CAMINA $p$-GROUPS THAT ARE GENERALIZED FROBENIUS COMPLEMENTS

by

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ABSTRACT

Let $P$ be a Camina $p$-group that acts on a group $Q$ in such a way that $C_P(x) \subseteq P'$ for all nonidentity elements $x \in Q$. We show that $P$ must be isomorphic to the quaternion group $Q_8$. If $P$ has class 2, this is a known result, and this paper corrects a previously published erroneous proof of the general case.

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Let $G$ be a finite group that is neither perfect nor abelian, and recall that $G$ is said to be a **Camina group** if every nontrivial coset of $G'$ is a conjugacy class of $G$, or equivalently, every nonlinear irreducible character of $G$ vanishes on $G - G'$. (That these conditions really are equivalent is immediate from the fact that for $x \in G$, we have $|C_G(x)| = \sum |\chi(x)|^2$, where the sum runs over $\chi \in \text{Irr}(G)$.)

The purpose of this note is to prove the following, which appeared with an incorrect proof as Theorem 2 of [3].

**THEOREM.** Let $P$ be a Camina $p$-group, and suppose that $P$ acts on a nontrivial group $Q$ in such a way that $C_P(x) \subseteq P'$ for all nonidentity elements $x \in Q$. Then $P$ is the quaternion group of order 8, and the action is Frobenius.

This theorem was used in [3] to give an alternative proof of a key step in the classification of Camina groups given in [1], where Dark and Scoppola proved that a Camina group must be either a $p$-group or a Frobenius group whose complement is either cyclic or $Q_8$. Unfortunately, as is explained in [3], the Dark-Scoppola proof ultimately relies on a flawed argument in [1]. Combining our result with Theorem 1 of [3], we now have what we hope is a correct (and simplified) proof of the Dark-Scoppola classification.

We begin with a fairly standard general result.

**LEMMA.** Let $A$ and $B$ be finite abelian groups, and suppose that there exists a non-degenerate bimultiplicative map $f : A \times B \to C$, where $C$ is a finite cyclic group. (This means that $f$ is a homomorphism in each variable and that the only elements $a \in A$ and $b \in B$ such that $f(a, B) = 1$ or $f(A, b) = 1$ are the identities of $A$ and $B$.) Then $A \cong B$.

**Proof.** Let $\mu$ be a faithful linear character of $C$. For each element $a \in A$, let $\lambda_a$ be the function on $B$ defined by $\lambda_a(b) = \mu(f(a, b))$. It is easy to check that the map $a \mapsto \lambda_a$ is an injective homomorphism from $A$ into the group $\hat{B}$ of linear characters of $B$. Then $|A| \leq |\hat{B}| = |B|$, and by symmetry, we have $|A| = |B|$. It follows that $A \cong \hat{B} \cong B$, as required.  

Next, we present a few easy results about Camina $p$-groups. Stronger versions of many of these are known, but they are scattered over a number of papers. (For example, see [1] and the references there.) Since the facts that we need can be established with elementary arguments, it seems reasonable to present the proofs here.

In the following, $P$ is always a Camina $p$-group.

**PROPOSITION 1.** $P/P'$ is elementary abelian.

**Proof.** Let $x \in P$. If $x^p \notin P'$, then $x \notin P'$, so $|P : C_P(x)| = |P'| = |P : C_P(x^p)|$, and thus $|C_P(x)| = |C_P(x^p)|$. Since $C_P(x) \subseteq C_P(x^p)$, we deduce that $C_P(x) = C_P(x^p)$. Now let $z \in Z(P)$ have order $p$, and note that $z \in P'$, so $x$ and $zx$ are conjugate. Then $x^t = zx$ for some element $t \in P$, and we have $(x^p)^t = (x^t)^p = (zx)^p = x^p$. It follows that $t \in C_P(x^p) = C_P(x)$, and this is a contradiction since $x^t = zx \neq x$. Thus $x^p \in P'$, as required.

**PROPOSITION 2.** Suppose $P$ has nilpotence class 2. Then $|P : P'| > |P'|$. 

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Proposition 2, we have into the cyclic group $\mathbb{Z}$ noncentral, we can choose since $P$ by Proposition 1, it contains both $x \in C_P(x)$. $\blacksquare$

¿From now on, we assume that $P$ has nilpotence class 3. (We mention that by the Theorem in [1], the nilpotence class of a Camina $p$-group cannot exceed 3. We will use this fact in the proof of our main result.)

PROPOSITION 3. $[P', P]$ is is elementary abelian.

In fact, more is true: $P'$ is elementary abelian. (See the remarks preceding and following Proposition 6, below.)

Proof of Proposition 3 Since $[P', P]$ is abelian and is generated by elements of the form $[u, x]$, where $u \in P'$ and $x \in P$, it suffices to show that $[u, x]^p = 1$. Since $[u, x]$ is central in $P$, we have $[u, x]^p = [u, x^p] = 1$, where the final equality holds since $P'$ is abelian, and by Proposition 1, it contains both $u$ and $x^p$. $\blacksquare$

In the following, $Z = Z(P)$ and $C = C_P(P')$, and we note that $Z \subseteq P' \subseteq C$.

PROPOSITION 4. Assume that $p > 2$. Then $C/Z$ is elementary abelian.

Actually, it is not really necessary to assume that $p > 2$ here or in Proposition 6. This is because by Theorem 3.1 of [4], Camina 2-groups never have nilpotence class exceeding 2. (This fact too will be used in our proof of the main result.)

Proof of Proposition 4. Let $c \in C$ and $x \in P$, and write $c^x = cu$ and $u^x = uv$, where $u \in P'$ and $v \in [P', P] \subseteq Z$. For positive integers $n$, it follows by induction that $c^{x^n} = cu^nv^n(n-1)/2$. Since $p > 2$ and $v^p = 1$ by Proposition 3, we have $c^{x^p} = cu^p$. Also $x^p \in P'$ by Proposition 1, and thus since $c \in C = C_P(P')$, we have $c^{x^p} = c$, and thus $u^p = 1$. Now $(c^x)^x = (c^x)^p = (cu)^p = c^p$, where the last equality holds because $c$ and $u$ commute and $u^p = 1$. Since $x \in P$ was arbitrary, it follows that $c^x \in Z$, and thus $C/Z$ has exponent $p$.

To see that $C/Z$ is abelian, note that $[P, C, C] \subseteq [P', C] = 1$, so it follows by the three-subgroups lemma that $[C', P] = 1$, and thus $C' \subseteq Z$. $\blacksquare$

PROPOSITION 5. Assume that $Z$ is cyclic. Then $|C : P'|$ is a square and $|C : P'| \geq p^2$.

Proof. Since $Z$ is cyclic, $P$ has a faithful irreducible character $\chi$, and we argue that $\chi$ vanishes $P - Z$, and thus $|P : Z| = \chi(1)^2$. To see this, observe that $\chi$ vanishes on $P - P'$ since $P$ is a Camina group, so it suffices to show that $\chi(x) = 0$ for $x \in P' - Z$. Since $x$ is noncentral, we can choose $t \in P$ with $x^t = xz$, where $z \in Z$ is some nonidentity element. Now $\chi_Z = \chi(1)\lambda$ for some faithful linear character $\lambda$ of $Z$, and thus $\chi(x) = \chi(x^t) = \chi(xz) = \lambda(z)\chi(x)$. Since $\lambda(z) \neq 1$, it follows that $\chi(x) = 0$, as wanted.

Now commutation defines a nondegenerate bimultiplicative map from $(P/C) \times (P'/Z)$ into the cyclic group $Z$, and thus $|P : C| = |P' : Z|$ by the lemma. Since $|P : Z|$ is a square, it follows that $|C : P'|$ is a square. Also, $P/Z$ is a class 2 Camina group, so by Proposition 2, we have $|P : P'| > |P' : Z| = |P : C|$, and thus $C > P'$, and we have $|C : P'| \geq p^2$. $\blacksquare$
PROPOSITION 6. Assume that $Z$ is cyclic and $p > 2$. Then $P'$ is elementary abelian.

As we have remarked, the assumption that $p > 2$ is redundant here. In fact, as we will explain following the proof, the assumption that $Z$ is cyclic is not really needed either.

Proof of Proposition 6. By Proposition 5, we can choose an element $c \in C - P'$. Now let $u \in P'$ be arbitrary, and choose an element $t \in P$ such that $c^t = uc$. Since $c^p \in Z$ by Proposition 4, we have $c^p = (c^t)^t = (c^t)^p = (uc)^p = u^p c^p$, where the last equality holds because $c$ and $u$ commute. Then $u^p = 1$, as required.

To see why it is not really necessary to assume that $Z$ is cyclic in Proposition 6, observe that it suffices to show for each character $\chi \in \text{Irr}(G)$ that the group $(P/\ker(\chi))'$ is elementary abelian. Now $P/\ker(\chi)$ is the centralizer of the action of $H$ on $Q$, and as such, $Q$ is either abelian, a Camina group of class 2 or a Camina group of class 3. If it is abelian, there is nothing to prove and if it has class 3, its derived subgroup is elementary by Proposition 6. Finally, it is easy to see using Proposition 1 that class 2 Camina $p$-groups always have elementary abelian derived subgroups.

Proof of Theorem. We proceed by induction on $|P|$. Observe first that the hypothesis guarantees that $P$ centralizes no nonidentity element of $Q$, and thus $|Q| \equiv 1 \mod p$, and hence $Q$ is a $p'$-group. Since $P$ is a Camina $p$-group, its nilpotence class is at least 2. If $P$ has class 2, the result follows by Lemma 3.1 of [2], so we can assume that the class of $P$ is at least 3. Since Camina 2-groups have nilpotence class 2 by Theorem 3.1 of [4], we deduce that $p > 2$, and we work to obtain a contradiction. Also, by the Theorem in [1], Camina $p$-groups have class at most 3, and hence the class of $P$ must be 3, exactly.

Since we can replace $Q$ by a nontrivial $P$-invariant subgroup, we can assume that $Q$ has no proper nontrivial $P$-invariant subgroup. It follows that $Q$ is an elementary abelian $q$-group for some prime $q \neq p$, and thus we can view $Q$ as an irreducible $F[P]$-module, where $F$ is the field of order $q$.

Let $K$ be the centralizer of the action of $G$ in the endomorphism ring of $Q$. Then $K$ is a finite division ring, and hence by Wedderburn’s theorem, $K$ is a field. We can view $Q$ as a vector space over $K$, and as such, $Q$ is an absolutely irreducible $K[P]$-module. (We stress that we have not changed $Q$; what has changed is our point of view.) Let $\chi$ be the irreducible $q$-Brauer character of $P$ corresponding to the absolutely irreducible $K[P]$-module $Q$, and observe that in fact, $\chi \in \text{Irr}(G)$ since $q$ does not divide $|P|$. We argue now that $\chi$ is faithful. Otherwise, there exists a nontrivial central subgroup $U \subseteq \ker(\chi)$, and hence $U$ acts trivially on $Q$. Also, $P/U$ is nonabelian since $P$ has class 3, and it follows that $P/U$ is a Camina $p$-group, so we can apply the inductive hypothesis to the action of $P/U$ on $Q$. Then $P/U$ has order 8, and this is a contradiction since we have established that $p > 2$.

Next, we show that no element of $P$ outside of $P'$ can have order $p$. To see this, suppose that $H \subseteq P$, where $|H| = p$ and $H \cap P' = 1$. Since $P$ is a Camina group, $\chi$ vanishes on the nonidentity elements of $H$, and hence $\chi_H$ has a principal constituent. The action of $H$ on $Q$ is completely reducible, however, and it follows that $H$ has nontrivial fixed points in $Q$, and hence by hypothesis, $H \subseteq P'$, and this is a contradiction.

Let $C = C_G(P')$ and $Z = Z(P)$, and note that $Z$ is cyclic since $\chi$ is faithful. Also $Z \subseteq P'$, and thus $|Z| = p$ by Proposition 6. If $c \in C - P'$, then $c^p \in Z$ by Proposition 4,
and since $c$ does not have order $p$, we deduce that $c^p$ is a generator of $Z$.

Now $C/P'$ is elementary abelian by Proposition 1, and it has order at least $p^2$ by Proposition 5, and thus we can choose two elements $b, c \in C$ that generate distinct subgroups of order $p$ modulo $P'$. Then $b^p$ and $c^p$ are generators of $Z$, and we can replace $c$ by a suitable power and assume that in fact, $b^p = c^{-p}$. Now write $[b, c] = z$, so $z \in Z$ by Proposition 4. Then $(bc)^p = b^p c^p z^{p(p-1)/2} = b^p c^p = 1$ because $p > 2$ and $z^p = 1$. Since $b$ and $c$ generate different subgroups of order $p$ in $C/Z$, it follows that $bc \notin P'$, and since $bc$ has order $p$, this is our final contradiction.

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