A general construction of partial Grothendieck transformations

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Abstract

Fulton and MacPherson introduced the notion of bivariant theories related to Riemann-Roch-theorems, especially in the context of singular spaces. This is powerful formalism, which is a simultaneous generalization of a pair of contravariant and covariant theories. Natural transformations of bivariant theories are called Grothendieck transformations, and these generalize a pair of ordinary natural transformations.

But there are many situations, where such a bivariant theory or a corresponding Grothendieck transformation is only "partially known": characteristic classes of singular spaces (e.g. Stiefel-Whitney or Chern classes), cohomology operations (e.g. singular Adams Riemann-Roch and Steenrod operations for Chow groups) or equivariant theories (e.g. Lefschetz Riemann-Roch). We introduce in this paper a simpler notion of partial (weak) bivariant theories and partial Grothendieck transformations, which applies to all these examples. Our main theorem shows, that a natural transformation of covariant theories, which commutes with exterior products, automatically extends uniquely to such a partial Grothendieck transformations of suitable partial (weak) bivariant theories ! In the above geometric situations one has for example to consider only morphisms, whose target is a smooth manifold, or more generally, a suitable "homology manifold" (and in the general bivariant language this is related to the existence of suitable strong orientations). We illustrate our main theorem for the examples above, relating it to corresponding known Riemann-Roch theorems.

Acknowledgements.

This paper should be seen as a continuation of the basic work of W.Fulton and R.MacPherson [FM] about bivariant theories and Grothendieck transformations.
Its origin goes back to the work [Br] of J.P. Brasselet and [Y1] ... [Y6] of S. Yokura on the theory of bivariant Chern classes. In particular, our main theorem goes back to a result of S. Yokura [Y6] about the construction (and uniqueness) of a suitable (in our notion "partial") bivariant theory of such Chern classes. The author realized the abstract bivariant background of this construction, and our aim is to show by some important examples the power of this abstract bivariant result!

In the examples of this paper we consider only the case of partial Grothendieck transformations between bivariant theories in the sense of Fulton-MacPherson. The general construction of new partial bivariant theories will be explained elsewhere. The author would like to thank J.P. Brasselet and S. Yokura for some remarks on this work.

1 Uniqueness of bivariant transformations

First we recall some notions of Fulton-MacPherson [FM] for the general theory of bivariant theories and Grothendieck transformations.

Let $\mathcal{C}$ be a category with a final object $pt$ and fiber-products. Fix in addition a class of confined maps (closed under composition and base-change, containing all identity maps), and a class of independent squares (which we always assume to be fiber squares), closed under "vertical" and "horizontal" composition (as in [FM, p.17]) and containing any square (or its "transpose") of the form

$$
\begin{array}{ccc}
X & \xrightarrow{id} & X \\
\downarrow f & & \downarrow f \\
Y & \xrightarrow{id} & Y .
\end{array}
$$

A bivariant theory $\mathbb{B}$ on the category $\mathcal{C}$ assigns to each morphism

$$
\begin{array}{ccc}
f : X & \to & Y \\
\end{array}
$$

in $\mathcal{C}$ an abelian group

$$
\mathbb{B}(f : X \to Y)
$$

together with three (linear) operations:

1. **product**: For morphisms $f : X \to Y$ and $g : Y \to Z$ a product

   $$
   \bullet : \mathbb{B}(f : X \to Y) \times \mathbb{B}(g : Y \to Z) \to \mathbb{B}(g \circ f : X \to Z) .
   $$

2. **push-down**: For morphisms $f : X \to Y$ and $g : Y \to Z$ with $f$ confined a push-forward

   $$
   f_* : \mathbb{B}(g \circ f : X \to Z) \to \mathbb{B}(g : Y \to Z) .
   $$
3. **pull-back**: For any independent square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

a pull-back

\[g^*: \mathbb{B}(f: X \to Y) \to \mathbb{B}(f': X' \to Y').\]

These three operations are required to satisfy the following seven compatibilities (compare [FM, PartI, 2] for details):

- **(A1)** Product is associative.
- **(A2)** Push-down is functorial.
- **(A3)** Pull-back is functorial.
- **(A12)** Product and push-down commute.
- **(A13)** Product and pull-back commute.
- **(A23)** Push-down and pull-back commute.
- **(A123)** The (bivariant) projection formula.

The bivariant theory \(\mathbb{B}\) is by definition **graded** \((\mathbb{Z}_2\text{-} or \mathbb{Z}\text{-}graded)\), if each group \(\mathbb{B}(f : X \to Y)\) has a grading

\[\mathbb{B}^i(f : X \to Y),\]

which is stable under push-down and pull-back as above and additive under the (bivariant) product:

\[\cdot : \mathbb{B}^i(f : X \to Y) \times \mathbb{B}^j(g : Y \to Z) \to \mathbb{B}^{i+j}(g \circ f : X \to Z).\]

As in [FM], we use for \(\alpha \in \mathbb{B}(f : X \to Y)\) in bivariant diagrams a symbol near the arrow of the morphism:

\[
X \xrightarrow{\alpha \ f} Y.
\]

We assume that the bivariant theory \(\mathbb{B}\) has a **unit** [FM, p.22] (i.e. an element \(1_X \in \mathbb{B}(id : X \to X)\), which is a unit with respect to all possible bivariant products, with \(g^*(1_X) = 1_{X'}\): for any map \(g : X' \to X)\).
Finally, a (graded) bivariant theory $B$ is called (skew-)commutative if, whenever the "transpose" of an independent square

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
$$

is also independent, then

$$g^*(\alpha) \bullet \beta = f^*(\beta) \bullet \alpha$$

(or

$$g^*(\alpha) \bullet \beta = (-1)^{|\alpha|\cdot|\beta|} f^*(\beta) \bullet \alpha$$

in the skew-commutative case. Here $| \cdot |$ denotes the degree).

Moreover, $B^*(X) := B(id : X \to X)$ is by definition the associated contravariant theory, with a cup-product

$$\cup : B^*(X) \times B^*(X) \to B^*(X)$$

induced from the bivariant product for the composition $X \xrightarrow{id} X \xrightarrow{id} X$ (with possible grading $B_i(X) := B_i(id : X \to X)$). The associated covariant theory is defined as $B_*(X) := B(X \to pt)$ (which is covariant for confined maps), with a cap-product

$$\cap : B^*(X) \times B_*(X) \to B_*(X)$$

induced from the bivariant product for the composition $X \xrightarrow{id} X \to pt$ (with possible grading $B_i(X) := B^{-i}(X \to pt)$).

Suppose $C$ and $\bar{C}$ are categories with classes of confined maps, independent squares and a final object, and consider a functor

$$\gamma : C \to \bar{C}$$

respecting these structures. Then a Grothendieck transformation $\gamma$ of bivariant theories $F$ on $C$ and $H$ on $\bar{C}$ is a collection of homomorphisms

$$\gamma_f : F(f : X \to Y) \to H(\bar{f} : \bar{X} \to \bar{Y}),$$

which commutes with product, push-down and pull-back operations.

Remark 1.1. Note that in general a Grothendieck transformation $\gamma$ need not preserve a possible grading of the bivariant theories!
Let us now discuss the uniqueness problem of a Grothendieck transformation. Since \( \gamma \) respects the bivariant product, one gets

\[
\gamma_{gf}(\alpha \bullet \beta) = \gamma_f(\alpha) \bullet \gamma_g(\beta)
\]

for all \( \alpha \in F(f : X \to Y) \) and \( \beta \in F(g : Y \to Z) \). So if we have a distinguished element \( e_g \in F(g : Y \to Z) \) such that \( \gamma_g(e_g) \) is a strong orientation for \( \bar{g} \), i.e.

\[
\gamma_g(e_g) : H(f' : X' \to \bar{Y}) \to H(\bar{g} \circ f' : X' \to \bar{Z})
\]

is an isomorphism for all morphisms \( f' : X' \to \bar{Y} \) in \( \bar{C} \) (compare [FM, Def. 2.7.1, p.29]), then \( \gamma_f(\alpha) \) is uniquely determined by \( \gamma_{gf}(\alpha \bullet e_g), \gamma_g(e_g) \) and equation (1).

In particular, if the corresponding covariant transformation

\[
\gamma_* : F_* \to H_*
\]

of the associated covariant functors is unique, then the bivariant transformation

\[
\gamma_f : F(f : X \to Y) \to H(\bar{f} : \bar{X} \to \bar{Y})
\]

is also unique provided there is some element \( e_Y \in F_*(Y) := F(Y \to pt) \) such that \( \gamma_*(e_Y) \) is a strong orientation.

The important example for us comes from the theory of characteristic classes of singular spaces, with \( \bar{Y} \) a smooth manifold. Here the confined maps are the proper maps (as in the following examples). By resolution of singularities, the corresponding covariant transformation \( \gamma_* : F_* \to H_* \) is unique, and one has the normalization

\[
\gamma_*(e_Y) = c^*(\bar{Y}) \cap [\bar{Y}], \tag{2}
\]

with \( c^*(\bar{Y}) := c^*(T\bar{Y}) \in H^*(\bar{Y}) := H(id : \bar{Y} \to \bar{Y}) \) the corresponding characteristic class of the tangent bundle \( T\bar{Y} \), and \( [\bar{Y}] \in H_*(\bar{Y}) := H(\bar{Y} \to pt) \) the fundamental class of the manifold \( \bar{Y} \).

Here one can work in one of the following cases:

**Example 1.1.** In the first examples \( F \) and \( H \) are defined on the same category (so that \( \bar{f} \) is the identity transformation).

1. In the algebraic context with \( F_*(Y) := K_0(Y) \) the Grothendieck group of coherent sheaves, \( e_Y \) the class of the structure sheaf,

\[
H_*(Y) := A_*(Y) \otimes \mathbb{Q}
\]

the Chow-group with rational coefficients (resp. \( H \) the operational bivariant Chow group with rational coefficients) and \( \gamma_* = \tau_* \) the Baum-Fulton-MacPherson transformation (compare [FM] Part II and [Ful] Chapter 17,18) so that \( c^*(T\bar{Y}) \) is the total Todd class of \( T\bar{Y} \).
2. In the algebraic context (over a ground field of characteristic zero) with $F_*(Y)$ the group of algebraically constructible functions, $e_Y := 1_Y$,

\[ H_*(Y) := A_*(Y) \]

the Chow-group (resp. $H$ the operational bivariant Chow group) and $\gamma_* = c_*$ the \textit{Chern-Schwartz-MacPherson transformation} \cite{full, ken} so that $c^*(TY)$ is the total \textit{Chern class} of $TY$.

3. In the complex analytic (algebraic) context with $F_*(Y)$ the group of analytically (algebraically) constructible functions, $e_Y := 1_Y$,

\[ H_{2*}(Y) := H^B_{2*}(Y, \mathbb{Z}) \]

the Borel-Moore Homology-group (resp. $H$ the bivariant Homology) in even degrees and $\gamma_* = c_*$ the \textit{Chern-Schwartz-MacPherson transformation} \cite{bri, bri2} so that $c^*(TY)$ is the total \textit{Chern class} of $TY$. If we consider only spaces which can be embedded into complex analytic (algebraic) manifolds, then we can use instead of $F_*(Y)$ the isomorphic theory $L(X)$ of (conic) Lagrangian cycles \cite{sab1, sab2, ken, sch1}, with $e_Y$ the zero-section of the cotangent bundle $T^*Y$ for $Y$ smooth. Then $\gamma_*$ corresponds to the \textit{Chern-Mather transformation} so that $c^*(TY)$ is again the total \textit{Chern class} of $TY$.

4. In the subanalytic (or pl-) context with $F_*(Y)$ the group of subanalytically (or pl-) constructible Euler-functions with values in $\mathbb{Z}_2$, $e_Y := 1_Y$,

\[ H_*(Y) := H^B_*(Y, \mathbb{Z}_2) \]

the Borel-Moore Homology-group (resp. $H$ the bivariant Homology) and $\gamma_* = w_*$ the \textit{Stiefel-Whitney transformation} \cite{fum, sch1} or \cite{su, fm, eh} so that $c^*(TY) := w^*(TY)$ is the total \textit{Stiefel-Whitney class} of $TY$.

5. In this last example we work on the category of complex spaces with a \textit{real structure}, i.e. with an antiholomorphic involution (or in the algebraic context over the ground field $\mathbb{R}$ so that the associated complex spaces have an induced real structure). In this case the transformation $\bar{\cdot}$ is given on the objects by

\[ X \mapsto X(\mathbb{R}) \]

with $X(\mathbb{R})$ the fixed point set of the real structure (of the associated complex space), and on morphisms it is just restriction to these subsets (of the associated holomorphic map). Here we consider as $F_*(Y)$ the group of analytically (algebraically) constructible functions, which are \textit{invariant} under the real structure, $e_Y := 1_Y$ and

\[ H_*(\bar{Y}) := H^B_*(Y(\mathbb{R}), \mathbb{Z}_2) \]
the Borel-Moore Homology-group (resp. \( H \) the bivariant Homology). The
transformation \( \gamma_* \) is given as the composition

\[
F(X) \xrightarrow{|\text{mod } 2} F(X(\mathbb{R})) \xrightarrow{w_*} H_*(X(\mathbb{R})),
\]

with \(|\text{mod } 2\) given by restriction and taking the values \( \text{mod } 2 \), and \( w_* \) is the Stiefel-Whitney transformation of the example before (note that \(|\text{mod } 2\)
maps to real analytically (algebraically) constructible Euler-functions, and it
is functorial with respect to push-down for proper equivariant holomorphic
maps). So for a smooth manifold \( Y \) one has

\[
\gamma_*(e_Y) = w^*(TY(\mathbb{R})) \cap [Y(\mathbb{R})] .
\]

Note that in all cases the fundamental class \( [\bar{Y}] \in H_*(\bar{Y}) \) is a strong orien-
tation and \( c^*([\bar{Y}]) \in H^*(\bar{Y}) \) is invertible so that \( \gamma_*(e_Y) = c^*([\bar{Y}]) \cap [\bar{Y}] \) is also a
strong orientation!

Remark 1.2. The above reasoning for the uniqueness of these bivariant character-
istic classes is just the argument of [Y1] written up in the bivariant language. In
that paper Yokura shows in particular the uniqueness of bivariant Chern classes
for morphisms with smooth target (extending a result of [Zhou1, Zhou2], compar-
ing the bivariant Chern classes of [Br] and [Sab2] for the case of a smooth one-
dimensional target). Moreover, our bivariant argument goes already back to [FM,
p.70], where they prove the uniqueness of the bivariant Stiefel-Whitney classes in
the pl-context. Their proof consists of three steps:

1. By resolution of singularities, the corresponding covariant transformation
\( w_* : F_* \to H_* \) is unique [FM end of p.70].

2. Uniqueness in the case of a smooth target [FM eq. (**), p.70], with the
same argument as we explained above.

3. Uniqueness in the general case by reducing it to (a pullback of) the case 2.

Only the last step does not generalize to the other bivariant characteristic classes!
Moreover, steps 1. and 2. also apply to suitable "equivariant versions" of the
cases given in example 1.1 (as will be explained somewhere else and compare with
[EG2]).

Remark 1.3. Our uniqueness argument applies also to a suitable singular space
\( \bar{Y} \), if for example the fundamental class \( [\bar{Y}] \in H_*(\bar{Y}) \) is a strong orientation with
\( c^*(\bar{Y}) \in H^*(\bar{Y}) \) invertible, where this cohomology class is defined by

\[
\gamma_*(e_Y) = c^*(\bar{Y}) \cap [\bar{Y}] .
\]

In example [FM] this is true in the case

1. for \( Y \) an Alexander scheme in the sense of [Vis, Ki2, Ki3].
2. If \( Y \) has a Chow-isomorphism \( \tilde{Y} \to Y \) in the sense of [Ki1, def.3.5, p.296], with \( \tilde{Y} \) a smooth manifold (e.g. \( Y \) is a cusp curve [Ki1, ex.3.7(2), p.297]).

3. \( Y \) is an oriented \( \mathbb{Z} \)-homology manifold.

4. \( Y \) is a \( \mathbb{Z}_2 \)-homology manifold.

5. \( Y(\mathbb{R}) \) is a \( \mathbb{Z}_2 \)-homology manifold.

The fundamental class \([ \tilde{Y} ]\) is in the last three cases a strong orientation by [FM, prop.7.3.2, p.85]. Moreover, in all these five cases is \( H^*(\tilde{Y}) \) a commutative ring with unit \( 1_{\tilde{Y}} \) such that \( c^*(\tilde{Y}) - 1_{\tilde{Y}} \) is nilpotent, i.e. \( \gamma_*(e_Y) = [\tilde{Y}] \) up to lower order terms!

But even if the corresponding covariant transformation \( \gamma_* \) maybe is not unique, one can use the same argument to describe the bivariant transformation

\[
\gamma_f : F(f : X \to Y) \to H(\tilde{f} : \tilde{X} \to \tilde{Y})
\]

in terms of \( \gamma_* \). Important examples come from cohomology operations (compare [FM, 4.1.4, 4.2, p.46-49]):

**Example 1.2.** 1. \( \gamma : H \to H \) is the Grothendieck transformation on \( \mathbb{Z}_2 \)-valued bivariant homology \( H \) induced by the Steenrod square [FM, 4.2.1, p.47]. Then one has for a smooth manifold \( Y \) with fundamental class \( e_Y := [Y] \in H_*(Y) \):

\[
\gamma_*([Y]) = c^*(TY) \cap [Y],
\]

with \( c^*(TY) = w^*(TY)^{-1} \) the inverse Stiefel-Whitney class of \( TY \).

2. One looks for a bivariant transformation

\[
\gamma = \psi(j) : K \to \mathbb{Z}[1/j] \otimes K,
\]

with \( K \) a suitable bivariant algebraic \( K \)-theory (as in [FM, Part II]) such that

\[
\gamma_* = \psi_j : K_* \to \mathbb{Z}[1/j] \otimes K_*
\]

is the covariant transformation described in the singular Adams Riemann-Roch theorem [FL, p.190] (\( j \in \mathbb{N} \)). Then one has for a smooth manifold \( Y \), with the fundamental class \( e_Y := [Y] \in K_*(Y) \) given by the class of the structure sheaf:

\[
\gamma_*([Y]) = c^*(TY) \cap [Y],
\]

with \( c^*(TY) = \theta^j(TY^\vee)^{-1} \) the inverse cannibalistic class of the dual of \( TY \).

3. We work in the algebraic context over a base field, whose characteristic is different from a fixed prime \( p \). Then one can ask for a bivariant transformation

\[
\gamma = C(p) : A^* \otimes \mathbb{Z}_p \to A^* \otimes \mathbb{Z}_p,
\]
with $A^*$ the bivariant Chow groups, whose associated covariant transformation

$$\gamma_* = C_* : A_* \otimes \mathbb{Z}_p \to A_* \otimes \mathbb{Z}_p$$

is the total Steenrod $p$-th power operation on Chow groups recently constructed in [Bro]. By [Bro, prop.9.4(iii)] we have for $Y$ smooth:

$$S_\bullet([Y]) = c^*(TY) \cap [Y],$$

with $c^*(TY) = w^*(TY)^{-1}$ the inverse of the characteristic class

$$w^*(TY) := \prod \left(1 + \lambda^p - 1_i\right)$$

defined in terms of the Chern roots $\lambda_i$ of $TY$ [Bro, p.1891]. □

Another class of important examples comes from equivariant theories and corresponding Lefschetz Riemann-Roch theorems:

**Example 1.3.** Here we consider for simplicity the complex algebraic context with schemes of finite type over $\text{spec}(\mathbb{C})$, together with an automorphism of finite order (of course the morphisms have to commute with these automorphisms. For another treatment of the first example in the algebraic context over an algebraically closed field compare with [BFQ]).

1. We consider only quasi-projective schemes and work in the context of "coherent sheaves" given in [FM] 10.1, p.103-105 with a bivariant transformation

$$\tau : K_{eq}^{alg}(f : X \to Y) \to H(\{f \mid X \to |Y|\} \otimes \mathbb{C}).$$

Here $K_{eq}^{alg}$ is the Grothendieck group of "equivariant $f$-perfect complexes". Moreover, $\cdot := \mid \mid$ is given by restriction to the fixed point set of the associated complex spaces, and these fixed point sets are assumed to be projective! Finally $H$ is the usual bivariant Homology. Then one has for a smooth manifold $Y$ and $e_Y \in K_{eq}^{alg}(Y \to pt)$ given by the structure sheaf (with its canonical isomorphism $\phi$ lifting the automorphism of $Y$):

$$\gamma_*(e_Y) = \left(\text{ct}(\lambda|Y|)^{-1} \cup t\text{d}^*(T|Y|)\right) \cap [Y].$$

Here

$$\lambda|Y| := \sum (-1)^i[A^iN] \in K_{eq}^{alg}(id : |Y| \to |Y|)$$

is the corresponding (invertible) "Euler class" of the equivariant conormal sheaf $N$ to $|Y|$ in $Y$ (and for the definition of the "Chern trace" $\text{ct}$ compare with [FM, p.104]).
2. We work in the context of "algebraically constructible sheaves" of vector spaces (over a field $k$), whose stalks are finite dimensional. Consider the Grothendieck group $K_{c}^{eq}(X \to pt)$ of equivariant algebraically constructible sheaves on $X$ and define the transformation $\gamma_{*} : K_{c}^{eq}(X \to pt) \to H_{*}(|X|) \otimes k$ as the composition

$$K_{c}^{eq}(X \to pt) \xrightarrow{tr_{|X|}} F(|X|) \otimes k \xrightarrow{c_{*} \otimes k} H_{*}(|X|) \otimes k.$$ 

Here $tr_{|X|}$ is given by taking stalkwise the trace of the restriction to the fixed point set, and $c_{*}$ is the Chern class transformation on the group $F(|X|)$ of algebraically constructible functions on $|X|$. Here we can choose for $H_{*}$ the Borel-Moore homology group or the Chow group. One can show that $\gamma_{*}$ (or $tr_{|X|}$) is a natural transformation for equivariant proper morphisms $f : X \to Y$, where

$$f_{*} : K_{c}^{eq}(X \to pt) \to K_{c}^{eq}(Y \to pt)$$

is defined by the (alternating sum of the classes of the equivariant) higher direct image sheaves. So one looks for a bivariant extension of $\gamma_{*}$. If $Y$ is a smooth manifold, one has for $e_{Y} \in K_{c}^{eq}(Y \to pt)$ given by the constant sheaf $k_{Y}$ (with its canonical isomorphism $\phi$ lifting the automorphism of $Y$):

$$\gamma_{*}(e_{Y}) = c^{*}(T|Y|) \cap ||Y||,$$

with $c^{*}(T|Y|)$ the Chern class of the tangent bundle of the fixed point manifold $|Y|$. □

Also we are mainly interested in applications to singular spaces, our results also apply to suitable categories of manifolds, especially to Riemann-Roch theorems in the framework of oriented cohomology (pre)theories as recently studied in [LM, Lev, Pal, PaSm] (and compare with [FM, p.46-52] for similar results in the context of differentiable manifolds, which are oriented with respect to suitable cohomology theories, e.g. for complex manifolds and multiplicative complex oriented cohomology theories):

An oriented cohomology (pre)theory is a suitable (contra-variant) functor

$$A : Sm \to Rings$$

on the category $Sm$ of smooth quasi-projective varieties over a field $k$ with values in the category of (commutative graded) rings with unit. $A$ is also covariant functorial with respect to proper morphisms (of constant relative dimension). Note that a proper morphism of quasi-projective varieties is projective!

These satisfy a projection-formula (i.e. the push-down for $f : X \to Y$ is a two-sided (!) $A(Y)$-module operator), and the base-change property $g^{*}f_{*} = f_{*}^{'}g^{*}$ for
any transverse cartesian diagram

\[
\begin{array}{c}
X' \xrightarrow{g'} X \\
\downarrow f' \quad \quad \downarrow f \\
Y' \xrightarrow{g} Y,
\end{array}
\]

with \(f, f'\) proper (of constant relative dimension). \(A\) has in addition to satisfy some other properties (which are not important for us), and these imply especially a corresponding theory of Chern classes with ("universally central") values in \(A\) \[Pa\] thm.2.2.2.

As we explain later on, we therefore get on \(Sm\) a simple bivariant theory \(A\) defined by \(A(f : X \to Y) := A(X)\). The cofined maps are the proper morphisms (of constant relative dimension), and the independent squares are the transverse cartesian diagrams, with the obvious push-down and pull-back transformations. Finally the bivariant product

\[
\bullet : A(f : X \to Y) \times A(g : Y \to Z) \to A(g \circ f : X \to Z)
\]

is just given by \(\alpha \bullet \beta := \alpha \cup f^*(\beta)\), with \(\cup\) the given product of the ring-structure. Then \(e_Y := 1_Y \in A(Y) = A(Y \to pt)\) is (trivially) a strong orientation.

If \(\phi : A \to B\) is a "nice" ring morphism of two such theories (compare [Pa] thm.2.5.4, p.46) for details), then one has a corresponding Riemann-Roch theorem saying that the composition \(\gamma_*:\)

\[
A_*(X) = A^*(X) \xrightarrow{\phi} B^*(X) \xrightarrow{\cup td_\phi(TX)} B^*(X) = B_*(X)
\]

is a natural transformation between the corresponding covariant theories (i.e. commutes with push-down). Here \(td_\phi\) is a "suitable" Todd genus associated to \(\phi\) [Pa] Def. 2.5.2, p.45]. Since \(\phi\) is a ring morphism we get \(\phi(1_Y) = 1_Y\), and therefore

\[
\gamma_*(e_Y) = c^*(Y) \cap [Y],
\]

with \(c^*(Y) := td_\phi(TY) \in B(Y) = B^*(Y)\) and \([Y] := 1_Y \in B(Y) = B_*(Y)\). Moreover \(c^*(Y) := td_\phi(TY)\) is by definition invertible and "universally central" (i.e. any pullback of it is central). □

Let us come back to the general bivariant context. Then one gets in all the preceding examples the following explicit (!) description of a corresponding bivariant transformation

\[
\gamma_f : F(f : X \to Y) \to H(f : X \to Y)
\]
in the case \( Y \) a smooth manifold (compare [Y1 thm.3.4, thm.3.7]):
\[
\gamma_f(\alpha) \bullet [\bar{Y}] = \bar{f}^*(e^*(\bar{Y}))^{-1} \cap \gamma_*(\alpha \bullet e_Y) \in H_*(\bar{X}).
\] (3)

Here \( \bar{f}^* : H^*(\bar{Y}) \to H^*(\bar{X}) \) is the pullback of the associated *contravariant* theory and
\[
\cap : H^*(\bar{X}) \times H_*(\bar{X}) \to H_*(\bar{X})
\]
is the corresponding *cap-product* [FM p.23]. This formula (3) is also true, with the same proof, for \( Y \) a singular space as in remark [FM p.23].

Since our bivariant theories \( H \) are commutative [FM p.22] (or \( c^*(Y)^{-1} \) is universally central in the context of oriented cohomology (pre)theories), this follows from the following commutative diagram (whose left square is independent):
\[
\begin{array}{ccc}
\bar{X} & \xrightarrow{\gamma_f(\alpha)} & \bar{Y} \\
\xrightarrow{f} & & \xrightarrow{\bar{f}} \\
\bar{f}^*(c^*(\bar{Y}))^{-1} & \xrightarrow{id_X} & c^*(\bar{Y})^{-1} \\
\xrightarrow{\varphi_Y} & & \xrightarrow{1_p = [\bar{p}]} \\
\bar{X} & \xrightarrow{\gamma_f(\alpha)} & \bar{Y} \\
\end{array}
\] (4)

since the *associativity* of the bivariant product implies
\[
\gamma_f(\alpha) \bullet [\bar{Y}] = \gamma_f(\alpha) \bullet \left( c^*(\bar{Y})^{-1} \bullet \gamma_*(e_Y) \right) = \left( \gamma_f(\alpha) \bullet c^*(\bar{Y})^{-1} \right) \bullet \gamma_*(e_Y) = \left( \bar{f}^*(c^*(\bar{Y}))^{-1} \bullet \gamma_f(\alpha) \right) \bullet \gamma_*(e_Y) = \bar{f}^*(c^*(\bar{Y}))^{-1} \bullet \left( \gamma_f(\alpha) \bullet \gamma_*(e_Y) \right) = \bar{f}^*(c^*(\bar{Y}))^{-1} \cap \gamma_*(\alpha \bullet e_Y).
\]

*Remark 1.4.* Again this is just the argument of [Y1] written up in the bivariant language. In the special case \( \alpha = 1_f := 1_X \) in the pl-context for the bivariant Stiefel-Whitney class, the formula (3) is already explained (implicitly) in [FM p.12/13]:
\[
w_*(X) = f^*(w^*(TY)) \cap \xi,
\]
with \( \xi := w_f(1_f) \bullet [Y] \in H_*(X, \mathbb{Z}_2) \). Similarly, if \( f : X \to Y \) is smooth (so that \( X \) is also smooth) one has for \( e_f \in F(f : X \to Y) \) equal to \( 1_X \) (or the class of the structure sheaf of \( X \), resp. the constant sheaf \( k_X \)) the relation \( e_f \bullet e_Y = e_X \). Therefore (3) implies
\[
\gamma_f(e_f) \bullet [\bar{Y}] = f^*(c^*(\bar{T})\bar{Y}))^{-1} \cap \left( c^*(T\bar{X}) \right) \cap [\bar{X}] = \left( \bar{f}^*(c^*(\bar{T})\bar{Y}))^{-1} \cup c^*(T\bar{X}) \right) \cap [\bar{X}] = c^*(T_f) \bullet [\bar{f}] \bullet [\bar{Y}] = \left( c^*(T_f) \bullet [\bar{f}] \right) \bullet [\bar{Y}].
\]
Here \(T_f\) is the relative tangent bundle and \([\tilde{f}]\in H(\tilde{f}: \tilde{X}\to \tilde{Y})\) the relative orientation class. Since \([\bar{Y}]\) is a strong orientation, one gets the well known formula

\[
\gamma_f(e_f) = c^*(T_{\tilde{f}}) \bullet [\tilde{f}] .
\]  

(5)

Compare [FM, prop. 6A, p.65] for the pl-Stiefel-Whitney class, [EOY, prop.3.7] for the Chern class and [FM, formula (*), p.124] for the Todd class in the complex algebraic context (which is stated there more generally for local complete intersection morphisms). In the last case our argument above works for any morphism \(f:X\to Y\) of smooth spaces \(X,Y\). More generally it works for a morphism \(f\) of manifolds, if we have an element \(e_f\in F(f:X\to Y)\) with \(e_f\bullet e_Y = e_X\), and a relative orientation class \([\tilde{f}]\in H(f: \tilde{X}\to \tilde{Y})\) with \([\tilde{f}]\bullet [\tilde{Y}] = [\tilde{X}]\).

A similar diagram as in (4) can also be used to reduce under suitable assumptions a bivariant product to the cup- and cap-product of the associated contra- and covariant theory. Consider a (skew-)commutative (graded) bivariant theory \(H\) and two objects \(Y,Z\) such that \(H^*(Y)\) and \(H^*(Z)\) contain a strong orientation \([Y]\) and \([Z]\). Then we have in particular isomorphisms

\[
\bullet[Y]: H^*(Y) \to H_*(Y) , \quad \bullet[Z]: H(f: X\to Y) \to H_*(X)
\]

and similarly for \(Z\). Then the commutative diagram (whose left square is independent)

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha\ f} & Y & \xrightarrow{\beta\ g} & Z \\
& f^*(\beta') & | & \downarrow & |
\end{array}
\]

(6)

implies that for \(\alpha\in H(f: X\to Y)\) and \(\beta\in H(g: Y\to Z)\) the bivariant product \(\alpha\bullet\beta\in H(g\circ f: X\to Z)\) corresponds under the isomorphism \(\bullet[Z]\) to \(\alpha'\cap f^*(\beta')\), with \(\alpha' := \alpha\bullet[Y]\in H_*(X)\):

\[
(\alpha\bullet\beta)\bullet[Z] = \alpha\bullet(\beta\bullet[Z]) = \alpha\bullet(\beta'\bullet[Y]) = (\alpha\bullet\beta')\bullet[Y] =
\]

\[
(-1)^{|\beta'|\cdot|\alpha|}(f^*(\beta')\bullet\alpha)\bullet[Y] = (-1)^{|\beta'|\cdot|\alpha|}f^*(\beta')\bullet(\alpha\bullet[Y]) =
\]

\[
(-1)^{|\beta'|\cdot|\alpha|}f^*(\beta')\cap\alpha' =: \alpha'\cap f^*(\beta') .
\]

Here \(|\cdot|\) denotes the degree (which in the commutative case should be set equal to zero in the above formula), and in the last equality we use the usual (graded) right module structure of a (graded) left module. Moreover, we assume in the skew-commutative case that all strong orientations in the above calculation have even degrees!
In the example of oriented cohomology (pre)theories this gives us back our definition of the bivariant product $\bullet$. Another important example is given by the bivariant Homology theory (or the operational bivariant Chow group) with $Y$ smooth and $[Y]$ the corresponding fundamental class so that the isomorphism

$$\bullet[Y] : H^*(Y) \to H_*(Y)$$

is just Poincaré or Alexander duality. Then the above equality implies for $Z := pt$ (and $[Z] := 1_{pt}$):

$$\alpha' \odot \beta := \alpha' \cap f^*(\beta') = \alpha \bullet \beta.$$  \hspace{1cm} (7)

So the product $\odot : H_*(X) \times H_*(Y) \to H_*(X)$ corresponds under the isomorphism induced by $\bullet[Y]$ to the bivariant product

$$\bullet : H(f : X \to Y) \times H_*(Y) \to H_*(X)$$

(compare [Y3, proof of thm.3.9]). In particular, it is associative (if defined).

2 Partial (weak) bivariant theories

We now explain (following ideas of Shoji Yokura [Y6]), how a covariant transformation $c_* : F_* \to H_*$ of bivariant theories can ”partially” be extended to a Grothendieck transformation of (partial) bivariant theories. Here we introduce the following notions:

**Definition 2.1.** Let $C$ be a category with classes of confined maps, independent squares and a final object (as in [FM, Part I,2]).

1. A weak bivariant theory $T$ assigns to each morphism $f : X \to Y$ in $C$ an abelian group $T(f : X \to Y)$ together with three operations product, push-forward and pull-back satisfying the compatibilities $(A1)-(A23)$ as in [FM, Part II], but not necessarily the (bivariant) projection formula $(A23)$.

2. Consider in addition a class of maps in $C$, called allowable maps, which is closed under composition such that also the composition $g \circ f$ is allowable for any confined $f$ and allowable $g$. A partial (weak) bivariant theory $T$ assigns to each allowable morphism $f : X \to Y$ in $C$ an abelian group $T(f : X \to Y)$ together with three operations product, push-forward and pull-back satisfying the compatibilities $(A1)-(A123)$ (or $(A1)-(A23)$ in the weak case) as in [FM, Part II], but this time only for all allowable maps: e.g. the push-forward

$$f_* : T(g \circ f : X \to Z) \to T(g : Y \to Z)$$

is only defined for $g : Y \to Z$ allowable and $f : X \to Y$ confined, and the pull-back

$$g^* : T(f : X \to Y) \to T(f' : X' \to Y')$$
is only defined for any independent square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
f'| & \downarrow & \downarrow f \\
Y' & \xrightarrow{g} & Y,
\end{array}
\]

with \(f, f'\) allowable (and in the bivariant projection formula (A123) of [FM, p.21/22] one has in addition to assume \(h \circ g\) is allowable).

3. The partial (weak) bivariant theory \(T\) is graded (\(\mathbb{Z}_2\)– or \(\mathbb{Z}\)-graded), if each group \(T(f : X \to Y)\) (for \(f\) allowable) has a grading

\[T^i(f : X \to Y),\]

which is stable under push-down and pull-back, and additive under the (bivariant) product. One defines the (skew-)commutativity of such a theory as in the usual bivariant context (e.g. one asks (skew) commutativity for independent squares, whose "transpose" is also independent, with all maps allowable).

4. If \(C, \bar{C}\) are two such categories with allowable maps, and \(\dashv : C \to \bar{C}\) is a functor respecting these structures (especially it transforms allowable maps in \(C\) to allowable maps in \(\bar{C}\), then a partial Grothendieck transformation \(t\) of partial (weak) bivariant theories \(T\) and \(U\) (on \(C, \bar{C}\)) is a collection of homomorphisms

\[t_f : T(f : X \to Y) \to U(\bar{f} : \bar{X} \to \bar{Y})\]

for each allowable map \(f : X \to Y\) in \(C\), which commutes with product, push-forward and pull-back operations.

All of our discussions so far (and also many of the arguments of [FM, part I, 2]) extend directly to partial (weak) bivariant theories. Here we introduced also the "weak notions", since most of our arguments in this paper work without the projection formula (A123).

Two important differences are the following:

(a) In general \(id_X : X \to X\) need not be an allowable map for an object \(X\) of \(C\). Especially, one does not have in general an associated contravariant (or similarly covariant) theory. If \(X \to pt\) is allowable for all objects \(X\) of \(C\), then one has at least a corresponding covariant theory \(T_*\) (covariant with respect to confined maps).

(b) The pull-back \(g^*\) maybe is also defined for a morphism \(g\) which is not allowable.
Here are some possible ways of constructing partial bivariant theories:

1. Any (weak) bivariant theory $F$ on $C$ "restricts" to a partial (weak) bivariant theory, e.g. one uses as allowable maps only those morphisms, whose target belongs to a fixed class of objects in $C$ (containing $pt$, e.g. "smooth manifolds").

2. Similarly, consider a (partial weak) bivariant theory $F$ on $C$ and suppose that one has for each allowable morphism $f : X \to Y$ a subgroup $F'(f : X \to Y)$ of $F(f : X \to Y)$, which is stable under product, push-down and pull-back. This gives then a partial (weak) bivariant theory $F'$, which we call a partial subtheory of $F$.

**Example 2.1.** Let $C$ be the category of complex algebraic varieties, with the proper maps the confined maps. Define $f : X \to Y$ to be allowable, if $Y$ is a smooth manifold, and consider the "restrictions" $A^*$ and $H^{2*}$ of the bivariant Chow groups and the bivariant Homology groups (in even degrees). Then the cycle map $cl : A_* \to H_{2*}$ [Ful, chap.19] of the associated covariant theories induces by (7) and [Ful, thm.19.2, p.380] a partial Grothendieck transformation $\gamma$ of these partial bivariant theories:

$$\gamma_f : A^*(f : X \to Y) \simeq A_*(X) \xrightarrow{cl} H_{2*}(X) \simeq H^{2*}(f : X \to Y).$$

Note, that a corresponding Grothendieck transformation of the original bivariant theories is not known.

We explain now a general way how a natural transformation $c_* : F_* \to H_*$ of associated covariant theories can be extended to a partial Grothendieck transformation.

Consider two partial (weak) bivariant theories $F$ on $C$ and $H$ on $\bar{C}$, together with a functor $\gamma : C \to \bar{C}$ respecting the underlying structures. We assume that all maps $X \to pt$ are allowable in $C$ (and the same for $\bar{C}$), and $F_*(pt)$ (resp. $H_*(\bar{pt})$) contains a unit $1_{pt}$ (or $1_{\bar{pt}}$) such that $\alpha \cdot 1_{pt} = \alpha$ for all $\alpha \in F_*(X)$ (and similarly for $1_{\bar{pt}}$).

Let in addition $c_* : F_* \to H_*$ be a natural transformation of the associated covariant theories, with $c_*(1_{pt}) = 1_{\bar{pt}} \in H_*(\bar{pt})$ the corresponding unit. We are searching for a subclass of the allowable maps in $C$, containing $X \to pt$ for all objects $X$, and a partial subtheory $F'$ of $F$ together with a partial Grothendieck transformation

$$\gamma : F' \to H$$

such that $F'(X \to pt) = F(X \to pt)$ for all objects $X$ and $\gamma_* = c_*$. 
If one can find such a partial extension $\gamma$ of $c_*$, then $c_*$ has to \textit{commute} with suitable \textit{external products}. Consider an independent square (in $C$)

$$
\begin{array}{ccc}
X' & \rightarrow & X \\
\downarrow g' & & \downarrow f \\
Y' & \rightarrow & pt \\
\end{array}
$$

with $f'$ \textit{allowable} and define the external product

$$
\times : F_*(Y') \times F_*(X) \rightarrow F_*(X')
$$

as in [FM, p.24] by

$$
\beta \times \alpha := g^*(\alpha) \bullet \beta
$$

If in addition $f'$ is also \textit{allowable} with respect to $F'$ (so that the pull-back $g^*$ maps $F_*(X) = F'_*(X)$ to $F'_*(X' \rightarrow Y') \subset F_*(X' \rightarrow Y')$), then one gets

$$
c_* (\beta \times \alpha) = \gamma_*(\beta \times \alpha) = \gamma_*(g^*(\alpha) \bullet \beta) = \\
\gamma_f (g^*(\alpha)) \bullet \gamma_*(\beta) = g^*(\gamma_f (\alpha)) \bullet \gamma_*(\beta) = \\
\gamma_*(\beta) \times \gamma_*(\alpha) = c_* (\beta) \times c_*(\alpha).
$$

Next we want to \textit{use} the uniqueness results of the beginning of this paper.

Fix a class of objects $Y$ of $C$, called \textit{orientable} (with respect to $c_*$), containing the final object $pt$ such that $F_*(Y)$ contains a distinguished element $e_Y$ with $c_*(e_Y)$ a \textit{strong orientation} in $H$ (and $e_{pt} = 1_{pt} \in F_*(pt)$). Then we call a morphism $f : X \rightarrow Y$ in $C$ \textit{o-allowable} iff $f$ is \textit{allowable} (with respect to $F$) and the target $Y$ is \textit{orientable}. We define for such an \textit{o-allowable} morphism the transformation

$$
\gamma_f : F(f : X \rightarrow Y) \rightarrow H(\bar{f} : \bar{X} \rightarrow \bar{Y})
$$

as in [1] by

$$
c_*(\alpha \bullet e_Y) = \gamma_f (\alpha) \bullet c_*(e_Y).
$$

For $Y = pt$ we get especially $\gamma_*= c_*$, since $e_{pt} = 1_{pt}$ and $c_*(1_{pt}) = 1_{pt}$. So we get a transformation $\gamma$ from $F$ restricted to these \textit{o-allowable} maps to $H$, but this need not to be a partial Grothendieck transformation.

Nevertheless, this transformation commutes automatically with \textit{pushdown}. Consider two morphisms $g : X \rightarrow Z$ and $h : Z \rightarrow Y$ in $C$, with $g$ \textit{confined} and $h$ \textit{o-allowable}. Then one gets for $f := h \circ g : X \rightarrow Y$:

$$
\bar{g}_* \circ \gamma_f = \gamma_h \circ g_* : F(f : X \rightarrow Y) \rightarrow H(\bar{f} : \bar{X} \rightarrow \bar{Y}),
$$

as in [1].
because
\[ \gamma_h\left(g_*(\alpha)\right) \bullet c_*(e_Y) \overset{(8)}{=} c_*(g_*(\alpha) \bullet e_Y) \overset{A12}{=} c_*(g_*(\alpha \bullet e_Y)) \] 
\[ \bar{g}_*\left(c_*(\alpha \bullet e_Y)\right) \overset{(8)}{=} \bar{g}_*\left(\gamma_f(\alpha) \bullet c_*(e_Y)\right) \overset{A12}{=} \bar{g}_*\left(\gamma_f(\alpha)\right) \bullet c_*(e_Y). \]

Here the equality (*) comes from the functoriality of \( c_* \).

The commutativity with product and pull-back will be built in by the following

**Definition 2.2.** For an \( o \)-allowable morphism \( f : X \to Y \) in \( C \), define the subgroup
\[ F'(f : X \to Y) \subset F(f : X \to Y) \]
to be the set of all \( \alpha \in F(f : X \to Y) \) satisfying for any independent square
\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{g^*(\alpha)} & \searrow{f} & \downarrow{\alpha} \\
Y' & \xrightarrow{g} & Y,
\end{array}
\]
with \( f' \) also \( o \)-allowable the following two conditions:
\[ c_*(g^*(\alpha) \bullet \beta) = \gamma_{f'}\left(g^*(\alpha)\right) \bullet c_*(\beta) \quad (10) \]
for any \( \beta \in F_*(Y') \), and
\[ \gamma_{f'}\left(g^*(\alpha)\right) = \bar{g}^*\left(\gamma_f(\alpha)\right). \quad (11) \]

The next lemma shows that condition \( \text{HE} \) implies also the commutativity of \( \gamma \) with products (we assume (!) that any commutative square as above with \( g = id_Y, g' = id_X \) is independent \[EM\] (B1, page 16)).

**Lemma 2.1.** Let \( f : X \to Y \) be \( o \)-allowable and assume \( \alpha \in F(f : X \to Y) \) satisfies
\[ c_*(\alpha \bullet \beta) = \gamma_f(\alpha) \bullet c_*(\beta) \quad (12) \]
for any \( \beta \in F_*(Y) \). Then one also has for \( \beta \in F(g : Y \to Z) \), with \( g \) \( o \)-allowable the equality
\[ \gamma_{g \circ f}(\alpha \bullet \beta) = \gamma_f(\alpha) \bullet \gamma_g(\beta). \quad (13) \]
This is an easy application of the definition of $\gamma$, the associativity of the bivariant product and the assumption (12):

$$
\gamma_{g \circ f}(\alpha \bullet \beta) \circ c_*(e_Z) \overset{(8)}{=} c_*\left((\alpha \bullet \beta) \bullet e_Z\right) \overset{A1}{=}
$$

$$
c_*\left((\alpha \bullet (\beta \bullet e_Z))\right) \overset{(12)}{=} \gamma_f(\alpha) \bullet c_*(\beta \bullet e_Z) \overset{(8)}{=}
$$

$$
\gamma_f(\alpha) \bullet \left(\gamma_g(\beta) \circ c_*(e_Z)\right) \overset{A1}{=} \left(\gamma_f(\alpha) \bullet \gamma_g(\beta)\right) \bullet c_*(e_Z).
$$

Now we are ready to prove the main result of this paper (following [Y6], and compare also with [Y5, thm.A, thm.3.10] for a similar construction of a Grothendieck transformation to an operational bivariant theory):

**Theorem 2.1.** $F'$ is a partial subtheory of $F$ and $\gamma : F' \to H$ is a partial Grothendieck transformation. Assume in addition that $c_* : F_* \to H_*$ commutes with external products for any independent square (in $C$)

$$
\begin{array}{ccc}
X' & \overset{g'}{\longrightarrow} & X \\
\downarrow f' & & \downarrow f \\
Y' & \overset{g}{\longrightarrow} & pt,
\end{array}
$$

with $f'$ o-allowable. Then $F'_* = F_*$ and $\gamma_* = c_*$. Moreover, $F'$ is then independent of the choice of $e_Y \in F_*(Y)$ (for $Y$ orientable). More precisely, consider another partial Grothendieck transformation $\gamma'' : F'' \to H$, with $F''_* = F_*$, $\gamma''_* = c_*$, which is also defined on all o-allowable maps $f : X \to Y$. Then $F''(f : X \to Y) \subset F'(f : X \to Y)$ and $\gamma''_f = \gamma_f$ for all such $f$.

**Proof.** For the first part, we only have to show that $F'$ is "stable" under pull-back, product and push-down (then $\gamma$ is a partial Grothendieck transformation by the results we already explained before).

1. **pull-back:** Consider two independent squares

$$
\begin{array}{ccc}
X'' & \overset{h'}{\longrightarrow} & X' \\
\downarrow f'' & & \downarrow f' \\
Y'' & \overset{h}{\longrightarrow} & Y,
\end{array}
$$

with $f, f', f''$ o-allowable. Then one gets for $\alpha \in F'(f : X \to Y)$ and for all $\beta \in F_*(Y'')$:

$$
c_*\left(h^*\left(g^*(\alpha)\right) \bullet \beta\right) \overset{A3}{=} c_*\left((g \circ h)^*\left(\alpha\right) \bullet \beta\right) \overset{(10)}{=}$$
\[
\gamma_{f^\nu} \left( (g \circ h)^*(\alpha) \right) \cdot c_* (\beta) \overset{A3}{=} \gamma_{f^\nu} \left( h^* (g^*(\alpha)) \right) \cdot c_* (\beta).
\]

And similarly
\[
\gamma_{f^\nu} \left( h^* (g^*(\alpha)) \right) \overset{A3}{=} \gamma_{f^\nu} \left( (g \circ h)^*(\alpha) \right) \overset{(11)}{=}
\]
\[
(g \circ h)^* \left( \gamma_f(\alpha) \right) \overset{A3}{=} h^* \left( g^* \left( \gamma_f(\alpha) \right) \right) \overset{(11)}{=} h^* \left( \gamma_{f^\nu}(g^*(\alpha)) \right).
\]

2. \textbf{product}: Consider two independent squares

\[
\begin{array}{c}
X' \xrightarrow{h^\nu} X \\
\downarrow f' \quad \downarrow f \\
Y' \xrightarrow{h'} Y \\
\downarrow g' \quad \downarrow g \\
Z' \xrightarrow{h} Z,
\end{array}
\]

with the maps \( f, f', g, g' \) \( o \)-allowable. Then one gets for \( \alpha \in F'(f: X \to Y) \), \( \beta \in F'(f: Y \to Z) \) and for all \( \delta \in F_*(Z') \):

\[
c_* \left( h^*(\alpha \bullet \beta) \bullet \delta \right) \overset{A13}{=} c_* \left( (h^*(\alpha) \bullet h^*(\beta)) \bullet \delta \right) \overset{A1}{=} 
\]
\[
c_* \left( h^*(\alpha) \bullet (h^*(\beta) \bullet \delta) \right) \overset{(10)}{=} \gamma_{f^\nu} \left( h^*(\alpha) \right) \overset{A1}{=} c_* \left( h^*(\beta) \bullet \delta \right) \overset{(10)}{=}
\]
\[
\gamma_{f^\nu} \left( h^*(\alpha) \right) \bullet \left( \gamma_{g^\nu}(h^*(\beta)) \bullet c_* (\delta) \right) \overset{A1}{=}
\]
\[
\gamma_{f^\nu}(h^*(\alpha)) \bullet \gamma_{g^\nu}(h^*(\beta)) \bullet c_* (\delta).
\]

But \( h^*(\alpha) \in F'(f': X' \to Y') \) by 1. so that we can apply lemma 2.1 and the above equalities continue as follows:

\[
\overset{(13)}{=} \gamma_{g' \circ f^\nu} \left( h^*(\alpha) \bullet h^*(\beta) \right) \overset{A13}{=} \gamma_{g' \circ f^\nu} \left( h^*(\alpha \bullet \beta) \right) \overset{A1}{=} c_* (\delta).
\]

And similarly (again using lemma 2.1)

\[
\gamma_{g' \circ f^\nu} \left( h^*(\alpha \bullet \beta) \right) \overset{A13}{=} \gamma_{g' \circ f^\nu} \left( h^*(\alpha) \bullet h^*(\beta) \right) \overset{(13)}{=}
\]
\[
\gamma_{f^\nu}(h^*(\alpha)) \bullet \gamma_{g^\nu}(h^*(\beta)) \overset{(11)}{=} h^* \left( \gamma_f(\alpha) \right) \overset{A13}{=}
\]
\[
\gamma_{g^\nu}(h^*(\beta)) \overset{(13)}{=} h^* \left( \gamma_{g^\nu}(\alpha \bullet \beta) \right).
\]

3. **push-down:** Consider the independent squares as in 2., with $g, g'$ o-allowable. Assume $f$ (and therefore also $f'$) is confined and fix $\alpha \in F^*(g \circ f : X \to Z)$. Then one gets for all $\beta \in F_*(Z')$:

\[
\begin{align*}
&c_*(h^*(f_*(\alpha)) \cdot \beta) \overset{A23}{=} c_*(f'_*(h^*(\alpha)) \cdot \beta) \overset{A12}{=} \\
&c_*(f'_*(h^*(\alpha) \cdot \beta)) \overset{(*)}{=} f'_*(c_*(h^*(\alpha) \cdot \beta)) \overset{(10)}{=} \\
&f'_*(\gamma_{g' \circ f'}(h^*(\alpha)) \cdot c_*(\beta)) \overset{A12}{=} f'_*(\gamma_{g' \circ f'}(h^*(\alpha))) \cdot c_*(\beta) \overset{(9)}{=} \\
&\gamma_{g'}(f'_*(h^*(\alpha))) \cdot c_*(\beta) \overset{A23}{=} \gamma_{g'}(h^*(f_*(\alpha))) \cdot c_*(\beta).
\end{align*}
\]

Here $(*)$ follows from functoriality of $c_*$. Finally we have again by $(9)$:

\[
\begin{align*}
&\gamma_{g'}(h^*(f_*(\alpha))) \overset{A23}{=} \gamma_{g'}(f'_*(h^*(\alpha))) \overset{(9)}{=} \\
&f'_*(\gamma_{g' \circ f'}(h^*(\alpha))) \overset{(11)}{=} f'_*(\gamma_{g \circ f}(h^*(\alpha))) \overset{A23}{=} \\
&\bar{h}^*(f'_*(\gamma_{g \circ f}(\alpha))) \overset{(9)}{=} \bar{h}^*(\gamma_{g}(f_*(\alpha))) \overset{(11)}{=}. 
\end{align*}
\]

Assume now that $c_*$ commutes with *external products* as stated in the theorem. Then one gets for any independent square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{\beta} & pt,
\end{array}
\]

with $f'$ o-allowable:

\[
c_*(\beta \times \alpha) = c_*(\beta) \times c_*(\alpha)
\]

for $\alpha \in F_*(X)$ and $\beta \in F_*(Y')$. For $\beta = e_{Y'}$, one gets especially

\[
c_*(g^*(\alpha) \cdot e_{Y'}) = \bar{g}^*(c_*(\alpha)) \cdot c_*(e_{Y'})
\]

and this implies by definition (i.e. by equation $(9)$):

\[
\gamma_{f'}(g^*(\alpha)) = \bar{g}^*(c_*(\alpha)).
\]

So $\alpha$ satisfies $(14)$, and together with $(14)$ this implies also the property $(10)$, i.e.

\[
F'_*(X) = F_*(X).
\]
The last statement of the theorem follows directly from the fact that the corresponding bivariant transformation
\[ \gamma'_f : F''(f : X \to Y) \to H(\bar{f} : \bar{X} \to \bar{Y}) \]
is uniquely defined by
\[ c_*(\alpha \cdot e_Y) = \gamma'_f(\alpha) \cdot c_*(e_Y), \]
since \( c_*(e_Y) \) is a strong orientation. Especially, two such subtheories \( F', F'' \) defined by different choices of the \( e_Y \in F_*(Y) \) (for \( Y \) orientable) have to agree. \( \square \)

Remark 2.1. The last part of theorem 2.1 shows especially, that one gets for \( c_* = \gamma''_* : F_* \to H_* \) the associated covariant functor of a partial Grothendieck transformation \( \gamma'' : F \to H \) of partial (weak) bivariant theories:
\[ \gamma = \gamma'' : F' = F \to H, \]
with \( F \) restricted to all \( o \)-allowable maps. So one gets in this case nothing new (as it should be).

The covariant transformation \( \gamma_* \) in our examples comes in the following cases from a Grothendieck transformation \( \gamma : F \to H \) of bivariant theories:

- **example 1.1.1**, if all schemes are quasi-projective over a fixed non-singular base \( S \). Here \( F(f) = K(f) \) is the Grothendieck group of \( f \)-perfect complexes and
  \[ \gamma := \tau : K \to H := A \otimes \mathbb{Q} \]
is the Grothendieck transformation of [Ful, thm., p.366].

- **example 1.1.3**, if we only consider "cellular" holomorphic maps between complex spaces, which can be embedded into smooth complex manifolds. Here \( \gamma := c \) is the bivariant Chern transformation constructed in [Br] (and compare with [Sab2] for a corresponding bivariant theory of Lagrangian cycles).

- **example 1.1.4**. for the pl-context. Here \( \gamma := w \) is the bivariant Stiefel-Whitney transformation of [FM, Part I, 6] (corrected in [EH]).

- **example 1.2.1** and **example 1.3.1**.

- Finally also the transformation \( \gamma_* \) in the context of oriented cohomology (pre)theories comes from a bivariant transformation \( \gamma : A \to B \) defined for a morphism \( f : X \to Y \) of smooth manifolds as
  \[ \gamma_f := \phi \cup td_\phi(TX) \cup f^*(td_\phi(TY))^{-1}. \]
Here \( \phi = \gamma^* : A = A^* \to B = B^* \) is the given "nice" ring morphism. That \( \gamma \) commutes with the bivariant product follows from the projection formula.
and the fact, that \( \phi \) is ring morphism. For the commutativity with push-down one has in addition to use the Riemann-Roch theorem \([Pa\text{, thm.2.5.4}]\). That \( \gamma \) commutes with pull-back follows finally from the fact, that all independent squares are transverse cartesian diagrams, e.g. the class

\[
TX - f^*TY \in K^0(X)
\]

behaves well under pullback in transverse cartesian diagrams (and \( td_\phi : K^0(X) \to B(X) \) commutes with pullback \([Pa\text{, prop.2.2.3}]\)).

Before we apply theorem \([22]\) to the rest of our previous examples, let us recall the main formal properties of bivariant theories \([PM]\), which remain true (with slight modifications) in the context of a partial (weak) bivariant theory \( F \) on the category \( C \):

- We assume that all maps \( X \to pt \) are allowable so that one has an associated covariant theory (functorial with respect to confined maps).

- For objects \( X \) of \( C \) with \( id : X \to X \) allowable, one has an associated group \( F^*(X) := F(id : X \to X) \), with a cap-product (given by the bivariant product for the composite of this identity morphisms). This is contravariant with respect to the class of co-confined maps, i.e. maps \( f : X \to Y \) such that the square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
id_X & \downarrow & \downarrow id_Y \\
X & \xrightarrow{f} & Y 
\end{array}
\]

is independent (with \( id_X, id_Y \) allowable). Moreover for such \( X \) one also has a cap-product

\[
\cap : F^*(X) \times F_*(X) \to F_*(X) .
\]

- Similarly, for \( f : X \to Y \) and \( id_Y : Y \to Y \) allowable, one has a right action

\[
\cap f^* := \bullet : F(f : X \to Y) \times F^*(Y) \to F(f : X \to Y) ,
\]

which makes \( F(f : X \to Y) \) into an (unitary) right \( F^*(Y) \)-module.

- For an independent square

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & pt \\
\alpha & & \beta 
\end{array}
\]

with \( f' \) allowable, one has an external product

\[
\times : F_*(Y') \times F_*(X) \to F_*(X')
\]

defined by \( \beta \times \alpha := g^*(\alpha) \bullet \beta \).
For an independent square

\[
\begin{array}{ccc}
X_{pt} & \longrightarrow & X \\
\downarrow & \downarrow f & \\
pt & \longrightarrow & Y \\
\end{array}
\]

with \(f\) allowable, one has a restriction to the fiber:

\[i^*: F(f : X \to Y) \to F_*(X_{pt}).\]

Any element \(\theta \in F(f : X \to Y)\) for \(f\) allowable induces Gysin homomorphisms

\[\theta^* : F_*(Y) \to F_*(X), \quad \theta^*(\alpha) := \theta \cdot \alpha,\]

and for \(f\) also confined

\[\theta_* : F^*(X) \to F^*(Y), \quad \theta_*(\beta) := f_*(\beta \cdot \theta).\]

These Gysin homomorphisms are functorial in \(\theta\) [PM, (G1), p.26].

If in addition \(\gamma : F \to H\) is a partial Grothendieck transformation of partial (weak) bivariant theories, then \(\gamma\) induces associated transformations \(\gamma^* : F^* \to H^*\) and \(\gamma_* : F_* \to H_*\), with the module properties

\[\gamma^*(\alpha) \cap \gamma^*(\beta) = \gamma^*(\alpha \cap \beta)\]

and

\[\gamma_f(\alpha) \cap \gamma_f^*(\gamma_*(\beta)) = \gamma_f(\alpha \cap f^*(\beta)),\]

if \(\cap\) or \(\cap f^*\) is defined. \(\gamma_*\) commutes also with exterior products (if defined):

\[\gamma_*(\alpha \times \beta) = \gamma_*(\alpha) \times \gamma_*(\beta),\]

restriction to fibers and Gysin homomorphisms.

### 3 Examples

Let us come back to the given examples of our paper. We would like to apply theorem [21] First we remark, that in all examples \(\gamma_*\) commutes with exterior products:

- For example [11] this follows from [Ful, Ex. 18.3.1, p.360].
- For example [11] 2-3 this follows from [Sch1] in the case of spaces that can be embedded into smooth manifolds. The general case follows by resolution of singularities (compare [Kw, KwY]).
• For example 1.1.4-5 in the subanalytic context this follows from [Sch1]. Here all subanalytic sets are assumed to be given in a real analytic manifold. The pl-context follows from the corresponding bivariant theory [FM] and goes back to [HT].

• For example 1.2.2 this is not explicitly stated in [FL, p.190], but follows from the sketched construction there. In the form sufficient for our applications (i.e. one factor is a smooth manifold), it follows also from SSR 1-3 of [FL] p.188-190.

• For example 1.2.3 this follows from [Bro, prop.10.4(ii)].

• Finally example 1.3.2 follows from the corresponding property of the Chern class transformation $c$ already explained above, and the simple fact that the transformation

\[ K_{eq}^{c}(X \to pt) \xrightarrow{\text{tr}_{|X|}} F(|X|) \otimes k \]

commutes with exterior products.

The final piece of information that we need about our examples, is the fact that for the covariant transformations $\gamma_* : F_* \to H_*$ given there, $F_*, H_*$ are just the associated covariant theories of suitable (weak) bivariant theories $F, H$ (together with a corresponding functor $\sim$ of the underlying categories). For $H_*$ this is already the case. So we only have to deal with $F_*$ in the cases, where $\gamma_*$ is not already induced from a Grothendieck transformation $\gamma$ of bivariant theories. For all these remaining cases, the following general construction of (what we call) simple (weak) bivariant theories applies (in the context of constructible functions this notion goes back to [Y2, Y3]).

Let $C$ be a category with classes of confined maps, independent squares and a final object (as in [FM, Part I, 2]). We make in addition the following assumptions:

• (SB1) We have a contravariant functor

\[ F : C \to \text{Rings} \]

with values in the category of rings with unit.

• (SB2) $F$ is also covariant functorial with respect to the confined maps (as a functor to the category of Abelian groups).

• (SB3) $F$ satisfies the projection-formula (i.e. the push-down for $f : X \to Y$ confined is a right $F(Y)$-module operator).

• (SB4) $F$ has the base-change property

\[ g^* f_* = f'_* g'^* : F(X) \to F(Y') \]
for any independent square

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y,
\end{array}
\]

with \(f, f'\) confined.

The example we have already seen was the case of an oriented cohomology (pre)theory. And as in that case one gets a simple weak bivariant theory \(F\) by \(F(X) := F(f : X \to Y)\), with the obvious push-down and pull-back transformations. Finally the bivariant product

\[ \bullet : F(f : X \to Y) \times F(g : Y \to Z) \to F(g \circ f : X \to Z) \]

is just given by \(\alpha \bullet \beta := \alpha \cup f^*(\beta)\), with \(\cup\) the given product of the ring-structure. We leave it to the reader to check that this defines a weak bivariant theory (with units) in the sense of [FM] (i.e. without property (A123)). Suppose in addition:

- (SB5) A commutative square is independent iff its transpose [FM p.17] is independent, and \(F\) satisfies the two-sided projection formula (i.e. the push-down for \(f : X \to Y\) confined is a two-sided \(F(Y)\)-module operator).

Then the simple theory \(F\) satisfies also the projection formula (A123) of [FML].

**Remark 3.1.** These simple (weak) bivariant theories are (skew-)commutative (and graded), if \(F\) is functor to the category of (skew-)commutative (and graded) rings (and the push-down for confined maps is degree preserving). Moreover,

\[ 1_f := 1_X \in F(X) := F(f : X \to Y) \]

is always a canonical and strong orientation.

In the very special case, that one considers only trivial independent squares:

\[
\begin{array}{ccc}
X & \xrightarrow{id} & X \\
\downarrow f & & \downarrow f \\
Y & \xrightarrow{id} & Y
\end{array}
\]

and all maps are confined, our assumptions (SB1-3) just reduce to the properties F1-F3 of [FL] p.28.

Another more important example for us comes from the theory of constructible sheaves and functions in the complex algebraic or (real sub-) analytic context, or in the algebraic context of seperated schemes of finite type over a field.
k of characteristic zero. Here the confined maps are the proper maps, and the independent squares are given by the cartesian diagrams. In the (sub)analytic context we assume (for simplicity) that all spaces are of bounded dimension.

Then one can work in all cases with the corresponding "bounded derived category of constructible sheaves" $D^b_c(X)$. Here we consider constructible sheaves of vector-spaces over a (suitable) field $R$ (in the algebraic context), with finite dimensional stalks (in the closed points). These are stable under the usual pullback $f^*$, the exact tensor-product $\otimes_R$ and for proper $f$ also under push-down $Rf_*$. These are related by a "projection-formula" and "proper base-change formula", which imply that the functor $F$ given by the Grothendieck group

$$F(X) := K_0(D^b_c(X))$$

with it induced transformations $f^*$, $\otimes$, $f_*$ satisfies our assumptions (SB1-5) (with the unit given by the class of the constant sheaf $R_\times$). This is also the Grothendieck group of the abelian category of constructible sheaves. Moreover, by taking stalk-wise the Euler-characteristic ($\mod 2$ in the subanalytic context), one gets a natural surjective transformation

$$\chi_X : K_0(D^b_c(X)) \to CF(X)$$

onto the group of constructible functions (on the set of closed points in the algebraic context). This induces the corresponding transformations (compare [Sch3, sec.2.3])

$$f^*, \cdot, f_*$$

on $CF$, which therefore also satisfy our assumptions (SB1-5). A similar reasoning applies also to the equivariant context studied in example 1.4 and 1.2. The group of constructible functions invariant under the real structure in example 1.5 is stable under the transformations $f^*, \cdot, f_*$ for equivariant maps, and in example 1.2 one has similar transformations on the corresponding Grothendieck group $K^e_c(X)$ of "equivariant constructible sheaves". These simple bivariant theories are all commutative.

So the simple bivariant group $CF$ of (invariant) constructible functions can be used in example 1.2-4 (or 5), and the simple bivariant (Grothendieck) group $K^e_c$ can be used in example 1.2.

But one cannot use in example 1.4 the simple bivariant group $CF$ of $\mathbb{Z}_2$-valued subanalytically constructible functions, because the Stiefel-Whitney transformation $w_*$ is only defined for those constructible functions, which are in addition self-dual (i.e. Euler functions [FMMC, def.4.2, p.823]). But this self-duality condition is only stable under proper push-down, but in general not under the transformations $f^*$ and $\cdot$. 
This is one of the reasons for introducing the example 1.1.5 (which is better behaved). Nevertheless, if we restrict the simple category $\text{CF}$ to the subcategory of \textit{smooth} subanalytic maps (i.e. submersions) between real analytic \textit{manifolds}, then the corresponding subcategory $\text{CF}_{\text{eu}}$ of Euler constructible functions is stable under the bivariant product, push-down and pull-back and defines on this restricted category a suitable bivariant theory for example 1.1.4.

\textbf{Remark 3.2.} Of course one can also make further restrictions on the simple bivariant theory $\text{CF}$ of constructible functions in the algebraic or complex analytic context. The restriction to \textit{smooth} holomorphic maps between \textit{complex manifolds} was for example used in \cite{Y2}.

One gets another kind of restriction, if one makes additional assumptions on the base-change map $g$ in the definition of independent squares: Here one can for example assume that $g$ belongs to a class of morphisms (containing all identity maps), which is stable under composition and base-change, e.g. the class of smooth or \textit{\acute{e}tale} morphisms, open inclusions or projection of products. The case of \textit{smooth} maps $g$ for independent squares was used in the complex analytic context in \cite{Y3}.

So we can apply theorem 2.1 in all the cases above to the corresponding transformation $\gamma_*$, and the next natural question is, which $\alpha \in F(f : X \to Y)$ (with $Y$ smooth) belong to the bivariant subgroup $F'(f : X \to Y)$ constructed in theorem 2.1.

For example 1.1.3 one has the \textit{conjecture} (\cite{Y2 Y3 Y6}):

$$F_{\text{eu}}(f : X \to Y) \subset F'(f : X \to Y),$$

with $F_{\text{eu}}(f : X \to Y)$ the group of constructible functions satisfying a suitable \textit{local Euler condition} (compare \cite{Br FM Sab2 EY1 EY2 Zhou1 Zhou2}).

Here we restrict ourself to the important special case of the function $e_f := 1_X \in F(f : X \to Y)$ (or the class $e_f$ of the constant sheaf, with its canonical isomorphism $\phi$, in the context of example 1.3.2) for a \textit{smooth} morphism $f$ (so that $X$ is also a manifold).

By remark 1.4 we know already the formula

$$\gamma_f(e_f) = c^*(T\bar{f}) \bullet [\bar{f}],$$

with $T\bar{f}$ the relative tangent bundle (i.e. in our context of a smooth map $\bar{f}$ this is just the class of the tangent bundle to the fibers in the Grothendieck group $K_0(\bar{X})$ of vector bundles on $\bar{X}$) and $[\bar{f}] \in H(\bar{f} : \bar{X} \to \bar{Y})$ the relative orientation class of the smooth morphism $\bar{f}$. Finally

$$c^* : K_0(\bar{X}) \to H^*(\bar{X})$$
is the corresponding characteristic class (Chern or Stiefel-Whitney class, depending on the example). This characteristic class behaves well under pullback in cartesian diagrams, since

$$T_{f'} = \bar{g}^* T_f$$

for $f$ smooth. Therefore $e_f$ satisfies condition (11) of definition 2.2. Moreover, the other condition (10) is equivalent to the Verdier Riemann-Roch formula (i.e. the commutativity of the following diagram)

$$F_*(Y) \xrightarrow{\gamma^*} H_*(\bar{Y})$$

$$\downarrow f^* \quad \quad \downarrow c^* (T_{f'}) \cap \bar{f}$$

$$F_*(X) \xrightarrow{\gamma^*} H_*(\bar{X}),$$

with $\bar{f}$ the corresponding smooth pull-back in homology.

By [Sch1, Sch2] and [Y4, thm.2.2], this Verdier Riemann-Roch formula is true in all our cases (for $\bar{f}$ smooth). So we get $e_f \in F'(f : X \to Y)$, and this defines in all examples a canonical orientation on the class of smooth maps (between manifolds).

**Remark 3.3.** This Verdier Riemann-Roch formula is also true for smooth morphisms of singular spaces (e.g. by resolution of singularities this can be reduced to the case of manifolds as in the proof of [Y4, thm.2.2]). Therefore all $\alpha \in F(X) = CF(f : X \to Y)$ satisfy the condition (11) of definition 2.2 for any independent square with a smooth base change map $\bar{g}$. So if one defines in the examples above the independent squares as cartesian diagrams with a smooth pullback map, then one only has to check the condition (10) of definition 2.2 (compare [Y2, Y3]).

In a sequel to this paper we will explain a general construction of partial bi-variant theories, which applies in particular directly to (equivariant) Chow-groups, oriented Borel-Moore homology theories, equivariant K-theory, or higher algebraic K-theory. Moreover, we will illustrate the relation of our main theorem 2.1 to corresponding known Riemann-Roch theorems as in [BFM, BFQ, EG2, Gi, Sou].

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