Abstract

A partial matrix over a field $\mathbb{F}$ is a matrix whose entries are either an element of $\mathbb{F}$ or an indeterminate and with each indeterminate only appearing once. A completion is an assignment of values in $\mathbb{F}$ to all indeterminates. Given a partial matrix, through elementary row operations and column permutation it can be decomposed into a block matrix of the form $\begin{bmatrix} W & \ast & \ast \\ 0 & S & 0 \\ 0 & 0 & T \end{bmatrix}$ where $W$ is wide (has more columns than rows), $S$ is square, $T$ is tall (has more rows than columns), and these three blocks have at least one completion with full rank. And importantly, each one of the blocks $W$, $S$ and $T$ is unique up to elementary row operations and column permutation whenever $S$ is required to be as large as possible. When this is the case $\begin{bmatrix} W & \ast & \ast \\ 0 & S & 0 \\ 0 & 0 & T \end{bmatrix}$ will be called a WST-decomposition. With this decomposition it is trivial to compute maximum rank of a completion of the original partial matrix: $\text{rows}(W) + \text{rows}(S) + \text{cols}(T)$. In fact we introduce the WST-decomposition for a broader class of matrices: the ACI-matrices.

1 Introduction

1.1 Preliminaries

The ACI-matrices were introduced in 2010 by Brualdi, Huang and Zhan [4] as a generalization of partial matrices (matrices whose entries are either a constant or an indeterminate and with each indeterminate only appearing once). Let $\mathbb{F}[x_1, \ldots, x_k]$ denote the set of polynomials in the indeterminates $x_1, \ldots, x_k$ with coefficients on a field $\mathbb{F}$. A matrix over $\mathbb{F}[x_1, \ldots, x_k]$ is an Affine Column Independent matrix or ACI-matrix if its entries are polynomials of degree at most one and no indeterminate appears in two different columns. A completion of an ACI-matrix $A$ is an assignment of values in $\mathbb{F}$ to all indeterminates so that it gives a constant matrix in $\mathbb{F}$. All definitions and most of the results work for any field $\mathbb{F}$, so we will usually omit in what field we are working on.

Definition 1.1. The Rank of an ACI-matrix $M$ is the set of all possible ranks of completions of $M$. The maxRank of $M$ is the maximum rank for a completion of $M$. The minRank of $M$ is the minimum rank for a completion of $M$. We say that $M$ is constantRank if $\text{maxRank}(M) = \text{minRank}(M)$.

The Rank of partial matrices has a substantial literature (see Section 1 of [4]). The constantRank partial matrices were studied in [7], and the constantRank ACI-matrices were studied in [1, 2, 3, 4, 6].

Multiplying an ACI-matrix by a constant square matrix on the left produces an ACI-matrix of the same size. If, in addition, the constant matrix is nonsingular then the new ACI-matrix will share the same Rank, minRank and maxRank with the old one. The same happens if we permute the columns of an ACI-matrix. Since we are concerned with the Rank of ACI-matrices, the following definition makes sense.

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Definition 1.2. Let $M$ be an $m \times n$ ACI-matrix. For any nonsingular constant matrix $R$ of order $m$ and for any permutation matrix $Q$ of order $n$, the ACI-matrix $RMQ$ is said to be equivalent to $M$. We represent this equivalence by $M \sim RMQ$.

In this work we are interested in the study of the Rank of a given ACI-matrix $M$. In order to do it we will consider the equivalence class of $M$ so that we can find a representative in the class with an easier structure that, for example, reveal directly its maxRank. This easier structure will be the WST-decomposition of $M$ as we will see in Section 6. It is important to point out that this definition of equivalence can not be applied to partial matrices since $RMQ$ will not be necessarily a partial matrix, it will be an ACI-matrix.

1.2 Basic definitions

The relation between the number of rows and the number of columns of ACI-matrices will play an important role, that is why we introduce the following terminology:

Definition 1.3. Let $M$ be an ACI-matrix.

- $\text{rows}(M)$ denotes the number of rows of $M$.
- $\text{cols}(M)$ denotes the number of columns of $M$.
- $M$ is wide if $\text{cols}(M) > \text{rows}(M)$.
- $M$ is tall if $\text{rows}(M) > \text{cols}(M)$.
- $M$ is square if $\text{rows}(M) = \text{cols}(M)$.

For technical reasons we will consider as ACI-matrices the ones without rows or/and without columns, namely: (i) the wide degenerate ACI-matrix $0 \times q$ with $q > 0$; (ii) the tall degenerate ACI-matrix $p \times 0$ with $p > 0$; and (iii) the square degenerate or void ACI-matrix $0 \times 0$.

A constant matrix $M$ is full row rank if $\text{rank}(M) = \text{rows}(M)$, is full column rank if $\text{rank}(M) = \text{cols}(M)$, and is full rank if $\text{rank}(M) = \min\{\text{rows}(M), \text{cols}(M)\}$. We will adapt this common terminology to the maxRank of ACI-matrices.

Definition 1.4. The ACI-matrix $M$ is

- Full Row maxRank or FRmR if $\text{maxRank}(M) = \text{rows}(M)$.
- Full Column maxRank or FCmR if $\text{maxRank}(M) = \text{cols}(M)$.
- Full maxRank or FmR if $\text{maxRank}(M) = \min\{\text{rows}(M), \text{cols}(M)\}$ or, equivalently, if $M$ has a completion with full rank.

Just to emphasize: (i) FRmR is wide or square FmR; (ii) FCmR is tall or square FmR; (iii) FmR is FRmR or FCmR or both. Again, for technical reasons we will consider a tall degenerate to be FRmR, a wide degenerate to be FCmR, and the void to be FRmR and FCmR.

The next proposition shows how to build new FmR ACI-matrices from known FmR ACI-matrices. Its proof is straightforward.

Proposition 1.5. Let $M$ be an ACI-matrix.

1. If $M$ is FRmR and $M \sim M'$ then $M'$ is FRmR.
2. If $M$ is FCmR and $M \sim M'$ then $M'$ is FCmR.
3. If $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ where $A$ and $C$ are FRmR then $M$ is FRmR.
4. If $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ where $A$ and $C$ are FCmR then $M$ is FCmR.
5. If $M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ where $A$ and $C$ are square FmR then $M$ is square FmR.
1.3 The main Theorem

The first important result in ACI-matrices appeared in the work where they were introduced.

**Theorem 1.6.** (see [4, Theorem 3]) Let $M$ be an $m \times n$ ACI-matrix and let $\rho$ be an integer such that $0 \leq \rho < \min\{m, n\}$. The following two statements are equivalent:

(i) $\text{maxRank}(M) \leq \rho$.

(ii) For some positive integers $r$ and $s$ with $\rho = (m - r) + (n - s)$ there exist a nonsingular constant matrix $R$ and a permutation matrix $Q$ such that $RMQ = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ where 0 is an $r \times s$ submatrix with all its entries equal to zero.

It is important to note that the values of $r$ and $s$ in Theorem 1.6 are not unique in general, and neither are $A$ and $C$ (see the example below). The inspiration of the present work has been to generalize Theorem 1.6 to find an analogous decomposition which is unique in some sense. In our main theorem, Theorem 6.1, we will show that any ACI-matrix is equivalent to an ACI-matrix

$$
\begin{bmatrix}
W & * & * \\
0 & S & * \\
0 & 0 & T
\end{bmatrix}
$$

where $W$ is wide FRmR or void, $S$ is square FmR or void and $T$ is tall FCmR or void. And importantly, each one of the ACI-matrices $W, S$ and $T$ are unique up to equivalence whenever $S$ is required to be as large as possible. When this is the case the ACI-matrix $\begin{bmatrix} W & S & * \\ 0 & 0 & T \end{bmatrix}$ is called a **WST-decomposition**. This decomposition even works for FmR matrices (note that Theorem 1.6 did not), but then at least one of the blocks $W$ or $T$ become void or degenerate ACI-submatrices.

The WST-decomposition will allow us to restrict the study of some properties of ACI-matrices (and for that matter partial matrices) to the case of FmR ACI-matrices. For instance, in this work we will be focused on the maxRank and if we know a WST-decomposition of an ACI-matrix it will be trivial to compute its maxRank: $\text{rows}(W) + \text{rows}(S) + \text{cols}(T)$. So we might ask how to find the WST-decomposition in practice. A work that explains an algorithm that computes efficiently the WST-decomposition is in preparation. The maxRank for partial matrices was treated in [5] where the authors provide an procedure to compute it. Our algorithm will permit us to compute the maxRank for the broader class of ACI-matrices.

**Example:** Below we present a $5 \times 5$ ACI-matrix (it is actually a partial matrix) with maxRank 4, and with three different block partitions that verify the condition (ii) of Theorem 1.6.

$$
M = \begin{bmatrix}
1 & x_1 & y_1 & z_1 & 1 \\
0 & 0 & y_2 & z_2 & t_1 \\
0 & 0 & 0 & z_3 & t_2 \\
0 & 0 & 0 & 0 & t_3 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & x_1 & y_1 & z_1 & 1 \\
0 & 0 & y_2 & z_2 & t_1 \\
0 & 0 & 0 & z_3 & t_2 \\
0 & 0 & 0 & 0 & t_3 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & x_1 & y_1 & z_1 & 1 \\
0 & 0 & y_2 & z_2 & t_1 \\
0 & 0 & 0 & z_3 & t_2 \\
0 & 0 & 0 & 0 & t_3 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

(1)

For $M$ a valid WST-decomposition is:

$$
\begin{bmatrix}
1 & x_1 & * & * & * \\
0 & 0 & y_2 & z_2 & * \\
0 & 0 & 0 & z_3 & * \\
0 & 0 & 0 & 0 & t_3 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

where $W = \begin{bmatrix} 1 & x_1 \end{bmatrix}$, $S = \begin{bmatrix} y_2 & z_2 \\ 0 & z_3 \end{bmatrix}$, $T = \begin{bmatrix} t_3 \\ 1 \end{bmatrix}$.
The property of \( S \) being as big as possible is required for the uniqueness of \( W, S \) and \( T \) up to equivalence. Because decompositions like the following meet all the other requirements

\[
\begin{bmatrix}
1 & x_1 & \ast & \ast & \ast \\
0 & 0 & y_2 & \ast & \ast \\
0 & 0 & 0 & z_3 & t_2 \\
0 & 0 & 0 & 0 & t_3 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

2 Zero blocks

Given an ACI-matrix, we will be interested in finding equivalent ACI-matrices which have a lot of zeros. A submatrix with all its entries equal to 0 will be referred as a zero block. A measure associated to the size of a zero block that we will frequently use is its number of rows plus its number of columns.

**Definition 2.1.** Let \([A \ B \ 0 \ C]\) be an \( m \times n \) ACI-matrix where the zero block 0 is of size \( r \times s \).

- The zero block in \([A \ B \ 0 \ C]\) is **Big** when \( r + s > \max\{m, n\} \).
- The zero block in \([A \ B \ 0 \ C]\) is **Medium** when \( r + s = \max\{m, n\} \).

Note that a Medium zero block measures one less than the smallest Big zero block. Again, for technical reasons we include the possibility for a Medium zero block to be degenerate. In our next result we provide equivalent and more intuitive definitions for Big and for Medium zero blocks.

**Proposition 2.2.** For an ACI-matrix \( M = [A \ B \ 0 \ C] \) with a zero block the following properties are satisfied:

(i) The zero block is Big if and only if \( A \) is wide and \( C \) is tall.

(ii) The zero block is Medium if and only if either (1) \( M \) is tall, \( A \) is square and \( C \) is tall; (2) \( M \) is square, \( A \) and \( C \) are square; or (3) \( M \) is wide, \( A \) is wide and \( C \) is square.

**Proof.** Let \( M = [A \ B \ 0 \ C] \) be an \( m \times n \) ACI-matrix with a \( r \times s \) zero block, that is,

\[
M = \begin{bmatrix}
A & 0 & B \\
0 & \ast & C \\
\end{bmatrix}
\]

(i) The zero block of \( M \) is Big if and only if

\[
r + s > \max\{m, n\} \iff \begin{cases}
s > m - r \\r > n - s
\end{cases} \iff \begin{cases}
cols(A) > \text{rows}(A) \\\text{rows}(C) > \text{cols}(C)
\end{cases} \iff \begin{cases}A \text{ is wide} \\C \text{ is tall}
\end{cases}
\]

(ii) The zero block of \( M \) is Medium if and only if

\[
r + s = \max\{m, n\} \iff \begin{cases}
s = m - r \\
r = n - s
\end{cases} \iff \begin{cases}
cols(A) = \text{rows}(A) \\\text{rows}(C) = \text{cols}(C)
\end{cases} \iff \begin{cases}A \text{ square} \\C \text{ square}
\end{cases}
\]
Let us see a consequence when an ACI-matrix has a Big zero block.

**Proposition 2.3.** An ACI-matrix with a Big zero block is not FmR.

**Proof.** Let \( M = [\begin{array}{cc} A & B \\ 0 & C \end{array}] \) be as in (2). If the zero block is Big then

\[
\text{maxRank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \leq \text{rows}(A) + \text{cols}(C) = (m - r) + (n - s) = (m + n) - (r + s) < \min\{m, n\}
\]

and therefore \( M \) is not FmR.

If a Big or Medium zero block is present in an ACI-matrix it makes it trivial to compute the maxRank when the diagonal blocks are FmR.

**Theorem 2.4.** If the ACI-matrix \( [\begin{array}{cc} A & B \\ 0 & C \end{array}] \) has a Big or Medium zero block then the following conditions are equivalent:

(i) \( A \) is FRmR and \( C \) is FCmR.

(ii) \( \text{maxRank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \text{rows}(A) + \text{cols}(C) \).

**Proof.** Since 0 is a Big or Medium zero block of \( [\begin{array}{cc} A & B \\ 0 & C \end{array}] \) then \( A \) is wide or square and \( C \) is tall or square.

(i) \( \Rightarrow \) (ii) Let \( \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{0} & \hat{C} \end{bmatrix} \) be a completion of \( [\begin{array}{cc} A & B \\ 0 & C \end{array}] \) such that \( \hat{A} \) and \( \hat{C} \) are full rank. Then we have

\[
\text{maxRank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \leq \text{rows}(A) + \text{cols}(C) = \text{rank}(\hat{A}) + \text{rank}(\hat{C}) \leq \text{rank} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{0} & \hat{C} \end{bmatrix} \leq \text{maxRank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}
\]

and the result follows.

(ii) \( \Rightarrow \) (i) We divide the proof in two parts:

(a) Note that \( \text{maxRank} \begin{bmatrix} B \\ 0 \end{bmatrix} \leq \text{cols}(C) \) and \( \text{maxRank}(A) \leq \text{rows}(A) \), so

\[
\text{maxRank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \leq \text{maxRank} \begin{bmatrix} A \\ 0 \end{bmatrix} + \text{maxRank} \begin{bmatrix} B \\ 0 \end{bmatrix} \leq \text{rows}(A) + \text{cols}(C).
\]

Since \( \text{maxRank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \text{rows}(A) + \text{cols}(C) \) then \( \text{maxRank}(A) = \text{rows}(A) \). So \( A \) is FRmR.

(b) Note that \( \text{maxRank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \leq \text{rows}(A) \) and \( \text{maxRank}(C) \leq \text{cols}(C) \), so

\[
\text{maxRank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \leq \text{maxRank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \leq \text{rows}(A) + \text{cols}(C).
\]

Since \( \text{maxRank} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \text{rows}(A) + \text{cols}(C) \) then \( \text{maxRank}(C) = \text{cols}(C) \). So \( C \) is FCmR.

\]

3 Factor and semifactor sets

Frequently, we will need to permute the columns of an ACI-matrix so that a certain set \( F \) of columns appear as the first \#F columns. This will be achieved by right multiplying the ACI-matrix by the appropriate permutation matrix.
Notation 3.1. Let \( F = \{f_1, \ldots, f_s\} \subset \{1, \ldots, n\} \) and let \( \overline{F} = \{1, \ldots, n\} - \{f_1, \ldots, f_s\} = \{g_1, \ldots, g_{n-s}\} \). We define the permutation \( \sigma_F \) of \( \{1, \ldots, n\} \) by

\[
\begin{align*}
\sigma_F(f_i) &= i \quad \text{for all } i \in \{1, \ldots, s\} \\
\sigma_F(g_j) &= s + j \quad \text{for all } j \in \{1, \ldots, n-s\}.
\end{align*}
\]

Note that \( \sigma_F(F) = \{1, \ldots, s\} \) and \( \sigma_F(\overline{F}) = \{s+1, \ldots, n\} \). Finally, \( Q_F \) denotes the permutation matrix of order \( n \) such that for each \( k = 1, \ldots, n \) the \( k \)-th column of any \( m \times n \) ACI-matrix \( M \) is equal to the \( \sigma_F(k) \)-th column of \( MQ_F \).

We now introduce two concepts associated to ACI-matrices: factor and semifactor sets. It will be crucial in this work to determine when an ACI-matrix has factor sets or has semifactor sets, and also to determine the relation between all of its factor sets or between all of its semifactor sets.

Definition 3.2. Let \( M \) be an \( m \times n \) ACI-matrix. The set \( F \subseteq \{1, \ldots, n\} \) is a factor set of \( M \) if there exists a nonsingular matrix \( R \) of order \( m \) such that

\[
RMQ_F = \begin{bmatrix}
A & B \\
0 & C
\end{bmatrix} \tag{3}
\]

where the zero block is Big, \( A \) is (wide) FRmR and \( C \) is (tall) FCmR. We will say that \( RMQ_F \) is an \( F \)-decomposition of \( M \).

Note that in \( (3) \) \( A \) is wide and \( C \) is tall since the zero block is Big. For completeness we put in Table 1 all cases that are possible in \( (3) \) for \( RMQ_F \) taking into account when degenerate ACI-submatrices appear.

\[
\begin{array}{c|c|c}
A \text{ wide non-degenerate} & A \text{ wide degenerate} \\
\text{and FRmR} & \\
\hline
C \text{ tall non-degenerate} & \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} & \begin{bmatrix} 0 & C \end{bmatrix} \\
\text{and FCmR} & \\
\hline
C \text{ tall degenerate} & \begin{bmatrix} A \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \end{bmatrix} \\
F = \{1, \ldots, n\} & F = \{1, \ldots, n\} \\
\end{array}
\]

Table 1: \( RMQ_F \) for factor sets.

Now we introduce the concept of semifactor sets, where the Medium zero blocks take the same role as the Big zero blocks for factor sets.

Definition 3.3. Let \( M \) be an \( m \times n \) ACI-matrix. The set \( F \subseteq \{1, \ldots, n\} \) is a semifactor set of \( M \) if there exists a nonsingular \( R \) of order \( m \) such that

\[
RMQ_F = \begin{bmatrix}
A & B \\
0 & C
\end{bmatrix} \tag{4}
\]

where the zero block is Medium, \( A \) is (wide or square) FRmR and \( C \) is (tall or square) FCmR. Then \( RMQ_F \) is called an \( F \)-semidecomposition of \( M \).

Note that in \( (4) \) \( A \) or/and \( C \) are square since the zero block is Medium. For completeness we put in Table 2 all cases that are possible in \( (4) \) for \( RMQ_F \) taking into account when degenerate ACI-submatrices appear.
| Case 1: | Case 2: | Case 3: |
|---------|---------|---------|
| \[ A B \] | \[ 0 C \] | Tall degenerate zero block $F = \emptyset$ |

Case 4: \[ A \] square since 0 is Medium $F = \{1, \ldots, n\}$

Case 5: \[ A \] Wide degenerate zero block $F = \{1, \ldots, n\}$

Table 2: $RMQ_F$ for semifactor sets.

Table 2 shows that $FmR$ ACI-matrices have at least one semifactor set. Note that if $M$ is $FmR$ then $M$ is wide FRmR or square FmR or tall FCmR. Now if $M$ is wide/square FRmR then $\{1, \ldots, n\}$ is a semifactor set since we can always take $M = A$ with $C$ void and the Medium zero block being wide degenerate (Case 5 in Table 2); and if $M$ is tall/square FCmR then $\emptyset$ is a semifactor set since we can always take $M = C$ with $A$ void and the Medium zero block being tall degenerate (Case 3 in Table 2).

In the next result we characterize when an ACI-matrix has a factor or a semifactor set. As we will see, the part corresponding to factor sets is quite related with Theorem 1.6.

**Proposition 3.4.** The following assertions about factor and semifactor sets are true:

(i) An ACI-matrix has a factor set if and only if it is not $FmR$.

(ii) An ACI-matrix has a semifactor set if and only if it is $FmR$.

**Proof.** Let $M$ be an $m \times n$ ACI-matrix.

(i) $\Rightarrow$ If $M$ has a factor set $F$ then there exists a nonsingular $R$ such that

$$RMQ_F = \left[ \begin{array}{cc} A & B \\ 0 & C \end{array} \right]$$

where the zero block is Big. So $RMQ_F$ is not $FmR$ (Proposition 2.3) and thus $M$ is not $FmR$.

$\Leftarrow$ If $M$ is not $FmR$ then $\max\text{rank}(M) < \min\{m, n\}$ and, by Theorem 1.6, for some positive integers $r$ and $s$ with $\max\text{rank}(M) = (m-r) + (n-s)$ there exist a nonsingular $R$ and a permutation $Q$ such that $RMQ = \left[ \begin{array}{cc} A & B \\ 0 & C \end{array} \right]$ where $0$ is an $r \times s$ zero block. Note that

$$\min\{m, n\} > \max\text{rank}(M) = (m-r) + (n-s) \Rightarrow r + s > m + n - \min\{m, n\} = \max\{m, n\}$$

so the zero block of $\left[ \begin{array}{cc} A & B \\ 0 & C \end{array} \right]$ is Big. On the other hand

$$\max\text{rank} \left[ \begin{array}{cc} A & B \\ 0 & C \end{array} \right] = \max\text{rank}(M) = (m-r) + (n-s) = \text{rows}(A) + \text{cols}(C)$$

which implies (see Theorem 2.3) that $A$ is FRmR and $C$ is FCmR. And so, the existence of a factor set for $M$ is proved.

(ii) $\Rightarrow$ If $M$ has a semifactor set $F$ then there exists a nonsingular $R$ such that

$$RMQ_F = \left[ \begin{array}{cc} A & B \\ 0 & C \end{array} \right]$$

(5)
where the zero block is Medium, \(A\) is FRmR and \(B\) is FCmR. By Proposition 2.2 we know that \(A\) or/and \(C\) are square, and so both are FRmR or both are FCmR. This implies (see Proposition 1.5) that \(RMQ_F\) is FRmR or FCmR. So \(RMQ_F\) is \(FmR\), and then \(M\) is FRmR.

\(\Leftarrow\) If \(M\) is \(FmR\) then, as we have seen just after Table 2, \(M\) has at least one semifactor set.

\(\Box\)

## 4 Linear independent rows

Let \(M\) be an ACI-matrix. Remember that each column of \(M\) has its own indeterminates. Suppose that the first column of \(A\) has indeterminates \(x_1, x_2, \ldots, x_i\); that the second column \(y_1, y_2, \ldots, y_j\); and so on. Now, let us represent the vector space where the entries of the first column lie by \(\mathbb{F} + \mathbb{F}x_1 + \ldots + \mathbb{F}x_i\), and for the second column \(\mathbb{F} + \mathbb{F}y_1 + \ldots + \mathbb{F}y_j\), and so on. All these sets are vector spaces over \(\mathbb{F}\). And the row vectors of \(M\) are in the vector space

\[
(F + \mathbb{F}x_1 + \ldots + \mathbb{F}x_i) \times (F + \mathbb{F}y_1 + \ldots + \mathbb{F}y_j) \times \ldots
\]

From now on when we talk about linear independence or linear dependence of the rows of an ACI-matrix \(M\), we are talking about the vector space given in (6).

The next Proposition and Remark expose the relation of ACI-matrices with linear independent rows and ACI-matrices which are FRmR. It is important to keep in mind this relation.

**Proposition 4.1.** An FRmR ACI-matrix has linear independent rows.

**Proof.** Suppose that \(M\) is an \(m \times n\) ACI-matrix with linear dependent rows. Then any completion of \(M\) is a constant matrix with linear dependent rows whose rank is less than \(m\). So \(\max\text{rank}(M) < \text{rows}(M)\) and so \(M\) is not FRmR.  

**Remark 4.2.** The linear independence of the rows of an ACI-matrix does not imply that it is FRmR. For example, over any field the ACI-matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & x \\
1 & 1 & 1 & y
\end{bmatrix}
\]

has linear independent rows and \(\max\text{rank}\) equal to 2. So it is not FRmR.

The next result will be relevant in the proof of a key result: Lemma 4.6. In this somewhat long statement the condition that we want to emphasize is that \(A_1\) as well as \(A_2\) have linear independent rows. This condition will reappear in Lemma 4.6 and actually it will be a central theme of many proofs of our work.

**Lemma 4.3.** Consider two ACI-matrices of size \(m \times n\) given by

\[
M_1 = \begin{bmatrix}
A_1 & B_1 \\
0 & C_1
\end{bmatrix}
\quad \text{and} \quad
M_2 = \begin{bmatrix}
A_2 & B_2 \\
0 & C_2
\end{bmatrix}
\]

where \(A_1\) and \(A_2\) have the same number \(n_1\) of columns. Let \(R\) be a nonsingular constant matrix of order \(m\) and let \(Q\) and \(Q'\) be permutation matrices of orders \(n_1\) and \((n-n_1)\) respectively, such that

\[
\begin{bmatrix}
A_2 & B_2 \\
0 & C_2
\end{bmatrix} = R \begin{bmatrix}
A_1 & B_1 \\
0 & C_1
\end{bmatrix} \begin{bmatrix}
Q & 0 \\
0 & Q'
\end{bmatrix}.
\]

If \(A_1\) as well as \(A_2\) have linearly independent rows then \(A_1 \sim A_2\) and \(C_1 \sim C_2\).
Proof. By hypothesis $A_1$ and $A_2$ have the same number $n_1$ of columns, but nothing is said about the number of rows. Nevertheless, as $R \left[ \begin{array}{c} A_1 \\ 0 \end{array} \right] = \left[ \begin{array}{c} A_2 \\ 0 \end{array} \right]$ where $A_1$ and $A_2$ have linearly independent rows then $A_1$ and $A_2$ also have the same number $m_1$ of rows. Therefore $A_1$ and $A_2$ have the same size $m_1 \times n_1$. Thus $C_1$ and $C_2$ also have the same size $(m - m_1) \times (n - n_1)$. Writing

$$R = \begin{bmatrix} S & T \\ U & V \end{bmatrix}$$

as a block matrix where $S$ is $m_1 \times m_1$ and $V$ is $(m - m_1) \times (m - m_1)$ we have

$$\left[ \begin{array}{c} A_2 \\ 0 \end{array} \right] \left[ \begin{array}{c} B_2 \\ C_2 \end{array} \right] = \left[ \begin{array}{c} S \\ U \end{array} \right] \left[ \begin{array}{c} A_1 \\ B_1 \end{array} \right] \left[ \begin{array}{c} Q \\ 0 \end{array} \right] = \left[ \begin{array}{c} \cdots \\ U A_1 Q \cdots \end{array} \right].$$

Since $U A_1 Q = 0$, $A_1$ has linearly independent rows, and $Q$ is a permutation then $U = 0$. So

$$\left[ \begin{array}{c} A_2 \\ 0 \end{array} \right] \left[ \begin{array}{c} B_2 \\ C_2 \end{array} \right] = \left[ \begin{array}{c} S \\ 0 \end{array} \right] \left[ \begin{array}{c} A_1 \\ 0 \end{array} \right] \left[ \begin{array}{c} Q \\ C_1 \end{array} \right] \left[ \begin{array}{c} 0 \\ Q' \end{array} \right] = \left[ \begin{array}{c} \cdots \\ S A_1 Q \cdots \end{array} \right].$$

As $S$ and $V$ are nonsingular then $A_1 \sim A_2$ and $C_1 \sim C_2$.

Let $M$ be an ACI-matrix. Imagine that we want to find out if $F$ is a factor or a semifactor set of $M$. It makes sense to try to find an equivalent ACI-matrix with as many zero rows as possible in the ACI-submatrix formed by the columns indexed by $F$. An efficient way to do this is the procedure of a sweep from bottom to top that we are going to introduce now.

Definition 4.4. Let $M$ be an $m \times n$ ACI-matrix and let $F \subseteq \{1, \ldots, n\}$. A sweep from bottom to top in $M$ is a procedure that transforms $M$ into an equivalent ACI-matrix that has all its nonzero rows linearly independent. It consists of $m - 1$ steps and step $i$ is the following:

**Step i:** If it is possible, make the $(m - i)$-th row equal to the zero row by adding linear combinations of rows $m - i + 1, \ldots, m$ below it.

A sweep from bottom to top in $M$ with respect to the columns of $F$ is a procedure as the previous one but only requiring to do zeros in the entries allocated in the columns corresponding to $F$.

Example 4.5. For the field or reals consider the ACI-matrix

$$M = \begin{bmatrix} x + 2 & 1 & z \\ x + 1 & 8y & 3z - 5 \\ x & 4y & z - 2 \\ 1 & 4y & 2z - 3 \end{bmatrix}.$$

If we do a sweep from bottom to top in $M$ then

$$\begin{bmatrix} x + 2 & 1 & z \\ x + 1 & 8y & 3z - 5 \\ x & 4y & z - 2 \\ 1 & 4y & 2z - 3 \end{bmatrix} \overset{\text{step 1}}{\rightarrow} \begin{bmatrix} x + 2 & 1 & z \\ x + 1 & 8y & 3z - 5 \\ x - 1 & 0 & -z + 1 \\ 1 & 4y & 2z - 3 \end{bmatrix} \overset{\text{step 2}}{\rightarrow} \begin{bmatrix} x + 2 & 1 & z \\ x & 4y & z - 2 \\ 1 & 4y & 2z - 3 \end{bmatrix}$$

and we have finished with an equivalent ACI-matrix whose nonzero rows are linearly independent.

If we do a sweep from bottom to top in $M$ with respect to $F = \{2\}$ then

$$\begin{bmatrix} x + 2 & 1 & z \\ x + 1 & 8y & 3z - 5 \\ x & 4y & z - 2 \\ 1 & 4y & 2z - 3 \end{bmatrix} \overset{\text{step 1}}{\rightarrow} \begin{bmatrix} x + 2 & 1 & z \\ x + 1 & 8y & 3z - 5 \\ x - 1 & 0 & -z + 1 \\ 1 & 4y & 2z - 3 \end{bmatrix} \overset{\text{step 2}}{\rightarrow} \begin{bmatrix} x + 2 & 1 & z \\ x - 1 & 0 & -z + 1 \\ x - 1 & 0 & -z + 1 \\ 1 & 4y & 2z - t \end{bmatrix}$$

and we have finished with an equivalent ACI-matrix whose nonzero rows in the second column are linearly independent. Note that the linear combinations employed to make zeros in the second column of $M$ were extended to the entire rows of $M$. 


The definition of factor (resp. semifactor) set just requires one decomposition to exist. If we know somehow that $F$ is a factor (resp. semifactor) set of $M$ and we perform a sweep from bottom to top in $M$ with respect to $F$, we might ask the following question: Do we always arrive, up to permutation of rows and columns, to an $F$-decomposition (resp. $F$-semidecomposition)? The answer is yes, as we will see in the next result where we assume that the sweep from bottom to top with respect to $F$ has already occurred. Then we only need to permute rows and columns to leave a zero block in the bottom left part.

**Lemma 4.6.** Let $M$ be an $m \times n$ ACI-matrix. Suppose that $F$ is a factor (resp. semifactor) set of $M$ and $P$ is a permutation matrix of order $m$ such that

$$PMQ_F = \begin{bmatrix} \sigma_F(F) \\ A & B \\ 0 & C \end{bmatrix}$$

with $A$ having linearly independent rows. Then (7) is an $F$-decomposition (resp. $F$-semidecomposition). That is: the zero block is Big (resp. Medium), $A$ is FRmR and $C$ is FCmR.

**Proof.** Note that $PMQ_F$ is obtained from $M$ by permuting its rows by $P$ and its columns by $Q_F$.

As $F$ is a factor (resp. semifactor) set of $M$ then there exists a nonsingular $R$ such that

$$RMQ_F = \begin{bmatrix} \sigma_F(F) \\ A' & B' \\ 0 & C' \end{bmatrix}$$

is an $F$-decomposition (resp. $F$-semidecomposition). Therefore $A'$ is FRmR and so, by Proposition 4.1 it has all its rows linearly independent. Note that

$$\begin{bmatrix} A' & B' \\ 0 & C' \end{bmatrix} = RMQ_F = R \left( P^{-1} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} Q_F^{-1} \right) Q_F = RP^{-1} \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}.$$

From Lemma 4.3 we conclude that $A \sim A'$ and $C \sim C'$. As $A'$ and $C'$ are FmR then $A$ and $C$ are also FmR. And since the zero block of (8) is Big (resp. Medium) then the zero block of (7) will also be Big (resp. Medium). So $PMQ_F$ is an $F$-decomposition (resp. $F$-semidecomposition).

### 5 The Union and Intersection of Factors and of Semifactor Sets

Most of the heavy lifting of the main result is done in this section. In the following three results we will study the relative position of two factor sets or of two semifactor sets of an ACI-matrix. It is important to recall (see Proposition 3.4) that an ACI-matrix has a factor set if and only if it is not FmR, and that an ACI-matrix has a semifactor set if and only if it is FmR.

**Lemma 5.1.** Two factor sets of an ACI-matrix can not be disjoint.

**Proof.** Suppose $F_1$ and $F_2$ are two disjoint factor sets of an $m \times n$ ACI-matrix $M$. As the empty set is not a factor set of any ACI-matrix then, up to permutation of columns, we can assume that $F_1 = \{1, \ldots, h\}$ and $F_2 = \{h+1, \ldots, k\}$ with $1 \leq h < k \leq n$. And let $U = \{1, \ldots, n\} \setminus (F_1 \cup F_2)$.

First we do a sweep from bottom to top in $M$ with respect to $F_1$. After reordering the rows we obtain

$$M' = \begin{bmatrix} F_1 \setminus \{1\} \\ A \\ F_2 \setminus \{h+1\} \\ B \\ U \setminus \{h\} \\ C \\ \overset{r}{\cdots} \\ D \\ \overset{t}{\cdots} \end{bmatrix}.$$
where $A$ has linearly independent rows. As $M \sim M'$ and $M'$ is obtained from $M$ without permuting columns, then $F_1$ and $F_2$ are factor sets of $M'$.

Now we do a sweep from bottom to top in $M'$ with respecto to $F_2$. After reordering the first $r$ rows and the last $t$ rows we obtain:

$$M'' = \begin{bmatrix} F_1 & F_2 & U \\ A^\alpha & 0 & C'' \\ A' & B' & C' \\ 0 & D' & E'' \\ 0 & 0 & E'' \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ t_1 \\ t_2 \end{bmatrix}$$

with $r_1 + r_2 = r$, $t_1 + t_2 = t$, and where $[B' D']$ has linearly independent rows. As $A \sim [A'' A']$ then also $[A'' A']$ has linear independent rows. As $M' \sim M''$ and $M''$ is obtained from $M'$ without permuting columns then $F_1$ and $F_2$ are factor sets of $M''$.

Now let us deduce some inequalities that will be key to our analysis. On one hand we have a zero block corresponding to the factor set $F_1$, it is formed by the two zeros of the first block column of $M''$. As $[A'' A']$ has linear independent rows then this zero block must be Big (see Lemma 4.6). So

$$t_1 + t_2 + \#F_1 > \max\{m, n\}. \quad (9)$$

On the other hand we have a zero block corresponding to the factor set $F_2$, it is formed by the two zeros of the second block column of $M''$. As $[B' D']$ has linearly independent rows then this zero block must be Big (see Lemma 4.6). So

$$r_1 + t_2 + \#F_2 > \max\{m, n\}. \quad (10)$$

Finally, the culprit of the contradiction will be the $(t_1 + t_2) \times (\#F_2 + \#U)$ ACI-submatrix of $M''$

$$\begin{bmatrix} D' & E' \\ 0 & E'' \end{bmatrix},$$

since we will prove that it is FmR and not FmR at the same time:

1. $[D' E']$ is FmR because $F_1$ is a factor set of $M''$ and $[A'' A']$ has linearly independent rows (see Lemma 4.6).

2. The zero block of $[D' E']$ is Big if $t_2 + \#F_2 > \max\{\text{rows } [D' E'], \text{cols } [D' E']\}$. Let us see that this inequality is true:

   (a) $t_2 + \#F_2 > t_1 + t_2$.

   From (10) it follows that

   $$r_1 + t_2 + \#F_2 > m = r_1 + r_2 + t_1 + t_2$$

   which implies the required inequality.

   (b) $t_2 + \#F_2 > \#F_2 + \#U$.

   From (9) it follows that:

   $$t_1 + t_2 + \#F_1 > m = r_1 + r_2 + t_1 + t_2 \implies \#F_1 > r_1. \quad (11)$$

   From (10) it follows that:

   $$r_1 + t_2 + \#F_2 > n = \#F_1 + \#F_2 + \#U. \quad (12)$$

   From (11) and (12) we obtain the required inequality.
So \([D' \ E']_0[E''_0]\) is not FmR because its zero block is Big (see Proposition 2.3).

Although the proof of the next Lemma 5.2 is very similar to the proof of Lemma 5.1 we will include it for clarity because the differences are subtle.

**Lemma 5.2.** Two semifactor sets of a wide ACI-matrix can not be disjoint.

**Proof.** Suppose \(F_1\) and \(F_2\) are two disjoint semifactor sets of a wide ACI-matrix \(M\) of size \(m \times n\). The first half of the proof of Lemma 5.1 is the same as this one with the difference that in this case we have two semifactor sets and so the zero blocks that will appear will be Medium instead of Big. So, after the two sweeps from bottom to top in \(M\) and reordering the rows we obtain:

\[
M'' = \begin{bmatrix}
F_1 & E'
A' & C''
B' & 0
0 & D'
0 & 0
\end{bmatrix}
\]

Now let us deduce some equalities that will be key to our analysis. On one hand we have a zero block corresponding to the semifactor set \(F_1\), it is formed by the two zeros of the first block column of \(M''\). As \([A'\ A'']\) has linear independent rows then this zero block must be Medium (see Lemma 4.6). So

\[
t_1 + t_2 + \#F_1 = \max\{m, n\} = n.
\]

On the other hand we have a zero block corresponding to the semifactor set \(F_2\), it is formed by the two zeros of the second block column of \(M''\). As \([B'\ C'']\) has linearly independent rows then this zero block must be Medium (see Lemma 4.6). So

\[
r_1 + t_2 + \#F_2 = \max\{m, n\} = n.
\]

Again the culprit of the contradiction will be the \((t_1 + t_2) \times (\#F_2 + \#U)\) ACI-submatrix of \(M''\)

\[
\begin{bmatrix}
D' & E'
0 & E''
\end{bmatrix},
\]

since we will prove that it is FmR and not FmR at the same time:

1. \([D' \ E']_0[E''_0]\) is FmR because \(F_1\) is a semifactor set of \(M''\) and \([A'\ A'']\) has linearly independent rows (see Lemma 4.6).

2. The zero block of \([D' \ E']_0[E''_0]\) is Big if \(t_2 + \#F_2 > \max\{\text{rows} [D' \ E'], \text{cols} [D' \ E']\}\). Let us see that this inequality is true:

   (a) \(t_2 + \#F_2 > t_1 + t_2\).

   From (14) it follows that:

   \[
   r_1 + t_2 + \#F_2 = n > m = r_1 + r_2 + t_1 + t_2
   \]

   which implies the required inequality.

   (b) \(t_2 + \#F_2 > \#F_2 + \#U\).

   From (13) it follows that:

   \[
   t_1 + t_2 + \#F_1 = n > m = r_1 + r_2 + t_1 + t_2 \implies \#F_1 > r_1.
   \]

From (14) it follows that:

\[
\begin{aligned}
\#F_1 &= \#F_1 \\& \#F_1 \\& \#U
\end{aligned}
\]

From (15) and (16) we obtain the required inequality.
So \( \begin{bmatrix} D' & E' \\ 0 & E'' \end{bmatrix} \) is not FmR because its zero block is Big (see Proposition 2.3).

\[ \square \]

**Lemma 5.3.** Two semifactor sets of a tall or square ACI-matrix can be disjoint or not disjoint.

**Proof.** We provide an example for each case. For tall ACI-matrices

\[
\begin{bmatrix}
F_1 & F_2 \\
1 & 0 & x \\
0 & 1 & y \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
F_1 \\
1 & 0 & 0 \\
x & 1 & y \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

and for square ACI-matrices it is enough to delete the last row on each one. \[ \square \]

In what follows our main objective will be to prove that the intersection and the union of two factor (resp. semifactor) sets is a factor (resp. semifactor) set. That is what Theorem 5.6 below says. Note that Lemmas 5.1, 5.2 and 5.3 conclude that two factor or two semifactor sets always overlap, except in the case of some semifactor sets of tall or square ACI-matrices. So in order to achieve our objective we will first study this exceptional case of disjoint semifactor sets (Theorem 5.4), and then the generic case of overlapping factor or semifactor sets (Theorem 5.5).

**Theorem 5.4.** The intersection and the union of two disjoint semifactor sets of a tall or square ACI-matrix are semifactor sets.

**Proof.** Let \( F_1 \) and \( F_2 \) be two disjoint semifactor sets of a tall or square ACI-matrix \( M \) of size \( m \times n \).

Tall and square FmR ACI-matrices are the only ACI-matrices for which the empty set is a semifactor set (see Table 2). Then \( F_1 \cap F_2 = \emptyset \) is a semifactor set of \( M \).

The proof for \( F_1 \cup F_2 \) starts again like the proof of Lemma 5.1. So after the two sweeps from bottom to top in \( M \) and reordering the rows we obtain:

\[
M'' = \begin{bmatrix}
F_1 & F_2 \\
A'' & 0 & C'' \\
A' & B' & C' \\
0 & D' & E' \\
0 & 0 & E''
\end{bmatrix}
\]

(17)

Focusing on the \( F_2 \) semifactor set, by Proposition 2.2 we know that \( \begin{bmatrix} B' \\ D' \end{bmatrix} \) is square and so \( \#F_2 \geq t_1 \).

(18)

Focusing on the \( F_1 \) semifactor set, the zero block formed by the two zeros of the first column of \( M'' \) is Medium. So according to Definition 2.1:

\[
t_1 + t_2 + \#F_1 = \max \{ m, n \} = m.
\]

(19)

Let \( Z \) be the zero block composed by the two zero blocks of the last row of \( M'' \). From (18) and (19)

\[
t_2 + \#F_1 + \#F_2 \geq \max \{ m, n \} = m
\]

(20)

and so \( Z \) is either a Big (if inequality is strict) or a Medium (if there is equality) zero block in \( M'' \). It can not be Big, otherwise \( M'' \) would not be FmR (Proposition 2.3) and this contradicts that an ACI-matrix has a semifactor set if and only if it is FmR (Proposition 3.4). So \( Z \) is a Medium zero block in \( M'' \). So now we know that there must be equality in (20), which in turn means that there is
equality in (18): \( \#F_2 = t_1 \). So \( D' \) is square, which implies that \( B' \) is wide degenerate and therefore \( r_2 = 0 \). So (17) is simplified into:

\[
M'' = \begin{bmatrix}
F_1
\begin{bmatrix}
A'' & 0 & C'' \\
0 & D' & E' \\
0 & 0 & E''
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\{ & r_1 \\
\{ & t_1 \\
\{ & t_2
\end{bmatrix}
\]

To prove that \( F_1 \cup F_2 \) is a semifactor set, apart from \( Z \) being a Medium zero block, we still need to prove that:

i. \( \begin{bmatrix} A'' & 0 & D' \end{bmatrix} \) is FRmR. Since \( F_1 \) is a semifactor set we know that \( \begin{bmatrix} A'' \\ A' \end{bmatrix} \) is FRmR, and since \( F_2 \) is a semifactor set we know that \( \begin{bmatrix} B' \\ D' \end{bmatrix} \) is FRmR. Now we apply Proposition 4.5 to deduce this item.

ii. \( E'' \) is FCmR. Since \( F_1 \) is a semifactor set we know that \( \begin{bmatrix} D' & E' \\ 0 & E'' \end{bmatrix} \) is FCmR. This together with \( D' \) being square implies that \( E'' \) is FCmR.

\[\square\]

For the proof of our next theorem it will be convenient to introduce the notion of complementary of an ACI-submatrix. Let \( A \) be an ACI-submatrix of an ACI-matrix \( M \), the complementary \( \overline{A} \) of \( A \) in \( M \) is obtained by deleting all the rows and columns of \( M \) that are involved in \( A \).

**Theorem 5.5.** The intersection and the union of two overlapping factor (resp. semifactor) sets of an ACI-matrix are factor (resp. semifactor) sets.

**Proof.** Let \( F_1 \) and \( F_2 \) be two overlapping factor (resp. semifactor) sets of an ACI-matrix \( M \).

If \( F_1 \subset F_2 \) or \( F_2 \subset F_1 \) the result is trivial. So, without loss of generality we can assume that \( F_1 = \{1, \ldots, k\} \) and \( F_2 = \{h + 1, \ldots, l\} \) with \( 1 \leq h < k < l \). We do a sweep from bottom to top in \( M \) with respect to \( F_1 \) and after reordering the rows we obtain

\[
M' = \begin{bmatrix}
A & B \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
C & D \\
E & F
\end{bmatrix}
\begin{bmatrix}
\{ & r \\
\{ & t
\end{bmatrix}
\]

where \( \begin{bmatrix} A & B \end{bmatrix} \) has linearly independent rows and \( \begin{bmatrix} E & F \end{bmatrix} \) is FCmR (see Lemma 4.6). Now we do a sweep from bottom to top in \( M' \) with respect to \( F_2 \) and after reordering the first \( r \) rows and the last \( t \) rows we obtain

\[
M'' = \begin{bmatrix}
A'' & 0 \\
A' & B' \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
0 & D'' \\
C' & D'
\end{bmatrix} & \begin{bmatrix}
0 \\
E' & F'
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\{ & r_1 \\
\{ & r_2 \\
\{ & t_1 \\
\{ & t_2
\end{bmatrix}
\]

with \( r_1 + r_2 = r \), \( t_1 + t_2 = t \), \( r_1, r_2, t_1, t_2 \geq 0 \), and where \( \begin{bmatrix} A'' & 0 \\ A' & B' \end{bmatrix} \) and \( \begin{bmatrix} B' & C' \\ 0 & E' \end{bmatrix} \) have linearly independent rows. Note that the fourth column of \( M'' \) could be tall degenerate with size \((r_1 + r_2 + t_1 + t_2) \times 0\) while the other three columns will never be tall degenerate since \( 1 \leq h < k < l \). Let us see that \( t_1 = 0 \) is impossible: if this was the case then the third row of \( M'' \) would be wide degenerate and since \( F_1 \) is a factor set then \( \begin{bmatrix} 0 & F'' \end{bmatrix} \) should be FCmR, but this is impossible since it has columns full of zeros. So, from now on \( t_1 > 0 \). The value \( r_2 \) might be positive or zero and the arguments we will provide hold for both.

Four possibilities appear depending on the values of \( r_1 \) and \( t_2 \).
Suppose \( r_1 > 0 \) and \( t_2 > 0 \).

i. Since \( F_2 \) is a factor (resp. semifactor) set and \( \begin{bmatrix} B' & C' \\ 0 & E \end{bmatrix} \) has linearly independent rows, then (see Lemma 4.6) its complementary \( \begin{bmatrix} A' & D' \\ 0 & F' \end{bmatrix} \) in \( M'' \) is FCmR. In particular \( A'' \) is FCmR.

ii. Since \( F_1 \) is a factor (resp. semifactor) set and \( \begin{bmatrix} A'' & 0 \\ A' & B' \end{bmatrix} \) has linearly independent rows, then (see Lemma 4.6) \( \begin{bmatrix} A'' & 0 \\ A' & B' \end{bmatrix} \) is FRmR. In particular \( A'' \) is FRmR.

iii. Since \( A'' \) is FCmR and FRmR at the same time then \( A'' \) is a square \( FmR \).

iv. Since \( F_1 \) is a factor (resp. semifactor) set and \( \begin{bmatrix} A'' & 0 \\ A' & B' \end{bmatrix} \) has linearly independent rows, then (see Lemma 4.6) \( \begin{bmatrix} A'' & 0 \\ A' & B' \end{bmatrix} \) is FRmR. In particular \( E' \) is FRmR.

v. Since \( F_2 \) is a factor (resp. semifactor) set and \( \begin{bmatrix} B' & C' \\ 0 & E \end{bmatrix} \) has linearly independent rows then (see Lemma 4.6) \( \begin{bmatrix} B' & C' \\ 0 & E \end{bmatrix} \) is FRmR. In particular \( E' \) is FRmR.

vi. Since \( E' \) is FCmR and FRmR at the same time then \( E' \) is a square \( FmR \).

vii. Since \( \begin{bmatrix} B' & C' \\ 0 & E \end{bmatrix} \) is FCmR and \( E' \) is square then \( F'' \) is FCmR.

viii. Since \( \begin{bmatrix} A'' & 0 \\ A' & B' \end{bmatrix} \) is FRmR and \( A'' \) is square then \( B' \) is FRmR.

ix. The complementary matrix of \( F'' \) in \( M'' \) is \( \overline{F''} = \begin{bmatrix} A'' & 0 & 0 \\ A' & B' & C' \\ 0 & 0 & E' \end{bmatrix} \sim \begin{bmatrix} B' & A' & C' \\ 0 & 0 & 0 \end{bmatrix} \). Since \( B' \), \( A'' \) and \( E' \) are FRmR (viii, ii and vii) then Proposition 1.5 implies that \( F'' \) is also FRmR.

x. The complementary matrix of \( B' \) in \( M'' \) is \( \overline{B'} = \begin{bmatrix} A'' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \). Since \( A'' \), \( E' \) and \( F'' \) are FCmR (ii, iv and vii) then Proposition 1.5 implies that \( \overline{B'} \) is FCmR.

xi. Consider the zero block \( Z \) obtained by joining together the three zero blocks in the second column of \( M'' \), and consider the zero block \( Z_1 \) corresponding to the factor (resp. semifactor) set \( F_1 \). Note that \( Z \) has size \( (r_1 + t_1 + t_2) \times \#(F_1 \cap F_2) \), and that \( Z_1 \) has size \( (t_1 + t_2) \times \#(F_1) \). The number rows + cols for \( Z \) and \( Z_1 \) is equal since \( A'' \) is square (iii). As \( Z_1 \) is Big (resp. Medium) by hypothesis and the number rows + cols is what determines if a zero block is Big (resp. Medium), then \( Z \) is Big (resp. Medium).

xii. Consider the zero block \( Z' \) obtained by joining together the three zero blocks in the last row of \( M'' \), and consider again the zero block \( Z_1 \) corresponding to the factor (resp. semifactor) set \( F_1 \). Note that \( Z' \) has size \( t_2 \times \#(F_1 \cap (F_2 \setminus F_1)) \) and that \( Z_1 \) has size \( (t_1 + t_2) \times \#(F_1) \). The number rows + cols for \( Z' \) and \( Z_1 \) is equal since \( E' \) is square (vi). As \( Z_1 \) is Big (resp. Medium) by hypothesis then \( Z' \) is Big (resp. Medium).

xiii. \( F_1 \cap F_2 \) is a factor (resp. semifactor) set of \( M'' \) since \( Z \) is a Big (resp. Medium) zero block (xii), \( B' \) is FRmR (viii) and \( \overline{B'} \) is FCmR (x). xiv. \( F_1 \cup F_2 \) is a factor (resp. semifactor) set of \( M'' \) since \( Z' \) is a Big (resp. Medium) zero block (xii), \( F'' \) is FCmR (vii) and \( \overline{F''} \) is FRmR (x).

Suppose \( r_1 > 0 \) and \( t_2 = 0 \). Then

\[
M'' = \begin{bmatrix}
\begin{bmatrix}
F_1 \\
\hline
A'' & 0 \\
A' & B' \\
0 & 0
\end{bmatrix}
& D'' \\
E & F
\end{bmatrix}
\begin{bmatrix} r_1 \\
\hline
r_2 \\
t - \frac{1}{2}
\end{bmatrix}
\]

i. Since \( F_2 \) is a factor (resp. semifactor) set and \( \begin{bmatrix} B' & C' \\ 0 & E \end{bmatrix} \) has linearly independent rows, then (see Lemma 4.6) its complementary \( \begin{bmatrix} A' & D' \\ 0 & F' \end{bmatrix} \) in \( M'' \) is FCmR. In particular \( A'' \) is FCmR.

ii. Since \( F_1 \) is a factor (resp. semifactor) set and \( \begin{bmatrix} A'' & 0 \\ A' & B' \end{bmatrix} \) has linearly independent rows, then (see Lemma 4.6) \( \begin{bmatrix} A'' & 0 \\ A' & B' \end{bmatrix} \) is FRmR. In particular \( A'' \) is FRmR.
iii. Since $A''$ is FCmR and FRmR at the same time then $A''$ is a square FmR.

iv. Since $F_1$ is a factor (resp. semifactor) set and $[A'' \begin{array}{cc} 0 \\ A' \end{array} B']$ has linear independent rows, then (see Lemma 4.6) its complementary $[E \begin{array}{cc} F' \\ 0 \end{array}]$ in $M''$ is FCmR. In particular $E$ is FCmR.

v. Since $F_2$ is a factor (resp. semifactor) set and $[B' \begin{array}{cc} 0 \\ 0 \end{array} E']$ has linear independent rows then (see Lemma 4.6) $[B' \begin{array}{cc} 0 \\ 0 \end{array} E']$ is FRmR. In particular $E$ is FRmR.

vi. Since $E$ is FCmR and FRmR at the same time then $E$ is a square FmR.

vii. Since $[A'' \begin{array}{cc} 0 \\ A' \end{array} B']$ is FRmR and $A''$ is square FmR then $B'$ is FRmR.

viii. In this step the arguments diverge significantly depending on $F_1$, $F_2$ being factor or semifactor sets, so we consider the cases separately:

a. $F_1$ and $F_2$ are factor sets.
Since $F_1$ is a factor set then the two zero blocks on the last row of $M''$ compose a Big zero block. This implies (see Proposition 2.2) that $[E \begin{array}{cc} F' \\ 0 \end{array}]$ is tall, which is impossible since $E$ is square (vi).

b. $F_1$ and $F_2$ are semifactor sets.
Since $F_1$ is a semifactor set then the two zero blocks on the last row of $M''$ compose a Medium zero block. This implies (see Proposition 2.2) that $[E \begin{array}{cc} F' \\ 0 \end{array}]$ is tall or square, and since $E$ is square (vi) this forces $F$ to be tall degenerate with size $t_1 \times 0$. So

\[
M'' = \begin{bmatrix}
A'' & 0 & 0 \\
A' & B' & C'' \\
0 & 0 & E
\end{bmatrix}
\]

\[
\begin{array}{c}
F_1 \\
F_2
\end{array}
\]

\[
r_1, r_2, t = t_1
\]

- In $M''$ the complementary of $B'$ is $B'' = [A'' \begin{array}{cc} 0 \\ 0 \end{array}]$. Since $A''$ and $E$ are square FmR (iii) and (vi) then Proposition 4.6 says that $B''$ is square FmR.
- Consider the zero block $Z$ obtained by joining together the zero blocks below and above $B'$. By Proposition 2.2 $Z$ is Medium since $B'$ is wide or square (vii) and $B'$ is square.
- As $A''$ is square (iii), $B'$ is wide or square (vii) and $E$ is square (vi) then $M''$ is wide or square. As $M''$ has semifactor sets then $M''$ is FmR (see Proposition 3.2). Moreover, $F_1 \cup F_2$ span all columns of $M''$. Recall the discussion after Definition 3.3 where it was explained that in a wide or square FRmR ACI-matrix the set $\{1, \ldots, n\}$ is a semifactor set. So $F_1 \cup F_2$ is a semifactor set.
- $F_1 \cap F_2$ is a semifactor set of $M''$ since $Z$ is a Medium zero block, $B'$ is FRmR (vii) and $B'$ is FCmR (it is square FmR).

• Suppose $r_1 = 0$ and $t_2 > 0$. Then

\[
M'' = \begin{bmatrix}
A & B & C & D \\
0 & 0 & E' & F'' \\
0 & 0 & 0 & F''
\end{bmatrix}
\]

\[
\begin{array}{c}
F_1 \\
F_2
\end{array}
\]

\[
r = r_2, t_1
\]

As $F_2$ is a factor (resp. semifactor) set of $M''$ and $[B' \begin{array}{cc} 0 \\ 0 \end{array} E']$ has linearly independent rows, then its complementary in $M''$ $[0 \begin{array}{cc} 0 \\ F'' \end{array}]$ is FCmR (see Lemma 4.6). Which is impossible because FCmR ACI-matrices can not have columns full of zeros.

• Suppose $r_1 = 0$ and $t_2 = 0$. Then $M'' = M'$ (see 2.1). As $[B' \begin{array}{cc} 0 \\ 0 \end{array} E']$ has linearly independent rows then $F_2$ is not a factor (resp. semifactor) set of $M''$. Contradiction.
As we explained before, Theorem 5.4 together with Theorem 5.5 add up to the following result.

**Theorem 5.6.** The intersection and the union of two factor (resp. semifactor) sets of an ACI-matrix are factor (resp. semifactor) sets.

### 6 The WST-decomposition for ACI-matrices

Note that the set of factor sets of a non FmR ACI-matrix is a partial order set where the order is given by set inclusion. Indeed, Theorem 5.6 tells us that this set is a lattice. Since it is a finite lattice then it is bounded. So there is a factor set that is the maximum or top factor set: the union of all factor sets which we will denote $F_T$. And there is another factor set that is the minimum or bottom factor set: the intersection of all factor sets which we will denote $F_{\bot}$.

The previous paragraph is also valid when we substitute factor sets of a non FmR ACI-matrix by semifactor sets of a FmR ACI-matrix.

**Theorem 6.1.** For any ACI-matrix $M$ there exists a nonsingular $R$ and a permutation matrix $Q$ such that $M$ can be decomposed as follows:

$$RMQ = \begin{bmatrix} W & * & * \\ 0 & S & * \\ 0 & 0 & T \end{bmatrix}$$

(22)

where $W$ is a wide FmR or void, $S$ is square FmR or void, and $T$ is a tall FmR or void.

Moreover, the ACI-matrices $W$, $S$ and $T$ in decomposition (22) are unique up to equivalence if we impose that $S$ is as large as possible for such a decomposition.

**Proof.** Recall that when $M$ is not FmR (resp. $M$ is FmR) then it has at least one factor (resp. semifactor) set. Then this factor (resp. semifactor) set provides a decomposition of type (22) where $S$ is void. In this way the existence is solved in a trivial way, but we want to be more demanding and give the decompositions where the ACI-submatrix $S$ is as large as possible because these decompositions will lead to uniqueness.

How do we find such decompositions? Suppose we are given a decomposition as in (22) where $W$ is a wide FmR or void, $S$ is square FmR or void, and $T$ is a tall FmR or void. Let $F_1$ be the set of columns corresponding to $W$, and $F_2$ be the set of columns corresponding to $W^*$. It is easy to check that the $2 \times 2$ block partition $\begin{bmatrix} W & * & * \\ 0 & S & * \end{bmatrix}$ is an $F_1$-decomposition, and $\begin{bmatrix} W & * & * \\ 0 & S & * \\ 0 & 0 & T \end{bmatrix}$ is an $F_2$-decomposition. Note that the order of $S$ corresponds to the difference $\#F_2 - \#F_1$. If we take $F_1 = F_{\bot}$ and $F_2 = F_T$ then we will see (Existence) that we obtain a decomposition of type (22). Moreover, as $F_T$ is the union of all factor (resp. semifactor) sets it is the largest and is unique, and as $F_{\bot}$ is the intersection of all factor (resp. semifactor) sets it is the smallest and is unique. Since we are taking the extreme sizes, then we will obtain the largest possible order for $S$.

**Existence.** Let $M$ be an $m \times n$ ACI-matrix. We will make a systematic analysis to be sure that nothing wrong happens even when some of the ACI-submatrices become degenerate or void:

**M is FmR.** We divide the proof into three cases:

- **M is tall.** As we saw after Definition 3.3 the empty set is a semifactor set, so $F_{\bot} = \emptyset$.
  
  Without loss of generality we can assume that $F_T = \{1, \ldots , k\}$ with $0 \leq k \leq n$. Three subcases are possible:
  
  (a) $\emptyset = F_{\bot} = F_T$. Take $W$ void, $S$ void, and $T = M$.


(b) $\emptyset = F_\perp \subsetneq F_\top \subsetneq \{1, \ldots, n\}$. We do a sweep from bottom to top in $M$ with respect to $F_\top$ and after reordering the rows we obtain

$$M \sim \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where $A$ is square with linearly independent rows and $C$ is tall (see Proposition 2.2). And from Lemma 4.6 $A$ is $\text{Fr}_m$ and $C$ is $\text{FcFr}_m$. Take $W$ is void, $S = A$ and $T = C$.

(c) $\emptyset = F_\perp \subsetneq F_\top = \{1, \ldots, n\}$. We do a sweep from bottom to top in $M$ with respect to $F_\top$ and after reordering the rows we obtain $M \sim \begin{bmatrix} A \\ 0 & C \end{bmatrix}$ where $A$ is square (see Proposition 2.2) and has linearly independent rows. And from Lemma 4.6 $A$ is $\text{FmR}$. Take $W$ void, $S = A$ and $T$ tall degenerate.

$M$ is wide. As we saw after Definition 3.3 the set $\{1, \ldots, n\}$ is a semifactor set, so $F_\top = \{1, \ldots, n\}$. Without loss of generality we can assume that $F_\perp = \{1, \ldots, k\}$ with $1 \leq k \leq n$. Note that $F_\perp = \emptyset$ is not possible since it will not generate a Medium zero block. Three subcases are possible:

(a) $\emptyset \neq F_\perp = F_\top = \{1, \ldots, n\}$. Take $W = M$, $S$ void and $T$ void.

(b) $\emptyset \neq F_\perp \subsetneq F_\top = \{1, \ldots, n\}$ and all the entries of the columns corresponding to $F_\perp$ are equal to zero. Then $M = \begin{bmatrix} 0 \\ C \end{bmatrix}$ where $C$ is square (see Proposition 2.2) and $\text{FmR}$. Take $W$ wide degenerate, $S = C$ and $T$ void.

(c) $\emptyset \neq F_\perp \subsetneq F_\top = \{1, \ldots, n\}$ and not all the entries of the columns corresponding to $F_\perp$ are equal to zero. We do a sweep from bottom to top in $M$ with respect to $F_\perp$ and after reordering the rows we obtain

$$M \sim \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

were $A$ is wide with linearly independent rows and $C$ is square (see Proposition 2.2). And from Lemma 4.6 $A$ is $\text{Fr}_m$ and $C$ is $\text{FmR}$. Take $W = A$, $S = C$ and $T$ void.

$M$ is square. Take $W$ void, $S = M$ and $T$ void.

$M$ is not $\text{FmR}$. Then $M$ has factor sets. Note that $F_\perp = \emptyset$ is not possible since it will not generate a Big zero block. Without loss of generality we can assume that

$$F_\perp = \{1, \ldots, h\} \quad \text{and} \quad F_\top = \{1, \ldots, k\}$$

with $0 < h \leq k \leq n$. We consider four cases:

(1) $0 < h < k < n$. We distinguish two possibilities:

(a) Not all the entries of the columns corresponding to $F_\perp$ are equal to zero. We do a sweep from bottom to top in $M$ with respect to $F_\top$ and after reordering the rows we obtain

$$M \sim M' = \begin{bmatrix} A & B & C \\ 0 & 0 & D \end{bmatrix}$$
where \([ A \ B ]\) has linearly independent rows. By Lemma 1.6 \([ A \ B ]\) is FRmR and \(D\) is FCmR. As \(M'\) is obtained from \(M\) without permuting columns, then \(F_t\) and \(F_T\) are factor sets of \(M'\). Now we do a sweep from bottom to top in \(M'\) with respect to \(F_{\perp}\) and after reordering the rows we obtain

\[
M' \sim M'' = \begin{bmatrix}
A'' & B'' & C'' \\
0 & B' & C' \\
0 & 0 & D
\end{bmatrix}
\]

(23)

where \(A''\) has linearly independent rows. By Lemma 1.6 \(A''\) is FRmR and \([ B' C' \ D']\) is FCmR. And thus \(B'\) is FCmR. On the other hand, as \([ A \ B ]\) is FRmR and \([ A \ B ]\) \(\sim [ A'' B'' ]\) then \([ A'' B'' ]\) is also FRmR. And thus \(B'\) is FRmR. Since \(B'\) is FRmR and FCmR then it must be square FmR. Take \(W = A'', S = B'\) and \(T = D\).

(b) All the entries of the columns corresponding to \(F_{\perp}\) are equal to zero. Then we proceed as in case (1)(a) although now it is not necessary to perform the second sweep. We finish with \(M' = [ B' C' \ D']\). Take \(W\) wide degenerate, \(S = B\) and \(T = D\).

(2) \(0 < h < k = n\). We distinguish two possibilities:

(a) Not all the entries of the columns corresponding to \(F_{\perp}\) are equal to zero. The argument is the same as in case (1)(a) although in this case the last column does not appear as \(F_T = \{1, \ldots, n\}\). So \(M' = [ A' B' \ D' ]\) and \(M'' = [ A'' B'' ]\). Take \(W = A''\), \(S = B'\) and \(T\) tall degenerate.

(b) All the entries of the columns corresponding to \(F_{\perp}\) are equal to zero. Then we proceed as in case (2)(a) although now it is not necessary to perform the second sweep. So we finish with \(M' = [ B' C' \ D']\). Take \(W\) wide degenerate, \(S = B\), and \(T\) tall degenerate.

(3) \(0 < h = k < n\). We distinguish two possibilities:

(a) Not all the entries of the columns corresponding to \(F_{\perp}\) are equal to zero. The argument is the same as (1)(a) although in this case the second column does not appear as \(F_{\perp} = F_T\). Then only one sweep is necessary. So \(M' = [ A' C' \ D' ]\). Take \(W = A\), \(S\) void and \(T = D\).

(b) All the entries of the columns corresponding to \(F_{\perp}\) are equal to zero. Then \(M = [ B' D' ]\). Take \(W\) wide degenerate, \(S\) void, and \(T = D\).

(4) \(0 < h = k = n\). We distinguish two possibilities:

(a) Not all the entries of the columns corresponding to \(F_{\perp}\) are equal to zero. The argument is the same as (3)(a) although in this case the last column does not appear as \(F_{\perp} = \{1, \ldots, n\}\). Then only one sweep is necessary. So \(M' = [ A' ]\). Take \(W = A\), \(S\) void and \(T\) tall degenerate.

(b) All the entries of the columns corresponding to \(F_{\perp}\) are equal to zero. Then \(M = [ 0 ]\). Take \(W\) wide degenerate, \(S\) void, and \(T\) tall degenerate.

**Uniqueness up to equivalence of \(W\), \(S\) and \(T\).** We will do the FmR case and the non FmR case together. Actually, we will do all subcases that were studied in the Existence part together, since at this point to adapt the general argument to the different subcases should be straightforward (for example, some of the submatrices \(P\), \(P'\) or \(P''\) involved in (24) can be void).
So assume that we have two different decompositions

\[
R_1MQ_1 = \begin{bmatrix}
F_T \\
W_1 & * & * \\
0 & S_1 & * \\
0 & 0 & T_1
\end{bmatrix}_{F_\perp} \text{ and } R_2MQ_2 = \begin{bmatrix}
F_T \\
W_2 & * & * \\
0 & S_2 & * \\
0 & 0 & T_2
\end{bmatrix}_{F_\perp}
\]

where \(R_1\) and \(R_2\) are nonsingular matrices, \(Q_1\) and \(Q_2\) are permutation matrices, \(W_1\) and \(W_2\) are wide \(FmR\) or void, \(S_1\) and \(S_2\) are square \(FmR\) or void, and \(T_1\) and \(T_2\) are tall \(FmR\) or void. Then

\[
\begin{bmatrix}
W_1 & * & * \\
0 & S_1 & * \\
0 & 0 & T_1
\end{bmatrix}_{F_\perp} = R_1R_2^{-1} \begin{bmatrix}
W_2 & * & * \\
0 & S_2 & * \\
0 & 0 & T_2
\end{bmatrix}_{F_\perp} Q_2^{-1}Q_1.
\]

Note that the three groups of columns \(F_\perp, F_T \setminus F_\perp\) and \(\{1, \ldots, n\} \setminus F_T\) do not change of position. But the columns of each group might get permuted so there are three permutation matrices \((P\) of order \(#F_\perp, P'\) of order \(#F_T - #F_\perp\), and \(P''\) of order \(n - #F_T\)) such that \(Q_2^{-1}Q_1 = \begin{bmatrix} P & 0 & 0 \\
0 & P'' & 0 \\
0 & 0 & P''\end{bmatrix}\).

So

\[
\begin{bmatrix}
W_1 & * & * \\
0 & S_1 & * \\
0 & 0 & T_1
\end{bmatrix}_{F_\perp} = R_1R_2^{-1} \begin{bmatrix}
W_2 & * & * \\
0 & S_2 & * \\
0 & 0 & T_2
\end{bmatrix}_{F_\perp} \begin{bmatrix} P & 0 & 0 \\
0 & P'' & 0 \\
0 & 0 & P''\end{bmatrix} \tag{24}
\]

where the lines define \(2 \times 2\) block ACI-matrices: we consider \(\begin{bmatrix} S_1 & * \\
0 & T_1 \end{bmatrix}\), \(\begin{bmatrix} S_2 & * \\
0 & T_2 \end{bmatrix}\) and \(\begin{bmatrix} P' & 0 \\
0 & P'' \end{bmatrix}\) as just one block. Since \(W_1\) and \(W_2\) are wide \(FRmR\) then they have linear independent rows and Lemma 4.3 implies that

\[
W_1 \sim W_2 \text{ and } \begin{bmatrix} S_1 & * \\
0 & T_1 \end{bmatrix} \sim \begin{bmatrix} S_2 & * \\
0 & T_2 \end{bmatrix}. \tag{25}
\]

In the proof of Lemma 4.3 we saw that \(R_1R_2^{-1} = \begin{bmatrix} R & * \\
0 & R' \end{bmatrix}\) where \(R\) is nonsingular of order rows(\(W_2\)) and \(R'\) is nonsingular of order rows(\(S_2\) + rows(\(T_2\)) with

\[
\begin{bmatrix} S_1 & * \\
0 & T_1 \end{bmatrix} = R' \begin{bmatrix} S_2 & * \\
0 & T_2 \end{bmatrix} \begin{bmatrix} P' & 0 \\
0 & P'' \end{bmatrix}.
\]

Since \(S_1\) and \(S_2\) are square \(FmR\) then they have linear independent rows and Lemma 4.3 implies that \(S_1 \sim S_2\) and \(T_1 \sim T_2\). Which finishes the uniqueness part.

Since the decomposition of Theorem 6.1 involve a Wide (or void) \(W\), a Square (or void) \(S\) and a Tall (or void) \(T\), we will denote this decomposition the \textbf{WST-decomposition} for ACI-matrices whenever \(S\) is as large as possible.

7 ACI-matrices of constantRank

As a application of the WST-decomposition we will refine the main theorem of [6, Theorem 5] by Huang and Zhan, which is a characterization of constantRank ACI-matrices. The version of the theorem that will be given below makes the degenerate cases much more explicit than the original one. The same version of the theorem was already used in our work on ACI-matrices [2] as “Theorem 2.4 (detailed version)”.

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Theorem 7.1. ([2, Theorem 5], see also [2, Theorem 2.4 (detailed version)]) Let $M$ be a $m \times n$ ACI-matrix of constantRank $\rho$ with $1 \leq \rho \leq \min\{m, n\}$ over a field $F$ with $|F| \geq \max\{m, n + 1\}$. Depending on $m$, $n$ and $\rho$ we have the following possibilities:

(i) $\rho = m < n$ if and only if $M \sim \begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{0 \times s}$. (26)

(ii) $\rho = m = n$ if and only if $M \sim \begin{bmatrix} 1 & * \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{0 \times s}$. (26)

(iii) $\rho = n < m$ if and only if $M \sim \begin{bmatrix} * & \cdots & * \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{0 \times s}$. (26)

(iv) $1 \leq \rho < \min\{m, n\}$ if and only if for some positive integers $r$ and $s$ with $r + s = m + n - \rho$

$$M \sim \begin{bmatrix} 1 \\ \vdots \\ 0 \\ \vdots \\ 1 & * & \cdots & * \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{0 \times s}$$

where the upper blocks do not appear if $r = m$ and the right blocks do not appear if $s = n$.

Finally, we present the refinement of Theorem 7.1 that we were talking about.

Theorem 7.2. Let $M$ be an $m \times n$ ACI-matrix $M$ over a field $F$ such that $|F| \geq \max\{m, n + 1\}$. Then $M$ is constantRank if and only if there exist a nonsingular $R$ and a permutation $Q$ such that

$$R'MQ = \begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

where instead of the ACI-submatrix $\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ there could be a wide degenerate or a void, instead of $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ there could be a void, and instead of $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ there could be a tall degenerate or a void. If we impose that $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ is as large as possible then the three blocks are unique up to equivalence.

Proof. The sufficiency is obvious. So we proceed with the necessity. Let $R'$ be a nonsingular constant matrix and $Q'$ a permutation matrix such that

$$R'MQ' = \begin{bmatrix} W & * & * \\ 0 & S & * \\ 0 & 0 & T \end{bmatrix}$$

is a WST-decomposition of $M$ where: $W$ is wide FRmR or void, $S$ is square FmR or void, $T$ is tall FCmR or void, and $S$ is as large as possible. Therefore

$$\max\text{Rank}(W) = \text{rows}(W), \quad \max\text{Rank}(S) = \text{rows}(S) \quad \text{and} \quad \max\text{Rank}(T) = \text{cols}(T).$$

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Moreover, as \( M \) is constantRank then \( R'^{\prime}M^{\prime}Q' \) is also constantRank and then
\[
\min \text{Rank} \begin{bmatrix} W & * & * \\ 0 & S & * \\ 0 & 0 & T \end{bmatrix} = \max \text{Rank} \begin{bmatrix} W & * & * \\ 0 & S & * \\ 0 & 0 & T \end{bmatrix} = \text{rows}(W) + \text{rows}(S) + \text{cols}(T). \tag{28}
\]

We will prove that \( M \) is equivalent to an ACI-matrix as in (27) in three steps. We will assume that \( W, S \) and \( T \) are not void nor degenerate, otherwise the proof simplifies.

(i) First we will prove that \( W, S \) and \( T \) are full constantRank, that is:
\[
\min \text{Rank}(W) = \max \text{Rank}(W), \tag{29}
\]
\[
\min \text{Rank}(S) = \max \text{Rank}(S) \tag{30}
\]
\[
\min \text{Rank}(T) = \max \text{Rank}(T). \tag{31}
\]

Suppose that \( \min \text{Rank}(T) < \max \text{Rank}(T) \). Let \( \begin{bmatrix} \hat{W} & * & * \\ 0 & \hat{S} & * \\ 0 & 0 & \hat{T} \end{bmatrix} \) be a completion of \( \begin{bmatrix} W & * & * \\ 0 & S & * \\ 0 & 0 & T \end{bmatrix} \) such that \( \text{rank}(\hat{T}) < \max \text{Rank}(T) = \text{cols}(T) \). Then
\[
\min \text{Rank} \begin{bmatrix} W & * & * \\ 0 & S & * \\ 0 & 0 & T \end{bmatrix} \leq \text{rank} \begin{bmatrix} \hat{W} & * & * \\ 0 & \hat{S} & * \\ 0 & 0 & \hat{T} \end{bmatrix} \leq \text{rows}(\hat{W}) + \text{rows}(\hat{S}) + \text{rank}(\hat{T}) < \text{rows}(W) + \text{rows}(S) + \text{cols}(T). \tag{32}
\]

As (28) and (32) are contradictory then (31) is true. Similar arguments prove (29) and (30).

(ii) Now, we apply Theorem 7.1 (i) to \( W \) to obtain a nonsingular \( R_1 \) and a permutation \( Q_1 \) such that
\[
R_1WQ_1 = \begin{bmatrix} 1 & \cdots & * \\ 0 & \cdots & 1 \end{bmatrix}.
\]
We apply Theorem 7.1 (ii) to \( S \) to obtain a nonsingular \( R_2 \) and a permutation \( Q_2 \) such that
\[
R_2SQ_2 = \begin{bmatrix} 1 & \cdots & * \\ 0 & \cdots & 1 \end{bmatrix}.
\]
And we apply Theorem 7.1 (iii) to \( T \) to obtain a nonsingular \( R_3 \) and a permutation \( Q_3 \) such that
\[
R_3TQ_3 = \begin{bmatrix} * & \cdots & * \\ 0 & \cdots & 1 \end{bmatrix}.
\]

(iii) Finally, if \( R := \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{bmatrix} \) \( R' \) and \( Q = Q' \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix} \) then we obtain the desired result since
\[
RMQ = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{bmatrix} R'M'Q' \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix} = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{bmatrix} \begin{bmatrix} W & * & * \\ 0 & S & * \\ 0 & 0 & T \end{bmatrix} \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix}
\]
\[
= \begin{bmatrix} R_1WQ_1 & * & * \\ 0 & R_2SQ_2 & * \\ 0 & 0 & R_3TQ_3 \end{bmatrix}
\]
is an ACI-matrix of type (27) where \( R_2SQ_2 \) is as large as possible.

The characterization of constantRank ACI-matrices of Theorem 7.1 has a caveat: there is a restriction on the number of elements of the field that can not be avoided. In [2, Theorem 2.5.] we extended the characterization without any restriction on the field. It is possible to apply the WST-decomposition to refine this extension analogously.
References

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