Oscillating Quantum Droplets From the Free Expansion of Logarithmic One-dimensional Bose Gases

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Abstract
We analyse some issues related to the stability and free expansion of a one-dimensional logarithmic Bose–Einstein condensate, particularly its eventual relation to the formation of quantum droplet-type configurations. We prove that the corresponding properties, such as the energy of the associated N-body ground state, differ substantially with respect to its three-dimensional counterpart. Consequently, the free velocity expansion also shows remarkable differences with respect to the three-dimensional system when logarithmic interactions are taken into account. The one-dimensional logarithmic condensate tends to form quantum droplet-type configurations when the external trapping potential is turned off, i.e., the self-sustainability or self-confinement appears as in three-dimensions. However, we obtain that for some specific values of the self-interaction parameters and the number of particles under consideration, the cloud oscillates during the free expansion around to a specific equilibrium size. These results show that we are able to describe scenarios in which the one-dimensional cloud reaches stable configurations, i.e., oscillating quantum droplets.

Keywords Static properties of condensates · Thermodynamical · Statistical and structural properties · Drops and bubbles · Other Bose-Einstein condensation phenomena

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1 Introduction

The emergence of quantum droplets in Bose–Einstein condensates (BECs) has stimulated many works concerning this fascinating phenomenon [1–5]. The formation of quantum droplets depends on the balance of attractive and repulsive interactions within the system, which in some cases stabilize the BEC against collapse or explosion. Additionally, it is well-known that quantum fluids can show liquid or gas behavior according to the corresponding interactions within the system, below some critical temperature. In the case of ultracold quantum gases, several mechanisms to stabilize the system have been proposed, for instance, quantum fluctuations and three-body correlations [5]. In the case of a mixture of two BECs with competing contact interactions, ultracold atomic droplets have also been observed [5–7]. While a single-component attractive condensate with only contact interactions collapses for a sufficiently large number of particles [8, 9], quantum fluctuations stabilize a two-component mixture with inter-component attraction and intra-component repulsion [10]. In the single-component scenario, quantum droplets’ formation has also been analyzed, in which three-body and higher-order interactions can be inserted as a logarithmic term in the corresponding interacting potential. Consequently, it can be proved that the so-called self-sustainability or self-confinement (that can be interpreted as quantum droplets) appears [11]. We must mention here that BECs with logarithmic interactions can play a central role in this scenario since a logarithmic interaction potential can account for multi-body interactions [11–13]. As a consequence, this type of interaction potentials can be used to describe some properties related to dense systems in which multi-body interactions can be representative. However, the results mentioned above have been obtained for three-dimensional systems. We have to mention here that the dynamics of one-dimensional quantum droplets have been also analyzed, see for instance, in Ref. [14]. Nevertheless, in the latter work, the authors study the behavior of one-dimensional binary Bose gases, nor a single component with a specific interaction potential, i.e., a logarithmic interaction potential that can encode the contributions of multi-body interactions in a single sample.

Let us remark that BECs in one or two spatial dimensions open up a very interesting scenario to analyse the free expansion of the cloud and its relation with the search of quantum-droplet type configurations. It is well-known that the physics related to low-dimensional BECs contains essential departures from its three-dimensional counterpart. Although (strictly speaking) one-dimensional BECs can never be achieved, it is possible to obtain quasi-one-dimensional BECs in the laboratory by using extremely anisotropic traps. For instance, the results reported in Refs. [15–18] fulfill the theoretical predictions in the full one-dimensional theoretical description under certain conditions. Also, it must be mentioned that one-dimensional BECs have some pathological behavior in the thermodynamic limit [19–22]. Finite-size effects in the system are required to make the condensation possible, and consequently, the ground state energy per particle contributions must be taken into account. Additionally, the analysis of one-dimensional BECs shows that these systems could lie in the high-density regime [22–24] and suggests some Bose–Fermi
duality in one-dimensional systems [25–29]. Thus, according to this point of view, three-body interactions (and clearly, higher-order interactions or multi-body interactions) can be relevant in the stability analysis of the one-dimensional cloud and could be also relevant in the eventual formation of quantum droplets in the one-dimensional regime for a single component.

Moreover, some properties associated with BECs, particularly those associated with their stability, can be significantly modified by the interatomic interactions. For instance, it is possible to tune the scattering length $a_S$ (which describes the two-particle interactions) by using Feshbach’s resonances. In other words, the interactions within the cloud can be tuned from the repulsive regime ($a_S > 0$) through the ideal gas ($a_S \sim 0$) and finally to the attractive regime ($a_S < 0$), where the gas becomes unstable and collapses when the number of particles is large enough [9, 30]. Thus, a logarithmic interaction potential, which generalizes the interactions within the system, must be relevant in the stability analysis of one-dimensional BECs, and a fundamental ingredient in the emergence of quantum droplets. Additionally, it is noteworthy to mention that one of the more interesting phenomena related to BECs is the free expansion of the condensate and the emergence of interference fringes of two overlapping BECs when the trapping potential is turned off [31]. Let us remark that when the trapping potential is turned off, the free velocity expansion of the three-dimensional BEC and also for the one-dimensional counterpart corresponds approximately to the velocity predicted by Heisenberg’s uncertainty principle in the ideal case, i.e., $a_S = 0$ [31, 32]. However, when interactions are present, this situation could be drastically affected, leading to the collapse or implosion of the system after the expansion under certain circumstances [9]. Moreover, as we describe in our work, interactions can stabilize the system, then the so-called self-sustainability or self-confinement can be reached. Therefore, the system can form stable configurations or quantum droplet-type configurations, even when the trapping potential is turned off.

In the present work, we investigate the stability conditions for a one-dimensional BEC when logarithmic interactions within the system are present. We analyse how the insertion of this type of logarithmic self-interaction potential modifies the dynamics of the free expansion of the cloud and its relation with the eventual formation of oscillating quantum droplet-type configurations. This paper is organized as follows: In Sect. 2, we present some properties related to one-dimensional systems. In Sect. 3, we analyse the stability conditions associated with one-dimensional BECs, and consequently, we explore the free expansion of the system when logarithmic self-interactions are present. In Sect. 4, we analyse the relation between free expansion and the formation of quantum droplet-type configurations. Finally, in Sect. 5, we present a discussion and the main results.
2 One-dimensional Logarithmic Bose–Einstein Condensate

The time-dependent behavior of BECs can be used to obtain relevant information concerning its dynamics, regarding for instance, with properties related to the free expansion of the corresponding cloud when the trapping potential is turned off. The Gross–Pitaevskii equation which describes the BEC’s dynamics, under certain considerations, can be obtained formally by using a variational formulation. For instance, in three-dimensions, the Gross–Pitaevskii equation can be obtained through the action principle

\[ \delta \int_{t_1}^{t_2} L dt = 0, \]

where \( L \) is the corresponding Lagrangian, given by

\[ L = \int dr \left\{ \frac{i\hbar}{2} \left( \Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t} \right) - E \right\}, \]

where \( \Psi \) is the BEC wave function or the so-called order parameter in the three-dimensional scenario. For our purposes, the energy \( E \) including the contribution of the logarithmic potential in the above equation is defined as follows

\[ E = \int dr \left\{ \frac{\hbar^2}{2m} |\nabla \Psi|^2 + V(r)|\Psi|^2 + \frac{1}{2} g_{3D} |\Psi|^4 - \beta |\Psi|^2 \left[ \ln \left( \alpha^3 |\Psi|^2 \right) - 1 \right] \right\}, \]

i.e., the BEC properties in three-dimensions with logarithmic interactions are governed by the later energy functional as in Ref. [33].

Notice that in the functional form of the logarithmic potential, namely

\[ V_\beta = -\beta |\Psi|^2 \left[ \ln \left( \alpha^3 |\Psi|^2 \right) - 1 \right], \]

we can identify \( |\Psi|^2 \) as the density of particles \( n \). Moreover, it must be mentioned that the logarithmic potential Eq. (4) is regular at \( n = 0 \), as was mentioned in Ref. [11], and always has the Mexican-hat shape as a function of the order parameter. Additionally, it is straightforward to show that the logarithmic potential Eq. (4) admits a power expansion around \( 1/\alpha^3 \), i.e., a perturbative limit that describes several orders in the self-interactions. However, the expansion mentioned above is not the general case. Thus, the logarithmic nonlinearity can be interpreted as a potential for describing the multi-body interactions within the system, and the analysis must be, in general, non-perturbative.

Consequently, after the variation of the Lagrangian Eq. (2), the corresponding logarithmic Gross–Pitaevskii equation (LogGPE) can be obtained as
\[
\frac{i\hbar}{\partial_t} \psi(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g_{3D}|\psi(\mathbf{r}, t)|^2 - \beta \ln \left( \alpha^3 |\psi(\mathbf{r}, t)|^2 \right) \right] \psi(\mathbf{r}, t),
\]

subject to the following normalization condition

\[
\int |\psi(\mathbf{r}, t)|^2 \, d\mathbf{r} = N. \tag{6}
\]

In the above equations, \( m \) is the mass of the corresponding boson, \( V(\mathbf{r}) \) is the external potential, \( g_{3D} = 4\pi\hbar^2 a_s/m \) is the interaction strength between any pair of bosons in three-dimensions, with \( a_s \) the s-wave scattering length of the corresponding gas, \( N \) is the number of particles, while \( \beta \) and \( \alpha^3 \) measure the strength of the nonlinear logarithmic interaction. We must mention at this point that the LogGPE preserves all the properties associated with density-dependent nonlinearities, which is, in fact, the case of the logarithmic potential, such as conservation of probability and invariance under permutation.

We consider here that the three-dimensional BEC is confined in a trap described by a harmonic oscillator potential of the form \( V(\mathbf{r}) = m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2 \), where \( \omega_{x,y,z} \) are the corresponding trap frequencies. One-dimensional BECs have very elongated, cylindrical geometries, i.e., they have a cigar-like shape. These geometries are produced by using harmonic traps with one frequency in the axial direction, \( \omega_z \), and other in the radial (transverse) direction, \( \omega_{\perp} = \omega_x = \omega_y \), such that \( \omega_z \ll \omega_{\perp} \), i.e., very anisotropic traps. Under these conditions, it is possible to freeze the motion of the bosons in the radial direction, so they occupy only the ground state of the harmonic oscillator in the transverse direction. However, the interaction energy should not excite other radial modes, which requires that \( g_{3D} n_0 \ll \hbar \omega_{\perp} \) and \( \beta \ll \hbar \omega_{\perp} \), where \( n_0 \) is the peak density of the condensate. Accordingly, the BEC wave function \( \psi(\mathbf{r}, t) \) in three-dimensions can be factorized as follows

\[
\psi(\mathbf{r}, t) = e^{-i\omega_{\perp} t} \psi_{\perp}(x, y) \psi(z, t), \tag{7}
\]

\[
\psi_{\perp}(x, y) = \frac{1}{\sqrt{\pi a_{\perp}^2}} e^{-((x^2 + y^2)/2a_{\perp}^2)}, \tag{8}
\]

where the radial wave function is appropriately normalized, \( \int d\mathbf{r}_{\perp} |\psi_{\perp}(x, y)|^2 = 1 \), and \( a_{\perp} = \sqrt{\hbar/m\omega_{\perp}} \) is the characteristic length of the trap in the radial direction.

We substitute Eq. (7) in Eq. (3) and then, integrating over the transverse direction, we are able to obtain the energy functional for this scenario, which becomes

\[
E = \hbar \omega_{\perp} + E_{1D},
\]

\[
E_{1D} = \int dz \left\{ \frac{\hbar^2}{2m} \left| \frac{d\psi(z, t)}{dz} \right|^2 + V(z)|\psi(z, t)|^2 + \frac{1}{2} g|\psi(z, t)|^4 \right. \left. - \beta |\psi(z, t)|^2 \left[ \ln \left( \frac{\alpha^3}{N\pi e a_{\perp}^2} |\psi(z, t)|^2 \right) - 1 \right] \right\}, \tag{9}
\]
is the one-dimensional energy. Then, the one-dimensional LogGPE and the associated normalization condition can be expressed as

\[ i\hbar \frac{\partial \psi(z, t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) + g|\psi(z, t)|^2 \right. \]

\[ \left. - \beta \ln \left( \frac{\alpha^3}{N\pi\varepsilon a_\perp^2} |\psi(z, t)|^2 \right) \right] \psi(z, t), \]  

(10)

\[ \int |\psi(z, t)|^2 \, dz = N. \]  

(11)

At this point, we must mention that the strong trapping in the transverse direction modifies the interaction potential between pairs of bosons in one-dimension, which becomes

\[ g = \frac{2\hbar^2 a_S}{ma_\perp^2} = 2\hbar\omega_\perp a_S. \]  

(12)

Remarkably, the integration in the transverse directions (or the dimensional reduction) shows that the trapping does not modify the factor \( \beta \), i.e., its value in one-dimension and in three-dimensions remains the same, unlike what occurs with the \( g_{3D} \) parameter as can be seen in Eq. (12). On the other hand, the corresponding integration (or the dimensional reduction) introduces several pre-factors upon the density in the logarithmic term. These terms can be re-absorbed in a new effective parameter \( \alpha \to \alpha^3/N\pi\varepsilon a_\perp^2 \) for one-dimensional clouds without loss of generalization, i.e., they can be interpreted as an energy shift and can be re-absorbed in the total energy. In other words, we recover the three-dimensional behavior for \( \alpha^3 \) and \( \beta \) reported in Ref. [33]. Consequently, the upper bound for \( \beta \leq 3.3 \times 10^{-15} \text{ eV} \) obtained in Ref. [34] can be taken as a good approximation for our model, and \( \alpha \) is also not relevant in the one-dimensional context as in Ref. [33].

Finally, notice that the time-independent version of the one-dimensional LogGPE can be obtained by using the stationary condition \( \psi(z, t) = \psi(z) \exp (-i\mu t/\hbar) \) with \( \mu \) the corresponding chemical potential, that is

\[ \left[ -\frac{\hbar^2}{2m} \frac{d}{dz} + V(z) + g|\psi(z)|^2 - \beta \ln \left( \alpha|\psi(z)|^2 \right) \right] \psi(z) = \mu \psi(z), \]  

(13)

which can be also derived from the time-independent one-dimensional energy functional, namely

\[ E(\psi) = \int dz \left[ \frac{\hbar^2}{2m} \left| \frac{d\psi(z)}{dz} \right|^2 + V(z)|\psi(z)|^2 + U(|\psi(z)|^2) \right], \]  

(14)

where the logarithmic potential in one-dimension is given by
\[ U(|\psi(z)|^2) = \frac{1}{2} \beta |\psi(z)|^4 - \beta |\psi(z)|^2 \left[ \ln \left( \alpha |\psi(z)|^2 \right) - 1 \right], \]  

(15)

together with \(|\psi(z)|^2\) the corresponding density of particles and the trapping potential described as \(V(z) = m\omega_z^2 z^2 / 2\).

### 3 Stability Conditions

We must mention here that to calculate the corresponding energy in Eq. (14), formally we have to solve the corresponding one-dimensional LogGPE Eq. (10). However, in order to simplify the calculations, we are able to employ an accurate expression for the total energy of the cloud that can be obtained by using, as usual, an ansatz of the form [32]

\[
\psi(z) = \frac{N^{1/2}}{\pi^{1/4}} \sqrt{l} \exp \left( -z^2 / 2l^2 \right) \exp(i\phi(z)).
\]

(16)

The ansatz Eq. (16) is the solution of the Schrödinger equation associated with non-interacting systems in one dimension, together with \(N\) the corresponding number of particles. Additionally, we interpreted the characteristic length \(l = \sqrt{\hbar / m\omega_z}\) as the initial size of the quasi-one-dimensional condensate in the non-interacting case. The choice of the ansatz Eq. (16), for the case of a one-dimensional LogBEC trapped in a one-dimensional harmonic oscillator potential seems to be a reasonable conjecture. In other words, it is clear that the ansatz Eq. (16) reflects the symmetry of the trap and, in the non-interacting case, is the exact solution of the corresponding equation of motion. Thus, the free velocity expansion can be calculated in this scenario without loss of generality by using the ansatz Eq. (16) at least to first-order approximation, in order to obtain the contributions caused by the parameters related to the logarithmic interaction potential. Let us mention at this point that the system’s analysis by solving directly the corresponding one-dimensional LogGPE deserves a more in-depth study that we will present elsewhere.

As was mentioned above, the ansatz Eq. (16) corresponds to the Schrödinger equation’s solution associated with non-interacting systems, where the phase \(\phi\) can be associated with particle currents [32]. Thus, by inserting the ansatz Eq. (16) in the energy functional Eq. (14), we can obtain the corresponding energy

\[ E = E_F + E_R, \]

(17)

where \(E_F\) is the kinetic energy associated with particle currents given by

\[ E_F = \frac{\hbar^2}{2m} \int \mathrm{d}z |\psi(z)|^2 \left( \frac{d}{dz} \phi(z) \right)^2. \]

(18)

Additionally, \(E_R\) in Eq. (17) can be interpreted as the energy associated with an effective potential equal to the condensate’s total energy when the phase \(\phi\) does not
vary in space. The term $E_R$ contains the contributions of the zero-point energy ($E_0$), the harmonic oscillator potential ($E_P$), and the contributions due to the interactions among the particles within the condensate ($E_I$), i.e.,

$$E_R = E_0 + E_P + E_I,$$

where

$$E_0 = \frac{\hbar^2}{2m} \int \mathrm{d}z \left| \frac{\mathrm{d}\psi(z)}{\mathrm{d}x} \right|^2,$$

$$E_P = \frac{1}{2} m \omega_z^2 \int \mathrm{d}z z^2 |\psi(z)|^2,$$

and

$$E_I = \frac{1}{2} g \int \mathrm{d}z |\psi(z)|^4 - \beta \int \mathrm{d}z |\psi(z)|^2 \left[ \ln(\alpha |\psi(z)|^2) - 1 \right].$$

Consequently, $E_R$ can be written as

$$E_R = \frac{\hbar^2 N}{8ml^2} + \frac{m \omega_z^2 l^2 N}{8} + \frac{g N^2}{4 \sqrt{2\pi} l} - \frac{\beta}{4} \left( 2 \ln \left( \frac{\alpha N}{l} \right) + \sqrt{2} - 2 - \ln(\pi) \right) N,$$

where we have used the trial function Eq. (16) together with Eqs. (2021)–(22) in order to obtain the above expression for $E_R$. The system’s equilibrium radius $l_0$ can be calculated by minimizing the energy $E_R$ in Eq. (23), that is, $(dE_R/dl)_{l=l_0} = 0$. Additionally, the kinetic energy contribution Eq. (18) is positive definite and is zero when the phase $\phi$ is constant [32].

Due to the total energy associated with the BEC is $E = \hbar \omega_\perp + E_{1D}$, it is clear that $\hbar \omega_\perp$ arises as characteristic energy that we can use to rewrite the energy $E_R$ in a dimensionless form. After some algebraic steps, we can express the energy per boson as

$$\frac{E_R}{N \hbar \omega_\perp} = \frac{r^{-2}}{8} + \frac{\lambda^2 r^2}{8} + \frac{1}{2 \sqrt{2\pi}} \left( N \frac{a_s}{a_\perp} \right) r^{-1} - \frac{\beta}{4 \hbar \omega_\perp} \left[ -2 \ln(r) + \sqrt{2} - 2 - \ln(\pi) \right],$$

where the ratio $r = l/a_\perp$ measures the BEC’s size and $\lambda = \omega_z / \omega_\perp$. Since the constant $\alpha$ is positive but arbitrary and has no physical relevance [33], we choose $\alpha = a_\perp / N$, so the contributions of the term $\ln(\alpha N / a_\perp)$ that arises in Eq. (23) can be neglected for all practical purposes.

Let us point out some substantial differences between the energy Eq. (24) compared with its counterpart in three-dimensions. The first significative difference relies on the functional form of the third right-hand term of Eq. (23), that is, the contribution caused by the two-body interactions when $\beta$ is set to be zero. Due to the system’s dimensionality, the above third right-hand term in Eq. (24) behaves as $r^{-1}$.
instead of the usual $r^{-3}$ behavior for the three-dimensional scenario [32]. This difference is significant when we consider a BEC with attractive interactions (such as Lithium). In the three-dimensional case, the interaction energy becomes the dominant term when $r \to 0$, since its absolute value eventually becomes bigger than the zero-point energy term, which behaves as $r^{-2}$. Then, for attractive interactions, the energy diverges to minus infinity, which is why the BEC collapses when the number of particles is large enough. However, in the one-dimensional scheme, the behavior is entirely different since the zero-point energy term becomes dominant as $r \to 0$. Even with attractive interactions, the energy reaches a global minimum and eventually becomes positive and diverges as $r$ gets smaller. Nevertheless, the energy can become negative if the number of bosons is large enough, an indicator of instability, similar to what occurs in a three-dimensional BEC.

Thus, according to our results, we show in Fig. 1a the energy associated with our one-dimensional LogBEC with the characteristics reported in Ref. [35] for an experiment with a gas of $^{23}$Na. The scattering length of $^{23}$Na is $a_S = 53.65a_0 = 2.80$
nm, being $a_0$ the Bohr radius, where the BEC is confined in a trap with $\omega_z/2\pi = 3.5$ Hz and $\omega_\perp/2\pi = 360$ Hz. Accordingly, we find that $a_\perp \sim 1.105 \mu$m, $a_z \sim 10.142 a_\perp \sim 11.207 \mu$m, and $a_S/a_\perp \sim 2.534 \times 10^{-3}$. On the other hand, optical diffraction experiments with neutrons [34] set an upper bound for the magnitude of the logarithmic nonlinearity in three-dimensions, which turns out to be $\beta \leq 3.3 \times 10^{-15}$ eV, i.e., $\beta \leq 2.2174 \times 10^{-12} (\hbar \omega_\perp)$.

The results obtained in Fig. 1a show how the number of particles in the condensate affects the corresponding energy landscape in the case of $^{23}$Na. We see that the one-dimensional LogBEC energy grows as the number of particles in the gas increases. The minimum energy corresponding to the equilibrium size $l_0$ shows that the cloud’s size also increases as $N$ grows. We also can notice that the logarithmic nonlinearity in the LogGPE increases the energy. Due to this fact, the cloud’s equilibrium size (filled circles) is slightly smaller than the equilibrium size when the logarithmic term is turned off, i.e., $\beta = 0$ (empty circles). All of these results are as expected since the interactions between the sodium particles are repulsive.

On the other hand, in Fig. 1b, we show the corresponding energy of a $^7$Li one-dimensional LogBEC, subject to a harmonic trap with the same characteristics used for the $^{23}$Na BEC in Fig. 1a. According to Ref. [36], the $^7$Li scattering length is $a_S = -27.3 a_0 \approx 1.444$ nm, so $|a_S|/a_\perp \approx 7.22 \times 10^{-4}$. It is clear how the attractive interactions lower the energy more and more as the number of bosons increases since the interaction energy is negative. It is expected that the energy becomes negative if $N$ becomes large enough, which would signal the one-dimensional LogBEC collapse. However, this situation cannot be reliably studied by using the one-dimensional LogGPE since the ansatz Eq. (7) used for the dimensional reduction procedure requires that $g_{3D} n_0 \ll \hbar \omega_\perp$, with $n_0$ being the peak density at the minimum of the trap, a condition that is equivalent to the following condition

$$\frac{N |a_S|}{l_0} \left( \frac{a_S}{a_\perp} \right)^{-1} \ll 1. \quad (25)$$

For a large number of bosons, this condition breaks down as the interaction energy becomes significant and comparable to $\hbar \omega_\perp$. Hence, the bosons start to populate the excited states in the radial direction, so the ansatz Eq. (7) is no longer appropriated. In other words, the procedure to obtain the one-dimensional LogGPE is no longer valid. For comparison purposes, in Fig. 1b, every energy curve fulfills Eq. (25) at its minimum, i.e., at the one-dimensional LogBEC equilibrium size, except for $N = 1500$ and $N = 2000$, where the condition Eq. (25) breaks down. Therefore, the one-dimensional LogGPE energy functional Eq. (9) becomes unsuitable for such a large number of bosons.

It must be mentioned here that the above results are consistent with the behavior of low dimensional BECs, in the sense that finite-size corrections upon the system must be taken into account in order to obtain a well-defined condensate at finite temperature. In other words, the system becomes unstable for a large number of particles, and it seems to be that finite size corrections upon the one-dimensional LogBEC are necessary to reach stability. Consequently, these finite-size corrections play a central role in the formation and stability of quantum droplet-type configurations.
We can conclude that the condition Eq. (25) guarantees that a one-dimensional Log-BEC in the regime discussed in the preceding paragraphs is stable and does not collapse even in the presence of attractive interactions for finite-size systems.

4 Free Expansion and Quantum Droplets

We are interested in analyzing the free expansion of a logarithmic one-dimensional BEC and its eventual relation to the emergence of quantum droplets. For this purpose, we assume that the trapping potential in the \( z \)-direction \( V(z) \) is turned off at some time \( t \), while in the transverse directions remains active (see, for instance, Ref. [37] for an experimental realization of a similar situation). In this scenario, the BEC is free to expand in the \( z \)-direction, while it remains constrained in the \( y \) and \( z \) directions. Thus, we can model the BEC from the very beginning as an isolated one-dimensional system, and we are able to use the one-dimensional equation for the energy, Eq. (9), together with the ansatz, Eq. (16), in order to analyse the free expansion of the system and the emergence of quantum droplets in one-dimension without loss of generality.

Thus, after the external potential \( V(z) \) is turned off, let us say, at \( t = 0 \), there is a force that changes the stability size of the cloud and produces an expansion of the BEC. In order to determine an equation for the dynamics of the system, we must deduce the corresponding kinetic energy \( E_F \), Eq. (18), as a function of time through its dependence on the radius \( l \) at any time. By changing \( l \) from its initial value to a new value \( \tilde{l} \) amounts to a uniform dilation of the system, since the new density distribution \( |\psi(z)|^2 \) may be obtained from the old one by changing the corresponding coordinate of each particle by a factor \( \tilde{l}/l \), see, for instance, Ref. [32]. Thus, the velocity of a particle can be expressed as follows,

\[
\dot{v}(z) = z \frac{\dot{l}}{l},
\]

where the dot means differentiation with respect to time. Then, it is straightforward to obtain the kinetic energy \( E_F \) by using the ansatz Eq. (16), with the result

\[
E_F = \frac{mN}{8} \dot{l}^2,
\]

which scales linearly with the number of particles \( N \). Moreover, assuming that the energy is conserved at any time, we are able to obtain the following energy conservation condition for our system

\[
\frac{\dot{l}^2 m}{8} + \frac{\hbar^2}{8 m l^2} + \frac{g N}{4 \sqrt{2 \pi} l} - \frac{\beta}{4} \left( 2 \ln \left( \frac{aN}{l} \right) \right) = \frac{\hbar^2}{8 m l_0^2} + \frac{g N}{4 \sqrt{2 \pi} l_0} - \frac{\beta}{4} \left( 2 \ln \left( \frac{aN}{l_0} \right) \right),
\]

where \( l_0 \) is the condensate’s radius at time \( t = 0 \), and \( l \) corresponds to the radius at time \( t \). Notice that in the ideal case, i.e., setting \( g = \beta = 0 \), we obtain the analytical result
where \( v_0 = \frac{\hbar}{ml_0} \) is the free velocity expansion of the condensate, corresponding to the velocity predicted by Heisenberg’s uncertainty principle for a particle confined a distance \( l_0 \). However, as we will see later in the manuscript, the contribution of parameters \( g \) and \( \beta \) modifies the free expansion properties. When parameters \( g \) and \( \beta \) are present, the dynamics of the system allows the emergence of quantum droplets, which are not allowed in the ideal case or even when the contributions of \( g \) alone are taken into account for a single component when the trap is turned off.

In order to analyse the LogBEC’s free expansion, we rewrite Eq. (28) in terms of its rescaled size, \( r = l/a_\perp \), together with its equilibrium size before turning off the trap, i.e., \( r_0 = r(t = 0) = l_0/a_\perp \), with the result

\[
\left( \frac{dr}{d\tau} \right)^2 + (r^{-2} - r_0^{-2}) + \frac{4}{\sqrt{2\pi}} \left( N \frac{a_S}{a_\perp} \right) (r^{-1} - r_0^{-1}) - \frac{4\beta}{\hbar \omega_\perp} \ln \left( \frac{r}{r_0} \right) = 0,
\]

where \( \tau = \omega_\perp t \) is the rescaled time in terms of the transverse trapping frequency. Notice that Eq. (30) shows that the one-dimensional LogBEC radius does not depend on \( a_\perp \). By solving numerically Eq. (30), we can analyse the size of the one-dimensional LogBEC under free expansion (keeping the parameter \( \beta > 0 \)) for several number of particles, considering that its physical characteristics are similar to

\[ \text{Fig. 2.} \text{ Size and velocity of the one-dimensional Log-BEC in free expansion as a function of time } \omega_\perp t \text{ for several numbers of particles } N. \text{ The scattering length is } a_S = 53.65 a_0 \text{, see Ref. [35], and } \beta = 2.2174 \times 10^{-3} (\hbar \omega_\perp) \text{ (Color figure online)} \]
Fig. 3 Size and velocity of the one-dimensional LogBEC in free expansion as a function of time $\omega_{\perp}t$ for several numbers of particles $N$. The scattering length is $a_S = -27.3a_0$, see Ref. [36], and $\beta = 2.2174 \times 10^{-3}(\hbar\omega_{\perp})$ (Color figure online).

Fig. 4 Size and velocity of the one-dimensional LogBEC in free expansion as a function of time $\omega_{\perp}t$ for several numbers of particles $N$. The scattering length is $a_S = -27.3a_0$, and $\beta = 4.4348 \times 10^{-3}(\hbar\omega_{\perp})$ (Color figure online).
those of $^{23}\text{Na}$ and $^7\text{Li}$. The results of this analysis are shown in Figs. 2, 3, 4 and 5. Notice that, qualitatively speaking, the corresponding $^{23}\text{Na}$ and the $^7\text{Li}$ free-expansions are very similar, in the sense that they oscillate around a specific size that corresponds to the equilibrium radius. Moreover, the oscillation frequency of the cloud depends on the number of particles, i.e., the oscillation increases for a larger number of particles. Additionally, we observe that they form confined clouds in both cases, i.e., for $^{23}\text{Na}$ and $^7\text{Li}$ the so-called self-sustainability or self-confinement appears. In other words, the system is able to form oscillating quantum droplet-type configurations. Specifically, we notice that in the $^{23}\text{Na}$ case (see Fig. 2), in which $a_S > 0$ and $\beta > 0$, the time range of expansion is significative larger, which translates into slower oscillations. Also, the cloud’s size is much bigger compared with the case $a_S < 0$ and $\beta > 0$ for $^7\text{Li}$ showed in Figs. 3, 4 and 5. In this scenario, we have taken two different values of $\beta$ compared to the upper value reported in Ref. [34]. We can also notice that different values for the parameter $\beta$ modify the size and frequency of the cloud oscillations, as shown in Figs. 4 and 5. However, the choice of several values of the parameter $\beta$ does not change the fact the that the system is able to form quantum droplet-type configurations. The quantum droplet-type configurations vanish when $\beta$ vanishes, as expected.

Let us point it out that the oscillation time range shown in Figs. 3, 4, and 5 is around 100 and 200 milliseconds, which means that a similar experiment could be performed on earth laboratory experiments [31] where the free expansion time is of the order of milliseconds. In comparison, the oscillation time shown in Fig. 2 corresponds to a physical time of approximately 3 seconds. Therefore, a
Fig. 6  Size and velocity of the one-dimensional LogBEC in free expansion as a function of time $\omega_{\perp}t$ for different values of the nonlinear interaction $\beta$ and $N = 200$. The scattering length is $a_s = 53.65a_0$ and $\beta_0 = 2.2174 \times 10^{-3}/(\hbar \omega_{\perp})$ (Color figure online)

Fig. 7  Size and velocity of the one-dimensional LogBEC in free expansion as a function of time $\omega_{\perp}t$ for different values of the nonlinear interaction $\beta$ and $N = 200$. The scattering length is $a_s = -27.3a_0$ and $\beta_0 = 2.2174 \times 10^{-3}/(\hbar \omega_{\perp})$ (Color figure online)
similar experiment could be performed in a free fall microgravity environment like in Refs. [38, 39], or in a microgravity setup orbit experiment like in Refs. [40, 41] if larger times in the free expansion of the cloud are necessary.

Finally, in Figs. 6 and 7, we show the system’s behavior under free expansion for several values of the parameter $\beta$ in the case of $^{23}$Na and $^7$Li with $N = 200$ particles, correspondingly. We immediately notice that both systems are stable for the chosen parameters. Consequently, the self-sustainability or self-confinement (or specifically speaking, the formation of oscillating quantum droplet-type configurations) also appears. We have restricted our analysis to positive values of $\beta$ since negative values of this parameter seem to have no physical relevance, as was pointed out in Refs. [34, 42]. However, negative values of $\beta$ could be analyzed in the present context since the only restriction, as far as we know, is the upper bound reported in Ref. [34]. Accordingly, a lower bound for $\beta$ could be estimated through free expansion experiments and its corresponding quantum droplets if they appear. Nevertheless, the study of negative values of $\beta$ deserves a more in-depth analysis and is out of the scope of the present work.

5 Conclusions

In the present work, we prove that the one-dimensional LogBEC under free expansion can form quantum droplet-type configurations. In other words, we show that our model predicts almost the same structural configuration, qualitatively speaking for the case of attractive and repulsive interactions encoded in the one-dimensional two-body interaction parameter $g$, keeping $\beta > 0$. Additionally, we also prove with our formalism that it is not necessary a mixture of BECs, i.e., a two-component system, to obtain self-sustaining or self-confined configurations or, more specifically, quantum droplets. On the other hand, it is clear that the results obtained here must be generalized, i.e., it is necessary to solve the full LogGPE equation, within its regime of validity, in order to obtain a more general description of the ground state wave function or the corresponding order-parameter, its corresponding energy, and its dynamics. The aforementioned general description would allow us to establish the validity of the ansatz used in the present work. In this sense, it could be interesting also to extend the formalism described in the present report in order to explore, for instance, the possibility of negative values of $\beta$ and how these values affect the free-expansion of the cloud together with its relation to the formation of quantum droplet-type configurations. Also, the eventual emergence of emission of matter-wave jets, as reported in Ref. [43], can be analyzed, in principle, without the requirement of Feshbach resonances in the case of dense enough one-dimensional systems. Following this line of thought, it also becomes quite exciting to study bosenova-type effects in three, two, and one-dimensional systems when logarithmic interactions are present. Finally, it would be interesting also to extend the study realized in the present work to explore these phenomena in the gravitational physics context. For
instance, we could analyse the stability of the so-called *boson stars* and the formation of matter jets in these systems. We could also analyse the formation of eventual *bosonovas* and explore the relation of BECs as boson stars with dark matter in the universe.

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