Harmonic Polynomials Via Differentiation

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Abstract. It is well-known that if \( p \) is a homogeneous polynomial of degree \( k \) in \( n \) variables, \( p \in \mathcal{P}_k \), then the ordinary derivative \( p(\nabla)(r^2-n) \) has the form \( A_{n,k}Y(x)r^{2-n-2k} \) where \( A_{n,k} \) is a constant and where \( Y \) is a harmonic homogeneous polynomial of degree \( k, Y \in \mathcal{H}_k \), actually the projection of \( p \) onto \( \mathcal{H}_k \). Here we study the distributional derivative \( p(\nabla)(r^2-n) \) and show that the ordinary part is still a multiple of \( Y \), but that the delta part is independent of \( Y \), that is, it depends only on \( p-Y \). We also show that the exponent \( 2-n \) is special in the sense that the corresponding results for \( p(\nabla)(r^\alpha) \) do not hold if \( \alpha \neq 2-n \).

Furthermore, we establish that harmonic polynomials appear as multiples of \( r^{2-n-2k-2k'} - 2k' \) when \( p(\nabla) \) is applied to harmonic multipoles of the form \( Y'(x)r^{2-n-2k'} \) for some \( Y' \in \mathcal{H}_k \).

Key Words: Harmonic functions, harmonic polynomials, distributions, multipoles.

AMS Subject Classifications: 46F10, 33C55

1 Introduction

It is well known [1, 7, 19] that any homogeneous polynomial of degree \( k, p \in \mathcal{P}_k \), can be decomposed, in a unique fashion, as

\[
p = Y + r^2q,
\]

(1.1)

where

\[
Y = \pi_k(p) \in \mathcal{H}_k, \quad q = \chi_k(p) \in \mathcal{P}_{k-2},
\]

(1.2)

the notation \( \mathcal{H}_k \) being used to denote the harmonic homogeneous polynomials of degree \( k \).
One can easily find the projections $\pi_k(p)$ and $\chi_k(p)$. For example, if we apply the Laplacian to (1.1) we readily obtain $\Delta p = \Delta(r^2q) = 2nq + 4(k-2)q = 2(n+2k-4)q$, so that

$$q = \frac{\Delta p}{2(n+2k-4)}, \quad Y = p - \frac{r^2\Delta p}{2(n+2k-4)}.$$  \hfill (1.3)

Interestingly, these projections appear in other, somewhat surprising places. Indeed, as explained in the section Spherical Harmonics via Differentiation of [1, Chapter 5], whenever a homogeneous differential operator of degree $k$ is applied to $r^{2-n}$ in $\mathbb{R}^n$ one obtains an expression of the form $u(x)r^{2-n-2k}$ where $u$ is not just homogeneous of degree $k$, but actually belongs to $\mathcal{H}_k$. In fact, more is true, since $u = (2-n)(-n)\cdots(-n-2k+4)Y$, that is, if $p \in \mathcal{H}_k$ and we denote $(2-n)(-n)\cdots(-n-2k+4)$ as $A_{n,k}$ then

$$p(\nabla) \left( \frac{1}{r^{n-2}} \right) = A_{n,k} \frac{Y(x)}{r^{n+2k-2}},$$  \hfill (1.4)

and in particular if $Y \in \mathcal{H}_k$ then

$$Y(\nabla) \left( \frac{1}{r^{n-2}} \right) = A_{n,k} \frac{Y(x)}{r^{n+2k-2}}.$$  \hfill (1.5)

Several further questions arise, however. First, since the function $r^{2-n}$ is singular at the origin, these formulas hold in $\mathbb{R}^n \setminus \{0\}$ but not in all $\mathbb{R}^n$, so what are the corresponding formulas for the distributional derivatives\footnote{Following Farassat [6] we denote distributional derivatives with an overbar, namely, $\nabla_i, \Delta, \partial/\partial x_i$, and so on.} $p(\nabla)(r^{2-n})$ and $Y(\nabla)(r^{2-n})$, that is, the corresponding formulas in the whole space\footnote{Distributional derivatives of this kind play an important role in Physics; the distributional derivatives $\nabla_i \nabla_j (1/r)$ were given by Frahm [8], and can be found in the textbooks [14].}. Curiously, while in general $p(\nabla)(r^{2-n})$ will contain extra terms, namely a delta part, the distributional expression $Y(\nabla)(r^{2-n})$ remains basically equal to (1.5) since $Y(\nabla)(r^{2-n})$ does not have a delta part; delta parts and ordinary parts of a distribution are explained in Section 2. We give two different proofs of the formula for $Y(\nabla)(r^{2-n})$, one by induction in Section 3 and another in Section 5. We also consider the distributional derivative $p(\nabla)(r^{2-n})$ in Section 4, showing that in general the ordinary part of this derivative depends only on $Y$, while the delta part depends only on $q$.

Furthermore, we show that harmonic polynomials are also obtained when we take the derivatives of multipoles\footnote{Such harmonic multipoles have received increasing attention in recent years [2]; see also [18].} of the form $Y'(x)/r^{2k+n-2}$ for some harmonic polynomial $Y' \in \mathcal{H}_{k'}$. Indeed we obtain formulas for the derivatives $p(\nabla) \left( p.v. \left( Y'(x)/r^{2k+n-2} \right) \right)$ of the principal value distribution $p.v. \left( Y'(x)/r^{2k+n-2} \right)$ and show that the ordinary part is a multipole of the form $Z(x)/r^{2k+2k+n-2}$ for some $Z \in \mathcal{H}_{k+k'}$.\footnote{Following Farassat [6] we denote distributional derivatives with an overbar, namely, $\nabla_i, \Delta, \partial/\partial x_i$, and so on.}