To the Spectral Theory of the Bessel Operator on Finite Interval and Half-Line

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Abstract

The minimal and maximal operators generated by the Bessel differential expression on the finite interval and a half-line are studied. All non-negative self-adjoint extensions of the minimal operator are described. Also we obtain a description of the domain of the Friedrichs extension of the minimal operator in the framework of extension theory of symmetric operators by applying the technique of boundary triplets and the corresponding Weyl functions, and by using the quadratic form method.

1 Introduction

The one-dimensional Bessel differential expression was investigated in the classical form

\[
\tau_\nu = -\frac{d^2}{dx^2} + \frac{\nu^2 - \frac{1}{4}}{x^2}, \quad \nu \in [0, 1) \setminus \{1/2\}
\] (1.1)
on the half-line \(\mathbb{R}_+\) in numerous papers. Here, the parameter \(\nu\) is the order of the Bessel functions involved. When \(\nu = \frac{1}{2}\), it is the regular case. In particular, some results of spectral analysis were investigated in works \([4, 9, 10, 11, 17]\). We especially mention papers of W.N. Everitt and H. Kalf \([9, 14]\) the most relevant to our interest. Here, Titchmarsh–Weyl \(m\)-coefficient was explicitly computed in \(L^2(\mathbb{R}_+)\) using the classical definition. From the Nevanlinna representation of this \(m\)-coefficient the spectral function \(\Sigma\) was obtained to describe the spectrum of the associated self-adjoint operator in \(L^2(\mathbb{R}_+)\). The additional analysis then yields the limit behaviour of the functions in the domain of the Friedrichs extension (see L. Bruneau, J. Dereziński and V. Georgescu \([4]\), W.N. Everitt and H. Kalf \([9, 14]\) and Krein extension (see \([4]\)).
In this paper, we consider Bessel operator (1.1). Under the above restriction ($\nu \in [0, 1]$) the endpoint 0 of the equation
\[-y''(x) + \frac{\nu^2 - \frac{1}{4}}{x^2} y(x) = \lambda y(x)\] (1.2)
is the singular limit-circle case, with respect to $L^2(\mathbb{R}_+)$ or $L^2(0,b)$, except for the regular case.

We study the minimal and maximal Bessel operators on a finite interval and a half-line. We prove that the domain of the minimal operator $A_{\nu,\infty\text{min}}$ associated with $\tau_{\nu}$ in $L^2(\mathbb{R}_+)$ is given by
\[\text{dom}(A_{\nu,\infty\text{min}}) = H^2_0(\mathbb{R}_+),\] (1.3)
and we prove similar formula for the operator on a finite interval.

We investigate spectral properties of the Bessel operator by applying the technique of boundary triplets and corresponding Weyl functions. This new approach to extension theory of symmetric operators developed during last three decades (see [6, 7, 12] and references in therein).

We construct a boundary triplet for the maximal operators in $L^2(\mathbb{R}_+)$ and $L^2(0,b)$ and compute the corresponding Weyl functions. We determine the domains of Friedrichs and Krein extensions. In addition, all self-adjoint and all nonnegative self-adjoint extensions of the minimal Bessel operator are described. We also obtain the Weyl functions on half-line as a limit of corresponding Weyl functions of the operator considered in the finite interval. In particular, we obtain other proofs of results of L. Bruneau, J. Dereziński and V. Georgescu [1], W.N. Everitt and H. Kalf [9, 14].

## 2 Preliminaries

### 2.1 Boundary triplets and self-adjoint extension.

In this section we briefly review the notion of abstract boundary triplets in the extension theory of symmetric operators.

Let $A$ be a closed densely defined symmetric operator in the separable Hilbert space $\mathcal{H}$ with equal deficiency indices $n_{\pm}(A) = \dim \ker (A^* \pm iI) \leq \infty$.

**Definition 2.1 ([12]).** A totality $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a boundary triplet for the adjoint operator $A^*$ of $A$ if $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \text{dom}(A^*) \to \mathcal{H}$ are linear mappings such that

(i) the following abstract second Green identity holds
\[(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \text{dom}(A^*);\] (2.1)

(ii) the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \text{dom}(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

First note that a boundary triplet for $A^*$ exists since the deficiency indices of $A$ are assumed to be equal. Moreover, $n_{\pm}(A) = \dim (\mathcal{H})$ and $A = A^* \upharpoonright (\ker (\Gamma_0) \cap \ker (\Gamma_1))$ hold. Note also that a boundary triplet for $A^*$ is not unique.

A closed extension $\tilde{A}$ of $A$ is called proper if $A \subseteq \tilde{A} \subseteq A^*$. Two proper extensions $\tilde{A}_1$ and $\tilde{A}_2$ of $A$ are called disjoint if $\text{dom}(\tilde{A}_1) \cap \text{dom}(\tilde{A}_2) = \text{dom}(A)$ and transversal if in addition $\text{dom}(\tilde{A}_1) + \text{dom}(\tilde{A}_2) = \text{dom}(A^*)$. The set of all proper extensions of $A$ is denoted by $\text{Ext} A$.

With a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for $A^*$ one associates two self-adjoint extensions $A_j := A^* \upharpoonright \ker (\Gamma_j)$, $j \in \{0, 1\}$. 

Proposition 2.2 ([6] [12]). Let \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( A^* \). Then the mapping

\[
\text{Ext}_A \ni \tilde{A} := A \Theta \rightarrow \Theta := \Gamma(\text{dom}(\tilde{A})) = \{ \Gamma_0 f, \Gamma_1 f \} : f \in \text{dom}(\tilde{A}) \tag{2.2}
\]

establishes a bijective correspondence between the set of all closed proper extensions \( \text{Ext}_A \) of \( A \) and the set of all closed linear relations \( \tilde{C}(\mathcal{H}) \) in \( \mathcal{H} \). Furthermore, the following assertions hold.

(i) The equality \((A \Theta)^* = A \Theta^*\) holds for any \( \Theta \in \tilde{C}(\mathcal{H}) \).

(ii) The extension \( A \Theta \) in (2.2) is symmetric (self-adjoint) if and only if \( \Theta \) is symmetric (self-adjoint).

(iii) If, in addition, extensions \( A \Theta \) and \( A_0 \) are disjoint, i.e., \( \text{dom}(A \Theta) \cap \text{dom}(A_0) = \text{dom}(A) \), then (2.2) takes the form

\[
A \Theta = A_B = A^* \upharpoonright \ker(\Gamma_1 - B \Gamma_0), \quad B \in \mathcal{C}(\mathcal{H}). \tag{2.3}
\]

2.2 Weyl functions and extension of nonnegative operator

It is known that the classical Weyl-Titchmarsh functions play an important role in the direct and inverse spectral theory of singular Sturm-Liouville operators. In [6] the concept of the classical Weyl–Titchmarsh \( m \)-function from the theory of Sturm-Liouville operators was generalized to the case of symmetric operators with equal deficiency indices. The role of abstract Weyl functions in the extension theory is similar to that of the classical Weyl–Titchmarsh \( m \)-function in the spectral theory of singular Sturm-Liouville operators.

Let \( \mathfrak{N}_z := \ker(A^* - z) \) be the defect subspace of \( A \).

Definition 2.3 ([6]). Let \( A \) be a densely defined closed symmetric operator in \( \mathfrak{H} \) with equal deficiency indices and let \( \Pi = \{ \mathcal{H}, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( A^* \). The operator valued functions \( \gamma : \rho(A_0) \rightarrow [\mathcal{H}, \mathfrak{H}] \) and \( M : \rho(A_0) \rightarrow [\mathcal{H}] \) defined by

\[
\gamma(z) := (\Gamma_0 \upharpoonright \mathfrak{N}_z)^{-1} \quad \text{and} \quad M(z) := \Gamma_1 \gamma(z), \quad z \in \rho(A_0), \tag{2.4}
\]

are called the \( \gamma \)-field and the Weyl function, respectively, corresponding to the boundary triplet \( \Pi \).

The \( \gamma \)-field \( \gamma(\cdot) \) and the Weyl function \( M(\cdot) \) in (2.4) are well defined. Moreover, both \( \gamma(\cdot) \) and \( M(\cdot) \) are holomorphic on \( \rho(A_0) \) and the following relations hold (see [6])

\[
\gamma(z) = (I + (z - \zeta)(A_0 - z)^{-1}) \gamma(\zeta), \tag{2.5}
\]

\[
M(z) - M(\zeta)^* = (z - \overline{\zeta})\gamma(\zeta)^* \gamma(z), \tag{2.6}
\]

\[
\gamma^*(\overline{\zeta}) = \Gamma_1 (A_0 - z)^{-1}, \quad z, \ z \in \rho(A_0). \tag{2.7}
\]

Identity (2.6) yields that \( M(\cdot) \) is an \( R_{\mathfrak{H}} \)-function (or Nevanlinna function), that is, \( M(\cdot) \) is an \( ([\mathcal{H}]\text{-valued}) \) holomorphic function on \( \mathbb{C} \setminus \mathbb{R} \) and

\[
\text{Im} z \cdot \text{Im} M(z) \geq 0, \quad M(z^*) = M(\overline{z}), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{2.8}
\]
Besides, it follows from (2.6) that $M(\cdot)$ satisfies $0 \in \rho(\text{Im} M(z))$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Since $A$ is densely defined, $M(\cdot)$ admits an integral representation (see, for instance, [7])

$$M(z) = C_0 + \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\Sigma_M(t), \quad z \in \rho(A_0),$$

where $\Sigma_M(\cdot)$ is an operator-valued Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{1}{1 + t^2} d\Sigma_M(t) \in [\mathcal{H}]$ and $C_0 = C_0^* \in [\mathcal{H}]$. The integral in (2.9) is understood in the strong sense.

In contrast to spectral measures of self-adjoint operators the measure $\Sigma_M(\cdot)$ is not necessarily orthogonal. However, the measure $\Sigma_M$ is uniquely determined by the Nevanlinna function $M(\cdot)$. The operator-valued measure $\Sigma_M$ is called the spectral measure of $M(\cdot)$. If $A$ is a simple symmetric operator, then the Weyl function $M(\cdot)$ determines the pair $\{A, A_0\}$ up to unitary equivalence (see [7]). Due to this fact, spectral properties of $A_0$ can be expressed in terms of $M(\cdot)$.

Assume that a symmetric operator $A \in \mathcal{C}(\mathcal{H})$ is nonnegative. Then the set $\text{Ext}_A(0, \infty)$ of its nonnegative self-adjoint extensions is non-empty (see [2]). Moreover, there is a maximal nonnegative extension $A_F$ (also called Friedrichs’ or hard extension) and there is a minimal nonnegative extension $A_K$ (Krein’s or soft extension) satisfying

$$(A_F + x)^{-1} \leq (\bar{A} + x)^{-1} \leq (A_K + x)^{-1}, \quad x \in (0, \infty), \quad \bar{A} \in \text{Ext}_A(0, \infty)$$

(for detail we refer the reader to [2]).

The following proposition characterizes the Friedrichs and Krein extensions in terms of the Weyl function.

**Proposition 2.4** ([6, 7]). Let $A$ be a densely defined nonnegative symmetric operator with finite deficiency indices in $\mathcal{D}$, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $A^*$. Let also $M(\cdot)$ be the corresponding Weyl function. Then the following assertions hold.

(i) Extensions $A_0$ and $A_K$ are disjoint ($A_0$ and $A_F$ are disjoint) if and only if

$$M(0) \in \mathcal{C}(\mathcal{H}) \quad (M(\infty) \in \mathcal{C}(\mathcal{H}), \text{ respectively}).$$

Moreover,

$$\text{dom}(A_K) = \text{dom}(A^*) \upharpoonright \text{ker}(\Gamma_1 - M(0)\Gamma_0)$$

$$\text{dom}(A_F) = \text{dom}(A^*) \upharpoonright \text{ker}(\Gamma_1 - M(\infty)\Gamma_0), \quad \text{respectively}.$$

(ii) $A_0 = A_K$ ($A_0 = A_F$) if and only if

$$\lim_{x \to 0^+} (M(x)f, f) = +\infty, \quad f \in \mathcal{H} \setminus \{0\}$$

$$\lim_{x \to -\infty} (M(x)f, f) = -\infty, \quad f \in \mathcal{H} \setminus \{0\}, \quad \text{respectively}.$$

(iii) The set of all non-negative self-adjoint extensions of $A$ admits parametrization (2.2), where $\Theta$ satisfies

$$\Theta - M(0) \geq 0 \quad (\Theta - M(\infty) \leq 0, \text{ respectively}).$$

(2.10)
2.3 Bessel functions

Consider the equation

\[ z^2 \frac{d^2 u}{dz^2} + z \frac{du}{dz} + (z^2 - \nu^2)u = 0. \]  

(2.11)

Solutions of the (2.11) are the Bessel functions of the first \( J_{\pm \nu} \) and second \( Y_{\nu} \) kind, respectively (see [1, Ch. 9], [2, App. 2], [19, p. 284–285]).

Recall that the asymptotic behaviors of the Bessel functions \( J_{\nu}(t) \) and \( J_{-\nu}(t) \) for \( t \to 0 \) have the form

\[ J_{\nu}(t) = \frac{t^{\nu}}{2^{\nu}\Gamma(1+\nu)}[1 + O(t^2)], \quad J_{-\nu}(t) = \frac{2^{\nu}}{\Gamma(1-\nu)}t^{-\nu}[1 + O(t^2)], \quad t \to 0, \]  

(2.12)

and the asymptotic behavior of the Bessel functions \( Y_{\nu}(t) \) for \( t \to 0 \) has the form

\[ Y_0(t) = \frac{2}{\pi} \left( \log \left( \frac{t}{2} \right) + \gamma \right) \cdot [1 + O(t^2)], \quad Y_{\nu}(t) = -\frac{\Gamma(\nu)}{\pi} \frac{2}{t} \cdot [1 + O(t^2)], \quad t \to 0, \]  

(2.13)

where \( \gamma \) is Euler’s constant.

Moreover as \( t \to \infty \) we have

\[
\begin{cases}
  J_{\nu}(t) = \left( \frac{2}{\pi t} \right)^{\nu} \cos \left( t - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) + O(|t|^{-\frac{3}{2}}), \\
  J_{-\nu}(t) = \left( \frac{2}{\pi t} \right)^{\nu} \cos \left( t + \frac{\nu\pi}{2} - \frac{\pi}{4} \right) + O(|t|^{-\frac{3}{2}}), \\
  Y_{\nu}(t) = \frac{2}{\pi t} \sin \left( t - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) + O(|t|^{-\frac{3}{2}}),
\end{cases}
\]  

(2.14)

Also, we need decomposition of the Bessel functions into Taylor series about zero (see [1, Formulas 9.1.10, 9.1.12, 9.1.13])

\[ J_{\nu}(z) = \left( \frac{1}{2}z \right)^{\nu} \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)}, \]  

(2.15)

\[ J_0(z) = 1 - \frac{1}{4}z^2 + \frac{1}{8}z^4 - \frac{1}{32}z^6 + \ldots; \]  

(2.16)

\[ Y_0(z) = \frac{2}{\pi} \left\{ \log \left( \frac{1}{2}z \right) + \gamma \right\} J_0(z) + 2 \left\{ \frac{1}{4}z^2 \frac{1}{(1!)^2} - \left( 1 + \frac{1}{2} \right) \frac{1}{(2!)^2} + \left( 1 + \frac{1}{2} + \frac{1}{3} \right) \frac{1}{(3!)^2} \right\}. \]  

(2.17)

We use the following properties of Bessel functions (see [1, Formula 9.1.28])

\[ J_0'(t) = -J_1(t), \quad Y_0'(t) = -Y_1(t). \]  

(2.18)

Also recall [2, App. 2] that the Bessel function \( Y_{\nu} \) of the second kind is given by

\[ Y_{\nu}(t) = \frac{J_{\nu}(t) \cos \pi \nu - J_{-\nu}(t)}{\sin \pi \nu}, \quad \nu \neq 0. \]  

(2.19)
Next, we need formulas (see [1, Formula 9.1.29])

\[
zf'_{\nu}(z) = lqz^q f_{\nu-1}(z) + (p - \nu q) f_{\nu}(z),
\]

\[
zf'_{\nu}(z) = -lqz^q f_{\nu+1}(z) + (p + \nu q) f_{\nu}(z),
\]

(2.20)

in which \(f_{\nu}(z) = z^p G_{\nu}(lz^q)\) where \(G_{\nu}(\cdot)\) is one of the Bessel functions \(J_{\nu}(\cdot), Y_{\nu}(\cdot)\) or a linear combination, and \(p, q, l\) do not depend on \(\nu\).

Applying formula (2.20) for \(l = 1, q = 1/2, p = 0\) to the functions \(f_{\nu} = x^{1/2} G_{\nu}(x\sqrt{z})\) where \(G_{\nu}(\cdot)\) is one of the Bessel functions \(J_{\nu}(\cdot), Y_{\nu}(\cdot),\) we obtain

\[
[f_{\nu}, x^{1/2+\nu}]_x = \sqrt{x} x^{1/2+\nu} f_{\nu+1}, \quad [f_{-\nu}, x^{1/2+\nu}]_x = -\sqrt{x} x^{1/2+\nu} f_{-\nu-1},
\]

(2.21)

and

\[
[f_{\nu}, x^{1/2-\nu}]_x = -\sqrt{x} x^{1/2-\nu} f_{\nu-1}, \quad [f_{-\nu}, x^{1/2-\nu}]_x = \sqrt{x} x^{1/2-\nu} f_{-\nu+1},
\]

(2.22)

where \([f, g]_x := f(x)g'(x) - f'(x)g(x)\), for all \(x \in \mathbb{R}_+\).

The general solution of the equation (1.2) is given by

\[
y(x; \lambda) = c_1 x^{1/2} J_{\nu}(x\sqrt{\lambda}) + c_2 x^{1/2} Y_{\nu}(x\sqrt{\lambda}),
\]

(2.23)

where \(c_1, c_2\) are arbitrary constants.

3 Bessel operator \(S_{\nu; b}\) on the interval

In what follows, we need the following auxiliary lemma.

Lemma 3.1 ([21, p. 318–319]). Let \(T_K\) be the operator in \(L^p[0, \infty)\) of the form

\[
T_K : f \mapsto \int_0^\infty K(x, t)f(t)dt,
\]

(3.1)

and its kernel \(K(x, t)\) has a degree of homogeneity \(-1\), i.e. \(K(\lambda x, \lambda t) = \lambda^{-1} K(x, t), \lambda > 0\). Then the operator \(T_K\) is bounded in \(L^p[0, \infty)\) and its norm is

\[
\|T_K\|_p := \|T_K\|_{L^p \to L^p} = \int_0^\infty |K(1, t)| t^{-1/p} dy.
\]

(3.2)

Suppose further that \(I\) is the operator of integration, \(I : f \mapsto \int_0^x f(t)dt\). Then

\[
I^2 f = \int_0^x (x - t)f(t)dt.
\]

(3.3)

Also assume that \(Q : f \mapsto \frac{1}{x} f(x)\).
Lemma 3.2. The operator $QI^2$

$$QI^2 : f \mapsto \frac{1}{x^2} \int_0^x (x - t)f(t)dt,$$  \hspace{1cm} (3.4)

is bounded in $L^2[0, b]$ for each $b \in (0, \infty)$, and $\|QI^2\|_2 = \frac{4}{3}$.

Proof. Let

$$K(x, t) = \begin{cases} \frac{1}{x}(1 - \frac{t}{x}), & t \leq x, \\ 0, & t > x. \end{cases}$$  \hspace{1cm} (3.5)

Noting that $K(\lambda x, \lambda t) = \lambda^{-1}K(x, t)$ and applying Lemma 3.1 to the operator $T_K = QI^2$, we obtain

$$\|QI^2\|_2 = \|T_K\|_2 = \int_0^\infty |K(1, t)|t^{-1/2}dt = \int_0^1 (1-t)t^{-1/2}dt = \frac{4}{3},$$  \hspace{1cm} (3.6)

Let $H^2[0, b]$ be the Sobolev space on $[0, b]$. Assume also

$$\tilde{H}^2_0[0, b] = \{ f \in H^2[0, b] : f(0) = f'(0) = 0 \}.$$  \hspace{1cm} (3.7)

Lemma 3.3. If $f \in \tilde{H}^2_0[0, 1]$, then the following relations hold:

$$f(x) = o(x^{3/2}), \quad f'(x) = o(x^{1/2}) \quad \text{as } x \to 0.$$  \hspace{1cm} (3.8)

Proof. Since $f \in \tilde{H}^2_0[0, 1]$, then $f'(x) = \int_0^x f''(t)dt$. Therefore by the Cauchy–Bunyakovsky inequality

$$|f'(x)| \leq \left( \int_0^x |f''(t)|dt \right)^2 \leq x \int_0^x |f''(t)|^2dt = x \cdot o(1) = o(x) \quad \text{as } x \to 0,$$  \hspace{1cm} (3.9)

i.e. $f'(x) = o(x^{1/2})$, which proves the second estimate in (3.8).

Further, since $f \in \tilde{H}^2_0[0, 1]$, we get $f(x) = \int_0^x f'(t)dt$. Hence,

$$|f(x)| \leq \int_0^x |f'(t)|dt \leq \int_0^x o(x^{1/2})dx = o(x^{3/2}) \quad \text{as } x \to 0.$$  \hspace{1cm} (3.10)

The first estimate in (3.8) is proved. \hspace{1cm} \square

Let $D^2_{\min}$ be a minimal differential operator of the 2nd order, generated in $L^2(0, b)$ by differential expression $-d^2/dx^2$,

$$\text{dom } (D^2_{\min}) = H^2_0[0, b] = \{ f \in H^2[0, b] : f(0) = f'(0) = f(b) = f'(b) = 0 \}.$$  \hspace{1cm} (3.11)

Let $S_{\nu h} := S_{\nu h_{\min}}$ and $S_{\nu h_{\max}}$ are the minimal and maximal operators, respectively, generated by the differential expression (1.1) in $L^2(0, b)$, $b < \infty$. 

Theorem 3.4. Let $\nu \in [0, 1)$. Then the following assertions hold.

(i) The operator $S_{\nu, b}$ is a non-negative and its deficiency indices are $n_{\pm}(S_{\nu, b}) = 2$.

(ii) The domain of the operator $S_{\nu, b}$ is given by

$$\text{dom} (S_{\nu, b}) = H^2_0[0, b].$$

(iii) $S_{\nu, b}^{\text{max}} = S^*_{\nu, b}$ and

$$\text{dom} (S^*_{\nu, b}) = \begin{cases} \tilde{H}^2_0[0, b] + \text{span}\{x^{1/2+\nu}, x^{1/2-}\nu\}, & \nu \in (0, 1), \\ \tilde{H}^2_0[0, b] + \text{span}\{x^{1/2}, x^{1/2}\log(x)\}, & \nu = 0. \end{cases}$$ (3.12)

Proof. (i)–(ii) The function $u \in \tilde{H}^2_0[0, b]$ admits the integral representation $u(x) = \int_0^x (x-t)u''(t)dt$. Therefore,

$$Qu(x) = \frac{1}{x^2}u(x) = \frac{1}{x^2} \int_0^x (x-t)u''(t)dt = (QT^2(D^2_{\text{min}} u))(x).$$ (3.13)

By virtue of Lemma 3.2, this yields

$$\|Qu\|_2 = \left\| \frac{1}{x^2}u \right\|_2 = \|QT^2D^2_{\text{min}} u\|_2 \leq \|QT^2\|_2 \cdot \|D^2_{\text{min}} u\|_2 = \frac{4}{3}\|D^2_{\text{min}} u\|_2 \leq \frac{4}{3}\|u\|_{H^2_0[0, b]}. \quad (3.14)$$

It is easy to see that $\nu^2 - \frac{1}{4}$ admits the representation $\nu^2 - \frac{1}{4} = \frac{3}{4}(1 - \varepsilon)$, where $\varepsilon > 0$. Then relation (3.13) implies the estimate

$$\left\| \left(\nu^2 - \frac{1}{4}\right) Qu \right\|_2 = \nu^2 - \frac{1}{4} \cdot \|Qu\|_2 \leq \frac{3}{4}(1 - \varepsilon) \cdot \frac{4}{3}\|u\|_{H^2_0[0, b]} = (1 - \varepsilon)\|u\|_{H^2_0[0, b]}, \quad u \in H^2_0[0, b].$$ (3.15)

Estimate (3.15) means that $Q$ is strongly $D^2_{\text{min}}$-bounded. Therefore, by the Kato–Rellich theorem (see [15]) $n_{\pm}(S_{\nu, b}) = n_{\pm}(D^2_{\text{min}}) = 2$ and dom $(S_{\nu, b}) = H^2_0[0, b]$.

(iii) Since $\tau_{\nu}x^{1/2+\nu} = 0$,

the equality is understood in the meaning of the theory of distributions, and $x^{1/2+\nu} \in L^2(0, b)$, then

$$\{x^{1/2+\nu}, x^{1/2-\nu}\} \subset \text{dom} (S_{\nu, b}^{\text{max}}) = \text{dom} (S^*_{\nu, b}),$$

and ker $(S^*_{\nu, b}) = \{x^{1/2+\nu}, x^{1/2-\nu}\} \subset L^2(0, b)$. In addition, it is clear that $\tilde{H}^2_0[0, b] \subset \text{dom} (S^*_{\nu, b})$ and $\dim (\tilde{H}^2_0[0, b]) / \dim (S^*_{\nu, b}) = 2$. On the other hand, since $n_{\pm}(S_{\nu, b}) = 2$, we have $\dim (\text{dom} (S^*_{\nu, b}) / \text{dom} (S_{\nu, b})) = 2n_{\pm}(S_{\nu, b}) = 2n_{\pm}(S_{\nu, b}) = 4$ by the first Neumann formula. Therefore, formula (3.12) is valid. \qed
Consider the quadratic form $\mathcal{S}_{\nu,b}$ associated with the operator $S_{\nu,b}$,

$$
\mathcal{S}_{\nu,b}[u] := (S_{\nu,b}u, u), \quad \text{dom}(\mathcal{S}_{\nu,b}) = \text{dom}(S_{\nu,b}) = H_0^2[0, b].
$$

It is clear that $S_{1/2,b} = -D_{\text{min}}^2$.

**Theorem 3.5.** Let $\nu \in [0, 1)$ and $S_{\nu,b,F}$ be the Friedrichs extension of the operator $S_{\nu,b}$. Also assume $\xi \in C_0^1[0, b]$ such that $\xi(x) = 1$ for $x \in [0, b/2]$ and $\xi(b) = 0$. Then:

(i) For $\nu \in (0, 1)$ the quadratic form $\mathcal{S}_{\nu,b}$ quadratic form associated with the Friedrichs extension $S_{\nu,b,F}$ takes the form

$$
\mathcal{S}_{\nu,b}[u] = \int_0^b |u'(x)|^2 dx + \left(\nu^2 - \frac{1}{4}\right) \int_0^b \frac{|u(x)|^2}{x^2} dx,
$$

$$
\text{dom}(\mathcal{S}_{\nu,b}) = H_0^1[0, b].
$$

(ii) For $\nu = 0$ the quadratic form $\mathcal{S}_{0,b}$ quadratic form associated with the Friedrichs extension $S_{0,b,F}$ takes the form

$$
\mathcal{S}_{0,b}[u] = \int_0^b \left|u'(x) - \frac{u(x)}{2x}\right|^2 dx,
$$

$$
\text{dom}(\mathcal{S}_{0,b}) \supset H_0^1[0, b] + \text{span}\{x^{1/2}(x-b), x^{3/2}\xi(x)\}.
$$

Wherein $\dim(\text{dom}(\mathcal{S}_{0,b})/H_0^1[0, b]) = \infty$.

(iii) The domain of the Friedrichs extension $S_{\nu,b,F}$ of the operator $S_{\nu,b}$ takes the form

$$
\text{dom}(S_{\nu,b,F}) = \begin{cases} 
H_0^2[0, b] + \text{span}\{x^{1/2+\nu}(x-b), x^2(x-b)\}, & \nu \in (0, 1), \\
H_0^2[0, b] + \text{span}\{x^{1/2}(x-b), x^{1/2}\xi(x)\}, & \nu = 0.
\end{cases}
$$

**Proof.** (i) By Hardy’s inequality for $\nu \in (0, 1)$ and $u \in H_0^1[0, b]$

$$
\mathcal{S}_{\nu,b}[u] = \|u'(t)\|_2^2 + (\nu^2 - 1/4) \int_0^b \frac{|u(t)|^2}{t^2} dt \leq \|u'(t)\|_2^2(1 + |4\nu^2 - 1|).
$$

Thus $H_0^1[0, b] \subset \text{dom}(\mathcal{S}_{\nu,b})$.

We prove the converse inequality. Suppose first that $\nu \in [1/2, 1)$. Then

$$
\mathcal{S}_{\nu,b}[u] = \|u'(t)\|_2^2 + (\nu^2 - 1/4) \int_0^b \frac{|u(t)|^2}{t^2} dt \geq \|u'(t)\|_2^2, \quad u \in H_0^1[0, b].
$$

If $\nu \in (0, 1/2)$, then for $u \in H_0^1[0, b]$ applying the Hardy’s inequality we obtain

$$
\mathcal{S}_{\nu,b}[u] = \|u'(t)\|_2^2 - (1/4 - \nu^2) \int_0^b \frac{|u(t)|^2}{t^2} dt \geq \|u'(t)\|_2^2 + (4\nu^2 - 1)\|u'(t)\|_2^2 = 4\nu^2\|u'(t)\|_2^2.
$$
So on $H^1_0[0,b]$ the energy norm of $S_{\nu,b}$ is equivalent to the norm of space $H^1_0[0,b]$. Since $H^2_0[0,b] = \text{dom}(S_{\nu,b})$ is dense in the energy space of the operator $S_{\nu,b}$, then dom($s_{\nu,b}$) and $H^1_0[0,b]$ coincide algebraically and topologically.

(ii) Let $u_1(x) = x^{1/2}(x-b)$ and $u_2(x) = x^{1/2}\xi(x)$ then

\[
\mathfrak{s}_{0,b}[u_1] = \int_0^b x\,dx < \infty, \quad \mathfrak{s}_{0,b}[u_2] = \int_0^b x(\xi'(x))^2\,dx < \infty.
\]

So $\{x^{1/2}(x-b), x^{1/2}\xi(x)\} \subset \text{dom}(\mathfrak{s}_{0,b})$.

(iii) We note that $H^2_0[0,b] \subset H^1_0[0,b]$. If $u(x) = x^{1/2+\nu}(x-b)$ then $u'(\cdot) \in L^2(0,b)$, but $u(\cdot) \notin \text{dom}(S_{\nu,b}) = H^2_0[0,b]$. By the construction of the Friedrichs extension and the equalities (3.12), we obtain

\[
\text{dom}(S_{\nu,b}^F) = \text{dom}(S_{\nu,b}^*) \cap \text{dom}(\mathfrak{s}_{\nu,b}) = \text{dom}(S_{\nu,b}^*) \cap H^1_0[0,b] = H^2_0[0,b] + \text{span}\{x^{1/2}(x-b), x^{1/2}\xi(x)\}.
\]

The case $\nu = 0$ is considered similarly.

The case $\nu \in [0, 1/\sqrt{2})$ in Proposition 3.5 can be treated by means of KLMN–theorem. Therefore, applying Hardy’s inequality for one gets

\[
\left(\nu^2 - \frac{1}{4}\right) \int_0^b \frac{|u(x)|^2}{x^2}\,dx \leq 4 \left|\nu^2 - \frac{1}{4}\right| \int_0^b |u'|^2\,dx \leq (1 - \varepsilon) t_{D_{\min}^2}[u], \quad u \in H^1_0[0,b]. \tag{3.25}
\]

Hence, the form $(\nu^2 - \frac{1}{4}) q$ is strongly $t_{D_{\min}^2}$–bounded, where $q[u] := \int_0^b \frac{|u(x)|^2}{x^2}\,dx$. By the KLMN-theorem 1.5 dom($s_{\nu,b}$) = dom($t_{D_{\min}^2}$) = $H^1_0[0,b]$. This argument was already used in our previous paper [3].

4 Bessel operator $A_{\nu,b}$ on the interval

Here, we consider the Bessel operator $A_{\nu,b}$ generated by the differential expression (1.1) in $L^2(0,b)$ with the domain

\[
\text{dom}(A_{\nu,b}) = \{f \in \text{dom}(S_{\nu,b}^*) : f(0) = f'(0) = f(b) = 0\}, \quad \nu \in [0,1). \tag{4.1}
\]

**Theorem 4.1.** Let $\nu \in [0,1)$. The following assertions hold:

(i) The operator $A_{\nu,b}$ has equal deficiency indices $n_\pm(A_{\nu,b}) = 1$;

(ii) dom($A_{\nu,b}$) = \{f $\in H^2[0,b]$ : f(0) = f'(0) = f(b) = 0\};

(iii) dom($A_{\nu,b}^*$) = \{f $\in \text{dom}(S_{\nu,b}^*)$ : f(b) = 0\}.

**Proof.** It is easily seen that $S_{\nu,b} \subset A_{\nu,b} \subset S_{\nu,b}^*$ and dim(dom($A_{\nu,b}$)/dom($S_{\nu,b}$)) = 1. But, by Proposition 3.4 $n_\pm(S_{\nu,b}) = 2$. Hence, by the Second Neumann formula implies $n_\pm(A_{\nu,b}) = 1$. Later on branch of the function $z^\nu$ selected in the plane $\mathbb{C}$ with a cut along the positive half–line $\mathbb{R}_+$ so $z^\nu = x^\nu$ for $z = x > 0$. 

\[
\mathfrak{s}_{0,b}[u_1] = \int_0^b x\,dx < \infty, \quad \mathfrak{s}_{0,b}[u_2] = \int_0^b x(\xi'(x))^2\,dx < \infty.
\]

So $\{x^{1/2}(x-b), x^{1/2}\xi(x)\} \subset \text{dom}(\mathfrak{s}_{0,b})$.
Proposition 4.2. Let $\nu \in [0, 1)$ and $b < \infty$. Also assume that $A_{\nu, b}$ be the Bessel operator generated by the expression $(1.1)$ in $L^2(0, b)$ with the domain $(1.1)$. Then:

(i) Boundary triplet of the operator $A_{\nu, b}^*$ can be selected in the form of

$$
\mathcal{H} = \mathbb{C}, \quad \Gamma_{0}^{\nu, b} f = [f, x^{\frac{1}{2} - \nu}]_0, \quad \Gamma_{1}^{\nu, b} f = \left\{ \begin{array}{ll}
-(2\nu)^{-1}[f, x^{\frac{1}{2} - \nu}]_0, & \nu \in (0, 1), \\
[f, x^{\frac{1}{2}} \log(x)]_0, & \nu = 0.
\end{array} \right.
$$

(ii) The Weyl function $M_{\nu, b}(\cdot)$ corresponding to the boundary triplet $(1.2)$ has the form:

$$
M_{\nu, b}(z) = \left\{ \begin{array}{ll}
\frac{\Gamma(1 - \nu)}{2\nu \Gamma(1 + \nu)} J_{\nu} (b \sqrt{z}) \cdot z^{\nu}, & \nu \in (0, 1), \\
- \log \left( \frac{\sqrt{z}}{2} \right) + \frac{\pi Y_0 (b \sqrt{z})}{2 J_0 (b \sqrt{z})} - \gamma, & \nu = 0,
\end{array} \right.
$$

where $\gamma$ is Euler’s constant.

Proof. (i) Let $f, g \in \text{dom} (A_{\nu, b}^*)$. Integrating by parts, we obtain

$$
(A_{\nu, b}^* f, g) - (f, A_{\nu, b}^* g) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{b} \left( -f''(x) \overline{g(x)} + \frac{\nu^2 - \frac{1}{2}}{x^2} f(x) \overline{g(x)} \right) dx - \int_{\varepsilon}^{b} f(x) \left( -g''(x) + \frac{\nu^2 - \frac{1}{2}}{x^2} g(x) \right) dx = \lim_{\varepsilon \to 0} \left\{ -f(\varepsilon) \overline{g'(\varepsilon)} + f'(\varepsilon) \overline{g(\varepsilon)} \right\}.
$$

On the other hand it is easily seen that

$$
(\Gamma_{1}^{\nu, b} f, \Gamma_{0}^{\nu, b} g) - (\Gamma_{0}^{\nu, b} f, \Gamma_{1}^{\nu, b} g) =
$$

$$
= \frac{1}{2\nu} \lim_{x \to 0} \left[ \left( \frac{1}{2} + \nu \right) x^{-\nu - \frac{1}{2}} f(x) - x^{\frac{1}{2} + \nu} f'(x) \right] \left( \left( \frac{1}{2} - \nu \right) x^{-\frac{1}{2} - \nu} \overline{g(x)} - x^{\frac{1}{2} - \nu} \overline{g'(x)} \right) - \left( \left( \frac{1}{2} - \nu \right) x^{-\nu - \frac{1}{2}} f(x) - x^{\frac{1}{2} - \nu} f'(x) \right) \left( \left( \frac{1}{2} + \nu \right) x^{-\frac{1}{2} + \nu} \overline{g(x)} - x^{\frac{1}{2} - \nu} \overline{g'(x)} \right)
$$

$$
= \frac{1}{2\nu} \lim_{x \to 0} 2\nu (f'(x) \overline{g(x)} - f(x) \overline{g'(x)}) = \lim_{x \to 0} \{ -f(x) \overline{g'(x)} + f'(x) \overline{g(x)} \}.
$$

Comparing this formula with the previous one we obtain the Green’s formula

$$
(A_{\nu, b}^* f, g) - (f, A_{\nu, b}^* g) = (\Gamma_{1}^{\nu, b} f, \Gamma_{0}^{\nu, b} g) - (\Gamma_{0}^{\nu, b} f, \Gamma_{1}^{\nu, b} g).
$$

The case $\nu = 0$ is considered similarly.

(ii.1) First we consider the case $\nu \in (0, 1)$.

By the asymptotic relations (2.12) $x^{1/2} J_\nu (x \sqrt{z}) \in L^2(0, b)$ and $x^{1/2} J_{-\nu} (x \sqrt{z}) \in L^2(0, b)$. Therefore

$$
f_z (x) := x^{\frac{1}{2}} \left( J_\nu (x \sqrt{z}) - \frac{J_\nu (b \sqrt{z})}{J_{-\nu} (b \sqrt{z})} J_{-\nu} (x \sqrt{z}) \right) \in L^2(0, b).
$$

It is easily seen that $f_z (b) = 0$, and hence, $f_z \in \text{dom} (A_{\nu, b}^*)$ and $(A_{\nu, b}^* - z) f_z = 0$. In other words, deficient space $\mathfrak{N}_z (A_{\nu, b})$ of the operator $A_{\nu, b}$ generated by the vector $f_z$. 

Using the asymptotic behavior of the Bessel functions (2.12) and formula (2.21) we obtain
\[
\begin{align*}
[x^{1/2} J_\nu(x \sqrt{z}), x^{1/2+\nu}]_0 &= \lim_{x \to 0} \left[ x^{1/2} x^{1+\nu} J_{\nu+1}(x \sqrt{z}) \right] \\
&= \lim_{x \to 0} \left[ \frac{z^{1+\nu/2} 2^{(1+\nu)}}{\Gamma(2 + \nu)} (1 + O(x^2 z)) \right] = 0,
\end{align*}
\]
(4.5) and formula (2.21)
\[
\begin{align*}
[x^{1/2} J_{-\nu}(x \sqrt{z}), x^{1/2+\nu}]_0 &= \lim_{x \to 0} \left[ -x^{1/2} x^{1+\nu} J_{-\nu-1}(x \sqrt{z}) \right] \\
&= \lim_{x \to 0} \left[ -\frac{x^{1-\nu/2} 2^{1+\nu}}{\Gamma(-\nu)} (1 + O(x^2 z)) \right] = -\frac{x^{1-\nu/2} 2^{1+\nu}}{\Gamma(-\nu)}.
\end{align*}
\]
Similarly, using the asymptotic behavior of the Bessel functions (2.12) and formula (2.22) we obtain
\[
\begin{align*}
[x^{1/2} J_\nu(x \sqrt{z}), x^{1/2-\nu}]_0 &= -\lim_{x \to 0} \left[ z^{1/2} x^{1-\nu} J_{\nu-1}(x \sqrt{z}) \right] \\
&= -\lim_{x \to 0} \left[ \frac{z^{\nu/2}}{2^{\nu-1} \Gamma(\nu)} (1 + O(x^2 z)) \right] = -\frac{z^{\nu/2}}{2^{\nu-1} \Gamma(\nu)},
\end{align*}
\]
(4.6)
\[
\begin{align*}
[x^{1/2} J_{-\nu}(x \sqrt{z}), x^{1/2-\nu}]_0 &= -\lim_{x \to 0} \left[ z^{1/2} x^{1-\nu} J_{-\nu-1}(x \sqrt{z}) \right] \\
&= -\lim_{x \to 0} \left[ \frac{z^{1-\nu} 2^{\nu-1} x^{2(1-\nu)}}{\Gamma(2 - \nu)} (1 + O(x^2 z)) \right] = 0.
\end{align*}
\]
From the formulas (4.7), (4.7) and (4.8), (4.10) we arrive at the relation
\[
\begin{align*}
\Gamma_0 \nu_b f_z &= \frac{2^{1+\nu}}{\Gamma(-\nu)} \cdot \frac{J_\nu(b \sqrt{z})}{J_{-\nu}(b \sqrt{z})} \cdot z^{-\frac{\nu}{2}}; \quad \Gamma_1 \nu_b f_z = \frac{1}{\nu 2^{\nu} \Gamma(\nu)} \cdot z^{-\frac{\nu}{2}}.
\end{align*}
\]
(4.7)
Hence, relation (4.7) and Definition 2.3 yield the fist part of formula (4.3).

By the asymptotic relations (2.12) and (2.13) \( x^{1/2} J_0(x \sqrt{z}) \in L^2(0, b) \) and \( x^{1/2} Y_0(x \sqrt{z}) \in L^2(0, b) \). Therefore
\[
f_z(x) := x^{1/2} \left( J_0(x \sqrt{z}) - \frac{J_0(b \sqrt{z})}{Y_0(b \sqrt{z})} Y_0(x \sqrt{z}) \right) \in L^2(0, b).
\]
(4.8)
It is easily seen that \( f_z(b) = 0 \), and, hence, \( f_z \in \text{dom} (A_{0,b}^*) \) and \( (A_{0,b}^* - z) f_z = 0 \). In other words, the deficiency space \( N_z(A_{0,b}) \) of the operator \( A_{0,b} \) generated by the vector \( f_z \).

Using the asymptotic behavior of the Bessel functions (2.12) and formula (2.21) we obtain
\[
\begin{align*}
[x^{1/2} J_0(x \sqrt{z}), x^{1/2}]_0 &= \lim_{x \to 0} \left[ x z^{1/2} J_1(x \sqrt{z}) \right] \\
&= \lim_{x \to 0} \left[ \frac{x z^{1/2}}{2} (1 + O(x^2 z)) \right] = 0,
\end{align*}
\]
(4.9)
\[
\begin{align*}
[x^{1/2} Y_0(x \sqrt{z}), x^{1/2}]_0 &= \lim_{x \to 0} \left[ x z^{1/2} Y_1(x \sqrt{z}) \right] \\
&= \lim_{x \to 0} \left[ -x \sqrt{z} \cdot \frac{2}{\pi \sqrt{z}} (1 + O(x^2 z)) \right] = -\frac{2}{\pi}.
\end{align*}
\]
Proposition 4.4. Let $\nu \in [0, 1)$ and $\Pi_{\nu,b} = \{\mathcal{H}, \Gamma_0^{\nu,b}, \Gamma_1^{\nu,b}\}$ be the boundary triplet of the form (4.12) for the operator $A_{\nu,b}^*$. Then:

(i) The domain of the Friedrichs extension $A_{\nu,b,F}$ of the operator $A_{\nu,b}$ has the form

$$\text{dom}(A_{\nu,b,F}) = \ker (\Gamma_0^{\nu,b}) = \left\{ f \in \text{dom}(A_{\nu,b}^*) : [f, x^{\frac{1}{2}+\nu}]_0 = 0 \right\}.$$  

(4.12)
(ii) The domain of the Krein extension $A_{\nu,b}^{\mathcal{K}}$ of the operator $A_{\nu,b}$ has the form

$$\text{dom} (A_{\nu,b}^{\mathcal{K}}) = \left\{ \begin{array}{ll}
\{ f \in \text{dom} (A_{\nu,b}^*) : (2\nu)^{-1}[f, b^{-2\nu}x^{1/2+\nu} - x^{1/2-\nu}]_0 = 0 \} , & \nu \in (0, 1), \\
\{ f \in \text{dom} (A_{0,b}^*) : [f, x^2 \log (x)]_0 = 0 \} , & \nu = 0.
\end{array} \right.$$  

(4.13)

Proof. (i) First we consider the case $\nu \in (0, 1)$.

Applying the asymptotic behavior of the Bessel functions (2.14) to the Weyl function (4.3), we obtain

$$M_{\nu,b}(-\infty) \cdot \frac{2\nu 4^{\nu} \Gamma(1+\nu)}{\Gamma(1-\nu)} = \frac{2\nu 4^{\nu} \Gamma(1+\nu)}{\Gamma(1-\nu)} \lim_{z \to -\infty} M_{\nu,b}(z) = \lim_{z \to -\infty} \frac{J_{-\nu}(b\sqrt{z})}{J_{\nu}(b\sqrt{z})} \cdot z^\nu$$

$$= - \lim_{x \to +\infty} J_{-\nu}(ib\sqrt{-x}) \cdot (-x)^\nu = - \lim_{x \to +\infty} \left[ \frac{\cos (ib\sqrt{-x} + \frac{\nu\pi}{2} - \frac{\nu}{4})}{\cos (ib\sqrt{-x} - \frac{\nu\pi}{2} - \frac{\nu}{4})} \right] \cdot (-x)^\nu$$

$$= - \frac{e^{-i\nu\pi}}{e^{i\nu\pi}} \lim_{x \to +\infty} x^\nu = - \lim_{x \to +\infty} x^\nu = -\infty.$$  

The case $\nu = 0$.

Applying the asymptotic behavior of the Bessel functions (2.14) to the Weyl function (4.3), we obtain

$$M_{0,b}(-\infty) = \lim_{z \to -\infty} M_{0,b}(z) = \lim_{z \to -\infty} \left[ -\log \left( \frac{\sqrt{z}}{2} \right) + \frac{\pi Y_0(b\sqrt{z})}{2 J_0(b\sqrt{z})} - \gamma \right]$$

$$= \lim_{x \to +\infty} \left[ -\log \left( \frac{i\sqrt{x}}{2} \right) + \frac{\pi Y_0(ib\sqrt{x})}{2 J_0(ib\sqrt{x})} - \gamma \right]$$

$$= \lim_{x \to +\infty} \left[ -\frac{\pi}{2} i - \log(\sqrt{x}) + \frac{\pi}{2} \cdot \frac{\sin (bi\sqrt{x} - \frac{\nu}{4})}{\cos (bi\sqrt{x} - \frac{\nu}{4})} - \gamma \right]$$

$$= \lim_{x \to +\infty} \left[ -\frac{\pi}{2} i - \log(\sqrt{x}) + \frac{\pi}{2} \cdot i - \gamma \right] = -\infty.$$  

So, by Proposition 2.4, relation (4.12) is valid.

(ii.1) First we consider the case $\nu \in (0, 1)$.

Applying the asymptotic behavior of the Bessel functions (2.12) to the Weyl function (4.3), we obtain

$$M_{\nu,b}(0) = \lim_{z \to 0} M_{\nu,b}(z) = \lim_{z \to 0} \left[ \frac{\Gamma(1-\nu)}{2\nu 4^{\nu} \Gamma(1+\nu)} \cdot \frac{J_{-\nu}(b\sqrt{z})}{J_{\nu}(b\sqrt{z})} \cdot z^\nu \right]$$

$$= - \lim_{z \to 0} \left[ \frac{\Gamma(1-\nu)}{2\nu \Gamma(1+\nu) 4^{\nu}} \cdot \frac{\Gamma(1+\nu) 4^{\nu}}{\Gamma(1-\nu)} \cdot b^{-2\nu}z^{-\nu} z^\nu \right] = - \frac{b^{-2\nu}}{2\nu}. \quad (4.14)$$

The first part of relation (4.13) follows from Proposition 2.4.
(ii.2) The case \( \nu = 0 \).

\[
M_{0,b}(0) = \lim_{z \to -0} M_{0,b}(z) = \lim_{z \to -0} \left[ -\log \left( \frac{\sqrt{z}}{2} \right) + \frac{\pi Y_0(b \sqrt{z})}{2 J_0(b \sqrt{z})} - \gamma \right] = \lim_{z \to -0} \left[ -\log \left( \frac{\sqrt{z}}{2} \right) + \frac{2}{\pi} \log \left( \frac{b \sqrt{z}}{2} \right) + \gamma \right] = \log(b).
\]

The second part of relation (4.13) follows from Proposition 2.4. \( \square \)

**Remark 4.5.** We note that by virtue of the formulas (4.12) and (4.13) domain of the Friedrichs extension does not depend on \( b \), and the Krein extension depends.

**Corollary 4.6.** (i) For \( \nu \in (0, 1) \) extension \( A_{\nu,b}h \) is non-negative, \( A_{\nu,b}h \geq 0 \) if and only if

\[
h \geq -\frac{b^{-2\nu}}{2\nu}.
\]

(ii) For \( \nu = 0 \) extension \( A_{0,b}h \) is non-negative, \( A_{0,b}h \geq 0 \) if and only if

\[
h \geq \log(b).
\]

*Proof. (i) By virtue of the Proposition 2.4 (ii), \( A_{\nu,b} \) is the Friedrichs extension. From (4.14) it follows that \( M_{\nu,b}(0) = -\frac{b^{-2\nu}}{2\nu} \) and then, by virtue of the Proposition 2.4 (iii), the extension \( A_{\nu,b}h \) is a non-negative, \( A_{\nu,b}h \geq 0 \) if and only if \( h \geq M_{\nu,b}(0) = -\frac{b^{-2\nu}}{2\nu} \).

(ii) By virtue of the Proposition 2.4 (ii), \( A_{0,b} \) is the Friedrichs extension. From (4.15) it follows that \( M_{0,b}(0) = \log(b) \) and then, by virtue of the Proposition 2.4 (iii), the extension \( A_{0,b}h \) is a non-negative, \( A_{0,b}h \geq 0 \) if and only if \( h \geq M_{0,b}(0) = \log(b) \). \( \square \)

**Remark 4.7.** Note that, for \( \nu \in (0, 1) \), the solution \( x^{1/2+\nu} \in \text{dom}(A_{\nu,b}F) \), while the solution \( x^{1/2-\nu} \notin \text{dom}(A_{\nu,b}F) \). So \( x^{1/2+\nu} \) is the principal solution at 0 (see [8, Def. 11.5]). Similarly, for \( \nu = 0 \), the solution \( x^{1/2} \) is the principal solution at 0, while \( x^{1/2} \log(x) \) is not.

Indeed,

\[
[x^{1/2+\nu}, x^{1/2-\nu}] = \lim_{x \to 0} \left\{ \left( \frac{1}{2} - \nu \right) x^{1/2+\nu} x^{-1/2-\nu} - \left( \frac{1}{2} + \nu \right) x^{1/2-\nu} x^{-1/2+\nu} \right\} = -2\nu \neq 0.
\]

Therefore, by Proposition 4.4 \( x^{1/2-\nu} \notin \text{dom}(A_{\nu,b}F) \).

The case \( \nu = 0 \) is considered similarly.

## 5 The Bessel operator \( A_{\nu,\infty} \) on half-line

Here, we consider the minimal Bessel operator \( A_{\nu,\infty} \) generated by the expression (1.1) in \( L^2(\mathbb{R}_+) \) for \( \nu \in [0, 1) \).

Let \( D_{\text{min}}^2 \) be a minimal differential operator of the 2nd order, generated in \( L^2(\mathbb{R}_+) \) by differential expression \(-d^2/dx^2\),

\[
\text{dom}(D_{\text{min}}^2) = H_0^2(\mathbb{R}_+) = \{ f \in H^2(\mathbb{R}_+) : f(0) = f'(0) = 0 \}.
\]
Theorem 5.1. Let $\nu \in [0,1)$. Then the following assertions hold:

(i) The operator $A_{\nu,\infty}$ has equal deficiency indices $n_{\pm}(A_{\nu,\infty}) = 1$.

(ii) The domain of the operator $A_{\nu,\infty}$ is given by

$$ \text{dom} \left( A_{\nu,\infty} \right) = H_0^2(\mathbb{R}_+) \quad \text{(5.2)} $$

(iii) $A_{\nu,\infty}\text{max} = A_{\nu,\infty}^*$ and

$$ \text{dom} \left( A_{\nu,\infty}^* \right) = \begin{cases} H_0^2(\mathbb{R}_+) + \text{span} \{ x^{1/2+\nu} \xi(x), x^{1/2-\nu} \xi(x) \}, & \nu \in (0,1), \\ H_0^2(\mathbb{R}_+) + \text{span} \{ x^{1/2}\xi(x), x^{1/2}\log(x)\xi(x) \}, & \nu = 0, \end{cases} $$

(5.3)

where $\xi \in C_0^1(\mathbb{R}_+)$ is a some function such that $\xi(x) = 1$ for $x \in [0,1]$.

Proof. (i)–(ii) The function $u \in \tilde{H}_0^2(\mathbb{R}_+)$ admits the integral representation $u(x) = \int_0^x (x-t)u''(t)dt$. Therefore,

$$ Qu(x) = \frac{1}{x^2}u(x) = \frac{1}{x^2} \int_0^x (x-t)u''(t)dt = (QI^2(D_{\min}^2)u)(x). $$

(5.4)

By virtue of Lemma 3.2, this yields

$$ \|Qu\|_2 = \left\| \frac{1}{x^2}u \right\|_2 = \|QT^2D_{\min}^2u\|_2 \leq \|QT^2\|_2 \cdot \|D_{\min}^2u\|_2 $$

$$ = \frac{4}{3}\|D_{\min}^2u\|_2 \leq \frac{4}{3}\|u\|_{H_0^2(\mathbb{R}_+)} \quad \text{(5.5)} $$

It is easy to see that $\nu^2 - \frac{1}{4}$ admits the representation $\nu^2 - \frac{1}{4} = \frac{3}{4}(1 - \varepsilon)$, where $\varepsilon > 0$. Then relation (5.5) implies the estimate

$$ \left\| \left( \nu^2 - \frac{1}{4} \right) Qu \right\|_2 = \left\| \nu^2 - \frac{1}{4} \right\| \cdot \|Qu\|_2 \leq \frac{3}{4}(1 - \varepsilon) \cdot \frac{4}{3}\|u\|_{H_0^2(\mathbb{R}_+,b]} = (1 - \varepsilon)\|u\|_{H_0^2(\mathbb{R}_+,b]} \quad u \in H_0^2[0,b] \quad \text{(5.6)} $$

Estimate (5.6) means that $Q$ is strongly $D_{\min}^2$-bounded. Therefore, by the Kato–Rellich theorem (see [15]) $n_{\pm}(A_{\nu,\infty}) = n_{\pm}(D_{\min}^2) = 1$ and $\text{dom} \left( A_{\nu,\infty} \right) = H_0^2(\mathbb{R}_+)$. (iii) Since

$$ \tau_{\nu} x^{1/2+\nu} \xi(x) = 0, $$

where the equality is understood in the meaning of the theory of distributions, and $x^{1/2+\nu} \xi(x) \in L^2(\mathbb{R}_+)$, then

$$ \{ x^{1/2+\nu} \xi(x), x^{1/2-\nu} \xi(x) \} \subset \text{dom} \left( A_{\nu,\infty}\text{max} \right) = \text{dom} \left( A_{\nu,\infty}^* \right), $$

and $\ker \left( A_{\nu,\infty}^* \right) = \{ x^{1/2+\nu} \xi(x), x^{1/2-\nu} \xi(x) \} \subset L^2(\mathbb{R}_+)$. In addition, it is clear that $H_0^2(\mathbb{R}_+) \subset \text{dom} \left( A_{\nu,\infty}^* \right)$ and dim $\left( H_0^2(\mathbb{R}_+) / \text{dom} \left( A_{\nu,\infty} \right) \right) = 2$. On the other hand, since $n_{\pm}(A_{\nu,\infty}) = 1$, we have dim $\left( \text{dom} \left( A_{\nu,\infty}^* \right) / \text{dom} \left( A_{\nu,\infty} \right) \right) = 2n_{\pm}(A_{\nu,\infty}) = 2$ by the first Neumann formula. Therefore, formula (5.3) is valid.

The case $\nu = 0$ is considered similarly. □

Remark 5.2. In [4, Proposition 4.11] proved that for $0 < \text{Re} \nu < 1$ for $f \in \text{dom}(A_{\nu,\infty})$ the relations $f(x) = o(x^{3/2})$, $f'(x) = o(x^{1/2})$ are valid for $x \to 0$, and for $\nu = 0$ the relations $f(x) = o(x^{3/2}\log(x))$, $f'(x) = o(x^{1/2}\log(x))$ are valid for $x \to 0$, which are easily follow from (5.2).
Next we compute the Weyl function and the corresponding spectral function of the operator $A_{ν,∞}$ using the boundary triplet technique.

**Proposition 5.3.** Let $ν ∈ [0, 1)$. Then:

(i) The boundary triplet of the operator $A_{ν,∞}^*$ can be selected in the form:

\[ \mathcal{H} = \mathbb{C}, \quad Γ_{0,∞}^ν f = [f, x^{\frac{1}{2}+ν}]_0, \quad Γ_{1,∞}^ν f = \begin{cases} -(2ν)^{-1}[f, x^{\frac{2}{2}-ν}]_0, & ν ∈ (0, 1), \\ [f, x^{\frac{1}{2}} log(x)]_0, & ν = 0. \end{cases} \]  

(5.7)

(ii) The corresponding Weyl function $M_{ν,∞}(·)$ has the form:

\[ M_{ν,∞}(z) = \begin{cases} e^{i(1-ν)π} \frac{Γ(1-ν)}{2νΓ(1+ν)} z^ν, & ν ∈ (0, 1), \\ -\log \left( \frac{\sqrt{x}}{2} \right) + \frac{\nu}{2} - γ, & ν = 0, \end{cases} \quad z ∈ \mathbb{C} \setminus \mathbb{R}_+, \]  

where $γ$ is Euler’s constant.

(iii) The spectral function $Σ_ν(t)$ of the operator $A_{ν,∞0} = A_{ν,∞}^* \mid \ker Γ_0^ν$ is given by

\[ Σ_ν(t) = \frac{\nu^ν}{2ν+1} \chi(0,∞)(t). \]  

(5.9)

**Proof.** (i) Let $f, g ∈ \text{dom} (A_{ν,∞}^*)$. Integrating by parts, we obtain

\[ (A_{ν,∞}^ν f, g) - (f, A_{ν,∞}^ν g) = \lim_{ε \to 0} \left[ \int_ε^∞ \left( -f''(x)g(x) + \frac{ν^2 - \frac{1}{4}}{x^2} f(x) \right) g(x) dx - \right. \]

\[ \left. - \int_ε^∞ f(x) \left( -g''(x) + \frac{ν^2 - \frac{1}{4}}{x^2} g(x) \right) dx \right] = \lim_{ε \to 0} \left\{ -f(ε)g'(ε) + f'(ε)g(ε) \right\}. \]

On the other hand, it is easily seen that

\[ (Γ_{1,∞}^ν f, Γ_{0,∞}^ν g) - (Γ_{0,∞}^ν f, Γ_{1,∞}^ν g) = \]

\[ = \frac{1}{2ν} \lim_{x \to 0} \left[ \left( \frac{1}{2} + ν \right) x^{ν-\frac{1}{2}} f(x) - x^{\frac{1}{2}+ν} f'(x) \right] \left( \frac{1}{2} - ν \right) x^{-\frac{1}{2}-ν} g(x) - x^{\frac{1}{2}-ν} g'(x) \]

\[ - \left( \frac{1}{2} - ν \right) x^{-ν-\frac{1}{2}} f(x) - x^{\frac{1}{2}-ν} f'(x) \left( \frac{1}{2} + ν \right) x^{-\frac{1}{2}+ν} g(x) - x^{\frac{1}{2}+ν} g'(x) \]

\[ = \frac{1}{2ν} \lim_{x \to 0} 2ν(f'(x)g(x) - f(x)g'(x)) = \lim_{x \to 0} \left\{ -f(x)g'(x) + f'(x)g(x) \right\}. \]

Comparing this formula with the previous one, we obtain the Green’s formula

\[ (A_{ν,∞}^ν f, g) - (f, A_{ν,∞}^ν g) = (Γ_{1,∞}^ν f, Γ_{0,∞}^ν g) - (Γ_{0,∞}^ν f, Γ_{1,∞}^ν g). \]

The case $ν = 0$ is considered similarly.
First, we consider the case \( \nu \in (0, 1) \).

By the asymptotic relations (2.12) and (2.13), \( x^{1/2}J_\nu(x\sqrt{z}) \in L^2(\mathbb{R}_+) \) and \( x^{1/2}Y_\nu(x\sqrt{z}) \in L^2(\mathbb{R}_+) \). Therefore
\[
f_z(x) = x^{4} \{ J_\nu(x\sqrt{z}) + iY_\nu(x\sqrt{z}) \} \in L^2(\mathbb{R}_+).
\]
(5.10)

It is easily seen that \( \lim_{x \to \infty} f_z(x) = 0 \). So \( f_z \in \text{dom}(A_{\nu, \infty}^*) \) and \( (A_{\nu, \infty}^* - z)f_z = 0 \). In other words, the deficiency space \( \mathcal{R}_z(A_{\nu, \infty}) \) of the operator \( A_{\nu, \infty} \) generated by the vector \( f_z \).

Using the asymptotic behavior of the Bessel functions (2.12) and formula (2.21), we obtain
\[
[x^{1/2}Y_\nu(x\sqrt{z}), x^{1/2+\nu}])_0 = \left[ \frac{x^{1/2}J_\nu(x\sqrt{z}) \cos(\nu\pi) - J_{-\nu}(x\sqrt{z})}{\sin(\nu\pi)} x^{1/2+\nu} \right]_0
= -\frac{\nu^2 \nu^{1+\nu}}{\sin(\nu\pi)\Gamma(1-\nu)} z^{-\nu/2}.
\]
Similarly, using the asymptotic behavior of the Bessel functions (2.12) and formula (2.22), we obtain
\[
[x^{1/2}Y_\nu(x\sqrt{z}), x^{1/2-\nu}])_0 = \left[ \frac{x^{1/2}J_\nu(x\sqrt{z}) \cos(\nu\pi) - J_{-\nu}(x\sqrt{z})}{\sin(\nu\pi)} x^{1/2-\nu} \right]_0
= -\frac{\nu \cos(\nu\pi)}{\sin(\nu\pi)2^{\nu-1}\Gamma(1+\nu)} z^{\nu/2}.
\]
(5.11)
(5.12)

From the formulas (4.5), (5.6), (5.7), (5.10), (5.11) and (5.12), we arrive at the relation
\[
\Gamma^\nu f_z = -\frac{i \nu^{\nu+1}}{\sin(\nu\pi)\Gamma(1-\nu)} z^{-\nu/2}; \quad (5.13)
\]
\[
\Gamma^\nu f_z = \left( 1 + \frac{\cos(\nu\pi)}{\sin(\nu\pi)} \right) \frac{z^{\nu/2}}{2^\nu \Gamma(1+\nu)} = \frac{e^{i(1-\nu)}}{i \sin(\nu\pi)} \frac{z^{\nu/2}}{2^\nu \Gamma(1+\nu)}.
\]
(5.14)

Hence, by (5.13), (5.14), and Definition 2.3, we obtain the first part of the formula (5.8).

(ii.2) The case \( \nu = 0 \).

By the asymptotic relations (2.12) and (2.13), \( x^{1/2}J_0(x\sqrt{z}) \in L^2(\mathbb{R}_+) \) and \( x^{1/2}Y_0(x\sqrt{z}) \in L^2(\mathbb{R}_+) \). Therefore
\[
f_z(x) = x^{4} \{ J_0(x\sqrt{z}) + iY_0(x\sqrt{z}) \} \in L^2(\mathbb{R}_+).
\]
(5.15)

It is easily seen that \( \lim_{x \to \infty} f_z(x) = 0 \). So \( f_z \in \text{dom}(A_{0, \infty}^*) \) and \( (A_{0, \infty}^* - z)f_z = 0 \). In other words, the deficiency space \( \mathcal{R}_z(A_{0, \infty}) \) of the operator \( A_{0, \infty} \) generated by the vector \( f_z \).

From formulas (4.9), (4.10), (5.7) and (5.15), we arrive at the relations
\[
\Gamma^0 f_z = -\frac{2}{\pi} i; \quad (5.16)
\]
\[
\Gamma^0 f_z = 1 + \frac{2i}{\pi} \left[ \log \left( \frac{\sqrt{z}}{2} \right) + \gamma \right].
\]
(5.17)
Hence, by (5.16), (5.17), and Definition 2.3, we get the second part of the formula (5.8).

(iii) Since $M_{\nu,\infty}(t + iy)$ is bounded in the rectangle $(0, \infty) \times (0, y_0)$, its representing measure is absolutely continuous. By Fatou’s Theorem for $\nu \in (0, 1)$

$$
\Sigma'_\nu(t) = \frac{1}{\pi} \text{Im} \ M_{\nu,\infty}(t + i0) = \frac{1}{\pi} \frac{\Gamma(1 - \nu)}{2\nu \Gamma(1 + \nu)} \text{Im} (e^{i(1-\nu)\pi t^\nu})
= \frac{1}{\pi} \frac{\Gamma(1 - \nu)}{2\nu \Gamma(1 + \nu)} t^\nu \text{Im}(e^{i(1-\nu)}) = \frac{(\nu + 1)t^\nu}{2^{2\nu + 1}\Gamma^2(1 + \nu)}.
$$

The case $\nu = 0$ is considered similarly.

Remark 5.4. In addition, for $\nu \in (0, 1)$, the Weyl function $M_{\nu,\infty}(\cdot)$ admits the integral representation

$$
M_{\nu,\infty}(z) = A_{\nu} + \frac{1}{2^{2\nu + 1}\Gamma^2(1 + \nu)} \int_{-\infty}^{\infty} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) t^\nu dt, \quad (5.18)
$$

where

$$
A_{\nu} = -\frac{\Gamma(1 - \nu)}{2\nu \Gamma(1 + \nu)} \cos\left(\frac{\nu\pi}{2}\right).
$$

Similarly, for $\nu = 0$, the Weyl function $M_{0,\infty}(\cdot)$ admits the integral representation

$$
M_{0,\infty}(z) = A_0 + \frac{1}{2} \int_{-\infty}^{\infty} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) dt, \quad (5.19)
$$

where the constant

$$
A_0 = -\frac{\pi}{4} - \gamma + \log(2).
$$

Remark 5.5. For the Bessel operators, the formulas similar to (5.7) have been obtained in [16, Theorem 2] for $\nu \in (0, 1/2) \cup (1/2, 1)$ and in [16, Theorem 3] for $\nu = 0$.

Proposition 5.6. Let $\nu \in [0, 1)$ and $\Pi_{\nu,\infty} = \{\mathcal{H}, \Gamma_{0,\infty}^{\nu}, \Gamma_{1,\infty}^{\nu}\}$ be a boundary triplet for the operator $A_{\nu,\infty}^*$ of the form (5.7). Then:

(i) The domain of the Friedrichs extension $A_{\nu,\infty,F}$ of the operator $A_{\nu,\infty}$ has the form

$$
\text{dom} (A_{\nu,\infty,F}) = \ker (\Gamma_{0,\infty}^{\nu}) = \left\{ f \in \text{dom} (A_{\nu,\infty}^*) : [f, x\frac{1}{2} + \nu]_0 = 0 \right\}. \quad (5.20)
$$

(ii) The domain of the Krein extension $A_{\nu,\infty,K}$ of the operator $A_{\nu,\infty}$ has the form

$$
\text{dom} (A_{\nu,\infty,K}) = \begin{cases} 
\{ f \in \text{dom} (A_{\nu,\infty}^*) : [f, x\frac{1}{2} - \nu]_0 = 0 \}, & \nu \in (0, 1), \\
\{ f \in \text{dom} (A_{0,\infty}^*) : [f, x\frac{1}{2}]_0 = 0 \} = \ker (\Gamma_{0,\infty}^0), & \nu = 0.
\end{cases} \quad (5.21)
$$

In particular, in the case of $\nu = 0$ the Friedrichs and Krein extensions coincide

$$
A_{0,\infty,F} = A_{0,\infty,K}.
$$
Proof. To prove these statements, we use [6].

(i.1) For \( \nu \in (0,1) \),

\[
M_{\nu,\infty}(-\infty) = \lim_{z \to -\infty} M_{\nu,\infty}(z) = \lim_{x \to \infty} \left[ e^{i(1-\nu)\pi} \frac{\Gamma(1-\nu)}{2\nu^4 \Gamma(1+\nu)} \cdot (-x)^\nu \right] = \lim_{x \to \infty} \left[ \frac{1}{(-1)^\nu} \cdot \frac{\Gamma(1-\nu)}{2\nu^4 \Gamma(1+\nu)} \cdot (-1)^\nu x^\nu \right] = -\infty.
\]

By Proposition 2.4, the first part of relation (5.20) is valid.

(ii.1) First, we consider the case \( \nu \in (0,1) \):

\[
M_{\nu,\infty}(0) = \lim_{z \to 0} M_{\nu,\infty}(z) = \lim_{z \to 0} \left[ \frac{i\pi}{2} - \gamma - \log \left( \frac{\sqrt{z}}{2} \right) \right] = \lim_{z \to 0} \left[ \frac{1}{(-1)^\nu} \cdot \frac{\Gamma(1-\nu)}{2\nu^4 \Gamma(1+\nu)} \cdot z^\nu \right] = 0.
\]

By Proposition 2.4, the first part of relation (5.21) is valid.

(ii.2) The case \( \nu = 0 \):

\[
M_{0,\infty}(0) = \lim_{z \to 0} M_{0,\infty}(z) = \lim_{z \to 0} \left[ \frac{i\pi}{2} - \gamma - \log \left( \frac{\sqrt{z}}{2} \right) \right] = \lim_{x \to \infty} \left[ -\gamma - \log \left( \frac{\sqrt{x}}{2} \right) \right] = -\infty.
\]

By Proposition 2.4, the second part of relation (5.21) is valid.

Remark 5.7. In [2] the formulas (5.8), (5.9) and (5.20) were obtained by W.N. Everitt and H. Kalf by using the classical definitions of the Weyl–Titchmarsh function and Friedrichs extension. The formulas (5.20), (5.21) can be found for example in [4]. However, we emphasize that their association with our formula (5.3) gives an explicit description of Friedrichs and Krein extensions.

Corollary 5.8. Let \( \nu \in [0,1) \) and

\[
A_{\nu,\infty} = -y''(x) + \left( \frac{\nu^2 - \frac{1}{4}}{x^2} - \frac{a}{x} \right) y(x)
\]

be the operator on the half-line \( \mathbb{R}_+ \), \( a > 0 \). Then following assertions hold:

(i) The domain of the operator \( A_{\nu,\infty} \) coincides with the domain (5.2) of \( A_{\nu,\infty} \).

(ii) The domain of the operator \( A^*_{\nu,\infty} \) coincides with the domain (5.3) of \( A^*_{\nu,\infty} \).

(iii) The Friedrichs and Krein extensions of the operator \( A^*_{\nu,\infty} \) coincides with the Friedrichs and Krein extensions of \( A^*_{\nu,\infty} \).
Proof. Since perturbations embedded
\[
\left\| \frac{a}{x} f \right\|_{L^2[0,1]} \leq \left( \nu^2 - \frac{1}{4} \right) \left\| \frac{1}{x^2} f \right\|_{L^2[0,1]},
\]
then \( \text{dom}(A_{\nu,\infty}) \supset \text{dom}(A_{\nu,\infty}) \). \qed

Corollary 5.9. (i) Let \( \nu \in (0,1) \). All self-adjoint extensions of the operator \( A_{\nu,\infty} \) described by the formula
\[
A_{\nu,\infty} = A^*_{\nu,\infty} \uparrow \text{dom}(A_{\nu,\infty}), \quad h \in \mathbb{R} \cup \{ \infty \};
\]
\[
\text{dom}(A_{\nu,\infty}) = \{ f \in \text{dom}(A^*_{\nu,\infty}) : [f, x^{\frac{1}{2} - \nu} + 2\nu h x^{\frac{1}{2} + \nu}]_0 = 0 \}. \tag{5.27}
\]

(ii) Let \( \nu \in (0,1) \). Extension \( A_{\nu,\infty} \) is non-negative, \( A_{\nu,\infty} \geq 0 \) if and only if \( h \geq 0 \).

Proof. (i) Using boundary triplet (5.7), we will prove the corollary, by applying Proposition 2.2 (iii).

(ii) By virtue of the Proposition 2.4 (ii), \( A_{\nu,\nu} \) is the Friedrichs extension. From (5.28) it follows that \( M_{\nu,\infty}(0) = 0 \) and then, by virtue of the Proposition 2.4 (iii), the extension \( A_{\nu,\infty} \) is a non-negative, \( A_{\nu,\infty} \geq 0 \) if and only if \( h \geq M_{\nu,\infty}(0) = 0 \). \qed

Theorem 5.10. Let \( \nu \in [0,1) \) and \( A_{\nu,\infty} \) be the Friedrichs extension of the operator \( A_{\nu,\infty} \). Also assume \( \xi \in C^1_0(\mathbb{R}^+) \), \( \xi(x) = \begin{cases} 1, & x \in (0,1/2), \\ 0, & x \geq 3/4. \end{cases} \)

(i) For \( \nu \in (0,1) \) the quadratic form \( a_{\nu,\infty} \) quadratic form associated with the Friedrichs extension \( A_{\nu,\infty} \) takes the form
\[
a_{\nu,\infty}[u] = \int_0^\infty |u'(x)|^2 dx + \left( \nu^2 - \frac{1}{4} \right) \int_0^\infty \frac{|u(x)|^2}{x^2} dx, \tag{5.28}
\]
\[
\text{dom}(a_{\nu,\infty}) = H^1_0(\mathbb{R}^+). \tag{5.29}
\]

(ii) For \( \nu = 0 \) the quadratic form \( a_{0,\infty} \) quadratic form associated with the Friedrichs extension \( A_{0,\infty} \) takes the form
\[
a_{0,\infty}[u] = \int_0^\infty \left| u'(x) - \frac{u(x)}{2x} \right|^2 dx, \tag{5.30}
\]
\[
\text{dom}(a_{0,\infty}) \supset H^1_0(\mathbb{R}^+) + \text{span} \left\{ x^{\frac{1}{2}} |\log(x)|^{-\alpha} \xi(x) : 0 < \alpha \leq \frac{1}{2} \right\}. \tag{5.31}
\]

Wherein \( \dim(\text{dom}(a_{0,\infty}) / H^1_0(\mathbb{R}^+)) = \infty \).

(iii) For \( \nu \in [0,1) \) the domain of the Friedrichs extension \( A_{\nu,\infty} \) takes the form
\[
\text{dom}(A_{\nu,\infty}) = A^*_{\nu,\infty} \uparrow \text{domd}(A_{\nu,\infty}), \quad \text{dom}(A_{F}(\nu; \infty)) = H^1_0(\mathbb{R}^+) + \text{span} \{ x^{\frac{1}{2} + \nu} \xi(x) \}. \tag{5.32}
\]

(iv) For the quadratic form \( a_{\nu,\infty} \) associated with the operator \( A_{\nu,\infty} \) the following decomposition is valid
\[
\text{dom}(a_{\nu,\infty}) = H^1_0(\mathbb{R}^+) + \text{span} \{ x^{1/2-\nu} \xi(x) \}, \quad \nu \in (0,1). \tag{5.33}
\]
Proof. (i) By Hardy’s inequality for \( \nu \in (0, 1) \), \( u \in H^1_0(\mathbb{R}_+) \), we have

\[
\mathbf{a}_{\nu, \infty}[u] = \|u'(t)\|^2_2 + (\nu^2 - 1/4) \int_0^\infty \frac{|u(t)|^2}{t^2} dt
\leq \|u'(t)\|^2_2 (1 + |4\nu^2 - 1|), \quad u \in H^1_0(\mathbb{R}_+). \tag{5.34}
\]

Thus \( H^1_0(\mathbb{R}_+) \subset \operatorname{dom}(\mathbf{a}_{\nu, \infty}) \).

We prove the converse inequality. Suppose firstly that \( \nu \in [1/2, 1) \). Then, for \( u \in H^1_0(\mathbb{R}_+) \),

\[
\mathbf{a}_{\nu, \infty}[u] = \|u'(t)\|^2_2 + (\nu^2 - 1/4) \int_0^\infty \frac{|u(t)|^2}{t^2} dt \geq \|u'(t)\|^2_2, \quad u \in H^1_0(\mathbb{R}_+). \tag{5.35}
\]

If \( \nu \in (0, 1/2) \), then for \( u \in H^1_0(\mathbb{R}_+) \) applying the Hardy’s inequality we obtain

\[
\mathbf{a}_{\nu, \infty}[u] = \|u'(t)\|^2_2 - (1/4 - \nu^2) \int_0^\infty \frac{|u(t)|^2}{t^2} dt \\
\geq \|u'(t)\|^2_2 + (4\nu^2 - 1)\|u'(t)\|^2_2 = 4\nu^2\|u'(t)\|^2_2. \tag{5.36}
\]

So, the energy norm of \( A_{\nu, \infty} \) on \( H^1_0(\mathbb{R}_+) \) is equivalent to the norm of the space \( H^1_0(\mathbb{R}_+) \). Since \( H^2_0(\mathbb{R}_+) = \operatorname{dom}(A_{\nu, \infty}) \) is dense in the energy space of the operator \( A_{\nu, \infty} \), then \( \operatorname{dom}(\mathbf{a}_{\nu, \infty}) \) and \( H^1_0(\mathbb{R}_+) \) coincide algebraically and topologically.

(ii) Let \( u_\alpha(x) = x^{1/2} |\log(x)|^{-\alpha} \xi(x) \), then

\[
\mathbf{a}_{0, \infty}[u_\alpha] = \int_0^{1/2} \left| u'_\alpha(x) - \frac{u_\alpha(x)}{2x} \right|^2 dx = \frac{\alpha^2 2^{2\alpha + 1}}{2\alpha + 1}.
\]

So \( \{x^{1/2} |\log(x)|^{-\alpha} \xi(x)\} \subset \operatorname{dom}(\mathbf{a}_{0, \infty}) \).

Let functions \( x^{1/2} |\log(x)|^{-\alpha} \xi(x) \) are linearly independent.

Conversely, \( \sum_{j=1}^n C_j x^{1/2} |\log(x)|^{-\alpha_j} \xi(x) = 0 \), for \( \alpha_j \in (0, 1/2] \), \( x \in (0, 1) \). We order degrees: \( \alpha_1 < \alpha_2 < \ldots < \alpha_n \). Then multiplying by the term with the smallest degree, we obtain

\[
C_1 + \sum_{j=2}^n |\log(x)|^{-\alpha_j + \alpha_1} = 0.
\]

Thus \( C_1 = 0 \).

Similarly, we obtain that \( C_j = 0 \). This is a contradiction.

(iii) We note that \( H^2_0(\mathbb{R}_+) \subset H^1_0(\mathbb{R}_+) \). If \( u(x) = x^{1/2+\nu} \xi(x) \) then \( u'(\cdot) \in L^2(\mathbb{R}_+) \), but \( u(\cdot) \not\in \operatorname{dom}(A_{\nu, \infty}) = H^2_0(\mathbb{R}_+) \). By the construction of the Friedrichs extension and the equalities \( \mathbf{a}_{\nu, \infty} \), we obtain

\[
\operatorname{dom}(A_{\nu, \infty, F}) = \operatorname{dom}(A^*_{\nu, \infty}) \cap \operatorname{dom}(\mathbf{a}_{\nu, \infty}[u]) = \operatorname{dom}(A^*_{\nu, \infty}) \cap H^1_0(\mathbb{R}_+) = \mathbb{H}^2_0(\mathbb{R}_+) + \operatorname{span}\{x^{1/2+\nu} \xi(x)\}.
\]

(iv) The proof follows from \( \square \)
Proposition 6.1. Let \( \nu \in (0, 1) \) be treated similarly. Namely, if \( [f, x^{\frac{1}{2}+\nu}]_0 = 0 \), then for \( f = x^{\frac{1}{2}}\xi(x) \), we obtain
\[
[f, x^{\frac{1}{2}+\nu}]_0 = \lim_{x \to 0} \left( \left( \frac{1}{2} + \nu \right) x^{2\nu} \xi(x) - \left( \frac{1}{2} + \nu \right) x^{2\nu} \xi(x) - x^{1+2\nu} \xi'(x) \right) = 0,
\]
where \( \xi \in C^1_0(\mathbb{R}_+) \), \( \xi(x) = 1 \) for \( x \in [0, 1] \).

6 Connection of the Weyl functions of the operators \( A_{\nu,b} \) and \( A_{\nu,\infty} \)

**Proposition 6.1.** Let \( A_{\nu,b} \) and \( A_{\nu,\infty} \) be the operators with domains (4.1) and (5.2), respectively. Assume that \( \Pi_{\nu,b} \) and \( \Pi_{\nu,\infty} \) be the boundary triplets of the form (4.2) and (5.7), \( M_{\nu,b}(z) \) and \( M_{\nu,\infty}(z) \) be the Weyl functions of the form (4.3) and (5.8). Then the relation
\[
\lim_{b \to +\infty} M_{\nu,b}(z) = M_{\nu,\infty}(z)
\]
holds uniformly on compact subsets of \( \mathbb{C}_+ \).

**Proof.** First, we consider the case \( \nu \in (0, 1) \). Since the Bessel functions \( J_\nu(t) \) and \( J_{-\nu}(t) \) for \( t \to \infty \) have the asymptotic behavior (2.14), we have
\[
\lim_{b \to +\infty} M_{\nu,b}(z) = - \lim_{b \to +\infty} \frac{\Gamma(1 - \nu)}{2\nu 4^\nu \Gamma(1 + \nu)} \cdot \frac{J_{-\nu}(b\sqrt{z})}{J_\nu(b\sqrt{z})} \cdot z^\nu =
\]
\[
= - \lim_{b \to +\infty} \left[ \frac{\Gamma(1 - \nu)}{2\nu 4^\nu \Gamma(1 + \nu)} \cdot \frac{\cos \left( b\sqrt{z} + \frac{\nu \pi}{2} - \frac{\pi}{4} \right)}{\cos \left( b\sqrt{z} - \frac{\nu \pi}{2} - \frac{\pi}{4} \right)} \cdot z^\nu \right] =
\]
\[
= - \frac{\Gamma(1 - \nu)}{2\nu 4^\nu \Gamma(1 + \nu)} \lim_{b \to +\infty} e^{-i(b\sqrt{z} + \frac{\nu \pi}{2} - \frac{\pi}{4})} \cdot e^{i(1-\nu)\pi} \frac{\Gamma(1 - \nu)}{2\nu 4^\nu \Gamma(1 + \nu)} \cdot z^\nu = M_{\nu,\infty}(z).
\]
The case \( \nu = 0 \) is treated similarly. Namely,
\[
\lim_{b \to +\infty} M_{0,b}(z) = \lim_{b \to +\infty} \left[ - \log \left( \frac{\sqrt{z}}{2} \right) + \frac{\pi Y_0(b\sqrt{z})}{2 J_0(b\sqrt{z})} - \gamma \right] =
\]
\[
= \lim_{b \to +\infty} \left[ - \log \left( \frac{\sqrt{z}}{2} \right) + \frac{\pi}{2} \cdot \frac{\sin \left( b\sqrt{z} - \frac{\pi}{4} \right)}{\cos \left( b\sqrt{z} - \frac{\pi}{4} \right)} - \gamma \right] = - \log \left( \frac{\sqrt{z}}{2} \right) + \frac{\pi}{2} - \gamma = M_{\nu,\infty}(z).
\]
It is easily seen that the convergence in both relations is uniform on compact subsets. \( \square \)
7 Singular Sturm-Liouville operators of the Bessel type

Here, we consider the Sturm-Liouville differential expression

\[ \tau u := -u'' + qu \]  \hspace{1cm} (7.1)

in \( L^2(\mathbb{R}_+) \) with certain potentials \( q \).

The minimal operator \( T_{\min} = T \) associated with (7.1) is the closure of the operator \( T' \) of the form

\[ T'u := \tau u, \quad \text{dom}(T') = \{ u : u \in D, u \text{ has the compact support in } (0, \infty) \}, \]  \hspace{1cm} (7.2)

where

\[ D := \{ u : u \in AC_{\text{loc}}(\mathbb{R}_+) \cap L^2(\mathbb{R}_+), u' \in AC_{\text{loc}}(\mathbb{R}_+), \tau u \in L^2(\mathbb{R}_+) \}; \]  \hspace{1cm} (7.3)

and \( T \) is a densely defined symmetric operator.

The maximal operator associated with (7.1) is \( T_{\max} = T^* = \tau \upharpoonright D \). \hspace{1cm} (7.4)

The following relations hold:

\[ T_{\min} = T = T'* = T''_{\max} = T_{\max}. \]

**Corollary 7.1.** Let \( q \in L^1_{\text{loc}}(\mathbb{R}_+) \) and

\[ q(x) \geq \frac{\beta}{x^2} - \mu, \quad (x \in \mathbb{R}_+) \]  \hspace{1cm} (7.5)

for some \( \beta > -\frac{1}{4} \) and \( \mu \geq 0 \). Then

(i) The closure \( t_q \) of the quadratic form \( t_q' \) associated with the operator \( T \) is

\[ t_q[u] = \int_0^\infty |u'(x)|^2dx + \int_0^\infty q(x) \cdot |u(x)|^2dx, \]  \hspace{1cm} (7.6)

\[ \text{dom}(t_q) = \{ u \in H^1_0(\mathbb{R}_+) : \int_0^\infty q(x) \cdot |u(x)|^2dx < \infty \} =: H^1_0(\mathbb{R}_+; q). \]

(ii) \hspace{1cm} The domain of the Friedrichs extension \( T_F \) of \( T \) is

\[ \text{dom}(T_F) = D \cap H^1_0(\mathbb{R}_+; q), \]  \hspace{1cm} (7.7)

where \( D \) is given by (7.3).

**Proof.** Without loss of generality, we can assume that \( \mu = 0 \). Let \( \beta = \nu^2 - \frac{1}{4} > -\frac{1}{4} \). Consider the quadratic form \( t_q \) associated with the operator \( T_F \). Since \( q(x) > \frac{\nu^2 - \frac{1}{4}}{x^2} \), we have

\[ \text{dom}(t_q') \subset \text{dom}(a_{\nu,\infty}) = H^1_0(\mathbb{R}_+), \]  \hspace{1cm} (7.8)
where \( a_{\nu, \infty} \) is given by (3.17).

Further, let \( u(\cdot) \in C^\infty_0(\mathbb{R}_+) \subset \text{dom} (T') \). Integrating by parts, we obviously have

\[
\mathbf{t}_q'[u] = (Tu, u) = \lim_{x \to \infty} \left[ u'(x)u(0) + \int_0^x u'(t)^2 dt + \int_0^x q(t) \cdot |u(t)|^2 dt \right]
\]

\[= \int_0^\infty |u'(x)|^2 dx + \int_0^\infty q(x) \cdot |u(x)|^2 dx.
\]

Taking the closure of these forms and (7.8) into account, we arrive at (7.6).

According to the construction of the Friedrichs extension and (7.3), we get

\[
\text{dom} (T_F) = \text{dom} (T^*) \cap \text{dom} (t_q) = \mathcal{D} \cap H^1_0(\mathbb{R}_+; q).
\]

The Corollary is proved.

\[\square\]

**Corollary 7.2.** Let \( q \in L^1_{\text{loc}}(\mathbb{R}_+) \) and

\[ q(x) \geq -\frac{1}{4x^2} - \mu, \quad (x \in \mathbb{R}_+) \]

for some \( \mu \geq 0 \). Then

(i) The closure \( t_q \) of the quadratic form \( t_q' \) associated with the operator \( T \) takes the form

\[
t_q[u] = \int_0^\infty |u'(x)|^2 dx + \int_0^\infty q(x) \cdot |u(x)|^2 dx,
\]

\[\text{dom} (t_q) = H^1_0(\mathbb{R}_+; q) + \text{span} \{ x^{1/2} \xi(x) \} \],

where \( \xi \in C^1_0(\mathbb{R}_+) \), \( \xi(x) = 1 \), for \( x \in [0, 1] \).

(ii) \( [14] \) The domain of the Friedrichs extension of \( T \) is

\[
\text{dom} (T_F) = \mathcal{D} \cap (H^1_0(\mathbb{R}_+; q) + \text{span} \{ x^{1/2} \xi(x) \})
\]

where \( \mathcal{D} \) is given by (7.3).

The proof is similarly to Corollary 7.1.

**Corollary 7.3.** Another description of the Friedrichs extension was obtained by H. Kalf in \( [17] \)

\[
\text{dom} (T_F) = \left\{ u : u \in \mathcal{D}, \quad \int_0^\infty \left| u' - \frac{u}{2x} \right| < \infty \right\}
\]

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