Cutting plane method in fuzzy environment

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Abstract
Linear programming is one of the most widely used decision making tools for solving real world problems. Real world situations are characterized imprecision rather than exactness. In this paper classical cutting plan method is extended to solve fuzzy number linear programming problem to find integer solution. The trapezoidal fuzzy numbers are defuzzificated by using linear ranking method proposed by Maleki [22]. This method is easy to apply. This method is explained with suitable numerical examples.

Keywords
Fuzzy linear programming problem, Trapezoidal fuzzy number, Ranking method.

AMS Subject Classification
03E72

1. Introduction
In many practical problems the decision variables make sense only if they have integer values. It is often necessary to assign men, machines, and vehicles to activities in integer quantities. A linear programming problem in which some or all of the decision variables are restricted to assume non-negative integer values is commonly referred to as integer linear programming problem.

Integer programming problem is of particular importance in business and industry where, quite often, the fractional solutions are unrealistic because the units are not divisible. Hence, a new procedure has been developed in this direction for the case of LPP subjected to the additional restriction that the decision variables must have integer values. A systematic procedure for solving pure integer programming problem was first developed by R.E. Gomory in 1958. Later on, he extended the procedure to solve mixed integer programming problem named as cutting plane algorithm. In this method, we first solve the IPP as ordinary LPP by ignoring the integer restriction and then introducing additional constraints one after the other to cut certain part of the solution space until an integer valued solution is obtained.

This paper is organized as follows. In section 2, some fundamental concepts on fuzzy number and the ranking functions are given. In section 3, Formulation of fuzzy number integer programming problem. In section 4, the procedure of the proposed fuzzy number Gomory’s method is discussed. In section 5, two numerical examples are given and in section 6, the paper is concluded.

2. Fundamental of fuzzy set theory
The term fuzzy was proposed by Zadeh [20] in 1962. In [21], he published the paper entitled fuzzy sets in the year 1965. The fuzzy set theory is developed to improve the oversimplified model, thereby developing a more robust and flexible model in order to solve real-world complex systems involving human aspects. Furthermore, it helps the decision maker not only to consider the existing alternatives under given constraints (optimize a given system), but also to develop new alternatives (design a system). The fuzzy set theory has been applied in many fields, such as operations research, management science, control theory, artificial intelligence/expert system, human behavior, etc. In this study we will concentrate on fuzzy number mathematical programming problems. To do so, we will first introduce the required knowledge of the fuzzy set theory, fuzzy arithmetic and linear ranking function.
Definition 2.1. Let $X$ be a collection of objects denoted generically by $x$, then a fuzzy set in $X$ is a set of ordered pairs $\Delta = \{x, \mu_\Delta(x) | x \in X, \mu_\Delta(x) \in [0, 1]\}$, where $\Delta$ is called the membership function.

Definition 2.2. The support of a fuzzy set $\Delta$ is the crisp set defined by $\Delta = \{x \in X | \mu_\Delta > 0\}$.

Definition 2.3. The core of a fuzzy set $\Delta$ is the crisp set of points $x \in X$ with $\mu_\Delta = 1$.

Definition 2.4. The boundary of a fuzzy set $\Delta$ are defined set of points $x \in X$ such that $0 < \mu_\Delta < 1$.

It is evident that the boundary is defined as the region of the universal set containing elements that have non-zero membership but not complete membership. The Fig. 1 illustrates the region.

Definition 2.5. A fuzzy set $\Delta$ is normal if and only there exists $x_i \in X$ such that $\mu_\Delta(x_i) = 1$.

Definition 2.6. A fuzzy set $\Delta$ is sub normal if $\mu_\Delta < 1$.

Definition 2.7. The $\alpha$-cut of a fuzzy set $\Delta$ denoted by $|\Delta|_\alpha$ and is defined by $|\Delta|_\alpha = \{x \in X | \mu_\Delta(x) \geq \alpha\}$. If $\mu_\Delta(x) > \alpha$, then $|\Delta|_\alpha$ is called strong $\alpha$-cut. It is clear that $\alpha$-cut (strong $\alpha$-cut) is a crisp set.

Definition 2.8. A fuzzy set $\Delta$ on $X$ is convex if for any $x_1, x_2$ and $\lambda \in [0, 1], \mu_\Delta(\lambda x_1 + (1 - \lambda) x_2) \geq \min\{\mu_\Delta(x_1), \mu_\Delta(x_2)\}$.

It is to be noted that a fuzzy set is convex if and only if its $\alpha$-cut is convex.

Definition 2.9. A fuzzy number is a fuzzy subset in universal set $X$ which is both convex and normal.

Definition 2.10. A fuzzy number $\Delta = \{a, b, c, d\}$ is said to be a trapezoidal fuzzy number if its membership function is given by

$$\mu_\Delta(x) = \begin{cases} (x-a), & q \leq x < b \\ (b-a), & 1, b \leq x < c \\ (x-d), & c < x \leq d \\ 0, & \text{otherwise}. \end{cases}$$

Definition 2.11. Let $\Delta = \{a, b, a', b'\}$ be the $\text{TrFN}$, where $(a', a, b')$ is the support of $\Delta$ and $(a', b')$ is the core of $\Delta$.

**Arithmetic on Trapezoidal Fuzzy Numbers**

Let $F(\mathbb{R})$ be set of all trapezoidal fuzzy numbers over the real line $\mathbb{R}$. The arithmetic operations on trapezoidal fuzzy numbers are defined as follows:

Let $a = (a_l, a_u, \alpha, \beta)$ and $b = (b_l, b_u, \gamma, \delta)$ $(\frac{\beta}{\gamma} - \theta)$ be two trapezoidal fuzzy numbers and $x \in \mathbb{R}$. We define

$$x > 0, x \in \mathbb{R}; xa = (xa_l, xa_u, x\alpha, x\beta),$$

$$x < 0, x \in \mathbb{R}; xa = (xa_l, xa_u, -x\beta, -x\alpha),$$

$$a + b = (a_l + b_l, a_u + b_u, \alpha + \gamma, \beta + \delta),$$

$$a - b = (a_l - b_l, a_u - b_u, \alpha + \gamma, \beta + \delta).$$

**Ranking Function**

Ranking is one of the effective method for ordering fuzzy numbers. Various types of ranking function have been introduced and some have been used for solving linear programming problems with fuzzy parameters. An effective approach for ordering the element of $F(\mathbb{R})$ is to define a ranking function.

Let $\mathbb{R} : F(\mathbb{R}) \rightarrow \mathbb{R}$. We define order on $F(\mathbb{R})$ as follow:

(i) $a \geq b$ if $\mathbb{R}(a) \geq \mathbb{R}(b)$,

(ii) $a > b$ if $\mathbb{R}(a) > \mathbb{R}(b)$,

(iii) $a = b$ if $\mathbb{R}(a) = \mathbb{R}(b)$,

(iv) $a \leq b$ if $\mathbb{R}(a) \leq \mathbb{R}(b)$.

Here $\mathbb{R}$ is the ranking functions, such that $\mathbb{R}(ka + b) = k\mathbb{R}(a) + \mathbb{R}(b)$.

Here, we introduce a linear ranking function that is similar to the ranking function adopted by Maleki [22]. For a trapezoidal fuzzy number $a = (a_l, a_u, \alpha, \beta)$, we use ranking function as follows:

$$\mathbb{R}(a) = \int_0^1 (\inf_{\alpha} a + \sup_{\alpha} a) d\alpha$$

which reduced to $\mathbb{R}(a) = (a_l + a_u) + \frac{1}{2}(\beta - \alpha)$.

For any trapezoidal fuzzy numbers $a = (a_l, a_u, \alpha, \beta)$ and $b = (b_l, b_u, \gamma, \delta)$, we have $a \geq b$ if and only if $a_l + a_u + \frac{1}{2}(\beta - \alpha) \geq b_l + b_u + \frac{1}{2}(\delta - \gamma)$.
3. Fuzzy number integer programming problem

Fuzzy number integer programming problem is defined as Maximize $Z = Cx$ subject to constraint $Ax \leq b$, $x \geq 0$ and integers.

4. Gomory’s method

Gomory’s cutting plane algorithm starts with solving fuzzy number linear programming problem by simplex method ignoring the restriction of integral values. In the optimum solution if all the variables have integer values, the current solution will be the desired optimum integer solution. Otherwise, we add a new constraint to the problem such that the new set of feasible solutions includes all the original feasible integer solution but does not include the optimum non-integer solution initially found. We then solve the revised problem using the simplex method and see if we can get an integer solution. If not we add another fractional cut and repeat the process until an integer solution is found. Since, we never eliminate any feasible integer solution from consideration when we add fractional cuts, the integer solution ultimately found must be optimum.

Algorithm:

Step 1: Let us consider FNIPP and convert the problem into maximization if it is in the minimization form. All the coefficients and constants should be integers.

Step 2: Ignoring the integer condition, find the optimum solution of the resulting FNLPP by simplex method.

Step 3: Test the integer condition of the optimum solution.

   (i) If all $X_{B_i} \geq 0$ and are integers, an optimum integer solution is obtained.

   (ii) If all $X_{B_i} \geq 0$ and at least one $X_{B_i}$ is not an integer, then go to the next steps.

Step 4: Rewrite each $X_{B_i}$ as $X_{B_i} = \lfloor X_{B_i} \rfloor + f_i$, where $\lfloor X_{B_i} \rfloor$ is the integral part of $X_{B_i}$ and $f_i$ is the positive fractional part of $X_{B_i}$, $0 \leq f_i \leq 1$. Choose the largest fraction of $X_{B_i}$’s that is choose Max $\{f_i\}$ . In case of a tie, select arbitrarily. Let Max $\{f_i\} = f_k$ corresponding to $X_{B_k}$ (then k th row is called as a source row).

Step 5: Express each of the negative fractions if any, in the k th row (source row) of the optimum simplex Table as the sum of the negative integer and a non-negative fraction.

Step 6: Find the fractional cut constraint (Gomorian constraint or secondary constraint).

   From the source row $\sum_{j=1}^{n} a_{kj} x_j = X_{B_k}$; that is, $\sum_{j=1}^{n} (a_{kj} + f_k) x_j = \lfloor X_{B_k} \rfloor + f_k$ in the form $\sum_{j=1}^{n} f_kx_j \geq f_k$ or, $\sum_{j=1}^{n} f_k x_j \geq -f_k$ or, $-\sum_{j=1}^{n} f_k x_j + s_1 = -f_k$, where $s_1$ is the Gomorian slack.

Step 7: Add the fractional cut constraint obtained in step 6 at the bottom of the optimum simplex table obtained in step 2. Find the new feasible optimum solution using dual simplex method.

Step 8: Go to Step 3 and repeat the procedure until an optimum integer solution is obtained.

5. Numerical Examples

To illustrate the proposed algorithm the followings fuzzy number integer programming problems are solved.

Example 5.1. Solve the fuzzy number integer programming problem:

Max $Z = (5,8,2,5) X_1 + (6,10,2,6) X_2$ subject to $2X_1 + 3X_2 \leq 6; 5X_1 + 4X_2 \leq 10; X_1, X_2 \geq 0$ and are integer.

Solution: Ignoring the integer restriction and applying simplex method for FNIPP the optimum simplex table is:

\[
\begin{array}{ccccccccc}
\text{C}_i & \text{Basic} & \text{X}_0 & \text{X}_1 & \text{X}_2 & \text{X}_3 & \text{X}_4 & \text{X}_5 & \text{X}_6 \\
\hline
\text{C}_1 & \text{Basic} & \text{X}_0 & \text{X}_1 & \text{X}_2 & \text{X}_3 & \text{X}_4 & \text{X}_5 & \text{X}_6 \\
(5,8,2,5) & \text{X}_1 & 10^{-7} & 0 & 1 & 5^{-7} & 2^{-7} & 0 & 0 \\
(0,10,2,6) & \text{X}_2 & 6^{-7} & 0 & 0 & 1 & 0 & 0 & 0 \\
(5,8,2,5) & \text{X}_1 & 6^{-7} & 0 & 0 & 1 & 0 & 0 & 0 \\
(0,10,2,6) & \text{X}_2 & 1^{-7} & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{G} & \text{Basic} & \text{X}_0 & \text{X}_1 & \text{X}_2 & \text{X}_3 & \text{X}_4 & \text{X}_5 & \text{X}_6 \\
(9,14,8,9) & \text{X}_1 & 1^{-7} & 0 & 0 & 0 & 0 & 0 & 0 \\
(0,2,5) & \text{X}_2 & 1^{-7} & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{G} & \text{Basic} & \text{X}_0 & \text{X}_1 & \text{X}_2 & \text{X}_3 & \text{X}_4 & \text{X}_5 & \text{X}_6 \\
(9,14,8,9) & \text{X}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(0,2,5) & \text{X}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Since all $R(Z_i - C_i) \geq 0$ the current solution is optimum. The optimum solution is $X_1 = \frac{5}{7}, X_2 = \frac{10}{7}$ and maximum $Z = \left(\frac{90}{7}, \frac{148}{7}, \frac{32}{7}, \frac{90}{7}\right), R(Z) = \frac{267}{7}$.

But the solution obtained is not an integer solution.

Construction of Gomory’s constraint:

Now,

\[
\frac{10}{7} = 1 + \frac{3}{7} \quad (f_1 = \frac{3}{7})
\]

\[
\frac{6}{7} = 0 + \frac{6}{7} \quad (f_2 = \frac{6}{7})
\]

and maximum of $\left\{\frac{5}{7}, \frac{6}{7}\right\} = \frac{6}{7}$.

Hence we choose 2nd row ($X_1$-row) for constructing the Gomory’s constraints.

Now $X_1$ equation is

\[
0 + \frac{6}{7} = 1X_1 + 0X_2 + \left(-1 + \frac{3}{7}\right)X_3 + \frac{3}{7}X_4.
\]

Hence, the Gomory’s constraint is $-\frac{3}{7}X_3 - \frac{3}{7}X_4 + X_5 = -\frac{6}{7}$, where, $X_5$ is the Gomory’s slack variable. Hence $X_5 = -\frac{6}{7}$. As the solution is infeasible, Let us apply dual simplex method after including the Gomory’s constraint equation in the above table.

\[
\begin{array}{ccccccc}
\text{C}_i & \text{Basic} & \text{X}_0 & \text{X}_1 & \text{X}_2 & \text{X}_3 & \text{X}_4 \\
(0,10,2,6) & \text{X}_1 & 10^{-7} & 0 & 1 & 5^{-7} & 2^{-7} & 0 \\
(5,8,2,5) & \text{X}_2 & 6^{-7} & 0 & 0 & 1 & 0 & 0 \\
(0,10,2,6) & \text{X}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
(5,8,2,5) & \text{X}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Since maximum of $\left\{\frac{36}{7}, \frac{8}{7}\right\} = \{12, -\frac{8}{7}\} = -\frac{8}{7}$. Corresponding to $X_4$:
Since all \( x_B \geq 0 \) and integer and \( \mathbb{R}(Z_j - C_j) \geq 0 \) the current solution is optimum and the optimum integer solution is \( x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 2, x_5 = 0 \) and Max \( Z = 12, 20, 4, 12, \mathbb{R}(Z) = 36 \).

**Example 5.2.** Max \( Z = (2, 5, 1, 2)X_1 + (8, 9, 2, 5)X_2 \) subject to \( X_1 + 2X_2 \leq 6; -X_1 + X_2 \leq 2; 2X_1 + X_2 \leq 6, X_1, X_2 \geq 0 \) and are integer.

**Solution:** Ignoring the integer restriction and applying simplex method for FNLP the optimal simplex table is: Since

\[
\begin{array}{cccccccc}
C_j & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \\
\hline
C_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Since maximum of \( \left\{ \frac{28}{3}, \frac{11}{3}, \frac{4}{3} \right\} \) \( = -28, -4 \) \( \Rightarrow \) \( Z_4 \).

Corresponding to \( X_4 \):

\[
\begin{array}{cccccccc}
C_j & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \\
\hline
C_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Since all \( x_B \geq 0 \) and integer and \( \mathbb{R}(Z_j - C_j) \geq 0 \) the current solution is optimum and the optimum integer solution is \( x_1 = 2, x_2 = 2 \) and Max \( Z = 20, 28, 6, 14, \mathbb{R}(Z) = 52 \).

### References

[1] Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Application*, Academic, New York, 1980.

[2] A. Jones, A. Kaufmann and H.-J. Zimmermann, *Fuzzy Sets Theory and Applications*, Reidel, Dordrecht, 1985.

[3] A. Kaufmann and M.M. Gupta, *Introduction to Fuzzy Arithmetic: Theory and Applications*, Van Nostrand Reinhold, New York, 1985.

[4] H.-J. Zimmermann, *Fuzzy Set Theory and its Application*, Kluwer, Hingham, 1985.

[5] T. J. Ross, *Fuzzy Logic with Engineering Applications*, John Wiley and Sons, 2004.

[6] E.R. Bellman and L.A. Zadeh, Decision making in a fuzzy environment, *Management Sciences*, 17(1970), 141–164.

[7] A. Ebrahimnejad, S.H. Nasser and F. HosseinzadehLotfi, Bounded linear programs with trapezoidal fuzzy numbers, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 18(3)(2010), 269–286.

[8] A. Ebrahimnejad, S.H. Nasser, F. HosseinzadehLotfi and M. Soltanifar, A primal-dual method for linear programming problems with fuzzy variables, *European Journal of Industrial Engineering*, 4(2)(2010), 189–209.

[9] A. Kaufmann. M.M. Gupta, *Introduction to Fuzzy Arithmetic: Theory and Application*, Van Nostrand Reinhold, New York, 1991.

[10] F. Herrera and J. L. Verdegay, Three models of fuzzy integer linear programming, *European J. Oper. Res.*, 83(1995), 581–589.

[11] H. Tanaka and K. Asai, *Fuzzy Linear Programming*
Problems with Fuzzy Numbers, *Fuzzy Sets and Systems*, 13(1984), 1–10.

[12] H.J. Zimmermann, Applications of fuzzy sets theory to mathematical programming, *Inform. Sci.*, 36(1985), 29–58.

[13] K.K. Mishra and C. Das, Applications of simplex method for solving linear programming problems with trapezoidal fuzzy numbers, *Advances in Fuzzy Sets and Systems*, 10(1)(2011), 57–75.

[14] Mohmed. S. Osman, Omar.M. Saad and Azza G. Hasan, Solving a special of Large-scale fuzzy multi objective integer linear programming problems, *Fuzzy Sets and Systems*, 107(1999), 289–297.

[15] N. Mahdavi-Amiri, S.H. Nasseri and A. Yazdani, Fuzzy primal simplex algorithms for solving fuzzy linear programming problems, *Iranian Journal of Operations Research*, 1(2009), 68–84.

[16] P. Pandian and M. Jayalakshmi, A new method for solving integer linear programming with fuzzy variables, *Appl. Math. Sci.*, 4(2010), 997–1004.

[17] T. Allahviranloo, K.H. Shamsolkotabi, N.A. Kuni and L. Alizadeh, Fuzzy integer linear programming problems, *Int. J. Contemp. Math. Sci.*, 2(2007), 167–181.

[18] T. J. Ross, *Fuzzy Logic with Engineering Applications*, John Wiley and Sons, 2004.

[19] R.R. Yager, A procedure for ordering fuzzy subsets of the unit interval, *Information Sciences*, 24(1981), 143–161.

[20] L.A. Zadeh, *Fuzzy Logic and Its Applications*, Academic Press, New York, 1965.

[21] L.A. Zadeh, Fuzzy sets, *Information and Control*, 8(3)(1965), 338–353.

[22] H.R. Maleki, Ranking functions and their applications to fuzzy linear programming, *Far East J. Math. Sci.*, 4(2002), 283–301.