A Family of Graph Distances Generalizing both the Shortest-Path and the Resistance Distances

Pavel Chebotarev

Institute of Control Sciences of the Russian Academy of Sciences
65 Profsoyuznaya Street, Moscow 117997, Russia

Abstract

A new family of distances for graph vertices is proposed. These distances reduce to the shortest path distance and to the resistance distance at the extreme values of the family parameter. The most important property of them is that they are graph-geodetic: \(d(i, j) + d(j, k) = d(i, k)\) if and only if every path from \(i\) to \(k\) passes through \(j\). The construction of the distances is based on the matrix forest theorem and the graph bottleneck inequality.

Keywords: Shortest path distance; Resistance distance; Forest distance; Matrix forest theorem; Spanning rooted forest; Graph bottleneck inequality; Laplacian matrix

AMS Classification: 05C12, 05C50, 05C05, 15A48

1 Introduction

The classical distance for graph vertices is the shortest path distance [1]. Another distance, which is almost classical, is the resistance distance [12] also called the commute-time distance [2, 15].

The forest distances \(\delta_\alpha(i, j)[10]\) form a one-parametric family converging to the discrete distance \((\delta_0(i, j) = 1\) whenever vertices \(i\) and \(j\) are distinct) as \(\alpha \to 0\) and becoming proportional to the resistance distance as \(\alpha \to \infty\). The parameter \(\alpha\) controls the relative influence of short and long paths between vertices on the distance between them.

In a recent paper [16] (see also [14]), the authors construct a parametric class of graph dissimilarity measures whose extrema are the shortest path distance and the resistance distance. It is noteworthy that in clustering tasks, the best performance is obtained with intermediate values of the family parameter. At the same time, the corresponding intermediate measures need not be distances as they break the triangle inequality.
Thus, there is a demand in certain applications (these include data analysis, computer science, mathematical chemistry and some others) for a family of graph distances whose extreme properties are similar to those of the dissimilarity measures in [16]. Such a family is introduced in this paper. It is the family of logarithmically transformed forest distances, and its construction is based on the matrix forest theorem [5] and the graph bottleneck inequality [5]. The logarithmic transformation not only leads to the shortest path distance at \( \alpha \to 0 \), but also, for every \( \alpha > 0 \), ensures the remarkable graph-geodetic property: \( d(i, j) + d(j, k) = d(i, k) \) if and only if every path from \( i \) to \( k \) passes through \( j \).

We now introduce the necessary notation. Let \( G \) be a weighted multigraph (a weighted graph, where multiple edges are allowed) with vertex set \( V(G) = \{1, \ldots, n\} \), \( n > 1 \) and edge set \( E(G) \). We assume that \( G \) has no loops. For \( i, j \in V(G) \), let \( n_{ij} \in \{0, 1, \ldots\} \) be the number of edges incident to both \( i \) and \( j \) in \( G \); for every \( p \in \{1, \ldots, n_{ij}\} \), \( w_{ij}^p > 0 \) is the weight of the \( p \)th edge of this type; let \( w_{ij} = \sum_{p=1}^{n_{ij}} w_{ij}^p \) (if \( n_{ij} = 0 \), we set \( w_{ij} = 0 \)) and \( W = (w_{ij})_{n \times n} \). \( W \) is the symmetric matrix of total edge weights of \( G \).

A rooted tree is a connected and acyclic weighted graph in which one vertex, called the root, is marked. A rooted forest is a graph all of whose connected components are rooted trees. The roots of those trees are, by definition, the roots of the rooted forest.

By the weight of a weighted graph \( H \), \( w(H) \), we mean the product of the weights of all its edges. If \( H \) has no edges, then \( w(H) = 1 \). The weight of a set \( S \) of graphs, \( w(S) \), is the total weight of the graphs belonging to \( S \); the weight of the empty set is zero. If the weights of all edges are unity, i.e. the graphs in \( S \) are actually unweighted, then \( w(S) \) reduces to the cardinality of \( S \).

For a given weighted multigraph \( G \), by \( \mathcal{F} = \mathcal{F}(G) \), \( \mathcal{F}_{ij} = \mathcal{F}_{ij}(G) \), and \( \mathcal{F}^{(p)} = \mathcal{F}^{(p)}(G) \) we denote the set of all spanning rooted forests of \( G \), the set of all forests in \( \mathcal{F} \) that have vertex \( i \) belonging to a tree rooted at \( j \), and the set of all forests in \( \mathcal{F}_{ij} \) that have exactly \( p \) edges. Let

\[
\mathbf{f} = w(\mathcal{F}), \quad f_{ij} = w(\mathcal{F}_{ij}), \quad \text{and} \quad f^{(p)}_{ij} = w\left(\mathcal{F}^{(p)}_{ij}\right), \quad i, j \in V(G), \quad 0 \leq p < n; \quad (1)
\]

by \( \mathbf{f} \) we denote the matrix \( (f_{ij})_{n \times n} \); \( \mathbf{f} \) is called the matrix of forests of \( G \).

Let \( L = (\ell_{ij}) \) be the Laplacian matrix of \( G \), i.e.,

\[
\ell_{ij} = \begin{cases} 
-w_{ij}, & j \neq i, \\
\sum_{k \neq i} w_{ik}, & j = i.
\end{cases}
\]

Consider the matrix

\[
Q = (q_{ij}) = (I + L)^{-1}.
\]

By the matrix forest theorem [7, 8, 6], for any weighted multidigraph \( G \), \( Q \) does exist and

\[
q_{ij} = \frac{f_{ij}}{f}, \quad i, j = 1, \ldots, n. \quad (2)
\]

Consequently, \( F = fQ = f \cdot (I + L)^{-1} \) holds. \( Q \) can be considered as a matrix providing a proximity (similarity) measure for the vertices of \( G \) [8, 4].
By $d^s(i, j)$ we denote the shortest path distance, i.e., the number of edges in a shortest path between $i$ and $j$ in $G$; by $d^r(i, j)$ we denote the resistance distance between $i$ and $j$ defined as follows:

$$d^r(i, j) = \ell_{ii}^+ + \ell_{jj}^+ - 2\ell_{ij}^+,$$

where $(\ell_{ij}^+)^{n \times n} = L^+$ is the Moore-Penrose generalized inverse of the Laplacian matrix $L$ of $G$. If $G$ is connected, then, due to [9, Proposition 8],

$$L^+ = (L + \bar{J})^{-1} - \bar{J},$$

where $\bar{J}$ is the $n \times n$ matrix with all entries $\frac{1}{n}$, and by [9, Theorem 3]

$$\ell_{ij}^+ = f^{(n-2)}_i - \frac{1}{nt}f^{(n-2)}_j, \quad i, j \in V(G),$$

holds, where $f^{(n-2)}$ is the total weight of spanning rooted forests with $n - 2$ edges and $t$ is the total weight of spanning trees in $G$. By virtue of (3) and (4) this yields

**Corollary 1** (to Proposition 8 and Theorem 3 of [9]). If $G$ is connected, then

$$d^r(i, j) = x_{ii} + x_{jj} - 2x_{ij} = \frac{f^{(n-2)}_i + f^{(n-2)}_j - 2f^{(n-2)}_{ij}}{nt}, \quad i, j \in V(G),$$

where $(x_{ij}) = (L + \bar{J})^{-1}$.

In Section 2 we introduce a new class of intrinsic graph distances and in Sections 3 and 4 we study its properties.

## 2 Logarithmic forest distances

Let

$$Q_\alpha = (I + \alpha L)^{-1},$$

where $L$ is the Laplacian matrix of $G$, $I$ the identity matrix, and $\alpha > 0$ a real parameter. Define the matrix $H_\alpha$ as follows:

$$H_\alpha = \gamma (\alpha - 1) \log_\alpha Q_\alpha,$$

where $\alpha \neq 1$, $\gamma$ is a positive factor, and $\varphi(Q_\alpha)$ stands for componentwise operations. Finally, consider

$$D_\alpha = \frac{1}{2}(h_\alpha 1' + 1h_\alpha') - H_\alpha,$$

where $h_\alpha$ is the column vector containing the diagonal entries of $H_\alpha$, $h_\alpha'$ is the transpose of $h_\alpha$, $1$ and $1'$ being the column of $n$ ones and its transpose. In Theorem 4 below we show that $D_\alpha$ is a matrix of distances between the vertices of $G$.

Since $\lim_{\alpha \to 1} ((\alpha - 1)/\ln \alpha) = 1$, we extend Eq. (7) to $\alpha = 1$ as follows:

$$H_1 = \gamma \ln \overrightarrow{Q},$$

which preserves continuity. This extension is assumed throughout the paper.
Theorem 1  For any connected multigraph $G$ and any $\alpha, \gamma > 0$, $D_\alpha = (d_{ij}(\alpha))_{n \times n}$ defined by (8) with extension (9) is a matrix of distances on $V(G)$.

Before proving Theorem 1, we represent the entries of the distance matrix $D_\alpha$ in terms of the weights of spanning forests.

For $\alpha > 0$, let $G_\alpha$ be the weighted multigraph resulting from $G$ by multiplying the weights of all edges by $\alpha$. Let

$$f_{ij}(\alpha) = w(F_{ij}(G_\alpha)), \quad i, j = 1, \ldots, n.$$  \hfill (10)

**Proposition 1**  For any connected multigraph $G$ and any $\alpha, \gamma > 0$, the matrix $D_\alpha = (d_{ij}(\alpha))$ defined by (8) with extension (9) exists and

$$d_{ij}(\alpha) = \begin{cases} \gamma (\alpha - 1) \log_{\alpha} \frac{\sqrt{f_{ii}(\alpha)f_{jj}(\alpha)}}{f_{ij}(\alpha)}, & \alpha \neq 1, \\ \gamma \ln \frac{\sqrt{f_{ii}(\alpha)f_{jj}(\alpha)}}{f_{ij}(\alpha)}, & \alpha = 1 \end{cases}, \quad i, j = 1, \ldots, n. \hfill (11)$$

**Proof.** Observe that $\alpha L$ is the Laplacian matrix of the weighted multigraph $G_\alpha$. Then applying the matrix forest theorem (2) to $G_\alpha$ one obtains that the matrix $Q_\alpha$ exists and its entries are strictly positive, provided that $G$ is connected. Therefore $H_\alpha$ and $D_\alpha$ also exist.

Let $H_\alpha = (h_{ij}(\alpha))$ and $Q_\alpha = (q_{ij}(\alpha))$. For any positive $\alpha \neq 1$ and $\gamma$, equations (6) to (8) and the matrix forest theorem (2) imply

$$d_{ij}(\alpha) = \frac{1}{2}(h_{ii}(\alpha) + h_{jj}(\alpha)) - h_{ij}(\alpha) = \gamma (\alpha - 1) \left[ \frac{1}{2}(\log_{\alpha} q_{ii}(\alpha) + \log_{\alpha} q_{jj}(\alpha)) - \log_{\alpha} q_{ij}(\alpha) \right] = \gamma (\alpha - 1) \log_{\alpha} \frac{\sqrt{q_{ii}(\alpha)q_{jj}(\alpha)}}{q_{ij}(\alpha)} = \gamma (\alpha - 1) \log_{\alpha} \frac{\sqrt{f_{ii}(\alpha)f_{jj}(\alpha)}}{f_{ij}(\alpha)}$$

for every $i, j = 1, \ldots, n$. If $\alpha = 1$, the required expression follows similarly using (9). \hfill $\Box$

**Proof of Theorem 1.** Proving the theorem amounts to showing that for every $i, j, k \in V(G)$:

(i) $d_{ij}(\alpha) = 0$ if and only if $i = j$;

(ii) $d_{ij}(\alpha) + d_{jk}(\alpha) - d_{ki}(\alpha) \geq 0$ (triangle inequality).

Let $\alpha \neq 1$. If $i = j$ then $\log_{\alpha} \frac{\sqrt{f_{ii}(\alpha)f_{jj}(\alpha)}}{f_{ij}(\alpha)} = \log_{\alpha} 1 = 0$, hence, by Proposition 1, $d_{ij}(\alpha) = 0$. Conversely, if $d_{ij}(\alpha) = 0$, then by Proposition 1, $f_{ii}(\alpha)f_{jj}(\alpha) = (f_{ij}(\alpha))^2$ holds. If $i \neq j$, then $f_{ij}(\alpha) < f_{jj}(\alpha)$, since, by definition, $F_{ij}(G_\alpha) \subseteq F_{jj}(G_\alpha)$ and $F_{jj}(G_\alpha) \setminus F_{ij}(G_\alpha)$ contains the trivial spanning rooted forest having no edges and weight unity. Similarly, $f_{ji}(\alpha) < f_{ii}(\alpha)$ and since $Q_\alpha$ is symmetric, $f_{ij}(\alpha) < f_{ii}(\alpha)$. Consequently, $i \neq j$ contradicts the assumption $d_{ij}(\alpha) = 0$, hence $i = j$.

To prove (ii), observe that by (1), (3), and (2), for any positive $\alpha \neq 1$ we have

$$d_{ij}(\alpha) + d_{jk}(\alpha) - d_{ki}(\alpha) = \frac{1}{2}(h_{ii}(\alpha) + h_{jj}(\alpha)) + h_{ij}(\alpha) + h_{kk}(\alpha) - h_{kk}(\alpha) - h_{ii}(\alpha))$$

$$-h_{ij}(\alpha) - h_{jk}(\alpha) + h_{ki}(\alpha)$$

$$= h_{jj}(\alpha) + h_{kk}(\alpha) - h_{ij}(\alpha) - h_{jk}(\alpha)$$

$$= \gamma (\alpha - 1) \log_{\alpha} \frac{f_{jj}(\alpha)f_{kk}(\alpha)}{f_{ij}(\alpha)f_{jk}(\alpha)}, \hfill (12)$$
From the symmetry of $Q_\alpha$ and the graph bottleneck inequality \cite[Corollary 1]{5}, $f_{jj}(\alpha) f_{ki}(\alpha) \geq f_{ij}(\alpha) f_{jk}(\alpha)$. Therefore \cite{12} implies that $d_{ij}(\alpha) + d_{jk}(\alpha) - d_{ki}(\alpha) \geq 0$. For $\alpha = 1$ (i) and (ii) are proved similarly.

Theorem 1 enables us to give the following definition.

**Definition 1** Suppose that $G$ is a connected multigraph and $\alpha > 0$. The logarithmic forest distance with parameter $\alpha$ on $G$ is the function $d_\alpha : V(G) \times V(G) \to \mathbb{R}$ such that $d_\alpha(i, j) = d_{ij}(\alpha)$, where $D_\alpha = (d_{ij}(\alpha))$ is defined by (8) with extension (9).

In Definition 1 the scaling factor $\gamma$ is regarded as an implicit, i.e., internal parameter of logarithmic forest distances. In Section 3, we show that all of them are graph-geodetic. In Section 4 distances with a specific value of $\gamma$ will be considered.

3 The logarithmic forest distances are graph-geodetic

The key property of the logarithmic forest distances is that they are graph-geodetic.

**Definition 2** For a multigraph $G$, a function $d : V(G) \times V(G) \to \mathbb{R}$ is graph-geodetic whenever for all $i, j, k \in V(G)$, $d(i, j) + d(j, k) = d(i, k)$ holds if and only if every path from $i$ to $k$ passes through $j$.

If $d(i, j)$ is a distance for graph vertices, then the property of being graph-geodetic is a natural condition of strengthening the triangle inequality to equality. The shortest path distance clearly possesses the “if” part of the graph-geodetic property; this property of the resistance distance was proved in \cite{12}. The ordinary distance in a Euclidean space satisfies a similar condition resulting by substituting “line segment” for “path.”

**Theorem 2** For every connected multigraph $G$ and every $\alpha > 0$, the logarithmic forest distance $d_\alpha(i, j)$ is graph-geodetic.

**Proof.** By the graph bottleneck equality \cite[Corollary 1]{5} and the symmetry of $F(\alpha) = (f_{ij}(\alpha))_{n \times n}$, $f_{jj}(\alpha) f_{ki}(\alpha) = f_{ij}(\alpha) f_{jk}(\alpha)$ is true if and only if every path in $G(\alpha)$ from $i$ to $k$ passes through $j$. Owing to (12) and the analogous expression for $\alpha = 1$, this equality is equivalent to $d_\alpha(i, j) + d_\alpha(j, k) - d_\alpha(k, i) = 0$. The desired statement follows.

Graph-geodetic functions have many interesting properties. One of them, as mentioned in \cite{12}, is a simple connection (such as that obtained in \cite{11}) between the cofactors and determinant of $G$’s distance matrix and those of the strong blocks of $G$. Another example is the recursive Theorem 8 in \cite{13}. Clearly, for a tree, all the $n(n - 1)/2$ values of a graph-geodetic distance are determined by the $n - 1$ values corresponding to the pairs of adjacent vertices. The logarithmic forest distances, as well as the limiting shortest path and resistance distances, need not be Euclidean, however, by Blumenthal’s “Square-Root” theorem, the corresponding “square-rooted” distances satisfy the 3-Euclidean condition (cf. \cite{13}).

It can be observed that the “ordinary” forest distances \cite{10} generally are not graph-geodetic. The graph bottleneck inequality \cite{5} underlying the proofs of Theorems 1 and 2 is actually a multiplicative counterpart of the triangle inequality for proximities \cite{8}. 

\[\Box\]
4 Asymptotic properties

Consider the subclass of logarithmic forest distances with the scaling factor

$$\gamma = \ln(\epsilon + \alpha^2).$$

(13)

We will prove that these generalize both the shortest path and the resistance distances.

**Proposition 2** For any connected multigraph \(G\) and every \(i, j \in V(G)\), \(d_\alpha(i, j)\) with scaling factor \(\gamma\) converges to the shortest path distance \(d^*(i, j)\) as \(\alpha \to 0^+\).

**Proof.** Denote by \(m\) the shortest path distance \(d^*(i, j)\) between \(i\) and \(j \neq i\). Observe that the weight of every forest that belongs to \(F_i(G_\alpha)\) and has at least one edge vanishes with \(\alpha \to 0^+\), whereas \(F_i(G_\alpha)\) contains one trivial forest without edges whose weight is unity. Taking this into account and using Proposition 1 and (1) one obtains

$$\lim_{\alpha \to 0^+} d_\alpha(i, j) = \lim_{\alpha \to 0^+} \left( -\log_\alpha \frac{\sqrt{1+1}}{\alpha^m(f_{ij}^{(m)} + o(1))} \right),$$

where \(o(1) \to 0\) as \(\alpha \to 0^+\). Consequently,

$$\lim_{\alpha \to 0^+} d_\alpha(i, j) = \lim_{\alpha \to 0^+} \left( m + \log_\alpha f_{ij}^{(m)} \right) = m = d^*(i, j).$$

\[ \square \]

**Proposition 3** For any connected multigraph \(G\) and every \(i, j \in V(G)\), \(d_\alpha(i, j)\) with scaling factor \(\gamma\) converges to the resistance distance \(d^*(i, j)\) as \(\alpha \to \infty\).

**Proof.** Observe that for every \(i, j \in V(G)\), \(f_{ij}^{(n)}\) is the total weight of all spanning trees in \(G\). Denote it by \(t\); since \(G\) is connected, \(t > 0\). By Proposition 1 one has

$$\lim_{\alpha \to \infty} d_\alpha(i, j) = \lim_{\alpha \to \infty} \left( \frac{2\alpha}{n} \ln \alpha (\ln \alpha)^{-1} \ln \frac{\sqrt{\alpha^{n-1}(t + \frac{1}{\alpha} f_{ii}^{(n-2)} + o(\frac{1}{\alpha})) \alpha^{n-1}(t + \frac{1}{\alpha} f_{jj}^{(n-2)} + o(\frac{1}{\alpha}))}}{\alpha^{n-1}(t + \frac{1}{\alpha} f_{ii}^{(n-2)} + o(\frac{1}{\alpha}))} \right),$$

where \(o(\frac{1}{\alpha})\) denotes expressions such that \(\alpha \cdot o(\frac{1}{\alpha}) \to 0\) as \(\alpha \to \infty\). Hence

$$\lim_{\alpha \to \infty} d_\alpha(i, j) = \frac{2}{n} \lim_{\alpha \to \infty} \ln \frac{\left(1 + \frac{f_{ii}^{(n-2)}}{\alpha t}\right)^\alpha \left(1 + \frac{f_{jj}^{(n-2)}}{\alpha t}\right)^\alpha}{\left(1 + \frac{f_{ij}^{(n-2)}}{\alpha t}\right)^\alpha} = \frac{2}{n} \ln \frac{\exp\left(\frac{f_{ii}^{(n-2)}}{t}\right) \exp\left(\frac{f_{jj}^{(n-2)}}{t}\right)}{\exp\left(\frac{f_{ij}^{(n-2)}}{t}\right)} = \frac{f_{ii}^{(n-2)} + f_{jj}^{(n-2)} - 2f_{ij}^{(n-2)}}{nt}.$$  

(14)

Consequently, by Corollary 1 presented in Section 1 \(\lim_{\alpha \to \infty} d_\alpha(i, j) = d^*(i, j)\). \[ \square \]
5 Concluding remarks

On intercomponent distances
Throughout the paper, we assumed that $G$ is connected. Otherwise, if $G$ has more than one component and $i$ and $j$ belong to different components, then, by the matrix forest theorem (2), $q_{ij} = f_{ij} = 0$. Consequently, if $\log_{\alpha}(\cdot)$ is considered as a function mapping to the extended line $\mathbb{R} \cup \{-\infty, +\infty\}$, then (5) leads to the distance $+\infty$ between $i$ and $j$, which seems quite natural.

On the parameter $\alpha$ and the length of paths between vertices
The parameter $\alpha$ of logarithmic forest distances controls the relative influence of short, medium, and long paths between vertices $i$ and $j$ on the distance $d_\alpha(i,j)$. As $\alpha \to 0$, only the shortest paths matter; the long paths have the maximum effect as $\alpha \to \infty$.

On the “mixture” of the shortest-path and resistance distances
The simplest way of “generalizing” both the shortest-path and the resistance distances is to consider the convex combination of the form $d_{\alpha'}(i,j) = \alpha d^r(i,j) + (1 - \alpha)d^s(i,j)$, where $\alpha \in [0,1]$. However, this family is quite poor from both theoretical and practical points of view. Let, for example, $G$ be a path on four vertices: $V(G) = \{1,2,3,4\}$ and $E(G) = \{(1,2), (2,3), (3,4)\}$. Then $d^s(1,2) = d^s(2,3) = d^r(1,2) = d^r(2,3) = 1$, and therefore $d_{\alpha'}(1,2) = d_{\alpha'}(2,3)$ for all $\alpha \in [0,1]$. On the other hand, in applications, there are models and intuitive heuristics that result in either $d(1,2) > d(2,3)$ or $d(1,2) < d(2,3)$. Indeed, suppose that the distance $d(i,j)$ should depend on the whole set of routes between $i$ and $j$: the shorter and more numerous the routes, the smaller the distance. Then the inequality $d(1,2) > d(2,3)$ is suggested by the observation that there are three routes of length 3 between vertices 2 and 3 (namely, $(2,3,2,3)$, $(2,1,2,3)$, and $(2,3,4,3)$) and only two routes of length 3 between vertices 1 and 2 ($(1,2,1,2)$ and $(1,2,3,2)$). However, if the relative numbers of routes are important, then the opposite inequality $d(1,2) < d(2,3)$ can be justified by the observation that $(1,2)$ is the unique route of length 1 starting at vertex 1, whereas $(2,3)$ and $(3,2)$ are not unique routes starting at vertices 2 and 3, respectively. It should be noted that the inequality $d(1,2) < d(2,3)$ holds true for the quasi-Euclidean graph distance [13]. The above example demonstrates that distances providing $d(1,2) = d(2,3)$ are insufficient for the applications of graph theory. As regards the forest distances, the logarithmic forest distances provide $d_\alpha(1,2) < d_\alpha(2,3)$, whereas with the “ordinary” forest distances [10], we have $\tilde{d}_\alpha(1,2) > \tilde{d}_\alpha(2,3)$.

The shortest-path and resistance distances in the framework of forest distances
In the view of H. Chen and F. Zhang [3], “...the shortest-path [distance] might be imagined to be more relevant when there is corpuscular communication (along edges) between two vertices, whereas the resistance distance might be imagined to be more relevant when the communication is wave-like.” However, they do not develop this idea in depth. As has been shown in this paper, the shortest-path and resistance distances are two extreme examples of
the logarithmic forest distances. The forest distance between vertices \( i \) and \( j \) is interpreted as the probability of choosing a forest partition separating \( i \) and \( j \) in the model of random forest partitions of graph \( G \) [10, Proposition 5]. As \( \alpha \rightarrow 0 \), transformation \( \beta \) preserves only those partitions that connect \( i \) and \( j \) by a shortest path and separate all other vertices; thereby the shortest path distance results, as we see in Proposition 2. As \( \alpha \rightarrow \infty \), this transformation preserves only the partitions determined by two disjoint trees, which leads to the resistance distance, as Proposition 3 states.

**Acknowledgement**

I am grateful to Marco Saerens for drawing my attention to the problem of generalizing the shortest-path and the resistance distances.

**References**

[1] F. Buckley, F. Harary, Distance in Graphs, Addison-Wesley, Redwood City, CA, 1990.

[2] A.K. Chandra, P. Raghavan, W.L. Ruzzo, R. Smolensky, P. Tiwari, The electrical resistance of a graph captures its commute and cover times, in: Proceedings of the 21st Annual ACM Symposium on Theory of Computing, ACM Press, Seattle, 1989, pp. 574–586.

[3] H. Chen, F. Zhang, Resistance distance and the normalized Laplacian spectrum, Discrete Appl. Math. 155 (2007) 654–661.

[4] P. Chebotarev, Spanning forests and the golden ratio, Discrete Appl. Math. 156 (2008) 813–821.

[5] P. Chebotarev, A graph bottleneck inequality, arXiv preprint math.CO/0810.2732, 2008. http://arxiv.org/abs/0810.2732. Submitted.

[6] P. Chebotarev, R. Agaev, Forest matrices around the Laplacian matrix, Linear Algebra Appl. 356 (2002) 253–274.

[7] P.Yu. Chebotarev, E. Shamis, On the proximity measure for graph vertices provided by the inverse Laplacian characteristic matrix, in: Abstracts of the conference “Linear Algebra and its Applications,” 10–12 July, 1995, University of Manchester, Manchester, UK, 1995, pp. 6–7, http://www.ma.man.ac.uk/~higham/laa95/abstracts.ps

[8] P.Yu. Chebotarev, E.V. Shamis, The matrix-forest theorem and measuring relations in small social groups, Autom. Remote Control 58 (1997) 1505–1514.

[9] P.Yu. Chebotarev, E.V. Shamis, On proximity measures for graph vertices, Autom. Remote Control 59 (1998) 1443–1459.
[10] P. Chebotarev, E. Shamis, The forest metrics for graph vertices, Electron. Notes Discrete Math. 11 (2002) 98–107.

[11] R.L. Graham, A.J. Hoffman, H. Hosoya, On the distance matrix of a directed graph, J. Graph Theory 1 (1977) 85–88.

[12] D.J. Klein, M. Randić, Resistance distance, J. Math. Chem. 12 (1993) 81–95.

[13] D.J. Klein, H.Y. Zhu, Distances and volumina for graphs, J. Math. Chem. 23 (1998) 179–195.

[14] A. Mantrach, L. Yen, J. Callut, K. Françoisse, M. Shimbo, M. Saerens, The Sum-over-paths covariance kernel: A novel covariance measure between nodes of a directed graph, IEEE Trans. Pattern Anal. Machine Intelligence, 2009, in press.

[15] P. Tetali, Random walks and the effective resistance of networks, J. Theoret. Probab. 4 (1991) 101–109.

[16] L. Yen, M. Saerens, A. Mantrach, M. Shimbo, A family of dissimilarity measures between nodes generalizing both the shortest-path and the commute-time distances, in The 14th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining, 2008, pp. 785–793.