STABILITY ANALYSIS FOR A FAMILY OF DEGENERATE SEMILINEAR PARABOLIC PROBLEMS

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Abstract. This paper deals with the initial value problem for a class of degenerate nonlinear parabolic equations on a bounded domain in \( \mathbb{R}^N \) for \( N \geq 2 \) with the Dirichlet boundary condition. The assumptions ensure that \( u \equiv 0 \) is a stationary solution and its stability is analysed. Amongst other things the results show that, in the case of critical degeneracy, the principle of linearized stability fails for some simple smooth nonlinearities. It is also shown that for levels of degeneracy less than the critical one linearized stability is justified for a broad class of nonlinearities including those for which it fails in the critical case.

1. Introduction. The content of the main results established in this paper can be conveyed rather quickly through following special case. For \( N \geq 3 \), let \( B = \{ x \in \mathbb{R}^N : |x| < 1 \} \) with \( \partial B = \{ x \in \mathbb{R}^N : |x| = 1 \} \) and consider the initial value problem

\[
\frac{\partial u(x,t)}{\partial t} - \sum_{i=1}^{N} \partial_{x_i} \left( |x|^{\tau} \partial_{x_i} u(x,t) \right) = \lambda u(x,t) + k(u(x,t)) \quad \text{for} \quad x \in B \text{ and } t > 0,
\]

\[
u(x,t) = 0 \quad \text{for} \quad (x,t) \in \partial B \times (0, \infty) \quad \text{and} \quad u(x,0) = u_0(x) \quad \text{for} \quad x \in B,
\]

where \( \tau \in [0, 2], \lambda \in \mathbb{R} \) and \( k \) satisfies the condition

(K) \( k \in C^1(\mathbb{R}) \) is odd, convex on \([0, \infty)\) with \( k'(0) = 0 \) and \( k'(\infty) < \infty \).

The left hand side of (1.1) is a parabolic operator which is degenerate at \( x = 0 \) when \( \tau \in (0, 2] \). A natural phase space for this problem is

\[ F_\tau = \{ u \in W_0^{1,1}(B) : \int_B |x|^\tau |\nabla u|^2 dx < \infty \} \]

and, for all \( u_0 \in F_\tau \), there is a unique solution \( u \in C([0, \infty), F_\tau) \cap C^1((0, \infty), F_\tau) \).

Since \( k(0) = 0 \), the function \( u \equiv 0 \) is a stationary solution and the objective here is to discuss its stability. For \( \tau \in (0, 2] \) the principle of linearized stability holds.

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Thus, for $u \equiv 0$ in $F_\tau$ for the equation (1.1) is the same as that for its linearization at $u \equiv 0$,

$$\partial_t u(x,t) - \sum_{i=1}^{N} \partial_{x_i} \{ |x|^\tau \partial_{x_i} u(x,t) \} = \lambda u(x,t).$$

(1.3)

However, this is not the case for $\tau = 2$ which will therefore be referred to as critical degeneracy, the cases where $\tau \in [0,2)$ being subcritical. To be a little more precise, we note that $F_\tau$ is continuously and densely embedded in $L^2(B)$ and that a self-adjoint operator $S_\tau : D(S_\tau) \subset L^2(B) \rightarrow L^2(B)$ is defined by

$$D(S_\tau) = \{ u \in F_\tau : \nabla \cdot (|x|^\tau \nabla u) \in L^2(B) \} \text{ and } S_\tau u = -\nabla \cdot (|x|^\tau \nabla u).$$

For all $u_0 \in F_\tau$, the solution of (1.1)(1.2) has the additional property that $u \in C((0,\infty),D(S_\tau))$. Denoting the infimum of the spectrum of $S_\tau$ by $\Lambda_\tau$, the linearization (1.3) is asymptotically stable for $\lambda < \Lambda_\tau$ and unstable for $\lambda > \Lambda_\tau$ for all $\tau \in [0,2]$ and this is also true for the solution $u \equiv 0$ of (1.1) if $\tau \in [0,2)$. When $\tau = 2$ the solution $u \equiv 0$ of (1.1) is asymptotically stable for $\lambda < \Lambda_\tau - k'(\infty)$ and unstable for $\lambda > \Lambda_\tau - k'(\infty)$, where we note that the condition (K) implies that $k'(\infty) \in (0,\infty)$ unless $k \equiv 0$. As the problem is presented in greater detail many other striking differences between the critical and subcritical situations will emerge.

We now state the more general version of the initial value problem which is the subject of this paper. Let $\Omega$ be an open, bounded, connected subset of $\mathbb{R}^N$ for $N \geq 2$ such that $0 \in \Omega$ and the boundary $\partial \Omega$ is Lipschitz, as defined in [3] for example. For a function $u$ defined on $\Omega$ or $\Omega \times J$ for some interval $J$, $\nabla u$ will denote its gradient ($\partial_{x_1} u, ... , \partial_{x_N} u$) with respect to $x \in \Omega$. We consider the nonlinear initial value problem for a solution $u = u(x,t) : \overline{\Omega} \times [0,T) \rightarrow \mathbb{R}$ of

$$\partial_t u - \nabla \cdot \{ A(x) \nabla u \} + |V(x) - \lambda| u + g(x,u) = 0 \text{ in } \Omega \times (0,T),$$

(1.4)

$$\int_{\Omega} u^2 + A|\nabla u|^2 dx < \infty \text{ and } u(\cdot,0)_{|\partial \Omega} = 0 \text{ for } t \in (0,T),$$

(1.5)

$$u(x,0) = u_0(x) \text{ for } x \in \Omega,$$

(1.6)

with an initial condition $u_0 : \overline{\Omega} \rightarrow \mathbb{R}$ satisfying the compatibility condition

$$\int_{\Omega} u_0^2 + A|\nabla u_0|^2 dx < \infty \text{ and } u_0 = 0 \text{ on } \partial \Omega.$$  

(1.7)

The potential $V \in L^\infty(\Omega)$ and the nonlinear term $g$ is of higher order near 0,

i.e. $g(x,0) \equiv \partial_0 g(x,0) \equiv 0 \text{ for all } x \in \Omega,$

(1.8)

so that the linearization of (1.4) at $u \equiv 0$ is

$$\partial_t u - \nabla \cdot \{ A(x) \nabla u \} + |V(x) - \lambda| u = 0.$$  

(1.9)

It is assumed that the coefficient $A$ in the divergence term satisfies the following condition.

$$(A)_\tau \text{ A} \in C(\overline{\Omega}) \text{ with } A(x) > 0 \text{ for all } x \neq 0 \text{ and there exist } \tau \in [0,2] \text{ and } a \in (0,\infty) \text{ such that } \lim_{x \rightarrow 0} \frac{A(x)}{|x|^\tau} = a.$$  

Hence there exist constants $C_2 \geq C_1 > 0$ such that

$$C_1 |x|^\tau \leq A(x) \leq C_2 |x|^\tau \text{ for all } x \in \overline{\Omega}.$$  

(1.10)

Thus, for $\tau \in (0,2]$ (1.4) and (1.9) are degenerate parabolic equations.
Under the hypothesis (A)$_\tau$ the conditions (1.5) and (1.7) are satisfied by considering solutions of (1.4) and (1.9) having the property that $u(\cdot,t) \in H_A$ for all $t \in [0,T]$, where $H_A$ is defined as follows.

**Definition.** Setting

$$\langle u, v \rangle_A = \int_\Omega A\nabla u \cdot \nabla v \, dx$$

and

$$\|u\|_A = \left\{ \int_\Omega A|\nabla u|^2 \, dx \right\}^{1/2},$$

the space $H_A$ is the completion of the space $C_0^\infty(\Omega)$ with respect to the norm $\| \cdot \|_A$.

It is proved in Section 2 that there exists a constant $C$ such that

$$\int_\Omega u^2 dx \leq C \int_\Omega A|\nabla u|^2 \, dx$$

for all $u \in H_A$.

so $(H_A, \langle \cdot, \cdot \rangle_A, \| \cdot \|_A)$ is a real Hilbert space which is continuously embedded in $L^2(\Omega)$. Many important properties of the space $H_A$ are collected in Proposition 2.1, including the identification of elements of the completion $H_A$ using generalized derivatives. A family of weighted spaces, $F_\tau$, is defined in Section 2.1 by

$$F_\tau = \{ u \in W^{1,1}_0(\Omega) : \int_\Omega |x|^\tau |\nabla u|^2 \, dx < \infty \}$$

for $\tau \in [0,2]$ and it is shown that when (A)$_\tau$ is satisfied, $H_A = F_\tau$ except in the case $N = \tau = 2$. In fact, for $N = 2$, $F_2$ is not complete for norms equivalent to $\| \cdot \|_A$ when $A$ satisfies (A)$_2$. A second family of weighted spaces, $E_\tau$, is introduced in Section 2.2 using generalized derivatives on $\Omega \setminus \{0\}$ rather than $\Omega$. In Proposition 2.2 it is shown that for all $N \geq 2$ and $\tau \in [0,2]$, $H_A = E_\tau$ when (A)$_\tau$ is satisfied.

In Section 3 it is shown that the assumptions (A)$_\tau$ and $V \in L^\infty(\Omega)$ ensure that a self-adjoint operator $S : D(S) \subset L^2(\Omega) \to L^2(\Omega)$ is defined by

$$Su = -\nabla \cdot \{ A\nabla u \} + Vu$$

on $D(S) = \{ u \in H_A : \nabla \cdot \{ A\nabla u \} \in L^2(\Omega) \}$.

The linearized problem (1.9) can be written as

$$\frac{d}{dt} u(t) + [S - \lambda] u(t) = 0$$

and its stability depends on the position of $\lambda$ relative to the spectrum of $S$. This operator is bounded below, the space $H_A$ is the domain of $|S|^{1/2}$ and $\| \cdot \|_A$ is equivalent to the graph norm of $|S|^{1/2}$ on $H_A$. Propositions 3.1 and 3.2 deal with the spectral theory of $S$. Denoting the spectrum and essential spectrum of $S$ by $\sigma(S)$ and $\sigma_e(S)$ respectively, it is shown in Proposition 3.1 that $\sigma_e(S) = \emptyset$ if $A$ satisfies (A)$_\tau$ with $\tau \in [0,2)$, whereas Proposition 3.2 concerns the case $\tau = 2$ where $\sigma_e(S) \neq \emptyset$.

The assumptions concerning the nonlinear term $g$ in (1.4) are set out in Section 4. For a Caratheodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and a measurable function $u : \Omega \to \mathbb{R}$ let $\tilde{g}$ denote the Nemytskii operator defined by

$$\tilde{g}(u)(x) = g(x, u(x))$$

for $x \in \Omega$.

Defining $G : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$G(x,s) = \int_0^s g(x,t) \, dt,$$

let $\psi(u) = \int_\Omega G(x,u(x)) \, dx$.

when $u : \Omega \to \mathbb{R}$ is measurable and $\tilde{G}(u)$ is integrable over $\Omega$.

For a coefficient $A$ which satisfies the condition (A)$_\tau$ two types of assumption, (G1)$_\tau$ and (G2)$_\tau$, on the function $g$ are introduced in order to ensure that $\tilde{g}$ maps


$H_A$ continuously and boundedly into $L^2(\Omega)$. Since the space $H_A$ becomes bigger as $\tau$ increases, these conditions become more restrictive with increasing $\tau$. For all $\tau \in [0, 2]$, the hypothesis $(G1)_\tau$ admits functions of the form $g(x, s) = k(s)$ under suitable restrictions on $k$. In particular, if $\pm k$ satisfies the condition (K), then $g$ satisfies $(G1)_\tau$ for all $\tau \in [0, 2]$. The conditions $(G2)_\tau$ accommodate stronger growth of $g(x, s)$ as $|s| \to \infty$ provided that $g(x, s) \to 0$ as $x \to 0$ sufficiently fast. In all cases these assumptions ensure the $\hat{g} : H_A \to L^2(\Omega)$ is locally Lipschitz continuous and bounded. In fact, for $\tau \in [0, 2]$ we even have $\hat{g} \in C^1(H_A, L^2(\Omega))$, but as Lemma 4.2 shows, this is not ensured by $(G1)_2$ and, for the case where $\tau = 2$ and $g(x, s) = \pm k(s)$ with $k$ satisfying (K), $\hat{g} : H_A \to L^2(\Omega)$ is Fréchet differentiable at 0 only when $k \equiv 0$. Hence in the critical case it is important treat situations where $\hat{g} : H_A \to L^2(\Omega)$ is not Fréchet differentiable at 0. It is in such cases that the principle of linearized stability is shown to fail in Section 7.2 due to the behaviour of the energy functional $\phi_\lambda$ associated with (1.4). It is defined by

$$
\phi_\lambda(u) = \frac{1}{2} \int_\Omega A|\nabla u|^2 + (V - \lambda)u^2dx + \psi(u) \text{ for } u \in H_A,
$$

(1.17)

where $\psi$ is defined in (1.16) and in all cases $\phi_\lambda \in C^1(H_A, \mathbb{R})$ and the stationary solution $u \equiv 0$ is a critical point of $\phi_\lambda$ for all $\lambda$. In the subcritical case $\phi_\lambda \in C^2(H_A, \mathbb{R})$ but this is not true for $\tau = 2$ when $\hat{g}$ is not Fréchet differentiable at 0. Nonetheless the second variation $\phi''_\lambda(0)$ of $\phi_\lambda$ at $u \equiv 0$ is well defined in all cases and it is the quadratic form

$$
\int_\Omega A|\nabla u|^2 + (V - \lambda)u^2dx,
$$

which is positive definite on $H_A$ if and only if $\lambda < m$ where $m = \inf \sigma(S)$. However, when $\tau = 2$ this does not imply that $u \equiv 0$ is a local minimum of $\phi_\lambda$ for $\lambda < m$. This issue is investigated in [34] and the criteria established there form the basis for our analysis of stability in the critical case.

The hypotheses of Sections 2 to 4 allow us to deal with strong solutions of the initial value problem (1.4) to (1.7) as a dynamical system on the space $H_A$. More precisely, for every $u_0 \in H_A$ there exist $T > 0$ and a unique function $u \in C([0, T), H_A) \cap C((0, T), D(S)) \cap C^1((0, T), H_A)$ such that $u(0) = u_0$ and the equation (1.4) is satisfied in the sense that

$$
\frac{d}{dt} u(t) + [S - \lambda]u(t) + \hat{g}(u(t)) = 0 \text{ for all } t \in (0, T),
$$

(1.18)

where this equation expresses the equality of elements in $L^2(\Omega)$. These conclusions are derived from the theory of analytic semigroups in Section 6. The underlying results are recalled in a simple but convenient abstract setting in Section 5.1 and then the relevant definitions of stability and instability of the solution $u \equiv 0$ are recalled in the same abstract setting in Section 5.2. In preparation for the treatment of the critical case in Section 7.2, Theorem 5.3 collects several conclusions about stability and instability obtained using the energy as a Lyapunov function. Although this is a classical approach, existing references do not seem to cover the precise setting required here so proofs are given.

Criteria for the stability and instability of the solution $u \equiv 0$ of the problem (1.4) to (1.7) are established in Section 7.1 for the subcritical case and in Section 7.2 for the critical one. The claims made about the special case presented at the beginning of this introduction are justified by Theorem 7.2 for $\tau \in [0, 2)$ and by Corollary 7.4
for \( \tau = 2 \), since Proposition 6.2 shows that \( \inf \sigma(S) = \inf \sigma_e(S) \) when \( \Omega = B(0,1) \), \( A(x) = |x|^2 \) and \( V \equiv 0 \).

Having summarized the main contents of this paper, it is time to comment on some limitations of this study. The stability analysis of stationary solutions is restricted to the solution \( u \equiv 0 \) and the case \( N = 1 \) is excluded. Concerning the first point, we observe that in the critical case which is our main focus, it is shown in Proposition 6.2 that in some circumstances the set of all stationary solutions is \( \{ (\lambda, u \equiv 0) : \lambda \in \mathbb{R} \} \). For the second issue we refer to [33] where it is shown that, in the critical case, it is more natural to deal with the problem on the intervals \((p,0)\) and \((0,q)\) separately rather than on \( \Omega = (p,q) \) where \( p < 0 < q \). It is in this form that bifurcation is studied for the stationary problem in [35] and we claim that the present approach to the parabolic problem could easily be adapted to that setting.

Let us end this introduction with a few remarks about the position of the problem treated here with respect to the general theory of degenerate second order elliptic and parabolic problems. The classical treatment of degenerate linear elliptic operators of second order in general divergence form is mainly due to Kruskov [26], Murthy and Stampacchia [29] and Trudinger [36]. When specialized to the operator

\[
-\nabla \cdot \{ A \nabla \} + V - \lambda u
\]

with \( A \) satisfying (A)\( \tau \) and \( V \in L^\infty(\Omega) \), their main hypotheses are satisfied if and only if \( 0 \leq \tau < 2 \) and their results deal with weak solutions in \( H_A = F_\tau \) of the elliptic Dirichlet problem

\[
-\nabla \cdot \{ A \nabla u \} + (V - \lambda)u = 0.
\]

(1.19)

Concerning the eigenvalue problem (1.19) it is known from [26], [29] and [36] that for \( \tau < 2 \), eigenfunctions of \( -\nabla \cdot \{ A \nabla \} + V \) are Hölder continuous and bounded on \( \Omega \) and the spectrum is discrete. The case \( \tau = 2 \) is not covered by these results and many of the conclusions obtained there fail in this case. As is shown in Section 3.2 below and in greater generality in Section 4 of [33], eigenfunctions may become unbounded as \( x \to 0 \) and the spectrum is no longer discrete, \( \Gamma_0^2 \) being the infimum of the essential spectrum. For the case where \( A \) satisfies (A)\( \tau \) with \( 0 \leq \tau < 2 \), existence theory for the linear parabolic equation (1.9) is treated by Ivanov in [25] in a setting analogous to that used for the stationary problem in [26], [29] and [36]. Again, the results in [25] do not cover the case \( \tau = 2 \), which is the subject of the present paper. Properties, such as Hölder continuity, of solutions of degenerate parabolic equations are discussed in [11, 12, 22, 23]. See also [1] and [14] for weak solutions of degenerate parabolic equations containing singular coefficients.

2. Properties of space \( H_A \). In this section the space \( H_A \) is considered in some detail under the assumption that \( A \) satisfies the condition (A)\( \tau \). The main objectives here are to characterize the elements of this completion and to establish some embeddings which are crucial for the subsequent discussion of the nonlinear term in (1.4). It turns out that the case \( N = \tau = 2 \) presents some particular features so it is dealt with in a separate subsection where a family of spaces \( E_\tau \) similar to those used in [17, 18, 34] is introduced.

Spaces like \( H_A \) and \( F_\tau \) appear in almost all work concerning degenerate elliptic problems often in much greater generality than treated here. See [26, 29, 36] for linear problems and [8, 9, 17, 18, 28] for nonlinear problems having features similar to the stationary version of (1.4) and (1.5). The following exposition contains some information about the critical case not found in these references and it incorporates the Caffarelli-Kohn-Nirenberg estimates in a way which will be useful when dealing with the nonlinear term in (1.4).
The following notation will be used repeatedly. For \( x \in \mathbb{R}^N \), \( r = |x| \) and, for \( \varepsilon > 0 \), \( \Omega_{\varepsilon} = \{ x \in \Omega : r > \varepsilon \} \), where it is always tacitly assumed that \( \varepsilon < d(0, \partial \Omega) \).

2.1. The spaces \( F_\tau \) and \( H_\alpha \). The first step is to establish some crucial properties of the space \( F_\tau \) defined by

\[
F_\tau = \{ u \in W^{1,1}_0(\Omega) : \int_\Omega r^\tau |\nabla u|^2 dx < \infty \}
\]

with

\[
\langle u, v \rangle_\tau = \int_\Omega r^\tau \nabla u \cdot \nabla v dx \quad \text{and} \quad \| u \|_\tau = \{ \int_\Omega r^\tau |\nabla u|^2 dx \}^{1/2}.
\]

\( F_\tau \) is a Banach space when considered with the norm \( \| \nabla u \|_{L^1} + \| u \|_\tau \). For \( u \in F_\tau \) and \( 1 \leq p < 2 \),

\[
\int_\Omega |\nabla u|^p dx = \int_\Omega r^{-\frac{Np}{2}} (r^\tau |\nabla u|^2)^{\frac{p}{2}} dx \leq \{ \int_\Omega r^{-\frac{Np}{2}} dx \}^{1/q} \| u \|_{F_\tau}^p,
\]

by Hölder’s inequality where \( q = 2/(2-p) \). Since \( r^{-\frac{Np}{2}} \in L^1(\Omega) \) for \( p < 2N/(N+\tau) \) and \( 2N/(N+\tau) > 1 \) except in the case \( N = \tau = 2 \), it follows that for \( 1 \leq p < 2N/(N+\tau) \) there exists a constant \( C \) such that \( \| \nabla u \|_{L^p} \leq C \| u \|_\tau \) for all \( u \in F_\tau \).

In the case \( \tau = 0 \), we have the better conclusion that \( \| \nabla u \|_{L^p} = \| u \|_0 \) for all \( u \in F_0 \). Hence, except for the case \( N = \tau = 2 \), the norm \( \| \cdot \|_\tau \) on \( F_\tau \) is equivalent to \( \| \nabla \cdot \|_{L^1} + \| \cdot \|_\tau \) and \( (F_\tau, \| \cdot \|_\tau) \) is a Hilbert space.

From (2.1) the following properties of \( F_\tau \) can be deduced, the case \( N = \tau = 2 \) being excluded, of course. Using Lemmas 7.12 and 7.14 of [20], there exists a constant \( C \) such that \( \| u \|_{L^p} \leq C \| \nabla u \|_{L^p} \) for all \( u \in W^{1,p}_0(\Omega) \) provided that \( p < N \) and \( 1 \leq p < q < Np/(N-p) \). Combining these observations it follows that \( \| \cdot \|_\tau \) is a norm on \( F_\tau \) with \( (F_\tau, \| \cdot \|_\tau) \) being continuously embedded in \( W^{1,p}(\Omega) \) for \( 1 \leq p < 2N/(N+\tau) \) and in \( L^q(\Omega) \) for \( 1 \leq q < 2N/(N-2+\tau) = 2_\tau^\ast \), except in the case \( N = \tau = 2 \). Furthermore, \( F_0 = W^{1,1}_0(\Omega) \cap W^{1,2}(\Omega) = H_0^1(\Omega) \) since \( \partial \Omega \) is Lipschitz.

Since \( \Omega \) is bounded, \( r^\alpha \in L^1(\Omega) \) for \( \alpha > -N \) and so the embedding of \( F_\tau \) in \( L^q(\Omega) \) and Hölder’s inequality show that there exists a constant \( C = C(\Omega, \tau, \gamma, p) \) such that

\[
\| r^\tau u \|_{L^p} \leq C \| u \|_\tau
\]

provided that \( -\frac{1}{2}(N+2-\tau) < \gamma \leq 0 \) and \( 1 \leq p < \frac{2N}{N-2+\tau-2\tau} \).

Some of these conclusions are sharpened in the following result which also covers the case \( N = \tau = 2 \) which was excluded from the preceding discussion.

**Proposition 2.1.** Consider \( N \geq 2 \) and \( \tau \in [0,2] \).

(1) Except for the case \( N = 2 \) with \( \tau = 0 \), \( r^\tau(u) \in L^2(\Omega) \) for all \( u \in F_\tau \) with

\[
(N-2+\tau)\| r^\tau u \|_{L^2} \leq 2 \| u \|_\tau \quad \text{and} \quad \int_\Omega r^{\tau-2} u^2 dx = -2 \int_\Omega r^{\tau-2} u(x \cdot \nabla u) dx.
\]

(2) \( C_\infty^\infty(\Omega) \) is a dense subset of \( (F_\tau, \| \cdot \|_\tau) \).

(3) Setting \( v = r^\tau u \) for \( u \in F_\tau \),

\[
v \in H_0^1(\Omega) \quad \text{and} \quad \int_\Omega |\nabla v|^2 dx + \frac{\tau}{2} (N-2+\tau) \int_\Omega r^{\tau-2} u^2 dx = \| u \|_{\tau}^2.
\]
(4) There exists a constant $C$ such that,
\[ \|r^\gamma u\|_{L^p} \leq C\|u\|_r \] for $\gamma - 2 \leq 2\gamma \leq \tau$, where
(2.5)
(i) except for $N = 2$ with $\tau = 0$, $1 \leq p \leq \frac{2N}{N-2+\tau-2\gamma}$ ($1 \leq p < \infty$ for $N = 2$ with $2\gamma = \tau$) and
(ii) for $N = 2$ with $\tau = 0$, $1 \leq p < \frac{2}{\gamma}$ ($1 \leq p < \infty$ for $\gamma = 0$).

Thus, setting $2^*_\tau = \frac{2N}{N-2+\tau}$, ($2^*_0 = \infty$ for $N = 2$ with $\tau = 0$) it follows that $(F_r, \| \cdot \|_r)$ is continuously embedded in $(L^p(\Omega), \| \cdot \|_{L^p})$ for $1 \leq p \leq 2^*_\tau$ in all cases except $N = 2$ with $\tau = 0$ where the embedding holds for $1 \leq p < \infty$.

In particular, $(F_r, \| \cdot \|_r)$ is continuously embedded in $L^2(\Omega)$ since $2^*_\tau \geq 2$ in all cases, with equality only for $\tau = 2$.

(5) For $1 \leq p < 2^*_\tau$, $(F_r, \| \cdot \|_r)$ is compactly embedded in $(L^p(\Omega), \| \cdot \|_{L^p})$.

(6) For any function $A$ satisfying the condition $(A)_\tau$, $F_r$ can be identified with a subset of $H_A$ and then the norms $\| \cdot \|_r$ and $\| \cdot \|_A$ are equivalent on $F_r$. Except for the case $N = \tau = 2$, $F_r = H_A$ and $(F_r, \| \cdot \|_r)$ is a Hilbert space.

(7) For $N = 2$, $(F_2, \| \cdot \|_2)$ is not complete but its completion is isomorphic to the Hilbert space $(H_A, \| \cdot \|_A)$ for any $A$ satisfying $(A)_2$. For such $A$, (2.4) holds for $u \in H_A$ and (2.5) extends to $u \in H_A$ for $\gamma \in (0, 1)$ with $1 \leq p \leq \frac{2}{1-\gamma}$ and for $\gamma = 1$ with $1 \leq p < \infty$.

Remark 2.1. The inequality (2.2) is proved directly for all $u \in F_r$ and then used in the proof of (2). These proofs extend without change to cover the spaces $E_r$ defined in Section 2.2. The inequality (2.5) is due to Caffarelli, Kohn and Nirenberg [7] for $u \in C_0^\infty(\mathbb{R}^N)$. It can be extended to all of $F_r$ by part (2) and (2.2) is then a special case of (2.5). See also [31] for very general results of this kind, including necessary conditions for such inequalities and many historical remarks. The C-K-N inequalities have inspired a large amount of work on degenerate and singular nonlinear elliptic equations, [10, 2, 6] being early examples of this.

Remark 2.2. For $N = 2$, the completion of $F_2$ mentioned in part (7) is identified concretely in Section 2.2 as the space $E_2$.

Proof. Consider $\varepsilon_0 \in (0, d(0, \partial \Omega))$ and for $\varepsilon \in (0, \varepsilon_0)$, let
\[ \Omega_\varepsilon = \{ x \in \Omega : |x| > \varepsilon \} \] and $S(\varepsilon) = \{ x \in \mathbb{R}^N : |x| = \varepsilon \}$.

(1) If $u \in F_r$, its restriction belongs to $W^{1,2}(\Omega_\varepsilon)$ and hence its trace is a bounded linear map into $L^2(\partial \Omega_\varepsilon)$. Then
\[ 2 \int_{\Omega_\varepsilon} r^{\gamma-2} u(x \cdot \nabla u) \, dx = \int_{\Omega_\varepsilon} r^{\gamma-2} x \cdot \nabla (u^2) \, dx \]
\[ = - \int_{S(\varepsilon)} r^{\gamma-1} u^2 \, ds - \int_{\Omega_\varepsilon} (N - 2 + \tau) r^{\gamma-2} u^2 \, dx \]  
(2.6)
(2.7)
since the trace of $u$ is zero on $\partial \Omega$. Hence
\[ (N - 2 + \tau) \int_{\Omega_\varepsilon} r^{\gamma-2} u^2 \, dx \leq -2 \int_{\Omega_\varepsilon} r^{\gamma-2} u (x \cdot \nabla u) \, dx \leq 2 \int_{\Omega_\varepsilon} (r^{\gamma-1}|u|)(r^{\gamma}|\nabla u|) \, dx \]
\[ \leq 2 \left( \int_{\Omega_\varepsilon} r^{\gamma-2} u^2 \, dx \right)^{1/2} \left( \int_{\Omega_\varepsilon} r^{\gamma}|\nabla u|^2 \, dx \right)^{1/2} \]
and (2.2) follows by letting $\varepsilon \to 0+$ since $r^\gamma |\nabla u|^2 \in L^1(\Omega)$. But then
\[ |r^{\gamma-2} u(x \cdot \nabla u)| \leq r^{\gamma-1}|u|r^{\gamma}|\nabla u| \]
also belongs to $L^1(\Omega)$ and (2.7) now shows that $\lim_{x \to 0} \int_{S(x)} r^{-1} u^2 ds$ exists and is finite. Since $r^{-2} u^2 \in L^1(\Omega)$ and

$$\int_{B(0,\varepsilon_0)} r^{-2} u^2 dx = \int_0^{\varepsilon_0} r^{-2} \int_{S(r)} u^2 ds dr,$$

this limit must be zero and (2.3) follows from (2.7).

(2) As noted already, $F_0 = W^{1,2}_0(\Omega)$ and $C_0^\infty(\Omega)$ is a dense subspace since $\partial\Omega$ is Lipschitz.

Suppose henceforth that $\tau \in (0, 2]$. To show that $C_0^\infty(\Omega)$ is dense in $F_\tau$, choose $\xi \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \xi(x) \leq 1$ for all $x \in \mathbb{R}^N$ with $\xi(x) = 1$ for $|x| \geq 2$ and $\xi(x) = 0$ for $|x| \leq 1$. Consider integers $n \geq 2/\varepsilon$ where $\varepsilon \in (0, \varepsilon_0)$ and set $\xi_n(x) = \xi(nx)$. Note that $|\nabla \xi_n(x)| \leq n\|\nabla\xi\|_{L^\infty}$ for all $x$. Then for any $u \in F_\tau$, $\xi_n u$ belongs to $H^1_0(\Omega)$ and

$$\|u - u_n\|^2_\tau = \int_\Omega \tau^2 (1 - \xi_n)^2 u^2 dx \leq 2 \int_\Omega \tau^2 (1 - \xi_n)^2 u^2 dx.$$ But

$$\int_\Omega \tau^2 (1 - \xi_n)^2 u^2 dx = \int_{|x| < \frac{\tau}{n}} \tau^2 (1 - \xi_n)^2 u^2 dx$$

$$\leq n^2 \tau^2 \int_{\frac{\tau}{n} < |x| < \frac{2\tau}{n}} \tau^2 (1 - \xi_n)^2 u^2 dx \leq 4 \tau^2 \int_{\frac{\tau}{n} < |x| < \frac{2\tau}{n}} \tau^{-2} u^2 dx$$

where

$$\lim_{n \to \infty} \int_{\frac{\tau}{n} < |x| < \frac{2\tau}{n}} \tau^{-2} u^2 dx = 0$$

since $\tau^{-2} u^2 \in L^1(\Omega)$ by (2.2).

Also,

$$\int_\Omega (1 - \xi_n)^2 |\nabla u|^2 dx = \int_{|x| \leq \frac{\tau}{n}} (1 - \xi_n)^2 |\nabla u|^2 dx \leq \int_{|x| \leq \frac{\tau}{n}} \tau^2 |\nabla u|^2 dx$$

where

$$\lim_{n \to \infty} \int_{|x| \leq \frac{\tau}{n}} \tau^2 |\nabla u|^2 dx = 0$$

since $\int_\Omega \tau^2 |\nabla u|^2 dx < \infty$.

Hence $\xi_n u \to u$ for the norm $\| \cdot \|_\tau$, showing that $H^1_0(\Omega)$ is dense in $(F_\tau, \| \cdot \|_\tau)$.

But $C^\infty_0(\Omega)$ is dense in $H^1_0(\Omega)$ with its Dirichlet norm

$$\|u\|_{H^1_0(\Omega)} = \{ \int_\Omega |\nabla u|^2 dx \}^{1/2}$$

and, since $\Omega$ is bounded, there exists a constant $D$ such that

$$\int_\Omega \tau^2 |\nabla u|^2 dx \leq D \int_\Omega |\nabla u|^2 dx \text{ for all } u \in H^1_0(\Omega).$$

This proves that $C^\infty_0(\Omega)$ is dense in $(F_\tau, \| \cdot \|_\tau)$.

(3) It suffices to consider the case $\tau \in (0, 2]$ since it was already shown that $F_0 = H^1_0(\Omega)$ and (2.4) is trivial for $\tau = 0$.

Noting that

$$|\nabla v|^2 = \frac{1}{2} \tau^2 r^{-2} u^2 + 2r^2 \tau^{-2} u^2 + \tau^2 r^{-2} u^2 (x \cdot \nabla u),$$

the identity (2.4) is obtained for $\tau \in (0, 2]$ using (2.3). It implies that $v \in W^{1,2}(\Omega)$ since $\Omega$ is bounded. But the trace of $v$ on $\partial\Omega$ is zero so $v \in H^1_0(\Omega)$ since $\partial\Omega$ is Lipschitz.
(4) For \( u \in C_0^\infty(\Omega) \) this is a special case of the Caffarelli, Kohn, Nirenberg inequalities [7] or [10]. By part (2) they also hold for \( u \in F_\tau \).

(5) To establish the compactness of the embedding it is enough to show that if \( \{ u_n \} \) is a sequence converging weakly to zero in \( F_\tau \) as \( n \to \infty \), then \( \| u_n \|_{L^p} \to 0 \) as \( n \to \infty \) for \( 1 \leq p < 2^*_\tau \). Since \( F_\tau \) is continuously embedded in \( L^2(\Omega) \), \( L^2(\Omega)^* \subset F^*_\tau \) and hence \( \{ u_n \} \) converges weakly to zero in \( L^2(\Omega) \). Consider \( \varepsilon \in (0, \varepsilon_0] \). If \( \{ u_n \} \) does not converge weakly to zero in \( W^{1,2}(\Omega) \), there exist \( f \in W^{1,2}(\Omega)^* \), a subsequence \( \{ u_{n_k} \} \) and \( \delta > 0 \) such that \( |f(u_{n_k})| \geq \delta \) for all \( n_k \). Passing to a further subsequence if necessary, we can suppose that \( \{ u_{n_k} \} \) converges weakly to an element \( u \) in \( W^{1,2}(\Omega) \). Thus \( \{ u_{n_k} \} \) converges weakly to \( u \) in \( L^2(\Omega) \) and so \( u = 0 \) a.e. on \( \Omega \) since \( \{ u_n \} \) converges weakly to zero in \( L^2(\Omega) \) and hence also on \( L^2(\Omega) \). But then \( f(u_{n_k}) \to f(u) = f(0) = 0 \) as \( n_k \to \infty \), contradicting the choice of \( \delta \). Hence \( \{ u_n \} \) converges weakly to zero in \( W^{1,2}(\Omega) \) and therefore \( \| u_n \|_{L^p(\Omega)} \to 0 \) as \( n \to \infty \) for \( 1 \leq p < 2N/(N-2) \). Noting that \( 2^*_\tau \leq 2N/(N-2) \) it follows that

\[
\limsup_{n \to \infty} \| u_n \|_{L^p}^p = \limsup_{n \to \infty} \int_{B(0, \varepsilon)} |u_n|^p \, dx
\]

for \( 1 \leq p < 2^*_\tau \). But for \( 1 \leq p < q < 2^*_\tau \),

\[
\int_{B(0, \varepsilon)} |u_n|^p \, dx \leq |B(0, \varepsilon)|^{(q-p)/q} \| u_n \|_q^p
\]

by Hölder’s inequality. The weak convergence of \( \{ u_n \} \) in \( F_\tau \) and (2.5) with \( \gamma = 0 \) imply that this sequence is bounded in \( L^q(\Omega) \) and so there exists a constant \( M \) such that

\[
\int_{B(0, \varepsilon)} |u_n|^p \, dx \leq M \varepsilon^{N(q-p)/q} \text{ for all } n.
\]

Letting \( \varepsilon \to 0^+ \) shows that \( \| u_n \|_{L^p} \to 0 \) as \( n \to \infty \), completing the proof.

(6) Identifying \( C_0^\infty(\Omega) \) with a subspace of \( H_A \), the equivalence of the norms on \( C_0^\infty(\Omega) \) follows immediately from (1.10). Then part (2) implies that \( F_\tau \subset H_A \).

Exclude now the case \( N = \tau = 2 \) and consider a Cauchy sequence \( \{ u_n \} \) in \( (F_\tau, \| \cdot \|_\tau) \).

It follows from (2.1) that it is also a Cauchy sequence in \( W^{1,1}_0(\Omega) \), showing that \( F_\tau \) is complete and hence \( F_\tau = H_A \).

(7) Fix \( \varepsilon \in (0, \varepsilon_0) \) with \( \varepsilon < 1 \) and consider the function

\[ u(x) = (\varepsilon - r)/(r \ln r) \text{ for } 0 < r \leq \varepsilon \text{ and } u(x) = 0 \text{ on } \Omega_\varepsilon, \]

and the sequence of truncations

\[ u_n(x) = u\left( \frac{x}{n|x|} \right) = \frac{1 - n \varepsilon}{\ln n} \text{ for } 0 < r \leq \frac{1}{n} \text{ and } u_n(x) = u(x) \text{ for } x \in \Omega_{\varepsilon}, \]

where \( n \in \mathbb{N} \) with \( n > 1/\varepsilon \). Then \( u_n \subset H^1_0(\Omega) \subset F_2 \) and a short calculation shows that

\[
\int_{\Omega} r^2 |\nabla (u_n - u)|^2 \, dx = \int_{B(0,1/n)} r^2 |\nabla u|^2 \, dx \to 0 \text{ as } n \to \infty.
\]

It follows that \( \{ u_n \} \) is a Cauchy sequence in \( (F_2, \| \cdot \|_2) \) and if it converges, its limit must be \( u \). However, a similar calculation shows that

\[
\int_{\Omega_{\frac{1}{n}}} |\nabla u| \, dx = \int_{\frac{1}{n} < |x| < \varepsilon} |\nabla u| \, dx \to \infty \text{ as } n \to \infty,
\]

so \( u \not\in W^{1,1}(\Omega) \) and hence \( \{ u_n \} \) does not have a limit in \( F_2 \). Therefore \( (F_2, \| \cdot \|_2) \) is not complete and so \( F_2 \neq H_A \) for \( N = 2 \).
2. The spaces \( E_\tau \) and \( H_A \). In Section 2.1 it is shown that for a coefficient \( A \) satisfying (A)\(_\tau\), \( H_A = F_\tau \), except in the case \( N = \tau = 2 \). In order to obtain a characterisation of the elements of \( H_A \) using generalized derivatives when \( N = \tau = 2 \), we begin by introducing a space \( E_\tau \) consisting of functions having generalized derivatives on \( \Omega^* = \Omega \setminus \{0\} \). To put the case \( N = \tau = 2 \) in proper perspective we consider \( N \geq 2 \) and \( \tau \in [0, 2] \).

Setting

\[
E_\tau = \{ u \in L^2(\Omega^*) : r^{\tau/2} |\nabla u| \in L^2(\Omega^*) \},
\]

where \( \nabla u \in [L_{loc}^{1,\infty}(\Omega^*)]^N \) consists of generalized derivatives of \( u \) on \( \Omega^* \), it is easy to check that \( E_\tau \) with the scalar product

\[
(u, v)_\tau = \int_{\Omega^*} uv + |x|^\tau \nabla u \cdot \nabla v \, dx
\]

is a Hilbert space and (by a slight abuse of notation) \( E_\tau \subset W^{1,2}(\Omega_\tau) \). Then \( E_\tau \) coincides with the space \( H \) defined in [17, 18] for the case \( N \geq 3 \) and \( \tau = 2 \). Now let

\[
E_\tau = \{ u \in E_\tau : \Gamma u = 0 \}
\]

where \( \Gamma : W^{1,2}(\Omega_\tau) \to L^2(\partial \Omega) \) is the usual trace operator, see [3] A 5.7 for example. The continuity of \( \Gamma \) ensures that \( (E_\tau, (\cdot, \cdot)_\tau) \) is a Hilbert space.

Since \( F_\tau \) is continuously embedded in \( L^2(\Omega) \) by Proposition 2.1, \( F_\tau \subset E_\tau \). The exact relationships between the spaces \( E_\tau, F_\tau \) and \( H_A \) is established in the following result. Note that part (ii) shows that if \( u \in E_\tau \) then \( u \) admits generalized derivatives on \( \Omega \), except in the case \( N = \tau = 2 \).

**Proposition 2.2.** Consider \( N \geq 2 \) and \( \tau \in [0, 2] \).

(i) The inequality (2.2) holds for all \( u \in E_\tau \), except in the case \( N = 2 \) with \( \tau = 0 \).

(ii) \( E_\tau = F_\tau \) except in the case \( N = \tau = 2 \).

(iii) In all cases, \( \| \cdot \|_\tau \) is a norm on \( E_\tau \) equivalent to \( (\cdot, \cdot)^{1/2}_\tau \) and \( C^\infty(\Omega) \) is dense in \( (E_\tau, \| \cdot \|_\tau) \).

(iv) If \( A \) satisfies the condition (A)\(_\tau\), \( H_A \) can be identified with \( E_\tau \) and then \( H_A \) is continuously embedded in \( L^2(\Omega) \). Hence \( H_A = E_\tau = F_\tau \) except in the case \( N = \tau = 2 \) where \( H_A = E_2 \neq F_2 \).

(v) With a slight abuse of notation, \( E_\tau \) is continuously embedded in \( W^{1,2}(\Omega_\tau) \) and weak convergence in \( E_\tau \) implies both weak convergence in \( W^{1,2}(\Omega_\tau) \) and strong convergence in \( L^p(\Omega_\tau) \) for \( 1 \leq p < 2^* \).

(vi) \( E_2 \) is continuously but not compactly embedded in \( L^2(\Omega) \).

**Remark 2.3.** For \( \alpha \in \mathbb{R} \), define a function \( w_\alpha \) by

\[
w_\alpha(x) = 0 \text{ for } x \in \Omega_\tau \text{ and } w_\alpha(x) = r^\alpha - \varepsilon^\alpha \text{ for } 0 < r < \varepsilon.
\]

Noting that for \( 0 < r < \varepsilon \), \( r^2 |\nabla w_\alpha(x)|^2 = \alpha^2 r^{2\alpha} \), we see that \( w_\alpha \in E_2 \) if and only if \( \alpha > -N/2 \) whereas \( w_\alpha \in L^p(\Omega) \) if and only if \( \alpha > -N/p \). Hence \( E_2 \notin L^p(\Omega) \) if \( p > 2 \). On the other hand, by part (i) of the proposition, \( N\|u\|_{L^2} \leq 2\|u\|_2 \) for all \( u \in E_2 \) and hence, if \( A \) satisfies the condition (A)\(_2\),

\[
\|u\|_{L^2} \leq \frac{2}{N\sqrt{\mathcal{C}_1}} \|u\|_A \text{ for all } u \in H_A,
\]

by (1.10).

**Proof.** For part (i) it suffices to copy the proof of part (1) of Proposition 2.1.
(ii) The inequality (2.1) shows that $|\nabla u| \in L^1(\Omega^*)$ for $u \in E_{\tau}$ except in the case $N = \tau = 2$. The proof of Lemma 6.1(i) in [17] now shows that $u$ admits a generalized derivative on $\Omega$. It follows that $E_{\tau}$ is continuously embedded in $W^{1,1}(\Omega)$ and hence that $E_{\tau} = F_{\tau}$.

(iii) To establish the equivalence of the norms it suffices to prove that there is a constant $C = C(N, \tau)$ such that $\|u\|_{L^2} \leq C\|u\|_{\tau}$ for all $u \in E_{\tau}$. For the case $N = \tau = 2$, this follows from part (i) and from (2.5) in all other cases since then $E_{\tau} = F_{\tau}$. The density of $C_0^\infty(\Omega)$ in $E_{\tau}$ is now implied by part (ii) of Proposition 2.1 except in the case $N = \tau = 2$. However, since the inequality (2.2) holds for $u \in E_2$, the proof of Proposition 2.1(ii) can be repeated simply replacing $F_{\tau}$ by $E_2$.

(iv) Combining part (ii) with parts (6) and (7) of Proposition 2.1, it is sufficient to show that in the case $N = \tau = 2$, $E_2$ is dense in $(E_2, \|\cdot\|_2)$. This follows from part (iii) since $C_0^\infty(\Omega) \subset E_2$.

(v) Recalling the convention concerning $\Omega_\varepsilon$, we have that

$$\int_{\Omega_\varepsilon} u^2 + |\nabla u|^2 \, dx \leq \max\{1, \varepsilon^{-\tau}\} \int_{\Omega} u^2 + r^\tau |\nabla u|^2 \, dx$$

for all $u \in E_{\tau}$ and so $E_{\tau}$ is continuously embedded in $W^{1,2}(\Omega_\varepsilon)$. The argument at the beginning of the proof of part (5) of Proposition 2.1 shows that weak convergence in $E_{\tau}$ implies weak convergence in $W^{1,2}(\Omega_\varepsilon)$. Strong convergence in $L^p(\Omega_\varepsilon)$ for $1 \leq p < 2^*$ then follows from the compactness of the Sobolev embedding since $\Omega_\varepsilon$ has a Lipschitz boundary.

(vi) The continuity of the embedding is proved in part (iv). To show that it is not compact choose any $u \in E_2$ such that $u \neq 0$ but $u = 0$ on $\Omega_\varepsilon$ and then, for $n \geq 1$, set $u_n = 0$ on $\Omega_\varepsilon/n$ and $u_n(x) = n^{N/2}u(nx)$ for $r \leq \varepsilon/n$. It is easy to check that

$$\int_{\Omega} u_n^2 \, dx = \int_{\Omega} u^2 \, dx$$

and

$$\int_{\Omega} r^2 |\nabla u_n|^2 \, dx = \int_{\Omega} r^2 |\nabla u|^2 \, dx$$

for all $n \geq 1$. Furthermore, for any $v \in E_2$,

$$|\langle u_n, v \rangle| \leq \int_{B(0, \varepsilon/n)} r^2 \nabla u_n \cdot \nabla v \, dx \leq \|u_n\|_2 \left\{ \int_{B(0, \varepsilon/n)} r^2 |\nabla v|^2 \, dx \right\}^{1/2}$$

$$= \|u\|_2 \left\{ \int_{B(0, \varepsilon/n)} r^2 |\nabla v|^2 \, dx \right\}^{1/2} \quad \text{for all } n \geq 1.$$

Since $r^2 |\nabla v|^2 \in L^1(\Omega)$,

$$\int_{B(0, \varepsilon/n)} r^2 |\nabla v|^2 \, dx \to 0 \quad \text{as } n \to \infty,$$

showing that $u_n \rightharpoonup 0$ weakly in $E_2$ as $n \to \infty$. But $\|u_n\|_{L^2} = \|u\|_{L^2} \neq 0$ for all $n \geq 1$ and so the embedding of $E_2$ in $L^2(\Omega)$ is not compact. \hfill \Box

3. The operators $S_A$ and $S$. Under the assumptions that $A$ satisfies the condition $(A)_\tau$ and that $V \in L^\infty(\Omega)$, two self-adjoint operators are introduced to deal with the differential expressions $-\nabla \cdot \{A \nabla\}$ and $-\nabla \cdot \{A \nabla\} + V$ and the boundary conditions (1.5).
3.1. The self-adjoint operator $S_A$. In this section we introduce a self-adjoint operator acting in $L^2(\Omega)$ associated with the expression $-\nabla \cdot [A \nabla]$ and the Dirichlet boundary condition on $\Omega$, in such a way that $H_A$ appears as its form space. To this end we use a version the Friedrich’s extension procedure which we now recall, following [37].

Let $(H, (\cdot, \cdot), \| \cdot \|)$ and $(H_1, (\cdot, \cdot)_1, \| \cdot \|_1)$ be real Hilbert spaces such that $H_1$ is a dense subset of $(H, (\cdot, \cdot))$ and there is a constant $C$ such that $\| \cdot \| \leq C \| \cdot \|_1$. Then there is a unique self-adjoint operator $T : D(T) \subset H \to H$ acting in $H$ such that $D(T) \subset H_1$ and $(Tu, v) = (u, v)_1$ for all $u \in D(T)$ and $v \in H_1$. It follows that, for all $u \in D(T)$, $\|u\|_1^2 \leq \|Tu\| \|u\| \leq C\|u\|_1\|Tu\|$ and hence $\|u\|_1 \leq C\|Tu\|$. This operator has the following additional properties.

(1) $D(T) = \{u \in H_1 \mid \text{there exists a constant } K_u \text{ such that } |(u, v)_1| \leq K_u \|v\| \text{ for all } v \in H_1\}$.

(2) $(Tu, u) \geq C^{-2}(u, u)$ and $\|u\| \leq C^2\|Tu\|$ for all $u \in D(T)$.

(3) $D(T^{1/2}) = H_1$ and $\|T^{1/2}u\| = \|u\|_1$ for all $u \in H_1$.

For parts (1) to (3), see Theorem 5.36 of [37], for example. In part (4), $T^{1/2}$ denotes the unique positive self-adjoint square root of $T$. The graph norm of $T^{1/2}$ is defined by $\|u\|_2^2 = \|u\|^2 + \|T^{1/2}u\|^2$ for $u \in D(T^{1/2})$. It is well known that $D(T)$ is a dense subspace of $(D(T^{1/2}), \| \cdot \|_1)$. For $u \in D(T)$, $\|T^{1/2}u\|^2 = (Tu, u) = (u, u)_1$ so $\|u\|_2^2 = \|u\|^2 + \|T^{1/2}u\|^2$ and hence $\|u\|_1 \leq \|u\| \leq (C^2 + 1)^{1/2}\|u\|_2$. That is, $\| \cdot \|$ and $\| \cdot \|_1$ are equivalent norms on $D(T)$. But $D(T^{1/2})$ is the completion of $D(T)$ using $\| \cdot \|_2$ and, by property (2), $H_1$ is the completion of $D(T)$ using $\| \cdot \|_1$. Thus $D(T^{1/2}) = H_1$ and $\|T^{1/2}u\| = \|u\|_1$ for all $u \in H_1$, justifying (4).

It follows from parts (iii) and (iv) of Proposition 2.2 that the Hilbert spaces $(L^2(\Omega), (\cdot, \cdot)_{L^2})$ and $(H_A, (\cdot, \cdot)_A)$ form an admissible pair for this construction provided that the condition (A)$_e$ is satisfied. Henceforth, $S_A : D(S_A) \subset L^2(\Omega) \to L^2(\Omega)$ will denote the unique self-adjoint operator associated with this pair through the Friedrich’s extension procedure. Using the properties (1) to (4) we have

\begin{align*}
D(S_A) & \text{ with the graph norm of } S_A \text{ is continuously embedded in } H_A, \quad (3.1) \\
D(S_A) & \text{ is a dense subset of } (H_A, \| \cdot \|_A), \quad (3.2) \\
(S_A u, v)_{L^2} = (u, v)_A = \int_\Omega A\nabla \cdot \nabla v \, dx \text{ for } u \in D(S_A) \text{ and } v \in H_A, \quad (3.3) \\
& \text{there exists } C > 0 \text{ such that } (S_A u, w)_{L^2} \geq C\|w\|_{L^2}^2 \text{ for } u \in D(S_A), \quad (3.4) \\
D(S_A^{1/2}) & = H_A \quad (3.5) \\
S_A^{1/2} : (H_A, \| \cdot \|_A) \to (L^2(\Omega), \| \cdot \|_{L^2}) \text{ is an isometric isomorphism.} \quad (3.6)
\end{align*}

To proceed further we need a more explicit characterisation of the space $D(S_A)$. As noted in property (1), $u \in D(S_A)$ if and only if $u \in H_A$ and

\begin{equation}
\int_\Omega A\nabla u \cdot \nabla v \, dx = \int_\Omega uv \, dx \text{ for all } v \in H_A. \quad (3.7)
\end{equation}

Since $u \in H_A$ ensures that $A\nabla u \in [L^1(\Omega)]^N$ and $C_0^\infty(\Omega) \subset H_A$, we see that $u \in D(S_A)$ if and only if

\begin{equation}
u \in H_A \text{ and } -\nabla \cdot \{A\nabla u\} = v \in L^2(\Omega) \text{ in the sense of distributions on } \Omega, \quad (3.8)
\end{equation}
even though in the case $N = \tau = 2$ we may have $|\nabla u| \notin L^1_{\text{loc}}(\Omega)$. In this sense we can write

$$S_A u = -\nabla \cdot (A \nabla u)$$

for $u \in D(S_A) = \{ u \in H_A : \nabla \cdot (A \nabla u) \in L^2(\Omega) \}$ in all cases.

Although $C^\infty_0(\Omega) \subset H_A$, we cannot expect $C^\infty_0(\Omega)$ to be contained in $D(S_A)$ without making assumptions about the differentiability of $A$ on $\Omega$. This will not be required here.

### 3.2. The operator $S = S_A + V$ and its spectrum.

For $V \in L^\infty(\Omega)$ multiplication by $V$ defines a bounded self-adjoint operator from $L^2(\Omega)$ into itself and hence by Theorem 9.1 of [37],

$$S = S_A + V : D(S) = D(S_A) \subset L^2(\Omega) \to L^2(\Omega)$$

is self-adjoint,

provided that $A$ satisfies the condition $(A)_\tau$. Furthermore, the graph norms of $S_A$ and $S = S_A + V$ are equivalent norms on $D(S)$. The graph norm of $S$ will be denoted by $\| \cdot \|_S$. Therefore,

$$\|u\|_S = \{ \| u \|_{L^2} + \| Su \|_{L^2} \}^{1/2}$$

for all $u \in D(S)$ is equivalent to $\| S_A u \|_{L^2}$ by (3.4).

Since the linearization of (1.4) about the solution $u \equiv 0$ is (1.9), more precisely (1.14), the location of the spectrum of $S$ can be expected to play an crucial role in the stability theory of the solution $u \equiv 0$.

Let $\sigma(S)$ and $\sigma_e(S)$ denote the spectrum and essential spectrum of $S$, respectively. For the simpler subcritical case the essential spectrum of $S$ is empty for all $V \in L^\infty(\Omega)$.

#### Proposition 3.1.

Suppose that $A$ satisfies the condition $(A)_\tau$ for some $\tau \in [0, 2)$ and that $V \in L^\infty(\Omega)$. A self-adjoint operator $S : D(S) \subset L^2(\Omega) \to L^2(\Omega)$ is defined by

$$D(S) = \{ u \in H_A : -\nabla \cdot (A \nabla u) + Vu \in L^2(\Omega) \} \text{ and } Su = -\nabla \cdot (A \nabla u) + Vu.$$ 

It is bounded below and has no essential spectrum. In fact, setting $m = \inf \sigma(S)$, $m$ is a simple eigenvalue of $S$ with a positive eigenfunction and

$$m = \inf \{ \frac{\int_{\Omega} A|\nabla u|^2 + Vu^2 \, dx}{\int_{\Omega} u^2 \, dx} : u \in H_A \setminus \{ 0 \} \} \geq \frac{C_A N^2}{4 R_{\Omega}^2 \tau} + \inf V,$$

where

$$C_A = \inf_{\Omega} \frac{A(x)}{|x|^\tau}, \quad R_{\Omega} = \sup_{\Omega} |x| \quad \text{and} \quad \inf V \text{ is the essential infimum of } V \text{ on } \Omega.$$ 

All eigenfunctions of $S$ are Hölder continuous on $\Omega$.

#### Proof.

The self-adjointness of $S$ has already been noted at the beginning of this subsection. Hence

$$m = \inf \{ \langle Su, u \rangle_{L^2} : u \in D(S) \text{ and } \| u \|_{L^2} = 1 \}$$

and, since $D(S) = D(S_A)$ is dense in $(H_A, \| \cdot \|_A)$ by (3.2) it follows from (3.3) that

$$m = \inf \{ \frac{\int_{\Omega} A|\nabla u|^2 + Vu^2 \, dx}{\int_{\Omega} u^2 \, dx} : u \in H_A \setminus \{ 0 \} \}.$$
But \( C_0^\infty(\Omega) \) is dense in \( (H_A, \| \cdot \|_A) \) by Proposition 2.1 so this yields
\[
m = \inf \left\{ \frac{\int_\Omega A|\nabla u|^2 + V u^2 dx}{\int_\Omega u^2 dx} : u \in C_0^\infty(\Omega) \setminus \{0\} \right\},
\]
even although \( C_0^\infty(\Omega) \) may not be contained in \( D(S) \). For \( u \in C_0^\infty(\Omega) \),
\[
\int_\Omega A|\nabla u|^2 + V u^2 dx \geq C_A \| u \|_2^2 + \inf V \| u \|_{L^2}^2 \geq C_A R_\Omega^{-2} \| u \|_{L^2}^2 + \inf V \| u \|_{L^2}^2
\]
by (2.2), proving the lower bound for \( m \). Since \( H_A = F_r \) is compactly embedded in \( L^2(\Omega) \) by Proposition 2.1, it is easily seen that \( S \) has a compact resolvent and consequently a discrete spectrum. The fact that \( \ker(S - m) \) is spanned by a positive eigenfunction can be proved by the usual minimization argument, as in part (iv) of Theorem 4.1 in [33]. The regularity theory in [29, 36] shows that the eigenfunctions of \( S \) are Hölder continuous on \( \overline{\Omega} \). See Corollaries 5.5 and 6.1 in [36], for example.

In the case of critical degeneracy the properties of the operator \( S \) are quite different. Its essential spectrum is not empty and since inf \( \sigma_e(S) \) plays a prominent role in the stability criteria in Section 7 it is important to have a sharp estimates for it. In fact, when the potential \( V \) has a limit as \( x \) tends to 0, inf \( \sigma_e(S) \) can be calculated exactly.

\( \text{(V) } V \in L^\infty(\Omega) \) and there exists a constant \( V_0 \) such that \( \lim_{r \to 0} \| V - V_0 \|_{L^\infty(B(0,r))} = 0 \), where \( B(0,r) = \{ x \in \mathbb{R}^N : |x| < r \} \).

The following properties of \( S \) are established as part of Theorem 4.1 in [33].

**Proposition 3.2.** Suppose that the conditions \( (A) \) and \( (V) \) are satisfied and consider the self-adjoint operator \( S = S_A + V \).

(i) If \( V_0 = 0 \), multiplication by \( V \) defines a compact linear operator from \( H_A \) into \( L^2(\Omega) \).

(ii) \( \sigma_e(S) = \sigma_e(S_A) + V_0 \).

Setting
\[
m = \inf \sigma(S) \text{ and } m_e = \inf \sigma_e(S),
\]
we have
\[
m = \inf \left\{ \frac{\int_\Omega A|\nabla u|^2 + V u^2 dx}{\int_\Omega u^2 dx} : u \in H_A \setminus \{0\} \right\} \geq \frac{N^2 C_A}{4} + \inf V \text{ and } (3.9)
m_e = \frac{N^2 a}{4} + V_0. \quad (3.10)
\]

(iii) For all \( \lambda \in \mathbb{R} \), \( \ker(S - \lambda) \subset D(S) \cap C(\Omega \setminus \{0\}) \). In fact, eigenfunctions belong to \( C^\alpha(K) \) for every \( \alpha \in (0,1) \) and every compact subset \( K \) of \( \Omega \setminus \{0\} \).

(iv) If \( m < m_e \), then \( m \) is a simple eigenvalue of \( S \) and there exists an element \( \varphi \in D(S) \cap C(\Omega \setminus \{0\}) \) such that \( \varphi > 0 \) on \( \Omega \setminus \{0\} \) and \( \ker(S - m) = \text{span} \{ \varphi \} \).

Proposition 6.2 provides conditions under which \( m = m_e \) and \( S \) has no eigenvalues. If \( S \) does have eigenvalues below the essential spectrum the corresponding eigenfunctions can be singular at the origin even when \( A \) is smooth and \( V \) is zero in a
neighbourhood of the origin. In the following example, \( \chi_J \) denotes the characteristic function of an interval \( J \).

**Example 3.1.** Consider \( \Omega = B(0, 1) \subset \mathbb{R}^N \) and the family of eigenvalue problems

\[
S_\alpha u = \lambda u \quad \text{for} \quad u \in D(S_\alpha), \quad \text{where} \quad S_\alpha u = -\nabla \cdot (|\tau|^{2} \nabla u) + V_\alpha
\]

with

\[
V_\alpha(x) = -2\alpha^2 N^2 \chi_{[r_\alpha, 1]}(|x|) \quad \text{and} \quad r_\alpha = e^{-\frac{2\pi}{\alpha N}} \quad \text{for} \quad \alpha > 0.
\]

The coefficients in the equation (3.11) are radially symmetric and radially symmetric solutions can be calculated explicitly since the resulting ordinary differential equation is of Euler type.

A positive, radially symmetric eigenfunction of \( S_\alpha \) with eigenvalue \( \lambda = N^2(\frac{1}{4} - \alpha^2) \) is given by

\[
u_\alpha(x) = \varphi_\alpha(|x|^N) \quad \text{where} \quad \varphi_\alpha(t) = t^{\frac{\alpha - \frac{1}{2}}{2}} \quad \text{for} \quad 0 < t < t_\alpha \equiv \alpha^N
\]

\[
= -\sqrt{2} e^{-\frac{2\pi}{\alpha N}} t^{\frac{\alpha - \frac{1}{2}}{2}} \sin(\alpha \ln t) \quad \text{for} \quad t_\alpha < t < 1.
\]

Observe that \( u_\alpha \) is singular at \( x = 0 \) for \( \alpha \in (0, 1/2) \). In connection with Proposition 3.2, note also that the conditions \((A)_2\) and \((V)\) are satisfied in this example with \( a = 1 \) and \( V_\alpha(0) = 0 \). Hence \( \inf \sigma_e(S_\alpha) = \frac{N^2}{\alpha^2} \) for all \( \alpha > 0 \) and \( m_\alpha \equiv \inf \sigma(S_\alpha) \leq N^2(\frac{1}{4} - \alpha^2) \), so it follows from part (iv) that \( m_\alpha \) is a simple eigenvalue of \( S_\alpha \) with a positive eigenfunction. From the orthogonality of eigenfunctions of \( S_\alpha \) associated with different eigenvalues we conclude that \( m_\alpha = N^2(\frac{1}{4} - \alpha^2) \).

4. **The nonlinearity.** The results of this section concern the properties of the term \( g(x, u) \) in (1.4), viewed as a mapping from \( H_A \) into \( L^2(\Omega) \), under the assumption \((A)_\tau\).

The first step is to formulate hypotheses on the function \( g \) which ensure that it generates a continuously differentiable Nemytskii operator from \( H_A \) into \( L^2(\Omega) \). Since the space \( H_A \) gets bigger as \( \tau \) is increased the assumptions required to ensure this become more restrictive as \( \tau \in [0, 2] \) increases. A lemma then shows that when \( \tau = 2 \) Fréchet differentiability is too restrictive to admit nonlinearities where \( g(x, \cdot) \) is independent of \( x \). For such cases only Gâteaux differentiability at 0 can be obtained and a weaker condition \((G1)_\tau^*\) is formulated to ensure this property.

\((G1)_\tau\) \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Caratheodory function such that

(i) \( g(x, \cdot) \in C^1(\mathbb{R}) \) with \( g(x, 0) = \partial_x g(x, 0) = 0 \) for \( x \in \Omega \), and

(ii) there exist constants \( K > 0 \) and \( \sigma \in [0, \frac{2 - \tau}{2 + \tau}] \) \( 0 \leq \sigma < \infty \) in the case \( N = 2 \) with \( \tau = 0 \) such that

\[
|\partial_s g(x, s)| \leq K(1 + |s|^\sigma) \quad \text{for} \quad s \in \mathbb{R} \quad \text{and} \quad x \in \Omega.
\]

For \( \tau = 0 \), this condition is the standard assumption used to ensure that \( \tilde{g} \in C^1(H^1_0(\Omega), L^2(\Omega)) \). For \( \tau > 0 \) the range of admissible growth of \( g(x, s) \) with respect to \( s \) can be extended at the expense of imposing suitable decay to zero as \( x \) tends to 0.

\((G2)_\tau\) \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Caratheodory function such that

(i) \( g(x, \cdot) \in C^1(\mathbb{R}) \) with \( g(x, 0) = \partial_x g(x, 0) = 0 \) for \( x \in \Omega \), and
(ii) there exist constants $K$, $\sigma \in \left(\frac{2-N}{N-2+\gamma}, \frac{2}{N-2}\right)$, $\alpha \geq \frac{\sigma(2-N+\gamma)-2}{2}$, such that

$$|\partial_s g(x, s)| \leq K(|x|^{\beta} + |x|^\sigma |s|^\sigma)$$

for $s \in \mathbb{R}$ and $x \in \Omega$, where $\beta = 0$ if $\tau \in (0, 2)$ and $\beta$ is some positive constant when $\tau = 2$.

For $\tau = 0$ this condition is vacuous. If the estimate (4.2) holds for some $\alpha > \frac{\sigma(2-N+\gamma)-2}{2}$, it also holds for $\alpha = \frac{\sigma(2-N+\gamma)-2}{2}$ (with a different value of $K$) since $\Omega$ is bounded. It should also be noted that, for a given value of $\tau \in (0, 2]$, it is easy to find functions which satisfy both (G1)$_\tau$ and (G2)$_\tau$. For some purposes the following stronger version of (G2)$_2$ will be required.

$$(G2)_2^\tau g$$

satisfies (G2)$_2$ and

$$|g(x, s)| \leq K\{x^{\frac{\alpha}{\gamma}}|s|^{1+\sigma_1} + |x|^{\frac{\sigma}{\gamma}}|s|^{1+\sigma}\}$$

for $s \in \mathbb{R}$ and $x \in \Omega$, where $0 < \sigma_1 \leq \sigma \leq 2/(N-2)$ ($0 < \sigma_1 \leq \sigma < \infty$ for $N = 2$).

Recall that $\tilde{g}$, $G$ and $\psi$ are defined by (1.15) and (1.16).

**Proposition 4.1.** Consider $N \geq 2$ and $\tau \in [0, 2]$.

(A) Suppose that a function $g$ satisfies either (G1)$_\tau$ or (G2)$_\tau$. Then $\tilde{G}(u)$ and $\tilde{g}(u)v \in L^1(\Omega)$ for all $u, v \in F_\tau$ and $\tau \in C^1(F_\tau, \mathbb{R})$ with

$$\psi'(u)v = \int_\Omega \tilde{g}(u)v \, dx \text{ for all } u, v \in F_\tau. \quad (4.4)$$

(B) Suppose that a function $g$ satisfies either (G1)$_\tau$ with $\tau \in [0, 2]$ or (G2)$_\tau$ with $\tau \in (0, 2]$. Then $\tilde{g}$ maps bounded subsets of $F_\tau$ into bounded subsets of $L^2(\Omega)$ and $\tilde{g}$ is in $C^1(F_\tau, L^2(\Omega))$ with $\tilde{g}(0) = 0$ and $\tilde{g}'(0) = 0$. Also, $\psi \in C^2(F_\tau, \mathbb{R})$ and $\tilde{g}$ maps bounded subsets of $F_\tau$ into bounded subsets of $B(F_\tau, L^2(\Omega))$. If $g$ satisfies (G2)$_2^\tau$, there exists a constant $C$ such that $\|\tilde{g}(u)\|_{L^2} \leq C\{\|u\|_{L^2}^{1+\sigma_1} + \|u\|_{L^2}^{1+\sigma}\}$ for all $u \in F_2$.

(C) For a coefficient $A$ satisfying the condition (A)$_\tau$, the conclusions of parts (A) and (B) also hold when the space $F_\tau$ is replaced by $H_A$.

**Remark 4.1.** Lemma 4.2 shows that Fréchet differentiability of $\tilde{g} : F_2 \to L^2(\Omega)$ may fail for $g$ satisfying (G1)$_2$.

**Proof.** We begin by recalling a more or less standard result concerning the differentiability of Nemytskii operators acting between Lebesgue spaces. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(\cdot, 0) \in L^q(\Omega)$ for some $q \in [1, \infty)$ and $f(x, \cdot) \in C^1(\mathbb{R})$ for $x \in \Omega$ with

$$|\partial_s f(x, s)| \leq K(b(x) + |s|^\sigma)$$

for $s \in \mathbb{R}$ and $x \in \Omega$, where $K$ and $\sigma$ are positive constants and $b \in L^{q(1+\sigma)/\sigma}(\Omega)$. Then $\partial_s f : \Omega \times \mathbb{R} \to \mathbb{R}$ is also a Carathéodory function and $\tilde{f} \in C^1(L^{q(1+\sigma)/\sigma}(\Omega), L^q(\Omega))$, with $\tilde{f} \in C(L^{q(1+\sigma)/\sigma}(\Omega), B(L^{q(1+\sigma)/\sigma}(\Omega), L^q(\Omega)))$ and $\tilde{f}'(u)v = \partial_s f(\cdot, u)v \in L^q(\Omega)$ for $u, v \in L^{q(1+\sigma)/\sigma}(\Omega)$. These conclusions are justified in Theorem 2.6 of [15], for example. By integration and Young’s inequality,

$$|f(x, s)| \leq \frac{K}{1+\sigma}\{\sigma b(x)^{(1+\sigma)/\sigma} + 2|s|^{1+\sigma}\} + |f(x, 0)|, \quad (4.6)$$

showing that $\tilde{f}$ maps bounded subsets of $L^{q(1+\sigma)/\sigma}(\Omega)$ into bounded subsets of $L^q(\Omega)$. If $b \equiv f(\cdot, 0) \equiv 0$, $|f(x, s)| \leq K|s|^{1+\sigma}/(1+\sigma)$ and this implies that $\|\tilde{f}(u)\|_{L^q} \leq K\|u\|_{L^{q(1+\sigma)/\sigma}(\Omega)}/(1+\sigma)$. 


(A) Suppose that $g$ satisfies the condition $(G1)_\tau$ and consider the Caratheodory function $G : \Omega \times \mathbb{R} \to \mathbb{R}$. Since
\[ |\partial_s G(x, s)| = |g(x, s)| \leq \frac{K}{1+\sigma} \{ \sigma + 2|s|^{1+\sigma} \}, \]
\( \tilde{G} \in C^1(L^{2+\sigma}(\Omega), L^1(\Omega)) \) and $\tilde{G}'(u)v = \tilde{g}(u)v \in L^1(\Omega)$ for all $u, v \in L^{2+\sigma}(\Omega)$. Noting that $2 + \sigma < 2^*_\tau$ ($2 + \sigma < \infty$ for $N = 2$ with $\tau = 0$) since $0 \leq \sigma \leq (2-\tau)/(N-2+\tau)$ ($0 \leq \sigma < \infty$ for $N = 2$ with $\tau = 0$), it follows from Proposition 2.1 that $\tilde{G} \in C^1(F_\tau, L^1(\Omega))$. But $\psi(u) = \int_\Omega \tilde{G}(u) \, dx$ for all $u \in F_\tau$ and so the conclusion follows in this case.

Now suppose instead that $g$ satisfies the condition $(G2)_\tau$. This means that $\tau > 0$ and $\sigma > 0$ and we can take $\alpha = \frac{\tau(N-2+\tau)}{2}$. Set $\gamma = \alpha/(2+\sigma)$ and consider the Caratheodory function defined by
\[ f(x, s) = r^{-\gamma}g(x, r^{-\gamma}s) \text{ where } r = |x|. \]
Then integrating (4.1) and using Young’s inequality,
\[ |f(x, s)| \leq K(1+\sigma)_\gamma|s| + \frac{\gamma^{1+\gamma}|s|^{1+\sigma}}{1+\sigma} \leq \frac{K}{1+\sigma}(\sigma^{1+\gamma} + 2|s|^{1+\sigma}). \]

By the restrictions on $\sigma$ and $\beta$, $r^{(\beta-2\gamma)} \frac{\gamma^{1+\gamma}}{1+\sigma} \in L^{2+\sigma}(\Omega)$ and it follows that $\tilde{F} \in C^1(L^{2+\sigma}(\Omega), L^1(\Omega))$, where $F(x, s) = \int_0^s f(x, t) \, dt$.

Here $\gamma > 0$ since $\sigma > (2-\tau)/(N-2+\tau)$. Also $2\gamma \leq \tau$ for $N \geq 3$ since $\sigma \leq 2/(N-2)$ and $2\gamma < \tau$ for $N = 2$. Setting $Tu = r^\gamma u$, it follows from (2.5) that $T \in B(F_\tau, L^{2+\sigma}(\Omega))$ since $2 + \sigma \leq 2N/(N-2+\tau-2\gamma)$ for $\tau \leq 2$. Hence $\tilde{F} \circ T \in C^1(F_\tau, L^1(\Omega))$. But $G(x, s) = F(x, r^\gamma s)$ so $\tilde{G} = \tilde{F} \circ \tilde{T}$ and the conclusion follows as in the previous case.

(B) Suppose that $g$ satisfies the condition $(G1)_\tau$ where $\tau \in [0, 2)$. Then (4.1) holds for some $\sigma \in (0, \frac{2-\tau}{N-2+\tau})$ and so $2 < 2(1+\sigma) \leq 2^*_\tau$ (for $N = 2$ with $\tau = 0$, $0 < \sigma < \infty$). Hence $\tilde{g} \in C^1(L^{2(1+\sigma)}(\Omega), L^2(\Omega))$ and $\tilde{g}$ takes bounded subsets of $L^{2(1+\sigma)}(\Omega)$ into bounded subsets of $L^2(\Omega)$. But $F_\tau$ is continuously embedded in $L^{2(1+\sigma)}(\Omega)$ by Proposition 2.1 so the conclusion follows.

Suppose now that $g$ satisfies the condition $(G2)_\tau$. Recalling that this means that $\tau$ and $\sigma$ are positive and that we can take $\alpha = \frac{\sigma(N-2+\tau)}{2}$, set
\[ \gamma = \frac{N\sigma - (1+\sigma)(2-\tau)}{2(1+\sigma)} = \frac{\alpha}{1+\sigma}. \]
Then the restrictions on $\sigma$ ensure that $0 < 2\gamma \leq \tau$ for $N \geq 3$ and $0 < 2\gamma < \tau$ for $N = 2$. Since
\[ \frac{2N}{N-2+\tau-2\gamma} = 2(1+\sigma) \]
it follows from (2.5) that $Tu = r^\gamma u$ defines a bounded linear operator from $F_\tau$ into $L^{2(1+\sigma)}(\Omega)$.

Now define a Caratheodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ by setting
\[ f(x, s) = g(x, r^{-\gamma}s) \text{ for almost all } x \in \Omega \text{ and } s \in \mathbb{R}. \]
Then, for $x \in \Omega$, $f(x, \cdot) \in C^1(\mathbb{R})$ and
\[ |\partial_x f(x, s)| = r^{-\gamma} |\partial_x g(x, r^{-\gamma}s)| \leq K \{ r^{\beta-\gamma} + r^{\alpha-\gamma}(1+\sigma) |s|^\alpha \} = K \{ r^{\beta-\gamma} + |s|^\alpha \}, \]
where \( r^{\beta-\gamma} \in L^{2(1+\sigma)/\gamma}(\Omega) \). Noting that \( f(x,0) = \partial_s f(x,0) = 0 \) a.e. on \( \Omega \), it follows that \( \tilde{f} \in C^1(L^{2(1+\sigma)/\gamma}(\Omega), L^2(\Omega)) \) and \( \tilde{f} \) takes bounded subsets of \( L^{2(1+\sigma)/\gamma}(\Omega) \) into bounded subsets of \( L^2(\Omega) \). Hence \( \tilde{g} \in C^1(F_\tau, L^2(\Omega) \) since \( f \circ T = \tilde{g} \).

If \( g \) satisfies (4.3), \( |g(x,s)|^2 \leq 2K^2 \{|r^{\gamma_1}s|^{2(1+\sigma_1)} + |r^{\gamma_2}s|^{2(1+\sigma_2)}\} \) where \( \gamma_1 = \frac{N\sigma_1}{2(1+\sigma_1)} \) and \( \gamma = \frac{N\sigma}{2(1+\sigma)} \) so

\[
\|\tilde{g}(u\|_{L^2} \leq K_1 \{\|r^{\gamma_1}u\|_{L^{2(1+\sigma_1)}}^{1+\sigma_1} + \|r^{\gamma_2}u\|_{L^{2(1+\sigma_2)}}^{1+\sigma_2}\}.
\]

Since \( 2(1+\sigma_1) = 2N/(N-2\gamma_1) \) and \( 2(1+\sigma) = 2N/(N-2\gamma) \) it follows from (2.5) that

\[
\|\tilde{g}\|_{L^2} \leq C\{\|u\|_2^{1+\sigma_1} + \|u\|_2^{1+\sigma}\} \text{ for } u \in F_2.
\]

(C) Except for the case \( N = \tau = 2 \) this is trivial since \( H_A = F_\tau \) by part (6) of Proposition 2.1. For \( N = \tau = 2 \), by part (7) of that result, (2.5) extends to \( H_A \) and so the proofs of parts (A) and (B) yield the conclusions by replacing \( F_2 \) by \( H_A \). \( \square \)

Let \( k \) be a function satisfying the condition (K) introduced at the beginning of Section 1. Setting \( g(x,s) = \pm k(s) \) for \( x \in \Omega \) and \( s \in \mathbb{R} \) it is clear that \( g \) satisfies the condition (G1), for all \( \tau \in [0,2] \) and so \( \tilde{g} \in C^1(F_\tau, L^2(\Omega)) \) for \( \tau \in [0,2] \) by Proposition 4.1 (B), but as the following result shows, \( \tilde{g} : F_\tau \rightarrow L^2(\Omega) \) is not Fréchet differentiable unless \( k \equiv 0 \). For \( k \neq 0 \), \( g \) does not satisfy (G2), for any \( \tau \in [0,2] \).

**Lemma 4.2.** Consider a Carathéodory function \( g : \Omega \times \mathbb{R} \) having the following properties.

(a) There exists \( C \) such that \( |g(x,s)| \leq C|s| \) for \( (x,s) \in \Omega \times \mathbb{R} \) and \( \lim_{s \rightarrow 0} \frac{g(x,s)}{s} = 0 \) for \( x \in \Omega \).

Then for \( \tau \in [0,2] \), \( \tilde{g} : F_\tau \rightarrow L^2(\Omega) \) is continuous and it is Gâteaux differentiable at \( 0 \) with \( \tilde{g}'(0) = 0 \). For \( \tau \in [0,2] \), it is Fréchet differentiable at \( 0 \). If \( \tilde{g} : F_2 \rightarrow L^2(\Omega) \) is Fréchet differentiable at \( 0 \), then

\[
\lim_{\rho \rightarrow 0+} \frac{1}{|B(0,\rho)|} \int_{B(0,\rho)} |g(x,s)| \, dx = 0 \text{ for all } s \in \mathbb{R}.
\]  

(4.7)

For a coefficient \( A \) satisfying the condition \((A)_\tau\), the same conclusions are valid when \( F_\tau \) is replaced by \( H_A \).

**Remark 4.2.** If \( g \) satisfies the condition (a) and \( g(\cdot,s) : \Omega \rightarrow \mathbb{R} \) is continuous at \( 0 \) for all \( s \in \mathbb{R} \), the necessary condition (4.7) is satisfied if and only if \( g(0,s) = 0 \) for all \( s \). A function satisfying the condition \((G1)_2\) has the property (a) in the lemma but may not satisfy the condition (4.7). In particular, if \( g(x,s) = \pm k(s) \) where the function \( k \) satisfies the condition (K), \( g \) satisfies \((G1)_2\) but (4.7) is satisfied if and only if \( k \equiv 0 \).

**Proof.** By the standard result on Nemitskii operators (Theorem 2.3 in [15], for example), \( \tilde{g} \in C(L^2(\Omega), L^2(\Omega)) \) and for any \( u \in L^2(\Omega) \), \( |g(x, tu(x))/t| \leq C|u(x)| \) and \( g(x, tu(x))/t \rightarrow 0 \) as \( t \rightarrow 0 \) for almost all \( x \in \Omega \). Hence \( \|\tilde{g}(tu)\|_{L^2}/t \rightarrow 0 \) as \( t \rightarrow 0 \) by dominated convergence, showing that \( \tilde{g} : L^2(\Omega) \rightarrow L^2(\Omega) \) is Gâteaux differentiable at \( 0 \) with \( \tilde{g}'(0) = 0 \). Since \( F_\tau \) is continuously embedded in \( L^2(\Omega) \) for \( \tau \in [0,2] \) by Proposition 2.1, the operator \( \tilde{g} : F_\tau \rightarrow L^2(\Omega) \) also has these properties.

Set \( h(x,s) = g(x,s)/s \) for \( s \neq 0 \) and \( h(x,0) = 0 \) for \( x \in \Omega \). Then, for \( \tau \in [0,2] \) choose \( p \in (2,2^*_\tau) \) and let \( q = 2p/(p-2) \). For any \( u \in L^p(\Omega) \), it follows from Hölder’s inequality that

\[
\|\tilde{g}(u)\|_{L^2} = \|\tilde{h}(u)\|_{L^2} \leq \|u\|_{L^p} \|\tilde{h}(u)\|_{L^q},
\]
where \( \tilde{h} \in C(L^p(\Omega), L^q(\Omega)) \) with \( \tilde{h}(0) = 0 \) by Theorem 2.3 of [15] since \( h \) is a Carathéodory function with \( |h(x,s)| \leq C \) and \( h(x,0) = 0 \) for \( x \in \Omega \) and \( s \in \mathbb{R} \).

Hence \( \|\tilde{g}(u)\|_{L^2}/\|u\|_{L^p} \to 0 \) as \( \|u\|_{L^p} \to 0 \), showing that \( \tilde{g} : L^p(\Omega) \to L^2(\Omega) \) is Fréchet differentiable at 0 with \( \tilde{g}'(0) = 0 \). (This argument fails for \( \tau = 2 \) because \( 2_2^* = 2 \).) Since \( F_\tau \) is continuously embedded in \( L^p(\Omega) \) for \( \tau \in [0, 2] \) by Proposition 2.1, the operator \( \tilde{g} : F_\tau \to L^2(\Omega) \) is also Fréchet differentiable at 0.

Suppose now that \( \tilde{g} : F_2 \to L^2(\Omega) \) is Fréchet differentiable at 0. Since its Gâteaux derivative is zero so is its Fréchet derivative and hence \( \|\tilde{g}(u)\|_{L^2}/\|u\|_{L^p} \to 0 \) as \( \|u\|_{L^p} \to 0 \). Choose \( R > 0 \) such that \( B(0,R) \subset \Omega \) and then \( w \in C_0^\infty(B(0,R)) \) such that \( w(x) = 1 \) for \( x \in B(0, R/2) \). Fix \( s \in \mathbb{R} \) with \( s \neq 0 \) and then for \( \rho \in (0, R/2) \) define a function \( u_\rho \) on \( \Omega \) by

\[
u_\rho(x) = sw\left(\frac{R}{2\rho}, x\right)\text{ for } |x| < 2\rho \text{ and } u_\rho(x) = 0 \text{ for } x \in \Omega \setminus B(0, 2\rho).
\]

Clearly, \( u_\rho \in C_0^\infty(\Omega) \subset F_2 \) and

\[
\|u_\rho\|_2^2 = \int_{|x| < 2\rho} \tau^2|\nabla u_\rho(x)|^2 \, dx = \int_{|x| < 2\rho} |x|^2 s^2 \langle \frac{R}{2\rho}, x \rangle^2 |\nabla w(\frac{R}{2\rho}, x)|^2 \, dx = \left(\frac{2^N}{R}\right)^N s^2 \int_{|y| < R} |y|^2 |\nabla w(y)|^2 \, dy.
\]

Hence, since \( |B(0, \rho)| = \omega_N \rho^N \),

\[
\|u_\rho\|_2^2 = D|B(0, \rho)| \text{ and } D = \frac{2^N}{\omega_N R^N} s^2 \int_{|y| < R} |y|^2 |\nabla w(y)|^2 \, dy > 0.
\]

Thus \( \|u_\rho\|_2 \neq 0 \) for all \( \rho \in (0, R/2) \) and \( \|u_\rho\|_2 \to 0 \) as \( \rho \to 0 \). But,

\[
\|\tilde{g}(u_\rho)\|_2^2 = \int_{|x| < 2\rho} g(x, u_\rho(x))^2 \, dx \geq \int_{|x| < \rho} g(x, s)^2 \, dx
\]

and so

\[
0 \leq \frac{\int_{B(0, \rho)} g(x, s)^2 \, dx}{|B(0, \rho)|} \leq \frac{D\|\tilde{g}(u_\rho)\|_2^2}{\|u_\rho\|_2^2}.
\]

Therefore the Fréchet differentiability of \( \tilde{g} \) at 0 implies that

\[
\frac{\int_{B(0, \rho)} g(x, s)^2 \, dx}{|B(0, \rho)|} \to 0 \text{ as } \rho \to 0 + .
\]

The necessary condition follows from this since

\[
\int_{B(0, \rho)} g(x, s) \, dx \leq \left\{ \int_{B(0, \rho)} g(x, s)^2 \, dx \right\}^{1/2} |B(0, \rho)|^{1/2}.
\]

Consider a function \( A \) satisfying (A)$_\tau$. To see that the same conclusions are valid for \( \tilde{g} : H_\tau \to L^2(\Omega) \) it is sufficient to recall the following properties established in Proposition 2.1. For \( \tau \in [0, 2] \), \( H_\tau \) is continuously embedded in \( L^2(\Omega) \) and \( F_\tau \subset H_\tau \), with equality except in the case \( N = \tau = 2 \). Also the norms \( \| \cdot \|_\tau \) and \( \| \cdot \|_A \) are equivalent on \( F_\tau \) for \( \tau \in [0, 2] \).

This lemma shows that for a function \( g(x, s) \) satisfying (G1)$_2$ which does not decay to zero as \( x \) tends to 0, Fréchet differentiability of \( \tilde{g} : H_\tau \to L^2(\Omega) \) cannot be obtained for a function \( A \) satisfying (A)$_2$. If the goal of ensuring Fréchet differentiability is abandoned and replaced by simply requiring Gâteaux differentiability at 0, it is possible to proceed using the following assumption (G1)$_2$ which is weaker than
(G1)$_2$. For $\tau < 2$, the corresponding modification of (G1)$_\tau$ still ensures Fréchet differentiability at 0.

(G1)$_\tau^*$ $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that

(i) $\lim_{s \to 0} \frac{g(x,s)}{s} = 0$ for $x \in \Omega$,

(ii) there exist $K > 0$ and $\sigma \in [0, 2 - \frac{\tau}{2 + \tau}]$ ($0 \leq \sigma < \infty$ for $N = 2$ with $\tau = 0$) such that

$$|g(x,s) - g(x,t)| \leq K\{1 + |s|^{\sigma} + |t|^{\sigma}\}|s - t|$$

for $x \in \Omega$ and $s \in \mathbb{R}$.

These conditions become more restrictive as $\tau$ increases and for $\tau \in [0, 2]$, the condition (G1)$_\tau^*$ implies (G1)$_\tau$. For $k$ satisfying the condition (K), the function $g(x,s) = \pm k(s)$ for $x \in \Omega$ and $s \in \mathbb{R}$ satisfies (G1)$_\tau$, and hence (G1)$_\tau^*$, for all $\tau \in [0, 2]$.

Since $g(x,0) = 0$ for all $x \in \Omega$ by part (i), it follows from part (ii) and Young’s inequality that there exist constants $C$ and $D$ such that

$$|g(x,s)| \leq C\{1 + |s|^{\sigma}\}|s| \leq D\{1 + |s|^{1+\sigma}\}$$

for $x \in \Omega$ and $s \in \mathbb{R}$. (4.8)

In the condition (G1)$_\tau^*$, $\sigma = 0$ and the functions $g(x,\cdot) : \mathbb{R} \to \mathbb{R}$ are uniformly Lipschitz continuous on $\mathbb{R}$ for $x \in \Omega$. In this case we set

$$\ell = \sup_{{x \in \Omega}} \frac{|g(x,s) - g(x,t)|}{|s - t|} ; x \in \Omega \text{ and } s,t \in \mathbb{R} \text{ with } s \neq t.$$ (4.9)

**Proposition 4.3.** Suppose that the functions $A$ and $g$ satisfy the conditions (A)$_\tau$ and (G1)$_\tau^*$ respectively, for some $\tau \in [0, 2]$. The mapping $\tilde{g} : H_A \to L^2(\Omega)$ has the following properties.

(I) For $0 \leq \tau \leq 2$, $\tilde{G}(u)$ and $\tilde{g}(u)v \in L^1(\Omega)$ for all $u, v \in H_A$ and $\psi \in C^1(H_A, \mathbb{R})$ with

$$\psi'(u)v = \int_\Omega \tilde{g}(u)v \, dx$$

for all $u, v \in H_A$. (4.10)

(II) Suppose that $0 \leq \tau < 2$. The mapping $\tilde{g} : H_A \to L^2(\Omega)$ is locally Lipschitz continuous and there exists a constant $C$ such that

$$\|\tilde{g}(u) - \tilde{g}(v)\|_{L^2} \leq C\{1 + \|u\|_A^\sigma + \|v\|_A^\sigma\}\|u - v\|_A$$

for all $u, v \in H_A$. (4.11)

This mapping is also Fréchet differentiable at 0 with $\tilde{g}'(0) = 0$. If (G1)$_\tau^*$ (ii) is satisfied for some $\sigma \in [0, 2 - \frac{\tau}{2 + \tau}]$, then $\tilde{g} : H_A \to L^2(\Omega)$ is compact.

(III) Suppose that $\tau = 2$. The mappings $\tilde{g} : L^2(\Omega) \to L^2(\Omega)$ and $\check{g} : H_A \to L^2(\Omega)$ are uniformly Lipschitz continuous with $\tilde{g}(0) = 0$ and, for $\ell$ defined by (4.9),

$$\|\tilde{g}(u) - \tilde{g}(v)\|_{L^2} \leq \ell\|u - v\|_{L^2}$$

for $u, v \in L^2(\Omega)$,

$$\|\tilde{g}(u) - \tilde{g}(v)\|_{L^2} \leq \frac{2\ell}{\sqrt{C_1}}\|u - v\|_A$$

for $u, v \in H_A$. (4.12)

Both mappings are Gâteaux differentiable at 0 with derivative 0.

Suppose that, in addition to the condition (G1)$_\tau^*$, the function $g$ has the following property.

(C) There exists a constant $\gamma$ such that for all $\varepsilon > 0$, there exist positive constants $K(\varepsilon)$ and $\delta(\varepsilon)$ such that

$$|g(x,s) - \gamma s| \leq K(\varepsilon) + \varepsilon|s|$$

for all $s \in \mathbb{R}$ and $x \in \Omega$ with $|x| < \delta(\varepsilon)$.

Then the mapping $\tilde{g} - \lambda I : H_A \to L^2(\Omega)$ is compact if and only if $\lambda = \gamma$. 
Remark 4.3. In part (III) the mappings \( \tilde{g} : L^2(\Omega) \to L^2(\Omega) \) and \( \tilde{g} : H_A \to L^2(\Omega) \) are even Hadamard differentiable at 0 with derivative 0. See [17, 32, 33, 34] for more information about this property. Noting that \((G1)^*_2\) implies the condition \((a)\) inLemma 4.2, Fréchet differentiability of \( \tilde{g} : H_A \to L^2(\Omega) \) will fail if the condition \((4.7)\) is not satisfied.

Remark 4.4. The condition \((C)\) implies that \( g(0,s)/s \to \gamma \) as \( |s| \to \infty \). In fact, when \( g \) satisfies the condition \((G1)^*_2\), the property \((C)\) is implied by the more intuitive assumption that

\((C^*)\) there exist \( R > 0 \) and a function \( \Gamma : B(0,R) \to \mathbb{R} \) which is continuous at 0 such that \( g(x,s)/s \to \Gamma(x) \) as \( |s| \to \infty \), uniformly for \( x \in B(0,R) \).

Indeed, for any \( \varepsilon > 0 \), \((C^*)\) implies that there exists \( S(\varepsilon) > 0 \) such that

\[
\left| \frac{g(x,s)}{s} - \Gamma(x) \right| < \varepsilon \quad \text{for} \quad |s| > S(\varepsilon) \quad \text{and} \quad x \in B(0,R),
\]

whereas for \( |s| \leq S(\varepsilon) \) and \( x \in B(0,R) \), it follows from \((G1)^*_2\) that

\[
|g(x,s) - \Gamma(x)s| \leq (\epsilon + |\Gamma(x)|)S(\varepsilon).
\]

Hence for all \( s \in \mathbb{R} \) and \( x \in B(0,R) \),

\[
|g(x,s) - \Gamma(0)s| \leq (\epsilon + |\Gamma(x)|)S(\varepsilon) + (\varepsilon + |\Gamma(x) - \Gamma(0)|)|s|.
\]

From the continuity of \( \Gamma \) at 0 it follows easily that the condition \((C)\) is satisfied with \( \gamma = \Gamma(0) \).

Proof. (I) Using \((4.8)\) the proof is the same as in part \((A)\) of Proposition 4.1 for \((G1)_r\).

(II) If \( \sigma = 0 \),

\[
\|\tilde{g}(u) - \tilde{g}(v)\|_{L^2} \leq 3K\|u - v\|_{L^2} \leq \frac{6K}{N \sqrt{C_1}} \|u - v\|_{H_A},
\]

by \((2.13)\). Suppose now that \( N = 2, \tau = 0 \) and \( 0 < \sigma < \infty \). For \( u,v \in H_A \) and \( p > 1 \),

\[
\int_{\Omega} |u|^{2\sigma}|u - v|^2 \, dx \leq \left\{ \int_{\Omega} |u|^{2\sigma_q} \, dx \right\}^{1/q} \left\{ \int_{\Omega} |u - v|^{2p} \, dx \right\}^{1/p},
\]

where \( q = p/(p-1) \). Taking \( q > \max\{1, \frac{1}{2\sigma} \} \), it follows that there exist constants \( C \) and \( D \) such that

\[
\int_{\Omega} |u|^{2\sigma}|u - v|^2 \, dx \leq C \|u\|^{2\sigma}_{A} \|u - v\|^{2}_{L^{2p}},
\]

since \( H_A \) is continuously embedded in \( L^{2\sigma q}(\Omega) \) and \( L^{2p}(\Omega) \) by Proposition 4.1. In all other cases, \( 0 < \sigma \leq \frac{2-\tau}{N-2+\tau} \) and, taking \( q = \frac{2\tau}{2\tau + \sigma} \) we have \( q > 1 \) and we obtain \((4.15)\) with

\[
2p = \frac{2N}{2 - \tau - \sigma(N - 2 + \tau) + N - 2 + \tau} \leq 2\tau.
\]

The estimate \((4.11)\) follows from \((4.13)\) and \((4.15)\). Noting that \( 2p < 2\tau \) in \((4.15)\) when \( 0 < \sigma < \frac{2-\tau}{N-2+\tau} \), \( H_A \) is compactly embedded in \( L^{2p}(\Omega) \) in this case, yielding the compactness of \( \tilde{g} : H_A \to L^2(\Omega) \).
To prove Fréchet differentiability at 0, define a Caratheodory function $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ by $h(x,0) = 0$ and $h(x,s) = g(x,s)/s$ for $s \neq 0$. For $u \in H_A$, $q > 1$ and $p = q/(q-1)$,

$$\|\tilde{g}(u)\|^2_2 \leq \left\{ \int_\Omega \tilde{h}(u)^{2q} \, dx \right\}^{1/q} \left\{ \int_\Omega u^{2p} \, dx \right\}^{1/p}.$$  

Since $|\tilde{h}(x,s)| \leq C(1 + |s|^\gamma)$ by (4.8), $\tilde{h} \in C(L^q(\Omega), L^{2q})$ for $\max\{1, 2\sigma q\} \leq \tau < \infty$. Except for the case $N = 2$ with $\tau = 0$, choosing $p = 2\tau/2$, we have $q = N/(2 - \tau)$ and $2\sigma q \leq 2\tau$. Hence $\tilde{h} \in C(H_A, L^{2\tau}(\Omega))$ and $\|\tilde{g}(u)\|_{L^2} \leq C\|\tilde{h}(u)\|_{L^{2\tau}}\|u\|_A$, from which it follows that $\|\tilde{g}(u)\|_{L^2}/\|u\|_A \rightarrow 0$ as $\|u\|_A \rightarrow 0$ since $h(0) = 0$. For the case $N = 2$ with $\tau = 0$, it suffices to choose $p = q = 2$ since $H_A$ is continuously embedded in $L^4(\Omega)$ for all $s \in [1, \infty)$.

(III) The inequalities (4.12) follow immediately from (G1) and (2.13). Gâteaux differentiability at 0 is proved in Lemma 4.2 since (G1) implies the condition (a). Hence only the statements concerning compactness remain.

The first step is to prove that $J = \tilde{g} - \gamma I : H_A \rightarrow L^2(\Omega)$ is compact. Let $\{u_n\}$ be a sequence in $H_A$ which converges weakly to $u$ in $H_A$. It is enough to prove that $\|J(u_n) - J(u)\|_{L^2} \rightarrow 0$ as $n \rightarrow \infty$. Since $\{u_n\}$ is bounded in $H_A$ it is also bounded in $L^2(\Omega)$ by Proposition 2.2. Suppose that $\|u\|_{L^2}$ and $\|u_n\|_{L^2} \leq M$ for all $n$.

For any $\varepsilon > 0$, it follows from the property (C) that there exist $K(\varepsilon) > 0$ and $\delta(\varepsilon) \in (0, d(0, \partial \Omega))$ such that for all $n$,

$$\int_{B(0, \delta(\varepsilon))} J(u_n) \, dx \leq 2 \int_{B(0, \delta(\varepsilon))} K(\varepsilon)^2 + \varepsilon^2 u_n^2 \, dx \leq 2K(\varepsilon)^2 \|B(0, \delta(\varepsilon))\| + 2\varepsilon^2 M^2$$

and the same inequality holds with $u_n$ replaced by $u$. Furthermore, with $\varepsilon$ fixed, $\delta(\varepsilon)$ can be chosen small enough so that $K(\varepsilon)^2 \|B(0, \delta(\varepsilon))\| < \varepsilon^2$ and then we obtain

$$\int_{B(0, \delta(\varepsilon))} J(u_n)^2 + J(u)^2 \, dx \leq 4\varepsilon^2(1 + M^2)$$

for all $n$.

On the other hand, by the property (G1)\$^2\$,

$$\int_{\Omega_{\delta(\varepsilon)}} \{J(u_n) - J(u)\}^2 \, dx \leq (\ell + |\gamma|) \int_{\Omega_{\delta(\varepsilon)}} (u_n - u)^2 \, dx.$$

From Proposition 2.2 (v),

$$\int_{\Omega_{\delta(\varepsilon)}} (u_n - u)^2 \, dx \rightarrow 0 \text{ as } n \rightarrow \infty$$

and hence

$$\limsup_{n \rightarrow \infty} \|J(u_n) - J(u)\|^2_2 \leq \int_{B(0, \delta(\varepsilon))} \{J(u_n) - J(u)\}^2 \, dx \leq 8\varepsilon^2(1 + M^2).$$

Since $\varepsilon > 0$ is arbitrary, this proves the compactness of $J = \tilde{g} - \gamma I : H_A \rightarrow L^2(\Omega)$. By Proposition 2.2(vi), $I : H_A \rightarrow L^2(\Omega)$ is not compact and so $\tilde{g} - \lambda I : H_A \rightarrow L^2(\Omega)$ cannot be compact for $\lambda \neq \gamma$.  

Propositions 4.1 and 4.3 provide hypotheses which are sufficient to ensure that the initial value problem (1.4) to (1.7) is well-posed as is shown in Section 6. The following results will be used in the analysis of the asymptotic stability of the stationary solution $u \equiv 0$.

For $0 < r \leq R$ set

$$\mu(\lambda, r, R) = \inf\{\|Su + \tilde{g}(u) - \lambda u\|_{L^2} : u \in D(S) \text{ and } r \leq \|u\|_A \leq R\}, \quad (4.16)$$
where $S$ is the self-adjoint operator defined in Section 3.2.

**Lemma 4.4.** Let the conditions $(A)_2$ and $(V)$ be satisfied and consider a nonlinearity $g = g_1 + g_2$ where the function $g_1$ satisfies the condition $(G1)_{g_1}$ and $g_2$ satisfies $(G2)_2$. Suppose also that $g_1(x, s)s \geq 0$ for all $x \in \Omega$ and $s \in \mathbb{R}$. Then for $\lambda < m$, there exists $R(\lambda) > 0$ such that $\mu(\lambda, r, R(\lambda)) > 0$ for all $r \in (0, R(\lambda)]$.

**Proof.** Given $\lambda < m$, choose $d \in (0, 1)$ such that $\lambda < m - d(m + \|V\|_{L^{\infty}})$. For $u \in D(S)$,

$$
\langle (S - \lambda)u, u \rangle_{L^2} = d\langle S_A u, u \rangle_{L^2} + d\langle Vu, u \rangle_{L^2} + (1 - d)\langle Su, u \rangle_{L^2} - \lambda\|u\|_{L^2}^2
$$

$$
\geq d\|u\|_{L^2}^2 - d\|V\|_{L^{\infty}}\|u\|_{L^2}^2 + (1 - d)m\|u\|_{L^2}^2 - \lambda\|u\|_{L^2}^2 \geq d\|u\|_{L^2}^2
$$

by (3.3) and the choice of $d$. It follows from Proposition 4.1 that there exists $\rho > 0$ such that

$$
|\langle \tilde{g}_2(u), u \rangle_{L^2}| \leq \frac{d}{2}\|u\|_{L^2}^2 \text{ for } u \in H_A \text{ with } \|u\|_{L^2} \leq \rho.
$$

Hence for $u \in D(S)$ with $\|u\|_{L^2} \leq \rho$,

$$
\langle (S - \lambda)u + \tilde{g}(u), u \rangle_{L^2} \geq \langle (S - \lambda)u + \tilde{g}_2(u), u \rangle_{L^2} \geq \frac{d}{2}\|u\|_{L^2}^2
$$

and so

$$
\frac{d}{2}\|u\|_{L^2}^2 \leq \|Su + \tilde{g}(u) - \lambda u\|_{L^2} \leq \frac{2}{N\sqrt{C_1}}\|u\|_{L^2}
$$

by (2.13). This proves that $\mu(\lambda, r, \rho) \geq \frac{rdN\sqrt{C_1}}{r} > 0$ for $r \in (0, \rho]$.

If $g_1$ satisfies the condition $(C)$ a similar conclusion can be obtained without restricting its sign.

**Lemma 4.5.** Let the conditions $(A)_2$ and $(V)$ be satisfied and consider a nonlinearity $g = g_1 + g_2$ where the function $g_1$ satisfies the conditions $(G1)_{g_1}$ and $(C)$ and the function $g_2$ satisfies the condition $(G2)_2$ for some exponents $\alpha > N\sigma/2$ and $0 < \sigma < 2/(N - 2)$.

- (A) The mapping $\tilde{g}_2 : H_A \to L^2(\Omega)$ is compact.
- (B) For $\lambda < m_\epsilon + \gamma$ where $\gamma$ is given by the condition $(C)$, the mapping $S + \tilde{g} - \lambda : D(S) \to L^2(\Omega)$ is proper when restricted to closed bounded subsets of $(D(S), \|\cdot\|)$. 
- (C) For $\lambda < \min\{m, m_\epsilon - \ell_1\}$ where $\ell_1$ is the Lipschitz constant for $g_1$ given by $(G1)_{g_1}$, there exists $R(\lambda) > 0$ such that $\mu(\lambda, r, R(\lambda)) > 0$ for all $r \in (0, R(\lambda)]$.

**Proof.** (A) Let $\{u_n\}$ be a sequence in $H_A$ which converges weakly to $u$ in $H_A$. It is enough to prove that $\|\tilde{g}_2(u_n) - \tilde{g}_2(u)\|_{L^2} \to 0$ as $n \to \infty$. Since $\{u_n\}$ is bounded in $H_A$ there is a constant $M$ such that $\|u_n\|_{H_A}$ and $\|u_n\|_{A} \leq M$ for all $n$.

By the assumption about $g_2$,

$$
|g_2(x, s)| \leq K\{r^\beta |s| + \frac{r^{\alpha - N\sigma} (r^{\gamma} |s|)^{1+\sigma}}{1+\sigma} \} \text{ for } x \in \Omega \text{ and } s \in \mathbb{R},
$$

where $\gamma = \frac{N\sigma}{2(1+\sigma)} < 1$. For $v \in H_A$ and $\epsilon \in (0, d(0, \partial\Omega))$ it follows that

$$
\int_{B(0, \epsilon)} \tilde{g}_2(v)^2 \ dx \leq 2K^2 \int_{B(0, \epsilon)} \epsilon^{2\beta} \int_{B(0, \epsilon)} \int_{B(0, \epsilon)} \epsilon^{2(\alpha - N\sigma)} |r^{\gamma} v|^2(1+\sigma) \ dx.
$$
Noting that \(2(1 + \sigma) = 2N/(N - 2\gamma)\), the estimate (2.5), which is valid for all \(u \in H_A\) by parts (6) and (7) of Proposition 2.1, shows that there is a constant \(C\) (independent of \(n\)) such that

\[
\int_{B(0,\varepsilon)} \tilde{g}_2(u_n)^2 + \tilde{g}_2(u)^2 \, dx \leq C\{\varepsilon^{2\beta}M^2 + \varepsilon^{2\alpha - N\sigma}M^{2(1+\sigma)}\} \quad \text{for all } n \in \mathbb{N}.
\]

Since \(0 < 2(1 + \sigma) < 2\), it follows from Proposition 2.2(v) that \(u_n \to u\) strongly in \(L^{2(1+\sigma)}(\Omega_e)\) as \(n \to \infty\). But (G2) and the boundedness of \(\Omega\) imply that there is a constant \(D\) such that \(|g_2(x, s)| \leq D(1 + |s|^{1+\sigma})\) for all \(x \in \Omega\) and \(s \in \mathbb{R}\). Hence \(\tilde{g}_2 \in L^{2(1+\sigma)}(\Omega_e)\) continuously into \(L^2(\Omega_e)\) and so

\[
\int_{\Omega_e} |\tilde{g}_2(u_n) - \tilde{g}_2(u)|^2 \, dx \to 0 \quad \text{as } n \to \infty.
\]

Thus, for all \(\varepsilon \in (0, d(0, \partial \Omega))\)

\[
\limsup_{n \to \infty} \|\tilde{g}_2(u_n) - \tilde{g}_2(u)\|_{L^2}^2 \leq 2C\{\varepsilon^{2\beta}M^2 + \varepsilon^{2\alpha - N\sigma}M^{2(1+\sigma)}\},
\]

where \(\beta > 0\) and \(2\alpha > N\sigma\). This implies that \(\|\tilde{g}_2(u_n) - \tilde{g}_2(u)\|_{L^2} \to 0\) as \(n \to \infty\).

(B) It is sufficient to show that if \(\{u_n\}\) is a bounded sequence in \((D(S), \| \cdot \|_S)\) such that the sequence \(\{Su_n + \tilde{g}(u_n) - \lambda u_n\}\) converges in \(L^2(\Omega)\), then \(\{u_n\}\) has a subsequence converging in \((D(S), \| \cdot \|_S)\). Recalling from (3.1) and the properties of \(S\) that \((D(S), \| \cdot \|_S)\) is continuously embedded in \((H_A, \| \cdot \|_A)\), \(\tilde{g} - \gamma = \tilde{g}_1 - \gamma + \tilde{g}_2\) is a compact mapping of \((D(S), \| \cdot \|_S)\) into \(L^2(\Omega)\) by Proposition 4.3 and part (A). Hence by passing to a suitable subsequence we can suppose that \(\{Su_n - (\lambda - \gamma)u_n\}\) converges in \(L^2(\Omega)\). But, since \(\lambda - \gamma < m_e\), \(S - (\lambda - \gamma) : D(S) \to L^2(\Omega)\) is a Fredholm operator of index zero and so this implies that \(\{u_n\}\) has a subsequence converging in \((D(S), \| \cdot \|_S)\) by Yood’s criterion. See Proposition 9.3 in Chapter 2 of [16] for example.

(C) Fix \(\lambda < \min\{m, m_e - \ell_1\}\). Suppose that \(\{u_n\} \subset D(S)\setminus\{0\}\) is a sequence such that \(Su_n + \tilde{g}(u_n) = \lambda u_n\) for all \(n \in \mathbb{N}\) and \(\|u_n\|_A \to 0\) as \(n \to \infty\). Then \(\lambda\) is a bifurcation point of weak solutions in the sense of Section 6.1 in [33]. By Remark 6.1(I) in [33] the condition (6.3) in Theorem 6.1 in [33] is satisfied since \(\lambda < m_e - \ell_1\) and so, by part (ii) of that result, \(\lambda\) must be an eigenvalue of \(S\). But this is not the case since \(\lambda < m\). Hence there exists \(R(\lambda) > 0\) such that \(Su + \tilde{g}(u) - \lambda u \neq 0\) for \(0 < \|u\|_A \leq R(\lambda)\). Fix \(r \in (0, R(\lambda))\) and suppose that there exists a sequence \(\{u_n\} \subset \subset D(S)\) such that \(r \leq \|u_n\|_A \leq R(\lambda)\) for all \(n\) and \(\|Su_n + \tilde{g}(u_n) - \lambda u_n\|_{L^2} \to 0\) as \(n \to \infty\). Since \(\tilde{g}\) maps any bounded subset of \(H_A\) into a bounded subset of \(L^2(\Omega)\) by Propositions 4.3 for \(g_1\) and 4.1(C) for \(g_2\), it follows that the sequence \(\{u_n\}\) is bounded in \((D(S), \| \cdot \|_S)\). From (G1)\(S\) and (C) we have that \(\gamma \leq \ell_1\) and hence \(\lambda < m_e + \gamma\). By part (B), there exist a subsequence \(\{u_{n_k}\}\) and \(w \in D(S)\) such that \(\|u_{n_k} - w\|_S \to 0\) as \(n_k \to \infty\). Then \(r \leq \|w\|_A \leq R(\lambda)\) and \(Sw + \tilde{g}(w) - \lambda w = 0\), contradicting the choice of \(R(\lambda)\). \(\square\)

5. **An initial value problem.** In the first part of this section, we recall a result concerning existence, uniqueness and regularity of solutions of an abstract initial value problem in a form which is convenient to deal with the problem (1.4) to (1.7) in Sections 6 and 7. The basic notions related to stability are defined at the beginning of the second part. Then conclusions about the stability of the stationary solution \(u \equiv 0\) of a gradient system are established in Theorem 5.3 by using the energy as a Lyapunov function. This result is used to treat the problem (1.4) to
(1.7) in the case of critical degeneracy in Section 7.2. For the subcritical situation, stability is determined in Theorem 7.1 using standard results on linearization.

5.1. An abstract IVP. Let \((H, \langle \cdot, \cdot \rangle, \|\cdot\|)\) be a real Hilbert space and \(A : D(A) \subset H \rightarrow H\) a self-adjoint operator which is bounded below. Let \((E_\alpha, \|\cdot\|_\alpha)\) denote \(D(|A|^\alpha)\) equipped with a norm \(\|\cdot\|_\alpha\), which is equivalent to the graph norm of \(|A|^\alpha\).

Let \[B_\alpha(u, r) = \{ v \in E_\alpha : \|u - v\|_\alpha < r \}.\]

Recall that \((E_\beta, \|\cdot\|_\beta)\) is densely and continuously embedded in \((E_\alpha, \|\cdot\|_\alpha)\) for \(\beta \geq \alpha \geq 0\).

A mapping \(F : E_\alpha \rightarrow H\) is locally Lipschitz continuous if, for every \(u \in E_\alpha\) there exist a radius \(r = r_u > 0\) and a constant \(L = L_u\) such that
\[
\|F(v) - F(w)\| \leq L\|v - w\|_\alpha \text{ for all } v, w \in B_\alpha(u, r).
\]

In this setting we consider the following abstract initial value problem
\[
\frac{d}{dt}u(t) + Au(t) = F(u(t)) \text{ for } t > 0 \text{ and } u(0) = u_0 \in E_\alpha.
\]

Given \(u_0 \in E_\alpha\), a solution of (5.1) is defined to be a function \(u : [0, T) \rightarrow H\), for some \(T > 0\), having the following properties.

\(u \in C([0, T), E_\alpha) \cap C^1([0, T), E_\alpha) \cap C((0, T), E_1)\) and (5.1) is satisfied.

Note that for a solution, \(F(u(\cdot)) \in C([0, T), H)\) and \(F(u(\cdot)) : (0, T) \rightarrow H\) is locally Lipschitz continuous. In particular, for \(t \in (0, T)\) the differential equation (5.1) expresses the equality of elements of \(H\) since \(E_1 = D(A)\).

The following result concerning the existence, uniqueness and regularity of solutions of (5.1) can be derived from the theory of analytic semigroups.

**Theorem 5.1.** Suppose that \(A : D(A) \subset H \rightarrow H\) is a self-adjoint operator which is bounded below and that \(F : E_\alpha \rightarrow H\) is locally Lipschitz continuous for some \(\alpha \in [0, 1)\).

A) For every \(u_0 \in E_\alpha\), there exists \(T > 0\) such that (5.1) has a unique solution on \([0, T)\). The solution \(u\) has the additional property that \(u \in C^1([0, T), E_\beta)\) for all \(\beta \in [0, 1)\).

B) For \(u_0 \in H_A\), let \(u(\cdot, u_0) : [0, T(u_0)) \rightarrow E_\alpha\) be the unique maximal solution of (5.1) for the initial condition \(u_0 \in H_A\). If \(F\) maps bounded subsets of \(E_\alpha\) into bounded subsets of \(H\), then \(\limsup_{t \rightarrow T(u_0)} \|u(t, u_0)\|_\alpha = \infty\) if \(T(u_0) < \infty\).

C) If there exists a constant \(C\) such that \(\|F(u)\| \leq C\{1 + \|u\|_\alpha\}\) for all \(u \in E_\alpha\), then \(T(u_0) = \infty\) for all \(u_0 \in E_\beta\).

The proof of Theorem 5.1 can be pieced together from the following results in [24]. See Theorem 3.3.3 for local existence and uniqueness, Theorem 3.3.4 for part (B) and Theorem 3.5.2 for additional regularity. Part (C) follows from Corollary 3.3.5.

**Remark 5.1.** Observe that in Theorem 5.1 there is no loss of generality in supposing that \(A\) is positive definite (i.e. there exists a constant \(c > 0\) such that \(\langle Au, u \rangle \geq c\|u\|^2\) for all \(u \in D(A)\)) since the hypotheses are still satisfied with \(A\) and \(F\) replaced by \(A + kI\) and \(F + kI\) for any constant \(k\) where \(I\) is the identity on \(H\).

**Remark 5.2.** Sometimes local Lipschitz continuity is taken to mean that for every \(r > 0\) there exists a constant \(L_r\) such that \(\|F(u) - F(v)\| \leq L_r\|u - v\|_\alpha\) for all \(u, v \in B_\alpha(0, r)\). This stronger assumption implies that \(F\) maps bounded subsets of
$E_\alpha$ into bounded subsets of $H$ and Theorem 5.1 is proved in Chapters 2 and 9 of [13] under this hypothesis. This definition of local Lipschitz continuity is also used in Theorem 3.1 of [5] where the conclusion in part (B) is sharpened to
\[
\lim_{t \to T(u_0)} \|u(t, u_0)\|_\alpha = \infty \text{ if } T(u_0) < \infty. \tag{5.2}
\]
In fact, the proof in [5] yields this conclusion under the hypotheses of Theorem 5.1.

The case where (5.1) is an infinite dimensional gradient system is particularly useful in dealing with (1.4) to (1.7) when there is critical degeneracy.

**Corollary 5.2.** Suppose that $A : D(A) \subset H \to H$ is a positive self-adjoint operator and that $F : E_{1/2} \to H$ is locally Lipschitz continuous. Suppose also that there exists a potential $\Pi \in C^1(E_{1/2}, \mathbb{R})$ such that
\[
\Pi'(u)v = \langle F(u), v \rangle \text{ for all } u, v \in E_{1/2}. \tag{5.3}
\]

Setting
\[
J(u) = \frac{1}{2}\|A^{1/2}u\|^2 - \Pi(u) \quad \text{for } u \in E_{1/2}, \tag{5.4}
\]
we have that $J \in C^1(E_{1/2}, \mathbb{R})$ with
\[
J'(u)v = \langle A^{1/2}u, A^{1/2}v \rangle - \langle F(u), v \rangle \quad \text{for all } u, v \in E_{1/2}. \tag{5.5}
\]
Furthermore, if $u : [0, T) \to E_{1/2}$ is a solution of (5.1), then $J(u(\cdot)) \in C([0, T), \mathbb{R}) \cap C^1((0, T), \mathbb{R})$ and
\[
\frac{d}{dt} J(u(t)) = -\|u'(t)\|^2 = -\|Au(t) - F(u(t))\|^2 \quad \text{for } 0 < t < T. \tag{5.6}
\]

**Proof.** Since $A^{1/2} \in B(E_{1/2}, H)$ and $u \in C([0, T), E_{1/2}) \cap C^1((0, T), E_{1/2})$, it follows that $J(u(\cdot)) \in C([0, T), \mathbb{R}) \cap C^1((0, T), \mathbb{R})$ and, for $t \in (0, T)$,
\[
\frac{d}{dt} J(u(t)) = \langle A^{1/2}u(t), A^{1/2}u'(t) \rangle - \langle F(u(t)), u'(t) \rangle.
\]

But $u(t) \in D(A)$ for $t \in (0, T)$ and so this yields
\[
\frac{d}{dt} J(u(t)) = \langle Au(t) - F(u(t)), u'(t) \rangle, \tag{5.7}
\]
where $u'(t) = -Au(t) + F(u(t))$ by (5.1). \hfill \Box

### 5.2. Stability and Instability for (5.1)

If, in addition to the hypotheses of Theorem 5.1, $F(0) = 0$, then the function $u(t) = 0$ for all $t \geq 0$ is a solution of (5.1) and its stability will be investigated in this part. Since most of the results are obtained using the energy $J$ introduced in Corollary 5.2 as a Lyapunov function only the case $\alpha = 1/2$ will be treated from now on and so the notation will be simplified as follows.

$$X = E_{1/2} \text{ and } \|\cdot\|_X = \|\cdot\|_{1/2}. \tag{5.7}$$

and it should be recalled that $(X, \|\cdot\|_X)$ is continuously embedded in $(H, \|\cdot\|)$. Under the hypotheses of Theorem 5.1 with $\alpha = 1/2$ and $F(0) = 0$, (5.1) generates a dynamical system on $X$ and in this context the usual definitions of stability for the stationary solution $u \equiv 0$ are now recalled for convenience. For $u_0 \in X$,
\[
u(\cdot, u_0) \subset C([0, T(\alpha)), X) \cap C^1((0, T(u_0)), X) \cap C((0, T(u_0)), D(A))
\]
denotes the unique maximal solution with $u(0) = u_0$.

The solution $u \equiv 0$ is said to be **stable** if there exists $\delta > 0$ such that for $\|u_0\|_X < \delta$, $T(u_0) = \infty$ and, for all $\varepsilon > 0$ there exists $\delta(\varepsilon) \in (0, \delta]$ such that
if \( \|u_0\|_X < \delta(\varepsilon) \), then \( \|u(t, u_0)\|_X < \varepsilon \) for all \( t \geq 0 \). Otherwise, it is said to be \textit{unstable}, in which case, there exist \( \varepsilon_0 > 0 \) and a sequence \( \{u_n\} \subset X \) such that \( \|u_n\|_X \to 0 \) as \( n \to \infty \) and \( \sup_{0 \leq t < T(u_n)} \|u(t, u_n)\|_X \geq \varepsilon_0 \) for all \( n \).

The solution \( u \equiv 0 \) is said to be \textit{asymptotically stable} provided that it is stable and that \( \delta > 0 \) can be chosen such that if \( \|u_0\|_X < \delta \), then \( T(u_0) = \infty \) and \( \|u(t, u_0)\|_X \to 0 \) as \( t \to \infty \).

In cases where the nonlinearity defines a Fréchet differentiable mapping from \( X \) into \( H \), the stability and instability of the solution \( u \equiv 0 \) follow immediately from standard results on nonlinear perturbation of linear systems. See Section 5 of [24], for example. Broadly speaking, the conclusion is that, provided \( 0 \neq \Sigma \equiv \inf \sigma(A - F'(0)) \), the stability of \( u \equiv 0 \) is the same as that of the linearized equation \( u'(t) + Au(t) = F'(0)u(t) \), namely, asymptotic stability if \( \Sigma > 0 \) and instability if \( \Sigma < 0 \). This situation is referred to as the principle of linearized stability in the Introduction. As is pointed out by Lemma 4.2, Fréchet differentiability is not always an appropriate property when dealing with (1.4), even for smooth nonlinearities.

An alternative way of studying stability is through the use of Lyapunov functionals and the rest of this section is devoted to developing this in a general setting which will cover the problem (1.4) to (1.7), even in non-Fréchet differentiable situations.

Under the hypotheses of Corollary 5.2, the energy \( J : X \to \mathbb{R} \) is continuously differentiable and if \( F(0) = 0 \), \( J'(0) = 0 \). In this case the stability of the stationary solution \( u \equiv 0 \) of (5.1) depends on nature of this critical point of \( J \). It lies in a \textit{potential well} of \( J \) provided that there exists \( \delta > 0 \) such that \( \inf\{J(u) : \|u\|_X = r\} > J(0) \) for all \( r \in (0, \delta) \). Clearly, this implies that \( J \) has a strict local minimum at 0 and the relationship is explored more closely in Proposition A.1 of [34].

The Lipschitz continuity that is assumed in Corollary 5.2 limits how steep a potential well of \( J \) can be. Let \( r > 0 \) and \( L \geq 0 \) be such that \( \|F(u) - F(0)\| \leq L\|u\|_X \) for all \( u \in B_X(0, r) = \{u \in X : \|u\|_X < r\} \). Then for \( u \in B_X(0, r) \),

\[
\|\Pi(u) - \Pi(0)\| = \left| \int_0^1 \frac{d}{dt} \Pi(tu) \, dt \right| = \left| \int_0^1 \langle F(tu), u \rangle \, dt \right| \leq \int_0^1 \|F(tu) - F(0, u)\| \, dt + \|F(0)\| \|u\| \leq L \|u\|_X \|u\| + \|F(0)\| \|u\|. 
\]  

Thus under the hypotheses of Corollary 5.2 and assuming that \( F(0) = 0 \), it follows easily that there exist \( r > 0 \) and \( K > 0 \) such that

\[
J(u) - J(0) \leq K\|u\|_X^\alpha \quad \text{for all } u \in B_X(0, r). 
\]  

If there exist \( r > 0 \) and \( \xi > 0 \) such that

\[
J(u) - J(0) \geq \xi\|u\|_X^\beta \quad \text{for all } u \in B_X(0, r),
\]  

then 0 is said to lie in a \textit{quadratic potential well} of \( J \). In [34] sufficient conditions for 0 to lie in a quadratic potential well are derived without requiring the potential to be twice Fréchet differentiable at 0. These criteria will be used in Section 7.2 to exploit the following result.

**Theorem 5.3.** Under the hypotheses of Corollary 5.2 set \( E_{1/2} = X \) and suppose in addition that \( F(0) = 0 \) and that \( F \) takes bounded subsets of \( X \) to bounded subsets of \( H \). Then \( u \equiv 0 \) is a stationary solution of (5.1).

(i) If 0 lies in a potential well of \( J \), then \( u \equiv 0 \) is stable. If in addition there exists \( \rho > 0 \) such that

\[
2\Pi(u) \geq \Pi'(u)u \quad \text{for all } u \in B_X(0, \rho),
\]
then there exists $\delta > 0$ such that $\|u(t, u_0)\| \to 0$ monotonically as $t \to \infty$ for all $u_0 \in B_X(0, \delta)$. In the case of a quadratic potential well, (5.11) implies that there exists $\eta > 0$ such that $\|u(t, u_0)\| \leq \|u_0\|e^{-\eta t}$ for all $t \geq 0$ if $u_0 \in B_X(0, \delta)$.

(ii) If $0$ lies in a potential well of $J$ and if there exists $\delta > 0$ such that

$$\inf\{\|Au - F(u)\| : u \in D(A) \text{ and } r \leq \|u\|_{X} \leq \delta\} > 0 \text{ for all } r \in (0, \delta], \quad (5.12)$$

then $u \equiv 0$ is asymptotically stable.

(iii) If $J$ does not have a local minimum at $0$ and if there exists $\varepsilon > 0$ such that

$$2\Pi(u) \leq \Pi'(u)u \text{ for all } u \in B_X(0, \varepsilon), \quad (5.13)$$

then $u \equiv 0$ is unstable. If (5.13) holds for all $\varepsilon > 0$, there exists a sequence $\{u_n\} \subset X$ such that $\|u_n\|_X \to 0$ as $n \to \infty$ and for all $n$, $J(u_n) < 0$, $\|u(t, u_n)\|^2 \geq \|u_n\|^2 + 4J(u_n)|t| \text{ for } 0 \leq t < T(u_n)$ and $\|u(t, u_n)\|_X \to \infty$ as $t \to T(u_n)$.

Proof. By Corollary 5.2, $J \in C^1(X, \mathbb{R})$ with $J'(0) = 0$ and $F(0) = 0$. Without loss of generality it can be assumed that $J(0) = 0$.

(i) By Proposition A.1 of [34] there exist $R > 0$ and a function $\mu \in C^1([0, R])$ such that $\mu(0) = \mu'(0) = 0$, $\mu'(r) > 0$ and $m(r) \geq \mu(r)$ for $0 < r \leq R$ where $m(r) = \inf\{J(u) : \|u\|_X = r\}$. Thus

$$J(u) \geq \mu(\|u\|_X) \text{ for } u \in B_X(0, R). \quad (5.14)$$

Since $J : X \to \mathbb{R}$ is continuous and $J(0) = 0$ there exists $\delta \in (0, R)$ such that $J(u) < \mu(R)/2$ for $u \in B_X(0, \delta)$. Combining these inequalities yields, $\mu(\delta) \leq \mu(R)/2$.

Suppose that there is an initial condition $u_0 \in B_X(0, \delta)$ such that

$$\sup\{\|t(u, u_0)\|_X : 0 \leq t < T(u_0)\} > \mu^{-1}(\mu(R)/2).$$

Since $u(\cdot, u_0) \in C([0, T(u_0)), X)$ and $\|u_0\|_X < \delta \leq \mu^{-1}(\mu(R)/2)$ there exists $s \in (0, T(u_0))$ such that $\|u(s, u_0)\|_X = \mu^{-1}(\mu(R)/2)$ and hence $J(u(s, u_0)) \geq \mu(\|u(s, u_0)\|_X) = \mu(R)/2$. But, $J(u(s, u_0)) \leq J(u_0)$ by (5.6) and $J(u_0) < \mu(R)/2$ since $u_0 \in B_X(0, \delta)$. From this contradiction, it follows that

$$\sup\{\|u(t, u_0)\|_X : 0 \leq t < T(u_0)\} \leq \mu^{-1}(\mu(R)/2)$$

for all $u_0 \in B_X(0, \delta)$ and hence that $T(u_0) = \infty$ by Theorem 5.1. By (5.14) and (5.6) this means that for any $u_0 \in B_X(0, \delta)$,

$$\mu(\|u(t, u_0)\|_X) \leq J(u(t, u_0)) \leq J(u_0) \text{ for all } t \geq 0$$

and consequently $\|u(t, u_0)\|_X \leq \mu^{-1}(J(u_0))$ for all $u_0 \in B_X(0, \delta)$ and $t \geq 0$. Noting that $\mu^{-1}oJ : B_X(0, \delta) \to \mathbb{R}$ is continuous and that $\mu^{-1}oJ(0) = 0$, it follows that for any $\varepsilon > 0$ there exists $\delta(\varepsilon) \in (0, \delta)$ such that $\mu^{-1}oJ(u_0) < \varepsilon$ for all $u_0 \in B_X(0, \delta(\varepsilon))$. This establishes the stability of the stationary solution $u \equiv 0$.

Now choose $\varepsilon = \min\{r, \rho, R\}$ where $r, \rho$ and $R$ are the radii in (5.9), (5.11) and (5.14), respectively. Since we are assuming that $J(0) = 0$ it follows from (5.9) and (5.14) that

$$0 < \mu(s) \leq Ks^2 \text{ for all } s \in (0, \varepsilon). \quad (5.15)$$

For this radius $\varepsilon$ the stability of $u \equiv 0$ ensures the existence of $\delta > 0$ such that $T(u_0) = \infty$ and $u(t, u_0) \in B_X(0, \varepsilon)$ for all $u_0 \in B_X(0, \delta)$ and $t \geq 0$. Consider $u_0 \in B_X(0, \delta)$ with $\|u_0\|_X \neq 0$ and simplify the notation by setting $u(t) = u(t, u_0)$ for all $t \geq 0$ and then $h(t) = \|u(t)\|^2$. From the uniqueness of the solution to the
initial value problem for (5.1), $h(t) > 0$ for all $t \geq 0$ and from the regularity of the solution $u$ it follows that $h \in C([0, \infty)) \cap C^1((0, \infty))$ with $u(t) \in B_X(0, \varepsilon)$ and
\[
h'(t) = 2\langle u(t), u'(t) \rangle = -2\langle u(t), Au(t) - F(u(t)) \rangle \\
= -4J(u(t)) - 4\Pi(u(t)) + 2\Pi'(u(t))u(t)
\]
for all $t > 0$. From (5.11) and (5.14) it follows that
\[
h'(t) \leq -4\mu(C^{-1}h^{1/2}(t))
\]
for $t > 0$, showing that $\|u(t)\|$ is strictly decreasing on $[0, \infty)$. Recalling that $\mu$ is increasing and that there exists a constant $C > 0$ such that $\| \cdot \| \leq C\| \cdot \|_X$ on $X$, this yields the differential inequality
\[
h'(t) \leq -4\mu(C^{-1}h^{1/2}(t))
\]
for all $t > 0$, and that there exists a constant $C > 0$ such that $\| \cdot \| \leq C\| \cdot \|_X$ on $X$, this yields the differential inequality
\[
h'(t) \leq -4\mu(C^{-1}h^{1/2}(t))
\]
for all $t > 0$, which can be written as
\[
\frac{2C^2k(t)k'(t)}{\mu(k(t))} \leq -4\mu(C^{-1}h^{1/2}(t))
\]
Integrating from $0$ to $t > 0$,
\[
2C^{-2}t \leq \int_{k(t)}^{k(0)} \frac{s}{\mu(s)} \, ds
\]
where $s/\mu(s)$ is positive and continuous on $[0, k(0)]$. It follows that $k(t) \to 0$ as $t \to \infty$.

If $0$ lies in a quadratic potential well of $J$, we can suppose that there exists $\xi > 0$ such that $J(u) \geq \xi\|u\|^2_X$ for all $u \in B_X(0, R)$ and so we can take $\mu(s) = \xi s^2$ for all $s \in [0, R]$. Then
\[
\int_{k(t)}^{k(0)} \frac{s}{\mu(s)} \, ds = \frac{\xi^{-1} \ln k(0)}{k(t)},
\]
from which it follows that $\|u(t)\| \leq \|u_0\|e^{-2C^{-2}t}$ for all $t \geq 0$ if $u_0 \in B_X(0, \delta)$.

(ii) The stability of $u \equiv 0$ is proved in part (i). Let $R$ and $\mu$ be as in (5.14) and let $\delta$ be given by (5.12). Setting $\varepsilon = \min\{R, \delta\}$, there exists $\rho > 0$ such that for all $u_0 \in B_X(0, \rho)$, $T(u_0) = \infty$ and $\|u(t, u_0)\|_X < \varepsilon$ for all $t \geq 0$. Given $u_0 \in B_X(0, \rho)$, either $\lim_{t \to \infty} \|u(t, u_0)\|_X > 0$ or there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and $\|u(t_n, u_0)\|_X \to 0$ as $n \to \infty$. In the first case, there exist $T > 0$ and $\beta \in (0, \varepsilon)$ such that $\beta \leq \|u(t, u_0)\|_X \leq \varepsilon$ for all $t \geq T$. From (5.12), this implies that for $t \geq T$, $\|Au(t, u_0) - F(u(t, u_0))\| \geq K$ for some $K > 0$. Using (5.6) it follows that for $t \geq T$,
\[
J(u(t, u_0)) = J(u(T, u_0)) - \int_{t}^{T} \|Au(s, u_0) - F(u(s, u_0))\|^2 \, ds \\
\leq J(u(T, u_0)) - K^2(t - T)
\]
and so $J(u(t, u_0)) \to -\infty$ as $t \to \infty$. But this is false since by (5.14), $J(u(t, u_0)) \geq \mu(\|u(t, u_0)\|_X) \geq 0$. Hence the first case cannot occur. In the second case,
\[
J(u(t_n, u_0)) \to J(0) = 0 \quad \text{as} \quad n \to \infty \quad \text{by the continuity of} \quad J : X \to \mathbb{R}
\]
and so $J(u(t, u_0)) \to 0$ as $t \to \infty$ since $J(u(t, u_0))$ is non-increasing on $[0, \infty)$ by (5.6).

From (5.14) it now follows that $\|u(t, u_0)\|_X \to 0$ as $t \to \infty$, proving the asymptotic stability of the stationary solution $u \equiv 0$.

(iii) Seeking a contradiction, suppose that the solution $u \equiv 0$ is stable. For the $\varepsilon$ given by (5.13), there exists $\delta > 0$ such that for all initial conditions $u_0 \in B_X(0, \delta)$,
\( T(u_0) = \infty \) and \( \|u(t, u_0)\|_X < \varepsilon \) for all \( t \geq 0 \). Since \( J : X \to \mathbb{R} \) does not have a local minimum at 0 there exists \( u_0 \in B_X(0, \delta) \) such that \( J(u_0) = J(0) = 0 \). For this \( u_0 \) simplify the notation by setting \( u(t) = u(t, u_0) \) for all \( t \geq 0 \) and then \( h(t) = \|u(t)\|^2 \).

Recalling that there exists a constant \( C > 0 \) such that \( \|\cdot\| \leq C \|\cdot\|_X \) on \( X \), \( h(t) \leq C^2 \|u(t)\|^2_X \leq C^2 \varepsilon^2 \) for all \( t \geq 0 \). As for (5.16), \( h \in C((0, \infty)) \cap C^1((0, \infty)) \) and

\[
    h'(t) = -4J(u(t)) - 4\Pi(u(t))u(t) + 2\Pi'(u(t))u(t) \geq -4J(u(t)) \quad (5.17)
\]

by (5.13) for all \( t > 0 \) since \( \|u(t)\|_X < \varepsilon \). But \( J(u(t)) \leq J(u_0) < 0 \) for all \( t \geq 0 \) by (5.6) and so \( h'(t) \geq 4J(u_0) > 0 \) for all \( t > 0 \) from which it follows that \( h(t) \to \infty \) as \( t \to \infty \). This contradiction shows that the solution \( u \equiv 0 \) is unstable.

Suppose now that (5.13) holds for all \( \varepsilon > 0 \). Since 0 is not a local minimum of \( J \) there exists a sequence \( \{u_n\} \subset X \) such that \( J(u_n) < J(0) = 0 \) for all \( n \) and \( \|u_n\|_X \to 0 \) as \( n \to \infty \). Let \( h_n(t) = \|u(t, u_n)\|^2 \). As before (5.17) holds for \( 0 < t < T(u_n) \) and yields

\[
    h_n(t) \geq h_n(0) + 4J(u_n)|t| \quad \text{for } 0 \leq t < T(u_n). \quad (5.18)
\]

If there is a subsequence \( \{u_{n_k}\} \) such that \( T(u_{n_k}) < \infty \) for all \( n \) it follows from (5.2) that

\[
    \lim_{n \to T(u_{n_k})} \|u(t, u_{n_k})\|_X = \infty, \text{ as required.}
\]

On the other hand, if there exists \( n_0 \) such that \( T(u_n) = \infty \) for all \( n \geq n_0 \) it suffices to let \( t \to \infty \) in (5.18). \( \square \)

The final result in this section gives another criterion for instability without requiring Fréchet differentiability at \( u \equiv 0 \). It will be useful even in the subcritical case where the nonlinearity is Fréchet differentiable at \( u \equiv 0 \) since it requires less regularity that similar results. For example, Theorem 5.1.3 in [24] requires what is sometimes called strict Fréchet differentiability at \( u \equiv 0 \).

**Theorem 5.4.** Let \( A : D(A) \subset H \to H \) be a self-adjoint operator which is bounded below and let \( X \) be given by (5.7). Suppose that \( F : X \to H \) is locally Lipschitz continuous on \( X \) and Gâteaux differentiable at 0 with \( F'(0) = L \) where \( L \in B(H, H) \) is self-adjoint. Suppose also that \( F - F'(0) = R_1 + R_2 \) for mappings \( R_1 \) and \( R_2 \) having the properties,

- \( R_1 : X \to H \) is locally Lipschitz continuous on \( X \) with \( \langle R_1(u) - R_1(v), u - v \rangle \leq 0 \) for all \( u, v \in X \) and
- \( R_2 : X \to H \) is Fréchet differentiable at 0 with \( R_2'(0) = 0 \).

If \( F(0) = 0 \) and \( \inf \sigma(A - L) < \min \{0, \inf \sigma_e(A - L)\} \), then the stationary solution \( u \equiv 0 \) of (5.1) is unstable.

**Remark 5.3.** The hypotheses imply that \( A - L : D(A) \subset H \to H \) is self-adjoint and that the remainder \( R = R_1 + R_2 : X \to H \) is Gâteaux differentiable at 0 with \( R'(0) = 0 \). Hence \( R_1 : X \to H \) is also Gâteaux differentiable at 0 with \( R_1'(0) = 0 \). Furthermore, \( R_2 : X \to H \) is locally Lipschitz continuous on \( X \).

**Proof** Referring to Remark 5.1, we can assume that there exists \( c > 0 \) such that \( \langle Au, u \rangle \geq c\|u\|^2 \) for all \( u \in D(A) \) and that \( A - L, R_1 \) and \( R_2 \) remain unchanged by this adjustment.

Choosing some \( \lambda > \|L\|_{B(H, H)} \), it follows from Theorem 9.1 in [37] that \( D(S) = D(A) \) and \( S = A - L + \lambda I \) define a self-adjoint operator \( S : D(S) \subset H \to H \). For \( u \in D(S) \), \( \langle Su, u \rangle \geq \|A^{1/2}u\|^2 + (\lambda - \|L\|)\|u\|^2 \geq \|A^{1/2}u\|^2 \) so \( S \) is positive definite and

\[
    \langle Su, u \rangle \leq \|A^{1/2}u\|^2 + (\lambda + \|L\|)\|u\|^2 \leq \{1 + (\lambda + \|L\|)/c\}\|A^{1/2}u\|^2.
\]
Hence \( \|A^{1/2}(\cdot)\| \) and \( \|S^{1/2}(\cdot)\| \) define equivalent norms on \( D(A) = D(S) \). It follows easily from this that \( D(S^{1/2}) = D(A^{1/2}) = X \) and that \( \|S^{1/2}(\cdot)\| \) is equivalent to \( \|\cdot\|_X \) on \( X \). Henceforth we consider the Hilbert space \((X, \langle \cdot, \cdot \rangle_1, \|\cdot\|_1)\) where

\[
\langle u, v \rangle_1 = \langle S^{1/2}u, S^{1/2}v \rangle \quad \text{and} \quad \|u\|_1 = \|S^{1/2}u\| \quad \text{for } u, v \in X.
\]

Setting \( m = \inf \sigma(S) = \inf \sigma(A - L) + \lambda \) and \( m_e = \inf \sigma_e(S) = \inf \sigma_e(A - L) + \lambda \), the hypotheses imply that \( \lambda > m \) and \( m < m_e \). Hence \( m \) is an isolated eigenvalue of \( S \) having finite multiplicity. Set \( E = \ker(S - mI) \) and let \( P \) and \( Q \) denote the orthogonal projections of \( H \) onto \( E \) and \( F = E^\perp \), respectively. Noting that \( Qu = u - Pu \in X \) if and only if \( u \in X \) since \( E \subset D(S) \subset X \), it is easily seen that \( P \) and \( Q \) are orthogonal projections on the space \((X, \langle \cdot, \cdot \rangle_1)\) onto \( E \) and \( F_1 = F \cap X \), respectively.

Choose \( \rho \in (m, \lambda) \) such that \( \sigma(S) \cap (-\infty, \rho) = \{m\} \) and let \( M = \inf[\sigma(S) \setminus \{m\}] \).

Then,

\[
\|u\|_2^2 = m\|u\|^2 \quad \text{and} \quad \langle (S - \rho)u, u \rangle = -\left(\frac{\rho}{m} - 1\right)\|u\|^2 \quad \text{for all } u \in E
\]

and

\[
\langle (S - \rho)u, u \rangle \geq (1 - \frac{\rho}{M})\|u\|^2 \quad \text{for all } u \in F \cap D(S).
\]

Setting \( \varepsilon(\rho) = \min\{\frac{\rho}{M} - 1, 1 - \frac{\rho}{M}\} \), we have that \( \varepsilon(\rho) > 0 \) since \( 0 < m < \rho < M \). For any \( u \in D(S) \),

\[
\langle (Q - P)u, (S - \lambda)u \rangle = \langle (Q - P)u, (S - \rho)u \rangle + (\rho - \lambda)\langle (Q - P)u, u \rangle
\]

\[
= \langle Qu, (S - \rho)Qu \rangle - \langle Pu, (S - \rho)Pu \rangle + (\rho - \lambda)\{\|Qu\|^2 - \|Pu\|^2\}
\]

\[
\geq (1 - \frac{\rho}{M})\|Qu\|^2 + \left(\frac{\rho}{m} - 1\right)\|Pu\|^2 + (\rho - \lambda)\{\|Pu\|^2 - \|Qu\|^2\}
\]

\[
\geq \varepsilon(\rho)\|u\|^2 + (\lambda - \rho)\{\|Pu\|^2 - \|Qu\|^2\},
\]

since \( P \) and \( Q \) are orthogonal projections in both \((H, \|\cdot\|)\) and \((X, \|\cdot\|_1)\). Furthermore, for \( u \in D(S) \),

\[
\langle (Q - P)u, R_1(u) \rangle = \langle u - 2Pu, R_1(u) \rangle
\]

\[
= \langle u - 2Pu, R_1(u) - R_1(2Pu) \rangle + \langle u - 2Pu, R_1(2Pu) \rangle
\]

\[
\leq \|u - 2Pu, R_1(2Pu)\| \leq \|(Q - P)u\|\|R_1(2Pu)\| = \|u\|\|R_1(2Pu)\|.
\]

Since \( R_1 : X \to H \) is Gâteaux differentiable at 0 with \( R_1'(0) = 0 \) and Lipschitz continuous on an open neighbourhood of 0, it is Hadamard differentiable at 0 ([19], page 259). This implies that its restriction to any finite dimensional subspace of \( X \) is Fréchet differentiable at 0 ([19], page 266). Hence \( R_1(2P(\cdot)) : X \to H \) is Fréchet differentiable at 0 with derivative 0 and it follows that there exists \( \delta_1 > 0 \) such that

\[
\|u\|\|R_1(2Pu)\| < \frac{\varepsilon(\rho)}{4}\|u\|^2 \quad \text{for } u \in B_1(0, \delta_1),
\]

where \( B_1(0, \delta_1) = \{u \in X : \|u\|_1 < \delta_1\} \).

By hypothesis, \( R_2 : X \to H \) is Fréchet differentiable at 0 with derivative 0 so there exists \( \delta_2 > 0 \) such that

\[
\|\langle (Q - P)u, R_2(u) \rangle\| < \frac{\varepsilon(\rho)}{4}\|u\|^2 \quad \text{for } u \in B_1(0, \delta_2).
\]
Combining these inequalities shows that, for \( u \in D(S) \) with \( \|u\|_1 < \delta = \min\{\delta_1, \delta_2\} \),
\[
\langle (Q - P)u, (S - \lambda)u - R(u) \rangle \geq \frac{1}{2} \zeta(\rho)\|u\|_1^2 + (\lambda - \rho)\{\|Pu\|^2 - \|Qu\|^2\} \\
\geq (\lambda - \rho)\{\|Pu\|^2 - \|Qu\|^2\}.
\]

Seeking a contradiction, suppose the \( u \equiv 0 \) is a stable solution of (5.1). Then there exists \( r > 0 \) such that for all \( u_0 \in B_1(0, r) \), \( T(u_0) = \infty \) and \( u(t, u_0)\|_1 < \delta \) for all \( t \geq 0 \).

For \( u_0 \in X \) with \( \|u_0\|_1 < r \) and \( t \geq 0 \), let
\[
\xi_{u_0}(t) = \frac{1}{2}\{\|Pu(t)\|^2 - \|Qu(t)\|^2\}, \text{ where } u(t) = u(t, u_0).
\]

Then,
\[
\xi_{u_0}(t) \leq \frac{1}{2}\|Pu(t)\|^2 = \frac{1}{2m}\|Pu(t)\|_1^2 \leq \frac{1}{2m}\|u(t)\|_1^2 \leq \frac{\delta^2}{2m}.
\]

On the other hand by (5.1), \( \xi_{u_0} \in C([0, \infty)) \cap C^1((0, \infty)) \) and for \( t > 0 \),
\[
\xi_{u_0}'(t) = \langle Pu(t), Pu'(t) \rangle - \langle Qu(t), Qu'(t) \rangle = \langle Pu(t) - Qu(t), u'(t) \rangle \\
= (\lambda - \rho)\{\|Pu(t)\|^2 - \|Qu(t)\|^2\} = 2(\lambda - \rho)\xi_{u_0}(t),
\]
since \( u(t) \in D(S) \) with \( \|u(t)\|_1 < \delta \) for all \( t > 0 \). This implies that \( \xi_{u_0}(t) \geq \xi_{u_0}(0)e^{(\lambda - \rho)t} \) for all \( t \geq 0 \).

Choosing \( u_0 \in E \) with \( 0 < \|u_0\|_1 < r \), \( \xi_{u_0}(0) = \frac{1}{2}\|u_0\|^2 > 0 \) and it follows that \( \xi_{u_0}(t) \to \infty \) as \( t \to \infty \). Since it was shown previously that \( \xi_{u_0}(t) \leq \delta^2/(2m) \) for all \( t \geq 0 \) if \( \|u_0\|_1 < r \), this contradiction means that the solution \( u \equiv 0 \) must be unstable. 

\[ \square \]

6. **The problem (1.4) to (1.7).** Throughout this section the initial value problem (1.4) to (1.7) is considered under the following assumptions. Either

\begin{itemize}
\item[(SC)] (subcritical case) \( A \) satisfies \((A)_\tau\) for some \( \tau \in [0, 2) \), \( V \in L^\infty(\Omega) \) and \( g = g_1 + g_2 \) where \( g_1 \) satisfies \((G1)_\tau\) and \( g_2 \) satisfies \((G2)_\tau\),
\item[(CC)] (critical case) \( A \) satisfies the condition \((A)_2\), \( V \) satisfies \((V)\) and \( g = g_1 + g_2 \) where \( g_1 \) satisfies \((G1)_2\) and \( g_2 \) satisfies \((G2)_2\).
\end{itemize}

The development is based on the results in Section 5 using the Hilbert space
\[
H = L^2(\Omega) \text{ with } \langle u, v \rangle_{L^2} = \int _\Omega uv \, dx \text{ and } \|u\|_{L^2} = \left\{\int _\Omega u^2 \, dx\right\}^{1/2}
\]
and the positive self-adjoint operator \( S_A = -\nabla \cdot \{A\nabla\} \), defined in Section 3. It is positive definite by (3.4) and and \( D(S_A^{1/2}) = H_A \) by (3.6). Furthermore, \( \|\cdot\|_A \) is equivalent to the graph norm of \( S_A^{1/2} \) on \( H_A \). Thus \( (H_A, \langle \cdot, \cdot \rangle_A, \|\cdot\|_A) \) plays the role of the space \( X \) in (5.7).

For all \( \lambda \in \mathbb{R} \), the mapping \( F_\lambda \) defined by
\[
F_\lambda(u)(x) = \lambda u(x) - V(x)u(x) - g(x, u(x)) \quad (6.1)
\]
is locally Lipschitz continuous from \( H_A \) into \( L^2(\Omega) \) by Propositions 4.1 and 4.3 and it takes bounded subsets of \( H_A \) into bounded subsets of \( L^2(\Omega) \). (These results also show that \( F_\lambda \) is locally Lipschitz continuous in the stronger sense mentioned
in Remark 5.2.) Hence under these assumptions it follows from Theorem 5.1 with 
\( \alpha = 1/2 \) that, for every \( u_0 \in H_A \), there exists \( T = T(u_0) > 0 \) and a unique function 
\[ u \in C([0, T), H_A) \cap C^1((0, T), H_A) \cap C((0, T), D(S_A)) \]  
(6.2)
such that \( u(0) = u_0 \) and \( u'(t) + S_A u(t) = F(\lambda u(t)) \) for \( t \in (0, T) \) as an equality in 
\( L^2(\Omega) \). This differential equation can be written as 
\[ u'(t) + S u(t) + \tilde{g}(u(t)) = \lambda u(t), \]  
(6.3)
where \( S = S_A + V \) is the self-adjoint operator with \( D(S) = D(S_A) \) discussed in 
Section 3.2. For an initial condition \( u_0 \in H_A \) the unique maximal solution of (6.3) 
satisfying (6.2) will be denoted by \( u(\cdot, u_0) : [0, T(u_0)) \to H_A \). 
It follows from part (B) of Theorem 5.1 and (5.2) that \( \lim_{t \to T(u_0)} \|u(t, u_0)\|_A = \infty \) if 
\( T(u_0) < \infty \). By part (C) of Theorem 5.1, \( T(u_0) = \infty \) for all \( u_0 \in H_A \) if \( \tau = 2 \) 
and \( g_2 \equiv 0 \).

Setting 
\[ \phi_\lambda(u) = \frac{1}{2} \int_\Omega A|\nabla u|^2 + (V - \lambda)u^2 dx + \psi_1(u) + \psi_2(u) \]  
for \( u \in H_A \), 
(6.4)
where 
\[ \psi_i(u) = \int_\Omega G_i(x, u(x)) dx \]  
and 
\[ G_i(x, s) = \int_0^s g_i(x, y) dy, \]  
(6.5)
for \( i = 1, 2 \), Propositions 4.1 and 4.3 show that \( \phi_\lambda \in C^1(H_A, \mathbb{R}) \) with 
\[ \phi'_\lambda(u)(v) = \int_\Omega A \nabla u \cdot \nabla v + (V - \lambda)uv + \tilde{g}(u)v dx \]  
for all \( u, v \in H_A \). 
(6.6)
By (3.6) this energy functional can also be written as 
\[ J_\lambda(u) = \frac{1}{2} \|S_A^{1/2} u\|_L^2 - \Pi_\lambda(u) \]  
for \( u \in H_A \), 
(6.7)
where 
\[ \Pi_\lambda(u) = -\frac{1}{2} \int_\Omega (V - \lambda)u^2 dx + \psi_1(u) + \psi_2(u) \]  
(6.8)
and 
\[ \Pi_\lambda(u)v = -\int_\Omega \{Vu - \lambda u + \tilde{g}(u)\}v dx = (F_\lambda(u), v)_{L^2} \]  
for \( u, v \in H_A \), 
(6.9)
by Propositions 4.1 and 4.3. Noting that \( F_\lambda(0) = 0 \), the basic hypotheses of Corollary 5.2 and Theorem 5.3 are satisfied. In particular, if \( u : [0, T) \to H_A \) satisfies 
(6.2) and (6.3), it follows from Corollary 5.2 that \( \phi_\lambda(u(\cdot)) \in C^1((0, T), \mathbb{R}) \) and 
\[ \frac{d}{dt} \phi_\lambda(u(t)) = -\|Su(t) + \tilde{g}(u(t)) - \lambda u(t)\|^2_{L^2} = -\|u'(t)\|^2_{L^2} \]  
for \( t \in (0, T) \). 
(6.9)
From the uniqueness of the solution of the initial value problem this means that 
\( \phi_\lambda(u(\cdot)) \) is strictly decreasing on \( (0, T) \) unless \( u \) is a stationary solution.

For future reference, the main conclusions, which have just been derived, concerning the degenerate parabolic problem (1.4) to (1.7) are collected the following result.

**Theorem 6.1.** Consider the problem (1.4) to (1.7) under the assumptions (SC) or 
(CC). For an initial condition \( u_0 \in H_A \) there is a unique solution in the sense of 
(6.2) and (6.3). For the maximal solution \( u(\cdot, u_0) : [0, T(u_0)) \to H_A \), \( \lim_{t \to T(u_0)} \)
\[ \|u(t, u_0)\|_A = \infty \text{ if } T(u_0) < \infty. \] Furthermore, for the energy functional defined by (1.17), \( \phi_\lambda(u(\cdot, u_0)) \in C([0, T(u_0)), \mathbb{R}) \cap C^1((0, T(u_0)), \mathbb{R}) \) and
\[
\frac{d}{dt} \phi_\lambda(u(t, u_0)) = -\|u'(t, u_0)\|_{L^2, 2}^2 \text{ for } 0 < t < T(u_0).
\]

6.1. Stationary solutions of (1.4). Under the hypotheses of Theorem 6.1 a stationary solution of the problem (1.4) to (1.7) is a pair \((\lambda, u_0)\) \(\in \mathbb{R} \times H_A\) such that the function defined by \(u(t) = u_0\) for all \(t \geq 0\) satisfies (6.2) and (6.3). That is,
\[
u_0 \in D(S) \text{ and } Su_0 + g(u_0) = \lambda u_0 \text{ in } L^2(\Omega),
\]
where \(S = -\nabla \cdot (A \nabla) + V\). This is a degenerate elliptic nonlinear eigenvalue problem for which it is also natural to consider weak solutions, defined as elements \((\lambda, u_0) \in \mathbb{R} \times H_A\) such that
\[
\int_{\Omega} A \nabla u_0 \cdot \nabla v + \{V u_0 + g(u_0) - \lambda u_0\} v \, dx = 0 \text{ for all } v \in C_0^\infty(\Omega).
\]
However, the hypotheses of Theorem 6.1 ensure that \(Vu_0 + g(u_0) - \lambda u_0 \in L^2(\Omega)\) for all \(u_0 \in H_A\) and so by (3.8), (6.12) implies that \(u_0 \in D(S_A) = D(S)\) and that \((\lambda, u_0)\) satisfies (6.11).

Clearly, \((\lambda, u_0 \equiv 0)\) is a stationary solution for all \(\lambda \in \mathbb{R}\) since \(g(x, 0) \equiv 0\). The existence of non-trivial stationary solutions can be established using bifurcation theory. In the subcritical case this is straightforward since \(\tilde{g} \in C^1(H_A, L^2(\Omega))\), if \(g_1\) satisfies (G1)\(_\tau\) instead of (G1)\(_\tau\), and \(S\) has discrete spectrum so the simple eigenvalue \(m\) is a bifurcation point (Theorem 4.1 in Chapter 5 of [4] for example).

In fact, by the well-known result concerning variational problems (Theorem 6.6 in [27] for example) there is bifurcation at every eigenvalue of \(S\) since \(\phi_\lambda \in C^2(H_A, \mathbb{R})\) if \(g_1\) satisfies (G1)\(_\tau\). The situation is much more complicated in the critical case since \(\sigma_e(S) \neq 0\) and \(\tilde{g}\) may not be Fréchet differentiable at 0. However a variety of sufficient conditions for bifurcation can be found in [32] and [33], where the conditions imposed on the nonlinearity are considerably weaker than those required here for Theorem 6.1. But there the also situations where the hypothesis (CC) of Theorem 6.1 is satisfied and \((\lambda, u_0 \equiv 0)\) are the only stationary solutions. The following result is a special case of Corollary 7.8 in [33].

Proposition 6.2. Consider the problem (6.11) under the following assumptions.

(i) The bounded open set \(\Omega\) is star-shaped with respect to 0 and its boundary \(\partial \Omega\) is of class \(C^2\).
(ii) \(A(x) = |x|^2 B(x)\) for \(x \in \bar{\Omega}\) where \(B \in C^1(\bar{\Omega})\) with \(B(x) > 0\) and \(x \cdot \nabla B(x) \geq 0\) for all \(x \in \bar{\Omega}\).
(iii) \(V \in C^1(\bar{\Omega})\) with \(x \cdot \nabla V(x) \geq 0\) for all \(x \in \bar{\Omega}\).
(iv) \(g(x, s) = -ck(s) + d|x|^\alpha s^\sigma\) for \((x, s) \in \Omega \times \mathbb{R}\), where \(c \geq 0, d \geq 0, \sigma \in (0, 2/(N - 2))\) and \(\alpha > N \sigma/2\) are constants and \(k\) is a function satisfying the condition (K) with \(k(s)/s\) strictly increasing on \((0, \infty)\).

Then the hypothesis (CC) of Theorem 6.1 is satisfied and \(m = m_e = \frac{N^2B(0)}{4} + V(0)\). If \((c, d) \neq (0, 0)\), \(\{(\lambda, u \equiv 0) : \lambda \in \mathbb{R}\}\) are the only stationary solutions of the nonlinear problem (1.4) to (1.7). If either \(x \cdot \nabla B(x) > 0\) on \(\Omega\) or \(x \cdot \nabla V(x) > 0\) on \(\Omega\), then the operator \(S\) has no eigenvalues.

7. Stability in \(H_A\) of the solution \(u \equiv 0\). By Theorem 6.1 the problem (1.4) to (1.7) generates a dynamical system on \(H_A\) and the objective is now to determine how
the stability of the stationary solution \( u \equiv 0 \) depends upon the real parameter \( \lambda \). Stability and instability are as defined in Section 5.2 with \((X, \|\cdot\|_X) = (H_A, \|\cdot\|_A)\).

7.1. Subcritical degeneracy. In the subcritical case \( H_A = F_\tau \) where \( \tau \in [0, 2) \) and the principle of linearized stability can be justified as follows.

**Theorem 7.1.** Consider the initial value problem (1.4) to (1.7) under the assumption (SC). Let \( S \) denote the self-adjoint operator treated in Proposition 3.1 with \( m = \inf \sigma(S) \).

The conclusions of Theorem 6.1 hold and the stationary solution \((\lambda, u \equiv 0)\) is asymptotically stable for \( \lambda < m \) and unstable for \( \lambda > m \).

**Remark 7.1.** Unlike the situation mentioned in Remark 7.3 where diffusion does not eliminate a singularity at the origin, the initial value problem does have a regularizing effect on all of \( \Omega \) in the subcritical case. See [30, 23].

**Proof.** Setting \( F_\lambda(u) = \lambda u - V u - \tilde{g}(u) \) for \( u \in H_A \), as in (6.1), Propositions 4.1 and 4.3 show that \( F_\lambda : H_A \to L^2(\Omega) \) is locally Lipschitz continuous. Setting \( L_\lambda u = (\lambda - V)u, L_A \in B(L^2(\Omega), L^2(\Omega)) \) is self-adjoint and these results also show that \( F_\lambda : H_A \to L^2(\Omega) \) is Fréchet differentiable at 0 with \( F_\lambda'(0) = L_\lambda \mid_{H_A} \). Furthermore, \( \sigma(S_A - L_\lambda) = \sigma(S) - \lambda \) and, by Proposition 3.1, \( \sigma_e(S) = \emptyset \). The statement about asymptotic stability now follows directly from Theorem 5.1.1 in [24]. Instability follows from Theorem 5.4 with \( R_1 \equiv 0 \) and \( R_2 = -\tilde{g} \) since \( \inf \sigma_e(S_A - L_\lambda) = \infty \).

7.2. Critical degeneracy. In this part the stability of the solution \( u \equiv 0 \) of the problem (1.4) to (1.7) is considered under the hypotheses (CC). By Theorem 6.1 we are now dealing with a dynamical system on \((H_A, \|\cdot\|_A)\) where \( H_A = F_2 = E_2 \) for \( N \geq 3 \) and \( H_A = E_2 \) is the completion of \( F_2 \) for \( N = 2 \), as discussed in Section 2.2. The quantities \( m \) and \( m_e \) appearing in the following criteria for stability are defined in Proposition 3.2. For a restricted class of nonlinearities the principle of linearized stability can still be justified.

**Theorem 7.2.** Consider the initial value problem (1.4) to (1.7) under the hypothesis (CC) with \( g_1 \equiv 0 \). The conclusions of Theorem 6.1 hold and the stationary solution \((\lambda, u \equiv 0)\) is asymptotically stable for \( \lambda < m \). It is unstable for \( \lambda > m \) provided that \( g \) is a finite sum of terms satisfying the condition \((G2)_2^*\).

**Proof.** Using Proposition 4.1, asymptotic stability follows from Theorem 5.1.1 in [24] and, under the stronger assumption \((G2)_2^*\), instability follows from Corollary 5.1.6 in [24].

The rest of this section is devoted to the more interesting situation where (CC) is satisfied and \( g_1 \not\equiv 0 \). As pointed out by Lemma 4.2, the condition \((G1)_2\) does not imply that \( \tilde{g}_1 : H_A \to L^2(\Omega) \) is Fréchet differentiable at 0 and the results used to prove Theorem 7.2 do not apply. In fact, they would lead to an incorrect conclusion in cases where the principle of linearized stability fails. Most of the following conclusions about stability and instability will be obtained from Theorem 5.3 with \( X = H_A \) by using the energy (6.4) as a Lyapunov functional. In verifying the hypotheses of Theorem 5.3 extensive use will be made of the results in [34] concerning the existence of a potential well for a functional whose gradient is not necessarily Fréchet differentiable.
Theorem 7.3. Consider the initial value problem (1.4) to (1.7) under the hypotheses (CC). Let
\[ \nu_1 = \inf_{x \in \mathbb{R}, s > 0} \frac{G_1(x, s)}{s^2} \text{ and } \gamma_1 = \limsup_{s \to \infty} \frac{G_1(x, s)}{s^2}, \]
where \( G_1(x, s) = \int_0^s g_1(x, y) \, dy \). Note that by \((G1)_2^∗\),
\[ -\nu_1 \leq \gamma_1 \leq \nu_1 \leq \ell_1, \]
where \( \ell_1 \) is the best Lipschitz constant for \( g_1 \) given by \((4.9)\).

The conclusions of Theorem 6.1 hold and \((\lambda, u) \equiv 0\) is a stationary solution for all \( \lambda \in \mathbb{R} \).

(A, stability) (i) For \( \lambda < \min\{m, \max\{m - |\nu_1|, m_c - \ell_1\}\} \), the solution \((\lambda, u) \equiv 0\) is stable.

(ii) If in addition to satisfying the condition \((G1)_2^∗\) the function \( g_1(x, \cdot) : \mathbb{R} \to \mathbb{R} \) is non-increasing for all \( x \in \Omega \), then \( G_1(x, s) \leq 0 \) for all \( x \in \Omega \) and \( s \in \mathbb{R} \) and \((\lambda, u) \equiv 0\) is stable for \( \lambda < \min\{m, m_c - |\nu_1|\} \).

(iii) If \( G_1(x, s) \geq 0 \) for all \( x \in \Omega \) and \( s \in \mathbb{R} \), then \((\lambda, u) \equiv 0\) is stable for all \( \lambda < m \).

In all three cases, \( 0 \) lies in a quadratic potential well of \( \phi_\lambda : H_A \to \mathbb{R} \) and, if \( 2G(x, s) \leq g(x, s)s \) for all \( x \in \Omega \) and \( s \in \mathbb{R} \), then \( G(x, s) \geq 0 \) and there exist \( \delta > 0 \) and \( \eta > 0 \) such that \( \|u(t, u_0)\|_2^2 \leq \|u_0\|_2^2 e^{-\eta t} \) for all \( t \geq 0 \) if \( \|u_0\|_A < \delta \).

(B, asymptotic stability) (i) Suppose that the function \( g_1 \) satisfies the conditions \((G1)_2^∗\) and (C) and that the function \( g_2 \) satisfies the condition \((G2)_2\) for some exponent \( \alpha > N\sigma/2 \) and \( 0 < \sigma < 2/(N-2) \).

For \( \lambda < \min\{m, m_c - \ell_1\} \), the solution \((\lambda, u) \equiv 0\) is asymptotically stable.

(ii) If \( g_1(x, s) \geq 0 \) for all \( x \in \Omega \) and \( s \in \mathbb{R} \), then \( G_1(x, s) \geq 0 \) for all \( x \in \Omega \) and \( s \in \mathbb{R} \) and \((\lambda, u) \equiv 0\) is asymptotically stable for all \( \lambda < m \).

(C, instability) (i) Suppose that \( 2G(x, s) \geq g(x, s)s \) for all \( x \in \Omega \) and \( s \in \mathbb{R} \). Then \( G(x, s) \leq 0 \) and \((\lambda, u) \equiv 0\) is unstable for \( \lambda > \min\{m, m_c + \gamma_1\} \). In fact, there exists a sequence \( \{u_n\} \subset H_A \) (depending on \( \lambda \)) such that \( \|u_n\|_A \to 0 \) as \( n \to \infty \), \( \phi_\lambda(u_n) < 0 \) and \( \lim_{n \to \infty} \sup_{t \in [0, T(u_n)]} \|u(t, u_n)\|_A = \infty \) for all \( n \).

(ii) If in addition to the assumption in part (i),
\[ \liminf_{\|x\| \to 0} \{ \inf_{x \in \Omega} \frac{G_2(x, s)}{s^2} \} > -\infty, \]
then the sequence given in part (i) has the stronger properties that
\[ \lim_{t \to T(u_n)} \|u(t, u_n)\|_{L^2} = \lim_{t \to T(u_n)} \|u(t, u_n)\|_{L^\infty} = \infty \text{ for all } n. \]

(iii) Suppose that \( m < m_c \) and that, in addition to satisfying the condition \((G1)_2^∗\),
the function \( g_1(x, \cdot) : \mathbb{R} \to \mathbb{R} \) is non-decreasing for all \( x \in \Omega \). Then \( G_1(x, s) \geq 0 \) and \((\lambda, u) \equiv 0\) is unstable for \( \lambda > m \).

Remark 7.2. Observing that \( 2G(x, s) \geq g(x, s)s \) for all \( s \in \mathbb{R} \) is equivalent to the property that \( s\partial_x \{ \frac{G(x, s)}{s^2} \} \leq 0 \) for all \( s \neq 0 \), it follows that \( G(x, s) \leq 0 \) for all \( s \in \mathbb{R} \) and consequently, \( g(x, s) \leq 0 \) for all \( s \in \mathbb{R} \). Similarly, if \( 2G(x, s) \leq g(x, s)s \) for all \( s \in \mathbb{R} \) then \( G(x, s) \) and \( g(x, s) \geq 0 \) for all \( s \in \mathbb{R} \).

Remark 7.3. The conclusion \( \lim_{t \to T(u_n)} \|u(t, u_n)\|_{L^\infty} = \infty \) in part C(ii) needs to be placed in the proper perspective since elements in \( H_A \), and even \( D(S_A) \), are not necessarily bounded in a neighbourhood of the origin. Hence it conceivable that
there is a solution of (6.2)(6.3) for which \(\|u(t, u_0)\|_{L^\infty} = \infty \) for all \(t \in [0, T(u_0))\). This can be demonstrated explicitly for the linearised equation (1.14). In the setting of Example 3.1, for any \(\lambda \in \mathbb{R}, u(t, u_\alpha) = e^{(\lambda - m_\alpha)t}u_\alpha\) for all \(t \geq 0\) is the unique solution of (6.2) and (1.14) with initial condition the eigenfunction \(u_\alpha\) associated with the eigenvalue \(m_\alpha\). For \(\alpha \in (0, 1/2), u(t, u_\alpha)\) has a singularity at \(x = 0\) for all \(t \geq 0\).

**Remark 7.4.** In view of Theorem 7.2 which concerns the case \(g_1 = 0\) it would seem reasonable to expect that the conclusion in part (C)(iii) also holds when \(m = m_\varepsilon\), but so far I have not found a proof of this.

**Proof.** (A) These conclusions about stability are obtained from part (i) of Theorem 5.3 by showing that \(u \equiv 0\) lies in a quadratic potential well of \(\phi_\lambda\) under the restrictions imposed upon \(\lambda\). For this we refer to Theorem 4 (II) in [34]. The condition (G1)$_2$ in (CC) coincides with (G1) in [34] but (G2)$_2$ is a little different from the condition (G2) in [34]. However, Proposition 4.1 shows that \(\psi_2 \in \mathcal{C}^2(H_A, \mathbb{R})\) with \(\psi_i(0) = 0\) for \(i = 0, 1, 2\) if \(g_2\) satisfies (G2)$_2$ and this is the only property of \(\psi_2\) used in the proof of Theorem 4.1 in [34]. Hence the conclusions of that result are valid under the hypothesis (CC). This proves that \(u \equiv 0\) lies in a quadratic potential well of \(\phi_\lambda \in \mathcal{C}^1(H_A, \mathbb{R})\) in cases (i) and (ii). That is to say, there exist \(\xi > 0\) and \(r > 0\) such that

\[
\phi_\lambda(u) \geq \xi \|u\|^2_\lambda \quad \text{for all} \quad u \in B(0, r),
\]

where \(B(0, r) = \{u \in H_A : \|u\|_A < r\}\). For case (iii), consider the functional \(\tilde{\phi}_\lambda\) defined by

\[
\tilde{\phi}_\lambda(u) = \phi_\lambda(u) - \psi_1(u) = \frac{1}{2} \int_{\Omega} A|\nabla u|^2 + (V - \lambda)u^2 dx + \psi_2(u).
\]

Since the conclusions of Theorem 4.1 in [34] hold for this functional, \(u \equiv 0\) lies in a quadratic potential well of \(\tilde{\phi}_\lambda\) for \(\lambda < m\). But in case (iii), \(\phi_\lambda \geq \tilde{\phi}_\lambda\) so (7.2) also holds for \(\lambda < m\) in this case.

The conclusions about stability now follow from part (i) of Theorem 5.3. For the exponential decay of the \(L^2\)-norm, note simply that for all \(\lambda \in \mathbb{R},\)

\[
2\Pi_\lambda(u) - \Pi'_\lambda(u)u = -\int_\Omega 2G(x, u(x)) - g(x, u(x))u(x)\, dx
\]

for all \(u \in H_A\) by (6.5), (6.7) and (6.8).

(B) Parts (A)(i) and (iii) show that \(0\) lies in a quadratic potential well of \(\phi_\lambda\). By part (ii) of Theorem 5.3 it suffices to show that there exists \(R(\lambda) > 0\) such that

\[
\inf\{\|Su + \tilde{g}(u) - \lambda u\|_{L^2} : u \in D(S)\} \geq R(\lambda) > 0 \quad \text{for all} \quad r \in (0, R(\lambda)].
\]

This is established in part (C) of Lemma 4.5 for the case (i) and in Lemma 4.4 for case (ii).

(C) (i) As explained in the proof of part (A), the conclusions of Theorem 4.1 in [34] are valid under the hypothesis (CC). For \(\lambda > \min\{m, m_\varepsilon + \gamma_1\}\), Theorem 4.1(iii) in [34] shows that there exists a sequence \(\{u_n\} \subset H_A\) such that \(\|u_n\|_A \to 0\) as \(n \to \infty\) and \(\phi_\lambda(u_n) < 0\) for all \(n\). From (7.3) it follows that the condition (5.13) is satisfied for all \(\varepsilon > 0\) and Theorem 5.3(iii) yields the conclusion.

(ii) The extra hypothesis means that there exist \(C > 0\) and \(S > 0\) such that \(G_2(x, s) \geq -Cs^2\) for all \(x \in \Omega\) and \(|s| > S\), whereas the condition (G2)$_2$ implies
that there is a constant $C_1 > 0$ such that

$$G_2(x, s) \geq -K \left\{ \frac{|x|^\beta s^2}{2} + \frac{|x|^\alpha |s|^\sigma + 2}{(\sigma + 1)(\sigma + 2)} \right\} \geq -C_1 s^2$$

for all $x \in \Omega$ and $|s| \leq S$, since $\Omega$ is bounded. Hence, using (G1)$_2$,

$$G(x, s) \geq -(\frac{\ell_1}{2} + C + C_1)s^2 = -Ds^2$$

for all $x \in \Omega$ and $s \in \mathbb{R}$.

Thus for the sequence given by part (i),

$$0 > \phi_\lambda(u_n) \geq \phi_\lambda(u(t, u_n)) \geq \frac{1}{2} \| u(t, u_n) \|_A^2 - (\| V \|_{L^\infty} + |\lambda| + 2D) \| u(t, u_n) \|_{L^2}^2$$

by (6.10) for $0 \leq t < T(u_0)$ and it follows that $\lim_{t \to T(u_0)} \| u(t, u_n) \|_{L^2} = \infty$ for all $n$. Recalling that $\Omega$ is bounded, this implies that $\lim_{t \to T(u_0)} \| u(t, u_n) \|_{L^\infty} = \infty$ for all $n$.

(iii) This follows from Theorem 5.4 with $F_\lambda$ defined by (6.1). As in the proof of Theorem 7.1, $L_\lambda = \lambda I - V$, but now $R_1 = -\hat{g}_1$ and $R_2 = -\hat{g}_2$. Since $\inf \sigma(S_A - L_\lambda) = m - \lambda$ and $\inf \sigma_2(S_A - L_\lambda) = m_e - \lambda$, the solution $(\lambda, u \equiv 0)$ is unstable for $\lambda > m$.

For a more restricted class of nonlinearities the criteria for stability and instability in Theorem 7.3 become sharp. To illustrate this consider a nonlinearity of the form $g(x, s) = \pm k(s) + g_2(x, s)$ where the function $k$ satisfies the condition (K) mentioned in the Introduction and $g_2$ satisfies the condition (G2)$_2$. As mentioned in the Introduction some real analytic functions such as $k(s) = s - \tanh s$ and $k(s) = s - \arctan s$ satisfy the condition (K), as do functions of the form

$$k(s) = C \frac{|s|^\sigma s}{1 + |s|^\sigma}$$

for $s \in \mathbb{R}$ with $C \geq 0$ and $0 < \sigma \leq 1$. (7.4)

**Corollary 7.4.** Let $k$ be a function satisfying the condition (K) and let $g_2$ be a function satisfying (G2)$_2$ for some exponents $\alpha > N\sigma/2$ and $0 < \sigma < 2(N - 2)$. Consider the problem (1.4) to (1.7) under the assumptions (A)$_2$ and (V).

(I) Let $g(x, s) = k(s) + g_2(x, s)$. The solution $u \equiv 0$ is asymptotically stable for $\lambda < m$ and, if $m < m_e$, it is unstable for $\lambda > m$.

(II) Let $g(x, s) = -k(s) + g_2(x, s)$. The solution $u \equiv 0$ is asymptotically stable for $\lambda < \min\{m, m_e - k'(\infty)\}$. If in addition, $2G(x, s) \geq g(x, s)s$ for all $x \in \Omega$ and $s \in \mathbb{R}$, then $u \equiv 0$ is unstable for $\lambda > \min\{m, m_e - k'(\infty)\}$.

In particular, if $g(x, s) = -k(s)$, then $u \equiv 0$ is asymptotically stable for $\lambda < \min\{m, m_e - k'(\infty)\}$ and unstable for $\lambda > \min\{m, m_e - k'(\infty)\}$.

(III) For $g(x, s) = \pm k(s) + g_2(x, s)$, the linearization (1.9) of (1.4) at $u \equiv 0$ is asymptotically stable when $\lambda < m$ and unstable when $\lambda > m$.

**Remark 7.5.** For $g(x, s) = \pm k(s)$, $T(u_0) = \infty$ for all $u_0 \in H_A$ since $|k(s)| \leq k'(\infty)|s|$ for all $s \in \mathbb{R}$.

Proof. Setting $g_1(x, s) = \pm k(s)$ the conditions (G1)$_2$ and (C) are satisfied by $g_1$ with $\ell_1 = k'(\infty)$ and $\gamma = \pm k'(\infty)$.

(I) In this case $g_1(x, s)s \geq 0$ for all $x \in \Omega$ and $s \in \mathbb{R}$. The conclusions follow from parts (B)(ii) and (C)(iii) of Theorem 7.3 since $k$ is non-decreasing.

(II) In this case, $\nu_1 = \nu_2 = -k'(\infty)$ so $\max\{m - |\nu_1|, m_e - \ell_1\} = \max\{m - k'(\infty), m_e - k'(\infty)\} = m_e - k'(\infty)$. Hence the conclusions follow from parts (B)(i) and (C)(i) of Theorem 7.3. Concerning the case $g(x, s) = -k(s)$, note that since
k satisfies the condition (K), the convexity of k on $[0, \infty)$ implies that $k(s)/s$ is non-decreasing on $[0, \infty)$ and hence for $s > 0$,

$$K(s) \equiv \int_0^s k(y) \, dy = \int_0^s \frac{k(y)}{y} \, y \, dy \leq \frac{k(s)}{s} \int_0^s y \, dy = \frac{1}{2} k(s)s.$$ 

Since $K$ is even, this shows that $2K(s) \leq k(s)s$ for all $s \in \mathbb{R}$ and hence

$$2G(x, s) - g(x, s)s = k(s)s - 2K(s) \geq 0 \text{ for all } x \in \Omega \text{ and } s \in \mathbb{R}.$$

(III) This follows from part (II) with $k \equiv 0$ and $g_2 \equiv 0$. In fact it also follows from Theorem 7.2.

**Remark 7.6.** In the case $g(x, s) = -k(s)$, parts (II) and (III) justify the remarks made at the beginning of the Introduction concerning the failure of the principle of linearized stability. Furthermore, Theorem 7.3 provides some additional information about the nature of the instability in this case. Namely, $T(u_0) = \infty$ for all $u_0 \in H_A$ and if $\lambda > \min\{m, m - k'(\infty)\}$, there is a sequence $\{u_n\} \subset H_A$ of initial conditions such that $\|u_n\|_A \to 0$ as $n \to \infty$ and $\lim\inf_{t \to \infty} \frac{\|u(t, u_n)\|_{L^2}}{\sqrt{t}} \geq |\phi_\lambda(u_n)| > 0$ for all $n$.

A final example illustrates many of the main features of the more general results established in this paper in a simple setting where they are particularly sharp.

**Example 7.1.** Consider the problem (1.4) to (1.7) where

(i) the bounded open set $\Omega$ is star-shaped with respect to $0$ and its boundary $\partial \Omega$ is of class $C^2$,

(ii) $A(x) \equiv |x|^2$,

(iii) $V \in C^1(\Omega)$ with $V(0) = 0$ and $x \cdot \nabla V(x) \geq 0$ for all $x \in \Omega$,

(iv) $g(x, s) = -k(s)$ where $k$ is a function satisfying the condition (K) with $k(s)/s$ strictly increasing on $(0, \infty)$ and hence $k'(\infty) > 0$.

Then (1.4) can be written as

$$\partial_t u - v^2 \Delta u + V(x)u = 2x \cdot \nabla u + k(u) + \lambda u$$  \hspace{1cm} (7.5)

and, for all $\lambda \in \mathbb{R}$ and $u_0 \in H_A$, the initial value problem has a unique solution $u \in C([0, \infty), H_A) \cap C^1((0, \infty), H_A) \cap C((0, \infty), D(S_A))$.

The only stationary solutions are $(\lambda, u \equiv 0)$ for $\lambda \in \mathbb{R}$.

For $\lambda < \frac{N^2}{4} - k'(\infty)$, $u \equiv 0$ is asymptotically stable, whereas for $\lambda > \frac{N^2}{4} - k'(\infty)$, $u \equiv 0$ is unstable.

The linearization of (7.5) at $u \equiv 0$ is

$$\partial_t u - v^2 \Delta u + Vu = 2x \cdot \nabla u + \lambda u$$  \hspace{1cm} (7.6)

and for this equation the initial value problem is asymptotically stable for $\lambda < \frac{N^2}{4}$ and unstable for $\lambda > \frac{N^2}{4}$.

**REFERENCES**

[1] B. Abdellaoui, E. Colorado and I. Peral, Existence and non-existence results for a class of linear and semilinear parabolic equations related to some Caffarelli-Kohn-Nirenberg inequalities, *J. Eur. Math. Soc.*, 6 (2004), 119–149.

[2] B. Abdellaoui and I. Peral, On quasilinear elliptic equation related to some Caffarelli-Kohn-Nirenberg inequalities, *Comm. Pure Appl. Anal.*, 2 (2003), 539–566.

[3] H. W. Alt, *Lineare Funktionalanalysis* 2nd ed., Springer Lehrbuch, Berlin, 1993.

[4] A. Ambrosetti and G. Prodi, *A Primer of Nonlinear Analysis*, Cambridge Studies in Adv. Math. No 34, C.U.P. Cambridge 1993.
[5] J. M. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, *Quart. J. Math. Oxford*, 28 (1977), 473–486.

[6] T. Bartsch, S. Peng and Z. Zhang, Existence and non-existence of solutions to elliptic equations related to Caffarelli-Kohn-Nirenberg inequalities, *Calc. Var. P.D.E.*, 30 (2007), 113–136.

[7] L. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequalities with weight, *Compositio Math.*, 53 (1984), 259–275.

[8] P. Caldiroli and R. Musina, On a variational degenerate elliptic problem, *NoDEA*, 7 (2000), 187–199.

[9] P. Caldiroli and R. Musina, On the existence of extremal functions for a weighted Sobolev embedding with critical exponent, *Calc. Var. P.D.E.*, 8 (1999), 365–387.

[10] F. Catrina and Z.-Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: Sharp constants, existence (and nonexistence) and symmetry of extremals, *Comm. Pure Appl. Math.*, 54 (2001), 229–258.

[11] F. Chiarenza and R. Serapioni, Degenerate parabolic equations and Harnack inequality, *Ann. Mat. Pura Appl.*, 137 (1984), 139–162.

[12] F. Chiarenza and R. Serapioni, Pointwise estimates for degenerate parabolic equations, *Applicable Anal.*, 23 (1987), 287–299.

[13] J. W. Cholowa and T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, LMS Lecture Notes No. 278, Cambridge University Press, Cambridge, 2000.

[14] A. Dall’Aglio, D. Giachetti and I. Peral, Results on parabolic equations related to some Caffarelli-Kohn-Nirenberg inequalities, *SIAM J. Math. Anal.*, 36 (2004), 691–716.

[15] D. G. De Figueiredo, *The Ekeland Variational Principle with Applications and Detours*, TIFR Lecture Notes, Springer-Verlag, Berlin, 1989.

[16] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[17] G. Evéquoz and C. A. Stuart, Bifurcation points of a degenerate elliptic boundary-value problem, *Rend. Lincei Mat. Appl.*, 17 (2006), 309–334.

[18] G. Evéquoz and C. A. Stuart, Bifurcation and concentration of radial solutions of a nonlinear degenerate elliptic eigenvalue problem, *Adv. Nonlinear Studies*, 6 (2006), 215–232.

[19] T. M. Flett, *Differential Analysis*, Cambridge University Press, Cambridge, 1980.

[20] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Grundlehren der Mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin-New York, 1977.

[21] Z. M. Guo, F. S. Wan and X. H. Guan, Embeddings of weighted Sobolev spaces and degenerate elliptic problems, *Science China Math.*, 60 (2017), 1399–1418.

[22] C. E. Gutierrez, Pointwise estimates for solutions of degenerate parabolic equations, *Rev. U. Mat. Argentina*, 37 (1991), 261–270.

[23] C. E. Gutierrez and R. Wheeden, Harnack’s inequality for degenerate parabolic equations, *Comm. P.D.E.*, 16 (1991), 745–770.

[24] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer Lecture Notes in Math., No 840, Springer-Verlag, Berlin, 1981.

[25] V. A. Ivanov, Boundary value problems for degenerate second order linear parabolic equations, *Sb. Math., N.S.*, 14 (1971), 22–43.

[26] S. N. Kruzkov, Boundary value problems for degenerate second order elliptic equations, *Mat. Sb. (N.S.),* 77 (1968), 299–334.

[27] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Appl. Math. Sci. No 74, Springer-Verlag, Berlin, 1989.

[28] D. Motreanu and V. Rădulescu, Eigenvalue problems for degenerate nonlinear elliptic equations in anisotropic media, *Boundary Value Problems*, 2 (2005), 107–127.

[29] M. K. V. Murthy and G. Stampacchia, Boundary value problems for some degenerate elliptic operators, *Ann. Mat. Pura Appl.*, 80 (1968), 1–122 and *Errata Corrige, Ibidem*, 90 (1971), 413–414.

[30] F. Nicolosi, Sulla limitatezza delle soluzioni deboli dei problemi al contorno per operatori parabolici degeneri, *Rendiconti Circ. Mat. Palermo, XXXI* (1982), 23–40.

[31] P. J. Rabier, Embeddings of weighted Sobolev spaces and generalized Caffarelli-Kohn-Nirenberg inequalities, *J. Anal. Math.*, 118 (2012), 251–296.

[32] C. A. Stuart, Bifurcation at isolated singular points for a degenerate elliptic eigenvalue problem, *Nonlinear Anal. TMA*, 119 (2015), 209–221.

[33] C. A. Stuart, Bifurcation points of a critically degenerate elliptic Dirichlet problem, in preparation.
[34] C. A. Stuart, Criteria for the existence of a potential well, *Nonlinear Anal. TMA*, **158** (2017), 83–108.

[35] C. A. Stuart, Bifurcation points of a singular boundary-value problem on $(0,1)$, *J. Diff. Equat.*, **260** (2016), 6267–6321.

[36] N. S. Trudinger, Linear elliptic operators with measurable coefficients, *Ann. Scuola Norm. Super. Pisa*, **27** (1973), 265–308.

[37] J. Weidmann, *Linear Operators in Hilbert Space*, Springer, Berlin, 1980.

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