Painlevé-type asymptotics for the defocusing Hirota equation in transition region

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Abstract

We consider the Cauchy problem for the classical Hirota equation on the line with decaying initial data. Based on the spectral analysis of the Lax pair of the Hirota equation, we first expressed the solution of the Cauchy problem in terms of the solution of a Riemann-Hilbert problem. Further we apply nonlinear steepest descent analysis to obtain the long-time asymptotics of the solution in the critical transition region \[ \left| \frac{t}{\alpha} - \frac{2}{3} \right| t^{2/3} \leq M, \]
M is a positive constant. Our result shows that the long time asymptotics of the Hirota equation can be expressed in terms of the solution of Painlevé II equation.

Keywords: Hirota equation, steepest descent method, Painlevé II equation, long-time asymptotics.

Subject Classification: 35Q51; 35Q15; 37K15; 35C20.
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1 Introduction

In 1973, Hirota first derived the following equation
\begin{equation}
  uu_t + \alpha uu_{xx} + \beta uu_{xxx} + 3i\gamma |u|^2 u_x + \delta |u|^2 u = 0,
\end{equation}
where $u(x, t)$ is a complex-valued scalar function, $\alpha$, $\beta$, $\gamma$ and $\delta$ are real constants which satisfy $\alpha \gamma = \beta \delta$. Especially for $\alpha = \frac{1}{2} \delta$ and $\beta = \frac{1}{2} \gamma$, the equation (1.1) reduces to the form
\begin{equation}
  uu_t + \alpha (uu_{xx} - 2|u|^2 u) + i\beta (uu_{xxx} - 6|u|^2 u_x) = 0,
\end{equation}
where real parameters $\alpha$ and $\beta$ stand for the second dispersion and the third dispersion, respectively. The equation (1.2) is integrable system which is the combination of complex mKdV equation and the NLS equation.

On account of the remarkable properties and the important role played in the scientific research, much work on a series of theoretical and practical work on various problems of this equation has been done. Hirota obtained the exact $N$-envelope solitons by applying the bilinear direct method. The relation between discrete surfaces with constant negative gaussian curvature and the Hirota equation was considered in [2]. The rogue wave solution and rational solution of the Hirota equation were further studied [3, 4]. Nevertheless, as for long-time asymptotic analysis, the nonlinear steepest descent method developed by Deift and Zhou has been proved to be one of the most effective method [5]. Based on the nonlinear steepest descent method,
many meaningful asymptotic analysis results have been investigated [6–16]. For example, Huang et al. analyzed the high order asymptotics for the Hirota equation via the Deift-Zhou high order theory [17]. Guo et al. first considered the long time asymptotic behaviour of the solution for the Hirota equation on the half line [18]. The asymptotic analysis on the high-order solitons was discussed by Ling [19]. Boutet de Monvel et al. discussed the Painlevé-type asymptotics for the Camassa-Holm equation by nonlinear steepest descent method [20]. Charlier and Lenells have carefully considered the Airy and Painlevé asymptotics for the mKdV equation in [21]. Recently, Huang and Zhang complete the extension from the Painlevé asymptotics analysis for the mKdV equation to that of the mKdV hierarchy[22].

To our knowledge, the Painlevé asymptotics of the Hirota equation in transition region for the Hirota equation are still not presented yet. So in present paper, we focus on the long-time asymptotic behavior for the Hirota equation in transition region \( \{(x, t) \in \mathbb{R}^2 \mid |x - \frac{t^2}{3}|^2/3 \leq M\} \) by applying the improved nonlinear steepest descent method.

The organization of this paper is as follows: In Section 1, we first recall the construction of the corresponding Riemann-Hilbert problem and introduce the Painlevé region \( \mathcal{P} \) related to the Hirota equation. In Section 2, we focus on the long-time asymptotic analysis for the Hirota equation in the sector \( \mathcal{P}_\leq \). First, we make an analytic approximation for the scattering data. Next, we do a series of contour deformation to convert the Riemann-Hilbert problem into the solvable model problem. Based on the above operations, we can finally obtain the asymptotic results of the solutions for the defocusing Hirota equation. In section 4, we investigate the asymptotic behavior of the Hirota equation in the sector \( \mathcal{P}_\geq \) using the same way as the last section. Finally, we find that the final asymptotic result can be given explicitly in terms of the real-valued solutions of the Painlevé-II equation.

## 2 Riemann-Hilbert problem

We consider the Cauchy problem for defocusing Hirota equation

\[
iu_t + \alpha(u_{xx} - 2|u|^2u) + i\beta(u_{xxx} - 6|u|^2u_x) = 0, \quad t > 0, \quad x \in \mathbb{R},
\]

\[
u(x, 0) = v_0(x) \in \mathcal{S}(\mathbb{R}),
\]

where \( \alpha \in \mathbb{R}, \beta \in \mathbb{R}^+ \). The Lax pair corresponding to (2.1) is given by

\[
\phi_x = P\phi, \quad \phi_t = Q\phi,
\]
where

\[ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad P = -ik\sigma_3 + U, \quad U = \begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix}, \]

\[ Q = -4i\beta\sigma_3k^3 - 2i\alpha\sigma_3k^2 + V, \quad V = V_2k^2 + V_1k + V_0, \]

\[ V_2 = 4\beta U, \quad V_1 = \begin{pmatrix} -2i\beta|u|^2 & 2i\beta u_x + 2\alpha u \\ -2i\beta|u|^2 & 2i\beta|u|^2 \end{pmatrix}, \]

\[ V_0 = \begin{pmatrix} -i\alpha|u|^2 + \beta(-u\overline{u} + u_x\overline{u}) & i\alpha u_x - \beta(u_{xx} - 2|u|^2u) \\ -i\alpha\overline{u} + \beta(-u_xu + 2|u|^2\overline{u}) & i\alpha|u|^2 - \beta(-u\overline{u} + u_x\overline{u}) \end{pmatrix}, \]

and \( \overline{u} \) denotes the complex conjugate of \( u \). Here \( \sigma_3 \) is one of the Pauli matrices defined by

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (2.4)

Considering the asymptotic property of \( \phi_{\pm} \), we make the transformation

\[ \phi = \Phi e^{-ikx - (4i\beta k^3 + 2i\alpha k^2)t}\sigma_3, \]

we then obtain the equivalent Lax representation

\[ \Phi_x + ik[\sigma_3, \Phi] = U\Phi, \]

\[ \Phi_t + (2i\alpha k^2 + 4i\beta k^3)[\sigma_3, \Phi] = V\Phi. \] (2.5)

We first define two solutions of the spectral problem

\[ \Phi_-(x, t, k) = 1 + \int_{-\infty}^{x} e^{-ik(x-y)\sigma_3} U(y, t)\Phi_-(y, t, k)dy, \] (2.6)

\[ \Phi_+(x, t, k) = 1 - \int_{x}^{\infty} e^{-ik(x-y)\sigma_3} U(y, t)\Phi_+(y, t, k)dy. \] (2.7)

There exists a continuous matrix function \( S(k) \) satisfying

\[ \Phi_+(x, t, k) = \Phi_-(x, t, k)e^{-it\theta(k)\sigma_3} S(k), \] (2.8)

where

\[ S(z) = \begin{pmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{pmatrix}, \quad \theta(k) = 4\beta k^3 + 2\alpha k^2 + k\xi, \quad \xi = \frac{x}{t}. \] (2.9)

Moreover, we have the critical symmetry \( S(k) = \sigma_1 S(k)\sigma_1 \), which implies that matrix function \( S(k) \) can be written as the following form

\[ S(k) = \begin{pmatrix} a(k) & b(k) \\ \overline{b(k)} & \overline{a(k)} \end{pmatrix}. \] (2.10)
Based on the Abel formula, we obtain that $\det S(k) = 1$. Moreover, we have $|a(k)|^2 = 1 + |b(k)|^2$, $k \in \mathbb{R}$. For technical reasons, we assume here that $a(k)$ has no singularity on the real axis. Next, we define

$$m(x, t, k) = \begin{cases} 
\frac{[\Phi_+^1]}{a(k)}, \frac{[\Phi_+]_2}{a(k)}, & \text{Im}(k) > 0, \\
\frac{[\Phi_-]_1}{a(k)}, \frac{[\Phi_-]_2}{a(k)}, & \text{Im}(k) < 0.
\end{cases}$$

(2.11)

According to the relation (2.8), we obtain that $m(x, t, k)$ satisfies the following jump condition

$$m_+(x, t, k) = m_-(x, t, k)J(x, t, k), \quad k \in \mathbb{R},$$

(2.12)

where

$$J(x, t, k) = \begin{pmatrix} 1 - |r(k)|^2 & -r(k)e^{-2i\theta(k)} \\
\frac{r(k)e^{2i\theta(k)}}{1} & 1 \end{pmatrix}, \quad r(k) = \frac{b(k)}{a(k)}$$

(2.13)

The potential $u(x, t)$ is given by

$$u(x, t) = 2i \lim_{k \to \infty} (km)_{12},$$

(2.14)

where $m$ is a $2 \times 2$ matrix-value function satisfying following Riemann-Hilbert problem:

**Theorem 1.** Given $r(k)$, the function $m(x, t, k)$ satisfies the matrix Riemann-Hilbert problem as follows:

- **Analyticity:** $m(x, t, k)$ is analytic in $k \in \mathbb{C} \setminus \mathbb{R}$.
- **Jump condition:** $m_+(x, t, k) = m_-(x, t, k)J(x, t, k)$, $k \in \mathbb{R}$.
- **Asymptotic property:** $m(x, t, k) \to 1$, as $k \to \infty$.

The signature table of $\text{Re}(i\theta)$ is shown in Figure 1, and stationary points as are given by

$$k_1 = \frac{-\alpha - \sqrt{\alpha^2 - 3\beta \xi}}{6\beta}, \quad k_2 = \frac{-\alpha + \sqrt{\alpha^2 - 3\beta \xi}}{6\beta}.$$  

(2.15)
Based on the signature table, the jump matrix $J(x, t, k)$ has following triangular factorization

$$
J(x, t, k) = \begin{pmatrix}
1 - |r(k)|^2 & -r(k)e^{-2it\theta(k)} \\
r(k)e^{2it\theta(k)} & 1
\end{pmatrix} = \begin{pmatrix}
1 & -r(k)e^{-2it\theta(k)} \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
r(k)e^{2it\theta(k)} & 1
\end{pmatrix}.
$$

We aim to find the asymptotics of $u(x, t)$ in the transition region defined as

$$
\mathcal{P} := \left\{(x, t) \in \mathbb{R}^2, \ 0 < |\xi - \alpha^2/3\beta|^{2/3} < M \right\},
$$

where $M > 0$ is a constant. We use the following notations

$$
\mathcal{P}_\leq := \mathcal{P} \cap \left\{\xi \leq \frac{\alpha^2}{3\beta}\right\}, \quad \mathcal{P}_\geq := \mathcal{P} \cap \left\{\xi \geq \frac{\alpha^2}{3\beta}\right\}
$$

(2.17)

to denote the left and right halves of $\mathcal{P}$, respectively.

## 3 Asymptotics in Sector $\mathcal{P}_{\leq}$

Suppose $(x, t) \in \mathcal{P}_{\leq}$. In this region, the two stationary points $k_1, k_2$ defined by (2.15) are real and close to $-\frac{\alpha}{3\beta}$ at least as the speed of $t^{-\frac{2}{3}}$ as $t \to \infty$.

### 3.1 Analytical approximation

Let $\Gamma \in \mathbb{C}$ denote the contour $\Gamma = (\cup_{j=1}^4 l_j) \cup \mathbb{R}$ oriented to the right as in Figure 2, where

$$
l_1 := \{k_1 + le^{\frac{i\pi}{6}}, \ l > 0\}, \quad l_2 := \{k_2 + le^{\frac{i\pi}{6}}, \ l > 0\},
$$

$$
l_3 := \{k_1 + le^{-\frac{i\pi}{6}}, \ l > 0\}, \quad l_4 := \{k_2 + le^{-\frac{i\pi}{6}}, \ l > 0\},
$$

(3.1)
and let

\[ D = \{ \arg(k - k_1) \in \left(5\pi/6, \pi \right) \} \cup \{ \arg(k - k_2) \in \left(0, \pi/6 \right) \}, \]

\[ D^* = \{ \arg(k - k_1) \in \left(-\pi, -5\pi/6 \right) \} \cup \{ \arg(k - k_2) \in \left(-\pi/6, 0 \right) \} \]

denote the open subsets shown in the same figure.

Figure 2: (colour online). The contour \( \Gamma \) and the sets \( D \) and \( D^* \) in the case of \( P \leq 0 \). The region where \( \Re(i\theta) > 0 \) is shaded.

Then we have an analytical approximation for \( r(k) \) as follows:

**Proposition 1.** There exists a decomposition

\[ r(k) = r_a(x, t, k) + r_r(x, t, k), \quad k \in (-\infty, k_1) \cup (k_2, \infty), \quad (3.2) \]

where \( r_a \) and \( r_r \) satisfy the following properties:

(i) For \((x, t) \in P \leq 0\), \( r_a(x, t, k) \) is defined and continuous for \( k \in D \) and analytic for \( k \in D^* \).

(ii) The function \( r_a(x, t, k) \) satisfies

\[ |r_a(x, t, k)| \leq \frac{C}{1 + |k|^2} e^{2\Re(2i\theta(k))}, \quad k \in D, \quad (3.3) \]

and

\[ |r_a(x, t, k) - r(k_j)| \leq C|k - k_j| e^{2\Re(2i\theta(k))}, \quad k \in D, \quad j = 1, 2. \quad (3.4) \]

(iii) The \( L^1, L^2 \) and \( L^\infty \) norms of the function \( r_r(x, t, \cdot) \) on \( \mathbb{R} \setminus (k_1, k_2) \) are \( O(t^{-\frac{3}{2}}) \) as \( t \to \infty \) uniformly for \( (x, t) \in P \leq 0 \).

*Proof.* See [23], Lemma 4.8 for more details.
3.2 Contour deformation

In what follows, we perform the contour deformation as follows:

\[
m^{(1)}(x, t, k) = m(x, t, k) \times \begin{cases} 
\left( \begin{array}{cc} 1 & 0 \\ -r_a(k)e^{2it\theta} & 1 \end{array} \right), & k \in D, \\
\left( \begin{array}{cc} 1 & -r_a(k)e^{-2it\theta} \\ 0 & 1 \end{array} \right), & k \in D^*, \\
1, & \text{elsewhere}.
\end{cases}
\] 

(3.5)

Thus, we find \( m^{(1)}(x, t, k) \) satisfies the new RH problem

\[
m^{(1)}_+(x, t, k) = m^{(1)}_-(x, t, k)J^{(1)}(x, t, k),
\]

where

\[
J^{(1)}(x, t, k) = \begin{cases} 
\left( \begin{array}{cc} 1 & 0 \\ r_a(k)e^{2it\theta} & 1 \end{array} \right), & k \in l_1 \cup l_2, \\
\left( \begin{array}{cc} 1 & -r_a(k)e^{-2it\theta} \\ 0 & 1 \end{array} \right), & k \in l_3 \cup l_4, \\
\left( \begin{array}{cc} 1 - |r(k)|^2 & -r(k)e^{-2it\theta(k)} \\ r(k)e^{2it\theta(k)} & 1 \end{array} \right), & k \in (k_1, k_2), \\
\left( \begin{array}{cc} 1 - |r_r(k)|^2 & -r_r(k)e^{-2it\theta(k)} \\ r_r(k)e^{2it\theta(k)} & 1 \end{array} \right), & k \in \mathbb{R}/(k_1, k_2).
\end{cases}
\] 

(3.7)

3.3 Local model

Considering \(| \xi - \frac{\alpha^2}{\beta} | t^{\frac{2}{3}} \leq C\), it is obvious that for \( t \to +\infty, k_j \to -\frac{\alpha}{\beta} \). The phase function can be approximated as \( t\theta(k) = t\theta(-\frac{\alpha}{\beta}) + s\hat{k} + \frac{4}{3}\hat{k}^3 \), where

\[
s = (3\beta)^{-1/3}(\xi - \frac{\alpha^2}{3\beta})t^{2/3}, \quad \hat{k} = (3\beta t)^{1/3}(k + \frac{\alpha}{6\beta}).
\] 

(3.8)

The coefficients in above formula have been fittingly chosen such that the form of the scaled phase function is the same as that of the RH problem for the Painlevé II equation. For a fixed \( \varepsilon > 0 \), let \( D_\varepsilon(-\frac{\alpha}{6\beta}) = \{ k \in \mathbb{C} | |k + \frac{\alpha}{6\beta}| < \varepsilon \} \), and let \( \Gamma^\varepsilon = (\Gamma \cap D_\varepsilon(-\frac{\alpha}{6\beta}))\backslash((-\infty, k_1) \cup (k_2, \infty)) \). Next, we define

\[
m^{(2)}(s, t, \hat{k}) = m^{(1)}(x, t, k)e^{-it\theta(-\frac{\alpha}{\beta})\sigma_3}, \quad k \in D_\varepsilon(-\frac{\alpha}{6\beta}) \backslash \Gamma.
\] 

(3.9)
Then the new jump matrix $J^{(2)}$ can be approximated as follows:

$$J^{(2)}(s, t, \hat{k}) \to \begin{cases} 
    \left( \begin{array}{cc}
    r \left( -\frac{\alpha}{6\beta} \right) e^{2i(s\hat{k} + \frac{4}{3}\hat{k}^3)} & 0 \\
    1 & 1 \\
    0 & 1 \\
    0 & 1
    \end{array} \right), & k \in (\Gamma^x)_1, \\
    \left( \begin{array}{cc}
    r \left( -\frac{\alpha}{6\beta} \right) e^{-2i(s\hat{k} + \frac{4}{3}\hat{k}^3)} & 1 \\
    1 & 1 \\
    0 & 1 \\
    0 & 1
    \end{array} \right), & k \in (\Gamma^x)_2, \\
    \left( \begin{array}{cc}
    r \left( -\frac{\alpha}{6\beta} \right) e^{2i(s\hat{k} + \frac{4}{3}\hat{k}^3)} & 0 \\
    1 & 1 \\
    0 & 1 \\
    0 & 1
    \end{array} \right), & k \in (\Gamma^x)_3,
\end{cases}$$

(3.10)

which is consistent with the jump matrix $\hat{J}$ defined by the model RH problem in terms of the solution of the Painlevé II equation with $s = i r(\alpha / 6\beta) - 1$ in Appendix A.2. of [21].

We write $\Gamma^x = \bigcup_{j=1}^{3} \Gamma^x_j$, where $\Gamma^x_j$ denotes the part of $\Gamma^x$ that maps into $j$, see Figure 3.

Figure 3: (colour online). The contour $\Gamma^x = \bigcup_{j=1}^{3} \Gamma^x_j$.

Thus we expect that $m^{(1)}(x, t, k)$ in $D_\varepsilon(-\frac{\alpha}{6\beta})$ approaches the solution $m^r(x, t, k)$ defined by

$$m^r(x, t, k) = e^{-i\theta(-\rho)^\alpha \hat{s}} \hat{m}(\rho, s, \hat{k})$$

as $t \to \infty$, where $\hat{m}(\rho, s, \hat{k})$ is a model RH problem in terms of the solution of the Painlevé II equation in Appendix A.2. of [21].

**Proposition 2.** For each $(x, t) \in P_\leq$, $m^r(x, t, k)$ is an analytic function of $k \in D_\varepsilon(-\frac{\alpha}{6\beta}) \backslash \Gamma^x$ such that $|m^r(x, t, k)| \leq C$. Across $\Gamma^x$, $m^r(x, t, k)$ has the jump condition $m^r_+ = m^r_- J^r$, where the jump matrix $J^r$ satisfies

$$\|J^{(1)} - J^r\|_{L^1 \cap L^2 \cap L^\infty(\Gamma^x)} \leq C t^{-\frac{1}{4}}.$$  

(3.11)
Furthermore, as \( t \to \infty \),
\[
\left\| (m^r)^{-1}(x, t, k) - I \right\|_{L^\infty(\partial D_\varepsilon(-\frac{\alpha}{6\beta}))} = O\left(t^{-\frac{1}{2}}\right),
\]
and
\[
\frac{1}{2\pi i} \int_{\partial D_\varepsilon(-\frac{\alpha}{6\beta})} \left((m^r)^{-1}(x, t, k) - I\right) dk = \frac{m^r_1(s)}{(3\beta t)^{1/3}} + O\left(t^{-\frac{5}{6}}\right),
\]
where
\[
m^r_1(s) = \left(\begin{array}{c}
\frac{1}{2} \int^{s} y^2(\zeta) d\zeta \\
-\frac{1}{2} e^{-2it(\gamma + \frac{\alpha}{6\beta})} e^{i\gamma s} y(s) \\
\frac{1}{2} \int^{s} y^2(\zeta) d\zeta
\end{array}\right), \quad \gamma = \arg r\left(-\frac{\alpha}{6\beta}\right),
\]
which is fixed by its asymptotics as \( s \to +\infty \),
\[
y(s) \sim \frac{1}{2\sqrt{\pi}} \left| r\left(-\frac{\alpha}{6\beta}\right) \right| s^{-1/4} \exp\left(-\frac{2}{3} s^{3/2}\right).
\]

Proof. The proof is similar to the proof of Lemma 4.2 in [21].

Assume that the boundary of \( D_\varepsilon(-\frac{\alpha}{6\beta}) \) is oriented counterclockwise. Define the approximate solution \( m^{app} \) by
\[
m^{app}(x, t, k) = \begin{cases} 
(m^r(x, t, k), k \in D_\varepsilon(-\frac{\alpha}{6\beta}), \\
I, \text{ elsewhere.}
\end{cases}
\]

Then the error function \( E_r(x, t, k) \) defined by
\[
E_r(x, t, k) = m^{(1)}(m^{app})^{-1}
\]
satisfies a small-norm RH problem with the jump relation \((E_r)_+ = (E_r)_- J_r \) across \( \bar{\Gamma} = \Gamma \cup \partial D_\varepsilon(-\frac{\alpha}{6\beta}) \), where the jump matrix \( J_r \) is given by
\[
J_r = \begin{cases} 
(m^{app}_{-})^{(1)}(m^{(app)}_{+})^{-1}, & k \in \Gamma \cap D_\varepsilon(-\frac{\alpha}{6\beta}), \\
(m^{app}_{-})^{(1)}(m^{(app)}_{+})^{-1}, & k \in \partial D_\varepsilon(-\frac{\alpha}{6\beta}), \\
J^{(1)}, & k \in \Gamma \setminus D_\varepsilon(-\frac{\alpha}{6\beta}).
\end{cases}
\]

We now denote \( \bar{\Gamma} \) as \( \bar{\Gamma} = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3 \cup \bar{\Gamma}_4 \), where \( \bar{\Gamma}_1 = \bar{\Gamma} \setminus (\mathbb{R} \cup D_\varepsilon(-\frac{\alpha}{6\beta})) \), \( \bar{\Gamma}_2 = \mathbb{R} \setminus [k_1, k_2] \), \( \bar{\Gamma}_3 = \partial D_\varepsilon(-\frac{\alpha}{6\beta}) \), \( \bar{\Gamma}_4 = \Gamma^c \).
Proposition 3. Let \( w_r = J^r - I \). For \((x, t) \in \mathcal{P}_\leq\), the following estimates hold:

\[
\|w_r\|_{L^1 \cap L^2 \cap L^\infty(\tilde{\Gamma}_1)} \leq Ce^{-ct}, \tag{3.19}
\]
\[
\|w_r\|_{L^1 \cap L^2 \cap L^\infty(\tilde{\Gamma}_2)} \leq Ct^{-\frac{d}{4}}, \tag{3.20}
\]
\[
\|w_r\|_{L^1 \cap L^2 \cap L^\infty(\tilde{\Gamma}_3)} \leq Ct^{-\frac{d}{8}}, \tag{3.21}
\]
\[
\|w_r\|_{L^1 \cap L^2 \cap L^\infty(\tilde{\Gamma}_4)} \leq Ct^{-\frac{d}{16}}. \tag{3.22}
\]

3.4 Asymptotics of the solution

In this subsection, we will derive the asymptotics formula of the solution for the Hirota equation in Sector \( \mathcal{P}_\leq \). Firstly, we set \( \tilde{\mathcal{C}} \) as the Cauchy operator associated with \( \tilde{\Gamma} \) and let \( \tilde{\mathcal{C}}_w f := \tilde{\mathcal{C}}_w (f w_r) \). Therefore, matrix error function \( E_r(k) \) can be rewritten as

\[
E_r(x, t, k) = I + \frac{1}{2\pi i} \int_{\tilde{\Gamma}} \frac{(\mu_r w_r)(x, t, \zeta)}{\zeta - k} \, d\zeta,
\]

where the 2 \times 2 matrix-valued function \( \mu_r(x, t, k) \) is defined by \( \mu_r = I + \tilde{\mathcal{C}}_w (\mu_r) \). Moreover, using the Neumann series, \( \mu_r(x, t, k) \) satisfies

\[
\|\mu_r - I\|_{L^2(\tilde{\Gamma})} = O(t^{-\frac{d}{4}}), \quad t \to \infty.
\]

It follows that

\[
\lim_{k \to +\infty} k(m(x, t, k) - I) = -\frac{1}{2\pi i} \int_{\tilde{\Gamma}} (\mu_r w_r)(x, t, \zeta) d\zeta. \tag{3.23}
\]
By (3.12), (3.16), (3.18) and (3.21), the contribution from \( \partial D_{\epsilon}(-\frac{\alpha}{6\beta}) \) to the right-hand side of (3.23) is

\[
-\frac{1}{2\pi i} \int_{\partial D_{\epsilon}} (\mu_r w_r)(x, t, \zeta) d\zeta = -\frac{1}{2\pi i} \int_{\partial D_{\epsilon}(-\frac{\alpha}{6\beta})} w_r d\zeta - \frac{1}{2\pi i} \int_{\partial D_{\epsilon}(-\frac{\alpha}{6\beta})} (\mu_r - I) w_r d\zeta = m_r^1(s) + O\left(t^{-\frac{2}{3}}\right).
\]

(3.24)

The contributions from \( \tilde{\Gamma}_1, \tilde{\Gamma}_2 \) and \( \tilde{\Gamma}_4 \) to the right-hand side of (3.23) are \( O\left(e^{-ct}\right) \), \( O\left(t^{-3/2}\right) \) and \( O\left(t^{-1/3}\right) \), respectively. Recalling the reconstruction formula (2.14), (3.23) and the definition of \( \phi \), we immediately obtain the asymptotic formula of \( u(x, t) \) as follows

\[
u(x, t) = -\frac{1}{(3\beta t)^{1/3}} \exp(-2i\theta(-\frac{\alpha}{6\beta}) - i\gamma)y(s) + O\left(t^{-\frac{2}{3}}\right),
\]

(3.25)

where \( s = (3\beta)^{-1/3}(\xi - \frac{\alpha^2}{3\beta})t^{2/3} \). So far, we completed the long time asymptotic analysis of the Hirota equation in the space-time region \( \{(x, t) \in \mathbb{R}^2 \mid -M \leq (\xi - \frac{\alpha^2}{3\beta})t^{2/3} \leq 0\} \).

4 Asymptotics in Sector \( \mathcal{P}_\geq \)

We now consider the asymptotics in sector \( \mathcal{P}_\geq \). In this sector, the two stationary points \( k_1, k_2 \) are complex number and approach to \( -\frac{\alpha}{6\beta} \) as the speed of \( t^{-\frac{1}{3}} \) as \( t \to \infty \). As in Section 3, we first decompose \( r \) into two parts. In this part, we define the contour \( \Gamma \) and the open subsets \( D, D^* \) as in Figure 5.

![Figure 5](image-url)
Proposition 4. There exists a decomposition

\[ r(k) = r_a(x, t, k) + r_r(x, t, k), \quad k \in \mathbb{R}, \quad (4.1) \]

where \( r_a \) and \( r_r \) satisfy the following properties:

(i) For \((x, t) \in \mathcal{P}_2\), \( r_a(x, t, k) \) is defined and continuous for \( k \in \overline{\mathcal{D}} \) and analytic for \( k \in \mathcal{D} \).

(ii) The function \( r_a(x, t, k) \) satisfies

\[ |r_a(x, t, k)| \leq \frac{C}{1 + |k|^2} e^{\frac{t}{4}|\text{Re}(2i\theta(k))|}, \quad k \in \overline{\mathcal{D}}, \quad (4.2) \]

and

\[ |r_a(x, t, k) - r(-\frac{\alpha}{6\beta})| \leq C|k + \frac{\alpha}{6\beta}| e^{\frac{t}{8}|\text{Re}(2i\theta(k))|}, \quad k \in \overline{\mathcal{D}}. \quad (4.3) \]

(iii) The \( L^1, L^2 \) and \( L^\infty \) norms of the function \( r_r(x, t, \cdot) \) on \( \mathbb{R} \) are \( O(t^{-\frac{3}{2}}) \) as \( t \to \infty \) uniformly for \( (x, t) \in \mathcal{P}_2 \).

Using this decomposition of \( r(x, t, k) \), we define \( m^{(1)} \) as (3.5), then the jump matrix \( J^{(1)} \) in (3.6) changes into

\[
J^{(1)}(x, t, k) = \begin{cases} 
\frac{1}{(r_a(k)e^{2it\theta} - 1)} & \text{if } k \in l_1 \cup l_2, \\
\frac{1}{(r_a(k) - r_r(k))e^{2it\theta}} & \text{if } k \in l_3 \cup l_4, \\
\frac{1}{r_r(k)e^{2it\theta}} & \text{if } k \in \mathbb{R}.
\end{cases} \quad (4.4)
\]

For \( |\xi - \frac{\alpha}{6\beta}| t^\frac{3}{2} \geq C \), as in Section 3, the phase function can also be approximated as \( t\theta(k) = t\theta(-\frac{\alpha}{6\beta}) + s\hat{k} + \frac{\beta}{2}\hat{k}^3 \), and \( s, \hat{k} \) are given by (3.8). Let \( \Sigma^\varepsilon = (\Gamma \cap D_e(-\frac{\alpha}{6\beta})) \setminus \mathbb{R} \). Define

\[
m^{(2)}(s, t, \hat{k}) = m^{(1)}(x, t, k)e^{-it\theta(-\frac{\alpha}{6\beta})}, \quad k \in D_e(-\frac{\alpha}{6\beta}) \setminus \Gamma. \quad (4.5)
\]

We write \( \Sigma^\varepsilon = \Sigma^\varepsilon_1 \cup \Sigma^\varepsilon_2 \), where \( \Sigma^\varepsilon_j \) denotes the part of \( \Sigma^\varepsilon \) that maps into \( j \), see Figure 6. Then the jump matrix \( J^{(2)} \) can be approximated as

\[
J^{(2)}(s, t, \hat{k}) = \begin{cases} 
\frac{1}{r(-\frac{\alpha}{6\beta})e^{2i(s\hat{k} + \frac{\beta}{2}\hat{k}^3)}} & \text{if } k \in \Sigma^\varepsilon_1, \\
\frac{1}{1 - r(-\frac{\alpha}{6\beta})e^{-2i(s\hat{k} + \frac{\beta}{2}\hat{k}^3)}} & \text{if } k \in \Sigma^\varepsilon_2.
\end{cases} \quad (4.6)
\]
Thus we expect that $m^{(1)}(x, t, k)$ in $D_{\varepsilon}\left(-\frac{\alpha}{6}\right)$ approaches the solution $m^r(x, t, k)$ defined by $m^r(x, t, k) = e^{-i\theta(-\frac{\alpha}{6})}\hat{m}(\rho, s, \hat{k})$ as $t \to \infty$, where $\hat{m}(\rho, s, \hat{k})$ is the solution of the model RH problem.

![Figure 6: (colour online). The contour $\Sigma^\varepsilon = \Sigma^\varepsilon_1 \cup \Sigma^\varepsilon_2$.](image)

**Proposition 5.** For each $(x, t) \in P_\geq$, $m^r(x, t, k)$ is an analytic function of $k \in D_{\varepsilon}\left(-\frac{\alpha}{6}\right) \setminus \Gamma^\varepsilon$ such that $|m^r(x, t, k)| \leq C$. Across $\Gamma^\varepsilon$, $m^r(x, t, k)$ has the jump condition $m^r_+ = m^r_-$, where the jump matrix $J^r$ satisfies

$$\|J^{(1)} - J^r\|_{L^1 \cap L^2 \cap L^\infty(\Gamma^\varepsilon)} \leq Ct^{-\frac{1}{3}}. \tag{4.7}$$

Furthermore, as $t \to \infty$,

$$\left\| (m^r)^{-1}(x, t, k) - 1 \right\|_{L^\infty(\partial D_{\varepsilon}(\pm \hat{k}))} = O(t^{-\frac{2}{3}}),$$

and

$$\frac{1}{2\pi i} \int_{\partial D_{\varepsilon}(\pm \hat{k})} \left( (m^r)^{-1}(x, t, k) - 1 \right) dk = -\frac{m^r_1(s)}{(3\beta t)^{1/3}} + O\left(t^{-\frac{2}{3}}\right), \tag{4.8}$$

where $m^r_1(s)$ is defined by (3.13).

Define $E_r(x, t, k)$ by (3.17), then $E_r$ satisfies the RH problem with the jump matrix $J_r$ given by (3.18). Denote $\tilde{\Gamma}$ as $\tilde{\Gamma} = \Gamma \cup \partial D_{\varepsilon}(\pm \hat{k}) = \tilde{\Gamma}_1 \cup \mathbb{R} \cup \partial D_{\varepsilon}(\pm \hat{k}) \cup \Sigma^\varepsilon$, $\tilde{\Gamma}_1 = \tilde{\Gamma}\setminus(\mathbb{R} \cup D_{\varepsilon}(\pm \hat{k}))$. 

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Proposition 6. Let $w_r = J^r - I$. For $(x, t) \in \mathcal{P}_\geq$, the following estimates hold:

\[
\begin{align*}
\|w_r\|_{L^1 \cap L^2 \cap L^{\infty}(\tilde{\Gamma}_1)} &\leq Ce^{-ct}, \\
\|w_r\|_{L^1 \cap L^2 \cap L^{\infty}(\mathbb{R})} &\leq Ct^{-\frac{3}{2}}, \\
\|w_r\|_{L^1 \cap L^2 \cap L^{\infty}(\partial D_\varepsilon(-\frac{\alpha}{\varepsilon}))} &\leq Ct^{-\frac{3}{4}}, \\
\|w_r\|_{L^1 \cap L^2 \cap L^{\infty}(\Sigma)} &\leq Ct^{-\frac{1}{4}}.
\end{align*}
\]  

The remainder of the proof to the asymptotic formula proceeds as in sector $\mathcal{P}_\leq$.

In this work, we have investigated the long time asymptotic solution of the Cauchy problem for defocusing Hirota equation with decaying data in the special transition region $|\frac{x}{t} - \frac{\alpha^2}{4\beta}|t^{2/3} \leq M$, $M$ is a positive constant. Based on the Riemann-Hilbert problem which is established in [17], we perform the nonlinear steepest descent method to analysis the asymptotic properties of the solution in the left and right transition regions, respectively. What is meaningful is that we find the solution of defocusing Hirota equation can be approximated in terms of the real-valued solution of Painlevé II equation with the error $O(t^{-\frac{2}{3}})$. What’s more, as for the focusing Hirota equation, we can derive the asymptotic formula of the solution in transition regions by nonlinear steepest descent method. It’s worth noting that the result is similar to that of defocusing case except that other errors caused by soliton asymptotics.
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