Light-Front Ensemble Projector Monte Carlo

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Abstract

A new method to perform numerical simulations of light-front Hamiltonians formulated on transverse lattices is introduced. The method is based on a DLCQ formulation for the (continuous) longitudinal directions. The hopping term in the transverse direction introduces couplings between fields defined on neighboring 1 + 1-dimensional sheets. Within each sheet, the light-cone imaginary time evolution operator is calculated numerically with high precision using DLCQ. The coupling between neighboring sheets is taken into account using an initial value random walk algorithm based on the ensemble projector Monte Carlo technique and a checkerboard decomposition for the time evolution operator. The structure functions of $\lambda\phi^4$ theory in 2 + 1 dimensions are studied as a trial application. The calculations are performed with up to 64 transverse lattice sites. No Tamm-Dancoff truncations are necessary.
I. INTRODUCTION

Euclidean lattice $QCD$ allows one to calculate ground state properties of hadrons, but real time response functions are somewhere between difficult and impossible. On the other hand, deep inelastic scattering gives us information about correlation functions on (or exceedingly close to) the light-cone. Light-front quantization seems to be a promising tool to describe the immense wealth of experimental information about structure functions for a variety of reasons: (1) correlation functions along the light-cone become “static” (i.e. equal $x^+ = (x^0 + x^3)/\sqrt{2}$) observables in this approach. (2) structure functions are easy to evaluate from the light-front wavefunctions. (3) these structure functions are easily interpreted as light-front momentum densities. However, before one can apply the light-front formalism to $QCD$ and other field theories, one has to remove the divergencies first (i.e. regularize and renormalize). One interesting idea in this direction is called the “transverse lattice” [1–3]. Instead of discretizing all four space-time directions (like in Euclidean lattice $QCD$) or the three space directions (like in the Hamiltonian formulation of lattice $QCD$) one discretizes only the two transverse ($x^1$ and $x^2$) directions while leaving the longitudinal directions ($x^0$ and $x^3$) continuous. On the one hand the transverse lattice thus provides a gauge invariant UV regularization scheme and on the other hand it is still possible to perform canonical light-front quantization — making it a promising approach towards performing non-perturbative calculations of deep inelastic structure functions.

There remains the question what one should do with the still continuous (i.e. infinitely many degrees of freedom) longitudinal directions. First one may be tempted to discretize the $x^- = (x^0 - x^3)/\sqrt{2}$ (the light-cone space-) direction as well. However, such an approach is suffering from a fundamental difficulty: the longitudinal momentum is not conserved on a longitudinal lattice. Due to Bragg reflections it is only conserved modulo $2p_{max}^+ = 2\pi/a_L$, where $a_L$ is the longitudinal lattice spacing. Normally (i.e. in normal coordinates) this is not a problem because the minimum of the kinetic energy occurs at $\vec{P} = \vec{0}$. However, the light-cone energy decreases with increasing momentum ($P^- = M^2/2P^+$, in the continuum)
i.e. a minimum is reached for \( P^+ = \infty \). Of course on a lattice the momentum cannot become infinite but still the minimum of the kinetic term occurs around (depending on the precise form of the lattice action) \( p^+ = p_{\text{max}}^+ / 2 = \pi / 2a_L \). Since the total momentum is not conserved this implies that particles tend to accumulate near this minimum. However, with such a large momentum the particles can resolve the granular structure of the lattice and no meaningful continuum limit will be obtained. Note that a similar pathology would occur in an unconstrained (total momentum allowed to vary) variational calculation of the light-front energy of a hadron in the continuum. It is conceivable that adding a Lagrange multiplier proportional to the total light-front momentum to the lattice action cures the problem (in the continuum this amounts to minimizing \( \tilde{P}^- = P^- + \lambda P^+ \) instead of \( P^- \)). However, this idea will not be pursued here any further.

Instead, I found it more useful to work in momentum space as far as the longitudinal direction is concerned because this allows one easily to maintain longitudinal momentum conservation — a crucial necessity for light-front calculations as we have seen above. One momentum space technique which has been widely applied to light-front quantized 1 + 1-dimensional field theories is discrete light-cone quantization (DLCQ) \[4\]. There one puts the system into a longitudinal box and imposes periodic or antiperiodic boundary conditions \( \phi(x^- + L) = \pm \phi(x^-) \). The momenta thus become discrete and solving the equations of motion for a fixed value of \( P^+ \) has been reduced to diagonalizing a finite matrix (note that all light-cone momenta are positive and thus there is only a finite number of states associated with a given value of \( P^+ \). The longitudinal continuum limit is achieved by making \( P^+ \) sufficiently large. In 1 + 1 dimensions this technique was very useful and effective \[5,6\]. For example, it has been used to demonstrate the existence of a nucleon-nucleon bound state in \( QCD_{1+1} \) with \( SU(2) \)-color and \( SU(2) \)-flavor in a calculation on the level of quarks — despite the small binding energy (\( \approx 1\% \)) of this “deuteron” \[5\]. One mayor obstacle for applications of DLCQ to 2 + 1 and 3 + 1 dimensional field theories has been the exponential growth of the number of basis states with the number of transverse degrees of freedom. For example, a DLCQ calculation for a scalar field (with antiperiodic boundary conditions in
the $x^-$ direction) on a (rather modest) $4 \times 4$ transverse lattice with a longitudinal momentum $P^+ = \frac{15}{2}$ has already a basis size of $779022208$. A more reasonable $8 \times 8$ transverse lattice with the same $P^+$ requires a basis of $6.27 \cdot 10^{15}$ states! These astronomical numbers clearly demonstrate that any direct matrix diagonalization approach or even a Lanczos type algorithm is doomed to fail because one is not even able to store the wavefunction in any available computer. Another numerical method for studying light-front Hamiltonians is the light-front Tamm-Dancoff approach [7] where one imposes severe truncations of the Fock space. Sometimes this method is combined with DLCQ, i.e. one formulates the light-front Hamiltonian in the DLCQ basis but restricts (ad hoc) the Fock space to few particle states. However, it is not clear to what extend this truncation modifies the dynamics and whether effective light-front Hamiltonians can be constructed by systematically eliminating higher Fock states. Because of these difficulties, all numerical studies of light-front Hamiltonians have been restricted to $1 + 1$ dimensions and/or severe Tamm-Dancoff truncations and/or perturbation theory.

In this work Monte Carlo techniques will be exploited to obtain approximate ground state energies and structure functions for DLCQ problems in $2+1$ and $3+1$ dimensions on a transverse lattice without any constraints on the Fock space other than those resulting form the discreteness of the momenta. Note that the algorithm which I will present requires that the interaction in the transverse direction is local (nearest neighbor interaction at most). That is why the theory will be formulated on a transverse coordinate space lattice. The algorithm would not work with a momentum space lattice. To avoid obscuring the Monte Carlo algorithm with other difficulties the technique will be explained for a real scalar field with $\phi^4$ coupling in $2 + 1$ dimensions, which is one of the most simple field theories one can formulate on a transverse lattice. The main reason to chose $\phi^4_{2+1}$ for illustrating this new method is that $QCD$ is too complicated for a “first study case” and for demonstration purposes. Other theories one might think of, like $QED_{3+1}$ of $\phi^4_{3+1}$ are not asymptotically free, i.e. there is no Bjorken scaling for deep inelastic structure functions. However, it should be emphasized that the technique is a priori applicable to any DLCQ Hamiltonian which is
formulated on a transverse lattice, provided the interactions satisfy locality in the transverse direction.

The paper is organized as follows. First the DLCQ Hamiltonian for $\phi^4$ in 2+1 dimensions on a transverse lattice will be constructed. In section II, the infinitesimal light-front time evolution operator will be approximated using a checkerboard decomposition and a path integral in the Fock space of DLCQ will be used to project out the ground state for given quantum numbers. Finally, in section IV, the path integrals will be evaluated using an initial value random walk algorithm based on a variation of the ensemble projector Monte Carlo method.

II. $\phi_{2+1}^4$ ON A TRANSVERSE LATTICE

The Minkowsky action for the $\phi_{2+1}^4$ model, in the continuum, reads

$$A_{\text{cont.}} = \int d^3x \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4.$$  (2.1)

After discretization in the transverse direction one thus obtains

$$A_{\perp \text{latt.}} = a \int dx^- \sum_n \left[ \partial_+ \phi_n \partial_- \phi_n - \frac{m^2}{2} \phi_n^2 - \frac{\lambda}{4!} \phi_n^4 - \frac{(\phi_{n+1} + 1 - \phi_n)^2}{2a^2} \right].$$  (2.2)

Upon rescaling $\varphi_n = \sqrt{a} \phi_n$ the interpretation of the transverse lattice action as the action of a “multiflavor” field theory ($n$ being the “flavor” index in this interpretation and $\varphi_n$ being a canonical field defined in 1+1 dimensions) becomes evident

$$A_{\perp \text{latt.}} = \int dx^+ dx^- \sum_n \left[ \partial_+ \varphi_n \partial_- \varphi_n - \frac{m^2}{2} \varphi_n^2 - \frac{\lambda}{4!} \varphi_n^4 - \frac{(\varphi_{n+1} - \varphi_n)^2}{2a^2} \right].$$  (2.3)

Light-front quantization of (2.3) is standard [8]: first one puts the system into an $x^-$-box of length $L$ with antiperiodic boundary conditions (the associated zero-mode effects and
implications for the renormalization are discussed in Ref. [9]). The canonical commutation relations

$$[\partial_+ \varphi_n(x), \varphi_m(y)]_{x^+ = y^+} = -\frac{i}{2} \delta_{nm} \delta(x^- - y^-)$$  \(2.4\)

as well as the antiperiodic boundary condition are satisfied for

$$\varphi_n(x^-) = \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} \frac{[a_n(k)e^{-ip_k^+x^-} + a_n^\dagger(k)e^{ip_k^+x^-}]}{\sqrt{k - \frac{1}{2}}}$$  \(2.5\)

where

$$p_k^+ = \frac{2\pi}{L} \left( k - \frac{1}{2} \right)$$  \(2.6\)

and the \(a_n(k)\) satisfy the usual commutation relations

$$[a_n(k), a_m^\dagger(q)] = \delta_{nm} \delta_{kq}.$$  \(2.7\)

Finally one obtains for the light-front momentum operator

$$P^+ = \sum_n \int dx^- : \partial_+ \varphi_n \partial^+ \varphi_n :$$

$$= \frac{2\pi}{L} \sum_n \sum_{k=1}^{\infty} a_n^\dagger(k)a_n(k) \left( k - \frac{1}{2} \right)$$  \(2.8\)

and for the light-front energy

$$P^- = \frac{L}{2\pi} \sum_n (T_n + V_n + V_{n,n+1})$$  \(2.9\)

where

$$T_n = \frac{m^2}{2} \sum_{k=1}^{\infty} \frac{a_n^\dagger(k)a_n(k)}{k - \frac{1}{2}}$$  \(2.10\)

is the usual light-front kinetic energy on each site and

$$V_n = \frac{\lambda \delta P_f P_i}{4\pi a^4} \sum_{k_1,k_2,k_3,k_4=1}^{\infty} \frac{(a_n^\dagger(k_1) + a_n(k_1)) (a_n^\dagger(k_2) + a_n(k_2)) (a_n^\dagger(k_3) + a_n(k_3)) (a_n^\dagger(k_4) + a_n(k_4))}{\sqrt{k_1 - \frac{1}{2}} \sqrt{k_2 - \frac{1}{2}} \sqrt{k_3 - \frac{1}{2}} \sqrt{k_4 - \frac{1}{2}}}$$  \(2.11\)
is, up to the factor $1/a$ the self interaction for $\varphi_{1+1}^4$. $\delta_{P_f,P_i}$ is a momentum conserving Kronecker $\delta$. Neighboring sites are coupled through the hopping term

$$V_{n,n+1} = \frac{1}{2a^2} \sum_{k=1}^{\infty} \left( a_{n+1}^\dagger(k) - a_{n+1}^\dagger(k) \right) \left( a_{n+1}(k) - a_n(k) \right) \frac{k - \frac{1}{2}}{k - \frac{1}{2}}. \quad (2.12)$$

Note that, as in all DLCQ problems, the length of the box factorizes completely. Thus we will in the following work with the rescaled operators

$$K = P^+/2\pi \quad (2.13)$$

and

$$H = P^-2\pi/L. \quad (2.14)$$

At least in principle one could now proceed as follows: for fixed $K$ ($K$ and $H$ commute) one diagonalizes $H$, yielding the eigenvalue $E_i$ and thus the invariant masses of the physical states $M_i^2 = 2KE_i$. Physical observables like structure functions are obtained by calculating the appropriate matrix element in these states. The continuum limit is reached by extrapolating to $K \to \infty$. For one or two sites this can be easily done. For four sites this is also still possible. However, beyond about 8 sites direct diagonalization methods (including the Lanczos algorithm) soon become useless due to the exponential growth of the required basis size. What is needed at this point is a Monte Carlo method that allows one to find the ground state (for given value of $K$) of the DLCQ Hamiltonian $H$ more efficiently. Developing and testing such an algorithm will be the main subject in the rest of this paper.

Before we come to the Monte Carlo algorithm I should add a few comments. First, although I have derived the transverse lattice DLCQ Hamiltonian only for $\varphi_{2+1}^4$, similar expressions can be derived for 3 + 1 dimensions and/or other field theories $\varphi_{3+1}^4$. Many aspects are completely general. For example, for many field theories (including $QCD_{3+1}$) the transverse “hopping” is provided by a nearest neighbor interactions. This is crucial for the algorithm which I will present in this paper because (i) one can easily approximate the infinitesimal light-front time evolution operator as will be explained in section (ii)
is possible to “locally update” the states in a random walk algorithm which thus provides a computational advantage on large transverse lattices. Second, I should discuss renormalization at this point. Besides the tadpoles (which are zero in light-front quantization [9,10]) there is only one divergent diagram in \( \phi^4_{2+1} \), namely the setting sun diagram (Fig.1). This diagram leads to a divergent self mass contribution while the associated wave function renormalization is finite. Therefore, for \( \phi^4_{2+1} \) it will be sufficient to add an appropriate mass counterterm to render the theory finite in the continuum limit. Here one has to be a little careful in DLCQ because the longitudinal momentum of the incoming line in (Fig.1) effectively determines which higher Fock states are allowed in the intermediate state. Hence the self mass, and thus the required mass counterterm, for (Fig.1) depend on the longitudinal momentum (of course, for large longitudinal momenta, this dependence asymptotically disappears). For the renormalization this leaves at least two options: one possibility is to calculate how the infinite part of the self energy depends on the longitudinal momentum by evaluating (Fig.1) within DLCQ. Or one allows the bare mass to depend on the longitudinal momentum in such a way that the physical mass of the lightest state is independent of the longitudinal momentum. This is possible if one uses the following “sequential” procedure: First one solves the DLCQ Hamiltonian for \( K = 1/2 \) (which is trivial) and adjusting the bare mass for \( K = 1/2 \) until one satisfies the renormalization condition. Then one repeats the same procedure for \( K = 3/2 \) while keeping the bare mass for \( K = 1/2 \) partons fixed and so on. In the continuum limit (large longitudinal momenta) both methods should be equivalent. However, I preferred to use the second method (sequential renormalization) because it seemed to converge faster numerically in \( K \). Furthermore, because one must verify numerical convergence in \( K \), it is anyway necessary to repeat the calculation for several \( K \). In addition, the computer time spent usually grows rapidly with \( K \), i.e. the numerical effort associated with additional calculations for small values of \( K \) in the sequential renormalization is negligible. Another advantage of renormalizing such that \( M_{phys} \) is independent of \( K \) is that multiparticle thresholds in two point functions appear at the correct energy values relative to the single particle pole. This issue has often been neglected in DLCQ.
calculations. Furthermore, while the first method works only for $\phi^4_{2+1}$ because there is only one divergent self-energy diagram in this theory, the sequential method is non-perturbative and can thus be applied to the DLCQ Hamiltonian for any field theory.

The coupling constant for $\phi^4_{2+1}$ receives only a finite renormalization. Nevertheless, in principle one should always fix the bare coupling by imposing another renormalization condition. Since the spectrum of $\phi^4_{2+1}$ consists only of the fundamental particle and its scattering states, and since the mass of the fundamental particle has already been used up to fix the bare mass term, one has to use physical observables other than the mass spectrum in this case. However, since this is a special feature for $\phi^4_{2+1}$ I decided not to renormalize the coupling at this point. In the following, all results will be quoted with the bare coupling constant for which they were calculated.

III. THE MONTE CARLO PROCEDURE

In this section, we will use the fact that repeated application of $\exp(-\epsilon H)$, where $H$ is the DLCQ Hamiltonian (2.14), on any state $|\psi(K)\rangle$ with given light-cone momentum $K$ (2.13) gives the ground state $|\psi_0(K)\rangle$ for this particular $K$ — of course provided $|\psi(K)\rangle$ is not orthogonal to the ground state $|\psi_0(K)\rangle$. The light-front imaginary time evolution operator $\exp(-\epsilon H)$ can be approximated using the Trotter formula [11]

$$e^{-\epsilon (H_a + H_b)} = e^{-\epsilon H_a/2}e^{-\epsilon H_b}e^{-\epsilon H_a/2} +$$

$$\frac{\epsilon^3}{24} \left\{ [H_a, [H_b, H_a]] - 2 [H_b, [H_a, H_b]] \right\} + ...$$

$$= e^{-\epsilon H_a/2}e^{-\epsilon H_b}e^{-\epsilon H_a/2} + O(\epsilon^3) \quad (3.1)$$

where the following choice is particularly useful

$$H_a = \frac{1}{2} \sum_n (T_n + V_n) + \sum_{n, odd} V_{n,n+1}$$

$$H_b = \frac{1}{2} \sum_n (T_n + V_n) + \sum_{n, even} V_{n,n+1}. \quad (3.2)$$

This choice is motivated because $H_a$ and $H_b$ individually separate into sublattices which contain only 2 sites each.
\[ H_a = \left[ \frac{1}{2} (T_1 + V_1 + T_2 + V_2) + V_{1,2} \right] + \]
\[ \left[ \frac{1}{2} (T_3 + V_3 + T_4 + V_4) + V_{3,4} \right] + ... \]
\[ = H_{1,2} + H_{3,4} + ... \]
\[ H_b = \left[ \frac{1}{2} (T_2 + V_2 + T_3 + V_3) + V_{2,3} \right] + \]
\[ \left[ \frac{1}{2} (T_4 + V_4 + T_5 + V_5) + V_{4,5} \right] + ... \]
\[ = H_{2,3} + H_{4,5} + ... \quad (3.3) \]

i.e. both \( H_a \) and \( H_b \) are sums of commuting operators and can be easily diagonalized and exponentiated numerically! This task is facilitated even more by the fact that \( H_{1,2}, H_{2,3} \) etc. each satisfy longitudinal momentum conservation, i.e. \( H_{1,2}, H_{2,3} \) etc. are all block diagonal in the DLCQ basis, where the blocks are labeled by the sum of momenta on sites 1 and 2, etc.!

Once one has constructed \( \exp(-\varepsilon H) \) one can proceed to evaluate physical observables in the ground state, using

\[ E_0(K) = \lim_{N \to \infty} \frac{\langle \psi_f(K) | H \left( e^{-\varepsilon H} \right)^N | \psi_i(K) \rangle}{\langle \psi_f(K) | (e^{-\varepsilon H})^N | \psi_i(K) \rangle} \quad (3.4) \]

for the ground state energy and

\[ \frac{\langle \psi_0(K) | \hat{O} | \psi_0(K) \rangle}{\langle \psi_0(K) | \psi_0(K) \rangle} = \lim_{M,N \to \infty} \frac{\langle \psi_f(K) | (e^{-\varepsilon H})^M \hat{O} (e^{-\varepsilon H})^N | \psi_i(K) \rangle}{\langle \psi_f(K) | (e^{-\varepsilon H})^{M+N} | \psi_i(K) \rangle} \quad (3.5) \]

for any other observable, provided \( \langle \psi_0(K) | \psi_i(K) \rangle \neq 0 \) and \( \langle \psi_f(K) | \psi_0(K) \rangle \neq 0 \). The results thus obtained still depend on \( \varepsilon \) because \( \exp(-\varepsilon H) \) has been approximated only up to \( \mathcal{O}(\varepsilon^3) \). The \( \varepsilon \to 0 \) limit can be obtained by extrapolation.

The above products are evaluated as follows. First one inserts a complete set of states at each \( \varepsilon \)-step, e.g.

\[ \langle \psi_f(K) | (e^{-\varepsilon H})^N | \psi_i(K) \rangle = \sum_{\vec{i}_1 \vec{k}_1} \langle \psi_f(K) | e^{-\frac{\varepsilon}{2} H_a} | \vec{i}_{3N+1} \vec{k}_{3N+1} \rangle <\vec{i}_{3N+1} \vec{k}_{3N+1} | e^{-\varepsilon H_b} | \vec{i}_{3N} \vec{k}_{3N} \rangle \]
\[ \times <\vec{i}_{3N} \vec{k}_{3N} | \ldots e^{-\frac{\varepsilon}{2} H_a} | \vec{i}_1 \vec{k}_1 \rangle <\vec{i}_1 \vec{k}_1 | \psi_i(K) \rangle, \quad (3.6) \]
where the $|\vec{i}_l \vec{k}_l > (l = 1, ..., 3N+1)$ are a direct product of complete sets of states at each site. The vector notation is used to label these states where the n-th component of $\vec{k}$ refers to the longitudinal momentum on the n-th site and the n-th component of $\vec{i}$ refers to the internal excitation level on this n-th site with a given momentum. For example for a lattice with 4 sites a randomly picked state could look like this:

$$\vec{k} = (0, 1, \frac{5}{2}, 3)$$
$$\vec{i} = (1, 1, 3, 2) \quad (3.7)$$

which would mean that site one carries 0 longitudinal momentum, site two 1 unit, etc. while the internal excitation numbers are 1 (for zero momentum there is only one state — the vacuum), 1, 3 and 2 respectively. Of course one can chose many other ways to enumerate the basis but I found the above method the easiest to implement in a computer code.

In this work, a free Fock space basis has been selected to represent the internal excitations at each site because structure functions are diagonal only in a free Fock space basis. However, many other choices are conceivable as well. In fact, in many cases it may be more efficient numerically to chose some kind of interacting basis on the sites instead of a free basis. Particularly if one is more interested in energy eigenvalues or observables other than structure functions.

Several Monte Carlo techniques are available to perform the summations in Eq.(3.6). I have tried the projector Monte Carlo method \[11\] as well as the ensemble projector Monte Carlo method \[12\]. In the projector Monte Carlo method, in each sweep one starts from a state $|\vec{i}_1 \vec{k}_1 >$ which is randomly picked with probability $<\vec{i}_1 \vec{k}_1 |\psi_i(K) >$. In the next step one picks a state $|\vec{i}_2 \vec{k}_2 >$ with probability $W(\vec{i}_1, \vec{k}_1 \rightarrow \vec{i}_2, \vec{k}_2)$. The probabilities chosen in this work are \[13\]

\[1\] After all there is still a $1 + 1$ dimensional field theory attached to each site, i.e. for each site-momentum many different states are possible.

\[2\] In principle, there is little restriction in the choice of the “probabilities”. However, I found
\[ W(\vec{i}_l, \vec{k}_l \rightarrow \vec{i}_{l+1}, \vec{k}_{l+1}) = \frac{|<\vec{i}_{l+1}, \vec{k}_{l+1}|U|\vec{i}_l, \vec{k}_l>|}{\sum_{\vec{i}, \vec{k}} |<\vec{i}, \vec{k}|U|\vec{i}_l, \vec{k}_l>|} \]  \hspace{1cm} (3.8)

where \( U = \exp(-\frac{\varepsilon}{2}H_a) \) for \( l = 3N \) or \( l = 3N + 1 \) and \( U = \exp(-\varepsilon H_b) \) for \( l = 3N + 2 \), \( N = 0,1,2, \ldots \). Since \( U \) factorizes into two-site sublattices, so does the transition probability \( W \). For example, suppose (on a lattice with 4 sites)

\[
\vec{i}_l = (i_1^l, i_2^l, i_3^l, i_4^l)
\]

and suppose \( U = \exp(-\frac{\varepsilon}{2}H_a) \). Then one first selects \( i_{l+1}^1, k_{l+1}^1, i_{l+1}^2, k_{l+1}^2 \) with probability proportional to

\[
|<i_{l+1}^1, k_{l+1}^1, i_{l+1}^2, k_{l+1}^2|e^{-\frac{\varepsilon}{2}H_1}|i_l^1, k_l^1, i_l^2, k_l^2>|. \hspace{1cm} (3.10)
\]

The actual selection can be done with a Metropolis algorithm. Then one applies the same procedure to sites 3 and 4. Having thus “updated” the entire lattice one proceeds to the next \( \varepsilon \)-step where \( U = \exp(-\varepsilon H_b) \). This time it is sites 2 and 3 as well as sites 4 and 1 which interact with another. And so on until one reaches \( \vec{i}_{3N+1}, \vec{k}_{3N+1} \). For such a “path” one then computes the product

\[ S = S_{f,3N+1} \times S_{3N+1,3N} \times \ldots \times S_{2,1}, \]  \hspace{1cm} (3.11)

where

\[
S_{f,3N+1} = <\psi_f(K)|e^{-\frac{\varepsilon}{2}H_a}||\vec{i}_{3N+1}, \vec{k}_{3N+1}> \]  \hspace{1cm} (3.12)

Eq. (3.8) useful numerically.

3 Note that two adjacent steps with \( U = \exp(-\frac{\varepsilon}{2}H_a) \) can be combined into one step with \( U = \exp(-\varepsilon H_a) \).

4 Note that momentum conservation requires \( k_{l+1}^1 + k_{l+1}^2 = k_l^1 + k_l^2 \) and thus restricts the possible choices.
and the “scores” at each step are the actual transition amplitudes (the matrix elements of $U$) divided by the “probabilities”

$$S_{t+1,l} = \frac{\langle \vec{r}_{t+1} k_{t+1} | U | \vec{r}_t k_t \rangle}{W(\vec{r}_t, k_t \rightarrow \vec{r}_{t+1}, k_{t+1})}.$$  \hspace{1cm} (3.13)

These products of scores (3.11) are then averaged over many sweeps. Similarly one computes the numerators in Eqs. (3.4) (3.5). For example, if one wants to calculate a structure function, one averages over the product of scores times the structure function measured somewhere near the middle of the path and divides the result by the average of the product of scores.

The disadvantage of the projector Monte Carlo method is that once a small “score” occurs along a path, the whole product for the path will contribute negligibly to the sum of paths, i.e. the path integral will be dominated by a few paths which do not contain any small element and the relative statistical fluctuations in the path thus increase linearly with the number of steps in the path \[12,13\]. Since on the other hand one is interested in making the path as long as possible in order to project onto the ground state while keeping $\varepsilon$ small to avoid systematic errors, the projector Monte Carlo method turns out to be rather inefficient.

A variation of this method, the ensemble projector Monte Carlo method suffers less from this problem. There one starts form an ensemble of states at step 1. The probabilities and scores are calculated for each member of the ensemble in the same way as for the projector Monte Carlo method. However, after each $\varepsilon$-step, when one calculates the scores, one replicates and deletes members of the ensemble by the following rule: first one calculates the average score $\bar{S}$ (ensemble average). Then one replicates each state $\nu$ in the ensemble with multiplicity

$$n_\nu = Int \left[ \frac{|S_\nu|}{\bar{S}} + \text{'random number $\in (0;1)$'} \right],$$  \hspace{1cm} (3.14)

where $S_\nu$ is the score for the transition to state $\nu$. This way, any path where a very small score occurs will most likely be eliminated (and no further computer time will be wasted on

\[5\]Here it becomes clear why a diagonal representation is preferable for an easy computation of structure functions.
this path) while important paths branch out and contribute with multiple weight. Note that the size of the population is not fixed because the states are replicated with a multiplicity that depends on a random number. In general, the population thus fluctuates. Sometimes, the population grows or shrinks slowly. In order to keep the population stable on the average one can counterbalance the growing/shrinking by choosing $\bar{S}$ in Eq. (3.14) a little larger/smaller than the average score.

For the path integral one only has to evaluate the sign of the scores

$$\tilde{S} = S_{f,3N+1} \times \text{sign} (S_{3N+1,3N} \times ... \times S_{2,1}).$$

(3.15)

The absolute values of the scores along the path are already taken into account since they determined the multiplicities in the random walk (3.14). Observables are calculated similarly as in the projector Monte Carlo Method except that one now averages over both the ensemble and the sweeps. For the efficiency of the algorithm, it is important to start the random walk with a good initial guess for the ground state $|\psi_i(K)\rangle$. For many light-front Hamiltonians the exact ground state wavefunction has a sizeable overlap with the valence configuration. As a first try, it may thus be useful to make a valence ansatz for $|\psi_i(K)\rangle$ and $|\psi_f(K)\rangle$. This will also be the choice in this work. For more complicated theories one can also try to improve the valence ansatz for $|\psi_i(K)\rangle$ by including higher Fock components perturbatively.

IV. NUMERICAL RESULTS

The Lagrangian for $\phi^{4}_{2+1}$ contains two dimensionful parameters: the bare mass $m_0$ and the coupling constant $\lambda$ (which also carries dimension of mass) and hence an arbitrary scale $\Lambda$, specifying the units in which these dimensionful parameters are measured. In the following, the scale will be fixed by demanding that $M_{phys} = \Lambda$ for the lightest physical particle. All other dimensionful quantities (e.g. $\lambda$ or $a^{-1}$) will be measured in these units. After fixing the physical mass scale, which determines the bare mass, there is still (in the continuum limit) one dimensionless parameter left: $\lambda/4\pi\Lambda$. In my numerical work I have
considered two extreme cases: $\lambda/4\pi\Lambda = 1$ and $\lambda/4\pi\Lambda = 10$ corresponding to intermediate and strong coupling respectively. The case $\lambda/4\pi\Lambda = 1$ turned out to be rather boring because the structure function of the lightest physical particle is strongly dominated by a “bare” excitation $|\psi_0(K) > \approx \sum_n a_n(K)|0 >$. Deviations from this pointlike structure can be well approximated by summing a chain of “setting suns” (Fig. 1). Although this result was reproduced in the Monte Carlo calculations, it will not be discussed here any further because the real strength of the Monte Carlo method lies more in the nonperturbative regime. $\lambda/4\pi\Lambda = 10$ will thus be chosen throughout the rest of this paper.

The numerical calculations were done as follows: after selecting the coupling constant ($\lambda/4\pi\Lambda = 10$) and choosing a transverse lattice (characterized by the spacing $a\Lambda$ and by the number of sites) and after selecting a value for the “damping parameter” $\varepsilon$, the bare masses were determined by imposing the renormalization condition $M_{\text{phys}}/\Lambda = 1$. Within the DLCQ formalism used here this yields bare masses which show some dependence on the longitudinal momentum. For $p = 1/2$ there is no interaction and $M_{\text{phys}}(1/2) = \Lambda$ implies $m^2_{\psi(1/2)} = \Lambda^2$. In the next step the bare mass $m^2_{\psi(3/2)}$ for $p = 3/2$ is determined by requiring $M_{\text{phys}}(3/2) = \Lambda$ (keeping $m^2_{\psi(1/2)}$ fixed). Then $m^2_{\psi(3/2)}$ while keeping $m^2_{\psi(3/2)}$ and $m^2_{\psi(1/2)}$ fixed and so on, up to $p = 15/2$ — the largest momentum used in this work. In this fine-tuning process, at every longitudinal momentum, the physical mass of the lightest particle was determined using the ensemble projector Monte Carlo technique with an ensemble size of 500 states and with 10000 $\varepsilon$-steps. For $\psi_i$ and $\psi_f$ a plane wave (zero transverse momentum) of bare ground state “mesons” was used

$$|\psi_i(K) > = |\psi_f(K) > = \sum_n a_n^\dagger(K)|0 > . \quad (4.1)$$

This choice (which corresponds to the valence approximation in $\phi^4$) was motivated by the fact that the physical ground state particle in $\phi^4_{2+1}$ can be interpreted as a dressed single particle state.
After 15 “thermalization steps” the energy was sampled every 5th \( \varepsilon \)-step \(^6\) (to insure statistically independent sampling) by separately taking the ensemble average of the numerator and the denominator in Eq. \( (3.4) \). The physical mass is obtained from

\[
M_{phys}^2 = 2K\bar{E},
\]

where \( \bar{E} \) is the average over all energy measurements \( (10000/5 = 2000 \) in the above procedure\). Typical (statistical) errors with these parameters where of the order of \( 1 - 3\% \) for \( \bar{E} \) and thus also for the bare masses. Whenever \( M_{phys} \), evaluated using Eq. \( (4.2) \) deviated significantly from \( \Lambda \) (the renormalization condition) the bare masses where adjusted accordingly.

This procedure was repeated for \( \varepsilon \Lambda = .3, .15, .075 \) and for lattices with \( N_{sites} = 4, 8, 16, 32 \) and 64 transverse sites and for transverse spacings \( a\Lambda = 1, \frac{1}{2}, \frac{1}{4} \). It turned out that the bare masses depend only very weakly on \( \varepsilon \) and \( N_{sites} \) which made the tuning rather easy. The tuning was facilitated even more by the fact that the path integral can still be summed up numerically exactly (without Monte Carlo) for lattices with four sites (and \( p \leq \frac{15}{2} \)). Furthermore, one can estimate the difference between 4-site lattices and larger lattices by perturbative methods before one starts the nonperturbative tuning using the Monte Carlo. For \( a = 1/4 \) and large \( K \) the energy measurement described above became too noisy due to the sign problem discussed at the end of section \( \[4] \). For these cases, I used a slightly different algorithm: Instead of evolving the initial state for \( 10000 \) \( \varepsilon \)-steps and sampling the energy every fifth step, the initial state was evolved only for \( 10 \) \( \varepsilon \)-steps. This procedure was then repeated 2000 times to obtain the same statistical sample size for the energies as in the first method. The advantage of the second procedure is the following. Since only a few (small) matrix elements of \( \exp(-\varepsilon H) \) are negative, it is not very likely to encounter a negative score in a given “path” \( (3.15) \) — unless the path is very long. With this scenario it is clear that procedure two (starting over and over again from the same initial state) has

\(^6\)Here each application of \( \exp(-\varepsilon H_a) \) or \( \exp(-\varepsilon H_b) \) is counted as one \( \varepsilon \)-step.
much less of a sign problem than procedure one (continued evolution). However, typically it takes more \( \varepsilon \)-steps to project onto the ground state from \( |\psi_i(K)\rangle \) than it takes to get uncorrelated energy measurements. It thus depends on the concrete example which of the two procedures is more efficient.

After completing the renormalization for a given set of parameters \((N_{\text{sites}}, a, \varepsilon, \lambda)\) one can proceed to evaluate physical observables. At this point let me introduce the “structure functions” for scalar fields. In analogy to definitions of parton distributions in QCD one can introduce

\[
f(x) = x \int_{-\infty}^{\infty} \frac{d\xi^-}{4\pi} e^{ix\xi^-p^+} <\psi(p^+)|\phi(0)\phi(\xi^-)|\psi(p^+)> \tag{4.3}
\]
as the light-cone momentum density of elementary quanta in the state \( |\psi(p^+)> \). The normalization is such that the momentum sum rule reads \( \int_0^1 dx x f(x) = 1 \). Upon discretizing the structure function can be expressed as (2.5)(2.6)

\[
f(x_p) = \frac{1}{K} \sum_{n=1}^{N_{\text{sites}}} <\psi(K)|a_n^\dagger(p)a_n(p)|\psi(K)> \tag{4.4}
\]
where \( x_p = (p - \frac{1}{2})/K, p = 1, \ldots, K + \frac{1}{2} \). This expression (4.4) is the form used in this work.

What should one expect \( f(x) \) to look like for \( \phi_{2+1}^4 \)? First, for trivial kinematical reasons, \( f(x) \) is nonzero only for \( x \in [0; 1] \). Second, since the wavefunction renormalization in \( \phi_{2+1}^4 \) is finite, there is a finite probability to find the physical “meson” as a bare state. Thus, in the continuum limit, \( f(x) \) should contain a \( \delta \)-function at \( x = 1 \) with finite coefficient (in the structure function plots in this work this point will always be excluded because it would lie outside the chosen plotframe). Besides the \( \delta \)-function one expects a continuum because the bare state can always split into three “partons” (via the \( \phi^4 \)-interaction) which can split again and so on. While lowest order perturbation theory suggests a structure function that is peaked around \( x = \mathcal{O}(\frac{1}{3}) \), higher order effects (multifragmentation) will shift the maximum towards smaller values of \( x \).

\(^7\)One can in fact imagine gedanken experiments that would allow one to measure \( f(x) \) but this point will not be discussed here any further.
Structure functions were evaluated by averaging over 10000 sweeps (except for \( a = 0.25 \) where structure functions were averaged over 20000 sweeps) where again an ensemble size of 500 was used. Each sweep consisted of 30 \( \varepsilon \)-steps. Also \( \psi_i = \psi_f = \text{Eq.}(1.1) \) was used again. The operator to measure the structure functions was inserted after the first 15 \( \varepsilon \)-steps for each member of the ensemble.\(^{[8]}\) For the remaining steps of the random walk, a record is kept of the result of this structure function measurement from which each subsequent ensemble member has evolved \([9]\). Finally, the result for the measurement of \( f(x) \) in this sweep is obtained by separately evaluating the ensemble average of the numerator and the denominator in Eq.(3.5). This result is then averaged over the sweeps. In order to investigate convergence with respect to the number of \( \varepsilon \)-steps in the final state, this number was kept variable (1 – 15). Typical results for such a structure function measurement, as a function of the number of \( \varepsilon \)-steps before taking the overlap with \( < \psi_f(K) | \rangle \), are shown in Fig.2. For all values of \( \varepsilon \) used in this work, the plateau sets in before 10 \( \varepsilon \)-steps, i.e. the ground state expectation value of the structure function can be extracted from these results.\(^{[9]}\) There are several reasons for this rapid convergence. First, even for \( \lambda/4\pi\Lambda = 10 \), the true ground state has a large overlap with the bare state \( |\psi_i \rangle \) (4.1). Furthermore, any contamination of \( |\psi_i \rangle \) with excited states is filtered out very efficiently because higher states are suppressed by the square of their masses

\[
\exp(-\varepsilon H) = \sum_n |n > < n | \exp(-\varepsilon \frac{M_n^2}{2K}) .
\]  

(4.5)

The lowest excited state with the same quantum numbers as \( |\psi_i \rangle \) is a scattering state consisting of 3 ground state mesons with a threshold at \( M_1 = 3M_0 \) (i.e. \( M_1^2 = 9M_0^2 \)). For example, after 15 steps with \( \varepsilon = 0.3 \), \( M_0^2 = 1 \) and \( K = \frac{15}{2} \) the ground state is enhanced

\(^{[8]}\)Note that the structure functions in DLCQ are defined for a discrete set of points only (8 points for \( K = \frac{15}{2} \)). Hence “measuring the structure function” means evaluating it at a few points (4.4).

\(^{[9]}\)Since \( \psi_i = \psi_f \), conclusions about convergence in the final state can also be applied to the convergence in the initial state as well.
by a factor of $e^{-0.3}/e^{-2.7} \approx 11$ compared to the first excited state. Since most of the “contamination” comes not from the threshold itself but from many higher excited states, the filtering process is even more efficient than this numerical example illustrates.

Since the structure functions turned out to have converged already after 10 $\varepsilon$-steps in the final state, the measurements for $f(x)$ with 10 – 15 final steps were then averaged. Note that these measurements are statistically correlated. The statistical error for the average result was estimated by taking the statistical error for the measurement after 10 $\varepsilon$-steps in the final state (the statistical error almost does not change from step 10 to 15).

The results are shown in Figs.3-6. First one has to make sure that the longitudinal momentum $K$ was large enough. In Fig.3 results with $K = \frac{9}{2}$ and $K = \frac{15}{2}$ are compared for some typical choices of the other parameters. One can easily imagine that the $K = \frac{9}{2}$ and the $K = \frac{15}{2}$ results lie almost on the same smooth curve, indicating that $K = \frac{15}{2}$ is large enough. In Fig.4 it is demonstrated how the results converged as a function of the number of sites. Note that (with the exception of $N_{\text{sites}} = 2$ and 4, which were done numerically exactly) although the same number of sweeps and size of the ensemble were used for all values of $N_{\text{sites}}$, the statistical error bars increase only very slowly with $N_{\text{sites}}$. The same is true for the CPU-time required for the Monte Carlo calculation. The reason is that the light-cone vacuum far away from physical particles is trivial. Thus there are no statistical fluctuations arising from “updating the vacuum” on huge lattices (much larger than the transverse size of the particles).

The $\varepsilon$-dependence is illustrated in Fig.5. One way to understand Fig.5 is to evaluate the double commutator which governs the $O(\varepsilon^3)$ corrections in Eq.3.1). The rather lengthy expression will not be given here, but one can immediately guess the basic features. First, it is the hopping term $V_{n,n+1} \propto a^{-2}$ which gives rise to a nonvanishing commutator in $[H_a, H_b]$.

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10Note that this means that 10 $\varepsilon$-steps would also have been sufficient in the initial state. However, this was not clear before the calculations were completed.
which “explains” the increase of the finite ε corrections with decreasing a: Fig.5 a) vs. b) (of course, in order to be quantitative one has to to evaluate the matrix elements which could show an a dependence as well from the wavefunctions). Furthermore, the double commutator contains terms which spread over up to 4 transverse sites. It is thus not surprising to find a difference between the finite ε effects on 4-site and larger lattices: Fig.5 a) vs. c). Finally one has to take the a → 0 limit (Fig.6). This is the most difficult part because one first has to make sure that everything else has converged for fixed a. As discussed above, this requires smaller ε for smaller a (hence more steps to project onto the ground state and hence more computer time). The required number of lattice sites also increases with decreasing a. Furthermore, since the coupling between sites goes like a−2, the statistical fluctuations induced by this term increase as a decreases. An additional reason for the increase of the statistical fluctuations with decreasing a may lie in the choice of probabilities (3.8). On a two site lattice, the coupling between the two sites is twice as strong as between adjacent sites on a larger lattice (one step on the lattice in either direction leads to the same result on a two-site lattice with periodic boundary conditions). Thus the two site amplitudes tend to overemphasize highly excited states (compared to the actual physical situation on a large lattice). Since this “driving force” away from the ground state increases with smaller a, an increase in the statistical fluctuations results. Evidently, there is still room to improve the algorithm chosen here. However, since the purpose of this work is only a feasibility study and since the most useful choice for the probabilities (3.8) is expected to depend on the theory under consideration, this point will not be elaborated on here any further.

In general, one would extrapolate the results first to N_{sites} → ∞ and ε → 0 and then analyze the a-dependence of the extrapolated results. Here this is not necessary because the above results demonstrate that aN_{sites} = 4 and ε = 0.075 are already close enough to the continuum limit for a ≥ 0.025. Thus one can directly use these results without having to extrapolate.

While the structure function still changes significantly as one goes from (a, N_{sites}) = (1, 4) to (a, N_{sites}) = (0.5, 8), there is only a slight difference (at very small x and the slight
“shoulder” around $x \approx 0.6$) between $(a, N_{sites}) = (0.5, 8)$ and $(a, N_{sites}) = (0.25, 16)$, i.e. for $a \leq 0.5$ the numerical results (Fig.6) are almost independent of $a$ which shows that the small $a$ scaling region has been reached. Notice that the wave function renormalization in $\phi_{2+1}^4$ is finite and thus structure functions scale to a finite limit as $Q^2 \rightarrow \infty$ (corresponding to $a \rightarrow 0$); i.e. there is no logarithmic evolution for $\phi_{2+1}^4$ (if there were logarithmic evolution then the probability to find the state in the valence configuration would tend to zero as $Q^2 \rightarrow \infty$).

The shape of these structure functions can be understood as follows. With lowest order perturbation theory (fragmentation: valence state $\rightarrow$ 3 partons) one obtains a structure function which has a maximum near $x = 1/3$. Once one includes the (nonperturbative) interactions within the three particle sector, the structure function becomes smeared out. The rise at very small $x$ can only be understood from multiple fragmentation processes. The “shoulder” near $x \approx 0.5 - 0.6$ is a nonperturbative effect. It arises because the repulsive $\phi^4$ interaction tends to enhance components of the wavefunction with a node in the longitudinal direction. This also explains why the shoulder is absent for lattices with 4 or less sites. The reason is that parity and Bose symmetry require that a node in the longitudinal direction is accompanied by a node in the transverse direction. On a small lattice, components of the wavefunction with a node in the transverse direction have a very large kinetic energy and are thus suppressed. The $N_{sites}$-dependence of the results (Fig.4) indicates that the physical states have a transverse extension of about $1 - 2$ in the above units (i.e. $4 - 8$ lattice spacings for $a = 0.25$) because for lattices with a larger physical volumes, there is no significant volume dependence of physical observables. However, for more detailed information one would have to measure transverse density density correlation functions.

V. SUMMARY AND CONCLUSIONS

I have shown that it is perfectly feasible to perform a Monte Carlo calculation for a light-front Hamiltonian formulated as a DLCQ problem on a transverse lattice. The transverse lattice was used to separate longitudinal and transverse dynamics. The dynamics within the
longitudinal “sheets” attached to each transverse lattice point was solved using DLCQ and conventional matrix diagonalization algorithms. The transverse dynamics was then included by means of Monte Carlo techniques. For this purpose, the light-cone imaginary time evolution operator $\langle \psi_f | \exp(-N \varepsilon P^-) | \psi_i \rangle$ was approximated by breaking up the light-front Hamiltonian $P^-$ into two terms ($P^- = P_{a}^- + P_{b}^-$), each of which contains only interactions between pairs of sites. $\exp(-N \varepsilon P^-)$ is then evaluated by alternating application of infinitesimal “evolution”-operators generated by $P_{a/b}^-$ respectively. For calculating the actual path-integral, a variation of the ensemble projector Monte Carlo technique was used.

In this whole program it was crucial that the longitudinal momentum $P^+$ was conserved, otherwise the light-front Hamiltonian $P^-$ has no minimum corresponding to a physically meaningful particle solution. This was the major reason to use DLCQ to solve the longitudinal dynamics and not, for example, a longitudinal lattice. It was furthermore crucial that the transverse lattice action is local, i.e. it involves only interactions between neighboring sites. On the one hand, due to the locality, it was thus possible to perform the abovementioned breakup of the light front Hamiltonian $P^- = P_{a}^- + P_{b}^-$ in such a way that $P_{a/b}^-$ each can be written as direct sums of Hamiltonians acting on 2-sites-lattices. Thus $P_{a/b}^-$ can be easily diagonalized and exponentiated numerically. The locality of the transverse dynamics was furthermore important when updating the states at each Monte Carlo step. This is because locality of the interaction allowed to formulate the updating of the states at each $\varepsilon$-step as a sequence of independent local updatings (this will also be important when running the algorithm on parallel computers).

The advantages of the algorithm introduced in this paper are as follows: most importantly, structure functions are diagonal in the DLCQ basis used here and are thus easy to evaluate numerically. Furthermore, since the light-front momentum $P^+$ is manifestly conserved, and since the vacuum has $P^+ = 0$, physical particle states are always manifestly orthogonal to the vacuum. Thus the Monte Carlo procedure will always converge to the particle solution with lowest invariant mass for that particular value of $P^+$ (and the same discrete quantum numbers as the initial state). In addition, since the light-front vacuum
is trivial, no computer time is “wasted” to solve for the vacuum surrounding a physical particle while one is interested in the particle only (an annoyance for very large euclidean lattices). Another advantage of using the light-front Hamiltonian in the Monte Carlo procedure is that excited states are suppressed by the square of their masses: \( \exp\left(-N\varepsilon P^{-}\right) = \sum_n |n> \exp\left(-N\varepsilon M_n^2/2P^+\right) <n| \) (instead of \( \exp\left(-\beta P^0\right) = \sum_n |n> \exp\left(-\beta M_n\right) <n| \) which one encounters in a conventional Hamiltonian formulation). Thus fewer steps are necessary to filter out the ground state. The transverse lattice formulation also avoids part of the species doubling problem for fermions because doublers occur only for the latticized transverse directions. Thus at most four species of fermions are generated if one starts from the naive fermion action in 3 + 1 dimensions. Hence, by means of staggering, one can easily get to two light flavors of fermions \[3\].

One of the main disadvantages of the new method are the occurrence of a few negative matrix-elements in the infinitesimal evolution operator (although its eigenvalues are of course still positive). Thus negative scores resulted occasionally which slightly increased the fluctuations of the signal. It is expected that this problem gets worse once fermions are introduced because of the minus sign in exchange terms (of course this then is nothing but the usual sign problem for fermions). Another disadvantage is the explicit breaking of the symmetry between longitudinal and transverse directions which makes it difficult to recover full Lorentz invariance in the continuum limit. Although this was no problem in \(\phi_2^{4}\), it will be difficult to obtain Lorentz invariant physical results in general (for theories where the fundamental particles carry spin).

Numerous extensions of this work are conceivable. First of all, there is no profound difficulty to extend the formalism from 2 + 1 to 3 + 1 dimensions. Of course, in 3 + 1 dimensions \( P^{-} = P_a^{-} + P_b^{-} \) has to be decomposed in such a way that \( P^{-}_{a/b} \) each can be written as direct sums of Hamiltonians on 1-plaquette lattices. Another difference is the dependence of the numerical results on the lattice spacing \(a\): since \(\phi_2^{4}\) has a finite wave function renormalization, scaling is exact which implies that the structure functions approach a finite limit for \(a \to 0\). Of course in a renormalizable theory (like QCD) this will not be the
case and the structure functions will diverge as $a \to 0$ — corresponding to the logarithmic evolution in $Q^2$. However, at least in principle, this is not a problem because one can always perform the Monte Carlo calculations with smaller and smaller spacing $a$ until one can match on to the perturbative evolution. Since scaling in $QCD_{3+1}$ sets in at moderate $Q^2$ values already, there is reason to expect that this is also possible in practice (i.e. numerically practical).

In this work I have investigated only the projector and the ensemble projector Monte Carlo method because these are rather straightforward to apply to DLCQ problems. For more complicated field theories, it may be necessary to use more efficient techniques, like guided random walks [13] in the Monte Carlo procedure. One could also imagine combining the Monte Carlo technique presented in this work with renormalization group techniques. On the one hand this means using perturbative renormalization group arguments to facilitate the determination of the effective coupling constants in the light-front Hamiltonian. On the other hand one could use the Monte Carlo procedure to perform nonperturbative studies of renormalization group flow for light-front Hamiltonians without having to resort to uncontrolled truncations of the Fock space. Similarly, it is conceivable that Monte Carlo results are helpful in determining the effective coupling constants for the Tamm-Dancoff approach to solving light-front field theories. Besides structure functions, one can also use LFEPMC to calculate valence wavefunctions which have many interesting applications to various exclusive hard scattering processes [14]. However, the main question is whether one can apply the light-front ensemble projector Monte Carlo to $QCD_{3+1}$. Here the main difficulty which remains is formulating compact (to render the transverse lattice action gauge invariant) $QCD$ on a transverse lattice using DLCQ or to construct another approximation to $QCD$ on a transverse lattice which is suitable for DLCQ. Once one knows the DLCQ Hamiltonian for $QCD$, it is straightforward to apply the Monte Carlo technique presented in this work.

24
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FIGURES

FIG. 1. Divergent self-energy diagram in $\phi^4_{2+1}$

FIG. 2. Result of structure function measurements for 16 transverse sites, a transverse spacing of $a = 0.5$ and a longitudinal momentum of $K = 15/2$ as a function of the number of $\varepsilon$-steps before taking the overlap with $\psi_f$. Because of the smaller value of $\varepsilon$ in b), more $\varepsilon$-steps are necessary to project on the ground state.

FIG. 3. Typical examples for structure function measurements to illustrate the (in)dependence of the results $K = 9/2$ and $K = 15/2$.

FIG. 4. Dependence of the structure function on the number of transverse sites for the smallest lattice spacing ($a = 0.25$) and the smallest $\varepsilon$-step used in this work. The longitudinal momentum is $K = 15/2$. Both plots correspond to the same structure function but at different x-values. In order to avoid overlapping symbols from different x-values, the results are displayed in two plots.

FIG. 5. $\varepsilon$-dependence of the structure functions for $K = 15/2$. a) for 4 sites and $a = 0.5$; b) same as a) but for $a = 0.25$; c) same as a) but for 8 sites. Note the statistically significant deviation of the results for $\varepsilon = 0.3$ in c) in the intermediate x region, while a) (same $a$ as c) shows no such deviations.

FIG. 6. $a$-dependence of the structure functions for $\varepsilon = 0.075$ and $K = 15/2$. Note that $aN_{\text{sites}}$ — the physical volume — is kept fixed in a), b) and c).
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