Automorphic Black Hole Entropy

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Abstract

Over the past few years the understanding of the microscopic theory of black hole entropy has made important conceptual progress by recognizing that the degeneracies are encoded in partition functions which are determined by higher rank automorphic representations, in particular in the context of Siegel modular forms of genus two. In this brief review some of the elements of this framework are highlighted. One of the surprising aspects is that the Siegel forms that have appeared in the entropic framework are geometric in origin, arising from weight two cusp forms, hence from elliptic curves.
1 Introduction

Automorphic black hole entropy arises in the context of $\mathcal{N} = 4$ supersymmetric black hole entropy in theories with higher curvature couplings. Such black holes turn out to be a useful testing ground, in particular in the extremal limit, because they are simple enough, but not too simple. A typical case leads to actions of the schematic form

$$A = \int d^4x \sqrt{-g} \left( \phi_R(a_i) R - \phi_{IJ}(a_i) F^I F^J - \tilde{\phi}_{IJ}(a_i) F^I \tilde{F}^J + \phi_{GB}(a_i) g_{GB} \right) + \cdots$$

where $g_{GB}$ collects fourth derivative curvature terms, e.g. the combination

$$g_{GB} = aR_{\mu\nu\kappa\rho} R^{\mu\nu\kappa\rho} + bR_{\mu\nu} R^{\mu\nu} + cR^2,$$

which for $(a,b,c) = (1,-4,1)$ reduces to the Gauss-Bonnet term. The functions $\phi_\ast$ describe the couplings of the scalar fields $a_i$ of the theory and the dots indicate further terms not emphasized in the following discussion. For actions of this type it is natural to expect that the black hole entropy decomposes into two parts

$$S_{BH}(Q) = S_{EM}(Q) + S_{GB}(Q)$$
where $Q$ collectively denotes the charges associated to the gauge fields $F^I$, where $I$ is a multiplet index. The first term gives the entropy due to the Einstein-Maxwell term, while the second term gives the dependence of the entropy on the higher curvature terms.

It turns out that these two parts of the action are quite different in structure: while the two-derivative action is quite insensitive to the details of the $\mathcal{N} = 4$ theory space, the four-derivative higher curvature part not only depends on the detailed structure of the different models, it does so in a highly restricted way. The interesting phenomenon that occurs in at least some classes of models is that the higher curvature coupling function $\phi_{\text{GB}}(a_i)$, and therefore the resulting black hole entropy, is determined by functions $f$ that are modular with respect to Hecke congruence subgroups of the group $\text{SL}(2, \mathbb{Z})$, where the level of these forms is a characteristic of the theory. Modular forms are good because they are functions on the complex plane whose structure is extremely restricted by their symmetries. They are in fact so constrained that a finite number of computations determines them completely. Even more interesting is that the classical forms $f(\tau)$ that appear in certain models lift to automorphic forms $\Phi_f(\tau_i)$ of higher rank with respect to the subgroups of the symplectic group and these automorphic forms determine the microscopic entropy. This framework therefore provides a tantalizing link between classical modular forms that appear in the effective action and higher rank automorphic forms that determine the microscopic partition functions for the entropy.

The appearance of such automorphic forms and their associated automorphic representations allows to link the framework of black hole entropy to the arithmetic Langlands program, in particular the reciprocity conjecture. The latter takes the point of view that automorphic forms and their representations can be viewed as structures that are determined by representations of Galois groups, objects that are of a much simpler nature than automorphic representations. This is of interest because of Grothendieck’s earlier insight that Galois representations admit a geometric interpretation in the framework of motives, geometric structures that can be viewed as the basic geometric building blocks of manifolds, defining ”geometric atoms” in the original sense of the word. Combining the three frameworks of automorphic black hole entropy, the arithmetic Langlands program, and Grothendieck’s theory of motives
makes it natural to ask whether the modular forms that appear in $\mathcal{N} = 4$ black hole entropy admit a geometric interpretation. This will be discussed below.

The main goal of the present review is to provide an overview of the results obtained over the past years on the automorphic structure of black hole entropy in a certain class of $\mathcal{N} = 4$ models, with the aim to provide sufficient background material to illustrate the geometric origin of the resulting automorphic forms. In order to keep the following discussion within the confines of a brief review several interesting and important topics had to be omitted, e.g. the problem of the moduli dependence of the entropy, in particular the phenomenon of wall crossing. Also omitted is the class of black holes with torsion. For more extensive discussions of not only these topics, but also concerning the specific details of the computation of $\mathcal{N} = 4$ entropies outlined here, the reader may consult the extensive review of Sen [1], or the more recent shorter review by Mandal and Sen [2]. An in-depth analysis of the simplest model in this class can be found in the review of Dabholkar and Nampuri [3], while the useful summary of Mohaupt [4] presents a slightly different point of view.

2 Macroscopic $\mathcal{N} = 4$ black hole entropy

Concrete results for $\mathcal{N} = 4$ black hole entropy have been known for some time, starting with the work of Dijkgraaf-Verlinde-Verlinde [5] (see also [6, 7, 8, 9, 10]), but more systematic results were obtained only about a decade later by Jatkar-Sen [11] and Govindajaran-Krishna [12] in the context of a special class of compactifications considered by Chaudhuri-Hockney-Lykken [13]. In the heterotic picture the models are constructed by considering toroidal quotients with respect to abelian discrete groups $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ of order $N$, while in the dual IIA picture the compact spaces are quotients of products $K^3 \times T^2$ of K3 surfaces and elliptic curves $T^2$. The CHL$_N$ models can therefore be viewed in the two dual pictures as

$$\text{CHL}_N : \quad \text{Het}(T^6/\mathbb{Z}_N) \cong \text{IIA}((K^3 \times T^2)/\mathbb{Z}_N).$$

(4)

(More details on CHL$_N$ dyons can be found in refs. [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].) In the context of $T^6$ quotients the action of $\mathbb{Z}_N$ factors into a transformation
of order $N$ on the conformal field theory compactified on the 4-torus $T^4 \subset T^6$ and an order $N$ shift along one of the remaining 1-cycles $S^1 \subset T^6$.

The charge lattice $\Lambda^{(N)}$ depends on the order $N$ of the quotient group, leading to rank $r_N$ for the lattice associated to CHL$_N$. The values of these $r_N$ are listed in Table 1.

| $N$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| $r_N$ | 28 | 20 | 16 | 14 | 12 | 12 | 10 | 10 |

**Table 1.** Ranks $r_N$ of the CHL$_N$ models.

The electric and magnetic charges $(Q_e, Q_m) = (Q_e^I, Q_m^I)$, $I = 1, ..., r_N$ arise from the vector multiplets associated to which are the moduli spaces given by the quotients

$$\mathcal{M}(r_N) = \text{SO}(6, r_N - 6, \mathbb{Z})/\text{SO}(6, r_N - 6, \mathbb{R}) \times \text{SO}(r_N - 6, \mathbb{R})$$

with respect to the discrete T-duality group $T_N = \text{SO}(6, r_N - 6, \mathbb{Z})$ as well as the maximal compact subgroup. The T-duality invariant norms $Q_e^2$ for the electric charge, $Q_m^2$ for the magnetic charge, and $Q_e Q_m$ for their combination in turn can be combined to an S-duality invariant combination takes the form $Q_e^2 Q_m^2 - (Q_e Q_m)^2$. The S-duality group in $\mathcal{N} = 4$ supersymmetric theories is in general a subgroup of the full modular group $\text{SL}(2, \mathbb{Z})$, a fact that was first emphasized by Vafa and Witten [29] in the context of the CHL$_2$ quotient model. For general $N > 1$ the relevant group has been identified in [30] to be given by the congruence subgroup $\Gamma_1(N)$ of the modular group $\text{SL}(2, \mathbb{Z})$, defined as

$$\Gamma_1(N) = \left\{ g \in \text{SL}(2, \mathbb{Z}) \mid g \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \pmod{N} \right\},$$

a subgroup of the Hecke congruence group $\Gamma_0(N)$. A detailed discussion can be found in [31].

Duality invariance thus makes it natural to expect that the black hole entropy in the lowest order of the effective theory takes the form [32, 33]

$$S_{\text{EM}} \simeq \pi \sqrt{Q_e^2 Q_m^2 - (Q_e Q_m)^2}.$$  \hspace{1cm} (7)

Thinking of black holes as probes of the underlying theory, this leading order entropy is not very sensitive to the specific structure of the compactification variety $X_N$. This lack of
sensitivity of the Einstein-Maxwell part of the action makes it natural to consider the effects of the higher curvature corrections in the action (1). In the $\mathcal{N} = 4$ context corrections have been computed in refs. [34, 35] and the first nonleading correction of the curvature part of the action that has been considered is of the form (1), with the standard Gauss-Bonnet term

$$\phi^{(N)}(a, S)_{\text{GB}} = \phi^{(N)}(a, S) \left( R_{\mu\nu\kappa\rho} R^{\mu\nu\kappa\rho} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right),$$

where the Gauss-Bonnet coupling $\phi^{(N)}(S, a)$ is a function of the axion $a$ and the dilaton $S = e^{-2\varphi}$. It is useful to write this coupling as a complex function $\phi^{(N)}(a, S) = \phi^{(N)}(\tau, \overline{\tau})$ in terms of the variable $\tau := a + iS \in \mathcal{H}$ in the upper half plane.

The most interesting part of higher derivative action is that these couplings are essentially determined by modular forms, i.e. functions $f$ on the upper half-plane $\mathcal{H} \subset \mathbb{C}$ which under subgroups $\Gamma \subset \text{SL}(2, \mathbb{Z})$ transform as

$$f(g\tau) = \epsilon(d)(ct + d)^w f(\tau), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

where $\epsilon$ is a character and the integer $w$ denotes the weight. For each of the CHL$_N$ models the coupling $\phi^{(N)}$ becomes a function of the modular form $f^{(N)}(\tau)$. The structure of these forms is obtained by a simple rationale because covariance under the duality group is quite restrictive. This can be seen as follows [36]. Recall first that the duality group $\text{SL}(2, \mathbb{Z})$ for the $N = 1$ model is broken to a level $N$ group for $N > 1$. For $N = 1$ the result of Dijkgraaf-Verlinde-Verlinde [5] shows that the modular form is the discriminant form $f^{(1)}(\tau) = \eta(\tau)^{24}$, where $\eta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta function, written in terms $q = e^{2\pi i \tau}$. The function $f^{(1)}(\tau)$ is the unique modular cusp form of weight twelve and level one. Assuming that the forms $f^{(N)}$ for $N > 1$ have level $N$ leads for prime orders $N = p$ to unique candidate cusp forms that admit closed expressions as eta-products of the following type

$$f^{(N)}(q) = \eta(q)^w \eta(q^N)^{w+2} \in S_{w+2}(\Gamma_0(N), \epsilon_N),$$

where for $N = p = 1, 2, 3, 5, 7$ the weight is determined as $w + 2 = \frac{24}{(N+1)}$, and the modular groups are the Hecke congruence subgroups $\Gamma_0(N) \subset \text{SL}(2, \mathbb{Z})$, defined as

$$\Gamma_0(N) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid c \equiv 0(\text{mod } N) \right\}.$$
Here the character \( \epsilon_N \) is trivial, except for \( N = 7 \), for which it is given by the Legendre character \( \epsilon_7(d) = \chi_{-7}(d) = (\frac{-7}{d}) \). Legendre characters are defined as

\[
\chi_N(p) = \left( \frac{N}{p} \right) = \begin{cases} 
1 & \text{if } x^2 \equiv N \pmod{p} \text{ is solvable} \\
-1 & \text{if } x^2 \equiv N \pmod{p} \text{ is not solvable} \\
0 & \text{if } p \mid N. 
\end{cases}
\] (12)

For the remaining composite values \( N = 4, 6, 8 \) of the CHL \( \mathcal{N} \) class of models the quotient \( 24/(N + 1) \) is neither integral nor half-integral, hence there are no modular forms with corresponding weights. It is natural to extend the prime sequence above by considering forms of weights

\[
w_N = \left\lfloor \frac{24}{N + 1} \right\rfloor
\] (13)

where \([a]\) denotes the next largest integral number obtained from the number \( a \). For \( N = 4, 6, 8 \) these values lead to weights 5, 4, 3, respectively. Assuming furthermore that the order \( N \) of the quotient group \( \mathbb{Z}_N \) again determines the level of the modular group leads to unique candidates forms given by eta-products. These forms and their characters, again given by Legendre symbols, are collected in Table 2. The uniquely determined forms obtain with the simple assumptions above are precisely the forms proposed by Jatkar and Sen [11] for prime orders and by Govindarajan and Krishna [12] for composite orders.

| \( N \) | \( p \leq 7 \) prime | 4 | 6 | 8 |
|---|---|---|---|---|
| \( f^{(N)} \) | \( \eta(\tau)^{24} \eta(p\tau)^{24} \) | \( \eta(\tau)^4 \eta(2\tau)^2 \eta(4\tau)^2 \) | \( (\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2 \) | \( \eta(\tau)^2 \eta(2\tau)\eta(4\tau)\eta(8\tau)^2 \) |
| Type | \( S_w(\Gamma_0(N), \epsilon_N) \) | \( S_5(\Gamma_0(4), \chi_{-1}) \) | \( S_4(\Gamma_0(6)) \) | \( S_3(\Gamma_0(8), \chi_{-2}) \) |

**Table 2. The classical modular forms of the CHL\( \mathcal{N} \) models.**

The coupling \( \phi^{(N)}(\tau, \overline{\tau}) \) of the Gauss-Bonnet term then is essentially given by modular cusp forms \( f^{(N)}(\tau) \) via

\[
\phi^{(N)}(\tau, \overline{\tau}) \cong \ln \left[ f^{(N)}(\tau) \cdot f^{(N)}(-\overline{\tau}) \cdot (\text{Im } \tau)^{w+2} \right],
\] (14)

where \( \text{Im } \tau \) is the holomorphic anomaly. For \( N = 1 \) the resulting weight twelve form \( f^{(1)}(\tau) = \Delta(\tau) := \eta^{24}(\tau) \) with respect to the full modular group \( \text{SL}(2, \mathbb{Z}) \) is familiar from the early days in string theory because it leads to the partition function of the bosonic string.
The next-to-leading order correction of the entropy beyond the Einstein-Maxwell system entropy $S_{EM}$ is determined in terms of charge expressions that do not change under the scaling

$$(Q_e, Q_m) \rightarrow (\lambda Q_e, \lambda Q_m)$$

and are given by

$$Q^{(1)} = Q_e Q_m$$
$$Q^{(2)} = \frac{1}{Q_m^2} \sqrt{Q_e^2 Q_m^2 - (Q_e Q_m)^2}.$$  

(16)

With $Q^{(i)}$ the corrected entropy term is of the form $S_{GB} \cong \phi^{(N)}(Q^{(1)}, Q^{(2)})$, i.e. the axion and dilaton pair $(a, S)$ are replaced by the corresponding charge expressions. More precisely, up to fourth order in the derivative expansion the entropy is given by

$$S_{BH}(Q_e, Q_m) = S_{EM}(Q_e, Q_m) + \phi^{(N)}\left(\frac{Q_e Q_m}{Q_e^2}, \frac{1}{Q_m^2} \sqrt{Q_e^2 Q_m^2 - (Q_e Q_m)^2}\right),$$

(17)

first implicitly derived implicitly in [6], but most easily obtained via the entropy function formalism developed by Sen in several papers (e.g. in refs. [37, 38]), and reviewed in detail in [1].

This result for the Gauss-Bonnet contribution to the entropy shows that higher derivative corrections are much more sensitive to the details of the theory than the entropy based on the two-derivative action. This leads to the idea that if black holes are viewed as experimental objects they could in principle be used as tools to test predictions of gravitational theories beyond Einstein’s general relativity [36]. Entropy corrections arising from terms higher than fourth derivative order have been considered in ref. [39].

The computation of the entropy in the effective theory raises the question whether the contributions $S_E$ and $S_{GB}$ admit a microscopic interpretation, i.e. whether there are functions that define microscopic degeneracies $d_{mic}(Q)$ such that

$$S_{mic} = \ln d_{mic}$$

(18)

produces the expressions above derived from the effective theory.
3 General structure of automorphic black hole entropy

As mentioned above, over the past fifteen years or so impressive progress has been made toward the resolution of a problem that is forty years old — the microscopic understanding of the entropy of black holes. It has proven useful to focus on black holes with extended supersymmetries because this leads to black holes that are simple, but not too simple. It was shown in particular that for certain types of black holes in $\mathcal{N} = 4$ supersymmetric theories their entropy is encoded in the Fourier coefficients of Siegel modular forms, automorphic objects which provide one of the simplest generalizations of classical modular forms of one variable with respect to congruence subgroups of the full modular group $SL(2, \mathbb{Z})$.

A general conceptual framework of automorphic entropy functions has not been established yet. A formulation that generalizes the existing examples can be outlined as follows [40]. Suppose we have a theory which contains scalar fields parametrized by a homogeneous space $\prod_i (G_i/H_i)$, where the $G_i$ are Lie groups. Associated to these scalar fields are electric and magnetic charge vectors $Q = (Q_e, Q_m)$, taking values in a lattice $\Lambda$ whose rank is determined by the groups $G_i$.

Assume now that the theory in question has a T-duality group $\prod_i D_i(\mathbb{Z})$, where the $D_i(\mathbb{Z}) \subset G_i(\mathbb{Z})$ denote subgroups of the Lie groups $G_i$, considered over the rational integers $\mathbb{Z}$. Suppose further that the charge vector $Q$ leads to norms $||Q||_i$, $i = 1, ..., r$ that are invariant under the T-duality group. Choose conjugate to these invariant charge norms complex chemical potentials

$$ (\tau_i, ||Q||_i), \quad i = 1, ..., r, \quad (19) $$

which generalize the upper half plane of the bosonic string. On the generalized upper half plane $\mathcal{H}^r$ formed by the variables $\tau_i$ one can consider automorphic forms $\Phi(\tau_i)$, and the idea is that with an appropriate integral structure $\mathbb{Z} \ni k_i \sim ||Q||_i, i = 1, ..., r$ associated to the charge norms, the Fourier expansion of these automorphic forms given by

$$ \Phi(\tau_i) = \sum_{k_n \in \mathbb{Z}} g(k_1, ..., k_r) q_1^{k_1} \cdots q_r^{k_r}, \quad (20) $$
in terms of $q_k = e^{2\pi i \tau_k}$, determines the automorphic entropy via the coefficients of the expansion of the automorphic partition function

$$Z(\tau_i) = \frac{1}{\Phi(\tau_i)} = \sum_{k_n} d(k_1, ..., k_r) q_1^{k_1} \cdots q_r^{k_r},$$

as

$$S_{\text{mic}}(Q) \simeq \ln d_{\text{mic}}(Q).$$

Here

$$d_{\text{mic}}(Q) := d(\|Q\|_1, ..., \|Q\|_r)$$

and $\tilde{\Phi}$ denotes a modification of the Siegel form $\Phi$ that is determined by the divisor structure of $\Phi$. The precise definition of $\tilde{\Phi}$ is motivated by the interplay between the automorphic discrete group and the goal to isolate the dominant poles in $Z$. If in leading order of the large charge expansion the degeneracies lead to the asymptotic result

$$d_{\text{mic}} \simeq e^{\pi \sqrt{F(||Q||_i)}},$$

where $F(||Q||_i)$ is a quadratic form in terms of the norms $||Q||_i$, the microscopic entropy

$$S_{\text{mic}} \simeq \pi \sqrt{F(||Q||_i)}$$

is structurally of the same type as the large charge limit of the macroscopic entropy.

4 Siegel modular black holes in $\mathcal{N} = 4$ theories

The automorphic-entropy-outline of the previous section accounts for the behavior of the entropy of black holes in certain $\mathcal{N} = 4$ compactifications obtained by considering $\mathbb{Z}_N$–quotients of the heterotic toroidal compactification Het($T^6$), a small class of models first considered by Chaudhuri-Hockney-Lykken models [13]. Specifically, it was shown in [5, 11, 12] that for these CHL$_N$ models the microscopic entropy of extreme Reissner-Nordstrom type black holes is described by Siegel modular forms $\Phi^N \in S_w(\Gamma_0^{(2)}(N))$, where the weight $w$ of $\Phi^N$ is determined by the order $N$ of the quotient group. In this case the dyonic charges $Q = (Q_e, Q_m)$ form
three integral norms $||Q||_i$ invariant under the T-duality group $\text{SO}(6, r_N - 6)$. Associating to the norms $||Q||_i$ conjugate complex variables

$$(\tau, \sigma, \rho) = (\tau_1, \tau_2, \tau_3) \leftrightarrow (||Q||_1, ||Q||_2, ||Q||_3)$$

leads to a three-dimensional domain for automorphic forms associated to CHL$_N$ models. This domain should generalize to dyonic black holes the upper half plane $\mathcal{H}$ on which the partition functions of purely electric and purely magnetic black holes are defined. For dimensional reasons $\text{GL}(n)$-type automorphic forms are excluded, but Siegel-type automorphic forms of genus two are natural candidates because the Siegel upper half plane $\mathcal{H}_2$

$$T = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} \in \mathcal{H}_2,$$

with $\mathbb{C} \ni \tau, \sigma > 0$ and $\det \text{Im}(T) > 0$, reduces in the degeneration $\tau_3 \to 0$ into the product of a pair of classical upper half planes

$$\mathcal{H}_2 \xrightarrow{\rho \to 0} \mathcal{H}_1 \times \mathcal{H}_1,$$

a fact that will be reflected in the behavior of the automorphic forms. A key motivating result in this direction, pointing toward the usefulness of Siegel forms, is that in the simplest example, given by the Igusa form $\Phi_{10}$ of weight ten for the full symplectic group $\text{Sp}(4, \mathbb{Z})$, the degeneration (28) leads to a factorization of the Igusa form as

$$\Phi_{10}(\tau, \sigma, \rho) \xrightarrow{\rho \to 0} (2\pi i)^2 \rho^2 \Delta(\tau) \Delta(\sigma),$$

where $\Delta(\tau)$ is precisely the modular form of weight twelve that appears in the Gauss-Bonnet coupling $\phi^{(N)}(\tau, \bar{\tau})$ of the effective action for the CHL$_1$ model.

The automorphic groups relevant for the CHL$_N$ models are Hecke type genus two congruence subgroups $\Gamma_0^{(2)}(N)$ of the symplectic group

$$\text{Sp}(4, \mathbb{Z}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid M^tJM = J, \ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\},$$

hence the associated forms are Siegel modular forms of genus two, i.e. functions $\Phi$ on $\mathcal{H}_2$ which transform with respect to $\Gamma_0^{(2)}(N) \subset \text{Sp}(4, \mathbb{Z})$ in a way analogous to classical modular
forms. A genus two Siegel modular form \( \Phi_w \) is said to be of weight \( w \) if for any
\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(2)}(N)
\]
with \((2 \times 2)\)-matrices \( A, B, C, D \) it transforms under
\[
T \mapsto MT := (AT + B)(CT + D)^{-1} 
\]
as
\[
\Phi_w(MT) = \det(CT + D)^w \Phi_w(T).
\]
For the CHL\(_N\) models the weight \( w \) of the associated Siegel form \( \Phi^{(N)} \) is given by
\[
w(\Phi^{(N)}) = w_N - 2,
\]
where \( w_N \) is the weight defined in (13).

The Siegel forms \( \Phi^{(N)} \) do not immediately define the partition functions in the models for \( N > 1 \). The problem that arises is as follows. First one notes that already for the \( N = 1 \) CHL model the diagonal divisor
\[
D_{\text{diag}} = \{ \rho^2 = 0 \},
\]
which arises in the limit \( \rho \to 0 \) mentioned above via the factorization of the Siegel form
\[
\Phi^{(N)}(\tau, \sigma, \rho) \xrightarrow{\rho \to 0} \sim \rho^2 f^{(N)}(\tau) f^{(N)}(\sigma)
\]
does not provide the dominant contribution to the entropy, but is suppressed in the large charge expansion of the degeneracies \( d(k, \ell, m) \). The leading contribution instead is given by the dominant divisor
\[
D_{\text{dom}} = \{ \rho^2 - \rho - \tau \sigma = 0 \},
\]
which can be obtained as an Sp\((4, \mathbb{Z})\) image of \( D_{\text{diag}} \). In the \( N = 1 \) model the symplectic group Sp\((4, \mathbb{Z})\) is the symmetry group of the theory, hence the Siegel form can be transformed as (33) for such a group element. For the models \( N > 1 \) the map considered in [11] is however not an element of the symmetry group \( \Gamma_0^{(2)}(N) \), hence the Siegel forms \( \Phi^{(N)}(T) \) do not transform
under this map. The practical way out is to introduce functions $\tilde{\Phi}(N)(T)$ in precisely such a way that the large charge limit agrees with the macroscopic theory. This leads to the definition of the Siegel type partition function

$$Z(\tau_i) = \frac{1}{\Phi(\tau_i)},$$

(38)

with $\tilde{\Phi}(\tau_i)$, obtained from $\Phi(\tau_i)$ by multiplication with a function $A(\tau, \sigma, \rho)$ determined by the map from the diagonal to the dominant divisor. The motivation for this becomes clear already in the analysis of the $N = 1$ case.

In the $N = 1$ model $\text{Het}(T^6)$ the Siegel form must transform with respect to the full symplectic group $\text{Sp}(4, \mathbb{Z})$ of genus two with weight $w = 10$. This uniquely determines the form $\Phi_{10}$ to be the Igusa modular form, which leads directly to the degeneracies that enter the entropy by expanding

$$Z(\tau, \sigma, \rho) = \frac{1}{\Phi_{10}(\tau, \sigma, \rho)} = \sum_{k, \ell, m} d(k, \ell, m) q^k r^\ell s^m,$$

(39)

where $q = e^{2\pi i \tau}, r = e^{2\pi i \sigma}, s = e^{2\pi i \rho}$ (for $N > 1$ the divisor induction modification $\tilde{\Phi}$ has to be considered). With [9]

$$d(Q) = (-1)^Q e^{Q_m+1} \int dT \frac{e^{-\pi i T Q}}{\Phi_w(T)}$$

(40)

the degeneracies of the (in general modified) inverse Siegel form define the microscopic entropy

$$S_{\text{mic}}(Q) = \ln d_{\text{mic}}(Q).$$

(41)

The integral (40) can be evaluated by computing first the Cauchy integral for the off-diagonal variable for the pole given by the Humbert divisor and then evaluating the remaining integral in a saddle point approximation.

Insight into how contact with the macroscopic entropy is made can be obtained without going through the whole computing by simply making explicit the result of the $\rho$-integral obtained by using the transformation (33) for the matrix $M = (\begin{smallmatrix} A & B \\ C & D \end{smallmatrix})$ mapping the diagonal divisor (35) to the dominant divisor (37), given in terms of the block matrices as $A = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}), B = (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}), C = (\begin{smallmatrix} 0 & 0 \\ 1 & -1 \end{smallmatrix}), D = (\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix})$. With $T' = MT$ one gets for a weight $w$
Siegel form $\Phi_w(T') = \det(CT + D)^w \Phi_w(T) = (\sigma')^{-w} \Phi_w(T)$. Taking the factorization limit in the $T$-variables and mapping with $M$ the diagonal divisor $\rho^2$ to the primed coordinates gives essentially the dominant divisor, since $\rho = -(\rho^2 - \rho' - \tau' \sigma')/\sigma'$. These maneuvers then lead to the transformed integral

$$d_{\text{dom}}(Q) \approx \int d\tau' d\sigma' d\rho' \frac{(\sigma')^w e^{-\pi Q' T' Q}}{(\rho^2 - \rho' - \tau' \sigma')^2 \Delta(\tau) \Delta(\sigma)},$$

where $(w + 2) = 12$ is the weight of the form $\Delta$ and $\tau = (\tau' \sigma' - \rho^2)/\sigma'$ and $\sigma = (\tau' \sigma' - (\rho' - 1)^2)/\sigma'$ arise from the transformation given by $M$. The residue integral leads to an integral of the form

$$d_{\text{mic}}(Q) \approx \int d\tau' d\sigma' e^{-2\pi i \Sigma_{\text{mic}}(\tau', \sigma')} J(\tau', \sigma'),$$

where the microscopic entropy function $\Sigma_{\text{mic}}(\tau, \sigma)$ and $J(\tau, \sigma)$ are somewhat unwieldy expressions, but the key is that $\Sigma_{\text{mic}}$ essentially contains a term

$$\psi(\tau', \sigma') \approx \ln \Delta(\tau) \Delta(\sigma) (\sigma')^{-(w+2)},$$

which looks reminiscent to the expression $\phi^{(1)}$ of the higher derivative effective action, and in fact becomes identical to $\phi^{(1)}$ in the saddle point evaluation.

In leading order of the large charge expansion the remaining saddle point evaluation leads to

$$d_{\text{mic}}(Q) \approx e^{\pi \sqrt{Q^2 Q_m^2 - (Q_e Q_m)^2}},$$

in agreement with the macroscopic entropy described above [5]. In subleading order the agreement between the macroscopic and microscopic entropy has been shown for the Gauss-Bonnet term in ref. [6] for the $N = 1$ model. The generalization to $N > 1$ has been discussed in [11] for prime orders, and for the remaining composite orders in [12] and the issue of the path dependence of the degeneracy integral has been addressed in [20, 21, 22]. The above outline for a microscopic interpretation of CHL$_N$ black hole entropy is valid for charge configurations which satisfy the constraint

$$t_Q := \gcd\{Q^I Q^J_m - Q^I_e Q^J_m \mid 1 \leq I, J \leq r_N\} = 1.$$

The integer $t_Q$ is called the torsion of the black hole, and the issue of $N = 4$ black holes with nontrivial torsion $t_Q > 1$ was first raised in [21], and further discussed in several papers.
The extensions and issues briefly mentioned above are reviewed in detail refs. [1, 2] where further references can be found.

## 5 From black hole Siegel forms to higher weight classical modular forms

The key feature of the Siegel modular forms that appear in the context of CHL\(_N\) black hole entropy is that they are not of general type, but belong to the Maaß Spezialschar, more precisely they are obtained via a combination of the Skoruppa lift [44], which maps classical modular forms to Jacobi forms, and the Maaß lift [45], which maps Jacobi forms to Siegel modular forms

\[
f(\tau) \in S_{w+2} \xrightarrow{\text{SL}} \varphi_{w,1}(\tau, \rho) \in J_w \xrightarrow{\text{ML}} \Phi_w(\tau, \sigma, \rho) \in S_w,
\]

where \(\tau = \tau_1, \sigma = \tau_2, \rho = \tau_3\). Here the Maaß-lift ML sends a Jacobi form \(\varphi_{w,1}\) of weight \(w\) and level 1 with a Fourier expansion

\[
\varphi_{w,1}(\tau, \sigma) = \sum_{\substack{k \in \mathbb{N}_0, \ell \in \mathbb{Z} \atop 4k - \ell^2 \geq 0}} c(k, \ell) q^k r^\ell
\]

to a Siegel form of weight \(w\) with the Fourier expansion

\[
\Phi_w(q, r, s) = \sum_{\substack{k, \ell \in \mathbb{N}_0, m \in \mathbb{Z} \atop 4k\ell - m^2 \geq 0}} g(k, \ell, m) q^k r^\ell s^m
\]

with coefficients

\[
g(k, \ell, m) = \sum_{d \mid (k, \ell, m)} \chi(d) d^{w-1} c \left( \frac{k\ell}{d^2}, \frac{m}{d} \right),
\]

with Legendre character \(\chi\).

The Skoruppa lift SL is a map that sends classical forms \(f \in S_w(\Gamma_0(N), \epsilon_N)\) of weight \(w\), level \(N\), and character \(\epsilon_N\) to Jacobi forms via the prime form

\[
K(\tau, \sigma) := \frac{\vartheta_1(\tau, \sigma)}{\eta^3(\tau)}
\]
given in terms of the theta series

\[ \vartheta_1(q, s) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(2n+1)^2} s^{n+\frac{1}{2}} \]  

and the Dedekind eta function defined above as

\[ \text{SL}(f) = \varphi_{w,1} := K^2 f. \]  

The combination

\[ \text{MS} = \text{ML} \circ \text{SL} \]  

defines a map from classical cusp forms \( f(\tau) \) of weight \( (w + 2) \) to Siegel forms of genus two of weight \( w \). The form \( f(\tau) \) whose Maaß-Skoruppa lift is the Siegel modular form \( \Phi_w = \text{MS}(f) \) is called the Maaß-Skoruppa root.

6 Geometric origin of automorphic black hole entropy

The fact that the automorphic black hole entropy of CHL\( _N \) models encoded in the Siegel forms \( \Phi^{(N)} \) is sensitive to the details of the theory, not only through the spectrum but also through the couplings, motivates the question what exactly the essential information is that is contained in these entropy functions. Put differently, the issue becomes what the irreducible structure is that determines the entropy of these models. This question can be raised independently of any specific picture in which the Siegel forms \( \Phi^{(N)} \) are constructed, be that via type II D-branes, or M-theory branes, or with other ingredients provided by different dual pictures. The point here is that any given construction can in principle introduce redundant structures and in the process might not point to the irreducible building blocks. In the following this question is motivated in the first subsection in a more general framework, while the second subsection constructs the irreducible building block of the CHL\( _N \) models, following [36]. The second subsection is structurally independent of the first.
6.1 Historical background

It has been known for more than a century that there is a connection between certain types of modular forms and certain types of geometric structures. This insight can be traced to the work of Klein and Fricke, and in more recent times this observation has been generalized in both the geometric and and the automorphic direction. The first generalization makes the geometric aspects much more detailed by refining the monolithic geometries considered by Klein, Fricke, Hecke, Eichler and Shimura into the much more detailed and precise motivic framework of Grothendieck [46]. The second extension generalizes the concept of classical modular forms, associated to subgroups of the modular group $\text{SL}(2,\mathbb{Z})$ acting on the upper half plane, to the notion of automorphic forms. The latter can be viewed as functions on higher dimensional half planes, or as functions on linear algebraic reductive groups. The link between these a priori completely independent objects, i.e. motives defined by algebraic cycles on the one hand, and automorphic forms and representations on the other, is provided by the concept of a Galois representation. It is in this context that the arithmetic Langlands program enters: the reciprocity conjecture posits a general relation between Galois representations and automorphic representations [47]. Combining Langlands‘ reciprocity conjecture with Grothendieck’s description of motives as Galois representations makes it possible to think of motives as carrying an automorphic structure. The direction from automorphic forms to motives is less clear because not all automorphic forms are motivic. It is generally expected though that the subclass of algebraic automorphic forms are in fact motivic.

The idea of automorphic motives makes it natural to ask whether the automorphic forms that appear in the context of $\mathcal{N} = 4$ black hole entropy have a geometric interpretation. This question can be raised independently of the picture associated to the purported geometry: whether this is interpreted as a motive that lives in the compact direction of spacetime or whether it is interpreted as (part of) a moduli space. In the present section the focus is on establishing a geometric interpretation of the black hole entropy, while the final section puts this result into a broader perspective by summarizing the general motivic structure which is conjectured to be associated to Siegel modular forms.
6.2 Lifts of weight two forms

The key to the identification of the motivic origin of the CHL$_N$ black hole entropy turns out to be an additional lift construction that interprets the Maaß-Skoruppa roots of weight $(w + 2)$ in terms of modular forms of weight two for all $N$, and hence in terms of elliptic curves [36]. These Maaß-Skoruppa roots decompose into two distinct classes of forms, one class admitting complex multiplication, the second class not. For this reason it is natural to expect that the lifts of weight two modular forms to the CHL$_N$ Maaß-Skoruppa roots involve two different constructions, depending on the type of the higher weight form. For forms without complex multiplication the lift interpretation of the MS root $f_{w+2}$ in terms of the weight two form $f_2 \in S_2$ can be written as

$$f_{w+2}(q) = f_2(q^{1/m})^m, \quad \text{with} \quad m = \frac{1}{2} \left\lfloor \frac{24}{N + 1} \right\rfloor . \quad (55)$$

The relation between the levels $\tilde{N}$ of the weight two forms $f_2$ and the order $N$ of the defining group $\mathbb{Z}_N$ is made explicit in Table 3.

| Order $N$ | 1   | 2   | 3   | 5   | 6   |
|-----------|-----|-----|-----|-----|-----|
| Level $\tilde{N}$ | 36  | 32  | 27  | 20  | 24  |

Table 3. The levels $\tilde{N}$ in terms of the orders $N$ of the CHL$_N$ models.

For the CHL$_N$ models with $N = 4, 7, 8$ the lift (55) cannot be applied because the MS roots $f_{w+2}$ have odd weight. It is therefore necessary to come up with a different type of reduction. Inspection of the forms $f_{w+2}$ shows that they admit complex multiplication. Intuitively, this means that the Fourier expansion of these functions are sparse. The vanishing of the Fourier coefficients $a_p$ for an infinite number of primes $p$ is determined by the splitting behavior of these primes in an associated number fields. The coefficients $a_p$ vanish for precisely those primes $p$ that do not split in the ring of integers $\mathcal{O}_{K_D}$, where $K_D = \mathbb{Q}(\sqrt{-D})$ is an imaginary quadratic field. This splitting behavior is controlled by the Legendre symbol $\chi_D$ and therefore a complex multiplication form $f \in S_w(\Gamma_0(N), \epsilon_N)$ can be defined through its Fourier expansion.
by the condition that there exists a field $K_D$ such that

$$\chi_D(p) a_p = a_p. \quad (56)$$

It is useful to change the point of view and consider instead of the Fourier series $f(q) = \sum_n a_n q^n$ of the form its associated $L$-series

$$L(f, s) = \sum_n \frac{a_n}{n^s}, \quad (57)$$

where $s$ is a complex variable. The lift for the class of MS roots with complex multiplication derives from the existence of algebraic Hecke characters $\Psi$ whose $L$–functions are the inverse Mellin transform of the MS roots

$$L(f, s) = L(\Psi, s). \quad (58)$$

More details can be found in ref. [36]. The weight two forms that correspond to the classical higher weight forms $f_{w+2}$ are described in Table 4.

| $N$ of $\text{CHL}_N$ | BH Form $f_N(q) \in S_{w+2}(\Gamma_0(N))$ | Motivic form $\tilde{f}_N(q)$ | Level $\tilde{N}$ of $E_{\tilde{N}}$ |
|---|---|---|---|
| 1 | $\eta(\tau)^{24}$ | CM $\eta(q^6)^4 \in S_2(\Gamma_0(36))$ | 16 |
| 2 | $\eta(\tau)^8\eta(2\tau)^8$ | CM $\eta(q^4)^2\eta(q^8)^2 \in S_2(\Gamma_0(32))$ | 32 |
| 3 | $\eta(\tau)^6\eta(3\tau)^6$ | CM $\eta(q^3)^2\eta(q^9)^2$ | 27 |
| 4 | CM $\eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^4$ | Sym$^2(f_2^{32})$ with $f_2^{32} \in S_2(\Gamma_0(32))$ | 32 |
| 5 | $\eta(\tau)^4\eta(5\tau)^4$ | $\eta(q^2)^2\eta(q^{10})^2 \in S_2(\Gamma_0(20))$ | 20 |
| 6 | $\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$ | $\eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau) \in S_2(\Gamma_0(24))$ | 24 |
| 7 | CM $\eta(\tau)^3\eta(7\tau)^3$ | Sym$^2f_2^{19}$ with $f_2^{19} \in S_2(\Gamma_0(49))$ | 49 |
| 8 | CM $\eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2$ | Sym$^2f_2^{256}$ with $f_2^{256} \in S_2(\Gamma_0(256))$ | 256 |

**Table 4. Elliptic motives associated to the electric modular forms of $\text{CHL}_N$ models.**

The interpretation of the Maaß-Skoruppa roots in terms of weight 2 modular forms $f_2^{\tilde{N}}$ via these two additional lifts for CM and non-CM forms shows that the motivic origin of the Siegel modular entropy of $\text{CHL}_N$ models is to be found in elliptic curves. This follows from the fact
that for all CHL\(_N\) models the geometric structure that supports the weight 2 forms is that of elliptic curves \(E_{\tilde{N}}\), whose conductor \(\tilde{N}\) varies with the order \(N\) of the quotient group \(\mathbb{Z}_N\). More precisely, the \(L\)-functions associated to both of these objects agree

\[
L(f_2^{\tilde{N}}, s) = L(E_{\tilde{N}}, s).
\]

(59)

Abstractly, this follows from the proof of the Shimura-Taniyama-Weil conjecture [48, 49, 50], but no such heavy machinery is necessary for the concrete cases based on the CHL\(_N\) models, where the elliptic curves can be determined explicitly for each \(N\). This shows that the motivic origin of the Siegel black hole entropy for the CHL\(_N\) models can be reduced to that of complex curves \(E_{\tilde{N}}\). It is the arithmetic structure of these elliptic curves that carries the essential information of the entropy. (A detailed analysis of this arithmetic structure of elliptic curves in a physical context can be found in ref. [51] and applications of elliptic curves as building blocks of Calabi-Yau threefolds appear in [52].)

7 Automorphic motives

The general framework of automorphic motives raises the natural question whether it is possible to provide a direct motivic interpretation of the Siegel modular forms that encode the microscopic nature of \(\mathcal{N} = 4\) black hole entropy in the context of CHL\(_N\) models. This would immediately lead to the picture of using black holes to extract geometric information if we were able to experiment with them in the laboratory. If the resulting motives were of spacetime origin one might expect that such automorphic black holes encode information about the geometry of the extra dimensions in string theory. This raises the question of how one can identify the motives of the variety which support the automorphic forms that appear in the entropy results.

Given that the entropy of black holes is described by automorphic forms, one can ask whether the spacetime structure of the compactification manifolds leads to motives which could support these automorphic forms. It is not expected that general automorphic forms are of motivic origin, however algebraic automorphic forms are conjectured to be motivic. Background
material for Siegel forms can be found in [53] and discussions of their conjectured motivic structure can be found in [54, 55]. In the special case of the genus two Siegel modular forms that appear in the context of CHL$_N$ black holes the conjectures concerning the motivic origin indicate that the compactification manifold cannot provide the appropriate motivic cycle structure in the way envisioned in the Siegel motivic literature. The easiest way to see this is as follows [36]. Suppose $M_\Phi$ is a motive whose $L$–function $L(M_\Phi, s)$ agrees with the spinor $L$–function $L_{sp}(\Phi, s)$ associated to a Siegel modular form $\Phi$ of arbitrary genus $g$ and weight $w$

$$L(M_\Phi, s) = L_{sp}(\Phi, s). \quad (60)$$

The weight $\text{wt}(M_\Phi)$ of such genus $g$ spinor motives follows from the (conjectured) functional equation of the $L$–function as

$$\text{wt}(M_\Phi) = gw - \frac{g}{2}(g + 1). \quad (61)$$

For the special case of genus 2 spinor motives the Hodge structure takes the form

$$H(M_\Phi) = H^{w-1,0} \oplus H^{w-2,w-1} \oplus H^{w-2,w-1} \oplus H^{0,2w-3}. \quad (62)$$

This Hodge structure only applies to pure motives. In the case of mixed motives it is possible, for example, that rank 4 motives can give rise to classical modular forms [56].

While the Hodge type (62) of $M_\Phi$ is that of a Calabi-Yau variety, the precise structure is only correct for modular forms of weight three. Inspection shows that for the class of CHL$_N$ models the weights of the Siegel modular forms take values in a much wider range $w \in [1, 10]$. If follows that for most CHL$_N$ models the Siegel modular form will be of the wrong weight to be induced directly by motives in the way usually envisioned in the conjectures of arithmetic geometry.

The same is the case for the classical Maass-Skoruppa roots, whose weights are given by $(w+2)$. The motivic support $M_f$ for such modular forms $f$ is of the form

$$H(M_f) = H^{w-1,0} \oplus H^{0,w-1}, \quad (63)$$
hence the only modular forms that can fit into heterotic compactifications have weight two, three, or four. This fact motivates the attempt to construct the Maaß-Skoruppa roots in terms of the simplest possible geometric modular forms, namely elliptic modular forms of weight two. This can be done as described in the previous section [36].

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