdot e_{i,j})(S_j \cdot e_{i,j}) + \sin(2nR r_{ij})(S_i \times S_j) \cdot e_{i,j} \left. \right) \right),$$ \hspace{1cm} (S7)

Here $\varepsilon = (1 - \alpha^2)/(1 + \alpha^2)$ represents the position of a YSR state in the superconducting gap (detailed in the next section), and in this work we set $\varepsilon = 0.2$. Although the derivation in Ref. [S1] does not consider the superconducting term $\Delta_{c0}$ and the Zeeman term $m_{||} \sigma_{||}$, we note that when two adjacent impurities are not too distant from each other, i.e., $k_F r < \xi/r$, the effect of superconducting pairing $\Delta$ is negligible [S3], and the electronic Zeeman term $m_{||}$ can also be omitted since it is much smaller than $\Delta$, thus Eqs. (S6) and (S7) are still valid. Here $\xi = v_F / \Delta$ is the superconducting coherence length, and $v_F = k_F / m$ is the Fermi velocity. In this paper we set $\xi = 1200$ nm. In the context, we use Eq. (S5) and (S7) to calculate the phase diagram of the impurities (Fig. 2 in the main text for $\varphi = 0$ and Fig. S1 for $\varphi = \pi/4$), where the constraint $k_F r < \xi/r$ is obeyed.

**S-2. HAMILTONIAN FOR SHIBA STATES**

In this section, we will provide a detailed derivation of the effective tight-binding Hamiltonian for Shiba lattices. The derivation of YSR states under a weak magnetic field is based on Ref. [S4], and the derivation of the Hamiltonian for Shiba lattices follows Refs. [S5, S6].

**A. YSR states under a weak magnetic field**

In the ferromagnetic regime, the magnetic impurities can be treated as fixed classical spins. As a result, the Hamiltonian for electrons $H_e$ can be decoupled from the impurity Hamiltonian $H_m$, which is given by

$$H_e = H^{(0)} + \Delta H + \sum_i H_i,$$ \hspace{1cm} (S8)
In our Shiba metal model, the in-plane magnetic field is sufficiently weak so that we can neglect the Zeeman coupling. Specifically, the Hamiltonian takes the form:

\[ H = H^{(0)} + \Delta H \]

where \( H^{(0)} \) is the unperturbed Hamiltonian and \( \Delta H \) is the magnetic field-induced correction. In the presence of a weak in-plane magnetic field, the corresponding Green’s function can be written as

\[ G(r; E) = \frac{1}{2} \sum_{\nu = \pm 1} \int \frac{dk}{(2\pi)^2} e^{i \mathbf{k} \cdot \mathbf{r}} \left[ \frac{E + \zeta (1 + \nu) + \Delta \tau}{E^2 - \zeta^2 - \Delta^2} \right] \left[ 1 + \nu \left( \frac{\nu}{k} \sigma_x - \frac{k_x}{k} \sigma_y \right) \right] \]

We first focus on the substrate Hamiltonian \( H^{(0)} \), which is in the Nambu representation with the basis being \( \{ \psi_\uparrow(\mathbf{k}), \psi_\downarrow(\mathbf{k}), \psi_\downarrow^\dagger(-\mathbf{k}), -\psi_\uparrow^\dagger(-\mathbf{k}) \} \). The Green’s function of the unperturbed Hamiltonian \( H^{(0)} \) in k-space is given by

\[ G_0(\mathbf{k}; E) = \frac{1}{2} \sum_{\nu = \pm 1} \int \frac{dk}{(2\pi)^2} \frac{E + \nu \zeta + \Delta \tau}{E^2 - \zeta^2 - \Delta^2} \left[ 1 + \nu \left( \frac{\nu}{k} \sigma_x - \frac{k_x}{k} \sigma_y \right) \right] \]

Specifically, the \( r = 0 \) case \( G_0(0; E) \) in the low energy regime \( E \ll \Delta \) takes a spin-independent form of

\[ G_0(0; E) \approx -\frac{m}{2} \frac{E + \Delta \tau}{\sqrt{\Delta^2 - E^2}} \]

The Green’s function \( G_0 \) depicts the propagation of electrons in the substrate when the magnetic field is absent. In the presence of a weak in-plane magnetic field, the corresponding Green’s function can be written as \( G = G_0 + \Delta G \), where the variation \( \Delta G = G_0 \Delta H G_0 + G_0 \Delta H G_0 \Delta H G_0 + \cdots \approx G_0 \Delta H G_0 \). In the real space, we have

\[ \Delta G(r; E) = \int dr' G_0(r - r'; E) \Delta H G_0(r'; E) \]

In our Shiba metal model, the in-plane magnetic field is sufficiently weak so that \( m_\parallel \ll \Delta \), thus we have \( \Delta H \ll H^{(0)}(r) \). The Green’s function \( G_0(r; E) \) is the inverse of \( E - H^{(0)} \), which gives

\[ G_0(r; 0) = -\int dr' G_0(r - r'; 0) H^{(0)}(r') G_0(r'; 0) \]

Comparing Eq. (S15) and (S16), it is obvious that \( \Delta G(r; 0) \) is negligible compared with \( G_0(r; 0) \).

Furthermore, let’s take a look at the \( r = 0 \) case to estimate the effect of a weak magnetic field on the Green’s function:

\[ \Delta G(0; E) = \int \frac{d\mathbf{k}}{(2\pi)^2} G_0(\mathbf{k}; E) \Delta H G_0(\mathbf{k}; E) \]
Denoting $A_\nu$ and $B_\nu$ as
\begin{align}
A_\nu &= \frac{E + \zeta_\nu \tau_z + \Delta \tau_x}{E^2 - \zeta_\nu^2 - \Delta^2}, \quad (S18) \\
B_\nu &= \nu E + \zeta_\nu \tau_z + \Delta \tau_x (\sin \psi_k \sigma_x - \cos \psi_k \sigma_y), \quad (S19)
\end{align}
where $\sin \psi_k = k_y / k$ and $\cos \psi_k = k_x / k$, we have
\begin{equation}
\Delta G(0; E) = \int_0^{2\pi} \frac{d\psi_k}{2\pi} \int_0^\infty \frac{dk}{2\pi} \frac{A_+ + A_- + B_+ + B_-}{2} m_{||\sigma}\sigma_{||}, \quad (S20)
\end{equation}
Crossing terms concerning $B_\nu$ vanish under the integral $\int d\psi_k \cdots$, so we arrive at
\begin{equation}
\Delta G(0; E) = \int_0^{2\pi} \frac{d\psi_k}{2\pi} \int_0^\infty \frac{dk}{2\pi} \frac{(A_+ + A_-)^2}{4} m_{||\sigma}. \quad (S21)
\end{equation}
Each matrix element of $A_\nu$ is a function of $k$, which reaches its peak at $k_F' = k_F(\sqrt{1 + \lambda^2} - \nu \lambda)$ with $\lambda = \alpha_R / v_F$ being the dimensionless Rashba coefficient. The width of this peak depends on $\Delta / E_F$, which is extremely narrow. We have numerically checked that the peaks in $A_+$ and $A_-$ are completely mismatched, which makes the crossing term $A_+ A_-$ negligible. Moreover, with $\alpha_R k_F \ll E_F$, we have $A_\nu(k) \approx A(k + \nu m\alpha R)$ where $A$ is obtained by replacing $\zeta_\nu$ with $\zeta$ in $A_\nu$. In this context, when $E \ll \Delta$ we have
\begin{align}
\Delta G(0; E) &\approx \int_0^{2\pi} \frac{d\psi_k}{2\pi} \int_0^\infty \frac{dk}{2\pi} \frac{A(k + m\alpha R)^2 + A(k - m\alpha R)^2}{4} m_{||\sigma} \\
&\approx \int_0^{2\pi} \frac{d\psi_k}{2\pi} \int_0^\infty \frac{dk}{4\pi} A(k)^2 m_{||\sigma} \\
&= \int_0^\infty \frac{dk}{4\pi} \left[ \frac{E + \zeta \tau_z + \Delta \tau_x}{E^2 - \zeta_\nu^2 - \Delta^2} \right]^2 m_{||\sigma} \\
&= \frac{m}{4\pi} \int_{-E_F}^\infty d\zeta \left[ \frac{E + \zeta \tau_z + \Delta \tau_x}{E^2 - \zeta_\nu^2 - \Delta^2} \right]^2 m_{||\sigma} \\
&\approx \frac{m}{4\pi} \int_{-\infty}^{\infty} d\zeta \left[ \frac{E + \zeta \tau_z + \Delta \tau_x}{E^2 - \zeta_\nu^2 - \Delta^2} \right]^2 m_{||\sigma} \\
&= m_{||\sigma} \frac{\Delta^2 + E \Delta \tau_x}{4(\Delta^2 - E^2)^{3/2}}. \quad (S22)
\end{align}
It is obvious from Eq. (S22) that near $E = 0$ we have $\Delta G \sim m_{||\sigma} / \Delta$, which is negligible.

Next we consider a single magnetic impurity on the substrate. The impurity Hamiltonian is given by
\begin{equation}
H_i(r) = -JS \sigma_i \delta(r - r_i), \quad (S23)
\end{equation}
where $S$ is the magnitude of the classical impurity spin, $\sigma_i = n_i \cdot \sigma$ with $n_i = S_i / S$, and $r_i$ denotes the position of the $i$th impurity atom. The single impurity system in the real space is described by
\begin{equation}
[H^{(0)}(r) + \Delta H + H_i(r)] \Psi(r) = E \Psi(r). \quad (S24)
\end{equation}
In searching for low energy subgap YSR states, we apply the Green’s function $G_0 + \Delta G = (E - H^{(0)} - \Delta H)^{-1}$ to Eq. (S24) and get
\begin{equation}
[G_0(r, r'; E) + \Delta G(r, r'; E)] H_i(r) \Psi(r) = \Psi(r) \delta(r - r'). \quad (S25)
\end{equation}
Integrating over $r$ and letting $r' = r_i$, we obtain
\begin{equation}
[G_0(r_i, r_i; E) + \Delta G(r_i, r_i; E)] (-JS \sigma_i) \Psi(r_i) = \Psi(r_i). \quad (S26)
\end{equation}
Note that $G_0(r_i, r_i; E) = G(0; E)$ and $\Delta G(r_i, r_i; E) = \Delta G(0; E)$, thus by Eq. (S14) and Eq. (S22) we have
\begin{equation}
\left( 1 - \frac{mJS}{2} \frac{E + \Delta \tau_x}{\sqrt{\Delta^2 - E^2}} \sigma_i + \frac{mJS}{2} \frac{\Delta^2 + E \Delta \tau_x}{2(\Delta^2 - E^2)^{3/2}} \sigma_i \right) \Psi(r_i) = 0. \quad (S27)
\end{equation}
Substituting $\sigma_i = \cos \theta \sigma_z + \sin \theta \sigma_\parallel$ and denoting $\sigma_\perp = -i \sigma_z \sigma_\parallel$, we arrive at

$$\left[ \frac{2}{mJS} - \frac{\Delta}{\sqrt{\Delta^2 - E^2}} \tau_x \sigma_i + \frac{m_\parallel}{2\Delta} \sin \theta - i \cos \theta \sigma_\perp \frac{1}{1 - (E/\Delta)^2} \right] \Psi(r_i) = \frac{E}{\sqrt{\Delta^2 - E^2}} \left[ \sigma_i - \frac{m_\parallel}{2\Delta} \sin \theta - i \cos \theta \sigma_\perp \frac{1}{1 - (E/\Delta)^2} \right] \Psi(r_i). \quad (S28)$$

In the $\tau_x = +1$ sector, we have

$$\left[ \frac{2}{mJS} \sqrt{1 - \frac{E^2}{\Delta^2}} - \sigma_i + \frac{m_\parallel}{2\Delta} \sin \theta - i \cos \theta \sigma_\perp \frac{1}{1 - (E/\Delta)^2} \right] \Psi(r_i) = \frac{E}{\Delta} \left[ \sigma_i - \frac{m_\parallel}{2\Delta} \sin \theta - i \cos \theta \sigma_\perp \frac{1}{1 - (E/\Delta)^2} \right] \Psi(r_i), \quad (S29)$$

and in the $\tau_x = -1$ sector, we have

$$\left[ \frac{2}{mJS} \sqrt{1 - \frac{E^2}{\Delta^2}} + \sigma_i + \frac{m_\parallel}{2\Delta} \sin \theta - i \cos \theta \sigma_\perp \frac{1}{1 - (E/\Delta)^2} \right] \Psi(r_i) = \frac{E}{\Delta} \left[ \sigma_i + \frac{m_\parallel}{2\Delta} \sin \theta - i \cos \theta \sigma_\perp \frac{1}{1 - (E/\Delta)^2} \right] \Psi(r_i). \quad (S30)$$

When $m_\parallel = 0$, Eq. (S29) and (S30) suggest that the $i$th impurity binds two in-gap YSR states $| \uparrow + \rangle = \psi_i(r) | \uparrow \rangle$ and $| \downarrow - \rangle = \psi_i^\dagger(r) | \downarrow \rangle$ with energy $E_{\pm} = \pm \varepsilon \Delta$, where $\varepsilon = (1 - \alpha^2)/(1 + \alpha^2)$ and $\alpha = mJS/2$. Here $\psi_i(r)$ describes the amplitude of the YSR state near the $i$th impurity in the real space, $| \uparrow \rangle$ and $| \downarrow \rangle$ denote the eigenstates of $\tau_x$ in the particle-hole subspace, and the spin polarization of unperturbed YSR states are aligned with the magnetic impurities, with $| \uparrow \rangle$ and $| \downarrow \rangle$ being exact eigenstates of $\sigma_z$, i.e. $\sigma_z | \uparrow \rangle = | \uparrow \rangle$ and $\sigma_z | \downarrow \rangle = - | \downarrow \rangle$. In the presence of the Zeeman perturbation, the spin parts of the YSR states become

$$| \uparrow' \rangle \approx | \uparrow \rangle - \frac{m_\parallel \cos \theta}{4\Delta(1 - \varepsilon^2)} | \downarrow \rangle \quad (S31)$$

and

$$| \downarrow' \rangle \approx | \downarrow \rangle + \frac{m_\parallel \cos \theta}{4\Delta(1 - \varepsilon^2)} | \uparrow \rangle, \quad (S32)$$

and the energy of the YSR states is given by $E'_{\pm} = \pm \varepsilon' \Delta$ with

$$\varepsilon' \approx \varepsilon + \frac{m_\parallel \sin \theta}{2\Delta} \frac{1 - \varepsilon}{1 + \varepsilon}. \quad (S33)$$

In Cartesian coordinates, $n_i = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, and the YSR state in the eigenbasis of $\sigma_z$ is

$$| \uparrow' \rangle = \left( \begin{array}{c} e^{-i\frac{\varphi}{2}} \cos \frac{\theta'}{2} \\ e^{i\frac{\varphi}{2}} \sin \frac{\theta'}{2} \end{array} \right), \quad | \downarrow' \rangle = \left( \begin{array}{c} -e^{-i\frac{\varphi}{2}} \sin \frac{\theta'}{2} \\ e^{i\frac{\varphi}{2}} \cos \frac{\theta'}{2} \end{array} \right) \quad (S34)$$

where $\theta' = \theta - \Delta \theta$ with the deviation $\Delta \theta \approx \frac{m_\parallel \cos \theta}{2\Delta(1 - \varepsilon^2)}$, which means that the Zeeman field perturbation slightly modifies the polar angle of the polarization in YSR state by $\Delta \theta$. Since $m_\parallel \ll \Delta$, the deviation in $\varepsilon$ and $\theta$ are both negligible.

**B. Effective Hamiltonian for Shiba lattices**

In a Shiba lattice which consists of multiple magnetic impurities, the YSR states at different sites are coupled with each other in the superconducting substrate. This coupling process is governed by $G(r \neq 0)$. For this reason, the first goal in this section is to obtain a numerically computable form of Eq. (S13), which can be divided into two branches $\nu = \pm 1$. Considering the SOC modified free electron energy $\zeta_s$, the corresponding modified Fermi wavevector is $k_F^\nu$. In each branch, the integral is mainly contributed by the regions where $\zeta_s(k) \sim 0$. We can linearize $\zeta_s(k)$ near the Fermi surface $\zeta_s(k_F) = 0$, yielding $\zeta_s(k) = v_F^\nu (k - k_F)$, where $v_F^\nu = \sqrt{1 + \lambda^2 v_F}$. The Green’s function Eq. (S13) can be put as

$$G_0(r; E) = \frac{m}{2} \sum_{\nu = \pm 1} \left( 1 - \nu \frac{\lambda}{\sqrt{1 + \lambda^2}} \right) \int d\psi_k d\zeta \frac{e^{i(k_F^\nu + \zeta \psi)\tau_x \cos(\psi \psi - \nu \sigma_z)}}{(2\pi)^2} \frac{E + \zeta \tau_x + \Delta \tau_x}{E^2 - \zeta^2 - \Delta^2} \left( 1 + \nu (\sin \psi \sigma_x - \cos \psi \sigma_y) \right), \quad (S35)$$
where \( \psi_k \) and \( \psi_r \) are polar angles of \( \mathbf{k} \) and \( \mathbf{r} \), respectively. Substituting \( \phi = \psi_k - \psi_r \), and neutralizing the odd part with respect to \( \phi \), we have

\[
G_0(\mathbf{r}; E) = \frac{m}{2} \sum_{\nu = \pm 1} \left( 1 - \nu \frac{\lambda}{\sqrt{1 + \lambda^2}} \right) \int d\phi d\zeta \frac{e^{i(k_F r + \frac{\zeta}{\xi_E}) r \cos \phi} E + \zeta \tau_z + \Delta \tau_x}{(2\pi)^2} \frac{E^2 - \zeta^2 - \Delta^2}{E^2 - \zeta^2 - \Delta^2} \left[ 1 + \nu \cos \phi(\sin \psi \sigma_x - \cos \psi \sigma_y) \right], \tag{S36}
\]

The Green’s function in this form can be expressed via Bessel functions using the following identity relations:

\[
\int_{-\infty}^{\infty} \frac{d\zeta}{\pi} \int_{0}^{2\pi} \frac{d\phi}{2\pi} e^{i(k_F r + \frac{\zeta}{\xi_E}) r \cos \phi} \cos \phi = -i \text{Re} \left[ iJ_1(k_F r + \frac{r}{\xi_E}) + \frac{2}{\pi} - H_1(k_F r + \frac{r}{\xi_E}) \right], \tag{S37}
\]

\[
\int_{-\infty}^{\infty} \frac{d\zeta}{\pi} \int_{0}^{2\pi} \frac{d\phi}{2\pi} e^{i(k_F r + \frac{\zeta}{\xi_E}) r \cos \phi} \sin \phi = \frac{i}{\sqrt{\Delta^2 - E^2}} \text{Im} \left[ iJ_0(k_F r + \frac{r}{\xi_E}) + \frac{2}{\pi} - H_1(k_F r + \frac{r}{\xi_E}) \right], \tag{S38}
\]

\[
\int_{-\infty}^{\infty} \frac{d\zeta}{\pi} \int_{0}^{2\pi} \frac{d\phi}{2\pi} e^{i(k_F r + \frac{\zeta}{\xi_E}) r \cos \phi} \sin \phi = -i \text{Im} \left[ J_0(k_F r + \frac{r}{\xi_E}) + iH_0(k_F r + \frac{r}{\xi_E}) \right], \tag{S39}
\]

\[
\int_{-\infty}^{\infty} \frac{d\zeta}{\pi} \int_{0}^{2\pi} \frac{d\phi}{2\pi} e^{i(k_F r + \frac{\zeta}{\xi_E}) r \cos \phi} \sin \phi = -\frac{1}{\sqrt{\Delta^2 - E^2}} \text{Re} \left[ J_0(k_F r + \frac{r}{\xi_E}) + iH_0(k_F r + \frac{r}{\xi_E}) \right]. \tag{S40}
\]

With the help of Eq. (S37)–(S40), we can rewrite \( G_0(\mathbf{r}; E) \) into a compact form:

\[
G_0(\mathbf{r}; E) = -\frac{m}{4} \left[ \frac{E + \Delta \tau_x}{\sqrt{\Delta^2 - E^2}} \text{Re} S(r) - \tau_z \text{Im} S(r) + i \left( \tau_x \text{Re} A(r) + \frac{E + \Delta \tau_x}{\sqrt{\Delta^2 - E^2}} \text{Im} A(r) \right) \right] \left( \sin \psi \sigma_x - \cos \psi \sigma_y \right), \tag{S41}
\]

where

\[
S(r) = \sum_{\nu = \pm 1} \left( 1 - \nu \frac{\lambda}{\sqrt{1 + \lambda^2}} \right) \left[ J_0(k_F r + \frac{r}{\xi_E}) + iH_0(k_F r + \frac{r}{\xi_E}) \right], \tag{S42}
\]

\[
A(r) = \sum_{\nu = \pm 1} \nu \left( 1 - \nu \frac{\lambda}{\sqrt{1 + \lambda^2}} \right) \left[ iJ_1(k_F r + \frac{r}{\xi_E}) + \frac{2}{\pi} - H_1(k_F r + \frac{r}{\xi_E}) \right]. \tag{S43}
\]

Here \( J_n \) and \( H_n \) are the \( n \)th order Bessel and Struve functions, respectively, and \( \xi_E = \frac{\mu_F r}{\sqrt{\Delta^2 - E^2}} \) corresponds to the superconducting coherence length. Since we are dealing with low-energy YSR states, we can let \( E = 0 \) and replace \( \xi_E \) by \( \xi \).

As we have already acquired the Green’s function \( G_0(\mathbf{r}; E) \), now we are able to handle the multi-impurity system, which is described by

\[
(H^{(0)} + \Delta H + \sum_i H_i)|\Psi\rangle = E|\Psi\rangle, \tag{S44}
\]

with \( H_i \) representing the Hamiltonian for the \( i \)th impurity. Using the Green’s function for \( H^{(0)} + \Delta H \) which is denoted by \( G \), Eq. (S44) can be derived to

\[
G \sum_i H_i |\Psi\rangle = |\Psi\rangle. \tag{S45}
\]

Since the perturbation of YSR states by the magnetic field is insignificant, especially in the small \( \theta \) case, we can adopt the unperturbed YSR states as the complete orthogonal basis, and write the wavefunction \( |\Psi\rangle \) as

\[
|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_i |\Psi_i\rangle, \tag{S46}
\]

where \( N \) is the normalization factor and \( |\Psi_i\rangle = a_i |\uparrow + i\rangle + b_i |\downarrow - i\rangle \) is the wave function on the \( i \)th impurity. By Eq. (S45) and Eq. (S46), we obtain

\[
(1 - GH_i)|\Psi_i\rangle = \sum_{j \neq i} GH_j|\Psi_j\rangle. \tag{S47}
\]
By projecting Eq. (S47) on $\delta(r - r_i)|\uparrow +\rangle$ and $\delta(r - r_i)|\downarrow -\rangle$ respectively, Eq. (S47) can be expressed in a matrix form:

$$
\begin{bmatrix}
-\langle \uparrow | -1 - G(0; E)H_{\text{imp}}|\uparrow +\rangle & -\langle \uparrow | -1 - G(0; E)H_{\text{imp}}|\downarrow -\rangle \\
\langle \downarrow -1 - G(0; E)H_{\text{imp}}|\uparrow +\rangle & -\langle \downarrow | -1 - G(0; E)H_{\text{imp}}|\downarrow -\rangle
\end{bmatrix}
\begin{bmatrix}
a_i \\
b_i
\end{bmatrix}
$$

$$
= \sum_{j \neq i}
\begin{bmatrix}
-\langle \uparrow + | G(r_{ij}; E)H_{\text{imp}}|\uparrow +\rangle -\langle \downarrow - | G(r_{ij}; E)H_{\text{imp}}|\downarrow -\rangle \\
\langle \downarrow + | G(r_{ij}; E)H_{\text{imp}}|\uparrow +\rangle -\langle \downarrow - | G(r_{ij}; E)H_{\text{imp}}|\downarrow -\rangle
\end{bmatrix}
\begin{bmatrix}
a_j \\
b_j
\end{bmatrix}
$$

(S48)

where $H_{\text{imp}} = -JS\sigma_i$. The matrix elements on the left hand side are related to $G(0; E)$, given by

$$
\langle \uparrow + | 1 - G(0; E)H_{\text{imp}}|\uparrow +\rangle = \frac{JS}{2}(\varepsilon\Delta - E),
$$

(S49)

$$
\langle \downarrow - | 1 - G(0; E)H_{\text{imp}}|\downarrow -\rangle = \frac{JS}{2}(\varepsilon\Delta + E),
$$

(S50)

$$
\langle \uparrow + | 1 - G(0; E)H_{\text{imp}}|\downarrow -\rangle = \langle \downarrow - | 1 - G(0; E)H_{\text{imp}}|\uparrow +\rangle = 0.
$$

(S51)

Then we can rewrite Eq. (S48) into a time-independent Schrödinger-like equation

$$
E \begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} \varepsilon & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix} + \frac{2\Delta}{JS} \sum_{j \neq i} \begin{bmatrix} -\langle \uparrow + | G(r_{ij}; E)H_{\text{imp}}|\uparrow +\rangle -\langle \downarrow + | G(r_{ij}; E)H_{\text{imp}}|\downarrow -\rangle \\ -\langle \downarrow - | G(r_{ij}; E)H_{\text{imp}}|\uparrow +\rangle -\langle \downarrow - | G(r_{ij}; E)H_{\text{imp}}|\downarrow -\rangle \end{bmatrix} \begin{bmatrix} a_j \\ b_j \end{bmatrix}.
$$

(S52)

The matrix elements on the right hand side are related to $G(r \neq 0; E)$. Since the coupling between YSR states is weak, this term can be treated perturbatively so that $G(r \neq 0; E) \approx G(r \neq 0; 0)$ in the low-energy regime. Using Eq. (S34) and $G \approx G_0$ (since $\Delta G$ is negligible compared with $G_0$), we have

$$
\langle \uparrow + | G(r_{ij}; 0)H_{\text{imp}}|\uparrow +\rangle = \frac{mJS}{4}\left[\text{Re}S(r_{ij}) + i\text{Im}A(r_{ij}) \sin \theta \sin(\psi_r - \varphi)\right],
$$

(S53)

$$
\langle \downarrow - | G(r_{ij}; 0)H_{\text{imp}}|\downarrow -\rangle = \frac{mJS}{4}\left[\text{Re}S(r_{ij}) - i\text{Im}A(r_{ij}) \sin \theta \sin(\psi_r - \varphi)\right],
$$

(S54)

$$
\langle \uparrow + | G(r_{ij}; 0)H_{\text{imp}}|\downarrow -\rangle = \frac{imJS}{4}\text{Re}A(r_{ij})[\cos \theta \sin(\psi_r - \varphi) + i\cos(\psi_r - \varphi)],
$$

(S55)

$$
\langle \downarrow - | G(r_{ij}; 0)H_{\text{imp}}|\uparrow +\rangle = \frac{-imJS}{4}\text{Re}A(r_{ij})[\cos \theta \sin(\psi_r - \varphi) - i\cos(\psi_r - \varphi)].
$$

(S56)

Then the effective Schrödinger equation can be written as

$$
E \Psi_i = \sum_j \left[ d_0(r_{ij}) + d(r_{ij}) \cdot \sigma \right] \Psi_j
$$

(S57)

where $\sigma$ denotes $2 \times 2$ Pauli matrix, $\Psi_i = (a_i, b_i)^T$, for $r \neq 0$,

$$
d_0(r) = -i\frac{\Delta}{2}\text{Im}A(r) \sin \theta \sin(\psi_r - \varphi),
$$

(S58)

$$
d_x(r) = -i\frac{\Delta}{2}\text{Re}A(r) \cos \theta \sin(\psi_r - \varphi),
$$

(S59)

$$
d_y(r) = i\frac{\Delta}{2}\text{Re}A(r) \cos(\psi_r - \varphi),
$$

(S60)

$$
d_z(r) = \frac{\Delta}{2}\text{Re}S(r),
$$

(S61)

and for $r = 0$,

$$
d_0(0) = d_x(0) = d_y(0) = 0, \quad d_z(0) = \varepsilon.
$$

(S62)

The k-space Hamiltonian for a square Shiba lattice is given by

$$
H(k) = d_0(k) + d(k) \cdot \sigma,
$$

(S63)

$$
d_n(k) = \sum_R e^{-ik \cdot R} d_n(R),
$$

(S64)

which constitute the foundation for investigation in Shiba metals.
S-3. RECIPROCAL LATTICE REFLECTION SYMMETRY

In this section, we will demonstrate how a reciprocal lattice reflection symmetry protects a first-order topological phase transition and show that such phase transitions are widespread in Shiba metals.

The first-order topological phase transitions are protected on a continuous set of points in the Brillouin zone, on which the deformation term \( d_0(k) \) is enforced to vanish. Specifically, the Schrödinger-like Eq. (S52) gives

\[
d_0(k) = -\frac{\Delta}{JSm} \sum_R e^{-ikR}[\langle \uparrow | G(R; 0) H_{\text{imp}} | \uparrow \rangle - \langle \downarrow | G(R; 0) H_{\text{imp}} | \downarrow \rangle],
\]

where \( R \) runs over all the coordinates of impurities. Using

\[
G(R; 0) = \int \frac{dk}{(2\pi)^2} e^{ikR} G(k; 0),
\]

we arrive at

\[
d_0(k) = -\frac{\Delta}{JSm} \sum_K [(\langle \uparrow | G(k + K; 0) H_{\text{imp}} | \uparrow \rangle - \langle \downarrow | G(k + K; 0) H_{\text{imp}} | \downarrow \rangle],
\]

where \( K \) runs over all reciprocal lattice vectors. Since \( e_{\varphi\perp} = e_z \times e_{\varphi\parallel} \) is perpendicular to the plane spanned by \( e_z \) and \( e_{\varphi\parallel} \), and \( \sigma_{\perp} = \sigma \cdot e_{\varphi\perp} \), we have

\[
\tau_y \sigma_{\perp} | \uparrow \rangle = | \downarrow \rangle.
\]

At \( E = 0 \) we have \( G(k; 0) = -H(k)^{-1} \). So we get

\[
\langle \uparrow | G(k; 0) | \sigma_{\uparrow} \rangle = -\langle \downarrow | -\tau_y \sigma_{\perp} H(k)^{-1} | \sigma_{\perp} \rangle | \downarrow \rangle
\]

and

\[
\langle \downarrow | G(k; 0) | \sigma_{\downarrow} \rangle = -\langle \downarrow | -H(k)^{-1} | \sigma_{\downarrow} \rangle | \downarrow \rangle,
\]

based on which we reduce Eq. (S67) to

\[
d_0(k) = \Delta m \sum_K (\langle \downarrow | -\tau_y \sigma_{\perp} H(k + K)^{-1} | \sigma_{\perp} \rangle - \langle \downarrow | -H(k + K)^{-1} | \sigma_{\perp} \rangle | \downarrow \rangle).
\]

In addition, the substrate Hamiltonian \( H = H^{(0)} + \Delta H \) in \( k \)-space can be put as:

\[
H(k) = \zeta(k) \tau_z + \alpha_R(k_{\varphi\parallel} \sigma_{\parallel} - k_{\varphi\parallel} \sigma_{\perp}) \tau_z + \Delta \tau_x + m | \sigma_{\parallel} |,
\]

where \( k_{\varphi\parallel} = k \cdot e_{\varphi\parallel} \) and \( k_{\varphi\perp} = k \cdot e_{\varphi\perp} \). With the help of the following identities:

\[
\sigma_{\perp} | \sigma_{\parallel} \rangle = -\sigma_{\parallel} | \sigma_{\perp} \rangle \quad \sigma_{\parallel} | \sigma_{\parallel} \rangle = \sigma_{\parallel} | \sigma_{\parallel} \rangle,
\]

we obtain

\[
\tau_y \sigma_{\perp} H(k) | \sigma_{\perp} \rangle = \zeta(k) | \sigma_{\parallel} \rangle - \alpha_R | (k_{\varphi\perp} \sigma_{\perp} + k_{\varphi\parallel} \sigma_{\parallel}) | \tau_z + \Delta \tau_x + m | \sigma_{\parallel} |\]

\[
\sigma_{\parallel} H(k_{\varphi\parallel}, k_{\varphi\perp}) = \tau_y \sigma_{\perp} H(k_{\varphi\parallel}, -k_{\varphi\perp}) | \tau_y \sigma_{\perp} \rangle
\]

\[
H^{-1}(k_{\varphi\parallel}, k_{\varphi\perp}) | \sigma_{\parallel} \rangle = \tau_y \sigma_{\perp} H^{-1}(k_{\varphi\parallel}, -k_{\varphi\perp}) | \sigma_{\parallel} \rangle | \tau_y \sigma_{\perp} \rangle.
\]

Substituting Eq. (S77) into Eq. (S71), we have

\[
d_0(k) = -\frac{\Delta}{m} \sum_K (\langle \downarrow | -H(k + K)^{-1} - H(M(k + K))^{-1} | \sigma_{\parallel} \rangle | \downarrow \rangle),
\]

where \( M \) is the reflection operator satisfying \( M(k_{\varphi\parallel}, k_{\varphi\perp}) = (k_{\varphi\parallel}, -k_{\varphi\perp}) \). Eq. (S78) tells us that when reciprocal lattice vectors \( K \) are symmetrically distributed based around the magnetic direction \( e_{\varphi\parallel} \), \( d_0(k_{\varphi\parallel}, 0) \) is protected to be 0. Because in this case we have \( M k = k \) and \( \sum_K = \sum_M K \). For square lattice, this confinement yields \( \varphi = n\pi/4 \) \( (n = 0, 1, 2, 3) \). It is worth mentioning that although we have neglected the magnetic field disturbance on our effective Shiba lattice Hamiltonian [Eq. (S63)], the above derivation is still valid in the presence of the magnetic field.
Next, we demonstrate that such protected phase transitions are widespread in Shiba metals. To be specific, the Shiba metal is characterized by a series of system parameters. Excluding the controlled variable $k_F$, these parameters can be grouped as a vector $P = \{a, \lambda, \xi, \varepsilon, \cdots\}$. At some parameter points $P_{RS}$, tuning $k_F$ leads to gap closing on the symmetry line $k = ke_p$ in the Brillouin zone, which is parallel to the magnetic field. In the following, we show that all these points $P_{RS}$ constitute some continuous regions with the same dimension as the parameter space.

At the band touching points $k = (k_x, k_y)$, $d_x(k) = d_y(k) = d_z(k) = 0$. For square lattices with the Zeeman perturbation omitted, the three constraints are expressed as

\[
\sum_{x,y} [\sin(k_xx) \cos(k_yy) \cos \psi_r \sin \varphi - \cos(k_xx) \sin(k_yy) \sin \psi_r \cos \varphi] \text{Re}A(r) = 0, \\
\sum_{x,y} [\sin(k_xx) \cos(k_yy) \cos \psi_r \cos \varphi + \cos(k_xx) \sin(k_yy) \sin \psi_r \sin \varphi] \text{Re}A(r) = 0, \\
\varepsilon - \frac{1}{2} \sum_{x,y} \cos(k_xx) \cos(k_yy) \text{Re}S(r) = 0. 
\]

Generally, in pursuit of a band touching point on a particular trajectory $[k_x(t_k), k_y(t_k)]$ while tuning the controlled parameter $k_F$, the three constraint Eq. (S79)–(S81) must be met simultaneously with only two tunable parameters $(t_k, k_F)$, which is hard to achieve. However, for the trajectory $[t_k \cos n\pi/4, t_k \sin n\pi/4]$ with $\varphi = n\pi/4$, the first constraint Eq. (S79) is always satisfied, leaving only two independent constraints, corresponding to two curves in the $(t_k, k_F)$ parameter plane. These two curves are sensitive to $P$ and thus intersect frequently in the parameter space. Moreover, when the two curves intersect at a certain $P_{RS}$, they must keep intersecting at $P_{RS} + dP$ where $P_{RS}$ is shifted by a small value. For this reason, there is a continuous region of parameters in which band touching happens at some momenta $k$ satisfying $\psi_r = n\pi/4$ while tuning $k_F$.

In fact, the elimination of the first constraint Eq. (S79) is rooted in the reflection symmetry, which is still valid in the presence of the Zeeman field perturbation. Specifically, when the energy gap closes at $k$, the off-diagonal elements in Eq. (S52) satisfy the equation

\[
\sum_R e^{-i k \cdot R} \langle \uparrow | G(R'; 0) H_{imp} | \downarrow \rangle = 0, 
\]

which is equivalent to

\[
\sum_K \langle \uparrow | G(k + K; 0) H_{imp} | \downarrow \rangle = 0. 
\]

This equation imposes two constraints corresponding to its real and imaginary parts. However, with aid of Eq. (S77), we find that

\[
\sum_K \langle \uparrow | G(k + K; 0) H_{imp} | \downarrow \rangle = -\sum_K \langle \uparrow | G(M(k + K); 0) H_{imp} | \downarrow \rangle^*. 
\]

With the reciprocal lattice reflection symmetry such that $M\{K\} = \{K\}$, when $k$ lies on the symmetry line, Eq. (S84) leads to

\[
\text{Re} \sum_K \langle \uparrow | G(k + K; 0) H_{imp} | \downarrow \rangle = 0, 
\]

which eliminates one constraint.

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