Lower estimates for the number of closed trajectories of generalized billiards

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1 Introduction

Closed billiard trajectories is a classical object first considered by George Birkhoff. A billiard is motion of a particle inside some domain when field of force is lacking, then the particle moves along a geodesic line and rebounds from the domain’s boundary making the angle of incidence be equal to the angle of reflection. Closed trajectories of such a motion are connected with different areas of mathematics. For example, closed billiard trajectories correspond to closed geodesics of the following space: we take two copies of the given domain and glue corresponding points of the boundaries. One can obtain an another example noticing that the minimal number of closed billiard trajectories is an invariant of a knot or, say, of a plane curve.

George Birkhoff stated and solved the following problem in [1]: given an integer \( k \), estimate from below the number of closed billiard trajectories with exactly \( k \) rebounds. More precisely, he proved that if \( k > 2 \) is an integer and \( T \subset \mathbb{R}^2 \) is a strictly convex domain, then there exist at least \( \varphi(k) \) closed billiard trajectories with exactly \( k \) rebounds. Here \( \varphi(k) \) is Euler’s function that is an amount of coprime with \( k \) integers not exceeding \( k \).

The billiard ball rebounds in this theorem from the boundary of two-dimensional domain that topologically is a circle embedded in the Euclidean plane. It’s not hard to replace this circle with an arbitrary manifold embedded in a Euclidean space of any dimension. In fact, the billiard ball rebounding from something of codimension greater than 1 can do it in infinitely many directions, but only finitely many of them make the ball get back to the manifold.

Many mathematicians tried to estimate the number of closed billiard trajectories. Morse theory was applied to this problem by Morse himself. In
Morse investigated the simplest case: a manifold is an \( m \)-sphere and closed billiard trajectories consist of only two points. The best known estimate for the number of closed billiard trajectories with two rebounds was found by P. Pushkar in [4].

Estimates for the number of closed trajectories of an arbitrary period for billiards in multi-dimensional convex domains were proved by M. Farber and S. Tabachnikov in [5] and [6].

In our paper [7] a general estimate for the number of closed trajectories of period 3 was obtained. Unfortunately, the paper [7] contains an error noticed by M. Farber and S. Tabachnikov: in fact, the estimate was proved there only for manifolds lying in a boundary of some strictly convex domain.

The main goal of the present paper is, first, to correct that mistake (lemma 3.1). Besides, we give a general estimate for closed trajectories of any prime period, when the billiard ball rebounds from an arbitrary submanifold of a Euclidean space (theorem 1).

2 Preliminaries

Let \( M \) be a smooth closed connected \( m \)-dimensional manifold embedded in the Euclidean space \( \mathbb{R}^n \) (so \( m < n \)), \( k \geq 2 \) an integer. The dihedral group \( D_k \) acts on \( M \)'s Cartesian power \( M^{\times k} \), this action is given by the cyclic permutation

\[
(x_1, x_2, \ldots, x_k) \rightarrow (x_2, x_3, \ldots, x_k, x_1)
\]

and the reflection

\[
(x_1, x_2, \ldots, x_k) \rightarrow (x_k, x_{k-1}, \ldots, x_1).
\]

The cyclic permutation corresponds to the fact that a closed polygon may be considered starting from any of its vertices, while the reflection means that the direction can be reversed. In fact, closed polygons are points of the quotient space \( M^{\times k}/D_k \).

**Definition** An ordered set of points \((x_1, \ldots, x_k) \in M^{\times k}\) considered up to the action of the dihedral group \( D_k \) is said to be a closed (periodic, or \( k \)-periodic) billiard trajectory if for any cyclic index \( i \) (we mean \( i = i + k \)) the following conditions hold:

1. \( x_i \neq x_{i+1} \),
Note that the second of these conditions is the same as the angle of incidence equals the angle of reflection.

Let us introduce the following notation:

\[
\tilde{\Delta} = \bigcup_{i \in \mathbb{Z}_k} \{x_i = x_{i+1}\} \subset M^{\times k},
\]

\[
\Delta = \tilde{\Delta} / D_k
\]

is the diagonal consisting of all the closed polygons, at least one of whose segments vanishes. We see that a closed billiard trajectory is a point of the space \((M^{\times k} \setminus \tilde{\Delta}) / D_k = (M^{\times k} / D_k) \setminus \Delta\).

In order to emphasize that \(\Delta\) consists of closed polygons with \(k\) segments, we’ll write \(\Delta_k\). If we need to accentuate that \(\Delta\) consists of closed polygons with vertices belonged to the manifold \(M\), we’ll write \(\Delta_M\).

Let

\[
l = \sum_{i \in \mathbb{Z}_k} \| x_i - x_{i+1} \| : M^{\times k} \to \mathbb{R}
\]

be the length function of a closed polygon, all of whose vertices lie on the manifold \(M\). Obviously, the function \(l\) is smooth outside the diagonal \(\tilde{\Delta}\). It’s easy to see that \(l\) is invariant under the action of the dihedral group \(D_k\), so it essentially is a function on the quotient space \(M^{\times k} / D_k\).

It is well known that closed billiard trajectories with \(k\) segments (or, more exactly, their inverse images under the natural projection \(M^{\times k} \to M^{\times k} / D_k\)) are exactly the critical points of the function \(l\) outside of the diagonal \(\tilde{\Delta}\).

**Definition** A embedding \(M \to \mathbb{R}^n\) is *generic* (or, more precisely, \(k\)-*generic*) if all the critical points of all the functions \(l_{k'}\) with \(k' < k\) outside of the corresponding diagonals \(\tilde{\Delta}_{k'}\) are non-degenerate.\(^1\)

Thus our problem is to estimate the minimal number \(BT_p(M)\) of critical points of the function \(l\) for all generic embeddings \(M \to \mathbb{R}^n\) (with fixed \(M\) and unfixed \(n\)). We do solve this problem only for prime \(k = p^2\).

Our main statement is

\(^1\)Such embeddings form an open dense set in the space of all embeddings, for details see [5].

\(^2\)First, if \(k = ab\), then among the all \(k\)-periodic billiard trajectories there are \(a\)-periodic ones repeated \(b\) times. Second, the action of the dihedral group is free only when \(k\) is prime, else the quotient space \((M^{\times k} / D_k) \setminus \Delta\) is not a smooth manifold.
Theorem 1  Let $M$ — be a smooth closed connected $m$-dimensional manifold, $p > 3$ is a prime integer. Put $k_i = \dim H_i(M; \mathbb{Z}_2)$, $B = \sum_{i=0}^m k_i$. Then the minimal number of closed $p$-periodic billiard trajectories for all generic embeddings of the manifold $M$ into a Euclidean space satisfy

$$BT_p(M) \geq \frac{(B-1)((B-1)^{p-1}-1)}{2p} + \frac{mB}{2}(p-1).$$

(5)

Proof. By the Morse inequalities (lemma 3.1), it follows that

$$BT_p(M) \geq \sum_{i=0}^{pm} \dim H_q(M^x/D_p, \Delta; \mathbb{Z}_2).$$

(6)

By the results of the paper [7], it follows that if the homology groups of spaces $X_1$ and $X_2$ are isomorphic, then

$$H_*(X_1^x/D_p, \Delta_{X_1}; \mathbb{Z}_2) \cong H_*(X_2^x/D_p, \Delta_{X_2}; \mathbb{Z}_2)$$

(7)

as well\(^4\). Hence,

$$\sum_{i=0}^{pm} \dim H_q(M^x/D_p, \Delta; \mathbb{Z}_2) = \sum_{i=0}^{pm} \dim H_q(X^x/D_p, \Delta_X; \mathbb{Z}_2),$$

(8)

where $X$ is the bouquet of spheres

$$S^m \lor S_1^{m-1} \lor \cdots \lor S_{m-1}^{m-1} \lor \cdots \lor S_1 \lor \cdots \lor S_1^{k_m}.$$  

(9)

Finally, lemma 4.2 implies that

$$\sum_{i=0}^{pm} \dim H_q(X^x/D_p, \Delta_X; \mathbb{Z}_2) \geq \frac{(B-1)((B-1)^{p-1}-1)}{2p} + \frac{mB}{2}(p-1).$$

(10)

This completes the proof. □

\(^3\)For $p = 2, 3$ this estimate can be strengthened. In fact, the number of closed billiard trajectories of period 2 is at least $\frac{B^2+(m-1)B}{2}$ (see [4]), for period 3 the estimate is $\frac{B^3+3(m-1)B^2+2B}{6}$ (see [7]).

\(^4\)This statement is proved in [7] only for $p = 2, 3$, but one can easily generalize it for the case of arbitrary $p$. 

4
3 Morse inequalities

Let us state the main lemma first and then proceed with all the propositions needed for its proof.

**Lemma 3.1** Let \( p \) be a prime integer, \( M \) be a smooth closed connected \( p \)-generic submanifold of the Euclidean space \( \mathbb{R}^n \). Then there exist at least

\[
\sum_{q=0}^{mp} \dim H_q(M^{\times p}/D_p, \Delta; \mathbb{Z}_2)
\]

\( p \)-periodic billiard trajectories for the manifold \( M \).

**Proof.** If \( p = 2 \) or if \( M \) lies in a boundary of a strictly convex domain, then the lemma is proved in [7].

If \( M \) does not, we can deform the embedding \( M \to \mathbb{R}^n \) slightly such that \( M \) gets to a boundary of a strictly convex domain. Indeed, \( M \subset \mathbb{R}^n \subset \mathbb{R}^{n+1} \) and \( \mathbb{R}^n \) can be deformed to a sphere in \( \mathbb{R}^{n+1} \). This deformation is small on \( M \) itself. Lemma 3.5 implies that the number of closed billiard trajectories remains the same. This completes the proof. \( \square \)

Since lemma 3.1 is already proved for \( p = 2 \), in this section we may assume that \( p > 2 \).

Let us introduce the following functions:

\[
f_2 = \langle a, x_1 - x_2 \rangle : S^{n-1} \times M \times M \to \mathbb{R},
\]

\[
f_k = \sum_{i \in \mathbb{Z}_k} \langle a_i, x_i - x_{i+1} \rangle : (S^{n-1})^k \times M^k \to \mathbb{R}, \; k \geq 3,
\]

where \( a, a_i \in S^{n-1} = \{u_1^2 + \cdots + u_n^2 = 1\} \subset \mathbb{R}^n \), \( x_i \in M \) and, as above, \( i = i + k \). Here \( S^{n-1} \subset \mathbb{R}^n \) and \( M \subset \mathbb{R}^n \), that’s why all the scalar products are well defined.

**Lemma 3.2** Suppose \((x_1, \ldots, x_k)\) is a closed billiard trajectory that is a critical point of the function

\[
l_k = \sum_{i \in \mathbb{Z}_k} \|x_i - x_{i+1}\| : M^k \to \mathbb{R}.
\]
Then
\[
P_0 = \left( \frac{x_1 - x_2}{\|x_1 - x_2\|}, \frac{x_2 - x_3}{\|x_2 - x_3\|}, \ldots, \frac{x_k - x_1}{\|x_k - x_1\|}, x_1, x_2, \ldots, x_k \right)
\]
(14)
is a critical point of the function \(f_k\). Similarly,
\[
P_0 = \left( \frac{x_1 - x_2}{\|x_1 - x_2\|}, x_1, x_2 \right)
\]
(15)
is a critical point of the function \(f_2\).

Proof. Let \(P_0 = (a_1, \ldots, a_k, x_1, \ldots, x_k)\) be a critical point of the function \(f_k\) such that \(x_i \neq x_{i+1}\) for any \(i \in \mathbb{Z}_k\). We have \(\frac{\partial f_k}{\partial a_i}(P_0) = 0\) and \(\frac{\partial f_k}{\partial x_i}(P_0) = 0\) for all \(i\). The first condition implies
\[
x_i - x_{i+1} \perp T_{a_i}S^{n-1},
\]
that is \(a_i \parallel x_i - x_{i+1}\). The second condition means that
\[
a_{i-1} - a_i \perp T_{x_i}M.
\]
Let now \((x_1, \ldots, x_k)\) be a critical point of the function \(l_k\). It follows that
\[
\frac{x_i - x_{i+1}}{\|x_i - x_{i+1}\|} - \frac{x_{i-1} - x_i}{\|x_{i-1} - x_i\|} \perp T_{x_i}M.
\]
(18)
Put \(a_i = \frac{x_i - x_{i+1}}{\|x_i - x_{i+1}\|}\). This completes the proof. \(\Box\)

Lemma 3.3 Let \(P_0 = (a_1, \ldots, a_k, x_1, \ldots, x_k)\) be a critical point of the function \(f_k\) with \(k > 2\) such that some of \(x_i\) coincide. Actually assume that \(\beta_1, \ldots, \beta_{k'}\) are integers such that
1. \(\beta_1, \ldots, \beta_{k'} \geq 1\),
2. \(\beta_1 + \cdots + \beta_{k'} = k\),
3. \(1 < k' < k\),
and put \(\alpha_i = \beta_1 + \cdots + \beta_i\). Suppose \(x_i\) to coincide as follows
\[
x_1 = \ldots = x_{\alpha_1} \neq x_{\alpha_1+1} = \ldots = x_{\alpha_2} \neq \cdots = x_{\alpha_{k'-1}+1} = \ldots = x_k \neq x_1
\]
(19)
Then the following conditions hold:
1. the point $P'_0 = (a_1, a_{\alpha_1+1}, \ldots, a_{\alpha_{k'}-1+1}, x_{\alpha_1}, \ldots, x_{\alpha_{k'}})$ is critical for the function $f_{k'}$.

2. the point $P_0$ belongs to a critical manifold $M_0$ given by

$$a_2 - a_1, \ldots, a_{\alpha_1} - a_{\alpha_1-1}, a_{\alpha_1+1} - a_{\alpha_1} \perp T_{x_1} M,$$

$$a_{\alpha_1+2} - a_{\alpha_1+1}, \ldots, a_{\alpha_2} - a_{\alpha_2-1}, a_{\alpha_2+1} - a_{\alpha_2} \perp T_{x_2} M,$$

$$\vdots$$

$$a_{\alpha_{k'-1}+2} - a_{\alpha_{k'-1}+1}, \ldots, a_{\alpha_{k'}} - a_{\alpha_{k'}-1}, a_1 - a_k \perp T_{x_k} M.$$  

(20)

Proof. Without loss of generality consider the simplest case: $x_1 = x_2$ and $x_i \neq x_{i+1}$ for $i \neq 1$. As above, we obviously have

$$a_i \parallel x_i - x_{i+1}, \ i \neq 1,$$

and

$$a_i - a_{i+1} \perp T_{x_i} M, \ i \in \mathbb{Z}_k.$$  

(21)

(22)

For $x_1 = x_2$ we obtain that $a_k - a_1 \perp T_{x_1} M$ and $a_1 - a_2 \perp T_{x_1} M$. Summing these two conditions, we get that $a_k - a_2 \perp T_{x_1} M$, while all possible $a_i$ form an $(n - m - 1)$-sphere. For $n = m + 1$ this sphere is just a couple of points, but, in fact, we do not need to consider this case very detailed, since in further we always have $n > m + 1$.

Notice that for $k' < k - 1$ these critical manifold can be products of spheres. □

Remark 3.1 Besides, there exists a critical manifold $M^{(0)}$ given by

$$x_1 = \cdots = x_k = x \in M,$$

$$a_i - a_j \perp T_x M, \ i \neq j.$$  

(23)

It is a bundle over $M$, the fiber is defined by the second of these conditions.

Lemma 3.4 Let $(x_1, \ldots, x_k)$ be a non-degenerate critical point of the function $l_k$ and $\mu$ be its Morse index. Then the corresponding critical point

$$P_0 = (a_1, \ldots, a_k, x_1, \ldots, x_k), \ a_i = \frac{x_i - x_{i+1}}{\|x_i - x_{i+1}\|},$$

(24)

of the function $f_k$ is also non-degenerate and its Morse index equals $\mu + k(n - 1)$ for $k > 2$ or $\mu + n - 1$ for $k = 2$. 7
Proof. Assume that \( P = (b_1, \ldots, b_k, y_1, \ldots, y_k) \) lies in a small neighborhood of the critical point \( P_0 \) being considered. Let us introduce coordinates in this neighborhood in the following way. Suppose that 
\[
y_i = y_i(t_i),
\]
where \( t_i \in \mathbb{R}^m \) is some parametrization for \( y_i \). Put 
\[
b_i = b_i(s_i),
\]
where the parametrization \( s_i \) for \( b_i \) is defined as follows. Let \( A_i \) be an orthogonal operator \( \mathbb{R}^{n-1} \rightarrow (y_i - y_{i+1})^\perp \) and \( s_i \in \mathbb{R}^{n-1} \) be our parameter. Put 
\[
b_i = \frac{\|s_i\| A_is_i + \frac{y_i - y_{i+1}}{\|y_i - y_{i+1}\|}}{\|s_i\| A_is_i + \frac{y_i - y_{i+1}}{\|y_i - y_{i+1}\|}} = \frac{\|s_i\| A_is_i + \frac{y_i - y_{i+1}}{\|y_i - y_{i+1}\|}}{\sqrt{1 + \|s_i\|^2}}.
\]
Evidently, \( \|b_i\| = 1 \) and \( b(0) = \frac{y_i - y_{i+1}}{\|y_i - y_{i+1}\|} \). Since \( A_is_i \perp y_i - y_{i+1} \), we have 
\[
\langle b_i, y_i - y_{i+1} \rangle = \frac{\|y_i - y_{i+1}\|}{\sqrt{1 + \|s_i\|^2}}.
\]
Thus the following condition holds:
\[
f_k(P) = \sum_{i \in \mathbb{Z}_k} \langle b_i, y_i - y_{i+1} \rangle = \sum_{i \in \mathbb{Z}_k} \frac{\|y_i - y_{i+1}\|}{\sqrt{1 + \|s_i\|^2}} = l_k(y_1, \ldots, y_k) - \frac{1}{2} \sum \|s_i\|^2 + \ldots
\]
This concludes the proof. \( \square \)

Remark 3.2 Suppose \( x_i = x_{i+1} \) for some \( i \). As we have showed above, in this case there are some critical manifolds corresponding to critical points of the function \( f_{k'} \), \( k' < k \). If \( (x_1, \ldots, x_{k'}) \) is a non-degenerate critical point of a function \( l_{k'} \), then the corresponding critical manifold is also non-degenerate. The critical manifold \( M(0) \) defined by \( x_1 = x_2 = \cdots = x_k \) is non-degenerate as well.

Lemma 3.5 Suppose \( F : M \times [0,1] \rightarrow \mathbb{R}^n \) is a smooth homotopy such that for every \( t \) the embedding \( F_t : M \rightarrow \mathbb{R}^n \) is generic, \( k \geq 2 \) an integer. Then the homotopy \( F \) keeps the number of closed billiard trajectories with \( k \) vertices.
Proof. Consider the homotopy $F : M \times [0,1] \to \mathbb{R}^n$. Denote $f_k$- and $l_k$-functions corresponding to an embedding $F_t$ by $f_{kt}$ and $l_{kt}$.

From the previous statements we know that the whole picture is as follows. Closed billiard trajectories (those are non-degenerate isolated critical points of the function $l_k$) correspond to non-degenerate critical points of the function $f_k$ that is a smooth function defined on the smooth manifold

$$(S^{n-1})^k \times M^k \times k. \quad (30)$$

We suppose every embedding $F_t : M \to \mathbb{R}^n$ to be generic, thus $f_{kt}$ has an amount of isolated critical points and several non-degenerate critical manifolds corresponding to isolated critical points of the functions $f_{k't}$ with $k' < k$.

Thus when the embedding $F_0 : M \to \mathbb{R}^n$ is being deformed, isolated critical points of the function $f_k$ could disappear and be born only from non-degenerate critical manifolds that is impossible. Indeed, suppose an isolated critical point is born at $t = t_0$. We mean that there exists $M_t$ — a non-degenerate critical manifold of the function $f_{kt}$ for $|t - t_0|$ small enough and for $t > t_0$ there exists an isolated critical point $P_t$ such that $\lim_{t \to t_0+0} P_t = P_0 \in M_{t_0}$. By Morse-Bott theory, there are coordinates $t, X^1, \ldots, X^N$ in a neighborhood $U \subset (S^{n-1})^k \times M^k \times [t_0 - \varepsilon, t_0 + \varepsilon]$ of the point $P_{t_0}$ such that $M_t$ is given by $X^1 = \cdots = X^r = 0$ and

$$f_{kt} = C(t) - (X^{r+1})^2 - \cdots - (X^{r+s})^2 + (X^{r+s+1})^2 + \cdots + (X^N)^2. \quad (31)$$

We see that in the neighborhood $U$ there are no other isolated critical points of the function $f_{kt}$. This contradiction completes the proof. $\square$

4 Computations for a bouquet of spheres

Lemma 4.1 Let $M$ be a smooth closed connected $m$-dimensional manifold, $k_i = \dim H_i(M; \mathbb{Z}_2)$, $i = 0, 1, \ldots, m$, $B = \sum_{i=0}^m k_i$. Then

$$\sum_{i=1}^m ik_i = \frac{mB}{2}. \quad (32)$$
Proof. Poincaré duality implies that:

\[
\sum_{i=1}^{m} ik_i = \sum_{i=0}^{m} ik_i = \frac{1}{2} \sum_{i=0}^{m} (ik_i + (m - i) k_{m-i}) = \frac{mB}{2}.
\]  \tag{33}

Lemma 4.2 Let \( M \) be a smooth closed connected \( m \)-dimensional manifold, \( p \) and odd prime, \( k_i = \dim H_i(M; \mathbb{Z}_2) \), \( B = \sum_{i=0}^{m} k_i \). Suppose

\[
X = S^m \vee S_1^{m-1} \vee \ldots \vee S_{k_{m-1}}^{m-1} \vee \ldots \vee S_{1}^{1} \vee \ldots \vee S_{k_1}^{1}.
\]  \tag{34}

Then

\[
\sum_{i=1}^{pm} \dim H_i(X \times p / D_p, \Delta X; \mathbb{Z}_2) \geq \frac{(B - 1)((B - 1)^{p-1} - 1)}{2p} + \frac{mB}{2} (p-1).
\]  \tag{35}

Proof. Consider the bouquet of spheres \( X \). By \( X_0 \) denote the common point of all the spheres. Let \( X_i \) be the \( i \)th sphere of the bouquet without the point \( X_0 \), so topologically \( X_i \) is a Euclidean space \( \mathbb{R}^q \) and \( X = X_0 \cup X_1 \cup \ldots \cup X_{B-1} \) is a cell decomposition.

Clearly we have

\[
X \times p = \bigcup_{i_1, \ldots, i_p} X_{i_1} \times \cdots \times X_{i_p}
\]  \tag{36}

is a cell decomposition of the Cartesian power \( X \times p \). What we do need is to construct its subdecomposition such that

- it is invariant under the action of the dihedral group \( D_p \),
- the diagonal \( \Delta \) is a cell subspace.

Note that if \( i_1 \neq i_2 \neq i_3 \neq \cdots \neq i_p \neq i_1 \), then \( X_{i_1 \ldots i_p} = X_{i_1} \times \cdots \times X_{i_p} \) does not intersect the diagonal. It follows that \( X_{i_1 \ldots i_p} \) is a cell of the decomposition being constructed and its boundary is zero.

Consider now \( X_{i_1 \ldots i_p} = X_{i_1} \times \cdots \times X_{i_p} \) such that \( i_{\alpha} = i_{\alpha+1} \) for some \( \alpha \in \mathbb{Z}_p \). First suppose that not all of the \( i_{\alpha} \) coincide. Without loss of generality we can
assume that $i_1 = \cdots = i_{\beta_1} \neq i_{\beta_1 + 1} = \cdots = i_{\beta_2} \neq \cdots \neq i_{\beta_u + 1} = \cdots = i_p \neq i_1$.

We construct a cell subdecomposition for all $X_i \times X_i \times \cdots \times X_i$ and the decomposition for the whole $X_{i_1} \cdots i_p$ would be their tensor product.

Each $X_i$ is topologically a Euclidean space $\mathbb{R}^q$. Thus we deal with the Cartesian power $(\mathbb{R}^q)^{\times \beta}$. Let the $j$th $\mathbb{R}^q$ have coordinates $x_1^j, \ldots, x_q^j$. A cell is given by the following conditions:

$$
\begin{align*}
&x_1^1 \epsilon_1^1 x_2^1 \epsilon_2^1 \cdots \epsilon_{\beta_1}^1 x_{\beta_1}^1, \\
&x_1^2 \epsilon_1^2 x_2^2 \epsilon_2^2 \cdots \epsilon_{\beta_2}^2 x_{\beta_2}^2, \\
&\vdots \\
&x_1^q \epsilon_1^q x_2^q \epsilon_2^q \cdots \epsilon_{\beta_q}^q x_{\beta_q}^q,
\end{align*}
$$

(37)

where each $\epsilon_\ast^\ast$ is one of the signs $<$, $>$, or $=$.

Now consider $X_{i_1} \cdots i_p$ having $i_1 = \cdots = i_p = i$. Then the cell subdecomposition for this thing is given by the same construction with inequalities $x_j^\beta \epsilon_j^\beta x_1^1$ added. Clearly, $\epsilon_1^j, \epsilon_2^j, \ldots, \epsilon_{\beta_j}^j$ should not be all $<$ or all $>$, since in this case the system of inequalities has no solutions at all.

We have just constructed the cell decomposition for the space $X^{\times p}$. Denote the corresponding chain complex by $C(X^{\times p})$. It induces the cell decomposition for the quotient $X^{\times p}/D_p$ with the diagonal contracted to a point. Let us denote the induced chain complex by $C(X^{\times p}/D_p, \Delta_X)$. Our goal is to calculate its homology

$$
H_*(C(X^{\times p}/D_p, \Delta_X)).
$$

(38)

First consider $X_{i_1} \cdots \cdots i_{p}$ for some fixed $i > 0$. Suppose dim $X_i = q$. Note that all $X_{i_1} \cdots i_{p}$ such that $i_\alpha$ is either $0$ or $i$ for all $\alpha = 1, \ldots, p$ form a chain subcomplex. Denote it by $C^+(X_{i_1} \cdots i_{p})$. Moreover, there is no cell outside $C^+(X_{i_1} \cdots i_{p})$ such that its algebraic boundary contains terms lying in $C^+(X_{i_1} \cdots i_{p})$. Hence $C^+(X_{i_1} \cdots i_{p})$ is a direct summand in $C(X^{\times p}/D_p, \Delta_X)$. Obviously, $C^+(X_{i_1} \cdots i_{p})$ coincides with a chain complex for a sphere $C((S^q)^{\times p}/D_p, \Delta_{S^q})$. By the results of M. Farber and S. Tabachnikov (see [5], [6]), it follows that

$$
\sum \dim H_\alpha(C^+(X_{i_1} \cdots i_{p}); \mathbb{Z}_2) = q(p - 1).
$$

(39)

Summing for all $i$ and using lemma 4.1, we obtain that the contribution to the sum 35 being calculated equals

$$
\frac{mB}{2}(p - 1).
$$

(40)
Now consider $X_{i_1 \ldots i_p} = X_{i_1} \times \cdots \times X_{i_p}$ for $i_1 \neq i_2 \neq i_3 \neq \cdots \neq i_p \neq i_1$. Each of these $X_{i_1 \ldots i_p}$ is a cell such that

- its algebraic boundary is zero,
- it is not contained in an algebraic boundary of any other cell.

Hence it forms a chain subcomplex in $C(X^{\times p}/D_p, \Delta_X)$ consisting of only one group with only one generator and zero boundary operator. Let us denote this chain complex by $C(X_{i_1 \ldots i_p})$. It contributes 1 to the sum 35.

It is well known from combinatorics that the number of all $X_{i_1 \ldots i_p}$ having $i_1 \neq i_2 \neq i_3 \neq \cdots \neq i_p \neq i_1$ equals

$$\left( B - 1 \right) \left( \left( B - 1 \right)^{p-1} - 1 \right). \quad (41)$$

Anyway let us prove it. Suppose $N(p)$ is the number of all $p$-tuples $(i_1, \ldots, i_p)$ such that $0 \leq i_\alpha \leq B - 1$ and $i_1 \neq i_2 \neq i_3 \neq \cdots \neq i_p \neq i_1$. Let now $p$ be not necessarily prime. Then we have

$$N(p) = B(B - 1)^{p-1} - N(p-1). \quad (42)$$

Indeed, $i_1$ may be chosen in $B$ ways. Each $i_\alpha$, $\alpha = 2, \ldots, p$ may be chosen in $B - 1$ ways to be different from $i_{\alpha-1}$. It gives $B(B - 1)^{p-1}$. If $i_1 = i_p$, then $(i_1, i_2, \ldots, i_{p-1})$ is a correct $(p - 1)$-tuple. Thus recalling $p$ is an odd prime we get

\[
N(p) = B(B - 1)^{p-1} - N(p - 1) = \\
B(B - 1)^{p-1} - B(B - 1)^{p-2} + N(p - 2) = \\
B(B - 1)^{p-1} - B(B - 1)^{p-2} + B(B - 1)^{p-3} - \cdots - B(B - 1) = \\
B(1 - B) \frac{1 - (1 - B)^{p-1}}{1 - (1 - B)} = (B - 1)(\left( B - 1 \right)^{p-1} - 1). \quad (43)
\]

This calculation gives us

$$\frac{(B - 1)((B - 1)^{p-1} - 1)}{2p} \quad (44)$$

after factorizing by the action of the dihedral group $D_p$. \qed
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