BI-MAGIC LABELINGS OF SOME CONNECTED AND DISCONNECTED GRAPHS

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Abstract. Various graph labeling that generalize the idea of a magic square have been discussed. In particular, a magic labeling on a graph with \( v \) vertices and \( e \) edges will be defined as a one to one map taking the vertices and edges onto integers \( 1, 2, \ldots, v+e \) with the property that the sum of the label on an edge and the labels of its end points is constant, independent of the choice of edge.

Introduction

In 1970, Kotzig Rosa defined a magic valuation of a graph \( G(V,E) \) as a bijection \( f \) from \( V \cup E \) to \( \{1, 2, \ldots, |V \cup E|\} \) such that for all edges \( xy \), \( f(x)+f(y)+f(xy) \) is constant (called a magic constant). It was later called edge magic labeling. Enomoto, Llado, Nakamigawa and Ringel called a graph \( G(V,E) \) with an edge magic total labeling that has the additional property that the vertex labels are \( 1 \) to \( |V| \) super edge magic labeling.

In 2004, Baskar Babujee introduced the notion of vertex bimagic labeling in which there exists two constants \( k_1 \) and \( k_2 \) such that the sums involved in a specified type of magic labeling is \( k_1 \) or \( k_2 \). An edge bimagic labeling of a graph \( G(V,E) \) with \( p \) vertices and \( q \) edges is a bijection \( f \) from the set of vertices and edges to such that for every edge \( uv \) in \( E \), \( f(u)+f(uv)+f(v) \) is one of two constants \( k_1 \) or \( k_2 \). In this paper, we discuss the bimagic labeling of the graphs: Helm, Bistar, Shell graph, Rooted tree and the disconnected graph \( nK_3 \).

Section 1 – Necessary definitions:

Definition 1.1: An edge bi-magic total labeling of a graph \( G(V,E) \) with \( p \) vertices and \( q \) edges is a bijection \( f \) from the set of vertices and edges to such that for every edge \( uv \) in \( E \), \( f(u)+f(uv)+f(v) \) is one of two constants \( k_1 \) or \( k_2 \), independent of the choice of the edge.

Definition 1.2: A graph \( G(V,E) \) with order \( p \) and size \( q \) is edge bimagic if there exists a bijection \( g: V \cup E \rightarrow \{1, 2, \ldots, p+q\} \) such that \( g(u)+g(v)+g(uv) \) is a constant either constant \( c_1 \) or \( c_2 \) for all edges \( uv \) in \( G \) and \( G \) is super edge bimagic if \( g \) also also satisfies \( g(V) = \{1, 2, \ldots, p\} \).

Section 2 – Edge bimagic labeling of some connected graphs

Definition 2.1: The helm \( H_n \) is a connected graph whose vertex set is \( \{v_0, v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\} \) and edge set is \( \{v_0v_i : i = 1 \text{ to } n\} \bigcup \{v_iv_{i+1} : i = 1 \text{ to } (n-1)\} \bigcup \{v_nv_1\}. \) It is obtained from a wheel by attaching a pendent edge at each vertex of the \( n \)-cycle. The vertex set and edge set of the \( n \)-cycle are \( \{v_0, v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\} \) and \( \{v_0v_i : i = 1 \text{ to } n\} \bigcup \{v_iv_{i+1} : i = 1 \text{ to } (n-1)\} \bigcup \{v_nv_1\}. \) The pendent edges are \( \{v_iv_{n+i} : i = 1 \text{ to } n\} \)
**Theorem 2.2:** The graph helm $H_n$ is bi-magic

**Proof:** One of the arbitrary labeling of the given graph is as follows in figure 1.

![Figure 1 – One of arbitrary labelings of vertices of Helm graph $H_n$](image)

Define $f : V(G) \rightarrow \{1,2,3,\ldots,p\}$ by
\[
f(v_i) = i+1 : i = 0 \text{ to } 2n
\]

Define $f : E(G) \rightarrow \{p+1,p+2,\ldots,p+q\}$ by
\[
f(v_0v_i) = 5n+2-i : i = 1 \text{ to } n
\]
\[
f(v_iv_{i+1}) = 4n+1-2i ; i = 1 \text{ to } (n-1)
\]
\[
f(v_iv_{n+i}) = 4n+2-2i ; i = 1 \text{ to } n
\]
\[
f(v_nv_1) = 4n+1.
\]

For all edges $uv$ in $G$, $f(u)+f(uv)+f(v) = f(uv) = 4n + 4$ or $5n + 4$. Fix $k_1 = 4n+4$; and $k_2 = 5n+4$. So $H_n$ satisfies the all conditions of bi-magic graph, and helm graph $H_n$ is bi-magic.

**Definition 2.3:** The graph $B_{n,n}$ is the connected graph obtained by joining the centre vertex of one of $K_{1,n}$ to the pendent vertex of the other $K_{1,n}$. Its vertex set is $\{v_0,v_1,v_2,\ldots,v_n,v_{n+1},\ldots,v_{2n+2}\}$ and edge set is $\{v_1v_i : i = 2 \text{ to } (n+1)\} \cup \{v_{n+2}v_i : i = n+3,n+4,\ldots,2n+2\} \cup \{v_1v_{n+3}\}$.

The vertex set and edge set of one of $K_{1,n}$ are $\{v_0,v_1,v_2,\ldots,v_n,v_{n+1}\}$ and $\{v_1v_i : i = 2 \text{ to } (n+1)\}$ and the other are $\{v_{n+2},\ldots,v_{2n+2}\}$ and $\{v_{n+2}v_1 : i = n+3,n+4,\ldots,2n+2\}$. They are joined by the edge $v_1v_{n+3}$.

**Theorem 2.4:** The graph bi-star $B(n,n)$ is bi-magic

**Proof:** One of the arbitrary labeling of the given graph is as follows.
Define $f : V(G) \rightarrow \{1,2,3,...,p\}$ by
\[ f(v_i) = i : i = 1 \text{ to } (2n+2) \]
Define $f : E(G) \rightarrow \{ p+1, p+2,...,p+q \}$ by
\[ f(v_1v_{n+3}) = 4n+3; \]
\[ f(v_1v_i) = 4n+4-i : i = 2,3,...,(n+1) \]
\[ f(v_{n+2}v_i) = 4n+5-i ; i = n+3,n+4,...,(2n+2). \]
For all edges $uv$ in $G$, $f(u)+f(uv)+f(v) = f(uv) = 4n + 5$ or $5n + 7$. Fix $k_1 = 4n + 5$; and  $k_2 = 5n + 7$. So $B_{n,n}$ fulfilled all conditions of bi-magic graph, and helm graph $B_{n,n}$ is bi-magic.

**Definition 2.5: (Shell graph):** The shell graph $C(n,k)$ denotes a connected graph which contains a cycle $C_n$ with $k$ chords sharing a common end point called the apex. Its vertex set is $\{v_1,v_2,...,v_n\}$ and edge set is $\{v_i v_{i+1} : i = 1 \text{ to } (n-1)\} \cup \{v_i v_1 : i = 3 \text{ to } (n-1)\} \cup \{v_1 v_i\}$. The vertex set of cycle $C_n$ is $\{v_1,v_2,...,v_n\}$ and edge set is $\{v_i v_{i+1} : i = 1 \text{ to } (n-1)\} \cup \{v_n v_1\}$. The chords are $\{v_i v_1 : i = 3 \text{ to } (n-1)\}$ and $v_1$ is the apex.

**Theorem 2.6:** The shell graph $C(n,k)$ is bi-magic ($n$ – even).

**Proof:** One of the arbitrary labeling of the given graph is as follows:

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**Figure 2 – One of arbitrary labeling of vertices of graph $B_{n,n}$.**

**Figure 3 – One of arbitrary labeling of vertices of shell graph $C(n,k)$.**
Define \( f: V(G) \rightarrow \{1,2,3,\ldots,p\} \) by
\[
f(v_i) = i : i = 1 \text{ to } n.
\]
Define \( f: E(G) \rightarrow \{p +1, p +2,\ldots,p+q\} \) by
\[
f(v_1v_i) = 2n+k-i+3; \ i = 3 \text{ to } (n-1)
f(v_1v_n) = 2n
f(v_i v_{i+1}) = (2n-1)-2(i-1) ; \ i =1 \text{ to } (n/2)
=2n-2i+1.
f(v_i v_{i+1}) = (2n-2)-(i -(n+2)/ 2) ; \ i = (n+2)/2 \text{ to } (n-1)
=(5n-2i-2)/2.
\]
For all edges \( uv \) in \( G \), \( f(u)+f(uv)+f(v) = f(uv) = 2n + 2 \text{ or } 3n + 1. \)

**Definition 2.7:** Rooted tree is joined of many paths at a point called its root. Each path is of odd length whose vertex set = \( \{v_0,v_1,…,v_{k_1},v_{k_1+1},…,v_{2k_1},v_{2k_1+1},…,v_{nk_1}\} \) and edge set =\( \{v_i v_{i+1} ; \ i =1,2,\ldots,k_1-1: i = k_1+1,\ldots,2k_1-1; \ i =2k_1+1,\ldots,3k_1-1; \ldots; i = (n-1)k_1+1,\ldots,nk_1-1\} \bigcup \{v_0v_j ; \ j=1,k_1+1,2k_1+1,\ldots,(n-1)k_1+1\}. \)

**Theorem 2.8:** Rooted tree is bi-magic.

**Proof:** One of the arbitrary labeling of the given graph is as follows;

![Figure 4 – One of arbitrary labeling of vertices of rooted tree](image)

Define \( f:V(G) \rightarrow \{1,2,\ldots,p\} \) by
\[
f(v_0) = 1;
\]
\[
f(v_j)=f(v_{lk_1+i})(v_{lk_1+i})
= l+2+n (i-1) \text{ where } l= 0,1,2,\ldots,(n-1) , \ i =1,2,3,\ldots,k_1.
\]
Define \( f:E(G) \rightarrow \{p+1,p+2,\ldots,p+q\} \) by
\[ f(v_0v_t) = f(v_0v_{(t-1)k+1})v_{(t-1)k+1+1} \]
\[ = 2nk_1 + 2 - t \quad \text{where } t=1,2,\ldots, \]
\[ f(v_{(t-1)k+s}v_{(t-1)k+s+1}) = (2nk_1 - (2s-1)n - 2(t-1))\quad ; \quad t=1,2,\ldots,n; \quad s=1,2,\ldots,(\frac{k-1}{2}) \]
\[ f(v_{(t-1)k+s}v_{(t-1)k+s+1}) = (2nk_1+1) - 2(t-1) - 2\left(s - \left(\frac{k+1}{2}\right)\right)\quad ; \quad t=1,2,\ldots,n; \quad s=\left(\frac{k+1}{2}\right),\ldots,k. \]

For all edges uv in G, \( f(u) + f(uv) + f(v) = f(uv) = 2^{\ell+4}n_1 + 4; \quad 2^{\ell+4}n_1 + 5 \). So all conditions of bi-magic are checked in given rooted tree, and it is bi-magic.

Section 3- Disconnected graph \( nK_3 \):

**Definition 3.1**: \( nK_3 \) are disconnected collection of \( k_3 \) graphs in which each \( k_3 \) is connected whose vertex set is \( \{ v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n}, v_{2n+1}, \ldots, v_{3n}\} \), and edge set is \( \{ v_i v_{i+1} : i = 1 \text{ to } n \} \cup \{ v_i v_{2i+1} : i = 1 \text{ to } n \} \cup \{ v_{i+n} v_{2i+1} : i = 1 \text{ to } n \} \).

**Theorem 3.2**: \( nK_3 \) (\( n \) is even) is bi-magic

**Proof**: One of the arbitrary labeling of the given graph is as follows;

![Figure 5 – One of arbitrary labeling of vertices of nk3.](image)

\[ f(v_i) = i \quad (i=1 \text{ to } n) \]
\[ f(v_{n+i}) = n+i \quad (i=1 \text{ to } n) \]
\[ f(v_{2n+i}) = 2n+i \quad (i=1 \text{ to } n) \]

Define f: \( E(G) \rightarrow \{ p+1, p+2, \ldots, p+q \} \) by

\[ f(v_i v_{i+n}) = 6n-2i+2 \quad (i=1 \text{ to } n) \]
\[ f(v_i v_{2i+n}) = 6n-2i-10 \quad (i=1 \text{ to } \frac{n}{2}) \]
\[ f(v_{i+n} v_{2i+n}) = 6n-11-2\left(i - \left(\frac{n+2}{2}\right)\right) \quad (i=\frac{n}{2}+1, \ldots, n) \]

For all edges uv in G, \( f(u) + f(uv) + f(v) = f(uv) = 7n + 2; \quad 8n + 1 \) So all conditions of bi-magic are satisfied in the graph \( nK_3 \) (\( n \) is even), and it is bimagic.
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