Universality in the entanglement structure of ferromagnets

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Systems of exchange-coupled spins are commonly used to model ferromagnets. The quantum correlations in such magnets are studied using tools from quantum information theory. Isotropic ferromagnets are shown to possess a universal low-temperature density matrix which precludes entanglement between spins, and the mechanism of entanglement cancellation is investigated, revealing a core of states resistant to pairwise entanglement cancellation. Numerical studies of one-, two-, and three-dimensional lattices as well as irregular geometries showed no entanglement in ferromagnets at any temperature or magnetic field strength.

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Quantum correlations are responsible for much of the challenge in understanding interacting many-body quantum systems, and it is therefore of fundamental importance to have quantitative knowledge of these correlations. Progress in quantum information theory has led to the development of new measures of the inseparability of a quantum state, and in the last few years these measures have been used to assess the quantum correlations in diverse physical systems. Concurrence \[1\] is an especially useful metric for such studies because it can be applied to mixed as well as pure states. It therefore can be used to quantitate the thermal entanglement in a system at nonzero temperature. It can also be applied to evaluate the inseparability of an equal incoherent mixture of degenerate energy eigenstates. However, concurrence is defined only for a pair of qubits. Since a qubit is formally equivalent to a spin-1/2 particle when only the spin degree of freedom of the latter is considered, this has led to several analyses of the thermal entanglement between a pair of interacting spin-1/2 particles. Two-spin systems studied in this way include the XXX, XXZ, and XYZ Heisenberg models, the X-X-model, \[3\], the XY-model, \[4\], the Heisenberg-Dzyaloshinski-Moriya model, \[5\], and the Ising model, \[6\]. Inseparability at zero and finite temperature has also been studied in two-spin subsystems of one-dimensional spin chains, including the XXX-chain, \[7\], the XY-chain, \[8\], \[9\], the Ising chain, \[10\], and the Majumdar-Ghosh chain, \[11\], as well as in the ground state of quasi-one-dimensional spin ladders, \[12\].

This paper will examine quantum correlations in isotropic ferromagnets without restrictions on the geometry of the spins (such as one-dimensionality or periodicity) or on the range of the spin-spin interaction. A suitably general model of a magnet, which will be called the Heisenberg spin graph, consists of \(N\) spins coupled by exchange interactions with arbitrary range and strength. It is defined by the Hamiltonian

\[
H_{\text{HSG}} = \sum_{i<j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j,
\]

This can be visualized as a graph with a spin-1/2 particle at each vertex, where the edge between spins \(i\) and \(j\), representing the exchange interaction \(\mathbf{S}_i \cdot \mathbf{S}_j\) between the two spins, is weighted by the coupling constant \(J_{ij}\). The summation runs over each pair of spins once. The operator \(\mathbf{S}_i = (S^x_i, S^y_i, S^z_i)\) is the spin operator associated with the spin at the \(i\)th vertex of the graph; its Cartesian components obey the usual angular momentum algebra. As a matter of notation, states in the \(2^N\)-dimensional Hilbert space \(\mathcal{H}\) of the model are specified with respect to basis states of the form \(|i,j,\ldots,k\rangle\), where the listed spins \(i,j,\ldots,k\) are aligned in the +z-direction ('up') and the remaining spins are antiparallel ('down'). \(|\emptyset\rangle\) denotes the state with all spins down. One might anticipate that the Heisenberg spin graph would be so general a model that nothing much could be said about it. It turns out that when the coupling constants \(J_{ij}\) are nonpositive (that is, ferromagnetic), certain features of Heisenberg spin graphs are universal, in the sense that they do not depend on the exact values of the coupling constants or even on the number of spins. This universality allows a complete analysis of low-temperature correlations in such ferromagnets.

Because the Heisenberg spin graph Hamiltonian \(H_{\text{HSG}}\) commutes with the total \(z\)-spin operator \(S^z = \sum_{i=1}^{N} S^z_i\), the two-qubit reduced density matrix for any two spins in an eigenstate of \(H_{\text{HSG}}\) always takes the restricted form

\[
\rho = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & \gamma & 0 \\ 0 & \gamma^* & \delta & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}.
\]

For such a density matrix, the concurrence is \(C(\rho) = \max (0, |\gamma| - \sqrt{\alpha \epsilon})\). Concurrence ranges from zero, for a separable state, to one, for a maximally entangled state. The entanglement of formation, another important entanglement measure, can be calculated directly from the concurrence, and is monotonically related to it. The method of calculating the concurrence for more general density matrices can be found in Wootters \[1\].

In Ref. \[2\] the authors have reported the absence of
From combinatoric reasoning we have a completely symmetric state, has the form:

\[ |\psirangle = \frac{1}{\sqrt{N}} (|0\rangle + \cdots + |N\rangle) \]

\[ |\psi_2rangle = \frac{1}{\sqrt{N(N-1)/2}} (|12\rangle + |13\rangle + \cdots + |(N-1)N\rangle) \]

\[ \vdots \]

\[ |\psi_Nrangle = |12 \ldots N\rangle \]

are eigenstates of \( H_{HSG} \) with the common eigenvalue \( \frac{1}{2} \sum_{i<j} J_{ij} \). These states span the ground subspace of the model; there are no ‘accidental’ degeneracies (see Appendix). At sufficiently low temperature, the thermal density matrix is an incoherent mixture of these \( N+1 \) states, the contribution from other states being negligible.

We choose any two spins in the solid and trace out the rest. Because of the linearity of the partial trace, we can perform the trace on each of the completely symmetric states separately. The total reduced density matrix \( \rho \) will then be

\[ \rho = \frac{1}{N+1} (\rho_0 + \cdots + \rho_N), \]

where \( \rho_n \), the reduced density matrix of the \( n \)th completely symmetric state, has the form:

\[ \rho_n = \begin{pmatrix} \alpha_n & 0 & 0 & 0 \\ 0 & \beta_n & \gamma_n & 0 \\ 0 & \gamma_n^* & \delta_n & 0 \\ 0 & 0 & 0 & \epsilon_n \end{pmatrix}. \]

From combinatoric reasoning we have

\[ \alpha_n = \frac{n(n-1)}{N(N-1)}, \]

\[ \beta_n = \gamma_n = \delta_n = \frac{n(N-n)}{N(N-1)}, \]

\[ \epsilon_n = \frac{(N-n)(N-n-1)}{N(N-1)}, \]

for the components of the density matrix \( \rho_n \).

Carrying out the summation we find that the ground state density matrix is

\[ \rho = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \]

Thus the low temperature density matrix for any two spins assumes a universal form, irrespective of the strength, geometry, or homogeneity of the interactions or the dimensionality of the solid. The concurrence of this universal density matrix is zero. This shows that, at least at low temperatures, no two qubits in an isotropic ferromagnet share any entanglement. The correlations present, such as \( \langle S^z_i S^z_j \rangle = 1/12 \), can be regarded as purely classical.

The form of the reduced density matrix given in Eq. (9) is of course strictly valid only in the limit of zero temperature. As the temperature \( T \) increases, the remaining eigenstates of \( H_{HSG} \) (those not listed in Eq. (3)) begin to contribute to the reduced density matrix \( \rho \) describing the state of the two chosen spins. As there are a finite number of such states, it can be seen that the elements \( \alpha, \beta, \ldots \), of the reduced density matrix are continuous functions of \( T \). Since at zero temperature \( |\gamma| - \sqrt{\alpha \epsilon} = -1/6 \), there must be some (possibly infinite) temperature interval, with \( T = 0 \) as one endpoint and closed if the interval is finite, in which the concurrence is zero. Thus the onset of entanglement in ferromagnets, if it occurs at all, happens as a phase transition at nonzero temperature.

Note that, individually, each \( \rho_n \) is an inseparable state, except for \( \rho_0 \) and \( \rho_N \), with concurrence

\[ C(\rho_n) = \frac{2}{N(N-1)} \times \left( n(N-n) - \sqrt{n(n-1)(N-n)(N-n-1)} \right). \]

It is the incoherent mixing of these inseparable states that washes out the quantum correlations. One might anticipate that this entanglement cancellation occurs pairwise, between states related by flipping the orientation of each spin, so that \( \rho'_n \equiv (\rho_n + \rho_{N-n})/2 \) would be separable; only the entanglement of \( \rho_{N/2} \) when \( N \) is even could not be eliminated in this way. Interestingly, this is
not quite correct. Rather, there are threshold values of \( n \) at \( n = (N \pm \sqrt{N})/2 \), below and above which pairwise cancellation occurs, but between which two-state mixing cannot account for the loss of entanglement. Analytically,

\[
C(\rho_n') = \max \left( 0, \frac{4n(N-n)}{N(N-1)} - 1 \right),
\]

with maximal entanglement of \( 1/(N-1) \) occurring at \( N/2 \) (\( N \) even) or of \( 1/N \) at \( (N+1)/2 \) (\( N \) odd). Thus it is the least entangled states which are most resistant to pairwise entanglement cancellation, as illustrated in Fig. 1.

Further, the lack of quantum correlations in the ground subspace cannot be explained by incoherent mixing of the states within the zone \( (N - \sqrt{N})/2 < n < (N + \sqrt{N})/2 \) among themselves \([14]\). This is already apparent from consideration of the \( N = 2 \) and \( N = 4 \) cases, where \( \rho_1 \) and \( \rho_2 \) respectively have no other states in their zones with which to mix. But it remains true at \( N = 6 \), where mixing of \( \rho_2 \), \( \rho_3 \), and \( \rho_4 \) results in an inseparable state \((C = 1/9)\). As shown in Fig. 2 the concurrence of the incoherent mixture of the states within this zone decays as \( \sim 1/N \). The separability of the degenerate ground state therefore arises from the mixing of this inseparable core with the surrounding sea of pairwise-separable states.

Whether any isotropic ferromagnet can become entangled at any temperature remains an open question. It seems unlikely. If the first excited state is nondegenerate, or nearly so, then it will change the low-temperature density matrix \( \rho \) (Eq. 4) by terms of order \( \exp(-\Delta E/kT)/N \), where \( \Delta E > 0 \) is the energy difference between the ground and first excited state. Such changes can be seen to be too small to result in an inseparable state, by considering the extreme case where the reduced density matrix for the excited state is a perfectly entangled triplet state. On the other hand, if the first excited state is highly degenerate (as in the Heisenberg spin chain, for example), or if there are many closely spaced levels just above the first excited state, then the thermal correction can be of order \( \exp(-\Delta E/kT) \), but in this case one expects that incoherent mixing among the degenerate (or almost degenerate) excited states will result in a separable state, as occurred when the degenerate ground states mixed. In either case, there appears to be no way to form an inseparable thermal state.

Since the obstacle to thermal entanglement seems to be the highly degenerate ground state, it is natural to wonder if the application of an external magnetic field would induce entanglement by relieving the degeneracy. To study this question, a simple extension of a model introduced by Majumdar and Ghosh \([17]\), defined by the Hamiltonian

\[
H_{XMG} = g_1 \sum_{i=1}^{N} S_i \cdot S_{i+1} + g_2 \sum_{i=1}^{N} S_i \cdot S_{i+2} + g_3 \sum_{i=1}^{N} S_i \cdot S_{i+3} + B \sum_{i=1}^{N} S_i^z,
\]

was studied numerically. Here \( B \) is a homogeneous external magnetic field directed along the \( +z \)-axis, and periodic boundary conditions are assumed, so that \( S_{m+N} = S_m \). The \( g_i \) are coupling constants, determining the strength of the interaction of a spin with its \( i \)th nearest neighbor. When \( g_3 = 0 \) the model reduces to the Majumdar-Ghosh spin chain; if \( g_2 = 0 \) as well, then it is the Heisenberg spin chain \([16]\). Without loss of generality the Hamiltonian can be rescaled so that \( g_1 = -1 \) for a ferromagnetic chain. Concurrences were calculated numerically for each qubit pair in spin chains of lengths \( N = 4 \) through 8. The coupling constants \( g_2 \) and \( g_3 \) were varied between \(-4 \) and \( 0 \), while the temperature

![FIG. 1: Concurrence of the unmixed symmetric state \( \rho_n \) (upper curve), Eq. 10, and of the pairwise mixed symmetric state \( \rho_n' \) (lower curve), Eq. 11 for \( N = 100 \).](image1)

![FIG. 2: Concurrence of the incoherently mixed symmetric states in the zone \((N - \sqrt{N})/2 < n < (N + \sqrt{N})/2 \) as a function of \( N \). Within this zone pairwise entanglement cancellation fails. The curve \( 1/N \) (dashed) is shown for comparison.](image2)
and magnetic field were varied between 0 and $N$. In no case did any qubit pair exhibit any entanglement. Similar numerical studies of other geometries were also carried out, including open spin chains (i.e. without periodic boundary conditions), nine spins arranged as a $3 \times 3$ grid with and without periodic boundary conditions, eight spins arranged as a cube, and several irregular geometries. These calculations also failed to reveal any entanglement between any qubit pair. Thus it appears reasonable to conjecture that ferromagnets cannot exhibit thermal entanglement at any temperature or applied field strength.

This paper has examined the quantum correlations present in isotropic ferromagnets. The ground subspace of an isotropic ferromagnet is spanned by the completely symmetric states, and it was proven that no other states intrude. As a result, the low-temperature two-spin reduced density matrix assumes a universal, separable form. Pairwise entanglement cancellation cannot entirely account for this separability, as there exists a core of states which remain inseparable even when mixed incoherently among themselves. Numerical evidence was provided supporting the conjecture that isotropic ferromagnetic spin systems do not exhibit thermal entanglement, even in the presence of an external magnetic field.

In fact, universality in the entanglement behavior of magnets is even more general than described here. Although anisotropic deformations of the exchange interaction can modify the structure of the ground subspace in several ways, universal behavior does reappear at low temperature even in anisotropic models. This will be discussed in detail in a subsequent paper.

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APPENDIX

Theorem: The ground subspace of the $N$-site connected ferromagnetic spin graph, defined by the Hamiltonian $H = \sum_{i<j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$ with $J_{ij} \leq 0$, is spanned by the $N + 1$ completely symmetric states given in Eq. (11) there are no ‘accidental’ degeneracies.

Proof: First consider the simpler case where $J_{ij} < 0$, that is, every spin interacts, though perhaps very weakly, with every other spin. The spin operator product $\mathbf{S}_i \cdot \mathbf{S}_j$ is equivalent to $\frac{1}{2} (P_{ij} - \frac{1}{2} I)$, where $P_{ij}$ is an operator which permutes spins $i$ and $j$, and $I$ is the identity operator. Each completely symmetric state is an eigenstate of each permutation operator separately, and so its energy eigenvalue is simply the sum of the coefficients of the individual permutation operators in the Hamiltonian: $E_0 = \langle H \rangle = \frac{1}{2} \sum_{i<j} J_{ij} \langle P_{ij} - \frac{1}{2} I \rangle = \frac{1}{2} \sum_{i<j} J_{ij}$. If a state $|\chi\rangle$ is not completely symmetric, however, there is by definition some permutation operator which does not leave this state invariant. In the special case under consideration, such a permutation has a nonzero coefficient in the Hamiltonian, say $J_{pq}$. Now, $P_{ij}^2 = I$, hence the maximum expectation value of a permutation operator in a normalized state is unity, so that $J_{pq} \langle \chi | P_{pq} | \chi \rangle > J_{pq}$, since the $J_{ij}$ are negative numbers. So $\langle \chi | H | \chi \rangle = \frac{1}{2} \sum_{i<j} J_{ij} \langle P_{ij} - \frac{1}{2} I | \chi \rangle > E_0$. Thus the completely symmetric states span the ground subspace, as claimed.

Now consider the more general case, where some of the coupling constants $J_{ij}$ may equal zero, but the resulting graph remains connected. Suppose once again that a state $|\chi\rangle$ is not symmetric under the interchange of $p$ and $q$. By the definition of connectedness, there is now a chain of spins $i_1, i_2, \ldots, i_n$ such that each of $J_{p_{i_1}}, J_{i_1 i_2}, \ldots, J_{i_n q}$ is nonzero. Then $H^{2n+1}$, the $(2n + 1)th$ power of the Hamiltonian, contains a product of permutation operators

$$P_{p_{i_1}} \cdots P_{i_{n-1}i_n} P_{i_{n}q} P_{i_{n-1}i_n} \cdots P_{i_1 i_2} P_{p_{i_1}}$$

which is equal to $P_{pq}$ (from the right, the first $n + 1$ permutations put the spin $p$ in the position $q$, while the remaining put spin $q$ into position $p$ and sort the intermediate spins of the chain back into their original positions). The above argument can now be repeated. Any completely symmetric state is an eigenstate of each term of $H^{2n+1}$, while $|\chi\rangle$ is not, and hence the expectation value of $\langle \chi | H^{2n+1} | \chi \rangle \neq E_0^{2n+1}$. Therefore $|\chi\rangle$ does not lie in the ground subspace. $\blacksquare$

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