Optimal investment-consumption problem post-retirement with a minimum guarantee

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Abstract
We study the optimal investment-consumption problem for a member of defined contribution plan during the decumulation phase. For a fixed annuitization time, to achieve higher final annuity, we consider a variable consumption rate. Moreover, to eliminate the ruin possibilities and having a minimum guarantee for the final annuity, we consider a safety level for the wealth process which consequently yields a Hamilton-Jacobi-Bellman (HJB) equation on a bounded domain. We apply the policy iteration method to find approximations of solution of the HJB equation. Finally, we give the simulation results for the optimal investment-consumption strategies, optimal wealth process and the final annuity for different ranges of admissible consumptions. Furthermore, by calculating the present market value of the future cash flows before and after the annuitization, we compare the results for different consumption policies.

Keywords: Defined contribution plan, Duccumulation phase, Portfolio optimization, Final annuity guarantee, HJB equation, Policy iteration method, Finite difference method
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1 Introduction
In this work, focusing on the decumulation phase, we fix the annuitization time and investigate the optimal investment-consumption strategies prior to annuitization in a Brownian market model with time dependent mortality rate. We follow the framework developed in [15] in which a target for the consumptions during the decumulation phase and a target for the terminal accumulated wealth is considered. Moreover, motivated from [7], we consider a minimum guarantee for the final annuity.
Assuming a fixed rate of consumption throughout the whole period of time before annuitization, which is usually a long period, is far from the optimality. On the other hand, it is quite reasonable to consider a minimum consumption
rate for a retiree to cover the essential expenses. Therefore, we consider the rate of consumption as a control variable which varies between the two limits, $C_1$ and $C_2$, in which $C_1 > C_2$. We will see from the simulation results that considering a variable consumption rate yields much higher final annuities. To compare the optimal portfolios obtained from different scenarios of the admissible consumptions, we take into account the present market value of the future cash flows before and after the annuity purchase.

Gerrard et al. [13] study the portfolio optimization problem post-retirement when the loss function is defined totally by the wealth process and the annuitization time and consumption rate are fixed. In [14], they violate the fixed consumption rate assumption. In a similar framework, Di Giacinto et al. [7] explore the optimal investment strategy when a minimum guarantee for the final wealth is assumed and the consumption rate is fixed. When the running cost is neglected, they obtain the closed form of solution. In this paper, in a same framework, we develop a numerical algorithm based on the policy iteration method to find approximations of the optimal investment-consumption strategies when the consumption rate is variable and a running cost for the loss function based on the consumptions is considered. The policy iteration method is a well studied method in finding approximations of solutions of optimal control problems, see [8], [20], in which the value function and the optimal policies are deriving iteratively to converge to the correct solution of the corresponding HJB equation.

The portfolio optimization problem post-retirement has been studied by many scholars by considering different utility or loss functions, different control variables and different wealth dynamics. Furthermore, different constraints are considered on the control variables and on the wealth dynamics. We mention here a few related works. Milevsky and Young [18] study extensively the optimal annuitization and investment-consumption problem for time dependent mortality function in the all or nothing market and also the more general anything anytime market, where gradual annuitization strategies are allowed. Milevsky et al. [17] derive the optimal investment and annuitization strategies for a retiree whose objective is to minimize the ruin probability when the consumption rate is fixed. Stabile [21] studies the optimal investment-consumption problem and investigates the optimal time for purchasing the annuity subject to a constant force of mortality by considering different utility functions defined on the consumption and the final annuity. Blake et al. [4] compare the immediate purchasing the annuity at the retirement age with distribution programs involving differing exposures to equities during retirement. Albrecht and Maurer [1] compare the immediate annuitization and the income drawdown option and determine the probability of running out of money before the uncertain date of death. Gerrard et al. [15], by considering targets for the consumptions during the decumulation phase and a target for the final wealth, investigate the optimal annuitization time together with the optimal consumption-investment strategies.

The article is organized as follows. In the next section, we identify the market model, the loss function and also the set of admissible strategies. In the third section, to apply the dynamic programming principle, we specify the value function. Furthermore, by considering a safety level for the wealth process we specify the set of admissible strategies and write the HJB equation obtained via the dynamic programming principle corresponding to the optimal control prob-
lem. In section 4, we express the numerical algorithm to find approximations of solution of the HJB equation. Finally in section 5, we present the simulation results for the final annuity, optimal investment-consumption strategies and optimal wealth process.

2 The Market Model

We consider a Brownian market model consists of a risky and a risk-less asset with the dynamics:

\[ dS_t = S_t(\mu dt + \sigma dB_t), \]
\[ dA_t = rA_t dt, \]

where \( B(\cdot) \) is a Brownian motion on the filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})\) and \( r \) is the fixed interest rate. So, the risky asset is a geometric Brownian motion with constant volatility \( \sigma \) and expected return \( \mu = r + \sigma \beta \), in which \( \beta \) is its Sharpe ratio.

At any time \( t \) and when the fund value is \( x \), let \( y(t, x) \) and \( 1 - y(t, x) \) denote the proportions of the fund’s portfolio that are invested in the risky and in the riskless asset, respectively. Moreover, denote the consumption rate at the point \((t, x)\) by \( c(t, x)\). Therefore, we have the following dynamics of the wealth process (or the fund value)

\[ dX_t = \left\{ y(\mu - r) + r \right\} X_t dt + \sigma y X_t dB_t, \quad (2.1) \]
\[ X(0) = x_0. \]

Let the decumulation phase be denoted by the time interval \([0, T]\). We consider the functions of two variables \( y : [0, T] \times [0, \infty) \rightarrow \mathbb{R} \) and \( c : [0, T] \times [0, \infty) \rightarrow [C_2, C_1] \) as control variables that must be determined by solving an optimal control problem. We let the variable \( y \) to be greater than one, \( y > 1 \), or to be negative, \( y < 0 \), which correspond to borrowing from the money market and short-selling of the risky asset, respectively. However, our numerical results justify that the short-selling does not appear as the optimal investment strategy. We restrict the consumption rate to the interval \([C_2, C_1]\). Actually \( C_1 \) is the desired rate of consumption and usually is set to be equal to the annuity that is purchasable by the accumulated wealth at the retirement and \( C_2 \) corresponds to the minimum required consumption during the decumulation phase.

Our market model can be extended in several directions. Among many other models, Boulier et al. \[5\] consider a model with stochastic interest rate. Han and Hung \[11\] equip the market model with the inflation rate which has important consequences on the optimal investment strategy. Gao \[12\], in a stochastic interest rate framework, considers a market model that consists of three assets, a risky asset, a risk-less asset and a bond. Hainaut and Declra investigate the optimal time for annuity purchase in a market model with jump-diffusion dynamics when the economic utility function is replaced by the expected present value operator. Considering the jump-diffusion dynamics, which is much more justified by the empirical data, is the aim of our ongoing work.

We consider a retiree with age 60 who is going to postpone the annuitization until the age 75, which means \( T = 15 \). The main purpose of a retiree in postponing the annuity purchase is to reach the desired annuity. Let \( F \) be
the target for the terminal accumulated wealth, $X(T)$, by which the retiree can purchase the desired annuity at the age 75. In other words, if $a_{75}$ is the actuarial value of the unitary lifetime annuity at the age 75, then $\frac{F}{a_{75}}$ would be the desired annuity. Moreover, during the decumulation phase, part of the retiree’s concern is on the consumption and he or she would like to have the maximum consumption rate $C_1$. Therefore, we write the loss function by two terms. One term, as the running cost, is formulated by the consumption and is considered as the distance from the desired rate, $C_1$. Moreover, this term is weighted by the impact factor $\kappa$. The second term is based on the final annuity and is written as the distance from the specified target $\frac{F}{a_{75}}$.

So, supposing the mortality rate after the retirement, $\mu(t), t \geq 60$, to be independent of the asset dynamics, we write the loss function as

$$\kappa \int_0^T e^{-\int_0^t (\rho + \mu(s))ds} (C_1 - c(t))^2 dt + e^{-\int_0^t (\rho + \mu(s))ds} \left( \frac{F - X(T)}{a_{75}} \right)^2, \tag{2.2}$$

in which the constant factor $\rho$ is the subjective discount factor.

For any $0 \leq t \leq T$, consider the filter $\mathbb{F}^t := (\mathcal{F}^t_s)_{s \in [t,T]}$ where $\mathcal{F}^t_s$ is the $\sigma$-algebra generated by the random variables $(B(u) - \bar{B}(t))_{u \in [t,s]}$. Then at any time $t \geq 0$, we choose the admissible strategies $\pi_1(\cdot) \in \mathcal{L}^2(\Omega \times [t,T]; \mathbb{R})$ and $\pi_2(\cdot) \in \mathcal{L}^2(\Omega \times [t,T]; [C_2, C_1])$, which are $\mathbb{F}^t$-progressively measurable. It should be noted that if the variables $y$ and $c$ in Eq. (2.1) be replaced by $\pi_1(\cdot)$ and $\pi_2(\cdot)$, respectively, the equation will have a unique strong solution, see [10] Section 5.6.C. We denote this solution by $X(.; t, x, \pi_1(\cdot), \pi_2(\cdot))$.

The main feature of our framework is designing a minimum guarantee for the final annuity or equivalently for the terminal wealth amount. So, if we denote $S$ as the safety level of the terminal wealth, then the set of admissible investment and consumption strategies reduces to

$$\tilde{\Pi}_{ad}(t) := \{ \pi_1(\cdot) \in \mathcal{L}^2(\Omega \times [t,T]; \mathbb{R}), \pi_2(\cdot) \in \mathcal{L}^2(\Omega \times [t,T]; [C_2, C_1]) \mid \pi_1(\cdot), \pi_2(\cdot) \text{ are } \mathbb{F}^t - \text{prog.meas., } X(T; t, x, \pi_1(\cdot), \pi_2(\cdot)) \geq S \text{ a.s.} \} \tag{2.3}$$

3 The HJB Equation

Considering the loss function (2.2), we define the objective functional $\tilde{J}$, for any $(t, x) \in [0, T] \times \mathbb{R}^+$, on the set of admissible strategies $\tilde{\Pi}_{ad}(t)$, as

$$\tilde{J}(t, x; \pi_1(\cdot), \pi_2(\cdot)) := \mathbb{E}^{t} \left[ \int_t^T e^{-\int_t^s (\rho + \mu(r))dr} (C_1 - \pi_2(s))^2 ds + e^{-\int_t^s (\rho + \mu(s))ds} \left( \frac{F - X(T)}{a_{75}} \right)^2 \right], \tag{3.1}$$

where $X(s), t \leq s \leq T$ is the wealth process obtained by employing the investment and consumption strategies $\pi_1(\cdot), \pi_2(\cdot) \in \tilde{\Pi}_{ad}(t)$ and $\mathbb{E}^t$ denotes the expectation subject to $X(t) = x$. Our goal is to find the admissible strategies that minimize the above functional. Therefore, to solve this stochastic optimal
control problem via the dynamic programming, we define the value function

\[
\tilde{V}(t,x) := \inf_{\pi_1(\cdot), \pi_2(\cdot) \in \Pi_{ad}(t)} \tilde{J}(t,x; \pi_1(\cdot), \pi_2(\cdot)).
\]  

(3.2)

The minimum guarantee constraint at the terminal time imposes consequently a constraint on the wealth process before the annuitization. Actually, the following curve is as a barrier for the wealth process,

\[
S(t) = C_2 - \frac{C_2}{r} - S e^{-r(T-t)}, \quad 0 \leq t \leq T.
\]  

(3.3)

Because, whenever the wealth process hits this curve, investing any portion of the wealth in the risky asset or consuming at any rate higher C_2, yields a positive probability for that final wealth to be less than S at the terminal time T. Therefore, thereafter, the only admissible investment and consumption policies will be the null strategy and consuming at the minimum rate, \(\pi_1(s) = 0\) and \(\pi_2(s) = C_2, s \geq t\), respectively, which keeps the wealth process on this curve until the terminal time T.

Moreover, at any time \(t \in [0, T]\), the wealth amount

\[
F(t) = \frac{C_1}{r} + (F - \frac{C_1}{r}) e^{-r(T-t)},
\]  

(3.4)

guarantees reaching the desired target F at the terminal time T, by investing the whole portfolio during the time interval \([t, T]\) in the risk-less asset and consuming at the maximum rate \(C_1\). This strategy keeps the wealth process on the curve \(\{F(t), 0 \leq t \leq T\}\) and yields zero cost. Therefore, it is the optimal strategy.

These observations indicate that the curves \(\{F(t), 0 \leq t \leq T\}\) and \(\{S(t), 0 \leq t \leq T\}\) are as the attractors of the wealth process. In the other words, if the initial value lies in \([S(0), F(0)]\) then the wealth process will remain in a domain bounded at the top and at the bottom by the curves \(\{F(t), 0 \leq t \leq T\}\) and \(\{S(t), 0 \leq t \leq T\}\), respectively.

Due to the definition of the value function \(\tilde{V}\), (3.2), the Bellman principle indicates the following HJB equation, see [19, Chapter 11],

\[
\inf_{y, c \in \mathbb{R}} \{\partial_t \tilde{V} + \tilde{A} \tilde{V}(t,x) + \kappa e^{-\int_0^t (\rho + \mu(s)) \, ds} (C_1 - c(t))^2\} = 0,
\]  

(3.5)

where \(\tilde{A}\) is the generator of the diffusion process \(2.1\),

\[
\tilde{A} = \{(y[\mu - r] + r)x - c\} \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2 y^2 x^2 \frac{\partial^2}{\partial x^2}.
\]

Additionally, the definition of \(\tilde{V}\) indicates the following boundary conditions

\[
\tilde{V}(T, x) = e^{-\int_0^T (\rho + \mu(s)) \, ds} \left(\frac{F - x}{a_{T5}}\right)^2, \quad x \in [S, F],
\]

\[
\tilde{V}(t, F(t)) = 0, \quad t \in [0, T],
\]

\[
\tilde{V}(t, S(t)) = e^{-\int_0^T (\rho + \mu(s)) \, ds} \left(\frac{F - S}{a_{T5}}\right)^2 + \kappa (C_1 - C_2)^2 \left(\int_t^T e^{-\int_0^t (\rho + \mu(r)) \, dr} \, ds\right), \quad t \in [0, T].
\]  

(3.6)
The above equation has been defined on an irregular domain, in which the top and the bottom borders are curved. Since we are going to apply the finite difference method as part of our algorithm, we apply a change of variable that converts the domain to a rectangle. To this end, we define the diffeomorphism \( L : C \to C' \), where \( C := \{(t, x)|t \in [0, T], S(t) \leq x \leq F(t)\}, C' := \{(t, z)|t \in [0, T], S \leq z \leq F\} \) and

\[
(t, x) \to (t, z) = L(t, x) = (t, \mathcal{L}_1(t, x)) := \left(t, xe^{r(T-t)} + \left[C_1 + (C_2 - C_1)\frac{F(t) - x}{F(t) - S(t)} \right] \frac{1 - e^{r(T-t)}}{r}\right).
\]

Notice that

\[
\mathcal{L}_1(t, S(t)) = S, \quad \mathcal{L}_1(t, F(t)) = F. \tag{3.7}
\]

Now, we define the process \( Z(\cdot) \) as

\[
Z(t) := \mathcal{L}_1(t, X(t)) = X(t)G(t) + H(t), \quad t \in [0, T],
\]

in which

\[
G(t) = e^{r(T-t)} + \frac{C_1 - C_2}{F(t) - S(t)} \frac{1 - e^{r(T-t)}}{r},
\]

\[
H(t) = \left(C_1 + \frac{F(t)(C_2 - C_1)}{F(t) - S(t)} \right) \frac{1 - e^{r(T-t)}}{r}.
\]

Applying the Itô formula, the process \( Z \) satisfies the dynamics

\[
dZ_t = X_t dG_t + G_t dX_t + dH_t
\]

\[
= X_t \left[ \left(\frac{C_1 - C_2}{F(t) - S(t)} - r\right) e^{r(T-t)} + \frac{(C_1 - C_2)(r(F_S) - C_1 + C_2)}{(F(t) - S(t))^2} \frac{1 - e^{-r(T-t)}}{r}\right]dt
\]

\[
+ G(t) [(y(\mu - r) + r)X_t - c]\ dt + \sigma_y X_t dW_t
\]

\[
+ \left[ C_1 + \frac{F(t)(C_2 - C_1)}{F(t) - S(t)} \right] e^{r(T-t)} dt
\]

\[
+ (C_2 - C_1) \frac{e^{-r(T-t)-1}}{r} \frac{(F(t) - S(t))(rF - C_1) - F(t)r(F - S) - C_1 + C_2}{(F(t) - S(t))^2} dt.
\]  

Moreover, by a few manipulations we get

\[
X_t = \frac{r(F(t) - S(t))e^{-r(T-t)}Z_t - [F(t)C_2 - S(t)C_1](e^{-r(T-t)} - 1)}{r(F(t) - S(t)) + (C_1 - C_2)(e^{-r(T-t)} - 1)}.
\]

We define for any \( \pi_1(\cdot), \pi_2(\cdot) \in \Pi_{ad}(t) \) the process

\[
Z(s; t, z, \pi_1(s), \pi_2(s)) := \mathcal{L}_1(s, X(s; t, x, \pi_1(s), \pi_2(s))), \quad t \leq s \leq T,
\]

in which \( Z(t) = z = \mathcal{L}_1(t, x) \). Then, because of the relations (3.7), the set of admissible strategies turns to

\[
\Pi_{ad}(t) = \{\pi_1(\cdot) \in \mathcal{L}^2(\Omega \times [t, T]; \mathbb{R}), \pi_2(\cdot) \in \mathcal{L}^2(\Omega \times [t, T]; [C_2, C_1]) \mid \pi_1(\cdot), \pi_2(\cdot) \text{ are } \mathbb{P}^t - \text{prog.meas.}, S \leq Z(s) \leq F, s \in [t, T]\}.
\]
Analogous to (3.1), we have for any \((t, z) \in C'\) the following functional defined on \(\Pi_{ad}(t)\) whose one term is represented by the process \(Z\left(\cdot; t, z, \pi_1(\cdot), \pi_2(\cdot) \right)\),
\[
J(t, z, \pi_1(\cdot), \pi_2(\cdot)) := \mathbb{E}^\pi \left[ \int_t^T e^{-\int_t^s (\rho + \mu(r)) \, dr} (C_1 - \pi_2(s))^2 \, ds + e^{-\int_t^T (\rho + \mu(s)) \, ds} \left( \frac{F - Z(T)}{a_{75}} \right)^2 \right].
\]
Now, defining the value function \(V\) as
\[
V(t, z) := \inf_{\pi_1(\cdot), \pi_2(\cdot) \in \Pi_{ad}(t)} J(t, z; \pi_1(\cdot), \pi_2(\cdot)), \quad (t, z) \in C',
\]
we get the following HJB equation on the domain \(C'\)
\[
\inf_{y, c \in \mathbb{R}} \left\{ \frac{\partial V}{\partial t} + \mathcal{A}V(t, z) + \kappa e^{-\int_0^t (\rho + \mu(s)) \, ds} (C_1 - c)^2 \right\} = 0, \tag{3.9}
\]
where \(\mathcal{A}\), the generator of the diffusion process \(Z(\cdot)\), is written as
\[
\mathcal{A} = \{ K(t, z) G'(t) + G(t) [y(\mu - r) + r] z - c + H'(t) \} \frac{\partial}{\partial z} + \frac{1}{2} G^2(t) \sigma^2 y^2 z^2 \frac{\partial^2}{\partial z^2}.
\]
In which
\[
K(t, z) = \frac{r(F(t) - S(t))e^{-r(T-t)}z - [F(t)C_2 - S(t)C_1](e^{-r(T-t)} - 1)}{r(F(t) - S(t)) + (C_1 - C_2)(e^{-r(T-t)} - 1)}.
\]
Since during the decumulation phase the loss function just depends on the consumptions, we get the following boundary conditions that are similar to (3.6).
\[
V(T, z) = e^{-\int_0^T (\rho + \mu(s)) \, ds} \left( \frac{F - z}{a_{75}} \right)^2, \quad z \in [S, F],
\]
\[
V(t, F) = 0, \quad t \in [0, T], \tag{3.10}
\]
\[
V(t, S) = e^{-\int_0^T (\rho + \mu(s)) \, ds} \left( \frac{F - S}{a_{75}} \right)^2 + \kappa (C_1 - C_2)^2 \int_t^T e^{-\int_t^s (\rho + \mu(r)) \, dr} \, ds, \quad t \in [0, T].
\]
The coefficients of Eq. (3.9) are Lipschitz continuous w. r. t. the state variables \((t, z)\) and the control variables \(y\) and \(c\). Therefore, they are bounded on the domain. Moreover, the control variable \(c\) is chosen from a compact interval. Although, we do not assume a bound for the variable \(y\), but practically it is located inside a bounded interval, too. Furthermore, the boundaries of the domain are the attractors of the wealth process. From these observations, we can conclude that the above equation has the non-degeneracy property. Now, Theorem 2.1] indicates the strong comparison property of Eq. (3.9) which implies the existence of viscosity solution for this equation.
4 Numerical Algorithm

We apply the policy iteration method in which the value function $V$ and the control variables $y, c$ are improved iteratively until they converge to the correct solution of the HJB Eq. (3.9). For the fixed control variable functions, we apply a fully implicit backward in time finite difference scheme to obtain the value function.

In discretizing the domain $C' = [0, T] \times [S, F]$, the time horizon $[0, T] = [0, 15]$ is divided to 780 subintervals with the length $\Delta t = \frac{1}{2\Delta t}$, which corresponds to the length of one week in a year. Moreover, the interval $[S, F]$ is divided to $N + 1$ subintervals with $N$ interior nodes, in which the length of each subinterval is $\Delta_z$. Starting from the last column $t = T - \Delta t$ inside the domain $C'$, we apply to each column the following algorithm.

1) Initial value: For the starting point, we can take the investment function $y = \frac{1}{2}$ and the consumption rate as $c = \frac{C_1 + C_2}{2}$.

2) Policy evaluation: For the given investment and consumption functions $y, c$, apply a fully implicit finite difference scheme to solve the PDE (3.9) for the value function $V$ with the corresponding boundary conditions (3.10).

3) Policy improvement: For the fixed $t$, at any node $(t, z_j)$, $2 \leq j \leq N(i) + 1$, which correspond to all nodes on the fixed column $t = t_i$ that are inside the domain $C'$, solve the static optimization problem (3.9) to find new optimal investment and consumption functions $y^\text{new}$ and $c^\text{new}$, respectively.

4) End of iteration: Applying the functions $y, c$ and $V$ obtained from the steps (II) and (III), whenever, at any node $(t, z_j)$, $2 \leq j \leq N + 1$, the absolute value of the left hand side of Eq. (3.9) is not less than $10^{-6}$ or one of the following convergence criterions does not satisfy, return to the step (II).

\[
\max_j |c^\text{new}(t, z_j) - c^\text{old}(t, z_j)| \leq \max_j |c^\text{new}(t, z_j)| \times 10^{-6},
\]

\[
\max_j |y^\text{new}(t, z_j) - y^\text{old}(t, z_j)| \leq \max_j |y^\text{new}(t, z_j)| \times 10^{-6},
\]

\[
\max_j |V^\text{new}(t, z_j) - V^\text{old}(t, z_j)| \leq \max_j |V^\text{new}(t, z_j)| \times 10^{-6}.
\]

5) Iteration policy in the preceding column: If $t > 0$ go to the time step $t - \Delta t$ and return to the step (II).

To apply the finite difference method in step (II), we consider the forward scheme for the time and the space first derivatives and the central difference scheme for the space second derivative. So, denoting $V(i, j) = V(t_i, z_j)$, the value function at the node $(t_i, z_j)$, $1 \leq i \leq 781, 1 \leq j \leq N + 2$, the discretization of the PDE (3.9) at the node $(t_i, z_j)$ that is inside the domain $C'$ is given by

\[
\frac{V(i + 1, j) - V(i, j)}{\Delta t} + \alpha e^{-\int_0^t (\rho + \mu(s))ds}v^2(C_1 - c(t_i, z_j))^2
\]

\[
+ \left( \frac{\alpha(i, j)}{\Delta z} + \frac{\beta(i, j)}{(\Delta z)^2} \right) V(i, j + 1) + \beta(i, j) V(i, j - 1)
\]

\[
- \left( \frac{\alpha(i, j)}{\Delta z} + \frac{\beta(i, j)}{(\Delta z)^2} \right) V(i, j) = 0,
\]

in which
\[\alpha(i, j) = K(t_i, z_j)G'(t_i) + G(t_i)[(y(\mu - r) + r)z_j - c] + H'(t_i),\]

\[\beta(i, j) = \frac{1}{2}G^2(t_i)\sigma^2 y^2 z_j^2.\]

Then, applying the fully implicit scheme for the nodes on a fixed column, \((t_i, z_j), 2 \leq j \leq N + 1\), we get the linear equation \(Aw = b\), in which \(w = (V(t_i, z_2), \ldots, V(t_i, z_{N+1}))\) is the unknown vector and \(A = [a_{ij}]_{N \times N}\) and \(b = [b_j]_{1 \times N}\) are a tridiagonal matrix and a positive vector, respectively, with the following entries:

\[a_{jj} = \frac{1}{\Delta t} + \frac{\alpha(i, j + 1)}{\Delta z} + \frac{2\beta(i, j + 1)}{(\Delta z)^2}, \quad 1 \leq j \leq N,\]

\[a_{j+1,j} = -\frac{\alpha(i, j + 1)}{\Delta z} - \frac{\beta(i, j + 1)}{(\Delta z)^2}, \quad 1 \leq j \leq N - 1,\]

\[a_{j-1,j} = -\frac{\beta(i, j + 1)}{(\Delta z)^2}, \quad 2 \leq j \leq N,\]

\[b_j = \frac{V(i + 1, j + 1)}{\Delta t} + \kappa e - \int_{t_i}^{t_i+\Delta t} \rho(s) + \mu(s) \, ds (C_1 - (c(t, z_j + 1))^2), \quad 2 \leq j \leq N,\]

\[b_1 = \frac{V(i + 1, 2)}{\Delta t} + \kappa e - \int_{t_i}^{t_i+\Delta t} \rho(s) + \mu(s) \, ds (C_1 - (c(t, z_2))^2) + \frac{\beta(i, 2)}{(\Delta z)^2} V(t_i, S).\]

It is clear from the definitions of the functions \(\alpha\) and \(\beta\) that the discrete representation (4.1) has the positive coefficient property, see [8] for the definition of this property, which consequently indicates that the matrix \(A\) is an M-matrix. The positive coefficient property and the boundary condition (3.10) easily yield the stability, consistency and the monotonicity of our numerical scheme, see [8]. Therefore, from [2] and [3], the convergence of our numerical approximations to the viscosity solution of Eq. (3.9) is guaranteed. Here we show just the stability property.

**Proposition 4.1.** The discretization (4.1) satisfies the \(l_\infty\)-stability property,

\[\|V(t_i, \cdot)\|_\infty \leq \left(\frac{F - S}{a_{75}}\right)^2 + \kappa(C_1 - C_2)^2, \quad 1 \leq i \leq 780.\]  

**Proof.** We have for every \(1 \leq i \leq 780\) and \(2 \leq j \leq N + 1\)

\[V(i, j) = V(i + 1, j) + \Delta t\alpha(i, j) \frac{V(i + 1, j + 1) - V(i, j)}{\Delta z} + \Delta t\beta(i, j) \frac{V(i + 1, j + 1) - 2V(i, j) + V(i, j - 1)}{(\Delta z)^2} + \Delta t\kappa e - \int_{t_i}^{t_i+\Delta t} \rho(s) + \mu(s) \, ds (C_1 - (c(t, z_j))^2).\]

So, we have
\[ |V(i,j)| \left( 1 + \frac{\Delta t}{\Delta z} \alpha(i,j) + 2 \frac{\Delta t}{(\Delta z)^2} \beta(i,j) \right) \leq \]
\[ \|V(i+1,\cdot)\|_\infty + \|V(i,\cdot)\|_\infty \left( \frac{\Delta t}{\Delta z} \alpha(i,j) + 2 \frac{\Delta t}{(\Delta z)^2} \beta(i,j) \right) + \Delta t \kappa e^{-\int_{t_i}^t (\rho + \mu(s)) \, ds} (C_1 - C_2)^2. \]

If \( V(i,j_1) = \max_{1 \leq j \leq N+2} V(i,j) = \|V(i,\cdot)\|_\infty \), then we can write
\[ \|V(i,\cdot)\|_\infty \left( 1 + \frac{\Delta t}{\Delta z} \alpha(i,j_1) + 2 \frac{\Delta t}{(\Delta z)^2} \beta(i,j_1) \right) \leq \]
\[ \|V(i+1,\cdot)\|_\infty + \|V(i,\cdot)\|_\infty \left( \frac{\Delta t}{\Delta z} \alpha(i,j_1) + 2 \frac{\Delta t}{(\Delta z)^2} \beta(i,j_1) \right) + \Delta t \kappa e^{-\int_{t_i}^t (\rho + \mu(s)) \, ds} (C_1 - C_2)^2, \]

which means
\[ \|V(i,\cdot)\|_\infty \leq \|V(i+1,\cdot)\|_\infty + \Delta t \kappa e^{-\int_{t_i}^t (\rho + \mu(s)) \, ds} (C_1 - C_2)^2. \]

Then, from the terminal condition of the value function we obtain the bound (4.2).

5 Simulation Results

For comparison purposes, we consider the same market parameters as in [13] and [7]. So, we consider the interest rate \( r = 0.03 \) and the expected return and the volatility of the risky asset \( \mu = 0.08 \) and \( \sigma = 0.15 \), respectively, which implies a Sharpe ratio equal to \( \beta = 0.33 \). Furthermore, we consider a retiree with age 60 and with the initial wealth \( x_0 = 100 \) and set the decumulation period equal to \( T = 15 \) years. The maximum consumption rate is set to be \( C_1 = 6.5155 \), which is assumed to be equal to the payments of a lifetime annuity purchasable by a retiree at age 60 with the wealth \( x_0 = 100 \), regarding the mortality rate given in this section.

We consider four different rates for the minimum consumption, \( C_2 = C_1, C_2 = \frac{3}{4} C_1, C_2 = \frac{1}{2} C_1 \) and \( C_2 = \frac{1}{3} C_1 \), which correspond to different minimum cost of living of a retiree. Moreover, we consider the target level \( F = 1.75 C_1 a_{75} \) and the safety level \( S = 0.5 C_1 a_{75} \) for the wealth process, which in the literature correspond to the medium level of risk aversion, see [13] and [7]. Actually, in this level, the final annuity that the retiree will get is at most 1.75 times \( C_1 \) and at least half of \( C_1 \).

By employing the dynamics (2.1) and the optimal investment and consumption functions obtained from the previous section, we simulate the optimal wealth process. To this end, we apply a same 5000 generated stream of pseudo random numbers for four different ranges of consumptions. Using the simulated optimal wealth processes when the impact factor of the running cost, \( \kappa \), is equal to 0.5, we present the histograms of the final annuities, Figs. 1, 2, 4 and 5 and reveal some percentiles of the optimal wealth amounts, Figs. 3, 6, 7 and 10.
and the optimal investment strategies, Figs. 8, 9, 11 and 12, during the decumulation period.

The graphs show that although the optimal strategies are similar regarding different scenarios of admissible consumption interval, we get higher final annuities when this interval is more restricted. The remarkable indication of the simulation results is the difference between the fixed consumption case and three other scenarios. Actually, the results show that by considering a variable consumption rate, although restricted, we can get much more valuable final annuity and higher percentiles of wealth amounts.

To compare our results corresponding to different scenarios, we consider the market present value of the cash flows that a retiree would have before and after the annuitization. Actually, the cash flow before the annuitization consists of withdrawals from the fund, that in [14] and [7] are supposed to be constant and in this paper vary between the two limits $C_2, C_1$. Furthermore, we consider the accumulated wealth at the time of death of an individual if this occurs before the annuitization time $T$. The cash flow after the annuitization consists of the constant payments of the lifetime annuity that has been purchased by the accumulated wealth at the terminal time $T$.

For the mortality rate, we consider the Gompertz-Makeham distribution, which for an individual of age $t$ is written as

$$\mu(t) = A + BC^t.$$  

The parameters $A, B$ and $C$ are those defined by the Belgian regulator for the pricing of life annuities purchased by males, as in [10]. So, we assume $A = 0.00055845$, $B = 0.000025670$ and $C = 1.1011$.

Therefore, considering $X(t)$ and $c(t)$ as the fund value and the consumption rate at time $t$, respectively, the market present value of the future cash flows is written as

$$P.V. = \int_0^{\tau_d \wedge 15} e^{-\rho t} c(t) dt + e^{-\rho \tau_d} X_{\tau_d} 1_{\{\tau_d < 15\}} + \int_{\tau_d \wedge 15}^{\tau_d} \frac{X(15)}{a_{75}} e^{-\rho t} dt,$$

in which $\tau_d$ is the time of death. Then, given that the time of death is independent of the filtration of financial returns, we get

$$P.V. = \int_0^{15} e^{-\int_0^t (\rho + \mu(60+s)) ds} (c(t) + \mu(60+t)X(t)) dt + \int_{15}^{T_m - 60} \frac{X(15)}{a_{75}} e^{-\int_t^{T_m - 60} (\rho + \mu(60+s)) ds} dt,$$

in which $T_m = 100$ is considered as the maximum lifespan and $\mu(t)$ is the mortality rate of an individual of age $t$. Table 1 gives more information concerning the distribution, the mean and standard deviation, of the final annuity and the market present value of the cash flows for different ranges of consumption and different impact factors of the running costs, $\kappa = 0.25, 0.5, 1$. Furthermore, it reports the mean of the consumptions before the annuitization and the probability
Figure 1: \( C_2 = \frac{1}{2} C_1 \)

Figure 2: \( C_2 = \frac{3}{4} C_1 \)

Figure 3: \( C_2 = \frac{1}{2} C_1 \)

Figure 4: \( C_2 = \frac{2}{3} C_1 \)

Figure 5: \( C_2 = C_1 \)

Figure 6: \( C_2 = \frac{2}{3} C_1 \)
Optimal Wealth Amount

Figure 7: \( C_2 = \frac{3}{4}C_1 \)

Optimal Wealth Amount

Figure 10: \( C_2 = C_1 \)

Optimal Investment Strategy

Figure 8: \( C_2 = \frac{1}{2}C_1 \)

Optimal Investment Strategy

Figure 11: \( C_2 = \frac{3}{4}C_1 \)

Optimal Investment Strategy

Figure 9: \( C_2 = \frac{3}{4}C_1 \)

Figure 12: \( C_2 = C_1 \)
Table 1: Distribution of final annuities, consumptions and cash flows

| $C_2 = \frac{1}{2} C_1$ | $C_2 = \frac{1}{4} C_1$ | $C_2 = \frac{1}{8} C_1$ | $C_2 = C_1$ |
|-------------------------|-----------------------|---------------------|------------|
| $\kappa = 0.25$       | $\kappa = 0.25$     | $\kappa = 0.25$    | $\kappa = 0.25$ |
| $\kappa = 0.5$         | $\kappa = 0.5$      | $\kappa = 0.5$     | $\kappa = 0.5$ |
| $\kappa = 0.75$        | $\kappa = 0.75$     | $\kappa = 0.75$    | $\kappa = 0.75$ |
| $\kappa = 1$           | $\kappa = 1$        | $\kappa = 1$       | $\kappa = 1$ |
| mean FA               | 9.59                 | 9.43               | 9.28      | 5.69   |
|                        | 9.08                 | 8.96               | 8.85      |
|                        | 8.77                 | 8.67               | 8.57      |
|                        | 8.54                 | 8.45               | 8.37      |
| sd. FA                | 2.14                 | 2.24               | 2.33      | 2.77   |
|                        | 2.43                 | 2.50               | 2.55      |
|                        | 2.58                 | 2.63               | 2.67      |
|                        | 2.68                 | 2.71               | 2.74      |
| mean PV               | 105.21               | 104.15             | 103.25    | 96.17  |
|                        | 105.34               | 104.59             | 103.92    |
|                        | 105.06               | 104.46             | 103.91    |
|                        | 104.71               | 104.22             | 103.76    |
| sd. PV                | 13.95                | 13.77              | 13.49     | 12.17  |
|                        | 14.11                | 14.14              | 14.08     |
|                        | 14.12                | 14.19              | 14.21     |
|                        | 14.16                | 14.18              | 14.21     |
| Prob(FA $> C_1$) (%)  | 89.62                | 88.10              | 86.22     | 37.44  |
|                        | 83.72                | 82.42              | 80.98     |
|                        | 80.10                | 79.08              | 77.92     |
|                        | 77.42                | 76.54              | 75.76     |
| Prob(FA = $\frac{1}{2}$ C_1) (%) | 1.62       | 1.94               | 2.16      | 6.94   |
|                        | 1.66                 | 2.60               | 2.86      |
|                        | 1.44                 | 1.94               | 3.24      |
|                        | 0.94                 | 2.18               | 3.26      |
| mean consumption      | 5.7752               | 5.7466             | 5.7341    | 6.5155 |
|                        | 5.9869               | 5.9698             | 5.9595    |
|                        | 6.0873               | 6.0756             | 6.0668    |
|                        | 6.1470               | 6.1398             | 6.1330    |

1 FA=Final Annuity
2 PV=Present Value
that a member get a final annuity that is higher than the annuity purchasable at the retirement, \( C_1 \).

As it is expected, when the minimum consumption rate, \( C_2 \), decreases or the admissible consumption interval becomes wider, we get more valuable final annuity and higher cash flows. Furthermore, in this case, the probability of getting a final annuity higher than \( C_1 \) increases.

The interesting point of this table is the big difference between the outcomes of the fixed consumption rate case, \( C_2 = C_1 \), and the three other cases. Actually, when we assume a fixed consumption rate, we get much less valuable final annuity and, moreover, the present value of the cash flows is remarkably less than the corresponding outcomes of the variable consumption rate. However, it is clear that, since in the fixed consumption case the cash flow before the annuitization, and of course before the time of death, is constant, the deviation of cash flows is less than the other cases.

For a fixed admissible consumption interval, when we compare the results corresponding to different impact factors of the running cost, we observe finer final annuity (higher mean with lower standard deviation) for lower impact factors. This is quite reasonable, since when the impact factor \( \kappa \) is small, more weight is devoted to the second term of the loss function which is based on the final annuity. The surprising result for a fixed admissible consumption interval is that the maximum present value of cash flows is attained when \( \kappa = 0.5 \). This can be interpreted by considering the two terms of loss function. Actually, for a smaller impact factor \( \kappa \), more weight of the loss function is devoted to the final annuity and therefore higher cash flow after the annuitization is expected. On the other side, for a greater impact factor \( \kappa \), more weight is devoted to the consumptions and therefore higher cash flow before the annuitization time is expected.

The last rows of the table report the mean of consumptions before the annuitization. It is clear that, on every fixed admissible consumption interval, the greater impact factor of running cost, the more weight to the first term of loss function and therefore the more consumptions. But, it is surprising that when the minimum consumption rate, \( C_2 \), decreases, the mean of consumptions slightly decreases. This can be interpreted by the previous rows in the table that when the admissible consumption interval becomes wider the probability of hitting the lower border slightly increases.

6 Conclusions

Studying the optimal investment-consumption problem post-retirement with a minimum guarantee for final annuity for a member of a defined contribution plan yields an HJB equation on a bounded domain. Applying the policy iteration method in solving this equation, the value function and the optimal strategies are improved iteratively. To find the approximation of the value function in one step of our algorithm, we apply the backward in time implicit scheme of finite difference method. Our scheme has the positive coefficient property which guarantees the convergence to the viscosity solution.

However, the simulation results show that by assuming variable, although very restricted, consumption rate, we can get much finer final annuity. Moreover, considering the cash flows before and after the annuitization, the variable
consumption rate yields higher present value of cash flows.
In the next work, we are going to consider the jump-diffusion dynamics which is more justified by the empirical data.

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