A METHOD HOW TO DETERMINE PARAMETERS ARISING IN A SMOLDERING EVOLUTION EQUATION BY IMAGE SEGMENTATION FOR EXPERIMENT’S MOVIES

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Abstract. We propose a simple and accurate procedure how to extract the values of model parameters in a flame/smoldering evolution equation from 2D movie images of real experiments. The procedure includes a novel method of image segmentation, which can detect an expanding smoldering front as a plane polygonal curve. The evolution equation is equivalent to the so-called Kuramoto–Sivashinsky (KS in short) equation in a certain scale. Our results suggest a valid range of parameters in the KS equation as well as the validity of the KS equation itself.

1. Introduction. Moving boundary problems have attracted many researchers and been extensively studied in the fields of not only applied mathematics but also physics such as combustion and fire research. In recent studies, spreading flame/smoldering fronts along a sheet of paper were tracked experimentally (see Figure 1 and [5]), and numerically (see Figure 2 and [4]) by applying the Kuramoto–Sivashinsky (KS in short) equation [3, 7, 12]. The purpose of this paper is to detect the fronts from two dimensional experimental images, and to determine the “real” values of the parameters that appear in the KS equation.

Mathematical formulation is as follows. We consider a moving Jordan curve \( \Gamma(t) \) in the plane \( \mathbb{R}^2 \) that represents an expanding flame/smoldering front in time \( t \) over a thin solid. The curve is parameterized by a smooth function \( \mathbf{x} : [0, 1] \times [0, \infty) \to \mathbb{R}^2 \) such that \( \Gamma(t) = \{ \mathbf{x}(u, t); u \in [0, 1] \} \) and \( \| \mathbf{x}_u \| > 0 \), where the positive direction of the parameter \( u \) is anti-clockwise. Here and hereafter, we denote \( F_u = \partial F(u, t)/\partial u \), \( F_t = \partial F(u, t)/\partial t \), \( \mathbf{F} = \partial F/u, t \) and \( ||\mathbf{a}||^2 = \mathbf{a} \cdot \mathbf{a} \) where \( \mathbf{a} \cdot \mathbf{b} \) is the Euclidean inner product between vectors \( \mathbf{a} \) and \( \mathbf{b} \). The unit tangent vector \( T \) at the point \( \mathbf{x}(u, t) \) can be defined as \( T(u, t) = \mathbf{x}_s \), where \( \mathbf{F}_s = \mathbf{F}_u/\|\mathbf{F}_u\| \) is the formal derivative of \( F \) with respect to the arc-length parameter \( s \). The unit outward normal vector

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Figure 1. The photographs depict snapshots from an experimental movie of spreading flame/smoldering front along a sheet of paper placed near the floor at 200th, 400th, 1000th, 1600th, ..., 4000th frames at the rate of 30 fps. Experiments were performed by the same method as [5].

is $N(u, t) = -T(u, t)^\perp$, where $(a, b)^\perp = (-b, a)$. The curve $\Gamma(t)$ moves by the evolution law

$$x_t = VN + WT,$$

where $V$ is the normal velocity and $W$ is the tangential velocity. It is known that the tangential velocity $W$ does not affect the shape of curve [2]. Therefore, the shape is thoroughly determined by the normal velocity $V$ only and we can choose the value of $W$ according to our convenience. The following equation is a version of the KS equation for gas combustion [3], which is also a model equation for flame/smoldering fronts along a sheet of paper [4, 5]:

$$V = V(0) + (\alpha_{\text{eff}} - 1)\kappa + \delta\kappa_{ss},$$

where $F_{ss} = (F_s)_s$ and $\kappa = \det(x_s, x_{ss})$ is the curvature, $V(0) > 0$ is a constant velocity independent of the front shape, and $\alpha_{\text{eff}}$ is a positive parameter close to but larger than 1. The second term $(\alpha_{\text{eff}} - 1)\kappa$ induces instability, but the third term $\delta\kappa_{ss}$ suppresses it, controlled by a positive parameter $\delta$, which is usually considered to be 4 at least when $\alpha_{\text{eff}}$ is close to 1 in the original KS equation.

The first purpose of this paper is to detect flame/smoldering fronts such as those shown in Figure 1 by means of a new method of image segmentation, which is an updated version of [1]. In general, the aim of image segmentation in the plane is to detect edges of the image by using plane curves. The image can be represented by its image intensity function, say $I$, and the edges correspond to the positions where the value of $|\nabla I|$ is large. We put an energy functional which takes a small value if $|\nabla I|$ is large. A simple idea of image segmentation method is to use a gradient flow of the energy. According to the gradient flow, the moving curve converges to an edge. The idea has been sophisticated as the so-called active contour models [6], and the method was generalized by [8, 9]. One can find the survey [13] for the whole picture of image segmentation.
**Figure 2.** The left figure depicts numerical solutions to (1), with the normal velocity (2) in which the parameters are given by the right table and \( W \) is chosen for controlling the grid-point spacing to be uniform (see section 3). The solution curves evolve from inside to outside. The initial curve is a circle with the diameter \( R = R_{\text{ini}} \) with 10% noise (see [4] in detail).

On the other hand, the energy-minimizing method was simplified by [1] and [10]. In their method, they use the curvature flow equation (3), in which sign of the forcing term is inward (resp. outward) if the curve is on a background image (resp. foreground image, the object to be segmented), and utilize the so-called the curvature-adjusted method for tangential velocity. This method is very fast and simple, but the exact stopping condition is difficult to realize numerically. Therefore, we will propose a new version in the next section, and here we explain basic concepts of image segmentation based on the method proposed in [1]. We define the image intensity function \( I(x) \in \{ w \in \mathbb{Z} ; 0 \leq w \leq 255 \} \) which is a piecewise constant in each pixel of the target image. When the color of the pixel is equal to the background (resp. foreground) color, we put \( I(x) = 0 \) (resp. \( I(x) = 255 \)). Image segmentation is performed by the following normal velocity:

\[
V = -\kappa - G(x),
\]

\[
G(x) = G_{\text{max}} - (G_{\text{max}} - G_{\text{min}}) \frac{I(x)}{255}, \tag{3}
\]

where \( G_{\text{min}} \) and \( G_{\text{max}} \) are given parameters satisfying \( G_{\text{min}} < 0 < G_{\text{max}} \). In [1], several examples of image segmentation with respect to images supplied in gray scale are shown. The method in [1] was successful except the stopping condition; it was difficult to realize \( V = 0 \) numerically, since images are associated with step-functions of pixel. To overcome this difficulty, we propose, say a step-function adjusted method below.

Before we show it, we explain the second purpose of the present paper. The second purpose is to determine the parameters \( V(0) \) and \( \alpha_{\text{eff}} \) which appear in (2). The time derivatives of the total length \( L \) and the enclosed area \( A \) of \( \Gamma(t) \) are

\[
\dot{L} = \int_{\Gamma(t)} \kappa V ds, \tag{4}
\]

\[
\dot{A} = \int_{\Gamma(t)} V ds, \tag{5}
\]
respectively. We substitute (2) into (4) and (5), respectively, and obtain the following:

\[ \dot{L} = 2\pi V(0) + (\alpha_{\text{eff}} - 1) \int_{\Gamma(t)} \kappa^2 ds + \delta \int_{\Gamma(t)} \kappa \kappa_{ss} ds, \]

(6)

\[ \dot{A} = V(0) L + 2\pi (\alpha_{\text{eff}} - 1). \]

(7)

The parameters \( V(0) \) and \( \alpha_{\text{eff}} \) can be determined numerically by estimating \( \kappa, \kappa_{ss}, \dot{A} \) and \( \dot{L} \) from the segmentation curves of experimental images such as those shown in Figure 1, and substituting them into (6) and (7).

The organization of this paper is as follows. In section 2, we propose an image segmentation method, called the step-function adjusted method, developed from [1]. In section 3, we show a numerical method for simulating the evolution of a polygonal curve that approximates a smooth curve moving at a normal velocity described in section 2. Also, to numerically evaluate the values of \( V(0) \) and \( \alpha_{\text{eff}} \) from (6) and (7), we need to calculate \( \kappa \) and \( \kappa_{ss} \) in a discrete sense, which can be obtained as the first variation of a prescribed energy such as the total length or the elastic energy. In section 4, our numerical algorithm for evaluating the "real" values of \( V(0) \) and \( \alpha_{\text{eff}} \) are presented, and some numerical experiments are performed. Future works and some remarks are summarized in the last section 5.

2. Image segmentation by a step-function adjusted method. As shown in Figure 1, the color of the burned region is regarded as black, whereas that of the unburned, i.e. the paper region, is regarded as white. A flame/smoldering front which is to be segmented is the interface between black and white. Experimental images are in color, so that a small flame near the interface remains as yellow- or red-colored pixels. In general, such yellowish or reddish colors are converted to gray scale by using their luminance or brightness. Therefore, it is impossible to distinguish such gray flame pixels from the burned region (black) or the unburned region (white). In our method, nevertheless, such an ambiguous region is regarded as burned if the pixel color changes appreciably as compared with its initial value. Thus, the image intensity function is given in the following manner.

1. We choose one frame (hereafter denoted as the initial frame) among early frames of an experimental movie, e.g., frame 200 in Figure 1. The maximum difference of each component of RGB color values between \( R_{\text{ini}} = (R_{\text{ini}}, G_{\text{ini}}, B_{\text{ini}}) \) at a pixel of the initial frame and \( R_j(x) = (R_j, G_j, B_j) \) at the same pixel of the \( j \)th frame to be image-segmented is defined as the intensity function \( I(x) \) at the center \( x \) of the pixel. In other words, the intensity function \( I(x) \) is defined by

\[ I(x) = \max\{|R_{\text{ini}} - R_j|, |G_{\text{ini}} - G_j|, |B_{\text{ini}} - B_j|\} \]

at the center \( x \) of each pixel.

2. The discrete value \( I(x) \) for the center \( x \) is extended to a continuous function \( I(x) \) for any position \( x \) in the image (except within the boundary of width of a half pixel) by the standard bilinear interpolation. Hereafter, we again denote \( I(x) \) as \( I(x) \).

Once \( I(x) \) is determined, the following procedure will be repeated, so that a polygonal segmentation curve will evolve at the normal velocity \( V \) defined by (9) below.
We decide the threshold value $I^*$ appropriately, and define that $x$ is in the “inside”, i.e., the burned region enclosed by the front if $I(x) > I^*$ holds, whereas $x$ is in the “outside” if $I(x) < I^*$ holds. The normal velocity for image segmentation is defined as follows:

$$V = -\gamma(x)\kappa + G(x),$$

$$\gamma(x) = (2J(x) - 1)^2, \quad J(x) = \min \left\{ \frac{I(x)}{2I^*}, 1 \right\},$$

$$G(x) = (2J(x) - 1)(G_0|2J(x) - 1| + 1),$$

where $G_0$ is a positive parameter. Note that $V = 0$ if $J(x) = 0.5$ ($I(x) = I^*$), and $V = -4\varepsilon^2\kappa + 4G_0\varepsilon|\varepsilon| + 2\varepsilon$ if $J(x) - 0.5 = \varepsilon$. Thus, when an evolving polygonal curve approaches sufficiently close to the front, the normal velocity $V$ becomes very small and the speed of the curve is slowed down. The stopping condition is $I(x) \simeq I^*$, in other words, $V \simeq 0$ at any points on the curve.

3. Discretization. The purpose of this section is to define in a discrete sense the “curvature” $\kappa$ and its second “derivative” $\kappa_{ss}$ arising in (6) and (7). We use a similar way of space discretization and computation of the tangential velocity to [4]. In the direct approach, a moving Jordan curve $\Gamma(t)$ is approximated by a moving Jordan polygonal curve, say $P(t)$ at time $t$, with $N$ vertices labeled $x_1, x_2, \ldots, x_N$ in the anti-clockwise order. Let $P_i$ be the $i$th edge $P_i = \{x_{i-1}, x_i\}$ $(i = 1, 2, \ldots, N; \ x_0 = x_N)$. Then the moving Jordan polygonal curve at time $t$ is $P(t) = \bigcup_{i=1}^{N} P_i(t)$. Our goal here is to construct a discretization of (1) in space, i.e., to derive a system of ordinary differential equations (ODEs in short) for $P(t)$: for $i = 1, 2, \ldots, N$

$$\dot{x}_i(t) = V_i(t)N_i(t) + W_i(t)T_i(t),$$

where $V_i$ is the normal $N_i$-component of the velocity at $x_i$, and $W_i$ is its tangential $T_i$-component that controls the grid-spacing to be asymptotically uniform.

![Figure 3](image-url)

**Figure 3.** (a) Jordan curve $\Gamma$ (b) Jordan polygonal curve $P$

The right-hand-side of (10) consists of several polygonal quantities on $P$ at time $t$, and all of them can be constructed from \{ $x_i$ \}_{i=1}^{N} through the following steps. In what follows, these are regarded as functions of time $t$ with $N$-periodic index, i.e., $F_0 = F_N$, $F_{N+1} = F_1$. **Table 1** shows discretizations of the total length, the normal vector, the tangent vector, and the normal velocity on $P_i$ and at $x_i$, respectively.
Table 1. Discretizations of length, normal/tangent vector, and normal velocity

| $r_i = \|\mathbf{x}_i - \mathbf{x}_{i-1}\|$ | The length of $P_i$ |
| $L = \sum_{i=1}^{N} r_i$ | The total length of $P$ |
| $t_i = (\mathbf{x}_i - \mathbf{x}_{i-1})/r_i$ | The unit tangent vector on $P_i$ |
| $n_i = -t_i^\perp$ | The outward unit normal vector on $P_i$ |
| $v_i$ | A given representative normal velocity on $P_i$ |
| $\phi_i = \text{sgn}(D_i) \arccos(t_i \cdot t_{i+1})$ | The angle between the adjacent edges $P_i$ and $P_{i+1}$ where $D_i = \det(t_i, t_{i+1})$ |
| $T_i = (t_i + t_{i+1})/(2\cos_i)$ | The unit tangent vector at $\mathbf{x}_i$ where $\cos_i = \cos(\phi_i/2) = \|t_i + t_{i+1}\|/2$ |
| $N_i = (n_i + n_{i+1})/(2\cos_i)$ | The outward unit normal vector at $\mathbf{x}_i$ |
| $V_i = (v_i + v_{i+1})/(2\cos_i)$ | The normal velocity at $\mathbf{x}_i$ |

To define the curvatures on $P_i$ and at $\mathbf{x}_i$, we use (4), because the curvature on edges is zero and the curvature at vertices cannot be defined in the usual sense. The curvature can be defined by the first variation of the total length $L$ from (4). From (10), the total length $L$, and $\hat{r}_i = V_i \sin_i + V_{i-1} \sin_{i-1} + W_i \cos_i - W_{i-1} \cos_{i-1}$, we obtain

$$\hat{L} = \sum_{i=1}^{N} \hat{\kappa}_i V_i \hat{r}_i,$$

where $\sin_i = \sin(\phi_i/2)$, $\cos_i = \cos(\phi_i/2)$, $\hat{r}_i = (r_i + r_{i+1})/2$, and $\hat{\kappa}_i = 2\sin_i/\hat{r}_i$ is the polygonal curvature at $\mathbf{x}_i$. Then it follows that

$$\hat{L} = \sum_{i=1}^{N} \kappa_i v_i r_i,$$

which is a discretization of (4), where $\kappa_i$ is the polygonal curvature on $P_i$. Therefore, we obtain

$$\kappa_i = \frac{\tan_i + \tan_{i-1}}{r_i} \quad \text{on} \quad P_i,$$

where $\tan_i = \sin_i/\cos_i$.

To compute $\langle \kappa_s^s \rangle_i$, we calculate the gradient flow of $E = \sum_{i=1}^{N} \kappa_i^2 r_i/2$, which is a discrete analogue of obtaining the Willmore flow equation from

$$\dot{E}(t) = -\int_{\Gamma(t)} \left( \kappa_{ss} + \frac{1}{2} \kappa^3 \right) V \, ds,$$

where $E(t) = \frac{1}{2} \int_{\Gamma(t)} \kappa^2 ds$ is the elastic energy. By a direct calculation, we have

$$\dot{E} = -\sum_{i=1}^{N} \left( \langle \kappa_s^s \rangle_i + \frac{1}{2} \langle \kappa^3 \rangle_i \right) v_i r_i + \text{err}_E,$$

where $\langle \kappa^3 \rangle_i = (\kappa_i^+ \kappa_{i+1}^2 + 2\kappa_i^2 \kappa_{i-1}^+ + \kappa_i^+ \kappa_{i-1}^2)/4$ is the average of $\kappa_i$ cubed on $P_i$ ($\kappa_i^+ = 2\tan_i/r_i$, $\kappa_i^- = 2\tan_{i-1}/r_i$, n.b. $\kappa_i = (\kappa_i^+ + \kappa_i^-)/2$), and $\text{err}_E$ is the remaining term. The term $\langle \kappa_s^s \rangle_i$ is extracted from (14), which is defined

$$\langle \kappa_s^s \rangle_i = \frac{\langle \kappa_{s+1} \rangle_i - \langle \kappa_{s-1} \rangle_i}{2r_i} \quad \text{on} \quad P_i,$$

as a correction term for (15).
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where

\[(F_s)_i = \frac{1}{r_i} \left( \frac{F_{i+1} + F_i}{2 \cos_i^2} - \frac{F_i + F_{i-1}}{2 \cos_i^2} \right). \tag{16}\]

Note that this difference operator is meaningful, since \((t_s)_i = -\kappa_i n_i\) holds, which is a discrete version of the Frenet formula \(T_s = -\kappa N\). Of course, the above argument is not the only way to discretize \(\kappa_{ss}\).

The normal velocity \(V_i\) at \(x_i\) is calculated from \(v_i, v_{i+1}, \cos\) as in Table 1. The velocities \(\{V_i\}\) and \(\{v_i\}\) satisfy the relation (11):

\[
\sum_{j=1}^{N} \kappa_i v_i r_i = \sum_{j=1}^{N} \hat{\kappa}_i V_i \hat{r}_i.
\]

\(v_i\) is a given representative normal velocity on \(P_i\) and the problem is characterized by \(v_i\). However, \(v_i\) is not necessarily equivalent to \(x_i \cdot n_i\). Note that, under the crystalline motion, \(v_i = x_i \cdot n_i\) holds. In the present paper, we use the following two kinds of velocities:

1. \(v_i = V^{(0)} + (\alpha_{eff} - 1) + \delta(\kappa_{ss})_i\) for the numerical computation of (2) (see Figure 2).
2. \(v_i = -\gamma(x)\kappa_i + G(x)\) for the image segmentation of (9) (see section 2), where \(x \in P_i\).

As mentioned in section 1, the tangential velocity \(W\) can be arbitrarily chosen based on our convenience. Hence, we apply the asymptotic uniform distribution method, that is, we use the tangential velocity

\[W_i = \frac{\Psi_i + c}{\cos_i}, \tag{17}\]

where

\[\psi_j = \frac{1}{N} \sum_{l=1}^{N} \kappa_l v_l r_l - V_j \sin_j - V_{j-1} \sin_{j-1} + \left( \frac{L}{N} - r_j \right) \omega \quad (j = 2, 3, \ldots, N),\]

\[\psi_1 = 0, \quad \Psi_i = \sum_{j=1}^{N} \psi_j, \quad c = -\frac{\sum_{j=1}^{N} \psi_j}{\sum_{j=1}^{N} \cos_j},\]

and \(\omega\) is a sufficiently large constant. Then, \(r_i \to L/N (t \to \infty)\) is satisfied (see [4, 8, 9, 11] in detail).

From the above, (10) can be summarized as the following ODEs:

\[
\dot{X} = F(X), \tag{18}
\]

where \(X = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^{2 \times N}\), and

\[
\begin{cases}
F = (F_1, F_2, \ldots, F_N) : \mathbb{R}^{2 \times N} \to \mathbb{R}^{2 \times N}; \\
\mathbb{R}^{2 \times N} \ni X \mapsto F_i(X) \in \mathbb{R}^2 \quad (i = 1, 2, \ldots, N).
\end{cases}
\]

4. Evaluation of parameters \(V^{(0)}\) and \(\alpha_{eff}\). We show the procedure for image segmentation of experimental movie and determining the values of parameters \(V^{(0)}\) and \(\alpha_{eff}\).

**Step 1:** We convert an experimental movie to obtain an uncompressed color bitmap image for each frame; we can easily obtain the RGB values of its each pixel by reading the uncompressed bitmap file as a binary file. The offset, which is the distance between the beginning of file to the beginning of the
data of RGB values, is stored in the first 11 to 14 bytes of the file, depending on the version of the bitmap file. Therefore, it is read first, and the values after the address specified by it are used as RGB values.

**Step 2:** The image intensity function is computed by (8) and the standard bilinear interpolation.

**Step 3:** We obtain the approximate total length of the front $L_{\text{pre}}$ as follows:
- Take a regular polygon enclosing the entire burned region. The number of its vertices is 64.
- Put $I^* = 40$. (This value is appropriate from a viewpoint of our image segmentation purpose.)
- Solve (10) by the explicit Euler method with the normal velocity (9) and the tangential velocity (17).
- Stop the iteration when the image intensity function satisfies $36 \leq I(x_i) \leq 44$ at all vertices $\{x_i\}_{i=1}^{64}$ or the number of iterations is large enough.
- Compute the approximate total length of the front
  \[ L_{\text{pre}} = \sum_{i=1}^{64} r_i, \]  \(19\)
where the unit length is the length of pixel edge.

**Step 4:** We get a segmentation curve as follows:
- Take a regular polygon enclosing the entire burned region. The number of its vertices is $N_{\text{ini}} = \lceil \log_2 L_{\text{pre}} \rceil$, \(20\)
  where $\lfloor x \rfloor = \max\{n \in \mathbb{Z}; n \leq x\}$.
- Put $I^* = 40$.
- Solve (10) by the same manner as in Step 3.
- Insert a new vertex at the midpoint of each edge when the image intensity function satisfies $36 \leq I(x_i) \leq 44$ at all vertices $\{x_i\}_{i=1}^{N}$ and the number of vertices are $N = N_{\text{ini}}$ at the first iteration and $N = 2N_{\text{ini}}$ at the second iteration.
- Stop the iteration when the number of vertices is $N = 4N_{\text{ini}}$ and the image intensity function satisfies $38 \leq I(x_i) \leq 42$ at all vertices $\{x_i\}_{i=1}^{N}$. \(\text{Figure 4}\) is the result of our method for the image of \text{Figure 1}.

**Step 5:** Since the segmentation curve thus obtained contains many noises, FFT is used to eliminate high frequency components. If we did not remove noises, $\kappa_i$ and $(\kappa_{ss})_i$ would also contain undesired noises.

**Step 6:** From the segmentation curve of each frame, we can compute the values of curvature $\tilde{\kappa}$, its second derivative with respect to the arc-length $\tilde{\kappa}_{ss}$, the total length of the front $\tilde{L}$, and the enclosed area $\tilde{A}$, each with an appropriate dimension. Assume that $\tilde{L}$ and $\tilde{A}$ are a linear and a quadratic functions of time, respectively, i.e.,
\[ \tilde{L} = \tilde{L}_1 t + \tilde{L}_0, \] \(21\)
\[ \tilde{A} = \tilde{A}_2 t^2 + \tilde{A}_1 t + \tilde{A}_0. \] \(22\)
Then, coefficients are determined by the method of least squares:
\[ \tilde{L} = 3.853295652 t + 47.0874054 \text{ [mm]}, \] \(23\)
\[ \tilde{A} = 1.296716905 t^2 - 15.00733847 t + 1375.969254 \text{ [mm}^2]. \] \(24\)
Figure 4. The upper-left figure depicts selected segmentation curves at frames: 400, 1000, 1600, 2200, 2800, 3400, 4000, summarizing the front evolution in the other photographs from left to right, upper to lower. The blue curve in each photograph is a segmentation curve, and the background vague region is the same as that in Figure 1.

Figure 5 compares the actual data from segmentation curves and functions (23) and (24).

Figure 5. (Left) The total length of front $\tilde{L}$[mm] vs. the actual time [second]. (Right) The enclosed area $\tilde{A}$[mm$^2$] vs. the actual time [second]. Blue points indicate the actual values and red curves are the graphs of (23) and (24), respectively.

Step 7: All quantities are nondimensionalized as follows:

\[
\begin{align*}
\tilde{L} &= \zeta_L L, \\
\dot{\tilde{L}} &= \zeta_V \dot{L}, \\
\dot{\tilde{A}} &= \zeta_L \zeta_V \dot{A}, \\
\tilde{\kappa} &= \frac{\kappa}{\zeta L}, \\
\tilde{\kappa}_{ss} &= \frac{\kappa_{ss}}{\zeta L},
\end{align*}
\]

\[\zeta_L = \frac{D_{th}}{u_r}, \quad \zeta_V = \frac{2u_r}{\beta m},\]

where $u_r = 1000$ [mm/s] is the reference oxidizer velocity at which the spread velocity without heat loss becomes zero, $D_{th} = 168.8808917$ [mm$^2$/s] is the
thermal diffusivity, $l_0 = D_{th}/u_r$ [mm], $\beta = 10$ is the Zel’dovich number and $m = 7.557354925776$ is the weighted density ratio of the gas to the solid phases. These data can be calculated from [5].

**Step 8:** Substituting (25) into (7) and

$$\dot{L} = 2\pi V^{(0)} + (\alpha_{\text{eff}} - 1) \sum_{i=1}^{N} \kappa_i^2 r_i + \delta \sum_{i=1}^{N} \kappa_i (\kappa_{ss})_i r_i,$$

(26)

which is discretization of (6), we can obtain the values of $V^{(0)}$ and $\alpha_{\text{eff}}$ of each frame (see Figure 6).

![Figure 6](image.png)

**Figure 6.** (Left) $V^{(0)}$ vs. time, (Right) $\alpha_{\text{eff}}$ vs. time.

**Step 8’:** Another way to obtain $V^{(0)}$ and $\alpha_{\text{eff}}$ is to use only (7) under the assumptions of (21) and (22); in other words, we substitute (21), (22) and (25) into (7) and obtain

$$V^{(0)} = \frac{2\tilde{A}_2}{\zeta_V \zeta_L}, \quad \alpha_{\text{eff}} = 1 + \frac{1}{2\pi} \left( \frac{\tilde{A}_1}{\zeta_V} - \tilde{L}_0 \right) \frac{V_0}{\zeta_L}.$$

(27)

In this way, we obtain the following values of parameters from (23) and (24):

$$V^{(0)} = 4.453 \times 10^{-4}, \quad \alpha_{\text{eff}} = 0.980236.$$

(28)

We consider the difference between two parameters obtained in **Step 8** and **Step 8’**. In **Step 8’**, the parameters were obtained by the method of equating the coefficients without using values of $\kappa$ and $\kappa_{ss}$, i.e. geometric information of solution curves. The way seems simple, but we need assumptions of forms $\tilde{L}$ and $\tilde{A}$, respectively, such as a linear form (21) and a quadratic form (22). On the other hand, in **Step 8**, we need to use the computed values $\tilde{L}$, $\tilde{A}$, $\kappa_i$, and $(\kappa_{ss})_i$ at each time, and assumptions of $\tilde{L}$ and $\tilde{A}$ were not necessary. For these reasons, the difference was seemingly caused by the computed values $\kappa_i$, $\kappa_{ss}$, and the necessity of the assumptions for each of the ways.

As can be seen from **Figure 6**, changes in the obtained parameters are smaller with time and approach constants, respectively.
5. **Conclusion.** We have proposed a new method of image segmentation, called the step-function adjusted method, taking the property of pixel image into account. The method was applied to experimental image data of a spreading flame/smoldering front along a paper sheet. The parameters of a model equation (the KS equation) were determined from the actual experimental data with the actual length- and time-scale dimensions. The obtained parameters in Step 8 and Step 8' were close to 1, not unreasonable values because the KS equation is valid for $\alpha_{\text{eff}} \simeq 1$. As to the future works, a new KS-like equation is desired, since in the phenomena of thin solid combustion $\alpha_{\text{eff}} > 1$ is expected. Nevertheless, we can say that our method has already indicated the application limit of the KS equation in a negative sense. Our method or strategy can be applied any experimental data to test the validity of model equations in other research fields.

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