Covariant $q$-differential operators and unitary highest weight representations for $U_q\mathfrak{su}_{n,n}$

Dmitry Shklyarov†  Genkai Zhang‡

†Department of Mathematics, Kansas State University
Manhattan, KS 66506, USA
‡Chalmers Tekniska Högskola/Göteborgs Universitet, Matematik
412 96, Göteborg, Sweden

e-mail: shklyarov@math.ksu.edu, genkai@math.chalmers.se

ABSTRACT: We investigate a one-parameter family of quantum Harish-Chandra modules of $U_q\mathfrak{sl}_{2n}$. This family is an analog of the holomorphic discrete series of representations of the group $SU(n, n)$ for the quantum group $U_q\mathfrak{su}_{n,n}$. We introduce a $q$-analog of "the wave" operator (a determinant-type differential operator) and prove certain covariance property of its powers. This result is applied to the study of some quotients of the above-mentioned quantum Harish-Chandra modules. We also prove an analog of a known result by J. Faraut and A. Koranyi on the expansion of reproducing kernels which determines the analytic continuation of the holomorphic discrete series.

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1 Introduction

We start with recalling classical known results about analytic continuation of the weighted Bergman spaces in the unit disk and their explicit realization.

Recall that the group $SU_{1,1}$ acts on the unit disk by fractional-linear transformations. Many important representations of the group are realized geometrically in various functional spaces on the disk and on the unit circle. In particular, representations of the discrete series admit a realization of that kind. Namely, consider the kernel $(1 - z\bar{w})^{-\lambda}$ in the unit disk. For $\lambda > 1$ it is the reproducing kernel for the so-called weighted Bergman space consisting of holomorphic functions that are square integrable with the weight $(1 - |z|^2)^{\lambda-2}dm(z)$ (here $dm(z)$ is the normalized Lebesgue measure). The group $SU_{1,1}$ acts in the space via change of variable and a multiplier:

$$\pi_\lambda(g)(f(z)) = f(g^{-1}z) \cdot (cz + d)^{-\lambda}, \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (1.1)$$

(for non-integer $\lambda$'s one should consider the universal covering $\widetilde{SU}_{1,1}$ instead of $SU_{1,1}$). Thus obtained unitary representation belongs to the discrete series and is said to be a representation of the holomorphic discrete series for $SU_{1,1}$ or $\widetilde{SU}_{1,1}$.

The reproducing kernel $(1 - z\bar{w})^{-\lambda}$ has analytic continuation in the parameter $\lambda$. This is obtained from the formula

$$(1 - z\bar{w})^{-\lambda} = \sum_{m=0}^{\infty} (\lambda)_m \frac{(z\bar{w})^m}{m!}, \quad (\lambda)_m = \lambda \cdot (\lambda + 1) \cdot \ldots \cdot (\lambda + m - 1). \quad (1.2)$$

For $\lambda > 0$ the kernel is still positive definite, and the $SU_{1,1}$-action $\pi_\lambda$ in the associated Hilbert space is also unitary. For $\lambda = 1$, the Hilbert space is the Hardy space of holomorphic function on the closed disk whose boundary value are square integrable on the circle.

For further study of the previous representations, it is convenient to pass to the corresponding Harish-Chandra modules. Consider the space $\mathbb{C}[z]$ of polynomials on $\mathbb{C}$. The representation $\pi_\lambda$ induces a representation of $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}_{1,1} \otimes \mathbb{C}$ on $\mathbb{C}[z]$ which may be defined for any $\lambda \in \mathbb{R}$ (and even for $\lambda \in \mathbb{C}$). Let us denote by $\mathcal{P}_\lambda$ the space $\mathbb{C}[z]$ endowed with the above-mentioned action of $\mathfrak{sl}_2(\mathbb{C})$. $\mathcal{P}_\lambda$ is irreducible for all positive $\lambda$'s. However, if $\lambda = 1 - l$ for some positive integer $l$ then $\mathcal{P}_\lambda = \mathcal{P}_{1-l}$ has the following composition series

$$\{0\} \subset \mathcal{P}^{(0)}_{1-l} \subset \mathcal{P}_{1-l} \quad (1.3)$$

where $\mathcal{P}^{(0)}_{1-l}$ is the submodule of polynomials of degree $\leq l - 1$. The natural problem is to study the quotient $\mathcal{P}_{1-l}/\mathcal{P}^{(0)}_{1-l}$. This is the point where covariant differential operators appear on the scene. They play an important role in an explicit realization of the quotient. Namely, one checks that the differential operator $\left(\frac{\partial}{\partial z}\right)^l$ intertwines the actions $\pi_1-l$ and $\pi_1+l$:

$$\left(\frac{\partial}{\partial z}\right)^l \cdot \pi_{1-l}(\xi) = \pi_{1+l}(\xi) \cdot \left(\frac{\partial}{\partial z}\right)^l, \quad \xi \in \mathfrak{sl}_2(\mathbb{C}). \quad (1.4)$$

Clearly, $\left(\frac{\partial}{\partial z}\right)^l$ induces an isomorphism from $\mathcal{P}_{1-l}/\mathcal{P}^{(0)}_{1-l}$ into $\mathcal{P}_{1+l}$, and this, in particular, proves unitarizability of the former module.

The unit disk is the simplest example of a bounded symmetric domain [1]. The above-mentioned results admit appropriate generalization for any such domain (of course, the group $SU_{1,1}$ is replaced by the group of biholomorphic automorphisms of the domain under consideration).
For the so-called tube domains, some generalizations of the covariance property (1.4) have been obtained by G. Shimura [19], J. Arazy [1], H.P. Jakobsen [9], H.P. Jakobsen and M. Vergne [13], H.P. Jakobsen and M. Harris [12]. For example, in the case of the tube domain of type $I_{n,n}$ (the unit ball in the space of complex $n \times n$-matrices) the analog of (1.4) is a statement about an intertwining property of powers of the operator $\Box = \det \left( \frac{\partial}{\partial z^a} \right)_{a,\alpha=1,...,n}$ with respect to certain ”twisted” action of the group $SU_{n,n}$ analogous to (1.1).

The generalized covariance property (1.4) has turned out to be useful beyond the problems we mentioned previously. It has been applied also to computing the Harish-Chandra homomorphism of invariant differential operators [30].

Now for symmetric bounded domains the expansion (1.2) has been found by Ørsted [18] for type I matrix domains and in general case by J. Faraut and A. Koranyi [5]. From this expansion one can read off the composition series analogous to (1.3); the covariant property of the intertwining operators is related to the classical Cayley-Capelli type formula. We note that the unitarity of the highest weight modules had been classified earlier by Jakobsen [8] using algebraic method; however the analytic approach as in [18] and [5] generated some other interesting analytic subjects and is related to many problems in special functions and orthogonal polynomials. For quantum groups the classification of unitary highest weight representations has also been done recently [10], and we believe however that an analytic and concrete approach deserves pursuing.

In the present paper we obtain analogs of (1.2), (1.3), and (1.4) for a quantum matrix ball, an analog of the tube domain of type $I_{n,n}$ which has been defined in framework of quantum group theory by L. Vaksman et al [21].

In [22], the authors defined analogs of the weighted Bergman spaces on the quantum matrix ball. Also, they constructed analogs of the corresponding reproducing kernels and the ”twisted” unitary action of the group $SU_{n,n}$. From the representation theoretic point of view, the paper [22] presents a $q$-analog of the holomorphic discrete series of the group $SU_{n,n}$ (more precisely, analogs of the associated Harish-Chandra modules).

The natural problem now is to investigate those representations, particularly, to define their "analytic continuation" and to study composition series of the resulting Harish-Chandra modules. In the case $n = 2$, these problems were treated in [23]. In the present paper, we deal with the case of arbitrary $n$.

The role of covariant differential operators in the classical theory of bounded symmetric domains and related Harish-Chandra modules is very well known [9, 12, 13, 19]. Our intention is to bring covariant $q$-differential operators into the study of quantum Harish-Chandra modules and thus to demonstrate their importance in the quantum setting as well\(^1\). We introduce a determinant-type $q$-differential operator similar to $\Box$ and prove a $q$-analog of the covariance property. In the last section, this result is applied to investigation of certain quotients of the above quantum Harish-Chandra modules.

Another goal of the paper is to obtain an analog of the aforementioned result by J. Faraut and A. Koranyi which has allowed them to solve the problem of analytic continuation of the holomorphic discrete series in the classical setting.

As we already mentioned, there is a complete classification of unitarizable highest-weight modules over quantum groups (see [10]). Thus, neither the holomorphic discrete series of the quantum group $SU(n,n)$, constructed in [22], nor its analytic continuation, obtained in the present paper, give us a new family of unitary modules. The principal aim of the present paper, as well as papers [22], [23], is to develop an "analytic and geometric" framework for studying quantum Harish-Chandra modules related to quantum Cartan domains, particularly, to show

\(^1\)Note that similar questions have been already treated in the literature (see [3] and, especially, [11]).
that there are substantial generalizations of known classical constructions and results connected with the holomorphic discrete series.

The paper is organized as follows. Sections 2 and 3 contain some preliminary material. In Section 2, we recall some basic notions and results of quantum group theory (particularly, the notion of quantum space of matrices and of the quantized universal enveloping algebra $U_q\mathfrak{sl}_n$). This is done mainly for the purpose to set the notation we use further. Also, we recall certain hidden quantum $U_q\mathfrak{sl}_2$-symmetry of the quantum matrix space discovered in [27].

In the end of Section 2 we describe a twisted action (depending on a parameter $\lambda$) of $U_q\mathfrak{sl}_2$ on the quantum matrix space. For $\lambda$ large enough the corresponding Harish-Chandra modules are unitarizable representations of $U_q\mathfrak{su}_{n,n}$, which we call the holomorphic discrete series due to the previous motivation. Section 3 is devoted to $q$-differential operators. We recall there the notion of a $q$-differential operator with constant coefficients and describe certain properties of the algebra of such operators. Also, we introduce an analog of the operator $\square$ and derive its "obvious" quantum symmetry which amounts to an intertwining property of the operator with an action of the Hopf subalgebra $U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_n \subset U_q\mathfrak{sl}_2$. This obvious symmetry is extended to a large hidden symmetry, namely, the intertwining property of the operator (and of its powers) with the twisted $U_q\mathfrak{sl}_2$-actions. This covariance property is formulated and proved in Section 4. In the course of the proof, we use a number of results from the theory of quantum bounded symmetric domains, in particular, those obtained in [20] and, especially, results of [29]. To keep the size of the paper reasonable, we have to be more sketchy in this part of the paper. We omit proofs of those results giving appropriate references instead. In the last section of the paper we investigate the holomorphic discrete series for $U_q\mathfrak{su}_{n,n}$. First of all, we use computations of Section 4 to produce an analog of the result by J. Faraut and A. Koranyi we mentioned earlier. Then we derive some applications of the covariance property.

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2 Quantum space of matrices and its symmetries

In this paper, the parameter $q$ is supposed to be a number from the interval $(0,1)$.

2.1 Quantum space of matrices

Let us start with the definition of the algebra $\mathbb{C}[M_n]_q$ of polynomials on the quantum matrix space. It is the unital algebra given by its generators $z_a^\alpha$ (here $a, \alpha = 1, \ldots, n$, $a$ is the column index and $\alpha$ is the row index) and the following relations

$$z_a^\alpha z_b^\beta = \begin{cases} q z_b^\beta z_a^\alpha & , a = b \& \alpha < \beta \text{ or } a < b \& \alpha = \beta \\ z_b^\beta z_a^\alpha & , a < b \& \alpha > \beta \\ z_b^\beta z_a^\alpha + (q - q^{-1}) z_b^\alpha z_b^\alpha & , a < b \& \alpha < \beta \end{cases}.$$  (2.1)

These commutation relations, along with the relation

$$\det_q(z) = \sum_{s \in S_n} (-q)^{(s)} z_{a_1}^{\alpha_{s(1)}} z_{a_2}^{\alpha_{s(2)}} \cdots z_{a_n}^{\alpha_{s(n)}} = 1,$$  (2.2)

This hidden symmetry was one of the first hints that there should be a substantial theory of $q$-bounded symmetric domains. These objects were invented a little later in [26].
appeared for the first time in [4] as the relations between generators in the algebra \( \mathbb{C}[SL_n]_q \) of regular functions on the quantum \( SL_n \). It was suggested in [3] to discard (2.2) from the list of relations and to regard (2.1) as the defining relations of the algebra of polynomials on the quantum space of matrices. The algebra \( \mathbb{C}[SL_n]_q \) is then the quotient of \( \mathbb{C}[M_n]_q \) by the two-sided ideal generated by the element \( \det_q(z) - 1 \) (note that the \( q \)-determinant \( \det_q(z) \) belongs to the center of \( \mathbb{C}[M_n]_q \) ([2] Section 7.3.B)). Also, the algebra \( \mathbb{C}[M_n]_q \) is used to define the algebra of regular functions on the quantum \( GL_n \). The latter is just the localization of the former with respect to the multiplicative system \( \det_q(z)^m, m = 1, 2, \ldots \).

The crucial observation concerning the algebra \( \mathbb{C}[M_n]_q \) was the discovery of the comultiplication

\[
\mathbb{C}[M_n]_q \to \mathbb{C}[M_n]_q \otimes \mathbb{C}[M_n]_q, \quad z_\alpha^i \mapsto \sum_j z_\alpha^i \otimes z_\alpha^j
\]

which, along with the initial multiplication, makes \( \mathbb{C}[M_n]_q \) into a bialgebra. The comultiplication maps the \( q \)-determinant \( \det_q(z) \) to \( \det_q(z) \otimes \det_q(z) \) and thus induces a comultiplication on the algebra \( \mathbb{C}[SL_n]_q \). The latter, along with certain antipode and counit, makes \( \mathbb{C}[SL_n]_q \) into a Hopf algebra.

All the above structures allow one to produce \( q \)-analogs of the left and right actions

\[
L(g) : f(z) \mapsto f(g^{-1} \cdot z), \quad R(g) : f(z) \mapsto f(z \cdot g)
\]

of \( SL_n \) in \( \mathbb{C}[M_n] \). These \( q \)-analogs are usually described in terms of comodule algebras [2]. However, it is more convenient for us to use an "infinitesimal" version of those actions which is based on the notion of the quantum universal enveloping algebra \( U_q\mathfrak{sl}_n \) due to Drinfeld [4] and Jimbo [15].

First, we recall the definition of \( U_q\mathfrak{sl}_n \) (we follow the notation of [14]). The quantum universal enveloping algebra \( U_q\mathfrak{sl}_n \) is the unital algebra generated by the elements \( E_i, F_i, K_i^{\pm 1}, i = 1, \ldots, n \), which satisfy the relations

\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,
\]

\[
K_i E_j = q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i
\]

\[
E_i F_j - F_j E_i = \delta_{ij}(K_i - K_i^{-1})/(q - q^{-1}),
\]

\[
E_i^2 E_j - (q + q^{-1})E_i E_j E_i + E_j E_i^2 = 0, \quad |i - j| = 1
\]

\[
F_i^2 F_j - (q + q^{-1})F_i F_j F_i + F_j F_i^2 = 0, \quad |i - j| = 1
\]

\[
[E_i, E_j] = [F_i, F_j] = 0, \quad |i - j| \neq 1
\]

with \( (a_{ij}) \) being the Cartan matrix of type \( A_{n-1} \). Moreover, \( U_q\mathfrak{sl}_n \) is a Hopf algebra. The comultiplication \( \Delta \), the antipode \( S \), and the counit \( \varepsilon \) are determined by

\[
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i, \quad (2.3)
\]

\[
S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1}, \quad (2.4)
\]

\[
\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1. \quad (2.5)
\]

It is observed in [4] that the Hopf algebras \( U_q\mathfrak{sl}_n \) and \( \mathbb{C}[SL_n]_q \) are dual to each other. This, in particular, allows one to use the language of \( U_q\mathfrak{sl}_n \)-module algebras instead of that of \( \mathbb{C}[SL_n]_q \)-comodule algebras mentioned above. This is what we do in the present paper.
Let us recall now what the terminology "\(U_q\mathfrak{sl}_n\)-module algebra" means. Let \(A\) be a Hopf algebra. A unital algebra \(F\) is said to be an \(A\)-module algebra if \(F\) is an \(A\)-module, the unit of \(F\) is \(A\)-invariant (which means \(\varepsilon(1) = \varepsilon(\xi) \cdot 1\) for any \(\xi \in A\)), and, finally, the multiplication \(F \otimes F \to F\) intertwines the \(A\)-actions (we recall that for any \(A\)-modules \(V_1, V_2\) their tensor product is endowed with an \(A\)-module structure via the comultiplication \(\Delta : A \to A \otimes A\)).

**Remark.** In the sequel, we shall sometimes consider Hopf algebras with an additional structure, namely, Hopf \(*\)-algebras (a Hopf \(*\)-algebra is a pair \((A, \ast)\) where \(A\) is a Hopf algebra and \(\ast\) is an involution in \(A\) with certain properties; see \([2]\)). In the case of Hopf \(*\)-algebras the above-mentioned definition includes an additional requirement. Namely, let \(A_0 = (A, \ast)\) be a Hopf \(*\)-algebra and \(F\) an algebra. Then \(F\) is said to be an \(A_0\)-module algebra if, first, \(F\) is an \(A\)-module algebra in the previous sense, and, second, \(F\) is involutive and the involutions in \(A\) and \(F\) agree as follows:

\[
(\xi(f))^\ast = S(\xi^\ast(f^\ast)), \quad \xi \in A, f \in F.
\]

(2.6)

(The notion of module algebras can be clarified in the classical setting of a Lie group \(G\) acting on a smooth \(G\)-space \(X\). Denote by \(\mathfrak{g}\) the Lie algebra of \(G\). Then the universal enveloping algebra \(U\mathfrak{g}\) acts on the space \(C^\infty(X)\) via differential operators. The usual Leibnitz rule means that \(C^\infty(X)\) is a \(U\mathfrak{g}\)-module algebra.)

Let us turn back to the quantum space of matrices. Now we are in position to describe the very well-known "infinitesimal version" of the left and right actions of the quantum group \(SL_n\) in \(\mathbb{C}[M_n]_q\). Note, however, that the left action we present below is not an analog of the classical one, mentioned earlier. It is more convenient for us to use an action that differs from the usual left one by a simple automorphism of \(U_q\mathfrak{sl}_n\).

**Proposition 2.1**  

i) There exists a unique structure of \(U_q\mathfrak{sl}_n\)-module algebra in \(\mathbb{C}[M_n]_q\) such that

\[
R(K_i)z_{a}^{\alpha} = \begin{cases} 
qz_{a}^{\alpha}, & a = i \\
q^{-1}z_{a}^{\alpha}, & a = i + 1 \\
z_{a}^{\alpha}, & \text{otherwise}
\end{cases},
\]

\[
R(F_i)z_{a}^{\alpha} = \begin{cases} 
q^{1/2}z_{a+1}^{\alpha}, & a = i \\
z_{a}^{\alpha}, & \text{otherwise}
\end{cases},
\]

\[
R(E_i)z_{a}^{\alpha} = \begin{cases} 
q^{-1/2}z_{a-1}^{\alpha}, & a = i + 1 \\
z_{a}^{\alpha}, & \text{otherwise}
\end{cases}.
\]

(2.7)

ii) There exists a unique structure of \(U_q\mathfrak{sl}_n\)-module algebra in \(\mathbb{C}[M_n]_q\) such that

\[
L(K_j)z_{a}^{\alpha} = \begin{cases} 
qz_{a}^{\alpha}, & \alpha = n - j \\
q^{-1}z_{a}^{\alpha}, & \alpha = n - j + 1 \\
z_{a}^{\alpha}, & \text{otherwise}
\end{cases},
\]

\[
L(F_j)z_{a}^{\alpha} = \begin{cases} 
q^{1/2}z_{a+1}^{\alpha}, & \alpha = n - j \\
z_{a}^{\alpha}, & \text{otherwise}
\end{cases},
\]

\[
L(E_j)z_{a}^{\alpha} = \begin{cases} 
q^{-1/2}z_{a-1}^{\alpha}, & \alpha = n - j + 1 \\
z_{a}^{\alpha}, & \text{otherwise}
\end{cases}.
\]

(2.9)

iii) For any \(\xi, \eta \in U_q\mathfrak{sl}_n\) the endomorphisms \(R(\xi), L(\eta)\) commute

\[
R(\xi)L(\eta)f = L(\eta)R(\xi)f, \quad f \in \mathbb{C}[M_n]_q.
\]

Note that by statement iii) in the above proposition, the algebra \(\mathbb{C}[M_n]_q\) is acted upon by the tensor product \(U_q\mathfrak{sl}_n \otimes U_q\mathfrak{sl}_n\):

\[
\xi \otimes \eta(f) = R(\xi)L(\eta)f.
\]
One can check that the $q$-determinant $\det_q(z)$ (2.2) is invariant with respect to both left and right $U_q \mathfrak{sl}_n$-actions, i.e.

$$R(\xi)\det_q(z) = L(\xi)\det_q(z) = \varepsilon(\xi) \cdot \det_q(z)$$

for any $\xi \in U_q \mathfrak{sl}_n$. Thus the formulas from Proposition 2.4 define left and right $U_q \mathfrak{sl}_n$-actions in $\mathbb{C}[SL_n]_q$. By analogy with the classical case, one has the following proposition (see [2]):

**Proposition 2.2** The $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n$-module $\mathbb{C}[SL_n]_q$ splits into direct sum of simple pairwise non-isomorphic submodules whose lowest vectors are given via $q$-minors as follows

$$(z^n_a)^{a_1} \left( z^{2(\{n-1,n\})} \right)^{a_2} \left( z^{3(\{n-2,n-1,n\})} \right)^{a_3} \cdots \left( z^{(n-1)(\{2,\ldots,n\})} \right)^{a_{n-1}}.$$

We recall that the $q$-minors are defined by

$$(z^k\{a_1,a_2,\ldots,a_k\})^{a_1,a_2,\ldots,a_k}_{\{a_1,a_2,\ldots,a_k\}} = \sum_{s \in S_k} (-q)^{l(s)} z_{a_1}^{a(s)(1)} z_{a_2}^{a(s)(2)} \cdots z_{a_k}^{a(s)(k)}$$

(2.11)

with $\alpha_1 < \alpha_2 < \ldots < \alpha_k$, $a_1 < a_2 < \ldots < a_k$, and $l(s)$ being the length of $s \in S_k$. In particular, $\det_q(z) = (z^{\{n\}})_{\{1,2,\ldots,n\}}$.

Let us denote the tensor product $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n$ with the canonical Hopf algebra structure by $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$. Thus, $\mathbb{C}[M_n]_q$ is a $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$-module algebra. It follows from the definition of the quantum universal enveloping algebra $U_q \mathfrak{sl}_n$ that there is an embedding of Hopf algebras $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n) \hookrightarrow U_q \mathfrak{sl}_{2n}$ determined by

$$1 \otimes E_i \mapsto E_i, \quad 1 \otimes F_i \mapsto F_i, \quad 1 \otimes K_i^{\pm 1} \mapsto K_i^{\pm 1}, \quad i = 1, \ldots, n - 1,$$

$$E_i \otimes 1 \mapsto E_{n+i}, \quad F_i \otimes 1 \mapsto F_{n+i}, \quad K_i^{\pm 1} \otimes 1 \mapsto K_{n+i}^{\pm 1}, \quad i = 1, \ldots, n - 1.$$  

This is a $q$-analog of the embedding $SL_n \times SL_n \hookrightarrow SL_{2n}$ given, in the matrix realization, by

$$(A,B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$ 

In the next subsection we shall extend the above $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$-module algebra structure in $\mathbb{C}[M_n]_q$ to a structure of $U_q \mathfrak{sl}_{2n}$-module algebra.

2.2 A structure of $U_q \mathfrak{sl}_{2n}$-module algebra on $\mathbb{C}[M_n]_q$

In this subsection we describe a "hidden" $U_q \mathfrak{sl}_{2n}$-module algebra structure in $\mathbb{C}[M_n]_q$. It was discovered in [27]. Its classical counterpart comes from an embedding of the matrix space $M_n$ into the Grassmannian $Gr_n(\mathbb{C}^{2n})$ as the affine cell $U \subset Gr_n(\mathbb{C}^{2n})$ defined by the inequality $t \neq 0$ with $t$ being a distinguished Plücker coordinate. A $q$-version of the embedding is described in [27], Proposition 0.7 (see also Proposition 5.4 from [21]).

Let us turn to the quantum case. The following statement was proved in [21], Section 2[.]

**Proposition 2.3** There exists a unique $U_q \mathfrak{sl}_{2n}$-module algebra structure in $\mathbb{C}[M_n]_q$ given on the Hopf subalgebra $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$ by the formulas from Proposition 2.4 and on the remaining generators $K_n^{\pm 1}$, $F_n$, $E_n$ by

$$K_n^{a_\alpha} = \begin{cases} q^{2a_\alpha}, & a = n \land \alpha = n \\ qz_\alpha a, & a = n \land \alpha \neq n \lor a \neq n \land \alpha = n \end{cases}$$

(2.12)
\[ F_n z_α^a = q^{1/2} \begin{cases} 1, & a = n \& α = n \\ 0, & \text{otherwise} \end{cases}, \quad E_n z_α^a = -q^{1/2} \begin{cases} q^{-1} z_α^m & a \neq n \& α \neq n \\ (z_α^m)^2, & a = n \& α = n \end{cases}. \quad (2.13) \]

Let us point out some straightforward but essential properties of this \( U_q sl_{2n} \)-action in \( C[M_n]_q \). Denote by \( U_q sl(gl_n \times gl_n) \) the Hopf subalgebra in \( U_q sl_{2n} \) derived from \( U_q(sl_n \times sl_n) \) by adding the generators \( K_{n,i}^{\pm 1} \). Clearly, elements of \( U_q sl(gl_n \times gl_n) \) preserve the natural \( \mathbb{Z}_+ \)-grading in \( C[M_n]_q \) given by powers of monomials. It is also obvious that the generators \( F_n, E_n \) act in \( C[M_n]_q \) as endomorphisms of degrees \(-1\) and \(1\), respectively. All this may be derived also from the following convenient description of the \( \mathbb{Z}_+ \)-grading in \( C[M_n]_q \):

\[ \deg f = N \iff \hat{K} f = q^{2N} f \]

where \( \hat{K} \) is the element of the center of \( U_q sl(gl_n \times gl_n) \) given by

\[ \hat{K} = (K_n)^n \cdot \prod_{j=1}^{n-1} (K_n K_{2n-j})^j. \]

For computational purposes, it is important to understand the structure of \( C[M_n]_q \) as a \( U_q sl(gl_n \times gl_n) \)-module in greater details. The following statement is a straightforward consequence of Proposition 2.2.

**Proposition 2.4** The \( U_q sl(gl_n \times gl_n) \)-module \( C[M_n]_q \) splits into direct sum of simple pairwise non-isomorphic submodules \( C[M_n]_q \otimes^{(k_1,k_2,...,k_n)} \), \( k_1 \geq k_2 \geq \ldots \geq k_n \geq 0 \), whose lowest vectors are given by

\[ (z_α^n)^{k_1-k_2} \left(z_α^{\wedge 2(n-1,n)}\right)^{k_2-k_3} \left(z_α^{\wedge 3(n-2,n-1,n)}\right)^{k_3-k_4} \ldots \left(\det_q z\right)^{k_n}. \]

In what follows, the above \( U_q sl_{2n} \)-action in \( C[M_n]_q \) will be sometimes called ‘the initial’ one, in contrast to a twisted action described in the next subsection.

Let us present another view on the above \( U_q sl_{2n} \)-action in \( C[M_n]_q \). The point is that the corresponding classical \( U sl_{2n} \)-action is well known in the theory of bounded symmetric domains (see, for instance, [1]). In framework of this theory, it is constructed as follows. The vector space \( M_n \) contains the so-called matrix ball (the boundary symmetric domain of type \( I_{n,n} \))

\[ D = \{ z \in M_n \mid zz^* < 1 \} \]

(with * being the hermitian conjugation and 1 the unit matrix). It is known that the real simple Lie group \( SU_{n,n} \) acts on \( D \) via biholomorphic automorphisms, and \( S(U_n \times U_n) \subset SU_{n,n} \) is the isotropy subgroup of the center \( 0 \in D \). Thus elements of the universal enveloping algebra \( U sl_{2n} \), and hence elements of its complexification \( U q sl_{2n} \), act on the space of holomorphic functions on \( D \) via differential operators. These differential operators have polynomial coefficients and, thus, preserve \( C[M_n] \). The resulting \( U sl_{2n} \)-action in \( C[M_n] \) is what we call the initial one. In framework of this approach, the result of Proposition 2.4 is just a \( q \)-analog of the famous Hua-Schmid decomposition [1] whereas the quantum enveloping algebra \( U q sl(gl_n \times gl_n) \) itself is an analog of the universal enveloping algebra of the complexified Lie algebra of the isotropy subgroup \( S(U_n \times U_n) \).
2.3 A twisted $U_q\mathfrak{sl}_{2n}$-action on $\mathbb{C}[M_n]_q$

In this subsection we introduce a one-parameter family $\pi_\lambda$, $\lambda \in \mathbb{R}$, of $U_q\mathfrak{sl}_{2n}$-actions in $\mathbb{C}[M_n]_q$ such that the initial $U_q\mathfrak{sl}_{2n}$-action, defined in the previous subsection, corresponds to $\lambda = 0$. In the classical case the corresponding twisted $U\mathfrak{sl}_{2n}$-action $\pi_\lambda$ for $\lambda \in \mathbb{Z}$ can be produced by trivializing the homogeneous line bundle $\mathcal{O}(\lambda)$ on the Grassmannian $\text{Gr}_n(\mathbb{C}^{2n})$ over the affine cell $U$. Namely, we identify the space of polynomials on $M_n$ with the space of sections $\Gamma(U, \mathcal{O}(\lambda))$ by $f(z) \sim f(z) \cdot t^{-\lambda}$ (here $t$ is the distinguished Plücker coordinate mentioned at the beginning of the previous subsection) and define the $U\mathfrak{sl}_{2n}$-action $\pi_\lambda$ as follows:

$$(\pi_\lambda(f))(z) = f(z) \cdot t^{-\lambda}$$

Note that, among the actions $\pi_\lambda$, the initial action $\pi_0$ is the only one that makes $\mathbb{C}[M_n]$ into a $U\mathfrak{sl}_{2n}$-module algebra. This is true in the $q$-setting as well.

Let us define a quantum version of the $U\mathfrak{sl}_{2n}$-action (2.16).

**Proposition 2.5** For any $\lambda \in \mathbb{R}$ the formulas

$$\pi_\lambda(K_j^{+1})f = \begin{cases} K_j^{+1}f, & j \neq n, \\ q^{\lambda}K_n^{1}f, & j = n \end{cases}$$

$$\pi_\lambda(F_j)f = \begin{cases} F_jf, & j \neq n, \\ q^{-\lambda}F_nf, & j = n \end{cases}$$

$$\pi_\lambda(E_j)f = \begin{cases} E_jf, & j \neq n, \\ E_nf - q^{1/2}1 - q^{2\lambda}(K_nf)z_n, & j = n \end{cases}$$

define a $U_q\mathfrak{sl}_{2n}$-action in $\mathbb{C}[M_n]_q$ (in the right-hand sides the initial $U_q\mathfrak{sl}_{2n}$-action is used).

This proposition was proved in [22] Proposition 6.2]. Note that $\pi_0$ coincides with the initial $U_q\mathfrak{sl}_{2n}$-action. For brevity, we shall denote the $U_q\mathfrak{sl}_{2n}$-module, corresponding to $\lambda$, by $P_\lambda$, namely $P_\lambda = (\mathbb{C}[M_n]_q, U_q\mathfrak{sl}_{2n}, \pi_\lambda)$.

The classical counterpart of the above twisted $U_q\mathfrak{sl}_{2n}$-action is also well known in the theory of bounded symmetric domains. The corresponding $SU_{n,n}$-action (more precisely, the action of the universal covering $\widetilde{SU}_{n,n}$) is defined by

$$\pi_\lambda(g) : f(z) \mapsto f(g^{-1}z) \cdot J_{g^{-1}}(z)$$

with $J_{g^{-1}}(z)$ being the Jacobian of the biholomorphic map $z \mapsto g^{-1}z$ (see [1] for details). For $\lambda > 2n - 1$ the action $\pi_\lambda$ on a weighted Bergman space defines a holomorphic discrete series representation of $SU_{n,n}$. In the last section of the paper we will describe a unitary structure on $P_\lambda$ which formally tends to the classical setting as $q \to 1$.

3 Some $q$-differential operators

3.1 Basic definitions

One of our results is connected with a $q$-analogue of the wave operator

$$\Box = \det \left( \frac{\partial}{\partial z^a} \right)$$

We start with some general consideration of $q$-differential operators with constant coefficients.
To produce $q$-analogues of the partial derivatives, we use certain known first order differential calculus over $\mathbb{C}[M_n]$, see \[2\]. Let $\Omega^1(M_n)_q$ be the $\mathbb{C}[M_n]_q$-bimodule given by its generators $dz_a^\alpha$, $a, \alpha = 1, \ldots, n$, and the relations

$$z_0^\beta dz_a^\alpha = \sum_{\alpha', \beta'} \sum_{\alpha, \beta = 1}^n R_{\beta \alpha}^{\beta' \alpha'} R_{ba}^{\alpha'} dz_a^\alpha \cdot z_b^{\beta'},$$

with

$$R_{ba}^{\alpha'} = \begin{cases} q^{-1}, & a = b = \alpha' = b' \\ 1, & a \neq b & a = \alpha' \& b = b' \\ q^{-1} - q, & a < b \& a = \alpha' \& b = b' \\ 0, & \text{otherwise} \end{cases}$$

The map $\mathbf{d} : z_a^\alpha \mapsto dz_a^\alpha$ can be extended to a linear operator $\mathbf{d} : \mathbb{C}[M_n]_q \to \Omega^1(M_n)_q$ satisfying the Leibnitz rule $\mathbf{d}(f_1 f_2) = \mathbf{d}(f_1) f_2 + f_1 \mathbf{d}(f_2)$. The pair $(\Omega^1(M_n)_q, \mathbf{d})$ is the first order differential calculus over $\mathbb{C}[M_n]_q$ we need.

The calculus itself has been known for a long time \[2\]. However, its hidden $U_q\mathfrak{sl}_2$-symmetry, was observed much later in \[27\]. To be more precise, it is proved in \[27\] that there exists a unique structure of a $U_q\mathfrak{sl}_2$-module in $\Omega^1(M_n)_q$ such that, first, the map $\mathbf{d}$ is a morphism of $U_q\mathfrak{sl}_2$-modules, and, second, the left and right multiplications

$$\mathbb{C}[M_n]_q \otimes \Omega^1(M_n)_q \to \Omega^1(M_n)_q, \quad \Omega^1(M_n)_q \otimes \mathbb{C}[M_n]_q \to \Omega^1(M_n)_q$$

are morphisms of $U_q\mathfrak{sl}_2$-modules. This is usually expressed by saying that the first order differential calculus $(\Omega^1(M_n)_q, \mathbf{d})$ is $U_q\mathfrak{sl}_2$-covariant. Before \[27\] appeared, only $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$-coherence of the calculus was known.

The first order differential calculus allows us to define the $q$-analogues of partial derivatives as follows: Set

$$\mathbf{d}f = \sum_{a=1}^n \sum_{\alpha=1}^n \frac{\partial f}{\partial z_a^\alpha} \cdot dz_a^\alpha, \quad f \in \mathbb{C}[M_n]_q.$$ 

Here the left-hand side defines the right-hand one.

It is quite reasonable to regard the unital subalgebra in $\text{End}(\mathbb{C}[M_n]_q)$ generated by all the derivatives as an analog of the algebra of differential operators with constant coefficients. This algebra seems to be interesting in itself. First of all, it admits a very explicit description. Namely, it is observed in \[20\] Section 2 that the map $z_a^\alpha \mapsto \frac{\partial}{\partial z_a^\alpha}$ may be extended to an algebra homomorphism $\Upsilon : \mathbb{C}[M_n]_q \to \text{End}(\mathbb{C}[M_n]_q)$ which means that the operators $\frac{\partial}{\partial z_a^\alpha}$ satisfy the same commutation relations as the generators $z_a^\alpha$ of $\mathbb{C}[M_n]_q$ do. Further, the algebra is invariant with respect to a certain natural $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$-action in $\text{End}(\mathbb{C}[M_n]_q)$ defined via a $q$-analogue of the commutator. Let us describe this latter observation in full details.

Endow the space $\text{End}(\mathbb{C}[M_n]_q)$ with a structure of $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$-module as follows: For $\xi \in U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$, $T \in \text{End}(\mathbb{C}[M_n]_q)$ put

$$\xi(T) = \sum_j \xi_j \otimes S^{-1}(\xi_j'),$$

where $\sum_j \xi'_j \otimes \xi_j = \Delta(\xi)$ (here $\Delta$ denotes the comultiplication in $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$), $S$ is the antipode of $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$, and the elements in the right-hand side are multiplied within $\text{End}(\mathbb{C}[M_n]_q)$. It is explained in \[20\] that the $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$-coherence of the first order differential calculus $(\Omega^1(M_n)_q, \mathbf{d})$ and the explicit formulas for the $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$-action in $\mathbb{C}[M_n]_q$, presented in Proposition \[27\] allow one to prove $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$-invariance of the linear span of all $\frac{\partial}{\partial z_a^\alpha}$ in
Clearly, the \( \text{End}(\mathbb{C}[M_n|_q]) \) and to describe the \( U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n) \)-action on the partial derivatives explicitly. The explicit description is based on the following intertwining property of the homomorphism \( \Upsilon \)

\[
\Upsilon(\xi f) = \omega(\xi)\Upsilon(f), \quad \forall \xi \in U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n), \forall f \in \mathbb{C}[M_n|_q]
\]

with \( \omega \) being the automorphism of \( U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n) \) (the Chevalley involution) given by

\[
\omega(E_i) = -F_i, \quad \omega(F_i) = -E_i, \quad \omega(K_i^{\pm 1}) = K_i^{\mp 1}.
\]

### 3.2 A q-wave operator

Here we apply the results from the previous subsection to study the q-wave operator given by

\[
\Box_q = \sum_{s \in S_n} (-q)^{(s)} \cdot \frac{\partial}{\partial z_1^{s(1)}} \cdot \frac{\partial}{\partial z_2^{s(2)}} \cdot \ldots \cdot \frac{\partial}{\partial z_n^{s(n)}}.
\]

Clearly, the q-wave operator belongs to the center of the algebra of quantum differential operators with constant coefficients since \( \Box_q = \Upsilon(\det_q(z)) \). The latter formula, together with \( U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n) \)-invariance of the q-determinant and the above intertwining property of \( \Upsilon \), implies also that the operator \( \Box_q \) commutes with the action of \( U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n) \) in \( \mathbb{C}[M_n|_q] \). Also, we can easily prove that

\[
K_n \cdot \Box_q = q^{-2} \Box_q \cdot K_n.
\]

Indeed, the degree of the operator \( \Box_q \) in \( \mathbb{C}[M_n|_q] \) is equal to \(-n\) which means \( \hat{K} \cdot \Box_q = q^{-2n} \cdot \Box_q \cdot \hat{K} \) (see (2.14)). The latter equality implies (3.1) since \( \Box_q \) commutes with all the \( K_i^{\pm 1} \)'s for \( i \neq n \).

### 4 A covariance property

#### 4.1 Formulation

The intertwining properties of the q-wave operator derived above may be written in a unified way as follows

\[
\Box_q^l \cdot \pi_{n-l}(\xi) = \pi_{n+l}(\xi) \cdot \Box_q^l, \quad \xi \in U_q\mathfrak{sl}_n \times \mathfrak{sl}_n.
\]

It turns out that this obvious symmetry of the operator \( \Box_q^l \) is a part of a large hidden symmetry:

**Theorem 4.1**

For any \( l \in \mathbb{N} \) the linear operator \( \Box_q^l : \mathbb{C}[M_n|_q] \rightarrow \mathbb{C}[M_n|_q] \) intertwines the \( U_q\mathfrak{sl}_{2n} \)-actions \( \pi_{n-l} \) and \( \pi_{n+l} \):

\[
\Box_q^l \cdot \pi_{n-l}(\xi) = \pi_{n+l}(\xi) \cdot \Box_q^l, \quad \xi \in U_q\mathfrak{sl}_{2n}
\]

(in other words, the map \( \Box_q^l : \mathcal{P}_{n-l} \rightarrow \mathcal{P}_{n+l} \) is a morphism of \( U_q\mathfrak{sl}_{2n} \)-modules).

We will proof Theorem 4.1 in subsection 4.5. The proof uses some results from the theory of quantum bounded symmetric domains which we recall in the subsequent three subsections. Very briefly, the idea is as follows (compare with [1]): We use the \( q \)-Cauchy-Szegö integral formula to rewrite the operator \( \Box_q^l \) as a \( q \)-integral operator; then, using some standard technique, we prove that the \( q \)-integral operator intertwines the \( U_q\mathfrak{sl}_{2n} \)-actions \( \pi_{n-l} \) and \( \pi_{n+l} \).
4.2 A \( q \)-analog of the Cauchy-Szegö integral representation

For any bounded symmetric domain there is a multivariable generalization of the famous Cauchy formula, the so-called Cauchy-Szegö integral representation [17]. This integral formula restores a holomorphic function on the domain from its boundary value on the Shilov boundary. In the case of the unit matrix ball the Cauchy-Szegö formula looks as follows

\[
f(z) = \int_{S(D)} \frac{f(\zeta)}{\det(1-z\zeta^*)^n} d\nu(\zeta).
\]

Here \( S(D) \) is the Shilov boundary of the unit matrix ball \( D \subset M_n \)

\[
S(D) = \{ z \in M_n \mid zz^* = 1 \},
\]

and \( d\nu \) is the unique \( U_n \)-invariant normalized measure on \( S(D) \) which, of course, coincides with the Haar measure under the identification \( S(D) = U_n \).

A \( q \)-analog of this formula was found in [29] in framework of quantum bounded symmetric domain theory. Particularly, in that paper \( q \)-analogs of the Shilov boundary \( S(D) \), the measure \( d\nu \), and the kernel \( \det(1-z\zeta^*)^{-n} \) were found. In this subsection we recall all these results. We omit proofs. An interested reader might want to look into [29] which is the main reference for this section.

The \( q \)-analog of the Shilov boundary is described by a (noncommutative) \#-algebra of functions on it. It is also natural to require the quantum Shilov boundary to be a homogeneous space of the quantum group \( SU_{n,n} \). Here is an explicit construction.

The localization of \( \mathbb{C}[M_n]_q \) with respect to the multiplicative system \( \det_q(z) \) is called the algebra of regular functions on the quantum \( GL_n \) and is denoted by \( \mathbb{C}[GL_n]_q \) (see subsection 2.1). It was observed in [29] Lemma 2.1 that there exists a unique involution \( * \) in \( \mathbb{C}[GL_n]_q \) such that

\[
(z_\alpha^*)^* = (-q)^{a+2n} \det_q(z)^{-1} \cdot z^{(n-1)J_n}, \quad (4.2)
\]

with \( J_n \) defined as \( \{1,2,\ldots,n\}\{e\} \) (here we use the notation (2.11)). The \( * \)-algebra \( \text{Pol}(S(D))_q = (\mathbb{C}[GL_n]_q,*) \) is a \( q \)-analog of the polynomial algebra on the Shilov boundary of the matrix ball \( D \). Note that

\[
\det_q(z) \det_q(z)^* = \det_q(z)^* \det_q(z) = q^{-n(n-1)}. \quad (4.3)
\]

Let’s describe a structure of homogeneous space of the quantum group \( SU_{n,n} \) on the quantum Shilov boundary. Recall that the \( q \)-determinant \( \det_q(z) \) belongs to the center of \( \mathbb{C}[M_n]_q \) and is a "relative invariant" with respect to the \( U_qsl_n \times gl_n \)-action:

\[
\xi \det_q(z) = \varepsilon(\xi) \cdot \det_q(z), \quad \xi \in U_qsl_n \times gl_n, \quad \det_q(z) = q^2 \det_q(z). \quad (4.4)
\]

Using (4.4), one can make \( \text{Pol}(S(D))_q \) into a \( U_qsl_n \times gl_n \)-module algebra. (More precisely, we can use the above formulas to define a \( U_qsl_n \times gl_n \)-action on negative powers of \( \det_q(z) \) which suffices to extend the \( U_qsl_n \times gl_n \)-action from \( \mathbb{C}[M_n]_q \) to \( \text{Pol}(S(D))_q \).)

In fact (see [29] Section 2), the above \( U_qsl_n \times gl_n \)-module algebra structure in \( \text{Pol}(S(D))_q \) may be extended to a structure of \( U_qsl_{2n} \)-module algebra which coincides on the subspace \( \mathbb{C}[M_n]_q \subset \text{Pol}(S(D))_q \) with the \( U_qsl_{2n} \)-module algebra structure described in Proposition 2.3.

Let us recall the definition of the "real form" \( U_qsu_{n,n} \) of the quantum universal enveloping algebra \( U_qsl_{2n} \). \( U_qsu_{n,n} \) is simply the pair \( (U_qsl_{2n},*) \) with * being an involution in \( U_qsl_{2n} \) determined by

\[
E_n^* = -K_n F_n, \quad F_n^* = -E_n K_n^{-1}, \quad (K_n^{\pm 1})^* = K_n^{\pm 1},
\]

\[
E_j^* = K_j F_j, \quad F_j^* = E_j K_j^{-1}, \quad (K_j^{\pm 1})^* = K_j^{\pm 1}, \quad \text{for } j \neq n.
\]
It is not difficult to verify that $U_q\mathfrak{su}_{n,n} = (U_q\mathfrak{sl}_{2n}, \ast)$ is a Hopf \ast-algebra (see \cite{2} for definitions). Evidently, the involution \ast keeps the Hopf subalgebras $U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)$ and $U_q(\mathfrak{gl}_n \times \mathfrak{gl}_n)$ invariant, and we shall denote the corresponding Hopf \ast-subalgebras in $U_q\mathfrak{su}_{n,n}$ by $U_q(\mathfrak{su}_n \times \mathfrak{su}_n)$ and $U_q\mathfrak{sl}(\mathfrak{u}_n \times \mathfrak{u}_n)$, respectively.

The crucial property of the involution \cite{12} is the following observation: it makes $\text{Pol}(S(\mathcal{D}))_q$ into a $U_q\mathfrak{su}_{n,n}$-module algebra (this is explained in \cite{29} after Proposition 2.7). It is in this sense that the quantum Shilov boundary is a homogeneous space of the quantum group $SU_{n,n}$.

To define a $q$-analog of the measure $dv$ on $S(\mathcal{D})$, we note (see \cite{29} Section 3) that the \ast-algebra $\text{Pol}(S(\mathcal{D}))_q$ is closely related to the \ast-algebra $\mathbb{C}[U_n]_q = (\mathbb{C}[GL_n]_q, \ast)$ of regular functions on the quantum group $U_n$ where, we recall, the involution \ast is defined by $(z_a^0)^\ast = (-q)^{a-\alpha}(\det q)z^{(n-1)J_0} \cdot z^a$.

It’s easy to check that $\ast = \theta^{-1} \cdot \ast \cdot \theta$ where $\theta : \mathbb{C}[GL_n]_q \to \mathbb{C}[GL_n]_q$ is an automorphism given by $\theta : z_a^0 \mapsto q^{a-n}z_a^0$. It is known that the compact quantum group $U_n$ possesses a unique normalized invariant integral, an analog of the Haar integral. The isomorphism $\theta$ of \ast-algebras $\text{Pol}(S(\mathcal{D}))_q \to \mathbb{C}[U_n]_q$ allows us to ”transfer" the invariant integral onto $\text{Pol}(S(\mathcal{D}))_q$. In this way we get a positive $U_q\mathfrak{sl}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant linear functional $\text{Pol}(S(\mathcal{D}))_q \to \mathbb{C}$, $f \mapsto \int_{S(\mathcal{D})} f dv$ which is the analog of the integral with respect to $dv$. The $U_q\mathfrak{sl}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariance means

$$\int_{S(\mathcal{D})} \xi f dv = \varepsilon(\xi) \cdot \int_{S(\mathcal{D})} f dv, \quad \forall \xi \in U_q\mathfrak{sl}(\mathfrak{gl}_n \times \mathfrak{gl}_n).$$

Finally let us describe the analog of the Cauchy-Szegö kernel $\det(1 - z\xi^\ast)^{-n}$.

Consider the algebra $\text{Pol}(M_n \times S(\mathcal{D}))_q = \mathbb{C}[M_n]_q^{op} \otimes \text{Pol}(S(\mathcal{D}))_q$ with ”op" indicating the change of the multiplication to the opposite one. Equip it with a $\mathbb{Z}_+$-grading by setting $\text{deg}(z_a^0 \otimes f) = 1$ for any $f \in \text{Pol}(S(\mathcal{D}))_q$. Its completion with respect to this grading is denoted by $\text{Fun}(M_n \times S(\mathcal{D}))_q$. The elements of $\text{Fun}(M_n \times S(\mathcal{D}))_q$ are $q$-analogs of kernels of integral operators, while the elements of the subalgebra $\text{Pol}(M_n \times S(\mathcal{D}))_q$ are $q$-analogs of polynomial kernels.

Let us comment on the replacement of the multiplication law in the first tensor multiplier in the definition of the algebra $\text{Pol}(M_n \times S(\mathcal{D}))_q$. Given a Hopf algebra $A$ and two $A$-module algebras $F_1, F_2$, $A$-invariant elements in $F_1 \otimes F_2$ do not form a subalgebra. However, they do form a subalgebra in $F_1^{op} \otimes F_2$ \cite{28}. Almost all the kernels we encounter in the present paper are $U_q\mathfrak{sl}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant in $\text{Pol}(M_n \times S(\mathcal{D}))_q$ or $\text{Fun}(M_n \times S(\mathcal{D}))_q$ and so, as we have explained, form a subalgebra.

Let us explain now why we are interested in $U_q\mathfrak{sl}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant kernels. It is not difficult to prove that there is a one-to-one correspondence between $U_q\mathfrak{sl}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant elements in $\text{Fun}(M_n \times S(\mathcal{D}))_q$ and endomorphisms of the $U_q\mathfrak{sl}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module $\mathbb{C}[M_n]_q$, explicitly given as follows: the element $K \in \text{Fun}(M_n \times S(\mathcal{D}))_q$ defines the morphism $f \mapsto (1 \otimes \int_{S(\mathcal{D})} f)(K \cdot (1 \otimes f))$. In other words, a linear operator on $\mathbb{C}[M_n]_q$ intertwines the $U_q\mathfrak{sl}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-action if and only if it is a q-integral operator with an invariant kernel.

It is convenient to choose some generators of $\text{Pol}(M_n \times S(\mathcal{D}))_q$ and express all other invariant kernels from $\text{Pol}(M_n \times S(\mathcal{D}))_q$ or $\text{Fun}(M_n \times S(\mathcal{D}))_q$ as (finite or formal) series in those generators.

Consider the elements $\chi_k \in \text{Pol}(M_n \times S(\mathcal{D}))_q$, $k = 1, \ldots, n$, given by

$$\chi_k = \sum z^{\wedge k}_J \cdot q^{Jn} \ast (z^{\wedge k'}_{J''} \cdot q^{J'n})$$

where the sum is taken over the pairs of subsets $J', J'' \subset \{1, 2, \ldots, n\}$ of cardinality $k$. It turns out that the elements $\chi_k$ are pairwise commuting and $U_q\mathfrak{sl}(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant \cite{22} Section 10.\footnote{The quantum group $U_n$ is one of the most well studied objects in quantum group theory. We refer to \cite{16} for basic definitions and facts about this quantum group. Of course, there are many other good references.}
Proposition 4.2 The elements $\chi_1, \ldots, \chi_n$ generate the subalgebra of $U_q(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant kernels in $\text{Pol}(M_n \times S(\mathcal{D}))_q$ (which is therefore a commutative algebra).

Sketch of a proof. Recall (see Proposition [24]) that the $U_q(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module $\mathbb{C}[M_n]_q$ splits into direct sum of simple pairwise non-isomorphic submodules $\mathbb{C}[M_n]_q^k$ with $k = (k_1, k_2, \ldots, k_n)$, $k_1 \geq k_2 \geq \ldots \geq k_n \geq 0$. Thus, any endomorphism of the $U_q(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-module $\mathbb{C}[M_n]_q$ is a (in general infinite) series of the form $\sum_k c_k \cdot \mathbf{P}^k$ where $\mathbf{P}^k$ stands for the projection in $\mathbb{C}[M_n]_q$ onto $\mathbb{C}[M_n]_q^k$ parallel to the sum of other $U_q(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-submodules and $c_k$ are complex numbers. It is sufficient to show that each projection $\mathbf{P}^k$ is a $q$-integral operator whose kernel is a function of $\chi_1, \ldots, \chi_n$. This may be done by using well known orthogonality relations for the quantum group $U_n$ (see [19]) and the precise relation between the quantum Shilov boundary and the quantum $U_n$ described previously.

The projection $\mathbf{P}^k$ can be written as a $q$-integral operator with a kernel $\mathbf{P}^k \in \text{Pol}(M_n \times S(\mathcal{D}))_q$. Namely, let $u^k$ be a polynomial such that $\mathbf{P}^k(z, \zeta^*) = u^k(\chi_1, \chi_2, \ldots, \chi_n)$. Consider the isomorphism

$$\text{Fun}(M_n \times S(\mathcal{D}))^U_q(\mathfrak{gl}_n \times \mathfrak{gl}_n) \rightarrow \mathbb{C}[[x_1, x_2, \ldots, x_n]] \otimes \mathbb{C}[\chi_1, \ldots, \chi_n], \quad \chi_k \mapsto \sigma_k, \quad k = 1, 2, \ldots n$$

from the subalgebra $\text{Fun}(M_n \times S(\mathcal{D}))^U_q(\mathfrak{gl}_n \times \mathfrak{gl}_n)$ of $U_q(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariants in $\text{Fun}(M_n \times S(\mathcal{D}))_q$ to the algebra of symmetric formal series of the variables $x_1, x_2, \ldots, x_n$, where $\sigma_k$ is the $i$-th elementary symmetric polynomial in $x_1, x_2, \ldots, x_n$. The image of $u^k(\chi_1, \chi_2, \ldots, \chi_n)$ under the above isomorphism differs only by a constant from the so-called Schur polynomial $s_k$ associated to the partition $k$ (see [17]), viz.

$$u^k(\sigma_1, \sigma_2, \ldots, \sigma_n) = C(k) \cdot s_k(x_1, x_2, \ldots, x_n).$$

We compute the coefficients $C(k)$ in the next subsection.

From now on, for a kernel $K \in \text{Fun}(M_n \times S(\mathcal{D}))_q$ we shall sometimes write

$$K(z, \zeta^*), \quad \int_{S(\mathcal{D})_q} K(z, \zeta^*) \cdot f(\zeta) d\nu(\zeta),$$

instead of $K$ and $(1 \otimes \int_{S(\mathcal{D})_q})(K \cdot (1 \otimes f))$, respectively.

Now we are ready to present the $q$-analog of the Cauchy-Szegö integral formula found in [29]. In short, it represents the identity operator on $\mathbb{C}[M_n]_q$ in the form of a $q$-integral operator.

Theorem 4.3 [29] Section 5] For any element $f \in \mathbb{C}[M_n]_q$ one has

$$f(z) = \int_{S(\mathcal{D})_q} C_q(z, \zeta^*) f(\zeta) d\nu(\zeta)$$

where $C_q = \prod_{j=0}^{n-1} \left(1 + \sum_{k=1}^{n} (-q^{2j})^k \chi_k \right)^{-1}$ (a $q$-analogue of the Cauchy-Szegö kernel).

4.3 $q$-Analogs of the kernels $\det(1 - z\zeta^*)^{-N}$

Consider the family $K_N$, $N = 1, 2, \ldots$, of $U_q(\mathfrak{gl}_n \times \mathfrak{gl}_n)$-invariant kernels given by

$$K_N = \prod_{j=0}^{N-1} \left(1 + \sum_{k=1}^{n} (-q^{2j})^k \chi_k \right)^{-1}.$$
Clearly, the $q$-Cauchy-Szegő kernel $C_q$ defined in the previous subsection coincides with $K_n$. One also has
\[
\lim_{q \to 1} K_N(z, \zeta^*) = \det(1 - z\zeta^*)^{-N}
\]
(the limit should be understood formally). The aim of this subsection is to study these kernels and the associated $q$-integral operators in details.

Let $K_N$ be the $U_q(su(n) \times su(n))$-intertwining $q$-integral operator corresponding to the kernel $K_N$:
\[
K_N f(z) = \int_{S^2} K_N(z, \zeta^*) \cdot f(\zeta) d\nu(\zeta).
\]
Then
\[
K_N = \sum_k c^k_N \cdot P^k
\]  
(4.10)
where $P^k$ stands for the projection defined in the previous subsection (see the proof of Proposition 4.2). We are interested in an explicit formula for the coefficients $c^k_N$.

One may write (4.10) as an equality of kernels:
\[
K_N(z, \zeta^*) = \sum_k c^k_N \cdot P^k(z, \zeta^*)
\]  
(4.11)
This approach, along with the formula (4.8), allows us to use some identities for the Schur functions [17] to compute $c^k_N$.

Our first step towards computing the coefficients in (4.10) consists in computing the constants $C(k)$ in (4.8). For that purpose, we note that
\[
K_n(z, \zeta^*) = \sum_k P^k(z, \zeta^*)
\]
(this is just another way to formulate Theorem 4.3). In view of the explicit form of the $q$-Cauchy-Szegő kernel (Theorem 4.3) and the isomorphism (4.7), the latter can be written as follows
\[
\prod_{j=0}^{n-1} \left(1 + \sum_{k=1}^{n} (-q^{2j})^k \sigma_k \right)^{-1} = \sum_k u^k(\sigma_1, \sigma_2, \ldots, \sigma_n),
\]
or, by taking into account (4.8)
\[
\prod_{j=0}^{n-1} \left(1 + \sum_{k=1}^{n} (-q^{2j})^k \sigma_k \right)^{-1} = \sum_k C(k) \cdot s_k(x_1, x_2, \ldots, x_n).
\]  
(4.12)
Recall, for any integer $N \geq 0$, the $q$-Pochhammer symbol $(x; q^2)_N = \prod_{j=0}^{N-1} (1 - xq^{2j})$. We have then,
\[
\prod_{j=0}^{N-1} \left(1 + \sum_{k=1}^{n} (-q^{2j})^k \sigma_k \right)^{-1} = \prod_{j=0}^{N-1} \left(\prod_{i=1}^{n} (1 - q^{2j}x_i) \right)^{-1} = \prod_{i=1}^{n} \left(\frac{1}{(x_i; q^2)_N} \right);
\]
in particular for $N = n$ the equality (4.12) reads
\[
\prod_{i=1}^{n} \frac{1}{(x_i; q^2)_n} = \sum_k C(k) \cdot s_k(x_1, x_2, \ldots, x_n).
\]
Now we are in position to make use of the following formula from [17]

\[
\prod_{i=1}^{n} \frac{(ax_i; q^2)_\infty}{(x_i; q^2)_\infty} = \sum_{k} C(k; a) \cdot s_k(x_1, x_2, \ldots, x_n)
\]  

(4.13)

where

\[
C(k; a) = \prod_{i=1}^{n} \frac{(aq^{2-2i}; q^2)_{k_i}}{(q^2; q^2)_{k_i+n-i}} \cdot \prod_{1 \leq i < j \leq n} (1 - q^{2k_i-2k_j-2i+2j}).
\]  

(4.14)

In our case \( a = q^{2n} \) and thus the equality (4.8) acquires the form

\[
u^k(\sigma_1, \sigma_2, \ldots, \sigma_n) = C(k; q^{2n}) \cdot s_k(x_1, x_2, \ldots, x_n),
\]  

(4.15)

where

\[
C(k; q^{2n}) = \prod_{i=1}^{n} \frac{q^{2(i-1)k_i}}{(q^2; q^2_{n-i})} \cdot \prod_{1 \leq i < j \leq n} (1 - q^{2k_i-2k_j-2i+2j}).
\]  

(4.16)

We turn back now to computing the coefficients in (4.10). Identifying \( K_N \) with its image under the isomorphism (4.7), we have, in view of (4.9),

\[
K_N = \prod_{j=0}^{N-1} \left( 1 + \sum_{k=1}^{n} (-q^{2j})^k \sigma_k \right)^{-1} = \prod_{i=1}^{n} \frac{1}{(x_i; q^2)_N}.
\]

(4.17)

By (4.13) and (4.15)

\[
K_N = \sum_{k} C(k; q^{2N}) \cdot s_k(x_1, x_2, \ldots, x_n) = \sum_{k} \frac{C(k; q^{2N})}{C(k; q^{2n})} u^k(\sigma_1, \sigma_2, \ldots, \sigma_n).
\]  

(4.18)

We have thus obtained

**Proposition 4.4** The coefficients in (4.10) are given by

\[
c^k_N = \frac{C(k; q^{2N})}{C(k; q^{2n})} = \prod_{i=1}^{n} \frac{(q^{2N+2-2i}; q^2)_{k_i}}{(q^{2n+2-2i}; q^2)_{k_i}}.
\]

4.4 A \( q \)-analog of the Fock inner product

The aim of this subsection is to describe some results on a \( q \)-analog of the Fock inner product in \( \mathbb{C}[M_n] \) obtained in [20]. At the end of the subsection we shall prove a \( q \)-analog of one known result by J. Faraut and A. Koranyi [5] which compares the Fock inner product with the one in the Hilbert space of square-integrable functions on the Shilov boundary of the matrix ball.

Recall that the Fock inner product in the space \( \mathbb{C}[M_n] \) is defined by

\[
(f_1, f_2)_F = \int_{M_n} f_1(z) \overline{f_2(z)} e^{-\text{tr}(zz^*)} \, dz
\]  

(4.17)

with \( dz \) being the Lebesgue measure on \( M_n \) normalized so that \((1, 1)_F = 1\). The inner product possesses the following remarkable property

\[
\left( \frac{\partial f_1}{\partial z^a}, f_2 \right)_F = (f_1, z^a f_2)_F \quad \forall a, \alpha.
\]  

(4.18)
This property, along with $S(U_n \times U_n)$-invariance of the inner product, is quite useful in explicit computations of various norms.

Below we present a $q$-analog of the Fock inner product. But first we have to explain what is understood by an invariance of an inner product in the $q$-setting.

Let $A_0 = (A,*)$ be a Hopf $*$-algebra. An inner product $(\cdot,\cdot)$ on an $A$-module $V$ is said to be $A_0$-invariant if for all $v_1,v_2 \in V$ and any $\xi \in A$

$$(\xi v_1, v_2) = (v_1, \xi^* v_2).$$

The following is one of the main results of [20].

**Proposition 4.5** There exists a (unique) $U_q\mathfrak{gl}(u_n \times u_n)$-invariant inner product $(\cdot,\cdot)_F$ in $\mathbb{C}[M_n]_q$ satisfying the properties

$$(1,1)_F = 1,$$

$$(\frac{\partial f_1}{\partial z^\alpha}, f_2)_F = (f_1, f_2 \cdot z^\alpha)_F \quad \forall a, \alpha.$$

Let $(\cdot,\cdot)_{S(D)}$ be the inner product in $\mathbb{C}[M_n]_q$ defined via the $U_q\mathfrak{gl}(u_n \times u_n)$-invariant integral on the quantum Shilov boundary

$$(f_1,f_2)_{S(D)} = \int_{S(D)_q} f_2(\zeta)^* f_1(\zeta) d\nu(\zeta).$$

Clearly, the inner product is $U_q\mathfrak{gl}(u_n \times u_n)$-invariant. This is a consequence of (4.5) and the condition (2.6). Since $(\cdot,\cdot)_F$ and $(\cdot,\cdot)_{S(D)}$ are $U_q\mathfrak{gl}(u_n \times u_n)$-invariant, the subspaces $\mathbb{C}[M_n]_q$ are pairwise orthogonal with respect to both inner products, and the corresponding norms are proportional by the Schur lemma. The proportionality constant is computed in the classical setting by J. Faraut and A. Koranyi [5 Corollary 3.5]. Here we present a $q$-analog of their result.

**Proposition 4.6**

$$(f_1,f_2)_F = \prod_{i=1}^n c_{q_2-2i}^{q_2} \cdot \left( f_1, f_2 \right)_{S(D)}, \quad f_1, f_2 \in \mathbb{C}[M_n]_q^k. \quad (4.19)$$

**Proof.** To prove the proposition, we need an explicit description of the inner product $(\cdot,\cdot)_F$.

Consider the algebra $\mathbb{C}[M_n \times M_n]_q = \mathbb{C}[M_n]_q \otimes \mathbb{C}[M_n]_q$. Equip it with the natural bigrading by setting $\text{deg}(f_1 \otimes f_2) = (\text{deg}(f_1), \text{deg}(f_2))$ for any $f_1, f_2 \in \mathbb{C}[M_n]_q$. Its completion with respect to this bigrading is denoted by $\mathbb{C}[[M_n \times M_n]]_q$.

Let

$$\hat{\chi}_k = \sum_{\substack{J, J' \subseteq \{1,2,\ldots,n\} \atop \text{card}(J') = \text{card}(J'') = k}} z^{k_{J'}^{J'\prime}} \otimes z^{k_{J'\prime}^{J'\prime}} \in \mathbb{C}[M_n \times M_n]_q, \quad k = 1, \ldots, n.$$

Note that $\hat{\chi}_k$ are similar to the kernels $\chi_k$ defined in (4.6). Since the latter pairwise commute, the elements $\hat{\chi}_k$ pairwise commute as well. Put

$$\hat{K}_\infty = \prod_{j=0}^{\infty} \left( 1 + \sum_{k=1}^n (-q^2)^k \hat{\chi}_k \right)^{-1} \in \mathbb{C}[[M_n \times M_n]]_q.$$
Let \( \langle \cdot, \cdot \rangle \) be the inner product in \( \mathbb{C}[M_n]_q \) so that \( \hat{K}_\infty \) is the reproducing kernel, namely, writing \( \hat{K}_\infty = \sum_j k'_j \otimes k''_j \) we have then
\[
f = \sum_j k'_j \cdot \langle f, k''_j \rangle
\].

**Lemma 4.7**
\[
\langle \frac{\partial f_1}{\partial z_a}, f_2 \rangle = \frac{1}{1 - q^2} \cdot \langle f_1, f_2 \cdot z_a^\alpha \rangle \quad \forall a, \alpha.
\]

**Sketch of a proof.** The inner product \( \langle \cdot, \cdot \rangle \) is described in a slightly different way in [20, Theorem 6.1]. The equivalence of the two definitions may be deduced from Theorem 9.1 in [22] via the limit \( \lambda \to \infty \).

The above lemma allows us to express the inner product \( \langle \cdot, \cdot \rangle \) via the \( q \)-Fock one
\[
\langle f_1, f_2 \rangle = (1 - q^2)^k \cdot \langle f_1, f_2 \rangle_F
\]
for \( f_1, f_2 \) homogeneous of degree \( k \). Thus, to prove the theorem it suffices to show that
\[
\langle f_1, f_2 \rangle = \prod_{i=1}^n (q^{2n+2i-2}; q^2)_{k_i} \cdot \langle f_1, f_2 \rangle_{S(D)}, \quad f_1, f_2 \in \mathbb{C}[M_n]_q
\]

It follows from the \( q \)-Cauchy-Szegő formula (Theorem 4.3) that the inner product \( \langle \cdot, \cdot \rangle_{S(D)} \) is the one associated to the kernel
\[
\hat{K}_n = \prod_{j=0}^{n-1} \left( 1 + \sum_{k=1}^n (-q^{2j})^k \hat{\chi}_k \right)^{-1}, \quad f_1, f_2 \in \mathbb{C}[[M_n \times M_n]]_q
\]
in the same sense as described above for \( \hat{K}_\infty \). It remains to use the same arguments as in the previous subsection and to compare the coefficients \( C(k; 0) \) and \( C(k; q^{2n}) \) (see (4.14)).

In the last section of the present paper we shall present more general result which is due to Ørsted [18] and J. Faraut and A. Koranyi [5] in the classical setting.

### 4.5 Proof of the covariance property

Now we are in position to prove Theorem 4.1.

In view of Theorem 4.3 \( \Box^l_q \) is the integral operator with the kernel \( \Box^l_q K_n(z, \zeta^*) \) (here and further we assume that the operator \( \Box_q \) acts on a kernel in the first argument \( z \)). We are going to compute the kernel explicitly.

Recall the notation \( P^k(z, \zeta^*) \) for the kernel of the \( q \)-integral operator \( P^k : \mathbb{C}[M_n]_q \to \mathbb{C}[M_n]_q^k \) (see subsection 4.3)
\[
P^k f(z) = \int_{S(D)_q} P^k(z, \zeta^*) f(\zeta) d\nu(\zeta).
\]

By Theorem 4.3
\[
C_q(z, \zeta^*) = \sum_k P^k(z, \zeta^*).
\]
Thus to compute the kernel \( \square_q^l C_q(z, \zeta^*) \) it suffices to compute \( \square_q^l P^k(z, \zeta^*) \). We observe that

\[
\square_q \left( \mathbb{C}[M_n]^k \right) = \begin{cases} \mathbb{C}[M_n]^{k-1}, & k_n \geq 1 \\ \{0\}, & \text{otherwise} \end{cases}.
\] (4.21)

(We use the notation \( k - l = (k_1 - l, k_2 - l, \ldots, k_n - l) \). Indeed, the operator \( \square_q \) is, in particular, a morphism of \( U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n) \)-modules, and all \( U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n) \)-modules, isomorphic to \( \mathbb{C}[M_n]^k \), have the form \( \mathbb{C}[M_n]^{k+m} \) for some (positive or negative) \( m \). The left hand side is then a subspace of the right hand side by (4.11). On the other hand, if \( k_n \geq 1 \) and \( \square_q \left( \mathbb{C}[M_n]^k \right) \nsubseteq \mathbb{C}[M_n]^{k-1} \) then \( \square_q \left( \mathbb{C}[M_n]^k \right) = \{0\} \) since \( \mathbb{C}[M_n]^k \) and \( \mathbb{C}[M_n]^{k-1} \) are simple isomorphic \( U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n) \)-modules. This, however, contradicts positive definiteness of the \( q \)-Fock inner product: we have \( \square_q(\det_q(z) \cdot f) = 0 \) for arbitrary element \( f \in \mathbb{C}[M_n]^{k-1} \) and so

\[
0 = (\square_q(\det_q(z) \cdot f), f)_F = (\det_q(z) \cdot f, \det_q(z) \cdot f)_F.
\]

The equality (4.21), together with (4.3) and (4.4), implies that for certain constant \( c(k, l) \)

\[
\square_q^l P^k(z, \zeta^*) = c(k, l) P^{k-l}(z, \zeta^*) \det_q(\zeta)^{\ast l}.
\]

Indeed, the \( q \)-integral operators with the kernels \( \square_q^l P^k(z, \zeta^*) \) and \( P^{k-l}(z, \zeta^*) \det_q(\zeta)^{\ast l} \) belong to

\[
\text{Hom}_{U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n)} \left( \mathbb{C}[M_n]^k, \mathbb{C}[M_n]^{k-l} \right),
\]

and thus differ by a constant (the latter space is one-dimensional since \( \mathbb{C}[M_n]^k \) and \( \mathbb{C}[M_n]^{k-l} \) are isomorphic irreducible \( U_q(\mathfrak{sl}_n \times \mathfrak{sl}_n) \)-modules). Our immediate aim is to compute \( c(k, l) \).

Given an inner product \( (\cdot, \cdot) \) in \( \mathbb{C}[M_n]_q \) and a kernel \( P(z, \zeta^*) = \sum_j p_j \otimes p_j^* \in \text{Pol}(M_n \times S(\mathcal{D}))_q \) we shall write \( (f(z), P(z, \zeta^*)) \) instead of \( \sum_j (f(z), p_j \otimes p_j^*) \).

Recall the notation \( (\cdot, \cdot)_{S(\mathcal{D})} \) for the inner product defined in subsection 4.4 via the invariant integral on the \( q \)-Shilov boundary. It is not difficult to observe that the reproducing property of the kernel \( P^k(z, \zeta^*) \) is equivalent to

\[
(f(z), P^k(z, \zeta^*))_{S(\mathcal{D})} = f, \quad \forall f \in \mathbb{C}[M_n]^k.
\]

We have

\[
\square_q^l P^k(z, \zeta^*) = c(k, l) P^{k-l}(z, \zeta^*) \det_q(\zeta)^{\ast l}
\]

or

\[
(f(z), \square_q^l P^k(z, \zeta^*))_{S(\mathcal{D})} = c(k, l) \cdot (f(z), P^{k-l}(z, \zeta^*) \det_q(\zeta)^{\ast l})_{S(\mathcal{D})},
\]

for any \( f \in \mathbb{C}[M_n]^{k-l} \). By Theorem 4.6

\[
\left( \frac{1 - q^2}{1 - q^2} \right)^{k_1 + k_2 + \ldots + k_n - 2n} \left( f(z), \square_q^l P^k(z, \zeta^*) \right)_F = c(k, l) \cdot (f(z), P^{k-l}(z, \zeta^*) \det_q(\zeta)^{\ast l})_F,
\]

and, due to the main property of the \( q \)-Fock product,

\[
\left( \frac{1 - q^2}{1 - q^2} \right)^{k_1 + k_2 + \ldots + k_n - 2n} \left( f(z) \det_q(z)^l, P^k(z, \zeta^*) \right)_F = c(k, l) \cdot (f(z), P^{k-l}(z, \zeta^*) \det_q(\zeta)^{\ast l})_F.
\]

Apply Theorem 4.6 once again:

\[
\left( \frac{1 - q^2}{1 - q^2} \right)^{k_1 + k_2 + \ldots + k_n - 2n} \left( f(z) \det_q(z)^l, P^k(z, \zeta^*) \right)_{S(\mathcal{D})} = c(k, l) \cdot (f(z), \det_q(\zeta)^l).
\]
The reproducing property of the kernel $P^k(z, \xi^*)$ implies that
\[
\frac{\prod_{i=1}^n (q^{2n+2-2i}; q^2)_{k_i}}{(1 - q^2)^{2kn}} \cdot f(\xi) \cdot \det_q(\xi)^l = c(k, l) \cdot f(\xi) \cdot \det_q(\xi)^l,
\]
consequently,
\[
c(k, l) = \frac{\prod_{i=1}^n (q^{2n+2-2i}; q^2)_{k_i}}{(1 - q^2)^{2kn}} \cdot \det_q(\xi)^l,
\]
We then get
\[
\square_q^l K_n(z, \xi^*) = \sum_{k: k_i \geq l} \frac{\prod_{i=1}^n (q^{2n+2-2i}; q^2)_{k_i}}{(1 - q^2)^{2kn}} \cdot P^{k-l}(z, \xi^*) \det_q(\xi)^l = \sum_{k} \frac{\prod_{i=1}^n (q^{2n+2-2i}; q^2)_{k_i+l}}{(1 - q^2)^{2kn}} \cdot P^k(z, \xi^*) \det_q(\xi)^l.
\]
Finally, Proposition 4.4 implies
\[
\square_q^l K_n(z, \xi^*) = \frac{\prod_{i=1}^n (q^{2n+2-2i}; q^2)_{l}}{(1 - q^2)^{2kn}} \cdot K_{n+l}(z, \xi^*) \cdot \det_q(\xi)^l.
\]
We have thus obtained

**Proposition 4.8** For any element $f \in \mathbb{C}[M_n]$ one has
\[
\square_q^l f(z) = \frac{\prod_{i=1}^n (q^{2n+2-2i}; q^2)_{l}}{(1 - q^2)^{2kn}} \cdot \int_{S(D)_{q}} K_{n+l}(z, \xi^*) \det_q(\xi)^* f(\xi) d\nu(\xi).
\]

Proposition 4.8 reduces the statement of Theorem 4.1 to the following proposition.

**Proposition 4.9** The integral operator
\[
f(z) \mapsto \int_{S(D)_{q}} K_{n+l}(z, \xi^*) \det_q(\xi)^* f(\xi) d\nu(\xi)
\]
intertwines the $U_q\mathfrak{sl}_{2n}$-actions $\pi_{n-l}$ and $\pi_{n+l}$.

**Proof.** We shall use a quantum version of the description (2.16) of the twisted action $\pi_\lambda$.

Let us extend the algebra $\mathbb{C}[M_n]_q$ by adding one more generator $t$ (an analog of the distinguished Plücker coordinate $t$ in (2.16)) such that
\[
t z_a^\alpha = q^{-1} z_a^\alpha t, \quad a, \alpha = 1, 2, \ldots, n.
\]
The localization of the resulting algebra with respect to the multiplicative system $t^{\mathbb{N}}$ will be denoted by $\mathbb{C}[M_n]_{q,t}$. It was noted in [21] that there exists a unique extension of the $U_q\mathfrak{sl}_{2n}$-module algebra structure in $\mathbb{C}[M_n]_q$ to the one in $\mathbb{C}[M_n]_{q,t}$ such that
\[
E_j t = F_j t = (K_j^{\pm 1} - 1)t = 0 \quad (j \neq n), \quad F_n t = (K_n^{\pm 1} - q^{-1})t = 0, \quad E_n t = q^{-1/2} t z_n^a. \quad (4.22)
\]
It is clear that the subspace $\mathbb{C}[M_n]_q \cdot t^{-\lambda} \subset \mathbb{C}[M_n]_{q,t}$ is $U_q\mathfrak{sl}_{2n}$-invariant for any $\lambda \in \mathbb{Z}$. The following is an equivalent definition of the $U_q\mathfrak{sl}_{2n}$-action $\pi_\lambda$:
\[
(\pi_\lambda(\xi) f) \cdot t^{-\lambda} = \xi (f \cdot t^{-\lambda}), \quad \xi \in U_q\mathfrak{sl}_{2n}, \quad f \in \mathbb{C}[M_n]_q.
\]
In other words, the linear map
\[ \mathbb{C}[M_n]_q \rightarrow \mathbb{C}[M_n]_q \cdot t^{-\lambda}, \quad f \mapsto f \cdot t^{-\lambda} \]
intertwines the \( U_q\mathfrak{sl}_{2n} \)-action \( \pi_\lambda \) and the natural \( U_q\mathfrak{sl}_{2n} \)-action in \( \mathbb{C}[M_n]_q \cdot t^{-\lambda} \).

We need also certain extension of the algebra \( \text{Pol}(S(D))_q \). Let us add to \( \text{Pol}(S(D))_q \) two generators \( t, t^* \) such that
\[ tt^* = t^* t, \quad t z_\alpha^\alpha = q^{-1} z_\alpha^\alpha t, \quad t^* z_\alpha^\alpha = q^{-1} z_\alpha^\alpha t^*, \quad a, \alpha = 1, 2, \ldots, n. \]

Denote this new algebra by \( \text{Pol}(\hat{S}(D))_q \) and its localization with respect to the multiplicative system \( (tt^*)^N \) by \( \text{Pol}(\hat{S}(D))_{q,x} \). The involution in \( \text{Pol}(S(D))_q \) can be extended to an involution in \( \text{Pol}(\hat{S}(D))_{q,x} \) by setting \( * : t \mapsto t^* \). It is proved in [29, Section 2] that there exists a unique structure of \( U_q\mathfrak{su}_{n,n} \)-module algebra in \( \text{Pol}(\hat{S}(D))_{q,x} \) which coincides with the original one on \( \text{Pol}(S(D))_q \subset \text{Pol}(\hat{S}(D))_{q,x} \) and satisfies (4.22). Following [29, Section 3], we equip \( \text{Pol}(\hat{S}(D))_{q,x} \) with a bigrading:
\[ \deg t = (0, 1), \quad \deg t^* = (1, 0), \quad \deg(z_\alpha^\alpha) = \deg(z_\alpha^\alpha)^* = (0, 0), \quad a, \alpha = 1, 2, \ldots, n. \]

Obviously, the homogeneous components
\[ \text{Pol}(\hat{S}(D))_{q,x}^{(i,j)} = \{ f \in \text{Pol}(\hat{S}(D))_{q,x} | \deg f = (i, j) \} = t^i \cdot \text{Pol}(S(D))_q \cdot t^j \]
are submodules of the \( U_q\mathfrak{sl}_{2n} \)-module \( \text{Pol}(\hat{S}(D))_{q,x} \).

Proposition [4.3] is an immediate consequence of the following statement

**Lemma 4.10** The linear operator from \( \text{Pol}(\hat{S}(D))_{q,x}^{(0,l-n)} \) to \( \mathbb{C}[M_n]_q \cdot t^{-l-n} \) given by
\[ f \cdot t^{-l-n} \mapsto \left( \int_{\hat{S}(D)_q} K_{n+l}(z, \xi^*) \text{det}_q(\xi)^* f(\xi) d\nu(\xi) \right) \cdot t^{-l-n} \]
is a morphism of \( U_q\mathfrak{sl}_{2n} \)-modules.

**Proof of the lemma.** The proof may be easily reduced to the following three statements:

(i) The linear map
\[ \text{Pol}(\hat{S}(D))_{q,x}^{(0,l-n)} \rightarrow \text{Pol}(\hat{S}(D))_{q,x}^{(l,n)}, \quad f(\xi) \cdot t^{-l-n} \mapsto \text{det}_q(\xi)^* f(\xi) \cdot t^l t^{-n} \quad (4.23) \]
is a morphism of \( U_q\mathfrak{sl}_{2n} \)-modules. This statement follows from the results of Sections 2.3 in [29].

(ii) Let \( K_{n+l}(z, \xi^*) = \sum_j k_j'(z) \otimes k_j''(\xi^*) \). Then the element
\[ \sum_j k_j'(z) \cdot t^{-l-n} \otimes t^s(-l-n) \cdot k_j''(\xi^*) \in (\mathbb{C}[M_n]_q \cdot t^{-l-n}) \otimes \text{Pol}(\hat{S}(D))_{q,x}^{(-l-n,0)} \]
is a \( U_q\mathfrak{sl}_{2n} \)-invariant (here the symbol \( \otimes \) has the same meaning as the one in the equality \( \mathbb{C}[M_n]_q \otimes \text{Pol}(S(D))_q \rightarrow \text{Fun}(M_n \times S(D)_q) \)). The statement is a consequence of results of Section 8 in [24]. This, together with statement (i), implies that the map
\[ \text{Pol}(\hat{S}(D))_{q,x}^{(0,l-n)} \rightarrow (\mathbb{C}[M_n]_q \cdot t^{-l-n}) \otimes \text{Pol}(\hat{S}(D))_{q,x}^{(-n,-n)}, \quad f(\xi) \cdot t^{-l-n} \mapsto \sum_j k_j'(z) \cdot t^{-l-n} \otimes t^s(-l-n) \cdot k_j''(\xi^*) \text{det}_q(\xi)^* f(\xi) \cdot t^l t^{-n} \quad (4.24) \]
is a morphism of $U_q\mathfrak{sl}_{2n}$-modules.

(iii) The linear functional

$$\text{Pol}(\tilde{S}(\mathcal{D}))_{q,x}^{(-n,-n)} \to \mathbb{C}, \quad t^s(-n) \cdot f \cdot t^{-n} \mapsto \int_{S(\mathcal{D})_q} f(\zeta) d\nu(\zeta)$$

is a $U_q\mathfrak{sl}_{2n}$-invariant integral. This is proved in Section 3 of [29]. As a consequence of this statement and statements (i), (ii) we get: the linear operator from $\text{Pol}(\tilde{S}(\mathcal{D}))_{q,x}^{(0,l-n)}$ to $\mathbb{C}[M_n]_q \cdot t^{-l-n}$ given by

$$f(\zeta) \cdot t^{l-n} \mapsto \sum_j k^j(\zeta) \cdot t^{-l-n} \cdot \int_{S(\mathcal{D})_q} \Theta_l \left( k^j(\zeta^*) \text{det}_q(\zeta)^s f(\zeta) \right) d\nu(\zeta) \tag{4.25}$$

is a morphism of $U_q\mathfrak{sl}_{2n}$-modules (here $\Theta_l$ means the automorphism of the algebra $\text{Pol}(S(\mathcal{D}))_q$ given by $f \mapsto t^s(-l) \cdot f \cdot t^s(l)$).

Lemma 4.10 follows from the latter statement and the equality

$$\int_{S(\mathcal{D})_q} \Theta_l(f(\zeta)) d\nu(\zeta) = \int_{S(\mathcal{D})_q} f(\zeta) d\nu(\zeta), \forall f$$

which is due to the simple observation that the functional $\int_{S(\mathcal{D})_q} f(\zeta) d\nu(\zeta)$ 'picks up' the constant term of $f$ and the constant terms of $\Theta_l(f)$ and $f$ are the same.

Proposition 4.9 consequently Theorem 4.1 is now proved.

5 Holomorphic discrete series for $U_q\mathfrak{su}_{n,n}$

In this last section we study the holomorphic discrete series representations for $U_q\mathfrak{su}_{n,n}$ and study their analytic continuation.

After giving a definition of the holomorphic discrete series for $U_q\mathfrak{su}_{n,n}$, we prove an analog of a classical known result by J. Faraut and A. Koranyi which allows one to express the inner product in a module of the holomorphic discrete series via the Fock inner product. We use the result to prove unitarizability of the modules $\mathcal{P}_\lambda$ with $\lambda > n - 1$; the discrete series parameters are $\lambda > 2n - 1$. We apply then the covariance property, proved earlier, to studying certain quotients of the modules $\mathcal{P}_\lambda$.

5.1 Definition of the holomorphic discrete series

We start by recalling the definition of the holomorphic discrete series for $\widetilde{SU}_{n,n}$. Fix $\lambda > 2n - 1$ and consider the Hilbert space of holomorphic functions on the unit matrix ball $\mathcal{D}$ which are square integrable with respect to the measure $\text{det}(1 - zz^*)^{\lambda - 2n} dz$ (here $dz$ is the normalized Lebesgue measure: $\int_{\mathcal{D}} dz = 1$). It is known [1] that the operators (2.17) are unitary on that Hilbert space. The corresponding representation of $\widetilde{SU}_{n,n}$ is said to be a representation of the holomorphic discrete series.

Now let us turn to the quantum setting. Suppose $A_0 = (A, \ast)$ is a Hopf *-algebra. An $A$-module $V$ is said to be a unitarizable $A_0$-module if there exists an inner product $(,)$ on $V$ such that for all $v_1, v_2 \in V$ and any $\xi \in A$

$$(\xi v_1, v_2) = (v_1, \xi^* v_2)$$
(that is, $V$ possesses an $A_0$-invariant inner product, see subsection 4.4).

Unitarizable $U_q \mathfrak{su}_{n,n}$-modules substitute unitary representations of $SU_{n,n}$ (or $\widetilde{SU}_{n,n}$) in the quantum setting. The following statement was proved in \cite{22}, Corollary 6.5.

**Proposition 5.1** For $\lambda > 2n - 1$ there exists a unique inner product $( , )_{\lambda}$ in $\mathcal{P}_\lambda$ such that for all $f_1, f_2 \in \mathcal{P}_\lambda$ and $\xi \in U_q \mathfrak{sl}_{2n}$

$$(\pi_\lambda(\xi)f_1, f_2)_{\lambda} = (f_1, \pi_\lambda(\xi^*)f_2)_{\lambda},$$

with the normalization $(1, 1)_{\lambda} = 1$.

Clearly, the unitarizable $U_q \mathfrak{su}_{n,n}$-modules $\mathcal{P}_\lambda$, $\lambda > 2n - 1$, are $q$-analogs of unitary representation of the holomorphic discrete series for $SU_{n,n}$.

The inner product $( , )_{\lambda}$ may be described explicitly as follows. Let us use the notation from the proof of Proposition 4.6. Consider the element

$$\hat{K}_\lambda = \frac{\prod_{j=0}^{\infty} (1 + \sum_{k=1}^{m} (-q^{2(\lambda+j)}k^2 \chi_k))}{\prod_{j=0}^{\infty} (1 + \sum_{k=1}^{m} (-q^{2j}k^2 \chi_k))} \in \mathbb{C}[[M_n \times M_n]]_q. \quad (5.1)$$

Then the inner product $( , )_{\lambda}$ is the one associated with the above element, i.e.

$$f = \sum_j k'_j \cdot (f, k''_j)_{\lambda} \quad (5.2)$$

provided $\hat{K}_\lambda = \sum_j k'_j \otimes k''_j$ (see Theorem 9.1 in \cite{22}).

### 5.2 A $q$-analog of a result by J. Faraut and A. Koranyi

In this subsection, we present a $q$-analog of Corollary 3.7 in \cite{5} where (in the classical setting) the Fock inner product and the inner products $( , )_{\lambda}$ are compared. We then apply the result to the problem of analytic continuation of the holomorphic discrete series for $U_q \mathfrak{su}_{n,n}$.

Using the arguments preceding Proposition 4.6 we deduce that the inner product $( , )_{\lambda}$ and the $q$-Fock inner product on a particular simple $U_q \mathfrak{sl}_n$-submodule $\mathbb{C}[M_n]_q^k \subset \mathbb{C}[M_n]_q$ are proportional. The proportionality constant is given by the following formula.

**Proposition 5.2** Let $\lambda > 2n - 1$. Then

$$(f_1, f_2)_F = \frac{\prod_{i=1}^{n} (q^{2\lambda+2-2i}; q^2)_k^i}{(1 - q^2)^{k_1 + k_2 + \ldots + k_n}} \cdot (f_1, f_2)_{\lambda}, \quad f_1, f_2 \in \mathbb{C}[M_n]_q^k. \quad (5.3)$$

**Proof.** Consider the reproducing kernel

$$K_\lambda = \frac{\prod_{j=0}^{\infty} (1 + \sum_{k=1}^{m} (-q^{2(\lambda+j)}k^2 \chi_k))}{\prod_{j=0}^{\infty} (1 + \sum_{k=1}^{m} (-q^{2j}k^2 \chi_k))}$$

associated to the element (5.1). The image of this kernel under the isomorphism (4.7) is given by $\prod_{i=1}^{n} (q^{2\lambda x_i}; q^2)_k^i$. By repeating the computation from subsection 4.3 one gets

$$(f_1, f_2)_{S(\mathfrak{g})} = \prod_{i=1}^{n} (q^{2\lambda x_i+2-2i}; q^2)_k^i \cdot (f_1, f_2)_{\lambda}, \quad f_1, f_2 \in \mathbb{C}[M_n]_q^k.$$
What remains is to apply Proposition 4.6.

The result of the above proposition has an important application to the problem of analytic continuation of the holomorphic discrete series. The point is that (5.3) allows one to define the sesquilinear form \((\cdot , \cdot)_\lambda\) on \(\mathbb{C}[M_n]_q\) for any \(\lambda \in \mathbb{R}\) for which all the multipliers \(\prod_{i=1}^k(q^{2\lambda+2-2i};q^2)_k\) are non-zero. It is not difficult to prove that the resulting form is still \(U_q\mathfrak{su}_{n,n}\)-invariant with respect to the corresponding twisted action. Indeed, the invariance is equivalent to the infinitely many equalities of the form

\[
(\pi_\lambda(\xi)f_1, f_2)_\lambda = (f_1, \pi_\lambda(\xi^*)f_2)_\lambda, \quad \xi \in U_q\mathfrak{sl}_2n, \; f_1, f_2 \in \mathbb{C}[M_n]_q.
\]

After simple transformations, each equality becomes an equality of two Laurent polynomials in \(q^\lambda\) which is known to hold for \(\lambda > 2n - 1\) by Proposition 5.1 and, thus, for any \(\lambda\). It is naturally to pose the problem of finding those \(\lambda\)'s for which the corresponding sesquilinear form is positive definite, i.e. the corresponding \(U_q\mathfrak{su}_{n,n}\)-modules are unitarizable. In the classical setting, such \(\lambda\)'s are said to belong to the continuous part of the Wallach set [1].

The proposition implies positive definiteness of the inner product \((f_1, f_2)_\lambda\) for any \(\lambda > n - 1\):

**Corollary 5.3** The \(U_q\mathfrak{su}_{n,n}\)-modules \(P_\lambda\) are unitarizable for \(\lambda > n - 1\).

For \(n = 2\), this statement was obtained in [23] (see Proposition 6.1).

### 5.3 Some consequences of the covariance property

In the previous subsection, we were able to deduce some irreducibility and unitarity property of \(U_q\mathfrak{sl}_2n\)-modules \(P_\lambda\) from the results obtained earlier. We will not pursue all the details here. It is immediate that \(P_\lambda\) is reducible for \(\lambda = n - 1, n - 2, \ldots\) for the following obvious reason: by the covariance property, \(P_n^{(0)} = \text{Ker} \Box^1_q\) is a submodule in \(P_{n-l}\). A related application of the covariance property is the following

**Proposition 5.4** For any \(l \in \mathbb{N}\) \(P_{n-l}/P_{n-l}^{(0)}\) is a unitarizable \(U_q\mathfrak{su}_{n,n}\)-module isomorphic to \(P_{n+l}\).

**Proof.** Unitarizability follows from the covariance property, Corollary 5.3 and injectivity of the induced morphism \(P_{n-l}/P_{n-l}^{(0)} \to P_{n+l}\) of \(U_q\mathfrak{sl}_2n\)-modules. Actually, this latter morphism is an isomorphism. To prove this, it suffices to show that the operator

\[
\Box^l_q : \mathbb{C}[M_n]_q \to \mathbb{C}[M_n]_q
\]

is surjective. In turn, it suffices to prove the latter statement for \(l = 1\), i.e. to show that \(\Box_q\) is surjective. But this follows from [1,21].

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