Perverse Schobers and Wall Crossing

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For a balanced wall crossing in geometric invariant theory (GIT), there exist derived equivalences between the corresponding GIT quotients if certain numerical conditions are satisfied. Given such a wall crossing, I construct a perverse sheaf of categories on a disk, singular at a point, with half-monodromies recovering these equivalences, and with behaviour at the singular point controlled by a GIT quotient stack associated to the wall. Taking complexified Grothendieck groups gives a perverse sheaf of vector spaces: I characterize when this is an intersection cohomology complex of a local system on the punctured disk.

1 Introduction

Kapranov and Schechtman [24] have initiated a program to define and study perverse sheaves of triangulated categories, known as perverse schobers. These new objects are expected to arise naturally in the field of categorified birational geometry, which has developed following the pioneering work of Bondal and Orlov in the 1990s [11]. In this article, I give a large class of examples in the form of “spherical pairs.” These occur as categorifications of perverse sheaves [7] of vector spaces on a disk, possibly singular at a point. I will explain, following Kapranov–Schechtman, different such categorifications, depending on a choice of “skeleton” consisting of a union of disjoint arcs from the disk boundary to the point: for two arcs, the categorification is a spherical pair; for one arc, it is a spherical functor.
My main result, Theorem A, constructs spherical pairs for many birational maps coming from geometric invariant theory (GIT) wall crossings. My simplest examples come from an orbifold flop of local $\mathbb{P}^1$ and an Atiyah flop of a resolved conifold. These are given in an expository Section 2: further examples, namely flops of local $\mathbb{P}^{\text{odd}}$, and standard flops, are given in the final Section 6.

**Remark.** Harder et al. [19] study a different notion of perverse sheaves of categories, in relation to rationality.

1.1 Discussion

I outline some of my results informally, before giving details in Section 1.2.

1.1.1 Spherical pairs

Take embeddings of a pair of categories of interest $\mathcal{E}_{\pm}$ into a single category $\mathcal{E}_0$. These data may be viewed as a perverse sheaf of categories on a disk as follows. Take $K$ to be a skeleton with two arcs joining $\pm 1$ to 0, as illustrated in Figure 1. For a perverse sheaf of vector spaces, possibly singular at 0, the local cohomology with support in $K$ is concentrated in some fixed degree by “purity”: the categories $\mathcal{E}_{\pm}$ and $\mathcal{E}_0$ should then be seen as categorifications of the stalks of this sheaf of local cohomology at $\pm 1$ and 0, respectively, and the embeddings as categorifications of maps between them. A spherical pair consists of such categories $\mathcal{E}_{\pm}$ and $\mathcal{E}_0$ along with embeddings satisfying natural conditions: these are given in Section 3.2.

1.1.2 Setup

To obtain an example of a spherical pair, consider a GIT wall crossing given by the data of a variety $X$ with an action of a group $G$ and a family of linearizations $\mathcal{M}_t$, for small $t \in \mathbb{R}$, such that the GIT quotients $X//_t G$ are constant for $t < 0$ and $t > 0$, respectively.
respectively, but there exist strict semistables for $t = 0$. Assume furthermore that this wall crossing is “simple balanced”: this implies in particular that the change in the GIT quotient due to crossing the wall is controlled by a single one-parameter subgroup $\lambda$ of $G$. In this case, assuming a numerical condition given in Theorem A(iii), there exist derived equivalences between the quotients $X//\pm$ by results of Halpern-Leistner [17] and Ballard et al. [5].

1.1.3 Construction

Given this setup, I construct a spherical pair or, equivalently, a perverse sheaf of categories on the disk where $\mathcal{E}_\pm$ are the bounded derived categories $D(X//\pm)$, and $\mathcal{E}_0$ is a subcategory $\mathcal{C}$ of the bounded derived category $D(X^{ss}(\mathcal{M}_0)/G)$ of the stack of semistables associated to the wall. These data are indicated in Figure 2.

Remark. It is natural to view the disk on which our perverse sheaf of categories is defined as lying in the complexification of the GIT parameter space $\mathbb{R}$: intriguingly, this suggests the existence of natural perverse schobers on (appropriate compactifications of) Bridgeland stability spaces. □

Remark. For comparison, note that Bodzenta and Bondal construct spherical pairs for certain varieties $Y_\pm$ related by flops of families of curves, some of which arise from GIT wall crossings as above, where $\mathcal{E}_\pm = D(Y_\pm)$, and $\mathcal{E}_0$ is obtained from the fibre product $Y_- \times_B Y_+$ over the base $B$ of the flop [10]. Current work of Bondal et al. [12] will generalize this construction to interesting webs of flops, in particular flops of Springer resolutions, using a notion of perverse schobers on $\mathbb{C}^n$ singular along a real hyperplane arrangement. □
1.1.4 Monodromy

Spherical pairs induce equivalences $\mathcal{E}_- \leftrightarrow \mathcal{E}_+$, categorifications of the half-monodromy functions for a perverse sheaf of vector spaces. (In the example above, these half-monodromy functors recover the known derived equivalences between the quotients $X//\pm$.) By composing these, the spherical pair also induce symmetries of the categories $\mathcal{E}_\pm$, categorifications of monodromy actions. Different choices of cut give different descriptions of this monodromy. For instance, it turns out that for a skeleton $K'$ with only one arc, as in Figure 3, the categorification is a spherical functor, and that spherical pairs determine spherical functors which recover the symmetry of $\mathcal{E}_+$ above as a spherical twist.

1.1.5 Spherical functor

The spherical functor determined by our spherical pair is shown in Figure 3. The category $\mathcal{D}$ is given in Theorem A(2) in terms of sheaves on the unstable locus of one of the GIT quotients; the functor itself is described in Corollary 4.10. These spherical functors have appeared previously in work of Halpern-Leistner and Shipman [18]: our spherical pair construction provides a new viewpoint on their results.

Remark. A technical advantage of working with spherical pairs over spherical functors is that the spherical pair conditions may be easier to verify in practice: we need only show that certain functors are equivalences, without the need to take troublesome functorial cones.

Remark. Halpern-Leistner and Shipman [18, Theorem 3.15] show that a general spherical functor yields a spherical pair, by directly constructing an appropriate category $\mathcal{E}_0$. We do not follow this approach here because, as we shall see, there already exists a candidate for such a category in our examples.
1.2 Results

Consider a GIT wall crossing given by the data of a projective-over-affine variety $X$ with an action of a connected reductive group $G$ and a pair of linearizations $\mathcal{M}_\pm$ in the sense of Section 3.5. The unstable loci $X - X^{ss}(\mathcal{M}_\pm)$ then come with non-canonical GIT stratifications. Require that this wall crossing is ‘simple balanced’ (Definition 3.18): informally, this means that only a single stratum is affected by crossing the wall. Associated to this stratum is the data of:

- a Levi subgroup $L$ of $G$;
- a one-parameter subgroup $\lambda$ of $L$; and
- an open subset $Z$ of the $\lambda$-fixed locus in $X$.

The following theorem, which is the main result of this article, constructs a spherical pair for such a wall crossing.

**Theorem A.** Take a GIT wall crossing for a variety $X$ with a $G$-action, as above, which is ‘simple balanced’ as in Definition 3.18. Assume that:

i. the variety $X$ is smooth in a $G$-equivariant neighbourhood of $Z$;
ii. the group $G$ is abelian; and
iii. the canonical sheaf $\omega_X$ has $\lambda$-weight zero on $Z$.

Let $\mathcal{M}_0$ denote a linearization on the wall and $X^{ss}(\mathcal{M}_0)$ the associated semistable locus in $X$. Then the following hold.

1. *(Theorem 4.4)* For each integer $w$, there exists a subcategory $\mathcal{C}$ of $D(X^{ss}(\mathcal{M}_0)/G)$ with embeddings

$$t_- : D(Z/L)^w \longrightarrow \mathcal{C}$$
$$t_+ : D(Z/L)^{w+\eta} \longrightarrow \mathcal{C}$$

2. *(Corollary 4.10)* The spherical pair $\mathcal{P}$ induces a spherical functor

$$S : \mathcal{D} \longrightarrow D(X^+/),$$

where $\mathcal{D}$ denotes the category $D(Z/L)^w$. 
3. **(Corollary 4.11)** Furthermore, there exist natural embeddings

\[ \epsilon_{\pm}: D(X//\pm) \rightarrow C \]

which determine a spherical pair in a sense dual to that of [24]. □

**Remark.** In many case, the quotients \( X//\pm \) corresponding to a wall crossing are related by a flip. This follows for instance if \( G = \mathbb{C}^* \) and the unstable loci have codimension at least two [36, Proposition 1.6]. Assumption (iii) then asserts that this flip is in fact a flop. □

**Remark.** It may seem strange to use the notation \( D(Z/L) \) when \( G \) is abelian, as the Levi subgroup \( L \) must coincide with \( G \) in this case. The notation is taken from Halpern-Leistner and Shipman [18, Section 2.1]: we retain it for convenience because, firstly, they give useful alternative descriptions of the categories \( D(Z/L)^w \) and, secondly, we will have cause to slightly amend one of their lemmas (Lemma 3.15). □

We will see that taking complexified Grothendieck groups of a spherical pair \( \mathcal{P} \) gives a perverse sheaf of vector spaces on a disk. The following theorem shows that this is an intersection cohomology complex when its monodromy satisfies an appropriate condition.

**Theorem B.** With the setup above, the following hold.

1. **(Proposition 3.7)** There exists a perverse sheaf \( ^k\mathcal{P} \) naturally associated to \( \mathcal{P} \) with generic fibre the complexified Grothendieck group \( K(X//+) \); and
2. **(Theorem 5.2)** Writing \( m \) for the monodromy action on \( K(X//+) \), \( ^k\mathcal{P} \) is an intersection cohomology complex of a local system on the punctured disk if and only if

\[ \text{rk}(m - 1) = \dim K(D(Z/L)^w). \]  \( (\ast) \)

I show in Proposition 5.4 that a sufficient condition for \((\ast)\) to be satisfied is that the codimension of \( Z \) in \( X \) is odd, and both spaces are \( G \)-equivariantly Calabi–Yau. Nevertheless, \((\ast)\) fails in many simple examples, including an Atiyah flop of a resolved conifold (Section 2.3).
1.3 Categorified intersection cohomology

I view the construction of a spherical pair in Theorem A as an instance of taking “categorified intersection cohomology.” Namely, the equivalences $E_\rightarrow \leftrightarrow E_\leftarrow$ naturally define a local system of categories on the punctured disk, and the spherical pair $\mathcal{P}$ extends this to a perverse sheaf of categories on the whole disk, thus categorifying the way in which intersection cohomology extends local systems of vector spaces to perverse sheaves of vector spaces. Theorem B shows that, under assumption (**), decategorification works as expected. A construction of Segal [33], which takes a general autoequivalence $\Psi : \mathcal{E} \to \mathcal{E}$ and expresses it as a twist of a spherical functor, may likewise be viewed as an instance of categorified intersection cohomology.

1.4 Fukaya categories

I briefly mention recent work using perverse schoberstounderstandFukayacategories: it would be interesting to link the present article to some of these studies. Kapranov and Schechtman’s program is related to Kontsevich’s proposal to localize Fukaya categories along singular Lagrangian skeleta, building on work of Seidel [34]: see [26, 27], and later [28]. This proposal has been extensively pursued by Nadler [30]. Dyckerhoff et al. [15] are currently developing a theory of perverse schobers on general Riemann surfaces, following previous work by Dyckerhoff and Kapranov studying topological Fukaya categories of surfaces [14], by Kapranov and Schechtman on perverse sheaves of vector spaces on surfaces [25], and by Pascaleff and Sibilla for punctured surfaces [29]. Soibelman [35] has discussed perverse schobers in relation to wall-crossing in the stability space associated to the Fukaya category. Perverse schobers have appeared furthermore in work of Nadler where they are used, for instance, to understand the Landau–Ginzburg A-model on $\mathbb{C}^n$ with superpotential $z_1 \ldots z_n$ [31].

2 Simple cases

In this section, I outline my construction and results in some concrete examples, namely an orbifold flop of local $\mathbb{P}^1$, and an Atiyah flop of a resolved conifold.

2.1 Perverse sheaves

These may be thought of as generalizations of the sheaves of solutions to differential equations. As motivation therefore, take a complex number $c$ and consider the equation

$$z \frac{df}{dz} = cf,$$
where $f$ is a function of a variable $z$ taking values in the punctured complex disk $\Delta - 0$. It has a local solution $f(z) = z^c$, however for non-integer $c$ this solution has monodromy around 0, so global solutions are sections of a local system $L$. This local system may be described by its fibres $E_{\pm}$ at $\pm 1$, along with half-monodromy morphisms between them.

To describe a perverse sheaf $P$ extending $L$ which is defined on all of $\Delta$, though possibly singular at 0, we may proceed as follows. Take the sheaf of (hyper)cohomology with support $H^1_K(P)$ on a cut $K$ as in Figure 4, and write $E_{\pm}$ and $E_0$ for its stalks at $\pm 1$ and 0, respectively. There are natural maps $E_0 \to E_{\pm}$, and dual maps in the reverse direction, as shown. These maps satisfy conditions given in Proposition 3.3: a spherical pair is then given by a diagram of categories satisfying analogous conditions given in Definition 3.6.

2.2 Example: local $\mathbb{P}^1$

Take a three-dimensional vector space $X$ with coordinates $(x_1, x_2, y)$, and a $\mathbb{C}^*$-action with weights $(1, 1, -2)$, whose orbits are shown in Figure 5. There are two GIT quotients $X/\!/\pm$ of this action, namely:

(-) the total space $\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2))$ of a line bundle on $\mathbb{P}^1$; and
(+) an orbifold $\mathbb{C}^2/C_2$, where the cyclic group $C_2$ acts by $\pm 1$.

These quotients arise as open substacks of the quotient stack $X/\mathbb{C}^*$, so we may consider restriction functors as follows.

$$\text{res}_{\pm}: D(X/\mathbb{C}^*) \to D(X/\!/\pm)$$

It was observed by Segal [32], following ideas of Kawamata, and Herbst et al. [20], later developed into a general theory in [5, 17], that these are equivalences on certain “window”
subcategories

\[ W_k = \langle \mathcal{O}(k), \mathcal{O}(k + 1) \rangle \]

in \( D(X/\mathbb{C}^*) \), generated by weight line bundles \( \mathcal{O}(k) \) on \( X/\mathbb{C}^* \). The key observation for us is that by taking a slightly larger window, for instance

\[ \mathcal{C} = \langle \mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1) \rangle, \]

we may then produce a diagram of categories as above, by composing the obvious embeddings of the \( W_k \) with equivalences, as follows:

\[
\begin{align*}
D(X//-) &\cong W_{-1} & \mathcal{C} &\cong W_0 &\cong D(X//+).
\end{align*}
\]

Functors in the reverse direction are given by adjoints, sketched in Figure 6.

The orthogonal subcategories \( ^{\perp}W_{-1} \) and \( ^{\perp}W_0 \) in \( \mathcal{C} \) are generated by objects \( \mathcal{O}_y(-1) \) and \( \mathcal{O}_x(1) \) supported respectively on the \( y \)-axis, and the \( (x_1, x_2) \)-plane: they yield the right-hand diagram in Figure 6 which gives a spherical pair \( \mathcal{P} \) in the sense of Kapranov and Schechtman: the left-hand diagram satisfies dual axioms which are made precise in Corollary 4.11.

Taking complexified Grothendieck groups of the left-hand diagram in Figure 6, we obtain a perverse sheaf of vector spaces \( P = k\mathcal{P} \). This has a nice description as
follows. By composing arrows in the diagram, we obtain a flop–flop functor $\mathcal{F}$ acting on $\mathcal{D}(X//+)$, and thence an endomorphism of the complexified Grothendieck group $K(X//+)$ which determines a local system of vector spaces on the punctured disk $\Delta - 0$: the intersection cohomology complex of this local system recovers $P$. This result is explained and generalized in Section 5.

2.3 Example: Resolved conifold

Now I briefly give a three-fold example, where the relation to intersection cohomology is less straightforward. Take a four-dimensional vector space $X$ with coordinates $(x_1, x_2, y_1, y_2)$ and a $\mathbb{C}^*$-action with weights $(1, 1, -1, -1)$. The two GIT quotients are then:

(-) the resolved conifold $\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2})$; and
(+) its flop along the zero section $\mathbb{P}^1$.

We again obtain a spherical pair, similarly to above. The categories $\mathcal{W}_k$ and $C$ are defined in just the same way. The orthogonals are now generated by $\mathcal{O}_{\vec{y}}(-1)$ and $\mathcal{O}_{\vec{x}}(1)$ supported respectively on the $(y_1, y_2)$-plane and $(x_1, x_2)$-plane.

In this case, however, taking complexified Grothendieck groups does not give the intersection cohomology complex of a local system on the punctured disk $\Delta - 0$: the reason is that the flop–flop functor acts trivially at the level of Grothendieck groups. See Example 6.2 for details.

Remark 2.1. This example is the simplest case where Bodzenta–Bondal construct a spherical pair [10]: it would be interesting to relate our constructions.
Remark 2.2. The examples in this section have natural extensions to higher dimensions, namely to an orbifold flop of local $\mathbb{P}^{\text{odd}}$ (Example 6.1) and to standard flops (Example 6.2).

3 Background

3.1 Perverse sheaves

This subsection summarizes results on perverse sheaves of vector spaces which will be used later. We first recall a standard description of the category $\text{Per}(\Delta, 0)$ of perverse sheaves on a disk $\Delta \subset \mathbb{C}$ containing 0, and possibly singular there.

Proposition 3.1 ([6] [16, Théorème II.2.3]). There is an equivalence from $\text{Per}(\Delta, 0)$ to the category of diagrams of vector spaces

$$
\begin{array}{c}
D_0 \leftarrow u \\
\downarrow \\
D_1
\end{array}
$$

such that the morphism $m = vu + 1$ is an isomorphism.

The above equivalence may be obtained by letting $D_0$ be vanishing cycles, and $D_1$ be nearby cycles in such a way that the morphism $m$ is realized as a monodromy operator acting on $D_1$.

The following will be used later, in Section 5.

Proposition 3.2. Given a local system $L$ on $\Delta - 0$ with fibre $F$, and monodromy $m$, the intersection cohomology complex $\text{IC}(L)$ is given by a diagram

$$
\begin{array}{c}
F/F^m \\
\downarrow \\
F
\end{array}
$$

where $F^m$ denotes the $m$-invariants, and $u$ is the quotient map.

Proof. This follows from a well-known computation of nearby and vanishing cycles, and the canonical maps between them.

We now give an alternative description of $\text{Per}(\Delta, 0)$, due to Kapranov–Schechtman: it is a special case of their remarkable characterization of perverse sheaves on a complex vector space singular with respect to a real hyperplane arrangement [23]. Their categorification of this description is given in Section 3.2.
Proposition 3.3 ([23, Section 9]). There is an equivalence from \( \text{Per}(\Delta, 0) \) to the category of diagrams of vector spaces

\[
\begin{array}{ccccc}
E_- & \xrightarrow{f_-} & E_0 & \xleftarrow{f_+} & E_+ \\
g_- & & g_+ & & 
\end{array}
\]

such that

1. \( g_\pm f_\pm = 1 \) and
2. \( g_\pm f_\mp \) are isomorphisms.

Monodromy actions on \( E_- \) and \( E_+ \) may be taken to be

\[
m_- = g_- f_+ g_- f_- \quad \text{and} \quad m_+ = g_+ f_- g_+ f_+.
\]

For \( P \) in \( \text{Per}(\Delta, 0) \) the spaces \( E_\pm \) and \( E_0 \) above are given by stalks at \( \pm 1 \) and 0 of the sheaf \( H^1_k(P) \) of cohomology with support on a skeleton \( K \) as discussed in Section 2.1. The maps \( g_\pm \) are generalization maps, as described for instance in [23, Section 1D], and the \( f_\pm \) are dual to them.

Proposition 3.4 ([23, Proposition 9.4]). There exist equivalences between the descriptions of \( \text{Per}(\Delta, 0) \) from Propositions 3.1 and 3.3, by taking diagrams as follows.

\[
\begin{array}{cccc}
D_1 & \xrightarrow{\begin{pmatrix} (v) \\ (0 1) \end{pmatrix}} & D_0 \oplus D_1 & \xleftarrow{\begin{pmatrix} (0) \\ (1 v) \end{pmatrix}} & D_1 \\
\ker g_- & \xleftarrow{f_- g_- - 1} & \text{Im } f_+ & & 
\end{array}
\]

Remark 3.5. By Proposition 3.4, \( E_0 \) is isomorphic to the direct sum of the nearby and vanishing cycles of the perverse sheaf, however this isomorphism is not canonical: see [23, Proof of Proposition 9.4].

3.2 Spherical pairs

This subsection gives categorifications of perverse sheaves of vector spaces on a disk, namely spherical pairs and functors, following Kapranov–Schechtman.

Consider a \( \mathbb{C} \)-linear triangulated category \( \mathcal{E} \) with admissible subcategories \( \mathcal{E}_\pm \) embedded by \( \delta_\pm: \mathcal{E}_\pm \rightarrow \mathcal{E} \). By definition, the functors \( \delta_\pm \) have left and right adjoints,
denoted by $\delta_\pm$ and $\delta_\pm^*$ respectively. Considering the right orthogonals $\mathcal{E}_\pm^\perp$ we also have embeddings $\gamma_\pm : \mathcal{E}_\pm^\perp \to \mathcal{E}$ with left adjoints $\gamma_\pm^*$.

**Definition 3.6** ([24, Definition 3.5]). The data $(\mathcal{E}_\pm, \mathcal{E})$, along with the embeddings above, gives a *spherical pair* if the following are equivalences:

a. $\delta_+^* \circ \delta_- : \mathcal{E}_- \to \mathcal{E}_+$

b. $\delta_-^* \circ \delta_+ : \mathcal{E}_+ \to \mathcal{E}_-

c. $\gamma_+^* \circ \gamma_- : \mathcal{E}_-^\perp \to \mathcal{E}_+^\perp$

d. $\gamma_-^* \circ \gamma_+ : \mathcal{E}_+^\perp \to \mathcal{E}_-^\perp$

In other words, a spherical pair is given by semi-orthogonal decompositions

$$\langle \mathcal{E}_-^\perp, \mathcal{E}_- \rangle = \mathcal{E} = \langle \mathcal{E}_+^\perp, \mathcal{E}_+ \rangle,$$

where $\mathcal{E}_\pm$ are admissible subcategories of $\mathcal{E}$, and the natural compositions (a)–(d) of embedding and projection functors are equivalences. Kapranov and Schechtman [24, Section 3B] write decompositions in the reverse order.

The following is implicit in Kapranov and Schechtman [24]: applying the complexified Grothendieck group construction to a spherical pair yields two perverse sheaves of vector spaces on a disk depending on whether the left- or right-hand components of (1) are taken. Write $K_0(\mathcal{E})$ for the Grothendieck group of a triangulated category $\mathcal{E}$, and $K(\mathcal{E})$ for the $\mathbb{C}$-algebra $K_0(\mathcal{E}) \otimes \mathbb{C}$.

**Proposition 3.7.** Given a spherical pair $\mathcal{P} = (\mathcal{E}_\pm, \mathcal{E})$ as above, two perverse sheaves $\mathcal{P}^\mathbb{C}$ and $\mathcal{P}^K$ on a disk $\Delta$ may be obtained, given as follows in the notation of Proposition 3.3.

$$K(\mathcal{E}_-) \xleftrightarrow{\delta_-} K(\mathcal{E}) \xleftrightarrow{\delta_+} K(\mathcal{E}_+)$$

$$K(\mathcal{E}_-^\perp) \xleftrightarrow{\gamma_-} K(\mathcal{E}) \xleftrightarrow{\gamma_+} K(\mathcal{E}_+^\perp)$$

Here the symbols $\delta$ and $\gamma$ for functors are reused for their images under $K(\mathcal{E})$.

**Proof.** First note that $\delta_\pm^* \circ \delta_\pm = \text{Id}_{\mathcal{E}_\pm}$, because $\delta_\pm$ are embeddings: see for instance [21, Corollary 1.22]. The conditions (1) and (2) of Proposition 3.3 for $\mathcal{P}^\mathbb{C}$ may then be verified immediately. The claim for $\mathcal{P}^K$ follows by a dual argument. \[\square\]
Spherical pairs yield spherical functors, as follows. Here we implicitly take enhanced triangulated categories, so that the functorial cones below make sense: see for instance [24, Appendix A].

**Proposition 3.8.** ([24, Propositions 3.7, 3.8]). Given a spherical pair \((E_\pm, E)\) as above, the functor

\[
S = \gamma_+ \circ \delta_- : E_\pm \to E_\perp
\]

is spherical in the sense of Anno [2] and Anno and Logvinenko [3], and we have that:

1. the twist acts on \(E_\perp\) by
   \[
   T_S := \text{Cone} \left( S \circ S^* \xrightarrow{\text{counit}} \text{Id} \right) \cong \gamma_+ \circ \gamma_- \circ \gamma_- \circ \gamma_+
   \]
   which is the composition (d) then (c) from Definition 3.6;
2. the cotwist acts on \(E_-\) by
   \[
   C_S := \text{Cone} \left( \text{Id} \xrightarrow{\text{unit}} S^* \circ S \right)[-1] \cong \delta_+ \circ \delta_- \circ \delta_+ \circ \delta_-
   \]
   which is the composition (a) then (b) from Definition 3.6. \(\square\)

Combining, we also have the following.

**Proposition 3.9.** Given a spherical pair \((E_\pm, E)\) as above, with associated spherical functor \(S\), we have that:

1. the action of \(T_S\) on \(K(E_\perp)\) is the monodromy around 0 of \(K_P\);
2. the action of \(C_S\) on \(K(E_-)\) is the monodromy around 0 of \(P^K\). \(\square\)

**Proof.** This follows from the definitions of \(P^K\) and \(K_P\) from Proposition 3.7, their monodromies given in Proposition 3.3, and the descriptions of the twist and cotwist in Proposition 3.8. \(\blacksquare\)

### 3.3 Results from GIT

This subsection tersely recalls results from GIT which will be used to study derived categories of GIT quotients. I follow treatments of Halpern-Leistner [17] and Ballard *et al.* [5].
**Setup 3.10.** Take a projective-over-affine variety $X$ with an action of a connected reductive group $G$, and a $G$-ample equivariant line bundle $\mathcal{M}$. □

The above data determines a semistable locus $X^{\text{ss}}(\mathcal{M}) \subseteq X$: we take the GIT quotient to be the quotient stack $X^{\text{ss}}(\mathcal{M})/G$. This may be a Deligne–Mumford stack: see Section 2 and Example 6.1 for simple examples. After certain choices, we obtain a stratification of $X - X^{\text{ss}}(\mathcal{M})$ by strata $S^i$. I now give the properties of this stratification needed in what follows, leaving further details to the references.

To each stratum $S^i$ is associated, by construction, a one-parameter subgroup $\lambda^i$, and the stratum contains an associated open subvariety of the $\lambda^i$-fixed locus, denoted by $Z^i$. Consider a single stratum $S$, dropping indices $i$ for simplicity. It contains furthermore a subvariety $Y \subseteq X$ flowing to $Z$ under $\lambda$, referred to as a “blade”, which will be important: writing the elements of $\lambda$ as $\lambda_t$ for $t \in \mathbb{C}^*$, this blade is given as follows.

$$Y = \left\{ x \mid \lim_{t \to 0} (\lambda_t \cdot x) \text{ lies in } Z \right\} \subseteq X.$$  

Note that $Z \subseteq Y$, and that $Y$ has a natural projection to $Z$. Notate morphism as below.

$$\xymatrix{ Z \ar[r]^\sigma \ar@{<-}[d]_\pi & Y \ar[r]^-j \ar@{<-}[d]_\pi & S \ar@{<-}[d]_\pi \ar[r]^j & X. }$$

These morphisms may be made equivariant, in the following manner. Write $L$ for the subgroup of elements of $G$ which commute with $\lambda_t$ for all $t$, and put

$$P = \left\{ g \mid \lim_{t \to 0} (\lambda_t \cdot g \cdot \lambda_t^{-1}) \text{ lies in } L \right\} \subseteq G.$$  

Observe that $L \subseteq P$, and that there is a map $P \to L$ given by

$$g \mapsto \lim_{t \to 0} (\lambda_t \cdot g \cdot \lambda_t^{-1}).$$

There are then morphisms of quotient stacks as follows, reusing the notation above.

$$\xymatrix{ Z/L \ar[r]^\sigma \ar@{<-}[d]_\pi & Y/P \ar[r]^-\sim & S/G \ar@{<-}[d]_\pi \ar[r]^j & X/G. }$$

The middle morphism is induced by the inclusion of $Y$ into $S$, and is an equivalence: see for instance [17, Section 2.1, Property (S2)].

**Remark 3.11.** If $G$ is abelian, the blade $Y$ coincides with the stratum $S$, and the subgroups $L$ and $P$ are equal to $G$. □
The following properties of the stratification will be used later.

**Proposition 3.12.** If \( X \) is smooth in a \( G \)-equivariant neighbourhood of \( Z \):

1. the sheaf \( \mathcal{N}_Y X \) restricted to \( Z \) has strictly positive \( \lambda \)-weights;
2. the variety \( X \) is smooth in a \( G \)-equivariant neighbourhood of \( Y \);
3. the subvarieties \( Y \) and \( Z \) are smooth;
4. the morphism \( \pi : Y \to Z \) is a locally trivial bundle of affine spaces;
5. the stratum \( S \) is regularly embedded in \( X \).

**Proof.** The definition of the blade \( Y \) yields (1). For the rest, we follow the approach of [17, Lemma 2.7]. To see (2), note that any \( G \)-invariant neighbourhood of \( Z \) contains \( Y \). Then (3) and (4) follow from the Białynicki-Birula decomposition [9, Theorem 4.1] applied to the action of \( \lambda \) on this neighbourhood, and (5) is given by an argument of Kirwan [22, Theorem 13.5].

3.4 Derived categories

We follow Halpern-Leistner [17], Halpern-Leistner and Shipman [18], and Ballard et al. [5]. Assume for simplicity that the stratification of the previous subsection consists of a single stratum with associated subvariety \( Z \), and that \( X \) is a smooth in a \( G \)-equivariant neighbourhood of \( Z \). We define windows in \( \mathcal{D}(X/G) \) which will be equivalent to the derived category \( \mathcal{D}(X^\text{ss}/G) \) of the GIT quotient, by measuring weights on \( Z \).

As notation, for \( F^\bullet \in \mathcal{D}(X/G) \), write \( \text{wt}_\lambda F^\bullet \) for the set of \( \lambda \)-weights which appear in some cohomology sheaf \( \mathcal{H}^k(\sigma^*j^*F^\bullet) \) on \( Z \). Then given an integer \( w \), a window \( \mathcal{G}^w \) is defined as the full subcategory of \( \mathcal{D}(X/G) \) with objects

\[
\mathcal{G}^w = \{ F^\bullet \in \mathcal{D}(X/G) \mid \text{wt}_\lambda F^\bullet \subseteq [w, w + \eta] \},
\]

where \( \eta \) is defined as follows.

**Definition 3.13.** The **window width** \( \eta \) is the \( \lambda \)-weight of \( \det \mathcal{N}_S^\vee X \) on \( Z \).

The category \( \mathcal{G}^w \) is equivalent to \( \mathcal{D}(X^\text{ss}/G) \): indeed we have the following more refined result.
**Theorem 3.14** ([5, 17]). Define a full subcategory

\[ C^w = \{ F^* \in D(X/G) \mid \text{wt}_i F^* \subseteq [w, w + \eta] \} \]

of \( D(X/G) \) and consider the restriction functor to the GIT quotient

\[ \text{res}: C^w \subset D(X/G) \longrightarrow D(X^{ss}/G). \]

This is an equivalence on the subcategories \( G^w \) and \( G^{w+1} \), and has adjoints

\[ \text{res}^*: D(X^{ss}/G) \longrightarrow G^w \subset C^w \quad \text{and} \quad \text{res}^*: D(X^{ss}/G) \longrightarrow G^{w+1} \subset C^w. \]

**Proof.** See [18, Lemma 2.4] for this statement.

The adjoint \( \text{res}^* \) is an embedding \( D(X^{ss}/G) \rightarrow C^w \). The following lemma and proposition give furthermore a semi-orthogonal decomposition of \( C^w \). Write \( D(Z/L)^w \) for the full subcategory of \( D(Z/L) \) whose objects have \( \lambda \)-weight \( w \).

**Lemma 3.15.** ([18]). For a given \( \lambda \)-weight \( w \), the functor

\[ \iota = j_* \pi^*(-): D(Z/L)^w \longrightarrow C^w \]

is an embedding, with left and right adjoints

\[ \iota = (\sigma^*j^*(-))^{w+\eta} \otimes \det N_{S/X|Z}[- \text{codim}_X S], \]

where superscripts denote projection to weight subcategories of \( D(Z/L) \).

**Proof.** This is [18, Lemma 2.3] with a shift \( [- \text{codim}_X S] \) added in the statement coming from Grothendieck duality for the morphism \( j \).

**Proposition 3.16.** ([5, 17]). There is a semi-orthogonal decomposition

\[ C^w = \langle D(X^{ss}/G), D(Z/L)^w \rangle \]

with embedding functors \( \text{res}^* \) and \( \iota \).

**Proof.** This follows from [5, 17] and may be found in this form in [18, Section 2, (3)].
3.5 Variation of GIT

This subsection sets up the necessary technology to study GIT wall crossings. Consider a GIT problem $X/G$ as in Setup 3.10. The results of the previous sections continue to hold if we replace the $G$-ample equivariant line bundle $\mathcal{M}$ used there with an element of $\text{NS}^G(X) \otimes \mathbb{R}$, namely the scalar extension to $\mathbb{R}$ of the equivariant Neron–Severi group of $X$. Elements of this group will henceforth be referred to as linearizations, so that we may consider continuous variation of linearization: see [5, Section 3.2] for a fuller treatment.

Take then linearizations $\mathcal{M}_0$ and $\mathcal{M}'$, and let

$$\mathcal{M}_t = \mathcal{M}_0 + t\mathcal{M}'$$

for each $t \in [-\epsilon, \epsilon]$ with $\epsilon$ small such that:

1. there exist strict semistables for $\mathcal{M}_0$;
2. $X^{ss}(\mathcal{M}_t)$ is constant for $t \in [-\epsilon, 0)$, with no strict semistables;
3. $X^{ss}(\mathcal{M}_t)$ is constant for $t \in (0, \epsilon]$, with no strict semistables.

In this situation, write $\mathcal{M}_\pm = \mathcal{M}_{\pm \epsilon}$, and say that the transition between $\mathcal{M}_\pm$ is a wall crossing, and $\mathcal{M}_0$ is a linearization on the wall. For brevity, we denote the GIT quotients corresponding to either side of the wall as follows.

$$X/\setminus G = X^{ss}(\mathcal{M}_\pm)/G.$$ 

The semistable locus for $\mathcal{M}_0$ may be related to the loci for $\mathcal{M}_\pm$ by the formulae

$$X^{ss}(\mathcal{M}_0) = X^{ss}(\mathcal{M}_\pm) \cup \bigcup_{i \in I_\pm} S^i_\pm,$$

where the $I_\pm$ are indexing sets for unstable strata under the respective linearizations $\mathcal{M}_\pm$ [13, Section 4]. We now recall a notion of Halpern-Leistner [17, Definition 4.4]: a similar situation is considered by Ballard et al. in [5, Condition 4.3.1].

**Definition 3.17.** A wall crossing is balanced if there is an identification $I_- = I_+$ under which one-parameter subgroups $\lambda_{\pm}$, associated $L_\pm$, and subvarieties $Z_\pm$ correspond as follows:

$$\lambda_+^i = (\lambda_-^i)^{-1}, \quad L_+^i = L_-^i, \quad \text{and} \quad Z_+^i / L_+^i = Z_-^i / L_-^i.$$

I make the following further restriction, for use in what follows.
**Definition 3.18.** A wall crossing is *simple balanced* if it is balanced, and furthermore $|I_-| = |I_+| = 1$. □

In this case, we drop indices, and write simply $Z$ and $L$ for the identified subvarieties $Z_\pm$, and subgroups $L_\pm$.

### 4 Spherical pairs

Assume given a simple balanced wall crossing as in Definition 3.18 for a GIT problem $X/G$. In this section, we construct a spherical pair in the sense of Definition 3.6, and describe the equivalences (a)–(d) which appear there. Let the wall crossing be between linearizations $\mathcal{M}_\pm$, with a linearization $\mathcal{M}_0$ on the wall, and an associated subvariety $Z$. We prove our theorem under the following assumption.

**Assumption 4.1.** Require that:

i. the variety $X$ is smooth in a $G$-equivariant neighbourhood of $Z$;
ii. the group $G$ is abelian; and
iii. the canonical sheaf $\omega_X$ has $\lambda_\pm$-weight zero on $Z$. □

**Remark 4.2.** Assumption (iii) is certainly satisfied, for instance, if $X$ is $G$-equivariantly Calabi–Yau. □

Recalling that the window width $\eta_\pm$ is defined to be the $\lambda_\pm$-weight of $\det \mathcal{N}_Y^\vee X$ on $Z$, we prove the following.

**Lemma 4.3.** For a simple balanced wall crossing satisfying Assumption 4.1 it follows that $\eta_+ = \eta_-$. □

**Proof.** Assumption (i) implies that $X$ is smooth in a neighbourhood of $Y_\pm$, and that $Y_\pm$ and $Z$ are smooth, using Propositions 3.12(2 and 3). The argument then proceeds as follows, as in [17, proof of Proposition 4.5]. Note that $\Omega_X$ on $Z$ is split by sign of $\lambda_\pm$-weights as

$$\mathcal{N}_Y^\vee X|_Z \oplus \Omega_Z \oplus \mathcal{N}_Y^\vee X|_Z. \quad (2)$$

An example is sketched in Figure 7. By Assumption (ii) that $G$ is abelian, the $Y_\pm$ are identified with the strata $S_\pm$. The claim is then shown by taking determinants and using Assumption (iii). ■
The following is the main theorem of the article, constructing and describing a spherical pair. Write $\eta$ for the common value of $\eta_{\pm}$, and recall that $X//\pm G$ are the GIT quotients corresponding to either side of the wall.

**Theorem 4.4.** Take a simple balanced wall crossing as in Definition 3.18 for a GIT problem $X/G$ satisfying Assumption 4.1. Fix an integer $w$, and consider the category

$$
C = \left\{ F^\bullet \in D(X^{ss}(\mathcal{M}_0)/G) \mid \text{wt}_{\lambda_-} F^\bullet \subseteq [w, w + \eta] \right\}.
$$

Then there exist embeddings

$$
\iota_- : D(Z/L)^w \longrightarrow C \\
\iota_+ : D(Z/L)^{-w-\eta} \longrightarrow C
$$

where the superscripts indicate $\lambda_-$-weight and $\lambda_+$-weight subcategories respectively, such that

1. the orthogonals $(\text{Im} \iota_{\pm})^\perp$ are equivalent to $D(X//\pm G)$ via the restriction functors $\text{res}_{\pm}$ and
2. the data $(\text{Im} \iota_{\pm}, C)$ give a spherical pair. $\square$

**Proof.** We first establish the claimed embeddings. Let $X^c = X^{ss}(\mathcal{M}_0)$ be the semistable locus for the linearization $\mathcal{M}_0$. We may replace $X/G$ with its open substack $X^c/G$, noting that Assumption 4.1 continues to hold. Because the wall crossing is simple balanced, the GIT stratifications associated to the linearizations $\mathcal{M}_{\pm}$ then consist of single unstable strata $S_{\pm}$, so that we are in the setting of Section 3.4. The equalities

$$
C = C^{w}_- = C^{w-\eta}_+
$$

are then immediate from the definitions in Theorem 3.14, using the fact that $\lambda_+ = \lambda_-^{-1}$. Hence the claimed embeddings $\iota_{\pm}$ may be defined using Lemma 3.15.
By the semi-orthogonal decomposition of Proposition 3.16, the orthogonals $(\text{Im} \, \iota_{\pm})^\perp$ are the images of the following embeddings from Theorem 3.14: claim (1) follows immediately.

\[
\text{res}^*_+ : D(X \sslash G) \xrightarrow{\sim} \mathcal{G}_w^- \subset \mathcal{C} \\
\text{res}^*_- : D(X \sslash G) \xrightarrow{\sim} \mathcal{G}^{-w-\eta}_+ \subset \mathcal{C}
\]

For $(\text{Im} \, \iota_{\pm}, \mathcal{C})$ to give a spherical pair, we require that $\iota_{\pm}$ have left and right adjoints: these are provided by Lemma 3.15. We require also that the embeddings $\text{res}^*_\pm$ have left adjoints: these are given by $\text{res}_\pm$. To conclude (2), it remains to show that the functors appearing in Definition 3.6 are equivalences: this is shown in the following Proposition 4.5.

The following proposition establishes and describes the equivalences associated to the spherical pair in Theorem 4.4.

**Proposition 4.5.** In the setting of Theorem 4.4, functors (a)–(d) of Definition 3.6 are equivalences given by

a. $- \otimes \det \mathcal{N}_{S+} X |_Z [\cdot - \text{codim}_X S_+] : D(Z/L)^w \rightarrow D(Z/L)^{w-\eta}_+$

b. $- \otimes \det \mathcal{N}_{S-} X |_Z [\cdot - \text{codim}_X S_-] : D(Z/L)^{-w-\eta}_+ \rightarrow D(Z/L)^w_-$

c. $\text{res}_+ \circ \text{res}^*_\pm : D(X \sslash G) \rightarrow D(X \sslash G)$

d. $\text{res}_- \circ \text{res}^*_\pm : D(X \sslash G) \rightarrow D(X \sslash G)$

with (c) and (d) factoring via window subcategories $\mathcal{G}_w^- \text{ and } \mathcal{G}^{-w-\eta}_+ \text{ of } D(X_{ss}(\mathcal{M}_0)/G)$.

**Remark 4.6.** The equivalences (c) and (d) are those obtained by Halpern-Leistner [17, Section 4.1] and Ballard et al. [5].

**Proof of Proposition 4.5.** Consider first the compositions (c) and (d). The functors $\text{res}^*_\pm$ go to

\[\mathcal{G}_w^- = \mathcal{G}_w^{-w-\eta+1} \quad \text{and} \quad \mathcal{G}^{-w-\eta}_+ = \mathcal{G}_w^{w+1}\]

so the compositions are equivalences because $\text{res}_+$ and $\text{res}_-$ restrict to equivalences on the latter subcategories, by Theorem 3.14.

Next consider the compositions (a) and (b) given as follows.
These expressions may be expanded using Lemma 3.15. Recall that morphisms are denoted as follows.

Then composition (b) is given by:

- applying \((\sigma^* j^* j_+ \pi^*_+ (-))^w \eta\) where the superscript denotes restriction to the \(\lambda_-\)-weight \(w + \eta\) subcategory of \(D(Z/L)\); then
- tensoring by \(\det N_{S_- X}[\codim X S_-]\).

To show that (b) is an equivalence with the form claimed, it therefore suffices to prove Lemma 4.7 below. The result for (a) follows similarly, and we are done.

Lemma 4.7. The following functors appearing in the proof of Proposition 4.5 are isomorphic to the identity, after identifying \(D(Z/L)_w\) and \(D(Z/L)_+\).

\[
\begin{align*}
\text{a. } & (\sigma^* j^* j_+ \pi^*_+ (-))^{-w} : D(Z/L)^w \to D(Z/L)^{-w} \\
\text{b. } & (\sigma^* j^* j_+ \pi^*_+ (-))^{w+\eta} : D(Z/L)^{-w-\eta} \to D(Z/L)^{w+\eta}
\end{align*}
\]

Here superscripts are used as before to denote restriction to \(\lambda_-\)-weight and \(\lambda_+\)-weight subcategories, and morphisms are as in (3) above.

Remark 4.8. I give a proof of this lemma using that the blades \(Y_{\pm}\) intersect transversally in \(Z\) for \(G\) abelian. For \(G\) non-abelian, the equality \(\eta_+ = \eta_-\) continues to hold because of a formula [17, (4)] of Halpern-Leistner, but I could not find a way to prove the lemma, or a counterexample: this matter did not seem worth pursuing further, as non-abelian examples may, at least in principle, be reduced to abelian ones by a trick of Thaddeus [36, Section 3.1].

Proof of Lemma 4.7. Recall that, for \(G\) abelian, the strata \(S_{\pm}\) are identified with the blades \(Y_{\pm}\), and \(L = P_{\pm} = G\), so that (3) simplifies to the following diagram of
I claim that $G$-equivariant base change around this Cartesian square holds, namely an isomorphism between equivariant derived functors

$$f^* j_{+*} \cong \sigma_- \sigma_+^*.$$  \hspace{1cm} (5)

Consider the non-equivariant base change first. By the splitting (2) of $\Omega_X$ on $Z$, we know that $Z$ is a transverse intersection of the $Y_\pm$, in particular it has the expected dimension. Furthermore, $X$ is smooth in a neighbourhood of $Y_\pm$, and $Y_\pm$ and $Z$ are smooth: these facts follow from Assumption 4.1(i) that $X$ is smooth in a $G$-equivariant neighbourhood of $Z$, by Propositions 3.12(2 and 3). The base change then follows from Lemma 4.9.

We claim furthermore a base change isomorphism (5) between equivariant derived functors. For this, we may use the framework of Bernstein and Lunts [8]. I give the key points of the argument. Recall that to specify an object $F$ of the equivariant derived category $D(X/G)$ [8, Section 2.4.5] we proceed as follows.

- Consider smooth resolutions $\phi: X' \to X$ where, by definition, $X'$ is a free $G$-space so that the quotient map $X' \to X'/G$ is a locally trivial fibration with fibre $G$ [8, Definition 2.1.1(b)], with $\phi$ a $G$-equivariant smooth map [8, Section 1.7].
- Take an object $F_\phi$ of the ordinary derived category of the free quotient $X'/G$ for each such resolution.

Now pulling back (4) along $\phi$ gives a $G$-equivariant square of resolutions, and we may then take quotients by free $G$-actions to get a Cartesian square as follows.

$$\sigma_+ \quad \sigma_- \quad \phi$$

\hspace{1cm} (6)
To prove equivariant base change (5) following the method of [8, Sections 3.3 and 3.4] we need to show, in particular, base change for the square (6). For this, we argue as in the non-equivariant case, using the following observations. The square (6) is an intersection of the expected dimension because: the resolution maps \( X' \to X \) and so on) have locally constant, and equal, fibre dimensions; and the quotient maps \( X' \to X'/G \) and so on) have constant fibre dimension \( \dim(G) \). The smoothness required follows from the smoothness shown in the non-equivariant setting, using that the quotient maps are locally trivial fibrations, and noting furthermore that the neighbourhood of \( Y_\pm \) may be taken to be \( G \)-equivariant.

Taking the functor (b) and applying base change (5), we are led to consider

\[
\sigma^* \sigma_* \pi^* + \pi^*
\]

and because \( \pi_+ \sigma_+ = 1d_Z \), this further reduces to

\[
\sigma^* \sigma_-
\]  

(7)

The normal bundle sequence associated to the inclusion \( \sigma_- \) is split because \( \pi_- \sigma_- = 1d_Z \): it follows, as in Arinkin and Căldăraru [4], that (7) is isomorphic to tensoring by the direct sum of

\[
\bigwedge^k N^\vee_{Z Y_-}[k]
\]

for integers \( k \in [0, \text{codim}_{Y_-} Z] \). The functor (b) therefore acts by tensoring with the \( \lambda_- \)-weight 0 subobject of this. The \( Y_\pm \) intersect transversally in \( Z \), so we have \( N^\vee_{Z Y_-} \cong N^\vee_{Y_\pm} X|_Z \). Proposition 3.12(1) gives that the \( \lambda_+ \)-weights of the latter are strictly positive, thence the \( \lambda_- \)-weights strictly negative, so this subobject is just

\[
\bigwedge^0 N^\vee_{Z Y_-} = O_Z,
\]

and therefore (b) is the identity as claimed. The proof for (a) is similar.

The following standard lemma is used to prove Lemma 4.7 above.

**Lemma 4.9.** Let a variety \( X \) be smooth in a neighbourhood of smooth subvarieties \( Y_\pm \) whose intersection \( Z \) has components of expected codimension \( \text{codim} Y_- + \text{codim} Y_+ \).
With notation as follows,

\[
\begin{array}{ccc}
Y_- & \xrightarrow{j_-} & X \\
Z & \searrow & \swarrow \\
& k_- & \ \\
& \nearrow & \ \\
Y_+ & \xleftarrow{j_+} & X
\end{array}
\]

there is an isomorphism

\[
j^* j_{+*} \cong k_{-*} k_{+*}.
\]

\[\square\]

**Proof.** This follows by the argument of [1, Appendix A] for instance, which gives the result in a more general setting where \(j_+\) is proper, \(j_-\) is arbitrary, and \(Y_\pm\) only required to be Cohen–Macaulay. The statement there requires \(X\) to be smooth globally, however the argument we need goes through using only smoothness in a neighbourhood of \(Y_\pm\). \[\blacksquare\]

Combining Theorem 4.4 with Proposition 3.8, we immediately have the following, recovering a result of Halpern-Leistner and Shipman [18, Section 3.2].

**Corollary 4.10.** In the setting of Theorem 4.4, there is a spherical functor

\[
S = \text{res}_+ \circ \iota_- : D \rightarrow D(X_{/\pm G})
\]

where \(D\) denotes the category \(D(Z/L)^\text{w}\). \[\square\]

Finally, note that there is a dual notion of spherical pair, given by swapping the roles of the functors in the definition: therefore, as an immediate corollary of the proof of Theorem 4.4, we have the following.

**Corollary 4.11.** In the setting of Theorem 4.4, the data \((D(X_{/\pm G}), C)\) with embeddings

\[
\text{res}_+^*: D(X_{/\pm G}) \rightarrow C
\]

gives a dual spherical pair, in the sense that it satisfies Definition 3.6 with the roles of the functors \(\delta\) and \(\gamma\) exchanged. \[\square\]
5 Intersection cohomology

Assume given a spherical pair $\mathcal{P}$ from Theorem 4.4. This section studies the perverse sheaves of vector spaces $\mathcal{P}^k$ and $^k\mathcal{P}$ obtained by taking complexified Grothendieck groups. Theorem 5.2 gives a necessary and sufficient condition for $^k\mathcal{P}$ to be an intersection cohomology complex of a local system on the punctured disk. Simple sufficient conditions are then given in Proposition 5.4. A dual result for $\mathcal{P}^k$ is established in Theorem 5.5.

We give the following preparatory lemma.

**Lemma 5.1.** For $P \in \text{Per}(\Delta, 0)$ given by a diagram

\[
\begin{array}{c}
E_- & \xrightarrow{f_-} & E_0 & \xleftarrow{f_+} & E_+ \\
\downarrow{g_-} & & \downarrow{g_+} & & \downarrow{f_+} \\
E_0 & \xleftarrow{f_-} & E_- & \xrightarrow{f_+} & E_+
\end{array}
\]

under the description from Proposition 3.3, there is an isomorphism

$$P \cong \text{IC}(M)$$

for $M$ a local system on $\Delta - 0$ if and only if

$$\dim \ker g_- = \text{rk}(m - 1), \quad (8)$$

where $m$ is the monodromy around 0. □

**Proof.** First note that the restriction of $\text{IC}(M)$ to $\Delta - 0$ is $M$, and so $M$ is necessarily isomorphic to the local system on $\Delta - 0$ obtained by restricting $P$. Under the description from Proposition 3.1, $P$ is given by

$$\ker g_- \xleftarrow{f_- g_- - I} \text{Im} f_+ \xrightarrow{f_+ g_+} \text{Im} f_+$$

using Proposition 3.4, and $\text{IC}(M)$ is given by

$$E_+ / E'^m_+ \xleftarrow{u} E_+ \xrightarrow{m-1} E_+$$

using Proposition 3.2, where $u$ is the quotient map. We seek a condition for these two objects to be isomorphic.
We have that \( \dim \frac{E_+}{E_+^m} = \text{rk}(m - 1) \), so the dimension assumption (8) is necessary. For sufficiency, observe that \( g_+ (\ker g_-) \) contains the image of \( m - 1 \) because

\[
m - 1 = g_+ f_- g_+ f_+ - g_+ f_+ = g_+ (f_- (g_- - 1) f_+),
\]
and \( g_+ f_- = 1 \), so that there is a commutative diagram as follows.

\[
\begin{array}{ccc}
\ker g_- & \rightarrow & \text{Im } f_+ \\
\downarrow \quad & & \quad \downarrow \quad f_+ \\
g_+ & \rightarrow & ? \\
\downarrow \quad & & \quad \downarrow \quad g_+ (\ker g_-) \quad \rightarrow \quad E_+ \\
\downarrow \quad m - 1 \\
E_+ / E_+^m & \leftrightarrow & E_+
\end{array}
\]

If the dimension assumption (8) holds, the left-hand morphisms are isomorphisms, so the rows give isomorphic objects in \( \text{Per}(\Delta, 0) \), and hence \( P \cong \text{IC}(M) \) as required. ■

The following is the main theorem of this section.

**Theorem 5.2.** In the setting of Theorem 4.4, consider the perverse sheaf \( K_P \) provided by Proposition 3.7, with description

\[
K(X//G) \xleftrightarrow{\text{res}_-} K(C) \xleftrightarrow{\text{res}_+} K(X//_+G),
\]

where unmarked arrows are given by right adjoints. Then:

1. the monodromy \( m \) around 0 on \( K(X//_+G) \) is given by the action of the twist \( T_S \) of a spherical functor

\[
S = \text{res}_+ \circ \iota_+: D \rightarrow D(X//_+G),
\]

where \( D \) denotes the category \( D(Z/L)^w \);

2. we have

\[
\text{rk}(m - 1) \leq \dim K(D);
\]
3. there is an isomorphism

$$K \mathcal{P} \cong IC(M)$$

for $M$ a local system on $\Delta - 0$ if and only if the inequality (10) is saturated. □

**Proof.** (1) This is an application of Proposition 3.9(1).

(2) Combining (1) with the definition of the twist in Proposition 3.8(1), it follows that

$$m = K(T_S) = 1 - K(S)K(S^*)$$

and hence $m - 1$ factors through the $K(D)$, implying the claim.

(3) By the semi-orthogonal decomposition of Proposition 3.16, we have that

$$K(C) = K(X//G) \oplus K(D),$$

where $\text{res}_-$ is the projection on to the first summand, and $\ker(\text{res}_-)$ is identified with $K(D)$. Applying Lemma 5.1 to $K\mathcal{P}$ therefore gives the result. ■

The inequality (10) may not be saturated, for indeed it often happens that $m = 1$ so that $\text{rk}(m - 1) = 0$ as Example 6.2 shows. On the other hand, we can give easily-checkable sufficient conditions for (10) to be saturated. Recall that Assumption 4.1 implied that $\eta_+ = \eta_-$, so that the $\lambda_\pm$-weights of $\det N_{S^\pm}X$ on $Z$ were equal: under the following slightly stronger condition the spherical pair obtained in Theorem 4.4 will be easy to control at the level of Grothendieck groups, so that these sufficient conditions for the saturation of (10) will become apparent.

**Lemma 5.3.** For $G$ abelian, the following conditions are equivalent.

1. The line bundle $\det N_SX$ on $Z$ is $G$-equivariantly trivial.
2. The line bundle

$$\det N_{S^+}X \otimes \det N_{S^-}X$$

(11)

is $G$-equivariantly trivial after restriction to $Z$.

These conditions hold if $X$ and $Z$ are $G$-equivariantly Calabi–Yau. □
Proof. The equivalence claim follows from the identification of the strata $S_{\pm}$ with the $Y_{\pm}$ and the splitting, as in (2), of $\Omega_X$ on $Z$ into

$$N_Y^X|_Z \oplus \Omega_Z \oplus N_Y^X|_Z,$$

combined with the definition of $N^X_Z$. The last claim follows from the adjunction formula

$$\det N^X_Z = \omega_Z \otimes \omega_Y^X|_Z.$$

$\blacksquare$

Proposition 5.4. In the setting of Theorem 4.4, assume furthermore that:

i. the equivalent conditions of Lemma 5.3 hold; and
ii. the codimension of $Z$ in $X$ is odd.

Then the inequality (10) is saturated, and

$$K^P \cong IC(M)$$

where $M$ is the local system on $\Delta - 0$ obtained by restricting $K^P$.

$\square$

Proof. I first calculate the action of the cotwist $\mathcal{C}_S$ of the spherical functor $S$ from (9) on $K(D)$ under the assumptions. From Proposition 3.8(2) this is given by the composition (a) then (b) from Definition 3.6. In our setting, by Proposition 4.5(a and b), this acts by tensoring by the line bundle (11) restricted to $Z$, which is $G$-equivariantly trivial by assumption, and a cohomological shift by the negative of

$$\text{codim}_X S_+ + \text{codim}_X S_-.$$

For $G$ abelian, the strata $S_{\pm}$ are identified with the $Y_{\pm}$, which intersect transversally in $Z$ as explained in the proof of Lemma 4.7, so this is just $\text{codim}_X Z$. If this is odd, then the action of $\mathcal{C}_S$ on $K(D)$ is therefore simply $-\mathbb{1}$. It then follows from the definition of $\mathcal{C}_S$ in Proposition 3.8(2) that

$$K(\mathcal{C}_S) = \mathbb{1} - K(S^+) K(S) = -\mathbb{1} \quad \text{and} \quad K(S^+) K(S) = 2 \cdot \mathbb{1}.$$

The map $2 \cdot \mathbb{1}$ is clearly bijective, and hence $K(S)$ is injective, and $K(S^+)$ surjective on to $K(D)$. Using Theorem 5.2(1) and the definition of $T_S$ from Proposition 3.8(1) we then
have that
\[
\text{rk}(m - 1) = \text{rk}(K(T_S) - 1) \\
= \text{rk}(- K(S) K(S^*)) \\
= \dim K(D),
\]
and hence the inequality (10) is saturated, and the claim follows by the proof of Theorem 5.2(3).

Finally, we record the following dual result to Theorem 5.2.

**Theorem 5.5.** In the setting of Theorem 4.4, consider the perverse sheaf \( P^K \) provided by Proposition 3.7, with description

\[
K\left(D\left(\frac{Z}{L}\right)_-^w\right) \xrightarrow{\iota_-} K(C) \xleftarrow{\iota_+} K\left(D\left(\frac{Z}{L}\right)_+^w\right),
\]

where unmarked arrows are given by left adjoints. Then:

1. the monodromy \( m' \) around 0 on \( K\left(D\left(\frac{Z}{L}\right)_-^w\right) \) is given by the action of the cotwist \( C_S \) of the spherical functor (9);
2. we have
\[
\text{rk}(m' - 1) \leq \dim K\left(X_{/+} G\right); (12)
\]
3. there is an isomorphism
\[
P^K \cong IC(M')
\]
for \( M' \) a local system on \( \Delta - 0 \) if and only if the inequality (12) is saturated.

**Proof.** The argument is dual to that of Theorem 5.2, using Proposition 3.9(2) in place of Proposition 3.9(1), and so on. 

6 Exponents

In this section, I give some examples of spherical pairs for flops in higher dimensions.
Example 6.1. Let $X = V \oplus \det V^\vee$ for $V$ a vector space of dimension $d = 2n$, with $\mathbb{C}^*$-action induced by dilation of $V$. Then $Z = 0$ with
\[ X^s_+ = X - (0 \oplus \det V^\vee) \quad \text{and} \quad X^s_- = X - (V \oplus 0), \]
so that the quotient $X/\mathbb{C}^*$ is the total space of the bundle
\[ O(-2n) \to \mathbb{P}V, \]
and $X/\mathbb{C}^*$ is its flop, the orbifold $V/C_{2n}$ where the cyclic group $C_{2n}$ acts on $V$ by scalars.

The space $X$ is smooth of dimension $2n + 1$ and equivariantly Calabi–Yau, so Assumption 4.1 is easily seen to be satisfied, and Proposition 5.4 may be applied. Hence we have a spherical pair $\mathcal{P}$, and $^k\mathcal{P}$ is an intersection cohomology complex of a local system on $\Delta - 0$. $\square$

Example 6.2. Let $X = V \oplus V^\vee$ for $V$ a vector space of dimension $d$, with $\mathbb{C}^*$-action induced by dilation of $V$. Then $Z = 0$ with
\[ X^s_+ = X - (0 \oplus V^\vee) \quad \text{and} \quad X^s_- = X - (V \oplus 0), \]
so that the quotient $X/\mathbb{C}^*$ is the total space of the bundle $V^\vee \otimes O(-1) \to \mathbb{P}V$, and $X/\mathbb{C}^*$ is its flop, the total space of $V \otimes O(-1) \to \mathbb{P}V^\vee$.

Once again, the space $X$ is smooth and equivariantly Calabi–Yau, so Assumption 4.1 is satisfied. Hence we have a spherical pair $\mathcal{P}$. However, we cannot apply Proposition 5.4 for dimension reasons and indeed the conclusion, that $^k\mathcal{P}$ is an intersection cohomology complex of a local system on $\Delta - 0$, does not hold.

To see this, let $w = 0$ so that $t^w_0(O_0) = O_{V \oplus 0}$, and observe (for instance by comparing terms in the respective Koszul resolutions) that this has the same Grothendieck group class, up to sign, as $O_{0 \oplus V^\vee} \otimes \det V$. The latter object is clearly in the kernel of res$, and because $O_0$ is a generator we may deduce that $K(S) = 0$ and
\[
\begin{align*}
\text{rk}(m - 1) &= \text{rk}(K(T_S) - 1) \\
&= \text{rk}(-K(S)K(S^*)) \\
&= 0.
\end{align*}
\]

Here $\mathcal{D} = D(Z/L)^w$ is generated by a single object $O_0$, so dim $K(\mathcal{D}) = 1$, the inequality (10) is not saturated, and thence $^k\mathcal{P}$ is not the intersection cohomology complex of a local system on $\Delta - 0$ by Theorem 5.2(3). $\square$
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