Some results on the reduced power graph of a group

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Abstract

Let $G$ be a group. The reduced power graph $\mathcal{R}P(G)$ of $G$ is the graph with vertex set $G$ and two vertices $u$ and $v$ are adjacent if and only if $u \neq v$ and $\langle v \rangle \subset \langle u \rangle$ or $\langle u \rangle \subset \langle v \rangle$. The proper reduced power graph $\mathcal{R}P^*(G)$ of $G$ is the subgraph of $\mathcal{R}P(G)$ induced on $G \setminus \{e\}$. In this paper, we classify the finite groups whose reduced power graph (resp. proper reduced power graph) is one of complete $k$-partite, acyclic, triangle free or claw-free (resp. complete $k$-partite, acyclic, triangle free, claw-free, star or tree). In addition, we obtain the clique number and the chromatic number of $\mathcal{R}P(G)$ and $\mathcal{R}P^*(G)$ for any torsion group $G$. Also, for a finite group $G$, we determine the girth of $\mathcal{R}P^*(G)$. Further, we discuss the cut vertices, cut edges and perfectness of these graphs. Then we investigate the connectivity, the independence number and the Hamiltonicity of the reduced power graph (resp. proper reduced power graph) of some class of groups.

Keywords: Reduced power graph, Groups, Clique number, Chromatic number, Independence number, Connectivity, Perfect graph, Hamiltonian graph.

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1 Introduction

Algebraic graph theory deals with the interplay between algebra and graph theory. The investigation of the algebraic properties of groups or rings by using the study of a suitably associated graph is a useful technique. One of the graph that has attracted considerable attention of the researchers is the power graph associated with a group. Kelarev and Quinn in [8] introduced the directed power graph of a semigroup. Later, Chakrabarty et al. in [6] defined the undirected power graph of a semigroup. We refer the reader to [4, 9] and the survey paper [7] for the detailed research on the power graphs associated to groups and semigroups. The \( \overrightarrow{P}(S) \) of a semigroup \( S \), is a digraph with vertex set \( S \), and for \( u, v \in S \), there is an arc from \( u \) to \( v \) if and only if \( u \neq v \) and \( v = u^n \) for some positive integer \( n \), or equivalently \( u \neq v \) and \( \langle v \rangle \subseteq \langle u \rangle \). The undirected power graph \( P(S) \) of \( S \) is the underlying graph of \( \overrightarrow{P}(S) \). Given a group \( G \), the directed proper power graph \( \overrightarrow{P}^*(G) \) of \( G \) (resp. undirected proper power graph \( P^*(G) \) of \( G \)) is defined as the subdigraph of \( \overrightarrow{P}(G) \) (resp. the subgraph of \( P(G) \)) induced on \( G \setminus \{e\} \), where \( e \) denotes the identity element of \( G \).

In order to reduce the complexity of the arcs and edges involved in the above graphs, in [10], the authors defined the following graphs: Let \( S \) be a semigroup. The directed reduced power graph \( \overrightarrow{RP}(S) \) of \( S \) is the digraph with vertex set \( S \), and for \( u, v \in S \), there is an arc from \( u \) to \( v \) if and only if \( u \neq v \), \( v = u^n \) for some positive integer \( n \) and \( \langle v \rangle \neq \langle u \rangle \), or equivalently \( u \neq v \) and \( \langle v \rangle \subset \langle u \rangle \). The (undirected) reduced power graph \( RP(S) \) of \( S \) is the underlying graph of \( \overrightarrow{RP}(S) \). That is, the vertex set of \( RP(S) \) is \( S \) and two vertices \( u \) and \( v \) are adjacent if and only if \( u \neq v \) and \( \langle v \rangle \subset \langle u \rangle \) or \( \langle u \rangle \subset \langle v \rangle \). Given a group \( G \), the directed proper reduced power graph \( \overrightarrow{RP}^*(G) \) of \( G \) (resp. (undirected) proper reduced power graph \( RP^*(G) \) of \( G \)) is defined as the subdigraph of \( \overrightarrow{RP}(G) \) (resp. the subgraph of \( RP(G) \)) induced on \( G \setminus \{e\} \).

Clearly, \( \overrightarrow{RP}(G) \) is a spanning subdigraph of \( \overrightarrow{P}(G) \). It is proved in [10] that the directed reduced power graph of a group \( G \) can be determined by its set of elements orders and consequently, some class of groups have determined by their directed reduced power graphs. This shows the importance of these graphs to the theory of groups. Moreover, some results
on the characterization of groups whose reduced power graph (resp. proper reduced power graph) is one of the following: complete graph, path, star, tree, bipartite, triangle-free (resp. complete graph, totally disconnected, path, star) were also established.

Though out this paper, we denote the set of all prime numbers by \( \mathbb{P} \). We use the standard notations and terminologies of graph theory following [1, 5].

The rest of the paper is organized as follows: In Section 2, we explore the structure of the reduced power graph and the proper reduced power graph of a group. We show that the reduced power graph of a torsion group (resp. proper reduced power graph) is \( (\Omega(n) + 1) \)-partite (resp. \( \Omega(n) \)-partite). In addition, we obtain the clique number, and the chromatic number of the reduced power graph (resp. proper reduced power graph) of any torsion group. As a consequence of this, we show that among all the reduced power graphs (resp. proper reduced power graphs) of finite groups of given order \( n \), \( \mathcal{RP}(\mathbb{Z}_n) \) (resp. \( \mathcal{RP}^*(\mathbb{Z}_n) \)) attains the maximum clique number and the maximum chromatic number. Then we classify the groups whose reduced power graphs (resp. proper reduced power graphs) is one of complete \( k \)-partite or triangle-free (resp. complete \( k \)-partite, or triangle-free, star or tree). Also, for a finite group \( G \), we determine the girth of \( \mathcal{RP}^*(G) \).

In Section 3, we show that for any finite group \( G \), \( \mathcal{RP}(G) \) and \( \mathcal{RP}^*(G) \) are perfect and not a cycle. Also we investigate the acyclicity and claw-freeness of \( \mathcal{RP}(G) \) and \( \mathcal{RP}^*(G) \) for a finite group \( G \).

In Section 4, we discuss the cut vertices and cut edges of \( \mathcal{RP}(G) \) and \( \mathcal{RP}^*(G) \) for a finite group \( G \).

In Section 5, we estimate the connectivity and the independence number of reduced power graph (resp. proper reduced power graph) of the following groups: finite cyclic groups, dihedral groups, quaternion groups, semi-dihedral groups and finite \( p \)-groups. Also we study the Hamiltonicity of these graphs.
2 \( k \)-partiteness, Clique number, Chromatic number, Girth

For a given graph \( \Gamma \), \( deg_{\Gamma}(v) \) denotes the degree of a vertex \( v \) in \( \Gamma \). The girth \( gr(\Gamma) \) of \( \Gamma \) is the length of shortest cycle in \( \Gamma \). The clique number \( \omega(\Gamma) \) of \( \Gamma \) is the maximum size of a complete graph in \( \Gamma \). The chromatic number \( \chi(\Gamma) \) of \( \Gamma \) is the minimum number of colors needed to color the vertices of \( \Gamma \) such that no two adjacent vertices gets the same color. If \( \omega(\Gamma) = \chi(\Gamma) \), then \( \Gamma \) is said to be weakly \( \chi \)-perfect. \( K_n \) denotes the complete graph on \( n \) vertices. \( P_n \) and \( C_n \), respectively denotes the path and cycle of length \( n \). The complete \( r \)-partite graph having partition sizes \( n_1, n_2, \ldots, n_r \) is denoted by \( K_{n_1,n_2,\ldots,n_r} \).

First, we start to investigate the structure of the proper reduced power graph of a group.

Let \( G \) be a torsion group. The order of an element \( x \) in \( G \) is denoted by \( o(x) \). Note that, if two vertices \( x \) and \( y \) in \( \mathcal{RP}^*(G) \) (resp. \( \mathcal{RP}(G) \)) are adjacent, then \( o(x) \mid o(y) \) or \( o(y) \mid o(x) \) according as \( \langle x \rangle \subset \langle y \rangle \) or \( \langle y \rangle \subset \langle x \rangle \). We frequently use the contrapositive of this to show the non adjacency of two vertices in \( \mathcal{RP}^*(G) \) (resp. \( \mathcal{RP}(G) \)). Let \( \pi_e(G) \) denotes the set of elements orders of \( G \). Let \( n \in \pi_e(G) \) be such that \( \Omega(n) \) is maximum, where \( \Omega(n) \) denotes the number of prime factors of \( n \) counted with multiplicity. For each \( i = 1, 2, \ldots, \Omega(n) \), let

\[
X_i := \{ x \in G \mid \Omega(o(x)) = i \},
\]

Clearly, \( \{X_i\}_{i=1}^{\Omega(n)} \) is a partition of the vertex set of \( \mathcal{RP}^*(G) \). Also for any \( x_1, x_2 \in X_i \), either \( o(x_1) = o(x_2) \) or \( o(x_1) \) and \( o(x_2) \) does not divide each other. Consequently, no two elements in the same partition are adjacent in \( \mathcal{RP}^*(G) \). Let \( x \in G \) be such that \( \Omega(o(x)) = \Omega(n) \). Let \( o(x) = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k} \), where \( p_i \)'s are distinct primes, \( \alpha_i \geq 1 \) for all \( i \). Then for each \( i = 1, 2, \ldots, \Omega(n) \), \( X_i \) has the elements in \( \langle x \rangle \) whose order is \( p_1^{\beta_1}p_2^{\beta_2} \cdots p_k^{\beta_k} \) with \( 0 \leq \beta_j \leq \alpha_j \) and \( \beta_1 + \beta_2 + \cdots + \beta_k = i \). For each \( i = 1, 2, \ldots, \Omega(n) \), we choose \( x_i \in \langle x \rangle \) as follows: First, we fix an element \( x_1 \) in \( X_1 \) of order \( q_1 \), where \( q_1 \in \{p_1,p_2,\ldots,p_k\} \). If \( \Omega(n) = 1 \), then we stop the procedure. If \( \Omega(n) \geq 2 \), then we choose an element \( x_2 \) in \( X_2 \) of order \( q_1q_2 \), where \( q_2 \in \{p_1,p_2,\ldots,p_k\} \) such that \( q_1q_2 \) divides \( o(x) \). Proceeding in this way, at the \( i^{th} \) stage \( (i = 3, 4, \ldots, \Omega(n)) \), we choose an element \( x_i \) in \( X_i \) of order \( q_1q_2 \cdots q_i \), where \( q_i \in \{p_1p_2,\ldots,p_k\} \) such that \( q_1q_2 \cdots q_i \) divides \( o(x) \). Then \( \langle x_r \rangle \subset \langle x_s \rangle \), where \( 1 \leq r < s \leq \Omega(n) \).
Let $G$ be a torsion group and let $n \in \pi_e(G)$ be such that $\Omega(n)$ is maximum. Then

(1) $\mathcal{RP}^*(G)$ is $\Omega(n)$-partite;

(2) $\omega(\mathcal{RP}^*(G)) = \Omega(n) = \chi(\mathcal{RP}^*(G))$; and so $\mathcal{RP}^*(G)$ is weakly $\chi$-perfect.

**Corollary 2.1.** Let $G$ be a torsion group and let $n \in \pi_e(G)$ be such that $\Omega(n)$ is maximum. Then

\[ \Omega(n) \text{ and so } x_1, x_2, \ldots, x_{\Omega(n)} \text{ forms a clique in } \mathcal{RP}^*(G). \text{ Also, since } \Omega(n) \text{ is maximum, it follows that } \Omega(n) \text{ is the minimal number such that } \mathcal{RP}^*(G) \text{ is a } \Omega(n)\text{-partite graph and so } \omega(\mathcal{RP}^*(G)) = \Omega(n). \text{ This implies that } \chi(\mathcal{RP}^*(G)) = \Omega(n). \]

The above procedure is illustrated in Figure 1 which shows the structure of $\mathcal{RP}^*(D_{24})$, where $a$ and $b$ are the generators of $D_{24}$.

The above facts are summarized in the following result.

**Theorem 2.1.** Let $G$ be a torsion group and let $n \in \pi_e(G)$ be such that $\Omega(n)$ is maximum. Then

(1) $\mathcal{RP}^*(G)$ is $\Omega(n)$-partite;

(2) $\omega(\mathcal{RP}^*(G)) = \Omega(n) = \chi(\mathcal{RP}^*(G))$; and so $\mathcal{RP}^*(G)$ is weakly $\chi$-perfect.

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\[ \Omega(n) \text{ and so } x_1, x_2, \ldots, x_{\Omega(n)} \text{ forms a clique in } \mathcal{RP}^*(G). \text{ Also, since } \Omega(n) \text{ is maximum, it follows that } \Omega(n) \text{ is the minimal number such that } \mathcal{RP}^*(G) \text{ is a } \Omega(n)\text{-partite graph and so } \omega(\mathcal{RP}^*(G)) = \Omega(n). \text{ This implies that } \chi(\mathcal{RP}^*(G)) = \Omega(n). \]

The above procedure is illustrated in Figure 1 which shows the structure of $\mathcal{RP}^*(D_{24})$, where $a$ and $b$ are the generators of $D_{24}$.

The above facts are summarized in the following result.
(1) $\mathcal{RP}(G)$ is $\Omega(n) + 1$-partite;

(2) $\omega(\mathcal{RP}(G)) = \Omega(n) + 1 = \chi(\mathcal{RP}(G))$; and so $\mathcal{RP}(G)$ is weakly $\chi$-perfect.

In the next result, we show that among all the reduced power graphs (resp. proper reduced power graphs) of finite groups of given order $n$, $\mathcal{RP}(\mathbb{Z}_n)$ (resp. $\mathcal{RP}^*(\mathbb{Z}_n)$) attains the maximum clique number and the maximum chromatic number.

**Corollary 2.2.** Let $G$ be a non-cyclic group of order $n$. Then $\omega(\mathcal{RP}(G)) < \omega(\mathcal{RP}(\mathbb{Z}_n))$ and $\chi(\mathcal{RP}(G)) < \chi(\mathcal{RP}(\mathbb{Z}_n))$. Also $\omega(\mathcal{RP}^*(G)) < \omega(\mathcal{RP}^*(\mathbb{Z}_n))$ and $\chi(\mathcal{RP}^*(G)) < \chi(\mathcal{RP}^*(\mathbb{Z}_n))$.

**Proof.** Let $m \in \pi_e(G)$ be such that $\Omega(m)$ is maximum. Then $m | n$ and since $G$ is non-cyclic, which forces $m \neq n$. This implies that $\Omega(m) < \Omega(n)$. By using this fact together with part (2) of Theorem 2.1 and Corollary 2.1 we get the result.

The classification of finite groups for which the reduced power graph is given in [10, Corollary 2.11]. As a consequence of Theorem 2.1(1), in the next result, we characterize the torsion groups whose proper reduced power graph is bipartite.

**Corollary 2.3.** Let $G$ be a torsion group. Then $\mathcal{RP}^*(G)$ is bipartite if and only if $\pi_e(G)$ has a non-prime number which is of the form either $p^2$ or $pq$, where $p, q$ are distinct primes.

The following result is an immediate consequence of Theorem 2.1(2).

**Corollary 2.4.** Let $G$ be a torsion group. Then

(1) $\mathcal{RP}^*(G)$ is triangle-free if and only if $\Omega(n) \leq 2$ for every $n \in \pi_e(G)$;

(2) $\mathcal{RP}(G)$ is triangle-free if and only if every non-trivial element of $G$ is of prime order.

In the following results we classify all finite groups whose reduced power graph (resp. proper reduced power graph) is complete $k$-partite.

**Theorem 2.2.** Let $G$ be a finite group and $p, q$ be distinct primes. Then

(1) $\mathcal{RP}^*(G)$ is complete bipartite if and only if $G \cong \mathbb{Z}_{p^2}, \mathbb{Z}_{pq}$ or $Q_8$;

(2) $\mathcal{RP}^*(G)$ is complete $k$-partite, where $k \geq 3$ if and only if $G \cong \mathbb{Z}_{p^k}$. 

Proof. By Theorem 2.1, $\mathcal{RP}^*(G)$ is $k$-partite, where $k = \Omega(n)$, $n \in \pi_e(G)$ is such that $\Omega(n)$ is maximum. Let $|G|$ has $t$ distinct prime divisors.

**Case 1.** If $t = 1$, then $|G| = p^n$, where $p$ is a prime, $n \geq 1$. Then $G$ has a maximal cyclic subgroup, say $H$, of order $p^k$ and so for each $i = 1, 2, \ldots, k$, the partition $X_i$ defined in (2.1) has all the elements in $G$ of order $p^i$. If $G$ has more than one cyclic subgroup of order $p^i$ for some $i < k$, then no non-trivial element in at least one of these subgroups is adjacent with any non-trivial element in $H$ and so $\mathcal{RP}^*(G)$ is not complete $k$-partite. If $G$ has a unique cyclic subgroup of order $p^i$ for each $i = 1, 2, \ldots, k - 1$, then by [11] Proposition 1.3, $G \cong \mathbb{Z}_{p^k}$ ($k \geq 2$) or $Q_8$. If $G \cong \mathbb{Z}_{p^k}$, then $|X_i| = p^{i-1}(p-1)$ for each $i$. So

$$\mathcal{RP}^*(\mathbb{Z}_{p^k}) \cong K_{p-1,p(p-1),\ldots,p^{k-1}(p-1)} \quad (2.2)$$

Clearly, $\mathcal{RP}^*(Q_8) \cong K_{1,6}$

**Case 2.** If $t = 2$, then $|G| = p_1^{n_1}p_2^{n_2}$, where $p_1$, $p_2$ are distinct primes, $n_1, n_2 \geq 1$.

If $G$ has an element of order $p_i^2$ for some $i = 1, 2$, then the elements in $X_2$ of order $p_i^2$ are not adjacent to any element in $X_1$ of order $p_j$ ($j \neq i$) in $\mathcal{RP}^*(G)$, so $\mathcal{RP}^*(G)$ is not complete $k$-partite. If every non-trivial element in $G$ is of prime order, then by [10] Theorem 2.9, $\mathcal{RP}^*(G)$ is totally disconnected, and so $\mathcal{RP}^*(G)$ is not complete $k$-partite. Now we assume that $G$ has an element of order $p_1p_2$. If $G$ has more than one cyclic subgroup of any one of the orders $p_1$, $p_2$ or $p_1p_2$, then $\mathcal{RP}^*(G)$ is not complete $k$-partite. If $G$ has a unique subgroup of each of the orders $p_1$, $p_2$ and $p_1p_2$, then $G \cong \mathbb{Z}_{p_1p_2}$ and in this case $\mathcal{RP}^*(\mathbb{Z}_{p_1p_2}) = K_{p_1+p_2-2,(p_1-1)(p_2-1)}$.

**Case 3.** Assume that $t \geq 3$. If $G$ has an element of order $p_ip_jp_l$, where $p_i$, $p_j$, $p_l$ are distinct prime divisors of $|G|$, then the elements in $X_2$ of order $p_ip_j$ are not adjacent to the elements in $X_1$ of order $p_l$ in $\mathcal{RP}^*(G)$, so $\mathcal{RP}^*(G)$ is not complete $k$-partite. If $G$ has an element of order $p_ip_j$, then the elements in $X_2$ of order $p_ip_j$ are not adjacent to the elements in $X_1$ of order $p_r$, where $r \neq i,j$ in $\mathcal{RP}^*(G)$, so $\mathcal{RP}^*(G)$ is not complete $k$-partite. If either $G$ has an element of order $p_i^2$ for some $i$ or every non-trivial element in $G$ of prime order, then by the argument used in Case 2, $\mathcal{RP}^*(G)$ is not complete $k$-partite.

Proof follows by combining all the above cases.
Corollary 2.5. Let $G$ be a finite group. Then $\mathcal{RP}(G)$ is complete $k$-partite, where $k \geq 3$ if and only if $G \cong \mathbb{Z}_{p^{k-1}}$.

The following three results are established in [9][10]:

**Theorem 2.3.** ([9, Corollary 4.1]) Let $G$ be a finite $p$-group, where $p$ is a prime. Then $\mathcal{P}^*(G)$ is connected if and only if $G$ is either cyclic or generalized quaternion.

**Theorem 2.4.** ([10, Theorem 2.15]) Let $G$ be a finite group of non-prime order or an infinite group. Then $\mathcal{RP}^*(G)$ is connected if and only if $\mathcal{P}^*(G)$ is connected.

**Lemma 2.1.** ([10, Lemma 2.16]) If $G$ is a group, and $x$ is a pendant vertex in $\mathcal{RP}^*(G)$, then $o(x) = 4$. Converse is true if $\langle x \rangle$ is either $\mathbb{Z}_4$ or a maximal cyclic subgroup of $G$ when $G$ is noncyclic.

The next result is an immediate consequence of Theorems 2.4 and 2.3.

**Corollary 2.6.** Let $G$ be a finite $p$-group, where $p$ is a prime. Then $\mathcal{RP}^*(G)$ is connected if and only if $G$ is either cyclic or quaternion.

**Theorem 2.5.** Let $G$ be a finite group. Then the following are equivalent.

1. $\mathcal{RP}^*(G)$ is a tree;
2. $\mathcal{RP}^*(G)$ is a star;
3. $G \cong \mathbb{Z}_4$ or $Q_8$.

**Proof.** (2) $\iff$ (3) is proved in [10, Theorem 2.17]. Clearly, (3) $\Rightarrow$ (1). Now we prove (1) $\Rightarrow$ (3). Let $|G| = p_1^{n_1}p_2^{n_2} \ldots p_k^{n_k}$, where $p_i$’s are distinct primes and $k \geq 1$, $n_i \geq 1$ for all $i = 1, 2, \ldots, k$. Since $\mathcal{RP}^*(G)$ is a tree, $\mathcal{RP}^*(G)$ has at least two pendant vertices. Then $G$ has an element of order 4, by Lemma 2.1 and this implies that $p_i = 2$ and $n_i \geq 2$ for some $i \in \{1, 2, \ldots, k\}$. Without loss of generality, we assume that $p_1 = 2$ and $n_1 \geq 2$.

**Case 1.** Let $\Omega(n) \geq 3$ for some $n \in \pi_0(G)$. Then by part (1) of Corollary 2.4 $C_3$ is a subgraph of $\mathcal{RP}^*(G)$. 

**Case 2.** Let for every \( n \in \pi_e(G) \), \( \Omega(n) \leq 2 \). If \( G \) has an element, say \( a \), of order \( p_i^2 \) \((i \neq 1)\) (resp. \( p_r p_s \) \((r \neq s)\)), then the elements of order \( p_i^2 \) (resp. \( p_r p_s \)) in \( \langle a \rangle \) together with the elements of order \( p_i \) (resp. \( p_r \) and \( p_s \)) in \( \langle a \rangle \) forms \( C_4 \) as a subgraph in \( \mathcal{RP}^*(G) \).

**Case 3.** Let \( \pi_e(G) = \{1, 4\} \cup \mathbb{P} \).

If \( k \geq 2 \), then the elements of order \( p_i \) \((i \neq 1)\) are isolated vertices in \( \mathcal{RP}^*(G) \), a contradiction to our assumption that \( \mathcal{RP}^*(G) \) is a tree. If \( k = 1 \), then \(|G| = 2^n\) with \( \pi_e(G) = \{1, 2, 4\} \). Then by Corollary 2.6, \( G \cong \mathbb{Z}_4 \) or \( Q_8 \), since \( \mathcal{RP}^*(G) \) is connected. Also, \( \mathcal{RP}^*(G) \cong K_{1,3} \) or \( K_{1,6} \), which are trees. This completes the proof.

The girth of the reduced power graph of a finite group is determined by the authors in [10, Corollary 2.12]. In the following result, we determine the girth of the proper reduced power graph of a finite group.

**Theorem 2.6.** Let \( G \) be a finite group and let \( n \in \pi_e(G) \) be such that \( \Omega(n) \) is maximum.

Then \( gr(\mathcal{RP}^*(G)) = \begin{cases} 3, & \text{if } \Omega(n) \geq 3; \\ 4, & \text{if } \Omega(n) = 2 \text{ and } \pi_e(G) \neq \{1, 4\} \cup \mathbb{P}; \\ \infty, & \text{otherwise}. \end{cases} \)

**Proof.** If \( \Omega(n) \geq 3 \), then by Corollary 2.4, \( gr(\mathcal{RP}^*(G)) = 3 \). If \( \Omega(n) = 2 \) and \( \pi_e(G) \neq \{1, 4\} \cup \mathbb{P} \), then by Theorem 2.1, \( \mathcal{RP}^*(G) \) is bipartite and \( G \) has an element of order \( p^2 \) \((p > 2)\) or \( pq \), where \( p, q \) are distinct prime factors of \(|G|\). Consequently, \( \mathcal{RP}^*(G) \) has \( C_4 \) as a subgraph. But the bipartiteness of \( G \) forces that \( gr(\mathcal{RP}^*(G)) = 4 \). If \( \Omega(n) = 2 \) and \( \pi_e(G) = \{1, 4\} \cup \mathbb{P} \), then \( \mathcal{RP}^*(G) \) is the union (not necessarily disjoint) of some copies of the graph \( K_{1,2} \) and isolated vertices. It follows that \( gr(\mathcal{RP}^*(G)) = \infty \). If \( \Omega(n) = 1 \), then \( \mathcal{RP}^*(G) \) is totally disconnected, so \( gr(\mathcal{RP}^*(G)) = \infty \). 

### 3 Perfectness, Acyclicity, Claw-freeness

The complement of a graph \( \Gamma \) is denoted by \( \overline{\Gamma} \). The join of two graphs \( \Gamma_1 \) and \( \Gamma_2 \) is denoted by \( \Gamma_1 + \Gamma_2 \). A graph \( \Gamma \) is perfect if for every induced subgraph \( H \) of \( \Gamma \), the chromatic number of \( H \) equals the size of the largest clique of \( H \). The strong perfect graph theorem [3] states that a graph is perfect if and only if neither the graph nor its complement contains an odd
cycle of length at least 5 as an induced subgraph. In the following two results we show that \( \mathcal{RP}(G) \) and \( \mathcal{RP}^*(G) \) are perfect for any finite group \( G \).

**Theorem 3.1.** For any finite group \( G \), \( \mathcal{RP}(G) \) is perfect.

*Proof.* If we show that neither \( \mathcal{RP}(G) \) nor its complement has an induced cycle of odd length at least 5, then by strong perfect graph theorem \([3]\), \( \mathcal{RP}(G) \) is perfect. Assume the contrary. Let \( C : u_1 - u_2 - \cdots - u_{2n+1} - u_1 \) be an induced cycle of odd length at least 5 in \( \mathcal{RP}(G) \). Then \( \langle u_i \rangle \neq \langle u_j \rangle, i \neq j \). This implies that \( C \) is also an induced cycle in \( \mathcal{P}(G) \), which is a contradiction to \([4\) Theorem 5], which states that \( \mathcal{P}(G) \) is perfect.

Suppose \( C' : v_1 - v_2 - \cdots - v_{2n+1} - v_1 \) be an induced cycle of odd length at least 5 in \( \mathcal{RP}(G) \). Then \( \langle v_i \rangle \neq \langle v_j \rangle, i \neq j \). Consequently, \( C' \) is also an induced cycle of odd length at least 5 in \( \mathcal{P}(G) \), since \( \mathcal{P}(G) \) is subgraph of \( \overline{\mathcal{RP}(G)} \), which is again a contradiction to the fact that \( \mathcal{P}(G) \) is perfect.

**Corollary 3.1.** For a finite group \( G \), \( \mathcal{RP}^*(G) \) is perfect.

*Proof.* The proof follows from the definition of perfect graph, Theorem 3.1 and the fact that \( \mathcal{RP}(G) = K_1 + \mathcal{RP}^*(G) \).

**Theorem 3.2.** Let \( G \) be a finite group and \( p, q \) be distinct primes. Then

1. \( \mathcal{RP}^*(G) \) is acyclic if and only if \( \pi_e(G) \subseteq \{1,4\} \cup \mathbb{P} \).

2. \( \mathcal{RP}(G) \) is acyclic if and only if \( G \) is either a p-group with exponent \( p \) or non-nilpotent group of order \( p^nq \) (\( n \geq 1 \)) with all non-trivial elements are of order \( p \) or \( q \).

*Proof.* (1) If \( \pi_e(G) \not\subseteq \{1,4\} \cup \mathbb{P} \), then by cases (1) and (2) of Theorem 2.5, \( \mathcal{RP}^*(G) \) has a cycle. Otherwise, \( \mathcal{RP}^*(G) \) is a disjoint union of some copies of star graphs and isolated vertices. So \( \mathcal{RP}^*(G) \) is acyclic.

(2) The proof follows from Corollary 2.4 (2) and by the classification of finite groups with non-trivial elements are of prime order given in \([2]\).

**Theorem 3.3.** Let \( G \) be a finite abelian group and \( p \) be a prime. Then

1. \( \mathcal{RP}^*(G) \) is acyclic if and only if \( G \cong \mathbb{Z}_p^n, \mathbb{Z}_4^n \) or \( \mathbb{Z}_4^m \times \mathbb{Z}_2^n \), where \( n, m \geq 1 \).
(2) \( \mathcal{RP}(G) \) is acyclic if and only if \( G \cong \mathbb{Z}_p^n \), where \( n \geq 1 \).

**Proof.** (1) By Theorem 3.2, it is enough to determine the finite abelian groups whose elements order set is \( \{1, 4\} \cup \mathbb{P} \). Assume that \( \pi_e(G) \subseteq \{1, 4\} \cup \mathbb{P} \). If \( G \) has elements of order \( p_1 \) and \( p_2 \), where \( p_1, p_2 \) are two distinct prime divisors of \( |G| \), then \( G \) has an element of order \( p_1 p_2 \). Consequently, \( \pi_e(G) = \{1, p\} \) or \( \{1, 2, 4\} \). In the former case we get \( G \cong \mathbb{Z}_p^n \), \( n \geq 1 \) and in the later case we get \( G \cong \mathbb{Z}_4^n \) or \( \mathbb{Z}_4^n \times \mathbb{Z}_2^n \), where \( n, m \geq 1 \).

(2) By Theorem 3.2, \( \pi_e(G) \subseteq \mathbb{P} \). If the non-trivial elements in \( G \) are of prime order, then \( G \cong \mathbb{Z}_p^n \) and so \( \mathcal{RP}(G) \cong K_{1,p^n-1} \), which is acyclic. \( \square \)

**Theorem 3.4.** Let \( G \) be a finite group. Then \( \mathcal{RP}(G) \) is claw-free if and only if \( G \cong \mathbb{Z}_n \), \( n = 2, 3, 4 \).

**Proof.** If \( G \) has an element \( a \) with \( o(a) \geq 5 \), then \( \mathcal{RP}(\langle a \rangle) \) has \( K_{1,3} \) as an induced subgraph of \( \mathcal{RP}(G) \). So we assume that the order of any element in \( G \) is at most 4. Then \( |G| \) is one of the following: \( 2^n, 3^m \) or \( 2^n 3^m, n, m \geq 1 \). Assume that \( |G| = 2^n 3^m, n, m \geq 1 \). Then the elements in \( G \) of order 2 and 3 together with the identity element in \( G \) forms \( K_{1,3} \) as an induced subgraph of \( \mathcal{RP}(G) \). Next we assume that \( |G| = p^n, p = 2 \) or 3 and \( n \geq 1 \). If \( G \) has more than one cyclic subgroup of order \( p \), then the elements of order \( p \) together with the identity element in \( G \) forms \( K_{1,3} \) as an induced subgraph of \( \mathcal{RP}(G) \). If \( G \) has a unique subgroup of order \( p \), then \( G \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4 \) or \( Q_8 \). If \( G \cong Q_8 \), then its elements of order 4 together with the element of order 2 forms \( K_{1,3} \) as an induced subgraph of \( \mathcal{RP}(G) \). In the remaining cases, we have \( \mathcal{RP}(\mathbb{Z}_2) \cong K_2, \mathcal{RP}(\mathbb{Z}_3) \cong K_{1,2} \) and \( \mathcal{RP}(\mathbb{Z}_4) \cong K_1 + K_{1,2} \), which are claw-free. \( \square \)

**Theorem 3.5.** Let \( G \) be a finite group. Then \( \mathcal{RP}^*(G) \) is claw free if and only if \( \pi_e(G) \subseteq \{1, 4\} \cup \mathbb{P} \) and any two cyclic subgroups of order 4 in \( G \) intersect trivially.

**Proof.** Let \( a \) be a non-trivial element in \( G \). If \( o(a) = p^2 \), where \( p > 2 \) is a prime, then the elements of order \( p \) and \( p^2 \) in \( \langle a \rangle \) are adjacent with each other in \( \mathcal{RP}^*(G) \) and so \( \mathcal{RP}^*(G) \) has \( K_{p-1,p(p-1)} \) as an induced subgraph. In particular, it has \( K_{1,3} \) as an induced subgraph.

If \( o(x) = 8 \), then the element of order 2 together with the elements of order 8 in \( \langle a \rangle \) forms \( K_{1,3} \) as an induced subgraph of \( \mathcal{RP}^*(G) \). Similarly, if \( o(a) = pq \), where \( p, q \) are distinct
primes, then the elements of order $pq$ together with the elements of order $p$ and $q$ in $\langle a \rangle$ forms $K_{1,3}$ as an induced subgraph of $\mathcal{RP}^*(G)$. Now we assume that $\pi_e(G) \subseteq \{1, 4\} \cup \mathbb{P}$. Let $H$ and $K$ be cyclic subgroups of $G$ of order 4. If they intersect non-trivially, then the non-trivial elements in $H \cup K$ forms $K_{1,3}$ as an induced subgraph of $\mathcal{RP}^*(G)$. If any two cyclic subgroups of order 4 in $G$ intersect trivially, then $\mathcal{RP}^*(G)$ is the disjoint union of the graph $K_{1,2}$ and isolated vertices and so $\mathcal{RP}^*(G)$ is claw free.

The following result shows that a cycle can not be realized as the reduced power graph (resp. proper reduced power graph) of any finite group.

**Theorem 3.6.** Let $G$ be a finite group. Then $\mathcal{RP}(G) \not\cong C_n$ and $\mathcal{RP}^*(G) \not\cong C_n$ for any $n \geq 3$.

**Proof.** If $\mathcal{RP}(G) \cong C_n$, $n \geq 3$, then $\deg_{\mathcal{RP}(G)}(a) = 2$ for each non-trivial element $a$ in $G$. Also $\deg_{\mathcal{RP}(G)}(e) = |G| - 1$. It follows that $|G| = 3$. Then $G \cong \mathbb{Z}_3$ and so $\mathcal{RP}(G) \cong P_3$, which is a contradiction to our assumption. If $\mathcal{RP}^*(G) \cong C_n$, $n \geq 3$, then it is claw-free. So by the proof of Theorem 3.5, $\mathcal{RP}^*(G)$ is the disjoint union of the graph $K_{1,2}$ and isolated vertices, which is a contradiction to our assumption that $\mathcal{RP}^*(G)$ is a cycle. \hfill $\square$

## 4 Cut vertices and Cut edges

A cut vertex (cut edge) of a graph is a vertex (edge) whose removal increases the number of components of the graph. A connected graph $\Gamma$ is said to be $k$—connected if there is no set of $k − 1$ vertices whose removal disconnects $\Gamma$.

Note that for any group $G$, if $x$ is a cut vertex of $\mathcal{P}^*(G)$, then $x$ is a cut vertex of $\mathcal{RP}^*(G)$, since $\mathcal{RP}^*(G)$ is a subgraph of $\mathcal{P}^*(G)$. But the converse of this statement is not true. For example, 2 is a cut vertex of $\mathcal{RP}^*(\mathbb{Z}_4)$, whereas $\mathcal{P}^*(\mathbb{Z}_4)$ has no cut vertex. In the next theorem, we show that the converse is true for all finite groups except $\mathbb{Z}_4$.

**Theorem 4.1.** Let $G$ be a finite group with $G \not\cong \mathbb{Z}_4$ and let $x$ be an element in $G$. Then $x$ is a cut vertex in $\mathcal{RP}^*(G)$ if and only if $x$ is a cut vertex in $\mathcal{P}^*(G)$.

**Proof.** Assume that $x$ is a cut vertex in $\mathcal{RP}^*(G)$. Then there exist $y, z \in G$ such that every $y − z$ path in $\mathcal{RP}^*(G)$ contains $x$. 
If $yz$ is an edge in $P^*(G)$, then $\langle y \rangle = \langle z \rangle$ and $y, z$ are pendent vertices in $RP^*(G)$. By Lemma 2.1, $o(y) = 4$ and $\langle y \rangle$ is maximal cyclic subgroup of $G$. Since $G \not\cong \mathbb{Z}_4$, $P^*(G) - \{x\}$ is disconnected and so $x$ is a cut vertex in $P^*(G)$.

Let $y$ be non adjacent to $z$ in $P^*(G)$. Suppose $x$ is not a cut vertex in $P^*(G)$, then we can choose a shortest $y - z$ path in $P^*(G)$, which does not contain $x$, let it be $P(y, z)$. Then $P(y, z) := y(= w_1) - w_2 - \cdots - w_{n-1} - (w_n =) z$, where $\langle w_i \rangle \neq \langle w_{i+1} \rangle$, $i = 1, 2, \ldots, k$. It follows that $P(y, z)$ is also a path in $RP^*(G)$ not containing $x$, a contradiction. Hence $x$ is a cut vertex in $P^*(G)$. Converse is clear.

**Corollary 4.1.** Let $G$ be a finite group and $G \not\cong \mathbb{Z}_4$. Then $RP^*(G)$ is 2–connected if and only if $P^*(G)$ is 2-connected.

Note that, for any group $G$, if $RP(G)$ has a cut vertex, then it must be the identity element in $G$. Converse of this statement is not true. For example, the identity element is not a cut vertex of $RP(\mathbb{Z}_4)$, since $RP(\mathbb{Z}_4) - \{e\} = K_{1,2}$. In the following theorem, we give a necessary condition for a vertex of $RP^*(G)$ to be its cut vertex.

**Theorem 4.2.** Let $G$ be a finite group. If an element $x$ in $G$ is a cut vertex in $RP^*(G)$, then $o(x) = 2$.

**Proof.** Let $x$ be a cut vertex in $RP^*(G)$. Then there exist $y, z \in G$ such that every $y - z$ path in $RP^*(G)$ contains $x$. Let $P(y, z)$ be a shortest $y - z$ path in $RP^*(G)$. Then $P(y, z) : y(= w_1) - w_2 - \cdots - w_i - x - w_{i+1} - \cdots - (w_n =) z$, where $\langle w_r \rangle \neq \langle x \rangle$, $r = 1, 2, \ldots, n$. Suppose $o(x) \neq 2$, then there exist $x' \in G$ such that $x \neq x'$ and $\langle x' \rangle = \langle x \rangle$. Consequently, $P'(y, z) : y(= w_1) - w_2 - \cdots - w_i - x' - w_{i+1} - \cdots - (w_n =) z$ is a $y - z$ path in $RP^*(G)$ not containing $x$, a contradiction to $x$ being a cut vertex of $RP^*(G)$. Therefore, $o(x) = 2$.

**Remark 4.1.** Converse of Theorem 4.2 is not true. For example, the element 3 in $\mathbb{Z}_6$ has order 2, but $RP^*(\mathbb{Z}_6) - \{3\} = K_{2,2}$ is connected.

**Corollary 4.2.** Let $G$ be a group of odd order. Then the following are equivalent.

1. $RP^*(G)$ is connected;
2. $RP^*(G)$ is 2–connected;
(3) \( \mathcal{P}^*(G) \) is 2-connected.

\textit{Proof.} (2) \( \Rightarrow \) (1) is obvious. Now we prove (1) \( \Rightarrow \) (2). Assume that \( \mathcal{R}\mathcal{P}^*(G) \) is connected. As a consequence of Theorem 4.2, \( \mathcal{R}\mathcal{P}^*(G) \) has no cut vertex and so it is 2-connected. The proof of (2) \( \iff \) (3) follows from Corollary 4.1. \( \square \)

**Theorem 4.3.** Let \( G \) be a finite cyclic group. Then \( \mathcal{R}\mathcal{P}^*(G) \) has a cut vertex if and only if \( G \cong \mathbb{Z}_4 \).

\textit{Proof.} Let \( G = \langle a \rangle \). Assume that \( \mathcal{R}\mathcal{P}^*(G) \) has a cut vertex, say \( x \). Then there exist \( y, z \in G \) such that every \( y - z \) path in \( \mathcal{R}\mathcal{P}^*(G) \) contains \( x \) and by Theorem 4.2, \( o(x) = 2 \). Suppose both \( y \) and \( z \) are non-generators of \( G \), then \( y - a - z \) is a \( y - z \) path not containing \( x \), which is a contradiction. Suppose \( y \) is a generator of \( G \) but not \( z \), then \( y \) and \( z \) are adjacent in \( \mathcal{R}\mathcal{P}^*(G) \), which is a contradiction. Now we assume that both \( y \) and \( z \) are generators of \( G \), then it forces that both \( y \) and \( z \) are adjacent only to \( x \). Consequently, \( o(y) = 4 = o(z) \), since \( o(x) = 2 \). This implies that \( G \cong \mathbb{Z}_4 \). Converse is clear. \( \square \)

**Theorem 4.4.** Let \( G \) be a finite group and let \( e' = xy \) be an edge in \( \mathcal{R}\mathcal{P}^*(G) \). Then \( e' \) is a cut edge in \( \mathcal{R}\mathcal{P}^*(G) \) if and only if at least one of \( \langle x \rangle \) or \( \langle y \rangle \) is either \( \mathbb{Z}_4 \) or a maximal cyclic subgroup of \( G \) of order 4 when \( G \) is noncyclic.

\textit{Proof.} Suppose \( e' = xy \) is a cut edge in \( \mathcal{R}\mathcal{P}^*(G) \). Then either \( x \) or \( y \) is a cut vertex in \( \mathcal{R}\mathcal{P}^*(G) \). Without loss of generality, we assume that \( x \) is a cut vertex in \( \mathcal{R}\mathcal{P}^*(G) \). By Theorem 4.2, \( o(x) = 2 \) and so \( o(y) \neq 2 \). Again by Theorem 4.2, \( y \) is not a cut vertex in \( \mathcal{R}\mathcal{P}^*(G) \). Therefore, \( y \) must be a pendent vertex in \( \mathcal{R}\mathcal{P}^*(G) \). By Lemma 2.1, \( y \) satisfies the requirements of the theorem. Consequently, \( o(y) = 4 = o(z) \), since \( o(x) = 2 \). This implies that \( G \cong \mathbb{Z}_4 \). Converse is clear. \( \square \)

**Corollary 4.3.** \( \mathcal{R}\mathcal{P}^*(\mathbb{Z}_n), n \geq 2 \) has a cut edge if and only if \( n = 4 \).

**Theorem 4.5.** Let \( G \) be a finite group and let \( e' = xy \) be an edge in \( \mathcal{R}\mathcal{P}(G) \). Then \( e' \) is a cut edge in \( \mathcal{R}\mathcal{P}(G) \) if and only if at least one of \( \langle x \rangle \) or \( \langle y \rangle \) is either \( \mathbb{Z}_p \), where \( p \) is a prime or a maximal cyclic subgroup of \( G \) of prime order when \( G \) is noncyclic.
Proof. The proof is similar to the proof of Theorem 4.4 and using the facts that if $\mathcal{RP}(G)$ has a cut vertex, then it must be an identity element in $G$ and any vertex which is adjacent to the identity element generates a maximal cyclic subgroup of prime order.

**Corollary 4.4.** $\mathcal{RP}(\mathbb{Z}_n)$, $n \geq 1$ has a cut edge if and only if $n = p$, where $p$ is a prime.

# 5 Independence number, Connectivity, Hamiltonicity

Let $\Gamma$ be a graph. The independence number $\alpha(\Gamma)$ of $\Gamma$ is the number of vertices in a maximum independent set of $\Gamma$. The connectivity $\kappa(\Gamma)$ of $\Gamma$ is the minimum number of vertices, whose removal results in a disconnected or trivial graph. A graph is Hamiltonian if it has a spanning cycle. It is known ([1, Theorem 6.3.4]) that if $\Gamma$ is Hamiltonian, then for every nonempty proper subset $S$ of $V(\Gamma)$, the number of components of $G - S$ is at most $|S|$. Also, if $\Gamma$ is 2-connected and $\alpha(\Gamma) \leq \kappa(\Gamma)$, then $\Gamma$ is Hamiltonian ([1, Theorem 6.3.13]). We use these facts for investigation of the Hamiltonicity of the reduced power graph of groups.

In this section, we explore the independence number, connectivity and Hamiltonicity of reduced power graph (resp. proper reduced power graph) of cyclic groups, dihedral groups, quaternion groups, semi dihedral groups and finite $p$-groups. Note that for any group $G$, $\mathcal{RP}(G) = K_1 + \mathcal{RP}^*(G)$, so it follows that $\alpha(\mathcal{RP}(G)) = \alpha(\mathcal{RP}^*(G))$.

## 5.1 Cyclic groups

**Theorem 5.1.** Let $n$ be a positive integer and let $\varphi(n)$ denotes its Euler’s totient function. Then

1. $\kappa(\mathcal{RP}(\mathbb{Z}_n)) = n - \varphi(n)$ if $2\varphi(n) \geq n$;
2. $\kappa(\mathcal{RP}(\mathbb{Z}_n)) \geq \varphi(n) + 1$ if $2\varphi(n) < n$. The equality holds for $n = 2p$, where $p$ is a prime.

**Proof.** Assume that $2\varphi(n) \geq n$. Let $S = \{g \in \mathbb{Z}_n \mid o(g) \neq n\}$. Then $|S| = n - \varphi(n)$. Also $\mathcal{RP}(\mathbb{Z}_n) - S = \mathcal{K}_{\varphi(n)}$. Therefore, $\kappa(\mathcal{RP}(\mathbb{Z}_n)) \leq n - \varphi(n)$. But the generator of $\mathbb{Z}_n$ are adjacent to all the vertices in $\mathcal{RP}(\mathbb{Z}_n)$. Hence $\kappa(\mathcal{RP}(\mathbb{Z}_n)) = n - \varphi(n)$. Assume that $2\varphi(n) < n$. Since the identity element and the generators of $\mathbb{Z}_n$ are adjacent to all the
vertices in $\mathcal{RP}(\mathbb{Z}_n)$, it follows that $\kappa(\mathcal{RP}(\mathbb{Z}_n)) \geq \varphi(n) + 1$. If $n = 2p$, where $p$ is a prime, then $\mathcal{RP}(\mathbb{Z}_{2p}) \cong K_{p-1,p+1}$. Consequently, $\kappa(\mathcal{RP}(\mathbb{Z}_{2p})) = p = \varphi(2p) + 1$.

**Corollary 5.1.** Let $n \geq 2$ be an integer. Then

1. $\kappa(\mathcal{RP}^*(\mathbb{Z}_n)) = n - \varphi(n) - 1$ if $2\varphi(n) \geq n$;
2. $\kappa(\mathcal{RP}^*(\mathbb{Z}_n)) \geq \varphi(n)$ if $2\varphi(n) < n$. The equality holds for $n = 2p$, where $p$ is a prime.

**Corollary 5.2.** $\kappa(\mathcal{RP}(\mathbb{Z}_{p^m})) = p^{m-1}$ and $\kappa(\mathcal{RP}^*(\mathbb{Z}_{p^m})) = p^{m-1} - 1$, where $p$ is a prime and $m$ is a positive integer.

**Theorem 5.2.** Let $n \geq 2$ be an integer. Then

1. $\alpha(\mathcal{RP}^*(\mathbb{Z}_n)) = p^{m-1}(p-1)$ if $n = p^m$, where $p$ is a prime and $m$ is a positive integer.
2. $\alpha(\mathcal{RP}^*(\mathbb{Z}_n)) \geq \varphi(n)$ if $n$ is not a prime power. The equality holds for $n = pq$, where $p$, $q$ are distinct primes with $p, q \neq 2$.

**Proof.** Clearly by (2.2), $\alpha(\mathcal{RP}^*(\mathbb{Z}_{p^m})) = p^{m-1}(p-1)$. Assume that $n$ is not a prime power. Since the generators of $\mathbb{Z}_n$ are not adjacent to each other in $\mathcal{RP}^*(\mathbb{Z}_n)$, it follows that $\alpha(\mathcal{RP}^*(\mathbb{Z}_n)) \geq \varphi(n)$. Finally, if $p, q$ are distinct primes with $p, q \neq 2$, then $\mathcal{RP}(\mathbb{Z}_{pq}) = K_1 + K_{\varphi(pq), p+q-1}$. Since $p, q \neq 2$, $\varphi(pq) = (p-1)(q-1) > p + q - 1$, we have $\alpha(\mathcal{RP}^*(\mathbb{Z}_{pq})) = \varphi(pq)$.

**Corollary 5.3.** Let $n$ be a positive integer. Then

1. $\alpha(\mathcal{RP}(\mathbb{Z}_n)) = p^{m-1}(p-1)$ if $n = p^m$, where $p$ is a prime and $m$ is a positive integer.
2. $\alpha(\mathcal{RP}(\mathbb{Z}_n)) \geq \varphi(n)$ if $n$ is not a prime power. The equality holds for $n = pq$, where $p$, $q$ are distinct primes with $p, q \neq 2$.

**Theorem 5.3.** Let $p$ be a prime number and $n \geq 1$. Then

1. $\mathcal{RP}(\mathbb{Z}_{p^n})$ is Hamiltonian if and only if $p = 2$ and $n \geq 2$;
2. $\mathcal{RP}^*(\mathbb{Z}_{p^n})$ is non-Hamiltonian.
Proof. (1) **Case 1.** Let $p \geq 3$. Take $S = G \setminus X_n$, where $X_n$ is given in (2.1). Then by (2.2), $\mathcal{R}(\mathbb{Z}_p^n) - S = K_{p^n - 1(p-1)}$. It follows that the number of components of $\mathcal{R}(\mathbb{Z}_p^n) - S$ is $p^n - 1 > p^n - 1 = |S|$. Therefore, $\mathcal{R}(G)$ is non-Hamiltonian.

**Case 2.** Assume that $p = 2$. Clearly, $\mathcal{R}(\mathbb{Z}_2^n) \cong K_2$, which is non-Hamiltonian. Now let $n \geq 2$. Then by Theorem 5.2 and Corollary 5.2, $\alpha(\mathcal{R}(\mathbb{Z}_2^n)) = 2^{n-1} = \kappa(\mathcal{R}(\mathbb{Z}_2^n))$. Also by (2.2), $\mathcal{R}^*(G)$ is connected. It follows that $\mathcal{R}(G)$ is 2-connected, since the identity element adjacent to all the elements of $\mathcal{R}(G)$. Therefore, $\mathcal{R}(G)$ is Hamiltonian.

(2) Proof follows by taking $S = G \setminus (X_n \cup \{e\})$ and following the similar argument as in the proof of part (1). \qed

### 5.2 Dihedral groups, Quaternion groups and Semi-dihedral groups

The dihedral group of order $2n$ ($n \geq 3$) is given by $D_{2n} = \langle a, b \mid a^n = e = b^2, ab = ba^{-1} \rangle$.

**Theorem 5.4.** For an integer $n \geq 3$,

1. $\kappa(\mathcal{R}(D_{2n})) = 1$ and $\kappa(\mathcal{R}^*(D_{2n})) = \kappa(\mathcal{R}^*(\mathbb{Z}_n))$;
2. $\alpha(\mathcal{R}(D_{2n})) = n + \alpha(\mathcal{R}(\mathbb{Z}_n))$;
3. $\mathcal{R}(D_{2n})$ and $\mathcal{R}^*(D_{2n})$ are non-Hamiltonian.

**Proof.** Since $D_{2n} = \langle a \mid a^n = e \rangle \cup \{a^ib \mid 1 \leq i \leq n\}$, it is easy to see that the structure of

![Figure 2: The graph $\mathcal{R}(D_{2n})$](image-url)
\(\mathcal{RP}(D_{2n})\) is as shown in Figure 2 and so
\[
\mathcal{RP}(D_{2n}) \cong K_1 + (\mathcal{RP}^*(\mathbb{Z}_n) \cup K_n), \quad (5.1)
\]
\[
\mathcal{RP}^*(D_{2n}) = \mathcal{RP}^*(\mathbb{Z}_n) \cup K_n
\]  
(5.2)

By (5.1), we can determine the values of \(\kappa(\mathcal{RP}(D_{2n}))\) and \(\alpha(\mathcal{RP}(D_{2n}))\). Moreover, \(\mathcal{RP}(D_{2n})\) has a cut vertex and so it is non-Hamiltonian. The proofs for \(\mathcal{RP}^*(D_{2n})\) follows from (5.2).

The quaternion group of order \(4n\) (\(n \geq 2\)) is given by
\[
\mathbb{Q}_4^n = \langle a, b \mid a^{2n} = e = b^4, bab^{-1} = a^{-1} \rangle.
\]

**Theorem 5.5.** For an integer \(n \geq 2\),

1. \(\kappa(\mathcal{RP}(Q_{4^n})) = 2\) and \(\kappa(\mathcal{RP}^*(Q_{4^n})) = 1\);
2. \(\alpha(\mathcal{RP}(Q_{4^n})) \in \{2n + \alpha(\mathcal{RP}^*(\mathbb{Z}_{2n})) - 1, 2n + \alpha(\mathcal{RP}(\mathbb{Z}_{2n}))\}\);
3. \(\mathcal{RP}(Q_{4^n})\) and \(\mathcal{RP}^*(Q_{4^n})\) are non-Hamiltonian.

**Proof.** Note that \(Q_{4^n} = \langle a \mid a^{2n} = e \rangle \cup \{a^i b \mid 1 \leq i \leq 2n\}\). It has \(\langle a^i b \rangle\) as a maximal cyclic subgroup of order 4 and \(\langle a^i b \rangle \cap \langle a^j b \rangle = \{e, a^n\}, i, j \in \{1, 2, \ldots, 2n\}, \) for all \(i \neq j\). So \(a^n\) is adjacent to all the elements of order 4 in \(\mathcal{RP}(Q_{4^n})\). From these facts, it is easy to see that the structure of \(\mathcal{RP}(Q_{4^n})\) is as in Figure 3.

![Figure 3: The graph \(\mathcal{RP}(Q_{4^n})\)](image)

So \(\mathcal{RP}(Q_{4^n})\) has no cut vertex and \(\mathcal{RP}(Q_{4^n}) - \{e, a^n\}\) is disconnected. Therefore, \(\kappa(\mathcal{RP}(Q_{4^n})) = 2\) and \(\kappa(\mathcal{RP}^*(Q_{4^n})) = 1\). It follows that \(\mathcal{RP}^*(Q_{4^n})\) is non-Hamiltonian.
Moreover, the number of components of $\mathcal{R}\mathcal{P}(Q_{4n}) - \{e, a^n\}$ is at least $2n + 1 > 2$. So, $\mathcal{R}\mathcal{P}(Q_{4n})$ is non-Hamiltonian. It can be seen from Figure 3 that if at least one of a maximal independent set of $\mathcal{R}\mathcal{P}(\mathbb{Z}_{2n})$ does not have the element of order 2 in $\mathbb{Z}_{2n}$, then $\alpha(\mathcal{R}\mathcal{P}(Q_{4n})) = 2n + \alpha(\mathcal{R}\mathcal{P}(\mathbb{Z}_{2n}))$; otherwise, $\alpha(\mathcal{R}\mathcal{P}(Q_{4n})) = 2n + \alpha(\mathcal{R}\mathcal{P}(\mathbb{Z}_{2n})) - 1$. 

The semi-dihedral group of order $8n$ ($n \geq 2$) is given by $SD_{8n} = \langle a, b \mid a^{4n} = e = b^2, bab^{-1} = a^{2n-1} \rangle$

**Theorem 5.6.** For an integer $n \geq 3$,

(1) $\kappa(\mathcal{R}\mathcal{P}(SD_{8n})) = 1 = \kappa(\mathcal{R}\mathcal{P}^*(SD_{8n}))$;

(2) $\alpha(\mathcal{R}\mathcal{P}(SD_{8n})) \in \{4n + \alpha(\mathcal{R}\mathcal{P}(\mathbb{Z}_{4n})) - 1, 4n + \alpha(\mathcal{R}\mathcal{P}(\mathbb{Z}_{4n})) \}$;

(3) $\mathcal{R}\mathcal{P}(SD_{8n})$ and $\mathcal{R}\mathcal{P}^*(SD_{8n})$ are non-Hamiltonian.

**Proof.** Note that

$SD_{8n} = \langle a \mid a^{4n} = e \rangle \cup \{a^kb \mid 1 \leq k \leq 4n \text{ and } k \text{ is even} \} \cup \{a^kb \mid 1 \leq k \leq 4n \text{ and } k \text{ is odd} \}$

Figure 4: The graph $\mathcal{R}\mathcal{P}(SD_{8n})$

Here for each $k$, $1 \leq k \leq 4n$, $k$ is odd, $\langle a^kb \rangle$ is a maximal cyclic subgroup of order 4 in $SD_{8n}$ and all subgroups of $SD_{8n}$ have an element of order 2 in $\langle a \rangle$ as common, so they are adjacent only to the identity element and the element of order 2 in $\langle a \rangle$; $\langle a^kb \rangle$ where $k$
is even, $1 \leq k \leq 4n$ is maximal cyclic subgroup of order 2, so they are adjacent only the identity element in $SD_{8n}$. Hence the structure of $\mathcal{RP}(SD_{8n})$ is as shown in Figure 4.

Moreover, the identity element is a cut vertex of $\mathcal{RP}(SD_{8n})$ and $a^{2n}$ is a cut vertex of $\mathcal{RP}^*(SD_{8n})$ and so these two graphs are non-Hamiltonian and their connectivity is 1. From Figure 4 it is not hard to see that if at least one of a maximal independent set of $\mathcal{RP}(Z_{4n})$ doesn’t have the element of order 2, then $\alpha(\mathcal{RP}(SD_{8n})) = 4n + \alpha(\mathcal{RP}(Z_{4n}))$; otherwise, $\alpha(\mathcal{RP}(SD_{8n})) = 4n + \alpha(\mathcal{RP}(Z_{4n})) - 1$.

5.3 Finite $p$-groups

Theorem 5.7. Let $G$ be a finite $p$-group, where $p$ is a prime. Then

$$\kappa(\mathcal{RP}(G)) = \begin{cases} p^{m-1}, & \text{if } G \cong \mathbb{Z}_{p^m}, m \geq 1; \\ 2, & \text{if } G \cong \mathbb{Q}_2^\alpha, \alpha \geq 3; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Proof follows from Theorem 5.5(1), Corollaries 2.6 and 5.2.

Theorem 5.8. Let $G$ be a finite $p$-group, where $p$ is a prime. Then

(1) $\mathcal{RP}(G)$ is Hamiltonian if and only if $G \cong \mathbb{Z}_{2^n}, n \geq 2$;

(2) $\mathcal{RP}^*(G)$ is non-Hamiltonian.

Proof. (1) If $G$ is a non-cyclic and non-quaternion $p$-group, then by Corollary 2.6, $\mathcal{RP}^*(G)$ is non-Hamiltonian. Moreover, the identity element in $G$ is a cut vertex of $\mathcal{RP}(G)$. Hence $\mathcal{RP}(G)$ is non-Hamiltonian. If $G \cong \mathbb{Z}_{p^n}$ or $\mathbb{Q}_2^\alpha$, where $p$ is a prime, $n \geq 1$ and $\alpha \geq 3$, then the proof follows by Theorems 5.3 and 5.5(3).

(2) Proof is similar to the proof of part (1).

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