Inflation with a massive vector field nonminimally coupled to gravity

J Páramos
Centro de Física do Porto, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal
E-mail: jorge.paramos@fc.up.pt

Abstract. The possibility that inflation is driven by a massive vector field with SO(3) global symmetry nonminimally coupled to gravity is presented. Through an appropriate Ansatz for the vector field, the behaviour of the equations of motion is studied through the ensuing dynamical system, focusing on the characterisation of the ensuing fixed points.

1. Introduction
Inflation is a key ingredient for solving the problems of the Cosmological Standard Model [1, 2, 3, 4, 5]. Most proposals imply that it is driven by a scalar field with a suitable potential; however, the possibility that inflation can result from the dynamics of vector fields is also interesting; this has been approached e.g. in Refs. [6, 7], resorting to a quadratic potential $V(A^\mu) \sim A^\mu A^\mu$, and more recently in Ref. [8] (see also Refs. [9, 10, 11, 12]).

This work describes the dynamics of a vector field first posited in Ref. [7] (see also Refs. [10, 12, 13, 14], but with a nonminimal coupling between the vector field and both the Ricci scalar and Ricci tensor [15]. The latter is also found in the similar context posited by the so-called bumblebee vector models for spontaneous breaking of Lorentz symmetry [16, 17, 18, 19, 20].

2. The Model
We consider the action for an SO(3)-invariant gauge group with a massive vector field nonminimally coupled to the curvature:

$$S = \int d^4 x \sqrt{-g} \left( \frac{R}{2k^2} + \frac{1}{8e^2} \text{Tr}[F_{\mu\nu}F^{\mu\nu}] + \frac{1}{2} \alpha \text{Tr}[A_{\mu}A^\mu] + \frac{\beta}{2} \text{Tr}[R_{\mu\nu}A^\mu A^\nu] \right),$$

where $k^2 = 8\pi G$, $e$ is the gauge coupling, $\alpha$ and $\beta$ are the strengths of the nonminimal couplings between the gauge field and the Ricci scalar and Ricci tensor, respectively [7]. The gauge field strength is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

The variation of the action with respect to the metric yields

$$\frac{1}{2k^2} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = -\frac{1}{k^2} R_{\mu\nu} + g_{\mu\nu} \mathcal{L} - m^2 A_\mu A_\nu - \frac{1}{2e^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \alpha [R_{\mu\nu}(A^\rho A_\rho) + RA_\rho A_\nu - \nabla_\mu \nabla_\nu (A_\rho A^\rho) + g_{\mu\nu} R (A_\rho A^\rho)] + \beta \left[ 2 \nabla_\beta \nabla_\mu (A_\nu) A^\beta - g_{\mu\nu} (\nabla_\alpha \nabla_\beta A^\alpha A^\beta) - \Box (A_\mu A_\nu) - 4 A^\alpha R_{\alpha(\mu} A_{\nu)} \right],$$

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
while the variation with respect to the gauge field gives the vector field equations of motion,
\[ \frac{1}{8c^2} \nabla_\mu (\nabla_\nu A^\nu) + \left( \frac{1}{2} m^2 + \frac{1}{3} \alpha R \right) A_\mu + \beta R_{\mu\nu} A^\nu = 0 \quad . \tag{3} \]

3. Cosmological Dynamics

We adopt the Robertson-Walker metric
\[ ds^2 = -N(t)^2 (dx^0)^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2 \quad , \tag{4} \]
where \( N(t) \) is the lapse function and \( a(t) \) is the scale factor. From the thorough discussion of Refs. [7, 15], it is found that the vector field takes the form
\[ A_0 = 0 \quad , \quad A_i = A_i^a L_a = \chi_0(t) \delta_i^a L_a \quad , \tag{5} \]
with \( \chi_0(t) \) being an arbitrary function and \( L_a \) the generators of the internal \( SO(3) \) group.

The cosmological field equations follow from the substitution of Eqs. (4) and (5) into Eqs. (2) and (3) or, since all constraints are respected, they can also be obtained by replacing the Ansatz Eqs. (4) and (5) into action (1) and discarding the infinite volume of the spatial hypersurface,
\[ S_{\text{eff}} = 3 \int dt \left[ -\frac{a \dot{a}^2}{k^2 N} + \frac{a}{4 N c^2} \left( \frac{\dot{\chi}_0^2}{2} - \frac{N^2}{a^2} V(\chi_0) \right) + \left( \frac{1}{4} N m^2 + \gamma \frac{\dot{a}^2}{N a} \right) \chi_0^2 \right] \quad , \tag{6} \]
where we define the quartic potential \( V(\chi_0) = \chi_0^4 / 8 \) and the composite coupling \( \gamma \equiv \alpha - \beta \), showing that the contribution of the two couplings between the vector field and the curvature have similar dynamical impact; notice that setting \( \beta = \alpha \) implies a vanishing effect from to the aforementioned couplings. In the remainder of this study, we thus consider that \( \gamma \neq 0 \).

Varying the above with respect to \( a(t) \), \( N(t) \) and \( \chi_0(t) \) and setting the gauge \( N(t) = 1 \) yields the Friedmann and Raychaudhuri equations and the equation of motion for the vector field:
\[ 4(a^2 - k^2 \chi_0^2) H^2 = \frac{k^2}{c^2} \left( \frac{\dot{\chi}_0^2}{2} + \frac{V}{a^2} \right) + k^2 m^2 \chi_0^2 \quad , \tag{7} \]
\[ (a^2 - k^2 \gamma \chi_0^2)(\dot{H} + H^2) = -H^2 a^2 + k^2 \left( 2 \gamma \chi_0 \dot{H} + \frac{m^2 \chi_0^2}{4} \right) \quad , \tag{8} \]
\[ \ddot{\chi}_0 + \dot{H} \chi_0 = -\frac{\dot{\chi}_0^3}{2 a^2} + 8 e^2 H^2 \gamma \chi_0 - 2 e^2 m^2 \chi_0 \quad , \tag{9} \]
where \( H = \dot{a}(t)/a(t) \) is the Hubble parameter.

To find general inflationary solutions, we must solve the dynamical system associated with Eqs. (7)–(9) and physically interpret the ensuing critical points. To do so, the following dimensionless variables are introduced,
\[ x = \frac{k \chi_0(t)}{a(t) \sqrt{1 - w^2}} \quad , \quad y = \frac{k^2 \dot{\chi}_0(t)}{2 \sqrt{2(1 - w^2)}} \quad , \quad z = kH \quad , \quad \tau = \frac{t}{k} \quad , \tag{10} \]
with the auxiliary function
\[ w = \sqrt{\gamma \frac{k \chi_0(t)}{a(t)}} = x \sqrt{\frac{\gamma}{1 + \gamma x^2}} \quad . \tag{11} \]
The Friedmann Eq. (7) yields the algebraic constraint
\begin{equation}
    z^2 = y^2 + \frac{1}{32} \frac{x^4}{1 + \gamma x^2} + \frac{1}{4} \mu^2 x^2 ,
\end{equation}
where \(\mu = mk\) is the reduced mass, and only two degrees of freedom remain. We calculate the derivative of the variables \((x, y)\) with respect to the dimensionless time \(\tau\), obtaining
\begin{align*}
    \frac{dx}{d\tau} &= (1 + \gamma x^2) \left[ y - x \sqrt{y^2 + \frac{1 - w^2}{32} x^4 + \frac{1}{4} \mu^2 x^2} \right] , \\
    \frac{dy}{d\tau} &= \frac{-1 - w^2}{4 \sqrt{2}} x^3 - \frac{\mu^2}{\sqrt{2}} x + \frac{4 \gamma}{\sqrt{2}} \left( 2 y^2 + \frac{1 - w^2}{32} x^4 + \frac{1}{4} \mu^2 x^2 \right) x - \\
    &\quad (\gamma x^2 + 2) y \sqrt{y^2 + \frac{1 - w^2}{32} x^4 + \frac{1}{4} \mu^2 x^2} .
\end{align*}
(13)

3.1. Finite critical points

Considering the dynamical system Eq. (13), we first analyse the trivial critical point \(F(0,0)\): the eigenvalues of the Jacobian matrix derived from Eq. (13) at this point are \(\lambda_{\pm} = \pm 2i \sqrt{2} \mu\), indicating that this is a stable critical point, as expected from the vanishing of the vector field.

Eight non-trivial critical points also arise, corresponding to the possible combinations \((\pm \sqrt{X_{\pm}}, \pm \sqrt{Y_{\pm}})\), defining for convenience
\begin{align*}
    X_{\pm} &= \frac{2 + \mu^2 \pm \sqrt{(1 - 16 \gamma)^2 \mu^4 + 4 (8 \gamma + 1) \mu^2 + 4}}{2 \gamma [1 + (8 \gamma - 1) \mu^2]} , \\
    Y_{\pm} &= \frac{1}{12 \gamma (4 \gamma - 1)} \left[ \mu^2 + \frac{2 + (1 + 4 \gamma) \mu^2 X_{\pm}}{8} \right] .
\end{align*}
(14)

Since the dynamical system is invariant under reflections \((x, y) \rightarrow (-x, -y)\), it suffices to analyse those presented in Table 1.

| Table 1. Non-trivial, finite critical points. |
|---------------------------------------------|
| \(x, y\) | \(H\) | \(x, y, z\) | \(H\) |
|---------------------------------------------|
| A \((\sqrt{X_+, Y_+})\) | \(H_+\) | B \((\sqrt{X_-, Y_+})\) | \(H_-\) |
| C \((\sqrt{X_-, Y_-})\) | \(H_+\) | D \((\sqrt{X_-, Y_-})\) | \(H_-\) |

Using the algebraic constraint Eq. (12) to read the Hubble parameter yields Eq. (15)
\begin{equation}
    H_{0\pm}^2 = \frac{2 + (1 + 8 \gamma) (mk)^2}{24 k^2 \gamma (4 \gamma - 1)} \left( 1 \pm \sqrt{1 + \frac{48 (4 \gamma - 1) \gamma (mk)^4}{[2 + (1 + 8 \gamma) (mk)^2]^2}} \right) ,
\end{equation}
(15)
so that the vector field can be read from the definition (11),
\begin{equation}
    A^2 = 4 \left[ (4 \gamma - 1) H_0^2 - m^2 \right] = \frac{8}{k^2 \left[ 12 \gamma + \left( \frac{m}{H_0} \right)^2 \right]} .
\end{equation}
(16)

Following a numerical procedure to evaluate the eigenvalues of the Jacobian matrix evaluated at the critical points \((A, B, C, D)\), one finds that these are saddle points [15]. We also find that the critical points \((C, D, G, H)\) are unphysical, as they lead to an imaginary Hubble parameter.
It does not suffice to require that \((x, y)\) are real-valued for the critical points \((A, B)\) to be physically meaningful: indeed, one must also consider the definition of the dimensionless variables Eq. \((10)\) to ensure that the physical vector field and its time derivative are well defined. This requires that \(1 - w^2 > 0 \rightarrow 1 + \gamma x^2 > 0\), which together with the requirement that \((x, y)\) are real translates into the condition \(\gamma > \frac{1}{4}\).

3.2. Critical points at infinity

![Figure 1. Degenerate eigenvalues of \(S_{\pm}\): real and negative (dark gray), imaginary (light gray).](image)

We now analyse putative critical points found at infinity: in order to do so, we introduce a new radial coordinate and time variable, together with the usual definition of polar angle,

\[
x = \frac{\rho \cos \theta}{1 - \rho}, \quad y = \frac{\rho \sin \theta}{1 - \rho}, \quad \frac{d\zeta}{d\tau} = \frac{1}{(1 - \rho)^2},
\]

(17)

where \(0 \leq \rho \leq 1\). By considering \(\rho = 1\) and approaching the ensuing dynamical system, it is possible to show that three additional fixed points arise \([15]\), depicted in Table 2 together with the eigenvalues of the related Jacobian matrix.

| Point | \(\theta\) | Eigenvalues |
|-------|-----------|-------------|
| \(N\) | \(\frac{\pi}{2}\) | \(\pm \frac{3\sqrt{1+64\gamma}}{2}\) |
| \(S_{\pm}\) | \(\arccos\left(\pm \sqrt{\frac{1}{1 - \frac{\gamma}{2\mu^2}} - \frac{\gamma}{2\mu^2}}\right)\) | \(\pm \frac{8\gamma \sqrt{-\gamma(\gamma \mu^2 + 1)}}{1 + 8\gamma(\mu^2 - 1)}\) |

3.2.1. Critical Point \(N\)  Inspection of Table 2 shows that the critical point \(N(1, \pi/2)\) is a saddle point, if \(\gamma \leq -1/8\); unstable, if \(-1/8 < \gamma < 1/64\); or a focus, if \(\gamma > 64\). Furthermore, the constraint \((12)\) shows that \(H \rightarrow \infty\) in all cases, thus yielding the possibility of a Big Rip.
3.2.2. Critical Points $S_\pm$  
By inspecting Table 2, we may ascertain the behaviour of the critical points $S_\pm$: we impose Eq. \[(12)\] for real critical points and vary the coupling $\gamma$ and reduced mass $\mu$ to determine the behaviour of the corresponding degenerate eigenvalues, as depicted in Fig. 1. We find that the latter are never positive, yielding the constraints
\[
\gamma < 0 \, : \, -\frac{1}{8\mu^2} < \gamma < 0 \, , \quad \Re(\gamma) = 0 \, : \begin{cases} 
\gamma < -\frac{1}{8\mu^2} \vee \gamma > \frac{1}{8(4-\mu^2)} , & \mu < 2 \\
\frac{1}{8(4-\mu^2)} < \gamma < -\frac{1}{8\mu^2} , & \mu > 2 .
\end{cases}
\] (18)

Again resorting to Eq. \[(12)\], we find that $H^2 = -1 /[32(k\gamma)^2] < 0$, so that the fixed point $N$ leads to an unphysical, imaginary expansion rate.

4. Conclusions
In this work, the dynamics of an $SO(3)$-invariant massive vector field \[(7)\] nonminimally coupled to the curvature are presented. The resulting system admits De Sitter solutions for a restricted region of the parameter space, $\gamma > 1/4$. The dynamical system arising from the equations of motion was studied, finding 9 finite critical points and 3 critical points at infinity. Aside from the trivial fixed point corresponding to no vector field, the other 8 non-trivial points are saddles leading to a constant Hubble parameter rate, with only 2 physical. Depending on the value of $\gamma$, it was found that the fixed point at infinity $N$ can behave as a saddle point, an unstable point or a focus — in which case it leads to a Big Rip. The other 2 critical points at infinity $S_\pm$ yield an imaginary Hubble parameter, and can thus be identified with an oscillating universe.

Acknowledgments
The author wishes to thank the organization of CPT and Lorentz Symmetry in Field Theory (Faro, July 2017) for its hospitality, and O. Bertolami for fruitful discussions.

References
[1] Starobinsky A A 1980 Phys. Lett. B 91 99
[2] Guth A 1981 Phys. Rev. D 23 347
[3] Albrecht A and Steinhardt P J Phys. Rev. Lett. 48 1220
[4] Linde A 1982 Phys. Lett. B 108 389
[5] Olive K A 1990 Physics Reports 190 307
[6] Ford L H 1989 Phys. Rev. D 40 967
[7] Bento M C, Bertolami O, Moniz P V, Mourão J M and Sá P M 1993 Class. Quantum Gravity 10 2
[8] Koivisto T S and Mota D F 2008 JCAP 0806 021
[9] Tartaglia A and Radicella N 2007 Phys. Rev. D 76 083501
[10] Golovnev A, Mulhanov V and Vanchurin V 2008 JCAP 0806 009
[11] Golovnev A and Vanchurin V 2009 Phys. Rev. D 79 103524
[12] Esposito-Farese G, Pitrou C and Uzan J P 2010 Phys. Rev. D 81 063519
[13] Bertolami O and Mota D F 1999 Phys. Lett. B 455 96
[14] Turner M S and Widrow L M 1988 Phys. Rev. D 37 2743
[15] Bertolami O, Bessa V and Páramos J 2016 Phys. Rev. D 93 064002
[16] Kostelecký V A 2004 Phys. Rev. D 69 105009
[17] Bluhm R and Kostelecký V A 2005 Phys. Rev. D 71 065008
[18] Jacobson T 2007 PoS QG-PH 020
[19] Bertolami O and Páramos J 2005 Phys. Rev. D 72 044001
[20] Capelo D and Páramos J 2015 Phys. Rev. D 91 104007