Tailoring discrete quantum walk dynamics via extended initial conditions

Germán J de Valcárcel¹, Eugenio Roldán¹ and Alejandro Romanelli²

¹ Departament d’Òptica, Universitat de València, Dr Moliner 50, 46100-Burjassot, Spain, EU
² Instituto de Física, Facultad de Ingeniería, Universidad de la República, CC 30, CP 11000, Montevideo, Uruguay
E-mail: german.valcarcel@uv.es, eugenio.roldan@uv.es and alejo@fing.edu.uy

New Journal of Physics 12 (2010) 123022 (13pp)
Received 17 June 2010
Published 13 December 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/12/123022

Abstract. We study the evolution of initially extended distributions in the coined quantum walk (QW) on the line. By analysing the dispersion relation of the process, continuous wave equations are derived whose form depends on the initial distribution shape. In particular, for a class of initial conditions, the evolution is dictated by the Schrödinger equation of a free particle. As that equation also governs paraxial optical diffraction, all of the phenomenology of the latter can be implemented in the QW. This allows us, in particular, to devise an initially extended condition leading to a uniform probability distribution whose width increases linearly with time, with increasing homogeneity.
1. Introduction

The discrete, or coined, quantum walk (QW) [1]–[9] is a process originally introduced as the quantum counterpart of the classical random walk (RW). In both cases, there is a walker and a coin: at every time step, the coin is tossed and the walker moves depending on the toss output. In the RW, the walker moves to the right or to the left, while in the QW, as the walker and coin are quantum in nature, coherent superpositions right/left and head/tail happen. This feature endows the QW with outstanding properties, such as making the standard deviation of the position of an initially localized walker grow linearly with time $t$, unlike the RW, in which this growth goes as $t^{1/2}$. This has strong consequences in algorithmics and is one of the reasons why QWs have received so much attention in the past decade. However, the relevance of QWs is being recognized to go beyond this specific arena and, for example, some simple generalizations of the standard QW have shown unsuspected connections with phenomena such as Anderson localization and quantum chaos [10]–[12]. Moreover, theoretical and experimental studies show that the QW finds applications in outstanding systems, such as Bose–Einstein condensates [13], atoms in optical lattices [14, 15], trapped ions [16, 17] or optical devices [18]–[24], just to mention a few; hence the QW could constitute a means for controlling the performance of those systems, at least under certain conditions. Apart from the discrete QW we consider here, continuous versions exist as well [25, 26], whose relationship with the coined QW has been discussed in [27].

Surprisingly enough, even the simplest version of the discrete QW has not been studied in all its extension. Specifically, we refer to the fact that in almost all studies to date, the initial state of the walker is assumed to be sharply localized at the line origin, with few exceptions [14, 27, 28]. In [27], it was shown that for wider initial distributions (an extended wavepacket), the evolution of the wavepacket is Gaussian like, not showing the two characteristic outer peaks appearing in the probability distribution for more sharply localized initial conditions. In [14] also, extended distributions, with a top-hat profile, were considered in the context of the superfluid-Mott insulator transition in optical lattices, but no general conclusions were drawn on the influence of these extended initial conditions on the long time state. It is this issue that we address in the present paper, and the results we obtain open the way to new types of distributions that the QW can exhibit, e.g. virtually flat ones, with an obvious impact on applications of this process.
2. The coined quantum walk on the line

In this QW, the walker moves at discrete time steps \( t \in \mathbb{N} \) along a one-dimensional lattice of sites \( x \in \mathbb{Z} \), with a direction that depends on the state of the coin (with eigenstates \( R \) and \( L \)). The state of the total system at \((x, t)\) can be expressed in the form

\[
|\Psi_{x,t}\rangle = \begin{pmatrix} R_{x,t} \\ L_{x,t} \end{pmatrix},
\]

where \( R_{x,t} \) and \( L_{x,t} \) are wave functions on the lattice. As \(|R_{x,t}|^2\) and \(|L_{x,t}|^2\) have the meaning of probability of finding the walker at \((x, t)\) and the coin in state \( R \) and \( L \), respectively, the probability of finding the walker at \((x, t)\) is

\[
P_{x,t} = |\langle \Psi_{x,t} | \Psi_{x,t} \rangle| = |R_{x,t}|^2 + |L_{x,t}|^2
\]

and \( \sum_x P_{x,t} = 1 \). The QW is ruled by a unitary map and a standard form is [29]

\[
\begin{align*}
R_{x,t+1} &= R_{x+1,t} \cos \theta + L_{x+1,t} \sin \theta, \\
L_{x,t+1} &= R_{x-1,t} \sin \theta - L_{x-1,t} \cos \theta,
\end{align*}
\]

where \( \theta \in [0, \pi/2] \) is a parameter defining the bias of the coin toss (\( \theta = \pi/4 \) for an unbiased, or Hadamard, coin).

2.1. The dispersion relation and the group velocity

Plane wave solutions to (3) exist in the form [30]

\[
|\Psi_{x,t}\rangle = \exp[i(kx - \omega^{(s)} t)]|\Phi_k^{(s)}\rangle,
\]

where \( s = \pm \), \( k \in [−\pi, +\pi] \),

\[
\omega^{(+)} = \omega, \quad \omega^{(−)} = \pi − \omega, \\
\omega = −\arcsin(\cos \theta \sin k) \in \left[−\frac{\pi}{2}, \frac{\pi}{2}\right],
\]

\[
|\Phi_k^{(±)}\rangle = N_\pm \left(\cos \theta \cos k \pm \cos \omega \right) e^{-ik \sin \theta}
\]

and \( N_\pm \) is a normalization factor (any \( N \) will have this meaning in the following). The dispersion relation \((5b)\) is represented in figure 1 together with the group velocity \( v_g^{(+)}(k) = d\omega^{(+)}/dk \) associated with \( |\Phi_k^{(+)}\rangle \). The group velocity corresponding to \( |\Phi_k^{(−)}\rangle \) is \( v_g^{(−)}(k) = d\omega^{(−)}/dk = −v_g^{(+)}(k) \). The QW group velocity has been used for determining hitting times [31], and it will allow us to make simple but relevant predictions about the QW dynamics when the initial state is a wavepacket close to some of the eigensolutions above, say

\[
|\Psi_{x, t = 0}\rangle = f_x^{(s)} e^{i k_0 x} |\Phi_{k_0}^{(s)}\rangle,
\]

with \( f_x^{(s)} \) a smooth envelope. In that case, as in any linear wave system, one must expect that, to leading order, the group velocity governs its propagation. Hence if \( k_0 = \pm \pi/2 \), a sufficiently extended wavepacket should stay at rest because \( v_g^{(s)}(\pm \pi/2) = 0 \), while if \( k_0 = 0 \), it should move...
Figure 1. Dispersion relation (blue full line) as given by equation \(5b\), and corresponding group velocity (red dashed line). \(\theta = \pi/4\).

with maximum velocity \(v^{(s)}(0) = -s \cos \theta\). If the initial condition projects onto both \(|\Phi_{k_0=0}^{(\pm)}\rangle\), we must expect that the initial wavepacket splits into two, moving at opposite velocities given by \(\mp \cos \theta\). Numerical simulations of the QW equations (3) confirm these notable effects and tell us that the dispersion relation is a powerful tool for predicting QW dynamics [31]. As an example, we show in figures 2 and 3 some numerical results for an initial Gaussian distribution given by

\[ |\Psi_{x,0}\rangle = N \exp \left[ -\frac{1}{2} (x/\sigma_0)^2 + i k_0 x \right] |C\rangle, \tag{7} \]

defined by a carrier wavenumber \(k_0\) and by a coin state \(|C\rangle\), taken equal at all sites as in (6). The initial width was sufficiently large (\(\sigma_0 = 50\)) in order to have a smooth envelope. In figure 2, we analyse the influence of \(k_0\) and \(|C\rangle\) on the QW dynamics. When \(k_0 = 0\) and \(|C\rangle = |\Phi_{k_0=0}^{(+)}\rangle\), figure 2(a), the wavepacket moves (without any appreciable distortion) at its maximum allowed speed; an analogous behaviour is seen in figure 2(b), now with a splitting of the wavepacket into two equal ones, when \(k_0 = 0\) and the coin \(|C\rangle = 2^{-1/2}(\downarrow_1 + \downarrow_2)\), which projects equally onto both eigenvectors \(|\Phi_{k_0=0}^{(\pm)}\rangle\), figure 2(b). Finally, in figure 2(c), we take \(k_0 = \pi/2\) and \(|C\rangle = |\Phi_{k_0=\pi/2}^{(+)}\rangle\), observing that the wavepacket remains at rest as expected, with a width that increases slowly with time (we shall come back to that point later on). To conclude this check, we plot in figure 3 the value of the measured group velocity (it is in fact the speed of the ‘centre of mass’ of the part of the distribution propagating rightwards or leftwards, never mind) as a function of \(k_0\) (for several widths \(\sigma_0\), see the caption). No difference is observed with respect to the actual group velocity \(v_g(k_0) = d\omega/dt\) following from \(5b\): compare with figure 1.

As we demonstrate in the next sections, the dispersion relation \(5b\) controls not only the velocity of the wavepacket, but also the evolution of its shape as time runs, which will allow us to make interesting predictions.

3. Broad initial distributions: wave equations in the continuum limit

The goal of this section is to find a wave equation for the wavepacket envelope with the help of discrete Fourier analysis. Given a function \(f_x\) on integers \(x \in \mathbb{Z}\), one can define its discrete Fourier transform (DFT) as

\[ \tilde{f}_k = \sum_x f_x e^{-ikx}, \tag{8} \]
Figure 2. Probability distribution as a function of position, calculated with the map equation (3) at three different times, $t = 0$ (black line), $t = 1000$ (red line) and $t = 2000$ (blue line). The initial condition is given by equation (7) with $\theta = \pi/4$, $\sigma_0 = 50$ and furthermore in (a) $k_0 = 0$ and $|C\rangle = (2\sqrt{2} + 4)^{-1/2}(1 + \sqrt{2}, 1)$, in (b) $k_0 = 0$ and $|C\rangle = 2^{-1/2}(1, -i)$, and in (c) $k_0 = \pi/2$ and $|C\rangle = 2^{-1/2}(1, -i)$.

which can be inverted as

$$f_x = \int_{-\pi}^{+\pi} \frac{dk}{2\pi} \tilde{f}_k e^{ikx}. \quad (9)$$

Applying this DFT to the map equation (3), it is straightforward to get an explicit solution to the QW given an arbitrary initial condition $|\Psi_{x,0}\rangle$. The result is

$$|\Psi_{x,t}\rangle = \sum_{s=\pm 1} |\Psi_{x,t}^{(s)}\rangle, \quad (10)$$

where

$$|\Psi_{x,t}^{(s)}\rangle = \int_{-\pi}^{+\pi} \frac{dk}{2\pi} \exp[i(kx - \omega^{(s)} t)] |\Phi_k^{(s)}\rangle \langle \Phi_k^{(s)}| \tilde{\Psi}_{k,0}\rangle, \quad (11)$$

and

$$|\tilde{\Psi}_{k,0}\rangle = \sum_x e^{-ikx} |\Psi_{x,0}\rangle \quad (12)$$
Figure 3. Dimensionless group velocity \( v_g \) as a function of dimensionless momentum, calculated using the map equation (3) with the initial condition given by equation (7) and \( \theta = \pi/4 \). This graph perfectly fits with the theoretical prediction \( v_g = \frac{\omega}{d} \), with \( \omega \) given by equation (5b). Additionally, the numerical calculations verify that \( v_g \) is independent of both \( \sigma_0 \) and \( |C\rangle \).

is the DFT of the initial condition. As stated, we are interested in initial conditions of the form

\[
|\Psi_{s,0}\rangle = \sum_{s=\pm 1} f_x^{(s)} e^{ik_0 x} |\Phi_{k_0}^{(s)}\rangle, \tag{13}
\]

where the envelopes \( f_x^{(s)} \) vary smoothly on \( x \), and \( k_0 \) is a (carrier) wavenumber. Then

\[
|\widetilde{\Psi}_{k,0}\rangle = \sum_{s=\pm 1} \tilde{f}^{(s)}_{k-k_0} |\Phi_{k_0}^{(s)}\rangle, \tag{14}
\]

which is peaked around \( k = k_0 \) as \( \tilde{f}^{(s)}_{k-k_0} \) is peaked around \( k - k_0 = 0 \) (low-frequency envelope). In such cases, equation (11) can be written as

\[
|\Psi_{s,t}\rangle = \exp\left[ \frac{i}{\hbar} \left( k_0 x - \omega_0^{(s)} t \right) \right] F_s(x, t) |\Phi_{k_0}^{(s)}\rangle + \mathcal{O}(\Delta k), \tag{15}
\]

where

\[
F_s(x, t) = \int_{-\pi}^{+\pi} \frac{dK}{2\pi} \tilde{f}^{(s)}_K e^{i(Kx - s\Omega t)} , \tag{16}
\]

\( \Delta k \ll 1 \) is the width of \( \tilde{f}^{(s)}_K \), \( K = k - k_0 \), \( \Omega = \omega - \omega_0 \), and we did not modify the limits of the integral because of the assumed smallness of \( \Delta k \). We have introduced two wave functions, \( F_{\pm}(x, t) \), in terms of which

\[
P_{s,t} = \sum_{s=\pm 1} |F_s(x, t)|^2 + \mathcal{O}(k) . \tag{17}
\]
We let \( F_s(x, t) \) be defined on the reals, as there is nothing against that in equation (16), so that it is straightforward setting a wave equation from that equation,

\[
i\partial_t F_s(x, t) = -is\omega_1 \partial_x F_s - \frac{1}{2} s^2 \omega_2 \partial_x^2 F_s + \cdots,
\]

(18)
after Taylor expanding \( \Omega \) around \( k_0 \), and where \( \omega_n = (d^n \omega/dk^n)_{k=k_0} \). This equation is to be solved under the initial condition \( F_s(x, 0) = f_s^{(0)} \) at integer \( x \).

Equation (18) is a main result of this paper. It evidences the role played by the dispersion relation (5b) as anticipated: for distributions whose DFT is centred around some \( k \), the local variations of \( \omega \) around \( k_0 \) determine the type of wave equation controlling the QW dynamics. The first term on the right-hand side gives the group velocity, already discussed, the second accounts for diffraction and so on.

3.1. Application to Gaussian initial distributions

Two cases of interest of equation (18) are analysed next, corresponding to \( k_0 = 0, \pi/2 \) as suggested by the analysis of the dispersion relation. First, the case \( k_0 = 0 \) yields, to leading order,

\[
\partial_t F_s = (s \cos \theta) \partial_x F_s.
\]

(19)

According to (19), if the (broad) initial condition projects onto both eigenspinors \(|\Phi^{(\pm)}_0\rangle\), two wavepackets (whose height will depend on the projections \( \langle \Phi^{(\pm)}_0 | \Psi_{s,0} \rangle \)) will propagate without distortion at equal but opposite velocities given by \( v^{(s)}(0) = -s \cos \theta \), in agreement with the numerical results shown in figures 2(a) and (b). The state \(|C\rangle\) of the coin (taken equal at any site) controls the projections \( \langle \Phi^{(\pm)}_0 | \Psi_{s,0} \rangle \). As already shown, the distortionless propagation at a velocity \( v^{(s)}(0) \) is observed in excellent agreement with the prediction, even for moderate values of \( \sigma_0 \). Nevertheless, in all cases, a deformation of the wavepackets is visible after some running time, which is longer for wider initial distribution. This deformation is controlled by the next, third-order derivative term, in which case one has an equation similar to that derived in [18, 33, 34], where the role of the third-order derivative was analysed. This type of approximation was shown to be quite good even for localized initial conditions, where the truncation of the dispersion relation is not so well justified as the initial condition projects over all \( k \) values.

The case \( k_0 = \pi/2 \) is more interesting for our purposes. It is described, to leading order, by

\[
i\partial_t F_s = -\frac{s}{2 \tan \theta} \partial_x^2 F_s,
\]

(20)

which is analogous to the Schrödinger equation for a free particle as well as to the equation of paraxial optical diffraction, and the pulse propagation equation in linear optical fibres. The solution to (20) under a Gaussian initial condition

\[
F_s(x, 0) = \mathcal{N} \exp \left[ -\frac{1}{2} \frac{(x-x_0)^2}{\sigma_0^2} \right],
\]

(21)

\[
\text{According to (16), } F_s(x, 0) = (2\pi)^{-1} \int_{-\infty}^{\infty} dK f_s^{(s)}(K) \exp(iKx) \sum_x f_s^{(s)}(x,y) \sin(x-y), \text{ where } \sin(x-y) = \sin(\pi x)/(\pi x) \text{ and at integer } x, \sin(x-y) = \delta_{x,y}. \text{ In general, the relation between } f_s^{(s)} \text{ and } F_s(x, 0) \text{ is not invertible, which means that not all of the choices for the latter correspond to a true } f_s^{(s)}. \text{ However, any choice of } F_s(x, 0) \text{ varying smoothly on the unit spatial scale corresponds to a true } f_s^{(s)} = F_s(x, 0) \text{ at integer } x.
\]
Figure 4. Dimensionless asymptotic velocity of standard deviation (blue full line) as a function of initial standard deviation $\sigma_0$ with $k_0 = \pi/2$ in log scale. It is calculated using equation (3) with the initial condition given by equation (7), $\theta = \pi/4$ and $|C| = |\Phi_{k_0=\pi/2}^{(+)}| = 2^{-1/2}(1, -i)$. The red dashed line shows the asymptotic velocity following from equation (25).

where $\sigma_0$ is the initial width and $x_0$ is the centre of the distribution, reads [36]

$$F_s(x, t) = \mathcal{N} q^{-1}_s(t) \exp\left[-\frac{1}{2} \left(\frac{x - x_0}{q_s(t)}\right)^2\right], \quad (22)$$

where the complex parameter

$$q_s(t) = \sqrt{\sigma_0^2 + ist/\tan \theta}. \quad (23)$$

Note that the centre of the distribution remains constant as the group velocity is null in the present case; see figure 1. The probability

$$P_s(x, t) = \mathcal{N} \exp\left[-\left(\frac{x - x_0}{\sigma(t)}\right)^2\right], \quad (24)$$

thus remains Gaussian with

$$\sigma(t) = \sqrt{\sigma_0^2 + \left(\frac{t}{\sigma_0 \tan \theta}\right)^2}, \quad (25)$$

the width of the distribution (note that these results are independent of $s$). At long times (say $t \gtrsim 5\sigma_0^2 \tan \theta$), $\sigma(t) = (\sigma_0 \tan \theta)^{-1} t$, hence the width increases linearly with time, the increase being faster for smaller $\sigma_0$. Again, numerical simulations of (3) for the initial condition (7) with $k_0 = \pi/2$ are in good agreement with the analytical prediction, equation (25), as can be seen in figure 4.
3.2. Achieving homogeneous distributions

In QWs, a most desired result is for the probability distribution to be as uniform as possible after a time. A hint towards that goal is given by light paraxial diffraction theory—that the QW follows for \( k_0 = \pi/2 \), equation (20). It is a textbook result that the far field corresponding to a light amplitude distribution

\[
sinc(x) = \sin(\pi x)/(\pi x)
\]  

(26)

is remarkably homogeneous within a certain spatial region [35].

Hence, in order to gain insight into the problem, we look at the solution to equation (20) under an initial condition

\[
F(x, 0) = N \text{sinc}(x/\sigma_0).
\]  

(27)

The solution is obtainable by Fourier transformation of the spatial coordinate [36]

\[
F_s(x, t) = \int_{-\infty}^{+\infty} dk \tilde{F}_s(k, 0) \exp(ikx - isk^2t/2 \tan \theta),
\]  

(28)

with

\[
\tilde{F}_s(k, 0) = (2\pi)^{-1} \int_{-\infty}^{+\infty} dx F_s(x, 0) \exp(-ikx).
\]  

(29)

In our case,

\[
\tilde{F}_s(k, 0) = (2\pi)^{-1} \sqrt{\sigma_0} \text{rect}(\sigma_0 k/\pi),
\]  

(30)

where \( \text{rect}(a) = 1 \) if \( |a| < 1 \) and 0 otherwise. Hence

\[
F_s(x, t) = (2\pi)^{-1} \sqrt{\sigma_0} \int_{-\pi/\sigma_0}^{+\pi/\sigma_0} dk \exp(-it\phi_s(k, \xi)),
\]  

(31)

with \( \phi_s(k, \xi) = k\xi - sk^2/2 \tan \theta \) and \( \xi = x/t \).

As we are interested in the behaviour of \( F_s \) at long times, we can use the stationary phase method [4, 32] to evaluate the leading dependence of the integral on \((x, t)\). As

\[
\partial_k\phi_s(k, \xi) = \xi - sk/\tan \theta,
\]  

(32)

the phase is stationary at \( k_s = s\xi \tan \theta \). Hence if \( k_s \not\in [-\pi/\sigma_0, +\pi/\sigma_0] \), i.e. for \( |x| > \pi t/(\sigma_0 \tan \theta) \), there is no stationary point inside the integration domain and then the integral decays sharply, that is, faster than any polynomial in \( t \). In contrast, if \( k_s \) lies inside the integration domain, i.e. for \( |x| < \pi t/(\sigma_0 \tan \theta) \), the integral decays as \( t^{-1/2} \). In this last case, as the exponential \( \exp[i t\phi_s(k, \xi)] \) oscillates very strongly slightly far from \( k = k_s \) (recall that we are considering long times), only the neighbourhood of \( k = k_s \) makes a net contribution to the integral and then its limits can be brought to \( \pm \infty \) without significant error, obtaining the result

\[
F_s(x, t) = w(t)^{-1/2} \exp \left[ -i \left( \frac{\pi}{4} + \frac{\tan \theta}{2t} x^2 \right) \right],
\]  

(33)
Figure 5. Probability distributions for $\theta = \pi/4$ and different initial conditions. In (a), the initial condition is $|\Psi_{x,0}\rangle = |\Phi_{\pi/2}^{(0)}\rangle$, fully localized at the origin, and the time run is $t = 6 \times 10^3$; the inset shows a magnification (odd sites have zero occupation probability). In (b), $|\Psi_{x,0}\rangle = \mathcal{N} \exp(i\pi x/2) \text{sinc}(x/\sigma_0) |\Phi_{\pi/2}^{(0)}\rangle$ is used, while in (c), additional Gaussians of widths (from top to bottom) $\sigma_G = 1.1\sigma_0$ (black line), $2\sigma_0$ (red line) and $3\sigma_0$ (blue line) multiply the initial condition. $\sigma_0 = 15$ and the time run is $t = 20 \times 10^3$.

with $w(t) = 2\pi t/(\sigma_0 \tan \theta)$. Hence the asymptotic analysis predicts a uniform probability
\[ P_s(x, t) = w(t)^{-1} \text{rect}(2x/w(t)), \]  
inside a segment of width $w(t)$, which does not depend on $s$. Thus we should expect, after a transient time (that can be estimated as $t \gtrsim 2\sigma_0^2 \tan \theta$), a flat distribution whose width increases linearly with time (its standard deviation is $\sigma(t) = w(t)/\sqrt{12}$).

Inspired by this result, we consider the actual QW initial condition
\[ |\Psi_{x,0}\rangle = \mathcal{N} \exp\left(i\pi x/2\right) \text{sinc}\left(x/\sigma_0^\prime\right) |\Phi_{\pi/2}^{(0)}\rangle, \]  
with $\sigma_0$ the initial width. Figure 5(b) shows the results of the simulation of (3), which are in qualitative agreement with the discussion above: a quite uniform distribution is attained. For comparison, the well-known result corresponding to an initially localized distribution is shown in figure 5(a). The main differences are the improved degree of uniformity in (b), which is free from the large outer peaks in (a), and the fact that in (a), even/odd sites have null occupation probability at odd/even times, unlike in (b). There is, however, a disgusting feature in (b),

New Journal of Physics 12 (2010) 123022 (http://www.njp.org/)
namely the high-frequency ripples that appear at the plateau. Nevertheless, the situation can be improved by multiplying the initial condition by a Gaussian of convenient width (this is a kind of smoothing processing, typical of optical diffraction [35], where it is known as apodization), as shown in figure 2(c). We see that the high-frequency ripples have been largely smoothed (although very high-frequency ripples can still be appreciated but these have a quite small modulation). This is another main result of this paper: highly uniform distributions (almost reaching a top-hat profile) can be obtained in the QW by making a judicious choice of the initial condition. We want to stress that these homogeneous distributions are so after a short transient, their homogeneity increasing with time.

The achievement of QW homogeneous distributions is a very desirable property. From the very beginning, the relatively high homogeneity of the probability distribution of the QW corresponding to a localized initial condition has been considered as a positive quality of this process for information purposes. In [37], the presence of some decoherence in the process was considered to be beneficial because it leads to more homogeneous distributions at a special time. In this sense, our finding may have relevance as we have seen that a judicious initial condition helps in achieving distributions with much larger and more permanent homogeneity than the initially localized case, even including decoherence, as figure 5(c) clearly demonstrates.

4. Conclusions and discussion

In this work, we have studied the influence of the initial condition on the discrete QW on the line guided by the QW dispersion relation and by its associated wave equations. Specifically, we have considered the evolution of initially extended probability distributions. We have shown that sufficiently wide Gaussian initial distributions propagate without distortion and small width increase at velocities that can be tuned, including null velocity, with a proper choice of the phases along the initial distribution as given by the phase factor $\exp(ik_0x)$. We have also devised an initial condition that leads, after a transient, to a homogeneous distribution whose width increases with time, remaining highly homogeneous at any later time. This result, figure 5(c), is a main result of this paper.

We further mention that any behaviour of light diffraction (or linear pulse propagation) can be transferred to the QW in the case of resting probability distributions ($k_0 = \pi/2$), as its continuous limit, equation (20), is nothing but the paraxial diffraction equation. For instance, a light pattern replicates at specific planes when it is periodic, the so-called Talbot effect [35, 38]. In our case, if $F_s(x + \lambda, 0) = F_s(x, 0)$, where $\lambda$ is the spatial period, then $|F_s(x, nT)|^2 = |F_s(x, 0)|^2$, where $T = (2\pi)^{-1}\lambda^2 \tan \theta$ is the Talbot period and $n$ is an integer. Other optical effects, such as those of lenses, can also be mimicked by introducing quadratic phase factors [35] in the initial condition. It is just the choice of that condition that can make any paraxial, linear optical phenomenon be reproduced with the QW, which can be useful for special implementations of this rich process.

We would like to mention finally the connection existing between the phenomenon studied here, the influence of the (extended) initial condition on the probability distribution achieved by the coined QW and the probability distribution of a particle in a box, a problem named quantum carpets (see e.g. [39]–[41] and references therein). A number of analytical tools have been applied to this last problem, which could be of use for the QW problem.
Acknowledgments

Continued discussions with C Navarrete–Benlloch and A Pérez are gratefully acknowledged. This work was supported by the Spanish Government and the European Union FEDER through Project FIS2008-06024-C03-01. AR acknowledges financial support from PEDECIBA, ANII, Universitat de València and Generalitat Valenciana.

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