A fluid generalization of membranes

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Abstract: In a certain sense a perfect fluid is a generalization of a point particle. This leads to the question as to what is the corresponding generalization for extended objects. Here the lagrangian formulation of a perfect fluid is much generalized by replacing the product of the co-moving vector which is a first fundamental form by higher dimensional first fundamental forms; this has as a particular example a fluid which is a classical generalization of a membrane; however there is as yet no indication of any relationship between their quantum theories.

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1. Introduction

There are three main types of dynamical system which have a lagrangian formulation: fields theories, extended objects, and fluids. Some other systems which have lagrangian formulation include relative motion systems [18]. Various relationships have been found between string theories and field theories [7, p. 26–27], also various string theories can be written as field theories [20]. Some relationships are known between extended objects and fluids [1, 6, 8, 11, 12], these seem to be of the form of extended objects producing fluids. In the present work it is fluids which are the more general object which reduce to extended objects. A generalization of string theory called M-theory is currently being sought, whether there is any relationship between this hypothetical generalization and fluids remains to be seen. The prospect of generalizing the lagrangian theory of fluids rests on the observations that a perfect fluid spacetime stress can be reduced to that of a congruence of point particles. It seems that imperfect fluids do not often have a lagrangian description, in particular heat conduction and anisotropic stress do not seem to be derivable from a lagrangian [19]. The lagrangian theory of fluids involves all three of: relativity, quantum theory and thermodynamics, although there has not yet been any connection established with quantum field theory on curved spacetime [2] which also involves all three.

Apart from the challenge of studying a formalism which has contact with the above three theories, there are at least four other motivations for studying lagrangian fluid theory.

- symmetry breaking, see [15],
- low temperature physics, so far there has been no contact at all of Lagrangian based theory with existing approaches, such as [4],
quantum cosmology, with the geometry coupled to a fluid, see for example [13],

- the study of irreversible process, see for example [10], a lagrangian form might give insight into the processes involved.

A perfect fluid has a gauge description [16]; this allows it to be canonically quantized producing a quantum fluid theory which has novel quantum algebra, see [16] and Section 5.

To describe the approach here from the technical point of view: if one considers a perfect fluid with stress

\[ T_{\mu\nu} = (p + \mu)V_\mu V_\nu + pg_{\mu\nu}. \] (1)

The \( V_\mu V_\nu \) is a geometric object called the first fundamental form of a one-dimensional surface, which can also be thought of as the tangent vector to the path of a point particle; in general the first fundamental form of a \( p + 1 \) surface and metric and projection tensor are equated by

\[ h^{\mu\nu} \equiv g^{\mu\nu} - \mathcal{X}^{\mu\nu}, \quad h = d - 1 - p, \] (2)

\( \mathcal{X}^{\mu\nu} \) being the generalization of \( V^\nu V^\mu \) where now the tangents are to a membrane, so the question arises as to what a fluid with \( p + 1 \) dimensional \( \mathcal{X} \) replacing the \( V \)'s would look like and this is approached here through the lagrangian method.

In Section 2 the properties of a perfect fluid are described. In Section 3 some of the properties of Dirac membranes are described, the treatment of the constraints here is a different from usual. In Section 4 the lagrangian description of a perfect fluid is much generalized, this involves putting internal indices on fluid objects, the indices label the object and do not usually involve differentiation. Up to adding similarly structured terms, it is hoped that the resulting \( f \)-fluid is the most general that can be derived by a spin-free lagrange method; however it is not general enough to incorporate heat conduction or anisotropic stresses. In Section 5 it is shown how to reduce the number of indices by using an internal metric, this produces the \( f \)-brane for which both lagrangian and metric stress have both the perfect fluid and membranes as examples. Reduction is taken to have happened when both the lagrangian and metric stress coincide.

The notation used is: signature (-+++), greek indices \( \alpha, \beta, \ldots, \mu, \ldots \equiv 0, 1, 2, 3 \) are spacetime indices, early latin indices \( a, b, c, \ldots \) are internal or fluid indices, middle greek indices \( i, \ldots \) index constraints if they are not otherwise indexed, middle latin indices \( i, j, k, \ldots \) are velocity potential indices, all indices are left out when it is hoped that the ellipsis is clear, \( \gamma \) is the auxiliary metric which is usually that of the internal space of a membrane, \( p \) is the pressure but \( p + 1 \) the dimensions of \( \gamma, \mu \) is the density, \( n \) is the particle number, \( \xi \) is the enthalpy but \( h \) is the trace of the projection tensor and \( h \) is Planck's constant divided by \( 2\pi \), \( W \) is a vector field which has been decomposed into clebsch velocity potentials, \( V \) is a unit timelike vector field constructed from \( W \) and the enthalpy \( \xi \). All other conventions are those of Hawking and Ellis [9].

2. The perfect fluid

In this section the lagrangian formulation of a perfect fluid [16] is recalled. The lagrangian of a perfect fluid is taken to be the pressure

\[ \mathcal{L} = \rho, \] (3)

and the hamiltonian is the density \( \mu \). The lagrangian (3) is varied via the first law of thermodynamics

\[ dp = nd\xi - nTds, \] (4)

where \( n \) is the particle number, \( \xi \) is the enthalpy, \( T \) is the temperature, and \( s \) is the entropy. The number of extra terms depends on the dimension of the spacetime involved. Clebsch's theorem is only local, so that when there are obstructions to the velocity \( V \) or unusual global properties it no longer holds; the usual way around this is to add yet more terms to the decomposition. For our purposes when the first three terms are known it is straightforward to add the extra terms, so that we use just three terms regardless of dimension and so on. Using the unit normalization of \( V \), \( V^2 = -1 \) and \( \xi V = W \) the first law (4) can be written

\[ dp = -nV_\mu dW^\mu - nTds, \] (6)

using

\[ p + \mu = n\xi, \] (7)
and varying with respect to the metric gives the metric
stress (1). The reduction to a congruence of point particles
is achieved using
\[
\mathcal{L} = \rho = -m\ell = -m\sqrt{-\chi^2},
\]
\[
\xi = \pm\ell, n = \pm m,
\]
\[
\mathcal{H} = \mu = 0,
\]
\[
V^\mu = \frac{\chi^\mu}{\ell},
\]
\[
T_{\mu\nu} = m\ell h_{\mu\nu}.
\]
(8)

The point particles equation of motion is \( \dot{V} = 0 \) is not
explicitly recovered, \( V^\alpha = V^\alpha_\text{ref} \) is just the acceleration
of the fluid, so that in addition to (8) the reduction requires
that the fluid is acceleration free. The reduction to a single
point particle is achieved by either using delta functions or
by varying the lagrangian with respect to the single
internal index, see [17]. Variation of (3) with respect to
the clebsch potentials give their equations of motion
\[
(nV^\alpha)_\mu = \dot{n} + n\Theta = 0, \quad \dot{s} = 0, \quad \dot{\theta} = T. \quad (9)
\]
The momenta are
\[
\Pi^\alpha = -n, \quad \Pi^0 = 0, \quad \Pi^i = -n\theta,
\]
(10)

and the constraints between the momenta are
\[
\phi_1 = \Pi^i - \theta\Pi^0, \quad \phi_2 = \Pi^\theta.
\]
(11)
The Poisson bracket is defined by
\[
\{A, B\} \equiv \frac{\delta A}{\delta q^i} \frac{\delta B}{\delta \Pi^i} - \frac{\delta B}{\delta q^i} \frac{\delta A}{\delta \Pi^i}.
\]
(12)
The Dirac matrix is defined by
\[
C_{\alpha\beta} \equiv \{\phi_1, \phi_2\},
\]
(13)

and the Dirac bracket is defined by
\[
\{A, B\}^{\star} \equiv \{A, B\} - \{A, \phi_1\}C_{\alpha\beta}^{-1}\{\phi_2, B\}.
\]
(14)
Quantization is achieved using the substitution of the
Dirac bracket by commutators
\[
\{A, B\}^{\star} \rightarrow \frac{1}{i\hbar}[\hat{A}\hat{B} - \hat{B}\hat{A}].
\]
(15)

For the perfect fluid
\[
C_{12} = \{\phi_1, \phi_2\} = \Pi^\alpha,
\]
\[
C_{\alpha\alpha} = -i\sigma^2\Pi^\alpha,
\]
\[
C_{\alpha\alpha}^{-1} = i\sigma^2\Pi^\alpha.
\]
(16)

where \( \sigma^2 \) is the Pauli matrix
\[
\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]
(17)

Forming the Dirac brackets between the fields and mo-
menta it turns out that it is possible to multiply by \( \Pi^\alpha \)
throughout. After applying the quantization (15) and drop-
ching the hats the equations between the fields and mo-
menta are
\[
[\sigma\theta - \sigma\dot{\theta}]\Pi^\alpha = \Pi^\alpha[\sigma\theta - \sigma\dot{\theta}] = -i\hbar\delta^4(\mathbf{x} - \mathbf{y}),
\]
\[
[\theta\sigma - \theta\dot{\sigma}]\Pi^\alpha = \Pi^\alpha[\theta\sigma - \theta\dot{\sigma}] = -i\hbar\delta^4(\mathbf{x} - \mathbf{y}),
\]
\[
[\Pi^\alpha \sigma - \sigma\Pi^\alpha] = -i\hbar\delta^4(\mathbf{x} - \mathbf{y}),
\]
\[
[\Pi^\alpha - \sigma\Pi^\alpha] = -i\hbar\delta^4(\mathbf{x} - \mathbf{y}).
\]
(18)

Using the substitutions
\[
v_1 = \sigma,
\]
\[
v_2 = \sigma,
\]
\[
v_3 = \theta,
\]
\[
v_4 = \Pi^\alpha,
\]
\[
v_5 = \Pi^\alpha,
\]
\[
v_6 = \Pi^\alpha,
\]
(19)

setting \( \hbar = 1 \), and then suppressing the delta function
\( \delta^4(\mathbf{x} - \mathbf{y}) \) gives
\[
v_4(v_3v_2 - v_2v_3) = -i,
\]
\[
v_6(v_4v_3 - v_3v_4) = -iv_3,
\]
\[
v_4v_3 - v_3v_4 = -i,
\]
\[
v_5v_2 - v_2v_5 = -i.
\]
(20)

\( v_6 \) does not occur. In the case when \( V \) is a gradient vector
\( V^\mu = \sigma^\mu \), only the second to last of these commutators
remains, so that the algebra is the same as that of the
point particle.
3. The Dirac membrane

The membrane lagrangian is [5]

\[ \mathcal{L} = k \sqrt{-y}, \]
\[ y_{ab} = x_{\mu} x_{\nu}^b, \]
\[ \sqrt{-y} = (-\det y_{ab})^{\frac{1}{2}}. \]  
(21)

In order to discuss variation of (21) it is necessary to introduce the geometric objects the projection tensor (2) and the first fundamental form

\[ \mathcal{N}^{\alpha\beta} = y^{ab} x^a_{\mu} x^b_{\nu}, \]
\[ \mathcal{N}^{\alpha\beta} y_\beta = \mathcal{N}^{\alpha\nu}, \]
\[ \mathcal{N}^{\alpha}_{\beta} = y^\nu_{\nu} = x^{\nu} x^\nu = p + 1. \]  
(22)

Varying the lagrangian (21) with respect to the metric gives the metric stress

\[ T^{\mu\nu} = k \sqrt{-y} ((p + 1) \mathcal{N}^{\mu\nu} + g^{\mu\nu}). \]  
(23)

Defining momenta by varying the lagrangian (21) with respect to velocities gives momenta and the constraints between

\[ P^\mu = -k \sqrt{-y} x^\mu, \quad \phi^{\mu\nu} \equiv P^\mu P^\nu + k^2 y x^\mu x^\nu, \]  
(24)

compare [14] and [3], the constraints in (24) have contractions

\[ \phi^{\mu\nu} = \phi^{\mu\rho} y_{\rho\nu}, \quad \phi^{\mu\nu} y^\nu = P^\rho P^\mu + k^2 y x^\mu x^\nu, \]
\[ \phi = \phi^\rho = \phi^{\mu\nu} = P^\mu P^\nu + k^2(p + 1)y. \]  
(25)

Using just the last of these it is possible to derive a Klein-Gordon equation with \( k^2 = m^2, \) see [18]; however in general one needs the other constraints. Defining

\[ X^{ab}_\mu \equiv \frac{\partial \phi^{ab}}{\partial P^\mu} = 2 \delta^a_b P^b, \]
\[ Y^{ab}_{\mu} \equiv \frac{\partial \phi^{ab}}{\partial x^\mu} = -2k^2(y x^\mu x^b, x^\nu) - 2k(y x^\mu x^b, x^\nu), \]
\[ = -2k \left( \sqrt{-y} (P^\mu x^\nu + P^\nu x^\mu) + \frac{\partial \phi^{ab}}{\partial x^\mu} \right). \]  
(26)

X has one more index than \( Y, \) so that it is not possible to form a Dirac matrix \( C \) as they are. Choosing \( c = \tau \) in \( X \) one can form \( C_{abcd} = \{ \phi^{ab}, \phi_{cd} \}, \) but it is not clear what the inverse of this should be. Choosing \( b = c = \tau \) in \( X \) and \( b = \tau \) in \( Y \) one sees that the constraints are limited to the \( (p + 1) \) first class constraints \( \phi^a = \phi^a \) and one can form Dirac matrix \( C_{ab} = \{ \phi_a, \phi_b \}. \) Restricting to the case of the string \( p = 1 \) and

\[ C_{ab} = i \sigma^a C_{a\tau} = i \sigma^a Z_{\tau\nu} \](\mu), \]
\[ C^{-1}_{ab} = -i \sigma^{\tau\nu} \](\mu), \]
\[ Z_{\tau\nu} \equiv Y_{\tau\nu} X_{\tau\nu} = Y_{\tau\nu} X_{\tau\nu}. \]  
(27)

The Dirac brackets between the coordinates and momenta are

\[ \{ x^\mu, x^\nu \} = \{ P^\mu, P^\nu \} = 0, \]
\[ \{ x^\mu, P^\nu \} = g^{\mu\nu} - i \sigma^{\mu\nu} \frac{Z_{\tau\nu}}{Z_{\tau\nu}}. \]  
(28)

One can get an explicit form of these Dirac brackets by using a specific form of the internal metric \( \gamma, \) say the Nambu-Goto choice, see for example [17], however from the present perspective the important point is that now the \( \{ x, P \}^* \) Dirac bracket is different from the Poisson bracket, whereas for the perfect fluid the \( \{ q, q \}^* \) bracket also differs.

4. The F-Fluid

The lagrangian (3) is generalized to depend on \( F \) pressures

\[ \mathcal{L} = f(\rho_1, \rho_2, \ldots, \rho_F). \]  
(29)

the pressures \( \rho \) and densities \( \mu \) are equated to the enthalpies \( \xi \) and particle numbers \( n \) by

\[ P_\mu + \mu_\mu = n_\mu \xi_\mu, \]  
(30)

which generalizes (5) the internal indices \( a, b, \ldots \) label distinct objects, for example \( \rho_1, \rho_2, \ldots \) and only indicate differentiation with respect to the index when it is on \( x \) or is outside a bracket (\( \xi_\mu \)). The unit timelike condition on \( V \) (5) generalizes to

\[ h_\mu V_\mu = W_\mu, \quad V_\mu V^\mu = -\delta^\mu_\mu, \]  
(31)

so that

\[ -\xi_{ab} = -\xi_\ell_{\ell ab} = \xi_{\ell ab} V^{\ell}_a V^\mu = V^\mu W_{\ell\mu}. \]  
(32)

(32) allows the first law (4) to be generalized to

\[ dp_\mu = n_\mu^\ell d\xi_\ell - n_\mu^\ell T_\ell d\xi_\ell, \]
\[ = -n_\mu^\ell V_\mu dW_\ell - n_\mu^\ell T_\ell d\xi_\ell. \]  
(33)
Varying (29) with respect to the spacetime metric gives the metric stress
\[
T_{\mu\nu} = f_{\phi\mu} n^b x^c V^\mu_\sigma V^\nu_\tau + f g_{\mu\nu}. \tag{34}
\]

The clebsch decomposition (5) of \( W \) generalizes to
\[
W^\mu_\sigma = \epsilon_\alpha^\mu \delta^{\sigma}_{\alpha\beta} s_\beta^\phi + \ldots. \tag{35}
\]

In the present case this decomposition is an assumption, unlike for the perfect fluid where it is a consequence of Clebsch’s theorem, \( \epsilon_\alpha^\mu \) is a new object. Varying with respect to the potential gives the equations of motion
\[
(V^\mu_\alpha n^b)_{\alpha} = 0, \quad n^b_{\alpha}\alpha_{\delta e} V^\mu_\delta s^\phi_\epsilon = 0, \quad V^\mu_\alpha (\alpha_{\delta e} \theta^e_\alpha)_{\alpha} - n^b_{\alpha} T_b = 0, \tag{36}
\]
for \( \sigma^\alpha, \theta^\alpha, s^\alpha \) respectively, compare (9). The momenta generalizing (10) are
\[
\Pi^a_{\alpha \beta c} = -n^b_{\alpha}, \quad \Pi^{0}_{\alpha} = 0, \quad \pi^{\alpha \beta c}_{\alpha} = -n^b_{\alpha} \alpha \cdot \theta, \tag{37}
\]
where \( \alpha_a = \epsilon_\alpha^\mu \). The constraints generalizing (11) are
\[
\phi^{b c}_{\alpha} \equiv \Pi^{a b c}_{\alpha} - \alpha \cdot \theta \Pi^{a b c}_{\alpha}, \quad \phi^{0}_{\alpha} \equiv \Pi^{0}_{\alpha}. \tag{38}
\]

In this form it is not possible to get an inverse Dirac matrix.

5. The F-brane

The F-brane is a particular case of the f-fluid. First reduce the number of indices by
\[
n^b_{\alpha} = \gamma^{b c} n^c, \quad \xi^{b c} = \frac{1}{p + 1} \xi^{b c}, \tag{39}
\]
where \( \gamma \) is an internal metric, this gives
\[
\rho_{\alpha} + \mu_{\alpha} = \xi n_{\alpha}, \quad \xi V^\mu_\sigma = W^\mu_\sigma, \quad V^\mu_\sigma V^{\nu}_\sigma = -\delta^\nu_\sigma. \tag{40}
\]

The first law is
\[
d\rho_{\alpha} = n_{\alpha} \gamma^{b c} d(\xi^{b c}) - \gamma^{b c} n_{\alpha} T_b d s_c, \quad \rho_{\alpha} = -n_{\alpha} V^\mu_\sigma d W^\sigma_\mu - n_{\alpha} T^c d c, \tag{41}
\]
where one can take \( \gamma \) through the d to get the second equality. The metric stress is
\[
T_{\mu\nu} = f_{\phi\mu} n^b \xi V^\mu_\sigma V^{\mu}_\nu + f g_{\mu\nu}. \tag{42}
\]

The best identification of \( V \) is not immediate. The problem is that for the point particle the explicit normalization condition is \( V^\mu_\sigma V^{\sigma}_\mu = -1 \), with \( \tau \) subscripted both times, for only one internal index this does not matter, but for more than one internal index one wants to preserve covariance, so that a different choice has to be made, choosing
\[
V^\mu_\sigma = \beta x^{\mu\nu}, \tag{43}
\]
the normalization condition (39) and the definition of \( \gamma \) require
\[
V^\mu_\sigma V^{\mu}_\sigma = \beta^2 x^{\mu\nu} x_{\mu\nu} = \beta^2 \gamma^{\epsilon \phi} = -\delta^{\epsilon \phi} \Rightarrow \beta = \epsilon. \tag{44}
\]

With this choice it is possible to introduce the first fundamental form into the metric stress
\[
T_{\mu\nu} = -f_{\phi\mu} n_{\alpha} \xi^{\alpha\beta c} n_{\sigma} + f g_{\mu\nu}, \tag{45}
\]
choosing the simplest lagrangian
\[
\mathcal{L} = f = \Sigma_{\rho \mu}, \tag{46}
\]
gives the metric stress of the f-brane
\[
T_{\mu\nu} = -\Sigma_{\rho \mu} n_{\alpha} \xi^{\alpha\beta c} n^c + \Sigma_{\rho \mu} n_{\alpha} g_{\mu\nu}. \tag{47}
\]

This is much as expected, the only other possibilities for its form would be terms summing the internal indices occurring across the pressure part and the \( x \) part. The f-brane reduces to the Dirac membrane when
\[
\Sigma_{\rho \mu} = k \sqrt{-\gamma}, \quad \Sigma_{\mu \rho} = p k \sqrt{-\gamma}. \tag{48}
\]

different choices can reduce to the stress of the conformal membrane [17, Section 4]. Varying with respect to the potentials the equations of motion are
\[
(V^\mu_\alpha n^b)_{\alpha} = 0, \quad n^b_{\alpha} \alpha^{b c} V^\mu_\sigma = 0, \quad V^\mu_\sigma (\alpha_{\delta e} \theta^e_\alpha)_{\alpha} - n^b_{\alpha} T_b = 0. \tag{49}
\]

The momenta are as for the f-fluid, however choosing \( \Pi^{a b c}_{\alpha} = \Pi^{a b c}_{\alpha} \) and similarly for \( \Pi^{0}_{\alpha} \) gives
\[
\Pi^{a b c}_{\alpha} = -n_{\alpha}, \quad \Pi^{0}_{\alpha} = 0, \quad \Pi^{a b c}_{\alpha} = -\alpha \cdot \theta n_{\alpha}, \tag{50}
\]
which have constraints
\[
\phi^{a}_{\alpha} \equiv \Pi^{a b c}_{\alpha} - \alpha \cdot \theta \Pi^{a b c}_{\alpha}, \quad \phi^{0}_{\alpha} \equiv \Pi^{0}_{\alpha}. \tag{51}
\]
These constraints give Dirac bracket
\[ C_{ab}^{12} = \{ \phi_a^4, \phi_b^5 \} = a_b \Pi^\alpha. \] (52)

To proceed it is necessary to find an inverse of this. The simplest way is to reduce it from a four indexed matrix to a two index matrix by tracing across the internal indices
\[ C_{\alpha}^{12} = \{ \phi_\alpha^4, \phi_\beta^5 \} = \alpha \cdot \Pi^\alpha, \] (53)

and then proceed as for the perfect fluid in Section 2. Defining \( v^a \) as in (19), except that this time they all are subscripted by an internal index, the following algebra is found
\[
\begin{align*}
\alpha \cdot v^1(v_3^3 v_1^2 - v_1^3 v_3^1) &= -i \alpha \cdot v^3 \delta_{ab}, \\
\alpha \cdot v^2(v_3^3 v_2^2 - v_2^3 v_3^2) &= -i \delta_{ab}, \\
\alpha \cdot v^4(v_3^3 v_4^2 - v_4^3 v_3^2) &= -i (a \cdot v^4 \delta_{ab} - a_b v^4), \\
v_1^2 v_2^1 - v_1^1 v_2^2 &= -i \delta_{ab}, \\
v_3^2 v_4^1 - v_3^1 v_4^2 &= -i \delta_{ab},
\end{align*}
\] (54)

which reduces to (20) when \( a, b, \ldots = 1 \). Note that \( v^6 \) occurs in (54) but not in (20).

6. Conclusion

Technically the classical perfect fluid can be reduced to a congruence of point particles using the reduction equations (8). The perfect fluid has one propagating degree of freedom, the particle number, and two independent functions in the stress, the pressure and density; whereas the congruence of point particles has one propagating degree of freedom, the particle number, and two independent functions (8). The perfect fluid has one propagating degree of freedom, and one independent function in the stress.

Technically the classical perfect fluid can be reduced to a congruence of point particles using the reduction equations (8). The perfect fluid has one propagating degree of freedom, the particle number, and two independent functions in the stress, the pressure and density; whereas the congruence of point particles has one propagating degree of freedom, the particle number, and two independent functions (8). The perfect fluid has one propagating degree of freedom, the particle number, and two independent functions in the stress.

Contact with existing fluid models of low temperature physics would require addition of more thermodynamical objects and it is not clear how this could be achieved in a lagrangian approach to a perfect fluid, there is the possibility that the more general fluids presented here might have emergent thermodynamical properties in some new reduction. The fluids presented here could be coupled to field equations such as those of general relativity in the hope of producing new quantum cosmologies, but it is hard to envisage what this would achieve. In many particle processes one could take the internal indices to correspond to a species of particle, this could lead to the embedded space being of higher dimension than space-time so that the geometric interpretation is lost.

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