Characteristic polynomials of modified permutation matrices at microscopic scale
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Abstract
We study the characteristic polynomial of random permutation matrices following some measures which are invariant by conjugation, including Ewens’ measures which are one-parameter deformations of the uniform distribution on the permutation group. We also look at some modifications of permutation matrices where the entries equal to one are replaced by i.i.d uniform variables on the unit circle. Once appropriately normalized and scaled, we show that the characteristic polynomial converges in distribution on every compact subset of $\mathbb{C}$ to an explicit limiting entire function, when the size of the matrices goes to infinity. Our findings can be related to results by Chhaibi, Najnudel and Nikeghbali on the limiting characteristic polynomial of the Circular Unitary Ensemble [4].

1 Introduction
1.1 Convergence of characteristic polynomials
Characteristic polynomials of random matrices have drawn much interest the last few decades. These objects encode the information of the whole spectrum of matrices. Moreover, in the case of unitarily invariant matrices (as Gaussian Unitary Ensemble or Circular Unitary Ensemble), the characteristic polynomial is believed to have a similar microscopic behavior as holomorphic functions which appear in number theory, as the Riemann zeta function. The characteristic polynomial of random matrices is also related to Gaussian fields, including the Gaussian multiplicative chaos introduced by Kahane [10].

On the macroscopic scale, Keating and Snaith [11], Hugues Keating and O-Connell [9], and then Bourgade Hugues Nikeghbali and Yor [1] study the logarithm of characteristic polynomial of unitary matrices following the Haar distribution, and prove in particular that its real and imaginary parts normalized by $\sqrt{\log n}$ converge jointly in law to independent centred and reduced Gaussian random variables. Hambly, Keevash, O-Connell and Stark [8] give a similar result for permutation matrices following the uniform measure. Zeindler [19] [20] generalizes this result for permutation matrices under Ewens measures, considering more general class functions than the characteristic polynomial, the so-called multiplicative class functions. Dehaye and Zeindler [6], and Dang and Zeindler [5] extend the study to some Weyl groups, and some wreath products involving the symmetric group.

On the microscopic scale, Chhaibi, Najnudel and Nikeghbali [4] show that the characteristic polynomial of unitary matrices following the Haar measure, suitably renormalized, converges to a limiting entire function. With the coupling of virtual isometries introduced by Bourgade, Najnudel and Nikeghbali [2], the authors get an almost sure convergence. Chhaibi, Hovhannisyan, Najnudel, Nikeghbali, and Rodgers [3] extend the study to the special orthogonal group, the symplectic group, and give a related result for the Gaussian Unitary Ensemble.

Our motivation in this paper is to prove similar results on the characteristic polynomial of some particular unitary matrices related to random permutations. More precisely:

- We focus on matrices belonging to two particular subgroups of the unitary group: the set of permutation matrices, and the wreath product $S^3 \wr S_n$ (which can be seen as the set of permutation matrices where entries equals to one are replaced by complex numbers of modulus one).

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• We tackle a large family of measures on the symmetric group, which are invariant by conjugation and verify a certain property of decay over the cycle lengths. This family includes the family of Ewens measures, as we shall see.

• We introduce a coupling method for generating sequences of modified permutations under these particular measures, by analogy of the notion of virtual isometries introduced in [2]. This coupling provides an almost sure convergence in our main result given below.

1.2 Generating random permutations

Before giving the construction of the random permutations we will deal with, let us recall the few following definitions and facts:

A **virtual permutation** is a sequence \((\sigma_n)_{n \geq 1}\) where for each \(n\), \(\sigma_n\) is an element of \(S_n\), which can be derived from \(\sigma_{n+1}\) by simply removing the element \(n+1\) from the decomposition into disjoint cycles of \(\sigma_{n+1}\). Let \(S\) denote the space of virtual permutations. We call **central measure** on \(S\) a probability measure which satisfies the following property:

\[
\forall n \geq 1, \forall \tau \in S_n, \sigma_n \overset{d}{=} \tau \sigma_n \tau^{-1}.
\]

For each \(n\), it is easy to notice that every central measure on \(S_n\) can be fully described by a distribution on the set

\[
\nabla^{(n)} := \left\{ (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n : \ell_1 \geq \ell_2 \geq \cdots \geq \ell_n, \sum_{j=1}^n \ell_j = n \right\}
\]

of partitions of the integer \(n\), and conversely, in such a way that there is a one-to-one correspondence. A highly less obvious result (Theorem 2.3 in [15]) is that there exists a natural one-to-one correspondence between the central measures on \(S\) and the probability measures on

\[
\nabla := \left\{ (x_1, x_2, \ldots) \in [0,1]^\infty : x_1 \geq x_2 \geq \cdots \geq \sum_{j=1}^\infty x_j \leq 1 \right\}.
\]

The following definition introduces a new notion which specifies the family of measures we are going to consider in the paper.

**Definition 1.** Let \(p\) be a probability measure on \(\nabla\).

- We say that \(p\) is a **measure with exponential decay** if it satisfies the following property: There exists \(r \in (0,1)\) and \(\nabla_1 \subset \nabla\) with \(p(\nabla_1) = 1\), such that for all \(x = (x_1, x_2, \ldots) \in \nabla_1\),

\[
\exists C > 0, \forall j \geq 1, x_j \leq Cr^j.
\]

- We say that a distribution on \(S\) is a **central measure with exponential decay** if its corresponding distribution on \(\nabla\) is a measure with exponential decay.

**Example 2.** The **Ewens measure** [7] of any arbitrary parameter \(\theta > 0\) on \(S\), denoted by Ewens(\(\theta\)), is a central measure with exponential decay.

Indeed, first recall that, given \(\theta > 0\), one can define Ewens(\(\theta\)) on \(S\) thanks to the family of Ewens measures of parameter \(\theta\) on \(S_n\), \(n \geq 1\), denoted by Ewens(\(n, \theta\)), and defined by the probability functions

\[
\forall \sigma \in S_n, \mathbb{P}_\theta^{(n)}(\sigma) = \frac{\theta^{K(\sigma)}}{\theta(\theta + 1) \cdots (\theta + n - 1)},
\]

where \(K(\sigma)\) denotes the total number of cycles of \(\sigma\) once decomposed into disjoint cycles. More precisely, the sequence of measures \((\text{Ewens}(n, \theta))_{n \geq 1}\) is coherent with the projections \(S_{n+1} \to S_n\). In other words, if \(\sigma_{n+1}\) follows Ewens(\(n+1, \theta\)), then the random permutation obtained by removing the element \(n+1\) from the cycle-decomposition of \(\sigma_{n+1}\) follows Ewens(\(n, \theta\)).

For each \(\theta > 0\), the fact that Ewens(\(\theta\)) is central on \(S\) immediately derives from the fact that Ewens(\(n, \theta\)) is central on \(S_n\) for all \(n\). It is also well-known that the corresponding distribution on \(\nabla\) of the central
measure Ewens(\(\theta\)) is the **Poisson-Dirichlet distribution** of parameter \(\theta\) (denoted by PD(\(\theta\))). Let \(y = (y_1, y_2, \ldots)\) be a random vector following PD(\(\theta\)). We know that \(Y\) has the same distribution as the order statistics \((Y_1), (Y_2), \ldots\) of the random vector \(Y = (Y_1, Y_2, \ldots)\) defined as follows: let \((V_k)_{k \geq 1}\) be a sequence of i.i.d Beta(1, \(\theta\)) random variables (with density function given by \(x \mapsto \theta(1-x)^{\theta-1}1_{(0,1)}(x)\)).

For all \(j \geq 2\), define \(Y_j := V_j \prod_{k=1}^{j-1} (1 - V_k)\), and \(Y_1 := V_1\). The distribution of \(Y\) is called GEM(\(\theta\)). In the literature, this method for generating such a vector \(Y\) with i.i.d random variables \((V_k)\) is called residual allocation model [16] or stick-breaking process [12]. With this representation it is easy to compute that for all \(j\),

\[
E(Y_j) = \frac{1}{1 + \theta} \left(\frac{\theta}{1 + \theta}\right)^{j-1} \leq r_\theta^j
\]

with \(r_\theta := \frac{1}{1 + \theta} < 1\). Hence for any arbitrary \(r \in (r_\theta, 1)\),

\[
P(Y_j > r^j) \leq \frac{E(Y_j)}{r^j} \leq \left(\frac{r_\theta}{r}\right)^j,
\]

which is summable over \(j\), and then the Borel-Cantelli lemma applies and gives that the number of \(j\) such that \(Y_j > r^j\) is almost surely finite. In other words, there exists a random number \(C > 0\) such that for all \(j\), \(Y_j \leq Cr^j\). Finally, coming back to \(y\) it remains to see that the same kind of inequality holds for its coordinates, which is a direct consequence of the fact that for all \(j\) we have \(Y_{(j)} \leq \left((Cr^j)_{k \geq 1}\right)_{(j)} = Cr^j\).

Then the Ewens measure is a central measure with exponential decay.

**Remark.** Note that the Ewens measures are particular central measures whom corresponding distributions on \(\nabla\) are supported on

\[
\nabla' := \left\{(x_1, x_2, \ldots) \in [0,1]^{\infty} : x_1 \geq x_2 \geq \ldots, \sum_{j=1}^{+\infty} x_j = 1\right\} \subset \nabla.
\]  

In the main body of the paper, we focus on central measures with exponential decay on \(\mathcal{S}\) whose corresponding distributions on \(\nabla\) are supported on \(\nabla'\).

Now, let us present the coupling we consider for generating random permutations, which is highly inspired from [18], [13], and [14]. Let \(\lambda = (\lambda_j)_{j \geq 1}\) be a sequence of decreasing real numbers summing to 1, and let \(E_\lambda = \bigcap_{j=1}^{\infty} C_j\) be the disjoint union of circles \(C_j\), where for all \(j\), \(C_j\) has perimeter \(\lambda_j\). Let \(x = (x_k)_{k \geq 1} \in (E_\lambda)^{\infty}\).

The coupling we are going to introduce is based on the following proposition:

**Proposition 3.** One can define a virtual permutation \(\sigma_\infty(\lambda, x) = (\sigma_n(\lambda, x))_{n \geq 1}\), where for all \(n\), for all \(k \in \{1, \ldots, n\}\), the image of \(k\) by \(\sigma_n(\lambda, x)\) is the index of the first encountered point in \(\{x_1, \ldots, x_n\}\) after \(x_k\) when exploring its circle counterclockwise. Moreover, if \(\lambda\) follows any arbitrary distribution \(p\) on \(\nabla'\), then \(\sigma_\infty(\lambda, x)\) follows the central measure on \(\mathcal{S}\) corresponding to \(p\).

**Example 4.** If \(\lambda\) follows the PD(\(\theta\)) distribution and if conditionally on \(\lambda\) the points \(x_k\) are i.i.d random variables uniformly distributed on \(E(\lambda)\), then \(\sigma_\infty(\lambda, x)\) follows Ewens(\(\theta\)).

Let \(p\) be a distribution on \(\nabla'\). Let \(y = (y_1, y_2, \ldots)\) be a random vector following \(p\) and let \(E_y\) be the disjoint union of circles \(C_j\) of perimeters \(y_j\). Assume that conditionally given \(y\), the \(x_k\) are i.i.d random variables uniformly distributed on \(E_y\). Finally, introduce the array of random variables \((\ell_{n,j})_{n,j \geq 1}\) defined by

\[
\ell_{n,j} := \#\{k \in \{1, \ldots, n\} : x_k \in C_j\},
\]

and denote \(y^{(n)} := \ell_{n,j} n\).

Then, as a consequence of Proposition 3, almost surely, \((y^{(n)}_1, y^{(n)}_2, \ldots)\) converges in distribution to \(y\). Moreover, conditionally on \(y\), for all \(j\),

\[
y_j^{(n)} = \frac{1}{n} \sum_{k=1}^{n} I_{x_k \in C_j} \xrightarrow{n \to \infty} y_j
\]
by the strong law of large numbers.

In this paper we also consider some modifications of permutation matrices, which will call modified permutation matrices, which are permutation matrices where the entries equal to one are replaced by complex numbers of modulus one. The set of modified permutation matrices of size \( n \) has a group structure and can be identified to the wreath product \( S^1 \wr S_n \), where \( S^1 \) denotes the unit circle. Let us denote by \( \mathcal{T}_n \) the subset of matrices of \( S^1 \wr S_n \) which do not have 1 as an eigenvalue. The next lemma provides a construction of sequences of elements of \( \mathcal{T}_n, n \geq 1 \), by analogy to the notion of virtual isometries introduced by Bourgade, Najnudel and Nikeghbali in [2].

**Lemma 5.** For all \( n \geq 1 \), for all \( M \in \mathcal{T}_{n+1} \), there exists a unique \( N \in \mathcal{T}_n \) such that

\[
\text{rank} \left( M - \begin{pmatrix} N & 0 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \right) = 1.
\]

Moreover, the permutation corresponding to \( N \) derived from the one of \( M \) by removing the element \( n+1 \) from its cycle-decomposition.

**Proof.** Let \( n \geq 1 \) and \( M \in \mathcal{T}_{n+1} \). Write \((w_1 \ w_2 \ldots \ w_{\ell+1} = n+1)\) the cycle of the corresponding permutation of \( M \) containing the element \( n+1 \). There exist \( z_1, \ldots, z_{\ell} \) and \( z_{n+1} \) some complex numbers of modulus one such that for all \( k \in \{1, \ldots, \ell\}, ME_{w_k} = z_k e_{w_{k+1}}\) and \( ME_{n+1} = z_{n+1} e_{a_1} \) where \((e_1, \ldots, e_{n+1})\) is the canonical basis of \( \mathbb{C}^{n+1} \).

Denote by \( M^{[n]} \) the top-left minor of size \( n \) of \( M \). Let \( N \in M_n(\mathbb{C}) \).

- If \( \ell = 0 \) (i.e \( n+1 \) is a fixed point of the associated permutation), then \( z_{n+1} \) is an eigenvalue of \( M \). By hypothesis, this implies \( z_{n+1} \neq 1 \). Hence \( \text{rank}(M - \text{diag}(N, 1)) = 1 \) if and only if \( N = M^{[n]} \) (since \( z_{n+1} - 1 \) is the only non-zero entry of the last row and last column of \( M \)). Moreover in this case, \( M = \text{diag}(N, z_{n+1}) \) we have \( N \in \mathcal{T}_n \), and the procedure amounts to remove the fixed point \( n+1 \) from the associated permutation of \( M \).

- If \( \ell \geq 1 \), then \( ME_{w_{\ell}} = z_\ell e_{n+1} \) and \( ME_{n+1} = z_{n+1} e_{w_1} \) with \( w_1 \neq n+1 \neq w_{\ell} \). The \((\ell+1)\)-th roots of \( z_1 \ldots z_\ell z_{n+1} \) are eigenvalues of \( M \). By hypothesis, it follows \( z_1 \ldots z_\ell z_{n+1} \neq 1 \). Moreover, \( \text{rank}(M - \text{diag}(N, 1)) = 1 \) if and only if \( N = M^{[n]} + z_\ell z_{n+1} E_{w_{\ell}w_1} \) where \( E_{ij} \) is the \( n \)-by-\( n \) matrix with 1 in row \( i \) column \( j \) and zeros elsewhere. In this case, \( NE_{w_k} = z_k e_{w_{k+1}} \) for all \( k \in \{1, \ldots, \ell-1\} \) (not considered when \( \ell = 1 \)) and \( NE_{w_1} = z_\ell z_{n+1} e_{w_1} \), so that the \( \ell \)-th roots of \( z_1 \ldots z_\ell z_{n+1} \) are eigenvalues of \( N \) (the corresponding cycle is \((w_1 \ w_2 \ldots \ w_{\ell})\)). As \( z_1 \ldots z_\ell z_{n+1} \neq 1 \) we deduce \( N \in \mathcal{T}_n \).

\( \square \)

**Definition 6.** We say that a sequence of matrices \((\widetilde{M}_n)_{n \geq 1}\) is a modified virtual permutation if for all \( n, \widetilde{M}_n \in \mathcal{T}_n \) and \( \text{rank}(\widetilde{M}_{n+1} - \text{diag}(\widetilde{M}_n, 1)) = 1 \).

**Remark.** Note that every modified virtual permutation is in particular a virtual isometry.

**Proposition 7.** Let \((\widetilde{M}_n)_{n \geq 1}\) be a modified virtual permutation. There exists a virtual permutation \((\sigma_n)_{n \geq 1}\) such that, for all \( n \geq 1 \), \( \widetilde{M}_n \) corresponds to the permutation \( \sigma_n \) and has a characteristic polynomial of the form

\[
\lambda_{\widetilde{M}_n}(X) := \det(XI - \widetilde{M}_n) = \prod_{j \geq 1} (X^{\ell_{n,j}} - u_j),
\]

where \((u_j)_{j \geq 1}\) is a sequence of elements of \( S^1 \setminus \{1\} \) and the \( \ell_{n,j} \) denote the cycle-lengths of \( \sigma_n \). Moreover, for all \( j \) and \( n \) such that \( \ell_{n,j} > 0 \), \( u_j \) can be defined as the product of the non-zero entries of \( \widetilde{M}_n \) corresponding to the cycle \( j \) of \( \sigma_n \).

**Proof.** Let \( n \geq 1 \). As in the proof of the previous lemma, let us denote by \((w_1 \ w_2 \ldots \ w_{\ell+1} = n+1)\) the cycle of \( \sigma_{n+1} \) containing the element \( n+1 \), and by \( z_1, \ldots, z_{\ell} \) and \( z_{n+1} \) the complex numbers of modulus
for all \( k \in \{1, \ldots, \ell\} \), \( \tilde{M}_{n+1}c_{w_k} = z_kc_{w_{k+1}} \) and \( \tilde{M}_{n+1}e_{n+1} = z_{n+1}e_1 \), where \((e_1, \ldots, e_{n+1})\) is the canonical basis of \( \mathbb{C}^{n+1} \). The characteristic polynomials of \( M_{n+1} \) and \( \tilde{M}_n \) satisfy the equality

\[
\chi_{\tilde{M}_{n+1}}(X) = \begin{cases} 
X^{\ell+1} - z_1 \cdots z_{\ell+1} & \text{if } \ell \geq 1 \\
(X-z_n)\chi_{\tilde{M}_n}(X) & \text{if } \ell = 0
\end{cases} \tag{8}
\]

Indeed, if \( \ell = 0 \), then \( \tilde{M}_{n+1} \) can be written \( \tilde{M}_{n+1} = \text{diag}(\tilde{M}[n], z_{n+1}) = \text{diag}(\tilde{M}_n, z_{n+1}) \) by the previous lemma, so that \( \chi_{\tilde{M}_{n+1}}(X) = (X-z_n)\chi_{\tilde{M}_n}(X) \).

Otherwise, there exists a permutation matrix \( P \) of size \( n+1 \) which fixes the element \( n+1 \), such that \( PM_{n+1}P^{-1} \) and \( P\text{diag}(M_n, 1)^{-1} \) are block diagonal matrices where:

- All the blocks are of the form \((\alpha_1)\) or \((0 \ldots 0 \alpha_j)\), with \( \alpha_1, \ldots, \alpha_j \in S^1 \).

- If \( k \) is the number of blocks of \( PM_{n+1}P^{-1} \), then \( P\text{diag}(M_n, 1)^{-1} \) has exactly \( k+1 \) blocks (including the bottom-right 1), and the \( k-1 \) first blocks of \( PM_{n+1}P^{-1} \) and \( P\text{diag}(M_n, 1)^{-1} \) are equal.

The last block of \( PM_{n+1}P^{-1} \) is

\[
\begin{pmatrix}
0 & \ldots & 0 & z_{n+1} \\
& \ddots & \vdots & \\
& & 0 & z_{\ell} \\
& & & z_{\ell-1}
\end{pmatrix}
\]

hence with the help of the previous lemma the penultimate block of \( P\text{diag}(M_n, 1)^{-1} \) is

\[
\begin{pmatrix}
0 & \ldots & 0 & z_{\ell}z_{n+1} \\
& \ddots & \vdots & \\
& & 0 & z_{\ell-1} \\
& & & z_{\ell-1}
\end{pmatrix}
\]

and we get

\[
\chi_{\tilde{M}_n}(X) = \chi_{P^{-1}\tilde{M}_{n}(P^{-1})}(X) = \frac{\chi_{P\text{diag}(\tilde{M}_n, 1)^{-1}}(X)}{X-1} = \frac{X^{\ell+1} - z_1 \cdots z_{\ell+1}}{X^{\ell+1} - z_1 \cdots z_{\ell}z_{n+1}}(X^{\ell} - z_1 \cdots z_{\ell}z_{n+1})
\]

which gives (8). As a consequence, by analogy with the Chinese restaurant process (see e.g. [17]), here the customers arrive one by one and choose a table according to its weight, regardless of the past, and when a new customer \( n+1 \) seats at a table (empty or not), it does not affect the element of \( S^1 \setminus \{1\} \) corresponding to this table. Hence we can assign a \( u_j \) to each table \( j \), independently of \( n \).

**Definition 8.** We call **random modified permutation matrix** a random matrix \( \tilde{M}_n \) such that:

- \( \tilde{M}_n \) corresponds to a random permutation \( \sigma_n \) generated with the procedure of Proposition 3 for a given distribution \( p \) on \( \nabla' \).

- The non-zero entries of \( \tilde{M}_n \) are i.i.d random variables uniformly distributed on the unit circle.

**Corollary 9.** Let \( (\sigma_n)_{n \geq 1} \) be a random virtual permutation, and let \( (u_j)_{j \geq 1} \) be a sequence of i.i.d uniform variables on the unit circle, independent of \( (\sigma_n)_{n \geq 1} \). One can couple \((\sigma_n)_{n \geq 1}, (u_j)_{j \geq 1}\) with a random modified virtual permutation \( (\tilde{M}_n)_{n \geq 1} \) such that, for all \( n \geq 1 \),

- \( \tilde{M}_n \) is a random modified permutation matrix corresponding to \( \sigma_n \).

- Denoting by \( \ell_{n,j} \) the cycle-lengths of \( \sigma_n \), then for all \( j \) and \( n \) such that \( \ell_{n,j} > 0 \), \( u_j \) is the product of the non-zero entries of \( \tilde{M}_n \) corresponding to the cycle \( j \) of \( \sigma_n \).
Proof. This immediately derives from Proposition 7 and the fact that the projection $M \mapsto N$ via $\ker(M - \text{diag}(N,1))$ is coherent with respect to the sequence of probability measures $(\mathcal{L}_n)$ defined for all $n$ as the law of a $n$-by-$n$ random modified permutation matrix. Indeed, if the non-zero entries of $\tilde{M}_{n+1}$, say $z_1, z_2, \ldots, z_{n+1}$, are i.i.d uniform variables on the unit circle, then the non-zero entries of $M_n$, say $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n$, satisfy the following rule: There exists $\pi \in \mathcal{S}_{n+1}$ such that for all $j \in \{1, \ldots, n-1\}$, $z_j = z_{\pi(j)}$, and $\tilde{z}_n = z_{\pi(n)}z_{\pi(n+1)}$. Consequently $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n$ are i.i.d uniform variables on the unit circle.

Notations

For all events $A$ and all random variables $X$, we will denote by $P_y(A) := \mathbb{E}(X_{A} | y)$ the conditional expectation of $X_A$ given $y$.

We will write $X_n \Rightarrow X$ for the convergence in distribution of the sequence of random variables $(X_n)$ to the random variable $X$. We will use the arrow $\Rightarrow$ to denote the convergence in law on the space of continuous functions from $\mathbb{C}$ to $\mathbb{C}$ equipped with the topology of uniform convergence on compact sets.

Finally, for all real numbers $x$, $\{x\} = x - \lfloor x \rfloor$ will denote the fractional part of $x$, and $\|x\|$ the distance from $x$ to the nearest integer.

1.3 Main results and outline of the paper

Let $\sigma = (\sigma_n)_{n \geq 1}$ be a random virtual permutation generated with the procedure given above. Let $(M_n)_{n \geq 1}$ be the sequence of random permutation matrices associated to $\sigma$, that is to say for each $n$ we define $M_n$ as the $n \times n$ matrix whose coordinates are given by

$$\forall 1 \leq i, j \leq n, \ (M_n)_{i,j} := 1_{i = \sigma_n(j)}.$$ (9)

Let $(\tilde{M}_n)_{n \geq 1}$ be a random modified virtual permutation generated by $\sigma$ and a sequence $(u_j)_{j \geq 1}$ of i.i.d uniform variables on the unit circle independent of $\sigma$ (see Corollary 9). In particular, using the notations of eq. (5), for all $n$ and $j$ such that $\ell_{n,j} > 0$, $u_j$ is the product of the non-zero entries of $\tilde{M}_n$ whom cycle is associated with the cycle $C_j$, so that $u_j$ does not depend on $n$.

For all $n \in \mathbb{N}^*$ and $z \in \mathbb{C}$, we consider the characteristic polynomials of $M_n$ and $\tilde{M}_n$, respectively, defined by

$$Z_n(z) := \det(zI - M_n)$$ (10)

and

$$\tilde{Z}_n(z) := \det(zI - \tilde{M}_n).$$ (11)

Let $\alpha$ be an irrational number between 0 and 1, and set

$$\tilde{\xi}_n(z) = \frac{\tilde{Z}_n(e^{2i\pi z/n})}{Z_n(1)}$$ (12)

and

$$\xi_{n,\alpha}(z) = \frac{Z_n(e^{2i\pi (z+\alpha)/n})}{Z_n(e^{2i\pi \alpha})}. $$ (13)

The characteristic polynomial $\tilde{\xi}_n$ can be written with the help of $(y_j^{(n)})_{j \geq 1}$ and $(u_j)_{j \geq 1}$ as

$$\tilde{\xi}_n(z) = \prod_{\ell_{n,j} \geq 1} \frac{e^{2i\pi y_j^{(n)}} - u_j}{1 - u_j}. $$ (14)

The characteristic polynomial $\xi_{n,\alpha}$ can be written as

$$\xi_{n,\alpha} = \prod_{\ell_{n,j} \geq 1} \frac{e^{2i\pi (z+\alpha)\ell_{n,j}}} {e^{2i\pi \alpha \ell_{n,j}} - 1}.$$ (15)
We will give more details about the expressions of \( \tilde{\xi}_n \) and \( \xi_{n,\alpha} \) in the next section.

Finally, let us recall that the type of any real number \( x \) is defined by

\[
\eta = \sup \{ \gamma \in \mathbb{R} : \liminf_{n \to +\infty} n^{\gamma} \| nx \| = 0 \} \in \mathbb{R}_+ \cup \{ +\infty \}.
\]  

We say that \( x \) is of \textit{finite type} if \( \eta \) is finite. A basic property is that if \( x \) is irrational then its type is greater or equal to one (and can be infinite).

We are ready to state the main result of the present paper:

**Theorem 10.** Assume that \( \sigma \) is generated by the coupling described above for a distribution \( p \) on \( \nabla' \) with exponential decay. Then we have the following convergences:

(i) Almost surely, \( \tilde{\xi}_n \) converges uniformly on every compact set to an entire function \( \tilde{\xi}_\infty \) defined by

\[
\tilde{\xi}_\infty(z) = \prod_{j=1}^{+\infty} \frac{e^{2i\pi z y_j} - u_j}{1 - u_j}.
\]

(ii) Assume \( \alpha \) is an irrational number of finite type. Then

\[
\xi_{n,\alpha} \Rightarrow \tilde{\xi}_\infty
\]

where \( \tilde{\xi}_\infty \) is the same entire function as above.

**Remark.** Without the coupling, the theorem still holds replacing the first point by:

\[\tilde{\xi}_n \Rightarrow \tilde{\xi}_\infty.\]

**Remark.** Note that the parameter \( \alpha \) is not allowed to be rational, otherwise some denominators in the product expression of \( \xi_{n,\alpha} \) could be zeros. Moreover, heuristically, the motivation to take \( \alpha \) irrational of finite type is to avoid a too fast accumulation of small denominators.

The article is organized as follows: In Section 2, we give a proof of Theorem 10 by showing the first point in Subsection 2.1, and the second point in Subsection 2.2. These two subsections are mutually independent. In Section 3, we give some estimates on the limiting function \( \tilde{\xi}_\infty \), and compare our results to the unitary case presented in [4]. Finally, in Section 4, we extend the study to more general central measures, removing the restriction to \( \nabla' \) for the support of their corresponding distributions on \( \nabla \).

## 2 Proof of the main theorem

### 2.1 Quotient of characteristic polynomials related to modified permutation matrices

Consider a distribution with exponential decay \( p \) on \( \nabla' \), giving a \( r \in (0, 1) \) as in (2). Let \( y = (y_1, y_2, \ldots) \) be a random vector following the distribution \( p \).

Let \( (M_n)_{n \geq 1} \) be a sequence of modified random permutation matrices generated by the coupling given by Corollary 9. For all \( n \in \mathbb{N}^* \) and \( z \in \mathbb{C} \), we consider the characteristic polynomial of \( M_n \), defined by (11).

As one is almost surely not a zero of \( \tilde{Z}_n \), the function \( \tilde{\xi}_n \) defined by (12) is an entire function.

Using Corollary 9, \( \tilde{\xi}_n \) can be reformulated with the help of a sequence \( (u_j)_{j \geq 1} \) of independent random variables that are uniformly distributed on the unit circle (and independent of the \( y_k^{(n)} \)) as

\[
\tilde{\xi}_n(z) = \prod_{j \geq 1 \atop \ell_n,j > 0} \frac{e^{2i\pi z y_j^{(n)}} - u_j}{1 - u_j}.
\]

\[= \prod_{j \geq 1 \atop \ell_n,j > 0} \left( 1 + \frac{1}{1 - u_j} (e^{2i\pi z y_j^{(n)}} - 1) \right).\]

\[\text{(17)}\]
The next lemmas aim to handle the tail of the infinite product in the expression of \( \tilde{\xi}_n(z) \), in order to apply a dominated convergence theorem and get the pointwise convergence of \( \tilde{\xi}_n \). Moreover, they provide a bound of \( \xi_n \) uniformly on compact sets, allowing to conclude with Montel theorem.

**Lemma 11.** Let \( \alpha > 2 \). For all \( k \in \mathbb{N}^* \), set \( m_k := \min_{1 \leq j \leq k} |1 - u_j| \). Then a.s there exists a random number \( C_1 > 0 \) such that for all \( k \),

\[
m_k > C_1 k^{-\alpha}.
\]

**Proof.** Let \( A \in (0,1) \). Let \( T \) be a random variable following the uniform distribution on \([0,1]\).

\[
\mathbb{P}(1 - e^{2\pi T} \geq A) \geq \mathbb{P}(\sin(\pi T) \geq A) \geq \mathbb{P}(2 \min(T, 1 - T) \geq A) = \mathbb{P}(T \geq A) = 1 - A.
\]

Then for all \( k \),

\[
\mathbb{P}(m_k \leq A) \leq 1 - (1 - A)^k \leq kA
\]

using the mean value inequality. Thus,

\[
\sum_{k=1}^{+\infty} \mathbb{P}(m_k \leq k^{-\alpha}) \leq \sum_{k=1}^{+\infty} k^{1-\alpha} < +\infty.
\]

Applying Borel-Cantelli lemma we deduce that the number of \( k \) such that \( m_k \leq k^{-\alpha} \) is a.s finite, i.e a.s there exists \( k_0 \in \mathbb{N}^* \) such that for all \( k > k_0 \), \( m_k > k^{-\alpha} \). Finally, \( C_1 := \min_{k \leq k_0} (j^n |1 - u_j|) \wedge 1 \) gives the claim.

**Lemma 12.** For all \( \rho \in (\sqrt{r},1) \), a.s there exists a random number \( C_2 > 0 \), such that for all \( j \geq 1 \),

\[
s_j := \sup_{n \geq 1} y_{j}^{(n)} \leq C_2 \rho^j.
\]

**Proof.** Let \( \rho \in (0,1) \). For each \( j \) and \( n \), write \( y_{j}^{(n)} \) as the empirical mean of \( n \) i.i.d Bernoulli random variables \( Z_{1,j}, Z_{2,j}, \ldots, Z_{n,j} \) of parameter \( y_j \). Fix \( j \).

If \( n \leq \rho^{-j} \), then it is easy to check that the events \( \{ y_{j}^{(n)} \geq \rho^j \} \) and \( \{ \exists k \in \{1,2,\ldots,n\} : Z_{k,j} = 1 \} \) are equal, hence

\[
\mathbb{E}\left( 1_{y_{j}^{(n)} \geq \rho^j} \right) = \mathbb{E}\left( 1_{\exists k \in \{1,2,\ldots,n\} : Z_{k,j} = 1} | y_j \right) \leq ny_j \leq \frac{y_j}{\rho^j}
\]

and then

\[
\sum_{n \leq \rho^{-j}} \mathbb{P}(y_{j}^{(n)} \geq \rho^j) \leq \frac{y_j}{\rho^j}.
\]

If \( n \geq \rho^{-j} \), then for any arbitrary \( \lambda > 0 \) we have the Chernoff bound

\[
\mathbb{E}\left( 1_{y_{j}^{(n)} \geq \rho^j} | y_j \right) \leq e^{-\lambda \rho^j} \mathbb{E}\left( e^{\lambda Z_{1,j}} | y_j \right)^n \\
= e^{-\lambda \rho^j} \left( 1 - y_j + y_j e^{\lambda} \right)^n \\
\leq e^{-\lambda \rho^j} \exp\left( ny_j \left( e^{\lambda} - 1 \right) \right).
\]

This inequality is optimized at point \( \lambda = n \log(\rho^j/y_j) \), which gives

\[
\mathbb{E}\left( 1_{y_{j}^{(n)} \geq \rho^j} | y_j \right) \leq e^{-n \rho^j \log\left( \frac{\rho^j}{y_j} \right) - y_j} \left( e^{\rho^j/y_j} - 1 \right).
\]

By assumption of exponential decay (2), conditionally on \( y = (y_1, y_2, \ldots) \), almost surely there exists a constant \( C > 0 \) such that for all \( j \geq 1 \), \( y_j \leq C r^j \). Thus, taking any arbitrary \( r' \in (r,1) \), for almost every \( y \) there exists an integer \( k \) such that for all \( j \geq k \), \( y_j \leq r'^j \). Fix a given \( y \), \( r' \) and \( k \). Then, setting \( \rho > \sqrt{r'} \),

\[
\log\left( \frac{\rho}{y_j} \right) \geq \log\left( \frac{\rho^j}{r'^j} \right) = j \log(\rho - \log r') \geq 2
\]
for all $j$ sufficiently large and greater than $k$, say for all $j \geq m$ ($m$ dependent on $y$). Then for all $j \geq m$,
\[
\sum_{n \geq \rho^{-j}} \mathbb{P}_y(y_j^{(n)} \geq \rho^j) \leq \frac{\frac{y_j}{\rho^j} e^{-\frac{y_j}{\rho^j} (\log \left( \frac{m}{\rho^j} \right) - 1)}}{1 - e^{-\rho^j (\log \left( \frac{m}{\rho^j} \right) - 1)}} \leq \frac{\frac{y_j}{\rho^j} e^{-\rho^j (\log \left( \frac{m}{\rho^j} \right) - 1)}}{1 - e^{-\rho^j}} \leq (2e) \frac{y_j}{\rho^j}.
\]
We deduce, for all $j \geq m$,
\[
\sum_{n=1}^{+\infty} \mathbb{P}_y(y_j^{(n)} \geq \rho^j) \leq (1 + 2e) \frac{y_j}{\rho^j} \leq (1 + 2e) \left( \frac{r'}{\rho^j} \right)^j
\]
and consequently, we get
\[
\mathbb{E} \left[ \sum_{j=m}^{+\infty} \sum_{n=1}^{+\infty} I_{y_j^{(n)} \geq \rho^j} | y \right] < +\infty
\]
which implies
\[
\mathbb{P}_y \left( \sum_{j=m}^{+\infty} \sum_{n=1}^{+\infty} I_{y_j^{(n)} \geq \rho^j} < +\infty \right) = 1
\]
for almost every $y$. Finally, taking the expectation we get
\[
\mathbb{P} \left( \sum_{j=m}^{+\infty} \sum_{n=1}^{+\infty} I_{y_j^{(n)} \geq \rho^j} < +\infty \right) = 1.
\]
In other words, the number of couples $(j, n)$ such that $j \geq m$ and $y_j^{(n)} \geq \rho^j$ is almost surely finite, which gives that the number of $j$ such that $\sup_{n \in \mathbb{N}} y_j^{(n)} \geq \rho^j$ is almost surely finite. \hfill \qed

**Lemma 13.** With the same notation as above, a.s,
\[
C_3 := \sum_{j=1}^{+\infty} \frac{y_j}{|1 - u_j|} < +\infty
\tag{20}
\]
and
\[
C_4 := \sum_{j=1}^{+\infty} \frac{8j}{|1 - u_j|} < +\infty.
\tag{21}
\]

**Proof.** Straightforward consequence of Lemma 11 and Lemma 12. \hfill \qed

Now, we are able to prove the first point of Theorem 10:

**Proof of Theorem 10 (i).** We begin to show the pointwise convergence. Let $z \in \mathbb{C}$. The idea of the proof consists in splitting the product in the expression of $\tilde{\xi}_n(z)$ as follows:
\[
\tilde{\xi}_n(z) = \prod_{j=1}^{j_0} \frac{e^{2i\pi z y_j^{(n)}}}{1 - u_j} - u_j \prod_{j=j_0+1}^{+\infty} \frac{e^{2i\pi z y_j^{(n)}}}{1 - u_j} - u_j,
\]
where $j_0$ is an integer depending on $|z|$ and on random numbers $C_1$ and $C_2$, chosen in such a way that for all $j > j_0$,
\[
\left| \frac{1}{1 - u_j} \left( e^{i \pi z y_j^{(n)}} - 1 \right) \right| \leq \frac{9}{10} < 1.
\]
Indeed we can chose such a $j_0$ as for all $j$,
\[
\left| \frac{1}{1 - u_j} \left( e^{i \pi z y_j^{(n)}} - 1 \right) \right| \leq \frac{1}{C_1} \left| e^{i \pi z y_j^{(n)}} \exp(2\pi |z| y_j^{(n)}) \right| \leq \frac{1}{C_1} \left| e^{i \pi z C_2 \rho^j} \exp(2\pi |z| C_2 \rho^j) \right|
\]

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and then it suffices to take \( j \) large enough such that 
\[
\max \left( \frac{1}{C_1 \rho^j z}, 2\pi |z| C_2 \rho^j \right) \leq \frac{1}{4},
\]
which provides a bound lower than \( \frac{1}{4^{1/2}} \approx 0.82 \leq \frac{a}{b} \). Thus, we can apply the logarithm to the product of terms for \( j > j_0 \) in the expression of \( \tilde{\xi}_n \), and furthermore it is straightforward to check that for all \( Z \in \mathbb{C} \) such that \(|Z| \leq \frac{1}{4} \) we have

\[
|\log(1 + Z)| = \left| \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} Z^k \right| \leq \frac{|Z|^k}{k} \leq \frac{|Z|}{1 - |Z|} \leq 10|Z|.
\]

Consequently for all \( j > j_0 \),

\[
\left| \log \left( 1 + \frac{1}{1 - u_j} (e^{2\pi i z y_j^{(n)}} - 1) \right) \right| \leq 10 \left| \frac{1}{1 - u_j} (e^{2\pi i z y_j^{(n)}} - 1) \right|
\]

\[
\leq \frac{10}{|1 - u_j|} \sum_{k=1}^{+\infty} \frac{|2\pi i z y_j^{(n)}|^k}{k!}
\]

\[
\leq \frac{10}{|1 - u_j|} y_j^{(n)} (e^{2\pi |z|} - 1)
\]

\[
\leq \frac{10 e^{2\pi |z|} C_2}{C_1} \rho^j j^3
\]

which is summable in \( j \). Moreover, as \( y_j^{(n)} \) converges a.s to \( y_j \), then by continuity \( \log \left( 1 + \frac{1}{1 - u_j} (e^{2\pi i z y_j^{(n)}} - 1) \right) \) converges to \( \log \left( 1 + \frac{1}{1 - u_j} (e^{2\pi i z y_j} - 1) \right) \). Hence by dominated convergence,

\[
\tilde{\xi}_n(z) \xrightarrow{n \to \infty} \tilde{\xi}_\infty(z).
\]

This holds true for every \( z \in \mathbb{C} \) so we get the pointwise convergence of \( \tilde{\xi}_n \) to \( \tilde{\xi}_\infty \) on \( \mathbb{C} \).

Now, consider an arbitrary compact set of \( \mathbb{C} \) included in \( \{ z \in \mathbb{C} : |z| \leq K \} \) for a certain fixed \( K > 0 \). For all \( z \) in this compact set and all \( n \geq 1 \),

\[
|\tilde{\xi}_n(z)| \leq \prod_{j=1}^{+\infty} \left( 1 + \frac{1}{|1 - u_j|} (e^{2\pi i z y_j^{(n)}} - 1) \right)
\]

\[
\leq \prod_{j=1}^{+\infty} \left( 1 + \frac{y_j^{(n)}}{|1 - u_j|} (e^{2\pi K} - 1) \right)
\]

\[
\leq \exp \left( \sum_{j=1}^{+\infty} \frac{y_j^{(n)}}{|1 - u_j|} (e^{2\pi K} - 1) \right)
\]

\[
\leq \exp \left( (e^{2\pi K} - 1) \sum_{j=1}^{+\infty} \frac{s_j}{|1 - u_j|} \right)
\]

\[
= \exp \left( C_4 (e^{2\pi K} - 1) \right).
\]

We deduce by Montel theorem the uniform convergence of \( \tilde{\xi}_n \) to \( \tilde{\xi}_\infty \) on all compact sets.

2.2 Quotient of characteristic polynomials related to permutation matrices (without modification)

Here, the problem of finding a suitable normalization of the characteristic polynomial in order to have a non-trivial limiting function is more difficult than the previous one. We precise below the nature of this problem and defend our choice of normalization.

Consider a distribution with exponential decay \( p \) on \( \nabla' \), giving a \( r \in (0,1) \) as in (2), and let \( y = (y_1, y_2, \ldots) \) be a random vector following the distribution \( p \).
Let \((M_n)_{n \geq 1}\) be a sequence of random permutation matrices generated by the coupling described in Subsection 1.2 with respect to \(y\). For all \(n \in \mathbb{N}^*\) and \(z \in \mathbb{C}\), we consider the characteristic polynomial of \(M_n\) defined by (10).

Contrarily to random permutation matrices with modification (the \(\tilde{M}_n\) defined in the previous subsection), for every \(n\), there are some points \(z\) on the unit circle such that \(\mathbb{P}(Z_n(z) = 0) > 0\). For instance, for all \(n\), the characteristic polynomial of \(M_n\) evaluated at \(z = 1\) is almost surely zero (since each \(j\)-cycle of the associated permutation corresponds to eigenvalues which are exactly the \(j\)-th roots of unity).

Thus, the function \(\tilde{\xi}_n\) of the previous section, replacing \(Z_n\) by \(Z_n\), i.e.

\[
\tilde{\xi}_n(z) = \frac{Z_n(e^{2i\pi z/n})}{Z_n(1)}
\]

is not well-defined on the whole complex plane here. Based on the fact that all eigenvalues of permutation matrices are roots of unity, then, for every irrational number \(\alpha\), \(z = e^{2i\pi \alpha}\) is almost surely not a zero of \(Z_n\) for all \(n\). Let \(\alpha\) be an irrational number between 0 and 1. It is natural to shift the random process of eigenangles by \(2\pi \alpha\), and consider the function \(\xi_{n,\alpha}\) defined by (13).

As \(\alpha\) is irrational, then \(\xi_{n,\alpha}\) is an entire function, and can be written as follows:

\[
\xi_{n,\alpha} = \prod_{j \geq 1, \ell_{n,j} > 0} \frac{e^{2i\pi(\frac{1}{j} + \alpha)}\ell_{n,j} - 1}{e^{2i\pi \alpha \ell_{n,j}} - 1} = \prod_{j \geq 1, \ell_{n,j} > 0} \left(1 + \frac{e^{2i\pi \alpha \ell_{n,j}}}{e^{2i\pi \alpha \ell_{n,j}} - 1} \left(e^{2i\pi \alpha y_j^{(n)}} - 1\right)\right)^{\ell_{n,j}}.
\]

(22)

Heuristically, the idea of considering \(\alpha\) of finite type in Theorem 10 is to get the denominators in the expression of \(\xi_{n,\alpha}\) not too close from 0 when \(n\) become large, by comparison to the factor \(e^{2i\pi \alpha y_j^{(n)}} - 1\) in the numerator. In the sequel we prove the result into two parts: first, we show the convergence in distribution for finite products, and then, we handle the remaining infinite product.

**Lemma 14.** For all \(k \in \mathbb{N}^*\), conditionally on \(y\),

\[
(\ell_{n,1}, \ell_{n,2}, \ldots, \ell_{n,k}, n - \ell_{n,1} - \ell_{n,2} - \cdots - \ell_{n,k}) \overset{d}{\rightarrow} \mathcal{M}(n, y_1, y_2, \ldots, y_k, 1 - y_1 - y_2 - \cdots - y_k),
\]

where \(\mathcal{M}(n, q_1, q_2, \ldots, q_m)\) denotes a multinomial random variable of parameters \(q_1, q_2, \ldots, q_m\).

**Proof.** Direct consequence of (5). \(\square\)

**Lemma 15.** For all \(k \in \mathbb{N}^*\), conditionally on \(y\),

\[
(\{\alpha \ell_{n,1}\}, \{\alpha \ell_{n,2}\}, \ldots, {\alpha \ell_{n,k}}, y_1^{(n)}, \ldots, y_k^{(n)}) \overset{d}{\rightarrow} (\Phi_1, \Phi_2, \ldots, \Phi_k, y_1, y_2, \ldots, y_k)
\]

where the \(\Phi_j\) are i.i.d random variables uniformly distributed on \([0, 1]\), independent of the \(y_j\).

**Proof.** Let \(j_1, j_2, \ldots, j_k\) be integers, and let \(\lambda_1, \ldots, \lambda_k\) be real numbers. We have:

\[
\mathbb{E}\left(e^{2i\pi j_1 (\alpha \ell_{n,1}) + \cdots + j_k (\alpha \ell_{n,k}) + \left(\lambda_1 \frac{\ell_{n,1}}{j_1} + \cdots + \lambda_k \frac{\ell_{n,k}}{j_k}\right) y}\right)
\]

\[
= \mathbb{E}\left(e^{2i\pi (j_1 \alpha + \frac{\lambda_1}{j_1}) \ell_{n,1} + \cdots + 2i\pi (j_k \alpha + \frac{\lambda_k}{j_k}) \ell_{n,k} + 0 \cdot (n - \ell_{n,1} - \cdots - \ell_{n,k})} \right) y
\]

\[
= \sum_{(\ell_1, \ldots, \ell_{k+1}) \in \mathbb{N}^{k+1}} \frac{n!}{\ell_1! \cdots \ell_{k+1}!} \left(y_1 e^{2i\pi (j_1 \alpha + \frac{\lambda_1}{j_1})} \cdots (y_k e^{2i\pi (j_k \alpha + \frac{\lambda_k}{j_k})} (1 - y_1 - \cdots - y_k)^{\ell_{k+1}}
\]

\[
= \left(y_1 e^{2i\pi (j_1 \alpha + \frac{\lambda_1}{j_1})} + \cdots + y_k e^{2i\pi (j_k \alpha + \frac{\lambda_k}{j_k})} + (1 - y_1 - \cdots - y_k)^n\right).
\]
If \( j_1 = j_2 = \cdots = j_k = 0 \), then this quantity converges to \( e^{i(\lambda_1 y_1 + \cdots + \lambda_k y_k)} \).

Otherwise, there exists \( m \in [1, k] \) such that \( j_m \neq 0 \). Since \( \alpha \) is irrational, \( \|j_m \alpha\| \neq 0 \). Hence there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \),

\[
\| j_m \alpha + \frac{\lambda_m}{2\pi n} \| \geq \frac{\| j_m \alpha \|}{2}.
\]

Consequently, \( y_1 e^{2i\pi(j_1 \alpha + \frac{\lambda_1}{2\pi n})} + \cdots + y_k e^{2i\pi(j_k \alpha + \frac{\lambda_k}{2\pi n})} + (1 - y_1 - \cdots - y_k) \) is a convex combination with a.s positive coefficients of points located on the unit circle, whose the distance between two points (namely the point 1 and the point \( e^{2i\pi(j_m \alpha + \frac{\lambda_m}{2\pi n})} \)) is bounded from below by a positive number which is independent of \( n \). Thus \( y_1 e^{2i\pi(j_1 \alpha + \frac{\lambda_1}{2\pi n})} + \cdots + y_k e^{2i\pi(j_k \alpha + \frac{\lambda_k}{2\pi n})} + (1 - y_1 - \cdots - y_k) \) is bounded, uniformly in \( n \), by a quantity strictly smaller than one, and finally the Fourier transform goes to 0, which gives the claim. \( \square \)

**Proposition 16.** For all \( k \in \mathbb{N}^* \), conditionally on \( y \),

\[
\prod_{1 \leq j \leq k, \ell_n > 0} \left( 1 + \frac{e^{2i\pi\alpha \ell_n j}}{e^{2i\pi\alpha \ell_n j} - 1} \left( e^{2i\pi y \ell_n j} - 1 \right) \right) \overset{n \to \infty}{\longrightarrow} \prod_{1 \leq j \leq k, \ell_n > 0} \left( 1 + \frac{e^{2i\pi \Phi_j}}{e^{2i\pi \Phi_j} - 1} \left( e^{2i\pi y \Phi_j} - 1 \right) \right).
\]

**Proof.** Let \( k \in \mathbb{N}^* \). The map \( (x_1, x_2, \ldots, x_{2k}) \mapsto \left( z \mapsto \prod_{j=1}^{2k} \left( 1 + \frac{e^{2i\pi x_j}}{e^{2i\pi x_j} - 1} \left( e^{2i\pi y x_j} - 1 \right) \right) \right) \) is defined and continuous on \( (R \setminus \mathbb{Z})^k \times \mathbb{R}^k \). Moreover, \( \mathbb{P}(\exists j \in [1, k], \Phi_j \in \mathbb{Z}) = 0 \). Thus, the convergence in distribution directly follows from the previous lemma and the continuous mapping theorem. \( \square \)

**Lemma 17.** Let \( p \in [0, 1/2] \) and let \( B_n \) be a random variable following the binomial distribution of parameters \( n, p \), for any \( n \in \mathbb{N}^* \). Then \( (B_n)_{n \geq 1} \) satisfies the following properties:

(i) For all \( k \in \mathbb{N}^* \),

\[
\sup_{n \in \mathbb{N}^*} \mathbb{P}(B_n = k) \ll \frac{1}{\sqrt{k}}.
\]

(ii) Let \( 1 \leq m \leq n \). Let \( E \) an ensemble of integers between 1 and \( n \) such that:

- For all distinct \( j_1, j_2 \in E, |j_1 - j_2| \geq m \).
- For all \( j \in E, j \geq m \).

Then

\[
\mathbb{P}(B_n \in E) = O \left( \frac{1}{\sqrt{m}} \right),
\]

where the \( O(\frac{1}{\sqrt{m}}) \) is independent of \( n \) and \( p \).

**Proof.** Proof of (i): Fix \( k \in \mathbb{N}^* \). Using Stirling’s formula and the fact that \( x^k(1 - x)^{n-k} \) is maximal for \( x = \frac{1}{2} \),

\[
\mathbb{P}(B_n = k) = \binom{n}{k} p^k (1 - p)^{n-k} \ll \frac{(\frac{1}{2})^{n+1/2}}{(n/e)^{n-k+1/2} k!} \left( \frac{k}{n} \right)^k \left( 1 - \frac{k}{n} \right)^{n-k} = \frac{k}{k!} \left( \frac{k}{e} \right)^k \sqrt{n \ln n} \ll \frac{1}{\sqrt{k}}.
\]

Hence, if \( 1 \leq k \leq \frac{3}{4}n \) we have \( \mathbb{P}(B_n = k) \ll \frac{1}{\sqrt{k}} \). If \( \frac{3}{4}n \leq k \leq n \), as \( x \to x^k(1 - x)^{n-k} \) is increasing on \([0, \frac{1}{2}]\) and \( p \leq \frac{1}{2} \leq \frac{1}{3} \) by hypothesis,

\[
\mathbb{P}(B_n = k) \leq \binom{n}{k} \left( \frac{1}{2} \right)^n \ll \left( \frac{1}{2} \right)^{1/4} \left( \frac{1}{3} \right)^{3/4} = \left( \frac{2}{3} \right)^{n} \leq \left( \frac{2}{3} \right)^{k} \ll \frac{1}{\sqrt{k}}.
\]
Proof of (ii): Fix \( n \) and denote \( f_k := \binom{n}{k} p^k (1 - p)^{n-k} \). For all \( 0 \leq k < n \), it is easy to check that
\[
\frac{f_{k+1}}{f_k} < 1 \iff k > (n + 1)p - 1,
\]
so the sequence \((f_k)_{k \in \{0, \ldots, n\}}\) is increasing for \( k \leq k_0 \), and decreasing for \( k \geq k_0 + 1 \), where \( k_0 := \lceil (n+1)p-1 \rceil \). If \( m \geq k_0 + 1 \), then
\[
\mathbb{P}(B_n \in E) = \sum_{k \in E} f_k \leq \sum_{r=0}^{\lfloor \frac{k}{m} \rfloor - 1} f_{km+mr} \leq f_m + \frac{1}{m} \sum_{j=0}^{\frac{k}{m} - 1} f_j \leq f_m + \frac{1}{m} \sum_{j=0}^{n} f_j = f_m + \frac{1}{m}
\]
and by (i) we deduce \( \mathbb{P}(B_n \in E) \ll \frac{1}{\sqrt{m}} \).

If \( m \leq k_0 \), then we look separately at the increasing part and the decreasing part: on the one hand,
\[
\sum_{k \in E} f_k \leq \sum_{r=0}^{\lfloor \frac{k}{m} \rfloor - 1} f_{k_0 - mr} \leq f_{k_0} + \frac{1}{m} \sum_{j=0}^{k_0} f_j \leq f_{k_0} + \frac{1}{m} \sum_{j=0}^{n} f_j,
\]
and on the other hand,
\[
\sum_{k \in E} f_k \leq \sum_{r=0}^{\lfloor \frac{k}{m} \rfloor - 1} f_{k_0+1 + mr} \leq f_{k_0+1} + \frac{1}{m} \sum_{j=k_0+1}^{\lfloor k_0/m \rfloor + 1} f_j \leq f_{k_0+1} + \frac{1}{m} \sum_{j=k_0+1}^{n} f_j.
\]
Thus,
\[
\mathbb{P}(B_n \in E) = \sum_{k \in E} f_k \leq f_{k_0} + f_{k_0+1} + \frac{1}{m} \sum_{j=0}^{n} f_j = f_{k_0} + f_{k_0+1} + \frac{1}{m},
\]
and by (i) we deduce \( \mathbb{P}(B_n \in E) \ll \frac{1}{\sqrt{k_0}} + \frac{1}{m} \ll \frac{1}{\sqrt{m}}. \)

\[\square\]

Lemma 18. Assume \( \alpha \) is an irrational number of finite type \( \gamma \geq 1 \). Let \((a_j)_{j \geq 1}\) be any arbitrary sequence of positive real numbers. For all \( j \), let \( E_j := \{ \ell \in \mathbb{N}^* : \| \alpha \ell \| \leq a_j \} \). Then for all \( \nu > \gamma \), there exists a number \( c_\nu \) depending only on \( \nu \) and \( a_1 \), such that for all \( j \),
\[
\sup_{n \in \mathbb{N}^*} \mathbb{P}_y(\ell_{n,j} \in E_j) \leq c_\nu a_j^{-\gamma}.
\]

Proof. Let \( \nu > \gamma \). From (16),
\[
\exists \ C > 0, \ \forall \ell \in \mathbb{N}^*, \ \| \alpha \ell \| \geq \frac{C}{\ell^\gamma}.
\]
Let \((a_j)_{j \geq 1}\) be a sequence of positive real numbers. For all \( j \in \mathbb{N}^* \), it is easy to check that the set \( E_j := \{ \ell \in \mathbb{N}^* : \| \alpha \ell \| \leq a_j \} \) satisfies
\begin{itemize}
\item \( \forall \ell \in E_j, \ \ell \geq C^{1/\nu} a_j^{-1/\nu} \).
\item \( \forall \ell_1 \neq \ell_2 \in E_j, |\ell_1 - \ell_2| \geq (\frac{C}{2})^{1/\nu} a_j^{-1/\nu} \).
\end{itemize}
Moreover, by construction, conditionally on \( y \) each random variable \( \ell_{n,j} \) follows a binomial distribution of parameters \( n, y_j \). Since almost surely the \( y_j \) sum to one and \( y_1 \) is the largest, then \( y_j \leq \frac{1}{2} \) for all \( j \neq 1 \), hence the previous lemma applies with \( m = (\frac{C}{2})^{1/\nu} a_j^{-1/\nu} \) and gives for all \( j \geq 2 \),
\[
\mathbb{P}_y(\ell_{n,j} \in E_j) = \mathcal{O}\left( a_j^{-\gamma} \right),
\]
where the \( \mathcal{O}\left( a_j^{-\gamma} \right) \) is independent of \( n \) and of the \( y_j \), \( j \geq 2 \). In other words, there exists a number \( b_\nu \) depending only on \( \nu \) and \( y \), such that for all \( j \geq 2 \),
\[
\sup_{n \in \mathbb{N}^*} \mathbb{P}_y(\ell_{n,j} \in E_j) \leq b_\nu a_j^{-\gamma}.
\]
Finally we get the result taking \( c_\nu := \max \left( b_\nu, a_1^{-\gamma} \right) \). \[\square\]
Proposition 19. Conditionally on \( y \), for all \( \varepsilon > 0 \) and for all compact subsets \( K \) of \( \mathbb{C} \),

\[
\sup_{n \in \mathbb{N}^*} \mathbb{P}_y \left( \sup_{\varepsilon \in K} \left| \prod_{j > k} \left( 1 + \frac{e^{2i\pi \alpha \ell_{n,j}}}{e^{2i\pi \alpha \ell_{n,j}} - 1} \left( e^{2i\pi y_j^{(n)}} - 1 \right) \right) - 1 \geq \varepsilon \right| \right) \xrightarrow{k \to \infty} 0.
\]

Proof. Let \( \varepsilon > 0 \) and let \( K \) be a compact subset of \( \mathbb{C} \). It suffices to show that conditionally on \( y \),

\[
\sup_{n \in \mathbb{N}^*} \mathbb{P}_y \left( \sum_{j > k} \frac{y_j^{(n)}}{\|\alpha \ell_{n,j}\|} \geq \varepsilon \right) \xrightarrow{k \to \infty} 0,
\]

Indeed, for all \( n, j \) such that \( \ell_{n,j} > 0 \),

\[
\left| \frac{e^{2i\pi \alpha \ell_{n,j}}}{e^{2i\pi \alpha \ell_{n,j}} - 1} \left( e^{2i\pi y_j^{(n)}} - 1 \right) \right| \leq \frac{1}{\min(\{\alpha \ell_{n,j} \}, 1 - \{\alpha \ell_{n,j} \})} \frac{C_K y_j^{(n)}}{\|\alpha \ell_{n,j}\|}
\]

where \( C_K \) is a constant number that only depends on \( K \).

Let \( n \geq 1 \). Let \( s, \rho \in (0, 1) \) such that \( r < \rho < s < 1 \). We have, denoting \( A := \left\{ \sum_{j > k} \frac{y_j^{(n)}}{\|\alpha \ell_{n,j}\|} \geq \varepsilon \right\} \) and \( B_j := \{ \|\alpha \ell_{n,j}\| \leq s^j, \ \ell_{n,j} > 0 \} \),

\[
\mathbb{P}_y(A) = \mathbb{P}_y(A \cap B_{k+1}) + \mathbb{P}_y(A \cap B_{k+2}) + \cdots + \mathbb{P}_y(A \cap (B_{k+1} \cup B_{k+2} \cup \ldots)^c) \\
\leq \left( \sum_{j > k} \mathbb{P}_y(B_j) \right) + \mathbb{P}_y(A \cap (B_{k+1} \cup B_{k+2} \cup \ldots)^c).
\]

On the one hand, for all \( j \), \( \mathbb{P}_y(B_j) = \mathbb{P}_y(\ell_{n,j} \in E_j) \) with \( E_j := \{ \ell \in \mathbb{N}^* : \|\alpha \ell\| \leq s^j \} \), then it follows from Lemma 18 that \( \mathbb{P}_y(B_j) = \mathcal{O} \left( s^j \right) \) independent of \( n \), hence

\[
\sup_{n \in \mathbb{N}} \sum_{j > k} \mathbb{P}_y(B_j) \xrightarrow{k \to \infty} 0.
\]

On the other hand,

\[
\mathbb{P}_y(A \cap (B_{k+1} \cup B_{k+2} \cup \ldots)^c) = \mathbb{P}_y(A \cap (\forall j > k, \ |\ell_{n,j}| > s^j \text{ or } \ell_{n,j} = 0)) \\
\leq \mathbb{P}_y(\tilde{A}),
\]

with \( \tilde{A} := \left\{ \sum_{j > k} \frac{y_j^{(n)}}{s^j} \geq \varepsilon \right\} \), and furthermore denoting \( \tilde{B}_j := \{ y_j^{(n)} > \rho^j \} \) it comes

\[
\mathbb{P}_y(\tilde{A}) \leq \left( \sum_{j > k} \mathbb{P}_y(B_j) \right) + \mathbb{P}_y(\tilde{A} \cap (\forall j > k, \ y_j^{(n)} < \rho^j)) \\
\leq \left( \sum_{j > k} \mathbb{E}(y_j^{(n)} | y) \right) + \mathbb{P}_y \left( \sum_{j > k} \frac{\rho^j}{y_j} \geq \varepsilon \right) \\
= \left( \sum_{j > k} \frac{y_j^{(n)}}{\rho^j} \right) + \mathbb{P}_y \left( \sum_{j > k} (\varepsilon)^j \geq \varepsilon \right).
\]
independent of \( n \), hence

\[
\sup_{n \in \mathbb{N}^*} P_y(A \cap (B_{k+1} \cup B_{k+2} \cup \ldots)\overline{C}) \longrightarrow 0.
\]

Consequently,

\[
\sup_{n \in \mathbb{N}^*} P_y \left( \sum_{j > k} \frac{y_j^{(n)}}{\|\alpha_{j,n}\|} \geq \varepsilon \right) \longrightarrow 0.
\]

**Proof of Theorem 10 (ii).** Combining Proposition 16 and Proposition 19, we deduce that conditionally on \( y \),

\[
\xi_{n,\alpha} \overset{\text{n} \to \infty}{\longrightarrow} \tilde{\xi}_\infty.
\]

In other words, for all functional \( F \) from \( C(\mathbb{C},\mathbb{C}) \) to \( \mathbb{C} \), continuous with respect to the topology of the uniform convergence on compact sets,

\[
E[F(\xi_{n,\alpha}) \mid y] \overset{\text{n} \to \infty}{\longrightarrow} E[F(\tilde{\xi}_\infty) \mid y].
\]

Hence by dominated convergence

\[
E[F(\xi_{n,\alpha})] \overset{\text{n} \to \infty}{\longrightarrow} E[F(\tilde{\xi}_\infty)].
\]

\[\square\]

## 3 Properties of the limiting function

In the following proposition we show that \( \tilde{\xi}_\infty \) has order one, in the sense of entire functions. The bound \( \mathcal{O}(\exp(|z|)) \) for this model can be compared to the one for the unitary case \( \mathcal{O}(\exp(c|z| \log |z|)) \) established by Chhaibi, Najnudel and Nikeghbali in [4].

**Proposition 20.** For all \( \varepsilon > 0 \), there exists a random number \( C_\varepsilon > 0 \) such that for all \( z \in \mathbb{C} \),

\[
|\tilde{\xi}_\infty(z)| \leq C_\varepsilon e^{(2\pi+\varepsilon)|z|}. \tag{23}
\]

**Proof.** The bound is clear for \( |z| < 1 \) since \( \tilde{\xi}_\infty \) is locally bounded. Assume \( |z| \geq 1 \). Let \( \eta > 0 \). Set \( k = \max\{j \geq 1 : 2\pi |y_j| \geq \eta\} \lor 0 \) and \( k_1 = \max\{j \geq 1 : 2\pi y_j \geq \eta\} \lor 0 \). Note that \( k \geq k_1 \), with \( k_1 \) independent of \( z \). We distinguish between two regimes depending on whether \( j \) is lower or greater than \( k \).

- \( j > k \):

\[
\prod_{j=k+1}^{+\infty} \left| \frac{e^{2\pi z y_j}}{1 - u_j} \right| \leq \prod_{j=k+1}^{+\infty} \left( 1 + \frac{2\pi |z| y_j}{|1 - u_j|} \exp(2\pi |z| y_j) \right)
\]

\[
\leq \prod_{j=k+1}^{+\infty} \left( 1 + \frac{2\pi |z| y_j}{|1 - u_j|} \exp(\eta) \right)
\]

\[
\leq \exp \left( 2\pi |z| \exp(\eta) \sum_{j=k+1}^{+\infty} \frac{y_j}{|1 - u_j|} \right)
\]

\[
\leq \exp \left( 2\pi |z| \exp(\eta) \sum_{j=k_1+1}^{+\infty} \frac{y_j}{|1 - u_j|} \right).
\]
Moreover we have seen from (20) that \( \sum_{j=1}^{+\infty} \frac{u_j}{1-u_j} < \infty \), and furthermore if \( \eta \) tends to zero then \( k_1 \) goes to infinity, so we can chose \( \eta \) sufficiently close to 0 such that \( 2\pi \exp(\eta) \sum_{j=k_1+1}^{+\infty} \frac{u_j}{1-u_j} \leq \frac{\varepsilon}{2} \) and then
\[
\prod_{j=k+1}^{+\infty} \left| \frac{e^{2\pi i y_j} - u_j}{1 - u_j} \right| \leq \exp\left( \frac{\varepsilon}{2} |z| \right).
\] (24)

- \( j \leq k \) (case to be considered only when \( k \neq 0 \)): As for all \( m \in \mathbb{N}^* \),
\[
\left| \frac{e^{2\pi i y_m} - u_m}{1 - u_m} \right| \leq \frac{e^{2\pi |z| y_m} + 1}{|1 - u_m|} \leq \frac{2e^{2\pi |z| y_m}}{|1 - u_m|},
\]
then
\[
\prod_{j=1}^{k} \left| \frac{e^{2\pi i y_j} - u_j}{1 - u_j} \right| \leq \left( \frac{2}{\min_{1 \leq j \leq k} |1 - u_j|} \right)^k \exp\left( 2\pi |z| \sum_{j=1}^{k} y_j \right) \leq \left( \frac{2}{\min_{1 \leq j \leq k} |1 - u_j|} \right)^k e^{2\pi |z|}.
\] (25)

It just remains to show that
\[
\left( \frac{2}{\min_{1 \leq j \leq k} |1 - u_j|} \right)^k \leq C_\varepsilon \exp\left( \frac{\varepsilon}{2} |z| \right)
\] (26)
where \( C_\varepsilon \) is a random number depending on \( \varepsilon \) but not on \( z \). Lemma 11 with \( \alpha = 3 \) gives
\[
\left( \frac{2}{\min_{1 \leq j \leq k} |1 - u_j|} \right)^k \leq \left( \frac{2}{C_1} \right)^k.
\]

Moreover, using the assumption on the sequence \( \{y_j\} \), there exists \( r \in (0, 1) \) and a (random) number \( C > 0 \) such that for all \( j, y_j \leq Cr^j \), thus
\[
k \leq \max \left\{ j \geq 1 : Cr^j \geq \frac{\eta}{2\pi |z|} \right\} \leq \frac{\log \left( \frac{2\pi C|z|}{\eta} \right)}{-\log r} \ll \log(|z| + 1)
\]
Hence
\[
\log \left( \left( \frac{2}{C_1} \right)^k \right) \ll \log(|z| + 1)(1 + \log \log(|z| + 1)) \rightarrow o(|z|),
\]
which gives the existence of \( C_\varepsilon \).
Consequently, (24), (25) and (26) jointly give (23).

\[\square\]

Up to \( \varepsilon \), the bound we provide in Proposition 20 is sharp. Indeed, we have the following result:

**Proposition 21.** For all \( \varepsilon \in (0, 2\pi) \), there exists a random number \( c_\varepsilon > 0 \) such that for all \( x \geq 0 \),
\[
|\tilde{c}_\infty(-ix)| \geq c_\varepsilon e^{(2\pi - \varepsilon)x}.
\] (27)

**Proof.** Let \( x \geq 0 \). For all \( k \in \mathbb{N}^* \), if \( x \geq \frac{1}{2\pi y_k} \), then
\[
\left| \frac{e^{2\pi x y_k} - u_k}{1 - u_k} \right| \geq 1 \geq \frac{1}{2} \left| e^{2\pi x y_k} - u_k \right| \geq \frac{1}{2} \left| e^{2\pi x y_k} - 1 \right| \geq \frac{1}{4} e^{2\pi x y_k}.
\]
Moreover, if \( \lambda \) following the uniform distribution on \( E \) to \( p \in \{ k \} \)

One can define a virtual permutation \( \sigma_\infty(n, x) = (\sigma_n(n, x))_{n \geq 1} \), where for all \( n \), for all \( k \in \{ 1, \ldots, n \} \),

- if \( x_k \) is on a circle, then the image of \( k \) by \( \sigma_n(n, x) \) is the index of the first encountered point in \( \{ x_1, \ldots, x_n \} \) after \( x_k \) when exploring its circle counterclockwise (it is \( k \) itself if there is no other index \( j \leq n \) such that \( x_j \) is on the same circle as \( x_k \)),

- if \( x_k \) is in \( S \), then \( k \) is a fixed point of \( \sigma_n(n, x) \).

Moreover, if \( \lambda \) follows any arbitrary distribution \( p \) on \( \nabla \), and if conditionally on \( \lambda \) the points \( x_k \) are i.i.d following the uniform distribution on \( E(\lambda) \), then \( \sigma_\infty(n, x) \) follows the central measure on \( \mathcal{E} \) corresponding to \( p \).

Let \( p \) be any probability measure on \( \nabla \), and let \( (y_j)_{j \geq 1} \) be a random vector following the distribution \( p \). Introduce the random variable \( y_0 := \sum_{j=1}^{+\infty} y_j \).

For all \( n \) and \( j \), denote \( \ell_{n, j} := \# \{ k \in \{ 1, \ldots, n \} : x_k \in C_j \} \), and \( p_n := \# \{ k \in \{ 1, \ldots, n \} : x_k \in S \} \).

Let \( \alpha \) be an irrational number between 0 and 1. The expressions of \( \tilde{\xi}_n \) and \( \xi_{n,\alpha} \) defined at the beginning of the paper become

\[
\tilde{\xi}_n(z) = \left( \prod_{j \geq 1}^{\ell_{n, j} > 0} \frac{e^{2\pi i y_j} - u_j}{1 - u_j} \right) \left( \prod_{k=1}^{p_n} \frac{e^{2\pi i \frac{\pi}{\alpha} - v_k}}{1 - v_k} \right)
\]

(28)

where the \( u_j \) and the \( v_k \) are independent random variables uniformly distributed on the unit circle, and

\[
\xi_{n,\alpha} = \left( \prod_{j \geq 1}^{\ell_{n, j} > 0} \frac{e^{2\pi i (\frac{\pi}{\alpha} + \alpha) y_j} - 1}{e^{2\pi i \alpha y_j} - 1} \right) \left( \frac{e^{2\pi i (\frac{\pi}{\alpha} + \alpha)} - 1}{e^{2\pi i \alpha} - 1} \right)^{p_n}.
\]

(29)
Theorem 23. Assume that $\sigma$ is generated by the coupling described above for a distribution with exponential decay $p$ on $\nabla$. Then we have the following convergences in distribution:

(i) \[ \tilde{\xi}_n(z) \xrightarrow{n \to \infty} \left( \prod_{j=1}^{+\infty} \frac{e^{2\pi i z y_j} - u_j}{1 - u_j} \right) e^{i\pi z(1-y_0)} \prod_{k \in \mathbb{Z}} \left( 1 - \frac{z}{w_k} \right) \]

where $\{w_k : k \in \mathbb{Z}\}$ are points of a Poisson process with intensity $1 - y_0$ on $\mathbb{R}$ (if $y_0 = 1$, we make the convention $\prod_{k \in \mathbb{Z}} \left( 1 - \frac{z}{w_k} \right) = 1$).

(ii) For all irrational number $\alpha$ of finite type,

\[ \xi_{n,\alpha}(z) \xrightarrow{n \to \infty} \left( \prod_{j=1}^{+\infty} \frac{e^{2\pi i z y_j} - u_j}{1 - u_j} \right) e^{i\pi z(1-y_0)} \left( 1 - \frac{z}{w_0} \right) \]

Remark. In Theorem 23, the product $\prod_{k \in \mathbb{Z}} \left( 1 - \frac{z}{w_k} \right)$ is not absolutely convergent. It has to be understood as

\[ \left( 1 - \frac{z}{w_0} \right) \prod_{k \geq 1} \left( 1 - \frac{z}{w_k} \right) \left( 1 - \frac{z}{w_{-k}} \right), \]

where the points of the Poisson process $\{w_k : k \in \mathbb{Z}\}$ are labelled as follows:

\[ \cdots < w_{-2} < w_{-1} < 0 < w_0 < w_1 < w_2 < \ldots. \]

4.1 Proof of Theorem 23 (i)

For all $k \geq 1$, write $v_k = e^{2\pi i \Phi_k}$ where the $\Phi_k$ are independent and uniformly distributed on $[0,1)$. Then

\[ \prod_{k=1}^{p_n} \frac{e^{2\pi i \Phi_k} - v_k}{1 - v_k} = \prod_{k=1}^{p_n} \frac{e^{2\pi i \Phi_k} - e^{2\pi i \Phi_k}}{1 - e^{2\pi i \Phi_k}} \]

\[ = e^{i\pi z} \prod_{k=1}^{p_n} \frac{\sin(\pi (\Phi_k - \frac{z}{n}))}{\sin(\pi \Phi_k)} \]

\[ = e^{i\pi z} \prod_{k=1}^{p_n} \frac{\Phi_k - \frac{z}{n}}{\Phi_k} \lim_{B \to +\infty} \prod_{0 < j \leq B} \left( 1 - \frac{\Phi_k - \frac{z}{n}}{1 - \frac{\Phi_k - \frac{z}{n}}{j}} \right). \]

Remark. The product in the last expression is not absolutely convergent. We write it like this for convenience of notation, and has to be understood as given by the expression above.

Denote by $w_{n,k}$ the points $n(\Phi_k - j)$ (defined for example as follows: for $k$ from 1 to $p_n$, $w_{n,k} := n\Phi_k$, and for all $j \in \mathbb{Z}$, $w_{n,k+jp_n} := w_{n,k} - jn$). The order of labelling for the points $w_{n,k}$ does not matter for what we use in the sequel.

Let $\mu_n := \sum_k \delta_{w_{n,k}}$ be the empirical measure associated with the point process of the $w_{n,k}$.

Lemma 24. The empirical measure $\mu_n$ converges vaguely to $\mu_\infty$, where $\mu_\infty$ is the empirical measure associated with the points of a Poisson process with intensity $1 - y_0$ on $\mathbb{R}$. The convergence holds for all compactly supported test functions from $\mathbb{R}$ to $\mathbb{C}$ (measurable but not necessarily continuous).

Proof. Let $f : \mathbb{R} \to \mathbb{C}$ such that $\text{supp} f \subset [-M,M]$, $M > 0$. Let $t \in \mathbb{R}$. Let $n > 2M$. First note that by periodicity of the points $w_{n,k},$

\[ \mathbb{E} \left( e^{it \sum_k f(w_{n,k})} \right) = \mathbb{E} \left( e^{it \sum_{k=1}^{X_n} f(k)} \right). \]
where \( X_n \) is a random variable which counts the number of the points \( w_{n,k} \) lying in \([-M, M]\), and where the \( \phi_k \) are i.i.d random variables uniformly chosen on \([-M, M]\), independently of \( X_n \). Moreover, \( X_n \) is binomial of parameters \( p_n \) and \( 2M/n \). Hence,

\[
E \left( \exp \left( \frac{u}{n} \sum_{k} f(w_{n,k}) \right) \right) = E \left( \exp \left( \frac{u}{n} \sum_{k} f(\phi_k) \right) X_n \right)
\]

\[
= \sum_{k=0}^{p_n} \frac{p_n}{k!} \left( \frac{2M}{n} \right)^k \left( 1 - \frac{2M}{n} \right)^{p_n-k} \left( 1 + \frac{2M}{n} \right)^{p_n-k} \left( 1 - \frac{2M}{n} \right)^{p_n-k}
\]

\[
= \left( 1 + \frac{2M}{n} (\exp(uf(\phi_k)) - 1) \right)^{p_n}
\]

\[
= \left( 1 + \frac{1}{n} \int_{-M}^{M} (\exp(f(x)) - 1) \, dx \right)^{p_n}
\]

\[
\xrightarrow{n \to +\infty} \exp \left( (1 - y_0) \int_{-M}^{M} (\exp(f(x)) - 1) \, dx \right) + o(1),
\]

since \( p_n/n \to 1 - y_0 \) almost surely. Thus, the Fourier transform of \( \sum_k f(w_{n,k}) \) converges to the Fourier transform of \( T := \sum_{x \in N} f(x) \), where \( N \) is a homogeneous Poisson point process of parameter \( 1 - y_0 \) (for example the expression of the Fourier transform of \( T \) can be provided using the Campbell theorem), which gives the claim.

**Proposition 25.** For all \( A \),

\[
\prod_{|w_{n,k}|<A} \left( 1 - \frac{z}{w_{n,k}} \right) \xrightarrow{n \to +\infty} \prod_{|w_k|<A} \left( 1 - \frac{z}{w_k} \right)
\]

where \( \{w_k : k \in \mathbb{Z}\} \) are points of a Poisson process with intensity \( 1 - y_0 \) on \( \mathbb{R} \).

**Proof.** Let \( A > 0 \). Let \( \mathcal{M} \) denote the space of locally finite measures of the form \( \sum_k \delta_{\alpha_k} \) for some arbitrary real numbers \( \alpha_k \). Let \( F \) be the functional defined from \( \mathcal{M} \) to \( C(\mathbb{C}, \mathbb{C}) \) by

\[
F \left( \sum_k \delta_{\alpha_k} \right) = \prod_k \left( 1 - \frac{z}{\alpha_k} \right) \mathbb{1}_{0 < |\alpha_k| < A}.
\]

\( F \) is continuous at every measure which does not charge \(-A, 0, \) and \( A \). Since almost surely \( \mu_\infty \) (the empirical measure associated with the Poisson process \( \{w_k : k \in \mathbb{Z}\} \) does not charge these three points, \( F \) is continuous in \( \mu_\infty \). By Lemma 24 and the continuous mapping theorem we deduce \( F(\mu_n) \to F(\mu_\infty) \), which gives the claim.

**Proposition 26.** For all \( \varepsilon > 0 \), for all compact subsets \( K \) of \( \mathbb{C} \),

\[
\sup_{n \in \mathbb{N}^*} \left( \sup_{z \in K} \left| \prod_{|w_{n,k}|\geq A} \left( 1 - \frac{z}{w_{n,k}} \right) - 1 \right| \geq \varepsilon \right) \xrightarrow{A \to +\infty} 0.
\]

**Proof.** Let \( K \) a compact subset of \( \mathbb{C} \) of diameter \( D \) for the uniform norm, and let \( A > 2D \). For all \( z \in K \),

\[
\prod_{|w_{n,k}|\geq A} \left( 1 - \frac{z}{w_{n,k}} \right) = \exp \left( -z \sum_{|w_{n,k}|\geq A} \frac{1}{w_{n,k}} \right) \exp \left( \mathcal{O}_K \left( \sum_{|w_{n,k}|\geq A} \frac{1}{w_{n,k}} \right) \right). \tag{30}
\]
The sum \( \sum_{\{w_{n,k}\} \geq A} \frac{1}{w_{n,k}} \) is not absolutely convergent. Let \( B > A \). Integrating by parts,

\[
\sum_{A \leq |w_{n,k}| \leq B} \frac{1}{w_{n,k}} = \int_{-B}^{-A} \frac{1}{x} d\mu_n(x) + \int_{A}^{B} \frac{1}{x} d\mu_n(x)
\]

\[
= \frac{\mu_n[A,B] - \mu_n[-B,-A]}{B} + \int_{A}^{B} \frac{\mu_n[A,x] - \mu_n[x,-A]}{x^2} dx.
\]

As for all \( a, b \in \mathbb{R} \), \( \mu_n[a, b] = \frac{(b-a)}{n} p_n \leq 2p_n \), then \( \lim_{B \to +\infty} \frac{\mu_n[A,B] - \mu_n[-B,-A]}{B} = 0 \), and we get

\[
\sum_{|w_{n,k}| \geq A} \frac{1}{w_{n,k}} = \int_{A}^{+\infty} \frac{\mu_n[A,x] - \mu_n[-x,-A]}{x^2} dx.
\]

Hence

\[
E \left| \sum_{|w_{n,k}| \geq A} \frac{1}{w_{n,k}} \right| \leq \int_{A}^{+\infty} \frac{1}{x^2} E[\mu_n[A,x] - \mu_n[-x,-A]] dx,
\]

with for all \( x > A \), by Cauchy-Schwarz inequality,

\[
(E[\mu_n[A,x] - \mu_n[-x,-A]])^2 \leq E((\mu_n[A,x] - \mu_n[-x,-A])^2)
\]

\[
= \text{Var}(\mu_n[A,x] - \mu_n[-x,-A])
\]

\[
\leq 2(\text{Var}(\mu_n[A,x]) + \text{Var}(\mu_n[-x,-A]))
\]

\[
= 4\text{Var}(\mu_n[A,x])
\]

since \( \mu_n[A,x] \) and \( \mu_n[-x,-A] \) are equally distributed (consequence of the fact that the \( \Phi_k \) are uniformly distributed on \([0,1])\).

As each interval of the form \( \{jn, (j+1)n\} \) contains exactly \( p_n \) points, and these points are uniformly distributed, then \( \text{Var}(\mu_n[A,x]) \) is the variance of a binomial random variable of parameters \( p_n \) and \( \{\frac{x-A}{n}\} \), that is

\[
\text{Var}(\mu_n[A,x]) = p_n \left( \frac{x-A}{n} \right) \left( 1 - \left\{ \frac{x-A}{n} \right\} \right)
\]

\[
\leq p_n \frac{x-A}{n} \leq x - A.
\]

We deduce

\[
E \left| \sum_{|w_{n,k}| \geq A} \frac{1}{w_{n,k}} \right| \leq 2 \int_{A}^{+\infty} \frac{\sqrt{x-A}}{x^2} dx = \frac{\pi}{\sqrt{A}} \to 0.
\]

Moreover,

\[
E \sum_{|w_{n,k}| \geq A} \frac{1}{w_{n,k}^2} = \int_{-\infty}^{-A} \frac{1}{x^2} n dx + \int_{A}^{+\infty} \frac{1}{x^2} n dx \leq 2 \frac{1}{A} \to 0.
\]

From (30) we deduce that \( z \mapsto \prod_{|w_{n,k}| \geq A} \left( 1 - \frac{z}{w_{n,k}} \right) \) converges in probability to \( z \mapsto 1 \) on every compact sets as \( A \) goes to \( +\infty \), uniformly in \( n \), which gives the claim. \( \square \)

Proposition 25 and Proposition 26 together show

\[
\prod_{k=1}^{p_n} \prod_{j \in \mathbb{Z}} \left( 1 - \frac{z}{\Phi_k - j} \right) \xrightarrow{n \to +\infty} \prod_{k \in \mathbb{Z}} \left( 1 - \frac{z}{\Phi_k} \right)
\]

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(see remark of Theorem 23 for the sense given to this last non-absolutely convergent product), hence

$$
\prod_{k=1}^{p_n} \frac{e^{2i\pi \frac{k}{y_k}} - \frac{1}{y_k}}{1 - \frac{1}{y_k}} \xrightarrow{n \to \infty} e^{i\pi z(1 - y_0)} \prod_{k \in \mathbb{Z}} \left( 1 - \frac{z}{w_k} \right).
$$

Finally, in the same manner as in the proof of point (i) of Theorem 10, we prove the almost sure convergence of $\prod_{j \geq 1}^{\ell_n > j \geq 0} \frac{e^{2i\pi zy_j} - u_j}{1 - u_j}$ to $\prod_{j = 1}^{\infty} \frac{e^{2i\pi zy_j} - u_j}{1 - u_j}$. Conditionally on $y$ and $(u_j)_{j \geq 1}$, Slutsky’s theorem applies on the functional space $C(\mathbb{C}, \mathbb{C})$, which allows to conclude using the dominated convergence theorem.

### 4.2 Proof of Theorem 23 (ii)

The proof of point (ii) is much simpler. Indeed, it suffices to see that for all $n$ and for all $z$ in any compact subset $K$ of $\mathbb{C}$,

$$
\left( \frac{e^{2i\pi \frac{k}{z + \alpha}} - 1}{e^{2i\pi \alpha} - 1} \right)^{p_n} = \left( 1 + \frac{e^{2i\pi \frac{k}{z + \alpha}} - 1}{e^{2i\pi \alpha} - 1} \right)^{p_n}
$$

\[= \exp \left( p_n \frac{2i\pi \epsilon^{2i\pi \alpha} z}{e^{2i\pi \alpha} - 1} + O \left( \frac{1}{n^2} \right) \right)\]

\[= \exp \left( p_n \frac{2i\pi \epsilon^{2i\pi \alpha} z}{n} + O \left( \frac{1}{n} \right) \right)\]

\[= \exp \left( (1 - y_0) \frac{2i\pi \epsilon^{2i\pi \alpha} z}{e^{2i\pi \alpha} - 1} + o(1) \right)\]

uniformly in $z \in K$, and for all $n$ large enough depending on $\alpha$ and $K$.

Finally, simplifying $\frac{2i\pi \epsilon^{2i\pi \alpha}}{n \sin(\pi \alpha)} = \frac{1}{\tan(\pi \alpha)} + i$ gives the claim.

### 4.3 Properties of the limiting functions

**Lemma 27.** Let $\epsilon > 0$. Almost surely, for all $k \in \mathbb{Z}$,

$$
w_k = \frac{k}{1 - y_0} + O \left( k^{\frac{1}{2} + \epsilon} \right). \quad (31)
$$

We omit the proof of this lemma since it is a classical result on Poisson processes.

**Proposition 28.** For all $\epsilon > 0$, there exists a random number $C > 0$ such that for all $z \in \mathbb{C}$,

$$
|\tilde{C}_\infty(z)| \leq e^{C|z| \log(2 + |z|)}. \quad (32)
$$

**Proof.** First, following the same reasoning as in Section 3, it is easy to check that for all $\epsilon > 0$, there exists a random number $C_\epsilon > 0$ such that for all $z \in \mathbb{C}$,

$$
\left| \prod_{j=1}^{\infty} \frac{e^{2i\pi zy_j} - u_j}{1 - u_j} \right| \leq C_\epsilon e^{(2\pi y_0 + \epsilon)|z|}.
$$

Hence there exists a random number $c > 0$ such that for all $z \in \mathbb{C}$,

$$
\left| \prod_{j=1}^{\infty} \frac{e^{2i\pi zy_j} - u_j}{1 - u_j} \right| e^{i\pi z(1 - y_0)} \leq e^{c|z|}.
$$

Thus, it is enough to show that there exists a random number $C > 0$ such that for all $z \in \mathbb{C}$,

$$
\left| \prod_{k \in \mathbb{Z}} \left( 1 - \frac{z}{w_k} \right) \right| \leq e^{C|z| \log(2 + |z|)}. \quad (33)
$$
To this end, we distinguish between two regimes of \( k \neq 0 \) in this product: \( |k| \geq |z| \), and \( 1 \leq |k| \leq |z| \).

For the first regime, using the previous lemma with \( \varepsilon = \frac{1}{3} \) gives

\[
(1 - \frac{z}{w_k})(1 - \frac{z}{w_{-k}}) = 1 + \mathcal{O}\left(\frac{|z|^2 k^2}{k^2} + |z|^2 k^{-2}\right),
\]

hence

\[
\prod_{|k| \geq |z|} \left| 1 - \frac{z}{w_k} \right| \left| 1 - \frac{z}{w_{-k}} \right| \leq \exp\left(\mathcal{O}\left(\sum_{|k| \geq |z|} |z| k^{-\frac{7}{6}} + |z|^2 k^{-2}\right)\right)
\]

(34)

For the second regime, as \( |\frac{z}{w_k}| \) is almost surely bounded from below (since \( \frac{z}{w_k} = \frac{1}{1 - y_0} + \mathcal{O}\left(k^{-\frac{7}{6}} + \varepsilon\right)\)), we have

\[
1 - \frac{z}{w_k} = 1 + \mathcal{O}\left(\left|\frac{z}{k}\right|\right)
\]

and it follows

\[
\prod_{1 \leq k < |z|} \left| 1 - \frac{z}{w_k} \right| \left| 1 - \frac{z}{w_{-k}} \right| \leq \exp\left(\mathcal{O}\left(\sum_{1 \leq k < |z|} \frac{|z|}{k}\right)\right)
\]

(35)

Furthermore,

\[
\left| 1 - \frac{z}{w_0} \right| \leq \exp\left(\frac{|z|}{|w_0|}\right) = \exp(\mathcal{O}(|z|))
\]

(36)

since \( w_0 \neq 0 \) almost surely. Combining (34), (35) and (36), we deduce the existence of a random number \( C > 0 \) such that for all \( z \in \mathbb{C} \) we have (33), and the proof is complete.

\[\blacksquare\]

**Proposition 29.** For all \( \varepsilon > 0 \), there exists a random number \( C_\varepsilon > 0 \) such that for all \( z \in \mathbb{C} \),

\[
|\xi_{\infty, \alpha}(z)| \leq C_\varepsilon e^{(\varepsilon+2\pi(y_0+(1-y_0)t_\alpha)|z|},
\]

(37)

where \( t_\alpha = \frac{1}{\sin(\alpha t_\alpha)} \in (\frac{1}{2}, +\infty) \).

The proof of the last proposition is very similar to the one for the case \( \sum_{j=1}^{+\infty} y_j = 1 \) almost surely. We omit it here and refer to Section 3.

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