Notes on the lattice of fuzzy rough sets

Dávid Gégény\textsuperscript{a,1,*}, László Kovács\textsuperscript{b,1}, Sándor Radeleczki\textsuperscript{a,1}

\textsuperscript{a}Institute of Mathematics, University of Miskolc, 3515 Miskolc-Egyetemváros, Hungary
\textsuperscript{b}Department of Information Technology, University of Miskolc, 3515 Miskolc-Egyetemváros, Hungary

Abstract
Since the theory of rough sets was introduced by Zdzislaw Pawlak, several approaches have been proposed to combine rough set theory with fuzzy set theory. In this paper, we examine one of these approaches, namely fuzzy rough sets, from a lattice theoretic point of view. We connect the lower and upper approximations of a fuzzy relation $R$ to the approximations of the core and support of $R$. We also show that the lattice of fuzzy rough sets corresponding to a fuzzy equivalence relation $R$ and the crisp subsets of its universe is isomorphic to the lattice of rough sets for the (crisp) equivalence relation $E$, where $E$ is the core of $R$. We establish a connection between the exact (fuzzy) sets of $R$ and the exact (crisp) sets of the support of $R$. Additionally, we examine some properties of a special case of a fuzzy relation.

Keywords:
- fuzzy rough set
- lower and upper approximation
- fuzzy equivalence
- uncertain knowledge
- regular double Stone lattice
- dually well-ordered set

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*Corresponding author

Email addresses: matgd@uni-miskolc.hu (Dávid Gégény),
kovacs@iit.uni-miskolc.hu (László Kovács), matradi@uni-miskolc.hu (Sándor Radeleczki)

URL: http://www.uni-miskolc.hu/~matgd/ (Dávid Gégény),
https://www.iit.uni-miskolc.hu/munkatarsak/kovacs-laszlo.html (László Kovács), http://www.uni-miskolc.hu/~matradi/ (Sándor Radeleczki)

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1. Introduction

The notion of fuzzy sets and rough sets both extend the concept of traditional (crisp) sets by incorporating that our knowledge may be uncertain or incomplete. However, these approaches address the problem of imperfect information in a different way.

Rough sets were introduced by Zdzisław Pawlak [1], and they use the lower and upper approximations of a (crisp) set based on the indiscernibility relation of the elements. Given a reference set \( A \) in a universe \( U \) and an equivalence relation \( R \subseteq U \times U \), the lower approximation of the set \( A \) is

\[
A_R = \{ x \in U \mid [x]_R \subseteq A \}
\]

and the upper approximation of \( A \) is

\[
A^R = \{ x \in U \mid [x]_R \cap A \neq \emptyset \},
\]

where \([x]_R\) is the \( R \)-equivalence class of an element \( x \). The pair \((A_R, A^R)\) is called the rough set corresponding to the reference set \( A \) and \((U, R)\) is called an approximation space. The rough sets corresponding to this approximation space \((U, R)\) can be ordered with respect to the component-wise inclusion, and they form a complete lattice with several particular properties denoted by \( \mathbb{RS}(U, R) \), see e.g. [2], [3] and [4].

The theory of fuzzy sets was introduced by Lotfi Zadeh [5]. Traditional (crisp) sets can either contain an element of the universe or not contain the element. Fuzzy sets define a membership function instead, the value of which can range from 0 to 1. Therefore, the membership function of a set \( A \) is given by a map \( \mu_A : U \rightarrow [0, 1] \). The membership degree 0 means that the element is certainly not a member of the set, and a membership degree of 1 means that the element is certainly in the set.

The first approach to integrate the two main theories of granular computing, the theories of fuzzy sets and rough sets, relates to the work of Dubois and Prade [6]. The proposed lower and upper approximations for fuzzy sets are defined using the t-norm Min and its dual co-norm Max. Using the symbolic notation introduced by Yao in [7], the fuzzy rough set of a fuzzy set \( \Gamma \) is defined with

\[
\mu_{\text{apr}_R(\Gamma)}(x) = \inf\{ \max(\mu_\Gamma(y), 1 - \mu_R(x, y)) \mid y \in U \},
\]

\[
\mu_{\text{apR}(\Gamma)}(x) = \sup\{ \min(\mu_\Gamma(y), \mu_R(x, y)) \mid y \in U \},
\]
where $U$ denotes the universe set, $\mathcal{R}$ is the symbol for a fuzzy similarity relation and the family of all fuzzy equivalence classes is denoted by $U/\mathcal{R}$. As the definition shows fuzzy rough sets are rough sets having fuzzy sets as lower and upper approximations. As crisp sets are special cases of fuzzy sets, the given definition can also be used to construct fuzzy rough sets for crisp sets of the universe too. A comparison of the two approaches can also be found in [8]. A different integration view is realized in the theory of rough fuzzy sets [7]. We also note that as Yao presented in his work [7], rough fuzzy sets are special cases of fuzzy rough sets as defined by Dubois and Prade.

The main application area of the fuzzy rough set theory relates to optimisation of knowledge engineering algorithms. Regarding the data preprocessing phase, the fuzzy rough set models are used mainly for attribute reduction [9], [10]. The main benefit of this approach, that fuzzy-rough feature extraction preserves the meaning, the semantics of the selected features after elimination of the redundant attributes. The FRFS method works with discovering dependencies between the elements of the attribute set. The fuzzy rough set model can also be be used for general data mining operations, like clustering or classification in the case of uncertain input domains [11].

The main focus of this paper is on fuzzy rough sets (representing a combination of the notions of rough set and fuzzy set), meaning that we examine crisp sets as reference set in a fuzzy approximation space. In [7] other two cases are discussed as well: rough fuzzy sets, which use fuzzy reference sets in a crisp approximation space, and a more general case that uses fuzzy reference sets in a fuzzy approximation space. The properties of the latter are also examined in [12], where an application in query refinement is also presented.

Let $(U, R)$ be a fuzzy approximation space, where $U$ is the universe and $R$ is a fuzzy equivalence relation defined by a mapping $\mu_R : U^2 \to [0, 1]$. A fuzzy equivalence relation is a reflexive, symmetric and transitive fuzzy relation. Since we are considering fuzzy relations, reflexive property means that $\mu_R(x, x) = 1$ for every $x \in U$ and symmetry means that $\mu_R(x, y) = \mu_R(y, x)$ for every $x, y \in U$. Furthermore, a fuzzy relation is transitive if $\min(\mu_R(x, y), \mu_R(y, z)) \leq \mu_R(x, z)$ for every $x, y, z \in U$.

Now, let $A \subseteq U$ be a crisp set. A fuzzy rough set with reference set $A$ is defined as a pair of two fuzzy sets corresponding to $A$ [7]. The lower approximation of $A$ is given by the membership function

$$
\mu_{[A]_R}(x) = \inf\{1 - \mu_R(x, y) \mid y \notin A\},
$$
and the upper approximation of $A$ is given by the membership function

$$\mu_{[A]^R}(x) = \sup\{\mu_R(x, y) \mid y \in A\}.$$ 

It is easy to check that $\mu_{[\emptyset]^R} = \mu_{[\emptyset]^R} = 0$, where 0 denotes the constant 0 mapping on $U$, and $\mu_{[U]^R} = \mu_{[U]^R} = 1$, where 1 stands for the constant 1 mapping on $U$ (Notice that $\sup \emptyset = 0$, $\inf \emptyset = 1$, and $\mu_R(x,x) = 1$, for any $x \in U$.) The fuzzy rough set corresponding to the crisp set $A$ is the pair $(\mu_{[A]^R}, \mu_{[A]^R})$. We know that the set of all rough sets in approximation space $(U,R)$ form a lattice structure with several interesting properties (see e.g. [2], [3], [4], [13]). The goal of this paper is to examine the structure of fuzzy rough sets for such favorable properties and draw a comparison to the case of traditional rough sets.

2. Preliminary observations

Let $R$ be a fuzzy relation with a mapping $\mu_R : U^2 \rightarrow [0,1]$. The set $S_R := \{\mu_R(x,y) \mid x,y \in U\}$ is called the spectrum of the fuzzy relation $R$. Clearly, $S_R$ is a subset of the interval $[0,1]$. We say that a fuzzy relation $R$ has a dually well-ordered spectrum, if any nonempty subset of $S_R$ has a maximal element. This is equivalent to the fact that for any $x \in U$ and any crisp set $B \subseteq U$, $B \neq \emptyset$ there exists at least one element $m_x \in B$ such that

$$\sup\{\mu_R(x,y) \mid y \in B\} = \max\{\mu_R(x,y) \mid y \in B\} = \mu_R(x,m_x).$$

Observe that this is the case when the spectrum $S_R$ of $R$ is a finite set. Clearly, if $R$ has a dually well-ordered spectrum, then for any crisp set $A \subseteq U$, $A \neq \emptyset$

$$\mu_{[A]^R}(x) = 1 - \max\{\mu_R(x,y) \mid y \notin A\},$$

$$\mu_{[A]^R}(x) = \max\{\mu_R(x,y) \mid y \in A\}.$$ 

A similar approach in case of finite (crisp) base sets can be found in [13], for decision attributes of decision tables in order to introduce distance measures on fuzzy rough sets. As we pointed out previously the fuzzy rough set corresponding to a crisp set $A$ is a pair of mappings $(\mu_{[A]^R}, \mu_{[A]^R})$. Let us denote the collection of these pairs by $\mathcal{RS}(U, R)$, i.e. let

$$\mathcal{RS}(U, R) := \{(\mu_{[A]^R}, \mu_{[A]^R}) \mid A \subseteq U\}.$$ 

The elements of $\mathcal{RS}(U, R)$ can be ordered by component-wise order as follows:
\[ (\mu_{[A,R]} \cdot \mu_{[A,R]}) \leq (\mu_{[B,R]} \cdot \mu_{[B,R]}) \iff \mu_{[A,R]}(x) \leq \mu_{[B,R]}(x) \text{ and } \mu_{[A,R]}(x) \leq \mu_{[B,R]}(x), \text{ for all } x \in U, \]

obtaining a poset \((RS(U, R), \leq)\) with least element \((0, 0)\) and greatest element \((1, 1)\). We will prove that for any fuzzy equivalence relation \(R\) with a dually ordered spectrum, this poset is a complete lattice.

For any number \(\alpha \in [0, 1]\), the crisp relation

\[ R_\alpha := \{(x, y) \in U^2 \mid \mu_R(x, y) \geq \alpha\} \]

is called an \(\alpha\)-section (\(\alpha\)-level) of the fuzzy relation \(R\). If \(R\) is a fuzzy equivalence, then it is well-known that \(R_\alpha\) is a crisp equivalence for any \(\alpha \in [0, 1]\). Denote by \(E\) the crisp equivalence \(R_1\), i.e. let \(E := \{(x, y) \in U^2 \mid \mu_R(x, y) = 1\}\). The \(E\)-equivalence class of an element \(x \in U\) will be denoted by \([x]_E\). Hence

\[ [x]_E = \{y \in U \mid (x, y) \in E\} = \{y \in U \mid \mu_R(x, y) = 1\}. \]

Now we prove

**Lemma 1.** For any \(y \in [x]_E\), and for each \(z \in U\) we have \(\mu_R(z, x) = \mu_R(z, y)\).

**Proof.** Since \(R\) is a fuzzy equivalence on \(U\) and \((x, y) \in E\), we can write:

\[ \mu_R(z, y) \geq \min(\mu_R(z, x), \mu_R(x, y)) = \mu_R(z, x), \text{ because } \mu_R(x, y) = 1, \]

and similarly,

\[ \mu_R(z, x) \geq \min(\mu_R(z, y), \mu_R(y, x)) = \mu_R(z, y), \text{ because } \mu_R(y, x) = 1. \]

Hence we obtain \(\mu_R(z, y) = \mu_R(z, x)\). \(\square\)

Now, let \(S\) be the support of the fuzzy equivalence relation \(R\) with membership function \(\mu_R\), i.e. let

\[ S = \{(x, y) \in U^2 \mid \mu_R(x, y) > 0\}, \]

where \(U\) is the universe of \(R\).

It is trivial that \(S\) is reflexive and symmetric. Now, assume that \((x, y) \in S\) and \((y, z) \in S\) for some \(x, y, z \in U\). This means that \(\mu_R(x, y) > 0\) and \(\mu_R(y, z) > 0\). Since \(R\) is a fuzzy equivalence relation, \(\mu_R(x, z) \geq \min(\mu_R(x, y), \mu_R(y, z)) > 0\). Therefore, \(\mu_R(x, z) > 0\), from which
$(x, z) \in S$, meaning that $S$ is an equivalence relation. As before, the $S$-equivalence class of an element $x$ will be denoted by $[x]_S$.

Using the above defined crisp equivalence relations $E \subseteq U \times U$ and $S \subseteq U \times U$, we can assign (crisp) rough sets to any reference set $A \subseteq U$, by defining its lower and upper approximation with respect to $E$ or $S$:

$$A_E = \{x \in U \mid [x]_E \subseteq A\} \quad \text{and} \quad A^E = \{x \in U \mid [x]_E \cap A \neq \emptyset\},$$

$$A_S = \{x \in U \mid [x]_S \subseteq A\} \quad \text{and} \quad A^S = \{x \in U \mid [x]_S \cap A \neq \emptyset\}.$$

**Lemma 2.** For any subset $A \subseteq U$ we have

(i) $A_E = \{x \in U \mid \mu_{[A]^R}(x) = 1\}$,
(ii) $A_E = \{x \in U \mid \mu_{[A]^R}(x) > 0\}$,
(iii) $A_S = \{x \in U \mid \mu_{[A]^R}(x) > 0\}$,
(iv) $A_S = \{x \in U \mid \mu_{[A]^R}(x) = 1\}$.

In other words, assertions (i) and (ii) in Lemma 2 mean that $A^E$ is equal to the core of the fuzzy set corresponding to $\mu_{[A]^R}$, while $A_E$ is equal to the support of the fuzzy set corresponding to $\mu_{[A]^R}$. Similarly, (iii) and (iv) mean that $A^S$ is equal to the support of the fuzzy set corresponding to $\mu_{[A]^R}$, while $A_S$ is equal to the core of the fuzzy set corresponding to $\mu_{[A]^R}$.

**Proof.** (i) If $x \in A_E$, then there exists a $y \in A$ with $(x, y) \in E$, i.e. $\mu_R(x, y) = 1$. Hence $\mu_{[A]^R}(x) = \sup\{\mu_R(x, y) \mid y \in A\} = 1$. Conversely, suppose that $\mu_{[A]^R}(x) = 1$ for some $x \in U$. Since $R$ has a dually well-ordered spectrum, this means that $\sup\{\mu_R(x, y) \mid y \in A\} = 1$, i.e. there exists an element $y_x \in A$, with $\mu_R(x, y_x) = 1$. Then $(x, y_x) \in E$, and hence $[x]_E \cap A \neq \emptyset$. This implies $x \in A^E$, proving (i).

(ii) If $x \in A_E$, then $[x]_E \subseteq A$. This means that there is no $y \notin A$ with $(x, y) \in E$, i.e. such that $\mu_R(x, y) = 1$. Since $R$ has a dually well-ordered spectrum, the set $\{\mu_R(x, y) \mid y \notin A\}$ has (at least one) maximal element $\mu_R(x, y_m)$, where $y_m \notin A$. Then $\mu_R(x, y_m) < 1$, and we obtain $\mu_{[A]^R}(x) = 1 - \sup\{\mu_R(x, y) \mid y \notin A\} = 1 - \mu_R(x, y_m) > 0$. Conversely, assume that $\mu_{[A]^R}(x) > 0$, for some $x \in U$. Then for any $y \notin A$ we get $1 - \mu_R(x, y) \geq \inf\{1 - \mu_R(x, y) \mid y \notin A\} = \mu_{[A]^R}(x) > 0$. This implies $\mu_R(x, y) < 1$, for each $y \notin A$. Hence there is no $y \notin A$ with $\mu_R(x, y) = 1$, i.e. with $(x, y) \in E$. This yields $[x]_E \subseteq A$. Hence $x \in A_E$, and this proves (ii).
(iii) Assume that \( x \in A^S \). This can only be if there exists a \( y_m \in A \) such that \( \mu_R(x, y_m) > 0 \). However, this implies that \( \mu_{[A]^R}(x) = \sup \{ \mu_R(x, y) \mid y \in A \} \geq \mu_R(x, y_m) > 0 \), yielding that \( x \) is in the support of \( \mu_{[A]^R} \). Conversely, assume that \( x \) is in the support of \( \mu_{[A]^R} \), meaning that \( \mu_{[A]^R}(x) = \sup \{ \mu_R(x, y) \mid y \in A \} > 0 \). This can only happen if there exists \( A \) such that \( \mu_R(x, y_m) > 0 \), implying in \( x \in A^S \).

(iv) Let \( x \in A_S \), i.e. \([x]_S \subseteq A \). By the definition of \( S \), it follows that for every \( y \notin A, \mu_R(x, y) = 0 \). Thus, \( \mu_{[A]^R}(x) = \inf \{ 1 - \mu_R(x, y) \mid y \notin A \} = 1 - 0 = 1 \), i.e. \( x \) is in the core of \( \mu_{[A]^R}(x) \). Conversely, assume that \( \mu_{[A]^R}(x) = 1 \), meaning that \( \mu_{[A]^R}(x) = \inf \{ 1 - \mu_R(x, y) \mid y \notin A \} = 1 \), which requires that for every \( y \notin A, \mu_R(x, y) = 0 \), so the \( S \)-equivalence class of \( x \), i.e. \([x]_S \) cannot contain elements outside of \( A \), meaning that \([x]_S \subseteq A \Rightarrow x \in A_S \). \( \square \)

3. Main results

**Proposition 1.** Let \( R \) be a fuzzy equivalence on \( U \) with a dually well-ordered spectrum and \( E := \{(x, y) \in U^2 \mid \mu_R(x, y) = 1\} \). Then for any set \( A \subseteq U \) we have

\[
\mu_{[A]^R} = \mu_{[A]^E_R} \quad \text{and} \quad \mu_{[A]^R} = \mu_{[A]^E_R}.
\]

**Proof.** Let \( x \in U \) be arbitrary. Since \( A \subseteq A^E \), by definition \( \mu_{[A]^R}(x) \leq \mu_{[A]^E_R}(x) \). Now consider \( \mu_{[A]^E_R}(x) = \max \{ \mu_R(x, y) \mid y \in A^E \} \). Then clearly, \( \mu_{[A]^E_R}(x) = \mu_R(x, y_x) \), for some \( y_x \in A^E \). \( y_x \in A^E \) yields that there exists an \( y_0 \in A \) such that \( (y_x, y_0) \in E \), i.e. \( y_0 \in [y_x]_E \). Now, in view of Lemma 1 we get \( \mu_R(x, y_x) = \mu_R(x, y_0) \leq \max \{ \mu_R(x, y) \mid y \in A \} = \mu_{[A]^R}(x) \). Hence \( \mu_{[A]^E_R}(x) = \mu_R(x, y_x) \leq \mu_{[A]^R}(x) \), and this proves \( \mu_{[A]^R}(x) = \mu_{[A]^E_R}(x) \).

Similarly, \( A_E \subseteq A \) yields \( \mu_{[A]^R}(x) \geq \mu_{[A]^E_R}(x) \), because \( y \notin A \) implies \( y \notin A_E \), and hence \( 1 - \max \{ \mu_R(x, y) \mid y \notin A \} \geq 1 - \max \{ \mu_R(x, y) \mid y \notin A_E \} \). In order to prove the converse inequality, consider the value

\[
\mu_{[A]^E_R}(x) = 1 - \max \{ \mu_R(x, y) \mid y \notin A_E \}.
\]

Then \( \mu_{[A]^E_R}(x) = 1 - \mu_R(x, y_x) \), for some \( y_x \notin A_E \). Observe that \( y_x \notin A_E \), means that there exists a \( z \notin A \) such that \( (y_x, z) \in E \). Since \( z \in [y_x]_E \), by Lemma 1 we get \( \mu_R(x, y_x) = \mu_R(x, z) \), whence \( \mu_{[A]^E_R}(x) = 1 - \mu_R(x, y_x) = 1 - \mu_R(x, z) \). Therefore, we obtain: \( \mu_{[A]^R}(x) = \inf \{ 1 - \mu_R(x, y) \mid y \notin A \} \leq 1 - \mu_R(x, z) = \mu_{[A]^E_R}(x) \). This proves \( \mu_{[A]^R}(x) = \mu_{[A]^E_R}(x) \), for all \( x \in U \), completing our proof. \( \square \)
In what follows, denote as usually by \((\text{RS}(U, E), \leq)\) the lattice of rough sets defined by the equivalence relation \(E\).

**Theorem 1.** Let \(R\) be a fuzzy equivalence on \(U\) with a dually well-ordered spectrum. Then \((\mathcal{RS}(U, R), \leq)\) is a complete lattice isomorphic to \((\text{RS}(U, E), \leq)\).

**Proof.** For each (crisp) rough set \((A_E, A^E) \in \text{RS}(U, E)\) we will assign the fuzzy rough set corresponding to the crisp set \(A\), i.e. the pair \((\mu_{[A]_R}, \mu_{[A]^R})\).

Observe, that the function
\[
f : \text{RS}(U, E) \to \mathcal{RS}(U, R), \quad f((A_E, A^E)) = (\mu_{[A]_R}, \mu_{[A]^R}),
\]
is well-defined, because \((A_E, A^E) = (B_E, B^E)\) for some \(A, B \subseteq U\) implies \(A_E = B_E, A^E = B^E\), and hence, in view of Proposition 1, we obtain
\[
f((A_E, A^E)) = (\mu_{[A]_R}, \mu_{[A]^R}) = (\mu_{[A]_R}, \mu_{[A]^R}) = f((B_E, B^E)).
\]

In addition, \(f\) is order-preserving because \((A_E, A^E) \leq (B_E, B^E)\) implies \(A_E \subseteq B_E, A^E \subseteq B^E\), and this yields \(\mu_{[A]_R} \leq \mu_{[B]_R}\) and \(\mu_{[A]^R} \leq \mu_{[B]^R}\). Thus we obtain:
\[
f((A_E, A^E)) = (\mu_{[A]_R}, \mu_{[A]^R}) \leq (\mu_{[B]_R}, \mu_{[B]^R}) = f((B_E, B^E)).
\]

Clearly, \(f\) is onto (surjective), since for any \((\mu_{[X]_R}, \mu_{[X]^R}) \in \mathcal{RS}(U, R), X \subseteq U\) is a crisp set, and hence \(f((X_E, X^E)) = (\mu_{[X]_R}, \mu_{[X]^R})\). Therefore, to prove that \(f\) is an order-isomorphism, it suffices to show that \(f((A_E, A^E)) \leq f((B_E, B^E))\) implies \((A_E, A^E) \leq (B_E, B^E)\), for any \((A_E, A^E), (B_E, B^E) \in \text{RS}(U, E)\).

Indeed, \(f((A_E, A^E)) \leq f((B_E, B^E))\) yields that \((\mu_{[A]_R}(x), \mu_{[A]^R}(x)) \leq (\mu_{[B]_R}(x), \mu_{[B]^R}(x))\), for all \(x \in U\). Hence we get \(\mu_{[A]_R}(x) \leq \mu_{[B]_R}(x)\) and \(\mu_{[A]^R}(x) \leq \mu_{[B]^R}(x)\), for any \(x \in U\). Now, in view of Lemma 1, we obtain:
\[
A_E = \{x \in U \mid \mu_{[A]_R}(x) = 1\} \subseteq \{x \in U \mid \mu_{[B]_R}(x) = 1\} = B_E,
\]
\[
A^E = \{x \in U \mid \mu_{[A]^R}(x) > 0\} \subseteq \{x \in U \mid \mu_{[B]^R}(x) > 0\} = B^E.
\]

Hence \((A_E, A^E) \leq (B_E, B^E)\), and this proves that \(f\) is an order-isomorphism. Since \((\text{RS}(U, E), \leq)\) is a complete lattice, we obtain that \((\mathcal{RS}(U, R), \leq)\) is also a complete lattice isomorphic to \((\text{RS}(U, E), \leq)\). \(\square\)

As an immediate consequence we obtain 2 [3] [13]:

**Corollary 1.** If \(R\) is a fuzzy equivalence on the set \(U\) with a dually well-ordered spectrum, then \((\mathcal{RS}(U, R), \leq)\) is a completely distributive regular double Stone lattice.
Proof. It is known that the rough set lattice \((\mathcal{RS}(U, E), \leq)\) is completely distributive regular double Stone lattice. Hence Corollary 1 is obtained by applying the isomorphism established in Theorem 1.

\[\Box\]

**Example 1.** Let the universe be \(U = \{a, b, c, d, e\}\) and the fuzzy equivalence relation \(R\) be given by Figure 1. On the figure, relationship degrees of 0 are not drawn, and relationship degrees between 0 and 1 are denoted by dashed lines. Note that the relation is also reflexive, but loops are not presented on the figure for simplicity (all loops have a relationship degree of 1 for fuzzy equivalence relations). The corresponding \(E = \{(x, y) \in U^2 \mid \mu_R(x, y) = 1\}\) relation can be seen on Figure 2 (again, loops are not noted). Table 1 shows the lower and upper approximations of fuzzy relation \(R\) and the (crisp) equivalence relation \(E\). Figure 3 shows the Hasse-diagram of the lattice \((\mathcal{RS}(U, R), \leq)\). Whenever the membership function of the lower (upper) approximation is the same for all elements of the universe, only a single value is written in the corresponding cell in Table 1. Otherwise, the cell is split up into five values for every element of the universe. Similarly, the lattice shown on Figure 3 only shows a single value if the membership function has the same value for all elements of \(U\). The top row inside the nodes corresponding to elements of the lattice represents the membership function of the upper approximation, and the bottom row represents the membership function of the lower approximation of \(R\).
Figure 2: The equivalence relation $E$ corresponding to $R$.

| $A$ | $A_E$ | $A^E$ | $\mu_{[A_E]}(x)$ | $\mu_{[A]}R(x)$ |
|-----|-------|-------|-----------------|-----------------|
|     |       |       | $a$  $b$  $c$  $d$  $e$ | $a$  $b$  $c$  $d$  $e$ |
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $0$ | $0$ |
| $a$ | $\emptyset$ | $ab$ | $0$ | $1$ $1$ $0.5$ $0$ $0$ |
| $b$ | $\emptyset$ | $ab$ | $0$ | $1$ $1$ $0.5$ $0$ $0$ |
| $c$ | $c$ | $c$ | $0$ | $0$ $0.5$ $0$ $0$ | $0.5$ $0.5$ $1$ $0$ $0$ |
| $d$ | $\emptyset$ | $de$ | $0$ | $0$ $0$ $0$ $1$ $1$ |
| $e$ | $\emptyset$ | $de$ | $0$ | $0$ $0$ $0$ $1$ $1$ |
| $ab$ | $ab$ | $ab$ | $0.5$ | $0.5$ $0$ $0$ | $1$ $1$ $0.5$ $0$ $0$ |
| $ac$ | $abc$ | $abc$ | $0$ | $0$ $0.5$ $0$ $0$ | $1$ $1$ $1$ $0$ $0$ |
| $ad$ | $\emptyset$ | $abde$ | $0$ | $1$ $1$ $0.5$ $1$ $1$ |
| $ae$ | $\emptyset$ | $abde$ | $0$ | $1$ $1$ $0.5$ $1$ $1$ |
| $bc$ | $c$ | $abc$ | $0$ | $0$ $0.5$ $0$ | $1$ $1$ $0$ $0$ $0$ |
| $bd$ | $\emptyset$ | $abde$ | $0$ | $1$ $1$ $0.5$ $1$ $1$ |
| $be$ | $\emptyset$ | $abde$ | $0$ | $1$ $1$ $0.5$ $1$ $1$ |
| $cd$ | $c$ | $cde$ | $0$ | $0$ $0.5$ $0$ | $0.5$ $0.5$ $1$ $1$ $1$ |
| $ce$ | $c$ | $cde$ | $0$ | $0$ $0.5$ $0$ | $0.5$ $0.5$ $1$ $1$ $1$ |
| $de$ | $de$ | $de$ | $0$ | $0$ $0$ $1$ $1$ | $0$ $0$ $0$ $1$ $1$ |
| $abc$ | $abc$ | $abc$ | $1$ | $1$ $1$ $0$ | $1$ $1$ $1$ $0$ $0$ |
| $abd$ | $abde$ | $abde$ | $0.5$ | $0.5$ $0$ $0$ | $1$ $1$ $0.5$ $1$ $1$ |
| $abc$ | $abde$ | $abde$ | $0.5$ | $0.5$ $0$ $0$ | $1$ $1$ $0.5$ $1$ $1$ |
| $acb$ | $c$ | $U$ | $0$ | $0$ $0.5$ $0$ | $1$ |
| $ace$ | $c$ | $U$ | $0$ | $0$ $0.5$ $0$ | $1$ |
| $bce$ | $c$ | $U$ | $0$ | $0$ $0.5$ $0$ | $1$ |
| $bcd$ | $de$ | $abde$ | $0$ | $0$ $0$ $1$ $1$ | $1$ $1$ $0.5$ $1$ $1$ |
| $bcde$ | $cde$ | $cde$ | $0$ | $0$ $0.5$ $1$ $1$ | $0.5$ $0.5$ $1$ $1$ $1$ |
| $abcd$ | $abc$ | $U$ | $1$ | $1$ $1$ $0$ | $1$ |
| $abce$ | $abc$ | $U$ | $1$ | $1$ $1$ $0$ | $1$ |
| $abde$ | $abde$ | $abde$ | $0.5$ | $0.5$ $0$ $1$ | $1$ $1$ $0.5$ $1$ $1$ |
| $acde$ | $cde$ | $cde$ | $0$ | $0$ $0.5$ $1$ $1$ | $1$ |
| $bcde$ | $cde$ | $cde$ | $0$ | $0$ $0.5$ $1$ $1$ | $1$ |
| $U$ | $U$ | $U$ | $0$ | $1$ | $1$ |

Table 1: Approximations on $U$ given by the equivalence $E$ and fuzzy relation $R$. 
In regular rough set theory, a rough set is called \textit{exact} if the lower approximation and the upper approximation of the set are equal (they are also equal to the set itself). This definition can be extended to fuzzy rough sets. A fuzzy rough set defined by the fuzzy equivalence relation \( R \) is \textit{exact} if for every \( x \in U \), \( \mu_{[A]_R}(x) = \mu_{[A]^R}(x) \) holds, where \( U \) is the universe of \( R \).

The following proposition describes the relationship between exact fuzzy rough sets and the support of the fuzzy equivalence relation.

\textbf{Proposition 2.} Let \( A \) be a (crisp) subset of \( U \). Then

\[ \mu_{[A]_R}(x) = \mu_{[A]^R}(x) \text{ for all } x \in U \iff A_S = A^S. \]

\textbf{Proof.} Suppose that \( x \in A_S = A^S = A \). This means that

\[ \mu_{[A]^R}(x) = \sup\{\mu_R(x, y) \mid y \in A\} = \mu_R(x, x) = 1, \]

since \( R \) is reflexive.

Now let us examine the lower approximation. If \( y \notin A \), then \( y \notin [x]_S \) either, because \( [x]_S \subseteq A \). Since \( y \notin [x]_S \), according to the definition of \( S \), it
follows that $\mu_R(x, y) = 0$. This is true for every $y \notin A$, yielding
\[ \mu_{[A]_R} = \inf\{1 - \mu_R(x, y) \mid y \notin A\} = 1. \]
So we obtain in this case that $\mu_{[A]_R}(x) = \mu_{[A]_R}(x) = 1$.

Now, suppose that $x \notin A_S = A^S = A$. Then $\mu_R(x, y) = 0$ for every $y \in A$. Thus we have,
\[ \mu_{[A]_R}(x) = \sup\{\mu_R(x, y) \mid y \in A\} = 0, \text{ and} \]
\[ \mu_{[A]_R}(x) = \inf\{1 - \mu_R(x, y) \mid y \notin A\} = 1 - \mu_R(x, x) = 0. \]
Hence in this case we obtain $\mu_{[A]_R}(x) = \mu_{[A]_R}(x) = 0$.

Therefore, we demonstrated that $A_S = A^S$ implies $\mu_{[A]_R}(x) = \mu_{[A]_R}(x)$, for all $x \in U$.

Conversely, assume that $\mu_{[A]_R}(x) = \mu_{[A]_R}(x)$, for all $x \in U$. Let $x \in A$ be arbitrary. Then
\[ \sup\{\mu_R(x, y) \mid y \in A\} = \inf\{1 - \mu_R(x, y) \mid y \notin A\} = 1, \]
because $\mu_R(x, x) = 1$ and $\mu_R(x, y) \leq 1$ for all $y \in A$. As a consequence, $\mu_R(x, y) = 0$ for all $y \notin A$, otherwise the right side would be strictly less than 1.

Assume $(x, y) \in S$, i.e. $y \in [x]_S$. Then, $\mu_R(x, y) > 0$ by definition of $S$, so $y \notin A$ is not possible. Thus, we get $y \in A$ and this implies $x \in A_S$. Hence, $A = A_S$. Then $A^S = (A_S)^S \subseteq A$ implies $A^S = A$. Thus, we obtain $A_S = A^S = A$. Hence we also proved that $\mu_{[A]_R}(x) = \mu_{[A]_R}(x)$ for all $x \in U$ implies $A_S = A^S$.

The next corollary is an immediate consequence of the above proof.

**Corollary 2.** Let $R$ be a fuzzy equivalence on the set $U$, $S$ be the support of $R$ and let $A \subseteq R$ be a crisp set on $U$. If $A_S = A^S$, then $\mu_{[A]_R} = \mu_{[A]_R} = \chi_A(x)$, where $\chi_A(x)$ is the characteristic function of $A$, i.e.
\[ \chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases} \]
Example 2. Let the universe be $U = \{a, b, c, d\}$ and let $R$ be a fuzzy relation given by Figure 4. The relation is represented on the figure as mentioned in the earlier example. Table 4 shows the three sets for which this relation yields exact sets as lower and upper approximations for $R$ and for $S$ (the support of $R$).

Remark 1. It can be verified that the containment relationship between a base set $A$ and its fuzzy rough approximations is similar to the containment relationship between the base set and its crisp rough approximations, namely:

$$\text{core}(\mu_{|A|_R}) \subseteq \text{support}(\mu_{|A|_R}) \subseteq A,$$
$$A \subseteq \text{core}(\mu_{|A|_R}) \subseteq \text{support}(\mu_{|A|_R}).$$

An important simple case should also be discussed: when we have imperfect information and we are uncertain about setting up the relation. In this simple case, the relationship between two elements can have three possibilities: the elements are certainly related; the elements are certainly not related; the elements might be related, but we are uncertain. We model this with a fuzzy relation $R$, for which

$$\mu_R(x, y) = \begin{cases} 
1, & \text{if } x \text{ and } y \text{ are certainly related} \\
0, & \text{if } x \text{ and } y \text{ are certainly not related} \\
\frac{1}{2}, & \text{if } x \text{ and } y \text{ might be related, but we are uncertain}
\end{cases}.$$
The lower and upper approximations of this special case can be characterized according to the following proposition.

**Proposition 2.** Let $R$ be a fuzzy equivalence relation on universe $U$, such that $\mu_R(x, y) \in \{0, 1, \frac{1}{2}\}$ for every $x, y \in U$. Additionally, let $A$ be a (crisp) subset of $U$. Then the membership function of the lower and upper approximations of $R$ can be given as

(i) $\mu_{[A]_R}(x) = \begin{cases} 
0, & \text{if } x \notin A_E \\
\frac{1}{2}, & \text{if } x \in A_E \setminus A_S \\
1, & \text{if } x \in A_S 
\end{cases}$

(ii) $\mu_{[A]^R}(x) = \begin{cases} 
0, & \text{if } x \notin A^S \\
\frac{1}{2}, & \text{if } x \in A^S \setminus A^E \\
1, & \text{if } x \in A^E 
\end{cases}$

**Proof.** By definition of relation $R$ it is clear that $\mu_{[A]_R}(x)$ and $\mu_{[A]^R}(x)$ can only have three possible values, $0, \frac{1}{2}$ and $1$.

(i) According to Lemma 2., $\mu_{[A]_R}(x)$ can only be $0$ if $x \notin A_E$ and can only be $1$ if $x \in A_S$. Whenever $x \in A_E \setminus A_S$, the value of $\mu_{[A]_R}(x)$ is positive, but not $1$, i.e. it is $\frac{1}{2}$ in our case.

(ii) By similar reasoning, according to Lemma 2., $\mu_{[A]^R}(x)$ can only be $0$ if $x \notin A^S$ and can only be $1$ if $x \in A^E$. Whenever $x \in A^S \setminus A^E$, the value of $\mu_{[A]^R}(x)$ is positive, but not $1$, i.e. it is $\frac{1}{2}$ in our case. □

**Remark 2.** It is important to mention that the value of $\frac{1}{2}$ is chosen here because it is halfway through being certainly related (included) and certainly not related (not included). However, this model could also be represented by using any value $\alpha \in (0, 1)$ instead of $\frac{1}{2}$.

4. Conclusions and further work

In this paper, we examined the lattice of fuzzy rough sets corresponding to a fuzzy equivalence relation $R$. We also investigated the relationship between the core/support of the approximations of a fuzzy rough set and the (crisp) approximations corresponding to the core/support of $R$. We have shown that the lattice of fuzzy rough sets is isomorphic to the lattice of rough sets corresponding to $E$, the core of $R$. We also proved that the membership function of an exact fuzzy set (where $A \subseteq U$ is a crisp set and
\[ \mu_{[A],R}(x) = \mu_{[A],R}(x) \] for every \( x \in U \) is the same as the characteristic function of a (regular) exact set corresponding to \( S \), the support of \( R \).

We can extend the investigation of the \( E \)-based approximation to the \( \alpha \)-cut \( E \)-approximation. The related \( E_\alpha \) crisp relation for different \( \alpha \)-levels can be defined in the following way:

\[ E_\alpha = \{(x, y) \mid R(x, y) \geq \alpha\}. \]

It is known that \( E_\alpha \) is also an equivalence relation whenever \( R \) is a fuzzy equivalence. Then we can give the following result:

\[ A^{E_\alpha} = \{x \in U \mid \mu_{[A],R}(x) \geq \alpha\}, \]
\[ A_{E_\alpha} = \{x \in U \mid \mu_{[A],R}(x) \geq 1 - \alpha\}. \]

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