Schubert polynomials and the inhomogeneous TASEP on a ring

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Abstract. Consider a lattice of \(n\) sites arranged around a ring, with the \(n\) sites occupied by particles of weights \(\{1, 2, \ldots, n\}\); the possible arrangements of particles in sites thus corresponds to the \(n!\) permutations in \(S_n\). The inhomogeneous totally asymmetric simple exclusion process (or TASEP) is a Markov chain on the set of permutations, in which two adjacent particles of weights \(i < j\) swap places at rate \(x_i - y_{n+1-j}\) if the particle of weight \(j\) is to the right of the particle of weight \(i\). (Otherwise nothing happens.) In the case that \(y_i = 0\) for all \(i\), the stationary distribution was conjecturally linked to Schubert polynomials by Lam-Williams, and explicit formulas for steady state probabilities were subsequently given in terms of multiline queues by Ayyer-Linusson and Arita-Mallick. In the case of general \(y_i\), Cantini showed that \(n\) of the \(n!\) states have probabilities proportional to double Schubert polynomials. In this paper we introduce the class of evil-avoiding permutations, which are the permutations avoiding the patterns 2413, 4132, 4213 and 3214. We show that there are \(\frac{(2+\sqrt{2})^{n-1}-(2-\sqrt{2})^{n-1}}{2}\) evil-avoiding permutations in \(S_n\), and for each evil-avoiding permutation \(w\), we give an explicit formula for the steady state probability \(\psi_w\) as a product of double Schubert polynomials. We also show that the Schubert polynomials that arise in these formulas are flagged Schur functions, and give a bijection in this case between multiline queues and semistandard Young tableaux.

Keywords: Schubert polynomials, TASEP, multiline queues

1 Introduction

In recent years, there has been a lot of work on interacting particle models such as the asymmetric simple exclusion process (ASEP), a model in which particles hop on a one-dimensional lattice subject to the condition that at most one particle may occupy a given site. The ASEP on a one-dimensional lattice with open boundaries has been linked to Askey-Wilson polynomials and Koornwinder polynomials [8, 3, 7], while the ASEP on a ring has been linked to Macdonald polynomials [5, 6]. The inhomogeneous totally...
asymmetric simple exclusion process (TASEP) is a variant of the exclusion process on the
ring in which the hopping rate depends on the weight of the particles. In this paper
we build on works of Lam-Williams [10], Ayyer-Linusson [2], and especially Cantini [4]
to give formulas for many steady state probabilities of the inhomogeneous TASEP on a
ring in terms of Schubert polynomials.

**Definition 1.1.** Consider a lattice with \( n \) sites arranged in a ring. Let \( \text{St}(n) \) denote the
\( n! \) labelings of the lattice by distinct numbers \( 1, 2, \ldots, n \), where each number \( i \) is called
a particle of weight \( i \). The inhomogeneous TASEP on a ring of size \( n \) is a Markov chain with
state space \( \text{St}(n) \) where at each time \( t \) a swap of two adjacent particles may occur: a
particle of weight \( i \) on the left swaps its position with a particle of weight \( j \) on the right
with transition rate \( r_{i,j} \) given by:

\[
r_{i,j} = \begin{cases} 
  x_i - y_{n+1-j} & \text{if } i < j \\
  0 & \text{otherwise}
\end{cases}
\]

In what follows, we will identify each state with a permutation in \( S_n \). Following [10, 4], we multiply all steady state probabilities for \( \text{St}(n) \) by the same constant, obtaining
"renormalized" steady state probabilities \( \psi_w \), so that

\[
\psi_{123...n} = \prod_{i<j} (x_i - y_{n+1-j})^{j-i-1}.
\]

(1.1)

See Figure 1 for the state diagram when \( n = 3 \).

In the case that \( y_i = 0 \), Lam and Williams [10] studied this model\(^1\) and conjectured
that after a suitable normalization, each steady state probability \( \psi_w \) can be written as a
monomial factor times a positive sum of Schubert polynomials, see Table 1 and Table 2. They also gave an explicit formula for the monomial factor, and conjectured that under
certain conditions on \( w \), \( \psi_w \) is a multiple of a particular Schubert polynomial. Subse-
quently Ayyer and Linusson [2] gave a conjectural combinatorial formula for the sta-
tionary distribution in terms of multiline queues, which was proved by Arita and Mallick
[1]. In [4], Cantini introduced the version of the model given in Definition 1.1\(^2\) with \( y_i \)
general, and gave a series of exchange equations relating the components of the stationary
distribution. This allowed him to give explicit formulas for the steady state probabilities
for \( n \) of the \( n! \) states as products of double Schubert polynomials.

In this paper we build on [4, 2, 1], and give many more explicit formulas for steady
state probabilities in terms of Schubert polynomials: in particular, we give a formula for
\( \psi_w \) as a product of (double) Schubert polynomials whenever \( w \) is evil-avoiding, that is, it

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\(^1\)However the convention of [10] was slightly different; it corresponds to labeling states by the inverse
of the permutations we use here.

\(^2\)We note that in [4], the rate \( r_{i,j} \) was \( x_i - y_j \) rather than \( x_i - y_{n+1-j} \) as we use in Definition 1.1.
Schubert polynomials and TASEP

\[ x_1 - y_1 \]
\[ x_2 - y_1 \]
\[ x_1 + x_2 - y_1 - y_2 \]
\[ x_1 - y_2 \]
\[ x_1 - y_1 \]
\[ x_1 + x_2 - y_1 - y_2 \]
\[ x_1 - y_1 \]
\[ x_1 - y_1 \]
\[ x_1 - y_2 \]
\[ x_1 - y_1 \]
\[ x_1 - y_2 \]
\[ x_2 - y_1 \]
\[ x_1 + x_2 - y_1 - y_2 \]
\[ x_1 - y_1 \]
\[ x_1 - y_1 \]
\[ x_1 - y_1 \]
\[ x_1 + x_2 - y_1 - y_2 \]

**Figure 1:** The state diagram for the inhomogeneous TASEP on \( \text{St}(3) \), with transition rates shown in blue, and steady state probabilities \( \psi_w \) in red. Though not shown, the transition rate \( 312 \to 213 \) is \( x_2 - y_1 \) and the transition rate \( 231 \to 132 \) is \( x_1 - y_2 \).

**Table 1:** The renormalized steady state probabilities for \( n = 4 \).

| State \( w \) | Probability \( \psi_w \) |
|--------------|--------------------------|
| 1234         | \((x_1 - y_1)^2(x_1 - y_2)(x_2 - y_1)\) |
| 1324         | \((x_1 - y_1)\mathcal{S}_{1432}\) |
| 1342         | \((x_1 - y_1)(x_2 - y_1)\mathcal{S}_{1423}\) |
| 1423         | \((x_1 - y_1)(x_1 - y_2)(x_2 - y_1)\mathcal{S}_{1243}\) |
| 1243         | \((x_1 - y_2)(x_1 - y_1)\mathcal{S}_{1342}\) |
| 1432         | \(\mathcal{S}_{1423}\mathcal{S}_{1342}\) |

avoids the patterns 2413, 4132, 4213 and 3214.\(^3\) We show that there are \( \frac{(2+\sqrt{2})^{n-1}+(2-\sqrt{2})^{n-1}}{2} \) evil-avoiding permutations in \( S_n \), so this gives a substantial generalization of Cantini’s previous result [4] in this direction. We also prove the monomial factor conjecture from [10]. Finally, we show that the Schubert polynomials that arise in our formulas are flagged Schur functions, and give a bijection in this case between multiline queues and semistandard Young tableaux.

In order to state our main results, we need a few definitions. First, we say that two states \( w \) and \( w' \) are equivalent, and write \( w \sim w' \), if one state is a cyclic shift of the other.

\(^3\)We call these permutations evil-avoiding because if one replaces \( i \) by 1, \( e \) by 2, \( l \) by 3, and \( v \) by 4, then evil and its anagrams vile, veil and leiv become the four patterns 2413, 4132, 4213 and 3214. Note that Leiv is a name of Norwegian origin meaning “heir.”
Table 2: The renormalized steady state probabilities for \( n = 5 \), when each \( y_i = 0 \). In the table, \( x^{(a,b,c)} \) denotes \( x_1^a x_2^b x_3^c \).

| State \( w \) | Probability \( \psi_w \) |
|------------|------------------|
| 12345      | \( x^{(6,3,1)} \) |
| 12354      | \( x^{(5,2,0)} S_{13452} \) |
| 12435      | \( x^{(4,1,0)} S_{14532} \) |
| 12453      | \( x^{(4,1,1)} S_{14523} \) |
| 12534      | \( x^{(5,2,1)} S_{12453} \) |
| 12543      | \( x^{(3,0,0)} S_{14523} S_{13452} \) |
| 13245      | \( x^{(3,1,1)} S_{15423} \) |
| 13254      | \( x^{(2,0,0)} S_{15423} S_{13452} \) |
| 13425      | \( x^{(3,2,1)} S_{15243} \) |
| 13452      | \( x^{(3,3,1)} S_{15234} \) |
| 13524      | \( x^{(2,1,0)} (S_{164325} + S_{25431}) \) |
| 13542      | \( x^{(2,2,0)} S_{15234} S_{13452} \) |
| 14235      | \( x^{(4,2,0)} S_{13542} \) |
| 14253      | \( x^{(4,2,1)} S_{12543} \) |
| 14325      | \( x^{(1,0,0)} (S_{1753246} + S_{265314} + S_{2743156} + S_{356214} + S_{364215} + S_{365124}) \) |
| 14352      | \( x^{(1,1,0)} S_{15234} S_{14532} \) |
| 14523      | \( x^{(4,3,1)} S_{12534} \) |
| 14532      | \( x^{(1,1,1)} S_{15234} S_{14523} \) |
| 15234      | \( x^{(5,3,1)} S_{12354} \) |
| 15243      | \( x^{(3,1,0)} (S_{146325} + S_{24531}) \) |
| 15324      | \( x^{(2,1,1)} (S_{15432} + S_{164235}) \) |
| 15342      | \( x^{(2,2,1)} S_{15234} S_{12453} \) |
| 15423      | \( x^{(3,2,0)} S_{12534} S_{13452} \) |
| 15432      | \( S_{15234} S_{14523} S_{13452} \) |

In the table, \( x^{(a,b,c)} \) denotes \( x_1^a x_2^b x_3^c \).

E.g. \( (w_1, \ldots, w_n) \sim (w_2, \ldots, w_n, w_1) \). Because of the cyclic symmetry inherent in the definition of the TASEP on a ring, it is clear that the probabilities of states \( w \) and \( w' \) are equal whenever \( w \sim w' \). We will therefore often assume, without loss of generality, that \( w_1 = 1 \). Note that up to cyclic shift, \( St(n) \) contains \( (n - 1)! \) states.

**Definition 1.2.** Let \( w = (w_1, \ldots, w_n) \in St(n) \). We say that \( w \) is a \( k \)-Grassmannian permutation, and we write \( w \in St(n, k) \) if: \( w_1 = 1 \); \( w \) is evil-avoiding, i.e. \( w \) avoids the patterns 2413, 3214, 4132, and 4213; and \( w^{-1} \) has exactly \( k \) descents, equivalently, there are exactly \( k \) letters \( a \) in \( w \) such that \( a + 1 \) appears to the left of \( a \) in \( w \).

**Definition 1.3.** We associate to each \( w \in St(n, k) \) a sequence of partitions \( \Psi(w) = (\lambda^1, \ldots, \lambda^k) \) as follows. Write the Lehmer code of \( w^{-1} \) as \( \text{code}(w^{-1}) = c = (c_1, \ldots, c_n) \);
Schubert polynomials and TASEP

since $w^{-1}$ has $k$ descents, $c$ has $k$ descents in positions we denote by $a_1, \ldots, a_k$. We also set $a_0 = 0$. For $1 \leq i \leq k$, we define $\lambda^i = (n - a_i)^{a_i} - (0, \ldots, 0, c_{a_i-1+1}, c_{a_i-1+2}, \ldots, c_{a_i})$.

See Table 3 for examples of the map $\Psi(w)$.

**Definition 1.4.** Given a positive integer $n$ and a partition $\lambda$ of length $\leq (n-2)$, we define an integer vector $g_n(\lambda) = (v_1, \ldots, v_n)$ of length $n$ as follows. Write $\lambda = (\mu^1_1, \ldots, \mu^l_l)$ where $k_i > 0$ and $\mu_1 > \cdots > \mu_l$. We assign values to the entries $(v_1, \ldots, v_n)$ by performing the following step for $i$ from 1 to $l$.

- (Step $i$) Set $v_{n-\mu_i}$ equal to $\mu_i$. Moving to the left, assign the value $\mu_i$ to the first $(k_i - 1)$ unassigned components.

After performing Step $l$, we assign the value 0 to any entry $v_j$ which has not yet been given a value.

Note that in Step 1, we set $v_{n-\mu_1}, v_{n-\mu_1-1}, \ldots, v_{n-\mu_1-k_1+1}$ equal to $\mu_1$.

**Example 1.5.**

\[
g_5((2, 1, 1)) = (0, 1, 2, 1, 0) \quad g_6((3, 2, 2, 1)) = (0, 2, 3, 2, 1, 0) \quad g_6((3, 1, 1)) = (0, 0, 3, 1, 1, 0).
\]

The main result of this paper is **Theorem 3.1**. We state here our main result in the case that each $y_i = 0$. The definition of Schubert polynomial can be found in Section 2.

**Theorem 1.6.** Let $w \in \text{St}(n, k)$ be a $k$-Grassmannian permutation, as in Definition 1.2, and let $\Psi(w) = (\lambda^1, \ldots, \lambda^k)$. Adding trailing 0’s if necessary, we view each partition $\lambda^i$ as a vector in $\mathbb{Z}_{\geq 0}^{n-2}$, and set $\mu := (\binom{n-1}{2}, \binom{n-2}{2}, \ldots, \binom{2}{2}) - \sum_{i=1}^k \lambda^i$. Then when each $y_i = 0$, the renormalized steady state probability $\psi_w$ is given by

\[
\psi_w = x^\mu \prod_{i=1}^k g_n(\lambda^i),
\]

where $g_n(\lambda^i)$ is the Schubert polynomial associated to the permutation with Lehmer code $g_n(\lambda^i)$, and $g_n$ is given by Definition 1.4.

Equivalently, writing $\lambda^i = (\lambda^i_1, \lambda^i_2, \ldots)$, we have that

\[
\psi_w = x^\mu \prod_{i=1}^k s_{\lambda^i}(X_{n-\lambda^i_1}, X_{n-\lambda^i_2}, \ldots),
\]

where $s_{\lambda^i}(X_{n-\lambda^i_1}, X_{n-\lambda^i_2}, \ldots)$ denotes the flagged Schur polynomial associated to shape $\lambda^i$, where the semistandard tableaux entries in row $j$ are bounded above by $n - \lambda^i_j$. 
We illustrate Theorem 1.6 in Table 3 in the case that $n = 5$.

**Proposition 1.7.** The number of evil-avoiding permutation in $S_n$ satisfies the recurrence $e(1) = 1, e(2) = 2, e(n) = 4e(n - 1) - 2e(n - 2)$ for $n \geq 3$, and is given explicitly as

$$
e(n) = \frac{(2 + \sqrt{2})^{n-1} + (2 - \sqrt{2})^{n-1}}{2}.
$$

(1.2)

This sequence begins as 1, 2, 6, 20, 68, 232, and occurs in Sloane’s encyclopedia as sequence A006012. The cardinalities $|\text{St}(n,k)|$ also occur as sequence A331969.

**Remark 1.8.** Let $w(n,h) := (h, h-1, \ldots, 2, 1, h+1, h+2, \ldots, n) \in \text{St}(n)$. In [4, Corollary 16], Cantini gives a formula for the steady state probability of state $w(n,h)$, as a trivial factor times a product of certain (double) Schubert polynomials. Note that our main result is a significant generalization of [4, Corollary 16]. For example, for $n = 4$, Cantini’s result gives a formula for the probabilities of three states – $(1, 2, 3, 4), (1, 3, 4, 2),$ and
(1, 4, 3, 2). And for \( n = 5 \), his result gives a formula for four states – \((1, 2, 3, 4, 5), (1, 3, 4, 5, 2), (1, 4, 5, 3, 2)\), and \((1, 5, 4, 3, 2)\). On the other hand, Theorem 1.6 gives a formula for all six states when \( n = 4 \) (see Table 1) and 20 of the 24 states when \( n = 5 \). Asymptotically, since the number of special states in \( S_n \) is given by (1.2), Theorem 1.6 gives a formula for roughly \( \frac{(2 + \sqrt{2})^{n-1}}{2} \) out of the \((n-1)!\) states of \( \text{St}(n) \).

Another point worth mentioning is that the Schubert polynomials that occur in the formulas of [4] are all of the form \( S_{\sigma(a,n)} \), where \( \sigma(a,n) \) denotes the permutation \((1, a+1, a+2, \ldots, n, 2, 3, \ldots, n)\). However, many of the Schubert polynomials arising as factors of steady probabilities are not of this form. Already we see for \( n = 4 \) the Schubert polynomials \( S_{1432} \) and \( S_{1243} \), which are not of this form.

Note that it is common to consider a version of the inhomogeneous TASEP in which one allows multiple particles of each weight \( i \). This is the version studied in several of the previous references, and also in [11] (which primarily considers particles of types 0, 1 and 2). We plan to work in this generality in our subsequent work. However, since our focus here is on Schubert polynomials, we restrict to the case of permutations.

2 Background on permutations and Schubert polynomials

We let \( S_n \) denote the symmetric group on \( n \) letters, which is a Coxeter group generated by the simple reflections \( s_1, \ldots, s_{n-1} \), where \( s_i \) is the simple transposition exchanging \( i \) and \( i+1 \). We let \( w_0 = (n, n-1, \ldots, 2, 1) \) denote the longest permutation.

For \( 1 \leq i < n \), we have the divided difference operator \( \partial_i \) which acts on polynomials \( P(x_1, \ldots, x_n) \) as follows:

\[
(\partial_i P)(x_1, \ldots, x_n) = \frac{P(\ldots, x_i, x_{i+1}, \ldots) - P(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.
\]

If \( s_{i_1} \ldots s_{i_m} \) is a reduced expression for a permutation \( w \), then \( \partial_{i_1} \ldots \partial_{i_m} \) depends only on \( w \), so we denote this operator by \( \partial_w \).

**Definition 2.1.** Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) be two sets of variables, and let

\[
\Delta(x, y) = \prod_{i+j \leq n} (x_i - y_j).
\]

To each permutation \( w \in S_n \) we associate the double Schubert polynomial

\[
\mathcal{S}_w(x, y) = \partial_{w^{-1}w_0}\Delta(x, y),
\]

where the divided difference operator acts on the \( x \)-variables.
Definition 2.2. A partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is a weakly decreasing sequence of positive integers. We say that $r$ is the length of $\lambda$, and denote it $r = \text{length}(\lambda)$.

Definition 2.3. The diagram or Rothe diagram of a permutation $w$ is

$$D(w) = \{(i, j) \mid 1 \leq i, j \leq n, w(i) > j, w^{-1}(j) > i\}.$$ 

The sequence of the numbers of the points of the diagram in successive rows is called the Lehmer code or code $c(w)$ of the permutation. We also define $c^{-1}(l)$ to be the permutation whose Lehmer code is $l$. The partition obtained by sorting the components of the code is called the shape $\lambda(w)$ of $w$.

Example 2.4. If $w = (1, 3, 5, 4, 2)$ then $c(w) = (0, 1, 2, 1, 0)$ and $\lambda(w) = (2, 1, 1)$.

Definition 2.5. We say that a permutation $w$ is vexillary if and only if there does not exist a sequence $i < j < k < \ell$ such that $w(j) < w(i) < w(\ell) < w(i)$. Such a permutation is also called 2143-avoiding.

Definition 2.6. We define the flag of a vexillary permutation $w$, starting from its code $c(w)$, in the following fashion. If $c_i(w) \neq 0$, let $e_i$ be the greatest integer $j \geq i$ such that $c_j(w) \geq c_i(w)$. The flag $\phi(w)$ is then the sequence of integers $e_i$, ordered to be increasing.

Definition 2.7. Let $X_i$ denote the family of indeterminates $x_1, \ldots, x_i$. For $d_1, \ldots, d_n$ a weakly increasing sequence of $n$ integers, we define the flagged Schur function

$$s_\lambda(X_{d_1}, \ldots, X_{d_n}) = \sum_T x^{\text{type}(T)},$$

where the sum runs over the set of semistandard tableaux $T$ with shape $\lambda$ for which the entries in the $i$th row are bounded above by $d_i$.

There is also a notion of flagged double Schur polynomials. One can define them in terms of tableaux or via a Jacobi-Trudi type formula [12, Section 2.6.5].

Theorem 2.8. [12, Corollary 2.6.10] If $w$ is a vexillary permutation with shape $\lambda(w)$ and with flags $\phi(w) = (f_1, \ldots, f_m)$ and $\phi(w^{-1}) = (g_1, \ldots, g_m)$, then we have

$$\mathcal{G}_w(x; y) = s_\lambda(w)(X_{f_1} - Y_{g_m}, \ldots, X_{f_m} - Y_{g_1}),$$

i.e. the double Schubert polynomial of $w$ is a flagged double Schur polynomial.
3 Main results

Let \( w \in S_n \) be a state. In what follows, we write \( a \rightarrow b \rightarrow c \) if the letters \( a, b, c \) appear in cyclic order in \( w \). So for example, if \( w = 1423 \), we have that \( 1 \rightarrow 2 \rightarrow 3 \) and \( 2 \rightarrow 3 \rightarrow 4 \), but it is not the case that \( 3 \rightarrow 2 \rightarrow 1 \) or \( 4 \rightarrow 3 \rightarrow 2 \).

\[
xyFact(w) = \prod_{i=1}^{n-2} \prod_{k>i+1} (x_1 - y_{n+1-k}) \cdots (x_i - y_{n+1-k}). \tag{3.1}
\]

The following is our main theorem; when each \( y_i = 0 \), it reduces to Theorem 1.6.

**Theorem 3.1.** Let \( w \in \text{St}(n,k) \), and write \( \Psi(w) = (\lambda^1, \ldots, \lambda^k) \). Then the (renormalized) steady state probability is given by

\[
\psi_w = xyFact(w) \prod_{i=1}^{k} g_n(\lambda^i),
\]

where \( g_n(\lambda^i) \) is the double Schubert polynomial associated to the permutation with Lehmer code \( g_n(\lambda^i) \), and \( g_n \) is given by Definition 1.4.

We also prove the monomial factor conjecture from [10]. Suppose that \( y_i = 0 \) for all \( i \). Given a state \( w \), let \( a_i(w) \) be the number of integers greater than \((i+1)\) on the clockwise path from \((i+1)\) to \( i \). Let \( \eta(w) \) be the largest monomial that can be factored out of \( \psi_w \). The following statement was conjectured in [10, Conjecture 2].

**Theorem 3.2.** Let \( w \in \text{St}(n) \). Then

\[
\eta(w) = \prod_{i=1}^{n-2} x_i^{a_i(w) + \cdots + a_{n-2}(w)}.
\]

4 Multiline queues and semistandard tableaux

It was proved in [1] that when each \( y_i = 0 \), the steady state probabilities \( \psi_w \) for the TASEP on a ring can be expressed in terms of the multiline queues of Ferrari and Martin [9]. On the other hand, we know from Theorem 1.6 that when \( w \in \text{St}(n,1) \) (i.e. \( w^{-1} \) is a Grassmann permutation and \( w_1 = 1 \)), \( \psi_w \) equals a monomial times a single flagged Schur polynomial. In this section we will explain that result by giving a bijection between the relevant multiline queues and the corresponding semistandard tableaux.

**Definition 4.1.** Fix positive integers \( L \) and \( n \). A multiline queue \( Q \) is an \( L \times n \) array in which each of the \( Ln \) positions is either vacant or occupied by a ball. We say it has content \( \mathbf{m} = (m_1, \ldots, m_n) \) if it has \( m_1 + \cdots + m_i \) balls in row \( i \) for \( 1 \leq i \leq n \). We number the rows from top to bottom from 1 to \( L \), and the columns from right to left from 1 to \( n \).
Definition 4.2. Given an $L \times n$ multiline queue $Q$, the bully path projection on $Q$ is, for each row $r$ with $1 \leq r \leq L - 1$, a particular matching of balls from row $r$ to row $r + 1$, which we now define. If ball $b$ is matched to ball $b_0$ in the row below then we connect $b$ and $b_0$ by the shortest path that travels either straight down or from left to right (allowing the path to wrap around the cylinder if necessary). Here each ball is assigned a class, and matched according to the following algorithm:

- All the balls in the first row are defined to be of class 1.
- Suppose we have matched all the balls in rows $1, 2, \ldots, r - 1$ and have assigned a class to all balls in rows $1, 2, \ldots, r$. We now consider the balls in rows $r$.
- Pick any order of the balls in row $r$ such that balls with smaller labels come before balls with larger labels. Consider the balls in this order; suppose we are considering a ball $b$ of class $i$ in row $r$. If there is an unmatched ball directly below $b$ in row $r + 1$, we let $M(b)$ be that ball; otherwise we move to the right in row $r + 1$ and let $M(b)$ be the first unmatched ball that we find (wrapping around from column 1 to $n$ if necessary). We match $b$ to ball $M(b)$ and say that $M(b)$ is of class $i$.
- The previous step gives a matching of all balls in row $r$ to balls below in row $r + 1$. We assign class $r + 1$ to any balls in row $r + 1$ that were not yet assigned a class. We now repeat the process and consider the balls in row $r + 1$.

After completing the bully path projection for $Q$, let $w = (w_1, \ldots , w_n)$ be the labeling of the balls read from right to the left in row $L$ (where a vacancy is denoted by $L + 1$). We say that $Q$ is a multiline queue of type $w$ and let $MLQ(w)$ denote the set of all multiline queues of type $w$. We also consider a type of row $r$ in $Q$ to be the labeling of the balls read from right to the left in row $r$ (where a vacancy is denoted by $r + 1$).

A vacancy in $Q$ is called $i$-covered if it is traversed by a path starting on row $i$, but not traversed by any path starting on row $i'$ such that $i' < i$.

See Figure 2 for an example.

![Figure 2: A multiline queue of type (1, 2, 4, 3, 5), and the corresponding semistandard tableau under the bijection in Proposition 4.7.](image)

We define a weight $wt(Q)$ for multiline queues. It was first introduced in [2].
Definition 4.3. Given an \( L \times n \) multiline queue \( Q \), let \( v_r \) be the number of vacancies in row \( r \) and let \( z_{r,i} \) be the number of \( i \)-covered vacancies in row \( r \). Set \( V_i = \sum_{j=i+1}^{L} v_j \). We define
\[
\text{wt}(Q) = \prod_{i=1}^{L-1} (x_i^{V_i}) \prod_{1 \leq i < r \leq L} (\frac{x_r}{x_i})^{z_{r,i}}.
\]

Example 4.4. The multiline queue \( Q \) in Figure 2 has a 1-covered vacancy in row 2, a 2-covered vacancy in row 3 and a 3-covered vacancy in row 4. The weight of \( Q \) is
\[
\text{wt}(Q) = x_3^{3+2+1}x_2^{2+1}x_1^{1}(\frac{x_2}{x_1})(\frac{x_3}{x_2})(\frac{x_4}{x_3}) = x_1^5x_2^3x_3x_4.
\]

The following result was conjectured in [2] and proved in [1].

Theorem 4.5. [1] Consider the inhomogeneous TASEP on a ring (with each \( y_i = 0 \)). We have
\[
\psi_w = \sum_{Q \in MLQ(w)} \text{wt}(Q).
\]

We now give a (weight-preserving up to a constant factor) bijection between multiline queues in \( MLQ(w) \) and certain semistandard tableaux, when \( w \in \text{St}(n,1) \), i.e. \( w^{-1} \) is a Grassmann permutation and \( w_1 = 1 \).

Definition 4.6. Given a partition \( \lambda = (\mu_1^{b_1}, \cdots, \mu_k^{b_k}, 0^c) \), such that \( \mu_1 > \cdots > \mu_k > 0 \) and \( b_i > 0, c \geq 0 \), we define a permutation \( w(\lambda) \) as follows. Identify \( \lambda \) with the lattice path from \( (\mu_1, \sum_{i=1}^{k} (b_i + c)) \) to \((0,0)\) that defines the southeast border of its Young diagram. Label the vertical steps of the lattice path from 1 to \( k \) from top to bottom, and then the horizontal steps in increasing order from right to left starting from \( k + 1 \). Reading off the numbers along the lattice path gives \( w(\lambda) \). See Figure 3.

![Figure 3: The partition \( \lambda = (2,2,1) \) and \( w(\lambda) = (1,2,4,3,5) \).](image)

Proposition 4.7. Given a partition \( \lambda = (\mu_1^{b_1}, \cdots, \mu_k^{b_k}, 0^c) \) as in Definition 4.6, let \( d = (d_1, \cdots, d_k) \) be the numbers assigned to horizontal steps right after vertical steps in the construction of \( w(\lambda) \). For example, in Figure 3, \( d = (4,5) \). Let \( d' \) be the vector
\[
d' = (d_1 - b_1, \cdots, d_1 - 1, d_2 - b_2, \cdots, d_2 - 1, \cdots, d_k - b_k, \cdots, d_k - b_k).
\]
Then there exists a bijection $f: \text{MLQ}(w) \rightarrow \text{SSYT}(\lambda, d')$ such that $wt(Q) = Kx^{\text{type}(f(Q))}$ for some monomial $K$, where $\text{SSYT}(\lambda, d')$ is the set of semistandard tableaux with shape $\lambda$ for which the entries in the $i$th row are bounded above by $d'_i$. In particular, we have

$$\psi_{w(\lambda)} = \sum_{Q \in \text{MLQ}(w(\lambda))} wt(Q) = K \sum_{T \in \text{SSYT}(\lambda, d')} x^{\text{type}(T)} = Ks_\lambda(X_{d'_1}, X_{d'_2}, \ldots).$$

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