CURVES OF INFINITE GENUS I
RIEMANN–ROCH THEOREM FOR SMALL DEGREE

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Abstract. The most useful and interesting line bundles over algebraic curves of a very high genus have the ratio $\delta$ of the degree to the genus close to half-integer values, usually $\delta \approx 0$, $\delta \approx 1/2$, or $\delta \approx 1$; the numeric properties are very different in these three cases. This leads to three different theories for curves of infinite genus.

For analytic curves of infinite genus, to get a theory parallel to algebraic geometry one needs to restrict attention to holomorphic sections satisfying some “conditions on growth at infinity”. Each such condition effectively attaches an “ideal point” to the curve; this process is similar to compactification.

The theory of holomorphic functions on curves with such “ideal points” is developed (the variant presented in the first part of the series is tuned to the case $\delta \approx 0$). Conditions on the “lengths of handles” of the curve are found which ensure the geometry to be parallel to algebraic geometry.

It turns out that these conditions give no restriction on the density of ideal points on the curve. In particular, such curves may have a dense set of ideal points; these curves have no smooth points at all, and have a purely fractal nature. (Such “foam” curves live near the “periphery” of the corresponding $g = \infty$ moduli space; one needs to study these curves too, since they may be included in the support of natural measures arising on the moduli spaces.)

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0. Introduction
0.1. **Motivations.** The desire to find a set of curves of infinite genus for which “most” theorems of algebraic geometry hold stems from the following:

1. The existence of algebro-geometric description of a dense set of special solutions of infinite-dimensional integrable systems; thus a hope to describe other solutions by extending the algebraic geometry;
2. The ability to describe the series of perturbation theory for string amplitudes as integrals over moduli spaces of algebraic curves; thus a hope to describe non-perturbative terms as integrals over some ambient space which includes all the moduli spaces of algebraic curves;
3. The hope that an appropriate completion of the union of compactified moduli spaces may have a simpler geometry than the moduli spaces themselves (stability phenomenon).

In [23] we introduced several heuristics to describe a possible candidate for such a completion; outline the principal ingredients:

1. one works with a pair of a curve $C$ and a line bundle $\mathcal{L}$ on $C$;
2. a curve of infinite genus is a small “deformation” of an algebraic curve; small “deformations” of a pair should change the space of global sections of $\mathcal{L}$ (and $H^1(C, \mathcal{L})$) in a minimal possible way;
3. the description should be as conformally invariant as possible;
4. given a long tube $A$ and a 1-form $G$ such that $\text{Supp} G$ is “deep inside” $A$ and $\int G \neq 0$, the set $\{ F|_{\partial A} \mid \partial F = cG, c \in \mathbb{C} \}$ depends very weakly\(^1\) on $G$ (robustness);
5. enumerative algebraic geometry “requires” compactification of algebraic curves; one can substitute compactification by growth conditions: the local sections of the line bundle $\mathcal{O}(n \cdot P)$, $P = \infty$, on $\mathbb{P}^1$ coincide with local sections of $\mathcal{O}$ on $\mathbb{A}^1$ with the growth condition $O(|z|^{\beta})$, $n \leq \beta < n + 1$;
6. Riemann–Roch theorem is a litmus test to check whether a procedure to translate the growth conditions (such as $\beta$) to degree $n$ is “correct”;
7. duality theorem is a litmus test to check whether a growth condition is “reasonable” (note the specialty of integer $\beta$ w.r.t. the change $\beta \rightarrow -\beta$ in the translation law $n \leq \beta < n + 1$).

In our theory the compactification process adds a very large (possibly uncountable) set; thus to obtain a working theory it is crucial to start with “reasonable” growth conditions. In this paper we introduce a class of curves and growth conditions which satisfies the heuristics above. We expect that (when defined) the moduli space of such curves may lead to answers to the questions at the beginning of this sections.

0.2. **Finite-genus cases.** Recall which “small deformations” are involved in the “usual completion” of the moduli spaces. If both the initial and resulting curves are smooth, it is a two-steps procedure. First, gluing: given a curve $C$ with a pair of

\(^1\)E.g., consider Fourier coefficients of $F$ on two boundary circles for $A = \{ 1/N < |z| < N \}, N \gg 1$. 
0.3. The dust. Since \( g(C^\varepsilon) = g(C) + 1 \), one needs infinitely many operations of Section 0.2 to obtain a curve of infinite genus. Each such operation is equivalent to cutting two small disks \( R_i, R'_i \) out of \( C \) (with radii satisfying \( r_i r'_i = \varepsilon \)), then identifying \( \gamma_i = \partial R_i \) with \( \gamma'_i = \partial R'_i \) (the angle of rotation is determined by \( \text{Arg} \, \varepsilon \)) on \( \partial D \). Denote the result of these identifications by \( C^* \).

The description above has a gaping hole. Indeed, if \( I \) is infinite, \( \partial D \) is strictly larger than \( \bigcup_i (\gamma_i \cup \gamma'_i) \); call the remaining part of \( \partial D \) the dust of \( C^* \). There are two different possibilities: glue \( C^* \) out of \( D \), or out of the interior \( D \). In the first case \( C^* \) is very non-smooth at points of the dust, and we need to define which functions are “holomorphic” at these points. In the other case \( C^* \) is not compact; compactification of \( C^* \) requires addition of the dust. By heuristics of Section 0.1, to define line bundles on \( C^* \) one needs a growth condition near each point of the dust. Thus either way leads to growth conditions.

0.4. Growth conditions and representations of \( \text{SL}_2(\mathbb{C}) \). As the example of \( \mathbb{P}^1 = \mathbb{A}^1 \cup \{ \infty \} \) of Section 0.1 shows, the definition of “appropriate” growth conditions and degree is delicate even if dust consists of isolated points; the situation should be much harder for “massive” dust. The principal tool of our approach comes from the following observation: given a typical Banach norm on the space of functions, the functions of finite norm have a restricted growth rate near every point.

Suppose that the norm has good properties w.r.t. gluing a function from its restriction on open subsets. Then it makes sense to ask whether a function has a finite norm “where it is analytic”; most of the commonly used norms satisfy this condition. If so, then finiteness of the norm is a growth condition at each point of the set of non-analyticity of the function. This gives us some growth conditions at the dust.

A selection principle for “reasonable” norms now includes conformal invariance (or at least some strong bounds on how the norm can change under conformal transformations). It also helps if the norm is Hilbert. Recall the action of \( \text{SL}_2(\mathbb{C}) \) on \( \mathbb{P}^1 \) allows (see [4, 22]) a family of unitary representations of \( \text{SL}_2(\mathbb{C}) \) on sections of the
line bundles\(^2\) \(\mathcal{L}_\alpha = (\omega \otimes \omega')^{\alpha/2}, \alpha \in (0, 1)\); this gives a family of conformally-invariant Hilbert norms \(\|\|_\alpha\). We call a section \(F\) on \(D \subset C\) \(\alpha\)-acceptable if \(F = G|_D\), here \(\|G\|_\alpha < \infty\). Now we can glue \(C^*\) out of \(\tilde{D}\), as in Section 0.3, taking \(\alpha\)-acceptability (on \(\tilde{D}\)) as a growth condition on the dust; this defines a line bundle\(^3\) \(\tilde{\mathcal{L}}_\alpha\) on \(C^*\).

Similarly, one can define \(\tilde{\mathcal{L}}_\alpha\) \((n \cdot P)\) for \(P\) in the dust by requiring that \((z - P)^n F\) is acceptable instead of \(F\); such modified growth conditions define, in our conventions, a different line bundle. Alternatively, \(n\) is the “multiplicity of a divisor” at the dust points, or the contribution of infinity to the degree of a divisor.

The Hilbert norms on sections of \(\mathcal{L}_\alpha\) are equivalent to the Sobolev norms on spaces \(H^{1-\alpha}(\mathbb{P}^1, \mathcal{L}_\alpha)\); thus one can work with analogues of these norms on any compact curve \(C\). Being geometrically-defined, bundles \(\mathcal{L}_\alpha\) have an added convenience of auto-gluing: any identification of curves \(\gamma\) and \(\gamma'\) on \(\mathbb{P}^1\) leads to identification of \(\mathcal{L}_\alpha|_\gamma\) and \(\mathcal{L}_\alpha|_{\gamma'}\). This provides a very convenient “base point” on the Jacobian. As we will see it later, a modification of the gluing between \(\mathcal{L}_\alpha|_\gamma\) and \(\mathcal{L}_\alpha|_{\gamma'}\) for infinite number of indices can lead to a failure of the Riemann–Roch theorem; thus the validity of the Riemann–Roch for this particular point of the Jacobian is a property of the curve itself.

Now one can define a line bundle on \(C^*\) as a modification of the line bundle \(\tilde{\mathcal{L}}_\alpha\) by divisors on \(\tilde{D}\), or by modifications of the gluings of \(\mathcal{L}_\alpha|_\gamma\) and \(\mathcal{L}_\alpha|_{\gamma'}\), or by modifications of growth conditions at several points of the dust (thus “a divisor on the dust”).

Remark 0.1. In fact, in the body of this paper we do not use the language of divisors at all. As it is easy to see, one can replace a divisor on \(\tilde{D}\) by a change of the gluings of \(\mathcal{L}_\alpha|_\gamma\) and \(\mathcal{L}_\alpha|_{\gamma'}\). While divisors at infinity cannot be translated to a similar change of the gluing, the necessary generalizations of our results are trivial, as far as the divisor at infinity is has finite support.

0.5. Three theories. Unfortunately, for \(\alpha \neq 0\) there is no naturally defined notion of a complex-analytic section of \(\mathcal{L}_\alpha\). However, for \(\alpha \in \frac{1}{2}\mathbb{Z}\), one can define similar growth conditions on sections of the line bundle \(\omega^\alpha\), which is complex-analytic, and the action of \(\text{SL}(2, \mathbb{C})\) on this bundle is “very similar” to the action on \((\omega \otimes \omega')^{\alpha/2}\). Additionally, the Hilbert norms allow limits when \(\alpha\) goes to 0 or 1; while the limits are not positive-definite, they induce Hilbert norms on an appropriate subspace (or a quotient space) of codimension 1. This gives 3 satisfactory theories: \(\alpha \in \{0, 1/2, 1\}\).

In the case \(g(C^*) < \infty\), the defined above “reference bundles” \(\tilde{\mathcal{L}}_\alpha\) on \(C^*\) in these three cases are \(O\), \(\omega^{1/2}\), and \(\omega\); the degrees are 0, \(g - 1\), and 2 \((g - 1)\). By our definition of a line bundle on \(C^*\), there is a well-defined notion of “its relative degree” w.r.t. \(\tilde{\mathcal{L}}_\alpha\). Thus in the case \(g(C^*) = \infty\) we get three theories: one \((\alpha = 0)\) with a well-defined degree \(d\) of a line bundle; another \((\alpha = 1/2)\) with a well-defined difference \(d - g\); the third \((\alpha = 1)\) with a well-defined difference \(d - 2g\).

\(^2\)For a real oriented line bundle \(\mathcal{L}\), \(\mathcal{L}^\alpha\) is well-defined for \(\alpha \in \mathbb{C}\); same for \(\mathcal{L} \otimes_{\mathbb{R}} \mathbb{C}\).

\(^3\)I.e., a “usual” line bundle on the complement to dust, plus growth conditions on the dust.
Out of these three, $\alpha = 1/2$ gives the most interesting, self-dual theory. Moreover, in this case the geometrically defined gluing of $L_\alpha|_{\gamma}$ and $L_\alpha|_{\gamma'}$ automatically “adds a pole” for each glued pair of circles (thus each added handle), similar to pole at $Q$ in Section 0.2. However, the corresponding norm on $H^{1/2}(C, \omega^{1/2})$ is non-local (as all Sobolev norms with fractional index), and is conformally-invariant only “approximately”.

0.6. **Case $\alpha = 0$.** In this paper of the series, we consider the case $\alpha = 0$ only. This simplifies the discussion, since the principal objects are functions, not (fractional-degree) differential forms. Moreover, the norms we consider are manifestly conformally-invariant, and defined by local formulae.

The drawbacks are: first, the duality theorem needs to be postponed until we consider the case $\alpha = 1$. Second, to get a positive-definite invariant norm, we need to consider quotient-spaces of functions by constants (the constants are in the kernel of the “norm”). Third, the auto-gluing in the case $\alpha = 0$ differs a lot from the “small deformation” of Section 0.2. Essentially, it corresponds to consideration of $L^\varepsilon$ without the added pole at $Q$; to get a “small deformation” (thus a hope to get a finite dimension of “global sections” after an infinite number of such steps), one needs to add some one-dimensional “slack”, something similar to allowing a pole at $Q$ in Section 0.2.

By robustness, it is not very important which slack we allow instead of allowing a pole. Allowing a pole at $Q$ is equivalent to replacing $\overline{\partial} F = 0$ by $\overline{\partial} F = c \delta_Q$; here $c \in \mathbb{C}$, $\delta_Q$ is the $\delta$-function at $Q$. Since we need to consider functions up to a constant anyway, it makes sense to allow a jump by an additive constant when we glue $\partial R_i$ and $\partial R_i'$. Thus the last two inconveniences of the case $\alpha = 0$ partially compensate each other.

This makes starting our consideration with the case $\alpha = 0$ very convenient: we can introduce principal concepts without unnecessary complications. Moreover, this case is needed anyway for the general theory: it is dual to the consideration of the partial period mapping $\Gamma(M, \omega) \to \mathbb{C}^g$ of taking periods of global holomorphic forms along $A$-cycles on the Riemann surface. Until the remaining papers of the series appear, an interested reader can refer to [23], where all three cases $\alpha \in \{0, 1/2, 1\}$ are considered (though with much stricter assumptions than in this paper).

0.7. **The principal results.** In this paper of the series we define the principal notions: an infinite-genus curve and a line bundle on it (with $\alpha = 0$ only), and discuss only the simplest possible properties of these objects: the Riemann–Roch theorem. The principal result is Theorem 4.30, (and the amplifications in Sections 5.4, 5.5 and 5.6) which show that for validity of the Riemann–Roch theorem the only condition is that the removed disks $R_i$, $R'_i$ (notations of Section 0.3) are “small” enough. (The formalization of the latter notion is the notion of conformal distance, see Definition 4.20.)
The striking corollary of this fact is that there is no restriction on the position of the disks, only on their “sizes”. In particular, dust can take arbitrary large proportion of the whole curve $C^*$; not excluding the case when the $C^*$ consists of dust only (no smooth points on $C^*$ at all).

This is the principal difference of the approach of this paper to one in [23]: the much stricter conditions of [23] required an annulus of smooth points around each cycle $\gamma_i, \gamma'_i$. The other significant difference is that we allow gluing the curve $C^*$ out of a (possibly infinite) collection of curves $C[k]$ (with removed regions $R_{k,i}$). This allows, e.g., a uniform consideration of curves of Section 0.3 together with\(^4\) (more traditional) curves glued out of an infinite collection of pants (spheres with 3 disks removed); see also Section 5.7 for the example of yet another useful type of curves.

Let us sketch the principal ingredients of our presentation: curves we consider are glued of model domains (Section 1.2); analytic functions are replaced by Sobolev-holomorphic functions on model domains (Section 2.1); the gluing rules are introduced in Section 3.1. Sections 3.3 and 3.4 introduce the translation rules from the “usual” ($g < \infty$) Riemann–Roch theorem to the language of Sobolev-holomorphic functions and gluing data.

Sections 4.1, 4.2 and 4.3 reduce the (translated) Riemann–Roch theorem to a condition of almost-transversality for two appropriately defined subspaces in the space of functions on the smooth part of the boundary of the model domains. (This is one of the key conceptual ingredients: since this smooth part of the boundary is enumerated by a discrete collection of indices, this should be considered as a kind of discretization of the initial problem.) The remaining part of Section 4 introduces the translation of this almost-transversality condition to the estimates of the norms (or of the essential spectrum) of certain infinite matrices. This section concludes by the simplest possible effective form of the Riemann–Roch theorem.

Section 5 introduces generalizations of this simplest form which are needed to study models of curves appearing in integrable systems, as well as those needed for general divisor–line-bundle correspondence; in Section 5.3 we show that the curves which satisfy the Riemann–Roch theorem may be of purely fractal nature. We also discuss special properties enjoyed by the bundle $\mathcal{O}$.

Finally, in the appendix (Section 6) we prove (and discuss the motivations) for the particular form of Fredholm theorem used in our treatment of almost transversality.

Note that the most of the statements of this paper are technically straightforward; thus the motivations and heuristics may be as important as the particular formulations of statements. Let us list less straightforward technical statements forming the foundation of our methods: Theorem 2.3 (and Theorem 2.7) allow the “discretization” mentioned above; Lemma 2.9 allows application of the above statements to real curves inside complex curves (as well as the generalization of principal results to the

\(^4\)Note that while we expect our conditions of Riemann–Roch theorem to be close to optimal when we glue $C^*$ out of one curve $C$, they must be very non-optimal in the case of gluing of pants.
case of many-Jordan-curves boundary in Section 5.6); Lemma 4.2 translates bundle gluing data to settings of Section 6 (in the context similar to Segal–Wilson’s Universal Grassmannian [21, 16]). Applying this translation for infinitely many curves requires a stronger “discretization” condition: fatness (see Section 5.5); it is achieved in Theorem 2.26 in quasi-circular case, estimates of Section 5.5 show that in “most of the cases” fatness follows from the other assumptions of this paper. Theorem 5.2 introduces examples of foam curves.

0.8. Moduli spaces and the Universal Grassmannian. Let us also mention the natural problems which we do not discuss in this part of the series. First of all, the duality theorem requires consideration of two values \( \alpha, \alpha' \) with \( \alpha + \alpha' = 1 \); in this paper \( \alpha = 0 \), so we do not consider the duality here (see [23] instead—with more assumptions than we require in this paper).

Second, the principal unit we consider here is a model, i.e., a curve together with its representation via gluing of finite-genus pieces. We do not consider the question when two models define “the same” curve. In other words, here we treat the question how big is the collection of curves corresponding to points of the moduli space, not what is exactly a point of the moduli space. Recall, however, that [23] introduces the mapping of curves with a distinguished “quasi-smooth” point to the Sato’s Universal Grassmannian ([19, 16]), which leads to the notion of “sameness” for two models. The same mapping is still defined for the curves we consider in this paper, this this approach works for the foam curves too.

Another topic missing in this paper is the divisor–line-bundle correspondence. It is more or less trivial to define \( L(D) \) for a line bundle \( L \) and a finite divisor \( D \); similarly, one can do the same for divisors with infinite support, as far as points with multiplicity \(-1\) are close enough to points of multiplicity \(1\). (This is similar to how [23] uses the results of Section 5.8 to show that any “bounded” line bundle of degree \(0\) may be realized using constant gluing function \( \psi_j \) of Section 4.2.) However, the “interesting” theory would work with divisors \( D \) of infinite degree; in such cases \( \alpha \) for \( L(D) \) is different from \( \alpha \) for \( L \). Again, we cannot discuss this topic until we have theories suitable for different values of \( \alpha \).

0.9. Historic remarks. The roots of this paper go back to Yu. I. Manin’s seminars of the spring of 1981 (see [23] for details), as well as McKean and Trubowitz work

\[\text{There are two possible reasons why different gluing data may define the same curve: to construct a model, one needs to cut the curve along a collection of cycles, and choose identification of the pieces with subsets of compacts curves. It is relatively easy to describe the possible ambiguities given a fixed choice of the homotopy classes of cuts.}\]

\[\text{However, choosing different homotopy classes of cuts may lead to a “significantly different” model of a curve. However, recall the conjecture of [23]: if two different choices of cuts both lead to the gluing data satisfying the conditions of Theorem 4.30, then they differ only for a finite number of cuts. The heuristic for this conjecture is that the cuts which are small cycles across thin long handles. Given infinitely many handles which are thinner and thinner, there is essentially no choice: all the “other” cycles are going to be too long.}\]
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[11] on the hyper-elliptic case. During the last several years Feldman, Knörrer and Trubowitz made a major breakthrough using an unrelated approach (cf. [2]). The relation of these curves to what we discuss here is explained in Section 5.7.

Numerous alternative approaches to curves of infinite genus exist, both in rigorous settings, and in papers written using physical level of arguments. One can break them into two different categories: one is restricted to curves which allow a well-behaved finite-sheet covering over \( \mathbb{CP}^1 \) (ramified at infinitely many points, and with a “significant singularity” over \( \infty \in \mathbb{CP}^1 \)); these curves are similar to hyperelliptic curves. Another deals with curves similar to those in our settings, but with severe restriction on the dust (e.g., with the dust which consists of one point, or a finite number of points, only). Such curves appear in study of, e.g., double-periodic solutions of KP equation.

The first approach is developed in [20, 12, 15, 1], and [5]. The paper dealing with second approach are [10], which describes the period matrix (this question is dual to what we investigate in this paper), [14] and [13], which investigate the Riemann–Roch problem, the Jacobian, and the divisor-bundle correspondence; the book [6] contains comprehensive studies of the geometry of infinite-genus curves which appear in particular problems of mathematical physics. As in the case of curves in [2], these particular curves are very special cases of the curves we consider in this paper (as well as in [23]). The paper [9] describes the class of curves which cannot be extended (so are analogues of compact curves). In the paper [8] holomorphic forms on a curve with a Schottky model are studied as Taylor series of parameters of this model. (The relation of curves studied in [9] and [8] and the curves we consider in this paper is not yet clear.)

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1. Preliminaries

1.1. Notations. Consider a collection of topological vector spaces \( V_i, i \in I \). Then \( \prod_{i \in I} V_i \) denotes the space of all collections \( (v_i \in V_i)_{i \in I} \) with the projective limit topology; \( \bigoplus_{i \in I} V_i \) denotes the space of collections with a finite number of non-0 terms with the inductive limit topology. For a collection with \( V_i \subset V \), \( \sum V_i \) denotes the closure of the image of \( \bigoplus V_i \) in \( V \). If all \( V_i \) are Hilbert spaces, \( \bigoplus_{i \in I} V_i \) denotes the subspace of \( \prod_{i \in I} V_i \) consisting of collections \( v = (v_i \in V_i)_{i \in I} \) with \( \|v\|^2 \triangleq \sum_i \|v_i\|^2 < \infty. \)

We say that a topological vector space \( V \) has a Hilbert topology, if there is a Hilbert norm on \( V \) which induces the topology of \( V \). We say that two subspaces \( S_1, S_2 \subset V \) are comparable if \( S_1 \cap S_2 \) has a finite codimension in both \( S_1 \) and \( S_2 \); then \( \text{reldim} (S_1, S_2) \) is the difference of these codimensions. Two subspaces \( S_1, S_2 \subset V \) are
quasi-complementary if \( \dim S_1 \cap S_2 < \infty \) and \( \text{codim} S_1 + S_2 < \infty \). The excess of a quasi-complementary pair of subspaces is \( \dim (S_1 \cap S_2) - \text{codim} (S_1 + S_2) \).

Since we need to consider both Sobolev spaces and spaces of cohomology, we reserve the letter \( H \) for the former, and \( \mathbf{H} \) for the latter.

In what follows we define several flavors of distortions. List them here for reference purposes: \( \Delta (E) \) in Section 2.2, \( \Delta (\gamma) \) in Section 2.3 (used in Definition 4.12 of quasi-circularity), \( \Delta (E, E', \gamma, \gamma', \varphi, \psi) \) in Section 4.1, and \( \Delta (\varphi) \) in Section 4.5. To simplify the discussion, most of the time we assume that these numbers are uniformly bounded (otherwise we would need to incorporate them into the estimates). Note that in most important particular cases these numbers are all 1.

1.2. The model domains. The complex curves (or their generalizations) we consider here are going to be glued of several pieces \( D_k, k \in K \), each piece being a closed subset of a compact complex curve \( C_k \). The subsets \( D_\bullet \) we consider here have the boundary consisting of the smooth part,\(^6\) which may have infinitely many connected components, and of the accumulation points of these components. Enumerate the smooth components of the boundaries of all the pieces \( D_k \) as \( \gamma_j, j \in J \). Given one such component \( \gamma_j \), the gluing process associates to it another component \( \gamma_j' \), and a smooth orientation-inverting identification \( \varphi_j : \gamma_j \to \gamma_j' \).

Describe the effect of these modifications when both sets \( K \) and \( J \) are finite. Each gluing either decreases the number of connected components by 1 (if \( \gamma_j, \gamma_j' \) were on different components), or increase the genus of one of the components by 1. As a result, out of \( p = |K| \) connected pieces of genera \( g_k \), with \( d_k \) components of boundary each, one can glue one compact connected curve of genus \( g = \sum g_k - p + 1 + \sum d_k/2 \), \( |J| = \sum d_k \). Call the collection \( D_\bullet \) the model of the resulting curve.

Assume for a moment that all \( g_k = 0 \). Then there are two different ways to increase \( g \); either by increasing \( p \) (assuming \( d_k \geq 3 \)), or by increasing \( d_k \). On the other hand, gluing two pieces of genus 0 along a pair of components of boundary gives a piece of genus 0 too; this decreases \( p \) by 1, and increases \( \sum d_k \) by 2. Consequently, it is possible to substitute an increment of \( p \) by an increment of \( \sum d_k \); eventually, one can replace a model by one with \( p = 1 \) (and possibly large \( |J| \)). As the motivations in [23] show, this substitution can be made to work also when one has much more information ("growth conditions at infinity") attached to the pieces too.

In some sense, what we do in this paper is the formalization of the last remark, if "the growth conditions at infinity" are understood as "the growth compatible with \( H^1 \)-Sobolev smoothness".

Remark 1.1. One can make similar arguments that one can cut a piece with \( d_1 + d_2 - 2 \) components of boundary into two pieces with \( d_1 \) and \( d_2 \) components of boundary correspondingly. However, as the examples of Section 5.3 show, this argument works only if \( d_k \) is finite: some pieces with an infinite number of "holes" cannot be cut into

\(^6\)Section 5.6 introduces modifications allowing Jordan curves as components of the boundary.
an infinite number of pieces with a finite number of “holes” each. Thus consideration of pieces with infinitely many “holes” leads to new effects which cannot be described by gluing together simpler pieces.

Due to these new effects, and a possibility to replace many “simple” pieces by one “complicated” piece, the pieces with an infinite number of holes are especially important for us. Up to Section 2.1, we describe what are “holomorphic functions” on such pieces.7

1.3. Generalized Sobolev spaces. For our purposes we need to slightly extend some standard notions of the theory of Sobolev spaces (compare with [3, 17]). First, recall the notions which we use without any modification.

The Sobolev $s$-norm, $s \in \mathbb{R}$, on smooth rapidly decreasing functions $f(x)$ on $\mathbb{R}^n$ is defined by $\|f\|_s^2 = \int \hat{f}(\xi) (1 + |\xi|^s)^2 d\xi$ (here $\hat{f}$ is the Fourier transform of $f$). The completion w.r.t. this norm gives a Hilbert space $H^s(\mathbb{R}^n)$ called the Sobolev space. Any element of $H^s(\mathbb{R}^n)$ may be identified with a generalized function $f$ on $\mathbb{R}^n$; these generalized functions are those for which the Fourier transform $\hat{f}$ is locally-$L^2$, and the integral of the Sobolev $s$-norm converges. Since multiplication by a smooth function with a compact support is a continuous operator in $H^s(\mathbb{R}^n)$, it makes sense to consider the generalized functions which are locally Sobolev, i.e., become Sobolev after a multiplication by any smooth function with a compact support. While the finiteness of the Sobolev norm reflects both the degree of smoothness of the function, and its decay at infinity, the property of being locally Sobolev reflects the smoothness only.

Locally Sobolev functions form a vector space $H^s_{\text{loc}}(\mathbb{R}^n)$; it has a natural topology (of the appropriate inverse limit). Moreover, this topological vector space is invariant w.r.t. diffeomorphisms of $\mathbb{R}^n$. This makes it possible to define the topological vector space $H^s_{\text{loc}}(M)$ for any manifold $M$; a generalized function $f$ on $M$ belongs to $H^s_{\text{loc}}(M)$ if for any coordinate chart $\mathbb{R}^n \supset V \xrightarrow{\phi} U \subset M$ and any smooth function $\psi$ on $U$ the (generalized) function $\phi^*(\psi f)$ is locally Sobolev on $\mathbb{R}^n$. Similarly, one can define the topological vector space $H^s_{\text{loc}}(M, \mathcal{L})$ of locally Sobolev sections of an arbitrary finite-dimensional vector bundle $\mathcal{L}$ on $M$.

The vector space $H^s_{\text{loc}}(M)$ has a natural topology of the inverse limit. Moreover, if $M$ is compact, then, as it is easy to see, this topology is equivalent to the topology given by a Hilbert norm. In such a case we use notation $H^s(M)$ instead of $H^s_{\text{loc}}(M)$.

Multiplications by smooth functions and diffeomorphisms induce continuous operators in $H^s(M)$. If $\psi f = 0$ for an appropriate smooth function $\psi$ such that $\psi(m) \neq 0$, one says that $f$ vanishes near $m \in M$. The support $\text{Supp } f \subset M$ consists of points $m \in M$ such that $f$ does not vanish near $m$; it is a closed subset of $M$.

The only deviation from the “classical” terminology is in the following

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7Section 4.10 describes additional conditions on the pieces to imply the Riemann–Roch theorem.
8See Section 1.1 on how we avoid the conflict with cohomology.
**Definition 1.2.** Consider a subset $U$ of the manifold $M$. Let $\tilde{\mathcal{H}}^s(U)$ denote the closure of the vector subspace $\{ f \in H^s(M) \mid \text{Supp } f \subset U \}$.

Consider $V \subset M$. Let $H^s(V \subset M) = H^s(M)/\tilde{\mathcal{H}}^s(M \setminus V)$.

The deviation from the standard definition is that we do not require that $U$ is closed, and $V$ is open. Obviously, $\tilde{\mathcal{H}}^s(U) \subset \tilde{\mathcal{H}}^s(\bar{U})$, but this inclusion may be proper, as the example below shows. Call spaces $\tilde{\mathcal{H}}^s(U)$ and $H^s(V \subset M)$ the **generalized Sobolev spaces**.

**Example 1.3.** Consider a disjoint family of open subsets $V_i \subset M$, let $\mathcal{V} = \bigcup V_i \setminus \bigcup V_i$. Suppose that

1. The natural mapping $\bigoplus \tilde{\mathcal{H}}^s(V_i) \xrightarrow{\iota} H^s(M)$ is a (continuous) monomorphism.
2. There is a function $g \in \tilde{\mathcal{H}}^s(M)$ with $\text{Supp } g \subset V$.

The first condition insures that the space $\tilde{\mathcal{H}}^s(\bigcup V_i)$ is the image of the mapping $\iota$. Hence any non-zero function $f \in \tilde{\mathcal{H}}^s(\bigcup V_i)$ satisfies the condition $\text{Supp } f \cap \bigcup V_i \neq \emptyset$.

Thus the function $g \in H^s(M)$ with $\text{Supp } g \subset \mathcal{V}$ satisfies $f \in \tilde{\mathcal{H}}^s(\bigcup V_i)$, but $f \notin \tilde{\mathcal{H}}^s(\bigcup V_i)$. One of the standard facts of the theory of Hausdorff dimension is that the second condition is satisfied if $\dim_{\text{Hausdorff}} V > \dim M + 2s$. Moreover, it works if $s = 0$ and $\mathcal{V}$ has a positive measure.

In Section 5.3 we show how to construct a family of disks $V_i$ which satisfy the first condition. The centers of these disks may be an arbitrary locally discrete set $\Delta$. Moreover, one can find $\Delta$ such that the corresponding set $\mathcal{V}$ does not depend on radii; any set with an empty interior can be obtained as such $\mathcal{V}$. Thus the above construction works for any $s \leq 0$ (in this part of the series we are most interested in the case $s = 0$).

On the other hand, if $U$ has smooth boundary, then $\tilde{\mathcal{H}}^s(U) = \tilde{\mathcal{H}}^s(\bar{U})$.

### 1.4. Norms on Sobolev spaces.

As explained above, the Sobolev space of sections of a line bundle is defined up to topological equivalence only, it has no canonical Hilbert norm. However, in application we will need to consider Hilbert direct sums, which require a specification of the **Hilbert norm** (as opposed to **Hilbert topology**, see Section 1.1). As explained in Section 0.4, the spaces $\tilde{H}^{1-\alpha}(C, \omega^a)$ have a canonically defined norm (or an appropriate approximation); gluing $\omega^a|_{D_k}$ for pieces $D_k \subset C_k$ allows a definition of a norm on the space of global sections via the Hilbert direct sum in $k$. In the case we consider here, $\alpha \in \{0, 1\}$, and the norm is canonically defined.

Given a compact oriented surface $C$ with a conformal structure, denote by $\omega_\perp$ the $90^\circ$-counterclockwise rotation of a section $\omega$ of $\Omega^1(C)$. Then $\|\omega\|^2 \overset{\text{def}}{=} \int_C \omega \omega_\perp$ gives a canonically defined Sobolev norm on $H^0(C, \Omega^1)$. Restricting to the components $\omega$, there are simpler examples, but this one gives a domain we are going to deal with; see Section 5.3.

10I.e., the image is closed, and the mapping is an isomorphism onto the image.
\( \bar{\omega} \) of \( \Omega^1 \otimes \mathbb{C} = \omega \oplus \bar{\omega} \), one obtains the norms \( \| \alpha \|^2 = -\frac{i}{2} \int_C \bar{\alpha} \alpha \) on \( H^0 (C, \bar{\omega}) \) and \( \| \alpha \|^2 = \frac{1}{2} \int_C \bar{\alpha} \alpha \) on \( H^0 (C, \omega) \).

While \( \bar{H}^1 (C, \mathcal{O}) \) does not carry a natural norm, for a compact \( C \) the mapping \( \bar{\partial} : H^1 (C, \mathcal{O}) \to H^0 (C, \bar{\omega}) \) identifies \( H^1 (C, \mathcal{O}) / \text{const} \) with a closed subspace of \( H^0 (C, \bar{\omega}) \). This provides a canonically defined norm on \( H^1 (C, \mathcal{O}) / \text{const} \).

**Remark 1.4.** One could define a similar norm using the operator \( \partial \) instead; however, the result is going to be the same due to

\[
\int \bar{\partial} \bar{f} \partial f = \int \bar{\partial} \bar{f} \partial f = - \int \bar{f} \partial \bar{f} = - \int \partial \bar{f} \bar{f} = - \int \bar{\partial} \bar{f} \bar{f};
\]

this identity comes very handy in Section 4.7. In particular, the norm of \( f \in H^1 (C, \mathcal{O}) / \text{const} \) is \( \| df \| / \sqrt{2} \).

The norm on \( H^1 (C, \mathcal{O}) / \text{const} \) induces canonically defined norms on \( H^1 (D \subset C, \mathcal{O}) / \text{const} \) and on \( H^0 (D \subset C, \bar{\omega}) \). Note that the Hilbert norm on \( H^0 (D \subset C, \bar{\omega}) \) coincides with \( \| \alpha \|^2 = -\frac{i}{2} \int_D \bar{\alpha} \alpha \).

Consider a subset \( D \subset C \) such that \( 1 \notin \bar{H}^s (D) \subset H^s (C) \), here \( 1 \) is considered as an element of \( H^s (C) \). Since \( \bar{H}^s (D) \) is closed in \( H^s (C) \), the mapping \( \bar{H}^s (D) \hookrightarrow H^s (C) / \text{const} \) is a monomorphism, thus a norm on \( H^s (C) / \text{const} \) induces a norm on \( \bar{H}^s (D) \).

**Lemma 1.5.** Consider a connected compact complex curve \( C \). Then \( 1 \notin \bar{H}^s (D) \) if \( C \setminus D \) has a non-empty interior; or if \( C \setminus D \) contains a smooth curve and \( s > 1/2 \); or if \( C \setminus D \) contains a connected component which is not a point, and \( s = 1 \); or if \( C \setminus D \) is non-empty and \( s > 1 \).

**Proof.** All the statements except the last but one follow from the continuity properties of the restriction to submanifolds. The remaining statement is equivalent to the following statement: Let \( \gamma \) be a connected subset of a complex curve \( C \), and \( \gamma \) is not a point. Consider a sequence \( (\psi_k) \) of smooth functions on \( C \), and a sequence \( U_k \) of neighborhoods of \( \gamma \) such that \( \psi_k \mid_{U_k} = 0 \). It is enough get a contradiction with \( \psi_k \to 1 \) in \( H^1 (C) \).

Since multiplication by a smooth function is continuous in \( H^1 (C) \), it is enough to find \( \varepsilon \) such that \( \| \Psi (1 - \psi_k) \|_{H^1 (C)} > \varepsilon \) for any \( k \); here \( \Psi \) is an appropriate cut-off function. This makes the question local on \( C \), so we may assume \( C = \mathbb{CP}^1 \); moreover, it is enough to show that \( \int_{C \setminus \gamma} \| d (\Psi (1 - \psi_k)) \|^2 > \varepsilon \); this is formulated completely in terms of holomorphic geometry of \( C \setminus \gamma \). Thus we may assume that \( C \setminus \gamma \) is a unit disk. Now one can apply the first part of the lemma.

This provides a canonically defined norm on \( \bar{H}^1 (D) \) if \( C \setminus D \) contains a Jordan curve. In what follows we use the canonically defined norms above unless specified otherwise.
2. SOBOLEV HOLOMORPHIC FUNCTIONS

2.1. Sobolev-holomorphic functions and decomposition for \( g = 0 \). Differential operators act on Sobolev spaces decreasing \( s \) by the degree of the operator, and do not increase the support. Thus given an element \( f \) of \( H^1(D \subset C, \mathcal{O}) \), \( \bar{\partial}f \) is a correctly defined element of \( H^0(D \subset C, \bar{\omega}) \).

**Definition 2.1.** Given a closed subset \( D \) of a compact complex curve \( C \), an \( H^1 \)-holomorphic function on \( D \) is an element \( f \) of \( H^1(D \subset C, \mathcal{O}) \) which satisfies the condition \( \bar{\partial}f = 0 \in H^0(D \subset C, \bar{\omega}) \). Denote the the space of \( H^1 \)-holomorphic functions on \( D \subset C \) by \( \mathcal{H}^1(D \subset C) \) (or just \( \mathcal{H}^1 \)).

Note that the Sobolev spaces in this definition are generalized ones. The norm on \( H^1(D \subset C) / \text{const} \) induces a canonically defined norm on \( \mathcal{H}^1(D \subset C) / \text{const} \).

**Remark 2.2.** The heuristic on the use of “generalized Sobolev” space is that \( f \in H^1(D \subset C, \mathcal{O}) \) is an equivalence class modulo functions with support “inside” \( C \setminus D \); in other words, we “keep” the information about \( f|_{\partial D} \). The equation \( \bar{\partial}f = 0 \) is again satisfied only up to functions with support “inside” \( C \setminus D \); in other words, the equations should be satisfied “also” on \( \partial D \).

Consider \( D \) and \( C \) as in the definition above. From now on assume that \( D \neq C \). Since \( C \setminus D \) is open, it may be represented as a disjoint union of open connected sets \( R_j \subset C \), \( j \in J \); here \( J \) is an appropriate set of indices. For each \( j \in J \) let \( D_j = C \setminus R_j \), \( D \subset D_j \subset C \). Then \( D = \bigcap_j D_j \). Consider the restriction mapping \( \mathcal{H}^1(D_j \subset C) \rightarrow \mathcal{H}^1(D \subset C) \), and the induced mapping of quotient spaces \( \mathcal{H}^1(D_j \subset C) / \text{const} \rightarrow \mathcal{H}^1(D \subset C) / \text{const} \). Taken together for every \( j \in J \), these mappings define a mapping \( \bar{\rho} : \bigoplus \mathcal{H}^1(D_j \subset C) / \text{const} \rightarrow \mathcal{H}^1(D \subset C) / \text{const} \).

**Theorem 2.3.** If \( C \) is of genus 0, the described above mapping \( \bar{\rho} \) extends continuously to a Fredholm mapping \( \rho : \bigoplus \mathcal{H}^1(D_j \subset C) / \text{const} \rightarrow \mathcal{H}^1(D \subset C) / \text{const} \). In fact \( \rho \) is a natural unitary mapping of Hilbert spaces.

**Proof.** Consider a complex curve \( C \) of arbitrary genus. Consider the mapping \( \bar{\partial} : H^1(C, \mathcal{O}) \rightarrow H^0(C, \bar{\omega}) \). Since \( \bar{\partial} \) is an elliptic operator of degree 1, it is Fredholm, and any function in Ker \( \bar{\partial} \) is smooth, similarly for Ker \( \bar{\partial}^* \). As a corollary, Ker \( \bar{\partial} \) is spanned by 1, and Ker \( \bar{\partial}^* \) consists of global holomorphic 1-forms. From now on assume \( C = \mathbb{C} \mathbb{P}^1 \). In particular, Coker \( \bar{\partial} = 0 \).

Due to the conventions on norms from Section 1.4, \( \bar{\partial} \) induces a unitary isomorphism \( H^1(C, \mathcal{O}) / \text{const} \rightarrow H^0(C, \bar{\omega}) \). Given \( f \in \mathcal{H}^1(D \subset C) \), consider two different liftings \( \tilde{f}_1, \tilde{f}_2 \) of \( f \) to elements of \( H^1(C, \mathcal{O}) \). By definition, \( \tilde{f}_1 - \tilde{f}_2 \in \bar{\partial}H^1(C \setminus D, \mathcal{O}) \). Moreover, \( \bar{\partial}\tilde{f}_{1,2} \in \bar{\partial}H^0(C \setminus D, \bar{\omega}) \). This implies that \( \bar{\partial}\tilde{f}_1 \) is a canonically defined element of \( \mathcal{R}_{D \subset C} \).
Since $\bar{\partial}$ is surjective, the mapping $\mathcal{H}^1(D \subset C) / \text{const} \to \mathcal{R}_{DC}: f + \text{const} \mapsto \bar{\partial} \tilde{f}_1$ is a unitary mapping of Hilbert spaces.

**Lemma 2.4.** Consider disjoint open subsets $R_j$, $j \in J$, of a compact complex curve $C$. Let $R = \bigsqcup_j R_j$. Then $\tilde{H}^0(R, \omega) \simeq \bigoplus_j \tilde{H}^0(R_j, \omega)$, $\tilde{H}^1(R, \mathcal{O}) \simeq \bigoplus_j \tilde{H}^1(R_j, \mathcal{O})$, the former natural isomorphism is unitary, the latter is an isomorphism of topological vector spaces.

**Proof.** By definition, an element $f \in \tilde{H}^s(R)$ may be approximated by an element $f'$ of $H^s(C)$ with $\text{Supp } f' \subset R$. Since $\text{Supp } f'$ is closed, $\text{Supp } f'$ is compact, thus $\text{Supp } f'$ is contained in a finite union of several domains $R_s$. Thus $\tilde{H}^s(R) \subset \bigcup_j \tilde{H}^s(R_j)$, which implies $\tilde{H}^s(R) = \bigcup_j \tilde{H}^s(R_j)$. For $s = 0$ the subspaces $\tilde{H}^0(R_j) \subset \tilde{H}^0(C)$ are obviously orthogonal, which proves one statement of the lemma.

Similarly, $\bar{\partial}$-images of $\tilde{H}^1(R_j, \mathcal{O})$ in $\tilde{H}^0(C, \omega)$ are orthogonal; thus images of $\tilde{H}^1(R_j, \mathcal{O})$ in $H^1(C, \mathcal{O}) / \text{const}$ are orthogonal. One may assume that $C$ is connected, and $|J| > 1$, thus $\tilde{H}^1(R, \mathcal{O})$ projects monomorphically to $H^1(C, \mathcal{O}) / \text{const}$. Since $\tilde{H}^1(R_j, \mathcal{O})$ lie in $\tilde{H}^1(R, \mathcal{O})$, this proves the remaining statement of the lemma. \qed

As a corollary, we can see that $\mathcal{R}_{DC}$ can be naturally identified with $\bigoplus_j \mathcal{R}_{D_j C}$, and that this identification is unitary. Applying the same arguments to $D_j$ instead of $D$, one can see that $\mathcal{H}^1(D \subset C) / \text{const}$ is isomorphic to $\bigoplus \mathcal{H}^1(D_j \subset C) / \text{const}$. Obviously, the restriction mapping $\mathcal{H}^1(D_j \subset C) / \text{const} \to \mathcal{H}^1(D \subset C) / \text{const}$ is compatible with this identification. This finishes the proof of the theorem. \qed

### 2.2. Decomposition for an arbitrary genus

To generalize the theorem above to the case of $C$ of arbitrary genus $g(C)$, we need to compensate for $\bar{\partial}: H^1(C, \mathcal{O}) \to H^0(C, \omega)$ being not surjective. Results of this section allow to make the statements of this paper slightly more general, the price being slightly more cumbersome formulations. The results of this paper can be weakened by assuming that all the pieces we glue our curve from are of genus 0; since these weaker results are still interesting, one can skip this section on the first reading, assuming that during the gluing of Section 3.2 all the finite-genus pieces are of genus 0, and ignoring all lower indices $E$.

The arguments of Section 2.1 presumed solvability of $\bar{\partial} f = \alpha$. Since $\text{Im } \bar{\partial} \subset H^0(C, \omega)$ has codimension $g$, for $g > 0$ some compensation is needed. Consider an arbitrary projection $p: H^0(C, \omega) \to \text{Im } \bar{\partial}$; then for any $\alpha \in H^0(C, \omega)$ the expression $f = \bar{\partial}^{-1}(p\alpha)$ is a correctly defined element of $H^1(C, \mathcal{O}) / \text{const}$. If $E = \text{Ker } p$, then $f$ is a solution of $\bar{\partial} f \equiv \alpha \mod E$.

This leads to the following definition:

**Definition 2.5.** A subspace $E \subset H^0(C, \omega)$ is an *excess space* if the natural pairing between $E$ and the space $\Gamma_{an}(C, \omega)$ of global holomorphic 1-forms is non-degenerate. Given an excess space $E$, let $\mathcal{H}^1_E(D \subset C)$ consists of $f \in H^1(D \subset C)$ with $\bar{\partial} f \in \pi E$, here $\pi$ is the projection from $H^0(C, \omega)$ to $H^0(D \subset C, \omega)$.
For a subspace \( V \subset H^0(C,\bar{\omega}) \) and \( D \subset C \), let \( V_D = \{ f \in V \mid \text{Supp} \, f \subset D \} \). An excess space \( E \) is \( D \)-supported if \( E = E_D \). Given a collection \( R_i, i \in I \), of disjoint subsets of \( C \), and \( R = \bigcup R_i \), an excess space \( E \) is \( R_\bullet \)-split if \( E_R = \sum E_{R_i} \).

Given an excess space \( E \), the distortion \( \Delta(E) \) is the norm of the projector to \( \text{Im} \, \bar{\partial} \) along \( E \).

Obviously, \( \dim E = g(C) \). Since the particular choice of \( E \) is not important for the following arguments, we use notation \( \mathcal{H}_E^1 \) without mentioning \( E \) otherwise. If \( C = \mathbb{CP}^1 \), then \( \mathcal{H}_E^1 = \mathcal{H}^1 \). Note that the spaces \( \mathcal{H}_E^1 / \text{const} \) are equipped with natural Hilbert norms induced from \( H^1 / \text{const} \).

**Remark 2.6.** Obviously, excess subspaces exist. It is easy to find a \( D_\bullet \)-split one for any collection \( \{ D_i \} \) of disjoint open subsets. In fact, given any closed subset \( D \subset C \) of non-zero measure, one can find a \( D \)-supported excess space \( E \).

**Theorem 2.7.** In notations of Section 2.1, suppose that \( E \) is \( R_\bullet \)-split. Then the mapping \( \rho : \bigoplus \mathcal{H}_E^1(D_j \subset C) / \text{const} \rightarrow \mathcal{H}_E^1(D \subset C) / \text{const} \) can be extended to an invertible mapping \( \rho : \bigoplus I_2 \mathcal{H}_E^1(D_j \subset C) / \text{const} \rightarrow \mathcal{H}_E^1(D \subset C) / \text{const} \). Moreover, \( \rho \) is unitary if \( g(C) = 0 \).

**Proof.** For a subspace \( V \) of \( H^0(C,\bar{\omega}) \) put \( V^\theta \overset{\text{def}}{=} V \cap \text{Im} \, \bar{\partial} \); use the same notation for subspaces of quotients of \( H^0(C,\bar{\omega}) \). Obviously, \( \bar{\partial} \) sends \( \mathcal{H}^1(D \subset C) / \text{const} \) to \( \mathcal{R}^\theta_{DCC} \). Similarly, \( \mathcal{H}_E^1(D \subset C) / \text{const} \) is identified with \( (\mathcal{R}_{DCC} + E)^\theta \), here \( E \subset H^0(C,\bar{\omega}) / \partial \bar{\partial} H^1(C \setminus D,\mathcal{O}) \) is the projection of \( E \subset H^0(C,\bar{\omega}) \). It is clear that \( \dim E = \dim E^\theta \).

Since \( V^\theta \subset V \subset H^0(C,\bar{\omega}) \) is defined by equations \( \langle \alpha, v \rangle = 0, \alpha \in \Gamma_{an}(C,\omega) \), the projection of \( (V + E)^\theta \) to \( V \) along \( E \) is an isomorphism if \( E \cap V = 0 \). Thus \( (V + E)^\theta \) is naturally identified with \( V/E^V \) via taking the quotient by \( E \); here \( E^V = V \cap E \). This identification preserves the topology.

If \( W \subset H^1(C,\mathcal{O}) \), the same argument works for \( V \subset H^0(C,\bar{\omega}) / \partial \bar{\partial} W \) (substituting \( E \) for \( E \)), since the pairing with \( \Gamma_{an}(C,\omega) \) vanishes on \( \partial \bar{\partial} W \). Fix \( R \subset C \); put \( V_R = H^0(R,\bar{\omega}) / \partial \bar{\partial} H^1(C \setminus D,\mathcal{O}) \). Then \( E^{V_R} \overset{\text{def}}{=} E_R \). Thus \( \mathcal{R}_{DCC}^\theta \) is identified with \( \mathcal{R}_{DCC} / \mathcal{E}_{C\setminus D}^\theta \).

The splitness property can be restated as \( E_{C\setminus D} = \bigoplus E_{R_i} \). Now \( \mathcal{R}_{DCC} = \bigoplus I_2 \mathcal{R}_{D_{iCC}} \) implies \( \mathcal{R}_{DCC} / E_{C\setminus D} = \bigoplus I_2 \mathcal{R}_{D_{iCC}} / E_{R_i} \), which finishes the proof. \( \square \)

**Remark 2.8.** The mapping \( \rho \) is not necessarily unitary if \( g(C) > 0 \). The non-unitary component of the identifications of the theorem is the projection along \( \bar{E} \) from \( (V + E)^\theta \) to \( V/E^V \). Consequently, it is enough to estimate the “distortion” of this projection; or the angles between \( E \) and \( (\mathcal{R}_{DCC} + E)^\theta \), and between \( E/E_{C\setminus D} \) and \( \mathcal{R}_{DCC} / E_{C\setminus D} \).

If \( E \) is split w.r.t. the collection \( \{(R_i), D\} \), then the latter subspaces are orthogonal. Thus the degree of non-unitarity of \( \rho \) is majorated by a function of the distortion \( \Delta(E) \).
2.3. **Riemann problem decomposition.** Recall that for a submanifold $N \subset M$ of codimension $d$ the restriction mapping $H^s(M) \to H^{s-\frac{d}{2}}(M)$ is continuous as long as $s > \frac{d}{2}.$

**Lemma 2.9.** Consider a compact smooth real curve $\gamma$ which is a submanifold in a real surface $C$. Then the restriction mapping $\tilde{\rho}: H^1(C) \to H^{1/2}(\gamma)$ induces an isomorphism $H^1(\gamma \subset C) \simeq H^{1/2}(\gamma)$ of topological vector spaces.

*Proof.* Since $\tilde{\rho} \rho = 0$ if $\text{Supp } f \cap \gamma = 0$; by continuity, $\tilde{\rho}$ vanishes on $\hat{H}^1(C \setminus \gamma)$, thus induces a continuous mapping $\rho: H^1(\gamma \subset C) \to H^{1/2}(\gamma)$. Obviously, the surjectivity and injectivity of $\rho$ are local properties; by the invariance of the Sobolev topology w.r.t. diffeomorphisms, it is enough to consider one particular curve $\gamma$.

Assume $\gamma = \partial D$, $D$ being the unit disk in $S^2 = \mathbb{CP}^1$. Since $\rho$ commutes with the action of the group $\mathbb{T}$ of rotations, it is enough to show that $\rho$ induces uniformly bounded (from above and from below) isomorphisms between the isotypical components of $\mathbb{T}$. For $k \geq 0$ let $f_k(z) = z^k$ if $|z| \leq 1$, $f_k(z) = z^{-k}$ if $|z| \geq 1$. For $k < 0$ define $f_k(z) = \bar{f}_{-k}(z)$. A simple calculation shows that the $H^1$-norms of $f_k$ grow as $|k|^{1/2}$, $k \neq 0$; similarly for $\|f_k\|_{H^{1/2}}$. Thus $\rho|f_k)$ is an isomorphism of topological vector spaces (here $(f_k)$ is the vector subspace spanned by $f_k$), thus $\rho$ is surjective.

To show injectivity, it is enough to consider isotypical components one-by-one. A function from such a component can be written as $z^\alpha \psi(z)$ near $\partial D$, here $\psi$ is rotation-invariant, thus $\tilde{\psi}(z) = \psi(|z|)$. Thus reduces the problem to the following 1-dimensional problem: show that a smooth function $\psi$ on $[0.5, 2]$ such that $\psi(1) = 0$ can be $H^1$-approximated by a function which vanishes near 1. In turn, this is obvious. \[\square\]

**Definition 2.10.** Given $f \in H^1(D \subset C, \mathcal{O}) / \text{const}$, define $\|f\|_{1, \text{int}}$ as $\|df\|_{L_2(D, \Omega^1)}$. Here $d$ is de Rham differential.

It is clear that $\|f\|_{1, \text{int}} \leq \sqrt{2}\|f\|_{C^1(D \subset C, \mathcal{O})}$ and the coefficient can be reduced to 1 if $f \in H^1(D \subset C, \mathcal{O}) / \text{const}.$

**Lemma 2.11.** Consider a closed subset $D \subset C$ of a compact real surface $C$ with a smooth boundary. Then the norm $\|f\|_{1, \text{int}}$ induces a Hilbert space structure on $H^1(D \subset C, \mathcal{O}) / \text{const}$ compatible with the natural Hilbert topology on this space.

*Proof.* Obviously, the norm $\|f\|_{1, \text{int}} \leq \|f\|_{H^1(D \subset C) / \text{const}}$; thus it is enough to show that one can majorate $\|f\|_{H^1(D \subset C) / \text{const}}$ given $\|f\|_{1, \text{int}}$. In other words, given a function $f$ defined in $D$ with $\int_D |df|^2 d\mu \leq 1$, it is enough to construct a continuation $g$ of this function to $C$ so that $\int_C |dg|^2 d\mu \leq M$ (for an appropriate $M$ which does not depend on $f$).

It is clear that the existence of such a continuation depends on the local properties of $f$ near $\partial D$, thus one may assume that $C$ is $\mathbb{CP}^1$, $D$ is the unit disk. Now the proof can proceed as in the previous lemma. \[\square\]
Lemma 2.12. Consider a closed measure-0 subset $\gamma$ of a complex curve $C$. Then $\mathcal{H}^1_E(\gamma \subset C) = \mathcal{H}^1(\gamma \subset C) = H^1(\gamma \subset C)$.

Proof. By definition of $\mathcal{H}^1$, it is enough to show that $H^0(\gamma \subset C) = 0$, or that any $L_2$-function on $C$ can be approximated by an $L_2$-function with (the closure of) the support inside $C \setminus \gamma$. In turn, this follows from the fact that there exists an open subset $U$, $\gamma \subset U \subset C$, with an arbitrary small measure. \hfill \qed

Corollary 2.13. If $\gamma$ is a smooth real curve inside a complex curve $C$, then $\mathcal{H}^1(\gamma \subset C) \simeq H^{1/2}(\gamma)$. Suppose that $\gamma$ breaks $C$ into two pieces $D_\pm$. Let an excess space $E$ be $D_\pm$-split, and let $H^{1/2}_E(\gamma)$ be images of $\mathcal{H}^1_E(\bar{D}_\pm)$ inside $H^{1/2}(\gamma)$. Then $H^{1/2}(\gamma)$ / const is a direct sum of $H^{1/2}_E(\gamma) / \text{const}$ and $H^{1/2}_{1-E}(\gamma) / \text{const}$ as a topological vector space. The mappings $\mathcal{H}^1_E(D_\pm) / \text{const} \to H^{1/2}_{1-E}(\gamma) / \text{const}$ are isomorphisms of topological vector spaces.

Proof. Apply Theorem 2.7 to $D = \gamma$, and $D_{1,2} = D_\pm$. Now the statement follows from the lemmas above. \hfill \qed

Remark 2.14. Heuristically, the first part of the corollary is similar to the following statements: any function from $H^{1/2}(\gamma)$ can be approximated by a function which is analytic near $\gamma$; if $C = \mathbb{C} \mathbb{P}^1$, any analytic near $\gamma$ function can be represented as a sum of two functions analytic in neighborhoods of $D_+$ and $D_-$ correspondingly. In other words, it is similar to a statement about density of analytic functions inside Sobolev spaces.

Remark 2.15. Since $\mathcal{H}^1_E(\bar{D}_\pm) / \text{const}$ carries a naturally defined Hilbert norm, so do $H^{1/2}_E(\gamma) / \text{const}$, thus $H^{1/2}(\gamma)$ / const. In this norm the subspaces $H^{1/2}_E(\gamma) / \text{const}$ are orthogonal. Call this norm on $H^{1/2}(\gamma)$ / const the embedding norm. In Definition 2.18 we define a different norm on $H^{1/2}(\gamma)$ / const.

Remark 2.16. The norm on $H^{1/2}(\gamma)$ / const and the decomposition

$$H^{1/2}(\gamma) / \text{const} = H^{1/2}_E(\gamma) / \text{const} \oplus H^{1/2}_{1-E}(\gamma) / \text{const}$$

depend on the inclusion $\gamma \hookrightarrow C$. Clearly, the subspaces $H^{1/2}_E(\gamma) / \text{const}$ depend only on inclusions $\gamma \hookrightarrow D_\pm$. However, the norms on $H^{1/2}_E(\gamma) / \text{const}$ depend also on the inclusions $D_\pm \hookrightarrow C$, since the norms on $\mathcal{H}^1(D_\pm \subset C) / \text{const}$ depend on these inclusions.

From now on we assign indices $+$ and $-$ to the parts $D_\pm$ into which an oriented real curve $\gamma \subset C$ breaks $C$ so that the orientation of $\gamma$ coincides with the orientation of the boundary of $D_-$. Typically, we will have several clockwise circles $\gamma_j$ bounding disjoint disks $D_{j+}$; the complement to these disks is $D = \bigcap_{j} D_{j-}$; it is this complement we are interested this, and $\bigcup_j \gamma_j$ is the properly oriented boundary of $D$. 

Remark 2.17. In the applications the parts $D_+$ and $D_-$ do not play symmetrical roles. Typically, $D_+$ is a “small” domain; moreover, if $C = \mathbb{CP}^1$, $D_+$ is often a small disk.

Below we use a trick to postpone one complicated calculation until Section 5.5. The trick boils down to changing the norm on the subspace $H_{1/2}^+(\gamma)/\text{const}$ in the following way:

**Definition 2.18.** Consider a real oriented connected curve $\gamma$ inside a complex curve $C$. Suppose that $\gamma$ breaks $C$ into two parts $D_{\pm}$. Consider the decomposition $H_{1/2}^+(\gamma)/\text{const} = H_{1/2}^+(\gamma)/\text{const} \oplus H_{1/2}^-(\gamma)/\text{const}$. Consider the norm on $H_{1/2}^+(\gamma)/\text{const}$ defined by its identification with $H_{E}^1(C_- \subset C)/\text{const}$ (with the natural norm), consider the norm on $H_{1/2}^-(\gamma)/\text{const}$ defined by its identification with $H_{E}^1(C_- \subset C)/\text{const}$ with the norm $\| \cdot \|_{1, \text{int}}$ from Definition 2.10. Call the induced direct sum norm on $H_{1/2}^+(\gamma)/\text{const}$ the $+$-skewed norm.

The reason to consider the $+$-skewed norm is Theorem 2.26.

Remark 2.19. Section 5.5 provides an alternative version of the theory which does not use the $+$-skewed norms. This allows dropping one of the conditions on the pieces (quasi-circularity), the price being slightly more complicated conditions on the “distance” between components of the boundary of the pieces.

In other words, consideration of the $+$-skewed norm provides some shortcuts in the discussion which follows, but should not significantly influence the class of curves allowed by these discussions.

Remark 2.20. Obviously, $\| \alpha \|_{1, \text{int}} \leq \| \alpha \|_1$ for $\alpha \in H^1$. If $D$ is a disk, it is easy to show that $\| \alpha \|_{1, \text{int}} = \| \alpha \|_1/\sqrt{2}$. Due to Lemma 2.11, for a domain $D$ with a smooth boundary $\gamma$, $\| \alpha \|_1 \leq c \| \alpha \|_{1, \text{int}}$ for an appropriate number $c$. Call the minimal such number the distortion $\Delta(\gamma)$ of the curve $\gamma$.

2.4. **Restriction to boundary curves.** Consider a complex curve $C$ with a closed subset $D$. Let $R_j, j \in J$, be the collection of connected components of $C \setminus D$. Let $\hat{D}_j$ be the interior of $C \setminus R_j$. An excess space $E$ is $D$-adjusted if it is $R_\bullet$-split, and is $\{ \hat{D}_j, R_j \}$-split for any $j \in J$. Starting from this section, we consider only $D$-adjusted excess spaces.

**Definition 2.21.** Consider a complex curve $C$ with a closed subset $D$. Suppose that one of the connected components $R_j \subset C$ of $C \setminus D$ is bounded by an oriented smooth curve $\gamma_j$. Denote the well-defined restriction mappings $H^1(D \subset C)/\text{const} \rightarrow H_{1/2}^+(\gamma_j)/\text{const}$ by $\beta_j$, the compositions of $\beta_j$ with the projections $H_{1/2}^+(\gamma_j)/\text{const} \rightarrow H_{1/2}^+(\gamma_j)/\text{const}$ by $\beta_j$.

**Definition 2.22.** Call a closed subset $R$ of a compact complex curve $C$ pseudo-smooth if all the connected components $D_j, j \in J$, of $C \setminus R$ have smooth boundaries $\gamma_j$. Call curves $\gamma_j$ the smooth boundaries of $R$. 
Definition 2.23. Suppose that $D$ is pseudo-smooth with smooth boundaries $\gamma_j$, $j \in J$. Denote by $\tilde{\beta}$ the mapping $\prod_j \beta_j : H^1(D \subset C) / \text{const} \to \prod_j H^{1/2}(\gamma_j) / \text{const}$, and by $\tilde{\beta}_\pm$ the mappings $\prod_j \beta_{j\pm} : H^1(D \subset C) / \text{const} \to \prod_j H^{1/2}_{\pm}(\gamma_j) / \text{const}$.

Since $H^{1/2}(\gamma_j) / \text{const}$ is equipped with a natural Hilbert norm, it makes sense to consider $\bigoplus \gamma_j H^{1/2}(\gamma_j) / \text{const}$. (This is the first place where the distinction between Hilbert norms and Hilbert topologies becomes important.)

Theorem 2.24. The mapping $\tilde{\beta}_-$ defined above sends $\mathcal{H}_E^1(D \subset C) / \text{const}$ into $V_- = \bigoplus \gamma_j H^{1/2}(\gamma_j) / \text{const}$. Then the induced mapping $\tilde{\beta}_-$ into $V_-$ is an invertible continuous mapping. Moreover, $\tilde{\beta}_-$ is unitary if $g(C) = 0$.

Proof. Indeed, $\beta$ is a composition of the mapping $\rho^{-1}$ of Theorem 2.7 with a direct sum of the mappings $H^1(D_j \subset C) / \text{const} \to H^{1/2}_{-}(\gamma_j) / \text{const}$; here $D_j = C - R_j$. By definition of $H^{1/2}$, the latter mapping is an isomorphism of Hilbert spaces.

Remark 2.25. What is important to us in this result is that though we assume that the boundary of each of $R_j$ is smooth, we do not assume that the whole boundary of $D$ is smooth. Indeed, if the number $|J|$ of connected components of $C - D$ is infinite, then in addition to $\bigcup_{j \in J} \partial R_j$, $\partial D$ contains also the dust: all the accumulation points of the curves $\partial R_j$. It is easy to construct examples when the dust is very massive; see Section 5.3. For example, it may have a positive measure. It may also coincide with $D$.

Theorem 2.26. The mapping $\tilde{\beta}$ defined above sends $\mathcal{H}_E^1(D \subset C)$ into $\bigoplus \gamma_j H^{1/2}(\gamma_j) / \text{const}$, here each $H^{1/2}(\gamma_j) / \text{const}$ is equipped with the $+$-skewed norm.

Proof. It is enough to show that $\text{Im} \tilde{\beta}_+ \subset \bigoplus \gamma_j H^{1/2}_{+}(\gamma_j) / \text{const}$. In other words, given an $H^1$-function $f$ on $C$, the sequence $(n_j)$ is in $l_2$, here $n_j \overset{\text{def}}{=} \| (f|_{\gamma_j})_+ \|_{H^{1/2}_{+}(\gamma_j) / \text{const}}$, and $g_\pm$ are the $\pm$-components of $g \in H^{1/2}(\gamma_j)$. In turn, by Lemma 2.11 this follows from the restriction mapping $H^1(C) / \text{const} \to \bigoplus \gamma_j H^1(D_j) / \text{const}$ having a norm $\leq 1$ if we consider norms $\| \cdot \|_{1,\text{int}}$ on $H^1(D_j) / \text{const}$. In turn, the latter statement follows from the definition of the norm $\| \cdot \|_{1,\text{int}}$. □

Definition 2.27. Let $\beta_E$ be the mapping $\mathcal{H}_E^1(D \subset C) \to \bigoplus \gamma_j H^{1/2}(\gamma_j) / \text{const}$ induced by $\tilde{\beta}$, let $\beta = \beta_E|_{\mathcal{H}_E^1(D \subset C)}$.

3. GLUING THE CURVE FROM THE PIECES

3.1. Gluing data and mismatch.

Definition 3.1. Consider two connected oriented closed real curves $\gamma_1$ and $\gamma_2$. The curve gluing data for the pair $(\gamma_1, \gamma_2)$ is a pair of mutually inverse diffeomorphisms $\varphi_1 : \gamma_1 \to \gamma_2$ and $\varphi_2 : \gamma_2 \to \gamma_1$ which reverse the orientations. The bundle gluing data
for the pair \((\gamma_{1,2})\) and the curve gluing data \((\varphi_{1,2})\) is a pair of smooth complex-valued functions \(\psi_i\) on \(\gamma_i, i = 1, 2\), such that \(\psi_1 \cdot \varphi_1^* (\psi_2) = 1\).

The degree of the bundle gluing data is\(^{11}\) \(\text{ind } \psi_1 = \text{ind } \psi_2\).

Clearly, the gluing data for a pair \((\gamma_1, \gamma_2)\) induces gluing data for a pair \((\gamma_2, \gamma_1)\).

**Definition 3.2.** Given the curve and bundle gluing data \((\varphi_{1,2}, \psi_{1,2})\) for a pair \((\gamma_{1,2})\), call a pair of functions \(f_j \in H^s (\gamma_j), j = 1, 2\), compatible with the gluing data if \(f_1 = \psi_1 \cdot \varphi_1^* (f_2)\). The mismatch of a pair \((f_{1,2})\) is a pair \((\delta_{1,2})\), \(\delta_j \in H^s (\gamma_j), j = 1, 2\), given by \(\delta_j = f_j - \psi_j \cdot \varphi_j^* (f_k)\), for \((j, k) = (1, 2)\) or \((2,1)\).

Clearly, if \((f_1, f_2)\) is compatible with gluing data for \((\gamma_1, \gamma_2)\), then \((f_2, f_1)\) is compatible with the corresponding gluing data for \((\gamma_2, \gamma_1)\). Similarly, if \((\delta_1, \delta_2)\) is the mismatch of \((f_1, f_2)\), then \((\delta_2, \delta_1)\) is the mismatch of \((f_2, f_1)\). Moreover, \(\delta_2 = -\psi_2 \cdot \varphi_2^* (\delta_1)\).

**Definition 3.3.** Consider a collection \(\mathcal{D} = (D_k \subset C_k)_{k \in K}\) of pseudo-smooth closed subsets of compact complex curves. Let \(J_k\) be the set of connected components of \(C_k \setminus D_k, J = \bigsqcup_k J_k\). Let \(\gamma_j, j \in J, \) be the boundary of the connected component which corresponds to \(j\). The gluing decomposition for \(\mathcal{D}\) is a decomposition of \(J\) into a disjoint union of pairs \(\{j, j'\}\). Define a mapping \(': J \rightarrow J: j \mapsto j': j' \mapsto j\). The curve and bundle gluing data for such a gluing is a gluing decomposition together with curve and bundle gluing data \((\varphi_j, \varphi_{j'}), (\psi_j, \psi_{j'})\) for the pair of curves \((\gamma_j, \gamma_{j'})\) for each pair \(\{j, j'\}\).

Suppose that \(\text{ind } \psi_j = 0\) for all but a finite number of \(j \in J\). Then the degree of the bundle gluing data is the sum of degrees over all pairs \((\gamma_j, \gamma_{j'})\).

Call a collection of functions \(F_k \in H^1 (D_k \subset C_k)\), \(k \in K\), compatible with the gluing data, if the pair \((f_j, f_{j'})\) is compatible with the gluing data for \((\gamma_j, \gamma_{j'})\) for each pair \(\{j, j'\}\); here \(f_j = F_k|_{\gamma_j}\) (assuming that \(\gamma_j \subset C_{k_j}\)). Define similarly the mismatch \((\delta_j), \delta_j \in H^{1/2} (\gamma_j), j \in J\), of such a collection.

It is clear that mismatches \(\delta_j, \delta_{j'}\) satisfy \(\delta_{j'} = -\psi_{j'} \cdot \varphi_{j'}^* (\delta_j)\).

### 3.2. The curve and the line bundle

Given the curve and bundle gluing data \((D_k \subset C_k, \varphi, \psi)\), one can associate to it some more or less familiar objects. The associated curve \(C\) is the set obtained from \(\bigsqcup_k D_k\) by identifying points on the smooth parts of the boundaries via \(\varphi\). As quotients do, \(C\) is equipped with a natural topology. A point \(m \in C\) has one or two preimages. A point \(m \in C\) is a dust point if one of its preimages on \(\bigsqcup_k D_k\) lies in the dust of the corresponding component \(D_k\).

The dust \(C_\infty\subset C\) consists of dust points; it is a closed subset of \(C\). Obviously, \(C_\infty\) is empty unless a complement to one of \(D_k\) has infinity many connected components. It is clear that \(C_{\text{fin}} \overset{\text{def}}{=} C \setminus C_\infty\) has a natural structure of a complex curve. However, \(C_{\text{fin}}\) may be empty.

\(^{11}\)Recall that \(\text{ind } \psi \overset{\text{def}}{=} \frac{1}{2\pi i} \int d \log \psi\); here \(\psi\) is a nowhere-0 function on \(S^1\).
If $\gamma_j$ and $\gamma_j'$ have no dust points on them, the common image of these curves on $C_{\text{fin}}$ is a smooth cycle on $C_{\text{fin}}$.

Similarly to $C$, one can glue a “set-theoretic\footnote{I.e., a set with a projection $\pi: \mathcal{L} \to C$; fibers of $\pi$ are one-dimensional vector spaces.} line bundle” $\mathcal{L}$ over $C$ starting from $\coprod_k D_k \times \mathbb{C}$ and gluing via $(\varphi_\bullet, \psi_\bullet)$. It may be not a topological line bundle; however, it is an analytic line bundle over $C_{\text{fin}}$. Usual definitions of the dual bundle, of the tensor product of bundles, of the line bundles $\omega$ and $\bar{\omega}$ work without any change in this situation.

Let $H^1_{\text{loc}}(C, \mathcal{L})$ consist of $H^1_{\text{loc}}$-sections of $\mathcal{L}$ on $C$: an $H^1_{\text{loc}}$-section is a collection of elements of $H^1(D_k \subset C_k)$ which are compatible with the gluing data. (One can naturally define the support of an $H^1_{\text{loc}}$-section, thus one can also define what is a section of $\mathcal{L}$ on $U \subset C$.) Similarly one can define $H^0_{\text{loc}}(C, \mathcal{L})$ (without any compatibility conditions on $\gamma_j$) and the operator $\bar{\partial}: H^1_{\text{loc}}(C, \mathcal{L}) \to H^0_{\text{loc}}(C, \mathcal{L} \otimes \bar{\omega})$. The vector space $\mathcal{H}^1_{\text{loc}}(C, \mathcal{L})$ of locally-$H^1$-holomorphic sections consists of $f \in H^1_{\text{loc}}(C, \mathcal{L})$ such that $\bar{\partial}f = 0$; here $\bar{\partial}f \in H^0_{\text{loc}}(C, \mathcal{L} \otimes \bar{\omega})$. Define similarly the space $\mathcal{H}^1_{E,\text{loc}}(C, \mathcal{L})$.

The motivation for this definition is the following:

**Lemma 3.4.** Consider a complex curve $C$ and a compact smooth real curve $\gamma \subset C$ which breaks $C$ into two parts $D_\pm$. Consider two functions $f_\pm \in H^1_{\text{loc}}(D_\pm \subset C)$ such that $f_+|_\gamma = f_-|_\gamma$ and $\bar{\partial}f_\pm = 0 \in H^0_{\text{loc}}(D_\pm \subset C)$. Then there is a unique function $f \in H^1_{\text{loc}}(C)$ such that $f_\pm = f|_{D_\pm}$. Moreover, $\bar{\partial}f = 0 \in H^0_{\text{loc}}(C)$. In particular, $f$ is analytic near $\gamma$.

**Proof.** If $f$ exists, then obviously $\bar{\partial}f \in L^2_{\text{loc}}$ should vanish. The uniqueness of $f$ is also obvious. Show the existence of $f$; we may drop the conditions $\bar{\partial}f_\pm = 0$.

We know already that the mapping of restriction to $\gamma$ is surjective, thus we may suppose $f_+|_\gamma = f_-|_\gamma = 0$. It is enough to show that $f_+$ allows a continuation-by-0 without losing its smoothness class $H^1$. This is a local statement, so we may assume that $C$ is a neighborhood of the unit circle $\gamma$ in $\mathbb{C}$.

Again, consider the action of the group of rotations. One may restrict the attention to one isotypical component in $H^1(C)$. This reduces the problem to one-dimensional: given an $H^1$-function $g(t)$ on $[0.5, 2]$ such that $g(1) = 0$, the extension-by-0 from $[0.5, 1]$ to $[0.5, 2]$ obviously has the same norm. \hfill \Box

In this paper we are most interested in line bundles $\mathcal{L}$ of small degree. As explained in Section 0.5, this leads to consideration of the operator $\bar{\partial}$ sending $H^1$ to $H^0$. This is the reason for our interest in $H^1$-holomorphic functions.

**Remark 3.5.** Above, the index $\text{loc}$ relates to having no restriction on how the sequence $\|f_k\|$ grows; here $f_k$ is the restriction of $f \in H^s(C, \mathcal{L})$ to $D_k \subset C$. Such an approach is sufficient if $K$ is finite (which is the most interesting case in our approach). However, it is easy to modify this to work with infinite collections $K$, see Section 3.4.

Recall (see Section 0.6) that to expect Riemann–Roch theorem to hold, one needs to add some slack, allowing some non-strictly holomorphic sections. By robustness,
it is not very important which non-holomorphic functions are allowed; we add slack by allowing a finite-dimensional mismatch at each gluing:

**Definition 3.6.** Given a curve and bundle gluing data \((\varphi_j, \psi_j)_{j \in J}\) for a collection \((D_k \subset C_k)_{k \in K}\) with boundary curves \((\gamma_{j})_{j \in J}\), the mismatch allowance is a collection \((V_{j})_{j \in J}\) consisting of vector subspaces \(V_{j} \subset H^{1/2}(\gamma_{j})\) such that \(V_{j'} = \psi_{j'} \cdot \varphi_{j'}^{*}(V_{j})\).

An \(H^{1}_{\text{loc}}\)-section \(F\) modulo \((V_{j})_{j \in J}\) is a collection \((F_{k})_{k \in K}\) such that \(F_{k} \in H^{1}(D_{k} \subset C_{k})\) and the mismatch \((\delta_{j})_{\psi \in J}\) of \((F_{k})\) satisfies \(\delta_{j} \in V_{j}\). Define similarly \(H^{1}_{\text{loc}}\)-sections and \(H^{1}_{E,\text{loc}}\)-sections modulo \((V_{j})\).

**Definition 3.7.** Consider an involution \(\psi\) of a set \(J\). Given a quantity \(t_{j}, j \in J\), such that \(t_{j} = t_{j}'\), let \(\sum_{\{j,j'\}} t_{j} = \frac{1}{2} \sum_{j \in J} t_{j}\). Similarly, if \(V_{j}\) is a vector space with a fixed isomorphism \(\varphi_{j}\) between \(V_{j}\) and \(V_{j}'\) such that \(\varphi_{j} \cdot \psi_{j} = \varphi_{j}^{-1}\), let \(\bigoplus_{\{j,j'\}} V_{j}\) is the subspace of \(\bigoplus_{j} V_{j}\) formed by sequences \((v_{j})_{j \in J}\) such that \(v_{j'} = \varphi_{j} v_{j}\).

If \(J = J_{0} \sqcup J_{0}'\), then \(\bigoplus_{\{j,j'\}} V_{j}\) is canonically isomorphic to \(\bigoplus_{j \in J_{0}} V_{j}\). Similarly, define \(\prod_{\{j,j'\}}\) etc.

**3.3. Finite-genus Riemann–Roch theorem via gluing data.** This theorem relates the dimension of two vector spaces. One is the space of global sections of a line bundle on a curve. Another is the first homology. The first step to formulate the infinite-genus variant is the translation of the case \(g < \infty\) to our notations.

**Theorem 3.8.** In the conditions of the previous section suppose that the set \(K\) is finite, and each subset \(D_k\) has a smooth boundary (thus its complement \(C_k \setminus D_k\) has finitely many connected components). Then the vector subspace of \(\prod_{\{j,j'\}} H^{1/2}(\gamma_{j}) = \bigoplus_{\{j,j'\}} H^{1/2}(\gamma_{j})\) formed by mismatches of elements of \(\prod_{k} H^{1}(D_{k} \subset C_{k})\) is a closed subspace of finite codimension. Denote this codimension by \(h^{1}\).

The vector space of global analytic sections is finite dimensional, denote its dimension by \(h^{0}\). Then \(h^{0} - h^{1} = d - g + 1\), here \(d\) is the degree of the bundle gluing data, and \(g = |J|/2 - |K| + 1 + \sum_{k} g(C_{k})\).

**Sketch of the proof.** Since in what follows we are going to prove significantly more general results, let us show only that this statement is a generalization of the “usual” Riemann–Roch theorem, which is formulated using the language of analytic sections, not Sobolev-class section. To simplify the discussion, assume that the gluing data \((\varphi_{j})\) and \((\psi_{j})\) consists of real-analytic functions, that no connected components of the boundary of the same piece \(D_k\) are glued together, and that the complex analytic curve \(C\) obtained after gluing is connected.

Since functions \(\varphi_{j}\) can be analytically extended to neighborhoods of \(\gamma_{j}\), we can glue \(C\) of neighborhoods \(\tilde{D}_{k}\) of \(D_k\). Images \(U_{k}\) of \(\tilde{D}_{k}\) in \(C\) form a covering of \(C\), \(U_{k} \cap U_{k}'\) is a union of annuli (thus Stein), and \(U_k\) is Stein (unless \(|K| = 1, |J| = 0\), when the theorem is obvious).
Moreover, $D_k$ is identified with a closed subset of $C$, so we may assume that $D_k \subset C$ and $\gamma_j \subset C$, and $\gamma_j = \gamma_{j'}$ up to orientation change. We may assume that the bundle gluing functions $\psi_j$ on $\gamma_j$ can be extended to the corresponding connected component $V_{\{j,j'\}}$ of $U_k \cap U_{k'}$, thus define a line bundle $\mathcal{L}$ over $C$; sections of $\mathcal{L}$ are represented by functions on $U_k$ with vanishing mismatches of boundary values. A simple calculation shows that $\deg \mathcal{L} = d$, $g(C) = g$. Everything being Stein, we can calculate cohomology by the Čech complex

$$0 \to \bigoplus_{k \in K} \Gamma_{\text{an}}(U_k, \mathcal{L}) \to \bigoplus_{\{j,j'\}} \Gamma_{\text{an}}(V_{j,j'}, \mathcal{L}) \to 0.$$ 

Moreover, $\mathcal{L}|_{U_k}$ is already trivialized, so we may substitute $\mathcal{O}$ instead of $\mathcal{L}$, with an appropriate modification of the differential of the complex. After this change the differential becomes the operator of taking the mismatch. We want to show that the cohomology of the complex above coincides with the cohomology of the complex

$$0 \to \bigoplus_{k \in K} H^1(D_k) \to \bigoplus_{\{j,j'\}} H^{1/2}(\gamma_j) \to 0.$$ 

Call the differential $\tilde{\mu}$ the operator of taking the mismatch.

There is natural inclusion $\mathcal{C}_{\text{an}} \hookrightarrow \mathcal{C}_H$, consider the induced mapping of cohomology. On the level of $H^0$ (here $H$ denotes cohomology) it is automatically an injection. On the other hand, as Lemma 3.4 shows, any function compatible with the gluing data $(\varphi_*, \psi_*)$ induces an $H^1$-section of $\mathcal{L}$. Thus the mapping of $H^0$ is an isomorphism.

Consider the spaces $H^1$. Due to the duality theorem, $H^1(C, \mathcal{L})^* = H^0(C, \omega \otimes \mathcal{L}^{-1})$. Given a section $\alpha$ of $\omega \otimes \mathcal{L}^{-1}$, the pairing with a 1-cocycle $c \in \bigoplus_{\{j,j'\}} \Gamma_{\text{an}}(V_{j,j'}, \mathcal{L})$ is $\int_\gamma \alpha c$, here $\gamma$ is a suitably oriented curve with connected components generating 1-homology of annuli $V_{j,j'}$. Taking $\gamma = \bigcup_{\{j,j'\}} \gamma_j$ shows that the linear functional on $H^1(\mathcal{C}_{\text{an}})$ induced by $\alpha$ can be passed through the mapping $\mathcal{C}_{\text{an}} \to \mathcal{C}_H$, thus $H^1(\mathcal{C}_{\text{an}}) \to H^1(\mathcal{C}_H)$ is an injection. Since $\text{Im} \mathcal{C}_{\text{an}}$ is dense in $\mathcal{C}_H$, to show the surjectivity it is enough to show that the image of each component $H^1(D_k) \to \bigoplus H^{1/2}(\gamma_j)$ of the differential is closed, which follows from Theorem 2.24.

3.4. Plan of the campaign. We have shown that the finite-genus case of Riemann–Roch theorem coincides with the calculation of the index of the operator $\tilde{\mu}$ of taking the mismatch. The target of this paper is to investigate the mismatch operator in the more general case of arbitrary (possibly infinite) genus—assuming that the degree remains finite. There are two obstacles to restate the above theorem in the infinite genus case: first, $g - d$ becomes infinite. Second, by Theorem 2.24, the natural topology on $\text{Im} \tilde{\mu}$ is the Hilbert space topology, which is very far from both the topology on $\prod_{\{j,j'\}} H^{1/2}(\gamma_j)$ and on $\bigoplus_{\{j,j'\}} H^{1/2}(\gamma_j)$, so there is no hope to get a finite-dimensional cokernel of $\tilde{\mu}$. 

\[\Box\]
The trick to tackle the first problem is the allowance subspaces we introduced above. Instead of considering the mapping $\tilde{\mu}$ to $\bigoplus_{\{j,j'\}} H^{1/2} (\gamma_j)$, consider the induced mappings into $\bigoplus_{\{j,j'\}} (H^{1/2} (\gamma_j) / V_j)$; here $V_j$ is an arbitrary finite-dimensional subspace of $H^{1/2} (\gamma_j)$. This would change the right-hand side of Riemann–Roch theorem to $d - g + 1 + \sum_{\{j,j'\}} \dim V_j$. Now if $\dim V_j$ “compensates” the contribution of $j \in J$ into $g$, then $d - g + 1 + \sum_{\{j,j'\}} \dim V_j$ makes sense as a finite number. This is so, for example, if $|K| < \infty$, and $\dim V_j = 1$ for all but a finite number of $j \in J$.

Similarly, to compensate for an infinite $K$ with $g(C_k) = 0$ for almost any $k \in K$, it is enough to consider $\bigoplus_{k \in K} \mathcal{H}^1 (D_k) / W_k$ instead of $\bigoplus_{k \in K} \mathcal{H}^1 (D_k)$; here $W_k$ is an arbitrary 1-dimensional subspace of $\mathcal{H}^1 (D_k)$ (it is convenient to assume that restrictions of functions from $W_k$ to any boundary component $\gamma_j$ of $C_k$ are in $V_j$). As we will see, it is most convenient to take $W_k$ consisting of constant functions.

Finally, one can also put correction terms if infinitely many curves $C_k$ are not rational. Suppose that $\mathcal{H}^1 (D_k) \subset \tilde{\mathcal{H}}^1 (D_k) \subset H^1 (D_k \subset C_k)$, and $\dim \tilde{\mathcal{H}}^1 (D_k) / \mathcal{H}^1 (D_k) = g(C_k)$. Then substitution of $\tilde{\mathcal{H}}$ instead of $\mathcal{H}$ leads to the index formula for $\tilde{\mu}$ which does not include $g(C_k)$. Moreover, one can take $\tilde{\mathcal{H}}^1 = \mathcal{H}_E^1$ for an appropriate $D$-adjusted and $D$-supported excess space $E$.

As a result, we obtain the following reformulation of the finite-genus Riemann–Roch theorem:

**Theorem 3.9.** In the conditions of the previous section suppose that the set $K$ is finite, and each subset $D_k$ has a smooth boundary (thus the complement $C \setminus D_k$ has finitely many connected components). Suppose also that $1 \in V_j$ for any $j$. Then the operator of taking the mismatch modulo $V_j$

$$
\bigoplus_{k \in K} \mathcal{H}_E^1 (D_k) / \text{const} \xrightarrow{\mu'} \bigoplus_{\{j,j'\}} H^{1/2} (\gamma_j) / V_j
$$

is Fredholm, and its index is equal to $d + \sum_{\{j,j'\}} (\dim V_j - 1)$.

The way to handle the second problem is suggested by Theorem 2.24: the space which appears in the theory in a natural way is $\bigoplus_{l_2} (H^{1/2} (\gamma_j) / \text{const})$; thus it is natural to replace $\bigoplus_{\{j,j'\}} H^{1/2} (\gamma_j) / V_j$ by $\bigoplus_{l_2,\{j,j'\}} H^{1/2} (\gamma_j) / V_j$, provided $V_j = \langle \text{const} \rangle$ for all but a finite number of indices. Similarly, one should replace $\bigoplus$ by $\bigoplus_{l_2}$ as the target of $\mu'$ as well.

Combining these two arguments, we need to investigate the operator

$$
\bigoplus_{l_2, k \in K} \mathcal{H}_{E_k} (D_k) / \text{const} \xrightarrow{\mu} \bigoplus_{l_2,\{j,j'\}} H^{1/2} (\gamma_j) / V_j
$$

of taking the mismatch; here we suppose that $1 \in V_j$, $E_k$ is $D_k$-supported, and $\dim V_j = 1$ for all but a finite number of $j \in J$. (Note that these conditions imply that $\psi_j = \text{const}$ for all but a finite number of $j \in J$.) We want to find the cases when
this operator is continuous, Fredholm, and of the index prescribed by Riemann–Roch theorem.

One part of this question is trivial to answer: by Theorem 2.24, \( \mu \) is continuous as far as we consider the \(+\)-skewed norms on \( H^{1/2}(\gamma_j)/V_j \). (Later we will see that this skewing may be replaced by appropriate assumptions about curves \( \gamma_j \), but for the time being restrict our attention to the \(+\)-skewed norm.) The other questions require more information about properties of the operator \( \mu \).

4. RIEMANN–ROCH THEOREM

4.1. The mapping of gluing. Here we show that the gluing condition on one particular pair of curves \( \gamma_j, \gamma'_j \) may be reformulated in terms of an operators sending the \(+\)-parts of a pair of functions to the \(-\)-part.

Definition 4.1. Consider a complex curve \( C \) with an excess space \( E \) and a real smooth oriented curve \( \gamma \subset C \) splitting \( C \) into two domains \( D_\pm \) (as usual, the orientation of \( \gamma \) is compatible with the orientation of \( D_- \)). If \( D_+ \) has no handles, and \( E \) is \( D_- \)-supported, call the inclusion \( \gamma \hookrightarrow C \) admissible.

Given two such pairs \( \gamma \subset C, \gamma' \subset C' \) with excess spaces \( E, E' \) which are \( D_- \)- and \( D'_- \)-supported, say that an orientation-inverting gluing \( \varphi: \gamma \rightarrow \gamma' \) is compatible with \( E, E' \) if \( E \oplus E' \) is an excess space for the complex curve \( C^* = D_- \cup_\varphi D'_- \).

Consider bundle gluing data \( \psi, \psi' \) for the identification \( \varphi \). It determines a line bundle \( \mathcal{L} \) over \( C^* \). Construct 3 subspaces \((V_+, V_-, L)\) of the vector space \( V = H^{1/2}(\gamma) \oplus H^{1/2}(\gamma') : V_\pm \) correspond to \( \pm \)-parts of \( H^{1/2} \), and \( L \) consists of pairs \((f, f') \in V \) which are compatible with the curve and bundle gluing data: \( f = \psi \cdot \varphi^*(f') \).

The following lemma technical lemma is a key tool for translating properties of the operator \( \mu \) of mismatch (from Section 3.4) to the usual Fredholm theorem. Heuristically, it states that these 3 subspaces are in general position, and calculates the corresponding relative dimensions:

Lemma 4.2. Let \( \bar{V} = H^{1/2}(\gamma)/\text{const} \oplus H^{1/2}(\gamma')/\text{const}, \bar{L} \) be the image of \( L \) w.r.t. the natural projection \( \pi: V \rightarrow \bar{V} \). Let \( V = V_+ \oplus V_- \) be the decomposition of \( V \) corresponding to the decompositions \( H^{1/2}/\text{const} = H^{1/2}_+/\text{const} \oplus H^{1/2}_-/\text{const} \). Then there is a continuous mapping \( A: \bar{V}_+ \rightarrow \bar{V}_- \) such that

1. the graph \( L_A \) of \( A \) is comparable with \( \bar{L} \);
2. \( \text{rel} \text{dim} (\bar{L}, L_A) = \text{ind} \psi + \delta \); here \( \delta \) is 0 if \( \psi \equiv \text{const} \), and is 1 otherwise.

If \( \varphi \) is compatible with \( E, E' \) and \( \psi \equiv \text{const} \), one can chose \( A \) so that \( L_A = \bar{L} \). In particular, this is so if \( C, C' \) are of genus 0.

Proof. By the closed graph theorem, it is enough to show that \( \bar{L} \) and \( \bar{V}_- \) are quasi-complementary, and calculate the excess. In turn, it is enough to do the same with \( L \) and the preimage \( V_- \) of \( \bar{V}_- \) w.r.t. the projection \( V \rightarrow \bar{V} \). (It is this step what introduces \( \delta \) into the statement.)
We defined a line bundle $\mathcal{L}$ on $C^*$; its (continuous) global section $f$ of $\mathcal{L}$ corresponds to two continuous functions $F$, $F'$ on $D_-$, $D'_-$, which satisfy the gluing relationship $f|_\gamma = \psi \cdot \varphi^*(f'|_\gamma)$. Obviously, $\deg \mathcal{L} = \text{ind} \psi$.

By Lemma 3.4, there is a natural holomorphic structure on $\mathcal{L}$. Let $\mathcal{C} = C \cup C'$, $\mathcal{D}_- = D_- \cup D'_-$, $\mathcal{\hat{\gamma}} = \gamma \cup \gamma'$, and $\alpha, \beta$ be the common image of $\gamma$, $\gamma'$ in $C^*$. Consider the complexes

$$\mathcal{C}_1: H^1(C^*, \mathcal{L}) \xrightarrow{\delta} H^0(C^*, \mathcal{L} \otimes \mathcal{\bar{\omega}}) / E$$

$$\mathcal{C}_2: H^1(\mathcal{D}_- \subset \mathcal{\hat{C}}, \mathcal{O}) \xrightarrow{\delta} H^0(\mathcal{D}_- \subset \mathcal{\hat{C}}, \mathcal{\bar{\omega}}) / \mathcal{E};$$

here $\mathcal{E} = E \oplus E'$; clearly, $\mathcal{\hat{E}}$ may be considered as a subspace of both the $H^0$-spaces. By the (finite-genus) Riemann–Roch theorem $\delta$ in $\mathcal{C}_1$ is Fredholm of index $\deg \mathcal{L} + 1$; additionally, $\text{Im} \delta$ is given by a finite number of independent equations: $\beta \in \text{Im} \delta$ iff $\langle \alpha, \beta \rangle = 0$, here $\langle \alpha, \beta \rangle = \int_C \alpha \beta$, and $\beta$ is a holomorphic section of $\mathcal{L}^* \otimes \mathcal{\bar{\omega}}$.

By the definition of $\mathcal{L}$, there is an inclusion $\iota$ of the first complex into the second one; denote components of $\iota$ by $\iota_\mathcal{O}$, $\iota_\mathcal{\bar{\omega}}$. Obviously, $\iota_\mathcal{\bar{\omega}}$ an isomorphism, and $\iota_\mathcal{O}$ is a closed inclusion. Since $\partial_\mathcal{C}_2$ is surjective, $\text{Ker} \partial_\mathcal{C}_2$ and the image $L^*$ of $H^1(C^*, \mathcal{L})$ in $H^1(\mathcal{\hat{D}}_- \subset \mathcal{\hat{C}}, \mathcal{O})$ are quasi-complementary of excess $\deg \mathcal{L} + 1$.

Let $W_\mathcal{L}$ and $W_\mathcal{O}$ be the subspaces of $H^1(C^*, \mathcal{L})$ and $H^1(\mathcal{\hat{D}}_- \subset \mathcal{\hat{C}}, \mathcal{O})$ consisting of functions vanishing at $\gamma^*$ and $\mathcal{\hat{\gamma}}$ correspondingly. By Lemma 3.4 $\iota_\mathcal{O}$ identifies $W_\mathcal{L}$ and $W_\mathcal{O}$. Since $L^* \subset W_\mathcal{O}$, the subspaces $(W_\mathcal{O} + \text{Ker} \partial_\mathcal{C}_2) / W_\mathcal{O}$ and $L^*/W_\mathcal{O}$ of $H^1(D_- \subset C, \mathcal{O}) / W_\mathcal{O}$ are quasi-complementary of excess $\deg \mathcal{L} - g(C^*) + 1$.

On the other hand, restriction to $\mathcal{\hat{\gamma}}$ identifies $H^1(\mathcal{\hat{D}}_- \subset \mathcal{\hat{C}}, \mathcal{O}) / W_\mathcal{O}$ with $H^{1/2}(\mathcal{\hat{\gamma}})$. Since this identification sends $(W_\mathcal{O} + \text{Ker} \partial_\mathcal{C}_2) / W_\mathcal{O}$ to $H^{1/2}(\mathcal{\hat{\gamma}})$, and $L^*/W_\mathcal{O}$ to $L$, this finishes the proof of the first part of the lemma.

It is clear that $L \cap V_-$ consists of global solutions of $\partial_\mathcal{\bar{\varphi}} \in \mathcal{\hat{E}}$ in sections of $\mathcal{L}$. Thus if $\varphi$ is compatible with $E$, $E'$ and $\psi \equiv \text{const}$, $L \cap V_-$ consists of constants, and $L \cap V_- = 0$. In other words, $\mathcal{\hat{L}}$ and $V_-$ are complementary, hence $\mathcal{\hat{L}}$ is a graph of a continuous mapping $V_+ \to V_-$. \hfill $\square$

**Definition 4.3.** If $\varphi$ is compatible with $E$, $E'$ and $\psi \equiv \text{const}$, the *distortion* $\Delta(E, E', \gamma, \gamma', \varphi, \psi)$ is the norm of the operator $A$ from the last statement of the lemma. Otherwise put $\Delta(E, E', \gamma, \gamma', \varphi, \psi)$ to be $1$.

In this definition the curves $\gamma$, $\gamma'$ are considered together with inclusions into complex curves $C, C'$ with a choice of excess spaces $E, E'$.

**Remark 4.4.** There exist excess spaces which are compatible with arbitrary diffeomorphisms $\varphi$. Indeed, given a curve $C$ with an admissible real curve $\gamma \subset C$, perform $g = g(C)$ cuts on $D_-$ along real disjoint curves $B_1, \ldots, B_q$ so that the resulting curve is of genus $0$. Let $W$ be the vector space of linear functionals on $L_2(D_-, \omega)$ spanned
by integrals along these curves. Call \( E \) cyclic if pairing with \( E \) induces the same space \( W \) of linear functionals on \( H^0(D \subset E, \omega) \) (here \( H^0 \) is \( \text{Ker} \partial \) in \( H^0 \)). Obviously, if \( E, E' \) are cyclic, they are compatible with arbitrary gluing \( \varphi \).

It is possible to construct cyclic excess spaces in any situation we deal with in this paper. Indeed, if there is a annulus \( U \) around the cycle \( B_s \) which lies completely inside \( D \subset C \), then the form \( \partial \arg(z) \) on \( U \) represents the integration along the cycle.

Later we see that in the foam situation, as in Section 5.3, such an annulus may not exist. Sketch how to deal with foam curves glued of curves of non-0 genus; since one needs no such arguments if \( g(C) = 0 \), and with additional cuts one can always achieve this, we do not discuss these arguments in more details. First, one can find an annulus \( U \subset C \) around a cycle \( B_s \) such that it contains any connected component of \( C \setminus D \) which intersects \( U \). One may assume \([7]\) that \( U \cap D \) is conformally equivalent to an annulus with cuts along concentric arcs. Now \( \partial \arg(z)|_U \) is the suitable element of \( E \).

4.2. Global sections. Recall that \( H^1_{\text{loc}}(C, \mathcal{L}) \) was defined as the vector subspace of \( \prod_k H^1(D_k \subset C_k, \mathcal{O}) \) consisting of collections of functions (\( F_k \)) which are compatible with the gluing data. As suggested by arguments in Section 3.4, introduce the following refinement of Definition 3.6:

**Definition 4.5.** Given a curve and bundle data \((C, \mathcal{L})\), and a mismatch allowance \((V_j)\) for these data such that \( \text{const} \in V_j \) for any \( j \), define a norm \( \|F\| \) of an \( H^1_{\text{loc}} \)-section \( F = (F_k) \) modulo \((V_j)\) as \( \|F\|^2 \overset{\text{def}}{=} \sum_k \|F_k\|^2 \); here \( \|F_k\| \) is taken in \( H^1(D_k \subset C_k, \mathcal{O})/\text{const} \); call \( F \) with \( \|F\| < \infty \) a global section modulo \((V_j)\). Denote the space of such global sections by \( H^1_{[\cdot, \cdot]}(C, \mathcal{L}) \). Define similarly \( H^0(C, \mathcal{L} \otimes \bar{\omega}) \overset{\text{def}}{=} \bigoplus_{l_2} H^0(D_k \subset C_k, \mathcal{O}) \) and \( H^0_{E}(C, \mathcal{L}) \overset{\text{def}}{=} \bigoplus_{l_2} H^0(D_k \subset C_k, \mathcal{O})/E_k \).

Obviously, the operator \( \bar{\partial} \) induces an operator \( H^1_{[\cdot, \cdot]}(C, \mathcal{L}) \to H^0(C, \mathcal{L} \otimes \bar{\omega}) \), and \( H^1_{[\cdot, \cdot]}(C, \mathcal{L}) \cap H^1_{\text{loc}}(C, \mathcal{L}) \) coincides with the kernel of this operator. We denote this operator by the same symbol \( \bar{\partial} \); however, note that this operator has a slightly different semantic than the operator \( \bar{\partial} \) acting on generalized functions. If \( F \in H^1_{[\cdot, \cdot]}(C, \mathcal{L}) \), then “honest” \( \bar{\partial}F \) can be represented as a sum of an \( L_2 \)-section of \( \mathcal{L} \otimes \bar{\omega} \), and of generalized functions with support on the curves \( \gamma_j \), \( j \in J \); each of these generalized functions corresponds to an element of \( V_j \). When we consider \( \bar{\partial} : H^1_{[\cdot, \cdot]}(C, \mathcal{L}) \to H^0(C, \mathcal{L} \otimes \bar{\omega}) \), we keep only the \( L_2 \)-component.

Similarly, \( H^1_{[\cdot, \cdot]}(C, \mathcal{L} \otimes \bar{\omega}) \cap H^1_{\text{loc,E}}(C, \mathcal{L}) \) is \( \text{Ker} \bar{\partial} : H^1_{[\cdot, \cdot]}(C, \mathcal{L}) \to H^0_{E}(C, \mathcal{L} \otimes \bar{\omega}) \).

In this flavor of the \( \bar{\partial} \)-operator we forget not only about the \( \delta \)-function components on the curves \( \gamma_j \), but also about the “components” of \( \bar{\partial}F \) in \( \bigoplus_{l_2} E_k \).

**Remark 4.6.** The consideration of the index of the latter flavor of the \( \bar{\partial} \)-operator is the principal target of this paper. Is is forgetting about \( \delta \)-function components and \( E \)-components which allows \( \bar{\partial} \) to have a finite index (in “good” situations we
are going to describe soon). Indeed, we expect the “honest” operator $\bar{\partial}$ to have an infinite negative index; taking a quotient of the target space increases this index by exactly an amount needed for it to become finite.

4.3. Reduction to boundary. Suppose that the excess subspaces $E_k$ for curves $C_k$ are $D_k$-supported. Here we introduce two subspaces $W_{E,an}$ and $W_{\varphi\psi V}$ of an appropriate Hilbert space $W$; the relative position of these subspaces is going to encode all the information about the operator $\bar{\partial}$ we need.

By Theorem 2.26, there is a natural bounded operator $\beta$ of taking the boundary value modulo constants,

$$\beta: \bigoplus_{l_2,k} H^1(D_k \subset C_k, \mathcal{O}) / \text{const} \rightarrow \bigoplus_{l_2,j} H^{1/2}(\gamma_j) / \text{const},$$

here each summand $H^{1/2}(\gamma_j) / \text{const}$ is equipped with the $+\text{-skewed}$ norm. Let $W = \bigoplus_{l_2,j} H^{1/2}(\gamma_j) / \text{const}$, let $\beta_{E,an}$ be the restriction of $\beta$ to $\bigoplus_{l_2,k} \mathcal{H}_E(D_k \subset C_k, \mathcal{O}) / \text{const}$, $W_{E,an} = \text{Im} \beta_{E,an}$.

The vector space $W$ is naturally decomposed into a sum of vector subspaces $W_\pm$, as in Section 2.3. If the distortions $\Delta(E_k)$ of the excess spaces are bounded, the component $\beta_-$ of $\beta_{an}$ corresponding to this decomposition is invertible; thus $\beta$ and $\beta_{an}$ are monomorphisms; in particular, $W_{E,an}$ is a closed subspace. Consider the vector subspace $W_{\varphi\psi V} \subset W$ consisting of functions compatible with the gluing conditions up to elements of allowance spaces. $W_{\varphi\psi V}$ is always closed as a direct sum of closed subspaces.

Note that a component $H^{1/2}(\gamma_j) / V_j$ of the target space of the mismatch operator $\mu$ from Section 3.4 can be identified with the quotient of $H^{1/2}(\gamma_j) \oplus H^{1/2}(\gamma_{j'})$ by $W_{\varphi\psi V}$, which is $V_j \oplus V_j$ summed with the graph of the operator $f \mapsto \psi_j \cdot \varphi_{j'}^* f$. This defines a new Hilbert norm on $H^{1/2}(\gamma_j) / V_j$. In what follows we are going to use this norm when we consider the $l_2$-sum $\bigoplus_{l_2,(j,j')} H^{1/2}(\gamma_j) / V_j$. Now this sum can be identified with $W/W_{\varphi\psi V}$.

Remark 4.7. Another reason to introduce the new norm is dictated by Theorem 4.10 (see also the discussion which follows the theorem). Note that if all the distortions $\Delta(E, E', \gamma, \gamma', \varphi, \psi)$ (defined in Section 4.1) are bounded, then this norm is equivalent to the old one.

Theorem 4.8. Suppose that all the spaces $V_j$ but a finite number are spanned by constants, and the distortions $\Delta(E_k)$ of the excess spaces are bounded. Then the following conditions are equivalent:

1. the operator $\bar{\partial}: H^1_{V'}(C, \mathcal{L}) \rightarrow H^0_E(C, \mathcal{L} \otimes \overline{\omega})$ is Fredholm of index $d$;
2. the mismatch operator $\mu: \bigoplus_{l_2,k} \mathcal{H}_E(D_k \subset C_k, \mathcal{O}) / \text{const} \rightarrow \bigoplus_{l_2,(j,j')} H^{1/2}(\gamma_j) / V_j$ is Fredholm of index $d$;
3. the vector subspaces $W_{E,an} \subset W$ and $W_{V,\varphi\psi} \subset W$ defined above are quasi-complementary of excess $d$. 
Proof. We know that $W_{E, an}$ is identified with $\bigoplus_{l_2, k} \mathcal{H}_E^1 (D_k \subset C_k, \mathcal{O}) / \text{const}$. This identifies the operator $\mu$ with the projection $W_{E, an} \to W/W_{\varphi \psi V}$, which proves the equivalence of the last two conditions.

On the other hand, the operator

$$\bar{\partial}_Y : Y = \bigoplus_{l_2, k} H^1 (D_k \subset C_k, \mathcal{O}) / \text{const} \to \bigoplus_{l_2} H^0 (D_k \subset C_k, \mathcal{O}) / E_k = H^0_E (C, \mathcal{L} \otimes \bar{\omega})$$

is an epimorphism. This the information about the operator $\bar{\partial} : H^1_{[\nu \cdot]} (C, \mathcal{L}) \to H^0_E (C, \mathcal{L} \otimes \bar{\omega})$ is encoded in the relative position of the subspaces $H^1_{[\nu \cdot]} (C, \mathcal{L})$ and $\ker \bar{\partial}_Y$. Note that $\ker \bar{\partial}_Y = \bigoplus_{l_2, k} \mathcal{H}_E^1 (D_k \subset C_k, \mathcal{O}) / \text{const}$. Using $\beta_-$, the intersection of these subspaces is identified with $W_{E, an} \cap W_{\varphi \psi V}$. Since $H^1 (C, \mathcal{L}) \subset Y$ coincides with $\beta^{-1} W_{\varphi \psi V}$, the sum of $H^1 (C, \mathcal{L})$ and $\ker \bar{\partial}_Y$ is closed if $W_{\varphi \psi V} + \beta (\ker \bar{\partial}_Y) = W_{\varphi \psi V} + W_{E, an}$ is closed; moreover, the corresponding codimensions coincide.

Consider the case when $V_j$ is spanned by 1 and $\psi_j$. If $\psi_j \equiv \text{const}$, $\dim V_j = 1$, otherwise $\dim V_j = 2$. In the notations above $\beta$ identifies the vector space of global analytic sections of $\mathcal{L}$ modulo $(V_j)$ with $W_{E, an} \cap W_{\varphi \psi V}$. Since the mismatch vanishes exactly on collections of functions compatible with the gluing data, the vector space of possible mismatches modulo $V_j$ of collections of functions from $\bigoplus_{l_2, k} H^1 (D_k \subset C_k, \mathcal{O}) / \text{const}$ may be identified with $W/W_{\varphi \psi V}$.

Use the notations of Section 3.4. By arguments of Section 3.4, the finite genus Riemann–Roch theorem may be rewritten in the following way: the vector subspaces $W_{E, an}$ and $W_{\varphi \psi V}$ are quasi-complementary of excess $d + \sum_{(j, j')} (\dim V_j - 1)$ (ind $\psi_j + \dim V_j - 1$); here $V_j$ is spanned by 1 and $\psi_j$.

**Definition 4.9.** Say that curve and bundle gluing data $(D_k \subset C_k, \gamma_j, \varphi_j, \psi_j)_{k \in K, j \in J}$ with excess spaces $(E_k)$ satisfy Riemann–Roch theorem if $\psi_j \equiv \text{const}$ for all but a finite number of $j \in J$, the distortions $\Delta (E_k)$ are uniformly bounded, and the conditions of Theorem 4.8 are satisfied with $d = \sum_{(j, j')} \text{(ind} \psi_j + \dim V_j - 1)$; here $V_j$ is spanned by 1 and $\psi_j$.

4.4. Riemann–Roch theorem and operators $\mathcal{C}$ and $\mathcal{R}$. Now, when we formulated the requirements of Riemann–Roch theorem, it makes sense to investigate when these conditions hold. It is the last alternative of Theorem 4.8 that we are going to check.

Recall that the vector space $W$ is naturally decomposed into a sum of vector subspaces $W_{\pm}$. Since $\beta_-$ is an isomorphism provided the distortions $\Delta (E_k)$ are bounded, $W_{an}$ is a graph of continuous mapping $\mathcal{C} = \beta_+ \circ \beta_- : W_- \to W_+$. By Lemma 4.2, if the excess spaces are compatible with the gluing mappings,$^{13}$ $W_{\varphi \psi V}$ is comparable with a graph of a closed mapping $\mathcal{R} : W_+ \to W_-$, with the relative

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$^{13}$In fact, it is enough if the excess spaces are compatible with all but a finite number of gluing mappings. We will not repeat this remark in what follows.
Moreover, note that for a bounded operator $A$ that this statement is not invertible, but is very close to be such (see Remark 6.9). Theorem until the Appendix (Section 6). What is important for us now is the fact that the corresponding curves $C_{j,k'}$, $\gamma_{j,k'}$, $\varphi_j$, $\psi_{j,j'}$ are uniformly bounded, the operator $R$ is continuous.

**Theorem 4.10** (abstract Riemann–Roch theorem). Consider two vector subspaces $V_{1,2} \subset H$ of a Hilbert space $H = H_1 \oplus H_2$. Suppose that $V_1$ is comparable with the graph of a closed mapping $A_1 : H_1 \to H_2$ and the relative dimension of $V_1$ and this graph is $d_1$. Suppose $V_2$ is comparable with the graph of a bounded mapping $A_2 : H_2 \to H_1$ and the relative dimension of $V_2$ and this graph is $d_2$. If $A_1 \circ A_2$ is defined everywhere, and either $A_1 \circ A_2$ is compact, or $A_2 \circ A_1 |_{\text{Dom } A_1}$ is compact, then $V_1$ and $V_2$ are quasi-complementary in the space $\text{Dom}(A_1) \oplus H_2$ with the excess $d_1 + d_2$.

We postpone the discussion of this (more or less trivial) generalization of Fredholm theorem until the Appendix (Section 6). What is important for us now is the fact that this statement is not invertible, but is very close to be such (see Remark 6.9). Moreover, note that for a bounded operator $A_1$ this is just a reformulation of Fredholm theorem. Until Section 5.4, we concentrate our attention on the cases when $A_1$ is bounded. (If $A_1$ is not bounded, then $\text{Dom}(A_1)$ is considered as a Hilbert space, the norm being $\|v\|_{A_1}^2 \equiv \|v\|^2 + \|Av\|^2$.)

Now, to show that a Riemann–Roch theorem holds, it is enough to investigate the compositions $R \circ C$ and $C \circ R$ for compactness. Consider the operators $R$ and $C$ separately.

### 4.5. Properties of the operator $R$

From now on, we suppose that the excess spaces are compatible with the gluing data, as defined in Section 4.1. In such a case the operator $R$ is defined up to addition of a finite-dimensional operator.

With respect to $j$-decomposition, the operator $R$ has a very simple block structure: with a suitable numeration of blocks (separately in $W_+$ and $W_-$) and merging of 2 blocks corresponding to $j$ and $j'$, the operator $R$ becomes block-diagonal. Each block of this operator depends only on $\gamma_j$, $\gamma_{j'}$ (considered with inclusions into the corresponding curves $C_k$, $C_{k'}$) and $\varphi_j$, $\psi_j$ for one pair $(j, j')$ of matching indices. By Lemma 4.2, each block is bounded. Recall that the norm of this block is denoted by $\Delta(\gamma, \varphi, \psi)$.

In this section we are interested in describing when the operator $R$ is bounded. Since considering this condition one can drop an arbitrary finite collection of blocks of $R$, it is enough to consider the case when $\psi_j \equiv \text{const}$.

It is clear that $\Delta(\gamma, \varphi, \psi) \leq \max(\psi, \psi^{-1}) \Delta(\gamma, \varphi, 1)$. Thus if all the constants $\psi_j$ are bounded, it is enough to consider the case when $\psi_j = 1$ for any $j$. There is one case when it is easy to estimate $\Delta(\gamma, \varphi, 1)$: when for any $j$ the identification $\varphi_j$ of curves $\gamma_j$ and $\gamma_{j'}$ can be extended to a conformal mapping of the corresponding complex curves $C_k$ and $C_{k'}$, both of genus 0. Obviously, in such a case the corresponding block of the operator $R$ can be written as a composition of two operators: the inclusion $t_{j,k}^{skew}$ of $H^{1/2}(\gamma_j) / \text{const}$ (with the $+$-skewed norm) into $H^{1/2}(\gamma_j) / \text{const}$ (with non-skewed...
norm); and the unitary identification \( \varphi_j^* \) of \( H_{1/2} (\gamma_j) / \text{const} \) and \( H_{1/2} (\gamma'_j) / \text{const} \) (with non-skewed norms) via \( \varphi_j \).

By Remark 2.20, the operators \( \iota_j^{\text{skew}} \) are proportional to unitary operators if all the curves \( \gamma_j \) are circles. In this case, \( R \) is bounded iff \( |\psi_j| \) are uniformly bounded.

**Definition 4.11.** Curve gluing data is **Schottky** if all diffeomorphisms \( \varphi_j \) may be extended to a conformal transformation from \( C_k \) to \( C_{k'} \); here \( \gamma_j \subset C_k, \gamma'_j \subset C_{k'} \).

Curve gluing data is **circular** if all the curves \( C_k \) are isomorphic to \( \mathbb{CP}^1 \) and all the curves \( \gamma_j \) are circles.

Give modifications of these notions more suitable to our situation:

**Definition 4.12.** Call the norm of the operator \( \varphi_j^* \) the **distortion** \( \Delta (\varphi_j) \) of the diffeomorphism \( \varphi_j \). Curve gluing data is **quasi-Schottky** if the set \( \{ \Delta (\varphi_j) \mid j \in J \} \) is bounded. Curve gluing data is **quasi-circular** if the set \( \{ \Delta (\gamma_j) \mid j \in J \} \) is bounded.

The bundle gluing data is **bounded** if all the functions \( \psi_j \) are uniformly bounded.

We conclude that the operator \( R \) is always a direct sum of bounded operators, thus is a closed operator. Moreover, for a bounded bundle gluing data for quasi-circular quasi-Schottky curve gluing data this operator is a bounded invertible operator. In such a case, the question of compactness of \( R \circ C \) and \( C \circ R \) is reduced to the question of compactness of \( C \).

One expects that curve gluing data being quasi-Schottky (or even Schottky) is not a very significant restriction: any “reasonable” curve should have such a “representation”. Later we show that quasi-circularity condition may be circumvented. On the other hand, unbounded bundle gluing data comes very naturally in the theory of divisors; we postpone discussion of such gluing data until Section 5.4.

**4.6. Strong Riemann–Roch conditions.** By definition, \( C = \beta_+ \circ \beta_-^{-1} \). The operator \( \beta_- \) is unitary if all the curves \( C_k \) are of genus 0; if \( K = \{ 1 \} \) (so there is only one curve \( C_1 \)) then one can bound the operators \( \beta_- \) and \( \beta_-^{-1} \) by a constant depending only on the distortion \( \Delta (E) \) of the excess spaces \( E \) for the curve \( C_1 \).

**Definition 4.13.** A collection of compact curves \( C_k, k \in K \), with excess spaces \( E_k \) is **tame** if the set \( \{ \Delta (E_k) \mid k \in K \} \) is bounded.

Since the operator \( \beta_+ \) is always a contraction (due to our choice of the +-skewed norm), we obtain:

**Proposition 4.14.** If the collection \( (C_k) \) is tame, then the operator \( C \) is continuous.

**Corollary 4.15.** Suppose the curves \( (C_k) \) form a tame quasi-Schottky quasi-circular collection, and the bundle gluing data is bounded. Then \( R \) and \( C \) are continuous operators, and the curve-bundle gluing data satisfies Riemann–Roch theorem iff \( RC - 1 \) is a Fredholm operator of index 0. In particular, this holds if 1 is not in the essential spectrum of the operator \( R \circ C \).
The property of 1 not being in the essential spectrum looks very fragile w.r.t. changes in the curve and bundle data. However, to construct the Jacobian of a curve, it is useful to know that all the line bundles which correspond to points of the Jacobian satisfy Riemann–Roch theorem. Remark 6.9 suggests another, much more robust criterion:

**Definition 4.16.** Suppose that excess spaces of the curve-bundle data have uniformly bounded distortions. Say that the curve-bundle gluing data satisfies the strong Riemann–Roch theorem if $R \circ C$ is defined everywhere, and is a compact operator.

**Theorem 4.17.** Suppose that the curve-bundle gluing data satisfies the strong Riemann–Roch theorem. Consider another curve-bundle gluing data which differs from the initial data only by replacing functions $\psi_j$ by functions $\psi'_j$. If functions $\psi'_j/\psi_j$ are uniformly bounded, then the modified curve-bundle data also satisfies the strong Riemann–Roch theorem.

*Proof.* As Section 4.5 shows, the modified operator $R'$ differs from $R$ by a multiplication by a bounded operator.

Note that the last part of Remark 6.9 cannot be applied to reverse Theorem 4.17. Indeed, the operators $R$ have a very special form only. Moreover, one can construct counter-examples (see Section 5.7) when $R$ and $C$ are bounded, the composition $R \circ C$ is not compact, but has the essential spectrum $\{0\}$ (e.g., $(R \circ C)^2$ may be compact). However, all the counter-examples we know are not stable w.r.t. appropriate modifications of curve gluing data: if two components $C_k, C_{\tilde{k}}$ are glued together by identification of $\gamma_j \subset C_k, \gamma_j' \subset C_{\tilde{k}}$, we can replace this pair by one curve $C^*$ which is result of this identification; similarly, one can break a component into two, or chain several such operations.

It is natural to conjecture that the “interesting” curves, those which correspond to points of moduli spaces lying in the support of “interesting” measures, can be described by gluing data satisfying the strong Riemann–Roch theorem. In what follows we consider such gluing data only.

### 4.7. Operator $C$ and bar-projectors.

Obviously, the operator $C$ depends only on the domains $D_k \subset C_k$ (and the corresponding excess spaces), but not on the gluing data; moreover, it breaks into a direct sum of the corresponding operators for the individual curves $C_k$. Restrict our attention to the operator $C$ for one such curve $C$; in other words, we may assume that $K$ contains one element 1 only, and $C_1 = C$.

Consider now the finer block structure of the operator $C$ due to the decomposition of $W_\pm$ into a direct sum over indices $j$ of curves $\gamma_j$. Let $C_{j\ell}$ be the component sending $H^{1/2}_-(\gamma_\ell) / \text{const}$ into $H^{1/2}_+(\gamma_j) / \text{const}$. It is clear that $C_{jj} = 0$.

From now on assume that the curves $\gamma_j$ do not intersect. In particular, the sets $D_k$ have no open subsets of dimension 1. Under this condition we can find convenient bounds for the blocks $C_{jk}, j \neq k$. 
First of all, recall that a choice of the excess space $E$ on $C$ allows one to construct the operator $\partial^{-1}_E$ mapping global sections of $\bar{\omega}$ to global sections of $\mathcal{O}$ modulo constant. Thus the operator $\partial \circ \partial^{-1}_E$ maps sections of $\bar{\omega}$ to sections of $\omega$.

**Definition 4.18.** Consider a complex compact curve $C$ with an excess space $E$, and two non-intersecting open subsets $R, R' \subset C$. Define the bar-projector $\mathcal{P}_{RR'} : H^0 (R', \bar{\omega}) \to H^0 (R, \omega)$ as the composition $\rho_R \circ \partial \circ \partial^{-1}_E \circ \varepsilon_R$; here $\varepsilon_R f$ is the extension of $f$ by 0 from $R'$, and $\rho_R g$ is the restriction of $g$ to $R$.

Let $D' = C \setminus R'$. Obviously, the bar-projector vanishes on $\partial \bar{H}^1 (R')$; call the induced operator from $H^0 (R', \bar{\omega}) / \partial \bar{H}^1 (R') = \mathcal{R}_{D' \subset C; E} = \mathcal{H}_E^1 (D') / \text{const}$ by the same term. When acting from $\mathcal{H}_E^1 (D') / \text{const}$, the bar-projector is identified with $\rho_R \circ \partial$, or, if $E$ is $C \setminus R$-supported, with $\rho_R \circ d_{\text{deRham}}$. On the other hand, $d_{\text{deRham}}$ is a unitary operator from $H^1_{\text{int}} (R)$ to its image in $L^2 (R, \Omega^1)$. In other words, the bar-projector may be also modeled as the restriction operator $\mathcal{H}_E^1 (D') / \text{const} \to H^1_{\text{int}} (R) / \text{const}$.

The arguments above show that $\mathcal{C}_{jk}$ differs from the bar-projector from $R_i$ to $R_j$ only by appropriate approximately-unitary transformation of the image and the preimage. However, the bar-projector is explicitly written as an off-diagonal block of a pseudo-differential operator $\partial \circ \partial^{-1}$; in particular, the bar-projector is an operator with a smooth Schwartz kernel, so is compact. We obtain

**Proposition 4.19.** The block $\mathcal{C}_{jk}$ is compact, and can be written as $\iota_1 \circ \mathcal{P}_{R_i R_j} \circ \iota_2$; here $\iota_{1,2}$ are invertible operators, and $\|\iota_{1,2}\|, \|\iota_{1,2}^{-1}\| \leq \Delta (E)$.

4.8. Conformal distance and estimates of bar-projectors.

**Definition 4.20.** Consider two non-intersecting regions $R, R'$ in a complex curve $C$ such that $C \setminus (R \cup R')$ is conformally equivalent to a cylinder $S^1 \times [0, \lambda]$ of radius 1 and length $\lambda \geq 0$. If $g (C) > 0$, suppose that the support of the excess space $E$ is in $R \cup R'$. Call $\lambda$ the conformal distance between $R$ and $R'$.

For general non-intersecting regions $R, R'$ the conformal distance is at least $\lambda$ if after increasing one of the regions the condition above holds.

The specific form of the last part of the definition is chosen to simplify the proof of Proposition 4.22, while allowing the construction of Section 5.3 to remain simple.

**Lemma 4.21.** Suppose $C = \mathbb{CP}^1$, and both $R$ and $R'$ are disks on $C$ with conformal distance $\lambda$. Then the norm of the bar-projector from $R'$ to $R$ is equal to $e^{-\lambda}$.

**Proof.** Since $E = 0$, the bar-projector is canonically defined. Due to its conformal invariance, we may suppose $R$ is given by $|z| < e^{-\lambda}$, $R'$ is given by $|z| > 1$. Due to rotation-invariance, we may consider each irreducible component of the action of $U (1)$ on $\mathcal{H}^1 (D')$, $D' = C \setminus R'$, one-by-one. Clearly, $d (z^k) \mid_R$ has the squared $L^2$-norm $\pi k e^{-2k\lambda}$. The continuation $f$ of $z^k \mid_{\partial R'}$ into $R'$ with the minimal $L^2$-norm of $\partial f$ is $z^{-k}$; its squared norm on $R'$ is $\pi k$. \[\square\]
Since the conformal distance between two disks in $\mathbb{C}$ of radii $r_{1,2}$ and the distance between centers $d$ is $\text{ch}^{-1} \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$, the norm of the bar-projector is comparable with $\frac{r_1r_2}{d(d-r_1-r_2)}$.

**Proposition 4.22.** There is a constant $c \geq 1$ such that if the conformal distance between simply-connected domains $R, R' \subset C$ is at least $\lambda$, then the norm of the bar-projector from $R'$ to $R$ is less or equal to $c \Delta (E) e^{-\lambda}$; here we suppose that the excess space $E$ for $C$ is $C \setminus (R \cup R')$-supported.

**Proof.** The operator $\bar{\partial} \bar{E}^{-1}$ is bounded by $\Delta (E)$. Since the bar-projector is a block of this operator, its norm is bounded by the norm of $\partial \bar{E}^{-1}$. Thus we may assume that $\lambda \geq 2$. The norm of the bar-projector is monotonic w.r.t. the domains, increase one of the domains so that the the first part of the definition of conformal distance holds. After this, one of the regions $R$ or $R'$ contains the support of $E$. At first, suppose that this domain is $R'$.

Our aim is to reduce the statement to one of Lemma 4.21. The obstacles are the presence of the excess space $E$ between centers $d$ that $\text{const}$ is $O (d)$ between centers $d$. Let $\tilde{\partial} \tilde{E}^{-1}$ be a cut-off function which is $0$ on $R \cup R'$, glue the disk $\{|z| \geq 1\}$ to $C \setminus R'$ via this identification; denote the resulting curve $C_{00}$. Show that one may replace $C$ by $C_{00}$.

Let $\tilde{R}'$ be the image of the annulus $\{e^{-1} \leq |z| \leq 1\}$ in $C$, $\tilde{R}_{00}'$ be the image of this annulus in $C_{00}$.

**Lemma 4.23.** Given $\alpha \in H^0 (R', \bar{\omega})$, one can find $\tilde{\alpha} \in H^0 \left( \tilde{R}', \bar{\omega} \right)$ such that $\tilde{\partial}^{-1} (\alpha - \tilde{\alpha})$ is constant on $R$, $\tilde{\alpha} \in \text{Im} \tilde{\partial}$ and $\tilde{\partial}^{-1} \tilde{\alpha}$ is constant on $R'$, and $\|\tilde{\alpha}\| = O (\Delta (E) \|\alpha\|)$.

**Proof.** Find a representative $f \in H^1 (C)$ of $\partial^{-1} \alpha \in H^1 (C) / \text{const}$. Normalize $f$ by requiring $\int_{\tilde{R}'} f \, dz \, d\bar{z} = 0$. Let $\Psi'$ be a cut-off function which is $0$ on $R'$, $1$ outside of $R' \cup \tilde{R}'$. Then $\Psi' f$ is holomorphic outside of $\tilde{R}'$. Since the norm of $f$ in $H^1 (C) / \text{const}$ is $O (\Delta (E) \|\alpha\|)$, to show a similar estimate for $\Psi' f$, it is enough to estimate $\|f\|_{L^2 (\tilde{R}')}$. This estimate follows from the estimate of $\|df\|_{L^2}$. Now take $\tilde{\alpha} = \tilde{\partial} (\Psi' f)$.

Now replace $R'$ by $R' \cup \tilde{R}'$, and $\alpha$ by $\tilde{\alpha}$. This decreases $\lambda$ by $1$; thus it is enough to show that the norm of the bar-projector reduced to $\text{Im} \tilde{\partial}$ can be estimated as $O (e^{-\lambda})$ (no factor $\Delta (E)$). Moreover, since $\text{Supp} \tilde{\alpha} \subset A$, one can identify $\tilde{\alpha}$ with a $1$-form on $C_{00}$; since $\tilde{\partial}^{-1} \tilde{\alpha}$ is constant on $R'$, one can transfer $\tilde{\partial}^{-1} \tilde{\alpha}$ to $C_{00}$ as well. Thus we may assume $C = C_{00}$, $\alpha = \tilde{\alpha}$; since $g (C_{00}) = 0$, there is no $E$ to care about.
Glue the disk \( \{|z| \leq e^{-\lambda}\} \) to \( C \setminus R \) via the identification of \( A \) with \( \{e^{-\lambda} \leq |z| \leq 1\} \); denote the resulting curve \( C_0 \). Let \( R_0 \subset C_0 \) be the image of the disk \( \{ |z| < e^{-\lambda} \} \), \( \tilde{R} \subset C \) and \( \tilde{R}_0 \subset C_0 \) be the images of the annulus \( \{ e^{-\lambda} \leq |z| \leq e^{-\lambda+1} \} \). By Lemma 4.21, the norm of \( \partial_{C_0} \bar{\partial}_{C_0}^{-1} \) on \( R_0 \cup \tilde{R}_0 \) is \( O (e^{-\lambda} \| \alpha \|) \). Using again the trick with a cut-off function, we can find \( \tilde{\alpha} \in \tilde{H}^0 (\tilde{R}_0, \tilde{\omega}) \) such that \( \tilde{\partial}^{-1} (\alpha - \tilde{\alpha}) \) is constant on \( R_0 \), and \( \| \tilde{\alpha} \| = O (e^{-\lambda} \| \alpha \|) \). Again, since \( \tilde{\partial}^{-1} (\alpha - \tilde{\alpha}) \) is constant on \( R_0 \), we can transfer this function to \( C \). Denote the resulting function \( f \); transfer similarly \( \tilde{\alpha} \) to \( C \). Then \( \tilde{\partial}^{-1} \tilde{\alpha} \) differs from \( \partial^{-1} \alpha \) by a constant only; moreover, \( \| \tilde{\partial}^{-1} \tilde{\alpha} \|_{\partial^{1/\text{const}}} = O (e^{-\lambda} \| \alpha \|) \).

What remains to prove is the other case, when \( \text{Supp} \ E \subset R \). Construct \( C_0 \) as above, replacing \( R \) by a disk \( R_0 \); then \( g(C_0) = 0 \). Given \( \alpha \) as above, apply the already proved case \( g = 0 \); there are a function \( f \) on \( C_0 \) and a 1-form \( \tilde{\alpha} \) with support on \( \tilde{R}_0 \) such that \( f|_{R_0} \) is constant, \( \tilde{f} = \alpha - \alpha_0 \), and \( \| \tilde{\alpha} \| = O (e^{-\lambda}) \). One can transfer \( f \) and \( \tilde{\alpha} \) to \( C \); since one can replace \( \alpha \) by \( \tilde{\alpha} \) and \( \lambda \) by \( 0 \), the estimate \( O (\Delta (E)) \) for \( \partial \partial^{-1} E \) we already used finishes the proof. 

This proof in fact implies a much stronger result:

**Amplification 4.24.** The same estimate holds if one considers the bar-projector acting into the space \( H^1 (R \subset C) \) instead of \( H^1_{\text{int}} (R) \).

**Remark 4.25.** In fact the norm of the bar-projector may be much smaller than what is given by Proposition 4.22. Assume that \( C = \mathbb{CP}^1 \), \( R' = \{|z| > 1\} \). Suppose that \( R \) sits inside the disk \( \{|z| < \rho\} \), \( \rho < 1 \). Since we know the (smooth) kernel \( \frac{c \, dz \, d\bar{z}'}{(z - z')^2} \) of the bar-projector, it is easy to calculate the Hilbert–Schmidt norm of this operator, thus estimate its norm. In fact, due to the conformal invariance of the bar-projector, the square of its Hilbert–Schmidt norm is proportional to the area of \( R \) in the hyperbolic metric of the disk \( \{|z| < 1\} \), thus the bar-projector is bounded by \( O \left( \sqrt{|R|}/(1 - \rho^2) \right) \); here \( |R| \) is the Euclidean area of \( R \).

This argument shows that the norm of the bar-projector can be made arbitrary small even under the requirement that \( R \) contains the interval \( [-\varepsilon, \varepsilon] \), \( \varepsilon < 1 \). However, the conformal distance between this interval and \( R' \) is finite.

Using the test-function \( z \) in \( \{|z| \leq 1\} \), one can show that the norm of the bar-projector is bounded from below by \( \sqrt{|R|} \).

**Remark 4.26.** Note that if \( g(C) = 0 \), the norm of the bar-projector from \( R' \) to \( R \) is equal to the one from \( R \) to \( R' \). Indeed, let \( X = \partial \partial^{-1} \). Then \( X \) manifestly satisfies \( X = X^{-1} \); moreover, \( X \) is unitary due to Remark 1.4. Thus the operator \( X \) equals its transposed. Since the bar-projector is a block of the operator \( X \), and the blocks which correspond to projectors from \( R' \) to \( R \) and from \( R \) to \( R' \) are transposed, this implies the result.
Remark 4.27. Sometimes it is possible to calculate the norm of the bar-projector explicitly. E.g., let \( C \) be the disk \( \{|z| \leq 1\} \) with \( z \) identified with \( z^{-1} \) for any point \( z \) of the boundary; \( C \) is a rational curve. Let \( R_+ = \{|z| < a\} \), \( R_- \) be the image of \( \{b < |z| \leq 1\} \) on \( C \); here \( a < b < 1 \).

Suppose that \( R \subset C \) has a smooth boundary. Given a function \( f \) in \( H^1 (C \setminus R) \), let \( \hat{f} \) be the harmonic extension of \( f|_{\partial R} \) into \( R \). It is clear that \( \hat{f} \overset{\text{def}}{=} \overline{\partial f} \) is the representative of \( f \) in \( H^0 (R, \overline{\omega}) \) with the minimal norm. Take \( R = R_+ \), \( f_n = -(b^{-4n} - 1) z^n \); since harmonic functions in \( R_- \) are linear combinations of \( z^l + z^{-l} \), \( \overline{z}^l + \overline{z}^{-l} \), we conclude that \( \hat{f}_n = (z^n + z^{-n}) - b^{-2n} (\overline{z}^n + \overline{z}^{-n}) \), thus \( \hat{f}_n = -nb^{-2n} (\overline{z}^n - \overline{z}^{-n}) d\overline{z} / \overline{z} \).

If \( n \neq n' \), \( f_n \) and \( f_{n'} \) are orthogonal in \( H^1 (C \setminus R_-) \) (since \( \hat{f}_n, \hat{f}_{n'} \) are orthogonal in \( H^0 (R_-, \overline{\omega}) \)). Since \( R_+ \) is rotation-invariant, thus \( \partial f_n \) and \( \partial f_{n'} \) are orthogonal in \( H^0 (R_+, \omega) \), one may consider the action of the projector on functions \( f_n \) one-by-one.

Since
\[
\|\hat{f}_n\| = 2\pi n^2 b^{-4n} \int_0^1 (r^{2n} + r^{-2n}) dr / r, \quad \|\partial f_n\| = 2\pi n^2 (b^{-4n} - 1)^2 \int_0^a r^{2n} dr / r,
\]
this component of the projector has the square of the norm \( (1 - b^{4n}) e^{-2n\lambda} \); here \( \lambda \) is the conformal distance \( \log (b/a) \) between \( R_+ \) and \( R_- \). Thus the norm of the projector from \( R_- \) to \( R_+ \) is \( \sqrt{1 - b^4} e^{-\lambda} \). This suggests that the constant \( c \) of Proposition 4.22 may be 1.

4.9. Block matrices.

Definition 4.28. Consider Hilbert direct sums \( V = \bigoplus_{l_2} V_l \), \( V' = \bigoplus_{l_2} V'_k \) of Hilbert spaces, and an operator \( A : V \to V' \). It induces operators \( A_{kl} : V_l \to V'_k \); call these operators the blocks of \( A \).

Lemma 4.29. Use the notations of the definition above.

1. if the matrix \( (\|A_{kl}\|) \) corresponds to a bounded operator \( l_2 \to l_2 \), then the operator \( A \) is bounded.
2. Suppose that all the blocks of \( A \) are compact operators. If the matrix \( (\|A_{kl}\|) \)
   corresponds to a compact operator \( l_2 \to l_2 \), then the operator \( A \) is compact.

Proof. If the matrix \( (\|A_{kl}\|) \) corresponds to a continuous operator \( l_2 \to l_2 \) with a norm \( \leq M \), then \( \|A\| \leq M \). Indeed, otherwise one could find \( v \in V \), \( v' \in V' \) with \( |v| = |v'| = 1 \), \( |(Av, v')| > M \). Let \( v = (v_l), v' = (v'_k) \); then \( \tilde{v}_l = |v_l|, \tilde{v}'_k = |v'_k| \) are in \( l_2 \), and \( |\tilde{v}_l| = |\tilde{v}'_k| = 1 \). Now \( |(Av, v')| = \sum_{kl} \|A_{kl}\| \tilde{v}_l \tilde{v}'_k \) leads to a contradiction.

If \( (\|A_{kl}\|) \) gives a compact operator, it may be approximated with arbitrary precision by replacing all but a finite number of entries by 0. Thus \( A \) may be approximated by replacing all but a finite number of blocks by 0 blocks. What remains is a finite sum of compact operators, thus compact. Since \( A \) may be approximated by compact operators, \( A \) is compact itself.
In the other direction, if $A$ is compact, so are its blocks. To show that the operator $l_2 \to l_2$ given by the matrix $(\|A_{kl}\|)$ can be approximated by finite-dimensional operators, approximate $A$ by a finite-dimensional operator; write down this operator as a sum of one-dimensional operators $\sum_n w_n' \otimes w_n$, where $w' \otimes w$ sends $v$ into $(v, w) w'$. One can approximate each of $w_n, w_n'$ by a finite sum of vectors in $V, V'$; hence $A$ may be approximated by an operator with only a finite number of non-0 blocks; consequently, $A$ may be approximated by a finite sum $\hat{A}$ of its blocks. Then $\|A - \hat{A}\|$ can be made arbitrarily small, thus $|\langle (A - \hat{A}) \sum \varepsilon_l v_l, \sum \varepsilon'_k v'_k \rangle|$ can be made arbitrarily small for any $(\varepsilon_l), (\varepsilon'_k)$ with $|\langle \varepsilon_l \rangle|_{l_2} = |\langle \varepsilon'_k \rangle|_{l_2} = 1$.

4.10. Practical criteria for Riemann–Roch theorem. Consider the estimates for the norms of blocks of the operator $C$ to obtain easy-to-check criteria that a curve satisfies a strong Riemann–Roch theorem. Assume that the gluing data is bounded and quasi-Schottky quasi-circular, so that the operator $R$ is bounded and invertible. In such a case all we need to show is that the operator $C$ is compact.

Given $j, \tilde{j} \in J$, $e^{-\lambda_j \tilde{j}}$ is well-defined if $\gamma_j$ and $\gamma_{\tilde{j}}$ are on the same curve $C_k$; here $\lambda_{j \tilde{j}}$ is the conformal distance between $\gamma_j$ and $\gamma_{\tilde{j}}$. Put $e^{-\lambda_j \tilde{j}} \overset{\text{def}}{=} 0$ otherwise.

**Theorem 4.30.** Consider a bounded quasi-Schottky quasi-circular curve and bundle gluing data with uniformly bounded distortions $\Delta (E_k)$ of the excess spaces. Suppose that the excess spaces are compatible with identifications $\varphi_j$ of smooth boundary components, and that the smooth components of boundaries of subsets $D_k$ do not intersect. These data satisfies the strong Riemann–Roch condition if the operator $l_2 \to l_2$ defined by the matrix $(e^{-\lambda_j \tilde{j}})$ is bounded and compact.

**Proof.** By Proposition 4.22, the blocks $C_{j \tilde{j}}$ of the operator $C$ are compact and have norms $O(e^{-\lambda_j \tilde{j}})$. By Lemma 4.29, in the conditions of the theorem the operator $C$ is compact. Now Corollary 4.15 implies the theorem. \qed

**Corollary 4.31.** Consider a bounded quasi-Schottky quasi-circular curve and bundle gluing data. Suppose that the excess spaces are compatible with identifications $\varphi_j$ of smooth boundary components, and that the smooth components of boundaries of subsets $D_k$ do not intersect. These data satisfies the strong Riemann–Roch condition if $\sum_{m \neq j} e^{-2\lambda_{mj}} < \infty$.

**Proof.** The conditions ensure that the operator with the matrix $(e^{-\lambda_j \tilde{j}})$ is Hilbert–Schmidt, thus compact. \qed

Later, in Section 5.5, we will see that the restriction of quasi-circularity may be dropped.

5. Geometry of quasi-algebraic curves

5.1. Quasi-algebraic curves and sheaves. Curve gluing data defines what are the points of the curve. In this way we obtain a topological space. Bundle gluing
data defines what are fibers over the points, what are germs of the sections of the
bundle, and what are global sections of the bundle; this defines a sheaf. Call the
resulting sheaf on the resulting topological space a quasi-algebraic sheaf if the gluing
data satisfies the Riemann–Roch theorem.

Taking the gluing functions $\psi_j$ to be 1, we obtain the sheaf $\mathcal{O}$. Call the topological
space obtained from a curve gluing data a quasi-algebraic curve if the sheaf $\mathcal{O}$ satisfies
the strong form of the Riemann–Roch theorem.

5.2. **Ideal points.** Let us use the criterion of Section 4.10 to show how far can be a
curve from the classical picture of a complex algebraic curve without losing the nice
properties usually associated with algebraic curves. To simplify the exposition, we
consider only circular Schottky curve gluing data with bounded bundle gluing data;
we require that these data satisfy the strong form of the Riemann–Roch theorem;
such examples already exhibit most of the peculiarities of the general theory.

Restrict our attention yet more, to gluing data satisfying conditions of Corol-
mary 4.31; for such curves the matching decomposition and the gluing mappings play
no role (as far as they remain Schottky and bounded), all what matters are confor-
mal distances between the disks bounding the curve. Consequently, the question boils
down to the following one: how “bad” may be a collection of non-intersecting disks
on $\mathbb{CP}^1$ such that the complement satisfies the strong form of the Riemann–Roch
theorem when equipped with bounded Schottky gluing data.

A natural measure of “badness” is how big is the dust (accumulation points of the
disks). We say that a point $z$ is an accumulation point of a collection of subsets if
any neighborhood of $z$ intersects with an infinite number of these subsets.

For example, the first investigated case of quasi-algebraic curves was the the hyper-
elliptic case: the spectral curves for the KdV equation [11]. When translated to our
language, such a curve corresponds to the disks accumulating to points $0, \infty \in \mathbb{CP}^1$;
the disks have centers $c_n, n \in \mathbb{Z}; c_n \approx n^2, c_{-n} \approx n^{-2}$ (for $n \gg 0$); the radii of the disks
are rapidly decreasing. Many other examples of infinite-genus curves so abundant in
the theory of integrable systems share the same property of having a finite set as the
dust; the conditions on radii of the disks are much less drastic.

**Remark 5.1.** As we discussed it in the introduction (see also [23]), the dust corre-
sponds to ideal points of the curve: the theory of divisors on the curve should include
divisors at the points of dust (though these points are not included in the smooth
part of the curve).

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\[14\] Of course, the results discussed in this paper do not completely confirm similarity of these
curves with algebraic curves. However, the results of [23] suggest that most of others results of
algebraic geometry are applicable to these curves as well: that paper discusses duality for line
bundles (including the case of degree=genus), the description of Jacobians as quotients of vector
spaces by lattices, and conditions on periods of global 1-forms.
5.3. **Foam curves.** The natural measure of how far is the curve from the classical theory is the Hausdorff dimension of the dust: it shows how “heavy” are ideal points with respect to “usual” points of the curve are. (Note that any non-dust point of the curve is smooth after the gluing of the boundary circles together. Dust points are “very non-smooth”—in any imaginable way.) Note that for any collection of domains $D_j$, the accumulation points form a closed form of $\coprod_k C_k$. Moreover, this closed subset has no interior.

It turns out that the strong Riemann–Roch condition adds no restrictions on the dust:

**Theorem 5.2.** Consider a collection $(C_k)_{k \in K}$ of compact curves such that $g(C_k) \neq 0$ only for a finite number of indices $k$. For any closed subset $D$ of $\coprod_k C_k$ without interior there is curve-bundle gluing data which has $D$ as the dust and satisfies the strong form of the Riemann–Roch theorem.

**Proof.** To simplify arguments, consider the case of $K = \{1\}$ and $C_1 = C = \mathbb{CP}^1$. (The key idea needed for the general case is in Remark 5.4.) Construct a sequence of non-intersecting closed disks $R_j$ with centers in $c_j$ and radii $r_j$, $j \in \mathbb{N}$, one by one.

Choose a sequence $(a_n)$ of points in $C \setminus D$ which has $D$ as the set of accumulation points (possible since $D$ has no interior). Start with an empty collection of disks. Given $R_j$ for $j < j_0$, let $c_{j_0}$ be the first point of the sequence which is in $C \setminus \bigcup_{j < j_0} R_j$.

Let $\tilde{r} = \text{dist} (c_{j_0}, D \cup \bigcup_{j < j_0} R_j)$. Let $r_{j_0}$ be the maximal number $\leq \tilde{r}/2$ satisfying the conditions $\sum_{j < j_0} e^{-2\lambda_{j_0}} \leq 2^{-j_0}$; here $\lambda_{jk}$ is the conformal distance between $R_j$ and $R_k$, $\text{ch} \lambda_{jk} = \frac{(c_j - c_k)^2 - r_j^2 - r_k^2}{2r_j r_k}$. Such $r_{j_0}$ is unique, since each summand increases when $r_{j_0}$ increases.

It is clear that $\bigcup_{j \leq j_0} R_j$ does not intersect $D$, the disks do not intersect, and have no accumulation points outside $D$. Moreover, the collection of disks satisfies Corollary 4.31. Now any point $a_n$ is either a center of a disk from the collection, or is inside one of the disks $R_1, \ldots, R_{n-1}$. Thus any accumulation point of the points $a_n$ is either in one of the disks $R_j$, or an accumulation point of the disks $R_n$.

In other words, there is no restriction on the dust except that it is a set of accumulation points of non-intersecting open subsets. In particular, the dust may have a positive measure. Moreover, there may be nothing but the dust on the curve:

**Theorem 5.3.** There is a closed subset $D \subset C = \mathbb{CP}^1$ such that

1. $\mathbb{CP}^1 \setminus D$ is a union of non-intersecting disks;
2. $D$ coincides with the set of accumulation points of these disk;
3. $D \subset \mathbb{CP}^1$ satisfies the strong form of the Riemann–Roch theorem when equipped with an arbitrary bounded Schottky gluing data.

**Proof.** Put $D = \emptyset$; enumerate points $z = x + iy$ with rational $x, y$; let $(a_n)$ be the corresponding sequence. Construct a sequence of closed disks in $\mathbb{CP}^1$ basing on these $D$ and $(a_n)$ as in the proof of Theorem 5.2.
This sequence of non-intersecting disks contains all the points $x + iy$ with $x, y \in \mathbb{Q}$. Let $R_m$ be the interior of the $m$-th disk. Put $D$ to be the complement of $\bigcup R_m$. All we need to show is that any point $z \in D$ is an accumulation point of the disks $R_m$. Since the closed disks $\bar{R}_m$ contain all the rational points $a_m$, there is a sequence of points of $\bigcup R_m$ which goes to $z$; if $z \in \partial R_m$, then we can additionally require that no point of this sequence is in $R_m$. It is clear that one can choose a subsequence such that all its points are in different disks $R_m$.

Remark 5.4. In the case of a curve $C$ of arbitrary genus we need also to construct a subset $\varepsilon$ of positive measure to make it into $\text{Supp } E$. Use notations of Theorem 5.2. Construct $\varepsilon$ recursively: let $\varepsilon_0 = C$; choose $\varepsilon_{j_0}$ as $\varepsilon_{j_0-1} \setminus \bar{R}_{j_0}$. Here $\bar{R}_{j_0}$ is a centered at $c_{j_0}$ disk such that $\text{measure } (\varepsilon_{j_0}) / \text{measure } (\varepsilon_{j_0-1}) > 1 - 2^{-j_0}$. Now choose $R_{j_0}$ as a disk in $\bar{R}_{j_0}$ of a very small relative radius.

Remark 5.5. There is a more customary description of such “foam” curves via a “limit process” involving compact curves of finite genus. Take $C_0 = \mathbb{CP}^1$. Consider a metric on $C_0$, $\varepsilon_0 > 0$, and a finite subset such that its $\varepsilon_0$-neighborhood coincides with $C_0$. Remove from $C_0$ the collection of small disks with centers in this subset. Break this collection into pairs; for each pair consider two circles which are boundaries of this pair of removed disks. Glue in a long cylinder along these two circles; this way we add a handle per a pair of removed disks. Call the resulting curve $C_1$.

Now repeat the same process for $C_1$, choosing $\varepsilon_1 \ll \varepsilon_0$. Note that one needs to remove the disks not only from the common part of $C_0$ and $C_1$, but also from the added cylinders. Call the resulting curve $C_2$ etc.

One can show that if the lengths of added cylinders increase quickly enough, then the “limit” of the curves $C_k$ is a well-defined quasi-algebraic curve which enjoys all the benefits of the results of algebraic geometry.

Remark 5.6. As the example above shows, it is possible that a quasi-algebraic curve has no smooth point at all. Moreover, it may happen that no point of the curve has a neighborhood homeomorphic to a disk.

Keep in mind that our definition of topology on a quasi-algebraic curve (and of the set of points of a quasi-algebraic curve) has no deep underlying reason: they come from the construction of the set of the functions we allow. This somewhat undermines the observation of the preceding paragraph. On the other hand, it was shown in [23] that by strengthening conditions on the pairwise conformal distances, one may ensure that there is a well-defined power-series asymptotic formula near each point of the curve. As asymptotic formulae do, it works outside some “sparse” subset of the neighborhood of a point; the complement of this subset is locally homeomorphic to a subset of $\mathbb{C}$.

To make this description into a theorem, we need first to describe which gluing data describe “the same” quasi-algebraic curve, and define a topology on the resulting moduli space. These topics are outside of the scope of this paper.
Remark 5.7. One of the incentives to consider quasi-algebraic curves is that they might describe non-perturbative amplitudes of string propagation. Recall that the genus $g$ curves are responsible for $g$-loops Feynman diagrams for string propagations; this provides a measure on the finite-genus moduli spaces. There is a hope that these measures may be just “residues” of a measure on the moduli space of curves of arbitrary (including infinite) genus.

To investigate the details of this measure one needs much more advanced knowledge of the theory of quasi-algebraic curves. Even the question about the support of this measure is not clear at all. Consider the analogy with the Wiener measure on functions $f(x)$: “the density” of this measure $\exp \left( -\int_0^1 f'(x)^2 \, dx \right)$ is well defined for $f \in H^1([0,1])$; however, the support of this measure involves much more singular functions. Similarly, one cannot a priori expect that “foam” curves miss the support of the above measure. A natural path to investigate this measure is to consider as general theory as possible; if one of the corollaries of this theory is that interesting measures do not involve the foam curves, then one may want to consider simplifications of the theory which work with much easier “manageable” curves only.

Remark 5.8. Current approaches to quantum gravity suggest that consideration of a foam space-time may be necessary. On the other hand, strings have much less singular distribution of the mass (“along a curve” instead of “at a point”), which alleviates the need to drastically change the topology of space-time. It may be a poetic justice if the non-perturbative approach to string propagation leads indeed to consideration of “foam strings”, thus moving the dust out from under the carpet into a different place. However, this place promises to be much more convenient to deal with dust than considerations of “foam” space-time.

Remark 5.9. Recall why string theory has a chance to be easier to deal with comparing to the particle theory. Indeed, a particle concentrates its mass in an arbitrary small region of space. Such a concentration causes a cloud of virtual particle-antiparticle pairs around this point, as well as the singularity of the metric (which leads to foam models of space-time). On the other hand, a string concentrates vanishingly small mass in small regions of space, thus has a better chance to avoid the serious singularities.

Judging by the apparent failure of the initial rosy expectations of the string theory boom, concentrating the mass along space-dimension one is not enough to avoid singularities. However, if the “string measure on the infinite-genus moduli space” is wild enough to be concentrated on the set of foam curves, the corresponding strings in space are fractal. Being fractal, they may have a Hausdorff dimension larger than 1 (or of $1 + 0$ type), thus may cause milder singularities than the smooth curves.

5.4. Why closed operators? Up to this place we applied Theorem 4.10 only in the case of bounded bundle gluing data, when both the operators $A_{1,2}$ are bounded.

\[ \text{This is the reason for why we try to make every statement as general as possible.} \]
To deal with this case one does not need the machinery of Section 6, one could apply the usual Fredholm theorem. However, simple heuristics show that to have a good relation between divisors and line bundles the bounded gluing data is not enough.

Indeed, consider the effect of removing a point \( z_0 \) from a divisor on the corresponding bundle gluing data. We want the global sections \( f(z) \) of the new bundle correspond to sections of the old bundle which vanish at this point. Suppose that the curve is glued of a subset \( D \subset C \subset \mathbb{CP}^1 \), then \( f \) is a section of the new bundle if \( (z - z_0) f(z) \) is a section of the old bundle. Thus the new bundle gluing data \( \tilde{\psi} \) may be obtained from the old data \( \psi \) via \( \tilde{\psi}_j(z) = \frac{\phi_j(z)}{z - z_0} \psi_j(z) \).

Suppose that a curve has a very long handle; in the language of gluing data this corresponds to \( D \) containing an annulus with a very large conformal distance between two components of the boundary. The robustness argument from the introduction suggests that moving a point of a divisor along such a handle has a very little effect on the sections of the corresponding line bundle (at least at the part of \( D \) “far” from this annulus). On the other hand, the formula above shows that the effect of such a movement on \( \psi_j \) is very large if \( \varphi_j \) send a contour “inside” the annulus to one “outside” it.

To obtain a “model” of the given curve as a subset of \( \mathbb{CP}^1 \) with some gluings of boundaries, one needs to make a meridional cut across each “handle” of the curve. To apply the preceding argument, the handle we consider should have two marked subsets: a long tube which will become the annulus, and the loop we made the cut across. They should not intersect; consequently, two sides of the cut will form two curves, one inside the annulus, another outside. Moreover, all curves \( \gamma_j \) but one are either all inside or all outside of the annulus on \( \mathbb{CP}^1 \). In other words, there is going to be exactly one gluing function \( \psi_j \) which is strongly affected by the movement of the point inside the annulus.

Thus minor changes to a line bundle may correspond to a giant change of the gluing functions. For curves which have infinitely many handles which get progressively longer, one should expect that one can combine infinitely many such changes (each corresponding to moving a point of the divisor inside one handle) so that the change to a line bundle is still negligible, but the gluing functions become unbounded. Thus to have a nice theory of divisors, one needs to consider unbounded operators \( \mathcal{R} \) too.

Apply the non-bounded case of Theorem 4.10 to the settings of Corollary 4.31: \[ 17 \]

**Amplification 5.10.** If the distortions \( \Delta(E_{k,k'}, \gamma_{j,j'}, \varphi_{j,j'}, \psi \equiv 1) \) are uniformly bounded, then in the conditions of Corollary 4.31 one can replace the boundness condition on the bundle gluing data together with the condition \( \sum_{m \neq j} e^{-2\lambda_{mj}} < \infty \) by the condition \( \sum_{m \neq j} e^{-2\lambda_{mj}} |\psi_j'|^2 < \infty \).

\[ ^{17} \]These gluing data do not satisfy the conditions that all but a finite number of functions \( \psi_j \) are constant. To preserve this condition one needs to replace \( z - z_0 \) by a slightly different function (but still vanishing at \( z_0 \)). Such a change would not influence the following arguments, so we skip it.
Remark 5.11. We do not know whether the line bundles which satisfy the strong form of the Riemann–Roch theorem provide a nice divisor-to-line-bundle correspondence. This question needs a separate investigation.

5.5. Non quasi-circular case. Every curve of genus 1 has a circular Schottky model; one can construct examples of curves of genus 2 which have no circular Schottky model. On the other hand, any curve of finite genus has a Schottky model; it is natural to expect that a similar property should hold for “reasonable” curves of infinite genus too. However, it is not readily obvious that these “reasonable” curves would satisfy the quasi-circularity condition.

Thus it may be vital to be able to drop the condition of quasi-circularity; it is especially useful since this comes in essentially no cost. The preceding arguments needed the condition of quasi-circularity since the operator $\mathcal{R}$ involves the operator $\iota_{j}^{\text{skew}}$ of Section 4.5. Moreover, this operator is needed since we need to skew the norm on $H^{1/2}_{+}(\gamma_j)$ so that Theorem 2.26 holds. Recall that the only role the skew norm plays in the proof of this theorem is the fact that the image of the direct product of the restriction mappings to an infinite collection of disjoint subsets is in fact in $\bigoplus_{l_2}$ of their individual images, as opposed to $\prod$ (in other words, the “correlation” of the restrictions is negligible).

Avoid the need to consider $\pm$-skewed norms by encoding the above fact into a definition:

Definition 5.12. We say that a quasi-smooth subset $D \subset C$ is fat if the restriction mapping $H^{1}(C)/\text{const} \to \prod H^{1}(R_j \subset C)/\text{const}$ has its image in $\bigoplus_{l_2} H^{1}(R_j \subset C)/\text{const}$. Here $R_j, j \in J$, are connected components of $C \setminus D$. The restriction-tolerance of $D$ is the norm of the corresponding mapping $H^{1}(C) \to \bigoplus_{l_2} H^{1}(R_j \subset C)/\text{const}$.

A collection of fat subsets $D_k \subset C_k$ is uniformly fat if their restriction-tolerances are uniformly bounded.

Obviously, if the subsets $D_k \subset C_k$ of the gluing data are uniformly fat, one can consider the (non-skewed) embedding norms on $H^{1/2}_{+}(\gamma_j)$ whenever we considered the skewed norms before. With such an approach there is no mapping $\iota_{j}^{\text{skew}}$ in the context of Section 4.5, thus we may drop the restriction of quasi-circularity from all the statements as far as fatness conditions hold.

On the other hand, the condition of fatness is very close to the conditions of Theorem 4.30. To show this, we start with a technical statement:

Proposition 5.13. Consider subspaces $H_i, i \in I$, of a Hilbert space $H$; denote their orthogonal complements by $H_i^{\perp}$. Then the following conditions are equivalent:

1. the product of projection mappings $H \to \prod_i H/H_i$ has it image in $\bigoplus_{l_2} H/H_i$;
2. the sum of embedding mappings $\bigoplus_{l_2} H_i^{\perp} \to H$ extends continuously to a mapping $\bigoplus_{l_2} H_i^{\perp} \to H$;
3. the block-Gram matrix $\mathcal{P}$ (with the blocks $\mathcal{P}_{ij}$ being the orthogonal projectors $H_j^{\perp} \to H_i^{\perp}$) defines a continuous operator in $\bigoplus_{l_2,i} H_i^{\perp}$. 
Proof. The first two statements are dual to each other. The second implies the third, since $\mathcal{P} = i^*i_!$.

To show that the third implies the second, note that $\mathcal{P}$ is Hermitian, and non-negative definite. Thus it induces a Hilbert structure on a completion $H'$ of a quotient of $\bigoplus_{l_2,i} H^i_{l_2}$. The Hilbert spaces $H^i_{l_2}$ are naturally isometrically embedded into $H'$; it is clear that the natural mappings $i, i'$ of $\bigoplus_{\partial} H^i_\partial$ into $H$ and $H'$ satisfy $\|i(h)\| = \|i'(h)\|$. This defines an isometric identification of $H'$ with a subspace of $H$; thus a continuous mapping $\bigoplus_{l_2,i} H^i_{l_2} \to H$. 

Remark 5.14. If Hilbert spaces $H$ and $\tilde{H}$ are Hilbert-dual to each other, then one can consider the subspaces $H^i_{l_2}$ as being subspaces of $\tilde{H}$. Since $H^s(R \subset C) = H^s(C)/H^s(C \setminus R)$, the natural duality between $H^s(C)$ and $H^{-s}(C, \omega \otimes \bar{\omega})$ makes the calculation of the orthogonal complement to $\tilde{H}^s(C \setminus R)$ especially simple: it is $\tilde{H}^{-s}(R, \omega \otimes \bar{\omega}) \subset H^{-s}(C, \omega \otimes \bar{\omega})$. Thus to check fatness it is enough to consider the block-Gram matrix formed by the orthogonal projectors $\tilde{H}^{-1}_{f=0}(R_i, \omega \otimes \bar{\omega}) \to \bigoplus_{\partial} H^i_{l_2}(R_j, \omega \otimes \bar{\omega})$; here $H^s_{f=0} \subset H^s$ consists of sections with the integral being 0.

Obviously, $H^{-1}_{\tilde{f}=0}(C, \omega \otimes \bar{\omega})$ has a natural Hilbert norm, since it is dual to $H^1(C)/\text{const}$. This norm is invariant with respect to automorphisms of $C$.

Call the orthogonal projector $\tilde{H}^{-1}_{f=0}(R, \omega \otimes \bar{\omega}) \to \bigoplus_{\partial} H^i_{l_2}(R_j, \omega \otimes \bar{\omega})$ the $\Delta$-projector between $R$ and $R'$. Consider a quasi-smooth subset $D \subset C$ of a compact curve $C$ with connected components $D_j$, $j \in J$, of the complement. Associate to it the infinite block-matrix $(\mathcal{P}_{jk})$, $j, k \in J$, with block $\mathcal{P}_{jk}$ being the $\Delta$-projector between $R_k$ and $R_j$. One can consider $(\mathcal{P}_{jk})$ as a matrix of an operator $\tilde{\mathcal{P}}: \bigoplus_{\partial} \tilde{H}^{-1}_{f=0}(R_j, \omega \otimes \bar{\omega}) \to \bigoplus_{\partial} \tilde{H}^{-1}_{f=0}(R_j, \omega \otimes \bar{\omega})$.

Lemma 5.15. If $D$ is fat, then the operator $\mathcal{C}$ of Section 4.4 (with the image in $\pm$-skewed spaces $H^1_{\pm}(\partial_j)$) is bounded.

Proof. Let $H_j = \tilde{H}^{-1}_{f=0}(R_j, \omega \otimes \bar{\omega})$, $H = \bigoplus_{l_2,j} H_j$. Let $H_{j_+} = \text{Im} \partial|_{H^0(R_j, \omega)}$, $H_{j_-} = \text{Im} \partial|_{H^0(R_j, \bar{\omega})}$. First, the component of $\mathcal{P}_{jk}$ acting from $H_{k_-}$ to $H_{j_-}$ vanishes, similarly for the component acting from $H_{k_+}$ to $H_{j_+}$.

Indeed, consider the operator $\Delta = i\partial\bar{\partial}: H^1(C)/\text{const} \to H^{-1}_{\tilde{f}=0}(C, \omega \otimes \bar{\omega})$. Being an elliptic self-dual operator with no kernel, it is invertible. The norm in $H^{-1}_{\tilde{f}=0}(C, \omega \otimes \bar{\omega})$ is defined via the duality with $H^1(C)/\text{const}$; $\|\alpha\|_{H^{-1}} = \max_f \frac{\|f\alpha\|_{H^1}}{\|f\|_{H^1}}$; here $f \in H^1(C)/\text{const}$. Let $g = \Delta^{-1}\alpha$; then $\int \bar{f}\alpha = -i\int \bar{f}\partial\bar{\partial}g = i\int \partial f\bar{\partial}g = i\int \overline{\partial f}\bar{\partial}g$. Since $\|\overline{\partial f}\|_{L_2} = \|f\|_{H^{1/2}}$, we can see that $\|\alpha\|_{H^{-1}} = \|\partial\bar{\partial}\|_{L_2} = \|g\|_{H^{1/2}}$. Thus $\Delta$ is a unitary operator, and $(\alpha_1, \alpha_2)_{H^{-1}} = (\overline{\partial \Delta^{-1}\alpha_1}, \overline{\partial \Delta^{-1}\alpha_2})_{L_2} = (\overline{\partial^{-1}\alpha_1}, \overline{\partial^{-1}\alpha_2})_{L_2}$ (here we choose $\partial^{-1}: H^{-1}_{\tilde{f}=0}(C, \omega \otimes \bar{\omega}) \to H^0(C, \bar{\omega})$ so that the image is orthogonal.
to global holomorphic forms). This shows that for \( g(C) = 0 \) the component of \( \Delta \)-projector acting from \( H_{k-} \) to \( H_{j-} \) vanishes; similarly for \(+\)-components. For general \( g(C) \) these components do not vanish; the corresponding norm on \( \bigoplus_{t} H_{k-} \) is not the direct sum norm, but its correction via the projection on the orthogonal complement to the image of \( \partial \). Thus this norm differs from the direct sum norm by a continuous finite-rank operator.

Similarly, if \( \alpha_1 = \bar{\partial} \beta_1 \), \( \alpha_2 = \partial \beta_2 \), and \( g(C) = 0 \), then \( (\alpha_1, \alpha_2)_{H^{-1}} \) is proportional to \( \int \bar{\beta}_1 \partial \bar{\beta}_2 \). Thus the identifications above identify the \( \Delta \)-projector from \( H_{k+} \) to \( H_{j-} \) with the bar-projector. Again, for general \( C \) the projector from \( \bigoplus H_{k+} \) to \( \bigoplus H_{k-} \) differs from the operator \( C \) by a continuous finite-rank operator.

Remark 5.16. If \( C \) is bounded, then all the the components of \( P \) between \( \bigoplus H_{k-} \) are bounded too. Moreover, \( H_{k+} + H_{k-} = H_{k} \). Indeed, the orthogonal complements to \( H_{k+} \) and \( H_{k-} \) coincide with \( \text{Ker } \bar{\partial} \) and \( \text{Ker } \partial \) in \( H^1(R_k \subset C) / \text{const} \); these subspaces do not intersect.

However, this does not yet imply that \( P \) is bounded. Indeed, the angle between \( H_{k+} \) and \( H_{k-} \) can be arbitrary small (as examples of long ellipses show). However, in the context of “practical criteria” of Section 4.10, \( C \) and \( P \) behave the same way:

Proposition 5.17. Consider two regions \( R, R' \subset C \) of conformal distance \( l \).

1. There is a constant \( c \) such that if \( C = \mathbb{C}P^1 \) and \( R, R' \) are disks, then the norm of the \( \Delta \)-projector between \( R \) and \( R' \) is \( ce^{-l} \).

2. There is a constant \( c' \) such that the norm of the \( \Delta \)-projector between \( R \) and \( R' \) is less than \( c'e^{-l} \).

Proof. The first part follows from Lemma 4.21 and the proof of Lemma 5.15; indeed, for circles the subspaces \( H_{k+} \) and \( H_{k-} \) of this proof are orthogonal.

The second part of the statement can be proved similarly to Proposition 4.22.

Corollary 5.18. In Theorem 4.30 and in Corollary 4.31 one can drop the condition of quasi-circularity.

Remark 5.19. One can combine this corollary and Amplification 5.10. However, to ensure that conditions of Theorem 4.10 hold, one needs to require that both \( \sum_{m \neq j} e^{-2\lambda_m} |\psi_j| < \infty \) and \( \sum_{m \neq j} e^{-2\lambda_m} < \infty \) are finite. The first part gives compactness of \( R \circ C \), the second boundness of \( C \).

5.6. Non-pseudo-smooth case. Recall that so far we assumed that the connected components \( R_i \) of the complement to the model domain \( D \subset C \) is a union of regions with smooth boundary. However, the only place when this assumption is crucial is Lemma 2.11; in turn, this lemma is needed for the introduction of \(+\)-skewed norms.

As the previous section shows, the consideration of \(+\)-skewed norms can be avoided in many cases. Consequently, in these cases one can weaken the assumptions on the components \( R_i \). For example, one can assume that the curves \( \gamma_i = \partial R_i \) are Jordan
curves (what is crucial, due to Lemma 2.12, is that $\partial R_i$ has measure 0). In such a case one considers Lemma 2.9 as the definition of $H^{1/2}(\gamma)$.

However, we need the gluing functions $\varphi_i$ and $\psi_i$ induce a mapping between $H^{1/2}(\gamma_i)$ and $H^{1/2}(\gamma_i')$. The simplest modification to allow this is to require $\varphi_i, \psi_i$ be defined not on $\gamma_i$, but on a neighborhood of $\gamma_i$, and require that the corresponding gluing operators act in $H^1$ of these neighborhoods.¹⁸ For the latter condition, it is enough to require that these functions are Lipschitz.

Remark 5.20. This modification is very welcome, since as Section 5.3 shows, our approach allows consideration of gluing data of fractal nature. It does not make a lot of sense to allow $\partial D$ to be fractal, while requiring that all the connected components of $C \setminus D$ have smooth boundaries.

Moreover, the typical description of a complex curve via the Schottky model leads to an invariantly defined Schottky group; our language requires a choice of a fundamental domain for this group. One should hope that weakening the requirements on $\partial R_i$ allows treating most of the “natural” Schottky groups using our language.

5.7. **Black-white curves.** The abstract form of the Riemann–Roch theorem 4.10 we use to establish the Riemann–Roch theorem needs the condition that 1 is not in the essential spectrum of an appropriate operator $B$ (which is $C \circ R$ or $R \circ C$). However, up to now we used a much weakened form where the operator $B$ is assumed to be compact. By analogy with Remark 6.9, one could expect that the compactness condition should be close to the strict Riemann–Roch condition. However, the very restrictive form of the operator $R$ allows curves to have a non-compact operator $C$ such that the Riemann–Roch theorem still holds.

Indeed, if $R$ is bounded (so $C \circ R$ is bounded), and $(C \circ R)^2$ is compact, then $C \circ R$ has only 0 in the essential spectrum. This may be achieved if $C$ can be made $2 \times 2$ block-diagonal with one block being compact, another bounded, and $R$ interchanging these blocks.

The following example of such curves is similar to the curves studied in [6, 2], and [14]; this is why we discuss it in details. A curve gluing data defines a black-white curve, if the set of indices $K$ enumerating compact curves $C_k$ is broken into two parts, $K_b$ and $K_w$, and no boundary components of the $w$-parts are glued together. In other words, the glued curve is colored in two colors, and the connected components of the white part consist of one domain $D_k, k \in K_w$.

It is easy to see that for such curves to satisfy the strong form of Riemann–Roch theorem with bounded bundle gluing data, it is enough to require that blocks $C_{jl}$ corresponding to black curves only form a compact matrix (with our “usual assumptions” about white curves, so that the corresponding white blocks $C_{jl}$ are bounded; one can also drop the “usual assumptions” on the white blocks if one simultaneously

¹⁸Recall that the arguments of [7] suggest that any reasonable curve should have a model with $\varphi_*$ being fraction-linear. As shown in [23], line bundles should be representable by constant functions $\psi_*$ (with an exception of one pair of curves $\gamma, \gamma'$ to allow non-zero degree).
replaces the conditions on the black curves by stronger restrictions than just compactness). In other words, there is no restriction on the white domains $D_k \subset C_k$, $k \in K_w$ (except the usual restrictions on boundedness of distortions of excess spaces).

In particular, given an arbitrary curve-bundle gluing data $(D_k \subset C_k', \varphi_*, \psi_*, V_*)$, $k \in K$, one can convert it to a black-white gluing data. To do so, let $K_w = K$, $K_b$ consists of $'\text{-orbits}$. Let $D_k$ for $k \in K_b$ be an annulus $\{1 < |z| < N_k\}$ with $N_k \gg 1$ appropriately embedded into $C_k \simeq \mathbb{C}P^1$. Glue this wide annulus (same as a long tube) $D_k$ between the corresponding curves $\gamma_j, \gamma_{j'}$. It is easy to see that the new gluing data is a black-white curve, and that with an appropriate choice of $N_k$, $k \in K_b$, this gluing data satisfies the Riemann–Roch theorem.

In fact the only thing to prove is that one can embed $D_k$ in $\mathbb{C}P^1$ and glue $\partial D_k$, $k \in K_b$, to $\gamma_j, \gamma_{j'}$ in such a way that the distortion $\Delta\left(E_{k,k'}\gamma_j, \gamma_{j'}, \varphi_{j,j'}, \psi \equiv 1\right)$ is bounded. The simplest way to do this is the following one: by definition, the real curve $\gamma_j$ is a boundary of the region $R_j$ inside the corresponding complex curve $C_k$. Choose $R_j' \subset R_j$ such that the conformal distance between $\partial R_j'$ and $\partial R_j$ is $N_k$; proceed similarly for $\gamma_{j'}$. Now identify the annuli $R_j \setminus R_j'$ and $R_{j'} \setminus R_j'$, so that $\partial R_j$ is identified with $\partial R_j'$; gluing $R_j$ and $R_{j'}$ using this identification gives a rational curve $C_k$ with two contractible regions $R_j', R_{j'}'$ with conformal distance $N_j$.

Moreover, the boundary of $D_k \defeq R_j \setminus R_j' = R_{j'} \setminus R_j'$ is naturally glued to $\gamma_j$ and $\gamma_{j'}$. Now check the distortion is bounded is tautological; thus the distortion $\Delta\left(E_{k,j,0}, \gamma_j, \varphi_{j,j'}, \psi \equiv 1\right) = 1$.

To take into account $\psi_{j,j'}$, it is enough to consider the case of constant $\psi_j$. Make $\psi_*$ corresponding to the gluing of $\gamma_j$ and $\partial R_j$ to be 1, and $\psi_*$ corresponding to the gluing of $\gamma_{j'}$ and $\partial R_{j'}$ to be $\psi_j$; this makes the corresponding $\Delta$ to become $|\psi_j|$. Note that we do not need boundness of $\{\psi_j\}$, an increase of $\psi_j$ can be compensated by an increase of the corresponding $N_{k(j)}$.

### 5.8. The bundle $\mathcal{O}$

When all the functions $\psi_j, j \in J$, are 1, the bundle gluing data describes the bundle $\mathcal{O}$; one can take the allowance spaces $V_j$ to be spanned by 1. In this important special case the mismatch operator $\mu$ of Section 3.4 not only has index 0, but also is an isomorphism:

**Theorem 5.21.** Suppose that $g(C_k) = 0$, $k \in K$, for all the curves $C_k$ of the curve gluing data. Suppose also that the result of gluing is connected. If the bundle $\mathcal{O}$ over this curve gluing data satisfies the Riemann–Roch theorem, then the mismatch operator is a bijection.

**Proof.** Since all the excess spaces vanish, the Riemann–Roch theorem states that $\mu$ has index 0. Thus it is enough to show that $\ker \mu = \{0\}$, or to show that for a function $f$ modulo const with $\partial f = 0$ and the vanishing mismatch one has $\|\partial f\|_{L^2} = 0$. In the case of domains $D_k \subset C_k$ with smooth boundary and smooth $f$, the following argument works: $\int_D \partial f \wedge \bar{\partial} f = \int_D \partial f \wedge \partial \bar{f} = \int_D df \wedge d\bar{f} = \int_D d f d \bar{f} = \int_{\partial D} f d \bar{f}$; here $D = \coprod_k D_k$. Now $\partial D$ is broken into pairs $\gamma_j, \gamma_{j'}$ with identifications between
them, and the pull-backs of $d\bar{f}$ to these pairs are compatible with the identifications. Thus $\int_{\gamma_j \cup \gamma_{j'}} f \, d\bar{f} = \int_{\gamma_j} \Delta_j f \, d\bar{f} = \Delta_j f \int_{\gamma_j} d\bar{f} = 0$; here $\Delta_j f$ is the (constant) jump of $f$ when $\gamma_j$ is identified with $\gamma_{j'}$.

In our, more general, situation $f$ may be extended to become an $H^1_{\text{loc}}$ function on $C = \bigsqcup_k C_k$, so $df, d\bar{f}$ are $L_2 = H^0$-sections of $\Omega^1 C$. In particular, $df \wedge d\bar{f}$ is a well-defined $L_1$-section of $\Omega^2 C$. Thus $\int_D df \wedge d\bar{f}$ may be written as $\sum_k \int_{C_k} df \wedge d\bar{f} - \sum_j \int_{R_j} df \wedge d\bar{f}$. What we achieved so far is to reduce the question to integration over compact smooth manifolds with boundaries, as above. However, the differential forms we need to consider are not smooth.

Given a surface $S$, a smooth curve $\gamma \subset S$, and $H^1$-functions $f$ and $g$ on $C$ define $\int_{\gamma} f \, dg$ as the result of the pairing of $f|_{\gamma} \in H^{1/2}(\gamma, \mathcal{O})$ and $dg \in H^{-1/2}(\gamma, \Omega^1 \gamma)$. Now for a subset $T \subset S$ with a smooth boundary the expressions $\int_{T} df \wedge dg$ and $\int_{\partial T} f \, dg$ are both well-defined and continuous in $f$ (or $g$). Since these expressions coincide for smooth $f$ and $g$, they coincide everywhere; this covers the case when the curves $\gamma_j$ are smooth.

In general, when the curves $\gamma_j$ may be Jordan curves, we need an extra argument. It is easy to reduce what we need to the following statement:

**Lemma 5.22.** Consider a surface $S$, a domain $D \subset S$ such that $\Gamma = \partial D$ is of measure 0, and a sequence of smooth embedded curves $(\Gamma_n)$ in $S$ such that $\Gamma_n$ bounds a domain $D_n$, and for any neighborhood $U$ of $\Gamma$ the symmetric difference of $D$ and $D_n$ is inside $U$ for large enough $n$. Then bilinear functionals $A_n : (f, g) \rightarrow \int_{\Gamma_n} f \, dg$ on $H^1(S)$ have a limit $A$ when $n \to \infty$. Moreover, $A(f, g) = 0$ if the image of $f$ in $H^1(\Gamma \subset S)$ vanishes.

**Proof.** Since the area between $\Gamma$ and $\Gamma_n$ goes to 0, the arguments above imply that the sequence $A_n(f, g)$ is fundamental for any fixed pair $(f, g)$, thus has a limit. Similarly, the limit is continuous. By definition, if the image of $f$ in $H^1(\Gamma \subset S)$ vanishes, one can approximate $f$ by a function $\bar{f}$ which vanish near $\Gamma$; then $A_n(\bar{f}, g) = 0$ for large $n$, so $A(\bar{f}, g) = 0$; by continuity, $A(f, g) = 0$.

This finishes the proof of the theorem. 

This theorem is the central statement in the description of the geometry of the Jacobian of the curve [23].

### 6. Appendix: Fredholm theorem

In this section we discuss details and motivations for Theorem 4.10.

6.1. **Quasi-complementary subspaces.** Continue using notations of Section 1.1. Note that if $V_1, V_2$ are quasi-complementary, then the natural mapping $V_1 \to V/V_2$ is a Fredholm mapping with the index being the excess of $V_1, V_2$. Recall that arguments
of Section 4.1 required finite-dimensional “corrections” to graphs of linear operators. First, show that relative dimension is invariant w.r.t. finite-dimensional variations.

**Proposition 6.1.** Suppose that $V_1, V_2$ are quasi-complementary with the excess $d$, and $V_k'$ is comparable with $V_k$ with the relative dimension $d_k$, $k = 1, 2$. Then $V_1', V_2'$ are quasi-complementary with the excess $d + d_1 + d_2$.

**Proof.** It is enough to consider the case when $V_2 = V_2'$, and $V_1' \supset V_1$ with codimension 1. Let $v \in V_1' \setminus V_1$.

If $v \in V_1 + V_2$, we need to show that the codimension of $V_1 \cap V_2$ in $V_1' \cap V_2$ is 1. Let $v = v_1 + v_2, v_k \in V_k, k = 1, 2$. Then $v_2 \notin V_1 \cap V_2$, but $v_2 = v - v_1 \in V_1' \cap V_2$. Thus codim $(V_1 \cap V_2 \subset V_1' \cap V_2) \geq 1$. If $w \in V_1 \cap V_2$, then $w - \tau v \in V_1$, thus $w - \tau v_2 \in V_1 \cap V_2$, thus codim $(V_1 \cap V_2 \subset V_1' \cap V_2) = 1$.

Similarly, if $v \notin V_1 + V_2 = V_1' + V_2$, then $V_1' \cap V_2 = V_1 \cap V_2, V_1' + V_2 = V_1 + V_2 + C v$, thus is closed, and codim $(V_1 + V_2 \subset V_1' + V_2) = 1$. □

**Remark 6.2.** Call subspaces $V_1, V_2$ weakly quasi-complementary if they are closed, dim $V_1 \cap V_2 < \infty$, and codim $(V_1 + V_2) < \infty$; similarly, define reldim $(V_1, V_2)$. The law of a change to comparable subspaces fails spectacularly for weakly quasi-complementary subspaces. Indeed, if in the proof above $v \notin V_1 + V_2$, but $v \notin V_1' + V_2$, then the excesses of $(V_1, V_2)$ and $(V_1', V_2)$ coincide. Thus the weak quasi-complementarity is preserved by changing spaces to comparable, but there is no way to control the excess. This is why in what follows we are not interested in weak quasi-complementarity.

**6.2. Closed operators.** As explained in Section 5.4, to get a satisfactory divisor-bundle correspondence, one needs to allow unbounded bundle gluing functions. This would lead to unbounded operator $\mathcal{R}$. Recall the settings of closed operators (see [18] for details).

A **partial operator** from $V$ to $V'$ is a linear operator $V_1 \to V'$ with $V_1 \subset V$. Such a partial operator $A$ is **densely defined** if $V_1$ is dense in $V$, and it has a **closed graph** if the graph of $A$ (which is a vector subspace of $V_1 \oplus V' \subset V \oplus V'$) is closed in $V \oplus V'$. A partial operator is **closed** if it is densely defined and it has a closed graph. Obviously, any continuous operator $V \to V'$ is closed. Such closed operators are called **bounded**.

Two partial operators $A: V_1 \to W$ (with $V_1 \subset V$) and $B: W_1 \to V^*$ (with $W_1 \subset W^*$) are in duality if $w \in W_1$ iff $\langle Av, w \rangle = \langle v, v' \rangle$ for an appropriate $v' \in V^*$, and $Bw = v'$. If $A$ is densely defined, there is a unique operator which is in duality with $A$, it is called the **dual operator**. The dual operator automatically has a closed graph. If $A$ is closed, then the dual operator $A^*$ is closed, $A^{**} = A$, and the graph of $-A^*$ (which is a vector subspace in $W^* \oplus V^* \simeq V^* \oplus W^*$) is the orthogonal complement to the graph of $A$ in $V \oplus W$.

A **composition** $A' \circ A$ of two partial operators $A: V_1 \to V'$ (with $V_1 \subset V$) and $A': V'_1 \to V''$ (with $V'_1 \subset V''$) is defined on $A^{-1}(V'_1)$ as $A' \circ A|_{A^{-1}(V'_1)}$. The composition of two closed operators is not necessarily densely defined, and the closure of the graph of the composition is not necessarily a graph of a partial mapping (in other words,
this closure may intersect $0 \oplus V'' \subset V \oplus V'').$ However, if $A'$ is bounded, then $A' \circ A$ is densely defined; if $A$ is bounded, then the closure of the graph of $A' \circ A$ is a graph of a partial operator.

If the composition $A' \circ A$ is densely defined, then $(A' \circ A)^*$ is defined on $v \in V''^*$ if $A'^* \circ A^*$ is defined on $v,$ and $(A' \circ A)^* v = (A^* \circ A')^* v.$ Note that if $A'$ is not bounded, the domain of $A'^* \circ A^*$ may be strictly smaller than the domain of $(A' \circ A)^*.$

We say that two partial operators $A'$ and $A$ have a compact composition $A' \circ A$ if the graph of $A' \circ A$ is a vector subspace of a graph of a compact operator $B: V \to V''.$ Similarly, say that $1$ is not in significant spectrum of $A' \circ A$ if there is a bounded “extension” $B: V \to V''$ with $1$ not in the SpecEss $(B)$.

6.3. Complementarity criterion. By Theorem 4.8, to get a sufficiently general version of Riemann–Roch theorem, one should be able to characterize a sufficiently large subset of the set of quasi-complementary pairs. Consider two closed vector subspaces $V_1, V_2 \subset H$ such that the projection of $V_i$ on $H_i$ has no null-space and a dense image. This means that one can consider $V_1$ as a graph of a closed mapping $A_1: H_1 \to H_2,$ similarly $V_2$ is a graph of a closed mapping $A_2: H_2 \to H_1.$

Lemma 6.3 (abstract finiteness). If $A_1$ and $A_2$ have a compact composition $A_1 \circ A_2,$ then $V_1 \cap V_2$ is finite dimensional. If $A_2^*$ and $A_1^*$ have a compact composition $A_2^* \circ A_1^*,$ then $V_1 + V_2$ has a finite codimension.

Proof. The projection of $V_1 \cap V_2$ to $H_2$ is a subspace of Ker $(A_1 \circ A_2 - 1),$ thus is finite-dimensional. The second part can be proven by taking orthogonal complements to $V_1$ and $V_2.$

Corollary 6.4 (weak form). If $A_1$ and $A_2$ have a compact densely defined composition $A_1 \circ A_2,$ then $V_1$ and $V_2$ are weakly quasi-complementary.

Proposition 6.5 (strong bounded form). If $A_1,$ $A_2$ are bounded, and $A_1 \circ A_2$ is compact, then $V_1$ and $V_2$ are quasi-complementary with the excess $0.$

Proof. It is enough to prove is that $V_1 + V_2$ is closed.

Let $v_l + v'_l \xrightarrow{l \to \infty} v \in H_1 \oplus H_2,$ here $v_l \in V_1,$ $v'_l \in V_2,$ $l \in \mathbb{N}.$ Then $v_l = (h_l, A_1h_l),$ $v'_l = (A_2h'_l, h'_l).$ Let $v = (x, x'),$ $x \in H_1,$ $x' \in H_2.$ Then $h_l + A_2h'_l \to x,$ $h'_l + A_1h_l \to x'.$

We need to show that

$$h + A_2h' = x,$$  \hspace{1cm} $h' + A_1h = x'$

(6.1)

has a solution.

One may substitute $x - A_2h'_l$ instead of $h_l,$ thus it is enough to show that if $(1 - A_1A_2)h'_l \to x',$ then $x' = (1 - A_1A_2)h'$ for an appropriate $h',$ which follows from closeness of Im $(1 - A_1A_2).$\qed
Remark 6.6. Similarly, one may require that $A_2 \circ A_1$ is compact. This is not equivalent to $A_1 \circ A_2$ being compact. Indeed, let $K$ be a compact operator. Consider block matrices $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Amplification 6.7. One can require instead that 1 is not in the essential spectrum of $A_1 \circ A_2$ or $A_2 \circ A_1$.

The proposition is an insignificant generalization of Fredholm theorem, however, since our target is to get as large a class of pairs of closed subspaces as possible, we need a much stronger result which allows $A_1$ to be closed instead of bounded. In applications $A_2$ is bounded, we have a possibility to control $A_2$, and can make it arbitrarily small. Investigate which conditions on $A_1$ imply quasi-complementarity.

First, consider the case when $A_2 = 0$. Then $A_1 A_2$ is automatically defined everywhere and compact. However, $V_1$ and $V_2$ being quasi-complementary is equivalent to $A_1$ being bounded! Thus there is no hope to generalize the above proposition literally. However, this example shows that to have $V_1 + V_2$ closed it is enough to strengthen the topology on $H_2$, so that $A_1$ becomes bounded. Let $\| \cdot \|_{A_1}$ be the norm on the domain of $A_1$ defined by identification of it with $V_1$, $\| v \|_{H_1}^2 = \| v \|_{H_1}^2 + \| A_1 v \|_{H_2}^2$, let $H_1^{(A_1)}$ be the domain of $A_1$ considered with this norm. Obviously, $H_1^{(A_1)}$ is complete, and $A_1$ can be extended to a bounded operator $H_1^{(A_1)} \to H_2$, and $V_1$ is closed in $H_1^{(A_1)} \oplus H_2$. Moreover, $(V_1, H_2)$ are quasi-complementary in $H_1^{(A_1)} \oplus H_2$ with the excess 0.

How to generalize this to the case $A_2 \neq 0$? It is clear that it is enough to require that $\text{Im} \, A_2 \subset H_1^{(A_1)}$, or that $A_1 \circ A_2$ is defined everywhere. This finishes the proof of Theorem 4.10.

Remark 6.8. There is another argument why it is not possible to consider closed operators $A_1$ without making some corrections. Indeed, $V_2$ being quasi-complementary with $V_1$ with the excess 0 heuristically implies that $\dim V_2 = \dim H_2$. A change of $V_2$ to a comparable subspace of $H_1 \oplus H_2$ of non-0 relative dimension would break this coincidence. On the other hand, if $A_2$ is a closed unbounded operator, there there is another closed operator $A'_2$ such that $\text{Graph} (A_2)$ is comparable with $\text{Graph} (A'_2)$ with an arbitrary relative dimension. Indeed, since considering $A'_2$ inverts the relative dimension, it is enough to build $A_2$ with $\text{Graph} (A'_2) \subset \text{Graph} (A_2)$ with codimension 1. Take a linear functional $\alpha$ on $H_1 \oplus H_2$ with $H_2 \subset \ker \alpha$. Let $V'_2 = V_2 \cap \ker \alpha$. Then $V_2$ is a closed subspace of $H_1 \oplus H_2$, and is a graph of a partial operator. Obviously, the domain of this operator is dense unless $\alpha|_{H_1}$ is in the domain of $A'_2$.

Remark 6.9. It is easy to amplify this theorem by replacing compact operators by some larger set of operators $\hat{K}$ such that $1 - \hat{K}$ is Fredholm. The strongest such result (which is almost tautological) corresponds to the class of operators of the form $K_1 + K_0$, with $1 \notin \text{Spec} K_1$, and $K_0$ compact. Another amplification, with the
condition which is easier to check, corresponds to operators of the form $K_1 + K_0$, with $K_0$ compact, and the spectral radius $R$ of $K_1$ satisfying $R < 1$.

However, the stated form has the advantage of being invariant w.r.t. the change $A'_2 = A_2 \circ B$. In fact, this is the strongest form which is invariant w.r.t. such changes. Indeed, it is enough to show that for any non-compact operator $A$ one can find a bounded operator $B$ such that $1$ is in the essential spectrum of $A \circ B$. In turn, by the polar decomposition one can assume $A$ to be self-adjoint; now the statement follows from the spectral theorem and the characterization of compact self-adjoint operators by their spectrum.

References

1. I. A. Bikchantaev, The Riemann problem on a finite-sheeted Riemann surface of infinite genus, Mat. Zametki 67 (2000), no. 1, 25–35. MR 2001d:30072
2. J. Feldman, H. Knörrer, and Trubowitz E., Infinite genus riemann surfaces, Canadian Mathematical Society 1945-1995 (James B. Carrell and Ram Murty, eds.), no. 3, Canadian Mathematical Society, Ottawa, 1996, pp. 91–112.
3. F. G. Friedlander, Introduction to the theory of distributions, Cambridge University Press, Cambridge, 1982. MR 86f:46002
4. I. M. Gel’fand and M. A. Naimark, Unitarnye predstavleniya klassičeskikh grup, Izdat. Nauk SSSR, Moscow-Leningrad, 1950. MR 13,722f
5. Fritz Gesztesy and Helge Holden, Darboux-type transformations and hyperelliptic curves, J. Reine Angew. Math. 527 (2000), 151–183. MR 2002b:37108
6. D. Gieseker, H. Knörrer, and E. Trubowitz, The geometry of algebraic Fermi curves, Perspectives in Mathematics, vol. 14, Academic Press Inc., Boston, MA, 1993.
7. Adolf Hurwitz and R. Courant, Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen, Interscience Publishers, Inc., New York, 1944. MR 6,148e
8. Takashi Ichikawa, Schottky uniformization theory on Riemann surfaces and Mumford curves of infinite genus, J. Reine Angew. Math. 486 (1997), 45–68.
9. Naondo Jin, On maximal Riemann surfaces, Hiroshima Math. J. 26 (1996), no. 2, 385–404.
10. G. A. Koshevoi, Curves of infinite genus that can be uniformized in the sense of Schottky, Uspekhi Mat. Nauk 44 (1989), no. 4(268), 239–240.
11. H. P. McKean and C. Trubowitz, Hill’s operator and hyperelliptic function theory in the presence of infinitely many branch points, Comm. Pure Appl. Math. 29 (1976), no. 1, 143–226.
12. H. P. McKean and K. L. Vaninsky, Action-angle variables for the cubic Schrödinger equation, Comm. Pure Appl. Math. 50 (1997), no. 6, 489–562. MR 98b:35183
13. Franz Merkl, An asymptotic expansion for Bloch functions on Riemann surfaces of infinite genus and almost periodicity of the Kadomcev-Petviashvilli flow, Math. Phys. Anal. Geom. 2 (1999), no. 3, 245–278. MR 2000m:14037
14. ———, A Riemann Roch theorem for infinite genus Riemann surfaces, Invent. Math. 139 (2000), no. 2, 391–437. MR 2001e:32030
15. W. Müller, M. Schmidt, and R. Schrader, Hyperelliptic Riemann surfaces of infinite genus and solutions of the KdV equation, Duke Math. J. 91 (1998), no. 2, 315–352. MR 98m:58060
16. Andrew Pressley and Graeme Segal, Loop groups, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1986, Oxford Science Publications.
17. Jeffrey Rauch, Partial differential equations, Springer-Verlag, New York, 1991. MR 94e:35002
18. Walter Rudin, Functional analysis, second ed., McGraw-Hill Inc., New York, 1991. MR 92k:46001
19. Mikio Sato and Yasuko Sato, *Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold*, Nonlinear partial differential equations in applied science (Tokyo, 1982), North-Holland, Amsterdam, 1983, pp. 259–271. MR 86m:58072

20. Martin U. Schmidt, *Integrable systems and Riemann surfaces of infinite genus*, Mem. Amer. Math. Soc. 122 (1996), no. 581, viii+111.

21. Graeme Segal and George Wilson, *Loop groups and equations of KdV type*, Inst. Hautes Études Sci. Publ. Math. (1985), no. 61, 5–65. MR 87b:58039

22. David A. Vogan, Jr., *Unitary representations of reductive Lie groups*, Princeton University Press, Princeton, NJ, 1987. MR 89g:22024

23. Ilya Zakharevich, *Quasi-algebraic geometry of curves I, The Riemann–Roch theorem and Jacobian*, http://xxx.lanl.gov/e-print/alg-geom/9710013, August 1997.

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