Critical behaviour of the $1d$ annihilation fission process $2A \rightarrow \emptyset, 2A \rightarrow 3A$

Géza Ódor  
Research Institute for Technical Physics and Materials Science,  
H-1525 Budapest, P.O.Box 49, Hungary

Numerical simulations and cluster mean-field approximations with coherent anomaly extrapolation show that the critical line of the $1d$ annihilation fission process is separated into two regions. In both the small and high diffusion cases the critical behaviour is different from the well known universality classes of non-equilibrium phase transitions to absorbing states. The high diffusion region seems to be well described by the cyclically coupled directed percolation and annihilating random walk. Spreading exponents show non-universal behavior.

PACS numbers: 05.70.Ln, 82.20.Wt.

Non-equilibrium phase transitions may take place even in one dimensional systems. However the ordered phase lacks fluctuations that could destroy the state. If the system has evolved into that state it will be trapped there. We call this transition to an absorbing state. First order transition among these is very rare, there have been only a few system found that exhibit a discontinuous jump from the active to the absorbing phase .

Continuous phase transitions have been found to belong to a few universality classes, the most robust of them is the directed percolation (DP) class. According to the hypothesis of all continuous phase transitions to a single absorbing state in homogeneous systems with short ranged interactions belong to this class provided there is no additional symmetry and quenched randomness present. The most prominent system exhibiting phase transition of this class is the branching and annihilating random walk with one offspring (BARW). Furthermore systems with infinitely many absorbing states were also found to belong to this class .

An important exception from the DP class is the so-called parity-conserving (PC) class where particles follow a branching and annihilating random walk with two offsprings (BARW2) $A \rightarrow 3A, 2A \rightarrow \emptyset$ that conserves the parity of their number . Particles following BARW2 may also appear as kinks between ordered domains in systems exhibiting two absorbing states . However it was realized that the BARW2 dynamics alone is not sufficient condition for a transition in this class but an exact $Z_2$ symmetry between the two absorbing states is necessary, too .

Recently a study on the annihilation fission (AF) process $2A \rightarrow \emptyset, 2A \rightarrow 3A$ suggested that neither the BARW2 dynamics nor the $Z_2$ symmetry but simply the occurrence of two absorbing states can result in a PC class transition. This model was introduced first by in the context of a renormalization group analysis of the corresponding bosonic field theory. The main idea was to interpolate between systems with real and imaginary noise. These studies predicted a non-DP class transition, but they could not tell to which universality class this transition really belongs.

A fermionic version of the AF process introduced is controlled by two parameters, namely the probability of pair annihilation $p$ and the probability of particle diffusion $d$. The dynamical rules are

$$
AA\emptyset, \emptyset AA \rightarrow AAA \text{ with rate } (1-p)(1-d)/2
$$
$$
AA \rightarrow \emptyset\emptyset \text{ with rate } p(1-d)
$$
$$
A\emptyset \leftrightarrow OA \text{ with rate } d.
$$

Carlon et al. suggested that the phase diagram can be separated into two regions. For low values of $d$ (less than approximately 0.3), they found a continuous phase transition belonging to the PC universality class. For large values of $d$, however, a first order transition was reported. This claim is based on mean-field, pair mean-field and density matrix renormalization group method (DMRG) calculations. An even more recent preliminary numerical study found non-PC class like critical exponents and posed the question if there is a new type of non-equilibrium phase transition occurring here.

In this paper I report more detailed numerical results: Simulations and generalized mean-field (GMF) approximations with coherent anomaly (CAM) extrapolations that give numerical evidence of a rich phase diagram with two different kinds of new universality classes in the low $d$ and high $d$ regions.

For simulations a parallel update version of the AF process was used since the model can effectively be mapped on a massively parallel processor ring . To avoid collisions by simultaneous updates the annihilation and exchange steps are done on two sub-lattices, while creation is performed on three sub-lattices (with a rate double as in eq.4). The critical point has been determined for $d = 0.05, 0.1, 0.2, 0.5, 0.9$ by following the decay of the particles from a uniform, random initial state ($\rho(t) \propto t^{-\delta}$). The local slopes curve of the density

$$
\delta_{eff}(t) = \frac{-\ln[\rho(t)/\rho(t/m)]}{\ln(m)} \quad (2)
$$

(where we use $m = 8$ usually) at the critical point goes to exponent $\delta$ by a straight line, while in sub(super)-critical cases it veers down(up) respectively. As Fig. shows at
$d = 0.2$ and $p = 0.24802$ this quantity scales without any relevant corrections (i.e. the straight line is horizontal) and in the $1/t \to 0$ limit goes to $\delta = 0.268(2)$. Very similar results has been obtained for $d = 0.05$ and $d = 0.1$ showing that the exponent $\delta$ is close to the corresponding value of the PC class ($0.285(10)$ [1]), but appears to be significantly smaller. This $\delta$ is in agreement within error margin with the value ($\sim 0.272(18)$) one can obtain from DMGR calculations [23] using scaling laws $\delta = \beta/\nu_{\parallel}/z$.

For stronger diffusion rates $d = 0.5$ and $d = 0.9$ one can observe slower density decays with exponent $\delta = 0.215(10)$ (Fig. 2). Note that I checked, that changing the abscissa on the local slopes graphs from $1/t$ to $\ln(t)/t$ does not eliminate the differences between small and large $d$ behaviors. Also by assuming strong correction to scaling I could not get coherent estimates for $\beta$ and $\delta$ that would be in agreement with DP or PC classes. This value of $\delta$ is in agreement with that of the coupled DP+annihilation random walk suggested by [23]. This model was proposed to explain the space-time behavior of the AF with the help of a multi-component model, where $A$ particles (corresponding to pairs in AF) perform BARW process: $A \to 2A$, $A \to 0$ and occasionally create $B$ particles (corresponding to lonely particles in AF) who follow random walk + annihilation : $2B \to A$ (ARW). The space-time evolution picture of this model looks similar to that of AF (and very different from that of BARW2) but the numerical simulations predicted somewhat different exponents as those of the AF [23,26,29,30]. By approaching $d = 1$ corrections to scaling are getting stronger and this might leaded to the conclusion of [23] that for $d > 0.3$ the transition is of first order. The present study can not confirm those predictions based on extrapolations to DMRG results of small system sizes ($L < 100$). The first order transition for $d > 0.3$ appears to be unlikely also because the mean-field approximation of the model exhibits continuous phase transition (with $\beta_{MF} = 1$) and that corresponds to the strong diffusion limit without fluctuations.

The differences from PC class and from first order transition can be seen even more clearly by direct measurements of the order parameter $\beta$ exponent ($\rho(\infty) \propto \epsilon^\beta$). By looking at the effective exponent defined as

$$\beta_{eff}(\epsilon) = \frac{\ln \rho(\epsilon) - \ln \rho(\epsilon_{-1})}{\ln \epsilon - \ln \epsilon_{-1}},$$

on Fig. 2 one can observe two regions again: for $d < 0.3$ the $\beta_{eff}$ tends to 0.58(2) as $\epsilon = (p_c - p) \to 0$, while for $d > 0.3$ it goes to $\sim 0.42$.

For small diffusion rates the value 0.58 is very different from that of the PC class ($0.94(10)$) [1,22]) and from that of DP class ($0.2765(1)$) [7], but is in agreement with preliminary simulation results of [20] and [34]. For strong diffusion it coincides fairly well with the value 0.38(6) that was found for the coupled DP+ARW model [23].

The results for the order parameter exponent has been verified by generalized mean-field (GMF) approximations [31,32] with coherent anomaly extrapolation (CAM) [33]. This method has been proven to give precise estimates for the DP [34] and PC [2] classes. The details of the method are described in [22] and a forthcoming paper [35] will discuss the results and show estimates for other exponents as well.

For technical reasons a variant of the AF model was investigated where the creation is symmetric: $AOA \to A A A$. This does not change the essence since the $AAO/OAA \to A AA$ process can be decomposed into a $AO \to OA$ exchange and a symmetric creation. One can easily check that the mean field approximation and result of it is the same

$$\rho(\infty) = (1 - 3p)/(1 - p)$$

![FIG. 1. Local slopes of the particle density decay in the parallel update model at $d = 0.2$. Different curves correspond to $p = 0.2481, 0.24807, 0.24805, 0.24802, 0.248, 0.24795$ (from bottom to top). The simulations were performed on a ring of size $L = 24000$ up to $4 \times 10^5$ MCS. Throughout the whole paper $t$ is measured in units of Monte-Carlo sweeps.](image1)

![FIG. 2. The same as Fig. 1 at $d = 0.5$. Different curves correspond to $p = 0.2796, 0.27958, 0.27955, 0.2795$ (from bottom to top).](image2)
Higher order approximations for $N = 2, 3, 7$ converge to the simulation results of this variant (Fig. 3) and the phase diagram qualitatively agrees with that of ref. [23]. The leading order singularity at critical point remains mean-field type i.e. $\beta_{GMF} = 1$. Furthermore dynamical MC simulation of this variant gives the same critical decay behavior as the original AF model. According to CAM the amplitudes of these singularities $a(N)$ scale in such a way that

$$a(N) \propto (p_c(N) - p_c)^{\beta - \beta_{MF}}$$

the exponent of true singular behavior can be estimated. The CAM extrapolation has been applied for the highest levels of approximations ($N = 4, 5, 6, 7$) and resulted in $\beta_{CAM}$ values is agreement with those of simulations (see summarizing Table I). Note, that the CAM extrapolation does not work for $d > 0.5$ and $d < 0.1$, because corrections to scaling are getting strong, and it is very difficult to find the steady state solutions of the GMF equations for $N > 7$.

Time dependent simulations from a single active seed for these models have been proven to be a very efficient method [36] since the slowly spreading clusters do not exceed the allowed system sizes hence do not feel finite size effects. Very precise estimates for the cluster "mass" exponent $N(t) \propto t^\eta$ and the cluster survival probability exponent $P(t) \propto t^{-\delta'}$ have been obtained in DP and PC classes (see references in [37]). However example in the case of systems with infinitely many absorbing states these exponents seem to depend on initial conditions [8,9]. A rigorous explanation is still missing. The simulation results for the cluster mass (Fig. 5) show strong dependence of $\eta$ on $d$. For $d = 0.5$ and $d = 0.9$, where $\beta$ and $\delta$ are roughly the same it changes from 0.23(2) to 0.48(1). The survival probability exponent $\delta'$ is roughly zero, expressing the fact that a single particle survives with finite probability (see table I).

The continuously changing exponent $\eta$ is in agreement with the preliminary observations of [26], however $\beta$ and the $\delta$ seem to be constant in the region: $0.05 \leq d \leq 0.2$,

| $d$ | 0.05 | 0.1 | 0.2 | 0.5 | 0.9 |
|-----|------|-----|-----|-----|-----|
| $p_c$ | 0.25078 | 0.24889 | 0.24802 | 0.27955 | 0.4324 |
| $\beta_{CAM}$ | -0.58(6) | -0.58(2) | 0.42(4) | - | - |
| $\beta$ | 0.57(2) | 0.58(1) | 0.58(1) | 0.4(2) | 0.39(2) |
| $\delta$ | 0.273(2) | 0.275(4) | 0.268(2) | 0.21(1) | 0.20(1) |
| $\eta$ | 0.10(2) | - | 0.14(1) | 0.23(2) | 0.48(1) |
| $\delta'$ | 0.004(6) | - | 0.004(6) | 0.008(9) | 0.01(1) |

TABLE I. Summary of results. The non-universal critical parameter $p_c$ of the parallel model is shown here.
and $0.5 \leq d \leq 0.9$, suggesting two different universality classes.

The non-universal behavior of the cluster exponent may not be so surprising since it was also observed in the $d = 0$ limit of the AF in the pair contact process $\mu$.

In conclusion numerical simulations and analytical GMF+CAM calculations suggest two new universality classes in the AF model along the phase transition line. For small diffusion single particles cannot escape clusters where pair annihilation and creation dominates. For larger diffusions these single particles will wander among pair creation-annihilation clusters and the coupled BARW + ARW description of $\delta$ will be valid. These new universal behaviors may sound uneasy following two decades when the DP hypothesis has been found to describe almost every continuous transition to absorbing state, but do not contradict to it at all. In the AF model the absorbing state is not singlet, there are two (non symmetric) absorbing states where the system can evolve with equal probabilities. One of them is not completely frozen, but a single particle diffuses in it. My results do not support the conclusions of $\eta$ for a PC class transition without BARW2 process and $Z_2$ symmetry nor the first order transition for large $d$, but the exponents do not contradict to those of $\delta$ within numerical precision. It is likely that the two exponent ratios determined by $\eta$ are close to those of PC class accidentally. The cluster exponent $\eta$ behaves non-universally like in the pair contact process, the $d = 0$ limit of the AF. Deeper understanding of this model that would reveal hidden symmetries responsible for the strange critical behavior of AF would be highly desirable.

Acknowledgements:

The author would like to thank E. Carlon, P. Grassberger, M. Henkel, H. Hinrichsen and J. F. F. Mendes for helpful discussions. Support from Hungarian research fund OTKA (Nos. T-25286 and T-23552) and from Bolyai (No. BO/00142/99) is acknowledged. The simulations were performed on Aspex’s System-V parallel processing system (www.aspex.co.uk).

[1] R. Dickman and T. Tomé, Phys. Rev. A 44, 4833 (1991);
[2] N. Menyhárd and G. Ódor, J. Phys. A 28, 4505 (1995).
[3] G. Ódor and A. Szolnoki, Phys. Rev. E 53, 2231 (1996);
[4] N. Menyhárd, G. Ódor, J. Phys. A 31, 6771 (1998).
[5] K. Oerding, F. van Wijland, J.-P. Leroy and H. J. Hilhorst, cond-mat/9910351.
[6] H. K. Janssen, Z. Phys. B 42, 151 (1981).
[7] P. Grassberger, Z. Phys. B 47, 365 (1982).
[8] I. Jensen and R. Dickman, Phys. Rev. E 48, 1710 (1993); I. Jensen, Phys. Rev. Lett. 70, 1465 (1993).
[9] J. F. F. Mendes, R. Dickman, M. Henkel and M. C. Marques, J. Phys. A 27, 3019 (1994).
[10] H. Takayasu and A. Yu. Tretyakov, Phys. Rev. Lett. 68, 3060, (1992).
[11] I. Jensen, Phys.Rev.E 50, 3623 (1994).
[12] D. ben Avraham, F. Leyvraz and S. Redner, Phys. Rev. E 50, 1843 (1994).
[13] J. Cardy and U. Täuber, Phys. Rev. Lett. 77, 4780 (1996).
[14] P. Grassberger, F. Krause and T. von der Twer, J. Phys. A:Math.Gen., L105 17 (1984).
[15] N. Menyhárd, J.Phys.A:Math.Gen., 27, 6139(1994)
[16] H. M. Kim and H. Park, Phys. Rev. Lett. 73, 2579, (1994).
[17] H. Hinrichsen, Phys. Rev. E 55, 219 (1997).
[18] K. E. Bassler and D. A. Browne, Phys. Rev. Lett. 77, 4094 (1996).
[19] H. Park and H. Park, Physica A 221, 97 (1995).
[20] N. Menyhárd and G. Ódor, J.Phys.A:Math.Gen. 29, 7739 (1996).
[21] G. Ódor and N. Menyhárd Phys. Rev. E 57 (1998) 5168.
[22] N. Menyhárd N. and G. Ódor, cond-mat/0001101
[23] E. Carlon, M. Henkel, and U. Schollwöck, Critical properties of the reaction-diffusion model $2A \to 3A, 2A \rightarrow \emptyset$, preprint cond-mat/9912347
[24] M. J. Howard and U. C. Täuber, J. Phys. A 30, 7721 (1997).
[25] I. Peschel, X. Wang, M. Kaulke, and K. Hallberg, editors, Density-Matrix Renormalization, Berlin, 1999, Springer.
[26] H. Hinrichsen, preprint cond-mat/0001177.
[27] G. Ódor, A. Krikelis, G. Vesztergombi and F. Rohrbach: Properties of the reaction-diffusion model $2A \to 3A, 2A \rightarrow \emptyset$, preprint cond-mat/9912347.
[28] H. Hinrichsen, Cyclically coupled spreading and pair annihilation, private communication.
[29] P. Grassberger, private communication.
[30] J. F. F. Mendes, private communication.
[31] H.A. Gutowitz, J.D. Victor and B.W. Knight, Physica 28D, (1987), 18
[32] R. Dickman, Phys.Rev. A38 (1988) 2588.
[33] M. Suzuki, J. Phys. Soc. Jpn. 55, 4205 (1986).
[34] G. Ódor, Phys. Rev. E51 (1995) 6261; G. Szabó and G. Ódor Phys.Rev. E59 (1994) 2764.
[35] G. Ódor in preparation.
[36] P. Grassberger and A. de la Torre, Ann.Phys.(NY) 122, 373 (1979).
[37] J. Marro and R. Dickman, Nonequilibrium phase transitions in lattice models, Cambridge University Press, Cambridge, 1999.