PERFECT REEB FLOWS AND ACTION-INDEX RELATIONS

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Abstract. We study non-degenerate Reeb flows arising from perfect contact forms, i.e., the forms with vanishing contact homology differential. In particular, we obtain upper bounds on the number of simple closed Reeb orbits for such forms on a variety of contact manifolds and certain action-index resonance relations for the standard contact sphere. Using these results, we reprove a theorem due to Bourgeois, Cieliebak and Ekholm characterizing perfect Reeb flows on the standard contact three-sphere as non-degenerate Reeb flows with exactly two simple closed orbits.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. In this paper, we investigate non-degenerate Reeb flows arising from contact forms with vanishing contact homology differential, referred to as perfect contact forms throughout. In particular, we prove, under minor additional conditions, upper bounds on the number of simple closed Reeb orbits for perfect forms on a broad class of contact manifolds and certain action-index resonance relations for the spheres $S^{2n-1}$ equipped with the standard contact structure.

To put this work in perspective, recall that a well-known conjecture in Reeb dynamics asserts the existence of at least $n$ simple closed Reeb orbits on the standard contact $S^{2n-1}$. (See, e.g., [EH, Lo, LZ, Wa] for the lower bounds on the number of such orbits in the convex case and the discussion below for $S^3$.) Furthermore,
hypothetically, the number of closed Reeb orbits on $S^{2n-1}$ is either $n$ or infinite. Moreover, drawing an analogy to several results and conjectures concerning Hamiltonian flows, it is not unreasonable to conjecture that there are infinitely many simple closed Reeb orbits whenever there are more such orbits than necessary (e.g., $n$ for $S^{2n-1}$). We will refer to the latter conjecture, which is the main motivation for the present work, as the contact HZ-conjecture, for, to the best of the author’s knowledge, the first written statement of the Hamiltonian HZ-conjecture is in [HZ]; see [BH, CKRTZ, Fr1, Fr2, FH, GG4, GHHM, Gü1, Gü2, Ke, LeC] for some results in the Hamiltonian setting. All of these conjectures are open in general, even under the non-degeneracy assumption.

In contrast with the Hamiltonian case, it is not clear how to express the threshold number of simple closed orbits in the contact HZ-conjecture in terms of the contact homology. However, one can certainly interpret the condition that the Reeb flow has no unnecessary closed Reeb orbits by requiring the contact differential to vanish, i.e., the Reeb flow to be perfect. Note that, although perfect flows are clearly exceptional (see Remark 1.11), many Reeb flows of interest are in this class; see Section 1.3 for examples. For $S^3$ such flows were studied in [BCE].

These conjectures lead to the question whether there is an upper bound on the number of simple closed orbits of a perfect Reeb flow on, say, the standard contact $S^{2n-1}$. In this work, we establish such upper bounds of contact homological nature for the sphere and some other contact manifolds under some additional assumptions; see Theorems 1.5 and 1.7. (In general, without extra conditions, it is not even known if a perfect flow on $S^{2n-1}$, $2n-1 \geq 5$, must have finitely many simple closed orbits.) These upper bounds, in particular, provide homological interpretation for the upper bound $n$ in the HZ-conjecture for $S^{2n-1}$. To the best our knowledge, these are the first results in this direction beyond the three-dimensional case considered in [BCE].

Another aspect of our investigation of perfect Reeb flows is a resonance relation asserting that all simple closed Reeb orbits $x$ of a perfect Reeb flow on the standard contact $S^{2n-1}$ have the same ratio $\Delta(x)/A(x)$, where $\Delta(x)$ is the mean index and $A(x)$ is the action. This result, Theorem 1.2, is a contact analog of the action-index resonance relations proved in [CGG, GG1] for Hamiltonian flows and is also related to the results from [EH] and, at least on the conceptual level, to [HR].

Using these results, we also give a short proof of a theorem from [BCE], characterizing perfect Reeb flows on the standard contact sphere $S^3$ as non-degenerate Reeb flows with exactly two simple closed orbits; this is Theorem 1.8.

It should be noted that much more is known about the number of simple closed Reeb orbits in dimension three. This is partly due to the availability of powerful, but strictly 3-dimensional, methods such as finite energy foliations and the embedded contact homology and also because the Conley-Zehnder index and local contact homology behave in a much simpler way in this case. For instance, the Weinstein conjecture has been established in dimension three, [Ta], and, moreover, it has been recently proved that a Reeb flow on any closed contact 3-manifold has at least two closed orbits, [CGH]; see also [GGo, GHHM, LL] for the case of the standard contact $S^3$. Furthermore, there exist either two or infinitely many closed characteristics on any strictly convex hypersurface in $\mathbb{R}^4$ and, under some additional assumptions, for any non-degenerate Reeb flow on the standard contact $S^3$; [HWZ2, HWZ3]. (See also, e.g., [Ho, HWZ1, WHL] and references therein for some other relevant results.)
1.2. Main results. The main object underlying our results is a perfect Reeb flow on a contact manifold.

**Definition 1.1.** A non-degenerate contact form is said to be perfect if the contact homology differential (for a chosen flavor of contact homology) vanishes.

The flow of such a contact form will be called a perfect Reeb flow. This notion, in general, depends on the kind of contact homology used, but once this choice is made, perfectness is independent of the almost complex structure and other auxiliary data. There are many examples of perfect Reeb flows of interest; here, deferring further examples to Section 1.3, we merely note that a non-degenerate Reeb flow on an irrational ellipsoid is perfect. It is worth pointing out that in all the examples known to us the Reeb flow is perfect even in a stronger sense: all closed Reeb orbits in every free homotopy class have the same Conley-Zehnder index modulo 2; let us call such Reeb flows geometrically perfect. Observe that a Reeb flow can be geometrically perfect whether or not the contact homology is defined. If, however, a flavor of contact homology is defined, a geometrically perfect flow is also perfect in our sense. In what follows "perfectness" for us will always be understood in the sense of Definition 1.1.

Our first result is an analog of Theorem 1.1 from [CGG] for Reeb flows (see also [GG1]). The Hamiltonian version of this result asserts in particular that for a Hamiltonian diffeomorphism of \( \mathbb{CP}^n \) with finitely many periodic orbits, there are \( n+1 \) distinct fixed points \( x_0, \ldots, x_n \) such that the so-called augmented actions for all of these points are equal: \( \tilde{A}(x_0) = \cdots = \tilde{A}(x_n) \). (In general, the augmented action of a contractible periodic orbit \( x \) is defined by \( \tilde{A}(x) = A(x) - \lambda/2\Delta(x) \), where \( A(x) \) and \( \Delta(x) \) are, respectively, the symplectic action and the mean index of \( x \) and \( \lambda \) is the rationality constant of the ambient symplectic manifold.) Similarly, for perfect Reeb flows on the standard contact sphere \( S^{2n-1} \) we have

**Theorem 1.2.** Let \( \alpha \) be a perfect non-degenerate contact form on \( (S^{2n-1}, \xi_{\text{std}}) \). Then, for any two closed Reeb orbits \( x \) and \( y \) of \( \alpha \), we have

\[
\frac{\Delta(x)}{A(x)} = \frac{\Delta(y)}{A(y)}.
\]

**Remark 1.3.** Notice that, in contrast with the Hamiltonian case, in Theorem 1.2 we make no assertion about the number of simple closed Reeb orbits. At this stage we cannot rule out that the Reeb flow of \( \alpha \) can have any number greater than or equal to two (including infinity) of simple closed orbits. (It is not hard to see that a non-degenerate contact form on the standard contact \( S^{2n-1} \) must have at least two simple closed Reeb orbits; see Remark 3.3 and also [Ka].)

As an immediate consequence of this theorem and a result from [BCE], reproved below (see Theorem 1.8), asserting that a non-degenerate contact form on the standard contact \( S^3 \) is perfect if and only if it has exactly two simple closed Reeb orbits, we obtain

**Corollary 1.4.** Let \( \alpha \) be a non-degenerate contact form on \( (S^3, \xi_{\text{std}}) \) with exactly two simple closed Reeb orbits, say, \( x \) and \( y \). Then \( \Delta(x)/A(x) = \Delta(y)/A(y) \).

The next two theorems below hold for a more general class of contact manifolds \( (M^{2n-1}, \xi) \) which we now specify. First, we require a version of contact homology
HC_*(M, ξ) to be defined for M. In this paper we will utilize the linearized contact homology, although cylindrical contact homology would equally well suit our purposes. (See also Remark 1.10.) To this end, assume that (M^{2n-1}, ξ) is closed and strongly fillable by an exact symplectically aspherical manifold, i.e., M is the boundary of some exact symplectic manifold (W^{2n}, \omega = d\alpha) such that c_1(TW) = 0 and ker d|_M = ξ with matching orientations. Under these conditions, the linearized contact homology HC_*(M, ξ) is defined and independent of the contact form on M supporting ξ (see, e.g., [Bo2, BO1, EGH]); we will use the notation HC_*(ξ) whenever M is clear from the context. Furthermore, we will work with contact homology defined for the entire collection of free homotopy classes of closed Reeb orbits, although everything below would also go through if we restricted the homology to a smaller collection which is closed under iterations; see Remark 2.1 for a brief discussion of the linearized contact homology.)

In addition, suppose that HC_*(ξ) satisfies the following condition:

$$\dim HC_*(ξ) < \infty \text{ for all } * \leq 2n-4 \text{ and } HC_*(ξ) = 0 \text{ for } * \ll 0. \quad (1.1)$$

This condition holds for a large variety of contact manifolds of interest, but, of course, not for all contact manifolds. Next, set

$$b = \limsup_{m \to \infty} \sum_{i=0}^{2n-2} \dim HC_{m+i}(ξ). \quad (1.2)$$

For instance, b = n for the sphere (S^{2n-1}, ξ_{std}). Also, whenever dim HC_m(ξ) ≤ b_0 with b_0 independent of the degree m, we have b ≤ b_0(2n-1) in general and b ≤ b_0 n if HC_m(ξ) ≠ 0 only in even degrees. Note that in (1.2) we have also assumed that dim HC_m(ξ) < ∞ for all large m. Otherwise, by definition, b = ∞ and the results below hold trivially. Finally, a simple closed orbit is called even if the number of real eigenvalues of the linearized Poincaré return map in the interval (-1, 0) is even; see Section 2. Otherwise, an orbit is said to be odd.

Under the above hypotheses on (M^{2n-1}, ξ) with b as in (1.2), we have

**Theorem 1.5.** Let α be a perfect non-degenerate contact form on (M^{2n-1}, ξ) with r simple even orbits. Then r ≤ b.

**Corollary 1.6.** A non-degenerate perfect Reeb flow on the standard contact sphere S^{2n-1} has at most n even closed Reeb orbits.

Furthermore, considering all simple closed orbits, we have

**Theorem 1.7.** Let α be a perfect non-degenerate contact form on (M^{2n-1}, ξ). Assume that α has r simple orbits (even or odd) such that the reciprocals of their mean indices are linearly independent over Q. Then r ≤ b.

As an application of Theorem 1.5, we reprove the following result from [BCE].

**Theorem 1.8 ([BCE]).** For a non-degenerate contact form α on (S^3, ξ_{std}), the following are equivalent:

(i) α is perfect.

(ii) α has exactly two simple closed Reeb orbits; these orbits are elliptic and their mean indices and actions are the same as those for an irrational ellipsoid.

(These orbits are also unknotted and have linking number one.)

(iii) α has exactly two simple closed Reeb orbits.
Remark 1.9. It is worth pointing out that it is unknown in general, even for the standard contact $S^{2n-1}$ when $2n - 1 \geq 5$, whether a perfect Reeb flow necessarily has finitely many simple closed orbits.

Remark 1.10. We should note that the paper relies on the theory of contact homology which is still to be fully rigorously established; see [HWZ4, HWZ5]. To circumvent the foundational difficulties, we could have used the $S^1$-equivariant symplectic homology rather than the linearized contact homology; see [BO2]. However, the proofs in that case would certainly be more involved and less transparent, and hence we preferred to employ contact homology.

Remark 1.11. Combining the argument from [GG2] and Theorem 1.7, it is not hard to show that for any contact manifold $(M, \xi)$ meeting the requirements of the theorem, non-perfect contact forms form a residual subset in the set of all contact forms supporting $\xi$, equipped with $C^\infty$-topology.

Remark 1.12. The linear independence condition from Theorem 1.7 is satisfied for many perfect Reeb flows. Moreover, one can even conjecture that the condition holds for a generic perfect Reeb flow; for instance, this is the case for ellipsoids. Note, however, that the resonance relation from [GK, Vi] asserts that the reciprocals of mean indices and 1 are never linearly independent over $\mathbb{Q}$ when the flow has finitely many simple closed orbits.

1.3. Examples. Most of the examples of perfect Reeb flows known to the author originate from contact torus actions. In the most general setting, whenever we have a contact torus action with only finitely many one-dimensional orbits and such that there exists a one-parameter subgroup in the torus with orbits transverse to the contact structure, the action of such a generic one-parameter subgroup gives rise to a perfect Reeb flow. (The closed Reeb orbits of this flow are exactly the one-dimensional orbits of the torus action, and moreover, as is easy to see, all these orbits are non-degenerate and elliptic.) Among the contact manifolds this observation applies to are, for instance, contact toric manifolds; see [AM, Le]. Moreover, one can further perturb this Reeb flow, making it ergodic and creating non-trivial dynamics by applying the results from [Ka].

Here, however, rather than exploring this construction in the most general framework, we focus on more specific examples which we find interesting and illuminating.

Example 1.13 (Prequantization circle bundles). Let $(B, \omega)$ be a closed symplectic manifold equipped with a Hamiltonian circle action with isolated fixed points and let $H$ be a Hamiltonian (a moment map) generating the action. Assume in addition that $B$ is integral, i.e., $\omega \in H_2(B, \mathbb{Z})/\text{Tor}$. Let $\pi: M \to B$ be the pre-quantization circle bundle over $(B, \omega)$, i.e., $M$ is an $S^1$-bundle over $B$ with $c_1 = [\omega]$. In other words, $\pi^*\omega = da$, where $a$ is a connection form on $M$, and the Lie algebra of $S^1$ and $\mathbb{R}$ are suitably identified. As is well-known, $a$ is a contact form. Denote by $b \in \mathbb{R}/\mathbb{Z}$ the action of $H$ on an orbit of the action. (Here viewing $a$ as an element of $\mathbb{R}/\mathbb{Z}$ rather than $\mathbb{R}$ eliminates the ambiguity in the definition of the action arising from the choice of a capping.) It is easy to see that $a$ is well-defined, i.e., independent of the orbit.

Let us now lift the $S^1$-action on $B$ to a contact $\mathbb{R}$-action on $M$, preserving the form $\alpha$. This lift is generated by the vector field $HR + \dot{X}_H$. Here $R$ is the Reeb vector field of $\alpha$, i.e., the vector field generating the principal bundle $S^1$-action on
\( M \), and \( \hat{X}_H \) is the horizontal lift of the Hamiltonian vector field \( X_H \). (We refer the reader to [GGK, Appendix A] for a discussion of group actions and pre-quantization circle bundles, including all the facts used in this example.) Alternatively, when \( H > 0 \), this vector field can be described as the Reeb vector field of the form \( H^{-1} \alpha \), where, abusing notation, we write \( H \) for the pull-back \( H \circ \pi \).

One can show that when \( a \) is irrational, i.e., \( a \notin \mathbb{Q}/\mathbb{Z} \), the only simple periodic orbits of the resulting Reeb flow are the fibers of \( \pi \) over the fixed points of the \( S^1 \)-action on \( B \), i.e., the critical points of \( H \). Under our assumptions, there are finitely many simple closed orbits and, in fact, the number of these orbits \( r \) is equal to the sum \( \beta \) of the Betti numbers of \( B \). Furthermore, all such orbits are elliptic, since the critical points of \( H \) are elliptic, and hence the Reeb flow is geometrically perfect. The requirements that \( H > 0 \) and that \( a \notin \mathbb{Q}/\mathbb{Z} \) can always be satisfied by adding a constant to \( H \). Clearly, one can replace the circle action by a torus action in this construction.

As a side remark, recall that when \( a \) is rational, i.e., \( a \in \mathbb{Q}/\mathbb{Z} \), the lifted action is periodic, i.e., it is an \( S^1 \)-action. However, the minimal period \( T \) of the lifted action need not be equal to the period of the \( S^1 \)-action on \( B \). More specifically, when \( a = p/q \) with \( (p, q) = 1 \) and the period of the \( S^1 \)-action on \( B \) is one, we have \( T = q \).

Let us now briefly examine how Example 1.13 fits in the framework of Theorem 1.5. In all examples of symplectic manifolds admitting Hamiltonian circle actions with isolated fixed points known to the author, \( c_1(TB) \neq 0 \). Under this assumption, \( c_1(\xi) = 0 \), where \( \xi = \ker \alpha \), only when \( B \) is (strictly) monotone or negative monotone. To fix a grading of \( HC_* (\xi) \), let us choose a non-vanishing section \( s \) of the square of the complex determinant line bundle of \( \xi \) (see Section 2). Then working with cylindrical contact homology and using a suitably chosen sequence of perfect contact forms, we have

\[
HC_* (\xi) = \bigoplus_{m=1}^{\infty} H_{s-2+m\Delta} (B, \mathbb{Q}),
\]

where \( \Delta \) is the mean index of the fiber of \( \pi \) evaluated with respect to \( s \); see [Boi] and also [KvK] and references therein. (For example, when the fiber is contractible and \( B \) is monotone, \( \Delta = 2N \), where \( N \) is the minimal Chern number of \( B \); [Boi, p. 100].) Now, it is clear that \( b = r = \beta \), the sum of Betti numbers of \( B \), when \( \Delta > 2n - 2 \). When \( \Delta \leq 2n - 2 \), we may have \( b > r = \beta \). In the next example, we will see explicitly that this indeed can be the case.

**Example 1.14 (Katok–Ziller flows)**. The pre-quantization example described above includes, in particular, the Katok-Ziller description of perfect geodesic flows for asymmetric Finsler metrics on the spheres and some other manifolds. For the sake of simplicity, let us focus on the sphere \( S^n \).

In this case, \( M = ST^* S^n \) is the unit cotangent bundle of \( S^n \) and \( R \) is the spray of the round metric on \( S^n \), and the geodesic flow is clearly a circle action. The oriented geodesics are just the great circles on \( S^n \), and taking the quotient by the action, we obtain the principle circle bundle \( \pi: M \to B \) where \( B = \text{Gr}^\perp (2, n+1) \), the Grassmannian of oriented 2-planes in \( \mathbb{R}^{n+1} \).

Consider now a linear \( S^1 \)-action on \( \mathbb{R}^{n+1} \). This action naturally gives rise to a Hamiltonian circle action on \( B \). When the "weights" of the \( S^1 \)-action on \( \mathbb{R}^{n+1} \) are relatively prime, the fixed points of the resulting action on \( B \) are isolated and,
choosing $H$ appropriately as in the previous example, we can lift this action to a perfect Reeb flow on $M$. One can show that this Reeb flow is, in fact, the geodesic flow of an asymmetric Finsler metric on $S^n$, described in [Ka, Zi]. The number of simple closed Reeb orbits in this case is $r = n + 1$ when $n$ is odd and $r = n$ when $n$ is even, while $b = n + 3$ when $n$ is odd and $b = n + 2$ when $n$ is even. (See [KvK] and references therein for the calculation of contact homology in this case.) This example shows that the inequality $r \leq b$ in Theorem 1.5 can be strict.

**Example 1.15 (GMSW–AM contact structures).** A sequence of geometrically perfect contact forms $\alpha_k$ on $S^2 \times S^3$ with exactly $r = 4$ simple closed Reeb orbits is constructed in [AM, GMSW]. These contact forms have vanishing first Chern class; their contact homology is concentrated in even degrees and $\dim \text{HC}_{2m}(\ker \alpha_k) = 2k + 2$ when $2m > 2$. It readily follows that $b = 6(k+1)$, and hence the discrepancy between $r$ and $b$ in Theorem 1.5 can be arbitrarily large.

**Remark 1.16.** A particular feature of the examples described above is that the Reeb flow generates a contact torus action. We emphasize that, in general, perfect Reeb flows need not have such a simple dynamics. For instance, the ergodic perturbations of perfect Finsler flows on $S^n$ constructed in [Ka] are still perfect and have the same periodic orbits as the original flows, but exhibit highly non-trivial dynamics. There are, however, examples of perfect Reeb flows of an entirely different nature.

**Example 1.17 (Hyperbolic geodesic and Reeb flows).** The geodesic flow (or its $C^1$-small perturbation) on the unit cotangent bundle $M$ to a manifold $B$ with negative sectional curvature is geometrically perfect (see Section 1.2). Indeed, there exists exactly one closed orbit in every free homotopy class of loops in $B$. More generally, any hyperbolic Reeb flow is necessarily geometrically perfect; [MP]. The reason is that the parity of the index of a hyperbolic orbit is determined by whether the orbit is even or odd, and an orbit is odd if and only if the restriction of the stable bundle to the orbit is not orientable. Now, it is clear that for a hyperbolic Reeb flow all orbits in a fixed free homotopy class have the same index mod 2. We refer the reader to [FoHa] for the constructions of “exotic” hyperbolic Reeb flows.

**Remark 1.18 (Morse inequalities).** The classical Morse inequalities give lower bounds for the number of critical points of a function in terms of the Betti numbers of a manifold, and the Morse inequalities turn into equalities when the Morse function is perfect. The non-degenerate Arnold conjecture, as we understand it now, is essentially a variant of the Morse inequalities for fixed points of Hamiltonian diffeomorphisms and Floer homology. One can wonder if there are analogs of the Morse inequalities for Reeb flows and contact homology. Obviously, providing a lower bound for all, not necessarily simple, closed Reeb orbits of a fixed index presents no difficulty. However, once the count is restricted to simple orbits, the situation becomes much more subtle. Asymptotic versions of contact Morse inequalities are established in [GK, GGGo]; cf. [HM] and also [GG3, Proposition 3.4] for a Hamiltonian variant of asymptotic Morse inequalities. (Strictly speaking, these results require the Reeb flow to have finitely many simple closed orbits, but one can expect the inequalities to hold, under some reasonable assumptions, without this requirement.) But, even in the perfect case, there seems to be no obvious analog of the Morse (in)equalities. Theorems 1.5 and 1.7 give an inequality going in the opposite direction, and the examples of this section show that one can have arbitrarily large homology generated by very few simple orbits.
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2. Preliminaries

Assume that \((M^{2n-1}, \xi)\) is a closed contact manifold strongly fillable by an exact symplectically aspherical manifold, i.e., \(M = \partial W\) for some exact symplectic manifold \((W^{2n}, \omega = d\alpha)\) such that \(c_1(TW) = 0\) and \(\ker \alpha|_M = \xi\) with matching orientations. The latter orientation condition simply means that the Liouville vector field \(Y\) along \(M\) determined by the requirement \(i_Y\omega = \alpha|_M\) points outward. As mentioned earlier, the linearized contact homology \(HC_\ast(M, \xi)\) is then defined and independent of the contact form on \(M\) supporting \(\xi\); see, e.g., [Bo2, BO1, EGH]. We will write \(HC_\ast(\xi)\) whenever \(M\) is clear from the context. (Note that \(HC_\ast(\xi)\) also depends on the filling, but for the sake of brevity this dependence is suppressed in the notation.)

Let \(\alpha\) be a non-degenerate contact form on \((M^{2n-1}, \xi)\), i.e., \(\ker \alpha = \xi\) and all closed Reeb orbits of \(\alpha\), including the iterated ones, are non-degenerate. A simple Reeb orbit of \(\alpha\) is called even (odd) if the number of real eigenvalues of the linearized Poincaré return map in the interval \((-1, 0)\) is even (odd). Moreover, an iterated orbit is said to be bad if it is an even iteration of an odd periodic orbit. Otherwise, an orbit is said to be good. (For instance, all simple orbits are good.) The contact homology \(HC_\ast(\xi)\) is the homology of a complex \(CC_\ast(\alpha)\) generated over \(\mathbb{Q}\) by good closed Reeb orbits of \(\alpha\). The contact homology complex is graded by the Conley-Zehnder index of the Poincaré return map plus \(n-3\), i.e., the degree of a generator \(x\) is \(|x| = \mu_{CZ}(x) + n - 3\). The differential, roughly speaking, counts the number of rigid punctured holomorphic cylinders (with holomorphic cappings of the punctures in \(W\)) in the symplectization; see, e.g., [Bo2, BO1]. The precise workings of the differential will be inessential for us since we will mainly focus on perfect Reeb flows.

In the presence of non-contractible orbits, to have a well-defined grading of the complex \(CC_\ast(\alpha)\) (or to evaluate the Conley-Zehnder index of a periodic orbit) one needs an extra structure. To this end, we fix a non-vanishing section \(s\) of the square of the complex determinant line bundle of \(TW\), which exists since \(c_1(TW) = 0\); see, e.g., [Es, GGo]. Then, to evaluate \(\mu_{CZ}(x)\) (or the mean index described below) of a periodic orbit \(x\), it suffices to choose a unitary trivialization of \(x^*TW\) such that the square of its top complex wedge is \(s|_x\). Such a trivialization is unique up to homotopy, and the Conley-Zehnder index (and the mean index) is well-defined. The grading of the contact homology complex \(CC_\ast(\alpha)\) and, in turn, the homology \(HC_\ast(\xi)\) depends on the choice of the section \(s\), unless, of course, only the contractible orbits are taken into account.

The mean index \(\Delta(x)\) of a closed Reeb orbit \(x\) (possibly degenerate) measures the total rotation of certain eigenvalues on the unit circle of the linearized Poincaré return map of the Reeb flow at \(x\); see [Es, Lo, SZ]. When \(x\) is non-degenerate,

\[0 \leq |\Delta(x) - \mu_{CZ}(x)| < n - 1.\]  \hspace{1cm} (2.1)

Furthermore, the mean index is homogeneous with respect to iteration:

\[\Delta(x^k) = k\Delta(x).\]  \hspace{1cm} (2.2)

Finally, the action of a closed Reeb orbit \(x\) is defined by \(A(x) = \int_x \alpha > 0\).
Remark 2.1. The contact homology splits into a direct sum of homology spaces over free homotopy classes of loops in $W$. Hence one has a contact homology group defined for any collection of free homotopy classes. (The grading is independent of the section $s$ for the “contractible part” of the contact homology, but, in general, it does depend on $s$.) In what follows, we will work with the entire contact homology; however, our results hold for the contact homology defined for any collection of free homotopy classes closed under iterations.

3. Proofs

In this section we prove the main results of this paper.

3.1. Proof of Theorem 1.2. First, observe that by our assumption that $\alpha$ is perfect, $\text{CC}_*(\alpha) = \text{HC}_*(S^{2n-1}, \xi_{std})$, where

$$
\text{HC}_*(S^{2n-1}, \xi_{std}) = \begin{cases} 
\mathbb{Q} & \text{for } * \geq 2n - 2 \text{ and even}, \\
0 & \text{otherwise};
\end{cases}
$$

see, e.g., [Bo2]. Hence there is exactly one periodic orbit in every even degree $\geq 2n - 2$, generating the homology in that degree. The closed Reeb orbits are, therefore, strictly ordered by the degree and, in turn, by the Conley-Zehnder index. Furthermore, these orbits are also ordered strictly by the action: the action decreases as one goes down from the generator of $\text{HC}_*$ to the generator of $\text{HC}_{* - 2}$.

Indeed, denote by $x$ and $y$ the generators of $\text{HC}_*$ and $\text{HC}_{* - 2}$, respectively. By the main theorem of [BO1], $x$ and $y$ are connected by a holomorphic curve in the symplectization, and $A(x) - A(y) > 0$ since the only holomorphic curve having zero $\omega$-energy is the cylinder over a Reeb orbit; see, e.g., [BEHWZ, Lemma 5.4]. The two orderings of the orbits coincide and we have

$$
\mu_{\text{CZ}}(x) > \mu_{\text{CZ}}(y) \iff A(x) > A(y).
$$

This observation is the key point of the proof of Theorem 1.2.

Let $k > 0$ be an integer such that $A(y^k) > A(x)$. We count the iterations of $x$ “between” $x$ and $y^k$ in two ways: by the action and by the index. The result of counting by the action is

$$
\left\lfloor \frac{A(y^k) - A(x)}{A(x)} \right\rfloor = \left\lfloor k \frac{A(y)}{A(x)} \right\rfloor - 1.
$$

On the other hand, using (2.1), the count by the index is

$$
\left\lfloor \frac{\mu_{\text{CZ}}(y^k) - \mu_{\text{CZ}}(x)}{\mu_{\text{CZ}}(x)} \right\rfloor = \left\lfloor \frac{\Delta(y^k) - \Delta(x)}{\Delta(x)} \right\rfloor + O(1) = \left\lfloor k \frac{\Delta(y)}{\Delta(x)} \right\rfloor - 1 + O(1),
$$

where $O(1)$ stands for an error bounded (in absolute value) by a constant independent of $k$. Equality of the two counts implies that

$$
\left\lfloor k \frac{A(y)}{A(x)} \right\rfloor = \left\lfloor k \frac{\Delta(y)}{\Delta(x)} \right\rfloor + O(1).
$$

Finally, dividing by $k$ and letting $k \to \infty$, we obtain

$$
\frac{\Delta(x)}{A(x)} = \frac{\Delta(y)}{A(y)}.
$$

□
Remark 3.1. Theorem 1.2, in particular, implies that the ordering of the closed Reeb orbits by the mean index is also strict and agrees with the orderings by the action and by the Conley-Zehnder index.

3.2. Proofs of Theorems 1.5 and 1.7. In this section we prove Theorems 1.5 and 1.7.

3.2.1. Proof of Theorem 1.5. Let $\alpha$ be a perfect non-degenerate contact form on $(M^{2n-1}, \xi)$ such that $HC_\ast(\xi)$ satisfies (1.1). The key to the proof of Theorem 1.5 is the following combinatorial lemma.

Lemma 3.2. Assume that $\alpha$ has a collection of $r$ geometrically distinct periodic orbits $x_1, \ldots, x_r$. Then for every sufficiently small $\epsilon > 0$, there exist positive integers $k_1, \ldots, k_r \geq 1$ such that the iterated orbits $x_1^{k_1}, \ldots, x_r^{k_r}$ all have degrees in an interval of length $2(n - 1) + \epsilon$.

Theorem 1.5 immediately follows from Lemma 3.2. Indeed, note that all (even) orbits and their iterations contribute to the homology since $\alpha$ is perfect. Furthermore, by the definition of $b$ in (1.2), there are at most $b$ such orbits with indices in an interval of length $2(n - 1) + \epsilon$ comprising the $2n - 1$ possible degrees. Thus by Lemma 3.2 the total number of even orbits cannot exceed $b$.

We will now prove Lemma 3.2.

3.2.2. Proof of Lemma 3.2. We begin by making a few observations. First, $\Delta_i := \Delta(x_i) > 0$ for each $i$. Indeed, the fact that $\Delta_i \neq 0$ follows from our assumption that $\dim HC_\ast(\xi) < \infty$ for $* \leq 2n - 4$. For, otherwise, by (2.1), all iterations $x_i^k$ would have degrees in the bounded interval $(-2, 2n - 4)$, resulting in infinite homology in some degree in this range. The positivity, $\Delta_i > 0$, is a consequence of the assumptions that $HC_\ast(\xi) = 0$ for $* \ll 0$ and $\alpha$ is perfect, for all iterations of an orbit with negative index contribute to the homology in degrees $< 0$.

Below we assume without loss of generality that $\Delta_1 = \max \{\Delta_i\}$ where $i = 1, \ldots, r$. Set $k = (k_1, \ldots, k_r)$ with $k_i \in \mathbb{Z}$ and consider the $(r-1)$ linear forms $L_1, \ldots, L_{r-1}$ in the variable $k \in \mathbb{Z}^r$ defined as $$L_i(k) = \Delta_i k_1 - \Delta_{i+1} k_{i+1}$$ for $i = 1, \ldots, r-1$.

In this setting, for any $\delta > 0$ given, by the Minkowski approximation theorem, [Ca, Corollary 3, Appendix B], there exists an integer vector $k \neq 0$ such that $$|L_i(k)| < \delta \text{ for all } i = 1,\ldots, r.$$ (3.1)

Refining this assertion, we will show that if $\delta$ is sufficiently small, there exists such $k \in \mathbb{Z}^r$ with components $k_i > 0$ for all $i = 1, \ldots, r$. To this end, choose $\delta < \min\{1, \Delta_1, \ldots, \Delta_r\}$ and let $k_j \neq 0$ be the first non-zero component of $k$. Let us show that $k_j$ is necessarily $k_1$. Indeed, if $k_1 = 0$, by (3.1) and our choice of $\delta$, we have $|L_i(k)| = |\Delta_i k_{i+1}| < \delta < |\Delta_{i+1}|$. Therefore, $|k_{i+1}| < 1$ for all $i = 1, \ldots, r-1$ and, since $k_{i+1} \in \mathbb{Z}$, we conclude that $k_{i+1} = 0$ for all $i = 1, \ldots, r-1$. As a result, $k = 0$ since we also have $k_1 = 0$. This contradiction shows that $k_1 \neq 0$. In fact, $k_1$ can be chosen to be positive. To see this, observe that $k$ satisfies (3.1) if and only if $-k$ does; thus, by replacing $k$ by $-k$ if necessary, we can assume that $k_1 > 0$.

Now, given any $\epsilon > 0$ satisfying $\epsilon < \min\{1, \Delta_1, \ldots, \Delta_r\}$, let $\delta = \epsilon/2$. Then the above discussion implies that $|\Delta_1 k_1 - \Delta_1 k_1| < \epsilon/2$ for all $k_1$. By our choice of $\epsilon$, since $k_1 \geq 1$ and $\Delta_1 > 0$, we have $\Delta_1 k_1 - \epsilon/2 > 0$ and hence $\Delta_1 k_1 > 0$. This implies that $k_1 > 0$ for all $i$ since $\Delta_i > 0$. Finally, note that we also have $|\Delta_i k_1 - \Delta_j k_j| < \epsilon$ for all $i, j$ since $\Delta_i \neq 0$. This completes the proof of Lemma 3.2.
for all \( i, j \in \{1, \ldots, r\} \), i.e., the indices of the orbits \( x_1^{k_1}, \ldots, x_r^{k_r} \) are at most \( \epsilon \) apart from each other. Therefore, by (2.1), we conclude that their degrees lie in an interval of length \( 2(n - 1) + \epsilon \). \( \square \)

3.2.3. Proof of Theorem 1.7. Note that, just as in the proof of Theorem 1.5, we can assume that mean indices \( \Delta_i > 0 \) for all \( i \), and we set \( \Delta_1 = \max_i \Delta_i \). Clearly, it suffices to establish a version of Lemma 3.2 applicable in the setting of Theorem 1.7, i.e., when odd orbits are also allowed. Recall from Section 2 that only the odd iterations of odd orbits contribute to the complex; hence, for a small enough \( \epsilon > 0 \), it suffices to find odd positive integers \( k_1, \ldots, k_r \) such that \( |\Delta_1 k_1 - \Delta_i k_i| < \epsilon / 2 \).

To this end, set \( k_i = 2m_i + 1 \) and write
\[
|\Delta_1 k_1 - \Delta_i k_i| = 2\Delta_i|m\theta_i - m_i - \alpha_i|,
\]
where \( m \in \mathbb{Z} \) and \( \theta_i = \Delta_1 / \Delta_i \) and \( \alpha_i = 1/2 - \Delta_1 / 2 \Delta_i \). Now, the first step is to find \( m \in \mathbb{Z} \) such that \( |m\theta_i - \alpha_i| < \delta \), where \( \delta < \min \epsilon / 4\Delta_i \) and the norm \( \| \cdot \| \) is the distance to the closest integer. It is easy to see that \( \theta_1, \ldots, \theta_r \) are linearly independent over \( \mathbb{Q} \), for \( 1/\Delta_i \)'s are so by the hypotheses of Theorem 1.7. Therefore, by Kronecker’s theorem, see, e.g., [Ca], the orbit \( \{ m \cdot (\theta_2, \ldots, \theta_r) \mid m \in \mathbb{Z} \} \) of \((\theta_2, \ldots, \theta_r)\) is dense in the torus \( \mathbb{T}^{r-1} \). In fact, both the positive and negative semi-orbits are dense. Hence, we can find arbitrarily large \( m \in \mathbb{Z} \) such that \( m \cdot (\theta_2, \ldots, \theta_r) \) is sufficiently close to \((\alpha_2, \ldots, \alpha_r)\), i.e., \( |m\theta_i - \alpha_i| < \delta \). This proves that there exist integers \( m_i \) for \( i = 1, \ldots, r \) such that \( |m\theta_i - m_i - \alpha_i| < \delta \).

To complete the proof, it suffices to show that \( m_i \)'s are positive when \( m \) is large. This is obvious since \( \theta_i > 0 \) and we have \( m_i \geq m\theta_i - \alpha_i - \delta > 0 \) for large \( m \). \( \square \)

3.3. Proof of Theorem 1.8. To set the stage for the proof, we begin with a few observations. (We emphasize that most of what follows holds only for \( S^3 \) and when all closed Reeb orbits are non-degenerate.) Observe that there are two types of closed Reeb orbits on \( S^3 \): elliptic and hyperbolic, and we call a hyperbolic orbit negative (positive) hyperbolic if the eigenvalues of the linearized Poincaré return map lie on the negative (positive) real axis. Clearly, elliptic and positive hyperbolic orbits, denoted below by \( x \), are even and negative hyperbolic orbits, denoted below by \( y \), are odd in the sense of Section 2. Moreover, the Conley-Zehnder index is odd for elliptic and negative hyperbolic orbits and even for a positive hyperbolic orbit.

Denote, as above, by \( \Delta(z) \) the mean index of an orbit \( z \). By non-degeneracy, \( \Delta(z) \not\in \mathbb{Q} \) if \( z \) is elliptic. When \( z \) is negative (positive) hyperbolic, \( \Delta(z) = \mu_{CZ}(z) \) is an odd (even) integer. Set
\[
\sigma(z) = -(-1)^{\mu_{CZ}(z)}.
\]
So, \( \sigma(z) \) is the Poincaré index of the orbit \( z \), modulo the factor \(-1\). Note that \( \sigma \) is 1 for elliptic and negative hyperbolic orbits and \(-1\) for positive hyperbolic orbits.

One of the key ingredients of the proof of Theorem 1.8 is a resonance relation from [GK, Vi], generalized to the degenerate case in [GGo, HM, WHL]. This relation holds in a much greater generality, but we will only need it for non-degenerate Reeb flows on the standard contact \( S^3 \). Let \( \alpha \) be a non-degenerate contact form on \((S^3, \xi_{std})\) with finitely many simple periodic orbits. Then the resonance relation in the original form as in [Vi] is
\[
\sum \frac{\sigma(x_i)}{\Delta(x_i)} + \frac{1}{2} \sum \frac{\sigma(y_i)}{\Delta(y_i)} = \frac{1}{2}, \quad (3.2)
\]
Here the first sum runs over simple even (i.e., elliptic or positive hyperbolic) orbits with positive mean index and the second sum runs over simple odd (i.e., negative hyperbolic) orbits with positive mean index.

Finally, recall that the degree of a closed Reeb orbit $x$ in the contact homology complex for $(S^3, \xi_{std})$ is $|x| = \mu_{CZ}(x) - 1$, and

$$\text{HC}_* (S^3, \xi_{std}) = \begin{cases} \mathbb{Q} & \text{for } * \geq 2 \text{ and even}, \\ 0 & \text{otherwise}. \end{cases}$$

We now turn to the proof of Theorem 1.8.

$(i) \Rightarrow (ii)$. Assume that $\alpha$ is a non-degenerate perfect contact form on $S^3$ supporting the standard contact structure.

Observe first that $\alpha$ has no positive hyperbolic orbits, for $\text{HC}_* (S^3, \xi_{std})$ is non-vanishing only in even degrees (i.e., when the Conley-Zehnder index is odd) and $\alpha$ is perfect. Secondly, by Theorem 1.5, there are at most two simple elliptic orbits.

As above, denote by $x$ the even simple orbits and by $y$ the odd simple orbits of $\alpha$. So, $x$’s are elliptic and $y$’s are negative hyperbolic. Recall from the proof of Theorem 1.5 that all mean indices are positive. We claim that there is at most one negative hyperbolic orbit of $\alpha$. To prove this claim, assume the contrary and let $y_1$ and $y_2$ be two distinct negative hyperbolic orbits with $\mu_{CZ}(y_i) = \Delta(y_i) = 2m_i + 1$ for $i = 1, 2$ and $m_i \geq 1$. Then, using (2.2), we have

$$\mu_{CZ}(y_1^{2m_2+1}) = (2m_1 + 1)(2m_2 + 1) = \mu_{CZ}(y_2^{2m_1+1}).$$

In other words, $\alpha$ has two distinct orbits contributing to the homology in degree $m = (2m_1 + 1)(2m_2 + 1) - 1$, which contradicts the fact that $\text{HC}_m (S^3, \xi_{std}) = \mathbb{Q}$.

To recap, we have established that $\alpha$ has finitely many simple periodic orbits, and among these at most two are elliptic and at most one is negative hyperbolic. Moreover, when $\alpha$ is perfect, the relation (3.2) is

$$\sum_{x_i} \frac{1}{\Delta(x_i)} + \frac{1}{2} \sum_{y_i} \frac{1}{\Delta(y_i)} = \frac{1}{2},$$

where $x_i$’s are elliptic and $y_i$’s are negative hyperbolic.

Let us now show that $\alpha$ has at least two simple periodic orbits. Indeed, assume that there is only one orbit, $z$. Clearly, $z$ cannot be elliptic by (3.3), for elliptic orbits have irrational mean index. Hence $z$ must be negative hyperbolic with $\mu_{CZ}(z) = 3$ and should generate the entire homology. However, it is easy to see using (2.1) that no odd iteration of $z$ can generate the homology $\text{HC}_4 (S^3, \xi_{std}) = \mathbb{Q}$. Therefore, $\alpha$ must have at least two simple closed Reeb orbits. It is worth pointing out that this assertion also follows from the more general and highly non-trivial results of [CGH] or from [GHHM, GG]; see also Remark 3.3.

At this point there are three possibilities; the composition of simple closed orbits of $\alpha$ could be two elliptic, one elliptic and one negative hyperbolic or two elliptic and one negative hyperbolic. We will next rule out the latter two alternatives by arguing that presence of two different types of orbits is impossible. To this end, assume that $\alpha$ has (at least) one elliptic orbit, $x$, and one negative hyperbolic orbit, $y$. Recall that $\Delta(x) \not\in \mathbb{Q}$ and $\Delta(y) = 2m + 1 \in \mathbb{N}$. (Though this will follow from the discussion below, notice that the second alternative is actually impossible by (3.3).) Set $\theta = \Delta(x)/2\Delta(y) \not\in \mathbb{Q}$. Then, for any $\delta > 0$, by Kronecker’s theorem,
see, e.g., [HaWr], there exist \( k, l \in \mathbb{N} \) such that \( |k\theta - l - 1/2| < \delta \). Therefore, for any \( \epsilon > 0 \), choosing \( \delta < \epsilon/2\Delta(y) \), we obtain
\[
|k\Delta(x) - (2l + 1)\Delta(y)| = |\Delta(x^k) - \Delta(y^{2l+1})| < \epsilon.
\]
Of course, this is equivalent to \( |\Delta(x^k) - \mu_{CZ}(y^{2l+1})| < \epsilon \) and, by \((2.1)\), implies that
\[
|\mu_{CZ}(x^k) - \mu_{CZ}(y^{2l+1})| < 1 + \epsilon.
\]
Now, recall that both indices in \((3.4)\) are odd. Since the distance between two distinct odd integers is at least two, we have \( \mu_{CZ}(x^k) = \mu_{CZ}(y^{2l+1}) \), provided that \( \epsilon < 1 \). But then the contact homology in degree \((2l + 1)(2m + 1) - 1\) would be two dimensional, which is impossible.

By the above discussion, \( \alpha \) has exactly two simple closed Reeb orbits and both of these orbits are elliptic. The indices of these orbits satisfy \((3.3)\), and all pairs of positive irrational numbers satisfying this relation clearly occur as the mean indices for irrational ellipsoids (cf. [Lo]). Moreover, Corollary 1.4 implies that the actions are also the same as those for a suitable irrational ellipsoid. The assertion that the orbits are unknotted and have linking number one follows from the machinery of [HWZ2] as in [BCE].

\((iii) \Rightarrow (i)\). Assume that \( \alpha \) has exactly two simple closed Reeb orbits. To conclude that \( \alpha \) is perfect, it suffices to show that both orbits have the same parity (of Conley-Zehnder indices). Thus we need to prove that \( \alpha \) cannot have simultaneously elliptic or negative hyperbolic orbits and positive hyperbolic orbits. The former case is impossible by \((3.2)\).

To see that the presence of one negative and one positive hyperbolic orbit is impossible, assume the contrary and let \( x \) be the positive hyperbolic orbit and \( y \) be the negative hyperbolic orbit. Then \( \mu_{CZ}(x) = \Delta(x) = 2k \) and \( \mu_{CZ}(y) = \Delta(y) = 2m + 1 \) for some non-zero integers \( k \) and \( m \). Now, since \( \text{HC}_*(S^3, \xi_{\text{std}}) \) is non-vanishing only in positive even degrees, both mean indices must be positive, and hence \( k > 0 \) and \( m \geq 0 \). Then the relation \((3.2)\) reads
\[
-\frac{1}{2k} + \frac{1}{2(2m + 1)} = \frac{1}{2}.
\]
In other words, \( 1/(2m + 1) - 1/k = 1 \), which is clearly impossible. This proves that both of the orbits have the same parity, and hence \( \alpha \) must be perfect.

Theorem 1.8 follows from the implications \((i) \Rightarrow (ii)\) and \((iii) \Rightarrow (i)\) along with the obvious implication \((ii) \Rightarrow (iii)\). \(\square\)

Remark 3.3. A non-degenerate contact form \( \alpha \) supporting the standard contact structure on \( S^{2n-1} \) must have at least two distinct simple closed Reeb orbits. Indeed, assume that the Reeb flow of \( \alpha \) has just one simple closed (non-degenerate) orbit \( x \). Then we necessarily have \( \mu_{CZ}(x) = n + 1 \) and \( \Delta(x) = 2 \). Now, by \((2.1)\), the orbit \( x \) must be degenerate which contradicts the the background non-degeneracy assumption. (In fact, \( x \) is a symplectically degenerate maximum, [GHHM, Proposition 3], and it also follows from the results of that paper that in this case the Reeb flow must have infinitely many periodic orbits.)

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