The equation of state of nuclear matter in the hadronic phase is of central interest in the study of supernovae and neutron star matter in astrophysical context and of heavy ion collisions in the laboratory. There are microscopic frameworks to deal with the nuclear strong interaction involved here, such as the variational [1, 2] and the Brueckner-Goldstone [3, 4, 5] methods, using NN potentials based on experimental scattering data. There are also empirical expressions [6, 7] for the Helmholtz free energy using a model for nuclei. However, these approaches are model-dependent, involving some theoretical uncertainties.

A more direct and simple approach based completely on observables was suggested long ago by Beth and Uhlenbeck [8, 9], who expressed the second coefficient in the virial expansion for pressure of a gas in terms of two-body scattering phase-shifts of its constituents. But it does not shed much light on the virial series as a whole, though in a highly non-trivial way, it gives rise to additional ideal gas terms from these particles to that order. As expected, the large binding energy of the alpha particle enhances its formation in the system lowering the pressure.

In the present work we consider the complete expression for the grand partition function [10], as applied to nuclear matter, including, in particular, all the massive species and the scattering channels formed by them. The only major approximation is that of resonance domination of S-matrix elements for scattering channels with massive particles. As shown by Dashen and Rajaraman [12, 13], it gives rise to additional ideal gas terms from these resonances.

We consider nuclear matter, consisting of protons and neutrons as two independent species of 'elementary' particles, interacting strongly in the limit of isospin symmetry. The object of study is the grand canonical partition function for this system,

$$Z = \text{Tr} e^{-\beta (H - \mu_p N_p - \mu_n N_n)} ,$$

where \( H \) is the total Hamiltonian, \( \hat{N}_{p,n} \) the nucleon num-
hier operators, $\beta$ the inverse of temperature $T$ of heat bath and $\mu_{p,n}$ the nucleon chemical potentials. The trace is taken over any complete set of states of all possible numbers of nucleons. Denoting the fugacities by $\zeta_p = e^{\beta \mu_p}$, $\zeta_n = e^{\beta \mu_n}$, the full trace can be decomposed as

$$Z = \sum_{Z,N=0}^{\infty} \zeta_p^Z \zeta_n^N \mathrm{Tr}_{Z,N} e^{-\beta H},$$

where $\mathrm{Tr}_{Z,N}$ is now taken over states of $Z$ protons and $N$ neutrons. For small $\zeta_p$ and $\zeta_n$, $\ln Z$ can be expanded in a virial double series,

$$\ln Z = \sum_{Z,N=1}^{\infty} A_{Z,N} \zeta_p^Z \zeta_n^N.$$  

Our task is to calculate the virial coefficients $A_{Z,N}$.

We work in natural units, $\hbar = c = 1$. For later reference, we note here the virial expansion for the ideal quantum gas of bosons and of fermions,

$$\ln Z^{(0)} = \mp g V \int \frac{d^3 p}{(2\pi)^3} \ln \left( 1 \mp \zeta e^{-\beta p^2/2m} \right) = g V A(m) \left( \zeta \pm \frac{\zeta^2}{2\sqrt{2}} + \frac{\zeta^3}{3\sqrt{3}/2} + \cdots \right).$$

Here in the superscript $(0)$ will indicate ideal quantum gas. The upper and lower signs correspond respectively to bosons and fermions. The degeneracy of the single particle states is denoted by $g A(m)$. $V$ is the volume of the system and $\lambda(m) = \sqrt{2\pi/(mT)}$, the thermal wavelength of a particle of mass $m$. The integral representation above is valid for all values of $\zeta$, but its logarithmic branch point at $\zeta = \pm 1$ makes the series converge only for $|\zeta| < 1$.

Though Coulomb interaction is assumed absent in nuclear matter, we shall include its effect on the binding energies of nuclei needed below. Our calculation can then be readily applied to a physical system, such as neutron star matter, by including the effect of added electrons necessary to make it electrically neutral.

We now follow Dashen et al.\cite{10} to write the grand partition function as the sum of two types of terms,

$$\ln Z = \ln Z^{(0)}_{\text{part}} + \ln Z_{\text{scat}},$$

corresponding to contributions from stable, single particle states and (multiparticle) scattering states respectively.

Let us first concentrate on the particle piece, $\ln Z^{(0)}_{\text{part}}$. If $Z$ protons and $N$ neutrons form a bound state (nucleus) of mass number $A = Z + N$, it has mass $Am$ and energy,

$$\epsilon_{Z,N} = \frac{p^2}{2Am} - B_{Z,N},$$

in the ground state, where $\vec{p}$ is its momentum and $B_{Z,N}$ the binding energy. From the condition of chemical equilibrium among different species, its chemical potential is $\mu_{Z,N} = Z \mu_p + N \mu_n$. Further, all these nuclei, particularly the heavy ones, have a large number of dense excited states above their ground states, which are stable in the absence of electromagnetic interaction. (Actually their radiation widths are very small, of the order of eV.) Accordingly we split the particle piece in Eq.(5) into two,

$$\ln Z^{(0)}_{\text{part}} = \ln Z^{(0)}_{gr} + \ln Z^{(0)}_{ex},$$

denoting respectively the contribution of the ground and the excited states.

The first term in Eq.(7) is a sum of ideal gas terms, one for each of the ground states of all the nuclei,

$$\ln Z^{(0)}_{gr} = \mp V \sum_{Z,N} g \int \frac{d^3 p}{(2\pi)^3} \left( 1 \mp e^{-\beta(p^2/2Am - B_{Z,N})/\mu_{Z,N}} \right)$$

where $Z$ and $N$ count the number of protons and neutrons in these species. As in Eq.(4) the $\mp$ sign correspond to nuclei with $A$ even and odd, obeying Bose and Fermi statistics respectively. The sum, of course, includes the original elementary particles, namely the proton and the neutron with their $B_{Z,N} = 0$.

The second term in Eq.(7) gives the ideal gas terms for each of the elements in the set of excited states of the nuclei. Here the individual gas terms are actually suppressed by an additional Boltzmann factor, $e^{-\beta E}$, $E$ being the excitation energy above the ground state. But the density of excited states is rather high in heavy nuclei with $A > 8$, say and can, in fact, more than compensate for this suppression. We write the contribution of the excited states of a single nucleus as an integral over $E$ of an ideal gas term multiplied by their level density $\omega(A,E)$, which we take as\cite{15,16},

$$\omega(A,E) = \frac{\sqrt{\pi}}{12a^{1/4}} \frac{e^{2\sqrt{\pi}E}}{E^{5/4}},$$

obtained from the Fermi gas model of non-interacting nucleons within the nucleus. We adopt the empirical value for the parameter, $a \equiv A/8 (\text{MeV})^{-1}$. We thus get

$$\ln Z^{(0)}_{ex} = \mp V \sum_{Z,N} \frac{a}{E_0} \int_{E_0}^{E_s} dE \omega(A,E) \times \int \frac{d^3 p}{(2\pi)^3} \ln \left( 1 \mp e^{-\beta(p^2/2Am + E - B_{Z,N} - \mu_{Z,N})} \right),$$

where the prime on $\sum$ denotes exclusion of the light nuclei ($A \leq 8$) from the sum. The lower limit $E_0$ is determined by the beginning of the excited states and the applicability of the level density formula at low energy. The upper limit $E_s$ is the smallest of separation energies of any particle within the nucleus, beyond which lies the continuum. We take $E_0 = 2$ MeV and $E_s = 8$ MeV.
We next consider the scattering piece, $\ln Z_{scat}$, which is formally given by [10],

$$
\ln Z_{scat} = \sum \int dEe^{-\beta(E-\mu)} \frac{1}{2\pi i} Tr \left( AS^{-1}(e) \frac{\partial}{\partial E} S(e) \right)
$$

(11)

where the sum is over all scattering channels, each having its chemical potential $\mu$ and formed by taking any number of particles from any of the stable species (proton, neutron and nuclei in their ground and excited states) and the trace is over all plane wave states for each of these channels. $S$ is the usual scattering operator and $\mathcal{A}$, the boson symmetrization and fermion antisymmetrization operator.

The subscript $c$ denotes only the connected parts of the expression in parenthesis. To restate Eq.(11) explicitly for nuclear matter, we have to characterize the scattering channels.

Consider the set of channels, in which the constituting particles have a total number of $Z_t$ protons and $N_t$ neutrons with mass number $A_t = Z_t + N_t$. Let $\sigma$ denote all other labels required to fix a channel within this set. Clearly the total mass and the chemical potential is independent of $\sigma$, depending only on $Z_t$ and $N_t$. If $\tilde{P}$ is the total momentum of all the particles in a channel, its non-relativistic energy is given by

$$
E_{Z_t,N_t,\sigma} = \frac{\tilde{p}^2}{2A_t m} - B_{Z_t,N_t,\sigma} + \epsilon,
$$

(12)

where $\epsilon$ is the kinetic energy in the c.m. frame. Here $B_{Z_t,N_t,\sigma}$ is the sum of individual binding energies of all the particles in the channel. With the channels so characterized, one can integrate over $\tilde{P}$ to get

$$
\ln Z_{scat} = V \sum_{Z_t,N_t} e^{\beta B_{Z_t,N_t}} \sum_{\sigma} e^{\beta B_{Z_t,N_t,\sigma}} \times

\int_0^{\infty} d\epsilon \ e^{-\beta\epsilon} \frac{1}{2\pi i} Tr_{Z_t,N_t,\sigma} \left( AS^{-1}(e) \frac{\partial}{\partial E} S(e) \right)
$$

(13)

the trace being now restricted to the channel $(Z_t, N_t, \sigma)$.

Eq.(13) brings out the nature of convergence of the virial series. Here the integral depends mildly on temperature. Then the factor $e^{\beta B_{Z_t}}$ makes channels with large binding energies more important, determining the leading behaviour of the series. For an estimate of the convergence domain, we take $B_{Z_t,N_t,\sigma} \approx B(Z_t + N_t)$ for such channels with $b \sim 8.5$ MeV, the binding energy per nucleon of moderately large nuclei. We thus see that Eq.(13) may be regarded as a series in powers of $e^{\beta(|p|)} + b$ and $e^{\beta(|p|+b)}$, rather than simply of $\zeta_p$ and $\zeta_b$, as appears from the original definition (3). If we ignore the excited states of the nuclei, the series converges for $|\zeta_p|, |\zeta_b| < e^{-\beta b}$, which, as expected, is smaller than that for the ideal gas. Below we shall find a more realistic criterion of convergence.

Having discussed the convergence of the virial series, we now try to calculate $\ln Z_{scat}$ from all the scattering channels. It is convenient to divide the channels into light ones, consisting of low mass particles $(A < 8$, say) and heavy ones, containing at least one high mass particle $(A \geq 8)$, so that we write

$$
\ln Z_{scat} = \ln Z_{light} + \ln Z_{heavy},
$$

(14)

as the sum of contributions from the light and the heavy channels. The integrals in Eq.(13) over the $S$-matrix elements appear difficult to evaluate at first sight. But the presence of the Boltzmann factor simplifies the task by limiting the range of integration to rather low energies, if the temperature is small enough.

Generally speaking, two-particle channels are expected to dominate over multiparticle channels with same $Z_t$ and $N_t$ from the binding energy consideration. As in Ref. [11] we neglect all light channels, except the three two-particle channels, namely, $NN$, $N\alpha$ and $\alpha\alpha$ to get

$$
\ln Z_{light} = \ln Z_{NN} + \ln Z_{N\alpha} + \ln Z_{\alpha\alpha}
$$

(15)

We shall write below the individual contributions in terms of phase shifts in the respective channels.

The restriction of the integrals in Eq.(13) to the low energy region is all the more helpful for estimating the scattering contribution from the heavy channels. The cross-sections in these channels are known experimentally to be dominated by a multitude of narrow resonances near their thresholds. The $S$-matrix elements may thus be well approximated by these resonances. We then have the elegant result by Dashen et al [12, 13]: The corresponding partition function becomes that of a number of ideal gases, one for each of these resonances. Assuming their level density to be of the same form as for the excited states [15], we thus see that $\ln Z_{heavy}$ is again given by the same formula (10) as for $\ln Z_{ex}(0)$, with the $E$-integral now running from $E_0$ to $E_r$, the end of the resonance region. We take $E_r \simeq 12$ MeV. We also terminate the series over $Z_t$ and $N_t$ at the same values that we choose to do for $Z$ and $N$ in Eqs.(8) and (10). Then the two contributions can be combined together by extending the variable $E$ from $E_0$ to $E_r$:

$$
\ln Z_{ex}(0) + \ln Z_{heavy} = \text{r.h.s. of Eq.(10) with } E_s \rightarrow E_r.
$$

(16)

Let us summarize our result at this stage for the grand partition function of nuclear matter as,

$$
\ln Z = \ln Z_{ex} + (\ln Z_{forc} + \ln Z_{heavy}) + \ln Z_{light}
$$

(17)

with the terms given by Eqs.(8), (16) and (15) respectively. It is observed that the first two terms represent simply the ideal gas terms.

We now write explicitly the contributions from the three light, two-body channels retained in Eq.(15) for $\ln Z_{light}$. These may be recovered from Ref. [11] (see [17], however), but the master formula (13) gives them immediately. As we are considering elastic two-body scattering, the trace in Eq.(13) becomes a sum over the derivative of phase-shifts of the appropriate partial waves. It
gives formulae of the same form as derived by Beth and Uhlenbeck [8] for the second virial coefficient for particles without spin and isospin. The results for each of these channels can be expressed in terms of an integral of the form,

$$\Delta_{AB}(\beta) = \frac{1}{\pi T} \int_0^\infty de^\beta e \delta_{AB}(e),$$  

(18)

where A and B are the scattering particles and \(\delta_{AB}\) is the sum over phase-shifts of the relevant partial waves. We shall also indicate the isospin with \(I\) for the NN channel by a superscript on \(\Delta\) and \(\delta_{AB}\).

For the NN channel,

$$\ln Z_{NN} = \frac{V}{\lambda^3(2m)} \left\{ \left(\zeta_p^2 + \zeta_n^2\right) \Delta_{I=1}^{NN} - \zeta_n \left(-3 + \Delta_{I=1}^{NN} + \Delta_{I=0}^{NN}\right) \right\},$$  

(19)

with

$$\delta_{I=1}^{NN} = 3 \delta_{S_1} + 3 \delta_{1} + 5 \delta_{P_1} + 5 \delta_{D_2} + \cdots,$$

$$\delta_{I=0}^{NN} = 3 \delta_{S_1} + 3 \delta_{1} + 5 \delta_{D_1} + 7 \delta_{D_2} + \cdots.$$  

(21)

The term with \(-3\) in Eq.(19) arises from partial integration in Eq.(13), as the phase-shift for the \(3S_1\) wave (containing the deuteron bound state) at threshold is not 0 but \(\pi\), by the Levinson theorem [18].

For the Nα channel [19],

$$\ln Z_{N\alpha} = \frac{V}{\lambda^3(5m)} \left( \zeta_p + \zeta_n \right) \zeta_p^2 \zeta_n^2 e^{\beta B_A} \Delta_{N\alpha},$$  

(22)

where \(B_A\) is the binding energy of the alpha-particle and

$$\delta_{N\alpha}(e) = \sum_{L,J} (2J + 1) \delta_{L,J}(e) = 2\delta_{S_1} + 2\delta_{1} + 4\delta_{P_1} + 4\delta_{D_1} + 6\delta_{D_2} + \cdots.$$  

(23)

Lastly we consider the αα channel with both spin and isospin equal to zero, where

$$\ln Z_{\alpha\alpha} = \frac{V}{\lambda^3(8m)} \zeta_p^4 \zeta_n^4 e^{2\beta B_A} \Delta_{\alpha\alpha},$$  

(24)

with

$$\delta_{\alpha\alpha}(e) = \sum_L (2L + 1) \delta_L(e) = \delta_S + 9\delta_D + \cdots.$$  

(25)

The integrals \(\Delta_{AB}(\beta)\) have also been evaluated in Ref.[11] with the available phase shifts in the low energy region for all the scattering channels.

Our result (17) for the grand partition function is not in the form of the virial series (13). However, we may expand each of the ideal gas terms in it again in a virial series following Eq.(4) to get the first two terms as

$$\ln Z_{gg}^{(0)} = V \sum_{\alpha\alpha} \frac{g}{\lambda^3(4m)} \left( \zeta_{\alpha\alpha} \pm \zeta_{\alpha\alpha}^2 + \cdots \right),$$  

(26)

$$\ln Z_{ee}^{(0)} + \ln Z_{heavy} = V \sum_{\alpha\alpha} \frac{g}{\lambda^3(4m)} \left( f_1 \zeta_{\alpha\alpha} \pm f_2 \zeta_{\alpha\alpha}^2 + \cdots \right),$$  

(27)

in powers of ‘effective fugacities’, \(\zeta_{\alpha\alpha} = e^{\beta(pZ_N + BZ_N)}\), similar to those already appearing in Eq.(13). The \(A\)-dependent constants, \(f_n(A)\), \(n=1,2,\cdots\)

$$f_n(A) = \int_{E_0}^{E_\alpha} dE \omega(A, E) e^{-n\beta E},$$  

(28)

are generally large for small \(n\), if the temperature is not too low, but decreases steadily with increasing \(n\), so that they make the series (27) converge faster. The ideal gas pieces may now be readily calculated, given the binding energies of nuclei. We take these binding energies from Ref.[19], where they are available for some 9000 nuclei in their ground states up to \(Z = 135\). The pressure \(P\) and the nucleon densities \(n_{p,n}\) are obtained from the familiar formulae,

$$P = T \frac{\ln Z}{V}, \quad n_i = \zeta_i \left( \frac{\partial \ln Z}{\partial \zeta_i} \right)_{V,T}, \quad i = p, n$$  

(29)

and the total baryon density \(n_B\) from \(n_B = n_p + n_n\), at different temperatures and fugacities.

We now consider symmetric nuclear matter, \(\zeta_\alpha = \zeta_n = \zeta[20]\). At each temperature the numerical evaluation of density and pressure shows a steep rise in both the quantities, as we increase the fugacity beyond a certain value. We may rely on our evaluation for fugacities below the onset of such rise. It turns out that the allowed range of fugacity so determined is also given by half the radius of convergence of the virial series obtained earlier in the absence of excited states of nuclei, that is, \(\zeta < (1/2)e^{-\beta}\). In Fig.1 we plot pressure against density for \(T = 2, 5\) and 10 MeV, where we draw the curves for this range of \(\zeta\) for the first two temperatures, while for the third we do so for a smaller range to remain well below the saturation density. It is found that for fugacities (densities) considered here, nuclei heavier than \(A = 15\) do not play any significant role.

For comparison we show in Fig.1 the results of Horowitz and Schwenk [11], who calculate \(Z\) including only one heavy species, namely the alpha particle, besides the original proton and neutron and the low mass scattering channels formed by them, denoted here by \(Z_{light}\).
FIG. 1: Plot of pressure of symmetric nuclear matter against density at $T = 2$, 5 and 10 MeV. The solid curves represent our calculation while the dashed ones are from the virial calculation of Ref.[11], both curves drawn for the same range of fugacity at each temperature. The dotted curve with crosses (data points) show the microscopic calculation of Ref.[1] for $T = 10$ MeV.

It is observed that our results for pressure, besides being somewhat lower at higher densities, extend over a wider region in density than those in Ref.[11]. The difference is to be attributed to nuclei heavier than the alpha particle. Also shown are the results of the microscopic calculation of Ref.[1] up to a density, beyond which the pressure turns out to be unphysical up to a significant region, the compressibility there becoming negative.

Our numerical evaluation shows that the contribution from light scattering channels is about an order of magnitude smaller than that from the ideal gas terms. We are thus led, to a good approximation, to a modified statistical equilibrium model for nuclear matter at moderate temperature, where we include the contributions from not only the nuclei and their excited states as in the conventional model, but also the heavy scattering channels, which reduce again to ideal gas terms, once the $S$-matrices in these channels are approximated by resonances present in the low energy region.

It remains to calculate other interesting quantities, like the relative abundance of nuclei and the energy density and also consider asymmetric systems. Further, the results may be extended to higher temperatures by including pions in the initial set of species. It is, of course, known that chiral symmetry restricts the pions themselves to contribute significantly to the pressure [21]. But the contribution from their scattering with the nucleons and the $\alpha$-particles may not be negligible.

To conclude, we have started from the complete virial series for the grand partition function of nuclear matter, expressed entirely in terms of observables. Besides the binding energies of the nuclei and the density of their excited states, the observables include the $S$-matrix elements for scattering of any number of stable nuclear particles. While the scattering amplitudes in a few low mass, two-body channels are estimated from the experimental scattering data, those in heavy mass channels are assumed to be dominated by low energy resonances. We thus derive in a simple way the equation of state of nuclear matter at moderate temperature, that includes all significant contributions and, at the same time, is free from any serious theoretical uncertainty. Further, the results are valid up to a higher density than found previously.

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