A GENERIC MULTIPLICATION IN QUANTISED SCHUR ALGEBRAS

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ABSTRACT. We define a generic multiplication in quantised Schur algebras and thus obtain a new algebra structure in the Schur algebras. We prove that via a modified version of the map from quantum groups to quantised Schur algebras, defined in [1], a subalgebra of this new algebra is a quotient of the monoid algebra in Hall algebras studied in [10]. We also prove that the subalgebra of the new algebra gives a geometric realisation of a positive part of 0-Schur algebras, defined in [4]. Consequently, we obtain a multiplicative basis for the positive part of 0-Schur algebras.

INTRODUCTION

Schur algebras $S(n,r)$ were invented by I. Schur to classify the polynomial representations of the complex general linear group $Gl_n(\mathbb{C})$. Quantised Schur algebras $S_q(n,r)$ are quantum analogues of Schur algebras. Both quantised Schur algebras $S_q(n,r)$ and classical Schur algebras $S(n,r)$ have applications to the representation theory of $Gl_n$ over fields of undescribing characteristics.

In [1] A. A. Beilinson, G. Lusztig and R. MacPherson gave a geometric construction of quantised enveloping algebras of type $A$. Among other important results they defined surjective algebra homomorphisms $\theta$, from the integral form of the quantised enveloping algebras to certain finite dimensional associative algebras. They first defined a multiplication of pairs of $n$-step partial flags in a vector space $k^r$ over a finite field $k$ and thus obtained a finite dimensional associative algebra. They also studied how the structure constants behave when $r$ increases by a multiple of $n$. Then, by taking a certain limit they obtained the quantised enveloping algebras of type $A$. J. Du remarked in [5] that the finite dimensional associative algebras studied in [1] are canonically isomorphic to the quantised Schur algebras studied by R. Dipper and G. James in [2].

The aim of this paper is to study a generic version of the multiplication of pairs of partial flags defined in [1]. By this generic multiplication we get an algebra structure in the quantised Schur algebras. We prove that a certain subalgebra of this new algebra is a quotient of the monoid algebra in Hall algebras studied by M. Reineke in [10]. Via a modified version of the surjective algebra homomorphism $\theta$, defined in [1], we prove that the subalgebra is isomorphic to a positive part of 0-Schur algebras, studied by S. Donkin in [4]. Thus we achieve a geometric construction of the positive parts of the 0-Schur algebras.

This paper is organized as follows. In Section 1 we recall definitions and results in [1] on the multiplication of pairs of partial flags defined in [1]. In this generic multiplication we get an algebra structure in the quantised Schur algebras. We prove that a certain subalgebra of this new algebra is a quotient of the monoid algebra in Hall algebras studied by M. Reineke in [10]. Via a modified version of the surjective algebra homomorphism $\theta$, defined in [1], we prove that the subalgebra is isomorphic to a positive part of 0-Schur algebras, studied by S. Donkin in [4]. Thus we achieve a geometric construction of the positive parts of the 0-Schur algebras.

This paper is organized as follows. In Section 1 we recall definitions and results in [1] on the multiplication of pairs of partial flags (see also [5, 6]). In Section 2 we recall definitions and results on the monoid given by generic extensions studied in [10]. In Section 3 we study a generic version of the multiplication of pairs of partial flags in [1], and prove that this generic multiplication gives us a new algebra structure in quantised Schur algebras. In Section 4 we prove results on connection between our new algebras and the monoid algebras given by generic extensions in [10] and to 0-Schur algebras. We also provide a multiplicative basis for a positive part of 0-Schur algebras. As a remark, we would like to mention that this multiplicative basis is related to Lusztig’s canonical basis.

1. q-SCHUR ALGEBRAS AS QUOTIENTS OF QUANTISED ENVELOPING ALGEBRAS

In this section we recall some definitions and results from [1] on $q$-Schur algebras as quotients of quantised enveloping algebras (see also [5, 6]).
1.1. **q-Schur algebras.** Denote by $\Theta_r$ the set of $n \times n$ matrices whose entries are non-negative integers and sum to $r$. Let $V$ be an $r$-dimensional vector space over a field $k$. Let $\mathcal{F}$ be the set of all $n$-steps flags in $V$:

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V.$$

The group $\text{GL}(V)$ acts naturally by change of basis on $\mathcal{F}$. We let $\text{GL}(V)$ act diagonally on $\mathcal{F} \times \mathcal{F}$. Let $(f, f') \in \mathcal{F} \times \mathcal{F}$, we write

$$f = V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$$

and

$$f' = V'_1 \subseteq V'_2 \subseteq \cdots \subseteq V'_n = V.$$

Let $V_0 = V'_0 = 0$ and define

$$a_{ij} = \dim (V_{i-1} + V_i \cap V'_j) - \dim (V_{i-1} + V_i \cap V'_{j-1}).$$

Then the map $(f, f') \mapsto (a_{ij})_{ij}$ induces a bijection between the set of $\text{GL}(V)$-orbits in $\mathcal{F} \times \mathcal{F}$ and the set $\Theta_r$. We denote by $\mathcal{O}_A$ the $\text{GL}(V)$-orbit in $\mathcal{F} \times \mathcal{F}$ corresponding to the matrix $A \in \Theta_r$.

Now suppose that $k$ is a finite field with $q$ elements. Let $A, A', A'' \in \Theta_r$ and let $(f_1, f_2) \in \mathcal{O}_{A''}$. Following Proposition 1.1 in [1], there exists a polynomial $g_{A, A', A''} = c_0 + c_1 q + \cdots + c_m q^m$, given by

$$g_{A, A', A''} = |\{(f, f) \in \mathcal{F} | (f_1, f) \in \mathcal{O}_A, (f, f_2) \in \mathcal{O}_{A'}\}|,$$

where $c_i$ are integers that do not depend on $q$, the cardinality of the field $k$, and $(f_1, f_2) \in \mathcal{O}_{A''}$.

Now recall that the $q$-Schur algebra $S_q(n, r)$ is the free $\mathbb{Z}[q, q^{-1}]$-module with basis $\{e_A | A \in \Theta_r\}$, and with an associative multiplication given by

$$e_A e_{A'} = \sum_{A'' \in \Theta_r} g_{A, A', A''} e_{A''}.$$ 

For a matrix $A \in \Theta_r$, denote by $\text{ro}(A)$ the vector $(\sum_j a_{1j}; \sum_j a_{2j}; \cdots; \sum_j a_{nj})$ and by $\text{co}(A)$ the vector $(\sum_j a_{j1}; \sum_j a_{j2}; \cdots; \sum_j a_{jn})$. By the definition of the multiplication it is easy to see that

$$e_A e_{A'} = 0 \text{ if } \text{co}(A) \neq \text{ro}(A').$$

Denote by $E_{ij}$ the elementary $n \times n$ matrix with 1 at the entry $(i, j)$ and 0 elsewhere. We recall a lemma, which we will use later, on the multiplication defined above.

**Lemma 1.1** ([1]). Assume that $1 \leq h < n$. Let $A = (a_{ij}) \in \Theta_r$. Assume that $B = (b_{ij}) \in \Theta_r$ such that $B - E_{h, h+1}$ is a diagonal matrix and $\text{co}(B) = \text{ro}(A)$. Then

$$e_A e_B = \sum_{p \geq h+1, p > 0} v^{2\sum_j a_{pj}} v^{2(a_{pj}+1)} \frac{v^{2(a_{hb}+1)}}{v^{2-1}} e_A + E_{h, p} - E_{h+1, p}.$$ 

1.2. **The map $\theta : U_A(\text{gl}_n) \to S_v(n, r)$.** Let $A = \mathbb{Z}[v, v^{-1}]$ and $v^2 = q$. Let $U_A(\text{gl}_n)$ be the integral form of the quantised enveloping algebra of the Lie algebra $\text{gl}_n$. Denote by $S_v(n, r)$ the algebra $A \otimes S_v(n, r)$. Let

$$\theta : U_A(\text{gl}_n) \to S_v(n, r)$$

be the surjective algebra homomorphism defined by A. A. Beilinson, G. Lusztig and R. MacPherson in [1]. Through the map $\theta$ we can view the Schur algebra $S_v(n, r)$ as a quotient of the quantised enveloping algebra $U_A(\text{gl}_n)$. We are interested in the restriction of $\theta$ to the positive part $U^+$ of $U_A(\text{gl}_n)$.

Unless stated otherwise, we let $Q$ be the linearly oriented quiver of type $A_{n-1}$:

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n - 1.$$

By a well-known result of C. M. Ringel (see [1] [2]), the algebra $U^+$ is isomorphic to the twisted Ringel-Hall algebra $H_q(Q)$, which is generated by isomorphism classes of simple $kQ$-modules via Hall multiplication.
Denote by $S_i$ the simple module of the path algebra $kQ$ associated to vertex $i$ of $Q$. By abuse of notation we also denote by $M$ the isomorphism class of a $kQ$-module $M$. For any $s \in \mathbb{N}$, denote by $D_s$ the set of diagonal matrices satisfying that the entries are non-negative integers and that the sum of the entries is $s$. For a matrix $A \in \Theta_r$, denote by $[A] = v^{-\dim O_A + \dim pr_1(O_A)} e_A$, where $pr_1$ is the natural projection to the first component of $\mathcal{F} \times \mathcal{F}$. Now the map $\theta$ can be defined on the twisted Ringel-Hall algebra as follows:

$$\theta : H_q(Q) \to S_u(n, r), \ S_i \mapsto \sum_{D \in D_{r-1}} [E_{i,i+1} + D].$$

2. A monoid given by generic extensions

In this section we briefly recall definitions and results on the monoid of generic extensions in $[10]$, and we let $k$ be an algebraically closed field. Results in this section work for any Dynkin quiver $Q = (Q_0, Q_1)$, where $Q_0 = \{1, \cdots, n\}$ is the set of vertices of $Q$ and $Q_1$ is the set of arrows of $Q$. We denote by $\text{mod} kQ$ and $\text{Rep}(Q)$, respectively, the category of finitely generated left $kQ$-modules and the category of finite dimensional representations of $Q$. We don’t distinguish a representation of $Q$ from the corresponding $kQ$-module.

Let $b \in \mathbb{N}^{n-1}$. Denote by

$$\text{Rep}(b) = \Pi_{i \in Q_1} \text{Hom}_k(k^{b_i}, k^{b_j})$$

the representation variety of $Q$, which is an affine space consisting of representations with dimension vector $b$. The group $\text{GL}(b) = \Pi_{i} \text{GL}(b_i)$ acts on $\text{Rep}(b)$ by conjugation and there is a one-to-one correspondence between $\text{GL}(b)$-orbits in $\text{Rep}(b)$ and isomorphism classes of representations in $\text{Rep}(b)$. Denote by $E(M, N)$ the subset of $\text{Rep}(b)$, containing points which are extensions of $M$ by $N$.

**Lemma 2.1 ([10]).** The set $E(M, N)$ is an irreducible subset of $\text{Rep}(b)$.

Thus there exists a unique open $\text{GL}(b)$-orbit in $E(M, N)$. We say a point in $E(M, N)$ is generic if it is contained in the open orbit. Generic points in $E(M, N)$ are also called generic extensions of $M$ by $N$.

**Definition 2.2 ([10]).** Let $M$ and $N$ be two isomorphism classes in $\text{mod} kQ$. Define a multiplication

$$M \ast N = G,$$

where $G$ is the isomorphism class of the generic points in $E(M, N)$.

Denote by $H_q(Q)$ the Ringel-Hall algebra over $\mathbb{Q}[q]$ and by $H_0(Q)$ the specialisation of $H_q(Q)$ at $q = 0$.

**Theorem 2.3 ([10]).** (1) $\mathcal{M} = \{\{M \mid M \text{ is an isomorphism class in } \text{mod } kQ\}, \ast\}$ is a monoid.

(2) $\mathbb{Q}\mathcal{M} \cong H_0(Q)$ as algebras.

3. A monoid given by a generic multiplication in $q$-Schur algebras

In this section we define a generic multiplication in the $q$-Schur algebra $S_q(n, r)$. Via this multiplication we obtain a new algebra in $S_q(n, r)$. We let $k$ be an algebraically closed field in this section.

Denote by

$$\Theta^u_r = \{A \in \Theta_r \mid A \text{ is an upper triangular matrix}\}.$$

Let $A, A' \in \Theta^u_r$, define

$$\mathcal{E}(A, A') = \{(f_1, f_2) \in \mathcal{F} \times \mathcal{F} \mid \exists f \text{ such that } (f_1, f) \in O_A \text{ and } (f, f_2) \in O_{A'}\}.$$

We denote by $M_{ij}$ the indecomposable representation of $Q$ with dimension vector $\sum_{l=i}^{j-1} e_l$, where $e_l$ is the simple root of $Q$ associated to vertex $i$, that is, $e_i$ is the dimension vector of the simple module $S_i$. By the module determined by a matrix $A \in \Theta^u_r$ we mean the module
$\bigoplus_{i<j} M(ij)^{a_{ij}}$. Note that for any $(f_1, f_2) \in \mathcal{E}(A, A')$, we have $f_2$ is a subflag of $f_1$. Note also that by omitting the last step of a partial flag $f$ in $\mathcal{F}$, we can view $f$ as a projective $kQ$-module and by abuse of notation we still denote the projective module by $f$. Now for $A \in \Theta^u_r$, suppose that $(f, h) \in \mathcal{O}_A$ and that $M$ is the module determined by $A$. We have a short exact sequence

$$0 \rightarrow h \rightarrow f \rightarrow M \rightarrow 0,$$

that is, $h \subseteq f$ is a projective resolution of $M$.

Let $b \in \mathbb{N}^{n-1}$ and denote by $k^b$ the $Q_0$-graded vector space with $k^b$ as its $i$-th homogeneous component, where $i$ is a vertex of $Q$. Denote by $\text{Hom}_{gr}(f, k^b)$ the set of graded linear maps between $f$ and $k^b$, where $f$ is a partial flag in $\mathcal{F}$ viewed as a $Q_0$-graded vector space by omitting its last step.

### 3.1. Relation between generic points in $\mathcal{E}(A, A')$ and in $\mathcal{E}(M, N)$

For $A, A' \in \Theta_r$, we write $A \preceq A'$ if $\mathcal{O}_{A'}$ is contained in the Zariski closure of $\mathcal{O}_A$. In this case we say that $(f_1, f_2) \in \mathcal{O}_A$ degenerates to $(f'_1, f'_2) \in \mathcal{O}_{A'}$. Lemma 3.7 in [1] implies the existence of generic points in $\mathcal{E}(A, A')$, in the sense that the closure of their orbit contains orbits of all the other points in $\mathcal{E}(A, A')$. That is, there is a unique open orbit in $\mathcal{E}(A, A')$. In this subsection we will show that there is a nice correspondence between generic points $\mathcal{E}(A, A')$ and generic points in subset $\mathcal{E}(M, N)$, where $M$ and $N$ are the modules determined by $A$ and $A'$, respectively.

Let $(f_1, f_2) \in \mathcal{F} \times \mathcal{F}$ with $f_2$ a subflag of $f_1$. Denote by $a_1$ and $a_2$, respectively, the dimension vectors of the projective modules $f_1$ and $f_2$. Let $b = a_1 - a_2$. We define some sets as follows.

$$\text{Inj}(f_2, f_1) = \{ \sigma \in \text{Hom}_{kQ}(f_2, f_1) | \sigma \text{ is injective} \};$$

$$S_1 = \{ (\sigma, \eta) \in \text{Inj}(f_2, f_1) \times \text{Hom}_{gr}(f_1, k^b) | \eta \text{ is surjective and } \eta \sigma = 0 \};$$

$$S'_1 = \{ (\sigma, \eta) \in \text{Inj}(f_2, f_1) \times \text{Hom}_{gr}(f_1, k^b) | \eta \text{ is surjective, }\cr \text{ker} \eta \text{ is a } kQ\text{-module and } \eta \sigma = 0 \};$$

$$S_2 = \{ \eta \in \text{Hom}_{gr}(f_1, k^b) | \eta \text{ is surjective and } \text{ker} \eta \text{ is a } kQ\text{-module} \};$$

$$S'_2 = \{ (M, \eta) \in \text{Rep}(b) \times \text{Hom}_{gr}(f_1, k^b) | \eta : f_1 \rightarrow M \text{ is a } kQ\text{-homomorphism} \};$$

$$\text{Inj}_{A, A'}(f_2, f_1) = \{ \sigma \in \text{Inj}(f_2, f_1) | \text{cok}(\sigma) \in \mathcal{E}(M, N) \},$$

where $M$ and $N$ are the modules determined by $A$ and $A'$, respectively.

For convenience we denote by $\text{Inj}(f_2, f_1)$ by $S_3$. We obtain some fibre bundles as follows.

**Lemma 3.1 ([S]).** The natural projection $\pi_1 : S'_1 \rightarrow S_2$ is a vector bundle.

**Lemma 3.2 ([S]).** The natural projection $\pi_2 : S_1 \rightarrow S_3$ is a principal $\text{GL}(b)$-bundle.

**Lemma 3.3 ([S]).** The natural projection $\pi_3 : S'_2 \rightarrow \text{Rep}(b)$ is a vector bundle.

Note that any $\eta \in S_2$ determines a unique module $M \in \text{Rep}(b)$ and this defines an open embedding of $S_2$ into $S'_2$. So we can view $S_2$ as an open subset of $S'_2$.

**Lemma 3.4.** (1) $\text{Inj}_{A, A'}(f_2, f_1) = \pi_2(S_1 \cap \pi_1^{-1}(S_2 \cap \pi_3^{-1}(\mathcal{E}(M, N))))$.

(2) $\text{Inj}_{A, A'}(f_2, f_1)$ is irreducible.

**Proof.** Following the definitions of $\pi_1$ and $\pi_3$, $\text{cok}\sigma \in \mathcal{E}(M, N)$ for any $(\sigma, \eta) \in S_1 \cap \pi_1^{-1}(S_2 \cap \pi_3^{-1}(\mathcal{E}(M, N)))$. Therefore $\sigma = \pi_2((\sigma, \eta)) \in \text{Inj}_{A, A'}(f_2, f_1)$. On the other hand, suppose that $\sigma \in \text{Inj}_{A, A'}(f_2, f_1)$. Then $\sigma \in \pi_2(S_1 \cap \pi_1^{-1}(S_2 \cap \pi_3^{-1}(X)))$, where $X$ is the module determined by $\eta$ for a preimage $(\sigma, \eta) \in \pi_2^{-1}(\sigma)$. This proves (1). Now (2) follows from (1) and Lemmas 3.1-3.3. \(\square\)
Let
\[ \mathcal{E}'(A, A') = \{(f_1, f) \in \mathcal{E}(A, A')| f \in \mathcal{F}\}. \]
Define
\[ \pi : \text{Inj}_{A,A'}(f_2, f_1) \rightarrow \mathcal{F} \times \mathcal{F}, \quad \sigma \mapsto (f_1, \text{Im} \sigma), \]
where \( \text{Im} \sigma \) can be viewed as a flag in \( \mathcal{F} \) with the last step the natural embedding of \( \text{Im} \sigma(f_1)n-1 \) into \( V \).

**Lemma 3.5.** (1) \( \text{Im} \pi = \mathcal{E}'(A, A') \).

(2) \( \mathcal{E}'(A, A') \) is irreducible.

**Proof.** Let \( \sigma \in \text{Inj}_{A,A'}(f_2, f_1) \). Then by the following diagram,

\[
\begin{array}{ccc}
I & \xrightarrow{f_2} & I \\
\downarrow & \sigma & \downarrow \\
\text{ker} \eta & \xrightarrow{f_1} & M, \\
\downarrow & & \downarrow \\
N & \xrightarrow{cok \sigma} & M
\end{array}
\]

where each square commutes and all rows and columns are short exact sequences, we know that \( (f_1, \text{Im} \sigma) \in \mathcal{E}'(A, A') \). On the other hand by the definition of \( \mathcal{E}'(A, A') \), for any \( (f_1, f) \in \mathcal{E}'(A, A') \), the natural embedding \( f_2 \cong f \subseteq f_1 \) is in \( \text{Inj}_{A,A'}(f_2, f_1) \). Now the irreducibility of \( \mathcal{E}'(A, A') \) follows from that of \( \text{Inj}_{A,A'}(f_2, f_1) \). \( \square \)

As a consequence of Lemma 3.5 we can see the existence of a unique dense open orbits in \( \mathcal{E}(A, A') \). Indeed, by Lemma 3.5 and the surjective map \( \mathcal{E}'(A, A') \times \text{GL}(r) \rightarrow \mathcal{E}(A, A') \), the set \( \mathcal{E}(A, A') \) is irreducible. Since there are only finitely many \( \text{GL}(r) \)-orbits in \( \mathcal{E}(A, A') \), there exists a unique dense open orbits in \( \mathcal{E}(A, A') \).

**Definition 3.6.** Let \( \mathcal{O}_{A''} \) be the dense open orbit in \( \mathcal{E}(A, A') \). We say that an injection \( \sigma : f' \rightarrow f \) is generic in \( \text{Inj}_{A,A'}(f', f) \) if the pair of flags \( (f, \text{Im} \sigma) \) is contained in \( \mathcal{O}_{A''} \).

**Proposition 3.7.** Let \( (\sigma, \eta) \in S_1, (f_1, f) \in \mathcal{O}_{A'} \) and \( (f, f_2) \in \mathcal{O}_{A''} \). Then \( \sigma \) is generic in \( \text{Inj}_{A,A'}(f_2, f_1) \) if and only if the module determined by \( \eta \) is generic in \( \mathcal{E}(M, N) \), where \( M \) and \( N \) are the modules determined by \( A \) and \( A' \), respectively.

**Proof.** Suppose that \( \mathcal{O}_{A''} \) is the dense open orbit in \( \mathcal{E}(A, A') \) and that \( \mathcal{O}_X \) is the dense open orbit in \( \mathcal{E}(M, N) \). By Lemmas 3.1-3.3, \( \pi_2(S \cap \pi_1^{-1}(S_2 \cap \pi_3^{-1}(\mathcal{O}_X))) \) is open in \( \text{Inj}_{A,A'}(f_2, f_1) \). By Lemma 3.5, \( \pi^{-1}(\mathcal{O}_{A''} \cap \mathcal{E}'(A, A')) \) is open in \( \text{Inj}_{A,A'}(f_2, f_1) \). Since \( \text{Inj}_{A,A'}(f_2, f_1) \) is irreducible, the intersection \( \pi_2(S \cap \pi_1^{-1}(S_2 \cap \pi_3^{-1}(\mathcal{O}_X))) \cap \pi^{-1}(\mathcal{O}_{A''} \cap \mathcal{E}'(A, A')) \) is non-empty. Therefore, \( X \) is the module determined by \( A'' \). This finishes the proof. \( \square \)

### 3.2. A generic multiplication in \( S_q(n, r) \)

We now define a multiplication, called a generic multiplication, by

\[ e_A \circ e_{A'} = \begin{cases} 
  e_{A''} & \text{if } \mathcal{E}(A, A') \neq \emptyset, \\
  0 & \text{otherwise},
\end{cases} \]

where \( \mathcal{O}_{A''} \) is the dense open orbit in \( \mathcal{E}(A, A') \).

**Proposition 3.8.** Let \( A, A', A'' \in \Theta_n^q \). Then \( (e_A \circ e_{A'}) \circ e_{A''} = e_A \circ (e_{A'} \circ e_{A''}) \)

**Proof.** By the definition of the multiplication \( \circ \), we see that \( (e_A \circ e_{A'}) \circ e_{A''} = 0 \) implies that \( e_A \circ (e_{A'} \circ e_{A''}) = 0 \) and vice versa. So we may assume that neither of them is zero. Let \( M, N, L \) be the module determined by \( A, A', A'' \), respectively. By Lemma 3.1 in [10], we know that \( (M * N) * L = M * (N * L) \). Now the proof follows from Proposition 3.7. \( \square \)

We can now state the main result of this section.
Theorem 3.9. $\mathbb{Q}(\{e_A | A \in \Theta^u_r \}, +, 0)$ is an algebra with unit $\sum_{D \in D_r} e_D$.

Proof. We need only to show that $\sum_{D \in D_r} e_D$ is the unit. Let $D$ be a diagonal matrix and let $(f_1, f_2) \in \mathcal{O}_D$. Then $f_1 = f_2$. For any $A \in \Theta^u_r$, by the definition of the generic multipilcation,

$$e_A \circ \sum_{D \in D_r} e_D = e_A \circ e_C,$$

where $C$ is the diagonal matrix $\text{diag}(\sum_j a_{1j}, \ldots, \sum_j a_{nj})$. Since $e_A e_C = \sum_B g_{A,C,B} e_B$, where $(f_1, f) \in \mathcal{O}_B$ and $g_{A,C,B} = |\{f | (f_1, f) \in \mathcal{O}_A, (f, f) \in \mathcal{O}_C\}|$, we see that $e_A e_C = e_A$. Therefore $e_A \circ e_C = e_A$. Similarly, $(\sum_{D \in D_r} e_D) \circ e_A = e_A$. Therefore $\sum_{D \in D_r} e_D$ is the unit. \hfill \Box

We denote the algebra $\mathbb{Q}(\{e_A | A \in \Theta^u_r \}, +, 0)$ in Theorem 3.9 by $S^+_0$.

4. The algebra $S^+_0$ as a quotient and 0-Schur algebras

We have two tasks in this section. We will first prove that a certain subalgebra of $S^+_0$ is a quotient of the monoid algebra defined in Section 2. We will then prove that this subalgebra gives a geometric realisation of a positive part of 0-Schur algebras.

It is well-known that the specialisation of $S_0(n, r)$ at $q = 1$ gives us the classical Schur algebra $S(n, r)$ of type A. Much about the structure and representation theory of $S(n, r)$ is known, see for example \cite{7}. A natural question is to consider the specialisation of $S_q(n, r)$ at $q = 0$, which is called 0-Schur algebra and denoted by $S_0(n, r)$. 0-Schur algebras have been studied in \cite{4, 9, 13}. In this section the 0-Schur algebras will be studied from a different point of view, that is, via a modified version of 0-Schur algebra.

We call $\theta (\mathbb{H}_q(Q))$ the positive part of the $q$-Schur algebra, and denote it by $S^+_q(n, r)$. Denote the specialisation of $S^+_q(n, r)$ at $q = 0$ by $S^+_0(n, r)$. Denote by $S^+_0$ the subalgebra of $S^+_0$, generated by $l_A r = \sum_D e_{A+D}$, where $A$ is a strict upper triangle matrix with its entries non-negative integers and the sum is taken over all diagonal matrices in $D_r - \sum_i a_{ij}$.

4.1. A modified version of $\theta$. For convenience we denote by $E_i$ the element $l_{E_{i,i+1}, r}$ in $S_0(n, r)$. We have the following result.

Proposition 4.1. The elements $E_1, \ldots, E_{n-1}$ satisfy the following modified quantum Serre relations:

$$E_i^2 E_j - (q+1)E_i E_j E_i + qE_j E_i^2 = 0 \quad \text{for } |i-j| = 1$$

$$E_i E_j - E_j E_i = 0 \quad \text{for } |i-j| > 1.$$

Proof. We only prove the first equation for $j = i + 1$. The remaining part can be done in a similar way. By Lemma \[\Box\] we have the following.

\begin{align*}
E_i E_{i+1} &= l_{E_{i,i+2}, r} + l_{E_{i,i+1}+E_{i+1,i+2}, r}, \\
E_{i+1} E_i &= l_{E_{i,i+1}+E_{i+1,i+2}, r}, \\
E_i E_i &= (q+1) l_{2E_{i,i+1}, r}, \\
E_i l_{E_{i,i+2}, r} &= q l_{E_{i,i+1}+E_{i+1,i+2}, r}, \\
E_i l_{E_{i,i+1}+E_{i+1,i+2}, r} &= l_{E_{i,i+1}+E_{i+1,i+2}, r} + (q+1) l_{2E_{i,i+1}+E_{i+1,i+2}, r}, \\
E_{i+1} E_i E_i &= (q+1) l_{2E_{i,i+1}+E_{i+1,i+2}, r}.
\end{align*}

Therefore,

$$E_i^2 E_{i+1} - (q+1)E_i E_{i+1} E_i + qE_{i+1} E_i^2 = E_i (E_i E_{i+1} - (q+1)E_{i+1} E_i) + qE_{i+1} E_i^2 = 0.$$ 

}\hfill \Box
Following Proposition 4.1 and [12], we can now modify the restriction \( \theta|_{\mathcal{H}_q(Q)} \) as follows.

\[
\theta : \mathcal{H}_q(Q) \rightarrow S_q(n, r),
\]

\[
S_i \mapsto E_i
\]

From now on, unless stated otherwise, by \( \theta \) we mean the modified map \( \theta|_{\mathcal{H}_q(Q)} \). The following result is a modified version of Proposition 2.3 in [6] and we will give a direct proof. For any two modules \( M, N \), recall that Hall multiplication of \( M \) and \( N \) is given by

\[
MN = \sum_X F_{MN}^X X,
\]

where \( F_{MN}^X = |\{ U \subseteq X | U \cong N, X/U \cong N \}| \) and where the sum is taken over all the isomorphism classes of modules.

**Proposition 4.2.** Let \( A \) be a strict upper triangular matrix with entries non-negative integers and let \( M \) be the module determined by \( A \). Then

\[
\theta(M) = \begin{cases} 
  l_{A,r} & \text{if } \sum_{i,j} a_{ij} \leq r, \\
  0 & \text{otherwise}.
\end{cases}
\]

**Proof.** Note that \( \mathcal{B} = \{ \Pi_{i,j} M(ij)^{x_{ij}} | x_{ij} \in \mathbb{Z}_{\geq 0} \} \) is a PBW-basis of \( \mathcal{H}_q(Q) \), where the product is ordered as follows: \( M(ij) \) is on the left hand side of \( M(st) \) if either \( i = s \) and \( j > t \), or \( i > s \). Let \( M \in \mathcal{B} \) be the module determined by \( A \). Suppose that \( M(st) \) is the left most term with \( x_{st} > 0 \). We can write \( M = M(st) \oplus M' \).

First consider the case \( M = M(st) \), that is, \( M \) is indecomposable. We may suppose that \( t - s > 1 \). Then \( M = M(s, t - 1)M(t - 1, t) - M(t - 1, t)M(s, t - 1) \). By induction on the length of \( M \), Lemma 4.1 and a dual version of it, we have

\[
\theta(M(s, t - 1)M(t - 1, t)) = l_{E_s,t,r} + l_{E_s,t-1+E_{t-1},t,r},
\]

\[
\theta(M(t - 1, t)M(s, t - 1)) = l_{E_s,t-1+E_{t-1},t,r}.
\]

Therefore \( \theta(M) = \theta(M(s, t - 1)M(t - 1, t)) - \theta(M(t - 1, t)M(s, t - 1)) = l_{E_s,t,r} \).

Now consider the case that \( M \) is decomposable. We use induction on the number of indecomposable direct summands of \( M \). By the assumption we have \( M = \frac{q^{-1}}{q^{st-1}} M(st) M' \).

Therefore

\[
\theta(M) = \frac{q^{-1}}{q^{st-1}} \theta(M(st)) \theta(M')
\]

\[
= \frac{q^{-1}}{q^{st-1}} \sum_{D \in D_{r-1}} e_{E_{st}+D} e_{A-E_{st}+D'},
\]

where \( D' \) is the diagonal matrix with non-negative integers as entries such that \( co(E_{st}+D) = ro(A - E_{st} + D') \). Suppose that \( e_B \) appears in the multiplication of \( e_{E_{st}+D} e_{A-E_{st}+D'} \) and \( (f, h) \in O_B \). Note that in the minimal projective resolution \( Q \rightarrow P \) of \( M' \), the projective module \( P_t \) is not a direct summand of \( P \). Therefore by the definition of the multiplication \( e_{E_{st}+D} e_{A-E_{st}+D'} \), we have \( f/h \cong M \), that is \( B = A \), and the coefficient of \( e_B \) is about the possibilities of choosing a submodule, which is isomorphic to \( P_s \), of \( P_{st} \). Hence the coefficient is \( \frac{q^{st-1}}{q-1} \) and so \( \theta(M) = \sum_{D} e_{A+D} = l_{A,r} \). This finishes the proof. \( \square \)

**Remark 4.3.** Proposition 2.3 in [6] has a minor inaccuracy. Indeed, there is a coefficient missing in front of the image \( \theta(M) \). For example, let \( n = 3 \), \( r = 2 \) and let \( M \) be the module determined by the elementary matrix \( E_{13} \). Then \( \theta(M) = vy_{X,r} \), but not \( y_{X,r} \), as stated in Proposition 2.3 in [6], here \( \theta \) is the original map from the quantised enveloping algebra to the Schur algebra \( S_n(3, 2) \) and \( y_{X,r} = \sum_{D \in D_1}[X + D] \).
4.2. A homomorphism of algebras $\Gamma : \mathcal{QM} \to S_0^+$. For a given module $M$, denote by $|M|_{\text{dir}}$ the number of indecomposable direct summands of $M$. Let $\Gamma : \mathcal{QM} \to S_0^+$ be the map given by

$$\Gamma(M) = \begin{cases} \theta(M) & \text{if } |M|_{\text{dir}} \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Let $X$ be a module and let $D = \text{diag}(d_1, \cdots, d_n)$ be a diagonal matrix. Write $X = \oplus_{i,j} M(i,j)^{x_{ij}}$. By $e_{X+D}$ we mean the basis element in $S_q(n,r)$ corresponding to the matrix with its entry at $(i,j)$ given by $x_{ij} + \delta_{ij}d_i$ for $i \leq j$ and 0 elsewhere, where $\delta_{ij}$ are the Kronecker data. Let $\sigma = (\sigma_i)_i : Q \to P$ be an injection of $Q$ into $P$, where $P$ and $Q$ are projective modules. Then $(P,Q)$ gives a pair of flags $(f_1,f_2)$ with $f_2$ a subflag of $f_1$. More precisely, the $i$-th step of $f_1$ is given by $\text{Im}P_{\alpha_{n-2}} \cdots P_{\alpha_i}$ for $i \leq n-2$ and the $(n-1)$-th step is given by the vector space associated to vertex $n-1$ of $P$, where $P_{\alpha_j}$ is the linear map on the arrow $\alpha_j$ from $j$ to $j+1$ for the module $P$. The $i$-th step of $f_2$ is given by $\text{Im}P_{\alpha_{n-2}} \cdots P_{\alpha_i}$. We have the following result.

**Theorem 4.4.** The map $\Gamma$ is a morphism of algebras.

**Proof.** The unit in $\mathcal{M}$ is the zero module. By the definition of $\Gamma$, it is clear that $\Gamma(0) = \sum_{D \in D_r} e_D$, the unit of $\Theta^n_\sigma$. We now need only to show

$$\Gamma(M \ast N) = \Gamma(M) \circ \Gamma(N). \quad (1)$$

Let $X$ be a generic point $\mathcal{E}(M,N)$. Denote the number of indecomposable direct summands of $X$, $M$, $N$ by $a$, $b$, and $c$, respectively. By the definition of $\Gamma$, we can write

$$\Gamma(X) = \sum_{D \in D_{r-a}} e_{X+D},$$

$$\Gamma(M) = \sum_{D' \in D_{r-b}} e_{M+D'},$$

$$\Gamma(N) = \sum_{D'' \in D_{r-c}} e_{N+D''}.$$

We first consider the case where $a > r$. Clearly, in this case $\Gamma(M \ast N) = 0$ and we claim that $e_{M+D'}e_{N+D''} = 0$ for any $D' \in D_{r-b}$ and $D'' \in D_{r-c}$. In fact suppose that $e_{L+D''}$ appears in the multiplication, where $L$ is a module and $D''$ is a diagonal matrix. Then $|L|_{\text{dir}} \leq r$, and $L$ is a degeneration of $X$. Since $Q$ is linearly oriented, $|L|_{\text{dir}} \geq a$. This is a contradiction. Therefore $e_{M+D'}e_{N+D''} = 0$, and so

$$(\sum_{D' \in D_{r-b}} e_{M+D'}) \circ (\sum_{D'' \in D_{r-c}} e_{N+D''}) = 0.$$  

This proves the equation (1) for the case $a > r$.

Now suppose that $a \leq r$. Note that if $(D', D'') \neq (C', C'')$, where $D', C' \in D_{r-b}$ and $D'', C'' \in D_{r-c}$, then $e_{M+D'}e_{N+D''} \neq e_{M+C'}e_{N+C''}$. By Proposition 3.7 we know that if $e_{M+D'}e_{N+D''} \neq 0$, then $e_{M+D'}e_{N+D''} = e_{X+D}$ for some $D \in D_{r-a}$.

On the other hand, we can show that for any $e_{X+D}$ appearing in the image of $X$ under $\Gamma$, there exist $D' \in D_{r-b}$ and $D'' \in D_{r-c}$ such that $e_{M+D'}e_{N+D''} = e_{X+D}$. Suppose that

$$0 \xrightarrow{Q} P \xrightarrow{\sigma} X \xrightarrow{0}$$

is the minimal projective resolution of $X$. Write $D = \text{diag}(d_1, \cdots, d_n)$ and let $Y$ be the projective module $\oplus_i P_i^{d_i}$. Then the pair of flags in $\mathcal{F} \times \mathcal{F}$, determined by $(P \oplus Y, Q \oplus Y)$, is in the orbit $Q_{X+D}$. We have the following diagram where each square commutes and all rows and columns are short exact sequences,
where $I$ is the identity map on $Y$, $K = \text{Ker}(\rho r 0)$ and $\lambda = (\tau 0)|_K$. Let $Z' \xrightarrow{I} Z'$ be the maximal contractible piece of the projective resolution

$$0 \to K \to P \oplus Y (\rho r 0) \to M \to 0$$

of $M$, and let $Z'' \xrightarrow{I} Z''$ be the maximal contractible piece of the projective resolution

$$0 \to \text{Ker}\lambda \to K \to N \to 0$$

of $N$. Write

$$Z' = \oplus_i P_i^{d_i'}$$

and $Z'' = \oplus_i P_i^{d_i''}$,

and let $D' = \text{diag}(d_1', \ldots, d_n')$ and $D'' = \text{diag}(d_1'', \ldots, d_n'')$.

Then $e_{M+D'} \circ e_{N+D''} = e_X + D$. Therefore,

$$\sum_{D' \in D_{r-b}} e_{M+D'} \circ \sum_{D'' \in D_{r-c}} e_{N+D''} = \sum_{D \in D_{r-a}} e_X + D.$$

This proves the equations (1), and so the proof is done. $\square$

The following result is a direct consequence of Theorem 4.4.

**Corollary 4.5.** $\text{Ker}\Gamma = \mathbb{Q}\text{-Span}\{M||M|_{\text{dir}} > r\}$.

4.3. A geometric realisation of 0-Schur algebras.

**Theorem 4.6.** $S_0^+(n, r) \cong \mathbb{Q} S_0^+$ as algebras.

**Proof.** Denote by $\theta_0$ the specialisation of $\theta$ to 0, that is, $\theta_0 : H_0(Q) \to S_0(n, r)$. We have $\text{Ker}\theta_0 = \mathbb{Q}\text{-Span}\{M||M|_{\text{dir}} > r\} = \text{Ker}\Gamma$, where $\Gamma$ is as in Theorem 4.4. Now the proof follows from the following commutative diagram.

$$\begin{array}{ccc}
\text{Ker}\theta_0 & \xrightarrow{\theta_0} & S_0^+(n, r) \\
\text{Ker}\Gamma & \xrightarrow{\Gamma} & S_0^+
\end{array}$$

$\square$

As a direct consequence of Theorem 4.6, we obtain a multiplicative basis of the positive part of 0-Schur algebras, in the sense that the multiplication of any two basis elements is either a basis element or zero.

**Corollary 4.7.** The elements in $\{l_{A, r} | A \text{ is an strictly upper triangular matrix in } \bigcup_{s \leq r} \Theta^s_n \}$ form a multiplicative basis of $S_0^+(n, r)$. 
Under the map $\Gamma$, this multiplicative basis $\{l_{A,r} | \text{for any } A \in \bigcup_{s \leq r} \Theta_s^u\}$ is the image of the multiplicative basis for $H_0(Q)$ studied in [10]. By Theorem 7.2 in [10], the multiplicative basis for $H_0(Q)$ is the specialisation of Lusztig’s canonical basis for a two-parameter quantization of the universal enveloping algebra of $\mathfrak{gl}_n$ given in [14]. Thus we can consider the basis $\{l_{A,r} | A \text{ is a strictly upper triangular matrix in } \bigcup_{s \leq r} \Theta_s^u\}$ as a subset of a specialization of the canonical basis.

Acknowledgement: The author would like to thank Steffen König for helpful discussions.

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