Abstract. If a real harmonic function inside the open unit disk $B(0,1) \subset \mathbb{R}^2$ has its level set \( \{ x : u(x) = u(0) \} \) diffeomorphic to an interval, then we prove the sharp bound \( \kappa \leq 8 \) on the curvature of the level set \( \{ x : u(x) = u(0) \} \) in the origin. The bound is sharp and we give the unique (up to symmetries) extremizer.

1. Introduction and statement of result

1.1. Introduction. Level sets of solutions of partial differential equations are well studied. One of the first results is given in Ahlfors [1] and states that the level curves of the Green function on a simply connected convex domain in the plane are convex Jordan curves. This result is a cornerstone and has been extended in various directions [2, 5, 6], there exists a vast literature on the subject. Results of a more qualitative nature usually state that provided a level set is convex, some notion of curvature (Gauss-curvature, principal curvature or a product expression containing either of them and some power of the norm of the gradient) assumes its minimum at the boundary. Early results in this direction are due to Ortel-Schneider [10] and Longelli [7, 8], recent results of this type were given by Chang-Ma-Yang [3], Ma-Ye-Ye [9] and Zhang-Zhang [11].

1.2. Statement. We are interested in the opposite question: bounds on the maximal curvature of a level set in the interior. We were surprised to learn that this question seems to not have been answered even in the most basic setting of harmonic functions inside the unit disk. We obtain the sharp result.

Theorem. Let \( u : B(0,1) \to \mathbb{R} \) be harmonic. If \( \gamma := \{ x : u(x) = u(0) \} \) is diffeomorphic to an interval, then the curvature of \( \gamma \) in the origin satisfies
\[
\kappa \leq 8.
\]
The bound is sharp and attained for the unique extremizer (up to symmetries)
\[
w(x,y) = \frac{(x^2 + y^2 - 1)(x - 2x^2 + x^3 - 4y^2 + xy^2)}{(1 - 2x + x^2 + y^2)^3}.
\]
The bound \( \kappa \leq 24 \) was recently proven by De Carli & Hudson [4]. The topological assumption on \( \gamma \) is necessary: if there is no bound on the number of connected components of the level set, then the curvature may be unbounded: for example, \( e^{ax} \cos (ay) \) with a large real parameter \( a \) (this example was communicated to me by Wei Zhang). Note that there are two symmetries (rotation and multiplication with scalars) that leave the curvature of the level sets invariant – the fact that dilation has an obvious effect on curvature implies that any extremizing function must have a singularity at the boundary of the disk (which implies that dilating the function would move the singularity in the interior such that the function would no longer be harmonic inside the disk).

1.3. Extensions. This result motivates many related questions: suppose \( u \) solves
\[
-\Delta u = f
\]
inside the unit disk and that, as before, the level set \( \{ x : u(x) = u(0) \} \) is diffeomorphic to an interval. Is there an inequality of the type \( \kappa \lesssim 1 + \| f \|_X \) for some suitable norm and \( f \) from a suitable function class? Our result suggests that for bounds on the interior curvature of a level set one cannot do without some topological assumptions on the structure of the level set. Other natural questions are bounds for arbitrary elliptic differential operators depending only on the ellipticity
constants, $L^p$-norms of the curvature function integrated over the entire ball, and so on.

We would like to draw attention to two elementary questions in particular.

1. Suppose for $B(0,1) \subset \mathbb{R}^n$ that $u : B(0,1) \to \mathbb{R}$ is harmonic and $\{ x : u(x) > u(0) \}$ is simply connected: what bounds can be proven on the $(n-1)$-dimensional measure of the level set going through the origin? The question already seems difficult for $n = 2$.

2. Suppose $u : B(0,1) \subset \mathbb{R}^2$ is harmonic and $\{ x : u(x) > u(0) \}$ is simply connected. Is it true that the best constant $c > 0$ in the inequality

$$\left| \{ x \in B(0,1) : u(x) \geq u(0) \} \right| \geq c \left| \{ x \in B(0,1) : u(x) \leq u(0) \} \right|$$

is assumed if and only if $u = w$ (i.e. the extremizer from our main result)?

1.4. Outline of the proof. We study harmonic functions inside the unit disc which might not have a continuous extension to the boundary. Indeed, the function that will turn out to be the extremizer

$$w(x, y) = \frac{(x^2 + y^2 - 1)(x - 2x^2 + x^3 - 4y^2 + xy^2)}{(1 - 2x + x^2 + y^2)^3},$$

has a singularity in $(1,0)$. This motivates a proof by contradiction: assume $u : B(0,1) \to \mathbb{R}$ to be a function harmonic inside the unit disc with $\{ x : u(x) = u(0) \}$ diffeomorphic to an interval and the curvature $\kappa$ of that curve in the origin satisfying $\kappa > 8$. Let $r = 1 - \varepsilon$ and

$$\tilde{u}(x, y) := u(rx, ry).$$

Then, for $\varepsilon > 0$, this yields a counterexample $\tilde{u}$ to the inequality $\kappa \leq 8$ for a function $\tilde{u}$ having a continuous extension to the boundary. This first reduction proves useful.

As for the underlying main argument, the way it should be understood is as follows: we base it on two monotonicity statements. Suppose we are given a harmonic function $u$: there is a connected set on the boundary on which it is positive. If we replace this positive mass by a Dirac point mass, then the curvature in the origin increases.

![Figure 1. The first monotonicity: concentrating the entire positive part of the mass in one point increases curvature.](image)

The second monotonicity statement says that it is equally advantageous to move the negative mass closer to the positive (point-)mass: this also increases the local curvature. This is the intuition that motivates the construction of the proof, which is slightly different in structure. Restricting to the boundary as well as doing a Poisson extension of boundary data to the entire are both linear operations. Adding two harmonic functions, if one so pleases, can thus be regarded as

$$u + v \quad \Rightarrow \quad u|_{\partial B} + v|_{\partial B} \quad \Rightarrow \quad P(u|_{\partial B} + v|_{\partial B}).$$

This interpretation may seem complicated at first but indeed has a few selective advantages: in particular, by choosing $v$ to be a harmonic function depending on the boundary data of $u$, we may thus identify the process of moving $L^1$-data of $u$ on the boundary with that of adding a harmonic function. The actual proof will proceed by showing that any harmonic function satisfying the requirements can be written as the extremizing function $w$ from the theorem and the superposition
of harmonic functions from a certain set \( F \). Note that if a function \( u \) is normalized to have its gradient in \((0, 0)\) be of the form \( \nabla u(0, 0) = (\sigma, 0) \) for some \( \sigma > 0 \), then the curvature can also be written as
\[
\kappa = 2 \sup \left\{ a \in \mathbb{R} : \lim_{y \to 0} \frac{u(ay^2, y)}{y^2} \leq 0 \right\}.
\]
This equation immediately leads to the final step in the argument: show that indeed the extremizer \( w \) as well as all functions in \( F \) have the curvature of their level set in the origin bounded by \( \pi \). The same must then hold true for any superposition. A more careful analysis shows that indeed the curvature of the level set of any element in \( F \) is smaller than \( \pi \): this gives the case of equality.

2. Proof of the theorem

2.1. Preliminary observations. We assume w.l.o.g. that \( u(0, 0) = 0 \) and that \( u \) has a continuous extension to the boundary. Furthermore, we can assume after possibly rotating the disk that \( \nabla u(0, 0) = (\sigma, 0) \) for some \( \sigma > 0 \). Let \( I \subset \partial B \) be the positivity set \( \{ x \in \partial B : u(x) \geq 0 \} \).

Claim 1. We have \((1, 0) \in I\).

Proof. We identify the boundary of the unit disk with the torus \( \mathbb{T} \) of length \( 2\pi \). Using the explicit representation of the Poisson kernel, the statement reduces to the following: let \( \phi : \mathbb{T} \to \mathbb{R} \) be continuous, of mean 0, with precisely two roots \( \phi(a) = 0 = \phi(b) \) for \( a < 0 < b \) and \( \phi(0) > 0 \).

\[
\text{Figure 2. An example: the function } \phi \text{ is positive on the bold part of the boundary and negative everywhere else. The averaged vector lies inside the cone.}
\]

Then the vector
\[
\left( \int_0^{2\pi} \phi(x) \cos(x) dx, \int_0^{2\pi} \phi(x) \sin(x) dx \right)
\]
is contained in the cone given by connecting the origin with \((\cos a, \sin a)\) and \((\cos b, \sin b)\). The result follows immediately from using the linearity of the scalar product. \( \square \)

Note, however, that there is no uniform lower bound on the norm
\[
\left\| \left( \int_0^{2\pi} \phi(x) \cos(x) dx, \int_0^{2\pi} \phi(x) \sin(x) dx \right) \right\|,
\]
which corresponds to the fact that the invariance of the level sets under multiplication with scalars requires any optimal statement to not demand any lower bound on \( \| \nabla u(0, 0) \| \) in the main statement.

2.2. The transport maps. We introduce a two-parameter family of harmonic functions \( g_{a,b} \) that will serve as a way for us to transport mass on the boundary. They are defined as a Poisson extension of measures on the boundary. Given two numbers \(-\pi < a < 0 < b < \pi\), we define the measure
\[
\mu_{a,b} = \delta_{(1,0)} - \left( \frac{1}{1 - \frac{\sin a}{\sin b}} \right) \delta_{(\cos a, \sin a)} - \left( \frac{\sin a}{\sin b} \right) \delta_{(\cos b, \sin b)}.
\]
where $\delta_{(x,y)}$ is the Dirac measure at $(x,y)$. The function $g_{a,b} : B(0,1) \to \mathbb{R}$ is defined as the Poisson extension of $\mu_{a,b}$. These functions have several properties (which are easily verified using the Poisson kernel):

1. $g_{a,b}(0,0) = 0$,
2. $\nabla u(0,0) = (\sigma, 0)$ for some $\sigma > 0$,
3. the level set $\{x : g_{a,b}(x) = 0\}$ is a curve connecting $(\cos a, \sin a)$ and $(\cos b, \sin b)$,
4. and the curvature of $\{x : g_{a,b}(x) = 0\}$ in the origin is $2(1 + \cos (a) + \cos (b) + \cos (a+b)) \leq 8$.

Note that the Poisson extension of the measure $\mu$ is given as

$$g_{a,b}(x,y) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - x^2 - y^2}{(x - \cos t)^2 + (y - \sin t)^2} d\mu$$

and since $\mu$ is merely the linear combination of three Dirac Delta measures, the function $g_{a,b}$ is a rational function.

### 2.3. An implication.

The purpose of this section is to show how the previously introduced transport maps can be used to prove an implication: instead of proving the main statement for all functions, it suffices to prove it for all functions from a restricted class $F$. We define $F$ as follows:

- let $I \subset \mathbb{T}$ be a closed interval containing some neighbourhood of the origin and let $\phi : \mathbb{T} \to \mathbb{R}$ be a continuous, nonpositive function that vanishes precisely on $I$ and let $u$ be given as the Poisson extension of the measure

$$\mu = \left( \int_{\mathbb{T}} -\phi(x)dx \right) \delta_{(1,0)} + \phi(x)dx,$$

and assume additionally that it satisfies $\nabla u(0,0) = (\sigma, 0)$ for some $\sigma > 0$. Then $u \in F$.

**Claim 2.** If the curvature of the level set through the origin is bounded by 8 for functions in $F$, then it holds true for all functions.

**Proof.** Let the function $u$ be real-harmonic with continuous extension $\phi$ to the boundary and the curvature of its level set through the origin exceeding 8. Let us suppose that the endpoints of the positivity interval $I = \{x \in \partial B : u(x) \geq 0\}$ are given by $(\cos a, \sin a)$ and $(\cos b, \sin b)$ for $a < 0 < b$. We consider a one-parameter family of functions $u_t$ (where $a < t < b$) defined as the Poisson extension of the measure

$$\mu_t = \left( \int_{\mathbb{T} \setminus I} -\phi(x)dx \right) \delta_{(\cos t, \sin t)} + \phi(x)dx.$$

The mean-value theorem suggests that there exists a $t$ such that $\nabla u_t(0,0)$ and $u(0,0)$ are scalar multiples of each other. We fix this $t$ and rotate the function $u$ in such a way that $t = 0$. We will use the symbol $u_0$ to denote the particular function in the set $\{u_t : a < t < b\}$ with the property that $\nabla u_t(0,0)$ and $u(0,0)$ are scalar multiples of each other after it has been rotated in the same way as $u$. 

![Figure 3](image.png)

**Figure 3.** The function $u_0$: the negative measure lives on an interval while all positive measure is concentrated in $(1,0)$. An additional assumption is that the gradient in the origin points towards $(1,0)$.
Now we start the transport of mass. Note that we do not require the transport to be ‘optimal’ in any way and have some flexibility with regards to which order one performs the operations: we describe one possibility completely. Set \( u_1 = u \). Each step of the operation consists of adding a (positively) weighted average over the set \( g_{a,b} \) to the function \( u_n \) to produce the function \( u_{n+1} \) in such a way that \( u_{n+1} \geq 0 \) on \( I \) holds. We start with

\[
  u_2 = u_1 + \int_0^a \max(0, \min(\phi_1(-a), \phi_1(a)))g_{-a,a}da.
\]

This produces a function \( u_2 \) with the property that it’s extension to the boundary \( \phi_2 \) is continuous and \( \phi_2(a) > 0 \implies \phi_2(-a) \leq 0 \). We continue the construction in the most natural way. Let \( (a_n) : \mathbb{N}_{\geq 2} \to \mathbb{N}_{\geq 1} \times \mathbb{N}_{\geq 1} \) be a bijective enumeration such that \( a_2 = (1,1) \). We will use the notation \( \max a_n = \max((a_n)_1, (a_n)_2) \) to denote the larger of the two entries. Now we construct \( u_{n+1} \) via

\[
  u_{n+1} = u_n + \int_0^{\max a_n} \max(0, \min(\phi_n(-(a_n)_1 z), \phi_n((a_n)_2 z)))g_{-(a_n)_1 z,(a_n)_2 z}dz.
\]

It is easy to see that every \( u_n \) is harmonic and has a continuous extension \( \phi_n \) to the boundary. It is also trivial from the definition of the integral to see that \( \phi_n \geq 0 \) on \( I \). We now claim that

\[
  \lim_{n \to \infty} \phi_n = \phi_0 \quad \text{in the sense of distributions.}
\]

It suffices to check that it is not possible for a positive portion of the \( L^1 \)-mass of \( \lim_{n \to \infty} \phi_n \) to stay outside of \((1, 0)\). If this was the case, then the structure of the rearrangements implies that this hypothetical chunk of positive \( L^1 \)-mass is either entirely contained in \( I \cap \{ (\cos a, \sin a) : -\pi < a < 0 \} \) or in \( I \cap \{ (\cos a, \sin a) : 0 < \pi < a \} \). This, however, contradicts the construction of \( u_0 \) via demanding the gradients in the origin to be collinear. By assumption, the curvature of the level set of \( u_0 \) in the origin is bounded by \( 8 \). Since we have only added functions with curvature bounded from above by \( 8 \), we have shown the statement.

**Remark.** Let us give a pictorial description of the proof. Suppose we are given \( L^1 \)-mass on an interval. Then we fix a point \( p \) and take mass simultaneously from the left of \( p \) and the right of \( p \) and move it towards \( p \). This procedure is continued while it is possible to do so. Of course, once all the remaining mass is to the left of \( p \) or to the right of \( p \), we must stop. The construction of \( u_0 \) was carried out in such a way (or, in this analogy, the point \( p \) was chosen) that the procedure can be carried out until there is no mass on either side of the point \( p \).

![Figure 4](image-url)

**Figure 4.** Moving mass from both sides of \( p \) towards \( p \).

### 2.4. Some orientation.

We now calculate a candidate for the extremizer. Considering the section above, the natural candidate is \( \lim_{a \to 0} g_{a,-a} \). This sequence of function tends weakly to zero in the interior of \( B(0, 1) \). However, using the explicit form of the Poisson kernel, it is not difficult to compute the asymptotic order of vanishing to be of order \( \sim a^2 \) and

\[
  \lim_{a \to 0} \frac{g_{a,-a}(x,y)}{a^2} = \frac{(x^2 + y^2 - 1)(x - 2x^2 + 3x^3 - 4y^2 + xy^2)}{(1 - 2x + x^2 + y^2)^3}.
\]

This function is now rather curious and can best be thought of as having a triple singularity in \((1, 0)\): it is the Poisson extension of a positive Dirac mass surrounded on each side with a negative Dirac mass of half the mass (and then, in the limit, moved infinitely close). A physical
interpretation seems to be the following: the mean-value property prohibits the positive $L^1$-mass on the boundary to outweigh the negative $L^1$-mass; in order to achieve the biggest curvature of the level set, it is optimal to concentrate all the positive $L^1$-mass in the smallest possible space and symmetrically surround it by negative $L^1$-mass. Metaphorically, if you have a space heater and two snowmen, the curvature of the isothermic through the origin is going to be maximal if one puts the snowmen to the immediate left and the right of the radiator.

2.5. The final steps. This section concludes the proof. As it turns out, we will be able to entirely copy the structure of the proof of Claim 2.

**Claim 3.** The statement is true for functions in $F$.

At this point, it might seem natural to proceed as before and define transport maps to move the negative mass closer to the point $(1, 0)$ in such a way that the direction of $\nabla u(0, 0)$ does not change. This could indeed be done as the curvature will indeed monotonically increase under this rearrangement process. However, there is a simpler argument.

![Figure 5. Moving negative mass closer to the positive mass and increasing curvature.](image)

We suspect the extremizer to be the limit of a positive Dirac surrounded symmetrically by negative Dirac’s of half the mass. The transport maps $g_{a,b}$ take mass from $(\cos a, \sin a)$ and $(\cos b, \sin b)$ and moved it to $(1, 0)$. We are now interested in new transport maps, which move the mass from $(\cos a, \sin a)$ and $(\cos b, \sin b)$ to $(\cos \varepsilon, \sin \varepsilon)$ and $(\cos \varepsilon, -\sin \varepsilon)$ and then, ideally, let $\varepsilon$ tend to 0. In the limit $\varepsilon \to 0$, these maps would then converge to the old transport maps $g_{a,b}$. It is thus possible to reuse the transport maps for this problem as well. Note that simply applying the transport maps $g_{a,b}$ to a function of $u$ to move negative measure to $(1, 0)$ would ultimately lead to a cancellation of the positive and the negative mass and leave us with the function, which is identically 0 everywhere in the interior. The problem of having to move the mass first very close to $(1, 0)$ and then – in a limit procedure – ‘infinitesimally close’ corresponds to the fact that we had to obtain our (so far putative) extremizers via a limit procedure.

**Proof of Claim 3.** We give a proof of Claim 3 that does not use the previously described shortcut but follows a construction analogous to the one employed in the proof of Claim 2. First we require new transport maps: we define $g_{a,b,\varepsilon}$ as the Poisson extension of the measure

$$
\mu_{a,b,\varepsilon} = \delta_{(\cos \varepsilon, \sin \varepsilon)} + \delta_{(\cos \varepsilon, -\sin \varepsilon)} - \left(\frac{2}{1 - \sin \varepsilon}ight) \delta_{(\cos a, \sin a)} - \left(-\frac{2 \sin a}{1 - \sin \varepsilon}\right) \delta_{(\cos b, \sin b)},
$$

which corresponds to moving the mass (in unequal terms) away from $(\cos a, \sin a)$ and $(\cos b, \sin b)$, subdividing it into two equal parts and moving it to $(\cos \varepsilon, \sin \varepsilon)$ and $(\cos \varepsilon, -\sin \varepsilon)$. These functions have the properties properties

1. $g_{a,b,\varepsilon}(0, 0) = 0$,
2. $\nabla g_{a,b,\varepsilon}(0, 0) = (\sigma, 0)$ for some $\sigma > 0$,
3. the level set $\{x : g_{a,b,\varepsilon}(x) = 0\}$ is a curve connecting $(\cos a, \sin a)$ and $(\cos b, \sin b)$
4. and the curvature of $\{x : g_{a,b,\varepsilon}(x) = 0\}$ in the origin is bounded from above by 8.
The first and third statement follow from the mean-value theorem while the second and the fourth statement follow from the following observation: we may write
\[
\mu_{a,b,\varepsilon} = 2\mu_{a,b} + \left(\delta_{(\cos\varepsilon, \sin\varepsilon)} + \delta_{(\cos\varepsilon, -\sin\varepsilon)} - 2\delta_{(1,0)}\right)
\]
and then study the measure in the bracket. Its Poisson extension \( h_{a,b,\varepsilon} \) can again be explicitely computed and we get that the curvature of the level set in the origin is given by
\[
\kappa(h_{a,b,\varepsilon}) = 2\left(1 - \cos 2\varepsilon\right) \frac{1 - \cos \varepsilon}{1 - \cos \varepsilon} \leq 8
\]
with equality only for \( \varepsilon = 0 \). This yields the desired properties.

Using these transport maps, we can reiterate the previous transport process with a minor modification. Fix some small \( \varepsilon > 0 \). We set \( u_1 = u \) and construct \( u_{n+1} \) via
\[
u_{n+1} = u_n + \int_0^{a_n} \max(0, \min(\phi_n(-(a_n)_1z + \varepsilon), \phi_n((a_n)_2z + \varepsilon))) g(-(a_n)_1z + \varepsilon, (a_n)_2z + \varepsilon) dz.
\]
This process moves negative mass in a symmetric way to \((\cos \varepsilon, \sin \varepsilon)\) and \((\cos \varepsilon, -\sin \varepsilon)\). We take a limit and up with boundary data that consists of a continuous part and three Dirac measures: one in \((1,0)\) and two negative one’s at \((\cos \varepsilon, \sin \varepsilon)\) and \((\cos \varepsilon, -\sin \varepsilon)\). Now we define a new variable \( \varepsilon_1 = \varepsilon/2 \) and reiterate the process.

\[\square\]

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