Dilational symmetry-breaking in thermodynamics

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Abstract. Using thermodynamic relations and dimensional analysis we derive a general formula for the thermodynamical trace $2\mathcal{E} - D\mathcal{P}$ for nonrelativistic systems and $\mathcal{E} - D\mathcal{P}$ for relativistic systems, where $D$ is the number of spatial dimensions, in terms of the microscopic scales of the system within the grand canonical ensemble. We demonstrate the formula for several cases, including anomalous systems which develop scales through dimensional transmutation. Using this relation, we make explicit the connection between dimensional analysis and the virial theorem. This paper is focused mainly on the non-relativistic aspects of this relation.

Keywords: conformal field theory, cold atoms, quantum gases, renormalisation group
1. Introduction

The quantity $2\mathcal{E} - DP$ for non-relativistic systems, or $\mathcal{E} - DP$ for relativistic systems, where $\mathcal{E}$ is the thermal energy density, $D$ the number of spatial dimensions, and $P$ the pressure, plays an important role in physics. For example, this quantity is the thermal analog of the trace of the improved stress-energy tensor which is a measure of dilational symmetry-breaking and which plays a central role in the renormalization group [1–3].

In non-relativistic physics, $2\mathcal{E} - DP$ can be used as a measure of deviations of real gases from ideal ones. Traditionally, such deviations are measured by giving the two systems the same value for two of their thermodynamic variables, and taking the difference between them for a third. For example, the difference in pressure between a real gas and an ideal one at the same temperature and density can be written approximately as the virial equation: $P_{\text{real}} - P_{\text{ideal}} = \rho kT (B_2(T)\rho + B_3(T)\rho^2 + ...)$ [4]. For ideal gases, and in general non-anomalous scale-invariant systems, $2\mathcal{E} - DP = 0$. Therefore at constant pressure and volume, one can define $(2\mathcal{E}_{\text{real}} - DP) = (2\mathcal{E}_{\text{real}} - DP) - (2\mathcal{E}_{\text{ideal}} - DP) = (2\mathcal{E}_{\text{real}} - 2\mathcal{E}_{\text{ideal}}) \equiv 2\mathcal{E}_{\text{res}}$, so that $(2\mathcal{E}_{\text{real}} - DP)$ equals twice the residual internal energy characterizing the departure of the system from ideal [5]. In other words, for any system, $2\mathcal{E} - DP$ is equal to the difference in its energy from any non-anomalous scale-invariant system’s energy at the same $V$ and $P$. For ultracold gases interacting via contact interaction, $2\mathcal{E} - DP$ is
proportional to the Tan contact $\lambda^2\langle\bar{\psi_i}\psi_i\bar{\psi}_j\psi_j\rangle$ [6]. Many universal relations depending on the contact exist, independent of the exact details of the experimental setup [7, 8].

For systems that are scale invariant at the level of the classical action, a non-zero value of $2\mathcal{E} - DP$ signifies a quantum anomaly, so that $2\mathcal{E} - DP$ measures quantum anomalies. Previously, it was shown that even in anomalous non-relativistic systems, $2\mathcal{E} - DP$ can be expressed as a functional determinant via use of Fujikawa’s path integral methods [9–12]. Therefore, one can potentially extract information about $\beta(C)$ and hence obtain information from or even solve the scattering problem by extracting information from the thermodynamic problem.

In this paper we derive a simple expression for $2\mathcal{E} - DP$ from dimensional analysis and thermodynamics, independent of quantum mechanics or field theory and independent of Noether’s theorem and canonical commutation relations. In this paper we will use units where $\hbar = m = k_B = 1$. The units for all quantities can then be written as $\hbar m^{j}k_B^{k}L^{l} = L^{l}$, where $L$ is a variable in the problem with units of length. We will define $[g_k] = \ell$, and call $\ell$ the dimensions of the variable $g_k$.

With this convention, $2\mathcal{E} - DP = \sum_k [g_k]g_k\frac{\partial P}{\partial g_k}$, where $\mathcal{E} = \frac{E}{V}$ is the thermal energy per unit volume, $P$ is the pressure, and $D$ is the number of spatial dimensions. In this formula $g_k$ are the microscopic parameters of the theory, and $[g_k]$ are the dimensions of these parameters. The derivatives w.r.t. microscopic parameters are taken at constant temperature $\beta^{-1}$, volume $V$, and chemical potential for each species $\mu_i$. The LHS is written in terms of pure macroscopic thermodynamic variables, while the RHS contains derivatives purely on the microscopic parameters. Such an equation can be seen as connecting thermodynamics on the LHS (variables characterizing the macrostate) and statistical mechanics on the RHS (microscopic variables that are system dependent). In particular, for a theory in which all the couplings are dimensionless (in the sense that they have no length dimension as defined above), $[g_k] = 0$, and one might expect the system to be scale invariant with $2\mathcal{E} - DP = 0$. However, for such systems, we show $2\mathcal{E} - DP = -\beta(C)\frac{\partial P}{\partial C}$. The microscopic parameters $g_k$ of a system usually appear in its Hamiltonian as coupling constants, except in the case of dimensional transmutation. The latter leads to a new scale appearing in the pressure $P$, and in the literature this is called a quantum anomaly [13]. Therefore $2\mathcal{E} - DP$ is also a measure of the anomaly for classically scale-invariant systems.

The relativistic generalization is $\mathcal{E} - DP = \sum_k [g_k]g_k\frac{\partial P}{\partial g_k}$. It was shown in [14, 15] that the trace of the improved stress-energy tensor in relativistic $\lambda\phi^4$ has the property $\theta^{00} - \sum_i \theta^{ii} = m^2\phi^2$, where the mass term represents a dilational symmetry-breaking term and the improved stress-energy tensor $\theta^{\mu\nu}$ is related to the canonical one $T^{\mu\nu}$ by $\theta^{\mu\nu} = T^{\mu\nu} + \frac{D-1}{4D} (g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\phi^2$ [16]. Identifying $\theta^{00}$ as $\mathcal{E}$ and $\sum_i \theta^{ii} = DP_H$, where $P_H$ is the hydrodynamic pressure, one derives the thermal analog, $\theta^{ii}$ is equal to the hydrodynamic pressure [17]; however, in equilibrium, the thermodynamic pressure $P$ equals $P_H$ via the virial theorem (although anomalies can complicate matters [18]). Therefore deriving this expression from the stress-energy tensor requires an improvement of the stress-energy tensor, and an identification of field variables with thermodynamic variables. Similarly, the canonical stress-tensor $T_{ij}$ for nonrelativistic fields can be improved to $\theta_{ij} = T_{ij} + \frac{D}{4(D-1)} (\delta_{ij}\nabla^2 - \partial_i\partial_j) (\psi^\dagger\psi)$ such that $2\theta_{00} - \theta_{ii}$, which can be
identified with $2\mathcal{E} - DP$, gives the divergence of the dilation current [19]. We avoid the complications of having to construct the improved stress-energy tensor, or having to work in the context of field theory, by working directly within thermodynamics. We show the consistency of the equation for a variety of cases, with and without anomalies, and then we show that starting from $2\mathcal{E} - DP = \sum_k [g_k] g_k \frac{\partial P}{\partial g_k}$, one can derive the virial theorem, further illustrating the correctness of the expression and showing the relationship between scaling and the virial theorem. The relativistic case is also considered.

While the emphasis of this paper is on structural aspects of the thermodynamical traces $2\mathcal{E} - DP$ and $\mathcal{E} - DP$ (nonrelativistic and relativistic cases respectively), we have included in section 4 a few examples that connect our results with the literature and illustrate how to apply the techniques developed here. In our recent work [9–11, 20] we have aimed at offering a different approach and new perspectives to the study of the type of systems presented here. We hope to use the insight gained to apply these techniques and concepts to other problems and systems [21].

2. Proof of relation

For ease of presentation, take the independent, dimensionful microscopic parameters $g_k$ of our theory, and form new parameters $E_k$ with dimensions of energy by raising $g_k$ to the appropriate power, and rewrite the pressure in terms of these new variables [1]. The grand potential $\Omega = \Omega(\beta, \mu, V, E_i)$ for a homogeneous system in $D$-spatial dimensions must have the form

$$\Omega(\beta, z_i, V, E_i) = V \beta^{-\frac{D}{2}} f(z_i, \beta E_i),$$

(1)

where $f(z_i, \beta E_i)$ is a dimensionless function of dimensionless variables, and $z_i = e^{\beta \mu_i}$ is the fugacity corresponding to $\mu_i$. The reason $\Omega$ must have this form is because $\beta$ and $\mu_i$ don’t depend on the absolute size of the system (they are intensive variables). If one doubles the system keeping $\beta$ and $\mu_i$ constant, then $\Omega$, being an extensive quantity, should double. So $\Omega$ must be proportional to $V$. To make up for the remaining dimension ($[\Omega] = -2$), we are free to pull out one of the dimensionful arguments of $\Omega$, and the rest of the arguments must be ratios with the argument we pulled out. We will pull out $\beta$. This is equivalent to choosing our scale as $\beta$ and measuring all other quantities in units of $\beta$.

Now take the derivative of equation (1) w.r.t. to $\beta$ at constant fugacity $z_i$ and volume $V$, and multiply times $\beta$:

$$\beta \left. \frac{\partial \Omega}{\partial \beta} \right|_{z_i, V} = \left. \left( -1 - \frac{D}{2} \right) \Omega + V \beta^{-\frac{D}{2}} \beta \frac{\partial f(z_i, \beta E_i)}{\partial \beta} \right|_{z_i}$$

$$= \left. \left( -1 - \frac{D}{2} \right) \Omega + V \beta^{-\frac{D}{2}} \beta \left[ \sum_k E_k \frac{\partial f(z_i, \beta E_i)}{\partial E_k} \right] \right|_{z_i}$$

$$= \left. \left( -1 - \frac{D}{2} \right) \Omega + \sum_k E_k \frac{\partial \Omega}{\partial E_k} \right|_{z_i}. \quad (2)$$

---

1 E.g. if one has a scattering length $a$, replace it with the variable $E_k = 1/a^2 = \hbar/ma^2$. 

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Now, we use the thermodynamic identity \( E = \frac{\partial (\beta \Omega)}{\partial \beta} \bigg|_{z_i, V} = \Omega + \beta \frac{\partial \Omega}{\partial \beta} \bigg|_{z_i, V} \).

\[
2E - DPV = 2 \left( \Omega + \beta \frac{\partial \Omega}{\partial \beta} \bigg|_{z_i, V} \right) - DPV
\]

\[
= 2 \left( \Omega + \left( -1 - \frac{D}{2} \right) \Omega + \sum_k E_k \frac{\partial \Omega}{\partial E_k} \bigg|_{z_i, V} \right) - DPV
\]

\[
= -2 \left( P + \left( -1 - \frac{D}{2} \right) P + \sum_k E_k \frac{\partial P}{\partial E_k} \bigg|_{z_i, V} \right) V - DPV
\]

\[
= -2 \sum_k E_k \frac{\partial P}{\partial E_k} V
\]

\[
2\mathcal{E} - DP = -2 \sum_k E_k \frac{\partial P}{\partial E_k} \quad (3)
\]

For 0-temperature, we lose \( \beta \) as a scale. Instead we use \( \mu_1 \), where \( \mu_1 \) is the chemical potential for one of the particles:

\[
\Omega = V \mu_1^{1+D/2} f \left( \frac{\mu_1}{E_i}, \frac{\mu_1}{\mu_{j \neq 1}} \right). \quad (4)
\]

The calculation is done in appendix B.

In general, as long as the theory has microscopic parameters \( g_i \) that have dimensions of length (and not necessarily energy or \( L^{-2} \)), then by forming appropriate dimensionless variables \( x_i = \beta^{-\frac{|g_i|}{2}} g_i \) for the argument of \( \Omega(\beta, z_i, V, g_i) = V \beta^{-1-D} f(z_i, \beta^{-\frac{|g_i|}{2}} g_i) \), then one gets:

\[
2\mathcal{E} - DP = \sum_k [g_k] g_k \frac{\partial P}{\partial g_k} \quad (5)
\]

Alternatively, one can note that \( E_k = g_k^{-\frac{2}{|g_k|}} \), and apply the chain rule to equation (3) to get equation (5).

3. Relativistic Systems

In relativistic theories, \( \hbar = c = k_B = 1 \), and mass attains a dimension equal to \( 1/L \). The units for all quantities can then be written as \( \hbar^i c^j k_B^k L^\ell = L^i \), and we define the dimensions of the parameter \( g_k \) as \([g_k] = \ell\). The grand potential \( \Omega \) has \([\Omega] = -1 \) rather than the NR case \([\Omega] = -2 \), and can be written as:

\[
\Omega(\beta, z_i, V, E_i) = V \beta^{-1-D} f(z_i, \beta E_i). \quad (6)
\]

Following more or less the same steps as before one derives:
\[ \mathcal{E} - DP = \sum_k [g_k] g_k \frac{\partial P}{\partial g_k}, \]  

(7)

where again as in the nonrelativistic case, the derivatives are taken w.r.t. constant \( \beta^{-1}, V, \) and \( \mu_i. \)

4. Examples

4.1. No anomalies, no dimensionful parameters

The Tonks–Girardeau gas \([22, 23]\) is a 1-dimensional gas of bosons interacting via a two-body contact potential \( V = 2g \delta(x_i - x_j) \) in the limit \( g \to \infty. \) At zero temperature, \( \mathcal{E} = \frac{\pi^2}{6} \rho^3 \) and \( P = \frac{\pi^2}{3} \rho^3. \) Since the only dimensional parameter grows without bound, the gas is scale invariant, and hence by equation (5):

\[ 2\mathcal{E} - P = 0. \]  

(8)

4.2. No anomalies, dimensionful parameters

For a contact-interaction Bose gas at zero temperature (i.e. \( \mathcal{L} = \psi^\dagger \left( i\partial_t + \frac{\nabla^2}{2} \right) \psi - \frac{g}{2} (\psi^\dagger \psi)^2 \)), in odd dimensions \( D = 2n + 1 \) (perfectly finite in dimensional regularization, no anomalies), one can make the following 1-loop calculation for small coupling \([24, 25]\)

\[ \Omega = \left( -\frac{1}{2} \frac{\mu^2}{g} - L_D \mu \frac{D+1}{2} \right) V, \]  

(9)

where \( \Omega \) is the grand potential, \( L_D = \frac{\Gamma(1 - \frac{D}{2}) \Gamma(\frac{D+1}{2})}{2 \pi^{\frac{D+1}{2}} \Gamma(\frac{D}{2}+2)} \) is a pure number that depends only on dimension. We will verify equation (5) by computing the LHS involving macroscopic thermodynamic parameters by using thermodynamic relations on equation (9). Then we will calculate the LHS of equation (5) by differentiation w.r.t. microscopic parameters of equation (9), and compare the two results.

For the LHS, the following thermodynamic identities will be used, true for any homogeneous system:

\[ \Omega = -PV; \]

\[ \Omega = E - TS - \mu N \Rightarrow E = \Omega + \mu N; \]

\[ N = -\frac{\partial \Omega}{\partial \mu}. \]  

(10)

Calculating \( N \) for equations (9) using (10):

\[ N = \left( \frac{\mu}{g} + L_D \left( \frac{D}{2} + 1 \right) \mu \frac{D}{2} \right) V. \]  

(11)
Therefore:
\[
2E - DPV = 2(\mu N - PV) - DPV = 2\mu N - (D + 2)PV
\]
\[
= 2\mu \left( \frac{\mu}{g} + L_D \left( \frac{D}{2} + 1 \right) \mu \frac{g}{2} \right) V + (D + 2) \left( -\frac{1}{2} \frac{\mu^2}{g} - L_D \mu \frac{g}{2} + 1 \right) V
\]
\[
= \left( \left[ 1 - \frac{D}{2} \right] \frac{\mu^2}{g} \right) V
\]
\[
2\mathcal{E} - DP = \left( \left[ 1 - \frac{D}{2} \right] \frac{\mu^2}{g} \right).
\tag{12}
\]

Now make the same calculation but using the microscopic scales. Since we restrict ourselves to \( D = 2n + 1 \), there is no renormalization scale as everything is perfectly finite, a feature peculiar to odd dimensions \cite{odd_dimensions}. However, there is a microscopic length scale associated with the coupling \( g \), where \( g = D - 2 \):
\[
\partial P \partial g (\frac{g}{g}) = -\partial \left( \frac{\mathcal{E} V}{g} \right) (D - 2) g = \left( \left[ 1 - \frac{D}{2} \right] \frac{\mu^2}{g} \right).
\tag{13}
\]

The above calculation was performed for small coupling \( g \) so that perturbation theory could be used, but the relationship is in fact general. The Lieb–Liniger model which describes \( N \) bosons in one dimension interacting via a two-body contact potential interaction \( H = -\sum_i \frac{1}{2} \frac{\partial^2}{\partial x^2} + \sum_{i<j} 2g\delta(x_i - x_j) \) is exactly solvable in quantum mechanics, and the thermodynamic limit \( N, L \to \infty \) with \( \frac{N}{L} = \rho = \text{constant} \) can be taken \cite{lieb_liniger}. Approximate closed-form solutions exist for large and small \( g \) at zero temperature. For large coupling:
\[
\mathcal{E} = \frac{\sqrt{2} \mu^{3/2}}{3\pi} + \frac{4\mu^2}{3\pi^2 g} + \frac{14\mu^{5/2}}{3\sqrt{2}\pi^3 g^2} + \frac{32(100 - 9\pi^2)\mu^3}{405\pi^4 g^3},
\]
\[
P = \frac{2\sqrt{2} \mu^{3/2}}{3\pi} + \frac{4\mu^2}{3\pi^2 g} + \frac{14\mu^{5/2}}{9\pi^3 g^2} + \frac{16(100 - 9\pi^2)\mu^3}{405\pi^4 g^3},
\]
\[
- g \frac{\partial P}{\partial g} = \frac{4\mu^2}{3\pi^2 g^2} + \frac{28\sqrt{2} \mu^{5/2}}{9\pi^3 g^3} + \frac{16(100 - 9\pi^2)\mu^3}{135\pi^4 g^4}.
\tag{14}
\]

In the limit \( \frac{g}{\sqrt{\rho}} \to \infty \), the residual energy goes to zero and one gets the Tonks–Girardeau gas of equation (8).

For small coupling:
\[
\mathcal{E} = \frac{\mu^2}{4g} + \frac{\mu^{3/2}}{3\pi},
\]
\[
P = \frac{\mu^2}{4g} + \frac{2\mu^{3/2}}{3\pi},
\]
\[
- g \frac{\partial P}{\partial g} = \frac{\mu^2}{4g},
\tag{15}
\]

which agrees with the Bogoliubov approximation of equation (12) when \( g \to g \frac{3}{2} \) and \( D = 1 \).
For fermions in 3-dimensions interacting via contact interactions
\[ \mathcal{L} = \psi^\dagger \left( i \partial_t + \frac{\nabla^2}{2} \right) \psi - 4\pi a \psi^\dagger \psi \psi^\dagger \psi, \quad [a] = 1: \]
\[ 2\mathcal{E} - 3P = [a]a \frac{\partial P}{\partial a}. \tag{16} \]

Now \( \beta PV = \ln \int [d\psi d\psi^\dagger] e^{-\int_0^\beta \mathcal{L}_0 + 4\pi a \psi^\dagger \psi \psi^\dagger \psi} \) so that differentiating the path integral w.r.t. \( a \) gives
\[ [a]a \frac{\partial P}{\partial a} = -4\pi a \langle \psi^\dagger \psi^\dagger \psi \psi \rangle. \tag{17} \]

Plugging this into equation (16), we get Tan’s pressure relation:
\[ 2\mathcal{E} - 3P = -4\pi a \langle \psi^\dagger \psi^\dagger \psi \psi \rangle = -\frac{C}{4\pi a}, \tag{18} \]
where \( C = (4\pi a)^2 \langle \psi^\dagger \psi^\dagger \psi \psi \rangle \) is the Tan contact [6]. Equation (18) was also derived using the Hellmann–Feynman theorem and dimensional arguments in the canonical ensemble in [28]. Indeed, the energy \( E \), Helmholtz energy \( F \), Gibbs energy \( G \), and grand potential \( \Omega \) are related by Legendre transformation that trades conjugate macroscopic variables but leaves the microscopic parameters alone. Therefore:
\[ \left. \frac{\partial \Omega}{\partial g} \right|_{\mu,V,T} = \left. \frac{\partial F}{\partial g} \right|_{N,V,T} = \left. \frac{\partial G}{\partial g} \right|_{N,P,T} = \left. \frac{\partial E}{\partial g} \right|_{S,V,T}. \tag{19} \]

In fact, for the Lieb–Liniger model, where the N-body system is exactly solvable, it is more natural to work with the density \( \rho \) instead of the chemical potential \( \mu \):
\[ \mathcal{E} = \frac{1}{6} \pi^2 \left( 1 - \frac{1}{3} 12 \gamma^2 + \frac{32 (\pi^2 - 15)}{15 \gamma^3} \right) \rho^3 \quad \text{(strong)}, \]
\[ \mathcal{E} = \frac{1}{2} \left( \gamma - \frac{4 \gamma^{3/2}}{3 \pi} \right) \rho^3 \quad \text{(weak)}, \tag{20} \]
where \( \gamma = \frac{2a}{\rho} \). By the third law of thermodynamics, along a zero temperature path, variations in the coupling \( \gamma \) occur at constant entropy \( S \), so twice the residual energy is:
\[ g \frac{\partial E}{\partial g} = \gamma \frac{\partial E}{\partial \gamma} = \frac{1}{6} \pi^2 \left( \frac{4}{\gamma} - \frac{24}{\gamma^2} - \frac{32 (\pi^2 - 15)}{5 \gamma^3} \right) \rho^3 = g \left( \langle \psi^\dagger (x) \rangle^2 \langle \psi (x) \rangle^2 \right) \quad \text{(strong)}, \]
\[ = \frac{1}{2} \left( \gamma - \frac{2 \gamma^{3/2}}{\pi} \right) \rho^3 = g \left( \langle \psi^\dagger (x) \rangle^2 \langle \psi (x) \rangle^2 \right) \quad \text{(weak)} \tag{21} \]

### 4.3. Anomalies, no dimensionful parameters

A Fermi-gas in \( D = 2 \) has no dimensionful parameters in the Lagrangian, \( \mathcal{L} = \psi^\dagger \left( i \partial_t + \frac{\nabla^2}{2} \right) \psi - C \psi^\dagger \psi \psi^\dagger \psi, \quad [C] = 0 \). Nevertheless, the system develops a bound
state via dimensional transmutation, and the derivation of equation (3) goes through with \( E_k = E_b \), the bound state energy. Using cutoff regularization, the \( T \)-matrix is [29]:

\[
\frac{1}{T(E)} = \frac{1}{C} - \frac{1}{4\pi} \ln \left( \frac{-E}{\Lambda^2} \right).
\] (22)

The bound state is special since \( T(E) \) blows up there, so that \( \frac{1}{T(-E_b)} = 0 \). Therefore plugging in \( E = E_b \) into equation (22) gives:

\[
\frac{1}{C} = \frac{1}{4\pi} \ln \left( \frac{E_b}{\Lambda^2} \right).
\] (23)

Taking the derivative w.r.t. \( E_b \) on both sides of equation (23):

\[
-\frac{dC}{dE_b} = \frac{1}{4\pi E_b} \quad \text{and} \quad \frac{dC}{dE_b} = -\frac{C^2}{4\pi E_b}.
\] (24)

Therefore:

\[
-2E_b \frac{\partial P}{\partial E_b} = -2E_b \frac{dC}{dE_b} \frac{\partial P}{dC} = \frac{C^2}{2\pi} \frac{\partial P}{\partial C}.
\] (25)

Now \( \beta PV = \ln \int [d\psi d\psi^\dagger] e^{-\int_0^\beta t \int d^2 x \left( \mathcal{L}_0 + C\psi^\dagger \psi \psi^\dagger \psi \right)} \) so that differentiating the path integral w.r.t. \( C \) we obtain:

\[
\frac{\partial P}{\partial C} = -\langle \psi^\dagger \psi \psi \psi \rangle.
\] (26)

Plugging this result into equation (25) and using \( 2\mathcal{E} - DP = -2E_b \frac{\partial P}{dE_b} \):

\[
2\mathcal{E} - 2P = -\frac{C^2}{2\pi} \langle \psi^\dagger \psi \psi \psi \rangle,
\] (27)

agreeing with [30]. The coupling is bare, but the RHS is finite, and both sides are RG-invariant.

In our example, for equation (5), the dimensionful parameter is the bound-state energy. If one has a pressure written in term of bare parameters and cutoff \( P = P(C, \Lambda) \) or renormalized with scale \( \mu, P = P(C_R, \mu) \), then it is not correct to regard \( \Lambda \) or \( \mu \) as a microscopic parameter with dimensions of momentum \( (L^{-1}) \), because \( \frac{dP}{d\Lambda} = \frac{dP}{d\mu} = 0 \), so that there is in fact no dependence on these parameters. For our particular example, from equation (23), it is true that \( 2E_b \frac{dC}{dE_b} = -\Lambda \frac{dC}{d\Lambda} = -\beta(C) \) where \( \beta(C) \) is the beta function of the theory, so that equations (25) into our (5) would give:

\[
2\mathcal{E} - DP = -\beta(C) \frac{\partial P}{\partial C},
\] (28)

and comparison with equation (27) allows us to read off \( \beta(C) = \frac{C^2}{2\pi} \).
5. Connection with virial theorem

In previous work [20] we derived the virial theorem via path integrals, and then used the virial theorem to derive equation (5). One can also work backwards from equation (5) to derive the virial theorem by following the argument backwards. We reproduce the argument here. For a two-body potential $U = \frac{1}{2} \int d^Dx d^Dy \psi^*(\tau, \vec{x}) \psi(\tau, \vec{x}) V(\vec{x} - \vec{y}) \psi^*(\tau, \vec{y}) \psi(\tau, \vec{y})$:

$$2E - DPV = V \sum_k [g_k] g_k \frac{\partial P}{\partial g_k}$$

$$= \sum_k [g_k] g_k \frac{1}{\beta} \ln \int [d\psi] \int [d\psi^\dagger] e^{-\int_0^\beta \int d\tau \int d\vec{x} \psi^*(\tau, \vec{x}) \psi(\tau, \vec{x}) V(\vec{x} - \vec{y}) \psi^*(\tau, \vec{y}) \psi(\tau, \vec{y})}$$

$$= \sum_k [g_k] g_k \left( -\frac{1}{2} \right) \left\langle \int \int d\tau d\vec{x} d\vec{y} \psi^*(\tau, \vec{x}) \psi(\tau, \vec{x}) \frac{\partial V}{\partial g_k} \psi^*(\tau, \vec{y}) \psi(\tau, \vec{y}) \right\rangle. \quad (29)$$

Denoting $r = |\vec{x} - \vec{y}|$, one can show that $- \sum_k [g_k] g_k \frac{\partial V}{\partial g_k} = r \frac{dV}{dr} + 2V$ (see appendix A). Plugging this into equation (29) gives:

$$2E - DPV = \frac{1}{2} \left\langle \psi^*(\tau, \vec{x}) \psi(\tau, \vec{x}) r \frac{dV}{dr} \psi^*(\tau, \vec{y}) \psi(\tau, \vec{y}) \right\rangle + 2 \langle U \rangle$$

$$DPV = 2KE - \frac{1}{2} \left\langle \int d^Dx d^Dy \psi^*(\tau, \vec{x}) \psi(\tau, \vec{x}) \left[ (\vec{x} - \vec{y}) \cdot \nabla V(\vec{x} - \vec{y}) \right] \psi^*(\tau, \vec{y}) \psi(\tau, \vec{y}) \right\rangle,$$

which is the virial theorem [31–33].

6. Conclusion

We have derived an expression for $2E - DP$ using only dimensional arguments, valid for classical and quantum systems, for use in the grand canonical ensemble. We worked directly within the framework of thermodynamics, not having to improve the stress-energy tensor and invoke hydrodynamics, but instead working directly with thermodynamic variables. In the case of quantum systems, since the microscopic scales appear as coupling constants, or in the case of dimensional transmutation appear via the coupling constants, $\sum_k [g_k] g_k \frac{\partial P}{\partial g_k}$ manifests itself as thermal expectation values of the operators multiplying the coupling constants in the system’s Hamiltonian, which is manifest in the path integral formalism. Finally, using the path integral, we’ve shown how dimensional analysis leads to the virial theorem.

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Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Appendix A

The potential $V(r)$ has dimensions $[V] = -2$, so can generically be written:

$$V(r) = \frac{f\left(\frac{g}{r^2}\right)}{r^2},$$

(A.1)

$f$ is a dimensionless function whose arguments are the ratios of the couplings $g_i$ of $V(r)$ to their length dimension $[g_i]$ expressed in units of $r$.

$$r \frac{dV}{dr} = -2V(r) + \frac{1}{r} \frac{df\left(\frac{g}{r^2}\right)}{dr}$$

$$= -2V(r) - \frac{1}{r^2} \sum_i [g_i] g_i \frac{\partial f \left(\frac{g}{r^2}\right)}{\partial g_i}$$

$$= -2V(r) - \sum_i [g_i] g_i \frac{\partial V}{\partial g_i}$$

(A.2)

Appendix B

At zero temperature for a homogeneous system, the grand potential $\Omega$ can be written as:

$$\Omega = V \mu_1^{1+D/2} f\left(\frac{\mu_1}{E_1}, \frac{\mu_1}{\mu_j \neq 1}\right),$$

(B.1)

where $f$ is dimensionless function and $\mu_1$ is the non-zero chemical potential of one of the species. We calculate the number of particles:

$$N_1 = -\left. \frac{\partial \Omega}{\partial \mu_1} \right|_{V, \mu_j \neq 1}$$

$$= - \left(1 + D/2\right) \frac{\Omega}{\mu_1} - V \mu_1^{1+D/2} \frac{\partial f\left(\frac{\mu_1}{E_1}, \frac{\mu_1}{\mu_j \neq 1}\right)}{\partial \mu_1}$$

$$= - \left(1 + D/2\right) \frac{\Omega}{\mu_1}$$

$$- V \mu_1^{1+D/2} \left[ - \sum_k \frac{E_k}{\mu_1} \frac{\partial f\left(\frac{\mu_1}{E_1}, \frac{\mu_1}{\mu_j \neq 1}\right)}{\partial E_k} - \sum_{\ell \neq 1} \frac{\mu_{\ell \neq 1}}{\mu_1} \frac{\partial f\left(\frac{\mu_1}{E_1}, \frac{\mu_1}{\mu_j \neq 1}\right)}{\partial \mu_{\ell \neq 1}} \right]$$

$$= - \left(1 + D/2\right) \frac{\Omega}{\mu_1} + \sum_k \frac{E_k}{\mu_1} \frac{\partial}{\partial E_k} \Omega + \sum_{\ell \neq 1} \frac{\mu_{\ell \neq 1}}{\mu_1} \frac{\partial}{\partial \mu_{\ell \neq 1}} \Omega,$$

$$N_1 \mu_1 = - \left(1 + D/2\right) \Omega + \sum_k E_k \frac{\partial}{\partial E_k} \Omega - \sum_{\ell \neq 1} N_{\ell \neq 1} \mu_{\ell \neq 1},$$

$$\sum_i N_i \mu_i = - \left(1 + D/2\right) \Omega + \sum_k E_k \frac{\partial}{\partial E_k} \Omega.$$  

(B.2)

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The energy $E$ of the system at zero temperature is given by $E = \sum_i N_i \mu_i - PV$ which follows from the thermodynamic identity $E - TS + PV = \sum_i N_i \mu_i$. Therefore

$$2E - DPV = 2 \left( \sum_i N_i \mu_i - PV \right) - DPV = 2 \sum_i N_i \mu_i - (D + 2)PV$$

$$= 2 \left( - (1 + D/2)\Omega + \sum_k E_k \frac{\partial \Omega}{\partial E_k} \right) - (D + 2)PV$$

$$= 2 \left( - (1 + D/2)(-PV) + \sum_k E_k \frac{\partial}{\partial E_k}(-PV) \right) - (D + 2)PV$$

$$= -2V \sum_k E_k \frac{\partial}{\partial E_k}P,$$

$$2\mathcal{E} - DP = -2 \sum_k E_k \frac{\partial P}{\partial E_k}. \quad (B.3)$$

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