Covariate balancing for causal inference on categorical and continuous treatments

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Abstract: We propose novel estimators for categorical and continuous treatments by using an optimal covariate balancing strategy for inverse probability weighting. The resulting estimators are shown to be consistent and asymptotically normal for causal contrasts of interest, either when the model explaining treatment assignment is correctly specified, or when the correct set of bases for the outcome models has been chosen and the assignment model is sufficiently rich. For the categorical treatment case, we show that the estimator attains the semi-parametric efficiency bound when all models are correctly specified. For the continuous case, the causal parameter of interest is a function of the treatment dose. The latter is not parametrized and the estimators proposed are shown to have bias and variance of the classical nonparametric rate. Asymptotic results are complemented with simulations illustrating the finite sample properties. Our analysis of a data set suggests a nonlinear effect of BMI on the decline in self reported health.

Key words: Average causal effects; dose-response; double robust; semiparametric efficiency bound.

1 Introduction

Encouraged by the recent booming development of the causal inference literature, we devise and study a novel inference tool for categorical and continuous treatments by using covariate balancing strategies for inverse probability weighting (e.g., Fan et al. 2020, Imai & Ratkovic 2014, Wang & Zubizarreta 2019). Our study is built on the fundamental idea on optimal
When estimating a causal effect on an outcome, weighting based on the propensity score (model for the probability of the treatment given observed pre-treatment covariates) is often used to construct optimal estimators by an augmentation using fitted models for the outcome given the covariates. These augmented inverse probability weighting estimators have robustness properties to the specification of models used, and are locally efficient (e.g., Robins & Rotnitzky 1995, Scharfstein et al. 1999). A vast majority of the literature on causal inference have focused on binary treatments, i.e. where the causal parameter of interest is a contrast between two treatments. Nevertheless, there is an increasing interest in multi-valued treatments (e.g., Fong et al. 2018, Kennedy et al. 2017, Yang et al. 2016) as often encountered in applied work, both in the medical and social sciences. Causal effects of categorical treatment were formalized by, e.g., Imbens (2000) and Robins (2000), while Cattaneo (2010) deduced the semiparametric efficiency bound; see also Yang et al. (2016) for a review. Causal effects of continuous treatments were formalized in, e.g., Robins (2000), van der Laan & Robins (2003), Hirano & Imbens (2004) and Galvao & Wang (2015). In contrast to previous works, Kennedy et al. (2017) proposed a double robust estimation strategy avoiding parametric specification of the dose-response curve.

We contribute to the somewhat less rich literature on robust estimation for categorical and continuous treatments by using an estimation strategy based on covariate balancing propensity score estimation for inverse probability weighting (e.g., Fong et al. 2018, Imai & Ratkovic 2014). Fan et al. (2020) recently obtained key results in the binary treatment case by specifying which covariate functions should be balanced for efficient inference: the propensity score model should be fitted through balancing a set of bases for the outcome models in the space spanned by the covariates. We provide corresponding results to the categorical and continuous treatment cases, hence completes the story. In particular, the procedures we proposed balance the “most suitable” functions of the covariates when the propensity score is correctly specified, in the sense that they minimize the variability of the causal effect estimation. When the propensity score is misspecified and the outcome basis functions are correct, the procedure looks for an approximate balance by minimizing the
squared bias of the resulting estimator. As other recent proposals for the binary treatment case \cite{Atheyetal2018, WangZubizarreta2019, WongChan2017, Zubizarreta2015}, the method presented here does not necessarily try to achieve exact balance when this is not possible, although in practice exact balance can always be targeted by enriching the assignment model.

For both the categorical and continuous treatment case, the proposed estimators are shown to be robust, i.e., consistent and asymptotically normal for causal contrasts of interest, either when the model explaining treatment assignment is correctly specified, or when the correct set of bases for the outcome models has been chosen and the propensity score model is sufficiently rich. For the categorical treatment case, we show that the estimator proposed attains the semiparametric efficiency bound when both the treatment assignment model and the outcome basis are correctly specified. For the continuous case, the causal parameter of interest is a function. The latter is not parametrized and the estimators proposed are shown to have bias and variance of the classical nonparametric order under typical regularity conditions, hence with a usual bias-variance trade-off.

The rest of the paper is organized as follows. Sections 2 and 3 deal with the categorical and the continuous treatment cases, respectively. In both sections, inverse probability weighting estimators are introduced, where a working model for the generalized propensity score is estimated by balancing basis functions for the outcome models. We establish the theoretical properties of the estimators. Simulation studies are conducted in Section 4 to illustrate the finite sample performance of our methods. In Section 5, we estimate the dose-response curve of BMI on the decline in self-reported health from baseline to a 9 year follow up in a population of ages 50 or older. Section 6 concludes the paper, while all proofs are relegated to the Appendix.

2 Categorical treatments

2.1 Balancing scores and preliminaries on estimation

Consider $K + 1$ treatments, $A = 0, 1, \ldots, K$, and their respective potential outcomes $Y^0, \ldots, Y^K$. We observe a random sample $(A_i, Y_i, X_i)$, $i = 1, \ldots, n$, where we assume
$Y_i = Y_i^k$ if $A_i = k$, and $X_i \in \mathbb{R}^d$ is a vector of pre-treatment covariates. We also assume ignorability of the treatment assignment, i.e. $E(Y_i^k \mid X_i, A_i) = E(Y_i^k \mid X_i) \equiv m(k, X_i)$ and $\text{pr}(A_i = k \mid X_i = x) \equiv \pi_0(k, x) > \delta > 0$ for all $k \in \{0, 1, \ldots, K\}$ and all $x$, where $\pi_0(k, x)$ is named generalized propensity score in the literature (Imbens 2000).

Let $\theta_k \equiv E(Y_i^k)$ for $k = 0, 1, \ldots, K$ be the average response to the different treatment levels. The parameters of interest are typically average causal effects between treatment levels, i.e. causal contrasts such as $\theta_k - \theta_0$, if $k = 0$ is a treatment level of reference. We consider a parametric working model $\pi(k, x, \beta)$ for $\pi_0(k, x)$, with $\beta \in \mathbb{R}^p$, and vectors of basis functions, $B(k, X) : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^q$, aiming at spanning $m(k, x)$. We assume $q$ does not depend on $k$ for notational simplicity. Thus, correct specification will imply that there exists a value $\beta_0$ with

$$
\pi(k, x, \beta_0) = \pi_0(k, x),
$$

and there exists $\alpha = (\alpha_{0}^T, \ldots, \alpha_{K}^T)^T$ with

$$
\alpha_{k}^T B(k, x) = m(k, x),
$$

for all $k$ and all $x$. Misspecification, i.e. situations when (1) or (2) does not hold for any value of $\beta$ and $\alpha$, will also be considered in the sequel. Note that one of the advantages of the herein studied balancing approach is that the parameter $\alpha$ does not need to be known or estimated. We hence do not use a subscript $0$ on $\alpha$ and $m(\cdot)$ to distinguish true parameter value and correct model since this will be clear from the context.

For estimating $\theta_k$ under the above assumptions one needs to control for the covariates $X_i$ by using one or both working models. In particular, $\pi(k, x, \beta)$ is a balancing score in the sense that $X_i \perp A_i \mid \pi(k, x, \beta_0)$ under (1) (Rosenbaum & Rubin 1983). Thus, for the binary case ($K = 1$), Imai & Ratkovic (2014) proposed to solve

$$
\sum_{i=1}^{n} \left\{ \frac{I(A_i = 1)}{\pi(1, X_i, \beta)} - \frac{I(A_i = 0)}{\pi(0, X_i, \beta)} \right\} b(X_i) = 0,
$$

where $b(X_i)$ is a vector valued function of the covariates. Based on the resulting fitted propensity score $\pi(k, X_i, \hat{\beta})$, an inverse probability weighting estimator for $\theta_k$ is

$$
\hat{\theta}_k = n^{-1} \sum_{i=1}^{n} \frac{I(A_i = k) Y_i}{\pi(k, X_i, \hat{\beta})}.
$$
Two issues arise regarding the above procedure. One is that if the propensity score model (1) is misspecified then, \( \hat{\theta}_k \) is generally biased. Two is the choice of \( b(X) \), which is largely left unsupervised. Fan et al. (2020) overcome these two issues in the binary case \((K = 1)\), and proposed an optimal choice for \( b(X) \), in the sense that the resulting treatment effect estimator is consistent when (1) is correct, or when (2) is correct and (1) has sufficient flexibility, and is efficient if both are correct.

We aim to achieve the same kind of optimality and robustness in the categorical treatment case. Two different estimators may be introduced with different properties, which we discuss heuristically below, before giving a formal treatment in the next section. The first possibility to estimate \( \beta \) is to solve the following balancing condition

\[
\sum_{i=1}^{n} \left[ \left\{ \frac{I(A_i = k)}{\pi(k, X_i, \beta)} - 1 \right\} B(k, X_i) - \left\{ \frac{I(A_i = 0)}{\pi(0, X_i, \beta)} - 1 \right\} B(0, X_i) \right] = 0 \tag{4}
\]

at all \( k = 1, \ldots, K \), i.e. a system of \( qK \) equations. GMM, as described below, can be used if \( qK \geq p \). This balancing condition is motivated by pushing the bias of the contrast estimator \( \hat{\theta}_k - \hat{\theta}_0 \) towards zero. In fact, it will be shown that the asymptotic bias of \( \hat{\theta}_k - \hat{\theta}_0 \) is equal to

\[
E \left[ \left\{ \frac{I(A_i = k)}{\pi(k, X_i, \beta)} - 1 \right\} m(k, X_i) - \left\{ \frac{I(A_i = 0)}{\pi(0, X_i, \beta)} - 1 \right\} m(0, X_i) \right].
\]

An alternative to setting the bias of \( \hat{\theta}_k - \hat{\theta}_0 \) to zero for \( k = 1, \ldots, K \), is to directly put the bias of \( \hat{\theta}_k \) to zero, for \( k = 0, \ldots, K \), by separately balancing both terms in (4), i.e. solving the condition

\[
\sum_{i=1}^{n} \left\{ \frac{I(A_i = k)}{\pi(k, X_i, \beta)} - 1 \right\} B(k, X_i) = 0 \tag{5}
\]

at all \( k = 0, \ldots, K \), i.e. a system of \( q(K + 1) \) equations. We will use GMM allowing for \( q(K + 1) \geq p \); see (6) below.

The two choices are not necessarily equivalent. In fact, the former choice allows for biased estimation of \( \hat{\theta}_k \) with the only aim to estimate the contrast \( \theta_k - \theta_0 \) without bias. We find that, if \( \hat{\theta}_k \) is indeed biased, then \( \hat{\theta}_k - \hat{\theta}_0 \) will not be efficient. This is because local efficiency holds when the the fitted propensity score is correctly specified and its parameters are consistently estimated, which is not the case when (5) does not hold. Due to this consideration, below we focus on solving (5) and show that the resulting estimator of \( \theta_k \)
has, under certain conditions, a robust property and, when all working models are correctly
specified, reaches the asymptotic semiparametric efficiency bound.

2.2 Asymptotic properties

We now establish a robustness property and the asymptotic distribution results of the
estimator in (3), where \( \beta \) is estimated through covariate balancing (5); see Appendix A.1
for proofs. To gain an intuitive understanding of the robustness property, we can verify
that when the propensity score model is correctly specified, i.e. when (1) holds for all \( k \) and
all \( \mathbf{x} \), \( \hat{\beta} \) is \( \sqrt{n} \)-consistent under the standard regularity conditions for GMM estimation
(Newey & McFadden 1994), and \( \pi(k, \mathbf{x}, \hat{\beta}) \to \pi(k, \mathbf{x}, \beta_0) = \rho_0(k, \mathbf{x}) \) in probability as \( n \)
tends to infinity. The consistency is a consequence of
\[
E \left\{ \frac{I(A_i = k)}{\pi(k, \mathbf{X}_i, \hat{\beta})} \right\} \to \frac{1}{\pi_0(k, \mathbf{X}_i)} = \theta_k,
\]
as \( n \to \infty \). On the other hand, when the outcome model basis is actually correctly specified,
i.e. when (2) holds for all \( k \) and \( \mathbf{x} \), then the propensity model (1) does not need be correct
as long as (5) has a solution. In such case, \( \hat{\beta} \) is consistent for some value \( \beta^* \), hence \( \pi(k, \mathbf{x}, \hat{\beta}) \)
converges to some function \( \pi(k, \mathbf{x}) \) in probability. We then have
\[
E(\hat{\theta}_k) = E \left\{ n^{-1} \sum_{i=1}^{n} \frac{I(A_i = k)Y_i}{\pi(k, \mathbf{X}_i, \hat{\beta})} \right\} \to E \left\{ \frac{I(A_i = k)Y_i^k}{\pi_0(k, \mathbf{X}_i)} \right\} = \theta_k,
\]
as \( n \to \infty \), where the last equality is the result of (2) and (5).

To be more formal, let
\[
f_{ki}(\beta) = \left\{ \frac{I(A_i = k)}{\pi(k, \mathbf{X}_i, \hat{\beta})} - 1 \right\} \mathbf{B}(k, \mathbf{X}_i),
\]
\[
f_{i}(\beta) = \{f_{0i}(\beta)^T, \ldots, f_{Ki}(\beta)^T\}^T, \quad \mathbf{V}(\beta) = E\{f_{i}(\beta)f_{i}(\beta)^T\}, \quad \hat{\mathbf{V}}(\beta) = n^{-1} \sum_{i=1}^{n} f_{i}(\beta)f_{i}(\beta)^T, \quad \mathbf{A}(\beta) = E \left\{ \frac{\partial f_{i}(\beta)}{\partial \beta^T} \right\} \quad \text{and} \quad \hat{\mathbf{A}}(\beta) = n^{-1} \sum_{i=1}^{n} \partial f_{i}(\beta)/\partial \beta^T.
\]
$g_{ki}(\beta) \equiv I(A_i = k) Y_i / \pi(k, X_i, \beta) - E\{m(k, X_i)\}$, $g_i(\beta) = \{g_{1i}(\beta), \ldots, g_{Ki}(\beta)\}^T$ and $B(\beta) \equiv E\{\partial g_i(\beta^*) / \partial \beta^*\}$. We solve for a solution of (5) by minimizing

$$\{\sum_{i=1}^n f_i(\beta)\}^T \hat{V}(\beta)^{-1} \{\sum_{i=1}^n f_i(\beta)\}. \quad (6)$$

We will use the following regularity conditions.

**A0.** $\beta^*$ is the unique solution of $E\{f_i(\beta)\} = 0$.

**A1.** The variance-covariance matrix $V(\beta^*)$ has bounded positive eigenvalues.

**A2.** $f_i(\beta)$ is differentiable with respect to $\beta$.

**A3.** The matrix $A(\beta^*)$ is bounded and has full column rank.

**A4.** $g_i(\beta)$ is differentiable with respect to $\beta$.

These are classical regularity conditions. Condition A0 requires the existence and uniqueness of a solution, where the uniqueness can be relaxed to local uniqueness. The existence requirement is automatic when the $\pi(k, x, \beta)$ model is correct. In this case $\beta^* = \beta_0$. It is also natural and standard when $(K + 1)q$, the number of equations in $E\{f_i(\beta)\}$ is not larger than $p$, the dimension of $\beta$, which is achievable through enriching the $\pi(k, x, \beta)$ model. Thus, regardless of whether $\pi(k, x, \beta)$ is correctly specified or not, we can always justify Condition A0.

**Theorem 1.** Assume that either (1) holds for all $k$ and $x$, or (2) holds for all $k$ and $x$. Then, under regularity conditions A0 to A4, $n^{1/2}(\hat{\theta} - \theta)$ has asymptotic normal distribution with mean zero and variance

$$\Sigma = B(\beta^*)\{A(\beta^*)^T V(\beta^*)^{-1} A(\beta^*)\}^{-1} B(\beta^*)^T + C(\beta^*)$$

$$- B(\beta^*)\{A(\beta^*)^T V(\beta^*)^{-1} A(\beta^*)\}^{-1} A(\beta^*)^T V(\beta^*)^{-1} D(\beta^*)$$

$$- D(\beta^*)^T [B(\beta^*)\{A(\beta^*)^T V(\beta^*)^{-1} A(\beta^*)\}^{-1} A(\beta^*)^T V(\beta^*)^{-1}]^T,$$

where $C(\beta^*) \equiv E\{g_i(\beta^*)^\otimes 2\}$ and $D(\beta^*) \equiv E\{f_i(\beta^*) g_i(\beta^*)^T\}$. 

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Theorem 1 highlights a robust property. On the one hand, if the propensity score is correctly specified then we will have a consistent estimator of the treatment contrast even if the outcome basis is misspecified. On the other hand, we can also afford to misspecify the propensity score model, provided that the outcome basis functions are correctly specified. In the latter case, Condition A0 plays a pivotal role and it is crucial to ensure it. An example is to use the model \( \pi(k, x, \beta) = \beta_T^T(k)B(k, x) \), where \( \beta = (\beta_T^T(0), \ldots, \beta_T^T(K))^T \) so that \( \beta \) has length \( p = q(K + 1) \). Then (5) is the derivative of the loss function

\[
\sum_{i=1}^{n} [I(A_i = k)\log\{\beta_T^T(k)B(k, X_i)\} - \beta_T^T(k)B(k, X_i)],
\]

for \( k = 0, \ldots, K \), hence the minimizer is a root of (5). The utilization of the same basis of functions for both nuisance models is used in Wang & Zubizarreta (2019) as well. To further accommodate one’s favorite propensity model, we can also make linear combination of this model and any candidate model in mind.

The asymptotic variance simplifies greatly when all models are correctly specified, and a local efficiency result is obtained.

**Corollary 1.** Assume that (1) and (2) hold for all \( k \) and \( x \) and let \( \text{var}(Y_i^k \mid X_i) = v(k, X_i) \). Then, under the regularity conditions of Theorem 1, \( n^{1/2}(\hat{\theta} - \theta) \) has asymptotic normal distribution with mean zero and variance

\[
\Sigma = C(\beta_0) - B(\beta_0)\{A(\beta_0)^T V(\beta_0)^{-1} A(\beta_0)\}^{-1} B(\beta_0)^T,
\]

where

\[
A_k(\beta_0) = E \left\{ -\frac{B(k, X_i)\pi_{\beta}^T(k, X_i, \beta_0)^T}{\pi(k, X_i, \beta_0)} \right\},
\]

\[
B_k(\beta_0) = E \left\{ -\frac{m(k, X_i)\pi_{\beta}^T(k, X_i, \beta_0)^T}{\pi(k, X_i, \beta_0)} \right\},
\]

\[
V_{kl}(\beta_0) = E \left\{ \frac{I(k = l)}{\pi(k, X_i, \beta_0)} - 1 \right\} B(k, X_i)B(l, X_i)^T,
\]

\[
C_{kl}(\beta_0) = E \left\{ I(k = l)\frac{m(k, X_i)^2 + v(k, X_i)}{\pi(k, X_i, \beta_0)} - m(k, X_i)m(l, X_i) \right\} + E \left\{ m(k, X_i) - E\{m(k, X_i)\} \right\} \left\{ m(l, X_i) - E\{m(l, X_i)\} \right\}.
\]
Remark 1. The variance $\Sigma$ may be estimated without knowing nor estimating $\alpha$, by approximating the original definitions of the matrices involved, i.e. $B(\beta_0) \equiv E\{\partial g_i(\beta_0)/\partial \beta_0^T\}$ and $C(\beta_0) \equiv E\{g_i(\beta_0)^{\otimes 2}\}$, instead of the expression involving $m(\cdot)$ and $v(\cdot)$ given in Corollary 1.

Corollary 2. Under the assumptions of Corollary 1, the variance of $\hat{\theta}$ attains the semi-parametric efficiency bound $\Sigma_{\text{eff}}$, where the $(k, l)$ entry of $\Sigma_{\text{eff}}$ is

$$
\Sigma_{\text{eff},k,l} = I(k = l)E\{v(k, X)/\pi(k, X)\} + E([m(k, X) - E\{m(k, X)\}][m(l, X) - E\{m(k, X)\}]].
$$

3 Continuous treatments

3.1 Balancing scores and preliminaries on estimation

We now consider a continually valued treatment $A$, say taking values $a$ in $[0, 1]$. In this case, it is reasonable to assume that the potential outcome $Y^a$ changes with $a$ smoothly.

We write $Y^a$ as $Y(a)$ in a more conventional notation. Note that the observed outcome for the $i$th observation, $Y_i$, is assumed to be $Y_i(a_i)$ when we observe $A_i = a_i$. We observe a random sample $(A_i, Y_i, X_i), i = 1, \ldots, n$, where $X_i \in \mathbb{R}^d$ is a vector of pre-treatment covariates observed for all units. Following the literature convention, we assume ignorability of the treatment assignment, in the sense that $E\{Y_i(a) \mid X_i, A_i\} = E\{Y_i(a) \mid X_i\}$, and the generalized propensity score is the conditional probability density function of the continuous treatment $A_i$ given the covariates $X_i$: $\pi_0(a, x) \equiv f_{A_i|X}(a, x) > 0$ for all $a \in [0, 1]$ and all $x$. We write the expected conditional potential outcome as $m(a, x) \equiv E\{Y_i(a) \mid X_i = x\}$.

In such case, the parameter of interest is the treatment response function or the dose-response function, denoted as $\theta(a) = E\{Y_i(a)\}$ for $a \in [0, 1]$. The average causal effects between two treatment doses, say $a$ and $b$ are obtained by taking their contrast $\theta(a) - \theta(b)$.

We consider a parametric working model $\pi(a, x, \beta)$ for the propensity score $\pi_0(a, x)$, where $\beta \in \mathbb{R}^p$, and consider a set of basis functions $B(a, x) : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^q$ aiming at spanning $m(a, x)$. Thus, correctly specified situations will be such that there exists $\beta_0$ so that

$$
\pi(a, x, \beta_0) = \pi_0(a, x),
$$

(8)
and there exists $\alpha$ such that

$$\alpha^T B(a, x) = m(a, x),$$

(9)

for all $a \in [0, 1]$ and all $x$. Misspecification, i.e. situations where one of (8) and (9) does not hold, will be allowed in the sequel.

The balancing consideration then leads us to the condition

$$\sum_{i=1}^{n} \left[ \left\{ \frac{K_i(A_i - a)}{\pi(a, X_i, \beta)} - 1 \right\} B(a, X_i) - \left\{ \frac{K_i(A_i - b)}{\pi(b, X_i, \beta)} - 1 \right\} B(b, X_i) \right] = 0$$

for two arbitrary $a, b$ values in $[0, 1]$. Following the same considerations as in Section 2, we strengthen the above requirement and consider the balancing equations

$$\sum_{i=1}^{n} \left\{ \frac{K_i(A_i - a)}{\pi(a, X_i, \beta)} - 1 \right\} B(a, X_i) = 0$$

(10)

at all $a \in [0, 1]$. Here, $K_i(\cdot) = l^{-1}K(\cdot/l)$, where $K(\cdot)$ is a kernel function and $l$ is a bandwidth. Practically, we propose to solve (10) at a set of chosen $a$ values, typically those observed for $A_i$, and minimize

$$\sum_{j=1}^{n} \left\| \sum_{i=1}^{n} \left[ \left\{ \frac{K_i(A_i - A_j)}{\pi(A_j, X_i, \beta)} - 1 \right\} B(A_j, X_i) \right] \right\|_2^2 \{ \sum_{i=1}^{n} K_i(A_i - A_j) \}$$

(11)

with respect to $\beta$ to get $\hat{\beta}$. Once we obtain $\hat{\beta}$, we estimate the causal parameter $\theta(a)$ with an inverse probability weighting estimator

$$\hat{\theta}(a) = n^{-1} \sum_{i=1}^{n} \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \hat{\beta})},$$

(12)

for any $a$ within the range of observed values for $A_i$. Here, $h$ is a bandwidth.

**Remark 2.** The nonparametric estimator (12) can be viewed as an approximation of

$$\frac{n^{-1} \sum_{i=1}^{n} Y_i K_h(A_i - a) / \pi(A_i, X_i, \hat{\beta})}{n^{-1} \sum_{i=1}^{n} K_h(A_i - a) / \pi(A_i, X_i, \hat{\beta})},$$

which is the solution to

$$\min_{c} \sum_{i=1}^{n} \frac{(Y_i - c)^2 K_h(A_i - a)}{\pi(A_i, X_i, \hat{\beta})}.$$

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Thus, we can understand (12) as a weighted local constant estimator of $\theta(a)$. Similar to the generalization from local constant to local polynomial estimators in nonparametrics, we can also generalize (12) to more sophisticated versions. For example, through obtaining $\hat{\theta}_0$ from

$$
\min_{c_0,c_1} \sum_{i=1}^{n} \frac{(Y_i - c_0 - c_1(A_i - a))^2 K_h(A_i - a)}{\pi(A_i, X_i, \hat{\beta})},
$$

we can obtain the weighted local linear estimator of $\theta(a)$.

### 3.2 Asymptotic properties

We now study the limiting properties of the estimator (12) using (11); see Appendix A.2 for proofs. Denote by $\beta^*$ the probability limit of $\hat{\beta}$. If model (8) is correct, $\beta^* = \beta_0$, otherwise $\beta^*$ is the value that minimizes (11) at the population level, i.e. it minimizes

$$
E_j \left( \left\| E_i \left[ \left\{ K_i(A_i - A_j) \frac{\pi(A_j, X_i, \beta)}{\pi(A_j, X_i, \beta)} - 1 \right\} B(A_j, X_i) \right] \right\|_2^2 \left\{ \sum_{i=1}^{n} K_i(A_i - A_j) \right\} \right)
$$

with respect to $\beta$. Here $E_j$ means taking expectation of the $j$th observation. We list the following regularity conditions.

**C0.** $\beta^*$ is the unique solution of $E \left[ \left\{ \frac{\pi_0(a, X)}{\pi(a, X, \beta)} - 1 \right\} B(a, X) \right] = 0$.

**C1.** The kernel function $K(\cdot) \geq 0$ is bounded, twice differentiable with bounded first derivative, symmetric and has support on $(-1, 1)$. It satisfies $\int_{-1}^{1} K(t) dt = 1$.

**C2.** The bandwidth $l$ satisfies $nl^4 \to 0$ and $nl^2 \to \infty$. The bandwidth $h$ satisfies $l \to 0$ and $nh \to \infty$.

**C3.** The basis function $B(a, X)$ is bounded.

**C4.** The propensity score $\pi(a, X, \beta)$ is differentiable with respect to $\beta$ and $a$, is bounded away from zero, and its derivative with respect to $a$ is bounded.

**C5.** $m(a, X_i)$ is bounded, twice differentiable with respect to $a$, and the first derivative is bounded.

**C6.** $\sigma^2(A_i, X_i) \equiv \text{var}(Y_i | A_i, X_i)$ is bounded.
These are typical regularity conditions. Similar to Condition A0 in the categorical treatment case, the uniqueness requirement in Condition C0 can be relaxed to local uniqueness. Moreover, with finite samples, C0 can be translated to: $\beta^*$ is the unique solution of $E_i \left( \frac{K_i(A_i - A_j)}{\pi(X_i, \beta)} - 1 \right) B(A_j, X_i) = 0$ for $j = 1, \ldots, n$, which is easier to fulfill. The existence of $\beta^*$ is guaranteed when the propensity model $\pi(a, x, \beta)$ is correctly specified, and is a standard requirement when the number of equations $qn$ is not larger than the length of $\beta$.

Thus, in the situation where we are not confident that a correct propensity model is used, we can always enrich the model to accommodate Condition C0. We start by giving the convergence rate of $\hat{\beta}$.

**Lemma 1.** Denote by $\beta^*$ the probability limit of $\hat{\beta}$. If model (8) is correct, $\beta^* = \beta_0$, otherwise $\beta^*$ is the value that minimizes (13). Under regularity conditions C0 to C4, $\hat{\beta} - \beta^* = O_p(n^{-1/2})$.

Condition C0 is not really necessary for Lemma 1. We can redefine $\beta^*$ as the unique minimum of (13) and Lemma 1 still holds. Because the nonparametric estimation convergence rate is slower than $O_p(n^{-1/2})$, Lemma 1 indicates that we can fix $\beta$ at $\beta^*$ in the following analysis as long as we let $nl^4 \to 0$, and the first order bias and variance property of $\hat{\theta}(a)$ will not be affected.

**Theorem 2.** Under regularity conditions C0 to C6, and if (8) holds, then the estimator $\hat{\theta}(a)$ defined by (12) has asymptotic normal distribution with asymptotic bias and variance:

$$E\{\hat{\theta}(a)\} - \theta(a) = \frac{h^2}{2} E \left[ \frac{\partial^2 \{\pi_0(a, X_i)m(a, X_i)\}}{\pi_0(a, X_i) \partial a^2} \right] \int t^2 K(t) dt + O(h^4 + n^{-1/2}), \quad (14)$$

$$\text{var}\{\hat{\theta}(a)\} = \frac{\int K^2(t) dt}{nh} E \left\{ \frac{m^2(a, X_i) + \sigma^2(a, X_i)}{\pi_0(a, X_i)} \right\} + O(n^{-1}h^{-1/2}), \quad (15)$$

where $\sigma^2(A_i, X_i) = \text{var}(Y_i | A_i, X_i)$.

**Theorem 3.** Under regularity conditions C0 to C6, and if (9) holds, then the estimator $\hat{\theta}(a)$ defined by (12) has asymptotic normal distribution with asymptotic bias and variance:

$$E\{\hat{\theta}(a)\} - \theta(a) = \frac{h^2}{2} E \left[ \frac{\partial^2 \{\pi_0(a, X_i)m(a, X_i)\}}{\pi(a, X_i, \beta^*) \partial a^2} \right] \int t^2 K(t) dt + O(h^4 + n^{-1/2}), \quad (16)$$

$$\text{var}\{\hat{\theta}(a)\} = \frac{\int K^2(t) dt}{nh} E \left[ \frac{\pi_0(a, X_i)\{m^2(a, X_i) + \sigma^2(a, X_i)\}}{\pi^2(a, X_i, \beta^*)} \right] + O(n^{-1}h^{-1/2}), \quad (17)$$
Theorems 2 and 3 together reflect a robust property of the proposed estimator, and give equivalent results when all nuisance models are correctly specified. Specifically, Theorem 2 describes the robustness to misspecification of the outcome models, in that as long as the propensity score is correctly specified, the estimation of the treatment response function is valid even if we do not assume a correct model for the outcome. This is because the propensity score balances any functions of the covariates. Theorem 3 allows for the misspecification of the propensity score, with the restriction that Condition C0 needs to hold. If we choose to ensure C0 through allowing sufficiently many model parameters, then $\beta$ will have length $p = qn$, which practically means that the propensity score is non-parametrically estimated. For example, we can let $\pi(a_j, x) = \beta^T_j B(a_j, x)$, where $\beta_j$ has dimension $q$. Then, solving (10) for all observed $a = a_j$ corresponds to minimizing the loss function

$$
\sum_{i=1}^{n} [K_t(A_i - a_j) \log \{\beta^T_j B(a_j, X_i) - \beta^T_j B(a_j, X_i)\},
$$

for $j = 1, \ldots, n$.

Finally, note here, that the dose response function $\theta(a)$ is estimated nonparametrically, and this estimation has bias of order $h^2$, although asymptotically vanishing, and there is the usual bias-variance trade-off. Next, we give a result useful for inference on a causal contrast $\theta(a) - \theta(b)$.

**Theorem 4.** Under regularity conditions C0 to C6, and if either (8) or (9) hold, then $\hat{\theta}(a) - \theta(a)$ defined by (12) is asymptotically a Gaussian process, and has asymptotic variance-covariance:

$$
cov\{\hat{\theta}(a), \hat{\theta}(b)\} = (nh)^{-1} E \int_0^1 K(t)K(t + c) \{m^2(a, X_i) + \sigma^2(a, X_i)\} \frac{\pi_0(a, X_i)}{\pi(a, X_i, \beta^*)} \pi(b, X_i, \beta^*) dt + n^{-1} E \int_0^1 K(t)K(t + c) \{2m(a, X_i)m'_a(a, X_i)\pi_0(a, X_i) + m^2(a, X_i)\pi''_0(a, X_i) + 2\sigma(a, X_i)\sigma'_a(a, X_i)\pi_0(a, X_i) + \sigma^2(a, X_i)\pi''_0(a, X_i)\} t/\{\pi(a, X_i, \beta^*)\pi(b, X_i, \beta^*)\} dt - n^{-1} \theta(a)\theta(b) + O(n^{-1}h + h^{-1}n^{-3/2}),
$$

where $c \equiv (a - b)/h$. 

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Note that when \( c \notin (-2, 1) \), \( K(t)K(t+c) = 0 \) for all \( t \). Therefore, the covariance has order \( O(n^{-1}) \) if \( c \notin (-2, 1) \) and \( O((nh)^{-1}) \) otherwise. Thus, comparing the term of order \( O((nh)^{-1}) \) in the covariance in Theorem 4 with the terms of the same order for the variances in Theorems 2 and 3, we see that when \( a \) and \( b \) are close to each other relative to \( h \), the variance of the contrast \( \hat{\theta}(a) - \hat{\theta}(b) \) is close to zero. On the contrary, when \( a \) and \( b \) are far apart, then the variance of the contrast is dominated by the variance of \( \hat{\theta}(a) \) and \( \hat{\theta}(b) \).

Theorems 2, 3 and 4 provide theoretical properties of the leading orders of the bias, variance and covariance properties of the nonparametric estimators. In large samples, these results can be used to perform inference. Practically, unlike for parameter estimation, because the next order of the nonparametric analysis is only slightly smaller than the leading order, inference based on these results is often not sufficiently precise. This phenomenon has been observed in many nonparametric or even semiparametric problems including quantile regression, survival analysis, etc., and bootstrap is often used instead.

4 Simulation Experiments

4.1 Categorical treatment

To investigate the finite sample performance of our method for the categorical treatment case, we performed a first simulation study. We generate a five dimensional covariate vector \( \mathbf{X} \), where \( X_1 = 1 \), and \( X_2 \) to \( X_5 \) are generated independently from a normal distribution with mean 3 and variance 4. We set \( K = 3 \) and the propensity score \( \pi_0(k, x) = \exp(\mathbf{x}^T \beta_k) / \{1 + \sum_{k=0}^2 \exp(\mathbf{x}^T \beta_k)\} \) for \( k = 0, 1, 2 \), and let \( \pi_0(3, x) = 1 - \sum_{k=0}^2 \pi_0(k, x) \). Here, \( \beta_0 = (0, -0.2475, -0.275, 0.1875, 0.075)^T \), \( \beta_1 = (0, -0.165, -0.15, 0.125, 0.05)^T \), and \( \beta_2 = 0 \). We set \( m(k, x) = \alpha_k^T x \), where \( \alpha_0 = (200, 0, 13.7, 13.7, 13.7)^T \), and \( \alpha_1 \) to \( \alpha_3 \) are set to be \( (200, 27.4, 13.7, 13.7, 13.7)^T \). We generated \( Y_i^k \)'s by adding a standard normal random noise to the true mean \( m(k, x_i) \).

In implementing the estimators, in addition to the ideal case where both the \( \pi(\cdot) \) model and the basis for the \( m(\cdot) \) model are correct, we also experiment with incorrectly specified models. In misspecifying the \( \pi(\cdot) \) models, we replace \( X_1 \) with \( e^{X_1} \), \( X_2 \) with \( X_1 X_2 \), \( X_3 \) with \( X_1^2 X_3 \), \( X_4 \) with \( X_1 + X_4 \) and \( X_5 \) with \( X_5 \sin(X_5)^2 \). In misspecifying the \( m(\cdot) \) models, we
replace $X_1$ with $X^2_1$, $X_2$ with $X_1X_2$, $X_3$ with $X_2X^2_3$ and $X_4$ with $(X_4-3)^3+3$. We investigate four different scenarios, when both models are correct, when the $\pi(\cdot)$ model is misspecified, when the $m(\cdot)$ model is misspecified and when both models are misspecified. Note that our design is such that correctly specifying the basis for $m(\cdot)$ corresponds to balancing the first moments of the covariates. For comparison, we also implemented the inverse probability weighting estimators (IPW) using maximum likelihood for the estimation of the propensity score, and its double robust augmented version using both the correct propensity score and outcome models; for the latter we use the R-package PSweight (Zhou et al. 2020). The results over 1000 replicates are displayed in Tables 1-3 (see Appendix A.7) for different sample sizes, where for each causal contrast $\theta_j - \theta_0$, $k = 1, 2, 3$, we provide bias, standard deviation, mean squared errors (MSE) as well as average estimated standard deviation, and empirical coverage of the resulting 95% confidence interval. See Remark 1 for how the inference is carried out.

Figure 1: Absolute bias and sd for the three contrasts $\theta_j - \theta_0$, $j = 1, 2, 3$, over 1000 replicates for the six estimators: $m, \pi$ correct (mT.piT), $m$ correct (mT.piF), $\pi$ correct (mF.piT), $m, \pi$ misspecified (mF.piF), IPW and augmented IPW (DR), and three sample sizes.
Biases and standard deviations are also displayed graphically in Figure 1. These numerical experiments confirm the theoretical robustness properties in the sense that much smaller biases are observed when at least one of the models is correctly specified compared to when both models $\pi(\cdot)$ and $m(\cdot)$ are misspecified. Increasing sample sizes improves biases and variances as expected, except when all models are misspecified. Moreover, compared to the maximum likelihood based inverse probability weighting method (ML-IPW), our estimator yields lower variance, and its MSE is smaller even when both models are misspecified. The classical augmented IPW (DR) should be considered as a benchmark, since in contrast with our estimator which only fits the propensity score, DR fits all models. Fitting the outcome models is, however, arguably not desirable (Rubin 2007), and it appears to yield lower finite sample bias and variance in the cases considered. The relative efficiency of our estimator compared to DR improves with increasing sample sizes although slowly. Empirical coverages match the nominal level of 95%, and this gets better with increasing sample size, except for when all models are misspecified as expected from theory.

4.2 Continuous treatments

To assess the performance of the proposed methods under continuous treatment, we experiment with both linear and nonlinear outcome models. In the nonlinear design, we generate a five dimensional covariate vector $X$, where $X_1 = 1$ and $(X_2, X_3, X_4, X_5)^T$ follows a multivariate standard normal distribution. Thus, these covariates have mean zero, variance 1 and are independent of each other. The true propensity score function is

$$
\pi_0(a, x) = \frac{\Gamma(15)}{\Gamma[15\lambda(x)]\Gamma[15\{1 - \lambda(x)\}]} \left(\frac{a}{20}\right)^{15\lambda(x)-1} \left(1 - \frac{a}{20}\right)^{15\{1-\lambda(x)\}-1} \frac{1}{20}.
$$

Note that this is the probability density function of $A$ when $A/20$ follows a beta distribution with parameters $15\lambda(x)$ and $15\{1 - \lambda(x)\}$, where $\text{logit}\{\lambda(x)\} = (-0.8, 0.1, 0.1, -0.1, 0.2)x$.

We further generate the response $Y$ from a Bernoulli distribution with probability $m_1(A, X) \equiv \text{expit}\{\mu(A, X)\}$, where $\mu(a, x) = (1, 0.2, 0.2, 0.3, -0.1)x + a(0.1, -0.1, 0, 0.1, 0)x - 0.13a^3$. This simulation design is identical to that of Kennedy et al. (2017). In the linear design, the response is generated from a normal distribution with mean $m_2(A, X)$ and variance 0.16, where $m_2(a, x) = \{\mu(a, x) + 15\}/20$. 

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Two different types of IPW estimators are implemented in both linear and nonlinear outcome cases, respectively a maximum likelihood based inverse probability weighting estimator and the proposed robust balancing estimator. For the former, we used a maximum likelihood approach to estimate the parameter of the propensity score. For the balancing estimator, \( l = 3n^{-1/3} \) is minimized where the bandwidth \( l \) was set to \( 3n^{-1/3} \). In the nonparametric estimation of \( \theta(a) \) in (12), both the local constant and local linear estimators given in Remark 2 are implemented and \( h \) was selected by the leave-one-out cross-validation and the one-sided cross-validation (Hart & Yi 1998). For comparison, the inverse probability weighted and the doubly robust estimator given in Kennedy et al. (2017) are also implemented using the R-package npcausal (github.com/ehkennedy/npcausal).

For the linear outcome case, the estimators are assessed in four different scenarios where both models are correct or either of the models is misspecified. We use the basis of \( \mu(a, x) \) as basis of the outcome model. In misspecifying either the \( \pi(\cdot) \) or \( m(\cdot) \) model, we replaced the covariates with \( x^* \) as in Kang & Schafer (2007), with

\[
x^* = \left\{ 1, e^{x_2/2}, \frac{x_3}{1 + \exp(x_2)} + 10, (x_2x_4/25 + 0.6)^3, (x_3 + x_5 + 20)^2 \right\}^T.
\]

In addition, the misspecified \( m_i(\cdot) \) \((i = 1, 2)\) has no cubic term of \( a \) in its bases. We in fact used the same construction for the nonlinear outcome model. However, we point out that this leads to the scenario that the outcome model basis is never correctly specified, while the propensity score model is either correct or incorrect.

We generated the simulated data with sample sizes \( n = 500, 1000, 2000 \) and the result is based on 1000 replicates. Figure 2 illustrates the simulated data with the nonlinear outcome model and the empirical coverage of the proposed estimator under \( n = 1000 \). We assessed the performance of each estimator by calculating the integrated absolute bias and the integrated root-mean-squared error (RMSE), where

\[
\text{bias} = \int_{A^*} \left| E\{\hat{\theta}(a)\} - \theta(a) \right| f_A(a) da,
\]

\[
\text{RMSE} = \int_{A^*} E \left[ (\hat{\theta}(a) - \theta(a))^2 \right]^{1/2} f_A(a) da,
\]

where \( A^* \) is a trimmed support of \( A \) which excludes 10\% mass on the boundaries.
Figure 2: Simulation in the continuous nonlinear outcome case. Rug: One simulated data set with $n=1000$; Solid: True outcome; Dotted: Mean of the estimates, i.e., $\frac{1}{T} \sum_{t=1}^{T} \hat{\theta}_t(a)$, using local constant estimation and CV, and $T = 1000$; Filled curves: 5% and 95% quantiles of $\hat{\theta}_t(a)$.

The results are given in Tables 4 and 5 (Appendix A.7). The integrated absolute bias and the integrated RMSE are numerically calculated and presented with the integrated RMSE in parentheses. For ease of presentation, both measures are multiplied by 100. These results confirm that the proposed estimator is robust. In addition, as seen in Table 4 we find that our estimator shows robust performance even under the nonlinear outcome design where (9) does not hold, which means that none of the four cases used the true basis of the outcome model. Among the balancing estimators, the variant using local linear fit and one-sided CV seems to perform best in terms of bias and RMSE when both all nuisance models are correctly specified. The balancing method has also both lower bias and RMSE than the IPW estimators. We note that the bias is most sensitive to specification of the propensity score model. In all cases, the proposed estimator outperforms the estimator by Kennedy et al. (2017) in terms of bias, although RMSE Kennedy’s double robust estimator has lowest RMSE. Here, as for the categorical case, this estimator can be considered a benchmark.
since it fits also outcome models in contrast with the introduced balancing estimators.

5 Effect of BMI on self reported health decline

As a case study, we investigate the effect of Body Mass Index (BMI) on self reported health (SRH) decline. This analysis is based on data from the Survey of Health, Aging and Retirement in Europe (SHARE). This is an interview based longitudinal survey of individuals of age 50 years or older (Börsch-Supan et al. 2013). Here we use data on women from three countries (Sweden, Netherland, Italy) that participate in waves 1 and 5 of the SHARE study. Wave 1 data collected in 2004 serve as the baseline, and individuals are followed up at wave 5, collected in 2013. We are interested in estimating the average causal effect of BMI (a continuous valued treatment with range 15.62-49.60 in the data) on SRH decline between baseline and follow-up. SRH is measured by asking the question “Would you say your health is: excellent, very good, good, fair or poor?” Despite its unspecific nature, SRH has been found to predict mortality well in many studies (Idler & Benyamini 1997), and is thus considered as an important health indicator. SRH decline is here defined as a binary variable which, for the respondents reporting “excellent, very good, or good health” at baseline, will take value one if they changed their answer to “fair or poor health” at follow-up, and 0 otherwise. The resulting sample of complete cases consists of 1530 participants. In Genbäck et al. (2018), predictors of SRH decline were investigated using logistic regression, and it was found that BMI measured at baseline was a significant (5% level) predictor of SHR decline. Here we aim at sharpening this analysis and study whether there is evidence that BMI is a causal agent of SRH decline by using the introduced covariate balancing procedure for causal inference. The covariates observed at baseline that we use for balancing are age (years), whether the participant responded to the SRH question at the beginning of the interview (or the end), socio-economic variables (education level, make ends meet easily), cognitive function variables (numeracy test, date orientation question), health variables (number of chronic diseases, number of mobility problems, depression measure, maximum grip strength, limitation in normal activities), and lifestyle variables (smoking habits, alcohol usage, physical activities). We refer to Genbäck et al. (2018) for a detailed
description of these covariates. Encouraged by Afshin, A. et al. (2017) and Ng et al. (2016), our analysis is based on the following model for \( A = (\text{BMI} - 15)/40 \) given the covariate vector \( x \):

\[
\pi_0(a, x) = \frac{\Gamma(\phi)}{\Gamma[\phi[\lambda(x)]\Gamma[\phi[1 - \lambda(x)]]]} a^{\phi[\lambda(x)-1]}(1 - a)^{\phi[1-\lambda(x)-1]},
\]

\[
\logit\{\lambda(x)\} = \gamma^T x,
\]

\[
\beta = (\gamma, \phi).
\]

The basis functions for the outcome model are chosen to be \( B(a, x) = (x, a, a^2, a^3) \). A value for \( \beta^{(0)} = (\gamma^{(0)}, \phi^{(0)}) \) is obtained by the maximum likelihood estimation and used as the starting value for solving the balancing equations (11), with the bandwidth 6n^{-1/3}. For nonparametric estimation of \( \theta(a) \) in (12), the local constant estimator given in Remark 2 is used for simplicity, where \( h \) was selected by one-sided cross-validation (Hart & Yi 1998).

Figure 3: Effect of BMI on SRH decline. Rug plot: the observations; solid line: the estimated average treatment effect curve; filled gray curve: the estimated pointwise confidence band.

Figure 3 displays the estimated effect curve of BMI on SRH decline. Confidence bands
are obtained using the variance estimates described in Appendix A.2.4. Overall, we observe a nonlinear effect curve. Specifically, we observe that BMI has no significant effect for values of BMI considered as normal (i.e. below 25) in that the confidence band of the probability of decline contains the flat line. However in the range of BMIs considered as overweight (BMI larger than 25), an increase in the probability of SRH decline is observed, reflecting the causal effect of the increase of BMI on the probability of SRH decline. The causal interpretation of this effect relies on the assumptions made. Mainly that all confounders have been observed, and that a well defined intervention on BMI corresponds to the effect measured (Hernan & Taubman 2008). Nevertheless, the results are in line with earlier studies pointing at a wide range of health risks from overweight and obesity (Afshin et al. 2017).

6 Discussion

We have introduced novel robust estimation and inference tools for multi-level treatments. For continuous treatments our proposal together with that of Kennedy et al. (2017) are, to the best of our knowledge, the only robust methods which model the causal dose-response curve nonparametrically. Our results expand the recent important developments given by Fan et al. (2020). For both the categorical and continuous treatment cases, we achieve robustness by balancing basis functions for the outcome models when fitting a generalized propensity score model which is either correct or sufficiently rich. While the estimator proposed is locally efficient for the categorical case, asymptotic efficiency is not relevant for the continuous case where the parameter of interest is a function of the dose and is estimated non-parametrically.

The proposal differs from earlier double robust estimation in that it does not need outcome models to be fitted. This is an advantage when outcome is not observed at the design stage of the study. Indeed, it is argued that observational studies should be designed without using observed outcomes even if available in order to mimic the “objectivity” of the designs of randomized trials; see (Rubin 2007) for a detail discussion. Our simulation results indicate that this is done at a cost in finite sample performance. Our work is
somewhat in contrast to the widespread practice of using simple (e.g., linear or logistic) models for the propensity score with matching estimators assuming that balance in the joint distribution of the covariates is achieved (e.g., Rubin & Thomas 2000, Waernbaum 2010). However, balancing the joint distribution is not necessary, and in exchange, more elaborate requirements are on the propensity score. From the results presented herein, it becomes transparent which functions of the covariates are sufficient to balance for in order to both obtain consistency and, in the categorical treatment case, local efficiency.

In high-dimensional settings ($d \approx n$), it has recently been shown that bias due to regularization in estimating correctly specified linear outcome models can be corrected by using relevant weights which are not necessarily based on the true propensity score (Athey et al. 2018); see also, e.g., Farrell (2015) and Dukes et al. (2020) for double robust estimation with many covariates. An interesting future direction of research is whether one can generalize the results presented herein to high-dimensional situations, balancing many basis functions for the outcome models by using, e.g., regularized GMM techniques (Belloni et al. 2018).

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References

Afshin, A. et al. (2017), ‘Health effects of overweight and obesity in 195 countries over 25 years’, New England Journal of Medicine 377, 13–27.

Athey, S., Imbens, G. W. & Wager, S. (2018), ‘Approximate residual balancing: debiased inference of average treatment effects in high dimensions’, Journal of the Royal Statistical Society: Series B (Statistical Methodology) 80, 597–623.

Belloni, A., Chernozhukov, V., Chetverikov, D., Hansen, C. & Kato, K. (2018), ‘High-dimensional econometrics and regularized gmm’, arXiv:1806.01888.

Börsch-Supan, A., Brandt, M., Hunkler, C., Kneip, T., Korbmacher, J., Malter, F., Schaan, B., Stuck, S. & Zuber, S. (2013), ‘Data Resource Profile: The Survey of Health, Ageing and Retirement in Europe (SHARE)’, International Journal of Epidemiology 42, 992–1001.
Cattaneo, M. D. (2010), ‘Efficient semiparametric estimation of multi-valued treatment effects under ignorability’, *Journal of Econometrics* **155**, 138 – 154.

Dukes, O., Avagyan, V. & Vansteelandt, S. (2020), ‘Doubly robust tests of exposure effects under high-dimensional confounding’, *Biometrics*. On-line ahead of print: 10.1111/biom.13231.

Fan, J., Imai, K., Liu, H., Ning, Y. & Yang, X. (2020), ‘Optimal covariate balancing conditions in propensity score estimation’, *Working paper*. URL: https://cpb-us-w2.wpmucdn.com/sites.coecis.cornell.edu/dist/3/72/files/2020/09/CBPStheory.pdf

Farrell, M. (2015), ‘Robust inference on average treatment effects with possibly more covariates than observations.’, *Journal of Econometrics* **189**, 1–23.

Fong, C., Hazlett, C. & Imai, K. (2018), ‘Covariate balancing propensity score for a continuous treatment: Application to the efficacy of political advertisements’, *Annals of Applied Statistics* **12**, 156–177.

Galvao, A. F. & Wang, L. (2015), ‘Uniformly semiparametric efficient estimation of treatment effects with a continuous treatment’, *Journal of the American Statistical Association* **110**, 1528–1542.

Genbäck, M., Ng, N., Stanghellini, E. & de Luna, X. (2018), ‘Predictors of decline in self-reported health: addressing non-ignorable dropout in longitudinal studies of aging’, *European Journal of Ageing* **15**, 211–220.

Hahn, J. (1998), ‘On the role of the propensity score in efficient semiparametric estimation of average treatment effects’, *Econometrica* **66**, pp. 315–331.

Hart, J. D. & Yi, S. (1998), ‘One-sided cross-validation’, *Journal of the American Statistical Association* **93**, 620–631.

Hernan, M. A. & Taubman, S. L. (2008), ‘Does obesity shorten life? the importance of well-defined interventions to answer causal questions’, *International journal of obesity* **32**, S8–S14.

Hirano, K. & Imbens, G. W. (2004), The propensity score with continuous treatments, in A. Gelman & X.-L. Meng, eds, ‘Applied Bayesian Modeling and Causal Inference from Incomplete-data Perspectives’, Wiley, New York, p. 73–84.

Idler, E. L. & Benyamini, Y. (1997), ‘Self-rated health and mortality: a review of twenty-seven community studies’, *J Health Soc Behav.* **38**, 21–37.

Imai, K. & Ratkovic, M. (2014), ‘Covariate balancing propensity score’, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **76**, 243–263.

Imbens, G. (2000), ‘The role of the propensity score in estimating dose-response functions’, *Biometrika* **87**, 706–710.
Kang, J. D. & Schafer, J. L. (2007), ‘Demystifying double robustness: A comparison of alternative strategies for estimating a population mean from incomplete data’, Statistical science **22**, 523–539.

Kennedy, E. H., Ma, Z., McHugh, M. D. & Small, D. S. (2017), ‘Non-parametric methods for doubly robust estimation of continuous treatment effects’, *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **79**, 1229–1245.

Newey, W. K. & McFadden, D. L. (1994), Large sample estimation and hypothesis testing, in R. F. Engle & D. L. McFadden, eds, ‘Handbook of Econometrics, Volume IV’, Elsevier Science, Amsterdam, chapter 36, pp. 2111–2245.

Ng, M., Liu, P., Thomson, B. & Murray, C. J. (2016), ‘A novel method for estimating distributions of body mass index’, *Population health metrics* **14**(1), 6.

Robins, J. (2000), ‘Marginal structural models and causal inference in epidemiology’, *Epidemiology* **11**, 550–560.

Robins, J. M. & Rotnitzky, A. (1995), ‘Semiparametric efficiency in multivariate regression models with missing data’, *Journal of the American Statistical Association* **90**, 122–129.

Rosenbaum, P. R. & Rubin, D. B. (1983), ‘The central role of the propensity score in observational studies for causal effects’, *Biometrika* **70**, 41–55.

Rubin, D. B. (2007), ‘The design versus the analysis of observational studies for causal effects: parallels with the design of randomized trials’, *Statistics in Medicine* **26**(1), 20–36.

Rubin, D. B. & Thomas, N. (2000), ‘Combining propensity score matching with additional adjustments for prognostic covariates’, *Journal of the American Statistical Association* **95**, 573–585.

Scharfstein, D. O., Rotnitzky, A. & Robins, J. M. (1999), ‘Adjusting for nonignorable drop-out using semiparametric nonresponse models’, *Journal of the American Statistical Association* **94**, 1096–1120.

van der Laan, M. J. & Robins, J. M. (2003), *Unified Methods for Censored Longitudinal Data and Causality*, Springer, Berlin.

Waernbaum, I. (2010), ‘Propensity score model specification for estimation of average treatment effects’, *Journal of Statististical Planning and Inference* **140**, 1948–1956.

Wang, Y. & Zubizarreta, J. R. (2019), ‘Minimal dispersion approximately balancing weights: asymptotic properties and practical considerations’, *Biometrika* **107**(1), 93–105.

Wong, R. K. W. & Chan, K. C. G. (2017), ‘Kernel-based covariate functional balancing for observational studies’, *Biometrika* **105**(1), 199–213.

Yang, S., Imbens, G. W., Cui, Z., Faries, D. E. & Z, K. (2016), ‘Propensity score matching and subclassification in observational studies with multi-level treatments’, *Biometrics* **72**, 1055–1065.
Zhou, T., Tong, G., Li, F., Thomas, L. E. & Li, F. (2020), ‘Psweight: An r package for propensity score weighting analysis’, arXiv:2010.08893.

Zubizarreta, J. R. (2015), ‘Stable weights that balance covariates for estimation with incomplete outcome data’, Journal of the American Statistical Association 110, 910–922.
Appendix

A.1 Categorical treatment: derivations

A.1.1 Asymptotic distribution and variance of $\hat{\theta}_k$’s

Let

$$f_{ki}(\beta) \equiv \left\{ \frac{I(A_i = k)}{\pi(k, X_i, \beta)} - 1 \right\} B(k, X_i),$$

$$f_i(\beta) \equiv \{f_{1i}(\beta)^T, \ldots, f_{Ki}(\beta)^T\}^T, \quad V(\beta) \equiv E\{f_i(\beta)f_i(\beta)^T\}, \quad \hat{V}(\beta) \equiv n^{-1}\sum_{i=1}^n f_i(\beta)f_i(\beta)^T,$$

$$A(\beta) \equiv E\{\partial f_i(\beta)/\partial \beta^T\} \text{ and } \hat{A}(\beta) \equiv n^{-1}\sum_{i=1}^n \partial f_i(\beta)/\partial \beta^T.$$

**Lemma 2.** Under regularity conditions $A0$, $A1$, $A2$ and $A3$, the GMM estimator $\hat{\beta}$ obtained by minimizing $\{\sum_{i=1}^n f_i(\beta)\}\hat{V}(\beta)^{-1}\{\sum_{i=1}^n f_i(\beta)\}$, is such that

$$n^{1/2}(\hat{\beta} - \beta^*) = -\{A(\beta^*)^T V(\beta^*)^{-1} A(\beta^*)\}^{-1} A(\beta^*)^T V(\beta^*)^{-1} \{n^{-1/2} \sum_{i=1}^n f_i(\beta^*)\} + O_p(n^{-1/2}).$$

When (1) holds $\beta^* = \beta_0$.

**Proof.** The GMM estimator $\hat{\beta}$ is obtained by minimizing $\{\sum_{i=1}^n f_i(\beta)\}\hat{V}(\beta)^{-1}\{\sum_{i=1}^n f_i(\beta)\}$.

This entails

$$0 = \hat{A}(\hat{\beta})^T \hat{V}(\hat{\beta})^{-1} \{n^{-1/2} \sum_{i=1}^n f_i(\hat{\beta})\} + \frac{n^{1/2}}{2} \left[ \left\{ \frac{1}{n} \sum_{i=1}^n f_i(\beta) \right\} \frac{\partial \hat{V}(\hat{\beta})^{-1}}{\partial \hat{\beta}_k} \right] \left\{ \frac{1}{n} \sum_{i=1}^n f_i(\beta) \right\}^p_{k=1},$$

$$= \hat{A}(\hat{\beta})^T \hat{V}(\hat{\beta})^{-1} \{n^{-1/2} \sum_{i=1}^n f_i(\hat{\beta})\} + O_p(n^{-1/2})$$

$$= A(\beta^*)^T V(\beta^*)^{-1} \{n^{-1/2} \sum_{i=1}^n f_i(\beta^*)\} + A(\beta^*)^T V(\beta^*)^{-1} A(\beta^*) n^{1/2}(\hat{\beta} - \beta) + O_p(n^{-1/2}),$$

hence

$$n^{1/2}(\hat{\beta} - \beta^*) = -\{A(\beta^*)^T V(\beta^*)^{-1} A(\beta^*)\}^{-1} A(\beta^*)^T V(\beta^*)^{-1} \{n^{-1/2} \sum_{i=1}^n f_i(\beta^*)\} + O_p(n^{-1/2}).$$
Proof of Theorem 1. Using Lemma 2 we can write

\[ n^{-1/2} (\hat{\theta} - \theta) = n^{-1/2} \sum_{i=1}^{n} g_i(\hat{\beta}) \]

\[ = n^{-1/2} \sum_{i=1}^{n} \{ g_i(\hat{\beta}) - g_i(\beta^*) \} + n^{-1/2} \sum_{i=1}^{n} g_i(\beta^*) \]

\[ = -B(\beta^*) \{ A(\beta^*)^T V(\beta^*)^{-1} A(\beta^*) \}^{-1} A(\beta^*)^T V(\beta^*)^{-1} \{ n^{-1/2} \sum_{i=1}^{n} f_i(\beta^*) \} \]

\[ + n^{-1/2} \sum_{i=1}^{n} g_i(\beta^*) + O_p(n^{-1/2}). \]

When either (1) and/or (2) hold, we already know that \( E\{ g_i(\beta^*) \} = 0 \). Thus, under regularity conditions, \( \sqrt{n}(\hat{\theta} - \theta) \) has asymptotic normal distribution with mean zero and variance

\[ \Sigma = \text{var} \left[ -B(\beta^*) \{ A(\beta^*)^T V(\beta^*)^{-1} A(\beta^*) \}^{-1} A(\beta^*)^T V(\beta^*)^{-1} f_i(\beta^*) + g_i(\beta^*) \right] \]

\[ = B(\beta^*) \{ A(\beta^*)^T V(\beta^*)^{-1} A(\beta^*) \}^{-1} B(\beta^*)^T + C(\beta^*) \]

\[ -B(\beta^*) \{ A(\beta^*)^T V(\beta^*)^{-1} A(\beta^*) \}^{-1} A(\beta^*)^T V(\beta^*)^{-1} D(\beta^*) \]

\[ -D(\beta^*)^T [ B(\beta^*) \{ A(\beta^*)^T V(\beta^*)^{-1} A(\beta^*) \}^{-1} A(\beta^*)^T V(\beta^*)^{-1} ]^T, \]

where \( C(\beta^*) \equiv E\{ g_i(\beta^*)^{\otimes 2} \} \) and \( D(\beta^*) \equiv E\{ f_i(\beta^*) g_i(\beta^*)^T \}. \)

Proof of Corollary 1. When all models are correctly specified, i.e. (1)–(2) hold, we have \( \beta^* = \)
hence we can solve $\sum f_i(\beta) = 0$ directly. As a consequence, we can write

$$0 = n^{-1/2} \sum_{i=1}^{n} f_i(\hat{\beta}) + O_p(n^{-1/2})$$

$$= n^{-1/2} \sum_{i=1}^{n} f_i(\beta^*) + A(\beta^*)n^{1/2}(\hat{\beta} - \beta) + O_p(n^{-1/2}),$$

hence

$$n^{1/2}(\hat{\beta} - \beta) = -A(\beta^*)^{-1}\left\{n^{-1/2} \sum_{i=1}^{n} f_i(\beta^*)\right\} + O_p(n^{-1/2}).$$
This leads to

\[
\begin{align*}
  n^{-1/2}(\hat{\theta} - \theta) &= n^{-1/2} \sum_{i=1}^{n} g_i(\hat{\beta}) \\
  &= n^{-1/2} \sum_{i=1}^{n} \{g_i(\hat{\beta}) - g_i(\beta_0)\} + n^{-1/2} \sum_{i=1}^{n} g_i(\beta_0) \\
  &= -B(\beta_0)A(\beta_0)^{-1}\{n^{-1/2} \sum_{i=1}^{n} f_i(\beta_0)\} + n^{-1/2} \sum_{i=1}^{n} g_i(\beta_0) + O_p(n^{-1/2}) \\
  &= -(I_{k+1} \otimes \alpha^T)\{n^{-1/2} \sum_{i=1}^{n} f_i(\beta_0)\} + n^{-1/2} \sum_{i=1}^{n} g_i(\beta_0) + O_p(n^{-1/2}).
\end{align*}
\]

Thus, \(\sqrt{n}(\hat{\theta} - \theta)\) has asymptotic normal distribution with mean zero and variance

\[
\Sigma = \text{var}\{g_i(\beta_0) - (I_{k+1} \otimes \alpha^T)f_i(\beta_0)\}
\]

\[
= \text{var} \left( \begin{bmatrix}
  \frac{I(\alpha=0)}{\pi(0, X)} - E\{m(0, X)\} \\
  \vdots \\
  \frac{I(\alpha=K)}{\pi(K, X)} - E\{m(K, X)\}
\end{bmatrix} - (I_{k+1} \otimes \alpha^T) \begin{bmatrix}
  \{ \frac{I(\alpha=0)}{\pi(0, X)} - 1 \} B(0, X) \\
  \vdots \\
  \{ \frac{I(\alpha=k)}{\pi(K, X)} - 1 \} B(K, X)
\end{bmatrix}\right)
\]

\[
= \text{var} \left( \begin{bmatrix}
  \frac{I(\alpha=0)}{\pi(0, X)} - E\{m(0, X)\} \\
  \vdots \\
  \frac{I(\alpha=K)}{\pi(K, X)} - E\{m(K, X)\}
\end{bmatrix} - \begin{bmatrix}
  \{ \frac{I(\alpha=0)}{\pi(0, X)} - 1 \} m(0, X) \\
  \vdots \\
  \{ \frac{I(\alpha=k)}{\pi(K, X)} - 1 \} m(K, X)
\end{bmatrix}\right)
\]

\[
= \text{var} \left( \begin{bmatrix}
  \frac{I(\alpha=0)}{\pi(0, X)} - E\{m(0, X)\} + m(0, X) - E\{m(0, X)\} \\
  \frac{I(\alpha=1)}{\pi(1, X)} - E\{m(1, X)\} + m(1, X) - E\{m(1, X)\} \\
  \vdots \\
  \frac{I(\alpha=K)}{\pi(K, X)} - E\{m(K, X)\} + m(K, X) - E\{m(K, X)\}
\end{bmatrix}\right)
\]

i.e., the \((k, l)\) entry of \(\Sigma\) is

\[
\Sigma_{kl} = I(k = l)E \left\{ \frac{v(k, X)}{\pi(k, X)} \right\} + E(\{m(k, X) - E\{m(k, X)\}\}[m(l, X) - E\{m(l, X)\}]).
\]

Compared to the semiparametric efficiency bound obtained in Section A.1.2 below, we see that the estimator is asymptotically efficient. \qed
The tangent space of (19) is
\[ \mathcal{T} = \mathcal{T}_\zeta + \mathcal{T}_\beta + \mathcal{T}_\alpha + \mathcal{T}_\gamma, \]
where \( \pi(k, x) \) satisfies \( 0 < \pi(k, x) < 1 \), \( \sum_{k=0}^{K} \pi(k, x) = 1 \) and \( f_{\epsilon(A,X)}(y - m(k, x), k, x) \) satisfies \( \int f_{\epsilon(A,X)}(\epsilon, k, x) d\epsilon = 1 \) and \( \int \epsilon f_{\epsilon(A,X)}(\epsilon, k, x) d\epsilon = 0 \) for all \( k = 0, \ldots, K \). The parameter of interest is \( \theta = (\theta_1, \ldots, \theta_K)^T \), where \( \theta_k = E\{m(k, X)\} \). Here, we sometimes write \( \epsilon = y - m(a, x) \) for convenience. Consider an arbitrary parametric submodel
\[ f_{X, A, Y}(x, a, \delta) = f_X(x, \zeta) \prod_{k=0}^{K} \left[ \pi(k, x, \beta) f_{\epsilon(A,X)}(y - m(k, x, \alpha), k, x, \gamma) \right]^{I(a=k)}, \]
where \( \delta = (\zeta^T, \beta^T, \alpha^T, \gamma^T)^T \). We get the score function \( S_\delta = (S_\zeta^T, S_{\beta}^T, S_{\alpha}, S_{\gamma}^T)^T \), where

\[ S_\zeta = \frac{\partial f_X(x, \zeta)}{f_X(x, \zeta)}; \]
\[ S_{\beta} = \sum_{k=0}^{K} \left( I(A = k) \frac{\partial \pi(k, x, \beta)}{\pi(k, x, \beta)} \right); \]
\[ S_{\alpha} = \sum_{k=0}^{K} I(A = k) \left[ \frac{\partial m(k, x, \alpha)}{\partial \alpha} \frac{f_{\epsilon(A,X)}(y - m(k, x, \alpha), k, x, \gamma)}{f_{\epsilon(A,X)}(y - m(k, x, \alpha), k, x, \gamma)} \right]; \]
\[ S_{\gamma} = \sum_{k=0}^{K} I(A = k) \left[ \frac{\partial f_{\epsilon(A,X)}(y - m(k, x, \alpha), k, x, \gamma)}{f_{\epsilon(A,X)}(y - m(k, x, \alpha), k, x, \gamma)} \right]. \]
The parameter of interest in the submodel is

$$\theta(\zeta, \beta, \alpha, \gamma) = [E\{m(0, X, \alpha)\}, \ldots, E\{m(K, X, \alpha)\}]^T,$$

where

$$E\{m(k, X, \alpha)\} = \int m(k, x, \alpha)f_X(x, \zeta)d\mu(x).$$

Thus,

$$\frac{\partial \theta(\zeta, \beta, \alpha, \gamma)}{\partial \alpha} = \left[ E\left\{ \frac{\partial m(0, X, \alpha)}{\partial \alpha} \right\}, \ldots, E\left\{ \frac{\partial m(K, X, \alpha)}{\partial \alpha} \right\} \right]^T \bigg|_{\alpha=\alpha_0},$$

$$\frac{\partial \theta(\zeta, \beta, \alpha, \gamma)}{\partial \zeta} = \left[ E\left\{ m(0, X)S_\zeta \right\}, \ldots, E\left\{ m(K, X)S_\zeta \right\} \right]^T \bigg|_{\zeta=\zeta_0},$$

while $\frac{\partial \theta(\zeta, \beta, \alpha, \gamma)}{\partial \beta} = 0$ and $\frac{\partial \theta(\zeta, \beta, \alpha, \gamma)}{\partial \gamma} = 0$.

Now consider

$$\phi = \left[ I(A = 0) \frac{Y - m(0, X)}{\pi(0, X)} + m(0, X_i), \ldots, I(A = K) \frac{Y - m(K, X)}{\pi(K, X)} + m(K, X_i) \right]^T.$$

Denote $\tilde{\phi}_k = I(A = k) \frac{Y - m(k, X)}{\pi(k, X)} + m(k, X_i)$. We can easily verify that

$$E(\tilde{\phi}_k S_\beta)$$

$$= E \left[ \left\{ I(A = k) \frac{Y - m(k, X)}{\pi(k, X)} + m(k, X_i) \right\} \left\{ \sum_{l=0}^{K} I(A = l) \frac{\partial \pi(l, x, \beta) / \partial \beta}{\pi(l, x, \beta)} \right\} \right]$$

$$= E \left[ \left\{ I(A = k) \frac{Y - m(k, X)}{\pi(k, X)} \frac{\partial \pi(k, x, \beta) / \partial \beta}{\pi(k, x, \beta)} \right\} \right] + E \left[ m(k, X_i) \left\{ \sum_{l=0}^{K} I(A = l) \frac{\partial \pi(l, x, \beta) / \partial \beta}{\pi(l, x, \beta)} \right\} \right]$$

$$= E \left[ \{Y^k - m(k, X)\} \frac{\partial \pi(k, x, \beta) / \partial \beta}{\pi(k, x, \beta)} \right] + E \left[ m(k, X_i) \left\{ \sum_{l=0}^{K} \frac{\partial \pi(l, x, \beta) / \partial \beta}{\pi(l, x, \beta)} \right\} \right]$$

$$= 0.$$
and

\[
E(\phi_k S_\gamma)
\]

\[
= E \left\{ I(A = k) \frac{Y - m(k, X)}{\pi(k, X)} + m(k, X) \right\} \times \left[ \sum_{l=0}^{K} I(A = l) \frac{\partial f_{\epsilon l(A,X)} \{Y - m(l, X, \alpha), l, X, \gamma \}}{f_{\epsilon l(A,X)} \{Y - m(l, X, \alpha), l, X, \gamma \}} \right]
\]

\[
= E \left( I(A = k) \frac{Y - m(k, X)}{\pi(k, X)} \left[ \sum_{l=0}^{K} I(A = l) \frac{\partial f_{\epsilon l(A,X)} \{Y - m(l, X, \alpha), l, X, \gamma \}}{f_{\epsilon l(A,X)} \{Y - m(l, X, \alpha), l, X, \gamma \}} \right] \right) + E \left( m(k, X) \sum_{l=0}^{K} I(A = l) \frac{\partial f_{\epsilon l(A,X)} \{Y - m(l, X, \alpha), l, X, \gamma \}}{f_{\epsilon l(A,X)} \{Y - m(l, X, \alpha), l, X, \gamma \}} \right)
\]

\[
= E \left\{ \frac{\partial}{\partial \gamma} \int f_{\epsilon l(A,X)}(\epsilon, k, X, \gamma) d\epsilon \right\} + E \left[ m(k, X) \left\{ \sum_{l=0}^{K} \pi(l, X) \frac{\partial}{\partial \gamma} \int f_{\epsilon l(A,X)}(\epsilon, l, X, \gamma) d\epsilon \right\} \right]
\]

\[
= 0.
\]

Hence \(E(\phi S_\beta^T) = 0\) and \(E(\phi S_\gamma^T) = 0\). Further,

\[
E(\phi_k S_\zeta) = E \left\{ I(A = k) \frac{Y - m(k, X)}{\pi(k, X)} + m(k, X) \right\} \frac{\partial f_X(x, \zeta)}{f_X(x, \zeta)}
\]

\[
= 0 + E \left( m(k, X) \frac{\partial f_X(x, \zeta)}{f_X(x, \zeta)} \right)
\]

\[
= E\{m(k, X)S_\zeta(X, \zeta)\},
\]
and

\[
E(\phi_k S_\alpha) = E\left( \{ I(A = k) \frac{Y - m(k, X)}{\pi(k, X)} + m(k, X) \} \right)
\]

\[
\times \left[ \sum_{l=0}^{K} I(A = l) \frac{-\partial m(l, X, \alpha)}{\partial \alpha} \frac{\partial f_{l|A,X}}{f_{l|A,X}} \left( Y - m(l, X, \alpha), l, X, \gamma \right) / \partial \{ Y - m(l, X, \alpha) \} \right]
\]

\[
= E\left[ \left\{ \frac{Y^k - m(k, X)}{Y^k - m(k, X) / \partial \alpha} \left( Y^k - m(k, X, \alpha), k, X, \gamma \right) / \partial \{ Y^k - m(k, X, \alpha) \} \right] \right]
\]

\[
+ E\left[ \frac{-\partial m(k, X, \alpha)}{\partial \alpha} \int \frac{\partial f_{l|A,X}}{f_{l|A,X}}(\epsilon, k, X, \gamma) d\epsilon \right]
\]

\[
+ E\left[ \sum_{l=0}^{K} \frac{-\partial m(l, X, \alpha)}{\partial \alpha} \int \frac{\partial f_{l|A,X}}{f_{l|A,X}}(\epsilon, l, X, \gamma) d\epsilon \right]
\]

\[
= E\left\{ \frac{-\partial m(k, X, \alpha)}{\partial \alpha} \right\},
\]

where \( \alpha, \zeta \) are evaluated at the true value \( \alpha_0, \zeta_0 \). Therefore,

\[
E(\phi^T S^T \zeta) = [E\{m(0, X)S_\zeta(X, \zeta)\}, \ldots, E\{m(K, X)S_\zeta(X, \zeta)\}]^T = \partial \theta(\zeta, \beta, \alpha, \gamma) / \partial \zeta^T.
\]

and

\[
E(\phi^T S^T _\alpha) = [E\{\partial m(0, x, \alpha) / \partial \alpha\}, \ldots, \partial E\{m(K, x, \alpha) / \partial \alpha\}]^T = \partial \theta(\zeta, \beta, \alpha, \gamma) / \partial \alpha^T.
\]

Thus, \( \phi \) satisfies \( E(\phi^T S^T \delta) = \partial \theta(\delta) / \partial \delta^T \). Because the submodel is arbitrary, \( \phi \) is an influence function of \( \theta \). We now try to obtain \( \Pi(\phi \mid T) \) so we can obtain the efficient influence
function. Further, we decompose $\phi$ as $\phi = (\phi_1 + \phi_2 + \phi_3 + c)$, where

$$\phi_1 = \begin{pmatrix}
\frac{I(A=0)}{\pi(0, X)} \left[ Y - m(0, X) + \nu(0, X) \frac{f_{\pi(A \mid X)}^l(Y-m(0, X), 0)}{f_{\pi(A \mid X)}(Y-m(0, X), 0)} \right] \\
\vdots \\
\frac{I(A=K)}{\pi(K, X)} \left[ Y - m(K, X) + \nu(K, X) \frac{f_{\pi(A \mid X)}^l(Y-m(K, X), K, X)}{f_{\pi(A \mid X)}(Y-m(K, X), K, X)} \right]
\end{pmatrix},$$

$$\phi_2 = -\begin{pmatrix}
\frac{I(A=0)}{\pi(0, X)} \nu(0, X) \frac{f_{\pi(A \mid X)}^l(Y-m(0, X), 0)}{f_{\pi(A \mid X)}(Y-m(0, X), 0)} \\
\vdots \\
\frac{I(A=K)}{\pi(K, X)} \nu(K, X) \frac{f_{\pi(A \mid X)}^l(Y-m(K, X), K, X)}{f_{\pi(A \mid X)}(Y-m(K, X), K, X)}
\end{pmatrix},$$

$$\phi_3 = \begin{pmatrix}
m(0, X) - E\{m(0, X)\} \\
m(K, X) - E\{m(K, X)\}
\end{pmatrix},$$

and $c = [E\{m(0, X)\}, \ldots, E\{m(K, X)\}]^T$, where $v(k, X) \equiv \text{var}(Y^k \mid X, A = k)$. We can verify that $\phi_1 \in \mathcal{T}_\gamma$, $\phi_2 \in \mathcal{T}_\alpha$, and $\phi_3 \in \mathcal{T}_\zeta$, while $c$ is a constant. Then $\phi - c$ is the efficient influence function. Thus, the efficient variance is $\Sigma_{\text{eff}} = \text{var}(\phi)$, where the $(k, l)$ entry of $\Sigma_{\text{eff}}$ is

$$\Sigma_{\text{eff},k,l} = I(k = l)E\{v(k, X)/\pi(k, X)\} + E([m(k, X) - E\{m(k, X)\}][m(l, X) - E\{m(k, X)\}]).$$

When $K = 1$, this agrees with the special case corresponding to the binary treatments (Hahn 1998), and when $K > 1$, with earlier results (Cattaneo 2010).

### A.2 Continuous treatment: derivations

We prove all results under a general weight function $w(A_j)$, where $w(A_j) = \sum_{i=1}^n K_i(A_i - A_j)$ in the main paper.
A.2.1 Convergence rate of $\hat{\beta}$

Proof of Lemma 1. From (13), $\beta^*$ satisfies

$$0 = E_j \left( \left[ E_i \left\{ \frac{K_i(A_i - A_j)\pi'_{\beta}(A_j, X_i, \beta^*)}{\pi^2(A_j, X_i, \beta^*)} B^T(A_j, X_i) \right\} \right] \right)$$

$$\times w(A_j) E_i \left[ \left\{ \frac{K_i(A_i - A_j)}{\pi(A_j, X_i, \beta^*)} - 1 \right\} B(A_j, X_i) \right]$$

$$= E_j \left( \left[ E_i \left\{ \frac{\pi_0(A_j, X_i)\pi'_{\beta}(A_j, X_i, \beta^*)}{\pi^2(A_j, X_i, \beta^*)} B^T(A_j, X_i) \right\} \right] \right)$$

$$\times w(A_j) E_i \left[ \left\{ \frac{\pi_0(A_j, X_i)}{\pi(A_j, X_i, \beta^*)} - 1 \right\} B(A_j, X_i) \right] + O(l^2),$$

$$= E_j (U(A_j, \beta^*) w(A_j)$$

$$\times E_i \left[ \left\{ \frac{\pi_0(A_j, X_i)}{\pi(A_j, X_i, \beta^*)} - 1 \right\} B(A_j, X_i) \right] + O(l^2)$$

$$= E_j \left( U(A_j, \beta^*) w(A_j) E_i \left[ \left\{ \frac{\pi_0(A_j, X_i)}{\pi(A_j, X_i, \beta^*)} - 1 \right\} B(A_j, X_i) \right] \right) + O(l^2),$$

where

$$U(a_j, \beta^*) \equiv E \left\{ \frac{\pi_0(a_j, X)\pi'_\beta(a_j, X, \beta^*)}{\pi^2(a_j, X, \beta^*)} B(a_j, X)^T \right\}.$$
We now investigate the convergence rate of \( \hat{\beta} \) from (11). We note that

\[
0 = \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{K_t(A_i - A_j)\pi_\beta'(A_j, X_i, \hat{\beta})}{\pi^2(A_j, X_i, \hat{\beta})} B^T(A_j, X_i) \right\} \right] \\
\times w(A_j) \left( \frac{1}{n^{1/2}} \sum_{i=1}^{n} \left\{ \frac{K_t(A_i - A_j)}{\pi(A_j, X_i, \hat{\beta})} - 1 \right\} B(A_j, X_i) \right)
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \left[ \mathbb{E} \left\{ \frac{K_t(A_i - a_j)\pi_\beta'(a_j, X_i, \beta^*)}{\pi^2(a_j, X_i, \beta^*)} B(a_j, X_i)^T \right\} + o_p(1) \right] \\
\times w(A_j) \left( \frac{1}{n^{1/2}} \sum_{i=1}^{n} \left\{ \frac{K_t(A_i - A_j)}{\pi(A_j, X_i, \beta^*)} - 1 \right\} B(A_j, X_i) \right)
\]

\[-\frac{1}{n} \sum_{j=1}^{n} \left( \left[ \mathbb{E} \left\{ \frac{K_t(A_i - a_j)\pi_\beta'(a_j, X_i, \beta^*)}{\pi^2(a_j, X_i, \beta^*)} B(a_j, X_i)^T \right\} \right]^\otimes 2 w(A_j) + o_p(1) \right) n^{1/2}(\hat{\beta} - \beta^*)
\]

\[
= \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{j=1}^{n} U(A_j, \beta^*) w(A_j) \\
\times \left\{ \frac{K_t(A_i - A_j)}{\pi(A_j, X_i, \beta^*)} - 1 \right\} B(A_j, X_i)
\]

\[-E \left[ \left\{ U(A_j, \beta^*) \right\}^\otimes 2 w(A_j) \right] n^{1/2}(\hat{\beta} - \beta^*) + o_p(1).
\]
We have

\[
\frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{j=1}^{n} U(A_j, \beta^*)w(A_j) \left\{ \frac{K_i(A_i - A_j)}{\pi(A_j, X_i, \beta^*)} - 1 \right\} B(A_j, X_i)
\]

\[
= \frac{1}{n^{1/2}} \sum_{j=1}^{n} U(a_j, \beta^*)w(a_j) E_i \left\{ \frac{K_i(a_i - a_j)}{\pi(a_j, X_i, \beta^*)} - 1 \right\} B(a_j, X_i)
\]

\[
+ \frac{1}{n^{1/2}} \sum_{i=1}^{n} E_j \left[ U(A_j, \beta^*)w(A_j) \left\{ \frac{K_i(a_i - A_j)}{\pi(A_j, X_i, \beta^*)} - 1 \right\} B(A_j, X_i) \right]
\]

\[
- n^{1/2}E_{ij} \left[ U(A_j, \beta^*)w(A_j) \left\{ \frac{K_i(A_i - A_j)}{\pi(A_j, X_i, \beta^*)} - 1 \right\} B(A_j, X_i) \right] + o_p(1)
\]

\[
= \frac{1}{n^{1/2}} \sum_{j=1}^{n} U(a_j, \beta^*)w(a_j) E_i \left\{ \frac{\pi_0(a_j, X_i)}{\pi(a_j, X_i, \beta^*)} B(a_j, X_i) - B(a_j, X_i) \right\}
\]

\[
+ \frac{1}{n^{1/2}} \sum_{i=1}^{n} \left[ f_A(a_i) U(a_i, \beta^*)w(a_i) \frac{B(a_i, X_i)}{\pi(a_i, X_i, \beta^*)} - E_j \left\{ U(A_j, \beta^*)w(A_j)B(A_j, X_i) \right\} \right]
\]

\[
- n^{1/2}E_{ij} \left[ U(A_j, \beta^*)w(A_j) E_i \left\{ \frac{\pi_0(A_j, X_i)}{\pi(A_j, X_i, \beta^*)} B(A_j, X_i) - B(A_j, X_i) \right\} \right]
\]

\[
+ o_p(1) + O_p(n^{1/2}l^2).
\]

Thus, when \( nl^4 \to 0 \), we get

\[
E \left\{ U(A_j, \beta^*)^{\otimes 2} w(A_j) \right\} n^{1/2}(\beta - \beta^*)
\]

\[
= \frac{1}{n^{1/2}} \sum_{i=1}^{n} U(a_i, \beta^*)w(a_i) E_i \left\{ \frac{\pi_0(a_i, X_i)}{\pi(a_i, X_i, \beta^*)} - 1 \right\} B(a_i, X_i)
\]

\[
+ \frac{1}{n^{1/2}} \sum_{i=1}^{n} \left[ f_A(a_i) U(a_i, \beta^*)w(a_i) \frac{B(a_i, X_i)}{\pi(a_i, X_i, \beta^*)} - E_j \left\{ U(A_j, \beta^*)w(A_j)B(A_j, X_i) \right\} \right]
\]

\[
- n^{1/2}E_{ij} \left[ U(A_j, \beta^*)w(A_j) E_i \left\{ \frac{\pi_0(A_j, X_i)}{\pi(A_j, X_i, \beta^*)} - 1 \right\} B(A_j, X_i) \right]
\]

\[
+ o_p(1) + O_p(n^{1/2}l^2)
\]

\[
= \frac{1}{n^{1/2}} \sum_{i=1}^{n} U(a_i, \beta^*)w(a_i) E_k \left\{ \frac{\pi_0(a_i, X_k)}{\pi(a_i, X_k, \beta^*)} - 1 \right\} B(a_i, X_k)
\]

\[
+ \frac{1}{n^{1/2}} \sum_{i=1}^{n} \left[ f_A(a_i) U(a_i, \beta^*)w(a_i) \frac{B(a_i, X_i)}{\pi(a_i, X_i, \beta^*)} - E_j \left\{ U(A_j, \beta^*)w(A_j)B(A_j, X_i) \right\} \right]
\]

\[
+ o_p(1).
\]

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Obviously,
\[
E_i \left( U(A_i, \beta^*) w(A_i) E_k \left( \left\{ \frac{\pi_0(A_i, X_k)}{\pi(A_i, X_k, \beta^*)} - 1 \right\} B(A_i, X_k) \right) \right) = O(l^2)
\]
due to the definition of $\beta^*$. Further, we can verify that
\[
E_i \left[ f_A(A_i) U(A_i, \beta^*) w(A_i) \frac{B(A_i, X_i)}{\pi(A_i, X_i, \beta^*)} \right] = E_j \left\{ U(A_j, \beta^*) w(A_j) B(A_j, X_j) \right\}
\]
\[
= E_{i,j} \left\{ U(A_j, \beta^*) w(A_j) \frac{\pi_0(A_j, X_i) B(A_j, X_i)}{\pi(A_j, X_i, \beta^*)} \right\} - E_{i,j} \left\{ U(A_j, \beta^*) w(A_j) B(A_j, X_i) \right\}
\]
\[
= E_j \left( U(A_j, \beta^*) w(A_j) E_i \left[ \left\{ \frac{\pi_0(A_j, X_i)}{\pi(A_j, X_i, \beta^*)} - 1 \right\} B(A_j, X_i) \right] \right)
\]
\[
= O(l^2)
\]
also due to the definition of $\beta^*$. Thus, as long as $n l^4 \rightarrow 0$, $\hat{\beta} - \beta^* = O_p(n^{-1/2})$. \hfill \Box

\subsection{Robustness and asymptotic bias and variance}

\textbf{Proof of Theorem} When model (8) holds, we can easily check that the expectation of the left hand side of (10) at the true parameter value $\beta_0$ and any function $m(a, x) = B(a, x)^T \gamma$ satisfies
\[
E \left[ \left\{ \frac{K_h(A_i - a)}{\pi_0(a, X_i)} - 1 \right\} m(a, X_i) \right]
\]
\[
= E \left[ \left\{ \frac{E \{ K_h(A_i - a) \} | X_i \} - 1 \right\} m(a, X_i) \right]
\]
\[
= E \left[ \left\{ \frac{\int K_h(A_i - a) \pi_0(A_i, X_i) dA_i}{\pi_0(a, X_i)} - 1 \right\} m(a, X_i) \right]
\]
\[
= E \left[ \left\{ \frac{\int K(t) \pi_0(a + ht, X_i) dt}{\pi_0(a, X_i)} - 1 \right\} m(a, X_i) \right]
\]
\[
= E \left[ \left\{ \frac{\int K(t) \pi_0(a, X_i) dt}{\pi_0(a, X_i)} - 1 \right\} m(a, X_i) \right] + O(h^2)
\]
\[
= O(h^2).
\]
Thus, because the nonparametric estimation convergence rate is slower than $O_p(n^{-1/2})$, by Lemma 1 we can fix $\beta$ at $\beta_0$ in the following analysis, and the first order bias and variance property of $\hat{\theta}(a)$ will not be affected.
Hence, for (12), we have
\[
E\{\hat{\theta}(a)\} = E\left\{ \frac{K_h(A_i - a)Y_i}{\pi_0(a, X_i)} \right\} + O(n^{-1/2})
\]
\[
= E\left\{ \frac{K_h(A_i - a)Y_i}{\pi_0(a, X_i)} \right\} + O(n^{-1/2})
\]
\[
= E\left\{ \frac{K_h(A_i - a)m(A_i, X_i)}{\pi_0(a, X_i)} \right\} + O(n^{-1/2})
\]
\[
= E\left[ m(a, X_i) + \frac{\partial^2\{\pi_0(a, X_i)m(a, X_i)\}}{\pi_0(a, X_i)\partial a^2} \right] h^2 \frac{1}{2} \int t^2 K(t) dt + O(h^4 + n^{-1/2})
\]
\[
\theta(a) + E\left[ \frac{\partial^2\{\pi_0(a, X_i)m(a, X_i)\}}{\pi_0(a, X_i)\partial a^2} \right] h^2 \frac{1}{2} \int t^2 K(t) dt + O(h^4 + n^{-1/2}).
\]

The variance is calculated as
\[
\text{var}\{\hat{\theta}(a)\} = \text{var}\left[ n^{-1} \sum_{i=1}^{n} \left\{ \frac{K_h(A_i - a)Y_i}{\pi_0(a, X_i)} \right\} + O_p(n^{-1/2}) \right].
\]

Now, recall that the variance of \( Y_i(A_i) \) conditional on \( X_i, A_i \) is denoted \( \sigma^2(A_i, X_i) \), then
\[
E \left[ \left\{ \frac{K_h(A_i - a)Y_i}{\pi_0(a, X_i)} \right\}^2 \right]
\]
\[
= E \left[ \left\{ \frac{K_h(A_i - a)}{\pi_0(a, X_i)} \right\}^2 \left\{ m^2(A_i, X_i) + \sigma^2(A_i, X_i) \right\} \right]
\]
\[
= \int K^2(t) dt \frac{1}{h} E \left\{ \frac{m^2(a, X_i) + \sigma^2(a, X_i)}{\pi_0(a, X_i)} \right\} + O(h).
\]

Thus,
\[
\text{var}\{\hat{\theta}(a)\} = \text{var}\left[ n^{-1} \sum_{i=1}^{n} \left\{ \frac{K_h(A_i - a)Y_i}{\pi_0(a, X_i)} \right\} + O_p(n^{-1/2}) \right]
\]
\[
= \int K^2(t) dt \frac{1}{nh} E \left\{ \frac{m^2(a, X_i) + \sigma^2(a, X_i)}{\pi_0(a, X_i)} \right\} + O(n^{-1}h + n^{-1} + n^{-1}h^{-1/2}).
\]

The asymptotic normality is shown in Section A.2.3 below. \qed
Proof of Theorem 3. When model (9) is correct, then \( \hat{\beta} \) converges to \( \beta^* \) at root-\( n \) rate (Lemma 1). Thus,

\[
E\{\hat{\theta}(a)\} = E \left\{ \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \beta^*)} \right\} + O(n^{-1/2})
\]

\[
= E \left\{ \frac{K_h(A_i - a)Y_i(A_i)}{\pi(a, X_i, \beta^*)} \right\} + O(n^{-1/2})
\]

\[
= E \left[ \frac{K_h(A_i - a)m(A_i, X_i)}{\pi(a, X_i, \beta^*)} \right] + O(n^{-1/2})
\]

\[
= E \left[ \frac{\pi_0(a, X_i)m(a, X_i)}{\pi(a, X_i, \beta^*)} \right] + \int \frac{t^2 K(t)dt}{2h^2} \frac{\partial^2 \{m(a, X_i)\pi_0(a, X_i)\}}{(\pi(a, X_i, \beta^*)\partial a^2} + O(n^{-1/2})
\]

\[
= E\left\{ \frac{\pi_0(a, X_i)}{\pi(a, X_i, \beta^*)} - 1 \right\}m(a, X_i) + E\{m(a, X_i)\}
\]

\[
+ \int \frac{t^2 K(t)dt}{2h^2} E \left[ \frac{\partial^2 \{m(a, X_i)\pi_0(a, X_i)\}}{\pi(a, X_i, \beta^*)\partial a^2} \right] + O(n^{-1/2})
\]

\[
= E\{m(a, X_i)\} + \int \frac{t^2 K(t)dt}{2h^2} E \left[ \frac{\partial^2 \{m(a, X_i)\pi_0(a, X_i)\}}{\pi(a, X_i, \beta^*)\partial a^2} \right] + O(n^{-1/2}).
\]

The variance is calculated as

\[
\text{var}\{\hat{\theta}(a)\} = \text{var} \left[ n^{-1} \sum_{i=1}^{n} \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \beta^*)} + O_p(n^{-1/2}) \right].
\]

Then

\[
E \left\{ \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \beta^*)} \right\}^2
\]

\[
= E \left\{ \frac{K_h(A_i - a)}{\pi(a, X_i, \beta^*)} \right\}^2 \left\{ m^2(A_i, X_i) + \sigma^2(A_i, X_i) \right\}
\]

\[
= \int \frac{K^2(t)dt}{h} E \left[ \frac{\pi_0(a, X_i)\{m^2(a, X_i) + \sigma^2(a, X_i)\}}{\pi^2(a, X_i, \beta^*)} \right] + O(h).
\]

Thus,

\[
\text{var}\{\hat{\theta}(a)\} = \text{var} \left\{ n^{-1} \sum_{i=1}^{n} \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \beta^*)} + O_p(n^{-1/2}) \right\}
\]

\[
= \int \frac{K^2(t)dt}{nh} E \left[ \frac{\pi_0(a, X_i)\{m^2(a, X_i) + \sigma^2(a, X_i)\}}{\pi^2(a, X_i, \beta^*)} \right] + O(n^{-1/h} + n^{-1} + n^{-1}h^{-1/2}).
\]
Proof of Theorem 4

\[
\text{cov} \left\{ n^{-1} \sum_{i=1}^{n} \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \beta)}, n^{-1} \sum_{i=1}^{n} \frac{K_h(A_i - b)Y_i}{\pi(b, X_i, \beta)} \right\} \\
= E \left[ \left\{ n^{-1} \sum_{i=1}^{n} \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \beta)} \right\} \left\{ n^{-1} \sum_{i=1}^{n} \frac{K_h(A_i - b)Y_i}{\pi(b, X_i, \beta)} \right\} \right] \\
- E \left\{ n^{-1} \sum_{i=1}^{n} \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \beta)} \right\} E \left\{ n^{-1} \sum_{i=1}^{n} \frac{K_h(A_i - b)Y_i}{\pi(b, X_i, \beta)} \right\} \\
= n^{-2} \sum_{i=1}^{n} E \left\{ \frac{K_h(A_i - a)K_h(A_i - b)Y_i^2}{\pi(a, X_i, \beta)\pi(b, X_i, \beta)} \right\} + n^{-2} \sum_{i \neq j, j=1}^{n} E \left\{ \frac{K_h(A_i - a)Y_i K_h(A_j - b)Y_j}{\pi(a, X_i, \beta)\pi(b, X_j, \beta)} \right\} \\
- E \left\{ \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \beta)} \right\} E \left\{ \frac{K_h(A_i - b)Y_i}{\pi(b, X_i, \beta)} \right\} \\
= n^{-1} E \left\{ \frac{K_h(A_i - a)K_h(A_i - b)Y_i^2}{\pi(a, X_i, \beta)\pi(b, X_i, \beta)} \right\} - n^{-1} E \left\{ \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \beta)} \right\} E \left\{ \frac{K_h(A_i - b)Y_i}{\pi(b, X_i, \beta)} \right\} \\
= n^{-1} E \left\{ \frac{K_h(A_i - a)K_h(A_i - b)Y_i^2}{\pi(a, X_i, \beta)\pi(b, X_i, \beta)} \right\} - n^{-1} \{ \hat{\theta}(a) \hat{\theta}(b) \} \\
= n^{-1} \{ \theta(a) \theta(b) + O(h^2) \}. 
\]

The asymptotic normality is shown in Section A.2.3 below. □
When $a$ and $b$ are sufficiently close, so that $c \equiv (a - b)/h \in (-2, 1)$, we have

\[
E \left\{ \frac{K_h(A_i - a)K_h(A_i - b)Y_i^2}{\pi(a, X_i, \hat{\beta})\pi(b, X_i, \hat{\beta})} \right\}
\]

\[
= E \left\{ \frac{K_h(A_i - a)K_h(A_i - b)\{m^2(A_i, X_i) + \sigma^2(A_i, X_i)\}}{\pi(a, X_i, \hat{\beta})\pi(b, X_i, \hat{\beta})} \right\}
\]

\[
= h^{-1}E \int_0^1 \frac{K(t)K(t + c)\{m^2(a, X_i) + \sigma^2(a, X_i)\}}{\pi(a, X_i, \hat{\beta})\pi(b, X_i, \hat{\beta})}\pi_0(a, X_i)dt
\]

\[
+ E \int_0^1 K(t)K(t + c) \{2m(a, X_i)m'(a, X_i)\pi_0(a, X_i) + m^2(a, X_i)\pi'_0(a, X_i) + 2\sigma(a, X_i)\sigma'_a(a, X_i)\pi_0(a, X_i) + \sigma^2(a, X_i)\pi'_0(a, X_i)\} t/\{\pi(a, X_i, \hat{\beta})\pi(b, X_i, \hat{\beta})\}\pi_0(a, X_i)dt + O(h)
\]

\[
= h^{-1}E \int_0^1 \frac{K(t)K(t + c)\{m^2(a, X_i) + \sigma^2(a, X_i)\}}{\pi(a, X_i, \hat{\beta}^*)\pi(b, X_i, \hat{\beta}^*)}\pi_0(a, X_i)dt
\]

\[
+ E \int_0^1 K(t)K(t + c) \{2m(a, X_i)m'(a, X_i)\pi_0(a, X_i) + m^2(a, X_i)\pi'_0(a, X_i) + 2\sigma(a, X_i)\sigma'_a(a, X_i)\pi_0(a, X_i) + \sigma^2(a, X_i)\pi'_0(a, X_i)\} t/\{\pi(a, X_i, \hat{\beta}^*)\pi(b, X_i, \hat{\beta}^*)\}\pi_0(a, X_i)dt 
\]

\[
+ O(h + h^{-1}n^{-1/2}).
\]

Note that when $c \notin (-2, 1)$, $K(t)K(t + c) = 0$ for all $t \notin [-1, 1]$ hence the above expression still holds. Thus, we obtain

\[
\text{cov}(\hat{\theta}(a), \hat{\theta}(b)) = (nh)^{-1}E \int_0^1 \frac{K(t)K(t + c)\{m^2(a, X_i) + \sigma^2(a, X_i)\}}{\pi(a, X_i, \hat{\beta}^*)\pi(b, X_i, \hat{\beta}^*)}\pi_0(a, X_i)dt
\]

\[
+ n^{-1}E \int_0^1 K(t)K(t + c) \{2m(a, X_i)m'(a, X_i)\pi_0(a, X_i) + m^2(a, X_i)\pi'_0(a, X_i) + 2\sigma(a, X_i)\sigma'(a, X_i)\pi_0(a, X_i) + \sigma^2(a, X_i)\pi'_0(a, X_i)\} t/\{\pi(a, X_i, \hat{\beta}^*)\pi(b, X_i, \hat{\beta}^*)\}\pi_0(a, X_i)dt
\]

\[
- n^{-1}\theta(a)\theta(b) + O(n^{-1}h + h^{-1}n^{-3/2}).
\]

The asymptotic normality is shown in Section A.2.3 below.
A.2.3 Asymptotic distribution of $\hat{\theta}(a)$

Proof of asymptotic normality, Theorems 2-4. When (8) is correct, define

$$\text{bias}\{\hat{\theta}(a)\} = \frac{h^2}{2} E \left[ \frac{\partial^2 \{\pi_0(a, X_i) m(a, X_i)\}}{\pi_0(a, X_i) \partial a^2} \right] \int t^2 K(t)dt.$$ 

On the other hand, when (9) is correct, define

$$\text{bias}\{\hat{\theta}(a)\} = \frac{h^2}{2} E \left[ \frac{\partial^2 \{\pi_0(a, X_i) m(a, X_i)\}}{\pi(a, X_i, \beta^*) \partial a^2} \right] \int t^2 K(t)dt.$$ 

Regardless (8) or (9) is correct, define

$$\text{var}\{\hat{\theta}(a)\} = \int K^2(t) dt \frac{\pi_0(a, X_i) \{m^2(a, X_i) + \sigma^2(a, X_i)\}}{\pi^2(a, X_i, \beta^*)}.$$ 

Note that when (8) is correct, it degenerates to

$$\text{var}\{\hat{\theta}(a)\} = \int K^2(t) dt \frac{\pi_0(a, X_i) \{m^2(a, X_i) + \sigma^2(a, X_i)\}}{\pi_0(a, X_i)}.$$ 

Then

$$\left[ \hat{\theta}(a) - \theta(a) - \text{bias}\{\hat{\theta}(a)\} \right] = n^{-1} \sum_{i=1}^{n} \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \beta^*)} - \theta(a) - \text{bias}\{\hat{\theta}(a)\} + O_p(n^{-1/2})$$

$$= n^{-1} \sum_{i=1}^{n} \left[ \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \beta^*)} - E \left\{ \frac{K_h(A_i - a)Y_i}{\pi(a, X_i, \beta^*)} \right\} \right] + O_p(h^4 + n^{-1/2}).$$

Thus, when $n \to \infty$, following the variance result, we get that

$$\sqrt{nh} \left[ \hat{\theta}(a) - \theta(a) - \text{bias}\{\hat{\theta}(a)\} \right]$$

converges to a normal distribution with mean zero and variance $nh\text{var}\{\hat{\theta}(a)\}$. 

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Consider an arbitrary linear combination $\sum_{j=1}^{J} c_j \hat{\theta}(a_j)$. Then
\[
\begin{align*}
\left[ \sum_{j=1}^{J} c_j \hat{\theta}(a_j) - \sum_{j=1}^{J} c_j \theta(a_j) - \text{bias} \left( \sum_{j=1}^{J} c_j \hat{\theta}(a_j) \right) \right] \\
= n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{J} c_j \frac{K_h(A_i - a_j)Y_i}{\pi(a_j, \mathbf{x}_i, \mathbf{\beta}^*)} - \sum_{j=1}^{J} c_j \theta(a_j) - \sum_{j=1}^{J} c_j \text{bias}\{\hat{\theta}(a)\} + O_p(n^{-1/2}) \\
= n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{J} c_j \left[ \frac{K_h(A_i - a_j)Y_i}{\pi(a_j, \mathbf{x}_i, \mathbf{\beta}^*)} - E \left\{ \frac{K_h(A_i - a_j)Y_i}{\pi(a_j, \mathbf{x}_i, \mathbf{\beta}^*)} \right\} \right] + O_p(h^4 + n^{-1/2})
\end{align*}
\] converges to a normal distribution with mean zero. To compute its variance, we compute $\text{cov}\{\hat{\theta}(a), \hat{\theta}(b)\}$ for arbitrary $a, b$ below.

Let $\text{cov}\{\hat{\theta}(a), \hat{\theta}(b)\}$ be given as the leading term in (18). Note that when (8) is correct, it degenerates to
\[
\text{cov}\{\hat{\theta}(a), \hat{\theta}(b)\} = (nh)^{-1} E \int_0^1 \frac{K(t)K(t+c)\{m^2(a, \mathbf{x}_i) + \sigma^2(a, \mathbf{x}_i)\}}{\pi_0(b, \mathbf{x}_i)} dt \\
+ n^{-1} E \int_0^1 K(t)K(t+c) \left\{ 2m(a, \mathbf{x}_i)m'_a(a, \mathbf{x}_i)\pi_0(a, \mathbf{x}_i) + m^2(a, \mathbf{x}_i)\pi'_0(a, \mathbf{x}_i) \right\} t/\{\pi_0(a, \mathbf{x}_i)\pi_0(b, \mathbf{x}_i)\} dt \\
- n^{-1} \theta(a)\theta(b).
\] Here $c = (a - b)/h$. Then the above analysis leads to that $\hat{\theta}(a) - \theta(a)$ is asymptotically a Gaussian process with mean given by $\text{bias}\{\hat{\theta}(a)\}$ and variance-covariance function given in $\text{cov}\{\hat{\theta}(a), \hat{\theta}(b)\}$. \hfill \Box

A.2.4 Variance estimation

By Theorem 3
\[
\text{var}\{\hat{\theta}(a)\} = (nh)^{-1} E \int_0^1 \frac{K(t)^2\{m^2(a, \mathbf{x}_i) + \sigma^2(a, \mathbf{x}_i)\}}{\pi(a, \mathbf{x}_i, \mathbf{\beta}^*)^2} \pi_0(a, \mathbf{x}_i) dt + O(n^{-1} + n^{-1}h + n^{-3/2}h^{-1}) \\
= \int_0^1 \frac{K(t)^2 dt}{nh} E \left[ \frac{\pi_0(a, \mathbf{x}_i)\{m^2(a, \mathbf{x}_i) + \sigma^2(a, \mathbf{x}_i)\}}{\pi(a, \mathbf{x}_i, \mathbf{\beta}^*)^2} \right] + O(n^{-1} + n^{-1}h + n^{-3/2}h^{-1}).
\]
Thus, an estimator of this variance is obtained as

\[ \widehat{\text{var}} \{ \hat{\theta}(a) \} = \frac{1}{nh} \int_0^1 K(t)^2 dt \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ K_h(A_i - a) \left\{ \tilde{m}^2(A_i, X_i) + \{Y_i - \tilde{m}(A_i, X_i)\}^2 \right\} \right] \right\} . \]

Let \( \tilde{m}(A_i, X_i) = Y_i \). Then the above estimator becomes

\[ \widehat{\text{var}} \{ \hat{\theta}(a) \} = \frac{1}{nh} \int_0^1 K(t)^2 dt \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{K_h(A_i - a) Y_i^2}{\pi(a, X_i, \hat{\beta})^2} \right\} . \]

Its expectation is

\[ \mathbb{E} \left\{ \frac{1}{nh} \int_0^1 K(t)^2 dt \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{K_h(A_i - a) Y_i^2}{\pi(a, X_i, \hat{\beta})^2} \right\} \right\} = \frac{1}{nh} \int_0^1 K(t)^2 dt \left\{ n^{-1} \sum_{i=1}^{n} \frac{K_h(A_i - a) Y_i^2}{\pi(a, X_i, \hat{\beta})^2} + O(n^{-1/2}) \right\} + O(n^{-3/2} h^{-1}) \]

\[ = \frac{1}{nh} \int_0^1 K(t)^2 dt \left\{ \frac{K_h(A_i - a) Y_i^2}{\pi(a, X_i, \hat{\beta})^2} + O(n^{-1/2}) \right\} + O(n^{-3/2} h^{-1}) \]

\[ = \frac{1}{nh} \int_0^1 K(t)^2 dt \left\{ \frac{K_h(A_i - a) \left\{ m^2(A_i, X_i) + \sigma^2(A_i, X_i) \right\}}{\pi(a, X_i, \hat{\beta})^2} \right\} + O(n^{-3/2} h^{-1}) \]

\[ = \frac{1}{nh} \int_0^1 K(t)^2 dt \left( \mathbb{E} \left[ \frac{\partial^2 \pi_0(a, X_i) \left\{ m^2(a, X_i) + \sigma^2(a, X_i) \right\}}{\pi(a, X_i, \hat{\beta})^2} \right] + O(n^{-3/2} h^{-1}) \right) \]

\[ = \frac{1}{nh} \int_0^1 K(t)^2 dt \left\{ \pi_0(a, X_i) \left\{ m^2(a, X_i) + \sigma^2(a, X_i) \right\} \right\} + O(n^{-1} h + n^{-3/2} h^{-1}) . \]

Following Remark 2, an alternative variance estimator is:

\[ \widehat{\text{var}} \{ \hat{\theta}(a) \} = \frac{1}{nh} \int_0^1 K(t)^2 dt \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{K_h(A_i - a)}{\pi(a, X_i, \hat{\beta})^2} \right\}^{-1} \sum_{i=1}^{n} \frac{K_h(A_i - a) Y_i^2}{\pi(a, X_i, \hat{\beta})^2} . \]
A.7 Simulation results: Tables

Table 1: Results based on 1000 replicates for the estimation of contrasts $\theta_k - \theta_0, k = 1, 2, 3$ with balancing estimator proposed using model $\pi(\cdot)$ and basis of $m(\cdot)$, which are either correctly specified or misspecified. Last blocks contain maximum likelihood based IPW (ML-IPW) and augmented IPW (DR) estimators. Sample size $n = 500$.

| $\theta_k - \theta_0$ | bias | sd | MSE | sd | 95% |
|------------------------|------|----|-----|----|-----|
| $m, \pi$ correct       |      |    |     |    |     |
| $k = 1$                | 0.3160 | 2.6185 | 6.9566 | 2.6078 | 0.9520 |
| $k = 2$                | 0.3211 | 2.6183 | 6.9586 | 2.6073 | 0.9510 |
| $k = 3$                | 0.3167 | 2.6173 | 6.9503 | 2.6075 | 0.9520 |
| $\pi$ correct          |      |    |     |    |     |
| $k = 1$                | 1.3666 | 7.4357 | 57.1567 | 6.2238 | 0.9110 |
| $k = 2$                | 1.2198 | 7.1377 | 52.4345 | 5.7876 | 0.8940 |
| $k = 3$                | 1.3181 | 7.0207 | 51.0281 | 5.7158 | 0.9000 |
| $m$ correct            |      |    |     |    |     |
| $k = 1$                | 2.1145 | 3.4709 | 16.5182 | 3.5342 | 0.9550 |
| $k = 2$                | 2.1204 | 3.4748 | 16.5701 | 3.5341 | 0.9560 |
| $k = 3$                | 2.1154 | 3.4711 | 16.5235 | 3.5339 | 0.9530 |
| $m, \pi$ misspecified  |      |    |     |    |     |
| $k = 1$                | 3.2163 | 7.8904 | 72.6030 | 7.0024 | 0.9150 |
| $k = 2$                | 3.0839 | 7.6868 | 68.5982 | 6.5804 | 0.9020 |
| $k = 3$                | 3.1900 | 7.4916 | 66.3006 | 6.5237 | 0.9060 |
| ML-IPW, $\pi$ correct |      |    |     |    |     |
| $k = 1$                | 0.0842 | 16.5578 | 274.1668 | 16.3236 | 0.9650 |
| $k = 2$                | 0.4053 | 14.3483 | 206.0379 | 14.0882 | 0.9530 |
| $k = 3$                | 0.1948 | 14.0600 | 197.7213 | 14.0238 | 0.9520 |
| DR, $m, \pi$ correct   |      |    |     |    |     |
| $k = 1$                | 0.040 | 2.352 | 5.533 | 2.451 | 0.962 |
| $k = 2$                | 0.045 | 2.351 | 5.529 | 2.450 | 0.962 |
| $k = 3$                | 0.041 | 2.349 | 5.520 | 2.450 | 0.964 |
Table 2: Results based on 1000 replicates for the estimation of contrasts $\theta_k - \theta_0, k = 1, 2, 3$ with balancing estimator proposed using model $\pi(\cdot)$ and basis of $m(\cdot)$, which are either correctly specified or misspecified. Last blocks contain maximum likelihood based IPW (ML-IPW) and augmented IPW (DR) estimators. Sample size $n = 1000$.

| $\theta_k - \theta_0$ | bias  | sd    | MSE   | sd    | 95%  |
|-----------------------|-------|-------|-------|-------|------|
|                       |       |       |       |       |      |
| $m$, $\pi$ correct    |       |       |       |       |      |
| $k = 1$               | 0.1233| 1.9123| 3.6720| 1.8477| 0.9380|
| $k = 2$               | 0.1273| 1.9111| 3.6686| 1.8472| 0.9370|
| $k = 3$               | 0.1233| 1.9092| 3.6604| 1.8471| 0.9380|
| $\pi$ correct         |       |       |       |       |      |
| $k = 1$               | 0.3756| 5.1489| 26.6518| 4.4066| 0.9160|
| $k = 2$               | 0.4287| 4.7061| 22.3316| 4.0946| 0.9070|
| $k = 3$               | 0.3302| 4.7935| 23.0868| 4.0950| 0.9110|
| $m$ correct           |       |       |       |       |      |
| $k = 1$               | 1.2285| 2.2205| 6.4397| 2.2226| 0.9360|
| $k = 2$               | 1.2325| 2.2225| 6.4588| 2.2222| 0.9350|
| $k = 3$               | 1.2284| 2.2206| 6.4400| 2.2220| 0.9360|
| $m$, $\pi$ misspecified|     |       |       |       |      |
| $k = 1$               | 1.4565| 5.4090| 31.3788| 4.6882| 0.9080|
| $k = 2$               | 1.5062| 4.9498| 26.7694| 4.3911| 0.9050|
| $k = 3$               | 1.4004| 5.0466| 27.4296| 4.3925| 0.9150|
| ML-IPW, $\pi$ correct|       |       |       |       |      |
| $k = 1$               | 0.0974| 11.5132| 132.5634| 10.8010| 0.9540|
| $k = 2$               | 0.2635| 10.2896| 105.9450| 9.4923| 0.9510|
| $k = 3$               | 0.0573| 10.4489| 109.1838| 9.4719| 0.9480|
| DR, $m$, $\pi$ correct|     |       |       |       |      |
| $k = 1$               | 0.048 | 1.747 | 3.056 | 1.737 | 0.947 |
| $k = 2$               | 0.052 | 1.747 | 3.054 | 1.736 | 0.947 |
| $k = 3$               | 0.048 | 1.746 | 3.050 | 1.736 | 0.949 |
Table 3: Results based on 1000 replicates for the estimation of contrasts \( \theta_k - \theta_0, k = 1, 2, 3 \) with balancing estimator proposed using model \( \pi(\cdot) \) and basis of \( m(\cdot) \), which are either correctly specified or misspecified. Last blocks contain maximum likelihood based IPW (ML-IPW) and augmented IPW (DR) estimators. Sample size \( n = 2000 \).

| \( \theta_k - \theta_0 \) | bias | sd  | MSE | sd  | 95% |
|-------------------------|------|-----|-----|-----|-----|
| \( m, \pi \) correct    |      |     |     |     |     |
| \( k = 1 \)             | 0.0147 | 1.2971 | 1.6826 | 1.3063 | 0.9490 |
| \( k = 2 \)             | 0.0147 | 1.2972 | 1.6830 | 1.3059 | 0.9510 |
| \( k = 3 \)             | 0.0125 | 1.2972 | 1.6830 | 1.3059 | 0.9520 |
| \( \pi \) correct       |      |     |     |     |     |
| \( k = 1 \)             | 0.1837 | 3.5871 | 12.9007 | 3.2328 | 0.9310 |
| \( k = 2 \)             | 0.1936 | 3.3857 | 11.5003 | 3.0257 | 0.9220 |
| \( k = 3 \)             | 0.1522 | 3.3617 | 11.3241 | 3.0269 | 0.9310 |
| \( m \) correct         |      |     |     |     |     |
| \( k = 1 \)             | 0.7568 | 1.4234 | 2.5987 | 1.4744 | 0.9450 |
| \( k = 2 \)             | 0.7566 | 1.4232 | 2.5980 | 1.4740 | 0.9460 |
| \( k = 3 \)             | 0.7541 | 1.4243 | 2.5975 | 1.4740 | 0.9460 |
| \( m, \pi \) misspecified |      |     |     |     |     |
| \( k = 1 \)             | 0.9441 | 3.6714 | 14.3704 | 3.3614 | 0.9190 |
| \( k = 2 \)             | 0.9392 | 3.4964 | 13.1066 | 3.1605 | 0.9140 |
| \( k = 3 \)             | 0.8885 | 3.4607 | 12.7659 | 3.1639 | 0.9290 |
| ML-IPW, \( \pi \) correct |      |     |     |     |     |
| \( k = 1 \)             | -0.0099 | 7.1859 | 51.6464 | 7.2091 | 0.9460 |
| \( k = 2 \)             | 0.1173 | 6.3511 | 40.3504 | 6.3572 | 0.9460 |
| \( k = 3 \)             | 0.1109 | 6.3369 | 40.1689 | 6.3598 | 0.9420 |
| DR, \( m, \pi \) correct |      |     |     |     |     |
| \( k = 1 \)             | -0.006 | 1.208 | 1.459 | 1.229 | 0.962 |
| \( k = 2 \)             | -0.006 | 1.209 | 1.461 | 1.228 | 0.958 |
| \( k = 3 \)             | -0.008 | 1.208 | 1.460 | 1.228 | 0.959 |
Table 4: Results based on 1000 replicates for continuous treatment case, and nonlinear outcome model. Integrated absolute bias and integrated RMSE (in parentheses). ML-IPW is the maximum likelihood based IPW estimator and CB-IPW the robust balancing-IPW method proposed (11-12).

\[
n = 500
\]

|                      | \(\pi, m\) correct | \(\pi\) correct | \(m\) correct | none correct |
|----------------------|---------------------|------------------|----------------|--------------|
| IPW of Kennedy       | na                  | 3.33 (4.95)      | na             | 3.00 (4.81)  |
| DR of Kennedy        | 1.09 (3.31)         | 2.05 (3.75)      | 1.07 (3.31)    | 2.55 (4.02)  |
|                      | \(\pi\) correct     |                  | none correct   |              |
| ML-IPW               | na                  | 0.52 (4.52)      | na             | 1.21 (4.40)  |
| Constant, CV         | na                  | 0.39 (4.23)      | na             | 1.49 (4.42)  |
| Constant, OSCV       | na                  | 0.40 (4.08)      | na             | 1.99 (4.45)  |
| Linear, OSCV         | 0.38 (4.24)         | 0.26 (4.32)      | 1.15 (4.18)    | 1.23 (4.25)  |
| CB-IPW               | 0.28 (4.05)         | 0.31 (4.18)      | 1.41 (4.26)    | 1.52 (4.35)  |
| Linear, OSCV         | 0.69 (3.91)         | 0.82 (4.09)      | 1.86 (4.22)    | 1.99 (4.34)  |

\[
n = 1000
\]

|                      | \(\pi, m\) correct | \(\pi\) correct | \(m\) correct | none correct |
|----------------------|---------------------|------------------|----------------|--------------|
| IPW of Kennedy       | na                  | 3.15 (4.11)      | na             | 2.80 (3.91)  |
| DR of Kennedy        | 0.97 (2.60)         | 1.88 (3.16)      | 0.94 (2.37)    | 2.36 (3.28)  |
|                      | \(\pi\) correct     |                  | none correct   |              |
| ML-IPW               | na                  | 0.39 (3.26)      | na             | 1.32 (3.30)  |
| Constant, CV         | na                  | 0.46 (2.88)      | na             | 1.42 (3.23)  |
| Constant, OSCV       | na                  | 0.48 (2.80)      | na             | 1.96 (3.41)  |
| Linear, OSCV         | 0.27 (3.08)         | 0.20 (3.15)      | 1.27 (3.13)    | 1.34 (3.19)  |
| CB-IPW               | 0.29 (2.78)         | 0.20 (2.89)      | 1.37 (3.08)    | 1.46 (3.17)  |
| Linear, OSCV         | 0.68 (2.72)         | 0.69 (2.88)      | 1.85 (3.20)    | 1.97 (3.32)  |

\[
n = 2000
\]

|                      | \(\pi, m\) correct | \(\pi\) correct | \(m\) correct | none correct |
|----------------------|---------------------|------------------|----------------|--------------|
| IPW of Kennedy       | na                  | 3.02 (3.62)      | na             | 2.65 (3.44)  |
| DR of Kennedy        | 0.79 (1.83)         | 1.76 (2.58)      | 0.78 (1.81)    | 2.37 (3.82)  |
|                      | \(\pi\) correct     |                  | none correct   |              |
| ML-IPW               | na                  | 0.33 (2.44)      | na             | 1.45 (2.76)  |
| Constant, CV         | na                  | 0.56 (2.09)      | na             | 1.41 (2.57)  |
| Constant, OSCV       | na                  | 0.54 (1.97)      | na             | 2.00 (2.89)  |
| Linear, OSCV         | 0.22 (2.30)         | 0.19 (2.43)      | 1.41 (2.59)    | 1.47 (2.66)  |
| CB-IPW               | 0.39 (1.95)         | 0.26 (2.12)      | 1.36 (2.39)    | 1.44 (2.49)  |
| Linear, OSCV         | 0.66 (1.91)         | 0.70 (2.15)      | 1.91 (2.68)    | 2.00 (2.81)  |

Note: “na” stands for “not applicable”.

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Table 5: Results based on 1000 replicates for continuous treatment case, and linear outcome model. Integrated absolute bias and integrated RMSE (in parentheses). ML-IPW is the maximum likelihood based IPW estimator and CB-IPW the robust balancing-IPW method proposed (11-12).

\[ n = 500 \]

|                     | \( \pi, m \) correct | \( \pi \) correct | \( m \) correct | none correct |
|---------------------|----------------------|-------------------|-----------------|--------------|
| IPW of Kennedy      | na                   | 3.02 (5.31)       | na              | 2.58 (4.02)  |
| DR of Kennedy       | 0.58 (2.60)          | 0.72 (2.69)       | 0.64 (2.55)     | 0.90 (2.64)  |
| ML-IPW              | Constant, CV         | na                | 0.26 (3.55)     | na           |
|                     | Constant, OSCV       | na                | 0.07 (3.64)     | na           |
|                     | Linear, OSCV         | na                | 0.18 (3.36)     | na           |
| CB-IPW              | Constant, CV         | 0.23 (3.29)       | 0.17 (3.34)     | 0.27 (3.21)  |
|                     | Constant, OSCV       | 0.12 (3.55)       | 0.21 (3.58)     | 0.53 (3.56)  |
|                     | Linear, OSCV         | 0.25 (3.23)       | 0.33 (3.27)     | 0.65 (3.26)  |

\[ n = 1000 \]

|                     | \( \pi, m \) correct | \( \pi \) correct | \( m \) correct | none correct |
|---------------------|----------------------|-------------------|-----------------|--------------|
| IPW of Kennedy      | na                   | 2.96 (4.82)       | na              | 2.55 (3.33)  |
| DR of Kennedy       | 0.44 (1.92)          | 0.62 (1.97)       | 0.48 (1.85)     | 0.78 (1.98)  |
| ML-IPW              | Constant, CV         | na                | 0.29 (2.55)     | na           |
|                     | Constant, OSCV       | na                | 0.10 (2.52)     | na           |
|                     | Linear, OSCV         | na                | 0.07 (2.31)     | na           |
| CB-IPW              | Constant, CV         | 0.23 (3.24)       | 0.19 (2.39)     | 0.26 (2.28)  |
|                     | Constant, OSCV       | 0.04 (2.43)       | 0.05 (2.46)     | 0.44 (2.46)  |
|                     | Linear, OSCV         | 0.15 (2.21)       | 0.16 (2.27)     | 0.56 (2.27)  |

\[ n = 2000 \]

|                     | \( \pi, m \) correct | \( \pi \) correct | \( m \) correct | none correct |
|---------------------|----------------------|-------------------|-----------------|--------------|
| IPW of Kennedy      | na                   | 2.93 (3.44)       | na              | 2.43 (2.97)  |
| DR of Kennedy       | 0.41 (1.45)          | 0.60 (1.55)       | 0.43 (1.40)     | 0.75 (1.57)  |
| ML-IPW              | Constant, CV         | na                | 0.22 (1.84)     | na           |
|                     | Constant, OSCV       | na                | 0.12 (1.79)     | na           |
|                     | Linear, OSCV         | na                | 0.09 (1.70)     | na           |
| CB-IPW              | Constant, CV         | 0.18 (1.72)       | 0.15 (1.74)     | 0.29 (1.66)  |
|                     | Constant, OSCV       | 0.08 (1.72)       | 0.06 (1.76)     | 0.40 (1.74)  |
|                     | Linear, OSCV         | 0.14 (1.61)       | 0.14 (1.65)     | 0.54 (1.68)  |

Note: “na” stands for “not applicable”. 

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