Supersymmetric relativistic quantum mechanics in time-domain

Felipe A. Asenjo, Sergio A. Hojman, Héctor M. Moya-Cessa and Francisco Soto-Eguibar

I. INTRODUCTION

Supersymmetry is one of the cornerstones of theoretical physics, being ubiquitous in almost every branch of physics. The standard way to proceed is to construct supersymmetric theories in a space-domain, where the superpartners and supercharge operators take into account the spatial variations of, for example, external potentials. In this work, it is not our aim to focus in those spatial supersymmetries, but instead to inquire if a temporal version of such theories is possible for massive fields in relativistic quantum mechanics.

Recently, in Refs. [33, 34], it was introduced the concept of supersymmetry in time-domain (T-SUSY) for Maxwell equations. This is a supersymmetry occurring in the temporal part of the massless field dynamics, completely uncoupled from the spatial evolution of the field. They studied the applications of T-SUSY in the realm of optics for dispersive media, showing the novel capabilities of this theory to introduce hypothetical materials with new optical features. Besides, they showed that this time-domain supersymmetry applies to any field described, in principle, by a d’Alambertian equation. In other words, they developed the bosonic version of the T-SUSY theory.

Supersymmetric theories for purely bosonic systems have been extensively developed. On the contrary, it is the purpose of this manuscript to show that a T-SUSY theory can also be obtained in relativistic quantum mechanics for fields described by the Dirac equation, thus complementing the usual spatial supersymmetry theory. In this case, we show that the simplest T-SUSY theory may be constructed for time-dependent massive fields satisfying Dirac equation, finding solutions for different possible time-dependent masses. Also, this theory is equivalent to its bosonic partner, allowing us to obtain a massive particle field behavior that is analogue to a light-like one.

Besides, this theory produces probability states oscillations as a consequence of its supersymmetry. Thus, it can be used to study the neutrino oscillation problem. In this way, we give a different perspective to the origin of neutrino oscillations through supersymmetry in time-domain.

II. T-SUSY FOR DIRAC EQUATION

Let us consider a bi-spinor field $\Psi$ satisfying the Dirac equation in flat spacetime for a field with mass $m$,

$$i\gamma^\mu \partial_\mu \Psi = m\Psi,$$

where $\partial_\mu$ are the covariant derivatives. Here, $\gamma^\mu$ are the gamma matrices in the Dirac representation, fulfilling $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} 1_{4\times 4}$, with the flat spacetime metric $\eta^{\mu\nu} = \text{diag}(1,-1,-1,-1)$, and with the identity matrix $1_{4\times 4}$. The cases with external potential are discussed in the last section, but for now it is enough to consider the T-SUSY theory for Eq. (1).

In the following construction for a relativistic quantum mechanical T-SUSY, from Eq. (1), a time-dependent mass is essential. In principle, a time-dependent mass can be understood as an interacting Lorentz scalar potential field. It is important to remark that conservation of probability associated to bi-spinor fields only require real mass, and not fields with constant masses.
Let us consider the following form for the bi-spinor
\[ \Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \]
in terms of spinors \( \Psi_\pm \). Thereby, Eq. \( (1) \) becomes
\[ \left( i \frac{\partial}{\partial t} + m \right) \Psi_\pm = -i \sigma^j \partial_j \Psi_\mp. \]
In order to bring the usual aspects of supersymmetry theories, let us perform a Wick rotation in time and space, \( t \to it \) and \( x \to ix \). This is equivalent to a change in the flat spacetime metric signature. Under this change, Eq. \( (3) \) becomes
\[ Q_\pm \Psi_\pm = -\sigma^j \partial_j \Psi_\mp, \]
with operators
\[ Q_\pm = \frac{\partial}{\partial t} \pm m. \]
Now, let us consider a separable spinor of the form
\[ \Psi_\pm(t, x^j) = \psi_\pm(t) \chi_\pm(x^j), \]
with time-dependent functions \( \psi_\pm \), and space-dependent spinor \( \chi_\pm \). Using \( (4) \) in Eq. \( (1) \), we obtain
\[ (Q_\pm \psi_\pm) \chi_\pm = -\psi_\mp \sigma^j \partial_j \chi_\mp, \]
where \( Q_\pm \) operates on the function \( \psi_\mp \). In this way, the T-SUSY formalism can be achieved for the functions \( \psi_\pm \) as they fulfill
\[ Q_\pm \psi_\pm = - \left( \frac{\chi^\dagger_\pm \sigma^j \partial_j \chi_\mp}{\chi^\dagger_\mp \chi_\mp} \right) \psi_\mp. \]
Although this equation is general, along this work and in order to extract the main physical information from the simplest model, we study the case \( \chi^\dagger_\pm \sigma^j \partial_j \chi_\mp = -k \chi^\dagger_\pm \chi_\mp \) for an arbitrary constant \( k \). In the Dirac basis, this can be easily solved by the following ansatz for the spinor
\[ \chi_\pm(x^j) = e_\pm \exp \left( -k_1 x^1 \mp ik_2 x^2 \right) f(x^3), \]
in terms of an arbitrary function \( f \), with \( k = k_1 + k_2 \), and spinors
\[ e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]
with the properties \( e_\dagger_\pm e_\pm = 1 \), and \( e_\dagger_\pm e_\mp = 0 \). This allows us to reduce the system \( (1) \) into the form
\[ Q_\pm \psi_\pm = k \psi_\mp, \]
where \( k \) can be shown to play the role of the momentum of the particle. Eqs. \( (1) \) correspond to a set of supersymmetric equations for quantum mechanics, obtained in the Dirac matrices basis. It is the simplest form of a T-SUSY theory in relativistic quantum mechanics, and it has an equivalent form to supersymmetric theory in space-domain \( (13) \).

This form of the T-SUSY theory requires that the mass be responsible for the origin of the superpotential of this supersymmetry. This occurs when the mass becomes time-dependent, \( m = m(t) \), and it can be seen using Eqs. \( (14) \) to calculate the equations for each function \( \psi_\pm \),
\[ H_\pm \psi_\pm = k^2 \psi_\pm, \]
where we have defined the super-partners Hamiltonians \( H_\pm \) in terms of superpotentials \( W_\pm \) as
\[ H_\pm = Q_\mp Q_\pm = \frac{d^2}{dt^2} + W_\pm, \]
where
\[ W_\pm = \mp \frac{dm}{dt} - m^2, \]
such that \( W_- - W_+ = 2 \frac{dm}{dt} \). Thereby, the difference between states lies in the time dependence of the mass.

In general, from Eq. \( (15) \), we can write the functions \( (2) \) as
\[ \psi_\pm(t) = \exp \left( - \int dt E_\pm \right), \]
where \( E_\pm \) are time-dependent functions that satisfy the Ricatti equation
\[ \frac{dE_\pm}{dt} = E^2_\pm + k^2 = W_\pm, \]
such that \( \psi_\pm(0) = 1 \); whenever \( \frac{dm}{dt} \neq 0 \), then \( E_+ \neq E_- \). In this way, for a specific time functionality of the mass, solutions for \( E_\pm \) can be found in order to satisfy this T-SUSY description. On the contrary, for constant mass, we obtain the solution \( E_\pm^2 = k^2 + m^2 \) (from where we identify \( E \) and \( k \) as the particle energy and momentum, respectively) and the T-SUSY aspects of the theory vanish.

All the framework for a standard supersymmetric theory in quantum mechanics can be utilized for this T-SUSY theory. For instance, its algebra is defined by the super-Hamiltonian matrix operator \( H \), and the supercharge matrix operators \( Q \), defined as
\[ H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ Q_+ & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & Q_- \\ 0 & 0 \end{pmatrix}, \]
and that have the closed algebra \( H = \{ Q, Q^\dagger \} \), \( [H, Q] = 0 = [H, Q^\dagger] \), \( \{ Q, Q \} = 0 = \{ Q^\dagger, Q^\dagger \} \), for the bosonic \( H \) and fermionic \( Q \) and \( Q^\dagger \) operators. In the same way, as \( \psi_\pm \) are the eigenfunctions of \( H_\pm \), the spectrum of the two Hamiltonians is thus degenerate, except that one of the Hamiltonians has an extra state at zero energy. Therefore, using Eqs. \( (16) \), a partner eigenfunction can be converted into the other one with the same energy, creating or destroying an extra node in that eigenfunction \( (18) \).
The non-relativistic limit of the above theory can be obtained directly from Eq. 1 by applying the \((i\partial_t \pm m)\) operator, we obtain \((-\partial_t^2 \mp i\partial_t m - m^2)\Psi_{\pm} = -\sigma^j \sigma_k \partial_j \partial_k \Psi_{\pm} = -\nabla^2 \Psi_{\pm}.\) The non-relativistic limit of this T-SUSY theory is obtained when \(\Psi_{\pm} = \zeta_{\pm} \exp(\mp i \int t \, dt),\) and the eikonal approximation is taken \(\partial^2 \zeta_{\pm} \ll m \partial_t \zeta_{\pm}.\) In such case, we get the usual free-particle Schrödinger equation \(\pm 2i m \partial_t \zeta_{\pm} + \nabla^2 \zeta_{\pm} = 0.\)

It is worth to remark that non-relativistic versions of spatial supersymmetry can be constructed to deal with time-dependent Hamiltonians \([30, 33]\). This is still a spatial SUSY theory, but considering the dynamics of time-dependent systems. This is different to what has been developed above for relativistic time-domain SUSY, or what was first described in Refs. \([33, 34]\).

Finally, on the other hand, we notice that this relativistic quantum mechanical T-SUSY theory can be put in analogue fashion to its bosonic (Klein-Gordon) counterpart developed in Ref. \([34]\). We can rewrite Eq. 12 in the form

\[
\left( \frac{d^2}{dt^2} - \frac{1}{n_{\pm}^2} \right) \psi_{\pm} = 0, \tag{18}
\]

where

\[
n_{\pm}(t) = (k^2 - W_{\pm})^{-1/2}. \tag{19}
\]

Eq. 18 is equivalent to the one describing the propagation of light in a medium with supersymmetric refraction indices \(n_{\pm}.\) In this T-SUSY theory, both refraction indices are related by

\[
\frac{1}{n_{\pm}^2} = \frac{1}{n_{\pm}^2} + 2 \frac{dm}{dt}. \tag{20}
\]

This relation shows the equivalence of fermionic T-SUSY with bosonic T-SUSY discussed in Ref. \([34].\)

### III. SIMPLE SOLUTIONS

We can get simple solutions for particle fields with time-dependent mass by first considering \(k \ll m(t).\) For this case, from Eq. \([16]\), we find

\[
E_{\pm}(t) \approx \mp \dot{m}(t). \tag{21}
\]

Therefore, the two mass states have different behavior stemming from just one time-dependent mass.

Similarly, another simple solution can be obtained in the light-mass (ultra-relativistic) case, when \(k \gg m(t).\) For this case, \(k\) corresponds (approximately) to the energy of the particle. In this case, from Eq. \([16]\), we can find an approximated form for \(E_{\pm}\) given by

\[
E_{\pm}(t) \approx k + e^{2\kt} \int dt \, W_{\pm} \, e^{-2\kt}. \tag{22}
\]

### IV. OSCILLATIONS IN T-SUSY

The above T-SUSY theory allows now to consider the phenomenon of oscillation of states in a different fashion. These T-SUSY oscillations have their origin in the subjacent supersymmetry due to the temporal dependence of the mass. In order to obtain these physical states, let us return to the physical time \(t\) from Eq. \([1]\), through an inverse Wick rotation applied to the previous solutions.

Considering the bi-spinor \([2]\), let us now define a bi-spinor with mixed states

\[
\Phi = \begin{pmatrix} \Phi_a \\ \Phi_b \end{pmatrix} = \begin{pmatrix} \cos \theta \, \mathbb{1}_{2 \times 2} \\ \sin \theta \, \mathbb{1}_{2 \times 2} \end{pmatrix} \Psi, \tag{23}
\]

with the spinor wavefunctions \(\Phi_a\) and \(\Phi_b\) defining new states, the mixing angle in vacuum \(\theta [40, 43],\) and the \(2 \times 2\) identity matrix \(\mathbb{1}_{2 \times 2}.\)

The amplitude of mixed states change, from \(\Phi_a(t = 0)\) to \(\Phi_b(t),\) can be calculated to be \(\text{Amp}(\Phi_a \rightarrow \Phi_b) = \sin 2\theta [\psi^+_a(0) \psi_b(t) - \psi^+_b(0) \psi_a(t)] / 2.\) Finally, the probability of mixed states change is given by

\[
P(\Phi_a \rightarrow \Phi_b) = \left| \text{Amp}(\Phi_a \rightarrow \Phi_b) \right|^2 = \sin^2 2\theta \, e^{-\alpha} \left( \sinh^2 \beta + \sin^2 \rho \right), \tag{24}
\]

where

\[
\alpha = \int dt \left( \text{Im} \{E_-\} + \text{Im} \{E_+\} \right),
\]

\[
\beta = \frac{1}{2} \int dt \left( \text{Im} \{E_-\} - \text{Im} \{E_+\} \right),
\]

\[
\rho = \frac{1}{2} \int dt \left( \text{Re} \{E_-\} - \text{Re} \{E_+\} \right). \tag{25}
\]

The terms proportional to \(\exp(-\alpha)\) and \(\sinh^2 \beta\) enter in the probability due to the contribution of the possible time-dependent amplitude of functions \([14].\) Their phases only contribute to the probability through \(\sin^2 \rho.\)

In this way, the T-SUSY oscillations between the mixed states, given by the above transition probability, are due only to the different solutions of Eq. \([16].\) This occurs if the field has a non-constant mass, with the T-SUSY properties.

#### A. Oscillations for massive fields

As an example, let us evaluate the above probability \([24]\) with the previous simple example for the solution of a massive field, namely \(k \ll m,\) and \(E_{\pm} \approx \mp \dot{m}.\) For this case, \(\text{Im} \{E_{\pm}\} \approx 0,\) and \(\alpha = 0 = \beta;\) then the probability \([24]\) reduces to

\[
P(\Phi_a \rightarrow \Phi_b) \approx \sin^2 2\theta \sin^2 \left( \int dt \, m \right). \tag{26}
\]

Thus, the T-SUSY oscillations for the massive case occurs because the mass can evolve in time. On the contrary, with constant mass, then \(E_+ = E_-\) and no oscillation can occur.
B. Oscillations for ultra-relativistic fields

Let us model the particle oscillation considering the previous simple solution \( (22) \) for an ultra-relativistic particle, with \( m \ll k \). We consider the simplest form \( m(t) = m_0 \sin(\Lambda t) \) for a time-varying mass, where constant \( m_0 \ll k \), and \( \Lambda \) is a constant measuring the time-scale variation of mass. In this way, in this model, the particle has a mass that oscillates between a null and a finite value, with temporal span of \( 0 \leq t \leq \pi/\Lambda \) (in order to avoid negative masses). Consequently, in this case, solution \( (22) \) gives

\[
E_{\pm} \approx k + \frac{m_0^2}{4k} \pm \frac{m_0 \Lambda^2}{4k^2 - \Lambda^2} \sin(\Lambda t) - \frac{km_0^2}{4(k^2 - \Lambda^2)} \cos(2\Lambda t)
\]

\[
\pm i \frac{2km_0}{4k^2 - \Lambda^2} \cos(\Lambda t) + i \frac{m_0 \Lambda}{4(k^2 - \Lambda^2)} \sin(2\Lambda t).
\]

In this way, the total probability for flavour states change can be calculated from Eq. \( (24) \) to be

\[
P(\nu_e \rightarrow \nu_\mu) = \sin^2 2\theta \exp \left( \frac{m_0^2}{4(k^2 - k^2)} (\cos(2\Lambda t) - 1) \right)
\times \left[ \sin^2 \left( \frac{2km_0}{4k^2 - \Lambda^2} \sin(\Lambda t) \right) + \sin^2 \left( \frac{m_0 \Lambda}{4(k^2 - \Lambda^2)} \cos(\Lambda t) - 1 \right) \right].
\]

C. Application to the two-neutrino oscillation problem

We now invite the reader to re-think the two-neutrino oscillation problem in terms of the T-SUSY theory developed above. Neutrino oscillations have been studied in the context of supersymmetry \( [44, 56] \), but not within the context of the time-domain theory (with time-dependent mass fields).

In this way, the electron and muon flavour eigenstates can now be identified by the bi-spinor \( (23) \), while the oscillation probability can be calculated from Eq. \( (25) \) under the following assumptions. Let us consider MeV neutrinos \( (k \sim 10^6 \text{ eV}) \), with an upper mass limit of the order \( m_0 \sim 10^{-1} \text{ eV} \) \( [57, 58] \), and oscillation distances of tenth of kilometers \( [58] \). Let us assume that \( \Lambda \ll m_0 \ll k \), in order that the arguments of \( \sin \) and \( \sin \) in probability \( (28) \) are small. Thereby, in this case, the probability \( (28) \) for the flavour states change reduces simply to

\[
P(\nu_e \rightarrow \nu_\mu) \approx \frac{m_0^2}{4k^2} \sin^2(\Lambda t) + \frac{m_0^2 \Lambda^2}{16k^4} (\cos(\Lambda t) - 1)^2
\approx \frac{m_0^2}{4k^2} \sin^2(\Lambda t),
\]

as the last term is negligible compared to the first one. The probability \( (29) \) resembles the standard result for the two-neutrino oscillation problem \( [40, 43] \) when we identify \( \Lambda = \Delta m^2/k \sim 10^{-10} \text{ eV} \), where the known neutrino mass difference is of the order \( \Delta m^2 \sim 10^{-4} \text{ eV} \) \( [58] \). This value for \( \Lambda \) is consistent with our initial assumption \( \Lambda \ll m_0 \ll k \); besides, it allows us to get that the oscillation in this model permit the neutrino to travel distances of the order \( L = ct \leq c\pi/\Lambda < 10^9 \text{ km} \), which is consistent with the measurements \( [58] \).

Because of the above result \( (29) \), one could infer that the neutrino T-SUSY oscillations are not produced by different mass states, but because of the appearing of the supersymmetry in time-domain due to the single neutrino non-constant mass. Under this T-SUSY theory, the oscillation occurs due to the supersymmetric nature of the neutrino in time-domain. It is always one neutrino, with a unique mass, that it is oscillating between supersymmetric mass states.

V. DISCUSSION

When the mass of a relativistic quantum mechanical field is time-dependent, then a supersymmetric in time-domain theory may be constructed. This T-SUSY is the time analogue of any relativistic supersymmetric quantum theory. Along this work, we have presented solutions for different time-dependent masses. A remarkable outcome is that this theory contains solutions that mimic the light-like behavior studied for the bosonic T-SUSY theory.

One of the main results extracted here is the possibility for probability oscillation between two states due to the temporal changes of field mass. These T-SUSY oscillations were applied to the two-neutrino oscillation problem, obtaining that the oscillation between flavour states can be explained by invoking only the time-dependence of a single neutrino mass (not different masses for different states). Furthermore, the oscillation problem for more states (such as the three-neutrino oscillation) can be straightforwardly generalized from Eq. \( (28) \). In this way, this theory for spin fields introduces a different and interesting perspective to understand the neutrino oscillation problem under the new light of supersymmetry in time-domain.

The T-SUSY theory presented here may be its simplest, yet nontrivial, version. The extension to other types of physical interactions is straightforward. For example, a T-SUSY version of the Dirac equation with Lorentz scalar potential \( [18] \) may be constructed in an analogous way to what was presented here (in fact, the time-dependent mass can be understood as the interacting scalar potential field). In such case, the non-relativistic limit will coincide with known theories \( [36–39] \). Additionally, in the specific case of minimal coupling to electromagnetism \( [18] \), no T-SUSY are found in a simple way as presented here. On the other hand, extensions of this theory to spinor fields in curved spacetimes can also be proposed \( [59] \), and it is currently under investi-
Finally, it is a matter of exploration to understand if the supersymmetric structures of this formalism may be closely linked to those appearing in time crystals [60].

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