SW action for the lattice Schwinger model*

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We study some aspects of the $\mathcal{O}(a)$ improved Sheikholeslami-Wohlert (SW) action for the lattice Schwinger model. We find some improvement concerning the distribution of eigenvalues of the Dirac operator and of the masses but little or no improvement for rotational invariance of correlators or dispersion relations.

1. INTRODUCTION

The $\mathcal{O}(a)$ improved action, as introduced by Symanzik \cite{Symanzik} and proposed for lattices fermions by Sheikholeslami and Wohlert \cite{Sheikholeslami_Wohlert} has become a popular and relatively simple tool for improving lattice predictions over the last years. It is therefore interesting to investigate how different observables are affected by the $\mathcal{O}(a)$ improvement of the standard Wilson action in a simple and well-known toy model.

Our testing ground is the 2-flavor Schwinger model (QED in 2D), which is well-known analytically and quite undemanding computationally. We perform a full fermion HMC simulation and study bosonic masses and dispersion relations, rotational invariance and the eigenvalue spectrum of the action.

The SW action is obtained from the naive fermion action by a rotation of the fermionic fields

$$
\Psi \rightarrow (1 - \frac{1}{2}\mathcal{D}) \Psi , \quad \bar{\Psi} \rightarrow \bar{\Psi} (1 + \frac{1}{2}\mathcal{D})
$$

and discarding the $\mathcal{O}(a^2)$ terms in the resulting action.

This results in the Wilson action plus an additional SW term

$$
S_{SW} = -\kappa c_{SW} \frac{i}{2} \sum_x F_{\mu\nu} \bar{\Psi}_x \sigma_{\mu\nu} \Psi_x.
$$

The coefficient $c_{SW}$ is 1 for tree-level improvement and has to be determined perturbatively or non-perturbatively in 4D. In our case of the superrenormalizable Schwinger model, the coupling $e$ has dimension $1/a$, so that in a perturbative expansion, where the next-to-leading order graph has two vertices, $c_{SW} = 1 + \mathcal{O}(a^2 e^2)$. Therefore for an $\mathcal{O}(a)$ improvement $c_{SW} = 1$.

2. RESULTS

2.1. Phase diagram

First we looked at the phase diagram comparing Wilson to SW action (Fig. 1). We determined $\kappa_c$ at a given $\beta$ by PCAC methods, i.e. measuring an observable, which is proportional to the effective fermion mass at five different values of $\kappa$ and finding $\kappa_c$ by interpolation (for more details cf. \cite{Wilson}).

As expected, the critical line moves closer to the continuum critical value of $\beta_c = 0.25$. However, we find that for both actions the leading correction goes like

$$
\kappa_c = 0.25 + \mathcal{O}(1/\beta) = 0.25 + \mathcal{O}(a^2).
$$

We found the volume dependence of $\kappa_c$ to be weak (see \cite{Wilson} for a discussion for the Wilson action).

2.2. Spectrum

We also determined the eigenvalue spectrum of the fermion matrix for different configurations.

As has been pointed out recently, the fixed point action \cite{Diakonov} as well as the overlap action \cite{Re Noise} have an eigenvalue spectrum of circular shape; this is related to the fact that they satisfy the Ginsparg-Wilson condition \cite{Ginsparg} and therefore have good chiral properties.

As can be seen in Fig.\cite{fig:2} the SW spectrum differs from the Wilson spectrum (see e.g.\cite{Wilson}) but
does not improve much towards a circular shape. The change of the spectrum is consistent with the observations made in [3] for 4D SU(2) theory.

### 2.3. Rotational invariance

Another interesting aspect is, whether one may observe any improvement in the rotational invariance of propagators. For this purpose, we measured

\[
\langle \bar{u}(0) \sigma_3 d(0) \bar{d}(x) \sigma_3 u(x) \rangle,
\]

for all distances across the lattice and plotted the correlation function vs. $|x|$ (where $u$ and $d$ denote the two fermion flavors). For a comparison, this was done at the same $\beta$ and for the same effective fermion mass for both actions. For the SW situation the fermion fields were also improved according to (1). Although the propagator is an off-shell quantity one might expect some improvement, since we also improved the fermion fields.

As can be seen from Fig. 3, there is no noticeable improvement in the rotational invariance for short distances. This is our first indication, that also high momentum observables are not improved.

### 2.4. Dispersion relations

The spectrum of the massless $N$-flavor Schwinger model contains one free massive (isosinglet-vector) meson with mass $m_S = \sqrt{N/\beta \pi}$ and $N^2 - 1$ free massless (isomultiplet-vector) mesons.

We measured the dispersion relation of both the massive and the massless bosons and found – compared to the results for the Wilson action – no improvement (Fig. 4). The higher momentum modes are clearly deviating from the continuum behavior. For the fixed point action drastically better behavior has been observed [5].

For the massive boson one expects improved scaling; indeed there seems to be evidence that the mass estimate at small $\beta$ is improved with the SW action. However, the statistics of our data does not allow conclusive statements at this point.

### 3. CONCLUSIONS

We studied the effect of $O(a)$ SW-improvement of the Wilson action in the unquenched 2-flavor
We found that the critical value of the hopping parameter \( \kappa_c \) moves closer to its continuum value 0.25. We observed a change in the eigenvalue spectrum of the Dirac operator but we found no significant improvement towards e.g. a circular shape. We found no improvement in the rotational symmetry of correlation functions or in the dispersion relations at higher momentum.

Since it is well established from QCD simulations that meson masses are improved by using the SW action, we conclude that the \( \mathcal{O}(a) \) SW-improvement works best for low momentum states and that there might be no relevant improvement for observables connected with higher momentum states.

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