The dilute $A_L$ models and the integrable perturbations of unitary minimal CFTs.

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Abstract

Recently, a set of thermodynamic Bethe ansatz equations is proposed by Dorey, Pocklington and Tateo for unitary minimal models perturbed by $\phi_{1,2}$ or $\phi_{2,1}$ operator. We examine their results in view of the lattice analogues, dilute $A_L$ models at regime 1 and 2. Taking $M_{5,6} + \phi_{1,2}$ and $M_{3,4} + \phi_{2,1}$ as the simplest examples, we will explicitly show that the conjectured TBA equations can be recovered from the lattice model in a scaling limit.

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1 Introduction

Since the breakthrough in the integrable perturbation theory of CFT [1, 2], there has been a lot of progress in the understanding of $\phi_{1,3}$ perturbation theory [3, 4]. On the other hand, although the remarkable example, the Ising model in a magnetic field, was treated in [1], the progress on the $\phi_{1,2}$ and $\phi_{2,1}$ perturbed theories has been steady but slow.

The systematic studies on the bootstrap procedure on $S$ matrix have been initiated in [5] and [6]. The latter approach, based on the scaling $q-$ state Potts field theory, has been further elaborated by Dorey et al [7]. Thanks to the Coleman-Thun mechanism, they argue that the contributions from spurious poles cancel and conclude the closed set of $S-$ matrices for a wide range of parameters.

The check of the results against a finite size system, however suffers from the non-diagonal nature of the scattering process. Due to the lack of a relevant string hypothesis, the diagonalization of the transfer matrix is far from trivial. In [8], conjectured is a set of thermodynamic Bethe ansatz equations (TBA) from the consideration on special cases of which they found similarity to the TBA for the sine-Gordon model. Roughly speaking, they proposed the TBA by gluing the “breather-kink” part and the “magnon” part in which the latter originates from the sine-Gordon model at specific coupling [9, 10]. Although the derivation is intuitive, the resultant equations pass many non trivial checks.

In this report, we shall examine the problem in view of a solvable lattice model. As a lattice analogue to $M_{L,L+1} + \phi_{1,2}, M_{L+1,L+2} + \phi_{2,1}$ we consider the $L-$ state RSOS model proposed in [11, 12], which will be referred to as the dilute $A_L$ model. There are several evidences for this correspondence, the central charge [11], the scaling dimensions of the leading perturbation [11, 13], universal ratios [14, 15, 16] and so on.

The question whether it shares the identical TBA to describe its finite temperature (size) property has not yet been fully answered. The purpose of this report is to present positive evidences for this inquiry.

There are already few examples to the demonstration of the equivalence. The common TBA of the dilute $A_3$ model at regime 2 and the $M_{3,4} + \phi_{1,2}$ case is firstly proved in [17]. There the most dominant solutions to the Bethe ansatz equation are explicitly identified in the form of the “string solution”, which leads to the famous $E_8$ TBA. In the case $L = 4, 6$, corresponding to the $E_7, E_6$ case, such explicit identification of string hypothesis seems not yet to be completed.

An alternative approach, based on the quantum transfer matrix (QTM) [18, 19], has been successfully applied to $L = 3, 4, 6$ [20, 21]. The functional relations among properly chosen QTMs play the fundamental role there and it enables to derive TBA without knowing the explicit locations of dominant solutions to Bethe ansatz equation.

For $L = 3, 4, 6$ cases, the underlying affine Lie algebraic structure ($E_8, E_7, E_6$, respectively) provides several clues in the investigation of the functional relations among QTMs. The remaining case, which seems to lose a direct connection to affine Lie algebra in general (see, however exceptions [8]). It might be thus challenging to the clarify the functional relation, and thereby see if the $Y$ system in is actually recovered. In this report, the last
“exceptional” case (in terminology of [8]) $M_{5,6}$ for the $\phi_{1,2}$ perturbation, and the first exceptional case $M_{3,4}$ for the $\phi_{2,1}$ perturbation are focused.

This paper is organized as follows. In the next section, we give a brief review on the dilute $A_L$ models and the QTM method. Section 3 is devoted to the discussion on the dilute $A_5$ model at regime 2 which is expected to be a lattice analogue of the $M_{5,6} + \phi_{1,2}$ theory. Fusion QTMs parameterized by skew Young diagrams are introduced and found to satisfy a set of closed functional relations. It will be shown that the conjectured TBA is naturally derived in a scaling limit. In case of the dilute $A_L$ model, $L$ even, a fundamental role seems to be played by a “kink” transfer matrix. As the simplest and the most well-known example, we treat $M_{3,4} + \phi_{2,1}$, corresponding to the Ising model off critical temperature, in section 4. We conclude the paper with brief summary and discussion in section 5.

2 The dilute $A_L$ model and the quantum transfer matrix

The dilute $A_L$ model is proposed in [11] as an elliptic extension of the Izergin-Korepin model [22]. The model is of the restricted SOS type with local variables $\in \{1, 2, \cdots , L\}$. The variables $\{a, b\}$ on neighboring sites should satisfy the adjacency condition, $|a-b| \leq 1$, which is often described by a graph in fig.1. In [11], the RSOS weights, satisfying the Yang-Baxter relation, have been found to be parameterized by the spectral parameter $u$ and the elliptic nome $q$. The crossing parameter $\lambda$ needs to be a function of $L$ for the restriction. The model exhibits four different physical regimes depending on parameters,

- regime 1. $0 < u < 3$, $\lambda = \frac{\pi L}{4(L+1)}$, $L \geq 2$
- regime 2. $0 < u < 3$, $\lambda = \frac{\pi(L+2)}{4(L+1)}$, $L \geq 3$
- regime 3. $3 - \frac{\pi}{\lambda} < u < 0$, $\lambda = \frac{\pi(L+2)}{4(L+1)}$, $L \geq 3$
- regime 4. $3 - \frac{\pi}{\lambda} < u < 0$, $\lambda = \frac{\pi L}{4(L+1)}$, $L \geq 2$.

We are interested in regimes 1 and 2.

The central charge and scaling dimension associated to leading perturbation evaluated in [11] suggests,

- The dilute $A_{L-1}$ model at regime 1 is an underlying lattice theory for $M_{L,L+1} + \phi_{2,1}$
The dilute $A_L$ model at regime 2 is an underlying lattice theory for $M_{L,L+1} + \phi_{1,2}$

There are also further evidences supporting this correspondence, as mentioned in the introduction.

One can introduce an associated 1D quantum system to the above 2D classical model. The Hamiltonian $H_{1D}$ for the former is defined from the row to row transfer matrix $T_{RTR}(u)$ of the latter, by

$$H_{1D} = \epsilon \frac{\partial}{\partial u} \log T_{RTR}(u) \bigg|_{u=0}.$$ 

We omit the explicit operator form of $H_{1D}$. The parameter $\epsilon = -1, (1)$ labels regimes 1 and 2 (3 and 4).

The thermodynamics of the 1D quantum system is the central issue in the following. We apply the method of QTM \[18, 19\] to this problem. Leaving details to references, we list the only relevant results for the following discussion.

A fundamental QTM is defined in a staggered manner

$$(T_{QTM}(u, x))_{\{b\}}^{\{a\}} = \prod_{j=1}^{N/2} a_{2j-1} b_{2j-1} \begin{array}{cccc} u+ix & a_{2j+1} & b_{2j+1} \\ a_{2j} & u-ix \\ b_{2j} \end{array}.$$ 

In the above, squares represent Boltzmann weights; four indices represent local variables and the spectral parameters are specified inside of them. The fictitious dimension $N$ (even), sometimes referred to as the Trotter number, is introduced. It has nothing to do with the real system size of the original 1D system. The real system size will not appear in our discussion as the quantities after taking the thermodynamic limit is of our interest.

It is vital that two (spectral) parameters $u, x$ exist and that only the latter concerns the commutative property of QTMs, $[T_{QTM}(u, x), T_{QTM}(u, x')] = 0$. The remaining parameter $u$ plays the role in intertwining the finite Trotter number ($N$) system and the finite temperature system ($\beta$) by $u = u^* = -\epsilon \beta N$. More concretely, the free energy per site is represented only by the largest eigenvalue $T_1(u, x)$ of $T_{QTM}$ at $x = 0$ and $u = u^*$,

$$\beta f = -\lim_{N \to \infty} \frac{1}{N} \log T_1(u^*, x = 0).$$

The eigenvalue $T_1(u, x)$ takes the form

$$T_1(u, x) = w \phi(x + \frac{3}{2}i) \phi(x + \frac{1}{2}i) \frac{Q(x - \frac{5}{2}i)}{Q(x - \frac{3}{2}i)} + \phi(x + \frac{3}{2}i) \phi(x - \frac{3}{2}i) \frac{Q(x + \frac{3}{2}i)}{Q(x - \frac{3}{2}i)} \frac{Q(x + \frac{5}{2}i)}{Q(x + \frac{3}{2}i)} + w^{-1} \phi(x - \frac{3}{2}i) \phi(x - \frac{1}{2}i) \frac{Q(x + \frac{5}{2}i)}{Q(x + \frac{3}{2}i)},$$

\begin{align}
Q(x) &:= \prod_{j=1}^{N} h[x - x_j] \\
\phi(x) &:= (h[x + (\frac{3}{2} - u)i]h[x - (\frac{3}{2} - u)i])^{N/2}, \quad h[x] := \vartheta_1(i\lambda x),
\end{align}
where \( w = \exp(i \frac{\pi \ell}{L+1}) (\ell = 1 \text{ for the largest eigenvalue sector}) \).

The parameters, \( \{x_j\} \) are solutions to “Bethe ansatz equation” (BAE),

\[
\frac{w}{w'} \phi(x_j + i) \frac{\phi(x_j - i)}{\phi(x_j - i)} = -\frac{Q(x_j - i)Q(x_j + 2i)}{Q(x_j + i)Q(x_j - 2i)}, \quad j = 1, \ldots, N. \tag{2}
\]

From now on we suppress the dependency on \( u \) which must be set as \( u = u^* \).

It has been shown in many examples [23], that the functional relations among “generalized” (fusion) QTMs offer a way to evaluate the free energy without precise knowledge on the locations \( \{x_j\} \). We adopt the same strategy here and shall discuss the functional relations realized among fusion QTMs of the dilute \( A_L \) model below.

### 3 QTM associated to skew Young diagrams and quantum Jacobi -Trudi formula

We introduce fusion QTMs associated to Young diagrams. The idea to connect Young diagrams and (eigenvalues of) QTM, originated in [24, 25, 26] is very simple. Let three boxes with letters 1,2 and 3 represent the three terms in eigenvalue of the quantum transfer matrix \( \mathbb{I} \),

\[
T_1(x) = \begin{matrix} 1 \vline & 2 \vline & 3 \end{matrix}_x.
\]

Obviously, the eigenvalue of a fusion QTM can be represented by a summation of products of “boxes” with different letters and spectral parameters, over a certain set. The point is that the set can be identified with semi-standard Young tableaux (SST) for \( \mathfrak{sl}_3 \). We state the above situation more precisely. Let \( \mu \) and \( \lambda \) be a pair of Young tableaux satisfying \( \mu_i \geq \lambda_i, \forall i \). We subtract a diagram \( \lambda \) from \( \mu \), which is called a skew Young diagram \( \mu - \lambda \). The usual Young diagram is the special case that \( \lambda \) is empty, and we will omit \( \lambda \) in the case hereafter. On each diagrams, the spectral parameter changes +2i from the left box to the right box and -2i from the top box to the bottom. We fix the spectral parameter associated to the right-top box to be \( x + i (\mu' + \mu_1 - 2) \) (or equivalently the spectral parameter associated to the left-bottom box to be \( x - i (\mu' + \mu_1 - 2) \)). Insert a letter \( \ell_{i,j} \) to the \( (i,j) \)-th box such that the semi-standard condition is satisfied. We denote its spectral parameter by \( x_{i,j} \). Then the product

\[
\prod_{i,j} \ell_{i,j} \]

is associated to the Young table. The summation over the tableaux satisfying the semi-standard condition then defines

\[
T^\nu_{\mu/\lambda}(x) = \sum_{\{\ell_{i,j}\} \in \text{SST}} \prod_{i,j} \ell_{i,j}, \tag{3}
\]

which is expected to be the eigenvalue of a fusion QTM.
The simplest subset of the above is the QTM based on Young diagrams of the rectangular shape. It was shown [20] that for any such member reduces to QTM of $1 \times m$ Young diagram, which is related to $m$-fold symmetric fusion. For later convenience, we introduce a renormalized $1 \times m$ fusion QTMs $T_m(x)$ by

$$T_m(x) = \frac{1}{f_m(x)} \sum_{i_1 \leq i_2 \leq \cdots \leq i_m} i_1 i_2 \cdots i_m.$$

The renormalization factor $f_m$, common to tableaux of width $m$, is given by

$$f_m(x) := \prod_{j=1}^{m-1} \phi(x \pm i(\frac{2m-1}{2} - j)).$$

Hereafter, for any function $f(x)$, we denote by $f(x \pm ia)$ the product $f(x+ia)f(x-ia)$.

Then the resultant $T_m$‘s are all degree 2 w.r.t. $h[x + \text{shift}]$, and have a periodicity due to Boltzmann weights; $T_m(x + Pi) = T_m(x)$, where

$$P = \begin{cases} \frac{4(L+1)}{L+2}, & \text{for regime 2} \\ \frac{4L}{L+1}, & \text{for regime 1} \end{cases}$$

Remarkably, $T_m(x)$ enjoys a “duality”

$$T_m(x) = \begin{cases} T_{2L-1-m}(x), & m = 0, \cdots, 2L, \text{for } L \text{ even} \\ T_{2L-1-m}(x + Pi), & m = 0, \cdots, 2L, \text{for } L \text{ odd} \end{cases}$$

This is deduced from the $a_2^{(2)}$ nature of the model and special choice of $\lambda$. We have at least checked the validity numerically and assume their validity in this report. The above two properties, the periodicity and the duality [5] play the fundamental role in the proof of the closed functional relations.

The real usefulness of $T_m(x)$ lies in the fact that any QTM associated to a skew Young diagram can be represented in terms of their products.

**Theorem 1.** Let $T_{\mu/\lambda}(x)$ be a renormalized $T_{\mu/\lambda}(x)$ in [3] by a common factor, $\prod_{j=1}^{\mu_1} f_{\mu_j-\lambda_j}(x + i(\mu_1 - \mu_1 + \mu_j + \lambda_j - 2j + 1))$. Then the following equality holds.

$$T_{\mu/\lambda}(x) = \det_{1 \leq j, k \leq \mu_1} (T_{\mu_j - \lambda_k - j + k}(x + i(\mu_1 - \mu_1 + \mu_j + \lambda_k - j - k + 1)))$$

where $T_{m<0} := 0$.

We regard this as a quantum analogue of the Jacobi-Trudi formula.

By this, apparently $T_{\mu/\lambda}(x)$ is an analytic function of $x$ due to BAE, and contains the quantity of our interest, $T_{1}(x)$ as a special case. The former assertion is not obvious from the original definition by the tableaux, but it is trivial from the quantum Jacobi-Trudi formula.

In the same spirit, we introduce $\Lambda_{\mu/\lambda}(x)$, which is analytic under BAE,

$$\Lambda_{\mu/\lambda}(x) := T_{\mu/\lambda}(x) / \{T_{m \geq 2L}(x) \to 0\}.$$
4 dilute $A_5$ model at regime 2 as a lattice analogue to $M_{5,6} + \phi_{1,2}$

For $M_{5,6} + \phi_{1,2}$, Dorey et al argued the existence of two kinds of particles, 2 kinks and 4 breathers. For diagonalization of scattering theory, they introduced 2 magnons (massless particles), in addition. Explicitly, the $Y-$ system reads,

\[ Y_{B_1}(x \pm \frac{3}{14}i) = \Xi_B(x), \quad Y_{B_3}(x \pm \frac{3}{14}i) = \Xi_B(x)\Xi_B(x) \]
\[ Y_{B_5}(x \pm \frac{3}{14}i) = \Xi_B(x)\Xi_B(x) \Xi_B(x)\Xi_B(x)\Xi_B(x) \]
\[ Y_{B_2}(x \pm \frac{3}{14}i) = \Xi_B(x)\Xi_B(x) \Xi_B(x)\Xi_B(x) \]
\[ Y_{K_2}(x \pm \frac{1}{14}i) = \Xi_B(x)\Xi_B(x) \Xi_B(x) \Xi_B(x) \]
\[ Y_{B_1}(x \pm \frac{1}{14}i) = \Xi_B(x)\Xi_B(x) \Xi_B(x) \Xi_B(x) \]

with

\[ \mathcal{L}_a(x) := \frac{1}{1 + Y_a(x)} \quad \Xi_a(x) := 1 + Y_a(x) \]

where $a$ takes one of $B_1, B_3, \cdots, 1, 2$. ($Y_1, Y_2$ are written as $Y^{(1)}, Y^{(2)}$ in \text{[8]}.)

We are not starting from $Y$ but rather from the QTM. Corresponding to breathers, we introduce “breather” QTM by

\[ T_{B_1}(x) := T_1(x) \]
\[ T_{B_3}(x) := \Lambda_{(8,1)}(x + \frac{13}{14}i)/\phi(x - \frac{12}{7}i) \]
\[ T_{B_5}(x) := \Lambda_{(15,8,8)/(7,7)}(x)/\phi(x + \frac{3}{2}i) \]
\[ T_{B_2}(x) := \Lambda_{(15,15,8,8)/(14,7,7)}(x + \frac{11}{14}i)/\phi(x - \frac{12}{7}i)\phi(x + \frac{9}{7}i) \]
\[ T_{B_1}(x) := T_1(x) \]
\[ T^{(6)}(x) := \Lambda_{(8,7)/(6)}(x + \frac{25}{14}i) \]

then the following relations, referred to as the “breather” $T$ system, hold.

\[ T_{B_1}(x \pm \frac{3}{14}i) = T_0(x \pm \frac{11}{14}i) + \phi(x - \frac{12}{7}i)T_{B_3}(x) \]
\[ T_{B_3}(x \pm \frac{3}{14}i) = T_0(x)T_0(x \pm \frac{8}{14}i) + T_{B_1}(x)T_{B_5}(x) \]
\[ T_{B_5}(x \pm \frac{3}{14}i) = T_0(x \pm \frac{3}{14}i)T_0(x \pm \frac{5}{14}i) + T_{B_3}(x)T_{B_5}(x) \]
\[ T_{B_2}(x \pm \frac{3}{14}i) = T_0(x \pm \frac{1}{14}i) + T^{(6)}(x) \]
where $T_0(x) = f_2(x)$. They are originated from the “hidden su(2)” discussed in [27].

In contrast to the dilute $A_3$ model (equivalently the $E_8$ case), the “hidden su(2)” structure is not enough to obtain a closed set of functional relations. We then introduce another set of functional relations, related to magnons.

To each nodes on the $D_4$ Dynkin diagram (see fig 2), we associate $t^{(a)}_m(x), (a = 1, 2, 3, 4, m \in Z_{\geq 0})$ and set $t^{(a)}_0(x) = 1$. Then we impose a $D_4$ related $T$ system among them, in terminology of [28],

$$t^{(a)}_m(x \pm \frac{i}{14}) = t^{(a)}_{m-1}(x)t^{(a)}_{m+1}(x) + g^{(a)}_m(x)\prod_{b \sim a} t^{(b)}_m(x)$$  \hspace{1cm} (7)

where $g^{(a)}_1(x \pm \frac{i}{14}) = g^{(a)}_2(x)$. In the above by $b \sim a$, we mean that $a$ and $b$ are connected on the Dynkin diagram.

Moreover we set an inhomogeneous truncation, $t^{(3)}_3 = t^{(4)}_3 = 0$ and put $g^{(3)}_1 = g^{(4)}_1 = 1.$ Unless one introduces some further condition, the set of functional relations (7) are not closed, so can not be solved. Then we demand

$$t^{(1)}_3(x) = t^{(3)}_2(x), \quad t^{(2)}_3(x) = t^{(3)}_2(x)T_{B_3}(x),$$

$$g^{(1)}_1(x) = T_0(x), \quad g^{(2)}_1(x) = T_0(x \pm \frac{2i}{7}).$$

The second relation glues the breather $T$ system to the $D_4$ related $T$ system system.

The above requirements seem to be rather artificial, but they lead to remarkable consequences. First, solutions to (7) can be given in terms of QTM appearing in the dilute $A_5$ model as follows.
\[ t_1^{(1)}(x) = T^{(6)}(x) \quad t_2^{(1)}(x) = T_{B_7}(x) \]

\[ t_1^{(3)}(x) = \Lambda_{12,8,7}/(5,4)(x) \quad t_1^{(4)}(x) = \Lambda_{5,1}(x + \frac{15}{14} i) / \phi(x - \frac{13}{2} i + \frac{15}{14} i) \]

\[ t_2^{(1)}(x) = T_{B_5}(x)T_{B_2}(x + \frac{i}{7}) \quad t_2^{(2)}(x) = T_{B_5}(x + \frac{i}{7})T_{B_2}(x) \]

\[ t_2^{(3)}(x) = T_{B_5}(x + \frac{i}{14})T_{B_2}(x + \frac{i}{14}) \quad t_2^{(4)}(x) = T_{B_5}(x)T_{B_2}(x). \]

The proof of the above statement is too lengthy to reproduce here. We hope to present them with the general discussion of \( L \) general [29].

Second, the following combination of \( T \) and \( t \) solves the \( Y \)– system for \( M_{5,6} + \phi_{1,2} \).

\[ Y_{B_1}(x) = \frac{\phi(x - \frac{12}{7} i)T_{B_5}(x)}{T_0(x + \frac{17}{14} i)} \quad Y_{B_3}(x) = \frac{T_{B_1}(x)T_{B_5}(x)}{T_0(x)T_0(x + \frac{8}{14} i)} \]

\[ Y_{B_2}(x) = \frac{T_{B_3}(x)T_{B_7}}{T_0(x + \frac{3}{14} i)T_0(x + \frac{5}{14} i)} \quad Y_{B_2}(x) = \frac{T^{(6)}(x)}{T_0(x + \frac{17}{14} i)} \]

\[ Y_{K_1}(x) = \frac{t_2^{(1)}(x)}{t_1^{(3)}(x)g_1^{(1)}(x)} \quad Y_{K_2}(x) = \frac{t_2^{(2)}(x)}{t_1^{(3)}(x)g_1^{(2)}(x)} \]

\[ Y_1(x) = \frac{t_2^{(3)}(x)}{t_1^{(4)}(x)g_1^{(2)}(x)T_{B_2}(x)} \quad Y_2(x) = \frac{t_2^{(4)}(x)}{t_1^{(3)}(x)}. \]

Third, the functions \( T, t, Y \) possess “nice” analytic properties. Before stating the properties, we need preparations. Note that the \( Y \)– system is invariant, for even \( N \), if \( Y \) is replaced by \( \tilde{Y} \), defined by

\[ \tilde{Y}_{B_1}(x) = \begin{cases} 
\frac{Y_{B_1}(x)}{\kappa(x \pm u'(1 + u') \frac{i}{14})} & \text{for } u < 0 \\
Y_{B_1}(x)\kappa(x \pm i(1 - u') \frac{3}{14}) & \text{for } u > 0 
\end{cases} \]

and all other cases, \( \tilde{Y}_a = Y_a \). The parameter \( u' \) stands for \( \frac{14}{3} u \). This is due to the definition of \( \kappa \),

\[ \kappa(x) = \left( i \frac{\partial_j(i \frac{7}{5} \pi x, \tau')}{\partial_j(i \frac{8}{7} \pi x, \tau')} \right)^N \]

which satisfies \( \kappa(x \pm i \frac{3}{14}) = 1 \). The elliptic nome \( q' = \exp(-\tau'), \tau' = 4\tau \) is introduced so as to respect the periodicity of the \( Y \) function on the real direction of \( x \). We denote a typical \( \tilde{Y} \) equation as

\[ \tilde{Y}_a(x \pm i\alpha) = \prod_b \Xi_b(x \pm i\gamma_b) \prod_c \mathcal{L}_c(x \pm i\gamma_c) \quad \text{(8)} \]

Our numerical data indicate that the rhs is analytic and nonzero in the strip \( \mathcal{R} x \in [-\alpha, \alpha] \). Each element in the lhs also satisfies the same in appropriate strips , i.e., \( \Xi_b(x) \) is
analytic and nonzero in the strip $\Im x \in [-\gamma_b, \gamma_b]$, and so on. These remarkable properties enable us to solve the coupled algebraic equation, like \( M \), in the Fourier space (to be precise, its logarithmic derivatives). Then the inverse Fourier transformation leads to the coupled integral equations which yield the explicit evaluation of $\log Y_a(x)$.

To make a direct contact with the TBA result, three further steps are needed. First take the Trotter limit $N \to \infty$, $uN = \beta$, $\epsilon = -1$). Second rewrite $\log \Xi_b(x)$ by $\log L_b(x)$.

Third, take a scaling limit. The step1 is executable analytically, which manifests one of the advantage of the present approach. The resultant equations no longer have dependency on a fictitious $N$ but only depends on the temperature variable, $\beta$. After the step 2, we obtain the equations, in the Fourier space,

\[
\tilde{M} \begin{pmatrix} \log Y_{B_1} \\ \log Y_{B_4} \\ \vdots \end{pmatrix} = 4\pi \beta \begin{pmatrix} 1 & 0 & \vdots \\ 0 & 1 & \vdots \\ \vdots & \vdots & \ddots \end{pmatrix} + \tilde{K}_0 \begin{pmatrix} \tilde{L}_{B_1} \\ \tilde{L}_{B_4} \\ \vdots \end{pmatrix}
\]

where $\tilde{L}_{B_1} = \log(1 + \frac{1}{Y_{B_1}})$ and similarly for others. The quantities with hat indicate that they are Fourier transformations. $\tilde{M}$ and $\tilde{K}_0$ are asymmetric matrices of which explicit forms are omitted here but can be easily obtained from the $Y$ system. The only first entry has a nonvanishing inhomogeneous term in the rhs. This reflects the fact that only $Y_{B_1}$ needs some trivial renormalization so as to have nice analytic properties. By multiplying $M^{-1}$ from the left, the kernel matrix of TBA, $M^{-1}K_0$ turns out to be symmetric, remarkably. This property is crucial in applying the dilogarithm technique to evaluate the central charge. The inhomogeneous term vector $4\pi \beta M^{-1}t(1, 0, \cdots)$ possesses six non vanishing elements.

\[
\tilde{d}_{B_1} = \frac{8\pi \beta \cosh \frac{1}{14} k}{(2 \cosh \frac{2}{14} k - 1) D(k)} \\
\tilde{d}_{B_2} = \frac{4\pi \beta (2 \cosh \frac{2}{14} k + 1)(2 \cosh \frac{4}{14} k - 1)}{D(k)} \\
\tilde{d}_{B_3} = \frac{16\pi \beta \cosh \frac{1}{14} k \cosh \frac{4}{14} k}{(2 \cosh \frac{2}{14} k - 1) D(k)} \\
\tilde{d}_{B_4} = \frac{8\pi \beta \cosh \frac{1}{14} k}{(2 \cosh \frac{2}{14} k - 1) D(k)} \\
\tilde{d}_{K_1} = \frac{4\pi \beta}{(2 \cosh \frac{2}{14} k - 1) D(k)} \\
\tilde{d}_{K_2} = \frac{8\pi \beta \cosh \frac{4}{14} k}{(2 \cosh \frac{2}{14} k - 1) D(k)}
\]

where we denote by $\tilde{d}_{B_i}$ for the drive term associated to $\log Y_{B_1}$ and so on. A common denominator $D(k)$ denotes

\[
D(k) = 2 \cosh \frac{12}{14} k + 2 \cosh \frac{10}{14} k - 2 \cosh \frac{6}{14} k - 2 \cosh \frac{4}{14} k + 1.
\]

We finally perform the step 3. In view of QFT, the bulk quantity is not of direct interest, rather the fluctuation is. We introduce $y_{B_1}(x) = Y_{B_1}(x + \tau’)$, for example, to evaluate quantities near the “fermi surface” with $\tau' = \frac{12\pi}{7\pi}$. Then take a limit $q \to 0$ such that $m_k R = \frac{8\pi \beta x}{2 \cosh \frac{21}{14} k} q^2$. By $r$ we mean the residue of $i/D(k)$ at $k = \pi/3i$. Two quantities

\[
10
\]
$M^{-1}$ and $K_0$ seem to carry the information of $S$ matrices; the elements of $M^{-1}K_0$ agree with the expression described in \[\text{in terms of } S \text{ matrices, under identification } x = 3\theta/\pi \text{ in the limit } q \to 0. \] The matrix $M^{-1}$ also encodes the information of the mass spectra. When taking the inverse Fourier transformation, the nearest zero to the real axis, $k = \pm i\frac{\pi}{3}$ of $D(k)$, is relevant in the “scaling” limit as $\tau$ tends to be infinity. Applying the Poisson’s summation formula, we found a most dominant term,

$$d_{K_1}(x) = \frac{8\pi\beta r}{2\cos \frac{\pi}{3}} e^{-4}\cosh \frac{\pi}{3} x = m_K R \cosh \theta,$$

for example, where $\frac{\pi}{3} x = \theta$. Note that the relation $m_K \propto q^4$ is consistent with the scaling dimension $\Delta_{1,2} = \frac{1}{8}$. One similarly verifies that all other drive terms also take the form $mR \cosh \theta$ and their mass ratio agree with those in \[\text{in terms of } S \text{ matrices, under identification } x = 3\theta/\pi \text{ in the limit } q \to 0. \]

Thus the TBA of $M_{5,6} + \phi_{1,2}$ theory is recovered from the scaling limit of the dilute $A_5$ model at regime 2.

Once $Y$ is fixed by TBA, we can also evaluate the free energy from

$$T_1(x \pm \frac{3}{14} i) = T_{B_1}(x \pm \frac{3}{14} i) = T_0(x \pm \frac{11}{14} i)(1 + Y_{B_1}(x)).$$

It is readily shown that a “fluctuation” part of the free energy $f$ is proportional to $\frac{1}{\beta} \sum_k \int m_k R \cosh \theta \log(1 + 1/y_k) d\theta$, which is the desired expression.

5 dilute $A_2$ model at regime 1 as a lattice analogue to $M_{3,4} + \phi_{2,1}$

We treat another example corresponding to $\phi_{2,1}$ perturbation theory, the simplest and most well studied case, the Ising model off critical temperature, $M_{3,4} + \phi_{2,1}$. The model is described by a free fermion, thus is rather trivial in a sense. In view of functional relations, however, it is not trivial to derive the simplest $Y$ system $Y(x \pm i\frac{\pi}{3}) = 1$ (in the present normalization of $x$), from $T_1(x)$ in \[\text{in terms of } S \text{ matrices, under identification } x = 3\theta/\pi \text{ in the limit } q \to 0. \] which consists of 3 terms. This model is actually one of the first examples, which require a more fundamental object than $T_1(x)$, a box, which seems to correspond to a fundamental breather $B_1$. 

11
We define
\[
\tau_K(x) = w\phi(x + 2i)\frac{Q(x + 2i)}{Q(x + i)} + \phi(x)\frac{Q(x)Q(x + 3i)}{Q(x + i)Q(x - i)}
\]  \hspace{1cm} (9)
\[
+ w^{-1}\phi(x - 2i)\frac{Q(x - 2i)}{Q(x - i)} \hspace{1cm} (10)
\]
which has a property common to \(T_1(x)\) namely, it is pole-free due to the Bethe ansatz equation.

More importantly, we have functional relations,
\[
\tau_K(x \pm \frac{1}{2}i) = T_1(x) + T_2(x) = 2T_1(x) \hspace{1cm} (11)
\]
\[
\tau_K(x \pm \frac{3}{2}i) = T_3(x) + T_0(x) - \phi(x \pm \frac{5}{2}i)(w^3 + \frac{1}{w^3}) = 2(\phi(x \pm \frac{1}{2}i) + \phi(x \pm \frac{5}{2}i)) \hspace{1cm} (12)
\]
The first equalities are directly verified by comparing both sides in forms of the ratio of \(Q\) functions. The second are consequences of the duality. One then reaches a desired relation \((12)\) after proper renormalizations. The first equation, \((11)\) seems to suggest \(\tau_K(x)\) is related to the kink in the theory; the bound state of kink produces a breather.

In general \(L = \text{even}\) case, we find that \(\tau_K(x)\) plays the most fundamental role, which will be a topic of a separate publication.

It is a nice exercise to recover from \((11)\) and \((12)\), the free fermion free energy, in the scaling limit. We shall remark the analytic property, supported by numerics, that \(\tau_K(x)\) being Analytic and Nonzero in the strip \(Imx \in [-\frac{3}{2}, \frac{3}{2}]\), for that purpose.

6 Summary and discussion

In this report, we demonstrate explicitly that TBA for \(M_{5,6} + \phi_{1,2}\) and \(M_{3,4} + \phi_{2,1}\), conjectured by Dorey et al, are realized in the scaling limit of lattice models. The crucial idea is to introduce fusion transfer matrices associated to skew Young tableaux and to investigate the functional relations among them. The proofs of functional relations are rather combinatorial and lengthy, thus omitted due to the lack of space. They will be supplemented in the subsequent paper which discusses the TBA behind the dilute \(A_L\) models, \(L\) general \([29]\).

There are still many open problems. The explicit identification of string solutions would be definitely one of the most important. The complete study on this will shed some lights on the way how to proceed for TBA in the case of perturbed non-unitary minimal models. We mention the first step in this direction in \([30]\).

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References

[1] A.B. Zamolodchikov, Adv. Stud. Pure. Math. 19 (1989) 641.
[2] A.B. Zamolodchikov, Int. J. Mod. Phys. A 4 (1989) 4235.
[3] T. Eguchi and S.K. Yang, Phys.Lett.B 235 (1990) 282-286.
[4] Al. B. Zamolodchikov, Nucl. Phys. B358 (1991) 497-523.
[5] F. A. Smirnov, Int. J. Mod. Phys. A6 (1991) 1407-1428.
[6] L. Chim and A. Zamolodchikov, Int. J. Mod. Phys. A 7 (1992) 5317-5335.
[7] P. Dorey, A. Pocklington and R. Tateo, Nucl. Phys. B661 425-463 [hep-th/0208111].
[8] P. Dorey, A. Pocklington and R. Tateo, Nucl. Phys. B661 464-513 [hep-th/0208202].
[9] R. Tateo, Int. J. Mod. Phys. A9 (1995) 1357-1376.
[10] P. Dorey, R. Tateo and K.E. Thompson, Nucl. Phys. B470 (1996) 317
[11] S.O.Warnaar, B. Nienhuis and K. A. Seaton, Phys. Rev. Lett. 69(1992) 710.
[12] S.O.Warnaar, B. Nienhuis and K. A. Seaton, Int. J. Mod. Phys. B 7 (1993) 3727.
[13] S.O.Warnaar, P.A. Pearce, B. Nienhuis and K. A. Seaton, J. Stat. Phys 74 (1994) 469.
[14] M.T. Batchelor and K.A. Seaton, J. Phys. A 30 (1997) L479.
[15] K.A. Seaton, J.Phys. A35 (2002) 1597-1604
[16] C. Korff and K.A. Seaton, Nucl.Phys. B636 (2002) 435-464
[17] V.V. Bazhanov, O. Warnaar and B. Nienhuis, Phys. Lett. B 322 (1994) 198.
[18] M. Suzuki, Phys. Rev. B. 31 (1985) 2957.
[19] A. Klümper, Ann. Physik 1 (1992) 540.
[20] J. Suzuki, Nucl Phys B528 (1998) 683.
[21] J. Suzuki, Progress in Math. 191 (2000) 217-247.
[22] A.G. Izergin and V. E. Korepin, Comm. Math. Phys. 79 (1981) 303.
[23] See e.g., A. Klümper, Z. Phys. B 91 (1993) 507,
    G. Jüttner, A. Klümper and J. Suzuki, Nucl. Phys. B 512 (1998) 581.
    A. Kuniba, K. Sakai and J. Suzuki, Nucl. Phys. B 525 (1998) 597-626.
[24] V.V. Bazhanov and N. Yu Reshetikhin, J.Phys. A 23 (1990) 1477.
[25] J. Suzuki, Phys. Lett. A 195 (1994) 190.
[26] A. Kuniba and J. Suzuki, Comm. Math. Phys. 173 (1995) 225.
[27] Y.K. Zhou, P.A. Pearce and U. Grimm Physica A 222 (1995) 261.
[28] A. Kuniba, T. Nakanishi and J. Suzuki, Int. J. Mod. Phys. A9 (1994) 5215-5266.
[29] P. Dorey, J. Suzuki and R. Tateo, in preparation.
[30] R.M. Ellem and V.V. Bazhanov Nucl. Phys. B647 (2002) 404-432