A Random Walk with Collapsing Bonds and Its Scaling Limit

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Abstract

We introduce a new self-interacting random walk on the integers in a dynamic random environment and show that it converges to a pure diffusion in the scaling limit. We also find a lower bound on the diffusion coefficient in some special cases. With minor changes the same argument can be used to prove the scaling limit of the corresponding walk in \( \mathbb{Z}^d \).

In this note we introduce a self-interacting random walk in a dynamic random environment, prove that it is recurrent and find its scaling limit. The environment evolves in time in conjunction with the random walk and is non-Markovian; however we will show that the walk only remembers the recent past. Various models of self-interacting random walks and random walks in dynamic random environments have been studied recently (see for instance [1, 2, 3, 4, 6, 7] and references therein).

Consider a particle performing a continuous-time nearest neighbor symmetric random walk on the integers lattice. We assume that the particle is initially at the origin. The times between successive jumps are independent exponential random variables with rate \( \lambda \). Anytime the particle jumps over the bond connecting two neighboring lattice sites, there is a probability \( 0 < p \leq 1 \) that the bond connecting the sites breaks. The particle is not able to jump over that bond until that bond is repaired. If at the time the particle attempts to jump, one of the bonds neighboring the particle is broken, the particle jumps over the other bond with probability one. If both bonds neighboring the particle are broken, the particle can’t jump when it attempts to do so. The repair times of bonds are independent exponential random variables with rate \( \mu \in (0, \infty) \). Initially, there are no broken bonds.
Let $X(t)$ denote the position of the particle at time $t \in [0, \infty)$. We assume that the jump times are such that almost surely, $X(t)$ is continuous from right and has a left limit at all $t$. We will show that for all values of $p$, $X(t)$ is a recurrent process and its scaling limit is a pure diffusion. We will find a lower bound for the diffusion coefficient in some special cases.

**Theorem 1** The process $\{X(t)\}_{t \geq 0}$ is recurrent.

Theorem 1 is a consequence of the following.

**Lemma 2** With probability 1, for any $s \geq 0$, there exists $t \geq s$ such that at time $t$, there are no broken bonds.

**Proof** The bonds break at a rate $\lambda p$, until the particle is trapped at a lattice site. At that time no more bonds break, until the particle is free to move again. Each bond is repaired at a rate $\mu$, independent of the other bonds. Thus, $b_t$, the number of broken bonds at time $t$, is less than or equal to $Q_t$, where $\{Q_t\}_{t \geq 0}$ is an M/M/$\infty$ queue with incoming traffic rate $\lambda p$ and service rate $\mu$.

It is straightforward to show that this queue is a recurrent Markov chain. Furthermore if, starting from zero customers, $T$ is the recurrence time of this queue back to zero customers, then $T$ is the sum of the idle time (which is an exponential random variable with rate $\lambda p$) and the busy period of this queue, and these two times are independent of each other. The busy period has a Laplace transform (see [8]). Thus $T$ is finite almost surely, has finite moments of all orders, and $\mathbb{E}(T) = \exp(\lambda p/\mu)/\lambda p$. Note that since $b_t \leq Q_t$, when $Q_t = 0$ we also have $b_t = 0$. Therefore, assuming there are no broken bonds at the staring time, if $\tau$ is the recurrence time of $b_t$ back to zero, then $\tau \leq T$. Thus $\tau$ is finite almost surely, has finite moments of all orders, and $\mathbb{E}(\tau) \leq \exp(\lambda p/\mu)/\lambda p$. Hence, 0 is a recurrent state for $b_t$. Therefore, for any $s \geq 0$, there is a $t \geq s$ such that there are no broken bonds at time $t$. This completes the proof of the Lemma.

Put $\tau_0 = 0$ and if $\tau_i$ has been defined, put

$$\sigma_{i+1} = \inf\{t > \tau_i : b_t > 0\}$$

and

$$\tau_{i+1} = \inf\{t > \sigma_{i+1} : b_t = 0\}.$$  \hspace{1cm} (1)

The following is immediate from the proof of Lemma 2.

**Lemma 3** The random variables $\tau_{i+1} - \tau_i$, $i \geq 0$ are i.i.d, finite almost surely, and have finite moments of all orders.

Let $N(t)$ be a Poisson process with parameter $\lambda$ which increases by 1 anytime the particle makes an attempt to jump. Note that there are times that the particle is trapped at a site and an attempt to jump fails. We increase $N(t)$ by 1 in such instances as well.

**Proof of Theorem 1**
By Lemma 3, $\tau_1$ is finite almost surely and has finite moments of all orders. Since there are no broken bonds at times $\tau_i$, $i \geq 0$, the sequence $\{X_{\tau_{i+1}} - X_{\tau_i}\}_{i \geq 0}$ is an i.i.d sequence. Note that for any $t \geq 0$, $|X(t)| \leq N(t)$. Thus, $E|X_{\tau_1}| \leq \lambda E(\tau_1) \leq \exp(\lambda p/\mu)/p$. By symmetry considerations, $E(X_{\tau_1}) = 0$. Thus, $X_{\tau_n} = \sum_{k=0}^{n-1}(X_{\tau_{k+1}} - X_{\tau_k})$ is a sum of mean zero, i.i.d. random variables. By standard arguments for recurrence of sums, $P(X_{\tau_n} = 0 \text{ i.o.}) = 1$ (see for instance Theorem 3.38 in [5].) This completes the proof of Theorem 1.

Next we will show that the scaling limit of $X(t)$ is a pure diffusion.

For all $n \in \mathbb{N}$ define a sequence of random continuous increasing functions $T_n(t)$, $0 \leq t \leq 1$ as follows. If $t = m/n$ for some $m$, $0 \leq m \leq n$ then

$$T_n(t) = \tau_m + n(\tau_{m+1} - \tau_m)(t - m/n).$$

Let $\{\tau_i\}_{i \geq 0}$ be as in (1). Put $\alpha = E(\tau_1)$. We observe that $X(\tau_{i+1}) - X(\tau_i)$, $i \geq 0$, are i.i.d. random variables with mean zero and $E(X(\tau_1)^2) \leq \lambda E(\tau_1) + \lambda^2 E(\tau_1)^2 < \infty$ for all $i \geq 0$. Put $\beta^2 = \text{Var}(X(\tau_1))$. Note that $\beta > 0$. To see this assume that $A$ is the event that a bond breaks at the first jump and the broken bond is fixed before the second jump. Then $P(A) = p\mu/(\lambda + \mu)$, and

$$P(|X_{\tau_1}| > 0) \geq P(|X_{\tau_1}| = 1) \geq P(A) > 0.$$ 

Define the sequence of random functions $\{X_n(\cdot)\}_{n \geq 1}$ by

$$X_n(t) = \frac{X(nt)}{\sqrt{n}},$$

and let $B(t)$, $0 \leq t \leq 1$ denote the standard one-dimensional Brownian motion. We will prove the following.

**Theorem 4**

$$X_n(t) \Rightarrow \frac{\beta}{\sqrt{\alpha}} B(t) \quad \text{as } n \to \infty.$$ 

Let us denote by $\mathbb{P}$ the probability measure on the underlying probability space $\Omega$. First we prove the following.

**Lemma 5**

$$\lim_{n \to \infty} \sup_{0 \leq t \leq 1} \left| \frac{T_n(t)}{n} - \alpha t \right| = 0 \quad \mathbb{P}\text{-a.s.}$$

**Proof** Let $T_n(t)$ be defined by equations (2) and (3). If $t \in (m/n, (m+1)/n)$ then

$$\left| T_n(t) - \tau_{n\lfloor t \rfloor} \right| \leq (\tau_{m+1} - \tau_m).$$
Therefore we have for all \( t \in (m/n, (m + 1)/n) \),
\[
\frac{1}{n} |T_n(t) - \tau_{[nt]}| \leq \frac{(\tau_{m+1} - \tau_m)}{n}.
\]

Using the fact that \( \tau_1 \) has finite third moments and the Borel-Cantelli lemma we can easily show that
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq 1} \frac{1}{n} |T_n(t) - \tau_{[nt]}| = 0 \quad \mathbb{P}\text{-a.s.} \tag{4}
\]

Let \( m \in \mathbb{N} \) be given and consider \( t \) of the form \( \frac{l}{m} \), \( 0 \leq l \leq m \). By the strong law of large numbers we have for all \( l, 0 \leq l \leq m \)
\[
\lim_{n \to \infty} \frac{\tau_{[nt]}}{n} = \frac{l}{m} \quad \mathbb{P}\text{-a.s.}
\]

Using (4) we have
\[
\lim_{n \to \infty} \frac{T_n\left(\frac{l}{m}\right)}{n} = \frac{l}{m} \quad \mathbb{P}\text{-a.s.}
\]

Therefore for \( \mathbb{P} \)-almost all \( \omega \in \Omega \) there exists an \( N_m \) such that
\[
\left| \frac{T_n\left(\frac{l}{m}\right)}{n} - \frac{l}{m} \right| < \frac{\alpha}{m} \quad \text{if } n \geq N_m.
\]

If \( l/m < t < (l + 1)/m \), then
\[
\frac{T_n(t)}{n} > \frac{T_n\left(\frac{l}{m}\right)}{n} > \frac{l}{m} - \frac{\alpha}{m} = \frac{l + 1}{m} - \frac{2\alpha}{m} > \alpha t - \frac{2\alpha}{m}.
\]

Similarly
\[
\frac{T_n(t)}{n} < \frac{T_n\left(\frac{l + 1}{m}\right)}{n} < \frac{l + 1}{m} + \frac{\alpha}{m} = \frac{l}{m} + \frac{2\alpha}{m} < \alpha t + \frac{2\alpha}{m}.
\]

From this it follows that
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq 1} \left| \frac{T_n(t)}{n} - \alpha t \right| = 0 \quad \mathbb{P}\text{-a.s.},
\]
which proves the Lemma.

Now we can prove Theorem 4

Proof of Theorem 4 For all \( n \in \mathbb{N} \) and \( 0 \leq t \leq 1 \) define \( X'_n(t) = \frac{X(\tau_{[nt]})}{\sqrt{n\beta}} \).

Since \( X(\tau_k) = \sum_{i=1}^{k} (X(\tau_{i+1}) - X(\tau_i)) \), and \( X(\tau_{i+1}) - X(\tau_i) \) are i.i.d random variables with finite second moment it follows from Donsker’s invariance principle that
\[
X'_n(t) \Rightarrow B(t) \quad \text{as } n \to \infty.
\]
Let \( \{Y(t)\}_{t \geq 0} \) be the process with continuous paths that is obtained by linearly interpolating between successive jumps of \( \{X(t)\} \). Define \( Y_n(t) = \frac{Y(\tau_k)}{\sqrt{n\beta}} \)
if \( t = k/n \) and \( k/n < t < (k + 1)/n \), define \( Y_n(t) \) by linear interpolation. Since
\[
\frac{|X(\tau_k) - Y(\tau_k)|}{\sqrt{n}} \leq \frac{1}{\sqrt{n}},
\]
it follows that \(|X_n(t) - Y_n(t)| \leq \frac{1}{\sqrt{n}}\beta\) for all \( t \in [0, 1] \) almost surely.
This proves that
\[
Y_n(t) \Rightarrow B(t) \quad \text{as } n \to \infty.
\]
For \( n \in \mathbb{N} \) and \( 0 \leq t \leq 1 \), define \( \lambda_n(t) = T_n(t)/(an) \). Note that by Lemma 6
\[
\lim_{n \to \infty} \lambda_n(t) = t \quad \text{uniformly } \mathbb{P}\text{-a.s.}
\]
Put
\[
Y'_n(t) = \frac{Y(\alpha nt)}{\sqrt{n\beta}},
\]
and
\[
Z_n(t) = Y'_n(\lambda_n(t)) = \frac{Y(T_n(t))}{\sqrt{n\beta}}.
\]
Notice that if \( t = k/n \), then \( Z_n(t) = \frac{Y(\tau_k)}{\sqrt{n\beta}} \). We will show that paths of \( Z_n(t) \)
and \( Y_n(t) \) are uniformly close in \([0, 1]\). We will make use of the fact that all
(high enough) moments of \( \tau_k \) are finite and that the jumps occur at Poisson
times. The functions \( Y_n(\cdot) \) and \( Z_n(\cdot) \) agree at times \( m/n \); while \( Y_n(t) \) linearly
interpolates between times \( k/n \) and \((k + 1)/n \), \( Z_n(t) \) follows the \( Y' \) process
between times \( \tau_k \) and \( \tau_{k+1} \) with rescaled time. If \( t \in [k/n, (k + 1)/n] \), then \( \tau_k \leq T_n(t) \leq \tau_{k+1} \), and
\[
\sup_{t \in [\frac{k}{n}, \frac{k+1}{n}]} \left| \frac{Y(\tau_k)}{\beta \sqrt{n}} - Z_n(t) \right| \leq \frac{1}{\sqrt{n\beta}} \sup_{\tau_k \leq t \leq \tau_{k+1}} (|Y(t) - Y(\tau_k)|)
\]
and
\[
\sup_{t \in [\frac{k}{n}, \frac{k+1}{n}]} \left| \frac{Y(\tau_k)}{\beta \sqrt{n}} - Z_n(t) \right| \leq \frac{1}{\sqrt{n\beta}} (|Y(\tau_{k+1}) - Y(\tau_k)|).
\]
Now
\[
P \left( \sup_{t \in [\frac{k}{n}, \frac{k+1}{n}]} |Y_n(t) - Z_n(t)| > \frac{1}{n^{1/4}} \right)
\]
\[
\leq P \left( \sup_{t \in [\frac{k}{n}, \frac{k+1}{n}]} \left| \frac{Y(\tau_k)}{\beta \sqrt{n}} - Y_n(t) \right| + \left| \frac{Y(\tau_k)}{\beta \sqrt{n}} - Z_n(t) \right| > \frac{1}{n^{1/4}} \right),
\]
5
which implies
\[
\mathbb{P} \left( \sup_{t \in \left[ \frac{k}{n}, \frac{k+1}{n} \right]} |Y_n(t) - Z_n(t)| > \frac{1}{n^{1/4}} \right) \leq \mathbb{P} \left( \sup_{t \in \left[ \frac{k}{n}, \frac{k+1}{n} \right]} \left| \frac{Y(t_k)}{\beta \sqrt{n}} - Y_n(t) \right| > \frac{1}{2n^{1/4}} \right) \\
+ \mathbb{P} \left( \sup_{t \in \left[ \frac{k}{n}, \frac{k+1}{n} \right]} \left| \frac{Y(t_k)}{\beta \sqrt{n}} - Z_n(t) \right| > \frac{1}{2n^{1/4}} \right).
\]

Therefore it follows that to show that \( Z_n \) and \( Y_n \) are uniformly close it is sufficient to estimate \( \sup_{\tau_k \leq t \leq \tau_{k+1}} (\frac{1}{\sqrt{n \beta}} |Y(t) - Y(t_k)|) \) which we proceed to do now.

Let
\[
M_k = \sup_{\tau_k \leq t \leq \tau_{k+1}} (|Y(t) - Y(\tau_k)|).
\]

We now estimate \( \mathbb{P}(M_k > \frac{\beta}{2} n^{1/4}) \). Since the argument for the estimate is the same for all \( k \), we estimate \( \mathbb{P}(M_1 > \frac{\beta}{2} n^{1/4}) \). Let \( C_k \) denote the kth moment of \( \tau_1 \) and \( N(t) \) be defined as in the paragraph before the proof of Theorem 1. We have
\[
\mathbb{P} \left( M_1 > \frac{\beta}{2} n^{1/4} \right) = \mathbb{P} \left( M_1 > \frac{\beta}{2} n^{1/4}; \tau_1 > n^{1/8} \right) + \mathbb{P} \left( M_1 > \frac{\beta}{2} n^{1/4}; \tau_1 \leq n^{1/8} \right)
\leq \frac{C_{17}}{n^{1/8}} + \mathbb{P} \left( N(n^{1/8}) > \frac{\beta}{2} n^{1/4}; \tau_1 \leq n^{1/8} \right)
\leq \frac{C_{17}}{n^{1/8}} + \mathbb{P} \left( N(n^{1/8}) > \frac{\beta}{2} n^{1/4} \right)
\leq \frac{C_{17}}{n^{1/8}} + Ae^{-Bn^{1/8}}
\]
for some \( A \) and \( B > 0 \), where the last inequality follows from applying the Chebyshev inequality to \( \exp(N(n^{1/8})) \). Now
\[
\mathbb{P} \left( \sup_{t \in [0,1]} |Z_n(t) - Y_n(t)| > \frac{1}{n^{1/4}} \right) \leq n \mathbb{P} \left( M_1 > \frac{\beta}{2} n^{1/4} \right).
\]
Since \( \sum_n \left( \frac{C_{17}}{n^{1/8}} + Ae^{-Bn^{1/8}} \right) n < \infty \), it easily follows from Borel-Cantelli lemma that
\[
\sup_{0 \leq t \leq 1} |Z_n(t) - Y_n(t)| \to 0 \quad \mathbb{P} \text{-a.s.}
\]

From this it follows that
\[
Z_n(t) \Rightarrow B(t) \quad \text{as } n \to \infty.
\]

While we have considered the time interval \([0,1]\), we can define all random functions discussed so far on the interval \([0,T]\). All the above arguments can be
Therefore if \( n \) which implies that \( \geq N \)

There exists an \( e \), take \( \tau_n \), and \( \lambda_n \) have the same distribution respectively as \( Y_n', Z_n, B, \tau_n, \) and \( \lambda_n \); \( Z_n(t) = Y_n'(\lambda_n(t)) \), and both \( \lambda_n(t) \to t \) and \( Z_n(t) \to B(t) \) uniformly \( \bar{P} \)-almost surely. Thus \( B \) is a \( C[0,2] \)-valued random variable on \((\tilde{\Omega}, \tilde{P})\) with the same distribution as \( B \). Let \( \epsilon > 0 \) be given and let \( \omega \in \Omega \) be such that

\[
\lim_{n \to \infty} \tilde{Z}_n(t, \omega) = \tilde{B}(t, \omega) \quad \text{uniformly},
\]

and

\[
\lim_{n \to \infty} \tilde{\lambda}_n(t, \omega) = t \quad \text{uniformly.}
\]

There exists an \( N_1 = N_1(\omega) \in \mathbb{N} \) such that if \( n \geq N_1 \), then \( \tilde{\lambda}_n(1, \omega) \in [0, 2] \).

Take \( n \geq N_1 \). Since \( \lambda_n(t, \omega) \) is a continuous increasing function, for all \( t \in [0, 1] \), there exists \( e_n(t, \omega) \in [0, 2] \) such that \( \tilde{\lambda}_n(e_n(t, \omega), \omega) = t \). Therefore \( Y_n'(t, \omega) = \tilde{Z}_n(e_n(t, \omega), \omega) \). By (5) there exists an \( N_2 = N_2(\omega) \in \mathbb{N} \), \( N_2 \geq N_1 \) such that if \( n \geq N_2 \) then

\[
\sup_{0 \leq t \leq 1} \left| \tilde{Z}_n(e_n(t, \omega), \omega) - \tilde{B}(e_n(t, \omega), \omega) \right| < \frac{\epsilon}{2}
\]

Since \( \tilde{B}(t, \omega) \in C([0,2]) \) there exists \( \delta(\omega, \epsilon) > 0 \) such that if \( s, s' \in [0,2] \) and \( |s - s'| < \delta \) then \( |\tilde{B}(s, \omega) - \tilde{B}(s', \omega)| < \epsilon/2 \). By (6) we have there exists an \( N_3 = N_3(\omega) \in \mathbb{N} \), \( N_3 \geq N_2 \) such that if \( n \geq N_3 \), then \( \sup_{0 \leq t \leq 1} |t - e_n(t, \omega)| < \delta \). Therefore if \( n \geq N_3 \),

\[
\left| \tilde{B}(t, \omega) - \tilde{B}(e_n(t, \omega), \omega) \right| < \frac{\epsilon}{2}
\]

From equations (7) and (8) we have that if \( n \geq N_3 \), for all \( t \in [0,1] \),

\[
\left| \tilde{Y}_n'(t, \omega) - \tilde{B}(t, \omega) \right| = \left| \tilde{Z}_n(e_n(t, \omega), \omega) - \tilde{B}(t, \omega) \right|
\leq \left| \tilde{Z}_n(e_n(t, \omega), \omega) - \tilde{B}(e_n(t, \omega), \omega) \right|
+ \left| \tilde{B}(e_n(t, \omega), \omega) - \tilde{B}(t, \omega) \right| < \epsilon.
\]

This proves that

\( \tilde{Y}_n'(t) \to \tilde{B}(t) \) uniformly in \([0,1] \) \( \bar{P} \)-a.s.,

which implies that

\( Y_n'(t) \Rightarrow B(t) \) as \( n \to \infty \).
Since $|X(t) - Y(t)| \leq 1$ for all $t \geq 0$, we have that
\[
\left| \frac{X(ant)}{\sqrt{n}} - \frac{Y(ant)}{\sqrt{n}} \right| \to 0 \quad \text{uniformly as } n \to \infty,
\]
and thus
\[
\frac{X(ant)}{\beta \sqrt{n}} \Rightarrow B(t),
\]
proving the theorem.

The following proposition establishes a lower bound for $\beta/\sqrt{\alpha}$ in some cases.

**Proposition 6** If $p = 1$ and $\mu$ is large enough, then

\[
E(X(\tau_1)^2) > \lambda E(\tau_1). \tag{9}
\]

**Proof** Since $p = 1$, the first jump happens at $\sigma_1$. Thus $P(X(\sigma_1) = 1) = P(X(\sigma_1) = -1) = 1/2$. Let $\zeta$ be the first time after $\sigma_1$ that either the particle jumps or the broken bond is fixed. Therefore

\[
E(\zeta - \sigma_1) = \frac{1}{\lambda + \mu}
\]

and

\[
E(\zeta) = \frac{1}{\lambda} + \frac{1}{\lambda + \mu}.
\]

Note that the probability that the broken bond is fixed before the particle jumps is $\mu/(\lambda + \mu)$. Therefore,

\[
P(X(\zeta) = 1 | X(\sigma_1) = 1) = \frac{\mu}{\lambda + \mu},
\]

and

\[
P(X(\zeta) = 2 | X(\sigma_1) = 1) = \frac{\lambda}{\lambda + \mu}.
\]

Hence we have

\[
E(X(\zeta)^2) = 1^2 \cdot \frac{\mu}{\lambda + \mu} + 2^2 \cdot \frac{\lambda}{\lambda + \mu} = \frac{\mu + 4\lambda}{\lambda + \mu}
\]

and

\[
E(X(\zeta)^2) - \lambda E(\zeta) = \frac{2\lambda}{\lambda + \mu} > 0. \tag{10}
\]

By definition of $\zeta$ and $\tau_1$, $P(\tau_1 = \zeta) = \mu/(\lambda + \mu)$. On $\{\tau_1 > \zeta\}$, there are two broken bonds at time $\zeta$. Given that $\{\tau_1 > \zeta\}$, the conditional probability that both these bonds get fixed before there is a third jump is

\[
\frac{2\mu}{\lambda + 2\mu} = 1 - O\left(\frac{1}{\mu}\right).
\]
Therefore, the chance that there is a third jump is of the order $1/\mu^2$. Hence, as $\mu \to \infty$, the quantity $1 - P(X(\tau_1) = X(\zeta))$ is at most of the order $1/\mu^2$, and

$$E\left(X(\tau_1)^2\right) + O\left(\frac{1}{\mu^2}\right) \geq E\left(X(\zeta)^2\right) \text{ as } \mu \to \infty. \quad (11)$$

Let $\theta$ be the time of the first jump after $\zeta$. Note that $\{\tau_1 > \theta\} \subset \{\tau_1 > \zeta\}$ and therefore, $P(\tau_1 > \theta)$ is at most of order $1/\mu^2$. Furthermore, $E(\theta - \zeta \mid \tau_1 > \theta) = 1/\lambda$ and so $E(\theta - \zeta)$ is at most of order $1/\mu^2$. Hence, $E(\min(\tau_1 - \zeta, \theta - \zeta))$ is at most of order $1/\mu^2$. Also, using a first step analysis, we can show that for all $\mu \geq 1$, $E(\tau_1 - \theta \mid \tau_1 > \theta) \leq 4(e^\lambda - 1)/\lambda$, and therefore $E(\tau_1 - \theta)$ is of order $1/\mu^2$. Since

$$\tau_1 - \zeta \leq \min(\tau_1 - \zeta, \theta - \zeta) + (\tau_1 - \theta),$$

$E(\tau_1 - \zeta)$ is also of order at most $1/\mu^2$. Along with (10) and (11), this establishes the Proposition.

Without loss of generality, for the rest of this discussion, we will assume that $\lambda = 1$. Note that if $U(t)$ is a continuous time nearest neighbor random walk with jump rate 1, then

$$\frac{U(nt)}{\sqrt{n}} \Rightarrow B(t).$$

On the other hand, Theorem 4 and Proposition 6 imply that under the conditions of Proposition 6, we have

$$\frac{X(nt)}{\sqrt{n}} \Rightarrow \frac{\beta}{\sqrt{\alpha}} B(t),$$

with $\beta/\sqrt{\alpha} > 1$. Thus, under the conditions of Proposition 6, the effect of bond breaking is to make the process move faster in the scaling limit.

Remark: We can prove the scaling limit for the corresponding walk in $d$ dimensions by obtaining the bound on the recurrence time $\tau$ in the same way using an $M/M/\infty$ queue and in the proof of Theorem 1 using the multidimensional version of Donsker’s invariance principle. The rest of the argument proceeds in the same way.

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