Combinatorial Problems in Finite Geometry and Lacunary Polynomials

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Abstract

We describe some combinatorial problems in finite projective planes and indicate how Rédei’s theory of lacunary polynomials can be applied to them.

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1. Introduction

In 1991 I wrote a survey paper called Extremal Problems in Finite Geometries [7]. It concerns among others problems of the following type:

Given a set $B$ of points in a finite projective plane $\Pi$, with the property that there is a restricted number of possibilities for the size of the intersection of a line with $B$. What can be concluded about the size and the structure of $B$.

The archetypal result is Segre’s theorem [27]:

If $\Pi = PG(2, q)$, $q$ odd, and $B$ has at most two points on a line, then $|B| \leq q + 1$ (this part is easy), and in case of equality $B$ consists of the points of a conic.

The problem becomes much more difficult when larger intersections are allowed. A subset $B$ of $PG(2, q)$ of size $k$ having at most $n$ points on a line, is called a $(k, n)$-arc.

A simple counting argument going back to Barlotti [5] gives that if $B$ is a $(k, n)$-arc, then $k \leq (n - 1)(q + 1) + 1 = nq - q + n$, and equality implies that $n|q$ and all lines intersect $B$ in 0 or $n$ points.

An arc $B$ meeting the above upper bound is called a maximal arc. The first non trivial case is $n = 2$, and $q$ is even. In this case $|B| = q + 2$ is possible and the maximal arc is called a hyperoval. In fact every $(q + 1)$-arc can be extended to a hyperoval by adding one point and classifying or trying to find new hyperovals is one of the very active areas in finite geometry. For a survey on the current situation we refer to the two papers by Hirschfeld and Storme [20, 21].

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For \( n = 3 \) non-existence of maximal arcs in \( PG(2, 9) \) was shown by Cossu \[16\], and later by Thas for general \( q \) \[29\]. Very little is known however in this case. In fact, if \( k(q) \) stands for the maximal size of a \((k, 3)\)-arc in \( PG(2, q) \) then it is unknown whether \( \limsup k(q)/q > 2 \) or \( \liminf k(q) < 3 \).

For \( n = 4 \), and in fact for all \( n = 2^a \) and \( q = 2^b \), \( b > a \) examples are known, most of them due to Denniston \[17\] and Thas \[30, 31\].

For odd \( n \) it was conjectured in \[29\] that maximal arcs don’t exist. This was finally proved by Ball, Blokhuis and Mazzocca in 1996 \[1\]. A simplified proof appeared a year later \[2\].

Now we turn to the case where \( B \) intersects every line in at least one point. Such a set is called a blocking set. It is called non-trivial if it does not contain a line. Here the classical result is due to Bruen \[14\]:

\begin{align*}
\text{A non-trivial blocking set } B \text{ in a projective plane of order } q & \text{ has size at least } q + \sqrt{q} + 1, \text{ with equality if and only if } q \text{ is a square, and } B \text{ is a Baer subplane.} \\
\text{Our understanding of the situation when } q \text{ is not a square has increased dramatically in the last 10 years, from knowing very little to more or less complete knowledge.}
\end{align*}

Starting point was the unexpectedly simple proof (in 1993) \[8\] that a non-trivial blocking set in \( PG(2, p) \), \( p \) prime, has size at least \( 3(p + 1)/2 \).

The proof is based on properties of a certain kind of lacunary polynomial (as introduced and studied by Rédei in \[26\]) associated to the blocking set. The importance of Rédei’s work in this area was realized soon after the appearance of his book in 1970 notably in papers by Bruen and Thas, \[15\], where his result on the number of directions determined by the graph of a function on a finite field was used.

In contrast to the case that \( B \) is an arc, we can still say something if we require that \( B \) intersects every line of the plane in at least \( t \) points, for some \( t > 1 \). If \( q \) is a square then a natural candidate for \( B \) is the union of \( t \) disjoint Baer subplanes. These can be found for all appropriate values of \( t \) because it is possible to partition \( PG(2, q) \) in \( q - \sqrt{q} + 1 \) disjoint Baer subplanes.

Building on earlier work by Ball \[3\], Gács, Szőnyi \[19\] and others Blokhuis, Storme and Szőnyi showed that for \( t < q^{1/6} \) a \( t \)-fold blocking set in \( PG(2, q) \) has size at least \( t(q + \sqrt{q} + 1) \) with equality if and only if \( B \) is the union of \( t \) disjoint Baer subplanes \[6\].

2. Directions

Let \( f : GF(q) \rightarrow GF(q) \). Define the set of directions determined by \( f \) to be

\[ D_f := \left\{ \frac{f(u) - f(v)}{u - v} \mid u, v \in GF(q), u \neq v \right\}. \]

We are interested in functions \( f \) for which the set \( D_f \) is small. If \( f \) is linear, then \( D_f \) just consists of the slope of the line defined by the graph of \( f \). Our starting point will be the following important result of Rédei \[26\], p. 237, Satz 24.
Theorem 2.1 [Rédei, 1970] Let \( f : GF(q) \to GF(q) \) be a nonlinear function, where \( q = p^n \), \( p \) prime. Then \( |D_f| \) is contained in one of the intervals

\[
\left(1 + \frac{q - 1}{p^e + 1}, \frac{q - 1}{p^e - 1}\right), \quad e = 1, \ldots, \lfloor n/2 \rfloor; \quad \left(\frac{q + 1}{2}, q\right).
\]

Examples of functions determining relatively few directions are given by:

1. \( f(x) = x^{\frac{q+1}{2}}, \) \( q \) odd, \( |D_f| = (q + 3)/2; \)
2. \( f(x) = x^{p^e}, \) where \( e | n, |D_f| = (q - 1)/(p^e - 1); \)
3. \( f(x) = \text{Tr}(GF(q) \to GF(p^e))(x), |D_f| = (q/p^e) + 1. \)

In all the examples \( |D_f| \) is contained in one of the Rédei intervals corresponding to a subfield of \( GF(q) \), i.e., \( e | n, \) and the obvious question was whether this could be proved. In [12] Blokhuis, Brouwer and Szőnyi slightly improved Rédei’s result but the real progress came with the paper [13] where not only it was proved that only the intervals with \( e | n \) occur, but the functions for which \( |D_f| < (q + 3)/2 \) where essentially characterized:

Theorem 2.2 [Ball, Blokhuis, Brouwer, Storme, Szőnyi, 1999] Let \( f : GF(q) \to GF(q) \), where \( q = p^n \), \( p \) prime, \( f(0) = 0 \). Let \( N = |D_f| \). Let \( e \) (with \( 0 \leq e \leq n \)) be the largest integer such that each line with slope in \( D_f \) meets the graph of \( f \) in a multiple of \( p^e \) points. Then we have one of the following:

1. \( e = 0 \) and \((q + 3)/2 \leq N \leq q + 1,\)
2. \( e = 1, \) \( p = 2 \) and \((q + 5)/3 \leq N \leq q - 1),\)
3. \( p^e > 2, \) \( e | n, \) and \( q/p^e + 1 \leq N \leq (q - 1)/(p^e - 1),\)
4. \( e = n \) and \( N = 1.\)

Moreover, if \( p^e > 3 \) or \( (p^e = 3 \text{ and } q = 3/3 + 1), \) then \( f \) is a linear map on \( GF(q) \) viewed as a vector space over \( GF(p^e). \)

Very recently this result has been perfected by Simeon Ball, removing the condition \( p^e > 2 \) in the third case (and thus getting rid of the second).

When we consider the set \( B \) formed by the \( q \) points of the graph of \( f \) together with the \( N = |D_f| \) points on the line at infinity corresponding to the directions determined by \( f, \) we get a blocking set. For if \( l \) has a slope determined by \( f \) then the infinite point of \( l \) belongs to \( B, \) and if not, then \( l \) and its parallels all contain precisely one point of the graph of \( f. \)

Conversely, if \( B \) is a blocking set in \( PG(2, q) \) of size \( q + N, \) and there is a line intersecting \( B \) in \( N \) points, then it arises from this construction. The blocking set is then called of Rédei type.

As mentioned in the introduction, the smallest non-trivial blocking sets were characterized by Bruen [14] to be Baer subplanes. They are of Rédei type and correspond to the function \( x \mapsto x\sqrt{q}. \)

If the blocking set \( B \) is of Rédei type, then as a consequence of the direction theorem above the structure is very special if \( N < (q + 3)/2, \) or equivalently if \( |B| \leq \frac{3}{2}(q + 1). \) An important step towards showing that this is true in general is the following result for planes of prime order already mentioned in the introduction [8].
Theorem 2.3 [Blokhuis, 1994] Let \( B \) be a blocking set in \( PG(2, p) \), \( p \) prime, not containing a line. Then
\[ |B| \geq \frac{3}{2}(p + 1). \]

The proof is based on properties of lacunary polynomials, introduced and studied by Rédei in [26]. In the same paper it is proved that a blocking set in \( PG(2, p^n) \) has size at least \( p^3 + p^2 + 1 \). Recently Polverino [24] has shown that small blocking sets in \( PG(2, p^3) \) are all of Rédei type, and the possible sizes are \( p^3 + p^2 + 1 \) (corresponding to examples 2 and 3 above). When mentioning possible sizes of blocking sets we will always tacitly assume that they are minimal, so deleting a point destroys the blocking property.

A very interesting and probably feasible problem is to characterize the sets that give equality in the bound for \( PG(2, p) \). For all (odd) \( p \) the graph of the function
\[ f : x \mapsto x^{(p+1)/2}, \]
(the first example) together with its \((p + 3)/2\) directions is an example, and it is the essentially unique one of Rédei type (this was proved already in 1981 by Lovász and Schrijver [23] who also gave an elementary proof of Rédei’s result for the case that \( q = p \) is prime). Only two examples (of size \((p + 1)/2\)) are known that are not of Rédei type, one (with 12 points) in the plane of order 7, it looks like a dual affine plane of order 3. The other (with 21 points) in the plane of order 13 was only found last year by Blokhuis, Brouwer and Wilbrink. Both are unique [11]. In the same paper it is shown that no other examples exist in planes of (prime) order less than 37, and it is extremely unlikely that this is different later on.

Motivated by these results we call blocking sets of size \(< \frac{3}{2}(q + 1)\) small, so small blocking sets only exist in planes of non-prime order. The structure of small blocking sets is restricted by the following theorem of Szőnyi [28]:

Theorem 2.4 [Szőnyi, 1997] Let \( B \) be a (minimal) small blocking set in \( PG(2, q) \), where \( q \) is a power of the prime \( p \). Then \(|B \cap l| = 1 \mod p\) for every line \( l \).

For a long time I was convinced, and even conjectured that small blocking sets were necessarily of Rédei type, but this turned out to be false. Nice examples of small non-Rédei type blocking sets were found by Polito and Polverino [24].

The basic idea is very simple. Consider \( PG(2, q^s) \). By definition its points and lines are the 1- and 2-dimensional subspaces of \( V = V(3, q^s) \), a 3 dimensional vector space over \( GF(q^s) \). When we consider \( V' \) which is just \( V \) as 3s-dimensional over \( GF(q) \) then points and lines correspond to certain \( s \)-, and 2s-dimensional subspaces of \( V' \). Now let \( W \) be any \( s + 1 \)-dimensional subspace of \( V' \). Let \( B(W) \) be the collection of points in \( PG(2, q^s) \) for which the corresponding \( s \)-space in \( V' \) intersects \( W \) non-trivially. One readily checks that \( B(W) \) is a blocking set (of size at most \( (q^{s+1} - 1)/(q - 1) \)), because in the 3s dimensional vector space \( V' \) an \( (s + 1) \)-space and a \( (2s) \)-space must intersect in at least a 1-space. Polito and Polverino give examples that are not of Rédei type in all planes \( PG(2, p^n) \), \( n > 3 \). The examples of small blocking sets of Rédei type also fall under this more general construction, by the direction theorem.
Next we consider multiple blocking sets. $B$ is called a $t$-fold blocking set if every line intersects $B$ in at least $t$ points. If $q$ is a square, then $PG(2,q)$ can be partitioned into Baer subplanes, and taking $t$ of them produces a set with the property that every line intersects it in either $t$ or $t + \sqrt{q}$ points (this makes it a two-intersection set). Again using the theory of lacunary polynomials it can be shown that for small $t$ these are the minimal examples.

Our knowledge on the structure of (relatively) small multiple blocking sets is summarized in the following

**Theorem 2.5** [Blokhuis, Storme, Szönyi, 1998] Let $B$ be a $t$-fold blocking set in $PG(2,q)$ of size $t(q+1) + c$. Let $c_2 = c_3 = 2^{-1/3}$ and $c_p = 1$ for $p > 3$.

1. If $q = p^{2d+1}$ and $t < q/2 - cpq^{2/3}/2$ then $c \geq cpq^{2/3}$.
2. If $4 < q$ is a square, $t < q^{1/4}/2$ and $c < cpq^{2/3}$, then $c \geq t\sqrt{q}$ and $B$ contains the union of $t$ disjoint Baer subplanes.

3. If $q = p^2$ and $s < q^{1/4}/2$ and $c < p^{1/2 + \sqrt{(p+1)/2}}$, then $c \geq t\sqrt{q}$ and $B$ contains the union of $t$ disjoint Baer subplanes.

What it essentially says is that for $t < q^{1/6}$ a $t$-fold blocking set has at least the size of $t$ disjoint Baer subplanes, and equality implies that it is just that. In the special case that $q$ is the square of a prime, then the same is true for $t < q^{1/4}/2$.

This result appears to be rather sharp in the following sense: In [4] Ball, Blokhuis and Lavrauw construct a two-intersection set with the same parameters as, but different from the union of $q^{1/4} + 1$ disjoint Baer subplanes, so the above characterization does no longer apply if $t > q^{1/4}$.

The construction is based on the Polito-Polverino idea. So the plane is $PG(2,q^s)$, and $V = V(3,q^s)$ and $V' = V(3s,q)$. Now we take for $W$ an $s + 2$-dimensional subspace of $V'$. If this has the additional property that intersections with the $s$-dimensional subspaces of $V'$ corresponding to projective points are at most 1-dimensional, then the corresponding set $B(W)$ is a $(q + 1)$-fold blocking set. To see this note that if $L$ is a $2s$-dimensional subspace of $V'$ corresponding to a line, then $W \cap L$ is at least 2-dimensional, but by assumption $W$ intersects $s$-spaces corresponding to points in at most 1 dimension, so it has to intersect at least $q + 1$ of them.

The question of whether it is possible to find such subspaces lead to the notion of scattered subspaces with respect to spreads. An $s$-spread in a vector space $V$ is a collection of $s$-dimensional subspaces partitioning the nonzero vectors of $V$. In order for $V$ to admit an $s$-spread it is necessary and sufficient that its dimension is a multiple of $s$. So in the above example the $s$-spaces in $V'$ corresponding to points of $PG(2,q^s)$ define an $s$-spread. Given a vector space $V$ together with an $s$-spread $S$ we say that the subspace $W$ is scattered by $S$ if $W$ intersect each spread element in at most 1-dimensional subspace. A natural question is what the maximal dimension is of a scattered subspace. Results on this question and related problems can be found in the thesis of Lavrauw [22].

A detailed survey of the many recent results on blocking sets and multiple blocking sets is contained in the paper by Hirschfeld and Storme [21]. Blocking sets of projective planes can also be considered as a special case of the more general
concept of covers in hypergraphs, extensively treated in the excellent (but not too recent) survey by Füredi [13].

To conclude this section let me mention two attractive problems (on which no progress has been made in the last 10 years).

The first concerns double blocking sets in $PG(2, p)$. A lower bound due to Blokhuis and Ball gives $|B| \geq 5(p + 1)/2$. A trivial example is formed by the union of three lines, of size $3p$. Could it be that this is the minimal size? It is true for $p = 2, 3, 5, 7$, but it might already be false for $p = 11$.

The second question has repeatedly been asked to me by Paul Erdös. Is there a universal constant $c$ (10 say), such that in any plane (or any $PG(2, p)$ say) there is a blocking set with at most $c$ points on every line. In all of the known examples there are some lines with many points of the blocking set. Results by Ughi show that it does not work to use for $B$ the union of a small set of algebraic curves of bounded degree [32], using for instance a union of conics one obtains blocking sets with $c \log(q)$ points on a line. On the other hand, in $PG(2, p^n)$ there is a blocking set with at most $p + 1$ point on every line.

3. Lacunary polynomials

We now turn to the main tool in the recent investigations on (multiple) blocking sets.

Let $f(X) \in GF(q)[X]$ be fully reducible, in other words, $f(X)$ factors into linear factors over $GF(q)$. In [26] Rédei investigates the case $f(X) = X^q + g(X)$ with $\deg(g) < q - 1$, and calls the polynomial lacunary. The problem is to characterize those $f$ where the degree of $g$ is small. As an easy example we prove:

**Theorem 3.1** [Rédei, 1970] Let $f(X) = X^q + g(X)$ be fully reducible in $GF(q)[X]$, where $q = p^n$ is prime. Then either $f(X) \in GF(q)[X^p]$, or $g(X) = -X$ or $\deg(g) \geq (q + 1)/2$.

**Proof:** Write $f = s.r$, where $s$ has the same zeroes as $f$, but with multiplicity one, and $r$ consists of the remaining factors. Then $s | X^q - X$, as well as $f$, so $s | X + g$, and $r | f' = g'$. Hence $f = s.r | (X + g)g'$ so either $(X + g)g' = 0$ or $\deg(g) + \deg(g') \leq \deg(f) = q$.

If $q = p$ is prime, then $g' = 0$ together with $\deg(g) < p$ imply that $g$ is constant.

Much of Rédei’s book is devoted to the classification of those $f$ with $\deg(g) = (q + 1)/2$. For us the case $g' = 0$ (and hence $f \in GF(q)[X^p]$) is more interesting however, also for our applications we need to consider polynomials of the form $f(X) = X^g q(X) + h(X)$, where both $g$ and $h$ have degree less than $q$.

The following theorem summarizes what we know in this case [10]:

**Theorem 3.2** [Blokhuis,Storme, Szönyi, 1998] Let $f \in GF(q)[X]$, $q = p^n$, $p$ prime, be fully reducible, $f(X) = X^g q(X) + h(X)$, where $(g,h) = 1$. Let $k = \max(\deg(g), \deg(h)) < q$. Let $e$ be maximal such that $f$ is a $p^e$-th power (so $f \in GF(q)[X^{p^e}]$). Then we have one of the following possibilities:

1. $e = n$ and $k = 0$;
Let $y$ is a bijection, and hence $r$ only if $(q - 1)/2 = 0$, this would have very useful applications.

5. $e = n/2$ and $k \geq p^e \left[\frac{1}{2} + \sqrt{(p^e + 1)/2}\right]$;
6. $n/2 > e > n/3$ and $k \geq p^{(n+\varepsilon)/2} - p^{n-\varepsilon} - p^e/2$, or if $3e = n + 1$ and $p \leq 3$,
then $k \geq p^{(e^2+1)/2}$;
7. $n/3 \geq e > 0$ and $k \geq p^e[(p^{n-e} + 1)/(p^e + 1)]$;
8. $e = 0$ and $k \geq (q + 1)/2$;
9. $e = 0$, $k = 1$ and $f(X) = a(X^q - X)$.

It would be very pleasant to have stronger information in the case $n/3 < e < n/2$, this would have very useful applications.

4. The connection

In this section we will illustrate the connection between the direction problem, (multiple) blocking sets, and lacunary polynomials.

Let $f : GF(q) \to GF(q)$ be any map, and let $D_f$ be its set of directions. Consider the auxiliary (Rédie) polynomial

$$R(X,Y) = \prod_{w \in GF(q)} (X - wY + f(w)),$$

introduced by Rédie. Let $y$ and $v \neq w \in GF(q)$. Then $vy - f(v) = wy - f(w)$ if and only if $(f(v) - f(w))/(v - w) = y$. It follows that for $y \notin D_f$ the map $w \mapsto wy - f(w)$ is a bijection, and hence $R(X,y) = \prod_{z \in GF(q)} (X - z) = X^q - X$. Write

$$R(X,Y) = X^q + r_1(Y)X^{q-1} + \cdots + r_{q-1}(Y)X + r_q(Y),$$

where $r_i$ is a polynomial of degree $< i$ in $Y$ (with the exception of $r_{q-1}$ of degree $q - 1$: it is clear that $r_i$ has degree at most $i$, but the coefficient of $Y^i$ is the $i$-th elementary symmetric polynomial in the elements of $GF(q)$, so this is 0 for $0 < i \neq q - 1$). If $y \notin D_f$ and $i \neq q - 1$ then $r_i(y) = 0$. So $r_i$ is identically zero for $i \leq |D_f|$. As a consequence we have for $y \in D_f$ that $R(X,y) = X^q + g(X)$ for some $g$ depending on $y$ with $\deg(g) \leq q - |D_f| - 1$. So the Rédie polynomial when specialized for $Y = y \in D_f$ is a lacunary polynomial and information on $\deg(g)$ gives results for $|D_f|$.

The extension of the Rédie polynomial from the graph of a function to point sets in general can be illustrated best in the case of an ordinary blocking set $B$ of $PG(2,q)$. We may coordinatize our plane in such a way that the line at infinity becomes a tangent, containing the point $(1 : 0 : 0)$ of the blocking set.

Let $|B| = q + 1 + d$ then the remaining $q + d$ points have certain affine coordinates $(a_i, b_i)$ and the Rédie polynomial associated to $B$ (in this position) can be defined
as:

\[ R[X, Y] := \prod_{i}(X - a_iY + b_i) = X^{q+d} + r_1(Y)X^{q+d-1} + \cdots + r_{q+d}(Y). \]

For \( y, c \in GF(q) \) consider the line \( \{(U, V) : V = yU + c\} \). It contains an affine point \((a, b)\) of the blocking set: \( c = b - ay \). Hence \( R[X, y] \) is divisible by \((X - c)\) for all \( c \in GF(q) \), in other words \( X^q - X \) divides \( R[X, y] \). It follows as before that \( r_i \) is identically zero for \( i = d + 1, \ldots, q - 2 \). As a consequence we obtain the lacunary polynomial

\[ f(X) = \prod(X - a_i) = X^q g(X) + h(X), \]

with \( g \) of degree \( d \) and \( h \) of degree at most \( d + 1 \), and information on \( f \) translates to information on \( B \). For the case that \( q = p \) is prime it not only gives that \(|B| \geq 3(p+1)/2 \), but in case of equality it also gives that each point of the blocking set is on exactly \((p - 1)/2 \) tangents, and by classifying the possible polynomials it gives all possibilities for the configuration of the tangents through a particular point. For small \( p \) (at most 37) the number of possibilities is sufficiently small to be handled by a computer, and to prove uniqueness of the minimal example (for \( p \neq 7, 13 \)).

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