Coherent State Approach to Time Reparameterization Invariant Systems

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Abstract

For many years coherent states have been a useful tool for understanding fundamental questions in quantum mechanics. Recently, there has been work on developing a consistent way of including constraints into the phase space path integral that naturally arises in coherent state quantization. This new approach has many advantages over other approaches, including the lack of any Gribov problems, the independence of gauge fixing, and the ability to handle second-class constraints without any ambiguous determinants. In this paper, I use this new approach to study some examples of time reparameterization invariant systems, which are of special interest in the field of quantum gravity.

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1 Introduction

The coherent state formulation of the path integral [1] has many advantages over a conventional Feynman path integral. Because coherent states describe a minimum uncertainty wave packet, there is a natural relation between the classical system and underlying quantum system. The coherent state path integral is intrinsically superior with regards to a canonical coordinate transform which just amounts to relabeling of the states [2]. Moreover, for coherent state path integral, it is possible to find a well regularized path integral measure (the pinned Weiner measure) [3].

Recent work [4] - [7] has included constraints into this formulation. The first approach was to consider a semi-classical constraint [4]. This constraint can be inserted by hand into each time step in the construction of the path integral. The result is the formal path integral where the action is dependent on the total Hamiltonian. Klauder [5] constructed a projection operator that maps states defined on the full phase space onto physical states. The resulting path integral is independent of the functional form of the Lagrange multiplier term and hence gauge invariant. Later, Klauder and Shabanov [7] generalized this approach to a coordinate-free formulation.

For constrained system, coherent states offer further advantages. Because the path integral is regularized, we are not required to gauge fix to remove the infinite volume term that normally appears. The result is an averaging over the gauge orbits. Without gauge fixing, there are no potential problems with Gribov ambiguities. Also we are not required to eliminate second-class constraints [5], nor is there the possibility of an ambiguous determinant for this case.

In this paper, we will review coherent state quantization of constrained system and compare the results of the author’s semi-classical construction of the path integral [4] with Klauder’s projection operator approach [5]. Then we will work through details of two time-reparameterization invariant systems. The first example will be the single harmonic oscillator. Although this is a very simple example it can give us some insight into the details of this formulation. The second example is the double harmonic oscillator.

The double harmonic oscillator is an important example in the study of quantum gravity [8]. This is a good toy model to help understand the “problem of time.” In essence, one harmonic oscillator can be used as a quantum “clock” to measure the other oscillator. In terms of coherent state quantization, the double harmonic oscillator show the importance of the geometry on the system [8] - [10]. The geometry of the phase space determines the natural kinematical operator in which the system should be quantized. For this case, the resulting reduced phase space is spherical so the kinematical operators are spin-like operators. Also this system has a potential Gribov problem that results in a difference between the ground state energies of the

\[\text{In a time-reparameterization invariant system, such as quantum gravity, the roll of the local time coordinate is difficult to understand. For more about this “problem of time,” Rovelli has written a series of papers [11].}\]
reduced and Dirac quantizations. This Gribov problem results because the gauge orbits form a twisted bundle over the constraint surface.

2 Coherent state quantization

The ordinary phase space has a natural Heisenberg-Weyl algebra structure that comes from the symplectic structure. This operator algebra can be used to construct a coherent state representation of the Hilbert space that is labeled by the classical phase space coordinates. So, we will begin by considering a set of $M$ pairs of Heisenberg operator $\{\hat{P}_j, \hat{Q}^k\}$. These operators obey the standard Heisenberg-Weyl commutation relations,

$$[\hat{P}_j, \hat{Q}^k] = -i\hbar \delta^k_j, \quad j, k = 1, \ldots, M. \quad (2.1)$$

The coherent state representation is then a unitary representation of the Heisenberg-Weyl group acting on some fiducial vector $|\eta\rangle$ chosen from the Hilbert space.

$$|p, q\rangle = e^{-if(p,q)} e^{-\frac{i}{\hbar}p^k \hat{Q}^k} |\eta\rangle. \quad (2.2)$$

In most cases, the fiducial vector is chosen such that the coherent state is “physical centered,” $\langle \eta | \hat{P}_j | \eta \rangle = \langle \eta | \hat{Q}^k | \eta \rangle = 0$. For this reason, we will choose the fiducial vector to be the ground state of the harmonic oscillator, $|\eta\rangle = |0\rangle$. This set of states does not form an orthonormal basis, as is seen in the overlap function,

$$\langle p', q' | p, q \rangle = \exp \left\{ -\frac{1}{4\hbar} [ |p' - p|^2 + |q' - q|^2] + \frac{i}{2\hbar} [p' \cdot q - p \cdot q'] \right\}. \quad (2.3)$$

However, for any choice of the fiducial vector $|\eta\rangle$, they do admit a resolution of unity,

$$\mathbb{I} = \int |p, q\rangle \langle p, q| \prod_{j=1}^N \frac{dp_j dq_j}{2\pi}. \quad (2.4)$$

In addition, these states form an (over)complete set of states on the Hilbert space. We can represent any vector in our Hilbert space as a function of $(p, q)$ by defining the function to be $\psi(p, q) \equiv \langle p, q | \psi \rangle$. The overlap function, for example eqn.(2.3), is the reproducing kernel $K(p', q'; p, q) \equiv \langle p', q' | p, q \rangle$ on the Hilbert space. This reproducing kernel has the following properties:

$$\psi(p', q') = \int K(p', q'; p, q) \psi(p, q) \prod_{j=1}^N \frac{dp_j dq_j}{2\pi}, \quad (2.5)$$

$$K(p'', q''; p, q) = \int K(p'', q''; p', q') K(p', q'; p, q) \prod_{j=1}^N \frac{dp_j dq_j}{2\pi}. \quad (2.6)$$
We can construct a path integral based on this representation of the Hilbert space. Unlike the normal path integral which integrates over the configurations space \( (q) \), the coherent state path integral naturally integrates over the phase space \( (p, q) \). To construct this path integral, let us start with the Hamiltonian evolution between two states \( |p, q\rangle \) and \( |p', q'\rangle \). The matrix element may be broken in \( N + 1 \) time steps. Then at each time step we can insert a resolution of unity.

\[
\langle p', q'|e^{-\frac{i}{\hbar}\hat{H}T}|q, p\rangle = \left( \int \cdots \int \prod_{n=1}^{N} d\mu(p_n, q_n) \right) \prod_{n=0}^{N} \langle p_{n+1}, q_{n+1}|e^{-\frac{i}{\hbar}\hat{H}}|p_n, q_n\rangle,
\]

\(|p', q'\rangle = |p_{N+1}, q_{N+1}\rangle, \quad |p, q\rangle = |p_0, q_0\rangle, \quad \varepsilon = (t' - t)/(N + 1). \quad (2.7)\]

The measure \( d\mu(p, q) \) is the same as the measure defined in the resolution of unity (2.4). In the limit \( (\varepsilon \to 0) \), if the paths are regarded as continuous and differentiable, then we can formally rewrite the above matrix element (2.7) in form of a path integral (see [1] for more details),

\[
\int \mathcal{D}\mu(p, q) \exp \left\{ \frac{i}{\hbar} \int (p\dot{q} - H(p, q)) dt \right\}, \quad (2.8)
\]

where the symbol \( H(p, q) = \langle p, q|H|p, q\rangle \). In the stationary phase approximation, this then leads to the standard Hamilton’s equations of motion.

Unlike ordinary configuration space path integrals, the phase space path integral can be given a natural regularization by inserting an additional term into the path integral [9]. This is done by changing the measure to a pinned Weiner measure. This measure originally arose in the study of Brownian motion. The probability density of a particle undergoing Brownian motion is governed by the diffusion equation. The fundamental solution of the diffusion equation for a flat metric is a spreading Gaussian,

\[
\rho(t''; t') = \frac{1}{2\pi\nu(t' - t')} \exp \left[ -\frac{(p'' - p')^2 + (q'' - q')^2}{2\nu(t'' - t')} \right]. \quad (2.9)
\]

This solution possesses a semi-group structure with the following product rule:

\[
\rho(t'''; t') = \int dp'' dq'' \rho(t'''; t'')\rho(t''; t'). \quad (2.10)
\]
To construct the Weiner measure, we proceed in a similar fashion as we did with the construction of the path integral. We can use the product rule (2.10) repeatedly to break the time in \( N + 1 \) steps.

\[
\rho(t''; t') = \int \left( \prod_{i=1}^{N} dp_i \, dq_i \right) \left( \frac{1}{2\pi \nu \varepsilon} \right)^N \left( \exp \sum_{i=0}^{N} \frac{(p_{i+1} - p_i)^2 + (q_{i+1} - q_i)^2}{2\nu \varepsilon} \right),
\]

\[
(q'', p'') = (q_{N+1}, p_{N+1}), \quad (q', p') = (q_0, p_0), \quad \varepsilon = (t'' - t')/(N + 1). \tag{2.11}
\]

Then, in the continuum limit, we have a formal expression for the Weiner measure,

\[
d\mu_W(p, q) = \mathcal{N} e^{-\frac{1}{2\nu} \int \dot{p}^2 + \dot{q}^2 \, dt} \, Dp \, Dq \tag{2.12}
\]

Note that the initial and final points of the paths are fixed (or pinned) on the phase space. Writing this in a more general way to include other choices of the metric and higher dimensions, the measure is

\[
d\mu'_W(p, q) = \mathcal{N} e^{-\frac{1}{2\nu} \int \left( \frac{d\sigma(p, q)}{dt} \right)^2 \, dt} \, \prod_{j=1}^{N} Dp_j Dq_j. \tag{2.13}
\]

The measure in (2.8) can now be replaced by the formally well defined pinned Weiner measure just by the addition of the extra factor,

\[
e^{-\frac{1}{2\nu} \int \left( \frac{d\sigma(p, q)}{dt} \right)^2 \, dt}. \tag{2.14}
\]

In the limit \( \nu \to \infty \), this term formally becomes unity and we are left with our original path integral (2.8). Unless the action is explicitly dependent on the measure of the phase space, this Weiner measure is the only place that the geometry of the phase space come into play in the path integral. This geometry (as we will see in our second example) determines the natural kinematical operators in which the system should be quantized (see \cite{9}).

3 Constrained Coherent State

In this section, I will review the methods for applying first class constraints to coherent state path integrals in general, and then restrict to the case of time-reparameterization invariant systems.

To begin with, let us consider a \( 2M \) dimensional phase space labeled by coordinates \((p_i, q^j)\) where \( i, j = 1, \ldots M \). On this phase space, the constraint surface can be defined in terms of a system of \( N \) equations \( \phi_\alpha(p, q) = 0 \), where \((2M - N)\) is the dimension of the constraint surface. The evolution on the constraint surface is generated by the total Hamiltonian,
\[ H_T(p, q) = H(p, q) + \lambda^a \phi_a(p, q), \quad \frac{dF}{dt} = \{ F, H_T \} \bigg|_{\phi_a = 0}. \quad (3.1) \]

For this paper, we are only interested in the dynamics of a system where the constraint functions, \( \phi_a = 0 \), are all first class functions,

\[ \{ \phi_a, \phi_b \} \approx 0, \quad a, b = 1, \ldots, N. \quad (3.2) \]

We will assume that the time derivatives will not introduce any new (secondary) constraints. Therefore the set \( \{ \phi_a \} \) is complete. This also means that the Hamiltonian is also a first class function,

\[ \frac{d\phi_a}{dt} = \{ \phi_a, H_T \} = \{ \phi_a, H \} + \lambda^b \{ \phi_a, \phi_b \} \approx 0. \quad (3.3) \]

Furthermore, because these commutators are all weakly vanishing, near the constraint surface, they are given as linear combinations of the constraint functions [12]. Thus the Hamiltonian and the constraints form a closed algebra,

\[ \{ H, \phi_a \} = h^c_a \phi_c, \quad (3.4) \]

\[ \{ \phi_a, \phi_b \} = C^c_{ab} \phi_c. \quad (3.5) \]

We can now go on to study solutions to the time evolution equation on our constraint surface,

\[ \frac{df}{dt} = \{ f, H_T \} = \{ f, H \} + \lambda^a \{ f, \phi_a \}. \quad (3.6) \]

We see that in general the solution to eqn. (3.6) will depend on the choice of the Lagrange multiplier \( \lambda^a \). However, a physical observable will not have any dependency on this choice. Therefore, any solution that differs only by changing the value of the Lagrange multiplier is defined to be equivalent. For first class constraints, these gauge transformation are generated by the constraint equations [12],

\[ \delta f = \delta \varepsilon^a \{ f, \phi_a \}, \quad (3.7) \]

and the dimension of the gauge transformations is the same as the number of constraints. Thus the reduced phase space (the manifold after applying the constraints and quotienting out the gauge orbits) is then a \((2M - 2N)\) dimensional manifold. This reduced phase space admits a local symplectic structure [12]. So, it is possible to locally find a canonical coordinate system such that

\[ \{ \tilde{p}_i, \tilde{q}^j \} = \delta^j_i, \quad i, j = 1, \ldots, N. \quad (3.8) \]
Let us consider, these two phase space. In general we may describe the dynamics of the system on either the full phase space in terms of the total Hamiltonian or on the reduced phase space in terms of the reduced Hamiltonian,

$$H_0 = H(p, q)|_{\phi_a = 0}. \quad (3.9)$$

So we already have the a set of coherent states on the full phase space,

$$|p, q\rangle = e^{-i\tilde{f}(p, q)}e^{-\frac{i}{\hbar}p^{\dagger}Q}e^{\frac{i}{\hbar}Q^{\dagger}P}|\eta\rangle, \quad (3.10)$$

where $\hat{P}, \hat{Q}$ obeys the standard Heisenberg-Weyl commutation relations $[\hat{P}_j, \hat{Q}_k] = i\hbar \delta_{jk}$. So now, let us take the naive approach that we can construct a coherent state on the reduced phase space in the following way. We can try to use the symplectic structure on the reduced phase space to define an other set of Heisenberg-Weyl operators $[\hat{P}_j^{'}, \hat{Q}_k^{'}] = i\hbar \delta_{jk}$ where $j, k = 1, \ldots, N$. Note that these operators may not be global well defined nor are they necessarily defined in terms of the Heisenberg operators from the full phase space. For a least for some covering space of a large patch, we can construct the coherent state,

$$|\tilde{p}, \tilde{q}\rangle = e^{-i\tilde{f}(\tilde{p}, \tilde{q})}e^{-\frac{i}{\hbar}\tilde{p}^{\dagger}\tilde{Q}^{'}\hat{\eta}}. \quad (3.11)$$

If the initial and final states, $|\tilde{p}', \tilde{q}'\rangle$ and $|\tilde{p}''', \tilde{q}'''\rangle$, can be lifted back up onto the full phase space, $|p', q'\rangle$ and $|p'', q''\rangle$, the the resulting dynamics should be equivalent (up to possible normalizations),

$$\langle p', q'|e^{-\frac{i}{\hbar}\hat{H}_T(t' - t)}|p, q\rangle \sim \langle \tilde{p}', \tilde{q}'|e^{-\frac{i}{\hbar}\hat{H}_0(t' - t)}|\tilde{p}, \tilde{q}\rangle. \quad (3.12)$$

As we can see, there are two problems that we need to deal with in comparing these two descriptions. One is that we need to construct a set of meaning coherent states on the reduced phase space. We will see how this is done when we consider the projection approach of Klauder’s [5]. The other problem is to understand the dynamics of the total Hamiltonian in terms of a system of coherent states.

To begin with, let us consider the evolution generated by the total Hamiltonian (3.3). We can use the resolution of unity on the full phase space to construct the path integral (2.7). This gives us the path integral,

$$N \int DpDq \exp -\frac{i}{\hbar} \left\{ i\hbar \langle p, q|\frac{d}{dt}|p, q\rangle + \langle p, q|\hat{H} + \lambda\phi_a(p, q) \right\}. \quad (3.13)$$

We can replace the operators in terms of either the “upper” or “lower” symbol depending on our construction of the path integral (see [1] for more about these symbols). The resulting path integral is then

$$N \int DpDq \exp -\frac{i}{\hbar} \left\{ \int p\dot{q} - H(p, q) - \lambda^a \phi_a(p, q) \right\}. \quad (3.14)$$
We see that the results of the path integral depends on the choice of the Lagrange multipliers \( \lambda^a(t) \). To fix this, we can extend our phase space to include the Lagrange multipliers (see [13] for an example of such an extension), or reconstruct the path integral by placing the constraints in at each time step by hand [4]. The resulting new path integral includes integrating over the Lagrange multiplier,

\[
\mathcal{N} \int DpDqD\lambda \exp \left\{ -\frac{i}{\hbar} \int p\dot{q} - H(p, q) - \lambda^a \phi^a(p, q) \right\},
\]

and when we integrate over \( \lambda^a \), the result is our constraint equation,

\[
\int D\lambda \exp \{ \lambda^a \phi^a \} = \delta(\phi^a).
\] (3.16)

Another method of imposing the constraints is to project the coherent states onto the physical states [5]. This projection operator for first class constraints commutes with the time evolution. Thus a physical state will evolve into another physical state. When this projection operator is included, the resulting path integral picks up an additional term which is just a normal integration over the Lagrange multiplier. The resulting path integral becomes independent of the functional form of the Lagrange multiplier.

In either case, we are still left to contend with the gauge degrees of freedom. Normally, when we integrate over these degrees of freedom, we will get the volume of the space of paths for the gauge orbits. In an ordinary path integral, this volume term would be infinite and we would have to include a gauge fixing term to remove this infinite redundancy. With a coherent state path integral, we can use the Weiner measure to regularize the path integral, and because of this, we are not forced to introduce any gauge fixing into the system. The result is just an well defined averaging over the gauge degree of freedom. Then because we not required to gauge fix the system, we avoid any possible Gribov problem.

Let us begin a more detailed construction of this path integral by considering Klauder’s projection operator approach [5]. We wish to find a projection operator that takes any state onto a state that is annihilated by the constraint operator (or physical states),

\[
|p, q\rangle_{phys} = \mathcal{P}|p, q\rangle.
\] (3.17)

As a standard projection operator \( \mathcal{P} \) must have the following properties:

\[
\mathcal{P}^2 = \mathcal{P} \quad \text{and} \quad \mathcal{P}^\dagger = \mathcal{P}.
\] (3.18)

We can construct an example of such a projection operator in terms of the constraint functions. As we have seen, the constraint functions form a Lie algebra [1.4]. Let

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2A Gribov problem or obstruction occurs when the the gauge fixing term can not be defined globally or that gauge fixing function intersects a gauge orbit more then once.
us assume that the we can find the corresponding constraint operators such that this algebra is carried over to the commutator algebra. We can use the group elements generated by these operators to form a projection operator,

\[ P = \int e^{i\lambda a\hat{\Phi}} d\mu(\lambda). \]  (3.19)

We will choose the measure to be normalized \( \int d\mu(\lambda) = 1 \). In addition, it must satisfy the above properties of the projection operator (3.18). For a compact group, such a measure is the left and right equivalent Haar measure (see [3] for more details). This projection operator then projects onto the states that obey the quantum operator equation \( \hat{\Phi}_a\psi = 0 \), which are the physical states.

For a non-compact group, finding a measure that is normalizable is a bit more difficult. Klauder [5] suggested the following idea. Let the measure take the form,

\[ P = \int e^{i\lambda a\hat{\Phi}} \left( \frac{2\sin \varepsilon \lambda}{\pi \lambda} \right) d\lambda. \]  (3.20)

This projects onto states where the constraints operator is within a small interval,

\[ ||\hat{\Phi}|p, q\rangle_{\text{phys}}|| \leq ||\varepsilon |p, q\rangle_{\text{phys}}||. \]  (3.21)

Then in the limit \( \varepsilon \to 0 \), we have a handle on how to regularize this measure.

In order to use this projection operator in our construction of a path integral, we note that because the measure is left invariant, the projection operator is invariant under gauge transformations that are generated by the constraint,

\[ e^{i\sigma a\hat{\Phi}} P = \int e^{i(\sigma a\hat{\Phi})} d\mu(\lambda) \]

\[ = \int e^{i\lambda a\hat{\Phi}} d\mu(\sigma^{-1} \cdot \lambda) \]

\[ = P. \]  (3.22)

In a similar fashion because the addition of the Hamiltonian operator into the algebra is also closed (see eqn. 3.4), the projection operator commutes with the time evolution operator,

\[ Pe^{-\frac{i}{\hbar}H} = e^{-\frac{i}{\hbar}H} P = Pe^{-\frac{i}{\hbar}H} P. \]  (3.23)

For the time evolution of the physical states, because the projection operator commutes with the evolution operator and the Lagrange multiplier term can be absorbed into the projection operator, the matrix element can written in terms of just the evolution of the physical state on the full phase space.
\[ \langle p', q'; t|p, q\rangle_{\text{phys}} = \langle p', q'|e^{-\frac{i}{\hbar}H_T} \mathbb{P}|p, q\rangle = \langle p', q'|e^{-\frac{i}{\hbar}H + i\sigma^a\Phi_a} \mathbb{P}|p, q\rangle = \langle p', q'|e^{-\frac{i}{\hbar}H} \mathbb{P}|p, q\rangle \]

Then we can place a resolution of unity between the projection and time evolution operator. After doing this, the first term becomes the evolution on the full phase space, for which we have already constructed the path integral (2.7). Formally, we have the following modified path integral (see [5] for more details):

\[
\int \exp \left\{ \frac{i}{\hbar} \int p\dot{q} - H(p, q) \right\} \langle p'', q''|\mathbb{P}|p, q\rangle DpDq. \tag{3.25}
\]

By not absorbing the Lagrange multiplier term into the projection operator, we can repeat the same process with the Hamiltonian replaced by the total Hamiltonian operator. The resulting path integral is

\[
\int \exp \left\{ \frac{i}{\hbar} \int p\dot{q} - H(p, q) - \lambda^a \phi_a(p, q) \right\} \langle p'', q''|\mathbb{P}|p, q\rangle DpDq. \tag{3.26}
\]

So even though this path integral appears to still depend on the choice of the functional form Lagrange multiplier, we see that in fact it is equivalent to the path integral without this term (3.25).

Now let us consider a different construction of the path integral. Let us work with only one constraint \( \phi(p, q) = 0 \). We have the projection operator given by

\[
\mathbb{P} = \int e^{-i\tau \hat{\Phi}} d\mu'(\tau). \tag{3.27}
\]

Let \( \tau(t_1, t_2) = \int_{t_1}^{t_2} \lambda(t) dt \). The operators are inherently time independent. So, we have

\[
\mathbb{P} = \int e^{-i \int_{t_1}^{t_2} \lambda(t) \hat{\Phi} dt} d\mu'(\tau(t_1, t_2)). \tag{3.28}
\]

Then using the properties of the projection operator (3.18), we can construct the simple product rule,

\[
\mathbb{P} = \mathbb{P}^2
= \int \int e^{-i \int_{t_1}^{t_2} \lambda(t) \hat{\Phi} dt} e^{-i \int_{t_2}^{t_3} \lambda(t) \hat{\Phi} dt} d\mu'(\tau(t_1, t_2)) d\mu'(\tau(t_2, t_3))
= \int e^{-i \int_{t_1}^{t_3} \lambda(t) \hat{\Phi} dt} d\mu'(\tau(t_1, t_2) + \tau(t_2, t_3))
\]
\[
= \int e^{-i \int_{t_1}^{t_3} \lambda(t) \phi dt} d\mu'(\tau(t_1, t_3)).
\]

This projection operator can be broken into \(N\) time segments as we did in the construction of the path integral (2.7). So we can repeat this construction to include the projection operator. The above constrained propagator (3.24) then can be written in terms of the discrete path integral. Let each time step be given by \(\varepsilon\), then

\[
\int_{t_n}^{t_{n+1}} \lambda(t) dt \approx \varepsilon \lambda_n
\]

and the measure is

\[
d\mu' = d\mu'(\varepsilon \lambda).
\]

The discrete path integral is

\[
\int \ldots \int \prod_{n=1}^{N} d\mu(p_n, q_n) \prod_{n=1}^{N} d\mu'(\varepsilon \lambda_n) \prod_{n=0}^{N} \langle p_{n+1}, q_{n+1} | e^{-\frac{i\varepsilon}{\hbar} H_e^{\varepsilon \lambda_n} \Phi} | p_n, q_n \rangle.
\]

Let us rescale the Lagrange multiplier \(\lambda \rightarrow \lambda/\hbar\). Then we see that have derived the time evolution operator in terms of the total Hamiltonian,

\[
\int \ldots \int \prod_{n=1}^{N} d\mu(p_n, q_n) \prod_{n=1}^{N} d\mu'(\varepsilon \lambda_n) \prod_{n=0}^{N} \langle p_{n+1}, q_{n+1} | e^{-\frac{i\varepsilon}{\hbar} \left( \hat{H} + \lambda_n \Phi \right)} | p_n, q_n \rangle.
\]

In the continuum limit, we wish to replace the ordinary measure above with the well defined Wiener measure. Certainly, we already know to do this for the first measure of the momentum and position, but we would also like to do the same for the Lagrange multiplier measure.

Looking carefully at (3.29), we see that in fact we already have a path integral. We can also see that if we let the measure be defined in terms of the fundamental solution of the diffusion equation (2.9), then we can write the measure as

\[
d\mu'(\tau(t_1, t_2)) = \rho(t_1, t_2) \ d\lambda(t_2)
\]

\[
\rho(t_1, t_2) = \sqrt{\frac{1}{2\pi \nu'(t_1 - t_2)}} \exp \left\{ -\frac{(\lambda(t_1) - \lambda(t_2))^2}{2\nu(t_1 - t_2)} \right\}.
\]

We see that this measure is normalized,

\[
\int d\mu(\tau) = \int \rho(t_1, t_2) d\lambda(t_2) = 1,
\]

and the product rule of the this measure (2.10) is consistent with the above product rule (3.29). Then in the formal limit, we should replace the measure with a Wiener measure. Note however that this Wiener measure is not pinned at both ends, but in fact we should integrate over the end terms. This integration is how the propagator loses its dependence on the Lagrange multiplier. So this measure can be taken into an unpinned Wiener measure. Formally we can write the path integral as
This path integral is discussed in earlier work by the author \[4\] in terms of the semi-classical construction of the path integral.

Now that we have constructed the various forms of the path integral for the constrained system (eqs. 3.25, 3.34), we would like to consider time-reparameterization invariant systems. A large class of time-reparameterization invariant systems may be written in terms of a single constraint, \(H_T = \lambda(\hat{H} - E)\). A rescaling of the time coordinate can be aborted into the definition of the Lagrange multiplier. So \(\lambda(t)\) is just a lapse function.

In terms of the coherent state quantization, the matrix element is quite simple. Because the Hamiltonian is zero there is no “time” evolution on the full phase space, the matrix element (3.24) is then just

\[
\langle p', q'; t | p, q \rangle_{\text{phys}} = \langle p' q' | \mathcal{P} | p, q \rangle = \int d\mu(\tau) \langle p'', q'' | e^{-i\tau \frac{\hat{H} - E}{\hbar}} | p', q' \rangle.
\] (3.35)

Because there is no dependence on the position and momentum through the Hamiltonian, the path integral (3.34) becomes trivial to integrate in this direction. The remaining path integral is just dependent on the Lagrange multiplier. Then we can use the product rule (3.29) to integrate along this direction. The resulting matrix element is the same as above (3.35). Note that the integration variable for this operator should be identified with the proper time \(\tau = \int_{t_1}^{t_2} \lambda(t) dt\). This was first noted by Govaerts \[6\] in his consideration of the free particle case.

Let us now consider two examples of time-reparameterization invariant systems; the single and double harmonic oscillator.

### 4 The single harmonic oscillator

The harmonic oscillator is a natural place to begin the study of time reparameterization invariant systems. In addition to being a simple system to work with it is also the natural setting in which coherent states first appeared\[3\]. In this section, we will compare the recent projection operator approach to standard Dirac and reduced phase space quantization.

We will begin with a quick review of the classical time reparameterization invariant harmonic oscillator. The total Hamiltonian for this model is given by

\[H = \frac{p^2}{2m} + \frac{m}{2} \omega^2 q^2\]

\[\text{where } \omega \text{ is the natural frequency of the harmonic oscillator.}\]

\[H_T = \lambda(\hat{H} - E)\]

\[\lambda(t)\] is just a lapse function.

\[\langle p', q'; t | p, q \rangle_{\text{phys}} = \langle p' q' | \mathcal{P} | p, q \rangle = \int d\mu(\tau) \langle p'', q'' | e^{-i\tau \frac{\hat{H} - E}{\hbar}} | p', q' \rangle.
\] (3.35)
\[ H_T = \lambda \left[ \frac{1}{2}(p^2 + \omega^2 q^2) - E \right]. \quad (4.1) \]

The action for this system is then
\[ L = \int pdq - \int \lambda \left[ \frac{1}{2}(p^2 + \omega^2 q^2) - E \right] dt. \quad (4.2) \]

The equations of motion can easily be solved by defining the proper time \( \tau = \int_0^t \lambda dt \). Then the equations of motion appear as the normal equations of motion for the harmonic oscillator with \( \tau \) replacing the time variable
\[ \frac{dp}{d\tau} = -\omega^2 q \quad \frac{dq}{d\tau} = p. \quad (4.3) \]

The solutions of these equations of motion are
\[ q = A \cos(\omega \tau + \phi) \quad p = A\omega \sin(\omega \tau + \phi). \quad (4.4) \]

In addition the equations of motion we also must satisfy the constraint equation,
\[ \frac{1}{2}(p^2 + \omega^2 q^2) - E = 0. \quad (4.5) \]

Substituting the equations of motion (4.4) into the constraint equation, we can solve for the amplitude,
\[ A = \frac{\sqrt{E}}{\omega}. \quad (4.6) \]

The remaining degree of freedom of this system \( \phi \) is just the gauge degree of freedom. To see this, let \( \lambda \to \lambda + \varepsilon \), then \( \phi \to \phi' = \phi + \varepsilon t + \mathcal{O}(\varepsilon^2) \). So the resulting reduced phase space is just a single point. Quantizing this system is trivial, because there is only one state. Note, however, that the energy \( E \) appears to be arbitrary.

Now, let us look at the Dirac quantization of this system. To begin with, we must find the operator corresponding to the constraint function. It is natural to choose a Hermitian operator. For convenience, we will switch to the complex coordinate \( \alpha = \sqrt{\frac{\omega}{\hbar}}q + i \sqrt{\frac{1}{2\omega\hbar}}p \), and we will replace the momentum and position operators by the standard harmonic oscillator raising and lower operator \( (a, a^\dagger) \). The constraint operator can be written
\[ \hat{\Phi} = \frac{\omega \hbar}{2} (aa^\dagger + a^\dagger a) - E \mathbb{I}. \quad (4.7) \]

Let us define \( E' = E/\omega \hbar - 1/2 \) and rescale \( \lambda \). Then the constraint operator can be written in terms of the number operator \( (a^\dagger a) \),
\[ \hat{\Phi} = a^\dagger a - E' \mathbb{I}. \quad (4.8) \]
Following Dirac quantization, the physical states are defined as the states that are annihilated by the constraint operator. Therefore, the physical state is an eigenstate of the number operator. This also imposes the condition that $E'$ is an integer.

$$a^\dagger a |\Psi\rangle_{\text{phys}} = E'|\Psi\rangle_{\text{phys}} \Rightarrow |\Psi\rangle_{\text{phys}} = |n\rangle, \quad E' = n.$$ (4.9)

The Dirac quantization also leads to the single state $|n\rangle$. However, it imposes the restriction that the energy is quantized $E = \hbar \omega (n + 1/2)$.

We would now like to consider this system in terms of coherent states. We can project the coherent state on the full phase space $|a\rangle$ onto the physical space by using the projection operator $P$.

$$|\alpha\rangle_{\text{phys}} = P|\alpha\rangle = \int e^{i\lambda(a^\dagger a - E')} \delta\lambda \left(e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle\right)$$

$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(\int e^{i\lambda(n-E')} \delta\lambda\right) |n\rangle$$ (4.10)

Using the measure for non-compact groups (3.20), the integral becomes

$$\int e^{i\lambda(n-E')} \left(\frac{2 \sin \varepsilon \lambda}{\pi \lambda}\right) d\lambda = \begin{cases} 1 & |E' - n| < \varepsilon, \\ 1/2 & |E' - n| = \varepsilon, \\ 0 & |E' - n| > \varepsilon. \end{cases}$$ (4.11)

We can choose $\varepsilon$ to be arbitrarily small. Therefore, we see that $E'$ must be arbitrarily close to an integer $m$ otherwise the physical vector is null. So, if we let $E' = m$, we can calculate the above sum (4.10).

$$|\alpha\rangle_{\text{phys}} = e^{-|\alpha|^2/2} \frac{\alpha^m}{\sqrt{m!}} |m\rangle$$ (4.12)

However, this physical state is not yet normalized in the new space. After normalizing, the physical state is the energy eigenstate with a phase factor out in front.

$$|\alpha\rangle_{\text{phys}}' = \frac{|\alpha\rangle_{\text{phys}}}{\langle \alpha | \alpha \rangle_{\text{phys}}} = \frac{\alpha^m}{|\alpha|^m} |m\rangle = e^{im\theta} |m\rangle.$$ (4.13)

This phase factor is obviously irrelevant to the physics of this system, and it is easy to see that, in fact, it is the gauge degree of freedom generated by the constraint (4.5). It is clear that the projection method in this system is equivalent to the Dirac quantization (4.9).

The “evolution” of this system in the reduced phase space is trivial since there is only one state. However, we would like consider the matrix element on the full phase
space so we can compare with the classical solutions. On the full phase space, we can look at the physical state on the full space. Let the physical state be labeled by

\[ |\phi_{\alpha'}\rangle = |\alpha'\rangle_{phys}, \]  

(4.14)

where \( |\alpha'\rangle \) is the state that is normalized in terms of the full phase space (4.12). The matrix element, which is also the wavefunction on the full phase space, is then

\[ \phi_{\alpha'}(\alpha'') = \langle \alpha''|\phi_{\alpha'} \rangle = \langle \alpha''|e^{-|\alpha'|^2/2} \frac{\alpha'^m}{\sqrt{m!}} |m\rangle \]

\[ = e^{-|\alpha''|^2/2} e^{-|\alpha'|^2/2} \frac{(\alpha'\bar{\alpha}'')^m}{m!}. \]  

(4.15)

Let \( \alpha' = r' e^{i\theta'} \) and \( \alpha'' = r'' e^{i\theta''} \), and let us renormalize the wavefunction such that it is approximately one at the peak, \( r'^2 = r''^2 = m \).

\[ \left| \phi_{\alpha'}(\alpha'') \right| = \sqrt{2\pi m} e^{-|\alpha''|^2/2} e^{-|\alpha'|^2/2} \frac{r'^m r''^m}{m!}. \]  

(4.16)

This normalization can be explained in terms of a "gauge fix". We want the phase factor from the initial physical state (4.13) to be one \( e^{im\theta'} = 1 \). Then, normalizing this function (4.16) over phase space, we have

\[ \int \left| \phi_{\alpha'}(\alpha'') \right|^2 | \delta (e^{im\theta'} - 1) | \left( \frac{d\alpha'd\bar{\alpha}'}{\pi} \right) \left( \frac{d\alpha''d\bar{\alpha}''}{\pi} \right) = 1. \]  

(4.17)

So there seems to be a natural choice for the form of the gauge fixing term in this system.

Ordinarily, we would have to construct a set of gauge invariant operators to work on the reduced phase space, but in this system, the gauge orbits are understood to be the phase of the state. Because we know the behavior of the gauge orbits, we can remain on the full phase space and consider the correlation between the physical state and the other coherent states on this space. Some of the important correlation functions of this system are

\[ \langle \alpha''|\hat{H}|\phi_{\alpha'} \rangle = \left( m + \frac{1}{2} \right) \langle \alpha''|\phi_{\alpha'} \rangle \]  

(4.18)

\[ \langle \alpha''|\hat{Q}|\phi_{\alpha'} \rangle = \sqrt{\frac{2\hbar}{\omega}} \left( \frac{m}{\alpha''} + \alpha'' \right) \langle \alpha''|\phi_{\alpha'} \rangle \]  

(4.19)

\[ \langle \alpha''|\hat{P}|\phi_{\alpha'} \rangle = i\sqrt{2\hbar\omega} \left( \frac{m}{\alpha''} - \alpha'' \right) \langle \alpha''|\phi_{\alpha'} \rangle. \]  

(4.20)
Once again these correlations are picked when the classical constraint functions are met, $|\alpha''| = |\alpha'| = m$. Then, the classical limit (expanding about the peak) gives

\[
\langle \alpha'' | \hat{H} | \phi_{\alpha'} \rangle = m + 1/2 + O(\hbar),
\]

(4.21)

\[
\langle \alpha'' | \hat{Q} | \phi_{\alpha'} \rangle = \frac{\sqrt{E}}{\omega} \cos(\theta'' - \theta') + O(\hbar),
\]

(4.22)

\[
\langle \alpha'' | \hat{P} | \phi_{\alpha'} \rangle = \sqrt{E} \sin(\theta'' - \theta') + O(\hbar).
\]

(4.23)

Then, we can identify $\theta'' - \theta' = \omega t + \phi$. The resulting corrections then give us back the classical equations of motion (eqs. 4.4, 4.6).

5 Double harmonic oscillators

Next, we would like to consider a system of two independent but identical harmonic oscillators. In addition to being a non-trivial example of a time-reparameterization invariant system (it still has two degrees of freedom remaining after applying the constraints), the double harmonic oscillator has been of interest in helping to understand the “problem of time” in quantum gravity. One of the oscillators can be thought of as a quantum clock. Then the other oscillator can be written in terms of the “time” that this clock reads (see [8] for more about this system).

In this system, we also encounter a potential Gribov problem. The constraint surface is topologically a three sphere $S^3$. The gauge orbits are topologically equivalent to a circle $S^1$. The resulting reduced phase space is the two sphere $S^2$. However the three sphere is not a trivial bundle over the two sphere, $S^3 \neq S^2 \times S^1$, but rather a twisted bundle. Therefore, we can’t find a global gauge fixing condition [12]. We can find a local gauge fixing and extend it to cover all but a single point of the gauge orbit. How we treat this point will determine the ground state energy for the reduced phase space quantization.

Once again, let us start by considering the classical system. We will choose each of the harmonic oscillators to have the same frequency $\omega_1 = \omega_2 = \omega$. Then the Hamiltonian for the double harmonic oscillator is given by

\[
H_T = \lambda \left( \frac{1}{2}(p_1^2 + \omega^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega^2 q_2^2) - E \right).
\]

(5.1)

The action of this system is given by

\[
S = \int p_1 dq_1 + p_2 dq_2 - \int H_T \, dt.
\]

(5.2)

As in the single harmonic oscillator case (4.3), the equations of motion are easily solved in terms of the proper time $\tau = \int_0^t \lambda(t) \, dt$.
\[
q_1 = A \cos(\omega \tau + \phi) \quad p_1 = A \omega \sin(\omega \tau + \phi)
\]
\[
q_2 = B \cos(\omega \tau + \phi') \quad p_2 = B \omega \sin(\omega \tau + \phi').
\] (5.3)

The constraint equation,
\[
\frac{1}{2}(p_1^2 + \omega^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega^2 q_2^2) = E,
\] (5.4)
limits the amplitudes to
\[
(A \omega)^2 + (B \omega)^2 = E.
\] (5.5)

If \(\lambda \to \lambda + \varepsilon\), the we see that the gauge transformation take both \(\phi \to \phi + \omega \varepsilon t\) and \(\phi' \to \phi' + \omega \varepsilon t\). So the degree of freedom that is independent of the gauge transformation is the difference between the initial phase of the two harmonic oscillator \(\Delta \phi = \phi - \phi'\). The resulting reduced phase space is two dimensional.

We can take for the coordinates on the reduced phase the momentum and position of the first harmonic oscillator. This set of coordinates inherits the symplectic structure from the full phase space \(\{q_1, p_1\} = 1\). However, the metric on this reduced phase space is no longer flat. Let us rescale the the momentum \(p_i \to \sqrt{\omega} \overline{\hbar} p_i\) and the position \(q_i \to \sqrt{\hbar/\omega} q_i\) such that they have the same units. Then the volume of the full phase space is given by
\[
Vol = \int \prod_{j=1}^{M} \left( \frac{dp_j dq_j}{\hbar} \right).
\] (5.6)

We have the standard Cartesian metric,
\[
d\sigma^2 = dq_1^2 + dp_1^2 + dq_2^2 + dp_2^2.
\] (5.7)

To find the metric of the reduced phase space, we will restrict the coordinates to the constraint equation and choose the local gauge fixing term \((\tan^{-1}(p_2/q_2) = \text{constant})\). This gauge fixing is not global because it is ill defined at \(p_2 = q_2 = 0\). In order to find the induced metric on the reduced phase space, it is easier to work in a set of two polar coordinates,
\[
q_1 = \frac{p_1}{r_1} \quad \theta_1 = \tan^{-1} \left( \frac{p_1}{q_1} \right)
\]
\[
q_2 = \frac{p_2}{r_2} \quad \theta_2 = \tan^{-1} \left( \frac{p_2}{q_2} \right)
\] (5.8)

The constraint equation (5.4), after our rescaling and change of coordinates, looks like
\[ S^2 = r_1^2 + r_2^2 = \frac{2E}{\omega \hbar}. \]  
(5.9)

After applying the constraint \( r_1dr_1 = -r_2dr_2 \) and the gauge fixing \( d\theta_2 = 0 \), the metric on the reduced phase space becomes
\[
d\sigma^2 = \left(1 - \frac{r_1^2}{S^2}\right)^{-1} dr_1^2 + r_1^2 d\theta_1^2. \]  
(5.10)

We see that this metric is a constant curvature metric \( R = 2/S^2 \). The metric is ill defined at \( r_1 = S \), which is also the same place that the gauge fixing term is ill defined \( (r_2 = p_2 = q_2 = 0) \).

We can use this induced metric on the reduced phase space to tell us a bit more about the system. Let us follow a similar system that Klauder discussed \[4\]. On the two sphere the total surface area must be quantized in order that the term \( \exp(i \oint pdq) \) for a closed path be unambiguous. Note, we can not include the possibility that \( S = 0 \) because the metric (and the gauge fixing) are ill defined. Hence
\[
2\pi n = \int dp \wedge dq = \int \sqrt{g} dpdq = \pi S^2 \quad n = 1, 2, 3, \ldots \]  
(5.11)

This implies that the energy is also quantized \( E = \hbar \omega n \) where \( n = 1, 2, 3, \ldots \). The reduced Hamiltonian is just zero so the resulting propagator on the reduced phase space depends only on the Weiner measure. We can see this in the path integral
\[
\mathcal{N} \int Dp_1 Dq_1 e^{-\frac{1}{2\nu} \int \frac{d^2\sigma'}{\alpha^2} e^{i \int p_1 dq_1}. \]  
(5.12)

Such a Weiner measure gives rise to spin-like kinematical operators \( S_i \) where \([S_i, S_j] = i\epsilon_{ijk}S_k \) \[3\]. We will not work through the details of the resulting spin system here because the details can be seen when we consider the Dirac quantization.

Now we would like to consider the Dirac quantization of this system (see \[5\] for a similar discussion). Let us begin by constructing the projection operator \((3.18)\). Similar to the single harmonic oscillator \((4.3)\) the constraint operator can be defined in terms of the raising and lowering operators for the independent oscillators.
\[
\hat{\Phi} = a^\dagger a + b^\dagger b - E', \quad E' = E/\omega \hbar - 1. \]  
(5.13)

Because each oscillator is independent (before the constraint is applied), the double harmonics oscillator has a complete set of vectors that is just the direct product of the eigenvalue of each of these number operators,
\[ |m, n\rangle = |m\rangle \otimes |n\rangle. \]  
(5.14)

\[ a^\dagger a|m, n\rangle = m|m, n\rangle, \quad b^\dagger b|m, n\rangle = n|m, n\rangle. \]  
(5.15)

The constraint on this basis then quantizes the energy \( E = \hbar \omega (m + n + 1) \). In terms of the Dirac quantization the energy would have to be an integer \( (E = m' = 1, 2, 3, \ldots) \). The resulting physical states would be given by

\[ |\Psi\rangle_{phys} = |n, m' - n\rangle. \]  
(5.16)

On this set of states, the raising and lowering operators form above (5.14) are not defined, in that \( (a|\Phi\rangle_{phys}) \) is not a physical state. So, we need to find another set of operators that are defined on this set of states. These operators are equivalent to the spin operators.

For convenience, let us continue to use the scaled momentum and position (5.6). In terms of these coordinates, we can define a new set of coordinates which have zero Poisson brackets with the constraint \( \{s_i, \phi\} = 0 \). They are

\[ s_1 = \frac{1}{2}(p_1 p_2 + q_1 q_2), \]  
(5.17)

\[ s_2 = \frac{1}{2}(p_2 q_1 - p_1 q_2), \]  
(5.18)

\[ s_3 = \frac{1}{4}(p_1^2 + q_1^2 - p_2^2 - q_2^2). \]  
(5.19)

This set of coordinates possess the standard \( SO(3) \) Lie algebra, \( \{s_i, s_j\} = \epsilon_{ijk}s_k \). The square of these three coordinates is the constraint surface radius

\[ s_1^2 + s_2^2 + s_3^2 = s_0^2 = \frac{1}{4}S^2. \]  
(5.20)

Because the coordinates have a zero Poisson bracket with the constraint, this set of coordinates is gauge invariant. However, they are not all linearly independent, so let the reduced phase space be described by \( s_1, s_2 \). The induced metric is flat \( d\sigma'^2 = s_0^2(d s_1^2 + ds_2^2) \), and the domain is just a disk \( (s_1^2 + s_2^2 \leq s_0^2) \).

We can write a set of operators that correspond to the classical coordinates above, eqs. (5.17) - (5.20), that preserves the \( SO(3) \) algebra in terms terms of our raising and lowering operators (5.14).

\[ \hat{S}_1 = \frac{1}{2}(a^\dagger b + a^\dagger b), \quad \hat{S}_2 = \frac{i}{2}(a^\dagger b - a^\dagger b) \]

\[ \hat{S}_3 = \frac{1}{2}(a^\dagger a - b^\dagger b), \quad \hat{S}_0 = \frac{1}{2}(a^\dagger a + b^\dagger b) \]  
(5.21)
Then, it is possible to map the physical states given above (5.16) onto the set of angular momentum eigenstates.

\[ \hat{S}_2 |n, m' - n \rangle = \frac{1}{4} m'(m' + 1) |n, m' - n \rangle \equiv j(j + 1) |j, m \rangle, \quad (5.22) \]

\[ \hat{S}_3 |n, m' - n \rangle = \frac{1}{2} (2n - m') |n, m' - n \rangle \equiv m |j, m \rangle. \quad (5.23) \]

This means that \( j = 2m' \) and \( m = n - j \). The raising and lowering operators for the angular momentum \( \hat{S}_\pm = \hat{S}_1 \pm i \hat{S}_2 \) act on this set of states in the normal way,

\[ \hat{S}_+ |j, m \rangle = \sqrt{(j - m)(j + m + 1)} |j, m + 1 \rangle \quad (5.24) \]

\[ \hat{S}_- |j, m \rangle = \sqrt{(j + m)(j - m + 1)} |j, m - 1 \rangle. \quad (5.25) \]

We can now construct the \( SO(3) \) coherent states form these operators (5.24) (see [16] for the details about this coherent state). The coherent state is then

\[ |\xi \rangle = \exp \left( \xi \hat{S}_+ - \bar{\xi} \hat{S}_- \right) |\eta \rangle. \quad (5.26) \]

The let us choose the lowest weight vector from above (5.24) as our fiducial vector \(|\eta \rangle = |j, -j \rangle\). Then we can rewrite the above coherent state representation (5.26) in terms of the above basis vectors (5.22).

\[ |\xi \rangle = (1 + |\xi|^2)^{-j} \sum_{m=-j}^{j} \sqrt{\frac{2j!}{(j+m)!(j-m)!}} \xi^{j+m} |j, m \rangle. \quad (5.27) \]

This coherent state has a resolution of unity,

\[ \mathbb{I} = \int \frac{(2j + 1)}{\pi} \frac{d\xi d\bar{\xi}}{(1 + |\xi|^2)^2} |\xi \rangle \langle \xi|. \quad (5.28) \]

With this resolution of unity, we can construct the path integral. In the continuum limit, this path integral appears as

\[ \mathcal{N} \int d\mu_W \exp \left\{ i \hbar \int \frac{j}{(1 + |\xi|^2)^2} \left( d\xi \hat{\xi} - \xi d\bar{\xi} \right) \right\}. \quad (5.29) \]

The Weiner measure for this system was described in [3]. We see that in fact the reduced phase and the Dirac quantization result in a spin system, where the energy of the system is mapped onto the total angular momentum.

On first appearances would seem that the reduced phase space and Dirac quantization lead to the same results. In both cases the the energy is quantized \( E = \omega \hbar (n + 1) \) where \( n = 0, 1, 2, \ldots \) (see eqs. 5.11, 5.16). However, this result is dependent on the
fact that we removed the point where the gauge fixing went bad from the reduced phase space. It is possible to include this point by using two coordinate patches instead of one, where each coordinate patch has its own gauge fixing. Then, we can map this gauge fixing across the boundary. In so doing, we can include the zero energy term in the reduced phase space (5.11). The result is that the two systems then have different ground state energies [15].

In the case of constrained coherent state path integral, we will end up integrating over the gauge orbits in effect averaging over all possible gauge orbits. Because of this, we will not have to fix a gauge and we will not encounter this Gribov problems.

Let us work through the projection operator approach to the constrained coherent states for this system. Extending the single state oscillator, the coherent state for the double harmonic oscillator can be written as

$$|\alpha, \beta\rangle = e^{-|\alpha|^2/2 - |\beta|^2/2} \sum_{m,n} \frac{1}{\sqrt{n!} \sqrt{m!}} \alpha^m \beta^n |m, n\rangle.$$  (5.30)

Then we can project this on the physical states.

$$|\alpha, \beta\rangle_{phys} = \int e^{i\lambda \hat{\Phi}} d\mu(\lambda) \left( e^{-|\alpha|^2/2 - |\beta|^2/2} \sum_{m,n} \frac{1}{\sqrt{n!} \sqrt{m!}} \alpha^m \beta^n \left( \int e^{i\lambda(n+m-E)} d\mu(\lambda) \right) |m, n\rangle \right).$$  (5.31)

We will again choose Klauder’s measure for non-compact groups (3.20) for the measure for this projection. Then similar to the single harmonic oscillator (4.11), the physical vector is null unless $E$ is arbitrarily close to an integer. So let $E = m' = m + n$. Then the physical vector is given by

$$|\alpha, \beta\rangle_{phys} = e^{-|\alpha|^2/2 - |\beta|^2/2} \sum_{n=0}^{m'} \sqrt{\frac{1}{n!(m' - n)!}} \alpha^n \beta^{m' - n} |n, m' - n\rangle.$$  (5.32)

Now we wish to normalize the physical vector. It is just a quick calculation to show that the normalized physical vector is

$$|\alpha, \beta\rangle_{phys} = \left(|\alpha|^2 + |\beta|^2\right)^{-m'/2} \sum_{n=0}^{m'} \sqrt{\frac{m'!}{n!(m' - n)!}} \alpha^n \beta^{m' - n} |n, m' - n\rangle.$$  (5.33)

It is easy to see that the gauge transformation generated by the constraint is $\alpha \rightarrow \alpha e^{i\theta}$ and $\beta \rightarrow \beta e^{i\theta}$. Like the single harmonic oscillator (4.13) this gauge transformation appears as an overall phase in front of the physical vector

$$|\alpha, \beta\rangle_{phys} \rightarrow e^{im'\theta}|\alpha, \beta\rangle_{phys}.$$  (5.34)
To remove the gauge dependence, let us then define $\xi = \alpha / \beta$. Writing the physical state in terms of this variable, we can factor out the gauge transformations which appear again as phase factor (4.13).

$$\langle \alpha, \beta \rangle_{\text{phys}} = \left( \frac{\beta}{|\beta|} \right)^{m'} \left( 1 + \frac{\alpha}{|\beta|} \right)^{-\frac{m'}{2}} \sum_{n=0}^{m'} \sqrt{\frac{m'}{n!(m'-n)!}} \left( \frac{\alpha}{\beta} \right)^n |n, m' - n\rangle$$

$$= e^{im'\theta} \left( 1 + |\xi|^2 \right)^{-\frac{m'}{2}} \sum_{n=0}^{m'} \sqrt{\frac{m'}{n!(m'-n)!}} \xi^n |n, m' - n\rangle. \quad (5.35)$$

It is easy to see that the physical coherent state maps onto the $SO(3)$ coherent state (5.27). The energy is mapped onto total angular momentum $j = 2E'$.

$$|\xi\rangle = (1 + |\xi|^2)^{-\frac{j}{2}} \sum_{m=-j}^{j} \sqrt{\frac{2j!}{(j+m)!(j-m)!}} \xi^{j+m} |j, m\rangle \quad (5.36)$$

The resulting reduced phase space agrees with the information that we were able to discern from the reduced phase space coherent state discussed earlier (5.12). It is clearly a spin system with the total angular momentum given by the the energy $(2j = E')$.

The propagator for this system is simply the overlap function of the $SO(3)$ coherent state,

$$\langle \xi' | \xi \rangle \equiv \langle \alpha', \beta' | \mathbb{P} | \alpha \beta \rangle$$

$$= (1 + |\xi'|^2)^{-j} (1 + |\xi|^2)^{-j} \xi^j \xi^j. \quad (5.37)$$

At any given “time”, the coherent state gives a minimum uncertainty wave packet $\langle J_1^2 \rangle \langle J_2^2 \rangle = 1/4 \langle J_0 \rangle^2$. The most probable matrix element (the classical solution) is simply $\xi' = \xi$. Any expectation value of the spin operators (5.21) is simple enough to calculate as well. From these expectation value the classical “dynamics” of the reduced phase space can be can deduced.

Returning to the full phase space, we would like to reconstruct the classical equations of motion as we did in the single oscillator case (4.21). Using the non-normalized physical vector (5.32), we can define the physical state on the full phase space as

$$|\phi_{\alpha', \beta'}\rangle = \mathbb{P} |\alpha', \beta'\rangle \quad (5.38)$$

The wave function of this state on the phase space is again the overlap function,
The energy for each of the oscillators in the classical limit is

$$\phi_{\alpha'\beta'}(\alpha'', \beta'') = \langle \alpha'', \beta'' | \phi_{\alpha'\beta'} \rangle$$

$$= e^{-\frac{1}{2}(|\alpha''|^2 + |\beta''|^2 + |\alpha'|^2 + |\beta'|^2) \frac{(\bar{\alpha}' + \bar{\beta}' \cdot \beta'' \cdot \alpha'')^m}{m!}}$$  (5.39)

We can renormalize the state such that at its peak, it is approximately one, as we did in the single oscillator case. Let \( \alpha' = r'e^{i\theta'} \), \( \beta' = \rho' e^{i\phi'} \), etc. Then,

$$\left| \langle \alpha'', \beta'' | \phi_{\alpha'\beta'} \rangle \right| = \sqrt{2\pi m} e^{-\frac{1}{2}(r''^2 + \rho''^2 + r'^2 + \rho'') \frac{(r''^2 + \rho''^2 + 2r'\rho'' \cos(\theta))^m}{m!}}$$  (5.40)

where \( \theta = \theta' - \theta'' - \phi' + \phi'' \). The peak of this function is at \( \theta = 0, r'' = \rho'' \), and \( r'^2 + \rho'^2 = m \). Note that \( \theta = 0 \) implies that the states evolve together as they do in the classical equation of motion (5.3). The correlation functions of the position and momentum are give by

$$\langle \alpha'', \beta'' | \hat{Q}_1 | \phi_{\alpha'\beta'} \rangle = i\sqrt{2\pi \alpha''} \left( \frac{m\alpha'}{(\alpha''_{\alpha'} + \beta''_{\beta'})} - \bar{\alpha}'' \right) \langle \alpha'', \beta'' | \phi_{\alpha'\beta'} \rangle;$$  (5.41)

$$\langle \alpha'', \beta'' | \hat{P}_1 | \phi_{\alpha'\beta'} \rangle = i\sqrt{2\pi \beta''} \left( \frac{m\beta'}{(\alpha''_{\alpha'} + \beta''_{\beta'})} + \bar{\beta}'' \right) \langle \alpha'', \beta'' | \phi_{\alpha'\beta'} \rangle;$$  (5.42)

Likewise for the position and momentum of the second oscillator. Expanding about the peak, we have

$$\langle \alpha'', \beta'' | \hat{Q}_2 | \phi_{\alpha'\beta'} \rangle = \frac{r' \cos(\theta' - \theta'')}{\omega} + O(h)$$  (5.43)

$$\langle \alpha'', \beta'' | \hat{P}_2 | \phi_{\alpha'\beta'} \rangle = \frac{\rho \sin(\theta' - \theta'')}{\omega} + O(h)$$  (5.44)

$$\langle \alpha'', \beta'' | \hat{E}_2 | \phi_{\alpha'\beta'} \rangle = \frac{\rho \sin(\phi' - \phi'')}{\omega} + O(h)$$  (5.45)

The energy for each of the oscillators in the classical limit is

$$\langle \phi_{\alpha'\beta'} | \hat{E}_1 | \phi_{\alpha'\beta'} \rangle = r'^2 + O(h), \quad \langle \phi_{\alpha'\beta'} | \hat{E}_2 | \phi_{\alpha'\beta'} \rangle = \rho'^2 + O(h)$$

$$\langle \phi_{\alpha'\beta'} | E_{\text{total}} | \phi_{\alpha'\beta'} \rangle = m' = r'^2 + \rho'^2 + O(h)$$  (5.47)

So, we see that we meet the constraint equation (5.3), and we get back the classical equations of motion (5.3) in the classical limit.
6 Discussion

It is interesting to note that the gauge degree of freedom for both of these systems comes out in a phase factor in front of the reduced phase space coherent state. Although this might just be an artifact of the harmonic oscillator(s), it is suggestive that the gauge should appear in this way in general. If this is the case, the only dependence on the gauge orbits in the path integral will appear in the one form,

$$\langle p, q | \frac{d}{dt} | p, q \rangle dt = pdq$$

$$\langle p, q | e^{-iJ(p, q)} \frac{d}{dt} e^{iJ(p, q)} | p, q \rangle dt = pdq + df$$ (6.1)

If there is not a boundary, this difference can just be integrated out as a total derivative. If there is a boundary, we pick up a boundary term that is still dependent on the gauge orbits. For example, such a gauge symmetry break term is seen in the relationship between the Chern-Simons actions and the Wess-Zumino-Witten action.

It should be noted that, the above correlations functions (4.18) and (5.41) are not physical observables. This correlations are still dependent on the gauge degree of freedom. However, they do show that the equations of motions are still embedded in the formulation of the coherent state on the reduced phase space. In terms of the double harmonics oscillator, it is interesting to note that the width of the correlation function in the angular direction is dependent on the energy of each oscillators,

$$\sigma \sim \frac{E_{total}}{E_1 E_2}$$ (6.2)

In terms of “quantum clocks,” this means that at low energies, the correlations between this clocks may become fuzzy. Certainly, a more precise statement in terms of observables needs to be considered.

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