Non-Relativistic Limits of Colored Gravity in Three Dimensions

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ABSTRACT: The three-dimensional non-relativistic isometry algebras, namely Galilei and Newton-Hooke algebras, are known to admit double central extensions, which allows for non-degenerate bilinear forms hence for action principles through Chern-Simons formulation. In three-dimensional colored gravity, the same central extension helps the theory evade the multi-graviton no-go theorems by enlarging the color-decorated isometry algebra. We investigate the non-relativistic limits of three-dimensional colored gravity in terms of generalized İnönü-Wigner contractions.
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1 Introduction

In the non-relativistic limit, Einstein’s general relativity reduces to Newtonian gravity. In analogy to Einstein-Cartan gravity, Newtonian gravity has a coordinate-independent formulation in terms of frame fields, namely Newton-Cartan gravity [1–5]. Since Einstein-Cartan gravity can be obtained by gauging the relativistic Poincaré algebra, one may attempt to derive Newton-Cartan gravity by gauging a non-relativistic symmetry algebra. The first candidate algebra would be the Galilei algebra $\mathfrak{g}_0$, which is the non-relativistic limit of the Poincaré algebra. However, it turns out that one instead needs to consider the Bargmann algebra $\mathfrak{g}_0^+$, which is an extension of the Galilei algebra with a central generator $M$ called “mass”. This mass generator is related to an additional $U(1)$ gauge field in Newton-Cartan gravity. If we introduce a non-zero cosmological constant, then the counterparts of the Galilei algebra $\mathfrak{g}_0$ and the Bargmann algebra $\mathfrak{g}_0^+$ are the Newton-Hooke algebra $\mathfrak{g}_\Lambda$ and the extended Newton-Hooke algebra $\mathfrak{g}_\Lambda^+$. Since the equations of motion of Newton-Cartan gravity can be obtained by gauging the Bargmann algebra $\mathfrak{g}_0^+$ [6], Newton-Cartan gravity can be considered as a gauge theory of the Bargmann algebra. Another way to derive Newton-Cartan gravity is to consider a limit of Einstein-Cartan gravity mimicking the İnönü-Wigner contraction of the relativistic algebra [7]. However, in general spacetime dimensions, these equations of motion do not have a simple action principle.

In three dimensions, the Bargmann algebra $\mathfrak{g}_0$ admits a second central extension by the “spin” generator $S$ [8, 9]. The resulting algebra is referred to as the extended Bargmann algebra $\mathfrak{g}_0^{++}$. In parallel, in the cosmological case, the three-dimensional Newton-Hooke algebra admits double extensions. We refer to this as the doubly extended Newton-Hooke (NH) algebra $\mathfrak{g}_\Lambda^{++}$. We summarize the zoo of three-dimensional non-relativistic algebras in Table 1. Note that the subscript $\Lambda$ denotes the cosmological constant and the superscript + or ++ means the central extension by $M$ or $M, S$. Due to the second central extension, both $\mathfrak{g}_0^{++}$ and $\mathfrak{g}_\Lambda^{++}$ have non-degenerate invariant bilinear forms [10, 11]. Consequently, the corresponding theories admit action principles through Chern-Simons formulation. These non-relativistic gravity theories are sometimes called extended Bargmann gravity and extended Newton-Hooke (NH) gravity, respectively. Along the same lines, the non-relativistic Chern-Simons theories admit various extensions according to the symmetry algebras [12–14].

| No extension | Flat | (A)dS |
|--------------|------|--------|
| Extended by $M$ | Galilei $\mathfrak{g}_0$ | Newton-Hooke (NH) $\mathfrak{g}_\Lambda$ |
| Extended by $M, S$ | Bargmann $\mathfrak{g}_0^+$ | Extended NH $\mathfrak{g}_\Lambda^+$ |
|               | Extended Bargmann $\mathfrak{g}_0^{++}$ | Doubly extended NH $\mathfrak{g}_\Lambda^{++}$ |

Table 1. Three-dimensional non-relativistic isometry algebras.

The presence of two central generators $M$ and $S$ are hence crucial to the action principle of three-dimensional non-relativistic gravities in the frame formulation. In fact, as we shall show later, these doubly extended non-relativistic algebras can be obtained from the
relativistic ones with two \( u(1) \) generators after suitable contractions (see also \([12, 15]\)). For instance, in AdS, the relativistic isometry algebra with two \( u(1) \)'s,

\[
so(2, 2) \oplus u(1) \oplus u(1) = \left( sl(2, \mathbb{R}) \oplus u(1) \right) \oplus \left( sl(2, \mathbb{R}) \oplus u(1) \right) ,
\]

contracts to the doubly extended NH \( \mathfrak{h}^{++}_\Lambda \) under a suitable scaling limit. In fact, both the relativistic and non-relativistic algebras can be split into left- and right-moving sectors, so the contraction can be performed in each sector. After the contraction, the \( u(1) \) generators become central elements, and the resulting algebra cannot be arranged as a direct sum anymore. The is the key mechanism for a non-degenerate bilinear form — hence for the action principle.

We notice that another peculiar utility of \( u(1) \) can be found in a different extension of three-dimensional gravity, namely the colored gravity which has been studied recently in \([16–18]\). Several no-go results are known for theories of multiple gravitons,\(^1\) but in three dimensions they can be evaded by extending the isometry algebra \( so(2, 2) \) with two \( u(1) \) generators. It is remarkable that the resolution of the problems in non-relativistic limit and color-decoration are the same: replace \( sl(2, \mathbb{R}) \) by \( sl(2, \mathbb{R}) \oplus u(1) \). This observation leads us to the following question: does the three-dimensional colored gravity have a natural non-relativistic limit?

Let us explain a few more details about the colored gravity. It can be constructed in the Chern-Simons formulation with an appropriate choice of gauge algebra. For the multi-graviton interpretation, this algebra should contain multiple copies of isometry generators, hence a direct product of the isometry \( so(2, 2) \) and the internal symmetry. Usually, the direct product of two Lie algebras does not define a Lie algebra, but the case of two associative algebras does so. The internal symmetry can be taken as \( U(N) \) which obviously has an associative structure. About the isometry algebra, we can extend the Lie algebra \( sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}) \) to an associative algebra \( gl(2, \mathbb{R}) \oplus gl(2, \mathbb{R}) \) by adding two copies of \( u(1) \). We refer to the internal symmetry as “color symmetry” for the reason which will become clear later, and the resulting gravity theory will be correspondingly referred to as “colored gravity”. The colored gravity has several interesting features: \( i) \) It has a non-trivial potential with many saddles points corresponding to the (A)dS vacua of different cosmological constants. These vacua can be classified according to how much the internal symmetry is preserved. \( ii) \) Around the color singlet vacuum, the theory has one graviton, \( (N^2 - 1) \) massless spin-two fields charged under the color symmetry and \( U(N) \times U(N) \) Chern-Simons gauge field. The color non-singlet spin-two fields are strongly interacting for large \( N \). \( iii) \) Around a color-symmetry breaking background, a Higgs-like mechanism takes place and one finds partially-massless spin-two fields \([21, 22]\) appearing in the spectrum.

The organization of the paper is as follows. In Section 2, we present a concise overview of several aspects of the 3d non-relativistic algebras. In Section 3, we derive the non-relativistic colored-gravity algebras from the \( \ddot{I}n\ddot{on}\ddot{u}-Wigner \) contractions of the relativistic ones. In Section 4, we construct the actions of non-relativistic colored gravities using

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\(^1\)See e.g. \([19]\) for the no-go result and \([20]\) for an exception.
the Chern-Simons formulation. In Section 5 we conclude the work with a brief summary. Appendix includes technical details and additional materials.

2 Non-relativistic Algebras and their Bilinear Forms

2.1 Various non-relativistic algebras in three dimensions

We will give a concise review of various three-dimensional non-relativistic algebras: the Galilei, Bargmann, extended Bargmann, Newton-Hooke (NH), extended NH, and doubly extended NH algebras, which have been briefly discussed in Introduction.

**Galilei algebra \( g_0 \)**

The three-dimensional Galilei algebra is an eight-dimensional Lie algebra with the non-trivial Lie brackets:

\[
[J, G_i] = 2 \epsilon_{ij} G_j, \quad [J, P_i] = 2 \epsilon_{ij} P_j, \quad [H, G_i] = 2 \epsilon_{ij} P_j. \tag{2.1}
\]

Here, \( H, P_i, J \) and \( G_i \) are the generators of time translations, spatial translations, spatial rotations and Galilean boosts, respectively. The Levi-civita symbol \( \epsilon_{ij} \) is defined as

\[
\epsilon_{12} = -\epsilon_{21} = 1, \quad \epsilon_{11} = \epsilon_{22} = 0. \tag{2.2}
\]

Remark that, in higher dimensions, there exist non-zero brackets between two spatial rotation generators.

**Newton-Hooke algebra \( g_\Lambda \)**

When the cosmological constant \( \Lambda \) is turned on, the Galilei algebra is deformed to the so-called Newton-Hooke algebra. As a result, we have an additional set of Lie brackets,

\[
[H, P_i] = -2 \Lambda \epsilon_{ij} G_j, \tag{2.3}
\]

besides those in (2.1).

**Bargmann algebra \( g_0^+ \) and extended NH algebra \( g_\Lambda^+ \)**

In general dimensions, both the Galilean and Newton-Hooke (NH) algebras can be centrally extended by the mass generator \( M \).\(^2\) The resulting algebras are known as the Bargmann algebra and the extended NH algebra, respectively. In three dimensions, the additional Lie brackets are

\[
[G_i, P_j] = 2 \epsilon_{ij} M. \tag{2.4}
\]

\(^2\)In a relativistic theory, the energy of a particle in the rest frame is proportional to the rest mass. However, in a non-relativistic theory, spacetime symmetries do not require energy-mass equivalence, so we can consider an independent generator for mass in the form of a central extension. Intuitively speaking, the energy-mass relation is lost in the large speed of light limit.
Extended Bargmann algebra $\mathfrak{g}_0^{++}$ and doubly extended NH algebra $\mathfrak{g}_\Lambda^{++}$

In three dimensions, the Bargmann and extended NH algebras admit a second central extension by the spin generators $S$.\(^3\) The extended Bargmann algebra, or equivalently the doubling extended Galilei algebra, has one more set of Lie brackets,

$$[G_i, G_j] = 2 \epsilon_{ij} S.$$  \hfill (2.5)

In the doubly extended Newton-Hooke algebra, the additional Lie brackets are (2.5) and

$$[P_i, P_j] = -2 \Lambda \epsilon_{ij} S.$$  \hfill (2.6)

All the non-relativistic algebras presented above can be obtained from the most general one, the doubly extended NH algebra, by various İnönü-Wigner contractions. The cosmological constant dependent terms can be removed by the usual İnönü-Wigner contraction. The mass and spin generators only appear in the right hand sides of the Lie brackets, hence can be eliminated by a suitable contraction: rescale them first by a contraction parameter, then send the parameter to zero.

2.2 Non-relativistic limits as contractions

It is useful to realize these non-relativistic limits as contractions. In fact, the three-dimensional relativistic isometry algebra with a non-zero cosmological constant and two additional $u(1)$ generators can contract to the doubly extended NH algebra (hence all non-relativistic algebras discussed in section 2.1). Let us provide more details on this point. The relativistic isometry algebra in three dimensions has the Lie brackets,

$$[\mathcal{J}_a, \mathcal{J}_b] = 2 \epsilon_{ab} \mathcal{J}_c, \quad [\mathcal{J}_a, \mathcal{P}_b] = 2 \epsilon_{ab} \mathcal{P}_c, \quad [\mathcal{P}_a, \mathcal{P}_b] = -2 \Lambda \epsilon_{ab} \mathcal{J}_c,$$  \hfill (2.7)

where $\mathcal{P}_a$ and $\mathcal{J}_a$ are the translation and Lorentz generators, and $\Lambda$ is the cosmological constant.

**Contraction to Newton-Hooke algebra**

For an illustration, let us see how the relativistic isometry algebra (2.7) contracts to the Newton-Hooke algebra $\mathfrak{g}_\Lambda$. We first redefine the generators using the parameter $c$ as

$$G_i = \frac{1}{c} \mathcal{J}_i, \quad P_i = \frac{1}{c} \mathcal{P}_i,$$  \hfill (2.8)

and

$$J = \mathcal{J}_0, \quad H = \mathcal{P}_0,$$  \hfill (2.9)

then derive their Lie brackets. Notice that the temporal components scale differently in $c$ from the spatial ones. The İnönü-Wigner contraction is performed in the $c \to \infty$ limit which effectively removes certain terms in the structure constants. One can easily check that the resulting algebra is the NH algebra $\mathfrak{g}_\Lambda$. Hence, one can view this contraction as the non-relativistic limit and interpret the parameter $c$ as the speed of light.

\(^3\)To interpret $S$ as a spin generator is subtle. See [23, 24] for more details.
Contraction to doubly extended Newton-Hooke algebra

Let us now show how the doubly extended NH algebra $g^{++}_\Lambda$ can be derived by a contraction. As the algebra $g^{++}_\Lambda$ has dimension eight due to the central generators $M$ and $S$, it cannot be obtained as a contraction of the relativistic isometry algebra which has dimension six. To fix the mismatch between the numbers of generators, we need to include two additional $u(1)$ generators. We refer to these generators as $Z_P$ and $Z_J$. In terms of them, we change the definitions of $H$ and $J$ to

$$J = Z_J + J_0, \quad H = Z_P + P_0,$$  \hspace{1cm} (2.10)

but keep those of $P_i$ and $G_i$ the same as (2.8). The mass and spin generators are given by the other combinations with different rescalings,

$$S = \frac{Z_J - J_0}{2c^2}, \quad M = \frac{Z_P - P_0}{2c^2}. \hspace{1cm} (2.11)$$

With these definitions, one can successfully reproduce the doubly extended NH algebra $g^{++}_\Lambda$ by taking the $c \to \infty$ limit. Note that in (2.10) and (2.11) the mixings with $Z_P$ and $Z_J$ are independent, hence the corresponding central extensions by $M$ and $S$ are also independent.

Chiral decomposition

When $\Lambda \neq 0$, the relativistic isometry algebra can be decomposed into two parts:

$$\mathcal{L}_a = \frac{1}{2} \left( J_a + \frac{1}{\sqrt{-\Lambda}} P_a \right), \quad \tilde{\mathcal{L}}_a = \frac{1}{2} \left( J_a - \frac{1}{\sqrt{-\Lambda}} P_a \right),$$  \hspace{1cm} (2.12)

and each of them forms an $sl(2)$:

$$[\mathcal{L}_a, \mathcal{L}_b] = 2 \epsilon_{ab}{}^c \mathcal{L}_c, \quad [\tilde{\mathcal{L}}_a, \tilde{\mathcal{L}}_b] = 2 \epsilon_{ab}{}^c \tilde{\mathcal{L}}_c.$$  \hspace{1cm} (2.13)

Accordingly, it is natural to decompose the $u(1) \oplus u(1)$ part as

$$\mathcal{Z} = \frac{1}{2} \left( Z_J + \frac{1}{\sqrt{-\Lambda}} Z_P \right), \quad \tilde{\mathcal{Z}} = \frac{1}{2} \left( Z_J - \frac{1}{\sqrt{-\Lambda}} Z_P \right).$$  \hspace{1cm} (2.14)

For the non-relativistic limit $c \to \infty$, one can check that the redefinitions (2.8), (2.10) and (2.11) are compatible with the chiral decomposition. More precisely, with the redefinitions,

$$K = Z + L_0, \quad L_i = \frac{1}{c} L_i, \quad Z = \frac{Z - L_0}{2c^2},$$  \hspace{1cm} (2.15)

$$\tilde{K} = \tilde{Z} + \tilde{L}_0, \quad \tilde{L}_i = \frac{1}{c} \tilde{L}_i, \quad \tilde{Z} = \frac{\tilde{Z} - \tilde{L}_0}{2c^2},$$  \hspace{1cm} (2.16)

the two copies of the relativistic chiral algebra $sl(2) \oplus u(1)$ contract to the following non-relativistic chiral algebras,

$$[K, L_i] = 2 \epsilon_{ij} L_j, \quad [L_i, L_j] = 2 \epsilon_{ij} Z,$$  \hspace{1cm} (2.17)

$$[\tilde{K}, \tilde{L}_i] = 2 \epsilon_{ij} \tilde{L}_j, \quad [\tilde{L}_i, \tilde{L}_j] = 2 \epsilon_{ij} \tilde{Z},$$  \hspace{1cm} (2.18)
where $Z$ and $\tilde{Z}$ are the centers of the chiral algebras. The chiral subalgebra $\mathfrak{h}$, spanned by $K, L_i$ and $Z$, can be viewed as a central extension of the two-dimensional Euclidean algebra where $K$ and $L_i$ can be interpreted as two-dimensional translation and rotation generators, respectively. One can equally interpret $\mathfrak{h}$ as a harmonic oscillator, where $L_1 \pm i L_2$ are creation/annihilation or position/momentum operators, $K$ is the Hamiltonian, and $Z$ is the Plank constant $\hbar$.

The doubly extended NH algebra $\mathfrak{g}^{++}_\Lambda$ can be directly decomposed into two copies of the chiral algebra $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$:

$$\mathfrak{g}^{++}_\Lambda = \mathfrak{h} \oplus \tilde{\mathfrak{h}},$$

(2.19)

where the explicit decomposition reads

$$G_i = L_i + \tilde{L}_i, \quad J = K + \tilde{K}, \quad S = Z + \tilde{Z},$$

$$P_i = \sqrt{-\Lambda} (L_i - \tilde{L}_i), \quad H = \sqrt{-\Lambda} (K - \tilde{K}), \quad M = \sqrt{-\Lambda} (Z - \tilde{Z}).$$

(2.20)

### 2.3 Invariant bilinear form

One may attempt to use the non-relativistic algebras in section 2.1 to construct non-relativistic gravity theories. In three dimensions, as in the relativistic case, one can rely on the Chern-Simons formulation where non-degeneracy of the bilinear form is important for the dynamics of the resulting theory. It turns out that, among the algebras that we have considered, only those with double extensions by $M$ and $J$ admit a non-degenerate bilinear form. In below, we focus on the doubly extended NH algebra and discuss its bilinear form. Since the doubly extended NH algebra $\mathfrak{g}^{++}_\Lambda$ admits the decomposition (2.19), we can consider the invariant bilinear form $\langle \cdot, \cdot \rangle$ for each chiral sector. The non-diagonal part are the $2 \times 2$ symmetric matrices,

$$B = \begin{pmatrix} \langle K, K \rangle & \langle Z, K \rangle \\ \langle K, Z \rangle & \langle Z, Z \rangle \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \langle \tilde{K}, \tilde{K} \rangle & \langle \tilde{Z}, \tilde{K} \rangle \\ \langle \tilde{K}, \tilde{Z} \rangle & \langle \tilde{Z}, \tilde{Z} \rangle \end{pmatrix},$$

(2.21)

Due to the rotational symmetry, the diagonal part is fixed as

$$\langle L_i, L_j \rangle = \ell \delta_{ij}, \quad \langle \tilde{L}_i, \tilde{L}_j \rangle = \tilde{\ell} \delta_{ij},$$

(2.22)

with undetermined constants $\ell$ and $\tilde{\ell}$, while the other bilinear forms $\langle L_i, K \rangle, \langle L_i, Z \rangle$ and their tilde counterparts vanish. Symmetry also determines the matrices $B$ and $\tilde{B}$ to be

$$B = \begin{pmatrix} \beta & \ell \\ \ell & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{\beta} & \tilde{\ell} \\ \tilde{\ell} & 0 \end{pmatrix},$$

(2.23)

where $\beta$ and $\tilde{\beta}$ are additional allowed constants. The non-degeneracy of the bilinear form requires $\ell \neq 0$ and $\tilde{\ell} \neq 0$, but $\beta$ and $\tilde{\beta}$ are left unconstrained.

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4The same algebra can be obtained from the subalgebra of Virasoro algebra spanned by $L_m/m^{3/2}$, $L_{-m}/m^{3/2}$ and $L_0/m$ (here $L_m$ are the standard generators of Virasoro algebra) in the $m \to \infty$ limit. The Virasoro central charge $c$ becomes the central element $Z$. 
The bilinear form of the doubly extended NH algebra (2.20) is given by those of the left and right chiral algebras $h$ and $\tilde{h}$, together with possible cross terms. However, one can show that the only cross term compatible with symmetry is

$$\langle K, \tilde{K} \rangle = \gamma.$$  \hspace{1cm} (2.24)

In the end, the invariant bilinear form of $g_{\Lambda}^{++}$ has five undetermined constants:

$$\ell, \tilde{\ell}, \beta, \tilde{\beta}, \gamma.$$  \hspace{1cm} (2.25)

The number of free parameters is consistent with the analysis of [13].

As in the usual relativistic Chern-Simons gravity, we set $\ell$ and $\tilde{\ell}$ as

$$\ell = -\tilde{\ell} = \frac{1}{2\sqrt{-\Lambda}},$$  \hspace{1cm} (2.26)

so the spatial part takes the standard form. The non-vanishing components of the bilinear form are

$$\langle J, M \rangle = \langle S, H \rangle = 1, \quad \langle P_i, G_j \rangle = \delta_{ij},$$  \hspace{1cm} (2.27)

and

$$\begin{pmatrix}
\langle H, H \rangle & \langle H, J \rangle \\
\langle J, H \rangle & \langle J, J \rangle
\end{pmatrix} = \begin{pmatrix}
-\Lambda (\beta + \tilde{\beta} - 2\gamma) \sqrt{-\Lambda} (\beta - \tilde{\beta}) \\
\sqrt{-\Lambda} (\beta - \tilde{\beta}) & \beta + \tilde{\beta} + 2\gamma
\end{pmatrix}.$$  \hspace{1cm} (2.28)

The above symmetric matrix remains completely arbitrary.

The bilinear form that we have derived using symmetry can also be obtained from the relativistic one by a contraction. To begin with, let us focus on the chiral sector. The bilinear form of $h = sl(2) \oplus u(1)$ is unique for each of the $sl(2)$ and $u(1)$ part, up to overall factors:

$$\langle Z, Z \rangle = \zeta, \quad \langle \tilde{Z}, \tilde{Z} \rangle = \tilde{\zeta},$$  \hspace{1cm} (2.29)

$$\langle L_a, L_b \rangle = \lambda \eta_{ab}, \quad \langle \tilde{L}_a, \tilde{L}_b \rangle = \tilde{\lambda} \eta_{ab}.$$  \hspace{1cm} (2.30)

Using (2.15) and (2.16), these give the bilinear form for the rescaled generators as

$$\langle L_i, L_j \rangle = \frac{\lambda}{c^2} \delta_{ij}, \quad \langle \tilde{L}_i, \tilde{L}_j \rangle = \frac{\tilde{\lambda}}{c^2} \delta_{ij},$$  \hspace{1cm} (2.31)

$$B = \begin{pmatrix}
\zeta - \lambda & (\zeta + \lambda)/(2c^2) \\
(\zeta + \lambda)/(2c^2) & (\zeta - \lambda)/(4c^4)
\end{pmatrix}, \quad \tilde{B} = \begin{pmatrix}
\tilde{\zeta} - \tilde{\lambda} & (\tilde{\zeta} + \tilde{\lambda})/(2c^2) \\
(\tilde{\zeta} + \tilde{\lambda})/(2c^2) & (\tilde{\zeta} - \tilde{\lambda})/(4c^4)
\end{pmatrix}.$$  \hspace{1cm} (2.32)

Hence by choosing

$$\zeta = c^2 \ell + \beta/2, \quad \tilde{\zeta} = c^2 \tilde{\ell} + \tilde{\beta}/2, \quad \lambda = c^2 \ell - \beta/2, \quad \tilde{\lambda} = c^2 \tilde{\ell} - \tilde{\beta}/2,$$  \hspace{1cm} (2.33)

we recover the bilinear form (2.23) in the contraction limit. Then we consider the brackets of the left and right sectors. We have the freedom to introduce the cross term,

$$\langle Z, \tilde{Z} \rangle = \gamma,$$  \hspace{1cm} (2.34)
which leads to (2.24) after the contraction.

We have seen that all the non-relativistic algebras and their bilinear forms can be obtained from the relativistic one by contractions. In addition, it is also sufficient to consider the chiral sector \( \text{sl}(2) \oplus \text{u}(1) \). At the level of Lie algebra, there is no relation between \( \text{sl}(2) \) and \( \text{u}(1) \), hence their bilinear forms (2.29) are also independent.

On the other hand, one simple way to introduce an invariant bilinear form is through an invariant linear form, i.e. trace:

\[
\langle X, Y \rangle = \text{Tr}(XY). \tag{2.35}
\]

A well-defined trace requires an associative product between \( X \) and \( Y \). This can be done by representing \( X \) and \( Y \) as matrices. But the price to pay is that we have to work with a larger space because the associative products of the original elements will generate additional elements. The enlargement can be minimized by choosing the smallest representation. For \( \text{sl}(2) \), the smallest representation is the fundamental representation, which is two dimensional and requires only one more element, i.e. the identity, to have the associative structure. The associative product of the relativistic chiral algebra reads

\[
\mathcal{L}_a \mathcal{L}_b = \eta_{ab} I + \epsilon_{abc} \mathcal{L}_c, \quad \tilde{\mathcal{L}}_a \tilde{\mathcal{L}}_b = \eta_{ab} \tilde{I} + \epsilon_{abc} \tilde{\mathcal{L}}_c, \tag{2.36}
\]

and we recover the bilinear form (2.30) using

\[
\text{Tr}(I) = \lambda, \quad \text{Tr}(\tilde{I}) = \tilde{\lambda}, \quad \text{Tr}(\mathcal{L}_a) = \text{Tr}(\tilde{\mathcal{L}}_a) = 0. \tag{2.37}
\]

There is no a priori reason to identify the identities \( I \) and \( \tilde{I} \) with the \( \text{u}(1) \) generators \( Z \) and \( \tilde{Z} \). However, if they coincide up to some factors \( z(c) \) and \( \tilde{z}(c) \), the coefficients \( \lambda, \tilde{\lambda} \) and \( \zeta, \tilde{\zeta} \) in (2.29), (2.30) will be related as

\[
Z = z(c) I \quad \Rightarrow \quad \zeta = z^2(c) \lambda, \tag{2.38}
\]

\[
\tilde{Z} = \tilde{z}(c) \tilde{I} \quad \Rightarrow \quad \tilde{\zeta} = \tilde{z}^2(c) \tilde{\lambda}. \tag{2.39}
\]

These relations together with (2.33) fix the form of the functions \( z(c) \) and \( \tilde{z}(c) \) as

\[
z^2(c) = \frac{c^2 \ell + \beta/2}{c^2 \ell - \beta/2}, \quad \tilde{z}^2(c) = \frac{\tilde{c}^2 \ell + \tilde{\beta}/2}{\tilde{c}^2 \ell - \tilde{\beta}/2}. \tag{2.40}
\]

Therefore, we fully recover the non-relativistic algebra \( h \) and \( \tilde{h} \), and their bilinear forms using the associative structure of the relativistic counterpart, \( \text{gl}(2) \). As we mentioned in the introduction, the associative structure of the isometry algebras plays a central role in the color decoration of three-dimensional gravity. From this observation, we are driven to look more closely the possible interplay between non-relativistic and color deformations of gravity in three dimensions.
3 Colored Gravity and its Non-relativistic Limits

If a theory contains more than one graviton and they are related by internal symmetry, the global symmetry should be extended from the usual isometry $g_i$ to a color-decorated one $g_i \otimes g_c$, where $g_c$ is a finite vector space generated by the color label. This leads to severe algebraic constraints: there is only trivial possibility if we assume the color-decorated algebra does not include additional generators, which means extra fields are necessary [16, 17]. Even if we find a good candidate of colored gravity algebras by including a reasonable amount of additional fields, the construction of equations of motion or/and actions is not guaranteed to be straightforward. In three dimensions, the situation is much simpler. First, thanks to the Chern-Simons formulation, we obtain the action directly from a given algebra. Second, a color-decorated algebra can be constructed by extending the isometry algebra with only two additional $u(1)$ generators.

Technically speaking, we want to find a Lie algebra structure on the tensor-product vector space $g_i \otimes g_c$. The Lie brackets between two elements of the above space may be written as

$$[M_X \otimes \mathcal{T}^I, M_Y \otimes \mathcal{T}^J] = \frac{1}{2} \{M_X, M_Y \} \otimes \{ \mathcal{T}^I, \mathcal{T}^J \} + \frac{1}{2} \{ M_X, M_Y \} \otimes [ \mathcal{T}^I, \mathcal{T}^J ] , \quad (3.1)$$

where $M_X$ and $M_Y$ belong to $g_i$ whereas $\mathcal{T}^I$ and $\mathcal{T}^J$ to $g_c$. Note that we have used both the commutator and anti-commutator of the generators of each space. Therefore, in order to have well-defined Lie brackets (3.1), we should begin with two associative algebras $g_i$ and $g_c$. For the color algebra $g_c$, we use the matrix algebra $u(N)$.\footnote{In principle, any finite-dimensional associative algebra can be used as the color algebra.} For the isometry algebra $g_i$, the minimal choice is $gl(2) \oplus gl(2)$. This is precisely the same as the setting (1.1) for the doubly extended NH algebra.

3.1 Chiral sector

Let us start from the non-relativistic limits of one chiral sector of the colored gravity algebra. It is generated by the following generators,

$$\mathcal{Z}, \quad \mathcal{L}_a, \quad \mathcal{T}^I, \quad \mathcal{L}_a^I, \quad (3.2)$$

where $\mathcal{L}_a$ and $\mathcal{Z}$ form the original relativistic algebra $sl(2) \oplus u(1) = gl(2)$ and $\mathcal{T}^I$ are the $su(N)$ generators:

$$[ \mathcal{T}^I, \mathcal{T}^J ] = 2i f^{IJK} \mathcal{T}^K . \quad (3.3)$$

The generators $\mathcal{L}_a^I$ are the color decoration of the spin-two generators and they transform covariantly under $sl(2)$ and $su(N)$:

$$[ \mathcal{L}_a, \mathcal{L}_b^I ] = 2 \epsilon_{ab}^c \mathcal{L}_c^I , \quad [ \mathcal{T}^I, \mathcal{L}_a^J ] = 2i f^{IJK} \mathcal{L}_a^K , \quad (3.4)$$

but the Lie brackets between them are non-trivial:

$$[ \mathcal{L}_a^I, \mathcal{L}_b^J ] = 2i \eta_{ab} f^{IJK} \mathcal{T}^K + 2 \epsilon_{ab}^c \left( \frac{1}{N} g^{IJ} \mathcal{L}_c + g^{IJ} g_{LK} \mathcal{L}_c^K \right) . \quad (3.5)$$
In (3.5), we used the associative algebra of $u(N)$
\[ \mathcal{T}^I \mathcal{T}^J = \frac{1}{N} \delta^{IJ} I + g^{IJK} \mathcal{T}^K + i f^{IJK} \mathcal{T}^K, \]
where $g_{IJK}$ is totally symmetric and $f_{IJK}$ totally antisymmetric. We assign its bilinear form as
\[ \langle \mathcal{T}^I, \mathcal{T}^J \rangle = \text{Tr}(\mathcal{T}^I \mathcal{T}^J) = \delta^{IJ}, \]
where we used $\text{Tr}(I) = N$ while $\text{Tr}(\mathcal{T}^I) = 0$.

We want to construct the non-relativistic analogs of the above colored gravity algebra, which we assume to have the same number of generators as the relativistic one. The non-relativistic algebra is spanned by
\[ Z, \ L_i, \ K, \ \mathcal{T}^I, \ L_i^I, \ K^I. \]
Their Lie brackets depend on the non-relativistic limits.

### Color decoration after contraction

One of the possible ways to obtain the non-relativistic colored gravity algebra is by colored-decorating the doubly extended NH algebra. For that, the latter algebra should admit an associative algebra structure. In the previous section, we have shown that it can be obtained from an associative relativistic algebra by a suitable contraction. However, we have not examined the full consistency of the associative structure of the resulting non-relativistic algebra. An associative product can be split into commutators, which reproduce the Lie brackets, and anti-commutators. Therefore, we have to check whether the contraction limit (2.15) is equally well-defined for the anti-commutators. From (2.36), (2.38) and (2.40), most of the anti-commutators are regular in the contraction limit:
\[ \{K, L_i\} = 2 L_i, \ \{K, Z\} = 2 Z, \ \{L_i, L_j\} = 2 \delta_{ij} Z, \ \{L_i, Z\} = 0, \ \{Z, Z\} = 0. \]

However, before taking the contraction limit, the $\{K, K\}$ anti-commutator reads
\[ \{K, K\} = 2 (K - 2 c^2 Z). \]
In the $c \to \infty$ limit, the anti-commutator (3.10) becomes ill-defined, so the corresponding associative product is problematic. The contraction of the $gl(2)$ gives a good Lie algebra structure and bilinear form or trace, but not the fully associative algebra structure.

This issue does not affect the Chern-Simons action of the doubly extended NH gravity. The reason we consider the anti-commutator $\{K, K\}$ here is that from (3.1) they appear in the Lie bracket of its color-decorated counterparts, $[K^I, K^J]$. We can avoid the divergence by defining the colored generators as
\[ K^I = \frac{1}{c} K \otimes \mathcal{T}^I, \quad Z^I = Z \otimes \mathcal{T}^I, \]
which lead to a regular commutator
\[ [K^I, K^J] = -4 i f^{IJK} Z^K. \]
Concerning the color decoration of $L_i$ generators, we consider two options:
• In the first option, by defining them as
\[ L_i = \frac{1}{c} L_i \otimes T^I, \] (3.13)
we have
\[ [K^I, L_i] = [K, L_i^I] = 2 \epsilon_{ij} L_j^I. \] (3.14)

• In the second option, with the definition
\[ L_i = L_i \otimes T^I, \] (3.15)
we obtain different Lie brackets,
\[ [K, L_i^I] = 2 \epsilon_{ij} L_j^I, \]
\[ [L_i, L_j] = 2 \epsilon_{ij} Z^I, \] (3.16)
\[ [L_i^I, L_j^I] = 2 \epsilon_{ij} \left( \frac{1}{N} \epsilon_{ij} \delta^{IJ} Z + 2 g^{IJ} Z^K \right) + 2 i \delta_{ij} f^{IJ} Z^K. \] (3.17)

The rest of the Lie brackets vanish in both cases.

In the above paragraph, we obtained two non-relativistic colored gravity algebras with suitable contractions. However, it turns out that they do not have non-degenerate invariant bilinear form due to the additional contraction related to the color decoration. This problem can be fixed by considering the mixing between the two chiral algebras, which will be explained in section 3.3. In the next subsection, we will introduce the generalized İnönü-Wigner contraction, whose concept is crucial to the systematic construction of non-relativistic algebras from the relativistic ones.

### 3.2 Generalized İnönü-Wigner contraction

İnönü-Wigner contraction [25] is a procedure for generating Lie algebras from a starting Lie algebra by taking certain limits of the group-contraction parameters. The contraction procedure involves singular changes of bases, so the contracted algebra is not isomorphic to the original algebra. If a Lie algebra g has a subalgebra h, we can decompose g into
\[ g = h \oplus i. \] (3.18)

We then rescale the generators of i by a contraction parameter c and introduce the new generators as
\[ j = \frac{1}{c} i. \] (3.19)

The Lie brackets become
\[ [h, h] \subset h, \quad [h, i] \subset \frac{1}{c} h \oplus j, \quad [j, i] \subset \frac{1}{c^2} h \oplus \frac{1}{c} j. \] (3.20)

In the singular limit where \( c \to \infty \), we obtain a new Lie algebra,
\[ G(g) = h \oplus j, \] (3.21)
with the Lie brackets,
\[ [h, h] \subset h, \quad [h, i] \subset j, \quad [j, i] \subset 0, \] (3.22)
where j corresponds to an Abelian ideal. As an example, the Poincaré algebra can be obtained from the (A)dS algebra by sending the radius of curvature to infinity, and the translation generators in the Poincaré algebra generate an Abelian ideal.
Three-level contraction

The standard contraction procedure can be generalized. Assume the original Lie algebra \( g \) admits a decomposition into three parts:

\[
g = i_0 \oplus i_1 \oplus i_2,
\]

with the condition,

\[
[i_0, i_0] \subset i_0, \quad [i_0, i_1] \subset i_0 \oplus i_1.
\]

If we rescale the generators \( i_1 \) and \( i_2 \) by \( c^{-1} \) and \( c^{-2} \), respectively:

\[
j_0 = i_0, \quad j_1 = \frac{1}{c} i_1, \quad j_2 = \frac{1}{c^2} i_2,
\]

the contraction limit \( c \to \infty \) gives a new algebra,

\[
\mathcal{G}(g) = j_0 \oplus j_1 \oplus j_2,
\]

with the Lie brackets,

\[
[j_0, j_0] \subset j_0, \quad [j_0, j_1] \subset j_1, \quad [j_0, j_2] \subset j_2, \quad [j_1, j_1] \subset j_2,
\]

and

\[
[j_1, j_2] \subset \emptyset, \quad [j_2, j_2] \subset \emptyset.
\]

Note that \( j_2 \) and \( j_1 + j_2 \) are two different ideals of the contracted Lie algebra. This contraction actually covers all the non-relativistic limits considered in this paper.

Multi-level contraction

We can further generalize the contraction procedure by considering arbitrary many “levels”, i.e. several integer rescaling dimensions. The starting point is again a decomposition of the original algebra

\[
g = i_0 \oplus i_1 \oplus \cdots \oplus i_L,
\]

with the condition

\[
[i_m, i_n] \subset i_0 \oplus \cdots \oplus i_{m+n}, \quad i_{n>L} = \emptyset.
\]

After the rescaling

\[
j_k = \frac{1}{c^k} i_k,
\]

and the contraction \( c \to \infty \), we obtain a new algebra,

\[
\mathcal{G}(g) = j_0 \oplus j_1 \oplus \cdots \oplus j_L,
\]

with the Lie brackets

\[
[j_m, j_n] \subset j_{m+n}, \quad j_{n>L} = \emptyset.
\]

When the levels of the elements are higher than \( L/2 \), their commutators will vanish as the maximum is \( L \). At least half of the ideals \( j_n \oplus j_{n+1} \oplus \cdots \oplus j_L \) are Abelian. We need \( L > 1 \) to have a non-abelian ideal.
Mathematically, the condition (3.30) is nothing but a filtration structure, for which it is more nature to work with
\[ f_n = i_0 \oplus \cdots \oplus i_n. \] (3.34)
Then, the decomposition (3.29) and the condition (3.30) can be recast as the definition of the filtered Lie algebra:
\[ f_0 \subset \cdots \subset f_{L-1} \subset f_L = g, \quad \text{with} \quad [f_m, f_n] \subset f_{m+n}. \] (3.35)
For a given filtered (Lie) algebra \( g \), there exists a graded (Lie) algebra \( G(g) \) with (3.33). The latter is referred to as \textit{associated graded (Lie) algebra}. In the mathematical physics literature, this generalized \( \mathbf{\ddot{I}\mathbf{n\ddot{o}\mathbf{n}\ddot{u}}-\mathbf{Wigner}} \) contraction has been studied in [26].

An example of multi-level contractions can be found in higher spin algebras. By definition, the vector space of a higher spin algebra consists of the spaces \( \mathcal{R}_s \) of spin \( s \) Killing tensors,
\[ \mathfrak{g}_{HS} = \bigoplus_s \mathcal{R}_s. \] (3.36)
Each \( \mathcal{R}_s \) is generated by the traceless tensor \( M_{a_1 \cdots a_{s-1}, b_1 \cdots b_{s-1}} \) with the permutation symmetry of two-row Young diagram \((s-1, s-1)\). The Lie brackets of the higher spin algebra satisfy
\[ [\mathcal{R}_s_1, \mathcal{R}_s_2] \subset \mathcal{R}_{s_1+s_2-2} \oplus \mathcal{R}_{s_1+s_2-3} \oplus \cdots. \] (3.37)
The spins on the right hand side of (3.37) is bounded from above because one cannot construct \((r, r)\) tensors with \( r \geq s_1 + s_2 - 2 \) in terms of \( M_{(s_1-1, s_2-1)} \) and \( M_{(s_2-1, s_1-1)} \). Hence, taking \( i_{s-2} = \mathcal{R}_s \), one can see that the higher spin algebra enjoys the filtration structure (3.30). An unusual point is that a higher spin algebra is typically infinite dimensional, so it involves infinitely many levels \((L = \infty)\). In three dimensions, one can consider finite dimensional higher spin algebras. For instance, \( \mathfrak{sl}(N, \mathbb{R}) \) is isomorphic to the chiral part of the higher spin algebras involving spin 2 to \( N \). The contracted higher spin algebra still satisfies all the higher spin covariance conditions, but does not admit an invariant bilinear form. It would be interesting to check whether a central extension of this contracted higher spin algebra may admit a non-degenerate bilinear form, as in the case of three-dimensional non-relativistic spin-two algebras.

### 3.3 Non-relativistic colored gravity algebra

In this subsection, we will study the non-relativistic limits of the three-dimensional colored gravity algebra using the generalized \( \mathbf{\ddot{I}\mathbf{n\ddot{o}\mathbf{n}\ddot{u}}-\mathbf{Wigner}} \) contraction. The guiding principle is that the limiting algebra should have a non-degenerate bilinear form. This non-degeneracy is related to the physical requirement that the kinetic terms are present.\(^7\) As we have seen before, this condition cannot be met in the chiral sector, so we will work with the full algebra from now on.

\(^6\)The case of \( r = s_1 + s_2 - 2 \) is ruled out by the antisymmetry of Lie bracket.

\(^7\)In a more general setting, we may consider degenerate bilinear form. Then some fields will not have kinetic terms and only appear in the cubic terms.
We shall define the generators of the non-relativistic colored gravity algebras in terms of the relativistic ones and the contraction parameter \( c \). For the color-singlet generators, we will use the same contraction ansatz as the doubly extended NH algebra \( g_+^{++} \), i.e. (2.8), (2.10) and (2.11). This choice corresponds to the three-level ansatz:

\[
\begin{align*}
\mathfrak{j}_0 &: \ Z + \mathcal{L}_0, \quad \bar{Z} + \bar{\mathcal{L}}_0, \\
\mathfrak{j}_1 &: \ \frac{\mathcal{L}_i}{c}, \quad \frac{\bar{\mathcal{L}}_i}{c}, \\
\mathfrak{j}_2 &: \ \frac{Z - \mathcal{L}_0}{c^2}, \quad \frac{\bar{Z} - \bar{\mathcal{L}}_0}{c^2}.
\end{align*}
\] (3.38)

The non-singlet generators of the colored gravity algebra should be appended to \( \mathfrak{j}_0 \), \( \mathfrak{j}_1 \) and \( \mathfrak{j}_2 \) such that the contracted algebra admits a non-degenerate bilinear form. From the symmetry consideration, the bilinear form of the non-singlet sector generators are determined by that of the singlet sector as

\[
\langle a \otimes \mathcal{T}^I, b \otimes \mathcal{T}^J \rangle = \langle a, b \rangle \delta^{IJ},
\] (3.39)

where \( a \) and \( b \) are elements of \( gl(2) \oplus gl(2) \) and their bilinear form is given in Section 2.3. Let us summarize it here for the reader’s convenience: due to the associative structure, the bilinear form should be diagonal, i.e. \( \langle Z, \bar{Z} \rangle = \gamma = 0 \), then we have

\[
\begin{align*}
\langle Z, Z \rangle &= c^2 \ell + \beta/2, \quad \langle \mathcal{L}_a, \mathcal{L}_b \rangle = \eta_{ab} (c^2 \ell - \beta/2), \\
\langle \bar{Z}, \bar{Z} \rangle &= c^2 \bar{\ell} + \bar{\beta}/2, \quad \langle \bar{\mathcal{L}}_a, \bar{\mathcal{L}}_b \rangle = \eta_{ab} (c^2 \bar{\ell} - \bar{\beta}/2).
\end{align*}
\] (3.40)

As one can see, they all scale as \( c^2 \) hence diverge unless we rescale them appropriately. About the spatial part of non-singlet generators, we take the analogous rescaling as the singlet ones:

\[
\begin{align*}
\mathfrak{j}_1 &: \ \frac{\mathcal{L}_i}{c} \otimes \mathcal{T}^I, \quad \frac{\bar{\mathcal{L}}_i}{c} \otimes \mathcal{T}^I,
\end{align*}
\] (3.41)

hence these generators belong to \( \mathfrak{j}_1 \). About the remaining generators, there are a few consistent possibilities. In the classification, an important guideline is the fact that \( \mathfrak{j}_0 \) must form a subalgebra for a consistent contraction. Moreover, due to the presence of \( su(N) \) part, the \( gl(2) \oplus gl(2) \) part of the non-singlet elements in \( \mathfrak{j}_0 \) should be closed under anti-commutator: let us write the non-singlet part of \( \mathfrak{j}_0 \) as \( \mathfrak{A} \otimes \mathcal{T}^I \), then \( \mathfrak{A} \) should satisfy

\[
\{ \mathfrak{A}, \mathfrak{A} \} \subset \mathfrak{A},
\] (3.42)

where \( \mathfrak{A} \) is a subspace of \( gl(2) \oplus gl(2) \). With the above condition, let us examine what are the possible candidates for the \( \mathfrak{A} \) elements. The first guess is the combinations analogous to the singlet sector:

\[
(Z + \mathcal{L}_0) \otimes \mathcal{T}^I, \quad (\bar{Z} + \bar{\mathcal{L}}_0) \otimes \mathcal{T}^I,
\] (3.43)

but one can easily see that neither \( Z + \mathcal{L}_0 \) nor \( \bar{Z} + \bar{\mathcal{L}}_0 \) is closed under anti-commutator. In addition, we cannot consider \( Z \) or \( \bar{Z} \) because their bilinear form will diverge. In fact, the only possible way to have finite bilinear form in \( \mathfrak{j}_0 \) part — that is, without rescaling the
generators — is to cancel the $c^2$ terms: for instance, $(Z + L_0, Z + L_0) = (Z, Z) + (L_0, L_0) = (c^2 \ell + \beta/2) - (c^2 \ell - \beta/2) = \beta$ in the singlet sector. Noticing this, we can see

$$j_0 : (Z + \tilde{Z}) \otimes T^I$$

(3.44)

has a finite bilinear form: $(Z + \tilde{Z}, Z + \tilde{Z}) = \frac{\beta + \tilde{\beta}}{2}$, and it is closed under anti-commutator: $(Z + \tilde{Z})^2 = Z + \tilde{Z}$. The similar combination $Z - \tilde{Z}$ has a finite bilinear form but is not closed. For the non-degeneracy of the bilinear form, the choice (3.44) should be accompanied by

$$j_2 : \frac{(Z - \tilde{Z}) \otimes T^I}{c^2}.$$  

(3.45)

Now there remain only two generators $L_0$ and $\tilde{L}_0$, and there are two non-trivial choices:

$$j_0 : (L_0 \pm \tilde{L}_0) \otimes T^I, \quad j_2 : \frac{(L_0 \mp \tilde{L}_0) \otimes T^I}{c^2}.$$  

(3.46)

Note that both signs give consistent contractions. There also exist rather trivial options: i) the entire non-singlet generators are in $i_1$; ii) we have (3.44) and (3.45) but $L_0 \otimes T^I$ and $\tilde{L}_0 \otimes T^I$ are in $i_1$. The latter two cases are mathematically consistent options, but they treat the color-decorated isometry generators $L_a \otimes T^I$ in the Lorentz invariant manner, hence they are unnatural for the interpretation as non-relativistic physics. In the following, we consider only the “non-trivial” cases. See Appendix C for the other two cases. Another implicit assumption is that the $su(N)$ symmetry is not broken in the non-relativistic limits. There will be more possibilities if we relax this assumption.

### 4 Chern-Simons Formulation

In this section, we derive the actions of the non-relativistic colored gravity theories. There are two ways to compute these non-relativistic actions. The first way is to directly evaluate the Chern-Simons actions whose gauge fields take values in the non-relativistic gauge algebras. The second way is to take the non-relativistic limits of the relativistic Chern-Simons action according to the contractions. We will use the first method in the current section and discuss the second one in Appendix B.

#### 4.1 Gauge algebras

Before starting our discussion, let us summarize the algebraic structure of non-relativistic colored gravity symmetry. For a more clear physical interpretation, we use “$J, P$” bases rather than the chiral and anti-chiral bases. Since the singlet sector is identical to the doubly extended NH algebra, we directly start with the non-singlet ones. The contractions we have considered in the previous section are

$$C^I = Z_J \otimes T^I, \quad D^I = \frac{1}{c^2} Z_P \otimes T^I,$$

(4.1)

$$G^I_i = \frac{1}{c} J_i \otimes T^I, \quad P^I_i = \frac{1}{c} P_i \otimes T^I.$$  

(4.2)
These generators transform covariantly under the NH symmetry as

\[ [J, G_i^I] = 2 \epsilon_{ij} G_j^I, \quad [J, P_i^I] = 2 \epsilon_{ij} P_j^I, \quad (4.3) \]

\[ [H, G_i^I] = 2 \epsilon_{ij} P_j^I, \quad [H, P_i^I] = -2 \Lambda \epsilon_{ij} G_j^I, \quad (4.4) \]

and they have non-trivial Lie brackets among themselves:

\[ [C^I, C^J] = 2 i f^{IJK} C^K, \quad [C^I, D^J] = 2 i f^{IJK} D^K, \quad (4.5) \]

\[ [C^I, G_i^J] = 2 i f^{IJK} G_i^K, \quad [C^I, P_i^J] = 2 i f^{IJK} P_i^K. \quad (4.6) \]

Besides \( C^I, D^I, G_i^I \) and \( P_i^I \), we also have the temporal and central generators. The two possibilities of the sign in (3.46) are translated into the following two cases:

\[ \text{Type A : } J^I = J_0 \otimes T^I, \quad M^I = \frac{1}{c^2} P_0 \otimes T^I, \quad (4.7) \]

\[ \text{Type B : } H^I = P_0 \otimes T^I, \quad S^I = \frac{1}{c^2} J_0 \otimes T^I. \quad (4.8) \]

Only one of the two central generators in the doubly extended NH algebra admits color decoration. Let us present the Lie brackets of the two cases.

**Type A**

The non-zero commutators involving one singlet and one non-singlet are

\[ [J^I, G_i] = 2 \epsilon_{ij} G_j^I, \quad [J^I, P_i] = 2 \epsilon_{ij} P_j^I, \quad (4.9) \]

\[ [G_i^I, P_j] = [G_i, P_j^I] = -2 \epsilon_{ij} M^I. \quad (4.10) \]

The non-zero commutators involving only the non-singlet sectors are

\[ [C^I, J^J] = 2 i f^{IJK} J^K, \quad [C^I, M^J] = [D^I, J^J] = 2 i f^{IJK} M^K, \quad (4.11) \]

\[ [J^I, J^J] = -2 i f^{IJK} C^K, \quad [J^I, M^J] = -2 i f^{IJK} D^K, \quad (4.12) \]

\[ [J^I, G_i^J] = 2 \epsilon_{ij} \left( \frac{1}{N} \delta^{IJ} G_j + g^{IJ} K G_j^K \right), \quad (4.13) \]

\[ [J^I, P_i^J] = 2 \epsilon_{ij} \left( \frac{1}{N} \delta^{IJ} P_j + g^{IJ} K P_j^K \right), \quad (4.14) \]

\[ [G_i^I, G_j^J] = -\frac{2}{N} [P_i^I, P_j^J] = \frac{2}{N} \epsilon_{ij} \delta^{IJ} S, \quad (4.15) \]

\[ [G_i^I, P_j^J] = \frac{2}{N} \epsilon_{ij} \delta^{IJ} M - 2 \epsilon_{ij} g^{IJK} M^K + 2 i \delta_{ij} f^{IK} K D^K. \quad (4.16) \]
Type B

The non-zero commutators containing one singlet and one non-singlet sectors are
\[
[H^I, G_i] = 2 \epsilon_{ij} P^j, \quad [H^I, P_i] = -2 \Lambda \epsilon_{ij} G^j, \quad (4.17) \\
[G_i, G^j_I] = -\frac{1}{\Lambda} [P_i, P^j_I] = -2 \epsilon_{ij} S^j. \quad (4.18)
\]

The non-zero Lie brackets involving only the non-singlet sector are
\[
[C^I, H^J] = 2 i f^{IJ}_K H^K, \quad [C^I, S^J] = -\frac{1}{\Lambda} [D^I, H^J] = 2 i f^{IJ}_K S^K, \quad (4.19) \\
[H^I, H^J] = 2 i \Lambda f^{IJ}_K C^K, \quad [S^I, H^J] = -2 i f^{IJ}_K D^K, \quad (4.20) \\
[H^I, G^j_I] = 2 \epsilon_{ij} \left( \frac{1}{N} \delta^{IJ} P_j + g^{IJ}_K P^j_K \right), \quad (4.21) \\
[H^I, P^j_I] = -2 \Lambda \epsilon_{ij} \left( \frac{1}{N} \delta^{IJ} G_j + g^{IJ}_K G^j_K \right), \quad (4.22) \\
[G^I_i, G^j_I] = -\frac{1}{\Lambda} [P_i^I, P^j_I] = \frac{2}{N} \epsilon_{ij} \delta^{IJ} S - 2 \epsilon_{ij} g^{IJ}_K S^I, \quad (4.23) \\
[G^I_i, P^j_I] = \frac{2}{N} \epsilon_{ij} \delta^{IJ} M + 2 i \delta_{ij} f^{IJ}_K D^K. \quad (4.24)
\]

Bilinear form

Let us also remind the reader that the invariant bilinear form reads
\[
\langle J, M \rangle = \langle S, H \rangle = 1, \quad \langle P_i, G^j_I \rangle = \delta_{ij}, \quad (4.25) \\
\langle C^I, D^J \rangle = \delta^{IJ}, \quad \langle P^I_i, G^j_I \rangle = \delta_{ij} \delta^{IJ}, \quad (4.26) \\
\langle J^I, M^J \rangle = -\delta^{IJ}, \quad \langle H^I, S^J \rangle = -\delta^{IJ}, \quad (4.27)
\]

for type A and B, respectively. Here, we took the choice $\beta = \tilde{\beta} = \gamma = 0$ for simplicity.

4.2 Actions

Now we consider the Chern-Simons actions of the non-relativistic colored gravities. The one-form gauge fields are

Type A : \[
\mathcal{A} = j J + s S + h H + m M + \Omega^I G_i + E^i P_i + \sigma_I C^I + \varsigma_I D^I + \vartheta_I J^I + \varsigma_I M^I + \theta_I G^I_i + \pi_I^j P^j_I, \quad (4.28)
\]

Type B : \[
\mathcal{A} = j J + s S + h H + m M + \Omega^I G_i + E^i P_i + \sigma_I C^I + \varsigma_I D^I + \vartheta_I S^I + \varphi_I H^I + \theta_I G^I_i + \pi_I^j P^j_I, \quad (4.29)
\]
which correspond to the non-relativistic colored gravity algebras constructed in Section 3.3. It is now sufficient to plug the above into the Chern-Simons action,

$$S = \frac{\kappa}{4\pi} \int (A \wedge dA) + \frac{2}{3} (A \wedge A \wedge A).$$

(4.30)

In terms of the component fields, this action can be decomposed into several pieces:

$$S = S_{\text{gravity}} + S_{\text{gauge}} + S_{\text{matter}}.$$  

(4.31)

Here, $S_{\text{gravity}}$ is the usual uncolored non-relativistic action,

$$S_{\text{gravity}} = \frac{\kappa N}{2\pi} \int E^i \wedge \nabla \Omega_i + m \wedge d \omega + h \wedge [d s + \epsilon_{ij} (\Omega^i \wedge \Omega^j - \Lambda E^i \wedge E^j)],$$

(4.32)

where the covariant derivative $\nabla$ acts on “spatial” one-form $\omega_i$ as

$$\nabla \omega_i = d \omega_i + \epsilon_{ij} (\omega^i \wedge j - j \wedge \omega^j).$$

(4.33)

About the other pieces of the action, it is convenient to introduce the $su(N)$-valued fields as

$$\sigma = \sigma I T^I, \quad \varsigma = \varsigma I T^I, \quad \vartheta = \vartheta I T^I, \quad \varpi = \varpi I T^I, \quad \theta = \theta I T^I, \quad \pi = \pi I T^I,$$

(4.34)

and write the action in terms of $\text{Tr}$, the trace in the adjoint representation. Then, the gauge field part $S_{\text{gauge}}$ takes the simple form,

$$S_{\text{gauge}} = \frac{\kappa}{2\pi} \int \text{Tr} \left[ \varsigma \wedge (d\sigma + \sigma \wedge \sigma) \right].$$

(4.35)

The matter field part $S_{\text{matter}}$ depends whether we consider the type A or B:

$$S_{\text{matter}} = S_{\text{matter,0}} + S_{\text{type A/B}}.$$  

(4.36)

The common part $S_{\text{matter,0}}$ is

$$S_{\text{matter,0}} = \frac{\kappa}{2\pi} \int \text{Tr} \left[ -\theta \wedge D \varpi + \theta^i \wedge D \pi_i + \epsilon_{ij} h \wedge (\theta^i \wedge \theta^j - \Lambda \pi^i \wedge \pi^j) \right],$$

(4.37)

where the covariant derivative $D$ acts on the $su(N)$ valued one-form $\omega$ as

$$D \omega = \nabla \omega + \sigma \wedge \omega + \omega \wedge \sigma,$$

(4.38)

and $\omega$ can have spatial indices.

Finally, the last piece which depends on the type of the non-relativistic limit is

$$S_{\text{type A}} = \frac{\kappa}{2\pi} \int \text{Tr} \left[ -\varsigma \wedge \vartheta \wedge \vartheta + 2 \epsilon_{ij} \vartheta \wedge (\pi^i \wedge \Omega^j + \theta^i \wedge E^j + \theta^j \wedge \pi^j) \right],$$

(4.39)

$$S_{\text{type B}} = \frac{\kappa}{2\pi} \int \text{Tr} \left[ \Lambda \varsigma \wedge \varpi \wedge \varpi + \epsilon_{ij} \varpi \wedge (2 \theta^i \wedge \Omega^j - 2 \Lambda \pi^i \wedge E^j + \theta^j \wedge \pi^i - \Lambda \pi^i \wedge \pi^j) \right].$$

(4.40)
5 Conclusion

In this work, we have studied the non-relativistic limits of three-dimensional colored gravity in terms of generalized Inönü-Wigner contractions. Let us conclude our work with a brief summary of what have been done. In Section 2, we revisited the three-dimensional non-relativistic isometry algebras and their bilinear forms. We reviewed various central extensions of the Galilei and Newton-Hooke algebras. Then we discussed their chiral decompositions, how they arise from the contractions of relativistic algebras, and the issue of invariant bilinear forms. In Section 3, we explained the obstruction in the chiral decompositions of non-relativistic colored gravity and constructed the consistent non-relativistic colored gravity algebras using generalized Inönü-Wigner contractions. In Section 4, we derived the non-relativistic colored gravity actions in the Chern-Simons formulation using the non-relativistic algebras constructed in Section 3.

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A Associative structure of relativistic symmetry

If we identify the identities $I$ and $\tilde{I}$ with $Z$ and $\tilde{Z}$, the associative products of $gl(2) \oplus gl(2)$ generators are

\begin{align}
J_a J_b &= -\frac{1}{\Lambda} P_a P_b = \eta_{ab} Z_J + \epsilon_{abc} J_c, \\
Z_J Z_J &= Z_J, \\
Z_J Z_P &= Z_P Z_J = Z_J, \\
Z_P Z_P &= -\Lambda Z_J,
\end{align}

(A.1)

\begin{align}
J_a Z_J &= Z_J J_a = J_a, \\
J_a Z_P &= Z_P J_a = P_a,
\end{align}

(A.2)

\begin{align}
P_a Z_J &= Z_J P_a = P_a, \\
P_a Z_P &= Z_P P_a = -\Lambda J_a.
\end{align}

(A.3)

This associative extension of the relativistic isometry algebra can be realized as a tensor product,

\begin{align}
J_a &= L_a \otimes I, \\
P_a &= L_a \otimes \psi, \\
Z_J &= I \otimes I, \\
Z_P &= I \otimes \psi,
\end{align}

(A.4)

with

\begin{align}
L_a L_b &= \eta_{ab} I + \epsilon_{abc} L_c, \\
I^2 &= I, \\
\psi^2 &= -\Lambda I, \\
I \psi &= \psi I = \psi.
\end{align}

(A.5)

The chiral decompositions of the relativistic and the non-relativistic algebras are due to the existence of projectors,

\begin{align}
P_L &= \frac{1}{2} \left( I + \frac{1}{\sqrt{-\Lambda}} \psi \right), \\
P_R &= \frac{1}{2} \left( I - \frac{1}{\sqrt{-\Lambda}} \psi \right),
\end{align}

(A.6)

with

\begin{align}
P_L^2 &= P_L, \\
P_R^2 &= P_R, \\
P_L P_R &= P_R P_L = 0.
\end{align}

(A.7)
B Contraction at action level

The one-form gauge field (4.28) or (4.29) of the non-relativistic colored gravities should match with the relativistic one,

\[ A = (A Z + \tilde{A} \tilde{Z} + \omega^a J_a + e^a P_a) \otimes I + (A_I Z + \tilde{A}_I \tilde{Z} + \tau^a_I J_a + \chi^a_I P_a) \otimes T^I. \]  

(B.1)

In this way, we are able to identify the relations between relativistic and non-relativistic fields. The singlet part is

\[ j = \frac{1}{4} (A + \tilde{A}) + \frac{1}{2} \omega^0, \quad s = \frac{c^2}{2} (A + \tilde{A}) - c^2 \omega^0, \]  

(B.2)

\[ h = \frac{1}{4\sqrt{-\Lambda}} (A - \tilde{A}) + \frac{1}{2} e^0, \quad m = \frac{c^2}{2\sqrt{-\Lambda}} (A - \tilde{A}) - c^2 e^0, \]  

(B.3)

\[ \Omega^i = c \omega^i, \quad E^i = c e^i. \]  

(B.4)

The non-relativistic su(N) gauge fields are

\[ \sigma_I = \frac{1}{2} (A_I + \tilde{A}_I), \quad \varsigma_I = \frac{c^2}{2\sqrt{-\Lambda}} (A_I - \tilde{A}_I), \]  

(B.5)

and the spatial components of the matter fields are

\[ \theta^i_I = c \tau^i_I, \quad \pi^i_I = c \chi^i_I. \]  

(B.6)

Finally, the temporal components of the matter fields are defined as

Type A : \[ \vartheta_I = \tau^0_I, \quad \varpi_I = c^2 \chi^0_I, \]  

(B.7)

Type B : \[ \vartheta_I = c^2 \tau^0_I, \quad \varpi_I = \chi^0_I. \]  

(B.8)

Substituting the relativistic fields with the non-relativistic ones, we can derive the non-relativistic actions in Section 4.2 from the relativistic colored-gravity action, by multiplying the action with \( c^2 \) and taking the limit \( c \to \infty \).

C Other contractions of colored gravity algebra

In Section 3.3, we mentioned two other cases of non-relativistic limits of the colored gravity algebra. In the first case, we treat all the non-singlet generators as the level-one elements, whereas in the second case we have additional level-zero and level-two elements, (3.44) and (3.45).

In the first case, the non-relativistic colored generators are defined as

\[ C^I = \frac{1}{c} Z_J \otimes T^I, \quad D^I = \frac{1}{c} Z_P \otimes T^I, \]  

(C.1)

\[ J^I = \frac{1}{c} \mathcal{J}_0 \otimes T^I, \quad H^I = \frac{1}{c} P_0 \otimes T^I. \]  

(C.2)
\[ G^I_i = \frac{1}{c} J_i \otimes T^I, \quad P^I_i = \frac{1}{c} P_i \otimes T^I. \]  

(C.3)

In the non-relativistic limit, the non-zero commutators involving colored generators are

\[ [J, G^I_i] = -\frac{1}{\Lambda} [H, P^I_i] = 2 \epsilon_{ij} G^I_j, \quad [J, P^I_i] = [H, G^I_i] = 2 \epsilon_{ij} P^I_j, \]  

(C.4)

\[ [G^I_i, G^J_j] = -\frac{1}{\Lambda} [P^I_i, P^J_j] = 2 \frac{\epsilon_{ij}}{N} \delta^{IJ} S, \quad [G^I_i, P^J_j] = 2 \frac{\epsilon_{ij}}{N} \delta^{IJ} M. \]  

(C.5)

The relations between relativistic and non-relativistic fields are

\[ \sigma_l = \frac{c}{2} (A_l + \tilde{A}_l), \quad \varsigma_l = \frac{c}{2\sqrt{-\Lambda}} (A_l - \tilde{A}_l), \]  

(C.6)

\[ \vartheta_l = c \tau^0_l, \quad \varpi_l = c \chi^0_l, \quad \theta^I_l = c \tau^I_l, \quad \pi^I_l = c \chi^I_l. \]  

(C.7)

The non-relativistic action is again given as (4.31). \( S_{\text{gravity}} \) is the same as (4.32), whereas \( S_{\text{gauge}} \) and \( S_{\text{matter}} \) are given by

\[ S_{\text{gauge}} = \frac{\kappa}{2\pi} \int \text{Tr} (\varsigma \wedge d\varsigma), \]  

(C.8)

\[ S_{\text{matter}} = \frac{\kappa}{2\pi} \int \text{Tr} \left[ -\theta \wedge d\varpi + \theta^I \wedge \nabla \pi^I + \epsilon_{ij} h \wedge (\theta^I \wedge \theta^J - \Lambda \pi^I \wedge \pi^J) \right]. \]  

(C.9)

In the second case, the level-zero and level-two sectors contain \( Z_J \otimes T^I \) and \( Z_P \otimes T^I \), respectively. The non-relativistic generators are defined as

\[ C^I = Z_J \otimes T^I, \quad D^I = \frac{1}{c^2} Z_P \otimes T^I, \]  

(C.10)

\[ J^I = \frac{1}{c} J_0 \otimes T^I, \quad H^I = \frac{1}{c} P_0 \otimes T^I. \]  

(C.11)

\[ G^I_i = \frac{1}{c} J_i \otimes T^I, \quad P^I_i = \frac{1}{c} P_i \otimes T^I. \]  

(C.12)

The non-zero Lie brackets involving the colored generators are

\[ [J, G^I_i] = -\frac{1}{\Lambda} [H, P^I_i] = 2 \epsilon_{ij} G^I_j, \quad [J, P^I_i] = [H, G^I_i] = 2 \epsilon_{ij} P^I_j, \]  

(C.13)

\[ [C^I, C^J] = 2 i f^{IJ} K C^K, \quad [C^I, D^J] = 2 i f^{IJ} K D^K, \]  

(C.14)

\[ [C^I, J^J] = 2 i f^{IJ} K J^K, \quad [C^I, H^J] = 2 i f^{IJ} K H^K, \]  

(C.15)

\[ [C^I, G^J_j] = 2 i f^{IJ} K G^K_j, \quad [C^I, P^J_j] = 2 i f^{IJ} K P^K_j, \]  

(C.16)

\[ [J^I, H^J] = -2 i f^{IJ} K D^K, \]  

(C.17)

\[ [G^I_i, G^J_j] = -\frac{1}{\Lambda} [P^I_i, P^J_j] = 2 \frac{\epsilon_{ij}}{N} \delta^{IJ} S, \]  

(C.18)

\[ [G^I_i, P^J_j] = 2 \frac{\epsilon_{ij}}{N} \delta^{IJ} M + 2 i \delta_{ij} f^{IJ} K D^K. \]  

(C.19)

The non-relativistic fields are related to the relativistic fields by

\[ \sigma_l = \frac{1}{2} (A_l + \tilde{A}_l), \quad \varsigma_l = \frac{c^2}{2\sqrt{-\Lambda}} (A_l - \tilde{A}_l), \]  

(C.20)

\[ \vartheta_l = c \tau^0_l, \quad \varpi_l = c \chi^0_l, \quad \theta^I_l = c \tau^I_l, \quad \pi^I_l = c \chi^I_l. \]  

(C.21)

Again, the action is given by (4.31), where \( S_{\text{gravity}} \) and \( S_{\text{gauge}} \) correspond to (4.32) and (4.35), while \( S_{\text{matter}} \) coincides with \( S_{\text{matter},0} \) in (4.37).
References

[1] E. Cartan, “Sur les variétés à connexion affine et la théorie de la relativité généralisée. (première partie),” Annales Sci. Ecole Norm. Sup. 40, 325 (1923).
[2] E. Cartan, “Sur les variétés à connexion affine et la théorie de la relativité généralisée. (première partie) (Suite),” Annales Sci. Ecole Norm. Sup. 41, 1 (1924).
[3] C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, “Bargmann Structures and Newton-cartan Theory,” Phys. Rev. D 31, 1841 (1985) doi:10.1103/PhysRevD.31.1841.
[4] X. Bekaert and K. Morand, “Connections and dynamical trajectories in generalised Newton-Cartan gravity I. An intrinsic view,” J. Math. Phys. 57, no. 2, 022507 (2016) doi:10.1063/1.4937445 [arXiv:1412.8212 [hep-th]].
[5] X. Bekaert and K. Morand, “Connections and dynamical trajectories in generalised Newton-Cartan gravity II. An ambient perspective,” arXiv:1505.03739 [hep-th].
[6] R. Andringa, E. Bergshoeff, S. Panda and M. de Roo, “Newtonian Gravity and the Bargmann Algebra,” Class. Quant. Grav. 28, 105011 (2011) doi:10.1088/0264-9381/28/10/105011 [arXiv:1011.1145 [hep-th]].
[7] E. Bergshoeff, J. Rosseel and T. Zojer, “Newton-Cartan (super)gravity as a non-relativistic limit,” Class. Quant. Grav. 32, no. 20, 205003 (2015) doi:10.1088/0264-9381/32/20/205003 [arXiv:1505.02095 [hep-th]].
[8] D. R. Grigore, “The Projective unitary irreducible representations of the Galilei group in (1+2)-dimensions,” J. Math. Phys. 37, 460 (1996) doi:10.1063/1.531402 [hep-th/9312048].
[9] R. Jackiw and V. P. Nair, “Anyon spin and the exotic central extension of the planar Galilei group,” Phys. Lett. B 480, 237 (2000) doi:10.1016/S0370-2693(00)00379-8 [hep-th/0003130].
[10] G. Papageorgiou and B. J. Schroers, “A Chern-Simons approach to Galilean quantum gravity in 2+1 dimensions,” JHEP 0911, 009 (2009) doi:10.1088/1126-6708/2009/11/009 [arXiv:0907.2880 [hep-th]].
[11] G. Papageorgiou and B. J. Schroers, “Galilean quantum gravity with cosmological constant and the extended q-Heisenberg algebra,” JHEP 1011, 020 (2010) doi:10.1007/JHEP11(2010)020 [arXiv:1008.0279 [hep-th]].
[12] E. A. Bergshoeff and J. Rosseel, “Three-Dimensional Extended Bargmann Supergravity,” Phys. Rev. Lett. 116, no. 25, 251601 (2016) doi:10.1103/PhysRevLett.116.251601 [arXiv:1604.08042 [hep-th]].
[13] J. Hartong, Y. Lei and N. A. Obers, “Nonrelativistic Chern-Simons theories and three-dimensional Horava-Lifshitz gravity,” Phys. Rev. D 94, no. 6, 065027 (2016) doi:10.1103/PhysRevD.94.065027 [arXiv:1604.08054 [hep-th]].
[14] E. Bergshoeff, D. Grumiller, S. Prohazka and J. Rosseel, “Three-dimensional Spin-3 Theories Based on General Kinematical Algebras,” JHEP 1701, 114 (2017) doi:10.1007/JHEP01(2017)114 [arXiv:1612.02277 [hep-th]].
[15] J. Hartong, Y. Lei, N. A. Obers and G. Oling, “Zooming in on AdS$_3$/CFT$_2$ near a BPS Bound,” arXiv:1712.05794 [hep-th].
[16] S. Gwak, E. Joung, K. Mkrtchyan and S. J. Rey, “Rainbow Valley of Colored (Anti) de Sitter Gravity in Three Dimensions,” JHEP 1604, 055 (2016)
doi:10.1007/JHEP04(2016)055 [arXiv:1511.05220 [hep-th]].

[17] S. Gwak, E. Joung, K. Mkrtchyan and S. J. Rey, “Rainbow vacua of colored higher-spin (A)dS$_3$ gravity,” JHEP 1605, 150 (2016) doi:10.1007/JHEP05(2016)150 [arXiv:1511.05975 [hep-th]].

[18] E. Joung, J. Kim, J. Kim and S. J. Rey, “Asymptotic Symmetries of Colored Gravity in Three Dimensions,” arXiv:1712.07744 [hep-th].

[19] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, “Inconsistency of interacting, multigraviton theories,” Nucl. Phys. B 597, 127 (2001) doi:10.1016/S0550-3213(00)00718-5 [hep-th/0007220].

[20] N. Boulanger and L. Gualtieri, “An Exotic theory of massless spin two fields in three-dimensions,” Class. Quant. Grav. 18, 1485 (2001) doi:10.1088/0264-9381/18/8/306 [hep-th/0012003].

[21] S. Deser and R. I. Nepomechie, “Anomalous Propagation of Gauge Fields in Conformally Flat Spaces,” Phys. Lett. 132B, 321 (1983) doi:10.1016/0370-2693(83)90317-9.

[22] S. Deser and A. Waldron, “Partial masslessness of higher spins in (A)dS,” Nucl. Phys. B 607, 577 (2001) doi:10.1016/S0550-3213(01)00212-7 [hep-th/0103198].

[23] C. Duval and P. A. Horvathy, “Spin and exotic Galilean symmetry,” Phys. Lett. B 547, 306 (2002) Erratum: [Phys. Lett. B 588, 228 (2004)] doi:10.1016/j.physletb.2004.04.001, 10.1016/S0370-2693(02)02783-1 [hep-th/0209166].

[24] C. R. Hagen, “Second central extension in Galilean covariant field theory,” Phys. Lett. B 539, 168 (2002) doi:10.1016/S0370-2693(02)02032-4 [quant-ph/0203109].

[25] E. Inonu and E. P. Wigner, “On the Contraction of groups and their representation,” Proc. Nat. Acad. Sci. 39, 510 (1953) doi:10.1073/pnas.39.6.510.

[26] E. Weimar-Woods, “Contractions of Lie algebras: generalized Inönü-Wigner contractions versus graded contractions,” J. Math. Phys. 36, 4519-4548 (1995); “Contractions, generalized Inönü and Wigner contractions and deformations of finite-dimensional Lie algebras,” Rev. Math. Phys. 12, 1505-1529 (2000).