Construction of the discrete breathers and a simple physical interpretation of their existence

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Abstract

We present a simple numerical method for the discrete breather construction based on the idea of the pair synchronization of the particles involved in the breather vibration. It can be used for obtaining exact breather solutions in nonlinear Hamiltonian lattices of different types. We illustrate the above method using chains of the coupled Duffing oscillators. With some additional approximation, the pair synchronization method leads to a very simple physical interpretation of the existence of the exact breathers as strictly time-periodic and spatially localized dynamical objects.

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I. INTRODUCTION

The concept of the discrete breathers or intrinsic localized modes in nonlinear Hamiltonian lattices have been proposed by Sievers and Takeno two decades earlier in [1]. Since that time, there appeared a vast number of papers considering different physical, mathematical and numerical aspects of the discrete breathers dynamics. The appropriate references can be found in many review papers (see, for example, [2, 3, 4, 5]).

Discrete breathers represent spatially localized and time-periodic excitations in nonlinear lattices. As we have already discussed in [6], the possibility of the existence of such dynamical objects is not obvious. Indeed, because of spatial localization, different particles of the lattices vibrate with essentially different amplitudes. On the other hand, it is typical for nonlinear systems that frequencies depend on the amplitudes of vibrating particles. Therefore, one may ask: "How can a discrete breather exist as a strictly time-periodic dynamical object despite of its spatial localization"? In the present paper, we will try to give an explicit answer to this question.

There exist several rigorous existence proofs for breathers in networks of weakly coupled anharmonic oscillators reviewed, for example, in [3]. The first existence proof of the discrete breathers was obtained by the principle of anticontinuity in [7]. The authors have demonstrated this approach for the chain of linear coupled nonlinear oscillators where anticontinual limit corresponds to zero coupling. The main idea of this approach is connected with application of the implicit function theorem in the space of time-periodic solutions at the discrete breather period.

There exist a number of methods for breathers construction at computer accuracy provided that their frequency (and its multiples) is far enough from resonance with linear phonon spectrum [3, 5]. As was emphasized by Aubry in [3], the Newton method is used practically in all such methods and "it is nothing but a numerical implementation of the implicit function theorem" used for rigorous proof of the DBs existence in [7]. As a certain variation, in some numerical procedures, the Newton method is used for finding the Fourier coefficients of the exact breathers solutions [5, 8].

Unfortunately, the above methods, as well as the rigorous existence proofs, do not lead (at least, from our point of view) to any simple physical interpretation of the DBs existence [14]).
In the present paper, we develop a new simple method for breather construction (Sec. II) which is able to produce breathers solutions with high precision (Sec. III) and, on the other hand, leads to an explicit physical interpretation of DBs existence (Sec. IV).

II. METHOD OF PAIR SYNCHRONIZATION

Below we present the method of pair synchronization (PS-method) for discrete breather construction in nonlinear Hamiltonian lattices. Despite of the applicability of this method for wide class of such lattices, we prefer to demonstrate it with a simple case of the monoatomic chain of the linear coupled Duffing oscillators for clarity of exposition [15]. The generalization of the PS-method for other lattices seems to be more or less straightforward.

The term "method of pair synchronization" is connected with the specific nature of its algorithm. Indeed, the PS-method represents an iterative procedure which implies subsequent synchronization of the vibrations of each pair of the breather’s particles.

The dynamical equations for the chain of the hard Duffing oscillators with linear coupling can be written as follows

\[ \ddot{x}_i + x_i + x_i^3 = \gamma(x_{i+1} - 2x_i + x_{i-1}), \quad i = 1..N. \]  

(1)

Introducing the periodic boundary conditions

\[ x_{N+1}(t) = x_1(t), \quad x_0(t) = x_N(t), \]  

(2)

we will consider the simple symmetric breathers in the chains (1) with \( N = 3, 5, 7, 9...\) etc. particles. Because of the periodic boundary conditions [2], we can imagine that our chain represents a ring depicted in its equilibrium state in Figs. 1, 5, 15.

We start with the case \( N = 3 \) (see Fig. 1). Let the first particle be the central particle of the breather and, therefore, its amplitude is greater than amplitudes of the peripheral particles, i.e.

\[ |x_1(t)| > |x_2(t)|, |x_3(t)|. \]  

(3)

Moreover, the relation \( x_2(t) = x_3(t) \) must hold as far as we search for the symmetric breather.

To construct such breather, the following initial conditions can be used for integrating the dynamical equations [1]:

\[ x_1(0) = x_2(0) = x_3(0) = 0, \quad \dot{x}_1(0) = \alpha_1, \quad \dot{x}_2(0) = \dot{x}_3(0) = \alpha_2. \]  

(4)
FIG. 1: Scheme of the initial conditions for symmetric breather in the chain with $N = 3$ particles and periodic boundary conditions: $x_1(0) = x_2(0) = x_3(0) = 0; \dot{x}_1(0) = \alpha_1, \dot{x}_2(0) = \alpha_2, \dot{x}_3(0) = \alpha_3$.

FIG. 2: Start of the pair synchronization method for the chain with $N = 3$ particles $[\dot{x}_1(0) = 2, \dot{x}_2(0) = \dot{x}_3(0) = 0]$.

FIG. 3: The functions $x_1(t)$ and $x_2(t)$ for different values of $\dot{x}_2(0) = \alpha_2$ for the model with $\gamma = 0.3 \ [\alpha^{(0)}_2 = -0.7, \alpha^{(1)}_2 = -0.4, \alpha^{(2)}_2 = -0.1]$.

Thus, we suppose that all particles of the chain are in their equilibrium positions at the
initial instant, but they possess certain velocities which we denote by $\alpha_i$ ($i = 1..N$).

There are two parameters from which our breathers depend on, namely, $\gamma$ entering the dynamical equations (1) and the initial velocity $\alpha_1$ of the central breather’s particle. Choosing some reasonable values of these parameters and supposing $\alpha_2 = 0$, we integrate equations (1) for $N = 3$ over time interval containing, at least, one zero of the function $x_1(t)$ and one zero of $x_2(t)$ (see Fig. [2]). Now let us vary the initial velocity $\dot{x}_2(0) = \alpha_2$ of the second particles aiming at coincidence of the first zero of $x_2(t)$ with that of $x_1(t)$. In other words, we try to synchronize the oscillations of the first particle $[x_1(t)]$ with those of two neighboring particles $[x_2(t) \equiv x_3(t)]$.

Different numerical methods can be applied for such purpose, for example, the shooting method using the idea of dichotomy. Zeros of the functions $x_1(t)$ and $x_2(t)$ can be found by the Newton-Raphson method or by some other well-known numerical methods.

Some comments are appropriate at this stage.

Let $i_1 = 1, 2, 3, ...$ labels the subsequent zeros of the function $x_1(t)$ and $i_2 = 1, 2, 3, ...$ labels those of the function $x_2(t)$. In principle, we can superpose zeros of the functions $x_1(t)$ and $x_2(t)$ with different values $i_1$ and $i_2$. As a consequence, we obtain discrete breathers of different types (see Sec. [III]).

In this section, we consider only the single-frequency breathers, i.e. the breathers for which all particles of the chain oscillate with one and the same frequency $\omega_j = \omega_b$ ($j = 1..N$). This is not a general case. Indeed, one can imagine two-frequency, three-frequency, etc. breathers whose particles possess different, but multiple frequencies (for example, if $\omega_1 = 2\omega_2$.
or $\omega_1 = 3\omega_2$ for the chain with $N = 3$ particles, we have two-frequency breather). In such a case, the breather frequency $\omega_b$ is equal to the *minimal* frequency among all possible $\omega_j$ ($j = 1..N$) of the individual particles. Therefore, all the particles vibrate with the frequencies which prove to be multiples of $\omega_b$.

In order to avoid some misunderstanding, let us note that the single-frequency breather is not *monochromatic*, i.e. although all the breather’s particles possess identical vibrational frequencies, the Fourier series of periodic functions $x_i(t)$ contains not one, but a certain set of different harmonics.

Some comments on the terminology are appropriate at this point. The terms single-frequency and many-frequency breathers, which we use in the present paper, disagree with those used in [8]. Indeed, in the latter paper, a single-frequency discrete breather means the conventional strictly time-periodic dynamical object, while a many-frequency breather describes a *quasiperiodic* motion on an $n$-dimensional torus ($n$ basic frequencies correspond to its Fourier decomposition). In contrast, considering only strictly *time periodic* breathers, we divide them into *different classes*: in single-frequency breathers, all the particles vibrate with one and the same frequency, while in many-frequency breathers, different particles possess *different*, but divisible frequencies (see below for more details).

Let us return to the discussion of the numerical procedure of the pair synchronization for the chain with $N = 3$ particles.

In Fig. 3 we depict the function $x_2(t)$ for several values $\alpha_2$ of the initial velocity of the second particle. From this figure, one can see how the position of the first zero of $x_2(t)$ changes with changing $\dot{x}_2(0) = \alpha_2$. Note that changing of $\alpha_2$ leads not only to a shift of the zero of $x_2(t)$, but also to a certain shift of $x_1(t)$, but the latter turns out to be rather small [16] and is not noticeable in Fig. 3.

It is obvious from Fig. 3 that $\alpha_2^{(1)}$ and $\alpha_2^{(2)}$ provide a "fork" to which we can apply the method of dichotomy to attain concidence of the zeros of the functions $x_1(t)$ and $x_2(t)$ with a chosen level of accuracy. The final result of the discussed synchronization is shown in Fig. 4, where one can see a single-frequency breather for the chain with $N = 3$ particles.

The synchronization of the vibrations of the first and second particles we call $S(1,2)$ procedure. Complete realization of this procedure for the case of the single-frequency breather leads to the full coincidence of all the zeros of the functions $x_1(t)$ and $x_2(t)$ with equal labels $i_1$ and $i_2$. In general case, the similar procedure of synchronization of the vibrations of $i$-th
and \( j \)-th particles will be called \( S(i, j) \) procedure.

Now, let us construct a single-frequency symmetric breather for the chain (1) with \( N = 5 \) particles (see Fig. 5). Choosing resonable values of \( \gamma \) and \( \alpha_1 \), we have to find such values of \( \alpha_2 \) and \( \alpha_3 \) which lead to the equal vibrational frequencies of all the particles of our chain. In other words, we want to synchronize vibrations of the central particle \([x_1(t)]\) with those of peripheral particles \([x_2(t) = x_5(t) \) and \( x_3(t) = x_4(t)\)].

Let us apply the method of pair synchronization for discrete breather construction in the considered chain. The algorithm of the method can be described as follows. We synchronize vibrations of the particles 1 and 2 with aid of the above described procedure \( S(1, 2) \). Then we synchronize vibrations of the particles 2 and 3 with the aid of the procedure \( S(2, 3) \). The sequence of these two procedures, \( S(1, 2), S(2, 3) \), forms one iterative cycle of the pair synchronization method. We must repeat such cycles, namely, \( S(1, 2); S(2, 3); S(1, 2); S(2, 3); S(1, 2); S(2, 3); ... \), up to attain a desirable level of accuracy. Indeed, the coincidence of zeros of \( x_1(t) \) and \( x_2(t) \), after finishing the procedure \( S(1, 2) \), will be slightly disturbed because of applying the procedure \( S(2, 3) \) which changes \( \alpha_2, \alpha_3 \), and we must recorrect the accuracy of this coincidence by the next iterative cycle. One can hope for convergence of such iterative process, at least, for the case of sufficiently strong space localization of the discrete breather.

The generalization of the considered method to a chain with arbitrary \( N \) is straightforward: we repeat iterative cycles

\[
\{S(1, 2); S(2, 3); S(3, 4); ... S(N - 1, N)\}
\]

up to obtaining the discrete breather with resonable precision. Note that the method of pair synchronization can be regarded as a version of the relaxation procedure used in many different numerical methods.
Note that considering chains with subsequently increasing number of particles \(N = 3, 5, 7, 9, 11,\ldots\), we can use the final set of initial velocities \((\alpha_1, \alpha_2, \alpha_3,\ldots)\) of the chain with \(N = m\) in order to begin calculations for the chain with \(N = m + 2\), assuming two last velocities equal to zero.

### III. SOME NUMERICAL RESULTS

Below, we present some results of the discrete breathers construction with the aid of the pair synchronization method for a number of mechanical models and for different kinds of discrete breathers. The application of this method to other oscillatory chains and multidimensional lattices will be discussed in future publications.

#### A. Linear coupled Duffing oscillatory chain

Let us consider the model (1) for different values of the coupling parameter \(\gamma\) and different values of the initial velocity \(\alpha_1\) of the central breather’s particle.

The appropriate initial velocities \(\dot{x}_i(0) = \alpha_i, \ i = 2..N\) are necessary for a start of the pair synchronization procedure. For \(\gamma \lesssim 1\) one can obtain these velocities from the approximate formula, which is derived in Sec. IV:

\[
\alpha_{i+1} = -\frac{\gamma}{\omega_b^2 - \omega_0^2} \alpha_i, \quad (i = 1..N - 1). \tag{6}
\]

Here \(\omega_b\) is the breather frequency, which can be roughly approximated as \(\omega_b = \omega(\alpha_1)\) with \(\omega(\alpha_1)\) being the frequency of a single Duffing oscillator, while \(\omega_0^2 = 1 + 2\gamma\). The function \(\omega(\alpha)\) is determined by the solution of the Duffing equation \(\ddot{x} + x + x^3 = 0\) and we depict it in Fig. 6 for obtaining rough estimates of the breather frequency \(\omega_b\).

In Table I we present the velocity and amplitude breather profiles, as well as the breather frequency \(\omega_b\), for the chains with different number of particles \(N = 3, 5, 7, 9, 11, 13\) for \(\gamma = 0.3, \ \alpha_1 = 2\). In the first column of each fragment of this table the indices \((n)\) of the particles are given (for the central breather particle \(n = 0\)). In the following columns, we present the initial velocity \((\alpha_n)\) and vibrational amplitude \((A_n)\) of each particle of the considered chain.

From this table, one can see how the discrete breather for the infinite chain acquires its shape step-by-step with increasing \(N\). We restrict our consideration to \(N = 13\) because,
FIG. 6: The function $\omega(\alpha)$ determining dependence of the Duffing oscillator frequency $\omega$ on the initial velocity $\alpha$.

FIG. 7: The amplitudes $A_n$ for the symmetric breathers in the Duffing chain (1) for different values $\gamma$ (in the logarithmic scale).

In the case $\gamma = 0.3$, the amplitudes of the particles which are the most distant from the breather’s center do not exceed $10^{-4}$ of the amplitude of the central particle.

In Table II, we present information similar to that in Table I for the chain with $N = 9$ particles, but for different values of the coupling parameter $\gamma$ ($0.3 \leq \gamma \leq 1.5$). From this table, one can see that a degree of localization becomes worse with increasing $\gamma$.

Because the amplitudes $A_n$ of the particles decrease rapidly with increasing their distance from the breather center, we depict them in the logarithmic scale for different $\gamma$ in Fig. 7. From this picture one can see the exponential decay of the amplitudes of the peripheral particles. The degree of the breather space localization decreases with increasing $\gamma$.

Let us note that the pair synchronization method can be often used not only for suffi-
TABLE I: Discrete breathers for the Duffing chain (1) for $\gamma = 0.3$ and different number of particles ($N$).

| $N = 3; \omega_b = 1.70714$ | $N = 5; \omega_b = 1.70670$ | $N = 7; \omega_b = 1.70711$ |
|-----------------------------|-----------------------------|-----------------------------|
| $n$ | $\alpha_n$ | $A_n$ | $n$ | $\alpha_n$ | $A_n$ | $n$ | $\alpha_n$ | $A_n$ |
| 0  | 1.261441 | 0 | -2 | 0.099350 | 0.058247 | 0 | 1.29353 | 0.075822 |
| -1 | 0.406807 | 0.240930 | -1 | 0.530612 | 0.313171 | -1 | 0.538053 | 0.317510 |
| 0  | 2 | 1.254590 | 0  | 2 | 1.254181 |
| 1  | 0.406807 | 0.240930 | 1  | 0.530612 | 0.313171 | 1  | 0.538053 | 0.317510 |
| 2  | 0.099350 | 0.058247 | 2  | 0.129353 | 0.075822 |
| 3  | 0.000339 | 0.000197 |

$N = 9; \omega_b = 1.70714$ | $N = 11; \omega_b = 1.70714$ | $N = 13; \omega_b = 1.70715$ |
|-----------------------------|-----------------------------|-----------------------------|
| $n$ | $\alpha_n$ | $A_n$ | $n$ | $\alpha_n$ | $A_n$ | $n$ | $\alpha_n$ | $A_n$ |
| 0  | 2 | 1.254159 | 0  | 2 | 1.254160 |
| 1  | 0.538494 | 0.317761 | 1  | 0.538519 | 0.317776 | 1  | 0.538520 | 0.317777 |
| 2  | 0.131215 | 0.076909 | 2  | 0.131221 | 0.076913 |
| 3  | 0.131113 | 0.076849 | 3  | 0.131221 | 0.076913 |
| 4  | 0.00581 | 0.003404 | 4  | 0.007555 | 0.004426 | 4  | 0.007555 | 0.004426 |
| 5  | 0.001403 | 0.000822 | 5  | 0.001825 | 0.001069 |
| 6  | 0.000339 | 0.000197 |

ciently small coupling between the oscillators of the lattice. From Table II we see that this method also works, in the case of the model (1), for $\gamma \sim 1$. Moreover, the pair synchronization method can be used for $\gamma > 1$ and even for $\gamma \gg 1$ (in such cases formula (6) cannot be used for obtaining an initial velocity profile). As an example, let
TABLE II: Discrete breathers for the Duffing chain (1) for different values of the coupling parameter $\gamma$ ($N = 9$).

| $\gamma = 0.3; \omega_b = 1.70714$ | $\gamma = 0.5; \omega_b = 1.85545$ | $\gamma = 0.7; \omega_b = 2.03582$ |
|---|---|---|
| $\begin{array}{c|c|c} n \backslash \alpha_n & A_n \end{array}$ | $\begin{array}{c|c|c} n \backslash \alpha_n & A_n \end{array}$ | $\begin{array}{c|c|c} n \backslash \alpha_n & A_n \end{array}$ |
| -4 | 0.005812 | 0.003404 | -4 | 0.065523 | 0.022544 | -4 | 0.100398 | 0.049322 |
| -3 | -0.031273 | 0.018319 | -3 | -0.172127 | 0.087583 | -3 | -0.350110 | 0.172155 |
| -2 | 0.131113 | 0.076849 | -2 | 0.428520 | 0.229276 | -2 | 0.759077 | 0.374592 |
| -1 | -0.538494 | 0.317761 | -1 | -1.022931 | 0.557273 | -1 | -1.417113 | 0.706940 |
| 0  | 2  | 1.254159 | 0  | 2  | 1.128836 | 0  | 2  | 1.013062 |
| 1  | -0.538494 | 0.317761 | 1  | -1.022931 | 0.557273 | 1  | -1.417113 | 0.706940 |
| 2  | 0.131113 | 0.076849 | 2  | 0.428520 | 0.229276 | 2  | 0.759077 | 0.374592 |
| 3  | -0.031273 | 0.018319 | 3  | -0.172127 | 0.087583 | 3  | -0.350110 | 0.172155 |
| 4  | 0.005812 | 0.003404 | 4  | 0.065523 | 0.022544 | 4  | 0.100398 | 0.049322 |

| $\gamma = 1; \omega_b = 2.29177$ | $\gamma = 1.2; \omega_b = 2.44835$ | $\gamma = 1.5; \omega_b = 2.66735$ |
|---|---|---|
| $\begin{array}{c|c|c} n \backslash \alpha_n & A_n \end{array}$ | $\begin{array}{c|c|c} n \backslash \alpha_n & A_n \end{array}$ | $\begin{array}{c|c|c} n \backslash \alpha_n & A_n \end{array}$ |
| -4 | 0.175463 | 0.076581 | -4 | 0.212225 | 0.086704 | -4 | 0.250808 | 0.094053 |
| -3 | -0.569282 | 0.248818 | -3 | -0.669348 | 0.273867 | -3 | -0.769750 | 0.289048 |
| -2 | 1.077299 | 0.472699 | -2 | 1.200646 | 0.492969 | -2 | 1.312950 | 0.494406 |
| -1 | -1.669057 | 0.737848 | -1 | -1.738912 | 0.717933 | -1 | -1.793009 | 0.677654 |
| 0  | 2  | 0.889146 | 0  | 2  | 0.828581 | 0  | 2  | 0.757359 |
| 1  | -1.669057 | 0.737848 | 1  | -1.738912 | 0.717933 | 1  | -1.793009 | 0.677654 |
| 2  | 1.077299 | 0.472699 | 2  | 1.200646 | 0.492969 | 2  | 1.312950 | 0.494406 |
| 3  | -0.569282 | 0.248818 | 3  | -0.669348 | 0.273867 | 3  | -0.769750 | 0.289048 |
| 4  | 0.175463 | 0.076581 | 4  | 0.212225 | 0.086704 | 4  | 0.250808 | 0.094053 |

us give the set of the initial velocities $\dot{x}_i(0) = \alpha_i$ obtained by this method for the single-frequency discrete breather in the chain (1) with $N = 7$ particles for $\gamma = 20$: $\{\alpha_1 = 2, \alpha_2 = -1.801427, \alpha_3 = 1.245828, \alpha_4 = -0.444403\}$. Existence and stability of the discrete breathers in chains with large coupling will be discussed elsewhere.
FIG. 8: Two-frequency symmetric breather in the Duffing chain \( \mathbf{1} \) for \( \gamma = 0.3, \ \alpha_1 = 7, \ N = 5 \).

B. Many-frequency discrete breathers

First of all let us comment on one point of the procedure of pair synchronization. Considering this procedure in Sec. II we have aimed at coincidence of those zeros of two functions, \( x_i(t) \) and \( x_j(t) \), which are most close to the initial instant \( t = 0 \). This is not a general case. Indeed, one can imagine that \( x_i(t) \) and \( x_j(t) \) possess equal periods \( T_i = T_j = T \), but that they do not go to zero simultaneously at any internal instant \( 0 < t < T \). Then we should aim at coincidence of the second zeros of the considered functions, i.e. the relation \( \dot{x}_i(T) = \dot{x}_j(T) = 0 \) must hold.

Now let us focus on constructing many-frequency discrete breathers. Let us remember that we coin this term for such spatially localized dynamical objects whose particles vibrate with several different but divisible frequencies. This means that many-frequency breather proves to be a time-periodic object whose period \( T \) is the largest of the multiple periods \( T_1, T_2, ..., T_N \) of all the particles of the chain.

In Fig. 8 we depict a two-frequency symmetric breather \( (\omega_1 = 2\omega_2 = 2\omega_3) \) in the chain \( \mathbf{1} \) with \( N = 5 \) particles for \( \gamma = 0.3 \). Indeed, the first particle \( [x_1(t)] \) vibrates with the frequency twice larger than those of all other particles \( [x_2(t), x_3(t)] \). This breather with \( \omega_b = \omega_2(= \omega_3) = 1.39615 \) corresponds to the following data: \( \gamma = 0.3, \ \alpha_1 = 7, \ \alpha_2 = -1.064224, \ \alpha_3 = 0.352247 \). Hereafter, we do not point out explicitly that all the initial displacements are zero: \( x_i(0) = 0, \ i = 1..N \).

Note that for the same \( \gamma \) and \( \alpha_1 \), we can find a single-frequency breather \( (\omega_1 = \omega_2 = \omega_3 = 2.77828) \), as well (see Fig. 9 which was obtained for \( \gamma = 0.3, \ \alpha_1 = 7, \ \alpha_2 = -0.379206, \)
FIG. 9: Single-frequency symmetric breather in the Duffing chain (1) for $\gamma = 0.3$, $\alpha_1 = 7$, $N = 5$. Vibrations of the third particle is not visible, because of its small amplitude.

FIG. 10: Two-frequency symmetric breather in the Duffing chain (1) for $\gamma = 0.3$, $\alpha_1 = 19$, $N = 5$ ($\omega_1 = 3\omega_2 = 3\omega_3$). This breather is unstable.

$\alpha_3 = 0.017876$, $N = 5$).

In Fig. 10 we depict the two-frequency breather with $\omega_b = \omega_2 = 1.48666$ whose particles possess the following frequencies: $\omega_1 = 3\omega_2 = 3\omega_3$. This breather corresponds to the following initial data: $\gamma = 0.3$, $\alpha_1 = 19$, $\alpha_2 = -1.593445$, $\alpha_3 = 0.456080$, $N = 5$.

In Fig. 11 a three-frequency breather with $\omega_1 = 2\omega_2$, $\omega_2 = 2\omega_3$ is shown. It corresponds to the following initial data: $\gamma = 0.3$, $\alpha_1 = 28$, $\alpha_2 = -7.018426$, $\alpha_3 = 1.472607$, $N = 5$. The breather frequency is $\omega_b = \omega_3 = 1.35108$.

Note that the pair synchronization method providing us with an initial breather profile does not guarantee the breather stability. For example, the breather depicted in Figs. 10, 11 proves to be unstable dynamical object, and this fact can be checked by straightforward
FIG. 11: Three-frequency symmetric breather in the Duffing chain (1) for $\gamma = 0.3$, $\alpha_1 = 28$, $N = 5$ ($\omega_1 = 2\omega_2 = 4\omega_3$). This breather is unstable.

integration of the dynamical equation (1), as well as with the aid of the Floquet method.

Finally, in Fig. 12, we demonstrate a nontrivial example of two-frequency breathers ($\gamma = 0.3$, $\alpha_1 = 4$, $N = 5$ ($\omega_1 = 2\omega_2 = 2\omega_3 = 2\omega_4$).

Note that, for simplicity, in many examples of this section we consider only chains with small number of particles ($N$). This is not an essential restriction because, in every such case, the synchronization procedure can be easily continued for much greater values of $N$.

Let us emphasize once more that, unlike the terminology of the paper [8], our many-
frequency breathers are time-periodic dynamical objects, but different particles vibrate with
different (divisible) frequencies $\omega_i$ ($i = 1..N$). We would like to note that one can found in [2]
a certain mention of multibreathers whose particles vibrate with different but commensurate
frequencies. However, we do not know any papers with detailed analysis of such dynamical
objects and it seems that our many-frequency breathers are not fully identical with the
above mentioned multibreathers.

C. Constructing of antisymmetric breathers

Up to this point, we have considered only the symmetric discrete breathers, i.e. breathers
of the form $\{\ldots x_3(t), x_2(t), x_1(t), x_0(t), x_1(t), x_2(t), x_3(t), \ldots \}$. Often they are called Sievers-
Takeno modes.

The pair synchronization method can be also used for constructing antisymmetric
breathers which possess the following form $\{\ldots -x_3(t), -x_2(t), -x_1(t), x_1(t), x_2(t), x_3(t), \ldots \}$. Such
breathers are usually called Page modes. In Table III we present initial velocities ($\alpha_i$) and
amplitudes ($A_i$) of all particles of the antisymmetric breather for the Duffing chain (1)
with $N = 8$ particles. One can see the time evolution of this breather in Fig. 13.
TABLE III: A single-frequency antisymmetric breather in the Duffing chain (1) for different values of the coupling parameter $\gamma$ ($N = 8$).

| $\gamma = 0.3; \omega_b = 1.74894$ | $\gamma = 0.5; \omega_b = 1.89609$ | $\gamma = 0.7; \omega_b = 2.05935$ |
|----------------------------------|----------------------------------|----------------------------------|
| $n$                              | $\alpha_n$                       | $A_n$                            | $n$                              | $\alpha_n$ | $A_n$ | $n$                              | $\alpha_n$ | $A_n$ |
| -4                               | 0.026418                         | 0.015106                         | -4                               | 0.137987 | 0.072790 | -4                               | 0.379916 | 0.184678 |
| -3                               | -0.102000                        | 0.058347                         | -3                               | -0.300923 | 0.158896 | -3                               | -0.604583 | 0.294409 |
| -2                               | 0.466790                         | 0.268490                         | -2                               | 0.806715 | 0.428828 | -2                               | 1.148316 | 0.563216 |
| -1                               | -2                               | 1.215605                         | -1                               | -2       | 1.100366 | -1                               | -2       | 1.000219 |
| 1                                | 2                                | 1.215605                         | 1                                | 2        | 1.100366 | 1                                | 2        | 1.000219 |
| 2                                | -0.466790                        | 0.268490                         | 2                                | -0.806715 | 0.428828 | 2                                | -1.148316 | 0.563216 |
| 3                                | 0.102000                         | 0.058347                         | 3                                | 0.300923 | 0.158896 | 3                                | 0.604583 | 0.294409 |
| 4                                | -0.026418                        | 0.015106                         | 4                                | -0.137987 | 0.072790 | 4                                | -0.379916 | 0.184678 |

| $\gamma = 1; \omega_b = 2.31566$ | $\gamma = 1.2; \omega_b = 2.48400$ | $\gamma = 1.4; \omega_b = 2.64741$ |
|----------------------------------|----------------------------------|----------------------------------|
| $n$                              | $\alpha_n$ | $A_n$ | $n$                              | $\alpha_n$ | $A_n$ | $n$                              | $\alpha_n$ | $A_n$ |
| -4                               | 0.939477 | 0.407302 | -4                               | 1.388243 | 0.562493 | -4                               | 1.873498 | 0.714152 |
| -3                               | -1.161233 | 0.504527 | -3                               | -1.540839 | 0.625280 | -3                               | -1.909497 | 0.728140 |
| -2                               | 1.577787 | 0.689037 | -2                               | 1.790982 | 0.728834 | -2                               | 1.961880 | 0.748509 |
| -1                               | -2       | 0.879356 | -1                               | -2       | 0.816066 | -1                               | -2       | 0.763356 |
| 1                                | 2        | 0.879356 | 1                                | 2        | 0.816066 | 1                                | 2        | 0.763356 |
| 2                                | -1.577787 | 0.689037 | 2                                | -1.790982 | 0.728834 | 2                                | -1.961880 | 0.748509 |
| 3                                | 1.161233 | 0.504527 | 3                                | 1.540839 | 0.625280 | 3                                | 1.909497 | 0.728140 |
| 4                                | -0.939477 | 0.407302 | 4                                | -1.388243 | 0.562493 | 4                                | -1.873498 | 0.714152 |

D. Other types of oscillatory chains

Chains of the coupled Duffing oscillators are used for some physical applications. For example in [9], the model of such a type is exploited for describing discrete breathers in cantilever arrays. Indeed, the authors of that paper consider an oscillatory chain with on-site and inter-site forces both containing linear and cubic terms. Now we want to demonstrate that the pair synchronization method can be applied for the breather construction in the different chains of such class of mechanical systems.
We will consider the following two models:

\[ \ddot{x}_i + x_i + x_i^3 = \gamma [(x_{i+1} - x_i)^3 - (x_i - x_{i-1})^3], \]  
\( (7) \)

and

\[ \ddot{x}_i + x_i^3 = \gamma [(x_{i+1} - x_i)^3 - (x_i - x_{i-1})^3], \]  
\( (8) \)

with periodic boundary conditions \( (2) \).

The model \( (7) \) contrary to \( (1) \) represents the chain of the Duffing oscillators with cubic coupling. The model \( (8) \) represents the so-called \( K_4 \) chain, i.e. the chain whose potential energy is a uniform function of the fourth order. Discrete breathers in the \( K_4 \) chains with and without on-site interactions were studied in many papers from different points of view \[8, 10, 11\]. This chain is very convenient for investigation because it allows separation of space and time variables. In our paper \[6\], the symmetry discrete breathers were found for the model \( (8) \) with high precision using the concept of the similar nonlinear normal modes introduced by Rosenberg in \[12\]. The detailed investigation of the breather stability is also presented in \[6\].

In Table IV, we present the velocity and amplitude profiles for the chain \( (7) \) for some values of the coupling parameter \( \gamma \) in the chain with \( N = 7 \) particles. It is worth comparing this table with that for the linear coupled Duffing oscillators (see Table II). One can see that the replacing of the linear coupling of the model \( (1) \) on the cubic coupling of the model \( (7) \) changes the degree of the breather spatial localization.

In Table V we present the similar information for the \( K_4 \) chain, i.e. for the chain with cubic force interactions and without the linear on-site forces. As was already mentioned, for the latter model there exist Rosenberg nonlinear normal modes. For these modes, the displacement of every particle of the chain at any instant \( t \) is proportional to the displacement of an arbitrary chosen particle, say, the first particle. Therefore, for the dynamical regime described by a Rosenberg mode one can write

\[ x_i(t) = k_i \cdot x_1(t), \quad i = 1..N, \]  
\( (9) \)

where \( k_i \) are the constant coefficients which can be determined from a certain systems of nonlinear algebraic equations (see \[6\] for the further details).

From \( (9) \), one can see that \( \dot{x}_j(t) = k_i \dot{x}_1(t), \quad i = 1..N \) and, therefore, \( \dot{x}_i(0) = k_i \dot{x}_1(0) \). In turn, this means that the initial velocity profile is proportional to the spatial profile of the
TABLE IV: Symmetric discrete breathers for the chain \([\mathbf{7}]\) for different values of the coupling parameter \(\gamma\) \((N = 7)\).

| \(\gamma = 0.3; \omega_b = 1.86714\) | \(\gamma = 0.5; \omega_b = 1.89609\) | \(\gamma = 0.7; \omega_b = 2.22539\) |
|---|---|---|
| \(n\) | \(\alpha_n\) | \(A_n\) | \(n\) | \(\alpha_n\) | \(A_n\) | \(n\) | \(\alpha_n\) | \(A_n\) |
| -3 | \(10^{-8}\) | \(10^{-7}\) | -3 | \(-1.7 \times 10^{-7}\) | \(9.3 \times 10^{-8}\) | -3 | \(-4 \times 10^{-7}\) | \(2 \times 10^{-7}\) |
| -2 | 0.0071987 | 0.0043733 | -2 | 0.017014 | 0.0094272 | -2 | 0.023577 | 0.012211 |
| -1 | -0.598577 | 0.363639 | -1 | -0.776917 | 0.430465 | -1 | -0.857977 | 0.444358 |
| 0 | 2 | 1.215014 | 0 | 2 | 1.108135 | 0 | 2 | 1.035827 |
| 1 | -0.598577 | 0.363639 | 1 | -0.776917 | 0.430465 | 1 | -0.857977 | 0.444358 |
| 2 | 0.0071987 | 0.0043733 | 2 | 0.017014 | 0.0094272 | 2 | 0.023577 | 0.012211 |
| 3 | \(10^{-8}\) | \(10^{-7}\) | 3 | \(-1.7 \times 10^{-7}\) | \(9.3 \times 10^{-8}\) | 3 | \(-4 \times 10^{-7}\) | \(2 \times 10^{-7}\) |

\[\gamma = 1; \omega_b = 2.41195\] \(\gamma = 1.2; \omega_b = 2.51505\] \(\gamma = 1.5; \omega_b = 2.64864\]}

Rosenberg mode which was found for some range of \(\gamma\) in \([\mathbf{6}]\). The results of that paper and those from Table \([\mathbf{V}]\) of the present paper, obtained by the method of pair synchronization, proved to be identical.

IV. A SIMPLE PHYSICAL INTERPRETATION OF THE PAIR SYNCHRONIZATION PROCEDURE

In Sec. \([\mathbf{III}]\) we have described the method of pair synchronization for discrete breather construction. At each step of this method, we force two particles, who vibrate with essentially different amplitudes, to have exactly identical frequencies. Keeping in mind that in nonlinear
TABLE V: Symmetric discrete breathers for the chain (8) for different values of the coupling parameter $\gamma$ ($N = 5$).

| $\gamma$ | $\omega_b$ | $\alpha_n$ | $A_n$ |
|----------|------------|------------|-------|
| $\gamma = 0.3$ | $\omega_b = 1.86714$ | $n$ | $\alpha_n$ | $A_n$ |
| -2 | 0.0071987 | 0.0043733 | -2 | 0.017014 | 0.010332 | -2 | 0.023577 | 0.013202 |
| -1 | -0.598577 | 0.363639 | -1 | -0.776917 | 0.471793 | -1 | -0.857977 | 0.480441 |
| 0 | 2 | 1.215014 | 0 | 2 | 1.214528 | 0 | 2 | 1.119939 |
| 1 | -0.598577 | 0.363639 | 1 | -0.776917 | 0.471793 | 1 | -0.857977 | 0.480441 |
| 2 | 0.0071987 | 0.0043733 | 2 | 0.017014 | 0.010332 | 2 | 0.023577 | 0.013202 |
| $\gamma = 0.5$ | $\omega_b = 1.97301$ | $\alpha_n$ | $A_n$ |
| -2 | 0.0071987 | 0.0043733 | -2 | 0.017014 | 0.010332 | -2 | 0.023577 | 0.013202 |
| -1 | -0.598577 | 0.363639 | -1 | -0.776917 | 0.471793 | -1 | -0.857977 | 0.480441 |
| 0 | 2 | 1.215014 | 0 | 2 | 1.214528 | 0 | 2 | 1.119939 |
| 1 | -0.598577 | 0.363639 | 1 | -0.776917 | 0.471793 | 1 | -0.857977 | 0.480441 |
| 2 | 0.0071987 | 0.0043733 | 2 | 0.017014 | 0.010332 | 2 | 0.023577 | 0.013202 |
| $\gamma = 0.7$ | $\omega_b = 2.13965$ | $\alpha_n$ | $A_n$ |
| -2 | 0.0071987 | 0.0043733 | -2 | 0.017014 | 0.010332 | -2 | 0.023577 | 0.013202 |
| -1 | -0.598577 | 0.363639 | -1 | -0.776917 | 0.471793 | -1 | -0.857977 | 0.480441 |
| 0 | 2 | 1.215014 | 0 | 2 | 1.214528 | 0 | 2 | 1.119939 |
| 1 | -0.598577 | 0.363639 | 1 | -0.776917 | 0.471793 | 1 | -0.857977 | 0.480441 |
| 2 | 0.0071987 | 0.0043733 | 2 | 0.017014 | 0.010332 | 2 | 0.023577 | 0.013202 |

regimes, particles vibrating with different amplitudes possess, as a rule, different frequencies, one can wonder what is the physical nature of the possibility of the above discussed pair synchronization? The understanding of this nature leads us to the explicit answer to the question formulated in the introduction of the present paper: "How can discrete breathers exist as space localized and, at the same time, strictly time-periodic dynamical objects"?

Let us consider the chain (1) for $N = 3$ and write down explicitly the corresponding dynamical equations:

\[
\ddot{x}_1 + x_1 + x_3^3 = \gamma (x_2 - 2x_1 + x_3), \\
\ddot{x}_2 + x_2 + x_3^3 = \gamma (x_3 - 2x_2 + x_1), \\
\ddot{x}_3 + x_3 + x_3^3 = \gamma (x_1 - 2x_3 + x_2). \tag{10}
\]

These equations describe the vibrations of three Duffing oscillators with linear coupling and parameter $\gamma$ determines the strength of this coupling. The relation $x_2(t) = x_3(t)$ holds, since we search for a symmetric breather. Therefore, three equations (10) reduce to the
following two equations:
\[ \ddot{x}_1 + x_1 + x_1^3 = 2\gamma(x_2 - x_1), \quad (11) \]
\[ \ddot{x}_2 + x_2 + x_2^3 = \gamma(x_1 - x_2). \quad (12) \]

We suppose
\[ 0 < \gamma \ll 1, \quad |x_2(t)| \ll |x_1(t)|. \quad (13) \]

Actually, the last relation is the condition of strong space localization of our three-particle breather. Then it is reasonable to neglect the term \( x_3^2 \) in Eq. (12) and the term \( 2\gamma x_2 \) in Eq. (11). As a consequence of such simplification, Eqs. (11,12) read \[18\]
\[ \ddot{x}_1 + (1 + 2\gamma)x_1 + x_1^3 = 0, \quad (14) \]
\[ \ddot{x}_2 + (1 + \gamma)x_2 = \gamma x_1. \quad (15) \]

Eq. (14) represents the Duffing equation whose solution can be expressed via Jacobi elliptic functions, while Eq. (15) is the equation of the harmonic oscillator with \textit{periodic external force} \( \gamma \cdot x_1(t) \).

Let us make a next step of simplification replacing periodic function \( x_1(t) \) in (15) by its first Fourier harmonic or, more precisely, let us assume
\[ x_1(t) = A\sin(\omega t) + B\cos(\omega t), \quad (16) \]
where the correct frequency \( \omega \) can be obtained as a result of solving the Duffing equation (14).

From the initial conditions
\[ x_1(0) = 0, \quad \dot{x}_1(0) = \alpha_1, \quad (17) \]
we find for the solution (16) that \( A = \frac{\alpha_1}{\omega}, \quad B = 0 \) and, therefore,
\[ x_1(t) = \frac{\alpha_1}{\omega} \sin(\omega t). \quad (18) \]

In Fig. 14 we depict the exact solution to Eq. (14) (by a solid line) and the approximate solution (18) (by a dotted line) for \( \gamma = 0.1, \quad \alpha_1 = 2 \). From this figure, one can see how close can be to each other the above two solutions.

Substituting (18) into r.h.s. of Eq. (15), we can write
\[ \ddot{x}_2(t) + \omega_0^2 \cdot x_2 = \frac{\gamma \alpha_1}{\omega} \sin(\omega t), \quad (19) \]
where $\omega_0^2 = 1 + \gamma$. Eq. (19) is the equation of the harmonic oscillator with fundamental frequency $\omega_0$ and with periodic driven force proportional to $\sin(\omega t)$.

Taking into account the initial conditions $x_2(0) = 0$, $\dot{x}_2(0) = \alpha_2$, we easily obtain the solution to Eq. (19):

$$x_2(t) = \left[ \alpha_2 - \frac{\gamma \alpha_1}{(\omega_0^2 - \omega^2)} \right] \sin(\omega_0 t) + \frac{\gamma \alpha_1}{\omega(\omega_0^2 - \omega^2)} \sin(\omega t).$$

(20)

Here the first term containing $\sin(\omega_0 t)$ represents a solution to the equation (19) without right-hand side, i.e. a solution of the free harmonic oscillator with frequency $\omega_0$, while the second term containing $\sin(\omega t)$ represents a particular solution to the full Eq. (19) with the periodic driven force in its r.h.s.

Let us analyze the solution (20). In general case, the frequencies $\omega_0$ and $\omega$ are incommensurable. Therefore, for existence of the discrete breathers as strictly time-periodic dynamical objects it is necessary to eliminate the contribution from the ”vibrational individuality” of the peripheral particles, i.e. the coefficient in front of $\sin(\omega_0 t)$ must be equal to zero. From this condition, we find the initial velocity $\alpha_2$ of the second particle via that ($\alpha_1$) of the central particle:

$$\alpha_2 = -\frac{\gamma \alpha_1}{\omega_0^2 - \omega^2}.$$ 

(21)

Note that $\omega > \omega_0$ since the frequency of the hard Duffing oscillator increases with increasing of the vibrational amplitude. Therefore, $\alpha_2$ and $\alpha_1$ possess opposite signs.

Certainly, Eq. (21) gives us only a certain approximation to $\alpha_2$, because we take into account only one harmonic from the Fourier decomposition of the function $x_1(t)$. Thus, our
approximate breather solution is

\[ x_1(t) = \frac{\alpha_1}{\omega} \cdot \sin(\omega t), \quad (22) \]

\[ x_2(t) \equiv x_3(t) = -\frac{\gamma \alpha_1}{\omega(\omega^2 - \omega_0^2)} \cdot \sin(\omega t). \quad (23) \]

It is essential, that the vibrational amplitude of \( x_2(t) \) is \( \gamma/(\omega^2 - \omega_0^2) \) times less than the amplitude of the central particle. In a certain sense, the relation \( \gamma \ll 1 \) means the "space localization" of our breather in the three-particle chain.

Actually, obtaining (22–23) we have applied the synchronization procedure \( S(1, 2) \) (see Sec. II) in the framework of our approximation to nonlinear dynamics of the Duffing chain (1) with \( N = 3 \). Considering the chain with \( N = 5 \) particles (see Fig. 5), we must also synchronize the vibrations of the particles 2 and 3. Obviously, all arguments which brought us to Eq. (22–23) can be applied to this synchronizing. Indeed, because of Eq. (23), the second particle \( x_2(t) = x_5(t) \) already vibrates with the frequency \( \omega \) of the central particle. Therefore, the procedure \( S(2, 3) \) reduces to the finding solution for \( x_3(t) = x_4(t) \) by solving the harmonic oscillator equation for \( x_3(t) \) with time-periodic force driving exerted by the function \( x_2(t) \) from Eq. (23). Demanding full suppression of the fundamental frequency contribution of the third particle [function \( x_3(t) \)] leads us to the relation similar to (21)

\[ \alpha_3 = -\frac{\gamma \alpha_2}{\omega^2 - \tilde{\omega}_0^2}, \quad (24) \]

where \( \tilde{\omega}_0^2 = 1 + \gamma \).

Thus, the vibrational amplitudes of the third and fourth particles \( \frac{\gamma}{\omega^2 - \tilde{\omega}_0^2} \) times less than that of the particles 2 and 5 (see Fig 5).

Obviously, all this argumentation can be continued to the consideration of the breathers in the chains with arbitrary number of particles. As a result, we conclude that vibrational amplitudes of all the peripheral particles decrease, at least, by the factor \( \gamma \ll 1 \). In turn, this means that our discrete breather possesses exponential decay in vibrational amplitudes when we pass on from the central particle to the more distant (peripheral) particles.

One additional comment, concerning the fundamental frequencies \( \omega_0 = \sqrt{1 + 2\gamma} \) and \( \tilde{\omega}_0 = \sqrt{1 + \gamma} \) is appropriate at this point.

For the symmetric breather in the chain with \( N = 5 \) particles (see Fig. 5), there hold relations \( x_2(t) = x_5(t), \quad x_3(t) = x_4(t) \). Substituting these relations into the dynamical
FIG. 15: Scheme of the initial conditions for the seven-particle chain.

equations (1), we obtain:
\[ \ddot{x}_1 + x_1 + x_1^3 = 2\gamma(x_2 - x_1), \]
\[ \ddot{x}_2 + x_2 + x_2^3 = \gamma(x_3 - 2x_2 + x_1), \]
(25)
\[ \ddot{x}_3 + x_3 + x_3^3 = \gamma(x_2 - x_3). \]

Then taking into account the relations \(|x_2(t)| \ll |x_1(t)|\) and \(|x_3(t)| \ll |x_2(t)|\), which are a consequence of the breather localization, one can reduce Eqs. (25) to the following form:
\[ \ddot{x}_1 + \omega_0^2 x_1 + x_1^3 = 0, \]
\[ \ddot{x}_2 + \omega_0^2 x_2 = \gamma x_1, \]
(26)
\[ \ddot{x}_3 + \tilde{\omega}_0^2 x_3 = \gamma x_2, \]
containing slightly different fundamental frequencies \(\omega_0\) and \(\tilde{\omega}_0\).

The similar procedure for the case \(N = 7\) with taking into consideration the relations (see Fig. 15) \(x_2(t) = x_7(t), \ x_3(t) = x_6(t), \ x_4(t) = x_5(t)\), brings us to the equations
\[ \ddot{x}_1 + \omega_0^2 x_1 + x_1^3 = 0, \]
\[ \ddot{x}_2 + \omega_0^2 x_2 = \gamma x_1, \]
\[ \ddot{x}_3 + \omega_0^2 x_3 = \gamma x_2, \]
\[ \ddot{x}_4 + \tilde{\omega}_0^2 x_4 = \gamma x_3. \]
(27)

Proceeding in such a way for the chains with greater \(N\), we conclude that all approximate equations contain fundamental frequency \(\omega_0 = \sqrt{1 + 2\gamma}\) excepting only the last equation which contains the fundamental frequency \(\tilde{\omega}_0 = \sqrt{1 + \gamma}\). Bearing in mind that \(\gamma \ll 1\), we
obtain, from the equations similar to Eqs. (26) and (27), the above formulated conclusion about exponential decay of the amplitudes of the peripheral breather’s particles.

Let us summarize the above results. It has been manifested that the existence of discrete breather is connected with the ”suppressing of the individuality” of all peripheral particles by vibrations of the central particle of our breather. In other words, the central particle forces upon the peripheral particles its own rhythm of vibrations. To obtain the correct discrete breather one must tune onto this rhythm with high precision by suppressing all terms in the solution to Eq. (1) corresponding to vibrations with different frequencies of all the peripheral particles. Therefore, an exact discrete breather seems to be a very unusual dynamical object and, in practice, we deal with quasibreathers rather than with exact breathers. For the former objects, it is typical that all breathers particles vibrate with slightly different frequencies and these frequencies slightly drift in time. In turn, this means that in the spatially localized solution to Eq. (1) there exist terms with slightly different fundamental frequencies of the peripheral particles, whose amplitudes are sufficiently small.

We believe that the presence of small amplitude terms with the frequencies $\omega_j$, different from the principal breather frequency $\omega_b$, peculiar to any quasibreather solutions, cannot be the cause of the loss of their stability (see, for example, Eq. (20)). In other words, the loss of the strict periodicity does not mean, in general, the loss of the spatial localization of the considered dynamical objects. However, at the present time, we cannot give a rigorous mathematical proof of this proposition.

In conclusion, let us make two additional comments.

1. For the chains with an uniform potential, one can construct the Rosenberg nonlinear normal modes. Our chain (1) of the Duffing oscillators does not belong to such class of mechanical systems. Nevertheless, the approximate solutions obtained in this section [see, for example, Eqs. (22-23)] prove to be the Rosenberg mode, since ratios $\frac{x_j(t)}{x_1(t)} = k_j$ ($j = 1..N$) do not depend on $t$. But if we take into account the next Fourier harmonics for $x_1(t)$, such ratios do depend on time. We illustrate this fact in Fig. 16 for the chain with $N = 3$ particles.

2. In previous sections, we consider not only the single-frequency breathers, but many-frequencies breathers, as well. The existence of such dynamical objects can be also understood in the framework of our approximate methods for the discrete breathers
FIG. 16: The ratio $\frac{|x_2(t)|}{|x_1(t)|}$ for the exact solution to Eqs. (11, 12) for $\gamma = 0.3$.

construction discussed in the present section. Indeed, let us return to the expression (20) for the chain with $N = 3$ particles. Now we will not annihilate the coefficient in front of $\sin(\omega_0 t)$. Instead of this, we choose such value of the initial velocity of the central particle that leads to the frequency $\omega$, which is a certain multiple with respect to the frequency $\omega_0$ of the peripheral particle: $\omega = m\omega_0$ with $m$ being an integer number \[19\]. Then the resulting breather possesses the frequency $\omega_0$, determined by the vibrations of two peripheral particles (2 and 3), while the frequency $\omega = m\omega_0$ of the central particle turns out to be higher multiple of $\omega_0$. Thus, we obtain a discrete breather whose particles vibrate with two different, but divisible, frequencies ($\omega_0$ and $\omega = m\omega_0$), i.e. we find a two-frequency breather. In this case, the strictly periodic discrete breather exists also only if we tune onto the relation $\omega = m\omega_0$ with high precision, otherwise we obtain a quasibreather with two incommensurable frequencies $\omega$ and $\omega_0$.

When this paper was finished, we became aware of the very interesting paper \[13\] by Ovchinnikov concerning the localization of the vibrational energy in strongly excited molecules and molecular crystals. The author has presented the elegant arguments for the possibility of the above localization using as an example a molecular dimer described by equations similar to Eqs. (11-12) of our paper. The everaging procedure was applied to obtain an approximate analytical solution to these dynamical equations in order to demonstrate the existence of the periodic solution with different vibrational amplitudes of the dimer particles. Above in this section, we have presented somewhat different arguments for
V. CONCLUSION

In this paper, we present a simple method for the discrete breather construction which has been called the pair synchronization method because of its transparent physical meaning. Indeed, it represents a certain iterative procedure which, at each step, synchronizes subsequently the motion of the pairs of particles involving in the breather vibration. We believe, that this method can be applied for construction exact breathers in the nonlinear lattices of different types. The Duffing oscillatory chains with linear and nonlinear coupling were considered for explanation how the pair synchronization method can be used in practice.

A comparison of efficiency of the pair synchronization method with other methods for the discrete breather construction based on the Newton procedure for solving nonlinear algebraic equations, on the steepest descent procedure etc. will be presented elsewhere.

With some additional approximation, the pair synchronization method leads to a very simple physical interpretation of the possibility of the exact breathers existence as strictly time-periodic and spatially localized dynamical objects in nonlinear Hamiltonian lattices. Indeed, constructing a strictly time-periodic breather, we actually demand annihilation of the contributions with natural vibrational frequencies from all the peripheral particles to the breather solution. This means the existence of some kind of dictatorship of the central particle (in general, of the breather core) which must "suppress any individuality" of the peripheral particles by above mentioned annihilation. In other words, all terms of the exact breather solution with frequencies different from that of the central particle (which appear from the natural frequencies of the peripheral particles) must be turn into zero in compulsory order. The breather core compels the peripheral particles vibrate with its own frequency. In Sec. IV we demonstrate, for the case of weakly coupled oscillatory chains, that this compulsion leads to the exponential decay of the vibrational amplitudes of the peripheral breather particles.

On the other hand, if we do not tune the initial conditions on the exact breather solution, i.e. if the vibrational contributions from the peripheral particles are not equal to zero, we obtain a certain quasibreather\[6\]. This spatially localized dynamical object is characterized by different frequencies appearing from the peripheral particles. The difference in frequen-
FIG. 17: An extremely long-lived quasibreather for the Duffing chain (1) with $N = 5$ particles which, possibly, is a stable dynamical object $[\gamma = 0.3, \alpha_1 = 2, \alpha_2 = -0.5760, \alpha_3 = 0.1]$.

cies is brought about by the phenomenon typical for nonlinear dynamics, namely, by the dependence of the frequency on the vibrational amplitude.

In [6], we proposed to characterize the proximity of the quasibreather to an exact time-periodic breather with the aid of the mean square deviations in frequencies of the individual particles and that of the fixed particle in time (there exist a certain temporal drift of the frequency of every chosen particle). We also presented there some arguments (at least, for the case of $K_4$ chain) for the possibility of the quasibreather stability despite of the absence of the time periodicity. However, we cannot now present any refine mathematical proof on quasibreather stability for infinite time.

On the other hand, one often reveals that the lifetime of a quasibreather, if it actually decays in time, can be exponentially large for small deviations from the exact breather. For example, in Fig. [17] we depict the time evolution of the quasibreather in the weakly coupled Duffing chain which lives, at least, up to $t \sim 10^6$.

Finally, since we cannot tune exactly on the strictly time periodic breather solution in any physical experiment (and even in numerical experiments), quasibreathers seem to be much more relevant spatially localized dynamical objects than the exact discrete breathers. Some additional detail for this point of view can be found in [6].
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[14] Nevertheless, let us note that a certain physical interpretation of the DBs existence can be found in [13]. We will comment on this work in Sec. IV of the present paper.
[15] This model belongs to the class of mechanical systems treated in [7] where the first rigorous proof of the discrete breathers existence was presented.
[16] This is a consequence of the relations $\gamma \ll 1$, $|\dot{x}_2(0)| \ll |\dot{x}_1(0)|$.
[17] Note that by performing this procedure we do not worry about vibrational frequencies of all the other particles.
[18] It is easy to prove numerically that, for reasonable values of the parameters $\gamma$ and $\alpha_1 = x_1(0)$, Eqs. (14-15) allow the breather solution, which is very close to that of the exact equations (10).

[19] It is possible owing to the specific dependence of $\omega$ on the vibrational amplitude for the Duffing oscillator (see Fig. 16).