Block circulant matrices with circulant blocks, weil sums and mutually unbiased bases, II. The prime power case

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To cite this version:

M. Combescure. Block circulant matrices with circulant blocks, weil sums and mutually unbiased bases, II. The prime power case. Journal of Mathematical Physics, American Institute of Physics (AIP), 2009, 50, pp.032104.

HAL Id: in2p3-00184037
http://hal.in2p3.fr/in2p3-00184037
Submitted on 30 Oct 2007

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Abstract

In our previous paper [7] we have shown that the theory of circulant matrices allows to recover the result that there exists $p + 1$ Mutually Unbiased Bases in dimension $p$, $p$ being an arbitrary prime number. Two orthonormal bases $B, B'$ of $\mathbb{C}^d$ are said mutually unbiased if $\forall b \in B, \forall b' \in B'$ one has that

$$|b \cdot b'| = \sqrt{\frac{d}{d}}$$

$(b \cdot b'$ hermitian scalar product in $\mathbb{C}^d$). In this paper we show that the theory of block-circulant matrices with circulant blocks allows to show very simply the known result that if $d = p^n$ ($p$ a prime number, $n$ any integer) there exists $d + 1$ mutually Unbiased Bases in $\mathbb{C}^d$. Our result relies heavily on an idea of Klimov, Muñoz, Romero [11]. As a subproduct we recover properties of quadratic Weil sums for $p \geq 3$, which generalizes the fact that in the prime case the quadratic Gauss sums properties follow from our results.

1 INTRODUCTION

The Mutually Unbiased Bases in dimension $d$ are a set $\{B_1, ..., B_N\}$ of orthonormal bases in $\mathbb{C}^d$ such that for any $b_j^{(k)} \in B_k, b_{j'}^{(k')} \in B_{k'}$ one has

$$|b_j^{(k)} \cdot b_{j'}^{(k')}| = \frac{1}{\sqrt{d}}, \quad \forall j, j' = 1, ..., d, \quad \forall k' \neq k = 1, ..., N$$

where $b \cdot b' = \sum_{j=1}^{d} b_j^{*} b'_{j}$ is the usual scalar product in $\mathbb{C}^d$. This notion of mutually unbiased bases emerged in the seminal work of Schwinger
and turned out to be a cornerstone in the theory of quantum information. Furthermore it is strongly linked with the theory of Hadamard matrices and to the Gauss Sums properties.

An important problem is the maximum number of mutually unbiased bases (MUB) in dimension $d$. The problem has been completely solved for $d = p^n$ where $p$ is a prime number, and $n$ any integer, in which case one can find $N = d + 1$ MUB’s.

In a previous paper (hereafter referred to as I) we have provided a construction of $d + 1$ MUB’s for $d$ a prime number using a new method involving circulant matrices. Then the MUB problem reduces to exhibit a circulant matrix $C$ which is a unitary Hadamard matrix, such that its powers are also circulant unitary Hadamard matrices. Then using the Discrete Fourier Transform $F_d$ which diagonalizes all circulant matrices, we have shown that a MUB in that case is just provided by the set of column vectors of the set of matrices $\{F_d, I, C, C^2, \ldots, C^{d-1}\}$. Properties of quadratic Gauss sums follow as a by-product of the method.

The present paper is a continuation of I, where we consider $d = p^n$. Here circulant matrices are replaced by a set of block-circulant with circulant blocks matrices. Again the discrete Fourier transform which in this case will be simply $F = F_p \otimes F_p \otimes \ldots \otimes F_p$ will play a central role since it diagonalizes all block-circulant with circulant blocks matrices. We follow an idea of [11] to define them. The new result developed here is that these block-circulant matrices with circulant blocks together with $F$ will solve the MUB problem in that case.

Let $B_k = \{b^{(k)}_0, b^{(k)}_1, \ldots, b^{(k)}_{d-1}\}$ be orthonormal bases. Then in any given base, they are represented by unitary matrices $B_k$. Taking $B_0$ to be the natural base, we have that

$$b^{(k)}_j \cdot b^{(k')}_{j'} = (B_k^* B_{k'})_{j,j'}$$

Thus in order that the bases $B_k$ be unbiased, we just need that all the unitary matrices $B_k^* B_{k'}$, $k \neq k'$ have matrix elements of modulus $d^{-1/2}$. Such matrices are known as unitary Hadamard Matrices ([9]).

2 THE SQUARE OF A PRIME

Let $p$ be a prime number. One defines a primitive $p$-th root of unity:

$$\omega = \exp \left( \frac{2\pi i}{p} \right)$$
The Discrete Fourier Transform in \( \mathbb{C}^p \) is

\[
F_p = \frac{1}{\sqrt{p}} \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega & \omega^2 & \ldots & \omega^{p-1} \\
1 & \omega^2 & \omega^4 & \ldots & \omega^{2(p-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^{p-1} & \omega^{2(p-1)} & \ldots & \omega^{(p-1)(p-1)}
\end{pmatrix}
\]

**Definition 2.1** Consider a \( d \)-periodic sequence \( a_0, a_1, \ldots, a_{d-1}, a_0, a_1, \ldots \).

(i) A \( d \times d \) matrix \( D \) is diagonal and called \( \text{diag}(a_0, \ldots, a_{d-1}) \) if its matrix elements satisfy

\[ D_{j,k} = a_k \delta_{j,k}, \quad \forall j, k = 0, 1, \ldots, d-1 \]

(ii) A \( d \times d \) matrix \( C \) is called circulant and denoted \( C = \text{circ}(a_0, \ldots, a_{d-1}) \) if its matrix elements satisfy

\[ C_{j,k} = a_{(d-1)j+k} \]

Thus it can be written as:

\[
C = \begin{pmatrix}
a_0 & a_1 & a_2 & \ldots & a_{d-1} \\
a_{d-1} & a_0 & a_1 & \ldots & a_{d-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_1 & a_2 & a_3 & \ldots & a_0
\end{pmatrix}
\]

(iii) A diagonal and circulant matrix must be a multiple of the identity matrix \( \mathbb{1} \).

(iv) A \( d^2 \times d^2 \) matrix is said to be block-circulant if it is of the form

\[
C = \text{circ}(C_0, C_1, \ldots, C_{d-1})
\]

where the \( C_j \) are \( d \times d \) matrices.

(v) It is block-circulant with circulant blocks if furthermore the \( C_j \) are circulant.

(vi) A \( d \times d \) matrix \( H \) is a unitary Hadamard matrix if

\[
|H_{j,k}| = d^{-1/2}, \quad \text{and} \quad \sum_{k=0}^{d-1} H^*_{j,k} H_{k,l} = \delta_{j,l}
\]

We define the following \( p \times p \) unitary matrices

\[
X = \text{circ}(0, 0, \ldots, 1) \quad Z = \text{diag}(1, \omega, \ldots, \omega^{p-1})
\]

They obey the \( \omega \)-commutation rule:

**Lemma 2.2** (i) \( X^p = Z^p = \mathbb{1} \)

(ii) \( ZX = \omega XZ \)

(iii) Furthermore one has \( F_p X F_p^* = Z \)
(i) and (ii) are obvious. For a proof of (iii) see [8].

**Proposition 2.3** Let $C = \text{circ}(a_0, \ldots, a_{p-1})$.

(i) One has

$$C = \sum_{k=0}^{p-1} a_k X^{p-k}$$

(ii) The discrete Fourier transform diagonalizes the circulant matrices:

$$F_p CF_p^* = \text{diag}(\tilde{a}_0, \ldots, \tilde{a}_{p-1})$$

where

$$\tilde{a}_j = \sum_{k=0}^{p-1} a_k \omega^{-jk}$$

(iii) The set of circulant $p \times p$ matrices is a commutative algebra.

Proof:

$$F_p CF_p^* = \sum_{k=0}^{p-1} a_k Z^{p-k}$$

But $Z^{-k} = \text{diag}(1, \omega^{-k}, \ldots, \omega^{-(p-1)})$, hence (ii) follows. (iii) is a consequence of (i).

**Corollary 2.4** If the sequence $\{a_k\}_{k \in \mathbb{F}_d}$ is such that

$$|a_k| = \frac{d-1}{2} \quad \text{and} \quad |\tilde{a}_k| = 1$$

then $C$ is a circulant unitary Hadamard matrix.

Proof: $C$ is unitarily equivalent to an unitary matrix if $|\tilde{a}_k| = 1$. Furthermore $|a_k| = \frac{d-1}{2}$, hence the result follows (see [7]).

The discrete Fourier transform in $\mathbb{C}^{p^2}$ is defined as follows

$$F = F_p \otimes F_p \quad (2.1)$$

It has the following important property (similar to the property that the discrete Fourier transform diagonalizes all circulant matrices):

**Proposition 2.5** (i) $F$ is an unitary Hadamard matrix.

(ii) All block-circulant matrices $C$ with circulant blocks are diagonalized by $F$:

$$FCF^* = D$$

where $D$ is a $p^2 \times p^2$ diagonal matrix.
For a proof of this result see [8].

We shall be interested in finding block-circulant with circulant blocks unitary matrices in $\mathbb{C}^{p^2}$ that are Hadamard matrices. An example is of course $C \otimes C'$ where $C, C'$ are unitary circulant Hadamard matrices.

For $p$ a prime number, denote by $\mathbb{F}_p$ the field of residues modulo $p$. The corresponding Galois field $GF(p^2)$ is defined as follows. For any $p$ there exists an irreducible polynomial of degree two, with coefficients in $\mathbb{F}_p$, so that if we denote by $\alpha$ a root of this polynomial,

$$GF(p^2) = \{m\alpha + n\}_{m,n \in \mathbb{F}_p}$$

The product $\theta \cdot \theta' \in GF(p^2)$ for $\theta, \theta' \in GF(p^2)$ is obtained using the irreducible polynomial which expresses $\alpha^2$ in terms of $\alpha$ and 1.

The additive characters $\chi(\theta)$ in $GF(p^2)$ are defined as follows:

**Definition 2.6** The additive characters on $GF(p^2)$ are:

$$\chi(\theta) = \exp \left(\frac{2i\pi}{p} \text{tr}\theta\right)$$

where

$$\text{tr}\theta = \theta + \theta^p$$

**Lemma 2.7** They satisfy:

(i) $\chi(\theta + \theta') = \chi(\theta)\chi(\theta')$

(ii) One has:

$$\sum_{\theta \in GF(p^2)} \chi(\theta) = 0$$

(iii)

$$\sum_{\theta' \in GF(p^2)} \chi(\theta \cdot \theta') = p^2\delta_{\theta,0} \quad (2.2)$$

We take as natural basis in $\mathbb{C}^{p^2}$ the set of states labelled by $\theta \in GF(p^2)$, in the following order:

$$\mathcal{B} \equiv \{|0\rangle, |\alpha\rangle, |2\alpha\rangle, ..., |1\rangle, |1 + \alpha\rangle, ..., |p - 1\rangle, ..., |p - 1 + (p - 1)\alpha\rangle\}$$

Labelled by $\theta \in GF(p^2)$ we define a set of unitary operators in $\mathbb{C}^{p^2}$ such that:

**Definition 2.8** (i) The set of operators $\mathcal{F} \equiv \{X_\theta\}_{\theta \in GF(p^2)}$ obeys

$$X_\theta|\theta'\rangle = |\theta + \theta'\rangle, \forall \theta' \in GF(p^2) \quad (2.3)$$

(ii) The set of diagonal operators $\mathcal{F}' \equiv \{Z_\theta\}_{\theta \in GF(p^2)}$ obeys

$$Z_\theta|\theta'\rangle = \chi(\theta \cdot \theta')|\theta\rangle, \forall \theta' \in GF(p^2) \quad (2.4)$$
They obey:

**Proposition 2.9 (i)**
\[ Z_\theta X_{\theta'} = \chi(\theta \cdot \theta') X_{\theta'} Z_\theta \]  
(2.5)

(ii) The operators in \( \mathcal{F} \), \( \mathcal{F}' \) obey the group commutative property:
\[ X_{\theta+\theta'} = X_\theta X_{\theta'} = X_{\theta'} X_\theta, \quad Z_{\theta+\theta'} = Z_\theta Z_{\theta'} = Z_{\theta'} Z_\theta \]  
(2.6)

(iii) \( Z_0 = X_0 = 1 \)

Proof: Take any \( \varphi \in GF(p^2) \). Then
\[ Z_\theta X_{\theta'} |\varphi\rangle = \chi(\theta \cdot (\theta' + \varphi)) \varphi = \chi(\theta \cdot \theta') X_{\theta'} Z_\theta |\varphi\rangle = \chi(\theta \cdot \theta') \chi(\theta \cdot \varphi) |\theta' + \varphi\rangle \]

We also have (ii):
\[ Z_{\theta+\theta'} |\varphi\rangle = \chi((\theta + \theta') \cdot \varphi) |\varphi\rangle = \chi(\theta \cdot \varphi) \chi(\theta' \cdot \varphi) |\varphi\rangle = Z_\theta Z_{\theta'} |\varphi\rangle \]

\[ \square \]

**Theorem 2.10 (i)**
\[ \mathcal{F} = \{ X^m \otimes X^n \}_{m,n \in \mathbb{F}_p} \]

More precisely one has
\[ X_{\alpha+n} = X^n \otimes X^m \]  
(2.7)

(ii) All operators in \( \mathcal{F} \) are represented by unitary block-circulant with circulant blocks \( p^2 \times p^2 \) matrices.

(iii) \[ \mathcal{F}' = \{ Z^m \otimes Z^n \}_{m,n \in \mathbb{F}_p} \]

(iv) All operators in \( \mathcal{F}' \) are represented by diagonal \( p^2 \times p^2 \) matrices.

(v) For any \( \theta' \in GF(p^2) \) there exists a \( \theta \in GF(p^2) \) such that
\[ FX_{\theta} F^* = Z_\theta \]  
(2.8)

Proof of (i): It is enough to see that \( X_\alpha = 1 \otimes X \) and \( X_1 = X \otimes 1 \) since the other matrices \( X_{\alpha+n} \) will be given by the chain rule:
\[ X_{\alpha+n} = X^m \alpha X^n_1 \]

But these are obviously block-circulant with circulant blocks matrices.

One has for \( \theta' = m\alpha + n \):
\[ FX_{\alpha+n} F^* = FX^m_\alpha X^n_1 F^* = (F_p \otimes F_p)(X^n \otimes X^m)(F^*_p \otimes F^*_p) = Z^n \otimes Z^m \]

which is \( Z_\theta \) for some \( \theta \in GF(p^2) \). \( \square \)

One recalls a famous result [8]:
Proposition 2.11 All block-circulant matrices with circulant blocks commute and are diagonalized by $F$.

Proof: It follows from (2.8) that if $C = \sum_{\theta} \lambda_{\theta}X_{\theta}$ is a block-circulant matrix with circulant blocks, one has

$$FCF^* = \sum_{\theta' \in GF(p^2)} \lambda_{\theta'}FX_{\theta'}F^* = \sum_{\theta' \in GF(p^2)} \lambda_{\theta'}Z_{f(\theta')},$$

which is a diagonal matrix.

To find the MUB’s in dimension $p^2$ it is enough to exhibit a partition of the set of unitary operators:

$$E \equiv \{Z_{\theta}X_{\theta'}\}_{\theta, \theta' \in GF(p^2)}$$

into a set of commutant families: We define

$$\mathcal{F}_0 = \mathcal{F} \setminus \{1\}$$

One wants:

$$E = \mathcal{F}_0 \bigcup_{\theta \in GF(p^2)} \mathcal{C}_\theta \cup \{1\}$$

The family $\mathcal{C}_\theta$ will be defined as follows:

Definition 2.12 Let for any $\theta \in GF(p^2)$

$$\mathcal{E}_\theta = \{Z_{\theta'}X_{\theta'\theta}\}_{\theta' \in GF(p^2)}$$

Define

$$\mathcal{C}_\theta = \mathcal{E}_\theta \setminus \{1\}$$

Proposition 2.13 (i) $\mathcal{E}_0 = \mathcal{F}'$

(ii) $\mathcal{E}_\theta$ is a commuting family $\forall \theta \in GF(p^2)$.

(iii) $E = \mathcal{F}_0 \bigcup_{\theta \in GF(p^2)} \mathcal{C}_\theta \cup \{1\}$ is a partition of $E$.

Proof: (i) is obvious.

(ii) $\forall \theta', \theta'' \in GF(p^2)$ one has

$$Z_{\theta}X_{\theta'\theta}Z_{\theta''}X_{\theta'\theta} = \chi(-\theta \cdot \theta' \cdot \theta'')Z_{\theta' + \theta''}X_{\theta,(\theta' + \theta'')} = Z_{\theta'\theta''}X_{\theta,\theta''}Z_{\theta'}X_{\theta''}\theta$$

(iii) $\mathcal{C}_\theta$ and $\mathcal{F}_0$ contain $p - 1$ elements. The classes $\mathcal{C}_\theta$ for different $\theta$ are disjoint. Therefore

$$\bigcup_{\theta \in GF(p^2)} \mathcal{C}_\theta$$

contains $p(p - 1)$ elements. One has:

$$p - 1 + p(p - 1) + 1 = p^2$$
which is the total number of elements in \( E \).

Since all the unitary operators in \( C_\theta \) commute, they can be diagonalized by the same operator \( R_\theta \). In the above cited work [11] they are defined as “rotation operators”. In fact we shall see that they are represented in the basis \( B \) by block-circulant with circulant block matrices. The first main result of this paper is the following:

**Theorem 2.14**

(i) There exists a set \( \{ R_\theta \}_{\theta \in \text{GF}(p^2)} \) of unitary operators which diagonalize all the operators of the class \( C_\theta \), \( \forall \theta \in \text{GF}(p^2) \).

(ii) The operators \( R_\theta \) for \( \theta \neq 0 \) are represented in the basis \( B \) by block-circulant with circulant block matrices which are unitary Hadamard matrices.

(iii) For \( p \geq 3 \) they obey the group law:

\[
R_{\theta+\theta'} = R_\theta R_{\theta'}, \ \forall \theta, \theta' \in \text{GF}(p^2)
\]

Proof: It is enough to show that for any \( \theta \in \text{GF}(p^2) \) \( \setminus \{ 0 \} \) the \( R_\theta \) can be expanded as

\[
R_\theta = \sum_{\theta' \in \text{GF}(p^2)} \lambda_\theta^{(\theta')} X_{\theta'} \tag{2.9}
\]

since they will automatically represented in the basis \( B \) by block-circulant with circulant blocks matrices. One has to check that

\[
R_\theta^{-1} Z_{\theta'} X_{\theta \cdot \theta'} R_\theta = \mu_{\theta, \theta'} Z_{\theta'}, \ \forall \theta' \in \text{GF}(p^2)
\]

Since the operators \( Z_{\theta'} X_{\theta \cdot \theta'} \) are unitary, the \( \mu_{\theta, \theta'} \) are necessarily complex numbers of modulus one. But

\[
Z_{\theta'} X_{\theta \cdot \theta'} \sum_{\theta''} \lambda_{\theta'}^{(\theta')} X_{\theta'} = Z_{\theta'} \sum_{\theta''} \lambda_{\theta'}^{(\theta')} X_{\theta' + \theta' - \theta''} = \mu_{\theta, \theta'} \sum_{\theta''} \lambda_{\theta'}^{(\theta')} X_{\theta''} Z_{\theta'} = \mu_{\theta, \theta'} Z_{\theta'} \sum_{\theta''} \chi(-\theta' \cdot \theta'') \lambda_{\theta''}^{(\theta')} X_{\theta''}
\]

Equating the coefficients of \( X_{\theta''} \) in both sides we get

\[
\lambda_{\theta'' - \theta, \theta'} = \mu_{\theta, \theta'} \chi(-\theta' \cdot \theta'') \lambda_{\theta''}^{(\theta')}
\]

Taking \( \theta'' = 0 \) and assuming that \( \lambda_{\theta}^{(\theta)} = p^{-1}, \ \forall \theta \in \text{GF}(p^2) \) we get

\[
\lambda_{\theta, \theta'}^{(\theta)} = p^{-1} \mu_{\theta, \theta'}
\]

or equivalently, since \( \theta \neq 0 \)

\[
\lambda_{\theta'}^{(\theta)} = p^{-1} \mu_{\theta, -\theta - \theta'}
\]

This proves that all the \( \lambda_{\theta'}^{(\theta)} \) must be of modulus \( p^{-1} \).

Therefore since all the \( X_\theta \) are represented by unitary matrices that have non-zero elements (actually 1) where all the others have zeros, and since every matrix element of \( R_\theta \) is of the form \( \lambda_{\theta'}^{(\theta)} \) for some \( \theta' \in \text{GF}(p^2) \), this proves that all the \( R_\theta \) are represented by Hadamard matrices.
Now we have to check the compatibility condition. We reexpress it in terms of \( \mu_{\theta, \theta'}. \)
Suppressing the index \( \theta \) in the \( \mu_{\theta, \theta'} \) for simplicity, we need to have \( \forall \theta', \theta'' \in GF(p^2) \)
\[
\mu_{-\theta - 1(\theta'' - \theta')} = \mu_{\theta'} \mu_{-\theta - 1(\theta'' - \theta)} \chi(-\theta' \cdot \theta'')
\]
or in other terms
\[
\mu_{\theta' + \theta''} = \mu_{\theta'} \mu_{\theta''} \chi(\theta \cdot \theta')
\]  
(2.10)
But this results easily from the group property of the \( X_{\theta} \)'s and \( Z_{\theta} \)'s (2.3, 2.4):
\[
R_{\theta}^{-1} Z_{\theta' + \theta''} X_{\theta(\theta' + \theta'')} R_{\theta} = \chi(\theta \theta' \theta'') R_{\theta}^{-1} Z_{\theta'} X_{\theta''} R_{\theta} R_{\theta}^{-1} Z_{\theta''} X_{\theta' + \theta''} R_{\theta} = \chi(\theta \theta' \theta'') \mu_{\theta, \theta'} \mu_{\theta', \theta''} Z_{\theta' + \theta''}
\]
In [11] it is shown that for \( p \geq 3 \) the solution of (2.11) with \( \mu_{\theta,0} = 1 \) is
\[
\mu_{\theta, \theta'} = \chi(2^{-1} \theta \cdot \theta'^2)
\]  
(2.11)
Thus we deduce that
\[
\lambda_{\theta}^{(\theta')} = p^{-1} \chi(2^{-1} \theta' \cdot (\theta')^2)
\]  
(2.12)
We now prove the unitarity of \( R_{\theta}. \) For \( \theta = 0 \) this is obvious since \( R_0 = \mathbb{1}. \) It is enough to check that for \( \theta \neq 0 \) one has :
\[
\sum_{\theta' + \theta''} (\lambda_{\theta}^{(\theta')})^* \lambda_{\theta' + \theta''} = \delta_{\theta',0}
\]
One has :
\[
\sum_{\theta' + \theta''} (\lambda_{\theta}^{(\theta')})^* \lambda_{\theta' + \theta''} = \frac{1}{p^2} \sum_{\theta' + \theta''} \mu_{\theta,\theta-1,\theta'} \mu_{\theta,\theta-1,\theta''} = \frac{1}{p^2} \sum_{\theta' + \theta''} \mu_{\theta,\theta-1,\theta'} \mu_{\theta,\theta-1,\theta''} \chi(\theta^{-1} \theta' \theta'') = \mu_{\theta,\theta-1,\theta''} \frac{1}{p^2} \sum_{\theta' + \theta''} \chi(\theta^{-1} \theta' \theta'') = \delta_{\theta',0}
\]
where we have used (2.10) and (2.2), together with the fact that \( \theta^{-1} \cdot \theta'' = 0 \ implies \theta'' = 0 \) due to the field property of \( GF(p^2). \) Thus one has
\[
R_{\theta}^{-1} = R_{\theta}^*, \forall \theta \in GF(p^2)
\]
(iii) The group law \( R_{\theta + \theta'} = R_{\theta} R_{\theta'} \) for \( p \geq 3 \) has been established in [11]. For \( p = 2 \) it has to be suitably modified as shown in [11] (see remark below). Let us see how it works for \( p \geq 3 : \)
\[
(R_{\theta}^*)^m Z_{\theta} X_{(n+ma) \theta'} R_{\theta}^m R_{\theta}^m = (R_{\theta})^n (R_{\theta})^m Z_{\theta'} X_{\theta-ma} R_{\theta}^m X_{\theta''} R_{\theta}^m = \mu(ma, \theta') (R_{\theta}^*)^n Z_{\theta'} X_{\theta''} R_{\theta}^m
\]
holds for \( p \geq 3 \) since
\[
\mu_{ma + n, \theta'} = \mu_{ma, \theta'} \mu_{n, \theta'}, \forall \theta' \in GF(p^2)
\]
which easily follows from (2.11) for \( p \geq 3, \) and the additivity of the characters.
Remark 2.15 For \( p = 2 \) the group law is not satisfied (see \([11]\)). One has instead a very similar property (modified group law):

\[
\begin{align*}
R_\alpha R_{\alpha+1} &= R_{\alpha+1}R_\alpha = R_1 \\
R_\alpha R_1 &= R_{\alpha+1}X_{\alpha+1} \\
R_{\alpha+1}R_\alpha &= R_\alpha X_\alpha \\
R_\alpha^2 &= X_{\alpha+1} \\
R_{\alpha+1}^2 &= X_\alpha \\
R_1^2 &= X_1
\end{align*}
\]

The second main result of this paper is the following:

**Theorem 2.16** The set of operators \( \{F, R_\theta\}_{\theta \in GF(p^2)} \) defines a set of \( p^2 + 1 \) MUB’s in \( \mathbb{C}^{p^2} \).

Proof: Each \( R_\theta \) is represented by an unitary Hadamard matrix, and so is \( F \). Due to the group property, \( R_\theta^* R_{\theta'} = R_{\theta - \theta'} \) thus is an unitary Hadamard matrix. It remains to show that \( R_\theta F^* \) is an unitary Hadamard matrix \( \forall \theta \in GF(p^2) \). But we have

\[
R_\theta F^* = F^* D_\theta, \quad \forall \theta \in GF(p^2)
\]

with \( D_\theta \) an unitary diagonal matrix, since \( F \) diagonalizes all block-circulant matrices with circulant blocks. The product of \( D_\theta \) with the unitary Hadamard matrix \( F^* \) is obviously an unitary Hadamard matrix.

In the case of dimension \( 2^2 \) one has instead of the group property that

\[
\forall \theta, \theta', \exists \theta'' \text{ such that } R_\theta^* R_{\theta'} = R_{\theta - \theta'} X_{\theta''}
\]

Since \( X_{\theta''} \) has exactly one non-vanishing element (1) on each line and column, the product \( R_{\theta - \theta'} X_{\theta''} \) is indeed an unitary Hadamard matrix. \( \square \)

3 THE CASE \( d = p^n \)

The case \( d = p^n \) with general power \( n \) can be treated similarly. There is a notion of block-circulant matrices with block-circulant blocks and building block \( p \times p \) matrices which are circulant which generalizes the case \( n = 2 \). These are diagonalized by the Discrete Fourier Transform \( F \) in \( \mathbb{C}^{p^n} \) which is

\[
F = F_p \otimes F_p \otimes \ldots \otimes F_p
\]

(\( n \) times) which is obviously an unitary Hadamard matrix.

The Galois field \( GF(p^n) \) is defined through the irreducible polynomial which is of
power $n$, with coefficients in $\mathbb{F}_p$. Thus elements of the Galois field $GF(p^n)$ are of the form

$$\theta = \sum_{i=0}^{n-1} c_i \alpha^i$$

where $c_i \in \mathbb{F}_p$ and $\alpha$ is a root of the characteristic polynomial. The characters are

$$\chi(\theta) = \exp \left( \frac{2i\pi}{p} \text{tr}(\theta) \right)$$

where

$$\text{tr}(\theta) = \theta + \theta^p + \ldots + \theta^{p^{n-1}}$$

For any $\theta \in GF(p^n)$ the operators $X_\theta$, $Z_\theta$, $R_\theta$ are defined in [11] and the $R_\theta$ obey a group law (resp. a modified group law) if $p \geq 3$ (resp. $p = 2$).

As previously we have $X_\theta \in \{X^{k_1} \otimes X^{k_2} \otimes \ldots \otimes X^{k_n} \}$, with $k_j \in \mathbb{F}_p$ and $Z_\theta \in \{Z^{k_1} \otimes Z^{k_2} \otimes \ldots \otimes Z^{k_n} \}_{k_i \in \mathbb{F}_p}$ and

$$R_\theta = \sum_{\theta' \in GF(p^n)} \lambda_{\theta'}^{(\theta)} X_{\theta'}$$

with $\lambda_{\theta'}^{(\theta)}$ of modulus $\frac{1}{\sqrt{p^n}}$. The $\lambda_{\theta'}^{(\theta)}$ are given for $p \geq 3$ similarly to (2.12) by

$$\lambda_{\theta'}^{(\theta)} = p^{-n/2} \chi(2^{-1} \theta^{-1} (\theta')^2)$$

(3.13)

All the results of the previous section are easily generalized.

**Theorem 3.1** The unitary Hadamard matrices $F, R_\theta$ for $\theta \in GF(p^n)$ define a set of $p^n + 1$ MUB’s in $\mathbb{C}^{p^n}$.

4 Weil sums for $d = p^n$, $p \geq 3$

The Weil sums in dimension $p^n$ are the equivalent of the Gauss sums for $d = p$ ($p$ prime number). The characters $\chi(\theta)$, $\theta \in GF(p^n)$ replace the powers $\omega^n$, $n \in \mathbb{F}_p$. Usually in the literature (see for example [18]), the Weil sums properties are used to solve the MUB problem. Here, as in [4], we do the converse. In the previous sections we have constructed the $d+1$ bases, and we shall deduce the Weil sums properties from this construction.

**Theorem 4.1** Let $p \geq 3$. Then for any $\theta \in GF(p^n) \setminus \{0\}$ and any $\theta' \in GF(p^n)$ we have

$$\left| \sum_{\theta'' \in GF(p^n)} \chi(\theta \cdot (\theta'')^2 + \theta' \cdot \theta'') \right| = \sqrt{p^n}$$

(4.14)
Proof: The matrix elements of a given row in $F$ are of the form

$$\frac{1}{\sqrt{p^n}}\chi(\theta' \cdot \theta'')\theta'' \in GF(p^n)$$

for some $\theta' \in GF(p^n)$. Since

$$R_\theta = \frac{1}{\sqrt{p^n}} \sum_{\theta'' \in GF(p^n)} \chi((2\theta)^{-1} \cdot (\theta'')^2)X_{\theta''}$$

the matrix elements of the first column of the matrices $R_\theta$, $\theta \neq 0$ are of the form

$$\frac{1}{\sqrt{p^n}}\chi(2^{-1}\theta^{-1} \cdot (\theta'')^2)\theta'' \in GF(p^n)$$

where we have used (3.13). But the elements $(2\theta)^{-1}$ when $\theta \in GF(p^n) \setminus \{0\}$ span all $\theta_1 \in GF(p^n) \setminus \{0\}$. We have established that the matrix $FR_\theta$ is Hadamard. This implies that all its matrix elements have modulus $p^{-n/2}$. All matrix elements of the first column of $FR_\theta$ are thus of the form

$$p^{-n} \sum_{\theta'' \in GF(p^n)} \chi(\theta_1 \cdot (\theta'')^2 + \theta' \cdot \theta'')$$

with $\theta_1 = (2\theta)^{-1}$. Writing that its modulus is $p^{-n/2}$ yields equ. (4.14).

References

[1] Albouy O., Kibler M., $SU_2$ nonstandard bases : the case of mutually unbiased bases, Symmetry, Integrability and Geometry : Methods and Applications, (2007)

[2] Bandyopadhyay S., Boykin P.O., Roychowdhury V., Vatan F., A new proof of the existence of mutually unbiased bases, Algorithmica, 34, 512-528, (2002)

[3] Berndt B. C., Evans R. J., Williams K. S., Gauss and Jacobi Sums, Canadian Mathematical Society Series of Monographs and Advanced Texts, Vol 21, Wiley, (1998)

[4] Björck G. Saffari B, New classes of finite unimodular sequences with unimodular Fourier transforms. Circulant Hadamard matrices with complex entries, C. R. Acad. Sci. Paris, 320 Serie 1, (1995), 319-324
[5] Chaturvedi S. Aspects of mutually unbiased bases in odd-prime-power dimensions, Phys. Rev. A 65, 044301, (2002)

[6] Combescure M. The Mutually Unbiased Bases Revisited, Contemporary Mathematics, (2007), to appear

[7] Combescure M., Circulant matrices, Gauss sums and the Mutually Unbiased Bases, I. The prime number case to appear (2007)

[8] Davis P. J., Circulant matrices, Wiley, (1979)

[9] Hadamard J., Résolution d’une question relative aux déterminants, Bull. Sci. Math. 17, 2460-246 (1893)

[10] Ivanovic I. D. Geometrical description of quantum state determination J. Phys. A 14, 3241-3245, (1981)

[11] Klimov A. B., Muñoz C., Romero J. L., Geometrical approach to the discrete Wigner function, arXiv:quant-ph/0605113, (2006)

[12] Klimov A. B., Sanchez-Soto L. L., de Guise H. Multicomplementary operators via finite Fourier Transform, Journal of Physics A 38, 2747–2760 (2005)

[13] Planat M., Rosu H., Mutually unbiased phase states, phase uncertainties, and Gauss sums, Eur Phys. J. D 36, 133-139, (2005)

[14] Schwinger J., Unitary Operator Bases, Proc Nat. Acad. Sci. U.S.A. 46, 560 (1960)

[15] Saffari B., Quadratic Gauss Sums, to appear

[16] Turyn R. Sequences with small correlation, In Error correcting codes, H. B. Mann Ed., Wiley (1968), 195-228
[17] Weyl H., *Gruppentheorie and Quantenmechanik*, Hirzel, Leipzig, (1928)

[18] Wootters W. K., Fields B. D., *Optimal State-Determination by Mutually Unbiased Measurements*, Ann. Phys. 191, 363-381, (1989)