Generation of Integrable Quantum Nonultralocal Models through Braided Yang-Baxter Equation
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Abstract

Formulating quantum integrability for nonultralocal models (NM) parallel to the familiar approach of inverse scattering method is a long standing problem. After reviewing our result regarding algebraic structures of ultralocal models, we look for the algebra underlying NM. We propose an universal equation represented by braided Yang-Baxter equation and able to derive all basic equations of the known models like WZWN model, nonabelian Toda chain, quantum mapping etc. As further useful application we discover new integrable quantum NM, e.g. mKdV model, anyonic model, Kundu-Eckhaus equation and derive SUSY models and reflection equation from the nonultralocal view point.

1. Introduction

Quantum integrable systems (QIS) can be divided into two broad classes, namely ultralocal and nonultralocal depending on an important property of their representative Lax operators. Ultralocal QIS are the standard and the most studied ones, which include well known models like Nonlinear Schrödinger equation (NLS), sine-Gordon model, Toda chain, etc. They exhibit a common ultralocality property that their Lax operators at different lattice points \( i \neq j \) commute: \([L_{1i}, L_{2j}] = 0\).

This fact is actively used in constructing their integrability theory expressed by the universal equation

\[
R(\lambda, \mu) \ L_{1j}(\lambda) \ L_{2j}(\mu) = L_{2j}(\mu) \ L_{1j}(\lambda) \ R(\lambda, \mu),
\]

known as the quantum Yang Baxter equation (QYBE). Specific choices of \( L \) and \( R \) yield from (1) the basic equations for concrete integrable models [1]. The underlying algebras for such models also exhibit common structure related to quantum algebras, the knowledge of which not only deepens our understanding of the system and adds beauty to the subject, but also helps to generate integrable models in a systematic way [2].

However, it should be mentioned that parallel development was not persuaded for the nonultralocal models (NM), characterised by the property \([L_{1i}, L_{2j}] \neq 0\), though many famous models, e.g. quantum KdV model, Supersymmetric models, nonlinear \( \sigma \) models, WZWM etc. belong to this class. A proposal in this line was made in
though only for models convertible to ultralocal ones. A general formulation of the integrability theory given through a representative universal Yang-Baxter like equation is seriously lacking for such models including the clear idea about the nature of the underlying algebraic structures. Our aim therefore is to look into such systems from a rather general point of view, where the SUSY models, anyonic models as well as models satisfying reflection type equation can also be considered as the representatives of the nonultralocal class and exploring the related algebra to construct an extended QYBE and generate concrete NM as particular cases, parallel to the ultralocal models.

2. Ultralocal models: underlying algebra

We look first into the established ultralocal systems for understanding the role played by the underlying algebra.

Consider standard matrices $A$ and $B$ satisfying the obvious property

$$A \otimes B = (A_1 B_2) = (B_2 A_1)$$

or $[A_1, B_2] = 0$ where $A_1 \equiv A \otimes 1$ and $B_2 = 1 \otimes B$.

Let us choose now $A = L_i(\lambda), B = L_i(\mu)$ as Lax operators at the same lattice points and check the above property. It is immediate, that it no longer holds due to the operator nature of the matrix elements of the quantum Lax operators and the equation in effect turns into the QYBE, where a matrix $R$ appears to compensate for the noncommutativity of matrix elements of $L$. This is the basic reason for the appearance of nontrivial algebras underlying such integrable systems.

On the other hand the choice $A = L_{i+1}(\mu), B = L_i(\lambda)$ does satisfy the above commutation relation due the ultralocality property.

Let us now look into the standard matrix multiplication rule

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$

and check again for the Lax operators as $A = L_{i+1}(\lambda), B = L_{i+1}(\mu), C = L_i(\lambda), D = L_i(\mu)$. Note that since the multiplication rule holds due to the commutativity of $B_2$ and $C_1$, that also remains valid for the ultralocal Lax operators yielding

$$(L_{i+1}(\lambda)L_{2i+1}(\mu))(L_1(\lambda)L_{2i}(\mu)) = (L_{i+1}(\lambda)L_{1i}(\lambda))(L_{2i+1}(\mu)L_{2i}(\mu))$$

Now coupling two properties of the local Lax operators one can derive a global QYBE, essential for proving the integrability which is a global property. Indeed, starting from at $i + 1$ point and multiplying with the same relation at $i$ and subsequently using one can globalise the QYBE and repeating the step for $N$ times obtain finally the global equation

$$R(\lambda, \mu) T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) R(\lambda, \mu),$$
for the monodromy matrix \( T = \prod_i^N L_i \). Trace \( tr_{12} \) factorises the equation yielding 
\[ [tr T(\lambda), tr T(\mu)] = 0 \] or the integrability condition \([C_n, C_m] = 0\), where \( C_n, n = 1, 2, \ldots, N \) are the conserved quantities generated by \( \tau(\lambda) = Tr T(\lambda) \) as expansion coefficients.

To extract the algebra independent of the spectral parameters one can take the ancestor Lax operator in the form
\[
L_{\text{anc}}^q(t)(\xi) = L_{ij}, \quad L_{11} = \xi \tau_1^- + \frac{1}{\xi} \tau_1^+, \quad L_{22} = \xi \tau_2^- + \frac{1}{\xi} \tau_2^+, \quad L_{12} = \tau_{21}, \quad L_{12} = \tau_{12}, \quad (5)
\]
along with the twisted trigonometric \( R(\lambda, \mu) \)-matrix, which yields the underlying quadratic algebra
\[
t \tau_{12} \tau_{21} - t^{-1} \tau_{21} \tau_{12} = -(q - q^{-1}) \left( \tau_1^+ \tau_2^- - \tau_1^- \tau_2^+ \right),
\]
\[
\tau_i^\pm \tau_{ij} = q^{\pm 1} t \tau_{ij} \tau_i^\pm, \quad \tau_i^\pm \tau_{ji} = q^{\mp 1} t^{-1} \tau_{ji} \tau_i^\pm, \quad (6)
\]
for \( i, j = (1, 2) \). This Hopf algebra exhibits a coproduct structure and the multiplication rule (2), which are linked with the transition from local to global QYBE. Taking different limits of \( q, t \) and proper realisations of the algebra (6), from (5) one can generate different classes of ultralocal models along with their exact Lax operators in a rather systematic way [2].

3. **Nonultralocal quantum systems: algebraic structure, extended QYBE and generation of models**

We see immediately that both the above matrix properties fail for NM, since now the Lax operators do not commute at the same as well as at different points. Therefore the quantised algebra like (3) with somewhat trivial multiplication property (2) needs generalisation in the form
\[
(A \otimes B)(C \otimes D) = \psi_{BC}(A(C \otimes B)D) \quad (7)
\]
where the noncommutativity of \( B_2, C_1 \) could be taken into account. However the coproduct structure, essential for the globalisation of QYBE must be preserved. Such idea realised in the braided algebra [3] was implemented for formulating the integrability of NM [1]. Here we look into this problem from a bit different angle and report on new results. Limiting ourselves to only two types of braidings (describing nearest neighbours by the matrix \( Z \) and other neighbours by a single braiding \( \tilde{Z} \)) we present the braided generalisation of the QYBE as
\[
R_{12}(u-v)Z_{21}^{-1}(u, v)L_{1j}(u)\tilde{Z}_{21}(u, v)L_{2j}(v) = Z_{12}^{-1}(v, u)L_{2j}(v)\tilde{Z}_{12}(v, u)L_{1j}(u)R_{12}(u-v). \quad (8)
\]
In addition this must be complemented by the braiding relations
\[ L_{2j+1}(v)Z_{21}^{-1}(u,v)L_{1j}(u) = \tilde{Z}_{21}^{-1}(u,v)L_{1j}(u)\tilde{Z}_{21}(v,u)L_{2j+1}(v) \tilde{Z}_{21}^{-1}(u,v) \] (9)
at nearest neighbour points and
\[ L_{2k}(v)\tilde{Z}_{21}^{-1}(u,v)L_{1j}(u) = \tilde{Z}_{21}^{-1}(u,v)L_{1j}(u)\tilde{Z}_{21}(v,u)L_{2k}(v)\tilde{Z}_{21}^{-1}(u,v) \] (10)
with \( k > j + 1 \) answering for the other neighbours. Note that along with the usual quantum \( R_{12}(u - v) \)-matrix additional \( \tilde{Z}_{12} \), \( Z_{12} \) matrices appear here, which can be (in-)dependent of the spectral parameters and satisfy a system of Yang-Baxter type relations [4].

The set of relations (8-10) represent the universal equations for the integrable NM within a certain class of braidings and particular choices for \( R, L, Z, \tilde{Z} \) derive concrete models of physical interest. It is readily seen that the trivial choice \( Z = \tilde{Z} = 1 \) reduces the above set into the standard QYBE (1) together with the ultralocality condition, while the nontrivial \( Z \)'s would lead to different types of NM. For example, the homogenous braiding \( Z = \tilde{Z} \) would correspond to SUSY, anyonic models etc., while the choice \( Z = 1 \) with \( \tilde{Z} \neq 1 \) should describe the models like WZWN, quantum mKdV, nonlinear \( \sigma \) models etc. with \( \delta' \) function appearing in their fundamental commutation relations. The case \( Z = 1 \) and nontrivial \( \tilde{Z} \) also appears to be consistent with the reflection equation [6].

For establishing integrability we have to construct first the QYBE for the global monodromy matrix, which however can be carried out now almost like the ultralocal case due to the changed multiplication rule given by the braiding relations (8) and (10). For periodic models, where \( L_1 \) and \( L_N \) become nearest neighbours some special care should be taken, which yields finally the global QYBE as

\[ R_{12}(u - v)Z_{21}^{-1}(u,v)T_1(u)Z_{12}^{-1}(v,u)T_2(v) = Z_{12}^{-1}(v,u)T_2(v)Z_{21}^{-1}(u,v)T_1(u)R_{12}(u - v). \] (11)

Due to appearance of \( Z \) matrices one faces initial difficulty in trace factorisation, which nevertheless can be bypassed in most cases by introducing a \( K(u) \) matrix and defining \( t(u) = tr(K(u)T(u)) \) [3, 4].

To demonstrate the applicability of the present scheme we mention briefly the particular explicit choices for the \( R, Z, \tilde{Z} \) matrices, which derives from the general relations (8-10) the basic equations for the known models found earlier and at the same time yield new results.

1. **Nonabelian Toda chain** [7]
   
   \( \tilde{Z} = 1 \) and \( Z_{12} = 1 + ih(e_{22} \otimes e_{12}) \otimes \pi \) and rational \( R(u) \) matrix.

2. **Current algebra in WZNW model** [10]
\[ \tilde{Z} = 1 \] and \( Z_{12} = R_{q12}^{-1} \).

3. **Coulomb gas picture of CFT** \[ \tilde{Z} = 1 \] and \( Z_{12} = q^{-\sum_i H_i \otimes H_i} \).

4. **Nonultralocal quantum mapping** \[ \tilde{Z} = 1 \] and \( Z_{12}(u_2) = 1 + \frac{h}{u_2} \sum_{N} e_{N_{\alpha}} \otimes e_{\alpha N} \) and rational \( R(u) \).

5. **Integrable model on moduli space** \[ \tilde{Z} = Z_{12} = R_{N} + q^{-1} \sum_{i} \sigma_i \otimes \sigma_i \] and trigonometric \( R(u) \) matrix.

4. **New results**

1. **Supersymmetric models** \[ \tilde{Z} = 1 \] and \( Z_{12} = \sum_{\eta} \eta_{\alpha\beta} g_{\alpha\beta} \), where \( \eta_{\alpha\beta} = e_{\alpha\alpha} \otimes e_{\beta\beta} \) and \( g = (-1)^{\tilde{\alpha}\tilde{\beta}} \) with supersymmetric grading \( \tilde{\alpha} \). \( R \)-matrix is either rational or trigonometric of \( SU(1, 1) \) type.

2. **Reflection equation** The reflection equation of \[ \tilde{Z} = 1 \] can also be derived from the nonultralocal point of view by considering a monodromy matrix \( T = \prod_{j} \tilde{L}_j \) such that \( \tilde{L}_j(u) = L_{j-N}(u) \), for \( j > N \), \( = K^{-1}(u) \), for \( j = N \), and \( = L_{N-j}^{-1} \), for \( j < N \).

Though this refers to a bit different braiding than that we considered here, the choice \( Z = 1, \tilde{Z} = R(u_1 + u_2) \) and without periodicity recovers most of the results.

3. **Anyonic SUSY model** Generalizing the SUSY model we choose \( Z = \tilde{Z} = \sum_{\eta} \eta_{\alpha\beta} \tilde{g}_{\alpha\beta} \), where an anyonic phase is included in \( \tilde{g}_{\alpha\beta} = e^{i\theta_{\alpha\beta}} \). Rational \( R \) matrix derives integrable anyonic models while the trigonometric one can describe the \( q \)-deformed anyons.

4. **Kundu-Eckhaus equation** \[ \tilde{Z} = 1 \] as a quantum model involves fields with a nice anyonic type representation:

\[
\psi_n^0 \psi_m^0 = e^{i\theta} \psi_m^0 \psi_n^0, \quad n > m; \quad [\psi_m^0, \psi_n^0] = 1.
\]  

Our scheme gives the solution \( \tilde{Z} = 1, Z = diag(e^{i\theta}, 1, 1, e^{i\theta}) \) and the rational \( R \) matrix. Though we are able to construct the braided QYBE, the trace factorisation problem could not be solved due to some peculiarity of the monodromy matrix.

5. **Quantum mKdV model** An exact quantum solution of this interesting model can be found following the present scheme with \( \tilde{Z} = 1, Z_{12} = Z_{21} = q^{-\frac{1}{4}\sigma_3 \otimes \sigma_3} \), and the trigonometric
$R(u)$ matrix. For details see [5], while the details of the other results will be given elsewhere.

The author expresses his sincere thanks to the Alexander von Humboldt Foundation for the resumption of fellowship (in the Phys. Inst., Bonn University) and the financial support.

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