Flow structures and shear-stress predictions in the turbulent channel flow over an anisotropic porous wall

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Abstract. This article identifies the main coherent structures driving the flow dynamics in the turbulent channel flow over anisotropic porous walls. Two different cases have been analyzed where the drag increases or decreases with respect to a channel with isotropic porous walls. Higher order dynamic mode decomposition (HODMD) is applied to analyze these data, identifying 20 and 15 high amplitude modes in the drag increasing (DI) and drag reducing (DR) cases, respectively, which well reflects the largest flow complexity in the former case. The frequency of 13 modes and the three-dimensional structure of the modes are similar in the DR and DI cases, suggesting the need of using more complex analyses to deepen our physical insight of these flows. The spatio-temporal HODMD analysis identifies a periodic solution along the spanwise direction (as imposed by the boundary conditions). The wavenumbers related to the modes with highest amplitude are \( \beta = 0 \) and \( \beta = 3 \) (\( L_z = \frac{2\pi}{3} \)). The rollers, groups of spanwise correlated structures, are mostly identified in the DI case near the wall, with \( \beta = 0 \), while the presence of the streaks, streamwise correlated structures are mostly identified in the DR case. Although, in areas far away from the wall it is possible to identify these two types of structures with \( \beta = 3 \) in both cases, depending on the temporal frequency of the DMD modes, the rollers and the streaks are related to high and low frequency DMD modes, respectively. Finally, a model is constructed to predict the temporal evolution of the wall shear, using the 6 most relevant DMD modes interacting near the channel wall: 6 low frequency modes for DR and 3 low and 3 high frequency modes for DI. In the DR case the wall shear is predicted for almost 300 time units with relative error \( \sim 2\% \), however, this error is larger in the DI case, \( \sim 6\% \), suggesting the need of using a larger number of modes to represent this more complex flow.

1. Introduction

Wall-bounded turbulent flows usually exert a much higher wall friction than laminar ones and many researchers have studied the flow over complex surfaces, especially over porous walls which are commonly found in many industrial and natural flows. Most of the past research focused on isotropic porous surfaces whose main effect is the destabilisation of the mean flow and the enhancement of the Reynolds shear stresses with a consequent increase in skin-friction drag in wall-bounded turbulent flows [1, 2]. Only recently anisotropic permeable coatings started to receive some attention: in particular, the linear stability analyses [3] have shown that porous walls characterised by an anisotropic permeability allow for an effective passive manipulation of the near-wall turbulent flow since one can either obtain a significant drag increase or decrease.
by tailoring the directionality of the permeability tensor, as later on verified by Rosti et al. [4] by means of direct numerical simulations. More recently, Gómez-de-Segura et al. [5] performed additional direct numerical simulations to study in detail the evolution of the turbulent flow and the drag reduction ability when the permeability of the flow varies and also these authors reported that increasing the wall-normal permeability the drag increases. The drag-increasing mechanism is related to the presence of some spanwise elongated structures, called as rollers, which are related to a KH-like instability that are so often reported over permeable substrates [6]. In previous work, [5, 7, 8, 9, 10] identified the onset of this KH-instability using local linear stability analysis.

The main goal of this work is to identify the most relevant flow structures related to the evolution of a turbulent flow over anisotropic porous walls, and relate those to the case of an isotropic porous wall in two different cases. In the first case, when the permeability along the streamwise direction is increased, leading to drag reduction, and in the second case, when the permeability along the wall-normal direction is increased, leading to drag increase. In the present work we will use a data-driven method, higher order dynamic mode decomposition (HODMD) [11], to identify the main mechanism triggering the drag increase and drag reduction capacity, using the database described in Ref. [4]. In contrast to [5], we cannot clearly see the spanwise correlated structures near the wall of the channel, which makes this research more challenging, although we expect also to find the rollers near the channel wall.

This article is organized as follows. Section 2 presents the methodology used for identifying the main flow structures. Section 3 introduces the database of the porous wall and the main results obtained from this analysis. Section 4 briefly explains how to construct a model for the porous wall using the main flow structures, and uses the model to predict wall shear in time. Finally, section 5 presents the main conclusions.

2. Methodology: temporal and spatio-temporal higher order dynamic mode decomposition

Higher order dynamic mode decomposition (HODMD)[11] is an extension of dynamic mode decomposition (DMD) [12] that has been introduced to analyze flow structures in complex chaotic flows [13], owing to the improved performance in the case of highly noisy configurations. Similarly to DMD, HODMD approximates the Koopman modes [14] identifying the main frequencies and growth rates of the modes driving the flow.

HODMD is a data-driven method that decomposes the original data \(v(x, y, z, t_k)\) (snapshot) as an expansion of DMD modes in the following way

\[
v(x, y, z, t_k) \approx \sum_{m=1}^{M} a_m u_m(x, y, z)e^{(\delta_m+i\omega_m)t_k}, \quad k = 1, \ldots, K, \tag{1}\]

where \(u_m(x, y, z)\), \(\omega_m\), \(\delta_m\) and \(a_m\) are the DMD modes and their associated frequencies, growth rates and amplitudes, respectively, evolving in time \(t\).

The main HODMD algorithm can be summarized in two steps. In the first step, standard singular value decomposition (SVD) or higher order singular value decomposition (HOSVD) [15] is applied to the original data collected in a snapshot matrix or in tensor form, respectively. The data are equidistant in time. At this step, the spatial redundancies, and the noise, is removed by means of a tunable tolerance, \(\varepsilon_1\). This tolerance reduce the spatial complexity of the data from a total number of \(J\) grid points to a few \(N\) singular values \(\sigma\), identified as \(\sigma_i/\sigma_1 > \varepsilon_1\), for \(i = 1, \ldots, N\) (\(\sigma_{N+1}/\sigma_1 < \varepsilon_1\)). The second step applies the DMD-d algorithm, which combines DMD with time delayed snapshots. A second tolerance \(\varepsilon_2\) determines the number \(M\) of DMD modes retained in the expansion (1), defined by their amplitude as \(a_i/a_1 > \varepsilon_2\), for \(M = i, \ldots, M\) (\(a_{M+1}/a_1 < \varepsilon_2\)).
Spatio-temporal HODMD, also known as sequential Spatio-Temporal Koopman Decomposition [16], decomposes the data as an expansion of modes that oscillate in both time and space, providing relevant information about the frequencies and wavenumbers characterising the flow. The algorithm is relatively simple. Starting from the DMD expansion (1), this method applies HODMD over the DMD modes along one spatial direction (i.e., spanwise $z$), as

$$u_m(x, y, z) \simeq \sum_{n=1}^{N} \hat{a}_{mn} u_{mn}(x, y) e^{(\nu_{mn} + i\beta_{mn})z},$$

where $u_{mn}(x, y)$, $\beta_{mn}$, $\nu_{mn}$ and $\hat{a}_{mn}$ are the spatio temporal modes and their corresponding wavenumbers, spatial growth rates and amplitudes.

Introducing (2) into (1) and defining $a_{mn} = a_m \hat{a}_{mn}$ it is possible to obtain the following spatio-temporal DMD expansion

$$v(x, y, z, t) \simeq \sum_{m=1}^{M} a_{mn} v_{mn}(x, y) e^{(\delta_m + i\omega_m)t_k + (\nu_{mn} + i\beta_{mn})z}, \quad k = 1, \ldots, K.$$

The data analyzed in this article are fully periodic along the spanwise direction, thus the spatial growth rate should be exactly equal to zero (although due to numerical errors this value is not exact, but very close to zero, $\nu_{mn} \simeq 0$). Moreover, this DMD spatial analysis is equivalent to perform a Fourier decomposition (expansion of periodic modes). Nevertheless, for simplicity, we have chosen using the spatio-temporal HODMD method to analyze the data presented.

3. Flow structures in the anisotropic porous wall
The main flow structures driving the flow near the wall in a turbulent channel flow with an anisotropic porous wall have been identified using temporal and spatio-temporal HODMD. The two different numerical databases presented in Ref. [4] have been analyzed. These databases consider a drag increasing (DI) and drag reducing (DR) case, which are compared with an isotropic porous wall. In DI, the permeability along the wall normal direction is higher than the one along the streamwise and spanwise direction ($\sigma_y = 4$ and $\sigma_{xz} = 0.25$ with $\sigma$ being the square root of the permeability divided by the channel half height) while in DR the permeability along the wall normal direction is lower than the one along the streamwise and spanwise direction ($\sigma_y = 0.0625$ and $\sigma_{xz} = 16$). The computational domain consists of a box with dimensions $L_x = 4\pi h$ and $L_z = 2\pi h$ along the streamwise and spanwise directions, respectively, and $2.4h$ along the wall normal direction, defined in the interval $y/h \in [-0.2, 2.2]$, with $0.4/2 = 0.2$ being the thickness of the porous wall at the top and bottom channel walls. Thus, the pure flow is defined in the area $y/h \in [0, 2]$. Periodic boundary conditions are imposed among the inlet and outlet surfaces of the domain (streamwise direction) and on the two sides of the domain (spanwise direction), using a Fourier decomposition. The bulk Reynolds number, defined as $Re = U h/\nu$, with $U$ the bulk streamwise velocity, $h$ the channel half-height and $\nu$ the kinematic viscosity of the fluid, is $Re = 2800$ corresponding to a frictional Reynolds number of $Re = 180$. The porosity is maintained constant as $\epsilon = 0.6$. The flow within the porous layers is simulated using the so called Volume Averaged Navier–Stokes equations [17] and details about the numerical simulations and the code can be found in [18, 4]. Figure 1 shows the computational domain (top) and compares the streamwise velocity in the isotropic porous wall with the DR and DI cases (bottom). Starting from the streaks identified in the isotropic case, we can see that their size increases in the DR case, where the complexity of the flow decreases, while in the DI case these structures break down, forming less organized patterns.

Before applying HODMD, it is necessary to set the proper parameters to calibrate the method. These parameters are set to ensure robustness of the results: we obtain the same DMD modes
Figure 1. Top: computational domain in the channel flow with a porous wall. Bottom: Streamwise velocity in a representative plane XZ extracted near the wall. Isotropic porous wall (left), drag reduction (middle) and drag increasing (right) cases.

Figure 2. Frequencies vs. Amplitudes calculated with the temporal HODMD in a turbulent channel flow with anisotropic porous wall in a drag increasing (DI) and drag reduction (DR) case. Dashed circles identify the modes with similar frequencies.

using different values of $d$ and tolerances $\varepsilon_1$ and $\varepsilon_2$. Due to the high complexity of these data, we use the iterative version of the method: HODMD is applied recursively over a set of data represented by the reconstruction of the original signal until the number of DMD modes retained in two consecutive iterations is the same. For a database composed by 50 snapshots, equi-distant in time $\Delta t = 5$ time units, a good choice of parameters is $d = 18$ and $\varepsilon_1 = \varepsilon_2 = 10^{-3}$. Figure 2 shows the frequencies vs. amplitudes of the DMD modes retained with amplitude larger than 20% (considering the highest amplitude mode as 100%) for the drag increasing and drag reduction case. As expected, the number of modes retained in the DI case (20 modes) is larger than in the DR case (15 modes), since the flow complexity is larger in the former case. It is also noteworthy that the frequency in 13 of a total of 15 modes of the DR case is similar to the DI case, suggesting that the frequency of the DMD modes driving the flow in an anisotropic porous wall does not depend on the permeability of the wall. Also the shape of these temporal modes (not shown for the sake of brevity) is quite similar in both cases and it is possible to identify the streaks near the channel wall in all the DMD modes. Consequently, it is necessary to carry out a spatio-temporal analysis to get a deeper physical insight on these dominant flow structures.

The spatio-temporal HODMD analysis has therefore been carried out along the spanwise direction. Since the flow is periodic along this direction (imposed), the minimum wavenumber expected is $\beta_{\text{min}} = \frac{2\pi}{L_z} \simeq 1$ (wavelength $L_z = 2\pi$). Since the flow analyzed is fully non-linear, we also obtain the multiple harmonics of this mode. Figure 3 shows the variations of the amplitude as function of the spanwise wavenumber for six representative temporal modes (3 low frequency and 3 high frequency modes). As seen, the amplitude of the spatio-temporal modes decreases...
increasing the wavenumber. The highest amplitude mode is the mode with $\beta = 0$ in all cases. This mode is followed by the 3rd harmonic ($\beta = 3$) in the DI case and the 3rd or 6th ($\beta = 6$) harmonic in the DR case. To study the differences between these two cases, we focus on the modes with $\beta = 0$ and $\beta = 3$ ($L_z = \frac{2\pi}{3}$).

Figure 4 shows the module of the streamwise component of the spatio-temporal DMD modes extracted at some representative XY and XZ planes in the DR and DI cases. Two representative modes have been selected, a low frequency mode with $\omega \simeq 0.019$ and a high frequency mode with $\omega \simeq 0.44$. A similar behaviour is found for modes with frequencies below and under $\omega = 0.2$, (not shown here for the sake of brevity). In other words, we divide all the modes identified in Figure 2 as low ($\omega < 0.2$) and high ($\omega > 0.2$) frequency modes. In both cases we can see structures correlated both in the spanwise and streamwise direction, since the flow is periodic in these directions. However, as expected from previous studies, we report a significant presence of the streaks at the interface in the DR case. The size of the streaks, as identified by the streamwise velocity component, decreases at positions further away from the wall (as expected), since they are breaking down. In the DI case, on the contrary, it is not possible to distinguish the streaks at the interface. We can see groups of structures correlated across the spanwise direction: these are the rollers. The presence of these structures is visible at positions further away from the near-wall layer in the high frequency mode. However, it is also possible to distinguish some structures correlated in the streamwise direction, which are even more evident in the low frequency mode. This analysis suggests two main conclusions: (i) low frequency modes are related to the streaks, and high frequency modes are related to the rollers, both type of structures could be present in the turbulent channel flow over a porous wall; (ii) the origin of the flow instability producing the DI is at the interface of the porous wall, as reported in Ref. [3]. The increase in the wall-normal permeability of the porous wall triggers the presence of the rollers, that are either maintained in regions far away from the wall or that evolve in structures streamwise correlated (streaks) (see also Ref. [1]).

A three-dimensional representation of the previous modes is reported in Figure 5. In the DI case we can identify the rollers with $\beta = 0$ in both modes. In the low frequency mode, the mode with $\beta = 3$ interacts with the rollers. This interaction could be related with the presence of
Figure 4. Module of the streamwise component of the spatio-temporal DMD modes calculated along the spanwise direction with frequency $\omega \simeq 0.019$ and $\omega \simeq 0.44$. Reconstruction with $\beta = 3$ ($L_z = \frac{2\pi}{3}$). XY plane extracted at $Z = L_z/2$ and XZ plane extracted at three different positions. Top: DR (two planes). Bottom: DI (two planes).

Figure 5. Three-dimensional reconstruction of the spatio-temporal DMD modes calculated along the spanwise direction. Iso-surface of the streamwise component of the module of the DMD modes with $\beta = 0$ in grey and $\beta = 3$ in blue. DMD modes with frequency $\omega \simeq 0.19$ (left) and $\omega \simeq 0.44$ (right). Top: DR. Bottom: DI.
streamwise correlated structures previously identified in the middle of the channel, where the rollers are not that evident. On the contrary, in the high frequency mode, the mode with $\beta = 3$ is organized forming spanwise correlated structures. This fact justifies the three-dimensional character of the rollers. In the DR case the mode with $\beta = 0$ is near the channel wall in the low frequency mode, while in the high frequency mode it is possible to distinguish some spanwise correlated structures in the middle of the channel. Thus, the low frequency mode is clearly representing the streaks.

Finally, one should remark the traveling character of these two modes. The real and imaginary component of the modes with $\beta = 3$, combined with the mode with $\beta = 0$, are presented in Figure 6. In both cases, it is possible to distinguish localized wavepackages, which are travelling along the streamwise direction, as revealed by the differences found between the real and imaginary components of the modes.

4. Temporal predictions of the shear stress

This section presents a model, based on the modes presented above, to predict the shear at the wall of the channel in the DR and DI cases. As first step, HODMD is applied in the area close to the wall, for $y \in [-0.1h, 0.1h]$ to identify the DMD modes driving the flow motion in this area. Figure 7 shows the 6 modes selected for the parameters $d = 18$ and $\varepsilon_1 = \varepsilon_2 = 10^{-3}$. As expected, most of the modes selected in the DR case are low frequency modes, in good agreement with the major influence of the streaks in this flow. On the contrary, in the DI case the method selects 3 high frequency and 3 low frequency modes, the former related with the rollers connected with the drag increasing mechanism and the latter connected with the presence of the streaks in areas further away from the wall, as explained in the previous section.

The wall shear is modelled as $\tau_{app} = \mu \frac{\partial u_{app}}{\partial y} - u_{app}v_{app}$, where $\mu$ is the fluid viscosity and $u_{app}$ and $v_{app}$ are the reconstruction of the original field using the 6 modes selected in the DMD expansion (1). A model to study the temporal evolution of the wall shear, defined as $\frac{d\tau}{dt}$, is
constructed subject to the following constriction

$$\min |\frac{d\tau}{dt} - \frac{d\tau_{\text{app}}}{dt}|,$$

where \(\tau\) is the wall shear obtained from the real velocity values of \(u\) and \(v\). This problem is solved by minimization of the least-squares-error in the approximation

$$\tau^k \simeq w_k \tau^k_{\text{app}}, \quad k = 1, \ldots, K'$$

(4)

where \(w_k\) are the coefficients (unknown) weighting the influence of \(\tau_{\text{app}}\) for each time step \(k\).

Summarizing the methodology, (i) the DMD expansion (1) is used to reconstruct \(K'\) snapshots of \(u_{\text{app}}\) and \(v_{\text{app}}\) using the 6 DMD modes previously selected; (ii) these two variables are used to construct \(\tau_{\text{app}}\) (also conformed by \(K'\) snapshots); (iii) the coefficients \(w_k\) are calculated minimizing the least-squares-error in (4); (iv) the equation

$$\frac{d\tau_{\text{app}}}{dt} \simeq \frac{1}{\Delta t} \sum_{k=2}^{K'} (w_k \tau^k_{\text{app}} - w_{k-1} \tau^{k-1}_{\text{app}}),$$

(5)

is integrated in time to predict wall shear. To construct this model, an initial training based on \(K'\) snapshots is carried out collecting information in the temporal interval \(t \in [0, t_r]\). Figure 8 compares the temporal evolution of the shear with the real solution in the DR and DI cases and shows the relative error made in this prediction. In DR the training is set in the interval \(t \in [0, 200]\) and the model is able to predict up to time \(t_r \simeq 580\) with a relative error smaller than 2%. On the contrary, in the DI case the error made is larger, \(\sim 6\%\), although the model is at least able to identify the averaged values of the wall shear. Improving this prediction implies using a larger number of DMD modes, which reflects the larger complexity of the drag-increased flow; however, this remains as open topic for future research.

5. Conclusions

This article studies in detail the main structures driving the flow in a turbulent channel flow over anisotropic porous walls. Two different cases have been studied taking as reference the channel with isotropic porous walls: a DI case, where the wall normal permeability is high and the streamwise-spanwise permeability is low, and a DR case, where the wall normal permeability is low and the streamwise-spanwise permeability is high. As a first step, the temporal HODMD is applied to analyze the data of these two test cases, identifying a larger number of modes.
in the DI (20) than in the DR (15) case, indicating the highest flow complexity in the former case. The frequency of 13 from a total of 15 modes is the same in both cases, and the shape of the temporal modes is also similar, suggesting the need to carry out the next step to better distinguish between the two cases; as a consequence, a spatio-temporal HODMD is applied along the spanwise direction to analyze the same data. The solution is periodic along this direction (imposed), and the modes with highest amplitude are those with wavenumber $\beta = 0$ and $\beta = 3$ in both cases. The spanwise rollers, characterised by $\beta = 0$ are identified in the DI case, while in the DR case a group of well-organized correlated structures can be related to the streaks. Although, it is remarkable that the presence of both rollers and streaks is found in the DR and DI cases in areas far away from the wall. These roller- and streak-shaped structures are related to high and low frequency DMD modes, respectively.

Finally, a model is constructed using the 6 DMD modes with highest influence on the flow near the channel walls, aiming to predict the time evolution of the wall shear stress. This model is created solving an optimization problem that minimizes the difference between the wall shear extracted from the simulations and the approximation using the DMD modes. In the DR case it is possible to predict the wall shear temporal evolution with relative errors $\sim 2\%$ for more than 300 time units; in the DI case, on the other hand, the error is larger, $\sim 6\%$, suggesting the need of using more than 6 modes to model the wall shear, as a consequence of the higher complexity of the flow in this configuration.

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6. References
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