Coherent oscillations and incoherent tunneling in one-dimensional asymmetric double-well potential

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Abstract

For a model 1d asymmetric double-well potential we calculated so-called survival probability (i.e. the probability for a particle initially localized in one well to remain there). We use a semiclassical (WKB) solution of the Schroedinger equation. It is shown that behavior essentially depends on transition probability, and on one dimensionless parameter $\Lambda$ which is a ratio of characteristic frequencies for low energy non-linear in-well oscillations and inter wells tunneling. For the potential describing a finite motion (double-well) one has always a regular behavior. For $\Lambda \ll 1$ there is well defined resonance pairs of levels and the survival probability has coherent oscillations related to resonance splitting. However for $\Lambda \gg 1$ no oscillations at all for the survival probability, and there is almost an exponential decay with the characteristic time determined by Fermi golden rule. In this case one may

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not restrict oneself to only resonance pair levels. The number of perturbed by tunneling levels grows proportionally to $\sqrt{\Lambda}$ (by other words instead of isolated pairs there appear the resonance regions containing the sets of strongly coupled levels). In the region of intermediate values of $\Lambda$ one has a crossover between both limiting cases, namely the exponential decay with subsequent long period recurrent behavior.

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I. INTRODUCTION

Double level systems and models appear in various contexts in physics, chemistry and biology. The recurrent interest to the topic is related mainly with fairly rich and interesting physics of the systems, and with the experimental activity on several classes of systems which can be viewed as good physical realization of double level models (including fashionable quantum dots, see e.g. [1]). Among the possible types of behavior we will particularly be concerned with coherent oscillations and incoherent (dissipative like) tunneling. Our goal is to propose a simple mathematical model to illustrate crossover from coherent oscillations to dissipative tunneling (decay or relaxation), which are also related to incoherent transitions in multidimensional oscillator systems. In a certain sense this crossover reveals many features of chaotic behavior. It is a common wisdom now that classical chaos is defined as extreme complexity of the trajectories in phase space, with the trajectories being very sensitive to small changes in the initial conditions [2], [3]. As well evidently that the state vector (wave function) of a closed quantum system strictly speaking does not exhibit chaotic motion, as a consequence of the unitary nature of time evolution. But in fact since in quantum mechanics trajectories in the phase space can not be introduced due to Heisenberg uncertainty principle, the standard classical concept of the stability becomes ambiguous (see e.g. [4], [5], [6], [7], [8]).

We put forward a simple (but yet non-trivial) model of 1d asymmetric double-well potential which can be used to describe under relatively weak assumptions a crossover from coherent oscillations (say mechanical behavior) to incoherent decay or dissipative tunneling (say ergodic behavior). The essential part of the model we will present is to illustrate this semiclassical quasi-chaotic behavior. In fact the illustration was made long ago by Fermi, Pasta, and Ulam [9]. They performed computer studies of energy sharing and ergodicity for weakly coupled systems of \( N \) oscillators. Later on, the results of [9] were confirmed and refined (see e.g. [10], [11]). But all these papers were devoted to systems with many degrees of freedom \((N \gg 1)\) dimensional phase space) for the cases where the motion is
nearly integrable and irregular in different energy regions. Level statistics for such kind of mixed systems (i.e. when behavior is regular and chaotic in different phase space regions) changes gradually from Poisson to Wigner type of distributions \cite{12}, \cite{13}, \cite{14}. Thus these systems become non-integrable when the energy exceeds a certain critical value. Just on the contrary we will propose and investigate in 1d a conservative system with time independent Hamiltonian which is evidently always integrable, and it does not generate classical chaos.

For the sake of completeness let us note that the tunneling in the mixed (i.e. regular-chaotic) systems has been studied as well for two level systems when one of the levels interacts with a chaotic state \cite{15}, \cite{16} (see also review \cite{17} and references therein). In the case of a resonance between the tunneling doublet and suitable chaotic states, the tunneling is enhanced (so-called chaos assisted tunneling) and has very strong resonance dependence on quantum numbers. Similar effects due to transverse vibrations take place for isolated Fermi resonances in tunneling systems \cite{18}.

Our paper has the following structure. Section II contains basic equations necessary for our investigation. Section III is devoted to the calculation of so-called survival probability. We use the semiclassical approach \cite{19} (see also \cite{20} and references herein). The last section IV contains the summary. The appendix to our paper is devoted to the technical and methodical details of the calculations.

II. ASYMMETRIC 1D DOUBLE-WELL POTENTIAL

The simple model studied in this paper consists of a quantum particle in one dimensional asymmetric double-well potential \( U(X) \) with one-parameter dependent shape. Using the tunneling distance \( a_0 \) and the characteristic frequency of the oscillations around the left minimum \( \Omega_0 \), we can introduced the so-called semiclassical parameter \( \gamma \equiv m\Omega_0a_0^2/\hbar \gg 1 \) (\( m \) is a mass of a particle, and further we will set \( \hbar = 1 \) measuring energies in the units of frequency), which is assumed to be sufficiently large, i.e. the tunneling matrix element should be small in \( \Omega_0 \) scale. The choice of the model potential is dictated by the principle
of minimal requirements. Our aim is to describe in the framework of one universal model crossover from symmetric double-well potential to so-called decay potential, and to do it we need a parameter to make the right well ($R$-well) deeper and wider than the left well ($L$-well).

Using $\Omega_0$ and $a_0$ to set corresponding scales, the model potential satisfying these minimal requirements can be written in the following dimensionless form

$$V(x) = \frac{1}{2} x^2 (1 - x) \left[ 1 + \frac{1}{b^2} x \right],$$

where $V \equiv U/(\Omega_0 \gamma)$, and $x \equiv X/a_0$. The dimensionless parameter $b$ allows us to change the shape of the right well ($R$-well), and to consider both limiting cases, namely a traditional symmetric double-well potential (for $b = 1$), and for $b \rightarrow \infty$ a decay potential (or by other words to change the level spacings from $\Omega_0^{-1}$ scale to zero ). In fact it can be shown (see below and the Appendix to the paper) that qualitatively all our results do not depend on the concrete form of the one parametric potential satisfying these requirements (only on the density of $R$-states). Behavior in both limiting cases are well known, and for $b = 1$ one has coherent quantum oscillations, typical for any two-level systems, while for $b \rightarrow \infty$ there is a continuum spectrum of eigen states for $x \rightarrow +\infty$ and one can find an ergodic behavior (incoherent decay). Our main goal in this section is to study crossover between both limits at variations of $b$.

The general procedure for searching semiclassical solutions of the Schroedinger equation with the model potential (1) has a tricky point. The fact is that in the $L$-well we have a discrete eigenvalue spectrum (stationary states) while for the $R$-well in the case $b \gg 1$ we have quasi-stationary states, which are characterized by wave functions $\Psi_n(X)$ exponentially increased in the region of $\varepsilon \gg V(X)$. Both kind of states are defined on different sheets of complex energetic surfaces [19], and to treat both kind a states one should use different tools, namely, the standard quantization of the stationary states from the discrete part of the spectra [19], and proposed long ago by Zeldovich [21] for quasi-stationary states the flux probability conservation law, which leads to the Lorentzian envelope for spectral distribution.
functions. Unlike \[21\] in our case we get the Lorentzian envelope filled by \(\delta\)-peaks of the final states.

The procedure is described in the Appendix, and it includes three steps (see \[19\], \[21\], and we will use notations from \[20\]):

- First one should find the action \(W_L\) in the classically allowed region (i.e. \(W_L\) between turning points) in the left well (\(L\)-well), and apply the semiclassical quantization. For the low energy states in the \(L\)-well it leads to the following relation

\[
\gamma W_L = \pi \left[ n + \frac{1}{2} + \chi_n \right] \equiv \pi \varepsilon_n ,
\]

where, integer numbers \(n\) numerate eigenvalues, \(\chi_n\) is determined by an exponentially small phase shift, and the last r.h.s. of (2) is in fact the definition for eigenvalues \(\varepsilon_n\).

- Second, the same should be done for the right well (\(R\)-well). The calculation is almost trivial in the limit \(b \gg 1\) (when the potential (1) becomes strongly asymmetric)

\[
\gamma W_R = \gamma W_R^{(0)} + \pi \beta \varepsilon ,
\]

where the dimensionless energy \(\varepsilon\) is counted from the bottom of the \(L\)-well, the action \(W_R^{(0)}\) is

\[
\gamma W_R^{(0)} = \frac{\pi}{16b} (b^2 - 1)^2(b^2 + 1),
\]

and

\[
\beta = \frac{b^2 + 1}{b} \simeq b, \text{ for } b \gg 1 ,
\]

Note that the parameter \(\beta = 2\Omega_0/\omega_R\) is proportional to the density of states in the \(R\)-well (\(\omega_R\) is the frequency of non-linear oscillations in \(R\)-well at \(\varepsilon = 0\)), and therefore knowing the magnitude \(\beta\) one can compute the density of states in the \(R\)-well, which grows proportional to \(b\) for \(b \gg 1\). It is convenient to rewrite (3) - (4) in the same form as (1)
\[ \gamma W_R = \pi \left[ n_R + \frac{1}{2} + \alpha_n + \beta \chi \right], \quad (6) \]

where \( n_R \) and \( \alpha_n \) are integer and correspondingly fractional parts of the quantity

\[ \frac{\gamma W_R^{(0)}}{\pi} + \beta \left( n + \frac{1}{2} \right) - \frac{1}{2}. \quad (7) \]

The physical meaning of \( \alpha_n \) is the deviation from a resonance between the \( n \)-th level in the \( L \)-well and the nearest level in the \( R \)-well. By the definition of a fractional part \( |\alpha_n| < 1/2 \beta \).

- And as the last step, again using the quantization rule, one can find the spectrum.

It turns out (see Appendix) that the spectrum and the behavior of the system depends crucially on the parameter \( \Lambda \equiv \beta R_n \), where

\[ R_n = \frac{2^{n+2} e^{n+1/2}}{\pi^{1/2} n!} \exp(-2\gamma W_B) \quad (8) \]

is the \( \beta \) independent decay rate of the \( n \)-th metastable state of the \( L \)-well at \( b \to \infty \) (\( W_B \) is the action in the classically forbidden (between turning points) region).

For \( \Lambda \ll 1 \) solving the quantization relation (A2), one can easily find

\[ \varepsilon_{n\pm} = n + \frac{1}{2} \pm \frac{1}{2\beta} \left[ \sqrt{\alpha_n^2 + \frac{4}{\pi^2} \Lambda} - \alpha_n \right]. \quad (9) \]

This expression (9) determines the resonance pairs of the levels, so-called two-level systems.

Besides from the same quantization rule (A2) we get analytically (i.e. for arbitrary values of \( \Lambda \)) eigenvalues for the \( R \)-well in the vicinity of the resonance doublet

\[ \varepsilon_{nm} = n + \frac{1}{2} + \frac{1}{2\beta} \left[ \sqrt{(m - \alpha_n)^2 + \frac{4}{\pi^2} \Lambda} - (m - \alpha_n) \right] \quad m = \pm 1, \pm 2, \ldots \quad (10) \]

These levels are numerated by the quantum number \( m \).

For \( \Lambda \ll 1 \) all displacements of the levels due to tunneling are small, and two-level system approximation is valid (i.e. there is well defined isolated resonance pairs of levels with splitting \( \propto (R_n/\beta)^{1/2} \)). The situation becomes completely different for \( \Lambda \geq 1 \). In the
limit $\Lambda \gg 1$ we get almost equidistant spectrum of mixed $L - R$ levels in the vicinity of the following values of $\chi$ (see Appendix for the details)

$$\chi \equiv \chi_{nm} = \pm \frac{m + 1/2 - \alpha_n}{\beta} \left[1 + \frac{1}{\pi \Lambda}\right]. \quad (11)$$

The given above expressions (10) - (11) show that the number of the perturbed by tunneling levels grows proportionally to $\sqrt{\Lambda}$. In Fig. 1 we have shown the displacements of the levels perturbed by tunneling. These displacements are decreased very rapidly for the levels with quantum numbers larger than $\sqrt{\Lambda}$. The scales in this figure are given by the semiclassical parameter $\gamma$ which relates to the $L$-well and the barrier. Once the scales are fixed the $R$-well is characterized by the eigenfrequency $\propto 1/b$ at $\varepsilon = 0$ (or what is the same by the density of states or by the action $W_R$ in the $R$-well).

Summarizing the results of this section, thus we have shown that instead of isolated two level systems taking place for $\Lambda \ll 1$, in the opposite limit $\Lambda \gg 1$ there appear the resonance regions containing the sets of strongly coupled levels. The resonance widths are determined by tunneling matrix elements ($H_{12}^2 = \omega_L \omega_R \exp(-2\gamma W_B)/4\pi^2 = R_n/\beta$). In spite of the fact that for any finite values of $\Lambda$ (and $b$) we have only the discrete spectrum of real eigenstates, found above mixing of $L - R$ states very closely resembles the representation of quasi-stationary states in terms of eigenstates of a continuous spectrum. This behavior can be formulated by other words in terms of the so-called recurrence time, i.e. the characteristic time when the system is returned to the initial state. For a finite motion (i.e. for a finite value of $b$) the behavior of the system remains regular. The recurrence time (i.e. in the case merely coherent oscillation period) is proportional to $1/H_{12}$ for $\Lambda \ll 1$, while for $\Lambda \gg 1$ this time scales as $1/\omega_R$ (as a long-period time scale).

## III. SURVIVAL PROBABILITY

The tunneling dynamics can be characterized by the time evolution of the initially prepared localized state $\Psi(0)$, and by the definition the survival probability of the state is
\[ P(t) \equiv |\langle \Psi(0) | \Psi(t) \rangle|^2. \] (12)

For the stationary states evidently \( P(t) = 1 \), while for quasi-stationary (decaying states), the survival probability reads

\[ P(t) = \exp(-\Gamma t), \] (13)

where \( \Gamma \) is the decay rate which should be found, and we use \( \omega_R^{-1} \) for the time scale.

The simplest case is the coherent tunneling dynamics of two-level states. Let us consider the \( n - n' \) resonance region. The eigenfunctions of isolated \( R \) and \( L \) wells, \( \Psi^L_n \), and \( \Psi^R_{n'} \). If one has the initial state

\[ \Psi(0) = \Psi^L_n, \]

the survival probability can be easily calculated

\[ P(t) = \frac{1}{2} \left[ 1 + \cos \left( 2t \sqrt{\frac{R_n}{\beta}} \right) \right]. \] (14)

Normalized wave functions in the \( L \)-well can be calculated trivially, and using standard semiclassical wave functions for the \( R \)-well, we are in a position to compute the survival probability for a general case as a function of \( \Lambda \). The results are shown in Fig. 2.

For \( \Lambda \ll 1 \), \( P(t) \) oscillates with characteristic time scales proportional to \( H_{12}^{-1} = \sqrt{\beta/R_n} \). In the region \( \Lambda \simeq 1 \) these oscillations are strongly suppressed. The reason for the suppression of oscillations is related to interference of the states with energies in the resonance region. As a result of the interference the total probability for the system to return back from the \( R \)-well is decreased, and low-frequency modulation of coherent tunneling is raised. The period of the modulation grows with \( \beta \), and in the limit \( \Lambda \gg 1 \) we get the dense spectrum of states in the \( R \)-well, and almost exponential decay for \( P(t) \) with \( \beta \)-independent relaxational time \( \tau \propto R_n^{-1} \). In this case the survival probability (i.e. the probability to keep the system in its initial state) for the time interval \( \ll 1/\omega_R \) decay almost exponentially with time, and the characteristic relaxation time \( \tau \) is determined by Fermi golden rule, i.e. \( \tau^{-1} \propto H_{12}^2/\omega_R \). This result is also conformed to van Hove statement [22] concerning quasi-chaotic behavior of semi-classical systems at time scales of the order of \( \omega_R/H_{12}^2 \).
We can relate the phenomenon described above (i.e. almost vanishing probability for back-flow from the \( R \) to \( L \) well) to the Fermi golden rule for a transition probability

\[
W_{fi} = 2\pi |H_{fi}|^2 \rho_f ,
\]

(15)

where \( H_{fi} \) is the matrix element between the initial state \( E_i \) and the final state \( E_f \), and \( \rho_f \) is the density of final states. For our case \( (H_{if} \equiv H_{12} = \sqrt{R_n}/\beta, \text{ and } \rho_f = \beta/2) \) we get easily

\[
W_{if} = \pi R_n ,
\]

which does not depend on \( \rho_f \). Therefore the Fermi golden rule corresponds to the limit when the back flow from the \( R \)-well is totally suppressed due to the interference.

The survival probability can be related also to spectral distribution of the initially localized in the \( L \)-well states. Indeed, by the definition of the spectral distribution \( S(E) \) of the initially prepared localized state is determined by the transition amplitudes in expansion over the eigenstates \( (\Psi_n, E_n) \):

\[
S(E) = \sum_n |\langle \Psi(0) | \Psi_n \rangle|^2 \delta(E - E_n) ,
\]

(16)

and therefore

\[
\langle \Psi(0) | \Psi(t) \rangle = \int_{-\infty}^{+\infty} S(E) \exp(-iEt)dE .
\]

(17)

For \( \Psi(0) \equiv \Psi_i^L \) the spectral distribution is a set of \( \delta \)-peaks with Lorentzian envelope

\[
S(E) = \frac{2}{\pi \beta} \frac{\sqrt{R_i \beta}}{(E - E_i)^2 + R_i} \delta(E - E_i) .
\]

(18)

Crossover from the coherent oscillations to exponential decay occurs when the Lorentzian envelope begins to fill up by \( \delta \)-peaks of the final states. Note that the width of the Lorentzian envelope \( (18) \) does not depend on the final state density (see Appendix and also \[21\]). We have shown the results of the calculation of the spectral distribution in Fig. 3.
IV. CONCLUSION

Let us sum up the results of our paper. We investigated the behavior of a quantum particle in 1d asymmetric double-well potential with one parameter dependent shape, which allows us to consider in the framework of one universal model the crossover from the traditional symmetric double well potential to the decay one. We have shown that behavior essentially depends on transition probability, and on dimensionless parameter Λ which is a ratio of characteristic frequencies for low energy non-linear in-well oscillations and inter wells tunneling. For the potential describing a finite motion (double-well) strictly speaking one has always a regular behavior. For Λ ≪ 1 there is well defined resonance pairs of levels and the survival probability has coherent oscillations related to resonance splitting. However for Λ ≫ 1 there are no oscillations at all for the survival probability, and there is almost an exponential decay with the characteristic time determined by Fermi golden rule. In this case one may not restrict oneself to only resonance pair levels. The number of perturbed by tunneling levels grows proportionally to $\sqrt{\Lambda}$ (by other words instead of isolated pairs there appear the resonance regions containing the sets of strongly coupled levels). In the region of intermediate values of Λ one has a crossover between both limiting cases, namely the exponential decay with subsequent long period recurrent behavior.

However a number of remarks related to our results are in order. Many features often classified as evidences of quantum chaos in fact as we have illustrated in our model can occur for well defined states possessing only discrete energy levels. The deviation from two level system behavior, taking place for Λ ≫ 1 has nothing to do with random or chaotic properties of the system. It means only that due to well known phenomenon of level repulsion the two level approximation is not adequate. Lorentzian envelope (see Fig. 3) we found arises from the interaction of a single level in $L$-well with a set of levels in the $R$-well and not with appearance of level widths (imaginary self-energy contributions).

One should distinguish between short-time and long-time behavior, and the boundary between them depends on the parameter Λ. Short-time returns ($\propto \beta$) are governed by
one or a small number of semiclassical paths, while long-time returns (∝ R_{n+1}) arise from interference between many paths. In the limit Λ ≪ 1, exponential decay occurs for short-time dynamics, while the system remains regular for long-time scales, in contrast with chaotic models we discussed in the Introduction. Nevertheless the tunneling in the limit of Λ ≫ 1 can induce vibrational relaxation for localized R-levels. The relaxation appears due to tunneling recurrences, and results in redistribution of initial energy over all levels coupled with a single L-level.

Main physical idea of our paper, namely that specific quasi-chaotic behavior is associated with the fact that one level in L-well in a certain condition (Λ ≫ 1) is coupled to a set of almost dense levels in the R-well, was discussed in the literature long ago [22] (see also [21]), mainly qualitatively. Our achievement is that we alone seem to have propose the concrete and tractable analytical model to illustrate and to investigate explicitly and quantitatively this statement.

In this respect our results are quite different from numerical investigations of billiard-type systems (see e.g. review article [17]), showing universal behavior of level spacings in finite chaotic systems. Our results (for the totally integrable 1d model) demonstrate that level spacing distribution is not a specific feature of quantum systems with chaotic classical counterpart limit. Our finding of the equidistant regular level distribution is a result of the interaction of the single L-level with several (of the order of 10 for our particular choice of the parameters) R-levels (which in own turn are regular ones). As well we should distinguish our model and dynamic tunneling ones [23], [24]. The latter assumed strong coupling of the tunneling system with an environment which destroys the coherence, whereas in our model the coherence is destroyed by the tunneling itself due to the high density of R-states, breaking two level approximation.

Note also at the very end of the paper that results presented here not only interested in their own right (at least in our opinion) but they might be directly tested experimentally since there are many molecular systems where investigated in the paper 1d asymmetric potential is a reasonable model for the reality. And not only molecular systems, for instance
recently as a controllable two-level system, double quantum dots are proposed for realizing a single quantum bit in solid state systems. Experimentally \cite{1} in these systems there observed two distinct regimes characterizing the nature of low-energy dynamics:

(i) relaxational regime, when an excited-state electron population decays exponentially in time with a rate correctly given by Fermi golden rule;

(ii) vibronic regime, when the population oscillates for some number of cycles before decaying.

And what’s more, at short times the averaged excited-state populations oscillates but has a decaying envelope. The similarity with the behavior we found in the paper is evident.\footnote{All characteristics of our model are not specific only for 1d case. For $\Lambda \gg 1$ one can expect similar behavior and for multidimensional systems.}

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\textbf{APPENDIX:}

The semi-classical wave function is represented in the well known WKB form \footnote{Equivalently it can be represented in the so-called instanton or minimum action tunneling path formalism \cite{27} (see also \cite{20}) in the form of $\Psi = \exp(-\gamma W_E)$, which is more efficient for classically inaccessible parts of phase space.}

$$\Psi = \exp (iW),$$

The action $W$ should satisfy to WKB equation
\[
\frac{1}{2} \left( \frac{dW}{dX} \right)^2 = \frac{\varepsilon}{\gamma} - V(X),
\]

(Eq. A1)

and two turning points, which are boundaries of classically allowed regions, are situated near zeros of \( V(X) - \varepsilon/\gamma \).

For the asymmetric double-well potential (1) the Bohr - Sommerfeld [19] quantization equations read

\[
tg(\gamma W_L)tg(\gamma W_R) = 4 \exp(2\gamma W_B),
\]

(Eq. A2)

where \( W_B \) is the action in the classically forbidden region in between the turning points \( X_1, X_2 \) in the left and right wells, and \( W_{L,R} \) are the coordinate independent actions in the classically allowed regions inside of the \( L \) (respectively \( R \)) well. Using the following expansion

\[
\tan z = \sum_{m=0}^{\infty} 2z \left[ z^2 - \pi^2 \left( m + \frac{1}{2} \right) \right]^{-1},
\]

one gets the almost equidistant spectrum of the mixed \( L - R \) levels, and in this condition the solution of (A2) leads to the expressions (9), (10) presented in the main text of the paper.

The time evolution of any initially prepared state can be described by a superposition of the eigenfunctions of the discrete and continuous spectra with time dependent phases. For the potential (1) with \( b \gg 1 \) the initial finite motion, i.e. the initial density distribution

\[
\rho(t) = \int_{X_1}^{X_2} |\Psi(X, t)|^2 dX
\]

(Eq. A3)

concentrated in the \( L \)-well at \( t = 0 \) decreases exponentially with time

\[
\rho(t) = \rho(0) \exp (-\eta t).
\]

(Eq. A4)

Eq. (A4) signifies that the wave functions of quasi-stationary states have the form

\[
\Psi_n(X, t) = \Psi_n(X) \exp \left( (i\varepsilon_n - \eta_n/2) t \right),
\]

(Eq. A5)

and the eigenvalues are complex and lies on the lower half-space of \( (\varepsilon, \eta) \) plane. The quantization of the stationary states of a discrete spectrum is performed by the requirement [19]

\[ |\Psi(X, t)|^2 \to 0, \text{ at } |X| \to \infty. \]
This condition is impossible to impose to quasi-stationary states, since the wave functions $\Psi_n(X)$ exponentially is increased in the region of $\varepsilon \gg V(X)$. The physically meaningful boundary condition as was noted first by Zeldovich [21] for quasi-stationary states can be written as a conservation law for the flux probability from the $L$-well through the barrier. The difference between stationary and quasi-stationary states disappears as it should be at $\eta \to 0$.

The expansion of the initially quasi-stationary state is dominated by the continuum spectrum eigenfunctions with the energies close to the real parts of the eigenvalues $\varepsilon_n$. These eigenfunctions have the form

$$
\Psi_k(X) = \begin{cases} 
A(k)\phi_k^0(X), & X < X_m \\
\sqrt{\frac{2}{\pi}}\sin(kX + \delta(k)), & X > X_m
\end{cases},
$$

(A6)

where $X_m$ is the left turning point of the $R$-well, the localized wave function $\phi_k^0$ is normalized to unity, and the phase is given

$$
\delta(k) = \delta_0 - \arctan \frac{k_2}{k - k_1},
$$

(A7)

and $\delta_0$ is $k$-independent component, $k_1 = \sqrt{2m\varepsilon_n}$, $k_2 = k_1\eta_n/4\varepsilon_n$. For the eigenfunctions with the energies $\varepsilon$ and $\varepsilon'$ close to $\varepsilon_n$ we get

$$
\int_{-\infty}^{X} \phi_k(X')\phi_{k'}(X')dX' = \frac{1}{2m} \left( \frac{1}{\varepsilon - \varepsilon'} \right) \left( \phi_k'\frac{d\phi_k}{dX} - \phi_{k'}\frac{d\phi_{k'}}{dX} \right).
$$

(A8)

From (A6), (A7), and (A8) in the limit $\varepsilon - \varepsilon' \to 0$ we get

$$
A^2(k) = \frac{2}{\pi} \sqrt{\frac{2\varepsilon_n}{m}} \frac{\eta_n}{4(\varepsilon - \varepsilon_n)^2 + \eta_n^2}.
$$

(A9)

Expressions (A7), (A9) are valid for a continuous spectrum, for discrete levels the phase shift as well is governed by the probability flux from the $R$-well into classically forbidden region, and instead of (A7) it leads

$$
\delta = \arctan \sqrt{R_n\beta} \frac{1}{\varepsilon_n - \varepsilon_{nm}},
$$

(A10)

and instead of (A9) one can easily find
\[ A^2(\varepsilon_{nm}) = \frac{2}{\pi \beta (\varepsilon_n - \varepsilon_{nm})^2 + R_n}, \quad (A11) \]

Note that \((A11)\) has almost the same form as \((A9)\), although it depends on discrete energy levels, and besides it has a different coefficient due to different normalization condition.

The relation \((A9)\) shows that the probability density of the continuous spectrum eigenstates exhibits the Lorentzian distribution around the real part of the quasi-stationary eigenvalues \(\varepsilon_n\). Expressions \((A9)-(A11)\) are equivalent to the spectral distribution \((18)\) presented in the main body of the paper.

Few words concerning numerical results presented in the main text in the figures 1 - 3. The calculations have been performed to check:

(i) semiclassical approximation for the model potential \((1)\);

(ii) the spectral distribution \((18)\).

We used the numerical diagonalization of the Hamiltonian matrix in the basis set of trial functions, which includes: so-called instanton wave functions of the \(L\)-well (see \([20]\)), and the WKB functions of \(R\)-well. This basis was orthonormalized by using standard Schmidt method \([26]\). For the \(L\)-well highly excited states near the barrier top have been also included. In all numerical calculations we set the value of \(\alpha_0\) (so-called defect of a resonance) as zero. All results presented on the figures do not depend on this particular choice.

The numerical results confirm that Eq. \((18)\) is quiet accurate in the whole range of \(\Lambda\) where the transition from coherent oscillations to exponential decay occurs. Note that since \(R\)-levels with the negative energy are not mixed with \(L\)-levels, and besides the resonance region is sufficiently narrow \((R_n = 0.01)\), we need not diagonalize huge matrices. For our purposes the diagonalization of the matrix 3000 × 3000 is more than sufficient to find eigenvalues in the resonance region around the \(n = 0\) \(L\)-level.
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Figure Captions.

Fig. 1

The eigenvalues as functions of $\Lambda$ for the zero-point level ($n = 0$) of the $L$-well. Dashed lines indicate the limits of $\Lambda \ll 1$, and $\Lambda \gg 1$; $\gamma = 10$, $\alpha_0 = 0$.

Fig. 2

The survival probability for different values of $\Lambda$ and $\gamma = 10$:

(a) $\Lambda = 0.02$, $b = 5$ (solid line); $\Lambda = 0.5$, $b = 116$ (dashed line);

(b) $\Lambda = 0.5$, $b = 116$ (solid line); $\Lambda = 4.0$, $b = 929$ (dashed line);

(c) $\Lambda = 4.0$, $b = 929$ (solid line); $\Lambda = 16.0$, $b = 3715$ (dashed line).

Fig. 3

The spectral distribution for different values of $\Lambda$ and $\gamma = 10$:

(a) $\Lambda = 0.02$, $b = 5$;

(b) $\Lambda = 4.0$, $b = 929$;

(c) $\Lambda = 20.0$, $b = 4644$. 
FIG. 3b
FIG. 3c