Stability and bifurcation phenomena in asymptotically Hamiltonian systems

Oskar A Sultanov

Institute of Mathematics, Ufa Federal Research Centre, Russian Academy of Sciences, 112, Chernyshevsky Street, Ufa, 450008, Russia

E-mail: oasultanov@gmail.com

Received 24 June 2020, revised 18 February 2022
Accepted for publication 1 April 2022
Published 28 April 2022

Abstract

The influence of time-dependent perturbations on an autonomous Hamiltonian system with an equilibrium of center type is considered. It is assumed that the perturbations decay at infinity in time and vanish at the equilibrium. In this case the stability and the long-term behaviour of trajectories depend on nonlinear and non-autonomous terms of equations. The paper investigates bifurcations associated with a change of Lyapunov stability of the equilibrium and the emergence of new attracting or repelling states in the perturbed non-autonomous system. The dependence of bifurcations on the structure of perturbations is discussed.

Keywords: non-autonomous systems, stability, asymptotics, bifurcation, Lyapunov function

Mathematics Subject Classification numbers: 34C23, 34D10, 34D20, 37J65.

(Some figures may appear in colour only in the online journal)

1. Introduction

The influence of perturbations on the stability of solutions is a classical problem in the qualitative theory of differential equations. For autonomous systems, the solution of such a problem is effectively covered by the theory of stability and bifurcations [1–3]. This paper is devoted to time-dependent perturbations decaying at infinity in time. In this case the perturbed system is asymptotically autonomous. Note that asymptotically autonomous systems were first considered in [4], where the relation between the solutions of the complete system and the solutions of the corresponding limiting autonomous system was discussed. Stability and almost periodicity of solutions for a special class of such systems on the plane were investigated in [5]. A more wide class of systems was considered in [6], where the almost periodic solutions were
approximated by solutions of the corresponding limiting systems. It was shown in [7] that under some conditions, the solutions of a complete system have the same asymptotic behavior as the solutions of the limiting system. However, this is not true in general. See [8] for examples of non-autonomous systems whose solutions behave completely differently than the solutions of the corresponding limiting systems.

Bifurcations in non-autonomous systems have been discussed in several papers. In particular, the change of pullback stability and the appearance of new stable states under variation of a parameter in scalar differential equations with time-dependent coefficients were studied in [9]. Similar equations were considered in [10], where the bifurcation was understood as a change in the structure of the pullback attractor. Bifurcations as a change in the structure of the domain of attraction were discussed for asymptotically autonomous equations in [11], where some conditions ensuring the transfer of bifurcations in limiting equations to complete equations were described. The elements of general theory for non-autonomous systems are contained in [12], where particular bounded solutions were considered as bifurcating objects and the bifurcation was understood as a branching of solutions.

The present paper considers a class of asymptotically Hamiltonian systems with the equilibrium and investigates the effects of decaying time-dependent perturbations on the stability of solutions. Such systems appear in a wide range of problems in mathematical physics, for example, in the study of the Painlevé equations and their perturbations [13], autoresonance models [14, 15] and synchronisation phenomena [16, 17]. However, to the best of our knowledge, the bifurcations associated with decaying terms in asymptotically Hamiltonian systems have not been thoroughly discussed.

The paper is organised as follows. In section 2, the mathematical formulation of the problem is given and the class of non-autonomous perturbations is described. The proposed method of stability and bifurcation analysis is based on a change of variables associated with a Lyapunov function for a complete asymptotically autonomous system. The construction of this transformation is described in section 3. Section 4 is devoted to bifurcations associated with a change of the stability of the equilibrium. Bifurcations associated with the emergence of close to periodic solutions are discussed in section 5. The results of sections 4 and 5 are applied in section 6 for a description of bifurcations in the complete system under various restrictions on the perturbations. The paper concludes with a brief discussion of the results obtained.

2. Problem statement

Consider the system of two differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= \partial_y H(x, y, t), \\
\frac{dy}{dt} &= -\partial_x H(x, y, t) + F(x, y, t), \quad t > 0.
\end{align*}
\]  

(1)

It is assumed that the functions \(H(x, y, t)\) and \(F(x, y, t)\) are infinitely differentiable and for every compact \(D \subseteq \mathbb{R}^2\) \(H(x, y, t) \to H_0(x, y)\) and \(F(x, y, t) \to 0\) as \(t \to \infty\) for all \((x, y) \in D\). The limiting autonomous system with the Hamiltonian \(H_0(x, y)\) is assumed to have the isolated fixed point \((0, 0)\) of center type. Without loss of generality, it is assumed that

\[
H_0(x, y) = \frac{x^2 + y^2}{2} + O(r^3), \quad r = \sqrt{x^2 + y^2} \to 0.
\]  

(2)

It is also assumed that the level lines \(\{(x, y) \in \mathbb{R}^2 : H_0(x, y) = E\}\) define a family of closed curves on the phase space \((x, y)\) parameterised by the parameter \(E\) for all \(E \in (0, E_0]\), \(E_0 = \text{const}\).
Non-autonomous perturbations of the limiting system are described by the functions with power-law asymptotics:

\[ H(x, y, t) - H_0(x, y) = \sum_{k=1}^{\infty} t^{-k \frac{q}{q+1}} H_k(x, y), \]

\[ F(x, y, t) = \sum_{k=1}^{\infty} t^{-k \frac{q}{q+1}} F_k(x, y), \quad t \to \infty, \quad q \in \mathbb{Z}^+. \quad (3) \]

It is assumed that the perturbations preserve the fixed point (0, 0):

\[ \partial_x H(0, 0, t) \equiv 0, \quad \partial_y H(0, 0, t) \equiv 0, \quad F(0, 0, t) \equiv 0. \]

The structure of the perturbations can be more complicated, for example, asymptotic series (3) can differ from power series, or the coefficients of asymptotics can explicitly depend on \( t \). Such perturbations, however, are not considered in the present paper.

Our goal is to describe possible asymptotic regimes in the perturbed system and to reveal the role of decaying perturbations in the corresponding bifurcations. Here, the bifurcations are associated with a change of Lyapunov stability of the equilibrium and the emergence of new attracting or repelling states.

Let us note that the decaying perturbations do affect the stability of the system. A simple example is given by the following equation:

\[ \frac{d^2 x}{dt^2} + x = \gamma t \frac{dx}{dt}, \quad \gamma, \kappa \in \mathbb{R}, \quad \gamma \neq 0, \quad \kappa > 0. \]

This equation in the variables \( x, y = \dot{x} \) has the form (1) with \( H(x, y, t) \equiv H_0(x, y) \equiv \frac{x^2 + y^2}{2} \) and \( F(x, y, t) \equiv \gamma t^{-\kappa} y \). It can easily be checked that the unperturbed autonomous equation (\( \gamma = 0 \)) has the following general solution: \( x(t; a, \varphi) = a \cos(\varphi + t) \). The long-term asymptotics for a two-parameter family of solutions to the perturbed equation (\( \gamma \neq 0 \)) is constructed with using WKB approximations [18]:

\[ x(t; a, \varphi) = a \left[ \cos(\varphi + t) + \mathcal{O}(t^{1-\kappa}) \right], \quad \kappa > 1; \]

\[ x(t; a, \varphi) = a t^{\kappa/2} \left[ \cos(\varphi + t) + \mathcal{O}(t^{-1}) \right], \quad \kappa = 1; \]

\[ x(t; a, \varphi) = a \exp \left( \frac{\gamma t^{1-\kappa}}{1-\kappa} \right) \left\{ \cos(\varphi + t + \mathcal{O}(t^{1-2\kappa}) + \mathcal{O}(\log t)) + \mathcal{O}(t^{-\kappa}) \right\}, \quad \kappa < 1, \]

where \( a, \varphi \in \mathbb{R} \) are arbitrary parameters. It follows that the stability of the trivial solution \( x(t) \equiv 0 \) or the fixed point (0, 0) depends on the parameters \( \gamma \) and \( \kappa \). In particular, if \( \kappa > 1 \), the fixed point is marginally stable. In this case, the solutions of the non-autonomous equation have the same behaviour as the solutions of the limiting equation. The fixed point becomes attracting if \( \gamma < 0 \) (polynomially stable when \( \kappa = 1 \) and exponentially stable when \( 0 < \kappa < 1 \)), and loses stability if \( \gamma > 0 \). In the general case, the long-term asymptotics for solutions are obtained not so easily, and the stability of the equilibrium depends on nonlinear terms of equations. The examples of nonlinear equations are contained in section 6.

3. Change of variables

The proposed method of study of asymptotic regimes in system (1) is based on the construction of appropriate Lyapunov functions. Recently, it was noted in [19–21] that such functions are...
effective in the asymptotic analysis of solutions to nonlinear non-autonomous systems. See also [22] for application of the second Lyapunov method to asymptotic analysis of equations with a small parameter. Here, a Lyapunov function is used as a new dependent variable. In this section, the construction of such function and the change of variables are presented in a form suitable for further bifurcation analysis of system (1).

First, consider the limiting system

\[
\frac{dx}{dt} = \partial_x H_0(x, y), \quad \frac{dy}{dt} = -\partial_y H_0(x, y).
\]

To each level line \((x, y) \in \mathbb{R}^2 : H_0(x, y) = E\), \(E \in (0, E_0]\) there correspond a periodic solution \(x_0(t, E), y_0(t, E)\) of system (4) with a period \(T(E) = 2\pi/\omega(E)\), where \(\omega(E) \neq 0\) for all \(E \in [0, E_0)\) and \(\omega(E) = 1 + \mathcal{O}(E)\) as \(E \to 0\). The value \(E = 0\) corresponds to the fixed point \((0, 0)\).

Define auxiliary \(2\pi\)-periodic functions \(X(\varphi, E) = x_0(\varphi/\omega(E), E)\) and \(Y(\varphi, E) = y_0(\varphi/\omega(E), E)\), satisfying the system:

\[
\omega(E) \frac{\partial X}{\partial \varphi} = \partial_X H_0(X, Y), \quad \omega(E) \frac{\partial Y}{\partial \varphi} = -\partial_Y H_0(X, Y).
\]

These functions are used for rewriting system (1) in the action-angle variables \((E, \varphi)\):

\[
x(t) = X(\varphi(t), E(t)), \quad y(t) = Y(\varphi(t), E(t)).
\]

From the identity \(H_0(X(\varphi, E), Y(\varphi, E)) \equiv E\) it follows that

\[
\left| \frac{\partial X}{\partial \varphi} \frac{\partial X}{\partial \varphi} \right| = \frac{1}{\omega(E)} \neq 0.
\]

The last inequality guarantees the reversibility of transformation (5) for all \(E \in (0, E_0]\) and \(\varphi \in \mathbb{R}\). It can easily be checked that in new variables \((E, \varphi)\) system (1) takes the form:

\[
\frac{dE}{dt} = f(E, \varphi, t), \quad \frac{d\varphi}{dt} = \omega(E) + g(E, \varphi, t),
\]

where

\[
f(E, \varphi, t) \equiv -\omega(E) \left( \partial_x H(X(\varphi, E), Y(\varphi, E), t) - F(X(\varphi, E), Y(\varphi, E), t) \right) \partial_x X(\varphi, E),
\]

\[
g(E, \varphi, t) \equiv \omega(E) \left( \partial_x H(X(\varphi, E), Y(\varphi, E), t) - 1 - F(X(\varphi, E), Y(\varphi, E), t) \right) \partial_x X(\varphi, E)
\]

are \(2\pi\)-periodic functions with respect to \(\varphi\). Since \((0, 0)\) is the equilibrium of system (1), we have \(f(0, \varphi, t) \equiv 0\). From (3) it follows that

\[
f(E, \varphi, t) = \sum_{k=1}^{\infty} t^{\frac{k}{2}} f_k(E, \varphi), \quad g(E, \varphi, t) = \sum_{k=1}^{\infty} t^{\frac{k}{2}} g_k(E, \varphi), \quad t \to \infty,
\]

where

\[
f_k(E, \varphi) \equiv -\omega(E) \left( \partial_x H_k(X(\varphi, E), Y(\varphi, E), t) - F_k(X(\varphi, E), Y(\varphi, E), t) \right) \partial_x X(\varphi, E),
\]

\[
g_k(E, \varphi) \equiv \omega(E) \left( \partial_x H_k(X(\varphi, E), Y(\varphi, E), t) - 1 - F_k(X(\varphi, E), Y(\varphi, E), t) \right) \partial_x X(\varphi, E).
\]
To simplify the first equation in (6), we consider the transformation of the variable $E$ in the form:

$$V_N(E, \varphi, t) = E + \sum_{k=1}^{N} t^{-\frac{k}{2}} v_k(E, \varphi),$$

(8)

where the coefficients $v_k(E, \varphi)$ are chosen in such a way that the right-hand side of the equation for the new variable $v(t) \equiv V_N(E(t), \varphi(t), t)$ does not depend on $\varphi$ at least in the first terms of the asymptotics:

$$\frac{dv}{dt} = \sum_{k=1}^{N} t^{-\frac{k}{2}} \Lambda_k(v) + R_{N+1}(v, \varphi, t), \quad R_{N+1}(v, \varphi, t) = \mathcal{O} \left( t^{-\frac{N+1}{2}} \right), \quad t \to \infty.$$  

(9)

Under the transformation $(E, \varphi) \mapsto (v, \varphi)$ the form of the second equation in (6) changes slightly:

$$\frac{d\varphi}{dt} = \omega(v) + G_N(v, \varphi, t), \quad G_N(v, \varphi, t) = \sum_{k=1}^{\infty} t^{-\frac{k}{2}} g_{N,k}(v, \varphi), \quad t \to \infty.$$  

(10)

Here, each function $g_{N,k}(v, \varphi)$ is $2\pi$-periodic with respect to $\varphi$ and is expressed through $v_1, \ldots, v_k$ and $g_1, \ldots, g_k$. For example,

$$g_{N,1}(v, \varphi) = g_1(v, \varphi) - \omega(v)v_1(v, \varphi),$$

$$g_{N,2}(v, \varphi) = g_2(v, \varphi) - \omega'(v)(v_2(v, \varphi) - \partial_v v_1(v, \varphi)v_1(v, \varphi)$$

$$- \partial_v g_1(v, \varphi)v_1(v, \varphi) + \omega''(v)v_1^2(v, \varphi).$$

Note that such a transformation is usually applied in the averaging of systems with a small parameter and is associated with a fast variable elimination [23]. Here, $\varphi$ can serve as an analogue of a fast variable. However, the presence of a small parameter is not assumed in the system, and the terms ‘fast’ and ‘slow’ variables are not appropriate.

Let us move on to the calculation of the coefficients $v_k(E, \varphi)$. The total derivative of the function $V_N(E, \varphi, t)$ with respect to $t$ along the trajectories of system (6) has the following form

$$\left. \frac{dV_N}{dr} \right|_{(6)} = \partial_t V_N(E, \varphi, t) + f(E, \varphi, t) \partial_E V_N(E, \varphi, t) + (\omega(E) + g(E, \varphi, t)) \partial_\varphi V_N(E, \varphi, t)$$

$$= \sum_{k=1}^{\infty} t^{-\frac{k}{2}} \left( \omega(E) \partial_\varphi v_k(E, \varphi) + f_k(E, \varphi) - \frac{k-q}{q} v_{k-q}(E, \varphi) \right)$$

$$+ \sum_{j=2}^{N} t^{-\frac{j}{2}} \sum_{j \geq k} (f_j(E, \varphi) \partial_E v_k(E, \varphi) + g_j(E, \varphi) \partial_\varphi v_j(E, \varphi)),\quad (11)$$

where it is assumed that $v_j(E, \varphi) \equiv 0$ for $j \leq 0$ and $j > N$. Substituting (8) into the right-hand side of (9) and the comparison of the result with (11) lead to the following chain of differential equations:

$$\omega(E) \partial_\varphi v_k = \Lambda_k(E) - f_k(E, \varphi) + Z_k(E, \varphi), \quad k = 1, 2, \ldots, N,$$

(12)

where each function $Z_k(E, \varphi)$ is expressed through $v_1, \ldots, v_{k-1}$. In particular,

$$Z_1 \equiv 0,$$

$$Z_2 \equiv v_1 \partial_E \Lambda_1 - \left( f_1 \partial_E v_1 + g_1 \partial_\varphi v_1 \right).$$
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\[ Z_3 \equiv v_2 \partial_E \Lambda_1 + v_1 \partial_E \Lambda_2 + \frac{1}{2} v_1^2 \partial_E^2 \Lambda_1 - \sum_{i+j+k} \left( f_j \partial_E v_i + g_j \partial_E v_i \right), \]

\[ Z_4 \equiv \sum_{j+\alpha_1+2\alpha_2+\cdots+\alpha_k = k} C_{\alpha_1, \alpha_2, \ldots, \alpha_k} v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_i^{\alpha_i} \partial_E^m \Lambda_j - \sum_{i+j+k} \left( f_j \partial_E v_i + g_j \partial_E v_i \right) + \frac{k-q}{q} v_{k-q}, \]

where \( C_{\alpha_1, \alpha_2, \ldots, \alpha_k} = \text{const}. \) Define

\[ \Lambda_k(E) = \langle f_k(E, \varphi) \rangle - \langle Z_k(E, \varphi) \rangle, \quad (13) \]

where

\[ \langle f_k(E, \varphi) \rangle \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} f_k(E, \varphi) d\varphi. \]

Hence, for every \( k \geq 1 \) the right-hand side of (12) is \( 2\pi \)-periodic function with respect to \( \varphi \) with zero average. By integrating (12) with respect to \( \varphi \), we obtain

\[ v_k(E, \varphi) = H_k(X(\varphi, E), Y(\varphi, E)) + \frac{1}{\omega(E)} \int_0^\varphi \Lambda_k(E) \]

\[ \quad - \omega(E) F_k(X(\varphi, E), Y(\varphi, E)) \partial_\varphi X(\varphi, E) + Z_k(E, \varphi) \]

for \( k = 1, \ldots, N \). It can easily be checked that each \( v_k(E, \varphi) \) is a smooth \( 2\pi \)-periodic function with respect to \( \varphi \) such that \( v_k(0, \varphi) = 0 \). From (13) it follows that \( \Lambda_k(v) = \mathcal{O}(v) \) as \( v \to 0 \). The function \( R_{N+1}(E, \varphi, t) \) has the following form:

\[ R_{N+1}(E, \varphi, t) \equiv \sum_{k=N+1}^{\infty} \frac{1}{t^k} \left( f_k - \frac{k-q}{q} v_{k-q} + \sum_{j=1}^{k-1} \left( f_{k-j} \partial_E v_j + g_{k-j} \partial_E v_j \right) \right). \]

It is clear that \( R_{N+1}(E, \varphi, t) \) is \( 2\pi \)-periodic functions with respect to \( \varphi \) such that \( R_{N+1}(0, \varphi, t) \equiv 0 \) and \( |R_{N+1}(0, \varphi, t)| = \mathcal{O}(t^{-(N+1)/q}) \) as \( t \to \infty \) for all \( v \in [0, d_0] \), \( \varphi \in \mathbb{R} \) with \( d_0 = (1-\sigma)E_0 \).

From (8) it follows that for all \( \sigma \in (0, 1) \) there exists \( t_0 > 0 \) such that

\[ (1-\sigma)E \leq V_0(E, \varphi, t) \leq (1+\sigma)E, \quad (1-\sigma) \leq \partial_E V_0(E, \varphi, t) \leq (1+\sigma) \]

(14)

for all \( E \in [0, E_0] \), \( \varphi \in \mathbb{R} \) and \( t \geq t_0 \). Hence, the transformation \( (E, \varphi) \mapsto (v, \varphi) \) is reversible.

Thus, we have

**Proposition 1.** There exists a reversible change of variables \( (x, y, t) \mapsto (v, \varphi, t) \): \( x = x(v, \varphi, t), y = y(v, \varphi, t), t = t \), where the dependence on \( (v, \varphi) \) is equicontinuous with respect to \( t \), such that system (1) is reduced to the form (9) and (10).

**4. Bifurcations of the equilibrium**

In this section, possible bifurcations of the fixed point \((0, 0)\) of (1) as well as the trivial solution of equation (9) are discussed. From the properties of the function \( R_{N+1}(v, \varphi, t) \) it follows that the leading terms of asymptotics for solutions of (9) does not depend on \( \varphi \). The long-term behaviour of solutions \( v(t) \) is determined by the functions \( \{\Lambda_k(v)\}_{k=1}^{N} \). Besides, from (10) it follows that \( \varphi(t) \to \infty \) as \( t \to \infty \), while \( v(t) \in [0, d_0] \). Let \( n \geq 1 \) be the least natural number such that \( \Lambda_n(v) \neq 0 \). Then equation (9) takes the form:
\[
\frac{dv}{dt} = \sum_{k=0}^{N} t^{-\frac{k}{2}} \Lambda_k(v) + R_{N+1}(v, \varphi, t), \quad t \geq t_0.
\]

From (2) and (5), it follows that \( E = r^2/2 + \mathcal{O}(r^3) \) as \( r \to 0 \). Combining this with (14), we see that \( V_m(E, \varphi, t) \) is positive definite function in the vicinity of the fixed point \((0,0)\). Thus, \( V_m(E, \varphi, t) \) in the variables \((x,y)\) can be used as a Lyapunov function candidate for system (1). If the total derivative of \( V_m(E, \varphi, t) \) with respect to \( t \) along the trajectories of system (6) is sign definite for \( E \) close to zero and \( \varphi \in \mathbb{R} \), then this function can be effectively used for the stability analysis of the equilibrium \((0,0)\). It can easily be seen that the right-hand side of (9) coincides with the total derivative of \( V_m(E, \varphi, t) \).

**Theorem 1.** Let \( n \geq 1 \) be an integer such that \( \Lambda_k(v) \equiv 0 \) for \( k < n \) and
\[
\Lambda_n(v) = \lambda_n v + \mathcal{O}(v^2), \quad v \to 0, \quad \lambda_n = \text{const} \neq 0.
\]

Then the equilibrium \((0,0)\) of system (1) is unstable if \( \lambda_n > 0 \) and is uniformly stable if \( \lambda_n < 0 \). Moreover, if \( \lambda_n < 0 \) and \( n < q \ (n = q) \), the equilibrium is uniformly exponentially (polynomially) stable.

**Proof.** Consider \( V_m(E, \varphi, t) \) with \( N = n \) as a Lyapunov function candidate for system (1). From (15) it follows that the function \( v(t) = V_m(E(t), \varphi(t), t) \) satisfies the equations:
\[
\frac{dv}{dt} = t^{-\frac{q}{2}} v \left( \lambda_n + \mathcal{O}(v) + \mathcal{O} \left( t^{-\frac{1}{2}} \right) \right)
\]
as \( t \to \infty \) and \( v \to 0 \) for all \( \varphi \in \mathbb{R} \). Hence, for all \( \sigma \in (0,1) \) there exist \( 0 < d_1 \leq d_0 \) and \( t_1 \geq t_0 \) such that
\[
\begin{align*}
\frac{dv}{dt} &\geq t^{-\frac{q}{2}} (1 - \sigma) |\lambda_n| v \quad \text{if } \lambda_n > 0, \\
\frac{dv}{dt} &\leq -t^{-\frac{q}{2}} (1 - \sigma) |\lambda_n| v \quad \text{if } \lambda_n < 0,
\end{align*}
\]
for all \( v \in [0,d_1], \varphi \in \mathbb{R} \) and \( t \geq t_1 \). Integrating the first estimate in (16) with respect to \( t \) yields the instability of the trivial solution \( v(t) \equiv 0 \) of equation (9) for all \( \varphi \in \mathbb{R} \). Indeed, there exists \( \epsilon \in (0,d_1/4) \) such that for all \( \delta \in (0,\epsilon) \) the solution \( v(t) \) with initial data \( v(t_1) = \delta \) exceeds the value \( \epsilon \) as \( t > t_* \), where
\[
t_* = t_1 \left( \frac{2\epsilon}{\delta} \right) \left( \frac{1}{1 - \sigma |\lambda_n|} \right)^{-\frac{1}{q - n}} \quad \text{if } \frac{n}{q} = 1;
\]
\[
t_*^{-\frac{q}{2}} = t_1^{-\frac{q}{2}} + \left( \frac{q - n}{1 - \sigma |\lambda_n| q} \right) \log \left( \frac{2\epsilon}{\delta} \right) \quad \text{if } \frac{n}{q} \neq 1.
\]

Similarly, from the second estimate in (16) it follows that for all \( \epsilon \in (0,d_1) \) there exists \( \delta \in (0,\epsilon) \) such that the solution \( v(t) \) of equation (9) with initial data \( 0 < v(t_1) < \delta \) cannot exceed the value \( \epsilon \) as \( t \geq t_1 \). Hence, the solution \( v(t) \equiv 0 \) of equation (9) is uniformly stable for all \( \varphi \in \mathbb{R} \) (see, for example, [24, section 4.5]). Moreover, by integrating this estimate with respect to \( t \), we obtain the following inequalities:
\begin{align*}
0 &\leq v(t) \leq v(t_1) \left( \frac{t}{t_1} \right)^{-(1 - \sigma |\lambda_n|)} \quad \text{if } \frac{n}{q} = 1, \\
0 &\leq v(t) \leq v(t_1) \exp \left( \frac{(1 - \sigma |\lambda_n|)}{q - n} \left( t^{-\frac{q}{2}} - t_1^{-\frac{q}{2}} \right) \right) \quad \text{if } \frac{n}{q} \neq 1,
\end{align*}
(17)
as \( t \geq t_1 \). It follows that the solution \( E(t) \equiv 0 \) of system (6) is uniformly asymptotically stable for all \( \varphi \in \mathbb{R} \). In particular, the stability is exponential if \( n < q \), polynomial if \( n = q \) and marginal if \( n > q \). Taking into account (5) and (14), we obtain the corresponding propositions on the stability of the equilibrium \((0,0)\) of system (1).

Note that when the equilibrium loses the stability, the solutions of equation (9) starting from the vicinity of zero either remain inside the domain \((0, d_0)\), or cross the boundary \( d_0 \) at \( t_{\text{exit}} > t_0 \). In the first case, the trajectories of the perturbed system can be attracted by periodic solutions of the limiting system. The conditions which guarantee the existence of such attracting or repelling states are discussed in the next section. In the second case, the trajectories of (1) may pass through a separatrix of the limiting system as \( t > t_{\text{exit}} \) and can be captured by another attractor. However, such global bifurcations of solutions are not discussed in this paper.

Thus, for non-autonomous perturbations satisfying the conditions of theorem 1, \( \lambda_n \) can be considered as a bifurcation parameter such that \( \lambda_n = 0 \) is a critical value.

Let us consider the case when the leading term of the right-hand side of equation (9) is nonlinear with respect to \( v \).

**Theorem 2.** Let \( 1 \leq m < n \) be integers such that \( \lambda_k(v) \equiv 0 \) for \( k < m \) and

\[
\Lambda_m(v) = \gamma_{m,n}v + O(v^{q+4}), \quad \Lambda_j(v) = O(v^q), \quad m \leq j < n, \quad \Lambda_n(v) = \lambda_n v + O(v^2), \quad v \to 0,
\]

where \( \gamma_{m,n}, \lambda_n = \text{const} \neq 0, s \in \mathbb{Z}, s \geq 2 \). Then the equilibrium \((0,0)\) of system (1) is

- uniformly stable if \( \lambda_n < 0 \) and \( \gamma_{m,n} < 0 \);
- unstable if \( \lambda_n > 0 \) and \( \gamma_{m,n} > 0 \).

**Proof.** As above, consider \( V_n(E, \varphi, t) \) as a Lyapunov function candidate. In this case, its total derivative has the form:

\[
\frac{dV_n}{dt}(E) \equiv \frac{dV_n}{dt}(v) = v^m [\Lambda_m(v) + g_m(v, \varphi, t)] + v^{n-3} [\Lambda_n(v) + g_n(v, \varphi, t)],
\]

where \( v(t) = V_n(E(t), \varphi(t), t), \quad |g_m(v, \varphi, t)| \leq Mt^{-1/4}v^q, \quad |g_n(v, \varphi, t)| \leq Mt^{-1/4}v \) as \( v \to 0 \), \( t \to \infty \) for all \( \varphi \in \mathbb{R} \) with \( M = \text{const} > 0 \). Therefore, for all \( \sigma \in (0, 1) \) there exist \( 0 < d_1 \leq d_0 \) and \( t_1 \geq t_0 \) such that

\[
\frac{dv}{dt} \leq -(1 - \sigma)f v^m |v| + (1 - \sigma)f \lambda_n v \leq 0 \quad \text{if} \quad \lambda_n < 0, \quad \gamma_{m,n} < 0,
\]

\[
\frac{dv}{dt} \geq (1 - \sigma)f v^m |v| + (1 - \sigma)f \lambda_n v \geq 0 \quad \text{if} \quad \lambda_n > 0, \quad \gamma_{m,n} > 0,
\]

for all \( v \in [0, d_1], \varphi \in \mathbb{R} \) and \( t \geq t_1 \). From the last estimates, as in the previous theorem, it follows that the solution \( E(t) \equiv 0 \) to system (6) is uniformly stable if \( \lambda_n < 0 \), \( \gamma_{m,n} < 0 \) and unstable if \( \lambda_n > 0, \gamma_{m,n} > 0 \). Combining (5) and (14), we obtain the corresponding propositions on the stability of the equilibrium \((0,0)\) of system (1).

Note that in some cases the last proposition can be improved. In particular, we have

**Theorem 3.** Under the conditions of theorem 2, we have

- in case \( m < q < n \), the equilibrium \((0,0)\) of system (1) is
  - uniformly exponentially stable if \( \lambda_n < 0 \);
  - polynomially stable if \( \lambda_n > 0 \) and \( \gamma_{m,n} < 0 \).

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• in case $m < n = q$, the equilibrium $(0,0)$ of system (1) is
  * uniformly polynomially stable if $\lambda_0 + \frac{d_E}{q(s-1)} < 0$;
  * polynomially stable if $\lambda_0 + \frac{d_E}{q(s-1)} > 0$ and $\gamma_{m,s} < 0$.
• in case $m < q < n$, the equilibrium $(0,0)$ of system (1) is polynomially stable if $\gamma_{m,s} < 0$;
• in case $q \leq m < n$, the equilibrium $(0,0)$ of system (1) is stable if $\gamma_{m,s} < 0$.

**Proof.** It can easily be checked that the derivative of the function $v(t) = V_n(E(t), \varphi(t), t)$ satisfies the asymptotic estimate:

$$
\frac{dV_n}{dt} = t^{-\frac{q}{s+q}} \left( \gamma_{m,s} + O(v) + O \left( t^{-\frac{1}{s+q}} \right) \right) + t^{-\frac{q}{s+q}} \left( \lambda_0 + O(v) + O \left( t^{-\frac{1}{s+q}} \right) \right)
$$

as $t \to \infty$, $v \to 0$ for all $\varphi \in \mathbb{R}$. The right-hand side of the last expression is not sign definite uniformly for all small $v$ and big $t$. Indeed, if $v \sim \epsilon$ and $t \sim \epsilon^{-\kappa}$, where $0 < \epsilon \ll 1$, $\kappa = \text{const}$, then the sign of $dv/dt$ is determined by $\lambda_0$ in case $\kappa < q(s-1)/(n-m)$, and by $\gamma_{m,s}$ in the opposite case. Therefore, $V_n(E, \varphi, t)$ cannot be used as a Lyapunov function for system (1).

Define

$$
U(E, \varphi, t) = t^\nu V_n(E, \varphi, t), \quad \nu = \frac{n-m}{q(s-1)} > 0.
$$

From (14) it follows that

$$
(1-\sigma)t^\nu E \leq U(E, \varphi, t) \leq (1+\sigma)t^\nu E \tag{18}
$$

for all $E \in [0, E_0]$, $\varphi \in \mathbb{R}$ and $t \geq t_0$. This function corresponds to the change of variable $u(t) = t^\nu v(t)$ such that equation (9) takes the form:

$$
\frac{du}{dt} = \nu t^{-1} u + \sum_{k=m}^{n} t^{-\frac{k}{s}} A_k \left( t^{-\nu} u \right) + \nu t^m R_{n+1} \left( t^{-\nu} u, \varphi, t \right). \tag{19}
$$

Hence the total derivative of the function $U(E, \varphi, t)$ with respect to $t$ along the trajectories of system (6) has the following asymptotics:

$$
\frac{dU}{dt} = \frac{du}{dt} = u \left( \nu t^{-1} + \nu t^{-\frac{n}{s+q}} \lambda_0 + \gamma_{m,s} u^{n-1} + O \left( t^{\frac{n+1}{s+q}} \right) \right) + \nu t^{m-s} R_{n+1} \left( t^{-\nu} u, \varphi, t \right) \tag{20}
$$

as $t \to \infty$ for all $u \in [0, U_0]$ and $\varphi \in \mathbb{R}$ with $U_0 = \text{const} > 0$.

Consider the case $m < n < q$. From (20) it follows that

$$
\frac{du}{dt} = u^{-\frac{n}{s}} \left( \lambda_0 + \gamma_{m,s} u^{n-1} + O(t^{-\nu}) + O \left( t^{-\frac{1}{s+q}} \right) \right)
$$

as $t \to \infty$. If $\lambda_0 < 0$, then for all $\sigma \in (0,1)$ there exist $0 < U_1 \leq U_0$ and $t_1 \geq t_0$ such that

$$
\frac{du}{dt} \leq -t^{-\frac{n}{s}} (1-\sigma) |\lambda_0| u \leq 0
$$

for all $u \in [0, U_1]$, $\varphi \in \mathbb{R}$ and $t \geq t_1$. Hence, for all $\epsilon \in (0, U_1)$ there exists $\delta \in (0, \epsilon)$ such that the solution $u(t)$ of equation (19) with initial data $0 < u(t_1) < \delta$ cannot exceed the value $\epsilon$ as $t \to \infty$. Moreover, by integrating the differential inequality with respect to $t$, we get an estimate of the form (17) as $t \geq t_1$. Combining this with (18), we get uniform exponential stability of the solution $E(t) \equiv 0$ of equation (6).
If $\lambda_n > 0$ and $\gamma_{m,t} < 0$, the leading term of $du/dt$ has a zero at $u = U_c$, $U_c = (\lambda_n/|\gamma_{m,t}|)^{1/(n-1)}$. Let us show that $U(E(t), \varphi(t), t) \rightarrow U_c$ as $t \rightarrow \infty$. Consider the change of variable $u(t) = U_c + z(t)$ in equation (19). Then $z(t)$ satisfies the equation:

$$\frac{dz}{dt} = t^{-\frac{q}{2}}(U_c + z) \left(\lambda_n + \gamma_{m,t} U_c + z\right)^{\nu - 1} + p(z, \varphi, t),$$

where $|p(z, \varphi, t)| \leq M t^{-n/q-\sigma}$ as $t \geq t_1$ for all $|z| \leq z_1$ and $\varphi \in \mathbb{R}$ with positive constants $M, z_1$, and $\sigma = \min\{\nu, 1/q\}$. It can easily be checked that the equation with $p(z, \varphi, t) \equiv 0$ has asymptotically stable solution $z(t) \equiv 0$. Let us show that this solution is stable with respect to the perturbation $p(z, \varphi, t)$. Consider $\ell(z) = z^2/2$ as a Lyapunov function candidate for equation (21). The total derivative of $\ell(z)$ has the form:

$$\frac{d\ell}{dt}(21) = t^{-\frac{q}{2}} \left(-\mu z + \mathcal{O}(z^2) + \mathcal{O}(t^{-\sigma})\right)$$

as $z \rightarrow 0$ and $t \rightarrow \infty$ with $\mu = |\gamma_{m,t}|(s-1)U_c^{s-1} > 0$. Therefore, for all $\sigma > 0$, there exist $0 < z_2 \leq z_1$ and $t_2 \geq t_1$ such that

$$\frac{d\ell}{dt}(21) \leq t^{-\frac{q}{2}} \left(-(1-\sigma)\mu z^2 + M \sigma z^{-\sigma}\right)$$

for all $|z| \leq z_2$, $\varphi \in \mathbb{R}$ and $t \geq t_2$. Let us fix $\epsilon \in (0, z_2)$, then

$$\frac{d\ell}{dt}(21) \leq 2t^{-\frac{q}{2}} \left(-(1-\sigma)\mu + \frac{M}{\delta} \sigma t^{-\sigma}\right) \leq 0$$

for all $\delta \leq |z| \leq \epsilon$ and $t \geq t_3$, where

$$\delta = \frac{2M\epsilon\sigma}{(1-\sigma)\mu}, \quad t_3 = \max\left\{\frac{4M}{\epsilon(1-\sigma)\mu} \right. \left. \frac{1}{\delta}, t_2\right\}.$$

Note that $0 < \delta < \epsilon$. Since $\ell(z) \leq \delta^2/2$ for all $|z| \leq \delta$, $\ell(z) = \epsilon^2/2$ for $|z| = \epsilon$, $d\ell/dt \leq 0$ for all $\delta < |z| < \epsilon$ and $t \geq t_3$, we see that the solution $z(t)$ of equation (21) with initial data $z(t_3) \leq \delta$ cannot exit from the domain $|z| \leq \epsilon$ as $t \geq t_3$. Moreover, from (22) it follows that $d\ell/dt \leq t^{-n/q}(-\mu t + M \sigma t^{-\sigma})$ as $t \geq t_3$. By integrating the last inequality, we get

$$\ell(z(t)) \leq \ell(z(t_3))e^{-\mu(s-1)t} + M\epsilon\sigma t^{-\sigma} + \int_{t_3}^{t} e^{\mu(s-1)e^{-\sigma}} \sigma t^{-\sigma} d\tau.$$

It follows that $\ell(z(t)) = \mathcal{O}(t^{-\sigma})$ as $t \rightarrow \infty$ for solutions with initial data $|z(t_3)| \leq \delta$. Hence, $U(E(t), \varphi(t), t) = U_c + \mathcal{O}(t^{-\sigma/2})$ and $E(t) = \mathcal{O}(t^{-\sigma})$ as $t \rightarrow \infty$. Therefore, the solution $E(t) \equiv 0$ of (6) is polynomially stable.

Consider the case $m < n = q$. It follows from (20) that

$$\frac{du}{dt} = u(t)^{-1} \left(\lambda_n + \nu + \gamma_{m,t} u^{-1} + \mathcal{O}(t^{-\sigma}) + \mathcal{O}\left(t^{-\frac{q}{2}}\right)\right)$$

as $t \rightarrow \infty$. If $\lambda_n + \nu < 0$, then for all $\sigma > 0$, there exist $0 < U_1 \leq U_0$ and $t_1 \geq t_0$ such that
\[ \frac{du}{dt} \leq t^{-1}(-|\lambda_n + \nu| + \sigma)u \leq 0 \]

for all \( u \in [0, U_1], \varphi \in \mathbb{R} \) and \( t \geq t_1 \). Hence, for all \( \epsilon \in (0, U_1) \) there exists \( \delta \in (0, \epsilon) \) such that the solution \( u(t) \) of equation (19) with initial data \( 0 < u(t_1) < \delta \) cannot exceed the value \( \epsilon \) as \( t \geq t_1 \). Integrating the last inequality yields \( 0 \leq v(t) = t^{-\beta}u(t) \leq v(t_1)(t/t_1)^{\lambda_n + \sigma} \) as \( t \geq t_1 \). Therefore, the solution \( E(t) \equiv 0 \) is uniformly polynomially stable. If \( \lambda_n + \nu > 0 \) and \( \gamma_{m,\delta} < 0 \), then, as in the previous case, it can be shown that \( U(E(t), \varphi(t), t) = U_{\nu} + \mathcal{O}(t^{-\nu/2}) \) as \( t \to \infty \), where \( U_{\nu} = (\lambda_n + \nu)/|\gamma_{m,\delta}|^{1/(\nu-1)} \). Taking into account the transformation of variables, we get polynomial stability of the solution \( E(t) \equiv 0 \) to system (6).

In the case \( m < q < n \), the function \( U(E, \varphi, t) \) cannot be used in the stability analysis. Consider the function

\[ W(E, \varphi, t) \equiv t^\beta V_\eta(E, \varphi, t), \quad \eta = \frac{q-m}{q(s-1)} > 0, \]

which corresponds to the change of variables in (9): \( v(t) = t^{-\beta}w(t) \). The total derivative of \( W(E, \varphi, t) \) has the asymptotics

\[ \frac{dw}{dt} = t^{-1}w \left( \eta + \gamma_{m,\delta} w^{\gamma_{m,\delta} - 1} + \mathcal{O}(t^\gamma) \right) \]

as \( t \to \infty \) for all \( w \in [0, W_0] \) and \( \varphi \in \mathbb{R} \) with \( W_0 = \text{const} > 0 \). If \( \gamma_{m,\delta} < 0 \), then the leading term of \( dw/dt \) has a zero at \( w = W_\epsilon \), where \( W_\epsilon = (\eta/|\gamma_{m,\delta}|)^{1/(\nu-1)} \). As above, it can be shown that \( W(E(t), \varphi(t), t) \to W_\epsilon \) as \( t \to \infty \). Hence, \( E(t) = \mathcal{O}(t^{-\nu}) \) as \( t \to \infty \) and the solution \( E(t) \equiv 0 \) to system (6) is polynomially.

Finally, consider the case \( q < m < n \). For all \( \sigma \in (0, 1) \) there exist \( 0 < d_1 < d_0 \) and \( t_1 > t_0 \) such that

\[ \frac{dv}{dt} \leq -\left(1 - \sigma\right)\gamma_{m,\delta}|v|^{\frac{2}{\sigma}} + Mt^{\frac{2}{\sigma}}v \]

for all \( v \in [0, d_1], \varphi \in \mathbb{R} \) and \( t \geq t_1 \). Let us fix \( 0 < \epsilon < d_1 \) and define

\[ t_2 = \max \left\{ \left( \frac{2\bar{M}}{\epsilon^{-1}(1-\sigma)|\gamma_{m,\delta}|} \right)^{\frac{1}{2}}, t_1 \right\}. \]

Then for all \( \epsilon < d_1 \) and \( t \geq t_2 \) we have

\[ \frac{dv}{dt} \leq -t^{\frac{\sigma}{2}}(1 - \sigma)|\gamma_{m,\delta}|v^{\sigma} \]

This implies that any solution \( v(t) \) of equation (9) with initial data \( 0 < v(t_2) < \epsilon/2 \) cannot exceed the value \( \epsilon \) as \( t \geq t_2 \). Hence, the solution \( E(t) \equiv 0 \) is at least neutrally stable. □

Let us remark that in the case \( n = q \) with \( \gamma_{m,\delta} > 0 \) and \( \lambda_n + \nu > 0 \), Lyapunov stability of the equilibrium is not justified. Moreover, from theorem 3 it follows that the trivial solution is weakly unstable with the weight \( \nu^2 \): there exists \( \epsilon > 0 \) such that for arbitrarily small initial data \( \exists t_0 > 0: E(t_0)^{\nu^2} \geq \epsilon \) as \( t \geq t_0 \). From (2) it follows that the equilibrium \((0, 0)\) is unstable with the weight \( \nu^2/2 \). Similarly, in the case \( m < q < n \) with \( \gamma_{m,\delta} > 0 \), the fixed point \((0, 0)\) of system (1) is unstable with the weight \( \nu^2/2 \).

For non-autonomous perturbations satisfying the conditions of theorem 2, the stability of the equilibrium is determined by two parameters \( \lambda_n \) and \( \gamma_{m,\delta} \) (see figure 1). The partition of the parameter plane depends on the ratio \( n/q \). Note that if \( \lambda_n > 0 \), the equilibrium becomes
unstable in the corresponding linearised system. However, the asymptotic stability can preserve in the complete system due to nonlinear terms of equations. In this case, the system has a Hopf bifurcation in the scaled variables.

Now let us consider the case when the right-hand side of equation (9) has no linear terms in \( v \).

**Theorem 4.** Let \( 1 \leq m < n \) be integers such that \( \Lambda_k(v) \equiv 0 \) for \( k < m \) and
\[
\Lambda_m(v) = \gamma_{m,s} v^s + O(v^{s+1}), \quad \Lambda_j(v) = O(v^j), \quad j < n,
\]
\[
\Lambda_n(v) = \gamma_{n,d} v^d + O(v^{d+1}), \quad \Lambda_i(v) = O(v^i), \quad i > n
\]
as \( v \to 0 \) with \( \gamma_{m,s}, \gamma_{n,d} = \text{const} \neq 0, s, d \in \mathbb{Z}, s, d \geq 2 \).

(a) In case \( s \leq d \), the equilibrium \((0,0)\) of system (1) is

1. uniformly stable if \( \gamma_{m,s} < 0 \);
2. unstable if \( \gamma_{m,s} > 0 \).

(b) In case \( s > d \), the equilibrium \((0,0)\) of system (1) is

1. uniformly stable if \( \gamma_{m,s} < 0 \) and \( \gamma_{n,d} < 0 \);
2. unstable if \( \gamma_{m,s} > 0 \) and \( \gamma_{n,d} > 0 \).

**Proof.** If \( s \leq d \), the total derivative of the function \( V_n(E, \varphi, t) \) has the asymptotics:
\[
\frac{dV_n}{dt} \bigg|_{(6)} \equiv \frac{dV}{dt} = t^{-\frac{s}{2}} v^s \left( \gamma_{m,s} + O(v) + O \left( t^{-\frac{s}{2}} \right) \right)
\]
as \( t \to \infty \) and \( v \to 0 \) for all \( \varphi \in \mathbb{R} \). If \( \gamma_{m,s} > 0 \), the total derivative is locally positive and the solution \( E(t) \equiv 0 \) of system (6) is unstable. In the opposite case, when \( \gamma_{m,s} < 0 \), the total derivative is locally negative. Moreover, for all \( \sigma \in (0,1) \) there exist \( 0 < d_1 \leq d_0 \) and \( t_1 \geq t_0 \) such that
\[
\frac{dv}{dt} \leq -t^{-\frac{s}{2}} (1 - \sigma) |\gamma_{m,s}| v^s \leq 0
\]
for all \( v \in [0, d_1] \), \( \varphi \in \mathbb{R} \) and \( t \geq t_1 \). Hence, for all \( \epsilon \in (0, d_1) \) there exists \( \delta \in (0, \epsilon) \) such that the solution \( v(t) \) of equation (9) with initial data \( 0 < v(t_1) < \delta \) cannot exceed the value \( \epsilon \) as \( t \to \infty \). In this case, the trivial solution is uniformly stable.
Let $s > d$, $\gamma_{\text{ms}} < 0$ and $\gamma_{\text{nld}} < 0$. Then for all $\sigma > 0$ there exist $0 < d_1 \leq d_0$ and $t_1 \geq t_0$ such that
\[
\frac{dv}{dt} \leq -t \frac{\alpha}{\tau} (1 - \sigma)v^d \left( |\gamma_{\text{ms}}| v^{-d} + |\gamma_{\text{nld}}| t^{-\frac{\tau}{\alpha}} \right) \leq 0
\]
for all $v \in [0, d_1], \varphi \in \mathbb{R}$ and $t \geq t_1$. Arguing as above, we conclude that the solution $E(t) \equiv 0$ to system (6) is uniformly stable. Similarly, if $\gamma_{\text{ms}} > 0$ and $\gamma_{\text{nld}} > 0$, the trivial solution is unstable. \hfill \square

5. Close to periodic solutions

Let us show that decaying non-autonomous perturbations may lead to the appearance of trajectories converging to periodic solutions of the corresponding limiting system.

**Theorem 5.** Let $n \geq 1$ be an integer such that $\Lambda_k(v) \equiv 0$ for $k < n$, $\Lambda_n(v) \neq 0$ and $V_0 \in (0, E_0)$ be a real number such that $\Lambda_n(V_0) = 0$ and $\Lambda'_n(V_0) < 0$. Then for all $\epsilon > 0$ there exists $\delta_0 > 0$ and $t_0 > 0$ such that $\forall (x_0, y_0): |H_0(x_0, y_0) - V_0| \leq \delta$ and $t_0 \geq t$, the solution $x(t), y(t)$ of system (1) with initial data $x(t_0) = x_0, y(t_0) = y_0$ satisfies the estimate $|H_0(x(t), y(t)) - V_0| < \epsilon$ for all $t \geq t_0$. Moreover, if $1 \leq n < q$, $H_0(x(t), y(t)) \to V_0$ as $t \to \infty$.

**Proof.** Consider the function $V_n(E, \varphi, t)$ on the trajectories of system (6). It follows that $v(t) = V_n(E(t), \varphi(t), t)$ satisfies the following equation:
\[
\frac{dv}{dt} = t^{\frac{\alpha}{\tau}} \Lambda_n(v) + \mathcal{O}(t^{\frac{\alpha}{\tau}})
\]
as $t \to \infty$ for all $v \in [0, d_0]$ and $\varphi \in \mathbb{R}$. The change of the variable $v(t) = V_c + z(t)$ leads to the following equation:
\[
\frac{dz}{dt} = t^{\frac{\alpha}{\tau}} \Lambda_n(V_c + z) + p(z, \varphi, t), \quad (23)
\]
where $p(0, \varphi, t) \neq 0, |p(z, \varphi, t)| \leq M t^{-\eta - 1/\eta}$ for all $z \in [-V_c, d_0 + V_c], \varphi \in \mathbb{R}$ and $t \geq t_0$ with positive constant $M > 0$. From $\Lambda_n(\equiv \Lambda'_n(V_0) < 0$ it follows that the trivial solution of (23) with $p(z, \varphi, t) \equiv 0$ is stable. Let us show that this solution is stable with respect to the perturbation $p(z, \varphi, t)$. Indeed, consider $\ell(z) = z^n/2$ as a Lyapunov function candidate for (23). Its total derivative has the form:
\[
\frac{df}{dt} \bigg|_{(23)} = t^{\frac{\alpha}{\tau}} \Lambda_n(V_c + z) + p(z, \varphi, t)z. \quad (24)
\]
First note that there exists $\delta_1 > 0$ such that $\Lambda_n(V_c + z)z \leq -|\lambda_n| z^2/2$ as $|z| \leq \delta_1$. Let us fix $0 < \epsilon < \delta_1$ and choose
\[
\delta_* = \frac{\epsilon}{2}, \quad t_* = \max \left\{ \left( \frac{4M}{\delta_* |\lambda_n|} \right)^{\eta}, t_0 \right\}.
\]
then
\[
\frac{df}{dt} \leq -t^{\frac{\alpha}{\tau}} z^2 \left( \frac{|\lambda_n|}{2} - M M^{\frac{1}{\eta}} t_*^{-\frac{\eta}{2}} \right) \leq -t^{\frac{\alpha}{\tau}} z^2 \frac{|\lambda_n|}{4}
\]
for all $\delta_* \leq |z| \leq \epsilon, \varphi \in \mathbb{R}$ and $t \geq t_*$. Hence, any solution $z(t)$ with initial data $|z(t_0)| \leq \delta_*$, $t_0 > t_*$ cannot leave the domain $\{|z| < \epsilon\}$ as $t \geq t_0$. Returning to the original variables, we obtain the result of the theorem.
Let us show that \( H_0(x(t), y(t)) \rightarrow V_c \) as \( t \rightarrow \infty \) if \( n \leq q \). From (24) it follows that \( \frac{d\ell}{dt} \leq t^{-\frac{n}{q}}(|\lambda_n| - M_0 + M_0 t^{-\frac{n}{q}}) \) as \( t \geq t_0 \). By integrating the last inequality in the case \( n = q \), we get
\[
0 \leq \ell(z(t)) - \ell(z(t_0)) + M\delta t^{-|\lambda_n| - \frac{1}{2} - 1} \int_{t_0}^t s^{\frac{1}{2} - \frac{1}{2} - 1} \, ds
\]
with \( |z(t_0)| \leq \delta_1 \). Similar estimates hold in the case \( n < q \). Hence, if \( n \leq q \), \( v(t) \rightarrow V_c \) as \( t \rightarrow \infty \).

**Theorem 6.** Let \( 1 \leq n \leq q \) be an integer such that \( \Lambda_k(v) \equiv 0 \) for \( k < n \), \( \Lambda_n(v) \equiv 0 \) and \( V_c \in (0, E_0) \) be a real number such that \( \Lambda_n(V_c) = 0 \) and \( \Lambda_n'(V_c) > 0 \). Then there exists \( \epsilon > 0 \) such that for all \( \delta_1 > 0 \), \( |H_0(x_0, y_0) - V_c| < \delta_1 \) and \( \tau_0 > 0 \) the solution \( x(t), y(t) \) of system (1) with initial data \( x(\tau_0) = x_0, y(\tau_0) = y_0 \) satisfies the estimate \( |H_0(x(t), y(t)) - V_c| \geq \epsilon \) at some \( t \geq \tau_0 \).

**Proof.** From (24) it follows that
\[
\frac{d\ell}{dt} \geq t^{-\frac{n}{q}} \left( \frac{|\lambda_n|}{2} - M\delta t^{-\frac{1}{2}} \right)
\]
for all \( \delta \leq |z| < \delta_1, \varphi \in \mathbb{R} \) and \( t \geq \tau_0 \). We choose \( \tau_0 = (2M\delta_1/|\lambda_n|)^{\varphi} \) so that \( \frac{d\ell}{dt} \geq t^{-\frac{n}{q}}/2|\lambda_n|/4 \). By integrating the last inequality in the case \( n = q \), we get \( \ell(z(t)) \geq \delta_1^2t^2/4(2 + |\lambda_n| \log(t/\tau_0)) \), where \( z(\tau_0) = \delta_1 \). Thus, there exists \( 0 < \epsilon < \delta_1 \) such that for all \( \delta_1 < \epsilon \) the solution \( z(t) \) satisfies the estimate \( |z(t)| > \epsilon \) at
\[
t = 2\tau_0 \exp\left( \frac{2\epsilon^2}{\delta_1^2(2 + |\lambda_n|)} \right).
\]
Similar estimate holds in the case \( n < q \).

**Corollary 1.** Let \( n \geq 1 \) be an integer such that \( \Lambda_k(v) \equiv 0 \) for \( k < n \), \( \Lambda_n(v) \equiv 0 \) and \( V_c \in (0, E_0) \), \( l = 1, \ldots, s \geq 1 \) be a set of real numbers such that \( \Lambda_l(V_c) = 0 \), \( \Lambda_l'(V_c) < 0 \). Then for any integer \( l \in [1, s] \) and for all \( \epsilon > 0 \) there exist \( \delta_1 > 0 \) and \( \tau_0 > 0 \) such that \( \forall (x_0, y_0) \)\: \( |H_0(x_0, y_0) - V_c| < \delta_1 \), \( \tau_0 > 0 \), the solution \( x(t), y(t) \) of system (1) with initial data \( x(\tau_0) = x_0, y(\tau_0) = y_0 \) satisfies the estimate \( |H_0(x(t), y(t)) - V_c| < \epsilon \) for all \( t \geq \tau_0 \). Moreover, if \( 1 \leq n \leq q \), \( H_0(x(t), y(t)) \rightarrow V_c \) as \( t \rightarrow \infty \).

### 6. Applications

In the previous two sections, we described possible asymptotics regimes in systems of the form (1). In this section, we give some conditions on the perturbations which guarantee the applicability of these results.

**Theorem 7.** Let \( l, h, n \) be positive integers such that \( l + h \geq n \) and the coefficients of the perturbations (3) satisfy the following conditions:
\[
H_i(x, y) \equiv 0, \quad 1 \leq i < h;
F_j(x, y) \equiv 0, \quad 1 \leq j < l;
\]
\[
\int_{E(0, E_0)} F_l(x, y) |\partial H_0(x, y)| \, dl = 0, \quad \forall E \in (0, E_0), \quad l \leq k < n;
\]
\[
H_0(x, y) = E
\]
\[ F_k(x, y) = \mathcal{O}(r^2), \quad l \leq k < n, \]
\[ F_n(x, y) = \lambda_n y + \mathcal{O}(r^2), \quad r \to 0, \] (27)

where \( \lambda_n = \text{const} \neq 0 \). Then the equilibrium \((0, 0)\) of system (1) is unstable if \( \lambda_n > 0 \) and is uniformly stable if \( \lambda_n < 0 \). Moreover, if \( \lambda_n < 0 \) and \( n < q (n = q) \), the equilibrium is uniformly exponentially (polynomially) stable.

**Proof.** Let us show that there exists a transformation \((x, y) \mapsto (E, \varphi) \mapsto (v, \varphi)\) such that equation (9) has \( \Lambda_\delta(v) = 0 \) for \( k < n \) and \( \Lambda_\delta(v) = \lambda_n v + \mathcal{O}(r^2) \) as \( v \to 0 \). In this case, theorem 1 is applicable.

To be definite, let \( h < l \). The proof in the case \( h \geq l \) is similar. From (25) it follows that \( f_k(E, \varphi) \equiv 0 \), \( g_k(E, \varphi) \equiv 0 \) for \( 1 \leq k < h \);
\[
 f_k(E, \varphi) \equiv -\omega(E)\partial_v H_k(X(\varphi, E), Y(\varphi, E)),
 g_k(E, \varphi) \equiv \omega(E)\partial_v H_k(X(\varphi, E), Y(\varphi, E))
\]

for \( h \leq k < l \). The functions \( f_k(E, \varphi), g_k(E, \varphi) \) have the form (7) for \( l \leq k \leq n \). Hence, a Lyapunov function candidate for system (1) should be considered in the following form:
\[
 V(E, \varphi, t) = E + \sum_{k=h}^{n} t^k v_k(E, \varphi).
\] (28)

We assume that \( v_i(E, \varphi) \equiv 0 \) for \( i < h \). According to the scheme described in section 3, the functions \( v_k(E, \varphi) \) are determined from system (12). Therefore, for \( h \leq k < l \), we have
\[
 Z_k(E, \varphi) \equiv 0, \quad \Lambda_k(E) \equiv (f_k(E, \varphi)) \equiv 0, \quad v_k(E, \varphi) \equiv H_k(X(\varphi, E), Y(\varphi, E)).
\]

For \( l \leq k \leq n - 1 \), condition (26) is used to guarantee the equality \( \Lambda_k(E) \equiv 0 \). Indeed, from (11) it follows that
\[
 Z_k(E, \varphi) = -\sum_{i+j=l} (f_i(E, \varphi)\partial_v v_j(E, \varphi) + g_i(E, \varphi)\partial_v v_j(E, \varphi))
 = -\sum_{j=h}^{l-h} (f_{l-j}(E, \varphi)\partial_v v_j(E, \varphi) + g_{l-j}(E, \varphi)\partial_v v_j(E, \varphi))
 = \omega(E)\sum_{j=h}^{l-h} (\partial_v H_{l-j} \partial_v H_j - \partial_v H_{l-j} \partial_v H_j) \equiv 0.
\]

Hence,
\[
 \Lambda_k(E) \equiv (f_k(E, \varphi)) \equiv \omega(E)(F_l(X(\varphi, E), Y(\varphi, E))\partial_v X(\varphi, E))
 \equiv (\partial_v H_0(X(\varphi, E), Y(\varphi, E))F_l(X(\varphi, E), Y(\varphi, E))) \equiv 0
\]
and \( v_j(E, \varphi) \equiv H_j(X(\varphi, E), Y(\varphi, E)) + \hat{v}_j(E, \varphi) \), where
\[
 \hat{v}_j(E, \varphi) = \int F_l(X(\varphi, E), Y(\varphi, E))\partial_v X(\varphi, E) d\varphi.
\]
For \( l + 1 \leq k \leq n - 1 \), we have
\[
Z_k(E, \varphi) = -\sum_{i+j=k} \left( f_j(E, \varphi) \partial_{E} v_j(E, \varphi) + g_j(E, \varphi) \partial_{\varphi} v_j(E, \varphi) \right).
\]

Note that \( f_j \) and \( g_j \) with \( j \geq l \) are not involved in these sums. Such functions have the multipliers \( \partial_{E} v_{k-l} \) and \( \partial_{\varphi} v_{k-l} \), correspondingly. If \( j \geq l \), then \( k - j \leq n - j - 1 \leq n - l - 1 \leq h - 1 \) and \( v_{k-j}(E, \varphi) \equiv 0 \). Similarly, the functions \( \partial_{\varphi} v_j \) and \( \partial_{E} v_j \) with \( j \geq l \) are not contained in these sums. Therefore,
\[
Z_k(E, \varphi) = -\sum_{i+j=k \atop h \leq i, j \leq l} \left( f_j(E, \varphi) \partial_{E} v_j(E, \varphi) + g_j(E, \varphi) \partial_{\varphi} v_j(E, \varphi) \right)
\]
\[
= -\sum_{j=h}^{k-h} \left( f_{k-j}(E, \varphi) \partial_{E} v_{j}(E, \varphi) + g_{k-j}(E, \varphi) \partial_{\varphi} v_{j}(E, \varphi) \right)
\]
\[
= \omega(E) \sum_{j=h}^{k-h} \left( \partial_{E} H_{k-j} - \partial_{E} H_{j} \right) \equiv 0
\]

and \( \Lambda_k(E) = 0 \). This implies that \( \nu_k(E, \varphi) \equiv H_k(X(\varphi, E), Y(\varphi, E)) + \dot{\nu}_k(E, \varphi) \), where
\[
\dot{\nu}_k(E, \varphi) = \int F_k(X(\varphi, E), Y(\varphi, E)) \partial_{E} X(\varphi, E) \, d\varphi.
\]

We see that \( \nu_k(E, \varphi) = O(E), \dot{\nu}_k(E, \varphi) = O(E^{1/2}) \) as \( E \to 0 \) for all \( \varphi \in \mathbb{R} \).

For \( k = n \), the function \( Z_n(E, \varphi) \) has the following form:
\[
Z_n(E, \varphi) = -\sum_{i+j=n \atop h \leq i, j \leq l} \left( f_j(E, \varphi) \partial_{E} v_j(E, \varphi) + g_j(E, \varphi) \partial_{\varphi} v_j(E, \varphi) \right)
\]
\[
= -\sum_{j=(n-l)}^{(n-h)} \left( f_{n-j}(E, \varphi) \partial_{E} v_{j}(E, \varphi) + g_{n-j}(E, \varphi) \partial_{\varphi} v_{j}(E, \varphi) \right)
\]
\[
= -\sum_{i+j=n \atop h \leq i, j \leq l-1} \left( f_j(E, \varphi) \partial_{E} v_j(E, \varphi) + g_j(E, \varphi) \partial_{\varphi} v_j(E, \varphi) \right)
\]
\[
\equiv Z^n_1(E, \varphi) + Z^n_2(E, \varphi).
\]

If \( l + h > n \), then \( Z^n_1(E, \varphi) \equiv 0 \). If \( l + h = n \),
\[
Z^n_1(E, \varphi) = \omega(E) \left( \partial_{E} H_{l}(\partial_{E} \dot{v}_{l} + F_{l} \partial_{E} X) - \partial_{E} H_{l}(\partial_{E} \dot{v}_{l} + F_{l} \partial_{E} X) \right). \tag{29}
\]

It can easily be checked that
\[
Z^n_2(E, \varphi) = \omega(E) \sum_{i+j=n \atop h \leq i, j \leq l-1} \left( -\partial_{E} H_{l} \partial_{E} H_{l} + \partial_{E} H_{l} \partial_{E} H_{l} \right) \equiv 0. \tag{30}
\]

Consequently,
\[
\Lambda_n(E) = (\omega(E)F_n(X(\varphi, E), Y(\varphi, E))\partial_{E} X(\varphi, E) - Z_n(E, \varphi))
\]

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and \( \dot{v}_n(E, \varphi) = H_n(E, \varphi) + \dot{v}_n(E, \varphi) \), where
\[
\dot{v}_n(E, \varphi) = \frac{1}{\omega(E)} \left( \lambda_n(E) + Z_n(E, \varphi) - \omega(E)F(X(\varphi, E), Y(\varphi, E)) \times \partial_x X(\varphi, E) \right) d\varphi.
\]
Since \( H_0(X(\varphi, E), Y(\varphi, E)) \equiv E \), then from (2) and (27) it follows that
\[
X(\varphi, E) = -\sqrt{2E} \cos \varphi + \mathcal{O}(E), \quad Y(\varphi, E) = \sqrt{2E} \sin \varphi + \mathcal{O}(E),
\]
and \( F_n(X(\varphi, E), Y(\varphi, E)) = \lambda_n \sqrt{2E} \sin \varphi + \mathcal{O}(E), \quad Z_n(E, \varphi) = \mathcal{O}(E^{3/2}) \)
as \( E \to 0 \) for all \( \varphi \in \mathbb{R} \). Thus
\[
\lambda_n(E) = \frac{\omega(E)}{\omega(E)} \left( \sqrt{2E} \sin \varphi + \mathcal{O}(E) \right) \left( \lambda_n \sqrt{2E} \sin \varphi + \mathcal{O}(E) - Z_n(E, \varphi) \right)
\]
\[
= \lambda_n E + \mathcal{O}(E^2), \quad E \to 0.
\]
This implies that system (1) satisfies the conditions of theorem 1 and the stability of the equilibrium is determined by the sign of \( \lambda_n \).

**Example 1.** The system
\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x + \frac{r}{t} \kappa \lambda \sin x + \frac{r}{t} \kappa \lambda y, \quad t \geq 1,
\]
where \( \kappa, \lambda \) are const, \( 0 < l < n \), has the form of (1) with \( H_0(x, y) = 1 - \cos x + y^2 / 2 \), \( F_i(x, y) \equiv \kappa \lambda y \sin x \), \( F_i(x, y) \equiv \lambda_n y \), \( H_i(x, y) \equiv 0 \) for \( i \neq 0 \) and \( F_j(x, y) \equiv 0 \) for \( j \notin \{n, m\} \). It can easily be checked that system (31) satisfies the conditions of theorem 7 with \( h = \infty \). Therefore, the stability of the equilibrium \((0, 0)\) is determined by the parameter \( \lambda_n \) and the ratio \( n / q \) (see figure 2).

**Theorem 8.** Let \( l, h, n, m \) be positive integers such that \( l + h \geq n \), \( n > m \geq l \) and the coefficients of the perturbations (3) satisfy (25), (27) and
\[
\oint_{H_0(x,y)=E} F_k(x,y) \partial_x H_0(x,y) \left| \nabla H_0(x,y) \right|^{-1} dl = 0, \quad \forall \ E \in (0, E_0), \quad l \leq k \leq m,
\]
\[
F_m(x,y) = y(\alpha m x^2 + \beta_m y^2) + \mathcal{O}(r^4), \quad r \to 0,
\]
where \( \alpha_m, \beta_m \) are const. The equilibrium of system (1) is
- uniformly stable if \( \lambda_n < 0 \) and \( \alpha_m + 3 \beta_m < 0 \);
- unstable if \( \lambda_n > 0 \) and \( \alpha_m + 3 \beta_m > 0 \).

**Proof.** The proof is similar to the proof of theorem 7. In this case, we show that theorem 2 is applicable with \( s = 2 \) and \( \gamma_{m,i} = (\alpha_m + 3 \beta_m) / 2 \). Note that the change of the variables based on (28) leads to equation (9) with \( \Lambda_i(E) \equiv 0 \) for \( k < m \),
\[
\Lambda_i(E) = \omega(E) \langle F \partial_x X \rangle = \mathcal{O}(E^2), \quad E \to 0, \quad m \leq i \leq n - 1,
\]
\[
\Lambda_m(E) = 4E^2 \left( \sin^2 \varphi (\alpha_m \cos^2 \varphi + \beta_m \sin^2 \varphi) \right) + \mathcal{O}(E^3) = \gamma_{m,2} E^2 + \mathcal{O}(E^3).
\]
The functions \( \{ \Lambda_i(E) \}_{i=m}^{n-1} \) are used in the calculating of \( Z_j(E, \varphi) \), \( j \geq m + h \geq n \). If \( j = m + h = n \), then \( Z_n(E, \varphi) = Z_n^0(E, \varphi) + Z_n^1(E, \varphi) + Z_n^2(E, \varphi) \), where \( Z_n^0(E, \varphi) \) and \( Z_n^2(E, \varphi) \)
are defined by formulas (29) and (30), $Z_n(E, \varphi) = \partial E \Lambda_n(E) \eta(E, \varphi) = O(E^2)$ as $E \to 0$. Therefore, $\Lambda_n(E) = \lambda_n E + O(E^2)$ as $E \to 0$. This means that system (1) satisfies the conditions of theorem 2.

**Corollary 2.** Under the conditions of theorems 8 and 3 is applicable to system (1) with $s = 2$ and $\gamma_{m,r} = (\alpha_m + 3\beta_m)/2$.

**Example 2.** The system
\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x + t - \frac{m}{q} \alpha_m x^2 y + t - \frac{n}{q} \lambda_n y, \quad t \geq 1, \quad (32)
\]
where $\alpha_m, \lambda_n = \text{const}, 0 < m < n$, demonstrates the inefficiency of linear stability analysis for non-autonomous systems of the form (1). The matrix of the linearised system $x' = y, y' = -x + \lambda_n t - n/y$ has the following eigenvalues:
\[
\mu_{\pm}(t) = \frac{1}{2} \left( \lambda_n t \frac{2}{2} \pm i \sqrt{4 - \lambda_n^2 t^2 \frac{2}{2}} \right).
\]
Let $\lambda_n > 0$, then $\Re \mu_{\pm}(t) > 0$ for all $t \geq 1$. From theorem 7 it follows that the equilibrium $(0,0)$ is unstable in the linearised system. On the other hand, system (32) satisfies the conditions of theorem 8 with $l = m$ and $F_m(x, y) = \alpha_m x^2 y$. From corollary 2 it follows that the equilibrium is stable if $\lambda_n > 0$ and $\alpha_m < 0$ (see figure 3).
The system

\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= -x + t \frac{\pi}{2} \delta_m x^2 y + t \frac{\pi}{2} \alpha_n x^2 y, \quad t \geq 1
\end{align*}
\]  

(33)

has the form of (1) with \( H_0(x,y) = 1 - \cos x + y^2/2 \), \( F_m(x,y) = \delta_m x^2 y \), \( F_p(x,y) = \alpha_n x^2 y \), \( H_j(x,y) \equiv 0 \) for \( i \neq 0 \), and \( F_j(x,y) \equiv 0 \) for \( j \notin \{m,n\} \). It can easily be checked that

\[
Z_k(E, \varphi) \equiv 0, \quad \Lambda_k(E) \equiv 0, \quad k < m,
\]

\[
Z_m(E, \varphi) \equiv 0, \quad \Lambda_m(E) = \int_{H_0(x,y)=E} F_m(x,y) \frac{\omega(E) \partial_2 H_0(x,y)}{2\pi|\nabla H_0(x,y)|} \, dl = \gamma_{m,3} E^2 (1 + O(E)),
\]

\[
Z_m(E, \varphi) = O(E^3), \quad \Lambda_m(E) = O(E^3), \quad m \leq j < n,
\]

\[
Z_n(E, \varphi) = O(E^3), \quad \Lambda_n(E) = \int_{H_0(x,y)=E} F_n(x,y) \frac{\omega(E) \partial_2 H_0(x,y)}{2\pi|\nabla H_0(x,y)|} \, dl - \langle Z_n(E, \varphi) \rangle
\]

\[
= \gamma_{n,2} E^2 (1 + O(E)),
\]

as \( E \to 0 \), where \( \gamma_{m,3} = \delta_m / 2 \) and \( \gamma_{n,2} = \alpha_n / 2 \). Therefore, system (33) satisfy the condition of theorem 4 with \( s = 3 \) and \( d = 2 \). This implies that the equilibrium \((0,0)\) is uniformly stable if \( \gamma_{m,3} < 0 \) and \( \gamma_{n,2} < 0 \) (see figure 4).

**Theorem 9.** Let \( n \) be a positive integer such that \( 1 \leq n \leq q \) and the perturbations (3) satisfy the following conditions:

\[
F_i(x,y) \equiv 0, \quad 1 \leq i < n; \quad F_n(x,y) \equiv \partial_2 H_0(x,y) (\lambda_n - \mu_n H_0(x,y)) + \hat{F}_n(x,y),
\]  

(34)

where \( \lambda_n, \mu_n = \text{const} \neq 0 \) and \( \hat{F}_n(x,y) \) satisfies (26). Then the equilibrium \((0,0)\) of system (1) is

- **unstable if** \( \lambda_n > 0 \);
- **uniformly asymptotically stable if** \( \lambda_n < 0 \).

If \( \lambda_n > 0, \mu_n > 0 \) and \( |\lambda_n/\mu_n| < E_0 \), then for all \( \epsilon > 0 \) there exist \( \delta_\epsilon > 0 \) and \( t_\epsilon > 0 \) such that \( \forall (x_0, y_0) \) with \( |H_0(x_0,y_0) - \lambda_n/\mu_n| \leq \delta_\epsilon \) and \( \tau_0 \geq t_\epsilon \) the solution \( x(t), y(t) \) of system (1) with

\[
\begin{align*}
\alpha_m &= 0, \lambda_m = 0.4, \\
\alpha_m &= -2, \lambda_m = 0.4
\end{align*}
\]

**Figure 3.** The evolution of \( x(t) \) for solutions of (32) with \( q = 2, m = 1 \) and \( n = 2 \). The dashed lines correspond to \( \pm \nu \gamma /2, \nu = (n - m) / q = 0.5 \).
**Figure 4.** The evolution of \((x(t), y(t))\) for solutions of (33) with \(q = 2, m = 1\) and \(n = 2\). The black points correspond to initial data.

*initial data* \(x(\tau_0) = x_0, y(\tau_0) = y_0\) satisfies the estimate \(|H_0(x(t), y(t)) - \lambda_n/\mu_n| < \epsilon\) for all \(t \geq \tau_0\). Moreover, \(H_0(x(t), y(t)) \to \lambda_n/\mu_n ast t \to \infty\).

**Proof.** The proof is based on the application of theorems 1 and 5. From (34) it follows that there exists the transformation \((x, y) \mapsto (E, \varphi) \mapsto (v, \varphi)\) with

\[
V(E, \varphi, t) = E + \sum_{k=1}^{n} t^{\frac{q+1}{q}} v_k(E, \varphi),
\]

which reduces system (1) to the following form:

\[
\frac{dv}{dt} = \sum_{k=1}^{n} t^{\frac{q+1}{q}} v_k(E, \varphi), \quad \frac{d\varphi}{dt} = \omega(v) + G(v, \varphi, t),
\]

\[
R_{n+1}(v, \varphi, t) = \mathcal{O}\left(t^{-\frac{n+1}{q}}\right), \quad G(v, \varphi, t) = \mathcal{O}\left(t^{-\frac{q}{q+1}}\right), \quad t \to \infty,
\]

\[
\Lambda_n(E) \equiv \omega(E)(F_n(X(\varphi, E), Y(\varphi, E))\partial_x X(\varphi, E)) \equiv (\lambda_n - \mu_n E)(\partial_x H_0)\partial_x.
\]

Note that \(\Lambda_n(E) = \lambda_n E + \mathcal{O}(E^2)\) as \(E \to 0\). Hence, the equilibrium is asymptotically stable if \(\lambda_n < 0\) and the equilibrium is unstable if \(\lambda_n > 0\).

If \(\lambda_n > 0\) and \(\mu_n > 0\), the equation \(\Lambda_n(E) = 0\) has the root \(E_c = \lambda_n/\mu_n\) such that \(\Lambda_n'(E_c) < 0\). In this case, from theorem 5 it follows that \(H_0(x(t), y(t)) \to E_c ast t \to \infty\) for solutions of (1) with initial data such that \(|H_0(x(\tau_0), y(\tau_0)) - E_c|\) is small enough. \(\square\)

**Remark 1.** If \(\lambda_n < 0, \gamma_n < 0\) and \(|\lambda_n/\mu_n| < E_0\), the fixed point \((0, 0)\) is asymptotically stable and the set \(\{(x, y) : H_0(x, y) = E_c\}\) with \(E_c = \lambda_n/\mu_n\) is unstable (in the sense of theorem 6). If \(E_c\) is the minimal nonzero root of \(\Lambda_n(E)\), then \(\{(x, y) : 0 < H_0(x, y) < E_c\}\) is the domain of attraction of the fixed point \((0, 0)\).
Example 4. The system
\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + t^\frac{3}{2} y \left( \lambda_n + \kappa_n x - \mu_n \left( x^2 + y^2 \right)^2 \right), \quad t \geq 1
\]
with \( \lambda_n, \mu_n, \kappa_n = \text{const} \) satisfies the conditions of theorem 9 with \( H_0(x,y) = (x^2 + y^2)/2 \) and \( F_n(x,y) = \kappa_n xy \). Therefore, if \( \lambda_n > 0 \) and \( \mu_n > 0 \), then \( H_0(x(t),y(t)) \to \lambda_n/\mu_n \) as \( t \to \infty \) for solutions of (35) with initial data sufficiently close to the set \( \{(x,y) : H_0(x,y) = \lambda_n/\mu_n \} \) (see figure 5).

7. Conclusion

Thus, we have described possible bifurcations in asymptotically Hamiltonian systems in the plane. The important feature of non-autonomous systems is the inefficiency of the linear stability analysis: there are examples of nonlinear systems whose solutions behave completely differently than the solutions of corresponding linearised system. Here, through a careful non-linear analysis based on the Lyapunov function method we have shown that depending on the structure of decaying perturbations the equilibrium of the limiting system can preserve or lose stability. Note also that if perturbations do not preserve the equilibrium of the Hamiltonian system, a particular solution of the perturbed system should be considered.

Acknowledgments

The research is supported by the Russian Science Foundation (Grant No. 20-11-19995).

ORCID iDs

Oskar A Sultanov © https://orcid.org/0000-0003-1970-3382
References

[1] Guckenheimer J and Holmes P 1983 Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields (New York: Springer)
[2] Glendinning P A 1994 Stability, Instability and Chaos: An Introduction to the Theory of Nonlinear Differential Equations (Cambridge: Cambridge University Press)
[3] Haulmann H 2007 Local and Semi-Local Bifurcations in Hamiltonian Dynamical Systems: Results and Examples (Lecture Notes in Mathematics vol 1893) (Berlin: Springer)
[4] Markus L 1956 Asymptotically autonomous differential systems, Contributions to the Theory of Nonlinear Oscillations III (Annals of Mathematics Studies vol 36) ed S Lefschetz (Princeton, NJ: Princeton University Press) pp 17–30
[5] Wong J S W and Burton T A 1965 Some properties of solutions of $u''(t) + a(t)f(u)g(u') = 0$. II Monatsh. Math. 69 368–74
[6] Grimmer R C 1968 Asymptotically almost periodic solutions of differential equations, SIAM J. Appl. Math. 17 109–15
[7] Thieme H R 1992 Convergence results and a Poincaré–Bendixson trichotomy for asymptotically autonomous differential equations, J. Math. Biol. 30 755–63
[8] Thieme H 1994 Asymptotically autonomous differential equations in the plane, Rocky Mt. J. Math. 24 351–80
[9] Langa J A, Robinson J C and Suárez A 2002 Stability, instability, and bifurcation phenomena in non-autonomous differential equations, Nonlinearity 15 887–903
[10] Kloeden P E and Siegmund S 2005 Bifurcations and continuous transitions of attractors in autonomous and nonautonomous systems, Int. J. Bifurcation Chaos 15 743–62
[11] Rasmussen M 2008 Bifurcations of asymptotically autonomous differential equations, Set-Valued Anal. 16 821–49
[12] Pötzsche C 2010 Nonautonomous bifurcation of bounded solutions: I. A Lyapunov–Schmidt approach, Discrete Contin. Dyn. Syst. B 14 739–76
[13] Fokas A S, Its A R, Kapaev A A and Novokshenov V Y 2006 Painlevé Transcendents: The Riemann–Hilbert Approach (Mathematical Surveys and Monographs, vol 128) (Providence, RI: American Mathematical Society)
[14] Kalyakin L A and Sultanov O A 2013 Stability of autoresonance models, Differ. Equ. 49 267–81
[15] Sultanov O A 2016 Stability of capture into parametric autoresonance, Proc. Steklov Inst. Math. 295 156–67
[16] Pikovsky A, Rosenblum M and Kurths J 2001 Synchronization: A Universal Concept in Nonlinear Sciences (Cambridge: Cambridge University Press)
[17] Kalyakin L A 2014 Synchronization in a nonisochronous nonautonomous system, Theor. Math. Phys. 181 1339–48
[18] Wasow W 1966 Asymptotic Expansions for Ordinary Differential Equations (New York: Wiley)
[19] Sultanov O 2018 Stability and asymptotic analysis of the autoresonant capture in oscillating systems with combined excitation, SIAM J. Appl. Math. 78 3103–18
[20] Sultanov O 2019 Lyapunov functions and asymptotic analysis of a complex analogue of the second Painlevé equation, J. Phys.: Conf. Ser. 1205 012056
[21] Sultanov O A 2020 Bifurcations of autoresonant modes in oscillating systems with combined excitation, Stud. Appl. Math. 144 213–41
[22] Hapaev M M 1993 Averaging in Stability Theory: A Study of Resonance Multi-Frequency Systems (Dordrecht: Kluwer)
[23] Arnold V I, Kozlov V V and Neishtadt A I 2006 Mathematical Aspects of Classical and Celestial Mechanics (Berlin: Springer)
[24] Khalil H K 2002 Nonlinear Systems (Englewood Cliffs, NJ: Prentice-Hall)