Localization in the constrained quantum annealing of graph coloring

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Constrained quantum annealing (CQA) is a quantum annealing approach that is designed so that constraints are satisfied without penalty terms. There is an analogy between the model used for the CQA of graph coloring and a disordered spin chain. In a strongly disordered spin chain, localization phenomena such as Anderson localization and many-body localization occurs. In the model for the CQA of graph coloring, disorder corresponds to the fluctuation of local effective fields that increase in a CQA process. Several measures of entanglement show how localization appears in a CQA process, depending on the fluctuation of effective fields. Localization in CQA is also considered as Anderson localization in Fock space. The CQA approach of graph coloring causes extra localization due to degenerated ground states of the problem Hamiltonian and a specific choice of the driver Hamiltonian.

I. INTRODUCTION

Quantum annealing (QA), which is essentially the same as adiabatic quantum computation, has attracted huge interest in recent years. QA is typically known as a quantum-mechanical approach for combinatorial optimization problems [1–2]. In a QA framework, the total Hamiltonian consists of a problem Hamiltonian and a driver Hamiltonian. The problem Hamiltonian describes an optimization problem and is to be minimized. The driver Hamiltonian corresponds to quantum fluctuation. The QA process starts with the ground state of the driver Hamiltonian. The proportion of the driver Hamiltonian in the total one decreases as time proceeds. Instead, that of the problem Hamiltonian increases, and the total Hamiltonian coincides with the problem Hamiltonian in the end. If the process is adiabatic, the solution of the problem, i.e., the ground state of the problem Hamiltonian, is obtained as the ground state of the total Hamiltonian at the end of the QA process. The computational efficiency of QA depends on the energy gap between the ground and first-excited states. Several approaches have been proposed to avoid exponentially small energy gaps [10–13]. They accelerate an adiabatic process and contribute to better performance of computation. Constrained quantum annealing (CQA) is a different approach to provide good performance [14–16]. In the CQA approach, the driver Hamiltonian is chosen so that hard constraints are naturally satisfied. Practical optimization problems often have hard constraints. To impose such constraints, in a standard approach, penalty terms are added to the problem Hamiltonian. Since extra terms may make an energy landscape more complicated, adding penalty terms is undesirable. The CQA approach is beneficial also because it restricts the Hilbert space to a subspace with a considerably small dimension. The dimension reduction enables us to perform real-time quantum simulations [16].

Understanding quantum dynamics in many-body systems is essential for the high performance of quantum computing. Especially, localization of a wavefunction may obstruct attaining possible solutions and lower the performance. For example, Altshuler et al. pointed out that a phenomenon similar to Anderson localization makes adiabatic quantum optimization fail [17]. The Hamiltonian of QA with N spins can be regarded as the system of a single quantum particle that moves between the vertices of an N-dimensional hypercube. Each spin configuration \( \{ \sigma_i^z \} \) corresponds to a vertex of the hypercube. The problem and driver Hamiltonians correspond to disordered potential and hopping terms, respectively. Hence, the total Hamiltonian of QA describes a sort of the well-known Anderson model. As localization in a quantum many-body system, many-body localization (MBL) has attracted great interest in recent years. In the MBL phase, where disorder is relatively strong compared with many-body interaction, quantum dynamics is nonergodic and breaks the eigenstate thermalization hypothesis (ETH) [17–19]. MBL can also be regarded as a sort of Anderson localization in Fock space [20–24]. In quantum spin systems, the basis in Fock space is taken as the spin-configuration basis. While CQA can also describe Anderson localization in Fock space, the CQA of graph coloring provides another viewpoint of localization. In the CQA of graph coloring, the total Hamiltonian is analogous to an ensemble of spin chains with disordered fields [10]. The system is considered to be almost independent spin chains interacting weakly with other chains in the early stages of CQA. In the late stage, as interchain interaction increases with time, spins are affected by effective fields arising from neighboring chains.

In this paper, we investigate localization phenomena in the CQA of graph coloring. Instead of discussing localization transitions, we focus on detailed quantum dynamics in small systems. Phase transitions in QA have been studied in NP-hard problems [25–27]. For discussing a phase transition in such a system, a problem is usually supposed to have a unique solution. However, the ground state is degenerated in the problem Hamiltonian of graph coloring, since there are at least \( q! \) solutions when a graph
is colorable with \( q \) colors. The degeneracy of the ground state sometimes leads to unexpected results. For example, due to the degeneracy, we are more likely to obtain a solution at the end of CQA, even if an annealing process is less adiabatic \[16\]. Localization in systems with such highly degenerate ground states is nontrivial. Real-time quantum simulation, which is impossible in a large system, will provide insights on localization phenomena in QA.

We employ two entanglement measures, concurrence and entanglement entropy, to discuss localization in a spin chain under effective fields. Entanglement plays a crucial role in quantum computing, although the role of entanglement in adiabatic quantum computation and QA is currently unclear \[10, 31, 33\]. Concurrence, which is a quantity to measure pairwise entanglement between two spin-1/2 spins, is often used for measuring entanglement in a mixed state \[34–36\]. Entanglement entropy is often employed to characterize the MBL transition \[36–47\]. These two measures provide some insights from the viewpoint of localization in a spin chain. We also employ the inverse participation ratio (IPR) to discuss localization in Fock space. The IPR and the population distribution based on the spin-configuration basis demonstrate non-trivial features of localization in this system.

The remainder of the paper is organized as follows. In Sec. II the model and numerical methods of the CQA of graph coloring are outlined, and the details of graph selection are mentioned. In Sec. III time evolution of concurrence and entanglement entropy illustrates localization in a spin chain under effective fields. In Sec. IV the IPR and the population distribution based on the spin-configuration basis demonstrate non-trivial features of localization in this system.

II. MODELS AND METHODS

A. Constrained quantum annealing

We focus here on the CQA of graph coloring; this model was introduced in Ref. \[16\]. Graph coloring consists of coloring the nodes of a graph such that nodes directly connected through an edge do not share the same color. When coloring a graph \( G = (V, E) \) with \( q \) available colors, the classical Hamiltonian is given by

\[
H_{cl} = \sum_{(ij) \in E} \sum_{a=1}^{q} \frac{S_{i,a} + 1}{2} S_{j,a} + \frac{1}{2}, \tag{1}
\]

where \( i \) and \( j \) represent nodes \( V = \{1, \ldots, N\} \), and \( (ij) \in E \) denotes the edge connecting the pair of nodes \( i, j \in V \). If node \( i \) is colored \( a \), \( S_{i,a} = 1 \); otherwise, \( S_{i,a} = -1 \). Thus, Eq. (1) counts the number of edges that connect nodes with the same color. The quantum version of Eq. (1) is the problem Hamiltonian, given by

\[
H_p = J \sum_{(ij) \in E} \sum_{a=1}^{q} \sigma_{i,a}^z \sigma_{j,a}^z, \tag{2}
\]

where \( \sigma_{i,a}^z \) denotes the Pauli matrix of the component \( z \), and \( J \) has a unit of energy \( (J = 1 \) in the simulations below). As each node can only have one color, the required constraint is \( \langle \sum_{a=1}^{q} \sigma_{i,a}^z \rangle = 2 - q \). In standard QA approaches, the penalty term to satisfy this constraint is often incorporated into the problem Hamiltonian \[48–52\]. In the CQA approach, however, we choose the driver Hamiltonian so that the constraint is satisfied consistently, instead of adding a penalty term. Here, we give the driver Hamiltonian for this problem as

\[
H_d = -J \sum_{i=1}^{N} \sum_{a=1}^{q} (\sigma_{i,a}^x \sigma_{i,a+1}^x + \sigma_{i,a}^y \sigma_{i,a+1}^y), \tag{3}
\]

where periodic boundary conditions are imposed for index \( a \). The total Hamiltonian is given as

\[
H(s) = sH_p + (1 - s)H_d, \tag{4}
\]

where \( s \) is a time-dependent parameter. The QA process starts at \( s = 0 \) and ends at \( s = 1 \). As the total Hamiltonian is a type of \( XXZ \) model, the magnetization \( \langle \sum_{a=1}^{q} \sigma_{i,a}^z \rangle \) is conserved for each \( i \).

The model is a kind of an ensemble of tight-binding chains. In Fig. 1, which is a schematic of the model, thick arrows represent local effective fields. The solid and dashed lines correspond to the problem Hamiltonian \[2\] and the driver Hamiltonian \[3\], respectively. If we focus on the solid line (spin chain) of \( i = 1 \), the nodes neighboring the chain play a role of local effective fields.

![FIG. 1. Schematic of the model. Thick arrows represent local effective fields. The solid and dashed lines correspond to the problem Hamiltonian (2) and the driver Hamiltonian (3), respectively. If we focus on the solid line (spin chain) of \( i = 1 \), the nodes neighboring the chain play a role of local effective fields.](image)
TABLE I. The number of solutions $N_{\text{sol}}$ for regular graphs of $N = 6$ and degree $c$.

| $c = 2$ | $c = 3$ | $c = 4$ |
|---------|---------|---------|
| type    | $N_{\text{sol}}$ | type    | $N_{\text{sol}}$ | type    | $N_{\text{sol}}$ |
| A       | 576     | A       | 264     | 96      |
| B       | 732     | B       | 420     |         |

![FIG. 2. Selected graphs of $N = 6$ and degree $c = 2, 3, 4$.](image)

Simulations, $2^c N$ dimensions are required for coloring a graph of $N$ nodes with $q$ colors. However, with the CQA approach described above, the number of required dimensions is reduced to only $q^N$.

To perform the CQA based on real-time quantum simulations, we just deal with the subspace that satisfies the constraint $(\sum_{a=1}^{q} \sigma_{i,a}^z) = 2 - q$. The initial state is taken as the lowest-energy state of $H_q$ in the subspace. Although this initial state is not the global ground state, the state can be prepared in the whole Hilbert space of the system by adding additional Zeeman term with an appropriate magnetic field $B$. In the following sections, we refer to the lowest energy state in the subspace as the ground state. Time evolution is calculated by solving the time-dependent Schrödinger equation using the fourth Runge-Kutta method. The annealing schedule is taken as $s(t) = t/\tau$, where $t$ is time and $\tau$ is the final time, so that $s(0) = 0$ and $s(\tau) = 1$. Time is measured in units of $\hbar/J$.

B. Graph selection

In this paper, we focus on regular graphs of $N = 6, 8$ nodes and degree $c = 2, 3, 4$ and consider coloring them with four colors ($q = 4$). Listing all combinations of edges for each $N$ and $c$, we compute the time evolution of effective fields, which is defined in Sec. III at a specific node for each graph. Graphs whose results are different from each other are selected.

For $N = 6$, there are only two different types among 70 graphs with $c = 2, 3$, and all the 15 graphs with $c = 4$ show the same results. The number of solutions $N_{\text{sol}}$, i.e., the degeneracy of $H_{p}$, and selected graphs are listed in Table I and Fig. 2, respectively. Types A and B of $c = 2$ are disconnected and connected (cycle) graphs, respectively. Types A and B of $c = 3$ are planar and bipartite ($K_{1,3}$) graphs, respectively. Note that type-B graphs of $c = 2, 3$ are colorable with only two colors.

For $N = 8$, the numbers of selected graphs are 4 (out of 3507) for $c = 2$ and 11 (out of 19355) for $c = 3, 4$. The graph types and the number of solutions are listed in Table II and selected graphs are shown in Fig. 3. Some types of graphs have the same number of solutions, although they show different time evolution of effective fields. The difference is based on the choice of the specific node at which an effective field is evaluated. For example, types B1 and B2 of $c = 2$ are the same graph located at different angles. Since the specific node is chosen as the top node of each graph, the node belongs to the smaller and the larger subgraphs in types B1 and B2, respectively.

TABLE II. The number of solutions $N_{\text{sol}}$ for regular graphs of $N = 8$ and degree $c$.

| $c = 2$ | $c = 3$ | $c = 4$ |
|---------|---------|---------|
| type    | $N_{\text{sol}}$ | type    | $N_{\text{sol}}$ | type    | $N_{\text{sol}}$ |
| A       | 7056    | A       | 576     | A       | 168     |
| B1, B2  | 5760    | B1, B2  | 1344    | B       | 216     |
| C       | 6564    | C1, C2, C3 | 1752 | C1, C2, C3 | 288     |
|         | D1, D2, D3 | 1920 | D1, D2, D3 | 456     |
|         | E       | E1, E2  | 2160    | E       | 744     |
|         | F       | F       | 2652    | F       | 1812    |

![FIG. 3. Selected graphs of $N = 8$ and degree $c = 2, 3, 4$.](image)

III. LOCALIZATION IN A SPIN CHAIN UNDER EFFECTIVE FIELDS

A. Effective fields

When $s > 0$, spins are affected by effective fields arising from the interaction between neighboring nodes in a given graph. Note that the problem Hamiltonian $H_{p}$ can be rewritten as

$$H_{p} = \sum_{i=1}^{N} \sum_{a=1}^{q} J_{ij} \sigma_{j,a}^{z} \sigma_{i,a}^{z},$$

where $J_{ij} = J/2$ for $(ij) \in E$, otherwise 0. The average of effective fields, which is time dependent, is defined as

$$\langle \hat{h}_{i,a}^{\text{eff}}(s) \rangle = \langle \Psi(s)|\hat{h}_{i,a}^{\text{eff}}(s)|\Psi(s) \rangle,$$
Similarly, the fluctuation of effective fields is defined as

$$\Delta_1 = s \frac{1}{\sqrt{c}} \sqrt{1 - (q - 2)^2/q^2}.$$  

If the probability at each site ($a = 1, \ldots, q$) in a chain is equal, the average of effective fields is proportional to Eq. (9) and increases with $s$ in a CQA process. The fluctuation of effective fields is expected to increase as $\Delta_1(s) = s \Delta_1$.

In numerical simulations, we take $i = a = 1$, where $i = 1$ corresponds to the top node of each graph in Figs. 2 and 3. The annealing time is taken as $\tau = 200$, which is long enough to keep the annealing process almost adiabatic, in the simulations in this paper. In fact, the fidelity (ground-state population) is kept larger than 0.95 for $\tau = 200$.

Although the average of effective fields grows linearly with time and the rate is equal to Eq. (9), the fluctuation of effective fields shows nonlinear growth. In Fig. 4 the fluctuation of effective fields $\Delta_1$ divided by $s \Delta_1$ changes with time. If $\Delta_1$ increases as expected, $\Delta_1/(s \Delta_1)$ should be unity. Considering the analogy between the model and a disordered spin chain, we can regard $\Delta_1$ as disorder strength. This implies that the disorder is relatively large in the region where $\Delta_1/(s \Delta_1) > 1$ in Fig. 4.

B. intra-chain concurrence

We here employ concurrence as a measure of pairwise entanglement in a spin chain. The concurrence $C_{i,j}$ in spins $i$ and $j$ is defined from the eigenvalues of the matrix $\rho_{ij} \rho_{ji}$, where $\rho_{ij}$ is the reduced density matrix, and $\rho_{ij} = \sigma^y \otimes \sigma^y \rho_{ij} \sigma^y \otimes \sigma^y$. Suppose that the eigenvalues are $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, then the concurrence is $C_{i,j} = \text{max}(\sqrt{\lambda_1 - \sqrt{\lambda_2 - \sqrt{\lambda_3 - \lambda_4}}}, 0)$. From the conservation of magnetization, the concurrence in the systems considered here can be expressed in a simple form [36, 53]:

$$C_{i,j} = 2 \max(|z| - \sqrt{xy}, 0),$$  

where $z = (\langle \downarrow \downarrow | \rho_{ij} | \downarrow \downarrow \rangle, x = (\langle \uparrow \uparrow | \rho_{ij} | \uparrow \uparrow \rangle$, and $y = (\langle \downarrow \uparrow | \rho_{ij} \downarrow \uparrow \rangle$.

In our system, a spin has two indices, i.e., node $i$ and color $a$. Let $C(i, a; j, b)$ denote the concurrence in spins $(i, a)$ and $(j, b)$. We define the intra-chain concurrence of the $i$th chain as

$$C_i = \frac{1}{2} \sum_{a=1}^{q} C(i; a; i, a + 1),$$  

which is scaled so that $C_i = 1$ at $s = 0$. Equation (12) describes the nearest-neighbor concurrence in the $i$th chain.

The intra-chain concurrence decreases monotonically, as shown in Fig. 5. Since only one spin is up and the others are down in each chain, the concurrence in the ground state at $s = 0$ is maximal and $C(i; a; i, a + 1) = 2/q$, which leads to $C_i = 1$. When chains start interacting with each other, the intra-chain concurrence decreases. The decrease in the concurrence tends to be suppressed for types that show a large fluctuation of effective fields $\Delta_1$. This implies that the population at one of two neighboring sites in a chain is likely to be much larger than that of the other when $\Delta_1$, i.e., disorder, is large. In other words, the entanglement between the two neighboring spins, i.e., the concurrence is large for a large disorder.

C. Entanglement entropy

Entanglement entropy is often used as an entanglement measure and is defined as

$$S_E = -\text{Tr}(\rho_A \ln \rho_A),$$  

FIG. 4. Time evolution of the fluctuation of effective fields $\Delta_1$ divided by $s \Delta_1$, where $\Delta_1$ is given by Eq. (10). The number of nodes is (a) $N = 6$ and (b)–(f) $N = 8$. Degree is (b) $c = 2$, (c)–(d) $c = 3$, and (e)–(f) $c = 4$.

where $|\Psi(s)\rangle$ is the wavefunction at $s$ and

$$\hat{h}_{i,a}(s) = s \sum_{j=1}^{N} J_{ij} \sigma_{j,a}^z.$$  

Similarly, the fluctuation of effective fields is defined as

$$\Delta_1(s) = \sqrt{\langle \langle \hat{h}_{i,a}(s) \rangle \rangle^2 - \langle \langle \hat{h}_{i,a}(s) \rangle \rangle^2}.$$  

Considering regular random graphs of degree $c$, the average of effective fields at $s = 1$ is estimated as

$$\langle \langle \hat{h}_{i,a} \rangle \rangle = \frac{cJ}{2} \left( 1 - \frac{q - 1}{q} \right) = -\frac{cJ(q - 2)}{2q}.$$  

Thus, the estimated value of the fluctuation at $s = 1$ is written as

$$\Delta_1 = \sqrt{c} \left( 1 - \frac{(q - 2)^2}{q^2} \right).$$  

In numerical simulations, we take $i = a = 1$, where $i = 1$ corresponds to the top node of each graph in Figs. 2 and 3. The annealing time is taken as $\tau = 200$, which is long enough to keep the annealing process almost adiabatic, in the simulations in this paper. In fact, the fidelity (ground-state population) is kept larger than 0.95 for $\tau = 200$.  

$$S_E = -\text{Tr}(\rho_A \ln \rho_A),$$  

$$C_i = \frac{1}{2} \sum_{a=1}^{q} C(i; a; i, a + 1),$$  

$$C_{i,j} = 2 \max(|z| - \sqrt{xy}, 0),$$  

$$z = (\langle \downarrow \downarrow | \rho_{ij} | \downarrow \downarrow \rangle, x = (\langle \uparrow \uparrow | \rho_{ij} | \uparrow \uparrow \rangle, and y = (\langle \downarrow \uparrow | \rho_{ij} \downarrow \uparrow \rangle.$$
where $\rho_A$ is the reduced density matrix of the subsystem $A$. Here, we take the subsystem $A$ as the chain (a dashed line in Fig. 11) with $i = 1$. Then, the entanglement entropy describes how the $i$th chain is entangled with the other chains that behave as effective fields. We here call this the one-chain entanglement entropy. We examine the time dependence of $S_E$ divided by the Page value [46, 53, 54]. The Page value is defined by

$$S_T = \sum_{k=n+1}^{m} \frac{1}{k} - \frac{m-1}{2n}, \quad (14)$$

where $m = q$ and $n = q^{N-1}$ in this case.

In contrast to the concurrence, one-chain entanglement entropy increases monotonically, as shown in Fig. 6. The one-chain entanglement entropy for the type whose concurrence is higher tends to be lower than that of others. Since the intra-chain concurrence is maximized at $s = 0$, entanglement between different chains cannot arise at first. For $s > 0$, the interaction between neighboring chains increases, and entanglement between them increases as well, which leads to the increase in one-chain entanglement entropy. If the intra-chain concurrence remains large, the entanglement between different chains cannot increase, which causes the suppression of an increase in the one-chain entanglement entropy.

IV. LOCALIZATION IN FOCK SPACE

To investigate localization in Fock space, we employ the IPR defined by

$$IPR = \sum_{\{|\sigma_i^+\rangle\}} |\langle|\sigma_i^+\rangle\rangle|^4, \quad (15)$$

where $|\{\sigma_i^+\}\rangle$ and $|\psi\rangle$ are the spin-configuration basis and wavefunction, respectively. Since localization makes the IPR high, the IPR is expected to increase in a CQA process.

Figure 7 demonstrates the time evolution of the IPR. The IPRs of graphs with the same number of solutions show the same results. Although the IPR increases with time as expected, the increase in IPR is unexpectedly large. If the population is distributed equally among the solutions (ground states) at $s = 1$, the IPR should be $1/N_{sol}$. However, that is the case for only type A of $N = 6$ and $c = 2$, and that of $N = 8$ and $c = 3$. For most types, the final value of IPR is much higher, about two times higher, than expected. This indicates that the population localizes to a part of solutions and that the probability to obtain each solution (spin configuration) is not equal.

To investigate the cause of a high IPR, we use the population distribution defined by

$$p(j) = |\langle|\sigma_j^+\rangle\rangle|^2, \quad (16)$$

where $j$ is the index of a basis. Note that IPR is written as $\sum_j p(j)^2$. The population distribution at $s = 1$ is

![Fig. 5. Time evolution of the intra-chain concurrence defined by Eq. (12).](image)

![Fig. 6. Time dependence of the one-chain entanglement entropy $S_E$ divided by the Page value $S_T$.](image)
shown in Fig. 5, where \( p(j) \) is sorted in descending order. In Fig. 5(a), we see that the population is equal among all the solutions for type A of \( N = 6 \) and \( c = 2 \). However, for the other types shown in Fig. 5 the population varies widely. Some solutions are likely to be observed, but other solutions scarcely realize. The large dispersion of the population causes a high IPR.

Figure 5 exhibits another nontrivial phenomenon. Since we set \( q = 4 \), at least \( 4! = 24 \) configurations are equivalent for a given graph. However, there are only four configurations that have the largest population for type B of Figs. 5(a) and (b). When we express a configuration as \( (a_1, a_2, \ldots, a_N) \), where \( a_i \) denotes the color of the \( i \)th node, the four configurations are \((3,1,1,3,3,1)\), \((4,2,2,4,4,2)\), \((1,3,3,1,1,3)\), \((2,4,4,2,2,4)\) for type B of Fig. 5(a) and \((1,3,3,3,1,1)\), \((2,4,4,4,2,2)\), \((3,1,1,1,3,3)\), \((4,2,2,2,4,4)\) for that of Fig. 5(b). Here, the nodes are numbered counter-clockwise from the top of each graph in Fig. 2.

If \((3,1,1,3,3,1)\) is a solution, for example, \((2,1,1,2,2,1)\) is also a solution. However, the population (probability to observe the solution) is different between them in Fig. 5(a). What is the difference between them? The difference is caused by the driver Hamiltonian. In the driver Hamiltonian, nearest-neighbor sites (corresponding to colors) are connected. Then, a neighboring configuration in Fock space is the configuration in which the color of node \( i \) is changed from \( a_i \) to \( a_i \pm 1 \). For type B of \( c = 2 \) in Fig. 2, all the configurations neighboring \((3,1,1,3,3,1)\) are also solutions. In contrast, a configuration that neighbors \((2,1,1,2,2,1)\) is not always a solution. For example, \((1,1,1,2,2,1)\) neighbors \((2,1,1,2,2,1)\) but is not a solution. Neighboring configurations depends on the choice of the driver Hamiltonian. The results in this section imply that CQA can cause extra localization in Fock space.

V. CONCLUSIONS

We investigated localization phenomena in the CQA of graph coloring from two viewpoints. One is localization in a spin chain under effective fields. Effective fields that arise from neighboring chains behave as disorder and cause localization during a CQA process. The fluctuation of effective fields, i.e., disorder strength, depends on the structure of a graph. When the fluctuation of effective fields is large, the intra-chain concurrence and one-chain entanglement entropy tend to be larger and smaller, respectively, than in small fluctuation cases. This tendency agrees with large concurrence and small entanglement entropy in the MBL phase of disordered quantum spin chains [36–47].

The other viewpoint is localization in Fock space. Degenerated ground states of the problem Hamiltonian cause nontrivial localization. The probabilities to observe those ground states differ widely. Especially in the CQA approach, the driver Hamiltonian reduces the number of neighboring states (spin configurations) in Fock space. In other words, hopping in Fock space is highly restricted by the driver Hamiltonian. The restricted hopping leads to extra localization in Fock space. We should note that solutions obtained from the CQA approach may be biased, although CQA is a useful and powerful optimization approach.

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