Ultralimits, Amenable actions and Entropy*

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Abstract

In this paper we show that the minimal value of Furstenberg entropy (along all measures, not restricting to stationary ones) for any amenable action is the same as for the action of the group on itself. Using the boundary amenability result of [1], this allows us to compute the minimal value of the entropy over all the measure classes in the boundary of the free group. Similar results are proved for the action of a hyperbolic group on its Gromov boundary. Our main tool is an ultralimit realization of the Poisson boundary of a time dependent matrix-valued random walk on the group. This extends and refines the results and tools of [22].

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1 Introduction

In this paper we continue the study initiated in [22] on the problem of entropy minimizing. We will relate this problem in the case of amenable actions (see Theorem A) and boundary actions (see Theorems B, C). In particular, we calculate the minimal entropy number for the action of the free group on its boundary (see Theorem D). We will give a new perspective on amenable actions by providing an ultralimit construction for them, starting with the action of the group on itself (see Theorem E). A secondary goal of this paper is to develop further the tool of ultralimits from a measure theoretic point of view. Building on [9], [17], [12] and dealing with unbounded functions. We will also relate it to the construction of ultralimits in [22].

1.1 The problem of minimizing entropy

Let $G$ be a discrete countable group. Throughout this introduction $\lambda$ will be a finitely supported, generating probability measure on $G$ and $f$ will be a convex function on $(0, \infty)$ with $f(1) = 0$.

In [22], we introduced the notion of Furstenberg $(\lambda, f)$-entropy. Namely, for a $G$-space $X$ and a quasi-invariant probability measure $\nu$ on $X$, we define

$$h_{\lambda, f}(X, \nu) := \sum_{g \in G} \lambda(g) D_f(g\nu\|\nu) = \sum_{g \in G} \lambda(g) \int_X f\left(\frac{d\nu}{d\nu}\right) d\nu$$

Our main object of interest is the minimal entropy number defined for a $G$-space $X$ as follows:

$$I_{\lambda, f}(X) = \inf_{\nu \in M(X)} h_{\lambda, f}(X, \nu)$$
Here, by a $G$-space we mean that $X$ is a Borel space equipped with a fixed measure class which is $G$-invariant, $M(X)$ denotes the collection of probability measures on $X$ of this measure class.

One has the following topological variant - let $X$ be a topological $G$-space and let $M_{\text{top}}(X)$ be the collection of all Borel measures. Define

$$I_{\lambda, f}^{\text{top}}(X) = \inf_{\nu \in M_{\text{top}}(X)} h_{\lambda, f}(X, \nu)$$

We note that although $\nu$ is not assumed to be quasi-invariant one can easily adapt the definition for this case (see subsection 4.1).

In [22], we proved that when $G$ is amenable one has $I_{\lambda, f}(G) = 0$. In the present paper we extend this result to amenable actions in the sense of Zimmer (e.g. [26]):

**Theorem A (See Theorem 4.2.1).** Let $G$ be a discrete countable group and let $S$ be an amenable $G$-space. Then for any $\lambda, f$ as above we have:

$$I_{\lambda, f}(S) = I_{\lambda, f}(G)$$

Our proof of Theorem A is based on the ideas of [22], an ultralimit realization of Poisson boundaries of time-dependent matrix-valued random walks, as well as Elliott-Giordano realization of amenable actions as such Poisson boundaries (e.g. [13]).

Theorem A combined with the work of Adams on boundary amenability of the Gromov boundary of hyperbolic groups (e.g. [1]) implies the following:

**Theorem B (See Theorem 4.3.2).** Let $G$ be a hyperbolic group and $\partial G$ be its Gromov boundary. Then for any $\lambda, f$ as above we have:

$$I_{\lambda, f}^{\text{top}}(\partial G) = I_{\lambda, f}(G)$$

Similarly we will get:

**Theorem C (See Theorem 4.3.3).** Let $G$ be a discrete subgroup in a semi-simple Lie group $G$. Let $B \leq G$ be a Borel subgroup and consider the flag space $X = G/B$, then for any $\lambda, f$ as above we have:

$$I_{\lambda, f}^{X}(X) = I_{\lambda, f}(G)$$

We now focus on the free group $F_d$ on $d$ generators $\{a_1, \ldots, a_d\}$. Denote by $\Delta_d$ the set of generating symmetric probability measures supported on $\{a_1^\pm 1, \ldots, a_d^\pm 1\}$. Given a $\mu \in \Delta_d$ we denote by $\nu_\mu$ the $\mu$-harmonic measure on the boundary $\partial F_d$.

Suppose additionally that $f$ is strictly convex and smooth. In [22, Definition 7.4] we introduced a bijection $T : \Delta_d \rightarrow \Delta_d$.

Let $\lambda \in \Delta_d$ and $\mu = T^{-1}(\lambda)$, in [22] we proved $I_{\lambda, f}(F_d) = h_{\lambda, f}(\partial F_d, \nu_\mu)$ and we showed that the measure $\nu_\mu$ on the boundary $\partial F_d$ minimizes $(\lambda, f)$-entropy in its measure class.

A natural question left open in [22] is whether $\nu_\mu$ minimizes the $(\lambda, f)$-entropy among all measures on the boundary, that is, whether $I_{\lambda, f}^{\text{top}}(\partial F_d) = h_{\lambda, f}(\nu_\mu)$. One of the main goals of this paper was to prove this equality:

**Theorem D (See Theorem 4.3.6).** In the notations above, for any $\lambda \in \Delta_d$ we have:

$$I_{\lambda, f}^{\text{top}}(\partial F_d) = h_{\lambda, f}(\partial F_d, \nu_\mu)$$

where $\mu = T^{-1}(\lambda)$. 

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1.2 Ultralimits

1.2.1 An ultralimit construction for amenable actions

Amenability of $G$-spaces was first studied by Zimmer, see [26],[25]. Since then this property was generalized to various settings (see [3]) and studied extensively. In order to prove Theorem A, we provide a new perspective on this notion. For this we use the construction of ultralimit of uniformly bounded quasi-invariant (BQI) $G$-spaces from [22, Definition 4.17]. Based on this, our main tool is an ultralimit construction for (ergodic) amenable actions:

**Theorem E** (See Theorem 3.2.6). *Let $G$ be a discrete countable group, and $S$ is an ergodic amenable $G$-space. Then there is a BQI measure $\nu$ on $S$ (in the measure class) and a sequence of uniformly BQI measures $(\lambda_n)_{n \geq 0}$ on $G \times \mathbb{N}$ such that for any non-principle ultrafilter $U$ on the natural numbers we have an isomorphism between the Radon-Nikodym factors:*

$$(S,\nu)_{RN} \cong U \lim_{RN} (G \times \mathbb{N},\lambda_n)$$

**Remark 1.2.1.** In [16] it was proved that unitary twisted $L^2$-representations induced from amenable actions are weakly contained in the regular representation. This was our motivation to speculate Theorem E.

To prove Theorem E, we first use the result of [13] that represents such ergodic amenable action as generalized Poisson boundary. That is, Poisson boundary of a matrix-valued time dependent random walk on $G$. Such a Poisson boundary is attached to a sequence $\sigma = (\sigma(t))$ consisting of stochastic matrices of measures on $G$. For the classical Furstenberg-Poisson boundary $B(G,\mu)$ we introduced an ultralimit construction [22, Theorem E]. In Theorem 3.1.9 we generalize this construction to the general case. In [22], the verification that the construction coincides with the Furstenberg-Poisson boundary was based on a classical maximality property of the Furstenberg-Poisson boundary among $\mu$-stationary systems. We provide an analogue of the classical theory (of Furstenberg-Glasner [14]) to this general case in Appendix B.

1.2.2 Ultralimits in measure theoretic setting

The tool of ultralimit in the setting of BQI $G$-spaces was introduced in [22] using the tool of ultralimit of $C^*$-algebras. We remark that one could also have used Von-Neumann algebras (equipped with a state) as in [6].

Those approach has the disadvantage that they deal mostly with bounded functions. However, considering actions $G \curvearrowright X$ it is very common to consider unbounded yet integrable functions or even quasi-invariant measure which are not bounded quasi-invariant.

This motivates us to develop the construction of ultralimit from a measure theoretic point of view. The construction of the ultralimit measure space is due to Loeb (see [17]) and we recall it briefly. We will prove a basic theorem about interchanging the order of integral with ultralimits. This is an analogue of Lebesgue-Vitali’s Theorem about uniform integrability.

To state and prove this result, we introduce a filtration on $L^1$ of any probability space by Banach spaces of ‘uniformly-integrable-functions’. The basic properties of those spaces will be established in Appendix A, and we will use this formalism for our Vitali’s type theorem as well as showing that this filtration behaves well under ultralimits (see Proposition 2.3.2).

This is used in the deduction of Theorem A from Theorem E – we need to be able to deal with all the measures in the measure class of $S$, that may not be in the same bounded measure class as $\nu$.
2 The method of ultralimits

In this section we will study the ultralimit of probability spaces and quasi-invariant $G$-spaces. We begin by reviewing the ultralimit construction for probability spaces in subsection 2.1. Next, in subsection 2.2 we prove a Lebesgue-Vitali type theorem about interchanging of integration and ultralimits (see Theorem 2.2.3). For this we appeal to Appendix A where we introduced a filtration of $L^1$ by spaces uniform integrability.

In subsection 2.3 we compare the ultralimit of probability spaces and of Banach spaces attached to these probability spaces.

In subsection 2.4 we consider the equivariant version for this construction.

2.1 The construction of ultralimit of probability spaces

In this section we will study the ultralimit of probability spaces. We follow [9] which is based on [12] and the work of Loeb [17].

Let $I$ be a set, let $\mathcal{U}$ be an ultrafilter on $I$. We let $\mathcal{X}_i = (X_i, \Sigma_i, \nu_i)$ be a collection of probability spaces parameterized by $i \in I$.

In this section we will construct a probability space $\lim \mathcal{X}_i = (X, \Sigma, \nu)$, the ultralimit of $\mathcal{X}_i$.

The underlying set $X$: define $X = \prod_{i \in I} X_i / \mathcal{U}$, that is, the product $\prod_{i \in I} X_i$ divided by the equivalence relation given by:

$$ (x_i) \sim_{\mathcal{U}} (y_i) \iff \{i \in I | x_i = y_i\} \in \mathcal{U} $$

We denote by $[x_i]$ the equivalence class of $(x_i)$.

The $\sigma$-algebra $\Sigma$: for $(A_i)_{i \in I}$ we define

$$ \mathcal{U} \lim A_i = \{[x_i] | x_i \in A, \ \mathcal{U}\text{-a.e.} \} = \{[x_i] | \{i \in I | x_i \in A_i\} \in \mathcal{U}\} $$

An equivalent description is

$$ 1_{\mathcal{U} \lim A_i}([x_i]) = \mathcal{U} \lim 1_{A_i}(x_i) $$

Here, in the right hand side we have used $\mathcal{U} \lim$ for sequences of numbers (e.g. [22, Definition 4.1]).

The collection $\mathcal{A} := \{\mathcal{U} \lim A_i | A_i \in \Sigma_i \}$ forms an algebra on $X$. We define $\Sigma := \sigma(\mathcal{A})$ to be the $\sigma$-algebra generated by $\mathcal{A}$.

The measure $\nu$: our next goal is to define the ultralimit measure.

Define the function $\nu_* : \mathcal{A} \to [0,1]$ by

$$ \nu_*(\mathcal{U} \lim A_i) = \mathcal{U} \lim \nu_i(A_i) $$

Here also, in the right hand side we have used $\mathcal{U} \lim$ for sequences of numbers. As $\mathcal{U} \lim A_i = \mathcal{U} \lim B_i \iff A_i = B, \ \mathcal{U}\text{-a.e.}$ we conclude that $\nu_*$ is well defined. It is easy to see that $\nu_*$ gives rise to a finitely-additive measure of total mass 1 on $\mathcal{A}$.

By Dynkin’s Lemma (see [24, Lemma 1.6]) an extension of $\nu_*$ to a probability measure $\nu : \Sigma \to [0,1]$ is unique if it exists. To show existence, we proceed as follows:

We will say that $N \subset X$ is a null set if for any $\epsilon > 0$ we can find $A \in \mathcal{A}$ with $N \subset A$, $\nu_*(A) < \epsilon$. Define $\mathcal{N}$ to be the collection of null sets. Note that $\mathcal{N}$ is closed under finite unions.

**Lemma 2.1.1.** Let $A^{(r)} \in \mathcal{A}$ and consider $v = \lim_{m \to \infty} \nu_* (\bigcup_{r=1}^m A^{(r)})$. Then there is $A \in \mathcal{A}$ with $\nu_*(A) = v$ and $A^{(r)} \subset A$ for all $r$. 

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Proof. Write $A^{(m)} = \mathcal{U}\lim A^{(m)}_i$ and let $B_i = \bigcup_{r=1}^{m} A_i^{(r)}$ so that $B^{(m)} = \mathcal{U}\lim B^{(m)}_i = \bigcup_{r=1}^{m} A^{(r)}$ and denote $v_m := \nu_*(B^{(m)})$ so that $v_m \uparrow v$. Define

$$T_m = \{ i \in \mathcal{I} \mid |\nu_i(B^{(m)}_i) - v_m| < \frac{1}{m} \}$$

by the definition of $\nu_*$ we have that $T_m \in \mathcal{U}$. For any $i \in \mathcal{I}$ define:

$$m(i) = \sup \{ m \mid i \in \bigcap_{r=1}^{m} T_r \} \in \mathbb{N} \cup \{ +\infty \}, \quad A_i := B_i^{(m(i))} = \bigcup_{m=1}^{m(i)} A_i^{(m)}$$

We will show that $A = \mathcal{U}\lim A_i \in \mathcal{A}$ satisfies the desired properties. For any positive integer $M$ we have

$$\{ i \in \mathcal{I} \mid m(i) \geq M \} = \bigcap_{m=1}^{M} T_m \in \mathcal{U}$$

Hence we get $A^{(M)}_i \in \mathcal{A}$ for $\mathcal{U}$-a.e. $i$. This implies $A^{(M)} \in \mathcal{A}$ for any positive integer $M$. We conclude from this that $v \leq \nu_*(A)$.

Moreover, for each fixed $M$ we have $v(A_i) \leq v_m(i) + \frac{1}{m(i)} \leq v + \frac{1}{M}$ for $\mathcal{U}$-a.e. $i$. Thus $\nu_*(A) \leq v + \frac{1}{M}$ for any positive integer $M$. We conclude $\nu_*(A) = v$ and hence the proof of the lemma.

\[\square\]

Corollary 2.1.2.

1. $\mathcal{N}$ is closed under countable union.
2. $\Sigma := \{ B \subset X \mid \exists A \in \mathcal{A} : A \triangle B \in \mathcal{N} \}$ is a $\sigma$-algebra.
3. The mapping $\nu : \Sigma \to [0,1]$ given by $\nu(B) = \nu_*(A)$, where $A \in \mathcal{A}$ is such that $A \triangle B \in \mathcal{N}$, is well defined. Furthermore, $\nu$ defines a probability measure extending $\nu_*$.

Proof. 1. Let $N^{(m)} \in \mathcal{N}$ and consider $N := \bigcup_{m} N_m$. We show $N \in \mathcal{N}$. Let $\epsilon > 0$, by assumption we have $A^{(m)} \in \mathcal{A}$ such that $N^{(m)} \in \mathcal{N}(A^{(m)})$, $\nu_*(A^{(m)}) \leq \frac{\epsilon}{2}$. Applying Lemma 2.1.1 yields $A \in \mathcal{A}$ such that $A^{(m)} \in \mathcal{A}$ and $\nu_*(A) = \lim_{m} \nu_*(\bigcup_{r=1}^{m} A^{(r)}) \leq \epsilon$. Thus $N \in \mathcal{A} \subseteq \mathcal{A}$, $\nu_*(A) \leq \epsilon$.

2. It is obvious that $\Sigma$ is an algebra. In view of item 1, it is enough to show that for $A^{(n)} \in \mathcal{A}$ we have $\bigcup_{n=1}^{m} A^{(n)} \in \Sigma$.

Using Lemma 2.1.1, we can find $A \in \mathcal{A}$ such that: $A \triangle (\bigcup A^{(n)}) = A \triangle (\bigcup A^{(n)}) \in \mathcal{N}$. Indeed, $A \triangle (\bigcup A^{(n)}) \subset A \triangle (\bigcup_{r=1}^{m} A^{(r)})$ and $\nu_*(A \triangle (\bigcup_{r=1}^{m} A^{(r)})) = \nu_*(A) - \nu_*(\bigcup_{r=1}^{m} A^{(r)}) \xrightarrow{m \to \infty} 0$.

3. Since $\mathcal{A} \cap \mathcal{N} = \{ A \in \mathcal{A} \mid \nu_*(A) = 0 \}$ we conclude that $\nu$ is well defined. To show that it is $\sigma$-additive, using item 2 and the proof of item 2 it is enough to show that if $A^{(r)}$ are $\nu_*$-almost disjoint and $A$ satisfies Lemma 2.1.1 then $\nu_*(A) = \sum_{r} \nu_*(A^{(r)})$. However this is obvious since $\nu_*(A) = \lim_{m \to \infty} \nu_*(\bigcup_{r \leq m} A^{(r)}) = \lim_{m \to \infty} \sum_{r \leq m} \nu_*(A^{(r)}) = \sum_{r} \nu_*(A^{(r)})$.

\[\square\]

Definition 2.1.3. Defining $\nu = \nu|_{\Sigma}$ and $\mathcal{X} = (X, \Sigma, \nu)$ we have defined the $\mathcal{U}$-ultralimit of the probability spaces $(\mathcal{X}_i)_{i \in \mathcal{I}}$. 

Remark 2.1.4. • The measure space completion of $\mathcal{X}$ is $(X, \Sigma, \nu)$.
The collection of negligible sets (that is, sets contained in a measurable set of measure 0) in $X$ is the $\sigma$-ideal $N$.

To show uniqueness of an extension for $\nu_*$, one can avoid Dynkin’s uniqueness theorem. Clearly item 2 of Corollary 2.1.2 implies this uniqueness.

**Definition 2.1.5.** Let $f_i : X_i \to [-\infty, \infty]$ be functions on $X_i$. Define 

$$ (U \lim_i f_i)([x_i]) := U \lim_i f_i(x_i) $$ 

**Claim 2.1.6.**

- If $f_i$ are Borel measurable functions on $X_i$, then $U \lim f_i$ is Borel measurable function on $X$.
- If for $U$-a.e. $i$ we have $f_i = g_i$, then $U \lim f_i = U \lim g_i$.
- If we have $f_i = g_i$ $\nu_i$-a.e., then $U \lim f_i = U \lim g_i$ $\nu$-a.e.

**Proof.** The first item follows from:

$$ \{ [x_i] \mid (U \lim f_i)([x_i]) > a \} = \{ [x_i] \mid U \lim f_i(x_i) > a \} = \bigcup_m \{ [x_i] \mid i f_i(x_i) > a + \frac{1}{m} \} \in U \} = \bigcup_m \{ x_i \mid f_i(x_i) > a + \frac{1}{m} \} $$

The second and third items are trivial. □

Let us consider the functoriality of the construction:

**Definition 2.1.7.** A factor map between probability spaces $T : (X, \nu) \to (Y, m)$ is a measurable map with $T_*\nu = m$. We will say that $Y$ is a factor of $X$ and that $X$ is an extension of $Y$.

**Lemma 2.1.8.** Let $X_i$, $Y_i$ be probability spaces and let $T_i : X_i \to Y_i$ be factors. Then $T := U \lim T_i$ defined by $T([x_i]) = [T_i(x_i)]$ is a factor $T : U \lim X_i \to U \lim Y_i$.

**Proof.** Note that

$$ T^{-1}(U \lim B_i) = \{ [x_i] \mid T([x_i]) \in U \lim B_i \} = \{ [x_i] \mid \forall i \in \{ x_i \in T_i^{-1}(B_i) \} \in U \} = U \lim T_i^{-1}(B_i) $$

Thus $T$ is measurable. To show $T_*\nu = m$ it is enough to check it on a generating subalgebra:

$$ (T_*\nu)(U \lim B_i) = \nu(T^{-1}(U \lim B_i)) = \nu(U \lim T_i^{-1}(B_i)) = U \lim \nu_i(T_i^{-1}(B_i)) = U \lim \nu_i(B_i) = U \lim m_i(B_i) = m(U \lim B_i) $$

□

The last thing we will do in this subsection is to discuss the general relation between ultralimit and integration. We begin with an analog of Fatou’s lemma:

**Lemma 2.1.9.** Suppose $f_i$ are measurable non-negative functions on $X_i$. Then we have the following inequality (of numbers in $[0, \infty]$):

$$ \int_X U \lim f_i \, d\nu \leq U \lim \int_{X_i} f_i \, d\nu_i $$

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Proof. We may suppose that \( \mathcal{U}\lim f_i, f_i \, d\nu_i < \infty \). Denote \( f = \mathcal{U}\lim f_i \).

It is enough to show that for any \( X\)-simple function \( 0 \leq \varphi \leq f \) we have \( \int_X \varphi \, d\nu \leq \mathcal{U}\lim \int_X f_i \, d\nu_i \).

Write \( \varphi = \sum_{m=1}^{N} c_m \cdot 1_{B^{(m)}} \) where \( c_m \geq 0 \), \( B^{(m)} \in \Sigma \) and let \( K := \sum_{m=1}^{N} c_m \).

By Corollary 2.1.2(2), there are \( A^{(m)} = \mathcal{U}\lim A^{(m)}_i \) with \( A^{(m)}_i \in \Sigma \) so that \( A^{(m)} = B^{(m)} \) are null sets. Define:

\[
\psi := \sum_{m=1}^{N} c_m \cdot 1_{A^{(m)}} = \mathcal{U}\lim \psi_i , \quad \psi_i := \sum_{m=1}^{N} c_m \cdot 1_{A^{(m)}_i}
\]

Note that \( \psi = \varphi \) \( \nu \)-a.e., thus we have \( \psi \leq f \) \( \nu \)-a.e. Hence for any \( m \geq 1 \)

\[
0 = \nu\left( \{ \mathcal{U}\lim f_i < \mathcal{U}\lim \psi_i \} \right) \geq \nu\left( \mathcal{U}\lim \{ f_i - \psi_i < -\frac{1}{m} \} \right) = \mathcal{U}\lim \nu\left( \{ f_i - \psi_i < -\frac{1}{m} \} \right)
\]

This means that for all \( \delta, c > 0 \) we have for \( \mathcal{U}\lim \)-a.e. \( i \) : \( \nu\left( \{ f_i - \psi_i < -c \} \right) < \delta \).

Let \( \epsilon > 0 \) and take \( \delta = \frac{\epsilon}{2K} \), \( c = \frac{\epsilon}{2} \), then for \( \mathcal{U}\lim \)-a.e. \( i \):

\[
\int_{X_i} (f_i - \psi_i) \, d\nu_i \leq \int_{\{f_i - \psi_i < -\frac{\epsilon}{2}\}} |\psi_i| \, d\nu_i + \frac{\epsilon}{2} \leq +\nu\left( \{ f_i - \psi_i < -\frac{\epsilon}{2} \} \right) ||\psi_i||_\infty + \frac{\epsilon}{2} \leq \delta \cdot K + \frac{\epsilon}{2} = \epsilon
\]

Thus we conclude that for every \( \epsilon > 0 \) we have for \( \mathcal{U}\lim \)-a.e. \( i \) that

\[
\int_{X_i} f_i - \psi_i \, d\nu_i \geq -\epsilon
\]

This yields that for any \( \epsilon > 0 \)

\[
\mathcal{U}\lim \int_{X_i} f_i \, d\nu_i - \sum_{m=1}^{N} c_m \nu(B^{(m)}) = \mathcal{U}\lim \left( \int_{X_i} f_i \, d\nu_i - \sum_{m=1}^{N} c_m \cdot \mathcal{U}\lim \nu_i(A^{(m)}_i) \right) \geq -\epsilon
\]

As this is true for any \( \epsilon > 0 \), we conclude

\[
\mathcal{U}\lim \int_{X_i} f_i \, d\nu_i \geq \int_X \varphi \, d\nu
\]

Which is what we wanted to show.

As in classical analysis, ultralimits and integration do not necessarily commute. That is, the equality \( \int_X \mathcal{U}\lim f_i \, d\nu = \mathcal{U}\lim \int_X f_i \, d\nu_i \) does not always hold. Another reflection of this complication is that there can be measures \( \eta_i, \nu_i \) on \( X_i \) such that \( \eta_i \ll \nu_i \) but \( \eta = \mathcal{U}\lim \eta_i \) is not absolutely continuous with respect to \( \nu = \mathcal{U}\lim \nu_i \). The next example illustrates this phenomena.

**Example 2.1.10.**

1. Take \( \mathcal{I} = \mathbb{N} \) and \( \mathcal{U} \) any non-principle ultra-filter, \( X_i = [0, 1] \) with the Lebesgue measure. Let \( f_n = n \cdot 1_{[0, \frac{1}{n}]} \) then \( \int_{X_n} f_n \, d\nu_n = 1 \) but \( \mathcal{U}\lim f_n \equiv 0 \) \( \nu \)-a.e.

2. In the setup of the previous example consider the measures \( \eta_n = \frac{1 + f_n}{n} \cdot \nu_n \). Then \( \eta = \mathcal{U}\lim \eta_n \) is not absolutely continuous with respect to \( \nu = \mathcal{U}\lim \nu_n \). Indeed the set \( E = \mathcal{U}\lim [0, \frac{1}{n}] \) has measure \( \frac{1}{2} \) for \( \eta \) and measure 0 for \( \nu \).

As suggested from ordinary real-analysis, the extra condition required to get equality results is uniform integrability. We study this notion in the next subsection.
2.2 Ultralimit and integration for uniformly integrable functions

We begin with defining the spaces of uniform integrability. This is a filtration of $L^1$ by Banach spaces for any probability space. For their basic properties see Appendix A.

**Definition 2.2.1** (see Definition A.0.1). The space of majorants $M$ is the set of functions $\rho : [0, 1] \to [0, 1]$ such that:

1. $\rho(0^+) = \rho(0) = 0$, $\rho(1) = 1$.
2. $\rho$ is concave: for any $x, y, t \in [0, 1]$ we have $\rho((1-t)x + ty) \geq (1-t)\rho(x) + t\rho(y)$.

Any $\rho \in M$ is continuous, non-decreasing, sub-additive and satisfies $\rho(t) \geq t$. The space $M$ is closed under composition, max and is convex (see Lemma A.0.3).

**Definition 2.2.2.** Let $\mathcal{X} = (X, \Sigma, \nu)$ be a probability space and let $\rho \in M$.

1. We define the normed space $C_\rho(\mathcal{X})$ to be the space of functions (mod a.e. 0 functions) in $L^1(\mathcal{X})$ such that

\[ \|f\|_\rho = \|f\|_{C_\rho(\mathcal{X})} := \sup_{A \in \Sigma : \nu(A) > 0} \frac{1}{\nu(A)} \int_A |f|d\nu < \infty \]

2. Given a probability measures $m$ on $(X, \Sigma)$, we say that $m$ is $\rho$-absolutely continuous with respect to $\nu$ if $m$ is absolutely continuous with respect to $\nu$ and $\|\frac{dm}{d\nu}\|_\rho \leq 1$. We denote this relation by $m \ll \nu$.

$C_\rho(\mathcal{X})$ are Banach spaces that form an exhaustive filtration of $L^1(\mathcal{X})$ – see Lemma A.0.7.

We return to the setup of ultralimit of probability spaces, keeping the notations of subsection 2.1. We are now ready to formulate and prove our version of Lebesgue-Vitali theorem:

**Theorem 2.2.3.** Let $f_i$ are measurable functions on $\mathcal{X}_i$. Assume that there is $\rho \in M$ with $f_i \in C_\rho(\mathcal{X}_i)$ and $\sup_i \|f_i\|_{C_\rho(\mathcal{X}_i)} < \infty$. Then $\mathcal{U}\lim f_i \in L^1(\mathcal{X})$ and

\[ \int_X \mathcal{U}\lim f_i \, d\nu = \mathcal{U}\lim \int_{\mathcal{X}_i} f_i \, d\nu_i \]

**Proof.** We may assume by linearity that $f_i \geq 0$. Denote $f = \mathcal{U}\lim f_i$ and $M = \sup_i \|f_i\|_\rho$. As $\int_{\mathcal{X}_i} f_i \, d\nu_i \leq M$ we get $\mathcal{U}\lim \int_{\mathcal{X}_i} f_i \, d\nu_i \leq M$ and in particular it is finite. Lemma 2.1.9 (Fatou’s lemma) implies that $f \in L^1(\mathcal{X})$ and

\[ \int_X f \, d\nu \leq \mathcal{U}\lim \int_{\mathcal{X}_i} f_i \, d\nu_i \]

It remains to prove the inequality in the second direction. Suppose to the contrary that there is $\epsilon > 0$ with:

\[ \mathcal{U}\lim \int_{\mathcal{X}_i} f_i \, d\nu_i > 3\epsilon + \int_X f \, d\nu \]

Then there is $I_0 \in \mathcal{U}$ such that for all $i \in I_0$ we have:

\[ \int_{\mathcal{X}_i} f_i \, d\nu_i > 3\epsilon + \int_X f \, d\nu \]

Let $\delta > 0$ be such that $\rho(2\delta) < \frac{\epsilon}{2M}$. Let $\psi$ be an $\mathcal{X}$-simple function with:
1. \( \psi = \sum_{m=1}^{N} c_m 1_{A^{(m)}} \) where \( A^{(m)} = \mathcal{U}\lim A_i^{(m)} \), \( c_m \geq 0 \), \( A_i^{(m)} \in \Sigma_i \).

2. \( \nu\text{-a.e.} \ \psi \leq f \).

3. \( f_X(f - \psi) d\nu \leq \frac{\epsilon \delta}{2} \).

Define \( \tilde{\psi}_i = \sum_{m=1}^{N} c_m 1_{A_i^{(m)}} \) so that \( \psi = \mathcal{U}\lim \tilde{\psi}_i \).

By items 1, 2

\[
\int_X f d\nu \geq \int_X \psi d\nu = \mathcal{U}\lim_j \int_{X_j} \psi_j d\nu_j
\]

Thus there is \( I_1 \in \mathcal{U} \) so that for any \( j \in I_1 \):

\[
\int_X f d\nu \geq \int_{X_j} \psi_j d\nu_j - \epsilon
\]

Hence, for \( i \in I_0 \cap I_1 \) we have:

\[
\int_{X_i} f_i d\nu_i > 3\epsilon + \int_X f d\nu \geq 2\epsilon + \int_{X_i} \psi_i d\nu_i \quad (0)
\]

On the other hand, by items 2, 3 and Markov’s inequality:

\[
\mathcal{U}\lim_j \nu_j(\{ f_j - \psi_j > \epsilon \}) = \nu(\mathcal{U}\lim_j \{ f_j - \psi_j > \epsilon \}) \leq \nu(\mathcal{U}\lim_j \{ f_j - \psi_j > \epsilon/2 \}) = \nu(\{ f - \psi > \epsilon/2 \}) \leq \delta
\]

This gives \( I_2 \in \mathcal{U} \) such that for all \( j \in I_2 \):

\[
\nu_j(\{ f_j - \psi_j > \epsilon \}) \leq 2\delta \quad (1)
\]

But since \( \| f_i \|_{\rho} \leq M \) we conclude that for \( i \in I_0 \cap I_1 \cap I_2 \neq \emptyset \):

\[
2\epsilon < \int_{X_i} f_i d\nu_i - \int_{X_i} \psi_i d\nu_i \leq \epsilon + \int_{X_i} f_i d\nu_i \leq \epsilon + \| f_i \|_{\rho} \cdot \rho(2\delta) < \epsilon + M \cdot \frac{\epsilon}{2M} = \frac{3\epsilon}{2}
\]

Which is a contradiction, proving \( \int_X \mathcal{U}\lim f_i d\nu = \mathcal{U}\lim \int_{X_i} f_i d\nu_i \).

\[\square\]

**Corollary 2.2.4.** Let \( \rho \in \text{M} \), \( f_i \in C_{\rho}(X_i) \) and \( \sup_i \| f_i \|_{\rho} < \infty \).
Then \( \mathcal{U}\lim f_i \in C_{\rho}(\mathcal{X}) \) and \( \| \mathcal{U}\lim f_i \|_{\rho} \leq \mathcal{U}\lim \| f_i \|_{\rho} \).

**Proof.** By Theorem 2.2.3, \( f = \mathcal{U}\lim f_i \) lies in \( L^1(\mathcal{X}) \). Let \( A \subset X \) be a measurable set and take \( A_i \subset X_i \) measurable so that \( A \Delta \mathcal{U}\lim A_i \) is a null set (this is possible by Corollary 2.1.2(2)). Then \( \nu\text{-a.e.} \ |f| \cdot 1_A = \mathcal{U}\lim |f_i| \cdot 1_{A_i} \). Using Theorem 2.2.3:

\[
\int_A |f| d\nu = \mathcal{U}\lim \int_{A_i} |f_i| d\nu_i \leq \mathcal{U}\lim \| f_i \|_{\rho} \cdot \rho(\nu(A_i)) = (\mathcal{U}\lim \| f_i \|_{\rho}) \cdot \rho(\nu(A))
\]

\[\square\]
Corollary 2.2.5. Let \( \rho \in \mathcal{M} \) and let \( \eta, \nu \) be probability measures on \((X, \Sigma)\). Consider \((X, \Sigma)\) with the ultralimit measures \( \eta, \nu \). If \( \eta \preccurlyeq \nu \) then \( \eta \preccurlyeq \nu \) and \( \frac{da}{dv} = U \lim \frac{da}{dv} \).

Proof. Let \( f_i = \frac{da}{dv}, f = U \lim f \). Then by Corollary 2.2.4 \( f \in \mathcal{C}_\rho(X, \nu) \) with \( \|f\|_\rho \leq 1 \) and moreover by Theorem 2.2.3 we have \( \int_X f \, d\nu = 1 \). Define \( \eta_0 = f \cdot \nu \) then \( \eta_0 \) is a probability measure on \((X, \Sigma)\) with \( \eta \preccurlyeq \nu \), \( \frac{da}{dv} = f \).

We will show that \( \eta = \eta_0 \) and conclude the corollary. To show this we may show it only on the generating sub-algebra \( \mathcal{A} \). Let \( A \) be a set of the form \( U \lim A \), then by Theorem 2.2.3:

\[
\eta_0(A) = \int_X f \cdot 1_A \, d\nu = \int_X U \lim f_i \cdot 1_{A_i} \, d\nu = U \lim \int_X f_i \cdot 1_{A_i} \, d\nu_i = U \lim \eta_i(A_i) = \eta(A)
\]

\( \square \)

2.3 Comparison with ultralimit of C*-algebras

We now give a comparison between this notion of ultralimit and the notion for C*-probability spaces as in [22]. For the convenience of the reader, we remind the necessary notation:

Given a collection of Banach spaces \((B_i)_{i \in \mathcal{I}}\), we define \( \ell^\infty(\mathcal{I}, \{B_i\}) \) to be the Banach space of bounded sequences in the product of \( B_i \). It has a closed subspace \( \mathcal{N}_U \) consisting of sequences \((\nu_i)\) with \( U \lim \|\nu_i\|_{B_i} = 0 \) ([22, Notation 4.4]). The quotient \( U \lim B_i := \ell^\infty(\mathcal{I}, \{B_i\})/\mathcal{N}_U \) is defined to be the ultralimit of those Banach spaces ([22, Definition 4.6]).

A C* probability space is a pair \((A, \nu)\) of a unital commutative C*-algebra with a faithful state \( \nu \) ([22, Definition 2.11]).

Given a collection of C*-probability spaces \((A_i, \nu_i)_{i \in \mathcal{I}}\), on \( U \lim A_i \) there is a canonical state \( \nu = U \lim \nu_i \). Taking a quotient by the ideal of elements \( a \) with \( \nu(a^*a) = 0 \) we get a C*-probability space denoted by \( U \lim(A_i, \nu_i) \) ([22, Definition 4.11]).

Proposition 2.3.1. Let \( \mathcal{X}_i \) be probability spaces, then we have a canonical isomorphism of C*-probability spaces:

\[
U \lim(L^\infty(\mathcal{X}_i), \nu_i) \cong (L^\infty(U \lim \mathcal{X}_i), \nu)
\]

Proof. First define \( \Phi : \ell^\infty(\mathcal{I}, \{L^\infty(\mathcal{X}_i)\}) \rightarrow L^\infty(U \lim \mathcal{X}_i) \) by \( \Phi((f_i)) = U \lim f_i \). This is a C* homomorphism. It is clear that if \( (f_i) \) are such that \( U \lim \|f_i\| = 0 \) then \( U \lim f_i \) is a.e. 0. Hence \( \Phi \) factors to \( U \lim L^\infty(\mathcal{X}_i) \rightarrow L^\infty(U \lim \mathcal{X}_i) \).

By Theorem 2.2.3 we conclude that \( \Phi^*(\nu) = U \lim \nu_i \), since \( \nu \) is faithful on \( L^\infty(U \lim \mathcal{X}_i) \) we conclude that \( \Phi^* \) factors to an injection \( U \lim(L^\infty(\mathcal{X}_i), \nu_i) \rightarrow (L^\infty(U \lim \mathcal{X}_i), \nu) \).

\( \Phi \) is surjective since its image is closed and contains all simple functions of \( U \lim \mathcal{X}_i \) by Corollary 2.1.2(2). Thus \( \Phi \) is an isomorphism. \( \square \)

The next result shows a compatibility of the Banach spaces \( \mathcal{C}_\rho \) and the operation of taking ultralimit. This is an important tool in the deduction of Theorem A from Theorem E (e.g. see corollary 4.1.14). The proof uses results in Appendix A, in particular Proposition A.0.12 about extracting the \( \rho \)-integrable part out of any integrable function.

Proposition 2.3.2. Let \( \mathcal{X}_i \) be probability spaces and let \( \mathcal{X} = U \lim \mathcal{X}_i \). Then for any \( \rho \in \mathcal{M} \), the mapping \( (f_i) \mapsto U \lim f_i \) induces a continuous linear operator

\[
\Phi_\rho : U \lim \mathcal{C}_\rho(\mathcal{X}_i) \rightarrow \mathcal{C}_\rho(\mathcal{X})
\]
that satisfies $\|\Phi_\rho\| \leq 1$ and commutes with integration, e.g. $\Phi_\rho^*(\nu) = \mathcal{U}\lim\nu_t$.

Moreover,

$$\ker \Phi_\rho = \{(f_i + \mathcal{N}_I) | \mathcal{U}\lim_{X_i} \int f_i \, d\nu_t = 0\}$$

and

$$\overline{\Phi}_\rho : \frac{\mathcal{U}\lim\mathcal{C}_\rho(X_i)}{\ker \Phi_\rho} \to \mathcal{C}_\rho(X)$$

is an isometric isomorphism.

**Proof.** By Corollary 2.2.4, $\Phi_\rho$ is well defined and $\|\Phi_\rho\| \leq 1$. By Theorem 2.2.3, $\Phi_\rho$ commutes with integration. Note that:

$$\Phi_\rho(f_i + \mathcal{N}_I) = 0 \iff \int \mathcal{U}\lim_{X_i} f_i \, d\nu = 0 \iff \mathcal{U}\lim_{X_i} \int f_i \, d\nu_t = 0$$

which shows the description of the kernel.

Let us show that $\overline{\Phi}_\rho$ is an isomorphism. First, we show the following claim:

- Suppose $\lambda \in \mathbf{M}$ and $f \in \mathcal{C}_\rho(X) \cap \text{Im}(\Phi_\lambda)$ then for any $\delta > 0$ there are $f_i \in \mathcal{C}_\rho(X_i)$ with $\|f_i\|_\rho \leq (1 + \delta)\|f\|_\rho$ and $f = \mathcal{U}\lim f_i$. Indeed (by Theorem 2.2.3 applied for $\lambda$):

$$M\rho(\nu(\mathcal{U}\lim B_i)) \geq \int_{\mathcal{U}\lim B_i} |f| \, d\nu = \mathcal{U}\lim_{B_i} \int |g_i| \, d\nu_t \geq \mathcal{U}\lim_{B_i} (1 + \delta)M\rho(\nu_t(B_i)) = (1 + \delta)M\rho(\nu(\mathcal{U}\lim B_i))$$

Thus $\nu(\mathcal{U}\lim B_i) = 0$ as required.

We return to the proof of the Proposition. The claim applied to the case $\lambda = \rho$ yields that $\overline{\Phi}_\rho$ is an isometric embedding. Thus we only need to show $\overline{\Phi}_\rho$ is surjective.

Take $\lambda = \rho^+$ then as we explained $\text{Im}(\Phi_\lambda) \subset \mathcal{C}_\lambda(X)$ is closed (as the image of an isometric embedding) and contains all simple functions. By Lemma A.0.7(4) we conclude it contains $\mathcal{C}_\rho(X)$. Thus by the claim we conclude $\overline{\Phi}_\rho$ is surjective, finishing the proof.

**Corollary 2.3.3.** Let $X_i$ be probability spaces and let $X = \mathcal{U}\lim X_i$.

1. For any $f \in L^1(X)$ there is $\rho \in \mathbf{M}$ such that for all $\delta > 0$ we can find $f_i \in \mathcal{C}_\rho(X_i)$ with $\sup\|f_i\|_\rho \leq (1 + \delta)\|f\|_\rho$ and $f = \mathcal{U}\lim f_i$.

2. If $\eta$ is a probability measure on $(X, \Sigma)$ absolutely continuous with respect to $\nu$ then there is $\rho \in \mathbf{M}$ and probability measures $\eta_i$ on $(X_i, \Sigma_i)$ with $\eta_i \sim \nu$ and $\eta = \mathcal{U}\lim \eta_i$.

**Proof.** Item 1 is immediate from Proposition 2.3.2. For item 2, consider $f = \frac{d\eta}{d\nu}$, by item 1 we can find $\rho_0 \in \mathbf{M}$ and $f_i \in \mathcal{C}_{\rho_0}(X)$ with $\sup\|f_i\|_{\rho_0} \leq 2\|f\|_{\rho_0} =: M$. Define $\rho(t) = \min(1, M \cdot \rho_0(t))$ then $\rho \in \mathbf{M}$ and $\|f_i\|_{\rho} \leq 1$. Thus $\eta_i := f_i \cdot \nu$ satisfies $\eta = \mathcal{U}\lim \eta_i$, $\eta_i \sim \nu$. □
2.4 Equivariant ultralimit

Let $G$ be a discrete countable group. In this section we define ultralimit of $G$-spaces. In the case where the spaces are quasi-invariant (QI) we provide a criterion (so called "uniform QI") for the ultralimit to be QI as well. This is a generalization of the construction given in [22, section 4.4] (see Proposition 2.4.13).

2.4.1 Quasi invariant $G$-spaces

Let us remind the basic notations from [22, Subsection 2.1].

Definition 2.4.1.

- A Borel $G$-space is a Borel space $\mathcal{X} = (X, \Sigma)$ together with a measurable action of $G$ on $(X, \Sigma)$. We denote by $\mathcal{M}(X, \Sigma)$ the collection of probability measures on $X$.
- A quasi invariant $G$-space (QI $G$-space) is a probability measure space $\mathcal{X} = (X, \Sigma, \nu)$ together with a measurable action of $G$ on $X$ such that for all $g \in G$ the measures $g\nu, \nu$ are in the same measure class (that is, has the same null sets).
- The Radon-Nikodym cocyle is defined by: $R_{\nu}(g; x) := \frac{d\nu(g \cdot x)}{d\nu(x)}$.

It is easy to see that $R_{\nu}$ is indeed a cocyle:

$$R_{gh}(x) = R_{h}(g^{-1}x) \cdot R_{g}(x).$$

Recall that $\mathcal{M}$ stands for the space of majorants (see Definition 2.2.1).

Definition 2.4.2. A majorant for a QI $G$-space $\mathcal{X}$ is a function $B : G \to \mathcal{M}$ such that for all $g \in G$ we have $g \cdot \nu \ll \nu$ (see Definition 2.2.2(2)).

We remind the notion of BQI $G$-spaces from [22, Definition 2.1].

Definition 2.4.3. We say that $\mathcal{X}$ is bounded quasi invariant $G$-space (BQI $G$-space) if it has a majorant $B$ of the form $B(g) = \min(1, M(g) \cdot t)$ for a function $M : G \to [1, \infty)$.

Lemma 2.4.4. Let $\mu$ be a probability measure on $G$ and let $B$ be a majorant for the QI $G$-space $\mathcal{X} = (X, \nu)$, then $B(\mu) := \sum_{g \in G} \mu(g) B(g) \in \mathcal{M}$ and $\mu \ast \nu \ll \nu$.

Proof. $B(\mu) \in \mathcal{M}$ by Lemma A.0.3(2), and:

$$(\mu \ast \nu)(A) = \sum_{g} \mu(g)(\nu \ast g\nu)(A) \leq \sum_{g} \mu(g)B(g)(\nu(A)) = B(\mu)(\nu(A))$$

Definition 2.4.5.

1. A factor $p : \mathcal{X} \to \mathcal{Y}$ between two QI $G$-spaces is a factor of probability spaces which is $G$-equivariant. We say $\mathcal{Y}$ is a factor of $\mathcal{X}$ and $\mathcal{X}$ is an extension of $\mathcal{Y}$.

2. A factor $p : \mathcal{X} \to \mathcal{Y}$ is said to be measure preserving if $\frac{d\nu\circ p}{d\nu} = d\nu \circ p$. 
3. The Radon-Nikodym factor of a QI $G$-space $\mathcal{X}$ is the final object in the category of measure preserving factors of $\mathcal{X}$. We denote it by $\mathcal{X}_{RN}$. More concretely, $\mathcal{X}_{RN}$ is equipped with a measure preserving factor $\pi : \mathcal{X} \to \mathcal{X}_{RN}$, and has the property that for any other measure preserving factor $p : \mathcal{X} \to \mathcal{Y}$ there is a unique factor $q : \mathcal{Y} \to \mathcal{X}_{RN}$ so that $\pi = q \circ p$.

**Remark 2.4.6.** The Radon-Nikodym factor of $\mathcal{X}$ indeed exists, it is the underlying QI $G$-space for a topological model of $(X, \Sigma_{RN}, \nu)$ where $\Sigma_{RN} = \sigma(\frac{da}{d\nu} \mid g \in G)$ the $\sigma$-algebra generated by the Radon-Nikodym cocycle. The universal property guarantees that the pair $(\mathcal{X}_{RN}, \pi)$ is unique up to a unique isomorphism.

The following lemma is useful for constructing majorsants to actions:

**Lemma 2.4.7.** Suppose that $\omega$ is a QI measure on $G$ with majorant $B : G \to M$. Suppose that $(X, \Sigma, \nu)$ is measurable $G$-space such that there is a probability measure $\nu_0$ on $(X, \Sigma)$ for which $\nu = \omega \ast \nu_0$. Then $\nu$ is QI with $B$ as a majorant.

**Proof.** Consider the action map $a : G \times X \to X$. Note that $a_*(\omega \times \nu_0) = \nu$, $a_*\left((g\omega) \times \nu_0\right) = g\nu$. By Lemma A.0.14(2) it is enough to show $(g\omega) \times \nu_0 \leq B(g) \omega \times \nu_0$. However, considering the projection $\pi : G \times X \to G$ we have $\frac{d((g\omega) \times \nu_0)}{d(\omega \times \nu_0)} = \pi^*(\frac{da}{d\nu})$. Thus we conclude the lemma by Lemma A.0.14(3).

### 2.4.2 Definition of ultralimit of $G$-spaces

In this subsection, $\mathcal{I}$ is a set and $\mathcal{U}$ is an ultrafilter on $\mathcal{I}$.

**Lemma 2.1.8** yields immediately the following:

**Lemma 2.4.8.** Let $(X_i, \Sigma_i)_{i \in 2}$ be a collection of Borel $G$-spaces and let $\nu_i \in M(X_i, \Sigma_i)$. Consider the ultralimit $\mathcal{U}\lim X_i, \Sigma_i = (X, \Sigma)$ with the ultralimit measure $\nu = \mathcal{U}\lim \nu_i$. Then $(X, \Sigma)$ posses a natural structure of a Borel $G$-space. Moreover, for any $g \in G$ we have that $\mathcal{U}\lim g\nu_i = g\nu$.

**Lemma 2.4.9.** Suppose $(X_i, \Sigma_i)$ are Borel $G$-spaces, let $\nu_i \in M(X_i, \Sigma_i)$ and consider $\nu = \mathcal{U}\lim \nu_i$ the ultralimit measure on $(X, \Sigma)$. Then for any probability measure $\mu$ on $G$ we have:

$$\mathcal{U}\lim \mu \ast \nu_i = \mu \ast \nu$$

**Proof.** Let $\varepsilon > 0$, we have a finite set $S \subset G$ with $\mu(G \setminus S) < \frac{\varepsilon}{2}$. By linearity of $\mathcal{U}\lim$ we conclude $\sum_{g \in G} \mu(g) g\nu = \mathcal{U}\lim \sum_{g \in G} \mu(g) g\nu_i$. Thus $||| \cdot |||$ denotes the total variation of a measure:

$$||| (\mu \ast \nu) - \mathcal{U}\lim (\mu \ast \nu_i) || = ||| (\mu \ast \nu) - \sum_{g \in G} \mu(g) \cdot (g\nu) + \mathcal{U}\lim \sum_{g \in G} \mu(g) (g\nu_i) - \mathcal{U}\lim (\mu \ast \nu_i) ||\leq ||\sum_{g \in G \setminus S} \mu(g) (g\nu)|| + ||\mathcal{U}\lim \sum_{g \in G \setminus S} \mu(g) (g\nu_i)|| \leq 2\mu(G \setminus S) < \varepsilon$$

Taking $\varepsilon \to 0$ we conclude $\mathcal{U}\lim (\mu \ast \nu_i) = \mu \ast \nu$. □

Next, we would like to know when the ultralimit of QI $G$-spaces is QI. For this we have the following definition and proposition:

**Definition 2.4.10.** We say that a collection of QI $G$-spaces is uniformly QI if there is a function $B : G \to M$ that is a common majorant for all of them.
Proposition 2.4.11. Let \((X_i)_{i \in I}\) be uniformly QI G-spaces, then \(\mathcal{U} \text{lim } X_i\) is a QI G-space. Moreover, for any probability measure \(\mu\) on \(G\) we have
\[
\mathcal{U} \text{lim } \frac{d\mu * \nu_i}{d\nu} = \frac{d\mu * \nu}{d\nu}
\]

Proof. Let \(B\) be a uniform majorant for \((X_i)\). Denote \(X = \mathcal{U} \text{lim } X_i\). For any \(g \in G\) we have a measurable map \(g : X_i \to X_i\), and by functionality (Lemma 2.1.8) the induced mapping \(g : \mathcal{U} \text{lim } X_i \to \mathcal{U} \text{lim } X_i\) given by \(g \cdot [x_i] = [g \cdot x_i]\) is measurable. Thus we get a measurable action of \(G\) on \(\mathcal{U} \text{lim } X_i\).

We have by functionality \(g \nu = \mathcal{U} \text{lim } g \nu_i\), which is, by Corollary 2.2.5, \(B(g)\)-absolutely continuous with respect to \(\nu\).

In conclusion, \(X\) is a QI-G-space with \(B\) as a majorant.

For the moreover part, by Lemma 2.4.4 we have \(\mu * \nu_i \ll \nu_i^B(\mu)\) and thus we conclude from Corollary 2.2.5 and Lemma 2.4.9 that \(\mu * \nu = \mathcal{U} \text{lim } (\mu * \nu_i) \ll \nu\) and
\[
\mathcal{U} \text{lim } \frac{d\mu * \nu_i}{d\nu_i} = \frac{d\mathcal{U} \text{lim } (\mu * \nu_i)}{d\nu} = \frac{d\mu * \nu}{d\nu}
\]
as required.

Definition 2.4.12. With the notations above, we will call \(\mathcal{U} \text{lim } X_i\) with the \(G\)-action (which is structure a QI-G-space) the ultralimit of the uniformly QI-G-spaces \(X_i\).

Recall that in [22, Definition 4.17] we defined an ultralimit for a collection of uniformly BQI G spaces, which we denoted by \(\mathcal{U} \text{lim } \text{big } X_i\) and its Radon-Nikodym factor by \(\mathcal{U} \text{lim } \text{RN } X_i\).

Proposition 2.4.13. Suppose \(X_i\) is a collection of uniformly BQI G-spaces. Then:

- \(\mathcal{U} \text{lim } \text{big } X_i\) is a compact Hausdorff space with an isomorphism of BQI-G-C*-spaces:
  \[
  C(\mathcal{U} \text{lim } \text{big } X_i) \cong L^\infty(\mathcal{U} \text{lim } X_i)
  \]
- \(\mathcal{U} \text{lim } \text{RN } X_i\) is the Radon-Nikodym factor of \(\mathcal{U} \text{lim } X_i\).

Proof. Follows immediately from 2.3.1.

3 Realizations via ultralimit

In this section we provide an ultralimit constructions for the Poisson boundary of a time dependent matrix-valued random walk on a discrete countable group \(G\). This is similar to [22, Theorem E]. Then we will review the notion of amenable actions and prove Theorem E.

3.1 Realization of the Poisson boundary of a time dependent matrix-valued random walk as an ultralimit

Let us recall the notations of [10], [13] about the Poisson boundary of a time dependent matrix-valued random walk – the Poisson boundary of matrix-valued random walk on a group. For more details see Appendix B.

Let \(\ell = (\ell_n)_{n \geq 0}\) be a sequence of positive integers, denote \(\ell_{-1} = 1\). Let \(\sigma = (\sigma(n))_{n \geq 0}\) be a \(\ell\)-stochastic...
sequence, that is a sequence where \( \sigma^{(n)} = (\sigma^{(n)}_{i,j})_{\ell_n \times \ell_n} \) is an \( \ell_n \times \ell_n \) stochastic matrix of measures on \( G \) (see Definition B.1.2). Here, \( \ell_n = \{0, \ldots, \ell - 1\} \).

We assume (as in [13]), that for any \( j \in [\ell_n] \), the measure \( \sigma^{(n)}_{0,j} \) is supported on all of \( G \). We also suppose (without loss of generality, see Remark B.1.6) \( \sigma^{(n)} \) has no zero columns.

We denote \( V_n = [\ell_n] \times G \) for \( n \geq 0 \). This is a \( G \)-space with trivial action on the first coordinate. We consider the random walk \((X_n)_{n \geq 0}\) (where \( X_n \) takes values in \( V_n \)) with transition probabilities:

\[
P_{V_{n-1} \rightarrow V_n}(i, g) \mapsto (j, h) = \sigma^{(n)}_{i,j}(g^{-1}h)
\]

We get a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) where \( \Omega = \bigsqcup_{n \geq 0} V_n \) is the space of paths of the random walk and the probability measure \( \mathbb{P} \) on \( \Omega \) is given by the Markov measure with transition probabilities as above and initial distribution \( \nu \)

Also note that \( \sigma^{(n)} \) is a standard \( \sigma \)-stationary \( \sigma \)-system (see Definitions B.1.5, B.1.16 and Lemma B.1.17).

We say that a \( \sigma \)-system \( \mathcal{X} = (X, \Sigma, \nu) \) is regular if the underlying Borel space \((X, \Sigma)\) is a standard Borel space (see Definition B.2.7).

One can see that \((\Omega, \mathcal{A}_\mathcal{G}, (\mathbb{P}^{(m)}))_m\) is a \( \sigma \)-system. A regular \( \sigma \)-system equivalent to it is called the Poisson boundary of \( \sigma \) and denoted by \( \mathcal{B}(G, \sigma) \) (see Definition B.1.9 and Lemma B.1.10).

The Poisson boundary equals its own Radon-Nikodym factor (see Lemma B.1.18).

Another basic property is that an extension of the Poisson boundary is necessarily measure preserving (see Proposition B.2.20).

We now give our ultralimit construction for this Poisson boundary of a time dependent matrix-valued random walk. Fix an ultrafilter \( \mathcal{U} \) on \((\frac{\mathbb{N}}{\mathbb{N}})\) such that \( \lim_{\mathcal{U}} a = 1 \), that is, \( \forall \epsilon > 0 \colon (1 - \epsilon, 1) \in \mathcal{U} \).

**Definition 3.1.1.** Consider the space \( E = \bigsqcup_{n \geq 0} \{n\} \times V_n = \bigsqcup_{n \in [\ell_n]} \{n\} \times [j] \times G \) which is a \( G \)-space by the left multiplication on each copy of \( G \).

Let \( K \) be a non-negative integer. For each \( 0 < a < 1, t \geq -1 \) and \( r \in [\ell_t] \), define a measure \( \nu^{(t)}_{r,a,K} \) on \( E \):

\[
\nu^{(t)}_{r,a,K}(n,j,g) = \frac{1}{a^{t+1+K}} \cdot \mathbf{1}_{n \geq t+1+K} \cdot a^n \cdot \sigma^{(t+1)} \cdot \ldots \cdot \sigma^{(n)}_{r,j}(g)
\]

Since \( \sigma^{(n)} \) are stochastic matrices of measures we get that \( \nu^{(t)}_{r,a,K} \) are probability measures on \( E \).

We have the following basic formula: for any \( t \geq 0, s \in [\ell_{t-1}] \), \( K \geq 0 \) we have

\[
\sum_{r \in [\ell_s]} \sigma^{(t)}_{s,r} \cdot \nu^{(t)}_{r,a,K} = \nu^{(t-1)}_{s,a,K+1}
\]

Also note that

\[
\nu^{(t)}_{r,a,K+1} \leq \frac{1}{a} \nu^{(t)}_{r,a,K}
\]
Lemma 3.1.3. On $\mathcal{U}\lim_a E$ the measure $\mathcal{U}\lim \nu_{r,a,K}^{(t)}$ is independent of $K$.

Proof. From, $\nu_{r,a,K+1}^{(t)} \leq \nu_{r,a,K}^{(t)}$ we conclude that $\mathcal{U}\lim_a \nu_{r,a,K+1}^{(t)} \leq \mathcal{U}\lim_a \nu_{r,a,K}^{(t)}$. As both sides are probability measures, we conclude the equality. \hfill \square

Lemma 3.1.4. For any $K \geq 0$ the collection of measures $\nu_{0,a,K}^{(-1)}$ for $0 < a < 1$ is uniformly QI.

Proof. Moreover, for any $t \geq 0$, $r \in [\ell_t]$, there is $\rho \in M$ so that for all $\frac{1}{2} < a < 1$ we have: $\nu_{r,a,K}^{(t)} \leq \nu_{r,a,K}^{(-1)}$.

Lemma 3.1.5. $\mathfrak{B}_\mathcal{U}(\sigma)$ is the space $\mathcal{U}\lim_a E$ equipped with the measures $\nu_i^{(t)} := \mathcal{U}\lim_a \nu_{i,a,0}^{(t)}$ and $G$-action.

Proposition 3.1.6. $\mathfrak{B}_\mathcal{U}(\sigma)$ is a $\sigma$-system.

Proof. Follows immediately from formula (!), Lemma 3.1.3 and Lemma 2.4.9. \hfill \square

Definition 3.1.7. We denote by $\mathfrak{B}_\mathcal{U}(\sigma)_{RN}$ the Radon Nikodym factor of $\mathfrak{B}_\mathcal{U}(\sigma)$ as a $\sigma$-system (see Lemma B.1.17 for the Radon-Nikodym factor of a $\sigma$-system).

Our next goal is to mimic the proof of [22, Theorem 5.9] for the Poisson boundary of a time dependent matrix-valued random walk.

Proposition 3.1.8. Let $\mathcal{X}$ be a regular $\sigma$-system. Then we have a $\sigma$-system $\mathcal{Y}$ and a diagram:

$$
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\mu} & \mathcal{X} \\
\downarrow^p & & \\
\mathfrak{B}_\mathcal{U}(\sigma)_{RN}
\end{array}
$$

Where $\mu$ is a factor and $p$ is measure preserving extensions.
Proof. Let $\mathcal{X} = (X, m)$. Consider $E \times X$ where the $G$-action is only on the first coordinate. We have the following diagram:

$$
\begin{array}{ccc}
E \times X & \xrightarrow{\mu} & X \\
p & & \downarrow \\
E & & 
\end{array}
$$

where $p$ is the projection, and $\mu((n, i, g), x) = g \cdot x$. Both $\mu, p$ are $G$-equivariant.

For $0 < a < 1$, $t \geq 1, r \in [t_1]$ we define the following probability measure $m^{(t)}_{r,a}$ on $E \times X$:

On $\{n\} \times \{p\} \times G \times X \subset E \times X$ we put $\frac{1}{a^n} \cdot 1_{n \geq t+1} \cdot a^n \cdot (\sigma^{(t+1)} \ast \ldots \ast \sigma^{(n)})_{r,p} \times m_p^{(n)}$.

Then we have:

$$
\mu_*(m^{(t)}_{r,a}) = \sum_{n \geq 0, p \in [t_1]} \frac{1-a}{a^{t+1}} 1_{n \geq t+1} \cdot a^n \cdot \mu_*((\sigma^{(t+1)} \ast \ldots \ast \sigma^{(n)})_{r,p} \times m_p^{(n)}) = \frac{1-a}{a^{t+1}} \sum_{n \geq t+1} a^n m_{r,a}^{(t)} = m_r^{(t)}
$$

On the other hand, we have that $p_* (m^{(t)}_{r,a}) = \nu^{(t)}_{r,a,0}$ and that for any $g \in G$

$$
d \frac{g m_{r,a}^{(t)}}{d m_{t,a}^{(t)}} = d \frac{g \nu_{r,a,0}^{(t)}}{d \nu_{t,a,0}^{(t)}} \circ p
$$

Consider the ultralimits diagram (Lemma 2.1.8) where we get:

$$
(U \lim_n E \times X, U \lim_n m^{(t)}_{r,a}) \xrightarrow{\mu} (U \lim_n X, U \lim_n m_n^{(t)})
$$

Using Lemma 3.1.4 and Corollary 2.2.5 we conclude

$$
d \frac{g U \lim_n m^{(t)}_{r,a}}{d U \lim_n m_n^{(t)}} = d \frac{g U \lim_n \nu^{(t)}_{r,a,0}}{d \nu_{t,a,0}^{(t)}} \circ p
$$

In particular we conclude that $\mathcal{Y} = (U \lim E \times X, (U \lim m^{(t)}_{r,a})_{r=1,\ldots,t_2} \to \mathcal{L}_m \lim m^{(t)}_{r,a})_{t \geq 0}$ give rise to a $\sigma$-system and that $p$ is a measure preserving extension. Thus we got:

$$
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\mu} & (U \lim X, U \lim m) \\
p & & \downarrow \Delta \\
\mathcal{B}(\sigma) & & \mathcal{B}(\sigma)_{RN}
\end{array}
$$

where $\Delta$ is defined using the fact that $X$ is a standard probability space and we have $\Delta : L^\infty(X) \to L^\infty(U \lim X)$ which sends each $U \lim m^{(t)}_{r,a}$ to $m^{(t)}_{r,a}$ [and applying [20, Theorem 2.1]].

Note that $p, q$ (and $\Delta$) are measure preserving extensions, proving the result. □
Finally we obtain the key tool of this paper - the construction of the Poisson boundary of a time dependent matrix-valued random walk in terms of ultralimits:

**Theorem 3.1.9.** The Poisson boundary $B(G, \sigma)$ of $\sigma$ is isomorphic to $\mathcal{B}_U(\sigma)_{RN}$.

**Proof.** Take $X = B(G, \sigma)$ to be the Poisson boundary and apply Proposition B.2.20 the factor $Y \to B(G, \sigma)$ is measure preserving. Since the Poisson boundary is its own Radon Nikodym factor (Lemma B.1.18) and $Y \to \mathcal{B}_U(\sigma)_{RN}$ is measure preserving we conclude:

$$B(G, \sigma) \equiv B(G, \sigma)_{RN} \equiv Y_{RN} \equiv \mathcal{B}_U(\sigma)_{RN}$$

\[\square\]

**Remark 3.1.10.** We now sketch another proof of Theorem [22, Theorem A] without appealing to Furstenberg's conjecture ([15, Theorem 4.3]).

Suppose $G$ is amenable, let $\ell_n = 1$. Consider $\sigma^{(0)}$ which is BQI. We define inductively an exhausting Følner sequence $F_n$. Indeed, take $F_n$ to be a Følner set for $(\varepsilon, S) = (\frac{1}{2}, F_1 \cdots F_{n-1})$ that contains $e$ (here, $F_1 \cdots F_{n-1}$ is the set of products).

Let $\sigma^{(n)} = \frac{1}{|F_n|} \delta_{F_n}$ for $n \geq 0$.

Consider the Poisson boundary $B(G, \sigma) = B,(m^{(n)})$. Note that $\sigma^{(n+1)} = \pi^{(n+1)} = m^{(n)}$ which implies that for $g \in F_1 \cdots F_n$ one has $||g \cdot m^{(n)} - m^{(n)}|| \leq ||g \cdot \sigma^{(n+1)} - \sigma^{(n+1)}|| \leq \frac{1}{2^n}$. Thus $||m^{(n)} - m^{(n)}|| = (g^{(1)} \cdots \sigma^{(n)} \cdot m^{(n)} - m^{(n)}|| \leq \frac{1}{2^n}$. Thus for any $g \in G$, there is $n_0$ so that for $n \geq n_0$ one has: $||m^{(0)} - g \cdot m^{(0)}|| < \frac{1}{2^n}$. In particular $m^{(0)}$ is $G$-invariant and thus $m^{(n)} = \sigma^{(n)} \cdot m^{(0)}$ is $G$-invariant.

In particular considering $\pi : G \times N \to G$ and $\lambda_k = \pi^{(k)}_{0,1} \cdots \pi^{(k)}_{0,1}$ we conclude that $\lambda_k$ are uniformly BQI measures on $G$ such that for any non-principle ultrafilter $U$ we have $U \lim(G, \lambda_k)$ is $G$-invariant.

In particular, the measures $\lambda_k$ are KL-almost invariant (they are also $D_f$-almost invariant for any f-divergence).

### 3.2 Amenable actions and ultralimits

We begin by giving the definition for amenability, we use [4, Section 3] as our main resource. We treat only the case of discrete groups, which simplifies the presentation. However we do not assume that our probability spaces are standard.

**Definition 3.2.1.** Given a factor of QI $G$-spaces $\pi : Y \to X$, a $G$-equivariant conditional expectation is a linear mapping $E : L^\infty(Y) \to L^\infty(X)$ such that:

1. $E$ is $G$-equivariant: for every $f \in L^\infty(Y), g \in G$ we have $E(f \circ g) = E(f) \circ g$.
2. $E$ is non-negative: if $L^\infty(Y) \ni f \geq 0$ then $E(f) \geq 0$.
3. $E$ is normalized: $E(1) = 1$.
4. $E$ is $L^\infty(X)$-linear: for $f \in L^\infty(Y), h \in L^\infty(X)$ we have $E(\pi^*(h) \cdot f) = h \cdot E(f)$.

If such a $G$-equivariant expectation exists, we say that the pair $(Y, X)$ is amenable.

We say that a QI $G$-space $S$ is amenable (in Zimmer’s sense) if the pair $(G \times S, S)$ is amenable. Here, $G \times S$ has the product measure class and diagonal action, and the factor is the projection.

**Remark 3.2.2.** Here, the actual measure on our QI $G$-spaces is immaterial, only its measure class will play a role.
Example 3.2.3. A factor of $\text{QI } G$-spaces $\pi : (Y, m) \to (X, \nu)$ is a measure preserving extension iff the conditional expectation $\mathbb{E}_{m, \pi} : L^\infty(Y) \to L^\infty(X)$ is $G$-equivariant. In particular, suppose $S$ is a $\text{QI } G$-space, then for any measure $\nu$ in the measure class, the pair $(S, T)$ is amenable, where $T = (S, \nu)_{\text{RN}}$ is the RN factor of $(S, \nu)$.

In the case where the $G$-spaces are standard, this notion of amenable action is the one from [13],[2]. This concept is equivalent to Zimmer's: he introduced his notion of amenability in the papers [26],[27],[28] based on the idea of a fixed point property, and proved the equivalence to the definition above. We remind that we consider non-standard probability spaces, so we need to check that some basic properties still hold in this setting.

Lemma 3.2.4. Let $\pi : S \to T$ be a factor of $\text{QI } G$-spaces.

1. If $(S, T)$ is an amenable pair and $S$ is an amenable $G$-space, then $T$ is an amenable $G$-space.

2. If $(S, T)$ is an amenable pair and suppose $X$ is a $\text{QI } G$-space such that there are factors $S \xrightarrow{\pi} X \xrightarrow{\nu} T$ with $\pi = p \circ \tau$. Then $(X, T)$ is an amenable pair.

Proof. 1. Let $E : L^\infty(S) \to L^\infty(T)$ be a $G$-equivariant conditional expectation. As $S$ is amenable, we have a $G$-equivariant conditional expectation $E_S : L^\infty(G \times S) \to L^\infty(S)$. Consider

$$E_T := E \circ E_S \circ (id_G \times \pi)^* : L^\infty(G \times T) \to L^\infty(G \times S) \to L^\infty(S) \to L^\infty(T)$$

it is easy to see that $E_T$ is a $G$-equivariant conditional expectation $L^\infty(G \times T) \to L^\infty(T)$ and thus $T$ is amenable.

2. Let $E : L^\infty(S) \to L^\infty(T)$ be a $G$-equivariant conditional expectation, then:

$$E \circ \tau^* : L^\infty(X) \to L^\infty(S) \to L^\infty(T)$$

is a $G$-equivariant conditional expectation.

The next lemma allows us to reduce to the case of standard probability spaces in problems of amenability.

Lemma 3.2.5. Let $(S, \nu)$ be a $\text{QI } G$-space and consider the Radon-Nikodym factor $T = (S, \nu)_{\text{RN}}$. Then $S$ is amenable iff $T$ is amenable.

Proof. If $S$ is amenable, by Lemma 3.2.4 and Example 3.2.3 we conclude that $T$ is amenable. For the other direction, we need to define $\Phi : L^\infty(G \times S) \to L^\infty(S)$. Let $(\alpha, \pi_\alpha)_{\alpha \in I}$ be the set of measure preserving factors of $(S, \nu)$ that are standard probability spaces (up to isomorphism). $I$ has a partial order $\leq$ defined by $\alpha \leq \beta \iff \pi_\alpha^* L^\infty(S) \subset \pi_\beta^* L^\infty(S)$. It is clear that $(I, \leq)$ is directed. Applying [2, Corollary C] for $S_\alpha \to T = (S, \nu)_{\text{RN}}$ and the amenability of $T$ we conclude that $S_\alpha$ is amenable for every $\alpha \in I$. Thus, there is a $G$-equivariant conditional expectation $\Phi_\alpha : L^\infty(G \times S_\alpha) \to L^\infty(S_\alpha)$. Using $\pi_\alpha$ we consider $L^\infty(S_\alpha), L^\infty(G \times S_\alpha)$ naturally as sub-algebras $L^\infty(S), L^\infty(G \times S)$ respectively. Extending $\Phi_\alpha$ by zero we obtain a function $\Phi_\alpha : L^\infty(G \times S) \to L^\infty(S)$. Note that $\Phi_\alpha$ defines an element of

$$E := \prod_{f \in L^\infty(G \times S)} [T^f]_{L^\infty(S)}(\|f\|)$$
We give $E$ the product topology of the $w^*$-topology. By Tychonoff\'s theorem we conclude that $E$ is compact. So that the net $(\Phi_\alpha)_{\alpha \in I}$ has a sub-net converging to $\Phi \in E$.

Note that for any $f \in L^\infty(G \times S)$ there is $\alpha_0 \in I$ so that for every $\alpha \geq \alpha_0$ we have $f \in L^\infty(S, \nu)$. Indeed, consider the $\sigma$-algebra generated by the Radon-Nikodym cocycle of $\nu$ (that is the functions $\frac{d\nu}{d\nu}(s)$ and $f(g \cdot h \cdot s)$ for $g, h \in G$). This $\sigma$-algebra is separable and $G$-invariant thus corresponds to $S_{\alpha_0}$ for some $\alpha_0 \in I$, which satisfies the required property.

This easily implies that $\Phi : L^\infty(G \times S) \to L^\infty(S)$ is a $G$-equivariant conditional expectation. 

Let us prove Theorem E (we are using Proposition 2.4.13 to translate it to the measurable ultralimit construction):

**Theorem 3.2.6.** Let $S$ be an ergodic amenable $G$-space. Then there is a BQI measure $\nu$ on $S$ (in the measure class) and a sequence of uniformly BQI measures $\lambda_n$ on $G \times \mathbb{N}$ such that for any non-principle ultrafilter $U$ on $\mathbb{N}$ we have that there is an isomorphism

$$(S, \nu)_{RN} \cong (U \lim G \times \mathbb{N}, \lim \lambda_n)_{RN}$$

**Proof.** We first reduce to the case where $S$ is standard. Let $T = (S, m)_{RN}$ for some measure $m$ on $S$ (in the measure class), then $T$ is standard. By Lemma 3.2.5, $T$ is amenable and since it is a factor of an ergodic action it is also ergodic. Moreover, the result for $T$ implies it for $S$. Indeed, for any BQI probability measure $\nu_0$ on $T$ there is a BQI probability measure $\nu$ on $S$ for which $(S, \nu) \to (T, \nu_0)$ is measure preserving (e.g. $\nu = (\frac{d\nu}{d\nu} \circ \pi) \cdot m$ where $\pi : S \to T$ is the factor map).

Thus we may assume that $S$ is a standard probability space. By [2, Theorem A], there is a probability measure $\nu$ on $S$, a sequence $\ell$ and an $\ell$-stochastic system of measures $\sigma$ on $G$, so that $(S, \nu)$ is isomorphic to the underlying QI $G$-space of the Poisson boundary of $\sigma$ – that is, to $(B, m^{(-1)})$ where we denote $B(G, \sigma) = (B, m)$.

By replacing $\sigma$ we may assume that there are no zero columns (see Remark B.1.6). By taking any BQI measure $\omega$ on $G$ and replacing $\sigma^{(0)}$ with $\omega \ast \sigma^{(0)}$ we may assume that $\sigma^{(0)}$ is BQI. This changes the Poisson boundary only by changing $m^{(-1)}$ to $\omega \ast m^{(-1)}$, so by changing $\nu$ to $\omega \ast \nu$ we keep the isomorphism between $(S, \nu)$ and the underlying QI $G$-space of the Poisson boundary of $\sigma$.

Take $a_n = 1 - \frac{1}{n}$ a sequence that goes to 1, an ultrafilter on $\mathbb{N}$ can be identified with an ultrafilter on $(\frac{1}{n}, 1)$, and being non-principle means that $U \lim_n a_n = 1$. Note that $\lambda_n = \nu^{(-1)}_{0, a_n, 0}$ are uniformly BQI (Lemma 3.1.9). By Theorem 3.1.9 we conclude:

$$(S, \nu)_{RN} \cong (B, m^{(-1)})_{RN} \cong (B(G(\sigma))_{RN}, \nu^{(-1)}_{0})_{RN} \cong (B(G(\sigma)), \nu^{(-1)}_{0})_{RN} \cong (U \lim G \times \mathbb{N}, \lim \lambda_n)_{RN}$$

**Remark 3.2.7.** In the displayed sequence of isomorphisms in the end of the proof above, the subscript $RN$ has two meanings. One when applied to a QI $G$-space, and the other when applied to a $\sigma$-system. The latter was used only once during the proof in the notation $B(G(\sigma))_{RN}$.

The mapping $(B(G(\sigma)), \nu^{(-1)}_{0}) \to (B(G(\sigma))_{RN}, \nu^{(-1)}_{0})$ is measure preserving (for QI $G$-spaces), and thus induces an isomorphism for RN factors.

**Remark 3.2.8.** We actually used [2, Theorem A] only for discrete groups, which is the main result of the earlier paper [13].

**Remark 3.2.9.** For the applications of the next section, we could use the following weaker version of the above results:

For any amenable ergodic and standard $S$, there is a BQI measure $\nu$ on $S$, a BQI $G$-space $(X, m)$ and a collection of uniformly BQI measures $\lambda_n$ on $G$ such that:
1. \((X, m)\) is a measure preserving extension of \((S, \nu)\).

2. \((X, m)\) is an extension of \((U \lim G, \lambda)_{RN}\).

To prove this weaker version, one need not use Theorem 3.1.9, rather it is enough to use Proposition 3.1.6 combined with a structure theorem for stationary actions. Indeed, one first uses [2, Theorem A] to exhibit \(S\) as a Poisson boundary, then apply Proposition 3.1.6 and Theorem B.2.23 (Taking \(\Lambda = B(G, \sigma)\)) to construct \(X\).

Remark 3.2.10. [22, Lemma 5.6] implies that for any \(G\)-space \(Y\), any measure \(\lambda\) on \(G\), there is a measure \(\nu\) on \(Y\) so that \((Y, \nu)\) is a factor of a space which is measure preserving extension of \((G, \lambda)\) (we can take \(\nu = \lambda \ast m\) for any \(m\)). In this sense, actions of \(G\) on \((G, \lambda)\) are the "biggest". In this approach, we see that amenable actions as actions are "big", in the sense that they can be obtained as limits of free actions. This perspective is motivated by Theorem A.

4 Applications to Entropy
In this section we prove the main theorems of the paper. In subsection 4.2 we prove Theorem A, in sub-subsection 4.3.1 we prove Theorems B,C and in sub-subsection 4.3.2 we prove Theorem D.

4.1 Recollection of entropy
We recall the basic notions regarding entropy from [22, Section 3].

Let us recall the definition of \(f\)-divergence (see [19, Chapter 6]):

**Definition 4.1.1.** Let \(f\) be a convex function on \((0, \infty)\) with \(f(1) = 0\).

We denote \(f'(\infty) = \lim_{t \to \infty} \frac{f(t)}{t}\) and \(f(0) = f(0^+)\) (these numbers exist in \(\mathbb{R} \cup \{+\infty\}\)).

Given a Borel space \((X, \Sigma)\) we denote by \(M(X, \Sigma)\) the collection of probability measures on \((X, \Sigma)\).

The \(f\)-divergence between two measures \(P, Q \in M(X, \Sigma)\) is defined in the following way:

\[
D_f(P||Q) = \int_{\{q>0\}} f\left(\frac{P}{q}\right) q \, dR + P(\{q = 0\}) \cdot f'(\infty)
\]

This is independent of the choice of \(R\) (we have the agreement here that \(0 \cdot \infty = 0\)).

Note that one can change \(f(t)\) by \(a \cdot (t - 1)\) for any \(a \in \mathbb{R}\) and it does not change \(D_f\). Thus we may assume \(f \geq 0\) and in particular the integral above gives rise to a well defined value in \([0, \infty]\).

We summarize the basic properties of \(f\)-divergence in the following:

**Lemma 4.1.2.** Let \(f\) be a convex function on \((0, \infty)\) with \(f(1) = 0\).

1. For any Borel space \((X, \Sigma)\), the function \(D_f : M(X, \Sigma) \times M(X, \Sigma) \to [0, \infty]\) is convex.

2. If \(P, Q \in M(X, \Sigma)\) are of the same measure class then:

\[
D_f(P||Q) = \int_X f\left(\frac{dP}{dQ}\right) dQ
\]
3. For a measurable mapping between Borel spaces \( \pi : (X, \Sigma_X) \to (Y, \Sigma_Y) \) for any \( P, Q \in M(X, \Sigma_X) \) we have:

\[
D_f(\pi_\ast P \| \pi_\ast Q) \leq D_f(P \| Q)
\]

4. Suppose \( P, Q \in M(X, \Sigma_X) \) and \( \nu \in M(Y, \Sigma_Y) \) and consider the measures \( P \times \nu, Q \times \nu \) on the product \( (X \times Y, \Sigma_X \otimes \Sigma_Y) \). Then we have:

\[
D_f(P \times \nu \| Q \times \nu) = D_f(P \| Q)
\]

**Remark 4.1.3.** If \( f'(\infty) = +\infty \) and \( D_f(P \| Q) < \infty \) then \( P \ll Q \).

Recall the definition of entropy [22, Definition 3.4]:

**Definition 4.1.4.** Let \( f \) be a convex function on \((0, \infty)\) with \( f(1) = 0 \) and let \( \lambda \) be a probability measure on \( G \). Given a probability measure \( \nu \) on a Borel \( G \)-space \((X, \Sigma)\), its Furstenberg’s \((\lambda, f)\)-entropy is defined by:

\[
h_{\lambda, f}(X, \nu) = \sum_{g \in G} \lambda(g)D_f(g\nu \| \nu)
\]

By Lemma 4.1.2 we conclude that the entropy \( h_{\lambda, f} \) is a convex function that decreases along factors:

**Proposition 4.1.5.** Let \( f \) be a convex function on \((0, \infty)\) with \( f(1) = 0 \) and let \( \lambda \) be a probability measure on \( G \).

1. For any Borel \( G \)-space \((X, \Sigma)\) the function \( h_{\lambda, f} : M(X, \Sigma) \to [0, \infty) \) is convex.

2. Given a measurable \( G \)-equivariant map \( \pi : (X, \Sigma_X) \to (Y, \Sigma_Y) \) between Borel \( G \)-spaces, for any \( \nu \in M(X, \Sigma_X) \) we have:

\[
h_{\lambda, f}(Y, \pi_\ast \nu) \leq h_{\lambda, f}(X, \nu)
\]

**Remark 4.1.6.** In [22] we only considered the case of \( \nu \) being quasi-invariant, however, using the definition of \( f \)-divergence for every pair of measures, there is no reason to do so. However, when \( f'(\infty) = +\infty \) and \( \text{Supp}(\lambda) \) is generating as a semi-group then \( h_{\lambda, f}(\nu) < \infty \) implies that \( \nu \) is quasi-invariant.

We come to defining the main object studied in this section – various versions of the minimal entropy number.

**Definition 4.1.7.** Let \( f \) be a convex function on \((0, \infty)\) with \( f(1) = 0 \) and let \( \lambda \) be a probability measure on \( G \).

1. Given a QI \( G \)-space \( X \), we denote by \( M(X) \) the probability measures on \( X \) in the measure class, and define the minimal entropy number of \( X \) to be:

\[
I_{\lambda, f}(X) = \inf_{\nu \in M(X)} h_{\lambda, f}(X, \nu)
\]

2. Given a measurable action \( G \acts (X, \Sigma) \) on a Borel space, we define the minimal entropy number of \((X, \Sigma)\) to be:

\[
I^\text{Borel}_{\lambda, f}(X, \Sigma) = \inf_{\nu \in M(X, \Sigma)} h_{\lambda, f}(X, \nu)
\]

We recall that \( M(X, \Sigma) \) denotes the collection of all probability measures on \((X, \Sigma)\).
3. Given a continuous action $G \rightarrow X$ where $X$ is a compact metrizable space, we define the topological minimal entropy number of $X$ to be:

$$I^\text{top}_{\lambda,f}(X) := I^\text{Borel}_{\lambda,f}(X, \mathcal{B})$$

Here, $\mathcal{B}$ is the Borel $\sigma$-algebra on $X$.

**Remark 4.1.8.** The concepts $I_{\lambda,f}(X), I^\text{top}_{\lambda,f}(X)$ were introduced in [22].

**Lemma 4.1.9.**

1. If $\pi : S \rightarrow T$ is a factor map between QI $G$-spaces then $I_{\lambda,f}(S) \geq I_{\lambda,f}(T)$.

Moreover, if there is a measure $m \in M(S)$ so that $\pi : (S, m) \rightarrow (T, \pi_*m)$ is measure preserving extension then $I_{\lambda,f}(S) = I_{\lambda,f}(T)$.

2. For any Borel $G$-space $(X, \Sigma)$ and $\omega \in M(G), \nu \in M(X, \Sigma)$ we have $h_{\lambda,f}(\omega \ast \nu) \leq h_{\lambda,f}(\omega)$.

3. For any QI $G$-space $S$ we have $I_{\lambda,f}(S) \leq I_{\lambda,f}(G)$.

4. Suppose $\lambda$ is a finitely supported probability measure on $G$. Let $(S, \nu)$ be a BQI $G$-space. Then:

$$I_{\lambda,f}(S) = \inf \{h_{\lambda,f}(S, m) | m = \omega \ast m_0 \text{ where } m_0 \in M(S) \text{ satisfies } \frac{d\nu}{dm_0} \in L^\infty(S) \text{ and } \omega \in M(G) \text{ is BQI}\}$$

**Proof.**

1. The inequality $I_{\lambda,f}(S) \geq I_{\lambda,f}(T)$ follows from [22, Lemma 3.6]. If $\pi : (S, m) \rightarrow (T, \pi_*m)$ is measure preserving extension then for any $\nu \in M(T)$ we can take $\kappa = \left(\frac{d\nu}{dm_0} \circ \pi\right) \cdot \nu \in M(S)$ and then $\pi : (S, \kappa) \rightarrow (T, \nu)$ is measure preserving. By [22, Lemma 3.6] we conclude $I_{\lambda,f}(S) \leq h_{\lambda,f}(\kappa) = h_{\lambda,f}(\nu)$, which shows the other direction.

2. The same proof as [22, Lemma 3.8]: Consider the Borel $G$-space $G \times X$ (with the product $\sigma$-algebra) where the $G$-action is only on the first coordinate. The action map $a : G \times X \rightarrow X$ is $G$-equivariant and maps $\omega \ast \nu$ to $\omega \ast \nu$ we conclude by Lemma 4.1.2(4) and 4.1.5(2):

$$h_{\lambda,f}(G, \omega) = h_{\lambda,f}(G \times X, \omega \ast \nu) \geq h_{\lambda,f}(X, \omega \ast \nu)$$

3. Follows from the previous item (this is also [22, Corollary 3.9]).

4. Denote by $I'$ the right hand side. Let $\kappa \in M(S)$, we need to show $h_{\lambda,f}(\kappa) \geq I'$.

Choose a BQI measure $\omega$ on $G$ and let $\epsilon > 0$.

Define $m_0 := (1 - \epsilon) \kappa + \epsilon \nu$ and $\omega := (1 - \epsilon) \sum_{n \geq 0} \epsilon^n \omega^n$ and $m := \omega \ast m_0$. Note that $\omega$ is a BQI measure and since $\frac{d\nu}{dm_0} \leq \epsilon^{-1}$ we conclude that $h_{\lambda,f}(m) \geq I'$.

Note that $\omega := (1 - \epsilon) \delta_x + \epsilon \omega \ast \omega$, using convexity of $h_{\lambda,f}$ and item 2 we get:

$$I' \leq h_{\lambda,f}(m) = h_{\lambda,f}\left((1 - \epsilon)m_0 + \epsilon \omega \ast m\right) \leq (1 - \epsilon)h_{\lambda,f}(m_0) + \epsilon h_{\lambda,f}(\omega \ast m) \leq (1 - \epsilon)^2 h_{\lambda,f}(\kappa) + (1 - \epsilon)h_{\lambda,f}(\nu) + \epsilon h_{\lambda,f}(\omega)$$

As $\omega$ and $\nu$ are BQI we conclude $h_{\lambda,f}(\omega), h_{\lambda,f}(\nu) < \infty$ (indeed $\lambda$ is finitely supported).

Taking $\epsilon \rightarrow 0$ we conclude that $h_{\lambda,f}(\kappa) \geq I'$.
The next lemma shows that our concept of entropy is consistent with ergodic decomposition. We recall, that given a QI $G$-space $X$ which is a standard probability space, one considers $L^\infty(X)^G \subset L^\infty(X)$ which is $w^*$-closed and thus of the form $L^\infty(E)$ for a standard probability space $E$. The inclusion above yields a measurable mapping $\pi : X \to E$ which is $G$-invariant (there is no $G$-action on $E$). Given a measure $\nu$ on $X$ we denote $P = \pi_* \nu$ and $\{\nu_e\}_{e \in E}$ be the disintegration, $\nu = \int_E \nu_e dP$.

For almost every $e \in E$ one has that $(X, \nu_e)$ is QI and ergodic. We call this the ergodic decomposition of $(X, \nu)$.

**Lemma 4.1.10.** Suppose $(X, \nu)$ is a QI $G$-space which is a standard probability space. Let $\pi : (X, \nu) \to (E, P)$ and let $\nu = \int_E \nu_e dP(\cdot)$ be the ergodic decomposition. Then:

1. For almost every $e \in E$ we have $\frac{d\nu}{d\nu_e} = \frac{d\nu_e}{\nu_e}$ [$\nu_e$-a.e.].

2. For any probability measure $\lambda$ on $G$ and any convex function $f$ with $f(1) = 0$ we have:

$$h_{\lambda, f}(\nu) = \int_E h_{\lambda, f}(\nu_e) dP(\cdot)$$

**Proof.**

1. Since there is a countable algebra of measurable sets of $X$ that generates its $\sigma$-algebra, we need only to show that for each $A \subset X$ we have for a.e. $e \in E$ that $\int_A \frac{d\nu}{d\nu_e} d\nu_e = g\nu_e(A)$. This is equivalent to $\varphi(e) := \int_X (\frac{d\nu}{d\nu_e} - 1_{\nu_e}(A)) d\nu_e = 0$ for all $e \in E$. To show this we will show that for any $T \subset E$ we have $\int_T \varphi dP = 0$. Indeed,

$$\int_T \left( \int_X \left( \frac{d\nu}{d\nu_e} 1_{\nu_e}(A) - 1_{\nu_e}(A) \right) d\nu_e \right) dP = \int_T \left( \frac{d\nu}{d\nu_e} 1_{\nu_e}(A) \right) d\nu_e = g\nu_e(A)\nu_e(A) = 0$$

where the last equality follows from $\pi^{-1}(T) = g^{-1}(\pi^{-1}(T))$. Thus we conclude the item.

2. Assume without loss of generality that $f \geq 0$. Using the previous item, for a.e. $e \in E$ we have: $D_f(g\nu_e || \nu_e) = \int_X I_f(\frac{d\nu}{d\nu_e}) d\nu_e$. Integrating over $E$ and applying Fubini-Tonelli we get:

$$D_f(g\nu || \nu) = \int_E D_f(g\nu_e || \nu_e) dP(e)$$

applying the monotone convergence theorem we get the result.

**Lemma 4.1.11.** Let $f$ be a convex function with $f(1) = 0$ and let $\lambda$ be a finitely supported measure on $G$. Then for any $G \acts (X, \Sigma)$ measurable action we have:

$$I^{\text{rel}}_{\lambda, f}(X, \Sigma) = \inf \{ h_{\lambda, f}(X, \nu) : \nu \in M(X, \Sigma) \text{ quasi-invariant} \}$$

**Proof.** Denote by $I'$ the right hand side. We need to show that for any $\nu \in M(X, \Sigma)$ we have $h_{\lambda, f}(\nu) \geq I'$. Take a BQI measure $\omega$ on $G$, then $h_{\lambda, f}(\omega) < \infty$. Define $\omega_\epsilon = (1 - \epsilon)\Sigma \nu + \epsilon \cdot \omega_{\epsilon} + (1 - \epsilon)\delta_\epsilon$. By Lemma 4.1.9(2) and Proposition 4.1.5(1), for any $\nu \in M(X, \Sigma)$ we have:

$$I' \leq h_{\lambda, f}(\omega_\epsilon \ast \nu) \leq \epsilon h_{\lambda, f}(\omega \ast \omega_{\epsilon} \ast \nu) + (1 - \epsilon)h_{\lambda, f}(\nu) \leq \epsilon h_{\lambda, f}(\omega) + (1 - \epsilon)h_{\lambda, f}(\nu)$$

Since $h_{\lambda, f}(\omega) < \infty$ taking $\epsilon \to 0$ we see $h_{\lambda, f}(\nu) \geq I'$. 

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4.1.1 Entropy minimal number and ultralimit

The following follows either from Theorem 2.2.3 (in the uniformly bounded case) or from [22, Corollary 4.19] (combined with Proposition 2.4.13)

**Proposition 4.1.12.** Suppose \((X, \nu_i)_{i \in \mathcal{I}}\) are uniformly BQI G-spaces. Let \(U\) be an ultrafilter on \(\mathcal{I}\) and let \((X, \nu) = \mathcal{U} \lim (X, \nu_i)\). Then for any convex function \(f\) with \(f(1) = 0\) and finitely supported probability measure \(\lambda\) on \(G\) we have:

\[
h_{\lambda, f}(X, \nu) = \mathcal{U} \lim h_{\lambda, f}(X, \nu_i)
\]

**Lemma 4.1.13.** Let \((X, \nu_i)_{i \in \mathcal{I}}\) be uniformly QI G-spaces, let \(U\) be an ultrafilter on \(\mathcal{I}\) and consider \((X, \nu) = \mathcal{U} \lim (X, \nu_i)\). Suppose \(\omega\) be a BQI measure on \(G\) and \(m_0 \in M(X)\) is a probability measure on \(X\) of the same measure class as \(\nu\). Assume that \(\frac{dm}{dm_0} \in L^\infty\) and let \(m = \omega * m_0\).

Then there are uniformly BQI measures \(\eta_i\) on \(X_i\) of the same measure class as \(\nu_i\) with \(\mathcal{U} \lim \eta_i = m\).

**Proof.** Consider \(f = \frac{dm}{dm_0} \in L^1(X, \nu)\), by Corollary 2.3.3, there is a majorant \(\rho \in M\) and \(f_i \in C_\rho(X, \nu_i)\) with \(\sup \|f_i\|_i < \infty\) and \(f = \mathcal{U} \lim f_i\).

By assumption there is \(a > 0\) for which \(f \geq a\). We may assume that \(f_i \geq \frac{a}{2}\) and \(\int_X f_i d\nu_i = 1\) for all \(i\). Indeed, otherwise consider \(g_i = \max(f_i, a)\) then \(\mathcal{U} \lim g_i = \max(\mathcal{U} \lim f_i, a) = f\), moreover, \(\mathcal{U} \lim \int_X g_i d\nu_i = \int_X \mathcal{U} \lim g_i d\nu_i = 1\) and thus \(\mathcal{U}\text{-a.e. } \frac{1}{2} \leq \int_X g_i d\nu_i \leq 2\). Define \(h_i = \frac{g_i}{\|g_i\|_i}\), then for \(\mathcal{U}\text{-a.e. } i\) we have \(h_i \geq \frac{a}{2}\). Hence we can make this assumption on \(f_i\). Thus, \(\kappa_i = f_i \cdot \nu_i\) are probability measures on \(X_i\) of the same measure class as \(\nu_i\) and \(\mathcal{U} \lim \kappa_i = m_0\). This implies \(\eta_i = \omega * \kappa_i\) are uniformly BQI (since \(\omega\) is BQI) and by Lemma 2.4.9 we have \(\mathcal{U} \lim \eta_i = \omega * m_0 = m\).

The following corollary plays a key role in our proof of Theorem A:

**Corollary 4.1.14.** Let \((X, \nu_i)_{i \in \mathcal{I}}\) be uniformly QI G-spaces, let \(U\) be an ultrafilter on \(\mathcal{I}\) and consider \((X, \nu) = \mathcal{U} \lim (X, \nu_i)\). Then for any convex function \(f\) with \(f(1) = 0\) and finitely supported probability measure \(\lambda\) on \(G\) we have:

\[
\mathcal{U} \lim I_{\lambda, f}(X_i) \leq I_{\lambda, f}(X)
\]

**Proof.** By replacing \(\nu_i\) by \(\omega * \nu_i\) for a BQI measure \(\omega\) on \(G\) we may assume that \((X, \nu)\) is BQI. Let \(I := \mathcal{U} \lim I_{\lambda, f}(X_i)\), by Lemma 4.1.9 it is enough that show that if \(m \in M(X)\) is of the form \(m = \omega * m_0\) where \(\omega\) is BQI and \(\nu \ll m_0\) then \(h_{\lambda, f}(m) \geq I\). However, using Lemma 4.1.13, there are uniformly BQI measures \(\eta_i\) on \(X_i\) of the same measure class as \(\nu_i\) with \(\mathcal{U} \lim \eta_i = m\). By Proposition 4.1.12 we conclude:

\[
h_{\lambda, f}(m) = \mathcal{U} \lim h_{\lambda, f}(\eta_i) \geq \mathcal{U} \lim I_{\lambda, f}(X_i) = I
\]

The following general proposition will not be used in the next subsections, however we include it here:

**Proposition 4.1.15.** Suppose \((X_i, \nu_i)_{i \in \mathcal{I}}\) is a collection of QI G-spaces. Let \(\lambda\) be a generating (as a semi-group) measure on \(G\) and \(f\) a convex function with \(f(1) = 0\) satisfying \(\lim_{t \to \infty} \frac{4t}{df} = \infty\). If \(\sup I_{\lambda, f}(X_i, \nu_i) < \infty\), then \((X_i, \nu_i)\) are uniformly QI. Moreover, for any ultrafilter \(U\) on \(\mathcal{I}\) the ultralimit \((X, \nu) = \mathcal{U} \lim (X_i, \nu_i)\) satisfies:

\[
h_{\lambda, f}(X, \nu) \leq \mathcal{U} \lim h_{\lambda, f}(X_i, \nu_i)
\]
Proof. We can change $f$ by a linear function to assume further $f \geq 0$. Let $M = \sup _{i} h_{\lambda, f}(X_i, \nu_i)$. For any $g \in G$ we have $E_{\nu_i}(f \frac{d\mu}{d\nu_i}) \leq \frac{M}{\lambda(g)}$ and thus by Lemma A.0.11 we conclude that for any $g \in \text{Supp}(\lambda)$ there is $\rho \in M$ and $K > 0$ for which $\| \frac{d\mu}{d\nu} \|_{c}(X_i, \nu_i) \leq K$ (Both $K, \rho$ depend on $g \in \text{Supp}(\lambda)$). Changing $\rho$ to $\min(1, K \cdot \rho)$ we get that $g\nu_i \ll \rho \nu_i$. Since $\text{Supp}(\lambda)$ is generating $G$ as a semi-group and using the fact that $M$ is closed under composition (see Lemma A.0.3) we conclude that $(X_i, \nu_i)$ are uniformly QI.

The inequality $h_{\lambda, f}(X, \nu) \leq U \lim \ h_{\lambda, f}(X_i, \nu_i)$ follows immediately from Proposition 2.4.11 (compatibility of Radon-Nikodym derivative with ultralimit in the uniformly QI case), the continuity of $f$ and Lemma 2.1.9 (our version of Fatou’s lemma).}

### 4.2 Entropy in amenable actions - proof of Theorem A

We can now complete the proof of Theorem A:

**Theorem 4.2.1.** Let $S$ be an amenable action of $G$. Then for any convex function $f$ with $f(1) = 0$ and finitely supported measure $\lambda$ on $G$ we have:

$$I_{\lambda, f}(S) = I_{\lambda, f}(G)$$

**Proof.** Note that by Lemma 4.1.9(3) we only need to show that $I_{\lambda, f}(S) \geq I_{\lambda, f}(G)$. Using Lemma 4.1.9(1) we may assume $S$ is a standard measure space and using Lemma 4.1.10(2) and [2, Corollary B] we may assume that the action of $G$ on $S$ is ergodic.

Take a non-principle ultrafilter $U$ on $\mathbb{N}$. By Theorem 3.2.6 there is a measure $\nu$ on $S$ and BQI measures $\eta_n$ on $G \times \mathbb{N}$ (supported everywhere) with

$$(S, \nu)_{\text{RN}} \equiv (U \lim G \times \mathbb{N}, U \lim \eta_n)_{\text{RN}}$$

Consider the QI $G$-spaces $Z := (S, \nu)_{\text{RN}} \equiv U \lim(G \times \mathbb{N}, U \lim \eta_n)_{\text{RN}}$ and $Y := U \lim(G \times \mathbb{N}, U \lim \eta_n)$. Then we conclude by Lemma 4.1.9(1) and Corollary 4.1.14:

$$I_{\lambda, f}(S) = I_{\lambda, f}(Z) = I_{\lambda, f}(Y) \geq U \lim_n I_{\lambda, f}(G \times \mathbb{N}) = I_{\lambda, f}(G \times \mathbb{N}) \geq I_{\lambda, f}(G)$$

proving the result. \hfill \Box

One can use the above theorem to conclude that certain actions do not posses a non-trivial amenable factor:

**Corollary 4.2.2.** Suppose $f$ is strictly convex function with $f(1) = 0$ and $\lambda$ generating probability measure on $G$. Suppose $(S, \nu)$ is a QI $G$-space with $h_{\lambda, f}(S, \nu) = I_{\lambda, f}(S)$. Then if $\pi : (S, \nu) \rightarrow (T, m)$ is a factor and $T$ is an amenable $G$-space, then $\pi$ is a measure preserving extension.

**Proof.** Using Theorem 4.2.1 for $T$ we see

$$I_{\lambda, f}(G) \geq I_{\lambda, f}(S) = h_{\lambda, f}(S, \nu) \geq h_{\lambda, f}(T, m) \geq I_{\lambda, f}(T) = I_{\lambda, f}(G)$$

This implies that we have equality everywhere, in particular $h_{\lambda, f}(S, \nu) = h_{\lambda, f}(T, m)$ which implies $\pi$ is measure preserving extension (by [22, Lemma 3.3(3)]). \hfill \Box

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4.3 Entropy in boundary actions

4.3.1 Entropy for hyperbolic groups and linear groups - proof of Theorems B, C

In [23], Spatzier and Zimmer proved that certain topological boundary actions are amenable with respect to all measure classes. Adams generalized their result to all hyperbolic groups in the paper [1]. The notion of topological amenability of a locally compact groupoid was introduced [21] as the existence of an approximate invariant mean. It implies the Zimmer amenability for any quasi-invariant measure. In [5] the converse was shown to be true. For a comprehensive study of the amenability in both settings for the general case of groupoids, see the book [3] (in particular [3, Appendix B] for the topological amenability for hyperbolic groups on their Gromov boundary).

**Theorem 4.3.1.** Suppose $G$ is a discrete countable group, and let $X$ be a compact $G$-space which is topologically amenable. Then for any convex function $f$ with $f(1) = 0$ and finitely supported measure $\mu$ on $G$

$$I_{\mu,f}(G) = I_{\mu,f}^{\text{top}}(X)$$

Proof. Lemma 4.1.9(2) yields $I_{\mu,f}(X) \leq I_{\mu,f}(G)$. By Lemma 4.1.11 we only need to show that for quasi invariant Borel measure $\nu \in M(X)$ we have $h_{\lambda,f}(X,\nu) \geq I_{\lambda,f}(G)$. Let $S$ be the quasi invariant $G$-space $X$ with the measure class of $\nu$. It is amenable. Thus, we may use Theorem 4.2.1 to conclude

$$h_{\lambda,f}(X,\nu) \geq I_{\lambda,f}(S) = I_{\lambda,f}(G)$$

From this (and the result by Adams [1] mentioned above) we get Theorem B:

**Theorem 4.3.2.** Let $G$ be a hyperbolic group, and let $\partial G$ be its Gromov boundary. Then for any convex function $f$ with $f(1) = 0$ and finitely supported measure $\lambda$ on $G$ we have:

$$I_{\lambda,f}(G) = I_{\lambda,f}^{\text{top}}(\partial G)$$

We also get Theorem C:

**Theorem 4.3.3.** Let $G$ be a discrete subgroup in a semi-simple Lie group $G$. Let $B$ be a Borel subgroup and consider the flag space $X = G/B$, then for any $\lambda, f$ as above we have:

$$I_{\lambda,f}^{\text{top}}(X) = I_{\lambda,f}(G)$$

Proof. It is clear since the action of $G$ on $X$ is topologically amenable, see [3, Example 5.2.2(2)] - the groupoid $G \rtimes X$ is equivalent to $B \rtimes (G/G)$ which is amenable since $B$ is an amenable group.

4.3.2 Entropy in the boundary of the free group - proof of Theorem D

We recall some notations and results from [22, Section 7].

Let $F = F_d$ be a free group on $d \geq 2$ generators, which we denote by $a_1, \ldots, a_d$. Denote $a_{-i} = a_i^{-1}$. We consider the boundary $X = \partial F$ as the space of infinite length reduced words, e.g. the subset of $(a_{s1}, \ldots, a_{sd})^N$ consisting of words $(w_n)$ with $w_{n+1} \neq w_n^{-1}$. The space $X$ is a compact metrizable space with a continuous $F$-action.

We denote by $\Delta_d$ the set of generating symmetric probability measures on $F$ supported on $\{a_{\pm1}\}_{i=1}^d$. Let $\mu \in \Delta_d$ and denote $p_i = \mu(a_i)$. We recall that there are unique $(q_i)_{i=1,\ldots,d}$ in $(0,1)$ with $q_j = p_j + q_j \sum_{j+i=1,\ldots,d} p_i q_i$. They satisfy $q_i = q_{-i}$, and if we define $v_i = \frac{1}{1-p_i}$, then $\sum_{i=1,\ldots,d} v_i = 1$.

Let $f$ be a strictly convex smooth function on $(0,\infty)$ with $f(1) = 0$. We denote $\Psi_f(z) = f(z) - zf'(z) + f'(\frac{1}{z})$.  

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Definition 4.3.4. We define $T : \Delta_d \to \Delta_d$ as follows - given $\mu \in \Delta_d$, denote $p_i = \mu(a_i)$ and let $q_i$ be as above, define $T(\mu) = \lambda \in \Delta_d$ by:

$$
\lambda(a_\pm j) = \lambda_j = \frac{c}{\Psi_f(q_j) - \Psi_f(\frac{1}{\lambda_j})}
$$

where $c$ is a normalization $c = \left(\frac{2^{2d}}{\sum_{i=1}^{d} \Psi_f(q_i) - \Psi_f(\frac{1}{q_i})}\right)^{-1}$.

We denote by $\nu_\mu$ the $\mu$-harmonic measure on $\partial F_d$. We showed in [22, Proposition 7.6]

**Lemma 4.3.5.** $T$ is a bijection and for $\lambda = T(\mu)$ we have that $\nu_\mu$ minimize $(\lambda, f)$-entropy in its measure class on $\partial F_d$.

Using Theorem B and the previous Lemma we conclude Theorem D:

**Theorem 4.3.6.** For any $\mu \in \Delta_d$, $\lambda = T(\mu)$ we have:

$$
I^\text{top}_{\lambda,f}(\partial F_d) = h_{\lambda,f}(\partial F_d, \nu_\mu)
$$

**Proof.** Consider the QI $F_d$-space $X$ which is $(\partial F_d, \nu_\mu)$ (that is considering the measure class of $\nu_\mu$). By Theorem 4.3.2, Lemma 4.1.9(3) and Lemma 4.3.5 we conclude:

$$
I^\text{top}_{\lambda,f}(\partial F_d) = I_{\lambda,f}(F_d) \geq I_{\lambda,f}(X) = h_{\lambda,f}(\partial F_d, \nu_\mu)
$$

As the other direction is trivial, the Theorem follows.

**Remark 4.3.7.** In [22, Theorem 7.11] we showed that $h_{\lambda,f}(\partial F_d, \nu_\mu) = I_{\lambda,f}(F_d)$ which implies Theorem 4.3.6 when we combine it with Theorem 4.3.2. However I wanted to stress that all the information from the previous paper we use is [22, Proposition 7.6], which is a direct computation.

A nice application of Corollary 4.2.2 is:

**Corollary 4.3.8.** Let $\mu \in \Delta_d$, then $(\partial F_d, \nu_\mu)$ has no amenable factors other then itself.

**Proof.** Since $(X, \nu_\mu)$ doesn’t posses any measure preserving extensions (e.g. being the Poisson boundary of $\mu$), the corollary follows from Proposition 4.2.2 and Lemma 4.3.5, taking $f(t) = t \ln(t)$ and $\lambda = T(\mu)$.

**Remark 4.3.9.** It is probable that the Corollary can be deduced from [18] using the embedding of the Free group as a lattice in a Lie group. Our argument avoids such an embedding, and might be useful in situations where one has a good boundary theory but no direct connection to a Lie group.

5 Further questions and outlook

5.1 Entropy minimizing

5.1.1 Spaces attaining the minimal entropy and uniqueness of minimizing measures

In light of Theorem D, a natural question is about uniqueness of minimizers. The following very special case is still open:
Problem 1. Consider the action $F_d \curvearrowright \partial F_d$, and let $\lambda \in \Delta_d$ be the uniform measure. Are there any measures $\nu$ on $\partial F_d$ other than $\nu_\lambda$ for which $h_{\lambda}(\nu)$ is the minimal value for the Furstenberg $\lambda$-entropy?

More generally, we formulate the following problem:

Problem 2. Suppose $\lambda$ is a finitely support generating probability measure on $G$, $f$ is smooth absolutely convex function with $f(1) = 0$ and $f'(\infty) = +\infty$.

- Is there an amenable action $(S,\nu)$ such that $h_{\lambda,f}(\nu) = I_{\lambda,f}(G)$?
- Is an action as above unique up to measure preserving extensions?

If the answer to Problem 5 is yes (in particular for exact groups, see Lemma 5.2.1), then the existence follows from Proposition 4.1.15. We have shown in [22, Lemma 3.10], that given ergodic $S$, such a measure $\nu$ on it is unique if it exist. It is also clear that a factor between minimizing amenable actions is a measure preserving extension.

5.1.2 Boundary actions that are not amenable

There are interesting boundary actions that are not topologically amenable, for example the action of $SL_n(\mathbb{Z}) \curvearrowright \mathbb{P}^{n-1}_\mathbb{R}$ for $n \geq 3$.

Problem 3. Suppose that $G$ is a lattice in a real semi-simple Lie group $G$ with no compact factors (or any Zariski-dense discrete subgroup). Let $P$ be a parabolic (but not a Borel) subgroup and consider the flag space $X = G/P$. Let $f$ be a convex function with $f(1) = 0$ and let $\lambda$ be a finitely supported measure on $G$.

1. Can we describe the numbers $I_{\lambda,f}^{\text{top}}(X)$ in terms of the group $G$?
2. Considering $X$ as a quasi-invariant $G$-space with the Lebesgue measure class, is $I_{\lambda,f}(X)$ the same as $I_{\lambda,f}^{\text{top}}(X)$?
3. Is there a minimizing measure of the Lebesgue measure class?
4. Is the minimizing measure on $X$ unique in this case?

5.1.3 Possible generalizations

The notion of amenability has a natural generalization to groupoids, see [3]. It is natural to ask if there is a version for entropy and our results in this setting.

Another question is about weakening the assumption of finite support on $\lambda$ in Theorem A. In the case of $f(t) = t \ln(t)$ the equality holds for $\lambda$ with finite entropy. Indeed, in order to prove this, one need to refine Lemma 4.1.9(4) to consider only $\omega$'s of the form $\lambda = (1-\epsilon)\sum n^\epsilon \lambda^n$, and to prove that the for each $0 < a < 1$ and a collection $(X_i,\nu_i)$ with $(X,\nu) = U \lim(X_i,\nu_i)$ one has: $U \lim h_{\lambda} h(\lambda,\nu) = h_{\lambda}(\lambda,a \ast \nu)$. The details are left to the reader.

Another natural problem is a relative version for Theorem D:

Problem 4. Let $\pi : X \to Y$ be a factor between $QI G$-spaces for which $(X,Y)$ is an amenable pair (see Definition 3.2.1), does the equality $I_{\lambda,f}(X) = I_{\lambda,f}(Y)$ hold?
5.2 Amenable actions

5.2.1 Amenable actions and ultralimits

The reason for the ergodicity assumption in Theorem E is that [2, Theorem A] deals only with ergodic actions. However, the Poisson boundary of a time dependent matrix-valued random walk is always an amenable action, but not necessarily ergodic - the ergodic component corresponds to the Poisson boundary of the random walk on the $[\ell_n]$ induced from $V_n \to [\ell_n]$.

If the generalized Poisson boundaries cover all amenable actions, one can easily eliminate the ergodicity assumption in our realization of amenable actions as ultralimits.

One can ask the following converse to Theorem E:

**Problem 5.** Suppose $(S_i, \nu_i)$ are uniformly QI amenable actions, is $U\lim(S_i, \nu_i)$ necessarily amenable?

**Lemma 5.2.1.** The answer to Problem 5 is yes for exact groups.

**Proof.** By [8, Theorem 3.16], if $G$ is exact, a $G$-space $S$ is amenable iff there is a u.c.p. (unital completely positive) $G$-map $\Phi : L^\infty(G) \to L^\infty(S)$. Suppose $(S_i, \nu_i)$ are amenable and let $(S, \nu) = U\lim(S_i, \nu_i)$. By assumption we get $G$-equivariant u.c.p mappings $\varphi_i : L^\infty(G) \to L^\infty(S_i)$. Since $\varphi_i$ are u.c.p. they are contractions. Define $\varphi : L^\infty(G) \to L^\infty(S)$ by $\varphi(f) = U\lim \varphi_i(f) \in L^\infty(S)$. Then $\varphi$ is a $G$-equivariant u.c.p. which implies that $S$ is amenable.

5.2.2 Amenable actions and entropy

Another natural problem is whether a converse to Theorem A holds, namely:

**Problem 6.** Suppose $S$ is a QI $G$-space with $I_{\lambda,f}(S) = I_{\lambda,f}(G)$ for any finitely supported $\lambda$ and convex $f$ with $f(1) = 0$. Is the $G$-space $S$ necessarily amenable?

Of course, a negative answer to Problem 5 is also a negative answer for this, however this Problem is interesting also for exact groups.

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A Spaces of uniform integrability

In this appendix, we introduce a filtration on $L^1$-space by a Banach-spaces of uniformly integrable functions, indexed by the convex set of “majorants”. This filtration form a clean packaging for uniform-integrability. The introduction of these spaces allows one to extract the “absolutely-integrable part” out of any function, which is very useful for our study of ultralimits.

**Definition A.0.1.** The space of majorants $\mathcal{M}$ is the set of functions $\rho : [0, 1] \to [0, 1]$ such that:
1. \( \rho(0^*) = \rho(0) = 0, \rho(1) = 1 \).

2. \( \rho \) is concave: for any \( x, y, t \in [0, 1] \) we have \( \rho((1-t)x + ty) \geq (1-t)\rho(x) + t\rho(y) \).

**Example A.0.2.** Given \( q \geq 1 \) the function \( \rho(t) = t^+ \) is in \( M \).

We have the following lemma:

**Lemma A.0.3.** The space of majorant \( M \) satisfies the following properties:

1. Any \( \rho \in M \) is continuous, non-decreasing, sub-additive and satisfies \( \rho(t) \geq t \).

2. \( M \) is strongly convex: If \((\rho_n)_{n \geq 0}\) are in \( M \) and \((p_n)_{n \geq 0}\) is a probability measure on \( N \) then \( \rho = \sum_{n=0}^{\infty} p_n \rho_n \) is in \( M \).

3. If \( \rho, \eta \in M \) then \( \rho \circ \eta \in M \).

4. If \( \rho, \eta \in M \) then \( \max(\rho, \eta) \in M \).

5. If \( \rho \in M \) and \( K \geq 1 \) then the function \( \rho_K(t) = \min(1, K \cdot \rho(t)) \) is in \( M \).

6. Let \( \rho_0 : [0, 1] \to [0, 1] \) be any function with \( \rho_0(0^*) = \rho(0) = 0 \). Then, there is a function \( \rho \in M \) with \( \rho_0(t) \leq \rho(t) \). Moreover, we can choose such \( \rho \in M \) such that \( \frac{\rho_0(t)}{\rho(t)} = 0 \).

**Proof.**

1. \( \rho \) non-decreasing- suppose \( t \leq s \), then \( s \in [t, 1] \) and we conclude by concavity that \( \rho(s) \geq \min(\rho(1), \rho(t)) = \rho(t) \).

   Note that the concavity implies \( \rho(tx) \geq t\rho(x) \) for \( x, t \in [0, 1] \). Since \( \rho(1) = 1 \) we get \( \rho(t) \geq t \).

   \( \rho \) is sub-additive: for \( x, y \in [0, 1] \) with \( x + y \leq 1 \) we have
   \[
   \rho(x) + \rho(y) = \rho(x+y) \geq \rho(tx + ty) = \rho(t(x+y)) \geq \rho(x) + \rho(y) = \rho(x+y)
   \]

   \( \rho \) is continuous: for any \( \epsilon > 0 \) there is \( \delta > 0 \) with \( \rho(\delta) < \epsilon \) and then for \( 0 < t < \delta \) we have for any \( s \in [0, 1-\epsilon] \) that \( \rho(s) \leq \rho(s+t) \leq \rho(s) + \epsilon \).

2. It is obvious that \( \rho \) is a concave function with \( \rho(0) = 0, \rho(1) = 1 \). To see that \( \rho(0^*) = 0 \), let \( \epsilon > 0 \) and take \( N \) with \( \sum_{n=N+1}^{\infty} p_n < \frac{\epsilon}{2} \). Since \( \rho_n(0^*) = 0 \) for all \( n \) we can find \( \delta > 0 \) with \( \rho_n(\delta) < \frac{\epsilon}{2^n} \). Hence \( \rho(\delta) < \sum_{n=1}^{\infty} p_n \delta^n + \sum_{n=N+1}^{\infty} p_n \rho_n \delta^n < \epsilon \).

3. By item 1 we conclude that \( \kappa = \rho \circ \eta \) is continuous, non-decreasing and concave. Thus \( \kappa(0^*) = \kappa(0) = 0, \kappa(1) = 1 \).

4. By item 1 we conclude that \( \lambda = \max(\rho, \eta) \) is continuous and concave.

5. For the concavity of \( \rho_K \), note that \( \{(t, y) : y \leq \rho_K(t)\} = \{(t, y) : y \leq K \rho(t)\} \cap [0, 1]^2 \) thus it is a convex set as the intersection of convex sets. Thus \( \rho_K \) is concave.

6. Consider \( G =\{(x, y) \in [0, 1]^2 | y \leq \rho_0(x) \} \subset \{(1, 1)\} \) and let \( C = \text{conv}(G) \) be the convex hull. As \( G \) is compact we get that \( C \) is compact. Define \( \rho_1(x) = \max\{y : (x, y) \in C\} \), then \( \rho : [0, 1] \to [0, 1] \) is concave, \( \rho(1) = 1 \) and \( \rho \geq \rho_0 \). Let us show that \( \rho(0) = \rho(0^*) = 0 \). Indeed: let \( \epsilon > 0 \), by assumption there is \( \delta > 0 \) such that for \( t \leq \delta \) we have \( \rho_0(t) < \epsilon \).

   Thus \( C \subset [0, \delta] \times [0, \epsilon] \cup \{\delta, 1\} \times [0, 1] \). This implies that \( C \subset \{(x, y) \in [0, 1]^2 | y \leq \epsilon + \frac{1}{\delta} x\} \). We conclude that for \( x \leq \epsilon - \delta \) we have \( \rho(x) \leq 2 \epsilon \). Thus \( \rho(0) = \rho(0^*) = 0 \).

   Thus \( \rho \in M \) satisfies \( \rho \geq \rho_0 \). For the moreover part we can take \( \rho(t) = \frac{\rho(t)}{\rho(t)} \in M \) and then \( \lim_{t \to 0^+} \frac{\rho(t)}{\rho(t)} = 0 \). 

\( \square \)

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Lemma A.0.7.

Note that Example A.0.6.

Definition A.0.4. Let $\mathcal{X} = (X, \Sigma, \nu)$ be a probability space and let $\rho \in \mathcal{M}$.

1. We define the normed space $C_\rho(\mathcal{X})$ to be the space of functions (mod a.e. 0 functions) in $L^1(\mathcal{X})$ such that

$$\|f\|_\rho = \|f\|_{C_\rho(\mathcal{X})} := \sup_{A \in \Sigma} \int_A |f| d\nu \leq \frac{1}{\rho} \int_A |f| d\nu < \infty$$

2. Given a probability measures $m$ on $(X, \Sigma)$, we say that $m$ is $\rho$-absolutely continuous with respect to $\nu$ if $m$ is absolutely continuous with respect to $\nu$ and $\|\frac{dm}{d\nu}\|_\rho \leq 1$. We denote this relation by $m \ll \nu$.

Remark A.0.5. Note that $m \ll \nu$ if and only if $m(A) \leq \rho(\nu(A))$ for every $A \in \Sigma$.

Example A.0.6.

- For $\infty > p > 1$ let $q = \frac{p}{p-1}$ be the Hölder conjugate and let $\rho(t) = t^\frac{1}{q}$. Then $\rho \in \mathcal{M}$ (see Example A.0.2). Note we have a contraction $L^p(\mathcal{X}) \subset C_\rho(\mathcal{X})$.

Indeed, by Hölder’s inequality:

$$\int_A |f| d\nu \leq \left( \int_A |f|^p d\nu \right)^\frac{1}{p} \left( \int_A 1 d\nu \right)^\frac{1}{q} = \|f\|_{L^p(\mathcal{X})}$$

- For $\rho(t) = t$ we have that $\|\cdot\|_{C_\rho} = \|\cdot\|_{\infty}$ and thus $L^\infty(\mathcal{X}) = C_{\rho}(\mathcal{X})$.

Indeed, the inequality $\|\cdot\|_{C_\rho} \leq \|\cdot\|_{\infty}$ is the same as the previous item with $p = \infty$. For the reverse inequality, take $A = \{ |f| \geq \|f\|_{C_\rho} + \epsilon \}$. If $\nu(A) \neq 0$ then $\frac{1}{\nu(A)} \int_A |f| = \|f\|_{C_\rho} + \epsilon$ which is a contradiction. Thus $f$ is essentially bounded by $\|f\|_{C_\rho}$.

The next lemma explains the role of the spaces $C_\rho(\mathcal{X})$:

Lemma A.0.7. Let $\mathcal{X}$ be a probability space.

1. For any $\rho \in \mathcal{M}$ the space $C_\rho(\mathcal{X})$ is a Banach space.

2. Let $\rho_0, \rho_1 \in \mathcal{M}$ and $K > 0$. Suppose that $\frac{\rho_0}{\rho_1} \leq K$, then the inclusion induces a continuous embedding $C_{\rho_0}(\mathcal{X}) \subset C_{\rho_1}(\mathcal{X})$ with operator norm bounded by $K$.

3. Let $\rho \in \mathcal{M}$ and $K \geq 1$, and let $\rho_K(t) = \min(1, K \cdot \rho(t))$. Then $\|f\|_{\rho_K} \leq \max (\|f\|_{L^1(\mathcal{X})}, \frac{1}{K} \|f\|_\rho)$.

4. Let $\rho_0, \rho_1 \in \mathcal{M}$ be such that $\frac{\rho_0}{\rho_1}(0^+) = 0$. Then $C_{\rho_0}(\mathcal{X})$ is contained in the closure of the simple functions in $C_{\rho_1}(\mathcal{X})$.

5. For any $f \in L^1(\mathcal{X})$ there is $\rho \in \mathcal{M}$ with $f \in C_\rho(\mathcal{X})$.

Proof. 1. Suppose $f_n \in C_\rho(\mathcal{X})$ with $\sum_{n=0}^{\infty} \|f_n\|_{C_\rho(\mathcal{X})} < \infty$, we will show their sum converge in $C_\rho(\mathcal{X})$. Notice that $\|f\|_{L^1} \leq \|f\|_\rho$ and thus we have that $\sum_{n=0}^{\infty} \int f_n d\nu < \infty$ and in particular $f = \sum_{n=0}^{\infty} f_n$ converges almost everywhere and in $L^1(\mathcal{X})$. Moreover $\int_A |f| \leq \sum_{n=0}^{\infty} \int_A |f_n| d\nu$ which shows $f \in C_\rho(\mathcal{X})$, $\|f\|_{C_\rho(\mathcal{X})} \leq \sum_{n=0}^{\infty} \|f_n\|_{C_\rho(\mathcal{X})}$.

Applying the same reasoning, we see that for any $N$, the a.e. converging sum $\sum_{n=N+1}^{\infty} f_n$ is in $C_\rho(\mathcal{X})$ with $\|\sum_{n=N+1}^{\infty} f_n\|_{C_\rho(\mathcal{X})} \leq \sum_{n=N+1}^{\infty} \|f_n\|_{C_\rho(\mathcal{X})}$. Thus

$$\limsup_{N \to \infty} \|f_n\|_{C_\rho(\mathcal{X})} = \limsup_{N \to \infty} \sum_{n=0}^{N} f_n \|f_n\|_{C_\rho(\mathcal{X})} \leq \sum_{n=N+1}^{\infty} \|f_n\|_{C_\rho(\mathcal{X})} = 0$$

Which yields $f = \sum_{n=0}^{\infty} f_n$ in $C_\rho(\mathcal{X})$, as desired.
2. Indeed:
\[ \int_{A} |f| \, d\nu \leq \|f\|_{\rho_{0}} \rho_{0}(\nu(A)) \leq K \cdot \|f\|_{\rho_{0}} \rho_{1}(\nu(A)) \implies \|f\|_{\rho_{1}} \leq K \|f\|_{\rho_{0}} \]

3. Indeed:
\[ \int_{A} |f| \, d\nu \leq \min \left( \|f\|_{L^{1}}, \|f\|_{\rho}(\nu(A)) \right) \leq \max \left( \|f\|_{L^{1}}, \frac{1}{K} \right) \cdot \min \left( 1, K \rho(\nu(A)) \right) \]

4. Let \( f \in C_{\rho_{0}}(X) \), we will show it can be approximated in \( C_{\rho_{1}}(X) \) by simple functions. We may assume \( f \not\equiv 0 \). Let \( \epsilon > 0 \) and take \( C \) large enough such that:
\[ \forall t \in [0, \nu(\{ f \geq C \})] : \quad \frac{\rho_{0}(t)}{\rho_{1}(t)} < \frac{\epsilon}{2 \|f\|_{\rho_{0}}} \]

Let \( \varphi \) be a simple function such that on the set \( \{ f \leq C \} \) we have \( f - \frac{\epsilon}{2} \leq \varphi \leq f \) and outside of this set it is 0. Then:
\[ \int_{A} |f - \varphi| \, d\nu \leq \frac{\epsilon}{2} \cdot \nu(A) + \int_{A \setminus \{ f > C \}} |f| \, d\nu \leq \frac{\epsilon}{2} \cdot \rho_{1}(\nu(A)) + \|f\|_{\rho_{0}} \rho_{1}(\nu(A \cap \{ f > C \})) \]
\[ \leq \rho_{1}(\nu(A)) \left( \frac{\epsilon}{2} + \|f\|_{\rho_{0}} \cdot \frac{\rho_{0}}{\rho_{1}}(\nu(A \cap \{ f > C \})) \right) \leq \epsilon \rho_{1}(\nu(A)) \]

we conclude \( \|f - \varphi\|_{\rho_{1}} \leq \epsilon \).

5. Without loss of generality we may assume \( 0 \neq f \geq 0 \). We define a function \( \rho_{0} : [0, 1] \to [0, 1] \) in the following way:
\[ \rho_{0}(t) = \frac{1}{\|f\|_{L^{1}}} \sup \left\{ \int_{A} f \, d\nu(\{ \nu(A) = t \}) \right\} \]

We obviously have \( \rho_{0}(0) = 0 \). By Lebesgue dominant convergence theorem, for any sequence of measurable sets \( (A_{n}) \) with \( \nu(A_{n}) \to 0 \) we have \( \int_{A_{n}} f \, d\nu \to 0 \). Thus \( \rho_{0}(0) = 0 \), using Lemma A.0.3(5) we can find \( \rho \in \mathcal{M} \) with \( \rho_{0} \leq \rho \). With this \( \rho \) we obviously have \( f \in C_{\rho}(X) \).

\[ \square \]

**Corollary A.0.8.** Given a measured space \((X, \Sigma)\) and two probability measures \(m, \nu\) on it. If \( m \ll \nu \) then there is some \( \rho \in \mathcal{M} \) with \( m \ll \nu \).

**Proof.** Using Lemma A.0.7(4) for \( f = \frac{dm}{d\nu} \) we find \( \rho_{0} \in \mathcal{M} \) with \( f \in C_{\rho}(X) \). Consider \( M = \|f\|_{\rho} \) (note \( M \geq 1 \)) and \( \rho(t) = \min(M \cdot \rho_{0}(t), 1) \), then it is easy to see \( \rho \in \mathcal{M} \) and that \( \|f\|_{\rho} \leq 1 \).

**Example A.0.9.** Item 4 in Lemma A.0.7 is indeed “sharp”, in the sense that it is not always true that the simple (bounded) functions are dense in \( C_{\rho}(X) \). Let us give an example:

Take \( \rho(t) = \sqrt{t} \) which is a majorant. Let \( X = [1, \infty) \) with the Borel measure \( \nu = \frac{dx}{x} \) and let \( f : X \to \mathbb{R} \) be the function \( f(x) = x \). We claim \( f \in C_{\rho}(X, \nu) \) but not in the closure of the bounded functions. The distribution function of \( f \) is \( \lambda(s) = \nu(\{ f > s \}) = \frac{1}{\sqrt{s}} \cdot 1_{[1, \infty]} \). Note that for any \( v \in (0, 1] \) we have:
\[ \sup_{\text{Borel with } \nu(A) = v} \frac{1}{\rho(\nu(A))} \int_{A} f \, d\nu = \frac{1}{\rho(v)} \int_{(f > \lambda^{-1}(v))} f \, d\nu = \frac{1}{\sqrt{v}} \int_{\lambda^{-1}(v)} \lambda(s) \, ds = \frac{1}{\sqrt{v}} \int_{v}^{\infty} \frac{ds}{s^{2}} = 1 \]
Thus we see \( \|f\|_\rho = 1 \). On the other hand, we claim that \( \|f - \varphi\| \geq 1 \) for any bounded \( \varphi \). Indeed, suppose \( \varphi \) is a function bounded by \( C \), then consider a small \( v > 0 \) and \( A = \{ f > \lambda^{-1}(v) \} \). We have by the previous computation:

\[
\|f - \varphi\|_\rho \geq \frac{1}{\rho(\nu(A))} \int_A |f - \varphi| d\nu \geq \frac{1}{\rho(v)} \int_{(f > \lambda^{-1}(v))} (f - C) d\nu \geq 1 - \frac{1}{\sqrt{v}} C \nu(\{ f > \lambda^{-1}(v) \}) = 1 - C \cdot \sqrt{v}
\]

Taking \( v \to 0 \) we get \( \|f - \varphi\|_\rho \geq 1 \).

**Lemma A.0.10.** Let \( \mathfrak{X} \) be a probability space and let \( \rho \in \mathbf{M} \). Suppose that \( f \in C_\rho(\mathfrak{X}) \) and that \( \varphi \in L^\infty(\mathfrak{X}) \) with \( 0 \leq \varphi \leq 1 \). Then:

\[
\left| \int \mathfrak{X} f \cdot \varphi \, d\nu \right| \leq \|f\|_\rho \cdot \rho \left( \int \mathfrak{X} \varphi \, d\nu \right)
\]

**Proof.** By replacing \( f \) with \( |f| \) we may assume that \( f \geq 0 \).

Consider \( X \times [0, 1] \) with the measure \( \nu \times \text{Leb} \) where \( \text{Leb} \) is the Lebesgue measure. By Fubini’s theorem and the concavity of \( \rho \) we conclude:

\[
\begin{align*}
\int \mathfrak{X} f \cdot \varphi \, d\nu &= \int \mathfrak{X} f(x) \text{Leb}(\{ t \in [0, 1] \, | \, t \leq \varphi(x) \}) \, d\nu(x) \\
&= \int \mathfrak{X} f(x) \, d(\nu \times \text{Leb})(x, t) \\
&= \int_0^1 \int \mathfrak{X} f(x) \, d\nu(x) \, dt \leq \|f\|_\rho \cdot \rho \left( \nu(\{ x \in X \, | \, \varphi(x) \geq t \}) \right) dt \\
\|f\|_\rho \cdot \rho \left( \int_0^1 \nu(\{ x \in X \, | \, \varphi(x) \geq t \}) \, dt \right) &= \|f\|_\rho \cdot \rho \left( \int \mathfrak{X} \varphi \, d\nu \right)
\end{align*}
\]

The next is a version of a lemma by de la Vallée-Poussin (see [11, Theorem 19, Chapter 2]).

**Lemma A.0.11.** Suppose \( G : [0, \infty) \to [0, \infty) \) is a convex function with \( \lim_{t \to \infty} \frac{G(t)}{t} = +\infty \) and let \( M > 0 \). Then there is \( \rho \in \mathbf{M} \) and \( K > 0 \) such that for any probability space \( \mathfrak{X} \) and a measurable function \( f \) one has:

\[
\mathbb{E}_\mathfrak{X}[G \circ |f|] \leq M \quad \Longrightarrow \quad \|f\|_{C_\rho(\mathfrak{X})} \leq K
\]

**Proof.** Define for any \( v \in (0, 1] \):

\[
\rho_1(v) := \sup \left\{ t \, | \, G\left( \frac{t}{v} \right) \leq \frac{M}{v} \right\}
\]

Since \( \lim_{t \to \infty} \frac{G(t)}{t} = +\infty \) we have \( \rho_1(v) < \infty \) and \( \rho_1(0^+) = 0 \).

Define \( K = \rho_1(1) \) and let \( \rho_0 = \frac{\rho}{\rho_1} \). Using Lemma A.0.3(5) we find \( \rho \in \mathbf{M} \) with \( \rho_0(t) \leq \rho(t) \).

Let us show \( \rho, K \) satisfy the properties. Suppose \( \mathfrak{X} = (X, \Sigma, \nu) \) and that \( \int_X G \circ |f| \, d\nu \leq M \). Then for any \( A \in \Sigma \) with \( \nu(A) = v > 0 \) we have using Jensen:

\[
G\left( \frac{1}{v} \int_A |f| \, d\nu \right) \leq \frac{1}{v} \int_A G \circ |f| \, d\nu \leq \frac{M}{v} \implies \int_A |f| \, d\nu \leq \rho_1(v) \leq K \cdot \rho_1(\nu(A))
\]

Thus \( \|f\|_{C_\rho(\mathfrak{X})} \leq K \).□
The following proposition shows that one can extract the absolutely-integrable part out of any function, for this, note that any majorant is sub-additive.

**Proposition A.0.12.** Let $\mathcal{X}$ be a probability space, and let $f \in L^1(\mathcal{X})$, $\rho \in \mathcal{M}$, $C > 0$. Then there is $B \in \Sigma$ satisfying:

1. $f \cdot 1_{B^c} \in C_p(\mathcal{X})$ and $\|f \cdot 1_{B^c}\|_{\rho} \leq C$.

2. If $B \neq \emptyset$ then $\int_B |f| \, d\nu > C \cdot \rho(\nu(B))$.

**Proof.** For the proof of the proposition we will call $A \in \Sigma$ **bad** if $\int_A |f| \, d\nu > C \rho(\nu(A))$. We begin by verifying the following two observations:

1. If $(A_n)_n$ are pairwise disjoint bad sets then $A = \bigcup_n A_n$ is bad.

2. For any $B \in \Sigma$ we have: $||f \cdot 1_{B^c}||_{\rho} > C$ if and only if there is a bad set $D$ disjoint from $B$.

Indeed:

1. Since $\rho$ is continuous and sub-additive:

   $$\int_A |f| \, d\nu = \sum_n \int_{A_n} |f| \, d\nu \geq \sum_n C \cdot \rho(\nu(A_n)) \geq C \cdot \rho(\sum_n \nu(A_n)) = C \cdot \rho(\nu(A))$$

2. By definition, $||f \cdot 1_{B^c}||_{\rho} > C$ is equivalent to the existence of $A \in \Sigma$ with $\int_{A \cap B^c} |f| \, d\nu > C \cdot \rho(\nu(A \cap B^c))$. The "if" direction follows from taking $A = D$. The "only if" direction follows by taking $D = A \cap B^c$.

We return to the proof the proposition. Via observation 2, we need to show that there is a bad or empty set $B \subset X$ that intersects every other bad set. Assume otherwise, then for any bad or empty $B \subset X$ we have

$$\delta(B) := \sup \{ \nu(D) \mid D \text{ bad, } D \subset B^c \} > 0$$

Choose $\mathcal{D}[B]$ to be a bad set disjoint from $B$ with $\nu(\mathcal{D}[B]) > \frac{\delta(B)}{2}$.

Define inductively:

$$
\begin{align*}
B_0 &= \emptyset & D_0 &= \mathcal{D}[B_0] \\
B_1 &= D_0 & D_1 &= \mathcal{D}[B_1] \\
&\vdots & &\vdots \\
B_{n+1} &= B_n \cup D_n = \bigcup_{i=0}^n D_i & D_{n+1} &= \mathcal{D}[B_{n+1}]
\end{align*}
$$

This is well defined: at stage $n+1$ of the construction $D_i$ ($i = 0, \ldots, n$) are disjoint bad sets so by observation 1 we conclude that $B_{n+1}$ is also a bad set which allows us to define $D_{n+1}$.

Consider $B = \bigcup B_n = \bigcup D_n$, by observation 1 this is a bad set. Let $D = \mathcal{D}[B]$, then $\nu(D) > 0$. Note that as $D_n$ are disjoint in a probability space we must have $\lim_{n \to \infty} \nu(D_n) = 0$. Let $N$ be such that $\nu(D_N) < 2^{-2} \nu(D)$, then, $D$ is a bad set, disjoint from $B_N$ and $\nu(D) > 2\nu(D_N) = 2\nu(\mathcal{D}[B_N]) > \delta(B_N)$. This is a contradiction to the definition of $\delta$.

This has the following corollary:
Corollary A.0.13. Let $X$ be a probability space, and let $f \in L^1(X)$, $\rho \in M$, $\epsilon, C > 0$ such that the following implication is true:

\[
(A \in \Sigma) \quad \int_A |f|d\nu > C\rho(\nu(A)) \implies \nu(A) \leq \epsilon
\]

Then, there is $B \in \Sigma$ with $\nu(B) \leq \epsilon$ such that $f \cdot 1_B \in C_\rho(X)$ and $\|f \cdot 1_B\|_\rho \leq C$.

Proof. The $B$ from Proposition A.0.12 is either empty or satisfies $\int_B |f|d\nu > C \cdot \rho(\nu(B))$. By the implication this means that $\nu(B) \leq \epsilon$.

The last thing we discuss in this section is some functoriality. Suppose $\pi : Y = (Y, \Sigma_Y, m) \to X = (X, \Sigma_X, \nu)$ is a factor of measure spaces. Then we have the composition with $\pi$ which we denote by $\pi^* : L^1(X) \to L^1(Y)$. We also have the conditional expectation $E_{m,\pi} := E_{m\cdot [\cdot | X]} : L^1(Y) \to L^1(X)$.

Lemma A.0.14. Suppose $\pi : Y \to X$ is a factor of probability spaces and let $\rho \in M$. Then:

1. We have a contraction: $E_{m,\pi} : C_\rho(Y) \to C_\rho(X)$.

2. If $m' \ll m$ on $Y$ then $\pi_* m' \ll \nu$ on $X$.

3. We have an isometric embedding: $\pi^* : C_\rho(X) \to C_\rho(Y)$.

Proof. 1. Suppose $f \in C_\rho(Y)$ and $A \in \Sigma_X$, then:

\[
\int_A |E_{m,\pi}(f)|d\nu = \int_{\pi^{-1}(A)} |f|dm \leq \|f\|_{C_\rho(Y)} \cdot \rho(m(\pi^{-1}(A))) = \|f\|_{C_\rho(Y)} \cdot \rho(\nu(A))
\]

Thus $\|E_{m,\pi}(f)\|_{C_\rho(X)} \leq \|f\|_{C_\rho(Y)}$.

2. Follows from the previous item as $\|\frac{dm_{m'}}{dm}\|_{C_\rho(X,\nu)} = \frac{\|dm_{m'}\|_{C_\rho(X,\nu)}}{\|dm\|_{C_\rho(X,\nu)}} \leq \frac{\|dm_{m'}\|_{C_\rho(Y,m)}}{\|dm\|_{C_\rho(Y,m)}} = 1$.

3. Suppose $f \in C_\rho(X)$ and $A \in \Sigma_Y$, then applying Lemma A.0.10

\[
\int_A |\pi^*(f)|dm = \int_Y 1_A \pi^*(|f|)dm = \int_X 1_A d\nu \leq \|f\|_{C_\rho(Y)} \rho\left(\int_X 1_A d\nu\right) = \|f\|_{C_\rho(Y)} \rho(m(A))
\]

Thus $\|\pi^* f\|_{C_\rho(Y)} \leq \|f\|_{C_\rho(X)}$. By item 1 we conclude that $\pi^*$ is an isometric embedding.

\[\square\]

**B Poisson boundary of a time dependent matrix-valued random walk**

In this section we recall the Poisson boundary of a time dependent matrix-valued random walk from the papers [10], [13]. The goal of this section is to develop some of its properties that are required here. In particular, we consider a notion of stationary spaces parallel to $\mu$-stationary systems for the classical $\mu$-Poisson-Furstenberg boundary. In Theorem B.2.23 we prove an analogue of result of Furstenberg-Glasner [14, Theorem 4.3]. Throughout this section, $G$ is a discrete countable group.
B.1 Basic Definitions

Notation B.1.1. For a non-negative integer \( \ell \), we set \([\ell] := \{0, \ldots, \ell - 1\}\).

Throughout, we will have the following notation:
Let \( \ell = (\ell_n)_{n \geq 0} \) be a sequence of positive integers, denote \( \ell^{-1} = 1 \).

Definition B.1.2. An \( \ell \)-stochastic sequence is a sequence \( \sigma = (\sigma^{(n)})_{n \geq 0} \) where \( \sigma^{(n)} \) is an \([\ell_{n-1}] \times [\ell_n] \) stochastic matrix of measures on \( G \).
In more details, \( \sigma^{(n)} = (\sigma^{(n)}_{i,j})_{i \in [\ell_{n-1}], j \in [\ell_n]} \) where each \( \sigma^{(n)}_{i,j} \) is a non-negative measure on \( G \), and we have for each \( n, i \in [\ell_{n-1}] \) that \( \sum_{j \in [\ell_n]} \sigma^{(n)}_{i,j}(G) = 1 \).
We will always assume (as in [13]), that for any \( j \in [\ell_0] \), the measure \( \sigma_0^{(j)} \) is supported on all of \( G \).

Notation B.1.3. Suppose \( \sigma \) is an \( \ell \)-stochastic sequence.
- We denote \( V_n = [\ell_n] \times G \) for \( n \geq 0 \). This is a \( G \)-space with trivial action on the first coordinate.
- We consider the random walk \( (X_n)_{n \geq 0} \) (where \( X_n \) takes values in \( V_n \)) with transition probabilities \( (n \geq 1) \):
  \[ \mathbb{P}_{V_{n-1} \rightarrow V_n}(i,g) \rightarrow (j,h) = \sigma^{(n)}_{i,j}(g^{-1}h) \]
We get a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) where \( \Omega = \prod_{n \geq 0} V_n \) is the space of paths of the radon walk and the probability measure \( \mathbb{P} \) on \( \Omega \) is given by the Markov measure with transition probabilities as above and initial distribution \( \sigma^{(0)} \).
- We define the following random variables on \( \Omega \):
  1. \( Y_n((i_k, g_k)_k) = g_n \) (this is \( G \)-valued)
  2. \( I_n((i_k, g_k)_k) = i_n \) (this is \([\ell_n] \)-valued)
In this notation, \( X_n = (I_n,Y_n) \)
- The \( G \)-action on \( V_n \) gives rise to a \( G \)-action on \( \Omega \) with:
  \[ g \cdot ((i_n, g_n)) = ((i_n, g \cdot g_n)), \quad I_n(g \omega) = I_n(\omega), \quad Y_n(g \omega) = g Y_n(\omega) \]
- Let \( \mathcal{F}_m = \sigma(X_k|k \geq m), \mathcal{G}_m = \sigma(X_k|k \leq m) \). These are \( \sigma \)-algebras on \( \Omega \).
- The asymptotic algebra of the random walk: \( \mathcal{A}_\Omega = \bigcap_m \mathcal{F}_m \).
- For each \( m, j \in [\ell_m] \) consider the random walk with the same transition probabilities starting at \( (j,g) \in V_m \). The Markov measure will be denoted by \( \mathbb{P}^{(m)}_{(j,g)} \) and we can consider it as a measure on \( (\Omega, \mathcal{F}_m) \).
  We define \( \mathbb{P}^{(m)} \) to be the column vector of probability measures \( (\mathbb{P}^{(m)}_{j,e})_{j \in [\ell_m]} \).

Note that in the notations above,
- \( \mathbb{P}^{(-1)} = \mathbb{P} \)
- \( \mathbb{P}^{(0)}_{(j,g)} = g \mathbb{P}_{(j,e)} \)
• $X_n = (I_n, Y_n)$ has distribution given by matrix multiplication:

$$
\mathbb{P}(I_n = i, Y_n = g) = (\sigma^{(0)} \cdots \sigma^{(n)})_i(g)
$$

**Definition B.1.4.** A bounded $\sigma$-harmonic function is a sequence $h = (h_m)_{m\geq 0}$ consisting of bounded complex valued functions $h_m : V_m \to \mathbb{C}$ with $\sup_m \|h_m\|_{L^\infty(V_m)} < \infty$ and such that for all $m \geq 1$:

$$
h_{m-1}(i, g) = \sum_{(j, h) \in V_m} h_m((j, h)) \cdot \sigma^{(m)}_{i,j}(g^{-1}h)
$$

The space of all bounded $\sigma$-harmonic functions will be denoted by $\mathcal{H}_\sigma^\infty(G)$. When equipped with the norm $\|h\| = \sup_m \|h_m\|_{L^\infty(V_m)}$, $\mathcal{H}_\sigma^\infty(G)$ is a Banach space. It has the following isometric left $G$-action

$$
g \cdot (h_m)_{m \geq 0} = (g \cdot h_m)_{m \geq 0}
$$

where: $(g \cdot f)(z) = f(g^{-1}z)$

We may rewrite this condition of harmonicity in the following ways, the first is:

$$
\forall v \in V_{m-1} : \ h_{m-1}(v) = \sum_{u \in V_m} h_m(u) \cdot \mathbb{P}_{V_{m-1} \to V_m}(v \mapsto u)
$$

And the second is to think of $h_m$ as a $\ell_m$-row vector and then in matrix notation:

$$
h_{m-1} = h_m \ast \tilde{\sigma}^{(m)}
$$

Here, for a matrix $\sigma = (\sigma_{i,j})$ we denote $\tilde{\sigma}(g) = (\sigma_{j,i}(g^{-1}))$.

**Definition B.1.5.** Let $\sigma$ be an $\ell$-stochastic sequence.

1. A $\sigma$-stationary space (or $\sigma$-system), $X = (X, \nu)$ is a measure space $X$ together with a measurable $G$-action, and a sequence $\nu = (\nu^{(n)})_{n=1}^\infty$ of $\ell_n$-column vectors $\nu^{(n)} = (\nu_j^{(n)})_{j \in [\ell_n]}$ such that any $\nu_j^{(n)}$ is a probability measure on $X$, and such that $\sigma^{(n)} \ast \nu^{(n)} = \nu^{(n-1)}$ for any $n \geq 0$.

2. A factor of $\sigma$-stationary spaces $\pi : X = (X, \nu) \to Y = (Y, m)$ is a measurable $G$-mapping $\pi : X \to Y$ such that $\pi_* (\nu^{(n)}_j) = m^{(n)}_j$. We say that $Y$ is a factor of $X$, and that $X$ is an extension of $Y$.

Since $\nu^{(-1)} = \sigma^{(0)} \ast \nu^{(0)}$ and $\sigma^{(0)}_{i,j}$ are QI we see that $(X, \nu^{(-1)})$ is a QI $G$-space. We will consider $X$ with this measure class as the underlying $G$-space for $X$ and denote $L^\infty(X) = L^\infty(X, \nu^{(-1)})$.

From now on, we will assume that the matrices $\sigma^{(n)}$ do not have a zero column. More formally, this assumption on $\sigma$ is: For any $n$ and $j \in [\ell_n]$ there is $i \in [\ell_{n-1}]$ with $\sigma^{(n)}_{i,j}(G) > 0$.

Under the assumption, it is easy to see that for each for any $n$, $j \in [\ell_n]$ we have $\nu^{(n)}_j \ll \nu^{(-1)}$.

**Remark B.1.6.** This assumption does not change much for any $\sigma$, by deleting the zero-columns (and their corresponding rows in the next matrix) we will obtain a new stochastic sequence $\tau$ (for a different $\ell$). For any $\sigma$-stationary space, considering only the relevant $\nu^{(n)}_j$ yields a $\tau$-stationary space. Their underlying $G$-space is the same.

Later we will consider the Poisson boundary (see Definition B.1.9), it will be clear that this operation sends the Poisson boundary of $\sigma$ to the one of $\tau$.

We can now define a version of Furstenberg-Poisson transform:
Proposition B.1.7. Let $\mathcal{X} = (X, \nu)$ be a $\sigma$-stationary space, for any $f \in L^\infty(\mathcal{X})$, define $P_X(f)$ by:

$$
(P_X(f))_{n}(i, g) := \int_{X} f(g \cdot \nu_i^{(n)}) = \int_{X} f(gx) d\nu_i^{(n)}(x)
$$

Then $P_X(f) \in \mathcal{H}_\sigma^\infty(G)$. Moreover,

1. $P_X : L^\infty(\mathcal{X}) \to \mathcal{H}_\sigma^\infty(G)$ is a $G$-equivariant linear contraction.
2. If $\pi : X \to Y$ is a factor of $\sigma$-stationary spaces then $P_X \circ \pi^* = P_Y$.

Proof. The equalities

$$
\sum_{(j, h) \in V_m} \sigma_{i, j}^{(m)} (g^{-1} h) (P_X(f))_{m}(j, h) = \int_{X} f \left( \sum_{(j, h) \in V_m} \sigma_{i, j}^{(m)} (g^{-1} h) h \nu_j^{(m)} \right) = \\
\int_{X} f \left( g \cdot \sum_{(j, s) \in V_m} \sigma_{i, j} (s) s \nu_j^{(m)} \right) = \\
\int_{X} f \left( g \cdot \left((\sigma^{(m)} \ast \nu^{(m)})_{i}\right) \right) = \int_{X} f \left( g \cdot \nu_i^{(m-1)} \right) = (P_X(f))_{m-1}(i, g)
$$

shows that $P_X(f) \in \mathcal{H}_\sigma^\infty(G)$.

For the moreover part: it is obvious that $\|P_X\| \leq 1$. For the $G$-equivariance:

$$
(g(P_X(f)))_{n}(i, h) = (P_X(f))_{n}(i, g^{-1} h) = \int_{X} f(g^{-1} h \cdot \nu_i^{(n)}) = \int_{X} (g \cdot f) d(h \cdot \nu_i^{(n)}) = (P_X(g \cdot f))_{n}(i, h)
$$

The naturality of $\mathcal{P}$ is trivial.

We call the operator $P_X$ the Poisson transform.

Our next goal is to define the Poisson boundary of $\sigma$ using the random walk described above and to show that the Poisson transform attached to the Poisson boundary is an isomorphism.

Proposition B.1.8. We have equality of vectors of measures on $(\Omega, \mathcal{F}_m)$

$$
\sigma^{(m)} \ast \mathbb{P}^{(m)} = \mathbb{P}^{(m-1)}
$$

In particular, $(\Omega, \mathcal{A}_\Omega, (\mathbb{P}^{(n)})_n)$ is a $\sigma$-stationary space.

Proof. We only need to show the equality for the generating algebra consisting of cylinder sets of
the form $A = \{X_{t+r} = (i_{m+r}, g_{m+r}), \ (0 \leq r \leq \ell)\}$. And indeed, for $i \in [\ell_{m-1}]
\mathbb{P}^{(m)}(A) = \sigma_{i,m}^{(m)}(g_m) \cdot \sigma_{i,m+1}^{(m+1)}(g_{m+1} g_{m+1}) \cdot \ldots \cdot \sigma_{i,\ell-1,m+1}^{(m+\ell)}(g_{m+1} g_{m+1}) =
\sum_{g \in G, j \in [\ell_m]} \sigma_{i,j}^{(m)}(g) \cdot \delta(j, g_m) \cdot \prod_{r=1}^{\ell} \sigma_{i,m+r-1,m+1}^{(m+r)}(g_{m+r-1} g_{m+1}) =
\sum_{g \in G, j \in [\ell_m]} \sigma_{i,j}^{(m)}(g) \cdot \delta(j, g_m) \cdot \prod_{r=1}^{\ell} \sigma_{i,m+r-1,m+1}^{(m+r)}((g_{m+r-1})^{-1} g_{m+1}) =
\sum_{g \in G, j \in [\ell_m]} \sigma_{i,j}^{(m)}(g) \cdot \mathbb{P}^{(m)}((X_{m+r} = (i_{m+r}, g_{m+1}^{-1} g_{m+1}), \ (0 \leq r \leq \ell))) =
\sum_{g \in G, j \in [\ell_m]} \sigma_{i,j}^{(m)}(g) \cdot \mathbb{P}^{(m)}(g_{m+1}^{-1} A) = \sum_{g \in G, j \in [\ell_m]} \sigma_{i,j}^{(m)}(g) \cdot \mathbb{P}^{(m)}(A) = \sum_{g \in G, j \in [\ell_m]} \mathbb{P}^{(m)}(A) = (\sigma^{(m)} \ast \mathbb{P}^{(m)})(A).

\qed

**Definition B.1.9.** A Poisson boundary of $\sigma$, denoted by $\mathcal{B}(G, \sigma) = (B, \Sigma, (\nu_j^{(m)})_{m=1}^{\infty})$ is a $\sigma$-stationary space with $(B, \Sigma)$ a standard Borel space together with a $G$-equivariant isomorphism $L^\infty(B, \Sigma, \nu^{(-1)}) \cong L^\infty(\Omega, \mathcal{A}_\Omega, \mathbb{P})$ for which $\nu_j^{(m)}$ corresponds to $\mathbb{P}^{(m)}$.

Using the separability of $L^\infty(\Omega, \mathcal{A}_\Omega, \mathbb{P})$ with respect to the $w^*$-topology and [20, Theorem 2.1] we conclude:

**Proposition B.1.10.**

1. A Poisson boundary $(B, \Sigma, \nu)$ of $\sigma$ exists.

2. A Poisson boundary is unique up to a unique isomorphism that commutes with the isomorphism $L^\infty(B, \Sigma, \nu^{(-1)}) \cong L^\infty(\Omega, \mathcal{A}_\Omega, \mathbb{P})$.

3. There is a factor map $\tau : \Omega \to B$ inducing the isomorphism $L^\infty(B, \Sigma, \nu^{(-1)}) \cong L^\infty(\Omega, \mathcal{A}_\Omega, \mathbb{P})$.

**Remark B.1.11.** It follows from Proposition B.1.14 below that the isomorphism (as $\sigma$-systems) is actually unique and the commuting assumption in the uniqueness of Poisson boundary is redundant.

**Proposition B.1.12.** Let $h \in \mathcal{H}_G^\infty(G)$ and consider

$$M_n(\omega) = h_n(X_n(\omega)) = (h_n)_{f_n(\omega)}(Y_n(\omega))$$

Then $M_n$ is a bounded martingale for $(\Omega, \{G_n\}, \mathbb{P})$.

**Proof.** Indeed, $M_n$ is $G_n$-measurable and bounded by $\|h\|$. We need to show that for $A \in G_n$ we have $\int_{\Omega} M_n \cdot 1_A d\mathbb{P} = \int_{\Omega} M_{n+1} \cdot 1_A d\mathbb{P}$. It is enough to show this for $A = \{X_r = v_r, \ (0 \leq r \leq n)\}$ where $v_r = (i_r, g_r) \in V_r$. Indeed:

$$\int_{\Omega} M_{n+1} \cdot 1_A d\mathbb{P} = \int_{\Omega} h_{n+1}(X_{n+1}(\omega)) \sum_{v_{n+1} \in V_{n+1}} 1_{A \cap (X_{n+1} = v_{n+1})}(\omega) d\mathbb{P}(\omega) =
\sum_{v_{n+1} \in V_{n+1}} h_{n+1}(v_{n+1}) \cdot \mathbb{P}\left(X_r = v_r, \ (0 \leq r \leq n+1)\right) =
\sum_{v_{n+1} \in V_{n+1}} h_{n+1}(v_{n+1}) \mathbb{P}_{V_{n+1} = v_{n+1}}(v_{n+1}) \mathbb{P}(A) = h_n(v_n) \mathbb{P}(A) = \int_{\Omega} M_n \cdot 1_A d\mathbb{P}$$
Lemma B.1.13. Let \( v = (g, j) \in V_n \). Let \( B \in \mathcal{G}_{n-1} \) and let \( A = B \cap \{ X_n = v \} \). Then for any \( f \in L^\infty(\Omega, \mathcal{F}_{n+1}) \) we have:
\[
P(A) \cdot \int_A f \, d(g_{j,n}^{(n)}) = \int_A f \, d\mathbb{P}
\]

**Proof.** It is enough to verify the identity in the case where \( f \) is the indicator function of the cylinder \( C = \{ X_{n+1} = v_{n+1}, \ldots, X_{n+\ell} = v_{n+\ell} \} \) for \( v_{n+r} = (i_{n+r}, g_{n+r}) \in V_{n+r} \), \( (r = 1, \ldots, \ell) \). Moreover, we may assume that \( B = \{ X_0 = v_0, \ldots, X_{n-1} = v_{n-1} \} \) for \( v_t = (i_t, g_t) \in V_t \), \( (t = 0, \ldots, n-1) \). In this case:
\[
P(A) \cdot \int_A f \, d(g_{j,n}^{(n)}) = P(A) \cdot P((\sigma_{0}^{(0)}(g_{0}) \sigma_{i_{0},j_{0}}^{(1)}(g_{1}) \cdots \sigma_{i_{n-1},j_{n-1}}^{(n)}(g_{n-1}) \sigma_{i_{n},j_{n}}^{(n+1)}(g_{n+1}) \sigma_{i_{n+1},j_{n+1}}^{(n+2)}(g_{n+1}g_{n+2}) \cdots \sigma_{i_{n+r},j_{n+r}}^{(n+r)}(g_{n+r}^{-1}g_{n+r+1})) \}
= P(A \cap C) = \int_A f \, d\mathbb{P}
\]

**Proposition B.1.14.** The Poisson transform defines an isometric isomorphism:
\[
\mathcal{P}_\Omega : L^\infty(\Omega, \mathcal{A}_\Omega, \mathbb{P}) \xrightarrow{\sim} \mathcal{H}^\infty(G)
\]
Whose inverse satisfies:
\[
\mathcal{P}_\Omega^{-1}(h) = \lim_{n \to \infty} h_n(X_n) \quad a.e.
\]

**Proof.** Let \( h \in \mathcal{H}^\infty(G) \), considering the martingale \( M_n = h_n(X_n) \) of Proposition B.1.12, by the martingale convergence theorem (see [24, Chapter 11]), we conclude that \( M_n \) converges a.e. and in \( L^1 \) to a function \( f \in L^1(\Omega, \mathbb{P}) \). Note that \( f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \) and \( \|f\|_\infty \leq \|h\|_\infty \).

For all \( m \) we have that \( M_n \) is \( \mathcal{F}_m \)-measurable for \( n \geq m \) and thus, \( f \) is \( \mathcal{F}_{\infty} \)-measurable for all \( m \). This implies \( \mathcal{M}(h) := f \in L^\infty(\Omega, \mathcal{A}_\Omega) \).

To prove the proposition we will show that the contractions \( \mathcal{M}, \mathcal{P}_\Omega \) are inverse to each other.

Let \( h \in \mathcal{H}^\infty(G) \) and consider \( f = \mathcal{M}(h) \). We wish to show \( \mathcal{P}_\Omega(f) = h \). Indeed, for any \( n \geq 0, j \in [i_n], g \in G \) consider \( A = \{ X_n = (j,g) \} \). Note that since \( \sigma^{(0)} \) is positive everywhere, we have \( P(A) > 0 \).

Using B.1.13 and the martingale property we get:
\[
h_n(j,g) = \frac{1}{P(A)} \int_A h_n(X_n) d\mathbb{P} = \frac{1}{P(A)} \lim_{n \to \infty} \int h_m(X_m) d\mathbb{P} = \frac{1}{P(A)} \int_A f d\mathbb{P} = \int_A f d\mathbb{P}(g_{j,n}^{(n)}) = \mathcal{P}_\Omega(f)(j,g)
\]

For the other composition, let \( f \in L^\infty(\Omega, \mathcal{A}_\Omega) \) and consider \( h = \mathcal{P}_\Omega(f) \). We need to show that \( f = \mathcal{M}(h) \), that is, that \( M_n = h_n(X_n) \) converges a.e. to \( f \).

Using Levy’s upward convergence for martingales (see [24, Chapter 14]) it is enough to show that \( M_n = \mathbb{E}_P(f|G_n) \). As \( M_n \) is \( \mathcal{G}_n \)-measurable we only need to check \( \int_A f \cdot 1_A d\mathbb{P} = \int_A M_n \cdot 1_A d\mathbb{P} \) for all \( A \in \mathcal{G}_n \). By additivity we may assume that \( X_n|A = (j,g) \) and then by Lemma B.1.13 we have:
\[
\int_\Omega f \cdot 1_A d\mathbb{P} = \int_A f d\mathbb{P} = P(A) \int_A f d\mathbb{P}(g_{j,n}^{(n)}) = P(A)h_n(j,g) = \int_\Omega h_n(X_n(\omega)) \cdot 1_A d\mathbb{P} = \int_\Omega M_n \cdot 1_A d\mathbb{P}
\]

concluding the result. 

\[
\Box
\]
Corollary B.1.15. The subspace spanned by \( \{ \frac{dg_i}{dp} : n \geq 0, i \in [\ell_n], g \in G \} \) is dense in \( L^1(\Omega, A_\Omega, \mathbb{P}) \).

Proof. Otherwise, by Hahn-Banach theorem, there would be a non zero \( f \in L^1(\Omega, A_\Omega, \mathbb{P}) \) that vanishes on all those functions, meaning that
\[
\int_{\Omega} f \cdot \frac{dg_i}{dp} \, dp = 0
\]
But this implies that \( P_{\Omega}(f) = 0 \in \mathcal{H}_\sigma^\infty(G) \). By Proposition B.1.14 we conclude that \( f = 0 \) which is a contradiction. \( \square \)

Definition B.1.16. We will say that a factor between \( \sigma \)-stationary spaces \( \pi : (X, \nu) \to (Y, m) \) is measure preserving extension if for any \( n, i \in [\ell_n], g \in G \) we have that
\[
\frac{dg_{ni}}{dm^{(n)}} = \frac{dg_{ni}}{dm^{(n)}} \circ \pi
\]
In particular, \( (X, \nu^{(-1)}) \to (Y, m^{(-1)}) \) is measure preserving in this case.

Lemma B.1.17. Let \( X \) be a \( \sigma \)-stationary space. Then there is a \( \sigma \)-stationary system \( X_{RN} \) along with a measure preserving extension \( \pi : X \to X_{RN} \) with the following property: for any measure preserving extension of \( \sigma \)-stationary spaces \( p : X \to Y \), we have that \( \pi \) factors through \( p \), that is, there is a measure preserving extension \( q : Y \to X_{RN} \) with \( \pi = q \circ p \).

Proof. Consider the \( \sigma \)-algebra \( E \) generated by the (countably many) functions \( \frac{dg_{ni}}{dm^{(n)}} \) and take any standard Borel space \( (X_{RN}, m^{(-1)}) \) with \( L^\infty(X_{RN}) \cong L^\infty(X, E) \). Then \( X_{RN} \) posses a natural Borel probability measures \( m_{ni} \) and a measurable \( G \)-action that makes it a \( \sigma \)-stationary space \( X_{RN} \). As \( X_{RN} \) is a standard Borel space, using [20, Theorem 2.1] we get a measure preserving factor \( X \to X_{RN} \) satisfying the universal property. \( \square \)

By abstract nonsense, the pair \( (X_{RN}, \pi) \) is unique up to a unique isomorphism. We call \( X_{RN} \) the Radon-Nikodym factor of the \( \sigma \)-stationary space \( X \).

Proposition B.1.18. The Poisson boundary \( \mathcal{B}(G, \sigma) \) is its own Radon-Nikodym factor.

Proof. Clear from Corollary B.1.15. \( \square \)

B.2 Conditional measures and a structure theorem

Throughout this subsection, \( \sigma \) denotes an \( \ell \)-stochastic sequence with no zero columns.

Definition B.2.1. A topological \( \sigma \)-system is a \( \sigma \)-stationary space \( (X, \Sigma, \nu) \) such that \( X \) is a compact metrizable space, \( \Sigma \) is the Borel \( \sigma \)-algebra on \( X \) and the \( G \)-action is continuous.

We use the following well known lemma from functional analysis:

Lemma B.2.2. Let \( E \) be a Banach space and let \( \lambda_n \in E^* \) be a sequence of continuous linear functionals satisfying \( \sup \| \lambda_n \|_{E^*} \leq 1 \). Then the set:
\[
L := \{ v \in E \mid \lim_{n \to \infty} \lambda_n(v) \}
\]
is a closed linear subspace, and the function \( \lambda : L \to \mathbb{C} \) defined by \( \lambda(v) = \lim_{n \to \infty} \lambda_n(v) \) is a bounded functional with \( \| \lambda \|_{L^\infty} \leq 1 \).
The following is the existence of stationary measures in the $\sigma$-stationary setting, generalizing the construction in the classical Poisson boundary (see e.g. [7, section 2, III–VII]).

We recall from Notation B.1.3 that $(I_n,Y_n)$ is the $n$-th step of the $\sigma$-random walk – these are random variables on $(\Omega,\mathcal{F},\mathbb{P})$. Throughout we let $(B,\mathfrak{m}) = \mathcal{B}(G,\sigma)$ be the Poisson boundary and let $\tau : \Omega \to B$ be the factor map.

**Proposition B.2.3.** Let $\mathcal{X} = (X,\nu)$ be a topological $\sigma$-stationary space, let $M(\mathcal{X})$ denote the space of Borel probability measures on $X$ with the $w^*$ topology. Then there is a unique (up to a.e. equality) measurable $G$-mapping $\Phi_X : B \to M(\mathcal{X})$ with the following integral factorization for any $n \geq -1$, $j \in [\ell_n]$:

$$\nu_j^{(n)} = \int_B \Phi_X d\nu_j^{(n)}$$

Moreover, we have the following limit in $M(\mathcal{X})$ for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\lim_{n \to \infty} Y_n(\omega) \cdot \nu_j^{(n)} = \Phi_X \circ \tau(\omega)$$

**Proof.** Uniqueness of $\Phi_X$: suppose $\Phi_1, \Phi_2$ both satisfy the integral factorization. Since $M(\mathcal{X})$ is separable with the $w^*$ topology it is enough to show that for any $f \in C(\mathcal{X})$ the equality a.e. of $\Phi_2(b)(f) = \Phi_1(b)(f)$. And indeed, we know that for any $n, j \in [\ell_n]$ and $g \in G$ we have for $k = 1, 2$:

$$\mathcal{P}_X(f)_n(j,g) = \int_X f \ d(\nu_j^{(n)}) = \int_X f \circ g \ d\nu_j^{(n)} = \int_B \left( \int_X f \circ g \ d\Phi_k(b) \right) d\nu_j^{(n)}(b) = \int_B \left( \int_X f \ d\Phi_k(b) \right) \nu_j^{(n)}(b) = \int_B \left( \int_X f \ d\Phi_k(b) \right) \nu_j^{(n)}(b) =$$

$$= \int_B \left( \int_X f \ d\Phi_k(b) \right) \nu_j^{(n)}(b) = \mathcal{P}_B((f,\Phi_k))_n(j,g)$$

Since the Poisson transform is an isomorphism for the Poisson boundary (Proposition B.1.14) we conclude $(f,\Phi_1) = (f,\Phi_2)$ a.e. as claimed.

Existence of $\Phi_X$: the algebra $A = C(\mathcal{X})$ is separable. Take $A_0 \subset A$ a countable $G$-invariant $\mathbb{Q}[i]$-algebra that is dense in $A$. For each $f \in A_0$, we consider $\mathcal{P}_X(f) \in \mathcal{H}_\sigma^w(G)$ and using $L^\infty(B) \cong \mathcal{H}_\sigma^w(G)$ we choose a measurable function $\varphi_f$ on $B$ bounded by $||f||$ such that $\mathcal{P}_X(f) = \mathcal{P}_B(\varphi_f)$. Given $b \in B$ consider the map $\nu_b : A_0 \to C$ given by $\nu_b(f) = \varphi_f(b)$. Since $A_0$ is countable, we get there is a co-null subset $D \subset B$ such that for any $b \in D$ we have $\nu_b$ is $\mathbb{Q}[i]$-linear, positive and $\nu_b(1) = 1$. These properties insures that for each $b \in D$, $\nu_b$ extends to a state of $A = C(\mathcal{X})$. Thus we may regard $\nu_b$ as an element $\nu_b \in M(\mathcal{X})$. We define $\Phi_X(b) = \nu_b$ for $b \in D$ (and for the null set $B \setminus D$ take $\Phi_X(b) = \nu^{(-1)}$ for example).

We now verify that $\Phi_X$ satisfies the required properties. First, let us show $\Phi_X$ is measurable. Indeed, for any $f \in A_0$ and $b \in D$ we have $(f,\Phi_X(b)) = \nu_b(f) = \varphi_f(b)$ which is a measurable function on $D$. Since $A_0$ generates the topology and thus the $\sigma$-algebra we conclude $\Phi_X$ is measurable. Next, note that for any $f \in A_0$ we have a.e. $(f,\Phi_X(b)) = \mathcal{P}_B^{-1} \circ \mathcal{P}_X(f)(b)$. This yields the $G$-equivariance of $\Phi_X$ since the Poisson transforms are $G$-equivariant.

Thus we constructed a measurable $G$-equivariant map $\Phi_X : B \to M(\mathcal{X})$ such that for any $f \in A_0$:

$$\int_X f d\nu_j^{(n)} = \mathcal{P}_X(f)_n(j,e) = \mathcal{P}_B((f,\Phi_X))_n(j,e) = \int_B (f,\Phi_X(b)) d\nu_j^{(n)}(b) = \int_X f d\left( \int_B (f,\Phi_X(b)) d\nu_j^{(n)}(b) \right)$$
Thus this holds for any \( f \in A = C(X) \) which yields the integral factorization:

\[
\nu_j^{(n)} = \int_B \Phi_X \, dm_j^{(n)}
\]

For the moreover part, we use Proposition B.1.14 to conclude that for any \( f \in A_0 \) we have a co-null set \( \Omega_f \subset \Omega \) so that for \( \omega \in \Omega_f \):

\[
\left( f, \Phi_X \circ \tau(\omega) \right) = \mathcal{P}^{-1}_B(\mathcal{P}_X(f))(\tau(\omega)) = \mathcal{P}^{-1}_B(\mathcal{P}_X(f))(\omega) = \lim_{n \to \infty} \mathcal{P}_X(f)_n(I_n(\omega), Y_n(\omega)) = \lim_{n \to \infty} \int_X f \, d\left( Y_n(\omega) \nu_j^{(n)} \right)
\]

The set \( \Omega' = \bigcap_{f \in A_0} \Omega_f \) is co-null. Using Lemma B.2.2 we see that for \( \omega \in \Omega' \) we have the following convergence in \( \omega^* \)-topology:

\[
\lim_{n \to \infty} Y_n(\omega)\nu_j^{(n)} = \Phi_X(\tau(\omega)),
\]

concluding the result. □

**Definition B.2.4.** Let \( \mathcal{X} = (X, \nu) \) be a topological \( \sigma \)-stationary space. The mapping \( \Phi_X \) is called the conditional measures (sometimes we will write \( \Phi_X \) when we want to be specific).

Our next goal is to show functoriality of the conditional measures. First, we prove the following lemma:

**Lemma B.2.5.** Let \( \pi : \mathcal{X} \to \mathcal{Y} \) be a factor between two topological \( \sigma \)-systems. Then, there is a topological \( \sigma \)-systems \( Z \) and factors \( \tau : Z \to \mathcal{X} \) and \( \theta : Z \to \mathcal{Y} \) such that \( \tau, \theta \) are continuous and \( \theta = \pi \circ \tau \). Moreover one can take \( \tau \) to be an isomorphism (up to null sets).

**Proof.** Consider the \( C^* \)-algebra \( A = C(X) + \pi C(Y) \subseteq L^\infty(X) \) and let \( Z \) be the Gelfand dual to \( A \). Since \( A \) is \( G \)-invariant and separable we conclude that \( Z \) has the structure of a topological \( \sigma \)-system. Since \( C(X), \pi C(Y) \subseteq C(Z) \) we get the continuous factors \( \tau, \theta \) as needed. As \( C(Z) \subseteq L^\infty(X) \) we conclude \( \tau \) is an isomorphism. □

**Proposition B.2.6.** Let \( \pi : (X, \nu) \to (Y, \eta) \) be a factor (of \( \sigma \)-systems) between topological \( \sigma \)-systems, then (a.e.):

\[
\pi_* \Phi_X = \Phi_Y
\]

**Proof.** We first consider the case where \( \pi \) is continuous. By the uniqueness in Proposition B.2.3 we only need to exhibit the integral decomposition for \( \pi_* \Phi_X \). Note that \( \pi_* : M(X) \to M(Y) \) is the restriction of the \( \nu^* \)-continuous linear operator \( (\pi^*)^* : C(Y)^* \to C(Y)^* \) and thus preserves integration. Thus:

\[
\nu_j^{(n)} = \pi_* \nu_j^{(n)} = \pi_* \left( \int_B \Phi_X \, dm_j^{(n)} \right) = \int_B \pi_* \Phi_X \, dm_j^{(n)}
\]

In the general case, take \( Z = (Z, \lambda) \) as in Lemma B.2.5, then since \( \tau, \theta \) are continuous, we have:

\[
\pi_* \Phi_X = \pi_* (\tau_* \Phi_Z) = (\pi \circ \tau)_* \Phi_Z = \theta_* \Phi_Z = \Phi_Y
\]

**Definition B.2.7.** We say that a \( \sigma \)-system \( \mathcal{X} = (X, \Sigma, \nu) \) is regular if the underlying Borel space \( (X, \Sigma) \) is a standard Borel space. We will denote by \( M(X, \Sigma) \) the collection of all probability measures on the Borel space \( (X, \Sigma) \).
Remark B.2.8. Above we identified two mappings \( \pi_1, \pi_2 : X \to Y \) if they are equal \( \nu^{(-1)} \)-a.e. Note that this does not imply that the mappings \((\pi_1)_*, (\pi_2)_* : M(X, \Sigma_X) \to M(Y, \Sigma_Y)\) are equal. However, if \((X, \Sigma_X)\) is standard and we have given a measurable mapping \( T : M(X, \Sigma_X) \to M(Y, \Sigma_Y) \), \( t \mapsto \nu_t \), where \( T \) posses a measure \( m \) with \( \int_T \nu_t \, dm \ll \nu^{(-1)} \), we have \((\pi_1)_* \nu_t = (\pi_2)_* \nu_t\) for \( m\)-a.e. \( t \in T \).

The following lemma is clear by taking a topological model:

Lemma B.2.9. If a \( \sigma \)-stationary space is regular then it is isomorphic to a topological \( \sigma \)-system.

Example B.2.10. For any \( \sigma \)-stationary space \( \mathcal{X} \), the proof of Lemma B.1.17 shows that the Radon-Nikodym factor \( \mathcal{X}_{RN} \) can be given as a regular \( \sigma \)-stationary space.

We now summarize the previous propositions and lemmas in a theorem regarding conditional measures:

Theorem B.2.11. Let \( \mathcal{X} = (X, \nu) \) be a regular \( \sigma \)-stationary space. Then there is a unique (up to a.e. equality) measurable \( G \)-mapping \( \Phi_X : B \to M(X, \Sigma_X) \) from the Poisson boundary with the following integral factorization for any \( n \geq 1, j \in [f_n], g \in G \):

\[
\nu_j^{(n)} = \int_B \Phi_X \, dg \nu_j^{(n)}
\]

Moreover, for every \( f \in L^\infty(X) \) we have for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \)

\[
\lim_{n \to \infty} \{Y_n(\omega) \cdot \nu_j^{(n)}, f\} = \{\Phi_X(\tau(\omega)), f\}
\]

The \( \Phi \) is natural: given a factor \( \pi : \mathcal{X} \to \mathcal{Y} \) we have that \( \pi_* \Phi_X = \Phi_Y \).

Remark B.2.12. In the theory of classical Poisson boundary (e.g. [14], [7]), the measure \( \Phi_X(\tau(\omega)) \) is commonly denoted by \( \nu_\omega \).

Definition B.2.13. Let \( \mathcal{X} = (X, \nu) \) be a regular \( \sigma \)-stationary space. The mapping \( \Phi_X \) is called the conditional measures.

We define a \( \sigma \)-system \( \Pi(X) := (M(X, \Sigma_X), (\Phi_X)_*, m) \).

Example B.2.14. For the Poisson boundary one has \( \Phi_B(b) = \delta_b \).

Definition B.2.15. A factor \( \pi : \mathcal{X} \to \mathcal{Y} \) between regular \( \sigma \)-systems is called proximal if for a.e. \( b \in B \) we have that \( \pi : (X, \Phi_X(b)) \to (Y, \Phi_Y(b)) \) is an isomorphism (as usual, modulo null sets). We say that \( \mathcal{X} \) is proximal if \( \mathcal{X} \to pt \) is proximal.

Remark B.2.16. By definition, a system is proximal iff the conditional measures are a.e. delta measures.

Lemma B.2.17. A regular \( \sigma \)-system is proximal iff it is a factor of the Poisson boundary.

In particular, for any regular \( \sigma \)-system \( \mathcal{X} \), the \( \sigma \)-system \( \Pi(\mathcal{X}) \) is proximal.

Proof. If a regular \( \sigma \)-system is a factor of the Poisson boundary, then by the functionality of conditional measures and Example B.2.14 we conclude that the system is proximal. On the other direction, if a regular system \( \mathcal{X} \) is proximal then for a.e. \( b \in B \) we have \( \Phi_X(b) = \delta_{\pi(b)} \). The mapping \( \pi : B \to X \) is measurable and \( G \)-equivariant and the integral decomposition in Theorem B.2.11 yields that \( \pi \) is a factor.
Remark B.2.18. Note that the proof of Lemma B.2.17 implies that being a factor of the Poisson boundary is a property of the system and not an extra structure. It follows that proximal systems does not possess nontrivial endomorphisms.

We need the following easy lemma in measure theory (no group action) relating disintegration and Radon-Nikodym derivative.

Lemma B.2.19. Suppose \( \pi : (X, \nu) \to (Y, m) \) is a factor between standard Borel spaces. Let \( \nu_0 \ll \nu \) and \( m_0 = \pi_*\nu_0 \). Suppose there is a measurable mapping \( \Phi : Y \to M(X, \Sigma_X) \) such that:

- For \( m \cdot \text{a.e. } y \in Y \) we have \( \pi_*\Phi(y) = \delta_y \).
- \( \nu = \int_Y \Phi \, dm \) and \( \nu_0 = \int_Y \Phi \, dm_0 \).

Then \( \frac{dm_0}{d\nu} = \frac{dm_0}{dm} \circ \pi \).

Proof. We first show that for any \( f \in L^\infty(X, \nu) \), \( g \in L^1(Y, m) \) we have for \( m \cdot \text{a.e. } y \in Y \) that \( \langle \Phi(y), f \cdot (g \circ \pi) \rangle = \langle \Phi(y), f \cdot g \rangle \). Indeed:

\[
\left| \langle \Phi(y), f \cdot (g \circ \pi) \rangle - \langle \Phi(y), f \rangle \cdot g(y) \right| = \left| \langle \Phi(y), f \cdot (\pi^*(g - g(y))) \rangle \right| \leq \|f\|_\infty \cdot \|\pi^*(g - g(y))\| = \|f\|_\infty \cdot \|\delta_y \cdot (g - g(y))\| = 0
\]

Thus, for any \( f \in L^\infty(X, \nu) \):

\[
\int_X f \, d\nu_0 = \int_Y \Phi(y) \, d\nu_0(y) = \int_Y \Phi(y) \cdot \left( \frac{d\nu_0}{dm} \right) (y) \, dm(y) = \int_Y \Phi(y) \cdot \left( \frac{d\nu_0}{dm} \circ \pi \right) \, dm(y) = \int_X f \cdot \left( \frac{d\nu_0}{dm} \circ \pi \right) \, d\nu
\]

proving the result. \( \Box \)

The following will be used in section 3 in order to prove that a certain ultralimit construction produces the Poisson boundary.

Proposition B.2.20. If \( \pi : X \to B(G, \sigma) \) is a factor, then \( \pi \) is a measure preserving extension.

Proof. Replacing \( X \) by a measure preserving regular factor (that still has the Poisson boundary as a factor), we may assume \( X \) is regular. Note that \( \Phi_X : B \to M(X, \Sigma_X) \) is a measurable function such that \( a.e. \pi_*\Phi_X(b) = \delta_b \). Using the integral formula in Theorem B.2.11 and applying Lemma B.2.19 we conclude that \( \pi \) is measure preserving. \( \Box \)

We conclude this section with a construction for Joining and a Furstenberg-Glasner type theorem (strongly inspired from [14]).

Definition B.2.21. Let \( \mathcal{X} = (X, \nu), \mathcal{Y} = (Y, \kappa) \) be two regular \( \sigma \)-systems. The Joining of \( \mathcal{X}, \mathcal{Y} \) is \( \mathcal{X} \times \mathcal{Y} = (X \times Y, \rho) \) where \( X \times Y \) is equipped with the diagonal action, and the vectors of measures \( \rho = (\rho^{(n)}) \) are defined by:

\[
\rho^{(n)} = \int_B \Phi_X(b) \times \Phi_Y(b) \, dm^{(n)}(b)
\]

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Note that
\[ g \rho^{(n)} = \int_B g(\Phi_X \times \Phi_Y) \, dm^{(n)} = \int_B \Phi_X \times \Phi_Y \, d(gm^{(n)}) \]
Thus, \( X \times Y \) is a regular \( \sigma \)-system. We sometimes denote \( \rho = \nu \vee \kappa \).
By definition, the conditional measures of \( X \times Y \) are \( \Phi_{X,Y}(b) = \Phi_X(b) \times \Phi_Y(b) \).
Also, it is clear that the projections \( X \times Y \to X, X \times Y \to Y \) are factors of \( \sigma \)-systems.

**Lemma B.2.22.** Let \( X, Y \) be regular \( \sigma \)-systems. If \( X \) proximal then the projection \( \pi : X \vee Y \to Y \) is proximal.

**Proof.** We need to show that the projection \( \pi : (X \times Y, \delta_{\rho(b)} \times \Phi_Y(b)) \to (Y, \Phi_Y(b)) \) is an isomorphism. Indeed the inverse is \( y \mapsto (p(b), y) \).

Now we can prove the corresponding Furstenberg-Glasner type theorem:

**Theorem B.2.23.** Let \( X \) be a regular \( \sigma \)-system. Let \( \Lambda(X) \) be a proximal \( \sigma \)-system, having \( \Pi(X) \) as a factor. Consider \( X^* = X \vee \Lambda(X) \). Then we have:

1. \( X^* \to X \) is a proximal extension.
2. \( X^* \to \Lambda(X) \) is a measure preserving extension.

**Proof.** Since \( \Lambda(X) \) is proximal, we conclude item 1 from Lemma B.2.22.
For item 2: we write \( X = (X, \nu), \Pi(X) = (M(X, \Sigma), \kappa), \Lambda(X) = (\Lambda, \lambda), X^* = (X \times \Lambda, \rho) \).
Let \( \varphi : \Lambda(X) \to \Pi(X) \) be the given factor and \( \pi : B(G, \sigma) \to \Lambda(X) \) be the factor provided by the proximality of \( \Lambda(X) \) and Lemma B.2.17. From the same lemma we know that the factor from the Poisson boundary to a proximal system is unique. Considering the composition
\[ B(G, \sigma) \xrightarrow{\pi} \Lambda(X) \xrightarrow{\varphi} \Pi(X) \]
we conclude \( \varphi \circ \pi = \Phi_X \) by the definition of \( \Pi(X) \). Thus for any \( n \geq -1 \), \( i \in [\ell_i] \), \( g \in G \):
\[ g\rho^{(n)} = \int_B \Phi_X \times \Phi_{\Lambda(X)} \, d(gm^{(n)}_i) = \int_B \varphi(\pi(b)) \times \delta_{\pi(b)} d(gm^{(n)}_i)(b) = \int_{\Lambda} \varphi(\xi) \times \delta_{\xi} d(g\lambda^{(n)}_i)(\xi) \]
Using Lemma B.2.19 we conclude that the extension \( (X \times \Lambda, \rho) \to (\Lambda, \lambda) \) is measure preserving.

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