Nonfragile Finite-Time Stabilization for Discrete Mean-Field Stochastic Systems

Tianliang Zhang, Member, IEEE, Feiqi Deng, Senior Member, IEEE, and Peng Shi, Fellow, IEEE

Abstract—In this article, the problem of nonfragile finite-time stabilization for linear discrete mean-field stochastic systems is studied. The uncertain characteristics in control parameters are assumed to be random satisfying the Bernoullí distribution. A new approach called the “state-transition matrix method” is introduced and some necessary and sufficient conditions are derived to solve the underlying stabilization problem. The Lyapunov theorem based on the state-transition matrix also makes a contribution to the discrete finite-time control theory. One practical example is provided to validate the effectiveness of the newly proposed control strategy.

Index Terms—Finite-time stabilization, nonfragile control, state-transition matrix (STM), stochastic systems.

I. INTRODUCTION

It is well known that the behavior of a single individual may affect the collective action. Conversely, in many physical or sociological dynamical processes, collective interactions can also change individual judgment and behavior. In order to investigate the influence from collective to single individual, mean-field theory has been naturally developed [10], [23], [26]. In particular, the high popularity of quantum computers highlights the rising importance of mean-field theory and its relevant applications because the mean-field method is a common and effective method to deal with quantum many-body problems [23]. The well-known mean-field type stochastic models depict the system equation incorporating the mean of the state variables. In recent years, many outstanding results on the control problem relating to mean-field type stochastic systems have been proposed in the following literature. For example, linear-quadratic optimal control problems were discussed in [7], [22], and [28]. Lin et al. [17] were concerned with the Stackelberg game issue for mean-field stochastic systems. The stochastic maximum principle was discussed in [4]. In addition, mean-field stochastic systems with network structure and time delay have attracted lots of scholars’ attention, we refer the interested readers to the work in [9] and [21] for further references. With respect to the stability and stabilization problems, Ma et al. [19] studied the mean-square stability and spectral assignment in a prescribed area region for linear discrete mean-field stochastic (LDMFS) systems via the spectrum of a generalized Lyapunov operator.

Finite-time stability focuses on the system state behavior only in a specified finite-time horizon instead of the whole time interval, which differentiates the finite-time stability from the classical Lyapunov stability studied in [12], [36], and [37] for discrete stochastic stability and [14] and [18] for stochastic stability of continuous Itô systems. In some practical applications, the considered operating duration of the controlled system is often limited [11], [20], so, in some cases, the transient characteristics of systems may be more important than the state convergence in an infinite-time horizon. It is well known that finite-time stability contains two kinds of different concepts: One is defined as in [1], [2], [3], [16], [24], [30], and [35], which is in fact finite-time bounded in some sense, whereas the other one is defined as in [5], [8], [20], [27], [29], [31], [32], and [34], where finite-time stability satisfies both “stability in Lyapunov sense” and “finite-time attractiveness.” Throughout this article, we study the first kind of finite-time stability and stabilization of LDMFS systems. So from now on, when we refer to finite-time stability and stabilization, they are in the finite-time boundedness sense. Finite-time stability and stabilization have been researched for deterministic systems [1], [2], [3], [16], [24] and stochastic systems [30], [35]. In [1], [2], and [3], based on the state transition matrix (STM) of deterministic linear systems, necessary and sufficient conditions have been obtained for finite-time stability and stabilization. In [16], Lyapunov-type conditions for finite-time stability of continuous-time nonlinear time-varying delayed systems were presented. For continuous-time nonlinear differential systems, Song et al. [24] proposed a suitable sliding mode control law to drive the state trajectory into the prescribed sliding surface within a finite time. Yan et al. [30] and Zhang et al. [35] discussed the finite-time stability and stabilization of continuous-time discrete-time stochastic systems, respectively. As can be seen, most existing results on finite-time stability and stabilization are about deterministic/stochastic differential systems. However, regarding finite-time stability or stabilization for LDMFS systems, no result has been reported so far. In fact, we can only find a few papers such as the work in [19] to investigate asymptotical mean-square stability and stabilizability. In addition, most results in stochastic systems are based on Lyapunov function/functional method to present sufficient conditions but not necessary conditions.

In practice, it is more likely to encounter some unexpected failures. Once that happens, the performance of control systems is certainly affected or even irreversible. Therefore, various design frameworks...
for reliable controllers have been proposed, in which the nonfragile control has attracted a remarkable research interest in recent years [13], [25], [33]. However, to the best knowledge of the authors, there is no work addressing the nonfragile finite-time controller design for LDMFS systems. Up to now, for LDMFS systems, we can only find a few works such as the work in [7] on linear quadratic optimal control problem and [38] about cooperative linear quadratic dynamic difference game.

In this article, we investigate the finite-time stabilization of LDMFS systems via nonfragile control. The basic approach is based on STMs of LDMFS systems. In [37], STMs of linear discrete stochastic systems were first presented and employed to investigate the exact observability, exact/uniform detectability, and Lyapunov-type theorems of the following classical stochastic system with multiplicative noises:

\[
\begin{align*}
    x_k &= H_k x_k + M_k x_k w_k \\
    y_k &= G_k x_k.
\end{align*}
\]

In [35], the method of STM was first used to discuss the finite-time stability of system (1). However, this method has not been applied to LDMFS systems, this is because that, in LDMFS systems, it is very difficult to establish the STM expressions. The contributions of this article are highlighted as follows.

1) Some specific expression forms of STMs have been established by iterative equations. Based on the linear transformation and the augmented system method, we establish an equivalent relationship between the original LDMFS system with uncertain parameters and a certain augmented nonmean-field time-varying discrete stochastic system with random coefficients.

2) Based on the STM approach, several necessary and sufficient conditions for the finite-time stabilization of the LDTMF system have been obtained.

3) With the increase of the length of the time interval of interest, the criteria obtained by STM on finite-time stabilization often leads to higher computational complexity. To reduce the computational complexity, we construct novel necessary and sufficient Lyapunov-type conditions by using the introduced STMs and obtain a sufficient condition to guarantee the finite-time stabilization in the form of linear matrix inequalities (LMIs), which is easier to use in designing the finite-time controller.

The rest of this article is organized as follows: In Section II, some useful definitions and lemmas are introduced. In Section III, we investigate the STM approach of LDMFS systems and its application to finite-time stabilization. By system reconfiguration, we transform the original system into a new discrete stochastic system with state-dependent noise. Necessary and sufficient conditions are presented to solve the stabilization problem. One example is given in Section IV to illustrate the effectiveness of the theoretical results obtained. The conclusion is drawn in Section VI.

For convenience, we present the notations used in this article here: \( \mathcal{R}^n \) denotes the \( n \)-dimensional real Euclidean vector space and \( \mathcal{R}^{n \times m} \) stands for the space of all \( n \times m \) real matrices. \( \| \cdot \| \) means the Euclidean norm. The notation \( C > 0 \) means that the matrix \( C \) is positive definite real symmetric and \( C < 0 \) means that the matrix \( C \) is negative definite real symmetric. \( C' \) stands for the transpose of the matrix or vector \( C \). \( I_n \) denotes the \( n \times n \) identity matrix. Given matrices \( F \) and \( G \), the notation \( F \otimes G \) stands for the Kronecker product of \( F \) and \( G \). Given a positive integer \( M \), \( M^r \) denotes the set \{0, 1, 2, ..., \( M \)\} and diag\((a_1, a_2, ..., a_m)\) means a diagonal matrix whose leading diagonal entries are \( a_1, a_2, ..., a_m \). The notation \( E \) denotes the mathematical expectation operator.

II. PRELIMINARIES

In this section, we will consider the following LDMFS system:

\[
\begin{align*}
    x_{k+1} &= A_1 x_k + A_2 x_k + B u^F_k + (C_1 x_k + C_2 x_k + D a_k) w_k \\
    x_0 &= \xi \in \mathcal{R}^n, \quad k \in \mathcal{N}_{T-1}
\end{align*}
\]

where \( x_k \in \mathcal{R}^n \) and \( u^F_k \in \mathcal{R}^m \) are the state vector and actuator output vector with fault at time \( k \), respectively. Suppose that the initial state \( x_0 \) is a deterministic real vector \( \{w_k\}_{k \in \mathcal{N}_{T-1}} \) stands for the system noise assumed to be a one-dimensional independent white noise sequence defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Assume that \( \mathbb{E}[w_k] = 0 \), \( \mathbb{E}[w_k w_l] = 0 \) when \( i \neq k \), and \( \mathbb{E}[w_k w_k] = 1 \) when \( i = k \), \( A_1, A_2, B, C_1, C_2, D \) are deterministic matrices with appropriate dimensions.

In most practical systems, the uncertain parameters in feedback coefficients usually cannot be ignored due to that the state feedback control may be extremely sensitive or fragile with respect to errors. So, we have to consider the following nonfragile state feedback control:

\[
\begin{align*}
    u^F_k &= (K_1 + \alpha_k D K_{1,1}) x_k + (K_2 + \alpha_k D K_{2,2}) \mathcal{E} x_k \\
    k &= 1, 2, ..., \infty
\end{align*}
\]

where \( K_1 \) and \( K_2 \) are the control gain matrices. \( D K_{1,1} \) and \( D K_{2,2} \) are the uncertain parameters satisfying \( D K_{1,1} \) and \( D K_{2,2} \) are known matrices of appropriate dimensions, and \( F_k \) is the uncertain matrix such that \( F_k = F_{k+1} \leq I \), which is a finite sequence of independent random variables satisfying Bernoulli distribution with \( P(\alpha_k = 1) = \bar{\alpha} \) and \( P(\alpha_k = 0) = 1 - \bar{\alpha} \), \( 0 \leq \bar{\alpha} \leq 1 \).

Definition 2.1: Given any positive integer \( T > 0 \), two positive numbers \( \epsilon_1 \) and \( \epsilon_2 \) with \( 0 < \epsilon_1 < \epsilon_2 \), and a sequence of positive definite symmetric matrices \( \{R_k\}_{k \in \mathcal{N}_{T}} \). If there exists a nonfragile controller \( u^F_k \) such that the following closed-loop system:

\[
\begin{align*}
    x_{k+1} &= A_1 + B K_1 + \alpha_k D K_{1,1} x_k + (A_2 + B K_2 + \alpha_k D K_{2,2}) \mathcal{E} x_k + [(C_1 + D K_{1,1}) x_k + (C_2 + D K_{2,2} + \alpha_k D) \mathcal{E} x_k] w_k \\
    x_0 &= \xi, \quad k \in \mathcal{N}_{T-1}
\end{align*}
\]

satisfies

\[
\begin{align*}
    x_k R_k x_k \leq \epsilon_1 \Rightarrow \mathcal{E}(x_k' R_k x_k) \leq \epsilon_2 \quad \forall k \in \mathcal{N}_T
\end{align*}
\]

then system (2) is said to be finite-time stabilizable with respect to \((\epsilon_1, \epsilon_2, T, \{R_k\}_{k \in \mathcal{N}_{T}})\). The following property will be used to prove our main results.

Lemma 2.1 (see [35]): For given matrices \( F, G, H, \) and \( M \) of suitable dimensions, the following holds:

\[
(F \otimes G)(H \otimes M) = (FH) \otimes (GM).
\]

III. STM AND FINITE-TIME STABILIZATION

In this section, we will first build the STM of LDMFS system (2) and then research the finite-time stabilization of LDMFS system (2) based on the STM approach. With that \( \alpha_k \) and \( x_k \) are independent of each other, taking the mathematical expectation in system (4), it follows that

\[
\begin{align*}
    \mathcal{E} x_{k+1} &= (A_1 + A_2 + B(K_1 + K_2) + \alpha_k D [D K_{1,1} + D K_{2,2}]) \mathcal{E} x_k + \mathcal{E} x_k w_k \\
    \mathcal{E} x_0 &= \xi, \quad k \in \mathcal{N}_{T-1},
\end{align*}
\]
Subtracting (7) from (4) and setting $\hat{x}_k = x_k - \mathcal{E}x_k$, we have

$$
\begin{align*}
\hat{x}_{k+1} &= (A_1 + BK_1 + \alpha_1 B\Delta K_{1,k})\hat{x}_k + [\alpha_1 B(\Delta K_{1,k} + \Delta K_{2,k})]x_k \\
&\quad + [(C_1 + DK_1 + \alpha_1 D\Delta K_{1,k})\hat{x}_k \\
&\quad + (C_2 + C_1 + DK_2 + DK_1 + \alpha_2 D\Delta K_{2,k})]x_{k-1} \\
&\quad + \alpha_2 D\Delta K_{2,k}\mathcal{E}x_k|w_k \\
\hat{x}_0 &= 0, \quad k \in \mathcal{N}_{T-1}.
\end{align*}
$$

Letting $\tilde{x}_k = [\mathcal{E}x_k \; \hat{x}_k]$, we can obtain the following augmented system with respect to $\tilde{x}_k$:

$$
\begin{align*}
\tilde{x}_{k+1} &= \tilde{A}_k \tilde{x}_k + \tilde{C}_k \tilde{x}_k w_k \\
\tilde{x}_0 &= [\xi \; 0], \quad k \in \mathcal{N}_{T-1}
\end{align*}
$$

where

$$
\begin{align*}
\tilde{A}_k &= \begin{bmatrix} A_1 + A_2 + BK_1 + K_2 + \alpha_1 B(\Delta K_{1,k} + \Delta K_{2,k}) \\
&\quad + [\alpha_1 B(\Delta K_{1,k} + \Delta K_{2,k})]x_k \\
&\quad + [(C_1 + DK_1 + \alpha_1 D\Delta K_{1,k})\hat{x}_k \\
&\quad + (C_2 + C_1 + DK_2 + DK_1 + \alpha_2 D\Delta K_{2,k})]x_{k-1} \\
&\quad + \alpha_2 D\Delta K_{2,k}\mathcal{E}x_k|w_k \\
0 &\quad A_1 + BK_1 + \alpha_1 B\Delta K_{1,k} \end{bmatrix} \\
\tilde{C}_k &= \begin{bmatrix} 0 & 0 \\
C_2 + C_1 + DK_2 + DK_1 + \alpha_2 D\Delta K_{2,k} & 0 \\
C_1 + DK_1 + \alpha_1 D\Delta K_{1,k} \end{bmatrix}.
\end{align*}
$$

Note that $\mathcal{E}|\tilde{x}_k|^2 = (\mathcal{E}x_k')^2 + \mathcal{E}(\hat{x}_k')\mathcal{E}(\hat{x}_k) = (\mathcal{E}x_k')^2 + \mathcal{E}|(x_k - \mathcal{E}x_k)(x_k - \mathcal{E}x_k)| = \mathcal{E}|x_k|^2$, $k \in \mathcal{N}_T$. To study the second-order moment of $\tilde{x}_k$ in system (8), we need to prove some lemmas.

**Remark 3.1:** If we set

$$
\begin{align*}
A_{1,k} &= \begin{bmatrix} A_1 + A_2 + BK_1 + K_2 + \alpha_1 B(\Delta K_{1,k} + \Delta K_{2,k}) \\
&\quad + [\alpha_1 B(\Delta K_{1,k} + \Delta K_{2,k})]x_k \\
&\quad + [(C_1 + DK_1 + \alpha_1 D\Delta K_{1,k})\hat{x}_k \\
&\quad + (C_2 + C_1 + DK_2 + DK_1 + \alpha_2 D\Delta K_{2,k})]x_{k-1} \\
&\quad + \alpha_2 D\Delta K_{2,k}\mathcal{E}x_k|w_k \\
0 &\quad A_1 + BK_1 + \alpha_1 B\Delta K_{1,k} \end{bmatrix} \\
A_{2,k} &= \begin{bmatrix} 0 & 0 \\
B(\Delta K_{1,k} + \Delta K_{2,k}) & B\Delta K_{1,k} \end{bmatrix} \\
C_1 &= \begin{bmatrix} 0 & 0 \\
C_2 + C_1 + DK_2 + DK_1 & C_1 + DK_1 \end{bmatrix} \\
C_2,k &= \begin{bmatrix} 0 & 0 \\
D(\Delta K_{2,k} + \Delta K_{1,k}) & D\Delta K_{1,k} \end{bmatrix}
\end{align*}
$$

then $\tilde{A}_k = A_{1,k} + \alpha_1 A_{2,k} = C_1 + \alpha_2 C_2,k$ and system (8) can be rewritten into

$$
\begin{align*}
\tilde{x}_{k+1} &= (A_{1,k} + \alpha_1 A_{2,k})\tilde{x}_k + (C_1 + \alpha_2 C_2,k)\tilde{x}_k w_k \\
\tilde{x}_0 &= [\xi \; 0], \quad k \in \mathcal{N}_{T-1}.
\end{align*}
$$

Denote

$$
\psi_{1,k} := \begin{bmatrix} \psi_{1,k+1}(\sqrt{\alpha}A_{1,k} + \sqrt{\alpha}A_{2,k}) \\
\psi_{1,k+1}(\sqrt{\alpha}C_1 + \sqrt{\alpha}C_2,k) \\
\psi_{1,k+1}(\sqrt{\alpha} - \sqrt{\alpha}C_1) \end{bmatrix}
$$

$l > k, \psi_{1,k} = I_{2n}, \forall k \in \mathcal{N}_T$. In addition, there exists another expression for $\psi_{1,k}$ that is denoted by $\phi_{1,k}$ in the following lemma. These two expressions are both needed in the proof process of our subsequent results.

**Lemma 3.1:** Set

$$
\phi_{1,k} = \begin{bmatrix} \sqrt{\alpha}A_{1,k-1} + \sqrt{\alpha}A_{2,k-1} \\
\sqrt{\alpha}C_1 + \sqrt{\alpha}C_{2,k-1} \\
\sqrt{\alpha} - \sqrt{\alpha}C_1 \end{bmatrix}
$$

$l > k, \phi_{1,k} = I_{2n}, \forall k \in \mathcal{N}_T$. Then, we have the following relation:

$$
\phi_{1,k} = \psi_{1,k}, \quad \forall l > k \in \mathcal{N}_{T}.
$$

**Proof:** The proof can be found in [39].

**Remark 3.2:** By Lemma 3.1, the matrices $\psi_{1,k}$ and $\phi_{1,k}$ actually represent the same matrix but the difference lies in different iterative expressions. $\phi_{j,k}$ is calculated in forward time, whereas $\psi_{j,k}$ is calculated in backward time. Therefore, $\phi_{j,k}$ is in line with the characteristics of the STM of deterministic linear discrete systems. The introduction of $\phi_{j,k}$ is one important contribution of this article. Lemma 3.1 is new even in nonmean-field stochastic systems and plays an important role in this article.

In the following, we uniformly denote $\psi_{j,k}$ and $\phi_{j,k}$ as $\psi_{j,k}$ for simplicity. Moreover, we define another matrix $\phi_{l,k}$ as

$$
\begin{align*}
&\phi_{l,k} = \begin{bmatrix} \psi_{1,k+1}(\sqrt{\alpha}A_{1,k-1} + \sqrt{\alpha}A_{2,k-1}) \\
\psi_{1,k+1}(\sqrt{\alpha}C_1 + \sqrt{\alpha}C_{2,k-1}) \\
\psi_{1,k+1}(\sqrt{\alpha} - \sqrt{\alpha}C_1) \end{bmatrix}
\end{align*}
$$

$l > k, \phi_{l,k} = I_{2n}, \forall k \in \mathcal{N}_T$. On the basis of Lemma 3.1, we further give the following result for the state transition of system (9) in the mean-square sense.

**Lemma 3.2:** For system (9), we have the following iterative relations:

$$
\mathcal{E}|\tilde{x}_l|^2 = \mathcal{E}|\phi_{l,k}\tilde{x}_k|^2, \quad l \geq k
$$

where $\phi_{l,k}$ is defined as in Lemma 3.1

$$
\mathcal{E}|\tilde{x}_l|^2 = \mathcal{E}|\phi_{l,k}\tilde{x}_k|^2, \quad l \geq k
$$

where $\phi_{l,k}$ is defined as in (11).

**Proof:** The proof can be found in [39].

**Remark 3.3:** The matrices $\phi_{j,k}$ and $\psi_{j,k}$ can be regarded as the STMs in the mean-square sense of discrete stochastic system (9) with random coefficients. Different from deterministic systems, STMs are not unique in discrete stochastic systems, which have several expression forms.
We are now in a position to make the connection between the finite-time stabilization of mean-field system (2) and another classical time-varying stochastic system. Set \( \tilde{R}_k = \text{diag}(R_k, R_k) \) and \( \bar{z}_k = \tilde{R}_k^2 \bar{x}_k \), then we have the following lemma.

**Lemma 3.3:** System (2) is finite-time stabilizable with respect to \((\epsilon_1, \epsilon_2, T, \{R_k\}_{k \in \mathbb{N}_T})\) if and only if (iff) the system

\[
\begin{align*}
\bar{x}_{k+1} &= (\bar{A}_{1,k} + \alpha_k \bar{A}_{2,k}) \bar{x}_k + (\bar{C}_{1,k} + \alpha_k \bar{C}_{2,k}) \bar{w}_k \\
\bar{x}_0 &= \begin{bmatrix} R_k^2 \xi \\ 0 \end{bmatrix}, k \in \mathbb{N}_{T-1}
\end{align*}
\]  

(14)

is finite-time stable with respect to \((\epsilon_1, \epsilon_2, T, I_{2n})\), where

\[
\bar{A}_{1,k} = \tilde{R}_k^2 \tilde{A}_{1,k} \tilde{R}_k^{-2}, \quad \bar{A}_{2,k} = \tilde{R}_k^2 \tilde{A}_{2,k} \tilde{R}_k^{-2}, \quad \bar{C}_{1,k} = \tilde{R}_k^2 \tilde{C}_{1,k} \tilde{R}_k^{-2}, \quad \bar{C}_{2,k} = \tilde{R}_k^2 \tilde{C}_{2,k} \tilde{R}_k^{-2}.
\]

Moreover, the corresponding STMs \( \bar{\psi}_{l,k} \) and \( \bar{\phi}_{l,k} \) are given by

\[
\begin{align*}
\bar{\phi}_{l,k} &= \begin{bmatrix} (I_{d^2-1} \otimes \sqrt{\bar{\alpha}_1 A_{1,k} + \sqrt{\bar{\alpha}_1 A_{2,k}}}) \bar{\psi}_{l-1,k} \\
& (I_{d^2-2} \otimes \sqrt{\bar{\alpha}_1 C_{1,k} + \sqrt{\bar{\alpha}_1 C_{2,k}}}) \bar{\psi}_{l-1,k} \\
& (I_{d^2-1} \otimes \sqrt{\bar{\alpha}_1 C_{1,k} + \sqrt{\bar{\alpha}_1 C_{2,k}}}) \bar{\psi}_{l-1,k}
\end{bmatrix} \\
\bar{\theta}_{k,l} &= I_{2n}
\end{align*}
\]

and

\[
\begin{align*}
\bar{\psi}_{l,k} &= \begin{bmatrix} \sqrt{\bar{\alpha}_1 A_{1,k} + \sqrt{\bar{\alpha}_1 A_{2,k}}} \\
& \sqrt{\bar{\alpha}_1 C_{1,k} + \sqrt{\bar{\alpha}_1 C_{2,k}}} \\
& \sqrt{\bar{\alpha}_1 C_{1,k} + \sqrt{\bar{\alpha}_1 C_{2,k}}}
\end{bmatrix}
\]

\[\bar{\psi}_{k,l} = I_{2n}\]

respectively.

**Proof:** The proof can be found in [39].

The next two lemmas are dedicated to finding the relationship between \( \bar{\psi}_{l,k} \) and \( \bar{\phi}_{l,k} \), and \( \bar{\psi}_{l,k} \) and \( \bar{\phi}_{l,k} \), respectively.

**Lemma 3.4:** For any \(0 \leq k \leq l\), assume that the matrices \( \bar{\psi}_{l,k} \) and \( \bar{\phi}_{l,k} \) are STMs of systems (9) and (14), respectively. Then, the following relation always holds:

\[
\bar{\psi}_{l,k}^*(I_{d^2} \otimes \tilde{R}_k) \bar{\phi}_{l,k} = \tilde{R}_k \bar{\psi}_{l,k} \bar{\phi}_{l,k} \tilde{R}_k^2.
\]

**Proof:** The proof can be found in [39].

**Lemma 3.5:** For any \(0 \leq k \leq l\), assume that the matrices \( \bar{\psi}_{l,k} \) and \( \bar{\psi}_{l,k} \) are STMs of systems (9) and (14), respectively. Then, the following relation always holds:

\[
\bar{\phi}_{l,k}^*(I_{d^2} \otimes \tilde{R}_k) \bar{\psi}_{l,k} = \tilde{R}_k \bar{\phi}_{l,k} \bar{\psi}_{l,k} \tilde{R}_k^2.
\]

**Proof:** The proof can be found in [39].

**Theorem 3.1:** For an integer \(T > 0\), two positive scalars \(\epsilon_1\) and \(\epsilon_2\) with \(0 < \epsilon_1 \leq \epsilon_2\), and a sequence of positive definite symmetric matrices \(\{R_k\}_{k \in \mathbb{N}_T}\), the following conditions are equivalent.

a) LDMFS system (2) is finite-time stabilizable with respect to \((\epsilon_1, \epsilon_2, T, \{R_k\}_{k \in \mathbb{N}_T})\).

b) \[
\bar{\psi}_{k,0}^*(I_{d^2} \otimes \tilde{R}_0) \bar{\phi}_{k,0} \leq \epsilon_2 I_{2n}, \ \forall k \in \mathbb{N}_T.
\]

c) \[
\bar{\phi}_{k,0}^*(I_{d^2} \otimes \tilde{R}_0) \bar{\psi}_{k,0} \leq \epsilon_2 I_{2n}, \ \forall k \in \mathbb{N}_T.
\]

d) \[
\bar{\psi}_{k,0}^*(I_{d^2} \otimes \tilde{R}_0) \bar{\psi}_{k,0} \leq \epsilon_2 I_{2n}, \ \forall k \in \mathbb{N}_T.
\]

**Proof:** The proof can be found in [39].

**Remark 3.4:** The necessary and sufficient conditions for finite-time stabilizability of LDMFS system (2) are presented in Theorem 3.1. When \(\bar{\psi}_{k,0} = 0\) for \(k \in \mathbb{N}_{T-1}\), then necessary and sufficient conditions for finite-time stability of the following unforced system:

\[
\begin{align*}
\bar{\psi}_{k,0} &\leq \epsilon_2 I_{2n} \forall k \in \mathbb{N}_{T-1}
\end{align*}
\]

are given. When system (2) degenerates into a standard linear discrete stochastic system without mean-field terms, similar results first appeared in [35]. The main difficulty to give necessary and sufficient conditions for finite-time stability and stabilizability of LDMFS systems exists in that it is not easy to obtain the STMs as seen above, which differs from linear deterministic systems [1], [2], [3]. In [1], [2], and [3], necessary and sufficient conditions have been given for finite-time stability of linear deterministic systems based on the STM.

**IV. CONSTRUCTION OF LYAPUNOV FUNCTION BASED ON STMS**

In Theorem 3.1, five criteria are given through STMs. These criteria are all necessary and sufficient conditions for finite-time stabilization, and the first four criteria are relatively simple in form. However, solving these inequalities in Theorem 3.1 is not easy when \(T\) is large enough. For example, when using (f) in Theorem 3.1 to verify the finite-time stabilization of LDMFS system (2), with the progressive increase of \(k\), the order of the solution matrix \(P_k\) keeps expanding and is \(2^{2k+1} n \times 2^{2k+1} n\). Next, we will find ways to simplify the calculation of Theorem 3.1 and find a novel Lyapunov-type theorem.

Let \(\Gamma\) denote the set of block matrices composed of \(r \times r\) square submatrices with the same dimension. For the block matrix \(A\) belongs to \(\Gamma\), with \(A_{ij}\) denoting its submatrix, we introduce an operator \(\text{Tr}(A_{ij}) = \sum_{i=1}^{r} A_{ii}\). As a generalization of the standard matrix trace, \(\text{Tr}\) can be called block trace. It is not difficult to find that \(\text{Tr}\) enjoys the following useful properties.

**Lemma 4.1:** For any block matrix \(A_{ij} |_{r \times r} \in \Gamma\), the following are true:

i) \(\text{Tr}(A^T) = \text{Tr}(A)\).
ii) For any $2n$ matrices $C_i$, $D_i$ with appropriate dimension, $i \in \{1, 2, \ldots, n\}$, there will always be

$$\text{Tr} \left( \left[ I_r \otimes \left[ \begin{array}{c} C_1 \\ \vdots \\ C_n \end{array} \right] \right] A \left[ I_r \otimes \left[ \begin{array}{c} D_1 \\ \vdots \\ D_n \end{array} \right] \right] \right) = \sum_{i=1}^{n} \text{Tr}(A_i D'_i).$$

**Proof:** (i) is obvious, so we only need to show (ii). Without loss of generality, set

$$\Theta(\Theta_{ij}) = \left( I_r \otimes \left[ \begin{array}{c} C_1 \\ \vdots \\ C_n \end{array} \right] \right) A \left( I_r \otimes \left[ \begin{array}{c} D_1 \\ \vdots \\ D_n \end{array} \right] \right),$$

then $\Theta(\Theta_{ij})$ is a block matrix with $1 \leq i, j \leq r \times n$, $\lfloor \cdot \rfloor$ stands for the floor function, i.e., $\lfloor \alpha \rfloor = \max \{ \beta \in \mathbb{N} | \beta \leq \alpha \}$. Meanwhile, $\lceil \cdot \rceil$ is the ceil function, i.e., $\lceil \alpha \rceil = \min \{ \beta \in \mathbb{N} | \beta \geq \alpha \}$. When $n_\nu = i - \lfloor \frac{i}{n} \rfloor \times n$ and $\mu_\nu = \lceil \frac{i}{n} \rceil$, we have $\Theta(ij) = C_i(\nu_{ij} = 0)_{x+nu} A_{\mu_\nu, \nu_{ij}} D_i(\nu_{ij} = 0)_{x+nu},$

Therefore

$$\text{Tr}(\Theta(\Theta_{ij})) = \sum_{i=1}^{r} \sum_{j=1}^{n} C_i A_{\mu_\nu, \nu_{ij}} D_{i}(\nu_{ij} = 0)_{x+nu}.$$

**Remark 4.1:** The properties of $\text{Tr}$ and the standard matrix trace $\text{tr}$ are not completely consistent, such as commutativity. Generally speaking, $\text{Tr}(AB) = \text{Tr}(BA)$ does not hold.

Based on Lemma 3.1 and Theorem 3.1, $\varphi_{k,0}(\varphi_{k,0}, 0) = \bar{\nu}_{k,0}(\bar{\nu}_{k,0}, 0) \leq \frac{\alpha}{4} I_{2n}$ is a necessary and sufficient condition for finite-time stabilizability, where

$$\varphi_{k,0} = \left( I_{4k-1} \otimes \begin{array}{c} \sqrt{\alpha} A_{1,k} - 1 + \sqrt{\alpha} A_{2,k} - 1 \\ \sqrt{1 - \alpha} A_{1,k} - 1 \\ \sqrt{\alpha} C_{1,k} - 1 + \sqrt{\alpha} C_{2,k} - 1 \\ \sqrt{1 - \alpha} C_{1,k} - 1 \end{array} \right) \varphi_{k-1,0}.$$

Note that $\epsilon_{\max}(\varphi_{k,0}(\varphi_{k,0}, 0)) = \epsilon_{\max}(\text{Tr}(\varphi_{k,0}(\varphi_{k,0}, 0)))$, where $\epsilon_{\max}$ means the maximum eigenvalue and $\varphi_{k,0}(\varphi_{k,0}, 0)$ belongs to $\Gamma_k$ and each submatrix in $\varphi_{k,0}(\varphi_{k,0}, 0)$ belongs to $\mathbb{R}^{2n \times 2n}$. Set $\dot{P}_k = \text{Tr}(\varphi_{k,0}(\varphi_{k,0}, 0))$. So, we can get that

$$\dot{P}_{k+1} = \text{Tr}(\dot{\varphi}_{k+1,0}(\varphi_{k+1,0}, 0))$$

$$= \left( I_{4k+1} \otimes \begin{array}{c} \sqrt{\alpha} A_{1,k} + \sqrt{\alpha} A_{2,k} \\ \sqrt{1 - \alpha} A_{1,k} \\ \sqrt{\alpha} C_{1,k} + \sqrt{\alpha} C_{2,k} \\ \sqrt{1 - \alpha} C_{1,k} \end{array} \right) \varphi_{k,0}.$$

From Lemma 4.1, the above equation means that

$$\dot{P}_{k+1} = \text{Tr}(\dot{\varphi}_{k,0}(\varphi_{k,0}, 0)).$$

**Theorem 4.1:** LDMFS system (2) is finite-time stabilizable with respect to $(e_1, e_2, T, \{ R_k \}_{k \in \mathbb{N}_T^*})$ if there are symmetric positive definite matrices $\{ P_k \}_{k \in \mathbb{N}_T^*}$ satisfying the following constrained Lyapunov-type equation:

$$\begin{align*}
&\dot{A}_{1,k} P_k A_{1,k} + \bar{A}_{1,k} P_k A_{2,k} + \bar{A}_{2,k} P_k A_{1,k} + \bar{A}_{2,k} P_k A_{2,k} \\
&+ \bar{C}_{1,k} P_k C_{1,k} + \bar{C}_{2,k} P_k C_{2,k} + \bar{C}_{2,k} P_k C_{2,k}
\end{align*}$$

**Remark 4.2:** Theorem 4.1 provides a more convenient way to determine the finite-time stabilization than directly calculating STMIs. Equation (20) can be regarded as a nonfragile mean-field stochastic version of general Lyapunov equation about finite time stabilization. When the system (2) degenerates into a classical deterministic system, (20) reduces to the corresponding results in [3]. This necessary and sufficient Lyapunov-type theorem is able to improve many existing works.

**Theorem 4.2:** LDMFS system (2) is finite-time stabilizable with respect to $(e_1, e_2, T, \{ R_k \}_{k \in \mathbb{N}_T^*})$ if there are symmetric positive definite matrices $\{ P_k \}_{k \in \mathbb{N}_T^*}$ satisfying the following Lyapunov-type inequality:

$$\begin{align*}
&\dot{A}_{1,k} P_k A_{1,k} + \bar{A}_{1,k} P_k A_{2,k} + \bar{A}_{2,k} P_k A_{1,k} + \bar{A}_{2,k} P_k A_{2,k} \\
&+ \bar{C}_{1,k} P_k C_{1,k} + \bar{C}_{2,k} P_k C_{2,k} + \bar{C}_{2,k} P_k C_{2,k}
\end{align*}$$

**Proof:** According to Theorem 4.1, we need to prove that the solvability of (20) and (21) is equivalent to each other. Through observation, it is not difficult to find that (20) can definitely deduce (21) by choosing $P_k = \bar{P}_k$.

Next, let us consider: (21) $\Rightarrow$ (20). Suppose there are $P_k$ satisfying (21). Then, $0 < H_0 = P_k^{-1} I$ must exist and $P_0 = I = H_0 P_0$. By induction, it is assumed that there exists symmetric positive definite matrix $H_j$ makes $P_j = I = H_j P_j$. Then, we need to prove there exists $H_{j+1}$ such that $P_{j+1} = I = H_{j+1} P_{j+1}$. From (20), denote $M_j$ as

$$M_j = \dot{A}_{1,k} P_k A_{1,k} + \bar{A}_{1,k} P_k A_{2,k} + \bar{A}_{2,k} P_k A_{1,k} + \bar{A}_{2,k} P_k A_{2,k}$$

+ $\bar{C}_{1,k} P_k C_{1,k} + \bar{C}_{2,k} P_k C_{2,k} + \bar{C}_{2,k} P_k C_{2,k}$.

From Lemma 4.1, the above equation means that

$$\begin{align*}
&\dot{P}_{k+1} = \text{Tr}(\dot{\varphi}_{k,0}(\varphi_{k,0}, 0))
\end{align*}$$

$$= \left( I_{4k+1} \otimes \begin{array}{c} \sqrt{\alpha} A_{1,k} + \sqrt{\alpha} A_{2,k} \\ \sqrt{1 - \alpha} A_{1,k} \\ \sqrt{\alpha} C_{1,k} + \sqrt{\alpha} C_{2,k} \\ \sqrt{1 - \alpha} C_{1,k} \end{array} \right) \varphi_{k,0}.$$
So, \( H_{k+1} = M_k P_{k+1}^{-1} \).

Next, we are able to transform the nonfragile finite-time stabilizable controller \( u_k^F \) design problem into a feasible solution problem for a set of LMIs based on Schur’s complement.

**Theorem 4.3:** LDMFS system (2) is finite-time stabilizable with respect to \((\varepsilon_1, \varepsilon_2, T, \{R_k\}_{k \in \mathbb{N}_T})\) via a nonfragile controller \( u_k^F \), if for a given positive scalar \( \gamma > 0 \), there exist matrices \( K_1 \) and \( K_2 \), positive definite matrices \( \{P_k\}_{k \in \mathbb{N}_T}, \{Q_k\}_{k \in \mathbb{N}_T} \) solving the following LMIs:

\[
\begin{bmatrix}
-P_{k+1} & \sqrt{\gamma} M_1 & \sqrt{\gamma} \bar{C}_{1,k} & \sqrt{1-\alpha} \bar{C}_{1,k} \\
* & -\frac{\alpha}{\gamma} I_{2n} & 0 & 0 \\
* & * & -\frac{\alpha}{\gamma} I_{2n} & 0 \\
* & * & * & -\frac{\alpha}{\gamma} I_{2n} \\
\end{bmatrix} \leq \begin{bmatrix}
0 & \Pi_2 & 0 & 0 \\
0 & 0 & \Pi_3 & 0 \\
0 & 0 & 0 & \Pi_4 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} < 0 \tag{22}
\]

where \( P_0 \geq I_{2n}, P_k \leq \frac{\alpha}{\gamma} I_{2n} \).

\[
\Pi_1 = \bar{R}_{k+1}^\dagger \begin{bmatrix} A_1 + A_2 + B(K_1 + K_2) & 0 \\ 0 & A_1 + BK_1 \end{bmatrix} \bar{R}_{k+1}^\frac{1}{2}
\]

\[
\Pi_2 = \bar{R}_{k+1}^\dagger \begin{bmatrix} \frac{\alpha}{\gamma} \bar{B}M & 0 \\ (\sqrt{\gamma} - \alpha)BM & \sqrt{\alpha}BM \\ \alpha \sqrt{1-\alpha}BM & 0 \\ -\alpha \sqrt{1-\alpha}BM & \alpha BM \end{bmatrix} \left( I_2 \otimes \bar{R}_{k+1}^\frac{1}{2} \right)
\]

\[
\Pi_3 = \bar{R}_{k+1}^\dagger \begin{bmatrix} \sqrt{\gamma} N_1 & 0 \\ 0 & -\sqrt{\gamma} N_1 \end{bmatrix} \left( I_2 \otimes \bar{R}_{k+1}^\frac{1}{2} \right)
\]

\[
\Pi_4 = \bar{R}_{k+1}^\dagger \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \left( I_2 \otimes \bar{R}_{k+1}^\frac{1}{2} \right)
\]

\[
\Pi_5 = \bar{R}_{k+1}^\dagger \begin{bmatrix} \sqrt{\gamma} N_1 & 0 \\ 0 & \sqrt{\gamma} N_1 \end{bmatrix} \left( I_2 \otimes \bar{R}_{k+1}^\frac{1}{2} \right).
\]

**Proof:** By Schur’s complement, we have a sufficient condition from (21) that

\[
\begin{bmatrix}
-P_{k+1} & \sqrt{\gamma} (\bar{A}_{1,k} + \bar{A}_{2,k}) & \sqrt{1-\alpha} \bar{C}_{1,k} & \sqrt{\gamma} (\bar{C}_{1,k} + \bar{C}_{2,k}) \\
* & -\frac{\alpha}{\gamma} I_{2n} & 0 & 0 \\
* & * & -\frac{\alpha}{\gamma} I_{2n} & 0 \\
* & * & * & -\frac{\alpha}{\gamma} I_{2n} \\
\sqrt{1-\alpha} \bar{C}_{1,k} & 0 & 0 & 0 \\
\end{bmatrix} \leq \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} < 0. \tag{23}
\]

By [15, Th. 2.7] and Schur’s complement, (23) is equivalent to (22).

**Remark 4.3:** Using the STM method to study the finite-time stability and stabilization of linear discrete stochastic systems comes from [35], which is a main motivation for this study. The technical novelties compared with nonmean-field linear stochastic systems are the following aspects.

1. The coefficient matrices of the closed-loop system (4) have uncertain parameters \( \Delta K_{1,k}, \Delta K_{2,k} \) and random variable \( \alpha_k \). Therefore, the STMs of (4) are more complex in both mathematical derivations and expression forms than the nonmean-field linear discrete stochastic system without nonfragile control.

2. Based on the new STMs and Lemma 4.1 about a new operator “Tr,” novel necessary and sufficient Lyapunov-type theorems (Theorems 4.1 and 4.2) are proved. Theorems 4.1 and 4.2 are easier to verify than Theorem 3.1, especially for larger \( T > 0 \). As a corollary of Theorem 4.2, Theorem 4.3 presents an LMI-based sufficient condition for finite-time stabilization with the nonfragile control \( u_k^F \), which is more easily verified.

**V. VERIFICATION EXAMPLE**

In the sequel, we present an application to the price control problem of a company’s stock market, which is proposed in [6] and [17]. In addition to being adjusted by major shareholders, the stock market is also affected by market fluctuations, changes in crude oil prices, etc.
Several kinds of STMs of LDMFS systems have been first presented in this article and the nonfragile finite-time stabilization problem has been studied. Based on the STM method, several necessary and sufficient conditions for nonfragile finite-time stabilization have been derived. The advantage of the STM method is nonconservative. The feasibility and effectiveness of the new design schemes have been confirmed by one example. We believe that the STM method will have more applications in stochastic stability and stabilization, which merits further study in our future work.

VI. CONCLUSION

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