Relative trace formulae toward Bessel and Fourier-Jacobi periods of unitary groups

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Abstract

We propose a relative trace formula approach and state the corresponding fundamental lemma toward the global restriction problem involving Bessel or Fourier-Jacobi periods of unitary groups $U_n \times U_m$, extending the work of Jacquet-Rallis for $m = n - 1$ (which is a Bessel period). In particular, when $m = 0$, we recover a relative trace formula proposed by Flicker concerning Kloosterman/Fourier integrals on quasi-split unitary groups. As evidence for our approach, we prove the fundamental lemma for $U_n \times U_n$ in positive characteristics.

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1 Introduction

Recently, Jacquet and Rallis [JR] propose a new approach to the Gan-Gross-Prasad conjecture for unitary groups $U_n \times U_{n-1}$ relating certain periods with special $L$-values (cf. [GGP]). It is based on a relative trace formula. In this paper, we extend this approach to all kinds of pairs $U_n \times U_m$ where $0 \leq m \leq n$. If $n - m$ is odd, the related period is a Bessel period. If $n - m$ is even, the related period is a Fourier-Jacobi period.

Notations. We denote by $\mathbb{C}^1$ the unit circle in the complex line $\mathbb{C}$. For a number field $k$, we denote by $M_k$ the set of places of $k$.

We denote by $M_{r,s}$ the affine group of $r \times s$ matrices and $M_r = M_{r,r}$. We denote by $1_r$ the identity matrix of rank $r$.

We denote by $|S|$ the cardinality of a finite set $S$ and by $1_T$ the characteristic function of any set $T$.

For a locally compact abelian topological group or a vector space $X$, we denote $X^\vee$ for its dual. For a smooth representation $\pi$, we denote $\tilde{\pi}$ for its contragredient representation.

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1.1 Periods and special values of $L$-functions

Let us consider a quadratic extension of number fields $k/k'$. Let $\tau$ be the nontrivial element in $\Gal(k/k')$ and $\eta$ the corresponding quadratic character of the idèle group $\mathbb{A}^{\times}$ of $k'$. Let $V$ be a (non-degenerated) hermitian space over $k$ (with respect to $\tau$) of dimension $n$ with the hermitian form $(-,-)$ and $W \subset V$ a subspace of dimension $m$ such that the restricted hermitian form $(-,-)|_W$ is non-degenerate.

Let $U_n = U(V)$ and $U_m = U(W)$ be the unitary groups. We identify $U_m$ as a subgroup of $U_n$ fixing all elements in the orthogonal complement $W^\perp \subset V$ of $W$. We define a unipotent subgroup $U' = U_{1r,m+1} \subset U_n$ (resp. $U' = U_{1r,m+1}'$) normalized and hence acted through conjugation by $U_m$ when $n - m$ is even (resp. odd). We define $H' = U' \times U_m$ which is a non-reductive group viewed as a subgroup of $U_n \times U_m$ via the embedding into the first factor and the projection onto the second factor (see Section 4.1 and 5.1 for the precise definitions). Let $\pi$ (resp. $\sigma$) be an irreducible tempered representation of $U_n(k')$ (resp. $U_m(k')$) which occurs with multiplicity one in the space of cusp forms $A_0(U_n)$ (resp. $A_0(U_m)$). We denote by $A_\pi$ (resp. $A_\sigma$) the unique irreducible $\pi$ (resp. $\sigma$)-isotypic subspace in $A_0(U_n)$ (resp. $A_0(U_m)$).

First, we consider the case of Bessel periods where we require that $n - m = 2r + 1$ is odd. We have a generic character $\nu'$ of $U'(k') \backslash U'(k')$ and extend it to a character of $H'(k') \backslash H'(k')$ trivially on $U_m(k')$. For $\varphi_\pi \in A_\pi$ and $\varphi_\sigma \in A_\sigma$, we define

$$B_{\nu'}(\varphi_\pi; \varphi_\sigma) = \int_{H'(k') \backslash H'(k')} \varphi_\pi \otimes \varphi_\sigma(h')(h')^{-1} dh'$$

to be a Bessel period of $\pi \otimes \sigma$. It is conjectured that there is a nonzero Bessel period of representations in the Vogan $L$-packet of $\pi \otimes \sigma$ if and only if the central special $L$-value $L(\frac{1}{2}, BC(\pi) \times BC(\sigma)) \neq 0$, where BC stands for the standard base change and the $L$-function is the one defined by the Rankin-Selberg convolution on general linear groups (cf. [JPSS83]).

Second, we consider the case of Fourier-Jacobi periods where we require that $n - m = 2r$ is even. We have an automorphic (essentially Weil) representation $\nu_{\psi'},\mu$ of $H'(k')$ by choosing a nontrivial character $\psi' : k' \backslash k' \rightarrow \mathbb{C}^1$ and a character $\mu : k^\times \backslash k^\times \rightarrow \mathbb{C}^\times$ such that $\mu|_{k^\times} = \eta$, realizing on certain space of Bruhat-Schwartz functions $S$. For $\varphi_\pi \in A_\pi$, $\varphi_\sigma \in A_\sigma$ and $\phi \in S$, we define

$$J_{\nu_{\psi'},\mu}(\varphi_\pi, \varphi_\sigma; \phi) = \int_{H'(k') \backslash H'(k')} \varphi_\pi \otimes \varphi_\sigma(h') \theta(h', \phi) dh'$$

to be a Fourier-Jacobi period of $\pi \otimes \sigma$ (with respect to $\mu$), where $\theta(h', \phi)$ is a certain theta series on $H'(k')$ attached to $\phi$. It is conjectured that there is a nonzero Fourier-Jacobi period of representations
in the Vogan $L$-packet of $\pi \otimes \sigma$ if and only if the central special $L$-value $L\left(\frac{1}{2}, BC(\pi) \times BC(\sigma) \otimes \mu^{-1}\right) \neq 0$.

There has been significant progress toward this conjecture in a series of papers of Ginzburg-Jiang-Rallis: [GJR09] for unitary groups, [GJR04] for symplectic groups and [GJR05] for orthogonal groups. In all these cases, they prove one direction: nontrivial Bessel or Fourier-Jacobi periods imply the non-vanishing of corresponding central $L$-values. For the other direction, they also obtain some conditional results. Their approach is to study the residue of certain Eisenstein series and some Fourier coefficients attached to it.

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But one can ask more about the precise relation between these periods and central special $L$-values, known as the Ichino-Ikeda conjecture in the context of $SO_n \times SO_{n-1}$ (cf. [II10]). The advantage of the relative trace formula approach is that it is possible to prove the explicit formula relating $|\mathcal{B}_{\nu}^\nu(\varphi_\pi, \varphi_\sigma)|^2$ (or $|\mathcal{F}_{\nu,\nu,\nu}(\varphi_\pi, \varphi_\sigma, \phi)|^2$) and the product of certain local periods of positive type, within $L$-values as the scaling factor of these two periods. In particular, one can prove the positivity of the corresponding central special $L$-value, for example, as in [JC01]. We will not pursue the formulation of this explicit relation in this paper, but instead, formulate the relative trace formulas toward it for both periods.

1.2 Relative trace formulae and fundamental lemmas

We briefly describe our relative trace formula. To be simple for the introduction, we only do this for the case of Bessel periods, i.e., we assume that $n - m = 2r + 1$ is odd.

Let $f_n \in \mathcal{H}(U_n(\mathbb{A'})$ (resp. $f_m \in \mathcal{H}(U_m(\mathbb{A'}))$) be a smooth function on $U_n(\mathbb{A'})$ (resp. $U_m(\mathbb{A'})$) with compact support. We introduce a distribution

$$\mathcal{J}_{\pi,\sigma}(f_n \otimes f_m) := \sum \mathcal{B}_{\nu}^\nu(\rho(f_n)\varphi_\pi, \rho(f_m)\varphi_\sigma)\mathcal{B}_{\nu}^\nu(\varphi_\pi, \varphi_\sigma)$$

where the sum is taken over orthonormal bases of $\mathcal{A}_\pi$ and $\mathcal{A}_\sigma$ and $\rho$ denotes the action by right translation.

As usual, we associate to $f_n \otimes f_m$ a kernel function on $(U_n(k') \setminus U_n(\mathbb{A'}) \times U_m(k') \setminus U_m(\mathbb{A'}))^2$:

$$K_{f_n \otimes f_m}(g_1, g_2; h_1, h_2) = \sum_{\zeta' \in U_n(\mathbb{A'})} f_n(g_1^{-1}\zeta' g_1^* \langle h_1, h_2 \rangle) \sum_{\zeta' \in U_m(\mathbb{A'})} f_m(g_2^{-1}\zeta' g_2^*)$$

and consider the following distribution which is formally the “sum” of $\mathcal{J}_{\pi,\sigma}$ over all $\pi \otimes \sigma$:

$$\mathcal{J}(f_n \otimes f_m) := \iint_{(H^\prime(k') \setminus H^\prime(\mathbb{A'}))^2} K_{f_n \otimes f_m}(h_1', h_2'; h_1', h_2') \nu(h_1^{-1}h_2'\varphi_\sigma)dh_1' dh_2'.$$

The above integral is not absolutely convergent in general and need to be regularized. It turns out that the regular part of this distribution has the following decomposition:

$$\mathcal{J}_{reg}(f_n \otimes f_m) = \sum_{\zeta' \in \mathcal{U}(U_n(k')_{reg})/H^\prime(\mathbb{A'})} \mathcal{J}_{\zeta'}(f_n \otimes f_m)$$

where $\mathcal{U}(U_n(k')_{reg})/H^\prime(\mathbb{A'})$ is the set of certain regular orbits in $U_n(k')$ which will be discussed in Section 4.3 and $f \in \mathcal{H}(U_n(\mathbb{A'})$) is obtained from $f_n \otimes f_m$. Moreover, each summand $\mathcal{J}_{\zeta'}$ is an adelic weighted orbital integral:

$$\mathcal{J}_{\zeta'}(f) = \int_{U_m(\mathbb{A'})/U_{1r,m+1}(\mathbb{A'})} f(g_1^{-1}u_1^{-1}\zeta' u_2g')\nu(u_1^{-1}u_2)du_1' du_2' dg'.$$

To connect the $L$-function, one need to go to the $GL_n \times GL_m$ side. Let $\Pi = BC(\pi)$ and $\Sigma = BC(\sigma)$ and assume that they remain cuspidal. We define similarly a unipotent subgroup $U_{1r,m+1,1r} \times GL_m$, a non-reductive group $H = U_{1r,m+1,1r} \times GL_m$ viewed as a subgroup of $GL_n \times GL_m$ and a character $\nu$ on
for precise definitions). For \( \varphi_\Pi \in A_\Pi \) and \( \varphi_\Sigma \in A_\Sigma \), consider the following general linear group version of the Bessel period:

\[
B_{r,r}^* (\varphi_\Pi, \varphi_\Sigma) := \int_{H(k) \backslash H(k)} \varphi_\Pi \otimes \varphi_\Sigma(h) \nu(h)^{-1} dh.
\]

We remark that the above integral is the usual Rankin-Selberg convolution for \( U_n \times U_m \) when \( r = 0 \), but not when \( r > 0 \). But in fact, it is also an integral presentation of \( L(s, \Pi \times \Sigma) \).

To single out the cuspidal representations being the standard base change from the unitary groups, we follow [JR]. Saying that \( n \) is odd, let

\[
\mathcal{P}_n(\varphi_\Pi) = \int_{Z_\Pi^r(k') \backslash GL_n(k')} \varphi_\Pi(g_1) dg_1; \quad \mathcal{P}_m(\varphi_\Sigma) = \int_{Z_\Sigma^m(k') \backslash GL_m(k')} \varphi_\Sigma(g_2) \eta(\det g_2) dg_2
\]

where \( Z_\Pi^r \) is the center of \( GL_{r,k} \).

Let \( F_n \in \mathcal{H}(GL_n(\mathbb{A})) \) and \( F_m \in \mathcal{H}(GL_m(\mathbb{A})) \). We introduce another distribution

\[
\mathcal{J}_{\Pi,\Sigma}(F_n \otimes F_m) := \sum B_{r,r}^*(\rho(F_n)\varphi_\Pi, \rho(F_m)\varphi_\Sigma) \mathcal{P}_n(\varphi_\Pi) \mathcal{P}_m(\varphi_\Sigma)
\]

where the sum is taken over orthonormal bases of \( A_\Pi \) and \( A_\Sigma \). Repeat the process for the unitary groups, we have the kernel function \( K_{F_n \otimes F_m} \) and the distribution \( \mathcal{J}(F_n \otimes F_m) \) whose regular part has the following decomposition:

\[
\mathcal{J}_{\mathrm{reg}}(F_n \otimes F_m) = \sum_{\zeta \in [S_n(k')_{\mathrm{reg}}]/H(k')} \mathcal{J}_\zeta(F)
\]

where \( [S_n(k')_{\mathrm{reg}}]/H(k') \) is the set of certain regular orbits in the symmetric space \( S_n(k') \) which will be discussed in Section 4.3 and \( F \in \mathcal{H}(S_n(k')) \) is obtained from \( F_n \otimes F_m \). Moreover, each summand \( \mathcal{J}_\zeta \) is an adelic weighted orbital integral:

\[
\mathcal{J}_\zeta(F) = \int_{GL_n(\mathbb{A})} \int_{U_{n,m+1}^r(k')} F(g^{-1}u^{-1}\zeta u^r g) \nu(u^{-1}) du dg.
\]

We prove in Proposition 4.11 that there is a natural bijection

\[
\mathcal{N} : [S_n(k')_{\mathrm{reg}}]/H(k') \sim \coprod_{W \subset V} [U_n(k')_{\mathrm{reg}}]/H'(k')
\]

where the disjoint union is taken over all isometry classes of \( W \subset V \). Hence, we pose the global matching condition for \( \mathcal{N}(\zeta) = \zeta' \) as \( \mathcal{J}_\zeta(F) = \mathcal{J}_{\zeta'}(f) \). The precise conjecture of smooth matching of functions is proposed as Conjecture 4.12.

As the most important and interesting problem in this kind of trace formula, we now discuss the corresponding fundamental lemma. Here, again to be simple, we only state it for the unit elements in the following cases: \( n-m \) odd, \( m = 0 \), or \( n = m \).

Now let \( k' \) be a non-archimedean local field and \( k/k' \) an unramified quadratic field extension. Let \( \mathfrak{o}' \) (resp. \( \mathfrak{o} \)) be the ring of integers of \( k' \) (resp. \( k \)). There are only two non-isomorphic hermitian spaces of dimension \( m > 0 \) over \( k \). Let \( U^+_m \subset U^+_n \) be the pair associate to \( W^+ \subset V^+ \) both with trivial discriminant and \( U^+_m \subset U^+_n \) be another one. Then \( W^+ \) will have a self-dual \( \mathfrak{o} \)-lattice \( L_W \) which extends to a self-dual \( \mathfrak{o} \)-lattice \( L_V \) of \( V^+ \). The unitary group \( U^+_m \) (resp. \( U^+_n \)) is unramified and has the order \( \mathfrak{o}' \). The group of \( \mathfrak{o}' \)-points \( U^+_m(\mathfrak{o}') \) (resp. \( U^+_n(\mathfrak{o}') \)) is a hyperspecial maximal subgroup of \( U^+_m(k') \) (resp. \( U^+_n(k') \)). We also denote by \( GL_n,L_V \) \( \cong GL_n(\mathfrak{o}') \) a hyperspecial maximal subgroup of \( GL_n(k) \) and \( S_n(\mathfrak{o}') := S_n(k') \cap K_n \).

When \( n-m \) is even, we define \( S_{n,m}(k') = S_n(k') \times M_{1,m}(k') \times M_{m,1}(k') \) and \( U_{n,m}(k') = U_n(k') \times M_{1,m}(k') \). There is also a notion of regular elements in both sets and we have a natural bijection \( \mathcal{N} : [S_{n,m}(k')_{\mathrm{reg}}]/H(k') \sim \coprod_{W \subset V} [U_{n,m}(k')_{\mathrm{reg}}]/H'(k') \) (cf. Section 5.3). We propose the following conjecture.
Conjecture 1.1 (The fundamental lemma for unit elements). (1) When \( n - m \) is odd or \( m = 0 \), we have

\[
\int_{\text{GL}_n(k')} \int_{U_{\nu,n}(k')} \mathbb{I}_{S_n(o')}(g^{-1}u^{-1}\zeta u'g)\nu(u^{-1})\eta(\det g)du dg = \begin{cases} t(\zeta) \int_{U_{\nu,n}(k')} \mathbb{I}_{U_{\nu,n}(o')}(g^{-1}u'^{-1}\zeta^+ u'g')\nu'(u')du'_1 du'_2 dg' & \mathcal{N}(\zeta) = \zeta^+ \in U_n^+(k)'; \\ 0 & \mathcal{N}(\zeta) = \zeta^- \in U_n^-(k') \end{cases}
\]

where \( t(\zeta) \in \{\pm 1\} \) is a certain transfer factor defined in (4.21). In particular, when \( m = 0 \), the second case of the above identity does not happen.

(2) When \( n = m \), we have

\[
\int_{\text{GL}_n(k')} \mathbb{I}_{S_n(o')}(g^{-1}\zeta g)\mathbb{I}_{M_{n,n}(o')}(xyg)\mathbb{I}_{M_{n,1}(o')}(g^{-1}y)\eta(\det g)dg = \begin{cases} t([\zeta, x, y]) \int_{U_{\nu,n}(k')} \mathbb{I}_{U_{\nu,n}(o')}(g^{-1}\zeta^+ g')\mathbb{I}_{U_{\nu,n}(o')}(zg')dg' & \mathcal{N}([\zeta, x, y]) = [\zeta^+, z] \in U_n^+(k)'; \\ 0 & \mathcal{N}([\zeta, x, y]) = [\zeta^-, z] \in U_n^-(k') \end{cases}
\]

where \( t([\zeta, x, y]) \in \{\pm 1\} \) is a certain transfer factor defined in (5.16). In particular, when \( m = 0 \), the second case of the above identity does not happen.

We have

**Theorem 1.2.** (1) The second case of the identity in both cases in Conjecture 1.1 holds;

(2) When \( n = m + 1 \), the fundamental lemma holds if \( \text{char}(k) = p > \text{max}\{n, 2\} \) or \( \text{char}(k) = 0 \) and the residue characteristic is sufficiently large with respect to \( n \);

(3) When \( n \leq 3, m = 0 \), the fundamental lemma holds if \( \text{char}(k) \neq 2 \);

(4) When \( n = m \), the fundamental lemma holds if \( \text{char}(k) = p > \text{max}\{n, 2\} \).

**Proof.** (1) is proved in Proposition 4.15 and Proposition 5.12 in this paper.

(2) is proved by Yun in [Yun09] where the transfer to characteristic 0 is accomplished by Gordon in the appendix.

(3) is proved by Jacquet in [J92] when \( n = 3 \); (essentially) proved by Ye in [Ye89] when \( n = 2 \); and trivial when \( n = 1 \).

(4) is proved in Theorem 5.15.

**Remark 1.3.** When \( n = m + 1 \), the fundamental lemma is just the one proposed by Jacquet-Rallis in [JR]. When \( m = 0 \), the fundamental lemma is the one proposed by Flicker in [F91], which is the unitary group version of the Jacquet-Ye fundamental lemma (cf. [JY92]). When \( m > 0 \) (and \( n - m \) is odd), the fundamental lemma is a kind of hybrid of the Jacquet-Rallis fundamental lemma and the Flicker fundamental lemma. We hope that there is a geometric method toward this fundamental lemma, which is also a kind of hybrid of those in [Yun09] by Yun and [Ng99] by Ngô.

When \( 0 < n - m < n \) is even, we also formulate a corresponding fundamental lemma for Fourier-Jacobi periods. See Chapter 5 for details.

The fundamental lemma for all elements in the spherical Hecke algebra when \( n = 3, m = 0 \) is also proved, by Mao in [Mao93].
1.3 Variants of Rankin-Selberg convolutions

As we see in the last section, we need to consider certain periods of Bessel and Fourier-Jacobi types on the general linear groups as well. We will generalize the notion of Bessel (resp. Fourier-Jacobi) models and periods for $GL_n \times GL_m$ for a pair $(r, r^*)$ of nonnegative integers such that $n = m + 1 + r + r^*$ (resp. $n = m + r + r^*$). When $r = r^*$, they are introduced and considered in [GGP].

For simplicity, let us only describe the first case. Let $(r, r^*)$ be as above, we introduce a unipotent subgroup $U_{1, r, m+1, r^*}$ of $GL_n$ and $H = U_{1, r, m+1, r^*} \times GL_m$ viewed as a subgroup of $GL_n \times GL_m$. We have a character $\nu$ of $H$ which is automorphic if $k$ is a number field.

Let $\pi$ (resp. $\sigma$) be an irreducible cuspidal automorphic representation of $GL_n(\mathbb{A})$ (resp. $GL_m(\mathbb{A})$). We introduce the Bessel integral

$$B_{r, r^*}(s; \varphi_{\pi}, \varphi_\sigma) := \int_{H(k) \backslash H(\mathbb{A})} \varphi_{\Pi} \otimes \varphi_{\Sigma}(h) \nu(h)^{-1} |\det h|_\mathbb{A}^{s-\frac{d}{2}} dh$$

and the Bessel period $B_{r, r^*}^*(\varphi_{\pi}, \varphi_\sigma) := B_{r, r^*}(\frac{1}{2}; \varphi_{\pi}, \varphi_\sigma)$ for $\varphi_{\pi} \in A_{\pi}, \varphi_\sigma \in A_{\sigma}$ and $s \in \mathbb{C}$. Then $B_{r, r^*}^*$ is the usual Rankin-Selberg convolution on $GL_n \times GL_m$ if and only if $r = 0$.

For general $(r, r^*)$ and also for the Fourier-Jacobi integrals $FJ_{r, r^*}(s; \varphi_{\pi}, \varphi_\sigma; \Phi)$, we have the following theorem. We would like to remark that the following result is actually known by Jacquet, Piatetskii-Shapiro and Shalika long time ago. But since it is not recorded in the literature, we would like to write it down with proof just for the completeness.

**Theorem 1.4** (see Section 2.2 and 3.2 for notations). (1) The Bessel integrals are holomorphic in $s$ and satisfy the following functional equation

$$B_{r, r^*}^*(s; \varphi_{\pi}, \varphi_\sigma) = B_{r, r^*}^*(1 - s; \rho(w_{n,m}) \varphi_{\pi}, \varphi_\sigma).$$

For $\varphi_{\pi} \in A_{\pi}$ and $\varphi_\sigma \in A_{\sigma}$ such that $W^\Sigma_{\varphi_{\pi}} = \otimes_v W_v$ and $W^\Sigma_{\varphi_{\sigma}} = \otimes_v W_v$ are factorizable,

$$B_{r, r^*}(s; \varphi_{\pi}, \varphi_\sigma) = L(\frac{1}{2}, \pi \times \sigma) \prod_{v \in M_k} \frac{\Psi_{v,r}(s; W_v, W_v^{-*})}{L_v(s, \pi_v \times \sigma_v)} \Bigg|_{s=\frac{1}{2}}$$

where in the last product almost all factors are 1. In particular, there is a nontrivial Bessel period of $\pi \otimes \sigma$ if and only if $L(\frac{1}{2}, \pi \times \sigma) \neq 0$.

(2) The Fourier-Jacobi integrals are holomorphic in $s$ (when $n > m$) and satisfy the following functional equation

$$FJ_{r, r^*}(s; \varphi_{\pi}, \varphi_\sigma; \Phi) = FJ_{r, r^*}(1 - s; \rho(w_{n,m}) \varphi_{\pi}, \varphi_\sigma; \hat{\Phi}).$$

For $\varphi_{\pi} \in A_{\pi}, \varphi_\sigma \in A_{\sigma}$ and $\Phi \in S(W^*(\mathbb{A}))$ such that $W^\Sigma_{\varphi_{\pi}} = \otimes_v W_v, W^\Sigma_{\varphi_{\sigma}} = \otimes_v W_v^{-*}$ and $\Phi = \otimes \Phi_v$ are factorizable,

$$FJ_{r, r^*}(s; \varphi_{\pi}, \varphi_\sigma; \Phi) = L(\frac{1}{2}, \pi \times \sigma \otimes \mu^{-1}) \prod_{v \in M_k} \frac{\Psi_{v,r}(s; W_v, W_v^{-*} \otimes \mu^{-1}_v; \Phi_v)}{L_v(s, \pi_v \times \sigma_v \otimes \mu^{-1}_v)} \Bigg|_{s=\frac{1}{2}}$$

where in the last product almost all factors are 1. In particular, there is a nontrivial Fourier-Jacobi period of $\pi \otimes \sigma$ for $\nu_\mu$ if and only if $L(\frac{1}{2}, \pi \times \sigma \otimes \mu^{-1}) \neq 0$.

**Remark 1.5.** The above theorem completely confirms [GGP, Conjecture 24.1] for split unitary groups, i.e., general linear groups.

It is clear that the Bessel (resp. Fourier-Jacobi) period defines an element in the space of invariant functionals $\text{Hom}_{H(\mathbb{A})}(\pi \otimes \sigma \otimes \nu, \mathbb{C})$ where $\nu$ is a character (resp. an infinite dimensional representation) of $H(\mathbb{A})$. It has a decomposition into spaces of local invariant functionals. The following multiplicity one result is a generalization of [GGP, Corollary 15.3, 16.3] for general linear groups.
Theorem 1.6. Let $k$ be a non-archimedean local field of characteristic zero. Let $\pi$ (resp. $\sigma$) be an irreducible admissible representation of $GL_n$ (resp. $GL_m$). Then $\dim_{\mathbb{C}} \text{Hom}_H(\pi \otimes \sigma \otimes \tilde{\nu}, \mathbb{C}) \leq 1$. Moreover, if $\pi$ and $\sigma$ are generic, then $\dim_{\mathbb{C}} \text{Hom}_H(\pi \otimes \sigma \otimes \tilde{\nu}, \mathbb{C}) = 1$.

The following is an outline of the paper.

In Chapter 2, we focus on the Bessel model and period on general linear groups. We prove the Bessel part of Theorem 1.4 and Theorem 1.6. The proof of the local multiplicity one result will occupy the first section, while we follow the idea of [GGP]. In particular, our proof includes the case $r = r^*$ which was left as an exercise to readers in [GGP]. In the second section, as we have said, we will give a proof for the global integral being Eulerian for the completeness of literature.

In Chapter 3, we focus on the Fourier-Jacobi model and period on general linear groups. We prove the Fourier-Jacobi part of Theorem 1.4 and Theorem 1.6.

After briefly recalling the Bessel model and period for unitary groups, we introduce the relative trace formula in Chapter 4. We prove the matching of orbits in Section 4.3 and the smooth matching of functions at split places. We formulate the fundamental lemma and prove the easy half in 4.4.

Chapter 5 is repeating the previous chapter, but for the Fourier-Jacobi model and period. What we do more is the proof of the full fundamental lemma for $U_n \times U_n$ in positive characteristics, which is achieved in Section 5.5. The proof is reduced to a known combinatorial identity proved in [Yun09].

The last chapter is an appendix on integrals of local Whittaker functions for general linear groups. We collect all the results we need in Chapter 2 and 3 from existing literature. In particular, we have to use all kinds of auxiliary local Whittaker integrals in the theory of Rankin-Selberg convolutions.

2 Bessel periods of $GL_n \times GL_m$

2.1 Bessel models

Let $k$ be a local field and $| |_k$ the normalized absolute value on $k$. Let $V$ be a $k$-vector space of dimension $n$. Suppose that $V$ has a decomposition $V = X \oplus W \oplus E \oplus X^*$ where $W$, $X$ and $X^*$ have dimension $m$, $r$ and $r^*$ respectively and $E = \langle e \rangle$ with $e \neq 0$, hence $n = m + r + r^* + 1$. We want to generalize the construction of the pair $(H, \nu)$ as in [GGP, Section 13]. Let $P_{r,m+1,r^*}$ be the parabolic subgroup of $GL(V)$ stabilizing the flag $0 \subset X \subset X \oplus W \oplus E \subset V$ and $U_{r,m+1,r^*}$ its maximal unipotent subgroup. Then $U_{r,m+1,r^*}$ fits into the following exact sequence:

$$0 \longrightarrow \text{Hom}(X^*, X) \longrightarrow U_{r,m+1,r^*} \longrightarrow \text{Hom}(X^*, W \oplus E) + \text{Hom}(W \oplus E, X) \longrightarrow 0.$$ 

We may write the above sequence as:

$$0 \longrightarrow (X^*)^\vee \otimes X \longrightarrow U_{r,m+1,r^*} \longrightarrow (X^*)^\vee \otimes (W \oplus E) + (W^\vee \otimes E^\vee) \otimes X \longrightarrow 0.$$ 

Let $\ell_X : X \to k$ (resp. $\ell_{X^*} : k \to X^*$) be any nontrivial homomorphism (if exists) and let $U_X$ (resp. $U_{X^*}$) be a maximal unipotent subgroup of $GL(X)$ (resp. $GL(X^*)$) stabilizing $\ell_X$ (resp. $\ell_{X^*}$). Moreover, let

$$\ell_W : (W \oplus E) + (W^\vee \otimes E^\vee) \longrightarrow k$$

be a bilinear form which is trivial on $W + W^\vee$ and nontrivial on $E$ and $E^\vee$. The composition of $\ell_X + \ell_{X^*}$ and $\ell_W$ defines a homomorphism

$$\ell : U_{r,m+1,r^*} \longrightarrow (X^*)^\vee \otimes (W \oplus E) + (W^\vee \otimes E^\vee) \otimes X \xrightarrow{\ell_X + \ell_{X^*}} (W \oplus E) + (W^\vee \otimes E^\vee) \xrightarrow{\ell_W} k$$

which is fixed by $(U_X \times U_{X^*}) \rtimes GL(W)$. Hence we can extend $\ell$ trivially to it and define a homomorphism from $H = U_{r,m+1,r^*} \rtimes ((U_X \times U_{X^*}) \times GL(W))$ to $k$. Let $\psi : k \to \mathbb{C}^1$ be a nontrivial character and $\lambda : U_X \times U_{X^*} \to \mathbb{C}^1$ a generic character which can be viewed as a character of $H$. Let $\delta_W$ be the modulus function of $GL(W)$ with respect to the adjoint action on $U_{r,m+1,r^*}$, i.e., $\delta_W(g) = |\det g|_k^{r^* - r}$. Then we
can form a character $\nu = (\psi \circ \ell) \otimes \lambda \otimes \delta_W^{1/2}$ of $H$. There is a natural embedding $\varepsilon : H \hookrightarrow \text{GL}(V)$ and a projection $\kappa : H \rightarrow \text{GL}(W)$ which together induce an injective morphism $(\varepsilon, \kappa) : H \hookrightarrow \text{GL}(V) \times \text{GL}(W)$. Then the pair $(H, \nu)$ is uniquely determined up to conjugacy in the group $\text{GL}(V) \times \text{GL}(W)$ by the pair $W \subset V$ and $(r, r^*)$. The following theorem generalizes the result in [GGP].

**Theorem 2.1.** Let $k$ be of characteristic zero. Let $\pi$ (resp. $\sigma$) be an irreducible admissible representation of $\text{GL}(V)$ (resp. $\text{GL}(W)$). Then $\dim_{\mathbb{C}} \text{Hom}_H(\pi \otimes \sigma, \nu) \leq 1$. Moreover, if $\pi$ and $\sigma$ are generic, then $\dim_{\mathbb{C}} \text{Hom}_H(\pi \otimes \sigma, \nu) = 1$.

The existence part is due to Corollary 6.3 (1). We consider the uniqueness part. The proof for $k$ non-archimedean is similar to that in [GGP, Section 15] with mild modifications for general linear groups and a general pair $(r, r^*)$. The proof for $k$ archimedean should follow similarly as in [JSZ10] which we omit.

Recall that we have $V = X \oplus W \oplus E \oplus X^*$. Let $E^+$ be a $k$-line generated by $e^+$. Let $v_0 = e + e^+$, $v_0^* = e - e^+$ and

$$Y = X \oplus \langle v_0 \rangle; \quad Y^* = X^* \oplus \langle v_0^* \rangle; \quad V^+ = V \oplus E^+.$$ 

Let $P_0$ be the parabolic subgroup of $\text{GL}(V^+)$ stabilizing the flag $\mathcal{F}_0 : 0 \subset Y \subset Y \oplus W \subset V^+$ and $M_0$ its Levi subgroup such that $M_0 \cong \text{GL}(Y) \times \text{GL}(W) \times \text{GL}(Y^*)$. The group $\text{GL}(V)$ embeds into $\text{GL}(V^+) \times \text{GL}(V)$ diagonally. Let $\tau$ (resp. $\tau^*$) be an irreducible supercuspidal representation of $\text{GL}(Y)$ (resp. $\text{GL}(Y^*)$) and let

$$I(\tau, \sigma, \tau^*) := \text{Ind}_{P_0}^{\text{GL}(V^+)}(\tau \otimes \sigma \otimes \tau^*)$$

be the unnormalized (smoothly) induced representation of $\text{GL}(V^+)$ of the representation $\tau \otimes \sigma \otimes \tau^*$ viewed as a representation of $P_0$ through the projection $P_0 \rightarrow M_0$. We have the following proposition which is similar to [GGP, Theorem 15.1].

**Proposition 2.2.** With the notations as above and let $k$ be non-archimedean, we have

$$\text{Hom}_{\text{GL}(V)}(I(\tau, \sigma, \tau^*) \otimes \pi, \mathbb{C}) = \text{Hom}_H(\pi \otimes \sigma \delta_W^{1/2}, \nu)$$

as long as $\pi$ does not belong to the Bernstein component of $\text{GL}(V)$ associated to the data $(\text{GL}(Y_0) \times M, \tau \otimes \varsigma)$ and $(M^* \times \text{GL}(Y_0^*), \varsigma^* \otimes \tau^*)$ where $M$ (resp. $M^*$) is any Levi subgroup of $\text{GL}(Z)$ (resp. $\text{GL}(Z^*)$) and $\varsigma$ (resp. $\varsigma^*$) is any irreducible supercuspidal representation of $M$ (resp. $M^*$). Here $V = Y_0 \oplus Z$ (resp. $V = Z^* \oplus Y_0^*$) with $\dim(Y_0) = \dim(Y) = r + 1$ (resp. $\dim(Y_0^*) = \dim(Y^*) = r^* + 1$).

**Proof.** We need to calculate the restriction $\Pi := I(\tau, \sigma, \tau^*)|_{\text{GL}(V)}$. By the Bruhat decomposition, there are six elements in the double coset $\text{GL}(V) \backslash \text{GL}(V^+)/P_0$ whose representatives are:

**Big cell:** $g_0 = 1_{n+1}$.

**Medium cells:** $g_1$ sends $\mathcal{F}_0$ to $\mathcal{F}_1 : 0 \subset Y_1 \subset Y_1 \oplus W_1$ with $Y_1 = Y$ and $E^+ \subset Y_1 \oplus W_1$; $g_2$ sends $\mathcal{F}_0$ to $\mathcal{F}_2 : 0 \subset Y_2 \subset Y_2 \oplus W_2$ with $Y_2 \subset V$ and $E^+ \not\subset Y_2 \oplus W_2 \subset V$.

**Small cells:** $g_3$ sends $\mathcal{F}_0$ to $\mathcal{F}_3 : 0 \subset Y_3 \subset Y_3 \oplus W_3$ with $E^+ \subset Y_3$; $g_4$ sends $\mathcal{F}_0$ to $\mathcal{F}_4 : 0 \subset Y_4 \subset Y_4 \oplus W_4$ with $Y_4 \subset V$ and $E^+ \subset Y_4 \oplus W_4$; $g_5$ sends $\mathcal{F}_0$ to $\mathcal{F}_5 : 0 \subset Y_5 \subset Y_5 \oplus W_5$ with $Y_5 \subset W_5 \subset V$.

Let $P_i$ ($i = 0, 1, 2, 3, 4, 5$) be the parabolic subgroup of $\text{GL}(V^+)$ stabilizing $\mathcal{F}_i$, $Q_i = P_i \cap \text{GL}(V)$ and $\pi_i = (\tau \otimes \sigma \otimes \tau^*)|_{Q_i}$. By Mackey theory, there is a filtration $0 \subset P_0 \subset P_1 \subset P_2 = \Pi$ such that

$$\Pi_0 \cong c\text{Ind}_{Q_0}^{\text{GL}(V)}(\pi_0); \quad \Pi_1/\Pi_0 = c\text{Ind}_{Q_1}^{\text{GL}(V)}(\pi_1) + c\text{Ind}_{Q_2}^{\text{GL}(V)}(\pi_2); \quad \Pi_2/\Pi_1 \cong c\text{Ind}_{Q_3}^{\text{GL}(V)}(\pi_3) + c\text{Ind}_{Q_4}^{\text{GL}(V)}(\pi_4) + c\text{Ind}_{Q_5}^{\text{GL}(V)}(\pi_5).$$
where $\text{clnd}$ means the (unnormalized smooth) induction with compact support. Applying the functor $\text{Hom}_{GL(V)}(-, \tilde{\pi})$, we have the following exact sequence:

$$0 \longrightarrow \text{Hom}_{GL(V)} \left( \bigoplus_{i=3}^{5} \text{cInd}_{Q_i}^{GL(V)} \pi_i, \tilde{\pi} \right) \longrightarrow \text{Hom}_{GL(V)}(\Pi, \tilde{\pi}) \longrightarrow \text{Hom}_{GL(V)}(\Pi_1, \tilde{\pi})$$

But for $i = 3, 4$ (resp. 4, 5), $\text{cInd}_{Q_i}^{GL(V)} \pi_i$ is of the form $\text{Ind}_{P_i}^{GL(V)} \varsigma^* \otimes \tau^*$ (resp. $\text{Ind}_{P_0}^{GL(V)} \tau \otimes \varsigma$) where $P_i$ (resp. $P_0$) is some parabolic subgroup whose Levi is $M^* \times \text{GL}(Y_0^*)$ (resp. $\text{GL}(Y_0) \times \text{M}$) and $\varsigma^*$ (resp. $\varsigma$) is some smooth representation of $M^*$ (resp. $M$). By our assumption on $\tau$ and $\tau^*$, the second and the fifth (last) terms in the above exact sequence are both zero. Hence $\text{Hom}_{GL(V)}(\Pi, \tilde{\pi}) \cong \text{Hom}_{GL(V)}(\Pi_1, \tilde{\pi})$.

By a result of Gelfand-Kazhdan, we have

$$\text{Hom}_{GL(V)}(\Pi, \tilde{\pi}) \cong \text{Hom}_{GL(V)}(\Pi_1, \tilde{\pi}) \cong \text{Hom}_{GL(V)}(\Pi_0, \tilde{\pi}) \cong \text{Hom}_{GL(V)}(\text{cInd}_{Q_0}^{GL(V)} \pi_0, \tilde{\pi}).$$

Recall that $P_{r,m+1,r}$ is the parabolic subgroup of $GL(V)$ stabilizing $0 \subset X \subset X \otimes W \otimes E \subset V$. Hence it contains $Q_0$ as a subgroup and moreover, is equal to $U_{r,m+1,r} \rtimes (\text{GL}(X) \times \text{GL}(W) \times \text{GL}(X^*))$. The natural projection: $P_0 \to \text{GL}(Y) \times \text{GL}(W) \times \text{GL}(Y^*)$ induces the following commutative diagram with exact rows and injective vertical arrows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N_0 & \longrightarrow & P_0 & \longrightarrow & \text{GL}(Y) \times \text{GL}(W) \times \text{GL}(Y^*) & \longrightarrow & 0 \\
0 & \longrightarrow & N_0 \cap Q_0 & \longrightarrow & Q_0 & \longrightarrow & P_Y \times \text{GL}(W) \times P_{Y^*} & \longrightarrow & 0 \\
0 & \longrightarrow & N_0 \cap Q_0 & \longrightarrow & U_{r,m+1,r} & \longrightarrow & \text{Hom}(\langle v_0 \rangle, X) \times \text{Hom}(X^*, \langle v_0^* \rangle) & \longrightarrow & 0
\end{array}
$$

where $N_0$ is the maximal unipotent subgroup of $P_0$, $P_Y \subset \text{GL}(Y)$ is the mirabolic subgroup stabilizing $X$ and $P_{Y^*} \subset \text{GL}(Y^*)$ is the mirabolic subgroup fixing $v_0^*$. The proof of this is similar to [GGP, Lemma 15.2]. By the diagram, we have

$$\tau|_{P_Y} \cong \text{Ind}_{U_Y}^{P_Y} \lambda; \quad \tau^*|_{P_{Y^*}} \cong \text{Ind}_{U_{Y^*}}^{P_{Y^*}} \lambda^*$$

By a result of Gelfand-Kazhdan, we have

$$\tau|_{P_Y} \cong \text{Ind}_{U_Y}^{P_Y} \lambda; \quad \tau^*|_{P_{Y^*}} \cong \text{Ind}_{U_{Y^*}}^{P_{Y^*}} \lambda^*$$

where $U_Y$ (resp. $U_{Y^*}$) is the unipotent radical of a Borel subgroup of $GL(Y)$ (resp. $GL(Y^*)$) satisfying $U_X \subset U_Y \subset P_Y$ (resp. $U_{Y^*} \subset U_{Y^*} \subset P_{Y^*}$), and $\lambda$ (resp. $\lambda^*$) is a generic character of $U_Y$ (resp. $U_{Y^*}$).

By our choice of unipotent radical, it is clear that, the pre-image of $U_Y \times \text{GL}(W) \times U_{Y^*}$ in $Q_0$ is the subgroup

$$H = U_{r,m+1,r} \rtimes (U_X \times \text{GL}(W) \times U_{X^*})$$
and the pull-back of the representation $\lambda \otimes \sigma \otimes \lambda^*$ is just $\sigma \delta_{W^*}^{-\frac{1}{2}} \otimes \tilde{\nu}$. Hence, by induction in stages, we have
\[
\text{clInd}_{Q_0}^{GL(V)} \pi_0 \cong \text{clInd}_H^{GL(V)} \sigma \delta_W^{-\frac{1}{2}} \otimes \tilde{\nu}
\]
and by Frobenius reciprocity, we conclude that
\[
\text{Hom}_{GL(V)} \left( I \left( \tau, \sigma, \tau^* \right) \otimes \pi, \mathbb{C} \right) \cong \text{Hom}_H \left( \pi \otimes \sigma \delta_W^{-\frac{1}{2}}, \nu \right).
\]

\[
\square
\]

**Proof of Theorem 2.1 for the uniqueness part when $k$ is non-archimedean.** We choose an irreducible supercuspidal representation $\tau$ (resp. $\tau^*$) of $GL(Y)$ (resp. $GL(Y^*)$) satisfying the assumption in the above proposition. After twisting unramified characters (which still satisfy the assumption), we may assume that the induced representation $I \left( \tau, \sigma \delta_W^\frac{1}{2}, \tau^* \right)$ is irreducible. Then by the above proposition and [AGRS10], we have
\[
\dim \text{Hom}_H \left( \pi \otimes \sigma, \nu \right) = \dim \text{Hom}_{GL(V)} \left( I \left( \tau, \sigma \delta_W^\frac{1}{2}, \tau^* \right) \otimes \pi, \mathbb{C} \right) \leq 1.
\]

\[
\square
\]

**Definition 2.3.** A nontrivial element in the space $\text{Hom}_H(\pi \otimes \sigma, \nu)$ is called an $(r, r^*)$-**Bessel model** of $\pi \otimes \sigma$. When $r = r^*$, hence $n - m$ is odd, it is just the one defined in [GGP].

### 2.2 Bessel integrals, functional equations and $L$-functions

In this section, we consider the global situation. Hence $k$ will be a number field and $A$ is its ring of adèles. Let $| \cdot | = \prod_{v \in M_k} | \cdot |_v$. For any $v \in M_k$, we denote $k_v$ the completion of $k$ at $v$. We denote $\mathfrak{o}$ (resp. $\mathfrak{o}_v$) the ring of integers of $k$ (resp. $k_v$ for $v$ finite). For any algebraic group $G$ over $k$, we denote $G_v = G(k_v)$ the local Lie group for $v \in M_k$. If $G$ is reductive, we denote $\mathcal{A}(G)$ (resp. $\mathcal{A}_0(G)$) the space of automorphic forms (resp. cusp forms) of $G$ which is a representation of $G(A)$ by right translation $\rho$.

We define the pair $(H, \nu)$ as in the local case. Hence $\ell_X$, $\ell_X^*$ and $\ell_W$ are defined over the number field $k$; $\psi$ is a nontrivial character of $k \backslash \mathcal{A}$; $\lambda$ is a generic character $(U_X \times U_X^*)(k) \backslash (U_X \times U_X^*) (\mathcal{A}) \rightarrow \mathbb{C}^1$. Let $\pi$ (resp. $\sigma$) be an irreducible cuspidal automorphic representation of $GL(V)(\mathcal{A})$ (resp. $GL(V^*)(\mathcal{A})$). Then $\pi$ (resp. $\sigma$) is isomorphic to a unique irreducible sub-representation $\mathcal{A}_\pi$ (resp. $\mathcal{A}_\sigma$) of $\mathcal{A}_0(GL(V))$ (resp. $\mathcal{A}_0(GL(V^*))$).

**Definition 2.4.** The following absolutely convergent integral is called an $(r, r^*)$-**Bessel period** of $\pi \otimes \sigma$ (for a pair $(H, \nu)$):
\[
B_{r, r^*}^\nu(\varphi_\pi, \varphi_\sigma) := \int_{H(k \backslash H(A))} \varphi_\pi(\varepsilon(h))\varphi_\sigma(\kappa(h))\nu(h)^{-1} dh, \quad \varphi_\pi \in \mathcal{A}_\pi, \varphi_\sigma \in \mathcal{A}_\sigma
\]
where $dh$ is the Tamagawa measure on $H(A)$. If there exist $\varphi_\pi, \varphi_\sigma$ such that $B_{r, r^*}^\nu(\varphi_\pi, \varphi_\sigma) \neq 0$, then we say $\pi \otimes \sigma$ has a nontrivial $(r, r^*)$-Bessel period.

It is obvious that $B_{r, r^*}^\nu$ defines an element in
\[
\text{Hom}_{H(A)}(\pi \otimes \sigma, \nu) = \bigotimes_{v \in M_k} \text{Hom}_{H_v}(\pi_v \otimes \sigma_v, \nu_v).
\]
Since the later space has the multiplicity one property, we expect that the Bessel period is Eulerian. We now show that this is true.

We can choose a basis $\{ v_1, ..., v_r \}$ of $X$ such that
• The homomorphism $\ell_X : X \to k$ is given by the coefficient of $v_r$ under the above basis;

• $U_X$ is the maximal unipotent subgroup of the parabolic subgroup $P_X$ stabilizing the complete flag $0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \ldots, v_r \rangle = X$;

• The generic character $\lambda|_{U_X}$ is given by

$$\lambda(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{r-1,r})$$

where

$$u = \begin{bmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,r-1} & u_{1,r} \\ & 1 & u_{2,3} & \cdots & u_{2,r-1} & u_{2,r} \\ & & \ddots & \ddots & \ddots & \ddots \\ & & & 1 & u_{r-1,r} \\ & & & & 1 & u_{r-1,r} \end{bmatrix} \in U_X(A) \quad (2.1)$$

under the above basis.

Similarly, we can also choose a basis $\{v^*_1, \ldots, v^*_r\}$ of $X^*$ such that

• The homomorphism $\ell_{X^*} : k \to X^*$ is given by $x \mapsto cxv^*_r$, for some $c \neq 0$ determined later;

• $U_{X^*}$ is the maximal unipotent subgroup of the parabolic subgroup $P_{X^*}$ stabilizing the complete flag $0 \subset \langle v^*_r \rangle \subset \langle v^*_r, v^*_r-1 \rangle \subset \cdots \subset \langle v^*_r, \ldots, v^*_1 \rangle = X^*$;

• The generic character $\lambda|_{U_{X^*}}$ is given by

$$\lambda(u^*) = \psi(u^*_{1, r-1} + u^*_{2, r-2} + \cdots + u^*_{2,1})$$

where

$$u^* = \begin{bmatrix} 1 & u^*_r & u^*_r-2 & \cdots & u^*_r-1 & u^*_r \\ & 1 & u^*_r-1 & \cdots & u^*_r-2 & u^*_r-1 \\ & & \ddots & \ddots & \ddots & \ddots \\ & & & 1 & u^*_2 & u^*_2 \\ & & & & 1 & u^*_2 \end{bmatrix} \in U_{X^*}(A) \quad (2.2)$$

under the above basis.

Moreover, we can choose a basis $\{w_1, \ldots, w_m\}$ of $W$ and $\{w_0\}$ of $E$ such that the bilinear form $\ell_W : (W \oplus E) \to k$ is given by $\ell_W(w_i) = \ell_W(w^*_i) = 0$ (1 ≤ $i$ ≤ $m$) and $\ell_W(w^*_0) = 1$, where $\{w^*_1, \ldots, w^*_m, w^*_0\}$ is the dual basis. Let $c = \ell_W(w_0)^{-1}$.

We write elements in $\text{GL}(V)$ in the matrix form under the basis:

$$\{w_1, \ldots, w_m, v_1, \ldots, v_r, w_0, v^*_1, \ldots, v^*_r\} \quad (2.3)$$

and view $\text{GL}(W)$ as a subgroup of $\text{GL}(V)$. Then the image of $H(A)$ in $\text{GL}(V)(A)$ consists of following
matrices:

\[
h = h(n, n^*, b; u, u^*; g) = \begin{bmatrix}
g & u & b \\
n_{1,1} & \cdots & n_{1,m} & n_{1,0} & \cdots & n_{1,1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n_{r,1} & \cdots & n_{r,m} & n_{r,0} & \cdots & n_{1,1} \\
u & n_{0,r^*} & \cdots & n_{0,1}
\end{bmatrix}
\]

where

\[
n = \begin{bmatrix}
n_{1,1} & \cdots & n_{1,m} & n_{1,0} \\
\vdots & \vdots & \vdots & \vdots \\
n_{r,1} & \cdots & n_{r,m} & n_{r,0}
\end{bmatrix} \in \text{Hom}(W \oplus E, X)(\mathbb{A}),
\]

\[
n^* = \begin{bmatrix}
n_{1,1}^* & \cdots & n_{1,1} \\
\vdots & \vdots & \vdots \\
n_{r,1}^* & \cdots & n_{r,1}
\end{bmatrix} \in \text{Hom}(X^*, W \oplus E)(\mathbb{A}),
\]

\[b \in \text{Hom}(X^*, X)(\mathbb{A}), u \in U_X(\mathbb{A}), u^* \in U_{X^*}(\mathbb{A}) \text{ and } g \in \text{GL}(W)(\mathbb{A}). \]

Hence \(u\) and \(u^*\) are upper triangular matrices as in (2.1) and (2.2); the character \(\nu\) on \(H(\mathbb{A})\) is given by

\[\nu(h) = \nu(h(n, n^*, b; u, u^*; g)) = |\det g|_k^{1-n^*} \psi\left(\sum_{i=0}^{r-1} x_i + n_{r,0} + n_0 u_i^* + u_{i-r^*}^* \right).
\]

Let \(U_{r,m+1,1^*} = U_{r,m+1} \times (U_X \times U_{X^*})\) be a unipotent subgroup of \(\text{GL}(V)\), then

\[U_{r,m+1,1^*}(\mathbb{A}) = \left\{u = u(n, n^*, b; u, u^*): h(n, n^*, b; u, u^*; \mathbb{A}) \right\}
\]

and we denote \(du\) the product measure. Then we have \(dh = |\det g|_k^{1-n^*} du\). Let us simply write

\[\psi(u) = \psi\left(\sum_{i=0}^{r-1} x_i + n_{r,0} + n_0 u_i^* + u_{i-r^*}^* \right)
\]

and identify \(GL(V)\) (resp. \(GL(W)\)) with \(\text{GL}_{n,k}\) (resp. \(\text{GL}_{m,k}\)) under the basis (2.3). Then

\[B_{r,r^*}(\varphi_\pi, \varphi_\sigma) = \int_{\text{GL}(k)\backslash \text{GL}(\mathbb{A})} \int_{U_{r,m+1,1^*}(\mathbb{A})} \varphi_\pi(ug) \varphi_\sigma(g) |\det g|_k^{1-n^*} \psi(u) du dg.
\]

We insert an \(s\)-variable as

\[B_{r,r^*}(s; \varphi_\pi, \varphi_\sigma) = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_\pi(ug) \varphi_\sigma(g) |\det g|_k^{1-n^*} \psi(u) du dg.
\]

and call it the \((r, r^*)\)-Bessel integral. We are going to use the Fourier transform. Let

\[L_{r+1} = \left\{u(n_0, n^*, b; u, u^*) \mid n_0 = \begin{bmatrix} 0 & \cdots & 0 & n_{1,0} \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & n_{r,0} \end{bmatrix} \right\}
\]
be a subgroup of $U_{1^r,m+1,1^r}$. Let

$$L_r = \{ l = (l_r; n_0, n^*, b; u, u^*) | l_r = (l_{1,r}, \ldots, l_{m,r}) \}$$

where the matrix $l(l_r; n_0, n^*, b; u, u^*)$ is the one obtained from $u(n_0, n^*, b; u, u^*)$ by adding the column $l_r$ above the entry $n_{1,0}$ as in (2.4). It is clear that $L_r / L_{r+1}$ is isomorphic to $k^m$ which may be identified with the set of column vector $l_r$. By the Fourier inverse formula for $(k\backslash k)^m$, we have

$$\int_{g} \int_{M_{r,m}(k\backslash k) \backslash L_r(k)} \varphi_n(lng) \varphi_\sigma(g) \det g_{\lambda}^{|s-\frac{1}{2}+r-r^*|} \psi \left( \sum_{i=1}^{m} \epsilon_i l_{i,r} \right) \psi(l) dl dg$$

(2.5)

where $\varphi$ represents the element $u(n, 0, 0; 1_r, 1_r)$ with

$$n = \begin{bmatrix} n_{1,1} & \cdots & n_{1,m} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ n_{r,1} & \cdots & n_{r,m} & 0 \end{bmatrix}$$

and we view $\psi$ as a character on $L_r(k)$ through the natural quotient $L_{r+1}(k)$. We also extend the measure on $L_{r+1}(k)$ to $L_r(k)$ by the dual measure on $k^m$. Let $\epsilon$ be an element like $\epsilon(\epsilon, 0, 0; 1_r, 1_r)$ where

$$\epsilon = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ \epsilon_1 & \cdots & \epsilon_m & 0 \end{bmatrix},$$

then if we conjugate $l$ by $\epsilon$ from left to right, we can incorporate $\psi(\sum \epsilon_i l_{i,r})$ into $\psi(l)$ and collapse the summation over $\epsilon_i \in k$. The result is

$$\int_{g} \int_{M_{r,m}(k\backslash k) \backslash L_r(k) \times M_{1,1}(k)} \varphi_n(lng) \varphi_\sigma(g) \det g_{\lambda}^{|s-\frac{1}{2}+r-r^*|} \psi(l) dl dg.$$

(2.6)

Similarly, we can introduce the subgroup $L_{r-1}, \ldots, L_1$ and repeat the process $r-1$ more times. Then we finally get

$$\int_{g} \int_{M_{r,m}(k) \backslash L_1(k) \times L_1(k)} \varphi_n(lng) \varphi_\sigma(g) \det g_{\lambda}^{|s-\frac{1}{2}+r-r^*|} \psi(l) dl dg.$$

(2.7)

Here,

$$L_1 = \{ l = (l_r; n_0, n^*, b; u, u^*) | l_1 = \begin{bmatrix} l_{1,1} & \cdots & l_{1,r-1} & l_{1,r} \\ \vdots & \ddots & \vdots & \vdots \\ l_{m,1} & \cdots & l_{m,r-1} & l_{m,r} \end{bmatrix} \}$$

where $l(l_r; n_0, n^*, b; u, u^*)$ is the one obtained from $u(n_1, n^*, b; u, u^*)$ by adding $l_1$ on the sub-row $[u_{1,2}, \ldots, u_{1,r}, n_{1,0}]$. In fact, $L_1$ is the maximal unipotent subgroup of the (standard) parabolic subgroup stabilizing the flag

$$0 \subset W \oplus \langle v_1 \rangle \subset W \oplus \langle v_1, v_2 \rangle \subset \cdots \subset V.$$
Now the inner double integral is just the usual Rankin-Selberg convolution (cf. [JPSS83]). The following calculation is well-known:

\[
\int_{GL_m(k)\backslash GL_m(\mathbb{A})} |\det g|^s \frac{\Lambda_{n-m}}{-2} \int_{L_1(k)\backslash L_1(\mathbb{A})} (\rho(\mathfrak{a})) \varphi_\chi(Kg) \varphi_\sigma(g) \psi(g) \psi(L) dg dl
\]

\[
= \int_{U_1(m)\backslash GL_m(k)} \sum_{\gamma \in U_1(m)\backslash GL_m(k)} W^\psi_{\rho(\mathfrak{a})} \left( \begin{array}{ccc} u & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1_{n-m} \end{array} \right) \varphi_\sigma(g) |\det g|^s \frac{\Lambda_{n-m}}{-2} dg
\]

\[
= \int_{U_1(m)\backslash GL_m(k)} W^\psi_{\rho(\mathfrak{a})} \left( \begin{array}{ccc} g & 0 & 0 \\ 0 & 1_{n-m} \end{array} \right) \int_{U_1(k)\backslash U_1(\mathbb{A})} \psi(u) \varphi_\sigma(g) |\det g|^s \frac{\Lambda_{n-m}}{-2} dg
\]

\[
= \int_{U_1(m)\backslash GL_m(k)} W^\psi_{\rho(\mathfrak{a})} \left( \begin{array}{ccc} g & 0 & 0 \\ 0 & 1_{n-m} \end{array} \right) W_\varphi^\psi (g) |\det g|^s \frac{\Lambda_{n-m}}{-2} dg
\]

\[
= \int_{U_1(m)\backslash GL_m(k)} W^\psi_{\rho(\mathfrak{a})} \left( \begin{array}{ccc} g & 0 & 0 \\ 0 & 1_{n-m} \end{array} \right) W_\varphi^\psi (g) |\det g|^s \frac{\Lambda_{n-m}}{-2} dg.
\]

Plug (2.11) into (2.9), we have

\[
(2.9) = \int_{U_1(m)\backslash GL_m(\mathbb{A})} \int_{M_{r,m}(\mathbb{A})} W^\psi_{\varphi_\sigma} \left( \begin{array}{ccc} g & 0 & 0 \\ x & 1_r & 0 \\ 0 & 0 & 1_{n-m-r} \end{array} \right) W_\varphi^\psi (g) |\det g|^s \frac{\Lambda_{n-m}}{-2} dx dg.
\]

We denote the above integral by \( \Psi_{r,v}(s; W^\psi_{\varphi_\sigma}, W_\varphi^\psi) \) which is absolutely convergent when \( \Re(s) > 0 \). If we assume that \( W^\psi_{\varphi_\sigma} = \oplus_v W_v \) and \( W_\varphi^\psi = \oplus_v W_v^\psi \) are factorizable and let

\[
\Psi_{r,v}(s; W_v, W_v^\psi) = \int_{U_1(m)\backslash GL_m(\mathbb{A})} \int_{M_{r,m}(\mathbb{A})} W_v \left( \begin{array}{ccc} g_v & 0 & 0 \\ x_v & 1_r & 0 \\ 0 & 0 & 1_{n-m-r} \end{array} \right) W_v^\psi (g_v) |\det g_v|^s \frac{\Lambda_{n-m}}{-2} dx_v dg_v.
\]

Then in summary, we have for \( \Re(s) \) large,

\[
\mathcal{B}_{r,v}(s; \varphi_\sigma, \varphi_\chi) = \Psi_r(s; W^\psi_{\varphi_\sigma}, W_\varphi^\psi) = \prod_{v \in \mathfrak{M}_k} \Psi_{r,v}(s; W_v, W_v^\psi)
\]

where \( W_v \in \mathcal{W}(\pi_v, \psi_v) \) (see Section 6 for notations); \( W_v^\psi \in \mathcal{W}(\psi_v, \psi_v) \) and for almost all finite places \( v \), \( W_v, W_v^\psi \) are unramified satisfying \( W_v(1_n) = W_v^\psi(1_m) = 1 \).

Now we discuss the functional equation of the Bessel integrals. First, let us introduce some Weyl elements:

\[
w_1 = 1_1; \quad w_n = \left[ \begin{array}{ccc} w_{n-1} & 1 \\ 0 & 0 \end{array} \right] \quad w_{n,m} = \left[ \begin{array}{ccc} 1_m & 0 \\ 0 & w_{n-m} \end{array} \right].
\]
We define \( \iota \) the outer automorphism of \( \text{GL}_n \) and \( \text{GL}_m \) by \( \iota(g) = g' := g^{-1} \). Let \( \tilde{\varphi}(g) = \varphi_\pi(g') = \varphi_\pi(w_n g') \) and \( \tilde{\varphi}(g) = \varphi_\sigma(g') = \varphi_\sigma(w_m g') \). Then

\[
\mathcal{B}_{r,r^\ast}^\nu(s; \varphi_\pi, \varphi_\sigma) = \int_{\text{GL}(k) \backslash \text{GL}(k)} \int_{U_{1^r,m+1^r}(k) \backslash U_{1^r,m+1^r}(k)} \varphi_\pi(\omega g') \varphi_\sigma(g') |\det g'|_{\frac{k}{2r} + s - \frac{r-1}{2}} \frac{\psi(u)}{dudg} \]

Let \( \widetilde{W}_r^\nu(g) = W_\nu^\psi(g) \in \mathcal{W}(\tilde{\pi}, \tilde{\psi}) \) and similar for the one on \( \text{GL}_m \). If \( W_\nu^\psi = \otimes_v W_v^\psi \) are factorizable, then \( \widetilde{W}_r^\nu = \otimes_v \tilde{W}_r^\nu \) and \( \widetilde{W}_r^\nu \) are also factorizable with \( \tilde{W}_r^\nu(g) = W_\nu^\psi(w_n g') \) and similar for \( W^\nu_r \). Then for \( \mathcal{R}(s) \) large,

\[
\mathcal{B}_{r,r^\ast}^\nu(s; \varphi_\pi, \varphi_\sigma) = \Psi_r(s; W_\nu^\psi, \tilde{W}_r^\nu) = \prod_{v \in \mathcal{M}_k} \Psi_v(s; W_v^\psi, \tilde{W}_r^\nu);
\]

\[
\mathcal{B}_{r,r^\ast}^\nu(s; \rho(w_n,m) \tilde{\varphi}(\sigma), \tilde{\varphi}(\sigma)) = \Psi_{r^\ast}(s; \rho(w_n,m) \tilde{W}_r^\nu, \tilde{W}_r^\nu) = \prod_{v \in \mathcal{M}_k} \Psi_v(s; \rho(w_n,m) \tilde{W}_r^\nu, \tilde{W}_r^\nu).
\]

In summary, we have the following

**Theorem 2.5.** The Bessel integrals are holomorphic in \( s \) and satisfy the following functional equation

\[
\mathcal{B}_{r,r^\ast}^\nu(s; \varphi_\pi, \varphi_\sigma) = \mathcal{B}_{r,r^\ast}^\nu(s - 1; \rho(w_n,m) \tilde{\varphi}(\sigma), \tilde{\varphi}(\sigma)).
\]

Let \( \widetilde{W}_r^\nu(g) = W_\nu^\psi(g) \in \mathcal{W}(\tilde{\pi}, \tilde{\psi}) \) and similar for the one on \( \text{GL}_m \). If \( W_\nu^\psi = \otimes_v W_v^\psi \) are factorizable, then \( \widetilde{W}_r^\nu = \otimes_v \tilde{W}_r^\nu \) and \( \widetilde{W}_r^\nu \) are also factorizable with \( \tilde{W}_r^\nu(g) = W_\nu^\psi(w_n g') \) and similar for \( W^\nu_r \). Then for \( \mathcal{R}(s) \) large,

\[
\mathcal{B}_{r,r^\ast}^\nu(s; \varphi_\pi, \varphi_\sigma) = \Psi_r(s; W_\nu^\psi, \tilde{W}_r^\nu) = \prod_{v \in \mathcal{M}_k} \Psi_v(s; W_v^\psi, \tilde{W}_r^\nu);
\]

\[
\mathcal{B}_{r,r^\ast}^\nu(s; \rho(w_n,m) \tilde{\varphi}(\sigma), \tilde{\varphi}(\sigma)) = \Psi_{r^\ast}(s; \rho(w_n,m) \tilde{W}_r^\nu, \tilde{W}_r^\nu) = \prod_{v \in \mathcal{M}_k} \Psi_v(s; \rho(w_n,m) \tilde{W}_r^\nu, \tilde{W}_r^\nu).
\]

In particular, when \( r = r^\ast \) (hence \( n - m \) is odd), the \( (r, r^\ast) \)-Bessel integral itself has a functional equation.

By Proposition (6.1) and (6.2), we have the following theorem, which confirms [GGP, Conjecture 24.1] for the Bessel periods of split unitary groups, i.e., general linear groups.

**Theorem 2.6.** (1) Let \( \pi \) (resp. \( \sigma \)) be an irreducible cuspidal automorphic representation of \( \text{GL}(V)(\mathbb{A}) \) (resp. \( \text{GL}(W)(\mathbb{A}) \)). For any \( (r, r^\ast) \) such that \( r + r^\ast = n - m - 1 \) and the automorphic representation \( \nu \) introduced above, we have, for \( \varphi_\pi \in \mathcal{A}_\pi \) and \( \varphi_\sigma \in \mathcal{A}_\sigma \) such that \( W_\nu^\psi = \otimes_v W_v^\psi \) and \( W_\nu^\psi = \otimes_v W_v^\psi \) are factorizable,

\[
\mathcal{B}_{r,r^\ast}(\varphi_\pi, \varphi_\sigma) = L^\nu(\frac{1}{2}, \pi \times \sigma) \prod_{v \in \mathcal{M}_k} \frac{\Psi_v(s; W_v^\psi, W_v^\psi)}{L_v(s, \pi_v \times \sigma_v)} \bigg|_{s = \frac{1}{2}}
\]

where in the last product almost all factors are 1, and the \( L \)-functions are the ones defined by Rankin-Selberg convolutions (cf. [JPSS83]).

(2) There is a nontrivial Bessel period of \( \pi \otimes \sigma \) if and only if \( L^\nu(\frac{1}{2}, \pi \times \sigma) \neq 0 \).
3 Fourier-Jacobi models of $\text{GL}_n \times \text{GL}_m$

3.1 Fourier-Jacobi models

Let $k$ be a local field and $V$ a $k$-vector space of dimension $n > 0$. Suppose that $V$ has a decomposition $V = X \oplus W \oplus X^*$ where $W$, $X$ and $X^*$ have dimension $m$, $r$ and $r^*$ respectively. Then $n = m + r + r^*$. Let $P_{r,m,r^*}$ be the parabolic subgroup of $\text{GL}(V)$ stabilizing the flag $0 \subset X \subset X \oplus W \subset V$ and $U_{r,m,r^*}$ its maximal unipotent subgroup. Then $U_{r,m,r^*}$ fits into the following exact sequence:

$$0 \longrightarrow \text{Hom}(X^*, X) \longrightarrow U_{r,m,r^*} \longrightarrow \text{Hom}(X^*, W) + \text{Hom}(W, X) \longrightarrow 0.$$ 

We may write the above sequence as:

$$0 \longrightarrow (X^*)^\vee \otimes X \longrightarrow U_{r,m,r^*} \longrightarrow (X^*)^\vee \otimes W + W^\vee \otimes X \longrightarrow 0.$$ 

Let $\ell_X : X \to k$ (resp. $\ell_{X^*} : k \to X^*$) be any nontrivial homomorphism (if exists) and let $U_X$ (resp. $U_{X^*}$) be a maximal unipotent subgroup of $\text{GL}(X)$ (resp. $\text{GL}(X^*)$) stabilizing $\ell_X$ (resp. $\ell_{X^*}$). Then the above exact sequence fits into the following commutative diagram:

$$
\begin{array}{c}
\begin{CD}
0 @>>> (X^*)^\vee \otimes X @>>> U_{r,m,r^*} @>>> W^\vee \otimes X + (X^*)^\vee \otimes W @>>> 0 \\
\ell_X @VVV \ell_{X^*} @VVV \ell_X + \ell_{X^*} @VVV \\
0 @>>> k @>>> H(W^\vee + W) @>>> W^\vee + W @>>> 0 \\
\end{CD}
\end{array}
$$

which is equivariant under the action of $U_X \times U_{X^*} \times \text{GL}(W)$, where $H(W^\vee + W) = k + W^\vee + W$ is the Heisenberg group of $W^\vee + W$ whose multiplication is given by

$$(t_1, w_1^\vee, w_1)(t_2, w_2^\vee, w_2) = \left(t_1 + t_2 + \frac{w_1^\vee(w_2) - w_2^\vee(w_1)}{2}, w_1^\vee + w_2^\vee, w_1 + w_2\right).$$

Given a nontrivial character $\psi : k \to \mathbb{C}^1$, there is a unique infinite dimensional irreducible smooth representation $\omega_\psi$ of $H(W^\vee + W)$ with central character $\psi$. We choose the following model. Let $S(W^\vee)$ be the space of Bruhat-Schwartz functions on $W^\vee$. For $\Phi \in S(W^\vee)$, let

$$(\omega_\psi(t, w^\vee, w)\Phi)(w^\circ) = \psi\left(t + w^\circ(w) + \frac{w^\vee(w)}{2}\right)\Phi(w^\circ + w^\vee)$$

for all $(t, w^\vee, w) \in H(W^\vee + W)$. Moreover, if we choose a character $\mu : k^* \to \mathbb{C}^*$, we have a Weil representation $\omega_\mu$ of $\text{GL}(W)$ on $S(W^\vee)$ by

$$(\omega_\mu(g)\Phi)(w^\circ) = \mu(\det g) |\det g|^\frac{1}{2} \Phi(w^\circ \cdot g)$$

where $g \in \text{GL}(W)$ acts on $W^\vee$ by $(w^\circ \cdot g) w = w^\circ(g \cdot w)$ for all $w \in W$. They together form a representation $\omega_{\psi, \mu}$ of $U_{r,m,r^*} \times \text{GL}(W)$ through the projection $U_{r,m,r^*} \to H(W^\vee + W)$ and hence a representation of $H := U_{r,m,r^*} \rtimes (\text{U}_X \times X^* \rtimes \text{GL}(W))$ by extending trivially to $U_X \times U_{X^*}$. As in the Bessel model, we choose a generic character $\lambda : U_X \times U_{X^*} \to \mathbb{C}^1$. Then we define the representation

$$\nu_\mu = \omega_{\psi, \mu} \otimes \lambda \otimes \delta_W^\perp$$

of $H$ which has the Gelfand-Kirillov dimension $m$. We also define

$$\nu_\mu^{-1} = \nu_\mu^\perp \delta_W = \omega_{\psi, \mu^{-1}} \otimes \lambda^\perp \otimes \delta_W^\perp.$$

As in the Bessel model, we have an injective morphism $(\varepsilon, \kappa) : H \hookrightarrow \text{GL}(V) \times \text{GL}(W)$. Then the pair $(H, \nu_\mu)$ is uniquely determined up to conjugacy in the group $\text{GL}(V) \times \text{GL}(W)$ by the pair $W \subset V$, $(r, r^*)$ and $\mu$. The following theorem generalizes the result in [GGP].
Theorem 3.1. Let $k$ be of characteristic zero and non-archimedean. Let $\pi$ (resp. $\sigma$) be an irreducible admissible representation of $GL(V)$ (resp. $GL(W)$). Then $\dim_c \text{Hom}_H(\pi \otimes \sigma \otimes \overline{\nu}_\mu, C) \leq 1$. Moreover, if $\pi$ and $\sigma$ are generic, then $\dim_c \text{Hom}_H(\pi \otimes \sigma \otimes \overline{\nu}_\mu, C) = 1$.

The existence part is due to Corollary 6.3 (2). We consider the uniqueness part whose proof is similar to that in [GGP, Section 16] with mild modifications for general linear groups and a general pair $(r, r^*)$.

Recall that we have $V = X \oplus W \oplus X^\ast$. Let $\tau$ (resp. $\tau^*$) be a supercuspidal representation of $GL(X)$ (resp. $GL(X^\ast)$) and let

$$I(\tau, \sigma, \tau^*) := \text{Ind}_{P_{r, m, r^*}}^{GL(V)}(\tau \otimes \sigma \otimes \tau^*)$$

be the induced representation. We have the following proposition which is similar to [GGP, Theorem 16.1].

Proposition 3.2. With $\psi$ a fixed additive character of $k$ which is non-archimedean, we have

$$\text{Hom}_{GL(V)}\left(I\left(\tau, \sigma \mu^{-1} \delta_{\omega, \tau^*} \otimes \pi, \omega_{V, \psi}\right) \right) = \text{Hom}_H(\pi \otimes \sigma, \nu_\mu)$$

as long as $\tau$ does not belong to the Bernstein component of $GL(V)$ associated to the data $(GL(X) \times M^\ast, \tau \otimes \xi^*)$ and $(M \times GL(X^\ast), \xi \otimes \tau^*)$ where $M$ (resp. $M^\ast$) is any Levi subgroup of $GL(X + W)$ (resp. $GL(W + X^\ast)$) and $\xi$ (resp. $\xi^*$) is any irreducible supercuspidal representation of $M$ (resp. $M^\ast$). Here, $\omega_{V, \psi}$ is the Weil representation of $GL(V)$.

Proof. Let $Mp(W + W^\vee)$ be the $C^\ast$-metaplectic cover of the symplectic group of the symplectic space $W + W^\vee$. We choose the Lagrangian $W^\vee$ and an additive character $\psi$, and hence get a model of Weil representation $\omega_{W, \psi}$. We fix a homomorphism $GL(W) \hookrightarrow Mp(W + W^\vee)$ (by choosing $\mu$ to be trivial) lifting the embedding $GL(W) \hookrightarrow Sp(W + W^\vee)$ such that

$$(\omega_{W, \psi}(g)) (w^\psi) = |\det g|^\frac{1}{2} \Phi(w^\psi g).$$

We consider another symplectic space $V_1 + V_1^\vee$, where $V_1 = X + (X^\ast)^\vee \oplus W$ and $V_1^\vee = W^\vee + X^\ast + X^\vee$. We choose the Lagrangian $V_1^\vee \subset W^\vee$ and consider the mixed model of the Weil representation $\omega_{V_1, \psi}$ of $Mp(V_1 + V_1^\vee)$. This model has a realization on the space $S((X + (X^\ast)^\vee)^\vee) \otimes S(W^\vee) = S(W^\vee + X^\ast + X^\vee)$.

We choose a lifting of the embedding $GL(V) \hookrightarrow Sp(V_1 + V_1^\vee)$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Mp}(W + W^\vee) & \hookrightarrow & \text{Mp}(V_1 + V_1^\vee) \\
\downarrow & & \downarrow \\
\text{Sp}(W + W^\vee) & \hookrightarrow & \text{Sp}(V_1 + V_1^\vee) \\
\downarrow & & \downarrow \\
\text{GL}(W) & \hookrightarrow & \text{GL}(V)
\end{array}
\]

The parabolic subgroup $P_{r, m, r^*}$ of $GL(V)$ is contained in the parabolic subgroup $P(V_1, W)$ of $Sp(V_1 + V_1^\vee)$ stabilizing the flag $0 \subset X + (X^\ast)^\vee \subset X + (X^\ast)^\vee \oplus W + W^\vee$. Elements in $P_{r, m, r^*}$ can be written as $p = p(n, n^*, b; h, h^*, g)$ with $n \in \text{Hom}(W, X)$, $n^* \in \text{Hom}(X^\ast, W)$, $b \in \text{Hom}(X^\ast, X)$, $h \in GL(X)$, $h^* \in GL(X^\ast)$ and $g \in GL(W)$. We have the following formula for the mixed model:

$$(\omega_{V_1, \psi}(p(0, 0, 0; h, h^*; g))) (w^\psi, x^\xi, x^\eta) = |\det hgh^{-1} x^\xi|^{\frac{1}{2}} \Phi(w^\psi g, h^{-1} x^\xi, x^\eta h);$$

$$(\omega_{V_1, \psi}(p(n, n^*, b; 1, 1, 1, 1))) (w^\psi, x^\xi, x^\eta) = \psi\left(x^\eta (b(x^\xi)) + w^\psi (n^*(x^\xi))\right) \Phi\left(w^\psi + n^*(x^\xi), x^\xi, x^\eta\right)$$

for $(w^\psi, x^\xi, x^\eta) \in W^\vee + X^\ast + X^\vee$. We denote by $\omega_{V_1, \psi} = \omega_{V_1, \psi}|_{GL(V)}$. Then there is a $P_{r, m, r^*}$-equivariant map:

$$\text{ev} : \omega_{V_1, \psi} \to |\det X|^\frac{1}{2} \otimes \omega_{W + X^\ast, \psi} \bigoplus \omega_{X + W, \psi} \otimes |\det X|^\frac{1}{2}\Phi(w^\psi, x^\xi, 0) \Rightarrow \left(\Phi(w^\psi, x^\xi, 0), \Phi(w^\psi, 0, x^\eta)\right).$$

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The kernel of $\ev$ is generated by the Schwartz functions $\Phi_1 \otimes \Phi_2 \otimes \Phi_3 \in S(W^*) \otimes S(X^*) \otimes S(X^\vee)$ such that $\Phi_2$ (resp. $\Phi_3$) is compactly supported on $X^*-\{0\}$ (resp. $X^\vee-\{0\}$). Since $\omega_{\tau,\psi} \cong \omega_{\tau,\psi}$, we have the following exact sequence of smooth $P_{r,m,r^*}$-representations:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Ind_{N_{r,m,r}}^{\GL(V)}(P_X \times GL(W) \times P_{X^*})[\det X]^{1/2} \otimes \omega_{W,\psi} \otimes [\det X^{\vee}]^{1/2} & \longrightarrow & 0
\end{array}
$$

where $P_X$ (resp. $P_{X^*}$) is the mirabolic subgroup of $\GL(X)$ (resp. $\GL(X^*)$) stabilizing $\ell_X$ (resp. $\ell_X^\vee$) and $\omega_{W,\psi}$ is a representation of $N_{r,m,r^*} \rtimes \GL(W)$ which coincides with $\nu_{\mu} \delta_{W}^{1/2}$.

Tensoring the above sequence with $\tau \otimes \sigma \mu^{-1} \delta_{W}^{1/2} \otimes \tau^*$ and inducing to $\GL(V)$, we get an exact sequence of $\GL(V)$-representations:

$$
\begin{array}{cccccc}
0 & \longrightarrow & A \longrightarrow & I \left( \tau, \sigma \mu^{-1} \delta_{W}^{1/2}, \tau^* \right) \longrightarrow & B \otimes B^* & \longrightarrow & 0
\end{array}
$$

where

$A = \Ind_{N_{r,m,r} \rtimes (P_X \times GL(W) \times P_{X^*})}^{\GL(V)}(\tau|_{P_X})[\det X]^{1/2} \otimes \omega_{W,\psi} \otimes (\tau^*|_{P_{X^*}})[\det X^{\vee}]^{1/2}$;

$B = \Ind_{P_{r,m,r^*}}^{\GL(V)}(\omega_{W+X^*,\psi} \otimes \sigma \mu^{-1} \delta_{W}^{1/2} \otimes \tau^*)$;

$B^* = \Ind_{P_{r,m,r^*}}^{\GL(V)}(\omega_{W+X,\psi} \otimes \tau \otimes \sigma \mu^{-1} \delta_{W}^{1/2} \otimes \tau^*)[\det X^{\vee}]^{1/2}$.

By our assumption on $\pi$, we have

$\Hom_{\GL(V)}(B, \tilde{\pi}) = \Hom_{\GL(V)}(B^*, \tilde{\pi}) = \Ext^1_{\GL(V)}(B, \tilde{\pi}) = \Ext^1_{\GL(V)}(B^*, \tilde{\pi}) = 0$.

Hence

$\Hom_{\GL(V)} \left( I \left( \tau, \sigma \mu^{-1} \delta_{W}^{1/2}, \tau^* \right), \tilde{\pi} \right) = \Hom_{\GL(V)} \left( \Ind_{N_{r,m,r} \rtimes (P_X \times GL(W) \times P_{X^*})}^{\GL(V)}(\tau|_{P_X})[\det X]^{1/2} \otimes \omega_{W,\psi} \otimes (\tau^*|_{P_{X^*}})[\det X^{\vee}]^{1/2}, \tilde{\pi} \right)$.

Again since $\tau$ and $\tau^*$ are irreducible supercuspidal, we have

$\tau|_{[\det X]^{1/2}} \cong \Ind_{U_X}^{\GL(V)}(\lambda); \quad \tau^*|_{[\det X^{\vee}]^{1/2}} \cong \Ind_{U_{X^*}}^{\GL(V)}(\lambda^*)$

and by induction in stages, we have

$\Ind_{N_{r,m,r} \rtimes (P_X \times GL(W) \times P_{X^*})}^{\GL(V)}(\tau|_{P_X})[\det X]^{1/2} \otimes \omega_{W,\psi} \otimes (\tau^*|_{P_{X^*}})[\det X^{\vee}]^{1/2} = \Ind_H^{\GL(V)}(\sigma \otimes \nu^*_\mu)$.

Thus the proposition follows by the Frobenius reciprocity. 

\[ \square \]

**Proof of Theorem 3.1 for the uniqueness part.** We choose an irreducible supercuspidal representation $\tau$ (resp. $\tau^*$) of $\GL(X)$ (resp. $\GL(X^*)$) satisfying the assumption in the above proposition. After twisting unramified characters, we may assume that the induced representation $I \left( \tau, \sigma \mu^{-1} \delta_{W}^{1/2}, \tau^* \right)$ is irreducible. Then by the above proposition and [GGP, Theorem 14.1(iv)], we have

$\dim_{C} \Hom_{\GL(V)}(\tau \otimes \sigma \otimes \nu^*_\mu, C) = \dim_{C} \Hom_{\GL(V)} \left( I \left( \tau, \sigma \mu^{-1} \delta_{W}^{1/2}, \tau^* \right) \otimes \pi, C \right) \leq 1$.

\[ \square \]
Definition 3.3. A nontrivial element in the space $\text{Hom}(\pi \otimes \sigma \otimes \nu_\mu, \mathbb{C})$ is called an $(r, r^*)$-Fourier-Jacobi model of $\pi \otimes \sigma$. When $r = r^*$, hence $n - m$ is even, it is just the one defined in [GGP].

3.2 Fourier-Jacobi integrals, functional equations and $L$-functions

Let $k$ be a number field, $(H, \nu_\mu)$ be a pair associated with nontrivial $\psi: k \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$, $\mu: k^\times \backslash \mathbb{A}^\times \rightarrow \mathbb{C}^\times$, $\lambda$ a generic character $(U_X \times U_X^*)(k) \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$ and $\nu_\mu$ realizing on the space $\mathcal{S}(W^\vee(\mathbb{A}))$.

For any $\Phi \in \mathcal{S}(W^\vee(\mathbb{A}))$, we define the theta series

$$
\theta_{\psi, \lambda, \mu}(h, \Phi) = \sum_{w^\mu \in W^\vee(k)} \lambda(h) \langle \omega_{\psi, \mu}(h) \Phi \rangle (w^\lambda)
$$

which is an automorphic form of $H$. Let $\pi$ (resp. $\sigma$) be an irreducible cuspidal automorphic representation of $\text{GL}(V)(\mathbb{A})$ (resp. $\text{GL}(W)(\mathbb{A})$).

Definition 3.4. If $n > m$, the following absolutely convergent integral is called an $(r, r^*)$-Fourier-Jacobi period of $\pi \otimes \sigma$ (for a pair $(H, \nu_\mu)$):

$$
\mathcal{R}_{r, r^*}^\pi, \sigma(\varphi_\pi, \varphi_\sigma; \Phi) := \int_{H(k) \backslash H(\mathbb{A})} \varphi_\pi(\varepsilon(h)) \varphi_\sigma(\kappa(h)) \langle \theta^{\psi, \lambda, \mu}_{\pi, \sigma} - 1(h, \Phi) \rangle \det h |_h^{r^*-r} \text{ d}h, \quad \varphi_\pi \in \mathcal{A}_\pi, \varphi_\sigma \in \mathcal{A}_\sigma.
$$

If there exist $\varphi_\pi, \varphi_\sigma$ and $\Phi \in \mathcal{S}(W^\vee(\mathbb{A}))$ such that $\mathcal{R}_{r, r^*}^\pi, \sigma(\varphi_\pi, \varphi_\sigma; \Phi) \neq 0$, then we say $\pi \otimes \sigma$ has a nontrivial $(r, r^*)$-Fourier-Jacobi period. The case $m = n$ will be discussed in Remark 3.6.

It is obvious that $\mathcal{R}_{r, r^*}^\pi, \sigma$ defines an element in

$$
\text{Hom}_{H(\mathbb{A})}(\pi \otimes \sigma \otimes \nu_\mu, \mathbb{C}) = \bigotimes_{v \in \mathbb{M}_k} \text{Hom}_{H_v}(\pi_\nu \otimes \sigma_\nu \otimes \nu_\mu_v, \mathbb{C}).
$$

Since the later space has the multiplicity one property, we expect that the Fourier-Jacobi period is also Eulerian. We now show that this is true.

We choose basis $\{v_1, \ldots, v_r\}$ (resp. $\{v_1^*, \ldots, v_r^*\}$) for $X$ (resp. $X^*$) in the same way as in Section 2.2 with $c = 1$. We also choose a basis $\{w_1, \ldots, w_m\}$ of $W$ with dual basis $\{w_1^\vee, \ldots, w_m^\vee\}$ of $W^\vee$. We identify $\text{GL}(V)$ with $\text{GL}_{n,k}$ and hence $\text{GL}(W)$ with $\text{GL}_{m,k}$ via the basis:

$$
\{w_1, \ldots, w_m, v_1, \ldots, v_r, v_1^\vee, \ldots, v_r^\vee\}.
$$

Then the image of $H(\mathbb{A})$ in $\text{GL}_{n,k}(\mathbb{A})$ consists of following matrices:

$$
h = h(n, n^*, b; u, u^*; g) = \begin{bmatrix}
  g & n_{1,r} & \cdots & n_{1,m} & u \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  n_{r,1} & \cdots & n_{r,m} & b \\
  u^* & & & & \\
\end{bmatrix}
$$

(3.2)

where $n$, $n^*$, $b$, $u$, $u^*$ and $g$ are similar to those in Section 2.2, but without entries related to $w_0$. Let $U_{1r,m,1r^*} = U_{r,m,r^*} \times (U_X \times U_X^*)$ be a unipotent subgroup of $\text{GL}_{n,k}$, then

$$
U_{1r,m,1r^*}(\mathbb{A}) = \{w = w(n, n^*, b; u, u^*) := h(n, n^*, b; u, u^*; 1_m) \}.
$$

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For \( \Phi \in \mathcal{S}(W^\psi(\mathbb{A})) \), we have
\[
(\nu_\mu(w)\Phi)(w^p) = \psi\left(u_{1,2} + \ldots + u_{r-1,r} + b_{r,r^*} + u_{r^*,r-1} + \ldots + u_{2,1}^* + w^b \left[n_{2,1}^*, \ldots, n_{m,r^*}^*\right]\right)
\]
\[
\Phi(w^b + [n_{r,1}, \ldots, n_{r,m}]).
\]
For simplicity, we denote \( \psi(u) = \psi(u_{1,2} + \ldots + u_{r-1} + b_{r,r^*} + u_{r^*,r-1} + \ldots + u_{2,1}^*) \). Then
\[
\mathcal{F}_r^\nu(\varphi_\pi, \varphi_\sigma; \Psi) = \mathcal{F}_r^\nu\left(0; \varphi_\pi, \varphi_\sigma; \Psi\right)
\]
where
\[
\mathcal{F}_r^\nu\left(s; \varphi_\pi, \varphi_\sigma; \Psi\right) = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_\pi(ug)\varphi_\sigma(g)\lambda_{\Psi}(ug, \phi)(0) \mu(\det g)^{s-r^*-\frac{1}{2}} \, dg \, d\mu(g).
\]
There are two cases.

**Case 1:** \( r > 0 \). In this case, we have
\[
\theta_{\psi, \lambda, \mu}(ug, \phi) = \sum_{w^p \in W^\psi(k)} \lambda(u) (\omega_{\psi, \mu}(ug)\Phi)(w^p) = \sum_{n_r \in k^m} \lambda(u) (\omega_{\psi, \mu}(n_r, ug)\Phi)(0)
\]
where \( n_r = g(n_r, 0, 0; 1_r, 1_r^*) \) and
\[
n_r = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \\ n_{r,1} & \cdots & n_{r,m} \end{bmatrix}.
\]
We define the subgroup \( L_{r+1} \) of \( U_{1, r, m, 1_r^*} \) in the similar way as before just by taking the matrix \( n = 0 \), then we separate \( n \) from \( u \) and summate the last row in \( n \) over \( k^m \). We get
\[
(3.3) = \int g \left. M_{r-1,m}(k) \right/ L_{r+1}(k) \int \int_{\mathbb{A}^m} \varphi_\pi(ug, \phi)\lambda(l) \left(\varphi_{\psi, \lambda, \mu}(lg)\Phi(n_r)\right) |\det g|_{\mathbb{A}}^{s-r^*-\frac{1}{2}} \, dl \, d\mu(g) \, d\mu(g)
\]
\[
= \int g \left. M_{r-1,m}(k) \right/ L_{r+1}(k) \int \int_{\mathbb{A}^m} \varphi_\pi(ug, \phi)\lambda(l) \left(\varphi_{\psi, \lambda, \mu}(lg)\Phi(n_r, g)\psi(l)\right) |\det g|_{\mathbb{A}}^{s-r^*-\frac{1}{2}} \, dl \, d\mu(g) \, d\mu(g)
\]
where in the last line, \( n_r, g \) is the last row of \( n \). If we repeat the process (2.6), (2.7), (2.8) and (2.9), then
\[
(3.4) = \int g \left. M_{r,m}(k) \right/ L_1(k) \int \mu(\det g)^{-1} |\det g|_{\mathbb{A}}^{s-r^*-\frac{1}{2}} \int (\rho(\gamma)\varphi_\pi)(\gamma) \phi(g, r^*)\Phi(n_r, \psi(l)) \mu(l) \, dl \, d\mu(g) \, d\mu(g)
\]
By the classical argument for the Rankin-Selberg convolution,
\[
(3.5) = \int g \left. M_{r,m}(k) \right/ L_1(k) \int \int U_{1,m}(k) \left|\mathcal{F}_r^\nu(\varphi_\pi, \varphi_\sigma; \Psi)\right| |\det g|_{\mathbb{A}}^{s-r^*-\frac{1}{2}} \, dl \, d\mu(g) \, d\mu(g)
\]
\[
|W^\psi_{\varphi_\pi}(g)\Phi(g, r^*)\mu(\det g)^{-1} |\det g|_{\mathbb{A}}^{s-r^*-\frac{1}{2}} \, dl \, d\mu(g) \, d\mu(g). \tag{3.6}
\]
If we denote (3.6) by $\Psi_r(s; W_\varphi^0, W_\varphi^{-}; \mu^{-1}; \Phi)$, then it is absolutely convergent for $\Re(s) \gg 0$. Moreover, if $W_\varphi^0 = \otimes_\nu W_\nu$, $W_\varphi^{-} = \otimes_\nu W_\nu^*$ and $\Phi = \otimes_\nu \Phi_\nu$ are factorizable, we have

$$\mathcal{F}_{r, \nu}^{\mu, \nu}(s; \varphi_\pi, \varphi_\sigma; \Phi) = \Psi_r(s; W_\varphi^0, W_\varphi^{-}; \mu^{-1}; \Phi) = \prod_{\nu \in \mathcal{M}_k} \Psi_{r, \nu}(s; W_\nu, W_\nu^{-}; H_{\mu, \nu}; \Phi_\nu)$$

where

$$\Psi_{r, \nu}(s; W_\nu, W_\nu^{-}; \mu^{-1}; \Phi_v) = \int_{U_{1,m-1}(k) \backslash U_{1,m-1}(A)} \int_{U_{1,m}(A) \backslash GL_{m,v}} \int_{M_{r-1,m,v} \backslash GL_{m,v}} \varphi_v(g_\nu) \Phi_v(g_\nu) \mu_g \det g_\nu^{s-\frac{m-1}{2}} \, dg_\nu, dx, dy_\nu.$$ 

**Case 2:** $r = 0$ but $r^* > 0$. Let $P_m$ be the standard mirabolic subgroup of $GL_m$ consisting of (invertible) matrices whose last row is $e_m = [0, ..., 0, 1] \in k^m$. Then

$$\theta_{\psi, \lambda, \mu}(g_\nu, \Phi) = \sum_{w^\nu \in W^\nu(k)} \lambda(w^\nu) \omega_{\psi, \mu}(g_\nu) (w^\nu)$$

where the first term (for $\theta_{\psi, \lambda, \mu}$) contributes 0 to $\mathcal{F}_{0,n-m}^{\mu, \nu}(s; \varphi_\pi, \varphi_\sigma; \Phi)$ since $\varphi_\pi$ is a cusp form. Hence,

$$\mathcal{F}_{0,n-m}^{\mu, \nu}(s; \varphi_\pi, \varphi_\sigma; \Phi) = \int_{U_{1,m-1}(k) \backslash U_{1,m-1}(A)} \int_{U_{1,m}(A) \backslash GL_{m}(A)} \varphi_\pi(g_\mu) \psi_\sigma(g_\mu) \psi(e_m g n^s e_m^*) \mu \det g_\mu^{s-\frac{m-1}{2}} \, dg_\mu, du.$$ (3.7)

Factoring the inner integral through $U_{1,m}(k) \backslash U_{1,m}(k)$ and incorporating this unipotent part into $\mu$, we get is the integral over $U_{1,m}(k) \backslash U_{1,m}(k)$ where $U_{1,n}$ is the standard maximal unipotent subgroup of $GL_n$. Moreover, if we change the order of $g$ and $u \in U_{1,n}(k) \backslash U_{1,n}(k)$, all terms involving $\psi$ will form a generic character $\psi$ of $U_{1,n}(k) \backslash U_{1,n}(k)$:

$$\psi(u) = \psi(u_{1,2} + \cdots + u_{n-1,n}).$$

In all,

$$\mathcal{F}_{0,n-m}^{\mu, \nu}(s; \varphi_\pi, \varphi_\sigma; \Phi) = \int_{U_{1,m}(k) \backslash GL_{m}(k) \backslash U_{1,m}(k)} \int_{U_{1,m}(k) \backslash U_{1,m}(k)} \varphi_\pi(n g) \psi(n) \psi(e_m g n^s e_m^*) \mu \det g_\mu^{s-\frac{m-1}{2}} \, dg_\mu, du.$$ (3.8)
If we denote (3.9) by \( \Psi_0(s; W^\psi, W^{\overline{\psi}} \otimes \mu^{-1}; \Phi) \), then it is absolutely convergent for \( \Re(s) > 0 \). Moreover, if \( W^\psi_{\varphi_e} = \otimes_v W^\psi_v, \ W^{\overline{\psi}}_{\varphi_e} = \otimes_v W^{\overline{\psi}}_v \) and \( \Phi = \otimes_v \Phi_v \) are factorizable, we have

\[
\mathcal{F}_{(0,n-m)}^{\psi}(s; \varphi_e, \varphi_e; \Phi) = \Psi_0(s; W^\psi_{\varphi_e}, W^{\overline{\psi}}_{\varphi_e} \otimes \mu^{-1}; \Phi) = \prod_{v \in M_k} \Psi_{v,0}(s; W_v^\psi, W^{\overline{\psi}}_v \otimes \mu_v^{-1}; \Phi_v)
\]

where

\[
\Psi_{v,0}(s; W_v, W^{\overline{\psi}}_v \otimes \mu_v^{-1}; \Phi_v) = \int_{U_{1,m} \setminus \text{GL}_{m,v}} W_v \left( \begin{bmatrix} g_v & 0 \\ 0 & 1_{n-m} \end{bmatrix} \right) W_v^\psi(g_v) \Phi_v(e_{m,v} g_v \mu_v(\det g_v)^{-1} | \det g_v|^{s - \frac{m}{2}}) \, dg_v.
\]

Now we discuss the functional equations of the \((r, r^*)\)-Fourier-Jacobi integrals \( \mathcal{F}_{(r, r^*)}^{\psi}(s; \varphi_e, \varphi_e; \Phi) \). First, there is a linear map \( \tilde{\psi} : S(W^\psi(\mathbb{A})) \rightarrow S(W(\mathbb{A})) \) given by

\[
\tilde{\psi}(u^\psi) = \int_{W^\psi(\mathbb{A})} \Phi(u^\psi) \psi \left( u^{\overline{\psi}}(w^{\overline{\psi}}) \right) \, dw^{\overline{\psi}}.
\]

If we identify \( W \) with \( W^\psi \) through the basis \( \{w_1, \ldots, w_m\} \), then \( \tilde{\psi} \) is an endomorphism of \( S(W^\psi(\mathbb{A})) \). Consider the group isomorphism \( \iota_{r, r^*} : H \rightarrow H \) given by \( \iota_{r, r^*}(ug) = w_{n,m}u^{-1} w_{n,m}g^r \). Then for any \( h \in H(\mathbb{A}) \), we have the following commutative diagram which can be checked directly:

\[
\begin{array}{ccc}
S(W^\psi(\mathbb{A})) & \xrightarrow{\sim} & S(W^\psi(\mathbb{A})) \\
\xrightarrow{\iota_{r, r^*}(h)} & & \xrightarrow{\lambda_{\varphi_e, \mu}(h)} \\
S(W(\mathbb{A})) & \xrightarrow{\iota_{r, r^*}(h)} & S(W(\mathbb{A}))
\end{array}
\]

Then

\[
\mathcal{F}_{(r, r^*)}^{\psi}(s; \varphi_e, \varphi_e; \Phi) = \int_{g \in G} \varphi_e(ug^r) \varphi_e(g^r) \theta_{\varphi_e, \mu^{-1}}(ug^r, \Phi) | \det g|^{s - \frac{1}{2} + \frac{r - \frac{1}{2}}{2}} \, dg,
\]

\[
= \int_{g \in G} (\rho(w_{n,m}) \varphi_e) (w_{n,m}u_w w_{n,m}g) \varphi_e(g) \theta_{\varphi_e, \mu^{-1}}(ug^r, \Phi) | \det g|^{s - \frac{1}{2} + \frac{r - \frac{1}{2}}{2}} \, dg,
\]

\[
= \int_{g \in G} \left( \int_{U_{1,m} \setminus \text{GL}_{m,v}} (\rho(w_{n,m}) \varphi_e)(w_{n,m}u_w w_{n,m}g) \varphi_e(g) \theta_{\varphi_e, \mu^{-1}}(ug^r, \Phi) | \det g|^{s - \frac{1}{2} + \frac{r - \frac{1}{2}}{2}} \, dg \right) \, dw^{\overline{\psi}}.
\]

But by the Poisson summation formula and (3.10),

\[
\theta_{\varphi_e, \mu^{-1}}(\iota_{r, r^*}(ug), \Phi) = \sum_{w^\psi \in W^\psi(\mathbb{A})} \lambda(\psi) \left( \theta_{\varphi_e, \mu^{-1}}(\iota_{r, r^*}(ug), \Phi) \right) (w^\psi)
\]

\[
= \sum_{w^\psi \in W^\psi(\mathbb{A})} \lambda(\psi) \left( \theta_{\varphi_e, \mu^{-1}}(\iota_{r, r^*}(ug), \Phi) \right) (w^\psi)
\]

\[
= \sum_{w^\psi \in W^\psi(\mathbb{A})} \lambda(\psi) \left( \psi_{\lambda, \mu}(ug, \Phi) \right) (w^\psi)
\]

\[
= \theta_{\psi, \lambda, \mu}(ug, \Phi).
\]
Hence,

\[(3.11) = \int_{g \mathcal{U}(\mathbb{C}) \setminus \mathcal{U}(\mathbb{C})} (\rho(w_{n,m})(\varphi_\sigma)) (\frac{1}{2}) \det g |_\mathcal{A}^{1-s-r} e^{\frac{s}{2} \text{Im}(\text{Re}(g))} dg + \mathcal{F}_r^{\mathcal{F}_r}(1-s; \rho(w_{n,m})(\varphi_\sigma, \varphi_\sigma; \Phi)). \]

In summary, we have the following

**Theorem 3.5.** Let \( n > m \), the Fourier-Jacobi integrals are holomorphic in \( s \) and satisfy the following functional equation

\[ \mathcal{F}_r^{\mathcal{F}_r}(s; \varphi_\pi, \varphi_\sigma; \Phi) = \mathcal{F}_r^{\mathcal{F}_r}(1-s; \rho(w_{n,m})(\varphi_\sigma, \varphi_\sigma; \Phi)). \]

Let \( \bar{W}_\pi(g) = W_\pi(g_{1}) \in W(\mathcal{F}, \mathcal{F}) \) and similar for the one on \( GL_m \). If \( W_\pi = \otimes_v W_v^\prime = \otimes_v W_v^\prime \) and \( \Phi = \otimes_v \Phi_v = \otimes_v \Phi_v^\prime \) are factorizable, then \( \bar{W}_\pi^\prime = \otimes_v \bar{W}_v^\prime \) and \( \hat{\Phi} = \otimes_v \hat{\Phi}_v \) are also factorizable with \( \bar{W}_v(g) = W_v(w_{n,m}) \), \( \bar{W}_v(g) = W_v(w_{m,n}) \) and for any \( s \) large,

\[ \mathcal{F}_r^{\mathcal{F}_r}(s; \varphi_\pi, \varphi_\sigma; \Phi) = \Psi_r(s; W_\pi^\prime (W_\pi^\prime \otimes \mu^{-1}; \Phi); \prod_{v \in \mathbb{M}_k} \Psi_v(s; W_v, W_v^\prime \otimes \mu^{-1}; \Phi_v); \]

\[ \mathcal{F}_r^{\mathcal{F}_r}(s; \rho(w_{n,m})(\varphi_\pi, \varphi_\sigma; \Phi) = \Psi_r(s; \rho(w_{n,m})(\varphi_\pi, \varphi_\sigma; \Phi); \prod_{v \in \mathbb{M}_k} \Psi_v(s; \rho(w_{n,m})(\varphi_\pi, \varphi_\sigma; \Phi). \]

In particular, when \( r = r^* \) (hence \( n - m \) is even), the \( (r, r) \)-Fourier-Jacobi integral itself has a functional equation.

**Remark 3.6 (The case \( n = m \)).** In this case, \( V = W \) and \( H = GL_n \). We will see that this is exactly the case of Rankin-Selberg convolution for \( GL_n \times GL_n \). For simplicity, we assume that \( \pi \boxtimes \sigma \boxtimes \mu^{-1} \) is unitary. We fix a basis \( \{ v_1, \ldots, v_n \} \) for \( V \) and identify \( V^\prime \) as the set of row vectors of length \( n \). In this case, there is no choice of \( \lambda \) anymore.

For any \( \Phi \in \mathcal{S}(V^\prime(\mathbb{A})) \) and any character \( \chi : \mathbb{A}^x \rightarrow \mathbb{C} \) such that \( \mu \cdot \chi \) is unitary, we define

\[ \theta_{\pi, \mu}^\prime(s; g, \Phi, \chi) = | \det g |_{\mathcal{A}}^{1-s} \int_{\mathcal{A}^x \setminus V^\prime(\mathbb{A})} (\omega_{\pi, \mu}(a \Phi)) (\varphi^\prime)^{-1} \chi(a) da \]

which is absolutely convergent when \( \mathfrak{N}(s) > 1 \). It has a meromorphic continuation to the entire complex plane which is holomorphic at \( s = \frac{1}{2} \) (cf. [JS81]). For any holomorphic point \( s \), it is in \( \mathcal{A}(GL_{n,k}) \) with central character \( \chi^{-1} \). Moreover,

\[ \theta_{\pi, \mu}^\prime(\frac{1}{2}; g, \chi, \chi) : (\nu_\mu, \mathcal{S}(V^\prime(\mathbb{A}))) \rightarrow \mathcal{A}(GL_{n,k}) \]

is \( GL_n(\mathbb{A}) \)-equivariant. We denote \( Z_n \) the center of \( GL_n \), \( \chi \) (resp. \( \chi_\sigma \)) the central character of \( \pi \) (resp. \( \sigma \)) and let

\[ \mathcal{F}_0^{\mathcal{F}_0}(s; \varphi_\pi, \varphi_\sigma; \Phi) = \int_{Z_n(\mathcal{A})GL_n(\mathbb{A})} \varphi_\pi(g)(\varphi_\sigma) (\delta_{\Phi}^\prime) \mathcal{F}_0^{\mathcal{F}_0}(s; g, \Phi, \chi_\pi \cdot \chi_\sigma) dg \]

be the usual Rankin-Selberg integral (cf. [JS81, JPSS83]) which is absolutely convergent at any holomorphic point \( s \). The Fourier-Jacobi period \( \mathcal{F}_0^{\mathcal{F}_0}(s; \varphi_\pi, \varphi_\sigma; \Phi) = \mathcal{F}_0^{\mathcal{F}_0}(\frac{1}{2}; \varphi_\pi, \varphi_\sigma; \Phi) \) defines an element in

\[ \text{Hom}_{GL_n(\mathbb{A})}(\pi \boxtimes \sigma \boxtimes \nu_\mu, \mathbb{C}) = \bigotimes_{v \in \mathbb{M}_k} \text{Hom}_{H_v}(\pi_v \boxtimes \sigma_v \boxtimes \nu_\mu_v, \mathbb{C}). \]
Unfolding $\theta_{\psi, \mu}^{-1}$, we see that

$$
\begin{align*}
(3.12) &= \int_{P_n(k) \backslash GL_n(A)} \varphi_{\pi}(g) \varphi_{\sigma}(g) \Phi(e_n g) \mu(\det g)^{-1} \det g |_{\mathbb{A}}^{1} dg \\
&= \int_{U_{1^n}(A) \backslash GL_n(A)} W^\psi_{\pi_v}(g) W^\psi_{\varphi_{\sigma}}(g) \Phi(e_n g) \mu(\det g)^{-1} \det g |_{\mathbb{A}}^{1} dg.
\end{align*}
$$

(3.13)

If we denote (3.13) by $\Psi_0(s; W^\psi_{\pi_v}, W^\psi_{\psi_{\sigma}} \otimes \mu^{-1}; \Phi)$, then it is absolutely convergent for $\Re(s) \gg 0$. Moreover, if $W^\psi_{\pi_v} = \otimes_v W_v$, $W^\psi_{\psi_{\sigma}} = \otimes_v W_v$ and $\Phi = \otimes_v \Phi_v$ are factorizable, we have

$$
\mathcal{F}T_{0,0}(s; \varphi_{\pi_v}, \varphi_{\sigma_v}; \Phi) = \Psi_0(s; W^\psi_{\pi_v}, W^\psi_{\psi_{\sigma}} \otimes \mu^{-1}; \Phi) = \prod_{v \in \mathbb{M}_k} \Psi_{v,0}(s; W_v, W_v^{-1} \otimes \mu_v^{-1}; \Phi_v)
$$

where

$$
\Psi_{v,0}(s; W_v, W_v^{-1} \otimes \mu_v^{-1}; \Phi_v) = \int_{U_{1^n}(A) \backslash GL_{n,v}} W_v(g_v) W_v^{-1}(g_v) \Phi_v(e_n g_v) \mu_v(\det g_v)^{-1} \det g_v |_{\mathbb{A}}^{1} dg_v.
$$

Moreover, we have the following well-known functional equation

$$
\mathcal{F}T_{0,0}(s; \varphi_{\pi_v}, \varphi_{\sigma_v}; \Phi) = \mathcal{F}T_{0,0}(1 - s; \varphi_{\pi_v}, \varphi_{\sigma_v}; \tilde{\Phi})
$$

and $\mathcal{F}T_{0,0}(s; \varphi_{\pi_v}, \varphi_{\sigma_v}; \Phi)$ will have possible simple poles at $s = -i\sigma$ and $s = 1 - i\sigma$ with $\sigma$ real only if $\pi \cong \sigma \otimes \mu |_{\mathbb{A}} |^{\sigma}.$

By Proposition (6.1) and (6.2), we have the following theorem, which confirms [GGP, Conjecture 24.1] for the Fourier-Jacobi periods of split unitary groups, i.e., general linear groups.

**Theorem 3.7.** (1) Let $\pi$ (resp. $\sigma$) be an irreducible cuspidal automorphic representation of $GL(V)(A)$ (resp. $GL(W)(\mathbb{A})$). For any $(r, r^*)$ such that $r + r^* = n - m$ and the representation $\nu_\mu$ introduced above, we have, for $\varphi_{\pi_v} \in \mathcal{A}_v$, $\varphi_{\sigma_v} \in \mathcal{A}_v$ and $\Phi \in \mathcal{S}(W^V(\mathbb{A}))$ such that $W^\psi_{\pi_v} = \otimes_v W_v$, $W^\psi_{\psi_{\sigma}} = \otimes_v W_v$ and $\Phi = \otimes_v \Phi_v$ are factorizable,

$$
\mathcal{F}T_{r, r^*}(\varphi_{\pi_v}, \varphi_{\sigma_v}; \Phi) = L(1, 2; \pi \times \sigma \otimes \mu^{-1}) \prod_{v \in \mathbb{M}_k} \left. \Psi_{v, r}(s; W_v, W_v^{-1} \otimes \mu_v^{-1}; \Phi_v) \right|_{v=\frac{1}{2}}
$$

where in the last product almost all factors are 1, and the $L$-functions are the ones defined by Rankin-Selberg convolutions (cf. [JPSS83]).

(2) There is a nontrivial Fourier-Jacobi period of $\pi \otimes \sigma$ for $\nu_\mu$ if and only if $L(\frac{1}{2}, \pi \times \sigma \otimes \mu^{-1}) \neq 0$.

4 A relative trace formula for $U_n \times U_m$: Bessel periods

4.1 Bessel models and periods

Let $k'$ be a field and $k/k'$ be a separable quadratic extension which may be split. Let $\tau$ be the unique nontrivial involution of $k$ fixing $k'$. We denote by $\Tr$ and $\Nm$ the trace and norm of $k/k'$, respectively. Let $k^{-} = \{ x \in k \mid x^r = -x \}$. We fix a nonzero element $\gamma \in k^{-}$ once and for all and define $\overline{\Tr}(x) = \gamma(x - x^r) \in k'$. If $k'$ is a local field, we denote $\gamma$ the character associated with $k/k'$ via the class field theory. Let $\text{Her}_n(k/k')$ be the set of all $n \times n$ matrices satisfying $^{t}g^r = g$ and $\text{Her}_n(k/k')^\times = \text{Her}_n(k/k') \cap \text{GL}_n(k)$; $\text{Her}_n^-(k/k')$ be the set of all $n \times n$ matrices satisfying $^{t}g^r = -g$ and $\text{Her}_n^-(k/k')^\times = \text{Her}_n^-(k/k') \cap \text{GL}_n(k)$.

\footnote{When $n = m$, to prevent the occurrence of a pole at $s = \frac{1}{2}$, we assume that the character $\chi_v \otimes \chi_v \otimes \mu^{-1}$ is unitary for simplicity.}
Bessel model is just the $(\cdot)^+$-subgroup of $GL(n)$ over $k$. For a vector space $X$ over $k$, we denote $X_\tau$ the vector space over $k$ with the same underlying set of $X$ but the action of $k$ twisted by $\tau$. We also simply denote $X^{\cdot+}$ by $X$. All hermitian or skew-hermitian spaces over $k$ are defined with respect to $\tau$ and are assumed to be non-degenerate.

Let us briefly recall the definition of Bessel models and periods for unitary groups in [GGP]. First, let us consider the local situation, hence $k'$ is a local field. Let $V$ be a hermitian space over $k$ of dimension $n$ with the hermitian form $(-,-)$ and $W \subset V$ a subspace of dimension $m$ such that the restricted hermitian form $(-,-)|_W$ is non-degenerate. We assume that the orthogonal complement $W^\perp = E \oplus X \oplus X^{\cdot*}$ such that $E$ is a non-degenerate line and $X, X^{\cdot*}$ are isotropic, perpendicular to $E$ and of dimension $\ell$. Hence $n = m + 2\ell + 1$. The hermitian form restricted on $W$ (resp. $X \oplus X^{\cdot*}$) identifies $W$ (resp. $X^{\cdot*}$) with $W$ (resp. $X$). We denote $U(V)$ (resp. $U(W)$) the unitary group of $V$ (resp. $W$) which is a reductive group over $k'$. Let $P_{r,m+1}'$ be the parabolic subgroup of $U(V)$ stabilizing $X$ and $U_{r,m+1}'$ its maximal unipotent subgroup. Then $U_{r,m+1}'$ fits into the following exact sequence:

$$0 \longrightarrow \Lambda_r^2 X \longrightarrow U_{r,m+1}' \longrightarrow \text{Hom}_k(W \oplus E, X) \longrightarrow 0$$

where $\Lambda_r^2 X \subset X_\tau \otimes X = \text{Hom}_k(X, X)$ consists of homomorphisms $b$ such that $b^{\cdot*} = -b$. Here $b^{\cdot*}$ is just $b^*: X^{\cdot*} \to X$ viewed as an element in $\text{Hom}_k(X^{\cdot*}, X)$.

Let $\ell': X \to k$ be any nontrivial homomorphism (if exists) and let $U_{X}'$ be a maximal unipotent subgroup of $GL(X)$ stabilizing $\ell'X$. Let $U_{W}' : k \to W \oplus E$ be a nontrivial homomorphism such that its image is contained in $E$. Hence we have a homomorphism

$$\ell' : U_{r,m+1}' \longrightarrow \text{Hom}_k(W \oplus E, X) \xrightarrow{(r_\nu, \epsilon')_X} k$$

which is fixed by $U_{X}' \times U(W)$. Hence we can extend $\ell'$ trivially to it and define a homomorphism from $H' = U_{r,m+1}' \times (U_{X}' \times U(W))$ to $k$. Let $\nu' : k' \to \mathbb{C}^1$ be a nontrivial character and $\lambda' : U_{X}' \to \mathbb{C}^*$ be a generic character. We define $\nu' = (\nu' \circ \text{Tr} \circ \ell') \otimes \lambda'$ which is a character of $H'$. We have an embedding $(\varepsilon, \kappa) : H' \hookrightarrow U(V) \times U(W)$. Then up to $U(V) \times U(W)$-conjugacy, the pair $(H', \nu')$ is uniquely determined by $W \subset V$.

Let $\pi$ (resp. $\sigma$) be an irreducible admissible representation of $U(V)$ (resp. $U(W)$). A nontrivial element in $\text{Hom}_{H'}(\pi \otimes \sigma, \nu')$ is called a Bessel model of $\pi \otimes \sigma$. In particular, when $k/k'$ is split, then the Bessel model is just the $(r,r)$-Bessel model for general linear groups introduced in Section 2.1. We have the following multiplicity one result.

**Theorem 4.1.** Let $k$ be of characteristic zero and $\pi, \sigma$ as above. Then $\dim \text{Hom}_{H'}(\pi \otimes \sigma, \nu') \leq 1$.

**Proof.** If $k$ is non-archimedean, this is due to Aizenbud-Gourevitch-Rallis-Schiffmann [AGRS10] for $m = n - 1$ and to [GGP, Section 15] for general $n, m$. If $k$ is archimedean, this is due to Sun-Zhu [SZ09] and Aizenbud-Gourevitch [AG09] for $m = n - 1$ and to Jiang-Sun-Zhu [JSZ10] for general $n, m$. 

Now, we discuss the global case. Let $k/k'$ be a quadratic extension of number fields. We have the notions, $\varepsilon', k'_\nu, \sigma'_\nu$ for $\nu' \in \mathbb{M}_{k'}$, and $\Lambda', \psi'$ similar to those for $k$. Let $\lambda' : U_{X}'(k) \backslash U_{X}'(k) \to \mathbb{C}^1$ be a generic character. Then we have the pair $(H', \nu')$ in the global situation.

Let $\pi$ (resp. $\sigma$) be an irreducible tempered representation of $U(V)(k')$ (resp. $U(W)(A')$) which occurs with multiplicity one in the space $A_0(U(V))$ (resp. $A_0(U(W))$). We denote by $A_\pi$ (resp. $A_\sigma$) the unique irreducible $\pi$ (resp. $\sigma$)-isotypic subspace in $A_0(U(V))$ (resp. $A_0(U(W))$).

**Definition 4.2.** The following absolutely convergent integral is called a Bessel period of $\pi \otimes \sigma$ (for a pair $(H', \nu')$):

$$B_{r'}(\varphi_\pi, \varphi_\sigma) := \int_{H'(k') \backslash H'(k')} \varphi_\pi(\varepsilon(h')) \varphi_\sigma(\kappa(h')) \nu'(h')^{-1} dh', \quad \varphi_\pi \in A_\pi, \varphi_\sigma \in A_\sigma.$$ 

If there exist $\varphi_\pi, \varphi_\sigma$ such that $B_{r'}(\varphi_\pi, \varphi_\sigma) \neq 0$, then we say $\pi \otimes \sigma$ has a nontrivial Bessel period.
It is obvious that $B_{\nu'}$ defines an element in

$$\text{Hom}_{H((\mathbb{A})/\mathbb{R})}(\pi \otimes \sigma, \nu') = \bigotimes_{v' \in \mathcal{M}} \text{Hom}_{H_{v'}}(\pi' \otimes \sigma_{v'}, \nu_{v'}).$$

In order to decompose the distribution $\mathcal{J}_{\psi'}$ in the next section. We choose a basis $\{v_1, \ldots, v_r\}$ of $X$ such that

- The homomorphism $\ell'_{X}: X \to k$ is given by the coefficient of $v_r$ under the above basis;
- $U'_{X}$ is the maximal unipotent subgroup of the parabolic subgroup $P'_{X}$ stabilizing the complete flag $0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \ldots, v_r \rangle = X$;
- The generic character $\lambda'$ is given by

$$\lambda'(u') = \psi' \left( \frac{\text{Tr}(u'_{1,2} + u'_{2,3} + \cdots + u'_{r-1,r})}{\psi_{0}} \right)$$

under the above basis.

We denote by $\{\tilde{v}_1, \ldots, \tilde{v}_r\}$ the dual basis of $\tilde{X}$. We also choose a basis $\{w_1, \ldots, w_m\}$ of $W$ and $\{w_0\}$ of $E$ such that the homomorphism $\ell'_{W}: k \to W \otimes E$ is given by $a \mapsto aw_0$. Let $\beta = [(w_i, w_j)]_{i,j=1}^m \in \text{Her}_m(k/k')^\times$, $\beta_0 = (w_0, w_0) \in k'^\times$ and

$$\beta' = \begin{bmatrix} \beta & \beta_0 \end{bmatrix} \in \text{Her}_{m+1}(k/k')^\times.$$  \hfill (4.1)

We identify $U(V)$ (resp. $U(W)$) with a unitary group of $n$ (resp. $m$) variables $U_n$ (resp. $U_m$) under the basis $\{v_1, \ldots, v_r, w_1, \ldots, w_m, w_0, \tilde{v}_r, \ldots, \tilde{v}_1\}$ and view $U_m$ as a subgroup of $U_n$. Let $U'_{r,m+1} = U_{r,m+1} \cap U'_X$ be a unipotent subgroup of $U_n$. Then the image of $H'((k')^\times)$ in $U_n((k')^\times)$ consists of the matrices $h' = h'(n', b'; u'; g') = w'(n', b'; u') \cdot g'$ where $g' \in U_m((k')^\times)$. Here,

$$w'(n', b'; u') = \begin{bmatrix} 1_r & n' & w_r(b' + \frac{n'n_0'}{2}) & 0 \\ 1_{m+1} & n_0' & 0 & 0 \\ 0 & 1_r & 0 & 0 \\ 0 & 0 & 0 & u' \end{bmatrix} \in U'_{r,m+1}((k')^\times)$$

where $n' \in M_{r,m+1}((k')^\times)$, $b' \in \text{Her}_{r}((k'/k')^\times)$ (similarly defined as for $k/k'$), $u' \in U'_X((k')^\times)$; $n_0' = -\beta'^{-1} n' w_r$ and $u' = w_r^{-1} u'^{-1} w_r$. The character $\nu'$ on $H'((k')^\times)$ is given by

$$\nu'(h') = \nu'(h'(n', b'; u'; g')) = \nu'(w') = \psi'(w') := \psi' \left( \frac{\text{Tr}(u'_{1,2} + \cdots + u'_{r-1,r} + n'_{r,m+1})}{\psi_{0}} \right)$$

Then the Bessel period

$$B_{\nu'}(\varphi_\pi, \varphi_\sigma) = \int_{U_m((k')^\times) \setminus U_n((k')^\times)} \int_{U'_{r,m+1}((k')^\times) \setminus U'_X((k')^\times)} \varphi_\pi(u'g') \varphi_\sigma(g') w'(u') du' dg'.$$

### 4.2 Decomposition of distributions

We describe the relative trace formula on the unitary pair relating the Bessel periods. Recall that $\pi$ (resp. $\sigma$) is an irreducible tempered representation of $U_n((k')^\times)$ (resp. $U_m((k')^\times)$) which occurs with multiplicity one in the space $A_\pi(U_n)$ (resp. $A_\sigma(U_m)$). For simplicity, we further assume that the standard base change $\Pi$ (resp. $\Sigma$) of $\pi$ (resp. $\sigma$) is an irreducible cuspidal automorphic representation of $\text{GL}_n((k')^\times)$ (resp. $\text{GL}_m((k')^\times)$).
Let \( f_n \in \mathcal{H}(U_n(\mathcal{A}')) \) (resp. \( f_m \in \mathcal{H}(U_m(\mathcal{A}')) \)) be a smooth function on \( U_n(\mathcal{A}') \) (resp. \( U_m(\mathcal{A}') \)) with compact support. We introduce a distribution

\[
\mathcal{J}_{\pi, \sigma}(f_n \otimes f_m) := \sum \mathcal{B}_r' (\rho(f_n) \varphi_\pi, \rho(f_m) \varphi_\sigma) \mathcal{B}_r' (\varphi_\pi, \varphi_\sigma)
\]

where the sum is taken over orthonormal bases of \( \mathcal{A}_\pi \) and \( \mathcal{A}_\sigma \). Here we use a model of \( \nu' \) as (4.2).

As usual, we associate to \( f_n \otimes f_m \) a kernel function on \( (U_n(\mathcal{k}') \setminus U_n(\mathcal{A}')) \times (U_m(\mathcal{k}') \setminus U_m(\mathcal{A}')) \):

\[
\mathcal{K}_{f_n \otimes f_m}(g_1', g_2'; g_3, g_4) = \sum_{\xi' \in U_n(\mathcal{k}')} f_n(g_1^{-1}_1 \xi' g_3) \sum_{\xi' \in U_m(\mathcal{k}')} f_m(g_2^{-1}_2 \xi' g_4)
\]

and consider the following distribution:

\[
\mathcal{J}(f_n \otimes f_m) := \iint_{(H'(\mathcal{k}'))^2} \mathcal{K}_{f_n \otimes f_m}(\varepsilon(h_1'), \kappa(h_2'); \varepsilon(h_3'), \kappa(h_4')) \nu'(h_1'^{-1} h_2') dh_1' dh_2'.
\]

The above integral is not absolutely convergent in general and need regularization. In order to see what we should expect for these distributions, We avoid this problem in this paper. Plug in (4.3), we have

\[
(4.4) = \iint_{h_1', h_2'} \sum_{\xi' \in U_n(\mathcal{k}')} f_n(\varepsilon(h_1')^{-1} \xi' \varepsilon(h_2')) \sum_{\xi' \in U_m(\mathcal{k}')} f_m(\kappa(h_1')^{-1} \xi' \kappa(h_2')) \nu'(h_1'^{-1} h_2') dh_1' dh_2'.
\]

If we write \( h_i' = u_i' g_i' \), then

\[
(4.5) = \iint_{h_1', h_2', \xi' \in U_n(\mathcal{k}') \cap U_m(\mathcal{k}') \cap U_{\pi, \sigma}(\mathcal{A}')} f_n(u_1'^{-1} \xi' \varepsilon(h_2')) f_m(g_1'^{-1} \kappa(h_2')) \psi'(u_1'^{-1}) \nu'(h_2') d u_1' d h_2'.
\]

Define a function \( f \in \mathcal{H}(U_n(\mathcal{A}')) \) by

\[
f(g') = \int_{U_n(\mathcal{A}')} f_n(g' _1 g') f_m(g' _2) d g' _1.
\]

Then

\[
(4.6) = \sum_{\xi' \in U_{\pi, \sigma}(\mathcal{A}') \cap U_{\pi, \sigma}(\mathcal{A}')} \iint_{(H'(\mathcal{k}'))^2} f(\kappa(h_2')^{-1} u_1'^{-1} \xi' \varepsilon(h_2')) \psi'(u_1'^{-1}) \nu'(h_2') d u_1' d h_2'.
\]
where $H'$ acts on $U'_1 \setminus U_n$ by conjugation and $(U'_1 \setminus U_{n+1}(k')) \setminus H'(k')$ is the set of conjugacy classes of $k'$-points. We introduce a $k'$-algebraic group

$$H' := H' \times_{U_m} H'$$

which acts on $U_n$ in the following way: for any $k'$-algebra $R$, $h' = h'(u_1', u_2', g') \in H'(R)$ with $u_i' \in U'_1 \setminus \ker(R)$, $g' \in U_n(R)$ and $g \in U_n(R)$, we define the right action $[g]h' := g^{-1}u_i'g$. We also denote by $[U_n(k')]/H'(k')$ the set of $k'$-orbits under this action. We define a character (also denoted by $\psi$) of $H'(k')$ by

$$\psi(h') = \psi'(h'(u_1', u_2'; g')) := \psi(u_i'^{-1}u_i).$$

Then

$$(4.7) = \sum_{\zeta' \in [U_n(k')]/H'(k'), \text{Stab}_{H'}(\zeta')} \int f([\zeta']h')\psi'(h')dH'$$

$$= : \sum_{\zeta' \in [U_n(k')]/H'(k')} \mathcal{J}_{\zeta'}(f) =: \mathcal{J}(f)$$

which is a decomposition of the distribution $\mathcal{J}$ according to the orbits.

We denote by $U_n(k')_{\text{reg}}$ the set of all regular $k'$-elements which will be defined in Section 4.3, Definition 4.8. In particular, the $H'$-stabilizer $\text{Stab}_{H'}(\zeta')$ is trivial for $\zeta' \in U_n(k')_{\text{reg}}$ by Proposition 4.11 and the corresponding term $\mathcal{O}(f, \zeta') := \mathcal{J}_{\zeta'}(f)$ is a weighted orbital integral. If $f = \otimes_{v'} f_{v'}$ is factorizable, then

$$\mathcal{O}(f, \zeta') = \prod_{v' \in M_v} \mathcal{O}(f_{v'}, \zeta')$$

where

$$\mathcal{O}(f_{v'}, \zeta') = \int_{H'_{v'}} f_{v'}([\zeta']h'_{v'})\psi'(h'_{v'})dH'_{v'}.$$ 

In summary,

$$\mathcal{J}(f) = \mathcal{J}_{\text{reg}}(f) + \mathcal{J}_{\text{arr}}(f)$$

$$= \sum_{\zeta' \in [U_n(k')_{\text{reg}}]/H'(k')} \mathcal{J}_{\zeta'}(f) + \mathcal{J}_{\text{arr}}(f)$$

$$= \sum_{\zeta' \in [U_n(k')_{\text{reg}}]/H'(k')} \prod_{v' \in M_v} \mathcal{O}(f_{v'}, \zeta') + \mathcal{J}_{\text{arr}}(f).$$

Now we discuss the relative trace formula on the product of two general linear groups. We identify $\text{GL}_{m,k} \subset \text{GL}_{n,k}$ with $\text{GL}(W) \subset \text{GL}(V)$ and view $\text{GL}_{n,k'} \subset \text{Res}_{k/k'}(\text{GL}_{m,k})$ (resp. $\text{GL}_{m,k'} \subset \text{Res}_{k'/k}(\text{GL}_{m,k})$) through the basis $\{v_1, \ldots, v_r, w_1, \ldots, w_m, w_0, \tilde{v}_r, \ldots, \tilde{v}_1\}$. Let $Z'_n$ (resp. $Z'_m$) be the center of $\text{GL}_{n,k'}$ (resp. $\text{GL}_{m,k'}$). We denote $\psi = \psi' \circ \text{Tr}$. Let $\Pi$ and $\Sigma$ be as above which are cuspidal. Since $\Pi$ and $\Sigma$ are the standard base change of representations of unitary groups, we need to introduce a period integral to single out such representations.

Until the end of this section, we assume that $n$ is odd, hence $m$ is even. Since the other case is similar and will lead to the same fundamental lemma, we omit it in the following discussion. As pointed out in [F91], [F92] and [GJR01], the central character of $\Pi$ (resp. $\Sigma$) should be trivial on $A'$ and $\Pi$ (resp. $\Sigma \otimes \eta$) should be distinguished by $\text{GL}_{n,k'}$ (resp. $\text{GL}_{m,k'}$). Hence we consider the following integrals as in
and consider the following distribution:

\[
P_n(\varphi_{\Pi}) = \int_{Z'_{n}(k')} \varphi_{\Pi}(g_1) dg_1
\]

\[
P_m(\varphi_{\Sigma}) = \int_{Z'_{m}(k')} \varphi_{\Pi}(g_2) \eta(\det g_2) dg_2
\]

where we recall that \( \eta \) is the quadratic character associated to \( k/k' \).

Let \( F_n \in \mathcal{H}(\text{GL}_n(A)) \) and \( F_m \in \mathcal{H}(\text{GL}_m(A)) \). We introduce another distribution

\[
\mathcal{J}_{\Pi, \Sigma}(s; F_n \otimes F_m) := \sum \mathcal{B}'_{r,p}(s; \rho(F_n) \varphi_{\Pi}, \rho(F_m) \varphi_{\Sigma}) \mathcal{P}_{\Pi}(\varphi_{\Pi}) \mathcal{P}_{\Sigma}(\varphi_{\Sigma})
\]

where the sum is taken over orthonormal bases of \( \mathcal{A}_{\Pi} \) and \( \mathcal{A}_{\Sigma} \). Here, we take a model of \( \nu \) in the following way (be cautious that since we change the coordinates, the matrix form of the following element changes from (2.4)):

\[
\nu(h) = \nu(u_0) = \nu(h(n, n^*; k; u, u^*; g)) = \psi(u)
\]

where

\[
\psi(u) = \psi(u_1, 2 + \cdots + u_{r-1}, r + u_{r, 0} + \beta_0 n^*_0 + u^*_{r-1} + \cdots + u^*_{2, 1}) \quad (4.9)
\]

We associate to \( F_n \otimes F_m \) a kernel function on \((\text{GL}_n(k) \backslash \text{GL}_n(A)) \times (\text{GL}_m(k) \backslash \text{GL}_m(A))^2\) (averaged by \( Z'_{n} \times Z'_{m} \)):

\[
\mathcal{K}_{F_n \otimes F_m}(g_1, g_2; g_3, g_4) = \int_{Z'_{n}(k')} \int_{Z'_{m}(k')} \sum_{\zeta \in \text{GL}_n(k)} F_n(g_1^{-1} \zeta g_3) d\zeta_1 \int_{Z'_{m}(k')} \int_{Z'_{m}(k')} \sum_{\xi \in \text{GL}_m(k)} F_m(g_2^{-1} \zeta_2 \xi g_4) d\zeta_2
\]

and consider the following distribution:

\[
\mathcal{J}(s; F_n \otimes F_m)
\]

\[
= \int_{Z'_{n}(k')} \int_{Z'_{m}(k')} \int_{Z'_{n}(k')} \int_{Z'_{m}(k')} \mathcal{K}_{F_n \otimes F_m}(\epsilon(h), \kappa(h); g_1, g_2) \nu(h^{-1}) \det h_K^{s-\frac{1}{2}} \eta(\det g_2) dh \det g_1 dg_2
\]

\[
= \int_{Z'_{n}(k')} \int_{Z'_{m}(k')} \int_{Z'_{n}(k')} \int_{Z'_{m}(k')} \mathcal{K}_{F_n \otimes F_m}(\epsilon(h), \kappa(h); g_1, g_2) \nu(h^{-1}) \det h_K^{s-\frac{1}{2}} \eta(\det g_2) d\zeta_1 d\zeta_2 dg_1 dg_2
\]

\[
= \int_{\text{GL}_n(k)} \int_{\text{GL}_m(k)} \int_{\text{GL}_n(k)} \int_{\text{GL}_m(k)} \int_{U_{1, r, m+1, 1}(k) \backslash \text{GL}_n(k)} \int_{U_{1, r, m+1, 1}(k) \backslash \text{GL}_m(k)} \sum_{\zeta \in \text{GL}_n(k)} F_n(\epsilon(h)^{-1} \zeta g_1) F_m(\kappa(h)^{-1} g_2) \nu(h^{-1}) \det h_K^{s-\frac{1}{2}} \eta(\det g_2) dh \det g_1 dg_2. \quad (4.10)
\]

We decompose \( h = u_0 \) and notice that the group \( H(A) \) is unimodular. Moreover, we change a variable \( g \mapsto g_2^{-1} g \). Then

\[
(4.10) = \int_{u_0} \int_{U_{1, r, m+1, 1}(k) \backslash U_{1, r, m+1, 1}(k)} \int_{\text{GL}_n(k)} \int_{\text{GL}_m(k)} \sum_{\zeta \in \text{GL}_n(k)} F_n(g_1^{-1} \zeta g_1) F_m(g_2^{-1} \psi(u_0^{-1}) \det g_1^{s-\frac{1}{2}} \det g_2 \eta(\det g_2) d\zeta_1 d\zeta_2 dg_1 dg_2. \quad (4.11)
\]
Define a function \( \tilde{F}_s \) on \( \text{GL}_n(A) \), which is holomorphic in \( s \), by
\[
\tilde{F}_s(g) = \int_{\text{GL}_n(A)} \tilde{F}_n(g^{-1} \bar{g}) F_m(g^{-1}) \det g_\mathfrak{A}^{s-\frac{1}{2}} dg.
\]

We also introduce the symmetric space \( S_n \subset \text{Res}_{k/k'}(\text{GL}_{n,k}) \) defined by the equation \( ss^\tau = 1_n \), hence
\[
S_n(k') = \{ s \in \text{GL}_n(k) \mid ss^\tau = 1_n \}.
\]

We have an isomorphism \( \text{GL}_{n,k}/\text{GL}_{n,k'} \cong S_n \) given by \( g \mapsto gg^\tau \). Define a linear map \( \sigma : \mathcal{H}(\text{GL}_n(A)) \to \mathcal{H}(S_n(k')) \) by
\[
(\sigma(F))(gg^\tau) = \int_{\text{GL}_n(k')} F(g\bar{g})d\bar{g}
\]
and let \( F_s = \sigma(\tilde{F}_s) \).

Combining these two operations together, we get
\[
(4.11) = \sum_{\zeta \in S_n(k')/\text{GL}_{n,k'}} \int_{\text{GL}_n(k')} \int_{U_{1',m+1,1'}(k')/U_{1',m+1,1'}(A)} F_s(g_{2|\mathfrak{A}}^{-1}u^{-1} \zeta_u^{-1} g_2) \psi(u^{-1}) \det g_2 |\mathfrak{A}^{s-\frac{1}{2}} \eta(\det g_2) d\bar{u} dg_2
\]

Similar to the unitary case, we introduce the following \( k' \)-algebraic group
\[
\mathcal{H} := \text{Res}_{k/k'}(U_{1',m+1,1'}) \rtimes \text{GL}_{m,k'}
\]
which acts on \( S_n \) in the following way: for any \( k' \)-algebra \( R \), \( h = h(g; u) \) with \( u \in U_{1',m+1,1'}(R \otimes k) \), \( g \in \text{GL}_{m}(R) \) and \( s \in S_n(R) \), we define a right action \( [s]h := g^{-1}u^{-1}s u^\tau g \). We also denote by \( [S_n(k')]/\mathcal{H}(k') \) the set of \( k' \)-orbits under this action. We define a character (also denoted by \( \psi \)) of \( \mathcal{H}(A') \) by
\[
\psi(h) = \psi(h(u)) = \psi(u^{-1})
\]
and \( h := \det g \). Then
\[
(4.13) = \sum_{\zeta \in [S_n(k')]/\mathcal{H}(k')/\text{Stab}_\mathcal{H}(\zeta)} \int_{[S_n(k')]/\mathcal{H}(k')} F_s([\zeta]h) \psi(h) |\det h|^{s-\frac{1}{2}} \eta(\det h) dh
\]

We denote by \( S_n(k')_{\text{reg}} \) the set of all regular \( k' \)-elements which will be defined in Section 4.3, Definition 4.8. In particular, the \( \mathcal{H} \)-stabilizer \( \text{Stab}_\mathcal{H} \) is trivial for \( \zeta \in S_n(k')_{\text{reg}} \) by Proposition 4.11 and the corresponding term \( \mathcal{O}(s; F_s, \zeta) := \mathcal{J}_\zeta(s; F_s) \) is a weighted orbital integral. If \( F_s = \otimes_{v'} F_{s,v'} \) is factorizable, then
\[
\mathcal{O}(s; F_s, \zeta) = \prod_{v' \in M_{k'}} \mathcal{O}(s; F_{s,v'}, \zeta)
\]
where
\[
\mathcal{O}(s; F_{s,v'}, \zeta) = \int_{\mathcal{H}_{v'}} F_{s,v'}([\zeta]h_{v'}) \psi_{v'}(h_{v'}) |\det h_{v'}|^{s-\frac{1}{2}} \eta(\det h_{v'}) dh_{v'}
\]

In summary,
\[
\mathcal{J}(s; F_s) = \mathcal{J}_{\text{reg}}(s; F_s) + \mathcal{J}_{\text{irr}}(s; F_s)
\]
\[
:= \sum_{\zeta \in [S_n(k')_{\text{reg}}]/\mathcal{H}(k')} \mathcal{J}_\zeta(s; F_s) + \mathcal{J}_{\text{irr}}(s; F_s)
\]
\[
= \sum_{\zeta \in [S_n(k')_{\text{reg}}]/\mathcal{H}(k')} \prod_{v' \in M_{k'}} \mathcal{O}(s; F_{s,v'}, \zeta) + \mathcal{J}_{\text{irr}}(s; F_s).
\]
When $s = \frac{1}{2}$, the terms involving $|\text{det}|$ disappear and we discard $s$ in all notations in this case. In particular,

$$\mathcal{J}_\zeta(F) = \mathcal{O}(F, \zeta) = \int_{\mathbf{H}(\mathbb{A})} F([\zeta|h])\psi(h)\eta(\det h)dh$$

which should be compared with

$$\mathcal{J}_{\zeta'}(f) = \mathcal{O}(f, \zeta') = \int_{\mathbf{H}(\mathbb{A})} f([\zeta'|h'])\psi'(h')dh'$$

assuming that $\zeta$ and $\zeta'$ are both regular.

**Remark 4.3.** If the original functions $F_n = \otimes_v F_{n,v}$ and $F_m = \otimes_v F_{m,v}$ are factorizable, then $\tilde{F}_n = \otimes_v \tilde{F}_{n,v}$ and $\tilde{F}_m = \otimes_v \tilde{F}_{m,v}$ are also factorizable. If for some (finite) place $v'$, $F_{m,v'}$ has the property that $|\text{det} g_{v'} | F_{m,v'}(g) \neq 0$) is a singleton, then $\tilde{F}_{n,v'} = F_{v'}$ are independent of $s$. In particular, this is the case for almost all $v'$.

We will introduce the notion of matching orbits in the next section. It is expected that we have enough pairs $(F, f)$ such that if $\zeta \leftrightarrow \zeta'$ are regular and match, then

$$\mathcal{O}(F, \zeta) = \mathcal{O}(f, \zeta').$$

The above relation should also be true place by place. In particular, let us consider the case where $v' \in \mathbb{M}_v$ splits into two places $v_0, v_1 \in \mathbb{M}_v$. Then we may identify $S_{n,v'}$ with the set of pairs $(g_{\bullet}, g_{\circ}) \in \text{GL}_{n,v_0} \times \text{GL}_{n,v_1}$ with $g_{\bullet} g_{\circ} = 1_n$, hence with $\text{GL}_{n,v'}$ by $(g_{\bullet}, g_{\circ}) \mapsto g_{\bullet}$. Then $F_{v'}$ becomes a function on $\text{GL}_{n,v'}$ and

$$\mathcal{O}(F_{v'}, \zeta) = \int_{\text{GL}_{n,v'}(U_{1r,m+1}, r,v')} \int_{\int_{\text{GL}_{n,v'}(U_{1r,m+1}, r,v')}^2} F_{v'}(g^{-1}u^{-1}_0 \tilde{\zeta}_0, g)\psi'(u^{-1}_0)du_0 dg$$

for the generic character

$$\psi'(u) = \psi'(j(u_{1,2} + \cdots + u_{r-1,r} + n_{r,0} + \beta_0 u_{r+1} + \cdots + u_{2,1})) \quad (4.15)$$

where $j = (j_i - j)$. On the other hand, we may identify $U_{n,v'}$ with the pairs $(g_{\bullet}, g_{\circ})$ such that $g_{\circ} = w_{\beta_0}^{-1}g_{\bullet}^{-1}w_{\beta_0}$, and with $\text{GL}_{n,v'}$. Here

$$w_{\beta_0} = \begin{pmatrix} \beta_0 & u_r \\ u_r & 1 \end{pmatrix}$$

where $\beta' = (\beta'_{\bullet}, \beta'_{\circ})$. Then $f_{v'}$ becomes a function on $\text{GL}_{n,v'}$ and

$$\mathcal{O}(f_{v'}, \zeta') = \int_{\text{GL}_{n,v'}(U_{1r,m+1}, r,v')} \int_{\int_{\text{GL}_{n,v'}(U_{1r,m+1}, r,v')}^2} f_{v'}(g^{-1}u^{-1}_0 \tilde{\zeta}_0' g)\psi'(u^{-1}_0)du_0 dg'.$$

Moreover, in this case, that $\zeta$ and $\zeta'$ match exactly means that $\zeta = \zeta' \in \text{GL}_{n,v'}$. Hence for

$$f_{v'}(g) = F_{v'}(g) = \int_{\text{GL}_{n,v'}} F_{v'}(g\tilde{g})\tilde{F}_{v'}(\tilde{g})d\tilde{g},$$

we have

$$\mathcal{O}(F_{v'}, \zeta) = \mathcal{O}(f_{v'}, \zeta') \quad (4.16)$$

for all $\zeta$ regular.
4.3 Matching of orbits and functions

Let $k'$ be a number field or a local field and $k/k'$ possibly split. We say two elements $\beta_1, \beta_2 \in \text{Her}_m(k/k')^\times$ are similar, denoted by $\beta_1 \sim \beta_2$ if $\exists g \in \text{GL}_m(k)$ such that $\beta_2 = \text{tr} g^\beta g$. We denote $[\text{Her}_m(k/k')^\times]$ the set of similarity classes. We write $W = W^\beta, V = V^\beta$ if the representing matrix of $W$ is in the class $\beta$, and $U^\beta_n (\text{resp. } U^\beta_n, H^\beta)$ for $U(W^\beta) (\text{resp. } U(V^\beta), H^\beta)$. We define

$$\epsilon(\beta) = \eta \left( (-1)^{\frac{m-1}{2}} \det \beta \right) \in \{ \pm 1 \}$$

to be the $\epsilon$-factor.

We first define the notion of pre-regular orbits. Recall that we have the action of $H$ (resp. $H'$), and hence its maximal unipotent subgroup $\text{Res}_{k/k} U_{1^r,m+1,1^r}$ (resp. $(U_{1^r,m+1})^2$), on $S_n$ (resp. $U^\beta_n$).

**Definition 4.4.** An element $\zeta \in S_n(k')$ (resp. $\zeta^\beta \in U^\beta_n(k')$) is called pre-regular if its stabilizer under the action of $\text{Res}_{k/k} U_{1^r,m+1,1^r}$ (resp. $(U_{1^r,m+1})^2$) is trivial.

Let us start with the symmetric space $S_n$. Let $B$ be the Borel subgroup of $\text{GL}_n$ consisting of upper-triangular matrices and $A \cong (\text{GL}_1)^n$ be the maximal torus consisting of diagonal matrices. Let $W_n$ be the Weyl group of $\text{GL}_n$, which is isomorphic to the product of $n$-permutations. We identify elements in $W_n$ with permutation matrices, and hence identity $W_n$ with a subgroup of $\text{GL}_n(k)$. Moreover, let $W_S^k \subset W_n$ be the subgroup consisting of elements whose square is $1$. Let $P$ be a standard parabolic subgroup of $\text{GL}_{n,k}$ whose unipotent radical is $U$. We also choose a standard Levi subgroup $M$ of it consisting of matrices with diagonal blocks. The group $\text{Res}_{k/k} P$ acts on $S_n$ from right by $[s]p = p^{-1}sp^\tau$. First, we have the following lemma.

**Lemma 4.5.** An element $\zeta \in S_n(k')$ has trivial stabilizer under the action of $U(k') \subset P(k)$ if and only if its orbit intersects $[w]M(k)$, where $w = w_n$ is the longest element in $W_n$. Moreover, the intersection contains at most 1 element.

**Proof.** By [Fl92, Proposition 3], we have a Bruhat decomposition for $S_n(k')$:

$$S_n(k') = \coprod_{w \in W_S^k} [w]B(k).$$

It implies that for general $P$, we have

$$S_n(k') = \bigcup_{w \in W_S^k} [w]P(k) = \bigcup_{w \in W_S^k} [w]M(k)U(k).$$

Hence in any $U(k)$-orbit, there is a representative of the form $[w]m$. We assume that $\zeta = [w]m = m^{-1}wm^\tau$. Then its stabilizer is trivial if and only if

$$\{ u^{-1}wu^\tau = w \mid u \in U(k) \} = \{ 1_n \}. \quad (4.17)$$

But $u^{-1}wu^\tau = w$ is equivalent to $ww = u^\tau$, hence if $w = w$ is the longest Weyl element, it will force $u = 1_n$, and the $[w]m$ is the only point where its orbit and $[w]M(k)$ intersect.

Conversely, we need to show that if (4.17) holds, then $w \in [w]W_M$, where $W_M \subset W_n \cap M(k)$ is the Weyl group of $M$. We observe that any $w \in W_S^k$ is a disjoint union of transpositions. We use induction on $n$. The case $n = 1$ is trivial and assume this for $< n$. If the transposition $(1, a)$ appears in $w$, then we reduce to the case of $n - 1$ and we are done. Otherwise $(a, a)$ will appear in $w$ with $1 < a < n$. Suppose that $M = \text{GL}_{n_1} \times \cdots \times \text{GL}_{n_t}$ (arranged from upper-left to lower-right) with $n = n_1 + \cdots + n_t$, $n_t > 0$ and $t > 1$ (otherwise, it is trivial). If $n - a < n_t$, then $w' = (a, n)$ is an element in $W_M \subset M(k)$. The conjugation $w^{-1}ww' \in W_S^k$ will contain the transportation $(1, n)$ and we are done. Otherwise, $n - a \geq n_t$, and consider the transportation $(b, n)$ in $w'$ with $1 < b \leq n$. If $b - 1 < n_t$, then we can conjugate $w$ by $(1, b) \in W_M$ and we are again done. The rest case is that $b - 1 \geq n_t$. Then we define an element $u \in U(k)$ whose entries are 1 at diagonals and positions $(1, b), (a, n), 0$ elsewhere. Then $wwu = u = u^\tau$ which contradicts (4.17). }

□
Applying the above lemma to \( P = P_{1,r,m+1} \) stabilizing the flag \( 0 \subset \{ v_1 \} \subset \cdots \subset X \subset X \oplus W \oplus E \subset X \oplus W \oplus E \oplus \{ v_r \} \subset \cdots \subset V \), which is standard according to our basis \( \{ v_1, \ldots, v_r, w_1, \ldots, w_m, v_r, \ldots, v_1 \} \) used in this Chapter. Since \( w \) normalizes \( P_{1,r,m+1} \), the \( U_{1,r,m+1}(k) \)-orbit of a pre-regular element \( \zeta \) must contain a unique element of the form

\[
\begin{pmatrix}
  t_1(\zeta) \\
  \vdots \\
  t_r(\zeta) \\
  \vdots \\
  t_1(\zeta)^{-1}
\end{pmatrix}
\]

with \( t_i(\zeta) \in k^\times \) and \( \text{Pr}(\zeta) \in S_{m+1}(k') \). We call it the normal form of \( \zeta \).

Now we consider the unitary group \( U_\alpha^\beta(k') \). We fix a minimal parabolic subgroup \( P_\alpha^\beta \) such that its maximal unipotent subgroup \( U_\alpha^\beta \) contains \( U_{1,r,m+1} \). Let \( A_0^\beta \) be a maximal torus inside \( P_\alpha^\beta \) and \( W_\alpha^\beta \) the Weyl group. Let \( P' \) be a general standard parabolic subgroup of \( U_\alpha^\beta \) whose maximal unipotent subgroup is \( U' \) and \( M' \supset A_0^\beta \) be a Levi subgroup. The group \( (P')^2 \) acts on \( U_\alpha^\beta \) from right by \( [g](p_1, p_2) = p_1gp_2 \). We have the following lemma similar to the case of symmetric spaces.

**Lemma 4.6.** An element \( \zeta' \in U_\alpha^\beta(k') \) has trivial stabilizer under the action of \( (U'(k'))^2 \subset (P'(k'))^2 \) if and only if its orbit intersects \( [w^\beta](M'(k'))^2 = M'(k')w^\beta M'(k') \), where \( w^\beta \) is the longest element in \( W_\alpha^\beta \). Moreover, the intersection contains at most one element.

**Proof.** We have the usual Bruhat decomposition:

\[
U_\alpha^\beta(k') = \bigsqcup_{w \in W_{M'} \setminus W_{M''}/W_{M''}} P'(k')wP'(k') = \bigsqcup_{w \in W_{M'} \setminus W_{M''}/W_{M''}} U'(k')M'(k')wM'(k')U'(k').
\]

Hence in any \( (U'(k'))^2 \)-orbit, there is a representative of the form \( m_1wm_2 \). We assume that \( \zeta' = m_1wm_2 \). Then its stabilizer is trivial if and only if

\[
wU'(k')w^{-1} \cap U'(k') = \{ 1 \}. \tag{4.19}
\]

Let \( R^+(A_0^\beta, U_\alpha^\beta) \) (resp. \( R^+(A_0^\beta, M') \)) be the set of positive roots of \( A_0^\beta \) (resp. in \( M' \)), then any double coset of \( W_{M'} \setminus W_{M''}/W_{M''} \) has a unique representative \( w \) satisfying \( w(\alpha) < 0 \) and \( w^{-1}(\alpha) < 0 \) for all \( \alpha \in R^+(A_0^\beta, M') \). Assuming that \( w \) satisfies (4.19) and the above condition, then \( w(\alpha) < 0 \) for any \( \alpha \in R^+(A_0^\beta, U_\alpha^\beta) \). Hence \( w = w^\beta \). Conversely, if \( w = w^\beta \), then (4.19) holds and the intersection is a singleton.

\[\Box\]

Applying the above lemma to \( P' = P_{1,r,m+1} \), the standard parabolic subgroup stabilizing the flag \( 0 \subset \{ v_1 \} \subset \cdots \subset X \subset V \). Since \( w^\beta \) normalizes \( P_{1,r,m+1} \), the \( (U_{1,r,m+1}(k'))^2 \)-orbit of a pre-regular element \( \zeta^\beta \) must contain a unique element of the form

\[
\begin{pmatrix}
  t_1(\zeta^\beta) \\
  \vdots \\
  t_r(\zeta^\beta) \\
  \vdots \\
  t_1(\zeta^\beta)^{-1}
\end{pmatrix}
\]

with \( t_i(\zeta^\beta) \in k^\times \) and \( \text{Pr}(\zeta^\beta) \in U_{m+1}(k') \), where \( U_{m+1} = U(W^\beta \oplus E) \). We call it the normal form of \( \zeta^\beta \).
Remark 4.7. There is a more intrinsic way to define the invariants $t_i$. For $\xi \in M_n(k)$, for $i = 1, \ldots, r$, let $\zeta_i$ be the matrix leaving the first $i$ columns and the last $r$ rows of $\zeta$ and $s_i(\zeta) = \det \zeta_i$ which is invariant under the action $\zeta \mapsto u\zeta u^t$ for $u, u' \in U_{1, m+1, 1}(k)$. Then $\xi \in S_n(k)$ (resp. $\zeta_i \in U_n^\beta(k)$) is pre-regular if and only if $s_i(\zeta) \in k^n$ (resp. $s_i(\zeta_i) \in k^n$), where we view $\zeta$ (resp. $\zeta_i$) as elements in $M_m(k)$ through the natural inclusion $S_n \subset \Res_{k/k} M_{n,k} = \End(V)$ and $U_n^\beta \subset \Res_{k/k} M_{n,k}$. Moreover, the invariants $t_i$ and $s_i$ are related by $t_i(\zeta) = s_{i-1}(\zeta)\gamma s_i(\zeta)^{r-1}$ ($s_0 = 1$) and similar for $\zeta_i$.

Now we are at the point to introduce regular orbits. We have natural inclusions $S_{m+1} \subset \Res_{k/k} M_{m+1,k} = \End(W \oplus E)$ and $U_{m+1}^\beta \subset \Res_{k/k} M_{m+1,k}$.

Definition 4.8. An element $\xi \in M_{m+1}(k)$ is called regular if it satisfies:

- $\xi$ is regular semisimple as an element of $M_{m+1}(k)$;
- the vectors $\{w_0, \xi w_0, \ldots, \xi^m w_0\}$ span $W \oplus E$;
- the vectors $\{w_0^\gamma, w_0^\gamma \xi, \ldots, w_0^\gamma \xi^m\}$ span $W^\gamma \oplus E^\gamma$.

An element $\zeta \in S_n(k)$ (resp. $\zeta_i \in U_n^\beta(k)$) is called regular, if it is pre-regular and the uniquely determined element $\Pr(\zeta) \in M_{m+1}(k)$ (resp. $\Pr(\zeta_i) \in U_{m+1}^\beta(k)$) is regular. An $H$-orbit $\zeta \in [S_n(k')]/H(k')$ (resp. $H^\gamma$-orbit $\zeta_i \in [U_n^\beta(k')]/H^\gamma(k')$) is called regular if any, hence all elements inside it are regular. We denote by $M_{m+1}(k)_{reg}, \GL_{m+1}(k)_{reg} := M_{m+1}(k)_{reg} \cap \GL_{m+1}(k), S_{n,k'}(k')_{reg}$ (resp. $[S_n(k')]/H(k')$) and $U_{n,k'}(k')_{reg}$ (resp. $[U_n^\beta(k')]/H^\gamma(k')$) the set of regular elements (resp. orbits).

To proceed, we need to recall some structural results in [RS07, Section 6], see also [JR], [Yun09] and [Zh]. To include the whole action of $H$ (resp. $H^\gamma$), we need to consider the conjugation action (from right) of $\GL_{m,k'}$ (resp. $U_n^\beta$). We consider more generally the conjugation action of $\GL_{m,k}$. Recall that by our choice of coordinates, the group $\GL_{m,k}$ embeds into $\GL_{m+1,k}$ via

$$g \mapsto \begin{bmatrix} g \\ 1 \end{bmatrix}.$$ 

Let $a_i(\xi) = \Tr \left( \zeta^i \right)$ for $1 \leq i \leq m+1$ and $b_i(\xi) = w_0^{\gamma^i} w_0$ for $1 \leq i \leq m$. Let $T_\xi = \det \left[ (w_0^{\gamma^i})^{w_j} \right]_{i,j=1}^{m+1}$, $D_\xi$ the matrix $\left[ (w_0^{\gamma^i} \zeta^{\gamma^{i-1}})(\zeta^{\gamma^{i-1}} w_0) \right]_{i,j=1}^{m+1}$ and $\Delta_\xi = \det D_\xi$. It is clear that if $\xi$ is regular, then $\Delta_\xi \neq 0$. Moreover, we have

Lemma 4.9. Two regular elements $\xi$ and $\xi'$ are conjugate under $\GL_m(k)$ if and only if $a_i(\xi) = a_i(\xi')$ and $b_i(\xi) = b_i(\xi')$. The $\GL_{m,k}$-stabilizer of a regular element is trivial.

Proof. See [RS07, Proposition 6.2 & Theorem 6.1] for the (equivalent version of the) first and second statements, respectively.

To include all unitary groups at the same time, we consider the set $U_{m+1}$ of pairs $(\beta, \xi^\beta)$ where $\beta \in \He_m(k/k')^\times$ and $\xi^\beta \in U_{m+1}(k')_{reg} := \{ \xi^\beta \in M_{m+1}(k)_{reg} \mid (\beta^{\xi^\beta})^t \beta^{\xi^\beta} = \beta^t \}$, where $\beta^t$ is defined in (4.1). The group $\GL_{m+1}(k)$ acts on $U_{m+1}$ by $(\beta, \xi^\beta) g = (g^\beta^t \beta g, g^\beta^t \xi^\beta g)$. For $\xi \in S_{m+1}(k')_{reg} := S_{m+1}(k') \cap M_{m+1}(k)_{reg}$, we denote by $\xi \leftrightarrow (\beta, \xi^\beta)$ if there exists $g \in \GL_m(k)$ such that $\xi = g^{-1} \xi^\beta g$. The following lemma is considered in [JR] and [Zh].

Lemma 4.10. For $\xi \in S_{m+1}(k')_{reg}$, there exists a pair $(\beta, \xi^\beta)$, unique up to the $\GL_m(k)$-action, such that $\xi \leftrightarrow (\beta, \xi^\beta)$. Conversely, for every pair $(\beta, \xi^\beta) \in U_{m+1}$, there exists an element $\xi \in S_{m+1}(k')_{reg}$, unique up to the $\GL_m(k')$-conjugation, such that $\xi \leftrightarrow (\beta, \xi^\beta)$.

Proof. We first point out that two elements $\xi, \xi' \in S_{m+1}(k')_{reg}$ are conjugate under $\GL_m(k)$ if and only if they are conjugate under $\GL_m(k')$. Assume $g^{-1} \xi g = \xi'$, then $g^{-1} \xi g = g^{-1} \xi \xi^{-1} g = g^{-1} \xi g^t$, hence $g = g^t$.
The following proof is due to Zhang [Zh]. It is easy to see that for $\xi \in M_{m+1}(k)_{\text{reg}}$, $\xi$ and $^t\xi$ have the same invariants $a_i, b_i$. Hence by the above lemma, there is a unique element $g \in GL_m(k)$ such that $g^{-1}^t\xi g = ^t\xi$. If $\xi \in S_{m+1}(k)_{\text{reg}}$, then $^t\xi \in S_{m+1}(k')_{\text{reg}}$. Hence we have $g^r = g$. But we also have

$$^t g^t \xi ^t g^{-1} = \xi; \quad g^t \xi g^{-1} = \xi$$

which implies that $g = ^t g$. Together, we have $g \in \text{Her}_m(k/k')$. Moreover, $^t\xi^r(g^{-1})\xi = g^{-1}$. Hence $\xi \in U_{m+1}^g$ and $\xi \leftrightarrow (g^{-1}, \xi)$. Conversely, given any $(\beta, \xi) \in U_{m+1}$, then

$$^t\xi^t \xi^r = \beta^r \implies \beta^r = \xi^r = \xi^r \xi^r.$$ 

Moreover, there is a $\gamma \in GL_m(k)$ such that $\gamma^{-1}\xi \gamma = ^t\xi$. We have $\beta^r \gamma^{-1}\xi^r \xi \gamma = \xi^r \gamma$, i.e, $(\gamma \beta^r)^{-1}\xi (\gamma \beta^r) = \xi^r \gamma$. By regularity, $\gamma \beta^r \in S_m(k')$. Hence there exists $g \in GL_m(k)$ such that $\gamma \beta^r = g^r \gamma$. Then

$$g^{-1} g g^r \gamma^r = 1_{m+1} \implies g^{-1} \xi \gamma^r \gamma^r = 1_{m+1} \implies \left(g^{-1} \xi \gamma \right)^r = 1_{m+1}$$

e.i., $g^{-1} \xi \gamma^r \gamma^r = 1_{m+1}$. The uniqueness part is obvious.

All these considerations lead to the following proposition:

**Proposition 4.11.** (1). There is a natural bijection

$$\frac{[S_n(k')_{\text{reg}}]}{H(k')} \leftrightarrow \bigotimes_{\beta \in \text{Her}_m(k/k')} \frac{[U_n^\beta(k')]}{H^\beta(k')}.$$ 

If $N \xi \beta = \zeta$, then we say that they match and denote by $\zeta \leftrightarrow \xi \beta$.

(2). The set $S_n(k')_{\text{reg}}$ (resp. $U_n^\beta(k')_{\text{reg}}$) is non-empty and Zariski open in $S_n$ (resp. $U_n^\beta$). Moreover, the $H$-stabilizer (resp. $H^\beta$-stabilizer) of regular $\xi$ (resp. $\xi \beta$) is trivial.

**Proof.** (1). Start with an element $\zeta \in S_n(k')_{\text{reg}}$ and consider its normal form. We get $r$ invariants $t_1(\zeta), \ldots, t_r(\zeta)$ and an element $\Pr(\zeta) \in S_{m+1}(k')_{\text{reg}}$. By Lemma (4.10), there is a pair $(\beta, \xi \beta)$ such that $\xi = g^{-1} \xi \gamma g$. We fix $\beta$, then $\xi \beta$ is uniquely determined up to the $U_n^\beta(k')$-conjugation. We define an element $\zeta \beta$ by

$$\zeta \beta = \begin{bmatrix} t_1(\zeta) & & \\
\xi & t_r(\zeta) & \\
t_1(\zeta) \gamma^r & & & \ddots & \\
\gamma^{-1} & & & & \ddots & \\
& \ddots & \ddots & & \\
& & & \ddots & & \\
& & & & \ddots & & \\
& & & & & \end{bmatrix} \in U_n^\beta(k')_{\text{reg}}.$$ 

By construction, $\zeta \mapsto \xi \beta$ defines a map

$$N : [S_n(k')_{\text{reg}}]/H(k') \leftrightarrow \bigotimes_{\beta \in \text{Her}_m(k/k')} [U_n^\beta(k')_{\text{reg}}]/H^\beta(k')$$

which is injective. The converse is similar.

(2). The openness is due to the fact that the pre-regular elements in both cases correspond to the unique open cell in the Bruhat decomposition and the fact that the three conditions in Definition 4.8 are open. The non-emptiness is essentially exhibited in [JR, Section 3] combining with the exponential map. For the last part, we prove it for $\zeta \in S_n(k')_{\text{reg}}$ and the case for unitary groups is similar. We write $\zeta$ in its normal form. If $g^{-1} u^{-1} \zeta u'^{-1} g = \zeta$, then $u_1^{-1} g^{-1} \zeta g u'^{-1} = \zeta$ with $u_1 = g^{-1} u \in U_{1', n+1, 1'}(k)$. We have $g^{-1} \zeta g = \zeta$, $g^{-1} \Pr(\zeta) g = \Pr(\zeta)$ which implies that $g = 1_n$ and hence $u = 1_n$. \[\square\]
It is clear from the above discussion that the regular orbit $\zeta \in [S_n(k')_{reg}/H(k')]$ or $[U_n^\beta(k')_{reg}/H^\beta(k')]$ is determined by its invariants $t_i(\zeta)$ $(i = 1, \ldots, r)$, $a_i(\zeta) := a_i(Pr(\zeta))$ $(i = 1, \ldots, m+1)$ and $b_i(\zeta) := b_i(Pr(\zeta))$ $(i = 1, \ldots, m)$, and $\zeta \leftrightarrow \zeta^\beta$ if and only if they have the same invariants. We also denote $T_{\zeta} = T_{Pr(\zeta)}$, $D_{\zeta} = D_{Pr(\zeta)}$ and $\Delta_{\zeta} = \Delta_{Pr(\zeta)}$.

Now let $k'$ be a local field and $n$ odd. As suggested by the case where $k/k'$ is split. We are going to formulate the conjecture on the matching of functions. First, let us define a suitable “transfer factor” $t$. Recall that we have a character $\eta : k^\times \rightarrow C^\times$. We define, for $\zeta \in [S_n(k')_{reg}/H(k')]$,
\[
t(\zeta) = \eta \left( T_{\zeta} \cdot (\det Pr(\zeta) - \frac{m}{2}) \right)
\]
which makes sense since $m$ is even and $T_{\zeta} \cdot (\det Pr(\zeta))^{-\frac{m}{2}} \in k^\times$.

**Conjecture 4.12 (Smooth matching).** Let $k'$ be as above. Given any smooth, compactly supported function $F \in \mathcal{H}(S_n(k'))$, there exist functions $(f^\beta)_\beta$ for each $\beta \in [\text{Her}_m(k/k')^\times]$ such that
\[
\mathcal{O}(F, \zeta) = t(\zeta)\mathcal{O}(f^\beta, \zeta^\beta)
\]
for all $\zeta \in [S_n(k')_{reg}/H(k')]$ and $\zeta \leftrightarrow \zeta^\beta$. Conversely, given any functions $f^\beta \in \mathcal{H}(U_n^\beta(k'))$ for each $\beta \in [\text{Her}_m(k/k')^\times]$, there exists a function $F \in \mathcal{H}(S_n(k'))$ such that the above identity holds. We say that such $F$ and $(f^\beta)_\beta$ match and denote by $F \leftrightarrow (f^\beta)_\beta$.

**Corollary 4.13** (of Section 4.2). If $k/k'$ is split, then the conjecture of smoothing matching holds.

### 4.4 The fundamental lemma

As usual, to establish the equality between two relative trace formulae, one need to prove the corresponding fundamental lemma. We now formulate our fundamental lemma. We allow $m$ to be any nonnegative integer.

Let $k'$ be a non-archimedean local field and $k/k'$ a separable quadratic field extension. There are only two non-isomorphic hermitian spaces of dimension $m > 0$ over $k$ which is distinguished by the factor $\epsilon(\beta)$. We will use the superscript $\pm$ instead of $\beta$ for $\epsilon(\beta) = \pm 1$ in the following notations.

We now assume that $k/k'$ is an unramified field extension and $\psi' : k' \rightarrow C^\times$ is an unramified character. As before, we write $\sigma'$ (resp. $\sigma$) the ring of integers of $k'$ (resp. $k$). We denote by $val : k^\times \rightarrow Z$ the valuation map. We also assume that $\beta_i \in \sigma'$. Then $V^+$ will have a self-dual $\sigma$-lattice $L_{V'}$ which extends to a self-dual $\sigma$-lattice $L'$ of $V^+$. The unitary group $U_m^+$ (resp. $U_n^+$) is unramified and has a model over $\sigma'$. The group of $\sigma'$-points $U_m^{\sigma'}$ (resp. $U_n^{\sigma'}$) of a $\sigma'$ is a hyperspecial maximal subgroup of $U_m^{\sigma}$ (resp. $U_n^{\sigma}$). We also identity $GL_n(\sigma)$ with $GL_n(L_{V'})$, a hyperspecial maximal subgroup of $GL_n(k)$ and let $S_n(\sigma') := S_n(k') \cap GL_n(\sigma)$.

We denote by $\mathcal{H}(U_n^+(k')/U_n^+(\sigma'))$ (resp. $\mathcal{H}(GL_n(k)/GL_n(\sigma))$) the spherical Hecke algebra of $U_n^+$ (resp. $GL_n,k$). There is a base change map $b : \mathcal{H}(GL_n(k)/GL_n(\sigma)) \rightarrow \mathcal{H}(U_n^+(k')/U_n^+(\sigma'))$ and recall that we have a linear map $\sigma : \mathcal{H}(GL_n(k)/GL_n(\sigma)) \rightarrow \mathcal{H}(S_n(k'))$ similarly defined as (4.12) for the local case. Moreover, we define
\[
t(\zeta) = \begin{cases} (-1)^{\text{val}(T_{\zeta} \Pi_{\zeta}^{\frac{1}{2}}(\zeta))} & \text{if } m \text{ is odd;} \\ (-1)^{\text{val}(T_{\zeta})} & \text{if } m \text{ is even.} \end{cases}
\]

**Conjecture 4.14 (The fundamental lemma).** For any element $\tilde{F} \in \mathcal{H}(GL_n(k)/GL_n(\sigma))$, the functions $F = \sigma(\tilde{F})$ and $(f^+, f^-)$, where $f^+ = b(\tilde{F})$ and $f^- = 0$. In particular, we have
\[
\mathcal{O}(1_{S_n(\sigma')}, \zeta) = \begin{cases} t(\zeta)\mathcal{O}(1_{U_n^+(\sigma')}, \zeta^+) & \zeta \leftrightarrow \zeta^+ \in U_n^+(k'); \\ 0 & \zeta \leftrightarrow \zeta^- \in U_n^+(k') \end{cases}
\]
Proof. Proposition 4.15. If \( \text{val}(\Delta_{\zeta}) \) is odd, then

\[
O(\mathbb{I}_{S_n'}(\omega'), \zeta) = 0.
\]

Proof. The following is a modification of an argument in [Zh]. Let

\[
w = \begin{bmatrix} w_{r-1} & \vdots & w_r \end{bmatrix},
\]

then it is easy to see that \( \mathbb{I}_{S_n'}(s) = \mathbb{I}_{S_n'}(w^t sw) \). If we write \( \zeta \) in its normal form \((4.18)\), then

\[
O(\mathbb{I}_{S_n'}(\omega'), \zeta) = \int_{U_{1', m+1, 1'}(k) \text{GL}_m(k')} \mathbb{I}_{S_n'}(w^t u^{-1} g^1 \zeta g^{-1} u^{-1} w) \psi(h^{-1}) \eta(\det g) dgdw.
\]

But \( w^t \zeta w \) and \( \zeta \) are both of normal form and have the same invariants. In the proof of Lemma 4.10, we see that there exists \( h \in \text{GL}_m(k') \) such that \( w^t \zeta w = h^{-1} \zeta h \) and \( \eta(\det h) = -1 \). Moreover, \( \psi(h) = \psi(w^t h^{-1} \zeta w) \). After changing variables \( w^t h^{-1} \zeta w \mapsto w^t h^{-1} g^{-1} \), we have

\[
(4.22) = - \int_{U_{1', m+1, 1'}(k) \text{GL}_m(k')} \mathbb{I}_{S_n'}(w^t h^{-1} \zeta g^1 \zeta g^{-1} \zeta g^1 \zeta g^{-1} \zeta g^1 \zeta g^{-1}) \psi(h^{-1}) \eta(\det g) dgdw,
\]

which implies that \( O(\mathbb{I}_{S_n'}(\omega'), \zeta) = 0. \)

\[\square\]

5 A relative trace formula for \( U_n \times U_m \): Fourier-Jacobi periods

5.1 Fourier-Jacobi models and periods

Let us briefly recall the definition of Fourier-Jacobi models and periods for unitary groups in [GGP]. First, let us consider the local situation, hence \( k' \) is a local field. Let \( V \) be a hermitian space over \( k \) of dimension \( n \) with the hermitian form \( (\cdot, \cdot) \) and \( W \subset V \) a subspace of dimension \( m \) such that the restricted hermitian form \( (\cdot, \cdot)|_W \) is non-degenerate. We assume that the orthogonal complement \( W^\perp = X \oplus X^* \) such that \( X, X^* \) are isotropic and of dimension \( r \). Hence \( n = m + 2r \). The hermitian form restricted on \( W \) (resp. \( X \oplus X^* \)) identifies \( W \) (resp. \( X^* \)) with \( W = W_r^\perp \) (resp. \( X \)). We denote \( U(V) \) (resp. \( U(W) \)) the unitary group of \( V \) (resp. \( W \)) which is a reductive group over \( k' \). Let \( P_{r, m} \) be the parabolic subgroup of \( U(V) \) stabilizing \( X \) and \( U_{r, m}' \), its maximal unipotent subgroup. Then \( U_{r, m}' \) fits into the following exact sequence:

\[
0 \rightarrow \bigwedge^2_r X \rightarrow U_{r, m}' \rightarrow \text{Hom}_k(W, X) \rightarrow 0.
\]
Let \( \ell'_X : X \to k \) be any nontrivial homomorphism (if exists) and let \( U'_X \) be a maximal unipotent subgroup of \( \text{GL}(X) \) stabilizing \( \ell'_X \). The homomorphism \( \ell'_X \) induces a homomorphism \( \wedge^2 \ell'_X : \wedge^2 X \to \wedge^2 k = k^{-} \) and a homomorphism
\[
\text{Res}_{k'/k} \ell'_X : \text{Hom}_k(W, X) \to \text{Hom}_k(W, k) = \text{Hom}_{k'} \left( \text{Res}_{k'/k} W, k' \right) = (\text{Res}_{k'/k} W)^{\vee}
\]
The hermitian pair \((-,-)\) of \( W \) induces a symplectic pair \( \overline{\text{Tr}} \) on the \( n \)-dimensional \( k' \)-vector space \( \text{Res}_{k'/k} W \). Hence \( (\text{Res}_{k'/k} W)^{\vee} \) is identified with \( \text{Res}_{k'/k} W \) through this pair. Again, let \( H (\text{Res}_{k'/k} W) \) be the Heisenberg group, then we have the following commutative diagram:
\[
\begin{array}{cccccc}
0 & \longrightarrow & \wedge^2 X & \longrightarrow & U'_{r,m} & \longrightarrow & \text{Hom}_k(W, X) & \longrightarrow & 0 \\
& & \Lambda^2 \ell'_X & \downarrow & \text{Res}_{k'/k} \ell'_X & & \downarrow & & \\
0 & \longrightarrow & k' & \longrightarrow & H (\text{Res}_{k'/k} W) & \longrightarrow & \text{Res}_{k'/k} W & \longrightarrow & 0
\end{array}
\]
For a nontrivial character \( \psi' : k' \to \mathbb{C}^1 \), we have a Weil representation \( \omega_{\psi'}^{\wedge} \) of \( H (\text{Res}_{k'/k} W) \times \text{Mp} (\text{Res}_{k'/k} W) \).
If we choose a character \( \mu : k^\times \to \mathbb{C}^\times \) such that \( \mu|_{k^\times} = \eta \), we will have a splitting map
\[
\text{Mp} (\text{Res}_{k'/k} W) \to U(W) \to \text{Sp} (\text{Res}_{k'/k} W)
\]
(cf. [HKS96, Section 1.2]). By restriction, we get a Weil representation \( \omega_{\psi', \mu}^{\wedge} \) of \( \text{Res}_{k'/k} W \times U(W) \), and hence a representation of \( U'_{r,m} \times U(W) \) through the middle vertical map in (5.1). Let \( \lambda' : U'_{r,m} \to \mathbb{C}^\times \) be a generic character. Then we define \( \nu_{\psi', \mu} = \omega_{\psi', \mu}^{\wedge} \otimes \lambda' \) which is a smooth representation of \( H' := U'_{r,m} \times (U_X \times U(W)) \). As before, we have an embedding \( H' \hookrightarrow U(V) \times U(W) \). Then up to conjugation by the normalizer of \( H' \) in \( U(V) \times U(W) \), \( \nu_{\psi', \mu} \) is determined by \( \psi' \) modulo \( \eta \) and \( \mu \).

Let \( \pi \) (resp. \( \sigma \)) be an irreducible admissible representation of \( U(V) \) (resp. \( U(W) \)). A nontrivial element in \( \text{Hom}_{H'} \left( \pi \otimes \sigma \otimes \nu_{\psi', \mu}^{\wedge}, \mathbb{C}^\times \right) \) is called a Fourier-Jacobi model of \( \pi \otimes \sigma \). In particular, when \( k/k' \) is split, then the Fourier-Jacobi model is just the \((r,r)\)-Fourier-Jacobi model for general linear groups introduced in Section 3.1. We have the following multiplicity one result.

**Theorem 5.1.** Let \( k \) be a non-archimedean local field of characteristic zero and \( \pi, \sigma \) as above. Then \( \dim_{\mathbb{C}} \text{Hom}_{H'} \left( \pi \otimes \sigma \otimes \nu_{\psi', \mu}^{\wedge}, \mathbb{C}^\times \right) \) is equal to \( 0 \).

**Proof.** See [GGP, Section 14].

Now, we discuss the global case. Let \( k/k' \) be a quadratic extension of number fields, \( \psi' : k' \to \mathbb{C}^1 \) nontrivial, \( \mu : k^\times \to \mathbb{C}^\times \) such that \( \mu|_{k^\times} = \eta \), and \( \lambda' : U'_X(k) \times U'_X(\mathbb{A}) \to \mathbb{C}^1 \) a generic character. Then we have the pair \((H', \nu_{\psi', \mu})\) in the global situation. To define a global period, we need to fix a model for the Weil representation. Let \( L \subset (\text{Res}_{k'/k} W)^{\vee} \) be a Lagrangian subspace. Let \( \mathcal{S}(L(\mathbb{A}')) \) be the space of Bruhat-Schwartz functions on \( L(\mathbb{A}) \).

For \( \phi \in \mathcal{S}(L(\mathbb{A}')) \), we define the theta series to be
\[
\theta_{\psi', \lambda', \mu}(h', \phi) = \sum_{w \in L(\mathbb{A}')} \lambda'(h')(\omega_{\psi', \mu}(h')\phi)(w)
\]
which is an automorphic form on \( H' \).

Let \( \pi \) (resp. \( \sigma \)) be an irreducible tempered representation of \( U(V)(\mathbb{A}') \) (resp. \( U(W)(\mathbb{A}') \)) which occurs with multiplicity one in the space \( \mathcal{A}_0(U(V)) \) (resp. \( \mathcal{A}_0(U(W)) \)). We denote by \( \mathcal{A}_\pi \) (resp. \( \mathcal{A}_\sigma \)) the unique irreducible \( \pi \) (resp. \( \sigma \))-isotypic subspace in \( \mathcal{A}_0(U(V)) \) (resp. \( \mathcal{A}_0(U(W)) \)).
**Definition 5.2.** The following absolutely convergent integral is called a Fourier-Jacobi period of $\pi \otimes \sigma$ (for a pair $(H', \nu_{\psi', \mu})$):

$$\mathcal{F}_{r}^{\nu_{\psi', \mu}}(\varphi, \varphi; \phi) := \int_{H'(k')/H(k')} \varphi_{\pi}(h') \varphi_{\sigma}(h') \theta_{\varphi_{\pi}, \varphi_{\sigma}, \mu}(1; \phi) dh', \quad \varphi_{\pi} \in \mathcal{A}_{\pi}, \varphi_{\sigma} \in \mathcal{A}_{\sigma}, \phi \in \mathcal{S}(L(A'))$$

where $dh'$ is the Tamagawa measure on $H'(A')$. If there exist $\varphi_{\pi}, \varphi_{\sigma}, \phi$ such that $\mathcal{F}_{r}^{\nu_{\psi', \mu}}(\varphi, \varphi; \phi) \neq 0$, then we say $\pi \otimes \sigma$ has a nontrivial Fourier-Jacobi period.

It is obvious that $\mathcal{F}_{r}^{\nu_{\psi', \mu}}$ defines an element in

$$\Hom_{H'(k')}(\pi \otimes \sigma \otimes \nu_{\psi', \mu}, C) = \bigotimes_{\nu' \in \mathcal{M}_{r}} \Hom_{H'(k')} \left( \pi_{\nu'} \otimes \sigma_{\nu'} \otimes \nu_{\psi', \mu, \nu'}, C \right).$$

We choose a basis $\{v_{1}, ..., v_{r}\}$ or $X$ as in Section 4.1. We denote by $\{\tilde{v}_{1}, ..., \tilde{v}_{r}\}$ the dual basis of $\tilde{X}$. We also choose a basis $\{w_{1}, ..., w_{m}\}$ of $W$. Let $\beta = [(w_{i}, w_{j})]_{i,j=1}^{m} \in \text{Her}_{m}(k/k)^{\times}$. We identify $U(V)$ (resp. $U(W)$) with a unitary group of $n$ (resp. $m$) variables $U_{n}$ (resp. $U_{m}$) under the basis $\{v_{1}, ..., v_{r}, w_{1}, ..., w_{m}, \tilde{v}_{1}, ..., \tilde{v}_{r}\}$ and view $U_{m}$ as a subgroup of $U_{n}$. Let $U'_{r,m} = U_{r,m} \times U_{X}$ be a unipotent subgroup of $U_{n}$. Then the image of $H'(A')$ in $U_{n}(A')$ consists of the matrices $h' = h'(n', b'; u'; g') = \underline{u}'(n', b'; u'; g')$ where $g' \in U_{m}(A')$. Here,

$$\underline{u}'(n', b'; u') = \begin{bmatrix} 1_{r} & n' & w_{r}(b' + n'n_{\beta}^{-2}) \\ 1_{m} & n_{\beta} & 1_{r} \end{bmatrix} \begin{bmatrix} u'_{m} \\ 1_{m} \end{bmatrix} \in U'_{r,m}(A')$$

where $n' \in M_{r,m}(A)$, $b' \in \text{Her}_{r}(A/A')$, $u' \in U_{X}(A)$, $n'_{\beta} = -\beta^{-1} n'_{r} w_{r}$ and $\tilde{u}' = w_{r}^{-1} u'_{r}^{-1} w_{r}$. If $r > 0$, let $U^{1}$ be the maximal unipotent subgroup of the parabolic subgroup of $U(\{v_{r}, \tilde{v}_{r}\} \oplus W)$ stabilizing the flag $0 \subseteq \{v_{r}\}$. Let $H^{1} = U^{1} \times U(W)$ (resp. $H^{1} = U(W)$) if $r > 0$ (resp. $r = 0$). Then there is a map $H' \rightarrow H^{1}$. Write $h' = u' g'$ to be the image of $h'$ under this map. Then we have

$$\nu_{\psi', \mu}(h') = \psi'((\underline{u}') w'_{\psi', \mu}(h') = \psi' \left( \Tr(u'_{1,2} + \cdots + u'_{r-1,r}) \right) \omega_{\psi', \mu}(h')$$

**5.2 Decomposition of distributions**

This time, we start from the relative trace formula on general linear groups. We identify $GL_{m,k} \subset GL_{m,k}$ with $GL(W) \subset GL(V)$ and view $GL_{m,k'} \subset \text{Res}_{k/k'}(GL_{m,k})$ (res. $GL_{m,k'} \subset \text{Res}_{k/k'}(GL_{m,k})$) through the basis $\{v_{1}, ..., v_{r}, w_{1}, ..., w_{m}, v_{r}, ..., \tilde{v}_{1}\}$.

Recall that the Weil representation $\omega_{\psi, \mu}$ realizes on the space $\mathcal{S}(W^{\vee}(A))$ where $\psi = \psi' \circ \Tr$. We let

$$W_{\psi}^{\bullet} = \bigoplus_{i=1}^{r} k' w_{i}^{\psi}; \quad W_{\psi}^{\circ} = \bigoplus_{i=1}^{r} k' w_{i}^{\psi}; \quad W_{\psi} = \bigoplus_{i=1}^{r} k' w_{i}; \quad W_{\psi} = \bigoplus_{i=1}^{r} k' w_{i}$$

which are vector spaces over $k'$. Then $W = W_{\psi}^{\bullet} \oplus W_{\psi}^{\circ}, W_{\psi}^{\circ} = W_{\psi}^{\bullet} \oplus W_{\psi}^{\circ}$. We also let $W^{\dagger} = W_{\psi}^{\bullet} \oplus W_{\psi}^{\bullet}$ be a vector space over $k'$. We define a linear map from $\mathcal{S}(W^{\vee}(A))$ to $\mathcal{S}(W^{\dagger}(A))$ by $\Phi \mapsto \Phi^{\dagger}$, where

$$\Phi^{\dagger}(w_{\psi}^{\bullet}, w_{\psi}^{\bullet}) = \int W_{\psi}^{\circ} \Phi(w_{\psi}^{\bullet}, w_{\psi}^{\circ}) \psi(w_{\psi}^{\circ}(w_{\psi}^{\bullet})) dw_{\psi}^{\circ}$$

for $\Phi \in \mathcal{S}(W^{\vee}(A))$ and the self-dual measure on $W_{\psi}^{\circ}(A')$, which is an isomorphism.
If $r > 0$, let $U^\dagger$ be the maximal unipotent subgroup of the parabolic subgroup of $GL \{v_r, \tilde{v}_r \} \otimes W$ stabilizing the flag $0 \subset \{v_r \} \subset \{v_r, \tilde{v}_r \} \otimes W$. Let $H^1 = U^\dagger \times GL_m$ (resp. $H^1 = GL_m$) if $r > 0$ (resp. $r = 0$). Then there is a map $H \to H^1$. We write $h^\dagger = U^\dagger g = U^\dagger(n^\bullet, n^\triangledown, n^\star, n^\diamond, b^\dagger) g$ to be the image of $h$ under this map, where

\[
U^\dagger(n^\bullet, n^\triangledown, n^\star, n^\diamond, b^\dagger) = \begin{bmatrix}
1 & n^\triangledown + n^\star & b^\dagger \\
1_m & + & n^\diamond \\
1 & 1
\end{bmatrix}
\]

with $n^\bullet \in M_{1,m}(k')$, $n^\triangledown \in M_{1,m}(k^-)$, $n^\star \in M_{1,m}(k')$, $n^\diamond \in M_{1,m}(k^-)$.

We define a representation $\omega^{\dagger\mu}_{\varphi, \pi}$ of $H(\mathbb{A})$ on $S(W^\dagger(\mathbb{A}')$) by

\[
\omega^{\dagger\mu}_{\varphi, \pi}(h)\Phi^\dagger = \left(\omega^{\dagger\mu}_{\varphi, \pi}(h)\Phi\right)^\dagger.
\]

It is easy to see that $\omega^{\dagger\mu}_{\varphi, \pi}$ factors through $H^1$ and

\[
\left(\omega^{\dagger\mu}_{\varphi, \pi}(h)\Phi^\dagger\right)(w^\bullet, w^\star) = \eta(\det g)\overline{\varphi}(b^\dagger + w^\bullet n^\triangledown + n^\star w^\diamond)\Phi^\dagger((w^\bullet + n^\star)g, g^{-1}(w^\star - n^\bullet)) \quad (5.3)
\]

Moreover, we have the Poisson summation formula:

\[
\sum_{w^\dagger \in W^\dagger(k')} \Phi^\dagger(w^\dagger) = \sum_{w^\bullet \in W^\triangledown(k)} \Phi(w^\bullet).
\]

Until the end of this section, we assume that $n$ is odd, hence $m$ is also odd. Since the other case is similar and will lead to the same fundamental lemma, we omit it in the following discussion. We proceed exactly as in Section 4.2 and take $\mu$ to be the one used in (5.2) which is unitary. Let $F_n \in \mathcal{H}(GL_m(\mathbb{A}))$, $F_m \in \mathcal{H}(GL_m(\mathbb{A}))$ and $\Phi \in S(W^\dagger(\mathbb{A}))$. We introduce the following distribution

\[
\mathcal{J}^\mu(s; F_n \otimes F_m; \Phi) := \sum \mathcal{F}^{\mu}_{F_n}(s; \rho(F_n)\varphi_{\Pi}, \rho(F_m)\varphi_{\Sigma}; \Phi)\overline{\mathcal{P}(\varphi_{\Pi})}\overline{\mathcal{P}(\varphi_{\Sigma})}
\]

where the sum is taken over orthonormal bases of $A_{\Pi}$ and $A_{\Sigma}$.

We associate to $F_n \otimes F_m$ a kernel function $K_{F_n \otimes F_m}(g_1, g_2; g_3, g_4)$ and consider the distribution

\[
\mathcal{J}^\mu(s; F_n \otimes F_m; \Phi) = \int_{Z_m(k')GL_m(k')} \int_{Z_m(k')GL_m(k')} \int_{GL_m(k')} \int_{GL_m(k')} K_{F_n \otimes F_m}(\varepsilon(h), \kappa(h); g_1, g_2)\theta_{\omega, \varphi, \pi}(h, \Phi) \det h_\mathbb{A}^{s - \frac{1}{2}} dh \frac{dg_1}{g_1} \frac{dg_2}{g_2}. \quad (5.4)
\]

Proceeding similarly as in (4.10), we have

\[
(5.4) = \int_{g_2} \int_{g_1} \int_{U_{1,r,-1}(k)} \int_{U_{1,r,-1}(\mathbb{A})} \int_{GL_m(k)} \int_{GL_m(k')} \int_{GL_m(k')} \sum_{\zeta \in GL_m(k)} F_n(g^{-1} g_2^{-1} \frac{dg_1 + \zeta^{-1} \zeta^{-1} g_1 F_m(g^{-1}) \theta_{\omega, \varphi, \pi}(u g_2 g, \Phi)}{dg_1} \frac{dg_1}{g_2})^{1/2} \frac{dg_1}{g_2} \frac{dg_2}{g_2} \frac{dg_1}{g_2} \frac{dg_2}{g_2}.
\]

\[
(5.5) = \int_{GL_m(k')} \int_{GL_m(k')} \int_{U_{1,r,-1}(k)} \int_{U_{1,r,-1}(\mathbb{A})} \int_{GL_m(k)} \int_{GL_m(k')} \sum_{\zeta \in GL_m(k')} \sigma(F_n)(g^{-1} g_2^{-1} \frac{dg_1 + \zeta^{-1} \zeta^{-1} g_2 g^{-1}}{dg_1} F_m(g^{-1}) \theta_{\omega, \varphi, \pi}(u g_2 g, \Phi)}{dg_1} \frac{dg_1}{g_2} \frac{dg_1}{g_2} \frac{dg_1}{g_2} \frac{dg_2}{g_2}.
\]
Unfolding \( U_{1^r,m,1^r}(k) \), we have

\[
(5.5) = \sum_{\zeta \in [S_n(k') \cup U_{1^r,m,1^r}(k)] \setminus GL_m(k') \setminus GL_m(H)} \int_{(k') \setminus GL_m(H)} \int_{(k') \setminus GL_m(H)} \int_{(k') \setminus GL_m(H)}
\]

\[
s(\sigma(F_n)(g^{-1}g_2) \Phi) = \sum_{\zeta \in [S_n(k') \cup U_{1^r,m,1^r}(k)] \setminus GL_m(k') \setminus GL_m(H)} \int_{(k') \setminus GL_m(H)} \int_{(k') \setminus GL_m(H)} \int_{(k') \setminus GL_m(H)}
\]

But we have

\[
\sigma(F_n)(g^{-1}g_2) = \sum_{\zeta \in [S_n(k') \cup U_{1^r,m,1^r}(k)] \setminus GL_m(k') \setminus GL_m(H)} \int_{(k') \setminus GL_m(H)} \int_{(k') \setminus GL_m(H)} \int_{(k') \setminus GL_m(H)}
\]

To proceed, we introduce a \( k' \)-variety

\[
S_{n,m} = S_n \times M_{1,m,k'} \times M_{m,1,k'}.
\]

As before, we let \( H = \text{Res}_{k'/k}(U_{1^r,m,1^r}) \times GL_m(k') \) which acts on \( S_{n,m} \) in the following way: for any \( k' \)-algebra \( R \), \( h = h(u,g) \) with \( u \in U_{1^r,m,1^r}(R \otimes k) \), \( g \in GL_m(R) \) and \( [s, x, y] \in S_{n,m}(R) \), we define a right action \( [s, x, y]h = [g^{-1}u^{-1}su^tg, xg, yg^{-1}] \). We also define \( \psi(h) = \psi(u^{-1}) \) and \( \det h = \det g \). Then

\[
(5.6) = \sum_{[s, x, y] \in [S_{n,m}(k')] \setminus GL_m(k')} \int_{(k') \setminus GL_m(k')} \int_{(k') \setminus GL_m(k')} \int_{(k') \setminus GL_m(k')} \]

We denote by \( S_{n,m}(k') \) the set of all regular \( k' \)-elements which will be defined in Section 5.3. In particular, the \( H \)-stabilizer \( \text{Stab}(H) \) is trivial for \( [x, y] \) regular and the corresponding term

\[
\mathcal{O}_\mu(s) = \mathcal{J}_\mu(s, F_n \otimes F_m; \Phi) \]

is a weighted orbital integral. If \( F_n = \bigotimes_{v'} F_{n,v'} \), \( F_m = \bigotimes_{v'} F_{m,v'} \) and \( \Phi = \bigotimes_{v'} \Phi_{v'} \) are factorizable, then

\[
\mathcal{O}_\mu(s, F_n \otimes F_m; \Phi) = \prod_{v' \in M_{k'}} \mathcal{O}_{\mu,v'}(s, F_{n,v'} \otimes F_{m,v'}; \Phi_{v'}, [x, y])
\]

where the local orbital integrals are defined similarly as in (5.7). In summary, we have

\[
\mathcal{J}_\mu(s, F_n \otimes F_m; \Phi) - \mathcal{J}_{\mu,v'}(s, F_{n,v'} \otimes F_{m,v'}; \Phi) = \prod_{v' \in M_{k'}} \mathcal{O}_{\mu,v'}(s, F_{n,v'} \otimes F_{m,v'}; \Phi_{v'}, [x, y]).
\]
When $s = \frac{1}{2}$, we discard $s$ in all notations. In particular,

$$
\mathcal{J}_x^s(F_n \otimes F_m; \Phi) = \mathcal{O}_\mu(F_n \otimes F_m; \Phi, [\zeta, x, y]) = \int \int \sigma(F_n) (g^{-1} [\zeta] h g^{-1}) F_m(g^{-1}) \left( \omega_{s, \mu}(h) \Phi^\dagger \right) (x, y) \psi(h) dh dh
$$

(5.8)

when $[\zeta, x, y]$ is regular.

Now we describe the relative trace formula for unitary groups. Let $f_n \in H(U_n(\mathbb{A}'))$, $f_m \in H(U_m(\mathbb{A}'))$ and $\phi_1 \in \mathcal{S}(\mathbb{A}')$ for $i = 1, 2$. We introduce a distribution

$$
\mathcal{J}_{x, \mu}^s(f_n \otimes f_m; \phi_1 \otimes \phi_2) := \sum \mathcal{J}_{x, \mu}^s(f_n \otimes f_m; \phi_1 \otimes \phi_2) \mathcal{K}_{f_n \otimes f_m}
$$

where the sum is taken over orthonormal bases of $\mathcal{A}_x$ and $\mathcal{A}_x$. As usual, we have a kernel function $K_{f_n \otimes f_m}$. Let

$$
\mathcal{J}_{x, \mu}^s(f_n \otimes f_m; \phi_1 \otimes \phi_2) := \int \int \mathcal{K}_{f_n \otimes f_m}(\varepsilon(h_1), \kappa(h_1), \varepsilon(h_2), \kappa(h_2)) \theta_{s, \mu}(h_1, \phi_1) \theta_{s, \mu}(h_2, \phi_2) dh_1 dh_2
$$

(5.9)

Collapsing the summation over $\xi'$ and changing variable $g_2^{-1} g_1' = g_1'$, then

$$
\mathcal{J}_{x, \mu}^s(f_n \otimes f_m; \phi_1 \otimes \phi_2) := \int \int \mathcal{K}_{f_n \otimes f_m}(\varepsilon(h_1), \kappa(h_1), \varepsilon(h_2), \kappa(h_2)) \theta_{s, \mu}(h_1, \phi_1) \theta_{s, \mu}(h_2, \phi_2) dh_1 dh_2
$$

(5.10)

Recall that we define an $k'$-algebraic group $H'$ in Section 4.2 which acts on $U_{n, m} := U_n \times \text{Res}_{k'/k} \cdot M_{1, m, k}$ in the following way: for any $k'$-algebra $R$, $h' = h'(u_1', u_2'; g') \in H'(R)$ and $[g, z] \in U_n(R) \times M_{1, m}(R \otimes k)$, we define the right action $[g, z] h' = [g'^{-1} u_1^{-1} g u_2', z g']$. We also define

$$
\psi'(h') = \psi'(h'(u_1', u_2'; g')) := \psi'(u_1'^{-1} u_2')
$$

and $\text{det} h' = \text{det} g'$. There is a map $H' \rightarrow H^\dagger$, where

$$
H^\dagger = H^\dagger \times U_m
$$

(resp. $U_m$) if $r > 0$ (resp. $r = 0$). We denote by $h^\dagger = h^\dagger(u_1^\dagger, u_2^\dagger; g')$ the image of $h' = h'(u_1', u_2'; g')$ under the above map, where

$$
u_1^\dagger = u_1^\dagger(n_1^\dagger, b_1^\dagger) = \begin{bmatrix} 1 & n_1^\dagger b_1^\dagger & -n_1^\dagger b_1^\dagger n_1^\dagger & \beta^{-1} v_1^\dagger \tau \\ 1 & n_1^\dagger b_1^\dagger & -n_1^\dagger b_1^\dagger n_1^\dagger & \beta^{-1} v_1^\dagger \tau \\ 1 & n_1^\dagger b_1^\dagger & -n_1^\dagger b_1^\dagger n_1^\dagger & \beta^{-1} v_1^\dagger \tau \\ 1 & n_1^\dagger b_1^\dagger & -n_1^\dagger b_1^\dagger n_1^\dagger & \beta^{-1} v_1^\dagger \tau \\ \end{bmatrix}
$$

for $n_1^\dagger \in W^\dagger(\mathbb{A})$ and $b_1^\dagger \in \mathbb{A}^-$. 

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Lemma 5.3. Let $\omega_{\psi^{'},\mu}$ be the tensor product $\omega_{\psi^{'}} \otimes \omega_{\psi^{'},\mu}$ viewed as a smooth representation of $H'$, then it factors through $H' \to H$. We have an intertwining isomorphism

$$-1 : (S(L(A')))^{\otimes 2} \to S \left( (\text{Res}_{k'/k} W)^{\vee} (A') \right) = S(W^{\vee} (A))$$

such that:

1. For $\phi \in S(W^{\vee} (A))$,

$$\left( \omega_{\psi^{'},\mu}(h') \phi \right) (z) = \left( \omega_{\psi^{'}}(h') \phi \right) (z) = \left( \omega_{\psi^{'}} \left( h' (u_m, w; g') \right) \phi \right) (z)$$

$$= \frac{1}{b_1^2 - b_2^2 + z\beta^{-1} \frac{1}{n_1^2} + \frac{1}{n_2^2}} \phi \left( \frac{z + n_1^2 + n_2^2}{2} g' \right)$$

(5.11)

which does not depend on $\mu$, hence justifying the notation.

2. We have

$$\theta_{\psi^{'},\mu} (u_m' g', \phi_1) \theta_{\psi^{'},\mu} (w' g', \phi_2) = \psi' (h') \sum_{z \in W^{\vee} (k)} \left( \omega_{\psi^{'},\mu}(h') \left( \phi_1 \otimes \phi_2 \right) \right) (z)$$

where $h' = h' (u_m', w', g')$.

Proof. The isomorphism is by [HKS96, Proposition 2.2 (i), (ii)]. Actually one can construct an explicit intertwining operator by a (partial) Fourier transform, which implies (1) and (2) by the Poisson summation formula.

By the above lemma and repeating the process in (5.5), (5.6), (5.7), we have

$$\tag{5.10} \sum_{[\kappa', z] \in [U_{n,m}(k')] / H'} \int_{H^{\vee} (k')} \int_{H^{\vee} (k')} f_n (g'^{-1} [\kappa'] h') f_m (g'^{-1}) \left( \omega_{\psi^{'},\mu}(h') \left( \omega_{\psi^{'},\mu}(g') \phi_1 \otimes \phi_2 \right) \right) (z) \psi' (h') dg' dh'$$

$$= \sum_{[\kappa', z] \in [U_{n,m}(k')] / H^{\vee} (k')} J^{\psi^{'},\mu}_{[\kappa', z]} (f_n \otimes f_m; \phi_1 \otimes \phi_2).$$

(5.12)

We denote by $U_{n,m}(k')_{\text{reg}}$ the set of all regular $k'$-elements which will be defined in Section 5.3. In particular, the $H'$-stabilizer $H^{\vee} (k')_{\text{Stab}}$ is trivial for $[\kappa', z]$ regular and the corresponding term

$$O_{\psi^{'},\mu} (f_n \otimes f_m; \phi \otimes \phi_2, [\kappa', z]) := J^{\psi^{'},\mu}_{[\kappa', z]} (f_n \otimes f_m; \phi_1 \otimes \phi_2)$$

is a weighted orbital integral. If $f_n = \otimes_{v'} f_{n,v'}$, $f_m = \otimes_{v'} f_{m,v'}$ and $\phi = \otimes_{v'} \phi_{1,v'}$, $\varphi$ are factorizable, then

$$O_{\psi^{'},\mu} (f_n \otimes f_m; \phi_1 \otimes \phi_2, [\kappa', z]) = \prod_{v' \in M_{k'}} O_{\psi^{'},\mu_{v'}} (f_{n,v'} \otimes f_{m,v'}; \phi_{1,v'} \otimes \phi_{2,v'}, [\kappa', z])$$

where the local orbital integrals are defined similarly as in (5.12). In summary, we have

$$J^{\psi^{'},\mu} (f_n \otimes f_m; \phi_1 \otimes \phi_2) - J^{\psi^{'},\mu}_{[\kappa', z]} (f_n \otimes f_m; \phi_1 \otimes \phi_2)$$

$$= \sum_{[\kappa', z] \in [U_{n,m}(k')]_{\text{reg}} / H^{\vee} (k')} \prod_{v' \in M_{k'}} O_{\psi^{'},\mu_{v'}} (f_{n,v'} \otimes f_{m,v'}; \phi_{1,v'} \otimes \phi_{2,v'}, [\kappa', z]).$$

In general, if $\phi = \sum_i \phi^{(i)}_1 \otimes \phi^{(i)}_2$ is a finite sum, then we simply define the distribution $J$ and the weighted orbital integral $O$ as the sum of the corresponding terms defined above.
5.3 Matching of orbits and functions

We first consider the following orbital problem which is a little bit different from the Jacquet-Rallis case, the one considered in Section 4.3.

We define the affine space \( \mathcal{M}_m := M_m \times M_{1,m} \times M_{m,1} \).

**Definition 5.4.** An element \([\xi, x, y]\in \mathcal{M}_m(k)\) is called regular if it satisfies:

- \(\xi\) is regular semisimple as an element of \(M_m(k)\);
- the vectors \(\{x, x\xi, \ldots, x\xi^{m-1}\}\) span the \(k\)-vector space \(M_{1,m}(k)\);
- the vectors \(\{y, \xi y, \ldots, \xi^{m-1} y\}\) span the \(k\)-vector space \(M_{m,1}(k)\).

Let \(a_i([\xi, x, y]) = \text{Tr} \left( \wedge^i \xi \right) \) for \(1 \leq i \leq m\), \(b_i([\xi, x, y]) = x\xi^i y\) for \(0 \leq i \leq m - 1\),

\[
T_{[\xi, x, y]} = \det \begin{bmatrix} x & x\xi & \cdots & x\xi^{m-1} \end{bmatrix},
\]

(5.13)

\(D_{[\xi, x, y]}\) be the matrix \(\left[ x\xi^{i+j} - 2 \right]_{m+1}^{m+1}\) and \(\Delta_{[\xi, x, y]} = \det D_{[\xi, x, y]}\). It is clear that \(\Delta_{[\xi, x, y]} \neq 0\) if \([\xi, x, y]\) is regular. The group \(\text{GL}_m\) acts on \(\mathcal{M}_m\) from right side by \([\xi, x, y]g = [g^{-1}\xi g, xg, g^{-1} y]\) and \(a_i, b_i, D_{[\xi, x, y]}\) and \(\Delta_{[\xi, x, y]}\) are all invariants under this action. We denote by \(\mathcal{M}_m(k)_{\text{reg}}\) the set of all regular elements.

We have

**Lemma 5.5.** Two regular elements \([\xi, x, y]\) and \([\xi', x', y']\) are in the same \(\text{GL}_m(k)\)-orbit if and only if they have the same invariants \(a_i\) and \(b_i\). The \(\text{GL}_m\)-stabilizer of a regular element is trivial.

**Proof.** See [RS07, Proposition 6.2 & Theorem 6.1] for the first and second statements, respectively. \(\square\)

We define two more spaces:

\[
\mathcal{S}_m = \{[\xi, x, y] \in \mathcal{M}_m(k)_{\text{reg}} \mid \xi \in S_m(k'), x \in M_{1,m}(k'), y \in M_{m,1}(k')\};
\]

\[
\mathcal{U}_m^\beta = \{[\beta; \xi, z, z^*] \mid \beta \in \text{Her}_m(k/k')^\times, [\xi, z, z^*] \in \mathcal{M}_m(k)_{\text{reg}}, \xi \in U_m^\beta(k'), z^* = \beta^{-1} \tau z\tau\}
\]

where \(U_m^\beta = U(W^\beta)\). For \(\mathcal{U}_m^\beta\), we also define a right \(\text{GL}_m(k)\)-action by \([\beta; \xi, z, z^*]g = [g^{-1}\beta g; g^{-1}\xi g, zg, g^{-1}z^*]\).

For \([\xi, x, y] \in \mathcal{S}_m\), we denote by \([\xi, x, y] \Leftrightarrow [\beta; \xi\beta, z, z^*]\) if there exists \(g \in \text{GL}_m(k)\) such that \([\xi, x, y] = [\xi\beta, z, z^*]g\). We have the following lemma which is similar to Lemma 4.10:

**Lemma 5.6.** For \([\xi, x, y] \in \mathcal{S}_m\), there exists an element \([\beta; \xi, z, z^*]\) \(\in \mathcal{U}_m^\beta\), unique up to the \(\text{GL}_m(k)\)-action, such that \([\xi, x, y] \Leftrightarrow [\beta; \xi\beta, z, z^*]\). Conversely, for any \([\beta; \xi\beta, z, z^*]\) \(\in \mathcal{U}_m^\beta\), there exists an element \([\xi, x, y] \in \mathcal{S}_m\), unique up to the \(\text{GL}_m(k)\)-action, such that \([\xi, x, y] \Leftrightarrow [\beta; \xi\beta, z, z^*]\).

**Proof.** We first point out that \([\xi, x, y], [\xi', x', y'] \in \mathcal{S}_m\) are conjugate under \(\text{GL}_m(k)\) if and only if they are conjugate under \(\text{GL}_m(k')\). Assume that \([\xi, x, y]g = [\xi', x', y']\), then \(g^{-1}\xi g = \xi'\); \(xg = x'\) implies that \(xg^{-1} = x'g^{-1}\); \(g^{-1}y = y'\) implies that \(g^{-1}y^{-1} = y^{-1}\), hence \(g = g^{-1}\).

It is easy to see that for \([\xi, x, y] \in \mathcal{M}(k)_{\text{reg}}, [\xi, x, y]\) and \([\xi, 1, x, 1]\) have the same invariants, hence there is a unique \(g \in \text{GL}_m(k)\) such that \(g^{-1}\xi g = \xi, xg = x, g^{-1}y = y\). Now if \([\xi, x, y] \in \mathcal{S}_m\), then \([\xi, 1, x, 1] \in \mathcal{S}_m\), which implies that \(g = g^{-1}\).

Moreover, we have \(g = g'\) and \(\xi\xi = \xi\). Hence \(g^{-1} \in \text{Her}_m(k/k')^\times\) and \(\xi \in U_m^\beta(k')\). We also have \(g = (g^{-1})^{-1} = g^{-1}\), which means that \([g^{-1}; \xi, x, y] \in \mathcal{U}_m^\beta\) and \([\xi, x, y] \Leftrightarrow [\xi, x, y]\).

Conversely, given any \([\beta; \xi, z, z^*] \in \mathcal{U}_m^\beta\). Since \(1\beta \xi \xi = \beta\xi, \beta^{-1} \xi \beta\xi = \xi^{-1}\), we have

\[
[\xi, 1, z^*, 1, z, \beta\xi] = [\xi^{-1}, z^*, (z^*)^{-1}]
\]
since $z^* = \beta^{-1} z^\tau$. Moreover, since $[\xi, z, z^*]$ and $[\xi, 1 z^*, 1 z]$ have the same invariants, there exists $\gamma \in \text{GL}_m(k)$ such that $[\xi, z, z^*] \gamma = [\xi, 1 z^*, 1 z]$, hence

$$[\xi, z, z^*] (\gamma \beta^*) = [\xi^{\tau^{-1}}, z^\tau, (z^*)^\tau]$$

which implies that $\gamma \beta^* \in S_m(k')$, hence $\gamma \beta^* = g g^{-1} z^\tau$ for some $g \in \text{GL}_m(k)$. Then

$$(g^{-1} \xi g) (g^{-1} \xi g)^{-1} = 1_m$$

and $z g g^{-1} z^\tau = z^\tau$ implies $z g = (z g)^{-1} g^{-1} z^* = (z^*)^\tau$ implies $g^{-1} z^* = (g^{-1} z^*)^\tau$. In all, $[\xi, z, z^*] g = [g^{-1} \xi g, z g, g^{-1} z^*] \in \mathfrak{S}_m$. The uniqueness is obvious.

Recall that we denote $[\text{Her}_m(k/k)^\times]$ the set of similarity classes. We write $W = W^\beta$, $V = V^\beta$ if the representing matrix of $W$ is in the class $\beta$, and $U^\beta_m$ (resp. $U^\beta_n$, $U^\beta_{n,m}$, $H^\beta$) for $U(W^\beta)$ (resp. $U(V^\beta)$, $U(V^\beta) \times \text{Res}_{k/k} M_{1,m,k}$, $H^\beta$). We have

**Definition 5.7.** An element $[\zeta, x, y] \in S_{n,m}(k')$ (resp. $[\zeta^\beta, z] \in U^\beta_{n,m}(k')$) is called pre-regular if the stabilizer of $\zeta$ (resp. $\zeta^\beta$) under the action of $\text{Res}_{k/k} U_{1,m,1'}$ (resp. $(U_{1,m,1'})^2$) is trivial.

Applying Lemma 4.5 to $P = P_{1,m,1'}$, the $U_{1,m,1'}(k)$-orbit of $\zeta$ for which $[\zeta, x, y] \in \mathfrak{S}_m$ must contain a unique element of the form (4.18) with $t_i(\zeta) \in k^\times$ and $\text{Pr}(\zeta) \in S_m(k')$. We call the triple $[\text{Pr}(\zeta), x, y] \in \mathfrak{M}_m(k)$ the normal form of $[\zeta, x, y]$. Applying Lemma 4.6 to $P' = P_{1',m'}$, the $(U_{1',m'})^2$-orbit of $\zeta^\beta$ for which $[\zeta^\beta, z] \in \mathfrak{S}_m$ must contain a unique element of the form (4.20) with $t_i(\zeta^\beta) \in k^\times$ and $\text{Pr}(\zeta^\beta) \in U^\beta_{n,m}(k')$. We call the quadruple $[\beta, \text{Pr}(\zeta^\beta), z, z^*] \in \text{Her}_m(k/k)^\times \times \mathfrak{M}_m(k)$ the normal form of $[\zeta^\beta, z]$.

**Definition 5.8.** An element $[\zeta, x, y] \in S_{n,m}(k')$ (resp. $[\zeta^\beta, z] \in U^\beta_{n,m}(k')$) is called regular if it is pre-regular and its normal form $[\text{Pr}(\zeta), x, y] \in \mathfrak{S}_m$ (resp. $[\beta; \text{Pr}(\zeta^\beta), z, z^*] \in U^\beta_{n,m}$). We have the notions $S_{n,m}(k')_{\text{reg}}$, $U^\beta_{n,m}(k')_{\text{reg}}$ for the sets of regular elements.

As before, we have the following proposition whose proof we omit.

**Proposition 5.9.** (1) There is a natural bijection

$$S_{n,m}(k')_{\text{reg}} / H(k') \leftrightarrow \prod_{\beta \in \text{Her}_m(k/k)^\times} U^\beta_{n,m}(k') / H^\beta(k').$$

If $N[\zeta^\beta, z] = [\zeta, x, y]$, then we say that $\zeta$ and $\beta$ match and denote by $[\zeta, x, y] \leftrightarrow [\zeta^\beta, z]$.

(2) The set $S_{n,m}(k')_{\text{reg}}$ (resp. $U^\beta_{n,m}(k')_{\text{reg}}$) is non-empty and Zariski open in $S_{n,m}$ (resp. $U^\beta_{n,m}$). Moreover, the $H$-stabilizer (resp. $H^\beta$-stabilizer) of regular $[\zeta, x, y]$ (resp. $[\zeta^\beta, z]$) is trivial.
By (5.3) and (5.8), we have

\[
O_{\mu', (F_{n,v'} \otimes F_{m,v'}; \Phi_{v'}, [\zeta, x, y])} = \int \int \int \int \sigma(F_{n,v'})(g^{-1} u^{-1} \zeta u g_0) F_{m,v'}(g_0) d\mu_{n,v'}(g_0^{-1})
\]

where

\[
\mu'(u) = \psi'(j (u_{1,2} + \cdots + u_{r-1,r} + u_{r-1,r} + \cdots + u_{2,1})); \quad j = (j, -j)
\]

and

\[
u = \begin{bmatrix}
1 & u_{1,2} & \cdots & n_{1,1} & \cdots & n_{1,m} & b_{1,1} & \cdots & b_{1,1} \\
\vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
1 & n_{r,1} & \cdots & n_{r,m} & b_{r,1} & \cdots & b_{r,1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & u_{r,r-1} & \cdots & n_{r,r-1} & n_{r} & \cdots & n_{r} \\
1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1
\end{bmatrix};
\]

\[n_r = \begin{bmatrix}
n_{r,1} & \cdots & n_{r,r} 
\end{bmatrix}; \quad n_r^* = \begin{bmatrix}
n_{1,r}^* & \cdots & n_{r,r}^* 
\end{bmatrix}.
\]

On the other hand, by (5.11) and (5.12), we have

\[
O_{\psi_{\mu'}, \mu'} (f_{n,v'} \otimes f_{m,v'}; \phi_{1,v'} \otimes \phi_{2,v'}, [\zeta, (x, y)\beta_o]) = \int \int \int \int f_{n,v'}(g_0^{-1} u^{-1} \zeta u g_0) f_{m,v'}(g_0^{-1})
\]

where

\[
\psi'(u) = \psi'(j (u_{1,2} + \cdots + u_{r-1,r} + u_{r-1,r} + \cdots + u_{2,1})); \quad j = (j, -j)
\]

and

\[
u = \begin{bmatrix}
1 & u_{1,2} & \cdots & n_{1,1} & \cdots & n_{1,m} & b_{1,1} & \cdots & b_{1,1} \\
\vdots & \ddots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\
1 & n_{r,1} & \cdots & n_{r,m} & b_{r,1} & \cdots & b_{r,1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & u_{r,r-1} & \cdots & n_{r,r-1} & n_{r} & \cdots & n_{r} \\
1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1
\end{bmatrix};
\]

\[n_r = \begin{bmatrix}
n_{r,1} & \cdots & n_{r,r} 
\end{bmatrix}; \quad n_r^* = \begin{bmatrix}
n_{1,r}^* & \cdots & n_{r,r}^* 
\end{bmatrix}.
\]

To compare (5.14) and (5.15), we need to invoke the original models of the Weil representations. In particular

\[
\psi'(u) = \mu((\det g_0^{-1} \zeta) \det g^*)_v \psi'((x + z)g_0, (x - z)g_0) \psi'((jyz)dz)
\]

where \(\mu = (\mu, \mu^{-1})\) with abuse of notation. If we identify L\(_{v'}\) with M\(_{1,m,v}\) (possibly through a Fourier transform), we have

\[
\psi'(u) = \mu((\det g_0^{-1} \zeta) \det g^*)_v \psi'((x + z)g_0, (x - z)\psi'((jyz)dz).
\]
Now if we suppose that $\Phi_{v'} = \Phi_1 \otimes \Phi_2$ is factorizable with respect to the two components. Let $F_{m,v} = F_{m,v'}$, $\Phi_1 = \phi_{1,v}$ and assume that the following function on $(h,w) \in \text{GL}_{n,v} \times M_{1,m,v'}$:

$$\int_{\text{GL}_{n,v}} \sigma(F_{n,v})(hg_o) F_{m,v}(g_o^{-1}) \Phi_2(wg_o) \mu(\det g_o) \det g_o | \det g_o|^{\frac{1}{2}} dg_o$$

is equal to $\sum_i f^{(i)}_{m,v} \otimes \phi^{(i)}_{2,v'}$, then

$$O_{\mu}(F_{m,v} \otimes F_{m,v'}; \Phi_1 \otimes \Phi_2, \eta) = \sum_i O_{\psi_i', \mu}(f^{(i)}_{m,v} \otimes f_{m,v'}; \phi_{1,v'} \otimes \phi^{(i)}_{2,v'}, (\zeta, (x, y \beta, z)))$$

hence the smooth matching (of functions) holds!

Remark 5.10. As we see in the above calculation, there are several differences in the case of Fourier-Jacobi periods:

- The data of test functions involve Schwartz functions on the groups as well as on the linear spaces;
- Even for the split case, smooth matching is not obvious and we need to choose linear combination of functions to make corresponding weighted orbital integrals equal.

For all almost all split places $v'$ where everything is unramified and the test functions are the characteristic functions on corresponding maximal compact subgroups or lattices, then the two orbital integrals are equal.

Inspired by the split case, we conjecture that, similar to Conjecture 4.12, the smooth matching of functions holds for all places $v'$. We omit the explicit form of this conjecture in the current case.

### 5.4 The fundamental lemma

We now state the fundamental lemma for the Fourier-Jacobi periods. We use all the notations in the beginning of Section 4.4, except that we define

$$t([\zeta, x, y]) = \begin{cases} (-1)^{\text{vol}(T_{[\zeta, x, y]})} \prod_{i=1}^{t} t_i(\zeta) & \text{if } m \text{ is even;} \\ (-1)^{\text{vol}(T_{[\zeta, x, y]})} & \text{if } m \text{ is odd.} \end{cases}$$

(5.16)

For simplicity, we only consider the fundamental lemma for unit elements which is

**Conjecture 5.11 (The fundamental lemma).** Assume that $k/k'$, $\psi'$, $\mu$ are unramified and $j \in \mathfrak{o}$. Then we have

$$O_{\mu}(\mathbb{I}_{S_n}(\mathfrak{o}'); \mathbb{I}_{M_{1,m}}(\mathfrak{o}') \otimes \mathbb{I}_{M_{1,m}}(\mathfrak{o}'), [\zeta, x, y])$$

$$= \begin{cases} t([\zeta, x, y])O_{\psi', \mu}(\mathbb{I}_{U^+_n}(\mathfrak{o}'); \mathbb{I}_{M_{1,m}}(\mathfrak{o}), [\zeta^+, z]) & [\zeta, x, y] \leftrightarrow [\zeta^+, z] \in U^+_{n,m}(k'); \\
0 & [\zeta, x, y] \leftrightarrow [\zeta^-, z] \in U^-_{n,m}(k') \end{cases}$$

where

$$O_{\mu}(\mathbb{I}_{S_n}(\mathfrak{o}'); \mathbb{I}_{M_{1,m}}(\mathfrak{o}') \otimes \mathbb{I}_{M_{1,m}}(\mathfrak{o}'), [\zeta, x, y]) = \int \mathbb{I}_{S_n}(\mathfrak{o}')(\zeta, h) \left( \sum_{\psi, \eta} (\mathbb{I}_{M_{1,m}}(\mathfrak{o}') \otimes \mathbb{I}_{M_{1,m}}(\mathfrak{o}')) \right) (x, y) \psi(h) dh;$$

$$O_{\psi', \mu}(\mathbb{I}_{U^+_n}(\mathfrak{o}'); \mathbb{I}_{M_{1,m}}(\mathfrak{o}), [\zeta^+, z]) = \int \mathbb{I}_{U^+_n}(\mathfrak{o}')(\zeta^+, h') \left( \sum_{\psi', \eta'} (\mathbb{I}_{M_{1,m}}(\mathfrak{o}') \otimes \mathbb{I}_{M_{1,m}}(\mathfrak{o}')) \right) (z) \psi'(h') dh'.$$

It is easy to see that $[\zeta, x, y] \leftrightarrow [\zeta^+, z] \in U^+_{n,m}(k')$ if and only if $\text{val}(\Delta_{[\zeta, x, y]})$ is even.
In particular, when \( n = m \), the above orbital integrals become the following much simpler ones

\[
\mathcal{O}_\mu(\|S_n(x')\| \|M_{n,1}(x')\|, [\zeta, x, y]) = \int_{\text{GL}_n(k')} \|S_n(x')(g^{-1}\zeta g)\| \|M_{n,1}(x')(g^{-1}y)\| \eta(\det g) \, dg;
\]

\[
\mathcal{O}_{\varphi', \mu}(\|U_n^+(x')\| \|M_{n,1}(x')\|, [\zeta', z]) = \int_{U_n^+(k')} \|U_n^+(x')(g^{-1}\zeta' g')\| \|M_{n,1}(x')(zg')\| \, dg'.
\]

**Proposition 5.12.** Let \( n = m \) or \( m = 0 \), if \( \text{val}(\Delta_{(\zeta, x, y)}) \) is odd, then

\[
\mathcal{O}_\mu(\|S_n(x')\| \|M_{n,1}(x')\|, [\zeta, x, y]) = 0.
\]

**Proof.** It follows from the similar argument in the proof of Proposition 4.15 by noticing that \( \|M_{n,1}(x')(x) = \|M_{n,1}(x')(t^x) \). \( \square \)

It seems that, when \( n > m \), the above vanishing result is not that easy to proof.

### 5.5 Proof of the fundamental lemma for \( U_n \times U_n \)

In this section, we prove the fundamental lemma for the case \( n = m \) in positive characteristics. The proof uses a similar idea of relating orbiting integrals to certain problems of counting lattices, following [Yun09], and then reduces to an identity which has already been proved by Yun. In other word, the fundamental lemmas for both the case \( n = m \) and \( n = m + 1 \) (the Jacquet-Rallis case) will be implied by the same identity.

Let \( k' \) be a \( p \)-adic local field with ring of integers \( \mathfrak{o}' \), uniformizer \( \varpi \), and \( q = |\mathfrak{o}'/\varpi\mathfrak{o}'| \). Let \( k/k' \) be an unramified quadratic field extension with ring of integers \( \mathfrak{o} \) and \( 0 \neq j \in \mathfrak{o} \) with \( j^2 = -j \). Let \( \text{val} \) be the valuation on \( k^\times \) normalized such that \( \text{val}(\varpi) = 1 \). For two full rank \( \mathfrak{o} \)-lattice \( \Lambda_1, \Lambda_2 \) in some finite dimensional \( k \)-vector space \( V \), we define

\[
\text{leng}_\mathfrak{o}(\Lambda_1 : \Lambda_2) := \text{leng}_\mathfrak{o}(\Lambda_1/\Lambda_1 \cap \Lambda_2) - \text{leng}_\mathfrak{o}(\Lambda_2/\Lambda_2 \cap \Lambda_2)
\]

where for \( \Lambda_1 \supset \Lambda_2 \),

\[
\text{leng}_\mathfrak{o}(\Lambda_1 : \Lambda_2) = \frac{|\Lambda_1/\Lambda_2|}{|\mathfrak{o}/\varpi\mathfrak{o}|} = \frac{|\Lambda_1/\Lambda_2|}{q^2}.
\]

We have also the same notion for \( \mathfrak{o}' \) (but of course replacing \( q^2 \) by \( q \)).

Let us first review some constructions in [Yun09, Section 2]. Let \( n \geq 1 \), for any pair \((a, b)\) with \( a = (a_i), b = (b_i) \in \mathfrak{o}^n \) and \( a_n \in \mathfrak{o}^\times \), we define an \( \mathfrak{o} \)-algebra

\[
\mathbf{R}_{a, \mathfrak{o}} = \mathfrak{o}[t, t^{-1}]/(t^n - a_1 t^{n-1} + \cdots + (-1)^n a_n).
\]

Hence \( Z_a' := \text{Spec} \mathbf{R}_{a, \mathfrak{o}} \) is a subscheme of \( \text{Spec} \mathfrak{o} \times \mathbb{G}_m \) which is finite flat over \( \text{Spec} \mathfrak{o} \) of degree \( n \). Let \( \mathbf{R}_{a, \mathfrak{o}}' = \text{Hom}_\mathfrak{o}(\mathbf{R}_{a, \mathfrak{o}}, \mathfrak{o}) \) and define an element \( b \in \mathbf{R}_{a, \mathfrak{o}}' \) from the datum \( b \) by the formula

\[
b : \mathbf{R}_{a, \mathfrak{o}} \longrightarrow \mathfrak{o}
\]

\[
t^i \mapsto b_i, \quad i = 0, \ldots, n - 1.
\]

It induces an \( \mathbf{R}_{a, \mathfrak{o}} \)-linear homomorphism \( \gamma_{a, b} : \mathbf{R}_{a, \mathfrak{o}} \rightarrow \mathbf{R}_{a, \mathfrak{o}}' \) by the pairing

\[
\mathbf{R}_{a, \mathfrak{o}} \otimes \mathbf{R}_{a, \mathfrak{o}} \longrightarrow \mathfrak{o}
\]

\[
(u, v) \mapsto b(uv).
\]
Let \( \vartheta \) be the involution on \( \text{Res}_{\mathfrak{o}/\mathfrak{o}'}(\text{Spec} \, \mathfrak{o} \times \mathbb{G}_m) \) which is the product of \( \tau \) on \( \text{Spec} \, \mathfrak{o} \) and the involution \( t \mapsto t^{-1} \) on \( \mathbb{G}_m \). The subscheme invariant under \( \vartheta \) is the unitary group (scheme) \( U_{1,1}/\mathfrak{o}' \), finite flat of degree \( n \) over Spec \( \mathfrak{o}' \). Let \( R_n \) be the coordinate ring of \( Z_n \) which is a finite flat \( \mathfrak{o}' \)-algebra of rank \( n \) such that \( R_n \otimes_{\mathfrak{o}'} \mathfrak{o} = R_n \otimes_{\mathfrak{o}'} \mathfrak{o}' \). The \( R_n \otimes_{\mathfrak{o}'} \mathfrak{o} \)-linear map \( \gamma_{a,b} \) descends to an \( R_n \)-linear map \( \gamma_{a,b} : R_n \to R_n \), where \( R_n = \text{Hom}_{\mathfrak{o}'}(R_n, \mathfrak{o}') \).

Let us assume that \( \Delta_{[\xi, x, y]} \neq 0 \), then the image of \( \gamma_{a,b} \) is co-finite in \( R_n \). If we identify \( R_n \) as a submodule of \( R_n \), then \( \text{length}_{\mathfrak{o}'}(R_n : R_n) = \text{val} (\Delta_{[\xi, x, y]}) \). For each \( 0 \leq i \leq \text{val} (\Delta_{[\xi, x, y]}) \), let

\[
M_{i, a, b} := \{ R_n \text{-lattices } \Lambda \mid R_n \subset \Lambda \subset R_n \text{ and } \text{length}_{\mathfrak{o}'}(R_n : \Lambda) = i \}.
\]

We remark that for the orbital integral \( O_n( \mathbb{H}_{M_{i}}(\mathfrak{o}') \otimes \mathbb{H}_{M_{n,1}}(\mathfrak{o}'), [\xi, x, y] ) \) to be nonzero, there must be an element locating in \( S_n(\mathfrak{o}') \times M_{i,n}(\mathfrak{o}') \times M_{n,1}(\mathfrak{o}') \) in the GL\( (k') \)-orbit of \( [\xi, x, y] \). Then we have

**Proposition 5.13.** Let \( [\xi, x, y] \in S_n(\mathfrak{o}') \times M_{i,n}(\mathfrak{o}') \times M_{n,1}(\mathfrak{o}') \) be regular and hence such that \( \text{val} (\Delta_{[\xi, x, y]}) \geq 0 \), then the orbital integral

\[
O_n( \mathbb{H}_{S_n}(\mathfrak{o}') \otimes \mathbb{H}_{M_{i,n}}(\mathfrak{o}') \otimes \mathbb{H}_{M_{n,1}}(\mathfrak{o}'), [\xi, x, y] ) = \sum_{i=0}^{\text{val}(\Delta_{[\xi, x, y]})} (-1)^i |M_{i,a,b}|
\]

where \( T_{[\xi, x, y]} \) is defined in (5.13).

**Proof.** Let \( V = M_{i,n}(\mathfrak{o}') \) be the \( \mathfrak{o}' \)-module and we identify \( V \) with \( M_{i,n}(\mathfrak{o}') \) by matrix multiplication. Then \( V = V(k) := V \otimes_{\mathfrak{o}'} k \). Recall that

\[
O_n( \mathbb{H}_{S_n}(\mathfrak{o}') \otimes \mathbb{H}_{M_{i,n}}(\mathfrak{o}') \otimes \mathbb{H}_{M_{n,1}}(\mathfrak{o}'), [\xi, x, y] ) = \int_{\text{GL}_n(k')} \mathbb{H}_{S_n}(\mathfrak{o}') \mathbb{H}_{M_{i,n}}(\mathfrak{o}') \mathbb{H}_{M_{n,1}}(\mathfrak{o}') (g^{-1} \zeta g) \mathbb{H}_{M_{i,n}}(\mathfrak{o}') (xg) \mathbb{H}_{M_{n,1}}(\mathfrak{o}') (g^{-1} y) \eta(\text{det} g) \, dg
\]

and the measure is the one such that \( \text{GL}_n(\mathfrak{o}') \) gets volume 1. We define

\[
X_{[\xi, x, y]}^i := \{ g \in \text{GL}_n(k') \mid g^{-1} \zeta g \in S_n(\mathfrak{o}'), xg \in V, \text{det} g = i \};
\]

\[
Y_{[\xi, x, y]}^i := \{ \text{\( \mathfrak{o}' \)-lattice } L \subset V(k') \mid \zeta L \subset \mathfrak{o} L, x, y \in L, \text{length}_{\mathfrak{o}'}(L : V) = i \}
\]

where \( L' = \{ v \in V(k') \mid \eta(v) \in \mathfrak{o}' \text{ for any } \mathfrak{o}' \in L \} \). There is a bijection \( X_{[\xi, x, y]}^i \sim \to Y_{[\xi, x, y]}^i \) given by \( g \mapsto g v \).

On the other hand, we define an \( k \)-linear map

\[
y' : R_n(k) \to V(k)
\]

\[
t' \mapsto \zeta' y
\]

which is bijective since \( [\xi, x, y] \) is regular. Here, \( R_n(k) \) is the underlying \( k \)-module of the \( k \)-algebra \( R_n \otimes_{\mathfrak{o}'} k \). By the definition of \( \gamma_{a,b} \), the following \( k \)-linear map

\[
x' = \gamma_{a,b} \circ (y'^{y'})^{-1} : R_n(k) \to R_n(k) \to V(k)
\]

is given by \( t' \mapsto x \zeta' t' \). It is clear that \( y' \) (resp. \( x' \)) descends to a \( k' \)-linear map \( y : R_n(k') \to V(k') \) (resp. \( x : R_n(k') \to V(k') \)).

For any \( L \in Y_{[\xi, x, y]}^i \), we claim that \( y^{-1}(L) \in M_{\text{val}(T_{[\xi, x, y]})-a,b} \). In fact, \( y^{-1}(L) \) is a lattice stable under \( R_n \) by the construction. Hence we only need to show that \( R_n \subset y^{-1}(L) \subset R_n \). Since \( y \in L \), \( y^{-1}(y) = 1_{R_n} \in y^{-1}(L) \), where \( 1_{R_n} \) is the identity element of the algebra \( 1_{R_n} \). Hence

\[
R_n = 1_{R_n} \cdot 1_{R_n} \subset 1_{R_n} \cdot y^{-1}(L) \subset y^{-1}(L).
\]
Similarly, since \( x \in L' \), \( x^{-1}(x) = 1_{\mathbb{R}_a} \in x^{-1}(L') = (y^{-1}(L))^\vee \). But \((y^{-1}(L))^\vee \) is also stable under \( \mathbb{R}_a \) which implies that
\[
\mathbb{R}_a = \mathbb{R}_a \cdot 1_{\mathbb{R}_a} \subset \mathbb{R}_a \cdot (y^{-1}(L))^\vee \subset (y^{-1}(L))^\vee
\]
i.e., \( y^{-1}(L) \subset \mathbb{R}_a^\vee \). Moreover
\[
\text{length}_{\mathcal{O}}(\mathbb{R}_a^\vee : y^{-1}(L)) = \text{length}_{\mathcal{O}}(\mathbb{R}_a^\vee : y^{-1}(V)) = \text{length}_{\mathcal{O}}(L : V) = \text{val}(T_{(\zeta,x,y)}) - i.
\]
Conversely, for any \( \Lambda \in \text{M}_{\text{val}(T_{(\zeta,x,y)}) - i} \), let \( L = y(\Lambda) \). Then \( \mathbb{R}_a \cdot \Lambda = (\Lambda \otimes \mathcal{O}) \otimes \mathcal{O} \subset \Lambda \otimes \mathcal{O} \), hence \( \zeta L \subset \mathcal{O} L \). The fact \( \mathbb{R}_a \subset \Lambda \) implies that \( y = y(1_{\mathbb{R}_a}) \in L \) and the fact \( \Lambda \subset \mathbb{R}_a^\vee \) implies that \( x = x(1_{\mathbb{R}_a}) \in L' \). The length part is clear. Hence, we prove the claim.

Then
\[
(5.19) = \sum_{i=0}^{\text{val}(\Delta_{[\zeta,x,y]})} (-1)^i \left| X_{i}^{[\zeta,x,y]} \right| = \sum_{i=0}^{\text{val}(\Delta_{[\zeta,x,y]})} (-1)^i \left| Y_{i}^{[\zeta,x,y]} \right|
\]

\[= (-1)^{\text{val}(\text{val}(T_{(\zeta,x,y)}))} \sum_{i=0}^{\text{val}(\Delta_{[\zeta,x,y]})} (-1)^i |M_{i,a,b}|. \]

Now we consider the orbital integral on the unitary group. We fix an element \( \beta^+ \in \text{Her}_n(k) \cap \text{GL}_n(\mathcal{O}) \) which defines a unitary group scheme \( U^+_n, \mathcal{O} / \mathbb{R} \), whose generic fibre \( U^+_{n,\mathbb{R}(\mathcal{O})} \times k' \cong U^+_n \) viewed as a subgroup of \( \text{Res}_{k'/k} \text{GL}_{n,b} \). Now we simply view \( U^+_n \) as defined over \( \mathcal{O}' \). For \([\zeta^+, z] \in U^+_n(\mathcal{O}') \times M_{1,n}(\mathcal{O})\), we recall that \( z^+ = (\beta^+)^{-1} z^+ \in M_{n,1}(\mathcal{O}) \) and the invariants \((a, b) \in \mathcal{O}^{2n}) \). Then the corresponding subscheme \( Z^+_a = \text{Spec} \mathbb{R}_a,\mathcal{O} \) is also stable under \( \vartheta \), hence determining a subscheme \( Z^+_a \) of \( U_{1,\mathcal{O} / \mathcal{O}} \). Let \( \mathbb{R}_a \) be the coordinate ring of \( Z^+_a \). We have an \( \mathbb{R}_a,\mathcal{O} \)-linear map \( \gamma_{a,b} : \mathbb{R}_a,\mathcal{O} \rightarrow \mathbb{R}_a^\vee,\mathcal{O} \) identifying \( \mathbb{R}_a,\mathcal{O} \) as a submodule of \( \mathbb{R}_a^\vee,\mathcal{O} \).

We define the following morphism between \( k \)-modules
\[
z^+ : \mathbb{R}_a(k) \longrightarrow V(k)
\]
\[
t^i \mapsto (\zeta^+)^i z^+
\]
which is an isomorphism if \([\zeta^+, z] \) is regular, which we will assume in the following discussion. The hermitian form on \( V(k) \) defined by \( \beta^+ \) induces the following hermitian form on \( \mathbb{R}_a(k) \) through \( z^+ \):
\[
\langle u, v \rangle_{\mathbb{R}} = b(uv^0)
\]
where \( b : \mathbb{R}_a(k) \rightarrow k \) is defined in (5.17).

For an \( \mathcal{O} \)-lattice \( \Lambda^+ \subset \mathbb{R}_a(k) \), the dual lattice under \((-,-)_{\mathbb{R}} \) is the \( \mathcal{O} \)-lattice
\[
\left( \Lambda^+ \right)^\vee := \left\{ v \in \mathbb{R}_a(k) \mid \langle v, \Lambda^+ \rangle_{\mathbb{R}} \subset \mathcal{O} \right\}.
\]
We call \( \Lambda^+ \) self-dual if \( (\Lambda^+)^\vee = \Lambda^+ \). We define
\[
\text{N}_{a,b} := \left\{ \text{self-dual } R_{a,b,\mathcal{O}} \text{-lattice } \Lambda^+ \mid \mathbb{R}_a,\mathcal{O} \subset \Lambda^+ \subset \mathbb{R}_a^\vee,\mathcal{O} \right\}.
\]
We remark that for the orbital integral \( O_{\psi,\mu}(\mathbb{R}_{U_n}(\mathcal{O}'); \mathbb{R}_{M_{1,n}}(\mathcal{O}'), [\zeta^+, z]) \) to be nonzero, there must be an element locating in \( U^+_{n}(\mathcal{O}') \times M_{1,n}(\mathcal{O}) \) in the \( U^+_{n}(k') \)-orbit of \([\zeta^+, z] \). Then we have

**Proposition 5.14.** Let \([\zeta^+, z] \in U^+_{n}(\mathcal{O}') \times M_{1,n}(\mathcal{O}) \) be regular and hence such that \( \text{val}(\Delta_{[\zeta^+, z]}) \geq 0 \) is even, then the orbital integral
\[
O_{\psi,\mu}(\mathbb{R}_{U_n}(\mathcal{O}'); \mathbb{R}_{M_{1,n}}(\mathcal{O}'), [\zeta^+, z]) = |N_{a,b}|.
\]

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Proof. Recall that
\[ O_{\phi',\mu}(\mathbb{I}_{\mathbb{Z}_{\phi}}; \mathbb{I}_{\mathbb{M}_{\mathbb{a}}(\phi')} \mathbb{I}_{\mathbb{M}_{\mathbb{a}}(\phi')}, [\zeta^+, z]) = \int_{U^+_n(k')} \mathbb{I}_{\mathbb{Z}_{\phi}}(g^{t-1}\zeta^+ g') \mathbb{I}_{\mathbb{M}_{\mathbb{a}}(\phi)}(zg') dg' \] (5.20)
and the measure is the one such that \( U^+_n(\phi') \) gets volume 1. We define
\[ X_{[\zeta^+, z]} = \{ g' \in U^+_n(k')/U^+_n(\phi') \mid g^{t-1}\zeta^+ g' \in U^+_n(\phi'), g^{-1}z^+ \in V(\phi) \}; \]
\[ Y_{[\zeta^+, z]} = \{ \text{self-dual } \mathfrak{o} \text{-lattice } L^+ \subset V(k) \mid \zeta^+ L^+ \subset \mathfrak{o} L^+, z^+ \in L^+ \}. \]
There is a bijection \( X_{[\zeta^+, z]} \overset{\sim}{\to} Y_{[\zeta^+, z]} \) given by \( g' \mapsto g'V(\phi) \). We have another bijection \( Y_{[\zeta^+, z]} \overset{\sim}{\to} N_{a,b} \) given by \( L^+ \mapsto (z^+)^{-1} (L^+) \) whose proof is similar to that in Proposition 5.13. Hence we have
\[ (5.20) = |X_{[\zeta^+, z]}| = |Y_{[\zeta^+, z]}| = |N_{a,b}|. \]

\[ \square \]

**Theorem 5.15.** If \( \text{char}(k) = p > \max \{ n, 2 \} \), the fundamental lemma for \( U_n \times U_n \) (cf. Conjecture 5.11), i.e., the following identity
\[ O_{\mu}(\mathbb{I}_{\mathbb{S}_{\phi}}; \mathbb{I}_{\mathbb{M}_{\mathbb{a}}(\phi')} \mathbb{I}_{\mathbb{M}_{\mathbb{a}}(\phi')}, [\zeta, x, y]) = \begin{cases} (-1)^{\text{val}(T_{\zeta,x,y})}O_{\phi',\mu}(\mathbb{I}_{\mathbb{Z}_{\phi}}; \mathbb{I}_{\mathbb{M}_{\mathbb{a}}(\phi')} \mathbb{I}_{\mathbb{M}_{\mathbb{a}}(\phi')}, [\zeta^+, z]) & [\zeta, x, y] \leftrightarrow [\zeta^+, z] \in U^+_n(k'); \\
0 & [\zeta, x, y] \leftrightarrow [\zeta^-, z] \in U^-_n(k') \end{cases} \]
holds for \([\zeta, x, y] \) regular.

**Proof.** The second case of the above identity has already been proved in Proposition 5.12. But Proposition 5.13 leads to another proof by noticing that, first \( \text{val} (\Delta_{[\zeta,x,y]}) \) is odd and \([\zeta, x, y] \) satisfies the assumption in that proposition (otherwise, the orbital integral will be 0 automatically), and second \( \Lambda \mapsto \Lambda' \) induces a bijection \( M_{i,a,b} \overset{\sim}{\to} M_{\text{val}(\Delta_{[\zeta,x,y]})-i,a,b} \).

For the first part, we need to prove the identity
\[ \sum_{i=0}^{\text{val}(\Delta_{a,b})} (-1)^i |M_{i,a,b}| = |N_{a,b}| \] (5.21)
assuming that \([\zeta, x, y] \) (resp. \([\zeta^+, z] \)) satisfies the assumption in Proposition 5.13 (resp. 5.14) and they have the same invariants \((a, b)\). Here, \( \Delta_{a,b} \) is the determinant of the map \( \gamma_{a,b} \) under the basis \( \{1, t, \ldots, t^{n-1}\} \) of \( R_\mathfrak{a}(k) \) and the dual basis of \( R_\mathfrak{a}'(k) \) and hence \( \text{val}(\Delta_{a,b}) = \text{val}(\Delta_{[\zeta,x,y]}) \). We remark that the equality (5.21) is a property purely of the \( \mathfrak{a}' \)-algebra \( R_\mathfrak{a} \) and the \( \mathfrak{a}' \)-linear map \( \gamma_{a,b} : R_\mathfrak{a} \to R_\mathfrak{a}' \).

By the argument in [Yun09, Proposition 2.6.1], we can find \( \bar{a}_i, \bar{b}_j \in j' \phi' \) and an isomorphism of \( \phi' \)-algebras \( \rho : R_{\mathfrak{a}} \overset{\sim}{\to} R_{\mathfrak{a}}' \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
R_{\mathfrak{a}} & \overset{\gamma_{\mathfrak{a}, \mathfrak{b}}}{\longrightarrow} & R_{\mathfrak{a}}' \\
\rho \downarrow & & \downarrow \\
R_{\mathfrak{a}} & \overset{\gamma_{\mathfrak{a}, \mathfrak{b}}}{\longrightarrow} & R_{\mathfrak{a}}'
\end{array}
\]

where the algebra \( R_{\mathfrak{a}} \) and the map \( \gamma_{\mathfrak{a}, \mathfrak{b}} \) are defined in [Yun09, Remark 2.2.5]. Since \([\zeta, x, y] \) is regular which means in particular that \( R_{\mathfrak{a}} \otimes k \) is an étale \( k \)-algebra, hence \( R_{\mathfrak{a}} \otimes k \) is also étale. Applying the fundamental result [Yun09, Corollary 2.7.2] (and also their notations) to \( R_{\mathfrak{a}} \) and \( \gamma_{\mathfrak{a}, \mathfrak{b}} \), we have that
\[ \sum_{i=0}^{\text{val}(\Delta_{a,b})} (-1)^i |M_{i,a,b}| = |N_{a,b}| \] (5.21)
for \( \text{char}(k) = p > \max \{ n, 2 \} \), which implies (5.21) immediately.

\[ \square \]
Remark 5.16. If char(k) = 0 and p is sufficiently large with respect to n, the transfer principle in [Yun09, Appendix] should also apply to our case and hence imply the fundamental lemma in characteristic 0.

6 Appendix: A brief summary on local Whittaker integrals

In the appendix, we summarize some facts about certain integrals of local Whittaker functions which will be used in this paper. All the results are contained in [JS81], [JPSS83], [JS90], [CPS04] and [J09].

Let $k$ be a local field, $\psi : k \to \mathbb{C}$ be a nontrivial character. We denote $| \cdot | = | \cdot |_k$, $M_{r,s} = M_{r,s}(k)$. Let $\pi$ (resp. $\sigma$) be an irreducible admissible representation of $GL_m = GL_m(k)$ (resp. $GL_m = GL_m(k)$). Let $W(\psi) = \text{Ind}_{U_{1,m}}^{GL_m}(\psi)$ be the space of all smooth functions $W(g)$ on $GL_m$ satisfying $W(ug) = \psi(u)W(g) := \psi(u_{1,2} + \cdots + u_{n-1,n})W(g)$ for all $u = (u_{ij}) \in U_1$, the group of upper-triangular matrices with all 1 on the diagonal. It is a smooth representation of $GL_m$ by right translation. Let $V_\pi$ be the space where $\pi$ realizes. If $k$ is archimedean, then we take $V_\pi$ as the canonical Casselman-Wallach completion of the corresponding Harish-Chandra module of $\pi$. A fundamental theorem of Gelfand-Kazhdan and Shalika posits that there is at most one $GL_n$-equivariant map, up to a constant multiple, from $V_\pi$ to $W(\psi)$. If it exists, then we say $\pi$ is generic. Being generic is independent of $\psi$ we choose. Same arguments apply to $\sigma$. In what follows, we will assume that $\pi$ and $\sigma$ are generic. We denote by $W(\pi, \psi)$ (resp. $W(\sigma, \psi)$) the nontrivial image of $V_\pi$ (resp. $V_\sigma$) in $W(\psi)$ (resp. $W(\psi)$). Moreover,

$$W(\pi, \psi) = \{ \tilde{W}(g) := W(w_n \cdot g^{-1}) \mid W \in W(\pi, \psi) \}$$

where $w_n = \begin{bmatrix} \vdots \end{bmatrix}$ is the longest Weyl element of $GL_m$. Moreover, we let $e_m = [0, \ldots, 0, 1] \in M_{1,m}$ and $w_{n,m} = \begin{bmatrix} \vdots \end{bmatrix}$.

Let $W \in W(\pi, \sigma)$, $W^- \in W(\sigma, \psi)$ and $\Phi \in \mathcal{S}(M_{1,m})$, we consider the following kinds of integrals

- For $n > m$ and $0 \leq r \leq n - m - 1$,

$$\Psi_r(s; W, W^-) = \int_{U_{1,m} \backslash GL_m M_{r,m}} \int W \left( \begin{bmatrix} g & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) W^-(g) \det g^{s - \frac{n-m}{2}} \, dg \, dx \, dy$$

(6.1)

- For $n > m$ and $1 \leq r \leq n - m$,

$$\Psi_r(s; W, W^-; \Phi) = \int_{U_{1,m} \backslash GL_m M_{r-1,m} M_{1,m}} \int \int W \left( \begin{bmatrix} g & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \right) W^-(g) \Phi(y) \det g^{s - \frac{n-m}{2}} \, dy \, dx \, dg$$

(6.2)

- For $n \geq m$,

$$\Psi_0(s; W, W^-; \Phi) = \int_{U_{1,m} \backslash GL_m} W \left( \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \right) W^-(g) \Phi(e_m g) \det g^{s - \frac{n-m}{2}} \, dg$$

(6.3)

We denote by

$$\mathcal{I}_r(\pi \times \sigma) = \{ \Psi_r(s; W, W^-) \mid W \in W(\pi, \sigma), W^- \in W(\sigma, \psi) \}; \quad 0 \leq r \leq n - m - 1$$

$$\mathcal{I}_0^+(\pi \times \sigma) = \{ \Psi_r(s; W, W^-; \Phi) \mid W \in W(\pi, \sigma), W^- \in W(\sigma, \psi), \Phi \in \mathcal{S}(M_{1,m}) \}; \quad 0 \leq r \leq n - m$$

which are linear spaces over $\mathbb{C}$.
Proposition 6.1 ([JPSS83], [JS90], [CPS04], [J09]). We only state the following results for $k$ non-archimedean, while the statement for $k$ archimedean can be found, for example, in [J09, Section 2]. Let $\pi$ and $\sigma$ be as above and $\chi_\sigma$ the central character of $\sigma$, then:
(1) Each element in $I_r(\pi \times \sigma)$ and $I^*_r(\pi \times \sigma)$ is absolutely convergent for $\Re(s)$ large and has a meromorphic continuation to the entire complex plane;
(2) There exists a unique function $L(s, \pi \times \sigma)$ of the form $P(q^{-s})^{-1}$ where $P \in \mathbb{C}[X]$ and $q$ is the cardinality of the residue field of $k$, such that
$$I_r(\pi \times \sigma) = I^*_r(\pi \times \sigma) = L(s, \pi \times \sigma) \mathbb{C}[q^{-s}, q^s]$$
for any $r$; in particular, for any $r$ and $s_0 \in \mathbb{C}$, there exist $W$, $W^-$ and possibly $\Phi$ such that $\Psi_r(s; W, W^-)/L(s, \pi \times \sigma)|_{s=s_0}$ or $\Psi_r(s; W, W^-; \Phi)/L(s, \pi \times \sigma)|_{s=s_0}$ is in $\mathbb{C}^\times$.
(3) There is a factor $\epsilon(s, \pi \times \sigma, \psi)$, only depending on $\pi$, $\sigma$ and $\psi$, of the form $q^{-fs}$ such that
$$\frac{\Psi_{n-m-1+r}(1-s; \widetilde{W}, \widetilde{W}^-)}{L(1-s, \pi \times \sigma)} = \chi_\sigma(-1)^{n-1}\epsilon(s, \pi \times \sigma, \psi) \frac{\Psi_r(s; W, W^-)}{L(s, \pi \times \sigma)}$$
for $n > m$ and
$$\frac{\Psi_{n-m-r}(1-s; \widetilde{W}, \widetilde{W}^-; \Phi)}{L(1-s, \pi \times \sigma)} = \chi_\sigma(-1)^{n-1}\epsilon(s, \pi \times \sigma, \psi) \frac{\Psi_r(s; W, W^-; \Phi)}{L(s, \pi \times \sigma)}$$
for $n \geq m$, where $\hat{\Phi}$ is the $\psi$-Fourier transform of $\Phi$:
$$\hat{\Phi}(y) = \int_{M_{1,m}} \Phi(x)\psi(x^t y)dx$$
with the self-dual measure $dx$.

Proof. The proof of these statements can be found in the literature mentioned above, except (6.4) when $n > m$. For completeness, we will give a proof of (6.4) when $n > m$ below, following [JPSS83].

By the functional equation, we can assume that $0 \leq r < n - m$. We let
$$W_1 = \int_{M_{1,m}} \rho \left( \begin{array}{ccc} 1_m & u & 0 \\ 1_r & 1 & 0 \\ 1 & 1 & 1_{n-m-1-r} \end{array} \right) W(x^{-1} u)$$
which is in $W(\pi, \psi)$. Then $\Psi_r(s; W_1, W^-) = \Psi_r(s; W, W^-; \Phi)$. To prove (6.4), we only need to prove that $\Psi_{n-m-1-r}(s; \rho(w_{n,m})W_1, W^-) = \Psi_{n-m-r}(s; \rho(w_{n,m})W, W^-; \Phi)$ which is true by the Fourier inverse formula.

When $k$ is archimedean or the representations $\pi$ and $\sigma$ are unramified, then its $L$-factor $L(s, \pi \boxtimes \sigma)$ and $\epsilon$-factor $\epsilon(s, \pi \boxtimes \sigma, \psi)$ are defined using the Langlands parameter. We have the following.

Proposition 6.2 ([JS81], [JS90], [CPS04], [J09]). (1) If $k$ is archimedean, then $L(s, \pi \times \sigma) = L(s, \pi \boxtimes \sigma)$ and $\epsilon(s, \pi \times \sigma, \psi) = \epsilon(s, \pi \boxtimes \sigma, \psi)$;
(2) If $k$ is non-archimedean with $\mathfrak{o}$ its ring of integers, let $\pi$ (resp. $\sigma$) be an unramified representation associated to a semisimple conjugacy class $\mathcal{A}_\pi \subset \text{GL}_m(\mathbb{C})$ (resp. $\mathcal{A}_\sigma \subset \text{GL}_m(\mathbb{C})$). Let $W_\mathfrak{o}$ (resp. $W_{\sigma, \mathfrak{o}}$) be the unique $GL_n(\mathfrak{o})$- (resp. $GL_m(\mathfrak{o})$-) fixed Whittaker functions such that $W(1_n) = 1$ (resp. $W^-(1_m) = 1$) and $\Phi_\mathfrak{o}$ be the characteristic function of $M_{1,m}(\mathfrak{o})$, then
$$\Psi_r(s; W_\mathfrak{o}, W^-) = \Psi_r(s; W_\sigma, W^-; \Phi_\mathfrak{o}) = \det (1 - q^{-s}A_\pi \otimes A_\sigma)^{-1} = L(s, \pi \times \sigma) = L(s, \pi \boxtimes \sigma)$$
for any possible $r$.  

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Proof. In (2), the proof for the integral $\Psi_0(s; W_\sigma, W_\nu^*)$ when $n > m$, and for the integral $\Psi_0(s; W_\sigma, W_\nu^*; \Phi_0)$ when $n = m$ can be found in [JS81]. The rest will follow easily as in Proposition 6.1.

Corollary 6.3. (1) When $n > m$, the Whittaker integral (6.1) defines a nonzero element in $\text{Hom}_H(\pi \otimes \sigma, \nu)$, hence an $(r, n-m-1-r)$-Bessel model for $\pi, \sigma$ generic, by choosing a suitable basis like in Section 2.2.

(2) When $n > m$ and $r > 0$ (resp. $n \geq m$ and $r = 0$), the Whittaker integral (6.2) (resp. (6.3)) (with $\sigma$ replaced by $\sigma \otimes \mu^{-1}$) defines a nonzero element in $\text{Hom}_H(\pi \otimes \sigma \otimes \mu^*, C)$, hence an $(r, n-m-r)$- (resp. $(0, n-m)$-) Fourier-Jacobi model for $\pi, \sigma$ generic, by choosing a suitable basis like in Section 3.2.

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