ψ-Poisson, q-Cigler, ψ-Dobinski, ψ-Rota and ψ-coherent states- with Cigler’s Remark on simplicity

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Abstract

The Cigler simple derivation of the q-Carlitz- Dobinski formula is recalled and it is noticed that the formula may be interpreted as the average of powers of random variable $X_q$ with the q-Poisson distribution. In parallel new q-Cigler-Dobinski and psi-Carlitz-Dobinski formulas are introduced.

At first let us anticipate with ψ-remark. ψ denotes an extension of $\langle \frac{1}{n!} \rangle_{n \geq 0}$ sequence to quite arbitrary one ("admissible") and the specific choices are for example: Fibononially -extended ($\langle F_n \rangle$ - Fibonacci sequence ) $\langle \frac{1}{F_n} \rangle_{n \geq 0}$ or just "the usual" $\langle \frac{1}{n!} \rangle_{n \geq 0}$ or Gauss q-extended $\langle \frac{1}{n!} \rangle_{n \geq 0}$ admissible sequences of extended umbral operator calculus - see more below. With such an extension we may ψ-mnemonic repeat with exactly the same simplicity and beauty what was done by Rota forty years ago. Forty years ago Gian-Carlo Rota proved the exponential generating function for Bell numbers $B_n$ to be of the form

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}(B_n) = \exp(e^x - 1) \quad (1)$$

using the linear functional $L$ such that

$$L(X^n) = 1, \quad n \geq 0 \quad (2)$$
Then Bell numbers (see: formula (4) in [1]) are defined by

\[ L(X^n) = B_n, \quad n \geq 0 \]  (3)

The above formula is exactly the Dobinski formula [2] if \( L \) is interpreted as the average functional for the random variable \( X \) with the Poisson distribution with \( L(X) = 1 \). It is Blissard calculus inspired umbral formula [1]. Recently an interest to Stirling numbers and consequently to Bell numbers was revived among "q-coherent states physicists" [3, 4, 5]. Namely the expectation value with respect to coherent state \( |\gamma > \) with \( |\gamma| = 1 \) of the \( n \)-th power of the number of quanta operator is "just" the \( n \)-th Bell number \( B_n \) and the explicit formula for this expectation number of quanta is "just" Dobinski formula [3],(4). The same is with the \( q \)-coherent states case [3] i.e. the expectation value with respect to \( q \)-coherent state \( |\gamma > \) with \( |\gamma| = 1 \) of the \( n \)-th power of the number operator is the \( n \)-th \( q \)-Bell number defined as the sum of \( q \)-Stirling numbers \( \{ \binom{n}{k} \}_q \) due to Carlitz as in [6, 3, 4, 5]. Note there then that for standard Gauss \( q \)-extension \( x_q \) of number \( x \) we have

\[ x_q^n = \sum_{k=0}^{n} \binom{n}{k}_q x_q^k \]  (4)

Hence the expectation value with respect to \( q \)-coherent state \( |\gamma > \) with \( |\gamma| = 1 \) of the \( n \)-th power of the number operator is exactly the popular \( q \)-Dobinski formula. It can be given via (3) Blissard calculus inspired umbral formula form and may be treated as definition of \( B_n(q) \)

\[ L_q(X^n_q) = B_n(q), \quad n \geq 0. \]  (5)

due to the fact that linear functional \( L_q \) interpreted as the average functional for the random variable \( X_q \) with the \( q \)-Poisson distribution with \( L_q(X_q) = 1 \) satisfies

\[ L_q(X^n_q) = 1, \quad n \geq 0. \]  (6)

We arrive to this simple conclusion using Jackson derivative difference operator in place of \( D = d/dx \) in \( q \) =1 case and the power series generating function \( G(t) \) for \( q \)-Poisson probability distribution:

\[ p_n = [\exp_q \lambda]^{-1} \frac{\lambda^k}{k_q!}, G(t) = \sum_{n \geq 0} p_n t^k, \]  (7)

\[ p_n = [\partial_q^n G(t)_{q}]_{t=0}, [\partial_q G(t)]_{t=1} = 1 \text{ for } \lambda = 1. \]  (8)
There are many $q$-extensions of Stirling numbers according to their weighted counting interpretation. For example $w(\pi) = q^{\text{cross}(\pi)}$, $w(\pi) = q^{\text{inv}(\pi)}$ from \cite{7} gives after being summed over the set of $k$-block partitions the Carlitz $q$-Stirling numbers or $w(\pi) = q^{\text{nin}(\pi)}$ from \cite{8} gives rise to Carlitz-Gould $q$-Stirling numbers after being summed over the set of $k$-block partitions or with $w(\pi) = q^{i(\pi)}$ in \cite{9} - we arrive at another combinatorial interpretation of $q$-extended Stirling numbers. $q$-Stirling numbers much different from Carlitz $q$-ones were introduced in the reference \cite{10} from where one infers the cigl-analog of (5). Let $\Pi$ denotes the lattice of all partitions of the set $\{0, 1, \ldots, n-1\}$. Let $\pi \in \Pi$ be represented by blocks $\pi = \{B_0, B_1, \ldots, B_i, \ldots\}$, where $B_0$ is the block containing zero: $0 \in B_0$. The weight adapted by Cigler defines weighted partitions’ counting according to the content of $B_0$. Namely $w(\pi) = q^{\text{cigl}(\pi)}$, $\text{cigl}(\pi) = \sum_{l \in B_0} l$, $\sum_{\pi \in A_{n,k}} q^{\text{cigl}(\pi)} \equiv \left\{ \begin{array}{c} n \\ k \end{array} \right\}_q$ therefore $\sum_{\pi \in \Pi} q^{\text{cigl}(\pi)} \equiv B_{n}(q)$. Here $A_{n,k}$ stays for subfamily of all $k$-block partitions. With the above relations one has defined the cigl-$q$-Stirling and the cigl-$q$-Bell numbers. The cigl-$q$-Stirling numbers of the second kind are expressed in terms of $q$-binomial coefficients and $q = 1$ Stirling numbers of the second kind \cite{10}. These are new $q$-Stirling numbers. The corresponding cigl-$q$-Bell numbers recently have been equivalently defined via cigl-$q$-Dobinski formula \cite{11} $L(X_n^q) = B_n(q), \quad n \geq 0, \quad X_n^q \equiv X(X + q - 1)(X - 1 + q^{n-1})$ interpreted as the average of this specific $n-th$ cigl-$q$-power random variable $X_n^q$ with the $q = 1$ Poisson distribution such that $L(X) = 1$. To this end note that in \cite{12}, \cite{13} a family of the so called $\psi$-Poisson processes was introduced. The corresponding choice of the function sequence $\psi$ leads to the $q$-Poisson process. Accordingly the extension of Dobinski formula with its elementary essential content and context to general case of $\psi$- umbral instead $q$-umbral calculi case only - is automatic in view of an experience from \cite{12}, \cite{13} (see corresponding earlier references there and necessary definitions). At first what you do is to replace index $q$ by $\psi$ in formulas (3), (4),..., (8). Then you have got started problems with not easy combinatorial interpretation if at all and... etc. $\psi$-Stirling numbers and $\psi$-Bell numbers are being then defined by (4) and (3) correspondingly with $q$ replaced by $\psi$. We get used to write these extensions in mnemonic convenient upside down notation \cite{12}, \cite{13}

\begin{align}
\psi_n &\equiv n_{\psi}, x_{\psi} \equiv \psi(x) \equiv \psi_{x}, n_{\psi}! = n_{\psi}(n-1)_{\psi}!, n > 0, \quad (9) \\
k_{\psi} x_{\psi} = x_{\psi}(x-1)_{\psi}(x-2)_{\psi}(x-k+1)_{\psi} \quad (10)
\end{align}
\[ x_\psi(x-1)_\psi \cdots (x-k+1)_\psi = \psi(x)\psi(x-1)\cdots \psi(x-k-1). \]  
(11)

You may consult for further development and use of this notation [12], [13] and references therein.

(*) Remark based on the remark of Professor Cigler (in private).

The Katriel’s claim [3] that his derivation of the Dobinski formula is the simplest possible may be confronted with the extremely simple derivation by Cigler (see p.104 in [14]). Note on the way that this derivation is ready for \( \psi \)-extensions [12] [13] as it as a matter of fact based on elementary properties of GHW (Graves-Heisenberg-Weyl) algebra: see [12] [13] and references therein.

Namely, let \( \hat{x} \) denotes the multiplication by \( x \) operator while \( D \) denotes differentiation - both acting on the prehilbert space \( P \) of polynomials. Then from recursion for Stirling numbers of the second kind and the identification (operators in \( P \)):

\[ \hat{x}(D+1) \equiv \frac{1}{\exp(x)}(\hat{x}D)\exp(x) \]

one gets for exponential polynomials

\[ \varphi_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k \]

the formula which becomes Dobinski one for \( x = 1 \)

\[ \varphi_n(x) = \frac{1}{\exp(x)}(\hat{x}D)^n \exp(x) \]

i.e.

\[ \varphi_n(x) = \frac{1}{\exp(x)} \sum_{0 \leq k} \frac{k^n x^k}{k!}. \]

The \( q \)-case as well as \( \psi \)-case is automatically retained with the mnemonics from [12] [13].

As perfectly properly indicated to the present author by Professor Cigler - the derivation of Dobinski or \( q \)-Dobinski formulas does not require an introduction of a completion of prehilbert space \( P \) and coherent states in order to derive the formulas. - Well. Anyhow, the extremely simple and umbral-beautiful Cigler’s derivation above [14] is immediately represented and then fruitfully interpreted via expectation values on broader grounds of combinatorics applications.
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