The $S^1$-Equivariant Signature for Semi-free Actions as an Index Formula

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Abstract
Lott (Math Ann 316(4):617–657, 2000) defined an integer-valued signature $\sigma_{S^1}(M)$ for the orbit space of a compact orientable manifold with a semi-free $S^1$-action but he did not construct a Dirac-type operator which has this signature as its index. We construct such operator on the orbit space and we show that it is essentially unique and that its index coincides with Lott’s signature, at least when the stratified space satisfies the so-called Witt condition. For the non-Witt case, this operator remains essentially self-adjoint (in contrast to the Hodge-de Rham operator) and it has a well-defined index which we conjecture will also compute $\sigma_{S^1}(M)$.

Keywords Index theory · Stratified spaces · Semi-free actions · Equivariant signature · Dirac operator

Mathematics Subject Classification 58C40 · 58J20 · 58J32 · 53C12

1 Introduction
In [25], John Lott studied a signature invariant for quotients of closed oriented $(4k + 1)$-dimensional manifolds $M$ by $S^1$-actions. This signature, denoted by $\sigma_{S^1}(M)$, is defined at the level of basic forms with compact support. If the action is semi-free, i.e., if the isotropy groups are either the trivial group or the whole $S^1$, the quotient space $M/S^1$ is a stratified space with singular stratum the fixed point set $M^{S^1}$. Locally, a neighborhood of the singular stratum is homeomorphic to the product $D^{4k−2N−1} \times C(P^N)$ for some $N$. Here $C(P^N)$ denotes a cone with link $C(P^N)$. Moreover, on the open and dense subset $M_0 := M − M^{S^1}$, the action is free. In this context, Lott proved that the following remarkable formula yields an $S^1$-homotopy invariant [25, Theorem 4]
\[ \sigma_{S^1}(M) = \int_{M_0/S^1} L \left( T(M_0/S^1), g^{T(M_0/S^1)} \right) + \eta(M^{S^1}), \] (1)

where \( L \left( T(M_0/S^1), g^{T(M_0/S^1)} \right) \) is the \( L \)-polynomial of the curvature form of the tangent bundle \( T(M_0/S^1) \) with respect to the quotient metric \( g^{T(M_0/S^1)} \), and \( \eta(M^{S^1}) \) is the eta invariant of the odd signature operator defined on the fixed point set. It is important to emphasize that part of the result is the convergence of the integral over \( M_0/S^1 \). The question that arises naturally is whether there exists a Fredholm operator whose index computes \( \sigma_{S^1}(M) \). This question was posed by Lott himself as a remark in his original work [25, Sect. 4.2]. A natural candidate is the Hodge-de Rham operator \( D_{M_0/S^1} := d_{M_0/S^1} + d^\dagger_{M_0/S^1} \) defined on the space of compactly supported differential forms \( \Omega^1_c(M_0/S^1) \). If the quotient metric is not complete, \( D_{M_0/S^1} \) might have several closed extensions. In order to understand this phenomenon better, it is necessary to study the form of the operator close to the fixed point set. Following Brüning’s work [10], one sees that \( D_{M_0/S^1} \), close to \( F \subset M^{S^1} \), is unitarily equivalent to an operator of the form

\[ \Psi^{-1} D_{M_0/S^1} \Psi = \gamma \left( \frac{\partial}{\partial r} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes A(r) \right). \] (2)

The operator \( A(r) \) can be written as

\[ A(r) := A_H(r) + \frac{1}{r} A_V, \] (3)

where \( A_H(r) \) is a first-order horizontal operator, well defined for \( r \geq 0 \). The coefficient \( A_V \) is a first-order vertical operator, known as the cone coefficient. Using the techniques developed in [13], Brüning showed in [10, Sect. 4] that the operator (2) has a discrete self-adjoint extension. In addition, if the cone coefficient satisfies the spectral condition

\[ |A_V| \geq \frac{1}{2}, \] (4)

then the operator is in fact essentially self-adjoint. In the Witt case, i.e., when there are no vertical harmonic forms in degree \( N \) (i.e., \( N \) odd), we can always achieve condition (4) by rescaling the vertical metric, which is an operation that preserves the index. To see this, one needs to understand the spectrum of \( A_V \). It was shown in [10, Theorem 3.1] that the essential eigenvalues, those invariant under the rescaling, are the ones obtained when restricting to the space of vertical harmonic forms. These eigenvalues are explicitly given by \( 2j - N \), for \( j = 0, 1, \ldots, N \). Observe that if \( N \) is odd, zero does not appear as an essential eigenvalue and \( |2j - N| \geq 1 \). On the other hand, if \( N = 2\ell \) then zero appears as an eigenvalue when \( j = \ell \) and the corresponding eigenspace is non-zero. For the Witt case we prove following the work of Brüning [10, Sect. 5] that \( \text{ind}(D^+_{M_0/S^1}) = \sigma_{S^1}(M) \), where \( D^+_{M_0/S^1} \) is the chiral Dirac operator with respect to the Clifford involution \( \star_{M_0/S^1} \).
Up to this point, the picture looks incomplete as Lott’s geometric proof of (1) works without any distinction on the parity of $N$. In contrast, for the analytical counterpart one needs to distinguish between the Witt and the non-Witt case since in the latter we are forced to impose boundary conditions. This motivates the following question: Does there exist an essentially self-adjoint operator on $M/S^1$, independent of the codimension of the fixed point set in $M$, whose index is precisely the $S^1$-signature? Hope for the existence of such operator relies on the fundamental work [11] of Brüning and Heintze, where the authors develop a machinery to “push-down” self-adjoint operators to quotients of compact Lie group actions. The key observation of their formalism is that, whenever a self-adjoint operator commuting with the group action is restricted to the space of invariant sections, it remains self-adjoint in the restricted domain. Once this result is established, Brüning and Heintze constructed a unitary map $\Phi_1$ between the space of square integrable invariant sections on the open set of principal orbits and the space of square integrable sections of a certain vector bundle defined on the quotient space. This construction seems appropriate for our case of interest because all geometric differential operators on $M$, defined on smooth forms, are essentially self-adjoint since $M$ is closed. The next question is to determine which operator to choose in order to apply Brüning and Heintze’s construction. Two natural candidates are the Hodge-de Rham operator and the odd signature operator. Implementing the procedure described above for these two operators, one obtains only partially satisfactory results. Concretely, the induced operators are indeed self-adjoint by construction, but the resulting potentials do not anti-commute with $\star_{M_0/S^1}$ [27, Sect. 4.3]. This is of course a problem since $\star_{M_0/S^1}$ is the natural involution which should split the desired push-down operator in order to obtain the $S^1$-signature. Nevertheless, going back to the construction of [11], one can see that it is enough to push down a transversally elliptic operator in order to obtain an elliptic operator on the quotient. Using this observation, which enlarges the pool of candidates for the operator, and by analyzing the concrete form of the unitary transformation $\Phi$ defined by Brüning and Heintze, we are able to find an essentially self-adjoint $S^1$-invariant transversally elliptic operator whose induced push-down operator satisfies the desired conditions. Indeed, consider the first-order symmetric transversally elliptic differential operator $B := -c(\chi)d + d^\dagger c(\chi) : \Omega^\epsilon_c(M_0) \to \Omega^\epsilon_c(M_0)$, where $c(\chi)$ denotes the left Clifford action on $\wedge T^*M$ by the characteristic 1-form of the induced foliation by the $S^1$-action. As $B$ commutes with the Gauß–Bonnet grading $\epsilon := (-1)^j$ on $\cdot$-forms, then we define $\mathcal{D}'$ through the following commutative diagram

\[
\begin{array}{ccc}
\Omega^\epsilon_c(M_0)^{S^1} & \xrightarrow{B^\epsilon} & \Omega^\epsilon_c(M_0)^{S^1} \\
\psi_{ev} \downarrow & & \downarrow \psi_{ev} \\
\Omega_c(M_0/S^1) & \xrightarrow{\mathcal{D}'} & \Omega_c(M_0/S^1),
\end{array}
\]

where $\psi_{ev}$ is a modification of the unitary transformation introduced in [12, Sect. 5].
Theorem 1 The operator $\mathcal{D}' : \Omega_c(M_0/S^1) \to \Omega_c(M_0/S^1)$ is given explicitly by
$$\mathcal{D}' = D_{M_0/S^1} + \frac{1}{2} c(\bar{\kappa}) \varepsilon - \frac{1}{2} \tilde{c}(\bar{\phi}_0)(1 - \varepsilon),$$
where $\bar{\kappa}$ is the mean curvature form and $\tilde{c}(\bar{\phi}_0)$ is a bounded endomorphism. In addition $\mathcal{D}'$ satisfies:

1. It anti-commutes with $\star_{M_0/S^1}$.
2. It is essentially self-adjoint.
3. It has the same principal symbol as the Hodge-de Rham operator $D_{M_0/S^1}$.
4. It is discrete.

By the Kato–Rellich theorem, it is enough to study the essentially self-adjoint operator
$$\mathcal{D} := D_{M_0/S^1} + \frac{1}{2} c(\bar{\kappa}) \varepsilon.$$
We call this operator the induced Dirac-Schrödinger operator. Since the mean curvature form can be written close to the fixed point set as $\bar{\kappa} = -d r / r$, one verifies that close to the fixed point set we can express similarly
$$\Psi^{-1} \mathcal{D} \Psi = \gamma \left( \frac{\partial}{\partial r} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes \left( A(r) - \frac{\varepsilon}{2r} \right) \right).$$
One can deduce from [10, Theorem 3.1] that
$$\text{spec} \left( A \varepsilon - \frac{1}{2} \varepsilon \right) \cap \left( -\frac{1}{2}, \frac{1}{2} \right) = \emptyset,$$
which verifies that $\mathcal{D}$ is indeed essentially self-adjoint. Furthermore, it is easy to verify that the parametrix’s construction of [10, Sect. 4] can be adapted to $\mathcal{D}$, which allows us to prove that this operator is discrete.

For the Witt case, we show that the index of $\mathcal{D}^+$ computes the signature invariant $\sigma_{S^1}(M)$.

Theorem 2 In the Witt case, we have for the graded Dirac-Schrödinger operator $\mathcal{D}^+$, the following index identity
$$\text{ind}(\mathcal{D}^+) = \sigma_{S^1}(M) = \int_{M_0/S^1} L \left( T(M_0/S^1), g^{T(M_0/S^1)} \right).$$

Here we have used the fact that in the Witt case the eta invariant of the odd signature operator of the fixed point set vanishes.

For the non-Witt case, the index computation has been so far elusive. Nevertheless, as $\text{ind}(\mathcal{D}^+)$ is still defined in this case and in view of Theorem 2, we expect an analogous result.
2 Brüning–Heintze Construction

The aim on this first section is to give a brief description of the construction, developed in the fundamental work [11] of Brüning and Heintze, of induced self-adjoint operators on quotients of compact Lie group actions.

2.1 The Isomorphism \( \Phi \)

Let \( G \) be a compact Lie group acting on a smooth-oriented Riemannian manifold \( M \) by orientation-preserving isometries. Denote by \( M^G \subset M \) the fixed point set and by \( M_0 \subset M \) the open and dense subset consisting of the union of all principal orbits in \( M \) [16, Theorem 2.8.5]. If \( M \) is connected then so is \( M_0 \). For \( x \in M \), we denote by \( Gx \) and \( G_x \) its orbit and its isotropy group, respectively. The orbit map \( \pi_G : M \to M/G \) induces a Riemannian structure on the orbit space \( M/G \) by requiring \( \pi_G \) to be a Riemannian submersion. We call it the quotient metric. In addition, let \( \pi_E : E \to M \) be a complex vector bundle with Hermitian metric \( \langle \cdot, \cdot \rangle_E \). This metric induces an inner product on the space of continuous sections with compact support \( C_c(M, E) \) by

\[
(s, s')_{L^2(E)} := \int_M \langle s(x), s'(x) \rangle_E \text{vol}_M(x),
\]

where \( s, s' \in C_c(M, E), x \in M \), and \( \text{vol}_M \) denote the Riemannian volume element on \( M \). Define the space \( L^2(E) \) as the Hilbert space completion of \( C_c(M, E) \) with respect to the inner product (2.1).

**Remark 2.1** By [9, Proposition IV.3.7] it follows that \( M - M_0 \) has measure zero with respect to the Riemannian measure, hence \( L^2(E) = L^2(E|_{M_0}) \).

Assume further that \( E \) is a \( G \)-equivariant vector bundle, i.e., the projection \( \pi_E \) commutes with a \( G \)-action on \( E \). In this context, there is an induced action of \( G \) on the space of continuous sections \( C(M, E) \) defined by the relation

\[
(U_g s)(x) := g(s(g^{-1}x)),
\]

where \( g \in G, s \in C(M, E), x \in M \). This action induces a unitary representation of \( G \) in \( L^2(E) \). We say that a section \( s \in C(M, E) \) is \( G \)-invariant if \( U_g s = s \) for all \( g \in G \) and we denote by \( C(M, E)^G \) and \( L^2(E)^G \) the \( G \)-invariant subspaces of \( C(M, E) \) and \( L^2(E) \), respectively.

**Example 2.2** (Exterior Algebra) As before, let \( M \) be an oriented Riemannian manifold on which \( G \) acts by orientation-preserving isometries. The action on \( M \) induces an action on the exterior algebra bundle \( E = \wedge C \cup T^* M := \wedge T^* M \otimes \mathbb{C} \) so that \( E \) becomes a \( G \)-vector bundle over \( M \). The action (2.2) on differential forms is simply given by \( U_g \omega = (g^{-1})^* \omega \).

Now consider the subset

\[
E' := \bigcup_{x \in M_0} E^G_x
\]

(2.3)
where $E_x^{G_x}$ denotes the elements of the fiber $E_x := \pi^{-1}_E(x)$ which are invariant under the $G_x$-action. If $M$ is connected then $E'$ is a $G$-equivariant subbundle of $E|_{M_0}$ [11, Lemma 1.2]. As $G$ acts on $E'$ with one orbit type, then it follows that $F := E'/G$ is a manifold [16, Theorem 2.6.7]. In addition, if $\pi_G' : E' \to F$ denotes the orbit map and $\pi_E' : E' \to M_0$ denotes the projection, then using the Slice Theorem [16, Theorem 2.4.1] one can show that $F$ is a vector bundle over $M_0/G$, of the same rank as $E'$, and the following diagram commutes:

\begin{equation}
\begin{array}{ccc}
E' & \xrightarrow{\pi_G'} & F \\
\downarrow{\pi_E'} & & \downarrow{\pi_F} \\
M_0 & \xrightarrow{\pi_G} & M_0/G.
\end{array}
\end{equation}

**Lemma 2.3** ([27, Lemma 1.13]) The bundle $F$ inherits a Hermitian metric $\langle \cdot , \cdot \rangle_F$ from $E'$ defined by the relation $\langle \pi'_G v_1 , \pi'_G v_2 \rangle_F(y) := \langle v_1 , v_2 \rangle_E(x)$, where $x \in M_0$, $v_1 , v_2 \in E'_x$, and $\pi'_G(x) = y$.

For $y \in M_0/G$ let $h(y) := \text{vol}(\pi'^{-1}_G(y))$ be the *volume of the orbit* containing a point in $\pi'^{-1}_G(y)$. We can consider the weighted inner product on $C_c(M_0/G , F)$ defined by the formula

\begin{equation}
(s , s')_{L^2(F,h)} := \int_{M_0/G} \langle s(y) , s'(y) \rangle_F h(y) \text{vol}_{M_0/G}(y),
\end{equation}

where $\text{vol}_{M_0/G}$ denotes the Riemannian volume element of $M_0/G$ with respect to the quotient metric. Analogously we define $L^2(F,h)$ to be the completion of $C_c(M_0/G , F)$ with respect to this inner product. We now state one of the most important results of [11]. The spirit of the proof relies on the construction above and on Remark 2.1.

**Theorem 2.4** ([11, Theorem 1.3]) There is an isometric isomorphism of Hilbert spaces

$$
\Phi : L^2(E)^G \to L^2(F,h).
$$

With $\pi'_G : E' \to F$ denoting the orbit map $\Phi$ is given by

$$
\Phi s_1 \circ \pi_G(x) = \pi'_G \circ s_1(x),
$$

where $s_1 \in C_c(M_0 , E)^G$ and $x \in M_0$. Its inverse map is given by

$$
\Phi^{-1} s_2(x) = s_2 \circ \pi_G(x) \cap E_x,
$$

where $s_2 \in C_c(M_0/G , F)$ and $x \in M_0$. 

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2.2 Induced Operators on the Principal Orbit Type

In the context of the subsection above, consider a self-adjoint operator \( R : \text{Dom}(R) \subseteq L^2(E) \rightarrow L^2(E) \) commuting with the \( G \)-action, that is, \( U_g(\text{Dom}(R)) \subseteq \text{Dom}(R) \) and \( U_g R(s) = RU_g(s) \) for all \( g \in G \) and \( s \in \text{Dom}(R) \).

Lemma 2.5 ([11, Lemma 2.2]) The operator \( S := R|_{\text{Dom}(R) \cap L^2(E)^G} \) is a well-defined self-adjoint operator on \( L^2(E)^G \) with \( \text{Dom}(S) = \text{Dom}(R)^G \).

From Theorem 2.4 and Lemma 2.5, we deduce the following remarkable result which allows to “push down” a self-adjoint operator commuting to \( G \) to the quotient.

Proposition 2.6 ([11, pp. 178–179]) The operator \( T : \text{Dom}(T) \subseteq L^2(F, h) \rightarrow L^2(F, h) \) defined by \( T := \Phi \circ S \circ \Phi^{-1}|_{\Phi(\text{Dom}(S))} \), with \( \text{Dom}(T) := \Phi(\text{Dom}(S)) \) is self-adjoint.

A particular case of interest is when \( R \) is generated by an elliptic differential operator \( D : C^\infty_c(M, E) \rightarrow C^\infty_c(M, E) \) so that \( C^\infty_c(M, E) \subseteq \text{Dom}(R) \) and \( R|_{C^\infty_c(M, E)} = D \). The following result states that in this case, the induced operator \( T \) of Proposition 2.6 is also generated by a differential operator of the same order.

Proposition 2.7 ([11, Theorem 2.4]) If \( R \) is generated by a differential operator \( D \) of order \( k \), then \( T \) is also generated by a certain differential operator \( D' \) of order \( k \). Their principal symbols are related by the formula \( \sigma_p(D')(y, \xi) = \pi_G^{-1}(y, \xi))(e) \), where \( y \in M_0/G, \xi \in T^*_y(M_0/G), \xi \in \pi_G^{-1}(y), \) and \( e \in E'_x \). In particular, the operator \( D' \) is elliptic if \( D \) is transversally elliptic.

3 Lott’s \( S^1 \)-Equivariant Signature Formula

3.1 Definition of the Equivariant \( S^1 \)-Signature

Let \( (M, \mathbf{g}^TM) \) be an \( 4k + 1 \) dimensional, closed, oriented Riemannian manifold on which the circle \( S^1 \) acts by orientation-preserving isometries. Let us denote by \( V \) the generating vector field of the action and by \( \iota : M^{S^1} \rightarrow M \) the inclusion of the fixed point set into \( M \). We can define two sub-complexes of the de Rham complex of \( M \),

\[
\Omega_{\text{bas}}(M) := \{ \omega \in \Omega(M) \mid L_V \omega = 0 \text{ and } \iota^* \omega = 0 \},
\]

\[
\Omega_{\text{bas}}(M, M^{S^1}) := \{ \omega \in \Omega(M)_{\text{bas}} \mid \iota^* \omega = 0 \},
\]

where \( L_V \) is the Lie derivative. Denote by \( H^*_\text{bas}(M) \) and \( H^*_\text{bas}(M, M^{S^1}) \) their respective cohomology groups. It can be shown that there exist isomorphisms [25, Proposition 1]

\[
H^*_\text{bas}(M, M^{S^1}) \cong H^*_\text{bas,c}(M - M^{S^1}) \cong H^*(M/S^1, M^{S^1}); \quad (3.1)
\]
where the subscript $c$ denotes cohomology with compact support. These cohomology groups are all $S^1$-homotopy invariant [25, Proposition 2].

Using the musical isomorphisms induced by the Riemannian metric, one defines the 1-form $\alpha$ on $M - M^{S^1}$ by $\alpha := V^b/\|V\|^2$ so that $\alpha(V) = 1$. Here we list some important properties of $\alpha$.

**Proposition 3.1** ([25, Sect. 2]) The following relations hold true:

1. The 2-form $d\alpha$ is basic.
2. The form $\alpha$ satisfies $L_V \alpha = 0$, that is, $\alpha$ is $S^1$-invariant.
3. If $\omega \in \Omega^{4k-1}_{bas,c}(M - M^{S^1})$ then
   $$\int_M \alpha \wedge d\omega = 0.$$

**Definition 3.2** ([25, Definition 4]) The equivariant $S^1$-signature $\sigma_{S^1}(M)$ of $M$ with respect to the $S^1$-action is defined as the signature of the symmetric quadratic form

$$H^{2k}_{bas,c}(M - M^{S^1}) \times H^{2k}_{bas,c}(M - M^{S^1}) \to \mathbb{R}$$

$$\langle \omega, \omega' \rangle \mapsto \int_M \alpha \wedge \omega \wedge \omega'.$$

**Remark 3.3** It was shown by Lott that $\sigma(M)_{S^1}$ is independent of the Riemannian metric [25, Proposition 5] and that if $f : M \to N$ is a orientation-preserving $S^1$-homotopy equivalence then $\sigma_{S^1}(M) = \sigma_{S^1}(N)$ [25, Proposition 6].

### 3.2 The Equivariant $S^1$-Signature Formula for Semi-free Actions

Let us assume now that the action is semi-free, which means that the isotropy groups are either the trivial group or the whole $S^1$. Let $M_0 := M - M^{S^1}$ be the set of principal orbits where the action is free. We equip the manifold $M_0/S^1$ with the quotient metric $g^{T(M_0/S^1)}$. In this context, the dimension of the fixed point set $M^{S^1}$ must be odd, so we can consider its associated odd signature operator [1, Equation 4.6] and the corresponding eta invariant $\eta(M^{S^1})$ [1, Equation 1.7]. One of the most important results in [25] is the following index-like formula for the equivariant $S^1$-signature.

**Theorem 3.4** ([25, Theorem 4]) Suppose $S^1$ acts effectively and semi-freely on $M$, then

$$\sigma_{S^1}(M) = \int_{M_0/S^1} L\left(T(M_0/S^1), g^{T(M_0/S^1)}\right) + \eta(M^{S^1}),$$

where $L\left(T(M_0/S^1), g^{T(M_0/S^1)}\right)$ denotes the L-polynomial in the curvature of $g^{T(M_0/S^1)}$. 

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Fig. 1 Mapping cylinder of the $\mathbb{C}P^N$-fibration $\pi_F : \mathcal{F} \rightarrow F$

Remark 3.5 It is important to emphasize that part of the conclusion of Theorem 3.4 is the convergence of the integral of the $L$-polynomial over the open manifold $M_0/S^1$.

3.2.1 Sketch of the Proof of the Signature Formula

We now give a brief description of the proof of Theorem 3.4 presented in [25, Sect. 2.3]. Let $F \subset M^{S^1}$ be a connected component of the fixed point set. As the action is orientation preserving the dimension of $F$ must be odd and can be written as $\dim F = 4k - 2N - 1$, for some $N \in \mathbb{N}_0$. Let $NF$ denote the normal bundle of $F$ in $M$.

The first step of the proof is to model a neighborhood of $F$ in $M/S^1$ as the mapping cylinder $C(F)$ of the projection of a Riemannian fiber bundle $\pi_F : F \rightarrow F$, where $F := S/S^1$ and $S := SNF$ denote the associated sphere bundle (Fig. 1). By [30, Lemma 2.2], it follows that the fibers of $\mathcal{F}$ are copies of $\mathbb{C}P^N$.

For $t > 0$, let $N_t(F)$ be the $t$-neighborhood of $F$ in $M/S^1$. Using [5, Proposition A.III.5] one can verify $\sigma_{S^1}(M) = \sigma(M/S^1 - N_t(F))$ for $r$ small enough. Applying the Atiyah–Patodi–Singer signature theorem, we get

$$\sigma(M/S^1 - N_t(F)) = \int_{M/S^1 - N_t(F)} L\left(T(M/S^1 - N_t(F)), g^{T(M/S^1 - N_t(F))}\right) + \int_{\partial N_t(F)} TL(\partial N_t(F)) + \eta(\partial N_t(F)).$$

The main idea of Lott’s proof is to study the behavior of the terms in (3.2) as $r \rightarrow 0$. Let us see how to do this for the first term. Let $\{e^i\}_{i=1}^{2N} \cup \{f^\alpha\}_{\alpha=1}^{4k-2N-1}$ be a local orthonormal basis for $T^*\mathcal{F}$ as in [7, Sect. III(c)]. Let $\omega$ be the connection 1-form of the Levi–Civita connection associated to this basis. The components of $\omega$ satisfy the structure equations

$$de^i + \omega^j_i \wedge e^j + \omega^j_i \wedge f^\alpha = 0,$$

$$df^\alpha + \omega^\beta_j \wedge e^j + \omega^\beta_j \wedge f^\beta = 0.$$

Let $\Omega$ denote the associated curvature 2-form. We can construct a local orthonormal basis for $T^*C(\mathcal{F})$ from the basis above as $\{dr\} \cup \{\hat{e}^u_i\}_{i=1}^{u} \cup \{\hat{f}^\alpha_{\alpha=1}^h\}$ for $0 < r < t$. 

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where $\hat{e}^i := re^i$ and $\hat{f}^\alpha := f^\alpha$. The associated connection 1-form $\hat{\omega}$ satisfies the structure equations

$$
\hat{\omega}^r_i \wedge \hat{e}^i + \hat{\omega}^\alpha_i \wedge \hat{f}^\alpha = 0,
$$
$$
d\hat{e}^i + \hat{\omega}^i_j \wedge \hat{e}^j + \hat{\omega}^i_\alpha \wedge \hat{f}^\alpha + \hat{\omega}^i_r \wedge dr = 0,
$$
$$
d\hat{f}^\alpha + \hat{\omega}^\alpha_j \wedge \hat{e}^j + \hat{\omega}^\alpha_\beta \wedge \hat{f}^\beta + +\hat{\omega}^\alpha_r \wedge dr = 0.
$$

It can be shown that the components of $\hat{\omega}$ are

$$
\hat{\omega}^i_j = \omega^i_j,
$$
$$
\hat{\omega}^i_r = e^i,
$$
$$
\hat{\omega}^i_\alpha = r\omega^i_\alpha,
$$
$$
\hat{\omega}^\alpha_\beta = \omega^\alpha_\gamma f^\gamma + r^2 \omega^\alpha_\beta e^i,
$$
$$
\hat{\omega}^\alpha_r = 0.
$$

(3.3)

Using these expressions one can compute the components of the curvature 2-form $\hat{\Omega}$ as $r \to 0$ to show that the limit of the first term in the right-hand side of (3.2) is well defined. The transgression term in (3.2) vanishes using equivariant methods [6, Proposition 7.35]. For the limit of the eta invariant one uses Dai’s formula [15, Theorem 0.3]. One can verify that both the eta form ([6, Sects. 9.4, 10.7], [18, Sect. 1.c]) and the tau invariant [8, pp. 170, 270] vanish.

### 3.2.2 The Witt Condition

Now we comment on an important topological interpretation of the $S^1$-signature in the context of intersection homology theory introduced by Goresky and MacPherson in [19]. As we saw in the proof of Theorem 3.4, each connected component $F \subset M^{S^1}$ of the fixed point set is an odd-dimensional closed manifold, whose dimension can be written as $\dim F = 4k - 2N - 1$. We now distinguish the two possible cases for $N$. We say that $M / S^1$ satisfies the Witt condition, if $N$ is odd, that is, the codimension of the fixed point set $M^{S^1}$ in $M$ is divisible by four. Note that in this case $\eta(M^{S^1}) = 0$. This follows from [1, Remark (3), (4), p. 61] since $4k - 2N - 1 = 2(2k - N) - 1$ and $2k - N$ is odd if and only if $N$ is odd. Stratified spaces satisfying that the middle dimensional cohomology of the links vanish are called Witt spaces [28]. This is of course consistent with the definition above since $H^N(\mathbb{C}P^N)$ vanishes if and only if $N$ is odd. For this kind of stratified spaces one can always construct the Goresky–MacPherson $L$-class following the procedure described in detail in [4, Sect. 5.3], for example. The following result describes an explicit form of the $L$-homology class for our case of interest.

**Proposition 3.6** (L-homology Class, [25, Proposition 8]) In the Witt case, the differential form $L(T(M / S^1))$ represents the $L$-homology class of $M / S^1$. 

\[ \text{Springer} \]
In addition, for Witt spaces there is a well-defined non-degenerate pairing in intersection homology \[4, \text{Sect.} \ 4.4\] which gives rise to a signature invariant.

**Corollary 3.7** (\[25, \text{Corollary} \ 1\]) In the Witt case,

\[
\sigma_{S^1}(M) = \int_{M_0/S^1} L(T(M_0/S^1)),
\]

equals the intersection homology signature of \(M/S^1\).

### 4 Induced Dirac–Schrödinger Operator on \(M_0/S^1\)

The goal of this section is to implement the construction described in Sect. 2 for the special case of a semi-free \(S^1\)-action, as discussed in Sect. 3.2.

#### 4.1 The Mean Curvature 1-Form

Let \(M\) be an \((n + 1)\)-dimensional oriented, closed Riemannian manifold on which \(S^1\) acts by orientation-preserving isometries (before we had \(n = 4k\), but here we treat the general case). Denote by \(\nabla\) its associated Levi–Civita connection. As in Sect. 3.1, let \(V\) be the generating vector field of the \(S^1\)-action. The flow of the vector field \(V\) generates a 1-dimensional foliation \(L\) on \(M_0\). We will denote by \(L^\perp\) the transverse distribution, i.e., \(TM_0 = L \oplus L^\perp\). The distribution \(L\) is always integrable and the corresponding integral curves are precisely the \(S^1\)-orbits. In contrast, the transverse distribution is not necessarily integrable.

**Definition 4.1** Let \(X := V/\|V\|\) be the unit vector field which defines the foliation \(L\). Using the musical isomorphism induced by the metric, we define (see \[29\])

1. The associated characteristic 1-form \(\chi := X^\flat\).
2. The mean curvature vector field \(H := \nabla_X X\).
3. The mean curvature 1-form 1-form \(\kappa := H^\flat\).

It is easy to see that the mean curvature vector field satisfies \(H \in C^\infty(L^\perp)\). As a consequence \(\kappa\) is horizontal, i.e., \(\iota_X \kappa = 0\).

**Lemma 4.2** (\[29, \text{Chapter} \ 6\]) The mean curvature form \(\kappa\) satisfies \(\kappa = L_X \chi\).

Let \(\alpha := V^\flat/\|V\|^2\) be the 1-form considered in Sect. 3.1. The following statement is a consequence of Proposition 3.1(2) and Lemma 4.2.

**Corollary 4.3** The characteristic 1-form \(\chi\) satisfies \(L_V \chi = 0\), that is \(\chi\) is \(S^1\)-invariant.

We can combine Cartan’s formula and Lemma 4.2 to get \(\kappa = \iota_X d\chi\), or equivalently

\[
d\chi + \kappa \wedge \chi := \varphi_0, \tag{4.1}
\]
where \( \varphi_0 \) satisfies \( \iota_X \varphi_0 = 0 \), i.e., \( \varphi_0 \) is horizontal. Equation (4.1) is known as Rummel’s formula and it holds for general tangentially oriented foliations ([6, Lemma 10.4], [29, Chapter 4]). Observe that the characteristic form \( \chi \) can be thought as the volume form on each leaf of the foliation as the vector field \( X \) satisfies \( X \in C^\infty(L) \), it is \( S^1 \)-invariant and \( \| \chi \| = 1 \). In particular, the volume of the orbit function \( h : M_0/S^1 \to \mathbb{R} \) used in (2.5) can be written explicitly as

\[
h(y) = \int_{\pi_{S^1}^{-1}(y)} \chi. \tag{4.2}
\]

**Lemma 4.4** The exterior derivative of the volume of the orbit function is \( dh = -h \kappa \). Thus, the mean curvature form \( \kappa \) measures the volume change of the orbits.

**Proof** We use [8, Proposition 6.14.1] and (4.1) to compute

\[
dh = d \int_{\pi_{S^1}^{-1}(y)} \chi = \int_{\pi_{S^1}^{-1}(y)} d\chi = -\int_{\pi_{S^1}^{-1}(y)} \kappa \wedge \chi + \int_{\pi_{S^1}^{-1}(y)} \varphi_0 = -h \kappa,
\]

where we have used that the integral of \( \varphi_0 \) is zero because this is a horizontal 2-form. \( \square \)

The next proposition shows that all the geometric quantities discussed above are encoded in the norm of the generating vector field \( V \).

**Proposition 4.5** ([27, Proposition 4.7]) In terms of \( \| V \| \), we can express

1. \( \chi = \| V \| \alpha \).
2. \( \kappa = -d \log(\| V \|) \).
3. \( \varphi_0 = \| V \| d\alpha \).

**Corollary 4.6** The mean curvature 1-form \( \kappa \) is closed and basic.

We end this subsection with some properties of the 2-form \( \varphi_0 \).

**Proposition 4.7** ([27, Proposition 4.9]) The following relations for \( \varphi_0 \) hold:

1. If \( Y_1, Y_2 \in C^\infty(L^\perp) \), then \( \varphi_0(Y_1, Y_2) = -\chi([Y_1, Y_2]) \).
2. \( d\varphi_0 + \kappa \wedge \varphi_0 = 0 \).
3. \( \varphi_0 \in \Omega^2_{\text{bas}}(M_0) \).

### 4.2 The Operator \( T(D) \)

Let us consider now the Hermitian \( S^1 \)-equivariant vector bundle \( E := \bigwedge C T^* M \) of Example 2.2. Recall that the action on differential forms is given by the pullback \( U_g \omega := (g^{-1})^* \omega \) for \( g \in S^1 \). As the Hodge star operator on \( M \) commutes with the \( S^1 \)-action on differential forms [27, Lemma 4.11], then by Proposition 4.11(4) we obtain the following known result.
Proposition 4.8  The Hodge-de Rham operator $D = d + d\dagger$ of $M$ defined on the core of differential forms $\Omega(M) := C^\infty(M, \wedge C T^* M)$ is $S^1$-invariant.

This shows that we are in position to apply the construction of Brüning and Heintze described in Sect. 2. The strategy is then as follows:

1. Construct explicitly the vector bundle $F \longrightarrow M_0/S^1$ and describe the $L^2$-inner product (2.5).
2. Understand the isomorphism $\Phi_1$ of Theorem 2.4.
3. Describe the self-adjoint operator $D : \Omega(M)^{S^1} \longrightarrow \Omega(M)^{S^1}$ of Lemma 2.5
4. Explicitly compute the self-adjoint operator $T$ of Proposition 2.6 and describe its properties. For example, compute its principal symbol (Proposition 2.7).

Remark 4.9 ([24, Chapter II.5]) The Hodge-de Rham operator is the associated Dirac operator of the Clifford bundle $\wedge C T^* M$ with left Clifford action $c(\omega) := \omega \wedge -\iota_\omega ^\star$, which satisfies the relations $c(\omega)^2 = -\|\omega\|^2$ and $c(\omega)^\dagger = -c(\omega)$. The corresponding right Clifford action is $\hat{c}(\omega) := \omega \wedge +\iota_\omega ^\star$.

4.2.1 Decomposition of $S^1$-Invariant Differential Forms

We begin with a decomposition result of the space of $S^1$-invariant forms in terms of the basic forms. Recall that we have the inclusion $\Omega_{\text{bas}}(M_0) \subset \Omega(M)^{S^1}$, from the Lie derivative vanishing condition.

Proposition 4.10 ([27, Corollary 3.14]) Any $S^1$-invariant form $\omega \in \Omega(M_0)^{S^1}$ can be uniquely decomposed as $\omega = \omega_0 + \omega_1 \wedge \chi$, where $\omega_0, \omega_1 \in \Omega_{\text{bas}}(M_0)$. With respect to this decomposition, we will represent the form $\omega$ as the column vector

$$\omega = \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix}.$$ 

4.2.2 Construction of the Bundle $F$

We start by pointing out some important remarks:

- The action on $M_0$ is free and therefore the $S^1$-invariant bundle $E'$ of (2.3) is nothing else but $E' = \wedge C T^* M_0$.
- As a consequence, by counting dimensions, we see that the rank of $F$ must agree with the rank of $E'$, which is $\text{rk}(E') = 2^{n+1}$.
- From [26, Lemma 6.44] it follows that for each basic form $\beta \in \Omega_{\text{bas}}'(M_0)$ there exists a unique $\tilde{\beta} \in \Omega'(M_0/S^1)$ such that $\pi_{S^1}^* \tilde{\beta} = \beta$. Thus, using Corollary 4.10 we can identify $\Omega(M_0)^{S^1} \cong \Omega(M_0/S^1) \otimes C^2$, via the orbit map $\pi_{S^1}$.

These observations indicate that $F := E'/S^1 = \wedge C T^* (M_0/S^1) \oplus \wedge C T^* (M_0/S^1)$. Indeed, given $x \in M_0$ and $\omega_x = \omega'_x + \omega''_x \wedge \chi_x \in \wedge C T^*_x M_0$ where $\iota_{V_x} \omega'_x = \iota_{V_x} \omega''_x = 0$, the orbit map on $E'$ is explicitly given by
\[ \pi_{S^1} : E' = \wedge_{S^1} T^* M_0 \longrightarrow F = \wedge_{S^1} T^*(M_0/S^1) \oplus \wedge_{S^1} T^*(M_0/S^1) \]

\[ \omega_x = \omega'_x + \omega''_x \wedge \chi_x \mapsto \left( \bar{\omega}'_y, \bar{\omega}''_y \right), \]

(4.3)

where \( \pi_{S^1}(x) = y \) and the form \( \bar{\omega}'_y \in \wedge_{S^1} T^*(M_0/S^1) \) (similarly for \( \bar{\omega}''_y \)) is defined by the relation \( \omega_x(v_x) = \bar{\omega}_y((\pi_{S^1})_x v_x) \) for all \( v_x \in T_x M_0 \). Hence, the diagram (2.4) becomes,

\[ \begin{array}{ccc}
E' = \wedge_{S^1} T^* M_0 & \xrightarrow{\pi_{S^1}} & F = \wedge_{S^1} T^*(M_0/S^1) \oplus \wedge_{S^1} T^*(M_0/S^1) \\
\pi_E & & \pi_F \\
M_0 & \xrightarrow{\pi_{S^1}} & M_0/S^1.
\end{array} \]

### 4.2.3 Description of the Isomorphism \( \Phi \)

Given an \( S^1 \)-invariant form with compact support \( \omega \in \Omega_c(M_0)^{S^1} \) there are two unique compactly supported basic differential forms \( \omega_0, \omega_1 \in \Omega_{bas,c}(M_0) \) such that \( \omega = \omega_0 + \omega_1 \wedge \chi \). With respect to the vector notation introduced in Corollary 4.10, we write

\[ \left( \begin{array}{c}
\omega_0 \\
\omega_1
\end{array} \right) = \left( \begin{array}{c}
\pi_{S^1}^* \bar{\omega}_0 \\
\pi_{S^1}^* \bar{\omega}_1
\end{array} \right), \]

where \( \bar{\omega}_0, \bar{\omega}_1 \in \Omega_c(M_0/S^1) \). This representation allows us to express the isomorphism \( \Phi \), on compactly supported forms, as

\[ \Phi : \Omega_c(M_0)^{S^1} \longrightarrow \Omega_{bas,c}(M_0) \oplus \Omega_{bas,c}(M_0) \longrightarrow \Omega_c(M_0/S^1) \oplus \Omega_c(M_0/S^1) \]

\[ \omega \longmapsto \left( \begin{array}{c}
\omega_0 \\
\omega_1
\end{array} \right) = \left( \begin{array}{c}
\pi_{S^1}^* \bar{\omega}_0 \\
\pi_{S^1}^* \bar{\omega}_1
\end{array} \right) \longmapsto \left( \begin{array}{c}
\bar{\omega}_0 \\
\bar{\omega}_1
\end{array} \right). \]

We can extend this map to \( \Phi : L^2(M)^{S^1} \longrightarrow L^2(F, h) \) by density.

### 4.2.4 Description of the Operator \( S(D) \)

Now we want to understand the operator \( S \) of Lemma 2.5 associated to the Hodge-de Rham operator \( D = d + d^\dagger \). First, recall that the formal adjoint of the exterior derivative can be written as \( d^\dagger = (-1)^n \star d \star \), where \( \star \) is the chirality involution, associated to the
Clifford bundle $\wedge \mathbb{C} T^* M$, which is defined on $j$-forms by $\star := i^{((m+1)/2)+2mj+j(j-1)}$. [6, Lemma 3.17].

**Proposition 4.11** ([6, Proposition 3.58]) *The chirality operator $\star$ satisfies:*

1. $\star^2 = 1$.
2. $\star^\dagger = \star$.
3. For $\alpha \in T^* M$, $\star (\alpha \wedge \star) = (-1)^{n+1} \iota_\alpha \#$.
4. $d^\dagger = (-1)^n d \star$.
5. $\star c(\omega) = (-1)^n c(\omega) \star$ and $\star \tilde{c}(\omega) = (-1)^{n+1} \tilde{c}(\omega) \star$.
6. $\star \varepsilon = (-1)^{n+1} \varepsilon \star$.
7. $\star D = (-1)^n D \star$.

The strategy is to study $S(D)$ through the decomposition of Corollary 4.10, that is

$$S(D) := \left( d + (-1)^n d \star \right) \bigg|_{\Omega_c(M_0)^{S^1}} : \bigoplus_{\Omega_c(M_0)^{S^1}} \bigoplus_{\Omega_{\text{bas},c}(M_0)^{S^1}} \Omega_{\text{bas},c}(M_0)^{S^1}.$$

For the decomposition of $\star$ we follow the techniques of [29, Chapter 7].

**Definition 4.12** The *basic Hodge star operator* is defined as the linear map

$$\bar{\star} : \Omega_{\text{bas}}^j(M_0) \rightarrow \Omega_{\text{bas}}^{n-j}(M_0),$$

satisfying the conditions

$$\bar{\star} \beta = (-1)^{n-j} \star (\beta \wedge \chi), \quad (4.4)$$
$$\star \beta = \bar{\star} \beta \wedge \chi, \quad (4.5)$$

where $\star$ is the Hodge star operator of $M$.

**Remark 4.13** Observe that the volume form can be written as $\text{vol}_{M_0} = \star 1 = \bar{\star} 1 \wedge \chi$.

**Lemma 4.14** The operator $\bar{\star}$ satisfies $\bar{\star}^2 = (-1)^{j(n-j)}$ on $j$-forms.

In view of this lemma, we can define a chirality operator on basic differential forms as in [21, Sect. 5]. The following result follows from Proposition 4.11.

**Proposition 4.15** The *basic chirality operator*

$$\bar{\star} : \Omega_{\text{bas}}^j(M_0) \rightarrow \Omega_{\text{bas}}^{n-j}(M_0)$$

defined by $\bar{\star} := i^{((n+1)/2)+2nj+j(j-1)} \star$ satisfies the relations analogous to Proposition 4.11.
Lemma 4.16 With respect to the decomposition of Corollary 4.10, we can express the operator $\star$ as
\[
\star \bigg|_{\Omega(M_0)^{S^1}} = i^{q(n)}(-1)^n \begin{pmatrix} 0 & -\varepsilon \star \\ \varepsilon \star & 0 \end{pmatrix},
\]
where $q(n) := (n - 1) \text{mod}(2)$.

Proof Recall that $[\cdot]$ denotes the integer part function. First observe the relation
\[
\left\lfloor \frac{n+1}{2} \right\rfloor + q(n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.
\]
For $\beta$ a basic $j$-form, we calculate
\[
\star \beta = i^{[n/2]+2(n+1)j+j(j-1)} \star \beta = i^{q(n)+2j+(n+1)/2+2nj+j(j-1)} \varepsilon \star \beta \wedge \chi = (i^{q(n)} \varepsilon \beta) \wedge \chi = (i^{q(n)}(-1)^n \varepsilon \star \beta) \wedge \chi.
\]
On the other hand using Proposition 4.11 we compute,
\[
\star (\beta \wedge \chi) = (\star \circ (\chi \wedge) \circ \varepsilon) \beta = (-1)^{n+1} (\iota_{\chi^z} \circ \star \circ \varepsilon) \beta = (\iota_{\chi^z} \circ \varepsilon \circ \star) \beta.
\]
Finally, using the first computation above, we conclude that
\[
(\iota_{\chi^z} \circ \varepsilon \circ \star) \beta = \iota_{\chi^z} \varepsilon ((i^{q(n)}(-1)^n \varepsilon \star \beta) \wedge \chi) = -i^{q(n)}(-1)^n \iota_{\chi^z}(\chi \wedge (\varepsilon \star \beta)) = -i^{q(n)}(-1)^n \varepsilon \star \beta.
\]
\[\square\]

We are now ready to describe the operator $S(D)$ of Lemma 2.5.

Theorem 4.17 With respect to the decomposition of Corollary 4.10, the exterior derivative decomposes as
\[
d \bigg|_{\Omega(M_0)^{S^1}} = \begin{pmatrix} d & \varepsilon \varphi_0 \wedge \\ 0 & d - \kappa \wedge \end{pmatrix}
\]
and its formal adjoint as
\[
d^\dagger \bigg|_{\Omega(M_0)^{S^1}} = \begin{pmatrix} (\iota_{\chi^z} \circ \varepsilon \circ \star) \beta = (i^{q(n)}(-1)^n \varepsilon \star \beta) \wedge \chi = -i^{q(n)}(-1)^n \iota_{\chi^z}(\chi \wedge (\varepsilon \star \beta)) = -i^{q(n)}(-1)^n \varepsilon \star \beta. \\ (\iota_{\chi^z} \circ \varepsilon \circ \star) \beta = \iota_{\chi^z} \varepsilon ((i^{q(n)}(-1)^n \varepsilon \star \beta) \wedge \chi) = -i^{q(n)}(-1)^n \iota_{\chi^z}(\chi \wedge (\varepsilon \star \beta)) = -i^{q(n)}(-1)^n \varepsilon \star \beta. 
\end{pmatrix}
\]
Hence, the restriction of the Hodge-de Rham operator $D$ to the space of $S^1$-invariant forms with respect to this decomposition is
\[
S(D) := D \bigg|_{\Omega(M_0)^{S^1}} = \begin{pmatrix} d + (\iota_{\chi^z} \circ \varepsilon \circ \star) \beta = (i^{q(n)}(-1)^n \varepsilon \star \beta) \wedge \chi = -i^{q(n)}(-1)^n \iota_{\chi^z}(\chi \wedge (\varepsilon \star \beta)) = -i^{q(n)}(-1)^n \varepsilon \star \beta. \\ (\iota_{\chi^z} \circ \varepsilon \circ \star) \beta = \iota_{\chi^z} \varepsilon ((i^{q(n)}(-1)^n \varepsilon \star \beta) \wedge \chi) = -i^{q(n)}(-1)^n \iota_{\chi^z}(\chi \wedge (\varepsilon \star \beta)) = -i^{q(n)}(-1)^n \varepsilon \star \beta. 
\end{pmatrix}
\]
Proof Let \( \omega_0 + \omega_1 \wedge \chi \in \Omega(M_0)^{S^1} \), using (4.1) we compute (cf. [6, Proposition 10.1]),

\[
d(\omega_0 + \omega_1 \wedge \chi) = d\omega_0 + d\omega_1 \wedge \chi - (\varepsilon \omega_1) \wedge \kappa \wedge \chi + (\varepsilon \omega_1) \wedge \varphi_0
\]

from where we obtain the desired decomposition for the exterior derivative. For the adjoint, we first calculate using the decomposition of \( d \) and Lemma 4.16,

\[
d\star \bigg|_{\Omega(M_0)^{S^1}} = i^q(n)(-1)^n \begin{pmatrix} d \quad \varepsilon \varphi_0 \wedge \kappa \wedge \chi & 0 & -\varepsilon \star \varphi_0 \\ 0 & d - \kappa \wedge \chi & 0 \end{pmatrix}
\]

\[
= i^q(n)(-1)^n \begin{pmatrix} \varphi_0 \wedge \star & -d\varepsilon \star \\ (d - \kappa \wedge \chi)\varepsilon \star & 0 \end{pmatrix}
\]

\[
= i^q(n)(-1)^n \begin{pmatrix} \varphi_0 \wedge \star & \varepsilon d \star \\ -\varepsilon (d - \kappa \wedge \chi)\star & 0 \end{pmatrix}.
\]

The result then follows from the relation \( (i^q(n))^2 = (-1)^{n+1} \) and Proposition 4.15.

\[\Box\]

4.2.5 Construction of the Operator \( T(D) \)

Now that we have described the isomorphism \( \Phi \) and the operator \( S(D) \), we can compute the self-adjoint operator \( T := \Phi \circ S \circ \Phi^{-1} \) of Proposition 2.6. As \( D \) is a first-order differential operator, Proposition 2.7 ensures \( T \) is also generated by a differential operator of the same order. Let us begin with the Hodge star operator. In view of Remark 4.13, we choose the sign of volume form \( \text{vol}_{M_0/S^1} \) on \( M_0/S^1 \) so that \( \pi^* \text{vol}_{M_0/S^1} := \tilde{\star}1 \). This means that we can express \( \text{vol}_{M_0} = \pi^* \text{vol}_{M_0/S^1} \wedge \chi \).

With this choice we can identify \( \tilde{\star} \), via the orbit map \( \pi_{S^1} \), with the Hodge star operator \( \star_{M_0/S^1} \) of \( M_0/S^1 \) with respect to the quotient metric. Moreover, the following diagram commutes [27, Corollary 4.25].

\[
\begin{array}{c}
\Omega_{\text{bas}}(M_0) \xrightarrow{\star} \Omega_{\text{bas}}(M_0) \\
\pi^*_{S^1} \downarrow \quad \pi^*_{S^1} \\
\Omega(M_0/S^1) \xrightarrow{\star_{M_0/S^1}} \Omega(M_0/S^1).
\end{array}
\]

Now let us study the zero-order terms. Since the forms \( \kappa \) and \( \varphi_0 \) are both basic then there exist unique \( \tilde{\kappa} \in \Omega^1(M_0/S^1) \) and \( \tilde{\varphi}_0 \in \Omega^2(M_0/S^1) \) such that \( \kappa = \pi^*_{S^1}(\tilde{\kappa}) \) and \( \varphi_0 = \pi^*_{S^1}(\tilde{\varphi}_0) \). Moreover, as pullback commutes with the wedge product, then the following diagram commute (similarly for \( \varphi_0 \wedge \)
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Next we consider the term $-\bar{\ast}(\phi_0 \wedge \bar{\ast})$ as an operator on the quotient space. In view of (4.6) and to lighten the notation, we are going to identify $\bar{\ast} \equiv \ast_{M_0/S^1}$.

**Proposition 4.18** ([27, Proposition 4.26]) With respect to the quotient metric on $M_0/S^1$ we have $(\bar{\phi}_0 \wedge \bar{\ast})^\dagger = -\bar{\ast}(\phi_0 \wedge \bar{\ast})$. In particular, the operator $\hat{c}(\bar{\phi}_0) := (\phi_0 \wedge + (\bar{\phi}_0 \wedge)^\dagger$ satisfies

1. $\hat{c}(\bar{\phi}_0) + \bar{\ast}\hat{c}(\bar{\phi}_0) = 0$.
2. $\hat{c}(\bar{\phi}_0) - \hat{c}(\bar{\phi}_0) = 0$.

Finally we treat the first-order terms of $S(D)$ in Theorem 4.17. For the exterior derivative, as it also commutes with pullbacks, we have an analogous commutative diagram,

$$
\begin{array}{ccc}
\Omega_{bas}(M_0) & \xrightarrow{\partial} & \Omega_{bas}(M_0) \\
\pi^*_{S^1} & & \pi^*_{S^1} \\
\Omega(M_0/S^1) & \xrightarrow{\partial_{M_0/S^1}} & \Omega(M_0/S^1),
\end{array}
$$

where $\partial_{M_0/S^1}$ is the exterior derivative of $M_0/S^1$. Hence, it remains to study the operator

$$
(-1)^{n+1}\bar{\ast}\partial \bar{\ast} : \Omega_{bas}^j(M_0) \rightarrow \Omega_{bas}^{j-1}(M_0).
$$

**Remark 4.19** Let $\partial^\dagger_{M_0/S^1} = (-1)^{n+1}\ast_{M_0/S^1} \partial_{M_0/S^1} \ast_{M_0/S^1}$ be the $L^2$-formal adjoint of $\partial_{M_0/S^1}$ with respect to the quotient metric (Proposition 4.11(4)). One might think that there is an analogous commutative diagram as (4.8) where $\partial^\dagger$ and $\partial^\dagger_{M_0/S^1}$ are placed instead. Note however that $\partial^\dagger$ does not preserve the space of basic forms, as it can be explicitly seen from Theorem 4.17, and in general $\partial^\dagger \circ \pi^*_{S^1} \neq \pi^*_{S^1} \circ \partial^\dagger_{M_0/S^1}$.

Observe that (4.6) and (4.8) can be combined to obtain the following commutative diagram.
\[ \Omega_{\text{bas}}(M_0) \overset{(-1)^{n+1}d^*}{\rightarrow} \Omega_{\text{bas}}(M_0) \]  

(4.9)

\[ \Omega(M_0/S^1) \overset{d^*_{M_0/S^1}}{\rightarrow} \Omega(M_0/S^1). \]

Altogether, from the discussion of Sect. 4.2.2, Theorem 4.17, (4.6), (4.7), (4.8), and (4.9), we can describe explicitly the operator \( T(D) \) of Proposition 4.8.

**Theorem 4.20** The operator \( T(D) \) of Proposition 2.6 for a semi-free \( S^1 \)-action can be written as

\[ T(D) = \left( \frac{D_{M_0/S^1} + \iota \vec{\kappa}}{\varepsilon(\vec{\varphi}_0 \wedge)} \varepsilon(\vec{\varphi}_0 \wedge) \right) \left( \frac{D_{M_0/S^1} - \vec{\kappa} \wedge}{\varepsilon(\vec{\varphi}_0 \wedge)} \right), \]

where \( D_{M_0/S^1} := d_{M_0/S^1} + d^\dagger_{M_0/S^1} \) is the Hodge-de Rham operator on \( M_0/S^1 \). The operator \( T(D) \), when defined on the core \( \Omega_{\text{c}}(M/S^1) \), is essentially self-adjoint.

### 4.3 Dirac–Schrödinger Operators

The operator \( T := T(D) \) of Theorem 4.20 is self-adjoint in \( L^2(F, h) \) but not in \( L^2(F) \), the \( L^2 \)-inner product without the weight \( h \). For example, the adjoint in \( L^2(F) \) of \( D_{M_0/S^1} + \iota \vec{H} \) is \( (D_{M_0/S^1} + \iota \vec{H})^\dagger = D_{M_0/S^1} + \vec{\kappa} \wedge \). To obtain a self-adjoint operator in \( L^2(F) \), we perform the following unitary transformation:

\[ \omega = \left( \begin{array}{c} \omega_0 \\ \omega_1 \end{array} \right) \mapsto U(\omega) := h^{-1/2} \left( \begin{array}{c} \omega_0 \\ \omega_1 \end{array} \right), \]  

(4.10)

for \( \omega_0, \omega_1 \in \Omega_{\text{c}}(M_0/S^1) \). Note that \( \|U(\omega)\|_{L^2(F, h)} = \|\omega\|_{L^2(F)} \). Using this transformation, we want to compute an explicit formula for the operator \( \hat{T} := U^{-1}TU \), defined on \( \text{Dom}(\hat{T}) := U^{-1}(\text{Dom}(T)) \).

**Lemma 4.21** The volume of the orbit function \( h : M_0/S^1 \rightarrow \mathbb{R} \) satisfies

\[ d(h^{\pm 1/2}) = \mp \frac{1}{2} h^{\pm 1/2} \vec{\kappa}. \]

**Proof** It follows directly from Lemma 4.4. \( \square \)

The transformation formula follows directly from this lemma and Theorem 4.20.

**Theorem 4.22** ([27, Theorem 4.31]) The operator \( \hat{T} \) is given by

\[ \hat{T} = \left( \frac{D_{M_0/S^1} + \frac{1}{2} \vec{\varepsilon}(\vec{\kappa})}{\varepsilon(\vec{\varphi}_0 \wedge)} \varepsilon(\vec{\varphi}_0 \wedge) \right) \left( \frac{D_{M_0/S^1} - \frac{1}{2} \vec{\varepsilon}(\vec{\kappa})}{\varepsilon(\vec{\varphi}_0 \wedge)} \right), \]
where \( \hat{c}(\bar{\kappa}) := \bar{\kappa} \wedge +t_{\bar{\kappa}z} \) is the right Clifford multiplication by the mean curvature form.

### 4.3.1 An Involution on \( F \)

As we are interested in Fredholm indices, we would like to find a self-adjoint involution which anti-commutes with \( \hat{T} \) in order to split this operator. Since the dimension of \( M_0/S^1 \) is \( n \) then \( \bar{\star} D_{M_0/S^1} + (-1)^n D_{M_0/S^1} \hat{\star} = 0 \), thus a first natural candidate is

\[
\hat{\star} := \begin{pmatrix} 0 & \hat{\star} \\ \hat{\star} & 0 \end{pmatrix}.
\]

From Proposition 4.11 and Proposition 4.18, we verify \( \hat{\star} \hat{T} = (-1)^{n+1} \hat{T} \hat{\star} \). This implies that if \( n \) is even then we can decompose

\[
\hat{T} = \begin{pmatrix} 0 & \hat{T}^- \\ \hat{T}^+ & 0 \end{pmatrix},
\]

with respect to the involution \( \hat{\star} \). Nevertheless, by studying the trivial case of spinning a closed manifold, it turns out that the index of \( \hat{T} \) is always zero [27, Example 4.33].

**Remark 4.23** (Induced Dirac–Schrödinger Geometric Operators) In [27, Sect. 4.3], two other natural geometric operators on \( M \) were pushed down to \( M_0/S^1 \) following the Brüning–Heintze approach: the positive signature operator and the odd signature operator. From the construction itself we know the resulting operators are elliptic and self-adjoint. Nevertheless, the induced potential (zero-order term) in both cases does not commute with the involution \( \hat{\star} \).

### 4.3.2 The Dirac–Schrödinger Signature Operator

In view of Proposition 4.10 and (4.10), we introduce, for \( j = 0, 1, \ldots 4k \), the unitary transformation

\[
\psi_j : \Omega^j_{c}(M_0/S^1) \oplus \Omega^j_{c}(M_0/S^1) \rightarrow \Omega^j_{c}(M_0)^{S^1}
\]

\[
(\omega_{j-1}, \omega_j) \rightarrow h^{-1/2} \left( \pi^*_S \omega_j + (\pi^*_S \omega_{j-1}) \wedge \chi \right).
\]

The following expressions follow immediately from Lemma 4.21 and Theorem 4.22.

**Lemma 4.24** For the maps \( \psi_j \), we have the relations

\[ \vdash \text{ Springer} \]
\[
d d\psi_j(\omega_{j-1}, \omega_j) = \psi_{j+1}\left(d_{M_0/S^1}\omega_{j-1} - \frac{1}{2}\bar{k} \wedge \omega_{j-1}, d_{M_0/S^1}\omega_j + \frac{1}{2}\bar{k} \wedge \omega_j + \varepsilon(\bar{\phi}_0) \wedge \omega_{j-1}\right),
\]

\[
d^\dagger\psi_j(\omega_{j-1}, \omega_j) = \psi_{j-1}\left(d^\dagger_{M_0/S^1}\omega_{j-1} - \frac{1}{2}\bar{k} \wedge \omega_{j-1} + \varepsilon(\bar{\phi}_0) \wedge \omega_j, d^\dagger_{M_0/S^1}\omega_j + \frac{1}{2}\bar{k} \wedge \omega_j\right).
\]

Now consider the transformations introduced in [12, Sect. 5],

\[
\psi_{ev} : \Omega_c(M_0/S^1) \rightarrow \Omega^e_c(M_0)^{S^1}
\]

\[
(\omega_0, \ldots, \omega_{4k}) \mapsto (\psi_0(0, \omega_0), \psi_2(\omega_1, \omega_2), \ldots, \psi_{4k}(\omega_{4k-1}, \omega_{4k})),
\]

\[
\psi_{odd} : \Omega_c(M_0/S^1) \rightarrow \Omega^e_{c, odd}(M_0)^{S^1}
\]

\[
(\omega_0, \ldots, \omega_{4k}) \mapsto (\psi_0(0, \omega_0), \psi_2(\omega_1, \omega_2), \ldots, \psi_{4k-1}(\omega_{4k-2}, \omega_{4k-1})).
\]

Motivated by Lemma 4.24, we define the operator

\[
\mathcal{D} := \psi_{odd}^{-1}d\psi_{ev} + \psi_{ev}^{-1}d^\dagger\psi_{odd} : \Omega_c(M_0/S^1) \rightarrow \Omega_c(M_0/S^1). \tag{4.12}
\]

Clearly \(\mathcal{D}\) is symmetric since \((\psi_{odd}^{-1}d\psi_{ev})^\dagger = \psi_{ev}^{-1}d^\dagger\psi_{odd}\). A simple computation using Theorem 4.17 allows us to compute the explicit form of \(\mathcal{D}\).

**Proposition 4.25** ([27, Proposition 4.39]) The operator \(\mathcal{D}\) defined in (4.12) is explicitly given by

\[
\mathcal{D} = D_{M_0/S^1} + \frac{1}{2}c(\bar{k})\varepsilon - \widehat{c}(\bar{\phi}_0) \left(1 - \frac{\varepsilon}{2}\right).
\]

Thus, it is a first-order elliptic operator. Moreover, it is symmetric on \(\Omega^e_c(M_0/S^1)\) and anti-commutes with the chirality operator \(\hat{*}\).

**Proof** Observe for the principal symbol that \(\sigma_P(\mathcal{D}) = \sigma_P(D_{M_0/S^1})\), so indeed \(\mathcal{D}\) is first-order elliptic. For the symmetry assertion, we verify \((c(\bar{k})\varepsilon)^\dagger = \varepsilon c(\bar{k})^\dagger = -\varepsilon c(\bar{k}) = c(\bar{k})\varepsilon\), and \((\widehat{c}(\bar{\phi}_0)(1 - \varepsilon))^\dagger = (1 - \varepsilon)\widehat{c}(\bar{\phi}_0) = \widehat{c}(\bar{\phi}_0)(1 - \varepsilon)\). For the last assertion, we calculate using Proposition 4.18,

\[
\hat{*}\mathcal{D} = -D_{M_0/S^1}\hat{*} - \frac{1}{2}c(\bar{k})\hat{*}\varepsilon + \widehat{c}(\bar{\phi}_0)\hat{*} \left(1 - \frac{\varepsilon}{2}\right)
\]

\[
= -D_{M_0/S^1}\hat{*} - \frac{1}{2}c(\bar{k})\varepsilon\hat{*} + \widehat{c}(\bar{\phi}_0) \left(1 - \frac{\varepsilon}{2}\right)\hat{*}.
\]

Due to the nature of the potential of the operator \(\mathcal{D}\) and in view of Theorem 4.22, we would expect \(\mathcal{D}\) to be essentially self-adjoint on the core \(\Omega^e_c(M_0/S^1)\). In order
to apply the construction of Brüning and Heintze, we need to find an operator on $M$, commuting with the $S^1$-action, so that when pushed down $M_0/S^1$ coincides with $\mathcal{D}'$. Let us explore how to find such an operator. In view of (4.12), we define $d := \psi^{-1}_{\text{odd}} \psi_{\text{ev}}$ and $d^\dagger := \psi_{\text{ev}}^{-1} d^\dagger \psi_{\text{odd}}$ so that $\mathcal{D}' = d + d^\dagger$. Observe that these operators fit in the commutative diagrams

$$
\begin{align*}
\Omega_{c}^{\text{ev}}(M_0)^{S_1} & \xrightarrow{d} \Omega_{c}^{\text{odd}}(M_0)^{S_1} & \xrightarrow{\psi_{\text{ev}} \psi_{\text{odd}}^{-1}} \Omega_{c}^{\text{ev}}(M_0)^{S_1} \\
\psi_{\text{ev}} & \uparrow & \psi_{\text{odd}} & \uparrow & \psi_{\text{ev}} \\
\Omega_{c}(M_0/S^1) & \xrightarrow{d} \Omega_{c}(M_0/S^1) & & = & \Omega_{c}(M_0/S^1),
\end{align*}
$$

and conclude that the operator $\mathcal{D}'$ is unitary equivalent to the operator $-c(\chi)d + d^\dagger c(\chi)$ when restricted to $\Omega_{c}^{\text{ev}}(M_0)^{S_1}$.

**Lemma 4.26** Let $c(\chi)$ be the left Clifford multiplication by the characteristic 1-form. Then $\psi_{\text{ev}} \psi_{\text{odd}}^{-1} = -c(\chi)$, and $\psi_{\text{odd}} \psi_{\text{ev}}^{-1} = c(\chi)$.

**Proof** This follows directly from the definition of $\psi_{\text{ev}}/\psi_{\text{odd}}$.

From this lemma we obtain the commutative diagram

$$
\begin{align*}
\Omega_{c}^{\text{ev}}(M_0)^{S_1} & \xrightarrow{-c(\chi)d+d^\dagger c(\chi)} \Omega_{c}^{\text{ev}}(M_0)^{S_1} \\
\psi_{\text{ev}} & \uparrow & \psi_{\text{ev}} \\
\Omega_{c}(M_0/S^1) & \xrightarrow{\mathcal{D}'} \Omega_{c}(M_0/S^1),
\end{align*}
$$

and conclude that the operator $\mathcal{D}'$ is unitary equivalent to the operator $-c(\chi)d + d^\dagger c(\chi)$ when restricted to $\Omega_{c}^{\text{ev}}(M_0)^{S_1}$.

**Proposition 4.27** The operator $B := -c(\chi)d + d^\dagger c(\chi)$ satisfies:

1. It is a transversally elliptic first-order differential operator with principal symbol $\sigma_{P}(B)(x, \xi) = -i(\langle \chi, \xi \rangle + \langle \xi \rangle c(\chi))$. 

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(2) It can be extended to $M$, and $B : \Omega(M) \to \Omega(M)$ is essentially self-adjoint when defined on this core.

(3) It commutes with the $S^1$-action on differential forms.

(4) It commutes with the Gauß–Bonnet involution $\varepsilon$ and therefore it can be decomposed as $B = B^{\text{ev}} \oplus B^{\text{odd}}$ where $B^{\text{ev/odd}}(M) : \Omega^{\text{ev/odd}} \to \Omega^{\text{ev/odd}}(M)$.

**Proof** To prove the first statement, recall the expressions for the principal symbols

$$\sigma_P(d)(x, \xi) = -i\xi \wedge,$$

$$\sigma_P(d^\dagger)(x, \xi) = i\iota\xi^\#.$$

Using the relation

$$c(\chi) \circ (\xi \wedge) = \chi \wedge \xi \wedge -\iota_X \circ (\xi \wedge) = \chi \wedge \xi \wedge -\langle \chi, \xi \rangle + \xi \wedge \iota_X$$

$$= -\xi \wedge c(\chi) - \langle \chi, \xi \rangle,$$

we calculate the principal symbol of the operator $B$,

$$\sigma_P(B)(x, \xi) = i \left( c(\chi) \circ (\xi \wedge) + (\iota\xi) \circ c(\chi) \right)$$

$$= i \left( -\xi \wedge c(\chi) - \langle \chi, \xi \rangle + (\iota\xi) \circ c(\chi) \right)$$

$$= -i \langle \chi, \xi \rangle + c(\xi)c(\chi)).$$

In particular we see that if $\langle \chi, \xi \rangle = 0$ then $\sigma_P(B)(x, \xi)^2 = \|\xi\|^2$, so we see that $B$ is a transversally elliptic first-order differential operator. To prove the second assertion, observe that the Clifford multiplication operator $c(\chi)$ has domain $\text{Dom}(c(\chi)) = \Omega_c(M_0).$ For $\omega \in \Omega_c(M_0)$, we have $\|c(\chi)\omega\|^2_{L^2(\wedge_M^g T^*M)} = \|\omega\|^2_{L^2(\wedge_M^g T^*M)}$; see we can extend the operator $c(\chi)$ to all $L^2(M, \wedge_M T^*M)$ by density. Using this fact, we see that $B$ is indeed densely defined with core $\Omega(M)$. Since $M$ is compact we can use Remark [20, Lemma 2.1] to conclude that $B$ is an essentially self-adjoint operator. The last two assertions follow easily from Proposition 4.8 and the fact that $\varepsilon$ anti-commutes with the left Clifford action. 

Now we can implement the construction described in Sect. 2.2 in this setting: the restriction of $B$ to the $S^1$-invariant forms remains essentially self-adjoint and since this operator is unitary equivalent to $D'$ through $\psi_{\text{ev}}$ we conclude that $D'$ is essentially self-adjoint with core $\Omega_c(M_0/S^1)$. We summarize these results in the next theorem.

**Theorem 4.28** The Dirac–Schrödinger operator $D'$ defined on $\Omega_c(M_0/S^1)$ is a first-order elliptic differential operator which is essentially self-adjoint. As the dimension of $M$ is odd then $D'$ anti-commutes with the chirality operator $\tilde{\chi}$ on $M_0/S^1$ and therefore we can define the operator $D'^+ : \Omega^+(M_0/S^1) \to \Omega^-(M_0/S^1)$, where $\Omega^\pm_c(M_0/S^1)$ is the $\pm 1$-eigenspace of $\tilde{\chi}$.

**Remark 4.29** We will later give a detailed description of the operator $D'$ close to a connected component of the fixed point set. We will see that the term containing $\hat{c}(\hat{\varphi}_0)$

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is actually bounded and therefore, by the Kato–Rellich theorem, it will be enough to consider the operator

\[ \mathcal{D} := D_{M_0/S^1} + \frac{1}{2} c(\bar{\kappa}) \varepsilon. \]

We will also see that the factor 1/2 is fundamental for the essential self-adjointness of \( \mathcal{D} \).

**Remark 4.30** ([27, Lemma 4.48]) It is easy to verify that \( \mathcal{D}^2 \) is a generalized Laplacian in the sense of [6, Definition 2.2].

Even though we are not going to make use of it, we want to finish this section by showing that the operator \( \mathcal{D}' \) is also essentially self-adjoint whenever \( M \) is a complete (not necessarily compact) manifold on which \( S^1 \) acts by orientation-preserving isometries. The strategy of the proof is inspired in the similar result for Dirac operators.

**Lemma 4.31** ([27, Lemma 4.50]) For \( f \in C^\infty(M) \), we have \( [B, f] = c(df)c(\chi) + \langle \chi, df \rangle \), where \([\cdot, \cdot]\) denotes the commutator.

**Proof** We use Proposition 4.11(3) to compute as in the proof of Proposition 4.27.

**Corollary 4.32** ([27, Corollary 4.51]) Let \( M \) be a complete Riemannian manifold. Then the operator \( B \) is essentially self-adjoint.

**Proof** If follows from 4.31 analogously as for the proof for Dirac operators given in [17, Chapter 4], [24, Theorem II.5.7] and [31].

**Remark 4.33** (The Basic Signature Operator) Motivation for the structure of the operator \( \mathcal{D}' \) of Theorem 4.28 comes from the work of Habib and Richardson on modified differentials in the context of Riemannian foliations [21]. In their setting, they defined the basic signature operator action on basic forms. In [27, Sect. 4.4], a detailed comparison between this operator and \( \mathcal{D}' \) was made. It was shown that, when pushing down to the quotient space, the basic signature operator can be identified with \( D_{M_0/S^1} \) and the twist of the de Rham differential can be seen as a consequence of the transformation (4.10).

**Example 4.34** (The 2-sphere) We consider the semi-free \( S^1 \)-action on the unit 2-sphere \( M = S^2 \subset \mathbb{R}^3 \) by rotations along the \( z \)-axis. The fixed point set is \( M^{S^1} = \{N, S\} \), where \( N \) and \( S \) denote the north and south pole, respectively. On the complement \( M_0 = S^2 - \{N, S\} \) the action is free. We equip \( S^2 \) with the induced metric coming from the Euclidean inner product of \( \mathbb{R}^3 \) which can be written in polar coordinates as

\[ g^{TS^2} = d\theta^2 + \sin^2 \theta d\phi^2. \] (4.14)

The quotient manifold \( M_0/S^1 \) can be identified with the open interval \( I := (0, \pi) \), which we equip with the flat metric \( g^{TI} = d\theta^2 \) so that the orbit map \( \pi_{S^1} : M_0 \to M_0/S^1 \) becomes a Riemannian submersion. The generating vector field of the action...
is clearly $V = \partial \phi$, the characteristic 1-from is $\chi = \sin \theta d\phi$, and the corresponding mean curvature form is $\kappa = -\cot \theta d\theta$. Using (4.1), we find that $\varphi_0 = 0$. As a result, the operator $\mathcal{D}'$ of Theorem 4.28 is

$$\mathcal{D}' = \mathcal{D} = D_I + \frac{1}{2} \cot \theta c(d\theta) \epsilon.$$

With respect to the degree decomposition, we can express it as

$$\mathcal{D} = \gamma \left( \frac{\partial}{\partial \theta} + \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \otimes \cot \theta \right), \quad \text{where} \quad \gamma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

When $\theta \to 0$ this operator takes the form

$$\gamma \left( \frac{\partial}{\partial \theta} + \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \otimes \frac{1}{\theta} \right).$$

In view of [12, Theorem 3.2], we see that this operator has the structure of a first-order regular singular operator, in the sense of Brüning and Seeley, and that indeed is essentially self-adjoint on the core $\Omega_c(I)$.

**Remark 4.35** We refer to [27, Sect. 5], to see more explicit examples of the construction of the operator $\mathcal{D}'$ in higher dimensions.

### 5 Local Description of $\mathcal{D}$ Close to $M^{S^1}$

#### 5.1 Mean Curvature 1-Form on the Normal Bundle

Recall from Sect. 3.2.1 that the strategy to proof the equivariant $S^1$-signature formula of [25, Theorem 4] is to decompose the quotient space as $M_0/S^1 = Z_t \cup U_t$ where $Z_t := M_0/S^1 - N_t(F)$ is a compact manifold with boundary and $U_t := N_t(F)$ is the $t$-neighborhood of $F$ in $M/S^1$, as schematically visualized in Fig. 2. One models $U_t$ as the mapping cylinder of a Riemannian fibration $\pi_F : F \to F$, with fiber $\mathbb{C}P^N$, whose total space $F$ comes as the quotient space $S/S^1$ of the sphere bundle $S := SNF$ of the normal bundle of $F$ in $M$.

Let $\pi_S : S \to F$ denote the projection and consider the decomposition of the tangent bundle $TS = TVS \oplus THS$ on which the metric decomposes as $g^{TS} = g^{TVS} \oplus g^{THS}$, where we identify $THS \cong TF$ via $\pi_S$. We choose a locally oriented
orthonormal basis for $TS$ of the form
\[ \{e_i\}_{i=0}^{2N} \cup \{f^\alpha\}_{\alpha=1}^{4k-2N-1}, \quad (5.1) \]
where $v := 2N$ and $h := 4k - 2N - 1$. Here $e_i$ and $f^\alpha$ are vertical and horizontal vector fields, respectively. Let $\{\hat{e}_i\}_{i=0}^{2N} \cup \{\hat{f}^\alpha\}_{\alpha=1}^{4k-2N-1}$ denote the associated dual basis. In view of Proposition [30, Lemma 2.2], we can assume without loss of generality that the generating vector field of the free $S^1$-action on $S$ is $V_S := e_0 \in C^\infty(S, TV_S)$. This implies the corresponding mean curvature form $\kappa_S$ vanishes by Proposition 4.5(2) since $\|e_0\| = 1$.

**Remark 5.1** We can assume that the 1-forms $\{\hat{e}_i\}_{i=0}^{2N} \cup \{\hat{f}^\alpha\}_{\alpha=1}^{4k-2N-1}$ are basic.

Let $\hat{\omega}$ be the connection 1-form of the Levi–Civita connection of the metric $g_{TS}$ associated with the orthonormal frame above. Recall that its components satisfy the structure equations
\[ de^i + \omega^i_j \wedge e^j + \omega^i_\alpha \wedge f^\alpha = 0, \]
\[ df^\alpha + \omega^\alpha_j \wedge e^j + \omega^\alpha_\beta \wedge f^\beta = 0. \]
Since the characteristic 1-form is $\chi_S = e^0$ and $\kappa_S = 0$, we can use these structure equations to compute the 2-form $\varphi_{0,S}$ from (4.1), namely,
\[ \varphi_{0,S} = d\chi_S = de^0 = -\omega^0_i \wedge e^i - \omega^0_\alpha \wedge f^\alpha. \quad (5.2) \]

Let $\hat{D} := D - F$ denote the disk bundle of the normal bundle without the zero section, where the induced $S^1$-action is still free. Now we calculate the corresponding mean curvature 1-form $\kappa_{\hat{D}}$ and the 2-form $\varphi_{0,\hat{D}}$ for the metric
\[ g^{\hat{T}\hat{D}} = dr^2 \oplus r^2 g_{TV_S} \oplus g_{TV_S}, \quad (5.3) \]
where $r > 0$ denotes the radial direction. From the orthonormal basis of $T^*S$ described above we can construct an orthonormal basis for $T^*\hat{D}$ as
\[ \{dr\} \cup \{\hat{e}^i\}_{i=0}^{2N} \cup \{\hat{f}^\alpha\}_{\alpha=1}^{4k-2N-1}, \quad (5.4) \]
setting $\hat{e}^i := re^i$ and $\hat{f}^\alpha := f^\alpha$. Here we regard $e^i$ and $f^\alpha$ as 1-forms on $\hat{D}$ by pulling them back from $S$ along the projection $\hat{D} \rightarrow S$. The generating vector field of the $S^1$-action on $\hat{D}$ is still $\hat{V}_{\hat{D}} = e_0 = r\hat{e}_0$ and therefore, by Proposition 4.5(2), we get
\[ \kappa_{\hat{D}} = -d \log(\|V_{\hat{D}}\|) = -d(\log r) = -\frac{dr}{r}. \]

Now we want to compute the 2-form $\varphi_{0,\hat{D}}$. First note that the associated characteristic 1-form is $\chi_{\hat{D}} = \hat{e}^0 = re^0$, thus we need to calculate $d\hat{e}^0$ in order to use (4.1).
Let \( \hat{\omega} \) be the connection 1-form corresponding to the Levi–Civita connection of the metric (5.3) associated with the basis (5.4). Using the structure equations

\[
\begin{align*}
    d\hat{e}^i + \hat{\omega}_j^i \wedge \hat{e}^j + \hat{\omega}_\alpha^i \wedge \hat{e}^\alpha + \hat{\omega}_r^i \wedge dr &= 0, \\
    d\hat{f}^\alpha + \hat{\omega}_j^\alpha \wedge \hat{e}^j + \hat{\omega}_\beta^\alpha \wedge \hat{f}^\beta + \hat{\omega}_r^\alpha \wedge dr &= 0, \\
    \hat{\omega}_f^j \wedge \hat{e}^j + \hat{\omega}_r^\alpha \wedge \hat{f}^\alpha &= 0,
\end{align*}
\]

we can proceed as in Sect. 3.2.1 to obtain the components of \( \hat{\omega} \) (see (3.3)),

\[
\begin{align*}
    \hat{\omega}_j^i &= \omega_j^i, \\
    \hat{\omega}_r^i &= e_i, \\
    \hat{\omega}_\alpha^i &= r \omega_\alpha^i, \\
    \hat{\omega}_\beta^\alpha &= r^2 \omega_\beta^\alpha f^\gamma, \\
    \hat{\omega}_r^\alpha &= 0.
\end{align*}
\]

From these equations we find

\[
\begin{align*}
    d\chi_{\hat{D}} &= -\omega_0^i \wedge (r e^i) - (r \omega_0^\alpha) \wedge f^\alpha - \varepsilon_0 \wedge dr \\
    &= r \left( -\omega_0^i \wedge e^i - \omega_0^\alpha \wedge f^\alpha \right) - \left( \frac{dr}{r} \right) \wedge (r e^0) \\
    &= r \varphi_{0,S} - \kappa_{\hat{D}} \wedge \chi_{\hat{D}}.
\end{align*}
\]

As a result, we conclude from (4.1) that \( \varphi_{0,\hat{D}} = r \varphi_{0,S} \). We summarize these results in the following proposition.

**Proposition 5.2** Let \( F \) be a connected component of the fixed point set of an effective semi-free \( S^1 \)-action on \( M \). Then, for the induced action on \( \hat{D} \), the mean curvature 1-form is \( \kappa_{\hat{D}} = -dr/r \) and the 2-form \( \varphi_{0,\hat{D}} \) defined by (4.1) is \( \varphi_{0,\hat{D}} = r \varphi_{0,S} \).

Now we proceed to study the \( S^1 \)-quotient. Recall from above that we have the following commutative diagram

\[
\begin{array}{ccc}
    S & \xrightarrow{\pi_{S^1}} & \mathcal{F} \\
    \downarrow{\pi_S} & & \downarrow{\pi_F} \\
    F & & \\
\end{array}
\]

where \( \pi_{S^1} : S \rightarrow \mathcal{F} \) is the orbit map and \( \pi_S, \pi_F \) are corresponding projections. The decomposition into the vertical tangent and an horizontal bundle \( TS = T_V S \oplus T_H S \).
induces a decomposition $T^*F = T_VF \oplus T_HF$ via the orbit map $\pi_{S^1}$ (cf. [10, Sect. 1]). Consequently, there is an induced splitting of the exterior algebra,

$$\wedge T^*F = \wedge T_H^*F \otimes \wedge T_V^*F \oplus \bigoplus_{r=p+q} \wedge^p T_H^*F \otimes \wedge^q T_V^*F.$$  (5.5)

We denote the space of sections by $\Omega^{p,q}(F) := C^\infty(F, \wedge^p T_H^*F \otimes \wedge^q T_V^*F)$.

By Remark 5.1 there exist 1-forms $\{e^i\}_{i=1}^v \cup \{f^\alpha\}_{\alpha=1}^h$, where $e^i \in T^*_V F$ and $f^\alpha \in T^*_H F$ such that $\pi_{S^1}^*e^i = e^i$ and $\pi_{S^1}^*f^\alpha = f^\alpha$. The set $\{e^i\}_{i=1}^v \cup \{f^\alpha\}_{\alpha=1}^h$ forms a local orthonormal basis for $T^*F$ and can be regarded as the basis considered in Sect. 3.2.1. We choose an orientation of $F$ so that $\{f_1, \ldots, f_h, e_1, \ldots, e_v\}$ is an oriented orthonormal basis. The following result is a direct consequence of Proposition 5.2.

**Proposition 5.3** Close to a connected component $F \subset M_{S^1}$ of the fixed point set:

1. The 1-form $\bar{\kappa}$ of diagram 4.7 is given by $\bar{\kappa} = -dr/r$.
2. There exists $\varphi_0, F \in \Omega^2(F)$ such that the 2-form $\bar{\varphi}_0$ defined by diagram and analogous diagram can be expressed as $\bar{\varphi}_0 = r \varphi_0, F$. In particular, the operator $\hat{c}(\bar{\varphi}_0)$ of Proposition 4.18 is bounded.

Combining this proposition with Theorem 4.28 and the Kato–Rellich Theorem ([23]), we can prove the claim of Remark 4.29.

**Corollary 5.4** The operator $\mathcal{D} : \Omega_c(M_0/S^1) \longrightarrow \Omega_c(M_0/S^1)$ defined by

$$\mathcal{D} := D_{M_0/S^1} + \frac{1}{2}c(\bar{\kappa})\varepsilon,$$

is essentially self-adjoint.

### 5.2 Local Description of $\mathcal{D}$

Now we analyze how the complete operator $\mathcal{D}$ can be written near the fixed point set. For the Hodge-de Rham operator this was done by Brüning in [10, Sect. 2]. For $t > 0$, define the model space $U_t := F \times (0, t)$ and let $\pi : U_t \longrightarrow F$ be the projection onto the first factor. Equip $U_t$ with the metric

$$g^{TU_t} := dr^2 \oplus g^{ThF} \oplus r^2 g^{T_vF},$$  (5.6)

for $0 < r < t$. We choose the orientation on $U_t$ defined by the oriented orthonormal basis $\{-\partial_r, \hat{f}_1, \ldots, \hat{f}_h, \hat{e}_1, \ldots, \hat{e}_v\}$. Consider the unitary transformation introduced in [10, Equation 2.12],

$$\Psi_1 : L^2((0, t), \Omega(F) \otimes \mathbb{C}^2) \longrightarrow L^2(U_t) \quad (\sigma_1, \sigma_2) \longmapsto \pi^* r^v \sigma_1(r) + dr \wedge \pi^* r^v \sigma_2(r),$$
where $\nu$ is the operator defined by

$$\nu := \nu d - \frac{\nu}{2} = \nu d - N.$$ 

For example, $\nu e_i = (1 - N)e_i$ and $\nu f_\alpha = -N f_\alpha$. Let us denote the horizontal and vertical chirality operators by

$$\tilde{\star}_H := i^{(h+1)/2}c(f^1)\ldots c(f^h),$$
$$\tilde{\star}_V := i^{(v+1)/2}c(e^1)\ldots c(e^v),$$

A straightforward computation shows that we can write the transformed chirality operator as [10, Lemma 2.4], [27, Remark 6.6]

$$\hat{\star} := \Psi_1^{-1} \tilde{\star} \Psi_1 = \begin{pmatrix} 0 & -\tilde{\star}_H \tilde{\star}_V \\ -\tilde{\star}_H \tilde{\star}_V & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \otimes (-\alpha), \quad (5.7)$$

where $\alpha := \varepsilon_H \tilde{\star}_H \otimes \tilde{\star}_V = \tilde{\star}_H \tilde{\star}_V$ and $\varepsilon_H/\varepsilon_V$ denote the horizontal/vertical Gauß–Bonnet involution. It is easy to verify from (5.7) that the unitary transformation

$$U := \frac{1}{\sqrt{2}} \begin{pmatrix} I & \alpha \\ -\alpha & I \end{pmatrix}$$

diagonalizes $\hat{\star}$ [10, Equation 2.27], i.e.,

$$\star := U^{-1} \hat{\star} U = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$ 

Theorem 5.5 ([10, Theorem 2.5]) Under the unitary transformation $\Psi := \Psi_1 U$ the Hodge-de Rham operator $D_{U_t}$ on $U_t$ is transformed as

$$\Psi^{-1} D_{U_t} \Psi = \gamma \left( \frac{\partial}{\partial r} + \star \otimes A(r) \right).$$

In particular, the transformed signature operator is

$$\Psi^{-1} D_{U_t}^+ \Psi = \frac{\partial}{\partial r} + A(r).$$

The operator $A(r)$ is defined by $A(r) := A_H(r) + r^{-1} A_V$. Here

$$A_H(r) := \left( \left( d_H^{(1)} + rd_H^{(2)} \right) + \left( d_H^{(1)} + rd_H^{(2)} \right)^\dagger \right) \alpha,$$
$$A_V := (d_V + d_V^\dagger)\alpha + \nu.$$
The operators $d_{V}, d_{H}^{(1)}$, and $d_{H}^{(2)}$ are obtained from the decomposition of the exterior derivative $d_{U_{t}}$ with respect to (5.5) (cf. [10, Lemma 2.1]). The first-order vertical differential operator $A_{V}$ is called the cone coefficient.

The following result, which is of fundamental importance for later purposes, is a consequence of Theorem 5.5 and the discussion of [10, Sect. 1].

**Lemma 5.6** ([10, Theorem 2.5]) The operator $A_{HV} := A_{H}(0)A_{V} + A_{V}A_{H}(0)$ is a first-order vertical operator, i.e., it only differentiates with respect to the vertical coordinates. If $A_{V}$ is invertible then, for $r$ small enough, there exists a constant $C > 0$ such that $A(r)^{2} \geq Cr^{-2}A_{V}^{2}$, in particular $A(r)$ is also invertible.

Using the relation $\varepsilon\alpha = -\alpha\varepsilon$, one verifies the transformation law

$$
\Psi^{-1}\left(-\frac{1}{2r}c(dr)\varepsilon\right)\Psi = \gamma\left(\frac{1}{2r}\left(-\varepsilon\ 0\right)\right).
$$

Combining this formula and Theorem 5.5, we obtain the following local description of $\mathcal{D}$ close to the fixed point set.

**Theorem 5.7** Under the unitary transformation $\Psi$ the operator $\mathcal{D}_{U_{t}}$ transforms as

$$
\Psi^{-1}\mathcal{D}_{U_{t}}\Psi = \gamma\left(\frac{\partial}{\partial r} + \bigstar \otimes \mathcal{A}(r)\right),
$$

where $\mathcal{A}(r) := A_{H}(r) + r^{-1}A_{V}$ and the associated cone coefficient is

$$
\mathcal{A}_{V} := A_{V} - \frac{1}{2}\varepsilon = A_{V} - \frac{1}{2r}\varepsilon_{H} \otimes \varepsilon_{V}.
$$

### 5.3 Spectral Decomposition of the Cone Coefficient

The cone coefficient $A_{V}$ is of fundamental importance for the study of the Hodge-de Rham operator $D_{M_{0}/S^{1}}$. The resolvent analysis by Brüning and Seeley ([10,13]) shows that if $|A_{V}| \geq 1/2$ then $D_{M_{0}/S^{1}}$ is discrete and essentially self-adjoint [10, Theorem 0.1]. Therefore, the understanding of the spectrum of $A_{V}$ is crucial for the theory. A detailed analysis on the subject is presented in [10, Sect. 3]. The main idea behind this analysis is to study $\text{spec}(A_{V})$ on the space of vertical harmonic forms $\mathcal{H}$ and its complementary space. On the first space, the eigenvalues of $A_{V}$ are given by $2j - N$ for $j = 0, 1, \ldots, N$ (here one uses the fact that the fiber $\mathbb{C}P^{N}$ has non-trivial cohomology groups in even dimensions). On the complement the eigenvalues have the form $\pm \sqrt{\lambda + (j - N - 1/2)^{2}}$ for $j = 0, 1, \ldots, 2N$ and $\lambda > 0$ is a eigenvalue of the vertical Laplacian. By rescaling the vertical component of the metric (5.6), one can make the eigenvalues $\lambda$ as large as necessary so that the spectrum condition of $A_{V}$ is satisfied [27, Remark 2.6]. Moreover, this rescaling does not change the index of $D_{M_{0}/S^{1}}$ [10, pp. 29–30]. This shows it is enough to study $\text{spec}(A_{V})$ on the space of vertical harmonic forms if we are interested in the index computation. We have

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two possible cases: if $N$ is odd (Witt case) then $|2j - N| \geq 1$ and if $N$ is even (non-Witt case) we get a zero eigenvalue on $H^{N/2}$. Hence, in the Witt case, by recalling the vertical metric if necessary, the Hodge-de Rham operator $D_{M_0/S^1}$ is essentially self-adjoint. In the non-Witt case, we are forced to impose boundary conditions.

From Theorem 4.28, we know that $D$ is essentially self-adjoint independently on the parity of $N$. We can also see this from the spectrum of the cone coefficient $A_V$. As before, up to rescaling, it is enough to study $\text{spec}(D)$ restricted to $H$. It follows directly from Theorem 5.7 that these eigenvalues are of the form $2j - N \pm 1/2$ for $j = 0, 1, \ldots, N$ [27, Theorem 6.22]. In particular, we see $|A_V| \geq 1/2$. In addition, due the nature of the potential of $D$, the resolvent analysis goes along the same lines as for $D_{M_0/S^1}$ and one can show that $D$ is also discrete [27, Sect. 7.2].

6 An Index Theorem

In this section, we compute the index of the operator $D^+$, at least in the Witt case, following the techniques used in [10, Sect. 5]. The analytic tools required to treat the index computation are based on the work [3] of Ballmann, Brüning, and Carron on Dirac–Schrödinger Systems. We begin by recalling some results on perturbation of regular projections ([3, Sects. 1.2, 1.6] and [10, Sect. 5]).

6.1 Perturbation of Regular Projections

Let $(H, \langle \cdot, \cdot \rangle)$ be a separable complex Hilbert space and let $A : \text{Dom}(A) \subset H \longrightarrow H$ be a discrete self-adjoint operator. For each Borel subset $J \subseteq \mathbb{R}$, we denote by $Q_J := Q_J(A)$ the associated spectral projection of $A$ in $H$. For $\Lambda \in \mathbb{R}$ we will use the notation $Q_{<\Lambda} := Q_{(-\infty, \Lambda)}$, $Q_{\geq\Lambda} := Q_{[\Lambda, \infty)}$, etc. For $\Lambda = 0$, we write $Q_> := Q_{>0}$, etc. In particular, $Q_0$ denotes the projection onto $\ker(A)$.

**Definition 6.1** (Sobolev chain) For $s \geq 0$, we define the space $H^s := H^s(A)$ to be the completion of $\text{Dom}(A) \subset H$ with respect to the inner product

$$\langle \sigma_1, \sigma_2 \rangle_s := \langle (I + A^2)^{s/2} \sigma_1, (I + A^2)^{s/2} \sigma_2 \rangle.$$

Note for example that $H^0 = H$ and $H^1 = \text{Dom}(A)$. For $s < 0$, we define $H^s$ to be the strong dual space of $H^{-s}$.

For a Borel subset $J \subseteq \mathbb{R}$, we denote by $H^s_J := Q_J(H^s)$ the image of the Sobolev space $H^s$ under the projection $Q_J$. In particular, we will use the notation $H^s_{<\Lambda} := Q_{<\Lambda}(H^s)$, etc.

**Definition 6.2** A bounded operator $S \in \mathcal{L}(H)$ is called $1/2$-smooth if it restricts to an operator $\hat{S} : H^{1/2} \longrightarrow H^{1/2}$ and extends to an operator $\tilde{S} : H^{-1/2} \longrightarrow H^{-1/2}$. The operator $\hat{S}$ will be called $(1/2)$-smoothing, or simply smoothing, if $\text{ran}(\hat{S}) \subset H^{1/2}$.

In addition to the operator $A$, let us assume that we are given $\gamma \in \mathcal{L}(H)$ such that

(1) $\gamma^* = \gamma^{-1} = -\gamma$.
(2) $\gamma A + A\gamma = 0$.

**Definition 6.3** A 1/2-smooth orthogonal projection $P$ in $H$ is called *regular* (with respect to $A$) if for some, or equivalently, for any $\Lambda \in \mathbb{R}$, we have

$$\sigma \in H^{-1/2}, \tilde{P}\sigma = 0, Q_{\leq \Lambda}(A)\sigma \in H^{1/2} \Rightarrow \sigma \in H^{1/2}.$$ 

A regular projection $P$ is called *elliptic* if $P\gamma := \gamma^*(1 - P)\gamma$ is also a regular projection.

**Example 6.4** (Spectral projections) Let $\Lambda \in \mathbb{R}$, then the spectral projection $Q_{\geq \Lambda}$ is an elliptic projection.

Let $B$ be a symmetric operator defined on $\text{Dom}(A)$ such that $\|B\sigma\| \leq a\|\sigma\| + b\|A\sigma\|$ for all $x \in \text{Dom}(A)$ and $a, b \in \mathbb{R}_+$ with $b < 1$. We call the operator $A + B$, defined on $\text{Dom}(A)$, a *Kato perturbation* of $A$. By Kato–Rellich Theorem, the operator $A + B$ is again self-adjoint and discrete [23, Theorem V.4.3]. Thus, we can consider the corresponding Sobolev chain $H^s(A + B)$. Moreover, we can identify as Hilbert spaces $H^s(A) \cong H^s(A + B)$ for all $s \in \mathbb{R}$ [27, Remark 8.15]. The following result describes how the ellipticity condition of a spectral projection behaves under a Kato perturbation.

**Theorem 6.5** ([10, Theorem 5.9]) Assume that $A + B$ is a Kato perturbation of $A$ with $b < 2/3$. Then $Q_{\geq}(A + B)$ is an elliptic projection with respect to $A$ and the subspaces $Q_{\leq}(A)(H) := \text{ran}(Q_{\leq}(A))$ and $Q_{\geq}(A + B)(H) := \text{ran}(Q_{\geq}(A + B))$ from a Fredholm pair. If $B$ is bounded and $|A| \geq \mu$ where $\mu > \sqrt{2}\|B\|$, then $\text{ind}(Q_{\leq}(A)(H), Q_{\geq}(A + B)(H)) = 0$.

### 6.2 The Index Formula for the Signature Operator in the Witt Case

The objective of this section is to study the techniques used in [10, Sect. 5] to compute the index of the signature operator $D^+$ on $M_0/S^1$ in the Witt case. The strategy, in view of [3, Theorem 4.17], is to consider compatible *super-symmetric Dirac systems* over $Z_t$ and $U_r$ (see Fig. 2) in the sense of [3, Sect. 3] and calculate the index contribution on each piece separately as $t \rightarrow 0$. In the subsequent section, we will then adapt these same techniques to compute the index of the operator $\mathcal{D}^+$ also in the Witt case.

In view of Theorem 5.5 we define, for $0 < t < t_0/2$ fixed, the operator

$$D_1 := \gamma \left( \frac{\partial}{\partial r} + \star \otimes A(t + r) \right),$$

where $0 < r < t$. From the relation $\gamma \star + \star \gamma = 0$, we see that $(D_1, \star)$ defines a super-symmetric Dirac system on $Z_t$. The corresponding graded operator is

$$D_1^+ = \frac{\partial}{\partial r} + A(t + r).$$
Remark 6.6 The Hilbert space on which $A(t)$ is defined is $H := L^2(\wedge^* T^* F_t)$ where
the metric on $F_t$ is $g^T_{F_t} = g_{T^* F_t} \oplus t^2 g_{T^* F}$. For $s \geq 0$, the associated Sobolev chain
is $H^s := \text{Dom}(|A(t)|^s)$.

Similarly, we define on $U_t$,

$$D_2 := -\gamma \left( \frac{\partial}{\partial r} - \star \otimes A(t - r) \right),$$

and we also see that $(D_2, \star)$ is a super-symmetric Dirac system with associated graded
operator

$$D_2^+ = - \left( \frac{\partial}{\partial r} - A(t - r) \right).$$

It is straightforward to verify that the super-symmetric Dirac systems $D_1 = (D_1, \star)$ and $D_2 = (D_2, \star)$ are compatible in the sense of [3, Sect. 3]. We
should think of $D_1$ and $D_2$ as the restrictions of the operator $D$ to $Z_t$ and $U_t$, respectively. For the operator $D_1$, we impose the APS-type boundary condition
$B_1 := \star Q_<(A(t))(\text{Dom}(|A(t)|^{1/2}))$, which we know is elliptic. Obviously, this boundary is invariant under $\star$, so we can split it as $B_1 = B_1^+ \oplus B_1^-$ where $B_1^+ := Q_<(A(t))(\text{Dom}(|A(t)|^{1/2}))$. For the operator $D_2^+$, we choose the comple-
mentary boundary condition $B_2^+ := Q_>(A(t))(\text{Dom}(|A(t)|^{1/2}))$. If we denote both
operators with their compatible elliptic boundary conditions by

$$D_{Z_t, Q_<(A(t))(H)}^+ := D_{1,B_1^+}^+,$$

$$D_{U_t, Q_>(A(t))(H)}^+ := D_{1,B_2^+}^+,$$

then from [3, Theorem 4.17], we obtain the following decomposition result for the index.

Theorem 6.7 ([10, Theorem 5.1]) The operators $D_{U_t, Q_>(A(t))(H)}^+$ and $D_{Z_t, Q_<(A(t))(H)}^+$ are Fredholm, and we have the index identity

$$\text{ind}(D^+) = \text{ind} \left( D_{Z_t, Q_<(A(t))(H)}^+ \right) + \text{ind} \left( D_{U_t, Q_>(A(t))(H)}^+ \right).$$

We first study the index contribution on $U_t$. Set $D_{t}^+ := D_{U_t, Q_>(A(t))(H)}$ to lighten the notation. From the discussion in Sect. 5.3, we know that

$$\mathcal{C}(D_{t}^+) := \{ \sigma \in C^1_c((0, t], H^1)) \mid Q_>(A(t))\sigma(t) = 0 \}$$

is a core for $D_{t}^+$. One of the main ingredients of the Brüning’s index computation in
[10] is the following remarkable vanishing result.
Theorem 6.8 ([10, Theorem 5.2]) For \( t \in (0, t_0) \), sufficiently small we have

\[
\text{ind} \left( D_{U_t, Q_2}(A(t))(H) \right) = 0.
\]

The key point of the proof is the fact that by the rescaling argument and since in our particular case the dimension of the fibers is even, we can always assume that

\[
\pm \frac{1}{2} \notin \text{spec}(A_V).
\]

This observation allows us to obtain an estimate which shows that \( \ker(D_t^+) = \{0\} \). Then, arguing analogously for the adjoint operator, one verifies the vanishing of the index. The concrete form of the claimed estimate is discussed in the following lemma. We revise its proof because it will inspire techniques to derive a similar vanishing result for the operator \( D^+ \).

Lemma 6.9 For \( t \) small enough and \( \sigma \in \mathcal{C}(D_t^+) \), we have the estimate

\[
\| D_t^+ \sigma \|_{L^2((0,t], H)} \geq \frac{1}{2t} \left\| \begin{pmatrix} A_V + \frac{1}{2} \end{pmatrix} \sigma \right\|_{L^2((0,t], H)}.
\]

Proof Let \( \sigma \in \mathcal{C}(D_t^+) \). We use the explicit form of \( A(r) \) to compute

\[
\begin{align*}
\| D_t^+ \sigma(r) \|_H^2 &= \| \sigma'(r) \|_H^2 + \| A_H \sigma(r) \|_H^2 + r^{-2} \| A_V \sigma(r) \|_H^2 \\
&\quad + r^{-1} \langle A_H V \sigma(r), \sigma(r) \rangle_H - \frac{d}{dr} \langle \sigma(r), A(r) \sigma(r) \rangle_H \\
&\quad + r^{-2} \langle \sigma(r), A_V \sigma(r) \rangle - \langle \sigma(r), A'_H(0) \sigma(r) \rangle_H.
\end{align*}
\]

From Lemma 5.6, it follows that \( L := A_H V(r) \left( A_V + 1/2 \right)^{-1} \) is a zero-order operator and therefore there exists a constant \( C_1 > 0 \) such that,

\[
\left\| A_H V(r) \left( A_V + \frac{1}{2} \right)^{-1} \right\|_H \leq C_1, \quad \text{for } r \in (0, t].
\]

It is straightforward to see that we can write

\[
\begin{align*}
\| D_t^+ \sigma(r) \|_H^2 &= \left( \| \sigma'(r) \|_H^2 - \frac{1}{4r^2} \| \sigma(r) \|_H^2 \right) + \frac{d}{dr} \langle \sigma(r), A(r) \sigma(r) \rangle_H \\
&\quad + r^{-2} \left( \left\| A_V + \frac{1}{2} \right\|_H \| \sigma(r) \|_H^2 + r^{-1} \left( \left\| A_V + \frac{1}{2} \right\|_H \sigma(r) \right) \right) \\
&\quad - \langle \sigma(r), A'_H(0) \sigma(r) \rangle_H + \| A_H(r) \sigma(r) \|_H^2.
\end{align*}
\]

Now we integrate (6.2) between 0 and \( t \) in order to compute the \( L^2 \)-norm. After integration, the first term in the right-hand side of this equation is positive by Hardy’s
inequality [22]. The second term in the right-hand side of (6.2) is also positive after integration as a result of the boundary condition at t. Finally, choose t small enough such that

\[ \| D_t^+ \sigma \|_{L^2((0,t], H)}^2 \geq \frac{1}{4} \int_0^t r^{-2} \left\| \left( A_V + \frac{1}{2} \right) \sigma (r) \right\|_H^2 \, dr \]

\[ \geq \frac{1}{4t^2} \left\| \left( A_V + \frac{1}{2} \right) \sigma \right\|_{L^2((0,t], H)}^2. \]

\[ \square \]

**Remark 6.10** Observe that the adjoint operator

\[ (D_t^+)^* = -\frac{\partial}{\partial r} + A(r), \]

has a core \( C((D_t^+)^*):= \{ \sigma \in C^1_c((0, t], H^1) \mid Q_\geq(A(t))\sigma(t) = 0 \}. \) Hence, we can compute similarly for \( \sigma \in C((D_t^+)^*), \) All together, we get the analogous estimate

\[ \| (D_t^+)^* \sigma \|_{L^2((0,t], H)} \geq \frac{1}{2t} \left\| \left( A_V - \frac{1}{2} \right) \sigma \right\|_{L^2((0,t], H)}^2. \]

From Theorem 6.8, we conclude that for t > 0 small enough

\[ \text{ind}(D^+) = \text{ind} \left( D_{Z_t,Q_<(A(t))}(H) \right), \]

which is just the index of the signature operator on the manifold with boundary \( Z_t \) with an APS-type boundary condition. In order to compute this index, we would like to use [1, Theorem 4.14]. Moreover, note that the left-hand side of (6.3) does not depend on t, thus we can study the behavior of the right-hand side in the limit \( t \to 0. \) This is of course motivated by the proof of Theorem 3.4. Nevertheless, we need to be aware of the following observations:

1. The metric close to the boundary is not a product. Still, we can modify it so that it becomes a product near \( r = t \) without changing the index [10, p. 32].
2. Note that \( Q_\geq(A(t))(H) \) is not the appropriate boundary used to derive [1, Theorem 4.14] since \( A(t) \) does not correspond to the tangential signature operator on \( \partial Z_t. \) To correct this, we just need to subtract a bounded term,

\[ A_0(t) := A(t) - \frac{V}{t}. \]

Combining observation (2) with the results of [1, Sect. 4], and Section 3.2.1, we obtain the formula

\[ \sigma_{S^1}(M) = \text{ind} \left( D_{Z_t,Q_<(A(t))}(H) \right) + \frac{1}{2} \dim \ker(A_0(t)). \]
The next result shows how the index changes when varying boundary conditions.

**Theorem 6.11** ([10, Theorem 5.3]) For \( t \) sufficiently small \( (Q_<(A(t))(H), Q_>(A_0(t))(H)) \) is a Fredholm pair in \( H \) and

\[
\text{ind} \left( D^+_{Z_t, Q_<(A(t))(H)} \right) = \text{ind} \left( D^+_{Z_t, Q_<(A_0(t))(H)} \right) + \text{ind}(Q_<(A(r))(H), Q_>(A_0(r))(H)).
\]

This theorem, combined with (6.3) and (6.4), implies the following result.

**Proposition 6.12** For \( t > 0 \) sufficiently small we have

\[
\text{ind}(D^+) = \sigma_{S^1}(M) - \frac{1}{2} \dim \ker(A_0(t)) + \text{ind}(Q_<(A(t))(H), Q_>(A_0(t))(H)).
\]

The computation of the Kato index in the proposition above is highly non-trivial. The main ingredient is the generalized Thom space \( T_\pi \) associated with the fibration \( \pi : \mathcal{F} \to F \) (cf. [14]). As we are assuming the Witt condition on \( M/S^1 \), then this is also true for the compact stratified space \( T_\pi \). In particular, it has a well-defined signature operator whose index computes the \( L^2 \)-signature \( \sigma_{(2)}(T_\pi) \). Using the vanishing result [10, Theorem 5.2] applied to \( T_\pi \), Brüning proved the following remarkable identity [10, Theorem 5.4]

\[
\text{ind}(Q_<(A(t))(H), Q_>(A_0(t))(H)) = \sigma_{(2)}(T_\pi) + \frac{1}{2} \dim \ker(A_0(t)).
\]

On the other hand, Cheeger and Dai showed, still restricted to the Witt case, that this \( L^2 \)-signature coincides with Dai’s \( \tau \) invariant of the fibration \( \pi_\mathcal{F} : \mathcal{F} \to F \), which is this case vanishes (cf. Sect. 3.2.1). We therefore conclude from Proposition 6.12 and Theorem 3.4 that the index of the signature operator on \( M_0/S^1 \), in the Witt case, computes Lott’s equivariant \( S^1 \)-signature.

**Theorem 6.13** Let \( M \) be a closed, oriented Riemannian \( 4k+1 \)-dimensional manifold on which \( S^1 \) acts effectively and semi-freely by orientation-preserving isometries. If the codimension of the fixed point set \( MS^1 \) in \( M \) is divisible by four, then \( M/S^1 \) is a Witt space and the index of the signature operator is

\[
\text{ind}(D^+) = \sigma_{S^1}(M) = \int_{M_0/S^1} \mathcal{L} \left( T(M_0/S^1), g^{T(M_0/S^1)} \right).
\]

Observe that we have used the fact that the \( \eta(MS^1) \) vanishes in the Witt case.

### 6.3 The Index Formula for the Dirac–Schrödinger Signature Operator

In this last section, we describe how to compute the index of the operator \( \mathcal{D}^+ \) using the techniques illustrated in Sect. 6.2. We obtain the complete index formula for this
operator in the Witt case. To begin with, we make no assumption on the parity of $N$, i.e., we do not distinguish between the Witt and the non-Witt case. As before, we use the geometric decomposition of $M_0/S^1$ in order to construct compatible super-symmetric Dirac systems

$$D_1 := \gamma \left( \frac{\partial}{\partial r} + \bigstar \otimes \mathcal{A} (t + r) \right),$$

$$D_2 := - \gamma \left( \frac{\partial}{\partial r} - \bigstar \otimes \mathcal{A} (t - r) \right),$$

where $\mathcal{A} (r)$ is the operator of Theorem 5.7. The corresponding graded operators, with respect to the super-symmetry $\bigstar$, are

$$D_1^+ = \frac{\partial}{\partial r} + \mathcal{A} (t + r),$$

$$D_2^+ = - \left( \frac{\partial}{\partial r} - \mathcal{A} (t - r) \right).$$

**Remark 6.14** In this case, we still set $H := L^2(\wedge T^*f_t)$ and consider the associated Sobolev chain of $\mathcal{A} (t)$ denoted by $H^s := \text{Dom}(|\mathcal{A} (t)|^s)$. Since for $t > 0$ fixed, $\mathcal{A} (t) - A (t)$ is a bounded operator, it follows by [27, Remark 8.15] that the Sobolev chains of $\mathcal{A} (t)$ and $A (t)$ are isomorphic.

As we did for the signature operator, we impose the complementary APS-type boundary conditions $B_1^+ := Q_{\prec} (\mathcal{A} (t)) (\text{Dom}(|\mathcal{A} (t)|^{1/2}))$ and $B_2^+ := Q_{\succ} (\mathcal{A} (t)) (\text{Dom}(|\mathcal{A} (t)|^{1/2}))$ on $Z_t$ and $U_t$, respectively. If we denote both operators with their compatible elliptic boundary conditions by

$$\mathcal{D}_{Z_t, Q_{\prec} (\mathcal{A} (t))} (H) := B_1^+, \mathcal{D}_{U_t, Q_{\succ} (\mathcal{A} (t))} (H) := B_2^+, \mathcal{D}_{Z_t, Q_{\prec} (\mathcal{A} (t))} (H) := B_1^+, \mathcal{D}_{U_t, Q_{\succ} (\mathcal{A} (t))} (H) := B_2^+, \mathcal{D}_{Z_t, Q_{\prec} (\mathcal{A} (t))} (H) := B_1^+, \mathcal{D}_{U_t, Q_{\succ} (\mathcal{A} (t))} (H) := B_2^+,$$

then, as in last section, we can apply [3, Theorem 4.17] to obtain decomposition formula

$$\text{ind} (\mathcal{D}^+) = \text{ind} \left( \mathcal{D}_{Z_t, Q_{\prec} (\mathcal{A} (t))} (H) \right) + \text{ind} \left( \mathcal{D}_{U_t, Q_{\succ} (\mathcal{A} (t))} (H) \right). \quad (6.5)$$

**6.3.1 The Index Formula for $\mathcal{D}^+$ in the Witt Case**

Now we restrict ourselves to the Witt case, i.e., $N$ is odd. In order to obtain a vanishing result for the index contribution of $U_t$ we need to modify the boundary conditions at $r = t$. As noted in Remark 6.14, for fixed $t > 0$ the operator $A (t)$ is a Kato perturbation of $\mathcal{A} (t)$. Hence, by Theorem 6.5 and Theorem [3, Theorem 4.17], we obtain the decomposition formula of the index

$$\text{ind} (\mathcal{D}^+) = \text{ind} \left( \mathcal{D}_{Z_t, Q_{\prec} (\mathcal{A} (t))} (H) \right) + \text{ind} \left( \mathcal{D}_{U_t, Q_{\succ} (\mathcal{A} (t))} (H) \right). \quad (6.6)$$
The following lemma relates these two boundary conditions at $r = t$.

**Lemma 6.15** The following index identity holds true,

$$\text{ind}(Q<(A(t))(H), Q_{\geq}(A_0(t))(H)) = \text{ind}(Q<(A(t))(H), Q_{\geq}(A_0(t))(H)) + \text{ind}(Q<(A'(t))(H), Q_{\geq}(A(t))(H)).$$

**Proof** Using similar arguments as in the proof of Theorem 6.11 one can see that

$$(Q<(A(t))(H), Q_{\geq}(A_0(t))(H)),$$

$$(Q<(A'(t))(H), Q_{\geq}(A(t))(H)),$$

are both Fredholm pairs. Now we apply [2, Lemma A.1] with $A_1 = A'(t)$, $A_2 = A_0(t)$, $\alpha_1 = \alpha_2 = 0$ to verify that

$$Q>(A(t)) - Q>(A_0(t)) = \frac{1}{2\pi i} \int_{\text{Re } z = 0} (A'(t) - z)^{-1} \left[ \frac{1}{t} \left( v - \frac{\epsilon}{2} \right) \right] (A_0(t) - z)^{-1} dz,$$

is compact for fixed $t > 0$. Indeed, both $A'(t)$ and $A_0(t)$ are discrete, so their resolvent is compact, and the perturbation term $(v - \epsilon/2)$ is a bounded operator. Similarly, we see that all the differences

$$Q<(A'(t)) - Q<(A_0(t)),$$

$$Q>(A'(t)) - Q>(A(t)),$$

$$Q>(A(t)) - Q>(A_0(t)),$$

are compact. Finally, we can use Lemma [2, Lemma A.1] with projections $P = Q<(A(r))$, $Q = Q<(A(t))$ and $B = Q_{\geq}(A_0(r))(H)$ to obtain the desired formula. $\square$

**Lemma 6.16** For $t > 0$ sufficiently small we have $\text{ind}(Q_{\leq}(A(t))(H), Q>(A'(t))(H)) = 0$.

**Proof** The idea is to apply Theorem 6.5 with $A = A(t)$ and $B = \epsilon/2t$ so that the sum $A + B = A'(t)$. By Lemma 5.6, we see that for $t$ small enough,

$$|A(t)| \geq \frac{\sqrt{C}}{t} |A_V| \geq \frac{\sqrt{C}}{t}.$$

Hence, the required condition for $\mu := \sqrt{C}/t$ is

$$\frac{\sqrt{C}}{t} > \sqrt{2} \left\| \frac{\epsilon}{2t} \right\| = \frac{\sqrt{2}}{2t},$$

that is, $C > 1/2$, which can be achieved. $\square$
From this lemma we conclude, via [3, Theorem 4.14], that in the Witt case the decompositions \((6.5)\) and \((6.6)\) are the same.

Now we describe the vanishing result for the index on \(U_t\). Analogously as before, consider the operator \( \mathcal{D}_t := \mathcal{D}_{U_t, Q_{\geq}}(A(t))(H) \) defined on the core

\[ C(D_t^+) := \{ \sigma \in C^1_c((0, t], H^1)) \mid Q_{\leq}(A(t))\sigma(t) = 0 \}. \]

\textbf{Theorem 6.17} \textit{In the Witt case, for} \( t \in (0, t_0] \) \textit{sufficiently small we have}

\[ \text{ind}\left( \mathcal{D}_t^+, Q_{\geq}(A(t))(H) \right) = 0. \]

To prove this theorem we use a deformation argument, as in the proof of [10, Theorem 5.2], splitting the index into a contribution of the space of vertical harmonic forms and a contribution of the complement. More precisely, denote by \( \Delta V \) the fiber Laplacian and choose \( \delta > 0 \) sufficiently small to define

\[ P_H(x) := \frac{1}{2\pi i} \int_{|z| = \delta} (\Delta V, x - z)^{-1} \overline{dz}, \quad \text{for} \ x \in F, \]

the projection onto the space of vertical harmonic \( \mathcal{H} \) and let \( P_{H^\perp} := I - P_H \) be the complementary projection.

\textbf{Lemma 6.18} (\cite[Lemma 8.63]{27}) \textit{The projection} \( I \otimes P_H \) \textit{commutes with} \( \mathcal{A}_V \) \textit{and with the principal symbol of} \( A_H(t) \).

Let us define

\[ \mathcal{A}^\delta(r) := (I \otimes P_H)\mathcal{A}(r)(I \otimes P_H) + (I \otimes P_{H^\perp})\mathcal{A}(r)(I \otimes P_{H^\perp}^\perp) \]

\[ := \mathcal{A}_H(r) + \mathcal{A}_{H^\perp}. \]

By Lemma 6.18 the difference \( C(r) := \mathcal{A}^\delta(r) - \mathcal{A}(r) \) is a uniformly bounded norm, i.e., there exists \( C > 0 \) such that \( \| C(r) \| \leq C \) for all \( r \in (0, t] \). We now consider the \textit{deformed} operator

\[ \mathcal{D}_t^+ := \frac{\partial}{\partial r} + \mathcal{A}^\delta(r), \]

defined on the core \( C(\mathcal{D}_t^+) := \{ \sigma \in C^1_c((0, t], H^1)) \mid Q_{\leq}(A^\delta(t))\sigma(t) = 0 \}, \) where

\[ A^\delta(r) := (I \otimes P_H)A(r)(I \otimes P_H) + (I \otimes P_{H^\perp})A(r)(I \otimes P_{H^\perp}^\perp) \]

\[ := A_H(r) + A_{H^\perp}(r). \]

Again, from Lemma 6.18 it follows that \( C(r) := A^\delta(r) - A(r) \) has uniformly bounded operator. Since \( \mathcal{D}_t^+ - \mathcal{D}_t^+ = C(r) \) is uniformly bounded then, by [3, Theorem 4.14],

\[ \text{ind}(\mathcal{D}_t^+) = \text{ind}(\mathcal{D}_t^+ + \mathcal{A}^\delta(r))(H), Q_{\geq}(A^\delta(t))(H)). \]
Now we introduce the operators
\[
\mathcal{D}_{t,H/H^\perp}^+ := \frac{\partial}{\partial r} + A_{H/H^\perp}(r)
\]
defined on the core \(C(\mathcal{D}_{t,H/H^\perp}^+) := \{\sigma \in C^1_c((0, t], H^1) \mid Q_<(A_{H/H^\perp}(t))\sigma(t) = 0\}\), which by orthogonality satisfy

\[
\text{ind}(\mathcal{D}_{t,H}^+) = \text{ind}(\mathcal{D}_{t,H}^+) + \text{ind}(\mathcal{D}_{t,H^\perp}^+). 
\]

and imply the index relation

\[
\text{ind}(\mathcal{D}_{t,H}^+) = \text{ind}(\mathcal{D}_{t,H}^+) + \text{ind}(Q_<(A(t))(H), Q_>(A^{\delta}(t))(H)).
\]

**Remark 6.19** In the Witt case, as \(A(t)\) is invertible for \(t > 0\) small enough, it is easy to see that this is also the case for the operator \(A^{\delta}(t)\).

Now we show these three contributions to the index vanish for \(t > 0\) sufficiently small.

**Proposition 6.20** For \(t > 0\) small enough, \(\text{ind}(Q_<(A(t))(H), Q_>(A^{\delta}(t))(H)) = 0\).

**Proof** As for the Witt case, we can assume \(|A_V| \geq 1\), Lemma 5.6 implies

\[
|A(t)| \geq \frac{\sqrt{C}}{t} |A_V| \geq \frac{\sqrt{C}}{t}.
\]

To apply the vanishing statement of Theorem 6.5 we require \(\sqrt{C}/t \geq \sqrt{2}\|C(t)\|\), which can always be achieved by making \(t\) small, since \(C(t)\) is uniformly bounded.

**Proposition 6.21** For \(t > 0\), small enough we have \(\text{ind}(\mathcal{D}_{t,H^\perp}^+) = 0\).

**Proof** As we are in the Witt case we can assume \(|A_V| \geq 1\). Hence, it is easy to see that for the operator

\[
D_{t,H^\perp}^+ := \frac{\partial}{\partial r} + A_{H^\perp}(r),
\]
defined on the core \(C(D_{t,H^\perp}^+)\), we can prove an analogue of Lemma 6.9. That is, for \(t > 0\) small enough and \(\sigma \in C(D_{t,H^\perp}^+)\), we can derive the estimate

\[
\left\| D_{t,H^\perp}^+ \sigma \right\|_{L^2((0,t],H)} \geq \frac{1}{2t} \left\| (A_{V,H^\perp} + \frac{1}{2}) \sigma \right\|_{L^2((0,t],H)},
\]

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where \(A_{V,\mathcal{H}^\perp} := (I \otimes P_{\mathcal{H}^\perp})A_V(I \otimes P_{\mathcal{H}^\perp})\). In particular, we can show as before that \(\text{ind}(D_t,\mathcal{H}^\perp) = 0\). Now, the strategy of the proof is to show that \(\mathcal{D}_t^+\) is a Kato perturbation of \(D_t,\mathcal{H}^\perp\) on \(C(\mathcal{D}_t^+,\mathcal{H}^\perp)\). If we are able to prove an estimate of the form
\[
\|\frac{\varepsilon}{2r} \sigma\|_{L^2((0,t],H)} \leq b \|D_t^+ \sigma\|_{L^2((0,t],H)},
\]
with \(b < 1\), then by [23, Theorem IV.5.22, p. 236] it would follow that \(\text{ind}(\mathcal{D}_t^+)=0\). From the proof of Lemma 6.9 we see that for an element \(\sigma \in C(\mathcal{D}_t^+,\mathcal{H}^\perp)\), we have
\[
\|D_t^+ \sigma(r)\|_{H}^2 = \left(\|\sigma'(r)\|_{H}^2 - \frac{1}{4r^2}\|\sigma(r)\|_{H}^2\right) + \frac{d}{dr}\langle\sigma(r),A_{\mathcal{H}^\perp}(r)\sigma(r)\rangle_H
\]
\[
+ r^{-2} \left(\left(A_{V,\mathcal{H}^\perp} + \frac{1}{2}\right)\sigma(r)\right)_{H}^2
\]
\[
+ r^{-1}\left(\left(A_{V,\mathcal{H}^\perp} + \frac{1}{2}\right)\sigma(r),L^*\sigma(r)\right)_H
\]
\[
- \langle\sigma(r),A'_{\mathcal{H}^\perp}(0)\sigma(r)\rangle_H + \|A_{\mathcal{H}^\perp}(r)\sigma(r)\|_{H}^2.
\]
Arguing as in the mentioned proof, we see that after integration between 0 and \(t\) the first term in brackets is non-negative by Hardy’s inequality and the term containing the total derivative with respect to \(r\) is also non-negative because \(\sigma\) has compact support and because the boundary condition at \(r = t\). Thus, it follows that for \(t\) small enough and \(0 < \beta < 1\),
\[
\left\|\frac{1}{r}\left(A_{V,\mathcal{H}^\perp} + \frac{1}{2}\right)\sigma\right\|_{L^2((0,t],H)} \leq (1+\beta)\|D_t^+ \sigma\|_{L^2((0,t],H)}.
\]
On the other hand, using the norm of the resolvent identity [23, Sect. V.5], we obtain the estimate
\[
\|\sigma\| \leq \left\|\left(A_{V,\mathcal{H}^\perp} + \frac{1}{2}\right)^{-1}\left(A_{V,\mathcal{H}^\perp} + \frac{1}{2}\right)\sigma(r)\right\|_{H} \leq \frac{1}{d^\perp}\left\|\left(A_{V,\mathcal{H}^\perp} + \frac{1}{2}\right)\sigma(r)\right\|_{H},
\]
where \(d^\perp := \text{dist}(-1/2, \text{spec}(A_{V,\mathcal{H}^\perp}))\). All together, it follows from (6.8) that
\[
\left\|\frac{\varepsilon}{2r} \sigma\right\|_{L^2((0,t],H)} \leq \left(\frac{1+\beta}{2d^\perp}\right)\|D_t^+ \sigma\|_{L^2((0,t],H)},
\]
i.e., we have shown the desired estimate with \(b := (1+\beta)/2d^\perp\). Recall, we require the condition \(b < 1\), which translates to \((1+\beta) < 2d^\perp\). This can always be achieved by rescaling the vertical metric.

\(\square\)
Proposition 6.22  For \( t \) small enough, we have \( \text{ind}(\mathcal{D}^+_t, \mathcal{H}) = 0 \).

Proof  We will proceed similarly as in the proof of [10, Theorem 5.2]. Since we can identify the first-order part of \( \mathcal{A}_\mathcal{H} \) with the odd signature operator \( A_F \) of \( F \) with coefficients in \( \mathcal{H} \) we know, by the discussion of [1, Remark (3), (4), p. 61], that in the Witt case there exists a self-adjoint involution \( \mathcal{U} \) such that \( \mathcal{U} \mathcal{A}_F \mathcal{U} = -\mathcal{A}_F \).

In particular, \( \mathcal{U} \) anti-commutes with the principal symbol of \( \mathcal{A}_\mathcal{H} \) and therefore \( \mathcal{U} \mathcal{A}_\mathcal{H}(r) \mathcal{U} = -\mathcal{A}_\mathcal{H}(r) + C_2(r) \), where \( \|C_2(r)\|_H \leq C_2 \) for \( r \in (0, t] \). Similarly, we get an analogous formula for \( A_{\mathcal{H}} \) since it has the same principal symbols as \( \mathcal{A}_\mathcal{H} \), i.e., \( \mathcal{U} A_{\mathcal{H}}(r) \mathcal{U} = -A_{\mathcal{H}}(r) + C_2(r) \), where \( \|C_2(r)\|_H \leq C_2 \) for \( r \in (0, t] \).

Observe now that the operator \( \mathcal{U} \mathcal{D}^+_t \mathcal{U} \) defined on the core \( \mathcal{C}(\mathcal{U} \mathcal{D}^+_t, \mathcal{U}) := \{ \sigma \in C^1_c((0, t], H^1) \mid Q_<(\mathcal{U} A_{\mathcal{H}}(t) \mathcal{U}) \sigma(t) = 0 \} \) is given by

\[
\left( \frac{\partial}{\partial r} + \mathcal{U} \mathcal{A}_H(r) \mathcal{U} \right) \sigma(r) = -\left( -\frac{\partial}{\partial r} + \mathcal{A}_H(r) - C_2(r) \right) \sigma(r).
\]

As we are in the Witt case we can use Lemma 5.6 and [23, Theorem V.4.10] to verify for \( t > 0 \) sufficiently small the relation \( Q_<(\mathcal{U} A_{\mathcal{H}}(t) + C_2(t)) = Q_<(\mathcal{U} A_{\mathcal{H}}(t)) = Q_>(A_{\mathcal{H}}) \) since \( C_2(r) \) is uniformly bounded. We now compare \( \mathcal{U} \mathcal{D}^+_t \mathcal{U} \) with the adjoint operator

\[
(\mathcal{D}^+_t, \mathcal{H})^* \sigma(r) = \left( -\frac{\partial}{\partial r} + \mathcal{A}_H(r) \right) \sigma(r),
\]

defined on the core \( \mathcal{C}(\mathcal{(D}^+_t, \mathcal{H})^*) := \{ \sigma \in C^1_c((0, t], H^1) \mid Q_>(A_{\mathcal{H}}(t)) \sigma(t) = 0 \} \).

In particular, \( \mathcal{U} \mathcal{D}^+_t \mathcal{U} = -(\mathcal{D}^+_t, \mathcal{H})^* \). Altogether,

\[
\text{ind}(\mathcal{D}^+_t, \mathcal{H}) = \text{ind}(\mathcal{U} \mathcal{D}^+_t, \mathcal{U}) = (\mathcal{D}^+_t, \mathcal{H})^* = -\text{ind}(\mathcal{D}^+_t, \mathcal{H}),
\]

which shows that \( \text{ind}(\mathcal{D}^+_t, \mathcal{H}) = 0. \)

Proof of Theorem 6.17  This follows now from the deformation argument described above, the vanishing results of Propositions 6.20, 6.21, and 6.22.

Regarding the index contribution of \( Z_t \) it is easy to see that, by deforming the metric close to \( r = t \), we can adapt the proof Theorem 6.11 to get the analogous formula,

\[
\text{ind} \left( \mathcal{D}^+_Z, Q_<(A(t))(H) \right) = \text{ind} \left( D^+_Z, Q_<(A_0(t))(H) \right) + \left( Q_<(A(r))(H), Q_>(A_0(r))(H) \right),
\]

where \( D^+_Z, Q_<(A_0(t))(H) \) is the signature operator on the manifold with boundary \( Z_t \). In particular, we can use (6.4) to conclude that for \( t > 0 \) small enough

\[
\text{ind} \left( \mathcal{D}^+_Z, Q_<(A(t))(H) \right) = \sigma_{S^1}(M) - \frac{1}{2} \dim \ker(A_0(t)) + \left( Q_<(A(r))(H), Q_>(A_0(r))(H) \right).
\]
Finally, from this formula and from the vanishing Theorem 6.17, we obtain a partial answer for the index of the operator $\mathcal{D}^+$.

**Theorem 6.23** Let $M$ be a closed, oriented Riemannian $4k + 1$-dimensional manifold on which $S^1$ acts effectively and semi-freely by orientation-preserving isometries. If the codimension of the fixed point set $M^{S^1}$ in $M$ is divisible by four, then $M/S^1$ is a Witt space and the index of the graded Dirac-Schrödinger operator is

$$\text{ind}(\mathcal{D}^+) = \sigma_{S^1}(M) = \int_{M_0/S^1} L \left( T(M_0/S^1), gT(M_0/S^1) \right).$$

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**References**

1. Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry. I. Math. Proc. Camb. Philos. Soc. 77, 43–69 (1975)
2. Ballmann, W., Brüning, J.: On the spectral theory of manifolds with cusps. J. Math. Pures Appl. 80(6), 593–625 (2001)
3. Ballmann, W., Brüning, J., Carron, G.: Regularity and index theory for Dirac-Schrödinger systems with Lipschitz coefficients. J. Math. Pures Appl. 89, 429–476 (2008)
4. Banagl, M.: Topological Invariants of Stratified Spaces. Springer Monographs in Mathematics. Springer, Berlin (2007)
5. Berger, M., Gauduchon, P., Mazet, E.: Le spectre d’une variété riemannienne. Lecture Notes in Mathematics, vol. 194. Springer, Berlin (1971)
6. Berline, N., Getzler, E., Vergne, M. Heat Kernels and Dirac Operators. Grundlehren Text Editions. Springer, Berlin (2004) (Corrected reprint of the 1992 original)
7. Bismut, J.-M., Lott, J.: Flat vector bundles, direct images and higher real analytic torsion. J. Am. Math. Soc. 8(2), 291–363 (1995)
8. Bott, R., Tu, L.W.: Differential forms in algebraic topology. Graduate Texts in Mathematics, vol. 82. Springer, New York (1982)
9. Bredon, G.: Introduction to Compact Transformation Groups. Pure and Applied Mathematics. Elsevier Science, Amsterdam (1972)
10. Brüning, J.: The signature operator on manifolds with a conical singular stratum. Astérisque 328, 1–44 (2009)
11. Brüning, J., Heintze, E.: Representations of compact Lie groups and elliptic operators. Invent. Math. 50(2), 169–203 (1978)
12. Brüning, J., Seeley, R.: An index theorem for first order regular singular operators. Am. J. Math. 110(4), 659–714 (1988)
13. Brüning, J., Seeley, R.: The expansion of the resolvent near a singular stratum of conical type. J. Funct. Anal. 95(2), 255–290 (1991)
14. Cheeger, J., Dai, X.: $L^2$-cohomology of spaces with nonisolated conical singularities and nonmultiplicativity of the signature. In: Riemannian Topology and Geometric Structures on Manifolds, vol. 271 of Progr. Math, pp. 1–24. Birkhäuser, Boston (2009)
15. Dai, X.: Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence. J. Am. Math. Soc. 4(2), 265–321 (1991)
16. Duistermaat, J., Kolk, J.: Lie Groups. Universitext (1979); Springer, Berlin (2000)
17. Friedrich, T.: Dirac Operators in Riemannian Geometry. Graduate Studies in Mathematics, vol. 25. American Mathematical Society, Providence, RI (2000) (Translated from the 1997 German original by Andreas Nestke)
18. Goette, S.: Equivariant $\eta$-invariants and $\eta$-forms. J. Reine Angew. Math. 526, 181–236 (2000)
19. Goresky, M., MacPherson, R.: Intersection homology theory. Topology 19(2), 135–162 (1980)
20. Grieser, D., Lesch, M.: On the $L^2$-Stokes theorem and Hodge theory for singular algebraic varieties. Math. Nachr. 246(247), 68–82 (2002)
21. Habib, G., Richardson, K.: Modified differentials and basic cohomology for Riemannian foliations. J. Geom. Anal. 23(3), 1314–1342 (2013)
22. Hardy, G.H.: Note on a theorem of Hilbert. Math. Z. 6(3–4), 314–317 (1920)
23. Kato, T.: Perturbation Theory for Linear Operators, 2nd ed. Springer, Berlin (1976) (Grundlehren der Mathematischen Wissenschaften, Band 132)
24. Lawson, H.B., Jr., Michelsohn, M.-L.: Spin Geometry. Princeton Mathematical Series, vol. 38. Princeton University Press, Princeton, NJ (1989)
25. Lott, J.: Signatures and higher signatures of $S^1$-quotients. Math. Ann. 316(4), 617–657 (2000)
26. Morita, S.: Geometry of Differential Forms, vol. 201 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI (2001) (Translated from the two-volume Japanese original (1997, 1998) by Teruko Nagase and Katsumi Nomizu, Iwanami Series in Modern Mathematics)
27. Orduz, J.C.: Induced Dirac-Schrödinger operators on $S^1$-semi-free quotients. arXiv:1711.04196 (2017)
28. Siegel, P.H.: Witt spaces: a geometric cycle theory for $KO$-homology at odd primes. Am. J. Math. 105(5), 1067–1105 (1983)
29. Tondeur, P.: Geometry of Foliations. Monographs in Mathematics. Springer, New York (1997)
30. Uchida, F.: Cobordism groups of semi-free $S^1$- and $S^3$-actions. Osaka J. Math. 7(2), 345–351 (1970)
31. Wolf, J.A.: Essential self-adjointness for the Dirac operator and its square. Indiana Univ. Math. J. 22, 611–640 (1972/73)