Bounds for theta sums in higher rank II*

Jens Marklof and Matthew Welsh

11 May 2023

Abstract

In the first paper of this series we established new upper bounds for multi-variable exponential sums associated with a quadratic form. The present study shows that if one adds a linear term in the exponent, the estimates can be further improved for almost all parameter values. Our results extend the bound for one-variable theta sums obtained by Fedotov and Klopp in 2012.

1 Introduction

For $M > 0$, a real $n \times n$ symmetric matrix $X$, and $x, y \in \mathbb{R}^n$, we define a theta sum as the exponential sum

$$\theta_f(M, X, x, y) = \sum_{m \in \mathbb{Z}^n} f(M^{-1}(m + x)) e\left(\frac{1}{2}mX^t m + m^t y\right),$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a rapidly decaying cut-off and $e(z) = e^{2\pi i z}$ for any complex $z$. If $f = \chi_B$ is the characteristic function of a bounded set $B \subset \mathbb{R}^n$ we have the finite sum

$$\theta_f(M, X, x, y) = \sum_{m \in \mathbb{Z}^n \cap MB-x} e\left(\frac{1}{2}mX^t m + m^t y\right).$$

In this case we will also use the notation $\theta_f = \theta_B$. In this paper we will focus on the case when $B$ is the open rectangular box $(0, b_1) \times \cdots \times (0, b_n) \subset \mathbb{R}^n$. The theorems below remain valid if $f = \chi_B$ is replaced by any function $f$ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ (infinitely differentiable, with rapid decay of all derivatives). The results in the latter case follow from a simpler version of the argument for the sharp truncation, so we do not discuss them here.

The principal result of part I [10] in this series is the following.

**Theorem 1.1.** Fix a compact subset $K \subset \mathbb{R}^n_{>0}$, and let $\psi : [0, \infty) \to [1, \infty)$ be an increasing function such that

$$\int_0^\infty \psi(t)^{-2n-2} dt < \infty.$$  

*Research supported by EPSRC grant EP/S024948/1
Then there exists a subset $\mathcal{X}(\psi) \subset \mathbb{R}_{\text{sym}}^{n \times n}$ of full Lebesgue measure such that

$$\theta_{\mathcal{B}}(M, X, x, y) = O_X\left(M^{\frac{n}{2}}\psi(\log M)\right)$$

(1.4)

for all $M \geq 1$, $b = (b_1, \ldots, b_n) \in \mathcal{K}$, $X \in \mathcal{X}(\psi)$, $x, y \in \mathbb{R}^n$. The implied constants are independent of $M$, $b$, $x$ and $y$.

For example, for any $\epsilon > 0$, the function $\psi(x) = (x + 1)^{\frac{1}{2n+2}} + \epsilon$ satisfies the condition (1.3), which produces the bound $M^{\frac{n}{2}}(\log M)^{\frac{1}{2n+4}} + \epsilon$ for almost every $X$ and any $x$ and $y$. This improved the previously best bound due to Cosentino and Flaminio [3] by a factor of $(\log M)^n$. Moreover, in the case $n = 1$, theorem 1.1 recovers the optimal result obtained by Fiedler, Jurkat and Körner [5].

In what follows we establish a stronger bound than (1.4), for example $M^{\frac{n}{2}}(\log M)^{\frac{1}{2n+4}} + \epsilon$, but now only valid for almost every $y$. In the case $n = 1$, theorem 1.2 recovers theorem 0.1 of Fedotov and Klopp [4].

**Theorem 1.2.** Fix a compact subset $\mathcal{K} \subset \mathbb{R}_{\geq 0}^n \times \mathbb{R}^n$, and let $\psi : [0, \infty) \to [1, \infty)$ be an increasing function such that

$$\int_0^{\infty} \psi(t)^{-2n-4} dt < \infty.$$

(1.5)

Then there exists a subset $\tilde{\mathcal{X}}(\psi) \subset \mathbb{R}_{\text{sym}}^{n \times n} \times \mathbb{R}^n$ of full Lebesgue measure such that

$$\theta_{\mathcal{B}}(M, X, x, y) = O_{X,y}\left(M^{\frac{n}{2}}\psi(\log M)\right)$$

(1.6)

for all $M \geq 1$, $(b, x) \in \mathcal{K}$, and $(X, y) \in \tilde{\mathcal{X}}(\psi)$. The implied constants are independent of $M$, $b$ and $x$.

The paper is organized as follows. In section 2 we review some basic properties of theta functions and the Jacobi group. The Jacobi group is defined as the semi-direct product $H \rtimes G$ of the Heisenberg group $H$ and the symplectic group $G = \text{Sp}(n, \mathbb{R})$, and, following a construction due to Lion and Vergne [8], the theta function associated to a Schwartz function $f \in S(\mathbb{R}^n)$ is a function $\Theta_f : H \rtimes G \to \mathbb{C}$ that, for appropriate $g \in G$ and $h \in H$, is a simple rescaling of the theta sums $\theta_f$. The theta functions $\Theta_f$ satisfy an automorphy equation, theorem 3.1, under a certain subgroup $\tilde{\Gamma} \subset H \rtimes G$. This subgroup, defined in section 3, projects to the discrete subgroup $\Gamma = \text{Sp}(n, \mathbb{Z}) \subset G$.

In order to exploit additional savings from the linear term parameterized by $y$, we found it necessary to have a better understanding of the shape of the cusp of $\Gamma \setminus G$ than in the first paper in this series [10]. For this reason we define in section 3.1 a new fundamental domain for $\Gamma \setminus G$ which has “box-shape” cusps, as explicates in section 3.2.

Section 4 contains the proof of theorem 1.2, which is based on a Borel-Cantelli type argument together with a multi-dimensional dyadic decomposition of the characteristic function of the open unit cube $(0, 1)^n$ that is naturally realized as an action of the diagonal subgroup of $G$. The execution of the Borel-Cantelli argument rests on a kind of “uniform continuity” property of a certain height function on $H \rtimes G$ that controls the theta function $\Theta_f$, see
corollary 4.1. The required property is proved in section 4.1, see lemma 4.4, whose proof is the motivation for the creation of the fundamental domain and the study of its cuspidal regions in sections 3.1 and 3.2. We remark that the interaction of the dyadic decomposition with the $H$ coordinate in the Jacobi group leads to additional complications not seen in [10], see section 4.2.

2 Theta functions and the Jacobi group

The theta function $\Theta_f$ associated to a Schwartz function $f \in S(\mathbb{R}^n)$ is a complex-valued function defined on the Jacobi group $H \rtimes G$, the semi-direct product of the Heisenberg group $H$ with the rank $n$ symplectic group $G = \text{Sp}(n, \mathbb{R})$. Here $H$ is the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with multiplication given by

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + \frac{1}{2}(y_1^t x_2 - x_1^t y_2)), \quad (2.1)$$

and $G$ is the group of $2n \times 2n$ real matrices $g$ preserving the standard symplectic form:

$${g \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} }^t g = {\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} } \quad (2.2)$$

with $I$ the $n \times n$ identity. Alternatively, writing $g$ in $n \times n$ blocks,

$$G = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A^t B = B^t A, \ C^t D = D^t C, \ A^t D - B^t C = I \right\}. \quad (2.3)$$

We note that $G$ acts on $H$ by automorphisms via

$$h^g = (xA + yC, xB + yD, t), \text{ where } h = (x, y, t), \ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (2.4)$$

so we may define the semidirect product $H \rtimes G$, the Jacobi group, with multiplication

$$(h_1, g_1)(h_2, g_2) = (h_1 h_2^{-1}, g_1 g_2). \quad (2.5)$$

The theta function is defined by

$$\Theta_f(h, g) = \sum_{m \in \mathbb{Z}^n} (W(h) R(g) f)(m), \quad (2.6)$$

where $W$ is the Schrödinger representation of $H$ and $R$ is the Segal-Shale-Weil (projective) representation of $G$. We refer the reader to [10] for details regarding these representations, including the slightly non-standard definition of $W$ and the unitary cocycle $\rho : G \times G \to \mathbb{C}$ satisfying $R(g_1 g_2) = \rho(g_1, g_2) R(g_1) R(g_2)$. We recall here that for

$$g = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} Y^{\frac{1}{2}} & 0 \\ 0 & (Y^{-\frac{1}{2}})^t \end{pmatrix} \in G, \quad (2.7)$$
we have

\[ \Theta_f((x, y, t), g) = (\det Y)^{\frac{1}{4}} e(-t + \frac{1}{2} x^t y) \sum_{m \in \mathbb{Z}^n} f((m + x)Y^{\frac{1}{2}}) e(\frac{1}{2}(m + x)X^t(m + x) + m^t y). \]  

(2.8)

For \( f(x) = \exp(-\pi x^t x) \) and \( h = (0, 0, 0) \), we recover \((\det Y)^{\frac{1}{4}}\) times the classical Siegel theta series that is holomorphic in the complex symmetric matrix \( Z = X + iY \). Here we choose \( Y^{\frac{1}{2}} \) to be the upper-triangular matrix with positive diagonal entries such that \( Y^{\frac{1}{2}}^t Y^{\frac{1}{2}} = Y \), and we emphasize that \( Y^{-\frac{1}{2}} \) is always interpreted as \((Y^{\frac{1}{2}})^{-1}\) and not \((Y^{-1})^{\frac{1}{2}}\).

For general \( g \in G \) we have the Iwasawa decomposition,

\[ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} Y^{\frac{1}{2}} & 0 \\ 0 & (Y^{-\frac{1}{2}})^t \end{pmatrix} \begin{pmatrix} \text{Re}(Q) & -\text{Im}(Q) \\ \text{Im}(Q) & \text{Re}(Q) \end{pmatrix}, \]

(2.9)

where \( X, Y \) are symmetric and \( Q \) is unitary. Explicitly, we have

\[ Y = (C^t C + D^t D)^{-1} \]
\[ X = (A^t C + B^t D)(C^t C + D^t D)^{-1} \]
\[ Q = (Y^{\frac{1}{2}})^t(D + iC). \]

(2.10)

We often further decompose \( Y = UV^t U \) with \( U \) upper-triangular unipotent and \( V \) positive diagonal, so \( Y^{\frac{1}{2}} = UV^{\frac{1}{2}} \). It is easy to express the Haar measure \( \mu \) on \( G \) in these coordinates,

\[ d\mu(g) = dQ \prod_{1 \leq i \leq j \leq n} dx_{ij} \prod_{1 \leq i < j \leq n} du_{ij} \prod_{1 \leq j \leq n} v_{j}^{-n-j-2} dv_{jj}, \]

(2.11)

where \( dQ \) is Haar measure on \( U(n) \) and \( dx_{ij}, du_{ij}, dv_{jj} \) are respectively the Lebesgue measures on the entries of \( X, U, V \). We can also express the Haar measure on the open, dense set of \( g \) which can be written as

\[ g = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} I & T \\ 0 & I \end{pmatrix} \]

(2.12)

with \( A \in \text{GL}(n, \mathbb{R}) \) and \( X \) and \( T \) symmetric. In these coordinates we have

\[ d\mu(g) = c(\det A)^{-2n-1} \prod_{1 \leq i \leq j \leq n} dx_{ij} \prod_{1 \leq i < j \leq n} da_{ij} \prod_{1 \leq i \leq n} dt_{ii} \]

(2.13)

where \( c \) is a positive constant and \( dx_{ij}, da_{ij}, dt_{ij} \) are respectively the Lebesgue measure on the entries of \( X, A, T \), see [10]. We note that the Haar measure \( \tilde{\mu} \) on the Jacobi group is simply

\[ d\tilde{\mu}(h, g) = dx \ dy \ dt \ d\mu(g), \]

(2.14)

with \( h = (x, y, t) \) and \( dx, dy, \) and \( dt \) the Lebesgue measures.
We often make use of the following refinements of the Iwasawa decomposition. For $1 \leq l \leq n$ and the same $Q$ as in (2.9), we write $g \in G$ as

$$
\begin{pmatrix}
I & R_l & T_l - S_l^tR_l & S_l \\
0 & I & tS_l & 0 \\
0 & 0 & I & 0 \\
0 & 0 & -tR_l & I
\end{pmatrix}
\begin{pmatrix}
U_lV_l^{1/2} & 0 & 0 & 0 \\
0 & Y_l^{1/2} & 0 & X_l^tY_2^{-1/2} \\
0 & 0 & tU_l^{-1}V_l^{1/2} & 0 \\
0 & 0 & 0 & tY_l^{-1/2}
\end{pmatrix}
\begin{pmatrix}
\Re(Q) & -\Im(Q) \\
\Im(Q) & \Re(Q)
\end{pmatrix},
$$

(2.15)

where $R_l$ and $S_l$ are $l \times (n-l)$ matrices, $T_l$ is $l \times l$ symmetric, $U_l$ is $l \times l$ upper-triangular unipotent, $V_l$ is $l \times l$ positive diagonal, $X_l$ is $(n-l) \times (n-l)$ symmetric, and $Y_l$ is $(n-l) \times (n-l)$ positive definite symmetric. We note that for $l = n$ we recover $X = T_l$ and the factorization $Y = U_lV_l^tU_l$. In what follows we use $g_l = g_l(g) \in \text{Sp}(n-l, \mathbb{R})$ to denote the matrix

$$
g_l = \begin{pmatrix}
I & X_l \\
0 & I
\end{pmatrix}
\begin{pmatrix}
Y_l^{1/2} & 0 \\
0 & tY_l^{-1/2}
\end{pmatrix}.
$$

(2.16)

These decompositions are closely related to the Langlands decompositions of the maximal parabolic subgroups $P_l$ of $G$. For $1 \leq l < n$, $P_l$ is the subgroup of $g \in G$ which can be written in the form

$$
\begin{pmatrix}
I & R_l & T_l - S_l^tR_l & S_l \\
0 & I & tS_l & 0 \\
0 & 0 & I & 0 \\
0 & 0 & -tR_l & I
\end{pmatrix}
\begin{pmatrix}
a_lI & 0 & 0 & 0 \\
a_l^{-1}I & 0 & 0 \\
0 & a_l^{-1}I & 0 \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
U_l & 0 & 0 \\
0 & A_l & B_l \\
0 & C_l & D_l
\end{pmatrix},
$$

(2.17)

where $R_l$ and $S_l$ are $l \times (n-l)$ matrices, $T_l$ is $l \times l$ symmetric, $a_l > 0$, $U_l \in \text{GL}(l, \mathbb{R})$ with $\det U_l = \pm 1$, and $g_l = \begin{pmatrix}A_l & B_l \\ C_l & D_l\end{pmatrix} \in \text{Sp}(n-l, \mathbb{R})$. The maximal parabolic $P_n$ is the subgroup of $g \in G$ that can be written as

$$
\begin{pmatrix}
I & T_n \\
0 & I
\end{pmatrix}
\begin{pmatrix}
a_nI & 0 \\
a_n^{-1}I & 0
\end{pmatrix}
\begin{pmatrix}
U_n & 0 \\
0 & tU_n^{-1}
\end{pmatrix},
$$

(2.18)

where $T_n$ is $n \times n$ symmetric, $a_n > 0$, and $U_n \in \text{GL}(n, \mathbb{R})$ with $\det U_n = \pm 1$. The factorizations (2.17), (2.18) are in fact the Langlands decompositions of $P_l$, $P_n$. The first paper in this series [10] contains more details on parabolic subgroups and their Langlands decompositions, and we refer the readers to [12], particularly sections 4.5.3 and 5.1, [7], particularly section 7.7, and the authors’ lecture notes [9] for further details.

3 The subgroups $\Gamma$ and $\tilde{\Gamma}$

We denote by $\Gamma$ the discrete subgroup $\Gamma = \text{Sp}(n, \mathbb{Z}) \subset G$. Recalling the notation of [10], for

$$
\gamma = \begin{pmatrix}A & B \\ C & D\end{pmatrix} \in \Gamma,
$$

(3.1)
we set $h_\gamma = (r, s, 0) \in H$ where the entries of $r$ are 0 or $\frac{1}{2}$ depending on whether the corresponding diagonal entry of $C^t D$ is even or odd, and the entries of $s$ are 0 or $\frac{1}{2}$ depending on whether the corresponding diagonal entry of $A^t B$ is even or odd. As in [10], we now define the group $\tilde{\Gamma} \subset H \rtimes G$ by

$$\tilde{\Gamma} = \{((m, n, t)h_\gamma, \gamma) \in H \rtimes G : \gamma \in \Gamma, m \in \mathbb{Z}^n, n \in \mathbb{Z}^n, t \in \mathbb{R}\}. \quad (3.2)$$

The relevance of the subgroup $\tilde{\Gamma}$ is made apparent by the following theorem, see theorem 4.1 in [10].

**Theorem 3.1.** For any $(uh_\gamma, \gamma) \in \tilde{\Gamma}$ and $(h, g) \in H \rtimes G$, there is a complex number $\varepsilon(\gamma)$ with $|\varepsilon(\gamma)| = 1$ such that

$$\Theta_f((uh_\gamma, \gamma)(h, g)) = \varepsilon(\gamma)\rho(\gamma, g)e\left(-t + \frac{1}{2}m^t n\right)\Theta_f(h, g), \quad (3.3)$$

where $u = (m, n, t)$.

A proof of this theorem is found in [8] but with $\Gamma$ replaced by the finite index subgroup for which $h_\gamma = (0, 0, 0)$. The automorphy under the full $\tilde{\Gamma}$ is proved in [11], but only for the special function $f(x) = \exp(-\pi x^t x)$. It is shown in [8] that this $f$ is an eigenfunction for all the operators $R(k(Q))$, with $R$ the Segal-Shale-Weil representation and $Q \in U(n)$, and it can be seen from the theory built in [8] that the automorphy for any Schwartz function follows from that for $\exp(-\pi x^t x)$. A self-contained proof along the lines of [8] is presented in the authors’ lecture notes [9].

### 3.1 Fundamental domains

We say that a closed set $\mathcal{D} \subset G$ is a fundamental domain for $\Gamma \backslash G$ if

- for all $g \in G$ there exists $\gamma \in \Gamma$ such that $\gamma g \in \mathcal{D}$ and
- if for $g \in \mathcal{D}$ there is a non-identity $\gamma \in \Gamma$ such that $\gamma g \in \mathcal{D}$, then $g$ is contained in the boundary of $\mathcal{D}$.

Similarly a closed set $\tilde{\mathcal{D}} \subset H \rtimes G$ is a fundamental domain for $\tilde{\Gamma} \backslash (H \rtimes G)$ if

- for all $(h, g) \in H \rtimes G$ there exists $\tilde{\gamma} \in \tilde{\Gamma}$ such that $\tilde{\gamma}(h, g) \in \tilde{\mathcal{D}}$ and
- if for $(h, g) \in \tilde{\mathcal{D}}$ there is a non-identity $\tilde{\gamma} \in \tilde{\Gamma}$ such that $\tilde{\gamma}(h, g) \in \tilde{\mathcal{D}}$, then $(h, g)$ is contained in the boundary of $\tilde{\mathcal{D}}$.

We note that if $\mathcal{D}$ is a fundamental domain for $\Gamma \backslash G$, then

$$\tilde{\mathcal{D}} = \left\{(x, y, 0) \in H : |x|, |y| \leq \frac{1}{2}\right\} \times \mathcal{D} \quad (3.4)$$

is a fundamental domain for $\tilde{\Gamma} \backslash (H \rtimes G)$. 6
From (2.10), we have for

\[ D \]

where

\[ \text{that} \]

Proof. We begin by showing that for

Proposition 3.2. \( D \)

In contrast to our previous paper [10], here we need to make careful use of the shape of our fundamental domain \( D \) in the cuspidal regions. Drawing inspiration for the fundamental domain for \( GL(n, \mathbb{Z}) \backslash GL(n, \mathbb{R}) \) constructed in [6] as well as from the reduction theory developed in [2] (see also [1]), we construct in this section a new fundamental domain \( D = D_n \) for \( \Gamma \backslash G \). In the following section we study the cuspidal region of \( D_n \).

For \( n = 1 \), we let \( D_1 \subset G \) denote the standard fundamental domain for \( \Gamma \backslash G = SL(2, \mathbb{Z}) \backslash SL(2, \mathbb{R}) \). That is,

\[
D_1 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y + \frac{i}{2} & 0 \\ 0 & y - \frac{i}{2} \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} : |x| \leq \frac{1}{2}, x^2 + y^2 \geq 1, 0 \leq \phi < 2\pi \right\}. \tag{3.5}
\]

We now define fundamental domains \( D_n \) inductively using the decomposition (2.15) for \( l = 1 \). Writing \( g \in G \) as

\[
g = \begin{pmatrix} 1 & r_1 & t_1 - s_1 & r_1 \\ 0 & I & 0 & s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -i r_1 & I \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & X_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} v_1^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & Y_1^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & v_1^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & tY_1^{\frac{1}{2}} \end{pmatrix} k(Q), \tag{3.6}
\]

where \( r = r(g) \in \mathbb{R}^{n-1}, s = s(g) \in \mathbb{R}^{n-1}, t_1 = t_1(g) \in \mathbb{R}, X_1 = X_1(g) \) is symmetric, \( v_1 = v_1(g) > 0, Y_1 = Y_1(g) \) is positive definite symmetric, and \( Q \in U(n) \), we define \( D_n \) as the set of all \( g \in G \) satisfying

- \( v_1(g) \geq v_1(\gamma g) \) for all \( \gamma \in \Gamma \),
- \( g_1(g) \in D_{n-1} \), see (2.16), and
- the entries of \( r_1(g), s_1(g), \) and \( t_1(g) \) are all less than or equal to \( \frac{1}{2} \) in absolute value with the first entry of \( r_1 \) greater than or equal to 0.

**Proposition 3.2.** \( D_n \) is a fundamental domain for \( \Gamma \backslash G \).

**Proof.** We begin by showing that for \( g \in G, \sup_{\gamma \in \Gamma} v_1(\gamma g) \) is indeed obtained by some \( \gamma \in \Gamma \). From (2.10), we have for

\[
\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \tag{3.7}
\]

that

\[
v_1(\gamma g)^{-1} = cY^t c + (cX + d)Y^{-1}t(cX + d) \tag{3.8}
\]

where

\[
g = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} Y^t & 0 \\ 0 & tY^t \end{pmatrix} k(Q) \tag{3.9}
\]

and \( c, d \) are the first rows of \( C, D \). Since \( Y \) is positive definite, there are only finitely many \( c \) such that \( cY^t c \), and hence \( v_1(\gamma g)^{-1} \), is below a given bound. Similarly, for a fixed \( c \), the
positive definiteness of $Y^{-1}$ implies that there are only finitely many $d$ such that $v_1(\gamma g)^{-1}$ is below a given bound. It follows that there are only finitely many $\gamma \in \Gamma_1 \setminus \Gamma$ such that $v_1(\gamma g)$ is larger than a given bound, where $\Gamma_1 = \Gamma \cap P_1$ and we recall $P_1$ is given by (2.17). As $v_1(\gamma g) = v_1(g)$ for $\gamma \in \Gamma_1$ it follows that $v_1(\gamma g)$ is maximized for some $\gamma \in \Gamma$.

Let $\gamma_0$ be so that $v_1(\gamma_0 g)$ is maximal. We now decompose an arbitrary $\gamma \in \Gamma_1$ as in (2.17),

$$\gamma = \begin{pmatrix} 1 & r_1 & t_1-s_1 t r_1 & s_1 \\ 0 & I & s_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t r_1 & I \end{pmatrix} \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & B_1 \\ 0 & 0 & \pm 1 & 0 \\ 0 & C_1 & 0 & D_1 \end{pmatrix}$$

(3.10)

with

$$\gamma_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \text{Sp}(n-1, \mathbb{Z}).$$

(3.11)

Proceeding inductively, there exists $\gamma_1$ such that $\gamma_1 g_1(\gamma_0 g) = g_1(\gamma_0 g) \in D_{n-1}$. Now, we can change $r_1(\gamma)$, $s_1(\gamma)$, $t_1(\gamma)$, and the $\pm$, noting that this does not change $g_1(\gamma_0 g)$, so that the entries of $r_1(\gamma_0 g)$, $s_1(\gamma_0 g)$ and $t_1(\gamma_0 g)$ are all $\leq \frac{1}{2}$ in absolute value and the first entry of $r_1(\gamma_0 g)$ is nonnegative. Therefore $\gamma_1 g \in D_n$ as required.

We now suppose that $g \in D_n$ and there is a non-identity $\gamma \in \Gamma$ such that $\gamma g \in D_n$. We set

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad g = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} Y^{1/2} & 0 \\ 0 & t Y^{-1/2} \end{pmatrix} k(Q).$$

(3.12)

By the maximality, we have $v_1(g) = v_1(\gamma g)$ and therefore

$$v_1^{-1} = c Y^t c + (cX + d) Y^{-1} t (cX + d)$$

(3.13)

where $c$ and $d$ are the first rows of $C$ and $D$. Let us first consider the case when $c \neq 0$. To show that $g$ is on the boundary of $D_n$ in this case, we consider

$$g_\epsilon = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} (1-\epsilon) Y^{1/2} \\ 0 \\ (1-\epsilon)^{-1/2} t Y^{-1/2} \end{pmatrix} k(Q)$$

(3.14)

for $0 < \epsilon < 1$. We have $v_1(g_\epsilon) = (1-\epsilon)v_1(g)$ and

$$v_1(\gamma g_\epsilon)^{-1} = (1-\epsilon)c Y^t c + (1-\epsilon)^{-1}(cX + d) Y^{-1} (cX + d)$$

$$= ((1-\epsilon) - (1-\epsilon)^{-1}) c Y^t c + v_1(g_\epsilon)^{-1}$$

(3.15)

by (3.13). Since $v_1(\gamma g_\epsilon) > v_1(g_\epsilon)$, we have that $g_\epsilon \notin D_n$. As $g_\epsilon$ can be made arbitrarily close to $g$, we conclude that $g$ is on the boundary of $D_n$.

If $c = 0$, then from (3.13) we have

$$v_1(g)^{-1} = (d^{(1)} - d^{(2)} t r_1)^2 v_1(g)^{-1} + d^{(2)} Y^{-1} t d^{(2)}$$

(3.16)
where \( \mathbf{d} = (d^{(1)} \quad d^{(2)}) \) are as above

\[
Y = \begin{pmatrix} 1 & r_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & Y_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t r_1 & I \end{pmatrix}.
\]

(3.17)

This time we consider

\[
g_\epsilon = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} Y_\epsilon^{\frac{1}{2}} & 0 \\ 0 & Y_\epsilon^{-\frac{1}{2}} \end{pmatrix} k(Q)
\]

with

\[
Y_\epsilon = \begin{pmatrix} 1 & r_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} (1 - \epsilon) v_1 & 0 \\ 0 & Y_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t r_1 & I \end{pmatrix}.
\]

(3.19)

We have

\[
v_1(g_\epsilon) = (1 - \epsilon)v_1(g)
\]

and

\[
v_1(\gamma g_\epsilon)^{-1} = (1 - \epsilon)^{-1}(d^{(1)} - d^{(2)} t r_1)^2 v_1(g)^{-1} + d^{(2)} Y_1^{-1} t d^{(2)}
\]

\[
= v_1(g_\epsilon)^{-1} + (1 - (1 - \epsilon)^{-1}) d^{(2)} Y_1^{-1} d^{(2)}
\]

(3.20)

from (3.16). If \( d^{(2)} \neq 0 \), then \( v_1(\gamma g_\epsilon) > v_1(g_\epsilon) \) and we conclude that \( g \) is on the boundary of \( \mathcal{D}_n \) as before.

When \( c = 0 \) and \( d^{(2)} = 0 \) we have \( d^{(1)} = \pm 1 \), and so \( \gamma \in \Gamma_1 \). We decompose \( \gamma \) as in (3.10) and define \( \gamma_1 \) as in (3.11). By the construction of \( \mathcal{D}_n \), we have \( g_1(g) \in \mathcal{D}_{n-1} \) and \( g_1(\gamma g) = \gamma_1 g_1(g) \in \mathcal{D}_{n-1} \). By induction, we have that either \( \gamma_1 \) is the identity or \( g_1(g) \) is on the boundary of \( \mathcal{D}_{n-1} \). In the latter case we have that \( g \) is on the boundary of \( \mathcal{D}_n \), and so it remains to consider

\[
\gamma = \begin{pmatrix} \pm 1 & r_1 & \pm t_1 \mp r_1 \mp s_1 \\ 0 & I & \pm t s_1 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & \mp r_1 & I \end{pmatrix}.
\]

(3.21)

If any of the entries of \( r_1(\gamma) \) or \( s_1(\gamma) \) is not zero, then the corresponding entry of \( r_1(g) \) or \( s_1(g) \) is \( \pm \frac{1}{2} \) and so \( g \) is on the boundary of \( \mathcal{D}_n \). Similarly if \( t_1(\gamma) \neq 0 \), we have \( t_1(g) = \pm \frac{1}{2} \) and again \( g \) is on the boundary of \( \mathcal{D}_n \). If all of \( r_1, s_1, t_1 \) are 0, the sign must be \( - \) as \( \gamma \) is not the identity, and it follows that the first entry of \( r_1(g) \) is 0 and \( g \) is again on the boundary of \( \mathcal{D}_n \). \( \square \)

The following proposition records some useful properties of \( \mathcal{D}_n \). It and its proof are very similar to the analogous statement for the different fundamental domain used in [10], see proposition 3.1 there.

**Proposition 3.3.** Let \( g \in \mathcal{D}_n \) and write

\[
g = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} Y^{\frac{1}{2}} & 0 \\ 0 & Y^{-\frac{1}{2}} \end{pmatrix} k(Q), \quad Y = UV^t U,
\]

(3.22)
where $X$ is symmetric, $Y$ is positive definite symmetric, $U$ upper triangular unipotent, $V$ positive diagonal, and $Q \in U(n)$, and
\[
V = \begin{pmatrix} v_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & v_n \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & r_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & Y_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_1 & I \end{pmatrix}.
\] (3.23)

Then we have
\begin{enumerate}
\item $v_n \geq \frac{\sqrt{3}}{2}$ and $v_j \geq \frac{3}{4} v_{j+1}$ for $1 \leq j \leq n-1$,
\item for all $x = (x^{(1)} \ x^{(2)}) \in \mathbb{R}^n$
\[
xY^t x \approx_n v_1 (x^{(1)})^2 + x^{(2)} Y_1^t x^{(2)}.
\] (3.24)
\end{enumerate}

Proof. For the first, we observe that by the inductive construction of $D_n$, we have that
\[
g_{n-1}(g) = \begin{pmatrix} 1 & x_{n-1}(g) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1^2 & 0 \\ 0 & v_n^2 \end{pmatrix} \in D_1.
\] (3.25)

As $D_1$ is the standard fundamental domain for $\text{SL}(2, \mathbb{Z}) \backslash \text{SL}(2, \mathbb{R})$, we conclude that $v_n \geq \frac{\sqrt{3}}{2}$.

To demonstrate that $v_j \geq \frac{3}{4} v_{j+1}$, we note that by the construction of $D_n$, it suffices to consider only $j = 1$. We start with
\[
v_1^{-1} \leq c Y^t c + (cX + d)Y^{-1} (cX + d)
\] (3.26)
for any $(c \ d) \in \mathbb{Z}^2$ nonzero and primitive. Choosing $c = 0$ and $d = (0 \ 1 \ 0 \ \cdots \ 0)$, we have
\[
v_1^{-1} \leq v_1^{-1} (r_1^{(1)})^2 + v_2^{-1},
\] (3.27)
where $r_1^{(1)}$ is the first entry of $r_1$. Since $0 \leq r_1^{(1)} \leq \frac{1}{2}$, we conclude that $v_1 \geq \frac{3}{4} v_2$.

To demonstrate the second part of the proposition, we let $y_1, \ldots, y_n$ denote the rows of
\[
Y_{\frac{1}{2}} = \begin{pmatrix} 1 & r_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} v_1^2 & 0 \\ 0 & Y_1^2 \end{pmatrix}.
\] (3.28)

Setting $y = x_2 y_2 + \cdots + x_n y_n$, where the $x_j$ are the entries of $x$, our aim is to prove that for some constants $0 < c_1 < 1 < c_2$ depending only on $n$,
\[
c_1 \left( ||y_1||^2 x_1^2 + ||y||^2 \right) \leq ||x_1 y_1 + y||^2 \leq c_2 \left( ||y_1||^2 x_1^2 + ||y||^2 \right),
\] (3.29)
from which the lower bound in (3.24) follows as $||y_1||^2 \geq v_1$. The upper bound in (3.24) follows from (3.29) and $v_1 \gg ||y_1||^2$, which is verified below, see (3.35). Expanding the expression in the middle of (3.29), we find that it is enough to show that
\[
2|x_1 y_1 y| \leq (1 - c_1) \left( ||y_1||^2 x_1^2 + ||y||^2 \right),
\] (3.30)
and
\[ 2|x_1 y_1 \cdot y| \leq (c_2 - 1) \left( ||y_1||^2 x_1^2 + ||y||^2 \right). \]  
(3.31)
The upper bound (3.31) is trivial if \( c_2 = 2 \), and the upper bound (3.30) would follow from
\[ ||y_1 \cdot y|| \leq (1 - c_1) ||y_1|| ||y||. \]  
(3.32)

We let \( 0 < \phi_1 < \pi \) denote the angle between \( y_1 \) and \( y \) and \( 0 < \phi_2 < \pi \) denote the angle between \( y_1 \) and the hyperplane \( \text{span}(y_2, \ldots, y_n) \). We have \( \phi_2 \leq \min(\phi_1, \pi - \phi_1) \), and so \( |\cos \phi_1| \leq |\cos \phi_2| \). We bound \( \cos \phi_2 \) away from 1 by bounding \( \sin \phi_2 \) away from 0.

We have
\[ \sin \phi_2 = \frac{||y_1 \wedge \cdots \wedge y_n||}{||y_1|| ||y_2 \wedge \cdots \wedge y_n||} = \frac{v_1^\frac{1}{2}}{||y_1||}, \]  
(3.33)
so it suffices to show that \( v_1^\frac{1}{2} \gg ||y_1|| \). Here \( \wedge \) denotes the usual wedge product on \( \mathbb{R}^n \) and the norm on \( \wedge^k \mathbb{R}^n \) is given by
\[ ||a_1 \wedge \cdots \wedge a_k||^2 = \det \begin{pmatrix} a_1 & \cdots & a_k \end{pmatrix} \begin{pmatrix} a_1^t & \cdots & a_k^t \end{pmatrix}. \]  
(3.34)

Using the inductive construction of \( D_n \) and the fact that the entries of \( r_1(Y), r_1(Y_1), \ldots \) are at most \( \frac{1}{2} \) in absolute value, we observe that \( U \) has entries bounded by a constant depending only on \( n \). We find that
\[ ||y_1||^2 \ll v_1 + \cdots + v_n \ll v_1 \]  
(3.35)
with the implied constant depending on \( n \).

### 3.2 Shape of the cusp

As explicated in [1] and [2], the cusp of \( \Gamma \backslash G \) can be partitioned into \( 2^n - 1 \) box-shaped regions. These regions are in correspondence with the conjugacy classes of proper parabolic subgroups of \( G \) and are formed as \( K \) times the product of three subsets, one for each of the components – nilpotent, diagonal, and semisimple – of the Langlands decomposition of \( P \).

In what follows we use the fundamental domain \( D_n \) constructed in section 3.1 to prove a variation of this fact, although only for the maximal parabolic subgroups \( (2.17), (2.18) \). Our main result for this section is proposition 3.6 which roughly states that if \( g \in G \) is close enough the boundary in a precise sense, then \( g \) can be brought into \( D_n \) by an element \( \gamma \) in some maximal parabolic subgroup which depends on the way \( g \) approaches the boundary.

For \( 1 \leq l < n \) we denote by \( \Gamma_{l,1} \) and \( \Gamma_{l,2} \) the subgroups of \( \Gamma_l = \Gamma \cap P_l \) given by
\[ \Gamma_{l,1} = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & t A^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} : A \in \text{GL}(l, \mathbb{Z}) \right\} \]  
(3.36)
and
\[
\Gamma_{l,2} = \left\{ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & I & 0 \\ 0 & C & 0 & D \end{pmatrix} : \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{Sp}(n-l, \mathbb{Z}) \right\}.
\] (3.37)

For \( l = n \), we set
\[
\Gamma_{n,1} = \left\{ \begin{pmatrix} A & 0 \\ 0 & t^{-1}A \end{pmatrix} : A \in \text{GL}(n, \mathbb{Z}) \right\},
\] (3.38)

and we let \( \Gamma_{n,2} \) be trivial. We now define for \( g \in G \) and \( 1 \leq l \leq n \),
\[
v_{l}(\Gamma_{l}g) = \min_{\gamma \in \Gamma_{l}} v_{l}(\gamma g) = \min_{\gamma \in \Gamma_{l,1}} v_{l}(\gamma g)
\] (3.39)

and, for \( 1 \leq l < n \),
\[
v_{l+1}(\Gamma_{l}g) = \max_{\gamma \in \Gamma_{l}} v_{l+1}(\gamma g) = \max_{\gamma \in \Gamma_{l,2}} v_{l+1}(\gamma g).
\] (3.40)

We note that in the proof of proposition 3.2, we saw that the maximum in (3.40) does exist. As for the minimum in (3.39), we simply note that
\[
v_{l}(AU_{l}V_{l}^{t}U_{l}^{t}A) = aU_{l}V_{l}^{t}U_{l}^{t}a
\] (3.41)

where \( a \) is the last row of \( A \in \text{GL}(l, \mathbb{Z}) \), so the positive definiteness of \( U_{l}V_{l}^{t}U_{l}^{t} \) implies that there are only finitely many values of \( v_{l}(AU_{l}V_{l}^{t}U_{l}^{t}A) \) below a given bound.

We now define a fundamental domain \( D'_{l} \) for the action of \( \text{GL}(l, \mathbb{Z}) \) on \( l \times l \) positive definite symmetric matrices. We set
\[
D'_{1} = \{ y > 0 \}
\]

and
\[
D'_{2} = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_{1} & 0 \\ 0 & v_{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} : 0 \leq r \leq \frac{1}{2}, \ r^{2} + \frac{v_{1}}{v_{2}} \geq 1 \right\},
\] (3.42)

the standard fundamental domain for \( \text{GL}(2, \mathbb{Z}) \) acting on \( 2 \times 2 \) positive definite symmetric matrices. The domain \( D'_{l} \) for \( l > 2 \) is then defined inductively as the set of all
\[
Y = \begin{pmatrix} 1 & r \\ 0 & I \end{pmatrix} \begin{pmatrix} v_{1} & 0 \\ 0 & Y_{1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}
\] (3.43)

such that
1. \( v_{1}(Y) \geq v_{1}(AY^{t}A) \) for all \( A \in \text{GL}(l, \mathbb{Z}) \),
2. \( Y_{1} \in D'_{l-1} \), and
3. \( |r_{j}| \leq \frac{1}{2} \) and \( 0 \leq r_{1} \leq \frac{1}{2} \) where \( r_{j} \) are the entries of \( r \).

This is in fact the set of \( Y \) such that \( Y^{-1} \) is in Grenier’s fundamental domain, see [6] and [12], so we do not prove that \( D'_{l} \) is a fundamental domain here. We do however record the following properties of \( D'_{l} \).
Lemma 3.4. Let $UV^tU \in \mathcal{D}_l'$ with

$$V = \begin{pmatrix} v_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & v_l \end{pmatrix}$$

(3.44)

positive diagonal and $U$ upper triangular unipotent. Then we have

1. $v_j \geq \frac{3}{4} v_{j+1}$ for $1 \leq j < l$,

2. for any $x \in \mathbb{R}^l$,

$$xUV^tU^tx \asymp xV^tx$$

(3.45)

with implied constant depending only on $l$, and

3.

$$\min_{A \in \text{GL}(l, \mathbb{Z})} v_l(AUV^tU^tA) \asymp v_l(UV^tU)$$

(3.46)

with implied constant depending only on $l$.

Proof. The first and second parts are proved in proposition 3.1 of [10]. To prove the third part, we note that with $a$ the last row of $A$,

$$v_l(AUV^tU^tA) = aUV^tU^ta \gg aV^ta,$$

(3.47)

by the second part of the lemma. Applying the first part of the lemma we have $aV^ta \gg v_l|a|^2 \geq v_l$, and (3.46) follows.

As the proof is almost identical to the proof of the third part of lemma 3.4 we record the following lemma for later use.

Lemma 3.5. If $g \in \mathcal{D}_n$, then for all $1 \leq l < n$,

$$v_l(\Gamma_l g) \asymp v_l(g)$$

(3.48)

with the implied constant depending only on $n$.

Proof. We recall from the second part of proposition 3.3 that for $x \in \mathbb{R}^l$,

$$xU_lV_l^tU_l^tx \gg xV_l^tx.$$  

(3.49)

We have

$$v_l(\Gamma_l g) = \min_{c \in \mathbb{Z}^l} cU_lV_l^tU_l^tc \gg \min_{c \in \mathbb{Z}^l, c \neq 0} cV_l^tc.$$  

(3.50)

Now as $c \neq 0$, we have $c_j^2 \geq 1$ for some $1 \leq j \leq l$, and so

$$v_l(\Gamma_l g) \gg v_j(g) \gg v_l(g)$$

(3.51)

by the first part of proposition 3.3.
We are now ready to prove the main result for this section.

**Proposition 3.6.** For \(1 \leq l \leq n\), there are constants \(a_l > 0\) such that for \(l < n\), if \(g \in G\) satisfies \(v_l(\Gamma_l g) = a_l v_{l+1}(\Gamma_l g)\), and for \(l = n\) if \(g \in G\) satisfies \(v_n(\Gamma_n g) = a_n\), then there exists \(\gamma \in \Gamma_l\) so that \(\gamma g \in D_n\). Moreover, for this \(\gamma\) we have \(v_l(\Gamma_l g) \approx v_l(\gamma g)\) and, for \(l < n\), \(v_{l+1}(\Gamma_l g) = v_{l+1}(\gamma g)\).

We remark that this proposition can be extended to any of the parabolic subgroups \(P_L\) of \(G\) by taking intersections of the maximal parabolics. However some care needs to be taken regarding the possible non-uniqueness of the \(\gamma\) bringing \(g\) into \(D_n\). Since it is unnecessary for our goals, we do not discuss this here.

**Proof.** By multiplying \(g\) by

\[
\gamma_1 = \begin{pmatrix} A' & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & t(A')^{-1} & 0 \\ 0 & C & 0 & D \end{pmatrix} \in \Gamma_l, \tag{3.52}
\]

we may assume that \(U_l V_l^t U_l \in D_l'\) and

\[
\begin{pmatrix} I & X_l \\ 0 & I \end{pmatrix} \begin{pmatrix} Y_l^\frac{1}{2} & 0 \\ 0 & tY_l^{-\frac{1}{2}} \end{pmatrix} \in D_{n-l}. \tag{3.53}
\]

We recall that for \(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\),

\[
v_l(\gamma)^{-1} = c Y^t c + (cX + d) Y^{-1} t(cX + d) \tag{3.54}
\]

where \(c, d\) are the first rows of \(C, D\). Now, writing \(c = (c^{(1)} \ c^{(2)})\), \(d = (d^{(1)} \ d^{(2)})\) and

\[
X = \begin{pmatrix} T_l + R_l X_l^t R_l & S_l + R_l X_l \\ tS_l + X_l^t R_l & X_l \end{pmatrix}, \tag{3.55}
\]

\[
Y = \begin{pmatrix} U_l & R_l \\ 0 & I \end{pmatrix} \begin{pmatrix} V_l & 0 \\ 0 & Y_l \end{pmatrix} \begin{pmatrix} U_l^t & 0 \\ 0 & I \end{pmatrix}, \tag{3.56}
\]

see (2.15), we obtain

\[
v_l(\gamma)^{-1} = c^{(1)} U_l V_l^t U_l ' c^{(1)} + (c^{(1)}R_l + c^{(2)}) Y_l^t (c^{(1)}R_l + c^{(2)})
+ (c^{(1)}(T_l - S_l^t R_l) + c^{(2)} t S_l + d^{(1)} - d^{(2)} t R_l) t U_l^{-1} V_l^{-1} U_l^{-1}
+ t (c^{(1)}(T_l - S_l^t R_l) + c^{(2)} t S_l + d^{(1)} - d^{(2)} t R_l)
+ (c^{(1)}(S_l + R_l X_l) + c^{(2)} X_l + d^{(2)}) Y_l^{-1}
+ t (c^{(1)}(S_l + R_l X_l) + c^{(2)} X_l + d^{(2)}). \tag{3.57}
\]
If \( c^{(1)} \neq 0 \), then, since \( U_l V_l U_l \in D'_l \), we have
\[
v_l(\gamma g)^{-1} \geq c^{(1)} U_l V_l U_l c^{(1)} \gg c^{(1)} U_l c^{(1)} \gg v_l
\] (3.58)
by the second part of lemma 3.4. Since, for \( l < n \),
\[
\begin{pmatrix}
I & X_l \\
0 & I
\end{pmatrix}
\begin{pmatrix}
Y_l^\frac{1}{2} & 0 \\
0 & tY_l^{-\frac{1}{2}}
\end{pmatrix} \in D_{n-l},
\] (3.59)
we have \( v_{l+1} \gg 1 \), see proposition 3.3, and so \( v_l \gg a_l \) by the hypothesis. For \( l = n \), we directly have \( v_n \gg a_n \) by hypothesis. Since also \( v_1 \gg v_l \) by lemma 3.4, we have \( v_1 v_l \gg a_l^2 \), so by taking \( a_l \) to be a sufficiently large constant, it follows that \( v_1 \geq v_1(\gamma g) \).

For \( l < n \), if \( c^{(1)} = 0 \) but \( (c^{(2)} \ d^{(2)}) \neq 0 \), then we have
\[
v_1(\gamma g)^{-1} \geq c^{(2)} Y_l c^{(2)} + (c^{(2)} X_l + d^{(2)}) Y_l^{-1} (c^{(2)} X_l + d^{(2)}) \geq v_{l+1}(\gamma g)^{-1}
\] (3.60)
since \( g_l(e) \in D_{n-l} \). We have \( v_{l+1}^1 \geq a_l v_{l-1}^1 \gg a_l v_{l-1} \), so \( v_{l+1}^1 \geq v_{l+1}^1 \) for \( a_l \) sufficiently large, and it follows that \( v_1 \geq v_1(\gamma g) \).

Now, if \( l = n \) or if \( c^{(1)} \), \( c^{(2)} \), and \( d^{(2)} \) are all 0, then we have \( d^{(1)} \neq 0 \) and
\[
v_1(\gamma g)^{-1} = d^{(1)} l U_l V_l U_l d^{(1)} \geq v_{l-1}^1
\] (3.61)
as \( U_l V_l U_l \in D'_l \). We have verified that for any \( \gamma \in \Gamma \), \( v_1 \leq v_1(\gamma g) \), which is the first condition defining the fundamental domain \( \mathcal{D}_n \).

Restricting to \( \gamma \in \Gamma_1 \), which fixes \( v_1(g) \), the same argument as above shows that \( v_2(\gamma g) \geq v_2(\gamma g) \) for all \( \gamma \in \Gamma_1 \). Continuing this way, we find that the \( v_j \), \( 1 \leq j \leq l \) are all maximal (over \( \Gamma_{j,2} \)), and so, by the construction of \( \mathcal{D}_n \), there is a \( \gamma \in \Gamma_1 \) with the form
\[
\gamma = \begin{pmatrix}
A & B \\
0 & tA^{-1}
\end{pmatrix},
\] (3.62)
where \( A \) is upper-triangular unipotent (so \( \gamma \in \Gamma_l \) for all \( l \)) such that \( \gamma g \in \mathcal{D}_n \).

4 Proof of the main theorem

In the following subsection we gather some technical lemmas regarding the height function needed in the proof of theorem 1.2, see section 4.2. This height function is motivated by the following corollary from [10].

**Corollary 4.1.** For a Schwartz function \( f \in S(\mathbb{R}^n) \) and \((h,g) \in \mathcal{D} \), and \( A > 0 \), we have
\[
\Theta_f(h,g) \ll f_A (\det Y)^{\frac{1}{2}} (1 + x Y x)^{-A}
\] (4.1)
where
\[
g = \begin{pmatrix}
I & X \\
0 & I
\end{pmatrix}
\begin{pmatrix}
Y^\frac{1}{2} & 0 \\
0 & tY^{-\frac{1}{2}}
\end{pmatrix} k(Q).
\] (4.2)

We remark that in [10] this is obtained as a consequence of full asymptotics of the theta function in the various cuspidal regions. We also remark that in [10] we use a slightly different fundamental domain, however an examination of the proof there shows that the fundamental domain can be replaced by any set satisfying the conclusions of proposition 3.3.
4.1 Heights and volumes

For a fixed $A > 0$ sufficiently large depending only on $n$, we define the function $D : \hat{\Gamma}(H \rtimes G) \to \mathbb{R}_{>0}$ by

$$D \left( \hat{\Gamma}(h, g) \right) = \det Y(\gamma g) \left( 1 + x(uh, h\gamma^{-1})Y(\gamma g)^{t}x(uh, h\gamma^{-1}) \right)^{-A}$$

where $(uh, \gamma) \in \hat{\Gamma}$ is so that $(uh, \gamma)(h, g) \in \hat{D}$. Here we write $h \in H$ as $h = (x(h), y(h), t(h))$. For completeness, in case there are more than one $(uh, \gamma) \in \hat{\Gamma}$ such that $(uh, \gamma)(h, g) \in \hat{D}$, then we define $D \left( \hat{\Gamma}(h, g) \right)$ to be the largest of the finite number of values $(4.3)$. This point is not essential as these values are within constant multiples of each other; see the argument in lemma 4.4 for how this can be proved.

We begin by analyzing the growth of the height function. We let $\hat{\mu}$ denote the Haar probability measure on $\hat{\Gamma}(H \rtimes G)$, which is $\mu$, the Haar probability measure on $\Gamma \setminus G$, times the Lebesgue measure on the entries of $h = (x, y, t)$.

**Lemma 4.2.** For $R \geq 1$ we have

$$\hat{\mu} \left\{ \left( \hat{\Gamma}(h, g) \in \hat{\Gamma}(H \rtimes G) : D(\hat{\Gamma}(h, g)) \geq R \right) \right\} \ll R^{-\frac{n+2}{2}}$$

with the implied constant depending only on $n$.

**Proof.** We recall that $g \in D_n$ is written as

$$g = \begin{pmatrix} U & X^tU^{-1} \\ 0 & tU^{-1} \end{pmatrix} \begin{pmatrix} V^\frac{1}{2} & 0 \\ 0 & V^{-\frac{1}{2}} \end{pmatrix} k(Q)$$

for $U$ upper-triangular unipotent, $X$ symmetric, $Q \in U(n)$, and

$$V = V(g) = \begin{pmatrix} v_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & v_n \end{pmatrix}$$

positive diagonal. The Haar measure $\mu$ on $G$ is then proportional to Lebesgue measure with respect to the entries of $X$ and the off-diagonal entries of $U$, $U(n)$-Haar measure on $Q$, and the measure given by

$$v_1^{-n-1}v_2^{-n} \cdots v_n^{-2}dv_1dv_2 \cdots dv_n$$

on $V$.

By proposition 3.3, we observe that the set in $(4.4)$ is contained in the set of $(h, g)$ satisfying $v_j \geq cv_{j+1}$ for all $1 \leq j < n$ and some $c > 0$ in addition to $\det Y \geq R$ and $xY^tx \leq R^{-\frac{1}{2}}(\det Y)^{\frac{1}{2}}$. Moreover, the variables $x, y, t$ as well as $U, X$ are constrained to compact sets, and so the measure of the set $(4.4)$ is

$$\ll R^{-\epsilon} \int \cdots \int_{v_{j} \geq cv_{j+1}} v_1^{-n-1+\epsilon}v_2^{-n-\frac{1}{2}+\epsilon} \cdots v_n^{-\frac{3}{2}+\epsilon}dv_1dv_2 \cdots dv_n,$$
where $\epsilon = \frac{n}{2A}$.

Changing variables $v_j = \exp(u_j)$, the integral in (4.8) is

$$
R^{-\epsilon} \int \cdots \int_{u_{j-1} + 1 \geq \log c} \exp \left( - (n + \frac{1}{2} - \epsilon)u_1 - (n - \frac{1}{2} - \epsilon)u_2 - \cdots - (\frac{3}{2} - \epsilon)u_n \right) du_1 du_2 \cdots du_n. \quad (4.9)
$$

We now make the linear change of variables $s_j = u_j - u_{j+1}$ for $j < n$ and $s_n = u_1 + \cdots + u_n$. This transformation has determinant $n$ and its inverse is given by

$$
u_j = -\frac{1}{n} \sum_{1 \leq i < j} is_i + \frac{1}{n} \sum_{j \leq i < n} (n - i)s_i + \frac{1}{n}s_n. \quad (4.10)
$$

We find that the exponent in (4.9) is then

$$
- \sum_{1 \leq j \leq n} (n - j + \frac{3}{2} - \epsilon)u_j = - \left( \frac{n + 2}{2} - \epsilon \right) s_n - \sum_{1 \leq j < n} \frac{j(n - j)}{2}s_j. \quad (4.11)
$$

As $\frac{j(n-j)}{2} > 0$ for $j < n$, the bound (4.4) follows.

Lemma 4.3. Let $g, g_0 \in G$ with $\|g_0 - I\| \leq 1$, then

$$
v_l(g) \approx v_l(g_0), \quad v_l(\Gamma_l g) \approx v_l(\Gamma_l g_0), \quad v_{l+1}(\Gamma_l g) \approx v_{l+1}(\Gamma_l g_0) \quad (4.12)
$$

for all $1 \leq l \leq n$ with implied constants depending only on $n$.

Proof. We first note that we may in fact work with $\|I - g_0\| \leq \epsilon$ as then the statement would follow by repeated application of the estimates. In fact, we may assume $\|I - g_0^{-1}\| \leq \epsilon$ as well. Now write

$$
g = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} Y^{\frac{1}{2}} & 0 \\ 0 & \imath Y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} R & -S \\ S & R \end{pmatrix}, \quad (4.13)
$$

with $R + iS \in U(n)$, so in particular $R^tR + S^tS = I$. With $g_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we have from (2.10) that

$$
Y(gg_0)^{-1} = \imath Y^{-\frac{1}{2}} \left( SA^tA^tS + RC^tA^tS + SA^tC^tR + RC^tCR \\
+ SB^tB^tS + RD^tB^tS + SB^tD^tR + RD^tD^tR \right) Y^{-\frac{1}{2}}. \quad (4.14)
$$
As $\|g_0 - I\| \leq \epsilon$, we have

$$^{t}Y(gg_0)^{-\frac{1}{2}} = ^{t}Y^{-\frac{1}{2}}(I + O(\epsilon)).$$  \hfill (4.15)

On the other hand, letting $y_j$ and $y_j'$ denote the rows of $^{t}Y^{-\frac{1}{2}}$ and $^{t}Y(gg_0)^{-\frac{1}{2}}$, we have

$$v_1(g)^{-\frac{1}{2}} = \|y_1\|, \quad v_1(gg_0)^{-\frac{1}{2}} = \|y_1'\|$$  \hfill (4.16)

and for $2 \leq l \leq n$,

$$v_l(g)^{-\frac{1}{2}} = \frac{\|y_1 \land \cdots \land y_l\|}{\|y_1 \land \cdots \land y_{l-1}\|}, \quad v_l(gg_0)^{-\frac{1}{2}} = \frac{\|y_1' \land \cdots \land y'_l\|}{\|y_1' \land \cdots \land y'_{l-1}\|},$$  \hfill (4.17)

and so $v_l(g) \asymp v_l(gg_0)$ follows.

Now let $\gamma \in \Gamma_l$ be so that $v_l(\Gamma_l g) = v_l(\gamma g)$. We have

$$v_l(\Gamma_l gg_0) \leq v_l(\gamma gg_0) \ll v_l(\gamma g) = v_l(\Gamma_l g),$$  \hfill (4.18)

and the reverse bound follows by switching the roles of $g$ and $gg_0$, and using $\|g_0^{-1} - I\| \leq \epsilon$. The final estimate in (4.12) is proved in the same way. \hfill \Box

**Lemma 4.4.** If $(h, g), (h_0, g_0) \in G$ with $\|g_0 - I\| \leq 1$ and $h_0 = (x_0, y_0, t_0)$ satisfies $\|x_0\|, \|y_0\| \leq 1$, then

$$D(\bar{\Gamma}(h, g)) \asymp D(\bar{\Gamma}(h, g)(h_0, g_0)).$$  \hfill (4.19)

**Proof.** We observe as in lemma 4.3 we may in fact assume

$$\|g_0 - I\| \leq \epsilon, \quad \|x_0\| \leq \epsilon, \quad \text{and} \quad \|y_0\| \leq \epsilon.$$  \hfill (4.20)

Moreover, it suffices to show that $D(\bar{\Gamma}(h, g)(h_0, g_0)) \gg D(\bar{\Gamma}(h, g))$ as the other inequality follows from switching $(h, g)$ and $(h, g)(h_0, g_0)$ as we may assume in addition that $(h_0, g_0)^{-1} = (h_0^{-g_0}, g_0^{-1})$ also satisfies (4.20).

Now let us suppose that $(h, g) \in \bar{D}$ so that

$$D(\bar{\Gamma}(h, g)) = (\det Y(g))(1 + x(h)Y(g)^{t}x(h))^{-A}.$$  \hfill (4.21)

Let $1 \leq l \leq n$ be the largest index such that $v_l(g) \geq av_{l+1}(g)$ (or $v_n(g) \geq a$ when $l = n$) where $a$ is a constant determined by the constants in proposition 4.6 and lemma 4.3. If no such $l$ exists, then we have $v_j(g) \asymp 1$ for all $j$, and lemma 4.3 implies that $v_j(gg_0) \asymp 1$ as well. The bounds

$$D(\bar{\Gamma}(h, g)(h_0, g_0)) \gg 1 \gg D(\bar{\Gamma}(h, g))$$  \hfill (4.22)

then follow immediately.

Now assuming that such a maximal $l$ exists, we have that $v_j(g) \asymp 1$ for all $j > l$. For these $j$, lemma 4.3 then implies that $v_j(gg_0) \asymp 1$, and it follows that $v_j(\gamma gg_0) \asymp 1$ for $\gamma \in \Gamma_l$ such that $g_l(\gamma gg_0) \in D_{n-l}$, see (2.16). By lemma 4.3 we have $v_l(\Gamma_l g) \gg v_l(g)$, and so

$$v_l(\Gamma_l g) \gg av_{l+1}(g) = av_{l+1}(\Gamma_l g)$$  \hfill (4.23)
since \( g_l(g) \in D_{n-l} \). Via lemma \[4.3\], this implies that \( v_l(\Gamma_1gg_0) \gg av_{l+1}(\Gamma_1gg_0) \), so \( a \) can be chosen large enough so that \( gg_0 \) satisfies the hypotheses of proposition \[5.6\], and we let \( \gamma \in \Gamma_l \) be so that \( \gamma gg_0 \in D \).

We write

\[
\gamma = \begin{pmatrix}
A_1 & * & * & * \\
0 & * & * & * \\
0 & 0 & * & 0 \\
0 & * & * & *
\end{pmatrix},
\]

\( (4.24) \)

where \( A_1 \in \text{GL}(l, \mathbb{Z}) \). From the estimates above, we have

\[
\det Y(\gamma gg_0) \simeq \det U_l(\gamma gg_0)V_l(\gamma gg_0)^tU_l(\gamma gg_0) = \det U_l(gg_0)V_l(gg_0)^tU_l(gg_0) \\
\simeq \det U_l(g)V_l(g)^tU_l(g) \approx \det Y(g),
\]

\( (4.25) \)

where the equality follows from the fact that \( \gamma \in \Gamma_l \) normalizes the first matrix in \( (2.15) \) and \( \det A_1 = \pm 1 \).

It now remains to consider the factors \( 1 + \mathbf{x}(\gamma gg_0)^t\mathbf{x}(\gamma gg_0) \) in the definition of the height function \( D \). Let \( u = (m, n, 0) \) with \( m, n \in \mathbb{Z}^n \) be so that \( (uh_\gamma, \gamma)(h, g)(h, g_0) \in \hat{D} \). Recalling the definition of \( h_\gamma = (r, s, 0) \) following \( (3.1) \), we have that \( r^{(1)} = 0 \) where \( r = (r^{(1)}, r^{(2)}) \). Moreover, writing \( \mathbf{x} = (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) \), we have \( \mathbf{x}^{(1)}((hh_0^{g^{-1}})^{g^{-1}}) = \mathbf{x}^{(1)}(hh_0^{g^{-1}})A_1^{-1} \).

Using proposition \[3.3\] together with the fact that \( u \) minimizes the absolute values of the entries of \( \mathbf{x}(uh_\gamma(hh_0^{g^{-1}})^{g^{-1}}) \), we have

\[
1 + \mathbf{x}(uh_\gamma(hh_0^{g^{-1}})^{g^{-1}})Y(\gamma gg_0)\mathbf{x}(uh_\gamma(hh_0^{g^{-1}})^{g^{-1}}) \\
\ll 1 + \mathbf{x}(h_\gamma(hh_0^{g^{-1}})^{g^{-1}})Y(\gamma gg_0)\mathbf{x}(h_\gamma(hh_0^{g^{-1}})^{g^{-1}}),
\]

\( (4.26) \)

and from the estimates above on the \( v_j(\gamma gg_0) \) for \( j > l \), we have

\[
1 + \mathbf{x}(h_\gamma(hh_0^{g^{-1}})^{g^{-1}})Y(\gamma gg_0)\mathbf{x}(h_\gamma(hh_0^{g^{-1}})^{g^{-1}}) \\
\asymp 1 + \mathbf{x}^{(1)}(h_\gamma(hh_0^{g^{-1}})^{g^{-1}})U_l(\gamma gg_0)V_l(\gamma gg_0)^tU_l(\gamma gg_0)^t\mathbf{x}(h_\gamma(hh_0^{g^{-1}})^{g^{-1}}).
\]

\( (4.27) \)

Using the expressions for \( h_\gamma \), \( (hh_0^{g^{-1}})^{g^{-1}} \), and that

\[
U_l(\gamma gg_0)V_l(\gamma gg_0)^tU_l(\gamma gg_0) = A_1U_l(gg_0)V_l(gg_0)^tU_l(gg_0)^tA_1,
\]

\( (4.28) \)

the right side of \( (4.27) \) is equal to

\[
1 + \mathbf{x}^{(1)}(hh_0^{g^{-1}})U_l(gg_0)V_l(gg_0)^tU_l(gg_0)^t\mathbf{x}^{(1)}(hh_0^{g^{-1}}) \asymp 1 + \mathbf{x}(hh_0^{g^{-1}})Y(gg_0)^t\mathbf{x}(hh_0^{g^{-1}})
\]

\( (4.29) \)

by the above bounds on \( v_j(gg_0) \) for \( j > l \).

Recalling that

\[
g = \begin{pmatrix}
I & X(g) \\
0 & I
\end{pmatrix} \begin{pmatrix}
Y(g)^{\frac{1}{2}} & 0 \\
0 & tY(g)^{-\frac{1}{2}}
\end{pmatrix} k(g)
\]

\( (4.30) \)
with \( k(g) \in K = G \cap \text{SO}(2n, \mathbb{R}) \), we set \( h_0' = h_0^{k(g)^{-1}} \) and note that
\[
||x(h_0')||^2 + ||y(h_0')||^2 = ||x(h_0)||^2 + ||y(h_0)||^2. \tag{4.31}
\]
Since \( Y(gg_0) = Y(g)\frac{1}{2}Y(k(g)g_0)^{\dagger}Y(g)^{\frac{1}{2}} \) and \( x(hh_0^{-1}) = x(h) + x(h_0')Y(g)^{-\frac{1}{2}} \), the right side of (4.29) is equal to
\[
1 + x(h)Y(g)^{\frac{1}{2}}Y(k(g)g_0)^{\dagger}Y(g)^{-\frac{1}{2}}x(h)
+ 2x(h)Y(g)^{\frac{1}{2}}Y(k(g)g_0)^{\dagger}x(h_0') + x(h_0')Y(k(g)g_0)^{\dagger}x(h_0'). \tag{4.32}
\]
We have that \( ||g_0 - I|| \leq \epsilon \) implies \( Y(k(g)g_0) = I + O(\epsilon) \) as in (4.14), so if (4.31) is at most \( \epsilon^2 \) as well, with \( \epsilon \) sufficiently small, then (4.32) is
\[
\approx 1 + x(h)Y(g)^{\dagger}x(h), \tag{4.33}
\]
where we have used
\[
2|x(h)Y(g)^{\frac{1}{2}}Y(k(g)g_0)^{\dagger}x(h_0')|
\leq \sqrt{|x(h_0')Y(k(g)g_0)^{\dagger}x(h_0')|} (x(h)Y(g)^{\dagger}x(h) + 1)
\ll (x(h)Y(g)^{\dagger}x(h) + 1) \tag{4.34}
\]
to bound the third term in (4.32). The bound \( D(\tilde{\Gamma}(h,g)(h_0, g_0)) \gg D(\tilde{\Gamma}(h,g)) \) now follows.

\section{Proof of theorem 1.2}

We recall the following lemma from [10].

**Lemma 4.5.** There exists a smooth, compactly supported function \( f_1 : \mathbb{R} \to \mathbb{R}_{\geq 0} \) such that
\[
\chi_1(x) = \sum_{j \geq 0} \left( f_1(2^j x) + f_1(2^j(1 - x)) \right), \tag{4.35}
\]
where \( \chi_1 \) is the indicator function of the open unit interval \((0, 1)\).

Now, following the method of [10], we define for a subset \( S \subset \{1, \ldots, n\} \) and \( j = (j_1, \ldots, j_n) \in \mathbb{Z}^n \) with \( j_i \geq 0 \),
\[
g_{j,S} = \begin{pmatrix} A_j E_S & 0 \\ 0 & A_j^{-1} E_S \end{pmatrix} \in G \tag{4.36}
\]
where \( E_S \) is diagonal with \((i, i)\) entry \(-1\) if \( i \in S \), \(+1\) if \( i \not\in S \), and
\[
A_j = \begin{pmatrix} 2^{j_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 2^{j_n} \end{pmatrix}. \tag{4.37}
\]
We also set \( h_S = (x_S, 0, 0) \in H \) where \( x_S \) has \( i \)th entry \(-1\) if \( i \in S \) and \( 0 \) if \( i \notin S \).

As in [10], we have

\[
\chi_B(x) = \sum_{j \geq 0} \sum_{S \subset \{1, \ldots, n\}} f_n ((xB^{-1} + x_S)A_jE_S),
\]

where \( \chi_B \) is the indicator function of the rectangular box \( B = (0, b_1) \times \cdots \times (0, b_n) \), \( B \) is the diagonal matrix with entries \( b_1, \ldots, b_n \),

\[
f_n(x_1, \ldots, x_n) = \prod_{1 \leq j \leq n} f_1(x_j),
\]

and the sums are over \( j \in \mathbb{Z}^n \) with nonnegative entries.

Let \( \psi : [0, \infty) \to [1, \infty) \) be an increasing function. Then for \( C > 0 \) we define \( G_j(\psi, C) \) to be the set of \( \Gamma(h, g) \in \tilde{\Gamma} \setminus (H \times G) \) such that

\[
D(\Gamma(h, g)(1, \begin{pmatrix} e^{-s}I & 0 \\ 0 & e^sI \end{pmatrix} I)(h_S, g_j, S))^{\frac{1}{4}} \leq C\psi(s)
\]

for all \( S \subset \{1, \ldots, n\} \) and \( s \geq 1 \).

**Lemma 4.6.** Suppose that \( \psi \) satisfies

\[
\int_0^\infty \psi(x)^{-(2n+4)} \, dx \leq C_\psi
\]

for some \( C_\psi \geq 1 \). Then

\[
\bar{\mu} \left( \tilde{\Gamma} \setminus (H \times G) - G_j(\psi, C) \right) \ll C_\psi C^{-(2n+4)}2^{j_1+\cdots+j_n}.
\]

**Proof.** Suppose that \( \Gamma(h, g) \notin G_j(\psi, C) \), so there exists \( S \subset \{1, \ldots, n\} \) and \( s \geq 1 \) such that

\[
D(\Gamma(h, g)(1, \begin{pmatrix} e^{-s}I & 0 \\ 0 & e^sI \end{pmatrix} I)(h_S, g_j, S))^{\frac{1}{4}} \geq C\psi(s).
\]

We let \( k \) be a nonnegative integer such that

\[
\frac{k}{K_j} \leq s < \frac{k + 1}{K_j},
\]

where \( K_j = K2^{j_1+\cdots+j_n} \) with \( K \) a constant to be determined. We have

\[
(1, \begin{pmatrix} e^{-s}I & 0 \\ 0 & e^sI \end{pmatrix} I)(h_S, g_j, S) = (1, \begin{pmatrix} e^{-K_j}I & 0 \\ 0 & e^{K_j}I \end{pmatrix} I)(h_S, g_j, S)(h_1, g_1),
\]
where, with $s' = s - \frac{k}{K_j}$,

$$h_1 = ((e^{s'} - 1)x_SA_j E_S, 0, 0), \quad g_1 = \begin{pmatrix} e^{-s'}I & 0 \\ 0 & e^{s'}I \end{pmatrix}. \quad (4.46)$$

As $|s'| \leq K_j^{-1}$, we can make $K$ sufficiently large so that $(h_1, g_1)$ satisfies the conditions of lemma 4.4. From this and the fact that $\psi$ is increasing, we have that

$$D(\tilde{\Gamma}(h, g)(1, \begin{pmatrix} e^{-\frac{k}{K_j}}I & 0 \\ 0 & e^{\frac{k}{K_j}}I \end{pmatrix}))(h_S, g_{j_S, S}) \gg C\psi \left( \frac{k}{K_j} \right). \quad (4.47)$$

By lemma 4.2 and the fact that right multiplication is volume preserving, we have that the set of $\tilde{\Gamma}(h, g)$ satisfying (4.47) has $\tilde{\mu}$-volume bounded by a constant times

$$C^{-2n-4} \psi \left( \frac{k}{K_j} \right)^{-2n-4}. \quad (4.48)$$

Bounding the volume of the set $\tilde{\Gamma}(H \rtimes G) - \mathcal{G}_j(\psi, C)$ by summing (4.48) over $S \subset \{1, \ldots, n\}$ and nonnegative $k \in \mathbb{Z}$, we obtain the bound

$$C^{-(2n+4)} \sum_{k \geq 0} \psi \left( \frac{k}{K_j} \right)^{-(2n+4)} \ll C^{-(2n+4)} \left( \psi(0) + \int_0^{\infty} \psi \left( \frac{x}{K_j} \right)^{-(2n+4)} dx \right) \quad (4.49)$$
as $\psi(x)$ is increasing. The bound (4.49) follows by changing variables.

We now proceed to the proof of theorem 1.2.

**Proof of theorem 1.2.** From (4.38) we express $\theta_S(M, X, x, y)$ as

$$\sum_{S \subset \{1, \ldots, n\}} \sum_{j \geq 0} \sum_{m \in \mathbb{Z}^n} f_n \left( \frac{1}{M}(m + x + MxSB)B^{-1}E_SA_j \right) e \left( \frac{1}{2}mx^t + \frac{1}{2}my^t \right). \quad (4.50)$$

We break the sum in (4.50) into terms $j$ such that $2^j b^{-1}_j \leq M$ for all $i$ and terms $\hat{j}$ such that $2^j b^{-1}_j > M$ for some $i$. Using (2.8), we write the first part as

$$e^{\frac{1}{2}xx^t} \frac{1}{M} \eta \left( \det B \right)^\frac{1}{2} \sum_{\hat{j} \geq 0} \sum_{2^j b^{-1}_j \leq M} 2^{-\frac{1}{2}(j_1 + \cdots + j_n)} \Theta_n \left( (h, g(MB, X))(h_S, g_{j_S, S}) \right). \quad (4.51)$$

where $h = (x, y - xX, 0)$ and

$$g(MB, X) = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} \frac{1}{M}B^{-1} & 0 \\ 0 & MB \end{pmatrix}. \quad (4.52)$$
Bounding this is the main work of the proof, but we first bound the contribution of the terms \( j \) with a large index.

Suppose that \( L \subset \{1, \ldots, n\} \) is not empty and that \( 2^i > b_j M \) for all \( l \in L \). Then the compact support of \( f_i \) implies that the sum over \( m^{(L)} \), the vector of entries of \( m \) with index in \( L \), has a bounded number of terms. We write

\[
mx^t m = m^{(L)}X^{(L,L')}^t m^{(L)} + 2m^{(L)}X^{(L',L')}^t m^{(L')} + m^{(L')}X^{(L',L')}^t m^{(L')},
\]

where \( L' \) is the complement of \( L \), and \( X^{(L,L')} \) is the matrix of entries of \( X \) with row and column indices in \( L_1 \) and \( L_2 \) respectively. We have (4.39) that \( f_n \left( \frac{1}{M}(m + x + MXS)B^{-1}ES \right) \) factors as

\[
f_{#L} \left( \frac{1}{M}(m^{(L)} + x(L) + MA(L))B^{(L,L)}^{-1}ES(L)L(A_{j(L,L)}(L,L)) \right)
\]
\[
\times f_{#L'} \left( \frac{1}{M}(m^{(L')} + x(L') + MA(L'))(B(L',L')^{-1}ES(L,L')(L',L')) \right),
\]

and so, by inclusion-exclusion and the boundedness of \( f_{#L} \), the terms \( j \) of (4.50) with \( j_i > b_j \) for some \( i \) is at most a constant times

\[
\sum_{L \subset \{1, \ldots, n\}} \sum_{S \subset L} \sum_{m^{(L)}} |\theta_{B(L)}(M, X^{L',L'}, x(L'), y(L') + m^{(L)}X^{L,L'})|,
\]

where the sum over \( m^{(L)} \) has a bounded number of terms, \( B^{(L)} \) is the edge of \( B \) associated to \( L' \), and we have used the decomposition (4.38) to express \( \theta_{B(L)}(M, X^{L',L'}, x(L'), y(L') + m^{(L)}X^{L,L'}) \) as

\[
\sum_{S' \subset L'} \sum_{j_{L'}} \sum_{m_{L'}} f_{#L'} \left( \frac{1}{M}(m^{(L')} + x(L') + MA(L'))(B(L',L')^{-1}ES(L',L')(L',L')) \right)
\]
\[
\times e \left( \frac{1}{2}m^{(L')X^{L',L'}}^t m^{(L')} + m^{(L')}^t(y^{(L')} + m^{(L)})X^{(L,L')} \right).
\]

When \( L = \{1, \ldots, n\} \), the corresponding part of (4.55) is clearly bounded. For any other \( L \), we may apply theorem 1.1 (emphasizing the importance of the uniformity in \( y \)) to conclude for any \( \epsilon > 0 \), there are full measure sets \( \mathcal{X}^{(n-\#L)} = \mathcal{X}^{(n-\#L)}(\epsilon) \) such that if \( X^{(L',L')} \in \mathcal{X}^{(n-\#L)} \), the corresponding part of (4.55) is \( \leq M^{\frac{n-\#L}{2}+\epsilon} \) for any \( \epsilon > 0 \). It follows that (4.55) is \( \leq M^{\frac{n}{2}} \) assuming that \( X \) is such that \( X^{(L',L')} \in \mathcal{X}^{(n-\#L)} \) for all nonempty \( L \subset \{1, \ldots, n\} \).

We now return to (4.51). We let \( \mathcal{X}_j(\psi, C) \) to be the set of \( (X, y) \) with all entries in the interval \((-\frac{1}{2}, \frac{1}{2}] \) such that there exist \( u \in (-\frac{1}{2}, \frac{1}{2})^n, A \in GL(n, \mathbb{R}) \) and \( T \in \mathbb{F}_{\text{sym}}^{n \times n} \) satisfying

\[
\sup_{B \in K} \left\| (B^{-1}A)^{-1} I \right\| A_j \leq \epsilon,
\]

(4.57)
\[ ||T|| \leq \varepsilon, \quad \text{and} \]
\[
\tilde{\Gamma} \left( (u, y - uX, 0), \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & tA^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ T & I \end{pmatrix} \right) (h_S, g_{j,S}) \in \mathcal{G}_j(\psi, C). \quad (4.58)
\]

Here we let \( \varepsilon > 0 \) be a sufficiently small constant, \( \mathcal{G}_j(\psi, C) \) is defined in \( (4.40) \), and \( K \) is the compact subset from the statement of theorem 1.2 identified with the compact subset of positive diagonal matrices \( B \) in the obvious way. We then set \( \mathcal{X}(\psi) \) to be the set of \( (X, y) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n \) such that
\[
(X + R, yR + s_R + s) \in \bigcup_{C > 0} \bigcap_{j \geq 0} \mathcal{X}_j(\psi, C 2^{a(j_1 + \cdots + j_n)})
\]
\[
\cap \bigcap_{L \subseteq \{1, \ldots, n\}, L \neq \emptyset} \{(X_1, y_1) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n : X_1^{(L', L')} \in \mathcal{X}^{(n - \#L)}\} \quad (4.59)
\]

for some \( (R, s) \in \mathbb{Z}^{n \times n} \times \mathbb{Z}^n \), where \( s_R \in \mathbb{R}^n \) has entries 0 or \( \frac{1}{2} \) depending on whether the corresponding diagonal entry of \( R \) is even or odd, and \( a > 0 \) is a constant to be determined.

We first verify that \( \mathcal{X}(\psi) \) has full measure, noting that it is enough to show that
\[
\bigcup_{C > 0} \bigcap_{j \geq 0} \mathcal{X}_j(\psi, C 2^{a(j_1 + \cdots + j_n)}) \quad (4.60)
\]
has full measure in the subset \( \mathcal{X}_0 \) of \( \mathbb{R}_n^{\text{sym}} \times \mathbb{R}^n \) having all entries in the interval \((-\frac{1}{2}, \frac{1}{2})\). Let us suppose that the Lebesgue measure of the complement of \( \mathcal{X}_j(\psi, C) \) in \( \mathcal{X}_0 \) is greater than some \( \delta > 0 \), which we assume is small. Now, with respect to the measure \( (\det A)^{-2n-1} \prod_{i,j} da_{ij} \) on \( \text{GL}(n, \mathbb{R}) \), the volume of the set of \( A \in \text{GL}(n, \mathbb{R}) \) satisfying \( (4.57) \) is within a constant multiple (depending on \( K \)) of \( 2^{-n(j_1 + \cdots + j_n)} \). Then, using the expression \((2.13), (2.14)\) for the Haar measure on \( H \rtimes G \), we have
\[
\tilde{\mu} \left( \tilde{\Gamma} \setminus (H \rtimes G) - \mathcal{G}_j(\psi, C) \right) \gg 2^{-n(j_1 + \cdots + j_n)}, \quad (4.61)
\]
with implied constant depending on \( K \). From lemma 4.6 it follows that
\[
\text{meas} (\mathcal{X}_0 - \mathcal{X}_j(\psi, C)) \ll C_{\psi} C^{-2n-4} 2^{(n+1)(j_1 + \cdots + j_n)} \quad (4.62)
\]
and we find that
\[
\text{meas} \left( \mathcal{X}_0 - \bigcup_{C > 0} \bigcap_{j \geq 0} \mathcal{X}_j(\psi, C 2^{a(j_1 + \cdots + j_n)}) \right) \ll \lim_{C \to \infty} C_{\psi} C^{-2n-4} \sum_{j \geq 0} 2^{((n+1) - a(2n+4))(j_1 + \cdots + j_n)} = 0 \quad (4.63)
\]
as long as \( a > \frac{n+1}{2n+4} \).
Now let us suppose that \((X, y) \in \mathcal{X}(\psi)\). By theorem 3.1 the size of the theta functions in (4.51) is invariant under the transformation on the left of (4.59), so we may assume that \(X \in \mathcal{X}_0\) as well. In particular, we have that \((X, y)\) is in \(\mathcal{X}_j(\psi, C2^{a(j_1+\cdots+j_n)})\) for some \(C > 0\) (independent of \(j\)) and all \(j \geq 0\). We have from corollary 4.1 and the definition of the height function \(D\) that

\[
\ll M^{\frac{n}{2}} \sum_{S \subseteq \{1, \ldots, n\}} \sum_{j \geq 0 \atop 2^j b_{ij}^1 \leq M} 2^{-\frac{j}{2}(j_1+\cdots+j_n)} D\left(\tilde{\Gamma}(h, g(MB, X))(h_s, g_j, s)\right) \quad (4.64)
\]

bounds (4.51). Now for all \(j \geq 0\) there is a \(\tilde{\Gamma}(h', g) \in \mathcal{G}_j(\psi, C2^{a(j_1+\cdots+j_n)})\) with \(g\) of the form

\[
g = \begin{pmatrix} I & X \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & tA^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \end{pmatrix} \quad (4.65)
\]

satisfying (4.57) and \(||T|| \leq \epsilon\) and \(h'\) having the for \((u, y - uX, 0)\) for some \(u \in (-\frac{1}{2}, \frac{1}{2})^n\).

We have

\[
(h', g)(1, \begin{pmatrix} 1/(M^\epsilon) & 0 \\ 0 & MI \end{pmatrix})(h_s, g_j, s) = (h, g(MB, X))(h_s, g_j, s)(h_1, g_1),
\]

where

\[
h_1 = \left(-x_S A_j E_S + x_S(BA)^{-1}A_j E_S + \frac{1}{M}(u - x)B^{-1}A_j E_j, 0, 0\right)
\]

and

\[
g_1 = g_{-1} \begin{pmatrix} BA & 0 \\ 0 & (BA)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \end{pmatrix} g_j, s.
\]

Recalling that \(2^{j_1} \leq M\), the conditions (4.57) and \(||T|| \leq \epsilon\) implies that \((h_1, g_1)\) satisfies the conditions of lemma 4.4 for all \(M\), which then implies

\[
D(\tilde{\Gamma}(h, g(MB, X))(h_s, g_j, s))^{\frac{1}{2}} \ll D\left(\tilde{\Gamma}(h', g)(1, \begin{pmatrix} 1/(M^\epsilon) & 0 \\ 0 & MI \end{pmatrix})(h_s, g_j, s)\right)^{\frac{1}{2}}
\]

\[
\ll C2^{a(j_1+\cdots+j_n)}\psi(\log M) \quad (4.69)
\]

since \((h', g) \in \mathcal{G}_j(\psi, C2^{a(j_1+\cdots+j_n)})\). Taking \(a = \frac{2n+3}{4n+8}\) so that \(\frac{n+1}{2n+4} < a < \frac{1}{2}\), it follows that (4.64) is bounded by

\[
\ll CM^{\frac{n}{2}}\psi(\log M) \sum_{j \geq 0} 2^{-\frac{j}{2}(j_1+\cdots+j_n)} \ll CM^{\frac{n}{2}}\psi(\log M),
\]

and theorem 1.2 follows. \(\square\)
References

[1] Armand Borel. *Introduction aux groupes arithmètiques*. Publications de l'Institut de Mathématique de l'Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341. Hermann, Paris, 1969.

[2] Armand Borel and Lizhen Ji. Compactifications of locally symmetric spaces. *J. Differential Geom.*, 73(2):263–317, 2006.

[3] Salvatore Cosentino and Livio Flaminio. Equidistribution for higher-rank Abelian actions on Heisenberg nilmanifolds. *J. Mod. Dyn.*, 9:305–353, 2015.

[4] Alexander Fedotov and Frédéric Klopp. An exact renormalization formula for Gaussian exponential sums and applications. *Amer. J. Math.*, 134(3):711–748, 2012.

[5] H. Fiedler, W. Jurkat, and O. Körner. Asymptotic expansions of finite theta series. *Acta Arith.*, 32(2):129–146, 1977.

[6] Douglas Grenier. Fundamental domains for the general linear group. *Pacific J. Math.*, 132(2):293–317, 1988.

[7] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[8] Gérard Lion and Michèle Vergne. *The Weil representation, Maslov index and theta series*, volume 6 of *Progress in Mathematics*. Birkhäuser, Boston, Mass., 1980.

[9] Jens Marklof and Matthew Welsh. Segal-Shale-Weil representation, theta functions, and applications. In preparation, 2022.

[10] Jens Marklof and Matthew Welsh. Bounds for theta sums in higher rank I. *J. d’Analyse Math.*, 2023.

[11] David Mumford. *Tata lectures on theta. I*, volume 28 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1983. With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman.

[12] Audrey Terras. *Harmonic analysis on symmetric spaces and applications. II*. Springer-Verlag, Berlin, 1988.

JM: School of Mathematics, University of Bristol, Bristol BS8 1UG, U.K.

MW: Department of Mathematics, University of Maryland, College Park, MD 20742, USA