Perturbation analysis of the matrix equation \( X - \sum_{i=1}^{m} A_i^* X^{p_i} A_i = Q \)

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Abstract
Consider the nonlinear matrix equation \( X - \sum_{i=1}^{m} A_i^* X^{p_i} A_i = Q \) with \( p_i > 0 \). Sufficient and necessary conditions for the existence of positive definite solutions to the equation with \( p_i > 0 \) are derived. Two perturbation bounds for the unique solution to the equation with \( 0 < p_i < 1 \) are evaluated. The backward error of an approximate solution for the unique solution to the equation with \( 0 < p_i < 1 \) is given. Explicit expressions of the condition number for the equation with \( 0 < p_i < 1 \) are obtained. The theoretical results are illustrated by numerical examples.

Keywords: nonlinear matrix equation, positive definite solution, perturbation bound, backward error, condition number

1. Introduction
In this paper the nonlinear matrix equation

\[
X - \sum_{i=1}^{m} A_i^* X^{p_i} A_i = Q
\]

(1.1)
is investigated, where \( A_1, A_2, \ldots, A_m \) are \( n \times n \) complex matrices, \( m \) is a positive integer, \( p_i > 0 \) \((i = 1, 2, \cdots, m)\) and \( Q \) is a positive definite matrix. Here, \( A_i^* \) denotes the conjugate transpose of the matrix \( A_i \).

When \( m > 1 \), Eq.(1.1) is recognized as playing an important role in solving a system of linear equations. For example, in many physical calculations, one must solve the system of linear equation

\[
Mx = f,
\]
where

\[
M = \begin{pmatrix}
I & 0 & \cdots & 0 & A_1 \\
0 & I & \cdots & 0 & A_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I & A_m \\
A_1^* & A_2^* & \cdots & A_m^* & -Q
\end{pmatrix}
\]

\begin{thebibliography}{9}

\end{thebibliography}
arises in a finite difference approximation to an elliptic partial differential equation (for more information, refer to [24]). We can rewrite \( M = M + D \), where

\[
\begin{pmatrix}
X^{-p_1} & 0 & \cdots & 0 & A_1 \\
0 & X^{-p_2} & \cdots & 0 & A_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & X^{-p_m} & A_m \\
A_1^* & A_2^* & \cdots & A_m^* & -Q
\end{pmatrix}
\]

\[
I - X^{-p_1} & 0 & \cdots & 0 & 0 \\
0 & I - X^{-p_2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I - X^{-p_m} & 0 \\
0 & 0 & \cdots & 0 & 0
\]

\( \bar{M} \) can be factored as

\[
\begin{pmatrix}
-I & 0 & \cdots & 0 & 0 \\
0 & -I & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -I & 0 \\
-A_1^*X^{p_1} & -A_2^*X^{p_2} & \cdots & -A_m^*X^{p_m} & -I
\end{pmatrix}
\begin{pmatrix}
-X^{-p_1} & 0 & \cdots & 0 & -A_1 \\
0 & -X^{-p_2} & \cdots & 0 & -A_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -X^{p_m} & -A_m \\
0 & 0 & \cdots & 0 & X
\end{pmatrix}
\]

if and only if \( X \) is a solution of equation \( X - \sum_{i=1}^{m} A_i X^{p_i} A_i = Q \). When \( m = 1 \), this type of nonlinear matrix equations arises in ladder networks, dynamic programming, control theory, stochastic filtering, statistics and so forth [1–3, 24, 25, 34].

Lemma 2.1. If \( A \succeq B > 0 \) and \( 0 \leq \gamma \leq 1 \), then \( A^\gamma \succeq B^\gamma \).
Lemma 2.2. [23]: For any Hermitian positive definite matrix $X$ and Hermitian matrix $\Delta X$, we have

(i) $X^q = \frac{\sin q\pi}{\pi} \int_0^\infty X^\dagger (\lambda I + X)^{-1} X^\dagger \lambda^{q-1} d\lambda, \quad 0 < q < 1$;

(ii) $X^q = \frac{\sin q\pi}{(1-q)\pi} \int_0^\infty X^\dagger (\lambda I + X)^{-1} X (\lambda I + X)^{-1} X^\dagger \lambda^{q-1} d\lambda, \quad 0 < q < 1$.

In addition, if $X + \Delta X \geq (1/\nu)X > 0$, then

(iii) $\|X^\dagger A^\dagger (X + \Delta X)^q - X^q A X^\dagger\| \leq (1 - q)(\|X^\dagger \Delta AX^\dagger\| + \nu\|X^\dagger \Delta AX^\dagger\|^2)\|X^\dagger AX^\dagger\|^2$.

Lemma 2.3. [2]. The matrix equation $X = \sum_{i=1}^m A_i^\dagger X^6 A_i = Q$ $(0 < |\delta_i| < 1)$ always has a unique positive definite solution $X$. The matrix sequence $X_k$:

$$X_{s+m+1} = Q + \sum_{i=1}^m A_i^\dagger X_{s+m}^6 A_i, \quad s = 0, 1, 2, \cdots,$$  \hspace{1cm} (2.1)

converges to the unique positive definite solution $X$ for arbitrary initial positive definite matrices $X_0, X_1, \cdots, X_m$.

3. Positive definite solutions of the matrix Eq.(1.1)

In this section, sufficient and necessary conditions for the existence of positive definite solutions of Eq.(1.1) are obtained.

Theorem 3.1. Eq.(1.1) has a positive definite solution $X$ if and only if the coefficient matrices $A_i$ can be factored as

$$A_i = (W^* W)^{-n/2} Y_i (W^* W)^{1/2},$$  \hspace{1cm} (3.1)

where $W, Y_i$ $(i = 1, 2, \cdots, m)$ are nonsingular matrices, $Z = Q^p (W^* W)^{-1/2}$ and

$$\begin{pmatrix} Z \\ Y_1 \\ \vdots \\ Y_m \end{pmatrix}$$

is column orthonormal. In this case $X = W^* W$ is a solution of Eq.(1.1).

Proof. If Eq.(1.1) has a positive definite solution $X$, then there exists a nonsingular matrix $W$, s.t. $X = W^* W$. Furthermore Eq.(1.1) can be rewritten as

$$W^* W - \sum_{i=1}^m A_i^\dagger (W^* W)^\dagger A_i = Q,$$

which implies that

$$(W^* W)^\dagger Q(W^* W)^\dagger + \sum_{i=1}^m (W^* W)^\dagger A_i^\dagger (W^* W)^\dagger A_i (W^* W)^\dagger = I.$$  \hspace{1cm} (3.2)
Let $Y = (W^*W)^{1/2}A_i(W^*W)^{-1/2}$, then $A_i = (W^*W)^{1/2}Y_i(W^*W)^{-1/2}$. Moreover, by $Z = Q^*(W^*W)^{-1/2}$, (3.2) turns into
$$
\begin{pmatrix}
Z \\
Y_1 \\
\vdots \\
Y_m
\end{pmatrix}^* 
\begin{pmatrix}
Z \\
Y_1 \\
\vdots \\
Y_m
\end{pmatrix} = I,
$$
which means that $\begin{pmatrix}
Z \\
Y_1 \\
\vdots \\
Y_m
\end{pmatrix}$ is column orthonormal.

Conversely, suppose that $A_i$ has the decomposition (3.1). Let $X = W^*W$. Then
\begin{align*}
X - \sum_{i=1}^m A_i^*X^0A_i &= W^*W - \sum_{i=1}^m (W^*W)^{1/2}Y_i^*(W^*W)^{-1/2}(W^*W)^{1/2}Y_i(W^*W)^{-1/2} \\
&= W^*W - \sum_{i=1}^m (W^*W)^{1/2}Y_i^*Y_i(W^*W)^{-1/2} \\
&= (W^*W)^{1/2}(I - \sum_{i=1}^m Y_i^*Y_i)(W^*W)^{-1/2} = (W^*W)^{1/2}Z^*Z(W^*W)^{-1/2} \\
&= (W^*W)^{1/2}Q^*Q^t(W^*W)^{-1/2} = Q,
\end{align*}
that is $X = W^*W$ being a positive definite solution of Eq. (1.1).

\section*{Theorem 3.2}
If $A_1, A_2, \ldots, A_m$ are invertible and $Q \in \mathcal{H}$, then Eq. (1.1) has a positive definite solution $X$ if and only if $A_i$ can be factored as
$$
A_i = (U^*MU)^{1/2}V_iNU,
$$
where $N > 0$, $U$ is unitary and $M > UQU^*$ is diagonal, $M - N^2 = UQU^*$ and $\begin{pmatrix}
V_1 \\
V_2 \\
\vdots \\
V_m
\end{pmatrix}$ is column orthonormal. In this case, $X = U^*MU$ is a solution of Eq. (1.1).

\begin{proof}
If Eq. (1.1) has a positive definite solution $X$, then $X$ can be factored as $X = U^*MU$, where $U$ is unitary and $M$ is diagonal. Therefore Eq. (1.1) can be rewritten as
$$
U^*MU = \sum_{i=1}^m A_i^*(U^*MU)^{1/2}A_i = Q,
$$
which implies that
$$
M - UQU^* = \sum_{i=1}^m UA_i^*(U^*MU)^{1/2}A_iU^*
$$

\end{proof}
Let \( N = (M - UQU^* \dagger)^\dagger \) and \( V_i = (U^* MU)^\dagger A_i U^*(M - UQU^*)^{-\dagger} \). Then \( A_i = (U^* MU)^{-\dagger} V_i NU \) and \( M - N^2 = UQU^* \). Eq. (3.4) turns into \( \sum_{i=1}^{m} V_i V_i = I \), which means that \( \frac{V_1}{V_2} \cdots \frac{V_{m}}{V_{m}} \) is column orthonormal.

Conversely, suppose that \( A_i \) has the decomposition (3.3). Let \( X = U^* MU \). Then

\[
X - \sum_{i=1}^{m} A_i^* X^0 A_i = U^* MU - \sum_{i=1}^{m} U^* N^2 V_i (U^* MU)^{\dagger} (U^* MU)^{-\dagger} V_i NU = U^*(M - N^2)U = Q,
\]

that is \( X = U^* MU \) is a solution of Eq. (1.1).

**Theorem 3.3.** If \( X \) is a solution of Eq. (1.1) with \( 0 < p_i < 1 \), then

\[
X \geq \left( \lambda_{\min}(Q) + \sum_{i=1}^{m} \lambda_{\min}(A_i^* A_i) \lambda_{\min}^0(Q) \right) I = \beta I.
\]

**Proof.** By Lemma 2.1, Eq. (1.1) with \( 0 < p_i < 1 \) always has a unique positive definite solution \( X \). Then \( X > 0 \), it follows that \( X^0 > 0 \). Therefore \( X \geq Q \). By Lemma 2.1 and Eq. (1.1), we have \( X \geq Q + \sum_{i=1}^{m} \lambda_{\min}(Q) A_i \geq \left( \lambda_{\min}(Q) + \sum_{i=1}^{m} \lambda_{\min}(A_i^* A_i) \lambda_{\min}^0(Q) \right) I = \beta I \).

4. Perturbation bounds of Eq. (1.1) with \( 0 < p_i < 1 \)

Here the perturbed equation

\[
\tilde{X} - \sum_{i=1}^{m} \tilde{A}_i^* \tilde{X}^0 \tilde{A}_i = \tilde{Q}, \quad 0 < p_i < 1,
\]

(4.1)
is considered, where \( \tilde{A}_i \) and \( \tilde{Q} \) are small perturbations of \( A_i \) and \( Q \) in Eq. (1.1), respectively. We assume that \( X \) and \( \tilde{X} \) are the solutions of Eq. (1.1) and Eq. (4.1), respectively. Let \( \Delta X = \tilde{X} - X \), \( \Delta Q = \tilde{Q} - Q \) and \( \Delta A_i = \tilde{A}_i - A_i \).

In this section two perturbation bounds for the solution of Eq. (1.1) with \( 0 < p_i < 1 \) are developed. The relative perturbation bound in Theorem 4.1 does not depend any knowledge of the actual solution \( X \) of Eq. (1.1). Furthermore, a sharper perturbation bound in Theorem 4.2 is derived.

The next theorem generalizes Theorem 4 in [20] with \( m = 1 \), \( \| \Delta Q \| = 0 \) to arbitrary integer \( m \geq 1 \), \( \| \Delta Q \| > 0 \).
Theorem 4.1. Let \( b = \beta + \|\Delta Q\| - \sum_{i=1}^{m} (1-p_i) \beta_i \|A_i\|^2, \) \( s = \sum_{i=1}^{m} \beta_i \|\Delta A_i\| (2\|A_i\| + \|\Delta A_i\|). \) If

\[
0 < b < 2(\beta - s) \quad \text{and} \quad b^2 - 4(\beta - s)(s + \|\Delta Q\|) \geq 0,
\]

then

\[
\frac{\|\tilde{X} - X\|}{\|X\|} \leq \varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\| \equiv \xi_1,
\]

where

\[
\varrho = \frac{2}{\sum_{i=1}^{m} \|\Delta A_i\|(b + \sqrt{b^2 - 4(\beta - s)(s + \|\Delta Q\|))}}, \quad \omega = \frac{2}{b + \sqrt{b^2 - 4(\beta - s)(s + \|\Delta Q\|))}.
\]

Proof. Let

\[
\Omega = \{\Delta X \in \mathcal{H}^n : \|X^{-1/2} \Delta XX^{-1/2}\| \leq \varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\| \}.
\]

Obviously, \( \Omega \) is a nonempty bounded convex closed set. Let

\[
f(\Delta X) = \sum_{i=1}^{m} (\tilde{A}_i^* (X + \Delta X)^n \tilde{A}_i - A_i^n X^n A_i) + \Delta Q, \quad \Delta X \in \Omega.
\]

Evidently, \( f : \Omega \mapsto \mathcal{H}^n \) is continuous. We will prove that \( f(\Omega) \subseteq \Omega. \)

For every \( \Delta X \in \Omega \), it follows \( \|X^{-1/2} \Delta XX^{-1/2}\| \leq \varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\|. \) Thus

\[
(\varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\|)I \geq \|X^{-1/2} \Delta XX^{-1/2}\| \geq (\varrho \sum_{i=1}^{m} \|\Delta A_i\| - \omega \|\Delta Q\|)I,
\]

\[
(1 + \varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\|)X \geq X + \Delta X \geq (1 - \varrho \sum_{i=1}^{m} \|\Delta A_i\| - \omega \|\Delta Q\|)X.
\]

According to (4.2) and (4.3), we have

\[
\omega \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\| = \frac{2(\|\Delta Q\| + s)}{b + \sqrt{b^2 - 4(\beta - s)(s + \|\Delta Q\|))} \leq \frac{2(\|\Delta Q\| + s)}{b} \leq \frac{b}{2(\beta - s)} < 1.
\]

Therefore

\[
(1 - \varrho \sum_{i=1}^{m} \|\Delta A_i\| - \omega \|\Delta Q\|)X > 0.
\]
From Lemma 2.2 and Theorem 3.3 it follows that

\[
\left\| X^{-\frac{1}{2}} \left[ \sum_{i=1}^{m} A_i^* (X + \Delta X)^{p_i} (X + \Delta X)^{p_i} - X^{-\frac{1}{2}} A_i \right] X^{-\frac{1}{2}} \right\| \\
\leq \left( \left\| X^{-\frac{1}{2}} \Delta XX^{-\frac{1}{2}} \right\| + \frac{\left\| X^{-\frac{1}{2}} \Delta XX^{-\frac{1}{2}} \right\|^2}{1 - \varrho \sum_{i=1}^{m} \| \Delta A_i \| - \omega \| \Delta Q \|} \right) \left( \sum_{i=1}^{m} (1 - p_i)\| \Delta A_i \| \right) (4.4)
\]

\[
\leq \left( \left\| X^{-\frac{1}{2}} \Delta XX^{-\frac{1}{2}} \right\| + \frac{\left\| X^{-\frac{1}{2}} \Delta XX^{-\frac{1}{2}} \right\|^2}{1 - \varrho \sum_{i=1}^{m} \| \Delta A_i \| - \omega \| \Delta Q \|} \right) \left( \sum_{i=1}^{m} \frac{1 - p_i}{\beta^i p_i} \| \Delta A_i \| ^2 \right)
\]

\[
\leq \left( \xi_1 + \frac{\xi_1^2}{1 - \xi_1} \right) \left( \sum_{i=1}^{m} \frac{1 - p_i}{\beta^i p_i} \| \Delta A_i \| ^2 \right) + \frac{s}{\beta} (1 + \xi_1) + \frac{\| \Delta Q \|}{\beta}
\]

\[
= \xi_1.
\]

That is \( f(\Omega) \subseteq \Omega \). By Brouwer’s fixed point theorem, there exists a \( \Delta X \in \Omega \) such that \( f(\Delta X) = \Delta X \). Moreover, by Lemma 2.3 we know that \( X \) and \( \tilde{X} \) are the unique solutions to Eq. (1) and Eq. (4.1), respectively. Then

\[
\frac{\| \tilde{X} - X \|}{\| X \|} = \frac{\| \Delta X \|}{\| X \|} = \frac{\| X^{1/2} (X^{-1/2} \Delta XX^{-1/2}) X^{1/2} \|}{\| X \|}
\]

\[
\leq \frac{\| X^{-1/2} \Delta XX^{-1/2} \|}{\| X \|} \leq \varrho \sum_{i=1}^{m} \| \Delta A_i \| + \omega \| \Delta Q \|.
\]
Lemma 4.1. Let \( Q \sum_{i=1}^{m} \| A_i \| + \omega \| Q \| = \frac{2(\sum_{i=1}^{m} \| A_i \|(2\| A_i \| + \| A_i \|) + \| Q \|)}{b + \sqrt{b^2 - 4(\beta - \sigma)(\beta + \| Q \|)}}. \)

we get \( Q \sum_{i=1}^{m} \| A_i \| + \omega \| Q \| \rightarrow 0 \) for \( Q \rightarrow 0 \), \( \| A_i \| \rightarrow 0 \) (\( i = 1, 2, \cdots, m \)). Therefore Eq. (4.7) is well-posed.

Next, a sharper perturbation estimate is derived.

Subtracting (1.1) from (4.1) we have

\[
\Delta X + \sum_{i=1}^{m} \frac{\sin p_i \pi}{\pi} \int_{0}^{\infty} [(\lambda I + X)^{-1} X^2 A_j] \Delta X [(\lambda I + X)^{-1} X^2 A_j] \lambda^{p_i - 1} d\lambda = E + h(\Delta X),
\]

where

\[
B_i = X^{p_i} A_i, \quad i = 1, 2, \cdots, m,
\]

\[
E = \sum_{i=1}^{m} (B_i A_i + A_i B_i) + \sum_{i=1}^{m} \Delta A_i X^{p_i} A_i + \Delta Q,
\]

\[
h(\Delta X) = \sum_{i=1}^{m} [A_i Z(\Delta X) A_i - \bar{A_i} V(\Delta X) A_i - \Delta A_i V(\Delta X) A_i],
\]

\[
Z_i(\Delta X) = \frac{\sin p_i \pi}{\pi} \int_{0}^{\infty} X^2 (\lambda I + X)^{-1} \Delta X (\lambda I + X + \Delta X)^{-1} X^2 \lambda^{p_i - 1} d\lambda,
\]

\[
V_i(\Delta X) = \frac{\sin p_i \pi}{\pi} \int_{0}^{\infty} X^2 (\lambda I + X)^{-1} \Delta X (\lambda I + X + \Delta X)^{-1} X^2 \lambda^{p_i - 1} d\lambda.
\]

Remark 4.1. With

\[
Q \sum_{i=1}^{m} \| A_i \| + \omega \| Q \| = \sum_{i=1}^{m} \| A_i \|(2\| A_i \| + \| A_i \|) + \| Q \|, \]

therefore Eq. (4.7) is well-posed.

Lemma 4.1. Let \( \sum_{i=1}^{m} \| A_i \|^2 < 1 \). Then the linear operator \( L : H^{\delta} \rightarrow H^{\delta} \) defined by

\[
L W = W + \sum_{i=1}^{m} \frac{\sin p_i \pi}{\pi} \int_{0}^{\infty} [(\lambda I + X)^{-1} X^2 A_j] W [(\lambda I + X)^{-1} X^2 A_j] \lambda^{p_i - 1} d\lambda, \quad W \in H^{\delta}.
\]

is invertible.

Proof. It suffices to show that the following equation

\[
L W = V
\]

has a unique solution for every \( V \in H^{\delta} \). Define the operator \( M : H^{\delta} \rightarrow H^{\delta} \) by

\[
M Z = \sum_{i=1}^{m} \frac{\sin p_i \pi}{\pi} \int_{0}^{\infty} X^2 A_i X^2 (\lambda I + X)^{-1} X^2 Z X (\lambda I + X)^{-1} X^2 A_i X^2 \lambda^{p_i - 1} d\lambda, \quad Z \in H^{\delta}.
\]

Let \( Y = X^{-1/2} W X^{-1/2} \). Thus (4) is equivalent to

\[
Y + M Y = X^{-1/2} V X^{-1/2}.
\]
According to Lemma 2.2, we have
\[
\|MY\| \leq \sum_{i=1}^{m} \left\| \sin \frac{pr_i}{\pi} \int_{0}^{\infty} X^{-\frac{1}{2}} A_i X^\frac{1}{2} (AI + X)^{-1} X(AI + X)^{-1} X^{-\frac{1}{2}} A_i X^{\frac{1}{2}} d\lambda \right\| \|Y\|
\]
\[
\leq \sum_{i=1}^{m} (1 - p_i) \|X^{-\frac{1}{2}} A_i X^{\frac{1}{2}}\|^2 \|Y\| \leq \sum_{i=1}^{m} \frac{\|A_i\|^2}{\beta^{1-p_i}} \|Y\| < \|Y\|
\]
which implies that \(\|M\| < 1\) and \(I + M\) is invertible. Therefore, the operator \(L\) is invertible. \(\square\)

Furthermore, we define operators \(P_i : C^{\omega\infty} \to H^{\omega\infty}\) by
\[
P_i Z_i = L^{-1}(B_i Z_i + Z_i B_i), \quad Z_i \in C^{\omega\infty}, \quad i = 1, 2, \ldots, m.
\]
Thus, we can rewrite (4.5) as
\[
\Delta X = L^{-1} \Delta Q + \sum_{i=1}^{m} P_i \Delta A_i + L^{-1} \left( \sum_{i=1}^{m} \Delta A_i X^p \Delta A_i \right) + L^{-1}(h(\Delta X)).
\tag{4.8}
\]
Define
\[
\|L^{-1}\| = \max_{W \in H^{\omega\infty}} \|L^{-1} W\|, \quad \|P_i\| = \max_{Z_i \in C^{\omega\infty}} \|P_i Z_i\|, \quad i = 1, 2, \ldots, m.
\]

Now we denote
\[
l = \|L^{-1}\|^{-1}, \quad \zeta = \|X^{-1}\|, \quad \xi_i = \|X^p\|, \quad n_i = \|P_i\|, \quad \theta = \frac{1}{2} \sum_{i=1}^{m} \xi_i \|A_i\|^2, \quad i = 1, 2, \ldots, m,
\]
\[
\epsilon = \frac{1}{l} \|\Delta Q\| + \sum_{i=1}^{m} (n_i \|A_i\| + \frac{\epsilon}{l} \|A_i\|^2), \quad \sigma = \frac{1}{l} \sum_{i=1}^{m} \xi_i (\|A_i\| + \|A_i\| \|\Delta A_i\|). \tag{4.9}
\]

**Theorem 4.2.** If
\[
\sigma < 1 \quad \text{and} \quad \epsilon < \frac{(1 - \sigma)^2}{\zeta + \sigma \zeta + 2\theta + 2 \sqrt{(\zeta + \theta)(\sigma \zeta + \theta)}} \tag{4.10}
\]
then
\[
\|\tilde{X} - X\| \leq \frac{2\epsilon}{1 + \epsilon \zeta - \sigma + \sqrt{(1 + \zeta \epsilon - \sigma)^2 - 4\epsilon (\zeta + \theta)}} \equiv \nu.
\]

**Proof.** Let
\[
f(\Delta X) = L^{-1} \Delta Q + \sum_{i=1}^{m} P_i \Delta A_i + L^{-1} \left( \sum_{i=1}^{m} \Delta A_i X^p \Delta A_i \right) + L^{-1}(h(\Delta X)). \tag{4.11}
\]
Obviously, \(f : H^{\omega\infty} \to H^{\omega\infty}\) is continuous. The condition (4.10) ensures that the quadratic equation \((\zeta + \theta)x^2 - (1 + \zeta \epsilon - \sigma)x + \epsilon = 0\) with respect to the variable \(x\) has two positive real roots. The smaller one is
\[
\nu = \frac{2\epsilon}{1 + \epsilon \zeta - \sigma + \sqrt{(1 + \zeta \epsilon - \sigma)^2 - 4\epsilon (\zeta + \theta)}}
\]
Define $\Omega = \{ \Delta X \in H^\infty : ||\Delta X|| \leq \nu \}$. Then for any $\Delta X \in \Omega$, by (4.10), we have

$$||X^{-1}\Delta X|| \leq ||X^{-1}|| ||\Delta X|| \leq \zeta \nu \leq \zeta \frac{2\epsilon}{1 + \zeta \epsilon - \sigma} = 1 + \frac{\zeta \epsilon + \sigma - 1}{1 + \zeta \epsilon - \sigma} \leq 1 + \frac{-2(1 - \sigma)(\zeta \epsilon + \theta)}{(\zeta + \sigma \zeta + 2\theta)(1 + \zeta \epsilon - \sigma)} < 1.$$ 

It follows that $I - X^{-1}\Delta X$ is nonsingular and

$$||I - X^{-1}\Delta X|| \leq \frac{1}{1 - ||X^{-1}\Delta X||} \leq \frac{1}{1 - \zeta ||\Delta X||}.$$

Using (4.6) and Lemma 2.2, we have

$$||Z_i(\Delta X)|| \leq (1 - \rho_i)||\Delta X||^2 ||X^{-1}||^2 ||I + X^{-1}\Delta X||^{-1} ||X^p|| \leq \xi \sigma \frac{||\Delta X||^2}{1 - \zeta ||\Delta X||},$$

$$||V_i(\Delta X)|| \leq ||X^p|| ||\Delta X|| ||X^{-1}|| ||I + X^{-1}\Delta X||^{-1} \leq \xi \zeta \frac{||\Delta X||}{1 - \zeta ||\Delta X||},$$

$$||h(\Delta X)|| \leq \sum_{i=1}^{m} \left( ||A_i||^2 ||Z_i(\Delta X)|| + (2||A_i|| + ||A_i|| ||A_i|| ||V_i(\Delta X)|| \right)$$

$$\leq \sum_{i=1}^{m} \left( \xi \sigma \frac{||\Delta X||^2}{1 - \zeta ||\Delta X||} + (2||A_i|| + ||A_i|| ||A_i|| ||\Delta X|| ||\xi \zeta \frac{||\Delta X||}{1 - \zeta ||\Delta X||} \right).$$

Noting (4.9) and (4.11), it follows that

$$||f(\Delta X)|| \leq \frac{1}{l}||Q|| + \sum_{i=1}^{m} (n_i ||A_i|| + \frac{\zeta}{l} ||A_i||^2) + \frac{1}{l}||h(\Delta X)||$$

$$\leq \epsilon + \frac{\sigma ||\Delta X||}{1 - \zeta ||\Delta X|| + \frac{\theta ||\Delta X||^2}{1 - \zeta ||\Delta X||} \leq \epsilon + \frac{\sigma \nu}{1 - \zeta \nu + \frac{\theta \nu^2}{1 - \zeta \nu} = \nu},$$

for $\Delta X \in \Omega$. That is $f(\Omega) \subseteq \Omega$. According to Schauder fixed point theorem, there exists $\Delta X \in \Omega$ such that $f(\Delta X) = X$. It follows that $X + \Delta X$ is a Hermitian solution of Eq. (4.1). By Lemma 2.3, we know that the solution of Eq. (4.1) is unique. Then $\Delta X = X - X$ and $||X - X|| \leq \xi \zeta$.  

**Remark 4.2.** From Theorem 4.2 we get the first order perturbation bound for the solution as follows:

$$||\tilde{X} - X|| \leq \frac{1}{l}||Q|| + \sum_{i=1}^{m} (n_i ||A_i|| + \frac{\zeta}{l} ||A_i||^2) + O \left( ||(\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q)||^2 \right),$$

as $(\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q) \to 0.$

Combining this with (3.3) gives

$$\Delta X = L^{-1} \Delta Q + L^{-1} \sum_{i=1}^{m} (B_i \Delta A_i + A_i^* B_i) + O \left( ||(\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q)||^2 \right).$$

as $(\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q) \to 0.$
5. Backward error of Eq. (1.1) with $0 < p_i < 1$

In this section, a backward error of an approximate solution for the unique solution to Eq. (1.1) with $0 < p_i < 1$ is developed.

**Theorem 5.1.** Let $\widetilde{X} > 0$ be an approximation to the solution $X$ of Eq. (1.1). If $\Sigma = \sum_{i=1}^{m}(1 - p_i)||\widetilde{X} - A_i\widetilde{X}^{-1}||^2 < 1$ and the residual $R(\widetilde{X}) \equiv Q + \sum_{i=1}^{m}A_i\widetilde{X}\theta_i A_i - \widetilde{X}$ satisfies

$$||R(\widetilde{X})|| < \frac{\theta_1}{2||X^{-1}||} \min[1, \frac{\theta_1}{2}], \text{ where } \theta_1 \equiv 1 + ||\widetilde{X}^{-1}||R(\widetilde{X})|| - \Sigma > 0,$$

then

$$||\widetilde{X} - X|| \leq \mu||R(\widetilde{X})||,$$

where $\mu = \frac{2||X||||\widetilde{X}^{-1}||}{\theta_1 + \sqrt{\theta_1^2 - 4||X^{-1}||R(\widetilde{X})||}}.$

**Proof.** Let

$$\Psi = \{\Delta X \in \mathcal{H}^{\text{exa}} : ||\widetilde{X}^{-1/2}\Delta X\widetilde{X}^{-1/2}|| \leq \theta_2||R(\widetilde{X})||\},$$

where $\theta_2 = \frac{1}{||X||}.$ Obviously, $\Psi$ is a nonempty bounded convex closed set. Let

$$g(\Delta X) = \sum_{i=1}^{m}A_i^*[(\widetilde{X} + \Delta X)\theta_i - \widetilde{X}]A_i + R(\widetilde{X}).$$

Evidently $g : \Psi \mapsto \mathcal{H}^{\text{exa}}$ is continuous. We will prove that $g(\Psi) \subseteq \Psi.$ For every $\Delta X \in \Psi,$ we have

$$||\widetilde{X}^{-1/2}\Delta X\widetilde{X}^{-1/2}|| \leq \theta_2||R(\widetilde{X})||.$$

Hence

$$\widetilde{X}^{-1/2}\Delta X\widetilde{X}^{-1/2} \geq -\theta_2||R(\widetilde{X})||I,$$

that is

$$\widetilde{X} + \Delta X \geq (1 - \theta_2||R(\widetilde{X})||)\widetilde{X}.$$

Using (5.1), one sees that

$$\theta_2||R(\widetilde{X})|| = \frac{2||X||||\widetilde{X}^{-1}||R(\widetilde{X})||}{\theta_1 + \sqrt{\theta_1^2 - 4||X^{-1}||R(\widetilde{X})||}} < \frac{2||X||||\widetilde{X}^{-1}||R(\widetilde{X})||}{\theta_1} < 1.$$ 

Therefore, $(1 - \theta_2||R(\widetilde{X})||)\widetilde{X} > 0.$

According to (4.4), we obtain

$$||\widetilde{X}^{-1/2}g(\Delta X)\widetilde{X}^{-1/2}|| \leq \left(\frac{||\widetilde{X}^{-1/2}\Delta X\widetilde{X}^{-1/2}||}{1 - \theta_2||R(\widetilde{X})||}\right)\Sigma + ||\widetilde{X}^{-1/2}R(\widetilde{X})\widetilde{X}^{-1/2}||$$

$$\leq \left(\frac{\theta_2||R(\widetilde{X})||}{1 - \theta_2||R(\widetilde{X})||}\right)\Sigma + ||\widetilde{X}^{-1}||R(\widetilde{X})||$$

$$= \theta_2||R(\widetilde{X})||.$$
By Brouwer fixed point theorem, there exists a $\Delta X \in \Psi$ such that $g(\Delta X) = \Delta X$. Hence $\tilde{X} + \Delta X$ is a solution of Eq. \((1.1)\). Moreover, by Lemma 2.3 we know that the solution $X$ of Eq. \((1.1)\) is unique. Then

$$ ||\tilde{X} - X|| = ||\Delta X|| \leq ||X||^2 ||\tilde{A}^T A X \tilde{X}^{-1}|| = \theta_2 ||X|| ||R(X)||. $$

\[ \square \]

### 6. Condition number

In this section, we apply the theory of condition number developed by Rice [27] to study condition numbers of the unique solution to Eq. \((1.1)\) with $0 < p_i < 1$.

#### 6.1. The complex case

Suppose that $X$ and $\tilde{X}$ are the solutions of the matrix equations \((1.1)\) and \((4.1)\), respectively. Let $\Delta A = \tilde{A} - A$, $\Delta Q = \tilde{Q} - Q$ and $\Delta X = \tilde{X} - X$. Using Theorem 4.2 and Remark 4.2, we have

$$ \Delta X = \tilde{X} - X = L^{-1} \Delta Q + L^{-1} \sum_{i=1}^{m} (B_i^T \Delta A_i + \Delta A_i^T B_i) + O \left( ||(\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q)||_F^2 \right), \quad (6.1) $$

as $(\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q) \to 0$.

By the theory of condition number developed by Rice [27], we define the condition number of the Hermitian positive definite solution $X$ to \((1.1)\) by

$$ c(X) = \lim_{\delta \to 0} \sup_{||\Delta A_1||_F, \cdots, \Delta A_m, \Delta Q < \delta} \frac{||\Delta X||_F}{\xi \delta}, \quad (6.2) $$

where $\xi$, $\rho$ and $\eta_i$, $i = 1, 2, \cdots, m$ are positive parameters. Taking $\xi = \eta_1 = \rho = 1$ in \((6.2)\) gives the absolute condition number $c_{ab}(X)$, and taking $\xi = ||X||_F$, $\eta_i = ||A_i||_F$ and $\rho = ||Q||_F$ in \((6.2)\) gives the relative condition number $c_{rel}(X)$.

Substituting \((6.1)\) into \((6.2)\), we get

$$ c(X) = \frac{1}{\xi} \max_{\Delta A_1, \cdots, \Delta A_m, \Delta Q \neq 0} \frac{||L^{-1} (\Delta Q + \sum_{i=1}^{m} (B_i^T \Delta A_i + \Delta A_i^T B_i))||_F}{||\Delta A_1, \cdots, \Delta A_m, \Delta Q||_F} $$

$$ = \frac{1}{\xi} \max_{E_1, E_2, \cdots, E_m, H \neq 0} \frac{||L^{-1} (\rho H + \sum_{i=1}^{m} \eta_i (B_i^T E_i + E_i^T B_i))||_F}{||E_1, E_2, \cdots, E_m, H||_F}. $$

Let $L$ be the matrix representation of the linear operator $L$. Then it is easy to see that

$$ L = I + \sum_{i=1}^{m} \sin \frac{p_i \pi}{\pi} \int_{0}^{\infty} [(A_l + X)^{-1} X^* A_i]^T \otimes [(A_l + X)^{-1} X^* A_i]^* d\theta - d \lambda. $$
Let

\[ L^{-1} = S + i \Sigma, \]

\[ L^{-1}(I \otimes B_i^*) = L^{-1}(I \otimes (X^0 A_i)^*) = U_{i1} + i \Omega_{i1}, \]

\[ L^{-1}(B_i^T \otimes I) = L^{-1}(X^0 A_i^T \otimes I) \Pi = U_{i2} + i \Omega_{i2}. \]

Then we have the following theorem.

\[ S_c = \begin{bmatrix} S & \Sigma \\ \Sigma & S \end{bmatrix}, \quad U_i = \begin{bmatrix} U_{i1} + U_{i2} \\ \Omega_{2i} - \Omega_{1i} \\ U_{i1} - U_{i2} \end{bmatrix}, \quad i = 1, 2, \ldots, m, \quad (6.3) \]

where \( x, y, a_i, b_i \in C^r, \quad S, \Sigma, U_{i1}, U_{i2}, \Omega_{1i}, \Omega_{2i} \in C^{n \times \infty}, \quad i = 1, 2, \ldots, m, \) \( \Pi \) is the vec-permutation matrix, such that

\[ \text{vec} A^T = \Pi \text{vec} A. \]

Then we obtain that

\[
c(X) = \frac{1}{\xi} \max_{M \neq 0} \frac{\|L^{-1}(\rho H + \sum_{i=1}^m \eta_i (B_i^* E_i + E_i^* B_i))\|_F}{\|\rho L^{-1} \text{vec} H + \sum_{i=1}^m \eta_i L^{-1} ((I \otimes B_i^*) \text{vec} E_i + (B_i^T \otimes I) \text{vec} E_i^*)\|_F}
\]

\[
= \frac{1}{\xi} \max_{M \neq 0} \frac{\|\rho L^{-1} \text{vec} H + \sum_{i=1}^m \eta_i L^{-1} ((I \otimes B_i^*) \text{vec} E_i + (B_i^T \otimes I) \text{vec} E_i^*)\|_F}{\|L^{-1}(\rho S + i \Sigma)(x + iy) + \sum_{i=1}^m \eta_i (U_{i1} + i \Omega_{i1})(a_i + i b_i) + (U_{i2} + i \Omega_{i2})(a_i - i b_i)\|_F}
\]

\[
= \frac{1}{\xi} \max_{M \neq 0} \frac{\|\rho S, \eta_1 U_1, \eta_2 U_2, \ldots, \eta_m U_m\|}{\|\|}
\]

Then we have the following theorem.

**Theorem 6.1.** The condition number \( c(X) \) defined by (6.2) has the explicit expression

\[
c(X) = \frac{1}{\xi} \| \rho S_c, \eta_1 U_1, \eta_2 U_2, \ldots, \eta_m U_m \|. \quad (6.4)
\]

where the matrices \( S_c \) and \( U_i \) are defined as in (6.3).

**Remark 6.1.** From (6.4) we have the relative condition number

\[
c_{\text{rel}}(X) = \frac{\| \|Q\|_{1} S_c, \quad \|A_1\|_{F} U_1, \quad \|A_2\|_{F} U_2, \ldots, \quad \|A_m\|_{F} U_m \|}{\|X\|_{F}}.
\]

13
6.2. The real case

In this subsection we consider the real case, i.e., all the coefficient matrices $A_i$, $Q$ of Eq. (1.1) are real. In such a case the corresponding solution $X$ is also real. Completely similar arguments as Theorem 6.1 gives the following theorem.

**Theorem 6.2.** Let $A_i$, $Q$ be real, $c(X)$ be the condition number defined by (6.2). Then $c(X)$ has the explicit expression

$$c(X) = \frac{1}{\xi} \| (\rho S_r, \eta_1 U_1, \eta_2 U_2, \cdots, \eta_m U_m) \|,$$

where

$$S_r = \left(I + \sum_{i=1}^{m} \frac{\sin p_{i} \pi}{\pi} \int_{0}^{\infty} \left[ (\lambda I + X)^{-1} X^{\frac{3}{2}} A_i^{*} \right]^{T} \otimes \left[ (\lambda I + X)^{-1} X^{\frac{3}{2}} A_i^{*} \right]^{T} \lambda^{p_{i}-1} d \lambda \right)^{-1},$$

$$U_i = S_r [I \otimes (A_i^{T} X^{p_i}) + ((A_i^{T} X^{p_i}) \otimes I) \Pi], \quad i = 1, 2, \cdots, m.$$

**Remark 6.2.** In the real case the relative condition number is given by

$$\epsilon_{rel}(X) = \frac{\| (\|Q\|F S_r, \|A_1\|F U_1, \|A_2\|F U_2, \cdots, \|A_m\|F U_m) \|}{\|X\|F}.$$

7. Numerical Examples

To illustrate the results of the previous sections, in this section three simple examples are given, which were carried out using MATLAB 7.1. For the stopping criterion we take $\epsilon_{k+1}(X) = \|X - \sum_{i=1}^{m} A_i^{*} X^{p_i} A_i - I\| < 1.0 \times 10^{-10}$

**Example 7.1.** We consider the matrix equation

$$X - A_1^{*} X^{\frac{3}{2}} A_1 - A_2^{*} X^{\frac{3}{2}} A_2 = I,$$

with

$$A_1 = \frac{1}{A} + 2 \times 10^{-2} A, \quad A_2 = \frac{1}{A} + 3 \times 10^{-2} A, \quad A = \begin{pmatrix} 2 & 0.95 \\ 0 & 1 \end{pmatrix}. $$

Suppose that the coefficient matrices $A_1$ and $A_2$ are perturbed to $\tilde{A}_i = A_i + \Delta A_i$, $i = 1, 2$, where

$$\Delta A_1 = \frac{10^{-j}}{||C^T + C||} (C^T + C), \quad \Delta A_2 = \frac{3 \times 10^{-j-1}}{||C^T + C||} (C^T + C)$$

and $C$ is a random matrix generated by MATLAB function `randn.`

We now consider the corresponding perturbation bounds for the solution $X$ in Theorem 4.7 and Theorem 6.2.

The conditions in Theorem 4.7 are

$$\text{con} 1 = 2(\beta - s) - b > 0, \quad \text{con} 2 = \beta + \| \Delta Q \| - \sum_{i=1}^{m} (1 - p_i) \beta \| \Delta A_i \| > 0,$$

$$\text{con} 3 = b^2 - 4(\beta - s)(s + \| \Delta Q \|) \geq 0.$$

14
The conditions in Theorem 4.2 are

\[ \text{con}4 = 1 - \sigma > 0, \quad \text{con}5 = \frac{(1 - \sigma)^2}{\zeta + \sigma \zeta + 2\theta + 2 \sqrt{(\zeta + \theta)(\sigma \zeta + \theta)}} - \epsilon > 0. \]

By computation, we list them in Table 1.

| \( j \) | 4    | 5    | 6    | 7    |
|---------|------|------|------|------|
| con1   | 1.1139 | 1.1141 | 1.1141 | 1.1141 |
| con2   | 0.9358 | 0.9358 | 0.9357 | 0.9357 |
| con3   | 0.8751 | 0.8755 | 0.8756 | 0.8756 |
| con4   | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| con5   | 0.7955 | 0.7957 | 0.7957 | 0.7957 |

The results listed in Table 1 show that the conditions of Theorem 4.1 and Theorem 4.2 are satisfied.

By Theorem 4.1 and Theorem 4.2, we can compute the relative perturbation bounds \( \xi_1, \xi_2 \) respectively. These results averaged as the geometric mean of 10 randomly perturbed runs. Some results are listed in Table 2.

| \( j \) | 4    | 5    | 6    | 7    |
|---------|------|------|------|------|
| \( \frac{\|X - \tilde{X}\|}{\|X\|} \) | 3.9885 \times 10^{-5} | 5.1141 \times 10^{-6} | 3.6513 \times 10^{-7} | 4.6136 \times 10^{-8} |
| \( \xi_1 \) | 1.9765 \times 10^{-4} | 2.3869 \times 10^{-5} | 1.8133 \times 10^{-6} | 2.1028 \times 10^{-7} |
| \( \xi_2 \) | 6.5069 \times 10^{-5} | 7.6524 \times 10^{-6} | 6.0514 \times 10^{-7} | 6.9911 \times 10^{-8} |

The results listed in Table 2 show that the perturbation bound \( \xi_2 \) given by Theorem 4.2 is fairly sharp, while the bound \( \xi_1 \) given by Theorem 4.1 which does not depended on the exact solution is conservative.

**Example 7.2.** We consider the matrix equation

\[ X - A_1^* X^{0.5} A_1 - A_2^* X^{0.25} A_2 = I, \]

with

\[ A_1 = \frac{\frac{1}{2} + 2 \times 10^{-2}}{\|A\|} A, \quad A_2 = \frac{\frac{3}{2} + 3 \times 10^{-2}}{\|A\|} A, \quad A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}. \]

Choose \( \tilde{X}_1 = A, \tilde{X}_2 = 2A. \) Let the approximate solution \( \tilde{X}_k \) of \( X \) be given with the iterative method (2.7), where \( k \) is the iterative number.

The residual \( R(\tilde{X}_k) \equiv I + A_1^* \tilde{X}_k^{0.5} A_1 + A_2^* \tilde{X}_k^{0.25} A_2 - \tilde{X}_k \) satisfies the conditions in Theorem 5.1.
By Theorem 5.1, we can compute the backward error bound for $\tilde{X}_k$

$$
\| \tilde{X}_k - X \| \leq \mu \| R(\tilde{X}_k) \|,
$$

where

$$
\mu = \frac{2\| \tilde{X}_k \| \| \tilde{X}_k^{-1} \|}{\theta_1 + \sqrt{\theta_1^2 - 4\| \tilde{X}_k^{-1} \| \| R(\tilde{X}_k) \|}}, \quad \theta_1 \equiv 1 + \| \tilde{X}_k^{-1} \| \| R(\tilde{X}_k) \| - (0.5\| \tilde{X}_k \| A_1 \tilde{X}_k^{-1} \| A_1 \tilde{X}_k^{-1} \|^2 + 0.75\| \tilde{X}_k \| A_2 \tilde{X}_k^{-1} \| A_2 \tilde{X}_k^{-1} \|^2).
$$

Some results are listed in Table 3.

| $k$  | 8   | 10  | 12  | 14  |
|------|-----|-----|-----|-----|
| $\| \tilde{X}_k - X \|$ | $6.1091 \times 10^{-4}$ | $4.0865 \times 10^{-5}$ | $2.6837 \times 10^{-6}$ | $1.7372 \times 10^{-7}$ |
| $\mu \| R(\tilde{X}_k) \|$ | $7.2094 \times 10^{-4}$ | $4.8224 \times 10^{-5}$ | $3.1670 \times 10^{-6}$ | $2.0506 \times 10^{-7}$ |

The results listed in Table 3 show that the error bound given by Theorem 5.1 is fairly sharp.

Example 7.3. We study the matrix equation

$$
X - A_1^* X A_1 - A_2^* X A_2 = Q,
$$

with

$$
A_1 = \begin{pmatrix} 0 & 0.55 + 10^{-k} \\ 0 & 0 \end{pmatrix}, \quad A_2 = \frac{1}{2} A_1, \quad Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
$$

By Remark 4.2, we can compute the relative condition number $c_{rel}(X)$. Some results are listed in Table 4.

| $k$  | 1   | 3   | 5   | 7   | 9   |
|------|-----|-----|-----|-----|-----|
| $c_{rel}(X)$ | 1.1888 | 1.1025 | 1.1019 | 1.1019 | 1.1019 |

The numerical results listed in the second line show that the unique positive definite solution $X$ is well-conditioned.

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