FINITE GENERATION OF LIE DERIVED POWERS OF
ASSOCIATIVE ALGEBRAS

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ABSTRACT. Let $A$ be an associative algebra over a field of characteristic $\neq 2$ that is generated by a finite collection of nilpotent elements. We prove that all Lie derived powers of $A$ are finitely generated Lie algebras.

1. INTRODUCTION AND MAIN RESULT

All algebras are considered over a field $F$ of characteristic $\neq 2$. An associative algebra $A$ gives rise to the Lie algebra $A^{(-)} = (A, [a, b] = ab - ba)$. In this note we address the question first raised by I. Herstein [4] (see also [1]):

given a finitely generated associative algebra $A$ when is the Lie algebra $[A, A]$ finitely generated?

Consider the derived series of the Lie algebra

$$A^{(-)}: A^{(-)} = A^{[0]} > A^{[1]} > \cdots, A^{[i+1]} = [A^{[i]}, A^{[i]}].$$

Recall that an element $a \in A$ is called nilpotent if there exists an integer $n(a) \geq 1$ such that $a^{n(a)} = 0$. We say that $A$ is a nil algebra if every element of $A$ is nilpotent.

E. S. Golod [3] and T. Lenagan, A. Smoktunowicz [6] constructed examples of infinite dimensional finitely generated nil algebras.

**Theorem 1.** Let $A$ be an associative algebra generated by a finite collection of nilpotent elements. Then an arbitrary derived power $A^{[i]}, i \geq 1,$ is a finitely generated Lie algebra.

**Corollary 1.** Let $A$ be a finitely generated nil algebra (see [3], [6]). Then the Lie algebras $A^{[i]}, i \geq 1,$ are finitely generated.

The key sufficient condition for finite generation of $A^{[i]}$ (see the Proposition 1 below) is essentially based on the work of C. Pendergrass (see [7]). In particular, the following lemma is a version of Lemma 3 in [7].

**Key words and phrases.** associative algebra, Lie algebra, finitely generated
Mathematics Subject Classification 2010: 17B60.
**Lemma 1.** Let $U$ be an ideal of the Lie algebra $A^{(-)}$. Then $[\text{id}_A([U, U]), A] \subseteq U$

**Proof.** Since $[U, U]$ is also an ideal of $A^{(-)}$ it follows that $\text{id}_A([U, U]) = [U, U] + [U, U]A$. Denote $x \circ y = xy + yx$. The following identities are well known:

1. \[ xy = \frac{1}{2}([x, y] + x \circ y) \]
2. \[ [z, x \circ y] = [z, x] \circ y + [z, y] \circ x \]
3. \[ [z, x \circ y] = [z \circ x, y] + [z \circ y, x] \]

By the identity (1)

\[ [U, U]A \subseteq [[U, U], A] + [U, U] \circ A. \]

Clearly $[[U, U], A] \subseteq U$ and $[[U, U], A], A] \subseteq U$.

By the identity (2)

\[ [U, U] \circ A \subseteq [U, U \circ A] + [U, A] \circ U \subseteq U + U \circ U. \]

By the identity (3)

\[ [U \circ U, A] \subseteq [U, U \circ A] \subseteq U, \]

which completes the proof of the lemma. \( \square \)

We say that an algebra $A$ is finitely graded if $A = A_1 + A_2 + \cdots$ is a direct sum, $A_i, A_j \subseteq A_{i+j}$, the algebra $A$ is generated by $A_1$ and $\dim_F A_1 < \infty$.

**Proposition 1.** Let $A$ be a finitely graded algebra. Suppose that all factor algebras $A/\text{id}_A(A^{[i]})$ are nilpotent. Then all derived powers $A^{[i]}, i \geq 1$, are finitely generated Lie algebras.

**Proof.** Let $i \geq 1$. Consider the ideal $U = A^{[i+1]} = [A^{[i]}, A^{[i]}]$ of the Lie algebra $A^{(-)}$. By Lemma 1 we have

\[ [\text{id}_A(A^{[i+2]}), A] \subseteq [A^{[i]}, A^{[i]}]. \]

By the assumption of the proposition there exists $n \geq 1$ such that $A^n \subseteq A^{[i+2]}$, and, therefore $\sum_{j \geq n} A_j \subseteq \text{id}_A(A^{[i+2]})$. We claim that the Lie algebra $A^{[i]}$ is generated by the finite dimensional subspace $A^{[i]} \cap \left( \sum_{k=1}^{2n-2} A_k \right)$. Indeed, since $A^{[i]}$ is a graded subspace of $A$ we need to check that if $m \geq 2n - 1$ then $A^{[i]} \cap A_m$ lies in the Lie subalgebra generated by $A^{[i]} \cap \left( \sum_{k=1}^{m-1} A_k \right)$. We have $A^{[i]} \cap A_m \subseteq \sum_{p+q=m} [A_p, A_q]$. Since $m \geq 2n - 1$ it follows that $p \geq n$ or $q \geq n$. If $p \geq n$ then $A_p \subseteq \text{id}_A(A^{[i+2]})$ and therefore

\[ [A_p, A_q] \subseteq [\text{id}_A(A^{[i+2]}), A] \subseteq [A^{[i]}, A^{[i]}]. \]
It implies
\[ A[i] \cap A_m \subseteq \sum_{k=1}^{m-1} A[k] \cap \sum_{k=1}^{m-1} A[k] \]
and completes the proof of the proposition.

\[
\square
\]

Proof of Theorem 1. Let the algebra \( A \) be generated by elements \( a_1, a_2, \ldots, a_m \) such that \( a_i^{n_i} = 0, n_i \geq 1, 1 \leq i \leq m \). Consider the algebra \( \tilde{A} \) presented by generators \( x_1, x_2, \ldots, x_m \) and relations \( x_i^{n_i} = 0, n_i \geq 1, 1 \leq i \leq m \). The algebra \( \tilde{A} \) is finitely graded and the algebra \( A \) is a homomorphic image of the algebra \( \tilde{A} \). It is sufficient to prove that for an arbitrary \( k \geq 1 \) the Lie algebra \( \tilde{A}[k] \) is finitely generated. Moreover, by Proposition 1 it is sufficient to prove that all factor algebras \( B_k = \tilde{A}/\text{id}_{\tilde{A}}(\tilde{A}[k]) \) are nilpotent. Consider the following elements of the free associative algebra

\[
f_1(x_1, x_2) = [x_1, x_2]
f_s(x_1, \ldots, x_{2^s}) = [f_{s-1}(x_1, \ldots, x_{2^{s-1}}), f_{s-1}(x_{2^{s-1}+1}, \ldots, x_{2^s})], \text{ for } s \geq 2.
\]

The algebra \( B_k \) is a finitely generated algebra that satisfies the polynomial identity \( f_k(x_1, \ldots, x_{2^k}) = 0 \). Hence the Jacobson radical \( J(B_k) \) is nilpotent (see [2]). The semisimple algebra \( B_k/J(B_k) \) is a subdirect product of primitive algebras satisfying the identity \( f_k = 0 \).

It follows from Kaplansky theorem (see [5]) that a primitive algebra satisfying the identity \( f_k = 0 \) is a field. Hence the algebra \( B_k/J(B_k) \) is commutative. The commutative algebra \( B_k/J(B_k) \) is generated by a finite collection of nilpotent elements. Hence the algebra \( B_k/J(B_k) \) is nilpotent, hence the algebra \( B_k = J(B_k) \) is nilpotent. This completes the proof of the theorem.

\[
\square
\]

Remark 1. In [1] we showed that if \( A \) is a finitely generated algebra with an idempotent \( e \) such that

\[
AeA = A(1-e)A = A \quad (\ast)
\]

then the Lie algebra \( A[1] = [A, A] \) is finitely generated. The condition (\ast) is equivalent to \( A \) being generated by \( eA(1-e) + (1-e)Ae \). Since elements from \( eA(1-e) \) and \( (1-e)Ae \) are nilpotent this result follows from Theorem 1.
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