Spatial organisation of physical form of an urban system, or city, both manifests and influences the way its social form functions. Mathematical quantification of the spatial pattern of a city is, therefore, important for understanding various aspects of the system. In this work, a framework to characterise the spatial pattern of urban locations based on the idea of entropy maximisation is proposed. Three spatial length scales in the system with discerning interpretations in terms of the spatial arrangement of the locations are calculated. Using these length scales, two quantities are introduced to quantify the system’s spatial pattern, namely mass decoherence and space decoherence, whose combination enables the comparison of different cities in the world. The comparison reveals different types of urban morphology that could be attributed to the cities’ geographical background and development status.

Keywords: Percolation; Urban morphology; Entropy; Complexity; Networks

Cities as complex systems [1–4] have been a topic of research beyond the traditional discipline of urban studies. The idea of complexity in cities arises from the fact that they comprise many entities interacting with one another locally and generating global emergent patterns. Those interactions and the associated patterns have been shown to exhibit properties e.g. [3] similar to those observed in theoretical models developed in the fields of Statistical Physics or Mathematics. Quantitative tools from these fields, therefore, can be fruitfully applied toward constructing a framework for Science of cities.

Of the many aspects of studying cities, the spatial organisation of physical form, i.e. infrastructure elements, in a city provides a fundamental understanding of the city’s way of life. Various methods have been employed to tackle the problem of characterising spatial patterns of urban systems, including fractal dimension [2], spatial entropy [4], and street networks [1, 8], or even entropy of population density [9]. Among them, percolation has proved to be a powerful and useful tool to study urban morphology [10]. In recent years, percolation method has becoming increasingly popular in analysing the spatial organisation of places in urban systems at various scales, from city [11, 12], to nation [13] and inter-country level [14]. The application of percolation in such studies has so far been mainly concerned with studying the evolution of the giant cluster formed when the distance threshold $\rho$ for inter-point interaction [13] changes. The growth of such cluster involves a transition from a segregate state where points are disconnected to an aggregate state in which a path exists between a pair of points located at opposite ends of the system. The identification of such transition regime is normally done via rate of growth of the giant cluster as the distance threshold increases. The profiles of such growth can be divided into 3 parts, namely slow growth, rapid growth and stabilisation (Fig. 1). At small values of distance threshold, most clusters are localised due to limited connections with other nearby points. As $\rho$ increases, points can have access to farther neighbours, making small clusters merge to form larger ones. When $\rho$ is sufficiently large, a dominant cluster emerges and rapidly grows within a narrow range of $\rho$, known as transition regime [16]. After this regime, the dominant cluster, also called a giant cluster, starts to stabilise as it has already grasped most of the points and only grows slowly until no further expansion is possible, i.e. all points now belong to a single, unified cluster with a path existing between any pair of points in the set.

Traditionally, theoretical study of percolation on regularly spaced lattices provides procedures to quantify the transition regime and characterise it in the framework of universality classes [16, 17]. Extending to continuous space, continuum percolation theory relaxes the position of points and studies their properties, including the conditions for existence of the giant cluster under different settings [18, 19]. While a number of studies have been devoted to estimate the value of percolation threshold, especially in thermodynamic limit for theoretical systems e.g. [20, 21]. much less focus has been put on determining the transition in a finite set of points, which could appear very fuzzy, especially in real data. Such result is particularly useful for practical applications like quantitative urban morphology, where data are always bounded.

This study, therefore, aims to present a framework to examine the transition in the context of continuum percolation of a finite set of points, by identifying different length scales associated with different states of the system as the distance threshold $\rho$ changes. Firstly, drawing upon the an important property of percolation that physical quantities (e.g. correlation length) diverge, i.e. lack of length scale, at the critical point, it could be paralleled that the variance of cluster size (number of points in a cluster) maximises when the system experiences the most abrupt change in its state. In other words, the val-
FIG. 1. Growth of largest cluster in a set of points as the range of interaction among points increases. Three stages of growth could be observed, namely slow growth, rapid growth and stabilisation. As will be discussed in the text later, the window of interaction among points increases. Three stages of growth can be determined by two length scales that signify the maximisation of entropy measures of connected clusters \((\rho_0, \rho_C, \rho_S)\) and robust components of the giant cluster \((\rho_S, \text{blue dotted curve})\) as range of interaction \(\rho\) changes. The rapid growth window also encompasses the transition of the system from segregate to aggregate state, represented by the percolation threshold \(\rho_C\) at which the sizes of the clusters are most diverse (green dash-dot curve).

Values of cluster size is most spread when the system transits across a “critical” point differentiating the aggregate and segregate state in the system \((\rho_C)\).

To make things concrete, let us consider a set of \(N\) points in a two-dimensional domain \(\mathbb{R}^2\). Given a distance threshold \(\rho\), the set is divided into \(n\) clusters of size \(\xi_i\), which sum up to \(N\), i.e.

\[
\sum_{i=1}^{n} \xi_i = N, \tag{1}
\]

and whose variance is given by \(\sigma^2 = \langle \xi^2 \rangle - \langle \xi \rangle^2\). The value of distance threshold at which the variance \(\sigma^2\) maximises is denoted \(\rho_C\) to mark the critical point in the transition of the system (see green dash-dot curve in Fig. I). This value is analogous with the percolation threshold in the classic percolation theory. As with percolation theory, the percolation threshold itself is not sufficient in characterising the phase transition in the system. Rather, the manner of transition is more important with many interesting properties. In what follows, it will be shown that the window of transition could be characterised by employing the measures of entropy. In particular, the measures of entropy can be used to quantify the pattern of clusters formed at every value of distance threshold and identify the length scales at which the entropy measures maximise. As will be argued later, these length scales correspond to the change of state of spatial agglomeration in the set of points.

For the clusters in Eq. (1), the probability of choosing a random point \(a\) that belongs to a cluster \(C_i\) of size \(\xi_i\), also the probability of picking the cluster \(C_i\) itself, is simply given by the fraction of points in that cluster, \(p_i = p(a \in C_i) = \frac{\xi_i}{N}\). With this, we can easily calculate the Shannon entropy of the particular cluster division in Eq. (1)

\[
S = -\sum_{i=1}^{n} p_i \log p_i = -\sum_{i=1}^{n} \frac{\xi_i}{N} \log \frac{\xi_i}{N}. \tag{2}
\]

It could be seen from Eq. (2) that when there is a dominant cluster \(C_{i^*}\) of very large size alongside several tiny clusters of vanishingly small sizes (which are yet to be absorbed into the giant cluster), the entropy is close to 0 since \(\log \frac{\xi_{i^*}}{N} \approx 0\) and \(\frac{\xi_i}{N} \approx 0\), \(\forall i \neq i^*\). This reflects the state of division that the set of \(N\) points is barely fragmented, where most of them belong to a single, unified cluster. On the other hand, it is a well-known fact for Shannon entropy formula that given \(n\) events, the respective entropy is maximised when each of them takes place with equal probability \(\frac{1}{n}\), which simply yields

\[
\max(S) = -\sum_{i=1}^{n} \frac{1}{n} \log \frac{1}{n} = \log n, \tag{3}
\]

i.e. the scenario of dividing the set of \(N\) points into \(n\) equal clusters. This is the state of maximal uncertainty since any of the clusters can be picked with equal probability. Equation (3) also indicates that the upper bound of entropy measure for \(n\) events increases with the number of events. This points to the fact that maximal possible entropy in the system is \(S_{\max} = \log N\) when there are \(N\) clusters, each of size 1 and being picked with equal probability \(\frac{1}{N}\). This corresponds to a state of being totally fragmented when each point forms its own cluster. In other words, the Shannon entropy in Eq. (2) can be interpreted as a measure of fragmentation in the set of \(N\) points. This is considered “first-order” measure in the sense that the formula operates directly on the size fraction of the individual clusters. In the following, we consider another measure that operates on the size distribution of the clusters.

We again consider a set of \(N\) points being divided into \(n\) clusters of size \(\xi_i\), which consist of points within dis-
tance \( \rho \) of one another. Let’s denote \( m(\xi) \) the number of clusters having size \( \xi \) so that we have

\[
\sum_{\xi=1}^{\xi_{\text{max}}} m(\xi)\xi = N, \tag{4}
\]

in which \( \xi_{\text{max}} \) is the size of the largest cluster. The probability of randomly choosing a point that belongs to a cluster of size \( \xi \) is given by \( P(\xi) = \frac{m(\xi)\xi}{N} \). With this, the entropy of cluster sizes could be calculated as

\[
\chi = -\sum_{\xi=1}^{\xi_{\text{max}}} P(\xi) \log P(\xi) = -\sum_{\xi=1}^{\xi_{\text{max}}} \frac{m(\xi)\xi}{N} \log \frac{m(\xi)\xi}{N}. \tag{5}
\]

When there are several tiny clusters of vanishingly small sizes alongside a dominant cluster \( C_{\rho} \) of very large size, i.e. very large \( \rho \), the entropy \( \chi \) is close to 0 since

\[
\log \frac{m(\xi_{\star})\xi_{\star}}{N} \approx 0 \quad (\text{for } m(\xi_{\star}) = 1 \text{ and } \xi_{\star} \approx N) \quad \text{and} \quad \frac{m(\xi)\xi}{N} \approx 0, \forall \xi \neq \xi_{\star}. \]

At the other extreme, when every point forms a cluster of its own, i.e. very small \( \rho \), the probability of choosing a random point that belongs to a cluster of size \( \xi \) is given by a Kronecker delta \( P(\xi) = \delta_{N,\xi} \) for which the entropy \( \chi \) is trivially 0. This points to the fact that at either extreme of cluster formation, the set of points is divided into a trivial pattern when the size of a randomly picked cluster is not uncertain, yielding vanishing entropy measure, i.e.

\[
\lim_{\rho \to 0} \chi = -P(m(\xi) = N, \xi = 1) \log P(m(\xi) = N, \xi = 1) = 0,
\]

\[
\lim_{\rho \to \infty} \chi = -P(m(\xi) = 1, \xi = N) \log P(m(\xi) = 1, \xi = N) = 0. \tag{6}
\]

From this, it can be seen that the measure of entropy \( \chi \) in Eq. (5) exhibits a maximum value at some finite value of \( \rho \) when the clusters are formed with various sizes at which the proportion of points in different cluster sizes are most uniform. At this juncture, it could be pictured that each point in a cluster of size \( \xi \) carries a label \( \xi \) and the division of \( N \) points into different label groups transits from trivial to non-trivial and back to trivial again, as \( \rho \) changes.

While the entropy \( S \) defined in Eq. (2) is interpreted as the measure of fragmentation of the clusters, the second entropy \( \chi \) defined in Eq. (5) could be interpreted as the measure of complexity of the clusters’ pattern. The pattern is simple when most of the points carry the same label, i.e. indistinguishable, whereas a more complex pattern is produced when many labels are needed to describe the points. This complexity measure is useful because we can employ it to mark the onset of giant cluster formation as the value of \( \rho \) changes. At small value of \( \rho \), many small clusters exist but the number of labels is limited as the largest cluster size remains small. When \( \rho \) increases, the labels become more diverse when more cluster sizes come to existence with the lifting of the largest cluster size. However, as \( \rho \) progresses further, the largest cluster starts to grow by absorbing smaller ones, reducing the number of labels needed, and hence, decreasing the complexity \( \chi \) of the clusters’ pattern. Once the giant cluster has been formed, it continues to (slowly) absorb other smaller clusters, further reducing the number of labels and decreasing the complexity \( \chi \), which eventually vanishes when only a single label is needed for all the points in a single cluster. The value of \( \rho \) at which the complexity measure attains its maximum \( \chi_{\text{max}} \) is denoted \( \rho_{\xi} \) to mark the onset of giant cluster formation (see red dashed curve in Fig. 1), as reasoned above.

The determination of clusters based on distance threshold indicates that there is a path between every pair of points of a cluster. It, however, does not tell how strongly connected the points are. In order to understand the internal structure of a cluster, pairwise connection between every pair of points in the cluster has to be taken into account. To this end, a network of points’ connections within a cluster could be constructed, where a link between a pair of points exists if and only if their distance is less than the threshold \( \rho \). The strength of connection between the pair is further taken into account in the form of the inverse of their distance, i.e. a distant pair is less connected than a closer one. With this network, it could be examined which parts of the cluster are only weakly connected to the rest, using a community detection method [23], the Louvain method [24] in particular. Once the communities within a cluster have been identified, one can then apply the measure of fragmentation introduced in Eq. (2) to determine the fragility of a cluster. If a cluster can be broken up into multiple tight-knit communities, it is said to be more fragile than a cluster that consists of only one or few closely connected communities. Applying this to the giant cluster, it could be conceived that when the giant cluster grows, it initially only contains a few points that are closely connected to one another, yielding low fragility (only one or few tight-knit communities). When the giant cluster grows further, more points are added to the cluster, whose ties are not yet strengthened, producing multiple communities, and hence, high fragility. This trend continues into the transition regime, with increasing fragility. After the transition regime, most of the points are now part of the giant cluster, slowing down the cluster’s growth. At this point, with sufficiently large value of distance threshold \( \rho \), points across different (distant) regions of the giant cluster can form links to strengthen the ties within the community they belong to, making the cluster more robust, or less fragile. The measure of fragmentation is useful in this case as the distance threshold \( \rho \) at which the entropy \( S \) peaks, denoted \( \rho_{S} \), is a good indicator of
the onset of stabilisation of the giant cluster (see blue dotted curve in Fig. 1). This is where the giant cluster is most fragile to be broken into components.

The two distance scales \( \rho_O \) and \( \rho_S \) discussed above can be used to determine the window of rapid growth of the giant cluster across the transition of the system from segregate to aggregate state. Further combination with the critical percolation distance \( \rho_C \) would enable calculation of the effective width of transition window, which characterises how the system transits from segregate to aggregate state. To do this, it is noted that a linear growth of the giant cluster between \( \rho_O \) and \( \rho_C \) should indicate a longer effective width than that of an exponential-like growth. For this, it is useful to use the ratio between the area \( F_1 \) under the growth curve of giant cluster and the change in cluster size \( \xi_C - \xi_O \) as the effective width (see Fig. 2). Similarly, the effective width after the critical distance, between \( \rho_C \) and \( \rho_S \) could be calculated in the same manner. Subsequently, the effective width \( \omega \) of the transition window \( \omega \) is simply the sum of widths both before and after the critical percolation threshold

\[
\omega = \delta(\rho_O, \rho_C) + \delta(\rho_C, \rho_S) = \frac{F_1}{\xi_C - \xi_O} + \frac{F_2}{\xi_S - \xi_C}. \tag{7}
\]

This effective width is useful for it characterises the sharpness of transition or the growth of the largest cluster, similar to the critical exponents that characterise the divergence of a system’s physical quantities (e.g. correlation length, average cluster size, etc…) in standard percolation theory. For the purpose of comparing different systems, a dimensionless width rescaled by the critical percolation threshold is used

\[
\epsilon = \frac{\omega}{\rho_C}. \tag{8}
\]

It could be seen that this quantity indeed provides a measure of “decoherence” of relative distance among points in a set. In other words, if the points are regularly spaced, their relative distances are mostly uniform, i.e. more coherent, yielding a small value of \( \epsilon \). However, if the points are scattered with inter-point distances ranging a wide spectrum, i.e. less coherent, the value of \( \epsilon \) would surge.

It should be noted that the discussion so far has been concerned with the measure of size (or mass) of the clusters formed in the continuum percolation process. As have been previously shown \[11, 12\], the area of clusters, i.e. their spatial extent, provides a different perspective to understand the (relative) spatial arrangement of points in a domain. The measures of size and area complement one another and their combination can be employed to distinguish different types of spatial point distribution. In the following discussion, \( \epsilon_A \) and \( \epsilon_\xi \) are used to denote the normalised spread in Eq. \[8\] calculated for cluster area and cluster size, respectively. Hereafter, the subscripts \( A \) and \( \xi \) also correspondingly denote other quantities with respect to cluster area and cluster size. On the one hand cluster size measures the amount of points contained in the cluster and can be interpreted as mass, on the other hand cluster area measures the (two-dimensional) space (continuously) occupied by the points of the cluster and can be interpreted as spatial extent. For that, we shall term \( \epsilon_A \) space decoherence and \( \epsilon_\xi \) mass decoherence.

In what follows, the two measures of space and mass decoherence are applied to a set of 39 cities in the world to compare the spatial patterns of their urban morphology. The set of 39 cities is drawn from the list of top 44 cities ranked by the Global Power City Index (GPCI) [25]. The data on spatial locations of the cities’ public transport nodes were either obtained from Open Street Map via Nextzen project [26] or from General Transit Feed Specification sources [27]. A small number (5) of cities were excluded since reliable data could not be obtained. Due to geographical features, some cities are divided into multiple parts by large water bodies (wide rivers or large bays or even open sea). As a result, the quantification of spatial pattern is reported for a total of 49 sets of points (see Fig. 3).

Using the measures of mass and space decoherence to quantify spatial patterns of points, 3 regions could be highlighted, namely highly coherent (\( \epsilon_A, \epsilon_\xi \lesssim 0.15 \)), coherent (\( 0.15 \lesssim \epsilon_A, \epsilon_\xi \lesssim 0.5 \)) and decoherent (\( \epsilon_A \gtrsim 0.5 \) or \( \epsilon_\xi \gtrsim 0.5 \)). When the points are decoherent, their pattern can further be classified as clustered or dispersed if one of the two measures is significantly smaller than the other (same reasoning for \( \sigma_A \) and \( \sigma_\xi \) in [11]).
or Seoul (7). The decoherent group with either patterns, all of which are from developed countries and Asian cities also belong to this group of coherent spatial appearance to support the perception that it is one of the lower GPCI rank) exhibit large space and/or mass decoherence in New York city possess patterns ranging from different morphologies. For example, the different boroughs in New York city are split into multiple parts due to geography. General patterns could be observed where American and European cities flock in the lower left corner of the plot together with highly developed Asian cities from Japan, South Korea and Singapore, while other cities are found scattered.

From the spatial pattern of 49 sets, it could be observed that most of American and European cities possess coherent spatial patterns with values of $\epsilon_A$ and $\epsilon_\xi$ not exceeding 0.5. The borderline case of Los Angeles ($\epsilon_\xi \approx 0.5$) appears to support the perception that it is one of the most sprawling cities in the U.S. [28]. Quite a number of Asian cities also belong to this group of coherent spatial patterns, all of which are from developed countries and ranked very high by GPCI, like Tokyo (3), Singapore (5) or Seoul (7). The decoherent group with either $\epsilon_A > 0.5$ or $\epsilon_\xi > 0.5$ contains cities mostly from developing countries like Egypt, India or those in Southeast Asia. It is worth mentioning that different parts of the same city divided by geography like water bodies can possess very different morphologies. For example, the different boroughs in New York city possess patterns ranging from high coherence of grid-like street pattern (Manhattan) to decoherence of unplanned Staten Island. Another interesting example is Istanbul where the Asian part east of the Bosporus strait appears more coherent than its European portion in the west, which has been noted in literature and could be explained by the major urban growth in Anatolian Istanbul in the later half of last century [29]. Further observations also suggest that cities known to be well-planned (and generally ranked high by GPCI) appear to possess small decoherence values, while the ones known for being sprawling (with tendency of lower GPCI rank) exhibit large space and/or mass decoherence, which could either have clustered ($\epsilon_\xi > \epsilon_A$) or dispersed ($\epsilon_\xi < \epsilon_A$) patterns.

In summary, this work illustrates that patterns of points embedded in two-dimensional space can be quantified using measures of complexity of the set of points and fragility of the giant cluster in the set, formed via a continuum percolation process. Although many sophisticated techniques have been developed for understanding different types of spatial data [30], the general framework of percolation proves to provide a powerful toolbox to study spatial organisation of point pattern, providing a different perspective from that of common techniques of point pattern analysis. While point pattern analysis mostly deals with whether a collection of points exhibit complete spatial randomness (CSR), uniform or clustered patterns, continuum percolation on the other hand, explores the structure of points’ locations based on global distribution, via the growth of the largest, dominating cluster in the system. This growth is typically characterised by three stages of initial and final slow expansion, sandwiching a rapid development region that embraces all the interesting properties of a phase transition. Here, it is shown that window of the transition could be determined by two length scales concerning the complexity measure of the entire system, which marks the onset of the existence of dominant cluster, and the component entropy of the giant cluster, which measures its fragmentation or fragility. The former is the point at which the complexity measure of connected clusters is maximum, while the latter is where the giant cluster is most fragile to be broken into components, from a perspective of network presentation of clusters. The two length scales together with a third length scale, at which clusters are most diverse in the spirit of critical phase transition, allow the characterisation of transition of the system across the critical regime, in the form of decoherence measure. The combination of mass decoherence (for amount of points accumulated) and space decoherence (for spatial extent of points accumulated) can be employed to quantify the pattern of a set of spatial locations, enabling comparison among different sets. Applying this framework to the set of public transport nodes in cities in the world from both developed and developing countries, different types of spatial pattern can be discerned and attributed to the cities’ economical and geographical backgrounds. It is also worth mentioning that the framework could be applied to any point data sets in urban context, e.g. building locations, road junctions etc., not necessarily restricted to public transport nodes.

As a final note, the term “complexity” in this work is inspired by a previous study of statistical complexity measure [31], in which two measures of metric entropy $h$ and statistical complexity $C_S$ were calculated for nonlinear dynamical systems. The measure of fragmentation $S$ in Eq. (2) is similar to $h$, measure of randomness, which maximises at one extreme of the system parameter and vanishes at the other; whereas the measure of cluster size entropy $\chi$ in Eq. (3) is similar to $C_S$, measure of complexity, which peaks at some intermediate value of...
the system parameter, suggesting the idea that a system is most complex at the interface of different states.

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