Authentication protocol based on polygamous nature of quantum steering

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It is well known that certain quantum correlations like quantum steering exhibit a monogamous relationship. In this paper, we exploit the asymmetric nature of quantum steering and show that there exist states which exhibit a kind of polygamous correlation, where the state of one party, Alice, can be steered only by the joint effort of the other two parties, Bob and Charlie. As an example, we explicitly single out a particular set of 3 qubit states which exhibit polygamous relationship and also provide a recipe to identify the complete set of such states. We also provide a possible application of such states to an information theoretic task, we term as quantum key authentication (QKA). QKA can also be used in conjunction with other well known cryptography protocols to improve their security and we provide one such example with quantum private comparison (QPC).

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Introduction:— Quantum mechanical correlations offer many surprises whenever one digs into the theory to understand its nature and differences from the classical world. Some examples include correlations arising from entanglement [1], Bell non-locality [2–4], contextuality [5, 6], coherence [7, 8] and steering [9]. Such correlations have the unique and surprising property of being monogamous [10–16]. Correlations between certain parties are said to be monogamous if they diminish when shared among more additional parties. A simple example is illustrated by Bell-CHSH inequality [14]: Two parties Alice and Bob share non-local correlations and are able to violate the Bell inequality. If the state of Alice is also entangled with a third party Charlie, the non-local correlations between Alice and Bob diminish as the correlations between Alice and Charlie increase. It is therefore implied that the Bell-CHSH correlations are monogamous. Monogamy of correlations has been extensively studied and has found widespread applications in information theoretic tasks like key distribution [17, 18].

Quantum steering [9] was first introduced for bipartite parties. In this scenario, Bob prepares an entangled state of joint systems A and B. He keeps the system B with himself and transmits the system A to Alice. Alice does not trust Bob, but believes that the system sent to her is quantum. Bob’s job is to convince Alice that the system she sent indeed belongs to an entangled state and it is possible for him to steer her state.

While it is well known that steering is monogamous [19, 20], a major aspect of it has not been addressed yet. Steerability at its core is asymmetric and although it’s monogamous from one side, the nature of correlations from the other side is yet to be studied.

Consider a scenario, where three parties Alice, Bob and Charlie share an entangled state \( \rho_{abc} \). From monogamy of steerability [19, 20] if Alice can steer Bob, she cannot steer Charlie and vice-versa. The idea originates from the resource theory of quantum correlations like entanglement [1, 21]. However, the question we address here is to check whether there exist states for which a particular party (say Alice) cannot be steered independently either by Bob or Charlie but rather only if they steer together. The scenario is exact opposite of what is generally considered to show monogamous nature of quantum steering.

In this paper, we provide a detailed analysis of quantum steering in such a scenario. We identify a set of states for which Alice can share a polygamous relationship with Bob and Charlie and also lay down the foundation for identifying the complete set of such states. Moreover, We show a possible advantage offered by a polygamous relationship much like its monogamous counterpart in the security of quantum key distribution (QKD) protocols. To that end we propose a new key distribution protocol, termed as quantum key authentication (QKA), which utilizes the GHZ state. The protocol is different than the standard QKD protocols and is made semi-device independent by the aforementioned polygamous relationship. The security of the protocol remains the same as for entanglement based QKD schemes [22–24]. QKA offers various advantages over QKD and is manifested in the example we provide where it is used in conjunction with quantum private comparison (QPC) protocols [25–27].

Polygamous steering:— Since we are solely interested in a subset of states for which Alice cannot be steered individually by Bob or Charlie but only by their joint efforts in a tripartite scenario, we start with a tripartite state \( \rho_{abc} \) prepared by Bob (or Charlie). Bob sends the subsystem A to Alice and C to Charlie. Since Alice does not believe Bob or Charlie, she asks them to perform a set
of measurements and send her the outcomes. Based on the measurement outcomes, she computes the coherence of her conditional states. We show that there exist states $\rho_{abc}$ for which Alice is steerable if and only if Bob and Charlie make an effort together but not otherwise. To find such a set of states $\{S(A \leftrightarrow B, C)\}$, we first find out a set of states $\{S(A \leftrightarrow B, C)\}$ for which Alice is steerable by Bob and Charlie together as well as individually. We then compute the union of the set of states $\{S(A \leftrightarrow B)\}$ for which Alice is steerable by Bob and Charlie individually. Our set of interest is the difference of the above two sets, i.e., $S_i = S(A \leftrightarrow B, C) \setminus \{S(A \leftrightarrow B)\}$.

We now explicitly find out a set of states which exhibit polygamous nature of quantum steering. First, we focus on to single out the set $\{S(A \leftrightarrow B, C)\}$. Alice will be convinced that her state is entangled if her system $A$ cannot be written by a local hidden state (LHS) model

$$\rho_{bc}^{BC} = \sum_{\lambda} \mathcal{P}(\lambda)\mathcal{P}(b, c|B, C, \lambda)\rho_{A}^{Q}(\lambda),$$

where $\{\mathcal{P}(\lambda), \rho_{A}^{Q}(\lambda)\}$ represents an ensemble of pre-existing local hidden states of Alice and $\mathcal{P}(b, c|B, C, \lambda)$ is Bob and Charlie’s joint stochastic map to convince Alice by preparing a state $\rho_{bc}^{BC}$. $\mathcal{P}(\lambda)$ forms a valid probability distribution such that $\sum_{\lambda} \mathcal{P}(\lambda) = 1$. The joint probability distribution on such states can be written as,

$$\mathcal{P}(a_i, b_j, c_k) = \sum_{\lambda} \mathcal{P}(\lambda)\mathcal{P}(b_j, c_k|B, C, \lambda)\rho_{A}^{Q}(a_i|\lambda),$$

where $\mathcal{P}(a_i, b_j, c_k)$ represents the probability to obtain outcome $a_i$ for the measurement of observables chosen from the set $\{A_i\}$ by Alice, outcome $b$ for the measurement of observables chosen from the set $\{B_j\}$ by Bob and outcome $c$ for the measurement of observables chosen from the set $\{C_k\}$ by Charlie.

We consider a tripartite state $\rho_{abc}$ distributed between Alice ($A$), Bob ($B$) and Charlie ($C$). Alice asks Bob and Charlie to perform projective measurements on their respective systems ($B$) and ($C$) on stated bases. We consider Bob and Charlie to perform projective measurements in Pauli eigenbases (or in general on a set of mutually unbiased bases) and communicate the results to Alice. Upon receiving the results, Alice measures coherences on her conditional states with respect to her Pauli eigenbases (or a mutually unbiased bases) with the choice of basis being based on the measurement results from Bob and Charlie. It is seen that Bob, together with Charlie can steer the state of Alice if any of the trailing inequalities are violated

$$\sum_{i \neq k, j, b, c} p(\rho_{A|I_i^k I_j^b}^C)C_k(\rho_{A|I_i^k I_j^b}^C) \leq 6\epsilon,$$  \hspace{1cm} (3)

$$\sum_{i, j \neq k, b, c} p(\rho_{A|I_i^k I_j^b}^C)C_k(\rho_{A|I_i^k I_j^b}^C) \leq 6\epsilon,$$  \hspace{1cm} (4)

$$\sum_{i = j \neq k, b, c} p(\rho_{A|I_i^k I_j^b}^C)C_k(\rho_{A|I_i^k I_j^b}^C) \leq \epsilon,$$  \hspace{1cm} (5)

$$\sum_{i \neq j = k, b, c} p(\rho_{A|I_i^k I_j^b}^C)C_k(\rho_{A|I_i^k I_j^b}^C) \leq 2\epsilon,$$  \hspace{1cm} (6)

$$\sum_{i \neq j \neq k, b, c} p(\rho_{A|I_i^k I_j^b}^C)C_k(\rho_{A|I_i^k I_j^b}^C) \leq 2\epsilon$$  \hspace{1cm} (7)

$$\sum_{i = k \neq j, b, c} p(\rho_{A|I_i^k I_j^b}^C)C_k(\rho_{A|I_i^k I_j^b}^C) \leq 2\epsilon.$$  \hspace{1cm} (8)

Unless otherwise stated we consider $\epsilon = \sqrt{6}$ as shown in $[25, 29]$ for $l_1$-norm measure of quantum coherence $C_1(\rho)$ of the state $\rho$ with respect to the basis $i$, $\{i, j, k\} \in \{0, 1, 2\}$ and $\{a, b, c\} \in \{0, 1\}$.

It is not difficult to guess other inequalities based on the permutations and combinations of Pauli bases. Now, to prove the criteria (3), we consider that the conditional states of Alice have a local hidden state model as given in Eq. (1), i.e., $\rho_{A|I_i^k I_j^b}^C = \frac{\rho_{A|I_i^k I_j^b}^{BC}}{p(\rho_{A|I_i^k I_j^b}^{BC})}$. Thus,

$$\sum_{i \neq k, j, b, c} p\left(\rho_{A|I_i^k I_j^b}^{BC} \mid \rho_{A|I_i^k I_j^b}^{BC}\right)C_k\left(\rho_{A|I_i^k I_j^b}^{BC} \mid \rho_{A|I_i^k I_j^b}^{BC}\right) \leq \sum_{i \neq k, j, b, c} \mathcal{P}(\lambda)\mathcal{P}(b, c|I_i^k I_j^b, \lambda)\rho_{A}^{Q}(\lambda)$$

$$\sum_{i \neq j = k, b, c} \mathcal{P}(\lambda)\mathcal{P}(b, c|I_i^k I_j^b, \lambda)C_k\left(\rho_{A}^{Q}(\lambda)\right)$$

$$\sum_{i \neq j \neq k, b, c} \mathcal{P}(\lambda)\mathcal{P}(b, c|I_i^k I_j^b, \lambda)C_k\left(\rho_{A}^{Q}(\lambda)\right)$$

$$\sum_{i = k \neq j, b, c} \mathcal{P}(\lambda)\mathcal{P}(b, c|I_i^k I_j^b, \lambda)C_k\left(\rho_{A}^{Q}(\lambda)\right)$$

$$\sum_{k, j, c, \lambda} \mathcal{P}(\lambda)\mathcal{P}(c|I_j^c, \lambda)C_k\left(\rho_{A}^{Q}(\lambda)\right)$$

Now we focus on to single out the second set, i.e., $S(A \leftrightarrow B) \cup S(A \leftrightarrow C)$. This is the union of sets of states for which Alice ($A$) can be steered individually by Bob ($B$) and Charlie ($C$). In this case, Alice ignores the results sent by one party while acknowledging the other. A set of steering inequalities in this two-qubit scenario, where Alice ignores the results of Charlie, can be constructed as $[25, 29]

$$\sum_{i = k, b} p(\rho_{A|I_i^k I_j^b}^C)C_k(\rho_{A|I_i^k I_j^b}^C) \leq \epsilon,$$  \hspace{1cm} (9)

$$\sum_{i \neq k, b} p(\rho_{A|I_i^k I_j^b}^C)C_k(\rho_{A|I_i^k I_j^b}^C) \leq 2\epsilon.$$  \hspace{1cm} (10)
\[ \sum_{j=k,c} p(\rho_{A|\Pi'_j})C_k(\rho_{A|\Pi'_j}) \leq \epsilon, \quad \text{and} \]
\[ \sum_{j\neq k,c} p(\rho_{A|\Pi'_j})C_k(\rho_{A|\Pi'_j}) \leq 2\epsilon. \]

We denote Eqs. (3)-(8) as the first set and Eqs. (9)-(12) as the second set of inequalities. It is our aim to look for a set of states which violate at least one of the first set but not the second set of inequalities. This would ensure that the state of Alice can only be steered by Bob and Charlie together but not individually.

It is well known fact that a state is deemed steerable if a steering inequality is violated. However, the converse is not always true. Thus, there is no definite way to single out the set of such unsteerable states as is required in the 2-qubit scenario to define the set of our interest \((S_i)\). To overcome this issue, we need to use the tightest steering inequalities with semi-definite programming and the free will to choose the bases. However, one may start with the set of states for which bi-partite entanglements i.e., \(E_{AB}\) and \(E_{AC}\) are zero. For such states, by definition, Alice cannot be steered in the 2-qubit scenarios.

For example, we consider a genuine entangled state such as a generalized GHZ state
\[ |\psi\rangle = \alpha|000\rangle + \sqrt{1-\alpha^2}|111\rangle, \]
where \(0 \leq \alpha \leq 1\). For the state, it can be shown that no inequality from the second set is violated. This is due to the fact that the entanglement between Alice-Bob \((E_{AB})\) and Alice-Charlie \((E_{AC})\) are zero for GHZ states. On the other hand, the inequality in Eq. (5) from the first set is violated for a certain range of \(\alpha\).

In this sense, a non-trivial example would be \(W\) state, i.e., \(|\psi\rangle_W = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)\), for which \((E_{AB})\) and \((E_{AC})\) are non-zero (see supplemental material [30]).

However, in the following sections, we will only be concerned about the maximal violation of inequalities (3)-(8). We consider the same set of measurement bases for Alice, Bob and Charlie. They follow certain protocols to choose a basis as depicted by these inequalities. For each inequality or protocol and a fixed set of measurement bases, there must be an unique state violating the inequality maximally. For example, the state \(|\psi\rangle_{13}\) shows the maximal violation at \(\alpha = \frac{1}{\sqrt{2}}\) only for the inequality (5) as shown in Fig. 1.

Quantum Key Authentication Protocol:— A practical advantage of polygamous nature of quantum steering is manifested in quantum key authentication protocols (QKA). Such protocols can be used as a standalone information processing task or in conjunction with other protocols like quantum private comparison (QPC) [25-27] to enhance their security. We propose one such QKA protocol and also exemplify its application with QPC.

Let us consider the scenario in which Bob and Charlie wish to share a secret key. They have fixed a protocol for the same, which includes them performing projective measurements in various predetermined basis. At the end of the protocol, it is desired that both the parties disclose the choice of basis of measurements but not the results which are kept as part of the key. They keep only those results as key for which their choice of basis matched. However, it may be the case that one of the parties claims to have performed a measurement in a particular basis, when in reality (s)he has not due to misalignment of their reference frames or maybe to cheat other. This issue is more relevant when their labs are far apart and there is no way to validate their reference frames. This scenario closely resembles online transactions between untrustworthy parties using an escrow service. To authenticate the claims of either party, they take help from a third trusted party, Alice, who should remain oblivious to the secret key. We term the protocol to authenticate the claims of either party as a quantum key authentication protocol (QKA).

The trusted third party Alice prepares the three qubit GHZ state,
\[ |\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \]
which is steerable maximally by Bob and Charlie together when they are in the same mutually unbiased bases but not individually as in Fig. 1. Alice distributes this state among Bob, Charlie and herself. Bob and Charlie perform projective measurements \(\Pi_i(\theta)\) (see supplemental material [30]) on their subsystems. Since Bob and Charlie do not trust each other, they ask Alice to authenticate whether the other party has indeed performed the measurements as claimed. Alice authenticates the key by
measuring the coherence on her subsystem and plugging in the various values in the steering inequality (5).

If Bob and Charlie indeed follow the protocol described by the steering inequality given in Eq. (6) and use the general measurements as shown in [30], the state of Alice is steered and the inequality (5) is violated maximally. However, if at least one of them had performed the measurement in a different set of bases, the inequality would not be maximally violated. Thus, Alice validates the key based on the violation of the steering inequality (5). Similarly, Alice can in principle choose any of the protocols listed in Eq. (6) and (8) and a state accordingly, which can violate the inequality maximally.

As a standalone quantum key distribution (QKD) protocol, it falls in the category of entanglement based schemes, for which the minimum secure key rate is given as

\[
    r_{\text{min}} = 1 + 2(1 - Q) \log_2(1 - Q) + 2Q \log_2 Q, \tag{15}
\]

where \(Q\) is the quantum bit error rate, and for entanglement based key distribution protocols is given as

\[
    Q = \frac{1}{2} \sum_{b+c=1}^{2} \sum_{i \neq j} \text{tr}(\Pi^b_i \otimes \Pi^c_j \rho_{abc}), \tag{16}
\]

where \(\rho_{abc} = |\psi\rangle\langle\psi|\). The quantum bit error rate \(Q\) is defined as the average probability for Bob and Charlie to not get the correct outcomes even when they perform measurements in the correct basis. For the particular case of GHZ states, \(Q = 0\) in the absence of an eavesdropper which gives \(r_{\text{min}} = 1\). However, the presence of an eavesdropper will increase the bit error rate and decrease the secure key rate. It should be noted that Bob and Charlie will be able to perform key distribution for any state of the form given in Eq. (13) with unit key rate in the ideal scenario. However, the keys can be authenticated with certainty only when Alice observes maximal violation of coherence complimentary relation (5). For the cases when a violation is not observed, the keys distributed between Bob and Charlie cannot not be authenticated. Although, the aforementioned protocol falls in the category of entanglement based QKD, it is quite different from the traditional ones. The protocol is semi-device independent, in the sense that the two interested parties do not trust each other, while another third trusted party enables secure key distribution. The third party, although essential to security, does not participate in key distribution: the key is only distributed among the interested parties. Furthermore, any attempt by the third party to eavesdrop (without losing trustworthiness) will correspond to quantum bit error rate (16) in the key, which can be measured.

The QKA protocol as detailed above can also be used in conjunction with other information processing tasks to increase their security. We provide an example where QKA used with QPC leads to a more robust version of QPC. The main aim of a QPC protocol is to compare the information coming from two parties. Examples include the millionaire’s problem and its variants [25]. Several proposed QPC protocols are based on the properties of EPR pairs or the GHZ state [26, 27]. Chen et. al.’s proposed protocol [27] based on GHZ states is tailor made to be used with QKA.

A standard QPC protocol comprises of two parties Bob and Charlie who wish to compare the equality of their information, without letting the other each other or anyone else have access to the information. They accomplish this task with the help of a third semi-honest party, Alice. Alice is tasked with comparison of the information, but she may also try to steal it. Therefore, the goal is to let Alice compare the information without letting her have access to it. An implicit assumption in such standard QPC protocols is that both the parties, Bob and Charlie are trustworthy. The devices with Bob and Charlie are assumed to be uncorrelated to any eavesdropper and perform the measurements as claimed. Our scheme is directed at bypassing this assumption, while maintaining that Alice does not gain any information. In this way QPC protocols can be elevated to the status of being semi-device independent.

Since our scheme utilizes GHZ states, it provides an added advantage in that existing QPC protocols like Chen’s protocol [27] can be easily modified. In Chen et. al.’s protocol, Alice prepares a sequence of GHZ states which she distributes among Bob, Charlie and herself. After performing a privacy amplification of the quantum channel, Bob and Charlie perform a projective measurement in the \(\sigma_x\) basis. Bob and Charlie declare the results of their measurements in the following manner: if the result corresponds to their secret message they announce 0, and 1 otherwise. Depending upon the announced results, Alice performs a unitary evolution of her subsystem followed by a projective measurement in the \(\sigma_x\) basis. The outcome to this measurement reveals the equality of Bob and Charlie’s information. No where in the protocol Bob and Charlie announce their information publicly.

In the above protocol it is implicitly assumed that Bob and Charlie’s devices are uncorrelated to an eavesdropper. Such a correlation with an eavesdropper corresponds to additional degrees of freedom which in turn implies that the devices can not essentially perform qubit measurements. In order to elevate the protocol to a semi-device independent status, an additional check in form of QKA can be implemented prior to the protocol. This single check serves two purposes: firstly, it can help to verify the security of the quantum channel to be used and secondly, it can help to authenticate the measurements of Bob and Charlie. Such cojoining of two information processing tasks, although requiring slightly higher number of resources (in the form of GHZ states), yields not one but two advantages over the standard QPC protocols.
In the modified protocol, Alice prepares a long sequence of GHZ states of length $L$ and distributes them amongst Bob, Charlie and herself. Bob and Charlie select a basis from the set of arbitrarily chosen mutually unbiased bases to perform the measurements. The selection made by them is not random, but is rather biased towards a particular basis. For a length $l < L$, which includes all the $\sigma_z$ measurements and a randomly selected equal number of $\sigma_x$ measurements, all the three parties perform QKA as detailed above. For the maximal violation of inequality [31], the coherence of Alice’s state is steered maximally, which in turn implies that Bob and Charlie’s devices indeed perform as claimed and they are not cheating. For the remaining sequence of states of length $L - l$, the parties perform QPC remain untouched as detailed in [27]. Thus with the help of QKA, we ensure privacy amplification of the quantum channel as well as elevate the protocol to a semi-device independent status, thereby providing an additional requirement of a third semi-honest party. Many contemporary key distribution scenarios deal with third parties; examples include a customer buying a product on an online web-portal. The web-portal hosts various distributors and upon a purchase, authenticates the transaction between customers and the distributors. Our scheme functions in a similar regard, albeit in a more secure manner. Furthermore, QKA can also be used in conjunction with other information processing tasks as exemplified above.

Conclusion:— In this paper we argue that another aspect of quantum steering exists in which two parties jointly steer the coherence of a third party while individual steering is not possible. We analyze conditions to check for such a polygamous steering scenario in a 3 qubit system and provide a general mechanism to compute states which exhibit this behaviour. We find that GHZ states are a good example of polygamous steering. Like standard quantum steering, polygamous steering too has its uses. We provide one such application in the form of a quantum key authentication (QKA) protocol. QKA proves helpful in detecting dishonest parties in key distribution scenarios and pushes a step ahead towards the semi-device independence with an additional requirement of a third semi-honest party. Many contemporary key distribution scenarios deal with third parties; examples include a customer buying a product on an online web-portal. The web-portal hosts various distributors and upon a purchase, authenticates the transaction between customers and the distributors. Our scheme functions in a similar regard, albeit in a more secure manner. Furthermore, QKA can also be used in conjunction with other information processing tasks as exemplified above.

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SUPPLEMENTAL MATERIAL

Example of states with zero bi-partite entanglements $E_{AB}$ and $E_{AC}$

Here, we first define Pauli matrices in an arbitrary $\Theta$ direction such that the eigenvectors for $\sigma_z(\Theta)$ are \( |z^+(\Theta)\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{-i\phi} \sin \frac{\theta}{2} \end{pmatrix} \) and \( |z^-(\Theta)\rangle = \begin{pmatrix} \sin \frac{\theta}{2} \\ -e^{-i\phi} \cos \frac{\theta}{2} \end{pmatrix} \), the eigenvectors for $\sigma_x(\Theta)$ are \( |x^+(\Theta)\rangle = \frac{|z^+(\Theta)\rangle + |z^-(\Theta)\rangle}{\sqrt{2}} \) and \( |x^-(\Theta)\rangle = \frac{|z^+(\Theta)\rangle - |z^-(\Theta)\rangle}{\sqrt{2}} \) and the eigenvectors for $\sigma_y(\Theta)$ are \( |y^+(\Theta)\rangle = \frac{|z^+(\Theta)\rangle + i|z^-(\Theta)\rangle}{\sqrt{2}} \) and \( |y^-(\Theta)\rangle = \frac{|z^+(\Theta)\rangle - i|z^-(\Theta)\rangle}{\sqrt{2}} \). We consider that Alice, Bob and Charlie— all start with the Pauli operators and perform measurements on the bases of Pauli operators but the reference frames of Bob and Charlie are aligned at an angle $\Theta \equiv (\theta, \phi)$ with respect to that of Alice.

Suppose Bob and Charlie measuring in $\theta, \phi$ direction. Inequality in Eq. (1) shows maximum violation for the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ measured in the basis with $\theta = \pi$ and $\phi = \pi$ violates the inequality in Eq. (2) maximally. Inequality in Eq. (3) is violated maximally by the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ for the measurement with $\theta = \frac{\pi}{2}$ and $\phi = 0$. Inequality in Eq. (4) is violated maximally by the state $\frac{1}{2}(|01\rangle + |10\rangle + |11\rangle + |00\rangle)$ for $\theta = \frac{\pi}{2}, \phi = \pi$. The next inequality in Eq. (5) shows maximum violation for the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $\theta = \frac{\pi}{2}, \phi = \frac{3\pi}{2}$. For the state $\frac{1}{2}(|01\rangle + |10\rangle + |11\rangle + |00\rangle)$ inequality in Eq. (6) is maximally violated for $\theta = 0$ and $\phi = \frac{3\pi}{2}$.

Example of states with non-zero bi-partite entanglements $E_{AB}$ and $E_{AC}$

Let’s take a generalized W state of the form $|\psi\rangle_{GW} = \frac{1}{\sqrt{5}}|001\rangle + \sqrt{\frac{2}{5}}|010\rangle + \frac{3}{5}|100\rangle$. This state gives violation for set 1 and set 2 as well. In set 1 first inequality gives a value of 17.4464, which is greater than $6\sqrt{6} \approx 14.6969$. Second one gives 14.5289 which is less than $6\sqrt{6} \approx 14.6969$. Third one is violated by 2.92952 which is greater than $\sqrt{6} \approx 2.4495$. Fourth one give 5.79661 which is more than $2\sqrt{6} \approx 4.8989$. Fifth and sixth one give 5.88952 which is also more than $2\sqrt{6} \approx 4.8989$. Now we will see what about the second set for W state. First one give 2.82029, which is greater than $\sqrt{6} \approx 2.4495$. For second one we get 5.64058 which is more than $2\sqrt{6} \approx 4.8989$. Third one give 0.893575 which is less than $\sqrt{6} \approx 2.4495$. The final one give 1.77809 which is again less than $2\sqrt{6} \approx 4.8989$.

Let’s take the three qubit W state $|\psi\rangle_W = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. For this state inequalities of set 1 are violated but not of set 2. In set 1 first inequality gives a value of 15.6835, which is greater than $6\sqrt{6} \approx 14.6969$. Second one gives 15.6835 which is also greater than $6\sqrt{6} \approx 14.6969$. Third one is violated by 2.86603 which is greater than $\sqrt{6} \approx 2.4495$. Fourth one give 5.4664 which is more than $2\sqrt{6} \approx 4.8989$. Fifth and sixth one give 5.69936 which is also more than $2\sqrt{6} \approx 4.8989$. Now we will see what about the second set for W state. After tracing out Charlie the entanglement of the reduced density matrix of Alice and Bob is $\frac{\sigma}{2}$. First one give 1.84424, which is less than $\sqrt{6} \approx 2.4495$. For second one we get 3.54606 which is again less than $2\sqrt{6} \approx 4.8989$. Now after tracing out Bob the concurrence of reduced density matrix of Alice and Charlie is again $\frac{\sigma}{2}$. Third one give 1.84424 which is less than $\sqrt{6} \approx 2.4495$. The final one give 3.54606 which is again less than $2\sqrt{6} \approx 4.8989$. 