THICK SUBCATEGORIES OF MODULES OVER COMMUTATIVE RINGS
(WITH AN APPENDIX BY SRIKANTH IYENGAR)

HENNING KRAUSE

Abstract. For a commutative noetherian ring $A$, we compare the support of a complex of $A$-modules with the support of its cohomology. This leads to a classification of all full subcategories of $A$-modules which are thick (that is, closed under taking kernels, cokernels, and extensions) and closed under taking direct sums.

1. Introduction

Let $A$ be a commutative noetherian ring. We consider the category Mod$_A$ of $A$-modules and the spectrum Spec$_A$ of prime ideals of $A$. Given a complex $X$ of $A$-modules, we wish to relate its support (in the sense of Foxby [6])

$$\text{Supp} X = \{ p \in \text{Spec} A \mid X \otimes_A \kappa(p) \neq 0 \}$$

to the support of its cohomology

$$\text{Supp} H^* X = \bigcup_{i \in \mathbb{Z}} \text{Supp} H^i X.$$

In some cases we have the equality

$$\text{Supp} X = \text{Supp} H^* X,$$

for example when $A$ is a Dedekind domain or when $H^* X$ is finitely generated over $A$. The failure of this equality is the main theme of recent joint work with Benson and Iyengar [2], which is motivated by the study of support varieties of modular representations. In this paper we establish a closely related classification of thick subcategories of Mod$_A$ and prove the following result.

**Theorem 1.1.** For a subset $\Phi$ of Spec$_A$ the following conditions are equivalent:

1. For every complex $X$ of $A$-modules we have

$$\text{Supp} X \subseteq \Phi \iff \text{Supp} H^* X \subseteq \Phi.$$

2. The $A$-modules $M$ with $\text{Supp} M \subseteq \Phi$ form a thick subcategory of Mod$_A$.

3. Every map $I^0 \to I^1$ between injective $A$-modules with $\text{Ass} I^i \subseteq \Phi$ ($i = 0, 1$) can be completed to an exact sequence $I^0 \to I^1 \to I^2$ such that $I^2$ is injective and $\text{Ass} I^2 \subseteq \Phi$.

Recall that a classical result of Gabriel [8] provides a bijection between the set of localizing subcategories of Mod$_A$ and the set of specialization closed subsets of Spec$_A$. More recently, a number of authors studied subcategories of Mod$_A$ in terms of subsets of Spec$_A$; see [9, 11, 16]. In this paper we generalize Gabriel’s result in two directions.
The first direction (via associated primes) is fairly elementary but seems to be new. The second direction (via support) leads to a classification of thick subcategories of Mod $A$ and corrects a result of Hovey in [11]. This classification is formulated in terms of subsets $\Phi \subseteq \text{Spec } A$ satisfying the equivalent conditions of Theorem 1.1. We call such subsets coherent and establish the following.

**Theorem 1.2.** For a commutative noetherian ring $A$ the following conditions are equivalent:

1. The Krull dimension of $A$ is at most one.
2. Every subset of $\text{Spec } A$ is coherent.
3. $\text{Supp } X = \text{Supp } H^* X$ for every complex $X$ of $A$-modules.

The proof of this theorem is illustrated by some explicit examples of subsets of $\text{Spec } A$ which are not coherent. It would be interesting to have a geometric interpretation of coherent subsets in terms of the Zariski topology on $\text{Spec } A$.

2. Subcategories via associated primes

Let $M$ be an $A$-module. Recall that a prime ideal $p$ is associated to $M$, if $A/p$ is isomorphic to a submodule of $M$. We denote by $\text{Ass } M$ the set of all prime ideals which are associated to $M$.

**Theorem 2.1.** The map sending a subcategory $C$ of $\text{Mod } A$ to

$$\text{Ass } C = \bigcup_{M \in C} \text{Ass } M$$

induces a bijection between the set of full subcategories of $\text{Mod } A$, which are closed under taking submodules, extensions, and direct unions, and the set of subsets of $\text{Spec } A$. The inverse map sends a subset $\Phi$ of $\text{Spec } A$ to

$$\text{Ass}^{-1} \Phi = \{ M \in \text{Mod } A \mid \text{Ass } M \subseteq \Phi \}.$$

The proof uses some basic facts about associated primes and the structure of injective modules. The injective envelope of a module $M$ is denoted by $E(M)$ and we observe that $\text{Ass } E(M) = \text{Ass } M$.

**Lemma 2.2.** Let $M$ be an $A$-module. Given a submodule $N \subseteq M$ and a family $(M_i)$ of submodules satisfying $M = \bigcup_i M_i$, we have

$$\text{Ass } N \subseteq \text{Ass } M \subseteq \text{Ass } N \cup \text{Ass } M/N \quad \text{and} \quad \text{Ass } M = \bigcup_i \text{Ass } M_i.$$

**Proof.** See [4, Chap. IV, §1].

For a prime ideal $p$ we denote by $k(p)$ its residue field.

**Lemma 2.3.** Let $I$ be an indecomposable injective $A$-module and $p$ its associated prime ideal. Then $I$ is obtained from $A/p$ by taking extensions and direct unions. More precisely,

1. $I$ is obtained from $k(p)$ by taking extensions and direct unions, and
2. $k(p)$ is a direct union of copies of $A/p$.
Proof. We sketch the argument and refer to [5, Chap. X, §8] for details. For each integer \( n \geq 0 \) let \( I_n \) denote the submodule of \( I \) consisting of all elements annihilated by \( p^n \). Then we have \( I = \bigcup_{n \geq 0} I_n \) and each factor \( I_{n+1}/I_n \) is isomorphic to a finite direct sum of copies of the residue field \( k(p) \). Now observe that \( k(p) \) is the field of fractions of \( A/p \) and therefore a direct union of the form \( k(p) = \bigcup_{0 \neq x \in A/p} x^{-1} A/p \). □

Proof of Theorem 2.1. Let \( \Phi \) be a subset of Spec \( A \). Then the subcategory \( \text{Ass}^{-1} \Phi \) is closed under taking submodules, extensions, and direct unions, by Lemma 2.2. Clearly, we have \( \text{Ass}(\text{Ass}^{-1} \Phi) = \Phi \). Now let \( C \) be a subcategory of \( \text{Mod}_A \), which is closed under taking submodules, extensions, and direct unions. We claim that \( \text{Ass}^{-1}(\text{Ass} C) = C \).

The inclusion \( \text{Ass}^{-1}(\text{Ass} C) \supseteq C \) is clear. Now suppose that \( M \) is a module contained in \( \text{Ass}^{-1}(\text{Ass} C) \). Then its injective envelope \( E(M) \) is a direct sum of copies of the form \( E(A/p) \) with \( p \in \text{Ass} C \), since \( \text{Ass} E(M) = \text{Ass} M \). But \( p \in \text{Ass} C \) implies \( A/p \in C \), and therefore \( E(A/p) \) belongs to \( C \), by Lemma 2.3. It follows that \( E(M) \) belongs to \( C \) and therefore \( M \in C \). This finishes the proof. □

We state a number of consequences of Theorem 2.1.

Corollary 2.4. For a full subcategory \( C \) of \( \text{Mod}_A \) the following conditions are equivalent.

1. \( C \) is closed under taking submodules, extensions, and direct unions.
2. There exists a subset \( \Phi \) of Spec \( A \) such that \( C \) consists of all \( A \)-modules \( M \) satisfying \( \text{Ass} M \subseteq \Phi \).
3. There exists an injective \( A \)-module \( I \) such that \( C \) consists of all \( A \)-modules which admit a monomorphism into a direct sum of copies of \( I \).

Proof. (1) \( \Leftrightarrow \) (2): This is an immediate consequence of Theorem 2.1.
(2) \( \Rightarrow \) (3): Take \( I = \bigoplus_{p \in \Phi} E(A/p) \). Then \( \text{Ass} M \subseteq \Phi \) for every submodule \( M \) of a direct sum of copies of \( I \). On the other hand, if \( \text{Ass} M \subseteq \Phi \), then \( \text{Ass} E(M) \subseteq \Phi \) and therefore \( E(M) \) is a submodule of a direct sum of copies of \( I \).
(3) \( \Rightarrow \) (1): Clear. □

Next we restrict the map \( C \mapsto \text{Ass} C \) to the category \( \text{mod}_A \) of all finitely generated \( A \)-modules.

Corollary 2.5 ([16, Theorem 4.1]). The map \( \mathcal{D} \mapsto \text{Ass} \mathcal{D} \) induces a bijection between the set of full subcategories of \( \text{mod}_A \), which are closed under taking submodules and extensions, and the set of subsets of Spec \( A \).

Proof. Consider the map \( C \mapsto C \cap \text{mod}_A \) between

(i) the set of full subcategories of \( \text{Mod}_A \), which are closed under taking submodules, extensions, and direct unions, and
(ii) the set of full subcategories of \( \text{mod}_A \), which are closed under taking submodules and extensions.
This map is bijective; its inverse sends a subcategory \( \mathcal{D} \) from (ii) to the full subcategory of \( \text{Mod} A \) consisting of all direct unions of modules in \( \mathcal{D} \). The composition of the first map \( \mathcal{C} \mapsto \mathcal{C} \cap \text{mod} A \) with \( \mathcal{D} \mapsto \text{Ass} \mathcal{D} \) is the bijection from Theorem 2.1. Thus \( \mathcal{D} \mapsto \text{Ass} \mathcal{D} \) is bijective. □

Recall that a full subcategory \( \mathcal{C} \) of \( \text{Mod} A \) is localizing if \( \mathcal{C} \) is closed under taking submodules, factor modules, extensions, and direct sums. A subset \( \Phi \) of \( \text{Spec} A \) is specialization closed if for any pair \( p \subseteq q \) of prime ideals, \( p \in \Phi \) implies \( q \in \Phi \).

**Corollary 2.6** ([8, p. 425]). The map \( \mathcal{C} \mapsto \text{Ass} \mathcal{C} \) induces a bijection between the set of localizing subcategories of \( \text{Mod} A \) and the set of specialization closed subsets of \( \text{Spec} A \).

**Proof.** Suppose that \( \mathcal{C} \) is localizing and let \( p \subseteq q \) be prime ideals with \( p \in \Phi = \text{Ass} \mathcal{C} \). Then \( A/p \in \mathcal{C} \) and therefore \( A/q \in \mathcal{C} \), because \( A/q \) is a factor module of \( A/p \). Thus \( q \in \Phi \), and we have that \( \Phi \) is specialization closed.

Now suppose that \( \Phi \subseteq \text{Spec} A \) is specialization closed and let \( N \subseteq M \) be \( A \)-modules with \( M \) in \( \mathcal{C} = \text{Ass}^{-1} \Phi \). Then \( M/N \in \mathcal{C} \), since \( \text{Ass} M/N \subseteq \{ p \in \text{Spec} A \mid (M/N)_p \neq 0 \} \subseteq \{ p \in \text{Spec} A \mid M_p \neq 0 \} \subseteq \Phi \) where the last inclusion uses that \( \Phi \) is specialization closed. Thus \( \mathcal{C} \) is localizing. □

### 3. Subcategories via support

A full subcategory \( \mathcal{C} \) of \( \text{Mod} A \) is called thick if for each exact sequence

\[
M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4 \rightarrow M_5
\]

of \( A \)-modules with \( M_i \in \mathcal{C} \) for \( i = 1, 2, 4, 5 \), the module \( M_3 \) belongs to \( \mathcal{C} \). Note that a thick subcategory is an abelian category and that the inclusion functor is exact.

We wish to classify all thick subcategories of \( \text{Mod} A \) which are closed under taking direct sums. For this a few definitions are needed.

Let \( M \) be an \( A \)-module. Following [6], the support of \( M \) is by definition

\[
\text{Supp } M = \{ p \in \text{Spec} A \mid \text{Tor}^A_4(M, k(p)) \neq 0 \}.
\]

For example, \( \text{Supp } I = \text{Ass } I \) for every injective \( A \)-module \( I \).

Let \( \Phi \) be a subset of \( \text{Spec} A \) and define the full subcategory

\[
\text{Inj}_\Phi A = \{ I \in \text{Mod} A \mid I \text{ is injective and } \text{Ass } I \subseteq \Phi \}.
\]

We call \( \Phi \) coherent if each morphism \( I^0 \rightarrow I^1 \) in \( \text{Inj}_\Phi A \) can be completed to an exact sequence \( I^0 \rightarrow I^1 \rightarrow I^2 \) with \( I^2 \) in \( \text{Inj}_\Phi A \). For example, each specialization closed subset of \( \text{Spec} A \) is coherent.

**Theorem 3.1.** The map sending a subcategory \( \mathcal{C} \) of \( \text{Mod} A \) to

\[
\text{Supp } \mathcal{C} = \bigcup_{M \in \mathcal{C}} \text{Supp } M
\]

1The term coherent refers to the characterizing property of a coherent ring that every morphism \( P_1 \rightarrow P_0 \) between finitely generated projective modules can be completed to an exact sequence \( P_2 \rightarrow P_1 \rightarrow P_0 \) such that \( P_2 \) is finitely generated projective.
induces a bijection between the set of full subcategories of $\text{Mod} A$, which are thick and closed under taking direct sums, and the set of coherent subsets of $\text{Spec} A$. The inverse map sends a subset $\Phi$ of $\text{Spec} A$ to

$$\text{Supp}^{-1} \Phi = \{ M \in \text{Mod} A \mid \text{Supp} M \subseteq \Phi \}.$$ 

Let us give an example of a set of prime ideals which is not coherent. This is based on an example from [2] and provides a counterexample to Hovey’s classification of thick subcategories closed under direct sums in [11, Theorem 5.2].

**Example 3.2.** Let $k$ be a field and $A = k[x, y]$. Then $\Phi = \text{Spec} A \setminus \{ m \}$ with $m = (x, y)$ is not coherent. To see this, let

$$0 \to A \to E(A) \to \bigoplus_{\text{ht}p=1} E(A/p) \to E(A/m) \to 0$$

be a minimal injective resolution of $A$. Then the morphism

$$E(A) \to \bigoplus_{\text{ht}p=1} E(A/p)$$

cannot be completed to an exact sequence lying in $\text{Inj}_{\Phi} A$, because its cokernel $E(A/m)$ does not belong to $\text{Inj}_{\Phi} A$.

It should be clear that one can construct such examples more generally for commutative noetherian rings of Krull dimension at least two. On the other hand, if $A$ is a Dedekind domain, then all subsets of $\text{Spec} A$ are coherent, because every factor module of an injective $A$-module is injective and therefore the cokernel of a morphism $I^0 \to I^1$ between injective $A$-modules is up to isomorphism a direct summand of $I^1$. We refer to Section 3 for details about coherent subsets.

The proof of Theorem 3.1 uses an alternative description of the support of a module. This involves the derived category $D(\text{Mod} A)$ of $\text{Mod} A$. Given two complexes $X$ and $Y$ of $A$-modules, we write $X \otimes A^L Y$ for their tensor product in $D(\text{Mod} A)$. Note that for every $A$-module $M$, we have

$$\text{Supp} M = \{ p \in \text{Spec} A \mid M \otimes A^L k(p) \neq 0 \}.$$ 

The following lemma is due to Foxby. The proof given here is inspired by Neeman’s work [14, §2]; see Proposition 5.1 for a more general statement.

**Lemma 3.3 ([6, Remark 2.9]).** Let $M$ be an $A$-module. Given a minimal injective resolution $I^*$ of $M$, we have

$$\text{Supp} M = \bigcup_{i \geq 0} \text{Ass} I^i.$$ 

**Proof.** First observe that we can pass from $M$ to the complex $I$, because the morphism $M \to I$ induces an isomorphism in $D(\text{Mod} A)$. Fix a prime ideal $p$. Recall that each injective $A$-module $J$ admits a unique decomposition

$$J = \bigoplus_{q \text{ prime}} \Gamma_q J$$

such that $\text{Ass} \Gamma_q J \subseteq \{ q \}$ for all $q$. We denote by $\Gamma_p I$ the complex which is obtained from $I$ by taking in each degree the component with associated prime $p$. To be precise, $\Gamma_p I$ is the subcomplex of $I_p = I \otimes A A_p$ supported at the closed point $p$. Note that the sequence
$I \to I_p \leftarrow \Gamma_p I$ of canonical morphisms is degreewise a split epimorphism, followed by a split monomorphism. In particular, it induces an isomorphism

$$I \otimes_A^L k(p) \cong I_p \otimes_A^L k(p) \cong \Gamma_p I \otimes_A^L k(p),$$

since $I' \otimes_A^L k(p) = 0$ for the kernel $I'$ of $I \to I_p$ and $I'' \otimes_A^L k(p) = 0$ for the cokernel $I''$ of $\Gamma_p I \to I_p$.

Suppose first that $I \otimes_A^L k(p) \neq 0$. Then $\Gamma_p I \neq 0$ and therefore $p \in \text{Ass } I^i$ for some $i$. Now suppose that $I \otimes_A^L k(p) = 0$. Then $\Gamma_p I = 0$ by [14, Lemma 2.14]. We want to conclude that $\Gamma_p(I^i) = (\Gamma_p I)^i = 0$ for all $i$. Here we need to use the minimality of $I$. Recall that a complex $J$ of injective $A$-modules is minimal if for all $i$ the kernel of the differential $J^i \to J^{i+1}$ is an essential submodule of $J^i$. If $J$ is minimal and $J^i = 0$ for $i \ll 0$, then $H^i J = 0$ for all $i$ implies $J^i = 0$ for all $i$. Observe that $\Gamma_p$ preserves minimality. Thus $\Gamma_p I = 0$ in $D(\text{Mod } A)$ implies $p \notin \text{Ass } I^i$ for all $i$, because $I$ is minimal.

**Lemma 3.4.** Let $\Phi$ be a coherent subset of $\text{Spec } A$. Then

$$\text{Supp}^{-1} \Phi = \{M \in \text{Mod } A \mid \text{Supp } M \subseteq \Phi\}$$

is a thick subcategory of $\text{Mod } A$.

**Proof.** We consider the full subcategory $C$ consisting of all $A$-modules $M$ which fit into an exact sequence

$$0 \to M \to I^0 \to I^1 \quad \text{with} \quad \text{Ass } I^i \subseteq \Phi \quad (i = 0, 1).$$

Without any assumptions on $\Phi$, it is clear that $C$ is an additive subcategory of $\text{Mod } A$ which is closed under taking kernels. An application of the horseshoe lemma shows that $C$ is closed under forming extensions. Next observe that $C$ is closed under taking cokernels. Here we use that $\Phi$ is coherent. By definition, the cokernel of a morphism between injective modules in $C$ belongs to $C$. A standard argument then shows that this property extends to arbitrary morphisms in $C$. It follows from Lemma [3.3] that $\text{Supp}^{-1} \Phi = C$, and therefore $\text{Supp}^{-1} \Phi$ is thick. \hfill \Box

**Lemma 3.5.** Let $C$ be a subcategory of $\text{Mod } A$ which is thick and closed under taking direct sums. Then the injective envelope $E(M)$ belongs to $C$ for every $M$ in $C$.

**Proof.** Fix $M$ in $C$. First observe that $\text{Tor}_i^A(M, N)$ belongs to $C$ for every $A$-module $N$ and every integer $i$. This is clear, because for any projective resolution $P$ of $N$, the complex $M \otimes_A P$ and therefore its cohomology lies in $C$. Given $p \in \text{Supp } M$, it follows that $k(p)$ belongs to $C$, since $\text{Tor}_i^A(M, k(p))$ is a direct sum of copies of $k(p)$. Then Lemma [2.3] implies that $E(A/p)$ belongs to $C$, and we conclude from Lemma [3.3] that $E(M)$ belongs to $C$. \hfill \Box

**Proof of Theorem 3.1.** Let $\Phi$ be a coherent subset of $\text{Spec } A$. Then the subcategory $\text{Supp}^{-1} \Phi$ is thick and closed under taking direct sums, by Lemma [3.4]. Clearly, we have $\text{Supp}(\text{Supp}^{-1} \Phi) = \Phi$.

Now let $C$ be a subcategory of $\text{Mod } A$, which is thick and closed under taking direct sums. Let $\Phi = \text{Supp } C$. First observe that $\text{Inj}_\Phi A \subseteq C$, by Lemma [3.3]. We claim that $\Phi$ is coherent. In deed, each morphism $I^0 \to I^1$ in $\text{Inj}_\Phi A$ can be completed to an exact
sequence $I^0 \to I^1 \to I^2$ in $\text{Inj}_A$ by taking for $I^2$ the injective envelope of a cokernel of $I^0 \to I^1$. Next we claim that

$$\text{Supp}^{-1}(\text{Supp}\ C) = C.$$ 

The inclusion $\text{Supp}^{-1}(\text{Supp}\ C) \supseteq C$ is clear. Now suppose that $M$ is a module contained in $\text{Supp}^{-1}(\text{Supp}\ C)$ and choose a minimal injective resolution $I^*$. Then $\text{Supp} I^i \subseteq \text{Supp} C$ for all $i$. Thus $I^0$ and $I^1$ belong to $C$, and we conclude that $M$ belongs to $C$. □

The classification of thick subcategories specializes to Gabriel’s classification of localizing subcategories.

Corollary 3.6 ([8, p. 425]). The map $C \mapsto \text{Supp}\ C$ induces a bijection between the set of localizing subcategories of $\text{Mod}_A$ and the set of specialization closed subsets of $\text{Spec} A$.

4. COHERENT SUBSETS OF Spec $A$

In this section we collect some basic properties of coherent subsets of $\text{Spec} A$. Let us fix some notation. Given a multiplicatively closed subset $S$ of $A$, let $\pi : A \to S^{-1}A$ denote the localization. Then we identify $\text{Spec} S^{-1}A$ via $\pi^{-1}$ with the subset of all prime ideals $p$ of $A$ satisfying $S \cap p = \emptyset$.

Proposition 4.1. Let $\Phi$ be a subset of $\text{Spec} A$.

1. Let $(\Phi_i)$ be a family of coherent subsets of $\text{Spec} A$. Then $\bigcap_i \Phi_i$ is coherent.
2. If $\Phi$ is specialization closed, then $\Phi$ is coherent.
3. If $q \subseteq \bigcup_{p \in \Phi} p$ implies $q \in \Phi$ for every prime ideal $q$, then $\Phi$ is coherent.
4. The subset $\Phi$ is coherent if and only if $\Phi \cap \text{Spec} A_p$ is a coherent subset of $\text{Spec} A_p$ for each prime ideal $p$.

Proof. We use that $\Phi$ is coherent if and only if the cokernel $C$ of each morphism $I^0 \to I^1$ between injective $A$-modules with $\text{Ass} I^i \subseteq \Phi$ ($i = 0, 1$) satisfies $\text{Ass} C \subseteq \Phi$.

1. Clear.
2. The assumption on $\Phi$ implies that for each pair $N \subseteq M$ of $A$-modules with $\text{Ass} M \subseteq \Phi$, we have that $\text{Ass} M/N \subseteq \Phi$.
3. The set

$$S = A \setminus \bigcup_{p \in \Phi} p = \bigcap_{p \in \Phi} A \setminus p$$

is multiplicatively closed. The assumption on $\Phi$ implies that the localization $A \to S^{-1}A$ identifies all injective $S^{-1}A$-modules with the injective $A$-modules $I$ satisfying $\text{Ass} I \subseteq \Phi$.

4. We write $\Phi_p = \Phi \cap \text{Spec} A_p$ for each prime ideal $p$. Suppose first that $\Phi$ is coherent and fix a prime ideal $p$. Let $I^0 \to I^1$ be a map in $\text{Inj}_{A_p} A$. There exist an exact sequence $I^0 \to I^1 \to I^2$ in $\text{Inj}_A A$ and localization at $p$ induces an exact sequence in $\text{Inj}_{A_p} A$. Thus $\Phi_p$ is coherent. Now suppose that $\Phi$ is not coherent. It follows that there exists an exact sequence

$$I^0 \to I^1 \to C \to 0$$

of $A$-modules with $I^i \in \text{Inj}_A A$ ($i = 0, 1$) but $\text{Ass} C \not\subseteq \Phi$. Let $p \in \text{Ass} C \setminus \Phi$. Then we localize at $p$ and obtain an exact sequence

$$I^0_p \to I^1_p \to C_p \to 0$$
of $A_p$-modules with $I_p^i \in \text{Inj}_{A_p} A$ ($i = 0, 1$) but
\[ \text{Ass } C_p = (\text{Ass } C) \cap \text{Spec } A_p \not\subseteq \Phi_p. \]
Thus $\Phi_p$ is a subset of $\text{Spec } A_p$ which is not coherent. \qed

Remark 4.2. (1) A subset $\Phi$ of $\text{Spec } A$ satisfies the condition (3) of Proposition 4.1 if and only if it is of the form $\text{Spec } S^{-1}A$ for some multiplicatively closed subset $S$.

(2) The union of two coherent subsets need not to be coherent. For instance, Example 3.2 provides a subset which is not coherent but Zariski open. Each Zariski open subset $U$ can be written as the finite union of basic open subsets. However, a basic open set is coherent because it is of the form $\text{Spec } S^{-1}A$ for some multiplicatively closed subset $S$.

Corollary 4.3. If the Krull dimension of $A$ is at most one, then every subset of $\text{Spec } A$ is coherent.

The converse of this statement is proved in the appendix of this paper.

Proof. We may assume that $A$ is local, by part (4) of Proposition 4.1, and we denote by $m$ the maximal ideal. Let $\Phi$ be a subset of $\text{Spec } A$. If $\Phi$ contains $m$, then $\Phi$ is specialization closed and therefore coherent, by part (2) of Proposition 4.1. If $m$ is not contained in $\Phi$, then all prime ideals in $\Phi$ are minimal and therefore the prime avoidance theorem implies that the condition in part (3) of Proposition 4.1 is satisfied. Thus $\Phi$ is coherent. \qed

Given a prime ideal $p$ of $A$, let
\[ V(p) = \{ q \in \text{Spec } A \mid p \subseteq q \} \quad \text{and} \quad A(p) = \{ q \in \text{Spec } A \mid q \subseteq p \}. \]
Subsets of the from $V(p)$ and $A(p)$ are coherent. They can be used to build new coherent subsets.

Corollary 4.4. Let $\Phi$ and $\Psi$ be subsets of $\text{Spec } A$ and suppose that $\Psi$ is finite. Then
\[ \bigcup_{p \in \Phi, q \in \Psi} V(p) \cap A(q) \]
is a coherent subset of $\text{Spec } A$.

Proof. We can express the set as the intersection of two coherent subsets:
\[ \bigcup_{p \in \Phi, q \in \Psi} V(p) \cap A(q) = \left( \bigcup_{p \in \Phi} V(p) \right) \cap \left( \bigcup_{q \in \Psi} A(q) \right). \] \qed

Example 4.5. Let $p_1, p_2$ be prime ideals. If $p_1 \subseteq p_2$, then $\{ q \mid p_1 \subseteq q \subseteq p_2 \}$ is coherent. If $p_1 \not\subseteq p_2$ and $p_2 \not\subseteq p_1$, then $\{ p_1, p_2 \}$ is coherent. In both cases the set is of the form
\[ \bigcup_{p, q \in \Phi} V(p) \cap A(q) \] with $\Phi = \{ p_1, p_2 \}$. 

5. Support of complexes

Let $X$ be a complex of $A$-modules. Following [6], the support of $X$ is by definition

$$\text{Supp } X = \{ p \in \text{Spec } A \mid X \otimes^L_A k(p) \neq 0 \}.$$ 

We use an alternative description of the support of $X$. A complex $I$ of injective $A$-modules together with a quasi-isomorphism $X \to I$ is called a minimal K-injective resolution of $X$, if $I$ is K-injective (that is, every morphism from an acyclic complex to $I$ is null homotopic) and $I$ is minimal (that is, for all $i$ the kernel of the differential $I^i \to I^{i+1}$ is an essential submodule of $I^i$).

One can show that each complex of $A$-modules admits a minimal K-injective resolution; see [15, Theorem 4.5] or [3, Application 2.4] for the existence of a K-injective resolution and [12, Proposition B.2] for the minimality. Note that each acyclic and K-injective complex is null homotopic. Moreover, if a minimal complex $I$ of injective $A$-modules is null homotopic, then $I^i = 0$ for all $i$.

The next proposition is the obvious generalization of Lemma 3.3 from modules to complexes of modules. The proof requires only minor modifications; it follows closely [14, §2].

**Proposition 5.1.** Let $X$ be a complex of $A$-modules and $X \to I$ a minimal K-injective resolution of $X$. Then we have

$$\text{Supp } X = \bigcup_{i \in \mathbb{Z}} \text{Ass } I^i.$$ 

**Proof.** First observe that we can pass from $X$ to the complex $I$, because the morphism $X \to I$ induces an isomorphism in $D(\text{Mod } A)$. Fix a prime ideal $p$. Recall that each injective $A$-module $J$ admits a unique decomposition $J = \bigoplus_{q \text{ prime}} \Gamma_q J$ such that $\text{Ass } \Gamma_q J \subseteq \{ q \}$ for all $q$. We denote by $\Gamma_p I$ the complex which is obtained from $I$ by taking in each degree the component with associated prime $p$. To be precise, $\Gamma_p I$ is the subcomplex of $I_p = I \otimes_A A_p$ supported at the closed point $p$. Note that the sequence $I \to I_p \leftarrow \Gamma_p I$ of canonical morphisms is degreewise a split epimorphism, followed by a split monomorphism. In particular, it induces an isomorphism

$$I \otimes^L_A k(p) \cong I_p \otimes^L_A k(p) \cong \Gamma_p I \otimes^L_A k(p),$$

since $I' \otimes^L_A k(p) = 0$ for the kernel $I'$ of $I \to I_p$ and $I'' \otimes^L_A k(p) = 0$ for the cokernel $I''$ of $\Gamma_p I \to I_p$.

Suppose first that $I \otimes^L_A k(p) \neq 0$. Then $\Gamma_p I \neq 0$ and therefore $p \in \text{Ass } I^i$ for some $i$. Now suppose that $I \otimes^L_A k(p) = 0$. Then $\Gamma_p I = 0$ by [14, Lemma 2.14], that is, $\Gamma_p I$ is acyclic. We want to conclude that $\Gamma_p (I^i) = (\Gamma_p I)^i = 0$ for all $i$. Here we need to use the minimality of $I$. We observe that $\Gamma_p$ preserves minimality. Also, $\Gamma_p I$ is a K-injective complex of injective $A$-modules. Thus $\Gamma_p I = 0$ in $D(\text{Mod } A)$ implies $p \notin \text{Ass } I^i$ for all $i$, because $I$ is minimal. □

**Theorem 5.2.** For a subset $\Phi$ of Spec $A$ the following conditions are equivalent:

1. $\Phi$ is coherent.
(2) For every complex $X$ of $A$-modules we have
$$\text{Supp } X \subseteq \Phi \iff \text{Supp } H^i X \subseteq \Phi \text{ for all } i \in \mathbb{Z}.$$  

(3) For every complex $X$ of $A$-modules we have
$$\text{Supp } X \subseteq \Phi = \iff \text{Supp } H^i X \subseteq \Phi \text{ for all } i \in \mathbb{Z}.$$  

Proof. (1) $\Rightarrow$ (2): Suppose that $\Phi$ is coherent. We use that the $A$-modules $M$ with $\text{Supp } M \subseteq \Phi$ form a thick subcategory, by Lemma 3.3. Now fix a complex $X$ of $A$-modules. If $\text{Supp } X \subseteq \Phi$, then we have a quasi-isomorphic complex $I$ of injective $A$-modules with $\text{Ass } I^i \subseteq \Phi$ for all $i$, by Proposition 5.1. It follows that $\text{Supp } H^i X = \text{Supp } H^i I \subseteq \Phi$ for all $i$. Now let $\text{Supp } H^\ast X \subseteq \Phi$ and $p \in \text{Supp } X$. A d´evissage argument shows that $\text{Supp } H^\ast (X \otimes_A Y) \subseteq \Phi$ for every complex $Y$. In particular,

$$\{p\} = \text{Supp } H^\ast (X \otimes_A k(p)) \subseteq \Phi$$

because $X \otimes_A k(p)$ is a direct sum of shifted copies of $k(p)$. Thus $\text{Supp } X \subseteq \Phi$.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): Let $I^0 \to I^1$ be a morphism of injective $A$-modules with $\text{Ass } I^i \subseteq \Phi$ for $i = 0, 1$. Viewing $I$ as a complex, we have $\text{Supp } I \subseteq \Phi$ by Proposition 5.1 and therefore $\text{Supp } H^0 I \subseteq \Phi$. It follows from Lemma 3.3 that we can complete $I^0 \to I^1$ to an injective resolution

$$0 \to H^0 I \to I^0 \to I^1 \to I^2 \to \cdots$$

of $H^0 I$ with $\text{Ass } I^i \subseteq \Phi$ for all $i$. Thus $\Phi$ is coherent.

Acknowledgements. A pleasant collaboration with Dave Benson and Srikanth Iyengar has been the starting point of this paper; in particular the Example 3.2 is taken from [2]. I would like to thank Amnon Neeman and Torsten Wedhorn for a number of helpful comments on this work.

Appendix A. Noncoherent subsets of $\text{Spec } A$

By Srikanth Iyengar

In this appendix, we establish the converse of Corollary 4.3. To this end, we recall some standard notions from commutative algebra; this serves also to fix notation.

Definition A.1. Let $A$ be a commutative noetherian ring, $a$ an ideal in $A$, and let $M$ be an $A$-module. The $a$-depth of $M$ is the number

$$\text{depth}_A(a, M) = \inf \{n \mid \text{Ext}_A^n(A/a, M) \neq 0\}.$$  

This invariant of $M$ can also be detected from its Koszul homology on a finite set of elements generating $a$, and also its local cohomology with respect to $a$; see, for instance, [7, Theorem 2.1]. When $M$ is finitely generated and $aM \neq M$, this number coincides with the length of the longest $A$-regular sequence in $a$; see [13, Theorem 28].

As usual, if $A$ is local, with maximal ideal $m$, we write $\text{depth}_A M$ for the $m$-depth of $M$, and call it the depth of $M$.

We record the following standard properties of depth, for ease of reference.
Lemma A.2. Let $a$ be an ideal in a commutative noetherian ring $A$, and let $M$ be an $A$-module. The following statements hold.

1. One has an equality, $\text{depth} \left( A \left( \sqrt{a}, M \right) \right) = \text{depth} \left( A(a, M) \right)$.

2. With $I^*$ the minimal injective resolution of $M$, for each prime ideal $p$ one has $\text{depth}_{A_p} M_p = \inf \{ n \mid E(A/p) \text{ is a direct summand in } I^n \}$.

3. If $A \to B$ is a homomorphism of rings and $N$ is a $B$-module, then viewing $N$ as an $A$-module by restriction of scalars, one has $\text{depth}_A(a, N) = \text{depth}_B(aB, N)$.

Proof. For (1) see, for instance, [7, Proposition 2.11], while (3) is evident, if one computes depth using Koszul complexes, or via local cohomology; see [7, Theorem 2.1]. Part (2) holds as $(I^*)_p$ is the minimal injective resolution of $M_p$ over $A_p$, so the complex $\text{Hom}_{A_p} \left( A_p/\mathfrak{p}A_p, (I^*)_p \right)$, whose first nonzero cohomology module occurs in degree $\text{depth}_{A_p} M_p$, has zero differential. □

We need the following result of Auslander and Buchsbaum, which is implicit in [1].

Proposition A.3. Let $A$ be a commutative noetherian ring with $\dim A \geq 1$. There exists a prime $p$ in $\text{Spec} A$ such that $\text{depth} A_p M_p = \dim A_p = \dim A - 1$.

Proof. The proof uses an induction on $\dim A$. When $\dim A = 1$, for $p$ one may take any minimal prime of $A$. This is the basis of the induction.

Assume $\dim A \geq 2$, and let $m$ be the maximal ideal of $A$. Since $\text{Ass} A$ is finite, the prime avoidance theorem implies that the following set is nonempty:

$$m \setminus \bigcup_{p \in \text{Ass} A \setminus \mathfrak{m}} p.$$

Choose an element $x$ in it. One then has that $\dim(A/Ax) = \dim A - 1$, so the induction hypothesis yields a prime $p$ in $A$, containing $x$, such that $\text{depth}(A_p/A_p x) = \dim(A_p/A_p x) = \dim(A/Ax) - 1 = \dim A - 2$.

Observe that $p \neq m$, so the choice of $x$ ensures that it is a nonzero divisor in $A_p$. Thus one has $\text{depth} A_p = \dim A_p = \dim A - 1$. This completes the induction argument. □

The gist of the result below is well-known; we provide a proof for lack of a suitable reference for this formulation.

Theorem A.4. Let $A$ be a commutative noetherian ring with $\dim A \geq 2$. There exists an $A$-module $M$ and a prime $p$ in $\text{Spec} A$ such that $\text{depth}_{A_p} M_p = \max \{ 2, \dim A - 1 \}$.

Proof. When $\dim A \geq 3$, we apply Proposition A.3 and take for $p$ any prime such that $\text{depth} A_p = \dim A - 1$ and set $M = A$. Henceforth, we assume $\dim A = 2$. Choosing a prime ideal $p$ in $A$ with $\dim A_p = 2$, and replacing $A$ with $A_p$, we may assume also that $A$ is local; the goal then is to find an $A$-module $M$ such that $\text{depth} A M = 2$. At this point, one may refer to, for instance, Hochster’s article [10], especially Section 3. We provide details, for completeness.

Let $\hat{A}$ be the completion of $A$ at its maximal ideal, say $m$. One has $\dim \hat{A} = 2$, so there exists a prime ideal $\mathfrak{a}$ in $\hat{A}$ with $\dim(\hat{A}/\mathfrak{a}) = 2$. Consider the canonical homomorphisms
of rings $A \to \hat{A} \to \hat{A}/a$. Observer that $m(\hat{A}/a)$ is the maximal ideal of $\hat{A}/a$, so, by Lemma A.2(3), for any module $M$ over $\hat{A}/a$, one has
\[
\text{depth}_A M = \text{depth}_{\hat{A}/a} M.
\]
Replacing $A$ with $\hat{A}/a$ one may assume $A$ is a complete local domain, with $\text{dim } A = 2$.

Let $B$ be the integral closure of $A$ in its field of fractions; the conditions on $A$ imply that $B$ is finite as an $A$-module, by [13, Corollary 2, pp. 234], so also a two dimensional noetherian ring, by [13, Theorem 20]. The finiteness of the extension $A \subseteq B$ implies that there exists a prime ideal $q$ in $B$ with $\text{dim } B_q = 2$ and $q \cap A = m$, the maximal ideal of $A$; see [13, Theorem 5(iii)]. The choice of $q$ ensures that $\sqrt{mB_q} = qB_q$, so part (3) and (2) of Lemma A.2 yield the first and second equalities below:
\[
\text{depth}_A B_q = \text{depth}_{B_q}(mB_q, B_q)
= \text{depth } B_q
= \text{dim } B_q
= 2
\]
The penultimate equality holds because $B_q$ is Cohen-Macaulay, by Serre’s criterion for normality; see [13, Theorem 39]. The $A$-module $B_q$ has thus the desired depth. \qed

The result below is a perfect converse to Corollary 10.

**Corollary A.5.** If $A$ is a commutative noetherian ring with $\text{dim } A \geq 2$, then there exists a subset $\Phi$ of $\text{Spec } A$ that is not coherent.

**Proof.** By Theorem A.4 there exists an $A$-module $M$ and a prime ideal $p$ in $A$ such that $d = \text{depth}_A M_p \geq 2$. Let $I^*$ be a minimal injective resolution of $M$, and set
\[
\Phi = \text{Ass}_A(I^{d-2}) \cup \text{Ass}_A(I^{d-1})
\]
Observe that $p$ is in $\text{Ass}_A(I^d)$ but not in $\Phi$, by Lemma A.2(2). Thus, the set $\Phi$ is not coherent, as the associated primes of $\text{Coker}(I^{d-2} \to I^{d-1})$ coincide with those of $I^d$. \qed

**References**

[1] M. Auslander and D. Buchsbaum: Homological dimension in noetherian rings. II. Trans. Amer. Math. Soc. **88** (1958), 194–206.
[2] D. Benson, S. Iyengar, and H. Krause: Local cohomology and support for triangulated categories. arXiv:math.KT/0702610
[3] M. Bökstedt and A. Neeman: Homotopy limits in triangulated categories. Compositio Math. **86** (1993) 209–234.
[4] N. Bourbaki: Algèbre commutative. Hermann, Paris, 1968.
[5] N. Bourbaki: Algèbre commutative. Chapitre 10. Masson, Paris, 1998.
[6] H.-B. Foxby: Bounded complexes of flat modules. J. Pure Appl. Algebra **15** (1979), 149–172.
[7] H.-B. Foxby and S. Iyengar: Depth and amplitude for unbounded complexes. Commutative algebra (Grenoble/Lyon, 2001), 119–137, Contemp. Math., **331**, Amer. Math. Soc., Providence, RI, 2003.
[8] P. Gabriel: Des catégories abéliennes. Bull. Soc. Math. France **90** (1962), 323–448.
[9] G. Garkusha and M. Prest: Classifying Serre subcategories of finitely presented modules. Preprint (2006).
[10] M. Hochster: Cohen-Macaulay modules. Conference on Commutative Algebra (Lawrence, Kansas 1972), 120–152, Lecture Notes Math. **311**, Springer-Verlag, 1973.
[11] M. Hovey: Classifying subcategories of modules. Trans. Amer. Math. Soc. **353** (2001), 3181–3191.
[12] H. Krause: The stable derived category of a noetherian scheme. Compos. Math. 141 (2005), 1128–1162.

[13] H. Matsumura: Commutative algebra. Second edition. Mathematics Lecture Note Series, 56, Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980.

[14] A. Neeman: The chromatic tower of $D(R)$. Topology 31 (1992), 519–532.

[15] N. Spaltenstein: Resolutions of unbounded complexes. Compositio Math. 65 (1988), 121–154.

[16] R. Takahashi: Classifying subcategories of modules over commutative noetherian rings. Preprint (2006).

SRIKANTH IYENGAR, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA, LINCOLN NE 68588, U.S.A.

E-mail address: iyengar@math.unl.edu

HENNING KRAUSE, INSTITUT FÜR MATHEMATIK, UNIVERSITÄT PADGBORN, 33095 PADERBORN, GERMANY.

E-mail address: hkrause@math.upb.de