Dunkl operators for arbitrary finite groups

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Abstract
The Dunkl operators associated with a necessarily finite Coxeter group acting on a Euclidean space are generalized to any finite group using the techniques of non-commutative geometry, as introduced by the authors to view the usual Dunkl operators as covariant derivatives in a quantum principal bundle with a quantum connection. The definitions of Dunkl operators and their corresponding Dunkl connections are generalized to quantum principal bundles over quantum spaces which possess a classical finite structure group. We introduce cyclic Dunkl connections and their cyclic Dunkl operators. Then, we establish a number of interesting properties of these structures, including the characteristic zero-curvature property. Particular attention is given to the example of complex reflection groups, and their naturally generalized siblings called groups of Coxeter type.

Keywords Dunkl operators · Finite groups · Quantum principal bundles

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1 Introduction

In our previous article [12], we showed how one can view the Dunkl differential-difference operators as covariant derivatives in a specific quantum principal bundle endowed with a specific quantum connection, whose (quantum!) curvature turns out to be zero. The zero curvature then implies that the Dunkl operators associated with a Dunkl connection commute among themselves. While this commutativity result dates back to Dunkl’s original paper [5] on this topic, the viewpoint established in [12] and continued here gives that result a geometric meaning. The Dunkl operators are associated with a finite Coxeter group which acts on a finite-dimensional Euclidean space as orthogonal transformations.

However, the general procedure used in [12] need not be restricted to Coxeter groups. Rather, the theory can be developed, as we show here, in much the same detail on the side of non-commutative geometry for arbitrary finite groups acting freely on a $C^\infty$ manifold which satisfies certain additional properties. This is the main topic of this paper, even though certain secondary topics, most especially cyclic structures as discussed later, will be necessary for our presentation.

This paper and [12] have opened up a bridge between theories that had been previously studied independently. On one side of this bridge, there is the non-commutative (or quantum) geometry of quantum principal bundles and their quantum connections. On the other side, there is the harmonic analysis which originally motivated and then continued to grow out of the Dunkl operators (see [5, 16]), but also their applications in probability, Segal–Bargmann analysis and other areas of mathematical physics such as Sutherland–Calogero–Moser models. (See [13, 17, 18] and references therein.) One can always cross the bridge back to other side of analysis to define and study the corresponding structures in harmonic analysis (such as a new generalization of the Fourier, or Dunkl, transform) and in mathematical physics, though this is more straightforward in the case of curvature zero. However, we leave these interesting and important topics for consideration in our future investigations.

Our paper is organized as follows. In the next section, we begin by reviewing for the reader’s convenience some background material on quantum differential calculus with an emphasis on finite groups, quantum principal bundles, quantum connections, and quantum covariant derivatives. We also give a new, quite general definition of quantum Dunkl connections, which generalizes the definition in [12]. In Sect. 3, we discuss the motivating example from [12] of the Dunkl operators associated with a Coxeter group acting on $\mathbb{R}^n$. In Sect. 4, we introduce a discrete geometry of points and lines in a finite set. This cyclic geometry is at the heart of the structure we wish to study. Our results come in Sect. 5 where we present a new construction, which enables us to define cyclic Dunkl connections for a quite large class of quantum principal bundles, including those with a quantum “total space” manifold having a finite structure group. We compute the curvature of these connections, and then, under certain general conditions, we prove that the curvature is zero in many cases of interest or, more generally, is equal to the curvature of the
initial background geometry. We also prove that these cyclic Dunkl connections possess an important multiplicative property.

As our main illustration, we explain in Sect. 6 how complex reflection groups and their associated Dunkl operators as introduced in [6] are included in this picture. In Sect. 7, we sketch a construction of a simple yet instructive class of examples of quantum principal bundles and their cyclic Dunkl connections, based on Cuntz algebras. These examples work with arbitrary finite groups, and are such that both the base manifold and the bundle are truly quantum objects.

The paper ends with three Appendices. In “Appendix A”, having an independent interest, we analyze properties of an underlying geometrical structure, a kind of primitive cyclic geometry. This structure is associated with the space of left-invariant elements of a bicovariant, $*$-covariant first-order differential calculus over a finite group, and it is closely related to the properties of quantum Dunkl connections. “Appendix B” presents some definitions, while “Appendix C” is a technical proof.

2 Background material

In this section, we review in some detail the results on which the rest of the paper is based. The references for this material are [8, 9, 12, 19]. All vector spaces are over the complex numbers. All maps are linear over the field of complex numbers, unless otherwise indicated. For example, if $V$ is a non-zero vector space with an involutive, additive conjugation $C : \psi \mapsto C(\psi) \equiv \psi^*$ satisfying $(\lambda \psi)^* = \lambda^* \psi^*$ for all $\psi \in V$ and $\lambda \in \mathbb{C}$, then $C$ is not linear, but rather is anti-linear. Also, a linear map $T : V \to W$ that satisfies $T(\psi^*) = T(\psi)^*$, where $V$ and $W$ are vector spaces with conjugation, is called a $*$-morphism or a hermitian map. The tensor product symbol $\otimes$ without a subscript means the context appropriate tensor product over the complex numbers $\mathbb{C}$. If $S$ is a (finite) set, we let $\text{Card}(S)$ denote its (finite) cardinal number of elements. We assume familiarity with Sweedler’s notation, which we sometimes use without explicit comment.

2.1 Quantum differential calculus on finite groups

We let $G$ denote a finite group. We put:

$$\mathcal{A} := \mathcal{F}(G) := \{ f : G \to \mathbb{C} \},$$

the set of all complex-valued functions with domain $G$. Then, $\mathcal{A}$ is a $*$-Hopf algebra with identity element $1_{\mathcal{A}} = 1$, the constant function. The multiplication in $\mathcal{A}$ is defined point-wise, that is as $(f_1 f_2)(g) := f_1(g) f_2(g)$ for $f_1, f_2 \in \mathcal{A}$ and $g \in G$, and so is commutative. A complex number $\lambda \in \mathbb{C}$ will be used at times to denote the element $\lambda 1_{\mathcal{A}} \in \mathcal{A}$.

The antipode $\kappa : \mathcal{A} \to \mathcal{A}$, which is defined as $(\kappa(f))(g) := f(g^{-1})$ for $f \in \mathcal{A}$ and $g \in G$, satisfies $\kappa^2 = \text{id}_\mathcal{A}$, the identity map on $\mathcal{A}$. So $\kappa$ is a bijection. We let $\phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ denote the co-product of $\mathcal{A}$, which is defined as the pull-back of the group multiplication function $G \times G \to G$. The co-product is co-commutative if and
only if the group $G$ is abelian. The $*$-operation on $\mathcal{A}$ is given by point-wise complex conjugation, $f^*(g) := \overline{f(g)}$. The co-unit $\varepsilon : \mathcal{A} \to \mathbb{C}$ is defined by $\varepsilon(f) := f(e)$, where $e \in G$ is the identity element in $G$.

The right adjoint co-action $\text{ad} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is defined for all $a \in \mathcal{A}$ by:

$$\text{ad}(a) := a^{(2)} \otimes \kappa(a^{(1)}) a^{(3)}$$

(2.1)

using Sweedler’s notation for the double co-product of $a \in \mathcal{A}$ and the co-associativity of the co-product $\kappa$, namely:

$$(\phi \otimes \text{id}) \phi(a) = (\text{id} \otimes \phi) \phi(a) = a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}.$$  

(2.2)

For each $g \in G$, let $\delta_g \in \mathcal{A}$ denote the Kronecker delta function with value 1 on $g$ and value 0 on all other elements of $G$. Then, the set $\{\delta_g \mid g \in G\}$ is a vector space basis of $\mathcal{A}$. We suppose that $S$ is a subset of $G$ satisfying these properties:

1. $S^{-1} = S$.
2. $g^{-1} S g = S$ for all $g \in G$.
3. $e \not\in S$, where $e$ denotes the identity element of $G$.
4. $S$ is non-empty.

At one extreme, we could take $S = G \setminus \{e\}$, where $G$ has at least two elements. At the other extreme, we could have $S = \{g_0\}$, where $g_0 \in G \setminus \{e\}$ is central and has order 2. An example of the latter is the multiplicative group of 8 quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$ with $g_0 = -1$.

Then, $S$ determines a unique first-order differential calculus (fodc), $d : \mathcal{A} \to \Gamma$, which is bicovariant and $*$-covariant. The vector space $\Gamma$ is an $\mathcal{A}$-bimodule, and the differential $d$ satisfies the Leibniz rule:

$$d(ab) = (da)b + a(db)$$

for $a, b \in \mathcal{A}$. Also, $\Gamma$ is generated as a vector space by all the elements of the form $a(db)$ for $a, b \in \mathcal{A}$. The bicovariance condition means that there is a canonical left co-action $\Phi_L : \Gamma \to \mathcal{A} \otimes \Gamma$ and a canonical right co-action $\Phi_R : \Gamma \to \Gamma \otimes \mathcal{A}$ of $\mathcal{A}$ that co-act on $\Gamma$ in a compatible way (meaning that these two co-actions commute); we say that $\Gamma$ is an $\mathcal{A}$-bicomodule. Also, the $*$-covariance means that the differential $d$ as well as that the canonical right and left co-actions are $*$-morphisms.

Without going into all the details, we note that $\Gamma$ is isomorphic as a left $\mathcal{A}$-module to the free left $\mathcal{A}$-module $\mathcal{A} \otimes (\ker \varepsilon/R)$ for the (two-sided) ideal $R \subset \ker \varepsilon \subset \mathcal{A}$ that is determined by $S$ by:

$$R := \{f \in \ker \varepsilon \mid f(s) = 0 \text{ for all } s \in S\}.  \tag{2.3}$$

(In the general theory of fodcs, $R$ is a right ideal, but since $\mathcal{A}$ is commutative, every right ideal is a two-sided ideal.) Here, $\varepsilon$ is the co-unit of $\mathcal{A}$. Under the isomorphism:

$$\Gamma \cong \mathcal{A} \otimes (\ker \varepsilon/R),$$

the right $\mathcal{A}$-module structure on $\Gamma$ is represented as:
\((a \otimes [b])c = (ac^{(1)}) \otimes [bc^{(2)}]\). \quad (2.4)

The canonical left co-action of \(A\) on \(\Gamma\) is identified with the map:

\[ \phi \otimes id : A \otimes (\ker \varepsilon / R) \to A \otimes A \otimes (\ker \varepsilon / R), \]

where \(\phi\) is the co-product of \(A\). The notation \(id\) here and always hereafter means the context appropriate identity map.

The subspace of left-invariant elements of \(\Gamma\) under its canonical left co-action \(\Phi_L\), defined as:

\[ \Gamma_{inv} := \{ \omega \in \Gamma \mid \Phi_L(\omega) = 1_A \otimes \omega \}, \]

is then identified with the subspace \(1_A \otimes (\ker \varepsilon / R)\) of \(\Gamma\). Therefore, we have:

\[ \Gamma_{inv} \cong \ker \varepsilon / R \quad \text{and} \quad \Gamma \cong A \otimes \Gamma_{inv}. \]

The projection \(\pi_{inv} : \Gamma \to \Gamma_{inv}\) is defined by \(\pi_{inv}(a \otimes \omega) := \varepsilon(a)\omega\) for every \(a \otimes \omega \in A \otimes (\ker \varepsilon / R)\). See [19] or [22] for more details about fodcs, but notice that in those references, \(\Gamma_{inv}\) is denoted as \(\pi_{inv}\Gamma\).

Moreover, the condition \(S \neq \emptyset\) is equivalent to the condition that the \(A\)-bimodule \(\Gamma\) is non-zero. It is important to point out that the finite group \(G\) is uniquely a \(C^\infty\) differential manifold in the discrete topology, in which case it has dimension zero and so its de Rham fodc (that is, the space of differential 1-forms on \(G\)) is the zero vector space. Therefore, the fode which we have constructed from \(S\) is not the de Rham fode of classical differential geometry, but rather is a quantum object. Therefore, the condition \(S \neq \emptyset\) is included here only to exclude the trivial case \(\Gamma = 0\). This condition is not included in Theorem 13.2 in [19] where the notation \(J\) corresponds to our notation \(S\).

There is a surjective linear map \(\pi : A \to \Gamma_{inv}\) defined by \(\pi := \pi_{inv} d\). This map, which is called the quantum germs map, provides us with a basis \(\{ \pi(\delta_s) \mid s \in S\}\) of the vector space \(\Gamma_{inv}\). In particular, \(\dim(\Gamma_{inv}) = \text{card}(S) \neq 0\), which is one way to understand the fact that \(\Gamma \cong A \otimes \Gamma_{inv} \neq 0\). The quantum germs map \(\pi\) is also closely connected to the differential \(d : A \to \Gamma\) by the identity for all \(a \in A\):

\[ da = a^{(1)} \pi(a^{(2)}), \]

where \(\phi(a) = a^{(1)} \otimes a^{(2)}\) in Sweedler’s notation. See Section 6.4 of [19] for this identity.

It is important to note that just the group \(G\) does not determine a unique bicovariant, \(\ast\)-covariant fode. In general, different choices of the subset \(S\) lead to different (i.e., non-isomorphic) fodcs. For example, as noted above, even the dimension of \(\Gamma_{inv}\) depends on \(S\).

The actual formula for \(d\) has its own interest. The action of \(d\) on the basis \(\{\delta_g \mid g \in G\}\) of \(A = \mathcal{F}(G)\) is given by:
\[ d(\delta_g) = \sum_{s \in S} (\delta_{gs^{-1}} - \delta_g) \pi(\delta_s). \] (2.7)

We remark that this is quite similar to a nearest-neighbor formula as frequently used in mathematical physics. However, here, the nearest-neighbor differences appear as the coefficients of distinct elements of a basis of \( \Gamma_{\text{inv}} \) rather than being added directly together. Therefore, the formula (2.7) looks more like a “gradient at \( g \)” associated with a family of “directional derivatives” \( \delta_{gs^{-1}} - \delta_g \) indexed by the elements \( s \in S \).

**Remark 2.1** In what follows, we shall use the same symbol for diverse mutually naturally identifiable objects. In particular, we shall commonly switch between \( g \) to denote an element of the group \( G \) and to denote its associated Kronecker delta function \( \delta_g \). We shall also write \( [g] \), or simply \( g \) when there is no risk of ambiguity, to denote the quantum germ \( \pi(g) \). In much the same spirit, we let \( \varepsilon \) denote the identity element in \( G \) as well the co-unit of \( A \).

### 2.2 Quantum principal bundles

In general, we say that a triple \( P = (B, A, F) \) is a *quantum principal bundle* if \( B \) is a \(*\)-algebra with identity element \( 1_B \), \( A \) is a \(*\)-Hopf algebra with identity element \( 1_A \), and \( F : B \to B \otimes A \) is right co-action of \( A \) on \( B \) that is also a unital, multiplicative, \(*\)-morphism of algebras. One also requires of \( F \) a technical property that is dual to that of an action being free. Next, the \(*\)-algebra \( \mathcal{V} \) of the “base space” is defined to be the right invariant elements in \( B \) under the right co-action by \( F \), namely:

\[
\mathcal{V} := \{ b \in B \mid F(b) = b \otimes 1_A \}. \tag{2.8}
\]

Suppose the finite group \( G \) acts freely on the right on \( E \), a \( C^\infty \) manifold. Then, the quotient map \( E \to E/G \) is a principal bundle with structure group \( G \), a zero-dimensional Lie group. So far, this is purely a construction in classical differential geometry.

We now modify this construction to get a quantum principal bundle in non-commutative geometry with the \(*\)-Hopf algebra \( A = \mathcal{F}(G) \) being its structure “quantum group”. In this construction, we follow the method used in [12] closely, though there are some details here which will be different. We construct a *quantum principal bundle* (QPB) with finite structure group \( G \) to be the triple \( P = (B, A, F) \), where \( A = \mathcal{F}(G) \) as already defined above and \( B := C^\infty(E) \), the \(*\)-algebra of complex-valued \( C^\infty \) functions \( f : E \to \mathbb{C} \), with \( E \) as above. The \(*\)-operation on \( B \) is given by point-wise complex conjugation of such a function \( f \). Finally, the linear map (which is a unital, multiplicative, \(*\)-morphism of algebras) \( F : B \to B \otimes A \) is defined as the pull-back of the right action map \( \varkappa : E \times G \to E \). Therefore, \( F \) is a right co-action of \( A \) on \( B \). Since \( \varkappa \) is a free action, the co-action \( F \) satisfies the technical condition dual to being a free action. This completely describes the QPB \( P = (B, A, F) \) with finite structure group \( G \).

Of course, a QPB with finite structure group \( G \) is a special case of a QPB for which an extensive theory has been developed. Much of what follows holds in the
context of QPB’s in general, though eventually we focus on the special case of a QPB with a finite structure group.

However, in general, just a QPB in and of itself does not have enough structure for our purposes. We also need to associate differential calculi to the algebras. We do this first in the case of any algebra \( A \) with identity element \( 1_A \), not necessarily \( B \) or \( \mathcal{F}(G) \). We start with a given fodc \( d : A \to \Gamma \). This is then associated with a graded algebra \( \Lambda = \bigoplus_{k=0}^{\infty} \Lambda^k \), meaning that \( \Lambda^k \cdot \Lambda^l \subset \Lambda^{k+l} \). Each homogeneous subspace \( \Lambda^k \) is also required to be an \( A \)-bimodule. We require that \( \Lambda \) have a differential \( d : \Lambda^k \to \Lambda^{k+1} \), meaning that \( d^2 = 0 \), which also satisfies for all \( \varphi_1 \in \Lambda^n \) the graded Leibniz rule:

\[
d(\varphi_1 \varphi_2) = d(\varphi_1) \varphi_2 + (-1)^n \varphi_1 d(\varphi_2).
\]

We require that \( \Lambda \) is an extension of the fodc \( \Gamma \) in the sense that \( \Lambda^0 = A \), \( \Lambda^1 = \Gamma \), and the map \( d : \Lambda^0 \to \Lambda^1 \) coincides with the differential \( d : A \to \Gamma \) of the given fodc, and hence, this justifies using the same notation \( d \). We require as well that the identity element \( 1_A \in A \) is also the identity element of \( \Lambda \). Any such object \( \Lambda \) is called a higher order differential calculus (hodc) that extends the given fodc. Moreover, in the case of an fodc over \( G \), we require that the hodc \( \Lambda \) must be generated as a differential algebra by \( \Lambda^0 \) or, in other words, every element in \( \Lambda^k \) can be written as a finite sum of elements of the form \( a_0 a_1 \ldots a_k \), where \( a_0, a_1, \ldots, a_k \in \Lambda^0 = A \) or, equivalently, of the form \( b_1 \ldots b_k \), where \( b_1, \ldots, b_k, b_0 \in \Lambda^0 = A \). If \( A \) is a Hopf algebra, we also require bicovariance of each \( \Lambda^k \). So far, the discussion in this paragraph does not involve \(*\)-operations and the corresponding \(*\)-covariance. However, these are also to be included as presented in Chapter 11 of [19]. For the special case of a QPB with finite structure group, the algebras \( A = \mathcal{F}(G) \) and \( B = C^\infty(E) \) are actually \(*\)-algebras, and so, it is natural to require a compatible \(*\)-operation on all the associated vector spaces \( \Lambda^k \) introduced above.

As an aside, we note that a graded algebra \( \Lambda = \bigoplus_{k=0}^{\infty} \Lambda^k \) is said to be graded commutative if \( \varphi_1 \varphi_2 = (-1)^{nm} \varphi_2 \varphi_1 \in \Lambda^{n+m} \) whenever \( \varphi_1 \in \Lambda^n \) and \( \varphi_2 \in \Lambda^m \). In particular, this implies that \( \Lambda^0 \) is commutative. We do not require this property of an arbitrary hodc, although the classical de Rham calculus is graded commutative.

A higher order differential calculus (hodc) for a QPB \( P = (B, A, F) \) is a triple \((\Omega(P), \Lambda, \widehat{F})\), where \( \Omega(P) \) is an hodc for \( B \) and \( \Lambda \) is a bicovariant, \(*\)-covariant hodc for \( A \). Finally, \( \widehat{F} : \Omega(P) \to \Omega(P) \otimes \Lambda \) is a grade preserving, right co-action of \( \Lambda \) on \( \Omega(P) \) which also extends the right co-action \( F : B \to B \otimes A \) and is a multiplicative, differential, unital \(*\)-morphism. See Section 12.4 of [19] for the definition of what it means for \( \widehat{F} \) to be a right co-action. The notation \( \widehat{F} \) is used in the papers of the first author such as [8], while the more cumbersome notation \( \Omega(P) \Psi \) is used for this right co-action in the book [19] of the second author.
2.3 A special case: the Hopf algebra of a finite group

We now study the case $\mathcal{A} = \mathcal{F}(G)$. For this case, when the fodc $d : \mathcal{A} \to \Gamma$ is bicovariant and $*$-covariant, there is a particular hodc $\Gamma^\wedge$ called the universal hodc associated with the fodc $\Gamma$, which itself has left and right co-actions of $\mathcal{A}$ co-acting on it. Every such hodc extending the fodc $d : \mathcal{A} \to \Gamma$ is a quotient of $\Gamma^\wedge$, thereby justifying the qualifier ‘universal.’ (See [7] or [19] for the construction of $\Gamma^\wedge$.) The theory can also be developed using another hodc, called the braided differential calculus of Woronowicz. (See [19] or [22].) The latter hodc, denoted as $\Gamma^\vee$, is based on the braided exterior algebra, which is associated with a canonical $\mathcal{A}$-bimodule braid-automorphism $\sigma : \Gamma \otimes \mathcal{A} \Gamma \to \Gamma \otimes \mathcal{A} \Gamma$. Its restriction $\sigma : \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}} \to \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}}$ is given for $\eta, \vartheta \in \Gamma_{\text{inv}}$ by:

$$
\sigma(\eta \otimes \vartheta) = \vartheta^{(0)} \otimes (\eta \circ \vartheta^{(1)}),
$$

(2.9)

where $\circ$ denotes a canonical right action of $\mathcal{A}$ on $\Gamma_{\text{inv}}$ defined for $b, c \in \mathcal{A}$ by:

$$
\pi(b) \circ c := \pi(bc - \varepsilon(b)c),
$$

(2.10)

where $\pi$ is the quantum germs map. In terms of the identification $\Gamma_{\text{inv}}$ with $\ker \varepsilon / \mathcal{R}$, this is the factor right $\mathcal{A}$-module structure. Also, $\text{ad} (\vartheta) = \vartheta^{(0)} \otimes \vartheta^{(1)}$ in Sweedler’s notation, where $\text{ad} : \Gamma_{\text{inv}} \to \Gamma_{\text{inv}} \otimes \mathcal{A}$ denotes the right adjoint co-action of $\mathcal{A}$ on $\Gamma_{\text{inv}}$. The relation of this ad with the right adjoint co-action $\text{ad}$ defined in (2.1) is given by the commutative diagram:

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\text{ad}} & \mathcal{A} \otimes \mathcal{A} \\
\pi \downarrow & & \downarrow \pi \\
\Gamma_{\text{inv}} & \xrightarrow{\text{ad}} & \Gamma_{\text{inv}} \otimes \mathcal{A}
\end{array}
$$

that is, the previously defined right adjoint co-action $\text{ad}$ passes to the quotient by the quantum germs map $\pi$. As a formula, this diagram reads as:

$$
\text{ad} \pi = (\pi \otimes \text{id}_{\mathcal{A}}) \text{ad}.
$$

(2.11)

Using (2.1), (2.2) and (2.11), we obtain the formula $\text{ad} \pi(a) = \pi(a^{(2)}) \otimes \kappa(a^{(1)}) a^{(3)}$ for all $a \in \mathcal{A}$. The long, technical proof of (2.9) is given in “Appendix C”.

2.4 A special type of hodc

Let us here present an important ‘intermediary’ context for an hodc. This will be highly relevant in our considerations of general Dunkl connections, because all of the considered fodc’s over finite groups are of this form. Therefore, we continue studying $\mathcal{A} = \mathcal{F}(G)$. Namely, let us assume that there exists an element $q \in \ker(\varepsilon) \setminus \mathcal{R}$ that satisfies:
\[(q - 1) \ker(e) \subseteq \mathcal{R},\]

where \(e: \mathcal{A} \to \mathbb{C}\) is the co-unit of the Hopf algebra \(\mathcal{A}\) and \(\mathcal{R}\) is the ideal in \(\ker(e)\) that determines the bicovariant, \(*\)-covariant fodec \(\Gamma\) on \(\mathcal{A}\) given by the subset \(S\), as defined in (2.3). Next, let us define:

\[
\tau := \pi(q) \in \Gamma_{\text{inv}}.
\]

Recalling that \(\ker \pi \cap \ker e = \mathcal{R}\), we see that \(\tau \neq 0\). Also, we remind the reader that \(\ker p = \mathcal{R} + \mathbb{C} 1_\mathcal{A}\).

First, we note for all \(a \in \mathcal{A}\) that:

\[
\pi(q(a - e(a))) = \pi((q - 1)(a - e(a))) + \pi(a - e(a)) = \pi(a),
\]

where we used \((q - 1)(a - e(a)) \in (q - 1) \ker e \subseteq \mathcal{R} \subseteq \ker \pi\) and \(\pi(1) = 0\). On the other hand, we have:

\[
\pi(q(a - e(a))) = \pi(qa - e(a)\pi(q) = \pi(qa - e(a)\tau.
\]

Combining the last two equalities, we arrive at:

\[
\pi(qa) = \pi(a) + e(a)\tau. \tag{2.12}
\]

Then, it is easy to see that:

\[
\tau \circ a = \pi(q) \circ a = \pi(qa - e(q)a) = \pi(qa) = \pi(a) + e(a)\tau, \tag{2.13}
\]

where we have used the definition (2.10) of \(\circ, q \in \ker e\) and (2.12).

We will also be using the identity:

\[
\theta a = a^{(1)}(\theta \circ a^{(2)}) \tag{2.14}
\]

for \(a \in \mathcal{A}\) and \(\theta \in \Gamma_{\text{inv}}\), where \(\phi(a) = a^{(1)} \otimes a^{(2)}\) is Sweedler’s notation for the co-product \(\phi(a)\) of \(a \in \mathcal{A}\). This is actually another way to describe the right \(\mathcal{A}\)-module structure (2.4) on \(\Gamma\). For the sake of completeness, we include a proof of this. Therefore, we have that:

\[
a^{(1)}(\theta \circ a^{(2)}) = a^{(1)}(\kappa(a^{(2)}) \theta a^{(3)}) = e(a^{(1)})\theta a^{(2)} = \theta e(a^{(1)})a^{(2)} = \theta a.
\]

Here, we used the co-associativity of the co-product expressed in Sweedler’s notation and basic Hopf algebra properties of the antipode \(\kappa\) and the co-unit \(e\) as well as this property of the action \(\circ:\n\)

\[
\theta \circ b = \kappa(b^{(1)}) \theta b^{(2)}
\]

for \(\theta \in \Gamma_{\text{inv}}\) and \(b \in \mathcal{A}\), where \(\phi(b) = b^{(1)} \otimes b^{(2)}\). See Section 6.4 and Appendix B of [19] for more details.

Next, for any \(a \in \mathcal{A}\), we compute:
\[ da = a^{(1)} \pi(a^{(2)}) = a^{(1)}(\tau \circ a^{(2)} - \varepsilon(a^{(2)}) \tau) = a^{(1)}(\tau \circ a^{(2)}) - a^{(1)} \varepsilon(a^{(2)}) \tau = \tau a - a \tau, \tag{2.15} \]

where we used the identity (2.6) for the differential \( d \), the Hopf algebra property \( a = a^{(1)} \varepsilon(a^{(2)}) \) of the co-unit \( \varepsilon \) as well as formulas (2.13) and (2.14).

This result, \( da = \tau a - a \tau \), is a sort of commutation relation, because it measures the difference between the left and right \( \mathcal{A} \)-module structures acting on the element \( \tau \in \Gamma_{\text{inv}} \subset \Gamma \) in the \( \mathcal{A} \)-bimodule \( \Gamma \). This justifies saying that \( d \) is an inner derivation with respect to the element \( \tau \). Then, any graded algebra \( \Omega(G) = \bigoplus_{k=0}^{\infty} \Omega^k(G) \) which extends the fodc \( d : \mathcal{A} = \mathcal{F}(G) \to \Gamma \) and in which \( \tau^2 = 0 \in \Omega^2(G) \) will automatically become an hodc with its extended differential being defined by the graded commutator with \( \tau \), namely:

\[ d \psi := \tau \psi - (-1)^k \psi \tau \in \Omega^{(k+1)}(G), \]

where \( \psi \in \Omega^k(G) \) is any homogeneous element of degree \( k \). The point is that one can readily verify that \( d^2 = 0 \) and that \( d \) satisfies the graded Leibniz rule. As we shall see, this is precisely what happens for bicovariant, \(*\)-covariant differential calculi over finite groups, where we can take \( q = 1_{\mathcal{A}} - \varepsilon \). Notice the similarity of this construction of an hodc with the extended bimodule method as used in [21, 22]. That method is also presented in Section 10.1 of [19].

The mere existence of the element \( q \in \ker(\varepsilon) \backslash \mathcal{R} \) implies that \( \mathcal{R} \cong \ker \varepsilon \) and so \( \Gamma_{\text{inv}} \cong \ker \varepsilon / \mathcal{R} \neq 0 \), which in turn implies that \( \Gamma \cong \mathcal{A} \otimes \Gamma_{\text{inv}} \neq 0 \). Therefore, the fodc is non-zero in this situation. Recall that \( \Gamma = 0 \) for the de Rham theory of the zero-dimensional manifold \( G \) of classical differential geometry. Therefore, the above discussion is only for the quantum case.

### 2.5 The hodc of the total space

Next, we will define an hodc for the “total space” \( B \) of the QPB \( P = (B, \mathcal{A}, F) \) with finite structure group \( G \) and with fodc \( d : \mathcal{A} \to \Gamma \) associated with \( G \) as above. More specifically, \( E \) is a \( C^\infty \) manifold on which \( G \) acts freely from the right and \( B = C^\infty(E) \). As a graded vector space, this is defined by this tensor product of graded vector spaces:

\[ \Omega(P) := \mathcal{D} \otimes \Gamma_{\text{inv}}^\wedge \quad \text{or explicitly} \quad \Omega^m(P) := \bigoplus_{k+l=m} \mathcal{D}^k \otimes \Gamma_{\text{inv}}^\wedge_l \quad (2.16) \]

for integers \( k, l, m \geq 0 \).

There is a lot to explain here. First, \( \mathcal{D} := \Omega_{\text{dR}}(E) \otimes \mathbb{C} \) is the complexified de Rham exterior calculus of the \( C^\infty \) manifold \( E \). Note that \( \Omega^0(P) = \mathcal{D}^0 = C^\infty(E) = B \) as it must be to have an extension. (As we shall see later, \( \Gamma_{\text{inv}}^\wedge_0 = \mathbb{C} \).) Second, \( \Gamma_{\text{inv}}^\wedge \) denotes the space of the left invariant elements in the universal hodc \( \Gamma^\wedge \) under its canonical left co-action by \( \mathcal{A} = \mathcal{F}(G) \).

Instead of using \( \Gamma^\wedge \), we could have used an acceptable hodc \( \Omega(G) \), which is defined to be an hodc extending \( d : \mathcal{A} \to \Gamma \) that is generated as a differential algebra.
by \( \Omega^0(G) = A = \mathcal{F}(G) \) and such that the co-multiplication \( \phi : A \to A \otimes A \) has a necessarily unique extension \( \widehat{\phi} : \Omega(G) \to \Omega(G) \otimes \Omega(G) \) that is a differential algebra morphism. Examples of acceptable hodcs are the universal enveloping hodc \( \Gamma^\wedge \) and the braided hodc \( \Gamma^\vee \). Also, using the partial order of hodcs’s (defined by \( \Omega'(G) \preceq \Omega(G) \) if \( \Omega'(G) \) is a quotient of \( \Omega(G) \)), all other acceptable hodcs are intermediates lying between the two extremes cases of \( \Gamma^\vee \) and \( \Gamma^\wedge \), which satisfy \( \Gamma^\vee \preceq \Gamma^\wedge \). This opens up a richness in the quantum theory that is not available with the unique, functorial de Rham hodc of classical differential geometry. Also, we claim that the very existence of such an extension \( \widehat{\phi} \) implies that the hodc being extended, namely \( d : A = \Omega^0(G) \to \Gamma = \Omega^1(G) \), is bicovariant, since the map

\[
\widehat{\phi} : \Gamma = \Omega^1(G) \to (\Omega(G) \otimes \Omega(G))^1 = (\Gamma \otimes A) \oplus (A \otimes \Gamma)
\]

has projections to the first (resp., second) summand which give a right (resp., left) co-action of \( A \) on \( \Gamma \). Moreover, these two co-actions are compatible, thereby making the hodc \( \Gamma \) bicovariant as claimed. However, hereafter, \( \Omega(P) \) means (2.16) with the universal hodc \( \Gamma^\wedge \) unless stated otherwise.

Moreover, we remark that \( \Omega(P) \), defined in (2.16), can be given four operations: a multiplication, a \( * \)-operation, a differential \( d_P \) of degree +1, and a right co-action:

\[
\widehat{F} : \Omega(P) \to \Omega(P) \otimes \Gamma^\wedge,
\]

which extends \( F : \mathcal{B} \to \mathcal{B} \otimes A \). All this gives us a graded \( * \)-algebra, such that \( d_P \) satisfies the graded Leibniz rule and is a \( * \)-morphism. Consequently, \( \Omega(P) \) is an hodc. The definition (2.16) of \( \Omega(P) \) seems to be a trivial product, but this is not so, because these four operations involve a non-trivial ‘twisting’ using the right co-action \( d_P \Phi \) defined below in (2.22). For the definitions of these four operations, see “Appendix B”. Finally, we remark that \( \Omega(P) \) is generated as a differential algebra by \( \Omega^0(P) = \mathcal{B} \). This is a kind of minimality condition for the calculus and ensures that the geometric and algebraic properties related to the differential calculus are uniquely defined by their restrictions to \( \mathcal{B} \).

Then, \( \Omega(P) \) is taken to be the hodc for the algebra \( \mathcal{B} = C^\infty(E) \) of the total space. Notice again that this is much more than simply the hodc \( \mathcal{D} \) of classical differential geometry.

### 2.6 The hodc of the base space

The hodc \( \Omega(M) \) of the \( * \)-algebra \( \mathcal{V} \) of the base space is defined to be the right invariant elements of \( \Omega(P) \), defined as the graded vector space:

\[
\Omega(M) := \{ \omega \in \Omega(P) \mid \widehat{F}(\omega) = \Omega(P) \Psi(\omega) = \omega \otimes 1_A \}.
\]

(2.17)

It turns out that \( \Omega(M) \subset \Omega(P) \) is closed under the multiplication, the \( * \)-operation, and the differential of \( \Omega(P) \) and so, with those operations, is an hodc in its own right. However, this hodc is not necessarily generated as a differential algebra by its elements in degree 0. However, do notice that the degree 0 elements of \( \Omega(M) \) give...
exactly the algebra \( \mathcal{V} \) of the “base space” as defined in (2.8). Also beware that the notation can be misleading. The hodc \( \Omega(M) \) is well defined for any QPB \( P \) with an hodc \( \Omega(P) \), even though there is no “base space” \( M \).

### 2.7 Horizontal forms

In classical differential geometry, a principal bundle has canonically associated vertical tangent vectors of the total space. However, the horizontal vectors of such a bundle are not uniquely determined, though any ‘reasonable’ choice of them is (or is equivalent to) an Ehresmann connection on the principal bundle. (See [20] for much more about this.) In the theory of non-commutative geometry, we typically have a space corresponding to 1-forms (namely, the fodc \( \Gamma \)) instead of a space of tangent vectors. Therefore, the quantum situation is dual to the classical situation, that is, a QPB has a canonically associated horizontal space \( \mathfrak{h} \), which does not require the existence of a (quantum) connection for its definition. Specifically, we use the definition in [8], namely:

\[
\mathfrak{h} = \{ \omega \in \Omega(P) \mid \tilde{F}(\omega) \in \Omega(P) \otimes \mathcal{A} \} = \tilde{F}^{-1}(\Omega(P) \otimes \mathcal{A}), \tag{2.18}
\]

which is a *-subalgebra of \( \Omega(P) \). We say that the elements of \( \mathfrak{h} \) are the horizontal forms of \( \Omega(P) \). Therefore, \( \Omega(M) \subset \mathfrak{h} \), possibly with a proper inclusion, and \( \mathfrak{h} \) inherits the structure of a graded vector space from \( \Omega(P) \), that is:

\[
\mathfrak{h} = \bigoplus_{k=0}^{\infty} \mathfrak{h}^k(P).
\]

The space \( \mathfrak{h} \) has many nice properties that justify calling it the horizontal space in \( \Omega(P) \). In this regard, see [8]. One such nice property is that \( \omega \in \mathfrak{h} \) implies that \( \tilde{F}(\omega) \in \mathfrak{h} \otimes \mathcal{A} \). Therefore, \( \tilde{F} \) restricted to \( \mathfrak{h} \) gives a right co-action of \( \mathcal{A} \) on the horizontal space \( \mathfrak{h} \). We want to emphasize that \( \mathfrak{h} \) is not always invariant under \( d_P \), the differential of \( \Omega(P) \). We will come back to this point when we discuss the covariant derivative.

For the case when we have a QPB with finite structure group \( G \) equipped with the hodc \( \Omega(P) \), as defined in (2.16), it turns out that:

\[
\mathfrak{h} = \mathcal{D} \otimes 1_{\mathcal{A}} \cong \mathcal{D};
\]

that is, the horizontal forms are identified with the complexified de Rham forms of the \( C^\infty \) manifold \( E \). (See Theorem 14.3 in [19].) This fundamental fact serves as a motivation and justification for the definitions in (2.16) and “Appendix B”.

### 2.8 Quantum connections

A (quantum) connection on a general QPB \( P \) with an hodc \( \Omega(P) \) is a linear map \( \omega : \Gamma_{inv} \rightarrow \Omega^1(P) \) satisfying these two conditions for all \( \theta \in \Gamma_{inv} \):
\[ \omega(\theta^*) = \omega(\theta)^* \quad \text{(optional)} \]  
\[ \widehat{F}(\omega(\theta)) = (\omega \otimes \text{id}_A) \text{ad}(\theta) + 1_B \otimes \theta. \]

As already mentioned, \( \text{ad} : \Gamma_{\text{inv}} \rightarrow \Gamma_{\text{inv}} \otimes A \) is the right adjoint co-action of \( A \) on \( \Gamma_{\text{inv}} \). We note that these conditions are \textit{not} linear in \( \omega \) due to the inhomogeneous term \( 1_B \otimes \theta \) in (2.20). The second condition (2.20) is an encoding of the properties of an Ehresmann connection in the dual context of non-commutative geometry. The property (2.19) is a reality condition that is usually satisfied in classical differential geometry, since in that context, the underlying scalar field is typically the reals \( \mathbb{R} \). Unlike (2.20), the property (2.19) is devoid of interesting geometric content. And this is why, (2.19) is an optional condition.

The object defined by the corresponding homogeneous conditions is known as a \textit{quantum connection displacement} or briefly a \( \text{QCD} \). Therefore, a QCD is a linear map \( \lambda : \Gamma_{\text{inv}} \rightarrow \Omega^1(P) \) satisfying for all \( \theta \in \Gamma_{\text{inv}} \) these two conditions:

\[ \lambda(\theta^*) = \lambda(\theta)^* \quad \text{(optional)} \]
\[ \widehat{F}(\lambda(\theta)) = (\lambda \otimes \text{id}_A) \text{ad}(\theta). \]

The second condition is simply the covariance of \( \lambda \) with respect to the two right co-actions, namely \( \widehat{F} \) and \( \text{ad} \), respectively, co-acting on the appropriate two spaces, namely \( \Omega^1(P) \) and \( \Gamma_{\text{inv}} \), respectively. This second condition plus the definition of the horizontal elements in \( \Omega(P) \) immediately implies using (2.11) that \( \lambda(\theta) \) is horizontal for all \( \theta \in \Gamma_{\text{inv}} \) or, in other words, \( \lambda(\theta) \in \text{hor}^1(P) \subseteq \Omega^1(P) \). The first condition is again an optional reality condition.

Given a QPB, the set of its quantum connections, if it is non-empty, is an affine space associated with the vector space (which always exists, though it may be zero) of quantum connection displacements.

The existence of a quantum connection on a QPB is a delicate question in general, but in this case, we can immediately define a quantum connection on the QPB \( P \) with finite structure group and with \( \text{hodc} \Omega(P) \) defined as above. (See [8] for a proof of the existence of a quantum connection on a QPB with compact matrix quantum group, which includes the present case.) We simply define \( \tilde{\omega} : \Gamma_{\text{inv}} \rightarrow \Omega^1(P) \) for \( \theta \in \Gamma_{\text{inv}} \) to be:

\[ \tilde{\omega}(\theta) := 1_B \otimes \theta. \]

Note that \( 1_B \otimes \theta \in \mathcal{B} \otimes \Gamma_{\text{inv}} = \mathcal{D}^0 \otimes \Gamma_{\text{inv}} \subseteq \Omega^1(P) \). First, \( \tilde{\omega} \) satisfies (2.19) by the definition of the \(*\)-operation as given in “Appendix B”. And second, (2.20) is shown in Theorem 14.4 in [19].

### 2.9 A right co-action

To establish notation and a definition for the next theorem, we note that there is a right co-action of \( A = \mathcal{F}(G) \) on \( \mathcal{D} \), denoted by:
and defined for \( \phi \in \mathcal{D} \) as the finite sum:

\[
\mathcal{D}(\phi) := \sum_{g \in G} (g \cdot \phi) \otimes \delta_g \in \mathcal{D} \otimes \mathcal{A},
\]

(2.22)

where \( g \cdot \phi \) denotes the left action of \( g \) on a classical, complexified differential \( k \)-form \( \phi \in \mathcal{D}^k \). This left action comes via pull-back from the right action of the group \( G \) on the \( C^\infty \) manifold \( E \). We also write \( \phi_g := g \cdot \phi \).

In the following, we also use Sweedler’s notation:

\[
\mathcal{D}(\phi) = \phi^{(0)} \otimes \phi^{(1)} \in \mathcal{D}^k \otimes \mathcal{A} \quad \text{for} \quad \phi \in \mathcal{D}^k = \mathfrak{hor}^k(P).
\]

(2.23)

Note that these two expressions (2.22) and (2.23) for \( \mathcal{D}(\phi) \) allow us to make these identifications: \( \phi^{(0)} = g \cdot \phi = \phi_g \) and \( \phi^{(1)} = \delta_g \), provided that we include an explicit sum over \( g \in G \).

### 2.10 The covariant derivative

**Definition 2.2** Let \( \omega \) be a quantum connection on the QPB \( P \) with finite structure group \( G \) and equipped with the hodc \( \Omega(P) \) as introduced above in (2.16). Then, we define the covariant derivative \( D_\omega \) of the quantum connection \( \omega \) by:

\[
D_\omega(\phi) := d_P(\phi) - (-1)^k \phi^{(0)} \omega(\pi(\phi^{(1)})) \in \Omega^{k+1}(P)
\]

(2.24)

for each \( \phi \in \mathfrak{hor}^k(P) \), where we use Sweedler’s notation in (2.23).

One important result here is that \( D_\omega : \mathfrak{hor}^k(P) \rightarrow \mathfrak{hor}^{k+1}(P) \). As we remarked earlier, in general, \( d_P(\phi) \) is not a horizontal vector for \( \phi \in \mathfrak{hor}(P) \) or, in other words, the \( \ast \)-subalgebra \( \mathfrak{hor}(P) \) of \( \Omega(P) \) is not necessarily invariant under \( d_P \), the differential of \( \Omega(P) \). So what is happening in the above definition of the covariant derivative is that a term is being subtracted off of \( d_P(\phi) \) to give us the horizontal vector \( D_\omega(\phi) \).

Continuing with this notation, we recall the following key result which is proved in Theorem 14.4 in [19].

**Theorem 2.1** The map \( \tilde{\omega} : \Gamma_{\text{inv}} \rightarrow \Omega^1(P) \) defined in Eq. (2.21) is a quantum connection on \( P \), and its associated covariant derivative \( D_{\tilde{\omega}} \) is given by:

\[
D_{\tilde{\omega}}(\phi) = d_P(\phi) - (-1)^k \phi^{(0)} \otimes \pi(\phi^{(1)})
\]

for \( \phi \in \mathfrak{hor}^k(P) \cong \mathcal{D}^k \).

Also, \( D_{\tilde{\omega}} = D_{\text{dR}} \), the complexified de Rham differential in \( \mathcal{D} = \Omega_{\text{dR}}(E) \otimes \mathbb{C} \), the complexified de Rham exterior differential calculus of the smooth manifold \( E \).

Let \( \omega = \tilde{\omega} + \lambda \), where \( \lambda \) is an arbitrary quantum connection displacement (QCD) on \( P \). Therefore, \( \omega \) is an arbitrary quantum connection on \( P \). Then, we can compute the covariant derivative of \( \omega \) as follows:
Theorem 2.2  For any \( \varphi \in \mathcal{D}^k \), we have:

\[
D_{\varphi} := D_{\varphi + \lambda} = D_{\varphi} + (-1)^k \sum_{s \in S} (\varphi - \varphi_s) \lambda(\pi(s)) \in \mathcal{D}^{k+1}.
\]  \( \text{(2.25)} \)

This formula is Eq. (14.6) in [19] and is an immediate precursor to the formula for the Dunkl operators, which is the case when \( k = 0 \) and the QCD \( \lambda \) is chosen adequately.

2.11 The quantum connection displacement

Now, the quantum connection displacement (QCD) \( \lambda \) is a linear map:

\[
\lambda : \Gamma_{\text{inv}} \rightarrow \mathfrak{hor}^1(P) \cong \mathcal{D}^1
\]

satisfying two conditions. However, we drop the first condition (namely, that \( \lambda \) is a \( * \)-morphism) as being inessential to the desired geometric structure. Being linear, \( \lambda \) is completely determined by its values on the basis elements \( \pi(s) \) of \( \Gamma_{\text{inv}} \), where \( s \in S \). Therefore, for each \( s \in S \), we define:

\[
\eta_s := \lambda(\pi(s)) \in \mathcal{D}^1,
\]

the complexification of the space of de Rham 1-forms on \( E \). The second condition on \( \lambda \) for it to be a QCD translates directly into a condition on the 1-forms \( \eta_s \in \mathcal{D}^1 \). Therefore, for all \( s \in S \), we must have:

\[
\widehat{F}(\eta_s) = (\lambda \otimes \text{id}_A) \text{ad}(\pi(s))
\]

\[
= (\lambda \otimes \text{id}_A)(\pi \otimes \text{id}_A) \text{ad}(\delta_s)
\]

\[
= (\lambda \pi \otimes \text{id}_A) \sum_{k \in G} \delta_{k^{k^{-1}}} \otimes \delta_k \quad \text{(2.26)}
\]

\[
= \sum_{k \in G} \eta_{k^{k^{-1}}} \otimes \delta_k \in \mathcal{D}^1 \otimes A \subset \Omega^1(P) \otimes A.
\]

Here, we used the identity (2.11) as well as the explicit formula for \( \text{ad}(\delta_g) \) for all \( g \in G \):

\[
\text{ad}(\delta_g) = \sum_{k \in G} \delta_{g^{g^{-1}}} \otimes \delta_k.
\]  \( \text{(2.27)} \)

For a proof of (2.27), see Eq. (13.3) in [19]. Note that (2.26) is a condition on each conjugacy class of \( S \). For example, if \( G \) is abelian or even if \( s \) is in the center of \( G \), then this condition becomes:

\[
\widehat{F}(\eta_s) = \sum_{k \in G} \eta_s \otimes \delta_k = \eta_s \otimes \sum_{k \in G} \delta_k = \eta_s \otimes 1_A;
\]

that is, \( \eta_s \in \mathcal{D}^1 \) is right invariant with respect to the right co-action \( \widehat{F} \). Therefore, the covariance condition (2.26) can be thought of as a generalization of right invariance.
2.12 Dunkl (quantum) connections

So far, we have the general condition on $\lambda$, so that $\omega = \tilde{\omega} + \lambda$ is a quantum connection, namely that $\lambda$ is a QCD. We now would like to add extra conditions on $\lambda$ to restrict to a class of quantum connections that is more closely related to the class of Dunkl connections as described in [12]. To do this, we first suppose from now on the following assumption and notations:

- Since the $C^\infty$ manifold $E$ is a covering space of its quotient $E/G$, it has an open coordinate neighborhood $U \subset E$ which projects diffeomorphically down to the quotient space $E/G$ to a coordinate neighborhood $U/G$. We fix a set of coordinate functions $x_i$ for $i = 1, \ldots, n = \dim E \geq 1$ in $U/G$ and use the same notation for the corresponding local coordinate functions defined on $U$. These coordinates define local vector fields $\partial/\partial x_i$ on the (possibly non-trivial) tangent bundle of $E/G$ and local forms $dx_i$ on the cotangent bundle of $E/G$. In particular, the local forms $dx_i$ are exact and thus closed. Also, the local vector fields $\partial/\partial x_i$ commute among themselves. From now on, we work in such a neighborhood $U/G$ with its distinguished coordinates functions $x_i$. In particular, we will use the complexification of these local trivializations, so that $dx_i$ is a holomorphic form and $\bar{\partial}/\partial x_i$ is the associated anti-holomorphic form. Similarly, we have the complex vector fields $\partial/\partial \bar{x}_i$ and $\bar{\partial}/\partial x_i$. We emphasize that we are not requiring globally defined coordinates on $E/G$.

An alternative setting is to assume that the $C^\infty$ manifold $E$ has a trivial tangent bundle and use globally defined vector fields and forms on it that are associated with a trivialization. In that case, local coordinates are not needed. As an example, we could take $E$ to be a Lie group with $G$ being any finite subgroup of $E$ that acts by right multiplication on $E$. We plan to present this situation in a future paper, since the associated Dunkl operators have quite different properties.

Next is a definition that imposes a covariance condition on a QCD $\lambda$.

**Definition 2.3** Suppose the following:

- For each $s \in S$, we have a non-zero element $x_s \in D^1 \cong \text{hor}^1(P)$, such that $x_s$ depends only on the conjugacy class of $s \in S$; that is, $x_{gs^{-1}} = x_s$ for all $g \in G$ and $s \in S$.
- For each $s \in S$, there exists a $C^\infty$ function $h_s : E \to \mathbb{C}$, such that:

$$h_{gs^{-1}}(x) = h_s(xg)$$

holds for all $x \in E$ and $g \in G$.

(Note that since $S$ is closed under conjugation, $gs^{-1} \in S$ and so $x_{gs^{-1}}$ and $h_{gs^{-1}}$ make sense. Also $xg$ denotes the right action of $g$ on $x$.) We then define $\lambda : \Gamma_{\text{inv}} \to \text{hor}^1(P) \cong D^1$ for $s \in S$ and $x \in E$ by:
\[ \lambda(\pi(\delta_s))(x) = h_s(x) \propto_s \in \mathfrak{hor}^1(P) \cong D^1 \]  

(2.28)
on the basis \( \{ \pi(\delta_s) \mid s \in S \} \) of \( \Gamma_{\text{inv}} \) and then extend linearly to \( \Gamma_{\text{inv}} \). We say that \( \lambda \) is a Dunkl (quantum) connection displacement or a Dunkl QCD.

In this case, we also say that \( \omega = \bar{\omega} + \lambda \) is a Dunkl (quantum) connection, where \( \bar{\omega} \) is defined in (2.21).

We remark that choosing \( h_s \equiv 0 \) for every \( s \in S \) shows that \( \lambda \equiv 0 \) is a Dunkl QCD, and hence, \( \bar{\omega} \) is a Dunkl connection.

This generalizes the definition given in [19] of a Dunkl connection in the context of finite Coxeter groups acting on Euclidean space, since the Definition 14.3 of a Dunkl QCD refers only to the Coxeter group case with a specific choice of the set \( S \). Consequently, Definition 14.4 of a quantum Dunkl connection in [19] also only refers to the Coxeter case. However, in this paper, we are dealing with an arbitrary finite group. The present theory also includes Coxeter groups, but with other choices for the set \( S \). For example, we can consider \( G = \Sigma_n \), the symmetric group on \( n \geq 3 \) letters acting on \( \mathbb{R}^n \) by permuting its coordinates (which is a Coxeter group), with \( S \) being the set of all \( k \)-cycles for \( 3 \leq k \leq n \), which does define a bicovariant, *-covariant fdc on \( \Sigma_n \). We remind the reader that the case when \( S \) is the set of all 2-cycles in \( \Sigma_n \) is a special case of the theory in [12].

Of course, this discussion would be besides the point without the following result whose proof is essentially the same as that of Proposition 14.2 of [19].

\textbf{Theorem 2.3} Every Dunkl QCD is a QCD. Consequently, every Dunkl quantum connection is a quantum connection.

\section*{2.13 Dunkl gradients}

Using the local coordinates introduced above, we write the covariant derivative \( D_{\omega} \) as a Dunkl gradient in \( U \), namely as:

\[ D_{\omega}(\varphi) = \sum_{j=1}^{n} D_{\omega j}^{i}(\varphi) \, dx_j \]  

(2.29)
for \( \varphi \in D^0 = C^\infty(E) \), where the components \( D_{\omega j}^{i} \) are called the generalized Dunkl operators associated with the local forms \( dx_j \) defined above. It follows from (2.25) that for \( \varphi \in D^0 = C^\infty(E) \) and for \( 1 \leq j \leq n \) the equation in \( U \):

\[ D_{\omega j}^{i}(\varphi) = \frac{\partial \varphi}{\partial x_j} + \sum_{s \in S} (\varphi - \varphi_s) \lambda_j(\pi(\delta_s)), \]  

(2.30)
(where \( \lambda_j \) is the \( j \)th component of \( \lambda \) with respect to the same local forms) gives us an explicit formula for the generalized Dunkl operator. Here, \( \partial \varphi / \partial x_j \) means the above-defined local vector field \( \partial / \partial x_j \) acting on the \( C^\infty \) function \( \varphi \in D^0 \).
2.14 How we build on this background

Nonetheless, to get interesting results, we can impose various other conditions, such as those in Definition 2.3, on the QCD $\lambda$. An example of this is given in the next section. In Sect. 5, we will introduce a new condition on a QCD for the case of a QPB with finite structure group. This is the central, new concept of this paper. Nonetheless, we could take the main result of [12] as motivation for saying that a generalized Dunkl operator is the covariant derivative of any quantum connection in any QPB, including the case when all the spaces are quantum spaces and the structure group is a quantum group. We take the point of view here that a generalized Dunkl operator should be of this type plus some extra structure such as a curvature zero condition, to name one possibility.

3 Basic example: the Dunkl operators

The motivating example for this paper is the case discussed in [12], where one takes $G$ to be a (necessarily finite) Coxeter group acting as orthogonal transformations on a finite-dimensional Euclidean space $V$ over $\mathbb{R}$. In that case, one takes:

$$S = \{ s \in G \mid s \neq e \text{ and } s^2 = e \},$$

which turns out to be the same as the set of reflections $\sigma_x$ for $x \in R \subset V$, where $R$ is the non-empty set of non-zero roots in $V$ which are used to define $G$. The orthogonal map $\sigma_x : V \to V$ fixes point-wise the hyperplane perpendicular to $x \neq 0$ and maps $x$ to $-x$. (See [14] or [15] for more definitions and basic properties of a Coxeter group.) One should proceed with a bit of caution when comparing the results of this paper with those of [12], since the mapping $x \mapsto \sigma_x$ is 2 to 1 from the set of roots $R$ (as used in [12]) onto the set $S$ used here. In this case, one also lets:

$$E = V \setminus \bigcup_{x \in R} H_x,$$

where $H_x \subset V$ is the hyperplane of fixed points of the reflection $\sigma_x$. Finally, one defines the QCD $\lambda$ by:

$$\eta_s(x) = \lambda(\pi(\delta_s))(x) = \frac{v(x)}{\langle x, x \rangle^2} x^*, \quad (3.1)$$

where $s = \sigma_x = \sigma_{-x}$, $x \in R$, $x \in E$ and $\nu : R \to [0, \infty)$ is a $G$-invariant function, meaning that $\nu$ is constant on each orbit of $G$. Then, $\nu$ called a multiplicity function. In particular, $\nu(-x) = \nu(x)$, since $\sigma_x(x) = -x$ and so $x$ and $-x$ lie in the same $G$-orbit. Also $x^*$ is the 1-form on $E$ defined via the inner product with the vector $x$. With these identifications, we obtain the Dunkl operators as originally introduced in [5], and then, the main result of [12] is a special case of this paper.

This section gives an example of the theory developed in Sect. 2. It is also an example of the new theory in Sect. 5 as we shall see.
4 Cyclic geometry on $S$

Our results for Dunkl operators admit elegant and far reaching generalizations to geometrical contexts where finite groups act on classical or quantum spaces. We shall start by extracting the key algebraic property of root vectors; that property is what enabled us to prove in [12] that the curvature of any Dunkl connection associated with a Coxeter group is always zero. In this paper, this topic has a auxiliary, though important, role. However, we find it to be of independent interest. More details on this are in “Appendix A”.

Let us begin by considering an interesting geometrical structure on the space $S$, any subset of a finite group $G$ closed under conjugation by all the elements of $G$. (The motivating example, of course, is the set $S$ defining a $*$-covariant, bicovariant $\text{fodc}$ for $\mathcal{F}(G)$ as described earlier. We will come back to this example, but for now, we could even take $S$ to be a finite subset of any group, provided that $S$ is closed under conjugation just by elements in $S$.) The canonical flip-over operator $\sigma$ acts naturally on $S \times S$. Explicitly, by definition, $\sigma$ and its inverse $\sigma^{-1}$ act on $(g, h) \in S \times S$ by:

$$\sigma : (g, h) \mapsto (ghg^{-1}, g) \in S \times S$$

$$\sigma^{-1} : (g, h) \mapsto (h, h^{-1}gh) \in S \times S.$$  \hfill (4.1)

For example, if $g$ and $h$ commute, then $\sigma(g, h) = (h, g)$. Therefore, we can think of $\sigma$ as a twisted interchange. Moreover, $\sigma^2(g, h) = (g, h)$ if and only if $g$ and $h$ commute. Another immediate observation is that $\sigma(g, h) = (g, h)$ if and only if $g = h$. When it is convenient, the elements $(g, h) \in S \times S$ will be identified with the tensor products $g \otimes h \equiv \delta_g \otimes \delta_h \in \mathcal{A} \otimes \mathcal{A}$, where $\mathcal{A} = \mathcal{F}(G)$.

Accordingly, the finite space $S \times S$ splits into finite orbits of this action. It is obvious from the above formula (4.1) for $\sigma$ that for all pairs in an orbit, the product of their two entries is the same element in $G$, which is then an invariant of the orbit. Each orbit consists of $n \geq 1$ distinct pairs in $S \times S$, say:

$$(q_1, q_2), (q_2, q_3), \ldots, (q_{n-1}, q_n), (q_n, q_{n+1}),$$  \hfill (4.2)

where each pair starting with $(q_2, q_3)$ is the image under $\sigma^{-1}$ of the previous pair and $\sigma^{-1}(q_n, q_{n+1}) = (q_1, q_2)$. This last equality implies that $q_{n+1} = q_1$. By the above remarks, we see that there are always orbits with $n = 1$ and, if $\text{card}(G) \geq 2$, for $n = 2$, as well.

We claim that $q_1, \ldots, q_n$ in the orbit (4.2) are $n$ distinct elements of $S$. Suppose to the contrary that there exists $k, l$ with $1 \leq k < l \leq n$, such that $q_k = q_l$. Then, by the invariance of the product along an orbit, we have that $q_kq_{k+1} = q_lq_{l+1}$. This together with $q_k = q_l$ implies that $q_{k+1} = q_{l+1}$, and therefore, $(q_k, q_{k+1}) = (q_l, q_{l+1})$; that is to say, the pairs in (4.2) are not all distinct. However, this contradicts that (4.2) is an orbit with $n$ distinct pairs. Therefore, the elements $q_1, \ldots, q_n$ of $S$ must be distinct. The fact that an orbit $\mathcal{O}$ lies in $S \times S$ implies that $\text{card}(\mathcal{O}) \leq k^2$, where $k = \text{card}(S)$. This is a very weak upper bound, since the result of this paragraph tells us that $\text{card}(\mathcal{O}) \leq k$. 

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In other words, every orbit of $\sigma$ in $S \times S$ projects into a certain ordered sequence $(q_1, q_2, \ldots, q_n)$ of $n$ distinct elements of $G$, where $n$ is the cardinality of the orbit. Since the same orbit is also associated with the ordered sequences $(q_2, q_3, \ldots, q_n, q_1)$, $(q_3, \ldots, q_n, q_1, q_2)$ and so forth for all cyclic permutations, we then define a \textit{cyclic set} to be the set whose elements are all of those ordered sequences arising from one orbit and which are so related by cyclic permutations. If we now consider $S$ together with the family $T$ of all of these cyclic sets, we get a kind of discrete geometry on $S$ with the cyclic sets being interpretable as ‘lines’ and the elements of $S$ as ‘points’. Cyclic sets are preferably called \textit{cyclic lines} to emphasize this geometrical interpretation. Notice that the cyclic line associated with $(q_1, q_2, \ldots, q_n)$ uniquely determines the orbit (4.2), since the second entries in those pairs are $(q_2, q_3, \ldots, q_n, q_1)$. We write the cyclic line $\ell$ associated with the ordered sequence $(q_1, q_2, \ldots, q_n)$ as $\ell \leftrightarrow (q_1, q_2, \ldots, q_n)$. This means that each orbit $O$ of $\sigma$ is also associated with a unique cyclic line $\ell \leftrightarrow (q_1, q_2, \ldots, q_n)$. We write this relation as $O \leftrightarrow (q_1, q_2, \ldots, q_n)$.

\textbf{Definition 4.1} We define the \textit{group invariant} of an orbit $O \leftrightarrow (q_1, q_2, \ldots, q_n)$ of $\sigma$ for $n \geq 2$ by:

$$\text{Inv}(O) := q_k q_{k+1} \in G,$$

where $k$ is any integer satisfying $1 \leq k \leq n$. As usual, we let $q_{n+1} = q_1$.

For $n = 1$, we define $\text{Inv}(O) := q_1^2$.

By our remarks above, we have that $\text{Inv}(O)$ is well defined; that is, its value does not depend on the particular choice of $k$. Also, it does not change value under cyclic permutations of $(q_1, q_2, \ldots, q_n)$ and so is a function of the orbit $O$.

For their own independent interest, these \textit{cyclic geometries} are discussed in more detail in “Appendix A”, where we have in particular defined the concepts of a \textit{cyclic space} and its representations.

We also wish to note that the flip-over operator $\sigma$ induces braidings. (Compare with the Woronowicz braid operator in (5.10).) We recall that the \textit{braid group} $B_n$ is defined for each integer $n \geq 2$ as the group generated by $\{g_1, \ldots, g_{n-1}\}$ with the relations $g_j g_k = g_k g_j$ whenever $|j - k| \geq 2$ and $g_j g_{j+1} g_j = g_{j+1} g_j g_{j+1}$ for $j = 1, \ldots, n-2$. We define the bijection $\sigma_{12}$ of $S \times S \times S$ as $\sigma$ acting on the first two factors, that is $\sigma_{12}(a, b, c) := (\sigma(a, b), c)$. Here, $a, b, c \in S$. Similarly, we define $\sigma_{23}(a, b, c) := (a, \sigma(b, c))$. One readily checks that the braid equation holds, namely:

$$\sigma_{12} \sigma_{23} \sigma_{12} = \sigma_{23} \sigma_{12} \sigma_{23}. \quad (4.3)$$

Next, letting $S^n = S \times \cdots \times S$ with $n$ factors, we define the bijection $\tau_k$ of $S^n$ for $1 \leq k < n$ to be $\sigma$ acting on the factors $k$ and $k + 1$ and to be the identity on the remaining factors. This defines an action of the braid group $B_n$ on $S^n$ for every integer $n \geq 2$, where we use (4.3) when $n \geq 3$. Since $\Gamma \cong \mathcal{A}[S]$, the free $\mathcal{A}$-module with basis $S$, and $\Gamma_{\text{inv}} \cong \mathbb{C}[S]$, the vector space with basis $S$, we also have induced representations of $B_n$ on $\Gamma_{S^n} \cong \Gamma \otimes \mathcal{A} \cdots \otimes \mathcal{A} \cong \mathcal{A}[S^n]$ and on $\Gamma_{\text{inv},S^n} \cong \Gamma_{\text{inv}} \otimes \mathbb{C} \cdots \otimes \mathbb{C} \cong \mathbb{C}[S^n]$, coming from the action of $B_n$ on the space $S^n$. 
5 A general picture

Let us incorporate all this into the context of quantum principal bundles. We shall assume that a quantum principal bundle \( P = (B, A, F) \) with a classical finite structure group \( G \) is given; that is, \( A = \mathcal{F}(G) \). We shall also assume here that a bicovariant, \( * \)-covariant fddc \( C^* \neq 0 \) on \( G \) is fixed, and as explained in Sect. 2, it is associated with a non-empty set \( S \subseteq G \backslash \{ e \} \) which is invariant under all conjugations by elements of \( G \) and under the inverse operation on \( G \). This is the set \( S \) which we will consider from now on. We let \( \sigma \) and \( T \) be as above for this particular subset \( S \) of the finite group \( G \).

In addition, we shall assume that \( B \) has been extended to the higher order differential calculus (hodc) \( \Omega(P) \) defined in (2.16). Then, \( \Omega(P) \) has its connection independent, horizontal forms \( \mathfrak{hor}(P) \) as defined in (2.18).

5.1 A regular ‘initial’ connection

We also assume the existence of a regular ‘initial’ connection \( \tilde{\omega} \) on \( \Omega(P) \) that has a covariant derivative \( D = D_{\tilde{\omega}} : \mathfrak{hor}(P) \to \mathfrak{hor}(P) \) which acts as a hermitian, degree +1 map satisfying the graded Leibniz rule and that also intertwines the right co-action \( \hat{F} \) of \( A \) on \( \mathfrak{hor}(P) \) or, colloquially speaking, that “intertwines the action of \( G \)”. Also, a connection \( \omega \) is said to be regular if, for all \( \varphi \in \mathfrak{hor}^k(P) \) and all \( \varphi \in \Gamma_{\text{inv}} \), we have:

\[
\omega(\varphi) = (-1)^k \varphi^{(0)} \omega(\varphi^{(1)}) \in \mathfrak{hor}^{k+1}(P),
\]

where we have used Sweedler’s notation for \( \hat{F}(\varphi) = \varphi^{(0)} \otimes \varphi^{(1)} \in \mathfrak{hor}^k(P) \otimes A \).

Remark 5.1 We remark that a connection \( \omega \) is regular if and only if its covariant derivative \( D_{\omega} \) satisfies the graded Leibniz rule. This is proved in [9]. Also see Theorem 12.14 in [19] where it is shown that regularity of \( \omega \) implies the graded Leibniz rule for \( D_{\omega} \).

Theorem 5.1 The connection \( \tilde{\omega} \) defined in (2.21) is regular.

Proof By the previous remark, this is equivalent to showing that \( D_{\tilde{\omega}} \) satisfies the graded Leibniz rule. However, by Theorem 2.1 \( D_{\tilde{\omega}} = D_{dR} \), the complexified de Rham differential, which indeed does satisfy the graded Leibniz rule. This is an abstract way to prove this result.

A computational proof starts with \( \varphi \in \mathfrak{hor}^k(P) \) and uses Sweedler’s notation for the right co-action \( d \Phi \) as follows:

\[
d \Phi(\varphi) = \varphi^{(0)} \otimes \varphi^{(1)} \in \mathfrak{hor}^k(P) \otimes A.
\]

Therefore, for \( \varphi \in \Gamma_{\text{inv}} \) and \( \varphi \in \mathfrak{hor}^k(P) \), we obtain:
\[\tilde{\omega}(\vartheta) \varphi = (1_B \otimes \vartheta)(\varphi \otimes 1_A) = (-1)^k 1_B \varphi^{(0)} \otimes (\vartheta \circ \varphi^{(1)}) 1_A \]
\[= (-1)^k \varphi^{(0)} \otimes (\vartheta \circ \varphi^{(1)}) = (-1)^k (\varphi^{(0)} \otimes 1_A)(1_B \otimes \vartheta \circ \varphi^{(1)}) \]
\[= (-1)^k \varphi^{(0)} \tilde{\omega}(\vartheta \circ \varphi^{(1)}) \]

using the definitions of \(\tilde{\omega}\) in (2.21) and of the product in \(\Omega(P)\) as given in “Appendix B”. We also used the isomorphism \(\mathcal{D} \cong \mathcal{D} \otimes 1_A\). □

### 5.2 Cyclic Dunkl displacements

In Definition 2.3, we defined a Dunkl QCD. The next definition is a similar, but different, concept.

**Definition 5.2** A cyclic Dunkl displacement with respect to a given (that is, ‘initial’) regular connection \(\tilde{\omega}\) on a quantum principal bundle \(P\) with a finite structure group \(G\) is a set function \(\lambda : S \rightarrow \text{hor}^1(P)\), where \(S \subset G\) has a cyclic geometry (with respect to the flip-over operator \(\sigma\)) whose set of lines is \(\mathcal{T}\) and which satisfies the following three properties:

(i) Cyclic property:

\[\lambda(\ell) := \lambda(s_1)\lambda(s_2) + \cdots + \lambda(s_{n-1})\lambda(s_n) + \lambda(s_n)\lambda(s_1) = 0 \in \text{hor}^2(P) \quad (5.2)\]

for every cyclic line \(\ell \leftrightarrow (s_1, \ldots, s_n) \in \mathcal{T}\), where \(s_1, \ldots, s_n \in S\). Notice that this condition is invariant under cyclic permutations of the chosen ordered sequence; that is, it depends only on the cyclic line \(\ell\) defined by \((s_1, \ldots, s_n)\).

(ii) Covariance property:

\[F^\wedge_g(\lambda(s)) = \lambda(gsg^{-1}) \in \text{hor}^1(P)\]

for every \(g \in G\) and \(s \in S\). Here, \(F^\wedge = (id \otimes g)F^\wedge\) and we identify points \(g \in G\) with their associated characters \(\chi_g\) on \(A\), where \(\chi_g(f) := f(g)\) for all \(f \in A\). Therefore, \(\chi_g : A \rightarrow \mathbb{C}\) is a unitary, multiplicative, \(\ast\)-morphism.

Here, \(F^\wedge := (id \otimes p_0) \tilde{F} : \Omega(P) \rightarrow \Omega(P) \otimes \mathcal{A}\), where \(\tilde{F} : \Omega(P) \rightarrow \Omega(P) \otimes \Gamma^\wedge\) is the right co-action of \(\Gamma^\wedge\) on \(\Omega(P)\) as introduced earlier. Also, \(p_0 : \Gamma^\wedge \rightarrow \mathcal{A}\) is the projection that maps all homogeneous elements of positive degree to 0 and is the identity on all homogeneous elements of degree 0. It follows that \(F^\wedge\) is a right co-action of \(\mathcal{A}\) on \(\Omega(P)\). Furthermore, \(F^\wedge_g : \text{hor}^1(P) \rightarrow \text{hor}^1(P)\). For more details, such as why \(F^\wedge_g[\lambda(s)] \in \text{hor}^1(P)\) holds, see [8].

The reader should be very aware that \(\tilde{F}\) and \(F^\wedge\) are not the same, but by definition \(F^\wedge = (id \otimes p_0) \tilde{F}\). It turns out that the map \(\partial \Phi\) defined in (2.22) is equal to \(F^\wedge\) restricted to \(\mathcal{D} = \text{hor}(P) \subset \Omega(P)\).

(iii) Closed-ness property:

\[D(\lambda(s)) = D_{\tilde{\omega}}(\lambda(s)) = 0 \in \text{hor}^2(P)\]

for every \(s \in S\). This property depends on the choice of the ‘initial’ quantum connection \(\tilde{\omega}\). When \(\lambda\) is extended linearly to \(L(S) \cong \Gamma_{\text{inv}}\), as discussed immediately.
below, then we will have \( D_\phi(\lambda([s])) = 0 \) for all the basis elements \([s]\) of \( \Gamma_{\text{inv}} \), and therefore, \( D_\phi \lambda = 0 \), the zero map \( \Gamma_{\text{inv}} \to \mathfrak{hor}^2(P) \). Notice that, in this definition, this is the only property which refers to the ‘initial’ connection \( \tilde{\phi} \).

**Remark 5.3** We remark that the constant function \( \lambda \equiv 0 \) is a cyclic Dunkl displacement.

**Remark 5.4** As we shall see shortly, every cyclic Dunkl displacement is indeed a QCD. It is worth observing that our main ‘cyclicity’ condition (5.2) is non-linear in \( \lambda \). It picks out an interesting intersection of a quadratic conic with the vector space of all Dunkl QCD’s.

The cyclic property \((i)\) says that \( \lambda \) is a representation of the cyclic space \((S, T)\) in the *-algebra \( \mathfrak{hor}(P) \). (See “Appendix A” for the appropriate definitions.) Since \( S \) labels the basis \( \{ \delta_s \mid s \in S \} \) of \( \Gamma_{\text{inv}} \), any set function \( \lambda : S \to \mathfrak{hor}(P) \) admits, by linearity, a unique linear extension to \( \lambda : L(S) \cong \Gamma_{\text{inv}} \to \mathfrak{hor}^1(P) \) (still denoted as \( \lambda \)), where \( L(S) \) is the abstract vector space whose elements are formal linear combination of elements in \( S \) with complex coefficients. This extended \( \lambda \) itself extends further to a unique unital algebra morphism \( \lambda^\otimes : \Gamma_{\text{inv}}^\otimes \to \mathfrak{hor}(P) \), where \( \Gamma_{\text{inv}}^\otimes \) is the space of left invariant elements in the tensor algebra \( \Gamma^\otimes = \bigoplus_{k=0}^\infty \Gamma^\otimes_k \). (See Definition 7.2 and Exercise 7.4 in [19] for the definition and properties of the left co-action of \( \mathcal{A} \) on \( \Gamma^\otimes \).) Here, \( \Gamma^\otimes_k = \Gamma \otimes \cdots \otimes \Gamma \) with \( k \) factors for \( k \geq 1 \) and \( \Gamma^\otimes_0 = \mathcal{A} \). It turns out that \( \Gamma_{\text{inv}}^\otimes := (\Gamma \otimes \cdots \otimes \Gamma)_{\text{inv}} = \Gamma_{\text{inv}} \otimes \cdots \otimes \Gamma_{\text{inv}} \) for \( k \geq 1 \) and that \( \Gamma_{\text{inv}}^\otimes_0 = \mathbb{C} \). Explicitly, \( \lambda^\otimes_k(\theta_1 \otimes \cdots \otimes \theta_k) := \lambda(\theta_1) \otimes \cdots \otimes \lambda(\theta_k) \) for \( k \geq 1 \), where \( \theta_1, \ldots, \theta_k \in \Gamma_{\text{inv}} \). Also, we put \( \lambda^\otimes_0(z) := z \mathbf{1}_\mathcal{G} \) for \( z \in \mathbb{C} \), since this must be an identity preserving map.

The next result is an essential step in this theory.

**Theorem 5.2** Every cyclic Dunkl displacement is a QCD.

**Proof** We have to show that any cyclic Dunkl displacement

\[
\lambda : L(S) \cong \Gamma_{\text{inv}} \to \mathfrak{hor}^1(P)
\]

is covariant with respect to the right co-actions of \( \mathcal{A} \) on \( \Gamma_{\text{inv}} \) and \( \mathfrak{hor}^1(P) \). These right co-actions are \( \text{ad} \) and \( \hat{F} \), respectively. Of course, this is a consequence of only the Covariance Property in Definition 5.2.

Since \( \lambda(s) \) is horizontal and \( \{ \delta_k \mid k \in G \} \) is a basis of \( \mathcal{A} \), we have:

\[
\hat{F}(\lambda(s)) = F^\lambda(\lambda(s)) = \sum_{k \in G} \omega_k \otimes \delta_k \in \mathfrak{hor}^1(P) \otimes \mathcal{A}
\]

for unique elements \( \omega_k \in \mathfrak{hor}^1(P) \). Then, by the Covariance Property, we obtain:

\[
\lambda(gs^{-1}) = F^g_\lambda(\lambda(s)) = (\text{id} \otimes g)F^\lambda(\lambda(s)) = \sum_{k \in G} \omega_k \otimes \delta_k(g) = \omega_g.
\]

Therefore, we conclude that:
\[
\hat{F}(\lambda(s)) = \sum_{k \in G} \lambda(ksk^{-1}) \otimes \delta_k. \tag{5.3}
\]

On the other hand:

\[
(\lambda \otimes \text{id}_A) \text{ad}(s) = (\lambda \otimes \text{id}_A) \sum_{k \in G} [ksk^{-1}] \otimes \delta_k = \sum_{k \in G} \lambda(ksk^{-1}) \otimes \delta_k,
\]

using our standard notation conventions and the identities (2.11) and (2.27). This proves the desired covariance.

5.3 Cyclic Dunkl connections

This theorem allows us to define a new generalization of a Dunkl connection as was originally defined in [12]. This is a central aspect of this paper.

**Definition 5.5** Let \( \lambda \) be a cyclic Dunkl displacement with respect to the ‘initial’ regular connection \( \tilde{\omega} \) as given above on the QPB \( P \) with finite structure group. By Theorem 5.2 \( \omega := \tilde{\omega} + \lambda \) is a quantum connection (dropping the optional hermitian condition). Then, we say that \( \omega \) is a cyclic Dunkl connection (with respect to \( \tilde{\omega} \)).

In particular, since \( \lambda \equiv 0 \) is a cyclic Dunkl displacement with respect to \( \tilde{\omega} \), it follows by this definition that \( \tilde{\omega} \) is a cyclic Dunkl connection (with respect to itself).

Another crucial fact is established in the next result.

**Theorem 5.3** Let \( \lambda : \Gamma_{\text{inv}} \rightarrow \mathfrak{h} \text{or}^1(P) \) be defined by formula (2.28), where the non-zero co-vectors satisfy \( \alpha_{ggs^{-1}} = \alpha_s \) for all \( g \in G \) and \( s \in S \). Then, \( \lambda \) is a Dunkl QCD if and only if \( \lambda \) satisfies the Covariance Property (ii).

**Proof** On one hand, we calculate:

\[
\lambda(gsg^{-1})(x) = \lambda(\pi(\delta_{gsg^{-1}}))(x) = h_{gsg^{-1}}(x)\alpha_{gsg^{-1}}
\]

for all \( x \in E, g \in G, \) and \( s \in S \). On the other hand, we have:

\[
F^\wedge_g(\lambda)(s) = (\text{id} \otimes g)F^\wedge(\lambda(s)) = (\text{id} \otimes g) \sum_{h \in G} h \cdot (\lambda(s)) \otimes \delta_h
\]

\[
= \sum_{h \in G} h \cdot (\lambda(s)) \otimes \delta_h(g) = g \cdot (\lambda(s)) \otimes 1 \cong g \cdot (\lambda(s)).
\]

Evaluating at \( x \in E \), we see that:

\[
F^\wedge_g(\lambda)(s)(x) = g \cdot (\lambda(s))(x) = \lambda(s)(xg) = h_s(xg)\alpha_s.
\]

Now, we assume that \( \lambda \) is a Dunkl QCD. Therefore, we use Definition 2.3 to obtain that:

\[
\lambda(gsg^{-1})(x) = h_{gsg^{-1}}(x)\alpha_{gsg^{-1}} = h_s(xg)\alpha_s = F^\wedge_g(\lambda)(s)(x).
\]

And this shows that \( \lambda \) satisfies the Covariance Property (ii).
Conversely, suppose that the Covariance Property (ii) holds for $\lambda$. Then, from the calculations above, we get that:

$$h_{g_{s}g^{-1}}(x) z_{s} = h_{s}(xg) z_{s}$$

holds for all $x \in E$, $g \in G$ and $s \in S$. By the hypothesis on the $z_{s}$s, we have:

$$h_{g_{s}g^{-1}}(x) z_{s} = h_{s}(xg) z_{s}.$$ 

Since $z_{s} \neq 0$, this implies $h_{g_{s}g^{-1}}(x) = h_{s}(xg)$, proving $\lambda$ is a Dunkl QCD. \hfill $\Box$

### 5.4 The fodc of $G$

Next, we present a more detailed description of the fodc on $G$. The following rather nice identity:

$$[\varepsilon] = - \sum_{s \in S} [s]$$

holds for every bicovariant, $*$-covariant fodc on $\mathcal{A} = \mathcal{F}(G)$. (See Section 13.4 and especially Corollary 13.2 in [19].) Recall that $[s] = \pi(s) = \pi(\delta_{s})$ and that the identity element $\varepsilon \in G$ also denotes its associated character $f \mapsto f(\varepsilon)$ for $f \in A$, which is also the co-unit for the Hopf algebra $\mathcal{A}$.

It is also worth recalling that $[\varepsilon] \in \Gamma$ is invariant under the right adjoint co-action of $\mathcal{A}$. We now prove this. Using $\varepsilon = \delta_{\varepsilon} \in \mathcal{A}$, where $\varepsilon \in G$ is the identity element in $G$ and (2.27), we get:

$$ad(\varepsilon) = ad(\delta_{\varepsilon}) = \sum_{k \in G} \delta_{\delta_{ek}^{-1}} \otimes \delta_{k} = \sum_{k \in G} \delta_{\varepsilon} \otimes \delta_{k} = \delta_{\varepsilon} \otimes \sum_{k \in G} \delta_{k} = \varepsilon \otimes 1_{\mathcal{A}}.$$ 

Next, using this and the identity (2.11), it follows that:

$$ad([\varepsilon]) = ad(\pi(\varepsilon)) = (\pi \otimes \text{id})ad(\varepsilon) = (\pi \otimes \text{id})(\varepsilon \otimes 1_{\mathcal{A}}) = \pi(\varepsilon) \otimes 1_{\mathcal{A}} = [\varepsilon] \otimes 1_{\mathcal{A}},$$

thereby showing that $[\varepsilon]$ is right adjoint invariant as claimed.

We also remark that for every $a \in \mathcal{A}$, we have:

$$[\varepsilon] \circ a = \varepsilon(a)[\varepsilon] - [a],$$

as follows from the definition (2.10) when we simplify after putting $b = \delta_{\varepsilon}$ and $c = a$ there. In particular, the identity $\delta_{\varepsilon}a = \varepsilon(a)\delta_{\varepsilon}$ is used.

Thus, $[\varepsilon]$ is a kind of “vacuum state”; that is, it is a right adjoint invariant, cyclic vector for the right $\mathcal{A}$-module $\Gamma_{\text{inv}}$.

Next, we note that the derivation $d : \mathcal{A} \rightarrow \Gamma$ of the fodc satisfies:
Here, we used the identity (2.6) for \( da \), Eqs. (5.6) and (2.14) as well as the Hopf algebra identity \( a = a^{(1)} \varepsilon(a^{(2)}) \) for the co-unit \( \varepsilon \). This result, written as

\[
da = [-\varepsilon]a - a[-\varepsilon], \tag{5.7}
\]

is the same sort of commutation relation as we saw earlier in (2.15), since it measures the difference between the left and right \( A \)-module structures acting on the element \([-\varepsilon] \in \Gamma_{\text{inv}} \) in the \( A \)-bimodule \( \Gamma \). Again, we say that \( d \) is an inner derivation, but now with respect to \([-\varepsilon] \), which also can be considered as a “vacuum state”.

As noted above (taking 1 to mean \( 1_A \)), we can put \( q = 1 - \varepsilon = 1 - \delta_e \in A \) and define \( \tau := \pi(q) = [q] = [-\varepsilon] \neq 0 \). Hence, \((q - 1)\ker(\varepsilon) = -\delta_e \ker(\varepsilon) = \{0\} \subset \mathcal{R} \) and \( q \in \ker(\varepsilon) \). We also claim that \( q \notin \mathcal{R} \). This is so since for any \( s \in S \), we have:

\[
q(s) = (1 - \delta_e)(s) = 1 - \delta_e(s) = 1 - 0 = 1 \neq 0.
\]

Here, we used that \( s 
eq e \) for all \( s \in S \). However, \( \mathcal{R} \) by definition is the ideal in \( \ker(\varepsilon) \) of those functions that annihilate \( S \). Therefore, \( q \notin \mathcal{R} \) as claimed. This shows that this differential \( d \) satisfies the general properties that gave us (2.15). Even though the identity (5.7) seems to be at odds with (2.7), both are identities for \( d : A \to \Gamma \).

All of this concerns the fode \( d : A \to \Gamma \). We will next investigate how this analysis can be extended to an hode that extends this fode.

### 5.5 Extending to the hode

We start out by remarking that \( \sigma \), the Woronowicz braid operator, satisfies:

\[
\sigma(\vartheta \otimes [\varepsilon]) = [\varepsilon] \otimes \vartheta \tag{5.8}
\]

for every \( \vartheta \in \Gamma_{\text{inv}} \), the space of left invariant elements of \( \Gamma \) with respect to its canonical left co-action by \( A \). To see that this is so we note that the canonical right co-action of \( A \) on \([\varepsilon] \in \Gamma_{\text{inv}} \) is equal to the right adjoint co-action of \( A \) on \([\varepsilon] \). This is the content of diagram (6.22) in [19]. Since \([\varepsilon] \) is right adjoint invariant, as shown in (5.5), it follows that \([\varepsilon] \) is right canonical invariant. However, \( \sigma(\omega_1 \otimes \omega_2) = \omega_2 \otimes \omega_1 \) provided that \( \omega_1 \) is left invariant and \( \omega_2 \) is right invariant with respect to the canonical left and right co-actions, respectively. (See Section 7.2 in [19].) And thus, we have proved (5.8). Then, by taking the special case \( \vartheta = [\varepsilon] \in \Gamma_{\text{inv}} \), we obtain:
\[ \sigma(\{g\} \otimes \{h\}) = [g] \otimes [h]. \]  

The action of the Woronowicz braid operator \( \sigma : \Gamma_{\text{inv}}^\otimes \rightarrow \Gamma_{\text{inv}}^\otimes \) is calculated for \( g, h \in S \) using (2.9), (2.11) and (2.27) to be:

\[ \sigma([g] \otimes [h]) = \sum_{k \in G} [khk^{-1}] \otimes ([g] \circ k) = [ghg^{-1}] \otimes [g]. \]

Here, we also used \([g] \circ k = 0\) for \( k \neq g \) and \([g] \circ g = [g]\), which follow from the definition (2.10). Notice that the set \( \{[g] \otimes [h] \mid g, h \in S\} \) is a basis of \( \Gamma_{\text{inv}}^\otimes \). Therefore, the Woronowicz braid operator corresponds to the permutation of the finite set \( S \times S \) known as the flip-over operator in (4.1). This justifies using the same notation \( \sigma \) for both of them.

There is a very interesting characterization of the cyclic property in terms of a natural higher order differential calculus on \( G \), which we are going to describe now. Let us define the algebra \( \Gamma^\sim \) as the quotient algebra of the tensor algebra \( \Gamma^\otimes = \bigoplus_{k=0}^{\infty} \Gamma^\otimes k \) over the ideal \( I_{\sigma_{\text{inv}}} \) in \( \Gamma^\otimes \) generated by \( \ker(\text{id} - \sigma) \), the vector space of all the \( \sigma \)-invariant elements in the degree-2 subspace \( \Gamma^\otimes 2 \), where \( \text{id} \) means the identity map of \( \Gamma^\otimes 2 \). An ideal, such as \( I_{\sigma_{\text{inv}}} \), generated by elements of degree 2 is called a quadratic ideal. In this quotient algebra \( [\varepsilon]^2 = 0 \), because of (5.9). Thus, we can consistently extend the differential \( d : A \rightarrow \Gamma \) to \( d : \Gamma^\sim \rightarrow \Gamma^\sim \) by setting:

\[ d\psi := -([\varepsilon]\psi - (-1)^{\partial\psi}[\varepsilon] \psi). \]

(In general, \( \partial\psi \) denotes the degree of a homogeneous element \( \psi \).) This says that the extended differential is a graded inner derivation with respect to the element \( -[\varepsilon] \). Our construction is covariant, and so we have that \( \Gamma^\sim \leftrightarrow A \otimes \Gamma^\sim \), where we define:

\[ \Gamma^\sim_{\text{inv}} := \Gamma_{\text{inv}}^\otimes / I_{\sigma_{\text{inv}}}. \]

Therefore, any bicovariant, \(*\)-covariant fdoc \( d : A \rightarrow \Gamma \) has a functorially associated quadratic hodc \( \Gamma^\sim \).

**Remark 5.6** It is easy to see from the covariance and \(*\)-interrelation properties of the flip-over operator \( \sigma \) that this calculus has both a compatible bicovariance structure and compatible \(*\)-operations. Moreover, the calculus admits (a necessarily unique) extension of the co-product to \( \phi^\sim : \Gamma^\sim \rightarrow \Gamma^\sim \otimes \Gamma^\sim \), and therefore, it is an acceptable hodc for the structure group \( G \) when considering quantum principal \( G \)-bundles. Indeed, if we consider a generic quadratic relation represented by

\[ \sum \theta \otimes \eta \in \ker(\text{id} - \sigma), \]

then a straightforward calculation shows:
\[
0 = \sum_{\partial \eta} \partial \eta \rightarrow \sum_{\partial \eta} \partial(0) \eta(0) \otimes \partial(1) \eta(1) + \sum_{\partial \eta} \partial(0) \otimes \partial(1) \eta - \sum_{\partial \eta} \eta(0) \otimes \partial(1) \\
+ \sum_{\partial \eta} 1 \otimes \partial \eta = \sum_{\partial \eta} \partial(0) \eta(0) \otimes \partial(1) \eta(1) + \sum_{\partial \eta} 1 \otimes \partial \eta = 0.
\]

The second and third terms cancel out because of the \(\sigma\)-invariance of the initial quadratic expression, and the first and last terms are both zero. This proves the compatibility property of the ideal, and thus the existence of \(\phi^\sim\).

Therefore, the hode \(\Gamma^\sim\) is situated, in terms of the richness of its generating relations, between the universal differential envelope \(\Gamma^\wedge\) of \(\Gamma\) and the braided exterior calculus \(\Gamma^\vee\) over \(\Gamma\). As we know from the general theory these two are the maximal and minimal hode’s over \(\Gamma\).

### 5.6 The cyclic property related to the hode

We are ready to present the characterization of the cyclic property in terms of the hode \(\Gamma^\sim\).

**Theorem 5.4** The cyclic property for \(\lambda\) is equivalent to:

\[
(m_{\Omega(P)} \lambda^{\otimes 2})(\ker(\id - \sigma)) = 0.
\]

In other words, in this case, \(\lambda^{\otimes}\) is projectable—or \(\lambda\) is extendible—to a unital algebra morphism \(\lambda^\sim : \Gamma^\sim_{\inv} \rightarrow \hor(P)\). Here, \(m_{\Omega(P)}\) denotes the multiplication in \(\Omega(P)\).

**Remark 5.7** Recall that \(\lambda^{\otimes}\) was defined just before Theorem 5.2.

**Proof** This follows from the fact that the \(\sigma\)-invariant elements of \(\Gamma^\inv_{\inv} \otimes \Gamma^\inv_{\inv}\) are precisely those expressible as linear combinations of certain elements associated with orbits or, equivalently, to cyclic lines as we now describe. Indeed, given a cyclic line \(\ell\) associated with the ordered sequence \((q_1, \ldots, q_n)\) of \(n\) distinct elements of \(S \subset \Gamma^\inv\), we can consider the tensor defined by:

\[
\tau(\ell) := q_1 \otimes q_2 + q_2 \otimes q_3 + \cdots + q_{n-1} \otimes q_n + q_n \otimes q_1 \in \Gamma^\inv_{\inv} \otimes \Gamma^\inv_{\inv},
\]

which does not change under cyclic permutation of the elements in the ordered sequence \((q_1, \ldots, q_n)\); that is, it only depends on \(\ell\). In other words, \(\tau(\ell)\) is the sum over the pairs of the corresponding orbit (4.2), where each such pair is interpreted as a tensor product. Each \(\tau(\ell)\) is clearly \(\sigma\)-invariant, that is, \(\tau(\ell) \in \ker(\id - \sigma)\). Moreover the set of all \(\tau(\ell)\) provides us with a basis for \(\ker(\id - \sigma)\) as can be easily checked. As we remarked above, the cyclic property says that \(m_{\Omega(P)} \lambda^{\otimes 2}\) annihilates every \(\tau(\ell)\). Therefore, \(\lambda\) satisfies the cyclic property if and only if \(m_{\Omega(P)} \lambda^{\otimes 2}\) annihilates \(\ker(\id - \sigma)\). \(\square\)
5.7 A geometrical property of cyclic Dunkl displacements

The following proposition shows us an interesting geometrical property of cyclic Dunkl displacements, namely that they turn out to be base space one-forms when evaluated in the canonical ad-invariant generator \([-\varepsilon]\). This can thus be interpreted as an ingredient of the base space geometry. It is worth recalling here that in the theory of quantum characteristic classes, in a similar spirit, we consider the cohomology classes of the closed forms on the base, expressible in terms of the connection and its differential.

**Proposition 5.1** For every cyclic Dunkl displacement \(\lambda\), the element \(\hat{\lambda}_0\) defined by:

\[
\hat{\lambda}_0 := \sum_{s \in S} \hat{\lambda}(s) = \hat{\lambda}(-\varepsilon) \in \text{hor}^1(\mathcal{P})
\]  

(5.11)

is right invariant under the right co-action \(\hat{\varphi}\). Therefore, \(\hat{\lambda}_0\) belongs to the ho \(\Omega(M)\) of the “base space”.

**Proof** We compute:

\[
\hat{\varphi}(\hat{\lambda}_0) = \sum_{s \in S} \hat{\varphi}(\hat{\lambda}(s))
\]

\[
= \sum_{s \in S} \sum_{k \in G} \hat{\lambda}(ksk^{-1}) \otimes \delta_k
\]

\[
= \sum_{k \in G} \sum_{s \in S} \hat{\lambda}(ksk^{-1}) \otimes \delta_k
\]

\[
= \sum_{k \in G} \sum_{s \in S} \hat{\lambda}(s) \otimes \delta_k
\]

\[
= \sum_{s \in S} \hat{\lambda}(s) \otimes \sum_{k \in G} \delta_k
\]

\[
= \hat{\lambda}_0 \otimes 1_A
\]

using the identity (5.3) in the second equality and the property \(gSg^{-1} = S\) in the fourth equality. This proves that \(\hat{\lambda}_0\) is right invariant. The last statement in the proposition follows immediately from the definition of \(\Omega(M)\).

5.8 The (quantum) curvature

The (quantum) curvature \(r_\omega : \mathcal{A} \to \text{hor}^2(\mathcal{P})\) of a quantum connection \(\omega\) is defined as:

\[
r_\omega(a) = d_P(\omega(\pi(a)) + \omega(\pi(a^{(1)})) \omega(\pi(a^{(1)})))
\]

(5.12)

for all \(a \in \mathcal{A}\), where \(\phi(a) = a^{(1)} \otimes a^{(2)}\) in Sweedler’s notation.

**Theorem 5.5** The curvature \(r_\omega\) of the cyclic Dunkl connection \(\omega = \tilde{\omega} + \lambda\) is given by \(r_\omega = r_{\tilde{\omega}}\).
Remark 5.8  This theorem says that the cyclic Dunkl displacement $\lambda$ does not change the non-commutative (or quantum) geometry described by the quantum curvature. Thinking of $\lambda$ as a certain type of perturbation, we can say that the curvature is invariant under such a perturbation.

Proof  The hypothesis is that $\lambda$ is a cyclic Dunkl displacement with respect to $\tilde{\omega}$. According to Eq. (3.10) in [12] as applied in this context, for all $k \in G$, we have:

$$r_{\tilde{\omega}}(\delta_k) = r_{\tilde{\omega}} + \lambda(\delta_k)$$

$$= r_{\tilde{\omega}}(\delta_k) + D_{\tilde{\omega}}(\lambda([\delta_k])) + \lambda([\lambda^{(1)}] \lambda([\lambda^{(2)}]))$$

$$= r_{\tilde{\omega}}(\delta_k) + D_{\tilde{\omega}}(\lambda([\delta_k])) + \sum_{gh=k} \lambda([\delta_g]) \lambda([\delta_h]).$$

(5.13)

We now analyze the images $\pi(\delta_k) = [\delta_k]$ of the basis $\{\delta_k \mid k \in G\}$ of $A$ under the quantum germs map $\pi$. By Section 13.4 in [19], we have these three exclusive and exhaustive cases:

- $[\delta_s]$ for $s \in S$, which form a basis of $\Gamma_{\text{inv}}$, and so, $[\delta_s] \neq 0$.
- $[\delta_e] = - \sum_{s \in S}[\delta_s] \neq 0$.
- $[\delta_k] = 0$ for $k \not\in S \cup \{e\}$.

By the third property, we see that $D_{\tilde{\omega}}(\lambda([\delta_k])) = 0$ for all $k \not\in S \cup \{e\}$. And by the first two properties and the Closed-ness property of $\lambda$, we see that $D_{\tilde{\omega}}(\lambda([\delta_k])) = 0$ for all $k \in S \cup \{e\}$. Consequently, $D_{\tilde{\omega}}(\lambda([\delta_k])) = 0$ for all $k \in G$. Therefore, (5.13) becomes:

$$r_{\tilde{\omega}}(\delta_k) = r_{\tilde{\omega}}(\delta_k) + \sum_{gh=k} \lambda([\delta_g]) \lambda([\delta_h]).$$

(5.14)

It is worth observing that our curvature formula (5.14) can be rewritten for every $a \in A$ in a more compact form as $r_{\tilde{\omega}}(a) = r_{\tilde{\omega}}(a) + m_{\Omega(P)}(\lambda^{-1}(\phi(a)))$, where $\phi$ is the co-multiplication of $A$ and $m_{\Omega(P)}$ is the multiplication in $\Omega(P)$.

Notice that the sum in (5.14) is over all pairs $(g, h)$, such that $gh = k$, where $k$ is the group element appearing on the left side. However, by the three properties given just above a term in this summation can only be non-zero if both $g \in S \cup \{e\}$ as well as $h \in S \cup \{e\}$. In turn, this breaks down into four mutually exclusive and exhaustive cases for the terms in the summation, recalling that $e \not\in S$. We now examine these cases. Throughout $k \in G$ is a given element.

Case 1  $g \in S$ and $h \in S$ with $gh = k$, that is, we are summing over the set:

$$\mathcal{M}_k := \{(g, h) \in S \times S \mid gh = k\} \subset S \times S.$$ 

(5.15)

We write the set $\mathcal{M}_k$ as a (disjoint!) union of certain orbits in $S \times S$ of the flip-over operator $\sigma$, namely:

$$\mathcal{M}_k = \bigcup \{O \mid O \text{ is an orbit of } \sigma \text{ satisfying } \text{Inv}(O) = k\}.$$ 

(5.16)

This is so since clearly $O \subset \mathcal{M}_k$ whenever $\text{Inv}(O) = k$. Conversely, if $(g, h) \in \mathcal{M}_k$, 

\[\vdots\]
then the $\sigma$-orbit $O$ of $(g, h)$ lies in $M_k$ and $\text{Inv}(O) = k$.

However, for any orbit $O \leftrightarrow (q_1, \ldots, q_n)$ of $\sigma$, we have that:

$$\lambda(O) := \sum_{j=1}^{n} \lambda([q_j]) \lambda([q_{j+1}]) = 0$$

by the cyclic property of $\lambda$.

Consequently, by the two previous equalities, we see that:

$$\sum_{(g, h) \in M_k} \lambda([\delta_g]) \lambda([\delta_h]) = \sum_{O: \text{Inv}(O) = k} \lambda(O) = 0.$$

**Case 2** $g = e$ and $h \in S$. We then have $k = gh = h \in S$, and so this case does not occur for $k \not\in S$. However, for $k \in S$, we do get one term, namely:

$$\lambda([\delta_e]) \lambda([\delta_k]).$$

**Case 3** $g \in S$ and $h = e$. Then, $k = gh = g \in S$ and so again, as in the previous case, there are no such terms if $k \not\in S$. However, for $k \in S$, we get again just one term which now is:

$$\lambda([\delta_k]) \lambda([\delta_e]).$$

**Case 4** $g = e$ and $h = e$. Therefore, $k = gh = g \not\in S$. Therefore, the case is vacuous if $k \neq e$. Note that when $k = e \not\in S$, we will have in general terms from Case 1, as well. If $k = e$, then this case gives us exactly one term in the summation, namely:

$$\lambda([\delta_e]) \lambda([\delta_e]) = 0,$$

since the product here is actually the wedge product of the de Rham differential calculus of elements in $\text{hor}^1(P) = \mathcal{D}^1$, the 1-forms on $E$.

Now, we have to add up the results from these four cases. If $k \in S$, then we add the results from Cases 1, 2, and 3 getting:

$$\sum_{gh = k} \lambda([\delta_g]) \lambda([\delta_h]) = \lambda([\delta_e]) \lambda([\delta_k]) + \lambda([\delta_k]) \lambda([\delta_e]) = 0,$$

since, again, the product in $\text{hor}^1(P)$ is anti-commutative. If $k \not\in S$, then we add up the results from Cases 1 and 4, and again, we get zero.

Therefore, by substituting this into (5.14), we find for all $k \in G$ that:

$$r_\omega(\delta_k) = r_\omega'(\delta_k).$$

Since the $\delta_k$’s form a vector space basis of $\mathcal{A} = \mathcal{F}(G)$, we conclude that $r_\omega = r_{\omega'}$. $\square$

We have an immediate consequence:

**Corollary 5.1** Under the hypotheses of the previous theorem, we have that the cyclic Dunkl connection $\omega$ has curvature zero provided that the initial connection $\omega'$ has curvature zero.
Because of this Corollary, the next result is relevant.

**Theorem 5.6** The connection \( \tilde{\omega} \) defined in (2.21) has zero curvature.

**Proof** This is a straightforward calculation. For \( a \in A \), we have:

\[
\tilde{\omega}(a) = d_P(\omega(\pi(a))) + \omega(\pi(a^{(1)}) \omega(\pi(a^{(2)}))
= d_P(1_B \otimes \pi(a)) + (1_B \otimes \pi(a^{(1)}))(1_B \otimes \pi(a^{(2)}))
= 1_B \otimes \pi(a^{(1)}) + 1_B \otimes \pi(a^{(2)})
= 1_B \otimes \left( d^\wedge \pi(a) + \pi(a^{(1)})\pi(a^{(2)}) \right)
= 0.
\]

In the last equality, we used the Maurer–Cartan identity. (See Section 10.3 in [19].) We also used the definitions given in “Appendix B” of \( d_P \) and of the multiplication in \( \Omega(P) \).

\[ \square \]

### 5.9 Multiplicativity of cyclic Dunkl connections

Another very important property of cyclic Dunkl connections is that they are always multiplicative relative to the acceptable hodc \( \Gamma_{\text{inv}} \). Here is the general definition:

**Definition 5.9** Suppose that \( \Gamma^\square \) is an acceptable hodc extending a bicovariant, \( * \)-covariant fdc \( d : A \to \Gamma \). Let \( P = (B, A, \mathcal{F}) \) be a QPB with hodc \( \Omega(P) \). Then, we say that a quantum connection \( \omega : \Gamma_{\text{inv}} \to \Omega^1(P) \) is multiplicative relative to the hodc \( \Gamma^\square \) if \( \omega \) extends to a unital, multiplicative morphism \( \omega^\square : \Gamma^\square_{\text{inv}} \to \Omega(P) \).

**Remark 5.10** In the special case of this paper, \( P = (B, A, \mathcal{F}) \) is a QPB for the finite group \( G \), where \( A = \mathcal{F}(G) \), and we define the hodc to be \( \Omega(P) := D \otimes \Gamma^\square_{\text{inv}} \) instead of using (2.16).

As another comment, we note that if \( \omega^\square \) exists, then it is unique, and consequently, if \( \omega \) is a \( * \)-morphism, then \( \omega^\square \) also is a \( * \)-morphism.

If \( \Omega(P) = D \otimes \Gamma^\square_{\text{inv}} \), where \( \Gamma^\wedge \) is the universal envelope of \( \Gamma \), then the quantum connection \( \omega : \Gamma_{\text{inv}} \to \Omega^1(P) \) is multiplicative if and only if:

\[
\omega(\pi(a^{(1)})) \omega(\pi(a^{(2)})) = 0 \in \Omega^2(P)
\]

for all \( a \in R \), the right ideal in \( \ker e \subseteq A \) used to define the fdc \( \Gamma \). Here, we are using Sweedler’s notation \( \phi(a) = a^{(1)} \otimes a^{(2)} \). This non-trivial result is shown in [7].

**Theorem 5.7** The connection \( \tilde{\omega} \) defined in (2.21) is multiplicative relative to the universal differential envelope \( \Gamma^\wedge \) of \( \Gamma \).

**Proof** Using the previous remark, we take \( a \in R \subseteq A \) and calculate using (2.21) and the definition of multiplication in \( \Omega(P) \) as defined in “Appendix B” to get:
\[\tilde{\omega}(\pi(a^{(1)})) \tilde{\omega}(\pi(a^{(2)})) = (1_G \otimes \pi(a^{(1)})) (1_G \otimes \pi(a^{(2)}))
= 1_G \otimes \pi(a^{(1)})\pi(a^{(2)}) = -1_G \otimes d^\wedge \pi(a) = 0,\]
since \(\mathcal{R} \subseteq \ker \pi\). Again, we used the Maurer–Cartan identity. \(\square\)

### 5.10 The main theorem

We now collect the results proved above into the main theorem of this paper:

**Theorem 5.8** Let \(P = (C^\infty(E), \mathcal{F}(G), F)\) be a QPB with finite structure group \(G\) and right co-action \(F\) induced by a right action of \(G\) on \(E\). Let \((\Omega(P), \Gamma^\wedge, \tilde{\mathcal{F}})\) be the hocd associated with \(\mathcal{f}\) odc \(d : A \to \Gamma\), as defined above. Let \(\tilde{\omega}\) be the regular quantum connection defined in (2.21).

Then, for every cyclic Dunkl displacement \(\lambda : \Gamma_{inv} \to \text{Hom}^1(P)\) with respect to \(\tilde{\omega}\), the cyclic Dunkl connection \(\omega = \tilde{\omega} + \lambda\) has curvature zero, \(r_\omega \equiv 0\).

Moreover, the coordinate cyclic Dunkl operators \(\{D^j_\omega \mid j = 1, \ldots, n = \dim E\}\), defined in (2.30) as the coordinates of the covariant derivative \(D_\omega\) of the quantum connection \(\omega\), commute among themselves.

**Proof** Only the last statement remains to be proved. First off, we note for all homogeneous elements \(\varphi \in D^k\) that we obtain a general formula relating the square of the covariant derivative with the curvature:

\[
D^2_\omega(\varphi) = d_P(D_\omega \varphi) - (-1)^{\tilde{\varphi}(D_\omega \varphi)} (D_\omega \varphi)^{(0)} \omega([D_\omega \varphi]^{(1)})
= d_P(D_\omega \varphi) - (-1)^{1+\tilde{\varphi}} (D_\omega \varphi)^{(0)} \omega([\varphi]^{(1)})
= d_P(D_\omega \varphi) + (-1)^{\tilde{\varphi}} (D_\omega \varphi)^{(0)} \omega([\varphi]^{(1)})
= d_P^2 \varphi - (-1)^{\tilde{\varphi}} d_P \left( \varphi^{(0)} \omega([\varphi]^{(1)}) \right)
+ (-1)^{\tilde{\varphi}} \left\{ d_P \varphi^{(0)} - (-1)^{\tilde{\varphi}(\varphi)} \varphi^{(0)} \omega([\varphi]^{(1)}) \right\} \omega([\varphi]^{(1)})
= d_P^2 \varphi - (-1)^{\tilde{\varphi}} d_P \left( \varphi^{(0)} \omega([\varphi]^{(1)}) \right) - (-1)^{\tilde{\varphi}} d_P \left( \omega([\varphi]^{(1)}) \right)
+ (-1)^{\tilde{\varphi}} \left\{ d_P \varphi^{(0)} - (-1)^{\tilde{\varphi}(\varphi)} \varphi^{(0)} \omega([\varphi]^{(1)}) \right\} \omega([\varphi]^{(1)})
= -\varphi^{(0)} d_P \left( \omega([\varphi]^{(1)}) \right) - \varphi^{(0)} \omega([\varphi]^{(1)}) \omega([\varphi]^{(1)})
= -\varphi^{(0)} d_P \left( \omega([\varphi]^{(1)}) \right) - \varphi^{(0)} \omega([\varphi]^{(1)}) \omega([\varphi]^{(1)})
= -\varphi^{(0)} \left( d_P \left( \omega([\varphi]^{(1)}) \right) + \omega([\varphi]^{(1)}) \omega([\varphi]^{(1)}) \right)
= -\varphi^{(0)} r_\omega(\varphi),
\]
where we used the definition (2.24) of \(D_\omega\) thrice, the covariance of \(D_\omega\) in the second equality (Theorem 12.12 in [19]), \(\tilde{\varphi} = \tilde{\varphi}^{(0)}\), the graded Liebniz rule for \(d_P\), \(d_P^2 = 0\), the co-action property in Sweedler’s notation, and the definition (5.12) of the curvature \(r_\omega\). Therefore, we have \(D^2_\omega(\varphi) = 0\), since \(r_\omega \equiv 0\).
However, on the other hand, a direct computation using (2.29) for $\varphi \in C^\infty(E)$ gives:

$$D^2_{\omega}(\varphi) = \sum_{k=1}^n D_{\omega}(D^k_{\omega}(\varphi \, dx_k)) = \sum_{k=1}^n D_{\omega}(D^k_{\omega}(\varphi))dx_k \quad \left(\text{more about this step later}\right)$$

$$= \sum_{j,k=1}^n D^j_{\omega}D^k_{\omega}(\varphi) \, dx_j \wedge dx_k = \sum_{1 \leq j < k \leq 1} (D^j_{\omega}D^k_{\omega}(\varphi) - D^k_{\omega}D^j_{\omega}(\varphi))dx_j \wedge dx_k.$$ 

For the last equality, we used that the horizontal forms are the complexified de Rham differential forms. Now, the forms $dx_j \wedge dx_k$ for $j < k$ are linearly independent, and so, it follows from $D^2_{\omega}(\varphi) = 0$ that:

$$D^j_{\omega}D^k_{\omega}(\varphi) = D^k_{\omega}D^j_{\omega}(\varphi)$$

for all $j, k \in \{1, \ldots, n\}$ and for all $\varphi \in \mathcal{D}^0 = C^\infty(E)$.

However, we still owe the reader more details about the second step in the above calculation. Among other things, this depends on the fact that $dx_k$ is a horizontal form, that is an element in $\Omega(M)$ or in other words that it is $\hat{F}$ invariant. (See (2.17).) Therefore, we examine some relations for a homogeneous form $w \in \Omega(M)$ and later specialize to the case when $w = dx_k$. We first note for all $\theta \in \Gamma_{\text{inv}}$ that:

$$\omega(\theta)w = (-1)^{\hat{\varphi}w} \omega(\theta), \quad (5.17)$$

which follows from the definition of multiplication in $\Omega(P)$. Next, we compute for any $\varphi \in D^k \cong \text{hor}^k(P)$ that:

$$D_{\omega}(\varphi w) = d_P(\varphi w) - (-1)^{\hat{\varphi} + \hat{\varphi}w}(\varphi w)^{(0)} \omega([\varphi w]^{(1)})$$

$$= d_P(\varphi w) - (-1)^{\hat{\varphi} + \hat{\varphi}w} \varphi^{(0)} w \omega([\varphi^{(1)}])$$

$$= d_P(\varphi w) - (-1)^{\hat{\varphi} + \hat{\varphi}w} \varphi^{(0)} \omega([\varphi^{(1)}]) w$$

$$= d_P(\varphi) w + (-1)^{\hat{\varphi} + \hat{\varphi}w} \varphi d_P w - (-1)^{\hat{\varphi} + \hat{\varphi}w} \varphi^{(0)} \omega([\varphi^{(1)}]) w$$

$$= D_{\omega}(\varphi) w + (-1)^{\hat{\varphi} + \hat{\varphi}w} \varphi dw,$$

where we used the definition of $D_{\omega}$, the formula for the co-action $\hat{F}$ acting on $\varphi w$ in the second equality (see “Appendix B”), (5.17) in the third equality, the Leibniz rule for $d_P$, the fact that $d_P$ on horizontal forms reduces to the de Rham differential $d$, and finally the definition of $D_{\omega}$ again. To conclude, we take $w = dx_k$ as we indicated earlier. Therefore, we get $d w = d dx_k = 0$ and therefore:

$$D_{\omega}(\varphi dx_k) = D_{\omega}(\varphi) dx_k,$$

which justifies the second step in the above argument. \qed

Remark 5.11 The main result of [12] can be understood as a consequence of the Dunkl QCD given here in (3.1) as being a cyclic Dunkl displacement.
5.11 Some more on multiplicativity

Theorem 5.9  Every cyclic Dunkl connection \( \omega = \tilde{\omega} + \lambda \) with \( \tilde{\omega} \) multiplicative relative to \( \Gamma^\wedge \) is itself multiplicative relative to \( \Gamma^\wedge \).

**Proof**  Let \( \omega^\oplus : \Gamma^\wedge_{\text{inv}} \to \Omega(P) \) be the unital multiplicative extension of \( \omega \). This is defined analogously to the definition given just before Theorem 5.2 of \( \lambda^\oplus \). According to Remark 5.10, we have to show that:

\[
\omega([[a^{(1)}]])\omega([[a^{(2)}]]) = m_{\Omega(P)} \omega^\oplus(\pi \otimes \pi)\phi(a) = 0 \tag{5.18}
\]

holds for all \( a \in \mathcal{R} \). Therefore, we take \( a \in \mathcal{R} \) and use \( \omega = \tilde{\omega} + \lambda \) to get:

\[
m_{\Omega(P)} \omega^\oplus(\pi \otimes \pi)\phi(a) = \omega([[a^{(1)}]])\omega([[a^{(2)}]]) = \tilde{\omega}([[a^{(1)}]])\tilde{\omega}([[a^{(2)}]]) + \lambda([[a^{(1)}]])\lambda([[a^{(2)}]]) + \lambda([[a^{(1)}]])\lambda([[a^{(2)}]])
\]

where \( \tilde{\omega} \) is multiplicative (Remark 5.10) as well as the compact notation \( [\cdot] = \pi(\cdot) \). We note that \( \mathcal{R} \) has a basis \( \mathcal{B}_\mathcal{R} := \{ \delta_g \mid g \not\in K\mathcal{R} \} \) by Corollary 13.1 in [19], where \( K\mathcal{R} := \{ h \in G \mid f(h) = 0 \text{ for all } f \in \mathcal{R} \} \). Since (5.18) is linear in \( a \in \mathcal{R} \), it suffices to prove it for the basis elements \( \delta_g \in \mathcal{B}_\mathcal{R} \). Now, for \( a = \delta_g \), we have:

\[
a^{(1)} \otimes a^{(2)} = \phi(\delta_g) = \sum_{hk = g} \delta_h \otimes \delta_k = \sum_{(h,k) \in \mathcal{M}_g} \delta_h \otimes \delta_k
\]

where \( \mathcal{O} \) is a \( \sigma \)-orbit and \( \text{Inv}(\mathcal{O}) \) is defined in 4.1. Also, \( \mathcal{M}_g \) is defined in (5.15) and satisfies (5.16). We will use explicit summation notation (here with \( h,k \in G \)) instead of Sweedler’s when the former is handier. It follows that:

\[
\lambda([[a^{(1)}]])\lambda([[a^{(2)}]]) = \sum_{O: \text{Inv}(O) \neq g} \sum_{(h,k) \in O} \lambda(\delta_h)\lambda(\delta_k) = 0,
\]

since \( \lambda \) satisfies the Cyclic Property and so the inner sum already is 0. Therefore, the equation above reduces to:

\[
m_{\Omega(P)} \omega^\oplus(\pi \otimes \pi)\phi(a) = \tilde{\omega}([[a^{(1)}]])\lambda([[a^{(2)}]]) + \lambda([[a^{(1)}]])\tilde{\omega}([[a^{(2)}]]) \tag{5.19}
\]

Next, we analyze the first term on the right side using \( \tilde{\omega} \) regular (see (5.1)) to get:

\[
\tilde{\omega}([[a^{(1)}]])\lambda([[a^{(2)}]]) = \sum_{hk = g} (-1)^1 \lambda([[\delta_k]])^{(0)} \tilde{\omega}(\delta_h) \lambda([[\delta_k]])^{(1)} \tag{5.20}
\]

And next, using that \( \lambda \) is a QCD in the second equality, we calculate:
\[
\lambda([\delta_k])^{(0)} \otimes \lambda([\delta_k])^{(1)} = \tilde{F}(\lambda([\delta_k])) = (\lambda \otimes \text{id}) \text{ad}([\delta_k]) \\
= \lambda([\delta_k^{(2)}]) \otimes \kappa(\delta_k^{(1)}) \delta_k^{(3)} = \sum_{lmn=k} \lambda([\delta_m]) \otimes \kappa(\delta_l) \delta_n,
\]

where \(l, m, n \in G\). We also used the formula (2.11) for \text{ad}. Then, (5.20) becomes:

\[
\tilde{\omega}([a^{(1)}])\lambda([a^{(2)}]) = -\sum_{hklm=g} \lambda([\delta_l]) \tilde{\omega}([\delta_h] \circ \kappa(\delta_k) \delta_m) \\
= -\sum_{hklm=g} \lambda([\delta_l]) \tilde{\omega}([\delta_h] \circ \delta_{k^{-1}} \delta_m) \\
= -\sum_{hm^{-1}lm=g} \lambda([\delta_l]) \tilde{\omega}([\delta_h] \circ \delta_m) \\
= -\sum_{hl=g} \lambda([\delta_l]) \tilde{\omega}([\delta_h]) = -\lambda([a^{(1)}])\tilde{\omega}([a^{(2)}]).
\]

In the second equality, we used \(\kappa(\delta_k) = \delta_{k^{-1}}\) as the reader can verify. To get the third equality, we summed on \(k\), and for the fourth, we summed on \(m\) and used the properties of the \(\circ\) operation mentioned just after (5.10). And in the last equality, we reverted back to Sweedler’s notation. Substituting back into (5.19), we get the desired result.

**Corollary 5.2** Every cyclic Dunkl connection \(\omega = \tilde{\omega} + \lambda\) with \(\tilde{\omega}\) multiplicative is extendible (in a necessarily unique way) to a unital algebra morphism:

\[
\omega^\sim : \Gamma^\sim_{\text{inv}} \rightarrow \Omega(P).
\]

**Remark 5.12** If we use the calculus \(\Gamma^\sim\) as the hodc over the fodc \(\Gamma\), then the above proposition can be restated as the multiplicativity of Dunkl connections. In particular, they are always multiplicative for the universal differential algebra \(\Gamma^\wedge\) over \(\Gamma\).

Therefore, cyclic Dunkl connections provide a nice class of examples for non-regular (in general) but nevertheless multiplicative connections. For example, the Dunkl operators as originally defined in [5] do not satisfy the Leibniz rule as is shown in [12]. Then, by Theorem 12.14 of [19], this implies that the corresponding Dunkl connection is not regular, but by what we have shown here they are multiplicative.

**Remark 5.13** For multiplicative connections \(\omega\), there is a very nice interpretation of the curvature as a measure of the deviation of \(\omega^\sim : \Gamma^\sim_{\text{inv}} \rightarrow \Omega(P)\) from being a differential algebra morphism. Indeed, for every \(a \in A\), we have:

\[
\left\{ d\omega^\sim - \omega^\sim \pi \right\} \pi(a) = d\omega \pi(a) + \phi(a^{(1)}) \omega \pi(a^{(2)}) = r_\omega(a),
\]

where we used the Maurer–Cartan formula \(d\pi(a) = -\pi(a^{(1)})\pi(a^{(2)})\) and where we write \(\phi(a) = a^{(1)} \otimes a^{(2)}\) in Sweedler’s notation.

In this case, there are no residual curvature terms for \(a \in R\) and the curvature
map $r_\omega : A \rightarrow \mathfrak{hor}^2(P)$ naturally projects down to $r_\omega : \Gamma_{\text{inv}} \rightarrow \mathfrak{hor}^2(P)$. As we know from the general theory [8, 9], the residual curvature terms are a manifestation of an interesting purely quantum phenomena, where a specific quadratic combination $\omega(\pi(a^1))\omega(\pi(a^2))$ of ‘vertical’ elements given by values of the connection form surprisingly turns out to be horizontal for $a \in \mathcal{R}$. The presence of these curvature terms can be understood as the obstacle to the multiplicativity of the connection.

6 Example: complex reflection groups

As an important special example of this construction, we consider complex reflection groups and their associated Dunkl operators. These operators were introduced in the paper [6].

6.1 Basic definitions

A complex reflection $s$ is a unitary transformation acting on a complex, finite-dimensional vector space $V$ with $\dim V = n \geq 1$ and with a Hermitian inner product, such that $s$ has finite order in the unitary group $U(V)$ of $V$ and exactly one of the eigenvalues of $s$ is not equal to 1 and has multiplicity 1. In particular, $s \neq e$, the identity.

Suppose that $S'$ is a non-empty set of complex reflections acting on $V$ and that $S'$ generates a finite subgroup $G$ of $U(V)$. Then, we say that $G$ is finite complex reflection group acting on $V$. Let $S$ be the smallest set containing $S'$, such that $S$ is closed under conjugation by arbitrary elements $g \in G$ and $S^{-1} = S$. Therefore, $S = \bigcup \{g^{-1}sg, g^{-1}s^{-1}g \mid g \in G, s \in S'\}$. It follows that $e \notin S$ and $S$ is non-empty. Therefore, $S$ determines a unique bicovariant, *-covariant fode $\Gamma$ for the Hopf algebra $A = \mathcal{F}(G)$. In particular, let $d : A \rightarrow \Gamma$ denote the differential of this fode.

For every one-dimensional complex subspace $L$ of $V$, let $\tilde{\xi}_L$ be the (singular) classical differential complex 1-form on $V$ defined by:

$$\tilde{\xi}_L(x) = \frac{1}{\langle x, \bar{z} \rangle} \sum_{j=1}^{n} \bar{z}_j d\bar{x}_j = \frac{\bar{z}}{\langle x, \bar{z} \rangle},$$

(6.1)

where $0 \neq \bar{z} = (z_1, \ldots, z_n) \in L$ is called a representative vector of $\tilde{\xi}_L$ and $x \in V$ satisfies $\langle x, \bar{z} \rangle \neq 0$; that is, $x \notin L^\perp$, the hyperplane orthogonal to $L$. In this section, we use the notation $\bar{z} = \sum_{j=1}^{n} \bar{z}_j d\bar{x}_j$ and the $d\bar{x}_j$’s are the standard (0, 1) one-forms associated with some given orthonormal basis of $V$.

Remark 6.1 It is easy to see that the above definition (6.1) does not depend on the representative vector $\bar{z}$, since our convention is that the inner product is linear in its second entry. And it also does not depend on the choice of the orthonormal basis. We include singular anti-meromorphic forms here, since the scalar product $\langle x, \bar{z} \rangle$ is zero for all points $x \in L^\perp \neq \emptyset$, the orthogonal complement of $L$. If $\dim V \geq 2$, then $L^\perp \neq 0$, as well. The exterior differential graded *-algebra structure remains well
defined for such singular 1-forms, although with implicitly defined domain restrictions.

6.2 Preliminary results

Proposition 6.1 For every three linearly dependent 1-dimensional subspaces $X$, $Y$, and $Z$ of $V$, we have:

$$\xi_X \xi_Y + \xi_Y \xi_Z + \xi_Z \xi_X = 0.$$  \hfill (6.2)

Proof The property is trivial if $X = Y$, $Y = Z$, or $Z = X$. Hence, we assume that $X \neq Y \neq Z \neq X$. Choose representative vectors $\alpha \in X$, $\beta \in Y$ and $\gamma \in Z$. These give us 3 distinct vectors, since $X$, $Y$, and $Z$ are 3 distinct one-dimensional subspaces. Every pair taken from $\{\alpha, \beta, \gamma\}$ forms a basis of the 2-dimensional subspace spanned by all 3 of them. We now compute:

$$\xi_X \xi_Y + \xi_Y \xi_Z = \frac{\bar{\alpha} \bar{\beta} \bar{\gamma}}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle \langle \gamma, \gamma \rangle} = \frac{\bar{\alpha} \bar{\gamma}}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} = -\xi_Z \xi_X,$$

where $\gamma = a\alpha + b\beta$ and $a, b \in \mathbb{C} \setminus \{0\}$, using the graded commutativity of classical 1-forms in the very last step.

Theorem 6.1 Let $n \geq 1$ be an integer. Then, for every family $W_1, \ldots, W_n$ of one-dimensional subspaces of $V$, every three of which are linearly dependent, we have that:

$$\xi_{W_1} \xi_{W_2} + \xi_{W_2} \xi_{W_3} + \cdots + \xi_{W_{n-1}} \xi_{W_n} + \xi_{W_n} \xi_{W_1} = 0.$$  \hfill (6.3)

Proof The hypothesis on the family $W_1, \ldots, W_n$ is vacuously true for every family in the cases $n = 1$ and $n = 2$. However, in those cases, the conclusion is trivially true. The case $n = 3$ was proved in Proposition 6.1. For $n \geq 4$, one argues inductively applying the cyclic property (6.2) for the three spaces $W_1, W_n, W_{n+1}$ and taking (6.3) as the induction hypothesis.

All of the subspaces $W_1, \ldots, W_n$ in the previous theorem are contained in the common subspace $U = W_1 + \cdots + W_n$ of $V$ and the dimension of $U$ is 1 or 2. We say that this subspace $U$ is the container of $W_1, \ldots, W_n$.

Let us note that $G$ acts faithfully on $V$ by unitary transformations, and every element of the set $S$ is a complex reflection. For every $s \in S$, let $W_s \subset V$ denote the eigenspace of $s$ whose eigenvalue is not equal to 1. Therefore, $W_s$ is one-dimensional. Then, we define:
\[ \Omega := \{ W_s \mid s \in S \}, \quad (6.4) \]

the set of these eigenspaces.

We also define \( \mu : S \rightarrow \Omega \) by \( \mu(s) := W_s \) for all \( s \in S \). Then, \( \mu \) is a surjection, though it need not be an injection. It is clear that \( G \) acts naturally on \( \Omega \) by \( W_g \cdot g = W_{g^{-1}sg} \) for \( s \in S \) and \( g \in G \). Also, \( G \) acts on \( S \) by \( s \cdot g := g^{-1}sg \in S \). These are right actions. Then, \( \mu \) is covariant with respect to these actions of \( G \). The cyclic structure on \( S \) projects to a cyclic structure on \( \Omega \), namely \( \sigma : \Omega \times \Omega \rightarrow \Omega \times \Omega \) is given by \( \sigma(W_s, W_t) = (W_{st}, W_s) \) for \( s, t \in S \).

Let us now observe that if \( \alpha \) and \( \beta \) are eigenvectors corresponding to complex reflections \( u \) and \( v \) (respectively) and whose eigenvalues are \( \neq 1 \), then we claim that:

\[ \beta - u(\beta) = a\alpha, \quad (6.5) \]

for some \( a \in \mathbb{C} \). Indeed, we can decompose \( \beta = \beta_{\parallel} + \beta_{\perp} \) into parallel and orthogonal components to \( \alpha \), and so:

\[ u(\beta) = c\beta_{\parallel} + \beta_{\perp}, \]

where \( u(\alpha) = c\alpha \) with \( c \neq 1 \). From this, the colinearity condition (6.5) follows from \( \beta - u(\beta) = \beta_{\parallel} + \beta_{\perp} - (c\beta_{\parallel} + \beta_{\perp}) = (1 - c)\beta_{\parallel} \) with \( 1 - c \neq 0 \). Using \( \beta_{\parallel} = b\alpha \) for some \( b \in \mathbb{C} \), then gives (6.5). As we shall now see, this condition implies that every cyclic line in \( \Omega \) possesses a one or 2-dimensional container, and hence, the cyclic relation (6.3) holds. We shall actually prove a more general result, regarding groups of Coxeter type which we now define.

### 6.3 Groups of Coxeter type

**Definition 6.2** A finite group \( G \), realized as a subgroup of the unitary group \( U(V) \), is of **Coxeter type**, if there is a (finite) generating subset \( S \) of \( G \) and a function \( \mu : S \rightarrow \text{CP}(V) \), the complex projective space of \( V \) (namely the space whose elements are the one-dimensional subspaces of \( V \)), such that:

(i) If \( u, v \in S \), then \( uvu^{-1} \in S \). This property implies that \( S \) is invariant under conjugations by arbitrary elements from \( G \). In other words, \( S \) splits into one or more entire conjugacy classes.

(ii) For \( u, v \in S \), we have \( \mu(uvu^{-1}) = u[\mu(v)] \). In particular, if \( u \) and \( v \) commute (for example, \( u = v \)), we see that \( \mu(v) \) is \( u \)-invariant, and thus, \( u \) acts on the subspace \( \mu(v) \) as multiplication by a unitary scalar.

(iii) The space \( u[\mu(v)] \) is contained in the linear span of \( \mu(u) \) and \( \mu(v) \). In other words, if \( 0 \neq \alpha \in \mu(u) \) and \( 0 \neq \beta \in \mu(v) \), then:

\[ u(\beta) = a\alpha + b\beta \quad (6.6) \]

for some \( a, b \in \mathbb{C} \).

Obviously, every complex reflection group is of Coxeter type. In this definition, we are requiring that the “generalized reflections” \( u \in S \) transform the “root
vector” representatives from $\mu(v)$ in a very local way, namely the transformed vector is always a linear combination of the initial vector $\beta$ and the reflection transformation vector $\alpha$.

6.4 The natural cyclic structure

We can now proceed exactly as above by first defining the family $\Omega := \text{Ran} \mu$, a finite subset of $\text{CP}(V)$, and then, the natural cyclic structure induced by $\mu$ on it.

Proposition 6.2 If $G$ is of Coxeter type, then every cyclic line in $\Omega$ possesses a one or 2-dimensional container. Thus, the above cyclic relation (6.3) holds.

Proof We consider $u$ and $v$ in $S$ (“generalized reflections”) and their associated cyclic line $\ell_{u,v}$. If $u = v$, the result is trivial. Therefore, we assume that $u \neq v$. The point preceding $u$ on $\ell_{u,v}$ is $uv^{-1}$. Let $\gamma \in \mu(uv^{-1}) \backslash \{0\}$ and choose non-zero vectors $\alpha$ and $\beta$ from $\mu(u)$ and $\mu(v)$, respectively. Since $\mu(uv^{-1}) = u[\mu(v)]$, the third condition implies that $\gamma = u(\beta) = a\alpha + b\beta$ for some $a, b \in \mathbb{C}$.

Now, we can proceed inductively to conclude that every one-dimensional subspace associated with the points on the cyclic line $\ell_{u,v}$ is contained in the subspace spanned by $\alpha$ and $\beta$.

6.5 The quantum principal bundle and its cyclic quantum displacement

Let $E$ be the dense open subset of $V$ consisting of all the vectors with trivial $G$-stabilizer. By definition, $G$ acts freely on $E$. Furthermore, every vector $x \in E$ is not orthogonal to any of the subspaces $W_s$ in $\Omega$ as defined in (6.4). As we have seen $P = (C^\infty(E), \mathcal{F}(G), F)$ is then a QPB, where $F$ is the pull-back of the right action of $G$ on $E$. Let $\Omega(P)$ be the associated hodc which was constructed as in [12] and reviewed in Sect. 2.2 and “Appendix B”.

Let us use the same symbol $\xi_W$ for the corresponding restricted differential, complex 1-form on $E$. Notice that $\xi_W$ when restricted to $E$ has no singularities. In other words, $\xi_W$ is a smooth (that is, $C^\infty$) section, defined on all of $E$, of the complexified cotangent bundle $T^*(E) \otimes \mathbb{C}$. In accordance with the above comments, $\xi_W$ is identified as a horizontal 1-form in $\Omega(P)$, namely $\xi_W \in \mathfrak{hor}^1(P)$. Let us define $\lambda : S \to \mathfrak{hor}^1(P)$ as:

$$\lambda(s) = v(s) \xi_{\mu(s)},$$

where $v : S \to \mathbb{C}$ is any function which is constant on each conjugation class of $S$.

Theorem 6.2 The map $\lambda$ is a cyclic Dunkl displacement for $P$, where $P$ is viewed as a quantum principal bundle with structure group $G$ and is equipped with the canonical connection $\tilde{\omega}$ defined in (2.21).

Proof We first prove that the form $\xi_W$ is closed with respect to the covariant derivative $D_{\tilde{\omega}}$. However, $D_{\tilde{\omega}} = d$, the complexified de Rham differential. Recall that $\xi_W(x) = \tilde{\alpha}/(x, \alpha)$, where $\alpha \in W \backslash \{0\}$. Then, we calculate that:
\[
d\bar{\zeta}_W = d\left(\sum_j \frac{x_j}{\langle x, x' \rangle} d\bar{x}_j \right) = \sum_j d\left[\frac{x_j}{\langle x, x' \rangle}\right] \wedge d\bar{x}_j \\
= \sum_j \sum_k \frac{\partial}{\partial \bar{x}_k} \left[\frac{x_j}{\langle x, x' \rangle}\right] \wedge d\bar{x}_j = -\sum_j \sum_k \frac{x_j x_k}{\langle x, x' \rangle^2} \wedge d\bar{x}_j = 0,
\]
where we used that \(\frac{\partial}{\partial \bar{x}_k} \left[\frac{x_j}{\langle x, x' \rangle}\right] = 0\), since the function inside the brackets is anti-holomorphic on its domain of definition. This proves that \(\bar{\zeta}_W\) is closed. Since \(v(s)\) is simply a complex number, it is immediate that \(d\lambda(s) = d(v(s) \bar{\zeta}_\mu(s)) = 0\); that is, \(\lambda(s)\) is closed for all \(s \in S\).

The cyclic property is a direct consequence of Proposition 6.2. For the covariance property, we see on one hand that:

\[
F^\lambda_g(\lambda(s)) = F^\lambda_g(v(s) \bar{\zeta}_\mu(s)) = v(s) F^\lambda_g(\bar{\zeta}_\mu(s)) = v(s)(\text{id} \otimes g)F^\lambda(\bar{\zeta}_\mu(s)) \\
= v(s)(\text{id} \otimes g) \sum_{k \in G} k \cdot \bar{\zeta}_\mu(s) \otimes \delta_k = v(s) \sum_{k \in G} k \cdot \bar{\zeta}_\mu(s) \otimes \delta_k(g) \\
= v(s)(g \cdot \bar{\zeta}_\mu(s)).
\]

On the other hand, using \(W_s \cdot g := W_{g^{-1}sg}\), we compute:

\[
\bar{\lambda}(gsg^{-1}) = v(gsg^{-1}) \bar{\zeta}_{\mu(gsg^{-1})} = v(s) \bar{\zeta}_{\mu(s)} \\
= v(s)(\bar{\zeta}_\mu(s)g^{-1}) = v(s)(g \cdot \bar{\zeta}_\mu(s)).
\]

In the last equality, we used \(g \cdot \bar{\zeta}_W = \bar{\zeta}_{Wg^{-1}}\) which holds, since the action by \(g\) is an orthogonal transformation.

This finishes the proof of the covariance property and of the theorem. \(\square\)

### 6.6 The covariant derivative

Let us now compute the covariant derivative of the Dunkl connection \(\omega = \bar{\omega} + \lambda\). According to the general theory, for \(\varphi \in \mathcal{D}^k\), we have that:

\[
D_\omega(\varphi) = D_{\bar{\omega}}(\varphi) - (-1)^k \varphi^{(0)} \lambda \pi(\varphi^{(1)}) = d\varphi(\varphi) - (-1)^k \sum_{g \in G}(\varphi_g \lambda \pi(\delta_g)) \\
= d\varphi(\varphi) + (-1)^k \sum_{s \in S}(\varphi - \varphi_s)\lambda[s] = d\varphi(\varphi) + (-1)^k \sum_{s \in S}(\varphi - \varphi_s)v(s)\bar{\zeta}\mu(s).
\]

This expression has the same abstract structure as the previously mentioned formula (2.25).

**Remark 6.3** It is worth recalling that we chose to deal with anti-meromorphic displacements, the meromorphic version is obtained by the simple complex conjugation of the relevant objects. We can write \(d\varphi = \partial \varphi + \bar{\partial} \varphi\) in the complex case, the decomposition into the holomorphic and anti-holomorphic differentials, satisfying \(\partial^2 = \bar{\partial}^2 = \partial \bar{\partial} + \bar{\partial} \partial = 0\). Each of these differentials can be taken...
as the representative of the differential calculus on $P$. Of course, in this case, we would lose the $\ast$-compatibility, but as already mentioned, this property was not essential for the present considerations. It is also important to observe that our expression holds in the very general context of all groups $G$ of Coxeter type.

### 6.7 The covariant derivative viewed as Dunkl operators

To make an explicit link with the Dunkl operators introduced in [6], let us fix for every $s \in S$, an element $\alpha_s \in \mu(s)$, and use the formula (6.1). For the classical differential, let us use the anti-holomorphic differential $\partial_P$. We have:

$$D_{\omega}(\varphi) = \partial_P(\varphi) + (-1)^k \sum_{s \in S} (\varphi - \varphi_s) v(s) \tilde{s}_s \langle x, \alpha_s \rangle,$$  \hspace{1cm} (6.7)

which shows that our operators effectively accommodate, as a special case and modulo trivial modifications and a change of notation, those of [6]. In particular, in [6], the set $S$ is naturally labeled by the hyperplanes associated with the complex reflections.

**Remark 6.4** It is also worth noticing that with such change $d_{\partial} \sim \partial_P$, the holomorphic (anti-holomorphic) forms on $P$ are fully preserved under the Dunkl covariant derivative action. If, in addition, $G$ is a complex reflection group, then the corresponding polynomial forms will be preserved, as the singular contributions of the type $\langle x, \alpha \rangle^{-1}$ will always be canceled out.

### 7 Example: Cuntz algebras

A rich class of truly quantum principal bundles equipped with cyclic Dunkl operators can be constructed from Cuntz algebras. We shall sketch here a kind of a universal construction for the full differential structure on these quantum principal bundles.

If a finite-dimensional unitary space $V$ is given, its Cuntz algebra $O(V)$ is defined as the $\ast$-algebra generated by the linear space $V$ and relations:

$$\psi_i^* \psi_j = \delta_{ij} \sum_{k=1}^n \psi_k \psi_k^* = 1,$$  \hspace{1cm} (7.1)

where $\psi_1 \ldots \psi_n$ is an arbitrary orthonormal basis of $V$ and $n = \dim V \geq 2$. The algebra $O(V)$ is independent of the choice of this orthonormal basis—it depends only on the scalar product on $V$.

The Cuntz algebras [2] naturally appear in the study of partial isometries in Hilbert spaces. Their enveloping $C^*$-algebras are simple, and play a fundamental role in constructive algebraic quantum field theory [3], where they provide a key structure to describe the superselection sectors, and related to this, the duality theory for compact groups [4]. The Cuntz algebras are also (together with their matrix
siblings) the building blocks for the quantum classifying spaces associated with compact matrix quantum groups [11].

If \( G \) is an arbitrary compact matrix quantum group unitarily acting on \( V \), via a unitary representation \( u : V \to V \otimes \mathcal{A} \), so that:

\[
u(w_{ij}) = \sum_{i=1}^{n} \psi_i \otimes u_{ij},
\]

then \( u \) uniquely extends to a \(*\)-homomorphism \( F : \mathcal{O}(V) \to \mathcal{O}(V) \otimes \mathcal{A} \) which is the action of \( G \) on \( \mathcal{O}(V) \).

If the representation \( u \) is faithful and such that the conjugate representation \( \bar{u} \) is contained as a subrepresentation of a tensor power \( u^\otimes m \), then the action \( F \) will be free (in the dual sense).

Indeed, in this case, the whole algebra \( \mathcal{A} \) is generated by the matrix entries \( u_{ij} \) (without involving the conjugates \( u_{ij}^* \)). On the other hand, the above defining relations give:

\[\psi_i^* F(\psi_j) = 1 \otimes u_{ij},\]

which can be iterated to prove that the relations of the form \( \sum_a q_a F(b_a) = 1 \otimes a \) hold for all the elements of \( \mathcal{A} \), which is precisely (the dual form of) the freeness condition for \( F \).

Interestingly, if \( G \) is any finite classical group, this condition is always satisfied. In other words, we have a quantum principal bundle \( \mathcal{P} \) based on \( \mathcal{O}(V) \). The “base space” algebra \( \mathcal{V} \) is given by the invariants of this action and it is highly non-commutative.

Let \( \mathcal{O}^+(V) \) be the subalgebra generated by \( V \). If \( S \subseteq G \setminus \varepsilon \) is the defining set for a bicovariant \(*\)-calculus \( \Gamma \) over \( G \), then we can always construct a map \( \zeta : \Gamma_{\text{inv}} \to \mathcal{O}^+(V) \) which intertwines the adjoint action \( \text{ad} \) and \( F \) and such that its values on \( S \) are all non-zero. This is a direct consequence of the above generating property for the representation \( u \) and the definition of the Cuntz algebras.

On the other hand, by universality, the action \( F \) extends uniquely to a homomorphism of graded differential \(*\)-algebras \( F : \Omega \mathcal{O}(V) \to \Omega \mathcal{O}(V) \otimes \mathcal{A} \), where \( \Omega \mathcal{O}(V) \) is the universal differential envelope of the algebra \( \mathcal{O}(V) \).

We would like to obtain here a cyclic Dunkl displacement. A natural candidate is the differential of \( \zeta \) in \( \Omega \mathcal{O}(V) \). If we thus define \( \lambda(s) := d\zeta(s) \), the Covariance and Closed-ness properties in the definition of cyclic Dunkl displacements are automatically satisfied. The only property which is possibly not fulfilled is the Cyclic Property. However, this is easily remedied by observing that the second-order elements of the form \( \lambda(\ell) = \lambda(s_1)\lambda(s_2) + \cdots + \lambda(s_{n-1})\lambda(s_n) + \lambda(s_n)\lambda(s_1) \) are all exact and transform covariantly, so they generate a graded differential \(*\)-ideal \( \mathcal{I} \) in \( \Omega \mathcal{O}(V) \), such that \( F(\mathcal{I}) \subseteq \mathcal{I} \otimes \mathcal{A} \). If we now set \( \text{hor}(\mathcal{P}) := \Omega \mathcal{O}(V)/\mathcal{I} \) and project \( F \) and \( \lambda \) from \( \Omega \mathcal{O}(V) \) to \( \text{hor}(\mathcal{P}) \), then such a projected \( \lambda \) is a cyclic Dunkl displacement for \( \text{hor}(\mathcal{P}) \).

We can then proceed with the general constructions to craft the complete calculus, as described in Sect. 5. It is interesting to observe that here \( \Omega \mathcal{O}(V) \) is used
to build the horizontal forms only, an appropriate full calculus on $P$ is constructed as a twisted tensor product $\Omega(P) \leftrightarrow \text{hor}(P) \otimes \Gamma_{\text{inv}}^\Lambda$. The projected differential $d : \text{hor}(P) \rightarrow \text{hor}(P)$ is interpreted as the covariant derivative of an associated ‘initial’ zero-curvature connection $\dot{\omega}$.

8 Concluding observations

This paper deals with the cyclic Dunkl operators exclusively from the point of view of purely algebraic aspects of non-commutative geometry. In a future paper, we will present the analytic properties of these new operators, such as a generalized Dunkl transform. We also intend to analyze in detail examples involving the groups of the Coxeter type, beyond the complex reflection groups. The geometry of the quantum principal bundle is essentially given by the choice of $S$, which determines the differential calculus.

Another interesting class of examples, which we leave for a future study, comes from appropriately quantized Euclidean spaces on which a finite group $G$ acts, essentially without changing the action on basic coordinates. Among other purely quantum phenomena in this context, we can mention an automatic freeness of the group action—as there are simply no points to possibly manifest themselves as elements in a non-trivial stabilizer. Also, as we mentioned earlier, the case when the tangent bundle of $E$ is trivial merits closer examination, something we plan to do in a future paper.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix A: Cyclic structures

Here, we shall explain how the basic geometry of oriented cycles, that naturally emerges in the context of differential calculi on finite groups, can be formalized, so that we can recover the group and the calculus from a simple set of axioms.

Let $\Omega$ be a finite set, equipped with a family $T$ of cyclically oriented subsets of $\Omega$. Every such set in $T$ is representable as some ordered $n$-tuple $(\omega_1, \ldots, \omega_n)$ with $n \geq 1$ mutually distinct elements $\omega_1, \ldots, \omega_n \in \Omega$. Here, we identify those $n$-tuples that can be obtained one from another using cyclic permutations.

The first property that we shall assume here is that the elements of $T$ behave like “oriented lines”. Specifically, for every ordered pair $(x, y) \in \Omega \times \Omega$ with $x \neq y$, we assume that there exists a unique $\ell_{x,y} \leftrightarrow (\omega_1, \ldots, \omega_n)$ in $T$, such that $\omega_1 = x$, $\omega_2 = y$ and $\omega_j \in \Omega$ for all $3 \leq j \leq n$.

In this case, we necessarily have a discrete ‘line’ with $n \geq 2$ points. To this, we add a kind of normalization property by also assuming that there exists a line $\ell_{x,x} = (x) \in T$ for every $x \in \Omega$. Therefore, in this case, the ‘line’ has exactly one point.
We can think of this as an oriented version of one of Euclid’s axioms: every oriented pair of points determines a unique oriented line on which they lie.

**Definition 8.1** Every pair \((\Omega, T)\) satisfying the above properties will be called a cyclic space.

We are mainly interested in the case \(\Omega = S\), a subset of a finite group \(G\) closed under conjugation by all \(g \in G\) and \(T\) being associated with the orbits of the action of the flip-over operator \(\sigma\) on \(S \times S\). However, there are different realizations of this simple scheme. In this regard it seems natural to introduce the following definition.

**Definition 8.2** Let \(\mathfrak{h}\) be an algebra and \((\Omega, T)\) a cyclic space. Then, we define a Dunkl representation of \((\Omega, T)\) in \(\mathfrak{h}\) to be a map \(\xi: \Omega \to \mathfrak{h}\) satisfying:

\[
\xi(w_1)\xi(w_2) + \xi(w_2)\xi(w_3) + \cdots + \xi(w_{n-1})\xi(w_n) + \xi(w_n)\xi(w_1) = 0 \quad (8.1)
\]

for every line \(\ell \leftrightarrow (w_1, \ldots, w_n)\) in \(T\). In particular, we always have \(\xi(w)^2 = 0\) for every \(w \in \Omega\). Also, if a line consists of two distinct points \(w_1\) and \(w_2\) only (which is equivalent to saying that \(w_1\) and \(w_2\) commute when \(\Omega \subset G\) as described in the main body of this paper), then (8.1) says that \(\xi(w_1)\) and \(\xi(w_2)\) anti-commute in \(\mathfrak{h}\).

In the main part of this paper, the example of a Dunkl representation in the algebra \(\mathfrak{h} = \mathfrak{so}(P)\) was given. In that example, it happens to be the case that \(\mathfrak{h}\) is actually a \(*\)-algebra.

For example, we can consider the Fano plane, consisting of the 7 imaginary units in the classical non-associative algebra of octonions \(\mathbb{O}\). In this case, \(T\) consists of 21 elements. The elements of \(T\) are the 7 lines \(\ell_{x,y}\) for each of the 7 points \(x\) of the Fano plane plus 14 oriented cycles which correspond to the 7 lines of the Fano plane, each line containing 3 points and taken with both of the two possible cyclic orientations. For this example to work out, one has to prove that for each of the \(7 \cdot 6 = 42\) ordered pairs of points \(x \neq y\) in the Fano plane, there exists a unique oriented cycle \((x,y,w_3)\) in \(T\). For more on the Fano plane and octonions, see [1].

On the other hand, as we shall see below, if we add a couple of simple additional properties, this entire context becomes equivalent to that of the quantum differential calculus as presented in the main part of this paper.

Given a cyclic space \((\Omega, T)\), we can introduce a natural action in the following way. We start with an ordered pair \((x,y) \in \Omega \times \Omega\) with \(x \neq y\) and consider the uniquely determined element \(\ell_{x,y} \in T\) represented by the ordered \(n\)-tuple \((x,y,\omega_2, \ldots, \omega_n)\). Then, for \(n \geq 3\), we define \(x \downarrow y := \omega_2\), while, for \(n = 2\), we define \(x \downarrow y := x\). Finally, for the diagonal elements \((x,x) \in \Omega \times \Omega\), we define \(x \downarrow x = x\). In short, we have defined a function \(\downarrow: \Omega \times \Omega \to \Omega\). We can think that \(y\) ‘acts’ upon \(x\) from the right in the expression \(x \downarrow y\).

**Proposition 8.1** Let \((\Omega, T)\) be a cyclic space. Then, we have this cancelation property: if \(a \downarrow x = b \downarrow x\), then \(a = b\).

**Proof** Let \(w := a \downarrow x = b \downarrow x\). Then, \(\ell_{x,w} \in T\) is uniquely determined. We first consider the case when \(x \neq w\). Therefore, \(\ell_{x,w}\) is represented by the \(n\)-tuple \((x,w,\ldots,w_{n-1},w_n)\) as well as by \((w_n,x,w,\ldots,w_{n-1})\). However, \(a \downarrow x = w\) means
that $\ell_{a,x}$ is represented by $(a, x, w, \ldots)$ for some $m$-tuple. However, the uniqueness of $\ell_{x,w}$ forces $\ell_{x,w} = \ell_{a,x}$. In particular, it follows that $a = w_n$. Similarly, $b \triangleleft x = w$ implies $b = w_n$. Consequently, $a = b$ as desired.

For the case when $x = w$, we have $a \triangleleft x = x$. This forces $a = x$. Similarly, $b \triangleleft x = x$ forces $b = x$ in this case. It follows that $a = b$. \hfill \Box

To every $x \in \Omega$, we can associate a map $(\cdot) \triangleleft x : \Omega \to \Omega$ by mapping $a \in \Omega$ to $a \triangleleft x \in \Omega$. In accordance with the above cancelation property, all these maps are injective. However, since $\Omega$ is finite, they are all actually permutations of $\Omega$. We let $\text{Perm}(\Omega)$ denote the finite group of permutations of the finite set $\Omega$. Let us now assume that a symmetric left cancelation property holds. Namely:

$$(\forall a \in \Omega) \quad a \triangleleft x = a \triangleleft y \Rightarrow x = y.$$ 

This means that the elements of $x \in \Omega$ are faithfully represented by the permutations $(\cdot) \triangleleft x$ of $\Omega$. To this, we shall add the following:

**Non-triviality assumption** Every such permutation $(\cdot) \triangleleft x$ is non-trivial; in other words, for every $x \in \Omega$, there exists at least one $a \in \Omega$, such that $a \triangleleft x \neq a$.

**Remark 8.3** In our context of quantum differential calculus on finite groups, this assumption is not necessarily the case, although it holds in the most interesting examples, where the group $G$ acts faithfully by conjugation on the basis set $S$ of $\Gamma_{\text{inv}}$.

Our next assumption deals with the compatibility between cyclical lines and the defined action. We postulate that:

$$(z \triangleleft x) \triangleleft y = (z \triangleleft y) \triangleleft (x \triangleleft y) \quad (8.2)$$

for every $x, y, z \in \Omega$.

**Remark 8.4** Algebraically, this equation says the right action $\triangleleft y$ distributes over the binary operation $\triangleleft$. Geometrically, it says that the ordered sequence of consecutive points $z, x, z \triangleleft x$ on their uniquely determined cyclic line transforms under the right action $(\cdot) \triangleleft y$ into the ordered sequence of consecutive points $z \triangleleft y, x \triangleleft y, (z \triangleleft x) \triangleleft y$ on their uniquely determined cyclic line.

This property trivializes for $x = y$ or $x = z$. On the other hand, in the special case $y = z$, it reduces to:

$$(y \triangleleft x) \triangleleft y = y \triangleleft (x \triangleleft y)$$

for every $x, y \in \Omega$, which is an associativity property for any two elements $x$ and $y$.

**Proposition 8.2** Let $(\Omega, T)$ be a cyclic space that satisfies (8.2). Then, for every cyclic line $\ell \in T$ with $\ell \leftrightarrow (w_1, \ldots, w_n)$ the composition:

$$x \mapsto (x \triangleleft w_k) \triangleleft w_{k+1}$$

of the right actions of any two cyclically consecutive elements $(w_k, w_{k+1})$ is a permutation of $\Omega$ that does not depend on $k$, but only depends on $\ell$. Here, $1 \leq k \leq n$ and $w_{n+1} = w_1$. 

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**Proof** The assertion is trivially true if \( n = 1 \). For \( n \geq 2 \), the statement of the theorem is equivalent to saying that:

\[
(x \downarrow w_k) \downarrow w_{k+1} = (x \downarrow w_1) \downarrow w_2 \tag{8.3}
\]

holds for all \( x \in \Omega \) and all \( 1 \leq k \leq n \). The case \( k = 1 \) is trivial. For \( k = 2 \), we compute:

\[
(x \downarrow w_2) \downarrow w_3 = (x \downarrow w_2) \downarrow (w_1 \downarrow w_2) = (x \downarrow w_1) \downarrow w_2,
\]

where we used first used \( w_3 = w_1 \downarrow w_2 \), and then, the identity (8.2) in the second equality. Then, (8.3) in general follows by an induction that uses (8.2) again.

Let \( G \) be the subgroup of \( \text{Perm}(\Omega) \) generated by all the permutations \( (\cdot) \downarrow x \), where \( x \in \Omega \). That is, \( G := \{ (\cdot) \downarrow x \mid x \in \Omega \} \subseteq \text{Perm}(\Omega) \), and so, \( G \) is a finite group. However, \( \Omega \) can also be viewed via \( x \sim (\cdot) \downarrow x \in G \) as a subset of \( G \). Therefore, we can extend the right action \( \downarrow : \Omega \times \Omega \rightarrow \Omega \) to a right action \( \downarrow : \Omega \times G \rightarrow \Omega \). In particular, for \( x, y, z \in \Omega \), we have \( (z \downarrow x) \downarrow y = z \downarrow (xy) \), where we are using the identifications of \( x \) and \( y \) as the permutations \( (\cdot) \downarrow x \) and \( (\cdot) \downarrow y \), respectively, and therefore taking \( xy \) to mean their product in \( \text{Perm}(\Omega) \).

**Proposition 8.3** Let \( (\Omega, T) \) be a cyclic space that satisfies (8.2) as well as the Non-triviality Assumption. Then, in terms of the above identification, we have:

\[
x \downarrow y = y^{-1}xy
\]

for every \( x, y \in \Omega \). In particular, the set \( \Omega \) is the disjoint union of conjugation classes of \( G \) and \( e \notin \Omega \).

**Proof** We have to prove that:

\[
z \downarrow (y^{-1}xy) = z \downarrow (x \downarrow y)
\]

for every \( x, y, z \in \Omega \). Without a lack of generality, we can replace \( z \) by \( z \downarrow y \), in which case the equality to prove becomes:

\[
z \downarrow (xy) = (z \downarrow y) \downarrow (x \downarrow y)
\]

because of \( (z \downarrow y) \downarrow (y^{-1}xy) = z \downarrow (xy) \). However, this is just a rewritten form of the property (8.2).

We see that \( \Omega \) is invariant under conjugations by elements of \( \Omega \) and, since \( \Omega \) generates \( G \), it follows that \( \Omega \) is invariant under all conjugations by elements in \( G \); that is, it is a disjoint union of conjugation classes of \( G \). The fact that \( e \notin \Omega \) is nothing but another way of expressing the non-triviality assumption for the elements of \( \Omega \).

In particular, the cyclical lines are precisely the projected orbits of the action of the inverse of the canonical flip-over operator \( \sigma^{-1} \) on \( \Omega \times \Omega \).

**Remark 8.5** It is worth observing that the condition (8.2) can be restricted only for those three elements \( x, y, \) and \( z \) belonging to a single line. This would include the
appropriate non-associative structures, like finite Moufang loops, in which every two elements generate an associative subgroup. As explained in [10], such non-associative objects can still be viewed as diagrammatic groups, within a more general framework of diagrammatic categories and collectivity structures.

Appendix B: Operations in \( \Omega(P) \)

We include here for the reader’s convenience the definitions of the four basic operations on \( \Omega(P) = D \otimes \Gamma^\wedge_{\text{inv}} \). In the following for \( \varphi \in \mathfrak{D} \), we use Sweedler’s notation:

\[
\mathfrak{D} \Phi(\varphi) = \varphi^{(0)} \otimes \varphi^{(1)} \in \mathfrak{D} \otimes \mathfrak{A}
\]

for the right co-action \( \mathfrak{D} \Phi \) defined in (2.22).

Then, for \( \psi \otimes \theta, \varphi \otimes \eta \in \Omega(P) = D \otimes \Gamma^\wedge_{\text{inv}} \) with \( \text{deg}(\theta) = k \) and \( \text{deg}(\varphi) = j \), we define their product by:

\[
(\psi \otimes \theta)(\varphi \otimes \eta) := (-1)^{jk}\psi\varphi^{(0)} \otimes (\theta \circ \varphi^{(1)})\eta. \tag{8.4}
\]

This bilinear expression defines a linear map, also called the multiplication, and denoted as \( m_{\Omega(P)} : \Omega(P) \otimes \Omega(P) \to \Omega(P) \).

The \( * \)-operation is defined for \( \text{deg}(\varphi) = j \) and \( \text{deg}(\theta) = k \) by:

\[
(\varphi \otimes \theta)^* := (-1)^{jk}\varphi^{(0)*} \otimes (\theta^* \circ \varphi^{(1)*}).
\]

The differential \( d_P \) in \( \Omega(P) \) is defined for \( \text{deg}(\varphi) = j \) by:

\[
d_P(\psi \otimes \theta) := D(\psi) \otimes \theta + (-1)^{ij}\varphi^{(0)} \otimes \pi(\varphi^{(1)})\theta + (-1)^{ij}\varphi \otimes d^\wedge(\theta),
\]

where \( d^\wedge : \Gamma^\wedge_{\text{inv}} \to \Gamma^\wedge_{\text{inv}} \) is the restriction of the differential \( d^\wedge \) defined on the acceptable algebra \( \Gamma^\wedge \). Also, \( D \) is the complexified de Rham differential.

Finally, there is a right co-action \( \widehat{F} : \Omega(P) \to \Omega(P) \otimes \Gamma^\wedge \) of \( \Gamma^\wedge \) on \( \Omega(P) \) that is explicitly defined by:

\[
\widehat{F}(\varphi \otimes \theta) := \varphi^{(0)} \otimes \theta^{(0)} \otimes \varphi^{(1)}\theta^{(1)}.
\]

Since \( \widehat{\phi} : \Gamma^\wedge \to \Gamma^\wedge \otimes \Gamma^\wedge \) restricts to \( \widehat{\phi} : \Gamma^\wedge_{\text{inv}} \to \Gamma^\wedge_{\text{inv}} \otimes \Gamma^\wedge_{\text{inv}} \), for \( \theta \in \Gamma^\wedge_{\text{inv}} \), we also are using Sweedler’s notation:

\[
\widehat{\phi}(\theta) = \theta^{(0)} \otimes \theta^{(1)} \in \Gamma^\wedge_{\text{inv}} \otimes \Gamma^\wedge_{\text{inv}}.
\]

The right co-action \( \widehat{F} \) extends \( F \) and is a differential, unital, degree zero \( * \)-morphism of graded algebras.

Hence, \( \Omega(P) \) is a graded differential, unital \( * \)-algebra. The differential \( d_P \) satisfies the graded Leibniz rule with respect to the product (8.4) on \( \Omega(P) \), is covariant with respect to the co-action \( \widehat{F} \) and is a \( * \)-morphism. And therefore, \( (\Omega(P), \Gamma^\wedge, \widehat{F}) \) is an hode which extends the QPB \( P = (C^\infty(E), \mathcal{F}(G), F) \).
It is worth mentioning that each of these four operations involves a ‘twisting’ coming from the right co-action $\triangleleft \Phi$. If this co-action is trivial (i.e., $\triangleleft \Phi(\phi) = \phi \otimes 1$ for all $\phi \in \Delta$), then these operations reduce to tensor product formulas, and so, in this particular case, it is correct to think that the structure of the total space is that of a tensor product. However, in general, these operations are not tensor products, and so, the total space is not simply a tensor product despite what definition (2.16) might otherwise suggest.

Appendix C: A technical proof

In this Appendix, we present a rather long technical proof of (2.9) to have a more complete presentation without interrupting the flow of the main body of this paper.

The motivation for considering the expression $\sigma(\eta \otimes \vartheta)$ for both entries being left invariant starts with the observation that the braiding operation $\sigma$ is the flip provided that $\eta$ is left invariant and $\vartheta$ is right invariant, namely $\sigma(\eta \otimes \vartheta) = \vartheta \otimes \eta$ in this case. This property of $\sigma$ is essentially its definition. However, in classical differential geometry, one considers either left invariant forms without ever mentioning right invariant forms (the usual convention) or, on the other hand, only right invariant forms without ever mentioning left invariant forms. Therefore, it becomes a matter of curiosity to understand how $\sigma$ acts in the ‘un-mixed’ case when both forms are left invariant. And the resulting identity then turns out to have its own utility.

We start out by establishing some notation. See Chapter 5 of [19] for more details on this notation and related properties. Throughout, we take $\Gamma$ to be a bicovariant fode over a Hopf algebra $\mathcal{A}$, though the result holds in the more general setting of Chapter 5 of [19]. We let $\{\omega_i \mid i \in I\}$ be a basis of the vector space $\Gamma_{\text{inv}}$ of left invariant forms in $\Gamma$. And we also let $\{\eta_i \mid i \in I\}$ be a basis of the vector space of right invariant forms in $\Gamma$. These bases are related by:

$$\omega_i = \sum_{j \in I} \eta_j R_{ji} \quad \text{and} \quad \eta_j = \sum_{i \in I} \omega_i \kappa(R_{ij})$$

for unique elements $R_{ij} \in \mathcal{A}$. (These identities are inverses of each other.)

Since we are assuming that both $\eta$ and $\vartheta$ are left invariant, we expand them in terms of the basis of $\Gamma_{\text{inv}}$ as $\eta = \sum_{j \in I} \lambda_j \omega_j$ and $\vartheta = \sum_{i \in I} \mu_i \omega_i$, where $\lambda_j, \mu_i \in \mathbb{C}$.

We have this identity for the right adjoint co-action $\text{ad}$ and the right canonical co-action $\triangleleft \Phi$, which are equal when evaluated on left invariant elements, namely that $\text{ad}(\omega_i) = \triangleleft \Phi(\omega_i) = \sum_{j \in I} \omega_j \otimes R_{ji}$. From this, it immediately follows that:

$$\text{ad}(\vartheta) = \vartheta^{(0)} \otimes \vartheta^{(1)} = \sum_i \mu_i \text{ad}(\omega_i) = \sum_{ij} \mu_i \omega_j \otimes R_{ji} = \sum_{ij} \omega_j \otimes \mu_i R_{ji}.$$

Some other identities that we will use are:

$$\omega_i b = \sum_j (f_{ij} \ast b) \omega_j, \quad \eta_i b = \sum_j (b \ast g_{ij}) \eta_j, \quad \phi(R_{ij}) = \sum_k R_{ik} \otimes R_{kj}.$$

Here, $b \in \mathcal{A}$ and $f_{ij}, g_{ij} : \mathcal{A} \to \mathbb{C}$ are a doubly indexed families of linear functionals.
known as the structure representations. (In this context, $f_{ij} = g_{ij}$ a fact we do not need.) Also, the symbol $\ast$ refers to two different convolution products between elements of $\mathcal{A}$ and linear functionals on $\mathcal{A}$.

Here is the derivation of (2.9). We take $\eta, \vartheta \in \Gamma_{\text{inv}}$ and compute using all of the above identities as follows:

$$\sigma(\eta \otimes \vartheta) = \sum_{i} \lambda_{i} \mu_{i} \sigma(\omega_{i} \otimes \omega_{i}) = \sum_{ijl} \lambda_{i} \mu_{i} \sigma(\omega_{l} \otimes \eta_{j} R_{ji})$$

$$= \sum_{ijkl} \lambda_{i} \mu_{l} \sigma(\omega_{i} \otimes (R_{ji} \ast g_{jk}) \eta_{k}) = \sum_{ijkl} \lambda_{i} \mu_{l} \sigma(\omega_{l}(R_{ji} \ast g_{jk}) \otimes \eta_{k})$$

$$= \sum_{ijklmn} \lambda_{i} \mu_{l} \left( f_{lm} \ast (R_{ji} \ast g_{jk}) \right) \sigma(\omega_{m} \otimes \eta_{k})$$

$$= \sum_{ijkl} \lambda_{i} \mu_{l} \left( f_{im} \ast (R_{ji} \ast g_{jk}) \right) \left( \eta_{k} \otimes \omega_{m} \right)$$

$$= \sum_{ijkl} \lambda_{i} \mu_{l} \left( \left( f_{im} \ast R_{ji} \right) \ast g_{jk} \right) \eta_{k} \otimes \omega_{m} = \sum_{ijkl} \lambda_{i} \mu_{l} \left( \eta_{j} \left( f_{im} \ast R_{ji} \right) \otimes \omega_{m} \right)$$

$$= \sum_{ijlm} \lambda_{i} \mu_{l} \eta_{j} \otimes \left( f_{im} \ast R_{ji} \right) \omega_{m} = \sum_{ij} \lambda_{i} \mu_{j} \eta_{j} \otimes \omega_{l} R_{ji}$$

$$= \sum_{ij} \mu_{i} \eta_{j} \otimes \sum_{l} \lambda_{l} \omega_{l} R_{ji} = \sum_{ij} \mu_{i} \eta_{j} \otimes \omega_{l} R_{ji}$$

$$= \sum_{ij} \eta_{j} \otimes \sum_{i} \mu_{i} R_{ji} = \sum_{jk} \omega_{k} \kappa(R_{kj}) \otimes \eta \sum_{i} \mu_{i} R_{ji}$$

$$= \sum_{jk} \omega_{k} \otimes \kappa(R_{kj}) \eta \sum_{i} \mu_{i} R_{ji} = \sum_{jk} \omega_{k} \otimes \sum_{i} \mu_{i} \sum_{j} \kappa(R_{kj}) \eta R_{ji}$$

$$= \sum_{ij} \omega_{k} \otimes \sum_{i} \mu_{i} \left( \eta \circ R_{ki} \right) = \sum_{ij} \omega_{k} \otimes \left( \eta \circ \mu_{i} R_{ki} \right) = \vartheta^{(0)} \otimes (\eta \circ \vartheta^{(1)}).$$

We also used the identity $\vartheta \circ a = \kappa(a^{(1)}) \vartheta a^{(2)}$ and an associativity property of the convolution operations.

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