Numerical analysis of backward erosion by soil-water interface tracking

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ABSTRACT

The backward erosion of soils, which is induced by seepage flows, is numerically simulated. To the end, the following three aspects need to be computed: Water flow fields, onset and speed of erosion and boundary tracking between the soil and the water phases. The authors employ the Darcy-Brinkman equations as the governing equations for the water flow fields around the soils, which easily enable the simultaneous analysis of the seepage flows in the porous media and the water flows in the fluid domain. The onset and the speed of the seepage-induced erosion is predicted by an empirical formula constructed from the flow velocity and the pressure gradient of the seepage water. The boundary tracking scheme based on the phase-field equation is applied for tracking the soil boundary changing with the erosion. This article shows that the combination of the above three aspects achieves the stable computation of the seepage-induced backward erosion of soils.

Keywords: backward erosion, Darcy-Brinkman equation, phase-field equation, finite volume method

1 INTRODUCTION

When erosion occurs around soil surface where a seepage water flow comes out, the erosion develops in the opposite direction to the water flow. This type of erosion is known as backward erosion, which leads to piping of soils. Recently, the damages and failures of soil structures, such as levees and small embankment dams for irrigation reservoirs, has occurred more frequently because of a greater chance of severe typhoons and localized heavy rains. The piping phenomenon, induced by the soil erosion due to the seepage flows, is known as a primary cause of embankment breaks. Actually, Foster et al. (2000) statistically investigated the failures and the incidents involving embankment dams around the world, and reported that the soil piping accounted for approximately 45% of these incidents. Therefore, the soil erosion is considered to be a major threat to the structures made of earth materials and the objective of this paper is to develop a numerical method to compute the seepage-induced erosion, especially the backward erosion.

To this end, this article begins with the simultaneous computation of seepage flows in porous media and regular flows in fluid domains, because the erosion of soils is affected by water flows inside and outside the soils, and these two flow fields need to be grasped for predicting how the erosion develops. As described in the next section, the authors employ the Darcy-Brinkman equations as the governing equations for this problem and propose a numerical method to simultaneously solve the Navier-Stokes flow in the fluid domains and the Darcy flow in the porous media. Combining the boundary tracking with the simultaneous analysis of the Darcy and the Navier-Stokes flows, the computation of the seepage-induced erosion is carried out. In order to determine the moving speed of the soil boundary due to the erosion, an empirical formula predicting the discharge rate of the soils is utilized. The numerical method proposed by Sun and Beckermann (2007) is applied to tracking the soil boundary changing with the erosion. The method enables the sharp interface tracking based on the phase-field equation. The numerical results presented in the end of this article show that the combination of the above numerical methods achieves the stable computation of the seepage-induced erosion.

2 GOVERNING EQUATIONS

2.1 Water flow

The Darcy-Brinkman equations are adopted as the governing equations for the coupled analysis of the Navier-Stokes and the Darcy flows;

\[
\frac{\partial u_i}{\partial x_i} = 0
\]

(1)
\[
\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i} \left( \frac{u_i u_j}{\lambda} \right) = -\frac{\lambda}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_i^2} - \frac{\lambda g}{k} u_i \quad (2)
\]

where \( u_i, p, \rho, \nu, \lambda, k, g, t \) and \( x_i \) denote the flow velocity, the piezometric pressure, the density of water, the kinematic viscosity of water, the porosity, the hydraulic conductivity, the gravitational acceleration, time and Cartesian coordinates, respectively.

Eq. (2) can describe the Navier-Stokes equations in the fluid phase by giving \( \lambda = 1.0 \) and \( k = 0 \), and can approximate the Darcy’s law in the Darcy phase when the hydraulic conductivity is sufficiently small. Therefore, the Darcy-Brinkman equations allow us to simulate the Darcy and the Navier-Stokes flows without employing the different governing equations between the fluid and the Darcy phases.

### 2.2 Boundary tracking

The governing equation based on the phase-field equation is used for boundary tracking. This method regards the zero contour of the phase-field variable \( \phi \) as the interface between two different phases, and the phase-field variable \( \phi \) has a hyperbolic tangent profile across the interface. The usual phase-field method allows us to track the interface by solving the following equation derived from the general interface advection equation.

\[
\frac{\partial \phi}{\partial t} + a \nabla \phi = b \left[ \nabla^2 \phi + \phi \frac{(1 - \phi^2)}{W} \right] \quad (3)
\]

where \( \phi, a, b \) and \( W \) denote the phase-field variable, the moving velocity normal to the interface, the curvature coefficient and a measure of the width of the hyperbolic tangent profile, respectively. However, this equation provides not only the normal interface motion but also the curvature driven motion which is not necessary for the alteration of the soil-water interface caused by the seepage-induced erosion. Hence, the following equation developed by Sun and Beckermann (2007) below is employed, which can avoid the curvature-driven interface motion.

\[
\frac{\partial \phi}{\partial t} + a \nabla \phi = b \left[ \nabla^2 \phi + \phi \frac{(1 - \phi^2)}{W} - \nabla \phi \cdot \nabla \left( \frac{\nabla \phi}{| \nabla \phi |} \right) \right] \quad (4)
\]

In the right hand side of Eq. (4), the curvature-driven motion induced by the right hand side of Eq. (3), \( \nabla^2 \phi + \phi \frac{(1 - \phi^2)}{W} \), is cancelled out by the counter term, \( \nabla \phi \cdot \nabla \left( \frac{\nabla \phi}{| \nabla \phi |} \right) \), whereby the interface motion is driven only by the velocity normal to the interface, denoted by \( a \) in the left hand side of Eq. (4).

### 3 Numerical Method

The numerical method developed by the authors, which achieves the stable computation of the Darcy-Brinkman equations, is explained. The detailed numerical procedures for the boundary tracking by Eq. (4) are well described in Sun and Beckermann (2007).

The method presented in this section is based on the one proposed by Kim and Choi (2000), which can solve the Navier-Stokes equations for incompressible fluids by the finite volume method with unstructured grids. Their method is characterized by the grid system. The velocity and the pressure are stored at the center of the finite volume cells and the flux \( U \) is additionally computed at the mid-point of each cell face, which has the following definition;

\[
U = (u_i)_{n+1} n_i \quad (5)
\]

Where \( (u_i)_{n+1} \) and \( n_i \) denote the flow velocity and outward-normal unit vector on the cell face, respectively. Applying a fractional step method and the Crank-Nicolson method to the time integration of Eq. (2), and spatially integrating it over the finite volume cells, the following equations are obtained;

\[
\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{\lambda g}{2k} (u_i^{n+1} + u_i^n) \quad (6)
\]

\[
\frac{\hat{u}_i - u_i^n}{\Delta t} = -\frac{1}{A} \int_0^1 \left( \hat{u}_i U_i^n + u_i^n \hat{u}_i n_i \right) dl \quad (7)
\]

\[
-\frac{1}{A} \int_0^1 \frac{\lambda}{\rho} p^n n_i dl + \frac{1}{2} \frac{\partial}{\partial n} (\hat{u}_i + u_i^n) dl \quad (8)
\]

\[
\frac{u_i^{n+1} - \hat{u}_i}{\Delta t} = \frac{1}{A} \frac{\lambda}{\rho} \int_0^1 p^n n_i dl \quad (9)
\]

\[
\int_0^1 \lambda \frac{\partial p^{n+1}_i}{\partial n} dl = \frac{1}{\Delta t} \int_0^1 U_i^n dl \quad (10)
\]

\[
\frac{U_i^{n+1} - U_i^n}{\Delta t} = -\frac{\lambda}{\rho} \int_0^1 \frac{\partial p^{n+1}_i}{\partial n} dl \quad (11)
\]

where \( A, l, n_i \) and \( \Delta t \) denote the area of the cell, the length of the cell faces, the outward normal unit vector at the cell faces and the time step size, respectively, and the superscript \( m \) implies the number of time steps. \( u_i^n, \hat{u}_i \) and \( u_i^{n+1} \) are the intermediate velocities between \( u_i^n \) and \( u_i^{n+1} \). \( U_i^n \) is also the intermediate flux defined at the cell edges as \( u_i^n n_i \). The numerical procedures for solving the Darcy-Brinkman equations are to conduct computation of Eqs. (6) to (11) in the above order. Eqs. (7) and (9) result in the linear systems for \( \hat{u}_i \) and \( p^{n+1}_i \), which require the inversion of the matrices.

In order to compute the integrals appearing in Eqs. (6) to (11), the velocities, the pressure and their directional derivatives need to be evaluated at the mid-point of each cell face. The values of the velocities and the pressure are interpolated from those at the centers of neighboring variables. Then, the manner of interpolating these variables plays an important role in the stable computation at the interface between the

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porous and the fluid domains. The simple linear interpolation of the variables may induce the physically unrealistic oscillations at the interface of the two different domains. In order to avoid the oscillations, the interpolation scheme described by the following equations is applied to the rectangular finite volume cells;

\[ p_f = \frac{k_a}{\delta_a} p_a + \frac{k_b}{\delta_b} p_b \]  \hspace{1cm} (12)

\[ \frac{\partial p}{\partial n_a} = \frac{p_b - p_a}{\delta_a + \delta_b} \] \hspace{4.5cm} (13)

\[ u_{i,f} = \frac{k_b}{\delta_b} u_{i,a} + \frac{k_a}{\delta_a} u_{i,b} \] \hspace{4cm} (14)

\[ \frac{\partial u_i}{\partial n_a} = \frac{u_{i,f} - u_{i,a}}{\delta_a} \] \hspace{5cm} (15)

where \( p_f \) and \( u_{i,f} \) denote the values of the pressure and the velocity at the interface, respectively, and \( \delta \) is the distance from the cell center to the interface. The subscripts \( a \) and \( b \) mean the indexes for neighboring cells (See Fig. 1 or 2). Eq. (12) means that when a cell face is located on the interface between the porous and the fluid domains, the value of the pressure stored in the cell of the fluid domain is given to the interface (See Fig. 1 and note \( k_b \) is infinity). On the other hand, the velocity is interpolated in an opposite manner by Eq. (14), i.e., the velocity of the porous domain is given to the interface (See Fig. 2). While the cell face is located in either domain, the pressure and the velocity are linearly interpolated onto the cell face.

Fig. 1. Interpolation of pressure at cell interface.

Fig. 2. Interpolation of velocity at cell interface.

4 NUMERICAL RESULTS

The moving speed of the interface denoted by \( a \) in Eq. (4), corresponds to the discharge rate of soils. The interface speed \( a \) can be estimated by the following formula;

\[ a = \frac{u_a}{\lambda} - \left\{ f + (1 - \lambda) \frac{\partial p}{\partial X} \right\} \frac{k}{\lambda^2 \rho g} \] \hspace{1cm} (16)

where \( u_a \), \( X \) and \( f \) denote the outward normal seepage flow velocity at the interface, the coordinate normal to the interface and the maximum resisting force exerted onto the soil particles. Eq. (16) was derived from the experiments by Fujisawa et al. (2012) and is based on the equilibrium of the forces exerted onto the soil particles in a direction perpendicular to the interface. Eq. (2) assumes that the effective stress of soils vanishes when the soil particles discharge, so it is applicable for the seepage erosion occurring on the soil surface. The profile of the boundary between the porous and the fluid domains can be updated by solving Eq. (4) at each time step after the values of \( a \) are calculated from the numerical solutions of the Darcy-Brinkman equations.

The numerical results of the backward erosion are presented herein. Fig. 3 shows the geometry and the boundary conditions. The porous domain of a 125 mm long soil block was installed at the middle of the computational domain and the rightward water flow was induced by the imposed boundary conditions. The right side of the porous domain was made concave, which intended to accelerate the seepage flow to the exit and to concentrate the seepage-induced erosion to the center of the porous domain. The other region was occupied by water, i.e., the fluid domain.

As for the boundary conditions, the horizontal velocity of 0.0016 m/s was given to the left side of the computational domain and the free outflow boundary.
The free-slip condition was imposed on both the upside and the downside. The hydraulic conductivity and the porosity of the porous domain were assumed to be 1.0E-3 m/s and 0.4, respectively. After the initial flow velocity and the initial water pressure were set to zero, the numerical computation was carried out until the penetration of the soil block occurred.

Fig. 4 shows the profile of the computed interface between the porous and the fluid domains changing due to the erosion. As seen in the figure, the boundary moves in a direction opposite to the seepage flow and the backward seepage erosion can be observed. The numerical results indicated that the flow velocity in the region where the soil was eroded became even greater than the other region, which concentrated the water flow into the eroded region and developed the seepage-induced erosion straightly upstream. After the soil block was penetrated, the erosion no longer proceeded because the water flow almost fully concentrated to the connected fluid domain.

Fig. 4. The profile of the interface altering with elapsed time due to the backward erosion.

5 CONCLUSIONS

This article has proposed a numerical method for the computation of the soil erosion induced by seepage flows. This method is built from the three parts, i.e., the simultaneous analysis of the Darcy and the Navier-Stokes flows, the estimation of the erosion rate and the computation of the boundary tracking. The authors employed the Darcy-Brinkman equations as the governing equations to achieve the simultaneous analysis of the Darcy and the Navier-Stokes flows. The erosion rate was estimated by the experimental formula of Fujisawa et al. (2012) and the tracking of the interface between the Darcy and the fluid phases was conducted by solving the phase-field equation modified by Sun and Beckermann (2007). The backward seepage erosion was computed by the proposed method, which has revealed that it can numerically predict the typical behavior of the seepage-induced erosion, which straightly develops upstream.

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