Model reduction for hybrid systems with state-dependent jumps

Giordano Scarciotti$^1$ and Alessandro Astolfi$^{1,2}$

$^1$Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, UK
$^2$Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma “Tor Vergata”, Via del Politecnico 1, 00133 Roma, Italy

Abstract: In this paper we present a model reduction technique based on moment matching for a class of hybrid systems with state-dependent jumps. The problem of characterizing the steady-state for this class of systems is studied and a result which allows to described the steady-state response of hybrid systems through the use of a hybrid mapping is given. Then a family of hybrid reduced order models which achieve moment matching and are easily parameterizable is provided. The special case of periodic input signals is analyzed and conditions for applying the technique are given for this class. A numerical simulation illustrates the results.

Keywords: Model order reduction; hybrid systems; switching systems.

1. INTRODUCTION

Hybrid systems are mathematical representations that combine behaviors described by continuous-time systems and behaviors described by discrete-time systems, see Goebel et al. [2009]. Many mechanical systems, e.g. walking robots and billiards, are subject to impacts that change the evolution of the system instantaneously, see e.g. Brogliato [1999], Moreau et al. [1988]. Biological systems, in which, for instance, neural impulsive behavior can be coupled with continuous cell dynamics, constitute another area of application, see e.g. Buck [1988], Pikovsky et al. [2001]. Since the problem of coupling continuous-time and discrete-time has been studied and analyzed from different research communities, different terminology and notation have been produced during the years. For instance, classes of hybrid systems are some times referred as hybrid automata, whereas other classes as switching systems, while some others as impulsive differential equations, see e.g. Heinzinger [1996], Liberzon and Morse [1999], Bainov and Simeonov [1989]. While some problems regarding hybrid systems have been recently solved (see Goebel et al. [2009]), some others, such as the problem of model reduction, remain open. The problem of model reduction consists in the construction of simplified mathematical descriptions, with respect to some notion of complexity, which preserve part of the behavior and properties of the original descriptions. While the literature of model reduction for continuous-time and discrete-time systems is rich and the field can be considered mature, see e.g. Antoulas [2005], Glover [1984], Moore [1981], Kimura [1986], Antoulas et al. [1990], Scherpen and Gray [2000], Gray and Verriest [2006], Hinze and Volkwein [2005], Willcox and Peraire [2002], Astolfi [2010], few attempts have been made to solve the problem of model reduction for hybrid systems. The paper Habets and van Schuppen [2002], in which the problem of reduction on polytopes has been addressed, is one of the first contributions in this area. A different approach based on the concept of abstraction has been presented in Mazzi et al. [2008], in which balanced truncation is used for the continuous-time subsystems and pseudo-equivalent location elimination is used for the discrete-time subsystem. Some methods have been proposed to address the problem of model reduction of discrete-time switching systems, see e.g. Gao et al. [2006], Wu and Zheng [2009], Zhang et al. [2008], switching systems of Markovian type, see e.g. Zhang et al. [2003], and switching systems with average dwell time, see e.g. Zhang and Shi [2008], Zhang et al. [2009]. An approach based on the notion of generalized Gramians has been presented in Shaker and Wisniewski [2009, 2011] to address the problem of model reduction of switching systems. Finally, the problem of reducing a hybrid system to a continuous-time model near a periodic orbit has been investigated in Burden et al. [2011].

This paper proposes a technique to solve the problem of model reduction of hybrid system, with linear dynamics and state-dependent jumps, based on the idea of matching the steady-state output response of the system to a selected class of signals. This work is inspired by Scarciotti and Astolfi [2016a], in which the moment matching approach has been extended to inputs produced by signal generators in explicit form, i.e. signals which may not be generated by smooth differential equations. herein, we propose a hybrid characterization of the mapping, coupled with the hybrid signal generator, describing the steady-state response of hybrid systems. We also specialize the results to the case of hybrid systems with periodic jumps (see Scarciotti and Astolfi [2016c] for some early results for this class of hybrid systems). Finally we provide a parameterized family of hybrid systems with state-dependent jumps. The parameters of this family can be easily exploited to set additional properties of the reduced order models, e.g. selecting the eigenvalues for both the continuous-time and
the discrete-time subsystems.

The rest of the paper is organized as follows. In Section 2 we give some preliminaries, we relate the present framework to the one of Scarciotti and Astolfi [2016a] and we formalize the problem addressed in the paper. In Section 3 the problem of model reduction for hybrid systems with state-dependent jumps is solved. In Section 4 the results are specialized to the case of periodic jumps. Section 5 provides a numerical simulation which is used to illustrate the results of the paper. Finally, Section 6 contains some concluding remarks and gives further directions of investigation.

Notation. We use standard notation. \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}_{>0} \) denotes the set of positive real numbers, \( \mathbb{C} \) denotes the set of complex numbers, \( \mathbb{N} \) denotes the set of non-negative integers, \( \mathbb{Z} \) denotes the set of integers and \( \mathbb{R}^{n \times n} \) denotes the set of \( n \times n \) matrices. \( \mathbb{I} \) denotes the identity matrix, \( \mathbb{O} \) denotes the null matrix, \( \mathbb{Z} \) denotes the set of complex numbers with strictly negative real part and \( \mathbb{C}_\pi \) denotes the set of complex numbers with modulus strictly smaller than one.

Given \( k \) matrices \( X_j \), with \( j = 1, \ldots, k \), \( \prod_{j=0}^{k} X_j \) and \( \prod_{j=0}^{k} X_j \) indicate the products \( X_1X_2 \cdots X_k \) and \( X_kX_{k-1} \cdots X_1 \), respectively.

### 2. PRELIMINARIES AND PROBLEM FORMULATION

In this section we introduce a few basic definitions and we formulate the problem addressed in the paper.

#### Definition 1. (Zadeh and Desoer [1963], Kalman et al. [1969]) Let \( x(t) \in \mathbb{R}^n \) be the state of a dynamical system \( \Sigma \). Let \( u(t) \in \mathbb{R}^m \) be the input of \( \Sigma \). Let \( t_0 \) and \( x_0 = x(t_0) \) be the initial time and the initial state, respectively. If there exists a function \( \phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n \) such that

\[
x(t) = \phi(t, t_0, x_0, u),
\]

for all \( t \geq t_0 \), we call equation (1) the representation in explicit form, or the explicit model, of \( \Sigma \).

Assume \( \phi(t, t_0, x_0, u) \) has a continuous derivative with respect to \( t \) for every \( t_0, x_0 \) and \( u \), and there exists a function \( f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \), continuous for each \( t \) over \( \mathbb{R}^n \times \mathbb{R}^m \), such that

\[
\dot{x} = f(x, u).
\]

We call the differential equation (2) the representation in implicit form, or the implicit model, of \( \Sigma \).

#### Definition 2. (Isidori [1995]) If \( x^* \in \mathcal{C} \subset \mathbb{R}^n \) and \( u^* \in \mathcal{U} \subset \mathbb{R} \) are such that

\[
\lim_{t \to \infty} \| \phi(t, t_0, x^*, u^*) - \phi(t, t_0, x^*, u^*) \| = 0
\]

for every \( x^* \in \mathcal{C} \), then the response \( \phi(x, u) \) is called the steady-state response of system \( \Sigma \) to the input \( u^* \).

We stress that the steady-state response is well-defined, under a few hypotheses, for systems in explicit form, i.e. it is not necessary that the steady-state response be the solution of a differential equation.

#### Definition 3. Consider the system \( \Sigma \) with an output mapping \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) and an input described by \( l(\omega) \), with \( \omega(\tau) = \omega(t_0 + \tau \omega) \in \mathbb{R}^n \) and \( \mathbb{I} : \mathbb{R}^n \rightarrow \mathbb{R} \). Suppose that the steady-state response \( \phi_{ss} \) to the input \( l(\omega) \) can be described as the graph of a map \( \pi : \mathbb{R}^r \rightarrow \mathbb{R}^n \). The moment of the system \( \Sigma \) at \( l(\omega) \) is defined as the mapping \( h \circ \pi \).

We can now formulate a general version of the problem of model reduction by moment matching.

#### Problem 1. Given a system \( \Sigma \) of order \( n \) equipped with a steady-state output mapping \( h \circ \pi \) at \( l(\omega) \), the problem of model reduction by moment matching consists in determining a system \( \Sigma' \) of order \( n' < n \) equipped with a steady-state output mapping \( \kappa \circ p \) at \( l(\omega) \) such that \( \kappa(\pi(\omega)) = \kappa(p(\omega)) \) for all \( \omega \in \mathbb{W} \).

For ease of exposition assume that \( t_0 = 0 \) and consider a linear, single-input, single-output, continuous-time, minimal, system described by the equations

\[
\dot{x} = Ax + Bu,

y = Cx,
\]

with \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R} \), \( y(t) \in \mathbb{R} \), \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times 1} \) and \( C \in \mathbb{R}^{1 \times n} \). It is well-known that the steady-state response of system (4) is described by the graph of a specific integral mapping.

#### Theorem 1. (Pavlov et al. [2006], Isidori and Byrnes [2008]) Consider system (4). Assume \( u \) is bounded and piecewise continuous, and that \( \sigma(A) \subset \mathbb{C}_{\pi} \). Then the steady-state response of system (4) is described by the equation

\[
x_{ss}(t) = \int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau.
\]

If in addition \( u = l(\omega) \), then there exists \( \pi \) such that

\[
x_{ss}(t) = \pi(l(\omega)).
\]

Equation (5) is very general (it holds for any bounded piecewise input, not necessarily \( l(\omega) \)), but it does not exploit any additional information regarding how \( u \) is generated. This makes difficult to evaluate the integral, since the whole trajectory of \( u \) has to be known. A first simplification can be achieved considering the class of signal generators in explicit form studied in Scarciotti and Astolfi [2015], namely

\[
\omega(t) = \Lambda(t)\omega(0), \quad u = Lu,
\]

with \( L \in \mathbb{R}^{1 \times n} \) and \( \Lambda(t) \in \mathbb{R}^{n \times n} \) such that \( \Lambda(0) = I \). Under certain assumptions, see Scarciotti and Astolfi [2016a], the steady-state response of system (4) interconnected with (6) can be described by \( x_{ss}(t) = \Pi(t)\omega(t) \) with

\[
\Pi(t) = \left( e^{A\Pi(0)} + \int_{0}^{t} e^{A(t-\tau)}BLA(\tau)d\tau \right) \Lambda(t)^{-1}.
\]
for a certain $\Pi(0) \in \mathbb{R}^{n \times \nu}$, or with
\[
\dot{\Pi}(t) = A\Pi(t) + BL - \Pi(t) \dot{A}(t) \Lambda(t) - 1,
\] (8)
for each $t \in T \subset \mathbb{R}_{>0}$ in which $\Lambda$ is differentiable. However, additional assumptions are needed to compute the initial conditions $\Pi(0)$ of (7) or (8) at the beginning of each of the intervals in which $\Lambda(t)$ is differentiable. The selection of the initial conditions $\Pi(0)$ is simplified when periodic signals, which are of special interest for applications (see Scarciotti and Astolfi [2016b]), are considered. In this case $\Lambda(t) = \Lambda(t - T)$ for all $t \geq T$ and $\Pi$ is given by
\[
\Pi(t) = \left( I - e^{-AT} \right)^{-1} \left[ \int_{T-t}^T e^{A(t-\tau)} BLA(\tau) d\tau \right] \Lambda(t) - 1.
\] (9)

With this characterization of moment, a family of reduced order models for the linear system (4) at periodic input signals generated by (6) can be defined. As hinted already in Scarciotti and Astolfi [2016a] and stressed in Scarciotti and Astolfi [2016c], this characterization allows to solve the problem of model reduction of linear systems at discontinuous input signals generated by hybrid systems with periodic jumps. Note also that an alternative formulation based on hybrid output regulation, but still inspired by the results of Scarciotti and Astolfi [2016a], has been given in Galeani and Sassano [2015]. However, when also the system to be reduced is hybrid it is preferable to give a characterization of $\Pi$ which is hybrid itself (this would allow to define a reduced order model that is hybrid). Moreover, if the discontinuous input signal is generated by a hybrid system which does not jump periodically, then equation (9) cannot be used at all and we have to fold back using equation (7), which has the additional problem of the determination of the initial conditions on each interval $T$.

It seems therefore natural to investigate the problem of determining these initial conditions exploiting the information that we are considering a hybrid generator. Hence, in the remaining of the paper we address the problem of model reduction by moment matching for linear hybrid systems. We first study the case of systems with state-dependent jumps. Then we show how the case of periodic jumps can be solved as a special case of the obtained results.

3. MODEL REDUCTION FOR HYBRID SYSTEMS WITH STATE-DEPENDENT JUMPS

Consider a linear, single-input, single-output, minimal, hybrid system described by the equations
\[
\begin{align*}
\dot{\omega}(t) &= S \omega(t), \quad \text{if } \omega(t) \in C(t), \\
\omega^+(t) &= J \omega(t), \quad \text{if } \omega(t) \in D(t),
\end{align*}
\] (10)
\[
\begin{align*}
u_c(t, k) &= L_c \omega(t), \\
u_d(t, k) &= L_d \omega(t),
\end{align*}
\]
with $\omega(t, k) \in \mathbb{R}^n$, $u_c(t, k) \in \mathbb{R}$, $u_d(t, k) \in \mathbb{R}$, $S \in \mathbb{R}^{n \times \nu}$, $J \in \mathbb{R}^{\nu \times \nu}$, $L_c \in \mathbb{R}^{1 \times \nu}$, $L_d \in \mathbb{R}^{1 \times \nu}$, $C(t)$ is a simply connected time-variant set and $D(t)$ is its time-variant boundary set, i.e. $i$ is such that for any given $t^*$, $D(t^*) = \overline{C(t^*) \setminus C(t^*)}$. We assume that $\omega$, $C(t)$ and $D(t)$ are such that $\omega$ cannot have multiple instantaneous jumps, i.e. once the flow of $\omega$ reaches $\overline{C(t)}$ it jumps according to the second equation in (10) in $C(t)$. The subsequent jumps and flows define an hybrid time domain $H \subset \mathbb{R} \times \mathbb{X}$ described by the union of infinitely many intervals of the form $[t_j, t_{j+1}] \times J$, with $0 = t_0 \leq t_1 \leq \cdots < +\infty$. In general, the jumping times $t_j$ are not known a priori and depend upon the initial condition $\omega_0 = \omega(0,0)$ of system (10). We also introduce the following assumption.

Assumption 1. The trajectories of system (10) are bounded forward and backward in time. The pairs $(L_c, S)$ and $(L_d, J)$ are observable. The matrix $J$ is such that $0 \notin \sigma(J)$.

As a consequence, the signals $u_c$ and $u_d$ are piecewise continuous and bounded.

Consider now a linear, single-input, single-output, minimal, hybrid system flowing and jumping according to $H$ described by the equations
\[
\begin{align*}
\dot{x}_c &= A_c x_c + B_c u_c, \quad \text{if } \omega(t) \in C(t), \\
x_d^+ &= A_x x_d + B_d u_d, \quad \text{if } \omega(t) \in D(t),
\end{align*}
\] (11)
\[
y = C x_c,
\]
with $x_c(t, k) \in \mathbb{R}^n$, $u_c(t, k) \in \mathcal{U}$, $u_d(t, k) \in \mathcal{U}$, $y(t, k) \in \mathbb{R}$, $A_c \in \mathbb{R}^{n \times n}$, $A_d \in \mathbb{R}^{n \times n}$, $B_c \in \mathbb{R}^{n \times 1}$, $B_d \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{n \times 1}$. We are assuming, for simplicity, that the jumps of system (11) are triggered by $\omega$. Note that this assumption can be easily relaxed defining $x$-dependent flow and jump sets for system (11) (with a non-trivial complication of the notation).

Theorem 2. Consider the interconnection of system (11) with the hybrid generator (10). Suppose the zero equilibrium of system (11) is asymptotically stable and that Assumption 1 holds. The steady-state response of the interconnected system is described by $x_{ss} = \Pi \omega$, where $\Pi(t, k) \in \mathbb{R}^{n \times \nu}$ is the unique solution of
\[
\begin{bmatrix}
\dot{\omega} \\
\Pi \\
\omega^+
\end{bmatrix} = \begin{bmatrix}
S \omega \\
A_c \Pi - I S + B_c L_c \\
J \omega
\end{bmatrix}, \quad \text{if } \omega(t) \in C(t),
\] (12)
\[
\begin{bmatrix}
\dot{\omega} \\
\Pi J \\
\omega^+
\end{bmatrix} = \begin{bmatrix}
A_d \Pi + B_d L_d \\
J \omega
\end{bmatrix}, \quad \text{if } \omega(t) \in D(t).
\]

Proof. We begin showing that the solution $\Pi$ of system (12) is well-defined and unique. We first prove that the solution of the first $\Pi$-equation in (12) over the interval $[t_k, t_{k+1}] \times k$ is
\[
\Pi(t, k) = \left( e^{A_c (t - t_k) \Pi_k} + \int_{t_k}^t e^{A_c (\tau - t_k) \Pi_k} B_c L_c e^{S(\tau-t_k) \Pi_k} + e^{-S(\tau-t_k) \Pi_k} \right) e^{-S(\tau-t_k) \Pi_k}.
\] (13)

To this end note that $\Pi(t, k)$ in (13) is always well-defined since the integral contains the product of exponential matrices. In the interval $[t_k, t_{k+1}] \times k$, $\Pi(t, k)$ is differentiable, yielding
\[
\Pi e^{S(t-t_k)} + I S e^{S(t-t_k)} = A_c e^{A_c (t-t_k) \Pi_k} + B_c L_c e^{S(t-t_k)} + A_c \int_{t_k}^t e^{A_c (\tau-t_k) \Pi_k} B_c L_c e^{S(\tau-t_k) \Pi_k} + e^{-S(\tau-t_k) \Pi_k} \right) e^{-S(\tau-t_k) \Pi_k}.
\] (14)

Note that to compute the last two terms in the right-hand side we have used the differentiation under the integral sign formula. Substituting the integral appearing in the last equation with the expression given in (13), yields
\[
\Pi e^{S(t-t_k)} + I S e^{S(t-t_k)} = A_c e^{A_c (t-t_k) \Pi_k} + B_c L_c e^{S(t-t_k)} + A_c \left( \Pi e^{S(t-t_k)} - e^{A_c (t-t_k) \Pi_k} \right) \Pi_k + \Pi + I S = B_c L_c + A_c \Pi,
\] (15)
proving that $\Pi(t,k)$ in (13) is the solution of the first \textit{P}-equation in (12) over the interval $[t_k,t_{k+1}] \times k$. We now show that the initial condition $\Pi(t_k,k)$ in the interval $[t_k,t_{k+1}] \times k$ is unique. Writing (13) for $t = t_{k+1}$, replacing the resulting expression of $\Pi(t_{k+1},k)$ in the right-hand side of the second $\Pi$-equation in (12) and multiplying on the right for $e^{S(t_{k+1}-t_k)}$, yields

$$\Pi_{k+1} = A_d e^{A_k (t_{k+1}-t_k)} \Pi_k + B_d L_d e^{S(t_{k+1}-t_k)} A_d \int_{t_k}^{t_{k+1}} e^{A_k (t_{k+1}-t')} B_d L_d e^{S(t_{k+1}-t')} dt', \quad (16)$$

First of all note that, since $J$ is invertible by hypothesis, given $\Pi_k$, i.e. the initial condition of (13) in the interval $[t_k,t_{k+1}] \times k$, we can compute $\Pi_{k+1}$, i.e. the initial condition of (13) in the interval $[t_{k+1},t_{k+2}] \times k + 1$. Thus, the last problem to address in proving that the solution of system (12) is well-defined and unique is to establish if one of such initial conditions exists. Writing equation (16) for $k = -1$ yields

$$\Pi_0 = A_d e^{A_{-1} (t_0-t_{-1})} \Pi_{-1} e^{-S(t_0-t_{-1})} J^{-1} + B_d L_d J^{-1} + A_d \int_{t_{-1}}^{t_0} e^{A_{-1} (t_0-t')} B_d L_d e^{-S(t_0-t')} dt', \quad (17)$$

Iterating and replacing the expression for $\Pi_{-1}, \Pi_{-2}, \ldots$, we obtain the relation

$$\Pi_0 = \prod_{k=0}^{\infty} \left( A_d e^{A_k (t_0-t_{k-1})} \right) \prod_{k=0}^{\infty} \left( e^{-S(t_0-t_{k-1})} J^{-1} + K \right), \quad (18)$$

with

$$K = B_d L_d J^{-1} + \sum_{j=0}^{\infty} \prod_{k=0}^{j} \left( A_d e^{A_k (t_0-t_{k-1})} \right) \times \left( B_d L_d J^{-1} + A_d \int_{t_{j-2}}^{t_j} e^{A_{j-1} (t_j-t')} B_d L_d e^{-S(t_j-t')} dt' \right) \times \prod_{k=0}^{j} \left( e^{-S(t_0-t_{k-1})} J^{-1} \right) + A_d \int_{t_{-1}}^{t_0} e^{-A_{-1} t'} B_d L_d e^{-S t'} dt'. \quad (19)$$

The second product sequence in (18) is bounded by hypothesis, whereas the first product sequence is converging to zero because the zero equilibrium of system (11) is asymptotically stable. The term $K$ is finite, because is the sum of constant terms multiplied on the left by increasingly longer sequences of $A_d e^{A_k (t_0-t_{k-1})}$ (for which the norm asymptotically converges to zero). As a result,

$$\Pi_0 = K.$$ \hspace{1cm}(20)

This concludes the proof that the solution of system (12) is well-defined and unique. It remains to show that the steady-state $x_{ss}(t,k)$ of the interconnected system can be expressed as $\Pi(t,k)\omega(t,k)$. Consider the equations

$$\begin{align*}
(x - \Pi \omega)^+ &= A_d x + B_d L_d \omega - \Pi \omega - \Pi S \omega \\
&= A_d x + B_d L_d \omega - A_d \Pi \omega + \Pi S \omega - B_d L_d \omega - \Pi S \omega \\
&= A_d (x - \Pi \omega), \quad (21)
\end{align*}$$

The response of the system of equations (20)-(21) can be written explicitly, namely

$$\varepsilon(t,k) = e^{A_{c} (t-t_0)} \prod_{j=0}^{k} \left( A_d e^{A_{c} (t_j-t_{j-1})} \right) (x_0 - \Pi_0 \omega_0).$$

As a consequence, the state response of the interconnection of system (11) with the hybrid generator (10) is

$$x(t,k) = \Pi(t,k) \omega(t,k) + \varepsilon(t,k).$$

Finally, note that the term $\varepsilon$ describes a transient response which decays to zero since the zero equilibrium of system (11) is asymptotically stable, proving the claim that $\Pi(t,k)\omega(t,k)$ describes the steady-state of the system.

Remark 1. From a computational point of view $\Pi_0$ can be approximated with arbitrary precision considering a finite number of terms in equation (19).

Remark 2. That, in case $C(t) \equiv \mathbb{R}^{n \times \nu}$, i.e. the system and the generator are purely continuous-time systems, equation (19) reduces to

$$\Pi_0 = \int_0^\infty e^{-A_{c} \tau} B_d L_d e^{-S \tau} d\tau,$$

which is consistent with equation (5).

Remark 3. The condition $\sigma \left( A_d e^{A_k (t_0-t_{k-1})} \right) \subset \mathbb{D}_{<1}$ for all $[t_k,t_{k+1}] \times k \in \mathcal{H}$ implies that the zero equilibrium of system (11) is asymptotically stable. However, this condition is rather restrictive in the hybrid framework.

Remark 4. The only role of $\omega$ in system (12) is to generate the hybrid time domain over which II is flowing and jumping.

Once we have the description of the steady-state $x_{ss}(t,k)$, the construction of a family of reduced order models of (11) can be easily given.

Proposition 1. Consider system (11) and the signal generator (10). Suppose the zero equilibrium of system (11) is asymptotically stable and that Assumption 1 holds. Then the system described by the equations

$$\begin{align*}
x &\in \mathbb{R}^{n}, \text{ if } \omega \in C(t) \\
\xi &\in \mathbb{R}^{n}, \text{ if } \omega \in C(t) \\
\psi &\in CI\omega(t), \text{ if } \omega \in D(t)
\end{align*}$$

with $\xi(t,k), \psi(t,k) \in \mathbb{R}^{c}, \phi(t,k) \in \mathbb{R}^{c}, G_c \in \mathbb{R}^{c \times 1}, G_d \in \mathbb{R}^{c \times 1}$ and II the unique solution of (12), is a model of system (11) at $\omega$, for any $G_c$ and $G_d$ such that the zero equilibrium of system (22) is asymptotically stable. System (22) is a reduced order model of system (11) at $\omega$ if $n < \nu$.

Proof. Under the hypotheses of the proposition and by Theorem 2, the interconnection of system (22) and the signal generator (10) has a steady-state output response described by $CIP\omega$, with $P$ the unique solution of

$$\begin{align*}
\dot{P} &= (S - G_d L_d) P - PS + G_c L_c \\
P^+ &= (J - G_d L_d) P + G_d L_d.
\end{align*}$$

We easily see that this unique solution has to be $P = I$. As a consequence the steady-state output response $\psi_{ss}$ of
system (22) becomes $C \nu = \omega$, i.e. the family of models (22) solves Problem 1.

**Remark 5.** The vectors $G_c$ and $G_d$ are free parameters and can be used to achieve additional properties for the family of reduced order models (22), see Astolfi [2010].

### 4. THE SPECIAL CASE OF PERIODIC SIGNALS

In this section we specialize the results to the case of hybrid systems in which the input signal is periodic, see Scarciotti and Astolfi [2016c] for some early results. The additional contribution of this section is to simplify the expression of $\Pi$.

Define the hybrid time domain $\mathcal{H}_p := \{(t,k) : t \in [kT, (k+1)T], k \in \mathbb{Z} \}$. Consider a linear, single-input, single-output, minimal, hybrid system flowing and jumping according to $\mathcal{H}_p$ described by the equations

$$
\dot{x} = A_c x + B_c u_c, \\
x^+ = A_d x + B_d u_d, \\
y = C x,
$$

and the signal generator described by the equations

$$
\dot{\omega} = S \omega, \\
\omega^+ = J \omega, \\
u_c = L_c \omega, \\
u_d = L_d \omega,
$$

with $\sigma(Je^{ST}) \subseteq \mathbb{D}_1$. Then Theorem 2 can be simplified as follows.

**Corollary 1.** Consider the interconnection of system (24) with the hybrid generator (25). Assume $\sigma (A_{d}e^{A_c T}) \subseteq \mathbb{D}_{<1}$ and $\sigma(Je^{ST}) \subseteq \mathbb{D}_1$. The steady-state response of the interconnected system is described by $x_{ss} = \Pi \omega$, where $\Pi(t,k) \in \mathbb{R}^{n \times n}$ is the unique solution of

$$
\dot{\Pi} = A \Pi - \Pi S + B_c L_c, 
$$

for all $t \neq kT$, with $k \in \mathbb{Z}$, and with the matrices $\Pi(kT,k)$ given by the unique solution of the Sylvester equation

$$
A_{d}e^{A_c T} \Pi_{kT} - \Pi_{kT} J e^{ST} = -B_d L_d e^{ST} - A_d \int_{0}^{T} e^{A_c ((T-\tau)} B_c L_c e^{ST} d\tau,
$$

where, with abuse of notation, $\Pi(kT,k) = \Pi_{kT}$.

**Proof.** From Theorem 2 we know that the steady-state $x_{ss}$ can be parametrized as $\Pi_0 \omega$. Since the input signal generated by (25) is periodic and the zero equilibrium of system (24) is asymptotically stable, the steady-state $x_{ss}$ is periodic. As a consequence $\Pi$ must be periodic as well, i.e. $\Pi((k+1)T,k+1) = \Pi(kT,k) = \cdots = \Pi(T,T) = \Pi_0$.

Writing equation (16) in the present scenario yields

$$
\Pi_{kT} J e^{ST} = A_{d}e^{A_c T} \Pi_{kT} + B_d L_d e^{ST} + A_d \int_{0}^{T} e^{A_c (T-\tau)} B_c L_c e^{ST} d\tau,
$$

where $\Pi_{kT} J e^{ST}$ has been replaced with $\Pi_{kT}$. Finally, note that equation (28) is a Sylvester equation which has a unique solution because the sets of eigenvalues of $Je^{ST}$ and $A_{d}e^{A_c T}$ are disjoint.

**Remark 6.** Usually, when hybrid systems with periodic jumps are considered the so-called condition of non-resonance i.e. $\sigma (Je^{ST}) \cap \sigma (A_{d}e^{A_c T}) = \emptyset$, plays a role in the definition of the steady-state. However, since in this section the signal generator is assumed to produce periodic trajectories and the system has an asymptotically stable zero equilibrium, the condition of non-resonance is automatically satisfied. In fact, we exclude signal generators which produce decaying signals, because in the problem of model reduction based on the steady-state notion of moment an input signal which decays to zero does not contribute any steady-state information, see Astolfi [2010].

Similarly, Proposition 1 can be easily specialized to the present scenario.

**Corollary 2.** Consider system (24) and the signal generator (25). Suppose $\sigma (A_{d}e^{A_c T}) \subseteq \mathbb{D}_{<1}$ and $\sigma(Je^{ST}) \subseteq \mathbb{D}_1$. Then the system described by the equations

$$
\dot{\xi} = (S - G_c L_c) \xi + G_c u_c, \\
\dot{\psi} = C \Pi \xi(t),
$$

with $\Pi$ the unique solution of (26), is a model of system (24) at $\omega$, for any $G_c$ and $G_d$ such that the zero equilibrium of system (29) is asymptotically stable. System (29) is a reduced order model of system (24) at $\omega$ if $\nu < n$.

### 5. A NUMERICAL EXAMPLE

In this section we present a numerical example to illustrate the results of the paper. Consider the flow set defined as

$$
\mathcal{C}(t) = \left\{ s \in \mathbb{R}_{>0} : s < \sin \left( \frac{3}{2} t \right) + \sin \left( \frac{7}{2} t \right) + 3 \right\}
$$

and the jump set $\mathcal{D}(t)$ as its boundary. Note that the jump set is defined as the sum of two sinussoids with frequencies that are not rational multiple of each other. As a consequence the sum of these two sinussoids, which are obviously periodic, is not itself periodic. Consider the signal generator (10) with matrices $S = 2$ and $J = 0.01$. It is easy to see that each initial condition $\omega_0 \in \mathbb{R}_{>0}$ produce a different hybrid time domain $\mathcal{H}$. Fig. 1 shows the time histories of the variable $\omega(t,k)$ (dotted line) for two
randomly generated initial conditions, $\omega_0 = 0.1953$ (top) and $\omega_0 = 1.266$ (bottom). We can see that the trajectories reach the jump set (solid line) at different times. The matrices of system (11) have been randomly generated with the functions $rss$ (continuous part) and $drrss$ (discrete part) of MATLAB$^TM$. To report the matrices in this paper we have selected $n = 6$ which produced

$$A_c = \begin{bmatrix} -1.280 & 0.953 & 0.080 & 0.646 & 1.207 & 1.152 \\ 0.124 & -1.635 & -0.007 & 0.369 & 1.077 & 1.529 \\ -0.037 & 0.179 & -0.891 & -0.046 & 0.243 & 0.758 \\ -1.146 & -0.634 & -0.591 & -0.490 & 1.255 & 0.706 \\ -0.796 & -1.232 & -0.533 & -1.315 & -0.980 & -0.754 \\ -1.467 & -0.889 & -0.172 & -1.359 & 0.695 & -0.953 \end{bmatrix}.$$
parameters $G_c$ and $G_d$ of the reduced order model achieve a partial recover of the transient behavior.

6. CONCLUSION

We have proposed a hybrid characterization of the mapping describing the steady-state response of the hybrid system with state-dependent jumps. Exploiting this result, we have provided a parameterized family of hybrid systems in which the parameters can be easily exploited to set additional properties of the reduced order models. We have also specialized the results to the case of hybrid systems with periodic jumps. A numerical simulation illustrates the results of the paper. The extension of this technique to hybrid systems in which the continuous-time and discrete-time subsystems are described by nonlinear differential and difference equations represents a natural direction of further investigation.

REFERENCES

A. Antoulas. Approximation of Large-Scale Dynamical Systems. SIAM Advances in Design and Control, Philadelphia, PA, 2005.
A. C. Antoulas, J. A. Ball, J. Kang, and J. C. Willems. On the solution of the minimal rational interpolation problem. Linear Algebra and Its Applications, Special Issue on Matrix Problems, 137-138:511–573, 1990.
A. Astolfi. Model reduction by moment matching for linear and nonlinear systems. IEEE Transactions on Automatic Control, 55(10):2321–2336, 2010.
D. D. Bainov and P. S. Simeonov. Systems with Impulse Effect: Stability, Theory and Applications. Ellis Horwood Limited, Chichester, UK, 1989.
B. Brogliato. Nonsmooth Mechanics. Communications and Control Engineering, Springer-Verlag, London, 1999.
J. Buck. Synchronous rhythmic flashing of fireflies. II. The Quarterly Review of Biology, 63(3):265–289, 1988.
S. Burden, S. Revzen, and S. S. Sastry. Dimension reduction near periodic orbits of hybrid systems. In Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, pages 6116–6121, Dec 2011.
S. Galeani and M. Sassano. Model reduction by moment matching at discontinuous signals via hybrid output feedback. To appear on IEEE Transactions on Automatic Control, 2016.
T. A. Henzinger. The theory of hybrid automata. In Proceedings of the 11th Annual IEEE Symposium on Logic in Computer Science, pages 278–292, Jul 1996.
M. Hinze and S. Volkwein. Proper orthogonal decomposition surrogate models for nonlinear dynamical systems: Error estimates and suboptimal control. In Dimension Reduction of Large-Scale Systems, Lecture Notes in Computational and Applied Mathematics, pages 261–306. Springer, 2005.
A. Isidori. Nonlinear Control Systems. Communications and Control Engineering. Springer, third edition, 1995.
A. Isidori and C. I. Byrnes. Steady-state behaviors in nonlinear systems with an application to robust disturbance rejection. Annual Reviews in Control, 32(1):1–16, 2008.
R. E. Kalman, P. L. Falb, and M. A. Arbib. Topics in mathematical system theory. International series in pure and applied mathematics. McGraw-Hill, 1969.
H. Kimura. Positive partial realization of covariance sequences. Modeling, Identification and Robust Control, pages 499–513, 1986.
D. Liberzon and A. S. Morse. Basic problems in stability and design of switched systems. IEEE Control Systems, 19(5):59–70, Oct 1999.
E. Mazzini, A. S. Vincentelli, A. Balluchi, and A. Birchi. Hybrid system reduction. In Proceedings of the 47th IEEE Conference on Decision and Control, pages 227–232, Dec 2008.
B. C. Moore. Principal component analysis in linear systems: controllability, observability, and model reduction. IEEE Transactions on Automatic Control, 26(1):17–32, 1981.
J. J. Moreau, P. D. Panagiotopoulos, and G. Strang. Topics in nonsmooth mechanics. Birkhäuser Verlag, 1988.
A. Pavlov, N. van de Wouw, and H. Nijmeijer. Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach. Systems & Control: Foundations & Applications. Birkhäuser Boston, 2006.
A. Pikovsky, M. Rosenblum, and J. Kurths. Synchronization: A Universal Concept in Nonlinear Sciences. Cambridge University Press, 2001. Cambridge Books Online.
G. Scarciotti and A. Astolfi. Characterization of the moments of a linear system driven by explicit signal generators. In Proceedings of the 2015 American Control Conference, Chicago, IL, July, pages 589–594, 2015.
G. Scarciotti and A. Astolfi. Model reduction by matching the steady-state response of explicit signal generators. To appear on IEEE Transactions on Automatic Control, 61(7), 2016a.
G. Scarciotti and A. Astolfi. Moment based discontinuous phasor transform and its application to the steady-state analysis of inverters and wireless power transfer systems. IEEE Transactions on Power Electronics (to appear), 2016b.
G. Scarciotti and A. Astolfi. Moments at “discontinuous signals” with applications: model reduction for hybrid systems and phasor transform for switching circuits. In 22nd International Symposium on Mathematical Theory of Networks and Systems, Minneapolis, MN, USA (submitted to), 2016c.
J. M. A. Scherpen and W. S. Gray. Minimality and local state decompositions of a nonlinear state space realization using energy functions. *IEEE Transactions on Automatic Control*, 45(11):2079–2086, Nov 2000.

H. R. Shaker and R. Wisniewski. Switched systems reduction framework based on convex combination of generalized gramians. *Journal of Control Science and Engineering*, 2009.

H. R. Shaker and R. Wisniewski. Generalised gramian framework for model/controller order reduction of switched systems. *International Journal of Systems Science*, 42(8):1277–1291, 2011.

K. Willcox and J. Peraire. Balanced model reduction via the proper orthogonal decomposition. *AIAA Journal*, 40(11):2323–2330, 2002.

L. Wu and W.X. Zheng. Weighted $H_\infty$ model reduction for linear switched systems with time-varying delay. *Automatica*, 45(1):186–193, 2009.

L. A. Zadeh and C. A. Desoer. *Linear system theory: The state space approach*. McGraw-Hill series in System Science. McGraw-Hill, 1963.

L. Zhang and P. Shi. $l_2 - l_\infty$ model reduction for switched LPV systems with average dwell time. *IEEE Transactions on Automatic Control*, 53(10):2443–2448, Nov 2008.

L. Zhang, B. Huang, and J. Lam. $H_\infty$ model reduction of Markovian jump linear systems. *Systems & Control Letters*, 50(2):103–118, 2003.

L. Zhang, P. Shi, E. Boukas, and C. Wang. $H_\infty$ model reduction for uncertain switched linear discrete-time systems. *Automatica*, 44(11):2944–2949, 2008.

L. Zhang, E. Boukas, and P. Shi. $\mu$-dependent model reduction for uncertain discrete-time switched linear systems with average dwell time. *International Journal of Control*, 82(2):378–388, 2009.