Quasi-triangular structures on Hopf algebras with positive bases

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Abstract

A basis $B$ of a finite dimensional Hopf algebra $H$ is said to be positive if all the structure constants of $H$ relative to $B$ are non-negative. A quasi triangular structure $R \in H \otimes H$ is said to be positive with respect to $B$ if it has non-negative coefficients in the basis $B \otimes B$ of $H \otimes H$. In our earlier work, we have classified all finite dimensional Hopf algebras with positive bases. In this paper, we classify positive quasi-triangular structures on such Hopf algebras. A consequence of this classification is a new way of constructing set-theoretical solutions of the Yang-Baxter equation.

1 Introduction

Consider a finite dimensional Hopf algebra $H$ with a basis such that all the structure constants with respect to this basis are non-negative. We call such a basis positive. In [LYZ1] we showed that any such Hopf algebra is isomorphic to the bicrossproduct Hopf algebra $H(G; G_+, G_-)$ coming from a unique factorization $G = G_+ G_-$ of a finite group $G$ (see Section 2 for the definition of $H(G; G_+, G_-)$). We also showed that such Hopf algebras are exactly the linearizations of Hopf algebras in the category of sets with correspondences as morphisms.

In this paper, we classify all quasi-triangular structures $R \in H \otimes H$ on the Hopf algebra $H = H(G; G_+, G_-)$ that are positive in the sense that the coefficients of $R$ in the basis $G \otimes G$ of $H \otimes H$ are non-negative. We also show that they are all quasi-equivalent to certain normal forms. It turns out that such an $R$ is set-theoretical, in the sense that it is the linearization of a bisection $\mathcal{R}$ of a product groupoid $\Gamma \times \Gamma$ satisfying the groupoid-theoretical Yang-Baxter equation. Consequently, for every $\Gamma$-set $X$, $\mathcal{R}$ induces a set-theoretical solution of the Yang-Baxter equation on $X$. This fact motivates our work in [LYZ2], where we give a general way of constructing set-theoretical solutions of the Yang-Baxter equation that includes as special cases the earlier ones by Weinstein and Xu in [WX] and by Etingof, Schedler and Soloviev in [ESS]. We also classify positive triangular structures on $H(G; G_+, G_-)$ and recover a construction of such structures by Etingof and Gelaki [EG].

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2 Positive quasi-triangular structures

A unique factorization $G = G_+ G_-$ of a group $G$ consists of two subgroups $G_+$ and $G_-$ such that any $g \in G$ can be written as $g = g_+ g_-$ for unique $g_+ \in G_+$ and $g_- \in G_-$. By
considering the inverse map, we see that for every \( g \in G \), there are unique \( g_+, \bar{g}_+ \in G_+ \) and \( g_-, \bar{g}_- \in G_- \) such that
\[
g = g_+ g_- = \bar{g}_- \bar{g}_+.
\]
We will denote \((g_+)^{-1} \in G_+\) and \((g_-)^{-1} \in G_-\) simply by \(g_+^{-1}\) and \(g_-^{-1}\).

The unique factorization induces the following actions of \(G_+\) and \(G_-\) on each other (from left and from right)
\[
\begin{align*}
G_- \times G_+ & \to G_+, \quad (\bar{g}_-, \bar{g}_+) \mapsto g_+ = \bar{g}_- \bar{g}_+ , \\
G_- \times G_+ & \to G_-, \quad (\bar{g}_-, \bar{g}_+) \mapsto g_- = \bar{g}_- \bar{g}_+ , \\
G_+ \times G_- & \to G_+, \quad (g_+, g_-) \mapsto \bar{g}_+ = g_+ g_- , \\
G_+ \times G_- & \to G_-, \quad (g_+, g_-) \mapsto \bar{g}_- = g_+ g_- .
\end{align*}
\]

By definition, we have \( g_+ g_- = (g_+ g_-) (g_+ g_-) \) and \( g_- g_+ = (g_- g_+) (g_- g_+) \). Moreover, the actions have the following properties
\[
\begin{align*}
\begin{cases}
g_+ (g_- h_-) = g_+ (g_- g_-) h_- , & (h_+ g_+)^{-1} g_- = h_+ (g_+ g_-) \bar{g}_+ g_- , \\
g_- (g_+ h_+) = g_- g_+ (g_+ g_-) h_+ , & (h_- g_-)^{-1} g_+ = h_- (g_- g_+) g_+ g_- .
\end{cases}
\end{align*}
\tag{1}
\]

\[
\begin{align*}
\begin{cases}
(g_+ g_-)^{-1} = (g_-^{-1}) g_-^{-1} , & (g_- g_+)^{-1} = g_+^{-1} (g_-^{-1}) , \\
(g_- g_+)^{-1} = (g_-^{-1}) g_-^{-1} , & (g_+ g_-)^{-1} = g_+^{-1} (g_-^{-1}) .
\end{cases}
\end{align*}
\tag{2}
\]

A Hopf algebra \( H(G; G_+, G_-) \) can be constructed from a unique factorization \( G = G_+ G_- \) of a finite group \( [M] \) \( [N] \). More precisely, \( H(G; G_+, G_-) \) is the vector space spanned by the set \( G \) together with the following Hopf algebra structure (we use \( \{g\} \) to denote the group element \( g \in G \) considered as an element of \( H(G; G_+, G_-) \)):
\[
\begin{align*}
\text{multiplication:} & \quad \{g\} \{h\} = \delta_{g \star h} \{gh\} \\
\text{unit:} & \quad 1 = \sum_{g \in G} \{g\} \\
\text{co-multiplication:} & \quad \Delta \{g\} = \sum_{h_+ \in G_+} \{g \star h_+^{-1} (h_+ g_-) \} \otimes \{h_+ g_-\} \\
\text{co-unit:} & \quad \epsilon \{g\} = \delta_{g, e} \\
\text{antipode:} & \quad S \{g\} = \{g^{-1}\}
\end{align*}
\]

We remark that the algebra structure on \( H(G; G_+, G_-) \) is that of the cross-product of the group algebra \( CG_- \) of \( G_- \) and the function algebra \( C(G_+) \) of \( G_+ \) with respect to the above right action of \( G_- \) on \( G_+ \), and similarly for its co-algebra structure. The Hopf algebra \( H(G; G_+, G_-) \) has \( G \) as the obvious positive basis. In [XYZ], we proved that all finite dimensional Hopf algebras with positive bases are of the form \( H(G; G_+, G_-) \).
Recall that a quasi-triangular structure on a Hopf algebra $H$ is an invertible element $R \in H \otimes H$ such that

$$
\tau \Delta(g) = R \Delta(g) R^{-1}, \quad \text{for all } g \in H
$$

$$
(\Delta \otimes \text{id}) R = R_{13} R_{23}, \quad (\text{id} \otimes \Delta) R = R_{13} R_{12}
$$

$$
(\varepsilon \otimes \text{id}) R = (\text{id} \otimes \varepsilon) R = 1,
$$

where $\tau(a \otimes b) = b \otimes a$. In the special case $H = H(G; G_+, G_-)$, we say that an element $R \in H \otimes H$ is positive if $R$ is a non-negative linear combination of the basis elements $\{g\} \otimes \{h\}$, $g, h \in G$.

**Theorem 1** Let $G = G_+ G_-$ be a unique factorization of a finite group $G$. Let $\xi, \eta : G_+ \to G_-$ be two group homomorphisms, and denote

$$
G'_+ = \{u \xi(u^{-1}) : u \in G_+\}, \quad G''_+ = \{u \eta(u^{-1}) : u \in G_+\},
$$

$$
F(u \xi(u^{-1})) = \eta(u) u^{-1} : G'_+ \to G''_+.
$$

Suppose that the following conditions are satisfied,

(a) Both $G'_+$ and $G''_+$ are normal subgroups of $G$;
(b) $F$ is a group isomorphism.

Then

$$
R = \sum_{u,v \in G_+} \{u (\eta(v) u)^{-1}\} \otimes \{v \xi(u)\}
$$

is a positive quasi-triangular structure. Conversely, every positive quasi-triangular structure on $H(G; G_+, G_-)$ is given by the construction above.

The following proposition rephrases the conditions (a) and (b) in a more concrete form. Theorem 1 is a consequence of this proposition and Propositions 2-5 of Section 3.

**Proposition 1** Let $G = G_+ G_-$ be a unique factorization. Let $\xi, \eta : G_+ \to G_-$ be two group homomorphisms. Then the conditions (a) and (b) in Theorem 1 are equivalent to

$$
\xi(u)^{v} = \xi(u \eta(v)),
$$

$$
^{u} \eta(v) = \eta(\xi(u)) v,
$$

$$
uv = (\xi(u)) v^{u},
$$

$$
\xi(xu)x^{u} = x \xi(u),
$$

$$
\eta(xu)x^{u} = x \eta(u),
$$

where $u, v, x \in G_+$. The conditions (6) - (10) are equivalent to (a) and (b) in Theorem 1.
for all \( u, v \in G_+ \), and \( x \in G_- \). Moreover, each of these properties is also equivalent to the corresponding property below

\[
\begin{align*}
\xi^v(u) &= \xi(\eta(v)u), \\
\eta(v)^u &= \eta(v^{\xi(v)}), \\
uv &= (\eta(u)v)(u^{\xi(v)}), \\
^ux\xi(u^x) &= \xi(u)x, \\
^ux\eta(u^x) &= \eta(u)x.
\end{align*}
\]

**Proof:** We first prove that (a) and (b) imply (6–10).

Since \( G'_+ \) is normal, for any \( u \in G_+ \) and \( x \in G_- \), we can find \( v \in G_+ \) such that \( xu\xi(u^{-1}) = v\xi(v^{-1})x \). By the unique factorization \( G = G_+G_- \), we have

\[
xu = v, \quad (x^u)\xi(u^{-1}) = \xi(v^{-1})x.
\]

This implies (9). Similarly, the fact that \( G''_+ \) is normal implies (10).

Since \( u\xi(u^{-1})v\xi(v^{-1}) = u(\xi(u^{-1})v)(\xi(u^{-1})x) \), and \( F \) is a homomorphism, we have \( (u(\xi(u^{-1})v))^{-1} = ((u^{-1})\eta(v))^{-1} \). This is exactly (8).

By (16) and the fact that \( G'_+ \) is a subgroup, we have

\[
\xi((u^{-1})\eta(v))^{-1} = (\xi(u^{-1})\xi(v^{-1}).
\]

Since \( u(\xi(u^{-1})v) \overset{8}{=} v((u^{-1})\eta(v))^{-1} \), the left side of (18) is \( \xi((u^{-1})\eta(v))\xi(v^{-1}) \). Therefore (18) is equivalent to \( \xi((u^{-1})\eta(v)) = \xi(u^{-1})v \), which is exactly (9). Similarly, the fact that \( G''_+ \) is a subgroup implies (4).

The converse that (6–10) implies (a) and (b) can be proved similarly, by making use of the same computations. The conditions (3), (4), and (8) imply that \( G'_+ \) and \( G''_+ \) are subgroups. The conditions (3) and (10) imply that \( G'_+ \) and \( G''_+ \) are normal subgroups. The condition (8) implies that \( F \) is a group isomorphism.

Finally, if (3) holds, then we have

\[
\xi^v(u) \overset{9}{=} (\xi(u^{-1})\eta(v))^{-1} \overset{10}{=} \xi((u^{-1})\eta(v))^{-1} \overset{8}{=} \xi(\eta(v)u),
\]

which is (11). The same idea shows that each of (12–15) is equivalent to the corresponding property in (7–10). 

\[\Box\]
Next we give an alternative description for the data \((G = G_+ G_-, \xi, \eta)\) used in Theorem 1.

Let \(G_-\) be a group acting on another group \(A\) as automorphisms, with the action denoted by \((x, a) \mapsto x \cdot a : G_- \times A \to A\). Then we have the semi-direct product group \(G = A \rtimes G_-\), with the group structure given by

\[
(ax)(by) = a(x \cdot b)xy, \quad a, b \in A, \quad x, y \in G_-.
\]

A map \(\zeta : A \to G_-\) is called a 1-cycle if

\[
\zeta(a) \zeta(b) = \zeta(a(\zeta(a) \cdot b)).
\] (19)

The 1-cycle condition (19) is equivalent to the fact that \(\{a\zeta(a) : a \in A\}\) is a subgroup of \(G\). Moreover, if \(\zeta\) is bijective, then \(\zeta^{-1} : G_- \to A\) is a 1-cocycle of \(G_-\) with coefficient in \(A\) as defined in \([EG]\).

**Theorem 2** There is a one-to-one correspondence between triples \((G = G_+ G_-, \xi, \eta)\) satisfying the conditions of Theorem 1 and the triples \((G = A \rtimes G_-, \zeta, F)\), where \(\zeta\) is a cycle of \(G_-\) with coefficients in \(A\) and \(F\) is an automorphism of \(G\), satisfying

(a) \(F(x) = x\) for any \(x \in G_-\);
(b) \(F(a)a \in G_-\) for any \(a \in A\).

Specifically, the correspondence is the following. Given \((G = G_+ G_-, \xi, \eta)\), we define

\[
A = G'_+ = \{u\xi(u^{-1}) : u \in G_+\};
\]

\[
F(u\xi(u^{-1}))x = \eta(u)u^{-1}x, \quad \text{for } u \in G_+ \text{ and } x \in G_-;
\]

\[
\zeta(u\xi(u^{-1})) = \xi(u), \quad \text{for } u \in G_+.
\] (20)

Moreover, since \(A\) is a normal subgroup, conjugations by elements in \(G_-\) give an action of \(G_-\) on \(A\) as automorphisms. Conversely, given \((G = A \rtimes G_-, \zeta, F)\), we define

\[
G_+ = \{a\zeta(a) : a \in A\};
\]

\[
\xi = P|_{G_+};
\]

\[
\eta = P \circ F^{-1}|_{G_+},
\] (21)

where \(P\) is the natural homomorphism \(G = A \rtimes G_- \to G_-\), \(ax \mapsto x\).

**Proof of Theorem 2** First we show that if \((G = G_+ G_-, \xi, \eta)\) satisfies the conditions of Theorem 1, then the construction (20) is as described in Theorem 2.

For \(a = u\xi(u^{-1})\), we have \(a\zeta(a) = u\). Therefore the subset \(\{a\zeta(a) : a \in A\} = G_+\) is a subgroup of \(G\). This implies \(\zeta\) is a 1-cycle.
By its very definition, $F$ is a homomorphism if and only if $F : G'_+ \to G''_+$ is an equivariant map with respect to the $G_-$-actions defined by conjugations. For $x \in G_-$ and $a = u\xi(u^{-1}) \in G'_+$, the action of $x$ on $a$ is

$$x \cdot a = xu\xi(u^{-1})x^{-1} = v\xi(v^{-1}), \quad v = (xu)_+ = xu,$$

and the action of $x$ on $F(a)$ is

$$x \cdot F(a) = x\eta(u)w^{-1}x = \eta(w)w^{-1}, \quad w = (xu)_+ = xu.$$

We conclude from this that $v = w$ and $F(x \cdot a) = x \cdot F(a)$.

Finally, the condition (a) follows from the definition, and (b) follows from

$$F(u\xi(u^{-1}))u\xi(u^{-1}) = \eta(u)u^{-1}u\xi(u^{-1}) = \eta(u)\xi(u^{-1}).$$

Now we turn to the construction ([21]).

First of all, since $\xi$ is a 1-cycle, we know $G_+$ is a subgroup of $G$. Moreover, for any $a \in A$ and $x \in G_-$, the decomposition $ax = (a\zeta(a))(\xi(a)^{-1}x)$ gives the unique factorization $G = G_+G_-.$

Since $P$ and $F^{-1}$ are homomorphisms, $\xi$ and $\eta$ are also homomorphisms.

We express an element in $G$ as $ux$ for unique $u \in G_+$ and $x \in G_-$. The element is in $A$ if and only if it is in the kernel of $P$. Since $P(ux) = P(u)P(x) = \xi(u)x$, we see that $A$ consists of elements of the form $u\xi(u^{-1})$, $u \in G_+$. In other words, we have $A = G'_+$, which in particular implies $G'_+$ is a normal subgroup. Similarly, by considering those elements $xu$ in the kernel of $P \circ F^{-1}$, we conclude that $G''_+ = F(A)$. Since $F$ is an automorphism, $G''_+$ is also a normal subgroup.

Since $F(G'_+) = G''_+$, for any $u \in G_+$, we can find $v \in G_+$ such that $F(u\xi(u^{-1})) = \eta(v)v^{-1}$. Then by condition (b), we have $\eta(v^{-1})vu\xi(u^{-1}) = F(u\xi(u^{-1}))u\xi(u^{-1}) \in G_-$. This implies $uv \in G_-$. On the other hand, $u, v \in G_+$ implies $uv \in G_+$. Therefore by the unique factorization $G = G_+G_-$, we have $uv = e$. Consequently, the formula $F(u\xi(u^{-1})) = \eta(u)u^{-1}$ holds.

\[\square\]

3 Proof of the classification theorem

We prove Theorem [14] in this section.

**Proposition 2** Suppose that $R \in H(G; G_+, G_-) \otimes H(G; G_+, G_-)$ is invertible and positive, and suppose that $R^{-1}$ is also positive. Then there is a subset $R \subseteq G \times G$ and a positive valued function $r : R \to \mathbb{R}^+$ such that

$$7$$
1. The restriction of the map \((g, h) \rightarrow (g_+, h_+) : G \times G \rightarrow G_+ \times G_+\) to \(R\) is a bijection;

2. \(R = \sum_{(g, h) \in R} r(g, h)\{g\} \otimes \{h\}\).

Proof: The positivity assumption implies that if the coefficients of \(\{g\} \otimes \{h\}\) in \(R\) and of \(\{k\} \otimes \{l\}\) in \(R^{-1}\) are non-zero, then either the multiplicability conditions \(g_+^a = k_+\) and \(h_+^b = l_+\) are not satisfied, or the coefficient of the product \((\{g\} \otimes \{h\})(\{k\} \otimes \{l\}) = \{gk\} \otimes \{hl\}\) in \(RR^{-1} = e \otimes e = \sum_{u,v \in G_+} u \otimes v\) is non-zero. Similarly for \(R^{-1}R\). The proposition is then a consequence of these two facts.

Instead of going through the details of the argument, we note that the proposition is a consequence of a general fact about bisections of groupoids. See the discussion after the proof of Proposition 7 for such a conceptual proof of the proposition.

Proposition 3 Suppose that \(\xi, \eta : G_+ \rightarrow G_-\) are two group homomorphisms. Suppose also that \(r : G_+ \times G_+ \rightarrow \mathbb{R}^{>0}\) is a function such that for any \(u, v, w \in G_+\), the equalities (11), (12), and the following are satisfied
\[
(r(uw, v) = r(u, v)r(w, v^\xi(u)),
\]
\[
(r(u, wv) = r(u, v)r(\eta(v)u, w).)
\]

Then
\[
R = \sum_{u,v \in G_+} r(u, v)\{u(\eta(v)^{-1})\} \otimes \{v^{\xi}(u)\}\]

satisfies (4). Conversely, if \(R\) is invertible, positive, satisfies (4), and \(R^{-1}\) is also positive, then \(R\) is given by the construction above.

Proof: Suppose that \(R\) is invertible, positive, and \(R^{-1}\) is also positive. Then Proposition 2 implies that
\[
R = \sum_{u,v \in G_+} r(u, v)\{u\phi(u, v)\} \otimes \{v\psi(u, v)\},
\]
where \(\phi, \psi : G_+ \rightarrow G_-\) are two maps, and \(r : G_+ \times G_+ \rightarrow \mathbb{R}^{>0}\) is a positively valued function. From (23), we have
\[
(\Delta \otimes id)R = \sum_{u,v,w \in G_+} r(u, v)\{uw^{-1}(w^\phi(u, v))\} \otimes \{w^\phi(u, v)\} \otimes \{v^\psi(u, v)\}
\]
and
\[
R_{13}R_{23} = \sum_{u,v,w \in G_+} r(u, v)r(w, v^\psi(u, v))\{u^\phi(u, v)\} \otimes \{w^\phi(w, v^\psi(u, v))\} \otimes \{v^\psi(u, v)\}.
\]
Then it is easy to see that \((\Delta \otimes id)R = R_{13}R_{23}\) means

\[
\begin{align*}
\Delta uw, v &= \Delta u, \psi(u,v) \\
\Delta u, \phi(u, v) &= \phi(u, v) \\
\Delta u, \phi(u, v) &= \phi(u, \phi(u,v)) \\
\psi(u, v) &= \psi(u, v)\psi(w, v^{\phi(u,v)})
\end{align*}
\]

(26)

for all \(u, v, w \in G_+\). Similarly, we see that \((id \otimes \Delta)R = R_{13}R_{12}\) means

\[
\begin{align*}
\Delta u, wv &= \Delta u, v \\
\Delta u, \phi(u, v) &= \phi(u, \phi(u,v)) \\
\Delta u, \psi(u, v) &= \psi(u, \phi(u,v), w) \\
\psi(u, wv) &= \psi(u, v)
\end{align*}
\]

(30)

for all \(u, v, w \in G_+\).

Equation (33) implies that

\[
\psi(u, v) = \xi(u)
\]

(34)

for a map \(\xi : G_+ \to G_-\). Then (29) becomes \(\xi(uw) = \xi(u)\xi(v)\), i.e., \(\xi\) is a group homomorphism.

Equation (27) implies that \(\phi(u, v) = u^{-1}\phi(e, v)\). Therefore we introduce \(\eta(v) = \phi(e, v)^{-1} : G_+ \to G_-\) and have

\[
\phi(u, v) = u^{-1}(\eta(v)^{-1}) \overset{(2)}{=} (\eta(v)^{-1})u.
\]

(35)

Moreover, we have

\[
u^{\phi(u,v)} = u^{(\eta(v)^{-1})} \overset{(1)}{=} \eta(v)u.
\]

(36)

Then by (33) and (36), equation (11) becomes

\[
\eta(wv)^u = \eta(w)^{(\eta(v))u}\eta(v)^u.
\]

Taking \(u = e\), we see that \(\eta\) is a group homomorphism. By making use of this fact, the equation above becomes (1), which is always satisfied.

By (34) and (36), equation (12) becomes (11). By (33), equation (28) becomes \(\eta(v)^{uw} = \eta(v)^{\phi(u,v)}w\). Applying the right action by \(w^{-1}\), we have (12). Finally, by (36), equations (26) and (31) become (22) and (23).

\[\square\]
Proposition 4 Suppose that $\xi$ and $\eta : G_+ \to G_-$ are two group homomorphisms. Suppose also that $r : G_+ \times G_+ \to \mathbb{R}^{>0}$ is a function such that for any $u,v,w \in G_+$ and $x \in G_-$, the equalities (8) and (14) and the following are satisfied

$$r(u,v) = r(u^x,v^{(x)},)$$

(37)

Then (24) is a positive quasi-triangular structure on $H(G;G_+,G_-)$. Conversely, any positive quasi-triangular structure on $H(G;G_+,G_-)$ is given by the construction above.

Proof: Suppose that $R$ is a positive quasi-triangular structure. Then $R^{-1} = (S \otimes id)R$, so that $R^{-1}$ is also positive. Consequently, Proposition 3 applies. In particular, $R$ is of the form (24), and we have properties (11) and (12). Note that by Proposition 4, we also have properties (11) and (12).

For $R$ given by (24), we have

$$\tau \Delta \{g\} R = \sum_{h_+ \in G_+} r(h_+g_-, (g_+h_-)^{-1}g_-)$$

(38)

$$\left\{h_+g_- \left(\eta((g_+h_-)^{-1}g_-)\right)^{-1} \right\} \otimes \left\{g_+h_+^{-1}(h_+g_-)\right\}.$$

On the other hand, in the product

$$R \Delta \{g\} = \left(\sum_{u,v \in G_+} r(u,v)\{u(\eta((v)v)^{-1}\} \otimes \{v \xi(u)\} \right) \left(\sum_{h_+ \in G_+} \{g_+h_+^{-1}(h_+g_-)\} \otimes \{h_+g_-\} \right),$$

we must have

$$h_+ = v^{\xi(u)}, \quad g_+h_+^{-1} = u^{(\eta((v)v)^{-1)} \otimes \eta((v)v)^{-1}} = u^{\eta((h_+)^{-1}}.$$

Therefore

$$u = (g_+h_+^{-1})^{\eta((h_+)^{-1}}$$

$$v = h_+^{\xi((g_+h_+^{-1})^{(h_+)^{-1}}) \otimes h_+^{((h_+)^{-1})^{(h_+)^{-1}}}}$$

$$\otimes h_+^{(h_+)^{-1}}(\xi((g_+h_+^{-1})^{(h_+)^{-1}}) \otimes (h_+)^{-1} \otimes (\xi((g_+h_+^{-1})^{(h_+)^{-1}}) \otimes g_+h_+^{-1}h_+,$$

$$\eta((v)^{-1}) = \eta((v)^{-1}) = \eta((h_+)^{-1})$$

$$\xi(u) = \xi((g_+h_+^{-1})^{(h_+)^{-1}}) \otimes \xi((g_+h_+^{-1})^{(h_+)^{-1}})$$

and

$$R \Delta \{g\} = \sum_{h_+ \in G_+} r((g_+h_+^{-1})^{\eta((h_+)^{-1}}), \xi((g_+h_+^{-1})^{(h_+)^{-1}})h_+)$$

(39)

$$\left\{(g_+h_+^{-1})^{\eta((h_+)^{-1}}) \otimes \left\{\xi((g_+h_+^{-1})^{(h_+)^{-1}})h_+\right\} \otimes \left\{\xi((g_+h_+^{-1})^{(h_+)^{-1}})h_+\right\} \otimes \left\{\xi((g_+h_+^{-1})^{(h_+)^{-1}})h_+\right\}.$$

10
Observe that in the $G_+$-parts of each term in $\tau \Delta \{g\}R$, we have $(g_+ h_+^{-1}) h_+ = g_+$. Therefore by $\tau \Delta \{g\}R = R \Delta \{g\}$ and $r > 0$, we see that for any $g_+, h_+ \in G_+$, we have

$$
(\xi(g_+ h_+^{-1}) h_+)((g_+ h_+^{-1}) \eta(h_+)) = g_+.
$$

This is exactly (8).

We now compare $\tau \Delta \{g\}R$ and $R \Delta \{g\}$ term by term. In order to avoid confusion, we change the index $h_+$ in $\tau \Delta \{g\}R$ to $\tilde{h}_+$.

The equality $\tau \Delta \{g\}R = R \Delta \{g\}$ and $r > 0$ suggests us to consider the map

$$
h_+ \mapsto \tilde{h}_+ = (g_+ h_+^{-1}) \eta(h_+).
$$

(40)

By using (3), (5), and (6), we can show that the map has the following inverse

$$
\tilde{h}_+ \mapsto h_+ = (g_+ \tilde{h}_+^{-1}) \xi(h_+).
$$

(41)

This means that the term in $\tau \Delta \{g\}R$ indexed by $\tilde{h}_+$ and the term in $R \Delta \{g\}$ indexed by $h_+$ must be equal. By comparing the coefficients, the $G_+$-components, and the $G_-$-components of the corresponding terms, we have

$$
r(\tilde{h}_+ g_-, (g_+ \tilde{h}_+^{-1})^{h_+ g_-}) = r((g_+ h_+^{-1}) \eta(h_+), \xi(g_+ h_+^{-1}) h_+).
$$

(42)

$$
g_- \left( \eta \left( (g_+ \tilde{h}_+^{-1})^{h_+ g_-} \right) \right)^{-1} = \eta(h_+)^{-1}(h_+ g_-).
$$

(43)

$$
g_+ \tilde{h}_+^{-1} = \xi(g_+ h_+^{-1}) h_+.
$$

(44)

$$
(h_+ g_-) \xi(\tilde{h}_+ g_-) = (\xi(g_+ h_+^{-1}) h_+) g_-.
$$

(45)

where $\tilde{h}_+$ is given by (40).

As pointed out earlier, (40) and (44) implies (8).

Let $h_+ = e$. Then from (40) we have $\tilde{h}_+ = g_+$, so that (45) becomes

$$(g_+ g_-) \xi(g_+ g_-) = \xi(g_+) g_-.
$$

This is (14).

We have from (6) and (10) that

$$
\xi(\tilde{h}_+) = \xi((g_+ h_+^{-1}) \eta(h_+)) = \xi(g_+ h_+^{-1}) h_+,
$$

so that

$$
h_+ \xi(\tilde{h}_+)^{-1} = h_+ (\xi(g_+ h_+^{-1}) h_+)^{-1} \xi(g_+ h_+^{-1}) h_+ (\xi(g_+ h_+^{-1}) h_+)^{-1} \left( h_+^{-1} (\xi(g_+ h_+^{-1}) h_+)^{-1} \right)^{-1} \xi(g_+ h_+^{-1}) h_+ (g_+ h_+^{-1}) h_+ \left( (g_+ h_+^{-1}) \eta(h_+) \right)^{-1} g_+ \tilde{h}_+^{-1}.
$$

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Thus \((g_{+}\bar{h}_{+}^{-1})\xi(h_{+}) = h_{+}\), and in (43) we have
\[
\eta \left( (g_{+}\bar{h}_{+}^{-1})^{(h_{+}g_{-})} \right)^{(h_{+}g_{-})} \quad \quad \text{(43)}
\]
\[
\eta \left( (g_{+}\bar{h}_{+}^{-1})^{(h_{+}g_{-})} \xi(h_{+}g_{-}) \right) \quad \quad \text{(14)}
\]
\[
\eta \left( (g_{+}\bar{h}_{+}^{-1})^{(h_{+}g_{-})} \right) = \eta(h_{+}g_{-}).
\]
Therefore (43) becomes
\[
\eta(h_{+})g_{-} = (h_{+}g_{-})\eta \left( (g_{+}\bar{h}_{+}^{-1})^{(h_{+}g_{-})} \right)^{(h_{+}g_{-})} = (h_{+}g_{-})\eta(h_{+}g_{-}).
\]
This is (15).

Finally, substituting (40) and (44) into (42) gives
\[
r(\bar{h}_{+}g_{-}, (g_{+}\bar{h}_{+}^{-1})^{(h_{+}g_{-})}) = r(\bar{h}_{+}, g_{+}\bar{h}_{+}^{-1}).
\]
This is (37).

This completes the proof that any positive quasi-triangular structure must be given by the construction in the proposition.

Conversely, given homomorphisms \(\eta, \xi\), and a function \(r\) satisfying the conditions of the proposition, we want to show that the formula (24) satisfies (3), (4), and (5).

First of all, \(R\) satisfies (4) by Proposition 3.

Secondly, we have
\[
(\epsilon \otimes \text{id})R = \sum_{v \in G_{+}} r(e, v)\{v\}.
\]
One other hand, putting \(u = e\) in (23) shows that \(r(e, ?) : G_{+} \rightarrow \mathbb{R}_{>0}\) is a group homomorphism. Since \(G_{+}\) is finite, we see that \(r(e, v) = 1\) for all \(v\). Consequently, we have \((\epsilon \otimes \text{id})R = 1\). Similarly, we have \((\text{id} \otimes \epsilon)R = 1\).

Finally, we may use the conditions (3), (7), and (8) of the proposition to show that (40) and (41) are inverse to each other, just as what we have done in the first part. This implies that (40) and (41) give a one-to-one correspondence between the terms in \(\tau\Delta\{g\}R\) and \(R\Delta\{g\}\). Thus in order to show \(\tau\Delta\{g\}R = R\Delta\{g\}\), it remains to verify (12-15). The detailed computation is almost the same as what we have done in the first part of the proof, except for (15). The following computation verifies (15):
\[
(\bar{h}_{+}g_{-})\xi(\bar{h}_{+}g_{-}) \quad \text{(4)} \quad \xi(\bar{h}_{+}g_{-}) \quad \text{(4)} \quad \xi((g_{+}\bar{h}_{+}^{-1})^{(h_{+}g_{-})})g_{-} \quad \text{(4)} \quad (\xi(g_{+}\bar{h}_{+}^{-1})^{h_{+}})g_{-}.
\]
This completes the proof of the converse.

\[\Box\]

It remains to study the conditions (22), (23), and (37) imposed on \(r\).
Proposition 5  In a positive quasi-triangular structure

\[ R = \sum_{u,v \in G_+} r(u, v) \{ u (\eta(v)u)^{-1} \} \otimes \{ v\xi(u) \} \]

on \( H(G; G_+, G_-) \), we must have \( r(u, v) = 1 \).

Proof: By Proposition 4, the maps \( \xi \) and \( \eta \) in the positive quasi-triangular structure must be homomorphisms satisfying (6-10). By Proposition 1, we conclude that we are in the situation described in Theorem 1. In particular, \( G'_+ \) is a subgroup.

The one-to-one correspondence

\[ u \mapsto u\xi(u^{-1}) : G_+ \rightarrow G'_+ \]

translates the group structure (16) on \( G'_+ \) to a group structure

\[ u \star v = u(\xi(u^{-1})v) \]

on \( G_+ \). We have

\[ r(u \star w, v) = r(u(\xi(u^{-1})w), v) r(u, v) r(\xi(u^{-1})w, v^{\xi(w)}). \]

Let \( x = w^{-1}\xi(u) \). Then

\[ w x = \xi(u), \quad w^x = w^{(w^{-1}\xi(u))} (w^{-1}\xi(u))^{-1} \xi(u^{-1})w. \]

Therefore

\[ r(u \star w, v) = r(u, v) r(w^x, v^{w^x}) r(u, v) r(w, v). \]

Thus for fixed \( v \), \( r(?, v) : (G_+, \star) \rightarrow \mathbb{R}^{>0} \) is a group homomorphism. Since \( G_+ \) is finite, we see that \( r = 1 \).

\[ \square \]

4  Comparing positive quasi-triangular structures

Let \( G = G_+G_- \) and \( G = G'_+G_- \) be two unique factorizations of a finite group \( G \). Then we have two Hopf algebra structures \( H(G; G_+, G_-) \) and \( H(G; G'_+, G_-) \) on \( CG \). We show that the two Hopf algebra structures are quasi-isomorphic.

The two unique factorizations give rise to a map \( \sigma : G_+ \rightarrow G_- \) such that

\[ G'_+ = \{ \sigma(u)u : u \in G_+ \}. \]
The fact that $G'_+$ is a subgroup implies that for any $u, v \in G_+$,
\[
\sigma((u^{\sigma(v)})v) = \sigma(u)(u^{\sigma(v)}). \tag{48}
\]

To avoid confusion about two structures on the same space $\mathbf{C}G$. We consider $G = G_+G_-$ as the “standard” factorization, and $G = G'_+G_-$ as the “shifting” of the standard factorization. All the notations in Section 2 refer to operations relative to the unique factorization $G = G_+G_-$ and the Hopf algebra structure $H(G; G_+, G_-)$. For any $g \in G$, we use $\{g\}$ and $\{g\}'$ to denote $g$ considered as an element in $H(G; G_+, G_-)$ and in $H(G; G'_+, G_-)$, respectively.

By solving
\[
\sigma(u)ux = y\sigma(v)v, \quad u, v \in G_+, \quad x, y \in G_-,
\]
for $v$ and $y$, we have
\[
v = u^x, \quad y = \sigma(u)(u^x)\sigma(v)^{-1} = \sigma(u)(u^x)\sigma(u^x)^{-1}.
\]

Therefore the left action of $G'_+$ on $G_-$ and the right action of $G_-$ on $G'_+$ are given by
\[
\begin{align*}
G'_+ \times G_- & \rightarrow G'_+, \quad (\sigma(u)u, x) \mapsto \sigma(u^x)u^x, \tag{49} \\
G'_+ \times G_- & \rightarrow G_-, \quad (\sigma(u)u, x) \mapsto \sigma(u)(u^x)\sigma(u^x)^{-1}. \tag{50}
\end{align*}
\]

**Proposition 6** Denote
\[
\phi\{\sigma(u)ux\}' = \{ux\} : \quad H(G; G'_+, G_-) \rightarrow H(G; G_+, G_-), \tag{51}
\]
and
\[
T = \sum_{u,v \in G_+} \{u\sigma(v)\} \otimes \{v\}. \tag{52}
\]

Then $(\phi, T)$ is a quasi-isomorphism of Hopf algebras.

**Proof.** We first show that $\phi$ is an isomorphism of algebras. From the action (48), we see that
\[
\{\sigma(u)ux\}'\{\sigma(v)vy\}' = \delta_{u^x,v}\{\sigma(u)uxy\}'.
\]
It follows from this that $\phi$ preserves the multiplication. It is also easy to see that $\phi$ preserves the unit.

Next we show that
\[
(\phi \otimes \phi)\Delta(\{g\}') = T(\Delta\phi(\{g\}'))T^{-1}. \tag{53}
\]
It is easy to verify that
\[ T^{-1} = \sum_{u,v \in G_+} \{ u\sigma(v)^{-1} \} \otimes \{ v \}. \]
Then for any \( g_+ \in G_+, g_- \in G_- \), we have
\[ T(\Delta \phi \{ \sigma(g_+)g_+g_- \} T) = \sum_{h_+ \in G_+} \left\{ \left( (g_+h_+) \sigma(h_+)^{-1} \right) \sigma(h_+)(h_+g_-)\sigma(h_-^{-1}) \right\} \otimes \{ h_+g_- \}. \]

On the other hand,
\[ \Delta \{ \sigma(g_+)g_+g_- \}^2 = \sum_{h_+ \in G_+} \left\{ \sigma(g_+)g_+\sigma(h_+)h_+^{-1}\sigma(h_+)(h_+g_-)\sigma(h_-^{-1}) \right\} \otimes \{ \sigma(h_+)h_+g_- \}.
\]
Since \( G'_+ \) is a subgroup, we have
\[ \sigma(g_+)g_+\sigma(h_+)h_+^{-1} = \sigma(w)w \]
for some \( w \in G_+ \). By considering the \( G_+ \)-components in the unique factorization \( G = G_-G_+ \), we find \( w = (g_+h_+^{-1})^{\sigma(h_+)}^{-1} \). Therefore
\[ \sigma(g_+)g_+\sigma(h_+)h_+^{-1} = \sigma \left( (g_+h_+^{-1})^{\sigma(h_+)}^{-1} \right) (g_+h_+^{-1})^{\sigma(h_+)}^{-1}, \]
and
\[ (\phi \otimes \phi)\Delta \{ \sigma(g_+)g_+g_- \} = \sum_{h_+ \in G_+} \left\{ \left( (g_+h_+^{-1})^{\sigma(h_+)}^{-1} \right) \sigma(h_+)(h_+g_-)\sigma(h_-^{-1}) \right\} \otimes \{ h_+g_- \}. \]
This completes the verification of (53).

Finally, it is easy to compute the following
\[ (T \otimes 1)(\Delta \otimes \text{id})T = \sum_{u,v,w \in G_+} \{ u\sigma(v)\sigma(w) \} \otimes \{ v\sigma(w) \} \otimes \{ w \}, \]
\[ (1 \otimes T)(\text{id} \otimes \Delta)T = \sum_{u,v,w \in G_+} \{ u\sigma(v)\sigma(w) \} \otimes \{ v\sigma(w) \} \otimes \{ w \}. \]
It then follows from (48) that \( (T \otimes 1)(\Delta \otimes \text{id})T = (1 \otimes T)(\text{id} \otimes \Delta)T \), so that the compatibility condition is verified.

Now we apply the proposition above to the special case in Theorem 1 with \( \sigma(u) = \xi(u^{-1}) \). Note that by taking inverse, \( G'_+ \) is the same as the one given in Theorem 1.
Thus the quasi-isomorphism constructed in Proposition 6 translates the quasi-triangular structure \( R \) on \( H(G;G_+,G_-) \) into another quasi-triangular structure
\[ R' = (\phi \otimes \phi)^{-1}(\tau T)RT^{-1}, \]
on $H(G; G'_+, G_-)$. An easy computation gives
\[
(\tau T)RT^{-1} = \sum_{u,v \in G_+} \{ u\eta(v)^{-1}\xi(v) \} \otimes \{ v \},
\]
so that
\[
R' = \sum_{u,v \in G_+} \{ \xi(u^{-1})u\eta(v)^{-1}\xi(v) \}' \otimes \{ \xi(v^{-1})v \}'. \quad (54)
\]
By Theorem 1, $R'$ is given by homomorphisms $\xi', \eta' : G'_+ \to G_-$. From the second component in (54), we have $\xi'(\xi(v^{-1})v) = e$. By (12), the triviality of $\xi'$ implies that the first component in (54) is of the form $\{ \xi(u^{-1})u\eta'(\xi(v^{-1})v)^{-1} \}'$. Therefore we conclude that
\[
\xi'(\xi(u^{-1})u) = e, \quad \eta'(\xi(v^{-1})v) = \xi(v)^{-1}\eta(v). \quad (55)
\]

**Definition** Let $R$ be a positive quasi-triangular structure on $H(G; G'_+, G_-)$ given by homomorphisms $\xi, \eta : G_+ \to G_-$ as in Theorem 1. We say that $R$ is normal if $\xi(u) = e$ for all $u \in G_+$. We say that the pair $(H(G; G'_+, G_-), R)$ is normal if $R$ is normal.

Thus in the special case described in Theorem 1, Proposition 6 implies the following.

**Theorem 3** Every pair $(H, R)$, where $H$ is a finite dimensional Hopf algebra with a positive basis and $R$ is a positive quasi-triangular structure in this basis, is quasi-isomorphic to a normal one.

\[\square\]

## 5 Positive triangular structures

A quasi-triangular structure $R$ is triangular if it further satisfies $(\tau R)R = 1 \otimes 1$. For the positive quasi-triangular structure $R$ given by Theorem 1, we have
\[
(\tau R)R = \sum \{ u\xi(u) (\eta(v)^u)^{-1} \} \otimes \{ u (\eta(v)^u)^{-1} \xi(\bar{u}) \},
\]
where the summation is over all $u, v, \bar{u}, \bar{v} \in G_+$ satisfying
\[
\nu^\xi(u) = \bar{u}, \quad \eta(v)^u = \bar{v}. \quad (56)
\]
Since $1 \otimes 1 = \sum_{u,v \in G_+} \{ u \} \otimes \{ v \}$, we see that $R$ is triangular if and only if (56) implies
\[
\xi(u) = \eta(\bar{v})^u, \quad \eta(v)^u = \xi(\bar{u}). \quad (57)
\]
Note that the first equality in (56) implies \( \eta(v)^u = \eta(v^{\xi(u)}) = \eta(\bar{u}). \) Therefore under the assumption (56), the second equality of (57) is equivalent to \( \xi = \eta. \) Furthermore, the property \( \xi = \eta \) and (56) imply

\[
\eta(\bar{v}) \bar{u} = \eta(\bar{v}) u \bar{u} = \xi(\eta(v) u) \bar{u} = (\xi(\eta(v)))(\eta(v^{\xi(u)}) = v \xi(u).
\]

In particular, the first equality of (57) also holds. Thus we conclude that \( R \) is triangular if and only if \( \xi = \eta. \)

**Theorem 4** There is a one-to-one correspondence between the following data

1. a finite dimensional Hopf algebras with a positive basis and a positive triangular structure;
2. a unique factorization \( G = G_+ G_- \) of finite group, and a homomorphism \( \xi : G_+ \to G_- \) such that \( A = \{u \xi(u^{-1}) : u \in G_+\} \) is an abelian normal subgroup;
3. a unique factorization \( G = G_+ G_- \) of finite group, and a homomorphism \( \xi : G_+ \to G_- \) satisfying \( uv = (\xi(u)v)(u^{\xi(v)}) \) and \( \xi(xu)x^u = x \xi(u); \)
4. a group \( G_- \), an abelian group \( A \) acted upon by \( G_- \) as automorphisms, and a 1-cycle \( \zeta \) of \( G_- \) with coefficient in \( A. \)

**Proof:** We already know the first item is equivalent to a unique factorization \( G = G_+ G_- \) of a finite group with a homomorphism \( \xi : G_+ \to G_- \) such that

\[
G'_+ = \{u \xi(u^{-1}) : u \in G_+\}, \quad G''_+ = \{\xi(u^{-1})u : u \in G_+\}
\]

are normal subgroups, and

\[
F(u \xi(u^{-1})) = \xi(u)u^{-1} : \ G'_+ \to G''_+
\]

is a homomorphism. Since \( \xi(u)u^{-1} = (u \xi(u^{-1}))^{-1}, \) we see that \( G'_+ = G''_+ \) and \( F(a) = a^{-1}. \) Therefore the condition is equivalent to \( A = G'_+ = G''_+ \) being an abelian normal subgroup. This proves the equivalence between the first two items.

By Proposition [1], if \( \xi \) induces a triangular structure, then the two conditions in the third item must be satisfied. Conversely, we need to show that the two conditions imply the other conditions in Proposition [1]. For example, the two conditions imply

\[
\xi(\xi(u)v)\xi(u)v = \xi(u)\xi(v) = \xi(uv) = \xi(\xi(u)v)\xi(u^{\xi(v)}).
\]

Canceling the left factor, we get (6). The other conditions can be verified similarly. This proves the equivalence to the third item.
Finally, the fourth item is the reinterpretation in terms of the alternative description. From the discussion above, we see that a positive triangular structure means $A = G'_+$ is abelian, and

$$F(ax) = a^{-1} x, \quad a \in A, \; x \in G_-.$$  

Since these already implies the conditions in the second item are satisfied, we see that there is no further condition on the 1-cycle $\zeta$. This proves the equivalence to the fourth item.

\[\square\]

Now let us apply the theory of Section 4 to normalize positive triangular structures. Since $\xi = \eta$, both $\xi'$ and $\eta'$ are trivial by (55). Another way to see the triviality is to use the fact that a quasi-isomorphism carries triangular structures to triangular structures. Therefore $(\xi', \eta')$ must induce a triangular structure, which implies $\xi' = \eta'$. Since $\xi'$ is trivial, so is $\eta'$. Thus, we find $R' = 1' \otimes 1'$, and the triangular Hopf algebra $(H(G; G'_+, G_-), R)$ is isomorphic to the twisting of the triangular Hopf algebra $(H(G; A, G_-), 1' \otimes 1')$, with the twist given by

$$T' = (\phi \otimes \phi)^{-1}(T) = \sum_{u,v \in G_+} \{\xi(u^{-1})u\xi(v^{-1})\}' \otimes \{\xi(v^{-1})v\}' = \sum_{a,b \in A} \{a\zeta(b^{-1})\}' \otimes \{b\}' .$$

Finally, we note that since $R' = 1' \otimes 1'$, $H(G; A, G_-)$ is cocommutative. Thus $H(G; A, G_-)$ is a group algebra, and we conclude that any positive triangular structure is the twisting of a group algebra. Explicit formulae for this conclusion can be found in Section 4 of [EG].

6 Groupoids and the set-theoretical Yang-Baxter equation

In [LYZ1], we have shown that the positivity condition on a Hopf algebra implies that the Hopf algebra is essentially set-theoretical. Theorem 1 says that positive quasi-triangular structures on such Hopf algebras are also set-theoretical. As a result, we expect our theory to lead to set-theoretical solutions of the Yang-Baxter equation. In this section, we explain how this happens.

Recall [MK] that a groupoid over a set $B$ (called base space) is a set $\Gamma$ (called total space) together with

1. two surjections $\alpha, \beta : \Gamma \to B$;
2. a product $\mu : (\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2$ in $\Gamma$, defined when $\beta(\gamma_1) = \alpha(\gamma_2)$;
3. an identity map $e : b \mapsto e_b, B \to \Gamma$;
4. an inversion map $\sigma : \gamma \mapsto \gamma^{-1}, \Gamma \to \Gamma$

such that

$$\alpha(\gamma_1 \gamma_2) = \alpha(\gamma_1), \quad \beta(\gamma_1 \gamma_2) = \beta(\gamma_2), \quad \alpha(e_b) = \beta(e_b) = b,$$

$$\alpha(\gamma^{-1}) = \beta(\gamma), \quad \beta(\gamma^{-1}) = \alpha(\gamma),$$

and the usual axioms similar to those for groups are satisfied. If $\Gamma$ is finite, then we have

an algebra structure on the vector space $\mathbb{C} \Gamma$

$$\{\gamma_1 \gamma_2\} = \begin{cases} \{\gamma_1 \gamma_2\} & \text{if } \beta(\gamma_1) = \alpha(\gamma_2) \\ 0 & \text{if } \beta(\gamma_1) \neq \alpha(\gamma_2) \end{cases}, \quad e = \sum_{b \in B} e_b,$$

called the linearization of the groupoid $\Gamma$ or the groupoid algebra of $\Gamma$.

For example, given a unique factorization $G = G_+ G_-$, we have the following groupoid $\Gamma_+$ with $G$ as the total space and with $G_+$ as the base space:

1. $\alpha_+: g \mapsto g_+, G \to G_+$, and $\beta_+: g \mapsto \bar{g}_+, G \to G_+$;
2. $\mu_+: (g, h) \mapsto gh_-$ when $\bar{g}_+ = h_+$;
3. $e_+: g_+ \mapsto g_+, G_+ \to G$;
4. $\sigma_+: g \mapsto \bar{g}_+ g^{-1} = \bar{g}_- g_+, G \to G$.

The linearization of $\Gamma_+$ is the algebra structure of $H(G; G_+, G_-)$.

We can read set-theoretical information about the groupoid from its linearization. Specifically, let $\Gamma$ be a (finite) groupoid over $B$. For a positive element

$$a = \sum_{\gamma \in \Gamma} r(\gamma) \{\gamma\}, \quad r(\gamma) \geq 0$$

of the groupoid algebra, we denote

$$L(a) = \{\gamma : r(\gamma) > 0\}.$$

Since $\Gamma$ is a positive basis of the groupoid algebra, $L$ has the following property: If $a_1$ and $a_2$ are two positive elements, then

$$L(a_1 a_2) = \{\gamma_1 \gamma_2 : \gamma_1 \in L(a_1), \gamma_2 \in L(a_2), \beta(\gamma_1) = \alpha(\gamma_2)\}. \quad (58)$$

If we define the product of two subsets $L_1, L_2 \subset \Gamma$ as

$$L_1 L_2 = \{\gamma_1 \gamma_2 : \gamma_1 \in L_1, \gamma_2 \in L_2, \beta(\gamma_1) = \alpha(\gamma_2)\}, \quad (59)$$

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then \((58)\) becomes \(L(a_1a_2) = L(a_1)L(a_2)\).

The subset

\[ E_\Gamma = \{ e_b : b \in B \} \subset \Gamma \]

is clearly a unit of the multiplication \((55)\). The next result tells us which subset of \(\Gamma\) is invertible.

**Proposition 7** Let \(L \subset \Gamma\) be a subset of a groupoid \(\Gamma\) over \(B\). Then the following are equivalent:

1. There is a subset \(K \subset \Gamma\) such that \(LK = E_\Gamma\) and \(KL = E_\Gamma\);
2. The restrictions \(\alpha|_L, \beta|_L : L \to B\) are bijections.

**Proof:** We show that the first statement implies the second.

Since \(LK = E_\Gamma\), for any \(b \in B\), there are \(\gamma_1 \in L\) and \(\gamma_2 \in K\) such that \(\beta(\gamma_1) = \alpha(\gamma_2)\) and \(\gamma_1\gamma_2 = e_b\). In particular, we have \(\alpha(\gamma_1) = \alpha(\gamma_1\gamma_2) = b\) and \(\beta(\gamma_2) = \beta(\gamma_1\gamma_2) = b\). Thus \(\alpha|_L\) and \(\beta|_K\) are surjective.

Now suppose we have \(\gamma_1, \gamma_1' \in L\) such that \(\alpha(\gamma_1) = \alpha(\gamma_1') = a\). By the surjectivity of \(\beta|_K\), we can find \(\gamma_2 \in K\) such that \(\beta(\gamma_2) = a\). Since \(KL = E_\Gamma\), we conclude that \(\gamma_2\gamma_1 = e_b\) and \(\gamma_2\gamma_1' = e_{b'}\) for some \(b, b' \in B\). Then

\[ \beta(\gamma_1) = \beta(\gamma_2\gamma_1) = b = \alpha(\gamma_2\gamma_1) = \alpha(\gamma_2) = \alpha(\gamma_2\gamma_1') = b' = \beta(\gamma_2\gamma_1') = \beta(\gamma_1'). \]

In particular, the products \(\gamma_1\gamma_2\) and \(\gamma_1'\gamma_2\) make sense. Since \(\gamma_1\gamma_2, \gamma_1'\gamma_2 \in LK = E_\Gamma\), we see that \(\gamma_1\gamma_2 = \gamma_1'\gamma_2 = e_a\). Combining this with \(\gamma_2\gamma_1 = e_b = e_{b'} = \gamma_2\gamma_1'\), we conclude that \(\gamma_1 = \gamma_2^{-1} = \gamma_1'\). This proves the injectivity of \(\alpha|_L\).

The bijectivity of \(\beta|_L : L \to B\) can be proved similarly.

Conversely, given the second statement, it is easy to verify that \(K = \{ \gamma^{-1} : \gamma \in L \}\) satisfies the first statement.

\(\square\)

A subset \(L \subset \Gamma\) of a groupoid satisfying the equivalent conditions of the proposition above is called a bisection. With product \((59)\), the collection \(U(\Gamma)\) of all bisections of a groupoid \(\Gamma\) form a group.

As an application of Proposition 7, let us prove Proposition 2. We note that the tensor algebra \(H(G; G_+, G_-) \otimes H(G; G_+, G_-)\) is the linearization of the product groupoid \(\Gamma_+ \times \Gamma_+\). Under the assumption of Proposition 4, both \(R\) and \(R^{-1}\) are positive elements. Then \(RR^{-1} = 1 \otimes 1 = R^{-1}R\) implies that \(L(R)L(R^{-1}) = L(1 \otimes 1) = E_{\Gamma_+ \times \Gamma_+} = L(R^{-1})L(R)\), i.e., \(L(R)\) is invertible. By Proposition 4, the restriction of \(\alpha_{\Gamma_+ \times \Gamma_+}(g, g) = (g_+, g_+)\) on \(\mathcal{R} = L(R)\) is bijective.
As another application, we consider the positive quasi-triangular structure $R$ in the classification Theorem 1. Denote

$$R = L(R) = \left\{ \left( u(\eta(v)^u)^{-1}, v \xi(u) \right) : u, v \in G_+ \right\}.$$  \hfill (60)

Then we clearly have

$$L(R_{12}) = R \times \{ e \} = R_{12}, \quad \text{etc.}$$

Applying $L$ to the Yang-Baxter equation satisfied by the positive quasi-triangular structure, we see that $R$ also satisfies the following groupoid-theoretical Yang-Baxter equation introduced in [WX]

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12},$$  \hfill (61)

which is an equality inside the group $\mathcal{U}(\Gamma_+ \times \Gamma_+ \times \Gamma_+)$ of bisections.

To get set-theoretical solutions of the Yang-Baxter equation from (61), we recall that the quasi-triangular structure $R$ induces a solution of the Yang-Baxter equation on any $H$-module. The set-theoretical analogue of modules is sets acted upon by groupoids.

Let $\Gamma$ be a groupoid over $B$. A (left) $\Gamma$-set consists of a set $X$, a map $J : X \to B$, and an action

$$(\gamma, x) \mapsto \gamma x \in X, \quad \text{for } \gamma \in \Gamma, x \in X, \text{ satisfying } \beta(\gamma) = J(x).$$

The action is required to satisfy

$$(\gamma_1 \gamma_2)x = \gamma_1(\gamma_2x), \quad e(Jx)x = x, \quad J(\gamma x) = \alpha(\gamma),$$

whenever the relevant actions are defined. The vector space $C_X$ has an obvious module structure over the linearization of $\Gamma$.

For any $L \in \mathcal{U}(\Gamma)$ and $x \in X$, the equation

$$Lx = \gamma x, \quad \text{for the unique } \gamma \in L \text{ satisfying } \beta(\gamma) = J(x)$$

defines a (left) action of the group $\mathcal{U}(\Gamma)$ of bisections on the set $X$. Now if $R \in \mathcal{U}(\Gamma \times \Gamma)$ satisfies the groupoid-theoretical Yang-Baxter equation (61), then the map $R_X : X \times X \to X \times X$ induced by $R$ is a set-theoretical solution of the Yang-Baxter equation over $X$.

Now we compute the set-theoretical solution of the Yang-Baxter equation induced by the action of the bisection (60) on the simplest $\Gamma_+$-set, the unit $\Gamma_+$-set $id : G_+ \to G_+$. In this case, $R_{G_+}$ is given by the following diagram

$$\begin{array}{ccc}
\beta_+ \times \beta_+ & \sqrt{u(v)^u} & v \xi(u) \\
(u(v)^u, v \xi(u)) & R_{G_+} & (u, v)
\end{array}$$

\hfill \alpha_+ \times \alpha_+
By $u^{(\eta(v))^{-1}} = \eta(v)u$, we have

$$R_{G_+}^{-1}(u, v) = (\eta(v)u, v\xi(u)).$$

Solving the equation, we get the set-theoretical solution

$$R_{G_+}(u, v) = (u\eta(v), \xi(u)v) \quad (62)$$

of the Yang-Baxter equation over $G_+$.

Direct computation shows that if $\xi$ and $\eta$ are two group homomorphisms satisfying (8), then (62) is already a set-theoretical solution of the Yang-Baxter equation. Moreover, in order for (8) and (62) to make sense, we do not even need to know anything about $G_-$. The only data we need are actions $(u, v) \mapsto \xi(u)v$ and $(u, v) \mapsto u\eta(v)$ of $G_+$ on itself. This is the basic data for the construction in [LYZ2].

In [LYZ2], we have an alternative description of our set-theoretical solution of the Yang-Baxter equation in terms of bijective 1-cocycles. If the solution is given by (62), then we may take the groups $G$ and $A$ in [LYZ2] to be $G_+$ and $G'_+$. Moreover, we may take the action of $G$ and $A$ in [LYZ2] to be the translation of the action $(u, v) \mapsto \xi(u)v$ of $G_+$ on itself under the bijection

$$\pi(u) = u\xi(u^{-1}) : G_+ \to G'_+.$$

Finally, $\pi$ also serves as the bijective 1-cocycle for the alternative description in [LYZ2]. This 1-cocycle is related to the 1-cycle $\zeta$ by

$$\zeta = \xi \circ \pi^{-1}.$$

We would like to end the section by mentioning that a comprehensive theory can be established for the Yang-Baxter equation on groupoids. Indeed, we can formulate the definitions of Hopf groupoids and quasi-triangular structures on them. We can further show that any unique factorization of a group induces a Hopf groupoid, and that all of its quasi-triangular structures are given by the bisections of the form (60). Moreover, we can introduce the notion of quasi-isomorphisms of Hopf groupoids and establish the set-theoretical analogue of Theorem 3. In particular, we conclude that actions of the braid group $B_n$ on the set $X^n$ induced by the solution $R_X$ is equivalent to the action induced by a normal solution. Note that by (8), the special solution (62) given by a normal quasi-triangular structure is of the form

$$R_{G'_+}(u, v) = (v^{-1}uv, v),$$

which is the conjugate solution (with respect to the new multiplication (16) coming from $G'_+$). Thus we conclude that the braid group action induced by any set-theoretical solution of the Yang-Baxter equation of the form (62) is equivalent to the action induced from a conjugate solution. All these considerations motivate our paper [LYZ2] on set-theoretical solutions of the Yang-Baxter equation.
7 An example

Any unique factorization $G = G_+G_-$ induces another unique factorization $\tilde{G} = G \times G = \tilde{G}_+\tilde{G}_-$, with

$$\tilde{G}_+ = \{(g_+,g_-) : g_+ \in G_+, g_- \in G_\}, \quad \tilde{G}_- = \{(g,g) : g \in G\}.$$ 

The Hopf algebra induced by this unique factorization is in fact the Drinfel’d double of $H(G; G_+, G_-)$ (see [LYZ1]).

Consider homomorphisms

$$\begin{align*}
\xi(g_+,g_-) &= (g_-,g_-) \\
\eta(g_+,g_-) &= (g_+,g_+) 
\end{align*}$$

The induced subgroups (as in Theorem 1) $\tilde{G}_+^\prime = G \times \{e\}$ and $\tilde{G}_-^\prime = \{e\} \times G$ are clearly normal. It is also easy to see that the map $F : \tilde{G}_+^\prime \to \tilde{G}_-^\prime$ is given by $F(a,e) = (e,a)$, which is clearly a group isomorphism. From this we get the standard quasi-triangular structure on the Drinfel’d double of $H(G; G_+, G_-)$.

To find the alternative description, we use the identification

$$\tilde{G}_+^\prime \cong G : (a,e) \leftrightarrow a; \quad \tilde{G}_-^\prime \cong G : (g,g) \leftrightarrow g.$$  \hspace{1cm} (63)

Then $(g,g)(a,e)(g,g)^{-1} = (gag^{-1}, e)$ implies that $G$ acts on $A = G$ by conjugations. Since $(a,e) = u\xi(u^{-1})$ for $u = (a_+, a_-)$, the 1-cycle is (after the identification (63))

$$\zeta(a) = a_+^{-1}.$$ 

Moreover, since $(e,a) = (a^{-1},e)(a,a)$ with respect to the unique factorization $\tilde{G} = \tilde{G}_+^\prime\tilde{G}_-^\prime$, the automorphism on $G \bowtie_{\text{conj}} G$ is

$$F(a \bowtie g) = F(a \bowtie e)F(e \bowtie g) = (a^{-1} \bowtie e)(e \bowtie a)(e \bowtie g) = a^{-1} \bowtie ag.$$ 

The quasi-triangular structure induces a solution of the Yang-Baxter equation on the set $\tilde{G}_+^\prime$. A detailed computation shows that this solution is the one given in Theorem 9.2 of [WX] (after the identification $\tilde{G}_+^\prime \cong G : (g_+g_-) \leftrightarrow g_+g_-^{-1}$).

To find the bijective 1-cocycle description of this solution, we use the construction near the end of last section. Thus we take the group $G$ and $A$ in [LYZ2] to be

$$\tilde{G}_+ = G_+ \times G_-, \quad G \cong G \times \{e\}.$$ 

The bijective 1-cocycle is

$$\pi(g_+,g_-) = (g_+,g_-)\xi(g_+,g_-)^{-1} = (g_+g_-^{-1},e) \in G \times \{e\} \leftrightarrow g_+g_-^{-1} \in G.$$ 

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Moreover, the action of $\tilde{G}_+$ on $G$ may be computed as follows: If $\xi(g_+, g_-)(a_+, a_-) = (b_+, b_-)(h, h)$, then $(g_+, g_-) \cdot \pi(a_+, a_-) = \pi(b_+, b_-)$. Since $g_- a_+ = b_+ h$ and $g_- a_- = b_- h$, we have

$$(g_+, g_-) \cdot (a_+ a_-^{-1}) = (b_+ h)(b_- h)^{-1} = (g_- a_+)(g_- a_-)^{-1} = g_- (a_+ a_-^{-1}) g_-^{-1}.$$ 

Therefore the action is given by a conjugation

$$(g_+, g_-) \cdot a = g_- a g_-^{-1}.$$ 

This interpretation of Weinstein and Xu’s solution of the Yang-Baxter appeared in [LYZ2].

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