Hamiltonian formulation for the classical EM radiation-reaction problem: application to the kinetic theory for relativistic collisionless plasmas.

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A notorious difficulty in the covariant dynamics of classical charged particles subject to non-local electromagnetic (EM) interactions arising in the EM radiation-reaction (RR) phenomena is due to the definition of the related non-local Lagrangian and Hamiltonian systems. As a basic consequence, the lack of a standard Lagrangian/Hamiltonian formulation in the customary asymptotic approximation for the RR equation may inhibit the construction of consistent kinetic and fluid theories. In this paper the issue is investigated in the framework of Special Relativity. It is shown that, for finite-size spherically-symmetric classical charged particles, non-perturbative Lagrangian and Hamiltonian formulations in standard form can be obtained, which describe particle dynamics in the presence of the exact EM RR self-force. As a remarkable consequence, based on axiomatic formulation of classical statistical mechanics, the covariant kinetic theory for systems of charged particles subject to the EM RR self-force is formulated in Hamiltonian form. A fundamental feature is that the non-local effects enter the kinetic equation only through the retarded particle 4-position. This permits, in turn, the construction of the related fluid equations, in which the non-local contributions carried by the RR effects are explicitly displayed. In particular, it is shown that the moment equations obtained in this way do not contain higher-order moments, allowing as a consequence the adoption of standard closure conditions. A remarkable aspect of the theory is related to the short delay-time asymptotic expansions. Here it is shown that two possible expansions are permitted. Both can be implemented for the single-particle dynamics as well as for the corresponding kinetic and fluid treatments. In the last case, they are performed *a posteriori*, namely on the relevant moment equations obtained after integration of the kinetic equation over the velocity space. Comparisons with the literature are pointed out.

I. INTRODUCTION

An open problem in relativistic theories is related to the Hamiltonian description of particle dynamics for which non-local interactions typically occur. In this regard, a basic difficulty which is usually met is the lack of a Hamiltonian formalism for non-local Lagrangian systems. In fact, for arbitrary non-local Lagrangians it is generally impossible to define the notion of Legendre transformation [1]. As a consequence even the phase-space itself may not be well-defined.

Most approaches to the construction of a Hamiltonian formalism for non-local first-order Lagrangians have tried to change the functional part of the Euler-Lagrange equations [2–5]. In principle this delivers infinite-order Euler-Lagrange equations and a corresponding infinite-dimensional phase-space. As an alternative, a finite dimensional phase-space can be recovered by introducing appropriate asymptotic approximations, i.e., truncating the expansion of the Lagrangian in terms of finite-order derivatives [4, 6].

A typical situation of this kind occurs for the relativistic equation of motion for single isolated charged particles, subject both to external and self EM forces, namely the radiation-reaction (RR) equation. There is an extensive literature devoted to this subject, most of which dealing with point charges. As remarked by Dorigo et al. [7], customary formulations based either on the LAD [8–10] or LL [11] equations are asymptotic, i.e., obtained by means of asymptotic expansions of different sort. In particular, as a consequence it follows that the LAD equation is represented by a third-order ODE, so that it does not admit a Hamiltonian formulation in the customary sense [12, 13]. The LL, instead, is intrinsically non-variational, although it is a second-order differential equation, being obtained by means of a one-step “reduction process” from the LAD equation [3]. As a consequence, the LAD equation does not define a dynamical system in the customary sense, since it requires, for non-rotating particles, a 12-dimensional phase-space involving also the particle acceleration. Therefore, for different reasons, both the LAD and LL equations are manifestly non-Hamiltonian. In particular, for the LL equation, this implies that the corresponding phase-space volume is not conserved. Moreover, within these treatments particles are treated as point-like, so that non-local EM effects produced by the RR self-interaction may remain undetermined.

Fundamental problems arise when attempting to formulate classical statistical mechanics (CSM) for systems of
relativistic charged particles based on the LAD or LL equations. In fact even the proper axiomatic formulation of the relativistic CSM for radiating particles is missing. This requires the precise identification of the corresponding phase-space and the definition of an invariant probability measure on this set. For a system of charged particles subject solely to an external EM field and the RR self-force this involves the construction of a Vlasov kinetic treatment. In this regard, important issues concern:

1) The lack of a standard Hamiltonian formulation of relativistic CSM based on such asymptotic equations, which implies the lack of a flow-preserving measure. This feature is shared by both the LAD and LL equations.

2) The proper definition of a phase-space. The problem is relevant for the LAD equation. In fact, although the construction of kinetic theory is still formally possible [14, 15], the corresponding fluid statistical description seems divergent.

3) The explicit dependence of the kinetic distribution function (KDF) in terms of the retarded EM self 4-potential are excluded. In fact, in the LAD and LL approximations the self-potential does not appear explicitly (see for example Refs. [16, 17]). Indeed, within the point-particle model, underlying both treatments, the retarded self-potential is divergent.

On the other hand, for the fluid description:

1) The precise form of the fluid closure conditions may depend on the approximations adopted in the kinetic description for the representation of the EM RR self-force. An example-case is provided by Ref. [17] where relativistic fluid equations are obtained based on the LL equation. As a result, it was found that, with the exception of the continuity equation, all moment equations involve higher-order fluid moments associated to the RR self-force. It is unclear whether this is an intrinsic physical feature of the RR interaction or simply a result of the approximations involved.

2) The fluid fields may in principle depend implicitly on the EM self 4-potential. In the framework of the LL equation it is unclear how such an effect can be dealt with. However, the treatment of such effects seems to present objective difficulties. In fact, in principle non-local effects might arise in this way in which retarded velocity contributions appear in the kinetic equation. In such a case the explicit construction of fluid equations would be ambiguous (and might involve an infinite set of higher-order moments).

The interesting question is whether these difficulties can be overcome in physically realizable situations, namely for exactly solvable classical systems (of particles) for which the relativistic equations of motion are both variational and non-asymptotic. The prerequisite is provided by the possibility of constructing an exact representation for the RR equation for a suitable type of classical charged particles. In the past, their precise identification with physically-realizable systems has remained elusive because of the difficulty of the problem. However, as recently pointed out (see Tessarotto et al. [18] and Cremaschini et al. [19], hereafter referred to as Paper I) in the framework of special relativity an exact variational RR equation can be obtained for classical finite-size charged particles. This refers to particles having a finite-size mass and charge distributions which are quasi-rigid, non-rotating, spherically symmetric and radially localized on a spherical surface \( \partial \Omega \) having a finite radius \( \sigma > 0 \) (see [20] and the related discussion in Paper I). In this formulation, contrary to the point-particle case, the retarded EM self 4-potential is well-defined, namely, it does not diverge, and can be determined analytically. As shown in Paper I, it follows that the RR equation is variational and the corresponding Hamilton variational functional is symmetric with respect to the non-local contributions. The latter are due to the retarded EM self interaction arising from the finite spatial extension of the charge distribution. As a consequence, the resulting exact RR equation is a second-order delay-type ODE which admits a Lagrangian formulation in standard form (see discussion below). Furthermore, under suitable conditions, the same equation defines a classical dynamical system (RR dynamical system).

In this paper we intend to prove that, based on the results of Paper I, the RR dynamical system admits also a Hamiltonian representation in terms of an effective non-local Hamiltonian function \( H^{eff} \). This implies that the exact RR equation can also be cast in the equivalent standard Hamiltonian form represented by first-order delay-type ODEs

\[
\begin{align*}
\frac{dr^\mu}{ds} &= \frac{\partial H^{eff}}{\partial P_\mu}, \\
\frac{dP_\mu}{ds} &= -\frac{\partial H^{eff}}{\partial r_\mu},
\end{align*}
\] (1)

with \( y = \{ r^\mu, P_\mu \} \) denoting, in superabundant variables, the particle canonical state which spans the eight-dimensional phase-space \( \Gamma \equiv \Gamma_r \times \Gamma_u \), where \( \Gamma_r \) and \( \Gamma_u \) are respectively the Minkowski \( M^4 \)-configuration space and the 4-dimensional velocity-space, both with metric \( \gamma_{\mu\nu} = \text{diag}(1, -1, -1, -1) \). Remarkably, here it is found that the Hamiltonian structure can be retained also after the introduction of a suitable short delay-time approximation of the RR force. The result is an intrinsic feature of the extended particle model adopted in the present treatment.

As a consequence, the statistical description of the RR dynamical system follows in a standard way. In particular, here we report both the exact and asymptotic kinetic and fluid formulations. These are developed for collisionless
relativistic plasmas in the Vlasov-Maxwell description, including consistently the contribution carried by the RR self-field. Applications of the theory here developed concern:

1) The kinetic and fluid treatments of relativistic astrophysical plasmas observed, for example, in accretion disks, relativistic jets, active galactic nuclei and mass inflows around compact objects.

2) The kinetic and fluid treatments of laboratory plasmas subject to ultra-intense and pulsed-laser sources.

### A. Goals and scheme of the presentation

In detail, the plan of the paper is as follows. In Section 2 we introduce the basic definitions concerning the Lagrangian formulation for non-local interactions and the concept of effective non-local Lagrangian function and the related covariance properties (THM.1 and Corollary). In Section 3 we provide an analogous generalization which permits to introduce the notion of non-local Hamiltonian formulation (THM.2 and Corollary). In particular, within the framework of the theory here developed, the standard form for the local Legendre transformation is retained, while the concepts of effective canonical momenta and effective Hamiltonian function are introduced. Then, in Section 4 the general case of a rotating finite-size and spherically-symmetric charged particle is discussed as a physical example of non-local interaction. It is shown that the corresponding Hamilton variational functional satisfies the requirements of THMs.1 and 2 (see THM.3) and therefore admits both Lagrangian and Hamiltonian formulations in standard form. In Section 5 the theory is applied to the specific case of a non-rotating particle. As a result, based on the analytical results of Paper I, the corresponding variational and effective Hamiltonian formulations are presented (THM.4). This provides the explicit form of the effective Hamiltonian function and a parameter-free representation for the retarded EM RR self-force. Then, in Section 6 a Hamiltonian asymptotic approximation of the RR equation is developed (THM.5), based on the retarded-time expansion holding in the short delay-time ordering. The approximation overcomes basic inconsistencies of the LAD and LL treatments, applying in the case of point-particles. As a consequence, in Section 7 the relativistic kinetic theory for a collisionless plasma with the inclusion of the EM RR effect is formulated in Hamiltonian form. The use of superabundant canonical variables allows a precise identification of the phase-space and the consequent axiomatic formulation of CSM with non-local EM RR interactions. This is based on the notion of invariant probability measure in such a setting. As a result, a relativistic Liouville-Vlasov kinetic equation is proved to hold for the KDF (THM.6). This permits to achieve a Vlasov-Maxwell description applicable to relativistic plasmas, in which the RR interaction is consistently taken into account also in the Maxwell equations both for the external and self EM fields. In Section 8 the corresponding fluid fields and fluid moment equations are computed in terms of 4-velocity integrals, which retain the standard conservative Eulerian form as in the absence of RR effects. The existence of both explicit and implicit non-local contributions arising from the RR effect in the fluid equations is discussed. It is shown that the former are associated to the EM force acting on the fluid, while the latter enter in the definition of the fluid fields through the effective momenta. In particular, the explicit dependence of the KDF on the retarded EM self 4-potential is discussed and an asymptotic estimation of the implicit contributions is presented. In Section 9 a Lagrangian formulation of the fluid equations is derived, which allows one to introduce an explicit parametrization of the non-local RR terms carried by the EM self 4-potential and the EM RR force. It is shown that the exact fluid equations with the inclusion of the RR interaction are of delay-type. Section 10 deals instead with the development of asymptotic approximations of the moment equations. This is motivated by the requirement of reducing the exact non-local fluid equations to a local form. Different asymptotic approximations are obtained for the non-local terms of the RR effect, based both on short-delay time expansions (THM.7) and an iterative procedure which holds under the assumption of weak RR self-force (in comparison with external EM and pressure forces). A detailed analysis of the basic physical properties of the kinetic and fluid treatments obtained here and a comparison with previous literature is reported in Section 11. Concluding remarks are presented in Section 12. Finally, in Appendix A a Green-function approach is developed for the calculation of the EM self 4-potential, while in Appendix B the connection with non-canonical representations is provided for the relativistic kinetic theory.

### II. NON-LOCAL LAGRANGIAN FORMULATION

The natural mathematical apparatus for an abstract description of Lagrangian and Hamiltonian mechanics is that of variational principles, whose methods have been studied for a long time by mathematicians and can be found in the textbooks. Nevertheless, actual problems of interest in classical relativistic dynamics involving the treatment of non-local interactions have escaped a solution. In particular, in the literature the prevailing view is that, while a non-local variational formulation is possible, a corresponding Hamiltonian representation is generally excluded. In the following we intend to point out that for a particular class of non-local Lagrangian systems the problem can be given a complete solution. The latter correspond to variational problems in which the variational functional is symmetric.
To this end, in this section we briefly recall basic notions holding for local and non-local Lagrangian systems. This task represents a necessary prerequisite for the establishment of a corresponding Hamiltonian formulation and for the subsequent investigation of the Hamiltonian dynamics of finite-size charged particles with the inclusion of the RR self-force.

**Definition #1 - Local and non-local Lagrangian systems.**

A local (respectively, non-local) Lagrangian system is defined by the set \( \{ \mathbf{x}, L \} \) such that the following conditions are satisfied.

1. \( \mathbf{x} \equiv \left( r^\mu (s), \frac{dr^\mu(s)}{ds} \equiv \dot{r}^\mu \right) \) is the Lagrangian state spanning the Lagrangian phase space \( \Gamma_L \subseteq \mathbb{R}^{2N} \).

2. The Lagrangian action functional \( S \) is a 4-scalar of the form
   \[
   S = \int_{s_1}^{s_2} dsL,
   \]
   with \( L \) to be referred to as variational Lagrangian function. In particular, the functional dependencies of \( S \) and \( L \) are respectively of the form:
   - \( S \equiv S_0(r) \) and \( L \equiv L_0(r, \frac{dr}{ds}) \) for local systems;
   - \( S \equiv S_1(r, [r]) \) and \( L \equiv L_1(r, \frac{dr}{ds}, [r]) \) for non-local systems, with \( [r] \) denoting non-local dependencies.

3. In the functional class
   \[
   \{ r^\mu \} \equiv \{ r^\mu(s) : r^\mu(s_i) = r^\mu_i, \text{ for } i = 1, 2, s_1, s_2 \in I, \text{ with } s_1 < s_2 \text{ and } r^\mu(s) \in C^2(I) \},
   \]
   the synchronous variations \( \delta r^\mu(s) \) are considered independent and vanish at the endpoints \( r^\mu(s_i) = r^\mu_i \). Hereafter \( \delta \) denotes, as usual, the Frechet functional derivative. For a synchronous variational principle the interval \( ds \) is such that \( \delta ds = 0 \) and is subject to the constraint
   \[
   ds^2 = g_{\mu\nu}dr^\mu(s)dr^\nu(s),
   \]
   where \( r^\mu(s) \) are the extremal curves.

4. The Lagrangian action \( S \) admits a unique extremal curve \( r^\mu(s) \) such that, for all synchronous variations \( \delta r^\mu(s) \) in the functional class \( \{ r^\mu \} \) the Hamilton variational principle
   \[
   \delta S = 0
   \]
   holds identically. For non-local systems the non-local Lagrangian must be suitably constructed in such a way that the extrema curves \( r^\mu(s) \) satisfy the constraint \( [r] \).

In particular, for local systems the extremal curves of \( S_0 \) are provided by the Euler-Lagrange (E-L) equations
   \[
   \frac{\delta S_0}{\delta r^\mu} = F_\mu(r)L_0 = 0,
   \]
   where, for an arbitrary set of Lagrange coordinates \( q^\mu \), \( F_\mu(q) \) denotes the E-L differential operator
   \[
   F_\mu(q) = \frac{d}{ds} \frac{\partial}{\partial q^\nu} - \frac{\partial}{\partial q^\mu},
   \]
   and \( \dot{q}^\mu \equiv \frac{d}{ds}q^\mu(s) \).

On the other hand, for non-local systems the extremal curves of the functional \( S_1 \) are provided by the Euler-Lagrange equations
   \[
   \frac{\delta S_1}{\delta r^\mu} = \frac{\delta S_1}{\delta r^\mu}_{[r]} + \frac{\delta S_1}{\delta [r]}_{[r]} = 0,
   \]
   where \( \frac{\delta S_1}{\delta r^\mu}_{[r]} \) and \( \frac{\delta S_1}{\delta [r]}_{[r]} \) carry respectively the contributions due to the local and non-local dependencies.
Definition #2 - Non-local Lagrangian systems in standard form.
A non-local Lagrangian system \( \{ \mathbf{x}, L_1 \} \) will be said to admit a standard form if the variational derivative (8) yields the E-L equations in the standard form:

\[
\frac{\delta S_1}{\delta r^\mu} + \frac{\delta S_1}{\delta [r^\mu]} \equiv F_\mu(r) L_{\text{eff}} = 0,
\]

with

\[
L_{\text{eff}} \equiv L_{\text{eff}} \left( r, \frac{dr}{ds}, [r] \right)
\]
denoting a suitable effective non-local Lagrangian.

On the base of these definitions, the following theorem holds.

**THM.1 - Non-local and Effective Lagrangian functions**
Given validity of the definitions #1 and #2, it follows that:

1. The non-local Lagrangian \( L_1 \) and the effective Lagrangian \( L_{\text{eff}} \) are generally different, namely

\[
L_1 \neq L_{\text{eff}}.
\]

2. If \( S_1(r,[r]) \) admits the general decomposition

\[
S_1(r,[r]) = S_a(r) + S_b(r,[r]),
\]

with \( S_a(r) \equiv \int_{s_1}^{s_2} ds L_a \left( r, \frac{dr}{ds} \right) \) and \( S_b(r,[r]) \equiv \int_{s_1}^{s_2} ds L_b \left( r, \frac{dr}{ds}, [r] \right) \), and moreover \( S_b(r,[r]) \) defines a symmetric functional such that

\[
S_b(r,[r]) = S_b([r],r),
\]

then the effective Lagrangian \( L_{\text{eff}} \) is related to the variational non-local Lagrangian \( L_1 \equiv L_a + L_b \) as

\[
L_{\text{eff}} = L_a + 2L_b = L_1 + L_b.
\]

**Proof - T1.1** The proof is an immediate consequence of Eqs. (8) and (9). In fact, by definition the E-L differential operator \( F_\mu(r) \) is a local differential operator that is required to preserve its form also for non-local systems. On the other hand, the variational derivative (8) is different from (6). Hence, in order to write the E-L equations associated to the non-local function \( L_1 \) in standard form, a suitable effective Lagrangian \( L_{\text{eff}} \) must be introduced, which must differ from \( L_1 \) and be expressed in such a way that the non-local dependencies contained in \( L_1 \) can be equivalently treated by means of \( F_\mu(r) \).

T1.2) The proof follows by inspecting the general definition (8). In this case, in view of the symmetry property (13), it follows manifestly that

\[
\frac{\delta S_1}{\delta r^\mu} \Big|_{[r]} + \frac{\delta S_1}{\delta [r^\mu]} \Big|_{r} = \frac{\delta S_a}{\delta r^\mu} \Big|_{[r]} + 2 \frac{\delta S_b}{\delta r^\mu} \Big|_{[r]} = 0.
\]

Then, by comparing this relation with the definitions both of the E-L differential operator (7) and the standard form representation of the E-L equations (9), it follows that the effective Lagrangian \( L_{\text{eff}} \) takes necessarily the form given in Eq.(14). This completes the proof of the statement.

Q.E.D.

A basic consequence of Definition #2 and THM.1 concerns the covariance property of the E-L equations (9). The result is stated in the following Corollary.

**Corollary 1 to THM.1 - Covariance of the E-L equations for arbitrary point transformations.**
The Euler-Lagrange equations (9) are covariant with respect to arbitrary point transformations

\[
r^\mu \rightarrow q^\mu(r)
\]
represented by a diffeomorphism of class \( C^k \), with \( k \geq 2 \), which requires they are of the form

\[
F_\mu(q)\tilde{L}_{eff} = \frac{\partial r^\nu}{\partial q^\mu} F_\nu(r)L_{eff} = 0,
\]

with \( \tilde{L}_{eff} \) denoting

\[
\tilde{L}_{eff} \left( q, \frac{dq}{ds}, [q] \right) \equiv L_{eff} \left( r, \frac{dr}{ds}, [r] \right).
\]

As a consequence, Eq. (17) satisfies also the covariance property with respect to arbitrary infinitesimal Lorentz transformations (Manifest Lorentz Covariance).

**Proof** - The Euler-Lagrange equations (9) are by definition covariant provided the variational Lagrangian \( L_1(r, \frac{dr}{ds}, [r]) \) is a 4-scalar (as it is by construction). Then, it is sufficient to represent the Lagrangian action in terms of the Lagrangian coordinates \( q^\mu \), yielding

\[
\tilde{S}_1(q, [q]) \equiv S_1(r, [r]),
\]

with \( \tilde{S}_1(q, [q]) \) denoting the transformed action

\[
\tilde{S}_1(q, [q]) = \int_{s_1}^{s_2} ds \tilde{L}_1 \left( q, \frac{dq}{ds}, [q] \right)
\]

and \( \tilde{L}_1 \) denoting the transformed variational non-local Lagrangian. Hence, the Hamilton variational principle \( \delta \tilde{S}_1(q, [q]) = 0 \) yields precisely the E-L equations (17). This proves the statement. The covariance property of Eqs. (9) with respect to point transformations (16) includes, as particular case, Lorentz transformations. Therefore, Eqs. (9) are also Manifestly Lorentz Covariant (MLC).

Q.E.D.

We notice the following remarkable features of this treatment:

1) In general, in absence of any kind of symmetry, a non-local Lagrangian system does not admit a standard form representation in terms of the effective Lagrangian \( L_{eff} \).

2) As shown in T12, the possibility of getting an explicit relationship between \( L_1 \) and \( L_{eff} \) is a consequence solely of the symmetry property (13) of the functional \( S_b \). This also proves the existence of \( L_{eff} \) and, as a consequence, of the standard form representation for non-local systems satisfying Eq. (13).

3) The symmetry assumption (13) can be effectively realized in physical systems. As it will be shown below, this condition is satisfied by the variational functional which describes the dynamics of finite-size classical charged particles with the inclusion of the RR effects associated to the interaction with the EM self-field.

**III. NON-LOCAL HAMILTONIAN FORMULATION**

In this section we deal with the basic features concerning the Hamiltonian formulation for non-local systems which admit a variational treatment in terms of non-local Lagrangian functions. This requires the introduction of the following preliminary definitions.

**Definition #3 - Local and non-local Hamiltonian systems.**

A local (respectively, non-local) Lagrangian system \( \{x, L\} \) is said to admit a local (non-local) Hamiltonian system \( \{y \equiv (r^\mu, p_\mu), H\} \) providing the following conditions are satisfied.

1. The *variational Hamiltonian* \( H \) is defined as the Legendre transformation of the local (non-local) variational Lagrangian \( L \)

\[
H = p_\mu \frac{dr^\mu}{ds} - L,
\]

with

\[
p_\mu = \frac{\partial L}{\partial \frac{dr^\mu}{ds}}
\]

being the corresponding canonical momentum, with corresponding action functional \( S_H \equiv \int_{s_1}^{s_2} ds \left[ p_\mu \frac{dr^\mu}{ds} - H \right]. \)
2. It is assumed that $H$ is respectively of the form:

- $H \equiv H_0 (r, p)$ for local systems;
- $H \equiv H_1 (r, p, [r])$ for non-local systems, namely it is a local function of $(r, p)$ and a functional of $[r]$.

The corresponding Hamilton action functionals are denoted respectively as

$$S_{H_0}(r, p) = \int_{s_1}^{s_2} ds \left[ p_\mu \frac{dr^\mu}{ds} - H_0 \right]$$

(23)

for local systems, and as

$$S_{H_1}(r, p, [r]) = \int_{s_1}^{s_2} ds \left[ p_\mu \frac{dr^\mu}{ds} - H_1 \right]$$

(24)

for non-local systems.

3. In the functional class

$$\{ y \equiv (r^\mu, p_\mu) \} \equiv \{ y(s) : y(s_i) = y_i, \text{ for } i = 1, 2, s_1, s_2 \in I, \text{ with } s_1 < s_2 \text{ and } y(s) \in C^2(I) \}$$

(25)

the synchronous variations $(\delta r^\mu(s), \delta p_\mu(s))$ are all considered independent and vanish at the endpoints $y(s_i) = y_i$. By assumption, synchronous variations imply that $\delta ds = 0$, with the interval $ds$ satisfying the constraint

$$ds^2 = g_{\mu\nu} dr^\mu(s) dr^\nu(s),$$

(26)

where $r^\mu(s)$ are the extremal curves.

4. The modified Hamilton variational principle

$$\delta S_H = 0$$

(27)

with variations $(\delta r^\mu(s), \delta p_\mu(s))$ is equivalent to the Hamilton principle \[4\], i.e., it yields the same extremal curves in the functional class \{y\}.

In particular, for local systems the extremal curves of $S_{H_0}$ can be cast in the standard Hamiltonian form as first-order ODEs

$$\frac{\delta S_{H_0}}{\delta p_\mu} = \frac{dr^\mu}{ds} = \frac{\partial H_0}{\partial p_\mu} = [r^\mu, H_0],$$

(28)

$$\frac{\delta S_{H_0}}{\delta r^\mu} = \frac{dp_\mu}{ds} = \frac{\partial H_0}{\partial r^\mu} = [p_\mu, H_0],$$

(29)

where the customary Poisson bracket formalism has been used.

On the other hand, for non-local systems the extremal curves of the functional $S_{H_1}$ are provided by the set of first-order ODEs

$$\frac{\delta S_{H_1}}{\delta p_\mu} = 0,$$

(30)

$$\frac{\delta S_{H_1}}{\delta r^\mu} = \left. \frac{\delta S_{H_1}}{\delta r^\mu} \right|_{[r]} + \left. \frac{\delta S_{H_1}}{\delta [r]} \right|_{r} = 0,$$

(31)

where $\left. \frac{\delta S_{H_1}}{\delta r^\mu} \right|_{[r]}$ and $\left. \frac{\delta S_{H_1}}{\delta [r]} \right|_{r}$ carry respectively the contributions due to the local and non-local dependencies.
Definition #4 - Non-local Hamiltonian systems in standard form.
A non-local Hamiltonian system \( \{ y, H_1 \} \) will be said to admit a standard form if the extremal first-order ODEs (39) and (41) can be cast in the standard Hamiltonian form in terms of the effective canonical momentum \( P_\mu \) and Hamiltonian function \( H_{\text{eff}} \) as

\[
\frac{\delta S_{H_1}}{\delta p_\mu} = \frac{dr_\mu}{ds} = \frac{\partial H_{\text{eff}}}{\partial P_\mu} = [r_\mu, H_{\text{eff}}],
\]

(32)

\[
\frac{\delta S_{H_1}}{\delta r_\mu} = \left. \frac{\delta S_{H_1}}{\delta r_\mu} \right|_{[r]} + \frac{\delta S_{H_1}}{\delta [r_\mu]} \bigg|_r = -\frac{dP_\mu}{ds} = \frac{\partial H_{\text{eff}}}{\partial r_\mu} = [P_\mu, H_{\text{eff}}].
\]

(33)

Here both \( H_{\text{eff}} = H_{\text{eff}}(r, P, [r]) \) and \( P_\mu \) must be defined in terms of the effective Lagrangian function introduced in Eq.(9) respectively as

\[
H_{\text{eff}} = P_\mu \frac{dr_\mu}{ds} - L_{\text{eff}}
\]

(34)

and

\[
P_\mu = \frac{\partial L_{\text{eff}}}{\partial \frac{dr_\mu}{ds}}.
\]

(35)

From this definition it follows that, if the non-local Hamiltonian system \( \{ y, H_1 \} \) admits a standard form, then the Poisson bracket representation holds for \( H_{\text{eff}} \) and \( P_\mu \).

The following theorem can be stated concerning the relationship between \( H_1 \) and \( H_{\text{eff}} \).

THM.2 - Non-local and Effective Hamiltonian functions

Given validity of the definitions #3 and #4 and the results of THM.1, if \( S_{H_1}(r, p, [r]) \) admits the general decomposition

\[
S_{H_1}(r, p, [r]) = S_{H_a}(r, p) + S_{H_b}(r, p, [r]),
\]

(36)

with

\[
S_{H_a}(r, p) = \int_{s_1}^{s_2} ds \left[ p_{a\mu} \frac{dr_\mu}{ds} - H_a(r, p) \right],
\]

(37)

\[
S_{H_b}(r, p, [r]) = \int_{s_1}^{s_2} ds \left[ p_{b\mu} \frac{dr_\mu}{ds} - H_b(r, p, [r]) \right],
\]

(38)

where the canonical momenta \( p_{a\mu} \) and \( p_{b\mu} \) are defined respectively as

\[
p_{a\mu} = \frac{\partial L_a}{\partial \frac{dr_\mu}{ds}}
\]

(39)

\[
p_{b\mu} = \frac{\partial L_b}{\partial \frac{dr_\mu}{ds}}
\]

(40)

and moreover \( S_{H_b}(r, p, [r]) \) defines a symmetric functional such that

\[
S_{H_b}(r, p, [r]) = S_{H_b}([r], p, r),
\]

(41)

then the effective Hamiltonian \( H_{\text{eff}} \) is related to the variational non-local Hamiltonian \( H_1 \equiv H_a + H_b \) as

\[
H_{\text{eff}} = H_a + 2H_b = H_1 + H_b,
\]

(42)

where, by definition

\[
H_1 = p_\mu \frac{dr_\mu}{ds} - L_1,
\]

(43)

\[
H_a = p_{a\mu} \frac{dr_\mu}{ds} - L_a,
\]

(44)

\[
H_b = p_{b\mu} \frac{dr_\mu}{ds} - L_b.
\]

(45)
Proof - The proof follows from THM.1 and by invoking the general definitions (32) and (33). In fact, in view of the symmetry property (41), it follows manifestly that
\[ \frac{\delta S_{H_1}}{\delta r^\mu} \equiv \frac{\delta S_{H_1}}{\delta [r]} \bigg|_r + \frac{\delta S_{H_a}}{\delta [r]} \bigg|_r + 2 \frac{\delta S_{H_b}}{\delta [r]} \bigg|_r . \] (46)

Then, by comparing this relation with the definitions (34)-(35) for the standard Hamiltonian form and using Eqs. (43)-(45), from the analogous results in THM.1 which concerns the relationship between \( L_1 \) and \( L_{\text{eff}} \) in the symmetric case, Eq. (42) is readily obtained.

Q.E.D.

Finally, as a basic consequence of Definition #4 and THM.2, the following Corollary can be stated concerning the covariance property of the Hamilton equations in standard form.

Corollary 1 to THM.2 - Covariance of the Hamilton equations for arbitrary point transformations.
The Hamilton equations (32)-(33) in standard form are covariant with respect to arbitrary point transformations
\[ r^\mu \rightarrow q^\mu(r) \] (47)
represented by a diffeomeorphism of class \( C^k \) with \( k \geq 2 \), which requires they are of the form

\[
\frac{dq^\mu}{ds} = \frac{\partial H_{\text{eff}}}{\partial P_{(q)^\mu}} = [q^\mu, H_{\text{eff}}],
\] (48)

\[
\frac{dP_{(q)^\mu}}{ds} = -\frac{\partial H_{\text{eff}}}{\partial q^\mu} = [P_{(q)^\mu}, H_{\text{eff}}],
\] (49)

with \( H_{\text{eff}} \) denoting

\[
H_{\text{eff}}(q, P, [q]) \equiv H_{\text{eff}}(r, P, [r]),
\] (50)

and \( P_{(q)^\mu} \) being the transformed canonical momentum. As a consequence, Eqs. (48) and (49) satisfy also the covariance property with respect to arbitrary infinitesimal Lorentz transformations (Manifest Lorentz Covariance).

Proof - In fact, for an arbitrary point transformation of the type (47), the corresponding transformation for the momenta \( P_\nu \) is
\[ P_{(q)^\mu} = \frac{\partial q^\nu}{\partial r^\mu} P_\nu, \] (51)
which yields

\[
\frac{\partial P_{(q)^\nu}}{\partial P_\mu} = \frac{\partial q^\mu}{\partial r^\nu},
\] (52)

\[
\frac{\partial P_{(q)^\mu}}{\partial P_\nu} = \frac{\partial r^\nu}{\partial q^\mu}.
\] (53)

Hence, it follows that

\[
\frac{dq^\nu}{ds} = \frac{\partial H_{\text{eff}}}{\partial P_{(q)^\mu}} \frac{\partial P_{(q)^\mu}}{ds} = \frac{\partial q^\nu}{\partial r^\mu} \frac{\partial H_{\text{eff}}}{\partial P_\nu},
\] (54)

\[
\frac{dP_{(q)^\mu}}{ds} = -\frac{\partial H_{\text{eff}}}{\partial q^\mu} = \frac{\partial P_{(q)^\mu}}{\partial P_\nu} \frac{dP_\nu}{ds} = \frac{\partial r^\nu}{\partial q^\mu} \frac{\partial H_{\text{eff}}}{\partial P_\nu},
\] (55)

which implies

\[
\frac{\partial H_{\text{eff}}}{\partial P_{(q)^\mu}} = \frac{\partial q^\nu}{\partial r^\mu} \frac{\partial H_{\text{eff}}}{\partial P_\nu},
\] (56)

\[
\frac{\partial H_{\text{eff}}}{\partial q^\mu} = \frac{\partial r^\nu}{\partial q^\mu} \frac{\partial H_{\text{eff}}}{\partial P_\nu},
\] (57)
where $\tilde{H}_{eff}$ is defined in Eq.(50) above. Therefore, the Hamilton equations in standard form for the Lagrangian coordinates $q^\mu$ and the canonical momenta $P_{(q)^\mu}$ are respectively covariant [Eq. (50)] and contravariant [Eq. (57)] with respect to the point transformation (47). This is true also for arbitrary infinitesimal Lorentz transformations, which proves the MLC of Hamilton equations Eqs.(48) and (49) in standard form.

Q.E.D.

IV. AN EXAMPLE OF NON-LOCAL INTERACTION: THE CLASSICAL EM RR PROBLEM

A crucial issue of the present investigation concerns the possible existence of physical systems subject to non-local interactions whose dynamics can be consistently described in terms of a variational action integral and which admit at the same time both Lagrangian and Hamiltonian formulations in standard form. In this section we prove that the EM RR problem for classical finite-size charged particles represents a physical example of non-local interactions of this kind. The reason behind the choice of considering extended particles is the necessity of avoiding the intrinsic divergences of the RR effect characteristic of the point-particle model.

In fact, consider the general form of the Hamilton action functional for the variational treatment of the dynamics of an extended charged particle in presence of an external EM field and with the inclusion of the RR self-interaction. This can be conveniently expressed as follows:

$$S_1 (z, [z]) = S_M (z) + S_C^{(ext)} (z) + S_C^{(self)} (z, [z]),$$

(58)

where $S_M, S_C^{(ext)}$ and $S_C^{(self)}$ are respectively the contributions from the inertial mass and the EM coupling with the external and the self fields. In particular, denoting by $J^{(self)} (r)$ the particle 4-current density generated by the particle itself and observed at a 4-position $r$, the two coupling action integrals are provided by the following 4-scalars:

$$S_C^{(ext)} (z) = \frac{1}{c^2} \int_1^2 d^4r A^{(ext)} (r) J^{(self)} (r),$$

(59)

$$S_C^{(self)} (z, [z]) = \frac{1}{c^2} \int_1^2 d^4r A^{(self)} (r) J^{(self)} (r),$$

(60)

where $A^{(ext)}$ and $A^{(self)}$ denote the 4-vector potentials of the external and the self EM fields and $z$ is a state to be suitably defined (see below). A clarification here is in order. The external EM 4-potential $A^{(ext)} (r)$ acting on the charged particle located at the 4-position $r$ is assumed to be produced only by prescribed “external” sources, namely, excluding the particle itself, by the remaining possible EM sources belonging to the configuration space $\Gamma_r$. Within the framework of special relativity, both the inertial term and the coupling term with the external field carry only local dependencies, in the sense that they depend explicitly only on the local 4-position $r$. They provide the classical dynamics of charged particles in absence of any RR effect. On the other hand, the functional $S_C^{(self)}$ associated to the EM self-interaction contains both local and non-local contributions. In particular, since the state $z$ of a finite-size particle must include a 4-position vector $r$, it follows that $S_C^{(self)}$ generally depends explicitly on two different 4-positions, $r$ and $[r]$, to be properly defined (see below). The non-local property of $S_C^{(self)}$ represents a characteristic feature of RR phenomena.

From the relationship (58) it follows that the Hamilton action functional for the treatment of the RR admits the decomposition (12) introduced by THM.1, namely it can be written as the sum of two terms, carrying respectively only local and both local and non-local dependencies. In order to prove that the same functional admits also a Lagrangian and a Hamiltonian representation in standard form it is sufficient to show that the self-coupling functional is symmetric in $z$ and $[z]$, in the sense defined in THM.1. For this purpose we need to determine explicitly the general expression of the 4-current and the self 4-potential for a rotating finite-size charged particle.

The first step consists in constructing a covariant representation for the 4-current density. We follow the approach presented by Nodvik [20]. Thus, we consider an extended charged particle with charge and mass distributions having the same support $\partial \Omega$, to be identified with a smooth surface. Denoting by $r^\mu (s)$ the 4-vector position (with proper time $s$) of a reference point belonging to the internal open domain $\Omega$ and by $\zeta^\mu$ a generic 4-vector of $\partial \Omega$, the displacement vector $\xi^\mu$ is defined as:

$$\xi^\mu \equiv \zeta^\mu - r^\mu (s).$$

(61)
The particle model is prescribed by imposing the constraints of rigidity of $\partial \Omega$, namely for all $\zeta^\mu$ and $r^\mu (s)$ [20]:

$$\xi^\mu \xi_\mu = \text{const.},$$

$$\xi_\mu u^\mu (s) = 0,$$

(62)
(63)

where $u^\mu (s) \equiv \frac{d}{ds} r^\mu (s)$. In particular, we shall assume that mass and charge distributions are spherically symmetric and therefore characterized by a form factor $f (\xi^\mu) \equiv f (\xi^\mu \xi_\mu)$. This allows one to identify $r^\mu (s)$ as the center-point of $\partial \Omega$. The extended particle can in principle exhibit both translational and rotational degrees of freedom. In particular, the translational motion can be described in terms of the Euler angles $\alpha$. Instead, the rotational dynamics, which includes both space-time rotations associated to the so-called Thomas precession and pure spatial rotations, can be described in terms of $\omega^\alpha (s)$, where $\omega^\alpha (s)$ is the antisymmetric angular velocity tensor [20], which depends on $s$ through the Euler angles $\alpha (s)$. The term $\frac{1}{c} \frac{d}{ds} x^\alpha$ contains the acceleration of $r^\mu (s)$ and represents the contribution associated to the Thomas precession effect. This can be formally eliminated by using the properties of the Dirac-delta function, implying the identity:

$$\delta (x^\alpha u_\alpha (s)) = \frac{1}{d [x^\alpha u_\alpha (s)] / ds} \delta (s - s_1) = \frac{1}{1 - \frac{d u_\alpha}{ds} x^\alpha} \delta (s - s_1),$$

(66)

where by definition $s_1 = s_1 (r)$ is the root of the algebraic equation

$$u_\mu (s_1) [r^\mu - r^\mu (s_1)] = 0.$$

As a result, the 4-current can be equivalently expressed as

$$j^{(self)} (r) = q c \int_{-\infty}^{+\infty} ds \left[ u^\mu \delta (s - s_1) - \frac{1}{c} \frac{d u_\alpha}{ds} x_\beta \delta (x^\alpha u_\alpha) \right] f (x^2).$$

(68)

The second step consists in constructing a Green-function representation for the EM self-potential $A^{(self)} (r)$ in terms of the 4-current $j^{(self)} (r)$. This technique is well-known. Thus, considering the Maxwell equations in flat space-time, in the Lorentz gauge $A^{(self)} = 0$, the self 4-potential must satisfy the wave equation

$$\Box A^{(self)} = \frac{4 \pi}{c} j^{(self)} (r),$$

(69)

where $\Box$ represents the D’Alembertian operator and $j^{(self)} (r)$ is given by Eq. (68). The formal solution of Eq. (69) is

$$A^{(self)} (r) = \frac{4 \pi}{c} \int d^4 r' G (r, r') j^{(self)} (r'),$$

(70)

where $G (r, r')$ is the retarded Green’s function corresponding to the prescribed charge density. By construction, it follows that $G (r, r')$ is symmetric with respect to $r$ and $r'$, and furthermore - since the particle is finite-size - both the 4-current and the self-potential are everywhere well-defined.

From these general results, it is immediate to prove the following theorem.

**THM.3 - Symmetry properties of $S^{(self)} (z, [\mathcal{C}])$**

Given validity of Eq. (68) for the covariant expression of the current density for a finite-size charged particle and of Eq. (70) for the general expression of the corresponding EM self-potential, it follows that:
The functional $S_C^{(self)}(z,[z])$, defined in Eq. (68), as an integral over the 4-volume element $d^4r$, can be written as a line integral of the form
\begin{equation}
S_C^{(self)}(z,[z]) = \int_{-\infty}^{\infty} ds L_C^{(self)}(z,[z]),
\end{equation}

where $L_C^{(self)}$ represents the Lagrangian of the coupling with the EM self-field. This is defined as
\begin{equation}
L_C^{(self)}(z,[z]) = \frac{4\pi q}{c^2} \int_1^2 d^4r \left\{ \left[ u^\mu \delta(s-s_1) - \frac{1}{c} \omega^{\mu\nu} x_{\nu} \delta(x^\alpha u_\alpha) \right] f(x^2) \int d^4r' G(r,r') j_\mu^{(self)}(r') \right\}.
\end{equation}

The functional $S_C^{(self)}(z,[z])$ contains both local and non-local dependencies in terms of the variational quantities $z \equiv z(s)$ and $[z] \equiv [z(s)]$. In particular, it is symmetric in these local and non-local variables, in the sense stated in THM.1, namely
\begin{equation}
S_C^{(self)}(z,[z]) = S_C^{(self)}([z],z).
\end{equation}

The functional $S_C^{(self)}(z,[z])$ contains at most only first-order derivatives of the variational functions $z(s)$.

Proof - T3.1) The proof of the first statement follows by noting that the action integral $S_C^{(self)}(z,[z])$ is a 4-scalar by definition. Hence, making explicit the expressions of $A^{(self)}_\mu$ and $j_\mu^{(self)}$ in Eq. (60) according to the results in Eqs. (61) and (68), by exchanging the order of the integrations and invoking the symmetry property of the Green function, the conclusion can be easily reached. In particular, the variational Lagrangian is found to be of the general form given in Eq. (72).

T3.2) To prove the second statement we first notice that in Eq. (71) both $z$ and $z'$ are integration variables, while by definition the variational quantities are identified with $z(s)$ and $[z(s)] \equiv [z(s')]$. These dependencies are carried respectively by the charge current densities $j^{(self)}_\mu(r)$ and $j^{(self)}_\mu(r')$. The result is then reached by noting that the functional carrying the self-coupling terms is symmetric with respect to the integrated quantities, and in particular with respect to $j^{(self)}_\mu(r)$ and $j^{(self)}_\mu(r')$. Hence, exchanging $(z, j^{(self)}_\mu(r))$ with $(z', j^{(self)}_\mu(r'))$ does not affect the form of the functional, with the consequence that Eq. (73) is identically satisfied.

The proof of the statement is an immediate consequence of the representation for the current density $j^{(self)}_\mu(r)$ given in Eq. (68). In fact, the term proportional to the acceleration $\frac{d^2 R}{ds^2}$ in Eq. (64) and which is associated to the Thomas precession, does not appear in Eq. (68), thanks to the property of the Dirac-delta function indicated above in Eq. (66).

Q.E.D.

An immediate consequence of THM.3 is that, thanks to THMs.1 and 2, the variational treatment of the dynamics of finite-size charged particles subject to the EM RR effect admits both Lagrangian and Hamiltonian representations in standard form. In particular, in this case, it follows that the following identification must be introduced:
\begin{equation}
L_b \equiv L_C^{(self)},
\end{equation}

where $L_b$ is the Lagrangian defined above in THM.1.

V. HAMILTONIAN THEORY FOR THE RR PROBLEM

In this section, based on THMs.1-3 and the theory developed in Paper I, we proceed constructing the Hamiltonian formulation for the RR problem. For this purpose, it is convenient to recall the explicit form of the EM self 4-potential obtained in Paper I. For the sake of comparison with traditional approaches based on point particle models, here we also propose an alternative approach based on the Green function method. Remarkably, as pointed out in Appendix A, for the spherically-symmetric and non-rotating extended particle considered here, the self-potential is proved to be formally analogous to the well-known solution valid for point charges. This result holds, however, only in the external domain (with respect to $\partial \Omega$), where $A^{(self)}_\mu(r)$ is found to admit the integral representation (see details in Appendix A):
\begin{equation}
A^{(self)}_\mu(r) = 2q \int_1^2 dr' \delta(R^\alpha \tilde{R}_\alpha).
\end{equation}
Here \( \hat{R}^\alpha = r^\alpha - r^\alpha(s') \), with \( r^\alpha \) and \( r^\alpha = r^\alpha(s') \) denoting respectively the generic 4-position and the 4-position of the center of the charge distribution at proper time \( s' \). As a fundamental consequence of the finite extension of the particle and the restrictions on the domain of validity of Eq.(75), the resulting variational functional and Faraday tensor for the self-field turn out to be completely different from the point-particle treatment. In particular, the action integral becomes now a non-local functional with respect to the 4-position \( r \). As pointed out in Paper I, this can be written as a line integral in terms of a variational Lagrangian \( L_1(r,[r]) \) as follows:

\[
S_1(r,[r]) = \int_{-\infty}^{+\infty} ds L_1(r,[r]).
\]

(76)

Here \( L_1(r,[r]) \) is defined as:

\[
L_1(r,[r]) = L_M(r) + L_C^{(ext)}(r) + L_C^{(self)}(r,[r]),
\]

(77)

where

\[
L_M(r,u) = \frac{1}{2} m_o c \frac{dr^\mu}{ds} \frac{dr^\nu}{ds},
\]

(78)

\[
L_C^{(ext)}(r) = \frac{q}{c} \frac{dr^\mu}{ds} \tilde{A}_\mu^{(ext)}(r),
\]

(79)

are the local contributions respectively from the inertial and the external EM field coupling terms, with \( \tilde{A}_\mu^{(ext)} \) denoting the surface-averaged external EM potential (see Paper I for its definition). On the other hand, \( L_C^{(self)} \) represents the non-local contribution arising from the EM self-field coupling, which is provided by

\[
L_C^{(self)}(r,[r]) = \frac{2 q^2}{c} \frac{dr^\mu}{ds} \int_1^2 dr' \delta(\hat{R}^\mu - \sigma^2),
\]

(80)

where the 4-scalar \( \sigma^2 \equiv \xi^\mu \xi_\mu \) is the radius of the surface distribution with respect to the center \( r^\mu(s) \) and \( \hat{R}^\mu \) is defined as

\[
\hat{R}^\alpha \equiv r^\alpha(s) - r^\alpha(s').
\]

(81)

Notice that \( \hat{R}^\alpha \) represents the displacement bi-vector between the actual position \( r^\alpha(s) \) of the charge center at proper time \( s \) and the retarded position \( r^\alpha(s') \) of the same point at the retarded proper time \( s' \). It is immediate to verify that the representation of \( S_C^{(self)} \) in terms of \( L_C^{(self)} \) given in Eq.(80) satisfies the hypothesis of THM.1, and therefore the solution admits a Lagrangian representation in standard form. As already shown in Paper I and according to THM.1, this is obtained by setting

\[
L_{\text{eff}} \equiv L_M(r) + L_C^{(ext)}(r) + 2 L_C^{(self)}(r,[r]),
\]

(82)

with \( L_M(r) \), \( L_C^{(ext)}(r) \) and \( L_C^{(self)} \) respectively given by Eqs. (78)-(80). Then, the corresponding E-L equation is provided by the following covariant 4-vector, second-order delay-type ODE:

\[
m_o c \frac{du^\mu(s)}{ds} = \frac{q}{c} F_{\mu\nu}^{(ext)}(r(s)) \frac{dr^\nu(s)}{ds} + \frac{q}{c} F_{\mu\nu}^{(self)}(r(s),r(s')) \frac{dr^\nu(s)}{ds},
\]

(83)

where

\[
u^\mu(s) \equiv \frac{dr^\mu(s)}{ds}.
\]

(84)

Here the notation is as follows. Denoting by \( F_{\mu\nu} \equiv F_{\mu\nu}^{(ext)} + F_{\mu\nu}^{(self)} \) the total Faraday tensor, \( L_{\mu\nu}^{(ext)} \) and \( F_{\mu\nu}^{(self)} \) are respectively the “external” and “self” Faraday tensors generated by \( A_{\nu}^{(ext)} \) and \( A_{\nu}^{(self)} \), which carry the contributions due to the external sources with respect to the charged particle and the particle EM self-interaction. In particular, the 4-tensor \( F_{\mu\nu}^{(ext)}(r(s)) \) denotes the surface-average of the Faraday tensor associated to the external EM field, to be identified with

\[
F_{\mu\nu}^{(ext)} \equiv \partial_\mu A_\nu^{(ext)} - \partial_\nu A_\mu^{(ext)},
\]

(85)
In the parameter-free representation this is given by
\[ s \text{ with } \chi \text{ the Hamiltonian vector field} \]

\[ F_{\mu k}^{(self)} (r, [r]) = -4q \int_1^2 \left[ dr'_{\mu} \frac{\partial}{\partial r'_{\mu}} \delta \left( \tilde{R}^\alpha \tilde{R}_\alpha - \sigma^2 \right) - dr'_{\mu} \frac{\partial}{\partial r_{\mu}} \delta \left( \tilde{R}^\alpha \tilde{R}_\alpha - \sigma^2 \right) \right]. \tag{86} \]

As pointed out in Paper I, \( F_{\mu k}^{(self)} \) can also be parametrized in terms of the particle proper time \( s \), by letting \( r \equiv r (s) \) and \([r] \equiv r (s') \) in the previous equation, which also implies \( dr'_{\mu} \equiv ds \frac{dr'_{\mu}}{ds} \). This means that the non-locality in Eq. (86) can be interpreted as non-locality in the particle proper time.

The remarkable feature of Eq. (86) is that the RR self-force (see the second term in the rhs of Eq. (83)) contains non-local effects only through the retarded particle 4-position and not through the 4-velocity. This feature is fundamental for the subsequent fluid treatment, since it permits the evaluation in the standard way of the velocity moments, retaining the exact form of the RR self-interaction.

The system of Eqs. (83) and (84) defines a delay-type ODE problem of the form
\[
\frac{ds}{ds} = X_H (y, [r]), \quad y (s_0) = y_0, \quad y (s_0) = y_{s_0}, \quad \forall s_0 \in I_{s_0, s_0 - s_{ret}},
\tag{87}
\]

with \( s_0 \) and \( s_{ret} \) denoting respectively the initial particle proper time and the causal retarded proper time (see Paper I), and \( X_H \) the Hamiltonian vector field
\[
X_H (y, [r]) = \left\{ \frac{\partial H_{eff} (r, P_s [r])}{\partial P_\mu}, - \frac{\partial H_{eff} (r, P_s [r])}{\partial r_\mu} \right\}.
\tag{88}
\]

Denoting by \( y (s) = \chi (y_0, \{ y_{s_0}, \forall s' \in I_{s_0, s_0 - s_{ret}} \}, s - s_0 \) the formal solution of the problem (87), in the reminder we shall assume that the map
\[
y_0 \rightarrow y (s)
\tag{89}
\]
is a diffeomorphism of class \( C^k \), with \( k \geq 1 \).

Based on these results, the Hamiltonian formulation is provided by the following theorem.

**THM. 4 - Non-local variational and effective Hamiltonian functions for the non-rotating particle**

Given validity of THMs. 1-3, it follows that:

**T4.1** The RR equation (83) for a non-rotating and spherically-symmetric charged particle admits the non-local Hamiltonian system \( \{ y \equiv (r^\mu, p_\mu), H_1 \} \). Here \( p_\mu \) and \( H_1 \equiv H_1 (r, p, [r]) \) are respectively the canonical momentum (22) defined with respect to the variational Lagrangian \( \mathcal{L}_1 \) given in Eq. (77), and the corresponding non-local variational Hamiltonian (27) defined as the Legendre transformation of \( \mathcal{L}_1 \). In particular, the variational non-local Hamiltonian (27) is identified with
\[
H_1 (r, p, [r]) = \frac{1}{2m_o c} \left( p_\mu - \frac{q}{c} A_\mu \right) \left( p^\mu - \frac{q}{c} A^\mu \right),
\tag{90}
\]

where \( A_\mu \) is the total EM 4-potential
\[
A_\mu (r, [r]) = A^{(ext)}_\mu (r) + A^{(self)}_\mu (r, [r]),
\tag{91}
\]

and from Eq. (80) \( A^{(self)}_\mu \) is the functional
\[
A^{(self)}_\mu (r, [r]) = 2q \int_1^2 dr'_{\mu} \delta (\tilde{R}^\alpha \tilde{R}_\alpha - \sigma^2).
\tag{92}
\]

**T4.2** There exist \( P_\mu \) and \( H_{eff} \), defined respectively by Eqs. (22) and (32), such that
\[
H_{eff} (r, P_s [r]) = \frac{1}{2m_o c} \left( P_\mu - \frac{q}{c} A_{(eff)}^\mu \right) \left( P^\mu - \frac{q}{c} A^{(eff)}_\mu \right),
\tag{93}
\]
with \( A_{(\text{eff})\mu} \) the non-local effective EM 4-potential

\[
A_{(\text{eff})\mu}(r, P) \equiv A^{(\text{ext})}_\mu(r) + 2A^{(\text{self})}_\mu(r, [r]) \quad (94)
\]

and \( A^{(\text{self})}_\mu \) defined in Eq. (92).

\( T^{(4,3)} \) The effective and variational Hamiltonian functions \( H_{\text{eff}} \) and \( H_1 \) coincide when expressed in terms of the 4-velocity \( \frac{dr^\mu(s)}{ds} \).

**Proof** - The proof of \( T^{(4,1)} \) and \( T^{(4,2)} \) follows immediately by applying THMs.1 and 2 with the variational Lagrangian \( L_1 \) given by Eq. (77). In particular, this yields

\[
p_\mu = m_o c \frac{dr_\mu(s)}{ds} + \frac{q}{c} \left[ A^{(\text{ext})}_\mu + A^{(\text{self})}_\mu \right] \quad (95)
\]

and

\[
P_\mu = m_o c \frac{dr_\mu(s)}{ds} + \frac{q}{c} \left[ A^{(\text{ext})}_\mu + 2A^{(\text{self})}_\mu \right] . \quad (96)
\]

The corresponding Legendre transformations then provide respectively Eq. (90) and Eq. (93). Finally, by direct substitution of Eq. (95) into Eq. (90) and Eq. (96) into Eq. (93), one obtains that

\[
H_{\text{eff}} = H_1 = \frac{m_o c}{2} \frac{dr_\mu(s)}{ds} \frac{dr^\mu(s)}{ds} , \quad (97)
\]

which proves also the last statement.

Q.E.D.

We remark that the Hamilton equation in standard form expressed in terms of \( H_{\text{eff}} \) and \( P_\mu \) are differential equations of delay-type, as a consequence of the non-local dependencies appearing in \( H_{\text{eff}} \) which are characteristic of the RR phenomenon. In this case, for the well-posedness of the solution the initial conditions in the interval \( I = [s_0 - s_{\text{ret}}, s_0] \) must be defined, with \( s_0 \) the initial proper time and \( s_{\text{ret}} \) a suitable retarded time. However, if the assumption of inertial motion in the proper time interval \( I_0 = [-\infty, s_0] \) holds, then the mapping

\[
T_{s_0,s} : y_0 \equiv y(s_0) \rightarrow y(s) \equiv T_{s_0,s}y_0, \quad (98)
\]

with \( y = (r^\mu, P_\mu) \), defines a classical dynamical system (see Paper I), and this dynamical system is Hamiltonian.

VI. A HAMILTONIAN ASYMPTOTIC APPROXIMATION FOR THE RR EQUATION

In this section we intend to carry out in detail a comparison of the present approach for extended particles with the customary point-particle treatments leading to the LAD and LL equations. For this purpose, asymptotic approximations of the exact RR self-force (86) are investigated.

The issue has been partially discussed in Paper I. As pointed out therein, an asymptotic approximation of the exact RR equation (83) can be obtained in validity of the short delay-time ordering, namely requiring

\[
0 < \epsilon \equiv \left| \frac{s_{\text{ret}}}{s} \right| \ll 1, \quad (99)
\]

where \( s_{\text{ret}} = s - s' \), with \( s \) and \( s' \) denoting respectively the present and retarded particle proper times. This permits two different possible strategies, respectively based on Taylor expansions performed with respect to \( s \) (present-time expansion) or \( s' \) (retarded-time expansion). In particular, adopting the present-time expansion for the RR self-force (86), the delay-type ODE (83) can be reduced, in principle, to an infinite-order differential equation. Instead, by truncating the Taylor expansion to first-order in \( \epsilon \), ignoring mass-renormalization terms and taking the point-particle limit \( \sigma \rightarrow 0 \), in this way the customary expression for the LAD equation is recovered (see THM.3 of Paper I).

As remarked in the Introduction, the resulting asymptotic approximation (given by the LAD equation) is non-variational and therefore non-Hamiltonian. In addition, contrary to the exact RR equation obtained here, the LAD equation, as well as the related LL approximation, both fail in the transient time intervals occurring when the external EM field acting on the particle is turned on and off. To elucidate this point, let us consider the dynamics of a charged
particle which is in inertial motion in the past for all \( s < s_0 \) and from \( s = s_0 \) is subject to the action of an external EM field. Then, by construction, it is immediate to show that in the transient time interval \( I_0 = [s_0, s_0 + s_{ret}] \) the exact RR self-force \( F_{µν}(s) \) is manifestly identically zero. In fact, in the case of inertial motion in the past (namely \( u_µ(s') = const. \) ) the RR self-force vanishes in such a time interval (see THM.1 in Paper I). In contrast, both the LAD and LL equations predict incorrectly a non-vanishing RR self-force. The same kind of inconsistency (for the LAD and LL equations) arises when the analogous transient time interval corresponding to the turning-off of the external EM field is considered.

Therefore, the issue arises whether an alternative asymptotic approximation can be determined (for the exact RR equation) which simultaneously:

1) overcomes this deficiency, by taking into account consistently relativistic finite delay-time effects characteristic of the RR phenomenon;

2) is variational and admits a standard Hamiltonian formulation.

In this section we propose a solution to this problem, by performing a retarded-time expansion, which provides an alternative to the LAD and LL equations.

A. The Hamiltonian approximation

For definiteness, let us assume that the external force acting on the particle is non-vanishing only in a finite proper-time interval \( I \equiv [s_0, s_1] \). Then, in validity of the ordering \( [99] \), we require that the external EM force is slowly varying in the sense that, denoting \( r' \equiv r^µ(s') \) and \( r \equiv r^µ(s) \),

\[
\frac{T^µν_{ext}(r') - T^µν_{ext}(r)}{c} \sim O(\epsilon), \quad (100)
\]

\[
\left( T^µν_{ext}(r') - T^µν_{ext}(r) \right)_{,h} \sim O(\epsilon), \quad (101)
\]

\[
\left( T^µν_{ext}(r') - T^µν_{ext}(r) \right)_{,hk} \sim O(\epsilon). \quad (102)
\]

Then, the retarded-time Hamiltonian approximation of the RR equation is obtained by performing a Taylor expansion in a neighborhood of \( s' \). The result is summarized by the following theorem.

**THM.5 - First-order, short delay-time Hamiltonian approximation (retarded-time expansion).**

Given validity of the asymptotic ordering \( [99] \) and the smoothness assumptions \( [100]-[102] \) for the external EM field, neglecting corrections of order \( \epsilon^n \), with \( n \geq 1 \) (first-order approximation), the following results hold:

1. **T5.1** The vector field

\[
G_µ \equiv \frac{q}{c} F^{(self)}_{µk}(r(s), r(s')) \frac{d^k(s)}{ds}, \quad (103)
\]

appearing in Eq. \( [58] \) can be approximated in a neighborhood of \( s' \) as

\[
g_µ(r(s')) = \left\{ -m_{aEM} \frac{d}{ds} u_µ(s') + g'_µ(r(s')) \right\}, \quad (104)
\]

to be referred to as retarded-time Hamiltonian approximation, in which the first term on the rhs identifies a retarded mass-correction term, \( m_{aEM} \equiv \frac{q^2}{c^2} \) denoting the leading-order EM mass. Finally, \( g'_µ \) is the 4-vector

\[
g'_µ(r(s')) = -\frac{1}{3} \frac{q^2}{c} \left[ \frac{d^2}{ds'^2} u_µ(s') - u_µ(s') u_k(s') \frac{d^2}{ds'^2} u_k(s') \right]. \quad (105)
\]

2. **T5.2** The corresponding RR equation, obtained replacing \( G_µ \) with the asymptotic approximation \( g_µ \) \( [104] \), is variational, Lagrangian and admits a standard Lagrangian form. Let us denote with \( r'_0 \equiv r_0(s') \) the extremal particle world-line at the retarded proper time \( s' \). Then, in this approximation the corresponding asymptotic variational Lagrangian and effective Lagrangian functions coincide. Both are defined in terms of the asymptotic approximation \( L^{(self)}_{C, asym}(r, r'_0) \), replacing \( L^{(self)}_{C} \). To leading-order in \( \epsilon \), this is found to be

\[
L^{(self)}_{C, asym}(r, r'_0) = g_µ(r'_0) r^µ. \quad (106)
\]
The asymptotic approximation given by Eq. (104) is also Hamiltonian. The asymptotic variational and effective Hamiltonian functions coincide and are given by

\[ H_{1,\text{asym}} = p_\mu \frac{dr_\mu}{ds} - L_{1,\text{asym}} \tag{107} \]

with

\[ L_{1,\text{asym}}(r, r'_0) = L_M(r) + L^{(\text{ext})}_C(r) + L^{(\text{self})}_C, \tag{108} \]

and now

\[ p_\mu = \frac{\partial L_{1,\text{asym}}}{\partial \frac{dr_\mu}{ds}}. \tag{109} \]

Proof - T51) The proof can be carried out starting from Eq. (83) and performing explicitly the Taylor expansion in a neighborhood of \( s' \equiv s - s_{\text{ret}} \). For a generic analytic function \( f(s) \), this yields the power series of the form

\[ f(s) = \sum_{k=0}^{\infty} \frac{(s - s')^k}{k!} \frac{d^k f(s')}{ds'^k}. \tag{110} \]

In particular, for the 4-vectors \( \frac{dr_\mu}{ds} \) and \( \tilde{R}^k \) one obtains respectively the asymptotic approximations

\[ \frac{dr_\mu}{ds} \approx \frac{dr_\mu(s')}{ds'} + (s - s') \frac{d^2 r_\mu(s')}{ds'^2} + \frac{(s - s')^2}{2} \frac{d^3 r_\mu(s')}{ds'^3} + O(\epsilon^3) \tag{111} \]

and

\[ \tilde{R}^k \approx (s - s') \frac{d^k r_\mu(s')}{ds'^k} + \frac{(s - s')^2}{2} \frac{d}{ds'} \tilde{r}^k(s') + \frac{(s - s')^3}{6} \frac{d^2}{ds'^2} \tilde{r}^k(s') + O(\epsilon^4), \tag{112} \]

while for the time delay \( s - s' \equiv s_{\text{ret}} \) the leading-order expression

\[ s - s' \approx \sigma + O(\epsilon^2) \tag{113} \]

holds. By substituting these expansions in Eq. (80), the asymptotic solution given by Eq. (104) can be recovered.

T52)-T53) The proof follows by first noting that \( L^{(\text{self})}_C \) contributes to the Euler-Lagrange equations only in terms of the local dependence in terms of \( r \). Then, in this approximation the canonical momentum becomes

\[ p_\mu = m_0 c \frac{dr_\mu(s)}{ds} + \frac{q}{c} A^{(\text{ext})}_\mu(r) = P_\mu, \tag{114} \]

while the asymptotic Hamiltonian reduces to

\[ H_{1,\text{asym}}(r, p, r'_0) = \frac{1}{2m_0 c} \left( p_\mu - \frac{q}{c} A^{(\text{ext})}_\mu(r) \right) \left( p^\mu - \frac{q}{c} A^{(\text{ext})}_\mu(r) \right) + g_\mu (r'_0) r^\mu. \tag{115} \]

The corresponding Lagrangian and Hamiltonian equations manifestly coincide with Eq. (83) once the approximation (104) is invoked for the vector field \( G_\mu \).

Q.E.D.

B. Discussion and comparisons with point-particle treatments

The asymptotic Hamiltonian approximation, here pointed out for the first time (see THM.5), preserves the basic physical properties of the exact RR force (83). In fact in both cases, the RR force:

1) is non-local, depending on the past history of the finite-size charged particle;
2) admits a variational formulation;
3) is both Lagrangian and Hamiltonian;
4) satisfies the Einstein Causality Principle and, when applicable, the Newton Principle of Determinacy (see also Paper I);
5) describes correctly the transient time intervals in which the external force is turned on and off (sudden force).

For these reasons, physical comparisons based on the retarded-time Hamiltonian asymptotic approximation are meaningful. In particular, here we remark that the present approach departs in several ways with respect to point-particle treatments based on the LAD and LL equations. More precisely:

1) The same type of asymptotic ordering is imposed, which is based on the short delay-time ordering (99). However, in contrast with the LAD and LL equations, the expansion adopted in THM.5 and leading to the retarded-time Hamiltonian approximation can only be performed based on the knowledge of the exact RR force for finite-size particles.

2) Unlike the LAD and LL equations, the asymptotic Hamiltonian approximation carries the information of the past dynamical history of the charged particle through the retarded time \( s' \). Therefore, the dynamical equation written adopting the approximation (104) is still a delay-type second-order ODE. The construction of its general solution becomes trivial in this case, since the self-force is considered as an explicit source term evaluated at proper time \( s' \).

3) The asymptotic approximation provided by Eq. (104) cannot be regarded as a point-particle limit. In fact, the retarded mass-correction term would diverge in this limit.

4) The exact RR equation satisfies identically by construction the kinematic constraint \( u_\mu u^\mu = 1 \). The same constraint is satisfied to leading-order in \( \epsilon \) also both by the retarded and present-time asymptotic expansions (and hence also the LAD equation).

5) The variational principle introduced in THM.5 is subject to the constraint that the past history is considered prescribed in terms of the extremal world-line. This requirement is consistent with the initial conditions for the RR equation, which is a delay-type ODE depending only on the past history of the particle. This requires that the world-line trajectory is prescribed in the past, namely in the time interval \( I = [-\infty, s_0] \). Since, however, the initial proper time \( s_0 \) is arbitrary, it follows that \( r(s) \) can be considered prescribed also in the time interval \( I' = [-\infty, s'] \).

In particular, if for all \( s < s_0 \) the motion is assumed to be inertial, the initial-value problem associated to the RR equation written in terms of the retarded asymptotic self-force (104) is well-posed, in the sense of the standard Newton Principle of Determinism, as discussed in Paper I (see in particular THM.4 presented there and dealing with the existence and uniqueness of solutions for the exact RR equation).

6) One might think that the same type of constrained variational principle, of the kind adopted in THM.5, could be adopted also for the exact RR equation. However, this belief is wrong. In fact, since the variational functional (77) is symmetric with respect to the local and non-local world-line trajectories, there is no distinction between past and future. Since future cannot be prescribed, such a constrained variational principle for the exact equation is forbidden. On the contrary, the extremal RR equation (88) is obtained by imposing also the Einstein Causality Principle, and therefore it depends only on the past history.

7) Despite some formal similarities between the retarded-time Hamiltonian approximation versus the corresponding LAD and LL equations, the latter cannot be recovered even in the framework of some kind of constrained variational principle. In fact this would require to consider prescribed for example, second or higher-order proper-time derivatives of the particle position vector (namely the acceleration and its derivatives). This viewpoint is manifestly unacceptable, because it would amount to constraint the present state of the particle at proper time \( s \).

8) The previous argument justifies, in turn, the introduction of the short delay-time asymptotic approximation given in THM.5. This is performed directly on the RR force, namely the 4-vector \( G_\mu \) entering the RR equation itself. In this way the variational character of the RR problem is preserved. It follows that the corresponding variational functional as well as the Lagrangian and Hamiltonian functions for the asymptotic RR equation are constructed only “a posteriori”.

9) Another advantage of the new representation (104) with respect to the customary LAD and LL equations is that it permits the approximate treatment of the solution also in the transient time intervals after the turning-on or the turning-off of the external EM field. In particular, in contrast to the LAD and LL equations, it predicts a vanishing RR self-force in the turning-on transient phase \( I_0 = [s_0, s_0 + s_{\text{ret}}] \).

10) Finally, it should be remarked that the retarded asymptotic self-force (104) cannot be trivially obtained from the corresponding local asymptotic representation performed at proper time \( s \) and leading to the LAD equation by simply exchanging \( s \) with \( s' \) (or by a further Taylor expansion). Indeed, the relationship between the two can only be established based on the exact form of the self-force.
VII. COLLISIONLESS RELATIVISTIC KINETIC THEORY FOR THE EM RR EFFECT - CANONICAL FORMALISM

In this section we proceed with the construction of the relativistic classical statistical mechanics (CSM) for a collisionless plasma with the inclusion of the EM RR effect. In particular we shall prove that the mathematical formalism introduced in the previous sections to deal with symmetric non-local interactions allows one to obtain a convenient formulation for the kinetic theory describing such a system and for the corresponding fluid representation. The derivation is based on the property of a symmetric non-local system represented by a finite-size charged particle of being Hamiltonian with respect to \( P_{\mu} \) and \( H_{\text{eff}} \).

In view of the peculiar features of the non-local RR phenomenon and the related delay-type differential Hamiltonian equations, it is instructive here to adopt an axiomatic formulation of the CSM for relativistic systems with the inclusion of such an effect. We shall assume that the latter are represented by a system of classical finite-size charged particles subject only to the action of a mean-field external EM force and a non-local self-interaction. We intend to show that, using the Hamiltonian representation in standard form given above, the explicit form of the relativistic Vlasov kinetic equation can be obtained for the kinetic distribution function describing the statistical dynamics of such a system. Therefore, the problem is reduced to a Vlasov-Maxwell description for a continuous distribution of relativistic charged particles.

For definiteness, let us consider the non-local Hamiltonian dynamical system in standard form \( \{ y, H_{\text{eff}} \} \) given above. This is characterized by the superabundant state vector \( y = (r^\mu, P_\mu) \) spanning the extended 8th-dimensional phase-space \( \Gamma \) and with essential state variables \( y_1(y) \) spanning the 6th-dimensional reduced phase-space \( \Gamma_1 \). Introducing the global proper time \( \hat{s} \), \( \Gamma_1(\hat{s}) \) is defined as

\[
\Gamma_1(\hat{s}) \equiv \left\{ y: y \in \Gamma, \ |u| = 1, \ s(y) = \hat{s}, \ ds(y) = \sqrt{g_{\mu\nu}dr^\mu dr^\nu} \right\},
\]

where \( |u| = \sqrt{u^\nu u_\nu} \) and \( s(y) \) is the world-line proper time uniquely associated to any \( y \). By assumption, \( \Gamma_1(\hat{s}) \) is an invariant set, i.e., \( \Gamma_1(\hat{s}) = \Gamma_1 \) for any \( \hat{s} \in \mathbb{R} \). Next, let us consider the Hamiltonian flow \( T_{s_0,s} \) defined in Eq. (119). By construction the dynamical system is autonomous, namely the flow is of the form

\[
T_{s_0,s} y_0 \equiv \chi(y_0, s - s_0).
\]

The existence of the dynamical system \( T_{s_0,s} \), for the state \( y(s) \) has been proved in Paper I. This requires that in the proper time interval \( I_0 = [-\infty, s_0] \) the motion of each charged particle is inertial, namely the external EM field vanishes in the same interval. As a result of Eq. (118), any point in the phase-space \( \Gamma \) spanned by \( y \) or \( y_0 \) is associated to a unique phase-space trajectory, namely such that \( y = y(s) \), for any \( y \in \Gamma \). Due to (118) there exists necessarily \( y_0 = y(s_0) \) which is mapped in \( y(s) \). Viceversa, for any \( s \in \mathbb{R} \) there exists a unique \( y = y(s) \). However, we notice here that for the axiomatic formulation of the CSM for the RR problem the assumption of existence of the dynamical system \( T_{s_0,s} \) is not a necessary condition. In fact, it is immediate to prove that the minimal requirement is actually provided only by the existence of the diffeomorphism \( [39] \) defined above.

Now, for a prescribed \( \hat{s}_0 \in \mathbb{R} \) let us consider the set \( B(\hat{s}_0) \subseteq \Gamma_1 \), with \( B(\hat{s}_0) \) an ensemble of states \( y_0 \), each one prescribed at the initial proper time \( s_0 = \hat{s}_0 \). Its image generated at any \( s = \hat{s} \in \mathbb{R} \) by the flow \( T_{s_0,s} \), for each trajectory, is

\[
B \equiv B(s) \equiv T_{s_0,s} B(s_0),
\]

where \( s \) and \( s_0 \) denote now the global proper times \( \hat{s} \) and \( \hat{s}_0 \).

We introduce the following axioms.

AXIOM #1: Probability on \( K(\Gamma) \).

Let \( K(\Gamma_1) \) be a family of subsets of \( \Gamma_1 \) which are \( L \)-measurable. We define the probability of \( B(s) \in K(\Gamma_1) \) as the function

\[
P(B) : K(\Gamma_1) \rightarrow [0, 1]
\]

such that it satisfies the constraints

\[
P(\Gamma_1) = 1,
\]

\[
P(\emptyset) = 0,
\]

\[
P(\bigcup_{i \in N} B_i) = \sum_{i = 0}^{\infty} P(B_i),
\]

for all \( B_i \) which are \( K(\Gamma_1) \)-measurable.
with \( \{ B_i \in K(\Gamma_1), i \in N \} \) being an arbitrary family of separate sets of \( K(\Gamma_1) \).

**AXIOM #2: Probability density.**

For any \( B(s) \in K(\Gamma_1) \) and for any state \( y \equiv (r^\mu, P_\mu) \) there exists a unique probability density \( \rho(y) > 0 \) on \( \Gamma_1 \) such that

\[
P(B(s)) = \int_{\Gamma} dy \rho(y) \delta(|u| - 1) \delta(s - s(y)) \delta_{B(s)}(y),
\]

where \( dy = dr^\mu dP_\mu \) is the canonical measure on \( \Gamma \) and \( \delta_{B(s)}(y) \) is the characteristic function of \( B(s) \). Furthermore, \( s(y) \) is a particle world-line proper time, while \( s \equiv s_0 + \Delta s \), with \( \Delta s \) an invariant proper time interval independent of \( s_0 \). We notice that \( s(y) \) can be equivalently parametrized in terms of the observer's coordinate time \( r^0 \), namely:

\[
ds(y) \equiv dr^0 \sqrt{g_{\mu\nu} \frac{dr^\mu}{dr^0} \frac{dr^\nu}{dr^0}}.
\]

**AXIOM #3: Equiprobability.**

Then, the equiprobability condition requires that, for all \( B(s_0) \) and for all \( s, s_0 \in I \subseteq \mathbb{R} \),

\[
P(B(s)) = P(B(s_0)).
\]

We remark that in the integral \([123]\) the two Dirac-delta functions can be interpreted as physical realizability conditions, required to reduce the dimension of the volume element \( dy \) defined on the extended phase-space \( \Gamma \).

We can now introduce the following theorem, concerning the validity of the Liouville equation for \( \rho(y) \).

**THM.6 - Relativistic Liouville equation for \( \rho(y) \).**

*Given a Hamiltonian system \( \{ y, H_{eff} \} \) and imposing the validity of Axioms #1-#3, it follows that the probability density \( \rho(y(s)) \) is a constant of motion, namely for any \( s, s_0 \in \mathbb{R} \) (to be intended now as world-line proper times) and for any \( y_0 \in \Gamma \)

\[
\rho(y(s)) = \rho(y_0),
\]

to be referred to as the integral Liouville equation. This can also be written equivalently as

\[
\frac{d}{ds} \rho(y(s)) = 0,
\]

to be referred to as the differential Liouville equation. As a consequence, introducing the kinetic distribution function (KDF) \( f(y) \)

\[
f(y) \equiv \rho(y) N,
\]

with \( N \) being the total number of particles in the configuration space of \( B \subseteq K(\Gamma) \), it follows that also \( f(y) \) satisfies the Liouville equation \([124]\).

*Proof -* We first notice that, from Axiom #1, by changing the integration variables we can write Eq.\([123]\) as

\[
P(B(s)) = \int_{\Gamma} dy y \rho(y) \delta(|u| - 1) \delta(s - s(y)) \delta_{B(s)}(y) = \int_{\Gamma} dy_0 \left| \frac{\partial y(s)}{\partial y_0} \right| \rho(y_0) \delta(|u| - 1) \delta(s - s(y)) \delta_{B(s_0)}(y(s_0)),
\]

with \( \left| \frac{\partial y(s)}{\partial y_0} \right| \) being the Jacobian of the variable transformation from \( y(s) \) to \( y_0 \). On the other hand, since the system \( \{ y, H_{eff} \} \) is Hamiltonian, it follows identically that \( \left| \frac{\partial y(s)}{\partial y_0} \right| = 1 \). Hence, invoking Axiom #2 we can write

\[
\int_{\Gamma} dy_0 \left[ \rho(y_0) \delta(|u| - 1) \delta(s - s(y)) - \rho(y_0) \delta(|u_0| - 1) \delta(s_0 - s(y_0)) \right] \delta_{B(s_0)}(y(s_0)) = 0,
\]

from which it must be that

\[
\rho(y(s)) \delta(|u| - 1) \delta(s - s(y)) = \rho(y_0) \delta(|u_0| - 1) \delta(s_0 - s(y_0)).
\]
On the other hand, by construction it follows that
\[
\delta \left( |u| - 1 \right) = \frac{1}{d|u|} \left( \delta \left( |u_0| - 1 \right) = \delta \left( |u_0| - 1 \right), \right.
\]
(132)
\[
\delta \left( s - s \left( y \right) \right) = \frac{1}{ds_0} \delta \left( s_0 - s \left( y_0 \right) \right) = \delta \left( s_0 - s \left( y_0 \right) \right). \quad (133)
\]
In fact, by definition the 4-velocity is normalized to 1 at all proper times, so that \( d|u| \left| d|u_0| \right| = 1 \). Furthermore, \( s \equiv s_0 + \Delta s \), with \( \Delta s \) being independent of the initial value \( s_0 \), and hence \( ds_0 \left| ds_0 \right| = 1 \) too.

Finally, because of these conclusions, from Eq. (131) it follows that
\[
\rho \left( y \left( s \right) \right) = \rho \left( y_0 \right), \quad (134)
\]
which represents the Liouville equation in integral form. By differentiating with respect to \( s \) the equivalent differential representation follows at once. An analogous equation holds manifestly also for the KDF \( f \left( y \right) \).

Q.E.D.

We conclude noting that, formally, the Liouville equation for non-local Hamiltonian systems in standard form is analogous to that characterizing local Hamiltonian systems. Such an equation can be viewed as a Vlasov equation for a relativistic collisionless plasma, in which each particle is subject only to the action of a mean-field EM interaction, generated respectively by the external and the self EM Faraday tensors. By definition, in this treatment the latter do not include retarded binary EM interactions. It follows that, in terms of the Lagrangian equation (127), the probability density \( \rho \left( y \left( s \right) \right) \) is parametrized in terms of the single-particle phase-space trajectory \( \{ y \left( s \right), s \in I \} \). Hence, it advances in (proper) time \( s \) by means of the canonical state \( y \left( s \right) \) as determined by the Hamiltonian equations of motion (53).

A. Vlasov-Maxwell description

To define a well-posed problem, the relativistic Vlasov equation (127) must be coupled to the Maxwell equations, which determine the total EM field produced by all the relevant sources. In particular, in order to determine the external Faraday tensor \( F_{\mu\nu}^{\text{ext}} \), the corresponding EM 4-potential \( A_{\nu}^{\text{ext}} \) must be determined. In the Lorentz gauge, this is prescribed requiring it to be a solution of the Maxwell equations
\[
\Box A^{\text{ext}}_{\mu} = \frac{4\pi}{c} j^{\text{ext}}_{\mu}(r), \quad (135)
\]
where \( j^{\text{ext}}_{\mu}(r) \) is identified with the total current density
\[
j^{\text{ext}}_{\mu}(r) \equiv q \int d^4 u \delta \left( |u| - 1 \right) u^\mu f \left( y \right) + j^{\text{conf}} \mu(r). \quad (136)
\]
Here, the first term is the Vlasov 4-current density, namely the velocity moment of \( f \left( y \right) \) carrying the non-local phase-space contributions which yield the collective field produced by the plasma. The second term, instead, is produced by possible prescribed sources located outside the plasma domain. Therefore, in the Vlasov-Maxwell description the total EM 4-potential acting on a single particle must be considered as represented by \( A_{\nu} = A_{\nu}^{\text{ext}} + A_{\nu}^{\text{self}} \), where \( A_{\nu}^{\text{self}} \) is given by Eq. (76) and \( A_{\nu}^{\text{ext}} \) is the solution of Eq. (135).

Therefore, the dynamical evolution of the KDF along a single-particle phase-space trajectory depends both explicitly, via \( A_{\nu}^{\text{self}} \), and implicitly, via the 4-current \( j^{\text{ext}}_{\mu}(r) \), on the whole Faraday tensor \( F_{\mu\nu} \equiv F_{\mu\nu}^{\text{ext}} + F_{\mu\nu}^{\text{self}} \). In this way contributions which are non-local both in configuration and phase-space are consistently included in the theory.
VIII. FLUID MOMENT EQUATIONS

We now proceed to compute explicitly the relativistic fluid moment equations which follow from the Liouville equation. To this aim, the relativistic Liouville equation is conveniently written as a PDE (Eulerian form)

\[ u^\mu \frac{\partial f(y)}{\partial y^\mu} + G^\mu(y) \frac{\partial f(y)}{\partial u_\mu} = 0, \]  

(137)

where \( G^\mu(y) \) is defined by Eq.(83), or as an ODE (Lagrangian form):

\[ \frac{dr^\mu}{ds} \frac{\partial f(y(s))}{\partial y^\mu} + \frac{du_\mu}{ds} \frac{\partial f(y(s))}{\partial u_\mu} = 0, \]  

(138)

with \( y(s) \) being the phase-space trajectory of a particle. Then, the relativistic fluid equations related to the Liouville equation are defined as the following integrals over the momentum space:

\[ \int d^4u \delta(|u| - 1) G \left[ u^\mu \frac{\partial f(y)}{\partial y^\mu} + G^\mu(y) \frac{\partial f(y)}{\partial u_\mu} \right] = 0. \]  

(139)

Similarly, the corresponding fluid fields are defined as

\[ \int d^4u \delta(|u| - 1) G f(y), \]  

(140)

with \( G = 1, u^\mu, u^\mu u^\nu, \ldots \) and \( u^\mu \) is the 4-velocity. In particular, we shall denote

\[ n(r) \equiv \int d^4u \delta(|u| - 1) f(y), \]  

(141)

\[ N^\mu(r) = n(r) U^\mu(r) = \int d^4u \delta(|u| - 1) u^\mu f(y), \]  

(142)

\[ T^{\mu\nu}(r) = \int d^4u \delta(|u| - 1) u^\mu u^\nu f(y), \]  

(143)

to be referred to as the number density, the 4-flow and the stress-energy tensor. It is immediate to prove that the corresponding moment equations are as follows.

**Continuity equation**

For \( G = 1 \) the Liouville equation provides the continuity equation

\[ \partial_\mu N^\mu(r) = 0. \]  

(144)

**Energy-momentum equation**

For \( G = u^\nu \) the Liouville equation provides the energy-momentum equation

\[ \partial_\mu T^{\mu\nu}(r) = F^{\nu\mu}_{(tot)}(r) N_\mu(r), \]  

(145)

where, from Eq.(83) we have that

\[ F^{\nu\mu}_{(tot)}(r) = F^{\nu\mu}_{(ext)}(r) N_\mu(r) + F_{(self)}^{\nu\mu}(r), \]  

(146)

is the total EM force, with \( F_{(self)}^{\nu\mu} \) containing the retarded non-local contributions arising from the EM RR effect.

We remark the following properties.

1) As a consequence of the Hamiltonian formulation in standard form, the fluid equations obtained from the kinetic equation with the inclusion of the RR effect are formally the same as in the usual treatment for local systems.

2) The contribution of the RR effect to the fluid equations is contained explicitly in the source term in the rhs of Eq.\((145)\), and also implicitly in the definition of the fluid fields. In fact, by assumption, the KDF is a function of the effective Hamiltonian state \( y \equiv (r^\mu, P_\mu) \), which depends on the retarded self-potential. Hence, the fluid fields defined by Eqs.\((141)-(143)\) must be interpreted as the fluid fields of the plasma which is emitting self-radiation and is therefore subject to the RR effect.
A. The implicit contribution of the RR self-force

It is worth discussing the features of the theory in connection with the implicit contribution of the RR effect contained in the definition of the fluid fields. In particular, here we show that such contribution can be made explicit and an analytical asymptotic estimation of it can be give provide some suitable assumptions are imposed on the physical system. This concerns the case in which the contribution of the self-potential is small in comparison with the external EM potential in the KDF. In these circumstances, the exact KDF can be Taylor expanded as follows:

\[ f(y) \approx f(y_{nc}) + (y - y_{nc}) \frac{\partial f(y)}{\partial y} \bigg|_{y=y_{nc}} + \ldots, \quad (147) \]

where \( y_{nc} \equiv (r^\mu, p_\mu) \) is the state which is canonical in absence of the EM self-field. It is clear that, by construction, only the canonical momenta are involved in this expansion, since the configuration state is left unchanged by the presence of the self-force. Therefore, from the form of the previous expansion it follows that the first term of the series, namely \( f(y_{nc}) \), does not contain any contribution from the RR self-field. Consider, for simplicity, the Taylor series to first order. Then, the corresponding fluid fields can be decomposed as follows:

\[
\begin{align*}
    n (r) & \simeq n_0 (r) + n_1 (r), \\
    N^\mu (r) & \simeq N_0^\mu (r) + N_1^\mu (r), \\
    T^{\mu \nu} (r) & \simeq T_0^{\mu \nu} (r) + T_1^{\mu \nu} (r),
\end{align*}
\]

where

\[
\begin{align*}
    n_0 (r) & \equiv \int d^4u \delta (|u| - 1) f(y_{nc}), \\
    n_1 (r) & \equiv \int d^4u \delta (|u| - 1) (y - y_{nc}) \frac{\partial f(y)}{\partial y} \bigg|_{y=y_{nc}} = \frac{2q\pi^2}{c} \frac{\Theta (s_{eff})}{\Delta} \int d^4u \delta (|u| - 1) \frac{\partial f(y)}{\partial P_\mu} \bigg|_{P_\mu = p_\mu},
\end{align*}
\]

and similar definitions hold for the other two fluid fields.

To illustrate the procedure, let us consider, for example, the case of a relativistic Maxwellian distribution of the form [21]

\[ f_M (y) \equiv \frac{1}{(2\pi\hbar)^3} \exp \left[ \frac{\mu - P^\mu U_\mu}{T} \right], \quad (153) \]

where \( \mu, P^\mu, U_\mu \) and \( T \) are respectively the chemical potential, the canonical momentum and the fluid 4-velocity and temperature. Then, in terms of the previous expansion, we obtain for the density

\[
\begin{align*}
    n_0 (r) & \equiv \frac{4\pi m^2 c T}{(2\pi\hbar)^3} K_2 \left( \frac{mc^2}{T} \right) \exp \left[ \frac{\mu}{T} - \frac{q\Theta (s_{eff}) U^\mu}{c T} \right], \\
    n_1 (r) & \equiv -\frac{2q A^{\mu (self)} U^\mu}{c T} n_0 (r),
\end{align*}
\]

with \( K_2 \left( \frac{mc^2}{T} \right) \) being the modified Bessel function of the second kind. As can be seen, the effect of the RR self-field appears only in \( n_1 (r) \) through the integral over the non-local dependencies contained in the potential \( A^{\mu (self)} \). It follows that for a Maxwellian KDF the 4-flow \( N^\mu (r) \) can be written as

\[ N^\mu (r) \simeq [n_0 (r) + n_1 (r)] U^\mu (r), \quad (156) \]

while the expansion terms of the stress-energy tensor \( T^{\mu \nu} (r) \) are given by

\[
\begin{align*}
    T_0^{\mu \nu} (r) & \equiv \frac{1}{c^2} n_0 c U^\mu U^\nu - p_0 \Delta^{\mu \nu}, \\
    T_1^{\mu \nu} (r) & \equiv \frac{1}{c^2} n_1 c U^\mu U^\nu - p_1 \Delta^{\mu \nu}.
\end{align*}
\]

\( \Box \)
Here the notation is as in Ref. [21]. Thus, $\Delta^{\mu \nu}$ is the projector operator $\Delta^{\mu \nu} \equiv \eta^{\mu \nu} - c^{-2} U^{\mu} U^{\nu}$, $e$ is the energy per particle

$$e = m c^2 \frac{K_1 \left(\frac{m c^2}{T}\right)}{K_2 \left(\frac{m c^2}{T}\right)} - T$$

(159)

and from the definition of the pressure as $p = n T$ it follows that

$$p_0 (r) = n_0 (r) T, \quad (160)$$

$$p_1 (r) = n_1 (r) T = - \frac{2q}{c} A^{\text{self}}_\mu U^\mu n_0 (r). \quad (161)$$

Finally, let us consider how the fluid equations are modified from the introduction of the series expansion (147). Substituting the relations (148)-(150) into the moment equations, for the continuity equation we get

$$\partial_\mu N^{\mu 0} (r) = - \partial_\mu N^{\mu 1} (r), \quad (162)$$

and for the momentum equation

$$\partial_\mu T^{\mu \nu} = F_{(\text{tot})}^{\nu \mu} N^\mu - \partial_\mu P^{\mu \nu}. \quad (163)$$

In this way, on the lhs we have isolated the terms of the “unperturbed fluid”, namely the physical observables corresponding to a charged fluid in absence of RR. On the other hand, the asymptotic contributions of the RR effect have been isolated on the rhs, which allows one to interpret them as source terms due to extra forces acting on the unperturbed fluid. In particular, the presence of the RR acts like a non-conservative collisional operator, if we interpret it as a sort of retarded scattering of the fluid (and therefore, of the single particles at the kinetic level) with itself.

IX. LAGRANGIAN FORMULATION OF THE FLUID EQUATIONS

An important issue concerns the treatment of the non-local contributions appearing in the fluid equations both in the definitions of the fluid fields and in the source term in the momentum equation. This requires, in particular, the explicit representation of the self-potential $A^{\text{self}}_\mu$ and the EM self-force $F^{\text{self}}_{\mu k}$ defined respectively in Eqs. (92) and (86). In fact, in the previous sections these non-local contributions have been written in a parameter-free representation (integral form), so that they do not depend on the retarded particle velocity. This allowed us to perform the velocity integrals in a straightforward way, only in terms of local 4-velocities, in agreement with the formalism adopted for the Hamiltonian formulation in standard form.

To treat these non-local terms it is first convenient to represent the fluid moment equations in Lagrangian form, describing the dynamics of fluid elements along their respective Lagrangian path (LP). By substituting the definition (142) in Eq.(144) we obtain the corresponding Lagrangian form of the continuity equation, given by

$$\frac{D}{D s} n + n \partial_\mu U^\mu = 0, \quad (164)$$

where $\frac{D}{D s} \equiv U^\mu (r (s)) \partial_\mu$ is the convective Lagrangian derivative along the LP of the fluid element parametrized in terms of the arc-length $s$, and $U^\mu (r (s)) = \frac{d r^\mu (s)}{d s}$. Similarly, writing the stress-energy tensor $T^{\mu \nu} (r)$ as $T^{\mu \nu} (r) = n U^\mu U^\nu + P^{\mu \nu} (r)$, with $P^{\mu \nu} (r) \equiv T^{\mu \nu} (r) - n U^\mu U^\nu$, the energy-momentum equation (145) can be represented in Lagrangian form as follows:

$$n \frac{D}{D s} U^\nu = n F^{\nu \mu}_{(\text{tot})} U_\mu - \partial_\mu P^{\mu \nu}. \quad (165)$$

Analogous results can be given for the asymptotic equations (162) and (163).

With the introduction of the LPs, the parametrization of the non-local contributions can be easily reached in terms of the LP arc-length $s$. Consider, for example, the self-potential $A^{\text{self}}_\mu$. This can be expressed as

$$A^{\text{self}}_\mu (r, [r]) \equiv 2 q \int_1^2 ds' \frac{d r'}{d s'} \delta (\vec{R'} - \vec{R}_\mu - \sigma^2), \quad (166)$$
where by definition now $\frac{dx'}{ds'} = U^\mu (r(s'))$ is defined along a fluid element LP. Then, by expressing the Dirac-delta function as

$$\delta\bigl(\bar{R}^\mu \bar{R}_\mu - \sigma^2\bigr) = \frac{1}{2\bar{R}^\alpha U_\alpha} \delta\bigl(s' - s + s_{ret}\bigr),$$

it follows that $\mathcal{A}_\mu^{(self)}$ can be equivalently written in the integrated form as

$$\mathcal{A}_\mu^{(self)} (r, [r]) = q \left[ \frac{U_\mu (r(s'))}{\bar{R}^\alpha U_\alpha (r(s'))} \right]_{s' = s - s_{ret}},$$

with $\bar{R}^\alpha$ being the displacement vector defined along a LP. In particular, in agreement with the Einstein Causality Principle, the retarded time $s_{ret} = s - s'$ is the positive root of the delay-time equation

$$\bar{R}^\mu \bar{R}_\mu - \sigma^2 = 0.$$ 

An analogous derivation can be carried out also for the self-force $F_{\mu k}^{(self)}$, giving the following result

$$F_{\mu k}^{(self)} (r, [r]) = -2q \left\{ \frac{1}{\bar{R}^\alpha U_\alpha (s')} \frac{D}{Ds'} X_{\mu k} (r(s')) \right\}_{s' = s - s_{ret}},$$

where

$$X_{\mu k} (r(s')) = \left[ \frac{U_\mu (r(s')) \bar{R}_k - U_k (r(s')) \bar{R}_\mu}{\bar{R}^\alpha U_\alpha (r(s'))} \right].$$

Again, this expression must be intended as a parametrization defined along a fluid element LP.

We conclude by commenting on the following remarkable aspects of the theory presented here.

1) The fluid equations with the inclusion of the non-local effect related to the EM RR have been derived in a closed analytical form in both Eulerian and Lagrangian formulations. In particular, it follows that the fluid dynamics of the non-local kinetic system is intrinsically non-local too.

2) Non-local contributions of the RR appear both in explicit and implicit contributions, through the definitions of the fluid fields as velocity moments of the KDF.

3) From the point of view of the fluid description, it follows that the natural setting for the treatment of the non-local fluid equations is given by the Lagrangian formulation and the concept of LPs. This is a consequence of the fact that the exact moment equations are of delay-type. In fact, in order to properly deal with the non-local contributions of the RR the parametrization of the retarded effects in terms of the arc-length of the corresponding LPs is needed. It follows that the dynamics of a generic fluid element along its LP is related to the EM RR effect produced at the retarded time along the LP itself.

X. ASYMPTOTIC APPROXIMATION

In the previous sections we derived an exact formulation for both kinetic and fluid theories describing systems of relativistic charged particles subject to the EM RR self-interaction. In particular, we have pointed out that the kinetic and fluid equations are of delay-type, and therefore intrinsically non-local, due to the characteristic feature of the RR effect of being a non-local retarded effect. The retarded proper time is determined by Eq.(169) in agreement with the causality principle. Notice that this equation has formally the same expression for the single-particle or the kinetic dynamics and for the fluid equations in Lagrangian form (see also Paper I). By inspecting Eq.(169) it is easy to realize that the order of magnitude of the delay-time is approximately $s_{ret} \sim \sigma/c$, and therefore very small for classical elementary particles. The smallness of the retarded time may represent a serious problem for the practical implementation of the exact theory presented here. In fact, the retarded time associated to the RR can be orders
of magnitude smaller than any other characteristic time for most of relevant physical situations. The question is of primary importance, for example, for the actual numerical integration of the exact fluid equations.

In view of these considerations, in this section we provide asymptotic estimations of the non-local terms appearing in the moment equations, which allow one to overcome the difficulty connected with the finite delay-time intervals carried by the RR phenomenon. This requires to introduce a suitable asymptotic expansion of the exact non-local terms by means of approximations in which the self-interaction contributions are all expressed only through local quantities. The result has potential interest also in relation to the use of Eulerian integration schemes for the fluid equations with the inclusion of the RR effect.

Specifically, the present analysis requires to develop an asymptotic approximation which involves the treatment of the delay-time \( s_{ret} \). This is accomplished within the short delay-time ordering approximation given by Eq. (99).

In the following we shall work adopting the Lagrangian representation form for the fluid equations. To perform the asymptotic expansion, we assume that both the external EM field acting on each fluid element and the macroscopic fluid fields associated to the kinetic system are smooth function of the coordinate 4-position vector \( r^\alpha \), namely they are of class \( C^k \), with \( k \geq 2 \). The result of the asymptotic approximation for the terms associated to the RR self-interaction is provided by the following theorem.

**THM.7 - First-order, short delay-time asymptotic approximation (present-time expansion).**

Given validity of the asymptotic ordering (99) and the smoothness assumptions for the external EM and the fluid fields, neglecting corrections of order \( \epsilon^n \), with \( n \geq 1 \) (first-order approximation), it follows that:

- \( T7_1 \) The retarded self-potential \( \tilde{A}_\mu^{\text{ret}} \) defined in Eq. (165) can be expanded in a neighborhood of \( s \) as follows:

  \[
  \tilde{A}_\mu^{\text{ret}}(s) = \tilde{A}_\mu^{\text{ret}}(s) \bigg|_s [1 + O(\epsilon)],
  \]

where the present-time leading-order contribution \( \tilde{A}_\mu^{\text{ret}}(s) \bigg|_s \) is given by

\[
\tilde{A}_\mu^{\text{ret}}(s) \bigg|_s = q \left[ \frac{1}{\sigma} U_\mu(r(s)) - \frac{D}{Ds} U_\mu(r(s)) \right],
\]

with \( \frac{D}{Ds} \) being the convective derivative along a fluid element Lagrangian path.

- \( T7_2 \) Concerning Eq. (165), let us define the vector field \( K_\mu \) as follows:

\[
K_\mu \equiv \frac{q}{m_0 c^2} \tilde{F}_{\mu\nu}^{\text{ret}} U^\nu,
\]

with \( \tilde{F}_{\mu\nu}^{\text{ret}} \) defined in Eq. (170). Then, in a neighborhood of \( s \), \( K_\mu \) can be expanded as follows:

\[
K_\mu = K_\mu \bigg|_s [1 + O(\epsilon)],
\]

where the present-time leading-order contribution \( K_\mu \bigg|_s \) is given by

\[
K_\mu \bigg|_s = \left\{-\frac{1}{\sigma m_0 c^2} \frac{D}{Ds} U_\mu(r(s)) + g_\mu \right\},
\]

with \( g_\mu \) denoting the 4-vector

\[
g_\mu = \frac{2 q^2}{3 m_0 c^2} \left[ \frac{D^2}{Ds^2} U_\mu - U_\mu(s) U^k(s) \frac{D^2}{Ds^2} U_k \right].
\]

**Proof** - The proof of \( T7_1 \) and \( T7_2 \) can be reached by introducing a Taylor expansion in terms of the retarded time \( s' \) for the relevant quantities appearing in Eqs. (168) and (170). In particular, for the 4-velocity \( U_\mu(r(s')) \) and the displacement vector \( \tilde{R}^k \) we obtain respectively

\[
U_\mu(r(s')) \cong U_\mu(r(s)) - (s - s') \frac{D}{Ds} U_\mu(r(s)) + \frac{(s - s')^2}{2} \frac{D^2}{Ds^2} U_\mu(r(s)) + O(\epsilon^3)
\]

and

\[
\tilde{R}^k \cong (s - s') U^k - \frac{(s - s')^2}{2} \frac{D}{Ds} U^k + \frac{(s - s')^3}{6} \frac{D^2}{Ds^2} U^k + O(\epsilon^4),
\]
while for the time delay $s - s' \equiv s_{\text{ret}}$ we get

$$s - s' \cong \sigma + O \left( \epsilon^2 \right).$$

(180)

By substituting these expansions in Eqs. (168) and (170), after straightforward calculations the asymptotic solutions (172) and (175) follow identically.

Q.E.D.

We notice that the asymptotic expansion of the self-potential illustrated in THM.7 is required to reduce the non-local dependencies which are implicit in the definition of the fluid fields through the KDF. On the other hand, within the approximation obtained in THM.7 for the 4-vector $K_\mu$, the RR equation (165) reduces to a local third-order ordinary differential equation. In particular, Eq. (170) in THM.7 represents the analogue of the LAD equation for the single-particle dynamics, which contains the first derivative of the particle 4-acceleration (see also related discussion in Paper I). In view of this similarity, the asymptotic solution (175) can be further simplified adopting a second reduction-step of the same kind of that which leads to the LL form of the self-force for single charged particles [11]. This is obtained by assuming that the RR effect is only a small correction to the motion of the fluid. As a consequence, an iterative approximation can be adopted which permits to represent the self-force in terms of the instantaneous local dependencies which are implicit in the definition of the fluid fields through the KDF. On the other hand, within the self-potential entering the definition of the fluid fields through the canonical momenta (172) and (175) follow identically.

By substituting these expansions in Eqs. (168) and (170), after straightforward calculations the asymptotic solutions (172) and (175) follow identically. We notice that the asymptotic expansion of the self-potential illustrated in THM.7 is required to reduce the non-local dependencies which are implicit in the definition of the fluid fields through the KDF. On the other hand, within the approximation obtained in THM.7 for the 4-vector $K_\mu$, the RR equation (165) reduces to a local third-order ordinary differential equation. In particular, Eq. (170) in THM.7 represents the analogue of the LAD equation for the single-particle dynamics, which contains the first derivative of the particle 4-acceleration (see also related discussion in Paper I). In view of this similarity, the asymptotic solution (175) can be further simplified adopting a second reduction-step of the same kind of that which leads to the LL form of the self-force for single charged particles [11]. This is obtained by assuming that the RR effect is only a small correction to the motion of the fluid. As a consequence, an iterative approximation can be adopted which permits to represent the self-force in terms of the instantaneous fluid forces. The latter include both the external EM field and the pressure forces. In particular, according to this method, to leading-order for the fluid 4-acceleration we have

$$D_s U^\mu = \frac{F_{\mu\nu}^{(ext)}}{m_o c^2} U_\nu - \frac{1}{n} \partial_\mu P^{\mu\nu},$$

(181)

where, for brevity we have introduced the notation

$$F_{\mu\nu}^{(ext)} = \frac{q}{m_o c^2} F_{\nu\sigma}^{(ext)} \sigma^\nu.$$  

(182)

The iteration gives

$$\frac{D^2}{D_s^2} U^\nu = \partial_\nu F_{\mu\nu}^{(ext)} U_\mu U^l + \frac{1}{n} \partial_\nu P_{\mu}^{\mu\nu} + \frac{1}{n} \partial_\nu P_{\mu}^{\mu\nu} U^l \ln n - \frac{1}{n} U^l \partial_\nu P_{\mu}^{\mu\nu} + \frac{2}{3} h_{\mu}^{(1)} + \frac{2}{3} h_{\mu}^{(2)}.$$  

(183)

Substituting this expansion in Eq. (175) and invoking the symmetry property of the Faraday tensor provides for the first-order term $K_\mu | s$ the following approximation:

$$K_\mu | s \cong \frac{q^2}{m_o c^2} \left\{ \frac{1}{\sigma} \left[ - \frac{q}{m_o c^2} F_{\mu\nu}^{(ext)} U^\nu - \frac{1}{n} \partial_\mu P_{\mu}^{\mu\nu} \right] + \frac{2q}{3m_o c^2} h_{\mu}^{(1)} + \frac{2}{3} h_{\mu}^{(2)} \right\},$$  

(184)

where the first term on the rhs represents the mass-renormalization contribution, and $h_{\mu}^{(1)}$ denotes the 4-vector

$$h_{\mu}^{(1)} = \partial_\nu \left( F_{\nu\mu}^{(ext)} U^\nu U^l - \frac{q}{m_o c^2} F_{\mu\nu}^{(ext)} F_{\nu\sigma}^{(ext)} U^l + \frac{q}{m_o c^2} F_{\nu\sigma}^{(ext)} U^l \right) \left( F_{\nu\sigma}^{(ext)} U^l \right).$$  

(185)

while $h_{\mu}^{(2)}$ is given by

$$h_{\mu}^{(2)} = \frac{1}{m_o c^2} \left[ \frac{1}{n} \partial_\nu \left( F_{\nu\mu}^{(ext)} \partial_\nu P_{\mu}^{\mu\nu} \right) - \frac{1}{n} U_\nu \partial_\nu \partial_\nu P_{\mu}^{\mu\nu} \right] + \frac{1}{n} \left( U_\nu U^k \partial_\nu P_{\mu}^{\mu\nu} \right).$$  

(186)

Eq. (184) represents the fluid analogue of the LL approximation of the self-force holding for single particle dynamics, with the mass-renormalization term retained. In particular here we notice that:

1) Eq. (184) provides a local approximation of the fluid self-force carrying the contribution of the RR effect. In contrast to Eq. (176), thanks to the iterative reduction procedure only second-order derivatives of the position vector appear in this approximation.

2) For consistency, Eq. (184) must be evaluated adopting the asymptotic expansion (172) also for the evaluation of the self-potential entering the definition of the fluid fields through the canonical momenta $P_\mu$ in the KDF.
3) Moreover, consistent with the approximation in which the RR self-potential is small with respect to the external EM potential, also the asymptotic approximation (147) can be adopted, which allows one to treat explicitly in an asymptotic way all the implicit RR contributions.

4) Finally, collecting together the analytical approximations provided by Eqs. (147), (172) and (184), the fluid equations are reduced to a set of asymptotic local second-order PDEs. This provides a convenient representation also for Eulerian implementation schemes of the same equations.

The detail comparison of Eqs. (183)-(186) with the literature is discussed in the next section.

A. Retarded-time asymptotic expansion

Despite the previous considerations, it is worth pointing out that, formally also for the fluid equations, an analogous result to THM.7 can be given. This is based on performing a Taylor expansion of the fluid RR force based on the retarded-time approximation. In this case, it is found that Eq. (168) is approximated as

\[ A_{\mu}^{(self)} = A_{\mu}^{(self)} \bigg|_{s'} [1 + O(\varepsilon)], \]  

where the retarded-time leading-order contribution \( A_{\mu}^{(self)} \bigg|_{s'} \) is simply given by

\[ A_{\mu}^{(self)} \bigg|_{s'} = \frac{q}{\sigma} U_{\mu} \left( r(s') \right), \]  

while Eq. (170) for the self-force, written in terms of \( K_{\mu} \) defined in Eq. (174), becomes

\[ K_{\mu} = K_{\mu} \bigg|_{s'} [1 + O(\varepsilon)], \]  

where the retarded-time leading-order contribution \( K_{\mu} \bigg|_{s'} \) is now given by

\[ K_{\mu} \bigg|_{s'} = \left\{ -\frac{q^2}{\sigma m_o c^2} \frac{D^2}{Ds'^2} U_{\mu} \left( r(s') \right) U_k \left( r(s') \right) U^k \left( r(s') \right) \right\}, \]  

with \( g_{\mu} \) denoting here the 4-vector

\[ g_{\mu} = \frac{1}{3} \frac{q^2}{m_o c^2} \left[ \frac{D^2}{Ds'^2} U_{\mu} \left( r(s') \right) U_k \left( r(s') \right) \frac{D^2}{Ds'^2} U_k \left( r(s') \right) \right]. \]  

This alternative expansion has the distinctive advantage (with respect to the present-time expansion) of retaining all the physical properties of the exact fluid equations for the treatment of RR delay-time effects. This alternative formulation is relevant for comparisons with the point-particle treatment.

XI. DISCUSSION AND COMPARISONS WITH LITERATURE

In this section we analyze in detail the physical properties of the kinetic and fluid theory developed for the EM RR problem, providing also a comparison with the literature. This concerns, in particular, the recent paper by Berezhiani et al. [17], where an analogous research program is presented for the relativistic hydrodynamics with RR based on the LL solution of the self-force.

A. Kinetic theory

Let us start by considering the kinetic theory. The solution here obtained has the following key features:

1) The kinetic theory adopts the Hamiltonian formulation of the RR problem here developed. The result is based on the exact analytical solution for the EM self-potential of finite-size charged particles, obtained in Paper I and Appendix A.

2) The kinetic theory is here developed for systems of charged particles subject to an external mean-field EM interaction and the RR self-interaction produced by the same particles. Due to the non-local property of the RR
interaction, the formulation of kinetic theory is non-trivial. For this purpose, in contrast to previous literature, an axiomatic formulation of CSM is adopted. Its key element is the introduction of a suitable definition for the Lorentz-invariant probability-measure in the particle extended phase-space. As a consequence, the corresponding Liouville-Vlasov kinetic equation with the inclusion of the exact RR effect is achieved in Hamiltonian form, namely in such a way to preserve the phase-space canonical measure. For comparison, instead, previous literature approaches dealt with measure non-preserving phase-space dynamics.

3) In particular, the kinetic theory has been developed within the canonical formalism representing the KDF in terms of the canonical state \( y \equiv (r^\mu, P_\mu) \). For reference, in Appendix B the connection with the corresponding non-canonical treatment is provided. This in turn implies that non-local contributions associated to the self-potential \( \Phi \) enter implicitly in the definition of the corresponding fluid moments \( \langle \cdots \rangle_1, \langle \cdots \rangle_2 \). This is made possible only within the framework of the present exact formulation, in which the analytical solution for the self-potential is by construction non-divergent. This feature departs from recent approaches where instead non-Hamiltonian formulations were adopted, based on the LL point-like approximation of the RR self-force. In such a case in fact, the explicit dependence of the KDF in terms of the EM self-potential cannot be retained.

4) Both the RR equation for single-particle dynamics and the kinetic equation for the KDF are of delay-type, reflecting the characteristic nature of the RR phenomenon. This property is completely missing from the previous literature on the subject, exclusively based on the LL local asymptotic approximation.

B. Fluid theory

For what concerns the fluid treatment, we notice that:

1) Both the fluid fields and the fluid moment equations retain the standard form (available in the absence of RR effects) and can be equivalently represented in Eulerian or Lagrangian form. This follows from the exact representation here adopted both for the RR self-potential and the RR self-force. In both cases the only non-local dependencies are those associated to the position 4-vector.

2) The exact fluid equations with the inclusion of the RR effect are delay-type PDEs. Because of this feature, their natural representation appears to be the Lagrangian form. In fact, the integration along the LPs must be in principle performed taking into account the retarded RR interaction.

3) From the exact theory presented here it follows that each fluid equation of a given order does not depend on fluid fields of higher orders. For example, the momentum equation contains only second-order tensor fields, identified respectively with the plasma stress-energy tensor and the EM Faraday tensor. This result contrasts with the treatment given in Ref.\[17\] where instead the asymptotic formulation based on the LL equation leads to moment equations involving higher-order tensor fields (for comparison, see also the related discussion in Appendix B).

4) If a kinetic closure is chosen, then the fluid moments appearing in the fluid equations are all uniquely determined. In particular, the stress-energy tensor is prescribed in terms of the KDF. This implies that both implicit and explicit contributions of the RR effect appear in the resulting equations, carried respectively by the fluid fields and the EM self-force in the momentum equation. Remarkably, kinetic closure is achieved prescribing solely the pressure contribution carried by the stress-energy tensor. Instead, in the approach of Ref.\[17\] the closure conditions involve generally also the specification of higher-order moments of the KDF.

5) An important feature of the exact fluid equations here obtained is that they can in principle be exactly implemented numerically adopting a Lagrangian scheme.

6) A remarkable aspect of the present theory is that the relevant asymptotic expansions are performed only “a posteriori” after integration over the velocity space. This means that the approximations involved are introduced only on the configuration space-variables (i.e., the fluid fields) and not on the phase-space KDF. In particular, a convenient approximation is the one obtained in the short delay-time ordering, which reduces the non-local dependencies to local terms. As a consequence, the introduction of higher-order moments is ruled out by construction.

C. Comparison with point-particle treatments

The relevant comparison here is represented by Ref.\[17\]. Such an approach is based on the adoption of the LL equation for the single-particle dynamics for the construction of the relativistic Vlasov-Maxwell description. The corresponding moment equations can be in principle adopted for the construction of a closed set of fluid equations. This requires however the specification of suitable closure-conditions. Let us briefly point out the novel features of the
current treatment for what concerns the adoption of the finite-size particle model in the construction of the kinetic and fluid descriptions. In detail:
1) Both in the kinetic and fluid treatments the RR force is taken into account by means of a non-local interaction. This is an intrinsic feature of the assumed finite extension of the charged particle. In the fluid treatment, in particular, as shown above, the RR force can be parametrized in terms of the past Lagrangian fluid velocity and position. This permits to treat consistently the causal delay-time effects due to the finite-size of the particles.
2) In validity of the asymptotic ordering given by Eq.(99), an asymptotic retarded-time Hamiltonian approximation of the RR force based on a retarded-time expansion has been given for the fluid equations. This approximation preserves the basic physical features of the solution based on the exact form of the RR self-force.
3) If the present-time asymptotic expansion is performed on the exact fluid moment equations, the resulting expression of the fluid RR force obtained adopting the finite-size charge model appears different from that given in Ref.[17].

These conclusions enable us to carry out a detailed comparison with the literature, emphasizing the basic differences between kinetic and fluid treatments based on finite-size and point particles.

A) Kinetic theory.
The kinetic equation adopted in Ref.[17] is based on the LL equation (see therein Eqs. 7 and 8). This means that the RR force in this approximation is non-conservative, non-variational and therefore non-Hamiltonian. In addition the LL equation: 1) does not retain finite delay-time effects characteristic of the RR phenomenon; 2) is not valid in the case of strong EM fields, where the iterative reduction scheme on which it is based, may fail; 3) ignores mass-renormalization effects (which are incompatible with the point-particle model). In contrast, the treatment of the relativistic Vlasov kinetic equation obtained here (see the Eulerian and Lagrangian equations (137) and (138)) is qualitatively different. In fact, even if the resulting RR equation remains a second-order ODE, it is conservative, variational, Hamiltonian and applies for arbitrary external EM fields. Further remarkable aspects are related to the adoption of the finite-size charge model, in which the charge and mass distributions have the same support. As a consequence, in this case the self 4-potential is everywhere well-defined, contrary to the point particle model. In addition, this is prescribed analytically (see Appendix A), a feature which allows one to treat consistently the RR delay-time effects.

B) Fluid theory.
The fluid treatment here obtained is provided by the Eulerian Eqs.(144)-(145) or the equivalent Lagrangian equations (164)-(165). The latter, considered as fluid equations, are manifestly not closed. However, the Hamiltonian formulation achieved here and holding for finite-size particles allows one to achieve a physically consistent kinetic closure condition, by prescribing uniquely the pressure tensor $P_{\mu\nu}$ in Eq.(165). We stress that in our treatment no higher-order moments need to be specified. In contrast, the corresponding Euler equation reported in Ref.[17] (see Eqs.11 and 12 therein) actually depends also on a third-order tensor moment, which must be prescribed (see comments in Sec.IIIA of Ref.[17]). Let us now consider the asymptotic fluid treatments based on the present theory. These can be achieved invoking either the present-time or the retarded-time asymptotic expansions (see Section X). The first expansion is mostly relevant for comparisons with Ref.[17] (given in THM.7) and enables one to achieve a local approximation of the delay-time effects carried by the RR force. However, remarkably, the resulting asymptotic fluid equations present a qualitatively different from the corresponding ones given in Ref.[17]. In particular: 1) no higher-order moments appear after performing the Taylor expansion and the iteration scheme discussed after THM.7; 2) a non-vanishing mass-correction contribution is now included (see first term on the rhs of Eq.(184)). Finally, we mention that the retarded-time asymptotic expansion given by Eqs.(187)-(191) provides a novel approximation which retains basic properties of the exact solution. In particular: 1) it only applies for finite-size particles; 2) it relies on the Hamiltonian formulation of the RR problem and of the Vlasov-Maxwell treatment; 3) it permits to retain transient-time and delay-time effects; 4) it takes into account retarded mass-correction effects; 5) in this approximation the natural fluid description is Lagrangian.

XII. CONCLUSIONS

In this paper, novel results have been obtained concerning the kinetic and fluid descriptions of relativistic collisionless plasmas with the inclusion of EM RR effects.

Relevant consequences of the variational form of the EM RR equation previously achieved for classical finite-size charged particles have been investigated. It has been shown that the non-local RR problem admits both Lagrangian and Hamiltonian representations in standard form, defined respectively in terms of effective Lagrangian and Hamiltonian functions. A remarkable novel feature of the theory concerns the development of a Hamiltonian retarded-time...
expansion of the RR force, which applies in validity of the short delay-time asymptotic ordering. On such a basis, the axiomatic formulation of classical statistical mechanics for relativistic collisionless plasmas with the inclusion of non-local RR effects has been presented. As a major result, the kinetic theory for such a system has been formulated in standard Hamiltonian form. The Liouville-Vlasov equation has been proved to hold in the extended phase-space, subject to non-local RR self-interactions. Remarkably, the non-local effects have been proved to enter the relativistic kinetic equation only through the retarded particle 4-position. As a consequence, the corresponding fluid moment equations can be determined in standard way by integration over the space of canonical momenta and cast both in Eulerian and Lagrangian forms. It has been pointed out that the exact relativistic fluid equations are intrinsically of delay-type and contain both implicit and explicit non-local contributions associated to the RR effect. The issue concerning the problem of fluid closure conditions has been discussed. In contrast with previous literature, it is found that in the present approach the closure conditions remain the standard ones, i.e., as in the absence of RR effects. Hence, the specification of higher-order moments of the KDF, for a given moment equation, is not required. Finally, appropriate approximations have been obtained for the fluid equations by employing “a posteriori” the relevant asymptotic expansions applicable in the short delay-time ordering. This allows one to reduce the exact non-local equations either to a set of local PDEs or to retarded PDEs still retaining finite delay-time effects.

The theory here developed has potential wide-ranging applications which concern the study of relativistic astrophysical plasmas for which RR emission processes are important. This involves, for example, plasmas in accretion disks, relativistic jets and active galactic nuclei. Other possible applications are also suggested for the case of laboratory plasmas generated in the presence of pulsed-laser sources.

XIII. APPENDIX A: INTEGRAL REPRESENTATION FOR \(A^{(self)}_{\mu}\) - CASE OF A NON-ROTATING SPHERICAL-SHELL CHARGED PARTICLE

In this Appendix we determine explicitly the integral representation of \(A^{(self)}_{\mu}\) for a non-rotating finite-size charged particle described by the model introduced in Paper I. We first remark that Eqs. (62) and (63) can be written for a spherically-symmetric charged particle of radius \(\sigma > 0\) as

\[
\xi^\mu \xi_\mu = -\sigma^2,
\]

\[
\xi_\mu u^\mu(s) = 0.
\]

Eq. (192) defines the boundary \(\partial \Omega\) on which the charge and mass are uniformly distributed, while Eq. (193) represents the constraint of rigidity for the finite-size particle. We can use the information from Eq. (192) to define the internal and the external domains with respect to the mass and charge distributions. In particular, in terms of the generic displacement 4-vector \(X^\mu \in \mathbb{M}^4\) defined as

\[
X^\mu = r^\mu - r^\mu(s)
\]

and subject to the constraint

\[
X^\mu u_\mu(s) = 0,
\]

the following relations hold:

\[
X^\mu X_\mu \leq -\sigma^2 : \text{external domain},
\]

\[
X^\mu X_\mu > -\sigma^2 : \text{internal domain},
\]

\[
X^\mu X_\mu = \xi^\mu \xi_\mu = -\sigma^2 : \text{boundary location}.
\]

As proved in Ref. [19], for the evaluation of the action integral \(S_C^{(self)}\) it is sufficient to know the solution of \(A^{(self)}_{\mu}\) in the external domain. In this domain the EM self 4-potential generated by the non-rotating finite-size particle must necessarily coincide with that of a point particle carrying the same total mass and charge. The retarded 4-potential of a point charge represents a well-known result in the literature [22]. This can be easily obtained by means of the Green function approach. In particular, introducing the retarded Green function of a point particle \(G(r - r')\), the self-potential \(A^{(self)}_{\mu}\) takes the form

\[
A^{(self)}_{\mu}(r) = \frac{4\pi}{c} \int d^4r' G(r - r') j^\mu(r'),
\]
where \( G(r - r') \) is symmetric in both \( r \) and \( r' \), is non-vanishing only for \( r^0 < r'^0 \) and satisfies in this domain the boundary-value problem

\[
\begin{cases}
\Box G(r - r') = \delta^{(4)} (r - r'), \\
G(r - r') = 0.
\end{cases}
\]  
(198)

As a consequence, written in integral form, the self-potential becomes

\[
A^{(\text{self})\mu}(r, q) = 2q \int_1^2 dr'^\mu \delta(\hat{R}^\alpha \hat{R}_\alpha),
\]  
(199)

with

\[
\hat{R}^\alpha \equiv r^\alpha - r'^\alpha(s).
\]  
(200)

This solution, derived for a point charged particle, also holds for the rigid finite-size non-rotating spherical shell in the external domain defined in Eq.(196). As can be seen, this coincides with the result in Ref.[19], where a complete covariant solution for the EM self 4-potential \( A^{(\text{self})\mu} \) holding in both internal and external domains has been obtained by adopting a derivation based on the principle of relativity and analogous to that outlined in Ref.[11] for the point charge case. Notice that, contrary to the case of point particle, the self-potential (199) is well-defined also on the support of the charge, namely the ensemble on which the charge is distributed.

**XIV. APPENDIX B: NON-CANONICAL REPRESENTATION**

In this appendix we present the equivalent representation of the kinetic theory developed in section adopting non-canonical variables. For definiteness, let us introduce an arbitrary non-canonical phase-space diffeomorphism from \( \Gamma \) to \( \Gamma_w \), with \( \Gamma_w \) denoting a transformed phase-space having the same dimension of \( \Gamma \),

\[
y \equiv (r^\mu, P_\mu) \rightarrow w \equiv w(y),
\]  
(201)

where, for example, \( w \) can be identified with the non-canonical state \( y_{nc} \equiv (r^\mu, p_\mu) \) defined in Eq.(147) or with \( y_u \equiv (r^\mu, u_\mu) \). In the second case the transformation, following from Eq.(96), is realized by

\[
r^\mu = r^\mu,
\]  
(202)

\[
u_\mu = P_\mu - \frac{q}{c} \left[ F^{(\text{ext})}_\mu + 2F^{(\text{self})}_\mu \right].
\]  
(203)

The transformed RR equation in the variables \( y_u \) becomes therefore:

\[
\frac{dr^\mu}{ds} = u^\mu,
\]  
(204)

\[
\frac{du_\mu}{ds} = F_\mu,
\]  
(205)

where \( F_\mu = \frac{\partial p_\mu}{\partial r^\nu} u^\nu - \frac{\partial u_\mu}{\partial P_\nu} \partial H_{\text{eff}}/\partial r^\nu \). Denoting now by

\[
f_1(w(s)) = \left| \frac{\partial y(s)}{\partial w(s)} \right| f(y(w(s)))
\]  
(206)

the KDF mapped onto the transformed phase-space \( \Gamma_w \) by the KDF \( f(y(s)) \), the differential Liouville-Vlasov equation (127) requires

\[
\frac{d}{ds} \left[ \frac{\partial w(s)}{\partial w_0} f_1(w(s)) \right] = 0,
\]  
(207)

where \( w_0 \equiv w(s_0) \). At the same time, Eq.(127) also implies, thanks to the chain rule:

\[
\frac{d}{ds} f(y(w(s))) = 0,
\]  
(208)
which for consistency delivers the well-known differential identity
\[
\frac{d}{ds} \left[ \begin{array}{c}
\partial y(s) \\
\partial w(s) \\
\frac{\partial w(s)}{\partial w_0}
\end{array} \right] = 0.
\] (209)

From Eq. (207) it follows
\[
\frac{d}{ds} f_1(w(s)) + f_1(w(s)) \frac{d}{ds} \ln \left( \left| \frac{\partial w(s)}{\partial w_0} \right| \right) = 0.
\] (210)

This equation can be represented, for example, in terms of \( w \equiv y_u \). In this case, due to the chain rule
\[
\frac{d}{ds} f_1(w(s)) = u^\mu \frac{\partial f_1(y_u)}{\partial r^\mu} + F_\mu \frac{\partial f_1(y_u)}{\partial u_\mu},
\] (211)

while, thanks to Liouville theorem
\[
\frac{d}{ds} \ln \left( \left| \frac{\partial w(s)}{\partial w_0} \right| \right) = \frac{\partial F_\mu}{\partial u_\mu}.
\] (212)

As an application of the result, it follows that, if the LL approximation is introduced for the 4-vector \( F_\mu \), namely Eqs. (204) and (205) are replaced with asymptotic equations of the form
\[
\frac{dr_\mu}{ds}_{LL} = u_\mu^{\prime LL},
\] (213)
\[
\frac{du_\mu}{ds}_{LL} = F_\mu^{LL},
\] (214)

where \( F_\mu^{LL} \) is the total EM force in this approximation, then Eq. (210) recovers the expression reported in Ref. [17]. This provides the connection with the exact canonical theory here developed. We remark, however, that since the LL equation is only asymptotic, the mapping between the canonical state \( y \equiv (r^\mu, P_\mu) \) and \( y_{LL} \equiv (r_\mu^{LL}, u_{LL\mu}) \) is also intrinsically asymptotic. Therefore, Eqs. (213) and (214) remain necessarily non-variational and non-canonical.

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