THE TRIVIALITY BOUND ON THE HIGGS MASS; ITS VALUE AND WHAT IT MEANS.

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Abstract
Older lattice work exploring the Higgs mass triviality bound is briefly reviewed. It indicates that a strongly interacting scalar sector in the minimal standard model cannot exist; on the other hand low energy QCD phenomenology might be interpreted as an indication that it could. We attack this puzzle using the $1/N$ expansion and discover a simple criterion for selecting a lattice action that is more likely to produce a heavy Higgs particle. Depending on the precise form of the limitation put on the cutoff effects, our large $N$ calculations, when combined with old numerical data, suggest that the Higgs mass bound might be around 750 GeV, which is higher than the $\sim 650$ GeV previously obtained. Preliminary numerical work indicates that an increase of at least 19% takes place at $N = 4$ on the $F_4$ lattice when the old simple action is replaced with a new action (still containing only nearest neighbor interactions) if one uses the lattice spacing as the physical cutoff for both actions. It appears that, while a QCD like theory could produce $M_H/F \sim 6$, a meaningful “minimal elementary Higgs” theory cannot have $M_H/F \gtrsim 3$. Still, even at 750 GeV, the Higgs particle is so wide ($\sim 290$ GeV), that one cannot argue any more that the scalar sector is weakly coupled.

OVERVIEW.

The aim of this section is to communicate the logical framework of “triviality” in as precise a fashion as possible and restrict the presentation of details to only the most important numbers. We believe that the logical framework has reached maturity but the specific numbers might still fluctuate by several percent before settling down within a year or so.

Consider a model with a scalar field transforming under an internal $O(4)$ symmetry group. The system is in the broken phase with three pions ($\pi$) and one unstable massive particle, denoted by $\sigma$ or $H$. The pion decay constant is denoted by $F$ ($F = 246$ GeV) and the scalar selfcoupling is defined by

$$g = \frac{3M_\sigma^2}{F^2},$$

where $M_\sigma$ is the real part of the complex pole representing the Higgs particle. The model also has an ultraviolet cutoff $\Lambda$.

When $g$ is small we have

$$\frac{M_\sigma}{\Lambda} \approx Cg^{\frac{1}{2}}e^{-\frac{1}{3}g} (1 + O(g)),$$
where
\[ b_1 = \frac{4}{(4\pi)^2}, \quad b_2 = -\frac{26}{3(4\pi)^4} \tag{3} \]

\( C \) is a finite but unknown constant.

If we try to take \( \Lambda \to \infty \) at fixed \( M_\sigma \) and equation (2) applies, \( g \) must go to zero and hence \( \frac{M_H}{\Lambda} \to 0 \), implying a vanishing physical mass for \( H \). Conversely, if we increase \( g \) and ignore the eventual large corrections to (2), then \( \frac{M_H}{\Lambda} \) increases. We cannot tolerate increases of \( \frac{M_H}{\Lambda} \) beyond some number of order 1 (\( \frac{1}{2} \) say) because cutoff effects will become all important at energies \( E \sim 2M_\sigma \) (say) and our model will lose the predictive power typical of renormalizable theories.

Two obvious improvements need to be made to turn the above into a quantitative estimate for \( g_{\text{max}} \) and, by definition, for the “triviality” bound on the Higgs mass.

- For small \( g \) we need the number \( C \).
- For larger \( g \) we need to replace (2) by a more accurate relation.

Both improvements require nonperturbative calculations and were implemented in 1988-90 by lattice field theory methods in some particularly simple models using

1. strong coupling and renormalization group improved perturbation theory on hypercubic lattices\(^1\).
2. Monte Carlo on hypercubic lattices\(^2\).
3. Monte Carlo on the \( F_4 \) lattice\(^3\).

In all models the simplest discretization of a \(|\phi|^4\) theory was adopted and the bounds were obtained in the nonlinear field limit. In the hypercubic work the result was \( M_H \leq 650 \text{ GeV} \) if \( \frac{M_H}{\Lambda} \equiv M_Ha < \frac{1}{2} \) (\( a \) is the lattice spacing), a result recently updated\(^4\) upwards by about 5-7\%. The \( F_4 \) lattice was investigated because it differs from the hypercubic lattice in that it does not suffer from contamination by Lorentz symmetry violations at order \( \frac{1}{\Lambda^2} \) and yielded \( M_H \leq 600 \text{ GeV} \) at cutoffs similar to the ones where the bound was obtained on hypercubic lattices. On the basis of these numbers, which amount to a \( g \) of about \( \frac{2}{3} \) of the unitarity bound\(^1\), a general feeling ensued that strong scalar selfcouplings in the minimal standard model are excluded by triviality.

So far we discussed the so called “obvious” improvements. One needs to take the analysis further by two additional steps:

- The vague requirement that \( \frac{M_H}{\Lambda} \) not become too large must be turned into a precise, regularization independent restriction. First steps in this direction were taken in references 1,3 and 4.
- One needs to gain an understanding of how the triviality bound obtained under a given physical restriction depends on the precise details of the cutoff scheme. There was an awareness to the issue already in 1987\(^5\) and some exploratory study was carried out in 1988\(^6\); also the work in reference 3 had some relevance to this issue. However, a serious investigation was started only in 1991\(^7\) and is still continuing. More recently, additional results relevant to this were presented in reference 4.

Again, both of the above steps are of a nonperturbative nature. Although the calculation of cutoff effects and the imposition of the related restriction can be carried out in the loop expansion, only tree level results in particularly simple cases are known and the reliability of the expansion is not at all clear. The second step is essential because without it we cannot infer a bound on the mass of the “real” Higgs particle; assuming that the minimal standard model applies in some energy range we know that there has to be a cutoff \( \Lambda \) but it is provided by an embedding theory of an unknown nature.

We are only interested in the situation where \( \frac{M_H}{\Lambda} \) is sufficiently small so that any process at energy \( E \lesssim 4M_H \) has only small \( \frac{1}{\Lambda^2} \) corrections. We admit some “fine tuning” in
the sense that $\frac{M_H}{\Lambda}$ is small, but since we are looking for an upper bound on $M_H$, $\frac{M_H}{\Lambda}$ will never be extremely small. However, we do exclude any excessive “fine tuning” that would make some of the generic $\frac{1}{\Lambda^2}$ corrections disappear. In other words, the scalar sector is assumed to be representable to order $\frac{1}{\Lambda^2}$ by an effective action

\[ L_{\text{eff}} = L_{\text{ren}} + \frac{1}{\Lambda^2} \sum_A c_A O_A, \quad (4) \]

where the sum over $A$ is finite, $\dim O_A \leq 6$ and order $\frac{1}{\Lambda^4}$ terms are considered negligible. The set of operators $\{O_A\}$ is restricted by symmetry requirements and the coefficients $c_A$ depend on the details of the embedding theory.

The cutoff $\Lambda$ is taken as some well defined quantity in the particular regularization, for example $\Lambda = a^{-1}$ where $a$ is the length of a bond on the hypercubic lattice. The operators $O_A$ are normalized in some reasonable way, for example, demanding the matrix elements of $O_A$ to be of order 1 between states of energy $E \sim M_H$.

The restriction on the cutoff effects translates into some limits on the $c_A$’s. If we vary the bare parameters in a given regularization, subjected to the above constraints on the $c_A$’s, a maximal value for $M_H$ ensues, typically at points where the limits on the $c_A$’s are saturated. Note that this process of extremization is highly nonlinear in the bare parameters and that this nonlinearity cannot be eliminated by RG improvement.

We do not know what the true embedding theory is\(^a\) but whatever its nature might be it will generate some $M_H$ and some $c_A$’s. The same $M_H$ and the same $c_A$’s can be generated by almost any reasonable cutoff model; such a model would agree with the true theory in the sense that it will generate the same physical effects as equation (4) does to order $\frac{1}{\Lambda^2}$.

Therefore, if we look at a large enough class of cutoff models it will quite likely contain a representative of the true embedding theory at the level of equation (4). The bound on $M_H$ in this “sufficiently large” class of theories will also have to be obeyed by the “true” Higgs particle.

What is a “sufficiently large” class of theories? The minimal requirement is that it contain a sufficient number of free parameters to vary all the $c_A$’s independently. The numerical data available to date does not yet satisfy this minimal requirement.

A further refinement enters at this point: We do not care about all the independent $c_A$’s that can appear in equation (4) because some combinations of $c_A$’s do not enter the S-matrix to order $\frac{1}{\Lambda^2}$. Eliminating the redundant operators, and absorbing the $c_A$’s that correspond to operators of dimension $\leq 4$ we are left with two measurable $c_A$’s that parametrize all observable cutoff effects at order $\frac{1}{\Lambda^2}$.

Just counting parameters isn’t yet quiet satisfactory; the bound depends on the bare cutoff action in a highly nonlinear way and we may worry that the range of allowable magnitudes for the $c_A$’s is limited by some restriction on the bare action, like, for example, that the Hamiltonian is bounded from below. One needs therefore to carry out some experimentation before a responsible estimate for the triviality bound can be obtained. It also helps to develop a physical intuition for which class of actions will be more likely to generate larger Higgs masses.

If we generalize the model to an $O(N)$ model and take $N$ to infinity it turns out that restrictions on the cutoff effects in the invariant amplitude for $\pi^{-} \pi^{+}$ scattering translate into a limitation on a single parameter $c$ thus reducing effectively the number of measurable

\(^a\)We also don’t know that one at all exists for the minimal standard model with an elementary Higgs, but this we have assumed.
parameters $c_A$ by one. The situation becomes as simple as it only could get and a comprehensive analysis of the bound was carried out recently in this framework\(^9\).

Combining the results from that investigation with presently available $N = 4$ numerical data, rough estimates for the $1/N$ corrections were obtained and approximate values for $N = 4$ bound were derived. For the Higgs mass it was found that a not too conservative bound is $M_H \leq 750 \, \text{GeV}$ with an expected accuracy of about 7\%. At 750 GeV combining $N = \infty$ results with a rough guess for the finite $N$ correction one gets an estimate for the width of 290 GeV significantly larger than the tree level result of 210 GeV.

This indicates that triviality does not quite exclude strong scalar selfinteractions in the minimal standard model. Still, triviality does rule out anything even remotely close to a QCD like theory where $\frac{M_f}{F} \sim 6$; for example there is no known instance of a regularized reasonable $|\phi|^4$ theory with $\frac{M_f}{F} = 4$ and cutoff effects no larger than about 4\% on $\pi-\pi$ scattering at CM energies of up to $2M_\pi$.

SOME EXAMPLES.

In this section several explicit examples of both numerical and analytical results will be described in some detail.

The typical result of a Monte Carlo or strong coupling analysis will look like the discrete points in Figure 1. For comparison we also show there the $N = \infty$ result rescaled to $N = 4$. The horizontal axis, labeled by $m_H$ gives the Higgs mass in lattice units, i.e. $m_H = \frac{M_H}{\Lambda} \approx aM_H$. Typically, cutoff effects on pion scattering become of the order of a few percent when $m_H \approx 0.5$. The vertical axis then gives the physical Higgs mass in units of $F = 246 \, \text{GeV}$. The parameter $\beta_2$ mentioned in the figure is a new parameter which when set to zero implies an action of the simplest type; such actions have been already quite thoroughly investigated.

The measurements entering the creation of graphs like in Figure 1. deal with two quantities: $f_\pi = F/\Lambda$ and $m_H = M_H/\Lambda$. The pion decay constant in lattice units is, in turn, determined by two other quantities: the wave function normalization of the pions, evaluated at zero momentum, and the vacuum expectation value of the order parameter.

The wave function normalization constant for the pions is typically determined by fitting the lattice pion propagator. The vacuum expectation value is extracted by direct measurement of the expectation value of the square of the order parameter and extrapolated to in-
finite volumes using soft pion induced finite volume corrections. The method also provides a check on the wave function normalization constant.

The mass is extracted by direct measurement of correlation decay in the imaginary time direction; this works in spite of the instability of the scalar due to kinematical constraints holding in sufficiently small volumes: The small volume makes the lightest pion pair heavy enough to prohibit the decay. This method is obviously imperfect and one needs independent checks on its results.

Typical checks are done by fits to the scalar propagator or by use of results of $\beta$–functions to extrapolate across the transition from the symmetric phase where measurements are cleaner conceptually and the strong coupling expansion is also available. The first nonperturbative results were obtained with this latter method.$^1$ Note that the finite volume also obscures the difference between the pion components and the scalar components and the field representatives of these particles have to be chosen with some care. The method of reference 1 never needs to employ finite volumes and agreement with its results provides a check on the validity of the infinite volume extrapolation one carries out in the Monte Carlo data analysis.

There is also a possibility to replace the measurement of $m_H$ by the measurement of a differently defined selfcoupling which is more convenient.$^8$ This leaves one with the problem to connect that coupling to $m_H / f_\pi$ which might be attempted in perturbation theory.$^4$ To date, the single method that seems to be able to allow (indirect) mass measurements in the case where the Higgs particle is expected to be really wide is that of reference 4. With all methods it is the overall consistency of the numbers one obtains that adds up to a believable result rather than the strength of one particular approach over others.

Cutoff effects are calculated analytically; in Figure 2, we show an example computed in the $1/N$ expansion for the $F_4$ lattice. Tree level estimates, where known, behave similarly. The plotted quantity is the percentage cutoff effect on the invariant $\pi-\pi$ scattering amplitude squared at various ratios of the center of mass energy to the Higgs mass.

There are particular simplifications that occur at infinite $N$ that make the calculation easier. One of these simplifications is that at infinite $N$ one can write down nontrivial universal expressions (for $\pi-\pi$ scattering for example) that have no cutoff dependence at all but are nonperturbatively dependent on the coupling constant. Triviality tells us that these expression cannot be made to represent the finite cutoff theory more and more accurately with an increasing cutoff and this is seen in the explicit formulae relating the bare parameters to the renormalized ones.

The so called universal expressions also know about triviality, and display unphysical tachyonic poles at energies of the order of the cutoff, again signaling the poorness of the approximation at higher energies. At low energies the universal expressions can be used however to unambiguously separate out the cutoff contribution to observables. Explicit calculation of these cutoff effects to order $1/\Lambda^2$ to $\pi-\pi$ scattering yields formulae that neatly factorize for all momenta into one parameter that depends on the bare action, but not on the coupling, and a universal factor that carries all the dependence on the momenta and couplings. This factor is common to all regularizations and we have checked several. The factorization is clear if one measures all momenta in units of one scale, the pion decay constant being the most natural choice.

From Figure 2, we see that the cutoff effect increases when the model is required to hold in
Figure 2. Leading order cutoff effects in the invariant $\pi - \pi$ scattering amplitude at 90° for the naïve action ($\beta_2 = 0$) and for the action with a four derivative term turned on to maximal allowed strength ($\beta_2 = -\beta_{2,\text{t.c.}}$). The dotted line represents center of mass energies $W = 2M_H$, the dashed line $W = 3M_H$ and the solid line $W = 4M_H$.

a larger energy range. The figure also shows that by changing the bare action the cutoff effects at fixed physical Higgs mass can be decreased, increasing the triviality bound. The cluster of lines labeled by $\beta_2 = 0$ corresponds to the simplest action for which numerical information is already available.

The horizontal line is in units of the physical Higgs mass and we see that the $N = \infty$ number at 4% violation is $\sim 680 \text{ GeV}$ with the simplest action. However, when we compare a graph like the one in Figure 1. to the data, we see that, at fixed mass in lattice units ($m_H$) the large $N$ result has a tendency to overestimate the $N = 4$ numbers; this has to be taken into account and is supposed to reduce the bound for the simplest action from 680 GeV at $N = \infty$ to 600 GeV at $N = 4$.

The most important thing to pay attention to is the amount of increase induced when the action is varied. One sees from Figure 2. that an increase of about 80 GeV in the bound is expected at $N = \infty$ and this is approximately confirmed at $N = 4$ by our preliminary numerical work. Our preliminary data is presented in Figure 3.

Variations between different actions are predicted by large $N$ to within less than a factor of 2 and always with the correct sign. This can be checked in several cases already and is useful to make projections of future results and guide the decision for what to simulate.

An important ingredient of the large $N$ study is that it helped us develop an intuitive understanding for the way the bound depends on the bare action. Firstly the bound appears always to be attainable in the nonlinear limit; this was already observed with the simplest actions and is not surprising because the bare coupling is maximized in this case. When the regularized model is nonlinear one can think about the Higgs resonance as a loose bound state of two pions in an $I = 0$, $J = 0$ state.
Pions in such a state attract because superposing the field configurations corresponding to individual pions makes the state look more like the vacuum and hence lowers the energy.

To see what the action tries to do expand it in slowly varying fields to the form

$$S_c = \int x \left[ \frac{1}{2} \vec{\phi} \cdot \left( -\partial^2 + 2b_0 \partial^4 \right) \vec{\phi} - \frac{b_1}{2N} \left( \partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi} \right)^2 - \frac{b_2}{2N} \left( \partial_\mu \vec{\phi} \cdot \partial_\nu \vec{\phi} - \frac{1}{4} \delta_{\mu,\nu} \partial_\sigma \vec{\phi} \cdot \partial_\sigma \vec{\phi} \right)^2 \right], \tag{5}$$

where $\vec{\phi}^2 = N\beta$.

There are four control parameters in the effective action of equation (5) but one is redundant since to this order it can be absorbed into the others by a field redefinition: The parameter $b_0$ can be absorbed into $b_1$ and $b_2$ by:

$$\vec{\phi} \rightarrow \frac{\vec{\phi} - b_0 \partial^2 \vec{\phi}}{\sqrt{\vec{\phi}^2 + b_0^2 (\partial^2 \vec{\phi})^2 - 2b_0 \vec{\phi} \partial^2 \vec{\phi}}} \sqrt{N\beta} \tag{6}$$

The four derivative term in the action can add or subtract to the pion–pion attraction in the $I = J = 0$ state. The smallest cutoff effects are obtained when the coupling $b_1$ of the four derivative term is set so that the term induces the maximal possible repulsion between the pions, postponing the appearance of the Higgs resonance to higher energies. This corresponds to trying to make $b_1$ as negative as possible; there is a limitation in trying to do so because a too large $b_1$ may induce a tendency for translational invariance to break spontaneously by overly enhancing some nonzero momentum mode of the field. This limitation was taken into account implicitly in all cases that we investigated at $N = \infty$.

This physical picture essentially describes the most relevant gross features of the phase diagrams of all the regularized models that we have studied at infinite $N$. The generic structure is displayed in Figure 4; and the main feature is that there is a tricritical point on the ordinary symmetry breaking line. To find the bound one has to be in the vicinity of a second order transition point that is as far away as possible from this tricritical point. The tricritical point is explained by corresponding to so much attraction between the two pions in the $I = J = 0$ state that they make a massless bound state, which condenses and couples to the energy momentum tensor, thus playing the role of a dilaton.

The tricritical point in the parameterization used in Figure 2. is at $\beta_2 = \beta_{2,t.c.}$ just opposite in sign to the point where the highest Higgs mass is obtained.

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Figure 4. Generic phase diagram containing qualitative features common to all models investigated.

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