DISTRIBUTED SADDLE-POINT PROBLEMS:
LOWER BOUNDS, NEAR-OPTIMAL AND ROBUST ALGORITHMS *

A PREPRINT

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July 11, 2022

ABSTRACT

This paper focuses on the distributed optimization of stochastic saddle point problems. The first part of the paper is devoted to lower bounds for the centralized and decentralized distributed methods for smooth (strongly) convex-(strongly) concave saddle-point problems as well as the near-optimal algorithms by which these bounds are achieved. Next, we present a new federated algorithm for centralized distributed saddle point problems – Extra Step Local SGD. Theoretical analysis of the new method is carried out for strongly convex-strongly concave and non-convex-non-concave problems. In the experimental part of the paper, we show the effectiveness of our method in practice. In particular, we train GANs in a distributed manner.

1 Introduction

Distributed algorithms have already become an integral part of solving many applied tasks, including machine learning problems [52, 33, 35]. This paper also deals with distributed methods, we study the saddle point problem (SPP):

$$\min_{x \in X} \max_{y \in Y} f(x, y) := \frac{1}{M} \sum_{m=1}^{M} f_m(x, y),$$

(1)

where parts of the function $f$ are distributed between $M$ nodes, while the function $f_m$ is known only to the node with the corresponding number $m$. SPPs or Min-Max problems, including distributed ones, have many applications. Here we can note the already classical and long-established applications in equilibrium theory, games and economics [12], as well as new and young trends in image deconvolution [6, 11], reinforcement and statistical learning [1, 20], adversarial training [32] and GANs [17]. In particular, a series of papers [8, 15, 36, 7, 28, 48] showed the connection of the theory for convex SPPs with the training of GANs and obtained insights and useful tips for the GANs community. From the point of view of machine learning, it can be interesting when $f_m$ is an empirical loss function of the model on the local data of the $m$th device. Therefore, we consider the statement of the problem (1) when we have access only to local stochastic oracle of $f_m(x, y) := E_{\xi_m \sim D_m} f_m(x, y, \xi_m)$, where the data $\xi_m$ follows unknown distributions $D_m$.

However, the main problem of the distributed learning tasks is not the stochasticity, but precisely the separation of the data from the devices. All $f_m$ have access only to their own data, while transferring data to other devices may be inefficient and, moreover, impossible for privacy reasons. Therefore, to solve (1), it becomes necessary to construct a distributed algorithm that combine local computations on each of the devices and communication between them. Such an Algorithm can be organized as follows: all devices communicate only with the main device (server). This approach is called centralized. The main problem is the importance of the server – it can crash and interrupt the whole process.

*The research of A. Beznosikov and A. Gasnikov was supported by a grant for research centers in the field of artificial intelligence, provided by the Analytical Center for the Government of the Russian Federation in accordance with the subsidy agreement (agreement identifier 000000D730321PSQ0002) and the agreement with the Moscow Institute of Physics and Technology dated November 1, 2021 No. 70-2021-00138.
Therefore, along with the centralized approach, the decentralized [14] one is also popular. In this case, all devices are equal and connected to a network, communication occurs along the edges of this network.

Both centralised and decentralised methods are well developed for minimization problems. But meanwhile, the direction of distributed algorithms for SPPs is much weaker. Our work makes the following contribution to this area:

1.1 Our contributions

- **Lower bounds** We present lower bounds for distributed stochastic smooth strongly-convex-strongly-concave and convex-concave SPPs. Both centralized and decentralized cases are considered.

- **Optimal algorithms** Next, we obtain the near-optimal algorithms. They are near-optimal from a theoretical point of view, because their upper bounds reach lower bounds (up to numerical constants and logarithmic factors). For the centralized problem, we construct our method based on Extra Step method [26][21] (classical and optimal method for non-distributed SPPs) with the right size of batches. In decentralized case, we also use Extra Step method as base, but communication takes place with the help of accelerated (gossip) consensus procedure [29].

To sum up and compare the lower and upper bounds see Table 1.

|        | lower | upper |
|--------|-------|-------|
|        | centralized |          |      |
| sc     | $\Omega \left( R_0^2 \exp \left( \frac{32\mu K}{L \Delta} \right) + \frac{\sigma^2}{\mu^2 MT} \right)$ | $\tilde{O} \left( R_0^2 \exp \left( -\frac{\mu K}{4L \Delta} \right) + \frac{\sigma^2}{\mu^2 MT} \right)$ |
| c      | $\Omega \left( \frac{L \Delta}{K} + \frac{\sigma_1}{\sqrt{MT}} \right)$ | $O \left( \frac{L \Delta}{K} + \frac{\sigma_1}{\sqrt{MT}} \right)$ |

|        |        |        |
|--------|-------|-------|
|        | decentralized |          |      |
| sc     | $\Omega \left( R_0^2 \exp \left( \frac{128\mu K}{L \chi} \right) + \frac{\sigma^2}{\mu^2 MT} \right)$ | $\tilde{O} \left( R_0^2 \exp \left( -\frac{\mu K}{8L \chi} \right) + \frac{\sigma^2}{\mu^2 MT} \right)$ |
| c      | $\Omega \left( \frac{L \Delta \chi}{K} + \frac{\sigma_1}{\sqrt{MT}} \right)$ | $O \left( \frac{L \Delta \chi}{K} + \frac{\sigma_1}{\sqrt{MT}} \right)$ |

Table 1: Lower and upper bounds for distributed smooth stochastic strongly-convex-strongly-concave (sc) or convex-concave (c) saddle point problems in centralized and decentralized cases. Notation: $L$ – smoothness constant of $f$, $\mu$ – constant of strong convexity-strong concavity, $R_0 = \| z_0 - z^* \|_2$, $\Omega$ – diameter of optimization set, $\Delta$, $\chi$ – diameter and condition number of communication set (condition number of gossip matrix), $K$ – number of communication rounds, $T$ – number of local calls of gradient oracle on each node. In the convex-concave case, the bounds are in terms of the gap function, in the strongly convex-strongly concave case – in terms of the (squared) distance to the solution.

- **Local method** We also present an extra-step modification of Local SGD [34][54], one of the most popular methods in Federated Learning [25][22]. More recently, other versions of the Local SGD methods for SPPs have appeared [2][18]. All of the methods presented in these papers are based on gradient descent-ascent, but it is known that such methods, even in the non-distributed case, diverge for the most common SPPs [16][8]. Our method is based on the classic method for smooth SPPs – Extra Step algorithm, which makes it stand out from the competitors.

- **Non-convex-non-concave analysis** We analyze our new algorithms: near-optimal and local, not only in convex-concave case, but even in the non-convex-non-concave case under minty assumption [37][10]. Minty is the weakest additional assumption for a non-convex-non-concave problem found in the literature. But even with the these minty assumption, there are not many analyses of distributed methods [31][30]. In particular, our analysis covers the estimates of the decentralized but deterministic method from [31], and also generalizes and overlaps the estimates for the stochastic method for homogeneous data ($f_m = f$) from [30].

- **Experiments** The first part of our experiments on classical bilinear problem is devoted to the comparison of the optimal centralized method and the method based on Local SGD, as well as comparison of our local method with competitors [9][18]. The second part is devoted to the use of Local SGD and Local Adam techniques for training GANs in a homogeneous and heterogeneous cases.
1.2 Related works

**SPPs.** First, we highlight two main non-distributed algorithms for SPPs. First algorithm – Mirror Descent [4], it is customary to use in the non-smooth case. For smooth problems, Extra Step/Mirror Prox is applied [26][42][21]. Also, the following methods [43][19][56] can be noted as popular for smooth SPPs.

**Lower bounds.** In the non-distributed case, the lower bounds for smooth strongly convex-strongly concave case SPPs are given in [58], for convex-concave - in [46]. In smooth stochastic convex optimization, we highlight works about lower bounds [43][13]. It is also important to note the works devoted to the lower bounds for centralized and decentralized distributed convex optimization [51][2].

**Distributed SPPs.** The following works are devoted to decentralized Min-Max: in the deterministic case [31][40][50], in the stochastic case [30]. Let us also highlight the local methods for SPPs [9][18] already noted earlier in Section 1.1.

2 Settings and assumptions

We consider problem (1), where the sets \( X \subseteq \mathbb{R}^{n_x} \) and \( Y \subseteq \mathbb{R}^{n_y} \) are convex sets. For simplicity, we introduce the set \( Z = X \times Y \), \( x \in X \) and the operators \( F_m \):

\[
F_m(z) := F_m(x, y) := \left( \nabla_x f_m(x, y) - \nabla_y f_m(x, y) \right). \tag{2}
\]

As mentioned above, we do not have access to the oracles for \( F_m(z) \), at each iteration our oracles gives only some stochastic realization \( F_m(z, \xi) \). Next, we introduce the following assumptions:

**Assumption 1(g)** \( f(x, y) \) is \( L \)-smooth, i.e. for all \( z_1, z_2 \in Z \)

\[
||F(z_1) - F(z_2)|| \leq L||z_1 - z_2||. \tag{3}
\]

**Assumption 1(l)** For all \( m \), \( f_m(x, y) \) is \( L_{max} \)-smooth, i.e. for all \( z_1, z_2 \in Z \)

\[
||F_m(z_1) - F_m(z_2)|| \leq L_{max}||z_1 - z_2||. \tag{4}
\]

**Assumption 2(sc)** \( f(x, y) \) is strongly-convex-strongly-concave with constant \( \mu \), if for all \( z_1, z_2 \in Z \)

\[
\langle F(z_1) - F(z_2), z_1 - z_2 \rangle \geq \mu ||z_1 - z_2||^2. \tag{5}
\]

**Assumption 2(c)** \( f(x, y) \) is convex-concave, if \( f(x, y) \) is strongly-convex-strongly-concave with \( \mu = 0 \).

**Assumption 2(nc)** \( f \) satisfies the minty assumption, if exists \( z^* \in Z \) such that for all \( z \in Z \)

\[
\langle F(z), z - z^* \rangle \geq 0. \tag{6}
\]

**Assumption 3** \( F_m(z, \xi) \) is unbiased and has bounded variance, i.e. for all \( z \in Z \) it holds that

\[
\mathbb{E}[F_m(z, \xi)] = F_m(z), \quad \mathbb{E}[||F_m(z, \xi) - F_m(z)||^2] \leq \sigma^2. \tag{7}
\]

**Assumption 4** \( Z \) is compact bounded, i.e. for all \( z, z' \in Z \)

\[
||z - z'|| \leq \Omega_z. \tag{8}
\]

Hereinafter, we use the standard Euclidean norm \( ||\cdot|| \). We also introduce the following notation \( \text{proj}_Z(z) = \min_{u \in Z} ||u - z|| \) – the Euclidean projection onto \( Z \).

We also assume that all devices are connected to each other in a network, which can be represented as an undirected graph \( G(V, E) \) with diameter \( \Delta \). As mentioned earlier, we are interested in several cases of distributed optimization: centralized, and decentralized. It is important to mention one of the most popular communication procedures in decentralized setup – the gossip protocol [23][5][41]. This approach uses a certain matrix \( W \). Local vectors during communications are “weighted” by multiplication by \( W \). The convergence of decentralized algorithms is determined by the properties of this matrix. Therefore, we introduce its definition:

**Definition 1** We call a \( M \times M \) matrix \( W \) a gossip matrix if it satisfies the following conditions: 1) \( W \) is symmetric positive semi-definite, 2) the kernel of \( W \) is the set of constant vectors: \( \text{ker}(W) = \text{span}(1) \), 3) \( W \) is defined on the edges of the network: \( W_{ij} \neq 0 \) only if \( i = j \) or \( (i, j) \subseteq E \).
Let $\lambda_1(W) \geq \ldots \geq \lambda_M(W) = 0$ the spectrum of $W$, and condition number $\chi = \chi(W) = \frac{\lambda_1(W)}{\lambda_M(W)}$. Note that in practical algorithms [5, 51, 29, 57] is used not the matrix $W$, but $\hat{W} = I - \frac{W}{\chi(W)}$. To describe the convergence, we introduce $\lambda_2(\hat{W}) = 1 - \frac{\lambda_1(\hat{W})}{\lambda_M(\hat{W})} = 1 - \frac{1}{\chi(W)} = 1 - \frac{1}{\chi}$. The next definition is necessary to describe a certain class of distributed algorithms, for which we will obtain lower bounds. We use a definition quite similar to [2, 51].

**Definition 2** Let introduce some procedure with two parameters $T$ and $K$, which we call Black-Box Procedure($T,K$). Each agent $m$ has its own local memories $M^x_m$ and $M^y_m$ for the $x$- and $y$-variables, respectively–with initialization $M^x_m = M^y_m = \{0\}$. $M^x_m$ and $M^y_m$ are updated as follows.

- **Local computation:** At each local iteration device $m$ can sample uniformly and independently random variable $\xi_m$ and adds to its $M^x_m$ and $M^y_m$ a finite number of points $x, y$, satisfying
  
  
  \[ x \in \text{span}\{x', \nabla_x f_m(x'', y'', \xi_m)\}, \quad y \in \text{span}\{y', \nabla_y f_m(x'', y'', \xi_m)\}, \]

  \[ \text{for given } x', x'' \in M^x_m \text{ and } y', y'' \in M^y_m. \]

- **Communication:** Based upon communication rounds among neighbouring nodes, $M^x_m$ and $M^y_m$ are updated according to
  
  \[ M^x_m := \text{span} \left\{ \bigcup_{(i,m) \in E} M^x_i \right\}, \quad M^y_m := \text{span} \left\{ \bigcup_{(i,m) \in E} M^y_i \right\}. \]

- **Output:** The final global output is calculated as:
  
  \[ \hat{x} \in \text{span} \left\{ \bigcup_{m=1}^M M^x_m \right\}, \quad \hat{y} \in \text{span} \left\{ \bigcup_{m=1}^M M^y_m \right\}. \]

We assume that each node makes no more than $T$ local iterations (for simplicity, that exactly $T$) during the operation of the algorithm. The number of communication rounds is also limited to a certain number of $K < T$.

## 3 Lower bounds

Following the classical results on obtaining lower bounds, it is sufficient to give an example of a «bad» function [45], and the «bad» partitioning of this function between nodes [51]. First, let us divide the original problem into two independent ones: deterministic and stochastic. Consider $f_m(x, y) = f_{m,\text{deter}}(x, y) + f_{m,\text{stoch}}(x, y)$, where the vectors $x, y \in \mathbb{R}^n$. Each agent $m$ and $y \in \mathbb{R}^n$. At the same time we have access to $F_m(x, y, \xi) = f_{m,\text{deter}}(x, y) + \nabla f_{m,\text{stoch}}(x, y, \xi)$. It means that for $f_{m,\text{deter}}$ we have a deterministic oracle and stochastic – for $f_{m,\text{stoch}}$. Such $f_m$ helps to rewrite the original problem [1] as follows:

\[ f_{m,\text{deter}}(x, y) + f_{m,\text{stoch}}(x, y). \]

\[ \min_{x \in X, y \in Y} \max_{\xi \in \Xi} \left\{ \frac{1}{M} \sum_{m=1}^M f_{m,\text{deter}}(x, y) + \min_{x \in X, y \in Y} f_{m,\text{stoch}}(x, y) \right\}. \]

Therefore, we separately prove the estimates for each of the problems, and then combine.

### 3.1 Deterministic lower bounds

In this part, we provide lower bounds for the centralized (Theorem 1) and decentralized (Theorem 2) cases.

**Theorem 1** For any $\lambda > \mu > 0$ and any connected graph with diameter $\Delta$, there exists a distributed saddle point problem (satisfying Assumptions 1 and 2) on $X \times Y = \mathbb{R}^n \times \mathbb{R}^n$ (where $n$ is sufficiently large) with $x^*, y^* \neq 0$, such that for any output $\hat{x}, \hat{y}$ of any procedure (Definition 2), the following estimates hold:

\[ \|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2 = 2 \left( \frac{\mu}{L - \mu} + \frac{K}{\Delta} \right) \|y_0 - y^*\|^2. \]

**Theorem 2** For any $L > \mu > 0$ and any $\chi \geq 1$, there exists a decentralized distributed saddle point problem (satisfying Assumptions 2 and 3) on $X \times Y = \mathbb{R}^n \times \mathbb{R}^n$ (where $n$ is sufficiently large) with $x^*, y^* \neq 0$ over a fixed network (Definition 2)
with a gossip matrix $W$ and characteristic number $\chi$, such that for any output $\hat{x}, \hat{y}$ of any procedure (Definition 2), the following estimates hold:

$$\|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2 = \Omega \left( \exp \left( -\frac{32\mu}{L - \mu} \cdot \frac{K}{\sqrt{\chi}} \right) \|y_0 - y^*\|^2 \right).$$

Regularization and convex-concave case. Note that in the convex-concave case the problem is usually considered on a bounded set (Assumption 4), moreover, the convergence criterion for algorithms is a gap function:

$$\text{gap}(x, y) := \max_{y' \in \mathcal{Y}} f(x, y') - \min_{x' \in \mathcal{X}} f(x', y).$$

Therefore the lower bounds are also needed in terms of (12). Following the inequality 6 of [58], we can rewrite the estimates from Theorems 1 and 2 as follows

$$\text{gap}(x, y) \geq \frac{\mu}{2} \|x - x^*\|^2 + \frac{\mu}{2} \|y - y^*\|^2.$$

Next, to obtain bounds for the convex-concave case from bounds for the strongly convex-strongly concave case, we use a regularization trick:

$$f_{\text{reg}}(x, y) = f(x, y) + \frac{\epsilon}{8\Omega_z^2} \|x - x_0\|^2 - \frac{\epsilon}{8\Omega_z^2} \|y - y_0\|^2,$$

where $\Omega_z$ is an Euclidean diameter of the set $\mathcal{X} \times \mathcal{Y}$. It turns out that if $f(x, y)$ is a convex-concave function, then $f_{\text{reg}}(x, y)$ is $\frac{\epsilon}{8\Omega_z^2}$ is strongly convex-strongly concave. The new problem is solved with an accuracy of $\epsilon/2$, then we find a solution to the original problem with an accuracy of $\epsilon$.

3.2 Stochastic lower bounds

Due to our choice of $f_m$ from (11), one can note that in order to obtain lower stochastic bounds, we need to consider the minimization problem rather than the SPP, moreover, the non-distributed minimization problem. Therefore, these bounds depend on the total number of local calls of the oracle, and this number is equal to $MT$. Let us formulate two theorems for the convex and strongly convex cases of $f^\text{stoch}$.

**Theorem 3** For any $L > \mu > 0$ and any $M, T \in \mathbb{N}$, there exists a stochastic minimization problem with $L$-smooth and $\mu$-strongly convex function such that for any output $\hat{x}$ of any BBP($T, K$) (Definition 2) with $M$ workers one can obtain the following estimate:

$$\|\hat{x} - x^*\|^2 = \Omega \left( \frac{\sigma^2}{MT\mu^2} \right).$$

**Theorem 4** For any $L > 0$ and any $M, T \in \mathbb{N}$, there exists a stochastic minimization problem with $L$-smooth and convex function such that for any output $\hat{x}$ of any BBP($T, K$) (Definition 2) with $M$ workers one can obtain the following estimate:

$$\mathbb{E} [f(\hat{x}) - f(x^*)] = \Omega \left( \frac{\sigma\Omega_x}{\sqrt{MT}} \right).$$

3.3 Connection of lower bounds

The connecting of deterministic and stochastic bounds follows from (11). The results for the centralized and decentralized cases are shown in Table 1. See Appendix B for complete proof of this part. To verify the tightness of our lower bounds, the next section designs algorithms that reach such bounds.

4 Optimal algorithms

This section focuses on theoretically near-optimal algorithms. It is easy to check that our algorithms satisfy the BBP definition.

4.1 Centralized case

We design our algorithm based on MiniBatch SGD and Extra Step. For this algorithm we introduce $r$ as a maximum distance from nodes to server.
Algorithm 1 Centralized Extra Step Method

**Parameters:** Stepsize $\gamma \leq \frac{1}{4L}$; communication rounds $K$, number of local steps $T$.

**Initialization:** Choose $(x^0, y^0) = z^0 \in \mathcal{Z}$, $k = \lfloor \frac{L}{\mu} \rfloor$ and batch size $b = \lfloor \frac{T}{2K} \rfloor$.

for $t = 0, 1, 2, \ldots, k - 1$ do

1. Generate batch $\xi^t_m$ on each machine independently

   - Each machine $m$ computes $g_m^t = \frac{1}{b} \sum_{i=1}^{b} F_m(z^t, \xi^t_m)$ and sends $g_m^t$ to server

   - Server computes $z^{t+1/2} = \text{proj}_Z(z^t - \frac{\gamma}{M} \sum_{m=1}^{M} g_m^t)$ and then sends $z^{t+1/2}$ to machines

2. Generate batch $\xi^{t+1/2}_m$ on each machine independently

   - Each machine $m$ computes $g_m^{t+1/2} = \frac{1}{b} \sum_{i=1}^{b} F_m(z^{t+1/2}, \xi^{t+1/2}_m)$ and sends $g_m^{t+1/2}$ to server

   - Server computes $z^{t+1} = \text{proj}_Z(z^t - \frac{\gamma}{M} \sum_{m=1}^{M} g_m^{t+1/2})$ and then sends $z^{t+1}$ to machines

end for

**Theorem 5** Let $\{z^t\}_{t \geq 0}$ denote the iterates of Algorithm 2 for solving problem 1. Let Assumptions 1(g), 3 be satisfied. Then, if $\gamma \leq \frac{1}{4L}$, we have the following estimates in

- $\mu$-strongly convex-strongly concave case (Assumption 2(sc)):
  \[
  \mathbb{E}[\Vert z^k - z^* \Vert^2] = \mathcal{O}\left(\Vert z^0 - z^* \Vert^2 \exp\left(-\frac{\mu K}{4L\Delta}\right) + \frac{\sigma^2}{\mu^2 MT}\right),
  \]

- convex-concave case (Assumptions 2(c) and 4):
  \[
  \mathbb{E}[\text{gap}(z^k_{avg})] = \mathcal{O}\left(\frac{L\Omega^2 \Delta}{K} + \frac{\sigma \Omega z}{\sqrt{MT}}\right),
  \]

- non-convex-non-concave case (Assumptions 2(nc) and 4):
  \[
  \mathbb{E}\left[\frac{1}{k} \sum_{t=0}^{k-1} \Vert F(z^t) \Vert^2\right] = \mathcal{O}\left(\frac{L^2 \Omega^2 \Delta}{K} + \frac{\sigma^2 K}{MT \Delta}\right),
  \]

where $z^k_{avg} = \frac{1}{k} \sum_{t=0}^{k-1} z^{t+1/2}$.

4.2 Decentralized case

The idea of Algorithm 2 combines three things: Extra Step, accelerated consensus - FastMix (see Algorithm 4 in Appendix C or [29, 57]) and the right size of batches.

**Theorem 6** Let $\{z^t_m\}_{t \geq 0}$ denote the iterates of Algorithm 2 for solving problem 1. Let Assumptions 1(g), 1(l), 3 be satisfied. Then, if $\gamma \leq \frac{1}{4L}$ and $P = \mathcal{O}\left(\sqrt{\chi} \log \frac{1}{\varepsilon}\right)$, we have the following estimates in

- $\mu$-strongly convex-strongly concave case (Assumption 2(sc)):
  \[
  \mathbb{E}[\Vert \bar{z}^k - z^* \Vert^2] = \mathcal{O}\left(\Vert z^0 - z^* \Vert^2 \exp\left(-\frac{\mu K}{8L\sqrt{\chi}}\right) + \frac{\sigma^2}{\mu^2 MT}\right),
  \]

- convex-concave case (Assumptions 2(c) and 4):
  \[
  \mathbb{E}[\text{gap}(z^k_{avg})] = \mathcal{O}\left(\frac{L\Omega^2 \sqrt{\chi}}{K} + \frac{\sigma \Omega z}{\sqrt{MT}}\right),
  \]

- non-convex-non-concave case (Assumptions 2(nc) and 4):
  \[
  \mathbb{E}\left[\frac{1}{k} \sum_{t=0}^{k-1} \Vert F(z^t) \Vert^2\right] = \mathcal{O}\left(\frac{L^2 \Omega^2 \sqrt{\chi}}{K} + \frac{\sigma^2 K}{MT \sqrt{\chi}}\right),
  \]
Algorithm 2 Decentralized Extra Step Method

**Parameters:** Stepsize $\gamma \leq \frac{1}{2\delta L}$; communication rounds $K$, number of local calls $T$, number of FastMix steps $P$.

**Initialization:** Choose $(x_0^0, y_0^0) = z_0 \in \mathcal{Z}$, $z_m^0 = z_0$, $k = \left\lceil \frac{K}{P} \right\rceil$ and batch size $b = \left\lfloor \frac{T}{2K} \right\rfloor$.

for $t = 0, 1, 2, \ldots, k - 1$ do

Generate batch $\xi_t^m$ on each machine independently

Each machine $m$ compute $z_{m}^{t+1/2} = z_t - \gamma \cdot \frac{1}{b} \sum_{i=1}^{b} F_m(z_t^i, \xi_{m}^{i})$

Communication: $z_{m}^{t+1/2}, \ldots, z_{M}^{t+1/2} = \text{FastMix}(z_{m}^{t+1/2}, \ldots, z_{M}^{t+1/2}, P)$

Each machine $m$ compute $z_{m}^{t+1/2} = \text{proj}_z(z_{m}^{t+1/2})$

Generate batch $\xi_{m}^{t+1/2}$ on each machine independently

Each machine $m$ compute $z_{m}^{t+1} = z_t - \gamma \cdot \frac{1}{b} \sum_{i=1}^{b} F_m(z_{t+1/2}^i, \xi_{m}^{i})$

Communication: $z_{m}^{t+1}, \ldots, z_{M}^{t+1} = \text{FastMix}(z_{m}^{t+1}, \ldots, z_{M}^{t+1}, P)$

Each machine $m$ compute $z_{m}^{t+1} = \text{proj}_z(z_{m}^{t+1})$

end for

where $\bar{z}^t = \frac{1}{M} \sum_{m=1}^{M} z_{m}^{t}$ and $z_{avg}^{k+1} = \frac{1}{M_T} \sum_{t=0}^{k-1} \sum_{m=1}^{M} z_{m}^{t+1/2}$.

**Discussions** Let us make some comments on our Algorithms:

- It is easy to see that our Algorithms are near-optimal – see Table 1 for details. However, there are several practical drawbacks of these Algorithms. The first is related to the fact that in Algorithm 2 we need to take multi consensus steps at each iteration. This approach does not always pay off in practice. On the other hand, the optimal decentralized algorithms for minimization problems also use FastMix – see literature review in [53]. Secondly, if $T \gg K$, at each iteration we collect a very large batch, in practice such batches do not make sense. Therefore, an idea arises to use these local computations of gradients more efficiently, for example, doing local steps. This brings us to Section 5.

- It can be noted that in the non-convex-non-concave case, we do not guarantee the convergence, when $T \approx K$. But the method converge sublinearly if $\sigma = 0$. In this case, we cover the deterministic results from [51]. In the stochastic case ($\sigma > 0$), convergence is also not guaranteed in [30, 53]. Therefore, we cover and even overlap their analysis, since they consider only the homogeneous case $(f_m = f)$.

5 New local algorithm

In this section, we work on sets $\mathcal{X} = \mathbb{R}^{n_x}$ and $\mathcal{Y} = \mathbb{R}^{n_y}$. Additionally, we introduce the following assumption:

**Assumption 5** The values of the local operator are considered sufficiently close to the value of the mean operator, i.e. for all $z \in \mathcal{Z}$

$$\|F_m(z) - F(z)\| \leq D. \quad (13)$$

This assumption is often called $D$ - heterogeneity.

Our algorithm is a combination of Local SGD and Extra Step. One can note that such an algorithm is BBP($T$, $K$).

**Theorem 7** Let $\{z_{m}^t\}_{t \geq 0}$ denote the iterates of Algorithm 3 for solving problem 1. Let Assumptions 3 and 5 be satisfied. Also let $H = \max P |k_{p+1} - k_{p}|$ is a maximum distance between moments of communication (for $P \in I$). Then we have the following estimates in

- **$\mu$-strongly convex-strongly concave case (Assumption 2(sc)) with $\gamma \leq \frac{1}{4L H_{\max}}$:**

$$\mathbb{E}[\|z^T - z^*\|^2] = \tilde{O}\left(\|z_0 - z^*\|^2 \cdot \exp\left(-\frac{\mu T}{42H L_{\max}}\right) + \frac{\sigma^2}{\mu^2 MT} + \frac{L_{\max}^2}{\mu^4 T^2} (HD^2 + \sigma^2)\right),$$

- **non-convex–non-concave case (Assumption 2(nc)) and with assumption that for all $t$, $\|z_t\| \leq \Omega$ with $\gamma \leq \frac{1}{4L_{\max}}$:**

$$\mathbb{E}\left[\frac{1}{T} \sum_{t=0}^{T-1} \|F(z^t)\|^2\right] = \tilde{O}\left(\frac{L_{\max}^2 \Omega^2}{T} + \frac{H L_{\max} \Omega (HD^2 + \sigma^2)^{2/3}}{T^{1/3}} + \frac{\sigma^2}{M} + L_{\max} \Omega \sqrt{H (HD^2 + \sigma^2)}\right).$$
Algorithm 3 Extra Step Local SGD

Parameters: stepsizes \( \gamma \leq \frac{1}{2HL_{\max}} \); number of local steps \( T \), sets \( I \) of communications steps (\(|I| = K\)).

Initialization: Choose \((x^0, y^0) = z^0 \in Z\), for all \( m \), \( x_m^0 = z^0 \) and \( \hat{z} = z^0 \).

for \( t = 0, 1, 2, \ldots, T - 1 \) do

- Generate random variable \( \xi_m^t \) on each machine independently
- Each machine \( m \) computes \( z_m^{t+1/2} = z_t^m - \gamma F_m(z_t^m, \xi_m^t) \)
- Generate random variable \( \xi_m^{t+1/2} \) on each machine independently
- Each machine \( m \) computes \( z_m^{t+1} = z_t^m - \gamma F_m(z_m^{t+1/2}, \xi_m^{t+1/2}) \)

if \( t \in I \) do

- Each machine sends \( z_m^{t+1} \) on server
- Server computes \( \hat{z} = \frac{1}{M} \sum_{m=1}^{M} z_m^{t+1} \), sends \( \hat{z} \) to machines
- Each machine gets \( \hat{z} \) and sets \( z_m^{t+1} = \hat{z} \)

end for

Output: \( \hat{z} \).

where \( \bar{z}^t = \frac{1}{M} \sum_{m=1}^{M} z_m^t \).

Discussions

- Compared to Algorithm 1, Algorithm 3 gives worse convergence guarantees. Why then Algorithm 3 is needed? For practical purposes. Local SGD or FedAvg is a fairly well-known and popular federated learning concept. We extend this concept to min-max problems, including non-convex-non-concave ones. In particular, the theory states that for Algorithm 1 step \( \gamma \sim \frac{1}{\mu^{1/4}} \), and for Algorithm 1, \( \gamma \sim \frac{1}{\mu^{1/2}} \), but in practice one can use the same steps (learning rates) for both Algorithms. It seems natural that Algorithm 3 can outperform Algorithm 1 in some regimes, simply because it takes more steps (see Section 6).

- As noted in Section 1.1, there are two more methods of the Local SGD type for SPPs [9, 18]. But these methods use Descent-Ascent instead of Extra Step as a base. Also, the stepsize of these methods is confusing, even in the strongly convex-strongly concave case, it is proposed to take \( \gamma = \frac{\mu}{H L_{\max}} \), which in practice is a very small number and provide a very slow convergence of the methods.

6 Experiments

6.1 Bilinear problem

Let us start with an experiment on the bilinear problem:

\[
\min_{x,y} \max_{[-1,1]^n} \frac{1}{M} \sum_{m=1}^{M} \left( x^T A_m y + b_m^T x + c_m^T y \right),
\]

where \( n = 100, M = 100 \), matrices \( A_m \geq 0 \) are randomly generated with \( \lambda_{\text{max}} = 1000 \) (then \( L = 1000 \)). Coordinates \( b_m, c_m \) are generated uniformly on \([-1000; 1000]\). Moreover, we add noise with \( \sigma^2 = 10000 \) to the gradients. Starting point is zero.

The purpose of the first experiment is to compare our local method (Algorithm 3) with the local approaches from papers [9, 18]. For all methods \( H = 3 \), and the step is chosen for best convergence. See Figure 1(a) for the results. Note that our Algorithm 3 outperforms the competitors. Moreover, methods from papers [9, 18] do not converge at all with any steps \( \gamma \). As noted above (Section 1.1), this is due to the fact that these methods are based on Descent-Ascent.

The next experiment is aimed at comparing Algorithm 3 with different communication frequencies \( H \). We take \( \gamma = \frac{\mu}{H L} \). From the point of view of communications (Figure 1(b)), we get a standard result for local methods: less often communications, the faster convergence (in communications), but worse solution accuracy. This is due to fluctuations during local iterations, which lead away from the solution of the global solution.

In the third experiment, we want to vary the step and compare Algorithm 3 with a frequency of 3 and Algorithm 1 with batch 6 (such parameters give that there are 6 local calls for one communication for both Algorithms). This problem
Results
Based on the results of experiments on bilinear problems (Section 6.1), it was expected that methods which connect to the server less frequently (but do the same number of local epochs) would outperform their competitors in terms of communication budget. This trend is observed in Figures 2 and 3 – methods making fewer communications do not lose in terms of FID and IS. Meanwhile the strongly increasing distance between the communications can affect the quality of training considerably, especially in the last epochs. Therefore, we recommend using local methods with...
long gap between communications only in the initial stages of training, then it is worth communicating more and more frequently.

For more experiments with Algorithm 3 and Local Adam on MNIST see Appendix E.

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Supplementary Material

A General facts and technical lemmas

Lemma 1 For arbitrary integer \( n \geq 1 \) and arbitrary set of positive numbers \( a_1, \ldots, a_n \) we have
\[
\left( \sum_{i=1}^{m} a_i \right)^2 \leq m \sum_{i=1}^{m} a_i^2. 
\] (15)

Lemma 2 Suppose given a convex closed set \( Z \), then the operator of the Euclidean projection onto this set is non-expansive, i.e. for all \( z, z' \in Z \),
\[
\| \text{proj}_Z(z) - \text{proj}_Z(z') \| \leq \| z - z' \|. 
\] (16)

B Proof of Theorems from Section 3

As mentioned in the main part of the paper we consider the following model of functions:
\[
f_m(x, y) = f_m^\text{deter}(x^\text{deter}, y) + f_m^\text{stoch}(x^\text{stoch}). 
\] (17)

Note that the function \( f_m^\text{deter} \) uses the vector \( x^\text{deter} \), and the function \( f_m^\text{stoch} \) uses another vector \( x^\text{stoch} \). The variables in the vectors \( x^\text{deter} \) and \( x^\text{stoch} \) do not intersect, but together \( x^\text{deter} \) and \( x^\text{stoch} \) form a complete vector \( x \), for example, according to the following rule: \( x_{2k} = x_{2k}^\text{deter} \) and \( x_{2k+1} = x_{2k+1}^\text{stoch} \) for \( k = 1, 2, \ldots \). At the same time, for \( f_m^\text{deter} \), we have access to \( \nabla_{x^\text{deter}} f_m(x, y), \nabla_{y^\text{deterministic}} f_m(x^\text{deter}, y), \) and for \( f_m^\text{stoch} \), to stochastic realizations \( \nabla_{x^\text{stochastic}} f_m^\text{stoch}(x^\text{stochastic}, \xi) \) that satisfy Assumption 3. Moreover, \( f_m^\text{deter} \) are different for each device, but \( f_m^\text{stoch} \) is the same.

We take «bad» functions with even \( n_{x^\text{stochastic}} = n_{x^\text{deterministic}} = n_y = n \). Moreover, \( n \) must be taken large enough, as stated in the Theorems.

B.1 Deterministic lower bounds

We begin with deterministic lower bounds. Our example builds on a splitting of the «bad» function for the non-distributed case from [58]. Next, we give an example of functions \( f_m^\text{deterministic}(x^\text{deterministic}, y) \) and their location on the nodes. For simplicity of notation, in this subsection we use \( f_m(x, y) \) instead of \( f_m^\text{deterministic}(x^\text{deterministic}, y) \).

We introduce some auxiliary arrangement of functions on the nodes, prove some facts for it, and then present the final "bad" examples and prove the lower bounds.

Let \( B \subset V - \) subset of nodes of \( G \). For \( d \in N \) we define \( B_d = \{ v \in V : d(B, v) \geq d \} \), where \( d(B, v) \) – distance between set \( B \) and node \( v \). Then we construct the following arrangement of bilinearly functions on nodes:

\[
f_m(x, y) = \begin{cases} 
1 \cdot 0 & \quad m \in B_d \\
1 \cdot -2 & \quad m \in B \\
\end{cases} 
\] (18)

where \( e_1 = (1, 0, \ldots, 0) \) and

\[
A_1 = \begin{pmatrix} 1 & 0 & 1 & \ldots & 1 \\ 1 & -2 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 1 & \ldots & 1 \\
\end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -2 & 1 & 0 & 1 & -2 \\ 1 & 0 & -2 & 1 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 1 & \ldots & 1 \\
\end{pmatrix}.
\]

In most cases, we want the simplest case with \( |B| = |B_d| = 1 \).
Lemma 3 Let Problem (18) be solved by any method that satisfies Definition 2. Then after $K$ communication rounds, only the first $\left\lfloor \frac{n}{2} \right\rfloor$ coordinates of the global output can be non-zero while the rest of the $n - \left\lfloor \frac{n}{2} \right\rfloor$ coordinates are strictly equal to zero.

Proof: We begin introducing some notation for our proof. Let

$$E_0 := \{0\}, \ E_K := \text{span}\{e_1, \ldots, e_K\}.$$  

Note that, the initialization from Definition 2 gives $M^x_0 = E_0, M^y_0 = E_0$.

Suppose that, for some $m$, $M^x_m = E_K$ and $M^y_m = E_K$, at some given time. Let us analyze how $M^x_m, M^y_m$ can change by performing only local computations.

Firstly, we consider the case when $K$ odd. After one local update, we have the following:

- For machines $m$ which own $f_1$, it holds

$$x \in \text{span}\{e_1, x', A_1y'\} = E_K,$$  

$$y \in \text{span}\{e_1, y', A_1^T x'\} = E_K,$$

for given $x', x'' \in M^x_m$ and $y', y'' \in M^y_m$. Since $A_1$ has a block diagonal structure, after one local computation, we have $M^x_m = E_K$ and $M^y_m = E_K$. The situation does not change, no matter how many local computations one does.

- For machines $m$ which own $f_2$, it holds

$$x \in \text{span}\{x', A_2y'\} = E_{K+1},$$  

$$y \in \text{span}\{y', A_2^T x'\} = E_{K+1},$$

for given $x' \in M^x_m$ and $y' \in M^y_m$. It means that, after local computations (at least one local computation), one has $M^x_m = E_{K+1}$ and $M^y_m = E_{K+1}$. Therefore, machines with function $f_2$ can progress by one new non-zero coordinate.

This means that we constantly have to transfer progress from the group of machines with $f_1$ to the group of machines with $f_2$ and back. Initially, all devices have zero coordinates. Further, after at least one local computation, machines with $f_1$ can receive the first non-zero progress (but only the first, the second is not), and the rest of the devices are left with all zeros. Next, we pass the first non-zero coordinate to machines with $f_2$. To do this, $d$ communication rounds are needed. By doing so, they can make the second coordinate non-zero, and then transfer this progress to the machines with $f_1$. Then the process continues in the same way. It remains to note that for this update in the number of non-zero coordinates, we need at least one local calculation for each non-zero coordinate. Note that the local computation budget is sufficient ($T > K$ – see Definition 2). This completes the proof.

Consider the problem with the global objective function:

$$f(x, y) := \frac{1}{M} \sum_{m=1}^{M} f_m(x, y) = \frac{1}{M} (|B_d| \cdot f_1(x, y) + |B| \cdot f_2(x, y) + (M - |B_d| - |B|) \cdot f_3(x, y))$$  

$$= \frac{L}{2} x^T A y + \frac{\mu}{2} \|x\|^2 - \frac{\mu}{2} \|y\|^2 + \frac{L^2}{4 \mu} e_1^T y, \text{ with } A = \frac{1}{2}(A_1 + A_2)$$

With the fact that $\|A\| \leq 2$, one can easy verify that (20) satisfies Assumptions 1(g) and 2(sc).

The previous lemma gives an idea of what the solution obtained using procedures that satisfy Definition 2. The next lemma is already to the approximate solution of the problem (20) and how it is closed to the real solution.

Lemma 4 (Lemma 3.3 from [58]) Let $\alpha = \frac{4\sqrt{2}}{\pi^2}$ and $q = \frac{1}{2} \left(2 + \alpha - \sqrt{\alpha^2 + 4\alpha}\right) \in (0; 1)$ – the smallest root of $q^2 - (2 + \alpha)q + 1 = 0$, and let introduce approximation $\bar{y}^*$

$$\bar{y}^*_i = \frac{q^i}{1 - q}.$$  

Then error between approximation and real solution of (20) can be bounded

$$\|\bar{y}^* - y^*\| \leq \frac{q^{n+1}}{\alpha(1 - q)}.$$
Proof: Let us write the dual function for (20):

\[ g(y) = -\frac{1}{2} y^T \left( \frac{L^2}{4\mu} A^T A + \mu I \right) y + \frac{L^2}{4\mu} e_1^T y. \]

where one can easy found

\[ AA^T = \begin{pmatrix}
1 & -1 \\
-1 & 2 & -1 \\
-1 & 2 & -1 \\
-1 & 2 & -1 \\
\cdots \\
-1 & 2 & -1 \\
-1 & 2 & -1
\end{pmatrix}. \]

The optimality of the dual problem \( \nabla g(y^*) = 0 \) gives

\[ \left( \frac{L^2}{4\mu} A^T A + \mu I \right) y^* = \frac{L^2}{4\mu} e_1, \]

or

\[ (A^T A + \alpha I) y^* = e_1. \]

Let us write in the form of a set of equations:

\[ \begin{cases}
(1 + \alpha)y_1^* - y_2^* = 1 \\
y_1^* + (2 + \alpha)y_2^* - y_3^* = 0 \\
\cdots \\
y_{n-2}^* + (2 + \alpha)y_{n-1}^* - y_n^* = 0 \\
y_{n-1}^* + (2 + \alpha)y_n^* = 0
\end{cases} \]

Note that the approximation (21) satisfies the following set of equations:

\[ \begin{cases}
(1 + \alpha)\bar{y}_1^* - \bar{y}_2^* = 1 \\
\bar{y}_1^* + (2 + \alpha)\bar{y}_2^* - \bar{y}_3^* = 0 \\
\cdots \\
\bar{y}_{n-2}^* + (2 + \alpha)\bar{y}_{n-1}^* - \bar{y}_n^* = 0 \\
\bar{y}_{n-1}^* + (2 + \alpha)\bar{y}_n^* = \frac{q^{n+1}}{1 - q}
\end{cases} \]

or in the short form:

\[ (A^T A + \alpha I) \bar{y}^* = e_1 + \frac{q^{n+1}}{1 - q} e_n. \]

Then the difference between the approximation and the true solution is

\[ \bar{y}^* - y^* = (A^T A + \alpha I)^{-1} \frac{q^{n+1}}{1 - q} e_n, \]

With the fact that \( \alpha^{-1} I \succeq (A^T A + \alpha I)^{-1} > 0 \), it implies the statement of Lemma.

□

Now we formulate a key lemma (similar to Lemma 3.4 from [58]).
Lemma 5 Consider a distributed saddle point problem in form (18), (20) with $B_d \neq \emptyset$. For any pairs $T, K (T > K)$ one can found size of the problem $n \geq \max \left\{ 2 \log_q \left( \frac{\alpha}{\sqrt{q}} \right), 2K \right\}$, where $\alpha = \frac{4\mu^2}{L^2}$ and $q = \frac{1}{2} \left( 2 + \alpha - \sqrt{\alpha^2 + 4\alpha} \right) \in (0; 1)$. Then, any output $\hat{x}, \hat{y}$ produced by any BBP($T, K$) satisfying Definition 2 after $K$ communications rounds and $T$ local computations, is such that

$$
\|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2 \geq q^{2K} \frac{\|y_0 - y^*\|^2}{16}.
$$

Proof: Lemma 3 states that after $K (K < T)$ communications only $k = \left\lfloor \frac{K}{\alpha} \right\rfloor$ coordinates in the output $\hat{y}$ can be non-zero. Therefore, by definition of $\hat{y}^*$ from (21), by $k \leq K \leq \frac{n}{2}$ and with $q < 1$, we have

$$
\|\hat{y} - \hat{y}^*\|^2 \geq \sum_{j=k+1}^{n} (\hat{y}_j^*)^2 = q^k \frac{1}{1-q} \sqrt{q^2 + q^4 + \ldots + q^{2(n-k)}}
$$

$$
\geq q^k \frac{1}{\sqrt{2(1-q)}} \sqrt{q^2 + q^4 + \ldots + q^{2n}} = q^k \frac{\sqrt{2}}{\sqrt{2}} \|\hat{y}^*\|^2 = q^k \frac{\sqrt{2}}{\sqrt{2}} \|y_0 - y^*\|^2.
$$

For $n \geq 2 \log_q \left( \frac{\alpha}{\sqrt{q}} \right)$ we can guarantee that $\hat{y}^* \approx y^*$ (for more detailed see [58]) and

$$
\|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2 \geq \|\hat{y} - y^*\|^2 \geq q^{2K} \frac{\|y_0 - y^*\|^2}{16} \geq q^{2K} \frac{\|y_0 - y^*\|^2}{16}.
$$

Building on the above preliminary results, we are now ready to prove our complexity lower bound as stated in Theorems 1 and 2.

B.1.1 Centralized case

Theorem 8 (Theorem 1). For any $L > \mu > 0$, any $\Delta$ and any $T, K \in \mathbb{N}$ with $T > K$, there exists a distributed saddle point problem of $\Delta + 1$ functions with centralized architecture. For which the following statements are true:

- the diameter of graph $G$ is equal to $\Delta$.
- $f = \frac{1}{M} \sum_{m=1}^{M} f_m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is $L$-smooth, $\mu$-strongly convex strongly concave,
- size $n \geq \max \left\{ 2 \log_q \left( \frac{\alpha}{\sqrt{q}} \right), 2K \right\}$, where $\alpha = \frac{4\mu^2}{L^2}$ and $q = \frac{1}{2} \left( 2 + \alpha - \sqrt{\alpha^2 + 4\alpha} \right) \in (0; 1)$.

Then for any output $\hat{x}, \hat{y}$ of any BBP($T, K$) (Definition 2), the following estimates hold:

$$
\|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2 = \Omega \left( \exp \left( - \frac{4\mu}{L - \mu} : \frac{K}{\Delta} \right) \|y_0 - y^*\|^2 \right).
$$

Proof: It suffices to consider a linear graph on $\Delta + 1$ vertices $\{v_1, \ldots, v_{\Delta+1}\}$ and apply Lemma 5 for problem (18), (20) with $B = \{v_1\}$ and $d = \Delta$. Then

$$
\left( \frac{1}{q} \right)^{\frac{2K}{\Delta}} \geq \frac{\|y_0 - y^*\|^2}{16(\|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2)}.
$$

Taking the logarithm of the two parts of the inequality, we get

$$
\frac{2K}{\Delta} \geq \ln \left( \frac{\|y_0 - y^*\|^2}{16(\|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2)} \right) \frac{1}{\ln(q^{-1})}.
$$
Next, we work with
\[
\frac{1}{\ln(q^{-1})} = \frac{1}{\ln(1 + (1 - q)/q)} \geq \frac{q}{1 - q} = \frac{1 + 2\mu^2/\ell^2 - 2\sqrt{\mu^2/\ell^2 + \left(\frac{\mu^2}{\ell^2}\right)^2}}{2\sqrt{\mu^2/\ell^2 + \left(\frac{\mu^2}{\ell^2}\right)^2} - 2\mu^2/\ell^2}
\]
\[
= 2\sqrt{\frac{\mu^2/\ell^2 + \left(\frac{\mu^2}{\ell^2}\right)^2}{4\mu^2/\ell^2}} - \frac{2\mu^2/\ell^2}{4\mu^2/\ell^2}
\]
\[
= \frac{1}{2}\frac{L^2}{\mu^2} + 1 - \frac{1}{2}.
\]
Finally, one can obtain
\[
\frac{2K}{\Delta} \geq \ln\left(\frac{\|y_0 - y^*\|^2}{16(\|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2)}\right) \cdot \frac{1}{2} \left(\frac{L}{\mu} - 1\right),
\]
and
\[
\exp\left(\frac{4\mu}{L - \mu} \frac{K}{\Delta}\right) \geq \frac{\|y_0 - y^*\|^2}{16(\|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2)},
\]
which completes the proof.

\(\square\)

B.1.2 Decentralized case

**Theorem 9** For any \(L > \mu > 0\), any \(\Delta\) and any \(T, K \in \mathbb{N}\) with \(T > K\), there exists a distributed saddle point problem with decentralized architecture and a gossip matrix \(W\). For which the following statements are true:

- a gossip matrix \(W\) have \(\chi(W) = \chi\).
- \(f = \frac{1}{M} \sum_{m=1}^{M} f_m : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) is \(L\)-smooth, \(\mu\) - strongly convex-strongly concave,
- size \(n \geq \max \left\{2 \log_q \left(\frac{\alpha}{\sqrt{\Delta}}\right) \cdot 2K\right\}\), where \(\alpha = \frac{L^2}{\mu^2}\) and \(q = \frac{1}{2} \left(2 + \alpha - \sqrt{\alpha^2 + 4\alpha}\right)\).

Then for any output \(\hat{x}, \hat{y}\) of any BBP\((T, K)\) (Definition 2), the following estimates hold:
\[
\|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2 = \Omega\left(\exp\left(-\frac{32\mu}{L - \mu} \frac{K}{\sqrt{\chi}}\right) \|y_0 - y^*\|^2\right).
\]

**Proof:** The proof follow similar steps as in the proof of [51 Theorem 2]. Let \(\gamma_M = \frac{1 - \cos \frac{\pi}{M}}{1 + \cos \frac{\pi}{M}}\) be a decreasing sequence of positive numbers. Since \(\gamma_2 = 1\) and \(\lim_{m} \gamma_M = 0\), there exists \(M \geq 2\) such that \(\gamma_M \geq \frac{1}{\gamma} > \gamma + 1\).

- If \(M \geq 3\), let us consider linear graph of size \(M\) with vertexses \(v_1, \ldots, v_M\), and weighted with \(w_{1,2} = 1 - a\) and \(w_{i,i+1} = 1\) for \(i \geq 2\). We apply Lemma 5 for problem \((18, 20)\) with \(B = \{v_1\}\) and \(d = M - 1\), then we have \(B_d = \{v_M\}\). Hence,
\[
\|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2 \geq q \frac{16\|y_0 - y^*\|^2}{\chi(\hat{W}_{\alpha})}.
\]
If \(\hat{W}_{\alpha}\) is the Laplacian of the weighted graph \(\hat{G}\), one can note that with \(a = 0\), \(\frac{1}{\chi(\hat{W}_{\alpha})} = \gamma_M\). Then \(\chi(\hat{W}_{\alpha}) \geq \gamma_M \geq \frac{2}{\left(\frac{M}{2}\right)^2}\), and \(M \geq \sqrt{2\chi - 1} \geq \frac{\sqrt{8}}{4}\). Finally, since \(M \geq 3\), we get \(d = M - 1 \geq \frac{M}{2} \geq \frac{\sqrt{8}}{8}\). Hence,
\[
\|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2 \geq q \frac{16\|y_0 - y^*\|^2}{\chi(\hat{W}_{\alpha})}.
\]
Similarly to the proof of the previous theorem
\[
\exp\left(\frac{32\mu}{L - \mu} \frac{K}{\sqrt{\chi}}\right) \geq \frac{\|y_0 - y^*\|^2}{16(\|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2)}.
\]
• If $M = 2$, we construct a fully connected network with 3 nodes with weight $w_{1,3} = a \in [0;1]$. Let $W_a$ is the Laplacian. If $a = 0$, then the network is a linear graph and $\rho(W_a) = \gamma_3 = \frac{1}{3}$. Hence, there exists $a \in [0;1]$ such that $\chi(W_a) = \chi$. Finally, $B = \{v_1\}$, $B_d = \{v_3\}$ and $d \geq 1 \geq \sqrt{\frac{2}{\chi^2}}$. Whence, it follows that in this case (22) is also valid.

\[ \square \]

**B.2 Stochastic lower bounds**

**B.2.1 Strongly-convex case**

We consider the following simple problem with function $f : \mathbb{R} \rightarrow \mathbb{R}$:

$$\min_{x \in \mathbb{R}} f(x) = \frac{\mu}{2} (x - x_0)^2,$$

(23)

where we do not know the constant $x_0 \neq 0$. $f(x)$ is a $\mu$-strongly-convex and $\mu$-smooth function. We minimize this function by using stochastic first order oracle

$$\nabla f(x, \xi) = \mu (x + \xi - x_0), \quad \text{where } \xi \in \mathcal{N} \left( 0, \frac{\sigma^2}{\mu^2} \right).$$

One can note that $\mathbb{E}[\nabla f(x, \xi)] = \mu (x - x_0) = \nabla f(x)$, and $\mathbb{E} \left[ |\nabla f(x, \xi) - \nabla f(x)|^2 \right] = \mathbb{E} \left[ \mu^2 |\xi|^2 \right] = \sigma^2$. We use some BPP($T, K$) (Definition 2), which calls the stochastic oracle $N = MT$ times in some set of points $\{x_i\}_{i=1}^N$, for these points oracle returns $y_i = \mu (x_i - x_0 + \xi_i)$, where all $\xi_i \in \mathcal{N}(0, \sigma^2/\mu^2)$ and independent. Using $x_i, y_i$, one can compute point $z_i = x_i - y_i/\mu = x_0 - \xi_i \in N(x_0, \sigma^2/\mu^2)$ and independent. Hence, the original problem (23) and the working of any BPP are easy to reformulate in the following way: after $N$ calls of the oracle we have set of pairs $\{(x_i, z_i)\}_{i=1}^N$, where $z_i \in N(x_0, \sigma^2/\mu^2)$ and independent. By these pairs we need to estimate the unknown constant $x_0$. One can do it by MLE:

$$x_N^{\text{MLE}} = \frac{1}{N} \sum_{i=1}^N z_i, \quad x_N^{\text{MLE}} \in \mathcal{N} \left( x_0, \frac{\sigma^2}{N\mu^2} \right).$$

Then

$$\mathbb{E} \left[ |x_N^{\text{MLE}} - x^*|^2 \right] = \mathbb{E} \left[ |x_N^{\text{MLE}} - x_0|^2 \right] = \text{Var} \left[ x_N^{\text{MLE}} \right] = \frac{\sigma^2}{N\mu^2},$$

or

$$\mathbb{E} \left[ f(x_N^{\text{MLE}}) - f(x^*) \right] = \frac{\mu}{2} \mathbb{E} \left[ |x_N^{\text{MLE}} - x_0|^2 \right] = \frac{\mu}{2} \text{Var} \left[ x_N^{\text{MLE}} \right] = \frac{\sigma^2}{2N\mu}.$$

We need to show that the estimate obtained with the MLE is the best in terms of $N$, for this we use the Cramer–Rao bound:

**Lemma 6** Suppose $x_0$ is an unknown parameter which is to be estimated from $\{z_i\}_{i=1}^N$, independent observations of $z$, each distributed according to some probability density function $f_{x_0}(z)$. Consider an estimator $\hat{x}_0 = \hat{x}_0(z_1, \ldots, z_N)$ with bias $b(x_0) = \mathbb{E}[\hat{x}_0 - x_0]$. Then, the estimator $\hat{x}_0$ satisfies

$$\mathbb{E} \left[ |\hat{x}_0 - x_0|^2 \right] \geq \frac{[1 + b'(x_0)]^2}{NI(x_0)} + b^2(x_0),$$

where $I(x_0) = \mathbb{E} \left[ \left( \frac{\partial \ln f_{x_0}(z)}{\partial x_0} \right)^2 \right]$ — Fisher information.

For normal distribution $I(x_0) = \frac{\mu^2}{2}$. Then

$$\mathbb{E} \left[ |\hat{x}_0 - x_0|^2 \right] \geq \frac{\sigma^2}{N\mu^2} \left[ 1 + b'(x_0) \right]^2 + b^2(x_0)$$

Suppose that there is such estimate that it is better than the MLE in terms of $N$. Hence,

$$[1 + b'(x_0)]^2 \sim \frac{1}{N^{2\alpha}}, \quad \text{where } \alpha > 0.$$

It means $-b'(x_0) \sim \frac{N^{\alpha - 1}}{N^{2\alpha}}$ or $\frac{N^{\alpha + 1}}{N^{2\alpha}}$. With enough big $N$ we get that $b'(x_0) \sim -1$ in terms of $N$. Then $b^2(x_0) \sim x_0^2$. We arrive at a contradiction in the existence of an estimate that asymptotically (in $N$) better than the MLE. Then we have the following theorem:
Theorem 10 (Theorem[3]) For any $L > \mu > 0$ and any $M, T \in \mathbb{N}$, there exists a stochastic minimization problem with $L$-smooth and $\mu$-strongly convex function such that for any output $\hat{x}$ of any BBP($T, K$) (Definition 2) with $M$ workers one can obtain the following estimate:

$$||\hat{x} - x^*||^2 = \Omega \left( \frac{\sigma^2}{MT\mu^2} \right).$$

B.2.2 Convex case

For convex case, we work with

$$\min_{x \in \left\{-\frac{1}{\Omega x}, \frac{1}{\Omega x}\right\}} \frac{\tilde{\varepsilon}}{\Omega x},$$

(24)

where $\tilde{\varepsilon}$ can only take two values $\varepsilon$ or $-\varepsilon$ with some positive $\varepsilon$. Of course, we do not know which of the two values $\tilde{\varepsilon}$ takes. For example, we can assume that at the very beginning $\tilde{\varepsilon}$ is chosen randomly and equally probable. It is easy to verify that (24) is convex and $L$-smooth for any $L$ and $\varepsilon$. The first order stochastic oracle returns $\nabla f(x, \xi) = \xi \in \mathcal{N}(\tilde{\varepsilon}/\Omega x, \sigma^2)$.

One can note that $\mathbb{E}[\nabla f(x, \xi)] = \tilde{\varepsilon}/\Omega x = \nabla f(x)$, and $\mathbb{E}[(\nabla f(x, \xi) - \nabla f(x))^2] = \sigma^2$. We use some procedure BBP($T, K$) (Definition 2), which calls oracle $\mathcal{N} = MT$ times in some set of points $\{x_i\}_{i=1}^N$. For these points oracle returns $\xi_i$, where all $\xi_i \in \mathcal{N}(\tilde{\varepsilon}, \sigma^2)$ and independent. Note that we can say in advance that $x^* = \frac{\Omega x}{\tilde{\varepsilon}}$ if $\tilde{\varepsilon} = -\varepsilon$ and $x^* = -\frac{\Omega x}{\tilde{\varepsilon}}$ if $\tilde{\varepsilon} = \varepsilon$. We have a rather simple task, from independent samples $\{\xi_i\}_{i=1}^N \in \mathcal{N}(\tilde{\varepsilon}/\Omega x, \sigma^2)$, we need to determine $\tilde{\varepsilon}$ from two equally probable hypothesis $H_1 : \tilde{\varepsilon} = \varepsilon$ or $H_2 : \tilde{\varepsilon} = -\varepsilon$. For these problem one can use likelihood-ratio criterion:

$$\delta(\xi_1, \ldots, \xi_N) = \begin{cases} H_1, & T(\xi_1, \ldots, \xi_N) < c \\ H_2, & T(\xi_1, \ldots, \xi_N) \geq c \end{cases}$$

T(\xi_1, \ldots, \xi_N) = \frac{f_{H_2}(\xi_1, \ldots, \xi_N)}{f_{H_1}(\xi_1, \ldots, \xi_N)},

(25)

where $f_H$ is a density function of a random vector $\xi_1, \ldots, \xi_N$ with distribution from the hypothesis $H$. The Neyman–Pearson lemma gives

Lemma 7 There is a constant $c$ for which the likelihood-ratio criterion (25) is

• minmax criterion. The number $c$ should be chosen so that the type I error and the type II error were the same;

• Bayesian criterion for given prior probabilities $r$ and $s$. The number $c$ is chosen equal to the ratio $r/s$.

Due to the symmetry of the hypotheses with respect to zero, as well as the prior probabilities can be considered equal to $1/2$, we have that $c = 1$ for minmax and Bayesian criterions. By simple transformations we can rewrite (25):

$$\delta(\xi_1, \ldots, \xi_N) = \begin{cases} H_1, & \sum_{i=1}^N \xi_i > 0 \\ H_2, & \sum_{i=1}^N \xi_i \leq 0 \end{cases}$$

$$\hat{x}_N = \begin{cases} -\frac{\Omega x}{2}, & \sum_{i=1}^N \xi_i > 0 \\ \frac{\Omega x}{2}, & \sum_{i=1}^N \xi_i \leq 0 \end{cases}.$$

This criterion is more than natural. Neyman–Pearson lemma says it is optimal. Next we analyse error of this criterion (we will consider only case with $\varepsilon = \tilde{\varepsilon}$, the other case one can parse similarly):

$$\mathbb{E}[f(\hat{x}_N) - f(x^*)] = \mathbb{E}\left[\frac{\varepsilon}{\Omega x} \hat{x}_N + \frac{\varepsilon}{2}\right] = \varepsilon \cdot \mathbb{P}\left\{\sum_{i=1}^N \xi_i \leq 0\right\} = \varepsilon \cdot \mathbb{P}\{S_N \leq 0\},$$

where $S_N = \sum_{i=1}^N \xi_i \in \mathcal{N}(\varepsilon N/\Omega x, \sigma^2 N)$, then $\frac{S_N - \varepsilon N/\Omega x}{\sigma \sqrt{N}} \in \mathcal{N}(0, 1)$. Finally, we get

$$\mathbb{E}[f(\hat{x}_N) - f(x^*)] = \varepsilon \mathbb{P}\left\{\frac{S_N - \varepsilon N/\Omega x}{\sigma \sqrt{N}} \leq -\frac{\varepsilon \sqrt{N}}{\Omega x \sigma}\right\} = \varepsilon \mathbb{P}\left\{S_N \geq \frac{\varepsilon \sqrt{N}}{\Omega x \sigma}\right\} \geq \varepsilon \cdot \frac{1}{3t} \exp\left(-\frac{t^2}{2}\right) \cdot \left(1 - \frac{1}{t^2}\right).$$

In last inequality we define $t = \frac{\varepsilon \sqrt{N}}{\Omega x \sigma}$ and use lower bound for tail of standard normal distribution. With $\varepsilon = \frac{\sigma \Omega x}{\sqrt{N}}$, we have $t = 2$ and then

$$\mathbb{E}[f(\hat{x}_N) - f(x^*)] \geq \frac{\varepsilon}{4t} \exp(-2) \geq \frac{1}{4 \exp(2)} \cdot \frac{\sigma \Omega x}{\sqrt{N}}.$$

Hence, we get the next theorem:
Theorem 11 (Theorem 4) For any $L > 0$ and any $M, T \in \mathbb{N}$, there exists a stochastic minimization problem with $L$-smooth and convex function such that for any output $\hat{x}$ of any $\text{BBP}(T, K)$ (Definition 2) with $M$ workers one can obtain the following estimate:

$$\mathbb{E} [f(\hat{x}) - f(x^*)] = \Omega \left( \frac{\sigma \Omega_x}{\sqrt{MT}} \right).$$
C Proof of Theorems from Section 4

C.1 Centralized case

We start our proof with the following lemma:

Lemma 8 Let \( z, y \in \mathbb{R}^n \) and \( Z \subset \mathbb{R}^n \) is convex compact set. We set \( z^+ = \text{proj}_z (z - y) \), then for all \( u \in Z \):

\[
\| z^+ - u \|^2 \leq \| z - u \|^2 - 2 \langle y, z^+ - u \rangle - \| z^+ - z \|^2.
\]

Proof: For all \( u \in Z \) we have \( \langle z^+ - (z - y), z^+ - u \rangle \leq 0 \). Then

\[
\| z^+ - u \|^2 = \| z^+ - z + z - u \|^2 \\
= \| z - u \|^2 + 2 \langle z^+ - z, z - u \rangle + \| z^+ - z \|^2 \\
= \| z - u \|^2 + 2 \langle z^+ - z, z^+ - u \rangle - \| z^+ - z \|^2 \\
= \| z - u \|^2 + 2 \langle z^+ - (z - y), z^+ - u \rangle - 2 \langle y, z^+ - u \rangle - \| z^+ - z \|^2 \\
\leq \| z - u \|^2 - 2 \langle y, z^+ - u \rangle - \| z^+ - z \|^2.
\]

Before proof the main theorems, we add the following notation:

\[
g_t = \frac{1}{M} \sum_{m=1}^{M} g_m^t, \quad g_t^{1/2} = \frac{1}{M} \sum_{m=1}^{M} g_m^{t+1/2}.
\]

C.1.1 Strongly convex-strongly concave problems

Theorem 12 (Theorem 5) Let \( \{z^t\}_{t \geq 0} \) denote the iterates of Algorithm 7 for solving problem 1. Let Assumptions 11, 20, 21, 22, and 3 be satisfied. Then, if \( \gamma \leq \frac{1}{4L} \), we have the following estimate for the distance to the solution \( z^* \):

\[
\mathbb{E} \left[ \| z^k - z^* \|^2 \right] = \mathcal{O} \left( \| z^0 - z^* \|^2 \exp \left( -\frac{\mu}{4L} \cdot \frac{K}{\Delta} \right) + \frac{\sigma^2}{\mu^2 MT} \right).
\]

Proof: Applying the previous Lemma with \( z^+ = z^{t+1}, u = z^t, y = z, \) and \( y = \gamma g^{t+1/2} \), we get

\[
\| z^{t+1} - z \|^2 \leq \| z^t - z \|^2 - 2\gamma \langle g^{t+1/2}, z^{t+1} - z \rangle - \| z^{t+1} - z^t \|^2,
\]

and with \( z^+ = z^{t+1/2}, z = z^t, u = z^{t+1}, y = \gamma g^t \):

\[
\| z^{t+1/2} - z^{t+1} \|^2 \leq \| z^t - z^{t+1} \|^2 - 2\gamma \langle g^t, z^{t+1/2} - z^{t+1} \rangle - \| z^{t+1/2} - z^t \|^2.
\]

Next, we sum up the two previous equalities

\[
\| z^{t+1} - z \|^2 + \| z^{t+1/2} - z^{t+1} \|^2 \leq \| z^t - z \|^2 - \| z^{t+1} - z^t \|^2 - 2\gamma \langle g^{t+1/2}, z^{t+1} - z \rangle - 2\gamma \langle g^t, z^{t+1/2} - z^{t+1} \rangle.
\]

A small rearrangement gives

\[
\| z^{t+1} - z \|^2 + \| z^{t+1/2} - z^{t+1} \|^2 \leq \| z^t - z \|^2 - \| z^{t+1} - z^t \|^2 - 2\gamma \langle g^{t+1/2}, z^{t+1} - z \rangle - 2\gamma \langle g^t, z^{t+1/2} - z^{t+1} \rangle + \gamma^2 \| g^{t+1/2} - g^t \|^2 + \| z^{t+1/2} - z^{t+1} \|^2.
\]

Then we substitute \( z = z^* \) and take the total expectation of both sides of the equation

\[
\mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] \leq \mathbb{E} \left[ \| z^t - z^* \|^2 \right] - \mathbb{E} \left[ \| z^{t+1} - z^t \|^2 \right] + \gamma^2 \mathbb{E} \left[ \| g^{t+1/2} - g^t \|^2 \right].
\]

(27)
Let work with $\mathbb{E} \left[ \| g^{t+1/2} - \hat{g}^t \|^2 \right]$: 

$$
\mathbb{E} \left[ \| g^{t+1/2} - \hat{g}^t \|^2 \right] = \mathbb{E} \left[ \| g^{t+1/2} - F(z^{t+1/2}) + F(z^t) - \hat{g}^t + F(z^{t+1/2}) - F(z^t) \|^2 \right]
$$

\[ \leq 2 \mathbb{E} \left[ \| g^{t+1/2} - F(z^{t+1/2}) \|^2 \right] + 3 \mathbb{E} \left[ \| F(z^t) - \hat{g}^t \|^2 \right] + 3 \mathbb{E} \left[ \| F(z^{t+1/2}) - F(z^t) \|^2 \right]
\]

\[ \leq 3 \mathbb{E} \left[ \left\| \sum_{m=1}^{M} \sum_{i=1}^{b} (F_m(z^{t+1/2}, \xi_m^{t+1/2, i}) - F_m(z^{t+1/2})) \right\|^2 \right] + 3 \mathbb{E} \left[ \left\| F_m(z^t, \xi_m^t) - F_m(z^t) \right\|^2 \right] + 3 L^2 \mathbb{E} \left[ \| z^{t+1/2} - z^t \|^2 \right]
\]

\[ = \frac{3}{(bM)^2} \mathbb{E} \left[ \left\| \sum_{m=1}^{M} \sum_{i=1}^{b} (F_m(z^{t+1/2}, \xi_m^{t+1/2, i}) - F_m(z^{t+1/2})) \right\|^2 \right] + 3 \mathbb{E} \left[ \left\| \sum_{i=1}^{b} (F_m(z^t, \xi_m^t) - F_m(z^t)) \right\|^2 \right] + 3 L^2 \mathbb{E} \left[ \| z^{t+1/2} - z^t \|^2 \right]
\]

Using that all $\{\xi_m^t \}_{i=1,m=1}^{b,M}$ and $\{\xi_m^{t+1/2, i} \}_{i=1,m=1}^{b,M}$ are independent, we get

$$
\mathbb{E} \left[ \| g^{t+1/2} - \hat{g}^t \|^2 \right] \leq \frac{3}{(bM)^2} \sum_{m=1}^{M} \sum_{i=1}^{b} \mathbb{E} \left[ \left\| F_m(z^{t+1/2}, \xi_m^{t+1/2, i}) - F_m(z^{t+1/2}) \right\|^2 \right]
$$

$$
+ \frac{3}{(bM)^2} \sum_{m=1}^{M} \sum_{i=1}^{b} \mathbb{E} \left[ \left\| F_m(z^t, \xi_m^t) - F_m(z^t) \right\|^2 \right] + 3 L^2 \mathbb{E} \left[ \| z^{t+1/2} - z^t \|^2 \right]
\]

$$
\leq \frac{3}{(bM)^2} \sum_{m=1}^{M} \sum_{i=1}^{b} \mathbb{E} \left[ \left\| F_m(z^{t+1/2}, \xi_m^{t+1/2, i}) - F_m(z^{t+1/2}) \right\|^2 \right] + 3 L^2 \mathbb{E} \left[ \| z^{t+1/2} - z^t \|^2 \right] + \frac{6 \sigma^2}{bM}.
\] (28)

Next we estimate $\mathbb{E} \left[ \langle g^{t+1/2}, z^{t+1/2} - z^* \rangle \right]$. To begin with, we use the independence of all $\xi$, as well as the unbiasedness of $\hat{g}^{t+1/2}$ with respect to the conditional m.o. by random variables $\{\xi_m^{t+1/2, i} \}_{i=1,m=1}^{b,M}$:

$$
\mathbb{E} \left[ \langle g^{t+1/2}, z^{t+1/2} - z^* \rangle \right] = \mathbb{E} \left[ \langle \xi_m^{t+1/2, i} \xi_m^t \rangle \mathbb{E} \left[ \langle g^{t+1/2}, z^{t+1/2} - z^* \rangle \right] \right]
\]

$$
= \mathbb{E} \left[ \langle F(z^{t+1/2}), z^{t+1/2} - z^* \rangle \right].
\] (29)

By property of $z^*$, we get

$$
\mathbb{E} \left[ \langle g^{t+1/2}, z^{t+1/2} - z^* \rangle \right] \geq \mathbb{E} \left[ \langle F(z^{t+1/2}) - F(z^*), z^{t+1/2} - z^* \rangle \right] \geq \mu \mathbb{E} \left[ \| z^{t+1/2} - z^* \|^2 \right].
\]

Let use a simple fact $\| z^{t+1/2} - z^* \|^2 \geq \frac{1}{2} \| z^t - z^* \|^2 - \| z^{t+1/2} - z^t \|^2$, then

$$
\mathbb{E} \left[ \langle g^{t+1/2}, z^{t+1/2} - z^* \rangle \right] \geq \frac{1}{2} \mathbb{E} \left[ \| z^t - z^* \|^2 \right] - \mu \mathbb{E} \left[ \| z^{t+1/2} - z^t \|^2 \right].
\] (30)

Combining 3 inequalities (27) with $z = z^*$, (28), (30):

$$
\mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] \leq (1 - \mu \gamma) \mathbb{E} \left[ \| z^t - z^* \|^2 \right] + (2 \mu \gamma + 3 \gamma^2 L^2 - 1) \mathbb{E} \left[ \| z^{t+1/2} - z^t \|^2 \right] + \frac{6 \sigma^2 \gamma^2}{bM}.
\]

In Algorithm 1 the step $\gamma \leq \frac{1}{4L}$, then

$$
\mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] \leq (1 - \mu \gamma) \mathbb{E} \left[ \| z^t - z^* \|^2 \right] + \frac{6 \sigma^2 \gamma^2}{bM}.
\]
Let us run the recursion from 0 to $k - 1$:
\[
\mathbb{E} \left[ \|z^k - z^*\|^2 \right] \leq (1 - \mu \gamma)^k \mathbb{E} \left[ \|z^0 - z^*\|^2 \right] + \frac{6\sigma^2 \gamma}{\mu b M}.
\]

Then we carefully choose $\gamma = \min \left\{ \frac{1}{4L}, \frac{\ln(\max(2bhM\mu \|z^0 - z^*\|^2/k/6\sigma^2))}{\mu k} \right\}$ and get (for more details one can see [55])
\[
\mathbb{E} \left[ \|z^{k+1} - z^*\|^2 \right] = \tilde{O} \left( \|z^0 - z^*\|^2 \exp \left( -\frac{\mu k}{4L} \right) + \frac{\sigma^2}{\mu^2 b M k} \right).
\]

Substitute the batch size $b$ and the number of iterations $k$ from the description of the Algorithm [1]
\[
\mathbb{E} \left[ \|z^{k+1} - z^*\|^2 \right] = \tilde{O} \left( \|z^0 - z^*\|^2 \exp \left( -\frac{\mu}{4L} \cdot \frac{K}{r} \right) + \frac{\sigma^2}{\mu^2 MT} \right).
\]

Finally, we remember that $r \leq \Delta$ and finish the proof.

\[
\square
\]

### C.1.2 Convex-concave problems

**Theorem 13 (Theorem 5)** Let $\{z^t\}_{t \geq 0}$ denote the iterates of Algorithm [1] for solving problem [1]. Let Assumptions [1(g), 2(c), 3] and [4] be satisfied. Then, if $\gamma \leq \frac{1}{4L}$, we have the following estimate:
\[
\mathbb{E}[\text{gap}(z_{avg}^k)] = O \left( \frac{L\|x\|^2 \Delta}{K} + \frac{\sigma \Omega}{\sqrt{MT}} \right).
\]

**Proof:** We have already shown some of the necessary estimates, namely, we need to use [26] with some small rearrangement
\[
2\gamma \langle F(z^{t+1/2}), z^{t+1/2} - z \rangle \leq \|z^t - z\|^2 - \|z^{t+1/2} - z\|^2 - \|z^{t+1/2} - z^t\|^2 + 2\gamma \langle F(z^{t+1/2}) - \bar{g}^{t+1/2}, z^{t+1/2} - z \rangle + \gamma^2 \|\bar{g}^{t+1/2} - \bar{g}^t\|^2.
\]

Next, we sum over all $t$ from 0 to $k - 1$
\[
\frac{1}{k} \sum_{t=0}^{k-1} (F(z^{t+1/2}), z^{t+1/2} - z) \leq \frac{\|z^0 - z\|^2 - \|z^{k+1} - z\|^2}{2\gamma k} + \frac{1}{k} \sum_{t=0}^{k-1} (F(z^{t+1/2}) - \bar{g}^{t+1/2}, z^{t+1/2} - z)
\]
\[
+ \frac{1}{2\gamma k} \sum_{t=0}^{k-1} \gamma^2 \|\bar{g}^{t+1/2} - \bar{g}^t\|^2 - \|z^{t+1/2} - z^t\|^2.
\]
\[
(31)
\]

Then, by $x_{avg}^k = \frac{1}{k} \sum_{t=0}^{k-1} x^{t+1/2}$ and $y_{avg}^k = \frac{1}{k} \sum_{t=0}^{k-1} y^{t+1/2}$, Jensen’s inequality and convexity-concavity of $f$:
\[
\text{gap}(z_{avg}^k) \leq \max_{y' \in Y} f \left( \frac{1}{k} \left( \sum_{t=0}^{k-1} x^{t+1/2} \right), y' \right) - \min_{x' \in X} f \left( x', \frac{1}{k} \left( \sum_{t=0}^{k-1} y^{t+1/2} \right) \right)
\]
\[
\leq \max_{y' \in Y} \frac{1}{k} \sum_{t=0}^{k-1} f(x^{t+1/2}, y') - \min_{x' \in X} \frac{1}{k} \sum_{t=0}^{k-1} f(x', y^{t+1/2}).
\]

Given the fact of linear independence of $x'$ and $y'$:
\[
\text{gap}(z_{avg}^k) \leq \max_{(x', y') \in \mathcal{Z}} \frac{1}{k} \sum_{t=0}^{k-1} \left( f(x^{t+1/2}, y') - f(x', y^{t+1/2}) \right).
\]
Using convexity and concavity of the function $f$:

\[
\text{gap}(z_{av}^k) \leq \max_{(x', y') \in \mathbb{Z}} \frac{1}{k} \sum_{t=0}^{k-1} \left( f(x_{t+1/2}, y') - f(x', y_{t+1/2}) \right)
\]

\[
= \max_{(x', y') \in \mathbb{Z}} \frac{1}{k} \sum_{t=0}^{k-1} \left( f(x_{t+1/2}, y') - f(x_{t+1/2}, y_{t+1/2}) + f(x_{t+1/2}, y_{t+1/2}) - f(x', y_{t+1/2}) \right)
\]

\[
\leq \max_{(x', y') \in \mathbb{Z}} \frac{1}{k} \sum_{t=0}^{k-1} \left( \langle \nabla_y f(x_{t+1/2}, y_{t+1/2}), y' - y_{t+1/2} \rangle + \langle \nabla_z f(x_{t+1/2}, y_{t+1/2}), x_{t+1/2} - x' \rangle \right)
\]

\[
\leq \max_{z \in \mathbb{Z}} \frac{1}{k} \sum_{t=0}^{k-1} \langle F(z_{t+1/2}), z_{t+1/2} - z \rangle.
\] (32)

Together with (32), (31) gives (additionally, we take a full expectation)

\[
\mathbb{E}[\text{gap}(z_{av}^k)] \leq \mathbb{E} \left[ \max_{z \in \mathbb{Z}} \frac{\|z_0 - z\|^2 - \|z_{av} - z\|^2}{2\gamma k} \right]
\]

\[
= \max_{z \in \mathbb{Z}} \mathbb{E} \left[ \frac{k-1}{k} \sum_{t=0}^{k-1} \langle F(z_{t+1/2}) - \bar{g}_{t+1/2}, z_{t+1/2} - z \rangle \right]
\]

\[
= \frac{\Omega^2}{2\gamma k} + \frac{1}{k} \mathbb{E} \left[ \max_{z \in \mathbb{Z}} \sum_{t=0}^{k-1} \langle F(z_{t+1/2}) - \bar{g}_{t+1/2}, z_{t+1/2} - z \rangle \right]
\]

\[
= \frac{\Omega^2}{2\gamma k} + \frac{1}{k} \mathbb{E} \sum_{t=0}^{k-1} 3\gamma^2 L^2 \|z_{t+1/2} - z_t\|^2 + \frac{6\gamma^2 \sigma^2}{bM} - \|z_{t+1/2} - z_t\|^2.
\]

With $\gamma = \frac{1}{4\epsilon}$ we get

\[
\mathbb{E}[\text{gap}(z_{av}^k)] \leq \frac{\Omega^2}{2\gamma k} + \frac{1}{k} \mathbb{E} \left[ \max_{z \in \mathbb{Z}} \sum_{t=0}^{k-1} \langle F(z_{t+1/2}) - \bar{g}_{t+1/2}, z_{t+1/2} - z \rangle \right] + \frac{3\gamma^2 \sigma^2}{bM}. \tag{33}
\]

To finish the proof we need to estimate $\mathbb{E} \left[ \max_{z \in \mathbb{Z}} \sum_{t=0}^{k-1} \langle F(z_{t+1/2}) - \bar{g}_{t+1/2}, z_{t+1/2} - z \rangle \right]$. Let define sequence $v$: $v^0 = z^{1/2}$, $v^{t+1} = \text{proj}_z(v^t - \gamma \delta_t)$ with $\delta_t = F(z_{t+1/2}) - \bar{g}_{t+1/2}$. Then we have

\[
\sum_{t=0}^{k-1} \langle \delta_t, z_{t+1/2} - z \rangle = \sum_{t=0}^{k-1} \langle \delta_t, z_{t+1/2} - v^t \rangle + \sum_{t=0}^{k-1} \langle \delta_t, v^t - z \rangle. \tag{34}
\]

By the definition of $v^{t+1}$, we have for all $z \in \mathbb{Z}$

\[
\langle v^{t+1} - v^t + \gamma \delta_t, z - v^{t+1} \rangle \geq 0.
\]

Rewriting this inequality, we get

\[
\langle \gamma \delta_t, v^t - z \rangle \leq \langle \gamma \delta_t, v^t - v^{t+1} \rangle + \langle v^{t+1} - v^t, z - v^{t+1} \rangle
\]

\[
\leq \langle \gamma \delta_t, v^t - v^{t+1} \rangle + \frac{1}{2} \|v^t - z\|^2 - \frac{1}{2} \|v^{t+1} - z\|^2 - \frac{1}{2} \|v^t - v^{t+1}\|^2
\]

\[
\leq \frac{\gamma^2}{2} \|\delta_t\|^2 + \frac{1}{2} \|v^t - v^{t+1}\|^2 + \frac{1}{2} \|v^t - z\|^2 - \frac{1}{2} \|v^{t+1} - z\|^2 - \frac{1}{2} \|v^t - v^{t+1}\|^2
\]

\[
= \frac{\gamma^2}{2} \|\delta_t\|^2 + \frac{1}{2} \|v^t - z\|^2 - \frac{1}{2} \|v^{t+1} - z\|^2.
\]
With (34) it gives
\[
\sum_{t=0}^{k-1} \langle \delta^t, z^{t+1/2} - z \rangle \leq \sum_{t=0}^{k-1} \langle \delta^t, z^{t+1/2} - v^t \rangle + \frac{\gamma}{2} \sum_{t=0}^{k-1} \left( \frac{\gamma^2}{2} \| \delta^t \|^2 + \frac{1}{2} \| v^t - z \|^2 - \frac{1}{2} \| v^{t+1} - z \|^2 \right)
\]
\[
\leq \sum_{t=0}^{k-1} \langle \delta^t, z^{t+1/2} - v^t \rangle + \frac{\gamma}{2} \sum_{t=0}^{k-1} \| \delta^t \|^2 + \frac{1}{2\gamma} \| v^0 - z \|^2
\]
\[
\leq \sum_{t=0}^{k-1} \langle \delta^t, z^{t+1/2} - v^t \rangle + \frac{\gamma}{2} \sum_{t=0}^{k-1} \| \delta^t \|^2 + \frac{1}{2\gamma} \| \xi_{l,m} \|^2 + \frac{\Omega^2}{2\gamma}.
\]

The right side is independent of \( z \), then
\[
\max_{z \in \mathbb{Z}} \sum_{t=0}^{k-1} \langle \delta^t, z^{t+1/2} - z \rangle \leq \sum_{t=0}^{k-1} \langle \delta^t, z^{t+1/2} - v^t \rangle + \frac{\gamma}{2} \sum_{t=0}^{k-1} \| F(z^{t+1/2}) - \bar{g}^{t+1/2} \|^2 + \frac{\Omega^2}{2\gamma}. \tag{35}
\]

Taking the full expectation and using independence \( v^t - z^{t+1/2}, \{\xi_{l,m}^{t+1/2,i} \}_{l=1,m=1}^{b,M} \), we get
\[
\mathbb{E} \left[ \max_{z \in \mathbb{Z}} \sum_{t=0}^{k-1} \langle \delta^t, z^{t+1/2} - z \rangle \right] \leq \mathbb{E} \left[ \sum_{t=0}^{k-1} \langle \delta^t, z^{t+1/2} - v^t \rangle \right]
\]
\[
+ \frac{\gamma}{2} \sum_{t=0}^{k-1} \mathbb{E} \left[ \| F(z^{t+1/2}) - \bar{g}^{t+1/2} \|^2 \right] + \frac{\Omega^2}{2\gamma}
\]
\[
= \mathbb{E} \left[ \sum_{t=0}^{k-1} \langle \mathbb{E} \xi_{l,m}^{t+1/2,i} \rangle_{b,M} \left[ F(z^{t+1/2}) - \bar{g}^{t+1/2} \right], z^{t+1/2} - v^t \rangle \right]
\]
\[
+ \frac{\gamma}{2} \sum_{t=0}^{k-1} \mathbb{E} \left[ \| F(z^{t+1/2}) - \bar{g}^{t+1/2} \|^2 \right] + \frac{\Omega^2}{2\gamma}
\]
\[
= \frac{\gamma}{2} \sum_{t=0}^{k-1} \mathbb{E} \left[ \| F(z^{t+1/2}) - \bar{g}^{t+1/2} \|^2 \right] + \frac{\Omega^2}{2\gamma}
\]
\[
\leq \frac{\gamma k}{2} \cdot 3\sigma^2 \cdot bM + \frac{\Omega^2}{2\gamma}.
\]

Then we can finish (33) and get
\[
\mathbb{E} [\text{gap}(z_{avg}^k)] \leq \frac{\Omega^2}{\gamma k} + \frac{\gamma}{2} \cdot \frac{5\sigma^2}{bM}.
\]

Let \( \gamma = \min \left\{ \frac{1}{4\gamma}, \Omega z \sqrt{\frac{2bM}{3\sigma^2}} \right\} \) then
\[
\mathbb{E} [\text{gap}(z_{avg}^k)] = O \left( \frac{L\Omega^2}{k} + \frac{\sigma \Omega z}{\sqrt{bMK}} \right).
\]

Substitute the batch size \( b \) and the number of iterations \( k \) from the description of the Algorithm with \( r \leq \Delta \):
\[
\mathbb{E} [\text{gap}(z_{avg}^k)] = O \left( \frac{L\Omega^2 \Delta}{k} + \frac{\sigma \Omega}{\sqrt{bMK}} \right).
\]

\( \square \)

### C.1.3 Non-convex-non-concave problems

**Theorem 14 (Theorem 5)** Let \( \{z^t\}_{t \geq 0} \) denote the iterates of Algorithm for solving problem 1. Let Assumptions 1(g), 2(m), 3, 4 be satisfied. Then, if \( \gamma \leq \frac{1}{4\gamma} \), we have the following estimates:
\[
\mathbb{E} \left[ \frac{1}{k} \sum_{t=0}^{k-1} \| F(z^t) \|^2 \right] = O \left( \frac{L^2\Omega^2 \Delta}{K} + \frac{\sigma^2 K}{MT\Delta} \right).
\]

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Proof: We start proof with combining (27), (28) and (29)

\[
\mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] \leq \mathbb{E} \left[ \| z^t - z^* \|^2 \right] - \mathbb{E} \left[ \| z^{t+1/2} - z^t \|^2 \right]
- 2\gamma \mathbb{E} \left[ F(z^{t+1/2}), z^{t+1/2} - z^* \right] + 3\gamma^2 L^2 \mathbb{E} \left[ \| z^{t+1/2} - z^t \|^2 \right] + 6\gamma^2 \sigma^2 / bM.
\]

Using minty assumption (6), we obtain

\[
\mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] \leq \mathbb{E} \left[ \| z^t - z^* \|^2 \right] - (1 - 3\gamma^2 L^2) \mathbb{E} \left[ \| z^{t+1} - z^t \|^2 \right] + 6\gamma^2 \sigma^2 / bM.
\]

With \( \gamma \leq \frac{1}{4L} \),

\[
\mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] \leq \mathbb{E} \left[ \| z^t - z^* \|^2 \right] - (1 - 3\gamma^2 L^2) \mathbb{E} \left[ \| z^{t+1} - z^t \|^2 \right] + 6\gamma^2 \sigma^2 / bM.
\]

The fact: \(-\| \bar{g}^t \|^2 \leq -\frac{3}{2} \| F(z^t) \|^2 + 2\| \bar{g}^t - F(z^t) \|^2 \), gives

\[
\mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] \leq \mathbb{E} \left[ \| z^t - z^* \|^2 \right] - \frac{\gamma^2}{4} \mathbb{E} \left[ \| F(z^t) \|^2 \right] + 2\gamma^2 \| \bar{g}^t - F(z^t) \|^2 + 6\gamma^2 \sigma^2 / bM.
\]

The term \( \| \bar{g}^t - F(z^t) \|^2 \) was estimated, when we deduced (28). Then

\[
\frac{\gamma^2}{2} \mathbb{E} \left[ \| F(z^t) \|^2 \right] \leq \mathbb{E} \left[ \| z^t - z^* \|^2 \right] - \mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] + \frac{8\gamma^2 \sigma^2}{bM}.
\]

Summing over all \( t \) from 0 to \( k - 1 \):

\[
\mathbb{E} \left[ \frac{1}{k} \sum_{t=0}^{k-1} \| F(z^t) \|^2 \right] \leq \frac{2 \mathbb{E} \left[ \| z^0 - z^* \|^2 \right]}{\gamma^2 k} + \frac{16\sigma^2}{bM}.
\]

Next we substitute \( \gamma = \frac{1}{4L}, k, b \) and finish the proof.

□
C.2 Decentralized case

First of all, we present the missing Algorithm 4:

**Algorithm 4 FastMix**

**Parameters:** Vectors $z_1, ..., z_M$, communic. rounds $P$.

**Initialization:** Construct matrix $z$ with rows $z_1^T, ..., z_M^T$, choose $z^{-1} = z$, $z^0 = z$, $\eta = \frac{1 - \sqrt{1 - \lambda_2(W)}}{1 + \sqrt{1 - \lambda_2(W)}}$.

for $h = 0, 1, 2, ..., P - 1$ do

$z^{h+1} = (1 + \eta)Wz^h - \eta z^{h-1}$,

end for

Output: rows $z_1, ..., z_M$ of $z^P$.

We introduce the following notation

$z^t = \frac{1}{M} \sum_{m=1}^{M} z_m^t$, $z^{t+1/2} = \frac{1}{M} \sum_{m=1}^{M} z_m^{t+1/2}$, $g^t = \frac{1}{M} \sum_{m=1}^{M} g_m^t$, $g^{t+1/2} = \frac{1}{M} \sum_{m=1}^{M} g_m^{t+1/2}$,

$z^t = \frac{1}{M} \sum_{m=1}^{M} \tilde{z}_m^t$, $\tilde{z}^{t+1/2} = \frac{1}{M} \sum_{m=1}^{M} \tilde{z}_m^{t+1/2}$, $\tilde{z}^t = \frac{1}{M} \sum_{m=1}^{M} \tilde{z}_m^t$, $\tilde{z}^{t+1/2} = \frac{1}{M} \sum_{m=1}^{M} \tilde{z}_m^{t+1/2}$.

Next, we introduce the convergence of FastMix [29, 57]:

**Lemma 9** Assume that $\{\tilde{z}_m^{t+1}\}_{m=1}^M$ are output of Algorithm 4 with input $\{\tilde{z}_m^{t+1}\}_{m=1}^M$. Then it holds that

$$\frac{1}{M} \sum_{m=1}^{M} \|\tilde{z}_m^{t+1} - \tilde{z}^{t+1}\|^2 \leq \left(1 - \frac{1}{\sqrt{\chi}}\right)^{2P} \left(\frac{1}{M} \sum_{m=1}^{M} \|\tilde{z}_m^{t+1} - \tilde{z}^{t+1}\|^2\right) \text{ and } \tilde{z}^t = \tilde{z}^t.$$

Let after $H$ iteration we get $\varepsilon_0$ accuracy of consensus, i.e.

$$\tilde{z}^t_m - \tilde{z}^t = \delta^t_m, \quad ||\delta^t_m|| \leq \varepsilon_0, \quad \tilde{z}^{t+1/2}_m - \tilde{z}^{t+1/2} = \delta^{t+1/2}_m, \quad ||\delta^{t+1/2}_m|| \leq \varepsilon_0. \quad \text{(36)}$$

Then let us estimate the number of iterations $H$ to achieve such $\varepsilon_0$ (how to choose this parameter we will talk later) accuracy:

**Corollary 1** To achieve accuracy $\varepsilon_0$ in terms of (36), we need to take $P$:

- in convex-concave (Assumptions 2(c) and 4) and non-convex-non-concave (Assumptions 2(nc) and 4) cases

$$P = \mathcal{O}\left(\sqrt{\chi} \log\left(1 + \frac{\Omega_z^2 + \frac{Q^2 + \sigma^2/L_{\text{max}}}{\varepsilon_0^2}}{\varepsilon_0^2}\right)\right),$$

- in strongly convex-strongly concave case (Assumption 2(sc))

$$P = \mathcal{O}\left(\sqrt{\chi} \log\left(1 + \frac{\left\|z^0 - z^*\right\|^2 + \frac{Q^2 + \sigma^2/L_{\text{max}}}{\varepsilon_0^2}}{\varepsilon_0^2}\right)\right),$$

where $Q^2 = \frac{1}{M} \sum_{m=1}^{M} \|F_m(z^*)\|^2$.

**Proof:** The proof is in a rough estimate of $\frac{1}{M} \sum_{m=1}^{M} \|\tilde{z}_m^{t+1} - \tilde{z}^{t+1}\|^2$. 

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\[
\frac{1}{M} \sum_{m=1}^{M} \| \hat{z}^{t+1}_m - \hat{z}^{t}_m \|^2 = \frac{1}{M} \sum_{m=1}^{M} \| z^t_m - \gamma g^{t+1/2}_m - z^t + \gamma g^{t+1/2} \|^2 \\
\leq \frac{2}{M} \sum_{m=1}^{M} \| z^t_m - z^t \|^2 + \frac{2\gamma^2}{M} \sum_{m=1}^{M} \| g^{t+1/2}_m - g^{t+1/2} \|^2 \\
\leq \frac{2}{M} \sum_{m=1}^{M} \| \text{proj}_Z(\hat{z}^{t}_m) - \frac{1}{M} \sum_{i=1}^{M} \text{proj}_Z(\hat{z}^{t}_i) \|^2 + \frac{2\gamma^2}{M} \sum_{m=1}^{M} \| g^{t+1/2}_m \|^2
\]

In last inequality we use property: \( \frac{1}{M} \sum_{m=1}^{M} \| g^{t+1/2}_m - g^{t+1/2} \|^2 \leq \frac{1}{M} \sum_{m=1}^{M} \| g^{t+1/2}_m \|^2 \).

\[
\frac{1}{M} \sum_{m=1}^{M} \| \hat{z}^{t}_m - \hat{z}^{t} \|^2 \leq \frac{4}{M} \sum_{m=1}^{M} \| \text{proj}_Z(\hat{z}^{t}_m) - \text{proj}_Z(\hat{z}^{t}) \|^2 + \frac{4\gamma^2\sigma^2}{b} \\
\leq \frac{8\epsilon_0^2 + 8\gamma^2 \sum_{m=1}^{M} \| F_m(\hat{z}^{t+1/2}_m) \|^2 + \frac{4\gamma^2\sigma^2}{b}}{8\epsilon_0^2 + 8\gamma^2 \sum_{m=1}^{M} \| F_m(z^*_m) \|^2 + \frac{4\gamma^2\sigma^2}{b}}
\]

The proof of the theorem follows from \( \gamma \leq \frac{1}{4\sigma\epsilon_0} \) and the fact that in the convex-concave and non-convex-non-concave cases we can bounded \( \| \hat{z}^{t+1/2}_m - z^* \| \leq \Omega_z \), in strongly convex-strongly concave \( \| \hat{z}^{t+1/2}_m - z^* \| \leq \| z^0 - z^* \| \).

We are now ready to prove the main theorems. Note we can rewrite one step of the algorithm as follows:

\[
z^{t+1/2} = \frac{1}{M} \sum_{m=1}^{M} z^{t+1/2}_m = \frac{1}{M} \sum_{m=1}^{M} \text{proj}_Z(\hat{z}^{t+1/2}_m + \delta^{t+1/2}_m) \\
= \text{proj}_Z(\hat{z}^{t+1/2}) + \frac{1}{M} \sum_{m=1}^{M} \text{proj}_Z(\hat{z}^{t+1/2}_m + \delta^{t+1/2}_m) - \text{proj}_Z(\hat{z}^{t+1/2}) \\
= \text{proj}_Z \left( \frac{1}{M} \sum_{m=1}^{M} z^{t}_m - \gamma g^{t}_m \right) + \Delta^t = \text{proj}_Z \left( z^{t} - \gamma g^{t} \right) + \Delta^t
\]

Here we add one more notation: \( \frac{1}{M} \sum_{m=1}^{M} \text{proj}_Z(z^{t+1/2}_m + \delta^{t+1/2}_m) - \text{proj}_Z(\hat{z}^{t+1/2}_m) = \Delta^{t+1/2} \). It is easy to see \( \| \Delta^{t+1/2} \| \leq \epsilon_0 \). We see that the step of the algorithm is very similar to step of Algorithm \( \square \) but with imprecise projection onto a set. Let us prove the following lemma:

**Lemma 10** Let \( z, y, \Delta \in \mathbb{R}^n \) and \( Z \subset \mathbb{R}^n \) is convex compact set. We set \( z^+ = \text{proj}_Z(z - y) + \Delta \), then for all \( u \in Z \):

\[
\| z^+ - u \| \leq \|z - u\|^2 - 2\langle y, z^+ - u \rangle - \| z^+ - z \|^2
\]

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Proof: Let $r = \text{proj}_Z(z - y)$. For all $u \in \mathcal{Z}$ we have $(r - (z - y), r - u) \leq 0$. Then

\[
\|z^+ - u\|^2 = \|z^+ - z + z - u\|^2 \\
= \|z - u\|^2 + 2(z^+ - z, z - u) + \|z^+ - z\|^2 \\
= \|z - u\|^2 + 2(z^+ - (z - y), z^+ - u) - 2(y, z^+ - u) - \|z^+ - z\|^2 \\
= \|z - u\|^2 + 2(r - (z - y), r - u) + 2\langle \Delta, r - u \rangle + 2\langle z^+ - (z - y), \Delta \rangle \\
- 2(y, z^+ - u) - \|z^+ - z\|^2 \\
\leq \|z - u\|^2 + 2\langle \Delta, z^+ - u \rangle + 2\langle \Delta, r - (z - y) \rangle - 2(y, z^+ - u) - \|z^+ - z\|^2 \\
\leq \|z - u\|^2 + 2\|\Delta\| \cdot \|z^+ - u\| + 2\|\Delta\| \cdot \|\text{proj}_Z(z - y) - \text{proj}_Z(z)\| + 2\|\Delta\| \cdot \|y\| \\
- 2(y, z^+ - u) - \|z^+ - z\|^2 \\
\leq \|z - u\|^2 + 2\|\Delta\| \cdot \|z^+ - u\| + 4\|\Delta\| \cdot \|y\| - 2(y, z^+ - u) - \|z^+ - z\|^2.
\]

\[
\square
\]

C.2.1 Convex-concave problems

**Theorem 15 (Theorem 6)** Let $\{z^t\}_{t \geq 0}$ denote the iterates of Algorithm 2 for solving problem 1. Let Assumptions \[1(\text{g})\] \[1(\text{i})\] \[2(nc)\] \[3\] and \[4\] be satisfied. Then, if $\gamma \leq \frac{1}{4L}$ and $P = O\left(\sqrt{\log \frac{1}{\Omega}}\right)$, we have the following estimates in

\[
\mathbb{E}[\text{gap}(z^k_{avg})] = \tilde{O}\left(\frac{L_\Omega^2 \sqrt{\mathcal{N}}}{K} + \frac{\sigma_{\Omega_x}}{\sqrt{MT}}\right).
\]

Proof: The same way as in Theorem 12 one can get

\[
\|z^{t+1} - z\|^2 \leq \|z^t - z\|^2 - \|z^{t+1/2} - z\|^2 - 2\gamma \langle g^{t+1/2}, z^{t+1/2} - z \rangle \\
+ \gamma^2 \|g^{t+1/2} - g^t\|^2 + 4\|\Delta^{t+1/2}\| \cdot \|z^{t+1} - z\| + 4\mathbb{E}\left[\|\Delta^{t+1/2}\| \cdot \|g^{t+1/2}\|\right] \\
+ 4\|\Delta^t\| \cdot \|z^{t+1/2} - z^{t+1}\| + 4\|\Delta^t\| \cdot \|\gamma g^t\| \\
\leq \|z^t - z\|^2 - \|z^{t+1/2} - z\|^2 \\
- 2\gamma \langle g^{t+1/2}, z^{t+1/2} - z \rangle + \gamma^2 \|g^{t+1/2} - g^t\|^2 \\
+ 4\varepsilon_0 \|z^{t+1} - z\| + 4\varepsilon_0 \|g^{t+1/2}\| + 4\varepsilon_0 \|z^{t+1/2} - z^{t+1}\| + 4\varepsilon_0 \|g^t\|.
\]

(37)

Here we use $\|\Delta^t\|, \|\Delta^{t+1/2}\| \leq \varepsilon_0$ and the triangle inequality. Next we use estimate on gap (32) and taking full expectation:

\[
2\gamma k \cdot \mathbb{E}[\text{gap}(z^k_{avg})] \leq 2\gamma \mathbb{E}\left[\max_{z \in \mathcal{Z}} \sum_{t=0}^{k-1} \langle F(z^{t+1/2}), z^{t+1/2} - z \rangle\right] \\
\leq \Omega_x^2 - \sum_{t=0}^{k-1} \mathbb{E}\left[\|z^{t+1/2} - z^t\|^2\right] + 2\gamma \mathbb{E}\left[\max_{z \in \mathcal{Z}} \sum_{t=0}^{k-1} \langle F(z^{t+1/2}) - g^{t+1/2}, z^{t+1/2} - z \rangle\right] \\
+ \gamma^2 \sum_{t=0}^{k-1} \mathbb{E}\left[\|g^{t+1/2} - g^t\|^2\right] + 4\varepsilon_0 \sum_{t=0}^{k-1} \mathbb{E}\left[\max_{z \in \mathcal{Z}} \|z^{t+1} - z\|\right] + 4\varepsilon_0 \gamma \sum_{t=0}^{k-1} \mathbb{E}\left[\|g^{t+1/2}\|\right] \\
+ 4\varepsilon_0 \sum_{t=0}^{k-1} \mathbb{E}\left[\|z^{t+1/2} - z^{t+1}\|\right] + 4\varepsilon_0 \gamma \sum_{t=0}^{k-1} \mathbb{E}\left[\|g^t\|\right].
\]

(38)
Let work with $\mathbb{E} \left[ \|g^{t+1/2} - g^t\|^2 \right]  
\quad = \quad \mathbb{E} \left[ g^{t+1/2} \right. \
\quad \quad - F(z^{t+1/2}) + F(z^t) - \frac{1}{M} \sum_{m=1}^{M} F_m(z_m^t) + \frac{1}{M} \sum_{m=1}^{M} F_m(z_m^t) - g^t 
\quad \left. + F(z^{t+1/2}) - F(z^t) \right|^2 \right] 
\quad \leq \quad 5\mathbb{E} \left[ \left\| \frac{1}{bM} \sum_{m=1}^{M} \sum_{i=1}^{b} \left( F_m(z_m^{t+1/2}, \xi_m^{t+1/2, i}) - F_m(z_m^{t+1/2}) \right) \right\|^2 \right] 
\quad + 5\mathbb{E} \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z_m^{t+1/2}) - F(z^{t+1/2}) \right\|^2 \right] 
\quad + 5\mathbb{E} \left[ \left\| F(z^t) - \frac{1}{M} \sum_{m=1}^{M} F_m(z_m^t) \right\|^2 \right] 
\quad + 5\mathbb{E} \left[ \left\| \frac{1}{bM} \sum_{m=1}^{M} \sum_{i=1}^{b} \left( F_m(z_m^t, \xi_m^i) - F_m(z_m^t) \right) \right\|^2 \right] 
\quad + 5\mathbb{E} \left[ \left\| F(z^{t+1/2}) - F(z^t) \right\|^2 \right]. 

Using that all $\left\{ g_{t,i} \right\}_{i=1,m=1}^{b,M}$ and $\left\{ \xi_{m,i} \right\}_{i=1,m=1}^{b,M}$ are independent, we get

$\mathbb{E} \left[ \|g^{t+1/2} - g^t\|^2 \right] \leq 5 \left( \frac{5}{(bM)^2} \sum_{m=1}^{M} \sum_{i=1}^{b} \mathbb{E} \left[ \left\| F_m(z_m^{t+1/2}, \xi_m^{t+1/2, i}) - F_m(z_m^{t+1/2}) \right\|^2 \right] 
\quad + \frac{5}{(bM)^2} \sum_{m=1}^{M} \sum_{i=1}^{b} \mathbb{E} \left[ \left\| F_m(z_m^t, \xi_m^i) - F_m(z_m^t) \right\|^2 \right] \right] + 10L^2\mathbb{E} \left[ \|z^{t+1/2} - z^t\|^2 \right] 
\quad + 5L^2 \mathbb{E} \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} (F_m(z_m^{t+1/2}) - F_m(z_m^{t+1/2})) \right\|^2 \right] 
\quad + 5\mathbb{E} \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} (F_m(z_m^t) - F_m(z^t)) \right\|^2 \right] 
\quad \leq 5L^2\mathbb{E} \left[ \|z^{t+1/2} - z^t\|^2 \right] + \frac{10\sigma^2}{bM} 
\quad + 5L^2 \max_{M} \mathbb{E} \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} \text{proj}_Z(z^{t+1/2} + \delta_m^{t+1/2}) - \frac{1}{M} \sum_{m=1}^{M} \text{proj}_Z(z^{t+1/2} + \delta_m^{t+1/2}) \right\|^2 \right] 
\quad + 5L^2 \max_{M} \mathbb{E} \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} \text{proj}_Z(z^t + \delta_m^t) - \frac{1}{M} \sum_{m=1}^{M} \text{proj}_Z(z^t + \delta_m^t) \right\|^2 \right] 
\quad + 5L^2 \max_{M} \mathbb{E} \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} \text{proj}_Z(z^t + \delta_m^t) - \frac{1}{M} \sum_{m=1}^{M} \text{proj}_Z(z^t + \delta_m^t) \right\|^2 \right].
\[ E \left[ \|g^{t+1/2} - g^t\|^2 \right] \leq 5L^2E \left[ \|z^{t+1/2} - z^t\|^2 \right] + \frac{10\sigma^2}{bM} \\
+ 10L^2_{\max} E \left[ \frac{1}{M} \sum_{m=1}^{M} \left\| \text{proj}_Z (\tilde{z}^{t+1/2} + \delta_m^{t+1/2}) - \text{proj}_Z (\tilde{z}^{t+1/2}) \right\|^2 \right] \\
+ 10L^2_{\max} E \left[ \frac{1}{M} \sum_{j=1}^{M} \left\| \text{proj}_Z (\tilde{z}^{t+1/2} + \delta_j^{t+1/2}) - \text{proj}_Z (\tilde{z}^{t+1/2}) \right\|^2 \right] \\
+ 10L^2_{\max} E \left[ \frac{1}{M} \sum_{m=1}^{M} \left\| \text{proj}_Z (\tilde{z}^{t} + \delta_m^t) - \text{proj}_Z (\tilde{z}^{t}) \right\|^2 \right] \\
+ 10L^2_{\max} E \left[ \frac{1}{M} \sum_{j=1}^{M} \left\| \text{proj}_Z (\tilde{z}^{t} + \delta_j^t) - \text{proj}_Z (\tilde{z}^{t}) \right\|^2 \right] \\
\leq 5L^2E \left[ \|z^{t+1/2} - z^t\|^2 \right] + \frac{10\sigma^2}{bM} + 40L^2_{\max} \varepsilon_0. \quad (39) \]

Next we estimate \[ E \left[ \max_{z \in Z} \sum_{t=0}^{k-1} (F(z^{t+1/2}) - g^{t+1/2}, z^{t+1/2} - z) \right]. \] To begin with, we use the same approach as in (34), (35) with sequence \( v_0 = z^{1/2}, v^{t+1} = \text{proj}_Z (v^t - \gamma(F(z^{t+1/2}) - g^{t+1/2})) \) and get

\[
\max_{z \in Z} \sum_{t=0}^{k-1} (F(z^{t+1/2}) - g^{t+1/2}, z^{t+1/2} - z) \leq \sum_{t=0}^{k-1} (F(z^{t+1/2}) - g^{t+1/2}, z^{t+1/2} - v^t) \\
+ \frac{\gamma}{2} \sum_{t=0}^{k-1} \|F(z^{t+1/2}) - g^{t+1/2}\|^2 + \frac{\Omega^2}{2\gamma}.
\]

To begin with, we use the independence of all \( \xi \), as well as the unbiasedness of \( g^{t+1/2} \) with respect to the conditional m.o. by random variables \( \{\xi^{t+1/2,i}^{m}, b, M\}_{i=1, m=1}^{k-1} \):

\[
E \left[ \max_{z \in Z} \sum_{t=0}^{k-1} (F(z^{t+1/2}) - g^{t+1/2}, z^{t+1/2} - z) \right] \\
\leq \sum_{t=0}^{k-1} E \left[ \sum_{i=1}^{M} \sum_{m=1}^{b} \left( E \left[ F(z^{t+1/2}) - g^{t+1/2}, z^{t+1/2} - v^t \right] \right) \right] \\
+ \frac{\gamma}{2} \sum_{t=0}^{k-1} \|F(z^{t+1/2}) - g^{t+1/2}\|^2 + \frac{\Omega^2}{2\gamma} \\
= \sum_{i=1}^{M} \sum_{m=1}^{b} \left[ E \left[ F(z^{t+1/2}) - g^{t+1/2}, z^{t+1/2} - v^t \right] \right] \\
+ \frac{\gamma}{2} \sum_{t=0}^{k-1} \|F(z^{t+1/2}) - g^{t+1/2}\|^2 + \frac{\Omega^2}{2\gamma} \\
= \sum_{i=1}^{M} \sum_{m=1}^{b} \left[ \frac{1}{M} \sum_{m=1}^{M} (F_m(z^{t+1/2}) - F_m(z^{t+1/2}), z^{t+1/2} - v^t) \right] \\
+ \frac{\gamma}{2} \sum_{t=0}^{k-1} \|F(z^{t+1/2}) - g^{t+1/2}\|^2 + \frac{\Omega^2}{2\gamma} 
\]
Next we combine \((38), (39), (40)\)
Then we use $\gamma \leq \frac{1}{4L}$ and Assumption 4

$$2\gamma k E[gap(z_{avg}^k)] \leq 2\Omega^2_z + \gamma^2 \sum_{t=0}^{k-1} E \left[ \|F(z_{t+1/2}^t) - g_{t+1/2}^t\|^2 \right]$$

$$+ \gamma^2 \frac{10k\sigma^2}{bM} + 4\gamma^2 k L^2_{max} \varepsilon_0^2 + 8(1 + \gamma L_{max}) \varepsilon_0 k \Omega_z$$

$$+ 4 \varepsilon_0 \gamma \sum_{t=0}^{k-1} E \left[ \|g_{t+1/2}^t\| \right] + 4 \varepsilon_0 \gamma \sum_{t=0}^{k-1} E \left[ \|g_t^t\| \right].$$

(41)

It remains to estimate $E \left[ \|g_t^t\| + \|g_{t+1/2}^t\| \right]$

$$E \left[ \|g_t^t\| \right] = E \left[ \|F(z^*) - F(z^t) + F(z^t) - F(z^t) + \frac{1}{M} \sum_{m=1}^{M} F_m(z_m^t) - \frac{1}{M} \sum_{m=1}^{M} F_m(z_m^t) + g_t^t \right]$$

$$\leq \|F(z^*)\| + E \left[ \|F(z^t) - F(z^t)\| \right] + E \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z_m^t) - F(z^t) \right\| \right]$$

$$+ E \left[ \left\| \frac{1}{bM} \sum_{m=1}^{M} \sum_{i=1}^{b} (F_m(z_m^t, \epsilon_m^t) - F_m(z_m^t)) \right\| \right].$$

From (39), we have that $E \left[ \left\| \frac{1}{bM} \sum_{m=1}^{M} \sum_{i=1}^{b} (F_m(z_m^t, \epsilon_m^t) - F_m(z_m^t)) \right\|^2 \right] \leq \frac{\sigma^2}{bM}$ and from (40)

$$E \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z_m^t) - F(z^t) \right\| \right] \leq 2L_{max} \varepsilon_0,$$ then

$$E \left[ \|g_t^t\| \right] \leq \|F(z^*)\| + E \left[ \|F(z^t) - F(z^t)\| \right] + 2L_{max} \varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \leq Q + L \Omega_z + 2L_{max} \varepsilon_0 + \frac{\sigma}{\sqrt{bM}},$$

where $Q^2 = \frac{1}{M} \sum_{m=1}^{M} \|F_m(z^*)\|^2$. Hence, we can rewrite (41):

$$E[gap(z_{avg}^k)] \leq \frac{\Omega^2_z}{2\gamma k} + \frac{5\sigma^2 \gamma}{bM}$$

$$+ 24 \gamma L^2_{max} \varepsilon_0^2 + 4 \left( \frac{1}{\gamma} + L_{max} \right) \varepsilon_0 \Omega_z + 4 \varepsilon_0 \left( Q + L \Omega_z + 2L_{max} \varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \right).$$

The same way as (39), one can estimate $E \left[ \|F(z_{t+1/2}^t) - g_{t+1/2}^t\|^2 \right]$

$$E[gap(z_{avg}^k)] \leq \frac{\Omega^2_z}{2\gamma k} + \frac{6\sigma^2 \gamma}{bM}$$

$$+ 24 \gamma L^2_{max} \varepsilon_0^2 + 4 \left( \frac{1}{\gamma} + L_{max} \right) \varepsilon_0 \Omega_z + 4 \varepsilon_0 \left( Q + L \Omega_z + 2L_{max} \varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \right).$$

Let $\gamma = \min \left\{ \frac{1}{4L}, \Omega_z \sqrt{\frac{bM}{12k\sigma^2}} \right\}$ and $\varepsilon_0 = O \left( \frac{\varepsilon}{\Omega_z L_{max} + Q} \right)$, where $\varepsilon = \max \left\{ \frac{L \Omega_z^2}{k}, \frac{\sigma \Omega_z}{\sqrt{bM}} \right\}$. Then for the output of the Algorithm 4 it is true

$$E[gap(z_{avg}^k)] = O \left( \frac{L \Omega_z^2}{k} + \frac{\sigma \Omega_z}{\sqrt{bMk}} \right).$$

Substitute the batch size $b$ and the number of iterations $k$ from the description of the Algorithm 2 and Corollary 1

$$E[gap(z_{avg}^k)] = \hat{O} \left( \frac{L \Omega_z^2 \sqrt{X}}{K} + \frac{\sigma \Omega_z}{\sqrt{MT}} \right).$$

□

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C.2.2 Strongly convex-strongly concave problems

Theorem 16 (Theorem 6) Let \( \{z^t_m \}_{t \geq 0} \) denote the iterates of Algorithm 2 for solving problem \([1]\). Let Assumptions 1(1), 1(II), 2(sc) and 3 be satisfied. Then, if \( \gamma \leq \frac{1}{2E} \) and \( F = O \left( \sqrt{\chi \log \frac{1}{\varepsilon}} \right) \), we have the following estimate for the distance to the solution \( z^* \):

\[
\mathbb{E} \left[ \|z^t - z^*\|^2 \right] = \mathcal{O} \left( \|z^0 - z^*\|^2 \exp \left( -\frac{\mu}{8L} \frac{K}{\sqrt{\chi}} \right) + \frac{\sigma^2}{\mu^2 M T} \right).
\]

Proof: We start with substituting \( z = z^* \) in (37) and taking full expectation. Then we use (39) and get

\[
2\gamma \mathbb{E} \left[ F(z^{t+1/2}), z^{t+1/2} - z^* \right] \leq \mathbb{E} [\|z^t - z^*\|^2] - \mathbb{E} [\|z^{t+1} - z^*\|^2] - \mathbb{E} [\|z^{t+1/2} - z^t\|^2] \\
+ 5L^2\gamma^2 \mathbb{E} [\|z^{t+1/2} - z^t\|^2] + \frac{10\sigma^2\gamma^2}{bM} \\
+ 4\varepsilon_0 \mathbb{E} \|z^{t+1} - z^*\| + 4\varepsilon_0 \gamma \mathbb{E} \|g^{t+1/2}\| + 4\varepsilon_0 \mathbb{E} \|z^{t+1/2} - z^{t+1}\| + 4\varepsilon_0 \gamma \mathbb{E} \|g^t\| \\
+ 4\varepsilon_0 \mathbb{E} \left( F(z^{t+1/2}) - g^{t+1/2}, z^{t+1/2} - z^* \right) + 40\gamma^2 L^2 \max \varepsilon_0^2.
\]

The same way as (40), one can get

\[
\mathbb{E} \left[ F(z^{t+1/2}) - g^{t+1/2}, z^{t+1/2} - z^* \right] \\
\leq \mathbb{E} \left[ \mathbb{E} \left( \{F_m(z^{t+1/2}) - F_m(z^*) \}_{m=1}^{M} \right) \right] \\
= \mathbb{E} \left[ \mathbb{E} \left( \{F_m(z^{t+1/2}) - g^{t+1/2}, z^{t+1/2} - z^* \}_{m=1}^{M} \right) \right] \\
= \mathbb{E} \left[ \frac{1}{M} \sum_{m=1}^{M} \|F_m(z^{t+1/2}) - F_m(z^*)\| \cdot \|z^{t+1/2} - z^*\| \right] \\
\leq \mathbb{E} \left[ \left( \frac{L_{\max}}{M} \sum_{m=1}^{M} \|z^{t+1/2} - z^{t+1/2}\| \right) \cdot \|z^{t+1/2} - z^*\| \right] \\
\leq \mathbb{E} \left[ \left( \frac{L_{\max}}{M} \sum_{m=1}^{M} \|\text{proj}_Z(z^{t+1/2} + \delta_m^{t+1/2}) - \frac{1}{M} \sum_{j=1}^{M} \text{proj}_Z(z^{t+1/2} + \delta_j^{t+1/2})\| \right) \cdot \|z^{t+1/2} - z^*\| \right] \\
\leq \mathbb{E} \left[ \left( \frac{L_{\max}}{M} \sum_{m=1}^{M} \|\text{proj}_Z(z^{t+1/2} + \delta_m^{t+1/2}) - \text{proj}_Z(z^{t+1/2})\| \right) \cdot \|z^{t+1/2} - z^*\| \right] \\
+ \mathbb{E} \left[ \left( \frac{L_{\max}}{M} \sum_{j=1}^{M} \|\text{proj}_Z(z^{t+1/2} + \delta_j^{t+1/2}) - \text{proj}_Z(z^{t+1/2})\| \right) \cdot \|z^{t+1/2} - z^*\| \right] \\
\leq 2L_{\max} \varepsilon_0 \mathbb{E} \left[ \|z^{t+1/2} - z^*\| \right].
\]

and then

\[
2\gamma \mathbb{E} \left[ F(z^{t+1/2}), z^{t+1/2} - z^* \right] \\
\leq \mathbb{E} [\|z^t - z^*\|^2] - \mathbb{E} [\|z^{t+1} - z^*\|^2] - \mathbb{E} [\|z^{t+1/2} - z^t\|^2] \\
+ 5L^2\gamma^2 \mathbb{E} [\|z^{t+1/2} - z^t\|^2] + \frac{10\sigma^2\gamma^2}{bM} \\
+ 4\varepsilon_0 \mathbb{E} [\|z^{t+1} - z^*\|] + 4\varepsilon_0 \gamma \mathbb{E} \|g^{t+1/2}\| + 4\varepsilon_0 \mathbb{E} [\|z^{t+1/2} - z^{t+1}\|] + 4\varepsilon_0 \gamma \mathbb{E} \|g^t\| \\
+ 4\gamma L_{\max} \varepsilon_0 \mathbb{E} [\|z^{t+1/2} - z^*\|] + 40\gamma^2 L^2 \max \varepsilon_0^2.
\]
Next, we work with

\[
\begin{align*}
z^{t+1} & = \frac{1}{M} \sum_{m=1}^{M} z^{t+1}_{m} = \frac{1}{M} \sum_{m=1}^{M} \text{proj}_{Z}(z^{t+1/2} + \delta^{t+1}_{m}) \\
& = \text{proj}_{Z}(z^{t+1}) + \frac{1}{M} \sum_{m=1}^{M} \text{proj}_{Z}(z^{t+1} + \delta^{t+1}_{m}) - \text{proj}_{Z}(z^{t+1}) \\
& = \text{proj}_{Z}\left(\frac{1}{M} \sum_{m=1}^{M} z^{t+1}_{m} - \gamma g^{t+1/2}_{m}\right) + \Delta^{t+1/2} = \text{proj}_{Z}\left(z^{t} - \gamma g^{t+1/2}\right) + \Delta^{t+1/2},
\end{align*}
\]

and get

\[
2\gamma \mathbb{E}\left[ \left( F(z^{t+1/2}), z^{t+1/2} - z^{*} \right) \right] \\
\leq \mathbb{E}\left[ \| z^{t} - z^{*} \|^2 \right] - \mathbb{E}\left[ \| z^{t+1} - z^{*} \|^2 \right] - \mathbb{E}\left[ \| z^{t+1/2} - z^{t} \|^2 \right] \\
+ 5L^2\gamma^2 \mathbb{E}\left[ \| z^{t+1/2} - z^{t} \|^2 \right] + \frac{10\sigma^2\gamma^2}{bM} \\
+ 8\varepsilon_0 \mathbb{E}\left[ \| z^{t+1} - z^{t} \| \right] + 4\varepsilon_0 \gamma \mathbb{E}\left[ \| g^{t+1/2} \| \right] + 4\varepsilon_0 \gamma \mathbb{E}\left[ \| g^{t} \| \right] \\
+ 4\varepsilon_0 (1 + \gamma L_{\max}) \mathbb{E}\left[ \| z^{t+1/2} - z^{t} \| \right] + 4\varepsilon_0 (1 + \gamma L_{\max}) \mathbb{E}\left[ \| z^{t} - z^{*} \| \right] + 40\gamma^2 L_{\max}^2 \varepsilon_0^2 \\
\leq \mathbb{E}\left[ \| z^{t} - z^{*} \|^2 \right] - \mathbb{E}\left[ \| z^{t+1} - z^{*} \|^2 \right] - \mathbb{E}\left[ \| z^{t+1/2} - z^{t} \|^2 \right] \\
+ 5L^2\gamma^2 \mathbb{E}\left[ \| z^{t+1/2} - z^{t} \|^2 \right] + \frac{10\sigma^2\gamma^2}{bM} \\
+ 8\varepsilon_0 \mathbb{E}\left[ \| \text{proj}_{Z}(z^{t} - \gamma g^{t+1/2}) \| + \Delta^{t+1/2} - z^{t} \| \right] + 8\varepsilon_0^2 \\
+ 4\varepsilon_0 (1 + \gamma L_{\max}) \mathbb{E}\left[ \| z^{t+1/2} - z^{t} \| \right] + 4\varepsilon_0 \gamma \mathbb{E}\left[ \| g^{t+1/2} \| \right] \\
+ 4\varepsilon_0 \gamma \mathbb{E}\left[ \| g^{t} \| \right] + 4\varepsilon_0 (1 + \gamma L_{\max}) \mathbb{E}\left[ \| z^{t+1/2} - z^{t} \| \right] + 40\gamma^2 L_{\max}^2 \varepsilon_0^2 \\
\leq \mathbb{E}\left[ \| z^{t} - z^{*} \|^2 \right] - \mathbb{E}\left[ \| z^{t+1} - z^{*} \|^2 \right] - \mathbb{E}\left[ \| z^{t+1/2} - z^{t} \|^2 \right] \\
+ 5L^2\gamma^2 \mathbb{E}\left[ \| z^{t+1/2} - z^{t} \|^2 \right] + \frac{10\sigma^2\gamma^2}{bM} \\
+ 8\varepsilon_0^2 + 4\varepsilon_0 (1 + \gamma L_{\max}) \mathbb{E}\left[ \| z^{t} - z^{*} \| \right] + 12\varepsilon_0 \gamma \mathbb{E}\left[ \| g^{t+1/2} \| \right] \\
+ 4\varepsilon_0 \gamma \mathbb{E}\left[ \| g^{t} \| \right] + 4\varepsilon_0 (1 + \gamma L_{\max}) \mathbb{E}\left[ \| z^{t+1/2} - z^{t} \| \right] + 40\gamma^2 L_{\max}^2 \varepsilon_0^2.
\]
It remains to estimate \(\mathbb{E} [\|g^{t+1/2}\| + \|g^t\|]:\)

\[
\mathbb{E} [\|g^t\|] = \mathbb{E} \left[ \|F(z^*) - F(z^t) + F(z^t) - F(z^t + 1/2)\| + \frac{1}{M} \sum_{m=1}^{M} F_m(z^t_m) - \frac{1}{M} \sum_{m=1}^{M} F_m(z^t_m) + g^t\right]
\]

\[
\leq \|F(z^*)\| + \mathbb{E} \left[ \|F(z^t) - F(z^*)\|\right] + \mathbb{E} \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z^t_m) - F(z^t)\right\|\right]
\]

\[
+ \mathbb{E} \left[ \left\| \frac{1}{bM} \sum_{m=1}^{b} \sum_{i=1}^{t} (F_m(z^t_m, c^t_{m,i}) - F_m(z^t_m))\right\|\right].
\]

From (39) we have that \(\mathbb{E} \left[ \left\| \frac{1}{bM} \sum_{m=1}^{M} F_m(z^t_m) - F(z^t)\right\|^2\right] \leq 2L_{\text{max}}\varepsilon_0,\) then

\[
\mathbb{E} [\|g^t\|] \leq \|F(z^*)\| + \mathbb{E} \left[ \|F(z^t) - F(z^*)\|\right] + 2L_{\text{max}}\varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \leq Q + L\mathbb{E} [\|z^t - z^*\|] + 2L_{\text{max}}\varepsilon_0 + \frac{\sigma}{\sqrt{bM}}.
\]

Substituting in (42):

\[
2\gamma \mathbb{E} \left[ \langle F(z^t+1/2), z^{t+1/2} - z^* \rangle \right]
\]

\[
\leq \mathbb{E} \left[ \|z^t - z^*\|^2\right] - \mathbb{E} \left[ \|z^{t+1/2} - z^*\|^2\right] - \mathbb{E} \left[ \|z^{t+1/2} - z^t\|^2\right]
\]

\[
+ 5L_2^2\gamma^2 \mathbb{E} \left[ \|z^{t+1/2} - z^t\|^2\right] + \frac{10\sigma^2\gamma^2}{bM} + 8\varepsilon_0^2 + 4\varepsilon_0 (1 + \gamma L_{\text{max}}) \mathbb{E} \left[ \|z^t - z^*\|\right]
\]

\[
+ 12\varepsilon_0 (Q + L\mathbb{E} [\|z^{t+1/2} - z^*\|] + 2L_{\text{max}}\varepsilon_0 + \frac{\sigma}{\sqrt{bM}})
\]

\[
+ 4\varepsilon_0 \gamma (Q + L\mathbb{E} [\|z^t - z^*\|] + 2L_{\text{max}}\varepsilon_0 + \frac{\sigma}{\sqrt{bM}})
\]

\[
+ 4\varepsilon_0 (1 + \gamma L_{\text{max}}) \mathbb{E} \left[ \|z^{t+1/2} - z^t\|\right] + 40\gamma^2 L_{\text{max}}^2 \varepsilon_0^2.
\]

By simple fact \(2ab \leq a^2 + b^2,\) we get

\[
2\gamma \mathbb{E} \left[ \langle F(z^t+1/2), z^{t+1/2} - z^* \rangle \right]
\]

\[
\leq \mathbb{E} \left[ \|z^t - z^*\|^2\right] - \mathbb{E} \left[ \|z^{t+1/2} - z^*\|^2\right] - \mathbb{E} \left[ \|z^{t+1/2} - z^t\|^2\right]
\]

\[
+ 5L_2^2\gamma^2 \mathbb{E} \left[ \|z^{t+1/2} - z^t\|^2\right] + \frac{10\sigma^2\gamma^2}{bM} + 20\varepsilon_0 (1 + \gamma L_{\text{max}}) \mathbb{E} \left[ \|z^t - z^*\|\right]
\]

\[
+ 16\varepsilon_0 \gamma (Q + 2L_{\text{max}}\varepsilon_0 + \frac{\sigma}{\sqrt{bM}}) + 40\gamma^2 L_{\text{max}}^2 \varepsilon_0^2 + 8\varepsilon_0^2
\]

\[
\leq (1 + 10\varepsilon_0) \mathbb{E} \left[ \|z^t - z^*\|^2\right] - \mathbb{E} \left[ \|z^{t+1/2} - z^*\|^2\right]
\]

\[
+ (5L_2^2\gamma^2 + 8\varepsilon_0 - 1) \mathbb{E} \left[ \|z^{t+1/2} - z^t\|^2\right] + \frac{10\sigma^2\gamma^2}{bM} + 20\varepsilon_0 (1 + \gamma L_{\text{max}}) \varepsilon_2^2 + 16\varepsilon_0 \gamma (Q + 2L_{\text{max}}\varepsilon_0 + \frac{\sigma}{\sqrt{bM}}) + 40\gamma^2 L_{\text{max}}^2 \varepsilon_0^2 + 8\varepsilon_0^2.
\]

By property of \(z^*,\) we get

\[
\mathbb{E} \left[ \langle F(z^t+1/2), z^{t+1/2} - z^* \rangle \right] \geq \mathbb{E} \left[ \langle F(z^t+1/2) - F(z^*), z^{t+1/2} - z^t \rangle \right] \geq \mu \mathbb{E} \left[ \|z^{t+1/2} - z^*\|^2\right].
\]

Let use a simple fact \(\|z^{t+1/2} - z^*\|^2 \geq \frac{1}{2} \|z^t - z^*\|^2 + \|z^{t+1/2} - z^t\|^2,\) then

\[
\mathbb{E} \left[ \langle F(z^t+1/2), z^{t+1/2} - z^* \rangle \right] \geq \frac{\mu}{2} \mathbb{E} \left[ \|z^t - z^*\|^2\right] - \mu \mathbb{E} \left[ \|z^{t+1/2} - z^t\|^2\right].
\]

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Then (44) gives
\[
\begin{align*}
E \left[ \|z^{t+1} - z^*\|^2 \right] &\leq (1 + 10\varepsilon_0 - \mu \gamma)E \left[ \|z^t - z^*\|^2 \right] + \frac{10\sigma^2\gamma^2}{bM} \\
&\quad + (5L^2\gamma^2 + 2\gamma\mu + 8\varepsilon_0 - 1)E \left[ \|z^{t+1/2} - z^t\|^2 \right] \\
&\quad + 20\varepsilon_0(1 + \gamma L_{\text{max}})^2 + 16\varepsilon_0\gamma \left( Q + 2L_{\text{max}}\varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \right) + 40\gamma^2 L_{\text{max}}^2 \varepsilon_0^2 + 8\varepsilon_0^2.
\end{align*}
\]

With \( \varepsilon_0 \leq \min \left( \frac{1}{5b}, \frac{\mu^2}{20} \right) \) and \( \gamma \leq \frac{1}{4L} \) we have
\[
E \left[ \|z^{t+1} - z^*\|^2 \right] \leq \left( 1 - \frac{\mu^2}{2} \right) E \left[ \|z^t - z^*\|^2 \right] + \frac{10\sigma^2\gamma^2}{bM} \\
&\quad + 20\varepsilon_0(1 + \gamma L_{\text{max}})^2 + 16\varepsilon_0\gamma \left( Q + 2L_{\text{max}}\varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \right) + 40\gamma^2 L_{\text{max}}^2 \varepsilon_0^2 + 8\varepsilon_0^2.
\]

Let’s run the recursion from 0 to \( k - 1 \):
\[
E \left[ \|z^k - z^*\|^2 \right] \leq \left( 1 - \frac{\mu^2}{2} \right)^k E \left[ \|z^0 - z^*\|^2 \right] + \frac{20\sigma^2\gamma}{\mu bM} \\
&\quad + \frac{2\varepsilon_0}{\mu\gamma} \left( 20(1 + \gamma L_{\text{max}})^2 + 16\gamma \left( Q + 2L_{\text{max}}\varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \right) + 40\gamma^2 L_{\text{max}}^2 \varepsilon_0^2 + 8\varepsilon_0^2 \right).
\]

Then we carefully choose \( \gamma = \min \left( \frac{1}{4L}, \frac{2\ln(\max\{2bM\mu^2\|z^0 - z^*\|^2/20\sigma^2\})}{\mu^2 bM} \right) \) and \( \varepsilon_0 = O \left( \varepsilon\mu\gamma(1 + Q + \gamma L_{\text{max}})^2 \right) \), where \( \varepsilon = \max \left( \|z^0 - z^*\|^2 \exp \left( -\frac{\mu k}{8L} \right) \right) \). Then the output of the Algorithm 4 it is true
\[
E \left[ \|z^k - z^*\|^2 \right] = \tilde{O} \left( \|z^0 - z^*\|^2 \exp \left( -\frac{\mu k}{8L} \right) + \frac{\sigma^2}{\mu^2 bM} \right).
\]

Substitute the batch size \( b \) and the number of iterations \( k \) from the description of the Algorithm 1
\[
E \left[ \|z^k - z^*\|^2 \right] = \tilde{O} \left( \|z^0 - z^*\|^2 \exp \left( -\frac{\mu k}{8L} \cdot \frac{K}{P} \right) + \frac{\sigma^2}{\mu^2 MT} \right).
\]

Corollary 1 ends the proof.

\[\Box\]

C.3 Non-convex-non-concave problems

**Theorem 17 (Theorem 6)** Let \( \{z^t\}_{t \geq 0} \) denote the iterates of Algorithm 2 for solving problem 1. Let Assumptions (g), (h), 2(nc), 3 and 4 be satisfied. Then, if \( \gamma \leq \frac{1}{4L} \) and \( P = O \left( \sqrt{\lambda} \log \frac{1}{\varepsilon} \right) \), we have the following estimate:
\[
E \left[ \frac{1}{k} \sum_{t=0}^{k-1} \|F(z^t)\|^2 \right] = \tilde{O} \left( \frac{L^2\Omega_z^2\sqrt{\lambda}}{K} + \frac{\sigma^2 K}{MT\sqrt{\lambda}} \right).
\]

**Proof:** We start from (43) with using diameter \( \Omega_z \):
\[
2\gamma E \left[ \langle F(z^{t+1/2}), z^{t+1/2} - z^* \rangle \right] \\
\leq E \left[ \|z^t - z^*\|^2 \right] - E \left[ \|z^{t+1} - z^*\|^2 \right] - (1 - 5L^2\gamma^2)E \left[ \|z^{t+1/2} - z^t\|^2 \right] + \frac{10\sigma^2\gamma^2}{bM} \\
+ 16\varepsilon_0\gamma \left( L\Omega_z + 2L_{\text{max}}\varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \right) + 8\varepsilon_0(1 + \gamma L_{\text{max}})\Omega_z + 8(1 + 5\gamma^2 L_{\text{max}})\varepsilon_0^2.
\]
With minty assumption it transforms to

\[
0 \leq \mathbb{E} \left[ \| z^t - z^* \|^2 \right] - \mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] - (1 - 5L^2\gamma^2)\mathbb{E} \left[ \| z^{t+1/2} - z^t \|^2 \right] + \frac{10\sigma^2\gamma^2}{bM} \\
+ 16\varepsilon_0 \gamma \left( Q + L\Omega_z + 2L_{max} \varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \right) + 8\varepsilon_0 (1 + \gamma L_{max})\Omega_z + 8(1 + 5\gamma^2L_{max}^2)\varepsilon_0^2
\]

After the choice of \( \gamma \leq \frac{1}{4L} \) we get

\[
0 \leq \mathbb{E} \left[ \| z^t - z^* \|^2 \right] - \mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] - \frac{\gamma^2}{2} \mathbb{E} \left[ \| g^t \|^2 \right] + \frac{10\sigma^2\gamma^2}{bM} \\
+ 16\varepsilon_0 \gamma \left( Q + L\Omega_z + 2L_{max} \varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \right) + 8\varepsilon_0 (1 + \gamma L_{max})\Omega_z + 8(1 + 5\gamma^2L_{max}^2)\varepsilon_0^2.
\]

The fact: \(-\|g^t\|^2 \leq -\frac{1}{2}\|F(z^t)\|^2 + \|g^t - F(z^t)\|^2\), gives

\[
0 \leq \mathbb{E} \left[ \| z^t - z^* \|^2 \right] - \mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] - \frac{\gamma^2}{4} \mathbb{E} \left[ \| F(z^t) \|^2 \right] + \frac{\gamma^2}{2} \mathbb{E} \left[ \| g^t - F(z^t) \|^2 \right] + \frac{10\sigma^2\gamma^2}{bM} \\
+ 16\varepsilon_0 \gamma \left( Q + L\Omega_z + 2L_{max} \varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \right) + 8\varepsilon_0 (1 + \gamma L_{max})\Omega_z + 8(1 + 5\gamma^2L_{max}^2)\varepsilon_0^2.
\]

The term \( \|g^t - F(z^t)\|^2 \) was estimated, when we deduced \((39)\). Then

\[
\frac{\gamma^2}{4} \mathbb{E} \left[ \| F(z^t) \|^2 \right] \leq \mathbb{E} \left[ \| z^t - z^* \|^2 \right] - \mathbb{E} \left[ \| z^{t+1} - z^* \|^2 \right] + \frac{11\sigma^2\gamma^2}{bM} \\
+ 16\varepsilon_0 \gamma \left( Q + L\Omega_z + 2L_{max} \varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \right) + 8\varepsilon_0 (1 + \gamma L_{max})\Omega_z + 8(1 + 6\gamma^2L_{max}^2)\varepsilon_0^2.
\]

Summing over all \( t \) from 0 to \( k - 1 \):

\[
\mathbb{E} \left[ \frac{1}{k} \sum_{t=0}^{k-1} \| F(z^t) \|^2 \right] \leq \frac{4\mathbb{E} \left[ \| z^0 - z^* \|^2 \right]}{\gamma^2k} + \frac{44\sigma^2}{bM} + \frac{64\varepsilon_0}{\gamma} \left( Q + L\Omega_z + 2L_{max} \varepsilon_0 + \frac{\sigma}{\sqrt{bM}} \right) \\
+ \frac{32\varepsilon_0}{\gamma^2} (1 + \gamma L_{max})\Omega_z + \frac{32}{\gamma^2} (1 + 6\gamma^2L_{max}^2)\varepsilon_0^2.
\]

Let \( \gamma = \frac{1}{4L} \) and \( \varepsilon_0 = O \left( \frac{\varepsilon}{\Omega_z L_{max} k} \right) \), where \( \varepsilon = \max \left( \frac{L^2\Omega_z^2}{k}, \frac{\sigma^2}{bM} \right) \). Then for the output of the Algorithm \( 4 \) it holds

\[
\mathbb{E} \left[ \frac{1}{k} \sum_{t=0}^{k-1} \| F(z^t) \|^2 \right] = O \left( \frac{\mathbb{E} \left[ L^2 \| z^0 - z^* \|^2 \right]}{k} + \frac{\sigma^2}{bM} \right).
\]

Substitute the batch size \( b \) and the number of iterations \( k \) from the description of the Algorithm \( 2 \) and Corollary \( 1 \)

\[
\mathbb{E} \left[ \frac{1}{k} \sum_{t=0}^{k-1} \| F(z^t) \|^2 \right] = O \left( \frac{L^2\Omega_z^2\sqrt{\chi}}{K} + \frac{\sigma^2K}{MT\sqrt{\chi}} \right).
\]

□

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D Proof of Theorems from Section 5

Here we present a theoretical analysis of the proposed method. To begin with, we introduce auxiliary sequences that we need only in theoretical analysis (Algorithm 3 does not compute them):

\[ z^t = \frac{1}{M} \sum_{m=1}^{M} z_m^t, \quad \bar{g}^t = \frac{1}{M} \sum_{m=1}^{M} F_m(z_m^t, \xi_m^t), \]
\[ z^{t+1/2} = z^t - \gamma \bar{g}^t, \quad z^{t+1} = z^{t} - \gamma \bar{g}^{t+1/2} \]

(45)

Such sequences are virtual, but at the communication moment \( \bar{x}_m^t \) or \( y_m^t \).

D.1 Strongly convex-strongly concave problems

Theorem 18 (Theorem 7) Let \( \{ z_m^t \}_{t \geq 0} \) denote the iterates of Algorithm 3 for solving problem (1). Let Assumptions 1[1], 2[sc], 3 and 5 be satisfied. Also let \( H = \max_p |k_{p+1} - k_p| \) be a maximum distance between moments of communication (\( k_p \in I \)). Then, if \( \gamma \leq \frac{1}{2HL_{\text{max}}} \), we have the following estimate for the distance to the solution \( z^* \):

\[ \mathbb{E}[\| \bar{z}^T - z^* \|^2] = \tilde{O} \left( \exp \left( -\frac{\mu K}{42HL_{\text{max}}} \right) \| z^0 - z^* \|^2 + \frac{\sigma^2}{\mu^2 MT} + \left( \frac{D^2 H + \sigma^2}{HL_{\text{max}}} \right)^2 \right). \]

Proof of Theorem: We start our proof with the following lemma.

Lemma 11 Let \( z, y \in \mathbb{R}^n \). We set \( z^+ = z - y \), then for all \( u \in \mathbb{R}^n \):

\[ \| z^+ - u \|^2 \leq \| z - u \|^2 - 2\langle y, z^+ - u \rangle - \| z^+ - z \|^2. \]

Proof: Simple manipulations give

\[ \| z^+ - u \|^2 = \| z^+ - z + z - u \|^2 \]
\[ = \| z - u \|^2 + 2\langle z^+ - z, z - u \rangle + \| z^+ - z \|^2 \]
\[ = \| z - u \|^2 + 2\langle z^+ - z, z^+ - u \rangle - \| z^+ - z \|^2 \]
\[ = \| z - u \|^2 + 2\langle z^+ - (z - y), z^+ - u \rangle - 2\langle y, z^+ - u \rangle - \| z^+ - z \|^2 \]
\[ = \| z - u \|^2 - 2\langle y, z^+ - u \rangle - \| z^+ - z \|^2. \]

Applying this Lemma with \( z = z^{t+1} \), \( \bar{z} = z^t \), \( u = z^* \) and \( y = \gamma \bar{g}^{t+1/2} \), we get

\[ \| z^{t+1} - z^* \|^2 = \| z^t - z^* \|^2 - 2\gamma \langle \bar{g}^{t+1/2}, z^{t+1} - z^* \rangle - \| z^{t+1} - z^t \|^2, \]

and with \( z = z^{t+1/2} \), \( z = z^t \), \( u = z^{t+1} \), \( y = \gamma \bar{g}^t \):

\[ \| z^{t+1/2} - z^{t+1} \|^2 = \| z^t - z^{t+1} \|^2 - 2\gamma \langle \bar{g}^t, z^{t+1/2} - z^{t+1} \rangle - \| z^{t+1/2} - z^t \|^2. \]

Next, we sum up the two previous equalities

\[ \| z^{t+1} - z^* \|^2 + \| z^{t+1/2} - z^{t+1} \|^2 = \| z^t - z^* \|^2 - \| z^{t+1/2} - z^t \|^2 \]
\[ - 2\gamma \langle \bar{g}^{t+1/2}, z^{t+1} - z^* \rangle - 2\gamma \langle \bar{g}^t, z^{t+1/2} - z^{t+1} \rangle. \]

A small rearrangement gives

\[ \| z^{t+1} - z^* \|^2 + \| z^{t+1/2} - z^{t+1} \|^2 = \| z^t - z^* \|^2 - \| z^{t+1/2} - z^t \|^2 \]
\[ - 2\gamma \langle \bar{g}^{t+1/2}, z^{t+1/2} - z^* \rangle + 2\gamma \langle \bar{g}^{t+1/2} - \bar{g}^t, z^{t+1/2} - z^{t+1} \rangle \]
\[ \leq \| z^t - z^* \|^2 - \| z^{t+1/2} - z^t \|^2 \]
\[ - 2\gamma \langle \bar{g}^{t+1/2}, z^{t+1/2} - z^* \rangle + \gamma^2 \langle \bar{g}^{t+1/2} - \bar{g}^t \|^2 \]
\[ + \| z^{t+1/2} - z^{t+1} \|^2, \]
Then we take the total expectation of both sides of the equation

\[ E \left[ \| \hat{z}^{t+1} - z^* \|^2 \right] = E \left[ \| \hat{z}^t - z^* \|^2 \right] - E \left[ \| \hat{z}^{t+1/2} - \hat{z}^t \|^2 \right] - 2\gamma E \left[ (\hat{g}^{t+1/2}, \hat{z}^{t+1/2} - z^*) \right] + \gamma^2 E \left[ \| \hat{g}^{t+1/2} - \hat{g}^t \|^2 \right]. \]

(46)

Further, we need to additionally estimate two terms \(-2\gamma (\hat{g}^{t+1/2}, \hat{z}^{t+1/2} - z^*) \) and \(\| \hat{g}^{t+1/2} - \hat{g}^t \|^2 \). For this we prove the following two lemmas, but before that we introduce the additional notation:

\[ \text{Err}(t) = \frac{1}{M} \sum_{m=1}^{M} \| \hat{z}^t - z_m^t \|^2. \]

(47)

**Lemma 12** The following estimate is valid:

\[ -2\gamma E \left[ (\hat{g}^{t+1/2}, \hat{z}^{t+1/2} - z^*) \right] \leq -\gamma \mu E \left[ \| \hat{z}^{t+1/2} - z^* \|^2 \right] + \frac{\gamma L^2_{\max}}{\mu} E \left[ \text{Err}(t + 1/2) \right]. \]

(48)

**Proof:** We take into account the independence of all random vectors \(\xi^i = (\xi_1^i, \ldots, \xi_m^i)\) and select only the conditional expectation \(E_{\xi^{t+1/2}}\) on vector \(\xi^{t+1/2}\)

\[ -2\gamma E \left[ (\hat{g}^{t+1/2}, \hat{z}^{t+1/2} - z^*) \right] = -2\gamma \left\langle \frac{1}{M} \sum_{m=1}^{M} E_{\xi^{t+1/2}} [F_m(z_m^{t+1/2}), \hat{z}^{t+1/2} - z^*] \right\rangle \]

\[ \leq -2\gamma \left\langle \frac{1}{M} \sum_{m=1}^{M} F_m(z_m^{t+1/2}, \hat{z}^{t+1/2} - z^*) \right\rangle \]

\[ = -2\gamma \left\langle \frac{1}{M} \sum_{m=1}^{M} F_m(\hat{z}^{t+1/2}, \hat{z}^{t+1/2} - z^*) \right\rangle \]

\[ + 2\gamma \left\langle \frac{1}{M} \sum_{m=1}^{M} [F_m(\hat{z}^{t+1/2} - F_m(z_m^{t+1/2})], \hat{z}^{t+1/2} - z^*] \right\rangle \]

\[ = -2\gamma \left\langle F(\hat{z}^{t+1/2}), \hat{z}^{t+1/2} - z^* \right\rangle \]

\[ + 2\gamma \left\langle \frac{1}{M} \sum_{m=1}^{M} [F_m(\hat{z}^{t+1/2} - F_m(z_m^{t+1/2})], \hat{z}^{t+1/2} - z^*] \right\rangle. \]

Using property of \(z^*\), we have:

\[ -2\gamma E \left[ (\hat{g}^{t+1/2}, \hat{z}^{t+1/2} - z^*) \right] \leq -2\gamma \left\langle F(\hat{z}^{t+1/2}) - F(z^*), \hat{z}^{t+1/2} - z^* \right\rangle \]

\[ + 2\gamma \left\langle \frac{1}{M} \sum_{m=1}^{M} [F_m(\hat{z}^{t+1/2} - F_m(z_m^{t+1/2})], \hat{z}^{t+1/2} - z^*] \right\rangle \]

\[ \leq -2\gamma \mu E \left[ \| \hat{z}^{t+1/2} - z^* \|^2 \right] \]

\[ + 2\gamma \left\langle \frac{1}{M} \sum_{m=1}^{M} [F_m(\hat{z}^{t+1/2} - F_m(z_m^{t+1/2})], \hat{z}^{t+1/2} - z^*] \right\rangle. \]
Lemma 13 The following estimate is valid:

\[
E \left[ \| \hat{g}^{t+1/2} - \tilde{g}^t \|^2 \right] \leq 5L_{\text{max}}^2 E \left[ \| \tilde{z}^{t+1/2} - \tilde{z}^t \|^2 \right] + \frac{10\sigma^2}{M} + 5L_{\text{max}}^2 E \left[ \text{Err}(t + 1/2) \right] + 5L_{\text{max}}^2 E \left[ \text{Err}(t) \right].
\]  

Proof:

\[
E \left[ \| \hat{g}^{t+1/2} - \tilde{g}^t \|^2 \right] = E \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z_{m}^{t+1/2}, \xi_{m}^{t+1/2}) - \frac{1}{M} \sum_{m=1}^{M} F_m(z_{m}^t, \xi_{m}^t) \right\|^2 \right]
\]

\[\leq 5E \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z_{m}^{t+1/2}, \xi_{m}^{t+1/2}) - \frac{1}{M} \sum_{m=1}^{M} F_m(z_{m}^t, \xi_{m}^t) \right\|^2 \right] + 5E \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z_{m}^{t+1/2}, \xi_{m}^{t+1/2}) - F_m(z_{m}^{t+1/2}) \right\|^2 \right]
\]

\[+ 5E \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z_{m}^t, \xi_{m}^t) - F_m(z_{m}^t) \right\|^2 \right] + 5E \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z_{m}^{t+1/2}) - F_m(z_{m}^t) \right\|^2 \right]
\]

\[+ 5E \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z_{m}^{t+1/2}, \xi_{m}^{t+1/2}) - F_m(z_{m}^{t+1/2}) \right\|^2 \right] + 5E \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z_{m}^t, \xi_{m}^t) - F_m(z_{m}^t) \right\|^2 \right]
\]

\[+ 5E \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z_{m}^{t+1/2}) - F_m(z_{m}^t) \right\|^2 \right] + 5E \left[ \left\| F(z_{m}^{t+1/2}) - F(z_{m}^t) \right\|^2 \right]
\]
We are now ready to combine (46), (48), (49) and get

**Lemma 14**

\[
\begin{align*}
E \left\| \bar{z}^{t+1/2} - \bar{z}^t \right\|^2 &\leq 5E \left[ \left\| \frac{1}{M} \sum_{m=1}^M F_m(z_m^{t+1/2}, \xi_m^{t+1/2}) - F_m(z_m^t) \right\|^2 \right] \\
&\quad + 5E \left[ \left\| \frac{1}{M} \sum_{m=1}^M [F_m(z_m^t, \xi_m^k) - F_m(z_m^t)] \right\|^2 \right] \\
&\quad + 5L_\max^2 E[\text{Err}(t + 1/2)] + 5L_\max^2 E[\text{Err}(t)] + 5L_\max^2 E \left[ \left\| \bar{z}^{t+1/2} - \bar{z}^t \right\|^2 \right] \\
&= 5E_{\xi_{t+1/2}} \left[ \left\| \frac{1}{M} \sum_{m=1}^M [F_m(z_m^t, \xi_m^t) - F_m(z_m^t)] \right\|^2 \right] \\
&\quad + 5E_{\xi_t} \left[ \left\| \frac{1}{M} \sum_{m=1}^M [F_m(z_m^t, \xi_m^t) - F_m(z_m^t)] \right\|^2 \right] \\
&\quad + 5L_\max^2 E[\text{Err}(t + 1/2)] + 5L_\max^2 E[\text{Err}(t)] + 5L_\max^2 E \left[ \left\| \bar{z}^{t+1/2} - \bar{z}^t \right\|^2 \right].
\end{align*}
\]

Using independence of each machine and (7), we get:

\[
E \left[ \left\| \bar{z}^{t+1/2} - \bar{g}^t \right\|^2 \right] \leq \frac{10\sigma^2}{M} + 5L_\max^2 E[\text{Err}(t + 1/2)] + 5L_\max^2 E[\text{Err}(t)] + 5L_\max^2 E \left[ \left\| \bar{z}^{t+1/2} - \bar{z}^t \right\|^2 \right].
\]

\(\square\)

We are now ready to combine (46), (48), (49) and get

\[
E \left[ \left\| \bar{z}^{t+1} - z^* \right\|^2 \right] \leq E \left[ \left\| \bar{z}^t - z^* \right\|^2 \right] - E \left[ \left\| \bar{z}^{t+1/2} - \bar{z}^t \right\|^2 \right]
\]

\[
- \gamma \mu E \left[ \left\| \bar{z}^{t+1/2} - z^* \right\|^2 \right] + \frac{\gamma L_\max^2}{\mu} E[\text{Err}(t + 1/2)]
\]

\[
+ 5\gamma^2 L_\max^2 E \left[ \left\| \bar{z}^{t+1/2} - \bar{z}^t \right\|^2 \right] + \frac{10\gamma^2 \sigma^2}{M}
\]

\[
+ 5\gamma^2 L_\max^2 E[\text{Err}(t + 1/2)] + 5\gamma^2 L_\max^2 E[\text{Err}(t)].
\]

Together with \(-\left\| \bar{z}^{t+1/2} - z^* \right\|^2 \leq \left\| \bar{z}^{t+1/2} - \bar{z}^t \right\|^2 - \left\| \bar{z}^t - z^* \right\|^2 \) it transforms to

\[
E \left[ \left\| \bar{z}^{t+1} - z^* \right\|^2 \right] \leq \left( 1 - \frac{\mu^2}{2} \right) E \left[ \left\| \bar{z}^t - z^* \right\|^2 \right] + \frac{10\gamma^2 \sigma^2}{M} + (\mu \gamma + 5\gamma^2 L_\max^2 - 1) \left\| \bar{z}^{t+1/2} - \bar{z}^t \right\|^2
\]

\[
+ \frac{\gamma L_\max^2}{\mu} E[\text{Err}(t + 1/2)] + 5\gamma^2 L_\max^2 E[\text{Err}(t + 1/2)] + 5\gamma^2 L_\max^2 E[\text{Err}(t)].
\]

Taking \(\gamma \leq \frac{1}{6HL_\max^2}\) gives

\[
E \left[ \left\| \bar{z}^{t+1} - z^* \right\|^2 \right] \leq \left( 1 - \frac{\mu^2}{2} \right) E \left[ \left\| \bar{z}^t - z^* \right\|^2 \right] + \frac{10\gamma^2 \sigma^2}{M}
\]

\[
+ \frac{7\gamma L_\max^2}{\mu} E[\text{Err}(t + 1/2)] + 5\gamma^2 L_\max^2 E[\text{Err}(t)].
\]

It remains to estimate \(E[\text{Err}(t + 1/2)]\) and \(E[\text{Err}(t)]\).

**Lemma 14** For \(t \in [t_p + 1; t_{p+1}]\) the following estimate is valid:

\[
E[\text{Err}(t + 1/2)] \leq 216(D^2 H + \sigma^2) H \gamma^2.
\]
Proof: First, let us look at the nearest past consensus point $t_p < t$, then $z^{t_{p+1}} = \bar{z}^{t_{p+1}}:

\begin{align*}
\mathbb{E} \left[ \text{Err}(t+1/2) \right] &= \frac{1}{M} \sum_{m=1}^{M} \mathbb{E} \| \bar{z}^{t+1/2} - z_{m}^{t+1/2} \|^2 \\
&= \frac{1}{M} \sum_{m=1}^{M} \mathbb{E} \| \bar{z}^{t+1/2} - \bar{z}^{t_p} + z_{m}^{t} - z_{m}^{t+1/2} \|^2 \\
&= \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left\| F_{m}(z_{m}^{t}, \xi_{m}^{t}) - \bar{g}^{t} + \sum_{k=t_p+1}^{t-1} F_{m}(z_{m}^{k+1/2}, \xi_{m}^{k+1/2}) - \bar{g}^{k+1/2} \right\|^2.
\end{align*}

Only $\bar{g}^{t}$ and $F_{m}(z_{m}^{k}, \xi_{m}^{k})$ depend on $\xi^{k}$, as well as the unbiasedness of $\bar{g}^{t}$ and $F_{m}(z_{m}^{k}, \xi_{m}^{k})$, we have

\begin{align*}
\mathbb{E} \left[ \text{Err}(t+1/2) \right] &= \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left( \frac{1}{M} \sum_{i=1}^{M} F_{i}(z_{i}^{t}) - \sum_{k=t_p+1}^{t-1} \frac{\gamma^{k+1/2}}{M} + \sum_{k=t_p+1}^{t-1} F_{m}(z_{m}^{k+1/2}, \xi_{m}^{k+1/2}) \right) \\
&\quad + \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left( \frac{1}{M} \sum_{i=1}^{M} F_{i}(z_{i}^{t}) - z_{m}^{t} - F_{m}(z_{m}^{t}, \xi_{m}^{t}) \right)^2.
\end{align*}

We want to continue the same way, but note that $z_{i}^{t}$ depends on $\xi^{k-1+1/2}$, then let us make the estimate rougher than in previous case

\begin{align*}
\mathbb{E} \left[ \text{Err}(t+1/2) \right] &\leq (1 + \beta_0) \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left( \frac{1}{M} \sum_{i=1}^{M} F_{i}(z_{i}^{t-1+1/2}) - \sum_{k=t_p+1}^{t-2} \bar{g}^{k+1/2} + \sum_{k=t_p+1}^{t-1} F_{m}(z_{m}^{k+1/2}, \xi_{m}^{k+1/2}) \right)^2 \\
&\quad + (1 + \beta_0^{-1}) \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left( \frac{1}{M} \sum_{i=1}^{M} F_{i}(z_{i}^{t}) + F_{m}(z_{m}^{t}, \xi_{m}^{t}) \right)^2 \\
&\quad + \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left( \frac{1}{M} \sum_{i=1}^{M} F_{i}(z_{i}^{t}) - \bar{g}^{t} - F_{m}(z_{m}^{t}, \xi_{m}^{t}) \right)^2.
\end{align*}

Here $\beta_0$ is some positive constant, which we define later. Then

\begin{align*}
\mathbb{E} \left[ \text{Err}(t+1/2) \right] &\leq (1 + \beta_0) \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left( \frac{1}{M} \sum_{i=1}^{M} F_{i}(z_{i}^{t-1+1/2}) - \sum_{k=t_p+1}^{t-2} \bar{g}^{k+1/2} \\
&\quad + F_{m}(z_{m}^{t-1+1/2}, \xi_{m}^{t-1+1/2}) + \sum_{k=t_p+1}^{t-2} F_{m}(z_{m}^{k+1/2}, \xi_{m}^{k+1/2}) \right)^2 \\
&\quad + (1 + \beta_0^{-1}) \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left( \frac{1}{M} \sum_{i=1}^{M} F_{i}(z_{i}^{t}) + F_{m}(z_{m}^{t}, \xi_{m}^{t}) \right)^2 \\
&\quad + (1 + \beta_0) \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left( \frac{1}{M} \sum_{i=1}^{M} F_{i}(z_{i}^{t-1+1/2}) - \bar{g}^{t-1+1/2} - F_{m}(z_{m}^{t-1+1/2}) \right)^2 \\
&\quad + \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left( \frac{1}{M} \sum_{i=1}^{M} F_{i}(z_{i}^{t}) - \bar{g}^{t} - F_{m}(z_{m}^{t}, \xi_{m}^{t}) \right)^2.
\end{align*}
and
\[
\mathbb{E}[\text{Err}(t+1/2)] \\
\leq (1 + \beta_0)(1 + \beta_1) \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left[ \left| - \sum_{k=t_p+1}^{t-2} \sum_{k=t_p+1}^{t-2} F_m(z_m^{k+1/2}) + F_m(z_m^{k+1/2}) \right|^2 \right] \\
+ (1 + \beta_0)(1 + \beta_1) \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left[ \left| \sum_{i=1}^{M} F_i(z_i^{t-1/2}) + F_m(z_m^{t-1/2}) \right|^2 \right] \\
+ (1 + \beta_0)^{-1} \frac{\gamma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left[ \left| \sum_{i=1}^{M} F_i(z_i^{t-1/2}) - \sum_{i=1}^{M} F_m(z_m^{t-1/2}) \right|^2 \right] .
\]

One can continue this way for all terms, setting \( \beta_i = \frac{1}{\alpha - i - 1} \), where \( \alpha = 4H \). Then for all \( i = 0, \ldots, (t - t_p - 1) \)
\[
(1 + \beta_0)(1 + \beta_1)(1 + \beta_2) \cdots (1 - \beta_i) = \frac{\alpha}{\alpha - i - 1} .
\]

Note that \( t - t_p \leq 2H \), hence for all \( i = 0, \ldots, (t - t_p - 1) \)
\[
(1 + \beta_0)(1 + \beta_1)(1 + \beta_2) \cdots (1 + \beta_i) \leq (1 + \beta_1)(1 + \beta_2) \cdots (1 + \beta_{t-t_p-1}) \leq \frac{\alpha}{\alpha - 2H} \leq 2 .
\]

Additionally, \( 1 + \beta_i^{-1} \leq \alpha \), then (\( \alpha = 4H \))
\[
\mathbb{E}[\text{Err}(t+1/2)] \\
\leq 2 \alpha \gamma^2 \frac{H}{M} \sum_{k=t_p+1}^{t-1} \sum_{m=1}^{M} \mathbb{E} \left[ \left| - \sum_{i=1}^{M} F_i(z_i^{k+1/2}) + F_m(z_m^{k+1/2}) \right|^2 \right] \\
+ 2 \alpha \gamma^2 \frac{H}{M} \sum_{m=1}^{M} \mathbb{E} \left[ \left| \sum_{i=1}^{M} F_i(z_i^{t-1/2}) - \sum_{i=1}^{M} F_m(z_m^{t-1/2}) \right|^2 \right] \\
+ 2 \alpha \gamma^2 \frac{H}{M} \sum_{k=t_p+1}^{t-1} \sum_{m=1}^{M} \mathbb{E} \left[ \left| \sum_{i=1}^{M} F_i(z_i^{k+1/2}) - \sum_{i=1}^{M} F_m(z_m^{k+1/2}) \right|^2 \right] \\
+ 8 \gamma^2 \frac{H}{M} \sum_{k=t_p+1}^{t-1} \sum_{m=1}^{M} \mathbb{E} \left[ \left| - \sum_{i=1}^{M} F_i(z_i^{k+1/2}) + F_m(z_m^{k+1/2}) \right|^2 \right] .
\]
It remains to estimate
\[
\frac{1}{M} \sum_{m=1}^{M} \left\| \frac{E}{M} \sum_{i=1}^{M} F_i(z_i^{k+1/2}) + F_m(z_m^{k+1/2}) \right\|^2 \leq \frac{3}{M} \sum_{m=1}^{M} \left\| \frac{E}{M} \sum_{i=1}^{M} F_i(z_i^{k+1/2}) + F_m(z_m^{k+1/2}) \right\|^2 + \frac{3}{M} \sum_{m=1}^{M} \left\| - F_m(z_m^{k+1/2}) + F_m(z_m^{k+1/2}) \right\|^2 + \frac{6}{M} \sum_{m=1}^{M} \left\| - F_m(z_m^{k+1/2}) + F_m(z_m^{k+1/2}) \right\|^2 + 3D^2
\]

and
\[
\frac{1}{M} \sum_{m=1}^{M} \left\| \frac{1}{M} \sum_{i=1}^{M} F_i(z_i^{k+1/2}) - g^{k+1/2} - F_m(z_m^{k+1/2}) + F_m(z_m^{k+1/2}, \xi_m^{k+1/2}) \right\|^2 \leq 2 \left[ E \left\| \frac{1}{M} \sum_{i=1}^{M} F_i(z_i^{k+1/2}) - g^{k+1/2} \right\|^2 + \frac{2}{M} \sum_{m=1}^{M} \left\| F_m(z_m^{k+1/2}) + F_m(z_m^{k+1/2}, \xi_m^{k+1/2}) \right\|^2 \right] \leq 4\sigma^2.
\]

Finally, we get
\[
E[\text{Err}(t + 1/2)] \leq 48\gamma^2 L_{\max}^2 H \sum_{k=t_p+1}^{t-1} E[\text{Err}(k + 1/2)] + 48\gamma^2 L_{\max}^2 H E[\text{Err}(t)] + 32(D^2H + \sigma^2) \sum_{k=t_p+1}^{t-1} \gamma^2 + 32\gamma^2 (\sigma^2 + D^2). \tag{54}
\]

The estimate for \( E[\text{Err}(t + 1/3)] \) is done in a similar way:
\[
E[\text{Err}(t)] \leq 48\gamma^2 L_{\max}^2 H \sum_{k=t_p+1}^{t-1} E[\text{Err}(k + 1/2)] + 32(D^2H + \sigma^2) \sum_{k=t_p+1}^{t-1} \gamma^2. \tag{55}
\]

Substituting \( E[\text{Err}(t)] \) to \( E[\text{Err}(t + 1/2)] \), we get
\[
E[\text{Err}(t + 1/2)] \leq 48\gamma^2 L_{\max}^2 H \sum_{k=t_p+1}^{t-1} E[\text{Err}(k + 1/2)] + 48\gamma^2 L_{\max}^2 H \left( 48\gamma^2 L_{\max}^2 H \sum_{k=t_p+1}^{t-1} E[\text{Err}(k + 1/2)] + 32(D^2H + \sigma^2) \sum_{k=t_p+1}^{t-1} \gamma^2 \right)
\]
\[
+ 32(D^2H + \sigma^2) \sum_{k=t_p+1}^{t-1} \gamma^2 + 32\gamma^2 (\sigma^2 + D^2). \]
With $\gamma \leq \frac{1}{21HL_{\max}}$

$$E[\text{Err}(t + 1/2)] \leq \frac{1}{8H} \sum_{k=t_p+1}^{t-1} E[\text{Err}(k + 1/2)] + 72(D^2H + \sigma^2)\gamma^2(t-t_p-1).$$

Let us run the recursion:

$$E[\text{Err}(t + 1/2)] \leq \frac{1}{8H} \left(1 + \frac{1}{8H}\right) \sum_{k=t_p+1}^{t-2} E[\text{Err}(k + 1/2)]$$

$$+ \frac{1}{8H} \cdot 72(D^2H + \sigma^2)\gamma^2(t-t_p-2)\gamma^2(t-t_p-1)$$

$$\leq 72(D^2H + \sigma^2)\gamma^2 \sum_{k=t_p+1}^{t-1} \left(1 + \frac{1}{8H}\right)^{t-1-j}.$$ 

Then one can note that $(1 + \frac{1}{8H})^{t-1-j} \leq (1 + \frac{1}{2H})^{2H} \leq \exp(1) \leq 3$ and then

$$E[\text{Err}(t + 1/2)] \leq 216(D^2H + \sigma^2) \sum_{k=t_p+1}^{t-1} \gamma^2 \leq 216(D^2H + \sigma^2)H\gamma^2.$$ 

Note that in the general case $E[\text{Err}(t + 1/3)]$ may be less than $E[\text{Err}(t)]$, but since recurrent (54) is stronger than (55), we assume for simplicity that $E[\text{Err}(k + 1/3)] \geq E[\text{Err}(k)]$. Then (52) can be rewritten as

$$E[\|\hat{z}^t - z^*\|] \leq \left(1 - \frac{\mu\gamma}{2}\right)E[\|\hat{z}^t - z^*\|] + \frac{10\gamma^2\sigma^2}{M}$$

$$+ \left(\frac{7\gamma L_{\max}^2}{\mu} + 5\gamma^2 L_{\max}^2\right)E[\text{Err}(t + 1/2)]$$

$$\leq \left(1 - \frac{\mu\gamma}{2}\right)E[\|\hat{z}^t - z^*\|] + \frac{10\gamma^2\sigma^2}{M}$$

$$+ \left(\frac{7\gamma L_{\max}^2}{\mu} + 5\gamma^2 L_{\max}^2\right)(216(D^2H + \sigma^2)H\gamma^2).$$

Running the recursion, we obtain:

$$E[\|\hat{z}^T - z^*\|] = O\left(\left(1 - \frac{\mu\gamma}{2}\right)^T \|z^0 - z^*\|^2 + \frac{\gamma^2(2H + \sigma^2)HL_{\max}^2}{\mu^2}\right),$$

or

$$E[\|\hat{z}^T - z^*\|^2] = O\left(\exp\left(-\frac{\mu\gamma T}{2}\right) \|z^0 - z^*\|^2 + \frac{\gamma^2(2H + \sigma^2)HL_{\max}^2}{\mu^2}\right).$$

Finally, we need tuning of $\gamma \leq \frac{1}{21HL_{\max}}$:

- If $\frac{1}{21HL_{\max}} \geq \frac{2\ln(\max\{2, \mu\|z^0 - z^*\|^2T\sigma/M^2\})}{\mu T}$ then $\gamma = \frac{2\ln(\max\{2, \mu\|z^0 - z^*\|^2T\sigma/M^2\})}{\mu T}$ gives

  $$\hat{O}\left(\exp\left(-\ln(\max\{2, \mu\|z^0 - z^*\|^2T\sigma/M^2\})\right) \|z^0 - z^*\|^2 + \frac{\sigma^2}{\mu^2 MT} + \frac{(D^2H + \sigma^2)HL_{\max}^2}{\mu^4 T^2}\right)$$

  $$= \hat{O}\left(\frac{\sigma^2}{\mu^2 MT} + \frac{(D^2H + \sigma^2)HL_{\max}^2}{\mu^4 T^2}\right)$$

- If $\frac{1}{21HL_{\max}} \leq \frac{2\ln(\max\{2, \mu\|z^0 - z^*\|^2T\sigma/M^2\})}{\mu T}$ then $\gamma = \frac{1}{21HL_{\max}}$ gives

  $$\hat{O}\left(\exp\left(-\frac{\mu T}{42HL_{\max}}\right) \|z^0 - z^*\|^2 + \frac{\sigma^2}{\mu^2 MT} + \frac{(D^2H + \sigma^2)HL_{\max}^2}{\mu^4 T^2}\right)$$

  $$\leq \hat{O}\left(\exp\left(-\frac{\mu T}{42HL_{\max}}\right) \|z^0 - z^*\|^2 + \frac{\sigma^2}{\mu^2 MT} + \frac{(D^2H + \sigma^2)HL_{\max}^2}{\mu^4 T^2}\right).$$
D.2 Non-convex-non-concave problems

**Theorem 19 (Theorem 7)** Let \( \{ z_m^t \}_{t \geq 0} \) denote the iterates of Algorithm 3 for solving problem 11. Let Assumptions 11(l), 2(nc), 3 and 5 be satisfied. Also let \( H = \max |k_p + 1 - k_p| \) be a maximum distance between moments of communication (\( k_p \in \mathbb{I} \)) and \( \| z^t \| \leq \Omega \) (for all \( t \)). Then, if \( \gamma \leq \frac{1}{4L_{\max}} \), we have the following estimate:

\[
\mathbb{E}
\left[
\frac{1}{T} \sum_{t=0}^{T-1} \| F(z^t) \|^2
\right]
= \mathcal{O}
\left(
\frac{L_{\max}^4 \| z^0 - z^* \|^2}{T}
+ \left(\frac{L_{\max} \Omega (D^2 H + \sigma^2) H^{2/3}}{T^{1/3}}\right)
+ \frac{\sigma^2}{M}
+ L_{\max} \sqrt{(D^2 H + \sigma^2) H}
\right).
\]

**Proof:** Most of the necessary estimates have already been made in the previous subsection. In particular, Lemmas 13 and 14 are valid for us. But Lemma 12 needs modification:

**Lemma 15** It holds:

\[
-2\gamma \mathbb{E}
\left[
\langle g^{t+1/2}, z^{t+1/2} - z^* \rangle
\right]
\leq
-2\gamma \mathbb{E}
\left[
\frac{1}{M} \sum_{m=1}^{M} \mathbb{E}_{\xi^{t+1/2}}[F_m(z_m^{t+1/2}, z_m^{t+1/2}), z^{t+1/2} - z^*]
\right]
\]

\[
= -2\gamma \mathbb{E}
\left[
\frac{1}{M} \sum_{m=1}^{M} F_m(z_m^{t+1/2}, z^{t+1/2} - z^*)
\right]
\]

\[
+ 2\gamma \mathbb{E}
\left[
\frac{1}{M} \sum_{m=1}^{M} [F_m(z_m^{t+1/2}) - F_m(z_m^{t+1/2}), z^{t+1/2} - z^*]
\right]
\]

\[
= -2\gamma \mathbb{E}
\left[
\frac{1}{M} \sum_{m=1}^{M} [F_m(z_m^{t+1/2}) - F_m(z_m^{t+1/2}), z^{t+1/2} - z^*]
\right]
\]

\[
\leq 2\gamma \mathbb{E}
\left[
\frac{1}{M} \sum_{m=1}^{M} [F_m(z_m^{t+1/2}) - F_m(z_m^{t+1/2}), z^{t+1/2} - z^*]
\right]
\]

\[
\leq 2\gamma \mathbb{E}
\left[
\frac{1}{M} \sum_{m=1}^{M} F_m(z_m^{t+1/2}) - F_m(z_m^{t+1/2})
\right]
\]

\[
\leq 2\gamma \mathbb{E}
\left[
\frac{1}{M} \sum_{m=1}^{M} \| F_m(z_m^{t+1/2}) - F_m(z_m^{t+1/2}) \|
\right]
\]

\[
\leq 2\gamma \mathbb{E}
\left[
\| z^{t+1/2} - z^* \| \cdot \frac{1}{M} \sum_{m=1}^{M} \| F_m(z_m^{t+1/2}) - F_m(z_m^{t+1/2}) \|
\right]
\]

\[
\leq 2\gamma \mathbb{E}
\left[
\| z^{t+1/2} - z^* \| \cdot \frac{1}{M} \sum_{m=1}^{M} \| z_m^{t+1/2} - z^{t+1/2} \|
\right]
\]

\[
\leq 2\gamma L_{\max} \| z^{t+1/2} - z^* \| \cdot \frac{1}{M} \sum_{m=1}^{M} \| z_m^{t+1/2} - z^{t+1/2} \|
\]
Then we have the same as (50):

\[ -2\gamma E\left(\langle \tilde{y}^{t+1/2}, \bar{z}^{t+1/2} - z^* \rangle\right) \]

\[ \leq 2\gamma L_{\max}E\left[\|\tilde{z}^t - z^*\| \cdot \frac{1}{M} \sum_{m=1}^{M} \left\|z_{m}^{t+1/2} - \bar{z}_{m}^{t+1/2}\right\| \right] \]

\[ + 2\gamma L_{\max}E\left[\|\tilde{z}_{m}^{t+1/2} - \bar{z}_{m}^{t+1/2}\| \cdot \frac{1}{M} \sum_{m=1}^{M} \left\|z_{m}^{t+1/2} - \bar{z}_{m}^{t+1/2}\right\| \right] \]

\[ \leq 2\gamma L_{\max}\sqrt{E[\|\tilde{z}^t - z^*\|^2]} \cdot \sqrt{E\left[\left(\frac{1}{M} \sum_{m=1}^{M} \left\|z_{m}^{t+1/2} - \bar{z}_{m}^{t+1/2}\right\| \right)^2 \right]} \]

\[ + \gamma L_{\max}E\left[\|\tilde{z}_{m}^{t+1/2} - \bar{z}_{m}^{t+1/2}\|^2 \right] + \gamma L_{\max}E\left[\left(\frac{1}{M} \sum_{m=1}^{M} \left\|z_{m}^{t+1/2} - \bar{z}_{m}^{t+1/2}\right\| \right)^2 \right]. \]

By (15) it is easy to see that

\[ E\left[\left(\frac{1}{M} \sum_{m=1}^{M} \left\|z_{m}^{t+1/2} - \bar{z}_{m}^{t+1/2}\right\| \right)^2 \right] \leq E\left[\frac{1}{M} \sum_{m=1}^{M} \left\|z_{m}^{t+1/2} - \bar{z}_{m}^{t+1/2}\right\|^2 \right]. \]

This completes the proof.

Then we have the same as (50):

\[ E[\|\tilde{z}_{m}^{t+1} - z^*\|^2] \leq E[\|\tilde{z}^t - z^*\|^2] - E[\|\tilde{z}_{m}^{t+1/2} - \bar{z}_{m}^{t+1/2}\|^2] \]

\[ + 2\gamma L_{\max}\sqrt{E[\|\tilde{z}^t - z^*\|^2]} \sqrt{E[\text{Err}(t + 1/2)]} \]

\[ + \gamma L_{\max}E[\|\tilde{z}_{m}^{t+1/2} - \bar{z}_{m}^{t+1/2}\|^2] + \gamma L_{\max}E[\text{Err}(t + 1/2)] \]

\[ + \gamma^2 \left(5\gamma^2 L_{\max}E[\|\tilde{z}_{m}^{t+1/2} - \bar{z}_{m}^{t+1/2}\|^2] + \frac{10\sigma^2}{M} + 5\gamma^2 L_{\max}E[\text{Err}(t + 1/2)] + 5L_{\max}^2E[\text{Err}(t)]\right). \]

Choosing \( \gamma \leq \frac{1}{4L_{\max}} \) gives

\[ \frac{1}{2} E[\|\tilde{z}_{m}^{t+1/2} - \bar{z}_{m}^{t+1/2}\|^2] \leq E[\|\tilde{z}^t - z^*\|^2] - E[\|\tilde{z}_{m}^{t+1} - z^*\|^2] \]

\[ + 2\gamma L_{\max}\sqrt{E[\|\tilde{z}^t - z^*\|^2]} \sqrt{E[\text{Err}(t + 1/2)]} \]

\[ + (5\gamma^2 L_{\max} + \gamma L_{\max})E[\text{Err}(t + 1/2)] + 5\gamma^2 L_{\max}E[\text{Err}(t)] + \frac{10\gamma^2 \sigma^2}{M}. \]
Next we work with
\[
\mathbb{E} \left[ \| \tilde{z}^{t+1/2} - \tilde{z}^t \|^2 \right] = \gamma^2 \mathbb{E} \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z^{t}_m, \xi^{t}_m) - F_m(z^{t}_m) + F_m(z^{t}_m) - \gamma L \|z^t - z^*\| \right\| \right]^2
\]
\[
\geq \frac{\gamma^2}{2} \mathbb{E} \left[ \| F(\tilde{z}^t) \|^2 \right] - \gamma^2 \mathbb{E} \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z^{t}_m, \xi^{t}_m) - F_m(z^{t}_m) \right\| \right]^2 - 2\gamma^2 \mathbb{E} \left[ \left\| \frac{1}{M} \sum_{m=1}^{M} F_m(z^{t}_m) - F_m(\tilde{z}^t) \right\| \right]^2
\]
\[
\geq \frac{\gamma^2}{2} \mathbb{E} \left[ \| F(\tilde{z}^t) \|^2 \right] - \frac{2\gamma^2 \sigma^2}{M} \sum_{m=1}^{M} \mathbb{E} \left[ \| z^t_m - \tilde{z}^t \|^2 \right] - 2\gamma^2 L_{\max}^2 \mathbb{E} [\text{Err}(t)] .
\]

Connecting with previous gives
\[
\frac{\gamma^2}{4} \mathbb{E} \left[ \| F(\tilde{z}^t) \|^2 \right] \leq \mathbb{E} \left[ \| \tilde{z}^t - z^* \|^2 \right] - \mathbb{E} \left[ \| \tilde{z}^{t+1} - z^* \|^2 \right] + 2\gamma L_{\max} \sqrt{\mathbb{E} \left[ \| \tilde{z}^t - z^* \|^2 \right]} \sqrt{\mathbb{E} [\text{Err}(t + 1/2)]}
\]
\[
+ (\gamma L_{\max} + 5\gamma^2 L_{\max}^2) \mathbb{E} [\text{Err}(t + 1/2)] + 6\gamma^2 L_{\max}^2 \mathbb{E} [\text{Err}(t)] + \frac{11\gamma^2 \sigma^2}{M}.
\]

With result of Lemma 14 we get
\[
\frac{\gamma^2}{4} \mathbb{E} \left[ \| F(\tilde{z}^t) \|^2 \right] \leq \mathbb{E} \left[ \| \tilde{z}^t - z^* \|^2 \right] - \mathbb{E} \left[ \| \tilde{z}^{t+1} - z^* \|^2 \right] + 2\gamma L_{\max} \sqrt{\mathbb{E} \left[ \| \tilde{z}^t - z^* \|^2 \right]} \sqrt{216(D^2 H + \sigma^2)H \gamma^2}
\]
\[
+ \frac{11\gamma^2 \sigma^2}{M} + 216(\gamma L_{\max} + 11\gamma^2 L_{\max}^2)(D^2 H + \sigma^2)H \gamma^2.
\]

Summing over all \( t \) from 0 to \( T - 1 \) and averaging gives:
\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} \| F(\tilde{z}^t) \|^2 \right] \leq \frac{4\| z^0 - z^* \|^2}{\gamma^2 T} + \frac{44\sigma^2}{M}
\]
\[
+ 1000(\gamma L_{\max} + 11\gamma^2 L_{\max}^2)(D^2 H + \sigma^2)H + \frac{120L_{\max} \sqrt{(D^2 H + \sigma^2)H}}{T} \sum_{t=0}^{T-1} \sqrt{\mathbb{E} \left[ \| \tilde{z}^t - z^* \|^2 \right]} .
\]

Under the additional assumption that \( \| z^* \| \leq \Omega \) and \( \| \tilde{z}^t \| \leq \Omega \), from (57), we obtain
\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} \| F(\tilde{z}^t) \|^2 \right] = \mathcal{O} \left( \frac{\| z^0 - z^* \|^2}{\gamma^2 T} + (\gamma L_{\max} + \gamma^2 L_{\max}^2)(D^2 H + \sigma^2)H
\]
\[
+ \frac{\sigma^2}{M} + L_{\max} \Omega \sqrt{(D^2 H + \sigma^2)H} \right).
\]

With \( \gamma = \min \left\{ \frac{1}{T L_{\max}} : \left( \frac{\| z^0 - z^* \|^2}{T L_{\max} (D^2 H + \sigma^2) H} \right) \right\}^{1/3} \) we have
\[
\mathbb{E} \left[ \frac{1}{T} \sum_{t=0}^{T-1} \| F(\tilde{z}^t) \|^2 \right] = \mathcal{O} \left( \frac{L_{\max}^2 \| z^0 - z^* \|^2}{T} + \frac{(L_{\max} \Omega (D^2 H + \sigma^2) H)^{2/3}}{T^{1/3}} + \frac{\sigma^2}{M} + L_{\max} \Omega \sqrt{(D^2 H + \sigma^2)H} \right).
\]
E Experiments

We implement all methods in Python 3.8 using PyTorch [47] and Ray [39] and run on a machine with 24 AMD EPYC 7552 @ 2.20GHz processors, 2 GPUs NVIDIA A100-PCIE with 40536 Mb of memory each (Cuda 11.3).

E.1 Federated GAN on MNIST

We continue with additional experiments on Generative Adversarial Networks. Before that, we make a small introduction. A simple GANs setup consists of two parts – the discriminator $D$ aimed at distinguishing real samples $x$ from adversarial ones by giving probability that sample is real, and the generator $G$ trying to fool the discriminator by generating realistic samples from random noise $z$. Following [17], the value function $V(D, G)$ used in such min-max game can be expressed as

$$
\min_G \max_D V(D, G) = \mathbb{E}_{x \sim p_{data}(x)}[\log D(x)] + \mathbb{E}_{z \sim p_z(z)}[\log(1 - D(G(z)))].
$$

(58)

As mentioned in main part, we use Deep Convolutional GAN [49]. As optimizers we use Algorithm 3 and a combination of Adam with Algorithm 5.

We create up to 3 or 4 replicas and train them for 200 epochs. In these experiments, we try to vary the synchronisation frequencies for the generator and discriminator separately. We split data in a following manner: the half of the dataset is shared uniformly by replicas, from another half we take only those digits, which are equal to the order number of the replica.

Usually, to get better performance, researchers vary the number of training steps done for the generator and the discriminator, pretrain one of the parts or use specific optimizer. We are more interested in numerical convergence, that is why we do not do such fine-tuning.

The results of the experiment for Algorithm 3 and Local Adam are reflected in Figures 4, 7, 5, 6, 8 and 9. Here $H_g, H_d$ – communication frequencies for generator and discriminator.

Figure 4: Digits generated by global generator during training. 4 replicas, Local SGD (left) and 4 replicas, Local Adam (right) $H_g = H_d = 20$.

The experiment shows that we have good global images despite the fact that the data is heterogeneous. The global (synced) discriminator converges to random guessing, what is shown by binary classification accuracy equal to 0.5, see Figures 7, 9. Based on [17], this behaviour is expected.
Figure 5: Generator and Discriminator Empirical Loss on MNIST during training, Local SGD, 3 replicas, $H_g = 10$, $H_d = 20$.

Figure 6: Generator and Discriminator Empirical Loss on MNIST during training, Local SGD, 3 replicas, $H_g = 20$, $H_d = 10$.

Figure 7: Accuracy on MNIST, Local SGD, 3 and 4 replicas, $H_g = H_d = 20$. 
Figure 8: Generator and Discriminator Empirical Loss on MNIST during training, Local Adam, 4 replicas, $H_g = 20$, $H_d = 10$.

Figure 9: Accuracy on MNIST, Local Adam, 4 replicas, $H_g = 20$, $H_d = 10$. 