Correlation function of null polygonal Wilson loops with local operators

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Abstract

We consider the correlator $\langle W_n O \rangle / \langle W_n \rangle$ of a light-like polygonal Wilson loop with $n$ cusps with a local operator (like the dilaton or a chiral primary scalar) in planar $\mathcal{N} = 4$ super Yang-Mills theory. As a consequence of conformal symmetry, the main part of such correlator is a function $F$ of $3n - 11$ conformal ratios. The first non-trivial case is $n = 4$ when $F$ depends on just one conformal ratio $\zeta$. This makes the corresponding correlator one of the simplest non-trivial observables that one would like to compute for generic values of the 't Hooft coupling $\lambda$. We compute $F(\zeta, \lambda)$ at leading order in both the strong coupling regime (using semiclassical $AdS_5 \times S^5$ string theory) and the weak coupling regime (using perturbative gauge theory). Some results are also obtained for polygonal Wilson loops with more than four edges. Furthermore, we also discuss a connection to the relation between a correlator of local operators at null-separated positions and cusped Wilson loop suggested in arXiv:1007.3243.

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1 Introduction

Recent remarkable progress in understanding the duality between planar $\mathcal{N}=4$ super Yang-Mills theory and superstring theory in $AdS_5 \times S^5$ based on integrability opens up the possibility of computing various observables exactly in 't Hooft coupling $\lambda$ or in string tension $\sqrt{\lambda}/2\pi$. Most of the progress was achieved for the scaling dimensions $\Delta_i$ of primary operators $O_i$ which determine the 2-point functions $\langle O(x^{(1)})O(x^{(2)})\rangle$ (for reviews see [1]). The next step is to understand 3-point functions $\langle O_i(x^{(1)})O_j(x^{(2)})O_k(x^{(3)})\rangle$ which, in addition to $\Delta_i$, are determined by non-trivial functions $C_{ijk}(\lambda)$. Higher-point correlation functions, though in principle dictated by the OPE, are much more complicated. For example, conformal invariance implies that a 4-point correlator $\langle O_1(x^{(1)})...O_4(x^{(4)})\rangle$ should, in general, contain a non-trivial function of the two conformal cross-ratios $u_1 = |x^{(1)}_2x^{(3)}|/|x^{(1)}_3x^{(2)}|$, $u_2 = |x^{(1)}_2x^{(4)}|/|x^{(1)}_4x^{(2)}|$, $u_3 = |x^{(1)}_3x^{(4)}|/|x^{(1)}_4x^{(3)}|$, $u_4 = |x^{(1)}_2x^{(3)}|/|x^{(1)}_3x^{(2)}|$.

Correlators of primary operators are natural observables in CFT. In addition, in a gauge theory, one may consider also expectation values of Wilson loops. An important class of these, related to gluon scattering amplitudes (see [2–4] and [5] for reviews), are expectation values of Wilson loops in the fundamental representation $\langle W_n \rangle$ corresponding to polygons built out of null lines with $n$ cusps (located at $x^{(i)}_m$, $i=1,...,n$ with $|x^{(i,i+1)}| = 0$, $x^{(n+1)} = x^{(1)}$). They were previously studied at weak [6] and at strong [2,7] coupling. Conformal invariance (broken by the presence of the cusps in a controllable fashion) implies [9] that for $n = 4,5$ these expectation values are fixed functions of $x^{(i)}$ (depending on a few $\lambda$-dependent coefficients, in particular, on the cusp anomalous dimension [10,11]) while for $n > 5$ they should depend on $3n - 15$ cross-ratios of the cusp coordinates. The first non-trivial example is $\langle W_6 \rangle$ which is expressed in terms of a function of $\lambda$ and three cross-ratios. For recent progress in computing this function at weak and at strong coupling see [12–14].

As suggested in [15], there is a close relation between certain correlators of local (BPS) operators and expectation values of cusped Wilson loops: a correlator $K_n = \langle \hat{O}(x^{(1)})...\hat{O}(x^{(n)})\rangle$ of primary operators (e.g., the highest weight part of 20’ scalar) located at positions of the null cusps is proportional to the expectation value of the null polygon Wilson loop in the adjoint representation (or to $\langle W_n \rangle^2$ in the planar approximation we will consider here). More precisely, $\lim_{|x^{(i,i+1)}| \to 0} K_n/K_{n0} = \langle W_n \rangle^2$, where $K_{n0} \sim \prod_{i=1}^n |x^{(i,i+1)}|^{-2} + ...$ is the most singular term in the tree-level ($\lambda = 0$) part of $K_n$.

In this paper we study a new observable that involves both a local operator and a cusped Wilson loop, i.e. $\langle W_n O(a) \rangle$ ($a$ will denote the position of the local operator). One moti-
vation is that such correlators may lead to new simple examples where one may be able to interpolate from weak to strong coupling. In particular, in the first non-trivial case \( n = 4 \) such correlator happens to be a function of just one non-trivial conformal ratio formed from the coordinates of the cusps \( x^{(i)} \) and the operator \( a_m \) (for \( n > 4 \) it will be a function of \( 3n-11 \) conformal ratios). For comparison, in the case of a circular Wilson loop (which, in fact, may be viewed as an \( n \to \infty \) limit of a regular null polygon) the dependence of the correlator \( \langle W_\infty O(a) \rangle \) on the location of the operator \( a \) is completely fixed [16, 19, 20] by conformal invariance. Determining such a function (both at weak and at strong coupling) should be easier than the function of the two conformal ratios in the 4-point correlator case or the function of the three conformal ratios in the 6-cusp Wilson loop case. We shall demonstrate this below by explicitly computing the leading contributions to \( \langle W_4 O(a) \rangle \) both at strong and at weak coupling (for \( O \) being the dilaton or a chiral primary operator).

Another motivation to study such “mixed” correlators is that they may shed more light on the relation [15] between the correlators of null-separated operators and cusped Wilson loops mentioned above. That relation was verified at weak coupling, but checking it explicitly at strong coupling remains an important open problem. For example, one may start with a \((n+1)\)-point correlator and consider a limit in which only \( n \) of the locations of the operators become null-separated and attempt to relate this limit to \( \langle W_n O(a) \rangle \) with \( a = x^{(n+1)} \).

More explicitly, since the derivative of a correlator over the gauge coupling brings down a power of the super YM action which is the same as the integrated dilaton operator, the relation \( \langle \hat{O}(x^{(1)})...\hat{O}(x^{(n)}) \rangle \sim \langle W_n \rangle^2 \) implies that

\[
\langle \hat{O}(x^{(1)})...\hat{O}(x^{(n)}) \int d^4a \ O_{\text{dil}}(a) \rangle \sim 2\langle W_n \rangle \langle W_n \int d^4a \ O_{\text{dil}}(a) \rangle .
\] (1.1)

Assuming that the integral over \( a \) can be omitted and, furthermore, the dilaton operator can be replaced by a generic local operator one may conjecture that \( \langle \hat{O}(x^{(1)})...\hat{O}(x^{(n)}) O(a) \rangle \sim \langle W_n \rangle \langle W_n O(a) \rangle \), i.e. that

\[
\lim_{|x^{(i)}-x^{(i+1)}| \to 0} \frac{\langle \hat{O}(x^{(1)})...\hat{O}(x^{(n)}) O(a) \rangle}{\langle \hat{O}(x^{(1)})...\hat{O}(x^{(n)}) \rangle} \sim \frac{\langle W_n O(a) \rangle}{\langle W_n \rangle} .
\] (1.2)

Finally, it would be very interesting to understand what is a counterpart of \( \langle W_n O(a) \rangle \) on the “T-dual” [2] scattering-amplitude side, e.g., if there is any relation to form factors given by \( \langle A(x^{(1)})...A(x^{(n)})O(a) \rangle \) where \( A \)’s stand for local fields like vector potential, cf. [22].

\[2\] The two are obviously related when the operator is at zero momentum (i.e. integrated over \( a \)), but in general one expects a complicated non-local relation involving a sum over contributions of different types of operators.
Let us briefly review the contents of this paper. In Section 2, we shall use general
symmetry considerations to determine the structure of the correlator (2.1) of a null n-polygon
Wilson loop and a conformal primary operator. We shall explicitly discuss the case of \( n = 4 \)
where the result will be expressed in terms of a function \( F \) of only one non-trivial conformal
ratio (2.13) depending on the locations of the operator and the cusps. Taking the \( |a| \to \infty \)
limit then determines the corresponding OPE coefficient [16].

In Section 3 we explicitly compute the \( n = 4 \) correlator at strong coupling using semi-
classical string theory methods [16,17], i.e. evaluating the vertex operator corresponding to
\( O \) on the string surface [2,7] ending on the null quadrangle. We shall explicitly determine
the strong-coupling form of the function \( F \) for the two cases: when \( O \) is the dilaton or is
a chiral primary operator. We shall also discuss the generalization to the case of non-zero
R-charge or angular momentum in \( S^5 \).

In Section 4 we note that since the string world surface ending on a null quadrangle
is related [23] (by a euclidean continuation and conformal transformation) to the surface
describing folded spinning string [25] in the infinite spin limit [26], the correlator computed
in Section 3 may be related to the strong-coupling limit of 3-point correlator of two infinite
spin twist-2 operators and a dilaton operator. The latter correlator may be computed [27,28]
using similar semiclassical methods [29,30]. We point out that while the integrands in the two
expressions are indeed the same, the ranges of integration are different. The two integrals,
however, are indeed proportional for a special choice of the locations of the twist-2 operators.

In Section 5 we discuss the computation of the correlator \( \langle W_n O(a) \rangle \) at strong coupling
for higher number of cusps \( n > 4 \). Unfortunately, the explicit form of the space-time solution
is not know in this case, but using the approach of [31] we are able to compute the correlator
numerically in the limit when the dilaton is far away from the null polygon, i.e. to find the
OPE coefficient corresponding to the dilaton in the expansion of the Wilson loop in its size.

In Section 6 we turn to the evaluation of this correlator at weak coupling, i.e. in perturba-
tive gauge theory. We explicitly see that the leading term in \( \langle W_4 O(a) \rangle \) has a form consistent
with the one expected on symmetry grounds with the function \( F(\zeta) = \lambda h_0 + O(\lambda^2) \), where
\( h_0 \) is a constant. We consider a generalization to \( n > 4 \) and compute the OPE coefficient
for the dilaton in the case of a regular null polygon for arbitrary \( n \). We also compute the
leading order \( \lambda \) term in \( F \) in the case of the regular null polygon with \( n = 6 \).

In Section 7 we summarize our results and mention some open questions. In Appendix
A we discuss the general structure of the correlator \( \langle W_4 O(a) \rangle \). In Appendix B we consider
some analytic results which can be obtained for \( n > 4 \) even number of cusps in the limit
when the dimension of the local operator is very large.
2 Structure of correlation function of cusped Wilson loop and a local operator

Below we will consider the correlation function

\[ C(W_n, a) = \frac{\langle W_n \mathcal{O}(a) \rangle}{\langle W_n \rangle}, \]  

(2.1)

where \( W_n \) is a polygonal Wilson loop made out of \( n \) null lines (see Figure 1) and \( \mathcal{O} \) is a local scalar operator inserted at a generic point \( a = \{ a_m \} = (a_0, a_1, a_2, a_3) \). While the expectation value \( \langle W_n \rangle \) of such Wilson loops is known to have UV divergences due to the presence of the cusps \([6,10,11]\) (enhanced in the null case) we will see that the ratio (2.1) is finite, i.e. does not require a regularization.

Figure 1: Cusped polygonal Wilson loop, in this figure, with six edges. Consecutive cusps, for instance \( x^{(1)} \) and \( x^{(2)} \), are null separated.

2.1 General considerations

As follows from conformal symmetry, the non trivial part of \( \langle W_n \rangle \) depends only on the conformally invariant ratios constructed using the coordinates of the cusps \([9]\). The number of such conformal ratios for \( n > 5 \) is \( 4n - n - 15 = 3n - 15 \). Here \( 4n \) stands for the total number of coordinates, \( n \) is the number of null conditions on the polygon lines and 15 is the dimension of the conformal group.\(^3\) Furthermore, we expect (2.1) to be finite, since divergences from the numerator will be canceled by divergences from the denominator.

The number \( 3n - 15 \) of independent conformal ratios is exactly the same as the one that would appear in a correlator of \( n \geq 4 \) primary operators located at the corners of a null polygon. In general, the structure of \( n \)-point correlator \( \langle \mathcal{O}(x^{(1)})...\mathcal{O}(x^{(n)}) \rangle \) is fixed by conformal symmetry up to a function of conformal ratios. The number of these conformal

\(^3\)In [9] this counting was found using anomalous Ward identities in the framework of perturbative gauge theory. In [13] the same result was found at strong coupling by counting the number of moduli of the Hitchin’s equation with certain boundary conditions determining the corresponding minimal surface in \( AdS_5 \).
ratios is always given by $c_n = 4n - \gamma_n$, where $4n$ is the total number of coordinates and $\gamma_n$ is the number of generators of the conformal group broken by the presence of the local operators. For $n = 2, 3, 4$ we have $\gamma_2 = 8, \gamma_3 = 12$ and $\gamma_4 = 14$ so that $c_2 = 0$, $c_3 = 0$, $c_4 = 2$.\footnote{This agrees with the familiar count of the conformal ratios $u^{(s)} = \prod_{i<j}^n |x_m^{(i)} - x_m^{(j)}|^{\nu_{ij}^{(s)}}$. Here $\nu_{ij}^{(s)}$ is a basis in the space of symmetric $n \times n$ matrices $\nu_{ij}$. Scaling invariance and inversion symmetry imply that one should have $\nu_{ii} = 0$, $\sum_{j=1}^n \nu_{ij} = 0$ for all $i = 1, ..., n + 1$. This leaves $\frac{1}{2} n(n + 1) - n - n = \frac{1}{2} n(n - 3)$ parameters \footnote{This can be seen, for example, as follows. For $n \geq 2$, we can fix translations and special conformal transformations by putting one point at the origin and one at infinity. This configuration of two points preserves dilatations and rotations which gives 7 parameters implying that the number of the broken generators for $n = 2$ is 8. If we add one more point at some arbitrary finite position we break dilatations and certain rotations. What survives is the subgroup of the Lorentz group which preserves one vector. This subgroup is 3-dimensional so that $\gamma_3 = 12$. If we add the fourth point the surviving subgroup has to preserve two vectors and, hence, is one-dimensional, $\gamma_4 = 14$. If we add one more point all the conformal group becomes broken.} (there are additional Gram determinant constraints).} A random configuration of $n > 4$ points breaks the conformal group completely, i.e. $\gamma_n = 15$ and thus for $n > 4$ we have $c_n = 4n - 15$.\footnote{We shall use the notation $\zeta_k$ to distinguish these conformal ratios from standard cross-ratios $u_k$ which appear in correlators of local operators at generic points.} If the operators are located at the corners of a null polygon we have to impose $n$ additional constraints which gives $d_n = 3n - \gamma_n$ for the number of conformal ratios, i.e. $d_4 = -2$, $d_5 = 0$ and thus $d_n = 3n - 15$ for $n > 4$.

Adding an operator $O$ in (2.1) at a generic point brings in 4 parameters so that $C(W_n, a)$ with $n \geq 4$ should be a non-trivial function of $3n - 11$ conformally invariant combinations $\zeta_k$ constructed out of the coordinates $x_m^{(i)}$ of the $n$ cusps and the point $a_m$.\footnote{We shall use the notation $\Delta_k(a)$ with dimension $\Delta$ under (i) dilatations and (ii) inversions, i.e. (i) $C \rightarrow h^{-\Delta} C$} This is, of course, the same as the number of conformal ratios parametrising a correlator of $n + 1$ operators with only $n$ points being null-separated,

$$c_{n+1} - n = 4(n + 1) - 15 - n = 3n - 11.$$ \hfill (2.2)

Like for correlators of primary operators or Wilson loops, the general structure of $C(W_n, a)$ in (2.1) should be determined by conformal invariance. We shall assume that in contrast to $\langle W_n \rangle$, which contains UV divergences, the correlator (2.1) should be UV finite (up to a possible renormalization of the operator $O$). As we shall argue below, in this case the conformal invariance together with the expected OPE property fixes $C(W_n, a)$ up to a single function $F$ depending on $3n - 11$ conformal ratios $\zeta_k$.

In general, $C(W_n, a)$ should be a function of $n$ distances $|a - x^{(k)}|$ and $\frac{1}{2} n(n - 3)$ non-zero “diagonals” of the null polygon $|x^{(i)} - x^{(j)}|, i \neq j \pm 1$.\footnote{We shall use the notation $|x - x'| = (x_m - x'_m)^2 = -(x_0 - x'_0)^2 + (x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2$.} It should also transform like the operator $O(a)$ with dimension $\Delta$ under (i) dilatations and (ii) inversions, i.e. (i) $C \rightarrow h^{-\Delta} C$.
under \( x^{(i)} \to h x^{(i)} \), \( a \to ha \), and (ii) \( C \to |a|^{2\Delta} C \) under \( a_m \to \frac{a_m}{|x|^2} x^{(i)} \to \frac{x^{(i)}}{|x^{(i)}|^2} \). The large \( |a| \) behavior of \( C \) can be fixed by consistency with the expected OPE expansion: for small Wilson loop one may represent it in terms of a sum of local operators \([16, 33]\)

\[
\frac{W_n}{\langle W_n \rangle} = 1 + \sum_k c_k r^{\Delta_k} O_k(0) + \ldots ,
\]

(2.3)

where \( r \) is the characteristic size of a loop, \( O_k \) are conformal primary operators with dimensions \( \Delta_k \), and dots stand for contributions of their conformal descendants.\(^8\) Taking the position \( a \) of the operator \( O \) to be far away from the null polygon one should then get

\[
\frac{\langle W_n O(a) \rangle}{\langle W_n \rangle} \bigg|_{|a| \to \infty} \sim \langle O^d(0) O(a) \rangle \sim \frac{1}{|a|^{2\Delta}} ,
\]

(2.4)

where \( O^d \) conjugate to \( O \) is among the operators present in (2.3). Since all distances \( |a - x^{(k)}| \) between the operator and the cusps should appear on an equal footing this suggests the following ansatz

\[
C(W_n, a) = \frac{\mathcal{F}(a, x^{(i)})}{\prod_{k=1}^n |a - x^{(k)}|^\frac{2}{n\Delta}} ,
\]

(2.5)

where \( \mathcal{F} \) is finite in \( |a| \to \infty \) limit, i.e. it may depend on \( |a - x^{(k)}| \) only through their ratios. The dependence of \( \mathcal{F} \) on \( |x^{(i)} - x^{(j)}| \) is constrained by the transformations under dilatations and inversions mentioned above which implies that under these two transformations we should have

\[
(i) \mathcal{F} \to h^\Delta \mathcal{F} , \quad (ii) \mathcal{F} \to (|x^{(i)}| \ldots |x^{(n)}|)^{-\frac{2}{n\Delta}} \mathcal{F} .
\]

(2.6)

These conditions are solved, e.g., by taking \( \mathcal{F} \sim \prod_{i<j}^{n} |x^{(i)} - x^{(j)}|^\mu \) with \( \mu = \frac{2}{n(n-3)} \Delta \). In addition, \( \mathcal{F} \) may contain a factor \( F \) depending only on conformal ratios \( \zeta \) which is manifestly invariant under the dilatations and inversions. As we argue in Appendix A, this is, in fact, the general structure of \( C \), i.e. we are led to the following expression for (2.1)

\[
C(W_n, a) = \frac{\prod_{i<j}^{n} |x^{(i)} - x^{(j)}|^\frac{2}{n(n-3)\Delta}}{\prod_{k=1}^n |a - x^{(k)}|^\frac{2}{n\Delta}} F(\zeta_1, \ldots, \zeta_{3n-11}) .
\]

(2.7)

In general, \( \Delta \) and \( F \) in (2.7) may depend also on the coupling \( \lambda \), i.e. they may look different at weak and at strong coupling, but the general structure (2.7) should be universal.

The same structure (2.7) follows also from the general form of the correlator of local operators if the relation (1.2) is assumed to be true. As is well know, conformal invariance

\(^8\)Since special conformal transformations are generated by translations and inversions, it is enough to consider only the transformation under the inversions. Note that under the inversions \( |x - x'|^2 \to |x - x'|^2 \).

\(^9\)For example, in pure YM theory [33]: \( W(C)_{C \to 0} = 1 + c_0 r^4 (O_{\text{dis}0}) + \ldots \), where \( O_{\text{dis}0} \sim \frac{1}{N} \text{tr} F^2_{mn} \), \( r^4 \) stands for the square of the area of a disc bounded by \( C \) and \( c_0 = a_1 g^2 + a_2 g^4 + \ldots \).
implies that a correlator of \( q \) primary operators \( O_1(x^{(i)}) \) of dimensions \( \Delta_i \) at generic positions should have the form

\[
\langle O_1(x^{(1)})...O_q(x^{(q)}) \rangle = T_q \ F_q(u_1,...,u_{c_q}) \ , \quad T_q \equiv \prod_{i<j} |x^{(i)} - x^{(j)}|^{-\gamma_{ij}} , \quad (2.8)
\]

\[
\gamma_{ij} = \frac{2}{q - 2} \left( \Delta_i + \Delta_j - \frac{1}{q - 1} \sum_{k=1}^{q} \Delta_k \right) , \quad c_4 = 2 , \quad c_{q>4} = 4q - 15 , \quad (2.9)
\]

where \( F_q \) is a function of conformably-invariant cross-ratios. Considering \( q = n + 1 \) with \( n \) operators being the same, \( O_k = \hat{O} \), \( \Delta_k = \hat{\Delta} \) and \( O_{n+1} = O \), \( \Delta_{n+1} = \Delta \) we find

\[
T_{n+1} = \prod_{i<j}^{n} |x^{(i)} - x^{(j)}|^{-\frac{2}{n} \left( \Delta - \frac{1}{n} \Delta \right)} \prod_{k=1}^{n} |x^{(n+1)} - x^{(k)}|^{-\frac{2}{n-1} \Delta} , \quad T_n = \prod_{i<j}^{n} |x^{(i)} - x^{(j)}|^{-\frac{2}{n-1} \Delta} , \quad (2.10)
\]

so that in the ratio of the two correlators in (1.2) we have

\[
\frac{T_{n+1}}{T_n} = \frac{\prod_{i<j}^{n} |x^{(i)} - x^{(j)}|^{\frac{2}{n} \left( \Delta - \frac{1}{n} \Delta \right)}}{\prod_{k=1}^{n} |a - x^{(k)}|^{\frac{2}{n-1} \Delta}} , \quad a \equiv x^{(n+1)} . \quad (2.11)
\]

To get a non-trivial expression in the null-separation limit \( |x^{(i)} - x^{(i+1)}| \to 0 \) we will need to assume that \( n \) of such vanishing factors in numerator of (2.11) get cancelled against similar factors in some cross-ratios contained in \( F_{n+1}/F_n \). That will change the powers of the remaining \( \frac{1}{2} n(n - 1) - n = \frac{1}{2} n(n - 3) \) non-zero factors \( |x^{(i)} - x^{(j)}| \) in (2.11) and also reduce the total number of non-trivial conformal ratios (now denoted by \( \zeta_r \)) by \( n \) as in (2.2). The result will then have the same form as in (2.7).

Indeed, the combination one needs to multiply (2.11) by to cancel the vanishing \( |x^{(i)} - x^{(i+1)}| \) factors in the numerator and to match the prefactor in (2.7) with \( \mu_{ij} = \frac{2 \Delta}{n(n-3)} \) is \( (x^{(n+1)} \equiv x^{(1)}) \)

\[
\frac{\prod_{i<j}^{n} |x^{(i)} - x^{(j)}|^{\frac{2}{n-1}(n-1) \Delta}}{\prod_{k=1}^{n} |x^{(k)} - x^{(k+1)}|^{\frac{2}{n-1} \Delta}} . \quad (2.12)
\]

One can check that this expression is invariant under both dilatations and inversions and can thus be expressed in terms of cross-ratios.

Let us note that one could, in principle, treat \( x^{(i)} \) and \( a \) on a different footing, aiming at determining the dependence on \( a_m \) for fixed positions of the cusps \( x^{(i)} \) viewed as given parameters. In this case the \( |x^{(i)} - x^{(j)}| \) dependent factor in (2.7) could be formally absorbed into the function \( F \).

One may wonder how the rational powers in (2.7) may appear in a weak-coupling perturbation theory. The point is that one can recover integer powers for an appropriate \( F \). We shall comment on this issue in Appendix A on the example of \( n = 5 \) and \( \Delta = 4 \).
2.2 \( n = 4 \) case

Let us now look in detail at the first non-trivial example: \( n = 4 \). Here the number of variables \( \zeta_k \) is \( 3 \times 4 - 11 = 1 \), i.e. \( F \) should be a function of a \textit{single} variable \( \zeta_1 \equiv \zeta \). For \( n = 5 \) the number of conformal ratios is already 4. This makes the correlation function (2.1) for \( n = 4 \) a particularly interesting and simple case to study.

As it follows from the above discussion, this variable \( \zeta \) can be viewed as the unique conformal ratio which one can build out of the coordinates \( x^{(i)}_m \) \((i = 1, \ldots, 4)\) of 4 cusps and the location \( a_m \) of the operator \( \mathcal{O} \). Assuming that the null quadrangle is ordered as \( x^{(1)} \), \( x^{(2)} \), \( x^{(3)} \), \( x^{(4)} \) (i.e. \( |x^{(1)} - x^{(2)}|^2 = |x^{(2)} - x^{(3)}|^2 = |x^{(3)} - x^{(4)}|^2 = |x^{(4)} - x^{(1)}|^2 = 0 \)) it is easy to see that the unique non-trivial conformally-invariant combination of these 5 points is

\[
\zeta = \frac{|a - x^{(2)}|^2 |a - x^{(4)}|^2 |x^{(1)} - x^{(3)}|^2}{|a - x^{(1)}|^2 |a - x^{(3)}|^2 |x^{(2)} - x^{(4)}|^2}.
\]  

(2.13)

In this case there is also a unique choice for the \( x^{(i)} \)-dependent factor in (2.7): \( \frac{(|x^{(1)} - x^{(3)}||x^{(2)} - x^{(4)}|)^{\Delta/2}}{\prod_{i=1}^{4} \frac{|a - x^{(i)}|^{\Delta/2}}{\Delta/6}} \) that ensures the right dimensionality of the result. We conclude that the correlation function (2.1) for \( n = 4 \) should have the following general form

\[
\mathcal{C}(W_4, a) = \frac{(|x^{(1)} - x^{(3)}||x^{(2)} - x^{(4)}|)^{\Delta/2}}{\prod_{i=1}^{4} \frac{|a - x^{(i)}|^{\Delta/2}}{\Delta/6}} F(\zeta),
\]  

(2.14)

where \( \Delta \) is the dimension of the operator \( \mathcal{O} \) and \( \zeta \) is given by (2.13).

As discussed above, the same conclusion applies also to a correlator of 4 equivalent null-separated operators and an extra operator \( \mathcal{O} \). Indeed, for \( n = 4 \) it is easy to see that (2.11) is to be multiplied, according to (2.12), by

\[
\frac{(|x^{(1)} - x^{(3)}||x^{(2)} - x^{(4)}|)^{\Delta/3}}{\prod_{k=1}^{4} \frac{|x^{(k)} - x^{(k+1)}|^{\Delta/6}}{\Delta/6}},
\]  

(2.15)

which is a product of two cross-ratios in power \( \Delta/6 \).

It is interesting to note that depending on just \textit{one} conformal ratio, the \( n = 4 \) correlator (2.14) is an “intermediate” case between a 3-point function \( \langle \mathcal{O}(x^{(1)})\mathcal{O}(x^{(2)})\mathcal{O}(x^{(3)}) \rangle \) which is completely fixed by conformal invariance (up to a function of the coupling) and a generic 4-point function \( \langle \mathcal{O}(x^{(1)})...\mathcal{O}(x^{(4)}) \rangle \) which depends on two conformal ratios.

In the limit when \( |a| \to \infty \) we get

\[
\mathcal{C}(W_4, a)_{|a|\to\infty} = \frac{C}{|a|^{2\Delta}},
\]  

(2.16)

\[
C \equiv \left( \frac{|x^{(1)} - x^{(3)}||x^{(2)} - x^{(4)}|}{|x^{(2)} - x^{(4)}|^2} \right)^{\Delta/2} F(\zeta_\infty), \quad \zeta_\infty = \frac{|x^{(1)} - x^{(3)}|^2}{|x^{(2)} - x^{(4)}|^2},
\]  

(2.17)

\[\text{The case of } n = 3 \text{ is trivial as there is no solution for coordinates of a null triangle in real 4d Minkowski space.}\]
where $C$ thus determines the corresponding OPE coefficient in (2.3).

Another special limit is when the position of the operator approaches the location of one of the cusps, e.g., $a \to x^{(1)}$. Setting $a_m = x^{(1)}_m + \epsilon\alpha_m$, $\epsilon \to 0$, and using that the vectors $x^{(1)} - x^{(2)}$ and $x^{(1)} - x^{(4)}$ are null we find from (2.13) that $\zeta$ is, generically, finite in this limit and is given by

$$
\zeta_{a \to x^{(1)}} = \frac{4\alpha \cdot (x^{(1)} - x^{(2)}) \cdot \alpha \cdot (x^{(1)} - x^{(4)})}{\alpha^2 |x^{(2)} - x^{(4)}|^2}, \quad a_m = x^{(1)}_m + \epsilon\alpha_m. \quad (2.18)
$$

Similarly, the limit of the pre-factor in (2.14) is

$$
\prod_{i=1}^4 |a - x^{(i)}|^{\Delta/2} \to 4\epsilon^\Delta \alpha \cdot (x^{(1)} - x^{(2)}) \cdot \alpha \cdot (x^{(1)} - x^{(4)}) |x^{(1)} - x^{(3)}|^2. \quad (2.19)
$$

Thus

$$
\mathcal{C}(W_4^{\text{reg}}, a)_{a \to x^{(1)}} \sim \frac{1}{|a - x^{(1)}|^{\Delta}}. \quad (2.20)
$$

Note that this is the same behavior that would be expected if the Wilson loop were replaced by a product of 4 same-type operators (e.g., scalar operators as in [15]) at the positions of the cusps: $\langle W_4\mathcal{O}(a) \rangle \to \langle \hat{\mathcal{O}}(x^{(1)})...\hat{\mathcal{O}}(x^{(4)})\mathcal{O}(a) \rangle$. Then the limit $a \to x^{(1)}$ would be determined by the OPE, $\hat{\mathcal{O}}(x^{(1)})\mathcal{O}(a) \sim \frac{1}{|a - x^{(1)}|^2}\hat{\mathcal{O}}(x^{(1)})$.

One may also consider a limit when $a$ does not approach a cusp but becomes null-separated from it, i.e. $|a - x^{(i)}| \to 0$. In this case the correlator will be divergent, i.e. having $|a - x^{(i)}| \neq 0$ is important for finiteness. This is analogous to the observation in [15] that keeping $|x^{(i)} - x^{(i+1)}|$ finite in the correlator of local operator effectively regularizes the null cusp divergences of the corresponding Wilson loop.

Below we will explicitly verify the general form (2.7),(2.14) of the correlator (2.1) at leading orders in the strong-coupling (section 3) and the weak-coupling (section 6) expansions and compute the corresponding function $F$.

## 3 Correlation function of 4-cusp Wilson loop with a local operator at strong coupling

In this section we will compute (2.1) for $n = 4$ corresponding to the 4-cusp Wilson loop at strong coupling. The result will have the expected form (2.14) and we will explicitly determine the function $F(\zeta)$. We shall always consider the planar limit of maximally supersymmetric Yang-Mills theory and assume that the operator $\mathcal{O}$ is such that for large ’t Hooft
coupling $\lambda$ its dimension $\Delta$ is much smaller than $\sqrt{\lambda}$.

In particular, $\mathcal{O}$ will be chosen as the dilaton operator or the chiral primary operator. We shall follow the same semiclassical string theory approach that was used in the case of the circular Wilson loop in [16, 17] (see also [20, 28–30, 34]).

In string-theory description the local operator $\mathcal{O}(a)$ is represented by a marginal vertex operator [35]

$$V(a) = \int d^2 \xi \, V[X(\xi); a], \quad (3.1)$$

where $X$ stands for the 2d fields that enter the $AdS_5 \times S^5$ superstring action. In general, (2.1) is then given by

$$\mathcal{C}(W_n, a) = \frac{1}{(W_n)} \int [dX] \, V(a) \, e^{-I[X]}. \quad (3.2)$$

Here $I$ is the string action proportional to the tension $T = \sqrt{\lambda}$ and the path integral is performed over the euclidean world-sheets with topology of a disc (we consider only the planar approximation) and the boundary conditions set out by the Wilson loop at $z = 0$. Considering the limit when $\sqrt{\lambda} \gg 1$ and assuming that the operator represented by $V$ is "light" [28] (i.e. the corresponding scaling dimension and charges are much smaller than $\sqrt{\lambda}$) one concludes that this path integral is dominated by the same semiclassical string surface as in the absence of $V$, i.e. as in the case of $(W_n)$. The resulting leading-order value of (3.2) is then given by (3.1) evaluated on this classical solution, i.e.

$$\mathcal{C}(W_n, a)_{\sqrt{\lambda} \gg 1} = \left( \int d^2 \xi \, V[X(\xi); a] \right)_{\text{semicl.}}. \quad (3.3)$$

### 3.1 Correlation function with dilaton operator

One simple case is when the local operator $\mathcal{O}$ is the dilaton operator $\mathcal{O}_{\text{dil}} \sim \text{tr}(F^2_{mn} Z^j) + ...$ (where we included also the R-charge $j$ dependence). The corresponding vertex operator has the form [28]

$$V_{\text{dil}}(a) = c_{\text{dil}} \int d^2 \xi \left[ \frac{z}{z^2 + (x_m - a_m)^2} \right]^\Delta X^j U_{\text{dil}}, \quad (3.4)$$

$$X^j = (\cos \theta \, e^{i\varphi})^j, \quad \Delta = 4 + j, \quad (3.5)$$

where $j \ll \sqrt{\lambda}$ is an angular momentum along $S^1$ in $S^5$. The operator $U_{\text{dil}}$ equals the $AdS_5 \times S^5$ Lagrangian

$$U_{\text{dil}} = \mathcal{L} = \mathcal{L}_{AdS_5} + \mathcal{L}_{S^5} + \text{fermions}, \quad \mathcal{L}_{AdS_5} = \frac{1}{z^2} [(\partial_a z)^2 + (\partial_a x_m)^2]. \quad (3.6)$$

---

11If the operator would carry charges that are of order $\sqrt{\lambda}$ at large coupling it would modify the minimal surface that determines the leading order of the semiclassical expansion.
Furthermore, \( c_{dil} \) is the normalization coefficient given by \([16, 28, 29]\)\(^{12}\)

\[
c_{dil} = \frac{\sqrt{\lambda}}{8\pi N} \sqrt{(j + 1)(j + 2)(j + 3)}. \tag{3.7}
\]

Below we shall mostly consider the case of \( j = 0 \) when

\[
j = 0 : \quad \Delta = 4, \quad c_{dil} = \frac{\sqrt{6} \sqrt{\lambda}}{8\pi N}, \tag{3.8}
\]

and return to the case of \( j \neq 0 \) at the end of this subsection.

### 3.1.1 Regular 4-cusp case

Let us start with the case when the Wilson loop is the regular (i.e. equal-sided) quadrangle with 4 cusps (Figure 2a). The classical euclidean world-sheet surface in \( AdS_5 \) ending on this

![Diagram](image)

Figure 2: \((x_1, x_2)\) plane projection of (a) regular and (b) irregular quadrangle Wilson loop.

Wilson loop was found in \([2]\) and is given by\(^{13}\)

\[
\begin{align*}
z &= \frac{r}{\cosh u \cosh v}, & x_0 &= r \tanh u \tanh v, \\
x_1 &= r \tanh u, & x_2 &= r \tanh v, & x_3 &= 0; & u, v &\in (-\infty, \infty). \tag{3.9}
\end{align*}
\]

Here \( z \) is the radial direction of the Poincare patch of \( AdS_5 \) and \( x_m = (x_0, x_1, x_2, x_3) \) are the coordinates on the boundary. The parameter \( r \) corresponds to the overall scale of the loop. To simplify later formulas we will set \( r = 1 \) (it is easy to restore \( r \) by simply replacing \( z \to r^{-1}z, x_m \to r^{-1}x_m \)). The cusps correspond to \((u, v) \to (\pm \infty, \pm \infty)\) and thus are located at

\[
\begin{align*}
x^{(1)} &= (1, 1, 1, 0), & x^{(2)} &= (-1, 1, -1, 0), \\
x^{(3)} &= (1, -1, -1, 0), & x^{(4)} &= (-1, -1, 1, 0). \tag{3.10}
\end{align*}
\]

\(^{12}\)\(N\) is the rank of gauge group here representing a factor of string coupling. Note that the normalization of the operator \( V \) is important in order to compute the correlation function (3.3) and this normalization is currently known only for the BPS operators \([16, 28, 29]\).

\(^{13}\)Here \((u, v)\) cover the full plane, but since infinity is not identified the world sheet has topology of a disc.
Substituting (3.10) into (2.13) gives the following explicit form of the conformal ratio $\zeta$ that is expected to appear in the correlator
\[
\zeta = \frac{\left(\frac{1}{2}q - a_0 - a_1 + a_2\right)\left(\frac{1}{2}q - a_0 + a_1 - a_2\right)}{\left(\frac{1}{2}q + a_0 - a_1 - a_2\right)\left(\frac{1}{2}q + a_0 + a_1 + a_2\right)},
\] (3.11)
\[
q \equiv 1 - a_0^2 + a_1^2 + a_2^2.
\] (3.12)

Substituting the classical solution (3.9) into (3.4) we obtain
\[
C_{\text{dil}}(W_4^{(\text{reg})}, a) = 2c_{\text{dil}} \int_{-\infty}^{\infty} dudv \left[\frac{(\cosh u \cosh v)^{-1}}{q - 2a_1 \tanh u - 2a_2 \tanh v + 2a_0 \tanh u \tanh v}\right]^4,
\] (3.13)
where $q$ is given by (3.12) and we used the fact that on the solution (3.9) one has $U_{\text{dil}} = 2$ in (3.17) (note also that here $\int d^2\xi = \int dudv$). The integral is straightforward to do by introducing the variables $U = \tanh u$, $V = \tanh v$ and we get
\[
C_{\text{dil}}(W_4^{(\text{reg})}, a) = c_{\text{dil}} \frac{16a_1a_2 - 8qa_0 - (q^2 + 4a_0^2 - 4a_1^2 - 4a_2^2)}{12(qa_0 - 2a_1a_2)^3} \log \zeta,
\] (3.14)
where we have used (3.11). The result is thus finite, in contrast to the area of the 4-cusp surface that requires a regularization [2]. Let us note that if we consider the dilaton operator at zero momentum, i.e. integrate over the point $a$, we will recover the divergent area expression as then the dilaton vertex operator with $\Delta = 4$ in (3.4) will become proportional to the string action (the $[\cdots]^\Delta$ factor in (3.4) effectively provides a regularization for $\Delta \neq 0$). This is, of course, related to the fact that an insertion of the zero-momentum dilaton is equivalent to taking a derivative over the string tension which brings down a factor of the string action.

Let us now show that the result (3.14) is indeed consistent with eq. (2.14) for $\Delta = 4$. We observe that
\[
q^2 + 4a_0^2 - 4a_1^2 - 4a_2^2 = \frac{1 + \zeta}{2}P_1P_2, \quad P_{1,2} \equiv q + 2a_0 \mp 2a_1 \mp 2a_2,
\]
\[
qa_0 - 2a_1a_2 = \frac{1 - \zeta}{8}P_1P_2.
\] (3.15)
If we substitute eqs. (3.15) into (3.14) we get (restoring the dependence on the scale parameter $r$ in (3.9))
\[
C_{\text{dil}}(W_4^{(\text{reg})}, a) = \frac{64r^4 c_{\text{dil}}}{3} \frac{1}{P_1^2P_2^2} \frac{1}{(\zeta - 1)^3}[-2(\zeta - 1) + (\zeta + 1) \log \zeta].
\] (3.16)
Finally, one can check that
\[
P_1^2P_2^2 = \zeta^{-1} \prod_{i=1}^{4} |a - x^{(i)}|^2, \quad |x^{(1)} - x^{(3)}|^2|x^{(2)} - x^{(4)}|^2 = 64r^4,
\] (3.17)
\[14\text{Below in this section the expression for a correlator will always stand for its leading } \sqrt{\lambda} \gg 1 \text{ value.} \]
where \(x^{(i)}\) are the locations (3.10) of the cusps in (3.9). We conclude that the correlator (3.13) is given by eq. (2.14) with

\[
F(\zeta) = \frac{c_{dil}}{3} \frac{\zeta}{(\zeta - 1)^3} [-2(\zeta - 1) + (\zeta + 1) \log \zeta].
\] (3.18)

In the limit \(|a| \to \infty\) (see (2.16)) we get \(\zeta_\infty = 1\) and thus

\[
C_{dil}(W_{4}^{(reg)}, a)_{|a|\to\infty} = \frac{32c_{dil} r^4}{9|a|^8}.
\] (3.19)

which determines the OPE coefficient of \(O_{dil}\) in the expansion (2.3) of the Wilson loop \(W_{4}^{(reg)}\).

Let us note also that one may consider a different limit when \(a\) does not approach the cusp \(x^{(1)}\) but becomes null separated from it, i.e. \(|a - x^{(1)}| \to 0\). In this case the correlator is logarithmically divergent: \(C_{dil}(W_{4}^{(reg)}, a) \sim \log |a - x^{(1)}|.\) If \(a\) becomes at the same time null-separated from the two adjacent cusps (say, \(x^{(1)}, x^{(2)}\)) then \(\zeta\) stays finite and the correlator has a power divergence from the prefactor: \(C_{dil}(W_{4}^{(reg)}, a) \sim |a - x^{(1)}|^{-2}|a - x^{(2)}|^{-2}.\)

### 3.1.2 Irregular 4-cusp case

The above calculation can be generalized to the case of an irregular quadrangle, i.e. the one with unequal diagonals \(s \neq t\) (Figure 2b). The corresponding solution can be found by applying a conformal transformation to (3.9) [2]

\[
z = f(u, v) \cosh u \cosh v, \quad x_0 = \sqrt{1 + b^2} f(u, v) \tanh u \tanh v, \\
x_1 = f(u, v) \tanh u, \quad x_2 = f(u, v) \tanh v, \quad x_3 = 0, \\
f(u, v) \equiv \frac{r}{1 + b \tanh u \tanh v}, \quad |b| \leq 1.
\] (3.21)

\[\text{15}\] The same expression can be obtained by taking \(|a|\) large directly in (3.4) and doing the resulting simple integral \(\sim |a|^{-8} \int d^2 \xi \ z^4\).

\[\text{16}\] Note that eq. (3.20) gives the general behavior near the cusp. However, if we approach it along a specific path we can have additional singularities which can all be found from (3.20).
\( b = 0 \) corresponds to the regular quadrangle case (3.9). The cusps are found by taking \((u,v) \to (\pm \infty, \pm \infty)\) and are located at (cf. (3.10); here we set \( r = 1 \))

\[
x^{(1)}_m = \left( \frac{\sqrt{1+b^2}}{1+b}, \frac{1}{1+b}, \frac{1}{1+b}, 0 \right), \quad x^{(2)}_m = \left( -\frac{\sqrt{1+b^2}}{1-b}, \frac{1}{1-b}, -\frac{1}{1-b}, 0 \right),
\]

\[
x^{(3)}_m = \left( \frac{\sqrt{1+b^2}}{1+b}, \frac{-1}{1+b}, \frac{-1}{1+b}, 0 \right), \quad x^{(4)}_m = \left( -\frac{\sqrt{1+b^2}}{1-b}, \frac{-1}{1-b}, \frac{1}{1-b}, 0 \right). \quad (3.22)
\]

The Wilson loop is the quadrangle \( x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)} \).\(^{17}\)

Since \( U_{\text{dil}} \) in (3.6) is invariant under \( SO(2,4) \), its value on this solution should be the same as for \( b = 0 \), i.e. \( U_{\text{dil}} = 2 \). Substituting (3.21) into (3.4) gives

\[
C_{\text{dil}}(W^4_{\text{irreg}}, a) = 2C_{\text{dil}} \int_{-\infty}^{\infty} \frac{dudv}{q - 2a_1 \tanh u - 2a_2 \tanh v + 2\tilde{a}_0 \tanh u \tanh v}^4, \quad (3.23)
\]

where \( \tilde{a}_0 \) is defined by

\[
\tilde{a}_0 = a_0 \sqrt{1+b^2} + \frac{1}{2} b (q-2),
\]

while \( q \) is again given by (3.12) (without the replacement \( a_0 \to \tilde{a}_0 \)). As (3.13) and (3.23) are related by replacing \( a_0 \to \tilde{a}_0 \) we get from (3.14),(3.11)

\[
C_{\text{dil}}(W^4_{\text{irreg}}, a) = C_{\text{dil}} \frac{16a_1a_2 - 8q\tilde{a}_0 - (q^2 + 4\tilde{a}_0^2 - 4a_1^2 - 4a_2^2) \log \zeta}{12(q\tilde{a}_0 - 2a_1a_2)^3},
\]

\[
\zeta = \frac{\left( \frac{1}{2}q - 2\tilde{a}_0 - a_1 + a_2 \right) \left( \frac{1}{2}q\tilde{a}_0 + a_1 - a_2 \right)}{\left( \frac{1}{2}q - a_0 - a_1 - a_2 \right) \left( \frac{1}{2}q + \tilde{a}_0 + a_1 + a_2 \right)}. \quad (3.26)
\]

It is straightforward to check, using the locations of the cusps in (3.22), that the argument of the logarithm in (3.26) is again the conformally-invariant ratio in (2.13). Note that while \( b \) may be interpreted as the parameter of a conformal transformation relating the regular and the irregular polygons, \( a \) is kept fixed under this transformation, so that \( \zeta \) in (3.26) now depends on \( b \) compared to the one in (3.11),(3.14).

After the same steps as in the case of the regular quadrangle we find that eq. (3.25) can indeed be written as (2.14), where \( \Delta = 4 \), \( x^{(i)} \)'s are given by (3.22) and

\[
|x^{(1)} - x^{(3)}|^2 |x^{(2)} - x^{(4)}|^2 = \frac{64r^4}{(1-b^2)^2}, \quad (3.27)
\]

\[
F(\zeta) = \frac{C_{\text{dil}}}{3} \frac{\zeta}{(\zeta-1)^3} [-2(\zeta - 1) + (\zeta + 1) \log \zeta]. \quad (3.28)
\]

\(^{17}\)It is easy to check that the vectors connecting the 4 cusps \( x^{(12)} = x^{(1)} - x^{(2)}, x^{(23)} = x^{(2)} - x^{(3)}, x^{(34)} = x^{(3)} - x^{(4)}, x^{(41)} = x^{(4)} - x^{(1)} \) are null. The two non-trivial parameters \( s \) and \( t \) are given by [2]

\[-(2\pi)^2 s = 2x^{(23)} \cdot x^{(34)} = |x^{(2)} - x^{(4)}|^2 = \frac{8}{(1+b)^2}, \quad -(2\pi)^2 t = 2x^{(12)} \cdot x^{(23)} = |x^{(1)} - x^{(3)}|^2 = \frac{8}{(1+b)^2}.\]
The function $F(\zeta)$ is thus the same as in (3.18), as expected. We find also that

$$C_{dil}(W_4^{(irreg)}, a)_{|a| \to \infty} = \frac{C_4}{|a|^8}, \quad C_4 = \frac{4c_{dil}}{3b^3} \left[ -2b + (1 + b^2) \log \frac{1 + b}{1 - b} \right],$$

which reduces to (3.19) in the limit $b \to 0$.

Let us note also that starting with the general expression for the correlator (2.14), (3.18) we may consider the case (obtained by a conformal transformation from (3.12) [24]) when two of the four cusps (e.g., $x^{(3)}, x^{(4)}$) are sent to infinity. Then (2.14) with $\Delta = 4$ takes the following form:

$$C_{dil}(W_4)_{x^{(3)}, x^{(4)} \to \infty} = \frac{1}{|a - x^{(1)}|^2 |a - x^{(2)}|^2} F(\tilde{\zeta}), \quad \tilde{\zeta} = \frac{|a - x^{(2)}|^2}{|a - x^{(3)}|^2}. \quad (3.30)$$

### 3.1.3 Case of dilaton with non-zero $S^5$ momentum

The above discussion can be generalized to the case when the dilaton operator (3.4) carries an angular momentum $j$ along $S^1 \subset S^5$. Again, we have to evaluate (3.4) on the solution (3.9) or (3.21). Since these solutions do not depend on the sphere coordinates the factor $X^j$ in (3.4) equals to unity so that the correlation function is given by the same expressions as in (3.13) or (3.23) with the power 4 replaced with $\Delta = 4 + j$. In the case of the more general solution (3.21) we get instead of (3.23)

$$C_{dil}(W_4^{(irreg)}, a_0) = 2c_{dil} \int_{-\infty}^{\infty} \frac{(\cosh u \cosh v)^{-1}}{q - 2a_1 \tanh u - 2a_2 \tanh v + 2a_0 \tanh u \tanh v} \Delta \quad (3.31)$$

where $a_0$ is defined in (3.24). Since the answer should be of the form (2.14), we are interested in computing the function $F(\zeta)$ of one variable only, so we may set $a_1 = a_2 = a_3 = 0$. Having computed $\tilde{F}(a_0) = F(\zeta(a_0))$ we may find $a_0 = a_0(\zeta)$ from (3.26) and thus restore $F(\zeta)$. Performing the integral (3.31) we obtain

$$C_{dil}(W_4^{(irreg)}, a_0) = \frac{2\pi c_{dil}}{(1 - a_0^2)^{\Delta}} \left( \frac{\Gamma(\frac{\Delta}{2})}{\Gamma(\frac{\Delta + 1}{2})} \right)^2 \left[ 2F_1(\frac{1}{2}, \frac{\Delta}{2}, \frac{\Delta + 1}{2}, \varphi^2) \right], \quad (3.32)$$

where $2F_1$ is the hypergeometric function and $\varphi$ is a function of $a_0$ given by

$$\varphi \equiv \frac{2a_0}{1 - a_0^2} = \frac{2a_0 \sqrt{1 + b^2} - b(1 + a_0^2)}{1 - a_0^2}. \quad (3.33)$$

To extract $F$ in (2.14) we have to multiply (3.32) by the factor

$$|x^{(1)} - x^{(3)}|^{-\Delta/2} |x^{(2)} - x^{(4)}|^{-\Delta/2} \prod_{i=1}^{4} |x^{(i)}_m - a_m|^{\Delta/2}.$$
This gives
\[ F(\zeta(\varrho)) = 2^{-\frac{3}{2}\Delta+1}\pi c_{\text{dil}} \left( \frac{\Gamma\left(\frac{\Delta}{2}\right)}{\Gamma\left(\frac{\Delta+1}{2}\right)} \right)^2 \left( 1 - \varrho^2 \right)^{\Delta/2} \binom{1}{\frac{\Delta}{2}, \frac{\Delta+1}{2}, \varrho^2}. \]  
(3.34)

Finally, we can express \( \varrho \) in terms of \( \zeta \) using (3.26):
\[ \varrho = \frac{1 - \sqrt{\zeta}}{1 + \sqrt{\zeta}}. \]  
(3.35)

One can check that setting \( j = 0 \), i.e. \( \Delta = 4 \), gives back our earlier expression (3.28).

3.1.4 Generalization to cusped Wilson loop with an \( S^5 \) momentum

One can formally repeat the above discussion in the case when the euclidean 4-cusp world surface (3.9) is generalized to the presence of a non-zero angular momentum in \( S^5 \) (we again set the scale \( r = 1 \))

\[ z = \frac{1}{\cosh k u \cosh v}, \quad x_0 = \tanh k u \tanh v, \]
\[ x_1 = \tanh k u, \quad x_2 = \tanh v, \quad x_3 = 0, \quad u, v \in (-\infty, \infty), \]
\[ \varphi = -i\ell u, \quad k^2 = 1 + \ell^2. \]  
(3.36)

This background (3.36) solves the string equations in conformal gauge. Here \( \varphi \) is an angle of a big circle in \( S^5 \) and \( \ell \) may be interpreted as a density of the corresponding angular momentum. We assume that \( u \) plays the role of a euclidean time; since \( v \) is non-compact, the total angular momentum is formally infinite. The presence of \( k \) does not influence the positions of the 4 cusps (3.10).

The correlation function with the dilaton operator is again given by (3.3),(3.4). Since now \( \varphi \) in (3.5) is non-zero\(^{19}\) there will be a non-trivial dependence on the product of the dilaton momentum \( j \) and the angular momentum density \( \ell \) (cf. (3.13),(3.23),(3.31))

\[ C_{\text{dil}}(W_{4,\ell}^{(\text{reg})}, a) = 2c_{\text{dil}} \int_{-\infty}^{\infty} du dv \left[ (\cosh k u \cosh v)^{-1} \right]^{4+j} e^{j\ell u}. \]  
(3.37)

We used that in (3.6) \( L_{\text{AdS}_5} = k^2 + 1, \quad L_{S^5} = (\partial_\alpha \varphi)^2 = -\ell^2 \) so that again \( U_{\text{dil}} = \mathcal{L} = 2 \). The resulting integral can be studied for generic \( j \) (it is useful to change variable \( u \rightarrow u' = ku \)).

\(^{18}\)This solution is related by an analytic continuation and a conformal transformation \([23]\) to the large spin limit of the folded \((S, J) \) spinning string \([26]\), see also section 4.

\(^{19}\)In (3.5) \( \theta = 0 \) as we consider rotation in a big circle of \( S^5 \).
For \( j = 0 \) we get back to the same expression as (3.13) with an extra factor of \( k^{-1} = (1 + t^2)^{-1/2} \), i.e.

\[
    j = 0 : \quad C_{\text{dil}}(W_{4,\ell}^{(\text{reg})}, a) = \frac{1}{\sqrt{1 + t^2}} C_{\text{dil}}(W_{4}^{(\text{reg})}, a) .
\]

(3.38)

It is also straightforward to discuss a generalization to irregular 4-cusp surface with \( b \neq 0 \) (cf. (3.28)).

### 3.2 Correlation function with chiral primary operator

Let us now consider a similar computation with the chiral primary operator \( \mathcal{O}_j = \text{tr} Z^j \) instead of the dilaton operator. The bosonic part of the corresponding vertex operator \([16,28,29]\) can be written in a form similar to (3.4)

\[
    V_j(a) = c_j \int d^2 \xi \left[ \frac{z}{z^2 + (x_m - a_m)^2} \right]^\Delta X^j U ,
\]

(3.39)

\[
    \Delta = j , \quad c_j = \frac{\sqrt{\lambda}}{8\pi N} \sqrt{j(j+1)} ,
\]

(3.40)

where \( X^j \) is the same as in (3.4) while the 2-derivative \( U \) part is more complicated \([34]\)

\[
    U = U_1 + U_2 + U_3 , \quad U_1 = \frac{1}{2} ( (\partial_a x_m)^2 - (\partial_a z)^2 ) - \mathcal{L}_{S^5} ,
\]

(3.41)

\[
    U_2 = \frac{8}{(z^2 + |x - a|^2)^2} \left[ |x - a|^2 (\partial_a z)^2 - [(x_m - a_m)\partial_a x_m]^2 \right] ,
\]

\[
    U_3 = \frac{8(|x - a|^2 - z^2)}{z(z^2 + |x - a|^2)^2} (x_n - a_n)\partial_a x_n \partial_a z .
\]

(3.42)

For simplicity, we will consider the case of the regular 4-cusp Wilson loop; the corresponding solution (3.9) does not depend on \( S^5 \) coordinates so \( X^j = 1 \).

To find the function \( F(\zeta) \) in (2.14) it is sufficient, as in section 3.1.3, to choose the special case of \( a = (a_0,0,0,0) \). Remarkably, in this case \( U \) can be put into the following simple form (cf. (3.33))

\[
    U_{a=(a_0,0,0,0)} = \frac{2}{\cosh^2 u \cosh^2 v} \frac{1 + \varrho^2 - (\sinh u \sinh v + \varrho \cosh u \cosh v)^2}{(1 + \varrho \tanh u \tanh v)^2} ,
\]

(3.43)

\[
    \varrho \equiv \frac{2a_0}{1 - a_0^2} .
\]

(3.44)

Substituting it into (3.3),(3.39) gives

\[
    C_j(W_4^{(\text{reg})}, a_0) = \frac{2c_j}{(1 - a_0^2)^j} \int_{-\infty}^{\infty} dudv \left[ \frac{(\cosh u \cosh v)^{-1}}{1 + \varrho \tanh u \tanh v} \right]^{j+2} \times \left[ 1 + \varrho^2 - (\sinh u \sinh v + \varrho \cosh u \cosh v)^2 \right] .
\]

(3.45)
For an arbitrary $j$ this integral is rather complicated but can be easily done for specific values of $j$. For instance, for $j = 2$ we obtain:

$$C_2(W_{4}^{\text{reg}}, a_0) = \frac{4c_2}{3(1 - a_0^2)^2} \log \frac{\varrho + 1}{\varrho - 1}.$$  (3.46)

To compute $F(\zeta)$ we have to multiply (3.46) by the factor (see (2.14))

$$\left(\left| x^{(1)} - x^{(3)} \right| \left| x^{(2)} - x^{(4)} \right| \right)^{-1} \prod_{i=1}^{4} |a - x^{(i)}| = \frac{1}{8} [(a_0 - 1)^2 + 2][(a_0 + 1)^2 + 2],$$  (3.47)

where the positions of the cusps $x^{(i)}$ are given by (3.10) and we used that $\Delta = j = 2$. Expressing $\varrho$ in terms of $\zeta$ in (3.11), (3.12). gives the same relation as in (3.35). As a result, we find (cf. (3.18))

$$j = 2 : \quad F(\zeta) = \frac{c_2}{3} \frac{\sqrt{\zeta}}{\zeta - 1} \log \zeta.$$  (3.48)

We can then restore the dependence of the correlator on all 4 coordinates of $a_m$ getting the following analog of (3.14)

$$C_2(W_{4}^{\text{reg}}, a) = -\frac{c_2}{3(1 - a_0^2)^2} \log \frac{\varrho + 1}{\varrho - 1},$$  (3.49)

where $q$ is given by (3.12) and $\zeta$ is defined in (3.11).

Taking the position of the operator $a$ to infinity we get the corresponding OPE coefficient (we restore the factor of the scale $r$ of the loop)

$$C_2(W_{4}^{\text{reg}}, a)|_{a|\rightarrow\infty} = \frac{8c_2}{3|a|^4}.$$  (3.50)

Here the power of $|a|$ reflects the value of the dimension $\Delta = 2$ (cf. (3.19)). We may also study the limit when $a_m$ is approaching the position of one of the cusps. Choosing $a_m$ as in (2.18) and expanding for small $\epsilon$ gives (cf. (3.20))

$$C_2(W_{4}^{\text{reg}}, a) \rightarrow \frac{1}{3\epsilon^2} \frac{\log \frac{\alpha_1^2 + \alpha_2^2 + \alpha_1^2 - \alpha_2^2}{2(\alpha_0 - \alpha_1)(\alpha_0 - \alpha_2)}}{-3\alpha_0^2 + \alpha_1^2 + (\alpha_1 - \alpha_2)^2 + 2\alpha_0(\alpha_1 + \alpha_2)}. \quad (3.51)$$

### 4 Comment on relation to 3-point correlator with two infinite spin twist-2 operators

As is well known, the anomalous dimension of the large spin $S$ twist-2 operator (the coefficient of the $\ln S$ term in it) is closely related to the UV anomaly in the expectation value of a

---

20Let us note that the same (up to a constant factor) expression is found by formally setting $\Delta = 2$ in (3.32),(3.34).
Wilson loop with a null cusp \([2, 5, 7, 11, 36, 37]\). At strong coupling, this relation can be understood [23] by relating the corresponding string world surfaces by a world-sheet euclidean continuation and an \(SO(2, 4)\) conformal transformation. This may be effectively interpreted as a relation between the 2-point function of twist-2 operators \(\langle \mathcal{O}^\dagger_S(x) \mathcal{O}_S(0) \rangle\) with \(S \to \infty\) and the singular part of the expectation of the cusped Wilson loop \(\langle W_4 \rangle\).

Below we shall discuss if such a relation may apply also if one includes in the respective correlators an extra “light” operator \(\mathcal{O}_\Delta\) \((\Delta \ll \sqrt{\lambda})\),

\[
\mathcal{C}(W_4, a) = \frac{\langle W_4 \mathcal{O}_\Delta(a) \rangle}{\langle W_4 \rangle} \quad \text{vs.} \quad K(x^{(1)}, x^{(2)}, a) = \frac{\langle \mathcal{O}^\dagger_S(x^{(1)}) \mathcal{O}_S(x^{(2)}) \mathcal{O}_\Delta(a) \rangle}{\langle \mathcal{O}^\dagger_S(x^{(1)}) \mathcal{O}_S(x^{(2)}) \rangle}.
\]

This relation may be expected only in the \textit{infinite} spin limit and one is to assume a certain correspondence between the locations of the 4 cusps \(x^{(i)}\) in \(W_4\) and the positions \(x^{(1)}, x^{(2)}\) of the twist-two operators.

The reason why this relation may be expected at strong coupling is the following. Computed at strong coupling with a “light” dilaton operator both correlators in (4.1) are given by the vertex operator corresponding to \(\mathcal{O}_\Delta\) evaluated on the semiclassical world surfaces associated, respectively, with \(\langle W_4 \rangle\) and with \(\langle \mathcal{O}^\dagger_S(x^{(1)}) \mathcal{O}_S(x^{(2)}) \rangle\). These two surfaces are closely related [23, 27]. As we shall see below, the integrands for the corresponding semiclassical correlators will be the same but the integration regions, however, will differ. For a special choice of the location of the “light” operator the results for the two integrals will be the same up to a factor of 2.

Indeed, the solution (3.9) is essentially the same as the semiclassical trajectory supported by the two twist-2 operators in the large spin limit \(S \gg \sqrt{\lambda}\)

\[
\begin{align*}
z &= \frac{1}{\cosh(\kappa \tau_e) \cosh(\mu \sigma)}, & x_0 &= \tanh(\kappa \tau_e) \tanh(\mu \sigma), \\
x_1 &= \tanh(\kappa \tau_e), & x_2 &= \tanh(\mu \sigma), & x_3 &= 0, \\
\kappa &= \mu \gg 1, & \kappa &= \frac{\Delta_S - S}{\sqrt{\lambda}}, & \mu &= \frac{1}{\pi} \log S.
\end{align*}
\]

The relation \(\kappa = \mu\) follows from the Virasoro condition which is also implied by the marginality of the twist 2 operator. This background is equivalent to the one found by a euclidean rotation \((\tau \to i \tau_e)\) of the large spin limit of the folded spinning string in \(AdS_5\).\footnote{To find the solution with the singularities prescribed by the two twist-2 operators one has to act on (4.2) with a two-dimensional conformal map which sends the cylinder to the plane with two marked points, see [27, 38] for details.} Here \(\tau_e \in (-\infty, \infty)\). In the original closed string solution \(\sigma \in [0, 2\pi]\); in fact, in (4.2) we have \(\sigma \in [0, \frac{\pi}{2}]\) and 3 other segments are assumed to be added similarly. In the infinite spin limit \(\mu \to \infty\) so we may formally set \(v = \mu \sigma \in (0, \infty)\). Introducing also \(u = \kappa \tau_e \in (-\infty, \infty)\) we conclude that (4.2) becomes equivalent to (3.9), up to the “halved” range of \(v\).
More precisely, the operators considered in [27] were defined on a euclidean 4-space and the corresponding surface had imaginary $x_2$; the real surface equivalent to (4.2) is obtained by the Minkowski continuation in the target space $x_2 \to ix_0$, $x_0 \to x_2$. Below we shall interchange the notation $x_1 \leftrightarrow x_2$ compared to [27]. Since in [27] the operators were assumed to be located at $R^4$ points $x^{(1,2)} = (\pm 1, 0, 0, 0)$ taking this continuation into the account we conclude that the solution (4.2) is the semiclassical trajectory saturating the two-point correlator of twist-2 operators located at the following special points in the Minkowski 4-space

\[
x^{(1)} = (0, 1, 0, 0), \quad x^{(2)} = (0, -1, 0, 0).
\]

Assuming for definiteness that $\mathcal{O}_\Delta$ is the dilaton operator we conclude that the leading strong-coupling term in the correlator of two infinite spin twist-2 operators and the dilaton $K_{\text{dil}} = \frac{\langle \mathcal{O}_\Delta(x^{(1)}) \mathcal{O}_\Delta(x^{(2)}) \mathcal{O}_{\text{dil}}(a) \rangle}{\langle \mathcal{O}_\Delta(x^{(1)}) \mathcal{O}_\Delta(x^{(2)}) \rangle}$ is then given by the same integral as in (3.13) with the only difference being that instead of the range of integration $v \in (-\infty, \infty)$ that we had in the Wilson loop case now in the closed string case we have $v \in [0, \infty)$ with the whole integral multiplied by 4. Equivalently, the spatial integral for the folded string solution is done over $\sigma \in [0, \frac{\pi}{2}]$ with the result multiplied by 4 [28]. The topology of the world sheet parametrized by $(u, v)$ should be a disc (or a plane with infinity removed) in the Wilson loop case and the cylinder (or a half-plane with infinity removed) in the folded closed string case.

Note that if we set $a_0 = a_2 = 0$ in the integrand in (3.13) it becomes an even function of $v$ and thus the integral over $v \in [0, \infty)$ is just half the integral over $v \in (-\infty, \infty)$. Thus for the special values of $a$ the two correlators in (4.1) are directly related.

For general $a$ we get for $K_{\text{dil}}$ an expression that is different from (3.14)

\[
K_{\text{dil}} = \frac{C_{\text{dil}}}{3(q^2 - 4a_1^2)(qa_0 - 2a_1a_2)^3} \left[ -4\left[ q^2 - 4a_1(a_0 + a_1) + 2qa_2 \right] (qa_0 - 2a_1a_2) 
+ (q^2 - 4a_1^2)(q^2 + 4a_0^2 - 4a_1^2 - 4a_2^2) \log \left( \frac{\frac{1}{2}q + a_1}{\frac{1}{2}q - a_0} \right) \frac{\frac{1}{2}q + a_0 - a_1 - a_2}{\frac{1}{2}q - a_0 + a_1 - a_2} \right]. \tag{4.5}
\]

This expression containing the logarithm of a coordinate ratio appears to be in conflict with the expected structure of the 3-point function of conformal primary operators ($x^{(ij)} = x^{(i)} - x^{(j)}$)

\[
\langle \mathcal{O}_\Delta(x^{(1)}) \mathcal{O}_\Delta(x^{(2)}) \mathcal{O}_\Delta(x^{(3)}) \rangle = C_{123} \bigg/ \prod_{x} |\Delta_1 + \Delta_2 - \Delta_3| |\Delta_2 + \Delta_3 - \Delta_1| |\Delta_3 + \Delta_1 - \Delta_2|. \tag{4.6}
\]

\footnote{Note that if $\mu \sigma$ in (4.2) were not extending to infinity, the boundary behavior of (4.2) would be different from a rectangular Wilson loop. The boundary would be reached only for $u \to \pm \infty$ and on the boundary we would have a piece of the null line $x_1 = x_0$ located at $x_2 = 1$ and a piece of the null line $x_1 = -x_0$ located at $x_2 = -1$. These null lines would no longer be connected on the boundary.}
In the present case with $\Delta S \gg \Delta_{\text{dil}} = 4$ and $a = x^{(3)}_i(4.6)$ can be explicitly written as

$$\langle O^\dagger_S(x^{(1)}) O_S(x^{(2)}) O_\Delta(a) \rangle \approx \frac{C}{|x^{(1)}_i - a|^{\Delta} |x^{(2)}_i - a|^{\Delta}}, \quad (4.7)$$

where $C$ is a constant which is finite in the $S \to \infty$ limit [28].

An explanation of this apparent puzzle is as follows. The standard argument leading to (4.6) assumes that the 3 points $x^{(i)}$ can be spatially separated (as is always the case in $\mathbb{R}^4$ but not in $\mathbb{R}^{1,3}$ we consider here). If we choose the dilaton operator insertion point $a$ to be away from the $(x_0, x_2)$-plane ($\{x_1, x_2\}$ plane of rotation of the original spinning string) we can spatially separate all the operators. For example, we may set $a_0 = a_2 = 0$. In this case the integrand in (3.13) becomes an even function of $v$ and the integral over $v \in [0, \infty)$ is just half the integral over $v \in (-\infty, \infty)$, i.e. (3.14) should be proportional to (4.7). Indeed, taking the limit $a_0 = a_2 = 0$ in (3.14), (4.7) we find that the logarithmic term goes away and we obtain

$$C_{\text{dil}}(W_4^{\text{reg}}, a)_{a_0=a_2=0} = \frac{1}{2} (K_{\text{dil}})_{a_0=a_2=0} = \frac{32 c_{\text{dil}}}{9} \frac{1}{((a_1 - 1)^2 + a_3^2)^2 ((a_1 + 1)^2 + a_3^2)^2}. \quad (4.8)$$

This is consistent with (4.7) if we recall the expression (4.4) for the locations of the twist-2 operators.

Similar analysis can be repeated for the correlation function with the chiral primary operator with $\Delta = j = 2$. Taking the limit $a_0 = a_2 = 0$ in (3.49) leads to a similar expression

$$C_2(W_4^{\text{reg}}, a)_{a_0=a_2=0} = \frac{8 c_2}{3} \frac{1}{((a_1 - 1)^2 + a_3^2)^2 ((a_1 + 1)^2 + a_3^2)^2}, \quad (4.9)$$

again consistent with (4.6),(4.7).

Finally, let us consider the generalized cusp solution with $S^5$ momentum (3.36). The generalization of the solution (4.2) to the case when large spin operators carry also a large angular momentum $J$ in $S^5$ is given by

$$z = \frac{1}{\cosh(\kappa \tau_e) \cosh(\mu \sigma)} , \quad x_0 = \tanh(\kappa \tau_e) \tanh(\mu \sigma) ,$$

$$x_1 = \tanh(\kappa \tau_e) , \quad x_2 = \tanh(\mu \sigma) , \quad x_3 = 0 , \quad \phi = -i \nu \tau_e , \quad \nu = \frac{J}{\sqrt{\lambda}} , \quad \kappa^2 = \mu^2 + \nu^2 . \quad (4.10)$$

Taking the scaling limit [26] when $\kappa, \mu, \nu \to \infty$ with fixed

$$k \equiv \frac{\kappa}{\mu} , \quad \ell \equiv \frac{\nu}{\mu} = \frac{\pi J}{\sqrt{\lambda} \ln S} , \quad k^2 = 1 + \ell^2 , \quad (4.12)$$
and setting \( v = \mu \sigma \in [0, \infty) \), \( u = \mu \tau \in (-\infty, \infty) \) we obtain the background (3.36), again up to a different range of \( v \). Taking the limit \( a_0 = a_2 = 0 \) in (3.38) with \( \ell \neq 0 \) gives

\[
C_{dil}(W_{4,\ell}^{(reg)}, a)_{a_0=a_2=0} = \frac{32 c_{dil}}{9 \sqrt{1 + \ell^2} \left[(a_1 - 1)^2 + a_3^2\right]^2 \left[(a_1 + 1)^2 + a_3^2\right]^2}.
\]

Comparing with the result [28] of the semiclassical computation of the corresponding correlator \( \langle O_S^{\dagger} J S O_S^{\dagger} J S O_{dil}^{(a)} \rangle / \langle O_S^{\dagger} J S O_S^{\dagger} \rangle \) we find again the agreement: the 3-point function coefficient in eq. (4.18) of [28] also scales as \( \frac{1}{\sqrt{1 + \ell^2}} \) in the limit (4.12).

The above discussion establishes a certain relation between the two correlators in (4.1) at strong coupling. An open question is whether a similar relation may hold also at weak coupling. Another natural question is whether this relation may generalize to correlators involving \( W_n \) with \( n > 4 \) and the corresponding number of large spin twist-2 operators.

## 5 On correlator of Wilson loops with \( n > 4 \) cusps and dilaton at strong coupling

For number of cusps \( n > 4 \) there are no explicit solutions like (3.9) or (3.21) known so far. This makes it difficult to generalize the analysis of the previous sections to \( n > 4 \). Below we will calculate numerically the large distance \( |a| \to \infty \) limit of the correlator (2.1),(3.3) with an even number of cusps \( n \) at strong coupling, i.e. the corresponding OPE coefficient.

Minimal surfaces ending on regular polygons with an even number of cusps \( n \) at strong coupling, i.e. the corresponding \( \text{OPE} \) coefficient.

Minimal surfaces ending on regular polygons with an even number of cusps \( n \) at strong coupling, i.e. the corresponding \( \text{OPE} \) coefficient. They can be embedded in \( \text{AdS}_3 \) and are described, using the Pohlmeyer reduction of the classical \( \text{AdS}_3 \) string equations, in terms of the two reduced-model fields: a holomorphic function \( p(\xi) \) and a field \( \alpha(\xi, \bar{\xi}) \) satisfying the generalized sinh-Gordon equation

\[
\partial \bar{\partial} \alpha + p \bar{p} e^{-2\alpha} - e^{2\alpha} = 0.
\]

Here \( \partial = \frac{\partial}{\partial \xi} \) and \( \xi \) is a complex coordinate on the world sheet \( \xi = u + iv \). In order to reconstruct the corresponding solution of the original string equations one is to solve the following two matrix auxiliary linear problems \( (\beta = (\xi, \bar{\xi})) \)

\[
(\partial_\beta + B^L_\beta) \psi_L = 0, \quad (\partial_\beta + B^R_\beta) \psi_R = 0,
\]

where the components of the flat connections are

\[
B^L_\xi = \begin{pmatrix} \frac{1}{2} \partial_\alpha & -e^\alpha \\ -e^{-\alpha} p(\xi) & \frac{1}{2} \partial_\alpha \end{pmatrix}, \quad B^L_\bar{\xi} = \begin{pmatrix} -\frac{1}{2} \bar{\partial}_\alpha & -e^{-\alpha} \bar{p}(\bar{\xi}) \\ -e^\alpha & \frac{1}{2} \bar{\partial}_\alpha \end{pmatrix},
\]

\[
B^R_\xi = \begin{pmatrix} -\frac{1}{2} \partial_\alpha & e^{-\alpha} p(\xi) \\ e^\alpha & \frac{1}{2} \partial_\alpha \end{pmatrix}, \quad B^R_\bar{\xi} = \begin{pmatrix} \frac{1}{2} \bar{\partial}_\alpha & -e^\alpha \\ e^{-\alpha} \bar{p}(\bar{\xi}) & -\frac{1}{2} \bar{\partial}_\alpha \end{pmatrix}.
\]
The flatness of these connections is equivalent to eq. (5.1) and the holomorphicity of $p(\xi)$. The solutions describing regular polygons correspond to the holomorphic function being a homogeneous polynomial of degree $k - 2$, with $n = 2k$ and $\alpha$ being a function of the radial coordinate $|\xi| = \sqrt{\xi \bar{\xi}}$ only, i.e.

$$p(\xi) = \xi^{k-2}, \quad \bar{p}(\bar{\xi}) = \bar{\xi}^{k-2}, \quad \alpha = \alpha(|\xi|).$$

(5.5)

In this case the sinh-Gordon equation (5.1) reduces to the Painleve III equation. The boundary conditions are that $\alpha$ is regular everywhere and $\hat{\alpha} = \alpha - \frac{1}{4} \log p \bar{p}$ vanishes at infinity. The solution to this equation can be written in terms of Painleve transcendentals.

While for computing the area of the world-sheet we only need to know the reduced fields, in order to compute the correlation function with the dilaton according to (3.3) we need an explicit expression for the space-time coordinates. The general strategy of finding the solution for the string space-time coordinates is as follows. One should first find the solutions $\psi_L, \psi_R$. Then the solution for the string $AdS_3$ embedding coordinates $Y_M$ is given by

$$Y_{a,\dot{a}} = (\psi^T_L)_a(\psi_R)_{\dot{a}} = \begin{pmatrix} Y_{-1} + iY_0 & Y_1 - iY_2 \\ Y_1 + iY_2 & Y_{-1} - iY_0 \end{pmatrix}.$$  

(5.6)

The form of the solution in Poincare coordinates is determined by

$$z = \frac{1}{Y_{-1}}, \quad x_0 = \frac{Y_0}{Y_{-1}}, \quad x_1 = \frac{Y_1}{Y_{-1}}, \quad x_2 = \frac{Y_2}{Y_{-1}}.$$  

(5.7)

Unfortunately, we do not know how solve the above linear problems for $\psi_{L,R}$, i.e. how to compute the coordinates in (5.7) and thus the integral in (3.3) for $n > 4$.

For that reason in this paper we shall focus on the simpler problem of computing the large distance limit $|a| \to \infty$ of the correlation function or the OPE coefficient. In this limit the integral in (3.4) takes a simpler form (here $X^j = 1$; cf. (2.16))

$$C_{\text{dil}}(W_n, a)_{|a|\to\infty} = \frac{C_n}{|a|^{2\Delta}}, \quad C_n = c_{\text{dil}} I_{n,\Delta},$$

(5.8)

$$I_{n,\Delta} = 8 \int du dv \frac{z^{\Delta} e^{2\alpha}}{Y_{-1}^{\Delta}},$$

(5.9)

where $\Delta = 4 + j$ (here we keep the angular momentum $j$ of the dilaton general). We used the fact that in the Pohlmeyer reduction the string Lagrangian or $U_{\text{dil}}$ in (3.4) is given by $8e^{2\alpha}$. The integral (5.9) (and, in fact, the full integral in (3.4)) can be computed exactly in

\footnotetext{23}{In \cite{13,22} and subsequent developments (see \cite{39} for a review) it was understood how to use integrability in order to compute the area of the world-sheet even without knowing its shape. It would be interesting to learn how to apply similar tricks to solve the problem at hand. On the other hand, following the methods of \cite{40} one could try to construct the shape of the world-sheet and then solve the current problem.}
the two cases: (i) $n = 4$ we already discussed above (see (3.32)) and (ii) $n \to \infty$ when the cusped polygon becomes a circular Wilson loop. In the $n = 4$ case we get using (3.4),(3.9) (cf. (3.1),(3.32))

$$I_{4,\Delta} = 2\pi \left( \frac{\Gamma[\frac{\Delta}{2}]}{\Gamma[\frac{\Delta+1}{2}]} \right)^2. \quad (5.10)$$

The world-surface ending on a circular loop is given (in conformal gauge) by [16]

$$z = \tanh u, \quad x_1 = \frac{\cos v}{\cosh u}, \quad x_2 = \frac{\sin v}{\cosh u}, \quad x_0 = x_3 = 0;$$

$$u \in [0, \infty), \quad v \in [0, 2\pi]. \quad (5.11)$$

This leads to

$$I_{\infty,\Delta} = \frac{4\pi}{\Delta - 1}. \quad (5.12)$$

For the case of general even $n$ and $j = 0$, i.e. $\Delta = 4$ we can compute the integral (5.9) numerically. The numerics is very accurate since the factor $z^4$ makes the integrand decay very fast. Let us simply present the results for $I_n \equiv I_{n,\Delta=4}$ in a few cases

$$I_4 = \frac{32}{9} \approx 3.5556$$
$$I_6 \approx 3.90901$$
$$I_8 \approx 4.04496$$
$$I_{10} \approx 4.10648$$
$$I_{\infty} = \frac{4\pi}{3} \approx 4.18879, \quad (5.13)$$

where we have also included the two special cases discussed above when $I_n$ is known exactly. Unfortunately, we could not find analytic expressions in agreement with these numbers.\textsuperscript{24}

\section{Correlation function of cusped Wilson loop with dilaton operator at weak coupling}

Let us now consider the computation of the correlator (2.1)

$$\mathcal{C}(W_n, a) = \frac{\langle W_n \mathcal{O}(a) \rangle}{\langle W_n \rangle}, \quad (6.1)$$

in the weakly coupled planar $SU(N)$ $\mathcal{N} = 4$ supersymmetric gauge theory. Here the expectation values are computed using gauge theory path integral and\textsuperscript{25}

$$W_n = \frac{1}{N} \text{tr} \mathcal{P} e^{ig \hat{f}_r A_{\mu} dx^\mu}, \quad (6.2)$$

\textsuperscript{24}$I_6$ is well approximated by $4\sqrt{\frac{3}{\pi}}$.

\textsuperscript{25}The additional coupling to the scalars in the locally-supersymmetric Wilson loop [41] drops out because the null polygon contour consists of null lines.
Here we rescaled the fields with the coupling constant $g$ (with $\lambda = g^2 N$) so that the $\mathcal{N} = 4$ Lagrangian is
\begin{equation}
\mathcal{L}_{\mathcal{N}=4} = -\frac{1}{4} \text{tr}(F_{mn}^2 + \ldots) \tag{6.3}
\end{equation}
with $g$ appearing only in the vertices. We use the conventions
\begin{equation}
A_m = A_m^r T^r, \quad \text{tr}(T^r T^s) = \delta^{rs}, \quad r, s = 1, \ldots, N^2 - 1. \tag{6.4}
\end{equation}
The path $\gamma$ in (6.2) is the union of $n$ null segments of the form
\begin{equation}
\gamma^{(i)}_m(t) = x^{(i)}_m + t(x^{(i+1)}_m - x^{(i)}_m), \quad t \in [0, 1], \tag{6.5}
\end{equation}
where $x^{(i)}_m$ ($i = 1, \ldots, n$) denote the locations of the cusps. The dilaton operator (which is a supersymmetry descendant of $\text{tr} Z^2$) is essentially the $\mathcal{N} = 4$ gauge theory Lagrangian up to a total derivative (see, e.g., [42])
\begin{equation}
\mathcal{O}_{\text{dil}} = \hat{c}_{\text{dil}} \text{tr}(F_{mn}^2 + \Phi^I \partial^2 \Phi^I + \bar{\psi} \gamma \cdot \partial \psi + \ldots), \tag{6.6}
\end{equation}
where $\Phi^I$ are the scalars and $\psi$ are the fermions and we did not write explicitly the terms of order $g$ and $g^2$. The normalization coefficient $\hat{c}_{\text{dil}}$ is given by [43]
\begin{equation}
\hat{c}_{\text{dil}} = \frac{\pi^2}{4 \sqrt{3} N}. \tag{6.7}
\end{equation}
The leading order contribution to (6.1) (to which we will refer as the “tree level” one) is proportional to $g^2$ as one can easily see from (6.1), (6.2). To compute (6.1) to this order we have to expand $W_n$ to order $g^2$. Hence, we can set $g = 0$ in the Lagrangian (6.3) and in the dilaton operator (6.6). Therefore, for the purpose of computing the leading order term in (6.1) we can take
\begin{equation}
\mathcal{O}_{\text{dil}} \rightarrow \hat{c}_{\text{dil}} \text{tr} F_{mn}^2 = 2\hat{c}_{\text{dil}} \left( \partial_m A^r_n \partial^m A^{nr} - \partial_m A^r_n \partial^n A^{mr} \right). \tag{6.8}
\end{equation}
The gluon propagator in the above conventions is
\begin{equation}
\langle A^r_m(x)A^s_n(0) \rangle = -\frac{1}{4\pi^2|x|^2} \eta_{mn} \delta^{rs}. \tag{6.9}
\end{equation}
We will see that just like at strong coupling, the weak coupling correlator (6.1) is finite, i.e. we do not need to introduce a UV regularization in (6.9). Also note that to compute (6.1) to order $g^2$ we can replace $\langle W_n \rangle$ in the denominator with unity. Therefore, we obtain
\begin{equation}
\mathcal{C}^{(g^2)}(W_n, a) = \langle W_n \mathcal{O}_{\text{dil}}(a) \rangle_{\text{tree}} \nonumber
\end{equation}
\begin{equation}
= -\frac{2\hat{c}_{\text{dil}} g^2}{N} \langle \mathcal{P} \int A^s_k(x)dx^k \int A^r_s(x')dx'^d (\partial_p A^p_q \partial^p A^{qr} - \partial_p A^p_q \partial^q A^{pr})(a) \rangle. \tag{6.10}
\end{equation}
\footnote{Up to the scalar and the fermion equation of motion terms $\mathcal{O}_{\text{dil}}$ is thus given by the YM Lagrangian plus the Yukawa and the quartic scalar interaction terms.}
The path ordering symbol $\mathcal{P}$ means that $x'$ in the second integral is placed between the origin (an arbitrary point along the loop, for instance one of the cusps) and $x$. Now using that

$$\langle A_k^\nu(x) \partial_\mu A_q^\rho(a) \rangle = -\frac{1}{4\pi^2} \frac{\partial}{\partial a^\rho} \frac{\eta_{kq} \delta^{rs}}{|a - x|^2} = -\frac{1}{2\pi^2} \frac{(a - x)_p \eta_{kq} \delta^{rs}}{|a - x|^4}, \quad (6.11)$$

and performing the Wick contractions we obtain ($\lambda = g^2 N$)

$$\mathcal{C}_{\text{dil}}(W_n^\text{reg}, a) = -\hat{c}_{\text{dil}} \frac{\lambda}{\pi^4} \mathcal{P} \left( \oint \oint \left[ \frac{(a - x) \cdot (x - x')}{|a - x|^4 |a - x'|^4} dx \cdot dx' - \frac{(a - x) \cdot dx' (a - x') \cdot dx}{|a - x|^4 |a - x'|^4} \right] \right). \quad (6.12)$$

So far our discussion have been general and applicable for any number of cusps $n$. Let us now specify to $n = 4$. Computing the $\mathcal{P}$-ordered integrals in (6.12) we obtain for generic locations of 4 null cusps

$$\mathcal{C}_{\text{dil}}(W_4^\text{reg}, a) = -\hat{c}_{\text{dil}} \frac{\lambda}{2\pi^4} \prod_{i=1}^4 \frac{|x^{(i)} - x^{(i+1)}|^2 |x^{(i+2)} - x^{(i+3)}|^2}{|a - x^{(i)}|^2}. \quad (6.13)$$

This agrees with the expected structure (2.14) of the correlator (for the dilaton $\Delta = 4$) with the leading weak-coupling term in the function $F(\zeta)$ thus being simply a constant

$$F(\zeta) = -\hat{c}_{\text{dil}} \frac{\lambda}{2\pi^4}. \quad (6.14)$$

Note that the structure of (6.13) is exactly the same as the one appearing in the 1-loop correction to the 4-cusp Wilson loop $\langle W_4 \rangle$ (given by a scalar box diagram). Indeed, integrating (6.13) over $a$ we get the integrated dilaton operator or gauge theory action insertion into the Wilson loop, which is proportional to derivative of $\langle W_4 \rangle$ over gauge coupling [9,11]. This observation may allow one to extract higher order corrections to (6.13) by comparing to integrands of higher-order corrections to $\langle W_4 \rangle$.

When computing the analogs of the integrals in (6.12) for $n > 4$ we have two different types of contributions. The first one is when the two line integrals are taken along the same segment. Let us call this contribution $T_{ii}$ where the $i$-th segment is parametrized by (6.5). After some computation we obtain (up to the obvious factor $-\hat{c}_{\text{dil}} \frac{\lambda}{\pi^4}$)

$$T_{ii}(a) = -\frac{1}{2} \frac{[(a - x^{(i)}) \cdot (x^{(i+1)} - x^{(i)})]^2}{[(a - x^{(i)}) \cdot (a - x^{(i+1)})]^2 ((a - x^{(i)}) \cdot (2x^{(i+1)} - a - x^{(i)}))^2}. \quad (6.15)$$

The other type of contribution appears when the two contractions are made in different
segments. In this case we obtain

\[
T_{ij} = \frac{(a - x^{(i)}) \cdot (a - x^{(j)}) \cdot (x^{(i+1)} - x^{(i)}) \cdot (x^{(j+1)} - x^{(j)})}{|a - x^{(i)}|^2 \cdot |a - x^{(j)}|^2 \cdot (a - x^{(i)}) \cdot (a + x^{(i)} - 2x^{(i+1)}) \cdot (a - x^{(j)}) \cdot (a + x^{(j)} - 2x^{(j+1)})}
\]

These expressions are completely general. Hence, the full answer (which is rather lengthy) will be the sum of such contributions.

Let us specify (6.15), (6.16) to the case of regular polygons with even \(n\) sides with the cusps located at\(^ {27}\)

\[
x^{(i)} = \left((-1)^i, \sqrt{\frac{1 - \cos \frac{2\pi}{n}}{1 + \cos \frac{2\pi}{n}}}, \frac{\cos \left(\frac{\pi}{n}(2i + 1)\right)}{\cos \frac{\pi}{n}}, \frac{\sin \left(\frac{\pi}{n}(2i + 1)\right)}{\cos \frac{\pi}{n}}, 0\right).
\]

The problem is purely combinatorial, but there does not seem to be a simple universal formula for generic \(n\). It is relatively easy, however, to compute the OPE coefficient by placing the operator very far from the loop: taking \(|a|\) large we obtain (cf. (2.4),(2.16),(5.8))

\[
C_{\text{dil}}(W_n^{\text{reg}}, a)|_{a \to \infty} = \frac{C_n}{|a|^8}, \quad C_n = -\frac{2\hat{c}_{\text{dil}}\lambda}{\pi^4} n^2 \tan^2 \frac{\pi}{n}. \tag{6.18}
\]

For generic location of the dilaton operator one can check that the result is consistent with the general expectation (2.7) (we have checked this explicitly for \(n = 5, 6\)). For instance, for \(n = 6\) and the case of a regular polygon the result depends on three conformal ratios (since the polygon is regular only three cross-ratios are independent) and we obtain

\[
F(\zeta_1, \zeta_2, \zeta_3) = -\frac{\hat{c}_{\text{dil}} \lambda}{2\pi^4} \frac{\zeta_1 \zeta_2 \zeta_3 (\zeta_3 - 1) + \zeta_3^2 - \zeta_2^3}{\left[\zeta_1 \zeta_2^2 \zeta_3^2 (\zeta_2 - \zeta_3)^2 (\zeta_1 \zeta_3 (\zeta_2 - 1) - \zeta_2^3 + \zeta_3)^{1/3}\right]}, \tag{6.19}
\]

where the conformal ratios are defined by

\[
\zeta_1 = \frac{|x^{(1)} - x^{(3)}|^2 |a - x^{(5)}|^2}{|x^{(1)} - x^{(5)}|^2 |a - x^{(3)}|^2}, \quad \zeta_2 = \frac{|x^{(2)} - x^{(4)}|^2 |a - x^{(6)}|^2}{|x^{(2)} - x^{(6)}|^2 |a - x^{(4)}|^2}, \quad \zeta_3 = \frac{|x^{(1)} - x^{(4)}|^2 |a - x^{(3)}|^2 |a - x^{(6)}|^2}{|x^{(3)} - x^{(6)}|^2 |a - x^{(1)}|^2 |a - x^{(4)}|^2}. \tag{6.20}
\]

\(^{27}\)Note that for \(\to \infty\) this null polygon becomes a unit circle in the (12) plane.
7 Concluding remarks

In this paper we considered the correlator (2.1) of a null $n$-polygon Wilson loop with a local operator, such as the dilaton ($O_{\text{dil}} \sim \text{tr} F^2_{mn} + ...$) or a chiral primary operator. Based on symmetry considerations we determined its general form (2.7), expressing it in terms of a function $F$ of $3n-11$ conformal ratios involving the position of the operator and the positions of the cusps.

In the first non-trivial case of $n = 4$ this function $F$ depends on just one conformal ratio $\zeta$ (defined in (2.13)), making the corresponding correlator (2.1), (2.14) one of the simplest non-trivial observables that one would like eventually to compute exactly for all values of the 't Hooft coupling $\lambda$. The value of $F$ determines, in particular, the corresponding OPE coefficient (2.17) in the expansion (2.3) of the Wilson loop in terms of local operators.

We have found the leading terms in $F$ both at strong coupling (using semiclassical string theory) and at weak coupling (using perturbative planar gauge theory). At leading order at strong coupling we find that $F \sim \sqrt{\lambda}$ and has non-trivial dependence on $\zeta$ (3.18) while at leading order in weak coupling $F \sim \lambda$ and is constant (6.14). In the case of more general dilaton operator with non-zero R-charge $j$ (with $\Delta = 4 + j$) the strong-coupling expression for $F$ is given by a hypergeometric function (3.34). Similar results were found in the case of the chiral primary operator (3.45), (3.48).

It would be important to compute subleading terms in the two respective expansions:

$$F_{\lambda \gg 1} = \frac{1}{N} \left[ \sqrt{\lambda} f_0(\zeta) + f_1(\zeta) + \frac{1}{\sqrt{\lambda}} f_2(\zeta) + ... \right], \quad (7.1)$$

$$F_{\lambda \ll 1} = \frac{1}{N} \left[ \lambda h_0 + \lambda^2 h_1(\zeta) + \lambda^3 h_2(\zeta) + ... \right]. \quad (7.2)$$

Another open problem is the extension to the case of the $n > 4$ cusped Wilson loop.

Let us note that in the case of the dilaton operator integrating (2.1) over the point $a$ we get the insertion of the action and so the resulting correlator should be proportional to a derivative over $\lambda$ of the logarithm of the null-polygon Wilson loop. Thus, in particular, the knowledge of $\langle W_n \rangle$ at higher orders in $\lambda$ provides a constraint on integral of (2.1) at lower order order in $\lambda$; in general, this is not, however, enough to determine the functions $h_n(\zeta)$ in (7.2).

Part of the original motivation for the present work was to shed more light on the relation [15] between a correlator of null-separated local operators and the square of corresponding cusped Wilson loop. We conjectured a more general relation (1.2) connecting correlators with one extra operator at an arbitrary position to the correlator (2.1) we considered in this paper. It would be interesting to try to verify the relation (1.2) for $n = 4$ at weak coupling.

There are several possible extensions of our present work. One may consider the case when the local operator $O$ is not “light” at strong coupling but is allowed to carry a large
charge (e.g., R-charge or angular momentum in \(S^5\) so that \(\Delta \sim \sqrt{\lambda}\)). As in the circular
loop case in [17], then the semiclassical surface will need to be modified to account for the
presence of the sources provided by the vertex operator \(V\) in the string path integral (see
also [20]).

One may consider also a correlator of a Wilson loop with several “light” \(\Delta \ll \sqrt{\lambda}\)
operators. At leading order in strong-coupling expansion such a correlator should factorize
like in the case of the correlators two “heavy” \(\Delta \sim \sqrt{\lambda}\) operators and several “light”
one [28, 34], i.e. \(\langle W_n \mathcal{O}(a_1) \mathcal{O}(a_2) \rangle \sim \langle W_n \mathcal{O}(a_1) \rangle \langle W_n \mathcal{O}(a_2) \rangle\). This follows from the fact
that for \(\sqrt{\lambda} \gg 1\) these correlators are found, like in (3.3), by evaluating the corresponding
vertex operators on the world surface ending on the null polygon that defines \(W_n\). The
study of such more general correlators may be of interest in trying to understand better
the relation [15] between the correlator of null-separated local operators and the square of
the corresponding cusped Wilson loop.

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A On the general structure of the correlator \(C(W_n, a)\)

Here we complete the proof in section 2.1 that eq. (2.7) gives the general expression for the
 correlator in (2.1). As we discussed in section 2.1, the conformal invariance implies that the

\[\langle W_n \mathcal{O}(a_1) \mathcal{O}(a_2) \rangle \sim \langle W_n \mathcal{O}(a_1) \rangle \langle W_n \mathcal{O}(a_2) \rangle\]

Note also that in general \(\langle W_n \mathcal{O}(a_1) \mathcal{O}(a_2) \rangle\) should depend, say for \(n = 4\), on \(3n - 11 + 4 = 5\) conformal
ratios, this strong-coupling factorization implies that the leading term depends only on \(2 \times (3n - 11) = 2\)
conformal ratios.

\[\langle W_n \mathcal{O}(a_1) \mathcal{O}(a_2) \rangle \sim \langle W_n \mathcal{O}(a_1) \rangle \langle W_n \mathcal{O}(a_2) \rangle\]

\[\langle W_n \mathcal{O}(a_1) \mathcal{O}(a_2) \rangle \sim \langle W_n \mathcal{O}(a_1) \rangle \langle W_n \mathcal{O}(a_2) \rangle\]
numerator function $F$ in (2.5), i.e. in
\[ C(W_n, a) = \frac{F(a, x^{(i)})}{\prod_{k=1}^{n} |a - x^{(k)}|^{\frac{2}{n}}}, \]  
(A.1)
should have dimension $\Delta$ and should transform under the inversions so that $C$ in (2.5) changes by factor of $|a|^{2\Delta}$, i.e. $F$ should transform according to (2.6). In general, one may start with $F$ written as a sum of several structures\(^{30}\)
\[ F = \sum_{I=1}^{M} f_I(|x^{(i)}|) F_I(\zeta), \]  
(A.2)
where $f_I$ are some functions of the distances $|x^{(i)}| = |x^{(i)} - x^{(j)}|$ between non-adjacent cusps having dimension $\Delta$ and the functions $F_I(\zeta)$ depend only on conformal ratios $\zeta_k(a, x^{(i)})$. Since all the functions $F_I$ are assumed to be independent, the structure of (A.1) implies that under the inversions $f_I$ should transform as
\[ f'_I = \left(\prod_{i<j}|x^{(i)} - x^{(j)}|\right)^{-\frac{2\Delta}{n}} f_I. \]  
(A.3)
Thus all $f_I$’s transform the same way under the conformal group. This means that the ratio of any two of them is conformally invariant and, hence, is a functions of conformal ratios. Therefore, we can represent $F$ as follows
\[ F = f_1 \left(1 + \sum_{I=2}^{M} \frac{f_I}{f_1} F_I\right) \equiv f_1 F. \]  
(A.4)
Here the combination in the brackets is conformally invariant and we denoted it as $F(\zeta)$. The coefficient $f_1$ can be chosen so that it has dimension $\Delta$ and transforms as (A.3) under inversions:\(^{31}\)
\[ f_1 = \prod_{i<j-1} |x^{(i)} - x^{(j)}|^\mu, \quad \mu = \frac{2}{n(n-3)} \Delta. \]  
(A.5)
Here the power $\mu$ is fixed by taking into account that $f_1$ should have dimension $\Delta$ (the number of diagonals of the polygon is $\frac{n(n-3)}{2}$). Note that with this power $f_1$ also has the right transformation property under the inversions. We thus arrive at the general expression (2.7) for $C(W_n, a)$ in terms of a single function $F$ depending on $3n - 11$ conformal ratios.

\(^{30}\)Note that this assumption is not a restriction on generality as this sum may be infinite. Alternatively, starting, say, with $F$ written as a ratio of similar sums one may repeat the argument below by observing that the numerator and denominator should have definite dimensions, etc.

\(^{31}\)Note that we have a freedom in determining $f_1$ since we can multiply it by an arbitrary conformally invariant function.
To get a better idea of the structure of (2.7) it is useful to look at specific examples. In section 2.2 we discussed the case of $n = 4$ cusps and here we will consider the $n = 5$ case. In this case there are 5 non-zero “diagonals” $|x^{(13)}|, |x^{(14)}|, |x^{(24)}|, |x^{(25)}|, |x^{(35)}|$ and $3n - 11 = 4$ non-trivial conformal ratios that may be chosen as (cf. (2.13))

$$
\zeta_1 = \frac{|a - x^{(1)}|^2 |x^{(24)}|^2}{|a - x^{(2)}|^2 |x^{(14)}|^2}, \\
\zeta_2 = \frac{|a - x^{(2)}|^2 |x^{(35)}|^2}{|a - x^{(3)}|^2 |x^{(25)}|^2}, \\
\zeta_3 = \frac{|a - x^{(3)}|^2 |x^{(14)}|^2}{|a - x^{(4)}|^2 |x^{(13)}|^2}, \\
\zeta_4 = \frac{|a - x^{(4)}|^2 |x^{(25)}|^2}{|a - x^{(5)}|^2 |x^{(24)}|^2},$$

(A.6)

so that (2.7) takes the following explicit form

$$
\mathcal{C}(W_5, a) = \frac{\left( |x^{(13)}| |x^{(14)}| |x^{(24)}| |x^{(25)}| |x^{(35)}| \right)^{\frac{1}{5} \Delta}}{( |a - x^{(1)}| |a - x^{(2)}| |a - x^{(3)}| |a - x^{(4)}| |a - x^{(5)}| )^{\frac{1}{5} \Delta}} F(\zeta_1, \zeta_2, \zeta_3, \zeta_4), \quad (A.7)
$$

For example, in the case of the dilaton operator with $\Delta = 4$ the prefactor in (A.7) contain distances in power 1/5; one may wonder how such powers may appear in a perturbative computation. To understand why there is no contradiction let us consider a model expression that has all the required symmetries and yet contains only integer powers of distances and then show that it can be put into the general form (A.7). Namely, with $n = 5$ and $\Delta = 4$ the following expression that mimicks the $n = 4$, $\Delta = 4$ one in (2.14) has the right scaling dimension and is covariant under the inversions:

$$
\tilde{\mathcal{C}}(W_5, a) = \frac{q_1 |x^{(13)}|^2 |x^{(24)}|^2}{|a - x^{(1)}|^2 |a - x^{(2)}|^2 |a - x^{(3)}|^2 |a - x^{(4)}|^2} + \frac{q_2 |x^{(14)}|^2 |x^{(25)}|^2}{|a - x^{(1)}|^2 |a - x^{(2)}|^2 |a - x^{(3)}|^2 |a - x^{(4)}|^2} + \frac{q_3 |x^{(25)}|^2 |x^{(24)}|^2}{|a - x^{(1)}|^2 |a - x^{(2)}|^2 |a - x^{(3)}|^2 |a - x^{(5)}|^2} + \frac{q_4 |x^{(35)}|^2 |x^{(14)}|^2}{|a - x^{(1)}|^2 |a - x^{(2)}|^2 |a - x^{(3)}|^2 |a - x^{(4)}|^2} + \frac{q_5 |x^{(35)}|^2 |x^{(25)}|^2}{|a - x^{(1)}|^2 |a - x^{(2)}|^2 |a - x^{(3)}|^2 |a - x^{(5)}|^2},
$$

where $q_i$ are conformal invariants (e.g., constants). It is straightforward to check that each term here can be rewritten in the form of prefactor in (A.7) multiplied by an appropriate factor of the conformal ratios (A.6). The result is then given by (A.7) with $F$ being a sum of $F_i$’s multiplied by powers of $\zeta_k$. Explicitly,

$$
F = q_1 \left( \zeta_1 \zeta_2 \zeta_3 \zeta_4 \right)^{-1/5} + ...
$$

(A.8)

**B Some analytic results for even $n$ for $\Delta \gg 1$**

In this Appendix, we briefly describe some analytic results for regular polygons with even number of sides $n$ in the limit of large $\Delta$. For larger and larger values of $\Delta$ the main
contribution to the integral comes from the region close to the origin. We can then solve the auxiliary linear problems (5.2) perturbatively around $\rho = 0$. For instance, we have

$$\alpha = c_0 + c_1 \rho^2 + c_2 \rho^4 + \ldots, \quad \rho = |\xi|, \quad (B.1)$$

where all the subsequent coefficients can be determined in terms of $c_0$. The coefficient $c_0$ is fixed by the boundary conditions at infinity and is given by [44]

$$e^{c_0} = 3^{1/3} \frac{\Gamma[\frac{2}{3}]}{\Gamma[\frac{1}{3}].} \quad (B.2)$$

Using this expansion, we can integrate the flat connections along the radial direction, for an arbitrary angle. This discussion is pretty general and applies to any $n$ (for the class of regular polygons which can be embedded into $AdS_3$). For definiteness, let us focus on the case of the hexagon, $n = 6$. In this case we obtain

$$Y_{-1} = 1 + 2e^{2c_0} \rho^2 + \frac{2}{3}e^{2c_0}(2c_1 + e^{2c_0})\rho^4 + \ldots. \quad (B.3)$$

After performing the integrals we get for the coefficient in (5.8)

$$I_{6,\Delta} = \frac{4\pi}{\Delta} + \frac{4\pi}{\Delta^2} + \frac{12 - 2e^{-6c_0}}{3\Delta^3} + \ldots \quad (B.4)$$

where $e^{c_0}$ is given by (B.2). As a check, for $\Delta = 4$ this expression reduces to $I_6 \approx 3.90536$ which is very close from the actual numerical value $I_6 = 3.90901$ in (5.13).
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