THE GODBILLON-VEY INVARIANT AS TOPOLOGICAL VORTICITY COMPRESSION AND OBSTRUCTION TO STEADY FLOW IN IDEAL FLUIDS

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Abstract. If the vorticity field of an ideal fluid is tangent to a foliation, additional conservation laws arise. For a class of zero-helicity vorticity fields the Godbillon-Vey (GV) invariant of foliations is defined and is shown to be an invariant purely of the vorticity, becoming a higher-order helicity-type invariant of the flow. GV \( \neq 0 \) gives both a global topological obstruction to steady flow and, in a particular form, a local obstruction. GV is interpreted as helical compression and stretching of vortex lines. Examples are given where the value of GV is determined by a set of distinguished closed vortex lines.

1. Introduction

In an ideal fluid the evolution of vorticity is given by

\[ \partial_t \omega + \mathcal{L}_U \omega = 0, \]

where \( U \) is the fluid velocity, \( \omega \) the vorticity and \( \mathcal{L} \) the Lie derivative. In a domain \( \Omega \), the fluid velocity is a vector field satisfying the conditions

\[ \nabla \times U = \omega, \quad \nabla \cdot U = 0, \quad U \parallel \partial \Omega, \]

which determine \( U \) up to the addition of a harmonic vector field. \[\text{[1]}\] implies that vortex lines flow along \( U \), they are ‘frozen’ in the field. Indeed, if \( \Phi^t \) is the diffeomorphism of \( \Omega \) generated by the fluid after a time \( t \), then the vorticity satisfies

\[ \omega_t = d\Phi^t(\omega_0). \]

This transport of vorticity leads to a number of conservation laws, in particular, as first observed by Woltjer [1] and Moreau [2], the helicity of the vorticity field

\[ H = \int_{\Omega} U \cdot \omega \, dV, \]

is conserved by the fluid. This can be seen explicitly,

\[ \frac{dH}{dt} = \int_{\Omega} (\partial_t + U \cdot \nabla)(U \cdot \omega) \, dV = \int_{\Omega} \nabla \cdot (F \omega) \, dV, \]

where \( F = U^2/2 - P \). Stokes’ theorem then gives a boundary term which vanishes as long as \( \omega \parallel \partial \Omega \), a condition we impose henceforth. \( H \) is a topological quantity invariant not only under the flow of \( U \) but any divergence-free vector field. First
understood by Moffatt [3], formalised by Arnold [4], this topological aspect of helicity manifests in its relation to the average linking of vortex lines, a fact revealed by writing $U$ in terms of the Biot-Savart operator.

For curves in three dimensions, linking is perhaps the most elementary topological property and integral invariants that measure other topological aspects of $\omega$ have long been sought [4], with qualified success. For vorticity fields supported on handlebodies, one can define invariants that measure higher-order topological invariants (such as triple-linking numbers) [6, 7, 8, 9, 10, 11, 12, 13], typically mirroring their development by Milnor through the use of Massey products [14], however these invariants have not been extended to general vorticity fields. Notably it has been shown by Enciso, Peralta-Salas, and Torres de Lizaur [15], building on the work of Kudryavtseva [16, 17], that any such invariant defined by a functional whose derivative can be written as an integral operator with a continuous kernel is a function of helicity. A manifestation of this result can be seen in the attempts to define asymptotic invariants other than helicity [18, 19, 20], which become functions of helicity.

As noted [15], this constraint can be evaded by considering invariants which are not continuous functionals on the space of volume-preserving vector fields, as in the case of the triple-linking invariants. Here we consider vorticity fields whose integral curves lie tangent to a (singular) codimension-1 foliation $\mathcal{F}$ of $\Omega$. This property is preserved under volume-preserving diffeomorphisms so that diffeomorphism invariants of $\mathcal{F}$ become invariants of the vorticity field. In particular we study vorticity fields admitting a vector potential $A$ satisfying $A \cdot \omega = 0$. This implies the helicity vanishes, when it is defined. In such cases we may define a higher order integral invariant, $GV$, related to the Godbillon-Vey invariant of foliations.

In this paper we discuss $GV$ for ideal fluid flows. We show, under assumptions on the singularities in $\mathcal{F}$, that it is defined purely in terms of the vorticity field, so is an integral invariant (conserved quantity) of the vorticity under general volume-preserving diffeomorphisms and hence the fluid flow. If $\Omega$ is closed and the foliation $\mathcal{F}$ is non-singular, then $GV$ is equal to the Godbillon-Vey invariant of $\mathcal{F}$. In general, $GV$ may be defined for fluids on manifolds with boundary with $\omega \parallel \partial \Omega$, where the Godbillon-Vey invariant cannot be defined.

As we will discuss, $GV$ is a topological invariant measuring the helical compression of the vortex lines. Inasmuch as it is a combination of twisting and squeezing of vorticity, we are tempted to refer to it as the ‘wring’ of a vorticity field, as in ‘wringing a towel’. While a local geometric meaning for $GV$ can be given fairly easily, the global topological interpretation is subtle [21]. Here we show, for a class of vorticity fields with $GV \neq 0$, that $GV$ is determined by the local structure of a distinguished set of closed vortex lines, associated to the Kupka phenomenon of integrable 1-forms [22, 23, 24]. These distinguished closed lines can be given local self-linking terms and $GV$ measures, in part, the failure of these self-linking terms to balance the linking terms from the other lines.

Helicity gives a lower bound for the magnetic energy of a vorticity field [5], and the goal has often been to give further bounds for the energy for higher-order helicities. $GV$ is dimensionless, and so cannot bound a dimensionful quantity such as the energy alone. We show, however, that $GV$ has a very natural relationship

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1Arnold and Khesin’s “dream” [5].
to Euler dynamics. In particular $GV \neq 0$ implies that the flow is not steady and

$$GV^2 \leq C \int_{\Omega} (\partial_t \omega)^2 dV,$$

where $C$ is a positive function depending on $\omega$ and the metric of $\Omega$ for which we give an explicit form. As $GV$ is invariant under volume-preserving diffeomorphisms, this gives a global topological obstruction to the existence of a metric making $\omega$ a steady solution of the Euler equations. We show further that in a particular metric-dependent form the density for $GV$ gives a local obstruction to $\partial_t \omega$ vanishing. In this form (6) can be further approximated, giving the relation

$$|GV| \lesssim \frac{L^7}{4E^2} \int_{\Omega} (\partial_t \omega)^2 dV,$$

where $L$ is a characteristic lengthscale of the system as $E$ is the total kinetic energy (of $U$). In this way, $GV$ can be seen to measure the ratio between the rate of change of vorticity to the kinetic energy of the flow. As discussed above, $GV$ has a geometric interpretation, measuring the average helical vortex compression (‘wring’) of the flow. Vorticity compression is the core nonlinearity of the Euler equations, given this as well as (6) we speculate that flows with $GV \neq 0$ will prove particularly interesting from a dynamical perspective.

The relevance of $GV$ in hydrodynamics has been noted previously [5, 25]. In particular, Tur and Yanovsky [26], establish the local conservation of $GV$, where it arises in the case of a hydrodynamic system described by a 1-form $S$, evolving as $(\partial_t + L_U)S = 0$. In Webb et al. [27] the potential application of $GV$ to ideal fluids is discussed. In general, the vector potential $A$ for $\omega$ satisfying $A \cdot \omega = 0$ may be written as $A = U + V$, where $U$ is the fluid velocity and $V$ is curl-free. Webb et al. discuss the special case $V = 0$, however, as we discuss below, $GV$ can be defined in more generality. Because $GV$ is invariant under all volume-preserving diffeomorphisms, it appears as a Casimir-type invariant in the Hamiltonian formulation of ideal fluids [28], which we discuss in another paper [29].

2. The Godbillon-Vey Invariant of a Vorticity Field

We consider a vorticity field $\omega$ defined on a 3-manifold $\Omega$ with volume-form $\mu$. We take all vector fields and differential forms to be smooth throughout. We impose $\omega \parallel \partial \Omega$ and

$$\text{div}(\omega) = \mathcal{L}_\omega \mu = 0,$$

hence $\iota_\omega \mu$ is closed, assumed exact so $\iota_\omega \mu = d\theta$. Then the helicity

$$H = \int_{\Omega} \theta \wedge d\theta,$$

is an invariant of volume-preserving diffeomorphisms. Helicity is invariant under a gauge transformation, $\theta \rightarrow \theta + df$, for any function $f$ and is invariant for any choice of $\theta$ only if it satisfies the fluxless condition [30] (see Appendix I). This condition is equivalent to the statement that

$$\int_S d\theta = 0,$$

for any surface $S \subset \Omega$, with $\partial S \subset \partial \Omega$. The fluxless condition is always satisfied if each component of $\partial \Omega$ is simply connected.
Here we will consider the special case where \( \theta \) can be chosen such that
\[
\theta \wedge d\theta = 0.
\]
\textit{i.e.} the helicity density vanishes. In the fluxless case this implies, but is stronger than, \( H = 0 \). For example, any simply-connected three-dimensional submanifold \( S \subset \Omega \) with \( \omega \parallel \partial S \) has a gauge-invariant helicity (\( S \) must also transform under any diffeomorphism of \( \Omega \)). Then it is quite possible to have \( \Omega = S_1 \cup S_2 \), \( H = H_1 + H_2 = 0 \) and \( H_1 \neq 0 \) in which case (11) cannot be satisfied. We also note that if the integral curves of \( \omega \) are closed then (11) does not imply that the pairwise linking numbers of the vortex lines vanish, for example consider a vorticity field supported on a set of solid tori each carrying flux \( \Phi_i \), with the vortex lines within each torus having zero mutual linking (zero self-linking term \([31]\)). Then (11) is equivalent to the statement that
\[
\sum_j \Phi_j Lk(i, j) = 0 \quad \forall i,
\]
which does not imply \( Lk(i, j) = 0 \) (as an example take all fluxes equal with the tori forming link \( 8_4 \) in the Rolfsen table \([32]\)).

The case we consider is of an integrable vorticity field, (11) implies the kernel of \( \theta \) defines a (singular) foliation \( F \) of \( \Omega \) with \( \omega \) tangent to the leaves. If \( \theta \) is unique, \( F \) is defined purely in terms of the vorticity field. In Ref. \([15]\) the integrable case is discussed and diffeomorphisms that act by transforming the pair \((F, \omega)\) are held distinct from those that act by transforming just the vorticity field. Because \( F \) is defined in terms of the vorticity, this issue is avoided.

For now we will assume that \( \theta \neq 0 \) and that \( \theta \) is unique, then there is a 1-form \( \eta \) satisfying
\[
d\theta = \theta \wedge \eta.
\]
Consider the integral
\[
GV = \int_\Omega \eta \wedge d\eta.
\]
The 1-form \( \eta \) is defined up to addition of \( f\theta \), \( f \) a function. Under such a transformation we have
\[
GV = \int_\Omega \eta \wedge d\eta - \int_\Omega df \wedge d\theta,
\]
the second term gives a boundary contribution which vanishes, \( \omega \) is tangent to \( \partial\Omega \) so \( d\theta|_{\partial\Omega} = 0 \) and \( GV \) does not depend on the choice of \( \eta \). By construction \( GV \) is invariant under any volume-preserving diffeomorphism and hence the fluid flow.

\( GV \) is essentially the Godbillon-Vey invariant \([33, 34]\) of the foliation defined by \( \theta \). In particular, if \( \Omega \) is closed, then \( GV \) is exactly the Godbillon-Vey invariant. If \( \Omega \) has a boundary then \( GV \) is defined for the pair \((\theta, \omega)\) whereas the Godbillon-Vey invariant may not be. For a foliation, one requires invariance under the transformation \( \theta \rightarrow h\theta \) for a non-zero function \( h \). This leads to a gauge transformation \( \eta \rightarrow \eta - d\log h \), and hence to a boundary term
\[
\int_{\partial\Omega} -\log h \, d\eta,
\]
which vanishes in general only if \( \partial\Omega \) is a leaf of the foliation (it is easily verified that \( \theta \wedge d\eta = 0 \)), and for an arbitrary integrable vorticity field this is not the case.
2.1. **Non-uniqueness of $\theta$.** Now suppose the choice of $\theta$ satisfying (11) is not unique. Then the difference between any two choices is a closed 1-form $\beta$ satisfying $\beta \wedge d\theta = 0$, and there is a 1-parameter family of 1-forms, $\theta + c\beta$, $c$ constant, satisfying (11). If $\theta \neq 0$ throughout the domain, then there is an open set $(a, b) \subset \mathbb{R}$, $a < 0 < b$, such that $\theta + c\beta \neq 0$ for $c \in (a, b)$. In general, non-uniqueness will lead to zeros in $\theta$. Zeros in $\theta$ can be accommodated for, and we will deal with them in part below, but for simplicity we will exclude the possibility for now.

Now restricting to non-zero choices of $\theta$, the nonuniqueness imposes sufficient structure on $\omega$ to force $GV = 0$. We have $\theta \wedge \beta = gd\theta$, where $g$ is a first integral of $\omega$. If $g$ is identically zero, then $\theta = h\beta$ for some non-zero function $h$, and we may take $\eta = -d\log h$, hence $GV = 0$. Now suppose the zero set of $g$ has positive codimension, then $\beta \wedge \eta = f d\theta$ and we find $g\eta = \beta - f\theta$. It follows for any choice of $\eta$ that $\eta \wedge d\eta = \frac{1}{g} df \wedge d\theta$, where $f$ depends on $\eta$. Now consider the set $S_\epsilon = \{x \in \Omega : |g(x)| > \epsilon\}$, note that $\omega \parallel \partial S_\epsilon$. Then observe that

$$ GV_\epsilon = \int_{S_\epsilon} \eta \wedge d\eta = \frac{1}{\epsilon} \int_{\partial S_\epsilon} f d\theta = 0, $$

in the limit $\epsilon \to 0$, $GV_\epsilon \to GV$ and we find $GV = 0$. Finally in the mixed case, where the zero set of $g$ includes a codimension-0 set $S$, $\omega$ must be tangent to the boundary of $S$, and one can treat the two regions separately. As a consequence, we find that the value of $GV$ depends purely on the vorticity field, rather than the 1-form $\theta$, if $\theta$ is not unique then $GV = 0$. Finally, note that this result also forces $GV = 0$ if $\omega$ has a first integral.

3. **GV and Dynamics**

3.1. **Global Obstruction to Steady Flow.** $GV$ has natural relation to the dynamics of the flow. First suppose the flow is steady, and that there is a choice of non-zero $\theta$ satisfying (11), then following Arnold [5], the flow must be one of three types:

1. $U \times \omega = X$, where $X$ is curl-free.
2. Non-constant Beltrami field, $\omega = \alpha U$, with $\alpha$ a non-constant function.
3. Constant Beltrami field, $\omega = \lambda U$ with $\lambda$ constant.

In the first two cases $\theta$ is not unique, the dual to $X$ is a closed 1-form $\beta$ satisfying $\beta \wedge d\theta = 0$, and the function $\alpha$ is a first integral of $\omega$, so $\alpha \wedge d\theta = 0$. In both these cases, $GV = 0$ by arguments in the previous section. In the third case, helicity is non-zero if $\lambda \neq 0$, so $GV$ cannot be defined. If $\lambda = 0$, then the flow is irrotational, and $GV = 0$ trivially.

Invariant under volume-preserving diffeomorphisms, $GV \neq 0$ is then a global obstruction to the existence of a metric making $\omega$ a steady solution of the Euler equations, under the assumptions given. This can be compared with results on the topology of the vorticity scalar in two dimensions obstructing steady flow [35]; results on the ‘Eulerisability’ of velocity fields [36, 37], where it is shown that such velocity fields cannot contain Reeb components; as well as results making use of contact topology, showing that a vector field without a closed orbit cannot be a solution to the Euler equations (for analytic vector fields on $S^3$) [38].

3.2. **Local Obstruction to Steady Flow.** We now give an alternate version of the connection between $GV$ and dynamics. For simplicity we assume that $\Omega$ is a
subset of $\mathbb{R}^3$ with the standard metric and total volume $V$. Then let the vector field $h$ be dual to $\eta$, giving

$$GV = \int_{\Omega} h \cdot \nabla \times h \, dV.$$  

For $h$ we make the choice

$$h = \frac{1}{U \cdot A} \omega \times U,$$

where $A$ is dual to $\theta$ and note that it is well-defined as per Section 2, provided $U \cdot A \neq 0$. If this is not the case, one can modify $h$ in the neighbourhood of such a zero. Then we have

$$h \cdot \nabla \times h = -\frac{1}{(U \cdot A)^2} (\omega \times U) \cdot \partial_t \omega.$$  

With this choice of $h$, $\partial_t \omega$ is non-zero wherever $h \cdot \nabla \times h$ does not vanish, the density of $GV$ gives a local obstruction to the flow being steady and implies the global result. This perspective also translates to arbitrary vector fields $V$ transporting the vorticity. One can replace $U$ in $h$ with any volume preserving vector field $V$, whereupon the density of $GV$ is related to the time derivative of $\omega$ under the flow of $V$.

### 3.3. Bounds on Vorticity Rate-of-Change.

Helicity is a dimensionful quantity, if $U$ has dimensions $[X]$, then helicity has dimensions $[X]^2[\text{Length}]^2$. Under a rescaling $\omega \rightarrow c \omega$, helicity transforms as $H \rightarrow c^2 H$ (along with orientation reversal, this is sufficient to establish a bijection between the sets of vector fields with fixed non-zero helicity) and it is well-known that helicity bounds the magnetic energy $\int_{\Omega} \omega^2 \, dV$ [4]. Conversely, $GV$ is dimensionless (note that $h$ always has dimensions of $\text{Length}^{-1}$) so can only bound ratios of physical quantities. First note that

$$\int_{\Omega} U \cdot A \, dV = \int_{\Omega} U^2 \, dV = 2E,$$

where $E$ is the kinetic energy of the flow, so we can write $U \cdot A = 2E/V + \delta$, expanding we find

$$GV = \frac{V^2}{4E^2} \int_{\Omega} dV (\omega \times U) \cdot (\partial_t \omega) \left( 1 - \delta \frac{V}{E} + O(\delta^2) \right).$$

Neglecting terms of order $\delta$ and above (constant energy density) we find

$$GV \approx \frac{V^2}{4E^2} \int_{\Omega} (\omega \times U) \cdot (\partial_t \omega) \, dV.$$  

Then using Cauchy-Schwarz and the Poincaré inequality we find

$$|GV| \lesssim \frac{L^7}{4E^2} \int_{\Omega} (\partial_t \omega)^2 \, dV,$$

where $L^7 = V^2/\sqrt{\lambda}$ and $\lambda$ is the minimal Laplacian eigenvalue, so that $GV$ is related to the ratio of the rate of change of the vorticity to the (squared) kinetic energy.
4. Zeros of $\theta$

So far we have required $\theta \neq 0$. Zeros in $\theta$ can be accommodated, and we will discuss some aspects here. Suppose $\theta$ has zero set $Z$, so that $\theta$ defines a singular foliation of $\Omega$. While this can be dealt with through the use of Haefliger structures [39], we will deal with the differential forms directly. Structurally stable, zeros of integrable 1-forms have a rich structure [23, 24]. For example, the 1-form

$$\theta = \alpha yzdx + \beta xzdy + \gamma xydx,$$

with $\alpha \neq \beta \neq \gamma$ is homogeneous of degree 2, satisfies $\theta \wedge d\theta = 0$ and has a $C^2$-stable zero at the origin. A related example has also been used to demonstrate the local non-existence of a Clebsch representation for velocity fields in the vicinity of vorticity zeros [40]. In general, in the smooth category, there are structurally stable examples such as (25) of arbitrarily high degree. Here we restrict our discussion to zeros of $\theta$ that are $C^1$ structurally stable, and note that an extension to higher-order zeros should be possible. $C^1$ structurally stable zeros come in two varieties [22].

The first are points where $\theta$ has the local form $fdg$, with $f \neq 0$ and $g$ having a Morse critical point. In a neighbourhood of such a point, one can write $\eta = -d\log f$ which is non-singular, so that $\eta$ can be defined.

The second type of zero has $\theta = 0$ and $d\theta \neq 0$, in this case the Kupka phenomenon occurs [22, 23, 24], and in a neighbourhood of such a zero there is a local system of coordinates such that

$$\theta = a(x,y)dx + b(x,y)dy,$$

where $a$ and $b$ have a simple zero at $x = y = 0$. In this case the zero becomes a line $L$, $\omega$ is tangent to $L$, which therefore cannot intersect $\partial \Omega$ transversely, making $L$ a distinguished closed orbit. On such a Kupka line $\eta$ cannot be defined, but $GV$ can still be computed. From (26) we can define a neighbourhood, $N_L$, such that $d\theta|_{\partial N_L} = 0$. The torus $\partial N_L$ separates the domain into two regions, as the vorticity is tangent to $\partial N_L$ both the exterior and interior (containing the Kupka line) have a well-defined $GV$, so we may write $GV = GV_e + GV_i$, for exterior and interior respectively. It is then simply a matter of checking (which can be done by a simple calculation), that $\eta$ may be defined on the interior of $\partial N_L$ such that $\eta \wedge d\eta$ is identically zero on the complement of $L$, and so we may identify $GV_i = 0$.

It is important to note that if we allow more complex zeros in $\theta$, then the analysis becomes more delicate. For example, suppose we allow (25). Then $\eta$ cannot be defined on the zero. In this case $GV$, as defined above, is invariant under $\eta \rightarrow \eta + f\theta$ only if $f \sim 1/\|x\|^a$, as $x \rightarrow 0$, where $a < 3$.

5. Local Structure

5.1. Local Conservation Law. Like helicity, $GV$ is a global property of the fluid and cannot be measured locally. For example, in a neighbourhood of any point where $\omega \neq 0$ we have a Clebsch representation [10],

$$U = f\nabla g + \nabla \phi.$$

Consequently, one may choose a vector potential for $\omega$ as well as a particular $\eta$ such that both the helicity density and the Godbillon-Vey density vanish on the neighbourhood. Despite this, there are several local analyses that are informative. First we will derive a local conservation law for $GV$, which obtains validity by
making a global choice for $\eta$. Suppose $\Omega \subset \mathbb{R}^3$ with the Euclidean metric. Consider the time derivative
\begin{equation}
(28) \quad (\partial_t + \mathcal{L}_U) \eta \wedge d\eta,
\end{equation}

note that $(\partial_t + \mathcal{L}_U) \theta = 0$. From this we have
\begin{equation}
(29) \quad (\partial_t + \mathcal{L}_U) \eta = f \theta
\end{equation}
for some function $f$. It follows that
\begin{equation}
(30) \quad (\partial_t + \mathcal{L}_U) \eta \wedge d\eta = d(f d\theta).
\end{equation}

In vector calculus notation we have
\begin{equation}
(31) \quad (\partial_t + U \cdot \nabla)(h \cdot \nabla \times h) = \nabla \cdot (f \omega)
\end{equation}
which gives an explicit local conservation law, and we find that $GV$ is carried by vorticity. One must choose $h$ to determine $f$. For example, taking $h$ as in (19) gives a simple calculation which leads the result
\begin{equation}
(32) \quad f = h^2 + \frac{1}{U \cdot A} h \cdot \nabla(P + U^2/2),
\end{equation}
where $P$ is the pressure. The analysis by Tur and Yanovsky \[26\] yields an alternate form for $f$ when the choice $h \cdot A = 0$ is made.

### 5.2. Interpretation as Helical Compression.

Due to the form of $GV$ it is tempting to attempt to interpret it as the linking of the integrals curves of the vector field $X$ associated to the 2-form $d\eta$ as $d\eta = \iota_X \mu$. This is not natural for two reasons. Firstly, $d\eta$ is not necessarily parallel to $\partial \Omega$, so integral curves of $X$ may end. Secondly, $d\eta$ is not invariant under the transformation $\eta \rightarrow \eta + f \theta$, so that the integral curves of $X$ change, and may reconnect locally. Instead, there is a local geometric interpretation of $GV$ as helical compression of vorticity. Vorticity compression is the core nonlinearity of the Euler equations, and given the strong relationship between between $GV$ and dynamics, we speculate that flows with $GV \neq 0$ will be particularly interesting from a dynamical perspective. The helical compression interpretation, essentially Thurston’s ‘helical wobble’ for foliations \[27\] \[41\], can be seen as follows. Suppose $\mathcal{F}$ is the foliation induced by $\omega$, and $N$ is a unit vector normal to it. Then the vector field $h_N = (N \cdot \nabla)N$ is a choice of $h$ for $\nabla \times N$, and is hence equal to a choice of $h$ for $\omega$ up to a gauge transformation. The helical compression interpretation is, essentially, to interpret $h_N \cdot \nabla \times h_N$ as a combination of twisting and compression, we are tempted to refer to $GV$ as the ‘wring’ of the vorticity.
Figure 1. Local geometric interpretation of $GV$ as helical vortex compression. Left: $N$ (red) is the normalised field to the foliation $\mathcal{F}$ to which the vorticity (blue) is tangent. The vector $h_N$ points in the direction of local compression of the surfaces of $\mathcal{F}$. Right: helical compression of $\mathcal{F}$. In this case the vector $h_N$ rotates about the vertical as one moves upwards, hence $h_N \cdot \nabla \times h_N \neq 0$.

6. Global Structure

For $GV \neq 0$ one must construct a vorticity field with global properties, much like the helicity requires the global linking of vortex lines. All known examples of vorticity fields with $GV \neq 0$ have a certain degree of complexity, and do not permit a simple expression. Here we will adapt Thurston’s construction [41] to give an example of a vorticity field where $GV$ is determined by certain Kupka lines – closed orbits of $\omega$.

First consider the link, $L$, in $S^3$ with $N$ components each denoted $L_i$, $i \in [1, N]$, shown in Figure 2 for $N = 3$. Thurston’s construction uses as its base the horocycle foliation of the unit tangent bundle of the hyperbolic plane $\mathbb{H}^2$ and creates a foliation on the complement of this link, defined by a 1-form $\theta$. The leaves of the foliation are transverse to the boundary and induce a slope $s_i$ on each link component (measured relative to the oriented longitudes as in Figure 2). In Thurston’s construction, the boundary components are spun onto tori and filled with Reeb components.

Equivalently, in our case, we can simply allow them to become zeros of $\theta$, where they become Kupka lines. Since the neighbourhood of each Kupka line has a well defined value of $GV = 0$, the value of $GV$ for the total vorticity field must be equal to the Godbillon-Vey invariant of the foliation, and is given by

\begin{equation}
GV = 4\pi^2 \left( N - 2 - \left( \frac{1}{s_1} + \sum_{i=2}^{N} s_i \right) \right).
\end{equation}

To understand the meaning of (33), we construct a $GV = 0$ vorticity field with the same set of closed vortex lines. First we define a closed 1-form $A$ on $S^3 \setminus L$. As $H^1(S^3 \setminus L; \mathbb{R}) \cong \mathbb{R}^N$, this form is defined by a set of $N$ fluxes $\phi_i$, and leads to a

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2 In the Poincaré disk model, this is defined by the 1-form $\alpha = 2(y - \sin \theta)dx + 2(\cos \theta - x)dy + (x^2 + y^2 - 1)d\theta$, where $\theta$ is the fiber coordinate. The leaves of the foliation are helicoids with axis on the boundary.
Figure 2. The link $L$ with 3 components in $S^3$. In the example given, each component of $L$ is a closed orbit of $\omega$. The local structure of $\theta$ in a neighbourhood of $L$ defines a slope for each component. The value of $GV$ is determined by the deviation of these slopes from the value they would have in the case when $\omega$ is singular, with each $L$ a flux line.

measured (singular) foliation of $S^3 \setminus L$. We can then multiply $A$ by a function $f$, zero on $L$, so that $A$ can be extended to a 1-form on $S^3$. The resulting singular foliation of $S^3$ has $GV = 0$ and $N$ Kupka lines, with slopes $s_1 = -\phi_1/\sum_{i \neq 1} \phi_i$ on $L_1$ and $s_i = -\phi_i/\phi_1$ on $L_i$, $i \neq 1$. Then note that $\sum_{i=2}^N s_i + 1/s_1 = 0$. Comparing to (33), we see that $GV$ measures, in part, the failure of the Kupka line slopes to ‘commute’ in the manner expected for singular vorticity field with flux lines on $L$.

The construction has at its core a foliation that is transverse to the fibers of a circle bundle, in which case $\eta \wedge d\eta$ can be integrated over the fibers [43]. Using similar techniques [44] it should be possible to produce similar examples to the one above with $L$ any link whose complement is an $S^1$ bundle over $S^2$, which have been classified [45].

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We consider the invariance of helicity

\[ H = \int_{\Omega} \theta \wedge d\theta, \]

under an arbitrary transformation, \( \theta \to \theta + \beta \), where \( \beta \) is a closed 1-form. Then the helicity integral gives a boundary term

\[ H \to \int_{\Omega} \theta \wedge d\theta + \int_{\partial \Omega} \theta \wedge \beta, \]

which in general does not vanish. By construction, \( d\theta|_{\partial \Omega} = 0 \), hence \( \theta|_{\partial \Omega} \) is a closed 1-form on \( \partial \Omega \) and defines a de Rahm cohomology class \([\theta] \in H^1(\partial \Omega; \mathbb{R})\).

Restriction to the boundary defines a map \( r : H^1(\Omega; \mathbb{R}) \to H^1(\partial \Omega; \mathbb{R}) \). \( d\theta \) is fluxless if \( [\theta] = 0 \in H^1(\partial \Omega; \mathbb{R})/\text{Im}(r) \). This condition is equivalent to the statement that

\[ \int_S d\theta = 0, \]

for any surface \( S \subset \Omega \), with \( \partial S \subset \partial \Omega \).

8. **Appendix II**

We start with the definition

\[ h = \frac{1}{U \cdot A} \omega \times U = \beta \omega \times U. \]

Then we assert that

\[ (\partial_t + L_U)h = fA, \]

where \( f \) is to be determined. Using coordinate notation (recall we are in Euclidean space), we have

\[ fA_i = \partial_i h_i + U_j \partial_j h_i + U_j \partial_i h_j. \]
Now by construction we have $U_i h_i = 0$, so this is rewritten as

$$f A_i = \partial_t h_i + U_j \partial_j h_i - h_j \partial_t U_j,$$

or

$$f A = \partial_t h - U \times \nabla \times h.$$

Expanding we find

$$f A = (\partial_t \beta) \omega \times U + \beta (\partial_t \omega) \times U + \beta \omega \times (\partial_t U) - U \times (\nabla \beta \times (\omega \times U)) + \beta U \times (\partial_t \omega),$$

which becomes

$$f A = ((\partial_t + U \cdot \nabla) \beta) \omega \times U + \beta \omega \times (\partial_t U).$$

Now, using the fact that

$$(\partial_t + U \cdot \nabla) A + (\nabla A) \cdot U = 0,$$

we find

$$(\partial_t + U \cdot \nabla) \beta = \beta (h \cdot A + \beta A \cdot \nabla (P + U^2/2)).$$

So we get

$$f A = ((h \cdot A) + \beta A \cdot \nabla (P + U^2/2)) h - \beta \omega \times \nabla (P + U^2/2) - \beta \omega \times (\omega \times U).$$

Then we find

$$f A = (h^2 + \beta h \cdot \nabla (P + U^2/2)) A,$$

so we may identify

$$f = h^2 + \beta h \cdot \nabla (P + U^2/2).$$