Abstract. The Yangian $Y(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ can be regarded as a deformation of two different Hopf algebras: the universal enveloping algebra of the current algebra $U(\mathfrak{g}[t])$ and the coordinate ring of the first congruence subgroup $O(G_1[[t^{-1}]]).$ Both of these algebras are obtained from the Yangian by taking the associated graded with respect to an appropriate filtration on $Y(\mathfrak{g}).$

Bethe subalgebras $B(C)$ in $Y(\mathfrak{g})$ form a natural family of commutative subalgebras depending on a group element $C$ of the adjoint group $G.$ The images of these algebras in tensor products of fundamental representations give all integrals of the quantum XXX Heisenberg magnet chain.

We describe the associated graded of Bethe subalgebras in the Yangian $Y(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ as subalgebras in $U(\mathfrak{g}[t])$ and in $O(G_1[[t^{-1}]]$) for all semisimple $C \in G.$ In particular, we show that the associated graded in $U(\mathfrak{g}[t])$ of the Bethe subalgebra $B(E)$ assigned to the unity element of $G$ is the universal Gaudin subalgebra of $U(\mathfrak{g}[t])$ obtained from the center of the corresponding affine Kac–Moody algebra $\hat{\mathfrak{g}}$ at the critical level. This generalizes Talalaev’s formula for generators of the universal Gaudin subalgebra to $\mathfrak{g}$ of any type. In particular, this shows that higher Hamiltonians of the Gaudin magnet chain can be quantized without referring to the Feigin–Frenkel center at the critical level.

Using our general result on the associated graded of Bethe subalgebras, we compute some limits of Bethe subalgebras corresponding to regular semisimple $C \in G$ as $C$ goes to an irregular semisimple group element $C_0.$ We show that this limit is the product of the smaller Bethe subalgebra $B(C_0)$ and a quantum shift of argument subalgebra in the universal enveloping algebra of the centralizer of $C_0$ in $\mathfrak{g}.$ This generalizes the Nazarov–Olshansky solution of Vinberg’s problem on quantization of the (Mishchenko–Fomenko) shift of argument subalgebras.
1. Introduction

1.1. Yangians and Bethe subalgebras

Let \( g \) be a simple complex Lie algebra, and \( G \) be the corresponding adjoint group. The Yangian \( Y(g) \) is the unique homogeneous Hopf algebra deformation of the universal enveloping algebra \( U(g[t]), \) see [D]. It is also a Hopf algebra deformation of the algebra \( \mathcal{O}(G_1[[t^{-1}]]) \) of functions on the first congruence subgroup \( G_1[[t^{-1}]] \subset G[[t^{-1}]] \) deforming the natural Poisson structure on \( G_1[[t^{-1}]], \) see [KWWY].

Bethe subalgebras are the family of commutative subalgebras of \( B(C) \subset Y(g) \) depending on a group element \( C \in G. \) The particular cases of Bethe subalgebras were defined in \([C], [D3], [KR], [MO], [M] \) and \([NO]. \) The most general definition of Bethe subalgebras goes back to Drinfeld: namely, one can define \( B(C) \) as the subalgebra generated by all Fourier coefficients of \( Tr_V(\rho(C) \otimes 1)(\rho \otimes \text{Id})(R(u)) \) for all finite dimensional representations \( \rho : Y(g) \to \text{End}(V), \) where \( R(u) \) is the universal \( R \)-matrix with spectral parameter. In [IR2] we gave a detailed description of these subalgebras using the \( RTT \)-realization of the Yangian from [D] and [W].

1.2. The (universal) Gaudin subalgebra

The universal enveloping algebra of the current algebra \( g[t] \) contains a large commutative subalgebra \( A_g \subset U(g[t]). \) This subalgebra comes from the center of the universal enveloping of the affine Kac–Moody algebra \( \hat{g} \) at the critical level and gives rise to the construction of higher hamiltonians generalizing those of Talalaev, Chervov and Molev, see [T], [CM].

1.3. The associated graded of a Bethe algebra

Let \( C \) be any element of a maximal torus \( T \subset G. \) Denote by \( Z_g(C) \) the centralizer of \( C \) in the Lie algebra \( g. \) It is a reductive Lie algebra containing the Cartan subalgebra \( h \subset g. \) The generators of Bethe subalgebra \( B(C) \subset Y(g) \) are invariant with respect to the adjoint action of \( Z_g(C). \)

Theorem A.

- The associated graded of \( B(C) \) in \( U(g[t]) \) is the universal Gaudin subalgebra \( A_{\mathfrak{z}_g(C)} \subset U(Z_g(C)[t]) \subset U(g[t]); \)
- The associated graded of \( B(C) \) in \( \mathcal{O}(G_1[[t^{-1}]]) \) is generated by all Fourier coefficients of the \( \mathbb{C}[t^{-1}] \)-valued functions \( \sigma(C)(g(t)) := Tr_V C \cdot g(t) \) (where \( g(t) \in G_1[[t^{-1}]] \)) for all finite dimensional \( G \)-modules \( V; \)
- The Bethe subalgebra \( B(C) \) is a maximal commutative subalgebra in \( Y(g)^{Z_g(C)}. \)

In particular, this gives a construction of the universal Gaudin subalgebra independent of the representation theory of \( \hat{g} \) at the critical level and for arbitrary simple \( g. \) We expect this leads to explicit type-free formulas for higher Gaudin Hamiltonians generalizing those of Talalaev, Chervov and Molev, see [T], [CM].
Remark. We believe that our Theorem A is a part of a more general picture describing all possible degenerations of the affine quantum group $U_q(\hat{\mathfrak{g}})$ at the critical level. In particular, according to Ding and Etingof [DE] the center of $U_q(\hat{\mathfrak{g}})$ at the critical level is generated by traces of the $R$-matrix, so it is natural to expect that both Bethe subalgebras in the Yangian and Gaudin subalgebras are degenerate versions of this center. We hope to return to this in forthcoming papers.

1.4. Limit Bethe subalgebras

Let $T^{\text{reg}} \subset T$ be the set of regular elements of the torus $T$. From Theorem A we see that the family of Bethe subalgebras $B(C) \subset Y(\mathfrak{g})$ is not flat, i.e., the Poincaré series of $B(C)$ is not constant in $C \in T$, because for non-regular $C \in T \setminus T^{\text{reg}}$, the subalgebra $B(C)$ becomes smaller. On the other hand a natural way to assign a commutative subalgebra of the same size as for $C \in T \setminus T^{\text{reg}}$ is by taking some limit of $B(C)$ as $C \to C_0$ (this idea goes back to Vinberg [V] and Shuvalov [Sh]). In general, such limit subalgebra $\lim_{C \to C_0} B(C)$ is not unique since it depends on the path $C(\varepsilon)$ such that $C(0) = C_0$. The second goal of this paper is to study the simplest limits of Bethe subalgebras corresponding to $C(\varepsilon) = C_0 \exp(\varepsilon \chi), C_0 \in T \setminus T^{\text{reg}}, \chi \in \mathfrak{h}$ as $\varepsilon \to 0$. It turns out that the resulting commutative subalgebra is the product of $B(C_0)$ and the quantum shift of argument algebra in the universal enveloping algebra $U(\mathfrak{J}_0(C_0)) \subset Y(\mathfrak{g})$.

1.5. Shift of argument subalgebras and Vinberg’s problem

The shift of argument subalgebras defined by Mishchenko and Fomenko in [MF] are (generically) maximal Poisson commutative subalgebras in $S(\mathfrak{g})$. For any $\chi \in \mathfrak{g}^*$ the corresponding shift of argument subalgebra $A_\chi \subset S(\mathfrak{g})$ can be described as the subalgebra generated by all the derivatives along $\chi$ of all adjoint invariant in $S(\mathfrak{g})$. More precisely, it is generated by all elements of the form $\partial^k_\chi \Phi_l$, for all generators $\Phi_l \in S(\mathfrak{g})^\mathfrak{g}$, $l = 1, \ldots, \text{rk} \mathfrak{g}$, $k = 0, 1, \ldots, m_l$, where $m_l = \deg \Phi_l - 1$ are the exponents of the Lie algebra $\mathfrak{g}$. Then the number of generators is $\sum_{l=1}^{\text{rk} \mathfrak{g}} (m_l + 1) = \frac{1}{2} (\dim \mathfrak{g} + \text{rk} \mathfrak{g})$, which is the maximal possible transcendence degree for Poisson commutative subalgebras in $S(\mathfrak{g})$.

Vinberg’s problem stated in [V] is the problem of lifting the Poisson commutative subalgebras $A_\chi \subset S(\mathfrak{g})$ to commutative subalgebras in the universal enveloping algebra $U(\mathfrak{g})$. In [NO] Olshansky and Nazarov construct the lifting of a shift of argument subalgebra $A_\chi \subset U(\mathfrak{g})$ as the image of Bethe subalgebra $B(\chi)$ in the (twisted) Yangian of $\mathfrak{g}$ under the evaluation homomorphism to $U(\mathfrak{g}(\mathfrak{l}))$. This works only for classical $\mathfrak{g}$ since for others there is no evaluation homomorphism from the Yangian to $U(\mathfrak{g})$.

In [R2] Vinberg’s problem was solved affirmatively for arbitrary simple $\mathfrak{g}$ and semisimple $\chi$ with the help of the Feigin–Frenkel center of $U(\hat{\mathfrak{g}})$ at the critical level. Namely, the lifting $A_\chi \subset U(\mathfrak{g})$, called quantum shift of argument subalgebra, was determined as the image of (a version of) the universal Gaudin subalgebra under some homomorphism. Moreover, it was proved that, for generic $\chi$, the subalgebras $A_\chi \subset S(\mathfrak{g})$ can be lifted to the universal enveloping algebra $U(\mathfrak{g})$ uniquely.

Our second main result is the following
Theorem B. Let $C(\varepsilon) = C_0 \exp(\varepsilon \chi), C_0 \in T \setminus T^{\text{reg}}$ with $\chi \in \mathfrak{h} \subset \mathfrak{g}_0(C_0)$ being a generic regular semisimple element of the centralizer of $C_0$ (i.e., belonging to some complement of countably many proper closed subsets in $\mathfrak{h}^{\text{reg}}$). Then

$$\lim_{\varepsilon \to 0} B(C(\varepsilon)) = B(C_0) \otimes Z(U(\mathfrak{g}_0(C_0))) A_\chi,$$

where $A_\chi \subset U(\mathfrak{g}_0(C_0))$ is the quantum shift of argument subalgebra corresponding to $\chi$.

Remark. Theorem B can be regarded as the closest approximation to the Olshansky–Nazarov solution of Vinberg’s problem for arbitrary simple $\mathfrak{g}$: indeed, now one can define the lifting of $A_\chi \subset S(\mathfrak{g})$ to the universal enveloping algebra as $A_\chi := U(\mathfrak{g}) \cap \lim_{\varepsilon \to 0} B(C(\varepsilon))$ for $C(\varepsilon) = \exp(\varepsilon \chi)$.

1.6. The paper is organized as follows. In Section 2 we study two classical limits of the Yangian and relations between them. In Section 3 we define Bethe subalgebras and give the lower bound for the size of a Bethe subalgebra. In Section 4 we define the universal Gaudin subalgebra and study some its properties. In Section 5 we prove Theorem A. In Section 6 we study some limits of Bethe subalgebras and prove Theorem B.

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2. Two classical limits of the Yangian

2.1. Notations and definitions
Let $\mathfrak{g}$ be a complex simple Lie algebra, $G$ be the corresponding connected adjoint group, $\hat{G}$ be the corresponding connected simply-connected group. Let $T \subset G$ be a maximal torus, $T^{\text{reg}} \subset T$ be the set of regular elements of $T$. Let $\mathfrak{h} \subset \mathfrak{g}$ be tangent Cartan subalgebra of $T$. Let $\langle \cdot, \cdot \rangle$ be the Killing form on $\mathfrak{g}$ and $\{x_a\}, a = 1, \ldots, \dim \mathfrak{g}$, be an orthonormal basis of $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$. We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using the Killing form. Let $m_i, i = 1, \ldots, \text{rk} \mathfrak{g}$ be the set of exponents of Lie algebra $\mathfrak{g}$. Let $O(G)$ and $O(\hat{G})$ be the algebras of polynomial functions on $G$ and $\hat{G}$ respectively.

Let $Y(\mathfrak{g})$ be the Yangian of $\mathfrak{g}$. Let $V = \bigoplus_{i=1}^{\text{rk} \mathfrak{g}} V(\omega_i, 0)$ be the direct sum of fundamental representations of $Y(\mathfrak{g})$. Let $R(u - v)$ be the image of the universal $R$-matrix in $\text{End}(V)^{\otimes 2}$. Using this data we define the $RTT$-realization $Y_V(\mathfrak{g})$ as follows. It turns out that $Y_V(\mathfrak{g}) \simeq Y(\mathfrak{g})$, see [D] and [W] for details.

Definition 2.2. The Yangian $Y_V(\mathfrak{g})$ is a unital associative algebra generated by the elements $t_{ij}^{(r)}, 1 \leq i, j \leq \dim V; r \geq 1$ with the defining relations

$$R(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u - v) \quad \text{in } \text{End}(V)^{\otimes 2} \otimes Y_V(\mathfrak{g})[[u^{-1}, v^{-1}]],$$
\[ S^2(T(u)) = T(u + \frac{1}{2} c_g), \]
where \( S(T(u)) = T(u)^{-1} \) is the antipode map and \( c_g \) is the value of the Casimir element of \( g \) on the adjoint representation.

Here
\[
T(u) = [t_{ij}(u)]_{i,j=1,\ldots,\dim V} \in \text{End} V \otimes Y_V(g),
\]
\[
t_{ij}(u) = \delta_{ij} + \sum_r t_{ij}^{(r)} u^{-r}
\]
and \( T_1(u) \) (resp. \( T_2(u) \)) is the image of \( T(u) \) in the first (resp. second) copy of \( \text{End} V \).

### 2.3. Two filtrations on the Yangian

The first filtration \( F_1 \) on \( Y_V(g) \) is determined by putting \( \deg t_{ij}^{(r)} = r \). More precisely, the \( r \)th filtered component \( F_1^{(r)} Y(g) \) is the linear span of all monomials \( t_{i_1j_1}^{(r_1)} \cdots t_{i_mj_m}^{(r_m)} \) with \( r_1 + \cdots + r_m \leq r \). By \( \text{gr}_1 \) we denote the operation of taking associated graded algebra with respect to \( F_1 \). From the defining relations we see that \( \text{gr}_1 Y(g) \) is a commutative algebra. Moreover, we have a Poisson algebra isomorphism \( \text{gr}_1 Y_V(g) \simeq \mathcal{O}(G_1[[t^{-1}]]) \) where the grading on \( \mathcal{O}(G_1[[t^{-1}]] \) is given by the \( \mathbb{C}^* \) action dilating \( t \) (see Section 2.13 for details):

**Theorem 2.4** ([IR2, Prop. 2.24]). There is an isomorphism of graded Poisson algebras \( \text{gr}_1 Y_V(g) \simeq \mathcal{O}(G_1[[t^{-1}]]). \)

**Corollary 2.5.** The Poincaré series of \( \text{gr}_1 Y_V(g) \) is
\[
\prod_{r=1}^{\infty} (1 - q^r)^{-\dim g}.
\]

The second filtration \( F_2 \) on \( Y_V(g) \) is determined by putting \( \deg t_{ij}^{(r)} = r - 1 \). Similarly, the \( r \)th filtered component \( F_2^{(r)} Y(g) \) is the linear span of all monomials \( t_{i_1j_1}^{(r_1)} \cdots t_{i_mj_m}^{(r_m)} \) with \( r_1 + \cdots + r_m \leq r + m \). By \( \text{gr}_2 \) we denote the operation of taking associated graded algebra with respect to \( F_2 \).

**Theorem 2.6** ([W]). \( \text{gr}_2 Y_V(g) \simeq U(g[t]) \) where the grading is given by the \( \mathbb{C}^* \) action dilating \( t \). Moreover, we have \( t^{-1} g \subset \text{span} \{ t_{ij}^{(r)} \} / F_2^{(r-2)} Y(g) \).

### 2.7. The associated bigraded algebra

The filtration \( F_1 \) on \( Y(g) \) produces a filtration on \( U(g[t]) = \text{gr}_2 Y(g) \) which we denote by the same letter \( F_1 \). Similarly, the filtration \( F_2 \) on \( Y(g) \) descends to a filtration \( F_2 \) on \( \mathcal{O}(G_1[[t^{-1}]] \) = \( \text{gr}_1 Y(g) \). The corresponding associated graded algebras \( \text{gr}_1 \text{gr}_2 Y(g) \) and \( \text{gr}_2 \text{gr}_1 Y(g) \) get a bigraded from the filtrations \( F_1 \) and \( F_2 \). We begin with some general facts about algebras with multiple filtrations.

For any algebra \( A \) endowed with two filtrations, \( F_1 \) and \( F_2 \), one can define the associated bigraded algebra of \( A \) as
\[
\text{bigr} A = \bigoplus_{i,j} (F_1^{(i)} A \cap F_2^{(j)} A) / (F_1^{(i-1)} A \cap F_2^{(j)} A + F_1^{(i)} A \cap F_2^{(j-1)} A).
\]

We also use the following notation:
\[
\text{gr}_{12} A = \text{gr}_1 \text{gr}_2 A, \quad \text{gr}_{21} A = \text{gr}_2 \text{gr}_1 A.
\]
Lemma 2.8. The associated bigraded algebra $\text{bigr} A$ is canonically isomorphic to $\text{gr}_{12} A$ and to $\text{gr}_{21} A$.

Proof. Consider the algebra $\text{gr}_1 A = F^{(0)}_1 A \oplus F^{(1)}_1 A / F^{(0)}_1 A \oplus \cdots$. The filtration $F_2$ produces a filtration $W_0 \subset W_1 \subset \cdots$ on $\text{gr}_1 A$ such that

$$W_i = \bigoplus F^{(i)}_2 A \cap F^{(j)}_1 A / F^{(i)}_2 A \cap F^{(j-1)}_1 A.$$

Note that

$$F^{(i)}_2 A \cap F^{(j)}_1 A / F^{(i)}_2 A \cap F^{(j-1)}_1 A \simeq F^{(i)}_2 A \cap F^{(j)}_1 A + F^{(j-1)}_1 A / F^{(j-1)}_1 A,$$

therefore $W_0 \subset W_1 \subset \cdots$ is indeed a filtration.

We have the following canonical isomorphisms

$$W_i / W_{i-1} = \bigoplus F^{(i)}_2 A \cap F^{(j)}_1 A / F^{(i-1)}_2 A \cap F^{(j)}_1 A \cap F^{(j-1)}_1 A.$$

Then associated graded algebra $\text{gr}_{21} A = W_0 \oplus W_1 / W_0 \oplus \cdots$ is canonically isomorphic to $\text{bigr} A$. It is also isomorphic to $\text{gr}_{12} A$ by the same argument. \qed

In contrast with the last lemma, if $U \subset A$ is a subspace, it is not true in general that $\text{gr}_{12} U = \text{gr}_{21} U$ as subspaces of $\text{bigr} A$, since the associated homomorphism of bigraded spaces $\text{bigr} U \rightarrow \text{bigr} A$ is not necessarily injective. Indeed, consider the algebra $A = \mathbb{C}[x, y]$ with two filtrations setting by $\deg_1 x = 1$, $\deg_1 y = 0$ and $\deg_2 x = 0$, $\deg_2 y = 1$ and take $U = \mathbb{C}[x + y]$. Then $\text{bigr} U$ is the polynomial algebra with one generator $z$ of the degree $(1, 1)$ (coming from $x + y$). We have $x + y \in F^{(1)}_1 A \cap F^{(0)}_2 A + F^{(0)}_1 A \cap F^{(1)}_2 A$, so the image of $z$ in

$$\text{bigr} U / \text{bigr} A = \mathbb{C} \cdot 1$$

is zero. Hence the image of $\text{bigr} U$ in $\text{bigr} A$ is $\mathbb{C} \cdot 1$. At the same time $\text{gr}_{12} U = \mathbb{C}[x]$, $\text{gr}_{21} U = \mathbb{C}[y]$.

On the other hand, the following is still true:

Proposition 2.9. Let $U$ be a vector subspace of $A$ such that $\text{gr}_{12} U \subset \text{gr}_{21} U$ as subspaces of $\text{bigr} A$. Then $\text{gr}_{12} U = \text{gr}_{21} U$. 

Proof. Suppose that we have an element \( x \in \text{gr}_{21}W \setminus \text{gr}_{12}W \). Suppose that \( \deg x = (k, l) \). Let \( \tilde{x} \) be a lifting of \( x \) to \( U \subset A \). Then \( \tilde{x} \in F_1^{(k)}A \cap F_2^{(l)}A + \sum_{k'<k} F_1^{(k')}A \cap F_2^{(l')}A \), where \( l' \) are some integers, so there is a presentation \( \tilde{x} = \sum_{i=1}^{N} \tilde{x}_i \) such that \( \tilde{x}_i \in F_1^{(k_i)}A \cap F_2^{(l_i)}A \) with \( k_1 = k, k_{i+1} < k_i \). Take such a presentation of \( \tilde{x} \) with the string \((k_1, k_2, k_3, \ldots, k_N)\) being lexicographically minimal among all such presentations. Then we have \( \tilde{x}_i \notin F_1^{(k_i-1)}A \cap F_2^{(l_i)}A + F_1^{(k_i)}A \cap F_2^{(l_i-1)}A \) and \( l_{i+1} > l_i \) for all \( i \). Moreover, we can assume that \( \tilde{x} \) is a lifting of \( x \) with the lexicographically minimal possible \((k_1, k_2, k_3, \ldots, k_N)\) among all liftings of \( x \) to \( U \). It is sufficient to show that \( N = 1 \); indeed, then \( \tilde{x} \in F_1^{(k)}A \cap F_2^{(l)}A \) and \( \text{gr}_{21} \tilde{x} = x \in \text{gr}_{12}A \).

Suppose that \( N > 1 \) and consider \( y = \text{gr}_{12} \tilde{x} \). It has degree \((k_N, l_N)\). Let \( \tilde{y} \in U \) be a lifting of \( y \) as an element of \( \text{gr}_{21}A \), i.e.,

\[
\tilde{y} \in F_1^{(k_N)}A \cap F_2^{(l_N)}A + \sum_{k'<k_N} F_1^{(k')}A \cap F_2^{(l')}A.
\]

Then, in the same way as before, we have \( \tilde{y} = \sum_{i=1}^{M} \tilde{y}_i \) such that \( \tilde{y}_i \in F_1^{(k_i)}A \cap F_2^{(l_i)}A \) with \( k_{i+1} < k_i \) and \( \tilde{y}_i \notin F_1^{(k_i-1)}A \cap F_2^{(l_i)}A + F_1^{(k_i)}A \cap F_2^{(l_i-1)}A \).

So \( \tilde{x} - \tilde{y} = \sum_{i=1}^{N-1} \tilde{x}_i - \sum_{j=N+1}^{M} \tilde{y}_j \) is a lifting of \( x \) such that to \( U \) such that the corresponding sequence of degrees of the summands \( x_1, \ldots, x_{N-1}, -y_{N+1}, \ldots, -y_M \) is \((k_1, \ldots, k_{N-1}, k_{N+1}, \ldots, k_M)\), hence lexicographically smaller than \((k_1, \ldots, k_N)\). This is a contradiction. \( \square \)

Suppose additionally that \( \text{gr}_1A \) is a commutative algebra. Then \( \text{bigr } A \) is also commutative and has a structure of a Poisson algebra. Let \( u \) and \( v \) be homogeneous elements of the degrees \((i_1, j_1)\) and \((i_2, j_2)\) respectively. Let \( \tilde{u}, \tilde{v} \) be their liftings to \( F_1^{(i_1)}A \cap F_2^{(j_1)}A \) and \( F_1^{(i_2)}A \cap F_2^{(j_2)}A \), respectively. Then we define

\[
\{u, v\} := […]\mod […]
\]

Also \( \text{gr}_1A \) is a Poisson algebra, and this give a Poisson algebra structure on \( \text{bigr } A \). It follows from definitions that these brackets are the same Poisson brackets as on the associated bigraded algebra.

Also on \( \text{gr}_{12}A \) one can obtain a Poisson bracket from the commutator on \( \text{gr}_2A \) which is also the same bracket as on the associated bigraded algebra. So we have the following

**Lemma 2.10.** If \( \text{gr}_1A \) is commutative then the associated bigraded algebra \( \text{bigr } A \) is canonically isomorphic to \( \text{gr}_{12}A \) and \( \text{gr}_{21}A \) as Poisson algebra.

Suppose that \( \dim F_1^{(i)}A/F_1^{(i-1)}A \) is always finite and let

\[
P_1(q) := \sum_{i=0}^{\infty} q^i \dim F_1^{(i)}A/F_1^{(i-1)}A
\]
be the Poincaré series of $\text{gr}A$. Let

$$P_{12}(q,z) = \sum_{i,j=0}^{\infty} q^i z^j \dim \left( F_1^{(i)} A \cap F_2^{(j)} A \right) / \left( F_1^{(i-1)} A \cap F_2^{(j)} A + F_1^{(i)} A \cap F_2^{(j-1)} A \right)$$

be the Poincaré series of $\text{bigr} A$. Then we have

**Lemma 2.11.** $P_1(q) = P_{12}(q,1)$.

**Proposition 2.12.** We have a bigraded Poisson algebra isomorphism $\text{gr}_{21}Y(\mathfrak{g}) \simeq S(\mathfrak{g}[t])$ where the bigrading on $S(\mathfrak{g}[t])$ is given by $\deg_1 x[r-1] = r$, $\deg_2 x[r-1] = r-1$ and the Poisson bracket on $S(\mathfrak{g}[t])$ is given by $\{x[r], y[s]\} = [x,y][r+s]$.

**Proof.** The first isomorphism $\text{gr}_{21}Y(\mathfrak{g}) \simeq \text{gr}_{21}Y(\mathfrak{g})$ is a particular case of Lemma 2.8. From Theorem 2.6 we have $x[r-1] \in F_1^{(r)}U(\mathfrak{g}[t])$. Hence the Poincaré series $P_{12}(q,z)$ is greater than or equal to $\prod_{r=1}^{\infty} (1-q^r z^{-1})^{-\dim \mathfrak{g}}$ (in the sense that every coefficient of the former is greater than or equal to the corresponding coefficient of the latter), and it is equal if and only if $x[r-1] \notin F_1^{(r)}U(\mathfrak{g}[t])$ for all $x \neq 0$. So according to Lemma 2.11, $P_1(q) = \prod_{r=1}^{\infty} (1-q^r)^{-\dim \mathfrak{g}}$ if and only if $x[r-1] \notin F_1^{(r-1)}U(\mathfrak{g}[t])$ for all $x \neq 0$. On the other hand, we have $P_1(q) = \prod_{r=1}^{\infty} (1-q^r)^{-\dim \mathfrak{g}}$ by Corollary 2.4. This completes the proof. □

**2.13. Congruence subgroup $G_1[[t^{-1}]]$ and its coordinate ring**

Let us give a few more details on $\text{gr}_1Y(\mathfrak{g}) = \mathcal{O}(G_1[[t^{-1}]])$. By definition, the proalgebraic group $G[[t^{-1}]]$ consists of $\mathbb{C}[[t^{-1}]]$ points of $G$. For any $g \in G[[t^{-1}]]$ we denote by $ev_g$ the corresponding homomorphism $\mathcal{O}(G) \rightarrow \mathbb{C}[[t^{-1}]]$. The first congruence subgroup $G_1[[t^{-1}]] \subset G[[t^{-1}]]$ is the kernel of the evaluation homomorphism at the infinity $G[[t^{-1}]] \rightarrow G$.

To any function $f \in \mathcal{O}(G)$ one can assign the $\mathbb{C}[[t^{-1}]]$-valued function $\tilde{f} : G_1[[t^{-1}]] \rightarrow \mathbb{C}[[t^{-1}]]$, $\tilde{f} = \sum_{r=0}^{\infty} f^{(r)} t^{-r}$ as follows: for any $g \in G_1[[t^{-1}]]$ we have

$$\tilde{f}(g) = ev_g(f).$$

The Fourier coefficients $f^{(r)}$ for all $f \in \mathcal{O}(G)$ generate the coordinate ring $\mathcal{O}(G_1[[t^{-1}]])$. Note that the group $G_1[[t^{-1}]]$ depends only on the formal group scheme assigned to $G$, so one can produce $f^{(r)}$ from any $f$ in the completion of $\mathcal{O}(G)$ with respect to the maximal ideal of $E \in G$.

There is a natural Poisson bracket on $\mathcal{O}(G_1[[t^{-1}]]$ coming from the Lie bialgebra structure on the loop algebra $\mathfrak{g}((t^{-1}))$ (or, equivalently, from the rational $r$-matrix). To write this bracket explicitly, we set, for any $x \in \mathfrak{g}$, the corresponding momenta vector fields of the left and right action on $G$, $\xi^L_x$ and $\xi^R_x$, respectively. Then for $f_1, f_2 \in \mathcal{O}(G)$, the bracket of corresponding $\mathbb{C}[[t^{-1}]]$-valued functions reads

$$\{\tilde{f}_1(u), \tilde{f}_2(v)\} = \frac{1}{u-v} (\xi^L_x f_1(u) \xi^L_x f_2(v) - \xi^R_x f_1(u) \xi^R_x f_2(v)). \tag{1}$$

The $\mathbb{C}^*$ action on $G_1[[t^{-1}]]$ by dilations of the variable $t$ determines a grading on $\mathcal{O}(G_1[[t^{-1}]]$ such that $\deg f^{(r)} = r$ for any $f \in \mathcal{O}(G)$. The Poisson bracket has degree $-1$ with respect to this grading. A more precise statement of Theorem 2.5 is the following
Proposition 2.14 ([IR2, Prop. 2.24]). There is an isomorphism of graded Poisson algebras $\text{gr}_1 Y_V(g) \simeq \mathcal{O}(G_1[[t^{-1}]])$ such that $\text{gr}_1 t_{ij}^{(r)} = \Delta_{ij}^{(r)}$ where $\Delta_{ij} \in \mathcal{O}(G)$ are the matrix elements of the representation $V$.

Any formal diffeomorphism $\varphi : (g, 0) \to (G, E)$ (i.e., any regular map $\varphi$ from the formal neighborhood of $0 \in g$ to the formal neighborhood of $E \in G$ such that $d_0 \varphi = \text{Id}_g$) determines an isomorphism $\Phi : t^{-1} g[[t^{-1}]] \to G_1[[t^{-1}]]$ which preserves the grading defined by the $C^*$ action by dilations on both sides. The coordinate ring of $t^{-1} g[[t^{-1}]]$ is the symmetric algebra of its graded dual space, i.e., $g[t]$ with the pairing given by

$$(x(t), y(t)) := \text{Res}_{t=0} \langle x(t), y(t) \rangle dt \quad \forall \ x(t) \in g[t], \ y(t) \in t^{-1} g[[t^{-1}]].$$

This means that any formal diffeomorphism $\varphi : (g, 0) \to (G, E)$ identifies the coordinate ring $\mathcal{O}(G_1[[t^{-1}]])$ with $S(g[t])$.

The filtration $F_2$ on $Y(g)$ induces a filtration on $\text{gr}_1 Y(g) = \mathcal{O}(G_1[[t^{-1}]])$. Slightly abusing notations, we denote this filtration by $F_2$ as well. Let $\mathcal{O}(G)_+$ be polynomial functions on $G$ consisting of $f \in \mathcal{O}(G)$ such that $f(E) = 0$.

**Proposition 2.15.** For any $f \in \mathcal{O}(G)$, we have $f^{(r)} \in F_2^{(r-1)} \mathcal{O}(G_1[[t^{-1}]])$. Moreover, if $f = f_1 \cdot f_2 \cdots f_k$ such that $f_1, \ldots, f_k \in \mathcal{O}(G)_+$ then $f^{(r)} \in F_2^{(r-k)} \mathcal{O}(G_1[[t^{-1}]])$.

**Proof.** It suffices to check the first assertion on generators of $\mathcal{O}(G)$. According to Peter-Weyl theorem, $\Delta_{ij}$ generate $\mathcal{O}(G)$ and we have

$$\Delta_{ij}^{(r)} = \text{gr}_1 t_{ij}^{(r)} \in F_2^{(r-1)} \mathcal{O}(G_1[[t^{-1}]])$$

by Proposition 2.14.

To prove the second assertion, notice that $f^{(r)}$ is a linear combination of $f_1^{(r_1)} \cdot f_2^{(r_2)} \cdots f_k^{(r_k)}$ with $r_i > 0$ and $r_1 + r_2 + \cdots + r_k = r$. $\square$

**Corollary 2.16.** Let $a, b \in \mathcal{O}(G)$ be a collection of functions such that $\{x_a = d_E f_a\}$ is the basis of $g = g^* = T^*_E G$. Then we have $f^{(r)}_a \in F_2^{(r-1)} \mathcal{O}(G_1[[t^{-1}]])$, and $\text{gr}_2 \mathcal{O}(G_1[[t^{-1}]])$ is freely generated by $\text{gr}_2 f_a^{(r)}$ with $a = 1, \ldots, n$, $r = 1, 2, \ldots$. Moreover, $\text{gr}_2 f_a^{(r)} = x_a[r-1]$.

**Corollary 2.17.** Let $\varphi_1 : (g, 0) \to (G, E)$ and $\varphi_2 : (g, 0) \to (G, E)$ be formal diffeomorphisms such that $d_0 \varphi_1 = d_0 \varphi_2$. Let $\Phi_1^*, \Phi_2^* : \mathcal{O}(G_1[[t^{-1}]]) \to S(g[t])$ be the corresponding ring isomorphisms. Then we have $\text{gr}_2 \Phi_1^* = \text{gr}_2 \Phi_2^*$.

This means that under any identification $\mathcal{O}(G_1[[t^{-1}]]) \simeq S(g[t])$ as above, the grading $F_1$ on $S(g[t])$ is given by $\deg x[r-1] = r$ and the filtration $F_2$ on $S(g[t])$ is given by $\deg x[r-1] = r-1$, for any $x \in g$. The Poisson bracket on $\mathcal{O}(G_1[[t^{-1}]]) \simeq S(g[t])$ descends to $\text{gr}_2 S(g[t]) = S(g[t])$. We denote the latter bracket by $\{\cdot, \cdot\}_t$.

**Lemma 2.18.** We have

$$\{x[m], y[l]\}_t = [x, y][n + m],$$

for any $x, y \in g, m, l \geq 0$. 
Proof. For any \( x \in \mathfrak{g} \), denote by \( \tilde{x}(u) \) the formal series

\[
\sum_{r=1}^{\infty} x[r-1]u^{-r} \in S(\mathfrak{g}[t])[u^{-1}].
\]

Let \( f_x \in \mathcal{O}(G) \) be a function such that \( d_E f_x = x \) under the identification \( \mathfrak{g} = \mathfrak{g}^* \). According to Corollary 2.16, we have \( f_x^{(r)} \in F_2^{(r-1)} \mathcal{O}(G_1[[t^{-1}]]) \) and \( \text{gr}_2 f_x = x[r-1] \). Slightly abusing notations we will write \( \text{gr}_2 f_x(u) = \tilde{x}(u) \).

For \( x, y \in \mathfrak{g} \) we take the functions \( f_x, f_y \in \mathcal{O}(G) \) as above and write the Poisson bracket

\[
\{ \tilde{x}(u), \tilde{y}(v) \} = \frac{1}{u-v} \left( \sum_{a=1}^{\dim \mathfrak{g}} \frac{2}{2} ((x_a, x)(x_a, y[v]) + [x_a, x(u)](x_a, y) \right.
\]

\[
- (x_a, x)[y(v), x_a] - [x(u), x_a](x_a, y)) \Big). \]

Since \( \sum_{a=1}^{\dim \mathfrak{g}} (x_a, x)[x_a, y] = - \sum_{a=1}^{\dim \mathfrak{g}} [x_a, x](x_a, y) = [x, y] \) we finally get

\[
\{ \tilde{x}(u), \tilde{y}(v) \} = [x(u), y(v)]. \quad \Box
\]

2.19.

From Proposition 2.10 we obtain that \( \text{gr}_{12} Y_{V}(\mathfrak{g}) \simeq \text{gr}_{21} Y_{V}(\mathfrak{g}) \) as Poisson algebras and are isomorphic to \( S(\mathfrak{g}[t]) \) with the standard Kirillov–Kostant Poisson bracket.

We identify \( \text{gr}_{12} Y_{V}(\mathfrak{g}) \) with \( S(\mathfrak{g}[t]) \) and \( \text{gr}_{21} Y_{V}(\mathfrak{g}) \) with \( S(\mathfrak{g}[t]) \) and thus obtain an automorphism of Poisson algebra \( \psi : S(\mathfrak{g}[t]) \to S(\mathfrak{g}[t]) \).

Lemma 2.20. \( \psi(x[k]) = c^k x[k], c \in \mathbb{C}^* \).

Proof. The filtrations \( F_1, F_2 \) are \( \mathfrak{g} \)-invariant, therefore \( \psi \) is \( \mathfrak{g} \)-invariant. Let us identify \( \text{gr}_{12} \) and \( \text{gr}_{21} \) with bi-graded quotient. We know that

\[
F_1^{(0)} Y(\mathfrak{g}) \cap F_2^{(0)} Y(\mathfrak{g}) = \mathbb{C} \cdot 1,
\]

\[
F_1^{(1)} Y(\mathfrak{g}) \cap F_2^{(0)} Y(\mathfrak{g})/F_1^{(0)} Y(\mathfrak{g}) \cap F_2^{(0)} Y(\mathfrak{g}) = \mathfrak{g},
\]

\[
(F_1^{(2)} Y(\mathfrak{g}) \cap F_2^{(1)} Y(\mathfrak{g}))/(F_1^{(1)} Y(\mathfrak{g}) \cap F_2^{(1)} Y(\mathfrak{g}) + F_1^{(2)} Y(\mathfrak{g}) \cap F_2^{(0)} Y(\mathfrak{g})) = t \cdot \mathfrak{g} \simeq \mathfrak{g}
\]
as \( \mathfrak{g} \)-modules. Note also that

\[
F_1^{(1)} Y(\mathfrak{g}) \cap F_2^{(0)} Y(\mathfrak{g})/F_1^{(0)} Y(\mathfrak{g}) \cap F_2^{(0)} Y(\mathfrak{g}) = \mathfrak{g}
\]
is a Lie algebra isomorphism with respect to the Poisson bracket on the left-hand side.

Using the fact that \( g \) is simple we see that the only isomorphism of \( g \) with itself is identity, and isomorphism of \( t \cdot g \) as \( g \)-module is the scalar of identity.

From Lemma 2.18 by induction on \( r \) we have

\[
\psi([x, y][r]) = \psi([x[r - 1], y[1]]) = \psi(x[r - 1], cy[1])
\]

\[
= \{c^{r-1}x[r - 1], cy[1]\} = c^r[x, y][r].
\]

Since \( g \) is simple, we have \([g, g] = g\) therefore \( \psi \) has the desired form. \( \square \)

It will be useful for what follows to have a way to write the leading term with respect to \( F_2 \) of any function of the form \( f^{(r)} \in \mathcal{O}(G_1[[t^{-1}]])) \) for any \( f \in \mathcal{O}(G) \). For this, we fix a formal coordinate system in the neighborhood of \( E \in G \), i.e., let \( \varphi : (g, 0) \to (G, E) \) be a formal diffeomorphism such that \( d_0 \varphi = \text{Id} \). Then to any function \( f \in \mathcal{O}(G) \) one can assign its Taylor expansion at \( E \in G \), namely a collection of homogeneous polynomials \( f_l \in S^l(g), \ l = 0, 1, \ldots \) such that \( \varphi^* f = \sum_{l=0}^{\infty} f_l \). We denote by \( D \) the derivation on \( S(g[t]) \) determined by

\[
D(x[r - 1]) = rx[r].
\]

**Lemma 2.21.** Suppose \( f_k \in S^k(g) \) is the first nonzero term in the Taylor series of \( f \in \mathcal{O}(\tilde{G})_+ \) at \( E \in G, \ k > 0 \).

Then we have

1. \( f^{(r)} = 0 \) for \( r < k \);
2. \( f^{(r)} \in F_2^{(r-k)} \mathcal{O}(G_1[[t^{-1}]])) \) and \( f^{(r)} \not\in F_2^{(r-k-1)} \mathcal{O}(G_1[[t^{-1}]])) \) for \( r \geq k \);
3. \( \text{gr}_2 f^{(r)} = \frac{1}{(r-k)!} D^{r-k} f_k \) where \( f_k \in S(g) \subset S(g[t]) \) as a polynomial of the \( x[0] \)'s.

**Proof.** The first assertion follows immediately from Proposition 2.15. The second one follows from Corollary 2.16. To show the last equality, note that \( \Phi^* \tilde{f}(u) = \sum_{i=k}^{\infty} \tilde{f}_i(u) \). According to the assertions (1–2) and by Corollary 2.17 the leading term of any Fourier coefficient with respect to the filtration \( F_2 \) is given by that of \( \Phi^* \tilde{f}_k(u) \). On the other hand for any \( x \in g \) the corresponding series \( \tilde{x}(u) = \sum_{r=1}^{\infty} x[r - 1]u^{-r} \) rewrites as \( \tilde{x}(u) = \exp(u^{-1}D)x[0] \). So we have \( \Phi^* \tilde{f}_k(u) = \exp(u^{-1}D)f_k \) as well, hence the assertion. \( \square \)

## 3. Bethe subalgebras in Yangian

### 3.1. Definition

Let \( \rho_i : Y_V(g) \to \text{End} V(\omega_i, 0) \) be the \( i \)th fundamental representation of \( Y(g) \). \( V(\omega_i, 0) \) is also a representation of \( U(g) \) (because \( U(g) \subset Y_V(g) \)) and hence can be regarded as a representation of \( \tilde{G} \). Slightly abusing notation we denote this group representation by the same symbol \( \rho_i \).

Let \( \pi_i : V \to V(\omega_i, 0) \)

be the projection.

Let \( T^i(u) = \pi_i T(u) \pi_i \) be the submatrix of \( T(u) \)-matrix, corresponding to the \( i \)th fundamental representation.
Definition 3.2. Let $C \in \tilde{G}$. Bethe subalgebra $B(C) \subset Y_V(\mathfrak{g})$ is the subalgebra generated by all coefficients of the following series with the coefficients in $Y_V(\mathfrak{g})$
\[ \tau_i(u, C) = \text{tr}_V(\omega_i, 0) \rho_i(C) T^i(u), \quad 1 \leq i \leq n. \]

Remark. In fact $B(C)$ depends only on the class of $C$ in $\tilde{G}/Z(\tilde{G})$, i.e., on an element of adjoint group $G$.

Proposition 3.3 ([IR2], [I]).

1. Bethe subalgebra $B(C)$ is commutative for any $C \in G$.
2. $B(C)$ is a maximal commutative subalgebra of $Y(\mathfrak{g})$ for $C \in T^{\text{reg}}$.

3.4. Bethe subalgebras in $\mathcal{O}(G_1[[t^{-1}]])$

Here we follow [IR2]. Let $\{V_{\omega_i}\}_{i=1}^n$ be the set of all fundamental representations of $\mathfrak{g}$. We also consider $\{V_{\omega_i}\}_{i=1}^n$ as a representations of the corresponding simply-connected group $\tilde{G}$. Let $\Lambda_i$ be some basis of $V_{\omega_i}$. For any $v \in \Lambda_i$ we denote the corresponding element of dual basis by $v^* \in \Lambda_i^*$. By $\Delta_{v,v^*} \in \mathcal{O}(\tilde{G})$ we denote the corresponding matrix coefficient of $V_{\omega_i}$.

Definition 3.5. Let $C \in \tilde{G}$. Bethe subalgebra $\tilde{B}(C)$ of $\mathcal{O}(G_1[[t^{-1}]])$ is the subalgebra generated by the coefficients of the following series:
\[ \sigma_i(u, C) = \text{tr}_{V_{\omega_i}} \rho_i(C) \rho_i(g) = \sum_{v \in \Lambda_i} \Delta_{v,v^*}(C g) = \sum_{r=0}^{\infty} \sum_{v \in \Lambda_i} \Delta_{v,v^*}^{(r)}(C g) u^{-r}, \]
where $\Lambda_i$ is some basis of $V_{\omega_i}$, $g \in G_1[[t^{-1}]]$.

Remark. This subalgebra depends only on the class of $C$ in $\tilde{G}/Z(\tilde{G})$ as well.

Remark. One can define the same subalgebra using all finite-dimensional representations of $\tilde{G}$.

Proposition 3.6 ([IR2]). We have $\text{gr}_1 B(C) = \tilde{B}(C)$ for any $C \in T^{\text{reg}}$.

We generalize this Proposition 3.6 to any $C \in T$ below.

3.7. Size of a Bethe subalgebra

Consider $B(C)$ with $C \in T$. In the next proposition we use the filtration $F_1$.

Proposition 3.8 (Lower bound for the size of Bethe subalgebra, see also [IR]).

Bethe subalgebra $B(C)$ contains $\text{rk} \mathfrak{g}$ infinite series of algebraically independent elements such that every series consist of elements with the degrees $m_i + 1, m_i + 2, \ldots$, where $m_i$ are the exponents of $\mathfrak{g}$, $i = 1, \ldots, \text{rk} \mathfrak{g}$.

Proof. Analogous to [IR2, Prop. 4.8] we have $\text{gr}_1 B(C) \supset \tilde{B}(C)$. We are going to find a set of algebraically independent elements in $\tilde{B}(C)$ of the same degrees as in the proposition statement with respect to the grading obtained from filtration $F_1$.

Let $\sigma_i(u, C)$ be generators of Bethe subalgebra $\tilde{B}(C) \subset \mathcal{O}(G_1[[t^{-1}]])$. One can extend $\sigma_i(u, C)$ to the group $G((t^{-1/2}))$ by means of Definition 3.5. Denote by $\sigma_i(C)^{(r)}$ the coefficient of $u^{-r}$ in $\sigma_i(u, C)$. 

For any coweight $\nu$ of the maximal torus $T \subset G$, we denote by $t^\nu$ the corresponding 1-parametric subgroup. Note that $t^\nu$ can be regarded as a $\mathbb{C}((t^{-1}))$-point of $T \subset G$, hence as an element of $G((t^{-1}))$. Consider also the element $t^{\tilde{\rho}} \in G((t^{-1/2}))$, $\tilde{\rho} = \sum_i \tilde{\omega}_i$, where $\tilde{\omega}_i$ are fundamental co-weights of $\mathfrak{g}_\mathfrak{g}(C)$. Note that this is a well-defined element because $2\tilde{\rho}$ belongs to the co-weight lattice of $\mathfrak{g}$.

Let $e$ be the principal nilpotent element of the reductive algebra $\mathfrak{g}_\mathfrak{g}(C)$. The differential of $\sigma_i(C)^{(r)}$ at the point $\exp(e) \in G((t^{-1/2}))$ is naturally a linear functional on the tangent space $T_{\exp(e)}G((t^{-1/2})) \simeq \mathfrak{g}((t^{-1/2}))$. Hereafter we identify $T_{\exp(e)}G((t^{-1/2}))$ with $\mathfrak{g}((t^{-1/2}))$ by the left $G((t^{-1/2}))$-action for any $g(t) \in G((t^{-1/2}))$.

Then we have

$$d_{\exp(t^{-1})} \sigma_i(C)^{(r)} = d_{t^{\tilde{\rho}} \cdot \exp(e)} t^{-\tilde{\rho}} \sigma_i(C)^{(r)} = (\text{Ad} \ t^{\tilde{\rho}}) d_{\exp(e)} \sigma_i(C)^{(r)}.$$

The last equality follows from the invariance of $\sigma_i^{(r)}(C)$ under conjugation by $t^{\tilde{\rho}}$.

We now consider the restriction of differentials to $T_{e}G_{1}[[t^{-1}]] \simeq t^{-1} \mathfrak{g}[[t^{-1}]]$. Let $\chi_{\omega_i}$ be characters of $\tilde{G}$-modules $V_{\omega_i}$, $i = 1, \ldots, \text{rk} \mathfrak{g}$.

**Lemma 3.9.** $d_{\exp(e)} \sigma_i(C)^{(r)}(xt^{-s}) = \delta_{r,s} d_{C^{-1} \exp(e)} \chi_{\omega_i}(x)$ for any $x \in \mathfrak{g}$.

**Proof.** We have

$$d_{\exp(e)} \sigma_i(C)^{(r)} = \sum_{v \in \Lambda_i} d_{\exp(e)} \Delta^{(r)}_{v,i} (C)(xt^{-s}) = \delta_{r,s} \text{tr} V_{\omega_i} \rho_i(C) \rho_i(x)$$

$$= \delta_{r,s} d_{C^{-1} \exp(e)} \chi_{\omega_i}(x)$$

for any $x \in \mathfrak{g}$. \hfill $\square$

Note that $C^{-1} \cdot \exp(e)$ is a regular element of $\tilde{G}$. As in [IR2] the key point here is the fact that differentials of characters of fundamental representations at regular points are linearly independent (see [St, Thm. 3, p. 119]).

**Lemma 3.10.** span $\langle d_{C^{-1} \exp(e)} \chi_{\omega_i} \rangle = \mathfrak{g}_\mathfrak{g}(C)(e)$ under the identification $\mathfrak{g}^* \simeq \mathfrak{g}$.

**Proof.** Note that $\mathfrak{g}_\mathfrak{g}(C(\exp(e))) = \mathfrak{g}_\mathfrak{g}(C)(e)$. It is sufficient to show now that span $\langle d_{C^{-1} \exp(e)} \chi_{\omega_i} \rangle = \mathfrak{g}(\exp(e))$. It is obvious that span $\langle d_{C^{-1} \exp(e)} \chi_{\omega_i} \rangle \subset \mathfrak{g}_\mathfrak{g}(C(\exp(e)))$ and dimensions coincide according to linear independence of differentials at regular points. \hfill $\square$

Under the correspondence from the previous lemma one can express eigenvector $v_j$ of $\tilde{\rho}$ with eigenvalue $m_j$ as a linear combination of $d_{C^{-1} \exp(e)} \chi_{\omega_i}$. Let $\sigma_{v_j}(C)^{(r)}$ be the corresponding linear combination of $\sigma_i(C)^{(r)}$, $i = 1, \ldots, \text{rk} \mathfrak{g}$.

**Lemma 3.11.** $(\text{Ad} \ t^{\tilde{\rho}}) d_{\exp(e)} \sigma_{v_j}(C)^{(r)}(xt^{-s}) = \delta_{r,s-m_j} \langle v_j, x \rangle$ for any $x \in \mathfrak{g}$.

**Proof.** It follows from Lemma 3.9 and the fact that $\sigma_{v_j}(C)$ is an eigenvector of $t^{\tilde{\rho}}$ with eigenvalue $t^{m_j}$. \hfill $\square$

From the last lemma the statement of the proposition follows. \hfill $\square$

**Remark.** We will also give another proof of Proposition 3.8 in Section 5, see Proposition 5.10.
4. Universal Gaudin subalgebra

4.1. Commutative subalgebra from the center on critical level

We regard the Lie algebra $\mathfrak{g}[t]$ as a “half” of the corresponding affine Kac–Moody algebra $\hat{\mathfrak{g}}$ which is a central extension of the loop Lie algebra $\mathfrak{g}(t^{-1})$. According to Feigin and Frenkel [FF], the local completion of the universal enveloping algebra $U(\hat{\mathfrak{g}})$ on the critical level $k = -h^\vee$ has a huge center $Z$. The image of natural homomorphism from $Z$ to the quantum Hamiltonian reduction

$$(U(\hat{\mathfrak{g}})/U(\hat{\mathfrak{g}})t^{-1}\mathfrak{g}[t^{-1}])(t^{-1}\mathfrak{g}[t^{-1}])$$

is a commutative subalgebra there. The latter naturally embeds into $U(\mathfrak{g}[t])$, so the image of $Z$ can be regarded as a commutative subalgebra $A_\mathfrak{g} \subset U(\mathfrak{g}[t])$, which we call the universal Gaudin subalgebra of $U(\mathfrak{g}[t])$.

Though there are no explicit formulas for the generators of the subalgebra $A_\mathfrak{g}$ in general, one can describe explicitly the associated graded subalgebra $A_\mathfrak{g} \subset S(\mathfrak{g}[t]) = O(t^{-1}\mathfrak{g}[t^{-1}])$. Namely, $A_\mathfrak{g}$ is freely generated by all Fourier components of $C[[t^{-1}]]$-valued functions $\Phi_t(x(t))$ for all generators $\Phi_t$ of the algebra of adjoint invariants $S(\mathfrak{g})^\theta$. The subalgebra $A_\mathfrak{g} \subset S(\mathfrak{g}[t])$ can be obtained via the Magri–Lenard scheme ([Ma]) from a pair of compatible Poisson brackets on $S(\mathfrak{g}[[t]])$ (see next subsection).

4.2. Two Poisson brackets on $S(\mathfrak{g}[[t]])$

Let $\mathfrak{g}[[t]]$ be a Lie algebra of formal power series with coefficients in $\mathfrak{g}$. Consider two Poisson brackets on $S(\mathfrak{g}[[t]])$:

$$\{x[n], x[m]\}_0 = [x, y][n + m];$$
$$\{x[n], y[m]\}_1 = [x, y][n + m + 1],$$

for any $x, y \in \mathfrak{g}$. Note that bracket $\{\cdot, \cdot\}_0$ is the bracket we obtain on $\text{gr }_2 Y_\mathfrak{g}(\mathfrak{g})$ if we restrict it to $S(\mathfrak{g}[t])$. Note also that $\mathfrak{g}[t]$ with $\{\cdot, \cdot\}_1$ is isomorphic to $t \cdot \mathfrak{g}[t]$ as a Lie algebra.

We call a pair of Poisson brackets on $S(\mathfrak{g}[[t]])$ compatible if every linear combination of them is also a Poisson bracket. The following lemma is well known.

**Lemma 4.3.**

1. Poisson brackets $\{\cdot, \cdot\}_0$ and $\{\cdot, \cdot\}_1$ are compatible.
2. Every linear combination of these brackets restricts to $S(\mathfrak{g}[[t]])^\theta$ (i.e., the bracket of $\mathfrak{g}$-invariant elements is $\mathfrak{g}$-invariant).

By $S(\mathfrak{g}[[t]])_{u, v}$ we denote a Poisson algebra $S(\mathfrak{g}[[t]])$ with Poisson bracket $u\{\cdot, \cdot\}_0 + v\{\cdot, \cdot\}_1$. From the pair of compatible Poisson brackets one can obtain a Poisson commutative subalgebra of $S(\mathfrak{g}[[t]])^\theta$ with respect to $\{\cdot, \cdot\}_0$ and $\{\cdot, \cdot\}_1$ at the same time, see, e.g., [R1]. The construction is as follows: the subalgebra is generated by all centers of $S(\mathfrak{g}[[t]])_{u, v}$ for $u, v \in \mathbb{C}$ except the case $u = 0, v = 1$.

4.4. Universal Gaudin subalgebra $A_\mathfrak{g}$

Consider the derivation $D$ of $S(\mathfrak{g}[t])^\theta$:

$$D(x[n]) = (n + 1)x[n + 1].$$

Let $\Phi_i, i = 1, \ldots, \text{rk} \mathfrak{g}$ be free generators of $S(\mathfrak{g}[0])^\theta$. 
Definition 4.5. Universal Gaudin subalgebra $A_g$ is the subalgebra generated by all $D^k \Phi_i, k \geq 0, i = 1, \ldots, \operatorname{rk} g$.

Proposition 4.6. Subalgebra $A_g$ is commutative and elements $D^k \Phi_i, k \geq 0, i = 1, \ldots, \operatorname{rk} g$ are free generators of $A_g$.

Proof. It is easy to check that the map
\[ \varphi_{1,v} : S(g[[t]])_{1,0} \to S(g[[t]])_{1,v}; \]
\[ x[m] \mapsto x[m] + \sum_{k=1}^{\infty} (-1)^k v^k x[m + k], \]
for all $x \in g, m \geq 0$, is an isomorphism of Poisson algebras. Indeed, the inverse map is
\[ x[m] \mapsto x[m] + vx[m + 1], \]
for all $x \in g, m \geq 0$. One can restrict $\varphi_{1,v}$ to $S(g[[t]])_{1,0}$ to obtain the isomorphism $S(g[[t]])^0_{1,0} \simeq S(g[[t]])^0_{1,v}$.

If $\Phi \in S(g[0])^0$ then it is central in $S(g[[t]])^0_{1,0}$ therefore $\varphi_{1,v}(\Phi)$ is central in $S(g[[t]])^0_{1,v}$. It implies that the elements of the form $\varphi_{1,v}(\Phi)$, $\Phi \in S(g[0])^0, v \in \mathbb{C}$ commutes with respect to any bracket $u \{ \cdot, \cdot \}_0 + v \{ \cdot, \cdot \}_1$. This implies that the coefficients of degrees of $v$ commute with respect to any bracket $u \{ \cdot, \cdot \}_0 + v \{ \cdot, \cdot \}_1$, in particularly $\{ \cdot, \cdot \}_0$ and $\{ \cdot, \cdot \}_1$. Note that the coefficients belongs to $S(g[t])^0$, hence these coefficients generate some commutative subalgebra of $S(g[t])^0$.

It is easy to see that for any $f \in S(g[0])$
\[ \varphi_{1,v}(f) = \exp(-vD)f. \]

Therefore the coefficient of $v^k$ of $\varphi_{1,v}(\Phi)$ is proportional to $D^k \Phi$. This means that our subalgebra coincides with $A_g$.

The statement that elements of the form $D^k \Phi_i$ are free generators of $A_g$ is well known, see, e.g., [BD, §2.4.1], [F, Prop. 9.3]. □

4.7. Properties of subalgebra $A_g$

By definition put
\[ \omega_g = \sum_a x_a[0]^2 \in S(g[0])^0, \]
\[ \Omega_g = \sum_a x_a[0]x_a[1] \in S(g[t])^0, \]
where $\{ x_a \}, a = 1, \ldots, \dim g$ is an orthonormal basis of $g$ with respect to $\langle \cdot, \cdot \rangle$.

Note that $\omega_g \in A_g$ by construction. Also $D \omega_g = 2 \Omega_g$ therefore $\Omega_g \in A_g$ too.

Proposition 4.8 ([R1]). Subalgebra $A_g$ is the centralizer of $\omega_g$ in $S(g[t])$ with respect to $\{ \cdot, \cdot \}_1$.

Proposition 4.9. Subalgebra $A_g$ is the centralizer of $\Omega_g$ in $S(g[t])^0$ with respect to $\{ \cdot, \cdot \}_0$. 
Proof. Let us again consider the isomorphism

$$\varphi_{1,v} : S(g[[t]])_{1,0} \to S(g[[t]])_{1,v}.$$

Note that

$$\varphi_{1,v}(\omega_g) = \omega_g + \Omega_g v + \cdots$$

and $\varphi_{1,v}(\omega_g)$ belong to the center of $S(g[[t]])_v^g$. Then for any $z \in S(g[t])^g \subset S(g[[t]])^g$ we have

$$\{\varphi_{1,v}(\omega_g), z\}_0 + v\{\varphi_{1,v}(\omega_g), z\}_1 = 0.$$

Considering the coefficient of $v$ we get

$$\{\Omega_g, z\}_0 + \{\omega_g, z\}_1 = 0.$$

Therefore the centralizer of $\omega_g$ in $S(g[[t]])^g$ with respect to $\{\cdot, \cdot\}_1$ coincides with the centralizer of $\Omega_g$ in $S(g[t])^g$ with respect to $\{\cdot, \cdot\}_0$. $\square$

Corollary 4.10. $A_g$ is a maximal commutative subalgebra of both $S(g[[t]])_{1,0}^g$ and $S(g[t])_{0,1}^g$.

Proposition 4.11. There exists no more than one lifting $A_g$ of $A_g$ to $U(g[[t]])^g$.

Proof. Up to scaling and additive constant there exists a unique lifting of $\Omega_g$ to $U(g[t])^g$. Moreover, any lifting of subalgebra $A_g$ is the centralizer of the lifting of the element $\Omega_g$. But the centralizer does not depend on a constant therefore the lifting is unique. $\square$

Remark. We will assign to any $C \in T$ the subalgebra $A_{3^g(C)} \subset S(3^g(C)) \subset S(g)$ and consider the elements $\omega_{3^g(C)}, \Omega_{3^g(C)}$ in it, i.e., consider the above objects for a reductive Lie algebra, not necessarily semisimple. All the statements and definitions of the present section remain the same for $3^g(C)$ with the following conventions: we take the restriction of $(\cdot, \cdot)$ to $3^g(C)$ as the invariant scalar product on $3^g(C)$, $\text{rk} \ 3^g(C) = \text{rk} \ g$, the exponents of $3^g(C)$ are the exponents of the semi-simple algebra $[3^g(C), 3^g(C)]$ plus additional $\text{rk} \ g - \text{rk} [3^g(C), 3^g(C)]$ of zeros.

5. Bethe subalgebras and universal Gaudin subalgebras

Let $E \in G$ be the identity element. We are going to prove Theorem A for $C = E$.

Theorem 5.1. $\text{gr}_2 B(E)$ is the universal Gaudin subalgebra, i.e., $\text{gr}_2 B(E) = A_g$.

Proof. We need the following

Lemma 5.2. The element $\Omega_g$ belongs to $\text{gr}_21 B(E)$ and to $\text{gr}_{12} B(E)$.

Proof of Lemma 5.2. Firstly we consider $\text{gr}_21 B(E) \subset S(g[t])^g$. From Proposition 3.8 we know that there are 2 algebraically independent elements of degree 3 for type $A$ and 1 element of degree 3 for other types in $\text{gr}_21 B(E)$. All degree 3 elements in $S(g[t])^g$ are from $S^3(g)^g + (g \cdot t g)^g$. In type $A$ the spaces $S^3(g)^g$ and $(g \cdot t g)^g$
are 1-dimensional. In other types we have $S^3(\mathfrak{g})^\theta = 0$ and $\dim(\mathfrak{g} \cdot t\mathfrak{g})^\theta = 1$. So $\text{gr}_{21} B(E)$ contains the spaces $S^3(\mathfrak{g})^\theta$ and $(\mathfrak{g} \cdot t\mathfrak{g})^\theta$.

Any element from $(\mathfrak{g} \cdot t\mathfrak{g})^\theta$ has the form $c \cdot \Omega_\theta, c \in \mathbb{C}^*$. Finally, $\Omega_\theta$ is homogeneous with respect to the second grading hence $\Omega_\theta \in \text{gr}_{21} B(E)$.

We identify $\text{gr}_{12} Y_V(\mathfrak{g})$ and $\text{gr}_{21} Y_V(\mathfrak{g})$ with $S(\mathfrak{g}[t])$ then obtain the automorphism $\psi : S(\mathfrak{g}[t]) \to S(\mathfrak{g}[t])$ of Poisson algebra. From Lemma 2.20 it follows that $\psi$ maps any graded (with respect to the first grading) vector subspace of $S(\mathfrak{g}[t])$ to itself.

We have $F_1^{(k)} Y_V(\mathfrak{g}) \cap F_2^{(l)} Y_V(\mathfrak{g}) = F_1^{(k)} Y_V(\mathfrak{g}) \cap F_2^{(k-1)} Y_V(\mathfrak{g})$ for $l \geq k$. Then any lifting of $\Omega_\theta$ to $Y_V(\mathfrak{g})$ belongs to

$$F_1^{(3)} \cap F_2^{(1)} + \sum_{k<3, l<k} F_1^{(k)} \cap F_2^{(l)} = F_1^{(3)} \cap F_2^{(1)}.$$

Then for any lifting $\tilde{\Omega}_\theta$ we have $\text{gr}_{12} \tilde{\Omega}_\theta = \Omega_\theta$. \Box

We also use the following

**Proposition 5.3.** $\text{gr}_{12} B(E) = A_\theta$.

**Proof of Proposition 5.3.** We know that $\Omega_\theta \in \text{gr}_{21} B(E)$ and that $A_\theta$ is the centralizer of $\Omega_\theta$. Moreover $\text{gr}_{21} B(E) \subset S(\mathfrak{g}[t])^\theta$ thus $\text{gr}_{21} B(E) \subset A_\theta$.

From Proposition 3.8 and the definition of $A_\theta$ we see that the Poincaré series of $\text{gr}_{21} B(E)$ and of $A_\theta$ coincide. Hence we have $\text{gr}_{21} B(E) = A_\theta$. Moreover, $\text{gr}_{12} B(E) \subset \text{gr}_{21} B(E)$, because $\Omega_\theta \in \text{gr}_{12} B(E)$ and $\text{gr}_{12} B(E)$ is Poisson commutative. Then by Proposition 2.9 we have $\text{gr}_{12} B(E) = \text{gr}_{21} B(E) = A_\theta$. \Box

To complete the proof of Theorem 5.1, note that from the last proposition and Proposition 4.11 it follows that $\text{gr}_2 B(E) = A_\theta$ and we are done. \Box

**Remark.** From Theorem 5.1 we get the construction of $A_\theta$ independent of center on the critical level of $\widehat{\mathfrak{g}}$.

**Corollary 5.4.** $B(E)$ is a maximal commutative subalgebra of $Y(\mathfrak{g})^\theta$.

### 5.5. Application to the Gaudin model

Let $z \in \mathbb{C}$. Let $ev_z : U(\mathfrak{g}[t]) \to U(\mathfrak{g})$ be an evaluation map. Let $z_i, i = 1, \ldots, n$ be different complex numbers. Let $d$ be the diagonal embedding

$$d : U(\mathfrak{g}[t]) \to U(\mathfrak{g}[t])^\otimes n.$$

We define a map

$$ev_{z_1, \ldots, z_n} := ev_{z_1} \otimes \ldots \otimes ev_{z_n} \circ d : U(\mathfrak{g}[t]) \to U(\mathfrak{g}) \otimes \ldots \otimes U(\mathfrak{g}).$$

**Definition 5.6.** Gaudin subalgebra $A(z_1, \ldots, z_n) \subset U(\mathfrak{g})^\otimes n$ is $ev_{z_1, \ldots, z_n}(A_\theta)$.

It is known (see [FFR]) that this subalgebra is commutative and gives the complete set of integrals for the quantum Gaudin magnet chain.

From Theorem 5.1 we have the following
Corollary 5.7. \( ev_{z_1,\ldots,z_n}(\text{gr}_2 B(E)) = A(z_1,\ldots,z_n) \).

Remark. Corollary 5.7 generalizes Talalaev’s formulas [T] for higher Gaudin Hamiltonians to \( \mathfrak{g} \) of arbitrary type modulo knowledge of the universal \( R \)-matrix \( \hat{R}(u) \) for the Yangian. Namely, suppose that we know the expression of the universal \( R \)-matrix of Yangian in PBW-generators with respect to the filtration \( F_2 \) (i.e., ordered monomials form a basis of \( Y(\mathfrak{g}) \) and their leading terms form a basis of \( U(\mathfrak{g}[t]) \), for example root generators in the “new” realization, see [L]). Then the leading terms of the generators of \( B(E) \), namely leading terms of \( \tau_i(u,E) = \text{tr}_V \rho_{\omega_i} \otimes \text{Id} \hat{R}(u), i = 1,\ldots, \text{rk} \mathfrak{g} \), are the generators of \( \text{gr}_2 B(E) \). In type A the standard RTT generators are PBW generators, and the entries of \( (\rho_{\omega_i} \otimes \text{Id} \hat{R}(u) \) are the quantum \( i \times i \) - minors, so we can write explicit formulas for the generators of \( \text{gr}_2 B(E) \) – and these are precisely Talalaev’s formulas.

5.8. Proof of Theorem A in the general case

Let \( C \in T \) and consider \( \text{gr}_2 B(C) \subset U(\mathfrak{g}[t]) \). As before \( \mathfrak{z}_G(C) \) is the infinitesimal centralizer of \( C \). In this subsection we are going to prove Theorem A in the full generality.

Theorem 5.9. \( \text{gr}_2 B(C) \) is the universal Gaudin subalgebra in \( U(\mathfrak{z}_G(C)[t])^{\mathfrak{z}_G(C)} \), i.e., \( \text{gr}_2 \hat{B}(C) = A_{\mathfrak{z}_G(C)} \).

Proposition 5.10. All generators of \( A_{\mathfrak{z}_G(C)} \) belong to \( \text{gr}_1 B(C) \).

Proof. The idea of the proof is as follows. Consider any linear combination of functions

\[
\sigma_i(C)(g) := \text{Tr}_{V_{\omega_i}} \rho_i(C) \rho_i(g), g \in \widetilde{G}
\]

and obtain from it the series of functions on \( \mathcal{O}(G_1[[t^{-1}]]) \). Then these functions by definition belong to \( \hat{B}(C) \subset \text{gr}_1 B(C) \). We will find all generators of \( A_{\mathfrak{z}_G(C)} \) using this construction.

Recall that \( \mathcal{O}(\widetilde{G})_+ \) is the set of polynomial functions on \( \widetilde{G} \) consisting of \( f \in \mathcal{O}(G) \) such that \( f(E) = 0 \). According to Lemma 2.21 it is sufficient to show that there are functions in \( \mathcal{O}(\widetilde{G})_+^{\mathcal{G}} \) such that the first non-zero term in the Taylor expansion at the point \( C^{-1} \) are \( \Phi_i, i = 1,\ldots, \text{rk} \mathfrak{g} \), where \( \Phi_i \) are free generators of \( S(\mathfrak{z}_G(C)[0])^{\mathfrak{z}_G(C)} \). Indeed, it is equivalent to find \( f \in \mathcal{O}(G) \) with \( \Phi_i \) being the first non-zero term in the Taylor expansion of \( f \) at the unity, as a linear combination of the functions \( \sigma_i(C) \).

We identify \( \mathfrak{g} \) with \( \mathfrak{g}^* \) by means of the Killing form, so we can identify the coordinate ring \( \mathcal{O}(\mathfrak{g}) \) with \( S(\mathfrak{g}) \). Consider the decomposition \( \mathfrak{g} = \mathfrak{z}_G(C) \oplus \mathfrak{n} \), where \( \mathfrak{n} \) is sum of eigenspaces corresponding to eigenvalues of \( \text{Ad}(C) \) not equal to 1. This decomposition is orthogonal with respect to the Killing form and thus \( \mathcal{O}(\mathfrak{z}_G(C)) \) gets identified with \( S(\mathfrak{z}_G(C)) \). We choose a formal coordinate system in the neighborhood of \( C^{-1} \in \widetilde{G} \) with the help of the map:

\[
\Psi : \mathfrak{z}_G(C) \oplus \mathfrak{n} \to G, (h,x) \mapsto \exp(-x)C^{-1}\exp(h)\exp(x).
\]

The differential of \( \Psi \) at \( C^{-1} \) is \( \text{Ad}C^{-1} - \text{Id} \) \( \oplus \text{Id} \) hence is non-degenerate.
Let $\tilde{S}(\mathfrak{g})$ be the completion of $S(\mathfrak{g})$ with respect to the maximal ideal of $0 \in \mathfrak{g}$. Consider the pullback $\Psi^* : \mathcal{O}(\tilde{G}) \to \tilde{S}(\mathfrak{g})$ which takes any function on $\tilde{G}$ to its Taylor series at $C^{-1}$ in our coordinates. Let $f \in \mathcal{O}(\tilde{G})^G$ be a central function. Since $f$ is constant on conjugacy classes its Taylor expansion $\Psi^* (f)$ does not depend on coordinates along $\mathfrak{n}$. So $\Psi^* (\mathcal{O}(\tilde{G})) \subset \tilde{S}(\mathfrak{g}_0(C))$. Let $Z_{\tilde{G}}(C)$ be a centralizer of $C$ in $\tilde{G}$. It is well known that the Lie algebra of $Z_{\tilde{G}}(C)$ is $\mathfrak{g}_0(C)$. The map $\Psi$ is $Z_{\tilde{G}}(C)$-equivariant with respect to the adjoint action on $\mathfrak{g}$ and the action by conjugation on $\tilde{G}$, so we have

$$\Psi^* : \mathcal{O}(\tilde{G})^G \to \tilde{S}(\mathfrak{g}_0(C))^{\mathfrak{g}_0(C)}.$$

Let $J = S(\mathfrak{g}_0(C))^{\mathfrak{g}_0(C)} = \{ f \in S(\mathfrak{g}_0(C))^{\mathfrak{g}_0(C)} \mid f(0) = 0 \}$. Let $e$ be the principal nilpotent of $\mathfrak{g}_0(C)$. According to Kostant [Ko], the differential at $e$ gives the isomorphism $d_e : J / J^2 \simeq \mathfrak{g}_0(C)(e)$. Consider the composite map

$$\Theta : \mathcal{O}(\tilde{G})^G_+ \to \tilde{S}(\mathfrak{g}_0(C))^{\mathfrak{g}_0(C)}_+ \to \tilde{S}(\mathfrak{g}_0(C))^{\mathfrak{g}_0(C)}_+ / \left( \tilde{S}(\mathfrak{g}_0(C))^{\mathfrak{g}_0(C)}_+ / \left( \tilde{S}(\mathfrak{g}_0(C))^{\mathfrak{g}_0(C)}_+ \right)^2 \right) \simeq J / J^2 \to \mathfrak{g}_0(C)(e).$$

Here the first arrow is $\Psi^*$, the second arrow is the projection and the last is taking the differential at $e$. The resulting map $\Theta$ is just taking the differential at $C^{-1} \exp(e)$. By Lemma 3.10 the map $\Theta$ is surjective.

Let $\{ e, h, f \}$ be the corresponding $\mathfrak{sl}_2$-triple in $\mathfrak{g}_0(C)$. One can split the centralizer $\mathfrak{g}_0(C)(e)$ into the eigenspaces of the operator $\frac{1}{2} \text{ad} h$:

$$\mathfrak{g}_0(C)(e) = \bigoplus_{i=1}^{\text{rk} \mathfrak{g}} V_i.$$

Note that $\text{dim} V_i$ is the number of algebraically independent generators of $S(\mathfrak{g}_0(C))^{\mathfrak{g}_0(C)}$ of degree $m_i + 1$, see [Ko]. For any $f \in \mathcal{O}(\tilde{G})^G_+$ whose Taylor series expansion starts from the $k$th term we have

$$\Theta(f) \in \bigoplus_{m_i \geq k-1} V_i.$$

By the surjectivity of $\Theta$ we have functions with Taylor series on $C^{-1} \exp(e)$ starting from $\Phi_i, i = 1, \ldots, \text{rk} \mathfrak{g}$ where $\Phi_i, i = 1, \ldots, \text{rk} \mathfrak{g}$ are free generators of $S(\mathfrak{g}_0(C))^{\mathfrak{g}_0(C)}$ and we are done. □

Proposition 5.10 implies that $A_{\mathfrak{g}_0(C)} \subset \mathfrak{g}_{21} B(C)$. To prove that in fact we have an equality we are going to prove that $A_{\mathfrak{g}_0(C)}$ is a maximal commutative subalgebra of $S(\mathfrak{g}[t])^{\mathfrak{g}_0(C)}$.

We mostly follow the argument of [R1] below. Let $\{ e, h, f \}$ be a principal $\mathfrak{sl}_2$-triple of $\mathfrak{g}$. Let us recall two classical facts.

**Proposition 5.11 ([Ko]).** Let $\pi$ be the restriction homomorphism

$$\pi : \mathbb{C}[\mathfrak{g}] \to \mathbb{C}[f + \mathfrak{g}_0(e)].$$

If we restrict $\pi$ to $\mathbb{C}[\mathfrak{g}]^\mathfrak{g}$ we obtain an isomorphism $\mathbb{C}[\mathfrak{g}]^\mathfrak{g} \simeq \mathbb{C}[f + \mathfrak{g}_0(e)]$.

The next proposition is well known.
Proposition 5.12. Let $\psi$ be the restriction homomorphism
\[ \psi : \mathbb{C}[g] \to \mathbb{C}[h]. \]
If we restrict $\psi$ to $\mathbb{C}[g]^0$ we obtain an isomorphism $\mathbb{C}[g]^0 \simeq \mathbb{C}[h]^W$, where $W$ is the Weyl group of $g$. Particularly, $\mathbb{C}[h]$ is an algebraic extension of $\mathbb{C}[g]^0$.

**Proposition 5.13.** $A_{3,g}(C)$ is a maximal commutative subalgebra of $S(g[t])_{3,g}(C)$ with respect to $\{\cdot, \cdot\}_0$. Moreover, $A_{3,g}(C)$ is centralizer of the element $\Omega_{3,g}(C)$.

**Proof.** Let $\psi : S(g[t]) \to S(h[t])$ be a $h$-invariant projection.

Let $g = A_{3,g}(C)(e) \oplus n$, where $n$ is any complement subspace. Consider $\pi : S(g[t]) \to S(A_{3,g}(C)(e)[t])$ such that $\pi(x[n]) = x[n], x \in A_{3,g}(C)(e), \pi(x[n]) = \delta_{0n} (x, f), x \in n.$

This map generalizes the map from Proposition 5.11.

**Lemma 5.14.**

1. $\pi(A_{3,g}(C)) \simeq S(A_{3,g}(C)(e)[t]);$
2. $A_{3,g}(C)$ is algebraically closed in $S(g[t]);$
3. $\psi(A_{3,g}(C)) \subset S(h[t])$ is an algebraic extension.

**Proof.** 1) Let $\Phi_i, i = 1, \ldots, \text{rk } g$ be algebraically independent generators of $S(A_{3,g}(C)[0])_{3,g}(C)$.

We have $\pi(\mathbb{C}[\Phi_1, \ldots, \Phi_k]) \simeq S(A_{3,g}(C)(e)[0])$ by Proposition 5.11. Moreover, $D$ commutes with $\pi$ hence we have $\pi(\mathbb{C}[D^s \Phi_1, \ldots, D^s \Phi_k]) \simeq S(A_{3,g}(C)(e)[s])$

and hence $\pi(A_{3,g}(C)) \simeq S(A_{3,g}(C)(e)[t]).$

2) Suppose that $A_{3,g}(C)$ is not algebraically closed. Let $a \in S(g[t])$ be an element which is algebraic over $A_{3,g}(C)$. Then by the first statement of this lemma we can assume that $\pi(a) = 0$. Suppose that $p_n a^n + \cdots + p_1 a + p_0 = 0$, where $p_i \in A_{3,g}(C)$ and $n$ is minimal. Then $\pi(p_0) = 0$, so we have $(p_n a^{n-1} + \cdots + p_1) a = 0$. But $S(g[t])$ does not have zero divisors and we have a contradiction with the minimality of $n$.

3) Note that $\psi(\mathbb{C}[\Phi_1, \ldots, \Phi_{\text{rk } g}]) \subset S(h[0])$ is the algebraic extension from Proposition 5.12. Using the fact that $D$ commutes with $\psi$ we see that $\psi(\mathbb{C}[\Phi_1, \ldots, \Phi_{\text{rk } g}, D\Phi_1, \ldots, D\Phi_{\text{rk } g}, \ldots, D^s \Phi_1, \ldots, D^s \Phi_{\text{rk } g}]) \subset S \left( \bigoplus_{i=0}^s h[i] \right)$ is the algebraic extension as well for any $s$. \(\square\)
From Lemma 4.8 it follows that the centralizer of $\omega_{3g}(C)$ in $S(\mathfrak{g}[t])_{0,1}^{3g}(C)$ contains the subalgebra $A_{3g}(C)$.

Now let us define the family of automorphisms of $S(\mathfrak{g}[t])$ with respect to the bracket $\{ \cdot, \cdot \}_{1}$. Let $\varphi_{s}(x[m]) = x[m] + s\delta_{0m} \langle h, x \rangle$. It is a straightforward computation that $\varphi_{s}$ is an automorphism. We use the notion of a limit subalgebra in the next lemma (see Subsection 6.2 for the definition). We also use the notion of a limit of a one-parametric family of subalgebras analogous to that of 6.4.

**Lemma 5.15.** We have $h[0] \in \lim_{s \to \infty} \varphi_{s}(A_{3g}(C))$.

*Proof.* Recall that an element $h$ is an element from principal $\mathfrak{sl}_{2}$-triple of $\mathfrak{g}$. It is a straightforward computation that

$$
\lim_{s \to \infty} \frac{\varphi_{s}(\omega_{3g}(C)) - s^{2}\langle h, h \rangle}{2s} = h[0].
$$

□

**Lemma 5.16 ([R1]).** The algebra $S(\mathfrak{h}[t])$ is the centralizer of $h[0]$ in $S(\mathfrak{g}[t])$ with respect to $\{ \cdot, \cdot \}_{1}$.

Now return to the proof of the proposition. The centralizer of $\Omega_{3g}(C)$ with respect to $\{ \cdot, \cdot \}_{0}$ coincides with the centralizer of $\omega_{3g}(C)$ with respect to $\{ \cdot, \cdot \}_{1}$. Suppose that we have some element $a \notin A_{3g}(C)$ in the centralizer of $\omega_{3g}(C)$ with respect to $\{ \cdot, \cdot \}_{1}$. From Lemma 5.14 it follows that $a$ should be transcendental over $A_{3g}(C)$ so we can assume that $\psi(a) = 0$. For some $k$ we have a non-zero limit $\tilde{a} = \lim_{s \to \infty} \varphi_{s}(a)/s^{k} \in \varphi_{s}(A_{3g}(C))$. This limit should lie in the centralizer of the element $h[0]$, and then lie in $S(\mathfrak{h}[t])$. It means that $\psi(a) \neq 0$ which is a contradiction and completes the proof. □

**Corollary 5.17.** We have $\text{gr}_{12}B(C) = A_{3g}(C)$ and $\text{gr}_{2}B(C) = A_{3g}(C)$.

*Proof.* Analogous to Lemma 5.2 we can find $\Omega_{3g}(C)$ in $\text{gr}_{12}B(C)$. Then from Proposition 5.13 and the fact that $\text{gr}_{12}B(C)$ is Poisson commutative it follows that $\text{gr}_{12}B(C) \subset \text{gr}_{21}B(C)$ and then by Proposition 2.9 we have $\text{gr}_{12}B(C) \subset \text{gr}_{21}B(C) = A_{3g}(C)$. From the uniqueness of lifting of $\Omega_{3g}(C)$ to $U(\mathfrak{g}[t])$ (up to scalar and additive constant) we see that $\text{gr}_{2}B(C) = A_{3g}(C)$. □

This finishes the proof of Theorem 5.9.

**Corollary 5.18.** $B(C)$ is a maximal commutative subalgebra of $Y(\mathfrak{g})_{0}^{3g}(C)$ and $\tilde{B}(C)$ is a maximal Poisson commutative subalgebra in $\mathcal{O}(G_{1}[[t^{-1}]])_{0}^{3g}(C)$.

*Proof.* This follows from Proposition 5.13 and Corollary 5.17, since if the associated graded of a commutative subalgebra is maximal Poisson-commutative then the original subalgebra is maximal commutative as well. □

**Corollary 5.19.** $\text{gr}_{1}B(C) = \tilde{B}(C)$ for any $C \in T$.

*Proof.* This is immediate from Corollary 5.18 since $\text{gr}_{1}B(C) \supset \tilde{B}(C)$ and $\tilde{B}(C)$ is maximal Poisson-commutative. □
6. Some limits of Bethe subalgebras

6.1. Closure of the family of subspaces in a vector space

Let \( \{U(m)\}_{m \in M} \) be a family of vector subspaces of the dimension \( k \) parameterized by a complex algebraic variety \( M \) (we allow \( U(m_1) = U(m_2) \) for different points \( m_1, m_2 \in M \)). Suppose that the resulting map \( \theta : M \to \text{Gr}(k, n) \) is regular. Consider the closure \( Z \) of \( \theta(M) \) in \( \text{Gr}(k, n) \). The variety \( Z \) naturally parameterizes a family of subspaces extending \( \{U(m)\}_{m \in M} \). According to the general principle (see, e.g., [Se]) the closure of an algebraic variety under a regular map with respect to Zariski topology coincides with its closure with respect to the analytic topology. We will use this fact in Subsection 6.4.

6.2. Closure of the family of subspaces in a graded space

Suppose that we have a graded vector space with finite-dimensional graded components \( V = \bigoplus_i V_i \) such that \( \dim V_i = n_i \) and a family of subspaces of the form \( U(m) = \bigoplus_i U_i(m) \) with fixed dimensions \( \dim U_i = k_i \) parametrized by a complex algebraic variety \( M, m \in M \). Suppose that all maps

\[
\theta_r : M \to \prod_{i=1}^r \text{Gr}(k_i, n_i), \quad m \mapsto (U_1(m), \ldots, U_r(m))
\]

are regular. Let \( Z_r \) be the closure of \( \theta_r(M) \). There are well-defined projections \( \zeta_r : Z_r \to Z_{r-1} \) for all \( r \geq 1 \). The inverse limit \( Z = \varprojlim Z_r \) is well defined as a pro-algebraic scheme and is naturally a parameter space for some family of graded vector subspaces of fixed dimension. Indeed, any point \( z \in Z \) is a sequence \( \{z_r\}_{r \in \mathbb{N}} \) where \( z_r \in Z_r \) and \( U(z) = \bigoplus_i U_i(z_i) \subset V \). We note also that the Poincaré series of \( U(z) \) coincides with the Poincaré series of \( U(m), m \in M \).

Suppose that we have a structure of an algebra on \( V \) and suppose that a family of graded subspaces is a family of subalgebras. Then any \( U(z), z \in Z \) is also a subalgebra because being a subalgebra is a Zariski-closed condition on the parameters. If all subalgebras of our family are commutative, then any \( U(z), z \in Z \) is commutative as well, by the same argument.

6.3. Closure of the family of subspaces in a filtered space

Suppose that we have a filtration on vector space \( V \) with finite-dimensional filtered components:

\[
V_0 \subset V_1 \subset V_2 \subset \cdots,
\]

where \( \dim V_i = n_i \). Suppose that we have a family of subspaces of the form

\[
U(m) = \bigcup_i U_i(m), U_i(m) = U(m) \cap V_i
\]

with fixed dimensions \( \dim U_i(m) = k_i \) parameterized by a complex algebraic variety \( M, m \in M \). Suppose that all maps \( \theta_r : M \to \prod_{i=1}^r \text{Gr}(k_i, n_i) \) are regular. As in the case of graded vector subspaces, we have the pro-algebraic scheme \( Z \). This scheme naturally parameterizes some family of subspaces in \( V \).
Indeed, every $z_r$ is a point in $\prod_{i=1}^r \text{Gr}(k_i, n_i)$ i.e., a collection of subspaces $U_i^{(r)}(z) \subset V_i$ such that $U_i^{(r)}(z) \subset U_{i+1}^{(r)}(z)$ for all $i < r$. Since $\zeta_r(z_r) = z_{r-1}$ we have $U_i^{(r)}(z) = U_{i-1}^{(r)}(z)$ for all $i < r$. We define the subspace corresponding to $z \in Z$ as $U(z) := \bigcup_{r=1}^\infty U_r^{(r)}(z)$.

In fact $Z$ parameterizes a bit richer data, namely, subspaces along with a filtration $\{U_i(z)\}$ such that $U_i(z) \subset U(z) \cap V_i$ (i.e., the inclusion is not necessarily equality since the dimension of $U(z) \cap V_i$ could jump). We define the Poincaré series of a subspace as the Poincaré series of the associated graded space. In contrast with the graded vector space the Poincaré series of $U(z)$ is not necessarily the same – but always not smaller lexicographically than that of $U(m), m \in M$.

Again, as in the case of the graded vector space if $V$ is an algebra and all $U(m), m \in M$ are (commutative) subalgebras, then any $U(z), z \in Z$ is a (commutative) subalgebra.

### 6.4. Definition of limit Bethe subalgebras

Let $C$ be an element of $G^{\text{reg}}$ or $T^{\text{reg}}$. Recall that the formula $\deg t_i^{(r)} = r$ defines the filtration $F_1$ on $Y_V(g)$. Recall that $F_1^{(r)} Y_V(g)$ is a $r$th filtered component. Consider $B^{(r)}(C) := F_1^{(r)} Y_V(g) \cap B(C)$. In the paper [IR2] (in the course of the proof of Theorem 2.6) it is proved that the images of the coefficients of $\tau_1(u, C), \ldots, \tau_n(u, C)$ freely generate the subalgebra $\text{gr} B(C) \subset \text{gr}_1 Y_V(g)$. Hence the dimension $d(r)$ of $B^{(r)}(C)$ does not depend on $C$. Now we can apply the results of Subsection 6.3 to this situation and obtain pro-algebraic schemes $Z_G$ and $Z_T$ which parameterize some families of commutative subalgebras.

**Proposition 6.5.** For any $z \in Z$ subalgebra $B(z)$ is a commutative subalgebra of $Y_V(g)$. The Poincaré series of $B(z)$ is (lexicographically) not smaller than the Poincaré series series of $B(C)$ for $C \in G^{\text{reg}}$. We call a subalgebra of the form $B(z), z \in Z$ a limit subalgebra.

Following [Sh] we want to explain a practical way to find some limit of one-parametric families of subalgebras and some elements belonging to them in the case of $Z_T$.

By definition put $k_i = \dim B^{(i)}(C), n_i = \dim F_1^{(i)} Y_V(g)$. Let $D$ be a small neighborhood of zero in $\mathbb{C}$. Suppose that $c : D \to T$ is an analytic map and that the restriction is a map $c' : D \setminus \{0\} \to T^{\text{reg}}$. Let $c'' = \theta_i \circ c' : D \to \text{Gr}(k_i, n_i)$.

**Lemma 6.6.** $c''_i$ (uniquely) extends to $D$ as an analytic map.

**Proof.** The map $c''_i$ is uniquely determined by the collection of Plücker coordinates of $c''_i(\varepsilon)$, which are Laurent series of $\varepsilon$ up to proportionality. So we can multiply all these coordinates by some power of $\varepsilon$ to make them Taylor series not all vanishing at the origin. This gives us the desired analytic map $D \to \text{Gr}(k_i, n_i)$. \[\Box\]

We define $\lim_{\varepsilon \to 0} B(c(\varepsilon)) := \bigcup_i c''_i(0)$. Now let us give a practical way to find some elements in $\lim_{\varepsilon \to 0} B(c(\varepsilon))$. Suppose that $d : D \to F_1^{(i)} Y_V(g)$ is an analytic map such that $d(\varepsilon) \in B(c(\varepsilon))$ for all $\varepsilon \neq 0$ and $d(\varepsilon) \neq 0$ for small enough $\varepsilon$. Let $n_i = \dim F_1^{(i)} Y_V(g)$. We obtain a map $d' : D \setminus \{0\} \to \mathbb{P}^{n_i-1}$. Analogous to the
previous lemma we know that there exists the unique continuation $d'' : D \rightarrow \mathbb{P}^{n_1-1}$ of $d'$. Practically, this means that we can divide $d$ by some power of $\varepsilon$ to make it nonzero at the origin. It is natural to call $d''(0)$ the limit of $d(\varepsilon)$ as $\varepsilon \to 0$, i.e., $d''(0) = \lim_{\varepsilon \to 0} d(\varepsilon)$.

**Lemma 6.7.** We have $\lim_{\varepsilon \to 0} d(\varepsilon) \subset \lim_{\varepsilon \to 0} B(c(\varepsilon))$.

**Proof.** Let $\text{diag} : D \setminus \{0\} \to D \setminus \{0\} \times D \setminus \{0\}$ be the diagonal embedding. The image of the map $c_i \times d' \circ \text{diag} : D \setminus \{0\} \to \text{Gr}(k_i, n_i) \times \mathbb{P}^{n_1-1}$ belongs to the closed subset given by the incidence condition. Hence the same is true for $c_i'' \times d'' : D \to \text{Gr}(k_i, n_i) \times \mathbb{P}^{n_1-1}$. So $d''(0) \in \lim_{\varepsilon \to 0} B(c(\varepsilon))$. \hfill $\Box$

6.8. Shift of argument subalgebras

Let $g$ be a reductive Lie algebra. We identify $g$ with $g^*$ by means of a non-degenerate invariant scalar product. To any $\chi \in g^* = g$ one can assign a Poisson-commutative subalgebra in $S(g)$ with respect to the standard Poisson bracket (coming from the universal enveloping algebra $U(g)$ by the PBW theorem). Let $ZS(g) = S(g)^0$ be the center of $S(g)$ with respect to the Poisson bracket. The algebra $A_\chi \subset S(g)$ generated by the elements $\partial^a \Phi$, where $\Phi \in ZS(g)$, (or, equivalently, generated by central elements of $S(g) = \mathbb{C}[g^*]$ shifted by $t \chi$ for all $t \in \mathbb{C}$) is Poisson-commutative and has maximal possible transcendence degree. More precisely, we have the following

**Theorem 6.9 ([MF]).** For regular semisimple $\chi \in g$ the algebra $A_\chi$ is a free commutative subalgebra in $S(g)$ with $\frac{1}{2} (\dim g + \text{rk } g)$ generators (this means that $A_\chi$ is a commutative subalgebra of maximal possible transcendence degree). One can take the elements $\partial^a \Phi_k$, $k = 1, \ldots, \text{rk } g$, $n = 0, 1, \ldots, \deg \Phi_k - 1$, where $\Phi_k$ are basic $g$-invariants in $S(g)$, as free generators of $A_\chi$.

**Theorem 6.10 ([R2]).** For any regular semisimple $\chi \in g$ there exists a lifting $A_\chi \subset U(g)$, i.e., a commutative subalgebra $A_\chi \subset U(g)$ such that $\text{gr } A_\chi = A_\chi$. Moreover, this lifting is unique for generic regular $\chi$.

**Theorem 6.11 ([R2]).** For generic (i.e., belonging to some complement of countably many proper closed subsets in $\mathfrak{h}^{\text{reg}}$) regular semisimple $\chi \in g$, the subalgebra $A_\chi$ is the centralizer of its quadratic part which is the linear span of the elements $\sum_{a \in \Phi} (\alpha, h) e_a e_{-\alpha}$ for all $h \in \mathfrak{h}$.

6.12. Certain limits of Bethe subalgebras

Let $E \in G$ be the identity element.

**Theorem 6.13.** Let $C(\varepsilon) = \exp(\varepsilon \chi), \chi \in \mathfrak{h}$ and $C(\varepsilon) \in T^{\text{reg}}$ if $\varepsilon \neq 0$. Then

$$\lim_{\varepsilon \to 0} B(C(\varepsilon)) = B(E) \otimes \text{Z}(U(g)) A_\chi$$

for generic $\chi \in \mathfrak{h}^{\text{reg}}$.

**Proof.** According to [I] the quadratic part of a Bethe subalgebra contains the following elements:
\[ \sigma_i(C) = 2J(t_{\omega_i}) - \sum_{\alpha \in \Phi^+} \frac{e^{\alpha}(C) + 1}{e^{\alpha}(C) - 1}(\alpha, \alpha_i)x_{\alpha}x_{\alpha}^- \in Y(\mathfrak{g}), \quad i = 1, \ldots, \text{rk} \mathfrak{g}. \]

Here \( J(t_{\omega_i}) \) is an element of \( Y(\mathfrak{g}) \) which does not depend on \( C \). In the limit \( \varepsilon \to 0 \), the leading term has the form

\[ \sum_{\alpha \in \Phi^+} \frac{2(\alpha, \alpha_i)}{(\alpha, \chi)}x_{\alpha}x_{\alpha}^-, \]

i.e., the quadratic part of the shift of the argument subalgebra \( A_\chi \) (see [V]). As we stated above, for a generic \( \chi \), the shift of argument subalgebra \( A_\chi \) is the centralizer of its quadratic part.

**Lemma 6.14.**

1. Suppose that \( \mathfrak{g} \) is a reductive Lie algebra, \( \mathfrak{g}_0 \) a reductive subalgebra of \( \mathfrak{g} \). Then the subalgebras \( Y(\mathfrak{g})^{\mathfrak{g}_0} \) and \( U(\mathfrak{g}_0) \) in \( Y(\mathfrak{g}) \) are both free \( U(\mathfrak{g}_0)^{\mathfrak{g}_0} \)-modules. Moreover, the product of these subalgebras in \( Y(\mathfrak{g}) \) is: \( Y(\mathfrak{g})^{\mathfrak{g}_0} \cdot U(\mathfrak{g}_0) \simeq Y(\mathfrak{g})^{\mathfrak{g}_0} \otimes_{U(\mathfrak{g}_0)^{\mathfrak{g}_0}} U(\mathfrak{g}_0); \)
2. \( \mathfrak{z}_Y(\mathfrak{g})(ZU(\mathfrak{g}_0)) = Y(\mathfrak{g})^{\mathfrak{g}_0} \otimes_{U(\mathfrak{g}_0)^{\mathfrak{g}_0}} U(\mathfrak{g}_0) \).

**Proof.** 1) Let us consider the associated bigraded algebra with respect to the filtrations \( F_1, F_2 \). Then \( \text{gr}_{21}Y(\mathfrak{g})^{\mathfrak{g}_0} = S(\mathfrak{g}[t])^{\mathfrak{g}_0} \), \( \text{gr}_{21}U(\mathfrak{g}_0)^{\mathfrak{g}_0} = S(\mathfrak{g}_0)^{\mathfrak{g}_0} \), \( \text{gr}_{21}U(\mathfrak{g}_0) = S(\mathfrak{g}_0) \).

Let \( f_n(t) \) be a polynomial of degree \( n \) with \( n \) pairwise different roots. Consider the quotient \( S_n(\mathfrak{g}[t]) := S(\mathfrak{g}[t])/f_n(t) \). From [IR, Lem. 4.8] it follows that

\[ S_n(\mathfrak{g}[t])^{\mathfrak{g}_0} \cdot S(\mathfrak{g}_0)^{\mathfrak{g}_0} \simeq S_n(\mathfrak{g}[t])^{\mathfrak{g}_0} \otimes_{S(\mathfrak{g}_0)^{\mathfrak{g}_0}} S(\mathfrak{g}_0). \]

We see that the statement of the lemma holds for any filtered component and hence for the whole algebra.

2) It follows from [K, Main Thm. (d)] if we consider the associated graded with respect to filtration \( F_2 \) and follow the proof of the first statement. \( \square \)

In our limit we have \( B(E) \subset Y(\mathfrak{g})^{\mathfrak{g}} \) and it is a maximal subalgebra of \( Y(\mathfrak{g})^{\mathfrak{g}} \) (Lemma 5.4). Moreover, \( Z(U(\mathfrak{g})) \) lie in \( B(E) \). From Lemma 6.14 we see that the limit subalgebra lies in \( B(E) \cdot U(\mathfrak{g}) \simeq B(E) \otimes_{Z(U(\mathfrak{g}))} U(\mathfrak{g}) \). But then it should lie in the centralizer of the quadratic part of the Bethe subalgebra in the latter tensor product, i.e., in \( B(E) \cdot A_\chi \simeq B(E) \otimes_{Z(U(\mathfrak{g}))} A_\chi \).

Subalgebra \( B(E) \otimes_{Z(U(\mathfrak{g}))} A_\chi \) has the same Poincaré series as \( B(C), C \in T^{\text{reg}} \). Then the limit coincides with \( B(E) \otimes_{Z(U(\mathfrak{g}))} A_\chi \). \( \square \)

**Theorem 6.15.** Let \( C(\varepsilon) = C_0 \exp(\varepsilon \chi), C_0 \in T \setminus T^{\text{reg}} \) with \( \chi \in \mathfrak{h} \subset \mathfrak{z}_\mathfrak{g}(C_0) \) being a generic regular semisimple element of the centralizer of \( C_0 \). Then

\[ \lim_{\varepsilon \to 0} B(C(\varepsilon)) = B(C_0) \otimes_{Z(U(\mathfrak{z}_\mathfrak{g}(C_0)))} A_\chi, \]

where \( A_\chi \subset U(\mathfrak{z}_\mathfrak{g}(C_0)) \) is the quantum shift of the argument subalgebra corresponding to \( \chi \).
Proof. The proof is the same as the proof of the previous theorem, with the only difference that we use Corollary 5.18 instead of Corollary 5.4. □

Remark. One can solve Vinberg’s problem of lifting the shift of argument subalgebras for generic $\chi \in \mathfrak{h}$ to the universal enveloping algebra by defining the lifting of $A_\chi \subset \mathcal{S}(\mathfrak{g})$ to the universal enveloping algebra as $A_\chi := U(\mathfrak{g}) \cap \lim_{\varepsilon \to 0} B(C(\varepsilon))$ for $C(\varepsilon) = \exp(\varepsilon \chi)$.

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