Quantization of moduli spaces of flat connections and Liouville theory

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Abstract. We review known results on the relations between conformal field theory, the quantization of moduli spaces of flat PSL(2, R)-connections on Riemann surfaces, and the quantum Teichmüller theory.

1. Introduction

The Teichmüller spaces $\mathcal{T}(C)$ are the spaces of deformations of the complex structures on Riemann surfaces $C$. The classical uniformization theorem gives an alternative picture as spaces of constant negative curvature metrics modulo diffeomorphisms. Such metrics naturally define flat PSL(2, R)-connections on $C$, relating the Teichmüller spaces to the moduli spaces $\mathcal{M}_{\text{flat}}(C)$ of flat PSL(2, R)-connections.

There are well-known connections between the Teichmüller spaces and the (complexified) Lie-algebra of smooth vector fields on the unit circle. Cutting out a disc from a Riemann surface $C$, and gluing it back after twisting by the flow generated by a given vector field may generate changes of the complex structure of $C$. Both the spaces of functions on the Teichmüller spaces and the dual to the space of vector fields on the unit circle have natural Poisson-structures which can be used to formulate quantisation problems. Quantisation of the dual to the space of vector fields on the unit circle gives the Virasoro algebra, the Lie algebra of symmetries of any conformal field theory. Despite the fact that the existence of a relation between the quantisation of the Teichmüller spaces and conformal field theory may seem natural from this point of view, it has turned out to be nontrivial to establish such connections more precisely. The goal in this article will be to outline what is currently known about the connections between quantized moduli spaces of flat PSL(2, R)-connections, quantum Teichmüller theory, and conformal field theory.

The resulting picture appears to be of certain mathematical interest. It can in particular be seen as a first example for non-compact generalisations of the known relations between conformal field theories, quantum groups and three-manifold invariants associated to compact (quantum-) groups. Indeed, many pieces of the resulting picture show deep analogies or even concrete relations to the harmonic analysis of non-compact groups. Relations with three-dimensional hyperbolic ge-

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1Here understood more precisely as representation theory of the Virasoro algebra with central charge $c > 1$, corresponding to what is often called Liouville theory in the physics literature.
ometry appear naturally, coming from known relations between three-dimensional hyperbolic geometry and Teichmüller theory. There are furthermore various connections with the theory of classical and quantum integrable models, including relations to the isomonodromic deformation problem. A unifying perspective was outlined in [T10], embedding such relations into a diamond of relations between conformal field theory, the (classical and quantized) Hitchin moduli spaces, and the geometric Langlands correspondence.

There exist various applications of the mathematical results described here in theoretical physics. They include relations to two-dimensional quantum gravity and matrix models, (non-critical) string theory and various relations to integrable models. Most strikingly, there even exist relations to four-dimensional $\mathcal{N} = 2$-supersymmetric gauge theories. Most direct seem to be the relations to work of Alday, Gaiotto and Tachikawa [AGT], Gaiotto, Moore and Neitzke [GMN09, GMN10], Nekrasov and Witten [NW] and Nekrasov, Rosly and Shatashvili [NRS]. These connections are described in [TV13].

This article concentrates on some mathematical aspects of the connections between quantization of moduli spaces of flat connections and conformal field theory.

2. Moduli of flat $\mathrm{PSL}(2,\mathbb{R})$-connections and Teichmüller theory

In this section we will briefly review the necessary background on the relevant moduli spaces $\mathcal{M}_{\text{flat}}(C)$ of flat connections and Teichmüller theory. The main goal will be to describe the algebra $\mathcal{A}(C) \equiv \text{Fun}^\text{alg}(\mathcal{M}_{\text{flat}}(C))$ of algebraic functions on $\mathcal{M}_{\text{flat}}(C)$ in terms of generators and relations in a way that will be useful for the quantization.

We will consider Riemann surfaces $C = C_{g,n}$ of genus $g$ with $n$ marked points called punctures. In this article we will consider connections having regular\(^2\) singularities at the punctures only.

2.1. Flat connections and uniformization. Let $\mathcal{M}_{\text{flat}}(C)$ be the moduli space of flat $\mathrm{PSL}(2,\mathbb{C})$-connections modulo gauge transformations. To each flat $\mathrm{PSL}(2,\mathbb{C})$-connection $\nabla = d + A$ we may associate its holonomies $\rho(\gamma)$ along closed curves $\gamma$ as $\rho(\gamma) = P \exp(\int_\gamma A)$. The map $\gamma \mapsto \rho(\gamma)$ defines a representation of the fundamental group $\pi_1(C)$ in $\mathrm{PSL}(2,\mathbb{C})$, defining a point in the character variety

$$\mathcal{M}^C_{\text{char}}(C) := \text{Hom}(\pi_1(C), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C}).$$

The space $\mathcal{M}^C_{\text{char}}(C)$ contains the real slice $\mathcal{M}^R_{\text{char}}(C)$ which is known to decompose into a finite set of connected components [Hit, Go88].

The uniformisation theorem allows us to represent any Riemann surface $C$ as a quotient of the upper half plane $\mathbb{H}$ by certain discrete subgroups $\Gamma$ of $\mathrm{PSL}(2,\mathbb{R})$.

\(^2\)The connection is gauge equivalent to a meromorphic connection with simple poles at the punctures.
called Fuchsian groups, \( C \simeq U/\Gamma \). The Fuchsian subgroups \( \Gamma \) define representations of \( \pi_1(C) \) in \( \text{PSL}(2, \mathbb{R}) \). There is a distinguished connected component \( \mathcal{M}^{\text{R,0}}(C) \) in \( \mathcal{M}^{\text{R}}(C) \) containing all the Fuchsian groups \( \Gamma \) uniformising Riemann surfaces. This component corresponds to a connected component \( \mathcal{M}^\text{flat}(C) \) in the moduli space \( \mathcal{M}^\text{flat}(C) \) of flat \( \text{PSL}(2, \mathbb{R}) \)-connections on \( C \). \( \mathcal{M}^\text{flat}(C) \) is called Teichmüller component as it is isomorphic to the Teichmüller space \( T(C) \) \cite{Hill, Go98}.

### 2.2. Coordinates associated to triangulations.

There exist useful systems of coordinates for \( \mathcal{M}^\text{flat}(C) \) associated to triangulations of \( C \) if \( C \) has at least one puncture. Coordinates of this type were introduced for \( T(C) \) in \cite{Pe}; the shear-coordinates introduced in \cite{F97} are closely related; there exists a natural complexification \cite{FG1}; the following formulation is due to \cite{GM09}.

Let \( \tau \) be a triangulation of the surface \( C \) such that all vertices coincide with marked points on \( C \). An edge \( e \) of \( \tau \) separates two triangles defining a quadrilateral \( Q_e \) with corners being the marked points \( P_1, \ldots, P_4 \). For a given connection \( \nabla = d + A \), let us choose four sections \( s_i, i = 1, 2, 3, 4 \) that are horizontal in \( Q_e \), \( \nabla s_i = (d + A)s_i = 0 \).

We shall furthermore assume that the sections \( s_i \) are eigenvectors of the monodromy around \( P_i \). Out of the sections \( s_i \) form

\[
X^\tau_e := -\frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_2 \wedge s_3)(s_4 \wedge s_1)},
\]

where all \( s_i, i = 1, 2, 3, 4 \) are evaluated at the same point \( P \in Q_e \). The ratio \( X^\tau_e \) does not depend on the choice of \( P \).

There is a natural Poisson structure on \( \mathcal{M}^\text{flat}(C) \) induced by the symplectic form \( \Omega_{AB} \) introduced by Atiyah and Bott. The Poisson bracket of the coordinates \( X^\tau_e \) becomes very simple,

\[
\{X^\tau_e, X^\tau_{e'}\} = n_{e,e'} X^\tau_{e'} X^\tau_e ;
\]

the definition of \( n_{e,e'} \in \{-2, -1, 0, 1, 2\} \) is best described in terms of the fat graph \( \acute{t} \) dual to the given triangulation \( t \): It is the total intersection index of the edges \( \acute{e} \) and \( \acute{e}' \) dual to \( e \) and \( e' \), respectively.

There furthermore exists a simple description of the relations between the coordinates associated to different triangulations. If triangulation \( \tau_{e'} \) is obtained from \( \tau_e \) by changing only the diagonal in the quadrangle containing \( e \), we have

\[
X^\tau_{e'} = \begin{cases} 
X^\tau_e (1 + (X^\tau_e)^{-\text{sgn}(n_{e,e'}))^{-n_{e,e'}} & \text{if } e' \neq e, \\
(X^\tau_e)^{-1} & \text{if } e' = e.
\end{cases}
\]

Poisson bracket \cite{FG1} and transformation law \cite{FG1} reflect the cluster algebra structure that \( \mathcal{M}^\text{flat}(C) \) has \cite{FG1}. 

Quantization of \( \mathcal{M}^\text{flat}(C) \) and conformal field theory
2.3. Trace functions. Useful coordinate functions for $M_{\text{flat}}(C)$ are the trace functions

$$L_\gamma := \nu_\gamma \text{tr}(\rho(\gamma)); \quad (2.6)$$

the signs $\nu_\gamma \in \{+1, -1\}$ will be chosen in such a way that the restriction of $L_\gamma$ to $M_{\text{flat}}^0(C)$ is positive and larger than two. It is possible to show that the length $l_\gamma$ of the geodesic on $H/\Gamma$ isotopic to $\gamma$ is related to $L_\gamma$ as

$$L_\gamma = 2 \cosh(l_\gamma/2).$$

If the curves $\gamma_r$ encircle the punctures $P_r$ on $C = C_{g,n}$ for $r = 1, \ldots, n$, we will identify the surface $C$ with the surface with constant negative curvature metric obtained by cutting out $n$ discs having the geodesics isotopic to $\gamma_r$ as boundaries.

There exists a natural complex structure on $M_{\text{flat}}(C)$ which is such that the trace functions $L_\gamma$ defined above are complex analytic.

2.3.1. Skein algebra. Let $A(C) \simeq \text{Fun}_{\text{alg}}^\text{alg}(M_{\text{flat}}(C))$ be the commutative algebra of functions on $M_{\text{flat}}(C)$ generated by the coordinate functions $L_\gamma$. We will explain how to describe $A(C)$ in terms of generators and relations

The well-known relation $\text{tr}(g)\text{tr}(h) = \text{tr}(gh) + \text{tr}(gh^{-1})$ valid for any pair of $SL(2)$-matrices $g, h$ implies that the trace functions satisfy the skein relations,

$$L_{\gamma_1}L_{\gamma_2} = L_{S(\gamma_1, \gamma_2)}, \quad (2.7)$$

where $S(\gamma_1, \gamma_2)$ is the curve obtained from $\gamma_1, \gamma_2$ by means of the smoothing operation, defined as follows. The application of $S$ to a single intersection point of $\gamma_1, \gamma_2$ is depicted in Figure 1. The general result is obtained by applying this rule at each intersection point, and summing the results.

2.3.2. Topological classification of closed curves. A Riemann surface $C$ of genus $g$ with $n$ punctures may be cut into pairs of pants by cutting along $h := 3g - 3 + n$ non-intersecting simple closed curves $\gamma = \{\gamma_1, \ldots, \gamma_h\}$ on $C$. It will be useful to supplement the collection of curves $\gamma$ specifying a pants decomposition by a three-valent graph $\Gamma$ on $C$ which has exactly one vertex inside each pair of pants, and the three edges emanating from a given vertex each intersect exactly one of the boundaries of the pair of pants. The pair of data $\sigma = (\gamma, \Gamma)$ will be called a pants decomposition.

With the help of pants decompositions one may conveniently classify all non-selfintersecting closed curves on $C$ up to homotopy [De]. Recall that there is a

\[\text{The graph } \Gamma \text{ allows us to distinguish pants decompositions related by Dehn-twists, the operation to cut open along a curve } \gamma_e \in \gamma, \text{ twisting by } 2\pi, \text{ and gluing back.}\]
unique curve $\gamma_e \in \gamma$ that intersects a given edge $e$ on $\Gamma$ exactly once, and which does not intersect any other edge. To a curve $\gamma_e \in \gamma$ let us associate the integers $(r_e, s_e)$ defined as follows. The integer $r_e$ is defined as the number of intersections between $\gamma$ and the curve $\gamma_e$. Having chosen an orientation for the edge $e$ we will define $s_e$ to be the intersection index between $e$ and $\gamma$.

Dehn’s theorem \cite{De} ensures that the curve $\gamma$ is up to homotopy uniquely classified by the collection of integers $(r, s) : e \mapsto (r_e, s_e)$, subject to the restrictions

\begin{align}
(i) \quad r_e &\geq 0, \\
(ii) \quad \text{if } r_e = 0 \Rightarrow s_e &\geq 0, \\
(iii) \quad r_{e_1} + r_{e_2} + r_{e_3} &\in 2\mathbb{Z} \quad \text{whenever } \gamma_{e_1}, \gamma_{e_2}, \gamma_{e_3} \text{ bound the same pair of pants}. \tag{2.8}
\end{align}

We will use the notation $\gamma_{(r,s)}$ for the geodesic which has parameters $(r, s) : e \mapsto (r_e, s_e)$.

2.3.3. Generators. As set of generators for $A(C)$ one may take the functions $L_{(r,s)} \equiv L_{\gamma_{(r,s)}}$. The skein relations allow us to express arbitrary $L_{(r,s)}$ in terms of a finite subset of the set of $L_{(r,s)}$. We shall now describe convenient choices for sets of generators.

Let us note that to each internal\footnote{An internal edge does not end in a boundary component of $C$.} edge $e$ of the graph $\Gamma$ of $\sigma = (\gamma, \Gamma)$ there corresponds a unique curve $\gamma_e$ in the cut system $C_\sigma$. There is a unique subsurface $C_e \hookrightarrow C$ isomorphic to either $C_{0,4}$ or $C_{1,1}$ that contains $\gamma_e$ in the interior of $C_e$. The subsurface $C_e$ has boundary components labeled by numbers $1, 2, 3, 4$ if $C_e \simeq C_{0,4}$, and if $C_e \simeq C_{1,1}$ we will assign to the single boundary component the label 0.

For each edge $e$ let us introduce the geodesics $\gamma^e_t$ which have Dehn parameters $(r^e_t, 0)$, where $r^e_0 = 2\delta_{e,e'}$ if $C_e \simeq C_{0,4}$ and $r^e_0 = \delta_{e,e'}$ if $C_e \simeq C_{1,1}$. The geodesics $\gamma^e_s$ and $\gamma^e_u$ are depicted as red curves on the left and right halves of Figures 2 and 4 respectively. There furthermore exist unique geodesics $\gamma^e_u$ with Dehn parameters $(r^e_s, s^e)$, where $s^e_0 = \delta_{e,e'}$. We will denote the trace functions associated to $\gamma^e_k$ by $L^e_k$, where $k \in \{s, t, u\}$. The set $\{L^e_s, L^e_t, L^e_u ; \gamma_e \in \gamma\}$ generates $A(C)$. 

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Figure 2. The geodesics $\gamma_s^e$ and $\gamma_t^e$ are the red curves on the left and right pieces of the figure. The change of pants decomposition from left to right is called F-move.
2.3.4. Relations. These coordinates are not independent, though. Further relations follow from the relations in $\pi_1(C)$. It can be shown (see e.g., [Go09] for a review) that any triple of coordinate functions $L_s^e$, $L_t^e$ and $L_u^e$ satisfies an algebraic relation of the form

$$P_e(L_s^e, L_t^e, L_u^e) = 0.$$ (2.9)

The polynomial $P_e$ in (2.9) is for $C_e \simeq C_{0,4}$ explicitly given as

$$P_e(L_s, L_t, L_u) := L_1L_2L_3L_4 + L_s^2 + L_t^2 + L_u^2 + L_1^2 + L_2^2 + L_3^2 + L_4^2 - 4$$ (2.10)

while for $C_e \simeq C_{1,1}$ we take $P$ to be

$$P_e(L_s, L_t, L_u) := L_s^2 + L_t^2 + L_u^2 - L_sL_tL_u + L_0 - 2.$$ (2.11)

In the expressions above we have denoted $L_i := \nu_{\gamma_i} \text{Tr}(\rho(\gamma_i))$, $i = 0, 1, 2, 3, 4$, where $\gamma_0$ is the geodesic representing the boundary of $C_{1,1}$, while $\gamma_i$, $i = 1, 2, 3, 4$ represent the boundary components of $C_{0,4}$. Generators $L_i^e$, $k \in \{s, t, u\}$, and relations (2.9) for all edges $e$ of $\Gamma$ describe $\mathcal{M}_{\text{flat}}(C)$ as an algebraic variety.

2.4. Poisson structure. There is also a natural Poisson bracket on $\mathcal{A}(C)$ [Go86], defined such that

$$\{L_{\gamma_1}, L_{\gamma_2}\} = L_{A(\gamma_1, \gamma_2)},$$ (2.12)

where $A(\gamma_1, \gamma_2)$ is the curve obtained from $\gamma_1$, $\gamma_2$ by means of the anti-symmetric smoothing operation, defined as above, but replacing the rule depicted in Figure 1 by the one depicted in Figure 4. The Poisson structure (2.12) coincides with the one induced from the symplectic form introduced by Atiyah and Bott.
The Poisson bracket \( \{ L^e_s, L^t_l \} \) can be written elegantly in the form [NRS]
\[
\{ L^e_s, L^t_l \} = \frac{\partial}{\partial L^e_u} P_e(L^e_s, L^t_l, L^e_u).
\] (2.13)

It is remarkable that the same polynomial appears both in (2.9) and in (2.13), which indicates that the symplectic structure on \( \mathcal{M}_{\text{flat}}(\mathbb{C}) \) is compatible with its structure as algebraic variety.

It is sometimes useful to introduce Darboux-coordinates \( e \mapsto (\ell_e, \kappa_e) \) such that \( \{ \ell_e, \kappa_{e'} \} = \delta_{ee'} \) and \( L^e_s = 2 \cosh(\ell_e/2) \). The Fenchel-Nielsen coordinates for \( T(\mathbb{C}) \) are such coordinates. There is a natural complexification of the Fenchel-Nielsen coordinates discussed in [NRS].

3. Quantization of \( \mathcal{M}^T_{\text{flat}}(\mathbb{C}) \)

We shall next review the quantization of the moduli spaces \( \mathcal{M}^T_{\text{flat}}(\mathbb{C}) \) that was constructed in [T05, TV13] based on the pioneering works [F97, Ka1, CF1].

3.1. Quantization of coordinates associated to triangulations.

The simplicity of the Poisson brackets (2.4) of the coordinates \( \mathcal{X}^t_e \) makes part of the quantization quite simple. To each edge \( e \) of a triangulation \( t \) of a Riemann surface \( \mathcal{C}_{g,n} \) associate the generator \( \mathcal{X}^t_e \) of a noncommutative algebra \( \mathcal{B}_t \) which has generators \( \mathcal{X}^t_e \) and relations
\[
\mathcal{X}^t_e \mathcal{X}^t_{e'} = e^{2\pi i b^2 n_{ee'}} \mathcal{X}^t_{e'} \mathcal{X}^t_e, \quad n_{ee'} = \frac{\pi i}{2} \left( (2a - 1)k^2 \mathcal{X}^t_e \mathsf{sgn}(n_{ee'}) - \mathsf{sgn}(n_{ee'}) \right), \tag{3.1}
\]
the integers \( n_{ee'} \) coincide with the structure constants of the Poisson algebra (2.4), and we have introduced the notation \( b^2 \) for the deformation parameter traditionally denoted \( \hbar \).

Note furthermore that the variables \( \mathcal{X}^t_e \) are positive for the Teichmüller component. This motivates us to consider representations \( \pi_t \) of \( \mathcal{B}_t \) in which the operators \( \mathcal{X}^t_e \) are positive self-adjoint. By choosing a polarization one may define representations \( \pi_t \) in terms of multiplication and finite difference operators on suitable dense subspaces of the Hilbert space \( \mathcal{H}_t \simeq L^2(\mathbb{R}^{3g-3+n}) \).

There exists a family of automorphisms which describe the relations between the quantized coordinate functions associated to different triangulations [F97, Ka1, CF1]. If triangulation \( t_e \) is obtained from \( t \) by changing only the diagonal in the quadrangle containing \( e \), we have
\[
\mathcal{X}^t_{e'} = \prod_{a=1}^{\left| n_{ee'} \right|} \left( 1 + e^{\pi i (2a - 1)k^2 \mathcal{X}^t_e \mathsf{sgn}(n_{ee'})} \right)^{-\mathsf{sgn}(n_{ee'})} \quad \text{if } e' \neq e, \quad \mathcal{X}^t_e^{-1} \quad \text{if } e' = e. \tag{3.2}
\]

Any two two triangulations \( t_1 \) and \( t_2 \) can be connected by a sequence of changes of diagonals in quadrilaterals. It follows that the quantum theory of \( \mathcal{M}^T_{\text{flat}}(\mathbb{C}) \) has the structure of a quantum cluster algebra [FG2].
It is possible to construct unitary operators \( T_{t_1 t_2} : \mathcal{H}_{t_1} \to \mathcal{H}_{t_2} \) that represent the quantum cluster transformations \( (3.2) \) in the sense that
\[
X_{t_2} = T_{t_1 t_2}^{-1} \cdot X_{t_1} \cdot T_{t_1 t_2}.
\] (3.3)

The operator \( T_{t_1 t_2} \) describes the change of representation when passing from the quantum theory associated to triangulation \( t_1 \) to the one associated to \( t_2 \). It follows that the resulting quantum theory does not depend on the choice of a triangulation in an essential way.

### 3.2. Quantization of the trace functions

There is a simple algorithm \([F97, FG1]\) for calculating the trace functions in terms of the variables \( X_t^e \) leading to Laurent polynomials in the variables \( X_t^e \) of the form
\[
L_t^\gamma = \sum_{\nu \in \mathbb{F}} C_t^\gamma(\nu) \prod_e (X_t^e)^{\frac{1}{2}\nu_e},
\] (3.4)

where the summation is taken over a finite set \( \mathbb{F} \) of vectors \( \nu \in \mathbb{Z}^{3g-3+2n} \) with components \( \nu_e \).

In order to define an operator \( L_t^\gamma \) associated to a classical trace function \( L_\gamma \) it has turned out \([CF1, CF2, T05]\) for some pairs \((\gamma, t)\) to be sufficient to simply replace \((X_t^e)^{\frac{1}{2}\nu_e}\) in \( (3.4) \) by \( \exp(\sum_e \frac{1}{2}\nu_e \log X_t^e) \). Let us call such pairs \((\gamma, t)\) simple.

In order to define \( L_t^\gamma \) in general \([T05]\) one may use the fact that for all curves \( \gamma \) there exists a triangulation \( t' \) such that \((\gamma, t')\) is simple, allowing us to define
\[
L_t^\gamma = T_{t't_1}^{-1} \cdot L_{t'}^{\gamma'} \cdot T_{t't_1}.
\] (3.5)

The operators \( L_t^\gamma \) defined thereby are positive self-adjoint with spectrum bounded from below by 2, as follows from the result of \([Ka3]\). Two operators \( L_{\gamma_1} \) and \( L_{\gamma_2} \) commute if the intersection of \( \gamma_1 \) and \( \gamma_2 \) is empty.

It turns out that \( (3.5) \) holds in general. It follows that we may regard the algebras generated by the operators \( L_t^\gamma \) as different representations \( \pi_t \) of an abstract algebra \( \mathcal{A}_{0^2}(C) \equiv \text{Fun}_{\text{alg}}(\mathcal{M}_{\text{flat}}^0(C)) \) which does not depend on the choice of a triangulation, \( L_\gamma \equiv \pi_t(L_\gamma) \) for \( L_\gamma \in \mathcal{A}_{0^2}(C) \). As in the classical case one may use pants decompositions to identify convenient sets of generators for \( \mathcal{A}_{0^2}(C) \) to be

set of generators: \( \{ L_t^e, \ i \in \{s,t,u\}, e \in \{\text{edges of } \Gamma\} \} \).

Important relations are
\[
\begin{align*}
P_{0,4}^{(a)}(L_s^e, L_t^e, L_u^e, L_t^e, L_s^e, L_u^e, L_4^e) &= 0, & \text{if } C_e \simeq C_{0,4}, \quad a = 2, 3, \\
P_{1,1}^{(a)}(L_s^e, L_t^e, L_u^e; L_0^e) &= 0, & \text{if } C_e \simeq C_{1,1}.
\end{align*}
\] (3.6)
where the polynomials $\mathcal{P}^{(a)}_{0,4}$ of non-commutative variables are defined as:

$$
\mathcal{P}^{(2)}_{0,4}(L_s, L_t, L_u; L_1, L_2, L_3, L_4) := e^{\pi i b^2} L_s L_t - e^{-\pi i b^2} L_t L_s
$$

(3.7)

\[ - (e^{2\pi i b^2} - e^{-2\pi i b^2}) L_u - (e^{\pi i b^2} - e^{-\pi i b^2})(L_1 L_3 + L_2 L_4). \]

$$
\mathcal{P}^{(3)}_{0,4}(L_s, L_t, L_u; L_1, L_2, L_3, L_4) := L_1 L_2 L_3 L_4 + L_1^2 + L_2^2 + L_3^2 + L_4^2

- e^{\pi i b^2} L_s L_t L_u + e^{2\pi i b^2} L_s^2 + e^{-2\pi i b^2} L_t^2 + e^{2\pi i b^2} L_u^2 - (2 \cos \pi b^2)^2

+ e^{\pi i b^2} L_s(L_3 L_4 + L_1 L_2) + e^{-\pi i b^2} L_t(L_2 L_3 + L_1 L_4) + e^{\pi i b^2} L_u(L_1 L_3 + L_2 L_4). \]

(3.8)

In the case $C_c \simeq C_{1,1}$ we have

$$
\mathcal{P}^{(2)}_{1,1}(L_s, L_t, L_u; L_0) := e^{\pi i b^2} L_s L_t - e^{-\pi i b^2} L_t L_s - (e^{\pi i b^2} - e^{-\pi i b^2}) L_u,
$$

(3.9)

$$
\mathcal{P}^{(3)}_{1,1}(L_s, L_t, L_u; L_0) := e^{\pi i b^2} (L_s^2 + e^{-2\pi i b^2} L_t^2 + L_u^2) - e^{2\pi i b^2} L_s L_t L_u + L_0 - 2 \cos \pi b^2. \]

(3.10)

The quadratic relations $\mathcal{P}^{(2)}_{g,n} = 0$ represent the deformation of the Poisson bracket (2.13), while the cubic relations $\mathcal{P}^{(3)}_{g,n} = 0$ are deformations of the relations (2.9).

One furthermore finds quantum analogs of the skein relations [CP1, CP2].

### 3.3. Representations associated to pants decompositions.

The operators $\mathbb{L}_\gamma$ and $\mathbb{L}_{\gamma'}$ associated to non-intersecting curves $\gamma$ and $\gamma'$ commute. It is therefore possible to diagonalise simultaneously the quantised trace functions associated to a maximal set $\gamma = \{\gamma_1, \ldots, \gamma_h\}$ of non-intersecting closed curves characterising a pants decomposition. This can be done by constructing operators $\mathbb{R}_{\gamma|\mathbb{L}}$ which map the operators $\mathbb{L}_{\gamma_e}$ associated to the curves $\gamma_e, e = 1, \ldots, h$, to the operators of multiplication by $2 \cosh(l_e/2)$, respectively [T05, TV13]. The states in the image $\mathcal{H}_\sigma$ of $\mathbb{R}_{\gamma|\mathbb{L}}$ can be represented by functions $\psi(l), l = (l_1, \ldots, l_h)$ depending on the variables $l_e \in \mathbb{R}^+$ which parameterise the eigenvalues of $\mathbb{L}_{\gamma_e}$. The operators $\mathbb{R}_{\gamma|\mathbb{L}}$ define a new family of representations $\pi_\sigma$ of $\mathcal{A}_{k/2}(C)$ via

$$
\pi_\sigma(L_\gamma) := \mathbb{R}_{\gamma|\mathbb{L}} \cdot \pi_L(L_\gamma) \cdot (\mathbb{R}_{\gamma|\mathbb{L}})^{-1}. \]

(3.11)

The representations $\pi_\sigma$ are naturally labelled by pants decompositions $\sigma = (\gamma, \Gamma)$. The unitary operators $\mathbb{R}_{\gamma|\mathbb{L}} : \mathcal{H}_\sigma \rightarrow \mathcal{H}_\sigma$ were constructed explicitly in [T05].

The operators $\pi_\sigma(L_\gamma)$ were calculated explicitly for the generators of $\mathcal{A}_{k/2}(C)$ in [TV13]. When $\sigma$ corresponds to the pants decomposition of $C = C_{0,4}$ depicted on the left of Figure 2 one finds, for example, $L_s := 2 \cosh(1/2), \]

$$
L_t := \frac{1}{2(\cosh l_s - \cosh 2\pi b^2)} \left(2 \cos \pi b^2 (L_2 L_3 + L_1 L_4) + L_s (L_1 L_3 + L_2 L_4) \right)

+ \sum_{c = \pm 1} \frac{1}{\sqrt{2 \sinh(1/2)}} e^{ck/2} \frac{\sqrt{C_{12}(L_s) C_{34}(L_s)}}{2 \sinh(1/2)} e^{ck/2} \frac{1}{\sqrt{2 \sinh(1/2)}}. \]

(3.12)

5Relations cubic in $L_s, L_t, L_u$. 

with operators \( l \) and \( k \) defined as \( l \psi_\sigma(l) = l \psi_\sigma(l) \), \( k \psi_\sigma(l) = -4\pi ib^2 \partial_\psi(l) \), respectively, while \( c_{ij}(L_s) \) is defined as \( c_{ij}(L_s) = L_{ij}^2 + L_{ij}^1 + L_{ij}^0 + L_{ij}L_{ij} - 4 \). \( L_{ij} \) is given by a similar expression \([TV13]\). The operators \( l \) and \( k \) can be identified as quantum counterparts of the Fenchel-Nielsen coordinates. In the general case one may use pants decompositions to reduce the description of the operators \( \pi_\sigma(L_i^e) \), \( i \in \{ s, t, u \}, e \in \{ \text{edges of } \Gamma \} \) to the cases \( C_e \simeq C_{0,1} \) and \( C_e \simeq C_{1,1} \).

The operators \( \pi_\sigma(L_i^e) \) are unbounded. The maximal domain of definition of \( \pi_\sigma(A_{b^2}(C)) \) defines a natural subspace \( \mathcal{S}_\sigma \subset \mathcal{H}_\sigma \) with topology given by the family of semi-norms \( \| \pi_\sigma(O) \|_i, O \in A_{b^2}(C) \). The topological dual \( \mathcal{D}_\sigma \) of \( \mathcal{S}_\sigma \) is a space of distributions canonically associated to \( (A_{b^2}(C), \pi_\sigma) \) such that \( \mathcal{S}_\sigma \subset \mathcal{H}_\sigma \subset \mathcal{D}_\sigma \).

### 3.4. Changes of pants decomposition

The passage between the representations \( \pi_{\sigma_1} \) and \( \pi_{\sigma_2} \) associated to two different pants decompositions can be described by

\[
U_{\sigma_2 \sigma_1} := R_{\sigma_2|t} \cdot (R_{\sigma_1|t})^{-1}.
\]

The passage between two pants decompositions \( \sigma_1 \) and \( \sigma_2 \) can always be decomposed into elementary "moves" called F-, S-, B- and Z-moves localized in subsurfaces with \( 3g - 3 + n \leq 1 \) \( [MS, BK1, FG1] \). We refer to \( [BK1, FG1] \) for precise descriptions of the set of generators. For future reference we have depicted the F- and S- moves in Figures 2 and 3, respectively. It is useful to formalize the resulting structure by introducing the notion of the Moore-Seiberg groupoid: The path groupoid of the two-dimensional CW-complex which has vertices identified with pants decompositions \( \sigma \), edges ("generators") called F-, S-, B- and Z-moves, and faces ("relations") being certain edge-paths localized in subsurfaces with \( 3g - 3 + n \leq 2 \) listed in \([MS, BK1, FG1]\).

The operators \( U_{\sigma_2 \sigma_1} \) intertwine the representation \( \pi_{\sigma_1} \) and \( \pi_{\sigma_2} \),

\[
\pi_{\sigma_2}(L_i^e) \cdot U_{\sigma_2 \sigma_1} = U_{\sigma_2 \sigma_1} \cdot \pi_{\sigma_1}(L_i^e).
\]

Explicit representations for the operators \( U_{\sigma_2 \sigma_1} \) have been calculated in \([NT, TV13]\) for pairs \( \{\sigma_2, \sigma_1\} \) related by the generators of the Moore-Seiberg groupoid. The B-move is represented as

\[
B \psi = B_{l_1 \Delta_1} \psi_s, \quad B_{l_1 \Delta}^{l_2} = e^{\pi i (\Delta_1 - \Delta_2) + \Delta_1},
\]

where \( \Delta_1 = (1 + b^2)/4b + (l/4\pi b)^2 \), and \( \psi \) is a generator for the one-dimensional space associated to \( C_{0,3} \). The F-move is represented in terms of an integral transformation of the form

\[
\psi_s(l_s) \equiv (F \psi_t)(l_s) = \int_{\mathbb{R}_+} dl_t \; F_{l_s l_t} \; [l_1 l_2] \psi_t(l_t).
\]

A similar formula exists for the S-move. The explicit formulae are given in \([TV13]\).

The operators \( U_{\sigma_1 \sigma_2} \) generate a projective unitary representation of the Moore-Seiberg groupoid,

\[
U_{\sigma_1 \sigma_2} \cdot U_{\sigma_1 \sigma_1} = \zeta_{\sigma_2 \sigma_1} U_{\sigma_2 \sigma_2},
\]

\( \\zeta_{\sigma_2 \sigma_1} \) is a space of semi-norms of \( \mathcal{D}_\sigma \).
where $\zeta_{\sigma_3,\sigma_2,\sigma_1} \in \mathbb{C}$, $|\zeta_{\sigma_3,\sigma_2,\sigma_1}| = 1$. The explicit formulae for the relations of the Moore-Seiberg groupoid in the quantization of $M_{\text{flat}}(C)$ are listed in [TV13]. The operators $U_{\sigma_2,\sigma_1}$ allow us to identify the spaces $\mathcal{S}_\sigma \subset \mathcal{H}_\sigma \subset \mathcal{D}_\sigma$ as different representatives of abstract spaces $\mathcal{S}(C) \subset \mathcal{H}(C) \subset \mathcal{D}(C)$ associated to $C$.

Having a representation of the Moore-Seiberg groupoid induces a representation of the mapping class group $\text{MCG}(C)$. Elements $\mu$ of $\text{MCG}(C)$ can be represented by diffeomorphisms of the surface $C$ not isotopic to the identity, and therefore map any pants decomposition $\sigma$ to another one denoted $\mu.\sigma$. Note that the Hilbert spaces $\mathcal{H}_\sigma$ and $\mathcal{H}_{\mu.\sigma}$ are canonically isomorphic, depending only on the combinatorics of the graphs $\sigma$, but not on their embedding into $C$. We may therefore define an operator $M_\sigma(\mu) : \mathcal{H}_\sigma \to \mathcal{H}_{\sigma}$ as

$$M_\sigma(\mu) := U_{\mu.\sigma,\sigma}. \hspace{1cm} (3.18)$$

It is automatic that the operators $M(\mu)$ define a projective unitary representation of $\text{MCG}(C)$ on $\mathcal{H}_\sigma$.

3.5. An analog of a modular functor. The description using representations associated to pants decompositions has the advantage to make manifest that we are dealing with an analog of a modular functor. This means in particular that the representations of the mapping class group associated to Riemann surfaces of varying topological type $C_{g,n}$ restrict to, and are generated by, the representations associated to embedded subsurfaces of simple topological type $C_{0,3}$, $C_{0,4}$ and $C_{1,1}$. This property can be seen as a locality property that is essential for having relations with conformal field theory. However, we are not dealing with a modular functor in the strict sense axiomatised in the mathematical literature (see e.g. [BK2, Tu]): The definition is restricted to stable surfaces $(2g - 2 + n > 0)$, and the vector spaces associated to such surfaces are infinite-dimensional in general. However, the theory described above still exhibits the most essential features of a modular functor, it is in many respects as close to a modular functor as it can be in cases where the vector spaces associated to surfaces $C_{g,n}$ are infinite-dimensional.

It would interesting to develop a generalised notion of modular functor that encompasses the quantum Teichmüller theory and the many conceivable generalizations. Some suggestions in this direction were made in [T08].

4. Conformal field theory

4.1. Definition of the conformal blocks. The Virasoro algebra $\text{Vir}_c$ has generators $L_n$, $n \in \mathbb{Z}$, satisfying the relations

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}. \hspace{1cm} (4.19)$$

We will consider irreducible highest weight representations $\mathcal{V}_\alpha$ of $\text{Vir}_c$ with $c > 1$ generated from vectors $e_\alpha$ annihilated by all $L_n$, $n > 0$, having $L_0$-eigenvalue $\alpha(Q - \alpha)$ if $c$ is parameterised as $c = 1 + 6Q^2$. 

Let $C \equiv C_{g,n}$ be a Riemann surface with genus $g$, $n$ marked points $P_1, \ldots, P_n$, and choices of local coordinates $t_r$, $r = 1, \ldots, n$ vanishing at $P_r$, respectively. It will be convenient to assume that the local coordinates $t_r$ are part of an atlas of local holomorphic coordinates on $C$ with transition functions represented by Möbius-transformations. Such an atlas defines a projective structure on $C$.

We associate highest weight representations $\mathcal{V}_r \equiv \mathcal{V}_{\alpha_r}$ of $\text{Vir}_c$ to $P_r$, $r = 1, \ldots, n$. The conformal blocks are linear functionals $\mathcal{F} : \otimes_{r=1}^n \mathcal{V}_r \to \mathbb{C}$ satisfying the invariance property

$$\mathcal{F}(\rho \chi v) = 0, \quad \forall v \in \otimes_{r=1}^n \mathcal{V}_r, \quad \forall \chi \in \mathfrak{V}_{\text{out}}(C),$$

where the notation $\mathfrak{V}_{\text{out}}(C)$ is used for the Lie algebra of meromorphic differential operators on $C$ which may have poles only at $P_1, \ldots, P_n$. The representation $\rho$ of $\mathfrak{V}_{\text{out}}(C)$ is defined on $\otimes_{r=1}^n \mathcal{V}_r$ via

$$\rho \chi = \sum_{r=1}^n \sum_{k \in \mathbb{Z}} \chi_k^{(r)} L_k^{(r)}, \quad L_k^{(r)} := \text{id} \otimes \ldots \otimes L_k \otimes \ldots \text{id},$$

where the $\chi_k^{(r)}$ are the Laurent coefficients of $\chi$ at $P_r$, $\chi(t_r) = \sum_{k \in \mathbb{Z}} \chi_k^{(r)} t_r^{k+1} \partial_t^{r}$, $t_r \in \mathbb{C}(t_r)\partial_t$. We may refer e.g. to [BF] for more details.

The vector space $\mathcal{CB}(C, \rho)$ of conformal blocks associated to the Riemann surface $C$ is the space of solutions to the defining invariance conditions (4.20). The space $\mathcal{CB}(C, \rho)$ is infinite-dimensional in general, being isomorphic to the space of formal power series in $3g - 3 + n$ variables.

**Example.** Let $n = 1$. Using the Weierstrass gap theorem it is straightforward to show that the defining condition (4.20) allows us to express the values $\mathcal{F}(v)$ for any $v \in \mathcal{V}_0$ in terms of the collection of complex numbers $\mathcal{F}(L_{-n_1} \ldots L_{-1} e_{\alpha_1})$, $n_k \in \mathbb{Z}_{\geq 0}$, $k = 1, \ldots, h$, where $h := 3g - 3 + 1$.

To any conformal block $\mathcal{F}$, let us associate the *chiral partition function* defined as the value

$$\mathcal{Z}(\mathcal{F}) := \mathcal{F}(e), \quad e := \otimes_{r=1}^n e_{\alpha_r}. \quad (4.22)$$

The vacuum representation $\mathcal{V}_0$ corresponding to $\alpha = 0$ plays a distinguished role. It can be shown that the spaces of conformal blocks with and without insertions of the vacuum representation are canonically isomorphic, see e.g. [BF], for a proof. Let the surface $C'$ be obtained from $C$ by introducing an additional marked marked point $P_0$. Let $\rho$ and $\rho'$ be the representations of $\mathfrak{V}_{\text{out}}(C)$ and $\mathfrak{V}_{\text{out}}(C')$, defined above on $\otimes_{r=1}^n \mathcal{V}_r$ and $\mathcal{V}_0 \otimes (\otimes_{r=1}^n \mathcal{V}_r)$, respectively. To each $\mathcal{F}' \in \mathcal{CB}(C', \rho')$ one may then associate a conformal block $\mathcal{F} \in \mathcal{CB}(C, \rho)$ such that

$$\mathcal{F}'(e_0 \otimes v) = \mathcal{F}(v). \quad (4.23)$$

for all $v \in \otimes_{r=1}^n \mathcal{V}_r$. This fact is often referred to as the “propagation of vacua”.

### 4.2. Deformations of the complex structure of $C$.

We shall now discuss the dependence of the spaces of conformal blocks on the choice of the Riemann surface $C$. The definition above defines sheaves of conformal blocks.
over \( \mathcal{M}_{g,n} \), the moduli space of complex structures on surfaces of genus \( g \) and \( n \) punctures. Let us consider a local patch \( \mathcal{U} \subset \mathcal{M}_{g,n} \) parameterised by local complex analytic coordinates \( q = (q_1, \ldots, q_{3g-3+n}) \), and represented by families \( C_q \) of Riemann surfaces with holomorphically varying projective structures.

A basic observation concerning the dependence of the space of conformal blocks on the complex structure is the existence of a canonical connection on the sheaves of conformal blocks over \( \mathcal{M}_{g,n} \). Let us define the infinitesimal variations

\[
\delta_\chi \mathcal{F}(v) := \mathcal{F}(\rho_\chi v),
\]

with \( \rho_\chi \) being defined via (4.21) for arbitrary \( \chi \in \oplus_{k=1}^n \mathbb{C}[[t_k]] \partial_k \). The “Virasoro uniformization theorem” (see e.g. [BF] for a proof) implies that the Teichmüller space, being the tangent space \( T \mathcal{M}_{g,n} \) to the space of complex structures \( \mathcal{M}(C) \) at \( C \) is isomorphic to the double quotient

\[
\mathcal{T}(C) \simeq \mathfrak{U}_{\text{out}}(C) \backslash \oplus_{k=1}^n \mathbb{C}[[t_k]] \partial_k / \mathfrak{U}_{\text{in}}(C); \tag{4.25}
\]

\( \mathbb{C}[[t_k]] \) denotes the space of finite Laurent series, while \( \mathfrak{U}_{\text{in}}(C) := \oplus_{k=1}^n \mathbb{C}[[t_k]] \partial_k \), with \( \mathbb{C}[[t_k]] \) being the space of finite Taylor series in the variable \( t_k \). Assuming temporarily \( \alpha_r = 0 \), \( r = 1, \ldots, n \), it follows from (4.25) that (4.24) relates the values \( \mathcal{F}(\rho_\chi e) \) to derivatives of the chiral partition functions \( Z(\mathcal{F}) \) with respect to the complex structure moduli of \( C \). More general cases for the parameters \( \alpha_r \) can be treated similarly. Using the propagation of vacua one may use (4.24), (4.25) to define a differential operator \( T(y) \) on \( \mathcal{T}(C) \) such that

\[
T(z_0)\mathcal{F}(v) = \mathcal{F}'(L_{-2\varepsilon_0} \otimes v). \tag{4.26}
\]

The defining conditions (4.20), (4.24) imply that the conformal blocks \( \mathcal{F} \) are fully characterised by the collection of all multiple derivatives of \( Z(\mathcal{F}) \).

There are two obstacles to the integration of the canonical connection on \( \mathcal{CB}(C, \rho) \), in general. The first problem is that the connection defined by (4.24) is not flat, but only projectively flat. One may, however, trivialize the curvature at least locally, opening the possibility to integrate (4.24) at least in open subsets \( \mathcal{U} \subset \mathcal{M}_{g,n} \). We will later define sections horizontal with respect to the canonical connection using the gluing construction of conformal blocks.

The other problem is that \( \mathcal{CB}(C, \rho) \) is simply way too big, multiple derivatives defined via (4.24) may grow without bound. However, there exist interesting subspaces of \( \mathcal{CB}(C, \rho) \) on which the canonical connection may be integrated. Let \( \mathcal{CB}^{an}(C, \rho) \) be the subspace of \( \mathcal{CB}(C, \rho) \) such that \( \mathcal{Z}(\mathcal{F}_{C_q}) \equiv \mathcal{Z}(\mathcal{F}_q) \) can be analytically continued over all of \( \mathcal{T}(C) \). Projective flatness of the canonical connection implies that we may in this way define a projective representation of the mapping class group on \( \mathcal{CB}^{an}(C, \rho) \). We will later briefly describe nontrivial evidence for the existence of a Hilbert-subspace \( \mathcal{H}_{\text{CFT}}(C, \rho) \) of \( \mathcal{CB}^{an}(C, \rho) \) which is closed under this action. The projective representation of the mapping class group on \( \mathcal{H}_{\text{CFT}}(C, \rho) \) will then define an infinite-dimensional unitary projective local system \( \mathcal{W}(C) \) over \( \mathcal{M}(C) \). This seems to be the best possible scenario one can hope for when the spaces of conformal blocks are infinite-dimensional.
It is known [FS] that the projectiveness of the local systems originating from the canonical connection on spaces of conformal blocks can be removed by tensoring with the projective line bundle $\mathcal{E}_c = (\lambda_H)^\sharp$, where $\lambda_H$ is the Hodge line bundle. It follows that $\mathcal{V}(C) := \mathcal{W}(C) \otimes \mathcal{E}_c$ is an ordinary holomorphic vector bundle over $\mathcal{M}(C)$. We are next going to describe how to construct global sections of $\mathcal{V}(C)$ by means of the gluing construction.

4.3. Gluing construction of conformal blocks. We are now going to construct large families of conformal blocks by means of the gluing construction.

4.3.1. Gluing Riemann surfaces. Let $C$ be a possibly disconnected Riemann surface, $q \in \mathbb{C}$ with $|q| < 1$, and $D_i(q) := \{ P \in C; |z_i(P)| < |q|^{-\frac{1}{2}} \}$, $i = 1, 2$ be non-intersecting discs with local coordinate $z_i(P)$ vanishing at points $P_{0,i}$, for $i = 1, 2$, respectively. Let us then define a new Riemann surface $C^2$ by identifying the annuli $A_i(q) := \{ P \in C; |q|^{\frac{1}{2}} < |z_i(P)| < |q|^{-\frac{1}{2}} \}$ iff the coordinates $z_i(Q_i)$ of points $Q_i \in A_i$ satisfy $z_1(Q_1)z_2(Q_2) = q$. The gluing parameter $q$ becomes part of complex structure moduli of $C^2$. By iterating this construction one may build Riemann surfaces $C_{g,n}$ of arbitrary genus $g$ and arbitrary number $n$ of punctures from three-punctured spheres $C_{0,3}$.

The surfaces $C_{g,n}$ obtained in this way come with a collection of embedded annuli $A_r(q_r)$, $r = 1, \ldots, h$, $h := 3g - 3 + n$. As the complex structure on $C_{0,3} \simeq \mathbb{P}^1 \setminus \{ 0, 1, \infty \}$ is unique, one may use $q = (q_1, \ldots, q_h)$ as local coordinates for $\mathcal{M}_{g,n}$ in a multi-disc centered around the boundary component in the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$ represented by the nodal surface obtained in the limit $\text{Im}(q_r) = 0$, $r = 1, \ldots, h$. It is possible to cover $\overline{\mathcal{M}}_{g,n}$ by local charts corresponding to the pants decompositions of $C$ [HV]. In order to get local coordinates for the Teichmüller spaces $\mathcal{T}(C)$ one may parameterise $q_r = e^{2\pi i \tau_r}$. Different local charts $\mathcal{U}_r \subset \mathcal{T}(C)$ defined by the gluing construction can be labelled by the pairs $\sigma = (\gamma, \Gamma)$ introduced in Section 2.3.2.

Using the coordinates around the punctures of $C_{0,3}$ coming from the representation as $\mathbb{P}^1 \setminus \{ 0, 1, \infty \}$ in the gluing construction one gets an atlas on $C$ with transition functions represented by Möbius-transformations defining a projective structure on $C_{g,n}$. By varying the gluing parameters $q_r$ one gets local holomorphic sections of the affine bundle $\mathcal{P}(C)$ of projective structures over $\mathcal{U}_r$.

4.3.2. Gluing conformal blocks. Let us first consider Riemann surfaces $C_{2g+1}$ obtained by gluing two surfaces $C_i$ with $n_i + 1$, $i = 1, 2$ boundary components, respectively. Let $n = n_1 + n_2$, and let $I_1$, $I_2$ be sets such that $I_1 \cup I_2 = \{ 1, \ldots, n \}$. Let $\mathcal{F}_{C_i} \subset \mathcal{CB}(C_i, \rho_i)$, $i = 1, 2$ be conformal blocks with $\rho_i$ acting on $\mathcal{V}^{[n_i]} = \mathcal{V}_\beta \otimes (\otimes_{\gamma \in I_i} \mathcal{V}_\gamma)$ for $i = 1, 2$, respectively. Let $(\ldots)_{\mathcal{V}_\beta}$ be the $\mathfrak{vir}_c$-invariant bilinear form on $\mathcal{V}_\beta$, and $(v_j; j \in \mathbb{I}_\beta), (\tilde{v}_j; j \in \mathbb{I}_\beta)$ be dual bases for $\mathcal{V}_\beta$ satisfying $\langle v_j, \tilde{v}_{j'} \rangle_{\mathcal{V}_\beta} = \delta_{j, j'}$. For given $v_i \in \otimes_{\gamma \in I_i} \mathcal{V}_\gamma$ let $V_i(v_i)$ be the vectors in $\mathcal{V}_\beta$ defined by

$$V_i(v_i) := \sum_{j \in \mathbb{I}_\beta} \tilde{v}_j \mathcal{F}_{C_i}(v_j \otimes v_i). \quad (4.27)$$
A conformal block associated to the surface \(C_2 \# C_1\) can then be constructed as
\[
\mathcal{F}_{C_2 \# C_1}^\beta (v_2 \otimes v_1) := \langle V_2(v_2), q^{\nu} V_1(v_1) \rangle_{\mathcal{V}_\beta}.
\] (4.28)

An operation representing the gluing of two boundary components of a single Riemann surface can be defined in a very similar way.

4.3.3. Gluing from pairs of pants. One can construct any Riemann surface \(C\) by gluing pairs of pants. Different ways of doing this are labelled by pants decompositions \(\sigma\). The building blocks, the conformal blocks associated to \(C_{0,3}\), are uniquely defined by the invariance property (4.20) up to the value of \(\mathcal{F}_{C_{0,3}}\) on the product of highest weight vectors
\[
N(\alpha_3, \alpha_2, \alpha_1) := \mathcal{F}_{C_{0,3}}(e_{\alpha_3} \otimes e_{\alpha_2} \otimes e_{\alpha_1}).
\] (4.29)

Using the gluing construction recursively leads to the definition of a family of conformal blocks \(\mathcal{F}_{\beta, q}^\sigma\) depending on the choice of pants decomposition \(\sigma = (\gamma, \Gamma)\), the coordinate \(q\) for \(\mathcal{U}_q \subset T(C)\) defined by the gluing construction, and an assignment \(\beta : e \mapsto \beta_e \in \mathbb{C}\) of complex numbers to the edges \(e\) of \(\Gamma\). The parameters \(\beta_e\) determine the Virasoro representations \(\mathcal{V}_{\beta_e}\) to be used in the gluing construction.

The partition functions \(Z_\sigma(\beta, q)\) defined from \(\mathcal{F}_{\beta, q}^\sigma\) via (4.22) represent local sections of \(\mathcal{V}(C)\) which are horizontal with respect to the canonical connection defined in Section 4.2.

4.3.4. Change of pants decomposition. It turns out that the partition functions \(Z_\sigma(\beta, q)\) constructed by the gluing construction in a neighborhood of the asymptotic region of \(T(C)\) that is determined by \(\sigma_1\) have an analytic continuation to the asymptotic region of \(T(C)\) determined by a second pants decomposition \(\sigma_2\). Based on [101, 103a] it was proposed in [101, 13] that the analytically continued partition functions \(Z_{\sigma_2}(\beta_1, q)\) are related to the functions \(Z_{\sigma_2}(\beta_2, q)\) by linear transformations of the form
\[
Z_{\sigma_1}(\beta_1, q) = E_{\sigma_1, \sigma_2}(q) \int d\mu(\beta_2) W_{\sigma_1, \sigma_2}(\beta_1, \beta_2) Z_{\sigma_2}(\beta_2, q).
\] (4.30)

The transformations (4.30) define the infinite-dimensional vector bundle \(\mathcal{V}(C) = \mathcal{E}_c \otimes \mathcal{W}(C)\). The constant kernels \(W_{\sigma_1, \sigma_2}(\beta_1, \beta_2)\) represent the transition functions of the projective local system \(\mathcal{W}(C)\), while the pre-factors \(E_{\sigma_1, \sigma_2}(q)\) can be identified as transition functions of the projective line bundle \(\mathcal{E}_c\).

It is enough to establish (4.30) for the cases \(C = C_{0,4}\) and \(C_{1,1}\) since the Moore-Seiberg groupoid is generated from the F-, S-, B- and Z-moves. A partly conjectural\(^6\) argument was proposed in [101, 103a] suggesting that the F-move can be realised by an integral transformation of the form
\[
Z_{\sigma_1}(\beta_1, q) = \int d\beta_2 F_{\beta_1, \beta_2}^{\alpha_3, \alpha_2} Z_{\sigma_2}(\beta_2, q);
\] (4.31)

\(^6\)The main conjecture is the integrability of the representation of the algebra [101, equation (201)], equivalent to the validity of the representation [101, equation (202)].
where \( S := \frac{Q}{2} + i \mathbb{R}^+ \). The relevant pants decompositions \( \sigma_s \) and \( \sigma_t \) are depicted on the left and right half of Figure 2 respectively. We assume that \( \beta_1 \in S \), and that the parameters \( \alpha_i \in S, i = 1, 2, 3, 4 \) label the representations assigned to the boundary components of \( C_{0,4} \) according to the labelling indicated in Figure 2.

It was shown in [HJS] that (4.31) implies the following realisation of the S-move

\[
Z_{\sigma_s}(\beta_1, q) = e^{\pi i \frac{Q}{2} \tau (\tau + 1/\tau)} \int_{S} d\beta_2 S_{\beta_1, \beta_2}(\alpha_0) Z_{\sigma_t}(\beta_2, q),
\]

(4.32)

The pants decompositions \( \sigma_s \) and \( \sigma_t \) for \( C = C_{1,1} \) are depicted in Figure 3.

5. Comparison with the quantization of the moduli spaces of flat connections

One may now compare the representation of the Moore-Seiberg groupoid obtained from the quantisation of \( \mathcal{M}_{\text{flat}}(C) \) to the one from conformal field theory. It turns out that one finds exact agreement if (i) the representation parameters are identified as

\[
\beta_e = \frac{Q}{2} + i \frac{l_e}{4\pi b}, \quad \alpha_e = \frac{Q}{2} + i \frac{l_r}{4\pi b}, \quad Q = b + b^{-1},
\]

where \( r = 1, \ldots, n \), respectively, and if (ii) a suitable normalisation constant \( N(\alpha_3, \alpha_2, \alpha_1) \) is chosen\(^7\) in (4.29). This implies that there are natural Hilbert-subspaces \( \mathcal{H}_{\text{CFT}}(C, \rho) \) of the spaces of conformal blocks \( \mathcal{CB}(C, \rho) \) on which the mapping class group action is unitary. These subspaces have (distributional) bases generated by the conformal blocks \( \mathcal{F}_{\beta, \gamma} \) constructed by the gluing construction with \( \beta_e \in S \) for all edges \( e \) of \( \sigma \). The Hilbert spaces \( \mathcal{H}_{\text{CFT}}(C, \rho) \) are isomorphic as representations of the Moore-Seiberg groupoid to the Hilbert spaces constructed in the quantisation of \( \mathcal{M}_{\text{flat}}(C) \) in Section 4.

In the rest of this section we will compare the representations of two algebras of operators that arise naturally in the two cases, respectively: The first is the algebra \( \mathcal{A}_{\text{flat}}(C) \) generated by the quantised trace functions. This algebra will be realised naturally on spaces of conformal blocks in terms of the so-called Verlinde loop operators [AGGTv, DGOT]. The second algebra of operators is the algebra of holomorphic differential operators on the Teichmüller spaces \( \mathcal{T}(C) \). This algebra is naturally realised on the conformal blocks via (4.24). We will briefly discuss, following [TV13], how a natural realisation is motivated from the point of view of the quantisation of \( \mathcal{M}_{\text{flat}}(C) \).

5.1. Verlinde line operators. We shall now define a family of operators \( L_\gamma \), called Verlinde line operators labelled by closed curves \( \gamma \) on \( C \) acting on spaces of conformal blocks. It will turn out that the operators \( L_\gamma \) generate a representation of the algebra \( \mathcal{A}_{\text{flat}}(C) \) on the spaces of conformal blocks isomorphic to the one

\(^7\) \( N(\alpha_3, \alpha_2, \alpha_1) = (C(Q - \alpha_3, \alpha_2, \alpha_1))^\frac{1}{2} \), where \( C(\alpha_3, \alpha_2, \alpha_1) \) is the function defined in [ZZ].
from the quantisation of $\mathcal{M}_{\text{flat}}(C)$. To define the operators $L_\gamma$ we will need a few preparations of interest in their own right.

5.1.1. Analytic continuation. The kernels $W_{\sigma_1,\sigma_2}(\beta_1,\beta_2)$ representing the transformations have remarkable analytic properties both with respect to the variables $\beta_2$, $\beta_1$, and with respect to the parameters $\alpha_r$, $r = 1,\ldots,n$ of the representations assigned to the marked points $[TV13]$. An argument has furthermore been put forward in [1035] indicating the absolute convergence of the series representing $Z_\sigma(\beta,q)$. If the normalisation constant in (4.29) is chosen to be $N(\alpha_3,\alpha_2,\alpha_1) = 1$, one may then show that the functions $Z_\sigma(\beta,q)$ are entire in the variables $\alpha_r$, and meromorphic in the variables $\beta_i$, having poles only if $\beta_i \in \mathbb{D}$, where $\mathbb{D} := \frac{b}{2}Z^{2,0} - \frac{1}{2b}Z^{2,0}$.

This suggests that one may embed the space $H_{\text{CFT}}(C,\rho)$ into a larger space $D_{\text{CFT}}(C,\rho)$ which contains in particular the conformal blocks constructed using the gluing construction for generic complex $\beta_e \notin \mathbb{D} := \frac{b}{2}Z^{2,0} - \frac{1}{2b}Z^{2,0}$. We will later characterise the spaces $D_{\text{CFT}}(C,\rho)$ more precisely. We may note, however, that the analytic properties of $W_{\sigma_1,\sigma_2}(\beta_1,\beta_2)$ and $Z_\sigma(\beta,q)$ ensure that the relations (4.30) can be analytically continued. The resulting relations characterise the realisation of the Moore-Seiberg groupoid on the spaces $D_{\text{CFT}}(C,\rho)$.

5.1.2. Degenerate punctures. The representations $\mathcal{V}_\alpha$ with $\alpha \in \mathbb{D}$ are called degenerate expressing the fact that the vectors in $\mathcal{V}_\alpha$ satisfy additional relations. Most basic are the cases where $\alpha = 0$, and $\alpha = -b^{\pm 1}/2$. In the first case one has $L_{-1} \mathcal{V}_0 = 0$, in the second case $(L^2_{-1} + b^{\pm 2} L_{-2}) \mathcal{V}_{-b^{\pm 2}/2} = 0$.

Let $C'$ be obtained from $C$ by introducing an additional marked point $z_0 \in C$. Analytically continuing conformal blocks with respect to the parameters $\alpha_r$, $r = 0,\ldots,n$ allows one, in particular, to consider the cases where, for example, $\alpha_0 \notin \mathbb{D}$. If $\alpha_0 = 0 \in \mathbb{D}$, it turns out that $D_{\text{CFT}}(C',\rho') \simeq D_{\text{CFT}}(C,\rho)$, as required by the propagation of vacua. In the cases $\alpha_0 = -b^{\pm 1}/2$ it can be shown that the partition functions $Z(F'_q)$ for $F'_q \in D_{\text{CFT}}(C',\rho')$ satisfy equations of the form

$$\left[ \partial^2_{z_0} + b^{\pm 2} \mathcal{T}(z_0) \right] Z(F'_q) = 0,$$  \hspace{1cm} (5.34)

where $\mathcal{T}(z_0)$ is a certain first order differential operator that transforms under changes of local coordinates on $C$ as a quadratic differential.$^8$ We will refer to these equations as the Belavin-Polyakov-Zamolodchikov (BPZ-) equations. It follows in particular that $D_{\text{CFT}}(C',\rho') \simeq C^2 \otimes D_{\text{CFT}}(C,\rho)$, with the two linearly independent solutions of (5.34) corresponding to the two elements of a basis for $C^2$.

5.1.3. Definition of the Verlinde line operators. Consideration of multiple degenerate punctures reveals some interesting phenomena. If, for example, $C''$ is obtained from $C$ by introducing two additional punctures at $z_0$ and $z_{-1}$ with $\alpha_0 = \alpha_{-1} = -b/2$ one finds a subspace of $D_{\text{CFT}}(C'',\rho'') \simeq C^2 \otimes C^2 \otimes D_{\text{CFT}}(C,\rho)$ naturally isomorphic to $D_{\text{CFT}}(C,\rho)$. This is similar (in fact related) to the fact that the tensor product of two two-dimensional representations of $\mathfrak{sl}_2$ contains

$^8$Remember that we had fixed a family of reference projective structures in the very beginning.
a one-dimensional representation. This phenomenon allows us to define natural embeddings and projections
\[ \iota : \mathcal{D}_{\text{CFT}}(C, \rho) \hookrightarrow \mathcal{D}_{\text{CFT}}(C'', \rho''), \]
\[ \varphi : \mathcal{D}_{\text{CFT}}(C'', \rho'') \rightarrow \mathcal{D}_{\text{CFT}}(C, \rho). \] (5.35)

Note furthermore that the mapping class group \( \text{MCG}(C) \) contains elements \( \mu_\gamma \) labelled by closed curves \( \gamma \) on \( C \), corresponding to the variation of the position of \( z_0 \) along \( \gamma \). This allows us to define a natural family of operators on the spaces \( \mathcal{D}_{\text{CFT}}(C, \rho) \) as
\[ L_\gamma := \varphi \circ M(\mu_\gamma) \circ \iota, \] (5.36)
where \( M(\mu) \) is the operator representing \( \mu \) on \( \mathcal{D}_{\text{CFT}}(C'', \rho'') \). The operators \( L_\gamma \) are called Verlinde line operators.

Comparing the explicit formulae for the Verlinde line operators calculated in [AGGT] with the formulae for the operators \( \pi_s(L_\gamma) \) found in [TV13] (see Section 3.3 above) one finds a precise match. This means that there is a natural action of the algebra \( A_{2d}(C) \) of quantised trace functions on spaces of conformal blocks. This action naturally defines dense subspaces \( \mathcal{S}_{\text{CFT}}(C, \rho) \subset \mathcal{H}_{\text{CFT}}(C, \rho) \) as maximal domains of definition for \( A_{2d}(C) \) such that \( \mathcal{D}_{\text{CFT}}(C, \rho) \) is the dual space of distributions forming a so-called Gelfand-triple \( \mathcal{S}_{\text{CFT}}(C, \rho) \subset \mathcal{H}_{\text{CFT}}(C, \rho) \subset \mathcal{D}_{\text{CFT}}(C, \rho) \). The spaces \( \mathcal{S}_{\text{CFT}}(C, \rho) \) and \( \mathcal{D}_{\text{CFT}}(C, \rho) \) are isomorphic as \( A_{2d}(C) \)-modules to the spaces \( \mathcal{S}(C) \) and \( \mathcal{D}(C) \) introduced in Section 5.3 respectively.

### 5.2. Kähler quantization of \( \mathcal{T}(C) \).

The relation between conformal field theory and the quantisation of \( \mathcal{T}(C) \) can be tightened considerably by considering an alternative quantisation scheme for \( \mathcal{T}(C) \) [TV03b,TV13] that we shall now discuss. Teichmüller theory allows one to equip \( \mathcal{T}(C) \) with natural complex and symplectic structures. The natural symplectic form \( \Omega_{\text{WP}} \) on \( \mathcal{T}(C) \) coincides with the restriction of the symplectic form \( \Omega_{\text{AB}} \) on \( \mathcal{M}_{\text{flat}}(C) \) to the Teichmüller component \( \mathcal{M}_{\text{flat}}(C) \). Natural functions on \( \mathcal{T}(C) \) are given by the values of the quadratic differential \( t(y) \equiv t(y|q, \bar{q}) \) defined from the metric of constant negative curvature \( e^{2y}dyd\bar{y} \) on \( C \) as \( t(y) = -(\bar{\partial} \varphi)^2 + \partial \varphi \). One may find a basis \( \{ \partial_r; r = 1, \ldots, h \} \) for the space \( H^0(C, K^2) \) of holomorphic quadratic differentials on \( C \) such that the functions \( H_r \equiv H_r(q, \bar{q}) \) on \( \mathcal{T}(C) \) defined via \( t(y) = \sum_{r=1}^{h} \partial_r(y)H_r \), are canonically conjugate to the complex analytic coordinates \( q_r \) on \( \mathcal{T}(C) \) in the sense that \( \{ H_r, q_s \} = \delta_{r,s} \) [TZST] [TT03].

In the corresponding quantum theory it is natural to realize the operators \( H_r \) corresponding to \( H_r \) as differential operators \( \partial^2 \partial_{q_r} \), and to represent states by holomorphic wave-"functions" \( \Psi^\sigma(q) \) [TV13]. The operator corresponding to the quadratic differential \( t(y) \) will be a differential operator \( \mathcal{T}(y) \). This operator coincides with the operator defined in (5.26). Recall that the space of conformal blocks \( \mathcal{CB}^{an}(C, \rho) \) can be identified with the space of holomorphic functions on \( \mathcal{T}(C) \). These observations suggest us to identify the space of states in the quantum theory of \( \mathcal{T}(C) \) with suitable subspaces of \( \mathcal{CB}^{an}(C, \rho) \).  

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More precisely holomorphic sections of the projective line bundle \( \mathcal{E}_s \). \( \Psi^\sigma(q) \) depends on the choice of a pants decomposition as the definition of the coordinates \( q \) depends on it.
It is natural to require that the mapping class group is represented on the wave-
"functions" $\Psi^\sigma(q)$ as deck-transformations $(M(\mu)\Psi^\sigma)(q) = \Psi^\sigma(\mu.q)$, where $\mu.q$ is
the image of the point $q$ in $T(C)$ under $\mu$. One may then show that $\Psi^\sigma_l(q)$, where $\sigma = (\gamma, \Gamma)$, $\gamma = (\gamma_1, \ldots, \gamma_h)$, $\Psi^\sigma_l(q)$ is the wave-function of an eigenstate of
the operators $L^{\gamma}$, $e = 1, \ldots, h$, and the variables are related via (5.33), respectively.

The observations made in this section indicate that conformal field theory is
nothing but another language for describing the quantum theories obtained by
quantisation of $\mathcal{M}_{\text{flat}}(C)$.

6. Further connections

The theory outlined above generalises and unifies various themes of mathematical
research. As an outlook we shall now briefly mention some of these connections,
some of which offer interesting perspectives for future research.

6.1. Relation with non-compact quantum groups. There is an
interesting non-compact quantum group called modular double of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ $\mathcal{F}_9$
which is on the algebraic level isomorphic to $\mathcal{U}_q(\mathfrak{sl}_2)$, and has a set of unitary
irreducible representations $P_s$, $s \in \mathbb{R}^+$ characterized by a remarkable self-duality
property: They are simultaneously representations of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ and $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$ with $\tilde{q} = e^{\pi i b^2}$ if $q = e^{\pi i b}$ $\mathcal{F}_9$. This family of representations is closed
under tensor products $\mathcal{P}$, and there exists a non-compact quantum group
$\mathcal{SL}^{+}_q(2, \mathbb{R})$ deforming a certain subspace of the space of functions on $\mathcal{SL}(2, \mathbb{R})$ which
has a Plancherel-decomposition into the representations $P_s$, $s \in \mathbb{R}^+$, $\mathcal{P}$.

There exists strong evidence $\mathcal{P}$ for an equivalence of braided tensor categories
of Kazhdan-Lusztig type $\mathcal{K}$ between the category of unitary representations of the Virasoro algebra having simple objects $\mathcal{V}_\alpha$, $\alpha \in \mathcal{S}$, with the category having the representations $\mathcal{P}_s$, $s \in \mathbb{R}^+$ of the modular double as simple objects. The
kernel representing the F-move coincides with the 6j-symbols of the modular double $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ $\mathcal{T}_1$. The complex numbers numbers representing the B-move coincide with the eigenvalues of the R-matrix of the modular double $\mathcal{B}$. The results of $\mathcal{TV}_1$ furthermore imply that the braided tensor category of unitary representations of the modular double has a natural extension to a modular
tensor category.

6.2. Relations to three-dimensional hyperbolic geometry. The
Teichmüller theory has numerous relations to hyperbolic geometry in three dimensions. Let us consider, for example, the Fenchel-Nielsen coordinates $(l_s, \kappa_s)$ and $(l_t, \kappa_t)$ associated to the pants decompositions on the left and on the right of Figure 3, respectively. It was observed in $\mathcal{NRS}$ that the generating function $W(l_s, l_t)$

\begin{equation}
Z_\sigma(\beta, q) = \Psi^\sigma_l(q),
\end{equation}

where $\sigma = (\gamma, \Gamma)$, $\gamma = (\gamma_1, \ldots, \gamma_h)$. The variables are related via (5.33), respectively.

The main open problem is the issue pointed out in Section $\mathcal{L}_3$.1
for the change of Darboux coordinates \((l_s, \kappa_s) \leftrightarrow (l_t, \kappa_t)\) for \(T(C)\), defined by the relations

\[
\kappa_s = \frac{\partial W}{\partial l_s}, \quad \kappa_t = -\frac{\partial W}{\partial l_t},
\]

(6.38)

coincides with the volume \(\text{Vol}_T(l)\) of the hyperbolic tetrahedron with edge lengths

\[l = (l_1, l_2, l_3, l_4, l_s, l_t)\], with \(l_i, \ i = 1, 2, 3, 4\) being the hyperbolic lengths of the boundaries of \(C_{0,4}\).

It is therefore not unexpected to find relations to hyperbolic geometry encoded within quantum Teichmüller theory. Considering the limit \(b \to 0\) of the kernel \(F(l_s, l_t) := F_{l_s l_t} \left[ l_1^{i_1} l_2^{i_2} \right]\) appearing in (3.16) one may show that \(\lim_{b \to 0} b^2 \log F(l_s, l_t)\) is equal to the volume \(\text{Vol}_T(l)\) of the hyperbolic tetrahedron considered above. This follows from the fact that (3.14) reduces to (6.38) in the limit \(b \to 0\). A closely related result was found in [TV13] by direct calculation.

Braided tensor categories of representations of compact quantum groups can be used to construct invariants of three-manifolds [Tu, BK2]. It should be interesting to investigate similar constructions using the modular tensor category associated to the modular double. It seems quite possible that the resulting invariants are related to the invariants constructed in [HK, DGLZ, AK]. If so, one would get an interesting perspective on the variants of the volume conjecture formulated in [HK, DGLZ, AK]. It could be a consequence of the relations between quantum Teichmüller theory and hyperbolic geometry pointed out above, which are natural consequences of known relations between Teichmüller theory and three-dimensional hyperbolic geometry.

### 6.3. Relations with integrable models.

There are several connections between the mathematics reviewed in this article and the theory of integrable models. We will here describe some connections to the theory of isomonodromic deformations of certain ordinary differential equations, for \(g = 0\) closely related to the equations studied by Painlevé, Schlesinger and Garnier. Further connections are described in [BT2, T10].

#### 6.3.1. Relations with isomonodromic deformations I

The limit \(b \to 0\) of the BPZ-equations (5.34) is related to isomonodromic deformations [T10].

Let us consider the case of a Riemann surface \(C \equiv C_{g,n+d+1}\) with \(n + d + 1\) marked points \(z_1, \ldots, z_n, u_1, \ldots, u_d\) and \(y\). For convenience let us assume that \(u_1, \ldots, u_d\) and \(y\) lie in a single chart of the surface \(C\) obtained from \(\hat{C}\) by filling \(u_1, \ldots, u_d\) and \(y\). The resulting loss of generality will not be very essential. We associate representations with generic value of the parameter \(\alpha_r\) to \(z_r\) for \(r = 1, \ldots, n\), degenerate representations with parameter \(-1/2b\) to the points \(u_1, \ldots, u_d\), and a degenerate representation with parameter \(-b/2\) to the point \(y\). The partition functions \(Z(q) \equiv Z(\mathcal{F}_q), \mathcal{F}_q \equiv \mathcal{F}_{C_q}\) will then satisfy a system of \(d + 1\) partial differential equations of the form,

\[
\begin{align*}
\left[ b^{k^2} \partial^2_{u_k} + T_k(u_k) \right] Z(\mathcal{F}_q) &= 0, \quad k = 1, \ldots, d, \quad (6.39a) \\
\left[ b^{-2} \partial^2_y + T_0(y) \right] Z(\mathcal{F}_q) &= 0. \quad (6.39b)
\end{align*}
\]
In the limit $b \to 0$ one may solve this system of partial differential equation with an ansatz of the form $Z(q) = \exp(\frac{1}{b}W(q'))\psi(y)(1 + O(b^2))$, where $W(q')$ does not depend on $y$. Equation (6.39b) implies that $\psi(y)$ satisfies $(\partial^2_y + t(y))\psi(y) = 0$, where $t(y) = b^2T_0(y)W(q')$. Equations (6.39) imply that $v_k := \partial_u, W$ satisfy

$$v_k^2 + t_{k,2} = u_k^2 + t_{k,2} = 0, \quad k = 1, \ldots, d,$$

(6.40)

with $t_{k,2}$ defined from $t(y) = \sum_{i=0}^\infty t_{k,i}(y - u_k)^{i-2}$. It follows that $\partial^2_y + t(y)$ has $d$ apparent singularities at $y = u_k$. Let $\vartheta_k(y)(dy)^2$ be a basis for $H^0(C, K^2)$, and let us define $H_k$ via $t(y) = \sum_{k=1}^d H_k \vartheta_k(y)$. In the case $d = 3g - 3 + n$ one has enough equations (6.40) to determine the $H_k \equiv H_k(u, v)$ as functions of $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$.

It is automatic that the monodromy of $\partial^2_y + t(y)$ will be unchanged under variations of the complex structure of $C$, which is equivalent to [Ok] [1c] to

$$\frac{\partial \vartheta_k}{\partial u_k} = \frac{\partial H_r}{\partial v_k}, \quad \frac{\partial \vartheta_k}{\partial v_k} = -\frac{\partial H_r}{\partial u_k},$$

(6.41)

$\{\vartheta_k: k = 1, \ldots, d\}$ being the basis for $TT(C)$ dual to the basis $\{\vartheta_k, k = 1, \ldots, d\}$ for $H^0(C, K^2) \simeq T^*T(C)$. It follows that the system of BPZ-equations (6.39) describes a quantisation of the isomonodromic deformation problem [110].

### 6.3.2. Relations with isomonodromic deformations II

A somewhat unexpected relation between conformal blocks and the isomonodromic deformation problem arises in the limit $c \to 1$. A precise relation between the tau-function for Painlevé VI and Virasoro conformal blocks was proposed in [GL]. A proof of this relation, together with its generalization to the tau-functions of the Schlesinger system was given in [LT]. The relations established in [LT] are

$$\tau(\lambda, \kappa; q) = \sum_{m \in \mathbb{Z}^N} e^{i\kappa \cdot m} Z_\sigma(\lambda + m, q),$$

(6.42)

where $N = n - 3$, $Z_\sigma(\beta, q)$ are the chiral partition functions associated to the conformal blocks defined using the gluing construction in the case $C = C_{0,n}$, and $\tau(\lambda, \kappa; q)$ is the isomonodromic tau-function, defined by $H_r = -\partial_{\vartheta_k} \tau(\lambda, \kappa; q)$, here considered as a function of the monodromy data parameterised in terms of Darboux coordinates $(\lambda, \kappa)$ for $M_{\text{mult}}(C)$ closely related to the coordinates used in [NRS].

In order to prove (6.42), the authors of [LT] consider partition functions $Z(F_q^{'m})$ of conformal blocks $F_q^{'m} \in D_{c_{\text{FT}}}(C'', \rho''')$ with two additional degenerate punctures as in Section 5.1. Recall that one gets an action of $\pi_1(C)$ on $D_{c_{\text{FT}}}(C'', \rho''')$ from monodromies of one of the degenerate punctures. The isomorphism $D_{c_{\text{FT}}}(C'', \rho'') \simeq \mathbb{C}^4 \otimes D_{c_{\text{FT}}}(C, \rho)$ allows us to represent the action of $\pi_1(C)$ on $D_{c_{\text{FT}}}(C'', \rho'')$ in terms of matrices having elements which are difference operators acting on $D_{c_{\text{FT}}}(C, \rho)$. The remarkable fact observed in [LT] is that the appearing difference operators can be diagonalised simultaneously by a generalised Fourier-transformation similar to (6.42) (provided that $y = 0$ and $c = 1$). This observation yields in particular a new and more effectively computable way to solve the classical Riemann-Hilbert problem [LT].
7. Outlook: Harmonic analysis on \( \text{Diff}(S^1) \) ?

Let \( \Pi_j \) be a unitary irreducible representation \( \Pi_j : G \to \text{End}(V_j) \) of a finite-dimensional Lie group on a Hilbert space \( V_j \) with scalar product \( (.,.) : V_j \otimes V_j \to \mathbb{C} \), \( j \) being a label for elements in the set of irreducible unitary representations of \( G \). Matrix elements such as \( (v_2, \Pi_j(g)v_1) \), \( v_i \in V_j \) for \( i = 1,2 \), play a fundamental role in the harmonic analysis of the Lie group \( G \). They allow us to realise the abstract Plancherel decomposition \( L^2(G) \approx \int_U d\mu(j) V_j \otimes V_j^\prime \) as a generalised Fourier-transformation

\[
f(g) = \int_U d\mu(j) \sum_{i,i'\in I_j} (v_i, \Pi_j(g)v_{i'}) \tilde{f}_{i'}(j),
\]

with \( \{v_i; i \in I_j\} \) being an orthonormal basis for \( V_j \). If the representations \( V_j \) contain unique vectors \( v_2^j, v_1^j \) invariant under subgroups \( H_2 \) and \( H_1 \), respectively, one may similarly represent functions on the double quotients \( H_2 \backslash G/H_1 \), as

\[
f(g) = \int_U d\mu(j) (v_2^j, \Pi_j(g)v_1^j) \tilde{f}(j).
\]

The functions \( \mathcal{Y}(j,g) := (v_2^j, \Pi_j(g)v_1^j) \) are called spherical or Whittaker functions depending on the type of subgroups \( H_2 \) and \( H_1 \) under consideration. Equation (7.44) expressed the completeness of the functions \( \mathcal{Y}(j,g) \) within \( L^2(H_2 \backslash G/H_1) \).

Turning back to conformal field theory let us consider the conformal blocks constructed by the gluing construction as described in Section 4.3.2. The partition function \( Z(\beta, g) \) can be represented as a matrix element in the form \( Z(\beta, g) = \langle V_2, q^{L_0}V_1 \rangle \). We could consider, more generally

\[
Z(\beta, g) = \langle V_2, \Pi_\beta(g)V_1 \rangle,
\]

where \( g \in \text{Diff}^+(S^1) \), and \( \Pi_\beta \) is the projective unitary representation of \( \text{Diff}^+(S^1) \) related to the representation \( V_\beta \) of the Virasoro algebra by exponentiation, \( \Pi_\beta(e^f) = e^{\pi a(T(f))} \), for \( f(\sigma)\partial_\sigma = \sum_{n\in\mathbb{Z}} f_n e^{i\pi \sigma} \partial_\sigma \) being a real smooth vector field on \( S^1 \), \( T[f] = \sum_{n\in\mathbb{Z}} f_n L_n \). Equation (7.44) will define a function on \( \text{Diff}^+(S^1) \) that has an analytic continuation to the natural complexification of \( \text{Diff}^+(S^1) \), the semi-group of annuli \( \mathfrak{A}n \) defined in [56]. One should note, however, that the states \( V_2 \) and \( V_1 \) will be annihilated by large sub-semigroups of \( \mathfrak{A}n_2 \) and \( \mathfrak{A}n_1 \) of \( \mathfrak{A}n \), obtained by exponentiation of the Lie-subalgebras of the Virasoro algebra generated by vector fields on \( S^1 \) that extend holomorphically to \( (C_i \backslash D_i) \cup A_i \), for \( i = 1, 2 \), respectively. This means that \( Z(\beta, g) \) will be a function on the double coset \( \mathfrak{A}n_2 \backslash \mathfrak{A}n / \mathfrak{A}n_1 \) which can be identified with an open subset of the Teichmüller space \( T(C) \).

This suggests to view the functions \( Z(\beta, g) \) as analogs of spherical or Whittaker functions. By taking certain collision limits where the punctures of \( C_{0,i} \) collide in pairs one may even construct honest Whittaker vectors of the Virasoro algebra from the states \( V_i \), \( i = 1, 2 \), making the analogy even more close. From this point of view it is intriguing to compare formula (4.31) with (7.44). It is
Quantization of $M_{\text{flat}}(C)$ and conformal field theory

\begin{equation}
\text{tempting to view formula (4.31) as an expression of the possible completeness of the functions within a - yet to be defined - space of “square-integrable” functions on } T(C), \text{ which in turn is related to a certain coset of the semigroup } \mathfrak{A} n \text{ according to the discussion above.}
\end{equation}

These remarks suggest that the relations between conformal field theory and the quantisation of the moduli spaces of flat PSL(2, $\mathbb{R}$)-connections observed in Section 5 should ultimately be understood as results of "quantisation commutes with reduction"-type. Quantisation of (a space containing) $T^*G$, $G = \text{Diff}^+(S^1)$ should produce an infinite-dimensional picture close to conformal field theory. The reduction to the finite-dimensional quantum theory of the Teichmüller spaces is a consequence of the invariances of the vectors $V_i$, $i = 1, 2$.

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