An Introduction to Yangian Symmetries

Denis Bernard
Service de Physique Théorique de Saclay
F-91191, Gif-sur-Yvette, France.

We review some aspects of the quantum Yangians as symmetry algebras of two-dimensional quantum field theories. The plan of these notes is the following:

1. The classical Heisenberg model:
   • Non-Abelian symmetries;
   • The generators of the symmetries and the semi-classical Yangians;
   • An alternative presentation of the semi-classical Yangians;
   • Digression on Poisson-Lie groups.

2. The quantum Heisenberg chain:
   • Non-Abelian symmetries and the quantum Yangians;
   • The transfer matrix and an alternative presentation of the Yangians;
   • Digression on the double Yangians.

Quantum integrable models are characterized by the existence of commuting conserved charges which one is free to choose as Hamiltonians. They usually also possess extra symmetries which are at the origin of degeneracies in the spectrum. These symmetries are often not a group but a quantum group. The Yangians are quantum group symmetries which (almost systematically) show up in integrable models invariant under an internal non-Abelian group. The most standard example of this kind is the Heisenberg model which we use in these notes for presenting an introduction to the Yangian algebras. Other examples of two-dimensional relativistic quantum field the-

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1 Talk given at the "Integrable Quantum Field Theories" conference held at Come, Italy, September 13-19, 1992
2 Laboratoire de la Direction des Sciences de la Matière du Commissariat à l’Energie Atomique.
ories which are Yangian invariant are provided e.g. by sigma models, two
dimensional massive current algebras, principal chiral models with or without
topological terms \([2, 3]\), etc... Usually the infinite dimensional quantum
symmetries are not compatible with periodic boundary conditions. However
more recently, it has been realized that a variant of the Heisenberg chain with
a long-range interaction was Yangian invariant in a way compatible with peri-
odic boundary conditions, making eigenstates countable and the degeneracy
easy to describe \([4]\). It is tempting to speculate that quantum integrable
theories could be solved only using their symmetry algebras in a way similar
to the solutions of two-dimensional conformal field theories. An introduc-
tion to quantum groups in relation with lattice models can be found in ref. \([5]\).
These notes arise essentially from a joint work with Olivier Babelon, ref. \([6]\).

1 The classical Heisenberg model

We start with the definition of the classical Heisenberg model. As a classical
Hamiltonian system, we need to introduce the phase space and its symplectic
structure as well as the Hamiltonian. The classical variables are the following
spin variables \(S(x)\):

\[
S(x) = \sum_{i=1}^{3} S^i(x) \sigma_i \quad \text{with} \quad \sum_{i=1}^{3} S^i(x)^2 = s^2
\]

where the \(\sigma_i\) are the Pauli matrices with \([\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k\) and \(tr(\sigma_j \sigma_k) = 2 \delta_{jk}\). Here and below, \(s\) is a fixed real number. The Poisson bracket are
defined by

\[
\{ S^i(x), S^j(y) \} = \epsilon^{ijk} S^k(x) \delta(x - y)
\] (1)

They are symplectic thanks to the constraint \(\sum_i S^i(x)^2 = s^2\). The Hamilton-
ian is

\[
H_1 = -\frac{1}{4} \int_0^L dx \, tr(\partial_x S \partial_x S)
\] (2)

The equations of motion deduced from this Hamiltonian are:

\[
\partial_t S = -\frac{i}{2} \left[ S, \partial_x^2 S \right] = \frac{i}{2} \partial_x \left[ S_x, S \right]
\] (3)

Notice that these equations are the conservation laws for a \(su(2)\)-valued cur-
rent. As is well known, the Heisenberg model is a completely integrable
model. Its integrability relies on the fact that the equations (3) can be writ-
ten as a zero curvature condition for an auxiliary linear problem. Namel-
y, consider the Lax connexion,

\[
A_x = \frac{i}{\lambda} S(x)
\]

\[
A_t = -\frac{2i s^2}{\lambda^2} S(x) + \frac{1}{2\lambda} [S(x), \partial_x S(x)]
\] (4)
then, the zero curvature condition, \([\partial_t + A_t, \partial_x + A_x] = 0\), is equivalent to the equations of motion. Note that the Lax connexion (4) is an element of the \(\tilde{su}(2)\) loop algebra \(\tilde{su}(2) = su(2) \otimes C [\lambda, \lambda^{-1}]\).

An important ingredient is the transfer matrix \(T(x, \lambda)\); it is defined as

\[
T(x, \lambda) = P \exp \left[ - \int_0^x A_x(y, \lambda) dy \right] \quad (5)
\]

The monodromy matrix \(T(\lambda)\) is simply \(T(L, \lambda)\). From its definition, \(T(x, \lambda)\) is analytic in \(\lambda\) with an essential singularity at \(\lambda = 0\). From eq.(5), we can easily find an expansion around \(\lambda = \infty\):

\[
T(x, \lambda) = 1 - \frac{i}{x} \int_0^x dy S(y) - \frac{1}{x^2} \int_0^x dy S(y) \int_0^y dz S(z) + \cdots
\]

This development in \(1/\lambda\) has an infinite radius of convergence. We will study it later in more details in relation with the non-Abelian symmetries of the model.

As is well known, the importance of the monodromy matrix lies in the fact that one can calculate the Poisson bracket of its matrix elements. One finds [7]:

\[
\{ T(\lambda) \otimes T(\mu) \} = \frac{1}{2} \left[ r(\lambda, \mu), T(\lambda) \otimes T(\mu) \right] \quad (6)
\]

where

\[
r(\lambda, \mu) = \frac{1}{\lambda - \mu} \sum_i \sigma_i \otimes \sigma_i
\]

(7)

From this result, it follows that \(\text{tr}(T(\lambda))\) is a generating function for quantities in involution.

Assuming periodic boundary conditions, the monodromy matrix \(T(\lambda)\) can be written as follows: \(T(\lambda) = \cos P_0(\lambda) \text{Id} + i \sin P_0(\lambda) \ M(\lambda)\), with \(M(\lambda)\) traceless. Therefore, the trace of the transfer matrix is: \(\text{tr}(T(\lambda)) = 2 \cos P_0(\lambda)\), and we can use \(P_0(\lambda)\) as a generating function for the commuting conserved quantities. The local conserved charges are found by expanding not around \(\lambda = \infty\) but around \(\lambda = 0\):

\[
P_0(\lambda) = -\frac{sL}{\lambda} + \sum_{n=0}^{\infty} \lambda^n I_n
\]

The quantities \(I_n\) are integral of local densities. The first two, \(I_0\) and \(I_1\), correspond to momentum and energy respectively.

### 1.1 Non-Abelian symmetries.

Clearly, since the Hamiltonian is a \(su(2)\) scalar, the equations of motion are \(su(2)\) invariant. Therefore, for any element \(v \in su(2)\), the transformation,

\[
\delta_v S(x) = i \left[ v , S(x) \right]
\]
is a symmetry of the equations of motion. Actually, the symmetry group is much bigger. For any \( v \in su(2) \) and any non-negative integer \( n \), the transformations \( S(x) \rightarrow \delta_v^n S(x) \) defined by:
\[
\begin{align*}
\delta_v^0 S(x) &= i\left[ v, S(x) \right] \\
\delta_v^n S(x) &= i\left[ Z_v^n(x), S(x) \right]
\end{align*}
\]  
(8)
where the functions \( Z_v^k(x) \) are recursively computed by
\[
\partial_x Z_v^k(x) + i\left[ S(x), Z_v^{k-1}(x) \right] = 0 \quad ; \quad Z_v^0 = v
\]  
(9)
are symmetries of the equations of motion.

Moreover, these transformations form a representation of the loop algebra (more precisely of the sub-algebra \( su(2) \otimes \mathbb{C}[\lambda] \) of the \( su(2) \) loop algebra); i.e. we have:
\[
\left[ \delta_v^n, \delta_w^m \right] = \delta_{[v,w]}^{n+m}
\]  
(10)
for any non-negative integers \( n \) and \( m \), and any \( v, w \in su(2) \). In other words, the symmetry group is the loop group (more precisely the sub-group of the loop group which consists of loops regular at zero).

This can be proved as follows. For any \( v \in su(2) \), let us define the functions \( Z_v^k(x) \) by:
\[
(TvT^{-1})(x, \lambda) = \sum_{k=0}^{\infty} \lambda^{-k} Z_v^k(x)
\]
where we have used the \( \left( \frac{1}{\lambda} \right) \) expansion of the transfer matrix. The differential equations satisfied by these functions are consequences of those satisfied by the transfer matrix \( T(x, \lambda) \). For any positive integer \( n \), we set: \( \Theta^n_v(x, \lambda) = i \sum_k \lambda^{-k} Z_v^k(x) \). Consider now the following gauge transformations acting on the Lax connexion:
\[
\begin{align*}
\delta_v^n A_x &= -[A_x, \Theta^n_v] - \partial_x \Theta^n_v \\
\delta_v^n A_t &= -[A_t, \Theta^n_v] - \partial_t \Theta^n_v
\end{align*}
\]
By construction, these transformations preserve the zero curvature condition, since they are gauge transformations. Therefore, they will be symmetries if the form of the components, \( A_t \) and \( A_x \), of the Lax connexion is preserved. It is a simple exercise to check this fact; e.g. for \( A_x \) we have:
\[
\begin{align*}
\delta_v^n A_x &= \lambda^{-1}[S, Z_v^n] - i \sum_{k=0}^{n-1} \left\{ \partial_s Z_v^{k+1} + i[S, Z_v^k] \right\} \lambda^{n-k-1} \\
&= \lambda^{-1}[S, Z_v^n]
\end{align*}
\]
The last sum vanishes by virtue of eq.(3) and we are left with \( \delta_v^n S(x) = i[Z_v^n(x), S(x)] \). One can check similarly that the form of \( A_t \) is unchanged, and its variation is compatible with eq.(8). This proves that eqs.(8) define symmetries of the equations of motion.
To these symmetries correspond an infinite number of conserved currents. Indeed, as we already remarked the equations of motion have the form of a conservation law $\partial_t J_t - \partial_x J_x = 0$ with $J_t = S$ and $J_x = \frac{i}{2}[S_x, S]$. Since the transformations are symmetries, transforming this local current produces new currents which form an infinite multiplet of currents:

$$
J_t^{n,v} = \delta_v^n S = i[Z_v^n, S] \\
J_x^{n,v} = -\frac{i}{2}[\partial_x[Z_v^n, S], S] - \frac{1}{2}[S_x, [Z_v^n, S]]
$$

(11)

for any $n \geq 0$ and $v \in su(2)$. Note that these currents are non-local. By construction they are conserved: $\partial_t J_t^{n,v} - \partial_x J_x^{n,v} = 0$. To them correspond charges which are defined by:

$$
Q_v^n = \int_0^L J_t^{n,v}(x) dx = Z_v^{n+1}(L)
$$

(12)

Since the currents are non-local, the charges are not conserved. We have

$$
\frac{d}{dt} Q_v^n = J_t^{n,v}(L) - J_t^{n,v}(0)
$$

In the infinite volume limit ($L \to \infty$), with an appropriate choice of the boundary conditions the charges can eventually be conserved. Even if they are not conserved, they are nevertheless important because, as we will see, they are the generators of the non-Abelian transformations.

1.2 The generators of the symmetries and the semi-classical Yangians.

We now discuss in which sense the non-local charges $Q_v^n$ are the generators of the non-Abelian symmetries (8); i.e. in which sense the infinitesimal transformation laws (8) of the spin variables are given by Poisson brackets between the non-local charges and the dynamical variables $S(x)$. As we will see, contrary to what happens for symplectic transformations, the infinitesimal variations $\delta_v^n S(x)$ are not linearly generated by the charges.

The charges $Q_v^n$ take values in the Lie algebra $su(2)$. Let us introduce their components, $Q_{ij}^n$ and $Q_i^n$, in the basis of the Pauli matrices $\sigma_i$:

$$
Q_{ij}^n = \frac{1}{2} \text{tr} \left( Q_v^n \sigma_j \right) \quad \text{and} \quad Q_i^n = \sum_{j,k=0}^3 \epsilon_{ijk} Q_j^k
$$

(13)

In particular, for $n = 0$ and $n = 1$, we have the simple formula:

$$
Q_i^0 = 4 \int_0^L dx \ S^i(x) \\
Q_i^1 = 4 \int_0^L dx \int_0^x dy \ \epsilon^{ijk} S^j(x) S^k(y)
$$

(14)

Notice that $Q_i^0$ are local whereas $Q_i^1$ are not. They generate the transformations $\delta_0^i S(y)$ and $\delta_1^i S(y)$ in the following way:

$$
\delta_0^i S(y) = \frac{1}{2} \{ Q_i^0, S(y) \} \\
\delta_1^i S(y) = \frac{1}{2} \{ Q_i^1, S(y) \} - \frac{1}{8} \epsilon^{ijk} Q_j^0 \{ Q_k^0, S(y) \}
$$

(15)
The first equation in (15) implies that the \( su(2) \) symmetry \( \delta_0^i S(y) \) is a symplectic action as it should be. The non-linearity in the second equation is the sign that these transformations are not symplectic but is characteristic to Lie-Poisson actions, a notion which generalize the notion of symplectic actions.

The non-linearity in eq. (15) has another echo. The Poisson algebra of the charges \( Q_i^0 \) and \( Q_i^1 \) is not the \( su(2) \) loop algebra but a deformation of it. Indeed recall that the \( su(2) \) loop algebra can be presented as the associative algebra generated by the elements \( \delta_0^i \) and \( \delta_1^i \) satisfying the following relations:

\[
\begin{align*}
[\delta_0^i, \delta_0^j] &= \epsilon_{ijk} \delta_k^0 \\
[\delta_0^i, \delta_1^j] &= \epsilon_{ijk} \delta_k^1 \\
[\delta_1^i, \delta_1^j] &= \epsilon_{ijk} \delta_k^1 \\
[\delta_1^i, [\delta_1^j, \delta_0^k]] - [\delta_1^j, [\delta_1^k, \delta_0^i]] &= 0 \\
[[\delta_1^i, \delta_1^j], [\delta_0^k, \delta_1^l]] + [[\delta_0^k, \delta_1^l], [\delta_0^i, \delta_1^j]] &= 0
\end{align*}
\]

(16)

On the other hand, given the explicit expressions of the charges \( Q_i^0 \) and \( Q_i^1 \) in terms of the spin variables, one can compute their Poisson brackets and check that they satisfy the following relations:

\[
\begin{align*}
\{Q_i^0, Q_j^0\} &= 4\epsilon_{ijk} Q_k^0 \\
\{Q_i^0, Q_j^1\} &= 4\epsilon_{ijk} Q_k^1 \\
\{Q_i^1, \{Q_j^0, Q_k^0\}\} - \{Q_i^0, \{Q_j^1, Q_k^1\}\} &= A_{ijk}^{lnm} Q_l^0 Q_m^0 Q_n^0 \\
\{\{Q_i^1, Q_j^0\}, \{Q_k^0, Q_l^1\}\} + \{\{Q_i^0, Q_j^1\}, \{Q_k^0, Q_l^1\}\} &= 8(A_{ij}^{lnp} \epsilon_kla + A_{kla}^{mnp} \epsilon_{ija}) Q_m^0 Q_n^0 Q_p^1
\end{align*}
\]

(17)

with \( A_{ijkl}^{nm} = \frac{2}{3}\epsilon_{ila}\epsilon_{jmkb}\epsilon_{knc}^{abc} \)

One sees that these relations form a deformation of those defining the \( su(2) \) loop algebra. Therefore the Poisson algebra of the charges is a deformation of the \( su(2) \) loop algebra. It is called the semi-classical \( su(2) \) Yangian. There is no extra relation between the charges \( Q_i^0 \) and \( Q_i^1 \) since there is no extra relation between the generators \( \delta_0^i \) and \( \delta_1^i \) in the \( su(2) \) loop algebra.

All the non-local charges \( Q_{ij}^n \) can be expressed as multiple Poisson brackets between the two first charges \( Q_i^0 \) and \( Q_i^1 \). Therefore, the Poisson algebra of the symmetries is generated by these two charges. (This fact is more easily proved by introducing the transfer matrix and its Poisson brackets as it will be done in the next section.)

Also, since the \( su(2) \) loop algebra is generated only by \( \delta_0^i \) and \( \delta_1^i \), all the infinitesimal transformations \( \delta_i^n S(y) \) can be expressed non-linearly in terms of the non-local charges \( Q_i^0 \) and \( Q_i^1 \), or alternatively, in terms of the charges \( Q_{ij}^n \). A close expression is:

\[
\delta_i^n S(y) = \frac{1}{2} \{Q_i^n, S(y)\} + \frac{1}{2} \sum_{p=0}^{n-1} \left[ Q^{n-p-1} - tr(Q^{n-p-1}) Id \right]_{ik} \delta_k^p S(y)
\]

This equation can be interpreted in two different ways: (i) this is a system of equations allowing to express recursively \( \delta_i^n S(y) \) in terms of Poisson brackets; or (ii) it allows to express the transformed spins \( \delta_i^n S \) in a non-linear way in terms of the charges \( Q_i \).
1.3 The transfer matrix and an alternative presentation of the semi-classical Yangians.

We now show that the knowledge of the charges is equivalent to the data of the monodromy matrix $T$. Let us introduce a notation for the matrix elements of the monodromy matrix:

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} ; \quad \text{Det } T(\lambda) = AD - BC = 1$$

Recall that from eq. (13), we have:

$$Q_{ij}(\lambda) = \delta_{ij} + \sum_{n=0}^{\infty} \lambda^{-n-1} Q^n_{ij} = \frac{1}{2} tr(T\sigma_i T^{-1} \sigma_j)$$

$$Q_i(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} Q^n_i = \epsilon_{ijk} Q_{jk}(\lambda)$$

Therefore, the quantities $Q_{ij}(\lambda)$ and $Q_i(\lambda)$ are quadratic functions of the matrix elements of $T(\lambda)$. We also point out that it is possible to express the charges $Q^n_{ij}$ in terms of $Q^n_i$.

But, one can invert these relations and express $T(\lambda)$ in terms of the generating functions $Q_i(\lambda)$. The relation between the transfer matrix and the non-local charges $Q_i(\lambda)$ is:

$$T(\lambda) = \frac{1}{2} W(\lambda) \ Id - i \frac{1}{2} W^{-1}(\lambda) \sum_i Q_i(\lambda) \sigma_i$$

with $W(\lambda) = \sqrt{2 + \sqrt{4 - \vec{Q}^2(\lambda)}}$.

So, the charges $Q_i(\lambda)$ contains the same amount of information as the monodromy matrix. It is interesting in this context to examine more closely the relation between the $Q_i(\lambda)$ and the matrix elements of $T(\lambda)$. We have

$$Q_+ (\lambda) = Q_1(\lambda) + i Q_2(\lambda) = 2i W(\lambda) C(\lambda)$$

$$Q_- (\lambda) = Q_1(\lambda) - i Q_2(\lambda) = 2i W(\lambda) B(\lambda)$$

Moreover, the trace of the transfer matrix is: $tr T(\lambda) = W[\vec{Q}^2(\lambda)]$. Therefore, $\vec{Q}^2(\lambda)$ is also a generating function for commuting quantities. Its relation with the generating function $P_0(\lambda)$ is:

$$\vec{Q}^2(\lambda) = 4 \sin^2 \left( 2P_0(\lambda) \right)$$

However, expanding $\vec{Q}^2(\lambda)$ around $\lambda = \infty$ gives non-local commuting quantities while expanding $P_0(\lambda)$ around $\lambda = 0$ gives the local commuting quantities. The links between them are hidden in the subtleties of the analytic properties of the monodromy matrix (these analytic properties should be encoded into the representation theory of the Yangian).

Since, the transfer matrix encodes the same amount of information as the charges, the semi-classical $su(2)$ Yangian can also be presented in
terms of the transfer matrix: the semi-classical $su(2)$ Yangians is the Poisson algebra generated by the transfer matrix, $T(\lambda)$, of unit determinant and with Poisson brackets:

$$\{ T(\lambda) \otimes T(\mu) \} = \frac{1}{2} \left[ r(\lambda, \mu), T(\lambda) \otimes T(\mu) \right]$$

The $\frac{1}{\lambda}$ expansion of $T(\lambda)$ is implicitly assumed in the definition. For the components $t^{(n)}_{ab}$ of $T(\lambda)$, defined by $T^{(n)}(\lambda) = \delta_{ab} + \sum_{n=0}^\infty t^{(n)}_{ab} \lambda^{-n-1}$, we get:

$$\{ t^{(n)}_{ab}, t^{(m)}_{cd} \} = \delta_{cb} t^{(n+m)}_{ad} - \delta_{ad} t^{(n+m)}_{cb} + \sum_{p=0}^{n-1} \left( t^{(m+p)}_{ad} t^{(n-1-p)}_{cb} - t^{(n-1-p)}_{ad} t^{(m+p)}_{cb} \right)$$

In particular, this last relation shows that the Poisson algebra is effectively generated by the two first charges $Q^0_i$ and $Q^1_i$.

Finally, let us describe how the monodromy matrix generates the transformations (8). By an explicit computation of the Poisson brackets between the monodromy matrix and the spin variables, one can easily check that the variation $\delta^n_v S(y)$ of the spin variables are given by following formula:

$$\delta^n_v S(y) = \int \frac{d\lambda}{2i\pi} \lambda^n \left( v T^{-1}(\lambda) \otimes 1 \right) \left\{ T(\lambda) \otimes 1, 1 \otimes S(y) \right\}$$

Here $tr_1$ denotes the trace over the first space in the tensor product. For $n = 0$ or 1, eq. (22) is equivalent to eq. (15). It indicates that $T(\lambda)$ is the generator of the non-Abelian symmetries and characterizes the transformations $S(y) \rightarrow \delta^n_v S(y)$ as Lie-Poisson actions.

### 1.4 Digression on Poisson-Lie groups.

In this section we present a few basic facts about Lie-Poisson actions [1, 8]. Let $M$ be a sympletic manifold. We denote by $\{ , \}_M$ the Poisson bracket in $M$.

Before describing Lie-Poisson actions, we recall some well known facts about Hamiltonian actions. Let $H$ be a Lie group and $\mathcal{H}$ its Lie algebra. The action of a one parameter subgroup $(h^t)$ of $H$ is said to be symplectic if for any functions $f_1$ and $f_2$ on $M$,

$$\{ f_1(h^t.x), f_2(h^t.x) \}_M = \{ f_1, f_2 \}_M(h^t.x)$$

Introducing the vector field $X$ on $M$ corresponding to the infinitesimal action, $\delta_X.f(x) = \frac{d}{dt}f(h^t.x)|_{t=0}$, the condition (23) becomes:

$$\{ \delta_X.f_1, f_2 \}_M + \{ f_1, \delta_X.f_2 \}_M = \delta_X.\{ f_1, f_2 \}_M$$

We have the standard property that the action of any one parameter subgroup of $H$ is locally Hamiltonian. This means that there exists a function $H_X$, locally defined on $M$, such that:

$$\delta_X.f = \{ H_X, f \}_M$$
The proof is standard. The global existence of \( H_X \) is another state of affair. The Hamiltonians \( H_X \) are used to define the moment map. These properties generalize to Lie-Poisson actions.

A Poisson-Lie group \( H \) is a Lie group equipped with a Poisson structure such that the multiplication in \( H \) viewed as map \( H \times H \rightarrow H \) is a Poisson mapping. Let us be more explicit. Any Poisson bracket \( \{,\} \) on a Lie group \( H \) is uniquely characterized by the data of a \( \mathcal{H} \otimes \mathcal{H} \)-valued function: \( h \in H \rightarrow \eta(h) \in \mathcal{H} \otimes \mathcal{H} \). Indeed, introducing a basis \( (e_a) \) of \( \mathcal{H} \), the Poisson bracket for any functions \( f_1 \) and \( f_2 \) on \( H \) can be written as:

\[
\{f_1, f_2\}_H(h) = \sum_{a,b} \eta^{ab}(h)(\nabla^R_a f_1)(h)(\nabla^R_b f_2)(h) \tag{25}
\]

where \( \eta(h) = \sum_{a,b} \eta^{ab}(h)e_a \otimes e_b \) and \( \nabla^R_a \) is the right-invariant vector field corresponding to the element \( e_a \in \mathcal{H} : \nabla^R_a f(h) = \frac{d}{dt} f(e^{te_a}h)|_{t=0} \). The antisymmetry of the Poisson bracket \( \{,\} \) requires \( \eta_{21} = -\eta_{21} \), and the Jacobi identity is equivalent to a quadratic relation for \( \eta \) which can be easily written down. The Lie Poisson property of the Poisson brackets \( \{,\} \) is the requirement that they transform covariantly under the multiplication in \( H \); it requires that \( \eta(h) \) is a cocycle \([1]\): \( \eta(hg) = \eta(h) + Ad \cdot \eta(g) \).

The bracket \( \{,\}_H \) can be used to define a Lie algebra structure on \( \mathcal{H}^* \) by \( \left[ d_e f_1, d_e f_2 \right]_{\mathcal{H}^*} = d_e \{ f_1, f_2 \}_H \), with \( d_e f \in \mathcal{H}^* \) the differential of the function \( f \) on \( H \) evaluated at the identity of \( H \). In a basis \( (e^a) \) in \( \mathcal{H}^* \), dual to the basis \( (e_a) \) in \( \mathcal{H} \), the differential at the identity can written as \( d_e f = \sum_a e^a(\nabla^L_a f)(e) \in \mathcal{H}^* \) where \( \nabla^L_a \) are the left-invariant vector fields on \( H \), and the Lie structure in \( \mathcal{H}^* \) is:

\[
\left[ e^a, e^b \right]_{\mathcal{H}^*} = f^{ab}_c e^c \tag{26}
\]

where the structure constants are \( f^{ab}_c = (\nabla^L_c \eta^{ab})(e) \). The Lie bracket eq.\([23]\) satisfies the Jacobi identity thanks to the Jacobi identity for the Poisson bracket in \( H \). We denote by \( \mathcal{H}^* \) the Lie group with Lie algebra \( \mathcal{H}^* \).

The action of a Poisson-Lie group on a symplectic manifold is a Lie-Poisson action if the Poisson brackets transform covariantly; i.e. if for any \( h \in H \) and any function \( f_1 \) and \( f_2 \) on \( M \),

\[
\{f_1(h.x), f_2(h.x)\}_H \otimes M = \{f_1, f_2\}_M(h.x) \tag{27}
\]

The Poisson structure on \( H \times M \) is the product Poisson structure.

Let \( X \in \mathcal{H} \) and denote also by \( \delta_X \) the vector field on \( M \) corresponding to the infinitesimal transformation generated by \( X \). Introducing two dual basis of the Lie algebras \( \mathcal{H} \) and \( \mathcal{H}^* \): \( e_a \in \mathcal{H} \) and \( e^a \in \mathcal{H}^* \) with \( < e^a, e_b > = \delta^a_b \), where \( <,> \) denote the pairing between \( \mathcal{H} \) and \( \mathcal{H}^* \), eq. \([27]\) becomes:

\[
\{\delta_{e_a}, f_1, f_2\}_M + \{f_1, \delta_{e_a}, f_2\}_M + f^{bd}_a(\delta_{e_b}, f_1)(\delta_{e_d}, f_2) = \delta_{e_a}, \{f_1, f_2\}_M \tag{28}
\]

It follows immediately from eq.\([28]\) that a Lie-Poisson action cannot be Hamiltonian unless the algebra \( \mathcal{H}^* \) is Abelian. However, in general, we have a non-Abelian analogue of the Hamiltonian action eq. \([24]\) \([1]\): There exists
a function $\Gamma$, locally defined on $M$ and taking values in the group $H^*$, such that for any function $f$ on $M$,

$$\delta_X f = \langle \Gamma^{-1} \{ f, \Gamma \}, X \rangle, \quad \forall \ X \in \mathcal{H} \quad (29)$$

We refer to $\Gamma$ as the non-Abelian Hamiltonian of the Lie-Poisson action.

The moment map $\mathcal{P}$ for the Lie-Poisson action is the map $\mathcal{P}$ from $M$ to $H^*$ defined by $x \mapsto \Gamma(x)$.

The proof is the following. Introduce the Darboux coordinates $(q^i, p^i)$.

Let $\Omega = e^a \Omega_a$ be the $H^*$-valued one-form defined by $\Omega_a = e^a_q dq^i - e^a_p dp^i$ where $e^a_q, e^a_p$ are the components of the vector field $e_a, e_a = e^a_q \partial_q + e^a_p \partial_p$.

Eq. (28) is then equivalent to the following zero-curvature condition for $\Omega$:

$$d \Omega + [\Omega, \Omega]_{H^*} = 0$$

Therefore, locally on $M$, $\Omega = \Gamma^{-1} d \Gamma$. This proves eq.(29). The converse is true: an action generated by a non-Abelian Hamiltonian as in eq.(29) is Lie-Poisson since then we have:

$$\delta_X \{ f_1, f_2 \} = \langle \Gamma^{-1} \{ f_1, \Gamma \}, \Gamma^{-1} \{ f_2, \Gamma \} \rangle_{H^*}, X \rangle$$

The non-Abelian symmetries we described above thus provide an example of Lie-Poisson actions, since they are generated by a non-Abelian Hamiltonian. They actually are a particular example of more general transformations called dressing transformations in solitons theories [10, 11].

2 The quantum Heisenberg chain

We introduce the quantum Heisenberg chain as a quantization of the discrete analogue of the classical Heisenberg model. Since, it is not more difficult to deal with the $su(p)$ algebra, we generalize it from $su(2)$ to $su(p)$. So let us assume that we have discretized the interval on which the model was defined into $N$ segments of identical length that we denote by $h$. The extremities of the segments form a chain of sites on which the $su(p)$ spin variables leave. We denote the $su(p)$ spin variables at the sites $j$ by $S_{ab}^j$ with $a, b = 1, \cdots, p$. They satisfy the commutation relations of the discretized $su(p)$ loop algebra:

$$[ S_{ab}^j, S_{cd}^k ] = \delta_{jk} ( \delta^{cb} S_{ad}^j - \delta^{ad} S_{bj}^c ) \quad (30)$$

This corresponds to the quantization of the Poisson brackets [11]. Approximating the derivatives by finite differences, and using the constraints $S_k^2 = s^2 = \text{const}$, the discretized version of the Hamiltonian (2) becomes, (up to a constant term):

$$H = \sum_{k=1}^{N} \sum_{ab} S_{ab}^{k} S_{ba}^{k+1} \quad (31)$$

Here, we have assumed periodic boundary conditions. As is well known, in order to preserve the integrability we have to choose the spin operators $S_{ab}^k$ to act on the fundamental vector representation of $su(p)$. So on each sites
there is a copy of $Q^p$ and the operator $S^a_{j}$ which acts only the $j^{th}$ copies of $Q^p$ is represented by the canonical matrix $|a\rangle\langle b|$

In the discretized quantized model, the monodromy matrix, which is a $p \times p$ matrix with operator entries, has matrix elements:

$$T_{ab}(\lambda) = \left. P \exp \left[ - \int A_x(y, \lambda) dy \right] \right|_{\text{discretized}}$$

$$= \sum_{a_1 \cdots a_{N-1}} \left[ 1 + \frac{\hbar}{\lambda} S_1 \right]^{a_{a_1}} \left[ 1 + \frac{\hbar}{\lambda} S_2 \right]^{a_{a_2}} \cdots \left[ 1 + \frac{\hbar}{\lambda} S_N \right]^{a_{N-1}}$$

As for the classical theory, the transfer matrix admits an expansion:

$$T_{ab}(\lambda) = \delta_{ab} + \frac{\hbar}{\lambda} \sum_k S^a_k + \frac{\hbar^2}{\lambda^2} \sum_{j<k} \sum_d S^ad\ S^db + \cdots$$

The important properties of this monodromy matrix is that we can compute the commutation relations between its matrix elements. These relations can be gathered into the famous relations of the algebraic Bethe ansatz, see e.g. [12, 13]:

$$R(\lambda - \mu) (T(\lambda) \otimes 1) (1 \otimes T(\mu)) = (1 \otimes T(\mu) ) (T(\lambda) \otimes 1) R(\lambda - \mu)$$

where $R(\lambda)$ is Yang’s solution of the Yang-Baxter equation:

$$R(\lambda) = \lambda - h \ P$$

with $P$ the exchange operator $P(x \otimes y) = y \otimes x$. For the matrix elements of the monodromy matrix this becomes:

$$(\lambda - \mu) [T_{ab}(\lambda), T_{cd}(\mu)] = h \left( T_{ad}(\lambda) T_{cb}(\mu) - T_{ad}(\mu) T_{cb}(\lambda) \right)$$

These relations are the quantum analogues of the Poisson brackets [8]. As in the classical theory, it implies that the trace of the monodromy matrix, or more accurately the logarithm of the trace of the monodromy matrix, form a generating function of commuting quantities. In the same as in the previous section, the local Hamiltonians are not found by expanding around $\lambda = \infty$ but around $\lambda = 0$. In particular, we recover the Hamiltonian by expanding the logarithm of the trace to first order: $H \propto \partial_\lambda \log \lambda^N T(\lambda) \big|_{\lambda=0}$.

2.1 Non-Abelian symmetries and the quantum Yangians.

The Hamiltonian is clearly $su(p)$ invariant. But, in the same as for the classical theories, the symmetry group is much bigger. The naive discrete quantum analogue of the classical charges $Q^0$ and $Q^1$ introduced in eq. (13) is:

$$Q^0_{ab} = \sum_k S^a_k$$

$$Q^1_{ab} = \frac{\hbar}{2} \sum_{j<k} \sum_d (S^ad\ S^db - S^ad\ S^db)$$

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This naive ansatz turns out to be correct: the charges $Q_{ab}^0$ commutes with the Hamiltonian and the charges $Q_{ab}^1$ formally commutes for chains of infinite length. For finite chain the commutation is broken by boundary terms. However, the corresponding currents, which are conserved on the lattice (of any size), were constructed in [14].

The charges (35) form a non-Abelian algebra, which is not a Lie algebra. They satisfy the following commutations relations:

$$\begin{align*}
[Q_{ab}^0, Q_{cd}^0] &= \delta_{cd}Q_{ad}^0 - \delta_{ad}Q_{cb}^0 \\
[Q_{ab}^0, Q_{cd}^1] &= \delta_{cd}Q_{ad}^1 - \delta_{ad}Q_{cb}^1 \\
[Q_{ab}^1, Q_{cd}^1] &= \delta_{cd}Q_{ad}^2 - \delta_{ad}Q_{cb}^2 + \frac{\hbar^2}{4} Q_{ad}^0 (\sum_e Q_{ce}^0 Q_{eb}^0) - \frac{\hbar^2}{4} (\sum_e Q_{ae}^0 Q_{ed}^0) Q_{cb}^0
\end{align*}$$

Here $Q_{ab}^2$ is a new operator whose explicit expression is irrelevant for our discussion. The remarkable fact is that the extra non-linear term in the last equation can be expressed only in terms of $Q_{ab}^0$. This last equation implies a relation involving only $Q_{ab}^0$ and $Q_{ab}^1$:

$$\begin{align*}
&\left[Q_{ab}^0, \left[Q_{cd}^1, Q_{ef}^1\right]\right] - \left[Q_{ab}^1, \left[Q_{cd}^0, Q_{ef}^0\right]\right] \\
&= \frac{\hbar^2}{4} \sum_{pq} \left(\left[Q_{ab}^0, \left[Q_{pq}^0, Q_{pd}^0, Q_{eq}^0 Q_{qf}^0\right]\right] - \left[Q_{ap}^0 Q_{pb}^0, \left[Q_{cd}^0, Q_{eq}^0 Q_{qf}^0\right]\right]\right)
\end{align*}$$

The associative algebra generated by elements $Q_{ab}^0$ and $Q_{ab}^1$ satisfying the relations (36) and (37) is called the $su(p)$ Yangians [1]. As it can be seen by comparing with eq. (16), this algebra is a deformation of the $su(p)$ loop algebra.

The $su(p)$ Yangian is not an Lie algebra but a Hopf algebra. It is therefore equipped with a comultiplication $\Delta$, which is a homomorphism from the algebra into the tensor product of two copies of the same algebra. For the $su(p)$ Yangians, the comultiplication is given by:

$$\begin{align*}
\Delta Q_{ab}^0 &= Q_{ab}^0 \otimes 1 + 1 \otimes Q_{ab}^0 \\
\Delta Q_{ab}^1 &= Q_{ab}^1 \otimes 1 + 1 \otimes Q_{ab}^1 + \frac{\hbar}{2} \sum_d \left(Q_{ad}^0 \otimes Q_{db}^0 - Q_{db}^0 \otimes Q_{ad}^0\right)
\end{align*}$$

It can be used to construct tensor products of representations. For $h = 0$ it reduces to the comultiplication of the $su(p)$ loop algebra.

2.2 The quantum transfer matrix and an alternative presentation of the Yangians.

The charges $Q_{ab}^0$ and $Q_{ab}^1$ appears as the first terms in the $\frac{1}{\lambda}$ expansion of the quantum monodromy matrix $T(\lambda)$. Therefore, the $su(p)$ Yangian can also be presented in terms of $T(\lambda)$, or more precisely, in terms of its components in an $\frac{1}{\lambda}$ expansion:

$$T_{ab}(\lambda) = \delta_{ab} + \hbar \sum_{n=0}^{\infty} \lambda^{-n-1} t_{ab}^{(n)}$$

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Therefore, the alternative presentation of the $su(p)$ Yangian is as the associative algebra generated by the elements $t^{(n)}_{ab}$ with relations:

$$
\left[ t^{(n)}_{ab}, t^{(m)}_{cd} \right] = \delta_{cb} t^{(n+m)}_{ad} - \delta_{ad} t^{(n+m)}_{cb} + \hbar \sum_{p=0}^{n-1} \left( t^{(m+p)}_{ad} t^{(n-1-p)}_{cb} - t^{(n-1-p)}_{ad} t^{(m+p)}_{cb} \right)
$$

(40)

These relations are equivalent to:

$$
\left[ t^{(0)}_{ab}, t^{(m)}_{cd} \right] = \delta_{cb} t^{(m)}_{ad} - \delta_{ad} t^{(m)}_{cb}
$$

(41)

$$
\left[ t^{(n+1)}_{ab}, t^{(m)}_{cd} \right] - \left[ t^{(n)}_{ab}, t^{(m+1)}_{cd} \right] = \hbar \left( t^{(m)}_{ad} t^{(n)}_{cb} - t^{(n)}_{ad} t^{(m)}_{cb} \right)
$$

which, in their turn, are equivalent to the fundamental commutation relations (33). In the quantum theory, the unit determinant constraint is modified into the condition that the so-called quantum determinant of the transfer matrix is one:

$$
Det_q T(\lambda) \equiv \sum_{\sigma \text{perm.}} \epsilon(\sigma) T_{\sigma(p)p}(\lambda) \cdots T_{\sigma(1)1}(\lambda + p\hbar) = 1
$$

(42)

The sum is over the permutation of $p$ objects and $\epsilon(\sigma)$ is the signature of the permutation $\sigma$. The quantum determinant commutes with all the components of the monodromy matrix, so this constraint can be imposed in a consistent way.

With the quantum determinant constraint, the $1/\hbar$ expansion of the monodromy matrix can be reconstructed from its two first components $t^{(0)}_{ab}$ and $t^{(1)}_{ab}$. Finally, the relation between these two components and the quantum charges $Q^0_{ab}$ and $Q^1_{ab}$ is:

$$
Q^0_{ab} = t^{(0)}_{ab}
$$

$$
Q^1_{ab} = t^{(1)}_{ab} - \frac{\hbar}{2} \sum_d t^{(0)}_{ad} t^{(0)}_{db}
$$

(43)

This shows that, as for the classical theory, the knowledge of the first two non-local charges $Q^0_{ab}$ and $Q^1_{ab}$ is equivalent to the knowledge of the $1/\hbar$ expansion of the monodromy matrix.

The comultiplication for the transfer matrix is given by:

$$
\Delta T_{ab}(\lambda) = \sum_d T_{ad}(\lambda) \otimes T_{db}(\lambda)
$$

For $Q^0_{ab}$ and $Q^1_{ab}$ it reduces to eq. (38). The adjoint action of the transfer matrix on an operator $\Phi$ is:

$$
[\text{Adj}. T_{ab}(\lambda)] \Phi = \sum_d T_{ad}(\lambda) \Phi T_{db}^{-1}(\lambda)
$$

where $T_{ab}^{-1}(\lambda)$ is the matrix of operators characterized by $\sum_d T_{ad}(\lambda) T_{db}^{-1}(\lambda) = \delta_{ab}$. This adjoint action is the quantum analogue of the semi-classical equation (22).
2.3 Digression on the double Yangians.

The Yangians are deformation of only half of the loop algebras, i.e. they are deformation of only the sub-algebra of loops regular at the origin. A quantum deformation of the complete loop algebra can be obtained by introducing the quantum double of the Yangian, following Drinfel’d [1]. In quantum field theory, the double Yangian was introduced in [15]. It can be presented in the following multiplicative form similar to eq. (33). Let $T_{ab}^+(\lambda)$ and $T_{ab}^-(\lambda)$ be two $p \times p$ matrices of operators with the following expansion:

\begin{align*}
T_{ab}^+(\lambda) &= \delta_{ab} + \hbar \sum_{n=0}^{+\infty} \lambda^{-n-1} t_{ab}^{(n)} \\
T_{ab}^-(\lambda) &= \hbar \sum_{n=0}^{+\infty} \lambda^n t_{ab}^{(-n-1)}
\end{align*}

Note that these expansions correspond to expansions into series regular at the origin or at infinity. The $gl(p)$ double Yangian is the algebra generated by the elements $t_{ab}^{(\pm n)}$ with relations:

\begin{align*}
R(\lambda - \mu)(T^\pm(\lambda) \otimes 1)(1 \otimes T^\pm(\mu)) &= (1 \otimes T^\pm(\mu))(T^\pm(\lambda) \otimes 1)R(\lambda - \mu) \\
R(\lambda - \mu)(T^+(\lambda) \otimes 1)(1 \otimes T^-(\mu)) &= (1 \otimes T^-(\mu))(T^+(\lambda) \otimes 1)R(\lambda - \mu)
\end{align*}

If in a finite dimensional representation of the Yangian the transfer matrix $T_{ab}^+(\lambda)$ is a polynomial of degree $N$ in $1/\lambda$, then the double Yangian is represented in the same vector space by $T_{ab}^-(\lambda) \propto \lambda^N T_{ab}^+(\lambda)$.

References

[1] V.G.Drinfeld, “Quantum Groups” Proc. of the ICM, Berkeley, (1986).

[2] M.Lüscher, K. Pohlmeyer, Nucl.Phys. B137 (1978) 46. M.Lüscher, Nucl.Phys. B135 (1978) 1.

[3] D. Bernard, Comm. Math. Phys. 137 (1991) 191-208, and “Quantum Symmetries in 2D Massive Field Theories”, to appear in Cargese ’92 proceedings.

[4] F.D.Haldane et al, Phys. Rev. Lett. 69 (1992) 2021.

[5] V.Pasquier, Quantum Groups in Lattice Models, to appear in Cargese ’92 proceedings.

[6] O.Babelon and D.Bernard, Comm. Math. Phys. 149 (1992) 279.
[7] E. Sklyanin, “On the Complete Integrability of the Landau-Lifshitz Equation”. Preprint LOMI E-3-79 Leningrad.

[8] M. Semenov-Tian-Shansky, Funct. Anal. Appl. 17 (1983) 259.

[9] J. H. Lu, “Multiplicative and Affine Poisson Structures on Lie Groups”. PhD Thesis, University of California at Berkeley (1990).

[10] V. E. Zakharov, A. B. Shabat, “Integration of Non Linear Equations of Mathematical Physics by the Method of Inverse Scattering II”. Funct. Anal. Appl. 13 (1979) 166.

[11] M. Semenov-Tian-Shansky, Publ. RIMS Kyoto Univ. 21 (1985) 1237.

[12] L. D. Faddeev, “Integrable Models in 1+1 Dimensional Quantum Field Theory”. Les Houches Lectures. Elsevier Science Publishers (1984).

[13] M. Gaudin, “La fonction d’onde de Bethe”, Masson (1983).

[14] D. Bernard, G. Felder, Nucl. Phys. B365 (1991) 98.

[15] A. LeClair, F. Smirnov, Int. J. Mod. Phys. A7 (1992) 2997.