PROBABILISTIC LOCAL WELL-POSEDNESS OF THE CUBIC NONLINEAR WAVE EQUATION IN NEGATIVE SOBOLEV SPACES

TADAHIRO OH, OANA POCOVNICU, AND NIKOLAY TZVETKOV

Dedicated to the memory of Professor Ioan I. Vrabie (1951–2017)

Abstract. We study the three-dimensional cubic nonlinear wave equation (NLW) with random initial data below $L^2(\mathbb{T}^3)$. By considering the second order expansion in terms of the random linear solution, we prove almost sure local well-posedness of the renormalized NLW in negative Sobolev spaces. We also prove a new instability result for the defocusing cubic NLW without renormalization in negative Sobolev spaces, which is in the spirit of the so-called triviality in the study of stochastic partial differential equations. More precisely, by studying (un-renormalized) NLW with given smooth deterministic initial data plus a certain truncated random initial data, we show that, as the truncation is removed, the solutions converge to 0 in the distributional sense for any deterministic initial data.

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1. Introduction

1.1. Main result. We consider the Cauchy problem for the defocusing cubic nonlinear wave equation (NLW) on the three-dimensional torus \( \mathbb{T}^3 = (\mathbb{R}/2\pi \mathbb{Z})^3 \):

\[
\begin{aligned}
\partial_t^2 u - \Delta u + u^3 &= 0 \\
(u, \partial_t u)|_{t=0} &= (u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3),
\end{aligned}
\tag{1.1}
\]

where \( u : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R} \) and \( \mathcal{H}^s(\mathbb{T}^3) = H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3) \). Here, \( H^s(\mathbb{T}^3) \) denotes the standard Sobolev space on \( \mathbb{T}^3 \) endowed with the norm:

\[
\|f\|_{H^s(\mathbb{T}^3)} = \|\langle n \rangle^s \hat{f}(n)\|_{L^2(\mathbb{Z}^3)},
\]

where \( \hat{u}(n) \) is the Fourier coefficient of \( u \) and \( \langle \cdot \rangle = (1+|\cdot|^2)^{\frac{1}{2}} \). The classical well-posedness result (see for example [43]) for (1.1) reads as follows.

**Theorem 1.** Let \( s \geq 1 \). Then, for every \( (u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3) \), there exists a unique global-in-time solution \( u \) to (1.1) in \( C(\mathbb{R}; \mathcal{H}^s(\mathbb{T}^3)) \). Moreover, the dependence of the solution map: \( (u_0, u_1) \mapsto u(t) \) on initial data and time \( t \in \mathbb{R} \) is continuous.

The proof of Theorem 1 follows from Sobolev’s inequality: \( H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3) \) and the conservation of the energy for (1.1). Recall that the scaling symmetry: \( u(t, x) \mapsto \lambda u(\lambda t, \lambda x) \) for (1.1) posed on \( \mathbb{R}^3 \) induces the scaling-critical Sobolev regularity \( s_{\text{crit}} = \frac{1}{2} \). By using the Strichartz estimates (see Lemma 2.4 below), one may indeed show that the Cauchy problem (1.1) remains locally well-posed in \( \mathcal{H}^s(\mathbb{T}^3) \) for \( s \geq \frac{1}{2} \) [28]. On the other hand, it is known that the Cauchy problem (1.1) is ill-posed for \( s < \frac{1}{2} \) [13, 10, 44, 35]. We refer to [43, 35] for the proofs of these facts.

One may then ask whether a sort of well-posedness of (1.1) survives below the scaling-critical regularity, i.e. for \( s < \frac{1}{2} \). As it was shown in the work [10, 11] by Burq and the third author, the answer to this question is positive if one considers random initial data. In this paper, we will primarily consider the following random initial data:

\[
\begin{aligned}
u_0^\omega &= \sum_{n \in \mathbb{Z}^3} g_n(\omega) e^{i n \cdot x} \\
u_1^\omega &= \sum_{n \in \mathbb{Z}^3} h_n(\omega) e^{i n \cdot x},
\end{aligned}
\tag{1.2}
\]

where the series \( \{g_n\}_{n \in \mathbb{Z}^3} \) and \( \{h_n\}_{n \in \mathbb{Z}^3} \) are two families of independent standard complex-valued Gaussian random variables on a probability space \( (\Omega, \mathcal{F}, P) \) conditioned that \( g_n = \overline{g_{-n}}, h_n = \overline{h_{-n}}, n \in \mathbb{Z}^3 \). More precisely, with the notation \( \mathbb{N} = \{1, 2, 3, \cdots \} \), we first define the index set \( \Lambda \) by

\[
\Lambda = (\mathbb{Z}^2 \times \mathbb{N}) \cup (\mathbb{Z} \times \mathbb{N} \times \{0\}) \cup (\mathbb{N} \times \{(0, 0)\}) \cup \{(0, 0, 0)\}. \tag{1.3}
\]

\(^1\)In particular, \( g_0 \) and \( h_0 \) are real-valued.
We then define \( \{g_n, h_n\}_{n \in \Lambda} \) to be a family of independent standard Gaussian random variables which are complex-valued for \( n \neq 0 \) and are real-valued for \( n = 0 \). We finally set \( g_n = g_{-n}, h_n = h_{-n} \) for \( n \in \mathbb{Z}^3 \setminus \Lambda \).

The partial sums for the series \((u_0^\omega, u_1^\omega)\) in (1.2) form a Cauchy sequence in \( L^2(\Omega; \mathcal{H}^s(\mathbb{T}^3)) \) for every \( s < \alpha - \frac{3}{2} \) and therefore the random initial data \((u_0^\omega, u_1^\omega)\) in (1.2) belongs almost surely to \( \mathcal{H}^s(\mathbb{T}^3) \) for the same range of \( s \). On the other hand, one may show that the probability of the event \((u_0^\omega, u_1^\omega) \in \mathcal{H}^{\alpha - \frac{3}{2}}(\mathbb{T}^3)\) is zero. See Lemma B.1 in [10]. As a result, when \( \alpha > \frac{5}{2} \), one may apply the classical global well-posedness result in Theorem 1 for the random initial data \((u_0^\omega, u_1^\omega)\) given by (1.2) since \((u_0^\omega, u_1^\omega) \in \mathcal{H}^1(\mathbb{T}^3)\) almost surely. For \( \alpha > 2 \), one may still apply the more refined (deterministic) local well-posedness result in Theorem 2 for the random initial data \((u_0^\omega, u_1^\omega)\) in (1.2) since \((u_0^\omega, u_1^\omega) \in \mathcal{H}^1(\mathbb{T}^3)\) almost surely. For \( \alpha \leq 2 \), however, the Cauchy problem (1.1) becomes ill-posed. Despite this ill-posedness result, the analysis in [11, 43] implies the following statement.

**Theorem 2.** Let \( \alpha > \frac{3}{2} \) and \( s < \alpha - \frac{3}{2} \). Let \( \{u_N\}_{N \in \mathbb{N}} \) be a sequence of the smooth global solutions\(^2\) to (1.1) with the following random \( C^\infty \)-initial data:

\[
\begin{align*}
  u_{0,N}^\omega(x) & = \sum_{|n| \leq N} \frac{g_n(\omega)}{(n)^\alpha} e^{inx} \\
  u_{1,N}^\omega(x) & = \sum_{|n| \leq N} \frac{h_n(\omega)}{(n)^{\alpha-1}} e^{inx},
\end{align*}
\]

where \( \{g_n\}_{n \in \mathbb{Z}^3} \) and \( \{h_n\}_{n \in \mathbb{Z}^3} \) are as in (1.2). Then, as \( N \to \infty \), \( u_N \) converges almost surely to a (unique) limit \( u \) in \( C(\mathbb{R}; \mathcal{H}^s(\mathbb{T}^3)) \), satisfying NLW (1.1) in a distributional sense.

Here, by uniqueness, we firstly mean that the entire sequence \( \{u_N\}_{N \in \mathbb{N}} \) converges to \( u \), not up to some subsequence. Compare this with the case of weak solution techniques, which usually only gives convergence up to subsequences. Furthermore, when we smooth the random initial data \((u_0^\omega, u_1^\omega)\) in (1.2) by mollification, it can be shown that the limit \( u \) is independent of the choice of mollification kernels. See Remark 1.1. Lastly, as we see in Subsection 1.3, the limit \( u \) admits a decomposition \( u = z_1 + v \), where \( v \) is the unique solution to the perturbed NLW:

\[
\begin{align*}
  L^* & = \left( v + z_1 \right)^3 = 0 \\
  \partial_t (v, \partial_t v) & = (0,0).
\end{align*}
\]

Similar comments apply to the limiting distribution \( u \) in Theorem 3 below.

For \( \alpha \leq \frac{3}{2} \), \( u_0^\omega \) in (1.2) is almost surely no longer a classical function and it should be interpreted as a random Schwartz distribution lying in a Sobolev space of negative index. Therefore for \( \alpha \leq \frac{3}{2} \), the study of (1.1) with the random initial data (1.2) is no longer within the scope of applicability of [11, 43]. The goal of this paper is to extend the results in [11, 43] to the random initial data when they are no longer classical functions. More precisely, we prove the following statement.

**Theorem 3.** Let \( \frac{5}{4} < \alpha \leq \frac{3}{2} \) and \( s < \alpha - \frac{3}{2} \). There exists a divergent sequence \( \{\alpha_N\}_{N \in \mathbb{N}} \) of positive numbers such that the following holds true; there exist small \( T_0 > 0 \) and positive constants \( C, c, \kappa \) such that for every \( T \in (0, T_0] \), there exists a set \( \Omega_T \) of complemental

\footnote{Theorem 1 guarantees existence of smooth global solutions \( \{u_N\}_{N \in \mathbb{N}} \) to (1.1).}
probability smaller than $C \exp(-c/T^*)$ such that if we denote by $\{u_N\}_{N \in \mathbb{N}}$ the smooth global solutions to

$$
\begin{cases}
\partial_t^2 u_N - \Delta u_N + u_N^3 - \alpha_N u_N = 0 \\
(u_N, \partial_t u_N)|_{t=0} = (u_{0,N}^\omega, u_{1,N}^\omega),
\end{cases}
$$

where the random initial data $(u_{0,N}^\omega, u_{1,N}^\omega)$ is given by the truncated Fourier series in (1.4), then for every $\omega \in \Omega_T$, the sequence $\{u_N\}_{N \in \mathbb{N}}$ converges to some (unique) limiting distribution $u$ in $C([-T, T]; H^s(\mathbb{T}^3))$ as $N \to \infty$.

As for the uniqueness statement, see Remark 1.6. See also Remark 1.1 below.

In view of the asymptotic behavior $\alpha_N \to \infty$, one may be tempted to say that the limiting distribution $u$ obtained in Theorem 3 is a solution to the following limit “equation”:

$$
\begin{cases}
\partial_t^2 u - \Delta u + u^3 - \infty \cdot u = 0 \\
(u, \partial_t u)|_{t=0} = (u_0^\omega, u_1^\omega),
\end{cases}
$$

where the random initial data $(u_0^\omega, u_1^\omega)$ is as in (1.2). The expression $\infty \cdot u$ is merely formal and thus a natural question is to understand in which sense $u$ satisfies the cubic NLW on $\mathbb{T}^3$. We will discuss this in the next two subsections. We also refer readers to [37] for a related discussion in the two-dimensional case.

Given fixed $N \in \mathbb{N}$, by adapting the classical argument, it is easy to see that the truncated equation (1.3) is globally well-posed in $H^s(\mathbb{T}^3)$ for $s \geq 1$. In particular, one needs to apply a Gronwall-type argument to exclude a possible finite-time blowup of the $H^1$-norm of a solution. The main issue here is that there is no good uniform (in $N$) bound for the solutions to (1.3). One may try to extend the local-in-time solutions constructed in Theorem 3 globally in time by using truncated energies in the spirit of the $I$-method, introduced in [14]. See [22] for such a globalization argument in the context of the two-dimensional stochastic NLW.

Our ultimate goal is to push the analysis in the proof of Theorem 3 to cover the case $\alpha = 1$, corresponding to the regularity of the natural Gibbs measure associated with the cubic NLW. In the field of singular stochastic parabolic PDEs, there has been a significant progress in recent years. In particular, a substantial effort [24, 19, 12, 25, 32, 2] was made to give a proper meaning to the stochastic quantization equation (SQE) on $\mathbb{T}^3$, formally written as

$$
\partial_t u - \Delta u = -u^3 + \infty \cdot u + \xi.
$$

(1.6)

Here, $\xi$ denotes the so-called space-time white noise. On the one hand, the randomization effects in the present paper are close in spirit to the works cited above. On the other hand, the deterministic part of the analysis in the context of the heat and the wave equations represent significant differences because, as it is well known, the deterministic regularity theories for these two types of equations are quite different. In fact, in order to extend Theorem 3 to lower values of $\alpha$, it is crucial to exploit dispersion at a multilinear level, a consideration specific to dispersive equations, and combine it with randomization effects. See, for example, a recent work [21] by Gubinelli, Koch, and the first author on the three-dimensional stochastic NLW with a quadratic nonlinearity. While such a consideration would allow us to improve Theorem 3, the $\alpha = 1$ case seems to be out of reach at this point. We plan to address these issues in a forthcoming work [36].
Remark 1.1. We say that $\eta \in C(\mathbb{R}^3; [0, 1])$ is a mollification kernel if $\int \eta dx = 1$ and $\text{supp} \eta \subset (-\pi, \pi]^3 \simeq \mathbb{T}^3$. Given a mollification kernel $\eta$, define $\eta_\varepsilon$ by setting $\eta_\varepsilon(x) = \varepsilon^{-3} \eta(\varepsilon^{-1} x)$. Then, $\{\eta_\varepsilon\}_{0<\varepsilon \leq 1}$ forms an approximate identity on $\mathbb{T}^3$. By slightly modifying the proof of Theorem 2, we can show that if we denote by $u_\varepsilon$, the solution to (1.1) with the initial data $(\eta_\varepsilon * u_0^\omega, \eta_\varepsilon * u_1^\omega)$, where $(u_0^\omega, u_1^\omega)$ is as in (1.2), then, for $\alpha > \frac{3}{2}$ and $s < \alpha - \frac{3}{2}$, $u_\varepsilon$ converges in probability to some (unique) limit $u$ in $C(\mathbb{R}; H^s(\mathbb{T}^3))$ as $\varepsilon \to 0$. Here, the limit $u$ is independent of the choice of mollification kernels $\eta$. Similarly, when $\frac{5}{4} < \alpha \leq \frac{3}{2}$, a slight modification of the proof of Theorem 3 shows that there exists a divergent sequence $\alpha_\varepsilon$ (as $\varepsilon \to 0$) such that the solution $u_{\varepsilon}$ to

$$
\begin{align*}
\partial_t^2 u_{\varepsilon} - \Delta u_{\varepsilon} + u_{\varepsilon}^3 - \alpha_\varepsilon u_{\varepsilon} &= 0 \\
(u_{\varepsilon}, \partial_t u_{\varepsilon})_{|_{t=0}} &= (\eta_{\varepsilon} * u_0^\omega, \eta_{\varepsilon} * u_1^\omega)
\end{align*}
$$

converges in probability to some (unique) limit $u$ in $C([-T_\omega, T_\omega]; H^s(\mathbb{T}^3))$, where $T_\omega > 0$ almost surely. Once again, the limit $u_{\varepsilon}$ is independent of the choice of mollification kernels $\eta$.

Remark 1.2. As in [10] [11], it is possible to consider a more general class of random initial data. Let a deterministic pair $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$ be given by the following Fourier series:

$$
\begin{align*}
u_0 &= \sum_{n \in \mathbb{Z}^3} a_n e^{in \cdot x} \quad \text{and} \quad u_1 = \sum_{n \in \mathbb{Z}^3} b_n e^{in \cdot x}
\end{align*}
$$

with the constraint $a_{-n} = \overline{a_n}$ and $b_{-n} = \overline{b_n}$, $n \in \mathbb{Z}^3$. We consider the randomized initial data $(u_0^\omega, u_1^\omega)$ given by

$$
\begin{align*}
u_0^\omega &= \sum_{n \in \mathbb{Z}^3} g_n(\omega) a_n e^{in \cdot x} \quad \text{and} \quad u_1^\omega = \sum_{n \in \mathbb{Z}^3} h_n(\omega) b_n e^{in \cdot x},
\end{align*}
$$

Then, by slightly modifying the proof of Theorem 3, it is easy to see that, for $s > -\frac{1}{6}$ (corresponding to $\alpha > \frac{4}{3}$ in (1.2)), we can introduce a time dependent divergent sequence $\{a_N\}_{N \in \mathbb{N}}$ with $a_N = a_N(t)$ such that the solution $u_N$ to (1.3) converges to some (unique) limit $u$ in $C([-T_\omega, T_\omega]; H^s(\mathbb{T}^3))$, where $T_\omega > 0$ almost surely. For this range of $s$, we need only the first order expansion. See the next subsection. For lower values of $s$, one may need to impose some additional summability assumptions on $\{a_n\}_{n \in \mathbb{Z}^3}$ and $\{b_n\}_{n \in \mathbb{Z}^3}$ (in particular to replicate the proof of Proposition 4.2) to obtain an analogue of Theorem 3.

1.2. Outline of the proof of Theorem 3. In the following, we present the main idea of the proof of Theorem 3. Fix $\alpha \leq \frac{3}{2}$. With the short-hand notation $\ell$,

$$
\mathcal{L} := \partial_t^2 - \Delta + 1,
$$

we denote by $z_{1,N} = z_{1,N}(t, x, \omega)$ the solution to the following linear Klein-Gordon equation:

$$
\mathcal{L} z_{1,N}(t, x, \omega) = 0
$$

with the random initial data $(u_{0,N}^\omega, u_{1,N}^\omega)$ given by the truncated Fourier series in (1.4). In the following, we discuss spatial regularities of various stochastic terms for fixed $t \in \mathbb{R}$. For simplicity of notation, we suppress the $t$-dependence and discuss spatial regularities. It is

\footnote{For our subsequent analysis, it will be more convenient to study the linear Klein-Gordon equation rather than the linear wave equation.}
easy to see from (1.4) that \( z_{1,N} \) converges almost surely to some limit \( z_1 \) in \( H^{s_1}(\mathbb{T}^3) \) as \( N \to \infty \), provided that
\[
s_1 < \alpha - \frac{3}{2}.
\] (1.9)
In particular, when \( \alpha \leq \frac{3}{2} \), \( z_{1,N} \) has negative Sobolev regularity (in the limiting sense) and thus \( (z_{1,N})^2 \) and \( (z_{1,N})^3 \) do not have well defined limits (in any topology) as \( N \to \infty \) since it involves products of two distributions of negative regularities.

Let \( u_N \) be the solution to the renormalized NLW (1.5) with the same truncated random initial data \( (u_{0,N}, u_{1,N}) \) in (1.4). By writing \( u = u_N + v_N \), (1.10)
\[
L v_N + v_N^3 + 3 z_{1,N} v_N^2 + 3 \left( (z_{1,N})^2 - \sigma_N \right) v_N + \left( (z_{1,N})^3 - 3 \sigma_N z_{1,N} \right) = 0
\]
where the parameter \( \sigma_N \) is defined by
\[
\sigma_N := \frac{\alpha_N + 1}{3}.
\]
As it is well known, the key point in the equation (1.11) is that the terms \( Z_{2,N} := (z_{1,N})^2 - \sigma_N \) and \( Z_{3,N} := (z_{1,N})^3 - 3 \sigma_N z_{1,N} \) are “renormalizations” of \( (z_{1,N})^2 \) and \( (z_{1,N})^3 \). Here, by “renormalizations”, we mean that by choosing a suitable renormalization constant \( \sigma_N \), the terms \( Z_{2,N} \) and \( Z_{3,N} \) converge almost surely in suitable negative Sobolev spaces as \( N \to \infty \).

The regularity \( s_1 < \alpha - \frac{3}{2} \) of \( z_{1,N} \) (in the limit) and a simple paraproduct computation show that if the expressions \( Z_{2,N} = (z_{1,N})^2 - \sigma_N \) and \( Z_{3,N} = (z_{1,N})^3 - 3 \sigma_N z_{1,N} \) have any well defined limits as \( N \to \infty \), then their regularities in the limit are expected to be
\[
s_2 < 2 \left( \alpha - \frac{3}{2} \right) \quad \text{and} \quad s_3 < 3 \left( \alpha - \frac{3}{2} \right),
\] (1.13)
respectively. In fact, by choosing the renormalization constant \( \sigma_N \) as
\[
\sigma_N := \mathbb{E} \left[ (z_{1,N}(t,x,\omega))^2 \right],
\] (1.14)
we show that \( Z_{j,N} \) converges in \( H^{s_j}(\mathbb{T}^3) \) almost surely. See Proposition 3.2. Note that the renormalization constant \( \sigma_N \) a priori depends on \( t, x \) but it turns out to be independent of \( t \) and \( x \). We will also see that, for \( N \gg 1 \), \( \sigma_N \) behaves like (i) \( \sim N^{3-2\alpha} \) when \( \alpha < \frac{3}{2} \) and (ii) \( \sim \log N \) when \( \alpha = \frac{3}{2} \). See (3.2) below.

Thanks to the Strichartz estimates (see Lemma 2.1 below), the deterministic Cauchy problem for
\[
L v + v^3 = 0
\]
is locally well-posed in $\mathcal{H}^s(T^3)$ for $s \geq \frac{1}{2}$. We may therefore hope to solve the equation (1.11) uniformly in $N \in \mathbb{N}$ by the method of [8, 10, 11], if we can ensure that the solution $v_N$ to the following linear problem:

$$\mathcal{L}v_N + \{(z_1, N)^3 - 3\sigma_N z_1, N\} = 0$$

(1.15)

with the zero initial data $(v_N, \partial_t v_N)_{|t=0} = (0, 0)$ remains bounded in $H^{\frac{3}{2}}(T^3)$ as $N \to \infty$.

Using one degree of smoothing under the wave Duhamel operator (see (2.6) below), we see that the solution to (1.15) is almost surely bounded in $H^{\frac{3}{2}}(T^3)$ uniformly in $N \in \mathbb{N}$, provided

$$3 \left( \alpha - \frac{3}{2} \right) + 1 > \frac{1}{2} \quad \implies \quad \alpha > \frac{4}{3}.$$  

Therefore, $\alpha = \frac{4}{3}$ seems to be the limit of the approach of [8, 10, 11].

In order to go below the $\alpha = \frac{4}{3}$ threshold, a new argument is needed. The introduction of such an argument is the main idea of this paper. More precisely, we further decompose $v_N$ in (1.10) as

$$v_N = z_{2, N} + w_N$$

(1.16)

for some residual term $w_N$, where $z_{2, N}$ is the solution to the following equation:

$$\begin{cases}
\mathcal{L}z_{2, N} + \{(z_1, N)^3 - 3\sigma_N z_1, N\} = 0 \\
(z_{2, N}, \partial_t z_{2, N})_{|t=0} = (0, 0).
\end{cases}$$

(1.17)

Thanks to the one degree of smoothing, we see that $z_{2, N}$ converges to some limit in $H^s(T^3)$, provided that

$$s = s_3 + 1 < 3 \left( \alpha - \frac{3}{2} \right) + 1$$

In terms of the original solution $u_N$ to (1.5), we have from (1.10) and (1.16) that

$$u_N = z_{1, N} + z_{2, N} + w_N.$$  

(1.18)

Note that $z_{1, N} + z_{2, N}$ corresponds to the Picard second iterate for the truncated renormalized equation (1.5).

The equation for $w_N$ can now be written as

$$\begin{cases}
\mathcal{L}w_N + (w_N + z_{2, N})^3 + 3z_{1, N}(w_N + z_{2, N})^2 + 3\{(z_1, N)^2 - \sigma_N\}(w_N + z_{2, N}) = 0 \\
(w_N, \partial_t w_N)_{|t=0} = (0, 0).
\end{cases}$$

(1.19)

By using the second order expansion (1.18), we have eliminated the most singular term $Z_{3, N} = (z_{1, N})^3 - 3\sigma_N z_{1, N}$ in (1.11). In the equation (1.19), there are several source terms\(^5\) and they are precisely the quintic, septic, and nonic (i.e. degree nine) terms added in considering the Picard third iterate for (1.5). As we see below, the most singular term in (1.19) is the following quintic term:

$$Z_{5, N} := 3\{(z_1, N)^2 - \sigma_N\}z_{2, N},$$

(1.20)

where $z_{2, N}$ is the solution to (1.17). As we already mentioned, the term $Z_{2, N} = (z_{1, N})^2 - \sigma_N$ and the second order term $z_{2, N}$ pass to the limits in $H^s(T^3)$ for $s < 2(\alpha - \frac{3}{2})$ and $s < \frac{3}{2}$.

\(^5\)Namely, purely stochastic terms independent of the unknown $w_N$. 

3(\(\alpha - \frac{3}{2}\)) + 1, respectively. In order to make sense of the product of \(Z_{2,N}\) and \(z_{2,N}\) in (1.20) by deterministic paradifferential calculus (see Lemma 2.1 below), we need the sum of the two regularities to be positive, namely

\[
2\left(\alpha - \frac{3}{2}\right) + 3\left(\alpha - \frac{3}{2}\right) + 1 > 0 \implies \alpha > \frac{13}{10}.
\]

Otherwise, i.e. for \(\alpha \leq \frac{13}{10}\), we will need to make sense of the product (1.20), using stochastic analysis. See Proposition 4.2. In either case, when the second factor in (1.20) has positive regularity \(3(\alpha - \frac{3}{2}) + 1 > 0\), i.e. \(\alpha > \frac{7}{6}\), we show that the product (1.20) (in the limit) inherits the regularity from \(Z_{2,N} = (z_{1,N})^2 - \sigma_N\), allowing us to pass to a limit in \(H^s(T^3)\) for

\[
s < 2\left(\alpha - \frac{3}{2}\right).
\]

Once we are able to pass the term \(Z_{5,N}\) in (1.20) in the limit \(N \to \infty\), the main issue in solving the equation (1.19) for \(w_N\) by the deterministic Strichartz theory is to ensure that

\[
\begin{align*}
\mathcal{L}w + 3\{(z_{1,N})^2 - \sigma_N\}z_{2,N} &= 0, \\
(w, \partial_t w)|_{t=0} &= (0, 0)
\end{align*}
\]

remains bounded in \(H^\frac{s}{2}(T^3)\) as \(N \to \infty\) (recall that \(s = \frac{1}{2}\) is the threshold regularity for the deterministic local well-posedness theory for the cubic wave equation on \(T^3\)). Using again one degree of smoothing under the wave Duhamel operator, we see that the solution to (1.21) is almost surely bounded in \(H^\frac{s}{2}(T^3)\), provided

\[
2\left(\alpha - \frac{3}{2}\right) + 1 > \frac{1}{2} \implies \alpha > \frac{5}{4}.
\]

This explains the restriction \(\alpha > \frac{5}{4}\) in Theorem 3. We point out that under the restriction \(\alpha > \frac{5}{4}\), we can use deterministic paradifferential calculus to make sense of the product of \(z_{1,N}\) and \(z_{2,N}^2\) appearing in (1.19), uniformly in \(N \in \mathbb{N}\).

In proving Theorem 3, we apply the deterministic Strichartz theory and show that \(w_N\) converges almost surely to some limit \(w\). Along with the almost sure convergence of \(z_{1,N}\) and \(z_{2,N}\) to some limits \(z_1\) and \(z_2\), respectively, we conclude from the decomposition (1.18) that \(u_N\) converges almost surely to

\[
u := z_1 + z_2 + w.
\]

By taking a limit of (1.19) as \(N \to \infty\), we see that \(w\) is almost surely the solution to

\[
\begin{align*}
\mathcal{L}w + (w + z_2)^3 + 3z_1(w + z_2)^2 + 3Z_2w + 3Z_5 &= 0, \\
(w, \partial_t w)|_{t=0} &= (0, 0),
\end{align*}
\]

where \(Z_2\) and \(Z_5\) are the limits of \(Z_{2,N}\) in (1.12) and \(Z_{5,N}\) in (1.20), respectively. This essentially explains the proof of Theorem 3.

Remark 1.3. The expansion (1.22) provides finer descriptions of \(u\) at different scales; the roughest term \(z_1\) is essentially responsible for the small scale behavior of \(u\), while \(z_2\) describes its mesoscopic behavior and the smoother remainder part \(w\) describes its large-scale behavior.
Remark 1.4. The argument based on the first order expansion (1.10) goes back to the work of McKean [30] and Bourgain [8] in the study of invariant Gibbs measures for the nonlinear Schrödinger equations on $\mathbb{T}^d$, $d = 1, 2$. See also [10]. In the field of stochastic parabolic PDEs, this argument is usually referred to as the Da Prato–Debussche trick [15].

As we explained above, the novelty in this paper with respect to the previous work [8, 10, 11] is that the proof of Theorem 3 crucially relies on the second order expansion (1.18). We also mention two other recent works [6, 38], where such higher order expansions were used in the context of dispersive PDEs with random initial data. The difference between the present paper and [6] is that, in this paper, we work in Sobolev spaces of negative indices, while solutions in [6] have positive Sobolev regularities. The higher order expansions used in [38] are at negative Sobolev regularity but they are related to a gauge transform, which is very different from the situation in the present paper.

For conciseness of the presentation, we decided to present only the simplest argument based on the second order expansion. There are, however, several ways for a possible improvement on the regularity restriction in Theorem 3. (i) In studying the regularity and convergence properties of the second order stochastic term $z_{2,N}$ in (1.17), we simply use a “parabolic thinking”, namely, we only count the regularity $s_1 < \alpha - \frac{3}{2}$ of each of three factors $z_{1,N}$ for $Z_{3,N}$ (modulo the renormalization) and put them together with one degree of smoothing coming from the wave Duhamel integral operator without taking into account the explicit product structure and the oscillatory nature of the linear wave propagator. See Proposition 4.1 below. In the field of dispersive PDEs, however, it is crucial to exploit an explicit product structure and study interaction of waves at a multilinear level to show a further smoothing property. In this sense, the argument presented in this paper leaves a room for an obvious improvement. (ii) In recent study of singular stochastic parabolic PDEs such as SQE (1.6) on $\mathbb{T}^3$, higher order expansions (in terms of the stochastic forcing in the mild formulation) were combined with the theory of regularity structures [24] or the paracontrolled calculus [12, 32, 2]. In fact, it is possible to employ the ideas from the paracontrolled calculus in studying nonlinear wave equations. See a recent work [21] on the stochastic NLW with a quadratic nonlinearity. In a forthcoming work [36], we plan to present a more refined approach by addressing the issues mentioned in (i) and (ii).

1.3. Factorization of the ill-posed solution map. In the following, let us consider initial data of the form:

$$\left. (u, \partial_t u) \right|_{t=0} = (w_0, w_1) + (u^\omega_0, u^\omega_1), \quad (1.23)$$

where $(w_0, w_1)$ is a given pair of deterministic functions in $\mathcal{H}^\frac{1}{2}(\mathbb{T}^3)$ and $(u^\omega_0, u^\omega_1)$ is the random initial data given in (1.2). Recall that the random initial data in (1.23) belongs almost surely to $\mathcal{H}^{\text{min}(\frac{3}{2}, s)}(\mathbb{T}^3)$ for $s < \alpha - \frac{3}{2}$. When $\alpha > 2$, the deterministic local well-posedness in $\mathcal{H}^\frac{1}{2}(\mathbb{T}^3)$ yields a continuous solution map $\Phi$:

$$\Phi : (w_0, w_1) + (u^\omega_0, u^\omega_1) \in \mathcal{H}^\frac{1}{2}(\mathbb{T}^3) \mapsto (u, \partial_t u) \in C([-T, T]; \mathcal{H}^\frac{1}{2}(\mathbb{T}^3)).$$

Here, the local well-posedness time is indeed random but we simply write it as $T$. The same comment applies in the following.
On the other hand, when \( \alpha \leq 2 \), the random initial data in (1.23) does not belong to \( \mathcal{H}^\frac{1}{2}(\mathbb{T}^3) \). In particular, the ill-posedness results in [11, 35] show that, given any \( (w_0, w_1) \in \mathcal{H}^\frac{1}{2}(\mathbb{T}^3) \), the solution map \( \Phi \) is almost surely discontinuous.

For \( \frac{3}{2} < \alpha \leq 2 \), the proof of Theorem 2 in [11] based on the first order expansion (1.10) yields the following factorization of the ill-posed solution map \( \Phi \):

\[
(w_0, w_1) + (u_0^0, u_1^0) \mapsto (w_0, w_1, z_1) \xrightarrow{\Psi_1} (v, \partial_t v) \in C([-T, T]; \mathcal{H}^\frac{1}{2}(\mathbb{T}^3)) \mapsto u = z_1 + v \in C([-T, T]; H^{s_1}(\mathbb{T}^3)),
\]

(1.24)

where \( z_1 \) is the solution to the linear equation (1.8) with the random initial data \( (u_0^0, u_1^0) \) in (1.2) and \( s_1 < \alpha - \frac{3}{2} \). Here, we view the first map in (1.24) as a lift map, where we use stochastic analysis to construct an enhanced data set \( (w_0, w_1, z_1) \), and the second map \( \Psi_1 \) is the deterministic solution map to the following perturbed NLW:

\[
\begin{aligned}
\mathcal{L}v + (v + z_1)^3 &= 0 \\
(v, \partial_t v)|_{t=0} &= (w_0, w_1),
\end{aligned}
\]

where we view \( (w_0, w_1, z_1) \) as an enhanced data set\(^7\). Furthermore, the deterministic map \( \Psi_1 : (w_0, w_1, z_1) \mapsto (v, \partial_t v) \) is continuous from

\[
\mathcal{X}^{s_1}_1(T) := \mathcal{H}^\frac{1}{2}(\mathbb{T}^3) \times C([-T, T]; W^{s_1, \infty}(\mathbb{T}^3))
\]

to \( C([-T, T]; \mathcal{H}^\frac{1}{2}(\mathbb{T}^3)) \).

Remark 1.5. In [11], using a conditional probability, Burq and the third author introduced the notion of probabilistic continuity and showed that the map: \( (w_0, w_1) + (u_0^0, u_1^0) \mapsto u \) in (1.24) is indeed probabilistically continuous when \( \frac{3}{2} < \alpha \leq 2 \). It would be of interest to investigate if such probabilistic continuity also holds for lower values of \( \alpha \).

For \( \frac{4}{3} < \alpha \leq \frac{3}{2} \), the first order expansion (1.10) along with renormalization yields the following factorization of the ill-posed solution map \( \Phi \):

\[
(w_0, w_1) + (u_0^0, u_1^0) \mapsto (w_0, w_1, z_2, Z_2) \xrightarrow{\Psi_2} (v, \partial_t v) \in C([-T, T]; \mathcal{H}^\frac{1}{2}(\mathbb{T}^3)) \mapsto u = z_1 + v \in C([-T, T]; H^{s_1}(\mathbb{T}^3)),
\]

(1.25)

where \( Z_2 \) and \( Z_3 \) are the limits of \( Z_{2,N} \) and \( Z_{3,N} \) in (1.12). With \( s_j, j = 1, 2, 3 \), as in (1.9) and (1.13), the second map \( \Psi_2 \) is the deterministic continuous map, sending an enhanced data set \( (w_0, w_1, z_2, Z_2, Z_3) \) in

\[
\mathcal{X}^{s_1, s_2, s_3}_2(T) := \mathcal{H}^\frac{1}{2}(\mathbb{T}^3) \times \prod_{j=1}^3 C([-T, T]; W^{s_j, \infty}(\mathbb{T}^3))
\]

to a solution \( (v, \partial_t v) \in C([-T, T]; \mathcal{H}^\frac{1}{2}(\mathbb{T}^3)) \) to the following perturbed NLW:

\[
\begin{aligned}
\mathcal{L}v + v^3 + 3z_1v^2 + 3Z_2v + Z_3 &= 0 \\
(v, \partial_t v)|_{t=0} &= (w_0, w_1).
\end{aligned}
\]

\(^7\)In particular, we view \( z_1 \) as a given deterministic space-time distribution of some specified regularity.
For \( \frac{13}{10} < \alpha \leq \frac{3}{2} \), the proof of Theorem 3 based on the second order expansion \( (1.18) \) yields the following factorization of the ill-posed solution map \( \Phi \):

\[
(w_0, w_1) + (u_0', u_1') \mapsto (w_0, w_1, z_2) \overset{\Psi_3}{\mapsto} (w, \partial_tw) \in C([-T, T]; \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3))
\]

\[
\mapsto u = z_1 + z_2 + w \in C([-T, T]; H^{s_1}(\mathbb{T}^3)),
\]

where \( z_2 \) is the limit of \( z_{2,N} \) defined in \( (1.17) \). Here, with \( s_4 = s_3 + 1 < 3(\alpha - \frac{3}{2}) + 1 \), the second map \( \Psi_3 \) is the deterministic continuous map, sending an enhanced data set \( (w_0, w_1, z_1, Z_2, z_2) \) in

\[
\Lambda_3^{s_1, s_2, s_4}(T) := \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3) \times \prod_{j \in \{1, 2, 4, 5\}} C([-T, T]; W^{s_j, \infty}(\mathbb{T}^3))
\]

to a solution \( (w, \partial_tw) \in C([-T, T]; \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)) \) to the following perturbed NLW:

\[
\begin{align*}
Lw + (w + z_2)^3 + & 3z_1(w + z_2)^2 + 3Z_2w + 3Z_2z_2 = 0 \\
(w, \partial_tw)|_{t=0} = (w_0, w_1).
\end{align*}
\]

Lastly, let us discuss the case \( \frac{3}{4} < \alpha \leq \frac{14}{10} \). In this case, the product \( Z_2z_2 \) in \( (1.27) \) can not be defined by deterministic paradifferential calculus and thus we need to define \( Z_5 \) as a limit of \( Z_{5,N} \) in \( (1.20) \). Then, the proof of Theorem 3 based on the second order expansion \( (1.18) \) yields the following factorization of the ill-posed solution map \( \Phi \):

\[
(w_0, w_1) + (u_0', u_1') \mapsto (w_0, w_1, z_1, Z_2, z_2, Z_5) \overset{\Psi_4}{\mapsto} (w, \partial_tw) \in C([-T, T]; \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3))
\]

\[
\mapsto u = z_1 + z_2 + w \in C([-T, T]; H^{s_1}(\mathbb{T}^3)).
\]

With \( s_5 < 2(\alpha - \frac{3}{2}) \), the second map \( \Psi_4 \) is the deterministic continuous map, sending an enhanced data set \( (w_0, w_1, z_1, Z_2, z_2, Z_5) \) in

\[
\Lambda_4^{s_1, s_2, s_4, s_5}(T) := \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3) \times \prod_{j \in \{1, 2, 4, 5\}} C([-T, T]; W^{s_j, \infty}(\mathbb{T}^3))
\]

to a solution \( (w, \partial_tw) \in C([-T, T]; \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)) \) to the following perturbed NLW:

\[
\begin{align*}
Lw + (w + z_2)^3 + & 3z_1(w + z_2)^2 + 3Z_2w + Z_5 = 0 \\
(w, \partial_tw)|_{t=0} = (w_0, w_1).
\end{align*}
\]

We point out that the last decomposition \( (1.28) \) with \( (1.29) \) can also be used to study the cases \( \frac{3}{4} < \alpha \leq \frac{5}{4} \) and \( \frac{13}{10} < \alpha \leq \frac{3}{2} \). For simplicity of the presentation, we only discuss the last decomposition \( (1.28) \) with \( (1.29) \) in this paper, while the previous decompositions \( (1.25) \) and \( (1.26) \) provide simpler arguments when \( \frac{13}{10} < \alpha \leq \frac{3}{2} \).

In all the cases mentioned above, we decompose the ill-posed solution map \( \Phi \) into

(i) the first step, constructing enhanced data sets by stochastic analysis and
(ii) the second step, where purely deterministic analysis is performed in constructing a continuous map \( \Psi_j \) on enhanced data sets, solving perturbed NLW equations.

Such decompositions of ill-posed solution maps also appear in studying rough differential equations via the rough path theory \cite{29, 17} and singular stochastic parabolic PDEs \cite{24, 19}.

**Remark 1.6.** By the use of stochastic analysis, the terms \( z_1, z_2, Z_2, \) and \( Z_5 \) are defined as the unique limits of their truncated versions. Furthermore, by deterministic analysis, we
prove that a solution $w$ to (1.29) is pathwise unique in an appropriate class (see the space $X_T$ defined in (2.5)). Therefore, under the decomposition $u = z_1 + z_2 + w$, the uniqueness of $u$ claimed in Theorem 3 refers to (i) the uniqueness of $z_1$ and $z_2$ as the limits of $z_{1,N}$ and $z_{2,N}$ and (ii) the uniqueness of $w$ as a solution to (1.29).

**Remark 1.7.** Given $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let $P_j$ be the (non-homogeneous) Littlewood-Paley projector onto the (spatial) frequencies $\{n \in \mathbb{Z}^3 : |n| \sim 2^j\}$ such that

$$f = \sum_{j=0}^{\infty} P_j f.$$ 

Given two functions $f$ and $g$ on $\mathbb{T}^3$ of regularities $s_1$ and $s_2$, we have the following paraproduct decomposition of the product $fg$ due to Bony [7]:

$$fg = f \otimes g + f \otimes g + f \otimes g$$

$$:= \sum_{j<k<2} P_j f P_k g + \sum_{|j-k| \leq 2} P_j f P_k g + \sum_{k<j<2} P_j f P_k g. \quad (1.30)$$

The first term $f \otimes g$ (and the third term $f \otimes g$) is called the paraproduct of $g$ by $f$ (the paraproduct of $f$ by $g$, respectively) and it is always well defined as a distribution of regularity $\min(s_1, s_1 + s_2)$. On the other hand, the resonant product $f \otimes g$ is well defined in general only if $s_1 + s_2 > 0$. See Lemma 2.1 below.

Let $\frac{5}{4} < \alpha \leq \frac{11}{10}$. In this case, the sum of the regularities $s_2 < 2(\alpha-\frac{3}{2})$ and $s_4 < 3(\alpha-\frac{3}{2})+1$ of $Z_2$ and $z_2$ is non-positive and thus we can not make sense of the product $Z_2 z_2$ by deterministic paradifferential calculus. As we pointed out above, however, the paraproducts $Z_2 \otimes z_2$ and $Z_2 \otimes z_2$ are well defined distributions. Hence, it suffices to define $Z_5^{\otimes}$ as a suitable limit of the resonant products $Z_2 \otimes z_2, N \otimes z_2, N$ in order to pass $3Z_2, N \otimes z_2, N$ to the limit

$$Z_5 = 3Z_2 \otimes z_2 + 3Z_5^{\otimes} + 3Z_2 \otimes z_2.$$ 

This shows that we can in fact replace the enhanced data set $(w_0, w_1, z_1, Z_2, z_2, Z_5)$ in (1.28) and $Z_5$ in (1.29) by $(w_0, w_1, z_1, Z_2, z_2, Z_5^{\otimes})$ and $3Z_2 \otimes z_2 + 3Z_5^{\otimes} + 3Z_2 \otimes z_2$, respectively. See also the proof of Proposition 4.2 and Remark 1.4 below.

**1.4. NLW without renormalization in negative Sobolev spaces.** We conclude this introduction by discussing a new instability phenomenon for NLW (1.1) (that is, without renormalization) in negative Sobolev spaces. This phenomenon is closely related to the so-called *triviality* in the study of stochastic PDEs [11, 26]. See Remark 1.8.

Fix a deterministic pair $(w_0, w_1) \in \mathcal{H}^\frac{3}{4}(\mathbb{T}^3)$. In the following, we study the (unrenormalized) NLW (1.1) with initial data of the form:

$$(u, \partial_t u)|_{t=0} = (w_0, w_1) + (0, w_0^r),$$

where $w_0^r$ is the random distribution given by (1.2). We consider this problem by studying the following truncated problem. Given $N \in \mathbb{N}$, let $u_N$ be the solution to the (unrenormalized) NLW (1.1) with the following initial data:

$$(u_N, \partial_t u_N)|_{t=0} = (w_0, w_1) + (\tilde{u}_0^N, \tilde{u}_1^N).$$
We denote by \( \tilde{w}_{0,N}^\omega, \tilde{w}_{1,N}^\omega \) the truncated random initial data given by

\[
\tilde{w}_{0,N}^\omega(x) = \sum_{|n| \leq N} \frac{g_n(\omega)}{(n)_N(n)^{\alpha-1}} e^{inx} \quad \text{and} \quad \tilde{w}_{1,N}^\omega(x) = \sum_{|n| \leq N} \frac{h_n(\omega)}{(n)_N(n)^{\alpha-1}} e^{inx},
\]

(1.31)

where \( \{g_n\}_{n \in \mathbb{Z}^3} \) and \( \{h_n\}_{n \in \mathbb{Z}^3} \) are as in (1.2) and

\[
\langle n \rangle_N = \sqrt{C_N + |n|^2}
\]

for some suitable choice of a divergent constant \( C_N > 0 \). Our goal is to study the asymptotic behavior of \( u_N \) as \( N \to \infty \).

Given \( N \in \mathbb{N} \), define the linear Klein-Gordon operator \( \mathcal{L}_N \) by setting

\[
\mathcal{L}_N := \partial_t^2 - \Delta + C_N.
\]

Then, \( u_N \) satisfies the following equation:

\[
\begin{cases}
\mathcal{L}_N u_N + u_N^3 - C_N u_N = 0 \\
(u_N, \partial_t u_N)|_{t=0} = (w_0, w_1) + (\tilde{w}_{0,N}^\omega, \tilde{w}_{1,N}^\omega).
\end{cases}
\]

(1.33)

We denote by \( \tilde{z}_{1,N} \) the solution to the following linear Klein-Gordon equation:

\[
\mathcal{L}_N \tilde{z}_{1,N} = 0
\]

(1.34)

with the truncated random initial data \( (\tilde{w}_{0,N}^\omega, \tilde{w}_{1,N}^\omega) \) in (1.31). Then, we have

\[
\tilde{z}_{1,N}(t, x, \omega) = \sum_{|n| \leq N} \frac{\cos(t\langle n \rangle_N)}{(n)_N(n)^{2(\alpha-1)}} g_n(\omega) e^{inx} + \sum_{|n| \leq N} \frac{\sin(t\langle n \rangle_N)}{(n)_N(n)^{2(\alpha-1)}} h_n(\omega) e^{inx}.
\]

(1.35)

In particular, for each fixed \( (t, x) \in \mathbb{R} \times \mathbb{T}^3 \), \( \tilde{z}_{1,N}(t, x) \) is a mean-zero Gaussian random variable with variance:

\[
\tilde{\sigma}_N := \mathbb{E}[(\tilde{z}_{1,N}(t, x))^2] = \sum_{|n| \leq N} \frac{(\cos(t\langle n \rangle_N))^2}{(n)_N(n)^{2(\alpha-1)}} + \sum_{|n| \leq N} \frac{(\sin(t\langle n \rangle_N))^2}{(n)_N(n)^{2(\alpha-1)}}
\]

(1.36)

In view of (1.33), we implicitly define \( C_N > 0 \) by

\[
C_N = 3\tilde{\sigma}_N = 3 \sum_{|n| \leq N} \frac{1}{(C_N + |n|^2)(n)^{2(\alpha-1)}}
\]

(1.37)

such that the subtraction of \( C_N u_N \) in (1.33) corresponds to renormalization of the cubic nonlinearity \( u_N^3 \). In Lemma 6.1 below, we show that for each \( N \in \mathbb{N} \), there exists unique \( C_N \geq 1 \) whose asymptotic behavior of \( C_N \) as \( N \to \infty \) is given by

\[
C_N \sim \begin{cases}
\log N, & \text{for } \alpha = \frac{3}{2}, \\
N^{3-2\alpha}, & \text{for } 1 \leq \alpha < \frac{3}{2},
\end{cases}
\]

for all sufficiently large \( N \gg 1 \). In particular, \( C_N \to \infty \) as \( N \to \infty \) and thus we see that \( (\tilde{w}_{0,N}^\omega, \tilde{w}_{1,N}^\omega) \) in (1.31) almost surely converges to \( (0, \tilde{u}_1^\omega) \) in a suitable topology.

We are now ready to state an instability result for the (un-renormalized) NLW (1.1) in negative Sobolev spaces.
Theorem 4. Let \( \frac{5}{4} < \alpha \leq \frac{3}{2} \) and \((w_0, w_1) \in H^\frac{3}{2}(\mathbb{T}^3)\). By setting \( C_N \) by (1.37), there exist small \( T_1 > 0 \) and positive constants \( C, c, \kappa \) such that for every \( T \in (0, T_1) \), there exists a set \( \Omega_T \) of complemental probability smaller than \( C \exp(-c/T^\kappa) \) such that if we denote by \( \{u_N\}_{N \in \mathbb{N}} \) the smooth global solutions to the defocusing cubic NLW (1.1) with the random initial data

\[
(u_N, \partial_t u_N)_{t=0} = (w_0, w_1) + (\bar{u}_0^\omega, \bar{u}_1^\omega),
\]

where \((\bar{u}_0^\omega, \bar{u}_1^\omega)\) is given by (1.31), then for every \( \omega \in \Omega_T \), the sequence \( \{u_N\}_{N \in \mathbb{N}} \) converges to 0 as space-time distributions on \([-T, T] \times \mathbb{T}^3\) as \( N \to \infty \).

The proof of Theorem 4 is based on the reformulation (1.33) and an adaptation of the argument employed in proving Theorem 3.

We point out that the instability stated in Theorem 4 is due to the lack of renormalization (in negative regularity). Indeed, let us briefly discuss the situation when a proper renormalization is applied. Consider the following renormalized NLW:

\[
\partial_t^2 u_N - \Delta u_N + u_N^3 - 3\bar{\sigma}_N u_N = 0
\]

with the random initial data in (1.38). First, note that the initial data in (1.38) gives rise to an enhanced data set

\[\Xi_N = (u_0, u_1, \tilde{z}_{1,N}, \tilde{Z}_{2,N}, \tilde{z}_{2,N}, \tilde{Z}_{5,N}),\]

where \( \tilde{Z}_{2,N} \) and \( \tilde{Z}_{5,N} \) are defined by

\[\tilde{Z}_{2,N} := (\tilde{z}_{1,N})^2 - \bar{\sigma}_N \quad \text{and} \quad \tilde{Z}_{5,N} := \{(\tilde{z}_{1,N})^2 - \bar{\sigma}_N\} \tilde{z}_{2,N}\]

and \( \tilde{z}_{2,N} \) is the solution to

\[
\begin{aligned}
\mathcal{L}_N \tilde{z}_{2,N} + \{(\tilde{z}_{1,N})^3 - 3\bar{\sigma}_N \tilde{z}_{1,N}\} &= 0 \\
(\tilde{z}_{2,N}, \partial_t \tilde{z}_{2,N})_{t=0} &= (0,0).
\end{aligned}
\]

In Section 6 we show that \( \Xi_N \) converges almost surely to the limiting enhanced data set

\[\Xi = (u_0, u_1, \tilde{z}_1, \tilde{Z}_2, \tilde{z}_2, \tilde{Z}_5),\]

emanating from the initial data \((w_0, w_1) + (0, w^\sigma_1)\). Then, by slightly modifying the proof of Theorem 3 we can show that the solutions \( u_N \) to (1.39) converges to some non-trivial limiting distribution \( u = \tilde{z}_1 + \tilde{z}_2 + w \), where \( w \) is the solution to

\[
\begin{aligned}
\mathcal{L} w + (w + \tilde{z}_2)^3 + 3\tilde{z}_1(w + \tilde{z}_2)^2 + 3\tilde{z}_2 w + \tilde{Z}_5 &= 0 \\
(w, \partial_t w)_{t=0} &= (w_0, w_1).
\end{aligned}
\]

Here, we see that \( u \neq 0 \) since the non-zero linear solution \( \tilde{z}_1 \) with initial data \((0, w^\sigma_1)\) does not belong to \( H^{\alpha-\frac{3}{2}}(\mathbb{T}^3) \) (for a fixed time) while \( z_2 + w \in H^{\alpha-\frac{3}{2}}(\mathbb{T}^3) \) almost surely. This shows the instability result stated in Theorem 4 is peculiar to the case without renormalization when we work in negative regularities.

Remark 1.8. The instability result in Theorem 4 essentially corresponds to triviality results in the study of stochastic PDEs, where the dynamics without renormalization trivializes (either to the linear dynamics or the trivial dynamics, i.e. \( u \equiv 0 \)) as smoothing on a singular random forcing is removed. See for example [11, 20]. In particular, our proof of Theorem 4 is inspired by the argument in [20] due to Hairer, Ryser, and Weber.
In the context of nonlinear Schrödinger type equations, such instability results without renormalization in negative Sobolev spaces are known even deterministically; see \[23, 39\].

**Remark 1.9.** In the discussion above, we needed to consider the random data \((\tilde{u}_{0,N}^\omega, \tilde{u}_{1,N}^\omega)\) in (1.31) in place of \((u_{0,N}, u_{1,N})\) in (1.4) such that \(C_N\) can be chosen to be time independent. Note that the distribution of \((\tilde{u}_{0,N}^\omega, \tilde{u}_{1,N}^\omega)\) in (1.31) is precisely an invariant measure for the linear dynamics: \(L_N u = 0\).

**Remark 1.10.** While the local-in-time results in Theorems 2 and 3 also hold in the focusing case, the proof of Theorem 4 only holds for the defocusing case. In the focusing case, we expect some undesirable behavior for solutions to the (un-renormalized) cubic NLW in negative Sobolev spaces but with a different mechanism.

1.5. **Organization of the paper.** The remaining part of this manuscript is organized as follows. In the next section, we state deterministic and stochastic tools needed for our analysis. In Sections 3 and 4, we study regularity and convergence properties of the stochastic terms from Subsection 1.2. In Section 5, we then use the deterministic Strichartz theory to study the equation (1.19) for \(w_N\) and present the proof of Theorem 3. In Section 6, by modifying the analysis from the previous sections, we prove Theorem 4.

2. Tools from deterministic and stochastic analysis

2.1. **Basic function spaces and paraproducts.** We define the \(L^p\)-based Sobolev space \(W^{s,p}(T^3)\) by the norm:

\[
\|f\|_{W^{s,p}} = \|F^{-1}(\langle n \rangle^s \hat{f}(n))\|_{L^p}
\]

with the standard modification when \(p = \infty\). When \(p = 2\), we have \(H^s(T^3) = W^{s,2}(T^3)\).

Next, we recall the regularity properties of paraproducts and resonant products, viewed as bilinear maps. For this purpose, it is convenient to use the Besov spaces \(B_{p,q}^s(T^3)\) defined by the norm:

\[
\|u\|_{B_{p,q}^s} = \left\|2^{sj}\|P_j u\|_{L^p}\right\|_{\ell^q_{\mathbb{N}_0}}.
\]

Note that \(H^s(T^3) = B_{2,2}^s(T^3)\).

**Lemma 2.1.** (i) (paraproduct and resonant product estimates) Let \(s_1, s_2 \in \mathbb{R}\) and \(1 \leq p, p_1, p_2, q \leq \infty\) such that \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.\) Then, we have

\[
\|f \diamond g\|_{B_{p,q}^{s_2}} \lesssim \|f\|_{L^{p_1}}\|g\|_{B_{p_2,q}^{s_2}}. \quad (2.1)
\]

When \(s_1 < 0\), we have

\[
\|f \diamond g\|_{B_{p,q}^{s_1+s_2}} \lesssim \|f\|_{B_{p_1,q}^{s_1}}\|g\|_{B_{p_2,q}^{s_2}}. \quad (2.2)
\]

When \(s_1 + s_2 > 0\), we have

\[
\|f \diamond g\|_{B_{p,q}^{s_1+s_2}} \lesssim \|f\|_{B_{p_1,q}^{s_1}}\|g\|_{B_{p_2,q}^{s_2}}. \quad (2.3)
\]

(ii) Let \(s_1 < s_2 < s_3\) and \(1 \leq p, q \leq \infty\). Then, we have

\[
\|u\|_{W^{s_1,p}} \lesssim \|u\|_{B_{p,q}^{s_2}} \lesssim \|u\|_{W^{s_3,p}}. \quad (2.4)
\]
The product estimates (2.1), (2.2), and (2.3) follow easily from the definition (1.30) of the paraproduct and the resonant product. See [31] for details of the proofs in the non-periodic case (which can be easily extended to the current periodic setting). The embeddings (2.4) follow from the \( \ell^s \)-summability of \( \{2^{(s_k - s_{k+1})j}\}_{j \in \mathbb{N}_0} \) for \( s_k < s_{k+1}, \ k = 1, 2 \), and the uniform boundedness of the Littlewood-Paley projector \( P_j \). Thanks to (2.4), we can apply the product estimates (2.1), (2.2), and (2.3) in the Sobolev space setting (with a slight loss of regularity).

2.2. Product estimates, an interpolation inequality, and Strichartz estimates. 
For \( s \in \mathbb{R} \), we set \( \langle \nabla \rangle^s : = (1 - \Delta)^{\frac{s}{2}} \). Then, we have the following standard product estimates. See [20] for their proofs.

**Lemma 2.2.** Let \( 0 \leq s \leq 1 \).
(i) Let \( 1 < p_j, q_j, r < \infty, j = 1, 2 \) such that \( \frac{1}{r} = \frac{1}{p_j} + \frac{1}{q_j} \). Then, we have
\[
\|\langle \nabla \rangle^s (fg)\|_{L^r_t(W^{s,r}_x(T^3))} \lesssim \|\langle \nabla \rangle^s f\|_{L^{p_j}_t(W^{s,p_j}_x(T^3))}\|g\|_{L^{q_j}_t(W^{s,q_j}_x(T^3))} + \|f\|_{L^{p_2}_t(W^{s,p_2}_x(T^3))}\|\langle \nabla \rangle^s g\|_{L^{q_2}_t(W^{s,q_2}_x(T^3))}.
\]
(ii) Let \( 1 < p, q, r < \infty \) such that \( s \geq 3\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{r}\right) \). Then, we have
\[
\|\langle \nabla \rangle^{-s} (fg)\|_{L^r_t(W^{s,r}_x(T^3))} \lesssim \|\langle \nabla \rangle^{-s} f\|_{L^p_t(W^{s,p}_x(T^3))}\|\langle \nabla \rangle^{-s} g\|_{L^q_t(W^{s,q}_x(T^3))}.
\]

Note that while Lemma 2.2 (ii) was shown only for \( s = 3\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{r}\right) \) in [20], the general case \( s \geq 3\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{r}\right) \) follows from a straightforward modification.

Next, we state an interpolation inequality. This lemma allows us to reduce an estimate on the \( L^\infty \)-norm in time to that with the \( L^q \)-norm in time for some finite \( q \).

**Lemma 2.3.** Let \( T > 0 \) and \( 1 \leq q, r \leq \infty \). Suppose that \( s_1, s_2, s_3 \in \mathbb{R} \) satisfy \( s_2 \leq s_1 \) and \( q(s_1 - s_3) > s_1 - s_2 \).

Then, we have
\[
\|u\|_{L^\infty([-T,T];W^{s_3,r}(T^3))} \lesssim \|u\|_{L^q([-T,T];W^{s_1,q}(T^3))}^{\frac{1}{q^*}}\|u\|_{W^{1,q}([-T,T];W^{s_2,q}(T^3))}^{\frac{1}{q^*}}
\]

Here, the \( W^{1,q}([-T,T];W^{s,r}(T^3)) \)-norm is defined by
\[
\|f\|_{W^{1,q}([-T,T];W^{s,r}(T^3))} = \|f\|_{L^q([-T,T];W^{s,r}(T^3))} + \|\partial_t f\|_{L^q([-T,T];W^{s,r}(T^3))}.
\]

The proof of Lemma 2.3 follows from duality in \( x \) and Gagliardo-Nirenberg’s inequality in \( t \) along with standard analysis based on (spatial) Littlewood-Paley decompositions. See the proofs of Lemmas 3.2 and 3.3 in [9] for the \( r = 2 \) case. The proof for the general case follows from a straightforward modification.

We now recall the Strichartz estimates. Let \( \mathcal{L} \) be the Klein-Gordon operator in (1.7). We use \( \mathcal{L}^{-1} = (\partial_t^2 - \Delta + 1)^{-1} \) to denote the Duhamel integral operator, corresponding to the forward fundamental solution to the Klein-Gordon equation:
\[
\mathcal{L}^{-1} F(t) := \int_0^t \frac{\sin((t - t')\langle \nabla \rangle)}{\langle \nabla \rangle} F(t') dt'.
\]

Namely, \( u := \mathcal{L}^{-1}(F) \) is the solution to the following nonhomogeneous linear equation:
\[
\begin{cases}
\mathcal{L} u = F \\
(u, \partial_t u)|_{t=0} = (0,0).
\end{cases}
\]
The most basic regularity property of $L^{-1}$ is the energy estimate:

$$\|L^{-1}(F)\|_{L^\infty([-T,T];H^s(T^3))} \lesssim \|F\|_{L^1([-T,T];H^{s-1}(T^3))}. \quad (2.6)$$

The Strichartz estimates are important extensions of (2.6) and have been studied extensively by many mathematicians. See [18, 28, 27] in the context of the wave equation on $\mathbb{R}^d$. Thanks to the finite speed of propagation, the Strichartz estimates on $T^3$ follow from the corresponding estimates on $\mathbb{R}^3$, locally in time. We now state the Strichartz estimates which are relevant for the analysis in this paper. We refer to [43] for a detailed proof.

**Lemma 2.4.** Let $0 < T \leq 1$. Then, the following estimate holds:

$$\|L^{-1}(F)\|_{L^4([-T,T] \times T^3)} + \|L^{-1}(F)\|_{L^\infty([-T,T];H^\frac{1}{2}(T^3))} \lesssim \min \left( \|F\|_{L^1([-T,T];H^{-\frac{1}{2}}(T^3))}, \|F\|_{L^4([-T,T] \times T^3)} \right). \quad (2.7)$$

For $T > 0$, we denote by $X_T$ the closed subspace of $C([-T,T];H^\frac{1}{2}(T^3))$ endowed with the norm:

$$\|u\|_{X_T} = \|u\|_{L^\infty([-T,T];H^\frac{1}{2}(T^3))} + \|u\|_{L^4([-T,T] \times T^3)}. \quad (2.8)$$

In the following, we use shorthand notations such as $L^q_T L^r_x := L^q([-T,T];L^r(T^3))$.

### 2.3. On discrete convolutions.

Next, we recall the following basic lemma on a discrete convolution.

**Lemma 2.5.**

1. Let $d \geq 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha + \beta > d$ and $\alpha, \beta < d$. Then, we have

$$\sum_{n=n_1+n_2} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{d-\alpha - \beta}$$

for any $n \in \mathbb{Z}^d$.

2. Let $d \geq 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy $\alpha + \beta > d$. Then, we have

$$\sum_{n=n_1+n_2 \atop |n_1| \sim |n_2|} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{d-\alpha - \beta}$$

for any $n \in \mathbb{Z}^d$.

Namely, in the resonant case (ii), we do not have the restriction $\alpha, \beta < d$. Lemma 2.5 follows from elementary computations. See, for example, Lemmas 4.1 and 4.2 in [33] for the proof.

### 2.4. Wiener chaos estimate.

Lastly, we recall the following Wiener chaos estimate [40, Theorem I.22]. See also [41, Proposition 2.4].

**Lemma 2.6.** Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of independent standard real-valued Gaussian random variables. Given $k \in \mathbb{N}$, let $\{P_j\}_{j \in \mathbb{N}}$ be a sequence of monomials in $\tilde{g} = \{g_n\}_{n \in \mathbb{N}}$
of degree at most k, namely, \( P_j = P_j(\bar{g}) \) is of the form \( P_j = c_j \prod_{i=1}^{k_j} g_{n_i} \) with \( k_j \leq k \) and \( n_1, \ldots, n_{k_j} \in \mathbb{N} \). Then, for \( p \geq 2 \), we have

\[
\left\| \sum_{j \in \mathbb{N}} P_j(\bar{g}) \right\|_{L^p(\Omega)} \leq (p-1)\frac{1}{2} \left\| \sum_{j \in \mathbb{N}} P_j(\bar{g}) \right\|_{L^2(\Omega)}.
\]

This lemma is a direct corollary to the hypercontractivity of the Ornstein-Uhlenbeck semigroup due to Nelson [34]. Note that in the definition of \( P_j \) above, we may have \( n_i = n_{\ell} \) for \( i \neq \ell \). Namely, we do not impose independence of the factors \( g_{n_i} \) of \( P_j \) in Lemma 2.6. In the following, we apply Lemma 2.6 to multilinear terms involving \( \{g_n\}_{n \in \mathbb{Z}^3} \) and \( \{h_n\}_{n \in \mathbb{Z}^3} \) in (1.2) by first expanding \( g_n \) and \( h_n \) into their real and imaginary parts.

3. On the Random Free Evolution and Its Renormalized Powers

Recall from (1.4) and (1.8) that \( z_{1,N}(t, x, \omega) \) denotes the solution to the linear Klein-Gordon equation:

\[
(\partial_t^2 - \Delta + 1) z_{1,N}(t, x, \omega) = 0
\]

with the truncated random initial data:

\[
z_{1,N}(0, x, \omega) = \sum_{|n| \leq N} g_n(\omega) e^{i n \cdot x} \quad \text{and} \quad \partial_t z_{1,N}(0, x, \omega) = \sum_{|n| \leq N} h_n(\omega) \frac{(n)_{\alpha-1}}{\langle n \rangle^\alpha} e^{i n \cdot x},
\]

where \( \{g_n\}_{n \in \mathbb{Z}^3} \) and \( \{h_n\}_{n \in \mathbb{Z}^3} \) are as in (1.2). Given \( t \in \mathbb{R} \), define \( g_n^t(\omega) \) by

\[
g_n^t(\omega) := \cos(t \langle n \rangle) g_n(\omega) + \sin(t \langle n \rangle) h_n(\omega).
\]

Then, we have

\[
z_{1,N}(t, x, \omega) = \cos(t \langle \nabla \rangle) \left( z_{1,N}(0, x, \omega) \right) + \sin(t \langle \nabla \rangle) \left( \partial_t z_{1,N}(0, x, \omega) \right) = \sum_{|n| \leq N} g_n^t(\omega) e^{i n \cdot x}.
\]

Using the definitions of the Gaussian random variables \( \{g_n\}_{n \in \mathbb{Z}^3} \) and \( \{h_n\}_{n \in \mathbb{Z}^3} \), we see that \( \{g_n^t\}_{n \in \mathbb{Z}^3} \) defined in (3.1) forms a family of independent standard complex-valued Gaussian random variables conditioned that \( g_0^t = g_0^{-t} \). Then, the renormalization constant \( \sigma_N \) defined in (1.14) is computed as

\[
\sigma_N = \mathbb{E} \left[ \left( z_{1,N}(t, x, \omega) \right)^2 \right] = \sum_{|n| \leq N} \frac{\mathbb{E}[|g_n^t(\omega)|^2]}{\langle n \rangle^{2\alpha}}
\]

\[
= \sum_{|n| \leq N} \frac{1}{\langle n \rangle^{2\alpha}} \sim \begin{cases} \log N, & \text{for } \alpha = \frac{3}{2}, \\ N^{3-2\alpha}, & \text{for } \alpha < \frac{3}{2}, \end{cases}
\]

which tends to \( \infty \) as \( N \to \infty \).

---

8In particular, \( g_0^t \) is real-valued.
Remark 3.1. From the definitions of the Gaussian random variables $g_n$ and $h_n$ and their rotational invariance, we see that
\[
\text{Law}(z_{1,N}(t,x)) = \text{Law}(z_{1,N}(0,0))
\]
for any $(t,x) \in \mathbb{R} \times T^3$. This also explains the independence of $\sigma_N$ from $t$ and $x$.

We now define the sequences $\{Z_{j,N}\}_{N \in \mathbb{N}}$, $j = 1,2,3$, by
\[
Z_{1,N} := z_{1,N}, \quad Z_{2,N} := (z_{1,N})^2 - \sigma_N, \quad \text{and} \quad Z_{3,N} := (z_{1,N})^3 - 3\sigma_N z_{1,N}.
\]
(3.3)
The main goal of this section is to prove the following proposition on the regularity and convergence properties of the stochastic terms $Z_{1,N}$, $Z_{2,N}$, and $Z_{3,N}$.

Proposition 3.2. Let $1 < \alpha \leq \frac{3}{2}$ and set
\[
s_1 < \alpha - \frac{3}{2}, \quad s_2 < 2(\alpha - \frac{3}{2}), \quad \text{and} \quad s_3 < 3(\alpha - \frac{3}{2}).
\]
(3.4)
Fix $j = 1,2, or 3$. Then, given any $T > 0$, $Z_{j,N}$ converges almost surely to some limit $Z_j$ in $C([-T,T];W^{s_j,\infty}(T^3))$ as $N \to \infty$. Moreover, given $2 \leq q < \infty$, there exist positive constants $C, c, \kappa, \theta$ such that for every $T > 0$, there exists a set $\Omega_T$ of complementary probability smaller than $C \exp(-c/T^\kappa)$ with the following properties; given $\varepsilon > 0$, there exists $N_0 = N_0(T,\varepsilon) \in \mathbb{N}$ such that
\[
\|Z_{j,N}\|_{L^q([-T,T];W^{s_j,\infty}(T^3))} \leq T^\theta
\]
(3.5)
and
\[
\|Z_{j,M} - Z_{j,N}\|_{C([-T,T];W^{s_j,\infty}(T^3))} < \varepsilon
\]
(3.6)
for any $\omega \in \Omega_T$ and any $M \geq N \geq N_0$, where we allow $N = \infty$ with the understanding that $Z_{j,\infty} = Z_j$.

We split the proof of this proposition into several parts. We first present preliminary lemmas and then prove Proposition 3.2 at the end of this section.

Lemma 3.3. Let $1 < \alpha \leq \frac{3}{2}$ and $s_j$, $j = 1,2,3$, satisfy (3.3). Then, given $2 \leq q < \infty$ and $2 \leq r \leq \infty$, there exists $\delta > 0$ such that the following estimates hold for $j = 1,2,3$:
\[
\|\langle \nabla \rangle^{s_j} Z_{j,N}\|_{L^p(\Omega;L^q([-T,T];L^r(T^3)))} \leq CT^{\frac{1}{q}p^*_r},
\]
(3.7)
\[
\|\langle \nabla \rangle^{s_j}(Z_{j,M} - Z_{j,N})\|_{L^p(\Omega;L^q([-T,T];L^r(T^3)))} \leq CN^{-\delta} T^{\frac{1}{q}p^*_r},
\]
(3.8)
for any $M \geq N \geq 1$, $T > 0$, and any finite $p \geq 1$, where the constant $C$ is independent of $M,N,T,p$.

Proof. In the following, we only prove the difference estimate (3.8) since the first estimate (3.7) follows in a similar manner.

When $r = \infty$, we can apply the Sobolev embedding theorem and reduce the $r = \infty$ case to the case of large but finite $r$ at the expense of a slight loss of spatial derivative. This, however, does not cause an issue since the conditions on $s_j$ are open. Hence, we assume $r < \infty$ in the following.

Let $p \geq \max(q,r)$. Since
\[
\langle \nabla \rangle^{s_1} Z_{1,N} = \sum_{|n| \leq N} \frac{g_n^i(\omega)}{\langle n \rangle^{a_i - s_1}} e^{in \cdot x},
\]
we see that \( \langle \nabla \rangle^{s_1} (Z_{1,N} - Z_{1,M}) (t,x) \) is a mean-zero Gaussian random variable for fixed \( t \) and \( x \). In particular, there exists a universal constant \( C > 0 \) such that

\[
\left\| \langle \nabla \rangle^{s_1} (Z_{1,M} - Z_{1,N}) (t,x) \right\|_{L^p(\Omega)} \leq C p^{\frac{1}{2}} \left\| \langle \nabla \rangle^{s_1} (Z_{1,M} - Z_{1,N}) (t,x) \right\|_{L^2(\Omega)}. \tag{3.9}
\]

Then, it follows from Minkowski’s integral inequality and (3.9) that

\[
\left\| \langle \nabla \rangle^{s_1} (Z_{1,M} - Z_{1,N}) \right\|_{L^p(\Omega)} \leq \left\| \langle \nabla \rangle^{s_1} (Z_{1,M} - Z_{1,N}) (t,x) \right\|_{L^p(\Omega)} \leq C T^\frac{1}{2} p^\frac{1}{2} \left( \sum_{N < |n| \leq M} \frac{1}{\langle n \rangle^{2(\alpha - s_1)}} \right)^\frac{1}{2} \leq C N^{-\delta} T^\frac{1}{2} p^\frac{1}{2} \tag{3.10}
\]

for some \( \delta > 0 \) under the regularity assumption (3.4). This proves (3.8) for \( j = 1 \).

Next, we turn to the \( j = 2 \) case. Let us write

\[
\langle \nabla \rangle^{s_2} Z_{2,N} = I_N + \Pi_N, \tag{3.11}
\]

where

\[
I_N(t,x) := \sum_{|n_1| \leq N, |n_2| \leq N \atop n_1 \neq -n_2} \frac{g_{n_1}^1(\omega) g_{n_2}^2(\omega)}{\langle n_1 + n_2 \rangle^{-s_2} \langle n_1 \rangle^\alpha \langle n_2 \rangle^\alpha} e^{i (n_1 + n_2) \cdot x}
\]

and

\[
\Pi_N(t,x) := \sum_{|n| \leq N} \langle n \rangle^{-2\alpha} \left( |g_n(\omega)|^2 - |g_n(\omega)|^2 \right) = \sum_{|n| \leq N} \langle n \rangle^{-2\alpha} \left( |g_n(\omega)|^2 - 1 \right).
\]

Fix \( (t,x) \in \mathbb{R} \times \mathbb{T}^3 \). By using the independence of \( \{ g_{n_k}^j(\omega) \}_{n_k \in \Lambda} \) with \( \Lambda \) as in (1.3) and Lemma 2.5, we have

\[
\left\| I_M(t,x) - I_N(t,x) \right\|_{L^2(\Omega)}^2 \lesssim \sum_{|n_1| \leq M, |n_2| \leq M \atop \max(|n_1|,|n_2|) > N} \frac{1}{\langle n_1 + n_2 \rangle^{-2s_2} \langle n_1 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha}} \tag{3.12}
\]

for some \( \delta > 0 \), with \( C \) independent of \( M \geq N \geq 1 \) and \( (t,x) \in \mathbb{R} \times \mathbb{T}^3 \), provided that \( 4\alpha - 2s_2 > 6 \). Namely, \( s_2 < 2(\alpha - \frac{3}{2}) \). Similarly, by using the independence of \( \{ |g_n(\omega)|^2 - 1 \}_{n \in \Lambda} \), we have

\[
\left\| \Pi_M(t,x) - \Pi_N(t,x) \right\|_{L^2(\Omega)}^2 \lesssim \sum_{N < |n| \leq M} \frac{1}{\langle n \rangle^{4\alpha}} \leq C N^{-\delta} \tag{3.13}
\]

for some \( \delta > 0 \), with \( C \) independent of \( M \geq N \geq 1 \) and \( (t,x) \in \mathbb{R} \times \mathbb{T}^3 \), provided \( 4\alpha > 3 \), which is guaranteed by the assumption \( \alpha > 1 \). Therefore, from (3.11), (3.12), and (3.13), we obtain

\[
\left\| \langle \nabla \rangle^{s_2} (Z_{2,M} - Z_{2,N}) (t,x) \right\|_{L^2(\Omega)} \leq C N^{-\delta}
\]

9Strictly speaking, in applying Lemma 2.5 when \( \alpha = \frac{3}{2} \), we need to replace \( 2\alpha \) in the exponent by \( 2\alpha - \varepsilon \) for some small \( \varepsilon > 0 \). This, however, does not affect the outcome since the condition on \( s_2 \) is open. The same comment applies to (3.15) below.
for some $\delta > 0$, with a constant $C$ independent of $M \geq N \geq 1$ and $(t, x) \in \mathbb{R} \times \mathbb{T}^3$. By the Wiener chaos estimate (Lemma 2.6), we then obtain
\[
\left\| \langle \nabla \rangle^{s_2} (Z_{2,M} - Z_{2,N}) (t, x) \right\|_{L^p(\Omega)} \leq C N^{-\delta} p^\frac{\alpha}{2}
\] (3.14)
for any finite $p \geq 2$. Then, arguing as in (3.10) with Minkowski’s integral inequality, the estimate (3.8) for $j = 2$ follows from (3.14).

Let us finally turn to (3.8) for $j = 3$. Write
\[
\langle \nabla \rangle^{s_3} Z_{3,N} = \mathbb{I}_N + \mathbb{IV}_N,
\]
where
\[
\mathbb{III}_N(t, x) := \sum_{|n| \leq N; j=1,2,3} \frac{g_{n_1}^j(\omega) g_{n_2}^j(\omega) g_{n_3}^j(\omega)}{\langle n_1 + n_2 + n_3 \rangle^{\alpha} \langle n_2 \rangle^{\alpha} \langle n_3 \rangle^{\alpha}} e^{i(n_1 + n_2 + n_3) \cdot x}
\]
and by the inclusion-exclusion principle
\[
\mathbb{IV}_N(t, x) := \sum_{|n| \leq N} \frac{|g_{n}^0(\omega)|^2 - \mathbb{E}[|g_{n}^0|^2]}{\langle n \rangle^{2\alpha}} \sum_{|m| \leq N} \frac{g_{m}^0(\omega)}{(m)^{\alpha - s_3}} e^{i m \cdot x} - \sum_{|n| \leq N} \frac{|g_{n}^0(\omega)|^2 g_{n_j}^j(\omega)}{\langle n \rangle^{3\alpha - s_3}} e^{i n_j \cdot x} + \frac{|g_{0}^0(\omega)|^2 g_{0_j}^j(\omega)}{\langle n \rangle^{\alpha}} e^{i 0_j \cdot x}
\]
Proceeding as above with Lemma 2.5, we have
\[
\left\| \mathbb{III}_M(t, x) - \mathbb{III}_N(t, x) \right\|_{L^2(\Omega)}^2 \leq \sum_{|n| \leq N} \frac{1}{\langle n_1 + n_2 + n_3 \rangle^{2\alpha} \langle n_2 \rangle^{2\alpha} \langle n_3 \rangle^{2\alpha}} \leq C N^{-\delta}
\] (3.15)
for some $\delta > 0$, with $C$ independent of $M \geq N \geq 1$ and $(t, x) \in \mathbb{R} \times \mathbb{T}^3$, provided $6\alpha - 2s_3 > 9$ and $\alpha > 1$. See Remark 3.6. Namely, $s_3 < 3(\alpha - \frac{5}{2})$ and $\alpha > 1$. Then, by the Wiener chaos estimate (Lemma 2.6), we obtain
\[
\left\| \mathbb{III}_M(t, x) - \mathbb{III}_N(t, x) \right\|_{L^p(\Omega)} \leq C N^{-\delta} p^\frac{\alpha}{2}
\] (3.16)
for any finite $p \geq 2$.

Let us now estimate IV_N. By Lemma 2.6 and Hölder’s inequality, we have
\[
\left\| \mathbb{IV}_N(t, x) \right\|_{L^p(\Omega)} \leq p^\frac{\alpha}{2} \sum_{|n| \leq N} \frac{1}{\langle n \rangle^{3\alpha - s_3}} + \left\| \sum_{|n| \leq N} \frac{|g_{n}^0(\omega)|^2 - \mathbb{E}[|g_{n}^0|^2]}{\langle n \rangle^{2\alpha}} \right\|_{L^p(\Omega)} \left\| \sum_{|m| \leq N} \frac{g_{m}^0(\omega)}{(m)^{\alpha - s_3}} e^{i m \cdot x} \right\|_{L^p(\Omega)}.
\]
The first sum on the right-hand side is convergent if $3\alpha - s_3 > 3$. Note that this condition is guaranteed under (3.4). Both factors in the second term on the right hand-side can be treated by the arguments presented above. We therefore have the bounds:
\[
\left\| \sum_{|n| \leq N} \frac{|g_{n}^0(\omega)|^2 - \mathbb{E}[|g_{n}^0|^2]}{\langle n \rangle^{2\alpha}} \right\|_{L^p(\Omega)} \leq C p
\] (3.17)
and
\[ \left\| \sum_{|m| \leq N} \frac{g_m(\omega)}{(m)^{\alpha - s_3}} e^{im \cdot x} \right\|_{L^p(\Omega)} \leq C p^{\frac{1}{2}} \] (3.18)
for any finite \( p \geq 2 \), provided that \( 4\alpha > 3 \) for (3.17) and \( 2\alpha - 2s_3 > 3 \) for (3.18). Note that the second condition is guaranteed under (3.4) with \( \alpha \leq \frac{3}{2} \). Then, by applying the Wiener chaos estimate (Lemma 2.6), this leads to
\[ \| IV_N(t, x) \|_{L^p(\Omega)} \leq C p^{\frac{3}{2}}. \]
A similar argument yields
\[ \| IV_M(t, x) - IV_N(t, x) \|_{L^p(\Omega)} \leq C N^{-\delta} p^{\frac{3}{2}} \] (3.19)
for some \( \delta > 0 \). Then, arguing as in (3.10) with Minkowski’s integral inequality, the estimate (3.8) for \( j = 3 \) follows from (3.16) and (3.19). This completes the proof of Lemma 3.3.

Thanks to Lemma 3.3, we already know that the sequences \( \{Z_{j,N}\}_{N \in \mathbb{N}}, j = 1, 2, 3 \), converge in \( L^p(\Omega; L^q([-T, T]; W^{s_j, r}(\mathbb{T}^3))) \) to some limits \( Z_j \). It turns out that the quantitative properties (3.8) of the convergence allow us to upgrade these convergences to almost sure convergences. See the proof of Proposition 3.2 below. In order to obtain convergence in \( C([-T, T]; W^{s_j, r}(\mathbb{T}^3)) \), however, we need to establish a difference estimate at two different times. The following lemma will be useful in this context.

**Lemma 3.4.** Let \( k \geq 1 \) be an integer. Then, we can write
\[ \prod_{j=1}^{k} g_{n_j}^\ell - \prod_{j=1}^{k} g_{n_j}^\tau = \sum_{\ell} c_{\ell}(t, \tau, n_1, \cdots, n_k) \prod_{j=1}^{k} g_{n_j}^\tau, \] (3.20)
where \( g_{n_j}^\ell \) is either \( g_{n_j}^{\ell} \) or \( h_{n_j}^{\ell} \) and the sum in \( \ell \) runs over all such possibilities. Furthermore, given any \( \delta > 0 \), there exists \( C_\delta > 0 \) such that
\[ |c_{\ell}(t, \tau, n_1, \cdots, n_k)| \leq C_\delta |t - \tau|^\delta \sum_{j=1}^{k} \langle n_j \rangle^\delta. \] (3.21)

**Proof.** From the definition (3.1) of \( g_{n}^\ell \), a typical term in the sum defining the right-hand side of (3.20) is given by
\[ \left( \prod_{j=1}^{k} H_j(t\langle n_j \rangle) - \prod_{j=1}^{k} H_j(\tau\langle n_j \rangle) \right) \prod_{j=1}^{k} g_{n_j}^\tau, \] (3.22)
where \( H_j(t\langle n_j \rangle) = \cos(t\langle n_j \rangle) \) (with \( g_{n_j}^{\ell} = g_{n_j}^{n} \)) or \( \sin(t\langle n_j \rangle) \) (with \( g_{n_j}^{\ell} = h_{n_j}^{n} \)). By the mean value theorem and the boundedness of \( H_j \), we have
\[ \left| \prod_{j=1}^{k} H_j(t\langle n_j \rangle) - \prod_{j=1}^{k} H_j(\tau\langle n_j \rangle) \right| \lesssim |t - \tau| \sum_{j=1}^{k} \langle n_j \rangle. \] (3.23)
We also have the trivial bound
\[ \left| \prod_{j=1}^{k} H_j(t\langle n_j \rangle) - \prod_{j=1}^{k} H_j(\tau\langle n_j \rangle) \right| \leq 2. \] (3.24)
By interpolating (3.23) and (3.24), we conclude that (3.22) satisfies the claimed bound (3.21). This completes the proof of Lemma 3.4. □

In view of Lemma 3.4, a slight modification of the proof of Lemma 3.3 yields the following statement.

**Lemma 3.5.** Let $1 < \alpha \leq \frac{1}{2}$ and $s_j$ satisfies (3.3), $j = 1, 2, 3$. Then, given $2 \leq r \leq \infty$, there exists $\delta > 0$ such that the following estimates hold for $j = 1, 2, 3$:

\[
\| (\nabla)^{s_j} \delta_h Z_{j,N}(t) \|_{L^p(\Omega; L^r(\mathbb{T}^3))} \leq C p^{\frac{1}{2}} |h|^\beta, \quad (3.25)
\]
\[
\| (\nabla)^{s_j} (\delta_h Z_{j,M}(t) - \delta_h Z_{j,N}(t)) \|_{L^p(\Omega; L^r(\mathbb{T}^3))} \leq C N^{-\delta} p^{\frac{1}{2}} |h|^\delta, \quad (3.26)
\]

for any $M \geq N \geq 1$, $t \in [-T,T]$, and $h \in \mathbb{R}$ such that $t + h \in [-T,T]$, where the constant $C$ is independent of $M, N, T, p, t,$ and $h$. Here, $\delta_h$ denotes the difference operator defined by

\[
\delta_h Z_{j,N}(t) = Z_{j,N}(t + h) - Z_{j,N}(t).
\]

In handling the renormalized pieces, we also need the following identity, which follows directly from (3.1):

\[
\left( |g_n^l|^2 - E[|g_n^l|^2] \right) - \left( |g_n^r|^2 - E[|g_n^r|^2] \right) = \left( \cos^2(t\langle n \rangle) - \cos^2(\tau\langle n \rangle) \right) \left( |g_n|\right)^2 - 1) + \left( \sin^2(t\langle n \rangle) - \sin^2(\tau\langle n \rangle) \right) \left( |h_n|\right)^2 - 1) + 2 \left( \cos(t\langle n \rangle) \sin(t\langle n \rangle) - \cos(\tau\langle n \rangle) \sin(\tau\langle n \rangle) \right) \cdot \text{Re}(g_n\overline{g_n}).
\]

The first two terms on the right-hand side can be treated exactly as in the renormalized pieces in the proof of Lemma 3.3 while the last term can be handled without any difficulty.

We conclude this section by presenting the proof of Proposition 3.2.

**Proof of Proposition 3.2.** Fix $2 \leq q < \infty$ and $j = 1, 2,$ or $3$. Passing to the limit $N \to \infty$ in (3.7) of Lemma 3.3 we obtain that the limit $Z_j$ of $Z_{j,N}$ satisfies

\[
\left\| Z_j \right\|_{L^q_t W^{s_j,\infty}_x} \left\|_{L^p(\Omega)} \leq C T^{\frac{1}{2}} p^\frac{1}{2} \right.
\]

for any finite $p \geq 1$. Then, it follows from Chebyshev’s inequality\(^{10}\) that there exists a set $\Omega_{T,\infty}^{(1)}$ of complemental probability smaller than $C \exp(-c/T^{\frac{1}{2}})$ such that

\[
\left\| Z_j \right\|_{L^q_t W^{s_j,\infty}_x} \leq \frac{1}{2} T^{\frac{1}{2q}} \quad (3.27)
\]

for any $\omega \in \Omega_{T,\infty}^{(1)}$. Similarly, given any $N \in \mathbb{N}$, it follows from (3.8) (with $M \to \infty$) that there exists a set $\Omega_{T,N}^{(1)}$ of complemental probability smaller than $C \exp(-cN^{\frac{2\beta}{T}}T^{\frac{1}{2q}})$ such that

\[
\left\| Z_j - Z_{j,N} \right\|_{L^q_t W^{s_j,\infty}_x} \leq \frac{1}{2} T^{\frac{1}{2q}} \quad (3.28)
\]

\(^{10}\)See for example Lemma 4.5 in [12] and the proof of Lemma 3/2.2 in [5].
for any $\omega \in \Omega^{(1)}_{T,N}$. Combining (3.27) and (3.28), we see that (3.5) holds for any $\omega \in \Omega^{(1)}_T$ defined by

$$\Omega^{(1)}_T := \bigcap_{N \in \mathbb{N} \cup \{\infty\}} \Omega^{(1)}_{T,N}$$  \hspace{1cm} (3.29)$$

whose complemental probability is smaller than $C \exp(-c/T^2\delta)$. Lemma 3.3 shows that the sequence $\{Z_{j,N}\}_{N \in \mathbb{N}}$ converges in $L^p(\Omega; L^q([-T, T]; W^{s_j, \infty}(\mathbb{T}^3)))$ to the limit $Z_j$. A slight modification of the proof of Lemma 3.3 shows that, given $t \in \mathbb{R}$, the sequence $\{Z_{j,N}(t)\}_{N \in \mathbb{N}}$ converges to the limit $Z_j(t)$ in $L^p(\Omega; W^{s_j, \infty}(\mathbb{T}^3))$ with the uniform bound:

$$\|Z_{j,N}(t)\|_{L^p(\Omega; W^{s_j, \infty})} \leq C p^{\frac{q}{p}}.$$  

We first upgrade this convergence to almost sure convergence. From (3.8) in Lemma 3.3 and Chebyshev’s inequality, we obtain that

$$P\left(\omega \in \Omega : \|Z_j(t) - Z_{j,N}(t)\|_{W^{s_j, \infty}} \geq \frac{1}{k}\right) \leq e^{-cnN^2\delta k^{-\frac{3}{2}}}$$

for $k \in \mathbb{N}$, where the positive constant $c$ is independent of $k$ and $N$. Noting that the right-hand side is summable in $N \in \mathbb{N}$, we can invoke the Borel-Cantelli lemma to conclude that there exists $\Omega_k$ of full probability such that for each $\omega \in \Omega_k$, there exists $M = M(\omega) \geq 1$ such that for any $N \geq M$, we have

$$\|Z_j(t; \omega) - Z_{j,N}(t; \omega)\|_{W^{s_j, \infty}} < \frac{1}{k}.$$  

Now, by setting $\Sigma = \bigcap_{k=1}^{\infty} \Omega_k$, we see that $P(\Sigma) = 1$ and that, for each $\omega \in \Sigma$, $Z_{j,N}(t)$ converges to $Z_j(t)$ in $W^{s_j, \infty}(\mathbb{T}^3)$. Note that the set $\Sigma$ is dependent on the choice of $t \in \mathbb{R}$.

We now prove that $\{Z_{j,N}\}_{N \in \mathbb{N}}$ converges to $Z_j$ almost surely in $C([-T, T]; W^{s_j, \infty}(\mathbb{T}^3))$. Fix $t \in [-T, T]$ and $h \in \mathbb{R}$ (such that $t + h \in [-T, T]$). From (3.25), (3.26), the almost sure convergence of $Z_{j,N}(t)$ to $Z_j(t)$, and the dominated convergence theorem, we obtain

$$\|\delta_h Z_{j_N}(t)\|_{L^p(\Omega; W^{s_j, \infty})} \leq C p^{\frac{q}{2}}|h|^\delta,$$  \hspace{1cm} (3.30)$$

$$\|\delta_h Z_j(t) - \delta_h Z_{j,N}(t)\|_{L^p(\Omega; W^{s_j, \infty})} \leq C N^{-\delta} p^{\frac{q}{2}}|h|^\delta$$  \hspace{1cm} (3.31)$$

for any $N \geq 1$. By choosing $p \gg 1$ sufficiently large such that $p\delta > 1$, it follows from Kolmogorov’s continuity criterion applied to (3.25) and (3.30) that $Z_{j,N}$, $N \in \mathbb{N}$, and $Z_j$ are almost surely continuous with values in $W^{s_j, \infty}(\mathbb{T}^3)$.

In the following, we only consider $[0, T]$. Let $Y_N = Z_j - Z_{j,N}$ and choose $p \gg 1$ sufficiently large such that $p\delta > 2$. Then, with $\theta \in (0, \delta - \frac{1}{p})$, it follows from Chebyshev’s inequality
and \(3.31\) that
\[
P\left( \sup_{N \in \mathbb{N}} \max_{j=1, \ldots, 2^\ell} N^{\frac{\ell}{2}} \left\| Y_N\left( \frac{j}{2^\ell} T \right) - Y_N\left( \frac{j-1}{2^\ell} T \right) \right\|_{W^{s_j, \infty}} \geq 2^{-\theta \ell} \right)
\]
\[
= P\left( \bigcup_{N \in \mathbb{N}} \bigcup_{j=1}^{2^\ell} \left\{ \left\| Y_N\left( \frac{j}{2^\ell} T \right) - Y_N\left( \frac{j-1}{2^\ell} T \right) \right\|_{W^{s_j, \infty}} \geq N^{-\frac{\delta}{2}} 2^{-\theta \ell} \right\} \right)
\]
\[
\leq \sum_{N=1}^{\infty} \sum_{j=1}^{2^\ell} P\left( \left\| Y_N\left( \frac{j}{2^\ell} T \right) - Y_N\left( \frac{j-1}{2^\ell} T \right) \right\|_{W^{s_j, \infty}} \geq N^{-\frac{\delta}{2}} 2^{-\theta \ell} \right)
\]
\[
\leq \sum_{N=1}^{\infty} \sum_{j=1}^{2^\ell} N^{\frac{\ell}{2}} 2^{\delta \ell} \prod_{j} \left[ \left\| Y_N\left( \frac{j}{2^\ell} T \right) - Y_N\left( \frac{j-1}{2^\ell} T \right) \right\|_{W^{s_j, \infty}} \right]^p
\]
\[
\leq C(p) \cdot 2^{(p(\theta - \delta) + 1) \ell} \sum_{N=1}^{\infty} N^{-\frac{\delta}{2}}
\]
\[
\leq C(p) \cdot 2^{(p(\theta - \delta) + 1) \ell},
\]
where we used the fact that \(p \delta > 2\) in the second to the last step. Note that \(p(\theta - \delta) + 1 < 0\).

Then, summing over \(\ell \in \mathbb{N}\), we obtain
\[
\sum_{\ell=0}^{\infty} P\left( \sup_{N \in \mathbb{N}} \max_{j=1, \ldots, 2^\ell} N^{\frac{\ell}{2}} \left\| Y_N\left( \frac{j}{2^\ell} T \right) - Y_N\left( \frac{j-1}{2^\ell} T \right) \right\|_{W^{s_j, \infty}} \geq 2^{-\theta \ell} \right) < \infty.
\]

Hence, by the Borel-Cantelli lemma, there exists a set \(\tilde{\Sigma} \subset \Omega\) with \(P(\tilde{\Sigma}) = 1\) such that, for each \(\omega \in \tilde{\Sigma}\), we have
\[
\sup_{N \in \mathbb{N}} \max_{j=1, \ldots, 2^\ell} N^{\frac{\ell}{2}} \left\| Y_N\left( \frac{j}{2^\ell} T \right; \omega) - Y_N\left( \frac{j-1}{2^\ell} T \right; \omega) \right\|_{W^{s_j, \infty}} \leq 2^{-\theta \ell}
\]

for all \(\ell \geq L = L(\omega)\). This in particular implies that there exists \(C = C(\omega) > 0\) such that
\[
\max_{j=1, \ldots, 2^\ell} \left\| Y_N\left( \frac{j}{2^\ell} T \right; \omega) - Y_N\left( \frac{j-1}{2^\ell} T \right; \omega) \right\|_{W^{s_j, \infty}} \leq C(\omega) N^{-\frac{\delta}{2}} 2^{-\theta \ell}
\]
(3.32)

for any \(\ell \geq 0\), uniformly in \(N \in \mathbb{N}\).

Fix \(t \in [0, T]\). By expressing \(t\) in the following binary expansion (dilated by \(T\)):
\[
t = T \sum_{j=1}^{\infty} \frac{b_j}{2^j},
\]
where \(b_j \in \{0, 1\}\), we set \(t_\ell = T \sum_{j=1}^{\ell} \frac{b_j}{2^j}\) and \(t_0 = 0\). Then, from \(3.32\) along with the continuity of \(Y_N\) with values in \(W^{s_j, \infty}(\mathbb{T}^d)\), we have
\[
\left\| Y_N(t; \omega) \right\|_{W^{s_j, \infty}} \leq \sum_{\ell=1}^{\infty} \left\| Y_N(t_\ell; \omega) - Y_N(t_{\ell-1}; \omega) \right\|_{W^{s_j, \infty}} + \left\| Y_N(0; \omega) \right\|_{W^{s_j, \infty}}
\]
\[
\leq C(\omega) N^{-\frac{\delta}{2}} \sum_{\ell=1}^{\infty} 2^{-\theta \ell} + \left\| Y_N(0; \omega) \right\|_{W^{s_j, \infty}}
\]
(3.33)
\[
\leq C'(\omega) N^{-\frac{\delta}{2}} + \left\| Y_N(0; \omega) \right\|_{W^{s_j, \infty}}
\]
for each $\omega \in \tilde{\Sigma}$. Note that the right-hand side of (3.33) is independent of $t \in [0, T]$. Hence, by taking a supremum in $t \in [0, T]$, we obtain

$$
\|Z_j(\omega) - Z_{j,N}(\omega)\|_{C([0,T];W^{s_j,\infty})} \leq C'(\omega)N^{-\frac{1}{2}} + \|Y_N(0;\omega)\|_{W^{s_j,\infty}} \rightarrow 0
$$

as $N \rightarrow \infty$. Here, we used the almost sure convergence of $\{Z_{j,N}(0)\}_{N \in \mathbb{N}}$ to $Z_j(0)$ in $W^{s_j,\infty}(\mathbb{T}^3)$. This proves almost sure convergence of $\{Z_{j,N}\}_{N \in \mathbb{N}}$ in $C([-T,T];W^{s_j,\infty}(\mathbb{T}^3))$.

Lastly, it follows from Egoroff’s theorem that, given $T > 0$, there exists $\Omega_T(2)$ of complementary probability smaller than $C\exp(-c/T^\kappa)$ such that the estimate (3.6) holds. Finally, by setting $\Omega_T = \Omega_T^{(1)} \cap \Omega_T^{(2)}$, where $\Omega_T^{(1)}$ is as in (3.29), we see that both (3.5) and (3.6) hold on $\Omega_T$. This completes the proof of Proposition 3.2.

**Remark 3.6.** The restriction $\alpha > 1$ in Proposition 3.2 appears in making sense of the renormalized cubic power $Z_3$ and it reflects the well-known fact that Wick powers of degree $\geq 3$ for the three-dimensional Gaussian free field do not exist. See for example Section 2.7 in [16].

4. On the second order stochastic term $z_{2,N}$

We first study the regularity and convergence properties of $z_{2,N}$ defined in (1.17). For notational convenience, we set

$$
Z_{4,N} := z_{2,N} = -L^{-1}\left((z_{1,N})^3 - 3\sigma_N z_{1,N}\right) = -L^{-1}Z_{3,N}.
$$

As a consequence of Proposition 3.2, we have the following statement.

**Proposition 4.1.** Let $1 < \alpha \leq \frac{3}{2}$ and set

$$
s_4 < 3(\alpha - \frac{3}{2}) + 1.
$$

Then, given any $T > 0$, $Z_{4,N}$ converges almost surely to some limit $Z_4$ in $C([-T,T];W^{s_4,\infty}(\mathbb{T}^3))$ as $N \rightarrow \infty$. Moreover, there exist positive constants $C$, $c$, $\kappa$, $\theta$ such that for every $T > 0$, there exists a set $\Omega_T$ of complementary probability smaller than $C\exp(-c/T^\kappa)$ such that given $\varepsilon > 0$, there exists $N_0 = N_0(T, \varepsilon) \in \mathbb{N}$ such that

$$
\|Z_{4,N}\|_{C([-T,T];W^{s_4,\infty}(\mathbb{T}^3))} \leq T^\theta
$$

and

$$
\|Z_{4,M} - Z_{4,N}\|_{C([-T,T];W^{s_4,\infty}(\mathbb{T}^3))} < \varepsilon
$$

for any $\omega \in \Omega_T$ and any $M \geq N \geq 1$, where we allow $N = \infty$ with the understanding that $Z_{4,\infty} = Z_4$.

**Proof.** Given $s_4$ satisfying (4.2), choose $\varepsilon > 0$ sufficiently small such that

$$
s_4 + 2\varepsilon < 3(\alpha - \frac{3}{2}) + 1.
$$

By Sobolev’s inequality, there exists finite $r \gg 1$ such that

$$
\|Z_{4,N}\|_{C^r T^{s_4,\infty}} \lesssim \|Z_{4,N}\|_{C^r T^{s_4+c,\infty}}.
$$

(4.4)
Furthermore, by Lemma \[\text{(2.3)}\] there exists finite \(q \gg 1\) such that
\[
\|Z_{4,N}\|_{C_T W^4_{x,z}} \leq \|Z_{4,N}\|_{L^1_T W^4_{x,z}} \lesssim \|Z_{4,N}\|_{L^1_T W^4_{x,z}} + \|\partial_t Z_{4,N}\|_{L^1_T W^4_{x,z}},
\]
where we applied Young’s inequality in the second step. From \(\text{(4.1)}\) with \(\text{(2.5)}\), we have
\[
\partial_t Z_{4,N} = -\int_0^t \cos((t - t')(\nabla))Z_{3,N}(t')dt'.
\]
Hence, from \(\text{(4.4)}\), \(\text{(4.5)}\), and \(\text{(4.6)}\) with \(\text{(4.1)}\), we obtain
\[
\|Z_{4,N}\|_{C_T W^4_{x,z}} \lesssim T^{-\frac{1}{q}} \sum_{\beta \in \{-1, 1\}} \|F^\beta_N(t, t')\|_{L^q_{t,t'}([-T,T]^2 W^4_{x,z})},
\]
where \(F^\beta_N\) is given by
\[
F^\beta_N(t, t') = e^{i\beta(t-t')\nabla} Z_{3,N}(t').
\]
Fix \((t, t', x) \in \mathbb{R}^2 \times \mathbb{T}\). Since the propagator \(e^{i\beta(t-t')\nabla}\) does not affect the computation done for \(Z_{3,N}\) in the proof of Lemma \(\text{(3.3)}\), we obtain
\[
\|F^\beta_N(t, t', x)\|_{L^p(\Omega)} \leq C p^{\frac{2}{q}},
\]
uniformly in \((t, t', x) \in \mathbb{R}^2 \times \mathbb{T}\). Therefore, given finite \(p \geq \max(q, r)\), from \(\text{(4.7)}\), Minkowski’s integral inequality, and \(\text{(4.8)}\), we have
\[
\left\|Z_{4,N}\right\|_{L^p(\Omega)} \lesssim T^{-\frac{1}{q}} \sum_{\beta \in \{-1, 1\}} \left\|F^\beta_N(t, t', x)\right\|_{L^q_{t,t'}([-T,T]^2 W^4_{x,z})}
\]
\[
\lesssim p^{\frac{2}{q}} T^{-\frac{1}{q}}
\]
thanks to the regularity restriction \(\text{(4.3)}\). Then, the rest follows from proceeding as in the proof of Proposition \(\text{(3.2)}\) (in addition to Lemma \(\text{(3.4)}\) one should take into account the trivial continuity property in \(t\) of the time integration in the definition of \(Z_{4,N}\)).

We also need to study the following quintic stochastic term:
\[
Z_{5,N} := \{(z_{1,N})^2 - \sigma_N\} z_{2,N}
\]
\[
= -\{(z_{1,N})^2 - \sigma_N\} \cdot \mathcal{L}^{-1}((z_{1,N})^3 - 3\sigma_N z_{1,N}).
\]
We have the following statement.

**Proposition 4.2.** Let \(1 < \alpha \lesssim \frac{3}{2}\) and set
\[
s_5 < \min(5\alpha - \frac{12}{q}, 2(\alpha - \frac{3}{2})).
\]
Then, given any \(T > 0\), \(Z_{5,N}\) converges almost surely to some limit \(Z_5\) in \(C([-T,T]; W^{s_5,\infty}(\mathbb{T}^3))\) as \(N \to \infty\). Moreover, there exist positive constants \(C, c, \kappa, \theta\) such that for every \(T > 0\), there exists a set \(\Omega_T\) of complemental probability smaller than \(C \exp(-c/T^\kappa)\) such that given \(\varepsilon > 0\), there exists \(N_0 = N_0(T, \varepsilon) \in \mathbb{N}\) such that
\[
\|Z_{5,N}\|_{C([-T,T]; W^{s_5,\infty}(\mathbb{T}^3))} \leq T^\theta
\]
and
\[
\|Z_{5,M} - Z_{5,N}\|_{C([-T,T]; W^{s_5,\infty}(\mathbb{T}^3))} < \varepsilon
\]
for any \( \omega \in \Omega_T \) and any \( M \geq N \geq 1 \), where we allow \( N = \infty \) with the understanding that \( Z_{5,\infty} = Z_5 \).

**Remark 4.3.** When \( \alpha \geq \frac{7}{6} \) (which in particular includes the case \( \alpha > \frac{3}{2} \)), the regularity condition (4.10) reduces to \( s_5 < 2(\alpha - \frac{3}{2}) \).

**Proof.** By the paraproduct decomposition (1.30), we have

\[
Z_{5,N} = Z_{2,N} z_{2,N} = Z_{2,N} \otimes z_{2,N} + Z_{2,N} \otimes z_{2,N} + Z_{2,N} \otimes z_{2,N}.
\]

Note that \( 2(\alpha - \frac{3}{2}) \leq \min \left( 0, 3(\alpha - \frac{3}{2}) + 1 \right) \) for \( \alpha \in (1, \frac{3}{2}] \). Then, from Lemma 2.1, we have

\[
\| Z_{2,N} \otimes z_{2,N}(t) \|_{W^{s_5, \infty}} \lesssim \| Z_{2,N}(t) \|_{W^{2(\alpha - \frac{3}{2}) - \varepsilon, \infty}} \| z_{2,N}(t) \|_{W^{3(\alpha - \frac{3}{2}) + 1 - \varepsilon, \infty}}
\]

for small \( \varepsilon > 0 \), provided that \( s_5 \) satisfies

\[
s_5 < 2(\alpha - \frac{3}{2}) + 3(\alpha - \frac{3}{2}) + 1 = 5\alpha - \frac{13}{2}. \tag{4.11}
\]

Similarly, for \( s_5 \) satisfying (4.11), Lemma 2.1 yields

\[
\| Z_{2,N} \otimes z_{2,N}(t) \|_{W^{s_5, \infty}} \lesssim \| Z_{2,N}(t) \|_{W^{2(\alpha - \frac{3}{2}) - \varepsilon, \infty}} \| z_{2,N}(t) \|_{W^{3(\alpha - \frac{3}{2}) + 1 - \varepsilon, \infty}}
\]

for small \( \varepsilon > 0 \), provided that \( 3(\alpha - \frac{3}{2}) + 1 - \varepsilon < 0 \) namely, \( \alpha \leq \frac{7}{6} \). On the other hand, when \( \alpha > \frac{7}{6} \), we see from Proposition 4.1 that \( z_{2,N} \) has a spatial positive regularity (for each fixed \( t \)). In this case, we have

\[
\| Z_{2,N} \otimes z_{2,N}(t) \|_{W^{s_5, \infty}} \lesssim \| Z_{2,N}(t) \|_{W^{2(\alpha - \frac{3}{2}) - \varepsilon, \infty}} \| z_{2,N}(t) \|_{L^\infty}
\]

as long as

\[
s_5 < 2(\alpha - \frac{3}{2}). \tag{4.12}
\]

Note that the condition (4.12) is stronger than (4.11) when \( \alpha > \frac{7}{6} \).

It remains to study the resonant product \( z_{2,N} \otimes Z_{2,N} \). When \( \alpha > \frac{13}{10} \), we have

\[
2(\alpha - \frac{3}{2}) + 3(\alpha - \frac{3}{2}) + 1 = 5\alpha - \frac{13}{2} > 0.
\]

and thus Lemma 2.1 yields

\[
\| Z_{2,N} \otimes z_{2,N}(t) \|_{W^{s_5, \infty}} \lesssim \| Z_{2,N}(t) \|_{W^{2(\alpha - \frac{3}{2}) - \varepsilon, \infty}} \| z_{2,N}(t) \|_{W^{3(\alpha - \frac{3}{2}) + 1 - \varepsilon, \infty}}
\]

for \( s_5 \) satisfying (4.11). Next, we consider the case \( 1 < \alpha \leq \frac{13}{10} \). Using the independence of \( \{ g_t \}_{n \in A} \), we have

\[
\sup_{N \in \mathbb{N}} \sup_{(t,x) \in \mathbb{R} \times \mathbb{T}^3} \mathbb{E} \left[ \langle \nabla \rangle^{s_5} (Z_{2,N} \otimes z_{2,N})(t, x) \right] \lesssim \sum_{n \in \mathbb{Z}^3} \langle n \rangle^{2s_5} \sum_{n = m_1 + m_2 \atop |m_1| \sim |m_2|} A(m_1) B(m_2),
\]

where \( A(m_1) \) and \( B(m_2) \) are given by

\[
A(m_1) = \sum_{m \in \mathbb{Z}^3} \frac{1}{\langle m \rangle^{2\alpha} \langle m_1 - m \rangle^{2\alpha}}.
\]
and

\[ B(m_2) = \frac{1}{\langle m_2 \rangle^2} \sum_{(n_1, n_2) \in \mathbb{Z}^2} \frac{1}{(n_1)^{2\alpha} (n_2)^{2\alpha} (m_2 - n_1 - n_2)^{2\alpha}} \]

In the following, we only consider the case \( \alpha < \frac{3}{2} \). We clearly have the bound

\[ A(m_1) \lesssim \frac{1}{\langle m_1 \rangle^{4\alpha - 3}}, \]

provided that \( \alpha > \frac{4}{3} \). Similarly, by Lemma 2.5, we have

\[ B(m_2) \lesssim \frac{1}{\langle m_2 \rangle^{6\alpha - 4}}, \]

provided that \( \alpha > 1 \). Hence, we obtain

\[ \sum_{n=m_1+m_2 \atop |m_1| \sim |m_2|} A(m_1) B(m_2) \lesssim \sum_{m_1 \in \mathbb{Z}^3 \atop |m_1| \sim |n-m_1|} \frac{1}{\langle m_1 \rangle^{4\alpha - 3} \langle n-m_1 \rangle^{6\alpha - 4}} \lesssim \frac{1}{\langle n \rangle^{10\alpha - 10}}, \]

where we crucially used the resonant restriction \( |m_1| \sim |n-m_1| \). Therefore, we obtain

\[ \sup_{N \in \mathbb{N}} \sup_{(t, x) \in [0, T] \times T^d} \mathbb{E} \left[ |\nabla|^s_5 (Z_{2,N} \otimes z_{2,N})(t, x)|^2 \right] \lesssim \sum_{n \in \mathbb{Z}^3} \frac{1}{\langle n \rangle^{-2s_5 + 10\alpha - 10}}, \]

where the last sum is convergent, provided that \( s_5 \) satisfies (4.11). With this bound in hand, we can proceed as in the proof of Proposition 3.2 (with Lemmas 3.3 and 3.5). This completes the proof of Proposition 4.2.

**Remark 4.4.** (i) When \( \alpha > \frac{13}{10} \), we made sense of the resonant product \( Z_{2,N} \otimes z_{2,N} \) in a deterministic manner. Namely, we only used the almost sure regularity properties of \( Z_{2,N} \) and \( z_{2,N} \) but did not use the random structure of these terms in making sense of their resonant product. On the other hand, when \( 1 < \alpha \leq \frac{13}{10} \), the sum of the regularities of \( Z_{2,N} \) and \( z_{2,N} \) is negative and thus their resonant product does not make sense in a deterministic manner. This requires us to make sense of the resonant product \( Z_{2,N} \otimes z_{2,N} \) via a probabilistic argument. Hence, when \( 1 < \alpha \leq \frac{13}{10} \), we need to view the limit \( Z_{3}^\oplus = Z_{2,\infty} \otimes z_{2,\infty} \) as part of a predefined enhanced data set, leading to a different interpretation of the equation for \( w = u - z_1 - z_2 \). See Subsection 1.3 for a further discussion. Lastly, we point out that the resulting regularity restriction (4.11) holds for both cases \( \alpha > \frac{13}{10} \) and \( 1 < \alpha \leq \frac{13}{10} \).

(ii) When \( \alpha = 1 \), there is a logarithmically divergent contribution in taking a limit of \( Z_{5,N} \) as \( N \to \infty \). In this case, we need to introduce another renormalization, eliminating a quartic singularity. For a related argument in the parabolic setting, see [15].

5. Proof of Theorem 3

5.1. Setup. Recall that \( u_N = z_{1,N} + z_{2,N} + w_N \), where \( w_N \) solves the equation (1.19). In Sections 3 and 4 we already established the necessary regularity and convergence properties of the sequences \( \{z_{j,N} \}_{N \in \mathbb{N}}, j = 1, 2 \). It remains to establish the convergence of the sequence \( \{w_N \}_{N \in \mathbb{N}} \). This will be done by (i) first establishing multilinear estimates via a purely deterministic method and then (ii) applying the regularity and convergence properties of the relevant stochastic terms from Sections 3 and 4.
With (3.3) and (4.9), we can write the equation (1.19) as
\[
\begin{aligned}
 & L w_N + F_0 + F_1(w_N) + F_2(w_N) + F_3(w_N) = 0 \\
 & \left( w_N, \partial_t w_N \right)_{t=0} = (0,0),
\end{aligned}
\]
where the source term\(^{11}\) is given by
\[
F_0 = 3Z_{5,N} + 3z_{1,N}(z_{2,N})^2 + (z_{2,N})^3,
\]
the linear term in \( w_N \) is given by
\[
F_1(w_N) = 3Z_{2,N}w_N + 6z_{1,N}z_{2,N}w_N + 3(z_{2,N})^2w_N,
\]
and the nonlinear terms in \( w_N \) are as follows:
\[
F_2(w_N) = 3z_{1,N}(w_N)^2 + 3z_{2,N}w_N^2 \quad \text{and} \quad F_3(w_N) = w_N^3.
\]

In the following, we study the Duhamel formulation for \( w_N \):
\[
w_N = L^{-1}(F_0) + L^{-1}(F_1(w_N)) + L^{-1}(F_2(w_N)) + L^{-1}(F_3(w_N)). \tag{5.1}
\]

In the next three subsections, we first establish estimates for each individual term in the \( X_T \)-norm defined in (2.8). In Subsection 5.5, we then combine these estimates with the regularity and convergence properties of the relevant stochastic terms from Sections 3 and 4 and prove almost sure convergence of the sequence \( \{w_N\}_{N \in \mathbb{N}} \). In the following, we fix \( 0 < T \leq 1 \).

5.2. On the nonlinear terms in \( w_N \). By the Strichartz estimate (2.7), we have
\[
\| L^{-1}(F_3(w_N)) \|_{X_T} \lesssim \| w_N^3 \|_{L^4_{t,x}} \leq \| w_N \|_{X_T}^3. \tag{5.2}
\]

We now turn to the analysis of \( L^{-1}(F_2(w_N)) \). By (2.7), we have
\[
\| L^{-1}(F_2(w_N)) \|_{X_T} \lesssim \| \langle \nabla \rangle^{-\frac{1}{2}} (z_{1,N}w_N^2 + z_{2,N}w_N^2) \|_{L^1_T L^2_x}. \tag{5.3}
\]

In the following, we first establish an estimate for fixed \( t \in [-T,T] \). Let \( \sigma_1 > 0 \). By Sobolev’s inequality,
\[
\| \langle \nabla \rangle^{-\frac{1}{2}} (z_{1,N}w_N^2)(t) \|_{L^2} \lesssim \| \langle \nabla \rangle^{-\sigma_1} (z_{1,N}w_N^2)(t) \|_{L^r},
\]
provided that
\[
\frac{1}{2} - \sigma_1 \geq \frac{3}{r} - \frac{3}{2}. \tag{5.4}
\]

By Lemma 2.2(ii), we have
\[
\| \langle \nabla \rangle^{-\sigma_1} (z_{1,N}w_N^2)(t) \|_{L^r} \lesssim \| \langle \nabla \rangle^{-\sigma_1} z_{1,N}(t) \|_{L^p} \| \langle \nabla \rangle^{\sigma_1} (w_N^2)(t) \|_{L^q},
\]
provided that \( 0 \leq \sigma_1 \leq 1 \) and
\[
\sigma_1 \geq \frac{3}{p} + \frac{3}{q} - \frac{3}{r}. \tag{5.5}
\]

\(^{11}\)Namely, the purely stochastic terms independent of the unknown \( w_N \).
In the following, we will choose \( p \gg 1 \) such that \( \sigma_1 > \frac{3}{q} - \frac{3}{r} \) guarantees (5.5). By Lemma 2.2(i) and Sobolev’s inequality, we have
\[
\| \langle \nabla \rangle^{\alpha_1} (w_N^2) (t) \|_{L^2} \lesssim \| \langle \nabla \rangle^{\alpha_1} w_N(t) \|_{L^{q_1}} \| w_N(t) \|_{L^4} \\
\lesssim \| \langle \nabla \rangle^{\frac{3}{2}} w_N \|_{L^2} \| w_N(t) \|_{L^4},
\]
provided that
\[
\frac{1}{q} = \frac{1}{4} + \frac{1}{q_1} \quad \text{and} \quad \frac{1}{2} - \sigma_1 \geq \frac{3}{2} - \frac{3}{q_1}.
\tag{5.6}
\]

In summary, if the conditions (5.4), (5.5), and (5.6) are satisfied, then we obtain the estimate
\[
\| \langle \nabla \rangle^{-\frac{3}{2}} z_{1,N} w_N^2 (t) \|_{L^2} \lesssim \| \langle \nabla \rangle^{-\alpha_1} z_{1,N} (t) \|_{L^p} \| \langle \nabla \rangle^{\frac{3}{2}} (w_N(t)) \|_{L^2} \| w_N(t) \|_{L^4}.
\tag{5.7}
\]
Let us now show that we may ensure (5.4), (5.5), and (5.6). Since \( p \gg 1 \), it suffices to ensure that
\[
\sigma_1 > \frac{3}{q} - \frac{3}{r} = \frac{3}{4} + \frac{3}{q_1} - \frac{3}{4} r \geq \frac{3}{4} + \frac{3}{2} - \left( \frac{1}{2} - \sigma_1 \right) - \frac{3}{r} \geq 2\sigma_1 - \frac{1}{4}.
\]
This shows that we can ensure (5.5) and (5.6) if \( \sigma_1 < \frac{1}{4} \). In this case, by (5.7), we arrive at the bound:
\[
\| \langle \nabla \rangle^{-\frac{3}{2}} (z_{1,N} w_N^2) \|_{L^1_t L^2_x} \lesssim \| \langle \nabla \rangle^{-\alpha_1} z_{1,N} \|_{L^p_t L^2_x} \| \langle \nabla \rangle^{\frac{3}{2}} w_N \|_{L^\infty_t L^2_x} \| w_N \|_{L^4_t L^4_x}.
\tag{5.8}
\]
Therefore, from (5.3) and (5.8) with the definition (2.3) of the \( X_T \)-norm, we obtain
\[
\| L^{-1} (F_2(w_N)) \|_{X_T} \lesssim T^\frac{4}{q} \left( \| z_{1,N} \|_{L^2_t W^{2,1}_x} + \| z_{2,N} \|_{L^2_t W^{2,1}_x} \right) \| w_N \|_{X_T}^2,
\tag{5.9}
\]
provided that
\[
s_1 = -\sigma_1 > -\frac{1}{4}.
\tag{5.10}
\]

5.3. On the linear terms in \( w_N \). Let us next turn to the analysis of the terms linear in \( w_N \). By the Strichartz estimate (2.7), we have
\[
\| L^{-1} (F_1(w_N)) \|_{X_T} \lesssim \| \langle \nabla \rangle^{-\frac{3}{2}} (Z_{2,N} w_N) \|_{L^1_t L^2_x} + \| \langle \nabla \rangle^{-\frac{3}{2}} (z_{1,N} z_{2,N} w_N) \|_{L^1_t L^2_x} + \| (z_{2,N})^2 w_N \|_{L^4_t L^4_x}.
\tag{5.11}
\]

We now evaluate each contribution on the right-hand side of (5.11). By Hölder’s inequality, we have
\[
\| (z_{2,N})^2 w_N \|_{L^4_t L^4_x} \leq T^\frac{1}{4} \| z_{2,N} \|_{L^2_t L^4_x} \| w_N \|_{L^4_t L^4_x} \leq T^\frac{1}{4} \| z_{2,N} \|_{L^2_t W^{2,1}_x} \| w_N \|_{X_T},
\tag{5.12}
\]
provided that
\[
s_4 \geq 0.
\tag{5.13}
\]

By Lemma 2.2(ii), we have
\[
\| \langle \nabla \rangle^{-\frac{3}{2}} (Z_{2,N} w_N) \|_{L^1_t L^2_x} \lesssim T^\frac{1}{4} \| \langle \nabla \rangle^{-\frac{3}{2}} Z_{2,N} \|_{L^2_t L^4_x} \| \langle \nabla \rangle^{\frac{3}{2}} w_N \|_{L^\infty_x L^2_x} \\
\lesssim T^\frac{1}{2} \| Z_{2,N} \|_{L^2_t W^{2,1}_x} \| w_N \|_{X_T},
\tag{5.14}
\]

Let\( s_2 \geq -\frac{1}{2} \) \( (5.15) \).

Finally, by applying Lemma 2.2 (ii) twice, we obtain
\[
\|
(\nabla)^{-\frac{1}{2}}(z_{1,N}z_{2,N}w_N)\|_{L^4_tL^4_x} \lesssim \|
(\nabla)^{s_1}(z_{1,N}z_{2,N})\|_{L^4_tL^4_x} \|
(\nabla)^{\frac{1}{2}}w_N\|_{L^\infty_tL^2_x} \\
\lesssim T^{\frac{1}{4}}\|z_{1,N}\|_{L^4_tW^{s_1,\infty}_x}\|z_{2,N}\|_{L^4_tW^{s_4,\infty}_x}\|w_N\|_{X_T},
\]
provided that
\[
\max\left(-\frac{1}{2}, -s_4\right) \leq s_1 \leq 0.
\]
Therefore, putting (5.11), (5.12), (5.14), and (5.16), we obtain
\[
\|L^{-1}(F_1(w_N))\|_{X_T} \lesssim T^\theta \left\{ \|z_{1,N}\|_{L^4_tW^{s_2,\infty}_x} + \|z_{1,N}\|_{L^4_tW^{s_1,\infty}_x}\|z_{2,N}\|_{L^4_tW^{s_4,\infty}_x} \\
+ \|z_{2,N}\|_{L^4_tW^{s_4,\infty}_x}\|z_{2,N}\|_{X_T} \right\}
\]
for some \( \theta > 0 \) and \( s_1, s_2, \) and \( s_4 \) satisfying (5.13), (5.15), and (5.17).

5.4. On the source terms. We now estimate the contributions from the source terms.

Let \( s_1 \) and \( s_4 \) satisfy (5.17). Then, by Lemma 2.2 (i) with Hölder’s inequality followed by Lemma 2.2 (i), we have
\[
\|
(\nabla)^{-\frac{1}{2}}(z_{1,N}(z_{2,N})^2)\|_{L^4_tL^4_x} \leq \|
(\nabla)^{s_1}(z_{1,N}(z_{2,N})^2)\|_{L^4_tL^4_x} \\
\lesssim \|
(\nabla)^{s_1}z_{1,N}\|_{L^4_tL^4_x} \|
(\nabla)^{s_4}(z_{2,N}^2)\|_{L^4_tL^4_x} \\
\lesssim T^{\frac{1}{4}}\|
(\nabla)^{s_1}z_{1,N}\|_{L^4_tL^4_x} \|
(\nabla)^{s_4}z_{2,N}\|_{L^4_tL^4_x} \|
(\nabla)^{s_4}z_{2,N}\|_{X_T}.
\]

Hence, from the Strichartz estimate (2.7) and (5.19), we obtain
\[
\|L^{-1}(F_0)\|_{X_T} \lesssim \|
(\nabla)^{-\frac{1}{2}}(z_{1,N}(z_{2,N})^2)\|_{L^4_tL^4_x} + \|
(\nabla)^{-\frac{1}{2}}(z_{1,N}(z_{2,N})^2)\|_{L^4_tL^4_x} + \|z_{2,N}\|_{L^4_tL^4_x}^3 \\
\lesssim T^\theta \left\{ \|z_{1,N}\|_{L^4_tW^{s_2,\infty}_x} + \|z_{1,N}\|_{L^4_tW^{s_1,\infty}_x}\|z_{2,N}\|_{L^4_tW^{s_4,\infty}_x}^2 \\
+ \|z_{2,N}\|_{L^4_tW^{s_4,\infty}_x}\|z_{2,N}\|_{X_T} \right\}
\]
for some \( \theta > 0 \), provided that \( s_1 \) and \( s_4 \) satisfy (5.17) and that \( s_5 \) satisfies
\[
s_5 \geq -\frac{1}{2}.
\]

5.5. End of the proof. Let \( s_1, s_2, s_4, \) and \( s_5 \) satisfy (5.10), (5.13), (5.15), (5.17), and (5.21). Then, from (5.11), (5.12), (5.9), (5.18), and (5.20), we have
\[
\|w_N\|_{X_T} \leq C T^\theta A_N^{(1)} + C T^\theta A_N^{(2)} \|w_N\|_{X_T} \\
+ C T^\theta \left( \sum_{j=1}^2 \|z_{j,N}\|_{L^4_tW^{s_{j,\infty}}_x} \right) \|w_N\|_{X_T} + C \|w_N\|_{X_T},
\]
where $A^{(1)}_N$ and $A^{(2)}_N$ are defined by

\[
A^{(1)}_N = \|Z_{5,N}\|_{L^2_t W^{s_5,\infty}_x} + \|Z_{1,N}\|_{L^2_t W^{s_1,\infty}_x} \|Z_{2,N}\|_{L^2_t W^{s_2,\infty}_x} + \|Z_{2,N}\|_{L^2_t W^{s_2,\infty}_x} + \|Z_{2,N}\|_{L^2_t W^{s_2,\infty}_x},
\]

\[
A^{(2)}_N = \|Z_{2,N}\|_{L^2_t W^{s_2,\infty}_x} + \|Z_{1,N}\|_{L^2_t W^{s_2,\infty}_x} \|Z_{2,N}\|_{L^2_t W^{s_2,\infty}_x} + \|Z_{2,N}\|_{L^2_t W^{s_2,\infty}_x}.
\]

Suppose that

\[
R(T) := \sup_{N \in \mathbb{N}} \max \left( \|z_{1,N}\|_{L^2_t W^{s_1,\infty}_x}, \|z_{2,N}\|_{L^2_t W^{s_2,\infty}_x}, \|Z_{5,N}\|_{L^2_t W^{s_5,\infty}_x}, \|Z_{2,N}\|_{L^2_t W^{s_2,\infty}_x} \right) \leq T^{\theta_0}
\]

for some $\theta_0 > 0$. Then, it follows from a standard continuity argument that there exists $T_0 > 0$ such that

\[
\|w_N\|_{X_T} \leq C(R) T^\theta
\]

for any $0 < T \leq T_0$, uniformly in $N \in \mathbb{N}$. Here, we used the fact that $(w, \partial_t w)|_{t=0} = (0, 0)$.

Let $M \geq N \geq 1$. Note that $F_j$, $j = 0, 1, 2, 3$, are multilinear in $w_N$ and the stochastic terms $z_{1,N}$, $z_{2,N}$, $Z_{2,N}$, and $Z_{5,N}$. Then, by proceeding as in Subsections 5.2, 5.3, and 5.4, we also obtain the following difference estimate:

\[
\|w_M - w_N\|_{X_T} \leq C(T^{\theta} B^{(1)}_{M,N} + C(T^{\theta} B^{(2)}_{M,N}) \|w_N\|_{X_T} + C(T^{\theta} A^{(2)}_N \|w_M - w_N\|_{X_T})
\]

\[
+ C\left( \sum_{j=1}^{2} \|z_{j,M} - z_{j,N}\|_{L^2_t W^{s_j,\infty}_x} \right) \|w_M\|_{X_T} + \|w_N\|_{X_T} \|w_M - w_N\|_{X_T}
\]

\[
+ C\left( \|w_M\|_{X_T}^2 + \|w_N\|_{X_T}^2 \right) \|w_M - w_N\|_{X_T}
\]

(5.24)

where $B^{(1)}_{M,N}$ and $B^{(2)}_{M,N}$ are defined by

\[
B^{(1)}_{M,N} = \|Z_{5,M} - Z_{5,N}\|_{L^2_t W^{s_5,\infty}_x} + \|Z_{1,M} - z_{1,N}\|_{L^2_t W^{s_1,\infty}_x} \|Z_{2,M}\|_{L^2_t W^{s_2,\infty}_x} + \|Z_{2,M}\|_{L^2_t W^{s_2,\infty}_x} + \|Z_{2,M}\|_{L^2_t W^{s_2,\infty}_x} + \|Z_{2,N}\|_{L^2_t W^{s_2,\infty}_x} + \|Z_{2,N}\|_{L^2_t W^{s_2,\infty}_x}.
\]

In addition to the assumption (5.23), we now suppose that as $N \to \infty$, $z_{1,N}$, $z_{2,N}$, $Z_{2,N}$, and $Z_{5,N}$ converge to the limits $Z_{1}$, $Z_{2}$, $Z_{2}$, and $Z_{4}$ in $C([-T, T]; W^{s_8,\infty}(\mathbb{T}^3))$ for $s = s_1, s_4, s_2$, and $s_5$, respectively. Then, from (5.24), we obtain

\[
\|w_M - w_N\|_{X_T} \leq C(R) T^\theta \|w_M - w_N\|_{X_T} + o_{M,N \to \infty}(1)
\]

Then, by possibly making $T_0 > 0$ smaller, we conclude that

\[
\|w_M - w_N\|_{X_T} \to 0
\]
for any $0 < T \leq T_0$ as $M, N \to \infty$. This implies that $w_N$ converges to some $w$ in $X_T$ as $N \to \infty$. Recalling the decomposition $u_N = z_1,N + z_2,N + w_N$, we conclude that $u_N$ converges to $u = z_1 + z_2 + w$ in $C([-T, T]; H^{s_1}(T^3))$ as $N \to \infty$.

It remains to check that the assumption (5.23) and the assumption on the convergence of $z_{1,N}, z_{2,N}, z_2, N$, and $Z_{5,N}$ hold true with large probability. By choosing $s_1 = \alpha - \frac{3}{2} - \varepsilon$, $s_2 = 2(\alpha - \frac{3}{2}) - \varepsilon$, $s_4 = 3(\alpha - \frac{3}{2}) + 1 - \varepsilon$, and $s_5 = 2(\alpha - \frac{3}{2}) - \varepsilon$ for some small $\varepsilon > 0$, it is easy to see that the conditions (5.10), (5.13), (5.15), (5.17), and (5.21) are satisfied for $\frac{5}{2} < \alpha \leq \frac{3}{2}$. (Note that the restriction $\alpha > \frac{3}{4}$ appears in (5.10), (5.15), (5.17), and (5.21).) Therefore, it follows from Proposition 3.2, 4.1, and 4.2 that there exists a set $\Omega$ of probability smaller than $C$ such that the conditions (5.10), (5.13), (5.15), (5.17), and (5.21) are satisfied for $\frac{5}{2} < \alpha \leq \frac{3}{2}$.

6. On the triviality of the limiting dynamics without renormalization

6.1. Reformulation of the problem. Fix $1 \leq \alpha \leq \frac{3}{2}$ and a pair $(w_0, w_1) \in H^3(T^3)$. Let $u_N$ be the solution to the (un-renormalized) NLW (1.1) with the following initial data:

$$(u_N, \partial_t u_N)|_{t=0} = (w_0, w_1) + (\bar{u}_{0,N}^\omega, \bar{u}_{1,N}^\omega),$$

(6.1)

where the random initial data $(\bar{u}_{0,N}^\omega, \bar{u}_{1,N}^\omega)$ is given by (1.31) with $C_N > 0$ implicitly defined as in (1.32). In this section, we present the proof of Theorem 3 by reformulating the Cauchy problem for $u_N$ as

$$\begin{cases}
L_N u_N + u_N^3 - C_N u_N = 0 \\
(u_N, \partial_t u_N)|_{t=0} = (w_0, w_1) + (\bar{u}_{0,N}^\omega, \bar{u}_{1,N}^\omega),
\end{cases}$$

(6.2)

where $L_N = \partial_t^2 - \Delta + C_N$ as in (1.32).

Since $C_N$ in (1.37) is implicitly defined, we first need to study the asymptotic behavior of $C_N$ as $N \to \infty$.

Lemma 6.1. Let $1 \leq \alpha \leq \frac{3}{2}$. Then, for each $N \in \mathbb{N}$, there exists a unique number $C_N \geq 1$ satisfying the equation (1.37). Moreover, we have

$$C_N = 3\sigma_N + R_N$$

(6.3)

for all sufficiently large $N \gg 1$, where $\sigma_N = \sum_{|n| \leq N} \langle n \rangle^{-2\alpha}$ is as in (6.2) and the error term $R_N$ satisfies

$$|R_N| \sim \begin{cases}
\log \log N, & \text{for } \alpha = \frac{3}{2}, \\
N^{\frac{3}{4}(3-2\alpha)^2}, & \text{for } 1 \leq \alpha < \frac{3}{2}.
\end{cases}$$

In particular, we have $R_N = o(\sigma_N)$ as $N \to \infty$.

Proof. Let $C_N$ be as in (1.37). As $C_N$ increases from 0 to $\infty$, the right-hand side of (1.37) decreases from $\infty$ to 0. Hence, for each $N \in \mathbb{N}$, there exists a unique solution $C_N > 0$ to (1.37).

Suppose that $C_N < 1$ for some $N \in \mathbb{N}$. Then, considering the contribution from $n = 0$ on the right-hand side of (1.37), we obtain $C_N \geq 3$, leading to a contradiction. Hence, we must have $C_N \geq 1$ for any $N \in \mathbb{N}$. 

We first consider the case $1 \leq \alpha < \frac{3}{2}$. Since $C_N \geq 1$, it follows from (1.37) that $C_N \lesssim N^{3-2\alpha}$. Using this upper bound on $C_N$, we estimate the contribution from $|n| \sim N$:

$$C_N \gtrsim \sum_{|n| \leq N} \frac{1}{(N^{3-2\alpha} + |n|^2)(n)^{2(\alpha-1)}} \gtrsim \sum_{|n| \sim N} \frac{1}{(n)^{2\alpha}} \sim N^{3-2\alpha},$$

where we used the assumption $\alpha \geq 1$ in the second step. This shows that $C_N \sim N^{3-2\alpha}$.

Using this asymptotic behavior with (3.2), we then obtain (6.3) with the error term $R_N$ given by

$$R_N = 3 \sum_{|n| \leq N} \frac{1}{(n)^{2\alpha}} \left( \frac{1}{C_N + |n|^2} - \frac{1}{|n|^2} \right), \quad \text{(6.4)}$$

By separately estimating the contributions from $\{|n| \ll N^{\frac{3}{2} - \alpha}\}$ and $\{N^{\frac{3}{2} - \alpha} \leq |n| \leq N\}$, we have

$$|R_N| = 3 \sum_{|n| \leq N} \frac{1}{(n)^{2\alpha}} \frac{C_N - 1}{(C_N + |n|^2)}$$

$$\sim N^{(\frac{3}{2} - \alpha)(3-2\alpha)}.$$

Next, we consider the case $\alpha = \frac{3}{2}$. Proceeding as above, we immediately see that $C_N \sim \log N$. The contribution to $R_N$ in (6.4) from $\{|n| \gtrsim \sqrt{\log N}\}$ is $O(1)$, while the contribution to $R_N$ in (6.3) from $\{|n| \ll \sqrt{\log N}\}$ is $O(\log \log N)$. This completes the proof of Lemma 6.1.

\[\square\]

6.2. On the Strichartz estimates with a parameter. In order to study the equation (6.2), we review the relevant Strichartz estimates for the Klein-Gordon operator with a general mass. Given $a \geq 1$, with a slight abuse of notation, define $\mathcal{L}_a$ by

$$\mathcal{L}_a := \partial_t^2 - \Delta + a.$$ 

Let $\mathcal{L}_a^{-1}$ be the Duhamel integral operator given by

$$\mathcal{L}_a^{-1} F(t) = \int_0^t \frac{\sin((t-t')\sqrt{a-\Delta})}{\sqrt{a-\Delta}} F(t') dt'.$$

Namely, $u := \mathcal{L}_a^{-1}(F)$ is the solution to the following nonhomogeneous linear equation:

$$\begin{cases}
\mathcal{L}_a u = F \\
(u, \partial_t u)|_{t=0} = (0, 0).
\end{cases}$$

Then, by making systematic modifications of the proof of Lemma 2.4 on $\mathbb{R}^3$ (see [13]) and applying the finite speed of propagation, we see that the same non-homogeneous Strichartz estimate as (2.7) holds, uniformly in $a \geq 1$:

$$\|\mathcal{L}_a^{-1}(F)\|_{X_T} \lesssim \min \left( \|F\|_{L^1([-T,T]; H^{-\frac{1}{2}}(\mathbb{T}^3))}, \|F\|_{L^4([-T,T] \times \mathbb{T}^3)} \right)$$

(6.5)

for any $0 < T \leq 1$, where the $X_T$-norm is defined in (2.8).

We also record the following lemma on the linear solution associated with $\mathcal{L}_a$, $a \geq 1$. 
Lemma 6.2. Given \( a \geq 1 \), define \( S_a(t) \) by
\[
S_a(t)(w_0, w_1) = \cos(t \sqrt{a - \Delta})w_0 + \frac{\sin(t \sqrt{a - \Delta})}{\sqrt{a - \Delta}}w_1.
\]
Then, there exists \( C > 0 \) such that
\[
\|S_a(t)(w_0, w_1)\|_{\mathcal{H}^1_{\frac{a}{2}}} \leq C\|(w_0, w_1)\|_{\mathcal{H}^1_{\frac{a}{2}}}
\]  
(6.6)
for any \((w_0, w_1) \in \mathcal{H}^1_{\frac{a}{2}}(\mathbb{T}^3)\) and \( 0 < T \leq 1 \), uniformly in \( a \geq 1 \). Moreover, \( S_a(t)(w_0, w_1) \) tends to 0 in the space-time distributional sense as \( a \to \infty \).

Proof. The estimate (6.6) follows easily from Hölder’s inequality in \( t \) and Sobolev’s inequality in \( x \) along with the boundedness of \( S_a(t) \) in \( \mathcal{H}^1_{\frac{a}{2}}(\mathbb{T}^3) \). As for the second claim, we only consider \( e^{it\sqrt{a-\Delta}}f \) for \( f \in L^2(\mathbb{T}^3) \). Note that, for each fixed \( n \in \mathbb{Z}^3 \), \( \sqrt{a} + |n|^2 - \sqrt{a} \) tends to 0 as \( a \to \infty \). Then, by the dominated convergence theorem (for the summation in \( n \in \mathbb{Z}^3 \) and the Riemann-Lebesgue lemma (for the integration in \( t \)), we have
\[
\lim_{a \to \infty} \left( \int e^{it\sqrt{a-\Delta}}f \phi(t, x) \right)(x) \, dx \, dt
\]
\[
= \lim_{a \to \infty} \int e^{it\sqrt{a}} \left( \sum_{n \in \mathbb{Z}^3} e^{it(\sqrt{a} + |n|^2 - \sqrt{a})} \hat{f}(n) \phi(t, n) \right) dt
\]
\[
= \lim_{a \to \infty} \int e^{it\sqrt{a}}(f, \phi(t))_{L^2_x} \, dt
\]
\[
= 0
\]
for any test function \( \phi \in C^\infty(\mathbb{R} \times \mathbb{T}^3) \) with a compact support in \( t \).

Remark 6.3. Let \( a \geq 1 \). Then, we have the following homogeneous Strichartz estimate:
\[
\|S_a(t)(w_0, w_1)\|_{L^q([0,1];L^r(\mathbb{T}^3))} \leq C\|(w_0, w_1)\|_{\mathcal{H}^1_{\frac{a}{2}} \times \mathcal{H}^1_{\frac{a}{2}-1}(\mathbb{T}^3)}
\]  
(6.7)
for \( 2 < q \leq \infty \) and \( \frac{1}{q} + \frac{1}{r} = \frac{1}{2} \), where the \( H^s_a \)-norm is defined by
\[
\|f\|_{H^s_a} = \left( \sum_{n \in \mathbb{Z}^3} (a + |n|^2)^s |\hat{f}(n)|^2 \right)^{\frac{1}{2}}.
\]
The proof of (6.7) follows from a straightforward modification of the standard homogeneous Strichartz estimate (i.e. \( a = 1 \)). For \( s > 0 \), the \( H^s_a \)-norm diverges as \( a \to \infty \) and hence the homogeneous Strichartz estimate (6.7) is not useful for our application.

6.3. Proof of Theorem 4. Let \( \tilde{z}_{1,N} \) and \( \tilde{\sigma}_N \) be as in (1.34) and (1.36). As in (3.3), (4.1), and (4.9), we define
\[
\tilde{Z}_{1,N} := \tilde{z}_{1,N}, \quad \tilde{Z}_{2,N} := (\tilde{z}_{1,N})^2 - \tilde{\sigma}_N, \quad \tilde{Z}_{3,N} := (\tilde{z}_{1,N})^3 - 3\tilde{\sigma}_N \tilde{z}_{1,N},
\]
\[
\tilde{Z}_{4,N} := \tilde{z}_{2,N} := -L_N^{-1}((\tilde{z}_{1,N})^3 - 3\tilde{\sigma}_N \tilde{z}_{1,N}),
\]
\[
\tilde{Z}_{5,N} := \{(\tilde{z}_{1,N})^2 - \tilde{\sigma}_N\} \tilde{z}_{2,N},
\]
(6.8)
where \( L_N \) is as in (1.32). Then, by repeating the arguments in Sections 3 and 4, we see that the analogues of Propositions 3.2, 4.1, and 4.2 hold for \( \tilde{Z}_{j,N}, j = 1, \ldots, 5 \). In the following lemma, we summarize the regularity and convergence properties of these stochastic terms.
Lemma 6.4. Let $1 < \alpha \leq \frac{3}{2}$ and $s_j$, $j = 1, \ldots, 5$, satisfy the regularity assumptions \ref{ass:1}, \ref{ass:2}, and \ref{ass:10}. Fix $j = 1, \ldots, 5$. Then, given any $T > 0$, $\tilde{Z}_{j,N}$ converges almost surely to 0 in $C([-T,T]; W^{s_j,\infty}(\mathbb{T}^3))$ as $N \to \infty$. Moreover, given $2 \leq q < \infty$, there exist positive constants $C$, $c$, $\kappa$, $\theta$ and small $\delta > 0$ such that for every $T > 0$, there exists a set $\Omega_T$ of complemental probability smaller than $C \exp(-c/T^\kappa)$ with the following properties; given $\varepsilon > 0$, there exists $N_0 = N_0(T, \varepsilon) \in \mathbb{N}$ such that
\begin{equation}
\|\tilde{Z}_{j,N}\|_{L^q([-T,T]; W^{s_j,\infty}(\mathbb{T}^3))} \leq C_N^{-\delta} T^{q/2} \tag{6.9}
\end{equation}
and
\begin{equation}
\|\tilde{Z}_{j,M} - \tilde{Z}_{j,N}\|_{C([-T,T]; W^{s_j,\infty}(\mathbb{T}^3))} < \varepsilon
\end{equation}
for any $\omega \in \Omega_T$ and any $M \geq N \geq N_0$, where we allow $N = \infty$ with the understanding that $\tilde{Z}_{j,\infty} = \tilde{Z}_j$. In particular, we have $\tilde{Z}_j = 0$ in $L^q([-T,T]; W^{s_j,\infty}(\mathbb{T}^3))$ for any $\omega \in \Omega_T$.

Proof. We only consider the case $j = 1$ since the other cases follow in a similar manner. With $\langle n \rangle_N = \sqrt{C_N + |n|^2}$, let
\begin{equation}
g_{N}^{i,N}(\omega) := \cos(t\langle n \rangle_N) g_n(\omega) + \sin(t\langle n \rangle_N) h_n(\omega).
\end{equation}
Then, from \ref{eq:6.5}, we have
\begin{equation}
\langle \nabla \rangle^{s_1} \tilde{Z}_{1,N} = \sum_{|n| \leq N} \frac{g_{N}^{i,N}(\omega)}{\langle n \rangle_N \langle n \rangle^{\alpha-1-s_1}} e^{in \cdot x},
\end{equation}
From Lemma 6.1 we have $\langle n \rangle_N \geq \langle n \rangle$ and thus we can repeat the computations in the proofs of Lemmas 3.3 and 3.5 and Proposition 3.2 to conclude that $\tilde{Z}_{1,N}$ converges almost surely to some $\tilde{Z}_1$ in $C([-T,T]; W^{s_1,\infty}(\mathbb{T}^3))$.

Next, we prove \ref{eq:6.9} and show that $\tilde{Z}_1 = 0$. Let $q, r < \infty$. Then, proceeding as in \ref{eq:3.10} with $\langle n \rangle_N \geq \max (C_N, \langle n \rangle)$, we have
\begin{equation}
\left\| \langle \nabla \rangle^{s_1} Z_{1,N} \right\|_{L^q_t L^r_x} \leq \left\| \langle \nabla \rangle^{s_1} Z_{1,N}(t,x) \right\|_{L^q_t L^r_x} \leq T^{1/2} p^{1/2} \left( \sum_{|n| \leq N} \frac{1}{\langle n \rangle_N \langle n \rangle^{\alpha-1-s_1}} \right)^{\frac{1}{2}} \lesssim C_N^{-\delta} T^{1/2} p^{1/2} \tag{6.10}
\end{equation}
for any $p \geq \max(q,r)$ and sufficiently small $\delta > 0$ such that $2(\alpha - s_1 - 2\delta) > 3$. Then, the estimate \ref{eq:6.9} for $j = 1$ (with a different $\delta > 0$) follows from \ref{eq:6.10} and Chebyshev’s inequality. From Lemma 6.1, we see that the right-hand side of \ref{eq:6.10} tends to 0 as $N \to \infty$. Then, by Fatou’s lemma and the uniqueness of the limit, we conclude from the asymptotic behavior $C_N \to \infty$ that $\tilde{Z}_1 = \lim_{N \to \infty} \tilde{Z}_{1,N} = 0$. \hfill \Box

With Lemma 6.4 in hand, we can proceed as in Section 5. Namely, given $(w_0, w_1) \in \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)$, let $u_N$ be the solution to the (un-renormalized) NLW \ref{eq:1.1} with the initial data in \ref{eq:6.1}:
\begin{equation}
(u_N, \partial_t u_N)|_{t=0} = (w_0, w_1) + (\tilde{w}_0^{N}, \tilde{w}_1^{N}),
\end{equation}
where $(\tilde{w}_0^{N}, \tilde{w}_1^{N})$ is the truncated random initial data defined in \ref{eq:1.31}. Now, we write
\begin{equation}
\tilde{u}_N = \tilde{z}_{1,N} + \tilde{z}_{2,N} + \tilde{w}_N, \tag{6.11}
\end{equation}
where \(z_{1,N}\) and \(z_{2,N}\) are as in (1.31) and (6.8), respectively. Recalling that \(u_N\) also satisfies (6.2), we see that \(\tilde{w}_N\) is the solution to

\[
\begin{align*}
\mathcal{L}_N \tilde{w}_N + \tilde{F}_0 + \tilde{F}_1(\tilde{w}_N) + \tilde{F}_2(\tilde{w}_N) + \tilde{F}_3(\tilde{w}_N) &= 0 \\
(\tilde{w}_N, \partial_t \tilde{w}_N)|_{t=0} &= (w_0, w_1),
\end{align*}
\]

(6.12)

where \(\mathcal{L}_N\) is as in (1.32) and \(\tilde{F}_j, j = 0, \ldots, 3\), are given by

\[
\begin{align*}
\tilde{F}_0 &= 3z_{5,N} + 3z_{1,N}(z_{2,N})^2 + (z_{2,N})^3, \\
\tilde{F}_1(\tilde{w}_N) &= 3z_{2,N}\tilde{w}_N + 6z_{1,N}\tilde{z}_{2,N}\tilde{w}_N + 3(z_{2,N})^2 \tilde{w}_N, \\
\tilde{F}_2(\tilde{w}_N) &= 3z_{1,N}(\tilde{w}_N)^2 + 3\tilde{z}_{2,N}\tilde{w}_N^2, \\
\tilde{F}_3(\tilde{w}_N) &= \tilde{w}_N^3.
\end{align*}
\]

Given \(N \in \mathbb{N}\), define \(S_N(t)\) by

\[
S_N(t)(w_0, w_1) = \cos(t\sqrt{C_N - \Delta})w_0 + \frac{\sin(t\sqrt{C_N - \Delta})}{\sqrt{C_N - \Delta}}w_1.
\]

Then, the Duhamel formulation of (6.12) is given by

\[
\tilde{w}_N = S_N(t)(w_0, w_1) + \mathcal{L}_N^{-1}(\tilde{F}_0 + \tilde{F}_1(\tilde{w}_N) + \tilde{F}_2(\tilde{w}_N) + \tilde{F}_3(\tilde{w}_N)),
\]

Define \(A^{(1)}_\theta, A^{(2)}_\theta\), and \(\hat{R}(T)\) by replacing \(z_{j,N}\) and \(Z_{j,N}\) in (5.22) and (5.23) with \(\tilde{z}_{j,N}\) and \(\tilde{Z}_{j,N}\). Then, by repeating the analysis in Section 5 with (6.5), we obtain

\[
\|\tilde{w}_N\|_{L^4_T H^\frac{1}{2}_x} \leq \|(w_0, w_1)\|_{H^\frac{1}{2}_x} + CT^\theta A^{(1)}_\theta + CT^\theta A^{(2)}_\theta \|\tilde{w}_N\|_{L^4_T X_T} \\
+ CT^\theta \left( \sum_{j=1}^{2} \|z_{j,N}\|_{L^4_T W^{s_j, \infty}_x} \right) \|\tilde{w}_N\|_{X_T}^2 + C\|\tilde{w}_N\|_{L^4_T X_T}^3
\]

(6.13)

and

\[
\|\tilde{w}_N\|_{L^4_T X_T} \leq \|S_N(t)(w_0, w_1)\|_{L^4_T X_T} + CT^\theta A^{(1)}_\theta + CT^\theta A^{(2)}_\theta \|\tilde{w}_N\|_{X_T} \\
+ CT^\theta \left( \sum_{j=1}^{2} \|z_{j,N}\|_{L^4_T W^{s_j, \infty}_x} \right) \|\tilde{w}_N\|_{X_T}^2 + C\|\tilde{w}_N\|_{L^4_T X_T}^4
\]

(6.14)

where the constants are independent of \(N \in \mathbb{N}\), thanks to the uniform Strichartz estimate (6.5). By Hölder's inequality in \(t\) and Sobolev's inequality in \(x\) (as in the proof of Lemma 6.2), we have

\[
\|S_N(t)(w_0, w_1)\|_{L^4_{t,x}} \lesssim T^\frac{1}{2} \|(w_0, w_1)\|_{H^\frac{1}{2}_x},
\]

(6.15)

uniformly in \(N \in \mathbb{N}\). Then, it follows from (6.13), (6.14), and (6.15) that there exists small \(T_1 > 0\) depending on \(\hat{R}(T)\) such that

\[
\|\tilde{w}_N\|_{L^4_T H^\frac{1}{2}_x} \leq 2\|(w_0, w_1)\|_{H^\frac{1}{2}_x}, \\
\|\tilde{w}_N\|_{L^4_T X_T} \leq (1 + C(\hat{R}(T))) T^\theta
\]

(6.16)
for any $0 < T \leq T_1$, uniformly in $N \in \mathbb{N}$. It follows from Lemma 6.4 that for each small $0 < T \leq T_1$, there exists a set $\Omega_T$ of complemental probability smaller than $C \exp(-c/T^\alpha)$ such that

$$\widetilde{R}(T) \leq C^{-\delta} T^\theta.$$  
(6.17)

In the following, we fix $\omega \in \Omega_T$ and show that $\tilde{w}_N$ tends to 0 as a space-time distribution as $N \to \infty$. From Lemma 6.4 with (6.16) and (6.17), we see that

$$\mathcal{L}_N^{-1}(\tilde{F}_0 + \tilde{F}_1(\tilde{w}_N) + \tilde{F}_2(\tilde{w}_N)) \to 0$$  
(6.18)

in $X_T$ as $N \to \infty$. On the other hand, by Sobolev’s inequality (with $\delta > 0$ sufficiently small) and Lemma 6.1, we have

$$\|\mathcal{L}_N^{-1}(\tilde{F}_3(\tilde{w}_N))\|_{L_\infty T^2 L_2^x} \leq C_N^{-\delta} \|\tilde{w}_N^3\|_{L_2^1 H^{1+2\delta}_x} \lesssim C_N^{-\delta} \|\tilde{w}_N\|_{L_4^4 T,x}^3 \to 0$$
(6.19)

as $N \to \infty$. Therefore from Lemmas 6.1 and 6.2 with (6.18) and (6.19), we conclude that $\tilde{w}_N$ tends to 0 in the space-time distributional sense.

Finally, from the decomposition (6.11), Lemma 6.4, and the convergence property of $\tilde{w}_N$ discussed above, we conclude that, for each $\omega \in \Omega_T$, $\tilde{u}_N$ converges to 0 as space-time distributions on $[-T,T] \times \mathbb{T}^3$ as $N \to \infty$. This completes the proof of Theorem 4.

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Tadahiro Oh, School of Mathematics, The University of Edinburgh, and The Maxwell Institute for the Mathematical Sciences, James Clerk Maxwell Building, The King’s Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom

E-mail address: hiro.oh@ed.ac.uk

Oana Pocovnicu, Department of Mathematics, Heriot-Watt University and The Maxwell Institute for the Mathematical Sciences, Edinburgh, EH14 4AS, United Kingdom

E-mail address: o.pocovnicu@hw.ac.uk

Nikolay Tzvetkov, Université de Cergy-Pontoise, 2, av. Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France

E-mail address: nikolay.tzvetkov@u-cergy.fr