A statistical mechanical interpretation of algorithmic information theory III: composite systems and fixed points†

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The statistical mechanical interpretation of algorithmic information theory (AIT for short) was introduced and developed in our previous papers Tadaki (2008; 2012), where we introduced into AIT the notion of thermodynamic quantities, such as the partition function $Z(T)$, free energy $F(T)$, energy $E(T)$ and statistical mechanical entropy $S(T)$. We then discovered that in the interpretation, the temperature $T$ is equal to the partial randomness of the values of all these thermodynamic quantities, where the notion of partial randomness is a stronger representation of the compression rate by means of program-size complexity. Furthermore, we showed that this situation holds for the temperature itself as a thermodynamic quantity, namely, for each of the thermodynamic quantities above, the computability of its value at temperature $T$ gives a sufficient condition for $T \in (0, 1)$ to be a fixed point on partial randomness. In this paper, we develop the statistical mechanical interpretation of AIT further and pursue its formal correspondence to normal statistical mechanics. The thermodynamic quantities in AIT are defined on the basis of the halting set of an optimal prefix-free machine, which is a universal decoding algorithm used to define the notion of program-size complexity. We show that there are infinitely many optimal prefix-free machines that give completely different sufficient conditions for each of the thermodynamic quantities in AIT. We do this by introducing the notion of composition of prefix-free machines into AIT, which corresponds to the notion of the composition of systems in normal statistical mechanics.

1. Introduction

Algorithmic information theory (AIT for short) is a framework for applying information-theoretic and probabilistic ideas to recursive function theory. One of AIT’s primary concepts is the program-size complexity (or Kolmogorov complexity) $H(s)$ of a finite binary

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string $s$, which is defined as the length of the shortest binary program for an optimal prefix-free machine to output $s$. Here an optimal prefix-free machine is a universal decoding algorithm. By definition, $H(s)$ is thought to represent the amount of randomness contained in a finite binary string $s$, which cannot be captured in an effective manner. In particular, the notion of program-size complexity plays a crucial role in characterising the randomness of an infinite binary string, or equivalently, a real.

In Tadaki (2008), we introduced and developed a statistical mechanical interpretation of AIT. In that paper, we introduced into AIT the notion of thermodynamic quantities at temperature $T$, such as the partition function $Z(T)$, free energy $F(T)$, energy $E(T)$ and statistical mechanical entropy $S(T)$. These quantities are real functions of a real argument $T > 0$. We then proved that if the temperature $T$ is a computable real with $0 < T < 1$, then, for each of these thermodynamic quantities, the partial randomness of its value is equal to $T$, where the notion of partial randomness is a stronger representation of the compression rate by means of program-size complexity. Thus, the temperature $T$ plays a role as the partial randomness of all the thermodynamic quantities in the statistical mechanical interpretation of AIT. In Tadaki (2008), we further showed that the temperature $T$ plays a role as the partial randomness of the temperature $T$ itself, which is itself a thermodynamic quantity. Specifically, we proved the fixed-point theorem on partial randomness†, which states that, for every $T \in (0, 1)$, if the value of partition function $Z(T)$ at temperature $T$ is a computable real, then the partial randomness of $T$ is equal to $T$, and therefore the compression rate of $T$ is equal to $T$, that is, $\lim_{n \to \infty} H(T |_n) / n = T$, where $T |_n$ is the first $n$ bits of the base-two expansion of $T$.

In our second paper on this interpretation, Tadaki (2012), we showed that a fixed-point theorem of the same form as that for $Z(T)$ also holds for each of the free energy $F(T)$, energy $E(T)$ and statistical mechanical entropy $S(T)$. Moreover, based on the statistical mechanical relation $F(T) = -T \log_2 Z(T)$, we showed that the computability of $F(T)$ gives completely different fixed points from the computability of $Z(T)$.

In this paper, we develop the statistical mechanical interpretation of AIT further and pursue its formal correspondence to normal statistical mechanics. As a result, we further unlock the properties of the sufficient conditions. The definitions of the thermodynamic quantities in AIT are based on the halting set of an optimal prefix-free machine. In this paper, we show in Theorem 4.2 that there are infinitely many optimal prefix-free machines that give completely different sufficient conditions for each of the thermodynamic quantities in AIT. We do this by introducing into AIT the notion of the composition of prefix-free machines, which corresponds to the notion of composition of systems in normal statistical mechanics.

Structure of the paper

We begin in Section 2 with some preliminaries on AIT and partial randomness. In Section 3, we review the results on the statistical mechanical interpretation of AIT that we

† The fixed-point theorem on partial randomness is called a fixed-point theorem on compression rate in Tadaki (2008).
have presented in a series of previous papers (Tadaki 2008; Tadaki 2010; Tadaki 2012). In Section 4, we present the main result of this paper. In Section 5, we introduce the notion of the composition of prefix-free machines into AIT, and then prove the main result based on it in Section 6. In Section 7, we conclude with a summary and an indication of the future direction of this work.

2. Preliminaries

2.1. Basic notation

We will begin by defining some notation we will use for numbers and strings in this paper. \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \) is the set of natural numbers, and \( \mathbb{N}^+ \) is the set of positive integers. \( \mathbb{Q} \) is the set of rationals and \( \mathbb{R} \) is the set of reals. Let \( f : S \to \mathbb{R} \) with \( S \subseteq \mathbb{R} \). We say that \( f \) is increasing (respectively, decreasing) if \( f(x) < f(y) \) (respectively, \( f(x) > f(y) \)) for all \( x, y \in S \) with \( x < y \). We use \( f' \) to denote the derived function of \( f \). Normally, \( o(n) \) denotes any function \( f : \mathbb{N}^+ \to \mathbb{R} \) such that \( \lim_{n \to \infty} f(n)/n = 0 \).

\( \{0, 1\}^* = \{\lambda, 0, 1, 00, 01, 10, 11, 000, \ldots\} \) is the set of finite binary strings, where \( \lambda \) denotes the empty string. For any \( s \in \{0, 1\}^* \), we use \( |s| \) to denote the length of \( s \). A subset \( S \) of \( \{0, 1\}^* \) is said to be prefix-free if no string in \( S \) is a prefix of another string in \( S \). For any function \( f \), the domain of definition of \( f \) is denoted by \( \text{dom } f \).

Let \( \alpha \) be an arbitrary real. We use \( \alpha^n \) to denote the first \( n \) bits of the base-two expansion of \( \alpha - \lfloor \alpha \rfloor \) with infinitely many zeros, where \( \lfloor \alpha \rfloor \) is the greatest integer less than or equal to \( \alpha \). For example, in the case of \( \alpha = 5/8 \), \( \alpha^6 = 101000 \). We say that a real \( \alpha \) is computable if there exists a total recursive function \( f : \mathbb{N} \to \mathbb{Q} \) such that \( |\alpha - f(n)| < 2^{-n} \) for all \( n \in \mathbb{N} \).

2.2. Algorithmic information theory

In this section we give a concise review of some of the definitions and results obtained for AIT (Chaitin 1975; 1987; Nies 2009; Downey and Hirschfeldt 2011). A prefix-free machine is a partial recursive function \( M : \{0, 1\}^* \to \{0, 1\}^* \) such that \( \text{dom } M \) is a non-empty prefix-free set. For each prefix-free machine \( M \) and each \( s \in \{0, 1\}^* \), \( H_M(s) \) is defined by

\[
H_M(s) := \min \{ |p| \mid p \in \{0, 1\}^* \& M(p) = s \}
\]

(which may be \( \infty \)). A prefix-free machine \( U \) is said to be optimal if for each prefix-free machine \( M \) there exists \( d \in \mathbb{N} \), which depends on \( M \), with the property that for every \( p \in \text{dom } M \) there exists \( q \in \{0, 1\}^* \) for which \( U(q) = M(p) \) and \( |q| \leq |p| + d \). It is then easy to see that there exists an optimal prefix-free machine. We choose a particular optimal prefix-free machine \( U \) as the standard one for use, and define \( H(s) \) as \( H_U(s) \), which is variously referred to as the program-size complexity of \( s \), the information content of \( s \) or the Kolmogorov complexity of \( s \) (Gács 1974; Levin 1974; Chaitin 1975). It follows that for every prefix-free machine \( M \) there exists \( d \in \mathbb{N} \) such that for every \( s \in \{0, 1\}^* \),

\[
H(s) \leq H_M(s) + d.
\]
For any $x \in \mathbb{R}$, we say that $x$ is weakly Chaitin random if there exists $c \in \mathbb{N}$ such that $n - c \leq H(x|_n)$ for all $n \in \mathbb{N}^+$ (Chaitin 1975; 1987). On the other hand, for any $x \in \mathbb{R}$, we say that $x$ is Chaitin random if $\lim_{n \to \infty} H(x|_n) - n = \infty$ (Chaitin 1975; 1987). It is obvious that for every $x \in \mathbb{R}$, if $x$ is Chaitin random, then $x$ is weakly Chaitin random. We can show that the converse also hold. Thus, for every $x \in \mathbb{R}$, $x$ is weakly Chaitin random if and only if $x$ is Chaitin random – see Chaitin (1987) for the proof and some historical detail.

### 2.3. Partial randomness

In Tadaki (1999; 2002), we generalised the notion of the randomness of a real so that the degree of the randomness, which is now often referred to as the partial randomness (Calude et al. 2006; Reimann and Stephan 2006; Calude and Stay 2006), can be characterised by a real $T$ with $0 \leq T \leq 1$ as follows.

**Definition 2.1 (Weak Chaitin $T$-randomness).** Let $T \in [0, 1]$ and $x \in \mathbb{R}$. We say that $x$ is weakly Chaitin $T$-random if there exists $c \in \mathbb{N}$ such that for all $n \in \mathbb{N}^+$, $Tn - c \leq H(x|_n)$.

When $T = 1$, the weak Chaitin $T$-randomness just results in the weak Chaitin randomness.

**Definition 2.2 ($T$-compressibility).** Let $T \in [0, 1]$ and $x \in \mathbb{R}$. We say that $x$ is $T$-compressible if $H(x|_n) \leq Tn + o(n)$ for all $n \in \mathbb{N}^+$, which is equivalent to

$$\limsup_{n \to \infty} \frac{H(x|_n)}{n} \leq T.$$

For every $T \in [0, 1]$ and every $x \in \mathbb{R}$, if $x$ is weakly Chaitin $T$-random and $T$-compressible, then

$$\lim_{n \to \infty} \frac{H(x|_n)}{n} = T. \quad (1)$$

The left-hand side of (1) is referred to as the compression rate of a real $x$ in general. Note, however, that (1) does not necessarily imply that $x$ is weakly Chaitin $T$-random. Thus, the notion of partial randomness is a stronger representation of the notion of the compression rate.

**Definition 2.3 (Chaitin $T$-randomness (Tadaki 1999; 2002)).** Let $T \in [0, 1]$ and $x \in \mathbb{R}$. We say that $x$ is Chaitin $T$-random if $\lim_{n \to \infty} H(x|_n) - Tn = \infty$.

When $T = 1$, the Chaitin $T$-randomness just results in the Chaitin randomness. It is obvious that for every $T \in [0, 1]$ and every $x \in \mathbb{R}$, if $x$ is Chaitin $T$-random, then $x$ is weakly Chaitin $T$-random. However, in 2005, Reimann and Stephan showed that the converse does not necessarily hold when $T < 1$ (Reimann and Stephan 2006). This contrasts with the equivalence between weak Chaitin randomness and Chaitin randomness, each of which corresponds to the case of $T = 1$. 

3. Previous results

In this section, we review some results for the statistical mechanical interpretation of AIT that were developed in our earlier work (Tadaki 2008; 2010; 2012). We will first introduce the notion of thermodynamic quantities into AIT as follows.

In statistical mechanics, the partition function $Z_{\text{sm}}(T)$, free energy $F_{\text{sm}}(T)$, energy $E_{\text{sm}}(T)$ and entropy $S_{\text{sm}}(T)$ at temperature $T$ are given as follows:

$$Z_{\text{sm}}(T) = \sum_{x \in X} e^{-\frac{E_x}{k_B T}}$$

$$F_{\text{sm}}(T) = -k_B T \ln Z_{\text{sm}}(T)$$

$$E_{\text{sm}}(T) = \frac{1}{Z_{\text{sm}}(T)} \sum_{x \in X} E_x e^{-\frac{E_x}{k_B T}}$$

$$S_{\text{sm}}(T) = \frac{E_{\text{sm}}(T) - F_{\text{sm}}(T)}{T} \tag{2}$$

where $X$ is a complete set of energy eigenstates of a quantum system and $E_x$ is the energy of an energy eigenstate $x$. The constant $k_B$ is called the Boltzmann Constant, and the $\ln$ denotes the natural logarithm.

Let $M$ be an arbitrary prefix-free machine. We introduce the notion of thermodynamic quantities into AIT using Replacements 3.1 below for the thermodynamic quantities (2) in statistical mechanics.

**Replacements 3.1.**

(i) Replace the complete set $X$ of energy eigenstates $x$ by the set $\text{dom} M$ of all programs $p$ for $M$.

(ii) Replace the energy $E_x$ of an energy eigenstate $x$ by the length $|p|$ of a program $p$.

(iii) Set the Boltzmann Constant $k_B$ to $1/\ln 2$.

Thus, motivated by the formulae (2) and taking into account Replacements 3.1, we introduce the notion of thermodynamic quantities into AIT as follows.

**Definition 3.2 (Thermodynamic quantities in AIT (Tadaki 2008)).** Let $M$ be any prefix-free machine and $T$ be any real with $T > 0$.

We first consider the case where $\text{dom} M$ is an infinite set. In this case we choose a particular enumeration $p_1, p_2, p_3, p_4, \ldots$ of the countably infinite set $\text{dom} M$.

‡ For background on the thermodynamic quantities in statistical mechanics, see, for example, Callen (1985, Chapter 16) or Toda et al. (1992, Chapter 2). To be precise, the partition function is not a thermodynamic quantity but a statistical mechanical quantity.

‡ The enumeration $\{p_i\}$ can be chosen quite arbitrarily, and the results of this paper are independent of the choice of $\{p_i\}$. This is because the sums $\sum_{i=1}^k 2^{-|p_i|/T}$ and $\sum_{i=1}^k |p_i| 2^{-|p_i|/T}$ in Definition 3.2 are positive term series and converge as $k \to \infty$ for every $T \in (0, 1)$. 

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We have:

(i) The partition function $Z_M(T)$ at temperature $T$ is defined as

$$
\lim_{k \to \infty} Z_k(T)
$$

where

$$
Z_k(T) = \sum_{i=1}^{k} 2^{-\frac{|p_i|}{T}}.
$$

(ii) The free energy $F_M(T)$ at temperature $T$ is defined as

$$
\lim_{k \to \infty} F_k(T)
$$

where

$$
F_k(T) = -T \log_2 Z_k(T).
$$

(iii) The energy $E_M(T)$ at temperature $T$ is defined as

$$
\lim_{k \to \infty} E_k(T)
$$

where

$$
E_k(T) = \frac{1}{Z_k(T)} \sum_{i=1}^{k} |p_i| 2^{-\frac{|p_i|}{T}}.
$$

(iv) The statistical mechanical entropy $S_M(T)$ at temperature $T$ is defined as

$$
\lim_{k \to \infty} S_k(T)
$$

where

$$
S_k(T) = \frac{E_k(T) - F_k(T)}{T}.
$$

In the case where $\text{dom } M$ is a non-empty finite set, the quantities $Z_M(T)$, $F_M(T)$, $E_M(T)$ and $S_M(T)$ are just defined as (3), (4), (5) and (6), respectively, where $p_1, \ldots, p_k$ is an enumeration of the finite set $\text{dom } M$.

For every optimal prefix-free machine $V$, note that we have $Z_V(1)$ is precisely the Chaitin halting probability $\Omega$ introduced in Chaitin (1975). Theorems 3.3 and 3.4 hold for these thermodynamic quantities in AIT.

**Theorem 3.3 (Properties of $Z(T)$ and $F(T)$ (Tadaki 1999; 2002; 2008)).** Let $V$ be an optimal prefix-free machine, and let $T \in \mathbb{R}$. Then:

(i) If $0 < T \leq 1$ and $T$ is computable, then each of $Z_V(T)$ and $F_V(T)$ converges and is weakly Chaitin $T$-random and $T$-compressible.

(ii) If $1 < T$, then $Z_V(T)$ and $F_V(T)$ diverge to $\infty$ and $-\infty$, respectively.

**Theorem 3.4 (Properties of $E(T)$ and $S(T)$ (Tadaki 2008)).** Let $V$ be an optimal prefix-free machine and $T \in \mathbb{R}$. Then:

(i) If $0 < T < 1$ and $T$ is computable, then each of $E_V(T)$ and $S_V(T)$ converges and is Chaitin $T$-random and $T$-compressible.
(ii) If $1 \leq T$, then both $E_V(T)$ and $S_V(T)$ diverge to $\infty$.

Theorems 3.3 and 3.4 show that if $T$ is a computable real with $T \in (0, 1)$, then the temperature $T$ is equal to the partial randomness (and therefore the compression rate) of the values of all the thermodynamic quantities in Definition 3.2 for an optimal prefix-free machine.

These theorems also show that the values of all the thermodynamic quantities diverge when the temperature $T$ exceeds 1. This phenomenon might be regarded as some sort of phase transition in statistical mechanics. Note here that the weak Chaitin $T$-randomness in Theorem 3.3 is replaced by the Chaitin $T$-randomness in Theorem 3.4 in exchange for the divergence at $T = 1$.

When we consider all of the thermodynamic quantities in statistical mechanics and thermodynamics, one of the most typical is temperature itself. Theorem 3.5 below shows that the partial randomness of the temperature $T$ can be equal to the temperature $T$ itself in the statistical mechanical interpretation of AIT.

We use $\mathcal{FP}_w$ to denote the set of all real $T \in (0, 1)$ such that $T$ is weakly Chaitin $T$-random and $T$-compressible, and $\mathcal{FP}$ to denote the set of all real $T \in (0, 1)$ such that $T$ is Chaitin $T$-random and $T$-compressible. It is obvious that $\mathcal{FP} \subset \mathcal{FP}_w$. Each element $T$ of $\mathcal{FP}_w$ is a fixed point on partial randomness, that is, it satisfies the property the partial randomness of $T$ is equal to $T$ itself, and therefore satisfies

$$\lim_{n \to \infty} \frac{H(T^1_n)}{n} = T. \quad (7)$$

This means that the compression rate of $T$ is equal to $T$ itself. Intuitively, we might interpret the meaning of (7) as follows: imagine a file of infinite size whose content is

“$The compression rate of this file is 0.100111001 \ldots$”

When this file is compressed, the compression rate of this file is actually equal to $0.100111001 \ldots$, as the content of this file says. This situation is self-referential and forms a fixed point.

Let $V$ be a prefix-free machine. We define the sets $\mathcal{Z}(V)$ by

$$\mathcal{Z}(V) = \{ T \in (0, 1) \mid Z_V(T) \text{ is computable} \}.$$

In the same way, we define the sets $\mathcal{F}(V), \mathcal{E}(V)$ and $\mathcal{S}(V)$ based on the computability of $F_V(T), E_V(T)$ and $S_V(T)$, respectively. We can then show the following theorem.

**Theorem 3.5 (Fixed points for partial randomness (Tadaki 2008; 2012)).** Let $V$ be an optimal prefix-free machine. Then $\mathcal{Z}(V) \cup \mathcal{F}(V) \subset \mathcal{FP}_w$ and $\mathcal{E}(V) \cup \mathcal{S}(V) \subset \mathcal{FP}$.

Theorem 3.5 is just a fixed-point theorem on partial randomness, where the computability of each of the values $Z_V(T), F_V(T), E_V(T)$ and $S_V(T)$ gives a sufficient condition for a real $T \in (0, 1)$ to be a fixed point on partial randomness. Thus, by Theorem 3.5, the above observation that the temperature $T$ is equal to the partial randomness of the values of the thermodynamic quantities in the statistical mechanical interpretation of AIT is further confirmed.
Note that for mathematical strictness, we used Definition 3.2 to introduce the thermodynamic quantities into AIT by performing Replacements 3.1 for the corresponding thermodynamic quantities (2) in normal statistical mechanics. However, if we do not require this degree of mathematical strictness, we can develop a total statistical mechanical interpretation of AIT that provides a perfect correspondence with normal statistical mechanics, as shown in Tadaki (2010). Generally speaking, in order to give a statistical mechanical interpretation to a framework that looks unrelated to statistical mechanics at first glance, it is important to identify a microcanonical ensemble in the framework. Once we can do this, we can easily develop an equilibrium statistical mechanics on the framework according to the theoretical development of normal equilibrium statistical mechanics. Here, the microcanonical ensemble is a certain sort of uniform probability distribution. In Tadaki (2010), we developed a total statistical mechanical interpretation of AIT that achieved a perfect correspondence with normal statistical mechanics. We did this by identifying a microcanonical ensemble in the framework of AIT. The microcanonical ensemble is based on the probability measure that gives Chaitin’s halting probability Ω the meaning of the actual halting probability. The arguments in Tadaki (2010) are not necessarily mathematically rigorous, but have the same level of mathematical strictness as normal statistical mechanics in physics. However, we can clarify the statistical mechanical meaning of the thermodynamic quantities in AIT, which were originally introduced by Definition 3.2 in a rigorous manner.

4. The main result

In this paper, we investigate the properties of the sufficient conditions for $T$ to be a fixed point on partial randomness in Theorem 3.5. Using the monotonicity and continuity of the functions $Z_V(T)$ and $F_V(T)$ on temperature $T$ and the statistical mechanical relation $F_V(T) = -T \log_2 Z_V(T)$, which holds from Definition 3.2, we can show the following theorem for the sufficient conditions in Theorem 3.5.

**Theorem 4.1 (Tadaki 2012).** Let $V$ be an optimal prefix-free machine. Then each of the sets $\mathcal{Z}(V)$ and $\mathcal{F}(V)$ is dense in $(0, 1)$ while $\mathcal{Z}(V) \cap \mathcal{F}(V) = \emptyset$.

Thus, for every optimal prefix-free machine $V$, the computability of $F_V(T)$ gives completely different fixed points from the computability of $Z_V(T)$. This also implies that $\mathcal{Z}(V) \subseteq \mathcal{FP}_w$ and $\mathcal{F}(V) \subseteq \mathcal{FP}_w$.

The aim of this paper is to investigate the structure of $\mathcal{FP}_w$ and $\mathcal{FP}$ in greater detail. Specifically, we will show in Theorem 4.2 that there are infinitely many optimal prefix-free machines that give completely different sufficient conditions in each of the thermodynamic quantities in AIT. We say that an infinite sequence $V_1, V_2, V_3, \ldots$ of prefix-free machines is recursive if there exists a partial recursive function $F: \mathbb{N}^+ \times \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that for each $n \in \mathbb{N}^+$ the following two conditions hold:

(i) $p \in \text{dom } V_n$ if and only if $(n, p) \in \text{dom } F$, and
(ii) $V_n(p) = F(n, p)$ for every $p \in \text{dom } V_n$.

Then the main result of this paper is given as follows.
Theorem 4.2 (Main result). There exists a recursive infinite sequence $V_1, V_2, V_3, \ldots$ of optimal prefix-free machines which satisfies the following conditions:

(i) For all $i, j$ with $i \neq j$,

$$Z(V_i) \cap Z(V_j) = \emptyset, \quad F(V_i) \cap F(V_j) = \emptyset, \quad E(V_i) \cap E(V_j) = \emptyset, \quad S(V_i) \cap S(V_j) = \emptyset.$$ 

(ii) $\bigcup_i Z(V_i) \subset F$ and $\bigcup_i F(V_i) \subset F$.

(iii) $\bigcup_i E(V_i) \subset F$ and $\bigcup_i S(V_i) \subset F$.

In the following sections we will prove the above theorems by introducing into AIT the notion of the composition of prefix-free machines, which corresponds to the notion of the composition of systems in normal statistical mechanics.

5. Composition of prefix-free machines

Definition 5.1 (Composition of prefix-free machines).
Let $M_1, M_2, \ldots, M_N$ be prefix-free machines. The composition

$$M_1 \otimes M_2 \otimes \cdots \otimes M_N$$

of $M_1, M_2, \ldots, M_N$ is defined as the prefix-free machine $D$ such that

(i) $\text{dom} \ D = \{p_1 p_2 \ldots p_N \mid p_1 \in \text{dom} \ M_1 \amp p_2 \in \text{dom} \ M_2 \amp \cdots \amp p_N \in \text{dom} \ M_N\}$; and

(ii) $D(p_1 p_2 \ldots p_N) = M_1(p_1)$ for every $p_1 \in \text{dom} \ M_1, p_2 \in \text{dom} \ M_2, \ldots, \text{and} \ p_N \in \text{dom} \ M_N$.

Theorem 5.2. Let $M_1, M_2, \ldots, M_N$ be prefix-free machines. If $M_1$ is optimal, then $M_1 \otimes M_2 \otimes \cdots \otimes M_N$ is also optimal.

Proof. We first choose particular strings $r_2, r_3, \ldots, r_N$ with $r_2 \in \text{dom} \ M_2, r_3 \in \text{dom} \ M_3, \ldots$, and $r_N \in \text{dom} \ M_N$. Let $M$ be an arbitrary prefix-free machine. Then, by the definition of the optimality of $M_1$, there exists $d \in \mathbb{N}$ with the property that for every $p \in \text{dom} \ M$ there exists $q \in \{0,1\}^*$ for which $M_1(q) = M(p)$ and $|q| \leq |p| + d$. It follows from the definition of the composition $M_1 \otimes M_2 \otimes \cdots \otimes M_N$ that for every $p \in \text{dom} \ M$ there exists $q \in \{0,1\}^*$ for which

$$(M_1 \otimes M_2 \otimes \cdots \otimes M_N)(qr_2 r_3 \ldots r_N) = M(p)$$

and

$$|qr_2 r_3 \ldots r_N| \leq |p| + |r_2 r_3 \ldots r_N| + d.$$ 

Thus $M_1 \otimes M_2 \otimes \cdots \otimes M_N$ is an optimal prefix-free machine.

We can prove the following theorem for the thermodynamic quantities in AIT in the same way as in normal statistical mechanics. In particular, the equations (9), (10) and (11) correspond to the fact that free energy, energy and entropy, respectively, are extensive parameters in thermodynamics.
Theorem 5.3. Let $M_1, M_2, \ldots, M_N$ be prefix-free machines. Then the following hold for every $T \in (0, 1)$:

\begin{align*}
Z_{M_1 \oplus \cdots \oplus M_N}(T) &= Z_{M_1}(T) \cdots Z_{M_N}(T), \\
F_{M_1 \oplus \cdots \oplus M_N}(T) &= F_{M_1}(T) + \cdots + F_{M_N}(T), \\
E_{M_1 \oplus \cdots \oplus M_N}(T) &= E_{M_1}(T) + \cdots + E_{M_N}(T), \\
S_{M_1 \oplus \cdots \oplus M_N}(T) &= S_{M_1}(T) + \cdots + S_{M_N}(T).
\end{align*}

Proof. Suppose $T \in (0, 1)$. First, the proof of (8) is given as follows:

\[
Z_{M_1 \oplus \cdots \oplus M_N}(T) = \sum_{p \in \text{dom}(M_1 \oplus \cdots \oplus M_N)} 2^{-|p|/T} = \sum_{p_1 \in \text{dom} M_1} \cdots \sum_{p_N \in \text{dom} M_N} 2^{-|p_1 \cdots p_N|/T} = \left( \sum_{p_1 \in \text{dom} M_1} 2^{-|p_1|/T} \right) \cdots \left( \sum_{p_N \in \text{dom} M_N} 2^{-|p_N|/T} \right) = Z_{M_1}(T) \cdots Z_{M_N}(T).
\]

Since $F_M(T) = -T \log_2 Z_M(T)$ for every prefix-free machine $M$, (9) follows immediately from (8). To show (10), first note that

\[
E_M(T) = \frac{1}{Z_M(T)} \sum_{p \in \text{dom} M} |p| 2^{-|p|/T}
\]  

for every prefix-free machine $M$. On the other hand, we see that

\[
\sum_{p_1 \in \text{dom} M_1} \cdots \sum_{p_N \in \text{dom} M_N} |p_1 \cdots p_N| 2^{-|p_1 \cdots p_N|/T} = \sum_{p_1 \in \text{dom} M_1} \cdots \sum_{p_N \in \text{dom} M_N} \left( |p_1| + \cdots + |p_N| \right) 2^{-|p_1 \cdots p_N|/T} = \left( \sum_{p_1 \in \text{dom} M_1} \cdots \sum_{p_N \in \text{dom} M_N} |p_1| 2^{-|p_1 \cdots p_N|/T} \right) + \cdots
\]

\[
+ \left( \sum_{p_1 \in \text{dom} M_1} \cdots \sum_{p_N \in \text{dom} M_N} |p_N| 2^{-|p_1 \cdots p_N|/T} \right) = \left( \sum_{p_1 \in \text{dom} M_1} |p_1| 2^{-|p_1|/T} \right) Z_{M_2}(T) \cdots Z_{M_N}(T) + \cdots
\]

\[
+ Z_{M_1}(T) \cdots Z_{M_{N-1}}(T) \left( \sum_{p_N \in \text{dom} M_N} |p_N| 2^{-|p_N|/T} \right).
\]
Thus, (10) follows from (12), (8) and the above. Finally, since
\[ S_M(T) = \frac{(E_M(T) - F_M(T))}{T} \]
for every prefix-free machine \( M \), (11) follows immediately from (10) and (9).

For any prefix-free machine \( M \) and any \( n \in \mathbb{N}^+ \), we use \( M \otimes^n \) to denote the prefix-free machine \( \underbrace{M \otimes \cdots \otimes M}_{n} \).

6. The proof of the main result

In order to prove the main result, Theorem 4.2, we will also introduce the notion of a physically reasonable prefix-free machine, which is motivated by the fact that a real quantum system has at least two energy levels.

**Definition 6.1 (Physically reasonable prefix-free machine).** For any prefix-free machine \( M \), we say that \( M \) is physically reasonable if there exist \( p, q \in \text{dom } M \) such that \( |p| \neq |q| \).

Note that every optimal prefix-free machine is physically reasonable.

**Theorem 6.2.** Let \( M \) be a physically reasonable prefix-free machine. Then each of the mappings

\[
\begin{align*}
(0,1) \ni T &\mapsto Z_M(T) \\
(0,1) \ni T &\mapsto E_M(T) \\
(0,1) \ni T &\mapsto S_M(T)
\end{align*}
\]
is an increasing and continuous real function. On the other hand, the mapping

\[
(0,1) \ni T \mapsto F_M(T)
\]
is a decreasing and continuous real function.

**Proof.** Let \( M \) be an arbitrary physically reasonable prefix-free machine. Since the mapping \( (0,1) \ni T \mapsto 2^{-L/T} \) is an increasing real function for each \( L \in \mathbb{N}^+ \), the mapping \( (0,1) \ni T \mapsto Z_M(T) \) is an increasing real function. Tadaki (2002, Theorem 3.3 (a)) shows that the mapping \( (0,1) \ni T \mapsto Z_V(T) \) is a function of class \( C^\infty \) and therefore continuous for every optimal prefix-free machine \( V \). We can also show in the same way that the mapping \( (0,1) \ni T \mapsto Z_M(T) \) is continuous.

To prove the remaining results, the following ‘thermodynamic’ relations (i) and (ii), can be proved in the same way parts (ii) and (iii), respectively, of Tadaki (2012, Theorem 4.4).

(i) \( F'_M(T) = -S_M(T), \ E'_M(T) = C_M(T), \) and \( S'_M(T) = C_M(T)/T \) for every \( T \in (0,1) \).

(ii) \( S_M(T), C_M(T) > 0 \) for every \( T \in (0,1) \).

Here \( C_M(T) \) is one of the thermodynamic quantities of AIT, called the specific heat at temperature \( T \), introduced in Tadaki (2008), in addition to the thermodynamic quantities of AIT given in Definition 3.2. The remaining results then follow immediately from the thermodynamic relations (i) and (ii). \( \square \)
In order to prove the main result, it is also convenient to use the notion of a computable measure machine, which was introduced in Downey and Griffiths (2004) to characterise the notion of Schnorr randomness of a real in terms of program-size complexity.

**Definition 6.3 (Computable measure machine (Downey and Griffiths 2004)).** A prefix-free machine \( M \) is called a computable measure machine if \( \sum_{p \in \text{dom } M} 2^{-|p|} \) (that is, \( Z_M(1) \)) is computable.

For the thermodynamic quantities in AIT, we can prove the following theorem using Theorem 6.2 and Lemma 6.5, which follow it.

**Theorem 6.4.** Let \( M \) be a physically reasonable, computable measure machine. Then, for every \( T \in (0, 1) \), the following conditions are equivalent:

(i) \( T \) is computable.
(ii) At least, one of \( Z_M(T) \), \( F_M(T) \), \( E_M(T) \) and \( S_M(T) \) is computable.
(iii) All of \( Z_M(T) \), \( F_M(T) \), \( E_M(T) \) and \( S_M(T) \) are computable.

**Lemma 6.5.** Let \( M \) be a computable measure machine. Then the following hold.

(i) There exists a partial recursive function \( f : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q} \) such that for every \( r \in \mathbb{Q} \cap (0, 1) \) and every \( n \in \mathbb{N} \), we have

\[
|f(r, n) - Z_M(r)| < 2^{-n}.
\]

The same holds for \( F_M(T) \).

(ii) For every \( T_1, T_2 \in \mathbb{Q} \) with \( 0 < T_1 < T_2 < 1 \), there exists a partial recursive function \( f : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q} \) such that for every \( r \in \mathbb{Q} \cap [T_1, T_2] \) and every \( n \in \mathbb{N} \), we have

\[
|f(r, n) - E_M(r)| < 2^{-n}.
\]

The same holds for \( S_M(T) \).

**Proof.** Let \( M \) be a computable measure machine. The results are obvious if \( \text{dom } M \) is a finite set, so we assume now that \( \text{dom } M \) is an infinite set. Let \( p_1, p_2, p_3, p_4, \ldots \) be a particular recursive enumeration of \( \text{dom } M \).

(i) Consider the following procedure:

Given \( r \in \mathbb{Q} \cap (0, 1) \) and \( n \in \mathbb{N} \), find \( k_0 \in \mathbb{N} \) such that

\[
Z_M(1) - Z_{k_0}(1) < 2^{-n-1}.
\]  \hspace{1cm} (13)

This is possible since \( Z_M(1) \) is computable and \( \lim_{k \to \infty} Z_k(1) = Z_M(1) \). We then calculate \( z \in \mathbb{Q} \) such that

\[
|z - Z_{k_0}(r)| < 2^{-n-1}.
\]  \hspace{1cm} (14)

This is also possible since \( r \in \mathbb{Q} \). Finally, we output \( z \).
It follows from (14) and (13) that
\[ |z - Z_M(r)| \leq |z - Z_{k_0}(r)| + \sum_{i=k_0+1}^{\infty} 2^{-|p_i|/r} < 2^{-n-1} + \sum_{i=k_0+1}^{\infty} 2^{-|p_i|} < 2^{-n-1} + 2^{-n-1} = 2^{-n}. \]

Thus, there exists a partial recursive function \( f : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q} \) such that for every \( r \in \mathbb{Q} \setminus (0, 1) \) and every \( n \in \mathbb{N} \), we have \( |f(r, n) - Z_M(r)| < 2^{-n} \). Since, from Definition 3.2, we have \( F_M(T) = -T \log_2 Z_M(T) \) for every \( T \in (0, 1] \), it is easy to show that there exists a partial recursive function \( f : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q} \) such that, for every \( r \in \mathbb{Q} \setminus (0, 1] \) and every \( n \in \mathbb{N} \), we have \( |f(r, n) - F_M(r)| < 2^{-n} \).

(ii) Let \( T_1, T_2 \in \mathbb{Q} \) with \( 0 < T_1 < T_2 < 1 \). First, it is easy to show that, for every \( k \in \mathbb{N}^+ \) and every \( T \in (0, 1) \),
\[ E_{k+1}(T) - E_k(T) = \frac{Z_k(T)|p_{k+1}| - W_k(T)}{Z_{k+1}(T)Z_k(T)} 2^{-|p_{k+1}|/r}, \]
where
\[ W_k(T) = \sum_{i=1}^{k} |p_i| 2^{-|p_i|/r}. \]
On the one hand, from (15) we have that for every \( k \in \mathbb{N}^+ \) and every \( T \in [T_1, T_2] \),
\[ E_{k+1}(T) - E_k(T) \geq \frac{Z_k(T_1)|p_{k+1}| - W_k(T_2)}{Z_{k+1}(T_2)Z_k(T_2)} 2^{-|p_{k+1}|/r}. \]
Also, since \( Z_k(T_1), Z_k(T_2) \) and \( W_k(T_2) \) converge as \( k \to \infty \) and \( \lim_{k \to \infty} |p_k| = \infty \), there exists \( k_1 \in \mathbb{N}^+ \) such that for every \( k \geq k_1 \) and every \( T \in [T_1, T_2] \),
\[ E_{k+1}(T) - E_k(T) > 0. \]
On the other hand, it follows from (15) that for every \( k \in \mathbb{N}^+ \) and every \( T \in [T_1, T_2] \),
\[ E_{k+1}(T) - E_k(T) < \frac{Z_k(T_2)|p_{k+1}|}{Z_{k+1}(T_1)Z_k(T_1)} 2^{-|p_{k+1}|/r_2}. \]
Since \( Z_k(T_1) \) and \( Z_k(T_2) \) converge as \( k \to \infty \), there exists \( a \in \mathbb{N} \) such that for every \( k \in \mathbb{N}^+ \) and every \( T \in [T_1, T_2] \),
\[ E_{k+1}(T) - E_k(T) < |p_{k+1}| 2^{-|p_{k+1}|/r_2 + a}. \]
Since \( T_2 < 1 \), we have \( l2^{-l/2_2+a} \leq 2^{-l} \) for all sufficiently large \( l \in \mathbb{N} \). So it follows from \( \lim_{k \to \infty} |p_k| = \infty \) that there exists \( k_2 \in \mathbb{N}^+ \) such that for every \( k \geq k_2 \) and every \( T \in [T_1, T_2] \),
\[ E_{k+1}(T) - E_k(T) < 2^{-|p_{k+1}|}. \]
Hence, by (16) and (17), we see that for every \( k \geq k_3 \) and every \( T \in [T_1, T_2] \),

\[
|E_M(T) - E_k(T)| < \sum_{i=k+1}^{\infty} 2^{-|p_i|},
\]

(18)

where \( k_3 = \max\{k_1, k_2\} \).

Now consider the following procedure:

Given \( r \in \mathbb{Q} \cap [T_1, T_2] \) and \( n \in \mathbb{N} \), we first find \( k_0 \) such that

\[
Z_M(1) - Z_{k_0}(1) < 2^{-n-1},
\]

(19)

and then calculate \( e \in \mathbb{Q} \) such that

\[
|e - E_{k_0}(r)| < 2^{-n-1}.
\]

(20)

This is possible since \( r \in \mathbb{Q} \). Finally, we output \( e \).

It follows from (20), (18) and (19) that

\[
|e - E_M(r)| \leq |e - E_{k_0}(r)| + |E_M(r) - E_{k_0}(r)|
\]

\[
< 2^{-n-1} + \sum_{i=k_0+1}^{\infty} 2^{-|p_i|}
\]

\[
< 2^{-n-1} + 2^{-n-1} = 2^{-n}.
\]

Thus, there exists a partial recursive function \( f : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q} \) such that for every \( r \in \mathbb{Q} \cap [T_1, T_2] \) and every \( n \in \mathbb{N} \), we have \( |f(r, n) - E_M(r)| < 2^{-n} \). Since, from Definition 3.2,

\[
S_M(T) = \frac{E_M(T) - F_M(T)}{T}
\]

for every \( T \in (0, 1) \), it follows from Lemma 6.5 (i) that there exists a partial recursive function \( f : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q} \) such that for every \( r \in \mathbb{Q} \cap [T_1, T_2] \) and every \( n \in \mathbb{N} \), we have \( |f(r, n) - S_M(r)| < 2^{-n} \). \( \square \)

**Proof of Theorem 6.4.** Let \( M \) be a physically reasonable, computable measure machine.

and let \( T \in (0, 1) \). We choose particular \( T_1, T_2 \in \mathbb{Q} \) with \( 0 < T_1 < T < T_2 < 1 \). It follows from Theorem 6.5 (ii) that there exist partial recursive functions \( c : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q} \) and \( d : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q} \) such that:

- \( c(r, n) \leq E_M(r) \leq d(r, n) \) for every \( r \in \mathbb{Q} \cap [T_1, T_2] \) and every \( n \in \mathbb{N} \); and
- \( \lim_{n \to \infty} c(r, n) = \lim_{n \to \infty} d(r, n) = E_M(r) \) for every \( r \in \mathbb{Q} \cap [T_1, T_2] \).

We now prove the equivalence of the three conditions in Theorem 6.4:

**(iii) ⇒ (ii):**

This is obvious.

**(i) ⇒ (iii):**

Suppose \( T \) is computable. We will show that \( E_M(T) \) is computable; we can use the same reasoning to show that \( Z_M(T), F_M(T) \) and \( S_M(T) \) are also computable.
Now, since $T$ is computable, there exist total recursive functions $a : \mathbb{N} \to \mathbb{Q}$ and $b : \mathbb{N} \to \mathbb{Q}$ such that:
- $T_1 \leq a(n) < T < b(n) \leq T_2$ for all $n \in \mathbb{N}$;
- $\lim_{n \to \infty} a(n) = \lim_{n \to \infty} b(n) = T$.

Since the mapping $(0,1) \ni t \mapsto E_M(t)$ is an increasing and continuous real function by Theorem 6.2, it is then easy to see that, given $k \in \mathbb{N}$, we can perform an exhaustive search and find $m,n \in \mathbb{N}$ such that:
- $d(a(m),n) < c(b(m),n)$; and
- $|d(b(m),n) - c(a(m),n)| < 2^{-k}$.

Since the mapping $(0,1) \ni t \mapsto E_M(t)$ is an increasing real function by Theorem 6.2 again, we see that

$$c(a(m),n) < E_M(T) < d(b(m),n)$$

and therefore

$$|E_M(T) - c(a(m),n)| < 2^{-k}.$$ 

Thus, there exists a total recursive function $f : \mathbb{N} \to \mathbb{Q}$ such that

$$|E_M(T) - f(k)| < 2^{-k}$$

for all $k \in \mathbb{N}$. Hence, $E_M(T)$ is computable.

(ii) $\Rightarrow$ (i):

We will show that $T$ is computable by assuming the computability of $E_M(T)$; we can use the same reasoning to show that $T$ is computable by assuming the computability of each of $Z_M(T)$, $F_M(T)$ and $S_M(T)$.

So we assume that $E_M(T)$ is computable. Then there exist total recursive functions $a : \mathbb{N} \to \mathbb{Q}$ and $b : \mathbb{N} \to \mathbb{Q}$ such that:
- $a(n) \leq E_M(T) \leq b(n)$ for all $n \in \mathbb{N}$; and
- $\lim_{n \to \infty} a(n) = \lim_{n \to \infty} b(n) = E_M(T)$.

Since the mapping $(0,1) \ni t \mapsto E_M(t)$ is an increasing real function by Theorem 6.2, it is then easy to see that, given $k \in \mathbb{N}$, we can perform an exhaustive search and find $r_1, r_2 \in \mathbb{Q}$ and $n \in \mathbb{N}$ such that:
- $T_1 \leq r_1 < r_2 \leq T_2$;
- $|r_2 - r_1| < 2^{-k}$;
- $d(r_1, n) \leq a(n)$; and
- $b(n) \leq c(r_2, n)$.

Since the mapping $(0,1) \ni t \mapsto E_M(t)$ is an increasing real function by Theorem 6.2 again, we see that $r_1 \leq T \leq r_2$ and therefore $|T - r_1| < 2^{-k}$. Thus, there exists a total recursive function $f : \mathbb{N} \to \mathbb{Q}$ such that $|T - f(k)| < 2^{-k}$ for all $k \in \mathbb{N}$. Hence, $T$ is computable.

Example 6.6. The following two prefix-free machines are examples of physically reasonable, computable measure machines.
Two-level system: Let $B$ be a particular prefix-free machine for which $\text{dom } B = \{1, 01\}$.
Then we see that, for every $T > 0$,
\begin{align*}
Z_B(T) &= 2^{-1/T} + 2^{-2/T} \\
F_B(T) &= -T \log_2 Z_B(T) \\
E_B(T) &= \frac{1}{Z_B(T)}(2^{-1/T} + 2 \cdot 2^{-2/T}) \\
S_B(T) &= (E_B(T) - F_B(T))/T.
\end{align*}

One-dimensional harmonic oscillator: Let $O$ be a particular prefix-free machine for which $\text{dom } O = \{0^l1 \mid l \in \mathbb{N}\}$.
Then we see that, for every $T > 0$,
\begin{align*}
Z_O(T) &= \frac{1}{2^{1/T} - 1} \\
F_O(T) &= T \log_2(2^{1/T} - 1) \\
E_O(T) &= \frac{2^{1/T}}{2^{1/T} - 1} \\
S_O(T) &= (E_O(T) - F_O(T))/T.
\end{align*}

Since $Z_B(1) = 3/4$ and $Z_O(1) = 1$, we see that $B$ and $O$ are physically reasonable, computable measure machines.

Before proving the main result of the paper, Theorem 4.2, we will prove a more general result in the form of Theorem 6.7, which is based on Theorems 5.3 and 6.4. The main result then follows immediately by applying Theorem 6.7 to a particular optimal prefix-free machine $V$ and physically reasonable, computable measure machine $M$.

**Theorem 6.7.** Let $V$ be an optimal prefix-free machine and $M$ be a physically reasonable, computable measure machine. For each $n \in \mathbb{N}^+$, we use $V_n$ to denote the prefix-free machine $V \ominus (M^\ominus n)$. Then the following hold:

(i) The infinite sequence $V_1, V_2, V_3, \ldots$ of prefix-free machines is a recursive infinite sequence of optimal prefix-free machines.

(ii) For all $i, j$ with $i \neq j$,
\begin{align*}
\mathcal{X}(V_i) \cap \mathcal{X}(V_j) &= \emptyset, \\
\mathcal{F}(V_i) \cap \mathcal{F}(V_j) &= \emptyset, \\
\mathcal{E}(V_i) \cap \mathcal{E}(V_j) &= \emptyset, \\
\mathcal{S}(V_i) \cap \mathcal{S}(V_j) &= \emptyset.
\end{align*}

(iii) $\bigcup_i \mathcal{X}(V_i) \subset \mathcal{F}_w$ and $\bigcup_i \mathcal{F}(V_i) \subset \mathcal{F}_w$.

(iv) $\bigcup_i \mathcal{E}(V_i) \subset \mathcal{F}$ and $\bigcup_i \mathcal{S}(V_i) \subset \mathcal{F}$.

**Proof.**

(i) Since $V$ is optimal, by Theorem 5.2, $V_n$ is optimal for every $n \in \mathbb{N}^+$. Furthermore, it is easy to see that the infinite sequence $V_1, V_2, V_3, \ldots$ of prefix-free machines is recursive.
(ii) From Theorem 5.3, for every $n \in \mathbb{N}^+$ and every $T \in (0,1)$, we have

\begin{align}
Z_{V_n}(T) &= Z_{V}(T)Z_{M}(T)^n \\
F_{V_n}(T) &= F_{V}(T) + nF_{M}(T) \\
E_{V_n}(T) &= E_{V}(T) + nE_{M}(T) \\
S_{V_n}(T) &= S_{V}(T) + nS_{M}(T).
\end{align}

(21)

Let $m$ and $n$ be arbitrary two positive integers with $m > n$. Then it follows from the equations (21) that

\begin{align}
Z_{V_m}(T) &= Z_{V_n}(T)Z_{M}(T)^{m-n} \\
F_{V_m}(T) &= F_{V_n}(T) + (m-n)F_{M}(T) \\
E_{V_m}(T) &= E_{V_n}(T) + (m-n)E_{M}(T) \\
S_{V_m}(T) &= S_{V_n}(T) + (m-n)S_{M}(T)
\end{align}

(22) (23) (24) (25)

for every $T \in (0,1)$. We will now use (23) to show that

$$\mathcal{F}(V_m) \cap \mathcal{F}(V_n) = \emptyset;$$

we can use similar reasoning using (22), (24) and (25) to show that

$$\mathcal{L}(V_m) \cap \mathcal{L}(V_n) = \emptyset; \mathcal{E}(V_m) \cap \mathcal{E}(V_n) = \emptyset; \mathcal{I}(V_m) \cap \mathcal{I}(V_n) = \emptyset$$

too.

We assume to show a contradiction that $\mathcal{F}(V_m) \cap \mathcal{F}(V_n) \neq \emptyset$. Then there exists $T_0 \in (0,1)$ such that both $F_{V_m}(T_0)$ and $F_{V_n}(T_0)$ are computable. It follows from (23) that

$$F_{M}(T_0) = \frac{1}{m-n} \left( F_{V_m}(T_0) - F_{V_n}(T_0) \right).$$

Thus, $F_{M}(T_0)$ is also computable. Since $M$ is a physically reasonable, computable measure machine, it follows from the implication (ii) $\Rightarrow$ (i) of Theorem 6.4 that $T_0$ is also computable. Therefore, since $V_m$ is optimal, it follows from Theorem 3.3 (i) that $F_{V_m}(T_0)$ is weakly Chaitin $T_0$-random. However, this contradicts the fact that $F_{V_m}(T_0)$ is computable. Thus we have $\mathcal{F}(V_m) \cap \mathcal{F}(V_n) = \emptyset$.

(iii) This follows immediately from Theorem 3.5 and the fact that $V_i$ is optimal for all $i \in \mathbb{N}^+$.

(iv) This follows immediately from Theorem 3.5 and the fact that $V_i$ is optimal for all $i \in \mathbb{N}^+$.  

7. Concluding remarks

As a sequel to our earlier work, in this paper we have further developed the statistical mechanical interpretation of AIT and pursued its formal correspondence with normal statistical mechanics. In particular, we have investigated the structure of the set of fixed points on partial randomness in greater detail. We did this by introducing into AIT the notion of the composition of prefix-free machines, which corresponds to the notion of the composition of systems in normal statistical mechanics.
In the definition of the composition $M_1 \circ M_2 \circ \cdots \circ M_N$ of prefix-free machines $M_1, M_2 \ldots M_N$, we gave the machine $M_1$ a special role relative to all other prefix-free machines (see Condition (ii) of Definition 5.1). However, all systems are dealt with on a completely equal basis in the original notion of the composition of systems in normal statistical mechanics. Thus, in order to realise a complete formal correspondence with normal statistical mechanics, it would be desirable to modify the definition of the composition of prefix-free machines so that all prefix-free machines $M_1, M_2 \ldots M_N$ play a completely equal role in the definition. The efforts required in seeking such a proper definition may well result in a further significant development of the statistical mechanical interpretation of AIT.

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