Simulating spacetime with indefinite causal order via Rindler observers

Aleksandra Dimić
Faculty of Physics, University of Belgrade, Studentski Trg 12-16, 11000 Belgrade, Serbia
E-mail: aleksandra.dimic@ff.bg.ac.rs

Marko Milivojević
Faculty of Physics, University of Belgrade, Studentski Trg 12-16, 11000 Belgrade, Serbia
E-mail: milivojevic@rcub.bg.ac.rs

Dragoljub Gočanin
Faculty of Physics, University of Belgrade, Studentski Trg 12-16, 11000 Belgrade, Serbia
E-mail: dgocanin@ipb.ac.rs

Časlav Brukner
Vienna Center for Quantum Science and Technology (VCQ), University of Vienna, Faculty of Physics, Boltzmanngasse 5, A-1090 Vienna, Austria
Institute for Quantum Optics and Quantum Information (IQOQI), Austrian Academy of Sciences, Boltzmanngasse 3, A-1090 Vienna, Austria
E-mail: caslav.brukner@univie.ac.at

Abstract. Realization of indefinite causal order, a theoretical possibility that even causal relations between physical events can be subjected to quantum superposition, apart from its general significance for the fundamental physics research, would also enable quantum information processing that outperforms protocols in which the underlying causal structure is definite. In this paper, we propose a way to simulate specific spacetime with indefinite metric structure by exploiting the equivalence between stationary observers sitting in the vicinity of the event horizon of a Schwarzschild black hole and Rindler observers in Minkowski space. Namely, by putting a Rindler observer, who resides in causally definite Minkowski background, in a state of quantum superposition of having two different values of proper acceleration, we can simulate the experience of a stationary observer in gravitational field with indefinite metric generated by a Schwarzschild black hole in a state of quantum superposition of being at two different spatial locations with respect to the observer. In this manner, a pair of entangled Rindler observers can be used to simulate quantum communication protocols such as gravitational quantum switch or the violation of Bell’s inequality for temporal order. We also discuss the possibility of experimental realization by means of optomechanical resonators.
1. Introduction

The principle of causality lays in the core of every physical theory and, depending on the context, it has various interpretations. From an operational point of view, causality can be understood as signaling/communication relations between physical observers (systems, in general), an information flow whose properties are intimately related to the nature of space and time, the notions of which have evolved through several stages. In the old Newtonian picture, space and time are two generically different entities, universal for all observers. There is a single flat Euclidean space and a single global time that enables us to universally distinguish between past, present and future. Together, they constitute an absolute, independent background structure relative to which every physical event takes place. Signals can propagate in space with unlimited speed (action at a distance) and, consequently, each event can be caused by any other in its present or past. Einstein’s theory of special relativity (SR) changed this paradigm: space and time became united into a four dimensional spacetime continuum - the Minkowski space - in which signals cannot travel faster than the speed of light, enforcing them to stay inside, or on the local light cone. Nevertheless, the structure of Minkowski space adhered the character of an independent, fixed background on which dynamical matter fields propagate. The radical change came with Einstein’s theory of general relativity (GR). The dynamics of gravitational field, that is, spacetime itself according to GR, is not given a priori, since it is coupled to the dynamics of matter fields. There is no prescribed, independent metric structure, no absolute background stage relative to which locations of physical events are to be defined, there are just dynamical fields, spacetime being one of them, and physical events are only located relative to one another. Every event has its own past and future, a class of events from which it can receive information and a class of events to which it can send information and the possibility of communication between different observers is completely determined by dynamical configuration of light cones. While in flat Minkowski space all light cones have the same slope, in curved spacetimes of GR, they can be tilted relative to each other, according to the distribution of matter. For a given observer in a spacetime with definite causal order, that is, an observer in a gravitational field with a definite metric structure, the causal relations between physical events (as they appear to the observer) are uniquely determined. Next logical question would be: can spacetime have indefinite causal order, that is, can it be in a state in which a particular observer experiences quantumly superposed metric structure?

It is generally expected that the unification of quantum mechanics (QM) and gravitational physics will provide us with some deeper insights concerning the nature of space and time at the microscopic (Planck) scale. That is the main subject of quantum gravity. However, the standard methods of quantization of matter fields employed in Quantum Field Theory (QFT) do not seem to work for Einstein gravity; it holds a status of a non-renormalizable theory with undetermined high-energy degrees of freedom. In order to transcend the traditional concepts established within GR and QFT, various radical approaches of “quantizing” gravity are proposed so far, stemming from String
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Theory, Quantum Loop Gravity, Noncommutative Field Theory, etc. Up to date, there has been no empirical evidence that would support or disprove any of the proposed “high-energy theories”. This state of affairs motivates us to take an operational point of view and reconsider in which sense and to what extent can the fundamental principles of QM be applied to gravity while adhering the tenets of GR [1, 2]. From a broader perspective, our goal is to find ways to “lure out”, in laboratory conditions, effects that would distinctly characterize gravity to have quantum features. An important question that arises out of these considerations is whether it is possible to impose the principle of quantum superposition upon the causal structure of spacetime, that is, can we have a quantum superposition of two (or more) macroscopically distinct metric structures?

In the work of Oreshkov, Costa and Brukner [3], it was found that it is possible to formulate quantum mechanics without any reference to a global causal structure. The resulting framework - the process matrix formalism - allows for processes incompatible with any definite order between operations performed on quantum systems. These indefinite causal structures are shown to be advantageous for quantum computing [4, 5] and quantum communication [6, 7, 8]. One particular example that has experimental demonstration is “quantum switch” [4, 9, 10, 11, 12, 13], where the main idea is to use an auxiliary quantum system which can coherently control the order in which certain operations are applied. In the case of the so called gravitational quantum switch (GQS) [14] the role of the control system is played by a gravitating object prepared in a state of quantum superposition of being at two different spatial locations. Due to entanglement with the gravitating object, the spacetime itself is expected to be in a state of quantum superposition of having two macroscopically distinct metrics generated by the gravitating object.

Here we propose a method for simulating some specific indefinite causal structures, potentially in laboratory conditions, by utilizing the equivalence between stationary observers in the vicinity of the event horizon of a Schwarzschild black hole and Rindler observers in Minkowski space [15]. It is based on the generalization of Einstein’s equivalence principle to spacetimes with indefinite metric structure by which we claim that quantum superposition of two macroscopically distinct metric structures of spacetime is locally equivalent to a “quantum reference frame” [16] in flat spacetime with two superposed proper accelerations. This allows us to understand the experience of an observer sitting in indefinite gravitational field, in particular, in two distinct “superposed Schwarzschild metrics”, in terms of a Rindler observer in flat Minkowski space in the state of superposition of having two different values of its proper accelerations. We present a Rindler-version of GQS and the protocol for violation of Bell’s inequality for temporal order that involves three Rindler observers. Finally, we discuss the possibility of experimental realization of these Rindler-protocols by the means of optomechanical oscillators [17, 18, 19, 20].
2. Rindler observers

Consider an arbitrary inertial observer in $(1+1)$-dimensional Minkowski space, and a light cone of a single event of its worldline. This observer defines global time $t$ running along its worldline. With respect to it, we introduce an observer that has constant proper acceleration of magnitude $\alpha$ in the $x$-direction, called the Rindler observer. The metric of the $(1+1)$-dimensional Minkowski space covered by globally inertial coordinates $x^\mu = (t, x)$ is given by $ds^2 = -dt^2 + dx^2$, where we set $c = 1$, implying the same dimensions of space and time. The worldline of a Rindler observer, parameterized by its proper time $\tau$, is given by the parametric equations:

$$t(\tau) = \frac{1}{\alpha} \sinh(\alpha \tau), \quad x(\tau) = \pm \frac{1}{\alpha} \cosh(\alpha \tau).$$

Thus, the shape of the Rindler observer’s worldline is a hyperbola, $t^2(\tau) - x^2(\tau) = -1/\alpha^2$, with branches embedded in the space-like separated wedges of the above mentioned light cone, called the left (L) and the right (R) Rindler wedge (see Fig. 1 (left panel)). Rindler observer with larger proper acceleration has more curved worldline. The structure of light cones in Minkowski space is such that Rindler observers in R-wedge can only witness the events from regions R and P, and so, the null surface $t = x$ acts as an event horizon for these observers. Regions L and R are causally disconnected from each other, meaning that Rindler observers in L-wedge cannot communicate with Rindler observers in R-wedge.

Consider now two Rindler observers in the R-wedge, with different proper accelerations $\alpha_1$ and $\alpha_2$. Let the second one be more curved than the first one, that is, let $\alpha_2 > \alpha_1$. A photon sent to the left from the source $S$, with spacetime coordinates $t_s = 0$ and $x_s = x_0 > 0$, intersects worldlines of these Rindler observers at proper times $\tau_1$ and $\tau_2$, respectively (see Fig. 1 (right panel)). At $t = 0$ both observers are closer to the origin than S. This implies that $\alpha_2 x_0 > \alpha_1 x_0 > 1$. This configuration has an interesting feature that will turn out to be important. Namely, given the values of $x_0$ and $\alpha_1$, there exists a unique value for $\alpha_2$, defined as the non trivial solution ($\alpha_2 \neq \alpha_1$) of the equation

$$\alpha_2 x_0 = (\alpha_1 x_0)^{\frac{\alpha_2}{\alpha_1}},$$

for which $\tau_1 = \tau_2$ (for details, see Appendix A).

3. Indefinite causal order via Rindler observers

Imagine that we have a simple system involving a Schwarzschild black hole, and a stationary observer sitting in his/her isolated laboratory that is well enough localized and has negligible effect on the gravitational field. We do not assume any fixed

‡ Having in mind that we are going to make a connection to Rindler observers in Minkowski space, it is more suitable to talk about a Schwarzschild black hole rather than some ordinary spherically symmetric gravitating object, e.g. a planet, because of the importance of an event horizon.
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**Figure 1.** Rindler observers. (Left panel): Hyperbolic worldlines of left and right Rindler observers in Minkowski space. Patches L and R, called the left and the right Rindler wedge, respectively, are causally disconnected. This feature disables left and right Rindler observer to communicate with each other. (Right panel): Photon’s worldline intersects the Rindlers. Two Rindler observers in R-wedge with different proper accelerations $\alpha_1$ and $\alpha_2$, $\alpha_1 < \alpha_2$. A photon sent from the point-like source $S$, with spacetime coordinates $t_s = 0$ and $x_s = x_0 > 0$, intersects worldlines of these Rindler observers at proper times $\tau_1$ and $\tau_2$, respectively.

Background spacetime to which we could refer to and define the locations of objects. Black hole and the observer are located relative to each other. Now suppose that the black hole and the observer are in a state of quantum superposition of being at two different relative distances from each other, where by relative distance we mean the physical proper distance between the observer and the black hole’s horizon, the length of a stationary observer’s meter stick if he/she would try to touch the horizon with it.

The observer would “feel” that he/she resides in a gravitational field with indefinite metric structure, a kind of a “quantum spacetime” the nature of which we want to comprehend. From an operational viewpoint, one could equivalently say that the observer is in a state of superposition of having two different radial distances from the horizon of a Schwarzschild black hole. In general, for every stationary ($r = \text{const.}$) observer in a gravitational field of a Schwarzschild black hole there is an equivalent Rindler observer in Minkowski space and vice versa (according to the Einstein’s principle of
equivalence gravitational field is *locally equivalent* to an accelerating reference frame in flat spacetime). Even more natural correspondence holds if a stationary Schwarzschild observer is close to the horizon, because spacetime metric *in the vicinity of the horizon* reduces to the metric of Minkowski space in Rindler coordinates. In that case, if the Schwarzschild observer’s proper distance from the horizon is $\rho$, then its corresponding Rindler observer’s distance from the origin will also be $\rho$. These two observers have the same proper acceleration, inversely proportional to $\rho$. For Rindler observer this relation between its proper acceleration and its distance from the origin always holds, but for a stationary Schwarzschild observer it holds only in the vicinity of black hole’s horizon, and so, this condition becomes important for obtaining genuine equivalence (see Appendix B for details). Hence, we can effectively transcribe the original system (an observer and a black hole in a state of quantum superposition of being at two different relative distances from each other) in terms of the corresponding Rindler observer in Minkowski space having two superposed values of its proper acceleration. This entails the claim that Einstein’s equivalence principle holds even in spacetime with indefinite metric structure, in other words, that quantum superposition of two macroscopically distinct metric structures of spacetime is *locally equivalent* to a “quantum reference frame” in flat spacetime with two superposed proper accelerations. It is an extension of Einstein’s equivalence principle that assumes its compatibility with the linearity of quantum mechanics applied to spacetime.

Let us now take two stationary observers, Schwarzschild-Amber ($A_S$) and Schwarzschild-Blue ($B_S$), sitting in their *isolated* laboratories, and a Schwarzschild black hole that is in a state of superposition of being at two different locations with respect to them. $A_S$ and $B_S$ then reside in a spacetime with indefinite metric structure. These two observers correspond to the pair of Rindler observers in Minkowski space, Rindler-Amber ($A_R$) and Rindler-Blue ($B_R$), with *entangled proper accelerations*. By examining all possible configurations of this system, we conclude that there are four nonequivalent cases (see Fig. 2). For example, the first configuration corresponds to the superposition of a state in which communication between observers $A_S$ ($A_R$) and $B_S$ ($B_R$) is impossible due to the presence of the horizon, and the other state in which they can communicate. The correspondence can be extended to the case of many observers residing in spacetime with indefinite metric structure, and here we will use it to simulate two simple quantum information protocols that can be naturally established using indefinite causal structures - gravitational quantum switch (for this the second configuration will be relevant) and the protocol for the violation of Bell’s inequality for temporal order (for which we need three observers).

§ Note that *amber* and *blue* are, conveniently, the actual names of the colors designating the two observers.
4. Simulating gravitational quantum switch via entangled Rindler observers

The idea that a gravitating object in spatial superposition can induce a superposition of two gravitational fields dates back to Feynman [21] and it was promoted, for example, in [14, 22, 23, 24]. Most importantly for this work, it was employed in [14] as a way of obtaining gravitational quantum switch. Basically, a gravitating object is prepared in a state of quantum superposition of being at two different spatial locations, thus producing, due to its entanglement with the gravitational field, a spacetime with indefinite metric structure. This opens a possibility of defining a communication protocol in which one can obtain a superposition of temporal order for two operationally defined physical events. The state of the gravitating object plays the role of a quantum control for the order of these events (see [14] for the complete account). We will now present a somewhat simpler version of gravitational quantum switch that can be more easily transcribed in terms of Rindler observers. The setup is illustrated in Fig. 3. It involves two observers, $A_S$ and $B_S$, sitting in their isolated laboratories, a photon source $S$, and a Schwarzschild black hole that is in the state of superposition of having two different locations relative to the observers. In the first case the black hole is closer to $A_S$, state $|L\rangle$, and in the other it is closer to $B_S$, state $|R\rangle$. In both superposed states, $A_S$ and $B_S$ lay on the same radial ray, they are sitting at some fixed (but different)
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Figure 3. Gravitational quantum switch. The system involves a photon source $S$, two stationary observers, $A_S$ and $B_S$, and a black hole in the state of superposition of having two different locations relative to them. Photon source shoots the photon towards the black hole, and so, the position of the black hole plays the role of a quantum control for the whole process. The photon is in a superposition of traveling in two opposite directions.

distance from the horizon, and have the same distance between each other. We can think of this as having two observers, with fixed relative distance, in a gravitational field with two superposed macroscopically distinct Schwarzschild metrics. A photon source is connected to the observers and it can send photons to them. The source is gravity-sensitive and it is adjusted so that it emits a photon in the polarization state $|\Psi\rangle$ towards the black hole. The position of the black hole thus plays the role of a quantum control for the whole process. Due to the fact that time runs slower for the observer closer to the horizon, we can arrange things so that the photon passes through both laboratories at the same moment of their local proper time (see Appendix C for details). This is analogous to the case of two Rindler observers from Section 2. When the photon gets inside the laboratory, instantaneously, a unitary transformation, $U_A$ or $U_B$, depending on the laboratory, is applied on its polarization state. The meeting of the photon and the laboratory $A_S$ and instantaneous application of unitary $U_A$ is the event $a$, and likewise, the meeting of the photon and the laboratory $B_S$ and instantaneous application of unitary $U_B$ is the event $b$.

If $A_S$ and $B_S$ are, in both superposed states, close to the horizon, than this whole system corresponds to the system involving two Rindler observers in Minkowski space with entangled proper accelerations (Fig. 4). In the Rindler-scenario, worldline of a photon emitted by the source $S$ intersects worldlines of the two Rindler observers $A_R$ and $B_R$ with entangled proper accelerations. On the left panel, the proper acceleration of $A_R$ ($\alpha_1$) is smaller than the proper acceleration of $B_R$ ($\alpha_2$) and on the right panel it is other way around, $A_R$ has larger proper acceleration ($\alpha_2$) and $B_R$ has smaller proper acceleration ($\alpha_1$). By choosing a suitable values for the proper accelerations $\alpha_1$ and $\alpha_2$, with $\alpha_1 < \alpha_2$, these meetings occur at the same proper time $\tau^*$ (see discussion in Section 2). The photon is in the polarization state $|\Psi\rangle$ and it is acted upon by the Rindler observers according to their “color” degree of freedom, “amber” ($A$) or “blue” ($B$), that identifies and distinguishes them. Observer $A_R$ performs a unitary

|| By “observer” we mean the internal dynamical degrees of freedom inside the laboratory, whatever they may be.
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Figure 4. Quantum switch via Rindler observers. Worldline of a photon emitted by the source $S$ intersects worldlines of the two Rindler observers $A_R$ and $B_R$ that have entangled proper accelerations. On the left panel, $A_R$ has smaller proper acceleration ($\alpha_1$) than $B_R$ ($\alpha_2$), and on the right panel, $A_R$ has larger proper acceleration ($\alpha_2$) than $B_R$ ($\alpha_1$). By choosing a suitable values for the proper accelerations, these meetings occur at the same proper time $\tau^\ast$. Rindler observers act upon the photon according to their "color" degree of freedom, "amber" ($A$) or "blue" ($B$), that identifies them. Observer $A_R$ performs a unitary transformation $U_A$ at $\tau^\ast$ without disturbing the trajectory of the photon, and observer $B_R$ performs a unitary transformation $U_B$ at $\tau^\ast$ also without disturbing the trajectory of the photon. When laboratories meet the photon, they instantaneously come to rest and remain that way until some particular moment $t_m$ of inertial observer $C$’s proper time at which a projective measurement is performed in order to disentangle the state of the photon from that of the laboratories. Observer $C$ eventually receives the photon.

transformation $U_A$ at $\tau^\ast$ without disturbing the trajectory of the photon, and observer $B_R$ performs a unitary transformation $U_B$ at $\tau^\ast$ also without disturbing the trajectory of the photon. In general, degrees of freedom of laboratory (its kinematic mode and internal degrees of freedom) can get entangled with the state of the photon. Internal state of the laboratories evolves according to the observer’s proper time, since it accounts for the physical rate of change. Thus, we will take a state $|\tau_\alpha\rangle$ of observer’s “clock” and its “ticking” (which depends on $\alpha$) to be abstractions of its entire actual state and its evolution, respectively, without getting into details of what are observer’s actual degrees of freedom and its hamiltonian. We now define event $a$ to be the meeting of the photon with $A_R$ and instantaneous application of the unitary $U_A(\tau^\ast)$, and likewise, we define event $b$ to be the meeting of the photon with $B_R$ and instantaneous application of the unitary $U_B(\tau^\ast)$. It would also be convenient to neutralize the accelerations of Rindler laboratories at the moment they meet the photon because kinematic state of the laboratory can also get entangled with the state of the photon. In this way we can avoid the difficulty of disentangling these kinematic degrees of freedom from the state.

\[\text{This is analogous to letting the corresponding Schwarzschild observers } A_S \text{ and } B_S \text{ to become free falling at the time they meet the radially falling photon.}\]
of the photon afterwards. Neutralization can be achieved by instantaneously putting to rest each of the laboratories when they meet the photon, making them inertial from that point on.

For the sake of reference, we will update the state of the whole system (Rindlers $\otimes$ photon) by using the proper time $t$ of inertial observer $C$ sitting at $x = 0$. If the system is prepared at $t = 0$ in the composite state $|\tau_{a_1}(0), A\rangle|\tau_{a_2}(0), B\rangle|\Psi\rangle$, the photon is first sent to the $A_R$ (left panel), and transmitted in the same direction to $B_R$. In the other case, when the state of the system is $|\tau_{a_1}(0), B\rangle|\tau_{a_2}(0), A\rangle|\Psi\rangle$ (right panel), the signal first gets to $B_R$ and then to $A_R$. If the system is prepared in a state of superposition of these two states, at $t = 0$ we have

$$
\frac{1}{\sqrt{2}} \left( |\tau_{a_1}(0), A\rangle|\tau_{a_2}(0), B\rangle + |\tau_{a_1}(0), B\rangle|\tau_{a_2}(0), A\rangle \right) |\Psi\rangle.
$$

At $t < t_1$ (where $t_1$ is the $C$’s time coordinate of the intersection of the photon’s worldline with the less curved Rindler worldline) the state is

$$
\frac{1}{\sqrt{2}} \left( |\tau_{a_1}(t), A\rangle|\tau_{a_2}(t), B\rangle + |\tau_{a_1}(t), B\rangle|\tau_{a_2}(t), A\rangle \right) |\Psi\rangle.
$$

When the photon passes through the laboratories, unitary transformation $U_A(\tau^*)$ or $U_B(\tau^*)$ is applied on it, depending on the laboratory. After the passage of the photon through both laboratories, at some instant $t$ such that $t > t_2$ (where $t_2$ is the $C$’s time coordinate of the intersection of the photon’s worldline with the more curved Rindler worldline) the state of the whole system is given by

$$
\frac{1}{\sqrt{2}} \left( |\tau^* + t - t_1, A\rangle|\tau^* + t - t_2, B\rangle U_B(\tau^*) U_A(\tau^*) |\Psi\rangle + |\tau^* + t - t_1, B\rangle|\tau^* + t - t_2, A\rangle U_A(\tau^*) U_B(\tau^*) |\Psi\rangle \right),
$$

where the differences $t - t_1$ and $t - t_2$ are the time intervals during which respective Rindler laboratories are at rest relative to $C$. Finally, we need to disentangle the state of the photon from the internal state of Rindler laboratories. To this end, at some fixed moment $t_m$ of the $C$’s global time, a projective measurement (postselection of the internal state of the Rindler laboratories) is performed in the superposition basis $\{|m_i\}, |m_i^+\rangle | i = 1, 2 \}$, separately for each laboratory. The basis states are given by

$$
|m_i\rangle = \frac{1}{\sqrt{2}} \left( |\tau^* + t_m - t_i, A\rangle + |\tau^* + t_m - t_i, B\rangle \right),
$$

$$
|m_i^+\rangle = \frac{1}{\sqrt{2}} \left( |\tau^* + t_m - t_i, A\rangle - |\tau^* + t_m - t_i, B\rangle \right).
$$

Postselection on any pair of possible measurement results leads to the following state of the photon

$$
\frac{1}{\sqrt{2}} (U_B(\tau^*) U_A(\tau^*) \pm U_A(\tau^*) U_B(\tau^*)) |\Psi\rangle.
$$

which corresponds to indefinite temporal order of events $a$ and $b$. 

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5. Violation of Bell’s inequality for temporal order

In Ref. [14] it was shown that by using a massive object in a spatial superposition (and by extension, superposition of spacetime with two different geometries) as a control system, one can realize events with “entangled temporal order”. This allows violation of Bell’s inequalities [25, 26, 27, 28, 29, 30, 31] for temporal order. Here we present an alternative realization of this protocol using Rindler observers. To this end, we consider the following situation: laboratory \( A_R \) is in the left Rindler wedge, laboratory \( B_R \) is in the right Rindler wedge and laboratory \( C_R \) is in the superposition of being in right and left Rindler wedge. This corresponds to the three stationary laboratories \( A_S, B_S \) and \( C_S \) residing in spacetime generated by a Schwarzschild black hole that is in a state of superposition of being between \( A_S \) and \( C_S \) and between \( C_S \) and \( B_S \). This corresponds to Rindler scenario where \( C_R \) is in a superposition of being in both wedges, left and right, while \( A_R \) and \( B_R \) have swapped magnitudes accelerations, while staying in the same respective wedge.

\[ \begin{align*}
\text{Figure 5.} & \quad \text{Protocol for the violation of Bell’s inequality. In the “gravitational” scenario, black hole is in a superposition of being between } A_S & \\
& \quad \text{and } C_S \text{ and between } C_S & \\
& \quad \text{and } B_S. \text{ This corresponds to Rindler scenario where } C_R \text{ is in a superposition of being in both wedges, left and right, while } A_R & \\
& \quad \text{and } B_R \text{ have swapped magnitudes accelerations, while staying in the same respective wedge.} & 
\end{align*} \]

The protocol for violation of Bell’s inequality goes as follows: two sources \( S_1 \) and \( S_2 \) send two photons (one photon each), which are initially in the uncorrelated state \(|\Psi_L\rangle \otimes |\Psi_R\rangle\), into right and left Rindler wedge, respectively. When photon meets a Rindler laboratory \( X_R \) (\( A_R, B_R \) or \( C_R \)), local unitary transformation \( U_{X}(\tau^*) \) is performed on it and the photon proceeds in the same direction. As in the previous protocol, after the passage of the photons, the states of the laboratories are decoupled from the states of the photons. Initial state of the laboratories and the photons is given by

\[ \frac{1}{\sqrt{2}} \left( |\tau^{-\alpha_2}_2\rangle |\tau^{C}_{\alpha_1}\rangle |\tau^{B}_{\alpha_2}\rangle + |\tau^{-\alpha_1}_1\rangle |\tau^{C}_{-\alpha_2}\rangle |\tau^{B}_{\alpha_1}\rangle \right) |\Psi_L\rangle |\Psi_R\rangle, \tag{10} \]

where \(|\tau^{\text{lab}}_{\beta}\rangle = |\tau_{\beta}(0)\rangle_{\text{lab}}\), \( \beta \in \{\alpha_1, \alpha_2, -\alpha_1, -\alpha_2\} \), \( \text{lab} \in \{A, B, C\} \). One can readily check that the joint state of the two photons, after performing appropriate measurements on the internal degrees of freedom of the Rindler laboratories\(^4\) is

\[ \frac{1}{\sqrt{2}} (U_A(\tau^*) |\Psi_L\rangle U_C(\tau^*) U_B(\tau^*) |\Psi_R\rangle \pm U_C(\tau^*) U_A(\tau^*) |\Psi_L\rangle U_B(\tau^*) |\Psi_R\rangle). \tag{11} \]

\(^4\) Trace out of the degrees of freedom of laboratories can be done, in principle, in many ways; one suggestion is given in the previous section.
If we perform unitary transformations such that states $U_A|\Psi_L\rangle$, $U_C U_B |\Psi_R\rangle$, respectively, then the state (11) is maximally entangled. One possible choice of states and operations is $|\Psi_L\rangle = |+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$, $|\Psi_R\rangle = |0\rangle$, $U_A = H = 1/\sqrt{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $U_B = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $U_C = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Finally, we can imagine two inertial observers sitting at $x = 0$, for example, that can choose suitable measurements on the corresponding photons and perform Bell’s test, thus confirming the violation of Bell’s inequality for temporal order.

6. Conclusion

To conclude, we applied an equivalence between stationary observers near the event horizon of a Schwarzschild black hole and Rindler observers in Minkowski space to simulate quantum information protocols in gravitational field with indefinite metric structure. We claim that such gravitational field is locally equivalent to a quantum non-inertial reference frame in Minkowski space that has superposed proper acceleration. An important example is gravitational quantum switch, where one uses a gravitating object in a state of superposition of being at two different spatial positions as a quantum control, for which we need two Rindler observers in Minkowski space with entangled proper accelerations. Likewise, the violation of Bell’s inequality for temporal order can be simulated by using three Rindler observers. Thus, we are able to “mimic” the experience of a stationary observer in spacetime with two superposed Schwarzschild metrics by preparing the corresponding Rindler observer in a state of superposition of having two different proper accelerations.

There is a growing effort in demonstrating quantum features of nano-to-mesoscale optomechanical systems. This may provide a challenging, yet feasible experimental realizations for the proposed Rindler protocols [17]. Recently, mesoscopic mechanical resonators were considered as quantum non-inertial reference frames [18, 19] and entanglement of two massive mechanical oscillators is achieved [20]. It has been proposed to utilize quantum optical fields in order to prepare and measure the quantum states of mechanical resonators, conceivably opening the possibility to quantumly control the acceleration of such quantum non-inertial reference frames [17].

A potential drawback of the proposed protocols might arise due to the Unruh effect, that is, the fact that Rindler observer experiences ordinary Minkowski vacuum as a thermal state. In this context, Rindler observer should be viewed as Unruh-DeWitt detector [32], which interacts with, for example, a scalar quantum field in Minkowski space. The temperature detected by the Rindler observer is related to its proper acceleration $\alpha$ by the relation $T = \hbar \alpha / 2\pi k_B c$. The increase of the thermal noise may affect the final state, such that it can no more be considered as a coherent superposition, but rather a (convex) classical mixture. However, since in our scheme we can choose $\alpha$ to be arbitrarily small by tuning the other parameters, the Unruh effect can always be
made negligible.

In future work, it would be interesting to explore further the possibility of transcribing quantum information protocols in general spacetimes with indefinite causal order in terms of equivalent quantum non-inertial reference frames in Minkowski space, with a goal of establishing the full correspondence. This could be a step towards a better understanding of quantum nature of spacetime.

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[1] L. Hardy, arXiv:0509120 [gr-qc].
[2] L. Hardy, J. Phys. A 40, 3081 (2007).
[3] O. Oreshkov, F. Costa, and Č. Brukner, Nat. Commun. 3, 1092 (2012).
[4] G. Chiribella, G. M. D’Ariano, P. Perinotti, and B. Valiron, Phys Rev. A 88, 022318 (2013).
[5] M. Araújo, F. Costa, and Č. Brukner, Phys. Rev. Lett. 113, 250402 (2014).
[6] A. Feix, M. Araújo, and Č. Brukner, Phys. Rev. A 92, 052326 (2015).
[7] P. A. Guérin, A. Feix, M. Araújo, and Č. Brukner, Phys. Rev. Lett. 117, 100502 (2016).
[8] D. Ebler, S. Salek and G. Chiribella, Phys. Rev. Lett. 120, 1205020 (2018).
[9] G. Chiribella, Phys. Rev. A 86, 040301(R) (2012).
[10] N. Friis, V. Dunjko, W. Dür, and H. J. Briegel, Phys. Rev. A 89, 030303(R) (2014).
[11] L. M. Procopio, A. Moqanaki, M. Araújo, F. Costa, I. A. Calafell, E. G. Dowd, D. R. Hamel, L. A. Rozema, Č. Brukner, and P. Walther, Nat. Commun. 6, 7913 (2015).
[12] T. M. Rambo, J. B. Altepeter, P. Kumar, and G. M. D’Ariano, Phys. Rev. A 93, 052321 (2016).
[13] G. Rubino, L. A. Rozema, A. Feix, M. Araújo, J. M. Zeuner, L. M. Procopio, Č. Brukner, and P. Walther, Sci. Adv. 3, e1602589 (2017).
[14] M. Zych, F. Costa, I. Pikovski, and Č. Brukner, arXiv: 1708.00248.
[15] F. Dahia, and P.J.F. da Silva, General Relativity and Gravitation, 43, 269 (2011).
[16] F. Giacomini, E. Castro-Ruiz, Č. Brukner, arXiv:1712.07207 [quant-ph].
[17] R. Kaltenbaek, M. Aspelmeyer, P. F. Barker, A. Bassi, J. Bateeman, K. Bongs, et al, EPJ Quantum
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Technology 3, 5 (2015).

[18] B. N. Katz, M. P. Blencowe, and K. C. Schwab, Phys. Rev. A 92, 042104 (2015).
[19] M. Abdi, P. Degenfeld-Schonburg, M. Sameti, C. Navarrete-Benlloch, and M. J. Hartmann, Phys. Rev. Lett. 116, 233604 (2016).
[20] C. F. Ockeloen-Korppi, E. Damskagg, J.-M. Pirkkalainen, M. Asjad, A. A. Clerk, F. Massel, M. J. Woolley and M. A. Sillanpaa, Nature 556, 478 (2018).
[21] R. Feynman, Chapel Hill Conference Proceedings, 1957.
[22] C. Anastopoulos and B.-L. Hu, Class. Quant. Grav. 32, 165022 (2015).
[23] M. Abdi, P. Degenfeld-Schonburg, M. Sameti, C. Navarrete-Benlloch, and M. J. Hartmann, Phys. Rev. Lett. 116, 233604 (2016).
[24] C. F. Ockeloen-Korppi, E. Damskagg, J.-M. Pirkkalainen, M. Asjad, A. A. Clerk, F. Massel, M. J. Woolley and M. A. Sillanpaa, Nature 556, 478 (2018).
[25] R. Feynman, Chapel Hill Conference Proceedings, 1957.
[26] C. Anastopoulos and B.-L. Hu, Class. Quant. Grav. 32, 165022 (2015).
[27] S. Bose, A. Mazumdar, G. W. Morley, H. Ulbricht, M. Toro, M. Paternostro et al., Phys. Rev. Lett. 119, 240401 (2017).
[28] C. F. Ockeloen-Korppi, E. Damskagg, J.-M. Pirkkalainen, M. Asjad, A. A. Clerk, F. Massel, M. J. Woolley and M. A. Sillanpaa, Nature 556, 478 (2018).
[29] R. Feynman, Chapel Hill Conference Proceedings, 1957.
[30] C. Anastopoulos and B.-L. Hu, Class. Quant. Grav. 32, 165022 (2015).
[31] M. Abdi, P. Degenfeld-Schonburg, M. Sameti, C. Navarrete-Benlloch, and M. J. Hartmann, Phys. Rev. Lett. 116, 233604 (2016).
[32] C. F. Ockeloen-Korppi, E. Damskagg, J.-M. Pirkkalainen, M. Asjad, A. A. Clerk, F. Massel, M. J. Woolley and M. A. Sillanpaa, Nature 556, 478 (2018).

APPENDIX

Appendix A. Equal proper time condition for two Rindler observers

Here we give a simple derivation of the relation between proper accelerations of two Rindler observers that must be satisfied so that a photon’s worldline intersects worldlines of the Rindler observers at the same moment of their individual proper time.

From (Fig. A1) we can see that
\[ x_0 - x_i = t_i, \] (A.1)

where \( i = \{1, 2\} \). Given the parametric equations of Rindler observer’s worldline,
\[ t_i = \frac{1}{\alpha_i} \sinh(\alpha_i \tau_i), \quad x_i = \frac{1}{\alpha_i} \cosh(\alpha_i \tau_i), \] (A.2)

we can deduce the instant of the observer’s proper time \( \tau_i \) in which he/she receives the signal from the point-source \( S \) at \( x_0 > 0 \). From (A.1) and (A.2) it follows that
\[ \frac{e^{\alpha_i \tau_i} + e^{-\alpha_i \tau_i}}{2\alpha_i} + \frac{e^{\alpha_i \tau_i} - e^{-\alpha_i \tau_i}}{2\alpha_i} = x_0, \] (A.3)

and so,
\[ \tau_i = \frac{1}{\alpha_i} \ln(\alpha_i x_0). \] (A.4)

Note that both the prefactor and the argument of the logarithm are positive. The condition of equality of the proper times \( \tau_1 \) and \( \tau_2 \) gives us the following relation between \( \alpha_1 \) and \( \alpha_2 \):
\[ \alpha_2 x_0 = (\alpha_1 x_0)^{\frac{\alpha_2}{\alpha_1}}. \] (A.5)
By introducing new variables, $X := \alpha_1 x_0$ and $Y := \alpha_2 x_0$, previous equation can be formulated as

$$Y = X^Y. \quad (A.6)$$

Numerical analysis of $(A.6)$ shows that solution $Y = X$, which exists for each value of $X$, is unique for $X < 1$ (Fig. A2 (a)). This solution is trivial, since it corresponds to a single Rindler observer (or two overlapping Rindler observers with $\alpha_1 = \alpha_2$). If $X > 1$, there are two possible solutions for $Y$, one which is trivial and the other that can be greater or less than $X$ (Fig. A2 (b,c)). Note that for every value of $x_0$ (position of the source) we have a continuous infinity of pairs of Rindler trajectories that satisfy $(A.5)$.

Figure A1. A photon intersecting two Rindler worldlines. Worldline of a photon sent from a point-like source $S$ intersects worldlines of two Rindler observers at instances $\tau_1$ and $\tau_2$ of their respective proper time. Inertial reference frame coordinates of the intersection points are denoted by $t_i, x_i, i = \{1, 2\}$.

Figure A2. Numerical analysis. Ratio of accelerations for Rindler observers. If we take that $X = \alpha_1 x_0$ and $Y = \alpha_2 x_0$ then $(A.5)$ becomes $Y = \Phi_X(Y) = X^Y$. We have found three classes of solutions, but we can regard only one class as relevant if we take an assumption $Y > X$. One trivial solution in all three cases is $Y = X$. In a), case $X < 1$ is illustrated, with $X = 1/2$, where we have only trivial solution. When $X \in (1, e)$, we have non-trivial solution, $Y > X$. That case is represented in b), for $X = e/2$. Finally, in the case $X > e$, we get two solutions - trivial one and the other for which $Y < X$. That case is illustrated in c), where $X$ is chosen to be $2e$. 
Appendix B. Schwarzschild metric in the vicinity of a horizon

Worldlines of stationary observers in curved spacetime do not conform to the geodesics defined by the spacetime metric. To maintain a fixed position they must oppose their inertia with some proper acceleration. Generally, 4-acceleration of an observer in curved spacetime is given by $a^\mu = U^\nu \nabla_\nu U^\mu$, where $U^\mu$ is its 4-velocity, and its proper acceleration by $a = \sqrt{g_{\mu\nu} a^\mu a^\nu}$. In particular, for a stationary observer in Schwarzschild metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_2^2,$$  \hspace{1cm} (B.1)

with $f(r) = 1 - \frac{R_S}{r}$ and $\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$, the metric of a 2-sphere $S^2$, we have

$$a = \frac{R_S}{2r^2 \sqrt{f(r)}}.$$  \hspace{1cm} (B.2)

Analogously to the proper time $d\tau = \sqrt{f(r)}dt$ of an observer sitting at $r = \text{const}$, we can introduce proper radial distance,

$$d\rho = \frac{dr}{\sqrt{f(r)}}.$$  \hspace{1cm} (B.3)

Integrating from $R_S$ to $r$ we get the proper radial distance of the stationary observer at some fixed $r$ from the event horizon,

$$\rho = r \left( 1 - \frac{R_S}{r} \right)^{1/2} + \frac{R_S}{2} \ln \left[ \frac{2r}{R_S} - 1 + \frac{2r}{R_S} \left( 1 - \frac{R_S}{r} \right)^{1/2} \right].$$  \hspace{1cm} (B.4)

From (B.4) it follows that $\rho \sim r$ in the limit $r/R_S \to +\infty$, and so from (B.2) we get $a \sim R_S/2\rho^2$, which is just the Newtonian inverse square law. Now, let $r = R_S + \epsilon$ for some small $\epsilon$. In the limit $\epsilon/R_S \to 0$ we have $\rho \sim 2\sqrt{R_S}\epsilon$, and the proper acceleration of a stationary observer in the vicinity of event horizon is inversely proportional to its proper distance from the horizon, $a \sim 1/\rho$. In the intermediate region, proper acceleration is some very complicated function of observers proper radial distance.

Schwarzschild metric near the horizon becomes

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 + R_S^2 d\Omega_2^2.$$  \hspace{1cm} (B.5)

where we introduced new time coordinate $\eta = \frac{t}{2R_S}$. The non-angular part of the above metric is the metric of $(1+1)$-dimensional Minkowski space, denoted by $\mathcal{M}_2$, in Rindler coordinates. This becomes evident if we start with the metric of $\mathcal{M}_2$ in Minkowski coordinates $(T, X)$

$$ds^2_{\mathcal{M}_2} = -dT^2 + dX^2,$$  \hspace{1cm} (B.6)

and introduce Rindler coordinates $(\rho, \eta)$ by $T = \rho \sinh \eta$ and $X = \rho \cosh \eta$ in which the above metric takes the form

$$ds^2_{\mathcal{M}_2} = -\rho^2 d\eta^2 + d\rho^2.$$  \hspace{1cm} (B.7)
Coordinate $\rho$ is time-like and $\eta$ is space-like. Since $X^2 - T^2 = \rho^2 \geq 0$, coordinates $(\rho, \eta)$ cover only part of $M_2$ - the right Rindler wedge. Rindler coordinates $(\rho, \eta)$ become singular at $\rho = 0$ but, using the Minkowski coordinates $(T, X)$, one could analytically continue them from the right Rindler wedge to the whole Minkowski space. Similarly, in the case of a Schwarzschild black hole, we use Kruskal coordinates to make an analytic continuation of Schwarzschild coordinates $(t, r)$ across the horizon thus obtaining their maximal extension. The event horizon of a black hole, defined by $\rho = 0$ as seen from (B.5), corresponds to the light cone $T = \pm X$, and near-horizon black hole geometry is $\text{Rindler} \times S^2$.

An observer at $r = \text{const}$ ($r \approx R_S$) in Schwarzschild metric corresponds to a uniformly accelerating observer with $\rho = \text{const}$ in the Rindler wedge, i.e. an observer in Minkowski space whose worldline is a hyperbola $X^2 - T^2 = \rho^2 = \text{const}$, whose constant proper acceleration is given by

$$\alpha = \frac{1}{\rho} = \frac{1}{2\sqrt{R_S\sqrt{r - R_S}}}.$$  

Figure C1. Schematic representation of black hole and laboratories $A_1$ and $A_2$. Radially falling photon passes through stationary laboratories $A_1$ and $A_2$ sitting at two different radial distances from the black hole. The different ticking rates of local clocks make it possible to arranged the relative distances so that meetings of the photon with the laboratories occur at the same moment of their local proper times.

Appendix C. Equal proper times for free falling photon

Start with the equation for radially free falling photon in Schwarzschild coordinates:

$$\frac{dt}{dr} = -\frac{1}{1 - \frac{R_S}{r}} = -\frac{r}{r - R_S},$$  

where $R_S$ is the Schwarzschild radius. A radially falling photon, emitted at $r = R_0$, passes through the laboratory $A_1$, sitting at $r = r_1$, at the time $t_1$. A simple calculation gives us that for $t_1$ we have

$$t_1 = R_0 - r_1 + R_S \ln \frac{R_0 - R_S}{r_1 - R_S}.$$  

(C.2)
The same photon passes through the laboratory $A_2$ sitting at $r = r_2$, at the time $t_2$ given by (see Fig. C1)

\[ t_2 = R_0 - r_2 + R_S \ln \frac{R_0 - R_S}{r_2 - R_S}. \]  
(C.3)

We are looking for the condition for both events to happen at the same local proper times measured in laboratories $A_1$ and $A_2$, i.e.

\[ \tau_1 = \tau_2. \]  
(C.4)

Proper times of the laboratories are related to the global time $t$ (proper time of a stationary observer at infinity) by

\[ \tau_i = \sqrt{1 - \frac{R_S}{r_i}} t_i = \sqrt{1 - \frac{R_S}{r_i}} \left( R_0 - r_i + R_S \ln \frac{R_0 - R_S}{r_i - R_S} \right). \]  
(C.5)

Numerical analysis of the previous equation shows that it is not possible to obtain the relation (C.4) for arbitrary values of $R_0$ and $R_S$. However, if the ratio $R_S/R_0 < 10^{-4}$, two possible solutions, pairs of $r_1$ and $r_2$ that satisfy the condition of equal proper times, are always present. It is important to mention that one position, $r_1$, is very close to the black hole, that is, $r_1 < 10R_S$, while the other position, $r_2$, can be further away.