Quantum Approximation II. Sobolev Embeddings

Stefan Heinrich  
Fachbereich Informatik  
Universität Kaiserslautern  
D-67653 Kaiserslautern, Germany  
e-mail: heinrich@informatik.uni-kl.de  
homepage: http://www.uni-kl.de/AG-Heinrich

Abstract

A basic problem of approximation theory, the approximation of functions from the Sobolev space $W^r_p([0,1]^d)$ in the norm of $L^q([0,1]^d)$, is considered from the point of view of quantum computation. We determine the quantum query complexity of this problem (up to logarithmic factors). It turns out that in certain regions of the domain of parameters $p,q,r,d$ quantum computation can reach a speedup of roughly squaring the rate of convergence of classical deterministic or randomized approximation methods. There are other regions where the best possible rates coincide for all three settings.

1 Introduction

We are concerned with the study of numerical problems of analysis in the quantum model of computation. A series of papers dealt with scalar valued problems, that is, with problems, whose solution is a single number. In [9] for the first time vector (function) valued problems were considered. The first analysis for such a type of problem with matching upper and lower bounds was carried out in [6].

The present paper is a continuation of [6]. We study one of the basic problems of approximation theory, the approximation of functions from the Sobolev class $W^r_p([0,1]^d)$ in the norm of $L^q([0,1]^d)$, a problem which has received much attention in the past in the classical settings (see the survey [3] and references therein).

Our results show that for $p < q$, the quantum model of computation can bring a speedup roughly up to a squaring of the rate in the classical (deterministic or randomized) setting. On the other hand, for $p \geq q$, the
optimal rate is the same for all three settings, so in these cases there is no speedup of the rate by quantum computation.

Our method of analyzing the function approximation problem is similar to the one developed in [5], namely, we discretize the Sobolev embedding problem and show that a sufficiently precise knowledge about the discrete building blocks, the embeddings of finite dimensional $L^N_p$ into $L^N_q$ spaces, leads to a full understanding of the infinite dimensional problem. Although in a completely different setting, this is close in spirit to Maiorov’s discretization technique from approximation theory [8]. These finite dimensional embeddings were studied in [6], the results of which will be exploited here. In this sense the present paper is related to [6] in a similar way, as a previous paper [5] on quantum integration in Sobolev spaces was related to results on summation [4, 7].

For an introduction and notation concerning the quantum setting of information-based complexity we refer to Section 2 of [6]. Some general results which we will use can be found in Section 3 of that paper. Finally, we also refer to [6] for comments on the bibliography. In Section 2 of the present paper, which contains the main result, we study approximation of the embeddings of Sobolev classes $W^r_p([0, 1]^d)$ into $L^q([0, 1]^d)$. In Section 3 we shortly discuss the cost of the algorithm in the bit model and compare the results to the classical settings.

2 Approximation of Sobolev Embeddings

Let $D = [0, 1]^d$ be the $d$-dimensional unit cube, let $C(D)$ denote the space of continuous functions on $D$, endowed with the supremum norm. For $1 \leq p \leq \infty$, let $L_p(D)$ be the space of real-valued $p$-integrable functions, equipped with the usual norm

$$
\|f\|_{L_p(D)} = \left( \int_D |f(t)|^p dt \right)^{1/p}
$$

if $p < \infty$, and

$$
\|f\|_{L_\infty(D)} = \text{ess sup}_{t \in D} |f(t)|.
$$

The Sobolev space $W^r_p(D)$ consists of all functions $f \in L_p(D)$ such that for all $\alpha = (\alpha_1, \ldots, \alpha_d) \in N_0^d$ with $|\alpha| := \sum_{j=1}^d \alpha_j \leq r$, the generalized partial derivative $\partial^\alpha f$ belongs to $L_p(D)$. The norm on $W^r_p(D)$ is defined as

$$
\|f\|_{W^r_p(D)} = \left( \sum_{|\alpha| \leq r} \|\partial^\alpha f\|_{L_p(D)}^p \right)^{1/p}.
$$
if $p < \infty$, and
$$\|f\|_{W^r(D)} = \max_{|\alpha| \leq r} \|\partial^\alpha f\|_{L^\infty(D)}.$$  

We always assume that $r/d > 1/p$. By the Sobolev embedding theorem (see [1], [10]), functions from $W^r_p(D)$ are continuous, and therefore function values are well-defined. Let $B(W^r_p(D))$ be the unit ball of the space $W^r_p(D)$ and $J_{pq}: W^r_p(D) \to L^q(D)$ the embedding operator $J_{pq}f = f$ ($f \in W^r_p(D)$).

Now we present the main result of this paper. To emphasize the essential parts of the estimates we introduce the following notation. For functions $a, b : \mathbb{N} \to [0, \infty)$, we write $a(n) \asymp b(n)$ if there are constants $c_1, c_2 > 0$, $n_0 \in \mathbb{N}$, $\alpha_1, \alpha_2 \in \mathbb{R}$ such that
$$c_1 (\log(n+1))^{\alpha_1} b(n) \leq a(n) \leq c_2 (\log(n+1))^{\alpha_2} b(n)$$
for all $n \in \mathbb{N}$ with $n \geq n_0$. Throughout the paper log means $\log_2$. Furthermore, we often use the same symbol $c, c_1, \ldots$ for possibly different positive constants (also when they appear in a sequence of relations). These constants are either absolute or may depend only on $p, q, r$ and $d$ – in all statements of lemmas, propositions, etc. this is precisely described anyway by the order of the quantifiers.

**Theorem 1.** Let $r, d \in \mathbb{N}$, $1 \leq p, q \leq \infty$ and assume $r/d > 1/p$. Then for $r/d \geq 2/p - 2/q$  
$$e^q_n(J_{pq}, B(W^r_p(D))) \asymp_{\log} n^{-r/d},$$
while for $r/d < 2/p - 2/q$
$$e^q_n(J_{pq}, B(W^r_p(D))) \asymp_{\log} n^{-2r/d+2/p-2/q}.$$  

Here $e^q_n(J_{pq}, B(W^r_p(D)))$ is the $n$-th minimal query error, that is, the minimal possible error among all quantum algorithms that use at most $n$ query calls to approximate $J_{pq}$ on $B(W^r_p(D))$, in the norm of $L^q(D)$ (see [6] for the definition of $e^q_n$). Theorem 1 is a direct consequence of Propositions 1 and 2 which are stated and proved below and which also contain the logarithmic factors. First we derive the upper bounds.

**Proposition 1.** Let $r, d \in \mathbb{N}$, $1 \leq p, q \leq \infty$ and assume $r/d > 1/p$. Then there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ with $n > 2$ the following hold: For $p < q$ and $r/d > 2/p - 2/q$,  
$$e^q_n(J_{pq}, B(W^r_p(D))) \leq cn^{-r/d}(\log n)^{2/p-2/q},$$  

(1)
for $p < q$ and $r/d = 2/p - 2/q,$

$$e_n^q(J_{pq}, B(W_p^r(D))) \leq cn^{-r/d}(\log n)^{4/p-4/q+1}(\log \log n)^{2/p-2/q},$$  \hspace{0.5cm} (2)

for $p < q$ and $r/d < 2/p - 2/q$,

$$e_n^q(J_{pq}, B(W_p^r(D))) \leq cn^{-2r/d+2/p-2/q},$$  \hspace{0.5cm} (3)

and for $p \geq q$,

$$e_n^q(J_{pq}, B(W_p^r(D))) \leq cn^{-r/d}.$$  \hspace{0.5cm} (4)

Proof. We need some preparations. We show that the discretization technique developed in [5], properly adapted, works also for the approximation problem. For the sake of completeness, we recall also needed details from [5]. For $l \in \mathbb{N}_0$ let

$$D = \bigcup_{i=0}^{2^d-1} D_{li}$$

be the partition of $D$ into $2^d$ congruent cubes of disjoint interior. Let $s_{li}$ denote the point in $D_{li}$ with the smallest Euclidean norm. Introduce the following operators $E_{li}$ and $R_{li}$ from $\mathcal{F}(D, \mathbb{R})$, the set of all real-valued functions on $D$, to $\mathcal{F}(D, \mathbb{R})$, by setting for $f \in \mathcal{F}(D, \mathbb{R})$ and $s \in D$

$$(E_{li}f)(s) = f(s_{li} + 2^{-l}s)$$

and

$$(R_{li}f)(s) = \begin{cases} f(2^l(s - s_{li})) & \text{if } s \in D_{li} \\ 0 & \text{otherwise.} \end{cases}$$

Let $P$ be any operator from $C(D)$ to $L_\infty(D)$ of the form

$$Pf = \sum_{j=0}^{\kappa-1} f(t_j)\varphi_j \quad (f \in C(D))$$

with $t_j \in D$ and $\varphi_j \in L_\infty(D)$. Assume furthermore that $P$ is the identity on $\mathcal{P}_{r-1}(D)$, that is,

$$Pf = f \quad \text{for all } f \in \mathcal{P}_{r-1}(D),$$  \hspace{0.5cm} (5)

where $\mathcal{P}_{r-1}(D)$ denotes the space of polynomials on $D$ of degree not exceeding $r - 1$. (For example, for $d = 1$ one can take Lagrange interpolation of appropriate degree and for $d > 1$ its tensor product.) Since $r > d/p$,
we have, by the Sobolev embedding theorem \[1], \[10], \( W^r_p(D) \subset C(D) \) and there is a constant \( c > 0 \) such that for each \( f \in W^r_p(D) \)

\[
\|f\|_{C(D)} \leq c \|f\|_{W^r_p(D)}.
\]

(6)

It follows that

\[
\|Pf\|_{L^q(D)} \leq \sum_{j=0}^{\kappa-1} \|f(t_j)\|_q \|\varphi_j\|_q \leq \sum_{j=0}^{\kappa-1} \|\varphi_j\|_q \|f\|_{C(D)} \leq c \|f\|_{W^r_p(D)}
\]

(7)

(in what follows the operator \( P \) will be fixed, hence \( \sum_{j=0}^{\kappa-1} \|\varphi_j\|_q \) can be considered as a constant). For \( f \in W^r_p(D) \) we denote

\[
|f|_{r,p,D} = \left( \sum_{|\alpha|=r} \|\partial^\alpha f\|_{L^p(D)}^p \right)^{1/p}
\]

if \( p < \infty \), and

\[
|f|_{r,\infty,D} = \max_{|\alpha|=r} \|\partial^\alpha f\|_{L^\infty(D)}.
\]

Now we use Theorem 3.1.1 in \[2\]: there is a constant \( c > 0 \) such that for all \( f \in W^r_p(D) \)

\[
\inf_{g \in P_{r-1}(D)} \|f - g\|_{W^r_p(D)} \leq c |f|_{r,p,D}.
\]

(8)

By \(5\), \(7\) and \(8\),

\[
\|f - Pf\|_{L^q(D)} \leq \inf_{g \in P_{r-1}(D)} \|(f - g) - P(f - g)\|_{L^q(D)} \leq c \inf_{g \in P_{r-1}(D)} \|f - g\|_{W^r_p(D)} \leq c |f|_{r,p,D}.
\]

(9)

For \( l \in \mathbb{N}_0 \) set

\[
P_l f = \sum_{i=0}^{2^{d_l-1}} R_{li} f E_{li} f = \sum_{i=0}^{2^{d_l-1}} \sum_{j=0}^{\kappa-1} f(s_{li} + 2^{-l}t_j) R_{li} \varphi_j.
\]

Note that

\[
\|R_{li} f\|_{L^p(D)} = 2^{-d_l/p} \|f\|_{L^p(D)} \quad (f \in L^p(D)).
\]

(10)
Then we have for \( u \in \{ p, q \} \) and all \( f \in W_r^p(D) \), using (9) and (10),

\[
\| f - P_l f \|_{L_u(D)} = \left\| \sum_{i=0}^{2^{d_l}-1} (R_{li}E_{li}f - R_{li}PE_{li}f) \right\|_{L_u(D)} \\
= \left( \sum_{i=0}^{2^{d_l}-1} \left\| (R_{li}(E_{li}f - P E_{li}f) \right\|_{L_u(D)}^u \right)^{1/u} \\
= \left( 2^{-dl} \sum_{i=0}^{2^{d_l}-1} \| E_{li}f - P E_{li}f \|_{L_u(D)}^u \right)^{1/u} \\
\leq c \left( 2^{-dl} \sum_{i=0}^{2^{d_l}-1} |E_{li}f|_{r,p,D}^p \right)^{1/p} \\
\leq c 2^{\max(1/p-1/u,0)dl} \left( 2^{-dl} \sum_{i=0}^{2^{d_l}-1} |E_{li}f|_{r,p,D}^p \right)^{1/p}
\]

and

\[
\left( 2^{-dl} \sum_{i=0}^{2^{d_l}-1} |E_{li}f|_{r,p,D}^p \right)^{1/p} = \left( 2^{-dl} \sum_{i=0}^{2^{d_l}-1} \sum_{|\alpha|=r} \int_D |\partial^\alpha f(s_{li} + 2^{-l}t)|^p dt \right)^{1/p} \\
= 2^{-rl} \left( 2^{d_l-1} \sum_{i=0}^{2^{d_l}-1} \sum_{|\alpha|=r} \int_{D_{li}} |\partial^\alpha f(t)|^p dt \right)^{1/p} \\
= 2^{-rl} \| f \|_{r,p,D} \leq 2^{-rl} \| f \|_{W_r^p(D)} \tag{11}
\]

(with the usual modifications for \( u = \infty \) or \( p = \infty \)). Consequently,

\[
\| f - P_l f \|_{L_u(D)} \leq c 2^{-rl+\max(1/p-1/u,0)dl} \| f \|_{r,p,D} \\
\leq c 2^{-rl+\max(1/p-1/u,0)dl} \| f \|_{W_r^p(D)}. \tag{12}
\]

Similarly to [5], we first approximate \( f \) by \( P_{l^*} f \) for some \( l^* \), giving the desired precision, but using a number of function values much larger than \( n \). This \( P_{l^*} \), in turn, will be split into the sum of a single operator \( P_{l_0} \), with number of function values of the order \( n \), which we compute classically, and a hierarchy of operators \( P_l^l \) \((l = l_0, \ldots, l^* - 1)\). We will show that the approximation of the \( P_l^l \) reduces to the approximation of appropriately
scaled embedding operators $J_{pq}^{N_l} : L_p^{N_l} \rightarrow L_q^{N_l}$ for suitable $N_l$. This enables us to apply the results of [6]. Define

$$ P' f := (P_1 - P_0) f = \sum_{i=0}^{2d-1} \sum_{j=0}^{\kappa-1} f(s_{1,i} + 2^{-1} t_j) R_{1,i} \varphi_j - \sum_{j=0}^{\kappa-1} f(t_j) \varphi_j, $$

(13)

which can be written as

$$ P' f = \sum_{j=0}^{\kappa'-1} \left( \sum_{k=0}^{\kappa''-1} a_{jk} f(t'_j) \right) \psi_j, $$

with

$$ \kappa', \kappa'' \leq \kappa(2^d + 1), $$

(14)

$a_{jk} \in \mathbb{R}$, $t'_j \in D$ ($j = 0, \ldots, \kappa' - 1$, $k = 0, \ldots, \kappa'' - 1$), and a linearly independent system $(\psi_j)_{j=0}^{\kappa'-1} \subseteq L_\infty(D)$. The linear independence implies that for $u \in \{p,q\}$ there are constants $c_1, c_2 > 0$ such that for all $\alpha_j \in \mathbb{R}$ ($j = 0, \ldots, \kappa' - 1$)

$$ c_1 \Vert (\alpha_j)_{j=0}^{\kappa'-1} \Vert_{L_u} \leq \sum_{j=0}^{\kappa'-1} \alpha_j \psi_j \Vert_{L_u(D)} \leq c_2 \Vert (\alpha_j)_{j=0}^{\kappa'-1} \Vert_{L_u^n}. $$

(15)

Put

$$ \psi_{lij} = R_{li} \psi_j, $$

(16)

and let

$$ \Pi_l = \text{span} \left\{ \psi_{lij} : i = 0, \ldots, 2^{dl-1}, j = 0, \ldots, \kappa' - 1 \right\} \subseteq L_\infty(D). $$

By the disjointness of the interiors of the $D_{li}$ and by (10) we have for $\alpha_{ij} \in \mathbb{R}$ ($i = 0, \ldots, 2^{dl-1}$, $j = 0, \ldots, \kappa' - 1$) and $1 \leq u < \infty

$$ \Vert \sum_{i=0}^{2^{dl-1}} \sum_{j=0}^{\kappa'-1} \alpha_{ij} \psi_{lij} \Vert_{L_u(D)}^u = \sum_{i=0}^{2^{dl-1}} \Vert \sum_{j=0}^{\kappa'-1} \alpha_{ij} \psi_{lij} \Vert_{L_u(D)}^u = \sum_{i=0}^{2^{dl-1}} \Vert R_{li} \sum_{j=0}^{\kappa'-1} \alpha_{ij} \psi_j \Vert_{L_u(D)}^u

= 2^{-dl} \sum_{i=0}^{2^{dl-1}} \sum_{j=0}^{\kappa'-1} \alpha_{ij} \psi_j \Vert_{L_u(D)}^u. $$

(17)
Let $N_l = \kappa'2^dl$. Then

$$2^{-dl}\sum_{i=0}^{2^dl-1} \|\alpha_{ij}\|_{L_{u_l}^{\kappa}}^\kappa' = \|\alpha_{ij}\|_{L_{u_l}^{N_l}},$$

(18)

where $(\alpha_{ij})$ stands for $(\alpha_{ij})_{i=0,j=0}^{2^dl-1,\kappa'-1}$. Combining (17), (18) and (15), we get

$$c_1\|\alpha_{ij}\|_{L_{u_l}^{N_l}} \leq \|\sum_{i=0}^{2^dl-1} \sum_{j=0}^{\kappa'-1} \alpha_{ij}\psi_{lij} \|_{L_{u_l}(D)} \leq c_2\|\alpha_{ij}\|_{L_{u_l}^{N_l}},$$

(19)

Relation (19) holds also for $u = \infty$, which can be proved with the usual modifications in the reasoning above. Define the operator $T_l : \Pi_l \to \mathbb{R}^{N_l}$ by

$$T_l \sum_{i=0}^{2^dl-1} \sum_{j=0}^{\kappa'-1} \alpha_{ij}\psi_{lij} = (\alpha_{ij}).$$

(20)

It follows from (19) that for $f \in \Pi_l$,

$$\|T_l f\|_{L_u^{N_l}} \leq c_1^{-1}\|f\|_{L_u(D)}.$$  

(21)

and for $g \in L_{u_l}^{N_l}$,

$$\|T_l^{-1} g\|_{L_u(D)} \leq c_2\|g\|_{L_{u_l}^{N_l}}.$$  

(22)

For $l \in \mathbb{N}_0$ and $f \in C(D)$ set

$$P_{li}' f = R_{li} P_{li} E_{li} f = \sum_{j=0}^{\kappa'-1} \sum_{k=0}^{\kappa''-1} a_{jk} f(s_{li} + 2^{-l}t_{j,k}'\psi_{lij}),$$

(23)

$$P_l' = \sum_{i=0}^{2^dl-1} P_{li}'.$$

(24)

It is readily verified that

$$P_{l+1} = \sum_{i=0}^{2^dl-1} R_{li} P_{li} E_{li},$$

and therefore

$$P_{l+1} - P_l = \sum_{i=0}^{2^dl-1} R_{li}(P_{li} E_{li} - P_0 E_{li})$$

$$= \sum_{i=0}^{2^dl-1} P_{li}' = P_l'.$$

(25)
From (11), (12) with \( u = p \) and (11), we get

\[
\| P' f \|_{L_p(D)} = \left( \sum_{i=0}^{2^{dl}-1} \| R_{ti} P' E_{li} f \|_{L_p(D)}^p \right)^{1/p}
\]

\[
= \left( 2^{-dl} \sum_{i=0}^{2^{dl}-1} \| P_1 E_{li} f - P_0 E_{li} f \|_{L_p(D)}^p \right)^{1/p}
\]

\[
\leq \left( 2^{-dl} \sum_{i=0}^{2^{dl}-1} \left( \| E_{li} f - P_1 E_{li} f \|_{L_p(D)} + \| E_{li} f - P_0 E_{li} f \|_{L_p(D)} \right)^p \right)^{1/p}
\]

\[
\leq c \left( 2^{-dl} \sum_{i=0}^{2^{dl}-1} | E_{li} f |_{L_{r,p,D}}^p \right)^{1/p} \leq c 2^{-r l \| f \|_{W^r_p(D)}}. \quad (26)
\]

We define operators \( U_l : W^r_p(D) \rightarrow L_{p}^{N_i} \) by

\[
U_l = T_l P'_l \quad (27)
\]

and \( V_l : L_{q}^{N_i} \rightarrow L_{q}(D) \) by

\[
V_l = T_l^{-1}. \quad (28)
\]

Then clearly

\[
V_l J_{p,q}^{N_i} U_l = P'_l, \quad (29)
\]

moreover, by (26) and (21) for \( u = p \),

\[
\| U_l f \|_{L_{p}^{N_i}} \leq c 2^{-r l \| f \|_{W^r_p(D)}} \quad (f \in W^r_p(D)) \quad (30)
\]

and, by (22) for \( u = q \)

\[
\| V_l g \|_{L_{q}(D)} \leq c \| g \|_{L_{q}^{N_i}} \quad (g \in L_{q}^{N_i}). \quad (31)
\]

Now we are ready to derive the upper bounds. It obviously suffices to prove them for

\[
n \geq \max(\kappa, 5). \quad (32)
\]

Define

\[
l_0 = \lfloor d^{-1} \log(n/\kappa) \rfloor. \quad (33)
\]
Then \( l_0 \geq 0 \). Furthermore, let

\[
l^* = \begin{cases} 
  l_0 & \text{if } p \geq q \\
  2l_0 & \text{if } p < q.
\end{cases}
\]  

(34)

By (25),

\[
P_l^* = P_{l_0} + \sum_{l=l_0}^{l^*-1} P'_l.
\]  

(35)

In the sequel we consider the \( P_l \) and \( P'_l \) as operators from \( W_p^r(D) \) to \( L_q(D) \).

By (33), \( \kappa 2^{dl_0} \leq n \), hence

\[
e^{-\nu_l/8} \leq \frac{1}{4}.
\]  

(37)

Put

\[
\tilde{n} = n + 2\kappa'' \sum_{l=l_0}^{l^*-1} \nu_l n_l
\]  

(38)

(if \( l^* = l_0 \), we do not define the numbers \( \nu_l, n_l \) and put \( \tilde{n} = n \)). From (12) above with \( u = q \) and Lemma 6(i) of [4], we get

\[
e_n^q(J_{pq}, B(W_p^r(D)))
\leq \sup_{f \in B(W_p^r(D))} \|J_{pq}f - P_l^*f\|_{L_q(D)} + e_n^q(P_{l^*}, B(W_p^r(D)))
\leq c2^{-n^* + \max(1/p-1/q,0)dl^*} + e_n^q(P_{l^*}, B(W_p^r(D))).
\]  

(39)

The upper bound for the case \( p \geq q \) follows directly from (39) and (36), since in this case \( l^* = l_0 \) and \( \tilde{n} = n \) (this is the trivial case where the optimal rate is already attained by a classical algorithm).

In the rest of the proof we assume \( p < q \). By Lemma 3 of [6] and (36),

\[
e_n^q(P_{l^*}, B(W_p^r(D)))
\leq e_n^q(P_{l_0}, B(W_p^r(D)), 0) + e_n^q(P_{l^*} - P_{l_0}, B(W_p^r(D)))
\leq e_n^q(P_{l^*} - P_{l_0}, B(W_p^r(D))).
\]  

(40)
From (35), (38), Corollary 3 of [6], and (37) we get
\[
e^q_{n-n}(P_1^* - P_{l_0}, \mathcal{B}(W_p^r(D))) = e^{q}_{2\kappa''} \sum_{l_0}^{l^*} e^q_{l, B}(W_p^r(D)) \leq 2 \sum_{l=0}^{l^*} e^q_{2\kappa''}(P_1^*, \mathcal{B}(W_p^r(D))).
\] (41)

Using Lemma 2 of [6], (29), and (31) we obtain
\[
e^q_{2\kappa''n_l}(P_1^*, \mathcal{B}(W_p^r(D))) = e^q_{2\kappa''n_l}(V, (N_i U_l, \mathcal{B}(W_p^r(D))) \leq c e^q_{2\kappa''n_l}(J_{N_i} U_l, \mathcal{B}(W_p^r(D))).
\] (42)

Joining relations (39)–(42), we infer
\[
e^q_{n}(J_{pq}, \mathcal{B}(W_p^r(D))) \leq c e^{-r^*+1/p-1/q} + c \sum_{l=0}^{l^*} e^q_{2\kappa''n_l}(J_{pq} U_l, \mathcal{B}(W_p^r(D))).
\] (43)

In a further reduction one would like to remove the $U_l$ in the last relation. This could be done on the basis of Corollary 1 of [5], if the $U_l$ were of the required form, which is not the case. Instead, we shall approximate the $U_l$ by appropriate mappings $\Gamma_l : \mathcal{B}(W_p^r(D)) \rightarrow L_{n_l}^{N_l}$ ($l_0 \leq l < l^*$). Note that by (20), (23), (24), and (27),
\[
U_l f(i,j) = \sum_{k=0}^{k''} a_{jk} f(s_{li} + 2^{-l'} t_{jk}).
\] (44)

Fix an $m^* \in \mathbb{N}$ with
\[
2^{-m^*/2} \leq (l^* + 1)^{-1/2} 1^{-r^*}
\] (45)
and
\[
2^{m^*/2-1} \geq c,
\] (46)
where $c$ is the constant from [5]. Hence,
\[
\|f\|_{C(D)} \leq 2^{m^*/2-1} \quad \text{for } f \in \mathcal{B}(W_p^r(D)).
\] (47)

Define $\beta : \mathbb{R} \rightarrow \mathbb{Z}[0, 2^{m^*})$ for $z \in \mathbb{R}$ by
\[
\beta(z) = \begin{cases} \\
0 & \text{if } z < -2^{m^*/2-1} \\
[2^{m^*/2}(z + 2^{m^*/2-1})] & \text{if } -2^{m^*/2-1} \leq z < 2^{m^*/2-1} \\
2^{m^*/2} - 1 & \text{if } z \geq 2^{m^*/2-1}
\end{cases}
\] (48)
and \( \gamma : \mathbb{Z}[0, 2^{m*}] \to \mathbb{R} \) for \( y \in \mathbb{Z}[0, 2^{m*}] \) as

\[
\gamma(y) = 2^{-m*/2}y - 2^{m*/2 - 1}.
\] (49)

Then we have for \(-2^{m*/2 - 1} \leq z \leq 2^{m*/2 - 1}\),

\[
\gamma(\beta(z)) \leq z \leq \gamma(\beta(z)) + 2^{-m*/2}.
\] (50)

Define \( \eta_{lk} : \mathbb{Z}[0, N_l] \to D \) \((k = 0, \ldots, \kappa'' - 1)\) by

\[
\eta_{lk}(i,j) = s_{li} + 2^{-l} \ell'_{jk} \quad (0 \leq i \leq 2^{dl} - 1, 0 \leq j \leq \kappa' - 1)
\]

(here \( \mathbb{Z}[0, N_l] \) stands for \( \{0, 1, \ldots, N_l - 1\} \) and we identify \( \mathbb{Z}[0, N_l] \) with \( \mathbb{Z}[0, 2^{dl}] \times \mathbb{Z}[0, \kappa'] \)). Next let \( \varrho_l : \mathbb{Z}[0, N_l] \times \mathbb{Z}[0, 2^{m*}]^{\kappa''} \to \mathbb{R} \) be given by

\[
\varrho_l((i,j), y_0, \ldots, y_{\kappa'' - 1}) = \sum_{k=0}^{\kappa'' - 1} a_{jk} \gamma(y_k).
\]

Finally, we define \( \Gamma_l : \mathcal{B}(W_p^r(D)) \to L^{N_l}_{p} \) by setting

\[
\Gamma_l(f)(i,j) = \varrho_l((i,j), (\beta \circ f \circ \eta_{lk}(i,j))_{k=0}^{\kappa'' - 1}).
\]

for \( f \in \mathcal{B}(W_p^r(D)) \). Note that \( \Gamma_l \) is of the form (4) of [5] needed to apply Corollary 1 of that paper, which we will do later on. We have

\[
\Gamma_l(f)(i,j) = \sum_{k=0}^{\kappa'' - 1} a_{jk} \gamma(\beta(f(s_{li} + 2^{-l} \ell'_{jk}))),
\]

hence, by \(44\), \(47\), \(50\), and \(45\)

\[
|(U_l f)(i,j) - \Gamma_l(f)(i,j)| 
\leq \sum_{k=0}^{\kappa'' - 1} |a_{jk}| |f(s_{li} + 2^{-l} \ell'_{jk}) - \gamma(\beta(f(s_{li} + 2^{-l} \ell'_{jk}))))|
\leq 2^{-m*/2} \sum_{k=0}^{\kappa'' - 1} |a_{jk}| \leq c 2^{-m*/2} \leq c(l^* + 1)^{-1} 2^{-rl^*},
\] (51)

and therefore, for all \( f \in \mathcal{B}(W_p^r(D)) \) and \( u \in \{p, q\}\),

\[
\|U_l f - \Gamma_l(f)\|_{L^u_{p}}
\leq \left((\kappa')^{-1} 2^{-dl} \sum_{i=0}^{2^{dl} - 1} \sum_{j=0}^{\kappa' - 1} |(U_l f)(i,j) - \Gamma_l(f)(i,j)|^u \right)^{1/u}
\leq c(l^* + 1)^{-1} 2^{-rl^*}.
\] (52)
Moreover, by (30) and (52) with \( u = p \),
\[
\|\Gamma_l(f)\|_{L^p_{N_l}} \leq \|U_l f\|_{L^p_{N_l}} + \|\Gamma_l(f) - U_l f\|_{L^p_{N_l}} \leq c 2^{-rl}.
\]
Consequently,
\[
\Gamma_l(B(W^r_p(D))) \subseteq c 2^{-rl} B(L^N_p).
\]
From (52) with \( u = q \) and Lemma 6(i) of [3]
\[
e_q^q(\epsilon_{pq}^q J_{pq}^N \Gamma_l, B(W^r_p(D))) \leq c_1^q (J_{pq}^N, c 2^{-rl} B(L^N_p)) \]
\[
\leq e_q^q(\epsilon_{pq}^q J_{pq}^N \Gamma_l, B(W^r_p(D))) + e_q^q(\epsilon_{pq}^q J_{pq}^N \Gamma_l, B(W^r_p(D))) \]
\[
\leq c 2^{-rl} e_q^q(\epsilon_{pq}^q J_{pq}^N \Gamma_l, B(L^N_p)).
\]
Corollary 1 of [5], relation (53) above and Lemma 6(iii) of [3] give
\[
e_q^q(\epsilon_{pq}^q J_{pq}^N \Gamma_l, B(W^r_p(D))) \leq e_q^q(\epsilon_{pq}^q J_{pq}^N \Gamma_l, B(L^N_p)) \]
\[
\leq c 2^{-rl} e_q^q(\epsilon_{pq}^q J_{pq}^N \Gamma_l, B(L^N_p)).
\]
From (54), (55) and (43), we conclude
\[
e_q^q(\epsilon_{pq}^q J_{pq}^N \Gamma_l, B(W^r_p(D))) \]
\[
\leq c 2^{-rl} + (\frac{1}{p} - \frac{1}{q}) d^r + c \sum_{l=0}^{l^* - 1} 2^{-rl} e_q^q(\epsilon_{pq}^q J_{pq}^N \Gamma_l, B(L^N_p)).
\]
Thus we reached the desired reduction and can now exploit the results for the finite dimensional case: By Proposition 2 of [3] and (56),
\[
e_q^q(\epsilon_{pq}^q J_{pq}^N \Gamma_l, B(W^r_p(D))) \]
\[
\leq c 2^{-rl} + (\frac{1}{p} - \frac{1}{q}) d^r + c \sum_{l=0}^{l^* - 1} 2^{-rl} e_q^q(\epsilon_{pq}^q J_{pq}^N \Gamma_l, B(L^N_p)).
\]
Recall that we consider the case \( p < q \). First we assume
\[
\frac{r}{d} > \frac{2}{p} - \frac{2}{q}. 
\]
Fix any \( \delta > 0 \) such that
\[
r > \left( \frac{2}{p} - \frac{2}{q} \right) (d + \delta)
\]
and put for $l = l_0, \ldots, l^* - 1$

$$n_l = \left\lfloor 2^{dl_0 - \delta(l-l_0)} \right\rfloor,$$  \hspace{1cm} (60)

$$\nu_l = \left\lfloor 8(2 \ln(l - l_0 + 1) + \ln 8) \right\rfloor.$$  \hspace{1cm} (61)

It follows from (61) that

$$\sum_{l=l_0}^{l^*-1} e^{-\nu_l} / 8 \leq \frac{1}{8} \sum_{l=l_0}^{l^*-1} (l - l_0 + 1)^{-2} < \frac{1}{4},$$  \hspace{1cm} (62)

so assumption (37) is satisfied. By (35), (61), (32), and (60),

$$\tilde{n} \leq n + 2n'' \sum_{l=l_0}^{l^*-1} \left\lfloor 8(2 \ln(l - l_0 + 1) + \ln 8) \right\rfloor \left\lfloor 2^{dl_0 - \delta(l-l_0)} \right\rfloor$$

$$\leq c 2^{dl_0} \leq cn.$$  \hspace{1cm} (63)

It follows from (67), (32), (33), (34), and (60) that

$$e_{\tilde{n}}^q(J_{p^q}, B(W^p_r(D)))$$

$$\leq c 2^{-rl^*/2} + c \sum_{l=l_0}^{l^*-1} 2^{-r(l-(\frac{d}{p} - \frac{2}{q})dl_0 + (\frac{2}{p} - \frac{2}{q})\delta(l-l_0) + (\frac{2}{p} - \frac{2}{q})d)} (l_0 + 1) \frac{2}{p} - \frac{2}{q}$$

$$= c 2^{-rl_0} + c 2^{-rl_0} (l_0 + 1) \frac{2}{p} - \frac{2}{q} \sum_{l=l_0}^{l^*-1} 2^{(-r(\frac{2}{p} - \frac{2}{q})(d+\delta))(l-l_0)}$$

$$\leq c 2^{-rl_0} (l_0 + 1) \frac{2}{p} - \frac{2}{q} \leq c n^{-\frac{2}{p} - \frac{2}{q}} (\log n)^{\frac{2}{p} - \frac{2}{q}}.$$  \hspace{1cm} (64)

Then (11) follows by an obvious scaling from (63) and (64). Next we assume

$$\frac{r}{d} < \frac{2}{p} - \frac{2}{q}.$$  

Here we take any $\delta > 0$ with

$$r < \left( \frac{2}{p} - \frac{2}{q} \right) (d - \delta)$$  \hspace{1cm} (65)

and put for $l = l_0, \ldots, l^* - 1$

$$n_l = \left\lfloor 2^{dl_0 - \delta(l^*-l)} \right\rfloor,$$  \hspace{1cm} (66)
\[ v_l = [8(2 \ln(l^* - l) + \ln 8)]. \] (67)

From (67) we see that (37) is satisfied, again:
\[ \sum_{l=l_0}^{l^*-1} e^{-v_l/8} \leq \frac{1}{8} \sum_{l=l_0}^{l^*-1} (l^* - l)^{-2} < \frac{1}{4}. \] (68)

By (38), (66), (67), (33), and (34),
\[ \tilde{n} \leq n + 2 \kappa'' \sum_{l=l_0}^{l^*-1} [8(2 \ln(l^* - l) + \ln 8)] \left[ 2^{d_{l_0} - \delta(l^* - l)} \right] \leq c 2^{d_{l_0}} \leq c n. \] (69)

Using (57), (32)–(34), (65), and (66), we get
\[ e^n_{\tilde{n}}(J_{pq}, B(W_p^r(D))) \leq c 2^{-\frac{2r}{d} + \frac{2}{p} - \frac{2}{q}}d_{l_0} \]
\[ + c 2^{-r l^* - \left( \frac{2}{p} - \frac{2}{q} \right) d_{l_0} + \left( \frac{2}{p} - \frac{2}{q} \right) \delta(l^* - l) + \left( \frac{2}{p} - \frac{2}{q} \right) d_l (l^* - l + 1) \frac{2}{p} - \frac{2}{q} } \leq c 2^{-\frac{2r}{d} + \frac{2}{p} - \frac{2}{q}}d_{l_0} \leq c n \frac{2r}{d} + \frac{2}{p} - \frac{2}{q}, \]
which, together with (69), gives (3). Finally we consider the case
\[ \frac{r}{d} = \frac{2}{p} - \frac{2}{q}. \]

Here we put for \( l = l_0, \ldots, l^* - 1 \)
\[ n_l = \left[ 2^{d_{l_0}} (l_0 + 1)^{-1} (\ln(l_0 + 2))^{-1} \right], \] (70)

and
\[ v_l = [8(\ln(l_0 + 2) + \ln 4)]. \] (71)

This way, relation (37) is valid:
\[ \sum_{l=l_0}^{l^*-1} e^{-v_l/8} \leq \frac{1}{4} \sum_{l=l_0}^{l^*-1} (l_0 + 2)^{-1} < \frac{1}{4}. \] (72)
From (38), (70), (71), (33), and (34), we get

\[ \tilde{n} \leq n + 2\kappa'' \sum_{l=l_0}^{l^*} [8(\ln(l_0 + 2) + \ln 4)] [2^{d\ell_0}(l_0 + 1)^{-1}(\ln(l_0 + 2))^{-1}] \]

\[ \leq c 2^{d\ell_0} \leq cn. \]  \hspace{1cm} (73)

By the help of (57), (32)–(34), and (70) we conclude that

\[ e^q_{\tilde{n}}(J_{pq}, B(W_p^r(D))) \]

\[ \leq c 2^{-r l^*/2} + c \sum_{l=l_0}^{l^*-1} 2^{-r l - (3/4 - 2/4) d\ell_0 + (3/4 - 2/4) d\ell (l_0 + 1)^{\frac{1}{p} - \frac{1}{q} + \frac{1}{2} \ln(l_0 + 2) + \frac{1}{2} \ln 4)} \]

\[ = c 2^{-r l_0} + c (l_0 + 1)^{\frac{1}{p} - \frac{4}{q}} (\ln(l_0 + 2))^{\frac{2}{p} - \frac{2}{q}} \sum_{l=l_0}^{l^*} 2^{-r l_0} \]

\[ \leq c 2^{-r l_0} (l_0 + 1)^{\frac{1}{p} - \frac{4}{q} + 1} (\ln(l_0 + 2))^{\frac{2}{p} - \frac{2}{q}} \]

\[ \leq cn^{-\frac{r}{2}} (\log n)^{\frac{1}{p} - \frac{4}{q} + 1} (\log \log n)^{\frac{2}{p} - \frac{2}{q}}. \]

This shows (3) and completes the proof. \[\square\]

**Proposition 2.** Let \( r, d \in \mathbb{N}, \ 1 \leq p, q \leq \infty, \) and suppose \( r/d > 1/p. \) Then there exists a constant \( c > 0 \) such that for all \( n \in \mathbb{N} \) with \( n > 4 \) the following holds: First assume \( r/d > 2/p - 2/q. \) If \( 2 < q \leq \infty, \) then

\[ e^q_{\tilde{n}}(J_{pq}, B(W_p^r(D))) \geq cn^{-r/d}, \]  \hspace{1cm} (74)

if \( q = 2, \) then

\[ e^q_{\tilde{n}}(J_{p,2}, B(W_p^r(D))) \geq cn^{-r/d} (\log \log n)^{-3/2} (\log \log \log n)^{-1}, \]  \hspace{1cm} (75)

and if \( 1 \leq q < 2, \) then

\[ e^q_{\tilde{n}}(J_{pq}, B(W_p^r(D))) \geq cn^{-r/d} (\log \log n)^{-2/q + 1}. \]  \hspace{1cm} (76)

Now assume \( r/d \leq 2/p - 2/q. \) Then

\[ e^q_{\tilde{n}}(J_{pq}, B(W_p^r(D))) \geq cn^{-2r/d + 2/p - 2/q}. \]  \hspace{1cm} (77)

**Proof.** Let \( \psi \) be a \( C^\infty \) function on \( \mathbb{R}^d \) with

\[ \text{supp } \psi \subset (0,1)^d, \quad \sigma_1 := \int_D \psi(t) \, dt > 0, \]
and denote $\|\psi\|_{W^r_p(D)} = \sigma_2$. Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $N = 2^{dk}$. Let $R_{ki}$ and $D_{ki}$ be as defined in the beginning of the proof of Proposition 1. Let 

$$
\psi_i(t) = R_{ki} \psi(i = 0, \ldots, N - 1).
$$

We have

$$
\int_{D_{ki}} \psi_i(t) \, dt = \sigma_1 \frac{N}{dki}
$$

and

$$
\|\psi_i\|_{W^r_p(D)} \leq 2^{(r-d/p)k} \|\psi\|_{W^r_p(D)} = \sigma_2 2^{(r-d/p)k}.
$$

Consequently, taking into account the disjointness of the supports of the $\psi_i$, for all $a_i \in \mathbb{R}$ ($i = 0, \ldots, N - 1$),

$$
\| \sum_{i=0}^{N-1} a_i \psi_i \|_{W^r_p(D)} = \left( \sum_{i=0}^{N-1} |a_i|^p \|\psi_i\|^p_{W^r_p(D)} \right)^{1/p} \leq \sigma_2 2^{rk} \| (a_i)_{i=0}^{N-1} \|_{L^p_N}
$$

(79)

\[ \text{(which holds also for } p = \infty) \]. Fix any $m^* \in \mathbb{N}$ with

$$
m^*/2 - 1 \geq dk/p. \quad (80)
$$

Let $\beta : \mathbb{R} \to \mathbb{Z}[0, 2^{m^*})$ and $\gamma : \mathbb{Z}[0, 2^{m^*}) \to \mathbb{R}$ be defined as in (48) and (49). For $f \in \mathcal{B}(L^N_p)$ we have

$$
|f(i)| \leq N^{1/p} = 2^{dk/p} \leq 2^{m^*/2 - 1}.
$$

Hence, by (80), for $0 \leq i < N$,

$$
\gamma(\beta(f(i))) \leq f(i) \leq \gamma(\beta(f(i))) + 2^{-m^*/2}. \quad (81)
$$

Define

$$
\Gamma : \mathcal{B}(L^N_p) \to W^r_p(D) \quad \text{by} \quad \Gamma(f) = \sum_{i=0}^{N-1} \gamma \circ \beta \circ f(i) \psi_i.
$$

By (79) and (81), for $f \in \mathcal{B}(L^N_p)$,

$$
\|\Gamma(f)\|_{W^r_p(D)} \leq \sigma_2 2^{rk} \|\gamma \circ \beta \circ f\|_{L^N_p}
\leq \sigma_2 2^{rk} \left( \|f\|_{L^N_p} + \|f - \gamma \circ \beta \circ f\|_{L^N_p} \right)
\leq \sigma_2 2^{rk} \left( 1 + 2^{-m^*/2} \right). \quad (82)
$$

17
Define $\Phi : L_q(D) \to L_q^N$ by

$$(\Phi f)(i) = N \int_{D_{ki}} f(t) dt.$$  

It follows from (78) that

$$\Phi \psi_i = \sigma_1 e_i,$$

where $e_i$ denotes the $i$-th unit vector in $L_p^N$. Moreover, by Hölder’s inequality,

$$\|\Phi f\|_{L_q^N}^q = N^{q-1} \sum_{i=0}^{N-1} \left( \int_{D_{ki}} |f(t)| dt \right)^q \leq N^{q-1} \sum_{i=0}^{N-1} \int_{D_{ki}} |f(t)|^q |D_{ki}|^{q-1} = \|f\|_{L_q(D)}^q.$$

Thus, since $\Phi$ is linear,

$$\|\Phi\|_{\text{Lip}} = \|\Phi\| \leq 1.$$  

Furthermore, by (83),

$$\Phi \circ J_{pq} \circ \Gamma(f) = \sum_{i=0}^{N-1} \gamma \circ \beta \circ f(i) \Phi \psi_i$$

$$= \sigma_1 \sum_{i=0}^{N-1} \gamma \circ \beta \circ f(i) e_i$$

$$= \sigma_1 J_{pq}^N (\gamma \circ \beta \circ f).$$  

Define $\eta : D \to \mathbb{Z}[0,N]$ by

$$\eta(s) = \min\{i \mid s \in D_{ki}\},$$

and

$$g : D \times \mathbb{Z}[0,2^m] \to \mathbb{R} \quad \text{by} \quad g(s, z) = \gamma(z) \psi_{\eta(s)}(s).$$

Then

$$\Gamma(f)(s) = \sum_{i=0}^{N-1} \gamma \circ \beta \circ f(i) \psi_i(s)$$

$$= \gamma \circ \beta \circ f(\eta(s)) \psi_{\eta(s)}(s)$$

$$= g(s, \beta \circ f \circ \eta(s)).$$
So Γ is of the needed form (see relation (4) of [6], with κ = 1) and, by (82), maps
\[ B(L^N_p) \] into \[ \sigma_2 2^{rk} \left( 1 + 2^{-m^*/2} \right) B(W^r_p(D)). \]

By (84), Lemma 2 and Corollary 1 of [6], and Lemma 6(iii) of [4],
\[ e^{q_2 n} (\Phi \circ J_{pq} \circ \Gamma, B(L^N_p)) \]
\[ \leq e^{q_2 n} (J_{pq}, B(W^r_p(D))) \]
\[ = \sigma_2 2^{rk} \left( 1 + 2^{-m^*/2} \right) e^{q_2 n} (J_{pq}, B(W^r_p(D))). \] (86)

Using (81) again, we infer
\[ \sup_{f \in B(L^N_p)} \| J_{pq} f - J^N_{pq} (\gamma \circ \beta \circ f) \|_{L^r_N} = \| f - \gamma \circ \beta \circ f \|_{L^N} \leq 2^{-m^*/2}, \]
and hence, by Lemma 6(i) and (ii) of [4], (85), and (86),
\[ e^{q_2 n} (J^N_{pq}, B(L^N_p)) \]
\[ \leq e^{q_2 n} (J_{pq}, \overline{\gamma \circ \beta}, B(L^N_p)) + 2^{-m^*/2} \]
\[ \leq e^{q_2 n} \left( J_{pq}, \sigma_2 2^{rk} \left( 1 + 2^{-m^*/2} \right) B(W^r_p(D)) \right) + 2^{-m^*/2} \]
\[ \leq c_N^{r/d} e^{q_2 n} (J_{pq}, B(W^r_p(D))) + 2^{-m^*/2}, \]
where \( \overline{\gamma \circ \beta} \) stands for
\[ (\gamma \circ \beta, \ldots, \gamma \circ \beta) : \mathbb{R}^N \rightarrow \mathbb{R}^N. \]

Since \( m^* \) can be made arbitrarily large, we get
\[ e^{q_2 n} (J^N_{pq}, B(L^N_p)) \leq c_N^{r/d} e^{q_2 n} (J_{pq}, B(W^r_p(D))). \] (87)

For the case \( r/d > 2/p - 2/q \), we choose \( k = \lceil d^{-1}(\log(n/c_0) + 1) \rceil \), where \( c_0 \) is the constant from Proposition 6 of [6], which can be assumed to satisfy 0 < \( c_0 \leq 1 \). It follows that
\[ c_0^{-d} N = c_0 2^{-d} 2^{dk} \leq 2n \leq c_0 2^{dk} = c_0 N. \] (88)

Now the lower bounds (74), (75), and (76) follow from (87), (88), and Proposition 6 of [6]. In the case \( r/d \leq 2/p - 2/q \), which implies \( p < q \), we set \( k = \lceil d^{-1}(\log(n^2/c_0) + 1) \rceil \). Consequently,
\[ c_0^{-d} N = c_0 2^{-d} 2^{dk} \leq 2n^2 \leq c_0 2^{dk} = c_0 N. \] (89)

Relation (77) results from (87), (89), and Proposition 4 in [6]. □

19
3 Comments

The algorithm we presented was optimal with respect to the number of queries. (Although parts of the algorithm occur only in an implicit way, through the use of properties of $e_\delta^3$ numbers, it is straightforward to transform all upper bound proofs into algorithmic details.) Let us now consider its cost in the bit model of computation. Here we assume that $n$ and $N$ are powers of 2. We use the respective remarks about bit cost made in Section 5 of [6].

For $p \geq q$ classical approximation suffices. For $p < q$ the problem splits into the classical computation of $P_{l_0}$ and the approximation of $J_{pq}$ for $l = l_0, \ldots, l^* - 1 = 2l_0 - 1 = \mathcal{O}(\log n)$ using $n_l$ queries (see the proof Proposition 4 for these numbers). To increase the respective success probabilities appropriately, we have to repeat these approximations $\nu_l$ times on level $l$, and we have $\nu_l = \mathcal{O}(\log n)$. The total number of queries is $n$ (or $\tilde{n} = \mathcal{O}(n)$ if considered before scaling), that of quantum gates is

$$\mathcal{O}\left(\sum_{l=l_0}^{l^*-1} \nu_l n_l \log N_l\right) = \mathcal{O}(n \log n).$$

The algorithm needs $\mathcal{O}(\log n)$ qubits and

$$\mathcal{O}\left(\sum_{l=l_0}^{l^*-1} \nu_l n_l^2 N_l^{-1} \max(\log(n_l/\sqrt{N_l}),1)^{-1}\right) = \mathcal{O}(n/\log n)$$

measurements. To compute $P_{l_0}f$ classically, we need $\mathcal{O}(n)$ function values and $\mathcal{O}(n \log n)$ classical bit operations. For the approximations on the levels a total of

$$\mathcal{O}\left(\sum_{l=l_0}^{l^*-1} \nu_l n_l^2 N_l^{-1} \log N_l\right) = \mathcal{O}(n \log n).$$

classical bit operations is required. This does not yet take into account the classical computation of the vector analogue of the median. Let us assume that we apply the constructive procedure described after Corollary 1 of [6]. At level $l$ we have to compute the norm of $\nu_l^2$ vectors in $L^{N_l}_{q}$ with at most

$$\mathcal{O}\left(n_l^2 N_l^{-1} \max(\log(n_l/\sqrt{N_l}),1)^{-2}\right)$$

non-zero coordinates. This amounts to

$$\mathcal{O}\left(\log n \sum_{l=l_0}^{l^*-1} \nu_l^2 n_l^2 N_l^{-1} \max(\log(n_l/\sqrt{N_l}),1)^{-2}\right) = \mathcal{O}(n \log \log n)$$
classical bit operations. We see that the overall quantum bit cost differs by at most a logarithmic factor from the quantum query cost $\Theta(n)$.

The concrete form of the output of the algorithm depends on the structure of $P$. If $P$ is, for example, tensor product Lagrange interpolation, then the output is a sum of $O(\log n)$ piecewise polynomial functions, with $O(n)$ pieces for the classical part $P_{l_0}f$ and

$$O\left(n^2 N^{l-1}_I \max(\log(n_I/\sqrt{N_I}), 1)^{-2}\right)$$

pieces not identical to zero on level $l$, that is, a total of

$$O\left(n + \sum_{l=l_0}^{l'-1} n^2 N^{l-1}_I \max(\log(n_I/\sqrt{N_I}), 1)^{-2}\right) = O(n)$$

nontrivial pieces, with each point of $D$ being contained in at most $O(\log n)$ pieces.

We summarize the results on the approximation of $J_{pq}$: $B(W^r_p([0,1]^d)) \rightarrow L^q([0,1]^d)$ in a table and compare them with the respective known quantities in the classical deterministic and randomized settings (see [3] and the bibliography therein). Recall that we always assume $r/d > 1/p$. The respective entries of the table give the minimal errors, constants and logarithmic factors are suppressed.

| $J_{pq}$ | deterministic | random | quantum |
|----------|---------------|--------|---------|
| $1 \leq p < q \leq \infty$, $r/d \geq 2/p - 2/q$ | $n^{-r/d+1/p-1/q}$ | $n^{-r/d+1/p-1/q}$ | $n^{-r/d}$ |
| $1 \leq p < q \leq \infty$, $r/d < 2/p - 2/q$ | $n^{-r/d+1/p-1/q}$ | $n^{-r/d+1/p-1/q}$ | $n^{-2r/d+2/p-2/q}$ |
| $1 \leq q \leq \infty$ | $n^{-r/d}$ | $n^{-r/d}$ | $n^{-r/d}$ |

We observe a possible improvement of $n^{-1}$ (for $p = 1, q = \infty$) over the classical deterministic and randomized case (which is essentially a squaring of the classical rate for $r/d$ close to 1). This is the maximal speedup over the randomized case observed so far in natural numerical problems (the same speedup was first found in [3] for integration of functions from $W^r_p(D)$). We also see that there are regions of the parameter domain where the speedup is smaller, and others, where there is no speedup at all.
References

[1] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[3] S. Heinrich, Random approximation in numerical analysis, in: K. D. Bierstedt, A. Pietsch, W. M. Ruess, D. Vogt (Eds.), Functional Analysis, Marcel Dekker, New York, 1993, 123 – 171.

[4] S. Heinrich, Quantum summation with an application to integration, Journal of Complexity 18 (2002), 1–50, see also http://arXiv.org/abs/quant-ph/0105116.

[5] S. Heinrich, Quantum integration in Sobolev classes, J. Complexity 19 (2003), 19–42, see also http://arXiv.org/abs/quant-ph/0112153.

[6] S. Heinrich, Quantum Approximation I. Embeddings of Finite Dimensional $L_p$ Spaces, 2003.

[7] S. Heinrich, E. Novak, On a problem in quantum summation, J. Complexity 19 (2003), 1–18, see also http://arXiv.org/abs/quant-ph/0109038.

[8] V. E. Maiorov, Discretization of the problem of diameters, Usp. Mat. Nauk 30, No. 6 (186) (1975), 179–180.

[9] E. Novak, I. H. Sloan, H. Woźniakowski, Tractability of approximation for weighted Korobov spaces on classical and quantum computers, 2002, see http://arXiv.org/abs/quant-ph/0206023.

[10] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, 2nd ed., Barth, Leipzig, 1995.