AN URYSOHN-TYPE THEOREM UNDER A DYNAMICAL CONSTRAINT

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Abstract. We address the following question raised by M. Entov and L. Polterovich: given a continuous map \( f: X \to X \) of a metric space \( X \), closed subsets \( A, B \subset X \), and an integer \( n \geq 1 \), when is it possible to find a continuous function \( \theta: X \to \mathbb{R} \) such that
\[
\theta f - \theta \leq 1, \quad \theta|A \leq 0, \quad \text{and} \quad \theta|B > n?
\]
To keep things as simple as possible, we solve the problem when \( A \) is compact. The non-compact case will be treated in a later work.

1. Introduction

We will consider a continuous map \( f: X \to X \) of a metric space \( X \). Suppose that \( A, B \subset X \) are closed subsets, with \( A \) compact, and \( n \geq 1 \) is an integer. In December 2014, M. Entov and L. Polterovich asked the author when is it possible to find a continuous function \( \theta: X \to \mathbb{R} \) such that
\[
\theta f - \theta \leq 1, \quad \theta|A \leq 0, \quad \text{and} \quad \theta|B > n.
\]
They observed that the condition \( B \cap \left( \bigcup_{i=0}^n f^i(A) \right) = \emptyset \) is obviously necessary. They needed the converse for their work [2]. This is the content of the following theorem.

Theorem 1.1 (Discrete case). Suppose \( f: X \to X \) is a continuous map defined on a metric space \( X \). Suppose that \( A, B \subset X \) are closed subsets, with \( A \) compact, and also that \( B \cap \left( \bigcup_{i=0}^n f^i(A) \right) = \emptyset \), where \( n \geq 0 \). Then we can find a Lipschitz function \( \theta: X \to [0, n+1] \) such that \( \theta f - \theta \leq 1 \) everywhere, the function \( \theta \) is identically 0 on a neighborhood of \( A \), and \( \theta \) is identically \( n+1 \) on a neighborhood of \( B \).

If \( X \) is locally compact metric and \( \sigma \)-compact, the map \( f: X \to X \) is proper, and \( B \) is also compact, then we can further assume that \( \theta \) has compact support.
**Corollary 1.2.** Under the hypothesis of Theorem 1.1, if we further assume that $X$ is a smooth manifold, then for every $\epsilon > 0$, we can find a $C^\infty$ function $\theta^\epsilon : X \to [0, n + 1]$ which is identically 0 in a neighborhood of $A$, identically $n + 1$ in a neighborhood of $B$, and satisfies $\theta^\epsilon f - \theta^\epsilon \leq 1 + \epsilon$. Moreover, if $B$ is also compact, we can assume that the smooth function $\theta^\epsilon$ also has a compact support.

In fact, in the paper [2], M. Entov and L. Polterovich were interested in the flow version of the theorem above, which we will obtain from the discrete case. We will only consider the case of autonomous flows, i.e., families of maps $\phi_t : X \to X$, $t \in \mathbb{R}$ which satisfy $\phi_0 = \text{Id}_X$, and $\phi_{t+s} = \phi_t \circ \phi_s$.

**Corollary 1.3 (Flow case).** Suppose $\phi_t : X \to X$, $t \in \mathbb{R}$, is a continuous flow defined on a metric space $X$. Let $A, B \subset X$ be closed subsets, with $A$ compact and $B \cap (\bigcup_{T=0}^T \phi_t(A)) = \emptyset$, where $T \geq 0$. We can find a continuous function $\theta : X \to [0, \infty)$, with $\theta$ identically 0 on a neighborhood of $A$, and $\theta$ identically constant and $> T$ on a neighborhood of $B$, such that the map $t \mapsto \theta \phi_t(x)$ has a continuous derivative in $t$ for every $x \in X$ with

$$\frac{d\theta \phi_t(x)}{dt} \leq 1, \text{ for all } x \in X, \text{ and all } t \in \mathbb{R}.$$ 

If $X$ is locally compact metric and $\sigma$-compact, and $B$ is also compact, then we can further assume that $\theta$ has compact support.

**Corollary 1.4 (Smooth flow case).** Under the hypothesis of Corollary 1.3, if we further assume that $X$ is a smooth manifold and the flow $\phi_t$ is $C^1$, generated by the vector field $V$, then we can find such a function $\theta$ which is moreover $C^\infty$ and satisfies $V \theta \leq 1$, where $V \theta$ is the derivative of $\theta$ in the direction of $V$.

The proof of these last two corollaries is deduced from the discrete case with the same idea that L. Buhovsky, M. Entov and L. Polterovich used to prove the corollary under the stronger hypothesis $B \cap (\bigcup_{T=-T}^T \phi_t(A)) = \emptyset$, see [1].

**Remark 1.5.** When $A$ is not compact, a slightly modified version of the results above remain true. To keep things simple, we postpone the statements and proofs for this non-compact case to another paper.

2. The Discrete Case

We will use the method from our previous work with Pierre Pageault, see [3]. Let us fix a metric defining the topology on $X$. For every $k > 0$, we define the cost $c_k : X \times X \to \mathbb{R}$ by

$$c_k(x, y) = kd(f(x), y) + 1.$$ 

Recall that a chain in $X$ is a sequence of points $(x_0, \ldots, x_n)$, where $n \geq 1$. We define the cost $C_k(x_0, \ldots, x_n)$ of the chain $(x_0, \ldots, x_n)$ by

$$C_k(x_0, \ldots, x_n) = \sum_{i=0}^{n-1} c_k(x_i, x_{i+1}) = \sum_{i=0}^{n-1} (kd(f(x_i), x_{i+1}) + 1).$$
Note that $C(x_0, \ldots, x_n) \geq n$. Therefore $n \leq |C(x_0, \ldots, x_n)|$, where $|r|$ is, as usual, the largest integer $\leq r \in \mathbb{R}$. We then define $\Gamma_k : X \times X \to \mathbb{R}$ by

$$\Gamma_k(x, y) = \inf \{C_k(x_0, \ldots, x_n) \mid x_0 = x, x_n = y\}.$$  

As in [3], it is not difficult to obtain the following properties of $\Gamma_k$.  

**Proposition 2.1.** The function $\Gamma_k$ satisfies the following properties:  

(i) for every $x, y \in X$, we have $1 \leq \Gamma_k(x, y) \leq kd(f(x), y) + 1$;  

(ii) for every $x, y, z \in X$, we have $\Gamma_k(x, y) \leq \Gamma_k(x, z) + \Gamma_k(z, y)$;  

(iii) for every $x \in X$, we have $\Gamma_k(x, f(x)) = 1$;  

(iv) for every $x, y \in X$, we have $\Gamma_k(x, f(y)) \leq \Gamma_k(x, y) + 1$;  

(v) for every $x, y, z \in X$, we have $|\Gamma_k(x, y) - \Gamma_k(x, z)| \leq kd(y, z)$, and $|\Gamma_k(x, y) - \Gamma_k(z, y)| \leq kd(f(x), f(z))$.  

In particular, the function $\Gamma_k$ is continuous, and uniformly Lipschitz in the second variable with Lipschitz constant $k$.  

**Proof of Theorem 1.1.** For $\epsilon > 0$, we recall that for a subset $S \subset X$, its closed $\epsilon$-neighborhood is

$$\hat{V}_\epsilon(S) = \{x \in X \mid d(x, S) \leq \epsilon\}.$$  

For $k > 0$, we define the function $\phi_k : X \to [0, \infty)$ by

$$\phi_k(x) = kd(x, \hat{V}_{1/k}(A)).$$  

Note that $\phi_k|\hat{V}_{1/k}(A) \equiv 0$. The function $\phi_k$ is $k$-Lipschitz. We now estimate from below the values of $\phi_k$ on $\hat{V}_{1/k}(B)$. It is not difficult to show that

$$d(\hat{V}_{1/k}(A), \hat{V}_{1/k}(B)) \geq d(A, B) - 2/k.$$  

This implies the inequality

$$\phi_k|\hat{V}_{1/k}(B) \geq kd(A, B) - 2.$$  

Since $A$ is compact and $B$ is closed, we have $d(A, B) > 0$. Hence $\inf_{\hat{V}_{1/k}(B)} \phi_k \rightarrow \infty$, as $k \to \infty$.

We next define $\theta_k : X \to [0, \infty)$ by

$$\theta_k(x) = \min \{\phi_k(x), \inf_{y \in X} \phi_k(y) + \Gamma_k(y, x)\}.$$  

We first observe that $\theta_k$ is $k$-Lipschitz, since $\phi_k$ is $k$-Lipschitz, and $\Gamma_k$ is uniformly $k$-Lipschitz in its second argument. We next show that $\theta_k f \leq \theta_k + 1$. If $x \in X$, by the definition of $\theta_k$, we have

$$\theta_k(f(x)) \leq \inf_{y \in X} \phi_k(y) + \Gamma_k(y, f(x)).$$  

Choosing $y = x$, we obtain

$$\theta_k(f(x)) \leq \phi_k(x) + \Gamma_k(x, f(x)) = \phi_k(x) + 1.$$  

Using $\Gamma_k(y, f(x)) \leq \Gamma_k(y, x) + \Gamma_k(x, f(x)) = \Gamma_k(y, x) + 1$, we also obtain

$$\theta_k(f(x)) \leq \inf_{y \in X} \phi_k(y) + \Gamma_k(y, x) + 1.$$
Therefore
\[ \theta_k(f(x)) \leq \min \{ \varphi_k(x) + 1, \inf_{y \in X} \varphi_k(y) + \Gamma_k(y, x) + 1 \} \\
= \theta_k(x) + 1. \]

Since \( \varphi_k|\tilde{V}_{1/k}(A) \equiv 0 \), we do have \( \theta_k|\tilde{V}_{1/k}(A) \equiv 0 \). We now show that for \( k \) large enough, we have \( \theta_k|\tilde{V}_{1/k}(B) \geq n + 1 \).

We argue by contradiction. If we assume that \( \theta_k|\tilde{V}_{1/k}(B) \geq n + 1 \) is not true for \( k \) large enough, we can find sequences \( k_\ell \to \infty \), and \( z_\ell \in \tilde{V}_{1/k_\ell}(B) \), such that
\[ \theta_{k_\ell}(z_\ell) < n + 1. \]

By \((2.1)\), we have \( \varphi_{k_\ell}(z_\ell) \geq kd(A, B) - 2 \). Therefore \( \varphi_{k_\ell}(z_\ell) \geq n + 1 \), for \( \ell \) large. Hence, without loss of generality, we can assume that
\[ \theta_{k_\ell}(z_\ell) = \inf_{y \in X} \varphi_{k_\ell}(y) + \Gamma_{k_\ell}(y, z_\ell) < n + 1. \]

By the definition of \( \Gamma_{k_\ell} \), it follows that for every \( \ell \), we can find a sequence, \( y_0^\ell, \ldots, y_{n_\ell}^\ell \), with \( y_{n_\ell}^\ell = z_\ell \), \( n_\ell \geq 1 \), and
\[
\varphi_{k_\ell}(y_0^\ell) + C_{k_\ell}(y_0^\ell, \ldots, y_{n_\ell}^\ell) = k_\ell d(y_0^\ell, \tilde{V}_{1/k_\ell}(A)) + \sum_{i=0}^{n_\ell-1} [k_\ell d(f(y_i^\ell), y_{i+1}^\ell) + 1]
= k_\ell d(y_0^\ell, \tilde{V}_{1/k_\ell}(A)) + n_\ell + \sum_{i=0}^{n_\ell-1} k_\ell d(f(y_i^\ell), y_{i+1}^\ell)
< n + 1.
\]

This yields
\[
n_\ell < n + 1, d(y_0^\ell, \tilde{V}_{1/k_\ell}(A)) < (n + 1)/k_\ell, \text{ and } \]
\[(2.2) \quad d(f(y_i^\ell), y_{i+1}^\ell) < (n + 1)/k_\ell, \text{ for } i = 0, \ldots, n_\ell - 1. \]

Therefore, extracting if necessary, we can assume that \( n_\ell = m \leq n \), with \( m \) independent of \( \ell \). Since \( k_\ell \to \infty \), the inequalities \((2.2)\) imply \( d(f(y_i^\ell), y_{i+1}^\ell) \to 0 \), as \( \ell \to \infty \), for \( i = 0, \ldots, m - 1 \), and also \( d(y_0^\ell, A) \to 0 \) as \( \ell \to \infty \). In particular, we can find a sequence \( x_\ell \in A \), such that \( d(y_0^\ell, x_\ell) \to 0 \). By compactness of \( A \), extracting further if necessary, we can assume \( x_\ell \to x \in A \). Hence \( y_0^\ell \to x \in A \), and from \( d(f(y_i^\ell), y_{i+1}^\ell) \to 0 \), by induction, we obtain \( y_{i+1}^\ell \to f^{i+1}(x) \), for \( i = 0, \ldots, m - 1 \). Since \( B \) is closed, and \( y_m^\ell = y_{n_\ell}^\ell = z_\ell \in \tilde{V}_{1/k_\ell}(B) \), we get \( f^m(x) = \lim_{\ell \to \infty} y_m^\ell \in B \).

But \( x \in A \) and \( m \leq n \). This contradicts the hypothesis \( B \cap (\cup_{i=0}^{n} f^i(A)) = \emptyset \).

Obviously, for \( k \) large, the function \( \theta = \min\{\theta_k, n + 1\} \) satisfies all the conditions in the first part of the theorem.

To prove the second part, we consider the Alexandrov (one point) compactification \( \tilde{X} = X \cup \{\infty\} \). Since \( X \) is locally compact, metric, and \( \sigma \)-compact, the compactification \( \tilde{X} \) is compact and metric.\(^1\) Since the map \( f : X \to \tilde{X} \) is proper, \(\)

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\(^1\)This follows from Urysohn’s metrization theorem, see [4, page 125], since it can be easily shown that the compact Hausdorff space \( \tilde{X} \) has a countable basis of open sets.
it can be extended continuously to $\tilde{X}$, with $f(\infty) = \infty$. If we set $\tilde{A} = A \cup \{\infty\}$, then both $\tilde{A}$ and $B$ are compact subsets of $\tilde{X}$, and they satisfy

$$B \cap \left( \bigcup_{i=0}^{n} f^i(\tilde{A}) \right) = \emptyset.$$ 

By the first part of the theorem, we can find a continuous function $\theta : \tilde{X} \to [0, n+1]$ which is equal to 0 in a neighborhood of $\tilde{A}$, equal to $n+1$ in a neighborhood of $B$, and satisfies $\theta f - \theta \leq 1$. Since $\infty \in \tilde{A}$, this implies that the restriction $\theta|X$ has compact support.

**Proof of Corollary 1.2.** This is a simple approximation argument. We pick up the continuous function $\theta : X \to [0, n+1]$ given by Theorem 1.1. For any given $\epsilon > 0$, we can find a smooth function $\theta^\epsilon : X \to [0, n+1]$ such that $\|\theta^\epsilon - \theta\|_\infty \leq \epsilon/2$. Using $\theta f - \theta \leq 1$, we obtain $\theta^\epsilon f - \theta^\epsilon \leq 1 + \epsilon$. Since $\theta \equiv 0$ (resp. $\theta \equiv n+1$) in a neighborhood of the closed set $A$ (resp. $B$), choosing $\theta^\epsilon$ carefully, we can assume that $\theta^\epsilon \equiv 0$ in a neighborhood of $A$, and $\theta^\epsilon \equiv n+1$ in a neighborhood of $B$. By a similar careful choice of $\theta^\epsilon$, we can assume that $\theta^\epsilon$ has compact support if $\theta$ has itself compact support.

3. **THE FLOW CASE**

In this section, we prove Corollary 1.3. The proof of this corollary is deduced from the discrete case using the well-known way to smooth a continuous function of one variable by averaging on an interval. In [1], L. Buhovsky, M. Entov and L. Polterovich used the same method to prove Corollary 1.2 under the stronger hypothesis $B \cap \left( \bigcup_{t=-T}^{T} \phi_t(A) \right) = \emptyset$. The easiest way to smooth a function $\theta$ along the orbits of a flow $\phi_t$ is to consider the function $\theta^\epsilon(x) = \int_0^\epsilon \theta(\phi_s(x)) \, ds$, where $\epsilon > 0$ is fixed. The following computation gives the derivative at $s = t$ of $s \mapsto \theta^\epsilon(\phi_s(x))$:

$$\frac{\theta^\epsilon(\phi_{t+h}(x)) - \theta^\epsilon(\phi_t(x))}{h} = \frac{1}{h} \left( \int_0^\epsilon \theta(\phi_{t+h+s}(x)) \, ds - \int_0^\epsilon \theta(\phi_{t+s}(x)) \, ds \right)$$

$$= \frac{1}{h} \left( \int_{t+h}^{t+h+\epsilon} \theta(\phi_s(x)) \, ds - \int_{t}^{t+\epsilon} \theta(\phi_s(x)) \, ds \right)$$

$$= \frac{1}{h} \left( \int_{t+h}^{t+h+\epsilon} \theta(\phi_s(x)) \, ds - \frac{1}{h} \int_{t}^{t+h} \theta(\phi_s(x)) \, ds \right).$$

Therefore if we let $h \to 0$, we obtain that the derivative in $t$ of $t \mapsto \theta^\epsilon(\phi_t(x))$ exists and is given by

$$d\theta^\epsilon(\phi_t(x)) \over dt = \theta(\phi_{t+\epsilon}(x)) - \theta(\phi_t(x)).$$

(3.1)
Proof of Corollary 1.3. Since $B \cap \bigcup_{t=0}^{T} \phi_t(A) = \emptyset$, using that $X \sim B$ is open, the continuity of the flow $\phi_t$ and the compactness of $A$ allow us to find $\alpha > 0$ such that

$$B \cap \left( \bigcup_{t=-\alpha}^{T+\alpha} \phi_t(A) \right) = \emptyset. \quad (3.2)$$

If we set $\tilde{B} = \bigcup_{t=0}^{\alpha} \phi_t(B)$, it follows from (3.2) that

$$\tilde{B} \cap \left( \bigcup_{t=0}^{T+\alpha} \phi_t(A) \right) = \emptyset. \quad (3.3)$$

Indeed, if we could find $t \in [0, T+\alpha]$ with $\tilde{B} \cap \phi_t(A) \neq \emptyset$, the definition of $\tilde{B}$ implies that there exists $s \in [0, \alpha]$ with $\phi_s(B) \cap \phi_t(A) \neq \emptyset$. This yields $B \cap \phi_{t-s}(A) \neq \emptyset$, but $t-s \in [-\alpha, T+\alpha]$. This contradicts (3.2).

We can find an integer $N \geq 1$ and $\epsilon \in [0, \alpha]$ such that

$$T+\alpha = (N+1)\epsilon. \quad (3.4)$$

We now define $\tilde{A} = \bigcup_{i=0}^{N} f^i(\tilde{A})$, and $f = \phi_\epsilon : X \to X$. We have

$$\bigcup_{i=0}^{N} f^i(\tilde{A}) = \bigcup_{i=0}^{N} \phi_i(\tilde{A}) = \bigcup_{t=0}^{(N+1)\epsilon} \phi_t(A) = \bigcup_{t=0}^{T+\alpha} \phi_t(A),$$

where the last equality follows from (3.4). Therefore, in view of (3.3), we obtain

$$\tilde{B} \cap \left( \bigcup_{i=0}^{N} f^i(\tilde{A}) \right) = \emptyset. \quad (3.5)$$

By Theorem 1.1, we can find a continuous function $\theta : X \to [0, N+1]$ and neighborhoods $V_{\tilde{A}}$ and $V_{\tilde{B}}$ of $\tilde{A}$ and $\tilde{B}$, respectively, such that

$$\theta|_{V_{\tilde{A}}} \equiv 0, \quad \theta|_{V_{\tilde{B}}} \equiv N+1, \quad \text{and} \quad \theta f - \theta = \theta \phi_\epsilon - \theta \leq 1. \quad (3.6)$$

We now introduce the function $\theta_\epsilon$ defined as above by

$$\theta_\epsilon(x) = \int_{0}^{\epsilon} \theta(\phi_s(x)) \, ds. \quad (3.7)$$

From (3.1), we know that $\theta(\phi_t(x))$ is differentiable in $t$, and the derivative is given by

$$\frac{d\theta_\epsilon(\phi_t(x))}{dt} = \theta(\phi_{t+\epsilon}(x)) - \theta(\phi_t(x)) = (\theta \phi_\epsilon - \theta)(\phi_t(x)). \quad (3.8)$$

Since $\theta \phi_\epsilon - \theta \leq 1$ by (3.6), we conclude that

$$\frac{d\theta_\epsilon \phi_t(x)}{dt} \leq 1 \text{ for all } x \in X \text{ and all } t \in \mathbb{R}.$$
Since \( \theta \) takes values in \([0, N + 1] \), the function \( \theta_c \) takes values in \([0, (N + 1)\epsilon] = [0, T + \alpha] \).

We now show that \( \theta_c \) is identically 0 (resp. \( T + \alpha \)) in a neighborhood of \( A \) (resp. \( B \)). Since \( \bigcup_{t=0}^\epsilon \phi_t(A) = \tilde{A} \) (resp. \( \bigcup_{t=0}^\epsilon \phi_t(B) \subset \tilde{B} \)) and \( V\tilde{A} \) (resp. \( V\tilde{B} \)) is a neighborhood of \( \tilde{A} \) (resp. \( \tilde{B} \)), using compactness of \( [0, \epsilon] \), we can find a neighborhood \( V_A \) (resp. \( V_B \)) of \( A \) (resp. \( B \)) such that

\[
\bigcup_{t=0}^\epsilon \phi_t(V_A) \subset V_{\tilde{A}} \quad \text{and} \quad \bigcup_{t=0}^\epsilon \phi_t(V_B) \subset V_{\tilde{B}}.
\]

By the choice of \( \theta \) and the definition of \( \theta_c \), we obtain that \( \theta_c \equiv 0 \) on \( V_A \) and \( \theta_c \equiv \epsilon(N + 1) = T + \alpha \) on \( V_B \). This finishes the proof that \( \theta_c \) is the function \( \theta \) we are looking for in the first part of the corollary.

It remains to prove the last statement about compact support. Using the assumptions of the last part of the corollary, we note that, by the last part of Theorem 1.1, we can further choose \( \theta \) whose support \( \text{Supp}(\theta) \) is compact. Obviously, from its definition, the function \( \theta_c \) is identically 0 outside of the compact set \( \bigcup_{t=-\epsilon}^\epsilon \phi_t[\text{Supp}(\theta)] \).

**Proof of Corollary 1.4.** We indicate here the changes that have to be done in the proof of Corollary 1.3. The choices of \( \alpha, \epsilon, N, f = \phi_c, \tilde{A}, \) and \( \tilde{B} \) are the same. Given \( \delta > 0 \), instead of applying Theorem 1.1, we apply Corollary 1.2, to obtain a \( C^\infty \) function \( \theta^\delta : X \rightarrow [0, N + 1] \), and neighborhoods \( V_{\tilde{A}} \) and \( V_{\tilde{B}} \) of \( \tilde{A} \) and \( \tilde{B} \), respectively, such that

\[
\theta^\delta|_{V_{\tilde{A}}} \equiv 0, \quad \theta^\delta|_{V_{\tilde{B}}} \equiv N + 1, \quad \text{and} \quad \theta^\delta f - \theta^\delta = \theta^\delta \phi_c - \theta^\delta \leq 1 + \delta.
\]

Again we define the function \( \theta^\delta_c \) by

\[
(3.10) \quad \theta^\delta_c(x) = \int_0^\epsilon \theta^\delta(\phi_s(x)) \, ds.
\]

Since \( \theta^\delta \) is \( C^\infty \), and \( \phi_t \) is \( C^1 \), the function \( \theta^\delta_c \) is also \( C^1 \). This implies

\[
V\theta^\delta_c(x) = \frac{d\theta^\delta_c(\phi_t(x))}{dt} \bigg|_{t=0} = (\theta^\delta \phi_c - \theta^\delta)(x) \leq 1 + \delta.
\]

Moreover, like in the proof above of Corollary 1.3, we can show that the function \( \theta^\delta_c \) takes values in \([0, T + \alpha] \) and is identically 0 (resp. \( T + \alpha \)) in a neighborhood of \( A \) (resp. \( B \)).

Since \( \theta^\delta_c \) is \( C^1 \), we can approximate it, in the fine Whitney \( C^1 \) topology, by a \( C^\infty \) function \( \tilde{\theta}^\delta : X \rightarrow [0, T + \alpha] \), which is still identically 0 (resp. \( T + \alpha \)) in a neighborhood of \( A \) (resp. \( B \)) and satisfies \(|V\tilde{\theta}^\delta - V\theta^\delta_c| \leq \delta \) everywhere. Therefore \( V\tilde{\theta}^\delta \leq 1 + 2\delta \). For \( \delta > 0 \) small enough, we still have \((1 + 2\delta)^{-1}(T + \alpha) > T\), and therefore the function \( \theta = (1 + 2\delta)^{-1}\tilde{\theta}^\delta \) indeed satisfies the conclusion of the corollary.

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