DISSIPATIVE OPERATORS AND OPERATOR LIPSCHITZ FUNCTIONS

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Abstract. The purpose of this paper is to obtain an integral representation for the difference \( f(L_1) - f(L_2) \) of functions of maximal dissipative operators. This representation in terms of double operator integrals will allow us to establish Lipschitz-type estimates for functions of maximal dissipative operators. We also consider a similar problem for quasicommutators, i.e., operators of the form \( f(L_1)R - Rf(L_2) \).

1. Introduction

It was discovered in Farforovskaya’s [F] paper that a Lipschitz function \( f \) on the real line \( \mathbb{R} \) (i.e., \( f \) satisfies the inequality \(|f(s) - f(t)| \leq \text{const} |s - t|, s, t \in \mathbb{R}\)) does not have to satisfy the inequality
\[
||f(A) - f(B)|| \leq \text{const} \|A - B\|
\]
for arbitrary bounded self-adjoint operators \( A \) and \( B \). Functions satisfying (1.1) are called operator Lipschitz functions on \( \mathbb{R} \). We refer the reader to [AP2] for a recent detailed survey of operator Lipschitz functions.

Operator Lipschitz functions can be defined on arbitrary closed subset \( \mathfrak{F} \) of the complex plane. A complex-valued function \( f \) on \( \mathfrak{F} \) is called operator Lipschitz if
\[
||f(N_1) - f(N_2)|| \leq \text{const} \|N_1 - N_2\|
\]
for arbitrary bounded normal operators \( N_1 \) and \( N_2 \) whose spectra are contained in \( \mathfrak{F} \). It turns out that if \( f \) is an operator Lipschitz function on a closed unbounded set \( \mathfrak{F} \), then inequality (1.2) also holds for not necessarily bounded normal operators \( N_1 \) and \( N_2 \) with spectra in \( \mathfrak{F} \); see Theorem 3.2.1 of [AP2].

In this paper we consider the class of operator Lipschitz functions on the closed upper half-plane \( \text{clos} \mathbb{C}_+ \overset{\text{def}}{=} \{ \zeta \in \mathbb{C} : \text{Im} \zeta \geq 0 \} \), and among operator Lipschitz functions on \( \text{clos} \mathbb{C}_+ \) we consider those that are analytic in the open half-plane \( \mathbb{C}_+ \). We denote the class of such functions by \( \text{OL}_A(\mathbb{C}_+) \).

The main purpose of this paper is to show that the class \( \text{OL}_A(\mathbb{C}_+) \) can be characterized as the maximal class of functions \( f \), for which the Lipschitz-type
estimate
\[(1.3)\quad \|f(L_1) - f(L_2)\| \leq \text{const} \|L_1 - L_2\|\]
holds for arbitrary maximal dissipative operators \(L_1\) and \(L_2\) with bounded difference.

We would like to mention that a similar result for functions of contractions was obtained in [KS2].

Note that the proofs of such results for dissipative operators are considerably more complicated than in the case of self-adjoint operators, unitary operators, or contractions. First of all, we deal with unbounded functions of unbounded operators. Secondly, unlike in the case of self-adjoint operators, we cannot use spectral projections onto subspaces on which the operators are bounded.

If \(f\) is an operator Lipschitz function on the real line and \(A\) and \(B\) are self-adjoint operators with bounded \(A - B\), then the following formula holds:
\[(1.4)\quad f(A) - f(B) = \iint_{\mathbb{R} \times \mathbb{R}} (\mathcal{D}f)(x,y) \, dE_A(x)(A - B) \, dE_B(y),\]
where \(E_A\) and \(E_B\) are the spectral measures of \(A\) and \(B\) and the divided difference \(\mathcal{D}f\) is defined on \(\mathbb{R} \times \mathbb{R}\) by
\[(1.5)\quad (\mathcal{D}f)(x,y) \overset{\text{def}}{=} \begin{cases} f(x) - f(y) / (x - y), & x \neq y, \\ f'(x), & x = y \end{cases} \]
(see [BS1] and the survey [AP2]). Recall that it follows from Theorem 4.1 of [JW] that the operator Lipschitz functions are differentiable everywhere on \(\mathbb{R}\) (see also the survey [AP2], §3.3). The expression on the right-hand side of (1.4) is a double operator integral; see [3].

In [4] we obtain an analog of this result for maximal dissipative operators. This will allow us to prove a Lipschitz-type estimate analogous to (1.3) for maximal dissipative operators \(L_1\) and \(L_2\) and functions \(f\) of class \(\text{OL}_A(\mathbb{C}_+^{\ast})\).

We obtain in [5] similar results for quasicommutators \(f(L_1)R - Rf(L_2)\) for maximal dissipative operators \(L_1\) and \(L_2\) and bounded operators \(R\).

Note that the above results for functions \(f\) in the Besov class \((B^1_{\infty,1})_+((\mathbb{R}))\) of functions analytic in \(\mathbb{C}_+\) were obtained earlier in [AP1]. However, the class \(\text{OL}_A(\mathbb{C}_+^{\ast})\) is bigger than \((B^1_{\infty,1})_+((\mathbb{R}))\). Indeed, the class \((B^1_{\infty,1})_+((\mathbb{R}))\) consists of continuously differentiable functions while the class \(\text{OL}_A(\mathbb{C}_+^{\ast})\) contain functions with discontinuous derivatives. Namely, it follows from Theorem 3.10.1 of [AP2] applied to the upper half-plane that the function \(z^2(e^{-i/z} - 1)\) is operator Lipschitz with derivative discontinuous at 0. Note that in the case of operator Lipschitz functions analytic in \(\mathbb{D}\) the existence of such functions was proved earlier in [KS1].

We give a brief introduction to dissipative operators in [2] and to double operator integrals in [3].

2. Dissipative operators

In this section we give a brief introduction in dissipative operators. We refer the reader to [SNF], [So], and [AP1] for more detailed information.
Let $\mathcal{H}$ be a Hilbert space. Recall that an operator $L$ (not necessarily bounded) with dense domain $\mathcal{D}_L$ in $\mathcal{H}$ is called dissipative if
\[ \text{Im}(Lu, u) \geq 0, \quad u \in \mathcal{D}_L. \]
A dissipative operator is called maximal dissipative if it has no proper dissipative extension.

The Cayley transform of a dissipative operator $L$ is defined by
\[ T \overset{\text{def}}{=} (L - iI)(L + iI)^{-1} \]
with domain $\mathcal{D}_T = (L + iI)\mathcal{D}_L$ and range $\text{Range} T = (L - iI)\mathcal{D}_L$ (the operator $T$ is not densely defined in general). It can easily be shown that $T$ is a contraction, i.e., $\|Tu\| \leq \|u\|$, $u \in \mathcal{D}_T$, 1 is not an eigenvalue of $T$, and $\text{Range}(I - T) \overset{\text{def}}{=} \{u - Tu : u \in \mathcal{D}_T\}$ is dense.

Conversely, if $T$ is a contraction defined on its domain $\mathcal{D}_T$, 1 is not an eigenvalue of $T$, and $\text{Range}(I - T)$ is dense, then it is the Cayley transform of a dissipative operator $L$ and $L$ is the inverse Cayley transform of $T$:
\[ L = i(I + T)(I - T)^{-1}, \quad \mathcal{D}_L = \text{Range}(I - T). \]
A dissipative operator is maximal if and only if the domain of its Cayley transform is the whole Hilbert space.

Every dissipative operator has a maximal dissipative extension. Every maximal dissipative operator is necessarily closed.

If $L$ is a maximal dissipative operator, then its spectrum $\sigma(L)$ is contained in the closed upper half-plane $\text{clos} \mathbb{C}_+$. If $L_1$ and $L_2$ are maximal dissipative operators, we say that the operator $L_1 - L_2$ is bounded if there exists a bounded operator $K$ such that $L_2 = L_1 + K$.

We now proceed to the construction of functional calculus for dissipative operators. Let $L$ be a maximal dissipative operator, and let $T$ be its Cayley transform. Consider its minimal unitary dilation $\bar{U}$, i.e., $\bar{U}$ is a unitary operator defined on a Hilbert space $K$ that contains $\mathcal{H}$ such that $T^n = P_{\mathcal{H}}\bar{U}^n|_{\mathcal{H}}, \quad n \geq 0,$
and $\mathcal{H} = \text{clos} \text{span}\{U^n h : h \in \mathcal{H}, n \in \mathbb{Z}\}$. Since 1 is not an eigenvalue of $T$, it follows that 1 is not an eigenvalue of $U$ (see [SNF] Ch. II, §6).

The Sz.-Nagy–Foiaş functional calculus allows us to define a functional calculus for $T$ on the Banach algebra
\[ C_{A,1} \overset{\text{def}}{=} \{g \in H^\infty : g \text{ is continuous on } T \setminus \{1\}\}. \]
If $g \in C_{A,1}$, we put
\[ g(T) \overset{\text{def}}{=} P_{\mathcal{H}}g(\bar{U})|_{\mathcal{H}}. \]
This functional calculus is linear and multiplicative and
\[ \|g(T)\| \leq \|g\|_{H^\infty}, \quad g \in C_{A,1} \]
(see [SNF] Ch. III]).

We can now define a functional calculus for our dissipative operator on the Banach algebra
\[ C_{A,\infty} = \{f \in H^\infty(\mathbb{C}_+) : f \text{ is continuous on } \mathbb{R}\}. \]
Indeed, if \( f \in C_{A,\infty} \), we put
\[
 f(L) \overset{\text{def}}{=} (f \circ \omega)(T),
\]
where \( \omega \) is the conformal map of \( \mathbb{D} \) onto \( \mathbb{C}_+ \) defined by
\[
 \omega(\zeta) \overset{\text{def}}{=} i(1 + \zeta)(1 - \zeta)^{-1}, \quad \zeta \in \mathbb{D}.
\]
We can now extend this functional calculus to the class \( \text{Lip}_A(\mathbb{C}_+) \) of Lipshitz functions on \( \text{clo}(\mathbb{C}_+) \) that are analytic in \( \mathbb{C}_+ \). Suppose that \( f \in \text{Lip}_A(\mathbb{C}_+) \). We have
\[
 f(\zeta) = f_i(\zeta)(\zeta + i)^{-1}, \quad \zeta \in \mathbb{C}_+,
\]
where \( f_i(\zeta) \overset{\text{def}}{=} f(\zeta)/(\zeta + i) \).
\[(2.1)
\]
Since \( f \in \text{Lip}_A(\mathbb{C}_+) \), the function \( f_i \) belongs to \( C_{A,\infty} \). The (possibly unbounded) operator \( f(L) \) can be defined by
\[
 f(L) \overset{\text{def}}{=} (L + iI)f_i(L)
\]
(see [SNF] Ch. IV, §1). It follows from Th. 1.1 of Ch. IV of [SNF] that
\[
 f(L) \supset f_i(L)(L + iI),
\]
and so \( D(f(L)) \supset D(L) \).

We now proceed to the definition of a resolvent self-adjoint dilation of a maximal dissipative operator. If \( L \) is a maximal dissipative operator on a Hilbert space \( \mathcal{H} \), we say that a self-adjoint operator \( A \) in a Hilbert space \( \mathcal{K} \), \( \mathcal{K} \supset \mathcal{H} \), is called a resolvent self-adjoint dilation of \( L \) if
\[
 (L - \lambda I)^{-1} = P_{\mathcal{H}}(A - \lambda I)^{-1} \big| \mathcal{H}, \quad \text{Im} \lambda < 0.
\]
The dilation is called minimal if
\[
 \mathcal{H} = \text{clo} \text{span} \left\{ (A - \lambda I)^{-1}v : v \in \mathcal{H}, \, \text{Im} \lambda < 0 \right\}.
\]
If \( f \in C_{A,\infty} \), then
\[
 f(L) = P_{\mathcal{H}}f(A) \big| \mathcal{H}, \quad f \in C_{A,\infty}.
\]
A minimal resolvent self-adjoint dilation of a maximal dissipative operator always exists (and is unique up to a natural isomorphism). Indeed, it suffices to take a minimal unitary dilation of the Cayley transform of this operator and apply the inverse Cayley transform to it.

Let us now define the semispectral measure of a maximal dissipative operator \( L \). Let \( T \) be the Cayley transform of \( L \), and let \( \mathcal{E}_T \) be the semispectral measure of \( T \) on the unit circle \( \mathbb{T} \) defined by
\[
 \mathcal{E}_T(\Delta) \overset{\text{def}}{=} P_{\mathcal{H}}E_U(\Delta) \big| \mathcal{H},
\]
where \( \Delta \) is a Borel subset of \( \mathbb{T} \) and \( E_U \) is the spectral measure of the minimal unitary dilation \( U \) of \( T \).

Then
\[
 g(T) = \int_{\mathbb{T}} g(\zeta) d\mathcal{E}_T(\zeta), \quad g \in C_{A,1}.
\]
We can now define the semispectral measure \( \mathcal{E}_L \) of \( L \) by
\[
 \mathcal{E}_L(\Delta) = \mathcal{E}_T(\omega^{-1}(\Delta)), \quad \Delta \text{ is a Borel subset of } \mathbb{R}.
\]
It follows easily from (2.4) that
\[
f(L) = \int_{\mathbb{R}} f(x) \, d\mathcal{E}_L(x), \quad f \in C_{A, \infty}.
\]
It can be easily verified that if \( A \) is the minimal self-adjoint resolvent dilation of \( L \) and \( E_A \) is the spectral measure of \( A \), then
\[
\mathcal{E}_L(\Delta) = \mathbb{P}_{\mathcal{H}} E_A(\Delta)\big|_{\mathcal{H}}, \quad \Delta \text{ is a Borel subset of } \mathbb{R}.
\]

3. Double operator integrals and operator Lipschitz functions

Double operator integrals
\[
\iint_{\mathcal{X} \times \mathcal{Y}} \Phi(x, y) \, dE_1(x)Q \, dE_2(y)
\]
were introduced by Yu. L. Dalecki˘ı and S. G. Kre˘ın; see [DK]. Later Birman and Solomyak elaborated their beautiful theory of double operator integrals [BS1] and [BS2] (see also [AP2] and the references therein). Here \( \Phi \) is a bounded measurable function, \( E_1 \) and \( E_2 \) are spectral measures on a separable Hilbert space defined on \( \sigma \)-algebras of subsets of sets \( \mathcal{X} \) and \( \mathcal{Y} \), and \( Q \) is a bounded linear operator.

The approach of Birman and Solomyak [BS1] starts with the case when \( Q \in S_2 \), i.e., \( Q \) is a Hilbert–Schmidt operator. Under this assumption double operator integrals can be defined for arbitrary bounded measurable functions \( \Phi \). Put
\[
E(\Lambda \times \Delta)Q = E_1(\Lambda)Q E_2(\Delta), \quad Q \in S_2,
\]
where \( \Lambda \) and \( \Delta \) are measurable subsets of \( \mathcal{X} \) and \( \mathcal{Y} \). Clearly, \( \mathcal{E} \) takes values in the set of orthogonal projections on the Hilbert space \( S_2 \). It was shown in [BS1] (see also [BS3]) that \( \mathcal{E} \) extends to a spectral measure on \( \mathcal{X} \times \mathcal{Y} \). If \( \Phi \) is a bounded measurable function on \( \mathcal{X} \times \mathcal{Y} \), then
\[
\iint_{\mathcal{X} \times \mathcal{Y}} \Phi(x, y) \, dE_1(x)Q \, dE_2(y) \overset{\text{def}}{=} \left( \iint_{\mathcal{X} \times \mathcal{Y}} \Phi \, d\mathcal{E} \right) Q.
\]
Clearly,
\[
\left\| \iint_{\mathcal{X} \times \mathcal{Y}} \Phi(x, y) \, dE_1(x)Q \, dE_2(y) \right\|_{S_2} \leq \left\| \Phi \right\|_{L^\infty} \| Q \|_{S_2}.
\]
If \( Q \) is an arbitrary bounded operator, then for the double operator integral to make sense, \( \Phi \) has to be a Schur multiplier with respect to \( E_1 \) and \( E_2 \) (see [Pe1], [AP2], and [Pi]) that \( \Phi \) is a Schur multiplier if and only if \( \Phi \) belongs to the Haagerup tensor product \( L_{E_1}^\infty \otimes_h L_{E_2}^\infty \), i.e., it admits a representation
\[
\Phi(x, y) = \sum_{n \geq 0} \varphi_n(x) \psi_n(y),
\]
where the \( \varphi_n \) and \( \psi_n \) are measurable functions such that
\[
\sum_{n \geq 0} |\varphi_n|^2 \in L_{E_1}^\infty \quad \text{and} \quad \sum_{n \geq 0} |\psi_n|^2 \in L_{E_2}^\infty.
\]
In this case
\[
\iint \Phi(x, y) \, dE_1(x)Q \, dE_2(y) = \sum_{n \geq 0} \left( \int \varphi_n(x) \, dE_1(x) \right) Q \left( \int \psi_n(y) \, dE_2(y) \right)
\]
and the right-hand side does not depend on the choice of a representation in (3.1).
In this paper we need double operator integrals with respect to *semispectral measures*

\[
\int \int \Phi(x, y) \, d\mathcal{E}_1(x) Q \, d\mathcal{E}_2(y).
\]

Such double operator integrals were introduced in [Pe2] (see also [Pe3]). By analogy with the case of double operator integrals with respect to spectral measures, double operator integrals of the form (3.3) can be defined for arbitrary bounded measurable functions \(\Phi\) in the case when \(Q \in S_2\) and for functions \(\Phi\) in \(L^\infty_{\mathcal{E}_1} \otimes_h L^\infty_{\mathcal{E}_2}\) in the case of an arbitrary bounded operator \(Q\).

We refer the reader to the recent surveys [AP2] and [Pe4] for detailed information.

Suppose now that \(f \in \text{OL}_A(\mathbb{C}_+).\) It follows from Theorem 4.1 of [JW] (see also the survey [AP2, §3.3]) that \(f\) is differentiable everywhere on \(\text{clos} \mathbb{C}_+.\) We define the divided difference \(\mathcal{D}f\) of \(f\) on \(\text{clos} \mathbb{C}_+ \times \text{clos} \mathbb{C}_+\) by

\[
(\mathcal{D}f)(z, w) \overset{\text{def}}{=} \begin{cases} f(z) - f(w) & z \neq w, \\ f'(z) & z = w. \end{cases}
\]

Let \(C_A(\mathbb{C}_+)\) be the set of functions analytic in \(\mathbb{C}_+\) and continuous in \(\text{clos} \mathbb{C}_+\) and having a finite limit at infinity. We need the following characterization of the divided differences \(\mathcal{D}f\) for functions in \(\text{OL}_A(\mathbb{C}_+)\) (see [AP2, Theorems 3.9.6]).

Let \(f\) be a function continuous on \(\text{clos} \mathbb{C}_+\) and analytic in \(\mathbb{C}_+\). Then \(f \in \text{OL}_A(\mathbb{C}_+)\) if and only if \(\mathcal{D}f\) admits a representation

\[
(\mathcal{D}f)(z, w) = \sum_{n \geq 1} \varphi_n(z) \psi_n(w), \quad z, w \in \text{clos} \mathbb{C}_+,
\]

where \(\varphi_n, \psi_n \in C_A(\mathbb{C}_+)\) such that

\[
\left( \sup_{z \in \mathbb{C}_+} \sum_{n \geq 1} |\varphi_n(z)|^2 \right)^{1/2} \left( \sup_{w \in \mathbb{C}_+} \sum_{n \geq 1} |\psi_n(w)|^2 \right)^{1/2} < \infty.
\]

If \(f \in \text{OL}_A(\mathbb{C}_+)\), then the functions \(\varphi_n\) and \(\psi_n\) can be chosen so that the left-hand side of (3.6) is equal to \(\|f\|_{\text{OL}_A}\).

Recall that

\[
\|f\|_{\text{OL}_A} \overset{\text{def}}{=} \sup \left\{ \frac{\|f(N_1) - f(N_2)\|}{\|N_1 - N_2\|} \right\},
\]

the supremum being taken over all distinct normal operators \(N_1\) and \(N_2\) with spectra in \(\text{clos} \mathbb{C}_+\) such that the operator \(N_1 - N_2\) is bounded.

This allows us to consider double operator integrals

\[
\int \int_{\mathbb{R} \times \mathbb{R}} (\mathcal{D}f)(x, y) \, d\mathcal{E}_1(x) Q \, d\mathcal{E}_2(y),
\]

where \(\mathcal{E}_1\) and \(\mathcal{E}_2\) are semispectral measures and \(Q\) is a bounded linear operator, and this double operator integral is equal to

\[
\sum_{n \geq 1} \left( \int_{\mathbb{R}} \varphi_n \, d\mathcal{E}_1 \right) Q \left( \int_{\mathbb{R}} \psi_n \, d\mathcal{E}_2 \right),
\]
where the functions $\varphi_n$ and $\psi_n$ satisfy (3.5) and (3.6). Moreover, the following inequality holds:

\begin{equation}
\left\| \int_\mathbb{R} \int_\mathbb{R} (\mathcal{D} f)(x, y) \, d\mathcal{E}_1(x) Q \, d\mathcal{E}_2(y) \right\| \leq \| f \|_{OL_A} \| Q \|,
\end{equation}

see [AP2].

Note that it can be deduced from Theorem 2.2.3 in [AP2] that if $f \in \text{Lip}_A(\mathbb{C}_+)$, then $f \in \text{OL}_A(\mathbb{C}_+) \text{ if and only if } f$ is an operator Lipschitz function on $\mathbb{R}$. Moreover, \( \| f \|_{\text{OL}_A(\mathbb{C}_+)} = \| f \|_{\text{OL}(\mathbb{R})} \), where

\[ \| f \|_{\text{OL}(\mathbb{R})} \overset{\text{def}}{=} \sup \| f(A) - f(B) \| \| A - B \|, \]

the supremum being taken over all distinct bounded self-adjoint operators $A$ and $B$.

### 4. The integral representation and Lipschitz-type estimates

In this section for function $f$ of class $\text{OL}_A(\mathbb{C}_+)$ and for maximal dissipative operators $L_1$ and $L_2$ with bounded difference, we obtain a representation of the difference $f(L_1) - f(L_2)$ in terms of a double operator integral. This allows us to obtain a Lipschitz-type estimate for functions of maximal dissipative operators.

Let $f \in \text{OL}_A(\mathbb{C}_+)$, and let $L$ be a maximal dissipative operator. It can easily be deduced from (2.2) and (2.3) that $f(L)(L + i I)^{-1} = f_1(L)$, where $f_1$ is defined in (2.1).

The following theorem holds for arbitrary maximal dissipative operators $L_1$ and $L_2$ without the assumption that $L_1 - L_2$ is bounded.

**Theorem 4.1.** Let $L_1$ and $L_2$ be maximal dissipative operators, and let $f \in \text{OL}_A(\mathbb{C}_+)$. Suppose that $\varphi_n$ and $\psi_n$ are functions in $C_A(\mathbb{C}_+)$ satisfying (3.5) and (3.6). Then

\begin{equation}
f_1(L_1)(L_2 + i I)^{-1} - (L_1 + i I)^{-1} f_1(L_2) = f(L_1)(L_1 + i I)^{-1}(L_2 + i I)^{-1} - (L_1 + i I)^{-1} f(L_2)(L_2 + i I)^{-1} = \sum_{n=1}^{\infty} \varphi_n(L_1)((L_2 + i I)^{-1} - (L_1 + i I)^{-1}) \psi_n(L_2)
\end{equation}

and the series on the right converges in the weak operator topology.

**Proof.** It follows from (3.5) that the following identity holds:

\begin{equation}
(f(z)(z + i)^{-1}(w + i)^{-1} - (z + i)^{-1} f(w)(w + i)^{-1}) = \sum_{n=1}^{\infty} \varphi_n(z)(z + i)^{-1} w(w + i)^{-1} \psi_n(w) - \sum_{n=1}^{\infty} \varphi_n(z)(z + i)^{-1} \psi_n(w)
\end{equation}
for all \( z, w \in \text{clos} \mathbb{C}_+ \). Let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be the semispectral measure of \( L_1 \) and \( L_2 \). We have

\[
\iint_{\mathbb{R} \times \mathbb{R}} (f(x)(x+i)^{-1}(y+i)^{-1} - (x+i)^{-1}f(y)(y+i)^{-1}) \, d\mathcal{E}_1(x) \, d\mathcal{E}_2(y)
\]

\[
= f(L_1)(L_2 + iI)^{-1} - (L_1 + iI)^{-1}f(L_2)
\]

\[
= f(L_1)(L_1 + iI)^{-1}(L_2 + iI)^{-1} - (L_1 + iI)^{-1}f(L_2)(L_2 + iI)^{-1}.
\]

Next,

\[
\iint_{\mathbb{R} \times \mathbb{R}} \left( \sum_{n=1}^{\infty} \varphi_n(x)(x+i)^{-1}(y+i)^{-1}\psi_n(y) \right) \, d\mathcal{E}_1(x) \, d\mathcal{E}_2(y)
\]

\[
= \sum_{n=1}^{\infty} \varphi_n(L_1)L_1(L_1 + iI)^{-1}(L_2 + iI)^{-1}\psi_n(L_2)
\]

\[
= \sum_{n=1}^{\infty} \varphi_n(L_1)(I - i(L_1 + iI)^{-1})(L_2 + iI)^{-1}\psi_n(L_2)
\]

and

\[
\iint_{\mathbb{R} \times \mathbb{R}} \left( \sum_{n=1}^{\infty} \varphi_n(x)(x+i)^{-1}y(y+i)^{-1}\psi_n(w) \right) \, d\mathcal{E}_1(x) \, d\mathcal{E}_2(y)
\]

\[
= \sum_{n=1}^{\infty} \varphi_n(L_1)(L_1 + iI)^{-1}L_2(L_2 + iI)^{-1}\psi_n(L_2)
\]

\[
= \sum_{n=1}^{\infty} \varphi_n(L_1)(L_1 + iI)^{-1}(I - i(L_2 + iI)^{-1})\psi_n(L_2).
\]

It follows that the integral of the right-hand side of (4.2) is equal to

\[
\sum_{n=1}^{\infty} \varphi_n(L_1)((I - i(L_1 + iI)^{-1})(L_2 + iI)^{-1} - (L_1 + iI)^{-1}(I - i(L_2 + iI)^{-1}))\psi_n(L_2)
\]

\[
= \sum_{n=1}^{\infty} \varphi_n(L_1)((L_2 + iI)^{-1} - (L_1 + iI)^{-1})\psi_n(L_2).
\]

To establish (4.2), it remains to equate the integral of the left-hand side of (4.2) with the integral of the right-hand side. \( \square \)

**Theorem 4.2.** Let \( f \in \text{OL}_A(\mathbb{C}_+) \). Then for arbitrary maximal dissipative operators \( L_1 \) and \( L_2 \) with bounded \( L_1 - L_2 \), the following formula holds:

\[
f(L_1) - f(L_2) = \iint_{\mathbb{R} \times \mathbb{R}} (\mathcal{D}f)(x,y) \, d\mathcal{E}_1(x)(L_1 - L_2) \, d\mathcal{E}_2(y),
\]

where \( \mathcal{E}_j \) is the semispectral measure of \( L_j \), \( j = 1, 2 \).

To be more precise, we mean that the operator \( f(L_1) - f(L_2) \) whose domain contains the dense set \( \mathcal{D}_{L_1} = \mathcal{D}_{L_2} \) is bounded and extends by continuity to the double operator integral on the right-hand side.
Proof. Let \( \varphi_n \) and \( \psi_n \), \( n \geq 1 \), be functions in \( C_A(\mathbb{C}_+) \) satisfying (3.5) and (3.6). Then the double operator integral is well defined and
\[
\int_{\mathbb{R} \times \mathbb{R}} (\mathcal{D} f)(x, y) \, d\mathcal{E}_1(x)(L_1 - L_2) \, d\mathcal{E}_2(y) = \sum_{n \geq 1} \varphi_n(L_1)(L_1 - L_2)\psi_n(L_2),
\]
where the series converges in the weak operator topology; see (3.2). Put
\[
Q \overset{\text{def}}{=} \sum_{n \geq 1} \varphi_n(L_1)(L_1 - L_2)\psi_n(L_2).
\]
Then \( Q \) is a bounded operator and
\[
(L_1 + i I)^{-1}Q(L_2 + i I)^{-1} = \sum_{n \geq 1} (L_1 + i I)^{-1}\varphi_n(L_1)(L_1 - L_2)\psi_n(L_2)(L_2 + i I)^{-1}
\]
\[
= \sum_{n \geq 1} \varphi_n(L_1)(L_1 + i I)^{-1}((L_1 + i I) - (L_2 + i I))(L_2 + i I)^{-1}\psi_n(L_2)
\]
\[
= \sum_{n \geq 1} \varphi_n(L_1)((L_2 + i I)^{-1} - (L_1 + i I)^{-1})\psi_n(L_2)
\]
\[
= f(L_1)(L_1 + i I)^{-1}(L_2 + i I)^{-1} - (L_1 + i I)^{-1}f(L_2)(L_2 + i I)^{-1}
\]
by Theorem 4.1.

Let us show that
\[
f(L_1)(L_1 + i I)^{-1}(L_2 + i I)^{-1} = (L_1 + i I)^{-1}f(L_1)(L_2 + i I)^{-1}.
\]
Indeed, \( \text{Range}(L_2 + i I)^{-1} = \mathcal{D}_{L_2} = \mathcal{D}_{L_1} \), and so we have to prove that
\[
f(L_1)(L_1 + i I)^{-1}\big|_{\mathcal{D}_{L_1}} = (L_1 + i I)^{-1}f(L_1)\big|_{\mathcal{D}_{L_1}}.
\]
This follows easily from the equalities \( f(L_1)(L_1 + i I)^{-1} = f_1(L_1) \) and \( f(L_1) = (L_1 + i I)f_1(L_1) \); see (2.2). Thus,
\[
(L_1 + i I)^{-1}Q(L_2 + i I)^{-1}
\]
\[
= (L_1 + i I)^{-1}f(L_1)(L_2 + i I)^{-1} - (L_1 + i I)^{-1}f(L_2)(L_2 + i I)^{-1}
\]
\[
= (L_1 + i I)^{-1}(f(L_1) - f(L_2))(L_2 + i I)^{-1}.
\]
Hence, \( Qu = f(L_1)u - f(L_2)u \) for all \( u \in \mathcal{D}_{L_1} = \mathcal{D}_{L_2} \). □

The following result establishes a Lipschitz-type estimate for functions of maximal dissipative operators.

**Theorem 4.3.** Let \( f \in \text{OL}_A(\mathbb{C}_+) \). Then
\[
\|f(L_1) - f(L_2)\| \leq \|f\|_{\text{OL}_A}\|L_1 - L_2\|
\]
for arbitrary maximal dissipative operators \( L_1 \) and \( L_2 \) with bounded \( L_1 - L_2 \).

**Proof.** This is an immediate consequence of Theorem 4.2 and inequality (3.7). □

**Remark.** In the case when the difference \( L_1 - L_2 \) is a Hilbert–Schmidt operator, the formula
\[
f(L_1) - f(L_2) = \int_{\mathbb{R} \times \mathbb{R}} (\mathcal{D} f)(x, y) \, d\mathcal{E}_1(x)(L_1 - L_2) \, d\mathcal{E}_2(y)
\]
holds for an arbitrary function \( f \) in \( \text{Lip}_A(C_+) \) which leads to the following Lipschitz-type estimate in the Hilbert–Schmidt norm:

\[
\|f(L_1) - f(L_2)\|_{S_2} \leq \{ \sup |f'(\zeta)| : \text{Im} \zeta > 0 \}\|L_1 - L_2\|_{S_2}.
\]

This is a consequence of Theorem 6.6 of [AP1] and the obvious equality

\[
f(L)u = \lim_{n \to \infty} (f\omega_n)(L)u = \lim_{n \to \infty} f(L\omega_n), \quad u \in D_L,
\]

for a maximal dissipative operator \( L \); see also §6 of [MNP]. Here

\[
\omega_n(z) \overset{\text{def}}{=} \frac{1}{\log n} \log \frac{z + in}{z + i}, \quad n \geq 2,
\]

and \( \log \) means the principal branch of logarithm.

5. Commutator Lipschitz estimates

The purpose of this section is to obtain Lipschitz-type norm estimates of (quasi) commutators of the form \( f(L_1)R - Rf(L_2) \) in terms of the norms of \( L_1R - RL_2 \), where \( L_1 \) and \( L_2 \) are maximal dissipative operators and \( R \) is a bounded linear operator.

We say that the operator \( L_1R - RL_2 \) is bounded if \( R(D_{L_2}) \subset D_{L_1} \) and

\[
\|L_1Ru - RL_2u\| \leq \text{const} \|u\| \quad \text{for every} \quad u \in D_{L_2}.
\]

The following result is an analog of Theorem 4.1. It also holds for arbitrary maximal dissipative operators \( L_1 \) and \( L_2 \) and for an arbitrary bounded operator \( R \) without any additional assumptions.

**Theorem 5.1.** Let \( L_1 \) and \( L_2 \) be maximal dissipative operators, let \( f \in \text{OL}_A(C_+) \), and let \( R \) be a bounded operator. Suppose that \( \varphi_n \) and \( \psi_n \) are functions in \( C_A(C_+) \) satisfying (3.5) and (3.6). Then

\[
(5.1) \quad f(L_1)(L_1 + iI)^{-1}R(L_2 + iI)^{-1} - (L_1 + iI)^{-1}Rf(L_2)(L_2 + iI)^{-1} = \sum_{n=1}^{\infty} \varphi_n(L_1)(R(L_2 + iI)^{-1} - (L_1 + iI)^{-1}R)\psi_n(L_2)
\]

and the series on the right converges in the weak operator topology.

**Proof.** We are going to use (4.2). We have

\[
\text{bezep} \int_{\mathbb{R}} \int_{\mathbb{R}} (x + i)^{-1}(f(x) - f(y))(y + i)^{-1} \, d\xi_1(x) \, d\xi_2(y)
\]

\[
= f(L_1)(L_1 + iI)^{-1}R(L_2 + iI)^{-1} - (L_1 + iI)^{-1}Rf(L_2)(L_2 + iI)^{-1}.
\]
On the other hand, this is equal to

\[ \int_R \int_R \sum_{n=1}^{\infty} \varphi_n(x)(x+i)^{-1}(y+i)^{-1} \psi_n(y) \, d\xi_1(x) R \, d\xi_2(y) \]

\[ - \int_R \int_R \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \varphi_n(x)(x+i)^{-1} y(y+i)^{-1} \psi_n(y) \, d\xi_1(x) R \, d\xi_2(y) \]

\[ = \sum_{n=1}^{\infty} \varphi_n(L_1)L_1(L_1+i)^{-1}R(L_2+i)^{-1} \psi_n(L_2) \]

\[ - \sum_{n=1}^{\infty} \varphi_n(L_1)(L_1+i)^{-1}RL_2(L_2+i)^{-1} \psi_n(L_2) \]

\[ = \sum_{n=1}^{\infty} \varphi_n(L_1)(I-i(L_1+i)^{-1})R(L_2+i)^{-1} \psi_n(L_2) \]

\[ - \sum_{n=1}^{\infty} \varphi_n(L_1)(L_1+i)^{-1}R(I-i(L_2+i)^{-1}) \psi_n(L_2) \]

\[ = \sum_{n=1}^{\infty} \varphi_n(L_1)(R(L_2+i)^{-1} - (L_1+i)^{-1}R) \psi_n(L_2). \] □

**Theorem 5.2.** Let \( f \in OL_A(C_+) \). Then for arbitrary bounded operator \( R \) and maximal dissipative operators \( L_1 \) and \( L_2 \) with bounded \( L_1R - RL_2 \), the following formula holds:

\[ f(L_1)R - Rf(L_2) = \int_{R \times R} (Df)(x, y) \, d\xi_1(x)(L_1R - RL_2) \, d\xi_2(y), \]

where \( \xi_j \) is the semispectral measure of \( L_j, j = 1, 2 \).

Note that the right-hand side is a bounded operator defined on the whole space, while the left-hand side is a bounded operator defined on \( D_{L_2} \), which extends by continuity to the operator on the right.

**Proof.** Let \( \varphi_n \) and \( \psi_n, n \geq 1 \), be functions in \( C_A(C_+) \) satisfying (3.5) and (3.6). Then the double operator integral is well defined and

\[ \int_{R \times R} (Df)(x, y) \, d\xi_1(x)(L_1R - RL_2) \, d\xi_2(y) = \sum_{n \geq 1} \varphi_n(L_1)(L_1R - RL_2) \psi_n(L_2). \]

Put

\[ Q \overset{\text{def}}{=} \sum_{n \geq 1} \varphi_n(L_1)(L_1R - RL_2) \psi_n(L_2). \]
Then $Q$ is a bounded linear operator and
\[
(L_1 + i I)^{-1} Q(L_2 + i I)^{-1} = \sum_{n \geq 1} (L_1 + i I)^{-1} \varphi_n(L_1) (L_1 R - RL_2) \psi_n(L_2) (L_2 + i I)^{-1}
\]
\[= \sum_{n \geq 1} \varphi_n(L_1) (L_1 + i I)^{-1} (L_1 R - RL_2) (L_2 + i I)^{-1} \psi_n(L_2)
\]
\[= \sum_{n \geq 1} \varphi_n(L_1) ((L_1 + i I)^{-1} L_1 R(L_2 + i I)^{-1} - (L_1 + i I)^{-1} RL_2(L_2 + i I)^{-1}) \psi_n(L_2).
\]
As we have observed in [41], for a maximal dissipative operator $L$, we have equality $L(L + i I)^{-1} = I - i(L + i I)^{-1}$. Using this equality, we find that
\[(L_1 + i I)^{-1} Q(L_2 + i I)^{-1} = \sum_{n \geq 1} \varphi_n(L_1) (R(L_2 + i I)^{-1} - (L_1 + i I)^{-1} R) \psi_n(L_2).
\]
By (5.1), the last expression is equal to
\[(5.2) \quad f(L_1)(L_1 + i I)^{-1} R(L_2 + i I)^{-1} - (L_1 + i I)^{-1} Rf(L_2)(L_2 + i I)^{-1}.
\]
Now it is time to use the fact that the operator $L_1 R - RL_2$ is bounded, and so $R(\mathcal{D}_L) \subset \mathcal{D}_L$. It follows that $Range R(L_2 + i I)^{-1} \subset \mathcal{D}_L$. Thus,
\[f(L_1)(L_1 + i I)^{-1} R(L_2 + i I)^{-1} = (L_1 + i I)^{-1} f(L_1) R(L_2 + i I)^{-1}.
\]
Hence, the expression in (5.2) is equal to
\[(L_1 + i I)^{-1} f(L_1) R(L_2 + i I)^{-1} - (L_1 + i I)^{-1} Rf(L_2)(L_2 + i I)^{-1}
\]
\[= (L_1 + i I)^{-1} (f(L_1) R - Rf(L_2))(L_2 + i I)^{-1},
\]
and we can conclude that
\[(L_1 + i I)^{-1} Q(L_2 + i I)^{-1} = (L_1 + i I)^{-1} (f(L_1) R - Rf(L_2))(L_2 + i I)^{-1}.
\]
Hence, $Qu = f(L_1)Ru - Rf(L_2)u$ for all $u \in \mathcal{D}_L$.

**Theorem 5.3.** Let $f \in \text{OL}_A(\mathbb{C}_+)$. Then
\[
\|f(L_1) R - Rf(L_2)\| \leq \|f\|_{\text{OL}_A} \|L_1 R - RL_2\|
\]
for an arbitrary bounded operator $R$ and arbitrary maximal dissipative operators $L_1$ and $L_2$ with bounded $L_1 R - RL_2$.

**Proof.** This is an immediate consequence of Theorem 5.2 and inequality (5.1).

**References**

[AP1] A. B. Aleksandrov and V. V. Peller, *Functions of perturbed dissipative operators* (Russian, with Russian summary), Algebra i Analiz 23 (2011), no. 2, 9–51, DOI 10.1090/S1061-0022-2012-01194-5; English transl., St. Petersburg Math. J. 23 (2012), no. 2, 209–238. MR2841671

[AP2] A. B. Aleksandrov and V. V. Peller, *Operator Lipschitz functions* (Russian, with Russian summary), Uspekhi Mat. Nauk 71 (2016), no. 4(430), 3–106, DOI 10.4213/rm9729; English transl., Russian Math. Surveys 71 (2016), no. 4, 605–702. MR3588921

[BS1] M. Š. Birman and M. Z. Solomjak, *Double Stieltjes operator integrals* (Russian), Probl. Math. Phys., No. I, Spectral Theory and Wave Processes (Russian), Izdat. Leningrad. Univ., Leningrad, 1966, pp. 33–67. MR0209872

[BS2] M. Š. Birman and M. Z. Solomjak, *Double Stieltjes operator integrals. III* (Russian), Problems of mathematical physics, No. 6 (Russian), Izdat. Leningrad. Univ., Leningrad, 1973, pp. 27–53. MR0348494
[BS3] M. Birman and M. Solomyak, Tensor product of a finite number of spectral measures is always a spectral measure, Integral Equations Operator Theory 24 (1996), no. 2, 179–187, DOI 10.1007/BF01193459. MR1371945

[DK] Yu. L. Dalecki˘ı and S. G. Kre˘ın, Integration and differentiation of functions of Hermitian operators and applications to the theory of perturbations (Russian), Voroneˇz. Gos. Univ. Trudy Sem. Funkcional. Anal. 1956 (1956), no. 1, 81–105. MR0084745

[F] Ju. B. Farforovskaja, An example of a Lipschitzian function of selfadjoint operators that yields a nonnuclear increase under a nuclear perturbation: Investigations of linear operators and the theory of functions, III (Russian), Zap. Nauˇcn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 30 (1972), 146–153. MR0336400

[JW] B. E. Johnson and J. P. Williams, The range of a normal derivation, Pacific J. Math. 58 (1975), no. 1, 105–122. MR0380490

[MNP] M. M. Malamud, H. Neidhardt, and V. V. Peller, Absolute continuity of spectral shift, J. Funct. Anal., DOI 10.1016/j.jfa.2018.05.011

[KS1] Edward Kissin and Victor S. Shulman, On a problem of J. P. Williams, Proc. Amer. Math. Soc. 130 (2002), no. 12, 3605–3608, DOI 10.1090/S0002-9939-02-06608-X. MR1920040

[KS2] E. Kissin and V. S. Shulman, On fully operator Lipschitz functions, J. Funct. Anal. 253 (2007), no. 2, 711–728, DOI 10.1016/j.jfa.2007.08.007. MR2370097

[Pe1] V. V. Peller, Hankel operators in the theory of perturbations of unitary and selfadjoint operators (Russian), Funktsional. Anal. i Prilozhen. 19 (1985), no. 2, 37–51, 96. MR800919

[Pe2] V. V. Peller, For which f does A – B ∈ Sp imply that f(A) – f(B) ∈ Sp?, Operators in indefinite metric spaces, scattering theory and other topics (Bucharest, 1985), Oper. Theory Adv. Appl., vol. 24, Birkh¨auser, Basel, 1987, pp. 289–294. MR903080

[Pe3] V. V. Peller, Differentiability of functions of contractions, Linear and complex analysis, Amer. Math. Soc. Transl. Ser. 2, vol. 226, Amer. Math. Soc., Providence, RI, 2009, pp. 109–131, DOI 10.1090/trans2/226/10. MR2500514

[Pe4] V. V. Peller, Multiple operator integrals in perturbation theory, Bull. Math. Sci. 6 (2016), no. 1, 15–88, DOI 10.1007/s13373-015-0073-y. MR3472849

[Pi] Gilles Pisier, Similarity problems and completely bounded maps, Includes the solution to “The Halmos problem”, Second, expanded edition, Lecture Notes in Mathematics, vol. 1618, Springer-Verlag, Berlin, 2001. MR1818047

[So] B. M. Solomyak, A functional model for dissipative operators. A coordinate-free approach (Russian, with English summary), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 178 (1989), no. Issled. Line˘ın. Oper. Teorii Funktsii. 18, 57–91, 184–185, DOI 10.1007/BF01095663; English transl., J. Soviet Math. 61 (1992), no. 2, 1981–2002. MR1037765

[SNF] Béla Sz.-Nagy and Ciprian Foia¸s, Harmonic analysis of operators on Hilbert space, Translated from the French and revised, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest, 1970. MR0275190

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