ON CURVATURES OF SECTIONS OF TENSOR BUNDLES.

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ABSTRACT. We consider natural differential operations acting on sections of tensor vector bundles. Arising problems can be reformulated as invariant theoretical problems (the IT-reduction). We give examples of usage of the IT-reduction. In particular, on a manifold with a connection and a Poisson structure we construct the canonical quantization.

1. INTRODUCTION.

Let $M$ be a differentiable $m$-dimensional manifold. The objects of our investigation are natural algebraic differential operations in the form

$$F_M : \Gamma(\text{Ass}_M(V))^{\text{reg}} \to \Gamma(\text{Ass}_M(U)),$$

where $\text{Ass}_M(V)$ and $\text{Ass}_M(U)$ are tensor bundles, $\Gamma(\text{Ass}_M(U))$ is the space of differentiable sections of the bundle $\text{Ass}_M(U)$, and $\Gamma(\text{Ass}_M(V))^{\text{reg}}$ is the space of differentiable nondegenerated sections of the bundle $\text{Ass}_M(V)$.

A tensor bundle $\text{Ass}_M(V)$ is a bundle associated to the frame bundle on $M$ [1], [2]. It corresponds to some linear representation $\rho : \text{GL}(m) \to \text{GL}(V)$. If $\varphi : M \to N$ is a diffeomorphism of $M$ onto an open submanifold in $N$, then

$$\varphi^*(\text{Ass}_N(V)) = \text{Ass}_M(V).$$

The typical examples of tensor vector bundles are tangent bundle $T(M)$, cotangent bundle $T^*(M)$, $k$-th exterior power of cotangent bundle $\wedge^k(M)$ and so on. The non-degeneracy condition is defined by some $r(v) \in \mathbb{R}[V]^{\text{GL}(m)}$. A section $s$ of the bundle $\text{Ass}_M(V)$ is called nondegenerated iff for any chart $Y$ we have

$$s_Y(y) \in V^{\text{reg}} \overset{\text{def}}{=} \{v \in V \mid r(v) \neq 0\} \subset V$$

for all $y \in Y$, where $s_Y : Y \to V$ is a presentation of $s$ at the chart $Y$. For example, if $r(v) \equiv \text{const} \neq 0$, then every section is nondegenerated.

A differential operation $F_M$ is called natural iff for any diffeomorphism $\varphi : M \to N$ of $m$-dimensional manifold $M$ onto an open submanifold of $N$ we have

$$F_M(\varphi^*(\eta)) = \varphi^*(F_N(\eta))$$

for all $\eta \in \Gamma(\text{Ass}_N(V))^{\text{reg}}$, where

$$\varphi^* : \Gamma(\text{Ass}_N(\cdot)) \to \Gamma(\text{Ass}_M(\cdot)).$$
is an induced by $\varphi$ mapping. This condition is equivalent to the following one. The presentation of $F_M$ in bundle charts of $\text{Ass}_M(V)$ and $\text{Ass}_M(U)$ is written by some universal formulas. These formulas depend on $m$ and on representations $\text{GL}(m): V, U$. The condition that $F_M$ is algebraic means that these universal formulas are algebraic. So for any chart $Y \subset M$ we have: $F_M(s_Y)$ is a polynomial function in $r(s_Y)^{-1}$ and

$$\left\{ \frac{\partial^{|i|} s_Y(y)}{\partial y_1^{i_1} \ldots \partial y_m^{i_m}} \right\}_{0 \leq |i| \leq k}.$$  

We consider in this article only algebraic differential operations. We omit the adjective "algebraic" in the sequel.

If $F_M$ is a universal differential operation and $s$ is a section of the corresponding tensor bundle, then $F_M(s)$ is called a curvature of that section.

The classical example of a natural differential operation is the exterior derivative

$$d : \Gamma(\wedge^n(M)) \to \Gamma(\wedge^{n+1}(M)).$$

$d$ is a linear differential operator of order 1. There is no other natural linear differential operations of order 1 (the Schouten theorem). Here are some other classical curvatures and natural differential operations: the commutator of two vector fields, the curvature of a Riemannian metric, the Laplacian on a Riemannian manifold [1].

In this article we propose the IT-reduction method. This method gives a reduction of problems about natural differential operations to the corresponding invariant theoretical problems. We show how to use the IT-reduction by considering several known problems. By the IT-reduction we obtain in §3 a description of curvatures of a Riemannian metric. (This description is known in the classical differential geometry [2]. In §4 we describe differential operations on a Riemannian manifold. In §5 we describe curvatures of a connection. In fact, a connection is not a section of any tensor bundle. But one can apply to connections the modified IT-reduction method. In §6 we describe differential operations on a manifold with a connection. As a corollary of this description we obtain in §7 that for any manifold with a connection $\theta$ and a Poisson structure $\omega$ there exists the canonical quantization $\ast((\theta, \omega))$. The word "canonical" means that if $N$ is a manifold with a connection $\theta$ and a Poisson structure $\omega$ and

$$\varphi : M \to N$$

is a diffeomorphism of a manifold $M$ onto an open submanifold of $N$, then

$$\ast((\varphi^*(\theta), \varphi^*(\omega)) = \varphi^*(\ast((\theta, \omega)).$$

Finally, we propose the following Conjecture.

**Conjecture.** Let $M$ be a manifold and $\Gamma(\text{Ass}_M(\wedge^2 V))^P$ be a set of all Poisson structures on $M$. Then there are no natural differential operations

$$\Xi_k : \Gamma(\text{Ass}_M(\wedge^2 V))^P \times C^\infty(M) \times C^\infty(M) \to C^\infty(M), \quad k = 1, 2, \ldots$$

such that the operation

$$\ast : C^\infty(M) \times C^\infty(M) \to C^\infty(M)[[h]],$$

$$(f, g) \mapsto fg + \Xi_1(\omega, f, g)h + \Xi_2(\omega, f, g)h^2 + \ldots.$$
is a quantization of the Poisson structure $\omega$.

Roughly speaking, this conjecture claims that for a canonical quantization of a Poisson structure on a manifold one needs some additional structure on that manifold.

2. INVARIANT THEORETICAL DESCRIPTION OF NATURAL DIFFERENTIAL OPERATIONS.

Let $x^1, \ldots, x^m$ and $z^1, \ldots, z^m$ be two copies of the standard basis of the space $\mathbb{R}^{m*}$. We consider $x^1, \ldots, x^m$ as coordinate functions in $\mathbb{R}^m$. Consider the linear space $\mathcal{E}_k$ of $k$-jets of germs of differentiable functions at 0 $\in \mathbb{R}^m$. We identify the space $\mathcal{E}_k$ and the space of polynomials in the variables $z^1, \ldots, z^m$ of degree $\leq k$. Consider the group $GL(m)_{k+1}$ of $(k+1)$-jets of germs of diffeomorphisms of a neighborhood of 0 in $\mathbb{R}^m$ onto a neighborhood of 0 in $\mathbb{R}^m$. By $GL(m)_{\infty}$ we denote the group of germs of diffeomorphisms of a neighborhood of 0 in $\mathbb{R}^m$ onto a neighborhood of 0 in $\mathbb{R}^m$.

We have the canonical group homomorphism

\[ \varphi_{k+1} : GL(m)_{k+1} \to GL(m), \]

where $\varphi_{k+1}(g)$ is the Jacobi matrix of $g$ at 0. Set

\[ N(m)_{k+1} = \{ g \in GL(m)_{k+1} \mid \varphi_{k+1}(g) = E_m \}, \]

where $E_m$ is the identity matrix of size $m \times m$. The group $GL(m)_{k+1}$ is a real linear algebraic group, $N(m)_{k+1}$ is its unipotent radical, and $GL(m)_{k+1}/N(m)_{k+1} \simeq GL(m)$.

Let $\rho : GL(m) \to GL(V)$ be a linear representation. Consider the space $\mathcal{E}_k \otimes V$ as a linear space of $k$-jets of germs of differentiable mappings of a neighborhood of 0 in $\mathbb{R}^m$ to $V$. By $\mathcal{E}_{\infty} \otimes V$ we denote the linear space of germs of differentiable mappings of a neighborhood of 0 in $\mathbb{R}^m$ to $V$. The group $GL(m)_{\infty}$ acts canonically on the space $\mathcal{E}_{\infty} \otimes V$:

\[ GL(m)_{\infty} : \mathcal{E}_{\infty} \otimes V, \quad (g \cdot v)(x) = \rho(J_g(g^{-1}(x))(v(g^{-1}(x))), \]

where $J_g(x)$ is the Jacobi matrix of $g$ at $x$. This action corresponds to the transition rule for local presentations of a section of the bundle $Ass_M(V)$ in bundle charts. It defines canonically the action

\[ GL(m)_{k+1} : \mathcal{E}_k \otimes V, \quad (g \cdot v)(z) = \{ k \text{-jet of the mapping } x \mapsto \rho(J_g(g^{-1}(x))(v(g^{-1}(x))) \}; \]

where $g = g(z) \in GL(m)_{k+1}$, $v = v(z) \in \mathcal{E}_k \otimes V$.

Note that for $a \geq b$ we have the canonical group homomorphism

\[ GL(m)_a \to GL(m)_b. \]

The representation $[2.1]$ and this homomorphism define the representation

\[ GL(m)_a : \mathcal{E}_b \otimes V \]

for all $a > b$.

We have the following $GL(m)_{k+1}$-equivariant linear mapping

\[ \varepsilon_k : \mathcal{E}_k \otimes V \to V, \quad v(z) \mapsto v(0). \]
Fix \( r(v) \in \mathbb{R}[V]^{GL(m)} \) and consider \( GL(m) \)-invariant subset
\[
V^{\text{reg}} = \{ v \in V \mid r(v) \neq 0 \} \subset V.
\]
This subset defines the natural algebraic nondegeneracy condition for sections of the tensor bundle \( \text{Ass}_M(V) \). Set
\[
(\mathcal{E}_k \otimes V)^{\text{reg}} = \varepsilon_k^{-1}(V^{\text{reg}}).
\]

**Theorem 2.1.** Let \( \sigma : GL(m) \to GL(U) \) be a linear representation. Then there is the canonical 1-1 correspondence between \( GL(m)_{k+1} \)-equivariant morphisms in the form
\[
\alpha : (\mathcal{E}_k \otimes V)^{\text{reg}} \to U = \mathcal{E}_0 \otimes U
\]
and the set of natural differential operations of \( k \)-th order in the form
\[
F_{\alpha,M} : \Gamma(\text{Ass}_M(V))^{\text{reg}} \to \Gamma(\text{Ass}_M(U)).
\]

**Proof.** First, we show that any \( GL(m)_{k+1} \)-equivariant morphism defines canonically a natural differential operation.

Let \( \alpha : \mathcal{E}_k \otimes V \to U \) be \( GL(m)_{k+1} \)-equivariant morphism. Consider local coordinates \( x = (x^1, \ldots, x^m) \) in some neighborhood of a point \( p \in M \). We assume that the local coordinates defines 1-1 mapping of an open subset \( p \in X \subset M \) and some neighborhood \( \widetilde{X} \) of 0 in \( \mathbb{R}^m \). Define the natural differential operation \( F_{\alpha,M} \) in the following way. Suppose that \( s_X : \widetilde{X} \to V \) is a presentation of a section \( s \) of the bundle \( \text{Ass}_M(V) \) in the chart \( X \). We define the section \( F_{\alpha,M}(s) \) of the bundle \( \text{Ass}_M(U) \) in the chart \( X \) by the following formula:
\[
(F_{\alpha,M}(s))_X : \widetilde{X} \to U,
\]
\[
x \mapsto \alpha \left( \sum_{0 \leq |i| \leq k} \frac{|i|!}{i_1! \ldots i_m!} \partial^{i_1}s_X(x) \partial(x^1)^{i_1} \ldots \partial(x^m)^{i_m}(z^1)^{i_1} \ldots (z^m)^{i_m} \right).
\]

Let us check that the section \( F_{\alpha,M}(s) \) is well-defined. Let \( y = (y^1, \ldots, y^m) \) be some other local coordinates in some neighborhood \( Y \) of the point \( p \). The transition formulas from the local coordinates \( x \) to the local coordinates \( y \) define a diffeomorphism \( g \) of a neighborhood of 0 in \( \mathbb{R}^m \) onto a neighborhood of 0 in \( \mathbb{R}^m \). Let \( g_{k+1} \) be \((k+1)\)-jet of the diffeomorphism \( g \) at \( 0 \in \mathbb{R}^m \). Suppose that \( s_Y : \widetilde{Y} \to V \) is a presentation of the section \( s \) in the chart \( Y \). By the transition rule we have
\[
(2.2) \quad s_Y(g(x)) = \rho(J(x))s_X(x),
\]
where \( J(x) \) is the Jacobi matrix of the mapping \( g \). We have to check that
\[
(F_{\alpha,M}(s))_Y(0) = \sigma(J(0))(F_{\alpha,M}(s)_X(0)).
\]
Consider \( k \)-jets of the left and the right sides of the equation \((2.2)\). We obtain

\[
\sum_{0 \leq |i| \leq k} |i|! \frac{\partial^{|i|} s_Y(y)}{\partial (y^1)^{i_1} \cdots \partial (y^m)^{i_m}} \bigg|_{y=0} \left( z^1 y_1 \cdots (z^m y_m) \right) = g_{k+1} \cdot \left( \sum_{0 \leq |i| \leq k} \frac{|i|!}{i_1! \cdots i_m!} \frac{\partial^{|i|} s_X(x)}{\partial (x^1)^{i_1} \cdots \partial (x^m)^{i_m}} \bigg|_{x=0} \left( z^1 i_1 \cdots (z^m i_m) \right) \right).
\]

By the \( \text{GL}(m)_{k+1} \)-equivalence of the morphism \( \alpha \) we obtain

\[
\alpha \left( \sum_{0 \leq |i| \leq k} \frac{|i|!}{i_1! \cdots i_m!} \frac{\partial^{|i|} s_X(x)}{\partial (x^1)^{i_1} \cdots \partial (x^m)^{i_m}} \bigg|_{x=0} \left( z^1 i_1 \cdots (z^m i_m) \right) \right) = \sigma(J(0))((F_{\alpha,M}(s))_Y(0)) = \sigma(J(0))(\alpha((F_{\alpha,M}(s)))_X(0)).
\]

Conversely, by the construction above any natural differential operation defines canonically the \( \text{GL}(m)_{k+1} \)-morphism.

Theorem 2.1 reduces the problem of description of natural differential operations of order \( k \) acting on \( \Gamma(\text{Ass}_M(V))^{\text{reg}} \) to an invariant theoretical problem. Namely, consider the regular action

\[
N(m)_{k+1} : (E_k \otimes V)^{\text{reg}}
\]

and the corresponding algebra of invariants

\[
\mathbb{R}[(E_k \otimes V)^{\text{reg}}]^{N(m)_{k+1}}.
\]

We have the canonical representation

\[
\text{GL}(m)_{k+1}/N(m)_{k+1} \simeq \text{GL}(m) : \mathbb{R}[(E_k \otimes V)^{\text{reg}}]^{N(m)_{k+1}}.
\]

Every \( \text{GL}(m) \)-embedding

\[
(2.3) \quad A_U : U \hookrightarrow \mathbb{R}[(E_k \otimes V)^{\text{reg}}]^{N(m)_{k+1}}
\]

of finite dimensional \( \text{GL}(m) \)-module \( U \) defines canonically \( \text{GL}(m)_{k+1} \)-morphism

\[
(2.4) \quad \alpha_U : (E_k \otimes V)^{\text{reg}} \to U^* = U^* \otimes E_0, \quad \alpha_U(v(z))(u) = A_U(u)(v(z)).
\]

Conversely, every \( \text{GL}(m)_{k+1} \)-morphism \((2.4)\) corresponds to some embedding \((2.3)\)
3. Curvatures of the Riemannian metrics.

Consider $m$-dimensional manifold $M$. Let $e_1, \ldots, e_m$ be the standard basis of $\mathbb{R}^m$ and $z^1, \ldots, z^m$ and $u^1, \ldots, u^m$ are two copies of the dual basis of the dual space $\mathbb{R}^{m*}$. We identify the space $\mathcal{E}_k$ and the space of polynomials in the variables $z^1, \ldots, z^m$ of degree $\leq k$. Riemannian metrics are nondegenerated sections of the bundle $\text{Ass}_M(S^2\mathbb{R}^{m*})$. Fix $k$ and consider the action

$$\mathcal{N}(m)_{k+1} : (\mathcal{E}_k \otimes S^2\mathbb{R}^{m*})^{\text{reg}}.$$

In this article we need a simplified concept of the Seshadri section (see [6]).

**Definition 3.1.** Let $G$ be a linear algebraic group, $G : X$ be a regular action on an affine variety. A closed subvariety $Y \subset X$ is called the nice Seshadri section, if

- $GY = X$,
- every $G$-orbit intersects transversally $Y$ at one point.

Suppose that $Y \subset X$ is a nice Seshadri section and $V$ is a vector space. Consider $V$ as a trivial $G$-module. Then we have the canonical 1-1 correspondence

$$\{\text{set of regular mappings of } Y \text{ to } V\} \rightarrow \{\text{set of regular } G \text{- mappings of } X \text{ to } V\},$$

$$\{\xi : Y \rightarrow V\} \rightarrow \{\xi : X \rightarrow V, \xi(x) = \hat{\xi}((G \cdot x) \cap Y)\}.$$

Our next purpose is to construct the nice Seshadri section for the action (3.1).

Consider the group $\mathcal{N}(m)_{k+1}$ as an affine variety. Then it is isomorphic to a linear space. Namely, we have the following isomorphism

$$\eta : (S^2\mathbb{R}^{m*} \otimes \mathbb{R}^m) \times (S^3\mathbb{R}^{m*} \otimes \mathbb{R}^m) \times \ldots \times (S^{k+1}\mathbb{R}^{m*} \otimes \mathbb{R}^m) \rightarrow \mathcal{N}(m)_{k+1},$$

$$(g_2, g_3, \ldots, g_{k+1}) \mapsto \eta(g_2, g_3, \ldots, g_{k+1}) = (E + g_{k+1}) \cdot \ldots \cdot (E + g_3) \cdot (E + g_2),$$

where $g_n = g_n^i(z) \otimes e_i \in S^n\mathbb{R}^{m*} \otimes \mathbb{R}^m$, $g_n(z) \in S^n\mathbb{R}^{m*}$, $E + g_n$ is $k$-jet at 0 of the mapping

$$\mathbb{R}^m \rightarrow \mathbb{R}^m, \quad c_ie_i \mapsto (c_i + g_n(c_1, \ldots, c_m))e_i.$$

We use the following identification

$$(\mathcal{E}_k \otimes S^2\mathbb{R}^{m*})^{\text{reg}} \simeq (S^2\mathbb{R}^{m*})^{\text{reg}} \times (\mathbb{R}^{m*} \otimes S^2\mathbb{R}^{m*}) \times \ldots \times (S^{k}\mathbb{R}^{m*} \otimes S^{2}\mathbb{R}^{m*}),$$

$$h = h_0 + h_1 + \ldots + h_k \sim (h_0, h_1, \ldots, h_k),$$

where $h_0 \in (S^2\mathbb{R}^{m*})^{\text{reg}}$, $h_n = h_n(z, u)$ is a bihomogeneous polynomial, $\deg_z(h_n) = n$, $\deg_u(h_n) = 2$, i.e., $h_n \in S^n\mathbb{R}^{m*} \otimes S^2\mathbb{R}^{m*}$.

Suppose

$$h = (h_0, h_1, \ldots, h_k) \in (\mathcal{E}_k \otimes S^2\mathbb{R}^{m*})^{\text{reg}}.$$

For $g = \eta(g_2, g_3, \ldots, g_{k+1}) \in \mathcal{N}(m)_{k+1}$ we have

$$g \cdot h = ((g \cdot h)_0, (g \cdot h)_1, \ldots, (g \cdot h)_k),$$

where $(g \cdot h)_n = \mu_n(g, h_j)$. From (2.1) it is not difficult to obtain that

$$\mu_n(g, h_j) = \mu_n'(g_{n+1}, h_0) + \mu_n''(g_2, \ldots, g_n, h),$$
where

\[ \mu_n^i(g_{n+1}, h_0) = -\frac{\partial^2 g_{n+1}}{\partial e_i \partial z_j} \frac{\partial h_0}{\partial u^i} w^j. \]

For \( n \geq 2 \) define GL\((m)\)-submodule

\[ L_n = \text{Ker}(\delta_n) \subset S^n \mathbb{R}^{m*} \otimes S^2 \mathbb{R}^{m*}, \]

where

\[ \delta_n : S^n \mathbb{R}^{m*} \otimes S^2 \mathbb{R}^{m*} \to S^{n+1} \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}, \]

\[ \delta_n(f(z) \otimes q(u)) = f(z)z^i \frac{\partial q(u)}{\partial u^i}. \]

The representation GL\((m) : L_n\) is irreducible. It is isomorphic to the representation GL\((m) : S_{(n,2)}(\mathbb{R}^{m*})\), where \( S_\lambda \) is the Schur functor corresponding to the partition \( \lambda \) (see [3]). Consider the subvariety

\[ \mathbb{L}_k \overset{\text{def}}{=} (S^2 \mathbb{R}^{m*})^{\text{reg}} \times \{0\} \times L_2 \times L_3 \times \ldots \times L_k \subset (E_k \otimes S^2 \mathbb{R}^{m*})^{\text{reg}}. \]

**Lemma 3.2.** Every \( N(m)_{k+1} \)-orbit intersects transversally \( \mathbb{L}_k \) at one point. In other words, \( \mathbb{L}_k \) is a nice Seshadri section for the action [3,1].

**Proof.** Suppose \( h \in (E_k \otimes S^2 \mathbb{R}^{m*})^{\text{reg}} \). For \( g = \eta(g_2, g_3, \ldots, g_{k+1}) \in N(m)_{k+1} \), the condition \( g \cdot h \in \mathbb{L}_k \) is equivalent to the equations \((E_1) - (E_k)\), where

\[ (E_1) ((E + g_2) \cdot h)_1 = 0, \]

\[ (E_n) ((E + g_{n+1}) \cdot \ldots \cdot (E + g_3) \cdot (E + g_2) \cdot h))_n \in L_n, \text{ where } 2 \leq n \leq k. \]

We claim that the equations \((E_1) - (E_k)\) for \( g \) have a unique solution. More precisely:

(*) One can find sequentially the elements \( g_2, \ldots, g_{k+1} \) in a unique way from the equations \((E_1), \ldots, (E_k)\) accordingly. Moreover, the equation \((E_n)\) for \( g_{n+1} \) with fixed (before defined) \( g_2, \ldots, g_n \) is a linear equation that has a unique solution.

Let us prove (*). First, by the nondegeneracy of \( h_0 \) and the GL\((m)\)-equivalence we can assume that \( h_0 = (u^1)^2 + \ldots + (u^m)^2 \).

Consider the equation \((E_1)\) for \( g_2 \). By [3,2] we can rewrite it in the following way:

\[ -2 \frac{\partial^2 g_2}{\partial e_i \partial z_j} \otimes u^i u^j + h_1 = 0. \]

It is easy to see that this equation for \( g_2 \) is a linear equation having a unique solution.

Suppose that we find \( g_2, \ldots, g_n \) from the equations \((E_1), \ldots, (E_{n-1})\). Consider the equation \((E_n)\) for \( g_{n+1} \). By [3,2] and the definition of \( L_n \) we can rewrite the equation \((E_n)\) in the following way:

\[ \delta_n \left( -2 \frac{\partial^2 g_{n+1}}{\partial e_i \partial z_j} \otimes u^i u^j + h_n' \right) = 0, \]

where \( h_n' = h_n'(h, g_2, \ldots, g_n) \in S^n \mathbb{R}^{m*} \otimes S^2 \mathbb{R}^{m*} \). Using the definition of \( \delta_n \) we simplify the equation [3,3] to the following one.

\[ -2 \left( \frac{\partial^2 g_{n+1}}{\partial e_i \partial z_j} + \frac{\partial^2 g_{n+1}}{\partial e_j \partial z_i} \right) z^i \otimes u^j + \delta_n(h_n') = 0. \]
It is not difficult to check that the equation (3.4) for \( g_{n+1} \) is a linear equation having a unique solution. \( \square \)

Consider the mapping
\[
\tilde{\alpha}_k : (\mathcal{E}_k \otimes S^2\mathbb{R}^{m*})_{\text{reg}} \to \mathbb{L}_k, \quad h \mapsto (N(m)_{k+1} \cdot h) \cap \mathbb{L}_k.
\]
We have the natural action of the group \( \text{GL}(m) \) on \( \mathbb{L}_k \). This action defines canonically the action \( \text{GL}(m)_{k+1} : \mathbb{L}_k \) such that the subgroup \( N(m)_{k+1} \) acts on \( \mathbb{L}_k \) trivially.

For \( n = 2, 3, \ldots, k \) let
\[
pr_n : \mathbb{L}_k \to L_n
\]
be the canonical projections. Then
\[
\alpha_n \overset{\text{def}}{=} pr_n \circ \tilde{\alpha}_k : (\mathcal{E}_k \otimes S^2\mathbb{R}^{m*})_{\text{reg}} \to L_n
\]
is \( \text{GL}(m)_{k+1} \)-equivariant morphism. By Theorem 2.1 \( \alpha_n \) corresponds to a natural differential operation
\[
A_n : \Gamma(\text{Ass}_M(S^2\mathbb{R}^{m*}))_{\text{reg}} \to \Gamma(\text{Ass}_M(L_n)).
\]
Lemma 3.2 implies the following statement.

**Theorem 3.3.** Let \( U \) be \( \text{GL}(m) \)-module and
\[
F : \Gamma(\text{Ass}_M(S^2\mathbb{R}^{m*}))_{\text{reg}} \to \Gamma(\text{Ass}_M(U))
\]
be a natural differential operation of order \( k \). Then
\[
F(h) = \tilde{F}(\det(h)^{-1}, h, A_2(h), \ldots, A_k(h)),
\]
where \( \tilde{F} \) corresponds to some polynomial \( \text{GL}(m) \)-mapping
\[
\tilde{f} : \mathbb{R} \times S^2\mathbb{R}^{m*} \times L_2 \times \ldots \times L_k \to U.
\]

**Remark 3.4.** The classical description of the operation \( A_n \) is the following: \( A_n(h)|_p \) is the homogeneous of degree \( n \) summand of the Taylor series of the metric \( h \) at the normal local coordinates with center \( p \) (see [2]).

**4. Natural differential operations on the Riemannian manifolds.**

**Theorem 4.1.** Let \( \rho : \text{GL}(m) \to \text{GL}(V) \) and \( \sigma : \text{GL}(m) \to \text{GL}(U) \) be linear representations. Then there is the canonical 1-1 correspondence between \( \text{GL}(m) \)-equivariant differential operations of bounded order in the form
\[
(4.1) \quad \xi : \mathbb{L}_k \times C^\infty(\mathbb{R}^m, V) \to C^\infty(\mathbb{R}^m, U),
\]
where \( k \in \mathbb{N} \) and the set of natural differential operations of bounded order in the form
\[
\Xi : \Gamma(\text{Ass}_M(V)) \to \Gamma(\text{Ass}_M(U))
\]
on a Riemannian manifold.
Proof. Let $z^1, \ldots, z^m$ be the standard basis of the space $\mathbb{R}^{m*}$. We identify $\mathcal{E}_k$ and the space of polynomials in the variables $z^1, \ldots, z^m$ of degree $\leq k$.

Suppose we have $\text{GL}(m)$-equivariant differential operation (4.1) of an order $\leq k$. Consider the regular action

$$N(m)_{k+1} : (\mathcal{E}_k \otimes S^2(\mathbb{R}^{m*}))^{\text{reg}} \times (\mathcal{E}_k \otimes V).$$

Note that the subvariety $L_k \times (\mathcal{E}_k \otimes V)$ is the nice Seshadri section for this action. Define $\text{GL}(m)$-mapping

$$\hat{\xi} : L_k \times (\mathcal{E}_k \otimes V) \to U, \quad (h, v(z)) \mapsto \xi(h, v(z))(0).$$

Extend the mapping $\hat{\xi}$ to the $\text{GL}(m)_{k+1}$-mapping

$$\xi : (\mathcal{E}_k \otimes S^2(\mathbb{R}^{m*}))^{\text{reg}} \times (\mathcal{E}_k \otimes V) \to U.$$

From Theorem 2.1 it follows that $\xi$ defines the natural differential operation

$$\tilde{\Xi} : \Gamma(\text{Ass}_M(S^2 \mathbb{R}^{m*}))^{\text{reg}} \times \Gamma(\text{Ass}_M(V)) \to \Gamma(\text{Ass}_M(U)).$$

Now suppose that $(M, h)$ is a Riemannian manifold. Define the operation

$$\Xi : \Gamma(\text{Ass}_M(V)) \to \Gamma(\text{Ass}_M(U)), \quad s \mapsto \tilde{\Xi}(h, s).$$

Conversely, by the construction above any natural differential operation on a Riemannian manifold defines canonically the $\text{GL}(m)$-equivariant differential operation (4.1) of bounded order. □

For example, the Laplacian

$$\Delta : \Gamma(\text{Ass}_M(V)) \to \Gamma(\text{Ass}_M(V))$$

corresponds to the following differential operation

$$\delta : S^2(\mathbb{R}^{m*})^{\text{reg}} \times C^\infty(\mathbb{R}^m, V) \to C^\infty(\mathbb{R}^m, V),$$

$$(h, v(x)) \mapsto (h^{-1})^{ij} \frac{\partial^2 v(x)}{\partial x^i \partial x^j},$$

where $h^{-1} \in S^2(\mathbb{R}^m)^{\text{reg}}$ is the dual to $h$ quadratic form.

Remark 4.2. From the proof of Theorem 4.1 we obtain the following description of the corresponding to $\xi$ differential operation $\Xi$. Let $M$ be a Riemannian manifold, $s$ be a section of $\text{Ass}_M(V)$, and $p \in M$. Take normal local coordinates with center $p$ and calculate $\Xi(s)(p)$ in that local coordinates by the formula (4.1).

5. Curvatures of connections.

Consider $m$-dimensional manifold $M$. Let $\text{Con}(M)$ be the set of all connections on $M$. In §1 we define natural differential operations acting on sections of tensor bundles. Analogously one can define natural differential operations acting on connections, pairs $(\theta, \eta)$, where $\theta$ is a connection and $\eta$ is a section of a tensor bundle.

By definition, a curvature of a connection $\theta \in \text{Con}(M)$ is $R(\theta)$, where

$$R : \text{Con}(M) \to \Gamma(\text{Ass}_M(U))$$

is some natural differential operation.
In this section we describe curvatures of a connection on a manifold.

Let $e_1, \ldots, e_m$ be the standard basis of $\mathbb{R}^m$, $x^1, \ldots, x^m, z^1, \ldots, z^m, u^1, \ldots, u^m$, and $v^1, \ldots, v^m$ be copies of the dual basis of the dual space $\mathbb{R}^{m*}$. We consider $x^1, \ldots, x^m$ as coordinate functions in $\mathbb{R}^m$. Define $E$, $\mathbf{GL}(m)_n$, $N(m)_n$ as in [27]. We identify $E_k$ and the space of polynomials in the variables $z^1, \ldots, z^m$ of degree $\leq k$.

A connection on $0 \in X \subset \mathbb{R}^m$ is a mapping

$$ \theta : X \to \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}, \quad x \mapsto \theta(x) = \theta_{ij}(x) e_i \otimes u^i \otimes v^j. $$

The mapping $\theta$ defines the connection operator $D_\theta$ in the following way:

$$ D_\theta(e_i) = \theta_{ij}(x) e_i \otimes u^j. $$

Let $\text{Con}(m)_\infty$ be the set of germs of connections at $0 \in \mathbb{R}^m$ and $\text{Con}(m)_k$ be the set of $k$-jets of germs of connections at $0 \in \mathbb{R}^m$. We use the following identification

$$ \text{Con}(m)_k \cong (\mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}) \cong (\mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}) \times (S^2(\mathbb{R}^{m*}) \otimes \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}) \times \ldots \times (S^k(\mathbb{R}^{m*}) \otimes \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}), $$

$$ \theta = \theta_0 + \theta_1 + \ldots + \theta_k \sim (\theta_0, \theta_1, \ldots, \theta_k), $$

where $\theta_n = \theta_n(z, e, u, v)$ is a polyhomogeneous polynomial of polydegree $(n, 1, 1, 1)$, i.e., $\theta_n \in S^n(\mathbb{R}^{m*}) \otimes \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$.

The group $\mathbf{GL}(m)_\infty$ acts canonically on $\text{Con}(m)_\infty$:

$$ \mathbf{GL}(m)_\infty : \text{Con}(m)_\infty, $$

$$(g \ast \theta)(x) = \left( J_g(x)_a^i \frac{\partial J_g^{-1}(x)_j^a}{\partial x^i} + J_g(x)_a^i \theta_{ab}^c(x) J_g^{-1}(x)_j^b \right) e_l \otimes u^i \otimes v^j, $$

where $J_g(x)$ is the Jacobi matrix of $g$ at $x$. This action corresponds to the transition rule for local presentations of a connection in charts. It defines canonically the action

$$ \mathbf{GL}(m)_{k+2} : \text{Con}(m)_k, \quad (g \cdot \theta)(z) = \{ k - \text{jet of the mapping} \} $$

$$ x \mapsto \left( J_g(x)_a^i \frac{\partial J_g^{-1}(x)_j^a}{\partial x^i} + J_g(x)_a^i \theta_{ab}^c(x) J_g^{-1}(x)_j^b \right) e_l \otimes u^i \otimes v^j, $$

where $g = g(z) \in \mathbf{GL}(m)_{k+2}$, $\theta = \theta(z) \in \text{Con}(m)_k$. The action $\mathbf{GL}(m)_{k+2} : \text{Con}(m)_k$ is an affine action. Consider the restriction of this action to the subgroup $N(m)_{k+2} \subset \mathbf{GL}(m)_{k+2}$:

$$ \text{N}(m)_{k+2} : \text{Con}(m)_k. $$

**Theorem 5.1.** Let $\rho : \mathbf{GL}(m) \to \mathbf{GL}(V)$ and $\sigma : \mathbf{GL}(m) \to \mathbf{GL}(U)$ be linear representations. Then there is the canonical 1-1 correspondence between $\mathbf{GL}(m)_{k+2}$-morphisms in the form

$$ \alpha : \text{Con}(m) \times (E_k \otimes V) \to U = E_0 \otimes U $$

and the set of natural differential operations of order $k$ in the form

$$ F_{\alpha, M} : \text{Con}(M) \times \Gamma(\text{Ass}_M(V)) \to \Gamma(\text{Ass}_M(U)). $$
The proof of this Theorem is the same as of Theorem 2.1.

Our next purpose is to construct the nice Seshadri section for the action (5.2).

We use the isomorphism \( \eta \) from the section 3. Suppose

\[ \theta = (\theta_0, \theta_1, \ldots, \theta_k) \in \text{Con}(m)_k, \]

where \( \theta_n \in S^n(\mathbb{R}^m^*) \otimes \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \). For \( g = \eta(g_2, g_3, \ldots, g_{k+2}) \in \text{N}(m)_{k+2} \) we have

\[ g \cdot \theta = ((g \cdot \theta)_0, (g \cdot \theta)_1, \ldots, (g \cdot \theta)_k), \]

where \( (g \cdot h)_n = \nu_n(\{g, h_j\}) \). From (5.1) it is easy to obtain that

\[ \nu_n(g, \theta_j) = \nu'_n(g_{n+2}) + \nu''_n(g_2, \ldots, g_{n+1}, \theta), \]

where

\[ \nu'_n(g_{n+2}) = -\frac{\partial^2 g_{n+2}}{\partial z^i \partial z^j} \otimes u^i \otimes v^j. \]

Define \( \text{GL}(m) \)-submodule

\[ C_n = \text{Ker}(\gamma_n) \subset S^n(\mathbb{R}^m^*) \otimes \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}, \]

where

\[ \gamma_n : S^n(\mathbb{R}^m^*) \otimes \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \to S^{n+2} \mathbb{R}^m^* \otimes \mathbb{R}^m, \]

\[ \gamma_n(f(z) \otimes e_i \otimes u^i \otimes v^j) = f(z) z^i z^j \otimes e_l. \]

Consider the subvariety

\[ \mathcal{C}_k \overset{\text{def}}{=} C_0 \times C_1 \times \ldots \times C_k \subset \text{Con}(m)_k. \]

**Lemma 5.2.** Every \( \text{N}(m)_{k+2} \)-orbit intersects transversally \( \mathcal{C}_k \) at one point. In other words, \( \mathcal{C}_k \) is a nice Seshadri section for the action (5.2).

**Proof.** Suppose \( \theta \in \text{Con}(m)_k \). For \( g = \eta(g_2, g_3, \ldots, g_{k+2}) \in \text{N}(m)_{k+2} \) the condition \( g \cdot \theta \in \mathcal{C}_k \) is equivalent to equations \( (D_0) \)-\( (D_k) \), where

\[ (D_n) \cdot ((E + g_{n+2}) \cdot \ldots \cdot (E + g_3) \cdot (E + g_2) \cdot \theta))_n \in C_n. \]

It is easy to see that the Lemma is a corollary of the following claim.

(*) One can find sequentially the elements \( g_2, \ldots, g_{k+2} \) in a unique way from the equations \( (D_0), \ldots, (D_k) \) accordingly. Moreover, the equation \( (D_n) \) for \( g_{n+2} \) with fixed (before defined) \( g_2, \ldots, g_{n+1} \) is a linear equation that has a unique solution.

Let us prove claim (*). By (5.3) and the definition of \( C_n \) we can rewrite the equation \( (D_n) \) in the following way:

\[ \gamma_n \left( -\frac{\partial^2 g_{n+2}}{\partial z^i \partial z^j} \otimes u^i \otimes v^j + \theta'_n \right) = 0, \]

where \( \theta'_n = \theta'_n(\theta, g_2, \ldots, g_{n+1}) \in S^n(\mathbb{R}^m^*) \otimes \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \). Using the definition of \( \gamma_n \) and the Euler theorem about homogeneous functions we get

\[ -(n + 2)(n + 1)g_{n+2} + \gamma_n(\theta'_n) = 0. \]

It is clear that this equation for \( g_{n+2} \) with a fixed (before defined) \( g_2, \ldots, g_{n+1} \) is a linear equation having a unique solution. \( \square \)
Consider the mapping 
\[ \tilde{\psi}_k : \text{Con}(m)_k \to \mathbb{C}_k, \quad \theta \mapsto (N(m)_{k+2} \cdot \theta) \cap \mathbb{C}_k. \]
The natural action of the group \( \text{GL}(m) \) on \( \mathbb{C}_k \) defines canonically the action \( \text{GL}(m)_{k+2} : \mathbb{C}_k \) such that the subgroup \( N(m)_{k+2} \) acts trivially. For \( n = 0, 1, \ldots, k \) let 
\[ \text{pr}_n : \mathbb{C}_k \to C_n \]
be canonical projections. Then 
\[ \psi_n \overset{\text{def}}{=} \text{pr}_n \circ \tilde{\psi}_k : \text{Con}(m)_k \to C_n \]
is \( \text{GL}(m)_{k+2} \)-morphism. From Theorem 5.1 it follows that \( \psi_n \) defines canonically a natural differential operation 
\[ \Psi_n : \text{Con}(M) \to \Gamma(\text{Ass}_M(C_k)). \]
From Lemma 5.2 we obtain the following statement.

**Theorem 5.3.** Let \( U \) be a \( \text{GL}(m) \)-module and 
\[ F : \text{Con}(M) \to \Gamma(\text{Ass}_M(U)) \]
be a natural differential operation of order \( k \). Then 
\[ F(\theta) = \tilde{F}(\Psi_0(\theta), \ldots, \Psi_k(\theta)), \]
where \( \tilde{F} \) corresponds to some polynomial \( \text{GL}(m) \)-mapping 
\[ \tilde{f} : C_0 \times C_1 \times \ldots \times C_k \to U. \]

**Remark 5.4.** From the construction of \( \Psi_n \) we obtain the following geometrical description of the curvature \( \Psi_n(\theta)|_p \) of the connection \( \theta \) on \( M \) at \( p \in M \). Take local coordinates \( (x^1, \ldots, x^m) \) with center \( p \) such that for the Taylor series 
\[ \theta = \theta_0 + \theta_1 + \theta_2 + \ldots \]
of the connection \( \theta \) in the local coordinates \( (x^1, \ldots, x^m) \) we have: \( \theta_l \in C_l \) for all \( 0 \leq l \leq n \). Then the local presentation at \( p \) in the local coordinates \( (x^1, \ldots, x^m) \) of the curvature \( \Psi_n(\theta) \) is \( \theta_n \).

**Remark 5.5.** It is easy to see that \( \Psi_0(\theta) \) is the torsion of the connection \( \theta \).

6. **Natural differential operations on a manifold with a connection.**

**Theorem 6.1.** Let \( \rho : \text{GL}(m) \to \text{GL}(V) \) and \( \sigma : \text{GL}(m) \to \text{GL}(U) \) be linear representations. Then there is the canonical 1-1 correspondence between \( \text{GL}(m) \)-equivariant differential operations of bounded order in the form 
\[ (6.1) \quad \xi : \mathbb{C}_k \times C^\infty(\mathbb{R}^m, V) \to C^\infty(\mathbb{R}^m, U), \]
where \( k \in \mathbb{N} \) and the set of natural differential operations of bounded order in the form 
\[ \Xi : \Gamma(\text{Ass}_M(V)) \to \Gamma(\text{Ass}_M(U)) \]
on a manifold with a connection.
Proof. Let $z^1, \ldots, z^m$ be the standard basis of the space $\mathbb{R}^{m*}$. We identify $E_k$ and the space of polynomials in the variables $z^1, \ldots, z^m$ of degree $\leq k$.

Suppose we have $\text{GL}(m)$-equivariant differential operations (6.1) of an order $\leq k$. Consider the regular action

$$N(m)_{k+2} : \text{Con}(m)_k \times (E_k \otimes V).$$

Note that the subvariety $C_k \times (E_k \otimes V)$ is a nice Seshadri section for this action. Define $\text{GL}(m)$-mapping

$$\hat{\xi} : C_k \times (E_k \otimes V) \rightarrow U, \quad (\theta, v(z)) \mapsto \xi(\theta, v(z))(0).$$

Extend the mapping $\hat{\xi}$ to the $\text{GL}(m)_{k+2}$-mapping

$$\xi : \text{Con}(m)_k \times (E_k \otimes V) \rightarrow U,$$

From Theorem 5.1 it follows that $\xi$ defines a natural differential operation $\tilde{\Xi} : \text{Con}(M) \times \Gamma(\text{Ass}_M(V)) \rightarrow \Gamma(\text{Ass}_M(U)).$

Now suppose that $(M, \theta)$ is a manifold with a connection. Define the operation

$$\Xi : \Gamma(\text{Ass}_M(V)) \rightarrow \Gamma(\text{Ass}_M(U)), \quad s \mapsto \tilde{\Xi}(\theta, s).$$

Conversely, by the construction above any natural differential operation of bounded order on a manifold with a connection defines canonically $\text{GL}(m)$-equivariant differential operation (6.1) of bounded order. □

Remark 6.2. From the proof of Theorem 4.1 we obtain the following description of the corresponding to $\xi$ differential operation $\Xi$. Let $M$ be a manifold with a connection $\theta$, $s$ be a section of $\text{Ass}_M(V)$, and $p \in M$. Take local coordinates $(x^1, \ldots, x^m)$ with center $p$ such that for the Taylor series $\theta = \theta_0 + \theta_1 + \ldots$ of the connection $\theta$ in the local coordinates $(x^1, \ldots, x^m)$ we have: $\theta_l \in C_l$ for all $0 \leq l \leq n$. Now calculate $\Xi(s)(p)$ in that local coordinates by the formula (6.1).

7. Canonical quantization of the Poisson structures on a manifold with a connection.

We need $*(\omega)$-product in $\mathbb{R}^m$.

Definition 7.1. Let $x = (x^1, \ldots, x^m)$ be coordinate functions in $\mathbb{R}^m$ and

$$\omega = \omega(x) = \omega^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

be a Poisson structure. $*(\omega)$-product in $\mathbb{R}^m$ is a mapping

$$*(\omega) : C^\infty(\mathbb{R}^m) \times C^\infty(\mathbb{R}^m) \rightarrow C^\infty(\mathbb{R}^m)[[\hbar]],$$

such that

- $*(\omega)$ is a quantization of the Poisson structure $\omega$. 


\[ \beta_k : C^\infty(\mathbb{R}^m, \wedge^2(\mathbb{R}^m)) \times C^\infty(\mathbb{R}^m) \times C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m) \]

is \( \text{GL}(m) \)-equivariant differential operation of bounded order, where \( k = 1, 2, \ldots \).

\( \star(\omega) \)-product in \( \mathbb{R}^m \) is a natural generalization of the Moyal \( \star \)-product. The first \( \star(\omega) \)-product in \( \mathbb{R}^m \) was constructed by Kontsevich in [4].

**Remark 7.2.** By using the IT-reduction it is not difficult to prove that there exists \( \star(\omega) \)-product in \( \mathbb{R}^m \). This proof is constructive: it gives an algorithm for calculation the operations \( \beta_k \).

**Theorem 7.3.** For a manifold with a connection and a Poisson structure there exists the canonical quantization.

**Proof.** Let \( M \) be a manifold with a connection \( \theta \) and a Poisson structure \( \omega \). Let the natural differential operation

\[ B_k : \Gamma(\text{Ass}_M(\wedge^2 \mathbb{R}^m)) \times C^\infty(M) \times C^\infty(M) \to C^\infty(M) \]

corresponds to \( \beta_k \), where \( k = 1, 2, \ldots \) (see Theorem 6.1). Set

\[ \star : C^\infty(M) \times C^\infty(M) \to C^\infty(M)[[[\hbar]]], \]

\[ (f, g) \mapsto fg + B_1(\omega, f, g)\hbar + B_2(\omega, f, g)\hbar^2 + \ldots. \]

(7.1)

We claim that the operation (7.1) is a quantization of the Poisson structure \( \omega \). To prove it we take a point \( p \in M \) and \( n \in \mathbb{N} \) and prove that the defined above operation \( \star \) gives a quantization of the Poisson structure \( \omega \) modulo \( O(\hbar^{n+1}) \). Suppose that \((x^1, \ldots, x^m)\) are local coordinates with center \( p \) such that for the Taylor series \( \theta = \theta_0 + \theta_1 + \ldots \) of the connection \( \theta \) in the local coordinates \((x^1, \ldots, x^m)\) we have: \( \theta_l \in C_l \) for all \( 0 \leq l \leq n \). By Remark 6.2 the operation \( \star \) at the point \( p \) in the local coordinates \((x^1, \ldots, x^m)\) coincides with the Moyal-Kontsevich \( \star(\omega) \)-product modulo \( O(\hbar^{n+1}) \). This concludes the proof. \( \square \)

**Remark 7.4.** From section 4 one can easily obtain the following quantization rule for a Riemannian manifold.

Let \( M \) be a Riemannian manifold with a Poisson structure \( \omega \), \( p \in M \). Take normal local coordinates \((x^1, \ldots, x^m)\) with center \( p \). Then define \( \star \)-product at \( p \) in coordinates \((x^1, \ldots, x^m)\) by the Moyal-Kontsevich formula for \( \star(\omega) \)-product.

**References**

[1] Besse A., *Einstein manifolds*. Springer-Verlag, 1987.
[2] Epstein D.B.A., *Natural tensors on Riemannian manifolds* J. Diff. Geom., 1975, 10, p. 631-645.
[3] Fulton W., Harris J., *Representation Theory*. Springer-Verlag, 1991.
[4] Kontsevich M., *Deformation quantization of Poisson manifolds I*. QA/9709040.
[5] Nomizu K., *Lie Groups and Differential Geometry*. Mathematical Society of Japan, Tokyo, 1956.
[6] Seshadri C.S., *On a theorem of Weitzenbök in invariant theory* J. Math. Kyoto Univ., 1962, 1, p. 403-409.

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