**P–Q “mixed” modular equations of degree 15**

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**Abstract**

Ramanujan in his second notebook recorded total of seven P–Q modular equations involving theta–function $f(−q)$ with moduli of orders 1, 3, 5 and 15. In this paper, modular equations analogous to those recorded by Ramanujan are obtained involving his theta–functions $ϕ(q)$ and $ψ(−q)$ with moduli of orders 1, 3, 5 and 15. As a consequence, several values of quotients of theta–function and a continued fraction of order 12 are explicitly evaluated.

**Keywords** Modular equations · Theta–functions

**Mathematics Subject Classification** 33E05 · 11F20

1 Introduction

Throughout, we assume that $|q|<1$, Ramanujan’s general theta–function $f(a, b)$ is defined by

$$f(a, b) := \sum_{n=−\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ |ab| < 1.$$  \hspace{1cm} (1.1)

Furthermore following Ramanujan we define three special cases of $f(a, b)$:

$$ϕ(q) := f(q, q) = \sum_{n=−\infty}^{\infty} q^{n^2},$$
$$ψ(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$
$$f(−q) := f(−q, −q^2) = \sum_{n=−\infty}^{\infty} (−1)^n q^{n(3n-1)/2}.$$
The ordinary or Gaussian hypergeometric function is defined by
\[ 2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad 0 \leq |z| < 1, \]
where \( a, b, c \) are complex numbers, \( c \neq 0, -1, -2, \ldots \), and
\[ (a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1) \text{ for any positive integer } n. \]

Now we recall the notion of a “mixed” modular equation. Let \( K(k) \) be the complete elliptic integral of the first kind corresponding, in pairs, to the moduli \( \sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma} \) and \( \sqrt{\delta} \), and their complementary moduli, respectively. Suppose that the equalities
\[ \frac{n_1 K'}{K} = \frac{L'_1}{L_1}, \quad \frac{n_2 K'}{K} = \frac{L'_2}{L_2} \quad \text{and} \quad \frac{n_3 K'}{K} = \frac{L'_3}{L_3}, \quad \text{(1.3)} \]
hold for some positive integers \( n_1, n_2 \) and \( n_3 \). Then the relation between the moduli \( \sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma} \) and \( \sqrt{\delta} \) that is induced by (1.3) is called as a “mixed” modular equation of composite degree \( n_3 = n_1 n_2 \). We say that \( \beta, \gamma \) and \( \delta \) are of degrees \( n_1, n_2 \) and \( n_3 \) respectively over \( \alpha \). The multipliers \( m = K/L_1 \) and \( m' = L_2/L_3 \) are algebraic relations involving \( \alpha, \beta, \gamma \) and \( \delta \).

Ramanujan [9] recorded total of seven \( P – Q \) modular equations involving theta–function \( f(-q) \) with moduli of orders 1, 3, 5 and 15. For example, he proved that
If \( P := \frac{f(-q^3) f(-q^5)}{q^{1/3} f(-q) f(-q^{15})} \) and \( Q := \frac{f(-q^6) f(-q^{10})}{q^{2/3} f(-q^2) f(-q^{30})} \), then
\[ P Q + \frac{1}{P Q} = \left( \frac{Q}{P} \right)^3 + \left( \frac{P}{Q} \right)^3 + 4. \quad \text{(1.4)} \]

The proof of above equation by using classical methods can be found in [5]. Recently, Mahadeva Naika et al. in [8], have derived a new class of modular identities relating \( P \) and \( Q_r \), where
\[ P := \frac{q^{1/3} f(-q) f(-q^{15})}{f(-q^3) f(-q^5)} \quad \text{and} \quad Q_r := \frac{q^{r/3} f(-q^r) f(-q^{15r})}{f(-q^{3r}) f(-q^{5r})}, \quad \text{(1.5)} \]
for \( r \in \{3, 4, 5, 7\} \) and using these modular relations they have explicitly evaluated several new cubic class invariants and cubic singular moduli.

The main goal of this paper is to establish new \( P – Q \) modular relations that involve \( \varphi(q) \) and \( \psi(q) \) with moduli of orders 1, 3, 5 and 15, which are not recorded by Ramanujan in his notebooks and also in his lost notebook. We use these modular relations to evaluate several new explicit evaluations of ratios of theta–functions \( \varphi(q) \) and \( \psi(q) \), also a continued fraction of order 12.

Ramanujan in his notebooks listed several explicit evaluations of \( \varphi(e^{-\pi \sqrt{n/k}}) \) for few rationals \( n \) and \( k \). Instigated by the works of Ramanujan, Jinhee Yi in [11], introduced two
parameterizations $h_{k,n}$ and $h'_{k,n}$ involving the theta-function $\varphi(q)$ and $\varphi(-q)$ as:

\begin{equation}
  h_{k,n} := \frac{\varphi(e^{-\pi \sqrt{n/k}})}{k^{1/4} \varphi(e^{-\pi \sqrt{n/k}})}
\end{equation}

and

\begin{equation}
  h'_{k,n} := \frac{\varphi(-e^{-\pi \sqrt{n/k}})}{k^{1/4} \varphi(-e^{-\pi \sqrt{n/k}})}.
\end{equation}

Yi systematically studied several properties of $h_{k,n}$ and $h'_{k,n}$, also explicitly evaluated the parameters for different positive rational values of $n$ and $k$. Motivated by works of Yi, in [3] and [12], the authors have defined two parameters $l_{k,n}$ and $l'_{k,n}$ involving the theta-function $\psi(-q)$ and $\psi(q)$ as follows:

\begin{equation}
  l_{k,n} := \frac{\psi(-e^{-\pi \sqrt{n/k}})}{k^{1/4} e^{-\frac{(k-1)\pi}{8} \sqrt{n/k}} \psi(-e^{-\pi \sqrt{n/k}})}
\end{equation}

and

\begin{equation}
  l'_{k,n} := \frac{\psi(e^{-\pi \sqrt{n/k}})}{k^{1/4} e^{-\frac{(k-1)\pi}{8} \sqrt{n/k}} \psi(e^{-\pi \sqrt{n/k}})}.
\end{equation}

They have also established several properties and some explicit evaluations of $l_{k,n}$ and $l'_{k,n}$ for different positive rational values of $n$ and $k$.

The work is organized as follows. In Sect. 2, we collect some relevant identities which are used in the subsequent sections. New $P–Q$ “mixed” modular identities of degree 1, 3, 5 and 15 involving $\varphi(q)$ and $\psi(-q)$ are established in Sect. 3. Several new explicit evaluations of quotients of $\varphi(q)$ and $\psi(-q)$ are evaluated in Sects. 4 and 5. In Sect. 6 explicit evaluations of a continued fraction of order 12 are established.

2 Preliminary results

To begin with we shall list equations which are helpful in proving our main results. For concise we set

\begin{equation}
  B_r := q^{r/3} f(-q^r) f(-q^{15r}) f(-q^{3r}) f(-q^{5r}).
\end{equation}

Lemma 2.1 [4, Ch. 16, Entry 24 (ii) & (iv), p. 39] We have

\begin{equation}
  f^3(-q) = \varphi^2(-q) \psi(q),
\end{equation}

\begin{equation}
  f^3(-q^3) = \varphi(-q) \psi^2(q).
\end{equation}

Lemma 2.2 The following identity holds

\begin{equation}
  B_1^3 + B_1^2 B_2^2 + B_2^3 = B_1 B_2.
\end{equation}

Proof Consider the identity (1.4) which is recorded by Ramanujan [5, Ch. 25, Entry 59, p. 214], factoring this identity, we get

\begin{equation}
  (B_1^3 - B_1 B_2 + B_1^2 B_2^2 + B_2^3) (B_1^3 + B_1 B_2 - B_1^2 B_2^2 + B_2^3) = 0.
\end{equation}
We find that the first factor of (2.5) vanishes and the second factor does not vanish for the sequence \( \{q_n\} = \left\{ \frac{1}{1 + n} \right\} \). Hence, first factor is identically equal to zero on \(|q| < 1\). This completes the proof. \( \square \)

**Lemma 2.3** [8] If \( P := B_1 B_3 \) and \( Q := \frac{B_1}{B_3} \), then

\[
\left( P^3 + \frac{1}{P^3} \right) \left( 46 + Q^6 + \frac{1}{Q^6} \right) - 9 \left( \sqrt{P^9} - \frac{1}{\sqrt{P^9}} \right) \left( \sqrt{Q^9} + \frac{1}{\sqrt{Q^9}} \right) \\
+ 9 \left( \sqrt{P^3} - \frac{1}{\sqrt{P^3}} \right) \left[ \sqrt{Q^9} + \frac{1}{\sqrt{Q^9}} + 2 \left( \sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) \right] = p^6 + \frac{1}{p^6} \\
+ Q^6 + \frac{1}{Q^6} + 92.
\] (2.6)

**Lemma 2.4** [8] If \( P := B_1 B_4 \) and \( Q := \frac{B_1}{B_4} \), then

\[
Q^3 + \frac{1}{Q^3} + 5 \left( Q^2 + \frac{1}{Q^2} \right) + \left( \sqrt{P^3} - \frac{1}{\sqrt{P^3}} \right) \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right) \\
+ 14 \left( Q + \frac{1}{Q} \right) + 18 = 0.
\] (2.7)

**Lemma 2.5** [8] If \( P := B_1 B_5 \) and \( Q := \frac{B_1}{B_5} \), then

\[
\left( Q^3 + \frac{1}{Q^3} \right) \left( 1 - P - \frac{1}{P} + P^2 + \frac{1}{P^2} \right) + 6 \left( P + \frac{1}{P} \right) - 21 \left( P^2 + \frac{1}{P^2} \right) \\
- 5 \left( \sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) \left[ \sqrt{P} - \frac{1}{\sqrt{P}} - 2 \left( \sqrt{P^3} - \frac{1}{\sqrt{P^3}} \right) + \sqrt{P^5} - \frac{1}{\sqrt{P^5}} \right] \\
+ 11 \left( P^3 + \frac{1}{P^3} \right) - \left( P^4 + \frac{1}{P^4} \right) + 8 = 0.
\] (2.8)

**Lemma 2.6** [8] If \( P := B_1 B_7 \) and \( Q := \frac{B_1}{B_7} \), then

\[
Q^4 + \frac{1}{Q^4} + 7 \left( Q^3 + \frac{1}{Q^3} \right) + 35 \left( Q^2 + \frac{1}{Q^2} \right) + 112 \left( Q + \frac{1}{Q} \right) - \left( P^3 + \frac{1}{P^3} \right) \\
- 7 \left( \sqrt{P^3} - \frac{1}{\sqrt{P^3}} \right) \left[ \sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} + 4 \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right) \right] + 168 = 0.
\] (2.9)

**Lemma 2.7** [1] If \( P := \frac{\varphi^4(q)}{\varphi^4(q^3)} \) and \( Q := \frac{\psi^4(-q)}{q \psi^4(-q^3)} \), then

\[
P + P Q = 9 + Q.
\] (2.10)

**Lemma 2.8** [1] If \( P := \frac{\varphi^2(q)}{\varphi^2(q^5)} \) and \( Q := \frac{\psi^2(-q)}{q \psi^2(-q^5)} \), then

\[
P + P Q = 5 + Q.
\] (2.11)
Lemma 2.9 \([2]\) If \(P := \frac{\varphi(q^3)}{\varphi(q^3)}\) and \(Q := \frac{\varphi(q^{15})}{\varphi(q^{15})}\), then
\[
(PQ)^2 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 + 5\left(\frac{Q}{P}\right)^2 + 5\left(\frac{P}{Q}\right)^2 + 5\left(\frac{Q}{P} - \frac{P}{Q}\right) - \left(\frac{P}{Q}\right)^3.
\] (2.12)

Lemma 2.10 \([5, \text{Ch. 25, Entry 67, p. 235}]\) If \(P := \frac{\varphi(q)}{\varphi(q^5)}\) and \(Q := \frac{\varphi(q^5)}{\varphi(q^{15})}\), then
\[
PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 + 3\left(\frac{Q}{P}\right) + 3\left(\frac{P}{Q}\right) - \left(\frac{P}{Q}\right)^2.
\] (2.13)

Lemma 2.11 If \(P := \frac{\varphi(q^3)\varphi(q^5)}{\varphi(q)\varphi(q^{15})}\) and \(Q := \frac{\psi(-q^3)\psi(-q^5)}{q^5\psi(-q)\psi(-q^{15})}\), then
\[
Q = \frac{1 + P}{1 - P}.
\] (2.14)

**Proof** Changing \(q\) to \(-q\) in the equation (2.4) and cubing the resultant equation on both sides, we have
\[
\begin{align*}
&u^3v^3 + v^9 - 6u^3v^6 + 6u^6v^3 - u^9 + u^6v^6 = 0,
\end{align*}
\] (2.15)

where \(u := B_1(-q)\) and \(v := B_2\).

By lemma 2.1, it is facile to observe that \(u^3 = P^2Q\) and \(v^3 = PQ^2\). By factoring the above equation (2.15), we obtain
\[
(PQ - P + 1 + Q)(P^2Q^2 + P^2Q + P^2 - PQ^2 - 3PQ + P + Q^2 - Q + 1) = 0. \tag{2.16}
\]

Observe the first factor of (2.16) vanishes and second factor does not vanish for the sequence \(\{q_n\} = \left\{\frac{1}{1+n}\right\}\). Hence, first factor is identically equal to zero on \(|q| < 1\). This completes the proof. \(\Box\)

### 3 P–Q “mixed” modular equations

In this section, we establish several new “mixed” modular equations involving Ramanujan’s theta–function \(\varphi(q)\). Throughout this section, we set
\[
A_r := \frac{\varphi(q^r)\varphi(q^{15r})}{\varphi(q^{3r})\varphi(q^{5r})} \quad \text{and} \quad C_r := \frac{q^r\psi(-q^r)\psi(-q^{15r})}{\psi(-q^{3r})\psi(-q^{5r})}.
\] (3.1)

**Theorem 3.1** If \(P := A_1A_2\) and \(Q := \frac{A_1}{A_2}\), then
\[
\begin{align*}
Q^2 + \frac{1}{Q^2} + P^2 + \frac{1}{P^2} + 6\left(P + \frac{1}{P}\right)\left[Q + \frac{1}{Q} - 4\right] - 8\left(Q + \frac{1}{Q}\right) & \\
+ 4\left(\sqrt{P} - \frac{1}{\sqrt{P}}\right)\left[\left(\sqrt{Q^3} + \frac{1}{\sqrt{Q^3}}\right) - 4\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right)\right] & \\
+ 4\left(\sqrt{P^3} - \frac{1}{\sqrt{P^3}}\right)\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) + 36 & = 0.
\end{align*}
\] (3.2)
Proof From lemma 2.1 and lemma 2.11, we have
\[
\left( \frac{f(-q^2)f(-q^{30})}{f(-q^6)f(-q^{10})} \right)^3 = u \left( \frac{u - 1}{u + 1} \right)^2, \quad \text{where } u = \frac{\varphi(q)\varphi(q^{15})}{\varphi(q^3)\varphi(q^5)}. \quad (3.3)
\]
Cubing the equation (2.4), we deduce that
\[
B_1^3 B_2^3 = B_1^9 + 6B_1^6 B_2^3 + 6B_1^3 B_2^6 + B_2^9 + B_1^6 B_2^6.
\]
Invoking (3.3) in (3.4), we get
\[
(\nu + u - 1 + v)(u^2 - u - 2uv + uv^2 + v^2)(u^2v + u^2 - 2uv - v + v^2)
\]
\[
= (1 - 4v - 4u + u^4 + v^4 + 36u^2v^2 + 16u^2v + 16uv^2 - 24uv + 6u^4v^2
\]
\[
- 16u^3v^2 + 4u^4v - 8u^3v + 4uv^4 - 8uv^3 + u^4v^3 + 4u^4v^3 + 6u^2v^4
\]
\[
- 16u^2v^3 + 4u^3v^4 - 24u^3v^3 - 4u^3 + 6u^2 + 6v^2 - 4v^3),
\]
where \(v = u(q^2)\).

We find that the last factor of (3.5) vanishes and other factors do not vanish for the sequence \(\{q_n\} = \left\{ \frac{1}{1 + n} \right\} \). Hence, last factor is identically equal to zero on \(|q| < 1\). By setting \(P := \nu v\) and \(Q := \frac{u}{v}\), we arrive at (3.2). This completes the proof. \(\square\)

Remark 1 Observe that the equation (3.3) can also be rewritten as follows:
\[
\left( \frac{f(-q^2)f(-q^{30})}{f(-q^6)f(-q^{10})} \right)^3 = u^2 \left( \frac{u + 1}{u - 1} \right)^2, \quad \text{where } u = \frac{q\psi(-q)\psi(-q^{15})}{\psi(-q^3)\psi(-q^5)}. \quad (3.6)
\]
Using (3.6) and adopting the same technique illustrated in Section 3 one can easily arrive at the modular relations connecting \(C_1\) with \(C_r\), for \(r \in \{2, 3, 5, \text{ and } 7\}\). For brevity these relations involving \(\psi(-q)\) are not included in this article.

Theorem 3.2 If \(P := A_1 A_3\) and \(Q := \frac{A_1}{A_3}\), then
\[
P^2 + \frac{1}{P^2} + 2 \left( P + \frac{1}{P} \right) + \left( Q^2 + \frac{1}{Q^2} \right) = 4 + 3 \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right)
\]
\[
\times \left[ \left( \sqrt{P^3} + \frac{1}{\sqrt{P^3}} \right) - 2 \left( \sqrt{P} - \frac{1}{\sqrt{P}} \right) \right] + \left( \sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right)
\]
\[
\times \left[ \left( \sqrt{P} + \frac{1}{\sqrt{P}} \right) - 3 \left( \sqrt{P} - \frac{1}{\sqrt{P}} \right) \right].
\]
Proof The proof of the equation (3.7) is similar to (3.2), except that in the place of the equation (2.4), (2.6) is used. \(\square\)

Theorem 3.3 If \(P := A_1 A_4\) and \(Q := \frac{A_1}{A_4}\), then
\[
Q^4 + \frac{1}{Q^4} = 96 \left( Q^3 + \frac{1}{Q^3} \right) + 1488 \left( Q^2 + \frac{1}{Q^2} \right) - 3522 \left( Q + \frac{1}{Q} \right) + 6692
\]
\[
P^4 + \frac{1}{P^4} + \left( P^3 + \frac{1}{P^3} \right) \left[ 28 \left( P + \frac{1}{P} \right) - 256 \right] - \left( P^2 + \frac{1}{P^2} \right) \left[ 96 \left( P + \frac{1}{P} \right) 
\]
\[
- 70 \left( P^2 + \frac{1}{P^2} \right) - 976 \right] + \left( P + \frac{1}{P} \right) \left[ 1064 \left( P + \frac{1}{P} \right) - 576 \left( P^2 + \frac{1}{P^2} \right) 
\]

\(\square\) Springer
The proof of the equation (3.8) is similar to (3.2), except that in the place of the equation (2.4), (2.7) is used.

\[ Q^4 + \frac{1}{Q^4} - \left( Q^3 + \frac{1}{Q^3} \right) + 7 \left( Q^2 + \frac{1}{Q^2} \right) + 5 \left( Q + \frac{1}{Q} \right) + 14 = \left( P^2 + \frac{1}{P^2} \right) \left( Q + \frac{1}{Q} \right) + \left( P + \frac{1}{P} \right) \left[ 6 - 2 \left( Q^2 + \frac{1}{Q^2} \right) - 7 \left( Q + \frac{1}{Q} \right) \right] + \left( \sqrt{P} - \frac{1}{\sqrt{P}} \right) \left[ \left( \sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) - 2 \left( \sqrt{Q^5} + \frac{1}{\sqrt{Q^5}} \right) - 7 \left( \sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) - 12 \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right) \right] + \left( \sqrt{P^3} - \frac{1}{\sqrt{P^3}} \right) \left[ 3 \left( \sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) + 5 \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right) \right]. \tag{3.9} \]

The proof of the equation (3.9) is similar to (3.2), except that in the place of the equation (2.4), (2.7) is used.

\[ P^4 + \frac{1}{P^4} + \left( P^3 + \frac{1}{P^3} \right) \left[ 5 \left( Q + \frac{1}{Q} \right) + 14 \right] + 42 = Q^3 + \frac{1}{Q^3} - 15 \left( Q + \frac{1}{Q} \right) + \left( P^2 + \frac{1}{P^2} \right) \left[ 4 + 10 \left( Q + \frac{1}{Q} \right) + 5 \left( Q^2 + \frac{1}{Q^2} \right) + \left( Q^3 + \frac{1}{Q^3} \right) \right] + \left( P + \frac{1}{P} \right) \left[ 6 + 10 \left( Q + \frac{1}{Q} \right) - \left( Q^3 - \frac{1}{Q^3} \right) \right] + 5 \left( \sqrt{P^7} - \frac{1}{\sqrt{P^7}} \right) \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right) + 15 \left( \sqrt{P^5} - \frac{1}{\sqrt{P^5}} \right) \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right) - 5 \left( \sqrt{P^3} - \frac{1}{\sqrt{P^3}} \right) \left[ 7 \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right) + 5 \left( \sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) + 5 \left( \sqrt{Q^5} + \frac{1}{\sqrt{Q^5}} \right) \right] + 11 \left( \sqrt{Q} + \frac{1}{\sqrt{Q}} \right) + 5 \left( \sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) + \left( \sqrt{Q^5} + \frac{1}{\sqrt{Q^5}} \right). \tag{3.10} \]

The proof of the equation (3.10) is similar to (3.2), except that in the place of the equation (2.4), (2.8) is used.

\[ \square \]
Theorem 3.6 If $P := A_1 A_7$ and $Q := \frac{A_1}{A_7}$, then
\[
P^3 + \frac{1}{P^3} + 14 \left( P^2 + \frac{1}{P^2} \right) - 35 \left( P + \frac{1}{P} \right) + 42 = Q^4 + \frac{1}{Q^4}
\]
\[
+ 14 \left( \sqrt{P} + \frac{1}{\sqrt{P}} \right)^2 + 7 \left( \sqrt{\frac{P}{Q}} + \frac{1}{\sqrt{\frac{P}{Q}}} \right) \left( \sqrt{\frac{P}{Q}} - \frac{1}{\sqrt{\frac{P}{Q}}} \right) - 2 \left( \sqrt{P^3} - \frac{1}{\sqrt{P^3}} \right)
\]
\[
+ \left( \sqrt{P^5} - \frac{1}{\sqrt{P^5}} \right) - 14 \left( \sqrt{Q^5} + \frac{1}{\sqrt{Q^5}} \right) \left( \sqrt{P} - \frac{1}{\sqrt{P}} \right)
\]
\[
+ 7 \left( \sqrt{Q^3} + \frac{1}{\sqrt{Q^3}} \right) \left( \sqrt{P^3} - \frac{1}{\sqrt{P^3}} \right) - \left( \sqrt{P} - \frac{1}{\sqrt{P}} \right) \right). \tag{3.11}
\]

Proof The proof of the equation (3.11) is similar to (3.2), except that in the place of the equation (2.4), (2.9) is used.

\[\Box\]

4 Explicit evaluation of ratios of theta–functions $\varphi(q)$

In the present section, we explicitly evaluate $h_{k,n}$ by using the modular relations established in Section 3. By using the definition of $h_{k,n}$ with $k = 3$ and $k = 5$ respectively the equations (2.12) and (2.13) respectively takes the form.

Lemma 4.1 The following identities hold for any positive real number $n$:
\[
3 \left( h_{3,n}^2 h_{3,25n}^3 + 5 \right) = \left( \frac{h_{3,325n}}{h_{3,n}} \right)^3 + 5 \left( \frac{h_{3,25n}}{h_{3,n}} \right)^3
\]
\[
+ 5 \left( \frac{h_{3,25n}}{h_{3,n}} - \frac{h_{3,n}}{h_{3,25n}} \right) - \left( \frac{h_{3,n}}{h_{3,25n}} \right)^3 \tag{4.1}
\]

and
\[
\sqrt{5} h_{5,n} h_{5,9n} + \sqrt{5} \frac{h_{5,9n}}{h_{5,n}} = \left( \frac{h_{5,9n}}{h_{5,n}} \right)^2 + 3 \left( \frac{h_{5,9n}}{h_{5,n}} \right) + 3 \left( \frac{h_{5,n}}{h_{5,9n}} \right) - \left( \frac{h_{5,n}}{h_{5,9n}} \right)^2. \tag{4.2}
\]

Lemma 4.2 [11] For all positive real numbers $k$, $a$, $b$, $c$, $m$, $n$, and $d$, with $ab = cd$, we have
\[
h_{a,b} h_{k,c,d} = h_{k,a,b} h_{c,d}, \tag{4.3}
\]
\[
h_{k,n/m} h_{m,nk} = h_{n,mk}. \tag{4.4}
\]

Theorem 4.1 We have
\[
h_{3,15} = 3^{1/4} (\sqrt{5} - 2)^{1/4} \left( \frac{\sqrt{5} - \sqrt{3}}{2} \right)^{1/2} (1 + \sqrt{3})^{1/2}, \tag{4.5}
\]
\[
h_{3,5/3} = 3^{-1/4} (2 + \sqrt{5})^{1/4} \left( \frac{\sqrt{5} - \sqrt{3}}{2} \right)^{1/2} (1 + \sqrt{3})^{1/2}. \tag{4.6}
\]
Proof Employing (1.6) in (3.7) with \( n = 1/15 \) and recalling that \( h_{3,n}h_{3,1/n} = 1 \), we obtain

\[
t^2 - 6t + 4 = 0, \quad \text{where } t := x - \frac{1}{x} \text{ and } x := h_{3,1/15}h_{3,3/5}.
\]

(4.7)

Since \( 0 < t < 1 \), we find that

\[
x - \frac{1}{x} = 3 - \sqrt{5}.
\]

(4.8)

As \( x > 1 \), on solving (4.8), we get

\[
h_{3,1/15}h_{3,3/5} = \frac{(\sqrt{5} + \sqrt{3})(\sqrt{3} - 1)}{2}.
\]

(4.9)

Now setting \( n = 1/15 \) in (4.1) and using (4.9), we deduce that

\[
(3s^2 - 2 - \sqrt{5})(s^2 + 6 - 3\sqrt{5}) = 0, \quad \text{where } s := h_{3,1/15}h_{3,5/3}.
\]

(4.10)

Since \( s > 1 \), we find that

\[
h_{3,1/15}h_{3,5/3} = \frac{(2 + \sqrt{5})^{1/2}}{\sqrt{3}}.
\]

(4.11)

Using (4.9) and (4.11), we obtain (4.5) and (4.6). \( \Box \)

Theorem 4.2 We have

\[
h_{3,20} = (\sqrt{2} - 1)\left(\sqrt{3} + 2b\right)^{1/2},
\]

(4.12)

\[
h_{3,4/5} = (\sqrt{2} + 1)\left(\sqrt{3} - 2b\right)^{1/2},
\]

(4.13)

\[
h_{5,12} = (\sqrt{2} - 1)\left(\sqrt{3} + 2a\right)^{1/2},
\]

(4.14)

\[
h_{5,4/3} = (\sqrt{2} + 1)\left(\sqrt{3} - 2a\right)^{1/2},
\]

(4.15)

\[
h_{4,15} = \sqrt{ab},
\]

(4.16)

\[
h_{4,5/3} = \sqrt[4]{b},
\]

(4.17)

where

\[
a = \sqrt{11689 - 3696\sqrt{10} - \sqrt{11688 - 3696\sqrt{10}}},
\]

\[
b = \sqrt{760 - 240\sqrt{10}} - \sqrt{759 - 240\sqrt{10}}.
\]

Proof Employing (1.6) in (3.8) with \( n = 1/20 \) and recalling that \( h_{3,n}h_{3,1/n} = 1 \), we obtain

\[
t^2 + 2 + 24t - 18 = 0, \quad \text{where } t := x - \frac{1}{x} \text{ and } x := h_{3,1/20}h_{3,4/5}.
\]

(4.18)

Since \( t > 1 \), we have

\[
x - \frac{1}{x} = 4\sqrt{10} - 12.
\]

(4.19)

On solving the above equation for \( x \) and noting that \( x > 0 \), we obtain

\[
h_{3,1/20}h_{3,4/5} = (\sqrt{2} + 1)^2(\sqrt{5} - 2).
\]

(4.20)
Now setting \( n = 1/20 \) in (4.1) and using (4.20), we deduce that
\[
(s^2 + (12\sqrt{10} - 40)s + 1)(s^2 + (40 - 12\sqrt{10})s + 1) = 0, \quad \text{where} \quad s = h_{3,1/20}h_{3,5/4}. \tag{4.21}
\]
Since \( s > 1 \), we have
\[
h_{3,1/20}h_{3,5/4} = \sqrt{760 - 240\sqrt{10}} + \sqrt{759 - 240\sqrt{10}}. \tag{4.22}
\]
Using (4.22) and (4.20), we arrive at (4.12) and (4.13).

Now we proceed to prove \( h_{5,12} \) and \( h_{5,3/4} \). Setting \( a = 5, \ b = 12, \ c = 3, \ d = 20 \) and \( k = 1/4 \) in (4.3) and using the fact \( h_{k,n} = h_{n,k} \), we find that
\[
h_{3,20}h_{3,5/4} = h_{5,12}h_{5,3/4}. \tag{4.23}
\]
From (4.20) and (4.23), we have
\[
h_{5,1/12}h_{5,4/3} = (\sqrt{2} + 1)^2(\sqrt{2} - 2). \tag{4.24}
\]
Now, setting \( n = 1/12 \) in the equation (4.2) and using (4.24), we deduce that
\[
s^2 + (48\sqrt{10} - 154)s + 1 = 0, \quad \text{where} \quad s := h_{5,1/12}h_{5,3/4}. \tag{4.25}
\]
Since \( s > 1 \), on solving the above equation (4.25), we get
\[
h_{5,1/12}h_{5,3/4} = \sqrt{11689 - 3696\sqrt{10}} + \sqrt{11688 - 3696\sqrt{10}}. \tag{4.26}
\]
By (4.26), (4.24) and the fact \( h_{k,n}h_{k,1/n} = 1 \), we obtain (4.14) and (4.15).
Again, setting \( k = 3, \ n = 4 \) and \( m = 5 \), in (4.4), we have
\[
h_{4,15} = h_{3,4/5}h_{5,12}. \tag{4.27}
\]
Setting \( k = 4, \ n = 5 \) and \( m = 3 \), in (4.4), we have
\[
h_{4,5/3} = h_{5,12}h_{3,1/20}. \tag{4.28}
\]
From the equations (4.27) and (4.28), we arrive at (4.16) and (4.17).

**Theorem 4.3** We have
\[
h_{5,15} = 2^{-1/6}(5^{1/6} - \sqrt{5^{1/3} - 2^{2/3}})^{1/2}\left(\frac{-5}{3} + \frac{10^{1/3}}{3} + \frac{10^{2/3}}{3}\right)^{1/2}, \tag{4.29}
\]
\[
h_{5,5/3} = 2^{-1/6}(5^{1/6} + \sqrt{5^{1/3} - 2^{2/3}})^{1/2}\left(\frac{-5}{3} + \frac{10^{1/3}}{3} + \frac{10^{2/3}}{3}\right)^{1/2}. \tag{4.30}
\]

**Proof** Employing (1.6) in (3.10) with \( n = 1/15 \) and recalling that \( h_{5,n}h_{5,1/n} = 1 \), we obtain
\[
t^3 + 8t - 2t^2 = 4, \quad \text{where} \quad t := x - \frac{1}{x} \quad \text{and} \quad x = h_{5,1/15}h_{5,5/3}. \tag{4.31}
\]
Since \( t > 0 \), we find that
\[
h_{5,1/15}h_{5,5/3} = 2^{-1/3}(5^{1/6} - \sqrt{5^{1/3} - 2^{2/3}}). \tag{4.32}
\]
Now setting \( n = 1/15 \) in (4.2) and using (4.32), we deduce that
\[
(3s + 5 - 10^{1/3} - 10^{2/3})(15s - 5 - 2(10)^{2/3} - 5(10)^{1/3}) = 0. \tag{4.33}
\]
where \( s = h_{5,1/15} h_{5,3/5} \). Since \( s > 1 \), we find that
\[
h_{5,1/15} h_{5,3/5} = \frac{1}{3} + 2 \frac{10^{2/3}}{15} + \frac{10^{1/3}}{3}.
\] (4.34)

Using (4.34) and (4.32), we obtain (4.29) and (4.30).

Theorem 4.4. We have
\[
h_{3,35} = 2^{-1/2}(9 - 4\sqrt{5})(\sqrt{21} + 2\sqrt{5})^{1/4}[(\sqrt{7} - \sqrt{5})(\sqrt{5} + \sqrt{3})]^{1/2},
\] (4.35)
\[
h_{3,7/5} = 2^{-1/2}(9 - 4\sqrt{5})(\sqrt{21} + 2\sqrt{5})^{1/4}[(\sqrt{7} + \sqrt{5})(\sqrt{5} - \sqrt{3})]^{1/2},
\] (4.36)
\[
h_{5,21} = 2^{-1} \left\{(3\sqrt{3} - 5)(3 + \sqrt{7})(\sqrt{7} - \sqrt{5})(\sqrt{5} + \sqrt{3})\right\}^{1/2},
\] (4.37)
\[
h_{5,7/3} = 2^{-1} \left\{(3\sqrt{3} - 5)(3 + \sqrt{7})(\sqrt{7} + \sqrt{5})(\sqrt{5} - \sqrt{3})\right\}^{1/2},
\] (4.38)
\[
h_{7,15} = 2^{-1/2}(9 - 4\sqrt{5})(\sqrt{21} - 2\sqrt{5})^{1/4}[(3\sqrt{3} - 5)(3 + \sqrt{7})]^{1/2},
\] (4.39)
\[
h_{7,5/3} = 2^{1/2}(9 + 4\sqrt{5})(\sqrt{21} + 2\sqrt{5})^{1/4}[(3\sqrt{3} - 5)(3 + \sqrt{7})]^{1/2}.
\] (4.40)

Proof. Employing (1.6) and (3.11) with \( n = 1/35 \) and recalling that \( h_{3,n} h_{3,1/n} = 1 \), we obtain
\[
t^2 + 2 - 10t + 2 = 0, \quad \text{where} \quad t := x - \frac{1}{x} \quad \text{and} \quad x := h_{3,1/35} h_{3,7/5}.
\] (4.41)

On solving the above equation for \( t \) and observe that \( 0 < t < 1 \), we have
\[
x - \frac{1}{x} = 5 - \sqrt{21}.
\] (4.42)

Again, solving the above equation for \( x \) and notice that \( x > 0 \), we get
\[
h_{3,1/35} h_{3,7/5} = \frac{(\sqrt{7} + \sqrt{5})(\sqrt{5} - \sqrt{3})}{2}.
\] (4.43)

Set \( n = 1/35 \) in (4.1) and with the help of (4.43), we deduce that
\[
(s^2 - 9\sqrt{21} + 18\sqrt{5} - 4\sqrt{105} - 40)(s^2 - 9\sqrt{21} - 18\sqrt{5} + 4\sqrt{105} + 40) = 0.
\] (4.44)

where \( s = h_{3,35} h_{3,7/5} \). Since \( 0 < s < 1 \), we get
\[
h_{3,35} h_{3,7/5} = \sqrt{(9 - 4\sqrt{5})(\sqrt{21} + 2\sqrt{5})}.
\] (4.45)

Using (4.43) and (4.45), we obtain (4.35) and (4.36).

Now we proceed to prove \( h_{5,21} \) and \( h_{5,7/3} \). Setting \( a = 5, b = 21, c = 3, d = 35 \) and \( k = 1/7 \) in (4.3) and using the fact \( h_{k,n} = h_{n,k} \), we find
\[
h_{3,35} h_{3,5/7} = h_{5,12} h_{5,3/7}.
\] (4.46)

From (4.43) and (4.46), we have
\[
h_{5,21} h_{5,3/7} = \frac{(\sqrt{7} - \sqrt{5})(\sqrt{5} + \sqrt{3})}{2}.
\] (4.47)

Now set \( n = 1/21 \) in (4.2) and using (4.47), we find
\[
(2s - 9\sqrt{3} + 5\sqrt{7} - 15 + 3\sqrt{21})(2s - 9\sqrt{3} + 5\sqrt{7} + 15 - 3\sqrt{21}) = 0.
\] (4.48)
where \( s := h_{5,1/21}h_{5,3/7} \). Note that \( s > 1 \), we find that

\[
h_{5,1/21}h_{5,3/7} = \frac{(3\sqrt{3} - 5)(3 - \sqrt{7})}{2}. \tag{4.49}
\]

By (4.47) and (4.49), we obtain (4.37) and (4.38).

Again Setting \( k = 3, n = 7 \) and \( m = 5 \), in (4.4), we have

\[
h_{7,15} = h_{3,7/5}h_{5,21}. \tag{4.50}
\]

Setting \( k = 7, n = 5 \) and \( m = 3 \), in (4.4), we have

\[
h_{7,5/3} = h_{5,21}h_{3,1/35}. \tag{4.51}
\]

From the equations (4.50) and (4.51), we arrive at (4.16) and (4.17).

\[\square\]

\section{5 Explicit evaluation of ratios of theta–functions \( \psi(-q) \)}

In this section, we explicitly evaluate \( l_{k,n} \) with the aid of following lemmas.

Lemma 5.1 \cite{3, 12} For all positive real number \( k, a, b, c, m, n, and d \), with \( ab = cd \), we have

\[
l_{a,b}l_{k,c,d} = l_{ka,kb,l_{c,d}}, \tag{5.1}
\]

\[
l_{k,n/m}l_{m,nk} = l_{n,mk}. \tag{5.2}
\]

Lemma 5.2 For all positive real number \( n \), we have

\[
\begin{align*}
    h_{3,n}^4 + 3h_{3,n}^4l_{3,n}^4 &= 3 + l_{3,n}^4, \\
    h_{5,n}^2 + \sqrt{5}h_{5,n}^2l_{5,n}^2 &= \sqrt{5} + l_{5,n}^2. 
\end{align*} \tag{5.3, 5.4}
\]

Proof Transcribing the equations (2.10) and (2.11) respectively by using the definition of \( h_{k,n} \) and \( l_{k,n} \) with \( k = 3 \) and \( k = 5 \) respectively, we arrive at (5.3) and (5.4).

\[\square\]

Theorem 5.1 We have

\[
l_{3,20} = \frac{(3 + \sqrt{5})(\sqrt{2} + 1)}{2^{3/4}} \left\{ \frac{\sqrt{394} + 120\sqrt{10} + \sqrt{390} + 120\sqrt{10}}{2^{3/4}} \right\}, \tag{5.5}
\]

\[
l_{3,4/5} = \frac{(3 - \sqrt{5})(\sqrt{2} - 1)}{2^{3/4}} \left\{ \frac{\sqrt{394} + 120\sqrt{10} + \sqrt{390} + 120\sqrt{10}}{2^{3/4}} \right\}, \tag{5.6}
\]

\[
l_{5,12} = \frac{(3 + \sqrt{5})(\sqrt{2} + 1)}{2} \left\{ \frac{(\sqrt{46} + 12\sqrt{10} + \sqrt{42} + 12\sqrt{10})}{2} \right\}^{1/2}, \tag{5.7}
\]

\[
l_{5,4/3} = \frac{(3 - \sqrt{5})(\sqrt{2} - 1)}{2} \left\{ \frac{(\sqrt{46} + 12\sqrt{10} + \sqrt{42} + 12\sqrt{10})}{2} \right\}^{1/2}, \tag{5.8}
\]

\[
l_{4,15} = 2^{-3/4}(2\sqrt{3} + \sqrt{10})^{1/2}(2\sqrt{3} + 1)^{1/2}5\sqrt{2} + 4\sqrt{3})^{1/4}(\sqrt{5} + \sqrt{3})^{1/4}, \tag{5.9}
\]

\[
l_{4,5/3} = 2^{-3/4}(2\sqrt{3} + \sqrt{10})^{1/2}(2\sqrt{3} + 1)^{1/2}(5\sqrt{2} - 4\sqrt{3})^{1/4}(\sqrt{5} - \sqrt{3})^{1/4}. \tag{5.10}
\]
Theorem 5.2  We have

Again setting \( k = 4 \), \( n = 5 \) and \( m = 3 \), in (5.2), we have

From the equations (5.11) and (5.12), we arrive at (5.9) and (5.10).

The continued fraction expression in (6.1) has been derived by Mahadeva Naika et al. [7].

6 A Continued fraction of order 12

A continued fraction \( H = H(q) \) for \(|q| < 1 \) is defined by

The continued fraction expression in (6.1) has been derived by Mahadeva Naika et al. [7]. For a summary of \( H(q) \) and for references to other work related to \( H(q) \) see [6][Ch. 12].

Lemma 6.1  For any positive rational \( n \), we have

Proof  Using the equation (5.3) and respective values of \( h_{3,n} \) for \( n = 20, 4/5 \), we arrive at (5.5) and (5.6) respectively. To prove (5.7) and (5.8), we use (5.4) and respective values of \( h_{5,n} \) for \( n = 12, 4/3 \) respectively.

Now we proceed to prove (5.9) and (5.10). Setting \( k = 3 \), \( n = 4 \) and \( m = 5 \), in (5.2), we have

\[
l_{4,15} = l_{4,45} l_{5,12}.
\]

Again setting \( k = 4 \), \( n = 5 \) and \( m = 3 \), in (5.2), we have

\[
l_{4,5/3} = l_{5,12} l_{3,1/20}.
\]

\[\]

The proof of Theorem 5.2 is similar to the proof of the Theorem 5.1, hence we omit the details.
Proof Using the definition of $h_{k,n}$ with $k = 3$ and (6.2), we arrive at (6.3).

We conclude this section, by tabulating few explicit evaluations of $H(e^{-\pi\sqrt{n}})$, for few positive rationals $n$ by using the values of $h_{3,n}$.

| $n$     | $H(e^{-\pi\sqrt{n}})$ |
|--------|------------------------|
| 1/45   | $(\sqrt{5} + 2)^{1/4}(\sqrt{3} - \sqrt{3})^{1/2}(\sqrt{3} - 1)^{1/2} - \sqrt{2}$ |
| 5/9    | $(\sqrt{5} + 2)^{1/4}(\sqrt{3} - \sqrt{3})^{1/2}(\sqrt{3} - 1)^{1/2} + \sqrt{2}$ |
| 20/3   | $3^{1/4}\left\{(\sqrt{5} + 2)\left(\sqrt{760 - 240\sqrt{10}} - \sqrt{759 - 240\sqrt{10}}\right)\right\}^{1/2} - (\sqrt{2} + 1)$ |
| 35/3   | $3^{1/4}\left\{(\sqrt{7} - \sqrt{3})(\sqrt{5} + \sqrt{3})\right\}^{1/2} - 2^{1/2}\left\{(9 + 4\sqrt{5})(\sqrt{21} - 2\sqrt{5})\right\}^{1/2}$ |
| 1/105  | $3^{1/4}\left\{(\sqrt{7} + \sqrt{3})(\sqrt{5} - \sqrt{3})\right\}^{1/2} - 2^{1/2}\left\{(9 - 4\sqrt{5})(\sqrt{21} + 2\sqrt{5})\right\}^{1/2}$ |

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