Abstract

Consider the set of sums of $m$’th powers of elements belonging to the Cantor middle third set $C$, and the question of the number of terms required to ensure we find a large open interval in this set. Also consider the question of finding open intervals in the product of Cantor sets. A broad general framework that makes it possible to deal with the first problem was outlined in a paper by Astels [1]. The question of finding the measure of $C \cdot C$ was considered recently in an article by Athreya, Reznick and Tyson [3]. Astels’ methods don’t immediately apply to the second problem, and Athreya, Reznick, Tyson’s methods become difficult in dealing with the first problem as $m$ becomes large.

With the same elementary dynamical technique, in this paper we are able to answer both these questions in a satisfactory way. In particular, with $t_m = 2 \cdot \left( \frac{3}{2} \right)^{m-1}$ many terms, we find an open interval of measure at least $2 \cdot \left( 1 - \left( \frac{2}{3} \right)^m \right)$ in our set of sums of $t_m$ many $m$'th powers, and the same elementary technique shows that the fourfold product of the Cantor set contains the interval $\left[ (8/9)^3, (8/9) \right]$.

1 Introduction

Let $C$ be the usual middle third Cantor set:

$$C = \left\{ \sum_{i=1}^{\infty} \frac{t_i}{3^i} : t_i \in \{0,2\} \right\}$$

The sets $C + C$ and $C - C$ have been well studied [6, 7, 8, 9].

One of the folklore proofs that shows that $\frac{1}{2}(C + C) = [0, 1]$ considers the ternary expansion of any real number $x \in [0, 1]$, and then shows constructively that there are always two elements belonging to $C$, whose average gives us $x$. Another elementary argument considers the set $C \times C \in \mathbb{R}^2$ and the lines $x + y = a$ with $a \in [0, 2]$, and uses compactness arguments to conclude that this line always contains a point of $C \times C$. In a recent article by Jayadev Athreya, Bruce Reznick, Jeremy Tyson [3], the measure of the set $C \cdot C$ was studied, and some obvious gaps were noted within $[0, 1]$ which could not be covered by the set $C \cdot C$. They also describe the set $\mathcal{U} := \left\{ \frac{u}{v} : u, v \in C, v \neq 0 \right\}$.

In their article they also pose the following conjecture [3]:

**Conjecture**: Every $u \in [0, 1]$ can be written as $x_1^2 + x_2^3 + x_3^2 + x_4^2$, with $x_i \in C$.

It turns out this conjecture is true, and the proof follows easily from earlier works of Sheldon Newhouse [10] and from a later and more exhaustive treatise by Astels [1]; the authors of [3] seem to be unaware. Theorem 2.2, Part 1 from Astels’ paper, or the more
any positive integers $t, m$ to leave out the interval $\left(2^k \tau, \frac{1}{3^k}\right)$ from the set $K_k$ at the $k$th stage, and the function $F(x, y) = x + y$, that $F(K^2_i) = [0, 2], \forall i \in \mathbb{N}$, and thus $\mathcal{C} + \mathcal{C} = [0, 2]$. By considering the function $F(x, y) = xy$, and considering the sets $F(K^2_i)$ for all $i$, we could find an estimate of the size of the set of common intersection, which has essentially been done in \cite{3, 4}.

One could use this same technique and consider $F(x_1, x_2, \ldots, x_t) = x_1^m + x_2^m + \ldots + x_t^m$ for any positive integers $t, m$, but then it becomes progressively more difficult to keep track of the intersections of the sets $F(K^m_i)$ for all $i$, as $m$ is made large. For $m = 2$ and $t = 4$ this has been done in \cite{5}, but this is where the generalized construction of a Cantor set by Astels, and his theorems prove to be effective in order to get results.

By considering only two square terms in the statement of the above conjecture, we’ll have to leave out the interval $(2/9, 4/9)$, with three terms we will have to leave out $(3/9, 4/9)$ while with four terms we would get an entire open interval $[0, 2^2]$. It is also clear that by considering $m$th powers, we would need to consider $2m$ many terms to get all of $[0, 2^m]$, and with lesser number of terms, we would have to leave out the obvious gaps.

An interesting next question is, for a given exponent $m$, how few terms $t$ in the sum of $m$th powers one can consider to get a large open interval contained in our sum. Theorem 2.4 Part 1 of Astels, modified slightly, would give us good answers here.

In the beginning of section 2 of Astels paper, the outline is made of the construction of a generalized Cantor set. For our specific purpose, the set $\mathcal{C}^{(m)} := \{x^m : x \in \mathcal{C}\}$ is a generalized Cantor set. Consider the $(k + 1)$th stage of construction, where we have $2^k$ many open intervals, $A^{(k)}_i$, $i = 1, 2, 3, \ldots, 2^k$. From each one of these, we delete an interval $O^{(k+1)}_i$, and are left with two remaining intervals $A^{(k)}_{i0}$ and $A^{(k)}_{i1}$. After suitable relabelling, these two new sets, for each $i$ ranging from 1 to $2^k$, get a superscript $(k + 1)$ and we end up with a labelling of the $2^{k+1}$ many remaining intervals in this $(k + 1)$th stage. We set

$$\tau(A^{(k)}_i) = \min \left( \frac{|A^{(k)}_{i0}|}{|O^{(k+1)}_i|}, \frac{|A^{(k)}_{i1}|}{|O^{(k+1)}_i|} \right)$$

The thickness of the Cantor set is defined as: $\tau(\mathcal{C}^{(m)}) = \inf_{A^{(k)}_i} \tau(A^{(k)}_i)$. Also consider the
number \( \gamma(C^{(m)}) = \frac{\tau(C^{(m)})}{1 + \tau(C^{(m)})} \). Astel’s theorem implies that with \( t_m := \lceil \frac{1}{\gamma(C^{(m)})} \rceil \) many terms, we would have an open interval in the sumset \( C^{(m)} + \ldots + C^{(m)} \) (added \( t_m \) times).

It is not difficult to see that in our case, \( \tau(C^{(m)}) = \frac{1}{2^m - 1} \cdot \gamma(C^{(m)}) = \frac{1}{2^m} \), and thus with \( 2^m \) many terms, we would contain an interval (in fact, all of \([0, 2^m]\) but that’s a separate matter).

However, if we were to only restrict to \( C^{(m)} \cap [\left(\frac{2}{3}\right)^m, 1] \), then we again have a generalized Cantor set \( C^{(m)} \), and in that case, the thickness clearly increases, and the \( \gamma \) factor becomes \( \frac{2^m - 6^m}{8^m - 6^m} \), and thus for large \( m \), we could do with approximately \( \lceil \left(\frac{2}{3}\right)^m \rceil \) many terms. In fact, as the conditions of part 2 of Astel’s theorem are satisfied, we would have that \( C^{(m)} + \ldots + C^{(m)} \) (added \( t_m \) times) is an interval that has measure approximately \( \lceil \left(\frac{2}{3}\right)^m \rceil (1 - \left(\frac{2}{3}\right)^m) \).

Clearly if we considered the set \( C^{(m)} \cap [\left(\frac{8}{9}\right)^m, 1] \), we would get an interval of smaller measure, but with even fewer number of terms.

As mentioned earlier however, it is not immediately clear how Newhouse and Astel’s methods would apply if we were to consider products of Cantor sets, in place of sums, since the product of Cantor sets do not have an easy description as a generalized Cantor set itself. On the other hand, as mentioned earlier, the methods of [3],[4] have the advantage on the problem of considering products of two Cantor sets, but become tedious on the question of the sums of \( C^{(m)} \).

In this paper, we construct an elementary dynamical argument that gives us an open interval contained within the set \( \mathcal{C} \cdot \mathcal{C} \cdot \mathcal{C} \cdot \mathcal{C} \), and also the same argument that gives us an open interval within the set \( C^{(m)} \) summed sufficiently many times. We state our two main results below.

**Theorem 2.** The set \([\left(\frac{8}{9}\right)^3, \frac{8}{9}] \) \( \subset \mathcal{C} \cdot \mathcal{C} \cdot \mathcal{C} \cdot \mathcal{C} \).

**Theorem 3.** Let \( m \geq 1 \) be an integer. Consider \( t_m = 2 \cdot \lceil (\frac{2}{3})^{m-1} \rceil \). Then the set \( \mathcal{C}^{(m)} + \mathcal{C}^{(m)} + \ldots + \mathcal{C}^{(m)} \), where \( \mathcal{C}^{(m)} \) is added \( t_m \) many times, contains the interval \( I = [(m+1)(\frac{2}{3})^m + (m-1)(\frac{2}{3})^m + (m+1)] \) of measure \( 2(1 - (\frac{2}{3})^m) \).

This method is a slight variation of the method of an earlier paper [2] that gives us \( \mathcal{C} + \mathcal{C} = [0, 2] \), which we restate in Section 2 of this paper. In Section 3, we outline the proof for the case of the product \( \mathcal{C} \cdot \mathcal{C} \cdot \mathcal{C} \cdot \mathcal{C} \). In Section 4, we outline our argument for the sum of the sets \( C^{(m)} \). In Section 5, we specifically deal with the case of \( m = 2 \), and expand the measure of the open set we obtain from Section 4 for the case \( m = 2 \).

Our method should also enable us to find open intervals in the sets of the form \( \mathcal{C}^{(m_1)} \cdot \mathcal{C}^{(m_2)} \ldots \mathcal{C}^{(m_k)} \), where \( m_1, m_2, \ldots \) are some suitable integers, and \( k \) is an integer dependent on these values of \( m_1, m_2, \ldots \), but we don’t pursue that problem here. As the values of \( m_1, m_2, \ldots \) become large, this is again a problem where Athreya, Reznick, Tyson’s methods become difficult.

In principle, our methods should also be applicable for questions on arbitrary \( \mathcal{C}_\lambda \) Cantor sets, (where \( C_\lambda \) is the Cantor set where at each stage a middle \( 1 - 2\lambda \) fraction open set is cut out from the existing intervals) and questions such as finding open intervals in product sets \( \mathcal{C}_\lambda_1 \cdot \mathcal{C}_\lambda_2 \cdot \ldots \cdot \mathcal{C}_\lambda_k \) should naturally be answerable with our methods, although we don’t deal with such a problem here. Although we don’t address the question here, we could also be able to find the Hausdorff dimension of such sets by using this method.
2 A proof that $C + C = [0, 2]$.

Here we give a proof of the fact that $\frac{1}{2}(C + C) = [0, 1]$. This is reproduced from the earlier paper by the author [2]. The idea of the proof is simply to consider any $x \in [0, 1]$ and the interval containing $x$ that is cut out from $[0, 1]$ at some stage while constructing $C$. We consider the endpoints of this interval, which both belong to the Cantor set, as two possible candidates whose average would give us $x$. If this average is slightly higher or lower, we keep moving the lower endpoint even lower, or the higher endpoint higher in an appropriately controlled way, and get two Cauchy sequences in the process, so that in the limit we find two elements belonging to the Cantor Set whose average is exactly $x$.

**Theorem 4.** Every $u \in [0,1]$ is the average of two real numbers each belonging to the Cantor’s middle third set.

**Proof.** Take an arbitrary real number $y \in [0, 1]$. In the process of creating the Cantor set from $[0, 1]$ by deleting the middle thirds, after a finite number ($k_0$) of steps, unless $y \in C$, $y$ would fall in the interior of, or on the boundary of an open set that’s cut out for the first time. Indeed if the ternary expansion of $y$ contains a 1, then $y$ falls in the interior or on the left boundary of a cut open third corresponding to the first time the 1 appears in the ternary expansion. Otherwise $y$ contains only 0, or 2, in its ternary expansion, and then $y \in C$ and we are done. The length of the interval cut out at this $k_0$th iteration is $1/3^{k_0}$.

(i) Let the closest end point to $y$ at this stage on the right be $a_1 \in [0, 1]$ at a distance $r_1 := |a_1 - y|$, and on the left be $b_1 \in [0, 1]$ at a distance $l_1 := |b_1 - y|$ (we have $a_1, b_1 \in C$). Consider the unique $k_1 > 0$ so that $1/3^{k_1+1} < |l_1 - r_1| \leq 1/3^{k_1}$. We have $k_1 \geq k_0$. Consider w.l.o.g $r_1 \geq l_1$.

(ii) To the left of $b_1$, we further iterate and remove successive middle thirds so that eventually there is a point $b_2 \in C$ to the left of $b_1$ with $l_2 - l_1 = 2/3^{k_1+1}$, where $l_2 := |b_2 - y|$. At this stage, take $a_2 = a_1$, and $r_2 = r_1$.

We have: $1/3^{k_1+1} - 2/3^{k_1+1} = -1/3^{k_1+1} < r_2 - l_2 = (r_2 - l_1) - (l_2 - l_1) \leq (1/3^{k_1} - 2/3^{k_1+1}) = 1/3^{k_1+1}$, and so $|r_2 - l_2| \leq 1/3^{k_1+1}$. Thus we can find a unique $k_2 > k_1$ so that $1/3^{k_2+1} < |r_2 - l_2| \leq 1/3^{k_2}$.

Now we perform steps exactly analogous to the steps (i) and (ii) above. It may happen that at the $k$th stage, we have $l_k > r_k$. In this case, corresponding to (ii), we would find a point $a_{k+1}$ to the right of $a_k$, while keeping $b_{k+1} = b_k$. The sequence $s_k = |r_k - l_k|$ in the $k$'th iterative step is bounded by a higher power of 1/3, and so $s_k \to 0$ as $k \to \infty$.

Here $\{a_i\}_{i=1}^{\infty}(\{b_i\}_{i=1}^{\infty})$ is bounded within $[0,1]$, is non-decreasing (non-increasing) and thus converges to a limit point $a_\infty(b_\infty)$ that also belongs to the Cantor set itself, the Cantor set being closed. In the limit, we thus have within the Cantor set two points that are equidistant from $y$ (with $r_\infty = |a_\infty - y| = |b_\infty - y| = l_\infty$), and this proves our assertion. \(\square\)

3 Open intervals in the set $C \cdot C \cdot C \cdot C$.

We demonstrate that the set $[(\frac{8}{9})^3, \frac{8}{9}] \subset C \cdot C \cdot C \cdot C$. The arguments of the proof can be repeated in a straightforward way to locate other closed intervals within $C \cdot C \cdot C \cdot C$, which is something we don’t do here. The method used here is a slight variation on the method of the previous section. As mentioned earlier, the analysis of the measure and structure of the set

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2When $l_1 > r_1$, the proof follows in an analogous way.
Proof. (Of Theorem 2) Consider any point \( x \in [(8/9)^3, 8/9] \) and take \( a_1 = b_1 = 8/9 \), and \( c_1 = d_1 = 1 \). Thus the entire dynamics is within the interval \([(8/9), 1]\). As in the previous proof, the idea is to construct four different Cauchy sequences \( a_k, b_k, c_k, d_k \) for \( k = 1, 2, 3, \ldots \), so that in the limit we have four points, \( a_\infty, b_\infty, c_\infty, d_\infty \) belonging to \( \mathcal{C} \) so that \( x = a_\infty \cdot b_\infty \cdot c_\infty \cdot d_\infty \). We also consider the product \( p_k := a_k \cdot b_k \cdot c_k \cdot d_k \) for every positive integer \( k \), and the difference \( \Delta_k := x - p_k \).

If \( x = (8/9)^3, (8/9)^2, \) or \( (8/9) \) we are done. Note that \( p_1 = (8/9)^2 \).

1. First assume that \((8/9)^3 < x < (8/9)^2\). Consider the value \( x_0 \in ((8/9), 1) \) so that \( x = a_1 \cdot b_1 \cdot x_0 \cdot d_1 = (8/9)^2 \cdot x_0 \). Unless \( x_0 \in \mathcal{C} \) in which case we are done, consider the open interval that \( x_0 \) falls in the interior or the boundary of, in the process of iteratively constructing \( \mathcal{C} \cap [8/9, 1] \) and take \( c_2 \) to be the left end point of this interval, and take \( a_2 = a_1, b_2 = b_1, d_2 = d_1 \). Then \( \Delta_2 = (8/9)^2(x_0 - c_2) < \frac{1}{3^2} \), since \((x_0 - c_2)\) is bounded by the length of the cut out interval under consideration, and the biggest possible such gap within \([8/9, 1]\) has length \( 1/3^3 \). Consider the unique \( k_2 \geq 3 \) so that \( -\frac{1}{3^{k_2+1}} < \Delta_2 = x - p_2 < \frac{1}{3^{k_2+1}} \).
The aim is to iteratively increase the sequences \( a_k, b_k \) and decrease the sequences \( c_k, d_k \) in a controlled way. Notice that there is room to decrease \( c_2 \) further by an amount \( 2/3^{k_2+1} \) or lesser, while there is room to decrease \( d_2 \) by an amount \( 2/3^3 \) or lesser, and room to increase \( a_2, b_2 \) by an amount \( 2/3^3 \) or lesser, when we try to mimic the method of proof from the previous section.

At the next stage, we try to increase either one or both of \( a_2, b_2 \), in order to decrease the difference between \( x \) and \( p_3 \) from the value \( \Delta_2 \). If we only take \( a_3 = a_2 + \frac{2}{3^{k_2+2}} \), and keep \( b_3 = b_2, c_3 = c_2, d_3 = d_2 \), then \((8/9)^3 \cdot \left(\frac{2}{3^{k_2+2}}\right) < p_3 - p_2 = b_2 \cdot c_2 \cdot d_2 \cdot \frac{1}{3^{k_2+1}} < \frac{8}{3^{k_2+1}} \). Call the number \((8/9)^3 \cdot \left(\frac{2}{3^{k_2+2}}\right) \approx 0.70 \) as \( \alpha \). Thus \( \Delta_3 = x - p_3 = (x - p_2) + (p_2 - p_3) \).

If \( \frac{1}{3^{k_2+1}} \leq (x - p_2) < \frac{1}{3^2} \), thus we have, \( -\frac{1}{3^{k_2+1}} \leq \Delta_3 < \frac{1}{3^2} - \frac{2\alpha}{3^{k_2+1}} \). In this process, the upper bound is greater than \( \frac{1}{3^{k_2+1}} \), and if in this process, \( \Delta_3 \) is greater than \( \frac{1}{3^{k_2+1}} \), i.e \( \frac{1}{3^{k_2+1}} < \Delta_3 < \frac{1}{3^2} - \frac{2\alpha}{3^{k_2+1}} \), then we take \( k_3 = k_2, b_4 = b_3 + \frac{1}{3^{k_2+1}}, a_4 = a_3, c_4 = c_3, d_4 = d_3 \). Again, \( \frac{1}{3^{k_2+1}} < \Delta_4 = (x - p_4) = \Delta_3 + (p_3 - p_4) < \left\frac{1}{3^2} - \frac{4\alpha}{3^{k_2+1}} \right\frac{1}{3^3} \), and thus \( \frac{1}{3^{k_2+1}} < \Delta_4 < \frac{1}{3^3} \).

Thus the absolute magnitude of the error term \( \Delta_4 \) is now bounded by \( \frac{1}{3^{k_2+1}} \). Choose the unique \( k_4 \geq k_3 + 1 \) so that now \( \frac{1}{3^{k_4+1}} \leq |\Delta_4| < \frac{1}{3^{k_4+1}} \). Now it should be clear how we would mimic the iteration of the proof from the previously written case, and further decrease the error term in the next iterations so that successively we get \( \frac{1}{3^{k_4+1}} < |\Delta_4| < \frac{1}{3^{k_4+1}} \) where necessarily \( k_l \geq k_{l-1} + 1 \). The \( \alpha \) factor remains the same as we progress along the iteration, and the bounds work in the same manner, as \( k \rightarrow \infty \). Thus clearly in the limit we find four elements \( a_\infty, b_\infty, c_\infty, d_\infty \in \mathcal{C} \) so that \( x = a_\infty \cdot b_\infty \cdot c_\infty \cdot d_\infty \).

2. The argument for the case where \((8/9)^2 < x < (8/9)\) proceeds in the similar manner; instead of decreasing \( c_1 \) from the value 1, we instead increase the value \( a_1 \) from \((8/9)\).
to the right endpoint of the cut out interval where $x_0$ is located, where similar to the previous case, we define $x_0$ to be the number so that $x = x_0 \cdot b_1 \cdot c_1 \cdot d_1$. The $\alpha$ factor of course remains the same as before, and then the argument from the previous case can thus be used here as well.

As mentioned before, this set $[(8/9)^3, (8/9)]$ can be expanded by similar other choices of the intervals and the starting values $a_1, b_1, c_1, d_1$. In particular it should be clear that we should be able to find smaller and smaller open chunks whose right end points are arbitrarily close to the value of 1, by considering $\mathcal{C} \cap [\frac{3\alpha}{3^k-1}, 1]$, for $k \geq 3$. In these cases, the $\alpha$ factor only goes closer to 1, and the algorithm runs as before. However, if we were to consider $\mathcal{C} \cap [2/9, 3/9]$, then the corresponding $\alpha$ factor becomes much smaller, and in that case we would require the product of more terms for the argument to work.

4 Sums of $\mathcal{C}^{(m)}$.

Now we finally come to the question of sums of $m$'th powers, with $m \geq 2$, and finding open intervals within them. This is a problem where Astels’ general methods gives stronger results, but where the methods of [3, 4] become difficult as $m$ grows larger.

We state our theorem:

Let $m \geq 1$ be an integer. Consider $t_m = 2 \cdot \lceil (\frac{2}{3})^{m-1} \rceil$. Then the set $\mathcal{C}^{(m)} + \mathcal{C}^{(m)} + . . . + \mathcal{C}^{(m)}$, where $\mathcal{C}^{(m)}$ is added $t_m$ many times, contains the interval $I = [(m+1)(\frac{2}{3})^m + (m-1)(\frac{2}{3})^m + (m+1)]$ of measure $2(1 - (\frac{2}{3})^m)$.

Proof. (of Theorem 3.) Write $s_m = \frac{1}{2} \cdot t_m$. Consider the sequences $a_1^{(1)}, a_2^{(1)}, . . . , a_{s_m}^{(1)}$, $b_1^{(1)}, b_2^{(1)}, . . . , b_{s_m}^{(1)}$ with $a_1^{(1)} = a_2^{(1)} = . . . = a_{s_m}^{(1)} = \frac{2}{3}$, and $b_1^{(1)} = b_2^{(1)} = . . . = b_{s_m}^{(1)} = 1$.

In this case, for each integer $k \geq 1$, we define

$$S_k = (a_1^{(k)})^m + . . . + (a_{s_m}^{(k)})^m + (b_1^{(k)})^m + . . . + (b_{s_m}^{(k)})^m$$

Note that $S_1 = m(\frac{2}{3})^m + 1$.

Consider any $x \in I$. If $x = S_1$ then of course we are done. Consider w.l.o.g, $x < S_1$. (The case for $x > S_1$ proceeds in an exactly similar way.)

In this case we consider the value $x_0 \in [2/3, 1]$ so that $x = (a_1^{(1)})^m + (a_2^{(1)})^m + . . . + (a_{s_m}^{(1)})^m + x_0^m + (b_1^{(1)})^m + . . . + (b_{s_m}^{(1)})^m$. As before, if $x_0 \in \mathcal{C}$ we are done, otherwise consider the open interval, of length some $1/3^{k_1}$ with $k_1 \geq 2$, that $x_0$ falls in the interior or the boundary of, in the process of iteratively constructing $\mathcal{C}$, and consider the left end point of this cut interval, and call it $b_2^{(1)}$, while keeping all other sequences constant between $k=1$ and $k=2$ stages of the sequence. Again consider the error term $\Delta_k = x - S_k$. In this situation, the difference $\Delta_2 = x - S_2$ is clearly bounded from above by $(b_2^{(1)})^m + (\frac{1}{3^{k_1}})^m - (b_2^{(1)})^m = \frac{1}{3^{k_1}} \cdot (...)$. where there are exactly $m$ terms in the bracket above, each of which is bounded from above by 1 (and bounded from below by $(\frac{2}{3})^{m-1}$). Consider the unique integer $k_2 \geq k_1$ so that $\frac{m}{3^{2+1}} \leq \Delta_2 < \frac{m}{3^{2+1}}$. Note that there is room to decrease $b_2^{(1)}$ by an amount $\frac{2}{3^{2+1}}$, or by an amount where the value of the exponent in the denominator greater than $k_2 + 1$.

Now we increase $a_2^{(1)}$, and consider, $a_3^{(1)} = a_2^{(1)} + \frac{2}{3^{2+1}}$, while keeping all the terms constant when passing from the $k=2$ to $k=3$ stages of the sequence. The increment in the value of
$S_3$ from the value $S_2$ is given by $(a_2^{(1)} + \frac{2}{3^{(k+1)}})^m - (a_2^{(1)})^m = \frac{2}{3^{(k+1)}} \cdots (...)$. There are $m$ terms in the bracket each of which is bounded from above by 1 and bounded from below by $(\frac{2}{3})^{m-1}$. The objective here is to make the error lower so that $|\Delta_3| < \frac{m}{3^{(k+1)}}$. It should now be clear from the above statements that at least one of the $a^{(i)}$, for $i$ varying from 1 to $s_m$, needs to be increased, and in the extreme case, we might possibly end up needing $u_m$ many terms where $u_m$ is the minimum integer so that we have $u_m \cdot 2 \cdot \lceil \frac{m}{3^{(k+1)}} \rceil > (\frac{m}{3^{(k+1)}} - \frac{m}{3^{(k+1)}}) = \frac{2m}{3^{(k+1)}}$, and thus we exactly have $u_m = \lceil \frac{m}{3^{(k+1)}} \rceil = s_m$.

In the next stages, we might have to decrease upto $s_m$ of the $b_k^{(i)}$ terms where $i$ ranges from 1 to $s_m$, or at a different step increase upto $s_m$ of the $a_k^{(i)}$ terms, in order to ensure that the absolute values of the errors terms $\Delta_k$ are bounded at each successive stage by a higher exponent of 1/3. Thus in total we need $2s_m = t_m$ many terms.

It should also be clear that the proof for the case where $x > S_1$ also follows in an identical way; in the beginning, we increase $a_1^{(1)}$ to $a_2^{(1)}$ by an appropriate amount, and then enforce our dynamical argument. In the end we find $t_m$ many limit points so that the sum of their $m$ th powers is exactly the value $x$.

Again we should remark that for $k \geq 3$, by focusing on a smaller interval $[1 - \frac{1}{k}, 1]$, with this method we should get, with $2 \cdot \lceil \frac{m}{3^{(k+1)}} \rceil$ terms, an open interval of length $2 \cdot (1 - (\frac{2}{3^{(k+1)}})^m)$.

5 Further open sets in $C^{(2)} + C^{(2)} + C^{(2)} + C^{(2)}$.

Notice that when $m = 2$, $s_m = 4$, and indeed with 4 elements you can actually get the entire interval $[0, 4]$.

In this section, we note specifically for the case $m = 2$ that you could get many more intervals by employing our technique from the previous section, in a slightly altered way, in that we now consider the end points of more than one closed interval, whereas in the earlier section, we considered the end points of only the specific closed interval $[2/3, 1]$.

We also note that if we first work within the interval $[36/81, 4]$, and find a subset $S'$ of $[36/81, 4]$ satisfying our property, then we also have the result for the set $S = \bigcup_{n=0}^{\infty} S'/n^h$, since if $u \in C$ then $\frac{u}{n} \in C$ as well. Of course, an analogous statement works for any $m \geq 2$.

The interval $[\frac{53}{81}, \frac{71}{81}]$: In this case, the reader can verify, that by choosing the initial points $a_1 = b_1 = 3/9$ and $c_1 = 2/9$, $d_1 = 6/9$, and confining attention to the intervals $[6/9, 1]$ and $[2/9, 3/9]$ we’ll get by a technique similar to the previous section, that all of $[53/81, 71/81]$ satisfies our property. For the sake of not being repetitive, we don’t produce that proof here.

We end by demonstrating a more complicated open interval that is contained in $C^{(2)} + C^{(2)} + C^{(2)} + C^{(2)}$. For any $k \geq 2$, the interval $[36/81, (8/9), (1/3^k + 1/3^k), (36/81 + 4/3^2 + 1/3^k + 2/3^k + 4/9), (1/3^k)]$ is contained in $C^{(2)} + C^{(2)} + C^{(2)} + C^{(2)}$.

In order to show this, we choose the intervals $[2/3], [1/3^k], [1/3^k]$ and the interval: $[2/3, 2/3 + 1/3^k]$, and with $a_1 = b_1 = 1/3^k$ and $c_1 = (2/3) \cdot (1/3^k)$, $d_1 = 2/3$.

So we have that $S_1 = a_1^2 + b_1^2 + c_1^2 + d_1^2 = 36/81 + 2/3^k + 4/9, (1/3^k)$, and for any $u \in (36/81 + (8/9), (1/3^k) + 1/3^k, 36/81 + 2/3^k + 4/9, (1/3^k))$, let us take the $x \in [0, 1]$ so that $u = 36/81 + (4/9), (1/3^k) + 1/3^k + x^2$. Similar to the previous case, we choose the value $a_2 \in C$ so that $\Delta_1 = (u - S_2)$ is bounded from above by at most $(2/3^{k+2}) \cdot (a_1 - 3/(2.3^{k+2}))$. 

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with \((2/3).(1/3^k) < (a_1 - 3/(2.3^{k+2})) < 1/3^k\). Thus the error bound from above is at most \(2/3^{k+t+2}\). In the general case, the maximal upper bound is clearly some \(2/3^{k+t+2}\) with \(t \geq k\). Whatever this maximum upper bound is, given the \(u\), consider the unique \(t\) such that \(2/3^{k+t+3} < \Delta_1 < 2/3^{k+t+2}\) with \(t \geq k\).

Now as before, in this next stage we’ll increase \(d_1\) from the value \(2/3\) to some value \(d_2\), and then also possibly increase \(c_1\) from \((2/3).1/3^k\) to some value \(c_2\) so that \(|\Delta_2| < 2/3^{2r+3}\). For this we simply first take \(d_2 = d_1 + 2/3^{k+t+3}\), in which case we have: \((2/3).1/3^{k+t+3}\) \(<\) \((d_1 + 2/3^{k+t+3})^2 - d_1^2 = (4/3^{k+t+3}).(d_1 + 1/3^{k+t+3}) < 4/3^{k+t+3}\). In addition, we might have to take \(c_2 = c_1 + (2/3^{t+3})\) in which case we have \((2/3).(4/3^{t+k+3}) < (c_2^2 - c_1^2) = (c_1 + 2/3^{t+3})^2 - c_1^2)\) \(<\) \((4/3^{t+3}).(c_1 + 1/3^{t+3}) < 4/3^{t+k+3}\). Now the algorithm runs as before, and in the next step, we ensure that \(|\Delta_3| < 2/3^{k+t+4}\).

In case that \(u \in \{36/81 + 2/3^{2k} + (4/9).(1/3^{2k}), (36/81 + 4/3^{2k+1} + 1/3^{4k}) + 2/3^{2k} + (4/9).(1/3^{2k})\}\), we will consider the \(x\) so that \(u = a_1^2 + b_1^2 + c_1^2 + x^2\), and then the difference \(0 < \Delta_1 = (S_2 - u)\) has the maximum upper bound \((d_1 + 2/3^{2k+1})^2 - (d_1 + 1/3^{2k+1})^2 = (2/3^{2k+1}).(d_1 + 3/2.(3^{k+1})) < 2/3^{2k+1}\). In general consider the unique \(t \geq 2k + 1\) so that \(2/3^{t+1} < \Delta_1 < 2/3^t\). In the next step, we will decrease \(a_1\) to the value \(a_2 = (a_1 - 2/3^{t-k-1})\) which is clearly possible, as well as possibly \(b_1\) to some value \(b_2 = (b_1 - 2/3^{t+1-k})\), so that \((2/3).(4/3^{t+1}) < a_2^2 - (a_1 - 2/3^{t-k-1})^2 = (4/3^{t-k+1}).(a_1 - 1/3^{t-k+1}) < 4/3^{t+1}\) and also \((2/3).(4/3^{t+1}) < b_2^2 - b_1^2 < 4/3^{t+1}\), and the algorithm runs as required from here on.

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