An HDG Method for Tangential Boundary Control of Stokes Equations I: High Regularity

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Abstract

We propose a hybridizable discontinuous Galerkin (HDG) method to approximate the solution of a tangential Dirichlet boundary control problem for the Stokes equations with an $L^2$ penalty on the boundary control. The contribution of this paper is twofold. First, we obtain well-posedness and regularity results for the tangential Dirichlet control problem on a convex polygonal domain. The analysis contains new features not found in similar Dirichlet control problems for the Poisson equation; an interesting result is that the optimal control has higher local regularity on the individual edges of the domain compared to the global regularity on the entire boundary. Second, under certain assumptions on the domain and the target state, we prove a priori error estimates for the control for the HDG method. In the 2D case, our theoretical convergence rate for the control is superlinear and optimal with respect to the global regularity on the entire boundary. We present numerical experiments to demonstrate the performance of the HDG method.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a Lipschitz polyhedral domain with boundary $\Gamma = \partial \Omega$. For a given target state $y_d$, we consider the following unconstrained Dirichlet boundary control problem for the Stokes equations:

$$
\min_{u \in U} J(u), \quad J(u) := \frac{1}{2} \| y - y_d \|^2_{L^2(\Omega)} + \frac{\gamma}{2} \| u \|^2_U,
$$

subject to

$$
-\Delta y + \nabla p = f \quad \text{in } \Omega,
\nabla \cdot y = 0 \quad \text{in } \Omega,
\n y = u \quad \text{on } \Gamma,
\n\int_{\Omega} p = 0,
$$

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where $U \subset L^2(\Gamma)$ is the control space and $\gamma > 0$ is a fixed constant.

Control of fluid flows modeled by the Stokes or Navier-Stokes equations is an important and active area of interest. After the pioneering works by Glowinski and Lions [26] and Gunzburger [25, 32, 38], many important developments have been made both theoretically and computationally in the past decades. For an extensive body of literature devoted to this subject we refer to, e.g., [3, 4, 7, 11, 23, 24, 39, 40, 50, 55, 62, 65, 70–72] and the references therein. Despite the large amount of existing work on numerical methods for fluid flow control problems, we are not aware of any contributions to the analysis and approximation of the tangential Stokes Dirichlet boundary control problem. Work on this problem is an essential step towards the analysis and approximation of similar Dirichlet boundary control problems for the Navier-Stokes equations and other fluid flow models.

In this work, we focus on the case where the control acts tangentially along the boundary through a Dirichlet boundary condition. This scenario has broad applications to optimal mixing and heat transfer problems. Omari and Guer in [60] conducted a numerical study of the effect of wall rotation on the enhancement of heat transport in the whole fluid domain. Gouillart et al. in [29–31, 68] studied in detail this crucial effect of moving wall on the mixing efficiency for the homogenization of concentration in a 2D closed flow environment. These problems naturally lead to the study of tangential boundary control and optimization of fluid flows. Recently, Hu and Wu in [41–43, 49] provided rigorous mathematical approaches for optimal mixing and heat transfer via an active control of Stokes and Navier-Stokes flows through Navier slip boundary conditions. Other tangential boundary control problems for fluid flows have been considered by Barbu, Lasiecka and Triggiani [5, 6, 53, 54] and Osses [61]. However, the authors are not aware of any existing work on approximation and numerical analysis for these problems.

Discontinuous Galerkin (DG) methods are widely used for fluid flow problems, since they can capture shocks and large gradients in solutions. However, most existing DG methods are commonly considered to have a major drawback: the memory requirement and computational cost of DG methods are typically much larger than the standard finite element method.

Hybridizable discontinuous Galerkin (HDG) methods were proposed by Cockburn et al. in [16] as an improvement of traditional DG methods. The HDG methods are based on a mixed formulation and utilize a numerical flux and a numerical trace to approximate the flux and the trace of the solution. The approximate flux and solution variables can be eliminated element-by-element. This process leads to a global equation for the approximate boundary traces only. As a result, HDG methods have significantly less globally coupled unknowns, memory requirement, and computational cost compared to other DG methods. Furthermore, HDG methods have been successfully applied to flow problems [14, 17, 18, 58, 64, 69], distributed optimal control problems [46, 48, 73], and Dirichlet boundary control problems [44, 45, 17].

For the Stokes tangential Dirichlet boundary control problem considered here, the Dirichlet boundary data $u \in L^2(\Gamma)$ takes the form $u = u\tau$, where $u$ is the control and $\tau$ is the unit tangential vector to the boundary. Formally, the optimal control $u \in L^2(\Gamma)$ and the optimal state
\[ y \in L^2(\Omega) \] minimizing the cost functional satisfy the optimality system

\[
\begin{align*}
-\Delta y + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot y &= 0 \quad \text{in } \Omega, \\
y &= u \tau \quad \text{on } \Gamma, \\
-\Delta z - \nabla q &= y - y_d \quad \text{in } \Omega, \\
\nabla \cdot z &= 0 \quad \text{in } \Omega, \\
z &= 0 \quad \text{on } \Gamma, \\
\partial_n z &= \gamma u \tau \quad \text{on } \Gamma.
\end{align*}
\]

We use an HDG method to approximate the solution of a mixed formulation of this optimality system. To do this, we first analyze the control problem for 2D convex polygonal domains in Section 2. We give precise meaning to the state equation (1.3a) for Dirichlet boundary data in \( L^2(\Gamma) \), and prove well-posedness and regularity results for the optimality system (1.3). The theoretical results for this problem share some similarities to results for Dirichlet boundary control of the Poisson equation on a 2D convex polygonal domain \cite{1}; however, there are new components to the analysis due to the mixed formulation and the regularity results for Stokes equations on polygonal domains \cite{20}. An interesting feature of our theoretical results is that the optimal control has higher local regularity (on each boundary edge) than global regularity (on the entire boundary \( \Gamma \)).

For the HDG method, we use polynomials of degree \( k+1 \) to approximate the velocity \( y \) and dual velocity \( z \), and polynomials of degree \( k \geq 0 \) for the fluxes \( L = \nabla y \) and \( G = \nabla z \), pressure \( p \) and dual pressure \( q \). Moreover, we also use polynomials of degree \( k \) to approximate the numerical trace of the velocity and dual velocity on the edges of the spatial mesh, which are the only globally coupled unknowns. We describe the HDG method and its implementation in detail in Section 3.

In Section 4, we prove a superlinear rate of convergence for the control in 2D under certain assumptions on the largest angle of the convex polygonal domain and the smoothness of the desired state \( y_d \). Similar superlinear convergence results for Dirichlet boundary control of the Poisson equation have been obtained in \cite{2, 13, 25, 44, 45, 47}. To give a specific example of our results, for a rectangular 2D domain, \( y_d \in H^2(\Omega) \), and \( k = 1 \), we obtain the following a priori error bounds for the velocity \( y \), adjoint velocity \( z \), their fluxes \( L \) and \( G \), pressure \( p \) and dual pressure \( q \) and the optimal control \( u \):

\[
\begin{align*}
\| y - y_h \|_{0, \Omega} &= O(h^{3/2-\varepsilon}), \\
\| L - L_h \|_{0, \Omega} &= O(h^{1-\varepsilon}), \\
\| p - p_h \|_{0, \Omega} &= O(h^{1-\varepsilon}), \\
\| z - z_h \|_{0, \Omega} &= O(h^{3/2-\varepsilon}), \\
\| G - G_h \|_{0, \Omega} &= O(h^{3/2-\varepsilon}), \\
\| q - q_h \|_{0, \Omega} &= O(h^{3/2-\varepsilon}),
\end{align*}
\]

and

\[
\| u - u_h \|_{0, \Gamma} = O(h^{3/2-\varepsilon}),
\]

for any \( \varepsilon > 0 \). The rate of convergence for the control \( u \) is optimal in the sense of the maximal global regularity of the control \( u \in H^{3/2-\varepsilon}(\Gamma) \). However, the numerical results presented in Section 5 show higher convergence rates than the rates predicted by our numerical analysis; this phenomenon might be caused by the higher local regularity of the optimal control mentioned above. The numerical convergence rates observed here are different than typical numerical results for Dirichlet boundary control of the Poisson equation.
We emphasize that the HDG method in this work is usually considered to be a superconvergent method. Specifically, if polynomials of degree \(k \geq 1\) are used for the numerical trace and the solution of the PDEs is smooth enough, then \(O(h^{k+2})\) error estimates can be obtained for the state variable; see, e.g., [48, 63, 64]. Hence, from the viewpoint of globally coupled degrees of freedom, this method achieves superconvergence for the scalar variable. For Dirichlet boundary control problems, to obtain the superlinear convergence rate, one usually needs a superconvergence mesh or higher order elements for the standard finite element method, see, e.g., [2, 22]. However, the HDG method considered here achieves the superlinear convergence rate without any special considerations.

In the second part of this work [27], we complete the numerical analysis of this HDG method for low regularity solutions of the optimality system. Specifically, we remove the assumptions made here on the boundary angles and the regularity of the target state. In this more general scenario, the flux \(\mathbb{L}\) and the pressure \(p\) may not have a well-defined superconvergent convergence rate, one usually needs a superconvergence mesh or higher order elements for the standard finite element method, see, e.g., [2, 22]. However, the HDG method achieves superconvergence for the scalar variable. For Dirichlet boundary control problems, \(\mathbb{V}_0^s(\Omega)\) is the subspace of \(\Pi^m_{i=1}H^s(\Gamma_i)\) satisfying certain compatibility conditions on the corners; see [22, Theorem 1.5.2.8]. For \(s = 3/2\), this definition would lead to ambiguities. Following [66, Section 2.1] we introduce the spaces

\[
\mathbb{V}^s(\Omega) = \{ y \in H^s(\Omega) : \nabla \cdot y = 0, [y \cdot \mathbf{n}, 1]_{\Gamma} = 0 \}, \quad \text{for } s \geq 0,
\]

\[
\mathbb{V}^s_0(\Omega) = \{ y \in H^s(\Omega) : \nabla \cdot y = 0, y = 0 \text{ on } \Gamma \}, \quad \text{for } s > 1/2,
\]

\[
\mathbb{V}^s(\Gamma) = \{ u \in H^s(\Gamma) : (u \cdot \mathbf{n}, 1)_\Gamma = 0 \}, \quad \text{for } 0 \leq s < 3/2.
\]
For $-3/2 < s < 0$, $V^s(\Gamma)$ is the dual space of $V^{-s}(\Gamma)$. For $s < -1/2$, $V^s(\Omega)$ is the dual space of $V_0^{-s}(\Omega)$ and for $-1/2 \leq s < 0$, $V^s(\Omega)$ is the dual space of $V^{-s}(\Omega)$.

Consider a target state $y_d \in H$, where $H \hookrightarrow V^0(\Omega)$ is a function space that will be specified later, and a Tykhonov regularization parameter $\gamma > 0$. Consider also a space $U \hookrightarrow V^0(\Gamma)$. We are interested in the optimal control problem

$$
\min_{u \in U} J(u) = \frac{1}{2} \| y_u - y_d \|^2_H + \frac{\gamma}{2} \| u \|^2_U,
$$

where $y_u \in V^0(\Omega)$ is the unique solution in the transposition sense of the Stokes system (see Definition 2.3 below)

$$
\begin{align*}
-\Delta y + \nabla p &= 0 \quad \text{in } \Omega, \\
\nabla \cdot y &= 0 \quad \text{in } \Omega, \\
y &= u \quad \text{on } \Gamma, \\
(p, 1)_\Omega &= 0.
\end{align*}
$$

Different choices of the spaces $H$ and $U$ appear in the related literature for Dirichlet control of Stokes and Navier-Stokes equations. In the early reference [34], $H = L^4(\Omega)$ and $U = V^1(\Gamma)$. The natural space for the controls to obtain a variational solution of the state equation (2.1) is $V^{1/2}(\Gamma)$. This is the choice in [21]. In that work, nevertheless, the Tykhonov regularization is done in the norm of $L^2(\Gamma)$. To prove existence of solution, the tracking is done in the space $H = V^1(\Omega)$. In the reference [50], the authors work in a smooth domain with $H = V^0(\Omega)$ and $U = V^0(\Gamma)$. This choice involves a harder analysis, but leads to an optimality system easier to handle. In polygonal domains, this approach leads to optimal controls that are discontinuous at the corners.

We assume throughout this work that the tracking term for the state is measured in the $L^2(\Omega)$ norm. We investigate the case $U = \{ u_\tau : u \in L^2(\Gamma) \}$, which corresponds to tangential boundary control; see [5,6]. We first precisely define the concept of solution for Dirichlet data in $V^0(\Gamma)$, prove precise regularity results, and use them to introduce a mixed formulation of the problem adequate for HDG methods.

### 2.1 Regularity results

The definition of very weak solution for data in $V^0(\Gamma)$ was introduced in [19, Appendix A] and is valid in convex polygonal domains; see also [67] and [50, Definition 2.1] for a similar definition for the Navier-Stokes equations and smooth domains. Also in smooth domains, very weak solutions can be defined for data in $V^{-1/2}(\Gamma)$; see [66, Appendix A]. We will show how to extend the concept to problems posed on nonconvex polygonal domains for data in $V^s(\Gamma)$ for some negative $s$. We will also prove that the optimal regularity $V^{s+1/2}(\Omega)$ expected for the solution can be achieved. In [56] a similar result is provided for convex polygonal domains, but only suboptimal regularity $V^{s+1/2-\epsilon}(\Omega)$ for all $\epsilon > 0$ is proved. We obtain a result comparable to the one given in [66, Appendix A] for smooth domains.

Let $\omega$ denote the greatest interior angle of $\Gamma$. Following [20, Theorem 5.5], we know there exists a number $\xi = \xi(\omega) \in (0.5, 4]$ that gives the maximal $H^s(\Omega)$ regularity for the problem (2.2). This means for very smooth $f$ and $h$ satisfying the compatibility condition $(h, 1)_\Omega = 0$ we can only expect that the variational solution $(z_{f,h}, q_{f,h}) \in H^1_0(\Omega) \times L^2_0(\Omega)$ of the compressible Stokes
problem

\[ -\Delta z + \nabla q = f \quad \text{in} \ \Omega, \]
\[ \nabla \cdot z = h \quad \text{in} \ \Omega, \]
\[ z = 0 \quad \text{on} \ \Gamma, \]
\[ (q, 1)_{\Omega} = 0, \]

(2.2)

satisfies \( z \in H^{3/2+s}(\Omega) \) and \( q \in H^{1/2+s}(\Omega) \) for \( s < \xi - 1/2 \). This singular exponent \( \xi \) is the smallest real part of all of the roots \( \lambda \) of the equation

\[ \frac{\sin^2(\lambda \omega) - \lambda^2 \sin^2 \omega}{\lambda^2(\lambda - 1)} = 0, \]

(2.3)

and satisfies that \( \omega \mapsto \xi \) is strictly decreasing, \( \xi > \pi/\omega \) if \( \omega < \pi \), and \( 0.5 < \xi < \pi/\omega \) if \( \omega > \pi \).

Let us denote

\[ s^* = \min\{\xi - 1/2, 1/2\}. \]

If \( f \in L^2(\Omega) \) and \( h \in H^1(\Omega) \) such that \( (h, 1)_\Omega = 0, \) [20, Theorem 5.5(a)] states that for all

\[ 0 < s < s^* = \min\{\xi - 1/2, 1/2\} \]

the solution of \( (2.2) \) satisfies \( z_{f,h} \in H^{3/2+s}(\Omega) \) and \( q_{f,h} \in H^{1/2+s}(\Omega) \). Moreover, we have that

\[ \|z_{f,h}\|_{H^{3/2+s}(\Omega)} + \|q_{f,h}\|_{H^{1/2+s}(\Omega)/\mathbb{R}} \leq C(\|f\|_{H^{t-1/2}(\Omega)} + \|h\|_{H^{t+1/2}(\Omega)/\mathbb{R}}). \]

(2.4)

Notice that although the pressure is uniquely determined as a function with the condition \( (q, 1)_\Omega = 0 \), the norm must be taken modulo constant functions. Another remarkable fact is that this result holds for \( s < 1/2 \). This means, in particular, that in nonconvex domains one cannot expect in general to have \( H^2(\Omega) \) regularity of \( z \).

**Remark 2.1.** To obtain this \( H^2(\Omega) \) regularity, an additional condition must be made on the divergence of \( z \). If, e.g., \( h \in H^1_0(\Omega) \), \( (h, 1)_\Omega = 0 \), then it follows from [20, Theorem 5.5(c)] or early reference [52] for convex polygonal domains that the result also holds for \( s = s^* \). In particular, in a convex domain we have \( z \in H^2(\Omega) \).

This fact was used both in [19] and in [56] to define very weak solutions in polygonal domains using \( h \in H^1_0(\Omega) \) as a test function. Although the approach works to define the transposition solution, it leads only to suboptimal regularity results for the solution of the Dirichlet problem.

For later reference, we state the regularity result for the case \( h \equiv 0 \) in a convex domain:

**Theorem 2.2.** [20, Theorem 5.5(b)] Suppose \( \Omega \) is convex and consider \( f \in H^{t-1}(\Omega) \) for some \( 1 \leq t < \xi \). Then, the unique solution of the incompressible Stokes problem

\[ -\Delta z + \nabla q = f \quad \text{in} \ \Omega, \]
\[ \nabla \cdot z = 0 \quad \text{in} \ \Omega, \]
\[ z = 0 \quad \text{on} \ \Gamma, \]
\[ (q, 1)_{\Omega} = 0. \]

(2.5)

satisfies \( z \in H^{t+1}(\Omega) \), \( q \in H^{t}(\Omega) \) and

\[ \|z\|_{H^{t+1}(\Omega)} + \|q\|_{H^{t}(\Omega)/\mathbb{R}} \leq C\|f\|_{H^{t-1}(\Omega)}. \]
Although we will pose our control problem for data in \( u \in V^0(\Gamma) \), the precise regularity results for the state equation will follow by interpolation; therefore we need a definition of very weak solution for data in \( u \in V^{-s}(\Gamma) \) for \( 0 < s < s^\ast \). The elements of this space do not always satisfy a condition analogous to \( \langle u \cdot n, 1 \rangle_\Gamma = 0 \), i.e., we may have \( [u, n]_\Gamma \neq 0 \), and it is necessary to take this into account to define a solution in the transposition sense. Following [66, Eq. (2.2)], we define for \((z, q) \in H^{3/2+s}(\Omega) \times H^{1/2+s}(\Omega), \) \( s > 0 \), the constant
\[
c(z, q) = \frac{1}{|\Gamma|} \langle q - \partial_n z \cdot n, 1 \rangle_\Gamma. \tag{2.6}
\]
This constant satisfies the relation
\[
\|\partial_n z - qn\|_{L^2(\Omega)/\mathbb{R}} = \|\partial_n z - qn + c(z, q)n\|_{L^2(\Gamma)}.
\]
Using this fact, usual trace theory and (2.4), we have that for \( 0 \leq s < 1/2 \)
\[
\|\partial_n z_{f,h} - q_{f,h} \cdot n + c(z_{f,h}, q_{f,h})n\|_{H^s(\Gamma)} \leq C(\|f\|_{H^{s-1/2}(\Omega)} + \|h\|_{H^{s+1/2}(\Omega)/\mathbb{R}}). \tag{2.7}
\]
The following definition makes sense:

**Definition 2.3.** Consider \( 0 \leq s < s^\ast \) and \( u \in V^{-s}(\Gamma) \). We say \( y_u \in V^0(\Omega), p_u \in (H^1(\Omega)/\mathbb{R})' \) is a solution in the transposition sense of (2.1) if \((y_u, p_u)\) satisfy
\[
(y, f)_\Omega - [p, h]_\Omega = [u, -\partial_n z_{f,h} + q_{f,h}n + c(z_{f,h}, q_{f,h})n]_\Gamma, \tag{2.8}
\]
for all \( f \in L^2(\Omega) \) and \( h \in H^1(\Omega)/\mathbb{R} \) such that \((h, 1)_\Omega = 0\), where \((z_{f,h}, q_{f,h}) \in H^1_0(\Omega) \times L^2_0(\Omega)\) is the unique solution of (2.2) and \(c(z_{f,h}, q_{f,h})\) is the constant given in (2.6).

Notice that if \( u \in V^0(\Gamma) \), equation (2.8) can be written as
\[
(y, f)_\Omega - [p, h]_\Omega = \langle u, -\partial_n z_{f,h} + q_{f,h}n \rangle_\Gamma. \tag{2.9}
\]

The definition follows integrating by parts twice the equation and once the null divergence condition. It can be written as two separate equations, one tested with \( f \) and the other one with \( h \), as in [50] or [66], or as single equation, cf. [19] or [67].

Next, we state a regularity result analogous to [66, Corollary A.1]. In that reference, a smooth domain is taken into consideration and the limit cases \( s = -1/2 \) and \( s = 3/2 \) can be achieved; however, this is not possible for polygonal domains so the cited result cannot be directly applied.

**Theorem 2.4.** Suppose \( u \in V^s(\Gamma) \) for \(-s^\ast < s < \min\{1/2 + \xi, 3/2\}\). Then the solution of (2.1) satisfies
\[
y_u \in V^{s+1/2}(\Omega) \text{ and } p_u \in \begin{cases} H^{s-1/2}(\Omega)/\mathbb{R} & \text{if } s \geq 1/2, \\ (H^{1/2-s}(\Omega)/\mathbb{R})' & \text{if } s \leq 1/2. \end{cases}
\]
Moreover, the control-to-state mapping \( u \mapsto y_u \) is continuous from \( V^s(\Gamma) \) to \( V^{s+1/2}(\Omega) \).

**Proof.** The proof follows by interpolation. The technique of proof is the same as in [66, Appendix A] or [1] Section 2, so we will just give a sketch of the proof and check some of the details that are different from those references.

We first do the regular case. Suppose \( 1/2 \leq s < \min\{1/2 + \xi, 3/2\} \). From the definition of \( V^s(\Gamma) \) we know that there exists \( Y \in H^{s+1/2}(\Omega) \) such that the boundary trace of \( Y \) equals \( u \). So we have that \( F = -\nabla Y \in H^{s-3/2}(\Omega) \) and \( H = \nabla \cdot Y \in H^{s-1/2}(\Omega) \). By linearity, we
have that \( y_u - Y = z_{F,H} \) and \( p_u = p_{F,H} \), where \( (z_{F,H}, p_{F,H}) \) is the variational solution of (2.2) for data \((F, H)\). From [20] Theorem 5.5(a), and using that \( s - 1/2 < \xi \), we have then that \( y_u - Y \in H^{s+1/2}(\Omega) \) and \( p_u \in H^{s-1/2}(\Omega)/\mathbb{R} \), and the result follows in a straightforward way.

Consider now \( -s^* < s < 0 \). Uniqueness follows testing (2.8) for the data \( u = 0 \) and the pairs \((y_u, 0)\) and \((0, h)\) for any \( h \in H^1(\Omega) \) such that \((h, 1)_{\Omega} = 0\); compare to [66, Theorem A.1(ii)] or [1] Theorem 2.5.

Existence follows by density arguments. Take \( u \in V^{1/2}(\Gamma) \), which is dense in \( V^s(\Gamma) \). Notice that \(-1/2 < -s < 1/2 < 0 \) and \( 1/2 < 1/2 - s < 1 \) and hence \( L^2(\Omega) \) is dense in \( H^{-s-1/2}(\Omega) \) and \( H^1(\Omega) \) is dense in \( H^{1/2-s}(\Omega) \). Therefore, we can consider

\[
\mathcal{F} = \{ f \in L^2(\Omega) : \| f \|_{H^{-1/2}(\Omega)} = 1 \}
\]

and

\[
\mathcal{H} = \{ h \in H^1(\Omega)/\mathbb{R} : \| h \|_{H^{1/2-s}(\Omega)/\mathbb{R}} = 1 \}
\]

to test the norms in \( H^{s+1/2}(\Gamma) \) and \((H^{1/2-s}(\Omega)/\mathbb{R})'\) respectively of the variational solution \((y_u, p_u)\) of (2.8). We obtain, using estimate (2.7),

\[
\| y_u \|_{H^{s+1/2}(\Gamma)} = \sup_{f \in \mathcal{F}} [f, y_u]_{H^{-1/2}(\Omega), H^{1/2}(\Omega)} = \sup_{f \in \mathcal{F}} (f, y_u)_{\Omega} \\
= \sup_{f \in \mathcal{F}} [u, -\partial_{\nu} z_{f,0} + q_{f,0} n + c(z_{f,0}, q_{f,0}) n]_{H^1(\Gamma), H^{-s}(\Gamma)} \\
\leq \sup_{f \in \mathcal{F}} \| u \|_{H^1(\Gamma)} \| f \|_{H^{-1/2}(\Omega)} = C \| u \|_{H^1(\Gamma)},
\]

and

\[
\| p_u \|_{(H^{1/2-s}(\Omega)/\mathbb{R})'} = \sup_{h \in \mathcal{H}} [p_u, h]_{(H^{1/2-s}(\Omega)/\mathbb{R})', H^{1/2-s}(\Omega)/\mathbb{R}} \\
= \sup_{h \in \mathcal{H}} [p_u, h]_{(H^1(\Omega)/\mathbb{R})', H^{1}(\Omega)/\mathbb{R}} \\
= \sup_{h \in \mathcal{H}} [u, -\partial_{\nu} z_{0,h} + q_{0,h} n + c(z_{0,h}, q_{0,h}) n]_{H^1(\Gamma)/\mathbb{R}, H^{-s}(\Gamma)/\mathbb{R}} \\
\leq \sup_{h \in \mathcal{H}} \| u \|_{H^1(\Gamma)} \| h \|_{H^{1/2-s}(\Omega)/\mathbb{R}} = C \| u \|_{H^1(\Gamma)}.
\]

The above proved estimates allow us to take a sequence \( u_n \) in \( V^{1/2}(\Gamma) \) converging to \( u \) in \( V^s(\Gamma) \) and obtain \( y_u \in V^{s+1/2}(\Omega) \) and \( p_u \in (H^{1/2-s}(\Omega)/\mathbb{R})' \) as the limits of the sequences \( y_{u_n} \) and \( p_{u_n} \); cf. [66, Theorem A.1(ii)] or [1] Theorem 2.5.

Finally, the case \( 0 \leq s < 1/2 \) follows by interpolation. \( \square \)

**Remark 2.5.** If \( u \in V^{1/2}(\Gamma) \), then the very weak solution and the variational solution are the same.

Next, we have to give some meaning to the mixed form. The main problem is that for data in \( u \in V^s(\Gamma), s < 1/2 \), the gradient of the state is not a function in \( L^2(\Omega) \).

We start with the regular compressible Stokes problem. Consider \( f \in L^2(\Omega) \) and \( h \in H^1(\Omega) \) such that \((h, 1)_{\Omega} = 0\) and denote \( z = z_{f,h} \) and \( q = q_{f,h} \) the (variational) solution of (2.2). If we
denote $G_{f,h} = \nabla z_{f,h} \in L^2(\Omega)$, we have that the triplet $(G_{f,h}, z_{f,h}, q_{f,h}) \in L^2(\Omega) \times H^1_0(\Omega) \times L^2_0(\Omega)$ is the unique solution of the weak formulation

\[
(G, T)_{\Omega} + (z, \nabla \cdot T)_{\Omega} = 0, \quad (2.10)
\]

\[
(G, \nabla v)_{\Omega} - (q, \nabla \cdot v)_{\Omega} = (f, v)_{\Omega}, \quad (2.11)
\]

\[
-(z, \nabla w)_{\Omega} = (h, w)_{\Omega}, \quad (2.12)
\]

\[
(q, 1)_{\Omega} = 0, \quad (2.13)
\]

for all $(T, v, w) \in H(\text{div}, \Omega) \times H^1_0(\Omega) \times H^1(\Omega)$. Moreover, it is clear that the regularity results stated above for (2.2) apply and $G_{f,h} \in H^{s-1/2}(\Omega)$ for $0 < s < s^*$. Notice also we can define analogously to (2.6)

\[
c(G, q) = \frac{1}{|\Gamma|} \langle q - (G n) \cdot n, 1 \rangle_\Gamma. \quad (2.14)
\]

Next we give a mixed formulation of problem (2.8) for Dirichlet data $u \in V^s(\Gamma)$ for $-s^* < s$.

**Definition 2.6.** For $-s^* < s$ and $u \in V^s(\Gamma)$, we say $y_u \in V^0(\Omega)$, $L_u = \nabla y_u \in (H^1(\Omega))'$, $p_u \in (H^1(\Omega)/\mathbb{R})'$ is a solution in the transposition sense of

\[
-\Delta y + \nabla p = f \quad \text{in } \Omega,
\]

\[
\nabla \cdot y = 0 \quad \text{in } \Omega,
\]

\[
y = u \quad \text{on } \Gamma,
\]

if $(y_u, L_u, p_u)$ satisfy

\[
[L, T]_{\Omega} = -(y, \nabla \cdot T)_{\Omega} + [u, T n]_\Gamma, \quad (2.15a)
\]

\[
(y, f)_{\Omega} - [p, h]_{\Omega} = [u, -G_{f,h} n + q_{f,h} n + c(G_{f,h}, q_{f,h}) n]_\Gamma, \quad (2.15b)
\]

for every $f \in L^2(\Omega)$, $h \in H^1(\Omega)$ such that $(h, 1)_{\Omega} = 0$ and $T \in H^1(\Omega)$, where $(G_{f,h}, z_{f,h}, q_{f,h}) \in L^2(\Omega) \times H^1_0(\Omega) \times L^2_0(\Omega)$ is the solution of (2.10)–(2.13) for data $(f, h)$.

The above definition simply incorporates an adequate definition for the gradient to the transposition solution defined in **Definition 2.3**. Nevertheless, this formulation is still not appropriate to use together with (2.10)–(2.13) in the context of hybridizable discontinuous Galerkin methods. Taking advantage of the regularity results stated in **Theorem 2.4**, we have $L_u \in H^{s-1/2}(\Omega)$ if $1/2 \leq s < \min\{1/2 + \xi, 3/2\}$ and $L_u \in (H^{1/2-s}(\Omega))'$ if $-s^* < s < 1/2$.

So we have that if $-s^* < s < 1/2$ and $u \in V^s(\Gamma)$, then there exists a unique solution $(L_u, y_u, p_u) \in (H^{1/2-s}(\Omega))' \times V^{1/2+s}(\Omega) \times (H^{1/2-s}(\Omega)/\mathbb{R})'$ of the problem

\[
[L, T]_{\Omega} + (y, \nabla \cdot T)_{\Omega} = [u, T n]_\Gamma, \quad (2.16a)
\]

\[
[L, G_{f,h}]_{\Omega} - [p, h]_{\Omega} = 0, \quad (2.16b)
\]

\[
[\nabla q_{f,h}, y]_{\Omega} = [u, q_{f,h} n]_\Gamma, \quad (2.16c)
\]

for every $f \in L^2(\Omega)$, $h \in H^1(\Omega)$ such that $(h, 1)_{\Omega} = 0$ and $T \in H^1(\Omega)$, where $(G_{f,h}, z_{f,h}, q_{f,h}) \in \bigcap_{t < s^*} H^{1/2+t}(\Omega) \times H^{1/2+t}(\Omega) \times H^{1/2+s}(\Omega)/\mathbb{R} \mapsto \bigcap_{t < s^*} H^{1/2-s}(\Omega) \times H^{3/2-s}(\Omega) \times H^{1/2-s}(\Omega)/\mathbb{R}$ is the solution of (2.10)–(2.13) for data $(f, h)$.

Taking all this into account we can summarize our results in the following theorem.
Theorem 2.7. For every \( u \in V^0(\Gamma) \), there exist a unique solution \( (\mathbb{I}_u, y_u, p_u) \in (\mathbb{H}^{1/2}(\Omega))' \times V^{1/2}(\Omega) \times (H^{1/2}(\Omega)/\mathbb{R})' \) of

\[
\begin{align*}
[\mathbb{I}, T]_\Omega + (y, \nabla \cdot T)_\Omega &= \langle u, Tn \rangle_\Gamma, \\
[\mathbb{I}, \nabla v]_\Omega - [p, \nabla \cdot v]_\Omega &= 0, \\
[\nabla w, y]_\Omega &= \langle u, wn \rangle_\Gamma,
\end{align*}
\] (2.17a) (2.17b) (2.17c)

for all \((T, v, w) \in \mathbb{H}^1(\Omega) \times \bigcap_{t<s^*} \mathbb{H}^{3/2+t}(\Omega) \cap H^1_0(\Omega) \times \bigcap_{t<s^*} H^{1/2+t}(\Omega)\). Moreover, if \( u \in V^s(\Gamma), -1/2 < s < s^* \), then \((\mathbb{I}_u, y_u, p_u) \in (\mathbb{H}^{1/2-s}(\Omega))' \times V^{1/2+s}(\Omega) \times (H^{1/2-s}(\Omega)/\mathbb{R})' \). Finally, the control-to-state mapping \( u \mapsto (\mathbb{I}_u, y_u, p_u) \) is continuous from \( V^s(\Gamma) \) to \((\mathbb{H}^{1/2-s}(\Omega))' \times V^{1/2+s}(\Omega) \times (H^{1/2-s}(\Omega)/\mathbb{R})' \) for \(-s^* < s < \min\{1/2 + \xi, 3/2\}\).

2.2 Well posedness and regularity of the tangential control problem

It is clear that \( U \hookrightarrow L^2(\Gamma) \) and there is no ambiguity in denoting by \( u \) the elements of \( U \). Hence the control-to-state mapping \( u \mapsto y_u \) is continuous from \( U \) to \( V^{1/2}(\Omega) \), and there exists a unique solution of the control problem

\[
(P_\tau) \quad \min J(u) = \frac{1}{2} \| y_u - y_d \|^2_{L^2(\Omega)} + \frac{\gamma}{2} \| u \|^2_{L^2(\Gamma)},
\]

where \( y_u \) is the solution of the state equation

\[
\begin{align*}
[\mathbb{I}, T]_\Omega + (y, \nabla \cdot T)_\Omega &= \langle u\tau, Tn \rangle_\Gamma, \\
[\mathbb{I}, \nabla v]_\Omega - [p, \nabla \cdot v]_\Omega &= 0, \\
[\nabla w, y]_\Omega &= 0,
\end{align*}
\] (2.18a) (2.18b) (2.18c)

for all \((T, v, w) \in \mathbb{H}^1(\Omega) \times \bigcap_{t<s^*} \mathbb{H}^{3/2+t}(\Omega) \cap H^1_0(\Omega) \times \bigcap_{t<s^*} H^{1/2+t}(\Omega)\). Notice that \((2.18a), (2.18b), (2.18c)\) is the weak formulation \((2.17a), (2.17b), (2.17c)\) obtained in Theorem 2.7 for the Stokes problem \((1.2)\) with Dirichlet datum \( u\tau \), where we have used that \( u\tau \cdot wn = 0 \) for any pair of functions \( u, w \) in \( L^2(\Gamma) \).

If \( \Omega \) is nonconvex, then the regularity of the optimal solution is limited mainly by the singular exponent related to the greatest nonconvex angle, and we would find discontinuous optimal controls that would lead to pressures and gradients of the state that are not functions. For convex domains, the regularity is better and we can write integrals instead of duality products. The main consequence is that we can formulate an HDG approximation method for the optimality system.

Theorem 2.8. Suppose \( \Omega \) is a convex polygonal domain and \( y_d \in H^{\min\{2, \xi\}}(\Omega) \). Let \( u \in L^2(\Gamma) \) be the solution of problem \((P_\tau)\). Then

\[
u \in H^s(\Gamma)
\]

for all \( 1/2 < s < \min\{3/2, \xi - 1/2\} \) and there exists

\[
y \in V^{s+1/2}(\Omega), \quad L \in \mathbb{H}^{s-1/2}(\Omega), \quad p \in H^{s-1/2}(\Omega) \cap L^2_0(\Omega),
\]

\[
z \in V^{s+1}_0(\Omega), \quad G \in \mathbb{H}^r(\Omega), \quad q \in H^r(\Omega) \cap L^2_0(\Omega),
\]
for all $1 < r < \min\{3, \xi\}$ such that

\begin{align}
(L, \nabla v)_\Omega - (p, \nabla \cdot v)_\Omega &= 0, \\
(\nabla u, y)_\Omega &= 0,
\end{align}

(2.19a)

\begin{align}
(L, T)_\Omega + (y, \nabla \cdot T)_\Omega &= (u_\tau, T n)_\Gamma,
\end{align}

(2.19b)

\begin{align}
(G, \nabla v)_\Omega + (q, \nabla \cdot v)_\Omega &= (y - y_d, v)_\Omega, \\
-(z, \nabla w)_\Omega &= 0,
\end{align}

(2.19c)

\begin{align}
(G, T)_\Omega + (z, \nabla \cdot T)_\Omega &= 0, \\
(\gamma u_\tau - G n, \mu_\tau)_\Gamma &= 0,
\end{align}

(2.19d)

\begin{align}
\gamma u_\tau = G n = \nabla z \in \Pi m_{i=1}^{m} H^{t-1/2}(\Gamma_i) \text{ for all } t \leq 3/2 \text{ such that } t < \xi.
\end{align}

(2.19e)

for all $(T, v, w, \mu) \in \mathbb{H}(\text{div}, \Omega) \times \mathbf{H}_0^1(\Omega) \times H^1(\Omega) \times L^2(\Gamma)$. Moreover,

\begin{align}
u \in \prod_{i=1}^{m} H^{r-1/2}(\Gamma_i) \text{ for all } r < \min\{3, \xi\}.
\end{align}

(2.20)

\begin{proof}
Optimality conditions follow in a standard way by computing the derivative of the functional with the help of the chain rule, the integration by parts formula and Definition 2.3. The regularity follows from a bootstrapping argument.

From Theorem 2.4 we have that $y \in V^{1/2}(\Omega)$. Using this and the regularity of the data $y_d$, we deduce from Theorem 2.2 that $z \in V^{1/2}(\Omega)$ and $q \in H^t(\Omega) \cap L^2(\Omega)$ for all $t \leq 3/2$ such that $t < \xi$. From the trace theory, it is clear that

\begin{align}
\nabla u_\tau = G n = \nabla z \in \Pi m_{i=1}^{m} H^{t-1/2}(\Gamma_i) \text{ for all } t \leq 3/2 \text{ such that } t < \xi.
\end{align}

Since $\xi > 1$, we notice that the gradient of the dual pressure $q$ is a function in $H^{t-1}(\Omega)$ with $t - 1 > 0$. So we have that each component $z^i, i = 1, 2$ of $z$, satisfies $\Delta z^i \in H^{t-1}(\Omega)$ and $z^i = 0$ on $\Gamma$. Therefore, we have that $\partial_n z^i(x_j) = 0$, $i = 1, 2$, for every corner $x_j$ (cf. [12, Appendix A], [10, Section 4]), and hence we also have (cf. [12, Lemma A.2]) that

\begin{align}
\nabla u_\tau = G n = \nabla z \in \Pi m_{i=1}^{m} H^{t-1/2}(\Gamma_i) \text{ for all } t \leq 3/2 \text{ such that } t < \xi.
\end{align}

Next, using that the pressure does not appear in the optimality condition (2.19g), we can write

\begin{align}
\gamma u_\tau = G n = \nabla z,
\end{align}

and therefore the Dirichlet datum of the state equation is also in the space $H^{t-1/2}(\Gamma)$ for all $t \leq 3/2$ such that $t < \xi$.

Repeating the argument, we obtain in a first step from Theorem 2.4 that $y \in V^1(\Gamma)$ for all $t \leq 3/2$ such that $t < \xi$, which leads, together with the maybe higher regularity of $y_d$ and Theorem 2.2 to $z \in V^{1/2+t_2}(\Omega)$ for $t_2 \leq 5/2, t_2 < \xi$. The normal trace argument leads to $u_\tau \in \Pi m_{i=1}^{m} H^{t-1/2}(\Gamma_i)$, but when we paste together the pieces with the help of the zero value at the corners, we cannot go further for the Dirichlet datum of the state equation than

\begin{align}
u \tau \in V^s(\Gamma) \text{ for } s < 3/2, s < \xi - 1/2.
\end{align}

The claimed regularity for the optimal control follows from the previous relation.

Taking the same argument for a third time, we obtain the regularity of the other involved variables.

\end{proof}
Notice that a higher regularity of the target state would not lead to a higher regularity of the solution, since it is mainly bounded by the singularities that appear due to the corners. Low regularity of the target would nevertheless lead to a low regularity solution. Suppose for instance that $\xi > 2 (\omega > 0.7\pi)$ and $y_d \in H^\alpha(\Omega)$, with $\alpha < 2$. If $\alpha < 1$, then the gradient of the dual pressure would not be a function, and the argument of the proof would not lead to any conclusion. If $1 \leq \alpha \leq 3/2$, then the argument would stop in the first step, obtaining regularity for the control $u \in H^{\alpha-1/2}(\Gamma)$. If $3/2 < \alpha < 2$, then the argument would finish in the second step obtaining again $u \in H^{\alpha-1/2}(\Gamma)$.

We use the following reformulation of the optimality system in our analysis of the HDG method:

**Corollary 2.9.** Suppose $\Omega$ is a convex polygonal domain and $y_d \in H^{\min\{2,\xi\}}(\Omega)$. The solution of the optimality system \(2.19a\)-\(2.19g\) also satisfies the following well-posed problem: find

\[
\begin{align*}
    u & \in H^{1/2}(\Gamma), & y & \in V^1(\Omega), & L & \in L^2(\Omega), & p & \in L_0^2(\Omega), \\
    z & \in V_0^1(\Omega), & G & \in L^2(\Omega), & q & \in L_0^2(\Omega),
\end{align*}
\]

such that $L - pI, G + qI \in L(\text{div}, \Omega)$ and

\[
\begin{align*}
    (L, T)_\Omega + (y, \nabla \cdot T)_\Omega &= \langle u\tau, Tn \rangle_\Gamma, \tag{2.21a} \\
    - (\nabla \cdot (L - pI), v)_\Omega &= 0, \tag{2.21b} \\
    (\nabla \cdot y, w)_\Omega &= 0, \tag{2.21c} \\
    (G, T)_\Omega + (z, \nabla \cdot T)_\Omega &= 0, \tag{2.21d} \\
    - (\nabla \cdot (G + qI), v)_\Omega &= \langle y - y_d, v \rangle_\Omega, \tag{2.21e} \\
    (\nabla \cdot z, w)_\Omega &= 0, \tag{2.21f} \\
    \langle \gamma u\tau - Gn, \mu\tau \rangle_\Gamma &= 0, \tag{2.21g}
\end{align*}
\]

for all $(T, v, w, \mu) \in L(\text{div}, \Omega) \times L^2(\Omega) \times L_0^2(\Omega) \times H^{1/2}(\Gamma)$.

### 3 HDG Formulation

Before we introduce the HDG method, we first define some notation. Let $\mathcal{T}_h$ be a collection of disjoint elements that partition $\Omega$. We denote by $\partial \mathcal{T}_h$ the set $\{\partial K : K \in \mathcal{T}_h\}$. For an element $K$ of the collection $\mathcal{T}_h$, $e = \partial K \cap \Gamma$ is the boundary face of the $d - 1$ Lebesgue measure of $e$ is non-zero. For two elements $K^+$ and $K^-$ of the collection $\mathcal{T}_h$, $e = \partial K^+ \cap \partial K^-$ is the interior face between $K^+$ and $K^-$ if the $d - 1$ Lebesgue measure of $e$ is non-zero. Let $\mathcal{E}_h^\partial$ and $\mathcal{E}_h^\partial$ denote the set of interior and boundary faces, respectively. We denote by $\mathcal{E}_h^\partial$ the union of $\mathcal{E}_h^\partial$ and $\mathcal{E}_h^\partial$. We introduce various inner products for our finite element spaces. We write

\[
\begin{align*}
    (\eta, \zeta)_{\mathcal{T}_h} &= \sum_{K \in \mathcal{T}_h} (\eta, \zeta)_K, & (\eta, \zeta)_{\partial \mathcal{T}_h} &= \sum_{K \in \partial \mathcal{T}_h} (\eta, \zeta)_{\partial K}, \\
    (\eta, \zeta)_{\mathcal{T}_h} &= \sum_{i=1}^d (\eta_i, \zeta_i)_{\mathcal{T}_h}, & (L, G)_{\mathcal{T}_h} &= \sum_{i,j=1}^d (L_{ij}, G_{ij})_{\mathcal{T}_h}, & (\eta, \zeta)_{\partial \mathcal{T}_h} &= \sum_{i=1}^d (\eta_i, \zeta_i)_{\partial \mathcal{T}_h},
\end{align*}
\]

where $(\cdot, \cdot)_K$ and $(\cdot, \cdot)_{\partial K}$ denote the standard $L^2$ inner products on the domains $K \subset \mathbb{R}^d$ and $\partial K \subset \mathbb{R}^{d-1}$.
Let \(\mathcal{P}^k(D)\) denote the set of polynomials of degree at most \(k\) on a domain \(D\). We introduce the following discontinuous finite element spaces

\[
\mathbb{K}_h := \{ L \in L^2(\Omega) : L |_K \in [P_k(K)]^{d \times d}, \forall K \in \mathcal{T}_h \},
\]

\[
\mathbb{V}_h := \{ v \in L^2(\Omega) : v |_K \in [P^{k+1}(K)]^d, \forall K \in \mathcal{T}_h \},
\]

\[
\mathbb{W}_h := \{ w \in L^2(\Omega) : w |_K \in \mathcal{P}^k(K), \forall K \in \mathcal{T}_h \},
\]

\[
\mathbb{M}_h := \{ \mu \in L^2(\varepsilon_h) : \mu |_e \in [\mathcal{P}^k(e)]^d, \forall e \in \varepsilon_h \},
\]

\[
\mathbb{M}_h := \{ \mu \in L^2(\varepsilon_h^0) : \mu |_e \in \mathcal{P}^k(e), \forall e \in \varepsilon_h^0 \},
\]

for the flux variables, velocity, pressure, boundary trace variables, and boundary control, respectively. Note that the polynomial degree for the scalar variable is one order higher than the polynomial degree for the flux variables and numerical trace. This combination of spaces has been used for the Navier-Stokes equations in [64]. The boundary trace variables will be used to eliminate the state and flux variables from the coupled global equations, thus substantially reducing the number of degrees of freedom.

Let \(\mathbb{M}_h(o)\) denote the space defined in the same way as \(\mathbb{M}_h\), but with \(\varepsilon_h\) replaced by \(\varepsilon_h^0\). Note that \(\mathbb{M}_h\) consists of functions which are continuous inside the faces (or edges) \(e \in \varepsilon_h\) and discontinuous at their borders. In addition, spatial derivatives of any functions in the finite element spaces are taken piecewise on each element \(K \in \mathcal{T}_h\). Finally, we define

\[
W^0_h = \left\{ w \in L^2(\Omega) : w |_K \in \mathcal{P}^k(K), \forall K \in \mathcal{T}_h \text{ and } (w, 1)_\Omega = 0 \right\}.
\]

### 3.1 The HDG Formulation

To approximate the solution of the mixed weak form \((2.21a)-(2.21g)\) of the optimality system, the HDG method seeks approximate fluxes \(L_h, G_h \in \mathbb{K}_h\), states \(y_h, z_h \in \mathbb{V}_h\), pressures \(p_h, q_h \in W^0_h\), interior element boundary traces \(\hat{y}_h^\circ, \hat{z}_h^\circ \in \mathbb{M}_h(o)\), and boundary control \(u_h \in \mathbb{M}_h\) satisfying

\[
(L_h, T_1)_T_h + (y_h, \nabla \cdot T_1)_T_h - (\hat{y}_h^\circ, T_1 n)_{\partial T_h \setminus \varepsilon_h^0} = (u_h T, T_1 n)_{\varepsilon_h^0},
\]

\[
(L_h, \nabla v)_T_h - (p_h, \nabla \cdot v)_T_h - (\hat{L}_h - p_h I) n, v_1)_{\partial T_h} = (f, v_1)_{T_h},
\]

\[
- (y_h, \nabla w_1)_T_h + (\hat{y}_h^0 \cdot n, w_1)_{\partial T_h \setminus \varepsilon_h^0} = 0,
\]

for all \((T_1, v_1, w_1) \in \mathbb{K}_h \times \mathbb{V}_h \times W^0_h\),

\[
(G_h, T_2)_T_h + (z_h, \nabla \cdot T_2)_T_h - (\hat{z}_h^0, T_2 n)_{\partial T_h \setminus \varepsilon_h^0} = 0,
\]

\[
(G_h, \nabla v_2)_T_h + (q_h, \nabla \cdot v_2)_T_h - ((G_h + q_h I) n, v_2)_{\partial T_h} = (y_h - y_d, v_2)_{T_h},
\]

\[
- (z_h, \nabla w_2)_T_h + (\hat{z}_h^0 \cdot n, w_2)_{\partial T_h \setminus \varepsilon_h^0} = 0,
\]

for all \((T_2, v_2, w_2) \in \mathbb{K}_h \times \mathbb{V}_h \times W^0_h\),

\[
\langle (\hat{L}_h - p_h I) n, \mu_1 \rangle_{\partial T_h \setminus \varepsilon_h^0} = 0,
\]

for all \(\mu_1 \in \mathbb{M}_h(o)\),

\[
\langle (G_h + q_h I) n, \mu_2 \rangle_{\partial T_h \setminus \varepsilon_h^0} = 0,
\]

for all \(\mu_2 \in \mathbb{M}_h(o)\).
for all $\mu_2 \in M_h(v)$,
\begin{equation}
\langle \widehat{G}_h \mathbf{n} - \gamma u_h \tau, \mu_3 \tau \rangle_{\epsilon_h^\partial} = 0,
\end{equation}
for all $\mu_3 \in M_h$. In contrast to Section 2, here we assume the forcing $f$ may be nonzero.

The numerical traces on $\partial \mathcal{T}_h$ are defined as
\begin{align}
\widehat{L}_h \mathbf{n} & = \mathbb{L}_h \mathbf{n} - h^{-1}(P_M y_h - \widehat{y}_h^o) \quad \text{on } \partial \mathcal{T}_h \setminus \epsilon_h^\partial, \quad (3.6j) \\
\widehat{L}_h \mathbf{n} & = \mathbb{L}_h \mathbf{n} - h^{-1}(P_M y_h - u_h \tau) \quad \text{on } \epsilon_h^\partial, \quad (3.6k) \\
\widehat{G}_h \mathbf{n} & = G_h \mathbf{n} - h^{-1}(P_M z_h - \widehat{z}_h^o) \quad \text{on } \partial \mathcal{T}_h \setminus \epsilon_h^\partial, \quad (3.6l) \\
\widehat{G}_h \mathbf{n} & = G_h \mathbf{n} - h^{-1}P_M z_h \quad \text{on } \epsilon_h^\partial, \quad (3.6m)
\end{align}

where $P_M$ denotes the standard $L^2$-orthogonal projection from $L^2(\epsilon_h^\partial)$ onto $M_h$. This completes the formulation of the HDG method.

As far as the authors are aware, this HDG method has not been used for the Stokes problems in the literature. The HDG discretization scheme and stabilization approach used above was motivated by the HDG method in [63] for the convection diffusion equation.

### 3.2 Implementation

In [57], the authors introduced two HDG approaches (with a different choice of spaces than considered here) for the Stokes equation. The first approach introduces the mean value of the pressure into the local solver, and the global system involves the trace of the velocity and the mean value of the pressure. The second method utilizes an augmented Lagrangian approach and introduces a time derivative of the pressure into the equations. In this approach the mean of the pressure is not required, and we obtain the new HDG formulation by discretizing the equations in time using the backward Euler method. This time discretization approach is unconditionally stable; therefore, we can choose an arbitrary time step. Moreover, we can express the pressure in terms of the velocity and eliminate the mean of the pressure from the local solver. This yields a globally coupled system in terms of the approximate velocity trace only. Although multiple linear systems need to be solved due to the time discretization, each system has has less degrees of freedom than the system from the first approach, which involves both the approximate velocity trace and the mean of the pressure.

In this section, we adopt the augmented Lagrangian approach for the boundary control problem. We introduce time derivatives of the pressure $p$ and dual pressure $q$ into the optimality system (1.3) and discretize in time and space. We show below that the resulting global system only involves the boundary traces of the velocity and dual velocity and the boundary control.

Given an initial guess for the pressure and dual pressure, $p_h^{(0)} \in L^2_0(\Omega)$ and $q_h^{(0)} \in L^2_0(\Omega)$, the augmented Lagrangian method generates the sequence $(\mathbb{L}_h^{(m)}, \mathbb{G}_h^{(m)}, y_h^{(m)}, z_h^{(m)}, p_h^{(m)}, q_h^{(m)}, y_h^{o,(m)}, z_h^{o,(m)}, u_h^{(m)})$ for $m = 1, 2, 3, \ldots$ by solving the system (3.7) below. We set the relative error tolerance for the pressure as $\text{tol} = 10^{-8}$, and we stop the iterations when
\begin{equation}
\frac{\|p_h^{(m)} - p_h^{(m-1)}\|_{\mathcal{T}_h}}{\|p_h^{(m)}\|_{\mathcal{T}_h}} + \frac{\|q_h^{(m)} - q_h^{(m-1)}\|_{\mathcal{T}_h}}{\|q_h^{(m)}\|_{\mathcal{T}_h}} < \text{tol}.
\end{equation}
The system to be solved at each iteration is

\begin{align}
(\mathbb{L}_h^{(m)}, T_1)\tau_h + (y_h^{(m)}, \nabla \cdot T_1)\tau_h - (y_h^{o,(m)}, T_1 n)_{\partial \Omega_h^\infty} & = 0, \\
(\mathbb{G}_h^{(m)}, T_2)\tau_h + (z_h^{(m)}, \nabla \cdot T_2)\tau_h - (z_h^{o,(m)}, T_2 n)_{\partial \Omega_h^\infty} & = 0, \\
-(\nabla \cdot \mathbb{L}_h^{(m)}, v_1)\tau_h + (\nabla p_h^{(m)}, v_1)\tau_h + (h^{-1} P_M y_h^{(m)}, v_1)_{\partial \Omega_h} & = 0, \\
-(h^{-1} y_h^{o,(m)}, v_1)_{\partial \Omega_h^\infty} & = 0, \\
-(\nabla \cdot \mathbb{G}_h^{(m)}, v_2)\tau_h - (\nabla q_h^{(m)}, v_2)\tau_h + (h^{-1} P_M z_h^{(m)}, v_2)_{\partial \Omega_h} & = 0, \\
-(\nabla \cdot \mathbb{L}_h^{(m)}, v_2)\tau_h + (y_h^{o,(m)}, v_2)_{\partial \Omega_h^\infty} & = 0,
\end{align}

for all \((T_1, T_2, v_1, v_2, w_1, w_2, \mu_1, \mu_2, \mu_3) \in \mathbb{K}_h \times \mathbb{K}_h \times V_h \times V_h \times W_h \times W_h \times M_h(o) \times M_h(o) \times M_h.

### 3.2.1 Matrix equations

Assume \(\mathbb{K}_h = \text{span}\{\mathbb{K}_j\}_{j=1}^{N_1}, V_h = \text{span}\{\mathbb{V}_j\}_{j=1}^{N_2}, W_h = \text{span}\{\mathbb{W}_j\}_{j=1}^{N_3}, M_h^o = \text{span}\{\mathbb{M}_j\}_{j=1}^{N_4}, M_h = \text{span}\{\mathbb{M}_j\}_{j=1}^{N_5}\). Then

\begin{align}
\mathbb{L}_h^{(m)} & = \sum_{j=1}^{N_1} L_j^{(m)} \mathbb{K}_j, & \mathbb{G}_h^{(m)} & = \sum_{j=1}^{N_1} G_j^{(m)} \mathbb{K}_j, & y_h^{(m)} & = \sum_{j=1}^{N_1} y_j^{(m)} \mathbb{V}_j, \\
z_h^{(m)} & = \sum_{j=1}^{N_2} z_j^{(m)} \mathbb{V}_j, & p_h^{(m)} & = \sum_{j=1}^{N_3} p_j^{(m)} \mathbb{W}_j, & p_h^{(m-1)} & = \sum_{j=1}^{N_3} p_j^{(m-1)} \mathbb{W}_j, \\
q_h^{(m)} & = \sum_{j=1}^{N_4} q_j^{(m)} \mathbb{W}_j, & q_h^{(m-1)} & = \sum_{j=1}^{N_4} q_j^{(m-1)} \mathbb{W}_j, & \tilde{y}_h^{o,(m)} & = \sum_{j=1}^{N_4} \alpha_j^{(m)} \mathbb{W}_j, \\
z_h^{o,(m)} & = \sum_{j=1}^{N_4} \gamma_j^{(m)} \mathbb{W}_j, & u_h^{(m)} & = \sum_{j=1}^{N_4} \xi_j^{(m)} \mathbb{W}_j.
\end{align}

Substitute (3.8) into (3.7a)-(3.7i) and use the corresponding test functions to test (3.7a)-(3.7i), respectively, to obtain the matrix equation

\[ A x^{(m)} = f^{(m)}, \]
where
\[
A = \begin{bmatrix}
A_1 & 0 & A_2 & 0 & 0 & 0 & -A_3 & 0 & -A_4 \\
0 & A_1 & 0 & A_2 & 0 & 0 & 0 & -A_3 & 0 \\
-A_2^T & 0 & A_5 & 0 & A_6 & 0 & -A_7 & 0 & -A_8 \\
0 & -A_2^T & -A_9 & A_5 & 0 & -A_6 & 0 & -A_7 & 0 \\
0 & 0 & -A_6^T & 0 & A_{10}/\Delta t & 0 & A_{11} & 0 & 0 \\
0 & 0 & 0 & -A_6^T & 0 & A_{10}/\Delta t & 0 & A_{11} & 0 \\
A_3^T & 0 & -A_7^T & 0 & -A_{11}^T & 0 & A_{12} & 0 & 0 \\
0 & A_3^T & 0 & -A_7^T & 0 & -A_{11}^T & 0 & A_{12} & 0 \\
0 & 0 & -A_{14} & 0 & 0 & 0 & 0 & 0 & -\gamma A_{15}
\end{bmatrix}
\]

and
\[
x^{(m)} = \begin{bmatrix}
\mathfrak{L}^{(m)} & \mathfrak{G}^{(m)} & \mathfrak{y}^{(m)} & \mathfrak{z}^{(m)} & \mathfrak{p}^{(m)} & \mathfrak{q}^{(m)} & \mathfrak{h}^{(m)} & \mathfrak{2}^{(m)} & \mathfrak{u}^{(m)}
\end{bmatrix}^T,
\]
\[
\mathfrak{b}^{(m)} = \begin{bmatrix}
0 & 0 & b_1 & -b_2 & b_3^{(m-1)} & b_4^{(m-1)} & 0 & 0 & 0
\end{bmatrix}^T.
\]

Here, \(\mathfrak{L}^{(m)}, \mathfrak{G}^{(m)}, \mathfrak{y}^{(m)}, \mathfrak{z}^{(m)}, \mathfrak{p}^{(m)}, \mathfrak{q}^{(m)}, \mathfrak{h}^{(m)}, \mathfrak{2}^{(m)} \), and \(\mathfrak{u}^{(m)} \) are the coefficient vectors for \(\mathbb{L}_h^{(m)}, \mathbb{G}_h^{(m)}, \mathbb{y}_h^{(m)}, \mathbb{z}_h^{(m)}, \mathbb{p}_h^{(m)}, \mathbb{q}_h^{(m)}, \mathbb{h}_h^{(m)}, \mathbb{2}_h^{(m)} \), respectively, and
\[
\begin{align*}
A_1 &= \left[ (\mathbb{K}_j, \mathbb{K}_i)_{\mathcal{T}_h} \right], \\
A_2 &= \left[ (\mathfrak{f}_j, \nabla \cdot \mathbb{K}_i)_{\mathcal{T}_h} \right], \\
A_3 &= \left[ (\mathfrak{f}_j, \mathbb{K}_i)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^3} \right], \\
A_4 &= \left[ (\mathfrak{f}_j, \mathbb{K}_i)_{\mathcal{E}_h^3} \right], \\
A_5 &= \left[ (\mathfrak{h}^{-1} \mathbb{P}_M \mathfrak{f}_j, \mathfrak{f}_i)_{\partial \mathcal{T}_h} \right], \\
A_6 &= \left[ (\mathfrak{f}_i, \mathfrak{f}_j)_{\mathcal{T}_h} \right], \\
A_7 &= \left[ (\mathfrak{f}_j, \mathfrak{f}_i)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^3} \right], \\
A_8 &= \left[ (\mathfrak{f}_j, \mathfrak{f}_i)_{\mathcal{E}_h^3} \right], \\
A_9 &= \left[ (\mathfrak{f}_j, \mathfrak{f}_i)_{\partial \mathcal{T}_h} \right], \\
A_{10} &= \left[ (\mathfrak{f}_j, \mathfrak{f}_i)_{\mathcal{T}_h} \right], \\
A_{11} &= \left[ (\mathfrak{f}_j, \mathfrak{f}_i)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^3} \right], \\
A_{12} &= \left[ (\mathfrak{f}_j, \mathfrak{f}_i)_{\mathcal{E}_h^3} \right], \\
A_{13} &= \left[ (\mathbb{K}_j, \mathbb{K}_i)_{\mathcal{E}_h^3} \right], \\
A_{14} &= \left[ (\mathfrak{f}_j, \mathfrak{f}_i)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^3} \right], \\
A_{15} &= \left[ (\mathfrak{f}_j, \mathfrak{f}_i)_{\mathcal{E}_h^3} \right],
\end{align*}
\]
\[
b_1 = \left[ (\mathfrak{f}_j, \mathfrak{v}_1)_{\mathcal{T}_h} \right], \quad b_2 = \left[ (\mathfrak{v}_d, \mathfrak{v}_2)_{\mathcal{T}_h} \right],
\]
\[
b_3^{(m-1)} = \frac{1}{\Delta t} (q_h^{(m-1)}, w_1)_{\mathcal{T}_h}, \quad b_4^{(m-1)} = \frac{1}{\Delta t} (q_h^{(m-1)}, w_1)_{\mathcal{T}_h}.
\]

Equation (3.9) can be rewritten as
\[
\begin{bmatrix}
B_1 & B_2 & 0 & B_3 \\
-B_4^T & B_4 & B_5 & B_6 \\
0 & B_7 & B_8 & B_9 \\
B_{10} & B_{11} & B_{12} & B_{13}
\end{bmatrix}
\begin{bmatrix}
\alpha^{(m)} \\
\beta^{(m)} \\
\gamma^{(m)} \\
\zeta^{(m)}
\end{bmatrix}
= \begin{bmatrix}
0 \\
\bar{b}_1 \\
\bar{b}_2^{(m-1)} \\
0
\end{bmatrix},
\tag{3.10}
\]
where \(\alpha^{(m)} = [\mathfrak{L}^{(m)}, \mathfrak{G}^{(m)}], \beta^{(m)} = [\mathfrak{y}^{(m)}, \mathfrak{z}^{(m)}], \gamma^{(m)} = [\mathfrak{p}^{(m)}, \mathfrak{q}^{(m)}], \zeta^{(m)} = [\mathfrak{h}^{(m)}, \mathfrak{2}^{(m)}; \mathfrak{u}^{(m)}], \bar{b}_1 = [b_1; -b_2], \bar{b}_2^{(m-1)} = [b_3^{(m-1)}; b_4^{(m-1)}] \) and \(\{B_i\}_{i=1}^{13} \) are the corresponding blocks of the coefficient matrix in (3.9).

Since \(\mathbb{K}_h, \mathbb{V}_h, \) and \(\mathbb{W}_h \) are discontinuous finite element spaces, the first three equations of (3.10) can be used to eliminate \(\alpha^{(m)}, \beta^{(m)} \) and \(\gamma^{(m)} \) in an element-by-element fashion. As a consequence, we can write system (3.10) as
\[
\begin{bmatrix}
\alpha^{(m)} \\
\beta^{(m)} \\
\gamma^{(m)}
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}
\zeta^{(m)} + \begin{bmatrix}
J_1^{(m)} \\
J_2^{(m)} \\
J_3^{(m)}
\end{bmatrix},
\tag{3.11}
\]
and

\[ B_{10}\alpha^{(m)} + B_{11}\beta^{(m)} + B_{12}\gamma^{(m)} + B_{13}\zeta^{(m)} = 0. \]  

(3.12)

We provide details on the element-by-element construction of \( F_1, F_2, F_3 \) and \( J_1^{(m)}, J_2^{(m)}, J_3^{(m)} \) in the appendix. Next, we eliminate \( \alpha^{(m)}, \beta^{(m)} \) and \( \gamma^{(m)} \) to obtain a reduced globally coupled equation for \( \zeta^{(m)} \) only:

\[ A\zeta^{(m)} = b^{(m)}, \]  

(3.13)

where

\[ A = B_{10}F_1 + B_{11}F_2 + B_{12}F_3 + B_{13} \]

\[ b^{(m)} = -B_{10}J_1^{(m)} - B_{11}J_2^{(m)} - B_{12}J_3^{(m)}. \]

After solving for \( \zeta^{(m)} \), the remaining coefficient vectors \( \alpha^{(m)}, \beta^{(m)} \) and \( \gamma^{(m)} \) can be easily computed using (3.11).

## 4 Error analysis

In this section, we perform a convergence analysis of the HDG method for the tangential Dirichlet boundary control for Stokes equations.

### 4.1 Main result

We assume throughout that there exists a unique solution of the optimality system \(^{(2.21a)-(2.21g)}\) satisfying the following “high regularity” condition:

\[ L \in H^{r_L}(\Omega), \quad y \in H^{r_y}(\Omega), \quad p \in H^{r_p}(\Omega), \quad G \in H^{r_G}(\Omega), \quad z \in H^{r_z}(\Omega), \quad q \in H^{r_q}(\Omega), \]

where

\[ r_y > 1, \quad r_z > 2, \quad r_L > 1/2, \quad r_G > 1, \quad r_p > 1/2, \quad r_q > 1. \]  

(4.1)

We assume \( r_L > 1/2 \) and \( r_p > 1/2 \) (instead of \( r_L > 0 \) and \( r_p > 0 \)) here in order to guarantee \( L \) and \( p \) have well-defined \( L^2 \) boundary traces. As mentioned earlier, in the second part of this work we consider the low regularity case and only require \( r_L > 0 \) and \( r_q > 0 \).

We can guarantee the high regularity condition holds in the 2D case by making an assumption on the domain \( \Omega \); see [Corollary 4.2](#) below. Regularity theory is not available in the 3D case, and therefore we do not know if high regularity solutions exist.

We also assume throughout that \( \Omega \) is convex and the family of meshes \( \{T_h\} \) is conforming and quasi-uniform.

Our main result is below:

**Theorem 4.1.** For

\[ s_L = \min\{r_L, k + 1\}, \quad s_y = \min\{r_y, k + 2\}, \quad s_p = \min\{r_p, k + 1\}, \]

\[ s_G = \min\{r_G, k + 1\}, \quad s_z = \min\{r_z, k + 2\}, \quad s_q = \min\{r_q, k + 1\}, \]

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we have
\[ \| L - L_h \|_{\mathcal{T}_h} \lesssim \varepsilon(h), \quad \| y - y_h \|_{\mathcal{T}_h} \lesssim h^{1/2} \varepsilon(h), \quad \| p - p_h \|_{\mathcal{T}_h} \lesssim \varepsilon(h), \]
\[ \| G - G_h \|_{\mathcal{T}_h} \lesssim h^{1/2} \varepsilon(h), \quad \| z - z_h \|_{\mathcal{T}_h} \lesssim h^{1/2} \varepsilon(h), \quad \| q - q_h \|_{\mathcal{T}_h} \lesssim h^{1/2} \varepsilon(h), \]
\[ \| u - u_h \|_{\mathcal{T}_h} \lesssim h^{1/2} \varepsilon(h), \]
where
\[ \varepsilon(h) = h^{s_L} \| L \|_{s, \Omega} + h^{s_y - 1} \| y \|_{s_y \Omega} + h^{s_p} \| p \|_{s_p \Omega} + h^{s_q - 1} \| q \|_{s_q \Omega}. \]

For the 2D case, we must restrict the domain to satisfy the limiting regularity requirements \( r_L > 1/2 \) and \( r_p > 1/2 \). Recall the singular exponent \( \xi = \xi(\omega) \) is the smallest real part of all of the roots of the equation \( (2.3) \). To ensure the high regularity condition \( (4.1) \) holds, by Theorem 2.8 we need \( \xi > 3/2 \), i.e., \( \omega \) satisfies
\[ \pi/3 \leq \omega \leq 0.839 \pi. \]

We note that this condition on the angle \( \omega \) is different from our earlier HDG works on Dirichlet boundary control for the Poisson and convection diffusion equations \([45, 47]\), where we needed \( \pi/3 \leq \omega < 2\pi/3 \). This different requirement here on the angle \( \omega \) is due to the different regularity results in Section 2 for the Stokes tangential Dirichlet boundary control problem.

**Corollary 4.2.** Suppose \( d = 2, f = 0 \), and \( y_d \in H^{\min(2, \xi)}(\Omega) \). Let \( \omega \in [\pi/3, 0.839 \pi) \) be the largest interior angle of \( \Gamma \), and define \( r_\Omega \) by
\[ r_\Omega = \min \left\{ \frac{3}{2}, \frac{\xi - 1}{2} \right\} \in (1, 3/2). \]

Then the regularity condition \( (4.1) \) is satisfied. Also, if \( k = 1 \), then for any \( r < r_\Omega \) we have
\[ \| L - L_h \|_{\mathcal{T}_h} \lesssim h^{r - 1/2}, \quad \| y - y_h \|_{\mathcal{T}_h} \lesssim h^r, \quad \| p - p_h \|_{\mathcal{T}_h} \lesssim h^{r - 1/2}, \]
\[ \| G - G_h \|_{\mathcal{T}_h} \lesssim h^r, \quad \| z - z_h \|_{\mathcal{T}_h} \lesssim h^r, \quad \| q - q_h \|_{\mathcal{T}_h} \lesssim h^r, \]
\[ \| u - u_h \|_{\mathcal{T}_h} \lesssim h^r. \]

Theorem 2.8 gives \( u \in H^r(\Gamma) \), and so the convergence rate for the control is optimal for \( k = 1 \) with respect to this global regularity result. However, Theorem 2.8 also gives the higher local regularity result \( (2.20) \) for the control: \( u \in H^r(\Gamma_i) \) for each boundary segment \( \Gamma_i \), where \( \kappa < \min\{3, \xi\} - 1/2 \). Our numerical results in Section 3 indicate that the actual convergence rate for \( k = 1 \) may indeed be restricted by the local regularity result instead of the global regularity result. A completely different method of proof is likely required to establish a sharper convergence rate for the control with respect to the local regularity result.

Also, Theorem 2.8 only yields global regularity results for the other variables. Our convergence rates for the flux \( L \) and pressure \( p \) are optimal for \( k = 1 \), but suboptimal for the other variables.

### 4.2 Preliminary material

We begin by defining the standard \( L^2 \) projections \( \Pi_K : L^2(\Omega) \to K_h, \Pi_V : L^2(\Omega) \to V_h \), and \( \Pi_W : L^2(\Omega) \to W_h \) satisfying
\[ (\Pi_K L, T)_{K} = (L, T)_{K} \quad \forall \ T \in [P_h(K)]^{d \times d}, \]  
\[ (\Pi_V y, v)_{K} = (y, v)_{K} \quad \forall \ v \in [P_{k+1}(K)]^d, \]  
\[ (\Pi_W p, w)_{K} = (p, w)_{K} \quad \forall \ w \in P_k(K). \]
For all faces $e$ of the simplex $K$, we also need the $L^2$-orthogonal projections $P_M$ and $P_M$ that map into $M_h$ and $M_h$, respectively:

$$\langle P_M u - u, \mu \rangle_e = 0, \quad \forall \mu \in M_h,$$

$$\langle P_M y - y, \mu \rangle_e = 0, \quad \forall \mu \in M_h.$$  

In the analysis, we use the following classical results:

$$\|\Pi_L - L\|_{\tau_h} \lesssim h^{\frac{4}{3}}\|L\|_{s_1, \Omega},$$

$$\|\Pi_V y - y\|_{\tau_h} \lesssim h^{s_y}\|y\|_{s_y, \Omega},$$

$$\|\Pi_L - L\|_{\partial \tau_h} \lesssim h^{\frac{1}{2}}\|L\|_{s_1, \Omega},$$

$$\|\Pi_V y - y\|_{\partial \tau_h} \lesssim h^{s_y}\|y\|_{s_y, \Omega},$$

$$\|P_M u - u\|_{\partial \tau_h} \lesssim h^{s_y}\|y\|_{s_y, \Omega}.$$  

(4.4a)  

(4.4b)  

(4.5a)  

(4.5b)  

(4.5c)  

(4.5d)

Similar projection error bounds hold for $G$, $z$ and $g$.

Define the HDG operator $\mathcal{B} : \mathbb{K}_h \times \mathbb{V}_h \times W^0_h \times M_h \times \mathbb{K}_h \times \mathbb{V}_h \times W^0_h \times M_h \rightarrow \mathbb{R}$ by

$$\mathcal{B}(L_h, y_h, p_h, \tilde{y}_h^0, T_1, v_1, w_1, \mu_1) = (L_h, T_1)_{\tau_h} + \langle y_h, \nabla \cdot T_1 \rangle_{\tau_h} - \langle \tilde{y}_h^0, T_1 n \rangle_{\partial \tau_h \setminus \epsilon^0_h}$$

$$+ (L_h, \nabla v_1)_{\tau_h} - (p_h, \nabla \cdot v_1)_{\tau_h} - \langle L_h n - p_h n - h^{-1} P_M y_h, v_1 \rangle_{\partial \tau_h}$$

$$- \langle h^{-1} \tilde{y}_h^0 v_1, \partial \tau_h \setminus \epsilon^0_h \rangle + \langle y_h, \nabla w_1 \rangle_{\tau_h} + \langle \tilde{y}_h^0 \cdot n, w_1 \rangle_{\partial \tau_h \setminus \epsilon^0_h}$$

$$+ \langle L_h n - p_h n - h^{-1} (P_M y_h - \tilde{y}_h^0), \mu_1 \rangle_{\partial \tau_h \setminus \epsilon^0_h}.$$  

(4.6)

This definition allows us to rewrite the HDG formulation of the optimality system (3.6): find $(L_h, G_h, y_h, z_h, p_h, q_h, \tilde{q}_h^0, u_h) \in \mathbb{K}_h \times \mathbb{K}_h \times \mathbb{V}_h \times \mathbb{V}_h \times W^0_h \times W^0_h \times M_h(o) \times M_h(o) \times M_h$ satisfying

$$\mathcal{B}(L_h, y_h, p_h, \tilde{y}_h^0, T_1, v_1, w_1, \mu_1) = \langle u_h \tau, T_1 n + h^{-1} v_1 \rangle_{\epsilon^0_h} + (f, v_1)_{\tau_h},$$  

(4.7a)  

$$\mathcal{B}(G_h, z_h, -q_h, \tilde{z}_h^0, T_2, v_2, w_2, \mu_2) = \langle y_h - y_d, v_2 \rangle_{\tau_h},$$  

(4.7b)  

$$\langle G_h n - h^{-1} P_M z_h, \mu_3 \rangle_{\epsilon^0_h} = \langle u_h, \mu_3 \rangle_{\epsilon^0_h},$$  

(4.7c)

for all $(T_1, T_2, v_1, v_2, w_1, w_2, \mu_1, \mu_2, \mu_3) \in \mathbb{K}_h \times \mathbb{K}_h \times \mathbb{V}_h \times \mathbb{V}_h \times W^0_h \times W^0_h \times M_h(o) \times M_h(o) \times M_h$.

We prove an energy identity for $\mathcal{B}$ that we use frequently in our analysis.

**Lemma 4.3.** For any $(T_h, v_h, w_h, \mu_h) \in \mathbb{K}_h \times \mathbb{V}_h \times W^0_h \times M_h$, we have

$$\mathcal{B}(T_h, v_h, w_h, \mu_h; T_h, v_h, w_h, \mu_h) = (T_h, T_h)_{\tau_h} + \langle h^{-1} (P_M v_h - \mu_h), P_M v_h - \mu_h \rangle_{\partial \tau_h \setminus \epsilon^0_h} + \langle h^{-1} P_M v_h, P_M v_h \rangle_{\epsilon^0_h}.$$

*Proof.* Using the definition of $\mathcal{B}$ in (4.6) gives

$$\mathcal{B}(T_h, v_h, w_h, \mu_h; T_h, v_h, w_h, \mu_h) = (T_h, T_h)_{\tau_h} + \langle v_h, \nabla \cdot T_h \rangle_{\tau_h} - \langle \mu_h, T_h n \rangle_{\partial \tau_h \setminus \epsilon^0_h}$$

$$+ (T_h, \nabla v_h)_{\tau_h} - (w_h, \nabla \cdot v_h)_{\tau_h} - \langle T_h n - w_h n - h^{-1} P_M v_h, v_h \rangle_{\partial \tau_h}$$

$$- \langle h^{-1} \mu_h, v_h \rangle_{\partial \tau_h \setminus \epsilon^0_h} - \langle v_h, \nabla w_h \rangle_{\tau_h} + \langle \mu_h \cdot n, w_h \rangle_{\partial \tau_h \setminus \epsilon^0_h}$$

$$+ \langle T_h n - w_h n - h^{-1} (P_M v_h - \mu_h), \mu_h \rangle_{\partial \tau_h \setminus \epsilon^0_h}$$

$$= (T_h, T_h)_{\tau_h} + \langle h^{-1} (P_M v_h - \mu_h), v_h - \mu_h \rangle_{\partial \tau_h \setminus \epsilon^0_h} + \langle h^{-1} P_M v_h, P_M v_h \rangle_{\epsilon^0_h}.$$  

\[ \square \]
Next, we give another property of the HDG operator \( B \) that is fundamental to our analysis.

**Lemma 4.4.** For all \((\mathbb{L}_h, G_h, y_h, z_h, p_h, q_h, \hat{y}_h^0, \hat{z}_h^0) \in K_h \times K_h \times V_h \times V_h \times W_h^0 \times W_h \times M_h(o) \times M_h(o),\) we have

\[
B(\mathbb{L}_h, y_h, p_h, \hat{y}_h^0; -G_h, z_h, q_h, \hat{z}_h^0) + B(G_h, z_h, -q_h, \hat{z}_h^0; \mathbb{L}_h, -y_h, p_h, -\hat{y}_h^0) = 0.
\]

**Proof.** By the definition of \( B \) in (4.6), we have

\[
B(\mathbb{L}_h, y_h, p_h, \hat{y}_h^0; -G_h, z_h, q_h, \hat{z}_h^0) + B(G_h, z_h, -q_h, \hat{z}_h^0; \mathbb{L}_h, -y_h, p_h, -\hat{y}_h^0)
= -(\mathbb{L}_h, G_h)_T_h - (y_h, \nabla \cdot G_h)_T_h + \langle \hat{y}_h^0, \nabla G_h \rangle_{\partial T_h \setminus e_h} + (L_h, \nabla z_h)_T_h
- (p_h, \nabla \cdot z_h)_T_h - \langle L_h n - p_h n - h^{-1} P_M y_h, z_h \rangle_{\partial T_h} - h^{-1} \langle \hat{y}_h^0, z_h \rangle_{\partial T_h \setminus e_h}^0
- (y_h, \nabla q_h)_T_h + \langle \hat{y}_h^0 \cdot q_h, q_h \rangle_{\partial T_h \setminus e_h} + \langle L_h n - p_h n - h^{-1} (P_M y_h - \hat{y}_h^0), \hat{z}_h^0 \rangle_{\partial T_h \setminus e_h}^0
+ (G_h, \mathbb{L}_h)_T_h + \langle z_h, \nabla \cdot L_h \rangle_{T_h} - \langle \hat{z}_h^0, L_h n \rangle_{\partial T_h \setminus e_h}^0 - \langle G_h, \nabla y_h \rangle_{T_h}
- (q_h, \nabla \cdot y_h)_T_h + \langle G_h n + q_h n - h^{-1} P_M z_h, y_h \rangle_{\partial T_h} + h^{-1} \langle \hat{z}_h^0, y_h \rangle_{\partial T_h \setminus e_h}^0
- (z_h, \nabla p_h)_T_h + \langle \hat{z}_h^0 \cdot n, p_h \rangle_{\partial T_h \setminus e_h} + \langle G_h n + q_h n - h^{-1} (P_M z_h - \hat{z}_h^0), \hat{y}_h^0 \rangle_{\partial T_h \setminus e_h}^0.
\]

Integration by parts gives the result. \( \square \)

**Proposition 4.5.** There exists a unique solution of the HDG discretized optimality system (4.7).

**Proof.** Since the system (4.7) is finite dimensional, we only need to prove solutions are unique. Therefore, we assume \( y_{h_1} = f = 0 \) and we show zero is the only solution of (4.7).

First, take \((T_1, v_1, w_1, \mu_1) = (-G_h, z_h, q_h, \hat{z}_h^0), (T_2, v_2, w_2, \mu_2) = (\mathbb{L}_h, -y_h, p_h, -\hat{y}_h^0), \) and \( \mu_3 = -u_h \) in the HDG equations (4.7a), (4.7b), and (4.7c) respectively, by Lemma 4.4 and sum to obtain

\[
B(\mathbb{L}_h, y_h, p_h, \hat{y}_h^0; -G_h, z_h, q_h, \hat{z}_h^0) + B(G_h, z_h, -q_h, \hat{z}_h^0; \mathbb{L}_h, -y_h, p_h, -\hat{y}_h^0)
= -(y_h, y_h)_T_h - \gamma(u_h, u_h)_{\varepsilon_h}^0
= 0.
\]

Since \( \gamma > 0 \), we have \( y_h = 0 \) and \( u_h = 0 \).

Next, taking \((\mathbb{T}_h, v_h, w_h, \mu_h) = (\mathbb{L}_h, y_h, p_h, \hat{y}_h^0) \) and \((\mathbb{T}_h, v_h, w_h, \mu_h) = (G_h, z_h, p_h, \hat{z}_h^0) \) in Lemma 4.3 shows \( \mathbb{L}_h, G_h, \) and \( \hat{y}_h^0 \) are all zero, \( P_M z_h = 0 \) on \( \varepsilon_h^0 \), and \( P_M z_h - \hat{z}_h^0 = 0 \) on \( \partial T_h \setminus \varepsilon_h^0 \). Then, since \( z_h = 0 \) on \( \varepsilon_h^0 \) we have

\[
P_M z_h - \hat{z}_h^0 = 0. \tag{4.8}
\]

Substituting (4.8) into (3.6d), and remembering again \( \hat{z}_h = 0 \) on \( \varepsilon_h^0 \), we get

\[
-(z_h, \nabla \cdot T_2)_T_h + \langle P_M z_h, T_2 n \rangle_{\partial T_h} = 0 \quad \text{for all } T_2 \in K_h.
\]

Use the property of \( P_M \) in (4.4), integrate by parts, and take \( T_2 = \nabla z_h \) to obtain

\[
(\nabla z_h, \nabla z_h)_T_h = 0.
\]

Therefore, \( z_h \) is constant on each \( K \in T_h \), and also \( z_h = P_M z_h = \hat{z}_h \) on \( \partial T_h \). Since \( \hat{z}_h = 0 \) on \( \varepsilon_h^0 \) and is single-valued on each face, we have \( z_h = 0 \) on each \( K \in T_h \), and therefore also \( \hat{z}_h = 0 \).
Lastly, since $L_h$, $y_h$, $u_h$, and $\tilde{y}_h^\alpha$ are all zero by equation (3.6b), we have
\[-(p_h, \nabla \cdot v_1)_{T_h} + \langle p_h n, v_1 \rangle_{\partial T_h} = 0 \quad \text{for all } v_1 \in V_h.\]
Integrating by parts and taking $v_1 = \nabla p_h$ gives
\[(\nabla p_h, \nabla p_h)_{T_h} = 0,\]
which implies $p_h$ is piecewise constant on each $K \in T_h$. By equation (3.6g) we have
\[\langle p_h n, \mu_1 \rangle_{\partial T_h \setminus \partial \Omega} = 0 \quad \text{for all } \mu_1 \in \mathcal{M}_h(\partial),\]
which implies $p_h$ is single-valued on the inter-element boundaries. Therefore, $p_h$ is constant on the whole domain $\Omega$.

Finally, $p_h = 0$ is obtained using $(p_h, 1)_{T_h} = 0$, and the same argument gives $q_h = 0$. This completes the proof. \(\square\)

### 4.3 Proof of the main result

In our proof of the main result, we use the following auxiliary HDG problem: for the optimal control $u$ fixed, find
\[
(L_h(u), G_h(u), y_h(u), z_h(u), p_h(u), q_h(u), \tilde{y}_h^0(u), \tilde{z}_h^0(u)) 
\in \mathbb{K}_h \times \mathbb{K}_h \times V_h \times V_h \times W_h^0 \times W_h^0 \times \mathcal{M}_h(\partial) \times \mathcal{M}_h(\partial)
\]
such that
\[
\mathcal{B}(L_h(u), y_h(u), p_h(u), \tilde{y}_h^0(u); \tau_1, v_1, w_1, \mu_1) = (f, v_1)_{\tau_h} + \langle P_M u \tau, h^{-1} v_1 + \tau_1 n \rangle_{\partial h}, \quad (4.9a)
\]
\[
\mathcal{B}(G_h(u), z_h(u), q_h(u), \tilde{z}_h^0(u); \tau_2, v_2, w_2, \mu_2) = (y_h(u) - y_d, v_2)_{\tau_h}, \quad (4.9b)
\]
for all $(\tau_1, \tau_2, v_1, v_2, w_1, w_2, \mu_1, \mu_2) \in \mathbb{K}_h \times \mathbb{K}_h \times V_h \times V_h \times W_h^0 \times W_h^0 \times \mathcal{M}_h(\partial) \times \mathcal{M}_h(\partial)$.

We split the proof of the main result, Theorem 4.1, into eleven steps. We first consider the solution of the mixed form (2.21a)-(2.21f) of the optimality system, and the solution of the auxiliary problem. We estimate the errors using $L^2$ projections. Define
\[
\delta^L = L - \Pi_K L, \quad \varepsilon^L_h = \Pi_K L - L_h(u),
\]
\[
\delta^y = y - \Pi_V y, \quad \varepsilon^y_h = \Pi_V y - y_h(u),
\]
\[
\delta^p = p - \Pi_W p, \quad \varepsilon^p_h = \Pi_W p - p_h(u),
\]
\[
\delta^\nu = y - P_MY, \quad \varepsilon^\nu_h = P_MY - \tilde{y}_h(u),
\]
\[
\hat{\delta}_1 = \delta^\nu n - \delta^p n - h^{-1} P_M \delta^\nu,
\]
where $\tilde{y}_h(u) = \tilde{y}_h^0(u)$ on $\varepsilon^\nu_h$ and $\tilde{y}_h(u) = P_M u \tau$ on $\varepsilon^0_h$, which gives $\varepsilon^\nu = 0$ on $\varepsilon^0_h$.

#### 4.3.1 Step 1: The error equation for part 1 of the auxiliary problem (4.9a).

Lemma 4.6. We have
\[
\mathcal{B}(\varepsilon^L_h, \varepsilon^y_h, \varepsilon^p_h, \varepsilon^\nu_h; \tau_1, v_1, w_1, \mu_1) = (\hat{\delta}_1, v_1)_{\partial T_h} - (\hat{\delta}_1, \mu_1)_{\partial T_h \setminus \partial \Omega}. \quad (4.11)
\]
Proof. Using the definition of \( \mathcal{B} \) \([4.6]\) gives
\[
\mathcal{B}(\Pi_K \mathbb{L}, \Pi_V y, \Pi_W p, P_M y; T_1, v_1, w_1, \mu_1)
= (\Pi_K \mathbb{L}, T_1)_{T_h} + (y, \nabla \cdot T_1)_{T_h} - (y, T_1 n)_{\partial T_h \setminus \epsilon_h^0} + (\Pi_K \mathbb{L}, \nabla v_1)_{T_h}
- (\Pi_K \mathbb{L}, \nabla \cdot v_1)_{T_h} - (\Pi_K \mathbb{L}, \nabla w_1)_{T_h} + (P_M y, \nabla v_1)_{T_h}
+ (\Pi_K \mathbb{L}, \nabla w_1)_{T_h} - (\Pi_K \mathbb{L}, \nabla w_1)_{T_h}
+ (P_M y, v_1)_{T_h} + (P_M y, w_1)_{T_h}.
\]
By properties of \( L^2 \) projections, we have
\[
\mathcal{B}(\Pi_K \mathbb{L}, \Pi_V y, \Pi_W p, P_M y; T_1, v_1, w_1, \mu_1)
= (\Pi_K \mathbb{L}, T_1)_{T_h} + (y, \nabla \cdot T_1)_{T_h} - (y, T_1 n)_{\partial T_h \setminus \epsilon_h^0} + (\Pi_K \mathbb{L}, \nabla v_1)_{T_h}
- (p, \nabla \cdot v_1)_{T_h} - (\Pi_K \mathbb{L}, \nabla w_1)_{T_h}
+ (\Pi_K \mathbb{L}, \nabla w_1)_{T_h} + (\Pi_K \mathbb{L}, \nabla w_1)_{T_h}
+ (h^{-1} P_M \delta^y, \mu_1)_{\partial T_h \setminus \epsilon_h^0} - (\delta^y n - \delta^p n, \mu_1)_{\partial T_h \setminus \epsilon_h^0}.
\]
The exact solution \((\mathbb{L}, y, p)\) satisfies
\[
(\mathbb{L}, T_1)_{T_h} + (y, \nabla \cdot T_1)_{T_h} - (y, T_1 n)_{\partial T_h \setminus \epsilon_h^0} = (u, T_1 n)_{\epsilon_h^0},
(\mathbb{L}, \nabla v_1)_{T_h} - (p, \nabla \cdot v_1)_{T_h} - (\Pi_K \mathbb{L}, \nabla w_1)_{T_h} = (f, v_1),
- (y, \nabla w_1)_{T_h} + (y \cdot n, w_1)_{\partial T_h \setminus \epsilon_h^0} = 0,
(\Pi_K \mathbb{L}, \nabla w_1)_{T_h} = 0,
\]
for all \((T_1, v_1, w_1, \mu_1) \in \mathbb{K}_h \times V_h \times W_h \times M_h(o).\) Therefore,
\[
\mathcal{B}(\Pi_K \mathbb{L}, \Pi_V y, \Pi_W p, P_M y; T_1, v_1, w_1, \mu_1)
= (u, T_1 n)_{\epsilon_h^0} + (f, v_1)_{T_h} + (\delta^y n - \delta^p n - h^{-1} P_M \delta^y, v_1)_{\partial T_h}
+ (h^{-1} P_M \delta^y, \mu_1)_{\partial T_h \setminus \epsilon_h^0} - (\delta^y n - \delta^p n, \mu_1)_{\partial T_h \setminus \epsilon_h^0} = (f, v_1)_{T_h} = (P_M u, \delta^y v_1 + T_1 n)_{\epsilon_h^0} + (\delta^y n, v_1)_{\partial T_h} - (\delta^y n, \mu_1)_{\partial T_h \setminus \epsilon_h^0}.
\]
Subtracting part 1 of the auxiliary problem \([4.9a]\) gives the result.

\[\square\]

### 4.3.2 Step 2: Estimate for \( \varepsilon_h^y \)

**Lemma 4.7.** We have
\[
\|\nabla \varepsilon_h^y\|_{T_h} + h^{-\frac{1}{2}} \|\varepsilon_h^y - \hat{\varepsilon}_h^y\|_{\partial T_h} \lesssim \|\varepsilon_h^y\|_{T_h} + h^{-\frac{1}{2}} \|P_M \varepsilon_h^y - \hat{\varepsilon}_h^y\|_{\partial T_h}. \tag{4.12}
\]

**Proof.** The proof closely follows a line of reasoning used to establish a similar result in \[63\], Lemma 3.2.

First, in the definition of \( \mathcal{B} \) \([4.6]\) take \((T_1, v_1, w_1, \mu_1) = (\varepsilon_h^y, 0, 0, 0)\) and use Lemma 4.6 to get
\[
(\nabla \varepsilon_h^y, \nabla \varepsilon_h^y)_{T_h} = (\varepsilon_h^y, \nabla \varepsilon_h^y)_{T_h} + (\varepsilon_h^y - \hat{\varepsilon}_h^y, \nabla \varepsilon_h^y n)_{\partial T_h},
\]
where we used \( \varepsilon_h = 0 \) on \( \partial \Omega \). Since \( \nabla \varepsilon_h \mathbf{n} \in M_h \) on each face \( e \in \partial \Omega \), we have
\[
\left\| \nabla \varepsilon_h \right\|_{T_h}^2 = (\varepsilon_h^y, \nabla \varepsilon_h^y)_{T_h} + \langle P_M \varepsilon_h^y - \varepsilon_y^h, \nabla \varepsilon_h^y \rangle_{\partial T_h} \\
\leq ||\varepsilon_h^y||_{T_h} \left\| \nabla \varepsilon_h^y \right\|_{T_h} + \|P_M \varepsilon_h^y - \varepsilon_y^h\|_{\partial T_h} \|\nabla \varepsilon_h^y\|_{T_h} \\
\leq \left( ||\varepsilon_h^y||_{T_h} + Ch^{-\frac{1}{2}} \|P_M \varepsilon_h^y - \varepsilon_y^h\|_{\partial T_h} \right) \|\nabla \varepsilon_h^y\|_{T_h},
\]
where we used a trace inequality and an inverse inequality. Moreover,
\[
\|\varepsilon_h^y - \varepsilon_y^h\|_{\partial T_h} = \|\varepsilon_h^y - P_M \varepsilon_h^y\|_{\partial T_h} + \|P_M \varepsilon_h^y - \varepsilon_y^h\|_{\partial T_h} \\
\leq ||\varepsilon_h^y - \varepsilon_y^h||_{\partial T_h} + \|P_M \varepsilon_h^y - \varepsilon_y^h\|_{\partial T_h} \\
\leq Ch^{-\frac{1}{2}} \|\nabla \varepsilon_h^y\|_{T_h} + \|P_M \varepsilon_h^y - \varepsilon_y^h\|_{\partial T_h} \\
\leq h^\frac{1}{2} ||\varepsilon_h^y||_{T_h} + \|P_M \varepsilon_h^y - \varepsilon_y^h\|_{\partial T_h},
\]
where we used the best approximation property of the \( L^2 \)-projection \( P_M \), \( \overline{\varepsilon} \) denotes the average of \( \varepsilon_h^y \) on each \( K \in T_h \), and we applied the Poincaré inequality on \( K \in T_h \). Combining the above inequalities gives the result.

\[ \square \]

**Lemma 4.8.** We have
\[
\left| \|\varepsilon_h^y\|^2_{T_h} + h^{-\frac{1}{2}} \|P_M \varepsilon_h^y - \varepsilon_y^h\|^2_{\partial T_h} \right| \leq h^{\frac{3}{2}L} \|L\|_{\partial^L_{\Omega}} + h^{\frac{1}{2}p} \|p\|_{\partial^p_{\Omega}} + h^{\frac{1}{2}p-1} \|y\|_{\partial^y_{\Omega}}.
\]

**Proof.** First, since \( \varepsilon_y^h = 0 \) on \( \partial_0 \), the energy identity for \( \mathcal{B} \) in Lemma 4.3 gives
\[
\mathcal{B}(\varepsilon_h^y, \varepsilon_h^y, \varepsilon_h^y, \varepsilon_h^y, \varepsilon_h^y, \varepsilon_h^y) = \|\varepsilon_h^y\|^2_{T_h} + h^{-\frac{1}{2}} \|P_M \varepsilon_h^y - \varepsilon_y^h\|^2_{\partial T_h}.
\]

Then taking \((T_1, v_1, w_1, \mu_1) = (\varepsilon_h^y, \varepsilon_h^y, \varepsilon_h^y, \varepsilon_h^y, \varepsilon_h^y, \varepsilon_h^y)\) in the error equation (4.11) gives
\[
\|\varepsilon_h^y\|^2_{T_h} + h^{-\frac{1}{2}} \|P_M \varepsilon_h^y - \varepsilon_y^h\|^2_{\partial T_h} = (\delta^y \mathbf{u} - \delta^p \mathbf{n} - h^{-1} P_M \delta^y \mathbf{u}, \varepsilon_h^y - \varepsilon_y^h)_{\partial T_h} \\
= (\delta^y \mathbf{u} - \delta^p \mathbf{n} - h^{-1} P_M \delta^y \mathbf{u}, \varepsilon_h^y - \varepsilon_y^h)_{\partial T_h} \\
= (\delta^y \mathbf{n} - \delta^p \mathbf{n} - h^{-1} P_M \delta^y \mathbf{u}, \varepsilon_y^h - \varepsilon_y^h, \varepsilon_h^y, \varepsilon_h^y, \varepsilon_h^y, \varepsilon_h^y)_{\partial T_h} \\
\leq h^\frac{1}{2} (\|\delta^y\|_{\partial T_h} + \|\delta^p\|_{\partial T_h}) \|\varepsilon_h^y - \varepsilon_y^h\|_{\partial T_h} + h^{-\frac{1}{2}} \|\delta^y\|_{\partial T_h} h^{-\frac{1}{2}} \|P_M \varepsilon_h^y - \varepsilon_y^h\|_{\partial T_h}.
\]

By Young’s inequality and Lemma 4.7, we obtain
\[
\|\varepsilon_h^y\|^2_{T_h} + h^{-\frac{1}{2}} \|P_M \varepsilon_h^y - \varepsilon_y^h\|^2_{\partial T_h} \leq h \|\varepsilon^2\|^2_{\partial T_h} + h \|\delta^p\|^2_{\partial T_h} + h^{-1} \|\delta^y\|^2_{\partial T_h} \\
\leq h^{2L} \|L\|^2_{\partial^L_{\Omega}} + h^{2p} \|p\|^2_{\partial^p_{\Omega}} + h^{2p-2} \|y\|^2_{\partial^y_{\Omega}}.
\]

\[ \square \]

**4.3.3 Step 3: Estimate for \( \varepsilon_h^p \)**

**Lemma 4.9.** We have
\[
\|\varepsilon_h^p\|_{T_h} \leq h^{\frac{3}{2}L} \|L\|_{\partial^L_{\Omega}} + h^{\frac{1}{2}p} \|p\|_{\partial^p_{\Omega}} + h^{\frac{1}{2}p-1} \|y\|_{\partial^y_{\Omega}}.
\]
Proof. We utilize an inf-sup proof strategy for the pressure; c.f., [17 Proposition 3.4], [64 Lemma 5.3]. We know [8] that for any function \( \vartheta \in L^2(\Omega) \) such that \( (\vartheta, 1)_{\Omega} = 0 \), we have

\[
\| \vartheta \|_{\Omega} \lesssim \sup_{v \in H^1_{\Omega}(\Omega) \setminus \{0\}} \frac{(\vartheta, \nabla \cdot v)_{\Omega}}{\|v\|_{H^1(\Omega)}}, \tag{4.15}
\]

Since

\[
(\varepsilon_h^p, 1)_{T_h} = (\Pi W p - p_h(u), 1)_{T_h} = (\Pi W p, 1)_{T_h} - (p_h(u), 1)_{T_h} = 0,
\]

we can take \( \vartheta := \varepsilon_h^p \) in (4.15). Then we have

\[
\| \varepsilon_h^p \|_{\Omega} \lesssim \sup_{v \in H^1_{\Omega}(\Omega) \setminus \{0\}} \frac{(\varepsilon_h^p, \nabla \cdot v)_{\Omega}}{\|v\|_{H^1(\Omega)}},
\]

and

\[
(\varepsilon_h^p, \nabla \cdot v)_{\Omega} = -(\nabla \cdot \varepsilon_h^p, \Pi V v)_{T_h} + \langle \varepsilon_h^p, v \cdot n \rangle_{\partial T_h}.
\]

Next, taking \((T_1, v_1, w_1, \mu_1) = (0, 0, 0, 0)\) in [Lemma 4.6] gives

\[
(\nabla \varepsilon_h^p, \Pi V v)_{T_h} = (\nabla \cdot \varepsilon_h^p, \Pi V v)_{T_h} - \langle h^{-1}(P_M \varepsilon_h^y - \varepsilon_h^y), \Pi V v \rangle_{\partial T_h} + \langle \delta_1, \Pi V v \rangle_{\partial T_h},
\]

where we used \( \varepsilon_h^y = 0 \) on \( \partial \Omega \). The above two equalities give

\[
(\varepsilon_h^p, \nabla \cdot v)_{\Omega} = -(\nabla \cdot \varepsilon_h^p, \Pi V v)_{T_h} + \langle h^{-1}(P_M \varepsilon_h^y - \varepsilon_h^y), \Pi V v \rangle_{\partial T_h} + \langle \varepsilon_h^p, v \cdot n \rangle_{\partial T_h} - \langle \delta_1, \Pi V v \rangle_{\partial T_h}.
\]

Next, we take \((T_1, v_1, w_1, \mu_1) = (0, 0, 0, P_M v)\) in [Lemma 4.6] since \( v \in H^1_{\Omega}(\Omega) \) we have

\[
\langle \varepsilon_h^p n - \varepsilon_h^p n, P_M v \rangle_{\partial T_h} = \langle h^{-1}(P_M \varepsilon_h^y - \varepsilon_h^y), P_M v \rangle_{\partial T_h} - \langle \delta_1, P_M v \rangle_{\partial T_h}.
\]

This implies

\[
(\varepsilon_h^p, \nabla \cdot v)_{\Omega} = (\varepsilon_h^p, \nabla v)_{T_h} + \langle h^{-1}(P_M \varepsilon_h^y - \varepsilon_h^y), P_M v \rangle_{\partial T_h} - \langle \delta_1, P_M v \rangle_{\partial T_h}.
\]

Apply the Cauchy-Schwarz inequality to obtain

\[
\| (\varepsilon_h^p, \nabla \cdot v)_{\Omega} \| \lesssim \| \varepsilon_h^p \|_{T_h} \| \nabla v \|_{T_h} + h^{-\frac{1}{2}} \| P_M \varepsilon_h^y - \varepsilon_h^y \|_{\partial T_h} \| \nabla v \|_{T_h} + h^{\frac{1}{2}} \| \delta_1 \|_{\partial T_h} \| \nabla v \|_{T_h}.
\]

Use [Lemma 4.8] to complete the proof:

\[
\| \varepsilon_h^p \| \lesssim \| \varepsilon_h^p \|_{T_h} + h^{-\frac{1}{2}} \| P_M \varepsilon_h^y - \varepsilon_h^y \|_{\partial T_h} + h^{\frac{1}{2}} \| \delta_1 \|_{\partial T_h} \lesssim h^s L \| \varphi \|_1 + h^s \| \varphi \|_{\partial T_h} + h^{s-1} \| \gamma \|_{\partial T_h}.
\]
4.3.4 Step 4: Estimate for $\varepsilon^y_h$ by a duality argument.

For any $\Theta \in L^2(\Omega)$, the dual problem is given by

\begin{align*}
A - \nabla \Phi &= 0 & \text{in } \Omega, \\
-\nabla \cdot A - \nabla \Psi &= \Theta & \text{in } \Omega, \\
\nabla \cdot \Phi &= 0 & \text{in } \Omega, \\
\Phi &= 0 & \text{on } \partial \Omega.
\end{align*}

(4.16)

Since the domain $\Omega$ is convex, we have the following regularity estimate

$$
\|\mathbb{A}\|_1 + \|\Phi\|_2 + \|\Psi\|_1 \leq C\|\Theta\|_\Omega.
$$

(4.17)

In the proof of the next lemma for estimating $\varepsilon^y_h$, we use the following notation:

\begin{align*}
\delta^h &= A - \Pi_A \mathbb{A}, & \delta^\Phi &= \Phi - \Pi_V \Phi, & \delta^\Psi &= \Psi - \Pi_W \Psi, & \delta^\Phi &= \Phi - P_M \Phi.
\end{align*}

(4.18)

Lemma 4.10. We have

$$
\|\varepsilon^y_h\|_{T_h} \lesssim h^{s_l+1} \|L\|_{s_l,\Omega} + h^{s_p+1} \|P\|_{s_p,\Omega} + h^{s_q} \|y\|_{s_q,\Omega}.
$$

(4.19)

Proof. We consider the dual problem (4.16) with $\Theta = \varepsilon^y_h$. In the definition of $B$ (4.6), we take $(T_1, v_1, w_1, \mu_1) = (-\Pi_A \mathbb{A}, \Pi_V \Phi, \Pi_W \Psi, P_M \Phi)$. Since $\Phi = 0$ on $\varepsilon^y_h$; $\varepsilon^y_\hat{h} = 0$ on $\varepsilon^y_h$; and $\nabla \cdot \Phi = 0$, we have

\begin{align*}
B(\varepsilon^y_h, \varepsilon^y_h, \varepsilon^y_h, \varepsilon^y_h, \varepsilon^y_h, \varepsilon^y_h, \varepsilon^y_h, \varepsilon^y_h, \varepsilon^y_h) &= -\langle \varepsilon^y_h, \Pi_A \mathbb{A} \rangle_{T_h} - \langle \varepsilon^y_h, \nabla \cdot \Pi_A \mathbb{A} \rangle_{T_h} + \langle \varepsilon^y_h, \Pi_A \mathbb{A}_h \rangle_{\partial T_h} - \langle \nabla \cdot \varepsilon^y_h, \Pi_V \Phi \rangle_{T_h} \\
&- \langle \varepsilon^y_h, \Pi_V \Phi \rangle_{T_h} + \langle \varepsilon^y_h, \Pi_A \mathbb{A} \rangle_{T_h} + \langle \varepsilon^y_h, \Pi_A \mathbb{A}_h \rangle_{\partial T_h} - \langle \nabla \cdot \varepsilon^y_h, \Pi_W \Psi \rangle_{T_h} \\
&+ \langle \varepsilon^y_h, \Pi_W \Psi \rangle_{\partial T_h} + \langle \varepsilon^y_h, \Pi_V \Phi \rangle_{\partial T_h} + \langle \varepsilon^y_h, \nabla \cdot \varepsilon^y_h, \Pi_V \Phi \rangle_{\partial T_h} \\
&- \langle \varepsilon^y_h, \nabla \cdot \varepsilon^y_h, \Pi_V \Phi \rangle_{\partial T_h} + \langle \varepsilon^y_h, \nabla \cdot \varepsilon^y_h, \Pi_V \Phi \rangle_{\partial T_h} \\
&- \langle \varepsilon^y_h, \varepsilon^y_h \rangle_{T_h} + \langle \varepsilon^y_h, \varepsilon^y_h \rangle_{\partial T_h} - \langle \varepsilon^y_h, \varepsilon^y_h \rangle_{T_h} + \langle \varepsilon^y_h, \varepsilon^y_h \rangle_{\partial T_h} \\
&= \langle \varepsilon^y_h, \nabla \cdot \varepsilon^y_h \rangle_{T_h} + \langle \varepsilon^y_h, \varepsilon^y_h \rangle_{T_h} - \langle \varepsilon^y_h, \varepsilon^y_h \rangle_{\partial T_h} + \langle \varepsilon^y_h, \nabla \cdot \varepsilon^y_h \rangle_{\partial T_h} \\
&- \langle \varepsilon^y_h, \nabla \cdot \varepsilon^y_h \rangle_{\partial T_h} + \langle \varepsilon^y_h, \varepsilon^y_h \rangle_{\partial T_h} - \langle \varepsilon^y_h, \varepsilon^y_h \rangle_{\partial T_h}.
\end{align*}

Here we used $\varepsilon^y_\hat{h} = 0$ on $\varepsilon^y_h$; $\mathbb{A} + \Psi \in \mathbb{H}(\text{div}, \Omega)$; and $(\varepsilon^y_\hat{h}, (\mathbb{A} + \Psi) \mathbb{I} \mathbb{I})_{\partial T_h} = 0$, which holds since $\varepsilon^y_\hat{h}$ is a single-valued function on interior edges and $\varepsilon^y_\hat{h} = 0$ on $\varepsilon^y_h$.

Next, integrate by parts to obtain

\begin{align*}
\langle \varepsilon^y_h, \nabla \cdot \delta^A \rangle_{T_h} &= -\langle \nabla \varepsilon^y_h, \delta^A \rangle_{T_h} + \langle \varepsilon^y_h, \delta^\Phi \rangle_{T_h} = \langle \varepsilon^y_h, \delta^\Phi \rangle_{T_h}, \\
\langle \varepsilon^y_h, \nabla \cdot \delta^\Phi \rangle_{T_h} &= -\langle \nabla \varepsilon^y_h, \delta^\Phi \rangle_{T_h} + \langle \varepsilon^y_h, \delta^\Phi \rangle_{T_h} = \langle \varepsilon^y_h, \delta^\Phi \rangle_{T_h}, \\
\langle \varepsilon^y_h, \nabla \delta^\Psi \rangle_{T_h} &= \langle \varepsilon^y_h, \delta^\Phi \rangle_{T_h} - \langle \nabla \cdot \varepsilon^y_h, \delta^\Psi \rangle_{T_h} = \langle \varepsilon^y_h, \delta^\Phi \rangle_{T_h}.
\end{align*}
Then
\[
\mathcal{B}(\varepsilon_h^y, \varepsilon_h^p, \varepsilon_h^\theta, -\Pi_K A, \Pi_V \Phi, \Pi_W \Psi, P_M \Phi) = \| \varepsilon_h^y \|_{T_h}^2 - (h^{-1}(P_M \varepsilon_h^y - \tilde{\varepsilon}_h^y), \delta^\Phi)_{\partial T_h} + (\varepsilon_h^y - \tilde{\varepsilon}_h^y, \delta^h n + \delta^\Psi n)_{\partial T_h}.
\]

On the other hand, using \( \Phi = 0 \) on \( \varepsilon_h^0 \) and the error equation (4.11) gives
\[
\mathcal{B}(\varepsilon_h^y, \varepsilon_h^p, \varepsilon_h^\theta, -\Pi_K A, \Pi_V \Phi, \Pi_W \Psi, P_M \Phi) = (\delta_1, \Pi_V \Phi - P_M \Phi)_{\partial T_h}.
\]

We note that
\[
(\delta_1, P_M \Phi)_{\partial T_h} = \langle -\Pi_K \mathbb{L} n + \Pi_W p n + h^{-1}P_M (\Pi_V y - y), P_M \Phi \rangle_{\partial T_h} + \langle \mathbb{L} n - p n, P_M \Phi \rangle_{\partial T_h} = \langle -\Pi_K \mathbb{L} n + \Pi_W p n + h^{-1}P_M (\Pi_V y - y), \Phi \rangle_{\partial T_h} + \langle \mathbb{L} n - p n, \Phi \rangle_{\partial T_h} = (\delta_1, \Phi)_{\partial T_h}.
\]

Here we used the fact \( \langle \mathbb{L} n - p n, \Phi \rangle_{\partial T_h} = 0 \) and \( \langle \mathbb{L} n - p n, P_M \Phi \rangle_{\partial T_h} = 0 \) since \( \mathbb{L} - p \mathbb{I} \in \mathbb{H}^{(\text{div}, \Omega)} \) and \( \Phi = 0 \) on \( \varepsilon_h^0 \).

Compare the above equations to obtain
\[
\| \varepsilon_h^y \|_{T_h}^2 \leq -\langle \varepsilon_h^y - \tilde{\varepsilon}_h^y, \delta^h n + \delta^\Psi n \rangle_{\partial T_h} - (\delta_1, \delta^\Phi)_{\partial T_h} + \langle h^{-1}(P_M \varepsilon_h^y - \tilde{\varepsilon}_h^y), \delta^\Phi \rangle_{\partial T_h} \\
\leq h^{-\frac{1}{2}} \| \varepsilon_h^y - \tilde{\varepsilon}_h^y \|_{\partial T_h} h^\frac{1}{2} (\| \delta^h n \|_{\partial T_h} + \| \delta^\Psi n \|_{\partial T_h}) + \| \delta_1 \|_{\partial T_h} \| \delta^\Phi \|_{\partial T_h} \\
+ h^{-\frac{1}{2}} \| P_M \varepsilon_h^y - \tilde{\varepsilon}_h^y \|_{\partial T_h} h^{-\frac{1}{2}} \| \delta^\Phi \|_{\partial T_h} \\
\leq (h^n + h^{s_y} + h^{s_p} \| y \|_{s_p, \Omega}) \| \varepsilon_h^y \|_{T_h}.
\]

\[
\square
\]

**Lemma 4.8** \[Lemma 4.9\] \[Lemma 4.10\] \[Lemma 4.11\] and the triangle inequality yield optimal convergence rates for \( \| \mathbb{L} - \mathbb{L}_h(u) \|_{T_h}, \| p - p_h(u) \|_{T_h} \) and \( \| y - y_h(u) \|_{T_h} \):

**Lemma 4.11.** We have
\[
\| \mathbb{L} - \mathbb{L}_h(u) \|_{T_h} \leq h^n \| y \|_{s_p, \Omega} + h^{s_p} \| p \|_{s_p, \Omega} + h^{s_y-1} \| y \|_{s_y, \Omega}, \quad (4.20a)
\]
\[
\| p - p_h(u) \|_{T_h} \leq h^n \| y \|_{s_p, \Omega} + h^{s_p} \| p \|_{s_p, \Omega} + h^{s_y-1} \| y \|_{s_y, \Omega}, \quad (4.20b)
\]
\[
\| y - y_h(u) \|_{T_h} \leq h^{n+1} \| y \|_{s_p, \Omega} + h^{s_p+1} \| p \|_{s_p, \Omega} + h^{s_y} \| y \|_{s_y, \Omega}. \quad (4.20c)
\]

### 4.3.5 Step 5: The error equation for part 2 of the auxiliary problem (4.9b).

We continue to focus on the solution of the auxiliary problem and the solution of the mixed formulation (2.21a)-(2.21d) of the optimality system. Next, we consider on the dual variables, i.e., \( G, z \) and \( q \). We estimate the errors using \( L^2 \) projections, and we use the following notation.

\[
\delta^G = \mathbb{G} - \Pi_K \mathbb{G}, \quad \varepsilon^G_h = \Pi_K \mathbb{G} - \mathbb{G}_h(u),
\]
\[
\delta^z = z - \Pi_V z, \quad \varepsilon^z_h = \Pi_V z - z_h(u),
\]
\[
\delta^q = q - \Pi_W q, \quad \varepsilon^q_h = \Pi_W q - q_h(u),
\]
\[
\delta^\hat{z} = \tilde{z} - P_M z, \quad \varepsilon^\hat{z}_h = P_M z - \tilde{z}_h(u),
\]
\[
\delta^\hat{n} = \delta^G + \delta^q - h^{-1}P_M \delta^z,
\]

where \( \tilde{z}_h(u) = z_h^0(u) \) on \( \varepsilon_h^0 \) and \( \tilde{z}_h(u) = 0 \) on \( \varepsilon_h^0 \), which gives \( \varepsilon^\hat{z}_h = 0 \) on \( \varepsilon_h^0 \).

The proof of the following error equation is similar to the proof of **Lemma 4.6** and is omitted.
Lemma 4.12. We have
\[ \mathcal{B}(\varepsilon_h^G, \varepsilon_h^\delta, -\varepsilon_h^q, \varepsilon_h^\varepsilon; \mathbb{T}_2, v_2, w_2, \mu_2) = \langle \tilde{\delta}_2, v_2 \rangle_{\partial \mathcal{T}_h} - \langle \tilde{\delta}_2, \mu_2 \rangle_{\partial \mathcal{T}_h \setminus \mathcal{T}_h} + (y - y_h(u), v_2)_{\mathcal{T}}. \] (4.22)

4.3.6 Step 6: Estimate for \( \varepsilon_h^G \) and \( \varepsilon_h^\varepsilon \).

To estimate \( \varepsilon_h^G \), we use the following discrete Poincaré inequality from [14] Proposition A.2.

Lemma 4.13. We have
\[ \| \varepsilon_h^G \|_{\mathcal{T}_h} \leq C(\| \nabla \varepsilon_h^G \|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \| \varepsilon_h^\delta - \varepsilon_h^G \|_{\partial \mathcal{T}_h}). \] (4.23)

Lemma 4.14. We have
\[ \| \varepsilon_h^G \|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \| P_M \varepsilon_h^\varepsilon - \varepsilon_h^\varepsilon \|_{\partial \mathcal{T}_h} \leq h^{\alpha+1} \| L \|_{s^i, \Omega} + h^{\alpha+1} \| p \|_{s^j, \Omega} + h^{\alpha} \| y \|_{s^y, \Omega} + h^{\alpha} \| G \|_{s^\varepsilon, \Omega} + h^{\alpha} \| q \|_{s^q, \Omega} + h^{\alpha-1} \| z \|_{s^z, \Omega}, \] (4.24a)
\[ \| \varepsilon_h^\varepsilon \|_{\mathcal{T}_h} \leq h^{\alpha+1} \| L \|_{s^i, \Omega} + h^{\alpha+1} \| p \|_{s^j, \Omega} + h^{\alpha} \| y \|_{s^y, \Omega} + h^{\alpha} \| G \|_{s^\varepsilon, \Omega} + h^{\alpha} \| q \|_{s^q, \Omega} + h^{\alpha-1} \| z \|_{s^z, \Omega}. \] (4.24b)

Proof. The inequality in Lemma 4.7 is valid with \((L, y, \hat{y})\) replaced by \((G, z, \hat{z})\), which gives
\[ \| \nabla \varepsilon_h^G \|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \| \varepsilon_h^G - \varepsilon_h^\varepsilon \|_{\partial \mathcal{T}_h} \leq \| \varepsilon_h^G \|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \| P_M \varepsilon_h^\varepsilon - \varepsilon_h^\varepsilon \|_{\partial \mathcal{T}_h}. \] (4.25)

Next, since \( \varepsilon_h^\varepsilon = 0 \) on \( \varepsilon_h^q \), the energy identity for \( \mathcal{B} \) in Lemma 4.3 gives
\[ \mathcal{B}(\varepsilon_h^G, \varepsilon_h^\delta, -\varepsilon_h^q, \varepsilon_h^G, \varepsilon_h^\varepsilon, -\varepsilon_h^q, \varepsilon_h^\varepsilon) = \| \varepsilon_h^G \|_{T_h}^2 + h^{-1} \| P_M \varepsilon_h^\varepsilon - \varepsilon_h^\varepsilon \|_{\partial \mathcal{T}_h}^2. \] (4.26)

Using \(( \mathbb{T}_2, v_2, w_2, \mu_2) = (\varepsilon_h^G, \varepsilon_h^\delta, \varepsilon_h^q, \varepsilon_h^\varepsilon)\) in the error equation (4.22) gives
\[ \| \varepsilon_h^G \|_{T_h}^2 + h^{-1} \| P_M \varepsilon_h^\varepsilon - \varepsilon_h^\varepsilon \|_{\partial \mathcal{T}_h}^2 = \langle \tilde{\delta}_2, \varepsilon_h^\varepsilon \rangle_{\partial \mathcal{T}_h} + (y - y_h(u), \varepsilon_h^\varepsilon)_{\mathcal{T}} =: T_1 + T_2. \]

For \( T_1 \), use (4.25) and Young’s inequality:
\[ T_1 = \langle \tilde{\delta}_2, \varepsilon_h^\varepsilon - \varepsilon_h^\varepsilon \rangle_{\partial \mathcal{T}_h} \leq Ch^\alpha \| \varepsilon_h^G \|_{\partial \mathcal{T}_h} + h^{-\frac{1}{2}} \| \varepsilon_h^\varepsilon - \varepsilon_h^\varepsilon \|_{\partial \mathcal{T}_h} \]
\[ + Ch^{-\frac{1}{2}} \| \delta^\varepsilon \|_{\partial \mathcal{T}_h} h^{-\frac{1}{2}} \| P_M \varepsilon_h^\varepsilon - \varepsilon_h^\varepsilon \|_{\partial \mathcal{T}_h} \]
\[ \leq Ch \| \delta^G \|_{\partial \mathcal{T}_h}^2 + h^{-\frac{1}{2}} \| \delta^\varepsilon \|_{\partial \mathcal{T}_h} + Ch^{-1} \| \delta^\varepsilon \|_{\partial \mathcal{T}_h}^2 + \frac{1}{4} \| \varepsilon_h^G \|_{\mathcal{T}_h}^2 + \frac{1}{4h} \| P_M \varepsilon_h^\varepsilon - \varepsilon_h^\varepsilon \|_{\partial \mathcal{T}_h}^2. \]

For the term \( T_2 \), apply Lemma 4.13 and (4.25) to obtain
\[ T_2 = \langle y - y_h(u), \varepsilon_h^\varepsilon \rangle_{\mathcal{T}} \leq \| y - y_h(u) \|_{\mathcal{T}} \| \varepsilon_h^\varepsilon \|_{\mathcal{T}_h} \]
\[ \leq C \| y - y_h(u) \|_{\mathcal{T}_h} (\| \nabla \varepsilon_h^G \|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \| \varepsilon_h^\varepsilon - \varepsilon_h^\varepsilon \|_{\partial \mathcal{T}_h}) \]
\[ \leq C \| y - y_h(u) \|_{\mathcal{T}_h} \| \varepsilon_h^G \|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \| P_M \varepsilon_h^\varepsilon - \varepsilon_h^\varepsilon \|_{\partial \mathcal{T}_h} \]
\[ \leq C \| y - y_h(u) \|_{\mathcal{T}_h}^2 + \frac{1}{4} \| \varepsilon_h^G \|_{\mathcal{T}_h}^2 + \frac{1}{4h} \| P_M \varepsilon_h^\varepsilon - \varepsilon_h^\varepsilon \|_{\partial \mathcal{T}_h}^2. \]

These estimates yield (4.24a). Then Lemma 4.13, (4.24a), and (4.25) give (4.24b). \( \square \)
4.3.7 Step 7: Estimate for $\varepsilon_h^q$.

**Lemma 4.15.** We have

$$
\|\varepsilon_h^q\|_{\Omega} \lesssim h^{s_q + 1} \|L\|_{s_q, \Omega} + h^{s_q + 1} \|p\|_{s_q, \Omega} + h^{s_q} \|y\|_{s_q, \Omega} + h^{s_q} \|q\|_{s_q, \Omega} + h^{s_q - 1} \|z\|_{s_q, \Omega}.
$$

(4.27)

**Proof.** By the same argument as in Lemma 4.9, we have

$$
\|\varepsilon_h^q\|_{\Omega} \lesssim \sup_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{(\varepsilon_h^q, \nabla \cdot v)_\Omega}{\|v\|_{H^1(\Omega)}},
$$

and

$$(\varepsilon_h^q, \nabla \cdot v)_\Omega = -(\nabla \varepsilon_h^q, \Pi_V v)_\Omega + (\varepsilon_h^q, v \cdot n)_{\partial \Omega}.$$

Next, taking $(T_2, v_2, w_2, \mu_2) = (0, \Pi_V v, 0, 0)$ in Lemma 4.12 gives

$$(\nabla \varepsilon_h^q, \Pi_V v)_{\Omega h} = - (\nabla \varepsilon_h^q, \Pi_V v)_{\Omega h} + (h^{-1}(P_M \varepsilon_h^q - \varepsilon_h^q), \Pi_V v)_{\partial \Omega h} - (\tilde{\delta}_2, \Pi_V v)_{\partial \Omega h} - (y - y_h(u), \Pi_V v)_{\Omega h}.$$

These equalities give

$$(\varepsilon_h^q, \Pi_V v)_{\Omega h} = -(\nabla \varepsilon_h^q, \Pi_V v)_{\Omega h} + (h^{-1}(P_M \varepsilon_h^q - \varepsilon_h^q), \Pi_V v)_{\partial \Omega h} + (\tilde{\delta}_2, \Pi_V v)_{\partial \Omega h} + (y - y_h(u), \Pi_V v)_{\Omega h}.$$

Next, using $(T_2, v_2, w_2, \mu_2) = (0, 0, 0, P_M v)$ in Lemma 4.12 and $v \in H_0^1(\Omega)$ gives

$$
(\varepsilon_h^q, \nabla \cdot v)_{\Omega h} = -(\nabla \varepsilon_h^q, \Pi_V v)_{\Omega h} + (h^{-1}(P_M \varepsilon_h^q - \varepsilon_h^q), P_M v)_{\partial \Omega h} - (\tilde{\delta}_2, P_M v)_{\partial \Omega h}.
$$

This implies

$$(\varepsilon_h^q, \nabla \cdot v)_{\Omega h} = -(\nabla \varepsilon_h^q, \Pi_V v)_{\Omega h} + (h^{-1}(P_M \varepsilon_h^q - \varepsilon_h^q), P_M v)_{\partial \Omega h} + (\tilde{\delta}_2, P_M v)_{\partial \Omega h} + (y - y_h(u), \Pi_V v)_{\Omega h}.$$

Applying the Cauchy-Schwarz inequality, we obtain

$$
\|\varepsilon_h^q\|_{\Omega h} \lesssim \|\varepsilon_h^q\|_{\Omega h} \|\nabla v\|_{\Omega h} + h^{-\frac{1}{2}} \|P_M \varepsilon_h^q - \varepsilon_h^q\|_{\partial \Omega h} \|\nabla v\|_{\Omega h} + h^{-\frac{3}{2}} \|\tilde{\delta}_2\|_{\partial \Omega h} \|\nabla v\|_{\Omega h} + \|y - y_h(u)\|_{\Omega h} \|v\|_{\Omega h}.$$

Since $v \in H_0^1(\Omega)$, the Poincaré inequality yields the result:

$$
\|\varepsilon_h^q\|_{\Omega h} \lesssim \|\varepsilon_h^q\|_{\Omega h} + h^{-\frac{1}{2}} \|P_M \varepsilon_h^q - \varepsilon_h^q\|_{\Omega h} + h^{-\frac{3}{2}} \|\tilde{\delta}_2\|_{\partial \Omega h} \|\nabla v\|_{\Omega h} + \|y - y_h(u)\|_{\Omega h} \lesssim h^{s_q + 1} \|L\|_{s_q, \Omega} + h^{s_q + 1} \|p\|_{s_q, \Omega} + h^{s_q} \|y\|_{s_q, \Omega} + h^{s_q} \|q\|_{s_q, \Omega} + h^{s_q - 1} \|z\|_{s_q, \Omega}.
$$

}\square
Lemma 4.16. We have
\[
\|G - G_h(u)\|_{\mathcal{T}_h} \lesssim h^{s+1} \|L\|_{s',\Omega} + h^{s+1} \|p\|_{s',\Omega} + h^s \|y\|_{s\Omega} + h^s \|q\|_{s\Omega} + h^{s-1} \|z\|_{s\Omega},
\]
(4.28a)
\[
\|q - q_h(u)\|_{\mathcal{T}_h} \lesssim h^{s+1} \|L\|_{s',\Omega} + h^{s+1} \|p\|_{s',\Omega} + h^s \|y\|_{s\Omega} + h^s \|q\|_{s\Omega} + h^{s-1} \|z\|_{s\Omega},
\]
(4.28b)
\[
\|z - z_h(u)\|_{\mathcal{T}_h} \lesssim h^{s+1} \|L\|_{s',\Omega} + h^{s+1} \|p\|_{s',\Omega} + h^s \|y\|_{s\Omega} + h^s \|q\|_{s\Omega} + h^{s-1} \|z\|_{s\Omega}.
\]
(4.28c)

4.3.8 Step 8: Estimate for \(\|u - u_h\|_{\varepsilon_h^0}\) and \(\|y - y_h\|_{\mathcal{T}_h}\).

Next, we consider the solution of the auxiliary problem and the solution of the HDG discretization of the optimality system (4.7). Our main result follows from bounding the errors between these solutions as well as Lemma 4.11 and Lemma 4.16.

Define
\[
\zeta_L = L_h(u) - L_h, \quad \zeta_y = y_h(u) - y_h, \quad \zeta_p = p_h(u) - p_h,
\]
\[
\zeta_G = G_h(u) - G_h, \quad \zeta_z = z_h(u) - z_h, \quad \zeta_q = q_h(u) - q_h,
\]
and
\[
\zeta\hat{y} = \hat{y}_h(u) - \hat{y}_h^0 \text{ on } \varepsilon_h^0, \quad \zeta\hat{y} = P_M u\tau - u_h\tau \text{ on } \varepsilon_h^0,
\]
\[
\zeta\hat{z} = \hat{z}_h(u) - \hat{z}_h^0 \text{ on } \varepsilon_h^0, \quad \zeta\hat{z} = 0 \text{ on } \varepsilon_h^0.
\]

Subtract the auxiliary problem and the HDG problem yields the error equations
\[
\mathcal{B}(\zeta_L, \zeta_y, \zeta_p, \zeta\hat{y}; \mathbb{T}_1, v_1, w_1, \mu_1) = \langle (P_M u - u_h)\tau, h^{-1}v_1 + \mathbb{T}_1 n \rangle_{\varepsilon_h^0},
\]
(4.29a)
\[
\mathcal{B}(\zeta_G, \zeta_z, -\zeta_q, \zeta\hat{z}; \mathbb{T}_2, v_2, w_2, \mu_2) = \langle \zeta_y, v_2 \rangle_{\mathcal{T}_h},
\]
(4.29b)
for all \((\mathbb{T}_1, \mathbb{T}_2, v_1, v_2, w_1, w_2, \mu_1, \mu_2) \in \mathbb{K}_h \times \mathbb{K}_h \times V_h \times V_h \times W^0_h \times W^0_h \times M_h(o) \times M_h(o).

Lemma 4.17. We have
\[
\gamma \|u - u_h\|_{\varepsilon_h^0} + \|\zeta y\|_{\mathcal{T}_h}^2 = \langle \gamma u\tau - G_h(u)n + h^{-1}P_M z_h(u), (u - u_h)\tau \rangle_{\varepsilon_h^0} - \langle \gamma u_h\tau - G_h n + h^{-1}P_M z_h, (u - u_h)\tau \rangle_{\varepsilon_h^0}.
\]
(4.30)

Proof. First,
\[
\langle \gamma u\tau - G_h(u)n + h^{-1}P_M z_h(u), (u - u_h)\tau \rangle_{\varepsilon_h^0} - \langle \gamma u_h\tau - G_h n + h^{-1}P_M z_h, (u - u_h)\tau \rangle_{\varepsilon_h^0} = \gamma \|u - u_h\|_{\varepsilon_h^0}^2 + \langle -\zeta_G n + h^{-1}P_M z_h, (u - u_h)\tau \rangle_{\varepsilon_h^0}.
\]

Next, Lemma 4.4 gives
\[
\mathcal{B}(\zeta_L, \zeta_y, \zeta_p, \zeta\hat{y}; -\zeta_q, \zeta z, \zeta q, \zeta\hat{z}) + \mathcal{B}(\zeta z, \zeta z; -\zeta y, \zeta\hat{y}) = 0.
\]
However, the error equations yield
\[
\mathcal{B}(\zeta_L, \zeta_y, \zeta_p, \zeta\tilde{y}; -\zeta_G, \zeta, -\zeta, -\zeta, -\zeta_y, \zeta_p, -\zeta\tilde{y}) + \mathcal{B}(\zeta, \zeta, -\zeta, -\zeta, -\zeta, -\zeta_y, \zeta_p, -\zeta\tilde{y}) = -\langle (u - u_h)\tau, -\zeta_G n + h^{-1}\zeta_M \zeta \rangle_{\varepsilon_h^0}.
\]

Comparing these equalities gives
\[
\langle (u - u_h)\tau, -\zeta_G n + h^{-1}P_M \zeta \rangle_{\varepsilon_h^0}.
\]

\[\square\]

**Theorem 4.18.** We have
\[
\|u - u_h\|_{\varepsilon_h^0}^2 \lesssim h^{s_L + \frac{1}{2}} \|L\|_{s_L, \Omega} + h^{s_p + \frac{1}{2}} \|P\|_{s_p, \Omega} + h^{s_y - \frac{1}{2}} \|\gamma\|_{s_y, \Omega},
\]

\[
\|y - y_h\|_{\varepsilon_h^0}^2 \lesssim h^{s_G - \frac{1}{2}} \|\mathcal{G}\|_{s_G, \Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s_q, \Omega} + h^{s_k - \frac{1}{2}} \|\zeta\|_{s_k, \Omega}.
\]

**Proof.** The optimality condition (2.21g) gives \(\langle \gamma u\tau - \mathcal{G} n, (u - u_h)\tau \rangle_{\varepsilon_h^0} = 0\). Also,
\[
\langle (\gamma u\tau - \mathcal{G} n, (u - u_h)\tau) \rangle_{\varepsilon_h^0} = \langle (\gamma u\tau - \mathcal{G} n, h^{-1}P_M \zeta) \rangle_{\varepsilon_h^0} = 0,
\]

where we used the HDG optimality condition (3.6) and (3.6m). Hence, we have
\[
\gamma \|u - u_h\|_{\varepsilon_h^0}^2 + \|\zeta\|_{\varepsilon_h^0}^2 \lesssim h^{s_G - \frac{1}{2}} \|\mathcal{G}\|_{s_G, \Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s_q, \Omega} + h^{s_k - \frac{1}{2}} \|\zeta\|_{s_k, \Omega},
\]

Next, since \(\tilde{z}_h(u) = z = 0\) on \(\varepsilon_h^0\) we have
\[
\|P_M \zeta_h(u)\|_{\varepsilon_h^0} = \|P_M \zeta_h(u) - P_M \Pi_V z + P_M \Pi_V z - P_M z + P_M z - \tilde{z}_h(u)\|_{\varepsilon_h^0} \lesssim \|P_M \epsilon_{\varepsilon_h^0} - \tilde{\epsilon}_{\varepsilon_h^0}\|_{\varepsilon_h^0} + \|\Pi_V z - z\|_{\varepsilon_h^0}.
\]

This yields
\[
\|u - u_h\|_{\varepsilon_h^0} + \|\zeta\|_{\varepsilon_h^0} \lesssim h^{s_G - \frac{1}{2}} \|\mathcal{G}\|_{s_G, \Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s_q, \Omega} + h^{s_k - \frac{1}{2}} \|\zeta\|_{s_k, \Omega} + h^{-1}\|P_M \epsilon_{\varepsilon_h^0} - \tilde{\epsilon}_{\varepsilon_h^0}\|_{\varepsilon_h^0} + h^{-1}\|\Pi_V z - z\|_{\varepsilon_h^0}.
\]

**Lemma 4.14** and properties of the \(L^2\) projection give
\[
\|u - u_h\|_{\varepsilon_h^0} + \|\zeta\|_{\varepsilon_h^0} \lesssim h^{s_G - \frac{1}{2}} \|\mathcal{G}\|_{s_G, \Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s_q, \Omega} + h^{s_k - \frac{1}{2}} \|\zeta\|_{s_k, \Omega},
\]

which proves (4.31a). The second estimate (4.31b) follows from Lemma 4.14 and properties of the \(L^2\) projection. \(\square\)
4.3.9 Step 9: Estimates for $\|G - G_h\|_{T_h}$ and $\|z - z_h\|_{T_h}$.

Lemma 4.19. We have

\[
\begin{align*}
\|\zeta_G\|_{T_h} & \lesssim h^{s_l + \frac{1}{2}} \|L\|_{s_l,\Omega} + h^{s_p + \frac{1}{2}} \|p\|_{s_p,\Omega} + h^{s_y - \frac{1}{2}} \|y\|_{s_y,\Omega} \\
& \quad + h^{s_\gamma - \frac{1}{2}} \|G\|_{s_\gamma,\Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s_q,\Omega} + h^{s_z - \frac{3}{2}} \|z\|_{s_z,\Omega}, \\
\|\zeta_z\|_{T_h} & \lesssim h^{s_l + \frac{1}{2}} \|L\|_{s_l,\Omega} + h^{s_p + \frac{1}{2}} \|p\|_{s_p,\Omega} + h^{s_y - \frac{1}{2}} \|y\|_{s_y,\Omega} \\
& \quad + h^{s_\gamma - \frac{1}{2}} \|G\|_{s_\gamma,\Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s_q,\Omega} + h^{s_z - \frac{3}{2}} \|z\|_{s_z,\Omega}.
\end{align*}
\]  

(4.32a) (4.32b)

Proof. Using the energy identity for $B$, Lemma 4.3, the error equation (4.29b), $\zeta_\varepsilon = 0$ on $\varepsilon_0^h$, the discrete Poincaré inequality in Lemma 4.13 and Lemma 4.7 gives

\[
B(\zeta_G, \zeta_z, -\zeta_q, \zeta_\varepsilon; \zeta_G, \zeta_z, -\zeta_q, \zeta_\varepsilon) = (\zeta_G, \zeta_G)_{T_h} + h^{-1} \|P_M \zeta_z - \zeta_\varepsilon\|_{\delta T_h}^2
\]

= $(\zeta_y, \zeta_y)_{T_h}$

$\lesssim \|\zeta_y\|_{T_h} \|\zeta_y\|_{T_h}$

$\lesssim \|\zeta_y\|_{T_h} (\|\nabla \zeta_z\|_{T_h} + h^{-\frac{1}{2}} \|\zeta_z - \zeta_\varepsilon\|_{\delta T_h})$

$\lesssim \|\zeta_y\|_{T_h} (\|\zeta_G\|_{T_h} + h^{-\frac{1}{2}} \|P_M \zeta_z - \zeta_\varepsilon\|_{\delta T_h})$.

This implies

\[
\begin{align*}
\|\zeta_G\|_{T_h} + h^{-\frac{1}{2}} \|P_M \zeta_z - \zeta_\varepsilon\|_{\delta T_h} \lesssim & \ h^{s_l + \frac{1}{2}} \|L\|_{s_l,\Omega} + h^{s_p + \frac{1}{2}} \|p\|_{s_p,\Omega} + h^{s_y - \frac{1}{2}} \|y\|_{s_y,\Omega} \\
& \quad + h^{s_\gamma - \frac{1}{2}} \|G\|_{s_\gamma,\Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s_q,\Omega} + h^{s_z - \frac{3}{2}} \|z\|_{s_z,\Omega}.
\end{align*}
\]

Using the discrete Poincaré inequality again completes the proof:

\[
\begin{align*}
\|\zeta_z\|_{T_h} \lesssim & \ \|\nabla \zeta_z\|_{T_h} + h^{-\frac{1}{2}} \|\zeta_z - \zeta_\varepsilon\|_{\delta T_h} \\
\lesssim & \ \|\zeta_G\|_{T_h} + h^{-\frac{1}{2}} \|P_M \zeta_z - \zeta_\varepsilon\|_{\delta T_h} \\
\lesssim & \ h^{s_l + \frac{1}{2}} \|L\|_{s_l,\Omega} + h^{s_p + \frac{1}{2}} \|p\|_{s_p,\Omega} + h^{s_y - \frac{1}{2}} \|y\|_{s_y,\Omega} \\
& \quad + h^{s_\gamma - \frac{1}{2}} \|G\|_{s_\gamma,\Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s_q,\Omega} + h^{s_z - \frac{3}{2}} \|z\|_{s_z,\Omega}.
\end{align*}
\]

The above lemma, the triangle inequality, Lemma 4.11 and Lemma 4.16 give the following result:

Theorem 4.20. We have

\[
\begin{align*}
\|G - G_h\|_{T_h} \lesssim & \ h^{s_l + \frac{1}{2}} \|L\|_{s_l,\Omega} + h^{s_p + \frac{1}{2}} \|p\|_{s_p,\Omega} + h^{s_y - \frac{1}{2}} \|y\|_{s_y,\Omega} \\
& \quad + h^{s_\gamma - \frac{1}{2}} \|G\|_{s_\gamma,\Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s_q,\Omega} + h^{s_z - \frac{3}{2}} \|z\|_{s_z,\Omega}, \quad (4.33a)
\end{align*}
\]

\[
\begin{align*}
\|z - z_h\|_{T_h} \lesssim & \ h^{s_l + \frac{1}{2}} \|L\|_{s_l,\Omega} + h^{s_p + \frac{1}{2}} \|p\|_{s_p,\Omega} + h^{s_y - \frac{1}{2}} \|y\|_{s_y,\Omega} \\
& \quad + h^{s_\gamma - \frac{1}{2}} \|G\|_{s_\gamma,\Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s_q,\Omega} + h^{s_z - \frac{3}{2}} \|z\|_{s_z,\Omega}. \quad (4.33b)
\end{align*}
\]
4.3.10 Step 10: Estimate for $\|q - q_h\|_{\mathcal{T}_h}$.

**Lemma 4.21.** We have
\[
\|\zeta_q\| \lesssim h^{-n/2} \|L\|_{s^{n},\Omega} + h^{s_y + \frac{1}{2}} \|p\|_{s^{y},\Omega} + h^{s_y - \frac{1}{2}} \|y\|_{s^{y},\Omega}
+ h^{s_0 - \frac{1}{2}} \|G\|_{s^{0},\Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s^{q},\Omega} + h^{s_z - \frac{1}{2}} \|z\|_{s^{z},\Omega}.
\]
(4.34)

**Proof.** By the same argument as in Lemma 4.9, we have
\[
\|\zeta_q\| \lesssim \sup_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{(\zeta_q, \nabla \cdot v)_{\Omega}}{\|v\|_{H^1(\Omega)}},
\]
and
\[
(\zeta_q, \nabla \cdot v)_{\Omega} = -(\nabla \zeta_q, \Pi V v)_{\mathcal{T}_h} + (\zeta_q, v \cdot n)_{\partial \mathcal{T}_h}.
\]
Next, taking $(T_2, v_2, w_2, \mu_2) = (0, \Pi V v, 0, 0)$ in the error equation (4.29b) and using $\zeta = 0$ on $\epsilon_h^0$ give
\[
(\nabla \zeta_q, \Pi V v)_{\mathcal{T}_h} = -(\nabla \cdot \zeta_G, \Pi V v)_{\mathcal{T}_h} + (h^{-1}(P_M \zeta - \zeta), \Pi V v)_{\partial \mathcal{T}_h} - (\zeta_y, \Pi V v)_{\mathcal{T}_h}.
\]
The above two equalities give
\[
(\zeta_q, \nabla \cdot v)_{\Omega} = (\nabla \cdot \zeta_G, \Pi V v)_{\mathcal{T}_h} - (h^{-1}(P_M \zeta - \zeta), \Pi V v)_{\partial \mathcal{T}_h}
+ (\zeta_q, v \cdot n)_{\partial \mathcal{T}_h} + (\zeta_q, \Pi V v)_{\mathcal{T}_h}
= (\nabla \cdot \zeta_G, v)_{\mathcal{T}_h} - (h^{-1}(P_M \zeta - \zeta), \Pi V v)_{\partial \mathcal{T}_h}
+ (\zeta_q, v \cdot n)_{\partial \mathcal{T}_h} + (\zeta_q, \Pi V v)_{\mathcal{T}_h}
= -(\zeta_G, \nabla v)_{\mathcal{T}_h} - (h^{-1}(P_M \zeta - \zeta), \Pi V v)_{\partial \mathcal{T}_h}
+ (\zeta_q n + \zeta_q \Pi M v)_{\partial \mathcal{T}_h \setminus \epsilon_h^0} + (\zeta_q, v)_{\mathcal{T}_h}.
\]
Next, take $(T_2, v_2, w_2, \mu_2) = (0, 0, 0, P_M v)$ in (4.29b) and use $v \in H_0^1(\Omega)$ to obtain
\[
(\zeta_G n + \zeta_q \Pi M v)_{\partial \mathcal{T}_h \setminus \epsilon_h^0} = (h^{-1}(P_M \zeta - \zeta), P_M v)_{\partial \mathcal{T}_h}.
\]
This implies
\[
(\zeta_q, \nabla \cdot v)_{\Omega} = -(\zeta_G, \nabla v)_{\mathcal{T}_h} - (h^{-1}(P_M \zeta - \zeta), \Pi V v - P_M v)_{\partial \mathcal{T}_h} + (\zeta_q, v)_{\mathcal{T}_h},
\]
and therefore
\[
|(\zeta_q, \nabla \cdot v)_{\Omega}| \lesssim \|\zeta_G\|_{\mathcal{T}_h} \|\nabla v\|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \|P_M \zeta - \zeta\|_{\partial \mathcal{T}_h} \|\nabla v\|_{\mathcal{T}_h} + \|\zeta_q\|_{\mathcal{T}_h} \|v\|_{\mathcal{T}_h}.
\]
Since $v \in H_0^1(\Omega)$, the Poincaré inequality gives the result:
\[
\|\zeta_q\| \lesssim \|\zeta_G\|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \|P_M \zeta - \zeta\|_{\partial \mathcal{T}_h} + \|\zeta_q\|_{\mathcal{T}_h}
\lesssim h^{-n/2} \|L\|_{s^{n},\Omega} + h^{s_y + \frac{1}{2}} \|p\|_{s^{y},\Omega} + h^{s_y - \frac{1}{2}} \|y\|_{s^{y},\Omega}
+ h^{s_0 - \frac{1}{2}} \|G\|_{s^{0},\Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s^{q},\Omega} + h^{s_z - \frac{1}{2}} \|z\|_{s^{z},\Omega}.
\]

The above lemma, the triangle inequality, and Lemma 4.16 give the following error bound:

**Theorem 4.22.** We have
\[
\|q - q_h\|_{\mathcal{T}_h} \lesssim h^{n+\frac{1}{2}} \|L\|_{s^{n},\Omega} + h^{s_y + \frac{1}{2}} \|p\|_{s^{y},\Omega} + h^{s_y - \frac{1}{2}} \|y\|_{s^{y},\Omega}
+ h^{s_0 - \frac{1}{2}} \|G\|_{s^{0},\Omega} + h^{s_q - \frac{1}{2}} \|q\|_{s^{q},\Omega} + h^{s_z - \frac{1}{2}} \|z\|_{s^{z},\Omega}.
\]

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4.3.11 Step 11: Estimates for $\|p - p_h\|_{\mathcal{T}_h}$ and $\|\mathbb{L} - \mathbb{L}_h\|_{\mathcal{T}_h}$.

Lemma 4.23. For $k \geq 1$, we have

$$\|\zeta_p\|_{\Omega} \lesssim \|\zeta_L\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|P_M\zeta_y - \zeta_{\tilde{g}}\|_{\mathbb{H}^1(\mathcal{T}_h) \setminus \mathcal{E}_h^0}$$

$$+ h^{-\frac{2}{3}}\|P_M\zeta_y\|_{\mathbb{H}^1(\mathcal{T}_h) \setminus \mathcal{E}_h^0} + h^{-\frac{2}{3}}\|P_Mu - u_h\|_{\mathbb{H}^1(\mathcal{T}_h) \setminus \mathcal{E}_h^0}. \tag{4.35}$$

Proof. As in the proof of Lemma 4.9 we have

$$\|\zeta_p\|_{\Omega} \lesssim \sup_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{(\zeta_p, \nabla \cdot v)_\Omega}{\|v\|_{H^1(\Omega)}},$$

and

$$(\zeta_p, \nabla \cdot v)_\Omega = - (\nabla \zeta_p, \Pi_V v)_{\mathcal{T}_h} + (\zeta_p, v \cdot n)_{\partial \mathcal{T}_h}.$$

Use $(T_1, v_1, w_1, \mu_1) = (0, \Pi_V v, 0, 0)$ in (4.29a) and $v \in H^1_0(\Omega)$ to obtain

$$(\nabla \zeta_p, \Pi_V v)_{\mathcal{T}_h} = (\nabla \cdot \zeta_L, \Pi_V v)_{\mathcal{T}_h} - h^{-1}(P_M\zeta_y - \zeta_{\tilde{g}}), \Pi_V v)_{\partial \mathcal{T}_h \setminus \mathcal{E}_h^0}$$

$$- (P_Mu - u_h)\tau, h^{-1}\Pi_V v)_{\epsilon_h^0}.$$
Lemma 4.24. If $k \geq 1$ holds, then
\[
\begin{align*}
\| \zeta_L \|_{T_h} &\lesssim h^{s_L} \| L \|_{\Omega} + h^{s_p} \| p \|_{s_p, \Omega} + h^{s_y} \| y \|_{s_y, \Omega} \\
&+ h^{s_{M1}} \| \mathbb{G} \|_{s_{V}, \Omega} + h^{s_y} \| q \|_{s_q, \Omega} + h^{s_{z2}} \| z \|_{s_z, \Omega}, \\
\| \zeta_p \|_{T_h} &\lesssim h^{s_L} \| L \|_{\Omega} + h^{s_p} \| p \|_{s_p, \Omega} + h^{s_y} \| y \|_{s_y, \Omega} \\
&+ h^{s_{M1}} \| \mathbb{G} \|_{s_{V}, \Omega} + h^{s_y} \| q \|_{s_q, \Omega} + h^{s_{z2}} \| z \|_{s_z, \Omega}.
\end{align*}
\] (4.36a)

Proof. By Lemma 4.3 and the error equation (4.29a), we have
\[
\begin{align*}
\mathcal{B}(\zeta_L, \zeta_y, \zeta_p, \zeta_{\hat{y}}; \zeta_L, \zeta_y, \zeta_p, \zeta_{\hat{y}}) &= (\zeta_L, \zeta_L)_{T_h} + \langle (h^{-1}(PM\zeta_y - \zeta_{\hat{y}}), \zeta_y - \zeta_{\hat{y}})_{\partial T_h \setminus \epsilon_h^\Omega} + \langle h^{-1}P_M\zeta_y, P_M\zeta_y \rangle_{\epsilon_h^\Omega} \\
&= \langle (P_M u - u_h) \tau, \zeta_L \cdot n + h^{-1}\zeta_y \rangle_{\epsilon_h^\Omega} \\
&= \langle (u - u_h) \tau, \zeta_L \cdot n + h^{-1}P_M\zeta_y \rangle_{\epsilon_h^\Omega} \\
&\lesssim \| u - u_h \|_{\epsilon_h^\Omega} \left( \| \zeta_L \|_{\epsilon_h^\Omega} + h^{-1} \| P_M\zeta_y \|_{\epsilon_h^\Omega} \right) \\
&\lesssim h^{-\frac{1}{2}} \| u - u_h \|_{\epsilon_h^\Omega} \left( \| \zeta_L \|_{T_h} + h^{-\frac{1}{2}} \| P_M\zeta_y \|_{\epsilon_h^\Omega}, \right)
\end{align*}
\]
which gives
\[
\begin{align*}
\| \zeta_L \|_{T_h} + h^{-\frac{1}{2}} \| P_M\zeta_y - \zeta_{\hat{y}} \|_{\partial T_h \setminus \epsilon_h^\Omega} + h^{-\frac{1}{2}} \| P_M\zeta_y \|_{\epsilon_h^\Omega} \\
&\lesssim h^{s_L} \| L \|_{s_L, \Omega} + h^{s_p} \| p \|_{s_p, \Omega} + h^{s_y} \| y \|_{s_y, \Omega} \\
&+ h^{s_{M1}} \| \mathbb{G} \|_{s_{V}, \Omega} + h^{s_y} \| q \|_{s_q, \Omega} + h^{s_{z2}} \| z \|_{s_z, \Omega}.
\end{align*}
\] (4.36b)

This bound together with Lemma 4.23 gives (4.36b). \(\Box\)

The above lemma, the triangle inequality, and Lemma 4.11 complete the proof of the main result:

Theorem 4.25. If $k \geq 1$, then
\[
\begin{align*}
\| p - p_h \|_{T_h} &\lesssim h^{s_L} \| L \|_{s_L, \Omega} + h^{s_p} \| p \|_{s_p, \Omega} + h^{s_y} \| y \|_{s_y, \Omega} \\
&+ h^{s_{M1}} \| \mathbb{G} \|_{s_{V}, \Omega} + h^{s_y} \| q \|_{s_q, \Omega} + h^{s_{z2}} \| z \|_{s_z, \Omega}, \\
\| L - L_h \|_{T_h} &\lesssim h^{s_L} \| L \|_{s_L, \Omega} + h^{s_p} \| p \|_{s_p, \Omega} + h^{s_y} \| y \|_{s_y, \Omega} \\
&+ h^{s_{M1}} \| \mathbb{G} \|_{s_{V}, \Omega} + h^{s_y} \| q \|_{s_q, \Omega} + h^{s_{z2}} \| z \|_{s_z, \Omega}.
\end{align*}
\]

5 Numerical Experiments

We consider two examples on square domains in $\mathbb{R}^2$. In the first example we have an explicit solution of the optimality system; in the second example an explicit form for the exact solution is not known.

For a square domain, the singular exponent is $\xi = 2.74$ and the local regularity result (2.20) gives $u \in H^k(\Gamma_i)$ for each $\kappa < \xi - 1/2 = 2.24$. 

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Table 1: Example 5.1 \( k = 1 \): Errors, observed convergence orders, and expected order (EO) for the control \( u \), pressure \( p \), dual pressure \( q \), state \( y \), adjoint state \( z \), and their fluxes \( L \) and \( G \).

| \( h / \sqrt{2} \) | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 | EO |
|------------------|-----|------|------|------|-------|-----|
| \( \| L - L_h \|_{T_h} \) | 1.72 | 5.63E-01 | 1.86E-01 | 6.29E-02 | 2.17E-02 | 1 |
| order            | -   | 1.62 | 1.60 | 1.56 | 1.54 |   |
| \( \| G - G_h \|_{T_h} \) | 1.41E-01 | 3.79E-02 | 9.74E-03 | 2.46E-03 | 6.19E-04 | 1.5 |
| order            | -   | 1.90 | 1.96 | 1.98 | 1.99 |   |
| \( \| y - y_h \|_{T_h} \) | 2.15E-01 | 2.84E-02 | 4.10E-03 | 7.18E-04 | 1.54E-04 | 1.5 |
| order            | -   | 2.92 | 2.79 | 2.51 | 2.22 |   |
| \( \| z - z_h \|_{T_h} \) | 2.78E-02 | 3.53E-03 | 4.44E-04 | 5.63E-05 | 7.41E-06 | 1.5 |
| order            | -   | 2.98 | 2.99 | 2.98 | 2.93 |   |
| \( \| p - p_h \|_{T_h} \) | 6.93E-01 | 2.24E-01 | 7.40E-02 | 2.50E-02 | 8.63E-03 | 1 |
| order            | -   | 1.63 | 1.60 | 1.56 | 1.54 |   |
| \( \| q - q_h \|_{T_h} \) | 7.78E-02 | 1.91E-02 | 4.71E-03 | 1.17E-03 | 2.92E-04 | 1.5 |
| order            | -   | 2.03 | 2.02 | 2.01 | 2.00 |   |
| \( \| u - u_h \|_{\varepsilon \partial \Omega} \) | 2.42E-01 | 6.29E-02 | 1.58E-02 | 3.96E-03 | 9.90E-04 | 1.5 |
| order            | -   | 1.94 | 1.99 | 2.00 | 2.00 |   |

Table 2: Example 5.1 The number of iterations in the augmented Lagrangian approach for various \( h \) and \( \Delta t \).

| \( h / \sqrt{2} \) | Time step |
|------------------|-----------|
|                  | \( \Delta t = 16 \) | \( \Delta t = 32 \) | \( \Delta t = 64 \) | \( \Delta t = 128 \) | \( \Delta t = 256 \) |
| 1/4              | 12        | 8         | 7          | 6          | 6          |
| 1/8              | 14        | 9         | 7          | 6          | 6          |
| 1/16             | 16        | 9         | 8          | 7          | 6          |
| 1/32             | 19        | 10        | 8          | 7          | 6          |
| 1/64             | 21        | 11        | 8          | 7          | 6          |

Example 5.1. The domain is the unit square, and the problem data \( f \) and \( y_d \) are chosen so that the exact solution of the optimality system is given by

\[

y_1 = -2\pi^2 \sin^2(\pi x_1) \cos(\pi x_2) - 2\pi^2 \sin(\pi x_1) \sin(2\pi x_2),
\]
\[
y_2 = 2\pi^2 \cos(\pi x_1) \sin^2(\pi x_2) + 2\pi^2 \sin(\pi x_2) \sin(2\pi x_1),
\]
\[
z_1 = \pi \sin^2(\pi x_1) \sin(2\pi x_2), \quad z_2 = -\pi \sin^2(\pi x_2) \sin(2\pi x_1),
\]
\[
p = \cos(\pi x_1), \quad q = \cos(\pi x_1), \quad \gamma = 1.
\]

The numerical results are shown in Table 1 for various values of \( h \) with \( \Delta t = 256 \) in the augmented Lagrangian solver. To compare the observed convergence orders in the numerical experiments with the theoretical rates, we mark the theoretical expected order (EO) in the last column.

Also, Table 2 shows the number of iterations required for the augmented Lagrangian method for various \( h \) and \( \Delta t \). From Table 2 we see that the number of iterations decreases when \( \Delta t \) increases and converges as \( h \) decreases.
Table 3: Example 5.2, \(k = 0\): Errors, observed convergence orders, and expected order (EO) for the control \(u\), pressure \(p\), dual pressure \(q\), state \(y\), adjoint state \(z\), and their fluxes \(L\) and \(G\).

Example 5.2. In this example, we set \(f = 0\), \(\gamma = 1\), \(\Delta t = 256\), the domain \(\Omega = [0, 1/8] \times [0, 1/8]\), and the desired state

\[
y_d = 200 \times 8^3 x_1^2 (1 - 8x_1)^2 x_2 (1 - 8x_2) (1 - 16x_2); \\
- x_1 (1 - 8x_1) (1 - 16x_1) x_2^2 (1 - 8x_2)^2.
\]

In [51], it shows that \(y_d\) has the form of a large vortex, see Figure 1.

Since we do not have an explicit expression for the exact solution, we solved the problem numerically for a triangulation with 524288 elements, i.e., \(h = \sqrt{2}/2^{12}\), and compared this reference solution against other solutions computed on meshes with larger \(h\). The numerical results are shown in Table 3 for \(k = 0\) and Table 4 for \(k = 1\). For illustration, we consider the problem on a square domain \(\Omega = [0, 1] \times [0, 1]\) on a relatively coarse mesh, i.e., \(h = \sqrt{2}/64\), and plot the computed optimal control in Figure 1.

The convergence rate for the control \(u\) in Example 5.1, Example 5.2, and other numerical tests not shown here are higher than predicted in our theoretical results; again, this may be due to the local regularity. For \(k = 0\), experimental orders of convergence are similar to the ones obtained for convection diffusion [45, Table 2]. Also, for \(k = 0\), we do not have guaranteed convergence rates for the pressure \(p\) and flux \(L\). Again, this is similar to the convection diffusion case [45]. For \(k = 1\), while the observed convergence rates for the dual variables are similar to those of [45, Table 1], the experimental orders of convergence for the optimal control, state, and flux are \(1/2\) better for the Stokes problem.

6 Conclusion

In this work, we considered a tangential Dirichlet boundary control problem for the Stokes equations. First, we established well-posedness and regularity results for the optimal control problem based on a weak mixed formulation of the PDE on polygonal domains. Next, we used an existing
Table 4: Example 5.2, \( k = 1 \): Errors, observed convergence orders, and expected order (EO) for the control \( u \), pressure \( p \), dual pressure \( q \), state \( y \), adjoint state \( z \), and their fluxes \( L \) and \( G \).
superconvergent HDG method to approximate the solution of the optimality system and established optimal superlinear convergence rates for the control under certain assumptions on the domain Ω and the desired state $y_d$. However, the numerical experiments show higher convergence rates than our theoretical results; this maybe due to the higher local regularity of the control on individual edges of the domain. This phenomenon is not present in Dirichlet boundary control problems for the Poisson equation.

In the second part of this work [27], we use the techniques in [44] to analyze the optimal control problems but remove the restrictions on the domain and the desired state.

As far as we are aware, this is the first work to explore the analysis of this tangential Dirichlet control problem of Stokes equations and the numerical analysis of a computational method for this problem. There are a number of topics that can be explored in the future, including using an energy space for the control (see [15, 59] for the Poisson equation), devising divergence free and pressure robust HDG schemes, and considering more complicated PDEs, such as the Oseen and Navier-Stokes equations. Also, it would be interesting to investigate the higher order convergence rate phenomenon observed here.

7 Appendix

By simple algebraic operations in equation (3.10), we obtain the following formulas for the matrices $F_1, F_2, F_3$ and vectors $J_1^{(m)}, J_2^{(m)}, J_3^{(m)}$ in (3.11):

\[
\begin{align*}
\alpha & = F_1 \zeta + J_1^{(m)} \\
\beta & = F_2 \zeta + J_2^{(m)} \\
\gamma & = F_3 \zeta + J_3^{(m)}
\end{align*}
\]

\[
G_1 = (B_4 + B_2^T B_1^{-1} B_2)^{-1}, \quad G_2 = (B_6 + B_2^T B_1^{-1} B_3), \quad G_3 = (B_7 - B_6 G_1 B_5)^{-1},
\]

\[
G_4 = B_9 - B_6 G_1 G_2, \quad F_1 = -B_1^{-1} B_2 (G_1 B_5 G_3 G_4 - G_1 G_2) - B_1^{-1} B_3,
\]

\[
F_2 = G_1 B_5 G_3 G_4 - G_1 G_2, \quad F_3 = -G_3 G_4, \quad H_1^{(m)} = G_3 (b_2^{(m-1)} - B_6 G_1 b_1),
\]

\[
H_2 = G_1 b_1, \quad J_1^{(m)} = -B_1^{-1} B_2 (H_2 - G_1 B_5 H_1^{(m)}),
\]

\[
J_2^{(m)} = H_2 - G_1 B_5 H_1^{(m)}, \quad J_3^{(m)} = H_1^{(m)}.
\]

We briefly describe how these matrices can be easily computed using the HDG method described in this work.

Since the spaces $X_h$, $V_h$ and $W_h$ consist of discontinuous polynomials, some of the system matrices are block diagonal and each block is small and symmetric positive definite (SSPD). The matrix $B_1$ is this type, and therefore $B_1^{-1}$ is easily computed and is also a matrix of the same type. Therefore, the matrices $G_1, G_2, H_1, \text{ and } H_2$ are easily computed if $B_4 + B_2^T B_1^{-1} B_2$ is also easily inverted.

First, it can be checked that $B_2$ is block diagonal with small blocks, but the blocks are not symmetric or definite. This implies $B_2^T B_1^{-1} B_2$ is block diagonal with small nonnegative definite blocks. Next, $B_4 = \begin{bmatrix} A_5 & 0 \\ -A_4 & A_5 \end{bmatrix}$, where $A_4$ and $A_5$ are both SSPD block diagonal. Due to the structure of $B_1$ and $B_2$, the matrix $B_2^T B_1^{-1} B_2 + B_4$ has the form $\begin{bmatrix} C_1 & 0 \\ -A_4 & C_2 \end{bmatrix}$, where $C_1$ and $C_2$
are SSPD block diagonal. The inverse can be easily computed using the formula

\[
\begin{bmatrix}
C_1 & 0 \\
-A_4 & C_2
\end{bmatrix}^{-1} = \begin{bmatrix}
C_1^{-1} & 0 \\
C_2^{-1}A_4C_1^{-1} & C_2^{-1}
\end{bmatrix}.
\]

Furthermore, \(C_1^{-1}, C_2^{-1}\) and \(C_2^{-1}A_4C_1^{-1}\) are both SSPD block diagonal.

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