On the Stability of Spherical Membranes in Curved Spacetimes

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(June 14, 2018)

Abstract

We study the existence and stability of spherical membranes in curved spacetimes. For Dirac membranes in the Schwarzschild–de Sitter background we find that there exists an equilibrium solution. By fine-tuning the dimensionless parameter $\Lambda M^2$, the static membrane can be at any position outside the black hole event horizon, even at the stretched horizon, but the solution is unstable. We show that modes having $l = 0$ (and for $\Lambda M^2 < 16/243$ also $l = 1$) are responsible for the instability. We also find that spherical higher order membranes (membranes with extrinsic curvature corrections), contrary to what happens in flat Minkowski space, do have equilibrium solutions in a general curved background and, in particular, also in the “plain” Schwarzschild geometry (while Dirac membranes do not have equilibrium solutions there). These solutions, however, are also unstable. We shall discuss a way of bypassing these instability problems, and we also relate our results to the recent ideas of representing the black hole event horizon as a relativistic bosonic membrane.

11.27.+d,04.70.Dy

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I. INTRODUCTION

The idea of representing physical objects by extended ones such as membranes, can be traced back to the pioneering work of Dirac [1]. In this model for the electron, tension forces tending to collapse the membrane were balanced by repulsive electric forces. Later, canonical [2,3] and semiclassical [4] quantization of membranes in a flat background was considered. On the other hand, starting with the works of ’t Hooft [5], a series of papers [6–8] considered the quantum degrees of freedom of a black hole, localized around its event horizon. It is also worth noting the classical approach to black holes by the, so called, “membrane paradigm”, summarized in Ref. [9].

In Ref. [10] it was considered the approach of effectively representing the quantum degrees of freedom of a black hole by a relativistic membrane. The total Lagrangian of the system was composed of the classical curved geometry plus a relativistic bosonic membrane. The membrane, located at the “stretched horizon”, was decomposed into a spherically symmetric term plus small fluctuations around it. The first order fluctuations have then been quantized and the energy spectrum was obtained. From this, the entropy of the membrane at the temperature of the black hole (given by the background geometry) was computed. It was however left open the question of the existence and stability of the zeroth order solution representing a spherical membrane at rest very close to the black hole event horizon. In the present paper we address the above question.

This paper is organized as follows: In Sec. II we study the Dirac membrane in its simplest form. No coupling to the electromagnetic field is considered, instead the membrane is let to freely move in a curved background. Since our final aim is to study black hole properties, and for the sake of simplicity, we consider spherically symmetric backgrounds. We derive the condition for the existence of spherical membranes to be in equilibrium at a given radius \( r_m \). We also find the stability condition and apply it to a membrane in the Schwarzschild–de Sitter space. Sec. III deals with a covariant description of small fluctuations around the equilibrium position \( r_m \). We find that the fluctuations are governed by a Klein–Gordon–like equation in the 2+1 dimensional world–volume swept by the membrane. We identify modes with angular momentum label \( l = 0, 1 \) as responsible for the instabilities in the Schwarzschild–de Sitter background. In Sec. IV we study a more general membrane described by a Lagrangian with up to quadratic terms in the extrinsic curvature. This introduces two more arbitrary constants \( A \) and \( B \), in addition to the membrane tension \( T \). This freedom allows us, in the plain Schwarzschild background, to choose \( r_m \) as close to the event horizon as we want (for instance at the stretched horizon). We study the radial small fluctuation equation (fourth order differential equation) and find the equilibrium and stability conditions. In Sec. V we discuss the way of extending the validity of the perturbative approach around a maximum of the potential, instead of a minimum, when we take the comoving system of reference to describe the zeroth order motion. In Appendix A we give the analytic expressions, corresponding to the plots of Fig. 1, for the cosmological and event horizon, and \( r_m \) in the Schwarzschild–de Sitter spacetime.
II. THE SPHERICAL DIRAC MEMBRANE

Let us consider the action of a Dirac membrane (with tension $T$ bearing dimension of length$^{-3}$) in a curved background

$$I_M = -T \int d\tau dp d\sigma \sqrt{-\det (\gamma_{ij})} , \quad (\text{II.1})$$

where $\gamma_{ij}$ is the induced metric on the world-volume

$$\gamma_{ij} = g_{\mu\nu} \partial_i x^\mu \partial_j x^\nu . \quad (\text{II.2})$$

The classical equations of motion derived from this action are

$$\Box_{\gamma} x^\mu + \gamma^{ij} \Gamma_{\kappa\lambda}^\mu (g) \partial_i x^\kappa \partial_j x^\lambda = 0 , \quad (\text{II.3})$$

where

$$\Box_{\gamma} = \frac{1}{\sqrt{-\gamma}} \partial_i \left( \sqrt{-\gamma} \gamma^{ij} \partial_j \right) \quad \text{and} \quad \gamma = \det (\gamma_{ij}) . \quad (\text{II.4})$$

We shall consider in this paper static and spherically symmetric backgrounds; this will simplify the analysis, but in principle one could carry out the study for more general backgrounds also. We then take

$$ds^2 = -a(r) dt^2 + b(r)^{-1} dr^2 + r^2 d\Omega^2 , \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2 . \quad (\text{II.5})$$

The zeroth order solution representing a spherical membrane can be conveniently described by the following spherically symmetric gauge choice

$$t = t(\tau) , \quad r = r(\tau) , \quad \theta = \rho , \quad \varphi = \sigma , \quad (\text{II.6})$$

so that the induced metric on the world-volume becomes

$$\gamma_{\tau\tau} = -at^2 + \dot{r}^2 / b(r) , \quad \gamma_{\rho\rho} = r^2 , \quad \gamma_{\sigma\sigma} = r^2 \sin^2 \rho , \quad (\text{II.7})$$

where a dot denotes derivative with respect to tau. Notice that the gauge choice \ref{II.6} does not fix completely the gauge for a spherical membrane. There is still one un-used reparametrization, which we shall return to in a moment.

The effective Lagrangian of the system takes the following simple form after integrating over $\rho$ and $\sigma$

$$L = -4\pi T r^2 \sqrt{a(r) \dot{t}^2 - \dot{r}^2 / b(r)} . \quad (\text{II.8})$$

The temporal component of the equations of motion \ref{II.3} is given by

$$\partial_\tau \left( \frac{4\pi T r^2 a(r) \dot{t}}{\sqrt{a(r) \dot{t}^2 - \dot{r}^2 / b(r)}} \right) = 0 . \quad (\text{II.9})$$
The corresponding integral, $E$, has dimension of $\text{length}^{-1} = \text{energy}$, when using units where $c = \hbar = 1$, but keeping Newton’s constant $G$ explicitly. Let us define the proper time $\tau$ by

$$\dot{\tau} = \frac{E}{(4\pi T)^{1/3}a(r)} \equiv \frac{\tilde{E}}{a(r)}. \quad (\text{II.10})$$

This definition fixes the remaining gauge freedom for the spherical membrane. Notice also that in a point-particle picture, the dimensionless constant $\tilde{E}$ would correspond to the energy per unit rest–mass (at least in an asymptotically flat spacetime). Introducing similarly the notation $\tilde{T} = T/(4\pi T)^{1/3}$, the equation (II.9) takes the form

$$r^2 = \frac{\tilde{E}^2 b}{a} \left(1 - \frac{(4\pi \tilde{T})^2 r^4}{\tilde{E}^2 a}ight) \equiv \tilde{E}^2 - \tilde{V}^2, \quad (\text{II.11})$$

hence the effective potential can be read off [11]. It is easy to check that the spatial components of the equations of motion (II.3) are consistent with (II.11), and they do not provide any further information. Thus the dynamics of the spherical membrane is fully determined by Eq. (II.11).

The condition of the existence of a static spherical membrane $r = r_m$ (an equilibrium solution) is given by $\dot{r} = 0$, $(\tilde{V}^2)' = 0$ at $r = r_m$, where the prime denotes derivative with respect to $r$, that is to say

$$4a(r_m) + r_m a'(r_m) = 0, \quad (\text{II.12})$$

while the condition for this membrane to be in stable equilibrium (stable with respect to radial perturbations; we shall consider arbitrary perturbations in the next section) is

$$(\tilde{V}^2)''(r_m) = (4\pi \tilde{T})^2 b(r_m) \frac{a(r_m)}{r_m^2} \left[-20a(r_m) + r_m a''(r_m)\right] > 0. \quad (\text{II.13})$$

It is remarkable that both conditions (II.12), (II.13) are actually independent of the function $b(r)$, at least as long as we stay to the regime where $r$ is spacelike ($b(r) > 0$).

It can now be easily shown by direct replacement into Eq. (II.12) that the simple assumption for the background metric given by the Minkowski, Schwarzschild or Reissner–Nordström metric, does not allow any spherical membrane to be in (stable or unstable) equilibrium outside the event horizon. This is easily understood from the physical point of view: the membrane tension acts in the direction of contracting the spherical membrane, so in the absence of any other internal degrees of freedom (which is the case for the Dirac membrane), a repulsive gravitational field would be necessary to support an equilibrium solution. The simplest physically interesting spacetime that admits solutions to Eq. (II.12) is de Sitter spacetime, which in static coordinates corresponds to $a(r) = b(r) = 1 - \Lambda r^2/3$.

The equilibrium solution is given by $r_m = \sqrt{2/\Lambda}$, and it is unstable according to Eq. (II.13).

However, as discussed in the Introduction, we are mostly interested in black hole spacetimes. In fact, let us take the Schwarzschild–de Sitter metric in static coordinates. From now on we use Planck units ($c = \hbar = G = 1$); we thus have

$$a(r) = b(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2. \quad (\text{II.14})$$
If (and only if) $\Lambda M^2 \equiv \tilde{\Lambda} \leq 1/9$, there is a black hole event horizon $r_{EH}$ and a cosmological horizon $r_{CH}$

$$r_{EH} \leq r_{CH}, \quad \text{(II.15)}$$

and there is a unique solution $r_m$ to Eq. (II.12) such that

$$r_{EH} \leq r_m \leq r_{CH} \quad \text{(II.16)}$$

Explicit expressions for $r_{EH}$, $r_m$ and $r_{CH}$ are given in Appendix A, and Fig. 1 shows the equilibrium membrane solution $r_m$, which always lies between the cosmological horizon $r_{CH}$ and the black hole event horizon $r_{EH}$. In Fig. 2 we show the effective potential $\tilde{V}^2 = (4\pi \tilde{T})^2 a(r)r^4$, and observe that the static spherical membrane sits at a local maximum of the potential. We thus see that the equilibrium membrane solution $r_m$ is unstable to radial perturbations. This can also be seen by checking that the condition (II.13) is not fulfilled for any of the values of $r_m$.

### III. COVARIANT PERTURBATIONS

To further understand the behavior of the membrane fluctuations and stability we will now adapt the covariant approach of Ref. [12] (see also Refs. [13,14]) for strings, to the present case.

We have, for a membrane in four dimensions, only one degree of freedom (as opposed to the two degrees of freedom for the string). This essentially represents the fluctuations of the membrane perpendicular to the hypersurface where it lies. The equation for this degree of freedom, that we will denote as $\phi(\tau, \rho, \sigma)$, takes the form of a simple Klein–Gordon equation [12] (which had been obtained by in Ref. [10] using the spherically symmetric gauge)

$$\Box_\gamma \phi + \mathcal{V}\phi = 0, \quad \text{ (III.1)}$$

where

$$\mathcal{V} = \Omega_{ij} \Omega^{ij} - \gamma^{ij} \partial_i x^\mu \partial_j x^\nu R_{\mu\lambda\kappa\nu} n^\lambda n^\kappa. \quad \text{ (III.2)}$$

In the above equation, $R_{\mu\lambda\kappa\nu}$ is the Riemann tensor of the background spacetime, while $\Omega_{ij}$ is the second fundamental form

$$\Omega_{ij} = g_{\mu\nu} n^\mu x_{\rho}^\nu \nabla_\rho x_{\lambda}^\nu. \quad \text{ (III.3)}$$

The normal vector $n^\mu$ is defined by

$$g_{\mu\nu} n^\mu x_{\lambda}^\nu = 0, \quad g_{\mu\nu} n^\mu n^\nu = 1, \quad \text{ (III.4)}$$

and it fulfills the completeness relation

$$g^{\mu\nu} = n^\mu n^\nu + \gamma^{ij} x_{i}^\mu x_{j}^\nu. \quad \text{ (III.5)}$$

Let us first consider a generic dynamical spherical membrane in a background of the form (II.5), as the zeroth order solution. The normal vector is
\[ n^\mu = \frac{\sqrt{b/a}}{\sqrt{E^2/a - \dot{r}^2/b}} \left( \dot{r}/b, \dot{E}, 0, 0 \right), \quad \text{(III.6)} \]

and then the components of the second fundamental form take the following explicit form for our background metric (I.5)

\[ \Omega_{\tau\tau} = \frac{\tilde{E} \sqrt{b/a}}{2a^2b \sqrt{E^2/a - \dot{r}^2/b}} \left[ 2a^2 \dot{r} + \tilde{E}^2(ba' - ab') \right] + \frac{\tilde{E} b' \sqrt{b/a}}{2b} \sqrt{E^2/a - \dot{r}^2/b}, \]

\[ \Omega_{\rho\rho} = -\frac{\tilde{E} r \sqrt{b/a}}{\sqrt{E^2/a - \dot{r}^2/b}} \right), \quad \Omega_{\sigma\sigma} = \frac{-\tilde{E} \sqrt{b/a}}{r(4\pi \tilde{T})} \sin^2 \rho, \quad \text{(III.7)} \]

where primes denote \( r \) derivatives. These expressions are valid off–shell, that is, we have not used the equation of motion (II.9). It is easy to check that the condition \( \gamma^{ij} \Omega_{ij} = 0 \) is equivalent to Eq. (II.9), as follows more generally from the Gauß–Weingarten equation

\[ D_i D_j x^\mu + \Gamma^\mu_{\kappa\lambda} \partial_i x^\kappa \partial_j x^\lambda = n^\mu \Omega_{ij}. \quad \text{(III.8)} \]

Here we are interested in the fluctuations around an on–shell spherical Dirac membrane. When using the equation of motion (II.11), the components (III.7) reduce to

\[ \Omega_{\tau\tau} = -2 \tilde{E} r (4\pi \tilde{T}) \sqrt{b/a}, \quad \Omega_{\rho\rho} \sin^2 \rho = \Omega_{\sigma\sigma} = -\tilde{E} \sqrt{b/a} r (4\pi \tilde{T}) \sin^2 \rho. \quad \text{(III.9)} \]

The first term of the potential (III.2) then becomes

\[ \Omega_{ij} \Omega^{ij} = \frac{6 \tilde{E}^2 b}{ar^6 (4\pi \tilde{T})}. \quad \text{(III.10)} \]

To compute the second term of the potential, we need explicit expressions for the non–vanishing components of the curvature tensor in the background metric (I.3)

\[ R_{\tau r \tau r} = -\frac{1}{2} a'' + \frac{a' b'}{4b} - \frac{(a')^2}{4a}, \quad R_{r \theta r \theta} = -\frac{r b'}{2b}, \quad R_{r \varphi r \varphi} = -\frac{r b'}{2b} \sin^2 \theta, \]

\[ R_{\theta \theta r r} = r b', \quad R_{\varphi \varphi r r} = r b' \sin^2 \theta, \quad R_{\theta \varphi r r} = r^2 (1 - b) \sin^2 \theta. \quad \text{(III.11)} \]

We have now all the elements to write down explicitly Eq. (III.1) since

\[ \mathcal{V}(r) = -\frac{b}{a} \left[ \frac{a''}{2} + \frac{a'}{r} + \frac{a'}{4ab} (ab' - ba') \right] + \frac{b}{a} \frac{\tilde{E}^2}{r^5 (4\pi \tilde{T})^2} \left[ \frac{6}{r} - \frac{ab' - ba'}{ab} \right], \quad \text{(III.12)} \]

and

\[ \Box_\gamma = -\frac{1}{r^4 (4\pi \tilde{T})^2} \partial_r^2 + \frac{1}{r^2 \sin^2 \rho} \left( \sin \rho \partial_\rho (\sin \rho \partial_\rho) + \partial_\sigma^2 \right), \quad \text{(III.13)} \]

Upon making the decomposition of the solution to Eq. (III.1) into spherical modes
\[ \phi(\tau, \rho, \sigma) = \sum_{l,m} \phi_l(\tau) Y_{lm}(\rho, \sigma), \quad (III.14) \]

where \( Y_{lm}(\rho, \sigma) \) are the usual spherical harmonics, we obtain

\[ \ddot{\phi}_l + r^4(4\pi \tilde{T})^2 \left( \frac{l(l+1)}{r^2} - \mathcal{V}(r) \right) \phi_l = 0. \quad (III.15) \]

This equation describes the fluctuations around an on-shell dynamical spherical membrane. For an equilibrium solution, \( r = r_m \), it is equivalent to the equation of motion of a harmonic oscillator with frequency \( \omega_l \)

\[ \omega_l^2 = r_m^4(4\pi \tilde{T})^2 \left( \frac{l(l+1)}{r_m^2} - \mathcal{V}(r_m) \right) \quad (III.16) \]

In the case of Schwarzschild–de Sitter spacetime (SdS) we get

\[ \mathcal{V}_{SdS}(r_m) = \frac{4r_m - 9M}{r_m^3}. \quad (III.17) \]

Since \( 0 < \mathcal{V}_{SdS}(r_m) < 4/r_m^2 \) we see that Eq. (III.15) for \( r = r_m \), will have real frequencies for \( l > 1 \). Modes with \( l = 0, 1 \) generally possess imaginary frequencies, and are then responsible for the instabilities we observed in the previous section. In fact, a similar observation was already made in Ref. [15] in the study of dynamical spherical membranes in de Sitter spacetime. There, however, modes \( l = 0, 1 \) were not relevant since they represented changes in the energy and linear momentum of the membrane, and these where symmetries of the background geometry. That is however not the case here, since the black hole breaks Lorentz invariance and the instability of the modes \( l = 0, 1 \) is physical. Notice, however, that for \( \tilde{\Lambda} \equiv \Lambda M^2 \geq 16/243 \), that is \( r_m \leq 9M/2 \), the mode with \( l = 1 \) is actually stable (and, by the way, this is the case we are formally interested in, c.f. the discussion in the Introduction, since it corresponds to the membrane being relatively close to the black hole event horizon, see Fig. 1), so eventually we are left with only the zero-mode \( l = 0 \) being unstable.

**IV. HIGHER ORDER MEMBRANE**

We have seen that a cosmological constant can provide the necessary amount of repulsive gravitational field to ensure the existence of spherical membranes in equilibrium in a black hole background. Furthermore, by appropriately choosing the dimensionless parameter \( \tilde{\Lambda} \equiv \Lambda M^2 \), the static membrane can be at any position outside the black hole event horizon. In particular, the static membrane can be at the stretched horizon, only a few Planck lengths outside the black hole event horizon. However, in the latter case we must have a very large cosmological constant. More generally, the fact that \( r_m \) rather follows the cosmological horizon and only approaches the black hole event horizon for large values of the cosmological constant, see Fig. 1, lead us to look for alternative ways of stabilizing a membrane. In this section we shall consider extrinsic curvature corrections to the membrane action [16]. They describe a membrane with finite thickness (the Dirac membrane is infinitely thin).
We write the action in curved spacetime as (where, in principle, \(T, A\) and \(B\) are arbitrary constants)

\[
I_M = \int d\tau d\rho d\sigma \sqrt{-\det(\gamma_{ij})} \left[ -T + A \left( \gamma^{ij} \Omega_{ij} \right)^2 + B \Omega^{ij} \Omega_{ij} \right], \tag{IV.1}
\]

where the second fundamental form is defined in (III.3) and where \(A\) and \(B\) have dimensions of \(\text{length}^{-1}\). In principle, we could also add a term proportional to the scalar curvature of the world-volume; it would be of the same order in derivatives as the two terms involving the second fundamental form. However, the purpose of the following analysis is to show that equilibrium solutions can now be obtained even without the presence of a repulsive gravitational field. That is to say, we will eventually consider membranes in plain Schwarzschild spacetime. In that case, because the spacetime is Ricci–flat, the scalar curvature of the world–volume is related to the two second fundamental form terms, via the Gauß–Codazzi equation

\[
R_\gamma = \left( \gamma^{ij} \Omega_{ij} \right)^2 - \Omega^{ij} \Omega_{ij}, \tag{IV.2}
\]

and therefore only two of these three terms are independent.

It is straightforward to derive the equations of motion from the action (IV.1). For a generic membrane, the trick is to use the Gauß–Weingarten equation (III.8) to eliminate the dependence on the normal vector. The resulting equations, which are not particularly enlightening for the purposes of our paper, contain up to four derivatives in the world–volume coordinates.

Here we shall follow instead a somewhat simpler approach directly adopted to the spherical membranes. We shall derive the effective Lagrangian for a higher order spherical membrane, i.e. the generalization of the Dirac membrane effective Lagrangian (II.8).

Furthermore, in this section we choose to work in the spherically symmetric rest gauge

\[
t = \tau , \quad r = r(\tau) , \quad \theta = \rho , \quad \varphi = \sigma , \tag{IV.3}
\]

compare with Eqs. (II.6) and (II.10). We restrict ourselves to the case when \(a = b\) in the metric (II.5). The induced metric on the world–volume then takes the form

\[
\gamma_{\tau\tau} = -a + \dot{r}^2/a , \quad \gamma_{\rho\rho} = r^2 , \quad \gamma_{\sigma\sigma} = r^2 \sin^2 \rho , \tag{IV.4}
\]

and the components of the second fundamental form are

\[
\Omega_{\tau\tau} = \frac{\dot{r} - aa'}{\sqrt{a - \dot{r}^2/a}} + \frac{3a'}{2 \sqrt{a - \dot{r}^2/a}}, \]
\[
\Omega_{\rho\rho} = \frac{-ar}{\sqrt{a - \dot{r}^2/a}} , \quad \Omega_{\sigma\sigma} = \frac{-ar}{\sqrt{a - \dot{r}^2/a}} \sin^2 \rho . \tag{IV.5}
\]

It is now straightforward to derive the effective Lagrangian corresponding to the action (IV.1)

\[
L = 4\pi r^2 \sqrt{a - \dot{r}^2/a} \left[ -T + \frac{A + B}{a - r^2/a} \left( \frac{\dot{r} - aa'}{a - \dot{r}^2/a} + \frac{3a'}{2} \right)^2 + \frac{2a^2}{r^2} \right] + \frac{A}{a - r^2/a} \left( \frac{4a}{r} \frac{\dot{r} - aa'}{a - \dot{r}^2/a} + \frac{3a'}{2} + \frac{2a^2}{r^2} \right) \tag{IV.6}
\]
The variational principle leads to the equation

$$\partial^2 \tau \left( \frac{\delta L}{\delta r} \right) - \partial_r \left( \frac{\delta L}{\delta \dot{r}} \right) + \frac{\delta L}{\delta r} = 0,$$

(IV.7)
i.e., it contains up to four derivatives with respect to time. In the first place, however, we will look for a static solution \( r_m \), corresponding to \( \partial_r \tau = \partial^2 \tau = \partial^3 \tau = \partial^4 \tau = 0 \). This condition reduces Eq. (IV.7) to

$$T \left[ 2ra + \frac{1}{2} r^2 a' \right] \bigg|_{r=r_m} = (A + B) \left[ 3aa' + \frac{1}{2} r(a')^2 + \frac{1}{2} r^2 a'' - \frac{r^2(a')^3}{8a} \right] \bigg|_{r=r_m}$$

$$+ A \left[ 5aa' + r(a')^2 + 2raa'' \right] \bigg|_{r=r_m}$$

(IV.8)

One can easily check that this condition reduces to Eq. (II.12) for \( A = 0 = B \), i.e. when no higher order terms are present. Furthermore, this equation has no solutions at all for \( r > 0 \) in the Minkowski background, i.e. when \( a = 1 \). This last result is in agreement with that of Ref. [19]. However, for generic \( a(r) \) we will find non-trivial solutions. This is a remarkable result: As shown in [19], the higher order terms cannot support a static spherical membrane in flat Minkowski spacetime, but we have now shown that they can actually do it in generic curved spacetimes, even in spacetimes where the gravitational field is purely attractive.

In particular, for the simple Schwarzschild metric (without cosmological constant!) corresponding to \( a(r) = 1 - 2M/r \), one does find \( r_m > 2M \) solutions where a spherical membrane can be in equilibrium: In this case Eq. (IV.8) reduces to

$$\alpha r_m^3 (2r_m - 3M) - 2\beta Mr_m = \frac{27M^3 - 26M^2 r_m + 6Mr_m^2}{r_m - 2M}.$$

(IV.9)

where

$$\alpha = \left( \frac{T}{A + B} \right) \quad \text{and} \quad \beta = \left( \frac{A}{A + B} \right).$$

(IV.10)

Treating \( T, A \) and \( B \) as arbitrary constants, we can take \( r_m \) at any value outside the event horizon and we even have a one-parameter family of ways to fulfill Eq. (IV.9). That is, we can for instance fix the membrane tension \( T \) from the beginning but still, choosing properly the otherwise arbitrary constants \( A \) and \( B \), pose this static membrane as close to the event horizon as we wish, even on the stretched horizon. In fact, for small \( \epsilon \) we have

$$r_m = 2M(1 + \epsilon), \quad \epsilon \approx \frac{1}{4(2\beta - 4M^2\alpha - 1)}.$$

(IV.11)

We have thus established the existence of the static solution. It remains open the question of the stability of this solution. A complete and covariant discussion of the stability properties of higher order membranes involves the expansion of the action (IV.1) up to second order in fluctuations around a particular solution to the equations of motion. In fact, that was how Eqs. (III.1), (III.2) were obtained for the Dirac membrane [12]. Some preliminary steps of the analog computation for the higher order membrane were taken in Ref.
[20], but the final equation determining the fluctuations around an arbitrary higher order membrane configuration embedded in an arbitrary curved spacetime has, to our knowledge, not yet been obtained in closed form.

Here we will address only the question of the stability of static spherical membranes (IV.8) against zero-mode fluctuations, that is, against radial fluctuations. We thus write

$$ r = r_m + \phi(\tau), $$

where $r_m$ is a solution to Eq. (IV.8). We insert this expression into the equation of motion (IV.7) and keep only terms linear in $\phi$. After some algebra, the resulting differential equation determining the radial fluctuations takes the general form

$$ \frac{d^4\phi}{d\tau^4} + F(r_m)\frac{d^2\phi}{d\tau^2} + G(r_m)\phi = 0, $$

where $F(r_m)$ and $G(r_m)$ are complicated functions carrying the information about the static zeroth order solution and of the curved spacetime. In the case of the Schwarzschild black hole background, $a(r) = 1 - 2M/r$, they are given by

$$ F(r_m) = \left(\frac{r_m}{M} - 1\right)\left[3\alpha \left(1 - \frac{2M}{r_m}\right)^2 - \frac{10Mr_m^2 - 45M^2r_m + 54M^3}{r_m^5}\right], $$

$$ G(r_m) = \frac{-3\alpha M}{r_m^3} \left(\frac{r_m}{M} - 1\right) \left(1 - \frac{2M}{r_m}\right)^2 + \frac{7M^2r_m^2 - 27M^3r_m + 27M^4}{r_m^8}, $$

and we have eliminated $\beta$ using Eq. (IV.9). The fluctuation equation (IV.13) is solved by

$$ \phi(\tau) = c_1 e^{d_1 \tau} + c_2 e^{d_2 \tau} + c_3 e^{d_3 \tau} + c_4 e^{d_4 \tau}, $$

where $(c_1, c_2, c_3, c_4)$ are arbitrary constants, while

$$ d_{(1,2,3,4)} = \pm \left(\frac{-F(r_m) \pm \sqrt{F^2(r_m) - 4G(r_m)}}{2}\right)^{1/2}. $$

The necessary and sufficient condition for stability is that $\phi(\tau)$ be oscillatory ($d_{(1,2,3,4)}$ purely imaginary). A necessary (but not sufficient !) condition for this, is that both $F(r_m) > 0$ and $G(r_m) > 0$. However, that condition leads to

$$ 10r_m^3 - 62Mr_m^2 + 126M^2r_m - 81M^3 < 0, $$

which can not be fulfilled outside the black hole event horizon. Thus any static higher order membrane in the plain Schwarzschild background is unstable with respect to radial fluctuations.

The main result of this section is that the higher order terms in the membrane action can ensure the existence of static spherical solutions in the background of a Schwarzschild black hole. Thus we do not need the cosmological constant of Secs. II–III, or any other kind of repulsive gravitational field, for that matter. However, the problem of instability (at least with respect to radial fluctuations) remained. In the next section we shall discuss a way of bypassing these instability problems.
V. DISCUSSION

In this section we will show that it is possible to avoid the instabilities by considering a dynamical (contracting) membrane instead of a static equilibrium membrane. We will perform the computations for a Dirac membrane in the Schwarzschild–de Sitter background, i.e. the model considered in Secs. II–III. However, the results of Sec. IV, concerning the higher order membrane (especially the existence of unstable static equilibrium solutions), strongly indicate that the following arguments can also be carried through for the higher order membrane in the plain Schwarzschild background, that is to say, the presence of the cosmological constant does not seem to be essential.

Let us return to the potential (II.11) in the case of the Schwarzschild–de Sitter spacetime. It is apparent from Fig. 2 that there are the following qualitatively different kinds of “orbits” for spherical membranes

If \( \tilde{E}^2 > \tilde{V}^2(r_m) \) membranes are always “over” the potential barrier, and either expand for ever or contract towards the black hole, ending trapped by the singularity at \( r = 0 \).

If \( \tilde{E}^2 < \tilde{V}^2(r_m) \) membranes might bounce if they are expanding and are “inside” the potential well, or if they are contracting but are “outside” the potential well.

We are particularly interested in the intermediate case, i.e. \( \tilde{E}^2 = \tilde{V}^2(r_m) \), of which we have analyzed the stability properties of the static equilibrium configuration \( r = r_m \) in Sec. II. However, for \( \tilde{E}^2 = \tilde{V}^2(r_m) \), there is also a contracting solution as well as an expanding solution. For the static solution we have:

\[
\tilde{E}^2 = (4\pi \tilde{T})^2 r_m^4 \left( 1 - \frac{2M}{r_m} - \frac{\Lambda}{3} r_m^2 \right),
\]

\[
2 - \frac{3M}{r_m} - \Lambda r_m^2 = 0.
\]

Now consider a contracting solution with ”energy” given by (V.1). We are interested in solutions contracting from a finite coordinate distance \( r_0 \) (where \( r_m < r_0 < r_{CH} \)) towards \( r_m \). The proper time to reach the summit of the potential barrier (\( r = r_m \)) happens to be logarithmically divergent, and can be derived as follows from Eq. (II.11)

\[
\int_{r_0}^{r_m} d\tau = -\int_{r_0}^{r_m} \frac{dr}{\sqrt{E^2 - V^2(r)}} = -\frac{1}{(4\pi \tilde{T})M} \int_{x_0}^{x_m} \frac{dx}{(x - x_m)\sqrt{f(x)}},
\]

where

\[
f(x) = \frac{1}{3} \left\{ \tilde{\Lambda} x^4 + 2\tilde{\Lambda} x^3 x_m + 3(\tilde{\Lambda} x_m^2 - 1)x^2 + 2(2\tilde{\Lambda} x_m^3 - 3x_m + 3)x + 5\tilde{\Lambda} x_m^4 - 9x_m^2 + 12x_m \right\},
\]

and we have taken the dimensionless variables \( x = r/M \) and \( \tilde{\Lambda} = \Lambda M^2 \). One can thus show that the proper time distance becomes infinite. In fact,

\[
\tau_m - \tau_0 = -\frac{1}{(4\pi \tilde{T})M \sqrt{f(x_m)}} \ln(x - x_m)\bigg|_{x_0}^{x_m} + ... \tag{V.5}
\]
where
\[ f(x_m) = x_m \left(6 - 6x_m + 5x_m^3 \tilde{\Lambda}\right), \]  
(V.6)
and the ellipsis stands for finite terms as \( x \to x_m \).

We have stressed that this infinity occurs in the proper time description. This is clearly different from the infinite “Schwarzschild” time it takes a particle to reach the event horizon (while it only takes a finite proper time, even to reach the singularity). In fact, the above logarithmic divergence arises at a coordinate radius \( r_m > r_{EH} \). This is a particular property of the Schwarzschild–de Sitter spacetime background, that generates a maximum in the effective potential of the membrane, and this does not occur for the Dirac membrane in the plain Schwarzschild geometry.

On the other hand, the above discussion applies quite generally to any potential barrier possessing a maximum. The infinite time to reach the top of the potential is well-known in classical mechanics, for instance, an ordinary pendulum experiences the same process when going towards the vertical (unstable) position with precisely the energy corresponding to being in the static, unstable equilibrium position. The results of Sec. IV show that the dynamics of the higher order membrane in the plain Schwarzschild background is also governed by some kind of finite potential barrier, thus also in that case we can expect to find the above mentioned type of solution.

We now consider fluctuations around the contracting Dirac membrane in the Schwarzschild–de Sitter background, using equation (III.15). In general, in the Schwarzschild–de Sitter spacetime
\[ V(r) = \Lambda + \frac{6\tilde{E}^2}{(4\pi \tilde{T})^2 r^6}, \]  
(V.7)
so that for the contracting membrane with \( \tilde{E} \) given by Eq. (V.1)
\[ V(r) = \Lambda + \frac{6r_m^4 a(r_m)}{r^6}, \quad r_m < r < r_0 \]  
(V.8)
where \( a(r) \) is given by (II.14) and \( r_m \) is the solution of (V.2). The ”frequency” in equation (III.15) now also depends on \( r \)
\[ \omega_1^2(r) = r^4 (4\pi \tilde{T})^2 \left( \frac{l(l+1)}{r^2} - \Lambda - \frac{6r_m^4 a(r_m)}{r^6} \right). \]  
(V.9)
By careful (and partly numerical) study of the formulas for \( r_m \) and \( r_{CH} \) (See Appendix A), one finds that:
\[ \omega_0^2(r) < 0 \]
\[ \omega_1^2(r) > 0 \quad \Rightarrow \quad \tilde{\Lambda} > 0.07737... \quad \text{for all } l \geq 2 \]  
(V.10)
and these results hold for all \( r_m \leq r \leq r_0 \). Notice that the condition for \( \omega_1^2 \) being positive is somewhat more restrictive than what was obtained at the end of Sec. III. The reason is that we now consider \( \omega_1^2 \) for arbitrary \( r \), not only for \( r = r_m \).
We are particularly interested in the case when the membrane approaches a position close to the Schwarzschild event horizon. This is the case when $\Lambda M^2 \equiv \tilde{\Lambda}$ is close to the maximal value $\tilde{\Lambda}_{\text{max}} = 1/9$, c.f. the discussion at the end of Sec. III. From the above results follow that for such contracting membranes, all $l \geq 1$-modes are stable everywhere (that is, for all $r$) during the contraction. At first sight it seems, on the other hand, that the zero-mode is always unstable. However, for a contracting spherical membrane, the zero-mode does not represent a real physical fluctuation. The zero-mode merely represents a time-translation, and therefore should be eliminated from the spectrum. We thus conclude that all physical fluctuations around the contracting membrane are stable.

To actually compute the classical and quantum spectrum of physical fluctuations ($l \geq 1$), we must solve the equation

$$\ddot{\phi}_l + \omega^2_l(r) \phi_l = 0,$$

where $\omega^2_l(r)$ is given by (V.9), and $r = r(\tau)$ is obtained by inverting (V.3). Eq. (V.3) can be solved explicitly in terms of Jacobi Elliptic Functions [21], and Eq. (V.11) is then of (generalized) Lamé-type. However, we shall not attempt here to obtain the exact solution for $\phi(\tau)$ in closed form. Instead we will use the result that the membrane takes infinite proper time to reach the summit of the potential, c.f. Eq. (V.3). Physically this means that after some finite time, the membrane is effectively always very close to the top of the barrier. We can thus approximate $\omega^2_l(r)$ by a constant which is essentially $\omega^2_l(r_m)$, as given by Eqs. (III.16) and (III.17). We are then left with an ordinary harmonic oscillator equation, which can be easily solved. In fact, in Ref. [10] it was assumed that this equilibrium point existed and from the quantization of the harmonic oscillations a discrete spectrum of energies for the membrane obtained. The entropy associated to this discrete energy levels, in the thermal bath of the background black hole, can be computed and one obtains a leading term proportional to the two-surface of the membrane plus a non-leading logarithmic term.

In conclusion, we have learned that in a curved background, as opposite to what happens in flat space, spherical membranes can exist in equilibrium. Higher order membranes provide enough arbitrariness to locate the membrane as close to the event horizon as we want, even in the plain Schwarzschild background. The issue of the stability of spherical membranes is delicate. We identified modes $l = 0$ and sometimes also $l = 1$ as responsible for the instabilities. We discussed, however, a certain range of validity of the perturbative description and eventual quantization of the fluctuations. Further study of the non-radial fluctuations of higher order membranes is needed, as well as of the situation in more general curved backgrounds (for example, to include rotation) to see if a “true” stability mechanism can be found.

**ACKNOWLEDGMENTS**

A.L.L. was supported by NSERC (National Sciences and Engineering Research Council of Canada), while C.O.L was supported by the NSF grant PHY-95-07719 and by research founds of the University of Utah.
APPENDIX A: HORIZONS AND STATIC MEMBRANE IN SCHWARZSCHILD–DE SITTER

Provided that $M > 0$ and $\Lambda \geq 0$, the equation

$$1 - \frac{2M}{r_H} - \frac{\Lambda}{3} r_H^2 = 0 \quad (A.1)$$

has real positive solutions if and only if $\Lambda M^2 \equiv \tilde{\Lambda} \leq 1/9$. In that case, the cosmological horizon $r_{CH}$ and the black hole event horizon $r_{EH}$ are given by:

$$\frac{r_{CH}}{M} = Z^{-1} + \frac{Z}{\tilde{\Lambda}}, \quad (A.2)$$

$$\frac{r_{EH}}{M} = -\frac{1}{2} \left[ \frac{1 - i\sqrt{3}}{Z} + \frac{(1 + i\sqrt{3})Z}{\tilde{\Lambda}} \right], \quad (A.3)$$

where

$$Z = \left( i\sqrt{\tilde{\Lambda}^3(1 - 9\tilde{\Lambda}) - 3\tilde{\Lambda}^2} \right)^{1/3}. \quad (A.4)$$

The radial coordinate of a static Dirac membrane, solution of Eq. (II.12),

$$2 - 3\frac{M}{r_m} - \Lambda r_m^2 = 0 \quad (A.5)$$

is explicitly given by

$$\frac{r_m}{M} = \frac{2}{W} + \frac{W}{3\tilde{\Lambda}}, \quad (A.6)$$

where

$$W = 2^{-1/3} \left( 3i\sqrt{3\tilde{\Lambda}^3(32 - 243\tilde{\Lambda}) - 81\tilde{\Lambda}^2} \right)^{1/3}. \quad (A.7)$$

In the limiting cases we have

$$\tilde{\Lambda} = 0 : \quad r_{EH} = 2M, \quad r_m = r_{CH} = \infty, \quad (A.8)$$

$$\tilde{\Lambda} = 1/9 : \quad r_{EH} = r_m = r_{CH} = 3M, \quad (A.9)$$

while in general

$$r_{EH} \leq r_m \leq r_{CH}. \quad (A.10)$$

These results are illustrated in Fig. [I].
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FIGURES

FIG. 1. We observe that the radius at which a spherical membrane is at equilibrium $r_m$, in Schwarzschild–de Sitter space, lies between the cosmological horizon $r_{CH}$ and the black hole event horizon $r_{EH}$. It rather follows the cosmological horizon and only approaches the event horizon for relatively large values of the cosmological constant. The three radii merge together at $r = 3M$ and $\Lambda = \Lambda M^2 = 1/9$. Higher values of the cosmological constant lead to a naked singularity, see Appendix A.

FIG. 2. The effective potential (normalized to its maximum value $V_m$) felt by a spherical membrane in Schwarzschild–de Sitter background for two different values of the cosmological constant $\Lambda = \Lambda M^2$. For the values of $\Lambda = 0.1, 0.01$ shown in the figure, the coordinates of the maximum are $r_m/M = 3.305, 13.32$, respectively. As $\Lambda$ decreases, $r_m$ increases and the potential moves to the right. In the other extreme, $\Lambda = 1/9$, we have $r_m = 3M$ and $V_m = 0$. 
Figure 1
Figure 2

\[ \frac{\tilde{V}^2}{\tilde{V}_m^2} \]

\[ \tilde{\Lambda} = 0.1 \]

\[ \tilde{\Lambda} = 0.01 \]