GEOMETRIC METHODS IN THE ANALYSIS OF NON-LINEAR FLOWS IN POROUS MEDIA

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ABSTRACT. Over the past few years, we developed a mathematically rigorous method to study the dynamical processes associated to nonlinear Forchheimer flows for slightly compressible fluids. We have proved the existence of a geometric transformation which relates constant mean curvature surfaces and time-invariant pressure distribution graphs constrained by the Darcy-Forchheimer law. We therein established a direct relationship between the CMC graph equation and a certain family of equations which we call $g$-Forchheimer equations. The corresponding results, on fast flows and their geometric interpretation, can be used as analytical tools in evaluating important technological parameters in reservoir engineering.

I. Background

The present report uses methods of differential geometry and integrable systems in modeling nonlinear flows in porous media. The classical flow, as defined by Darcy’s equation \[ \frac{D}{q} = \frac{1}{K} \] represents a linear relation between the gradient of pressure and velocity field. Darcy’s formulation of the momentum equation of motion essentially simplifies some complicated hydrodynamics in porous media, by using a permeability tensor as the major characteristic of the media at a point (pore, block, layer, etc) in space.

Most of the up-scaling techniques in heterogeneous reservoirs are based on the assumption that on each scaling level, the constitutive momentum relation between the gradient of pressure and velocity field remains invariant, while the only parameter that is reevaluated is the permeability tensor.

Philipp Forchheimer, one of the founders of groundwater hydrology at the beginning of the 20th century, observed that for high velocity rates, Darcy’s law is no longer valid. On the other hand, on a fine level, it is well-known that even slow flows can deviate from the linear ones\[14\].

Some newer theories suggest that it is precisely the inertial term that causes the deviation from the linear Darcy law, but the nature of these inertial forces is not fully understood. As it was pointed by Bear,\[22\] (see also \[23, 24, 25\]): “Most of the experiments indicate that the actual turbulence occurs at a Reynolds number at least one order of magnitude higher than the Reynolds value at which the deviation from Darcy’s law is observed”. In some of the experiments, a deviation from Darcy’s law was observed for Reynolds number \[Re = \frac{qd}{\mu} \simeq 10\] (where \(d\) is some length dimension of the porous matrix, \(q\) a specific discharge per unit cross-sectional area normal to the direction of the flow, and \(\mu\) is viscosity of the liquid). In some recent works, such as \[25\], it was experimentally observed that Darcy’s law is not verified even for \(Re \simeq 1\), for samples of the porous rocks containing fracture. Latest results
suggest that even some low velocity flows in a highly heterogeneous reservoir may deviate from Darcy’s linearity principle.

In reservoir engineering, there are three most popular non-linear approximations of the field data, establishing a formula for the pressure drop $\Delta P$ in terms of the production rate $Q$:
- the “two term” law $- AQ + BQ^2 = \Delta P$;
- the “power” law $CQ^n + aQ = \Delta P, \quad 1 \leq n \leq 2$;
- the “three term” cubic law $AQ + BQ^2 + CQ^3 = \Delta P$.

All three equations were originally introduced by Forchheimer in works published at the beginning of the 20th century.

Following [24, 4, 1, 3] and references herein we have established three types of generalized nonlinear Darcy law, with permeability tensor depending on the gradient of the pressure function. We will show that these constitutive equations correspond to a generalized Forchheimer equation. Under some assumption about fluids, the generalized Darcy-Forchheimer equations enable a reduction of the system of equations that governs the flow - namely to one parabolic non-linear equation for the pressure function. This parabolic equation displays some similarity to the constant mean curvature (CMC-graph) equation for surfaces. In this work we use this similarity to find a constraint on the Forchheimer flows, such that the pressure function can be regarded as a surface with given constant mean curvature, after a certain geometric transformation. Conversely, a graph with prescribed constant mean curvature can be interpreted as a pressure distribution of the flow subject to a nonlinear Darcy law, similar to the Forchheimer equation. This geometric interpretation provided us with a simple algorithm to compute the productivity index of the well, in a structured heterogeneous reservoir. The productivity index ($PI$) of the well is one of the fundamental concepts in reservoir engineering, defined as the ratio between the production rate and the difference between the well pressure and the average pressure in the reservoir. The $PI$ characterizes the well performance with respect to the geometry of the hydraulic system. It is shown that, for some specific condition that is reasonable to either impose or approximate, the productivity index of the well in the inhomogeneous reservoir can be computed using a solution of a corresponding CMC graph equation.

II. Introduction.

II.1. General Forchheimer equations. Darcy’s law (for viscous fluid laminar flows) assumes that the total discharge is equal to the product between the medium permeability, the cross-sectional area of the flow, and the pressure drop, divided by the dynamic viscosity. After dividing both sides of this equation by the area, one obtains another way to express Darcy’s law: namely that the filtration velocity (or Darcy flux) is proportional to the pressure gradient, through a certain permeability coefficient. Darcy’s equation, the continuity equation and the equation of state serve as the framework to model processes in reservoirs [14, 8]. For a slightly compressible fluid, the original PDE system reduces to a scalar linear second order parabolic equation for the pressure only. The pressure function is a major feature of the oil or gas filtration in porous media, which is bounded by the well surface and the exterior reservoir boundary. Different boundary conditions on the well correspond to different regimes of production, while the condition on the exterior boundary models flux or absence of flux into the drainage area. All together, the
linear parabolic equation, boundary conditions and some assumptions or guesses about the initial pressure distribution form an IBVP.

There are different approaches for modeling non-Darcy’s phenomena [11, 13, 24, 18, 20]. It can be derived from the more general Brinkman-Forchheimer’s equation [18, 7], or from mixture theory assuming certain relations between velocity field and “drag-like” forces due to fluid to solid friction in the porous media [19]. It can be also derived using homogenization arguments [21], or assuming some functional relation and then match the experimental data. In the current report, we just postulate a general constitutive equation relating the velocity vector field and the pressure gradient. We will introduce constraints on the momentum equation and on the fluid density. This will allow the reduction of the original system to a scalar quasi-linear parabolic equation for the pressure only.

Hereafter, we use the following notations and basic definitions:

• $v(x,t)$ represents the velocity field; $x$ is the spatial variable in $\mathbb{R}^d$, $d = 2$ or 3; $t$ denotes time; $p(x,t)$ is the pressure distribution; $y \in \mathbb{R}^d$ are the variable vectors related to $\nabla p$; $s, \xi$ represent scalar variables;

• The notations $C, C_0, C_1, C_2, \ldots$ denote generic positive constants not depending on the solutions.

Current studies of flows in porous media widely use the three specific Forchheimer laws which we already mentioned (two-term, power, and three-term, respectively). Darcy and Forchheimer laws can be written in vector forms as follows:

- **Darcy’s law:**
  \[
  \alpha v = -\nabla p,
  \]
  where $\alpha = \frac{\mu}{k}$ with $k$, in general, represents the permeability non-homogeneous function depending on $x$ subjected to the condition: $k_2^{-1} \geq k \geq k_2$, $1 \geq k_2 > 0$. Here, the constant $\mu$ is the viscosity of the fluid.

- **The Forchheimer two-term law:**
  \[
  \alpha v + \beta |v|v = -\nabla p,
  \]
  where $\beta = \frac{\rho F \Phi}{k^{1/2}}$, $F$ is Forchheimer’s coefficient, $\Phi$ is the porosity, and $\rho$ is the density of the fluid.

- **The Forchheimer power law:**
  \[
  \alpha v + c^n |v|^{n-1}v = -\nabla p,
  \]
  where $n$ is a real number belonging to the interval $[1, 2]$. The strictly positive and bounded functions $c$ and $a$ are found empirically, or can be taken as $c = (n-1)\sqrt{\beta}$ and $a = \alpha$. By this way, $n = 1$ and $n = 2$ reduce the power law (3) to Darcy’s law and to the Forchheimer two-term law, respectively.

- **The Forchheimer three-term law:**
  \[
  \mathcal{A}v + \mathcal{B}|v|v + \mathcal{C}|v|^2v = -\nabla p.
  \]
  Here $\mathcal{A}, \mathcal{B},$ and $\mathcal{C}$ are empirical constants.

We now introduce a **general form for the Forchheimer equations**.

**Definition II.1 (g-Forchheimer Equations).**

\[
 g(x, |v|) \, v = -\nabla p,
\]
here \( g(x, s) > 0 \) for all \( s \geq 0 \). We will refer to (7) as \( g \)-Forchheimer (momentum) equations.

Under isothermal condition the state equation relates the density \( \rho \) to the pressure \( p \) only, i.e. \( \rho = \rho(p) \). Therefore the equation of state has the form:

\[
\frac{1}{\rho} \frac{d\rho}{dp} = \frac{1}{\kappa},
\]

where \( 1/\kappa \) is the compressibility of the fluid. For slightly compressible fluid, such as compressible liquid, the compressibility is independent of pressure and is very small, hence we obtain

\[
\rho = \rho_0 \exp \left( \frac{p - p_0}{\kappa} \right),
\]

where \( \rho_0 \) is the density at the reference pressure \( p_0 \) (see [5] Sec. 2.3, and also [14] Sec. 3.4). Substituting Eq. (6) in the continuity equation

\[
\frac{d\rho}{dt} = -\nabla \cdot (\rho v),
\]

yields

\[
\frac{d\rho}{dp} \frac{dp}{dt} = -\rho \nabla \cdot v - \frac{d\rho}{dp} v \cdot \nabla p,
\]

\[
\frac{dp}{dt} = -\kappa \nabla \cdot v - v \cdot \nabla p.
\]

Since for most slightly compressible fluids in porous media the value of the constant \( \kappa \) is large, following engineering tradition we drop the last term in (9) and study the reduced equation:

\[
\frac{dp}{dt} = -\kappa \nabla \cdot v.
\]

II.2. Boundary conditions. Let \( U \subset \mathbb{R}^d \) be a \( C^1 \) domain modeling the drainage area in the porous media (reservoir), bounded by two boundaries: the exterior boundary \( \Gamma_e \), and the accessible boundary \( \Gamma_i \).

The exterior boundary \( \Gamma_e \) models the geometrical limit of the well impact on the flow filtration and is often considered impermeable. This yields the boundary condition:

\[
v \cdot N |_{\Gamma_e} = 0,
\]

where \( N \) is the outward normal vector on the boundary \( \Gamma = \Gamma_i \cup \Gamma_e \). Other types of boundary conditions on the exterior boundary are discussed in [3].

The accessible boundary \( \Gamma_i \) models the well and defines the regime of filtration inside the domain. On \( \Gamma_i \), consider a given rate of production \( Q(t) \), or a given pressure value \( p = \varphi(x, t) \), or a combination of both. It is very important from a practical point of view to build some “baseline” solutions capturing significant features of the well capacity and analyze the impact of the boundary conditions on these solutions. This analysis will be used to forecast the well performances and tune the model to the actual data.

On the boundary \( \Gamma_i \), a “split” condition of the following type is of particular interest:

\[
p = \psi(x, t) = \gamma(t) + \varphi(x),
\]
where the time and space dependence of $p$ are separated. This type of condition models wells which have conductivity much higher than the conductivity inside the reservoir. The limiting homogeneous case $\psi(x) = \text{const}$ corresponds to the case of infinite conductivity on the well.

In case the flow is controlled by a given production rate $Q(t)$, the solution is not unique.

Two important cases are:
(a) pressure distribution of the form $-At + \varphi(x)$, and
(b) constant total flux $Q = \text{const}$.

The particular solutions of IBVP with boundary conditions (a) and (b) are “time-invariant” (see Section IV) and are used actively by engineers in their practical work.

III. Non-Linear Darcy Equation and Monotonicity properties

In order to simplify the notation, we will further omit the $x$-dependence of $g$ in (5). Thus, one has

$$g(|v|) = g(x, |v|).$$

From (5) one has

$$g(|v||v| = -|\nabla p|, \text{ for } s \geq 0.$$

To make sure one can solve (14) for $|v|$, we impose the following conditions: function $g$ belongs to $C([0, \infty))$ and $C^1((0, \infty))$, and satisfies

$$g(0) > 0, \text{ and } g'(s) \geq 0 \text{ for all } s > 0.$$

Under this condition one has $sg(s)' = sg'(s) + g(s) \geq g(0) > 0$, for any positive value $s$. Therefore function $sg(s)$ is monotone and one can find $|v|$ as a function of $|\nabla p|$.

$$|v| = G(|\nabla p|)$$

Substituting equation (16) into (5) one obtains the following alternative form of the $g$-Forchheimer momentum equation (5):

**Definition III.1. (Non-linear Darcy Equation)**

$$v = \frac{-\nabla p}{g(G(|\nabla p|))} = -K(|\nabla p|)\nabla p,$$

where the function $K : [0, \infty) \rightarrow [0, \infty)$ is defined by

$$K(\xi) = K_g(\xi) = \frac{1}{g(G(\xi))}, \quad \xi \geq 0.$$

Substituting (17) for $u$ into (10) one derives the degenerate parabolic equation for the pressure:

$$\frac{dp}{dt} = \nabla \cdot (K(|\nabla p|)\nabla p).$$

where the compressibility constant $\kappa$ has been included in the non linear coefficient $K(|\nabla p|)$.

Some properties of the function $K$ can be found in [1]. It turns out that the function $y \rightarrow K(|y|)y$ associated with the non-linear potential field on the RHS of equation (17) is monotonic. This monotonicity and related properties are crucial in
the study of the uniqueness and qualitative behavior of the the solutions of initial value problems (see e.g. [12]).

We illustrate the function $K$ for the particular case of two-term Forchheimer’s equation. This is one of the few cases when the function $K$ can be found explicitly.

**Example III.2.** For the Forchheimer two-term law (2), let $g(s) = \alpha + \beta s$, then one has $sg(s) = \beta s^2 + \alpha s$ and $s = G(\xi) = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta \xi}}{2\beta}$. Thus

$$K(\xi) = \frac{1}{\alpha + \beta G(\xi)} = \frac{2}{\alpha + \sqrt{\alpha^2 + 4\beta \xi}}.$$

We now introduce the notion of generalized polynomial with positive coefficients and positive exponents, abbreviated as GPPC, as it is a useful tool in this study.

**Definition III.3.** We say that a function $g(s)$ is a GPPC if

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_k s^{\alpha_k} = \sum_{j=0}^{k} a_j s^{\alpha_j},$$

where $k \geq 0$ represents a natural number, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k$ represent real values, and the coefficients $a_0, a_1, \ldots, a_k$ are real and positive. The largest exponent $\alpha_k$ is the degree of $g$ and is denoted by $\deg(g)$.

Class (GPPC) is defined as the collection of all GPPC.

If the function $g$ in Definition II.1 belongs to class (GPPC) then we call it the $g$-Forchheimer polynomial.

**Lemma III.4.** Let $g(s)$ be a function of class (GPPC) as in (20). Then $K(\xi) = K_g(\xi)$ is well-defined, is decreasing and satisfies

$$C_0 \frac{1}{1 + \xi^a} \leq K(\xi) \leq C_1 \frac{1}{1 + \xi^a}, \forall \xi \geq 0,$$

where $a = \alpha_k / (\alpha_k + 1) \in [0, 1)$, and $C_0$ and $C_1$ are positive numbers depending on $a_i$‘s and $\alpha_j$‘s.

The proof of the previous Lemma can be found in [1] and makes use of a condition that is automatically satisfied by a GPPC.

Note that $a = 0$ corresponds to the linear Darcy’s case, while in the limiting case $a \to 1$ the largest exponent $\alpha_k$ diverges.

**IV. Pseudo Steady State Solutions and Productivity Index**

In engineering and physics, it is often essential to identify special time-dependent pressure distributions that generate flows which are time-invariant. In this section, we introduce the class of so-called pseudo-steady state (PSS) solutions which is used frequently by reservoir and hydraulic engineers to evaluate the “capacity” of the well (see. [4, 3, 15] and references therein).

**Definition IV.1.** A solution $\overline{p}(x,t)$ of the equation (19) in domain $U$, satisfying the Neumann condition on $\Gamma_e$ is called a pseudo steady state (PSS) with respect to a constant $A$ if

$$\frac{\partial \overline{p}(x,t)}{\partial t} = \text{const.} = -A \quad \text{for all} \quad t.$$
Equation (19) then implies

\[
\frac{\partial p(x,t)}{\partial t} = -A = \nabla \cdot (K(|\nabla p|)\nabla p).
\]

Using Green’s formula and the Neumann boundary condition on \( \Gamma_e \) one derives

\[
A[U] = -\int_{\Gamma_i} (K(|\nabla p|)\nabla p) \cdot N d\sigma = \int_{\Gamma_i} v \cdot N d\sigma = Q(t).
\]

Therefore, the total flux of a PSS solution is time-independent

\[
Q(t) = A[U] = Q = \text{const.}, \quad \text{for all } t.
\]

The PSS solutions inherit two important features, which we will explore further. On one hand, the total flux is defined by stationary equation (23) and is given. On the other hand, the trace of the solution on the boundary is split \textit{a priori}. Namely re-writing the PSS solution as

\[
\varphi(x,t) = -At + B + u(x),
\]

one has \( \nabla \varphi = \nabla u \), hence \( u \) and \( p \) satisfy the same boundary condition on \( \Gamma_e \). On \( \Gamma_i \), in general, we consider

\[
\varphi(x,t) = -At + B + \phi(x), \quad \text{on } \Gamma_i,
\]

where \( \phi(x) \) is given and the constant \( B \) is selected such that

\[
\int_{\Gamma_i} \phi d\sigma = 0.
\]

Therefore \( u(x) \) satisfies

\[
-A = \nabla \cdot K(|\nabla u|)\nabla u,
\]

\[
\frac{\partial u}{\partial N} = 0 \quad \text{on } \Gamma_e,
\]

\[
u = \phi \quad \text{on } \Gamma_i.
\]

Of particular interest is the case \( \phi(x) = 0 \) on \( \Gamma_i \). From a physical point of view, it relates to the constraint that conductivity inside well is non-comparably higher than in the porous media.

We call \( u(x) \) the profile of PSS corresponding to \( A \) and the boundary profile \( \phi(x) \).

**Remark IV.2.** Note that for a PSS as in (25), we have the quantity

\[
J(t) = \frac{Q(t)}{\frac{1}{|U|} \int_U p(x,t)dx - \frac{1}{|\Gamma_i|} \int_{\Gamma_i} p(x,t)d\sigma} = \frac{Q}{\frac{1}{|U|} \int_U u(x)dx}.
\]

It represents the production rate versus the pressure drawdown (the difference between averages in the domain and on the boundary \( \Gamma_i \)), and is independent of time. This quantity is called Productivity Index, and it is widely used by engineers to test the performances of a well/reservoir system (see [4, 3, 15]).
V. Geometric Interpretation

In the PDE literature equation (see Serrin G., 1967, Gilbarg, D. & Trudinger, N.1977, Evans L. 1999)

\[
\text{div} \left( \left( 1 + |\nabla u|^2 \right)^{-1/2} \nabla u \right) = -2H
\]

is referred as homogeneous CMC equation were \( u \) is a graph defined in the domain \( U \subset \mathbb{R}^2 \). We will also refer to it as a CMC graph equation. We noted that this equation looks somewhat similar to

\[
\text{div} \left( K(|\nabla u|) \nabla u \right) = -A,
\]

which has been previously introduced in defining the basic PSS profile for the non-linear Forchheimer equation.

It is not difficult to see that the non linear term in (31) is “about” \((1 + |\nabla u|)^{-1}\), which means that there exist constants \(C_0\) and \(C_1\) such that

\[
\frac{C_0}{1 + |\nabla u|} \leq \frac{1}{\left( 1 + |\nabla u|^2 \right)^{1/2}} \leq \frac{C_1}{1 + |\nabla u|}.
\]

We also showed in Lemma [I3.4] that the non linear term \(K(|\nabla p|)\) in (32) is, in the GPPC case, “about” \((1 + |\nabla u|)^{-1}\). Here \(a = \alpha_k/(\alpha_k + 1) < 1\), where \(\alpha_k \geq 0\) is the degree, \(deg(g)\), of the specific GPPC polynomial. Then in the limiting case \(a \to 1\), we can expect the two non-linear coefficients in Eqs. (31) and (32) to have the same structure. Therefore one cannot expect the PSS Forchheimer equation to “survive” in limiting case \(a = 1\). At the same time it is worth mentioning that, for the case \(a < 1\), the weak solution of the PPS Forchheimer equation is unique (see [1]) and exists in the corresponding Sobolev space \(W^{1,2-a}\). This result was treated in detail in [1].

In the next section we will introduce a few basic geometric notions and definitions, in order to show the robust link between these two objects. The actual relationship between the two equations is far from being straight-forward, in spite of their similarity. In particular, we will show that the pressure function can be interpreted as a CMC graph under some constraints and appropriate geometric transformations.

VI. The Mean Curvature Equation for a Graph.

Any \(C^2\) map \(r : D \subset \mathbb{R}^2 \to \mathbb{R}^3\) whose differential map \(dr\) is injective is called an immersion (surface immersion) in \(\mathbb{R}^3\). Equivalently, the map \(r\) represents an immersion if its Jacobian has rank 2, or all points are regular. If an immersion \(r\) is 1-1, it is sometimes called an embedding. Any immersed surface can be endowed with a general Riemannian metric \(g = g(x, y)\) ([22], page 418). In the particular case when the metric \(g\) is defined using the usual velocity vector fields, we will call it naturally induced metric (i.e., naturally induced by the immersion). In the most usual notation, \((M, g)\) represents a Riemannian manifold of Riemannian metric \(g\), and \(M = r(D)\).

Consider a smooth surface that can represented as a graph \(z = u(x, y)\) of an open domain \(D \subset \mathbb{R}^2\). This surface is parameterized via the map \(r : D \subset \mathbb{R}^2 \to \mathbb{R}^3\),

\[
r(x, y) = (x, y, u(x, y)).
\]
Definition VI.1. (Natural metric) We will call naturally induced metric the following quadratic differential form:

\[ dr^2(x, y) = |r_x|^2 dx^2 + 2 < r_x, r_y > dx dy + |r_y|^2 dy^2, \]

which can be rewritten as

\[ dr^2(x, y) = (1 + u_x^2) dx^2 + 2 u_x u_y dx dy + (1 + u_y^2) dy^2, \]

where the coefficients

\[ g_{11} = 1 + u_x^2, \quad g_{12} = u_x u_y, \quad g_{22} = 1 + u_y^2 \]

represent the entries of the corresponding matrix \( g \).

Definition VI.2. (Gauss Map) We will call Gauss map the usual unit normal vector field \( N : D \to S^2 \) defined as:

\[ N = \frac{r_x \times r_y}{||r_x \times r_y||} = \frac{-u_x i - u_y j + k}{\sqrt{u_x^2 + u_y^2 + 1}}. \]

Definition VI.3. (Second Fundamental form) The second fundamental form is defined by

\[ d\sigma^2(x, y) = h_{11} dx^2 + 2 h_{12} dx dy + h_{22} dy^2, \]

with the following coefficients

\[ h_{11} := <N, r_{xx}>, \quad h_{12} := <N, r_{xy}>, \quad h_{22} := <N, r_{yy}>, \]

and so it can be expressed as

\[ d\sigma^2(x, y) = \frac{u_{xx}}{\sqrt{u_x^2 + u_y^2 + 1}} dx^2 + 2 \frac{u_{xy}}{\sqrt{u_x^2 + u_y^2 + 1}} dx dy + \frac{u_{yy}}{\sqrt{u_x^2 + u_y^2 + 1}} dy^2. \]

Note that once the local coordinates are given, one can identify the first and second fundamental forms with their corresponding \( 2 \times 2 \) matrices, \( g \) and \( h \), respectively. The matrix operator \( S = g^{-1}h \) can be viewed as a linear operator from the tangent plane to the surface at a given point, to the same tangent plane. \( S \) is usually called the Weingarten map or shape operator (see [22], vol. II). Note that \( g^{ij} \cdot h_{ij} \) represents its trace. The following result is a classical result of differential geometry, which can be found in any text-book (e.g., [22], vol. II), and whose proof is elementary.

Proposition VI.4. For any first and second fundamental forms defined for an immersion \( r = r(x, y) \), the following formula is satisfied:

\[ g^{ij} \cdot h_{ij} = 2H, \]

where \( g \) and \( h \) are matrices corresponding to the first and second fundamental form, and \( H \) represents the mean curvature, defined as the arithmetic mean of the principal curvatures of the immersion \( r \).

This equation is frequently referred to as mean curvature equation.
Definition VI.5. (Laplace-Beltrami operator) The Laplace-Beltrami operator $\Delta_g u$ corresponding to the graph $z = u(x,y)$ and the metric defined in eq (36) is defined as (see [9]):

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \cdot g^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j}.$$

It is worthwhile noting that the Laplace-Beltrami operator is frequently defined without the factor $\frac{1}{\sqrt{\det g}}$; on the other hand, this factor plays an important role in our work. Historically speaking, its original definition is the same as ours. Also, it is important to clarify that the Laplace Beltrami operator of a general surface immersion $r$ is defined component-wise and represents a vector $\Delta_g(r)$, which in our case it reduces to its last component, $\Delta_g(u)$.

[9] was the first well-known reference to make the observation that for surfaces immersed in $\mathbb{R}^3$ the operator $g^{ij} \cdot h_{ij}$ acting at each point coincides with the Laplace-Beltrami operator. In view of this observation and the previous definition, the mean curvature equation (13) can be rewritten as

$$\Delta_g u = 2H.$$

A $C^2$ immersion $r : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with vanishing Laplace-Beltrami operator is said to be harmonic in a generalized (Riemannian) sense.

Remark that for the case when the surface metric is the flat Euclidean one $g$ represents the identity matrix and the Laplace-Beltrami operator becomes the usual Laplace operator $\Delta = \partial_{xx} + \partial_{yy}$.

A well-known result of geometric surface theory stated that an immersion $r$ as above is harmonic if and only if it is minimal [9]. This result agrees with our previous Proposition and Definition.

The mean curvature equation will be referred to as constant mean curvature equation for the case when $H$ is constant. The case of $H = \text{const} \neq 0$ represents the case of CMC surfaces (to be distinguished from the minimal case $H = 0$).

We showed in [2] that one can independently modify the coordinate functions of the velocity vector, in a way that links the CMC equation to the Forchheimer-type equation.

We started from the partial velocities $r_x$ and $r_y$ at the point $P(x,y)$ of the initial graph $r(x,y) = (x,y,u(x,y))$, and we applied the following transformation:

$$r_x \rightarrow \tilde{r}_x := (\chi, 0, \mu(x,y)u_x(x,y))$$

$$r_y \rightarrow \tilde{r}_y := (0, \chi, \mu(x,y)u_y(x,y)),$$

where $\chi$ is a scaling constant and $\mu(x,y)$ is a smooth function.

Note that the notations $\tilde{r}_x, \tilde{r}_y$ do not a priori assume the existence of an immersion $\tilde{r}$ whose partial velocity fields are written in this form. We have studied the conditions in which there exists such a parametric surface. This represents an easy case of the Frobenius theorem, and the existence condition of such an immersion $\tilde{r}$ reduces to the compatibility condition $(\tilde{r}_x)_y = (\tilde{r}_y)_x$ being satisfied for the vectors defined above. For details, see Example 1.2.3, from [10], based on successively applying Picard’s Theorem in the $x$ and $y$-directions, respectively. Assuming that this compatibility condition is satisfied, Example 1.2.3, from [10], shows that for any fixed initial position $P_0$ at the origin $(0,0)$ (or another fixed base point), there
exists a unique solution \( \tilde{r} = (\tilde{x}, \tilde{y}, \tilde{u}(x, y)) \), as an immersion whose partial velocity vectors are \( \tilde{r}_x \) and \( \tilde{r}_y \).

The change \( u(x, y) \to \tilde{u}(x, y) \) represents a smooth deformation of the graph along the \( z \)-axis, while the change \( (x, y) \to (\tilde{x}, \tilde{y}) \), with \( \tilde{z} = \chi x \) and \( \tilde{y} = \chi y \) represents the rescaling of the graph domain from \( D \) to \( \tilde{D} \). This non-trivial transformation modifies the Gauss map (the tangent plane), as well as the shape operator.

**Proposition VI.6.** Consider a smooth graph in \( \mathbb{R}^3 \) that is viewed as an immersion from an open simply connected planar domain \( D \) into the Euclidean space \( \mathbb{R}^3 \) via \( r(x, y) = (x, y, u(x, y)) \).

At every point on \( M = r(D) \) consider the modified velocity vectors defined as:

\[
\tilde{r}_x = (\chi, 0, \mu(x, y)u_x) \quad \text{and} \quad \tilde{r}_y = (0, \chi, \mu(x, y)u_y),
\]

where \( \chi \) is constant and \( \mu(x, y) \) is an arbitrary smooth function such that \( \tilde{r}_x \) and \( \tilde{r}_y \) are non-vanishing and linearly independent and such that the compatibility condition \( \tilde{r}_x y = (\tilde{r}_y)_x \) is satisfied (or equivalently, \( \mu_x u_y = \mu_y u_x \)).

Consider a fixed initial position \( P_0 \) at the origin \((0,0)\) (or base point). Let \( \tilde{r}(x, y) = (\tilde{x}, \tilde{y}, \tilde{u}(x, y)) \) represent the uniquely determined integral surface having \( \tilde{r}_x \) and \( \tilde{r}_y \) as partial velocity vectors, such that \( \tilde{r}(0,0) = P_0 \). These vectors naturally induce the first fundamental form \( \tilde{g} \). The corresponding acceleration vectors together with the Gauss map \( \tilde{N}(x, y) \) naturally induce the second fundamental form \( \tilde{h} \).

Then the corresponding Laplace-Beltrami operator can be interpreted in terms of the trace of the shape operator, that is,

\[
\Delta_{\tilde{g}} \tilde{u} = \tilde{g}^{ij} \tilde{h}_{ij} = 2\tilde{H}.
\]

**Remark VI.7.** Note that \( \tilde{r} \) actually represents a family of immersions of parameter \( \mu \), a real valued smooth function of two variables. The initial immersion \( r \) belongs to this family, corresponding to the case \( \chi = \mu = 1 \). We will call \( z = \tilde{u}(x, y) \) generalized graph, and \( \tilde{g} \) its naturally induced metric.

Example 1.2.3, from [16], based on successively applying Picard’s Theorem in the \( x \) and \( y \)-directions, provides an explicit solution for \( \tilde{r} \) above. Assuming that the compatibility condition is satisfied, we denote by \( \tilde{r} = (\tilde{x}, \tilde{y}, \tilde{u}(x, y)) \) an immersion whose partial velocity vectors are \( \tilde{r}_x \) and \( \tilde{r}_y \). Some computational details can be found in [2]. We therein collect the main information on the first and second fundamental forms of the generalized graph, namely:

We first derive the expressions of the data corresponding to the immersion \( \tilde{r} \). The induced metric is given by

\[
d\tilde{r}^2(x, y) = (\chi^2 + \mu^2 u_x^2) \, dx^2 + 2\mu^2 u_x u_y \, dx \, dy + (\chi^2 + \mu^2 u_y^2) \, dy^2.
\]

The unitary normal vector field is given by

\[
\tilde{N} = \frac{\tilde{r}_x \times \tilde{r}_y}{||\tilde{r}_x \times \tilde{r}_y||} = \frac{-\mu \, u_x \, \mathbf{i} - \mu \, u_y \, \mathbf{j} + \chi \, \mathbf{k}}{\sqrt{\chi^2 + \mu^2 (u_x^2 + u_y^2)}}.
\]
The coefficients of the second fundamental form, as entries of the corresponding matrix $\hat{h}$, are:

$$\hat{h}_{11} = \frac{\chi (\mu u_{xx} + u_x \mu_x)}{\sqrt{\chi^2 + \mu^2 (u_x^2 + u_y^2)}}$$

$$\hat{h}_{12} = \frac{\chi (\mu u_{xy} + u_x \mu_y)}{\sqrt{\chi^2 + \mu^2 (u_x^2 + u_y^2)}} = \frac{\chi (\mu u_{xy} + u_y \mu_x)}{\sqrt{\chi^2 + \mu^2 (u_x^2 + u_y^2)}}$$

$$\hat{h}_{22} = \frac{\chi (\mu u_{yy} + u_y \mu_y)}{\sqrt{\chi^2 + \mu^2 (u_x^2 + u_y^2)}}$$

The corresponding Laplace-Beltrami operator is given by

$$\Delta_{\tilde{y}} = \frac{\mu (\chi^2 + \mu^2 u_y^2) u_{xx} - 2\mu^3 u_x u_y u_{xy} + \mu (\chi^2 + \mu^2 u_x^2) u_{yy}}{\chi (\chi^2 + \mu^2 (u_x^2 + u_y^2))^{3/2}} + \frac{\chi (u_x \mu_x + u_y \mu_y)}{(\chi^2 + \mu^2 (u_x^2 + u_y^2))^{3/2}}$$

which can be rewritten as

$$\Delta_{\tilde{y}} = \nabla \cdot \left( \frac{\mu \nabla u}{\chi \sqrt{\chi^2 + \mu^2 |\nabla u|^2}} \right) = \tilde{\nabla} \cdot \left( \frac{\tilde{\nabla} \tilde{u}}{\sqrt{1 + |\tilde{\nabla} \tilde{u}|^2}} \right) = 2 \tilde{H}.$$ 

Here $\tilde{\nabla} \cdot$ and $\tilde{\nabla}$ are the divergence and the gradient operator in the new reference system $(\tilde{x}, \tilde{y})$.

**Remark VI.8.** In case $\tilde{H}$ is constant, $\tilde{u}$ is a CMC graph with respect to the domain $\tilde{D}$, provided a solution to (52) exists.

The following theorem is an immediate consequence (see [2] for details of the proof):

**Theorem VI.9.** Consider a smooth graph $u(x, y)$ in the domain $D(x, y)$, solution of

$$\nabla \cdot (K(|\nabla u|) \nabla u) = -A,$$

and such that

$$(u_x u_{xy} + u_y u_{yy}) u_x = (u_x u_{xx} + u_y u_{yx}) u_y.$$ 

Let $\tilde{D}(\tilde{x}, \tilde{y})$ be the scaled domain obtained by the conformal mapping $\tilde{x} = \chi x$ and $\tilde{y} = \chi y$. Let $\tilde{u}(\tilde{x}, \tilde{y})$ be the stretched graph, which is parameterized as the immersion $\tilde{r}$ with partial velocities $(\tilde{r}_x, \tilde{r}_y)$ given by (43) and (44) and

$$\mu(\chi, |\nabla u|) = \frac{-\chi K(|\nabla u|)}{\sqrt{1 - \chi^2 K(|\nabla u|)^2 |\nabla u|^2}},$$

$$\chi < \chi_{\text{max}} = \frac{1}{|K(|\nabla u|) \nabla u|_{\max}} = \frac{1}{|v|_{\max}}.$$ 

Then $\tilde{u}(\tilde{x}, \tilde{y})$ is solution of the corresponding CMC equation

$$\tilde{\nabla} \cdot \left( \frac{\tilde{\nabla} \tilde{u}}{\sqrt{1 + |\tilde{\nabla} \tilde{u}|^2}} \right) = A.$$
Remark VI.10. Condition (57) assures \( \mu \) to be a real smooth valued function.

Condition (54) assures the compatibility condition

\[
(\mu(\chi, |\nabla u|)u_x)_y = (\mu(\chi, |\nabla u|)u_y)_x
\]

which are necessary and sufficient conditions for \( \tilde{u} \) to exist. Reformulated, (54) states that each level curve of the graph of \( u \) (at \( z = c_1 \)) represents a level curve of the graph of \( |\nabla u| \) (at \( z = c_2 \)).

Replacing the generic function \( \mu \) with (55) specified in the statement of this theorem into equation (52) immediately gives

\[
\hat{\nabla} \cdot \left( \frac{\hat{\nabla} \tilde{u}}{\sqrt{1 + |\nabla \tilde{u}|^2}} \right) = -\nabla \cdot (K(|\nabla u|)\nabla u) = A
\]

This equality practically maps solutions of the Forchheimer equation (53) to solutions of the CMC equation (52) through an explicit transformation, under a certain natural assumption.

Proposition VI.11. Consider a smooth graph \( u(x,y) \) solution of the PSS Forchheimer equation (53) and satisfying (54). Consider the associated CMC graph \( \tilde{u}(x,y) \) solution of (57) with \( \mu \) given by (55) and \( \chi < \chi_{max} \). Let \( \eta = |\nabla u|, \xi = |\nabla \tilde{u}|, \) and \( \tau = \frac{\xi}{\sqrt{1 + \xi^2}} \), then

\[
\eta = g(|v(\tau, \chi)|)|v(\tau, \chi)|,
\]

where

\[
|v(\tau, \chi)| = \frac{\tau}{\chi}.
\]

Here \( v \) is the velocity of the fluid flow in the porous media. As a consequence

\[
u_x = -\frac{\eta u_x}{\xi}, \\
u_y = -\frac{\eta u_y}{\xi}
\]

Proof. The proof is based on the vector identity

\[
\mu(|\nabla u|)\nabla u = \nabla \tilde{u},
\]

which implies the scalar equality

\[
(\mu(|\nabla u|)|\nabla u|^2 = |\nabla \tilde{u}|^2.
\]

Taking into account equation (55) for \( \mu \), and substituting \( \eta \) for \( |\nabla u| \) and \( \xi \) for \( |\nabla \tilde{u}| \), we obtain

\[
\frac{(\chi K(\eta)\eta)^2}{1 - (\chi K(\eta)\eta)^2} = \xi^2.
\]

Let us consider \( \tau = \chi K(\eta)\eta \), which is positive. Substituting and solving for \( \tau \) in (63), we obtain

\[
\tau(\xi) = \frac{\xi}{\sqrt{1 + \xi^2}}.
\]
Recollecting Eq. (17) for the velocity field, it follows
\begin{equation}
|v(\tau,\chi)| = K(\eta)\eta = \frac{\tau}{\chi}.
\end{equation}

Now by using the g-Forchheimer Eq. (II.1), it follows Eq (59). Clearly, the correspondence between \(\xi\) and \(\eta\) is one-to-one. \(\square\)

VI.1. Application to reservoir engineering. The following proposition was proved in [4]. We recall it here only because the proposed solution strategy for the evaluation of the productivity index is among the direct applications of the results hereby presented.

**Proposition VI.12.** For the GPPC case the PSS Productivity Index can be computed as
\begin{equation}
(66) \quad PI(a_i,\alpha_i,|v|) = Q^2 \int_U g(|v|)|v|^2dx = \frac{Q^2}{\int_U \sum_{j=0}^{k} a_j|v|^\alpha_j + 2dx}.
\end{equation}

Here \(PI\) is time invariant, and it depends explicitly on \(a_i,\alpha_i,|v|\) and domain \(U\).

This result combined with previous theorem (in particular with Eq. (60) for the velocity) expresses the fact that the Productivity Index of the well can be evaluated in the following way, provided that condition (54) is verified.

**Solution strategy for the evaluation of the Productivity Index:**

1. The factor \(\chi\) is selected to generate the scaled domain \(\tilde{D}\).
2. The CMC equation (57) is solved for \(\tilde{u}(x,y)\) on the domain \(\tilde{D}\) for \(Q\), with the appropriate boundary conditions.
3. The gradient norm \(\xi = |\nabla \tilde{u}|\) is evaluated.
4. The coefficients \(a_i,\alpha_i\), \(i = 0,\cdots,k\) of the GPPC polynomial are selected.
5. The norm of the velocity \(|v|\) is explicitly evaluated through Eq. (60).
6. The Productivity Index \(PI(a_i,\alpha_i,|v|)\) is evaluated by using formula (66).

**Remark VI.13.** Points 1 and 2 do not depend by the choice of \(a_i\) and \(\alpha_i\) which means that the evaluation of \(\xi\) in 3 is \(a_i\) and \(\alpha_i\) independent. We need to solve just one BVP. The Productivity Index can be evaluated a posteriori for any choice \(a_i\) and \(\alpha_i\).

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