THEOREMS MOTIVATED BY FOUNDATIONS OF QUANTUM MECHANICS AND SOME OF THEIR APPLICATIONS

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February 4, 2022

ABSTRACT

This paper provides theorems aimed at shedding light on issues in the foundations of quantum mechanics. These theorems can be used to propose new interpretations to the theory, or to better understand, evaluate and improve current interpretations. Some of these applications include: (1) A proof of the existence of pilot-wave theories that are fully equivalent to standard quantum mechanics in a path-wise sense. This equivalence is stronger than what is entailed from the more traditional requirements of equivariance, or good mixing properties, and is necessary to assure proper correlations across time and proper records of the past. (2) A proposal for a minimalistic ontology for non-collapse quantum mechanics, in which Born’s rule provides the proper predictions. (3) The observation of a close relationship between Born’s rule and a version of the superposition principle.

Keywords Hilbert space · projections · commutation · probabilistic description · projection valued measure · non-collapse quantum mechanics · pilot-wave theories · consistent histories · superposition principle · Born rule · ontology · decoherence

1 Introduction

Problems in the foundations of quantum mechanics involve Physics, Philosophy and Mathematics. Here we state and prove theorems motivated by such issues. These theorems have implications in the evaluation and understanding of various distinct interpretations of quantum mechanics, and can also be used to propose new interpretations.

A separation between the mathematical work presented here and the discussion of its uses to evaluate, or build interpretations seems appropriate and beneficial. This is so because mathematical work can be understood and judged based solely on its correctness, in an objective way. In contrast, applications to interpretations seem to unavoidably lead to less precise and less objective ideas. For this reason, applications of the theorems will be separated from the purely mathematical part of this paper, and presented in the final section.

A few words about the relation between the theorems presented here and quantum mechanics and its foundations are nevertheless in order in this introduction, to help the reader understand our motivation and purpose, before engaging with the mathematics.

Our mathematical setting will correspond to a quantum mechanical universe described in the Heisenberg picture, in which the wave function does not evolve, while operators evolve in time. We assume that no collapse of the wave function ever occurs, so that the state of this universe is given by an unchanging vector $\Psi$ in a Hilbert space, and no probabilistic postulates are introduced.

One of our main goals is in producing a coherent interpretation of quantum mechanics in which there is no collapse of the wave function, and no special notion of “observation” or “experiment”, but in which predictions coincide with those of standard textbook quantum mechanics, based on Born’s rule and the associated collapses of the wave function.
Our work is in the tradition of Everettian approaches, \cite{11, 29, 39, 35}, and follows our contribution in \cite{31} and \cite{32}. The current project was in part motivated by the desire, expressed in \cite{31} and \cite{32}, of providing an adequate ontology for such a theory, which would be compatible with the prediction postulate introduced there. This is done in Subsection 8.11 based on Theorem 1 from Section 2 and further elaborated in the following three subsections, using also results from other sections of the paper.

Readers will notice that the right-hand side of the probability formula in part (b) of Theorem 1 corresponds to the usual quantum mechanics prescription based on Born’s rule with collapses after each observation. (See also Subsection 8.3.) And the theorems in Sections 3 and 4 indicate, modulo intuitive assumptions that include a version of the superposition principle, and which are explained in Subsections 8.9 and 8.15 that this probability should indeed correctly predict which experiences we can have, and which ones are ruled out, in a universe without wave function collapse.

Readers familiar with the literature on foundations of quantum mechanics will have no difficulty in seeing various additional ways in which several of the theorems in this paper relate to quantum mechanics and issues in its foundations.

In particular, the setting in which these theorems are formulated is clearly related to the “Consistent, or Decoherent, Histories” approach, \cite{25, 15, 17}. In this respect, the main thing to keep in mind is that no consistency condition will be assumed. Rather, we will see that the Hilbert space can be decomposed into two orthogonal subspaces, with very distinct properties. These properties suggest (again, modulo considerations from \cite{31, 32} and Subsection 8.9 of this paper) that the component of $\Psi$ in one of these subspaces does not affect our experiences. On the other hand, the other component, that is therefore responsible for our experiences, satisfies very strong consistency conditions, in the form of commutation of a large class of relevant operators, and yields predictions in full equivalence with textbook quantum mechanics. It will be argued, in Subsection 8.10 that this splitting of the Hilbert space into two orthogonal subspaces is associated to the concept of environmental decoherence, \cite{21, 40, 25, 15, 30, 1}.

Better understanding “Pilot-Wave” theories, \cite{13, 7} (Chapter 5), \cite{10, 23, 9, 10, 34, 3, 33, 12, 33}, was also one of the original motivations for this paper. Part (b) of Theorem 1 can be seen as providing a sort of “Pilot-Wave” picture, though in what we will call “T-space”, rather than physical space. This can, nevertheless, be used to produce pilot-wave theories in physical space. (See Subsections 8.4 and 8.16.) And those display stronger probabilistic agreement with quantum mechanics, in a path-wise sense, than the minimal agreement associated to the concept of equivariance (sometimes strengthened by requiring good mixing properties, \cite{35}), fulfilled by various versions of pilot-wave theories, including Bohmian mechanics. Full agreement with standard quantum mechanics in a path-wise sense is needed to assure that the path of the process produced by the pilot-wave theory displays correlations across time in accordance with quantum mechanics, and in particular yields appropriate records of its own past. In this way problems with some pilot-wave theories, pointed out in Section 10.2 of \cite{22}, Section 5 of \cite{23}, and Section 5 of \cite{2} are avoided.

Theorem 1 presented in Section 2, extends substantially the mathematical results in \cite{18} and \cite{37}. And Theorems 2 and 3 presented in Sections 5 and 4 extend substantially the mathematical results in \cite{32} (and its longer version \cite{31}). The main difference with respect to these papers is that there all the inputs were in the form of operators (usually projections) assumed, explicitly or implicitly, to commute with each other, while here no such assumption is made. We will see that this leads to the decomposition of the Hilbert space into the two subspaces mentioned above. Commutativity will turn out to be restricted then to the subspace accessible to our experiences.

The purely mathematical sections of this paper (which are all but the last one) can be read without knowledge of quantum mechanics. Requisites for their reading are knowledge of the basic theorems about Hilbert spaces, with emphasis on projection operators (Chapters 1 and 2 of \cite{19} provide an excellent presentation of this material), and measure theory (see, e.g., Chapters 11 and 12 of \cite{27}, Chapter 1 of \cite{14}, and Chapters 1 and 2 of \cite{6}). Additional knowledge of probability theory and the theory of (unbounded) operators in Hilbert spaces will be needed occasionally, and are covered in the books listed in the bibliography.

Sections 2, 3, and 4 of the paper contain the core mathematical results. Section 5 can be read independently of these and is included here to provide tools used in the first two core sections, and also a brief introduction to projection valued measures. Sections 6 and 7 contain additional results, related to the material in the core sections. Section 8 contains examples, remarks (both of a mathematical and philosophical nature) and applications of the theorems to issues in the foundations of quantum mechanics.

The fastest way to learn what the content of this paper is is to read the material in the beginning of each section, from Section 2 to 7 (possibly skipping Section 5), stopping after the statement of the first theorem in each section. And then reading subsections 8.1 to 8.4, 8.6, and 8.9 to 8.16.

The logic dependence of the proofs in the various sections is as follows. Section 5 is independent of the other sections. Section 2 depends on Section 5. Section 3 depends on Section 2 and at one place on Section 5. Section 4 depends on Sections 2 and 5. Sections 6 and 7 depend on Section 2.
2 Basic results

Let $\mathcal{H}$ be a Hilbert space (not necessarily separable) over the complex field. For $\varphi, \psi \in \mathcal{H}$ their inner product, assumed to be linear in the first argument and conjugate linear in the second, will be denoted by $\langle \varphi, \psi \rangle$. The norm of $\varphi$ will be denoted by $||\varphi||$. If $\varphi \neq 0$, we denote by $\hat{\varphi} = ||\varphi||^{-1}\varphi$ its normalized counterpart. The topological closure of a set $V \subset \mathcal{H}$ will be denoted by $\overline{V}$. By a subspace of $\mathcal{H}$ we will mean a subset of $\mathcal{H}$ that is linearly and topologically closed. Subsets of $\mathcal{H}$ that are linearly closed will be referred to as vector spaces, or linear spaces. By a projection $p$ we will mean a self-adjoint ($\langle p, \varphi \rangle = \langle \varphi, p \varphi \rangle$, idempotent ($p^2 = p$) operator. Given a family $\{p_a\}$ of projections, $\land_a p_a$ will denote the projection on the intersection of the ranges of the $p_a$, called the “meet”, or the “infimum” of these projections (see Section 30 of [19]). If $\{Q\}$ is a countable family of bounded operators in $\mathcal{H}$, and $Q$ is another bounded operator in $\mathcal{H}$, the statement $\sum Q_i = Q$ will always mean that $\sum_i Q_i \varphi = Q \varphi$, for all $\varphi \in \mathcal{H}$. The concept of projection valued measures will play a major role in this paper. Section 5 includes their definition and a brief introduction to their basic properties. (They are called “spectral measures” in [19], and “resolutions of the identity” in [27].) We will make extensive use of theorems from [19], when referring to Theorem x in Section y in that text, we will indicate it by Thm.H.y.x.

Let $S$ be an arbitrary set. In applications to quantum mechanics, we will take $S \subset \mathbb{R}$ and think of elements of $S$ as moments in time, but in this paper no structure or constraint needs to be assumed on $S$, unless when stated otherwise (as will happen in Section 3 where $S$ will often be supposed to be countable, and in Section 6 where $S$ will be supposed to be totally ordered). Still, we will refer to elements of $S$ as “times”. For each $t \in S$, let $\Gamma(t)$ be a countable index set. To each $t \in S$ and $a \in \Gamma(t)$ we associate a projection $p^t_a$ in $\mathcal{H}$. We assume that, for each $t \in S$,

$$\sum_{a \in \Gamma(t)} p^t_a = I,$$

(1)

where $I$ is the identity operator, so that $\{p^t_a : a \in \Gamma(t)\}$ is a partition of the identity. This assumption implies (see Thm.H.28.2 and Thm.H.27.4) the orthogonality condition $p^t_a p^t_b = p^t_b p^t_a = 0$, if $a \neq b$. Set

$$\pi = \{p^t_a : t \in S, a \in \Gamma(t)\}.$$  

The range of $p^t_a$ will be denoted by $\mathcal{H}^{t,a}_t$, and for $k = 2, 3, \ldots$ we extend this definition by setting $\mathcal{H}^{t_1, t_2, \ldots, t_k}_t = \cap_{i=1}^k \mathcal{H}^{t_i}_{t_i}$. The orthogonality condition mentioned above is equivalent to the statement that for each $t_1, \ldots, t_k$,

$$\mathcal{H}^{t_1, \ldots, t_k}_t \perp \mathcal{H}^{t_1, \ldots, t_k}_t, \quad \text{if} \quad (a_1, \ldots, a_k) \neq (b_1, \ldots, b_k).$$

(2)

This orthogonality condition allows us to define the direct sum

$$\mathcal{H}^{t_1, \ldots, t_k}_t = \bigoplus_{a_1, \ldots, a_k} \mathcal{H}^{t_1, \ldots, t_k}_{a_1, \ldots, a_k}.$$  

The projection on $\mathcal{H}^{t_1, t_2, \ldots, t_k}_t$ will be denoted by $p^{t_1, \ldots, t_k}_{a_1, \ldots, a_k, a_{k+1}} = \land_{i=1}^{k+1} p^{t_i}_{a_i}$. It is clear that for $k < l$, $\mathcal{H}^{t_1, \ldots, t_k}_t \subset \mathcal{H}^{t_1, \ldots, t_l}_t$ and therefore

$$p^{t_1, \ldots, t_k}_{a_1, \ldots, a_k, a_{k+1}} \varphi = 0, \quad \text{whenever} \quad p^{t_1, \ldots, t_k}_{a_1, \ldots, a_k} \varphi = 0.$$  

(3)

We introduce now a measurable space in the following fashion. Define the Cartesian product $\Omega = \times_{t \in \Gamma(t)} \mathcal{H}(t)$. In applications in which we think of elements of $S$ as moments in time, $\Omega$ can be thought of as the set of trajectories, or histories, in “$\Gamma$ space”. And let $\Sigma$ be the smallest sigma-algebra of subsets of $\Omega$ that contains all the sets of the form $\{\omega \in \Omega : \omega_t = a_t\}$, $t \in S$, $a_t \in \Gamma(t)$. Define the functions $X_t, t \in S$, from $\Omega$ to $\Gamma(t)$ by $X_t(\omega) = \omega_t$. We will often use probabilistic terminology and notation, so elements of $\Sigma$ will sometimes be called “events”, the $X_t$ will sometimes be called “random variables” and we abbreviate $\{\omega \in \Omega : \omega_t = a_t, i = 1, \ldots, k\} = \{X_{t_i} = a_{t_i}, i = 1, \ldots, k\}$. The events of the form $\{(X_{t_1}, \ldots, X_{t_k}) \in G\}$, for some $t_1, \ldots, t_k$ and $G \subset \Gamma(t_1) \times \cdots \times \Gamma(t_k)$ form an algebra $\mathcal{A}$, which generates the sigma-algebra $\Sigma$. The class of events obtained by countable unions of events in $\mathcal{A}$ will be denoted by $\mathcal{A}_c$, and the class of events obtained by countable intersections of events in $\mathcal{A}$ will be denoted by $\mathcal{A}_b$.

We turn now to the definition of a subspace of $\mathcal{H}$ that will play a central role in this paper. Define $\mathcal{W}$ as the set of operators on $\mathcal{H}$ that are products of finitely many elements of $\pi$. And for $W, V \in \mathcal{W}$, write $W \sim V$ in case $W$ and $V$ are obtained from the same elements of $\pi$, but possibly multiplied in different orders. Given a vector $\varphi \in \mathcal{H}$ we say that “$\varphi$ commutes on $\varphi$” if $W \varphi = V \varphi$, whenever $W \sim V$. And we define

$$\mathcal{H}_\pi = \{\varphi \in \mathcal{H} : \pi \text{ commutes on } \varphi\}.$$  

It is clear that $\mathcal{H}_\pi$ is a subspace of $\mathcal{H}$ and that for each $t \in S$ and $a \in \Gamma(t)$,

$$p^t_a \mathcal{H}_\pi \subset \mathcal{H}_\pi,$$

(4)
i.e., \( \mathcal{H}_\pi \) is invariant under the projections in \( \pi \). Whenever \( \mathcal{H}_\pi \) is invariant under a projection \( p \), we will denote by \( \tilde{p} \) the restriction of \( p \) to \( \mathcal{H}_\pi \). We will denote by \( p_\pi \) the projection on \( \mathcal{H}_\pi \). And for \( \varphi \in \mathcal{H} \) we define \( \varphi_\pi = p_\pi \varphi \).

Define also

\[
\mathcal{H}_\pi' = \{ \varphi \in \mathcal{H} : p_{a_1 \ldots a_k}^{t_1 \ldots t_k} \varphi = p_{a_1 \ldots a_k}^{t_1 \ldots t_k} \varphi \text{ for all } t_1, \ldots, t_k \text{ and } a_1, \ldots, a_k \},
\]

\[
\mathcal{H}_\pi'' = \{ \varphi \in \mathcal{H} : \sum_{a_1, \ldots, a_k} p_{a_1 \ldots a_k}^{t_1 \ldots t_k} \varphi = \varphi \text{ for all } t_1, \ldots, t_k \},
\]

\[
N = \{ \varphi \in \mathcal{H} : \text{for some } t_1, \ldots, t_k, p_{a_1 \ldots a_k}^{t_1 \ldots t_k} \varphi = 0 \text{ for all } a_1, \ldots, a_k \}.
\]

For examples of the setting above in a quantum mechanical context, see Subsections 8.1 and 8.2. For the relationship between \( H(d) \) or every \( \phi \), we define a subspace \( \mathcal{H}_\phi \) to each measurable function. And for every \( A \) for every \( \phi \), \( \mathcal{H}_\phi \) is invariant under the projections in \( \pi \). And for every \( \pi \), \( \mathcal{H}_\pi \) is a vector space and there corresponds a subspace \( \mathcal{H}_\pi^{-} \) in \( \mathcal{H}_\pi \) such that

\[
\mathcal{H}_\pi^{-} = \{ \varphi \in \mathcal{H} : \text{for all } t_1, \ldots, t_k \text{ and } a_1, \ldots, a_k, \pi_{a_1 \ldots a_k}^{t_1 \ldots t_k} \varphi = 0 \}.
\]

For remarks on the meaning of \( \pi \), see Subsection 8.10. For remarks on the meaning of the right-hand side of (6) in the following theorem, see Subsection 8.3.

**Theorem 1**

(a) \( N \) is a vector space and

\[
\mathcal{H}_\pi = \mathcal{H}_\pi' = \mathcal{H}_\pi'' = N^\perp.
\]  

(b) For any \( \varphi \in \mathcal{H}_\pi \setminus \{0\} \), there exists a unique probability measure \( \mathbb{P}_\varphi \) on \( (\Omega, \Sigma) \), such that

\[
\mathbb{P}_\varphi(X_{t_i} = a_i, i = 1, \ldots, k) = ||p_{a_1 \ldots a_k}^{t_1 \ldots t_k} \varphi||^2 = ||p_{a_1}^{t_1} \varphi||^2,
\]

for every \( t_1, \ldots, t_k \) and \( a_1, \ldots, a_k \).

(c) To each \( A \in \Sigma \) there corresponds a subspace \( \mathcal{H}_A \subset \mathcal{H}_\pi \), with the property that if \( p_A \) is the projection on \( \mathcal{H}_A \) and \( \tilde{p}_A \) is its restriction to \( \mathcal{H}_\pi \), then \( \{	ilde{p}_A : A \in \Sigma \} \) is the unique projection valued measure (p.v.m.) from \( \Sigma \) to projections in \( \mathcal{H}_\pi \) such that

\[
\tilde{p}(X_{t_1} = a_1, \ldots, X_{t_k} = a_k) \varphi = p_{a_1 \ldots a_k}^{t_1 \ldots t_k} \varphi,
\]

for all \( t_1, \ldots, t_k, a_1, \ldots, a_k \) and \( \varphi \in \mathcal{H}_\pi \). In particular, \( \mathcal{H}_\Omega = \mathcal{H}_\pi \), and for any \( A \in \Sigma \),

\[
\mathcal{H} = \mathcal{H}_A + \mathcal{H}_{A^c} + \mathcal{H}_\pi^{-} = \mathcal{H}_A + \mathcal{H}_{A^c} + \mathcal{N}.
\]  

(d) For every \( \varphi \in \mathcal{H}_\pi \setminus \{0\} \) and every \( A \in \Sigma \),

\[
\mathbb{P}_\varphi(A) = ||p_A \varphi||^2.
\]

And for every \( A \in \Sigma \),

\[
\mathcal{H}_A = \{ \varphi \in \mathcal{H}_\pi : \varphi = 0 \text{ or } \mathbb{P}_\varphi(A) = 1 \},
\]

and

\[
\mathcal{H}_A^\perp = \{ \varphi \in \mathcal{H} : \varphi_\pi = 0 \text{ or } \mathbb{P}_{\varphi_\pi}(A) = 0 \}.
\]

(e) For every \( \varphi \in \mathcal{H}_\pi \setminus \{0\} \) and every \( A, B \in \Sigma \) such that \( \mathbb{P}_\varphi(A) \neq 0 \) we have the following conditional probability formula

\[
\mathbb{P}_\varphi(B|A) = \mathbb{P}_{p_A \varphi}(B).
\]

(f) To each measurable function \( f \) from \( \Omega \) to \( \mathbb{R} \) (endowed with the Borel sigma-algebra \( \mathcal{B} \)) there corresponds a self-adjoint operator in \( \mathcal{H}_\pi \), denoted by \( Q_f \), with domain \( \mathcal{D}_f = \{ \varphi \in \mathcal{H}_\pi : \varphi = 0, \text{ or } \int f^2 d\mathbb{P}_\varphi < \infty \} \), such that

\[
\int f d\mathbb{P}_\varphi = \langle \varphi, Q_f \varphi \rangle,
\]

for every \( \varphi \in \mathcal{D}_f \setminus \{0\} \). The spectral decomposition of \( Q_f \) is given by the p.v.m.

\[
\{p_{f \in B} : B \in \mathcal{B} \}.
\]

All such operators \( Q_f \) commute with each other, in the sense ([26], Section VIII.5) that the projections in their spectral decompositions all commute with each other.

**Remark on item (f):** If \( f \) is bounded, then \( \mathcal{D}_f = \mathcal{H}_\pi \), and \( Q_f \) is a bounded operator. If also \( g \) is a bounded measurable function from \( \Omega \) to \( \mathbb{R} \), then the commutation stated in the theorem takes the usual form \( Q_f Q_g = Q_g Q_f \), thanks to the spectral theorem.

Before proving Theorem [11], we collect some technical results in two propositions.
Proposition 1 Suppose that \( p \) is the projection on the subspace \( S_p \) and \( q \) is the projection on the subspace \( S_q \). Then the following are equivalent:

(i) \( pS_q \subseteq S_q \).
(ii) \( pS_q^\perp \subseteq S_q^\perp \).
(iii) \( p \) and \( q \) commute.
(iv) \( qS_p \subseteq S_p \).
(v) \( qS_p^\perp \subseteq S_p^\perp \).

**Proof:** Thm.H.23.2 implies that (i) and (ii) are equivalent, because \( p \) is self-adjoint. Thm.H.27.2 implies that (i) and (ii) together are equivalent to (iii). This establishes the equivalence of (i), (ii) and (iii). Interchanging the roles of \( p \) and \( q \), we obtain also the equivalence of (iv) and (v) with (iii), completing the proof. \( \square \)

Proposition 2 For every \( t_1, \ldots, t_k \) and \( a_1, \ldots, a_k \),

(a) \[ p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \mathcal{H}_\pi \subseteq \mathcal{H}_\pi \quad \text{and} \quad p_{(a_1 \cdots a_k)}^{t_1} \mathcal{H}_\pi^\perp \subseteq \mathcal{H}_\pi^\perp. \]

(b) \( p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \) commutes with \( p_\pi \).

(c) The restriction of \( p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \) to \( \mathcal{H}_\pi \), \( p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \mathcal{H}_\pi \), is the projection in \( \mathcal{H}_\pi \) on the subspace \( \mathcal{H}_\pi(a_1 \cdots a_k) \cap \mathcal{H}_\pi \).

(d) For every \( \varphi \in \mathcal{H}_\pi \), \( p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \varphi = p_{(a_1 \cdots a_k)}^{t_1} \cdots p_{(a_1 \cdots a_k)}^{t_k} \varphi \).

(e) For every \( \varphi \in \mathcal{H}_\pi \),
\[
\sum_{a_i \in \Gamma(t_i)} p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \mathcal{H}_\pi \varphi = p_{(a_1 \cdots a_k)}^{t_1} \cdots p_{(a_1 \cdots a_k)}^{t_k} \varphi.
\]

**Proof:** (a) and (b): From (4) and Proposition 1, we have \( p_\pi \mathcal{H}_\pi \subseteq \mathcal{H}_\pi \), for each \( t \in S \) and \( \alpha \in \Gamma(t) \). Therefore \( p_\pi \mathcal{H}_\pi(a_1 \cdots a_k) \subseteq \mathcal{H}_\pi \). And invoking Proposition 1 again we complete the proof of (a) and (b).

(c): The operator \( p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \) in \( \mathcal{H}_\pi \) inherits self-adjointness and idempotency from \( p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \), so that it is indeed a projection in \( \mathcal{H}_\pi \).

Range \( (p_{a_1}^{t_1} \cdots p_{a_k}^{t_k})^{t_1} = \{ \varphi \in \mathcal{H}_\pi : p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \varphi = \varphi \} = \{ \varphi \in \mathcal{H} : p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \varphi = \varphi \} \cap \mathcal{H}_\pi = \mathcal{H}_\pi(a_1 \cdots a_k) \cap \mathcal{H}_\pi \).

(d): By the definition of \( \mathcal{H}_\pi \), the projections \( p_{a_1}^{t_1} \), \( t \in S \), \( \alpha \in \Gamma(t) \) commute with each other. Therefore Thm.H.29.1 implies that
\[
\prod_{i=1}^k p_{a_i}^{t_i} = p_{a_1}^{t_1} \cdots p_{a_k}^{t_k}.
\]

From (c) above, we have
\[
\text{Range} \left( \bigwedge_{i=1}^k p_{a_i}^{t_i} \right) = \bigcap_{i=1}^k \text{Range} \left( p_{a_i}^{t_i} \right) = \bigcap_{i=1}^k \left( \mathcal{H}_{a_i}^{t_i} \cap \mathcal{H}_\pi \right) = \mathcal{H}_{(a_1 \cdots a_k)}^{t_1} \cap \mathcal{H}_\pi = \text{Range} \left( p_{(a_1 \cdots a_k)}^{t_1} \right),
\]

i.e.,
\[
\bigwedge_{i=1}^k p_{a_i}^{t_i} = p_{(a_1 \cdots a_k)}^{t_1}.
\]

Combining (16) and (17), we have, for \( \varphi \in \mathcal{H}_\pi \),
\[
p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \varphi = p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \varphi = p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \varphi = p_{a_1}^{t_1} \cdots p_{a_k}^{t_k} \varphi.
\]

(e): Using (d) above and (1), we have, for \( \varphi \in \mathcal{H}_\pi \),
\[
\sum_{a_i \in \Gamma(t_i)} p_{a_1}^{t_1} \cdots p_{a_i-1}^{t_{a_i-1}} p_i^{t_i} p_{a_{i+1}}^{t_{a_{i+1}}} \cdots p_{a_k}^{t_k} \varphi = \sum_{a_i \in \Gamma(t_i)} p_{a_1}^{t_1} \cdots p_{a_i-1}^{t_{a_i-1}} p_i^{t_i} p_{a_{i+1}}^{t_{a_{i+1}}} \cdots p_{a_k}^{t_k} \varphi
\]
\[
= p_{a_1}^{t_1} \cdots p_{a_i-1}^{t_{a_i-1}} \left( \sum_{a_i \in \Gamma(t_i)} p_i^{t_i} \right) p_{a_{i+1}}^{t_{a_{i+1}}} \cdots p_{a_k}^{t_k} \varphi
\]
\[
= p_{a_1}^{t_1} \cdots p_{a_i-1}^{t_{a_i-1}} p_{a_{i+1}}^{t_{a_{i+1}}} \cdots p_{a_k}^{t_k} \varphi
\]
\[
= p_{a_1}^{t_1} \cdots p_{a_i-1}^{t_{a_i-1}} t_{a_{i+1}}^{t_{a_{i+1}}} \cdots t_{a_k}^{t_k} \varphi,
\]
The second equality is justified by Thm.H.28.1, which states that since the sum inside the parenthesis is well defined, we can exchange the order of the operations, as done there. □

**Proof of Theorem**: (a) $N$ is clearly closed with respect to multiplication by scalars. That it is closed with respect to sums is a simple consequence of (3). Indeed, if $\varphi', \varphi'' \in N$, we can combine the corresponding sets $t_1', ..., t_k'$, and $t_1'', ..., t_k''$, whose existence is implied by these assumptions, to produce a set $t_1, ..., t_k$ for which $\psi_{a_1', ..., a_k'} \varphi' = 0$ and any $a_1, ..., a_k$. Hence also $\psi_{a_1', ..., a_k'} (\varphi' + \varphi'') = 0$, completing the proof that $N$ is a vector space.

Next we will prove that

$$\mathcal{H}_\pi \subset \mathcal{H}_\pi' \subset \mathcal{H}_\pi'' \subset \mathcal{H}_\pi.$$  
(18)

The first of these claims is a restatement of part (d) of Proposition 2.

Suppose now that $\varphi \in \mathcal{H}_\pi'$. Then, using (1) and Thm.H.28.1 (as in the proof of part (e) of Proposition 2),

$$\sum_{a_1, ..., a_k} p_{a_1', ..., a_k} \varphi = \sum_{a_1, ..., a_k} p_{a_1} (\varphi (\sum_{a_k} t_{a_k}^{t_{a_k}})) \varphi = (\sum_{a_1} t_{a_1}^{t_{a_1}}) \varphi = \kappa \varphi = \varphi,$$

implying that $\mathcal{H}_\pi' \subset \mathcal{H}_\pi''$.

Finally, to prove the last claim in (18), suppose that $\varphi \in \mathcal{H}_\pi''$. Suppose that $W = p_{a_1}^{t_{a_1}} ... p_{a_k}^{t_{a_k}}$. We need to show that $W \varphi$ does not depend on the order of the factors defining $W$. The times $s_1, ..., s_i$ may include repetitions, so let $t_1, ..., t_k$ be the same set of times, but without the repetitions. We will use the equation $\varphi = \sum_{a_1} t_{a_1}^{t_{a_1}} p_{a_1} \varphi$. Apply $W$ to both sides of this equation, and use the following two facts. First $\mathcal{H}_{\mathcal{X}} \supset \mathcal{H}_{\mathcal{X}_{1, ..., k}}$, if $c = a_i$, so that in this case $p_{a_1}^{t_{a_1}} ... p_{a_k}^{t_{a_k}} = p_{a_1}^{t_{a_1}} ... p_{a_k}^{t_{a_k}}$. Second, $\mathcal{H}_{\mathcal{X}} \perp \mathcal{H}_{\mathcal{X}_{1, ..., k}}$, if $c \neq a_i$, so that in this case $p_{a_1}^{t_{a_1}} ... p_{a_k}^{t_{a_k}} = 0$. This gives us $W \varphi = 0$, in case $W$ includes two factors $p_{a_1}^{t_{a_1}}$ with distinct $c$, and $W \varphi = p_{a_1}^{t_{a_1}} ... p_{a_k}^{t_{a_k}}$ in case every factor $p_{a_i}^{t_{a_i}}$ in $W$ has $c = a_i$. In either case $W \varphi$ does not depend on the order of the factors that define $W$. Hence $\varphi \in \mathcal{H}_\pi''$.

This completes the proof of (18) and hence of the first two equalities in (5).

The orthogonality in (2) implies that the statement $p_{a_1}^{t_{a_1}} ... p_{a_k}^{t_{a_k}} \varphi = 0$ for all $a_1, ..., a_k$, in the definition of $N$, is equivalent to the statement that $\sum_{a_1, ..., a_k} p_{a_1}^{t_{a_1}} ... p_{a_k}^{t_{a_k}} \varphi = 0$. But this is equivalent to the statement that $\varphi$ is orthogonal to the range of the projection $\sum_{a_1, ..., a_k} p_{a_1}^{t_{a_1}} ... p_{a_k}^{t_{a_k}}$, which is $\mathcal{H}_{t_1, ..., t_k}$. Therefore

$$N = \bigcup \left\{ (\mathcal{H}_{t_1, ..., t_k})^\perp : t_1, ..., t_k \in S \right\},$$

And hence

$$N^\perp = \bigcap \left\{ \mathcal{H}_{t_1, ..., t_k} : t_1, ..., t_k \in S \right\} = \mathcal{H}_{\pi''},$$
finishing the proof of (5).

(b): We will use Kolmogorov’s extension theorem, also called Kolmogorov’s existence theorem. (See, e.g., Section 36 of [6], or Section 4 of Chapter 9 of [14]. To apply the theorem as usually stated, for real valued random variables, identify each $\Gamma (i)$ with a subset of the set $\left\{ 0, 1, ..., \right\} \subset \mathbb{R}$, so that the $X_i$ can be seen as real valued random variables.)

For this purpose, we first define probability measures on smaller spaces, corresponding to finitely many moments in time. For each $t_1, ..., t_k \in S$, define $\Omega_{t_1, ..., t_k} = \Gamma (t_1) \times ... \times \Gamma (t_k)$, and for each $G \subset \Omega_{t_1, ..., t_k}$ and $\varphi \in \mathcal{H}_\pi \setminus \{ 0 \}$ define

$$\mu_{\varphi}^{t_1, ..., t_k} (G) = \left\| \left( \sum_{(a_1, a_2) \in G} p_{a_1}^{t_{a_1}} ... p_{a_k}^{t_{a_k}} \varphi \right) \right\|^2.$$

The sum in this expression is well defined since, by Thm.H.28.2, a sum of orthogonal projections is a projection, and orthogonality comes from (2). Since $\varphi \in \mathcal{H}_\pi = \mathcal{H}_{\pi''}$, we have $\mu_{\varphi}^{t_1, ..., t_k} (\Omega_{t_1, ..., t_k}) = ||\varphi||^2 = 1$, so that these are indeed probability measures. These probability measures satisfy the following two consistency conditions.

First, suppose $G = G_1 \times ... \times G_k$, with $G_i \subset \Gamma (t_i)$, $i = 1, ..., k$. Let $\kappa$ be a permutation of the elements of the set $\{ 1, ..., k \}$, and set $\kappa (G) = G_{\kappa (1)} \times ... \times G_{\kappa (k)}$. Then $\mu_{\varphi}^{t_1, ..., t_k} (\kappa (G)) = \mu_{\varphi}^{t_{\kappa (1)}, ..., t_{\kappa (k)}} (G)$, simply because $p_{a_1}^{t_{\kappa (1)}} ... p_{a_k}^{t_{\kappa (k)}} = p_{a_1}^{t_{a_1}} ... p_{a_k}^{t_{a_k}}$.

Second, suppose that $G \subset \Omega_{t_1, ..., t_k}$, $t_{k+1} \in S \setminus \{ t_1, ..., t_k \}$, and set $G' = G \times \Gamma (t_{k+1}) \subset \Omega_{t_1, ..., t_{k+1}}$. Then (15) implies that
\[ \mu_{\varphi}^{t_1 \ldots t_{k+1}}(G') = \left( \sum_{(a_1, \ldots, a_k) \in G} \sum_{a_{k+1} \in \Gamma(t_{k+1})} \mu^{t_1 \ldots t_k}(G) \right)^2 = \left( \sum_{(a_1, \ldots, a_k) \in G} \mu^{t_1 \ldots t_k} \varphi \right)^2 = \mu_{\varphi}^{t_1 \ldots t_k}(G). \]

These consistencies establish, thanks to Kolmogorov’s extension theorem, the existence of a probability measure \( P_\varphi \) on \((\Omega, \Sigma)\) that satisfies \( P_\varphi((X_{t_1}, \ldots, X_{t_k}) \in G) = \mu_{\varphi}^{t_1 \ldots t_k}(G) \) and in particular the first equality in (6). The second equality there is a consequence of \( \varphi \in \mathcal{H}_\pi = \mathcal{H}' \).

Uniqueness of \( P_\varphi \) is proved as follows. If (6) holds, then, for any \( G \subset \cap \Gamma(t_1) \times \ldots \times \Gamma(t_k) \) and \( \varphi \in \mathcal{H}_\pi \setminus \{0\} \),

\[
P_\varphi((X_{t_1}, \ldots, X_{t_k}) \in G) = \sum_{(a_1, \ldots, a_k) \in G} ||p_{a_1 \ldots a_k}^{t_1 \ldots t_k} \varphi||^2 = \sum_{(a_1, \ldots, a_k) \in G} ||p_{a_1 \ldots a_k}^{t_1 \ldots t_k} \varphi||^2 = \mu_{\varphi}^{t_1 \ldots t_k}(G), \quad (19)
\]

where in the second equality we used the orthogonality \( (7) \). This defines uniquely the probability measure \( P_\varphi \) restricted to the algebra \( A \). And hence it can only have a unique extension to the sigma-algebra \( \Sigma \) generated by \( A \).

(c): For each \( A \in A \) set

\[
\mathcal{H}_A = \{ \varphi \in \mathcal{H}_\pi : \varphi = 0 \text{ or } P_\varphi(A) = 1 \}. \quad (20)
\]

We will show that each \( \mathcal{H}_A \) is a subspace and that if \( \bar{p}_A \) is the projection on \( \mathcal{H}_A \) and \( \bar{p}_A \) is its restriction to \( \mathcal{H}_\pi \), then \( \{ \bar{p}_A : A \in A \} \) is the unique p.v.m. from the algebra \( A \) to projections in \( \mathcal{H}_\pi \) that satisfies \( (7) \). The extension of these claims, from a p.v.m. on the algebra \( A \) to a p.v.m. on the sigma-algebra \( \Sigma \), as stated in the theorem, then follows from Theorem 5 in Section 5.

If \( A \in A \), then \( A = \{ (X_{t_1}, \ldots, X_{t_k}) \in G \} \), for some \( t_1, \ldots, t_k \) and \( G \). Since (6) has already been proved, (19) is also true and it implies the following. (Recall that \( \bar{p}_{a_1 \ldots a_k}^{t_1 \ldots t_k} \) denotes the restriction of \( p_{a_1 \ldots a_k}^{t_1 \ldots t_k} \) to \( \mathcal{H}_\pi \))

\[
\mathcal{H}_A = \left\{ \varphi \in \mathcal{H}_\pi : \sum_{(a_1, \ldots, a_k) \in G} ||\bar{p}_{a_1 \ldots a_k}^{t_1 \ldots t_k} \varphi||^2 = ||\varphi||^2 \right\} = \text{Range} \left( \sum_{(a_1, \ldots, a_k) \in G} \bar{p}_{a_1 \ldots a_k}^{t_1 \ldots t_k} \right). \quad (21)
\]

where, to obtain the second equality, we used the fact that a sum of orthogonal projections is a projection (Thm.H.28.2) combined with (2), and the characterization of the range of a projection as the set of vectors whose norms are not affected by the projection (Thm.H.26.2). As the range of a projection in \( \mathcal{H}_\pi \), the right-hand side of this equation is a subspace of \( \mathcal{H}_\pi \), and hence so is \( \mathcal{H}_A \). And the equation means that

\[
\bar{p}_A = \sum_{(a_1, \ldots, a_k) \in G} \bar{p}_{a_1 \ldots a_k}^{t_1 \ldots t_k}. \quad (22)
\]

Moreover, combining (19) and (22), we obtain that for any \( \varphi \in \mathcal{H}_\pi \setminus \{0\} \) and \( A \in A \),

\[
P_\varphi(A) = \left( \sum_{(a_1, \ldots, a_k) \in G} ||\bar{p}_{a_1 \ldots a_k}^{t_1 \ldots t_k} \varphi||^2 \right)^2 = ||\bar{p}_A \varphi||^2 = ||p_A \varphi||^2. \quad (23)
\]

If \( A, B \in A \) there exists \( t_1, \ldots, t_k \) such that \( A = \{ (X_{t_1}, \ldots, X_{t_k}) \in G_A \} \) and \( B = \{ (X_{t_1}, \ldots, X_{t_k}) \in G_B \} \), for some \( G_A \) and \( G_B \). If also \( A \cap B = \emptyset \), then \( G_A \cap G_B = \emptyset \) and the ranges in the right-hand side of (21) with \( G = G_A \) or \( G = G_B \) will be orthogonal to each other, due to (2), implying that \( \mathcal{H}_A \perp \mathcal{H}_B \). (In Section 5 this property is called (PVM6).)

We will prove next the two conditions that define \( \{ \bar{p}_A : A \in A \} \) as a p.v.m. (See Section 5 for this definition.)

(PVM1) From (20), it is immediate that \( \mathcal{H}_\Omega = \mathcal{H}_\pi \), as required.

(PVM2) If \( A_1, A_2, \ldots \) are disjoint events in \( A \), and also \( A = \cup_{i=1}^{\infty} A_i \in A \), then

\[
\mathcal{H}_A = \{ \varphi \in \mathcal{H}_\pi : \varphi = 0 \text{ or } P_\varphi(A) = 1 \} = \{ \varphi \in \mathcal{H}_\pi : \varphi = 0 \text{ or } \sum_i P_\varphi(A_i) = 1 \} = \left\{ \varphi \in \mathcal{H}_\pi : \sum_i ||\bar{p}_{A_i} \varphi||^2 = ||\varphi||^2 \right\} = \text{Range} \left( \sum_i \bar{p}_{A_i} \right), \quad (24)
\]
where in the third equality we used (24), in the fourth the equality we used the orthogonality due to the disjointness of the events $A_i$, (PVM6), proved above, and in the fifth equality we used the same kind of arguments used to justify the second equality in (21). Equation (24) implies that $\hat{p}_A = \sum_i p_{A_i}$, as required.

Uniqueness is immediate, since, by (PVM2) a p.v.m. that satisfies (7) must satisfy (22).

(d): Since $\{\hat{p}_A : A \in \Sigma\}$ is a p.v.m. in $\mathcal{H}_\pi$ and $\hat{\varphi} \in \mathcal{H}_\pi$, the right-hand side of (9) defines a measure on $(\Omega, \Sigma)$. Equation (25) states that this measure coincides with $\mathbb{P}_\varphi$ on the algebra $\mathcal{A}$. Hence it must coincide with $\mathbb{P}_\varphi$ on the sigma-algebra $\Sigma$, generated by $\mathcal{A}$, proving (9).

From Thm.H.26.3, that characterizes the range of a projection as the set of vectors whose norm is not affected by the projection, and (9) we have

$$\mathcal{H}_A = \{ \varphi \in \mathcal{H}_\pi : ||p_A \varphi||^2 = ||\varphi||^2 \} = \{ \varphi \in \mathcal{H}_\pi : \varphi = 0 \text{ or } ||p_A \varphi||^2 = 1 \} = \{ \varphi \in \mathcal{H}_\pi : \varphi = 0 \text{ or } \mathbb{P}_\varphi(A) = 1 \},$$

proving (10).

To prove (11), observe that the statement $\varphi \in \mathcal{H}_A^\perp$ is equivalent to the statement that $p_A \varphi = 0$. And since $\mathcal{H}_A \subset \mathcal{H}_\pi$, this last statement is equivalent to the statement that $\varphi = 0$, or $\mathbb{P}_{\varphi^*}(A) = ||p_A \varphi||^2/||\varphi||^2 = ||p_A \varphi||^2/||\varphi||^2 = 0$.

(e): Set $\eta = p_A \varphi$. Clearly $\eta \in \mathcal{H}_\pi$ and since $\mathbb{P}_\varphi(A) \neq 0$, we have, from (9) that $\eta \neq 0$. Therefore $\mathbb{P}_\eta(B)$ is well defined. Using now (9) and the property $p_{AB} = \hat{p}_{BA}$ of projection valued measures (called (PVM8) in Section 5), we have

$$\mathbb{P}_{\varphi}(B|A) = \mathbb{P}_{\varphi}(AB)/\mathbb{P}_{\varphi}(A) = ||p_{AB} \varphi||^2/||p_A \varphi||^2 = ||p_{BA} \varphi||^2/||p_A \varphi||^2 = ||p_B \eta||^2/||\eta||^2 = \mathbb{P}_\eta(B).$$

(f): All the statements follow from the fact that $\{p_A : A \in \Sigma\}$ is a p.v.m. in $\mathcal{H}_\pi$ that satisfies (9), by applying these results to Theorems 13.24 and 13.28 of [28].

$\square$

3 A characterization of $\mathcal{H}_A$ when $S$ is countable

For $A \subset \Omega$, we define

$$F_A = \{ \varphi \in \mathcal{H} : \text{for any } \omega \in A \text{ there is } t_1, ..., t_k \text{ such that } p_{\omega_{t_1}, ..., \omega_{t_k}} \varphi = 0 \}.$$ 

If $S = \{s_1, s_2, \ldots\}$ is countable, we can use (3) to see that

$$F_A = \{ \varphi \in \mathcal{H} : \text{for any } \omega \in A \text{ there is } k \in \{1, 2, \ldots\} \text{ such that } p_{s_1, \ldots, s_k} \varphi = 0 \}.$$ 

Recall that $p_\pi$ is the projection on $\mathcal{H}_\pi$ and that $\varphi_\pi = p_\pi \varphi$. Our goal in this section is to prove

**Theorem 2** If $S$ is countable and $A \in \Sigma$, then

(a) $\mathcal{H}_A = \mathcal{F}_A^\perp$ and $\mathcal{H}_A = \mathcal{F}_A$.

(b) $\varphi \in \mathcal{F}_A$ if and only if $\varphi = 0$, or $\mathbb{P}_{\varphi^*}(A) = 0$.

(c) $\mathcal{H}_A = \mathcal{F}_A \cap \mathcal{H}_\pi = \mathcal{F}_{A^*} \cap \mathcal{H}_\pi$.

This theorem will be partially extended to arbitrary $S$ in Section 4 building on the results in this section.

The proof of Theorem 2 will rely on several lemmas. First we collect some elementary properties of $F_A$ in a proposition:

**Proposition 3** $F_A$ decreases as $A$ increases. For any family $\{A_\alpha\}$ of subsets of $\Omega$, $F_{\bigcup_\alpha A_\alpha} = \bigcap_\alpha F_{A_\alpha}$. And for any $A \subset \Omega$, $F_A$ is a vector space, and $N \subset F_{\Omega} \subset F_A$.

The proof that $F_A$ is a vector space is analogous to that used for $N$ (Theorem 1 item (a)). The other statements are immediate. Note that $N$ relates to $F_\Omega$ by an interchange in the order of quantifiers, amounting to uniformity in the choice of $t_1, ..., t_k$ in the definition of $N$. (Subsection 8.7 explores this distinction.)
Lemma 1  For any $A \subset \Omega$, if $\varphi \in F_A$, then $\varphi_\pi \in F_A$.

Proof: For each $\omega \in A$ there exists $t_1, \ldots, t_k$ such that $p^{t_1, \ldots, t_k}_{\omega_1, \ldots, \omega_k} \varphi = 0$. Using part (b) of Proposition 2 we have

$$p^{t_1, \ldots, t_k}_{\omega_1, \ldots, \omega_k} \varphi_\pi = p^{t_1, \ldots, t_k}_{\omega_1, \ldots, \omega_k} p_\pi \varphi = p_\pi p^{t_1, \ldots, t_k}_{\omega_1, \ldots, \omega_k} \varphi = 0.$$  

This shows that $\varphi_\pi \in F_A$. $\square$

Lemma 2  For every $A \subset \Omega$,

$$F_A = \{ \varphi \in H : \varphi_\pi \in F_A \cap H_\pi \} = F_A \cap H_\pi \oplus H_\pi^\perp,$$  

(26)

and

$$F_A^\perp = (F_A \cap H_\pi)^\perp \cap H_\pi,$$  

(27)

and

$$F_A \cap H_\pi = F_A \cap H_\pi.$$  

(28)

Proof: The second equality in (26) is immediate, (27) follows from (26), by taking the orthogonal complement, and (28) follows from (26) by taking the intersection with $H_\pi$ on both sides. So we only need to prove the first equality in (26). This will be done in two steps:

\(\square\) If $\varphi \in F_A$, then there is a sequence $(\varphi_i)_{i=1,2,\ldots}$, such that $\varphi_i \in F_A$, $\varphi_i \to \varphi$. If we set $\xi_i = p_\pi \varphi_i$, then, by Lemma 1 $\xi_i \in F_A$, and hence $\xi_i \in F_A \cap H_\pi$. Since projections are continuous, $\xi_i \to p_\pi \varphi = \varphi_\pi$, and we conclude that $\varphi_\pi \in F_A \cap H_\pi$.

\(\square\) Suppose $\varphi$ is such that $\varphi_\pi \in F_A \cap H_\pi$. Then $\varphi_\pi \in F_A^\perp$. But also $\varphi - \varphi_\pi \in H_\pi^\perp = \varnothing \subset F_A$, where we used part (a) of Theorem 1 and Proposition 3. Therefore, $\varphi = \varphi_\pi + (\varphi - \varphi_\pi) \in F_A$. $\square$

Lemma 3  If $S$ is countable and $A \in A$, then $F_A \cap H_\pi = H_{A^c}$.

Proof: Since $A \in A$, it can be represented as $A = \{ (X_1, \ldots, X_k) \in G \} = \{ \omega \in \Omega : (\omega_1, \ldots, \omega_k) \in G \}$, for appropriate $t_1, \ldots, t_k \in S$ and $G \subset \Gamma(t_1) \times \ldots \times \Gamma(t_k)$. And from (21),

$$H_{A^c} = \{ \varphi \in H_\pi : \sum_{(a_1, \ldots, a_k) \in G^c} p^{t_1, \ldots, t_k}_{a_1, \ldots, a_k} \varphi = 0 \} = \{ \varphi \in H_\pi : \sum_{(a_1, \ldots, a_k) \in G} p^{t_1, \ldots, t_k}_{a_1, \ldots, a_k} \varphi = 0 \} = \{ \varphi \in H_\pi : p^{t_1, \ldots, t_k}_{a_1, \ldots, a_k} = 0 \text{ for every } (a_1, \ldots, a_k) \in G \}, \quad \text{for every } (a_1, \ldots, a_k) \in G \},$$  

(29)

where in the second equality we used the fact that $H_\pi \cap H_\pi^\perp = \varnothing$, from part (a) of Theorem 1 and in the third equality we used (2) and the fact that a sum of orthogonal vectors can only be 0 if all these vectors are 0.

It is clear from (29) that $H_{A^c} \subset F_A \cap H_\pi$.

Suppose now that $\varphi \not\in H_{A^c}$. We need to prove that then $\varphi \not\in F_A \cap H_\pi$. If $\varphi \not\in H_\pi$ we are done, so we will also assume now that $\varphi \in H_\pi$. Then (29) implies that there must exist some $(a_1, \ldots, a_k) \in G$, such that $p^{t_1, \ldots, t_k}_{a_1, \ldots, a_k} \varphi \neq 0$. Enumerate the elements of $S$ as $s_1, s_2, \ldots$, starting with $s_1 = t_1, \ldots, s_k = t_k$, and continuing in an arbitrary fashion. Using (15) we see that there exists $a_{k+1}$ such that $p^{t_1, \ldots, t_k}_{a_1, \ldots, a_k} \varphi \neq 0$. Proceeding inductively in this fashion, we conclude that there exists a sequence $a_1, a_2, \ldots$ such that $p^{t_1, \ldots, t_k}_{a_1, \ldots, a_l} \varphi \neq 0$, for every $l \geq k$. This extends to all $l \geq 1$, thanks to (3). If we define $\omega$ by $\omega_{a_i} = a_i$, $i = 1, 2, \ldots$, then $\omega \in A$, and, recalling (25), we have just proved that $\varphi \not\in F_A$. $\square$

Lemma 4  If $S$ is countable and $A \in A_\pi$, then $F_A \cap H_\pi = H_{A^c} = H_{A^c}^+ \cap H_\pi$, and $F_A^+ = H_A$.

Proof: We have $A = \bigcup A_i$, for a countable disjoint collection of sets $A_i \in A$. Now,

$$F_A \cap H_\pi = (\cap_i F_{A_i}) \cap H_\pi = \cap_i (F_{A_i} \cap H_\pi) = \cap_i H_{A_i^c}$$

$$= \cap_i (H_{A_i^c}^+ \cap H_\pi) = (\cap_i H_{A_i^c}^+) \cap H_\pi = ((\oplus_i H_{A_i})^+) \cap H_\pi = H_{A^c}^+ \cap H_\pi = H_{A^c}.$$
where in the first equality we used Proposition 3, in the third equality we used Lemma 3, in the fourth and eighth equalities we used (8), and in the sixth and seventh equalities we used properties of projection valued measures (called, respectively, (PVM6) and (PVM2) in Section 5).

Combining this result with (27), we obtain

\[ F_A^\perp = (H_A^\perp \cap H_\pi)^\perp \cap H_\pi = (H_A \oplus H_A^\perp) \cap H_\pi = H_A. \]

\[ \square \]

For \( \varphi \in H \), define

\[ \Omega(\varphi) = \left\{ \omega \in \Omega : \text{for all } t_1, \ldots, t_k, p_{\omega t_1}, \ldots, p_{\omega t_k} \varphi \neq 0 \right\}. \]

Then

\[ F_A = \{ \varphi \in H : A \subset \Omega^c(\varphi) \}. \quad (30) \]

**Lemma 5** If \( S \) is countable, then \( \Omega(\varphi) \in A_\delta \), for every \( \varphi \in H \).

**Proof:** Enumerate the elements of \( S \) as \( s_1, s_2, \ldots \). Then, (3) implies (as in (25)) that

\[ \Omega(\varphi) = \left\{ \omega \in \Omega : \text{for every } k \in \{1, 2, \ldots\}, p_{s_1, \ldots, s_k} \varphi \neq 0 \right\} = \bigcap_{k=1}^{\infty} \Omega_k(\varphi), \]

where \( \Omega_k(\varphi) = \{ \omega \in \Omega : p_{s_1, \ldots, s_k} \varphi \neq 0 \} \in A. \ \square \]

**Lemma 6** If \( S \) is countable, then, for every \( \varphi \in H \) and \( A \subset \Omega \),

(a) \( \Omega(\varphi) \subset A \iff \Omega(\varphi) \subset B \), for some \( B \in A_\delta \), \( B \subset A \).

(b) \( A \subset \Omega^c(\varphi) \iff C \subset \Omega^c(\varphi) \), for some \( C \in A_\sigma \), \( A \subset C \).

**Proof:**

(a): The \((\iff)\) part is obvious, since \( \Omega(\varphi) \subset B \subset A \).

For the \((\implies)\) part, set \( B = \Omega(\varphi) \), which belongs to \( A_\delta \) by Lemma 5. By assumption \( B \subset A \) and tautologically \( \Omega(\varphi) \subset B \).

(b): Apply (a) to \( A^c \) in place of \( A \) and set \( C = B^c \). \( \square \)

**Proof of Theorem 2**

(a): Since \( F_A \) is a vector space, the two statements are equivalent. We will prove the first one. Combining (30) with part (b) of Lemma 6 we have, for every \( A \subset \Omega \),

\[ F_A = \bigcup \{ F_C : C \in A_\sigma, A \subset C \}. \]

Taking the orthogonal complement on both sides and using Lemma 4 we obtain

\[ F_A^{\perp} = \bigcap \{ F_C^{\perp} : C \in A_\sigma, A \subset C \} = \bigcap \{ H_C : C \in A_\sigma, A \subset C \}. \]

In case \( A \in \Sigma \), this amounts to \( F_A^{\perp} = H_A \), as claimed, thanks to (45) in Theorem 5.

(b): Combine part (a) with (11) in part (d) of Theorem 1.

(c): From (8) we have \( H_A = H_\pi \cap H_\pi \). Hence

\[ H_A = H_A^\perp \cap H_\pi = F_A^\perp \cap H_\pi = F_A^\perp \cap H_\pi, \]

where in the second equality we used part (a) above, and the third equality is (28). \( \square \)

### 4 Partial extensions of Theorem 2 to arbitrary \( S \)

In this section we will partially extend Theorem 2 to arbitrary \( S \) in the following ways:
Theorem 3 If \( A \in \Sigma \), then

(a) \( \mathcal{H}_A^\perp \subset \overline{F_A} \).

(b) \( \varphi_\pi = 0 \), or \( \mathbb{P}_{\varphi_\pi}(A) = 0 \) \( \implies \) \( \varphi \in \overline{F_A} \).

If also \( \Gamma(t) \) is finite for all \( t \in S \), then

(c) \( \mathcal{H}_A = F_A^\perp \) and \( \mathcal{H}_A^\perp = \overline{F_A} \).

(d) \( \varphi \in \overline{F_A} \iff \varphi_\pi = 0 \), or \( \mathbb{P}_{\varphi_\pi}(A) = 0 \).

(e) \( \mathcal{H}_A = F_A^\perp \cap \mathcal{H}_\pi = \overline{F_A} \cap \mathcal{H}_\pi \).

The proof of Theorem 3 will build on the work done in Sections 2 and 3. The proof of the statements (c), (d) and (e) will use Zorn’s Lemma and therefore depend on the acceptance of the Axiom of Choice. Incidentally, if one does not accept this axiom, the very nature of the set \( \Omega \) becomes unclear, when \( S \) is not countable (see, e.g., [14], Section 2 in the Prologue).

It seems natural to conjecture that the limitation of parts (c), (d) and (e) of this theorem to \( \pi \) with finite \( \Gamma(t) \), for all \( t \in S \), is purely technical, so that Theorem 2 should fully extend to arbitrary \( \pi \).

We start with a few observations and new definitions. Given \( D \subset S \), let \( \Sigma^D \) be the sigma-algebra generated by \( \{X_t : t \in D\} \). Theorem 36.3.(ii) of [6] states that

\[
\Sigma = \bigcup \{\Sigma^D : D \subset S, D \text{ is countable}\}. \tag{31}
\]

This important fact can easily be proved, by noting that the right-hand side is a sigma-algebra and that any sigma-algebra that contains all the sets \( \{X_t = a\}, t \in S, a \in \Gamma(t) \), must contain this right-hand side.

Given \( D \subset S \), we define also

\[
F_A^D = \left\{ \varphi \in \mathcal{H} : \text{for any } \omega \in A \text{ there is } t_1, \ldots, t_k \in D \text{ such that } p_{t_1^\omega} \cdots t_k^\omega \varphi = 0 \right\},
\]

so that \( F_A^S = F_A \). Proposition 3 extends to \( F_A^S \), with the exception that \( N \) and \( F_A^D \) are not comparable when \( D \) is a proper subset of \( S \):

**Proposition 4** \( F_A^D \) decreases as \( A \) increases and increases as \( D \) increases. For any \( D \subset S \) and any family \( \{A_a\} \) of subsets of \( \Omega \), \( F_{\cup_a A_a}^D = \cap_a F_{A_a}^D \). And for any \( A \subset \Omega \) and \( D \subset S \), \( F_A^D \) is a vector space.

Again with \( D \subset S \), we also set

\[
\pi(D) = \{p_a^\omega : t \in D, a \in \Gamma(t)\}. \tag{32}
\]

And, in a self-explanatory fashion, we denote by \( \Omega(D) \), \( \Sigma(D) \) and \( \mathbb{P}_D^D \), \( \varphi \in \mathcal{H} \), the corresponding objects associated with \( \pi(D) \). Clearly

\[
\mathcal{H}_\pi \subset \mathcal{H}_{\pi(D)}. \tag{33}
\]

The next proposition collects some facts that are relatively easy consequences of the results in Sections 2 and 3. Note that, thanks to (31), for every \( A \in \Sigma \) there is some \( D \) in the conditions of this proposition. Note that some of the statements in this proposition do not involve \( D \); these are identical to statements (a) and (b) of Theorem 3. Part (a) of this proposition will be used in the proof of the other statements in Theorem 5.

**Proposition 5** If \( D \subset S \) is countable and \( A \in \Sigma^D \), then

(a) \( \mathcal{H}_A = F_A^D \cap \mathcal{H}_\pi = \overline{F_A} \cap \mathcal{H}_\pi \).

(b) \( F_A^D \subset \mathcal{H}_A \subset \overline{F_A} \).

(c) \( \varphi_\pi = 0 \), or \( \mathbb{P}_{\varphi_\pi}(A) = 0 \) \( \implies \) \( \varphi \in \overline{F_A} \).

If also \( \mathcal{H}_{\pi(D)} = \mathcal{H}_\pi \), then

(d) \( \mathcal{H}_A^\perp = F_A^D \).
(e) $\varphi \in \overline{F^D_A} \iff \varphi_\pi = 0$, or $P_{\varphi_\pi}(A) = 0$.

It is easy to produce examples in which the left-hand side containment in part (b) is not tight. For instance, making $\mathcal{H}_\pi = \{0\} \neq \mathcal{H}$ and taking $D$ with a single element. If $A = \Omega$, then $\overline{F^D_A} = \{0\} \neq \mathcal{H} = \mathcal{H}_\pi^+ = \mathcal{H}_\pi^\perp$.

For interesting examples in which the extra assumption needed in parts (d) and (e) holds, see Subsection 8.1 in which $S = \mathbb{R}$, and suppose that $D$ contains all the rationals (natural assumptions in applications to quantum mechanics).

To prove Proposition 5 we need one more concept. Given $D \subset S$, we define an equivalence relation in $\Omega$, by declaring as $D$-equivalent elements of $\Omega$ that have identical restrictions to $D$. We say that a set $A \subset \Omega$ is $D$-determined if any two $D$-equivalent elements of $\Omega$ either both belong to $A$, or neither one does. When $A \subset \Omega$ is $D$-determined we define

$$A(D) = \{ \omega \in \Omega(D) : \omega_t = \omega'_t \text{ for all } t \in D \text{ and some } \omega' \in A \},$$

so that

$$A = \{ \omega \in \Omega : (\omega_t)_{t \in D} \in A(D) \}.$$ 

Note that if $A$ is $D$-determined, then also $A^c$ has this property and

$$A^c(D) = (A(D))^c. \quad (34)$$

And if $\{A_\alpha\}$ is a family of disjoint $D$-determined subsets of $\Omega$, then also the sets $A_\alpha(D)$ are disjoint, $\bigcup_\alpha A_\alpha$ is $D$-determined and

$$(\bigcup_\alpha A_\alpha)(D) = \bigcup_\alpha (A_\alpha(D)). \quad (35)$$

Also, if $A$ is $D$-determined,

$$F_{A(D)} = \{ \varphi \in \mathcal{H} : \text{for any } \omega \in A(D) \text{ there is } t_1, \ldots, t_k \in D \text{ such that } p_{t_1}^{(1)} \ldots p_{t_k}^{(k)} \varphi = 0 \}$$

$$= \{ \varphi \in \mathcal{H} : \text{for any } \omega \in A \text{ there is } t_1, \ldots, t_k \in D \text{ such that } p_{t_1}^{(1)} \ldots p_{t_k}^{(k)} \varphi = 0 \} = F^D_A. \quad (36)$$

(Since $A(D) \subset \Omega(D)$, the notation $F_{A(D)}$ should be understood as identical to $F^D_{A(D)}$.)

**Lemma 7** For any $D \subset S$, if $A \in \Sigma^D$, then $A$ is $D$-determined, $A(D) \in \Sigma(D)$, and $P_{\varphi_\pi}(A(D)) = P_{\varphi}(A)$, for any $\varphi \in \mathcal{H}_\pi \setminus \{0\}$.

**Proof:** The proof is a simple application of the $\pi$-$\lambda$ Theorem (see, e.g., Theorem 3.2 of [6]).

Consider the following two classes of subsets of $\Omega$.

$$\mathcal{P} = \{ X_{t_1} = a_1, \ldots, X_{t_k} = a_k : t_1, \ldots, t_k \in D, a_1 \in \Gamma(t_1), \ldots, a_k \in \Gamma(t_k) \} \cup \{ \emptyset \}.$$

$$\mathcal{L} = \{ A \subset \Omega : A \text{ is } D\text{-determined, } A(D) \in \Sigma(D) \text{ and } P_{\varphi_\pi}(A(D)) = P_{\varphi}(A), \text{ for any } \varphi \in \mathcal{H}_\pi \setminus \{0\} \}.$$

It is clear that $\mathcal{P}$ is closed with respect to finite intersections, meaning that it is a $\pi$-system. And, using (34) and (35), it is also clear that $\mathcal{L}$ has the three properties that are required to be a $\lambda$-system: it contains $\Omega$, is close with respect to taking the complement and with respect to taking countable disjoint unions.

It is also not difficult to see that $\mathcal{P} \subset \mathcal{L}$. Each $A \in \mathcal{P}$ is clearly $D$-determined, $\emptyset(D) = \emptyset \in \Sigma(D)$, for $A = \{ X_{t_1} = a_1, \ldots, X_{t_k} = a_k \}$, $A(D) = \{ \omega \in \Omega(D) : \omega_{t_1} = a_1, \ldots, \omega_{t_k} = a_k \} \in \Sigma(D)$, and, if $\varphi \in \mathcal{H}_\pi \setminus \{0\}$, then also $\varphi \in \mathcal{H}_\pi(D) \setminus \{0\}$, by (35), and

$$P_{\varphi_\pi}(A(D)) = \| p_{t_k}^{(k)} \ldots p_{t_1}^{(1)} \varphi \| ^2 = P_{\varphi}(A),$$

where we used (6) in part (b) of Theorem 11 (for $\pi$ and for $\pi(D)$). And since the sigma-algebra generated by $\mathcal{P}$ is $\Sigma^D$, the $\pi$-$\lambda$ Theorem implies that $\Sigma^D \subset \mathcal{L}$. \(\square\)

**Lemma 8** For any $D \subset S$, if $A \in \Sigma^D$, then

$$\mathcal{H}_A = \mathcal{H}_{A(D)} \cap \mathcal{H}_\pi.$$
The next lemma is a counterpart for \( \omega \) with equality in case \( \varphi \). Our goal is to show that there is a good extension of \( \omega \). It is weaker than the latter one, because \( N \) and \( F_A^D \) are not comparable sets, when \( D \) is a proper subset of \( S \).

**Lemma 9** For any \( D \subset S \) and \( A \subset \Omega \),

(a) If \( \varphi \in F_A^D \), then \( \varphi \in F_A^D \).

(b) \( F_A^D \cap \mathcal{H}_\pi = F_A^D \cap \mathcal{H}_\pi \).

**Proof:** The proof of part (a) is analogous to the proof of Lemma 1. And part (b) is a simple consequence of part (a), since if \( \varphi \in \mathcal{H}_\pi \) can be approximated by \( \varphi_i \in F_A^D \), then it can also be approximated by \( p_\pi \varphi_i \in F_A^D \cap \mathcal{H}_\pi \). □

**Proof of Proposition 5**

(a):

\[
\mathcal{H}_A = \mathcal{H}_{A(D)} \cap \mathcal{H}_\pi = (\mathcal{F}_A^D(D) \cap \mathcal{H}_{\pi(D)}) \cap \mathcal{H}_\pi = \mathcal{F}_A^D(D) \cap \mathcal{H}_\pi = \mathcal{F}_A^D \cap \mathcal{H}_\pi,
\]

where the first equality is from Lemma 8, the second one is from part (c) of Theorem 2 (applied to \( \pi(D) \)), combined with (33), the third one is from (33), the fourth one is from (36), which can be used thanks to Lemma 7 the fifth one is from and Lemma 9.

(b) - (e): Using (3) and part (a) above,

\[
\mathcal{H}_A = \mathcal{H}_{A(D)} \cap \mathcal{H}_\pi = \left( \mathcal{F}_A^D(D) \cap \mathcal{H}_{\pi(D)} \right) \cap \mathcal{H}_\pi = \mathcal{F}_A^D \cap \mathcal{H}_\pi = \mathcal{F}_A^D \cap \mathcal{H}_\pi.
\]

But \( F_A^D \subset F_A \) and also \( N \subset F_\Omega \subset F_A \). Therefore we obtain \( \mathcal{H}_A \subset \mathcal{F}_A \).

Combining this with (17) in part (d) of Theorem 1 we obtain the statement in part (c) of the proposition.

By Lemma 8, parts (c) and (a) of Theorem 2 (applied to \( \pi(D) \)), and (36) again,

\[
\mathcal{H}_A = \mathcal{H}_{A(D)} \cap \mathcal{H}_\pi \subset \mathcal{H}_{A(D)} = \mathcal{F}_A^D(D) = (F_A^D)^{\perp},
\]

with equality in case \( \mathcal{H}_\pi = \mathcal{H}_{\pi(D)} \). Taking the orthogonal complement, we complete the proof of (b) and (d). Part (e) follows from part (d) and (17). □

The main technical work in the proof of parts (c), (d) and (e) of Theorem 3 is contained in the proof of the following theorem, that is interesting also in its own right.

**Theorem 4** Suppose that \( \Gamma(t) \) is finite, for every \( t \in S \). For any \( D \subset S \), if \( A \subset \Omega \) is \( D \)-determined, then

\[
F_A \cap \mathcal{H}_\pi = F_A^D \cap \mathcal{H}_\pi.
\]

**Proof:** Clearly we only have to prove that the left-hand side is contained in the right-hand side. So we suppose that \( \varphi \notin F_A^D \cap \mathcal{H}_\pi \) and will prove that \( \varphi \notin F_A \cap \mathcal{H}_\pi \). If \( \varphi \notin \mathcal{H}_\pi \) we are done, so we also assume \( \varphi \in \mathcal{H}_\pi \), which implies that we are assuming that \( \varphi \notin F_A^D \).

This assumption states that there is \( \omega = (\omega_t)_{t \in D} \in A(D) \) such that \( p_{\omega_1,\ldots,\omega_t} \varphi \neq 0 \), for any \( t_1, \ldots, t_k \in D \). Since \( A \) is \( D \)-determined, any extension of this \( \omega \) to \( (\omega_t)_{t \in S} \) will be an element of \( A \).

We will be considering extensions of \( \omega \) to \( (\omega_t)_{t \in T} \), \( D \subset T \subset S \). Such an extension will be said to be “good” if \( p_{\omega_1,\ldots,\omega_k} \varphi \neq 0 \), for any \( t_1, \ldots, t_k \in T \). Our goal is to show that there is a good extension of \( \omega \) on \( T = S \).
We proceed now in typical Zorn-Lemma-application fashion. Partially order the good extensions of \( \omega \), by declaring 
\[(\omega^1_t)_{t \in T^1} \leq (\omega^2_t)_{t \in T^2}, \text{ for all } t \in T^1, \text{ whenever } T^1 \subseteq T^2 \text{ and } \omega^1_t = \omega^2_t, \text{ for all } t \in T^1. \]
Given a linearly ordered family of good extensions, \( \{\omega^\lambda : \lambda \in \Lambda\} \), where \( \Lambda \) is some index set, we can present an upper bound for it as follows. Let \( T^\Lambda \) be the domain of \( \omega^\Lambda \). Set \( T^\Lambda = \bigcup_{\Lambda \in \Lambda} T^\Lambda \), and define \( \omega^\Lambda \) as the extension of \( \omega \) on \( T^\Lambda \) given by \( \omega^\Lambda_t = \omega^\Lambda_t \), where \( \lambda \) is such that \( t \in T^\Lambda \). The fact that \( \{\omega^\lambda : \lambda \in \Lambda\} \) is a linearly ordered family assures the consistency of this definition. It is clear that \( \omega^\Lambda \) is a good extension of \( \omega \) and that it is an upper bound for the family \( \{\omega^\lambda : \lambda \in \Lambda\} \).

Zorn’s Lemma therefore implies the existence of a maximal good extension of \( \omega \), that we denote by \( \omega^M \), and whose domain we denote by \( T^M \).

If \( T^M = S \), then the existence of the good \( \omega^M = (\omega^M_t)_{t \in S} \in A \) means that \( \varphi \notin F_A \), and we are done.

So suppose instead that \( T^M \neq S \). Then there exist \( s \in S \setminus T^M \). And for any such \( s \), any extension of \( \omega^M \) to \( T^M \cup \{s\} \) must not be good, by the maximality of \( \omega^M \) in the class of good extensions of \( \omega \). For \( a \in \Gamma(s) \), let \( \omega^a \) be the extension of \( \omega^M \) to \( T^M \cup \{s\} \) defined by \( \omega^a_s = a \). As \( \omega^M \) is good and \( \omega^a \) is not good, there exists \( k(a) \) and \( t_1^a, \ldots, t_{k(a)}^a \in T^M \) such that

\[
p_{\omega^M_{t_1^a}, \ldots, \omega^M_{t_{k(a)}^a}, \varphi} = 0. \tag{38}\]

Let \( \{t_1, \ldots, t_k\} = \bigcup_{a \in \Gamma(s)} \{t_1^a, \ldots, t_{k(a)}^a\} \). Since \( \Gamma(s) \) is finite, this set is also finite, and using (3) and (38), we obtain

\[
p_{\omega^M_{t_1}, \ldots, \omega^M_{t_k}, \varphi} = 0, \]

for each \( a \in \Gamma(s) \). Summing over \( a \in \Gamma(s) \), using (15), applicable since \( \varphi \in H_1 \), we obtain

\[
p_{\omega^M_{t_1}, \ldots, \omega^M_{t_k}, \varphi} = 0. \]

Since \( t_1, \ldots, t_k \in T^M \), this is in contradiction with the fact that \( \omega^M \) is good. This contradiction shows that the maximality of \( \omega^M \) implies \( T^M = S \), and concludes the proof of (37). \( \square \)

**Proof of Theorem 5** Parts (a) and (b) are contained in Proposition 5.

Since \( A \in \Sigma \), (31) implies that there exists \( D \subset S \) countable such that \( A \in \Sigma^D \). We can therefore combine part (a) of Proposition 5 with Theorem 4 and part (b) of Lemma 3 to prove part (e):

\[
\mathcal{H}_A = F_A^\perp \cap H_\pi = F_A^\perp \cap H_\pi = F_A^\perp \cap H_\pi.
\]

We can now use (26) to prove one of the equivalent statements in part (c):

\[
F_A = F_A^\perp \cap H_\pi + H_A^\perp = H_A^\perp + H_A^\perp = H_A^\perp,
\]

where the last step is from (8).

Finally, part (d) follows from part (a) and (11) in part (d) of Theorem 1. \( \square \)

## 5 Extension of projection valued measures

In this section we will prove an analogue of Carathéodory’s extension theorem for projection valued measures. (For the classical Carathéodory’s extension theorem for measures, see, e.g., Section 4 of Chapter 1 of [14], or Section 2 of Chapter 12 of [27], or Section 3 of Chapter 1 of [3].) Our setting includes a Hilbert space \( \mathcal{H} \), an arbitrary set \( \Omega \), and a family \( \mathcal{A} \) of subsets of \( \Omega \) that form an algebra. Those do not have to be the ones that appeared in other sections of this paper. This section of the paper is independent of the other sections, except in the first paragraph of Section 2.

Let \( \{p_A : A \in \mathcal{A}\} \) be a set of projections in \( \mathcal{H} \), and, for each \( A \in \mathcal{A} \), denote by \( \mathcal{H}_A \) the range of \( p_A \). We say that \( \{p_A : A \in \mathcal{A}\} \) is a projection valued measure (p.v.m.; called a “spectral measure” in [19], and a “resolution of the identity” in [27]), if it satisfies the following two axioms:

(PVM1) \( p_\Omega = I \), the identity operator.

(PVM2) If \( A_i \in \mathcal{A}, i = 1, 2, \ldots \) are disjoint sets in \( \mathcal{A} \) and also \( A = \bigcup_{i=1}^\infty A_i \in \mathcal{A} \), then \( p_A = \sum_{i=1}^\infty p_{A_i} \).

(If \( \mathcal{A} \) is a sigma-algebra, the condition \( A \in \mathcal{A} \) in (PVM2) is redundant. Often one reserves the name p.v.m. only for this case. But for the purpose in this paper it is more natural to also define a p.v.m. indexed by an algebra \( \mathcal{A} \), as done above.)

From the axioms (PVM1) and (PVM2) a number of other properties can be deduced, including the following, where all sets are assumed to be in \( \mathcal{A} \):
(PVM3) $p_\emptyset = 0$, the operator that maps every vector to the 0 vector.

(PVM4) If $A_1, \ldots, A_n$ are disjoint sets, then $p_A = \sum_{i=1}^n p_{A_i}$.

(PVM5) $p_A + p_{A^c} = I$, or equivalently, $\mathcal{H}_{A^c} = \mathcal{H}_A^\perp$.

(PVM6) If $A \cap B = \emptyset$, then $H_A \perp H_B$, or equivalently, $p_A p_B = p_B p_A = 0$.

(PVM7) If $A \subset B$, then $H_A \subset H_B$.

(PVM8) $p_{A \cap B} = p_A p_B = p_B p_A$, so that in particular all the $p_A$, $A \in \mathcal{A}$, commute with each other.

(PVM3) follows from (PVM2) by taking $A_i = \emptyset$, for all $i$. (PVM4) follows from (PVM2) and (PVM3) by taking $A_i = \emptyset$ for $i > n$. (PVM5) follows from (PVM1) and (PVM4). (PVM6) follows from (PVM4) and Thm. H.28.2, according to which a sum of projections is a projection if and only if the added projections are orthogonal to each other. (PVM7) follows from (PVM4) and (PVM6), as they imply $H_B = H_A \oplus H_B \cap A^c$. (PVM8) follows from using (PVM4) for writing $p_A = p_{A \cap B} + p_{A \cap B^c}$, then multiplying both sides by $p_B$, once on the left, once on the right, and then using (PVM6) for $B$ and $A \cap B^c$, and (PVM7) for $B$ and $A \cap B$.

Later, when dealing with more than one algebra, we will use the notation $\mathcal{A}$-(PVM$x$) to indicate the statement (PVM$x$) for sets assumed to be in $\mathcal{A}$.

For each $\varphi \in \mathcal{H}$ and $A \in \mathcal{A}$ define

$$M_{\varphi}(A) = ||p_A \varphi||^2.$$  (39)

Then we have, from (PVM1), (PVM2), (PVM3) and (PVM6), that

(M1) $M_{\varphi}(\emptyset) = 0$ and $M_{\varphi}(\Omega) = ||\varphi||^2$.

(M2) If $A_i \in \mathcal{A}$, $i = 1, 2, \ldots$ are disjoint sets and also $A = \bigcup_{i=1}^\infty A_i \in \mathcal{A}$, then $M_{\varphi}(A) = \sum_{i=1}^\infty M_{\varphi}(A_i)$.

Therefore $\{M_{\varphi} : A \in \mathcal{A}\}$ is a finite measure on $\mathcal{A}$.

Given $\varphi \in \mathcal{H}$, for each $A \subset \Omega$, we define its outer measure relative to $M_{\varphi}$ by

$$M^*_\varphi(A) = \inf \left\{ \sum_{i=1}^\infty M_{\varphi}(A_i) : A \subset \bigcup_{i=1}^\infty A_i, A_i \in \mathcal{A}, i = 1, 2, \ldots \right\}.$$  (40)

The set of $M^*_\varphi$-measurable sets is defined as

$$\mathcal{M}_{\varphi} = \left\{ A \subset \Omega : M^*_\varphi(B) = M^*_\varphi(B \cap A) + M^*_\varphi(B \cap A^c), \text{ for all } B \subset \Omega \right\},$$

and turns out to be a sigma-algebra that contains $\mathcal{A}$. Therefore, if we denote by $\Sigma$ the sigma-algebra generated by $\mathcal{A}$, we have $\mathcal{A} \subset \Sigma \subset \mathcal{M}_{\varphi}$, for each $\varphi \in \mathcal{H}$.

For each $A \in \mathcal{A}$, we have

$$M_{\varphi}(A) = M^*_\varphi(A),$$  (41)

so that this equality can be extended consistently as a definition of $M_{\varphi}(A)$, for $A \in \mathcal{M}_{\varphi}$.

Carathéodory’s extension theorem states that, for each $\varphi \in \mathcal{H}$, $\{M_{\varphi}(A) : A \in \mathcal{M}_{\varphi}\}$ is a measure, which extends the measure $\{M_{\varphi}(A) : A \in \mathcal{A}\}$. Furthermore, uniqueness holds on $\Sigma$, in that $\{M_{\varphi}(A) : A \in \Sigma\}$ is the only extension of $\{M_{\varphi}(A) : A \in \mathcal{A}\}$ to a measure on $\Sigma$.

Denote by $\mathcal{A}_{\varphi}$ the family of subsets of $\Omega$ that can be expressed as countable unions of sets in $\mathcal{A}$. It is clear from Carathéodory’s extension theorem and the definitions above that, for $A \in \mathcal{M}_{\varphi}$,

$$M_{\varphi}(A) = \inf \{ M_{\varphi}(B) : B \in \mathcal{A}_{\varphi}, A \subset B \}.$$  (42)

Our goal in this section is to prove a counterpart to Carathéodory’s extension theorem and the identity (42) for p.v.m. Define $\mathcal{M} = \cap_{\varphi \in \mathcal{H}} \mathcal{M}_{\varphi}$. Then $\mathcal{M}$ is also a sigma-algebra and $\mathcal{A} \subset \mathcal{M} \subset \mathcal{M}$. Our main result in this section is:

**Theorem 5** Suppose that $\{p_A : A \in \mathcal{A}\}$ is a p.v.m. Then there exists a p.v.m. $\{p_A : A \in \mathcal{M}\}$ that extends it to $\mathcal{M}$. For any $\varphi \in \mathcal{H}$ and $A \in \mathcal{M}$ we have

$$||p_A \varphi||^2 = M_{\varphi}(A).$$  (43)

For $A \in \mathcal{M}$ the range of $p_A$ is

$$\mathcal{H}_A = \{ \varphi \in \mathcal{H} : M_{\varphi}(A) = ||\varphi||^2 \},$$  (44)
and the following relation holds:
\[ H_A = \bigcap \{ H_B : B \in \mathcal{A}_\sigma, A \subset B \}. \] (45)

Furthermore, \( \{ p_A : A \in \Sigma \} \) is the unique \( p.v.m. \) that extends \( \{ p_A : A \in \mathcal{A} \} \) to \( \Sigma \).

Before we can prove this theorem, we need to prove some properties of projections, the first of which is well known, but for which we could not find a reference.

Given a sequence of subspaces \( (S_i)_{i=1,2,...} \), we will indicate with \( S_i \not\supset A \) the statement that \( S_i \subset S_{i+1}, i = 1,2,..., \) and \( S = \bigcup_{i=1}^{\infty} S_i. \) And we will indicate with \( S_i \not\subset A \) the statement that \( S_{i+1} \subset S_i, i = 1,2,..., \) and \( S = \bigcap_{i=1}^{\infty} S_i. \)

**Proposition 6** Suppose that \( S_i, i = 1,2,... \) and \( S \) are subspaces and denote by \( p_i \) the projection on \( S_i \) and by \( p \) the projection on \( S \). Suppose that \( \varphi \in H \).

(a) If \( S_i \not\supset A \), then
\[
\lim_{i \to \infty} p_i \varphi = p \varphi.
\] (46)

(b) If \( S_i \not\subset A \), then (46) holds as well.

**Proof:** (a): Since \( S_i \subset S \)
\[ \|p \varphi - p_i \varphi\| = \|p \varphi - p(p \varphi)\| = \text{dist}(p \varphi, S_i), \]
where the right-hand side is the distance between the point \( p \varphi \) and the subspace \( S_i \).

Since \( p \varphi \in S = \bigcup_{i=1}^{\infty} S_i \), for any \( \epsilon > 0 \) there is \( \zeta \in \bigcup_{i=1}^{\infty} S_i \) such that \( \|p \varphi - \zeta\| \leq \epsilon \). But this implies that, there is \( j \) such that \( \zeta \in S_j \) and hence \( \text{dist}(p \varphi, S_j) \leq \epsilon \). Since \( S_j \) increases with \( i \), we conclude that for \( i \geq j \),
\[ \text{dist}(p \varphi, S_i) \leq \epsilon. \] (48)

Combining (47) with (48) proves (46).

(b): Apply part (a) to \( S_i^\perp \) and \( S^\perp \). \( \square \)

**Theorem 6** Suppose that \( \{ S_\alpha \}_{\alpha \in \Lambda} \) is a family of subspaces of \( H \), where \( \Lambda \) is an arbitrary index set. Assume that it satisfies the following condition: For any \( \alpha, \beta \in \Lambda \), there exists \( \gamma \in \Lambda \) such that \( S_\gamma \subset S_\alpha \cap S_\beta \). For each \( \alpha \in \Lambda \), denote by \( p_\alpha \) the projection on \( S_\alpha \), and let \( p_\Lambda \) be the projection on \( \bigcap_{\alpha \in \Lambda} S_\alpha \). Then, for any \( \varphi \in H \),
\[ \|p_\Lambda \varphi\| = \inf_{\alpha \in \Lambda} \|p_\alpha \varphi\|. \] (49)

**Remark:** The need for some condition on the family \( \{ S_\alpha \}_{\alpha \in \Lambda} \) in this theorem is made clear by a simple counter-example in which the family contains only two orthogonal subspaces \( S \) and \( S^\perp \), and \( \varphi \) is not contained in either one of these. In this case the left-hand side of (49) is 0, while the right-hand side is positive.

**Proof:** There exists a sequence of indices \( (\alpha_i)_{i=1,2,...} \) such that \( \|p_\alpha \varphi\| \to \inf_{\alpha \in \Lambda} \|p_\alpha \varphi\|, \) as \( i \to \infty \). Set \( \beta_1 = \alpha_1 \), and for \( i = 2,3,... \), recursively choose \( \beta_i \) such that \( S_{\beta_i} \subset S_{\alpha_i} \cap S_{\beta_{i-1}} \). Since \( S_{\beta_i} \subset S_{\alpha_i} \), we have \( \inf_{\alpha \in \Lambda} \|p_\alpha \varphi\| \leq \|p_{\beta_i} \varphi\| \leq \|p_{\alpha_i} \varphi\| \), and hence
\[ \lim_{i \to \infty} \|p_{\beta_i} \varphi\| = \inf_{\alpha \in \Lambda} \|p_\alpha \varphi\|. \] (50)

Let \( q \) be the projection on \( \bigcap_{i=1}^{\infty} S_{\beta_i} \), and \( \eta = q \varphi. \) Since \( S_{\beta_i} \subset S_{\beta_{i-1}}, i = 2,3,... \), we have from part (b) of Proposition 6 that \( \eta = \lim_{i \to \infty} p_{\beta_i} \varphi \), and therefore, using (50),
\[ \|\eta\| = \inf_{\alpha \in \Lambda} \|p_\alpha \varphi\|. \] (51)

Since \( \bigcap_{\alpha \in \Lambda} S_\alpha \subset \bigcap_{i=1}^{\infty} S_{\beta_i} \), we have
\[ \|p_\Lambda \varphi\| = \|p_\Lambda q \varphi\| = \|p_\Lambda \eta\|. \] (52)

If we had \( \eta \in \bigcap_{\alpha \in \Lambda} S_\alpha \), we would have \( p_\Lambda \eta = \eta \), and then from (52) and (51),
\[ \|p_\Lambda \varphi\| = \|\eta\| = \inf_{\alpha \in \Lambda} \|p_\alpha \varphi\|. \]

Therefore, for (49) to be false, there must exist \( \gamma \in \Lambda \) such that \( \eta \notin S_{\gamma} \). Assuming this to be the case, choose \( \delta_1 \) such that \( S_{\delta_1} \subset S_{\beta_1} \cap S_{\gamma} \), and for \( i = 2,3,... \), recursively choose \( \delta_i \) such that \( S_{\delta_i} \subset S_{\beta_i} \cap S_{\delta_{i-1}} \). Let \( r \) be the projection on \( \bigcap_{i=1}^{\infty} S_{\delta_i} \). We would then have
\[ \inf_{\alpha \in \Lambda} \|p_\alpha \varphi\| \leq \lim_{i \to \infty} \|p_{\delta_i} \varphi\| = \|r \varphi\| \leq \|r q \varphi\| = \|r \eta\| \leq \|p_\gamma \eta\| < \|\eta\|, \] (53)
where in the second step we used part (b) of Proposition 6 and the fact that \( S_{\delta_i} \subset S_{\delta_{i-1}}, \ i = 2, 3, \ldots \), in the third step we used the fact that \( S_{\delta_i} \subset S_{\delta_1}, \ i = 1, 2, \ldots \), and therefore \( \bigcap_{i=1}^{\infty} S_{\delta_i} \subset \bigcap_{i=1}^{\infty} S_{\delta_1} \), in the fifth step we used the fact that \( S_{\delta_1} \subset S_i, \ i = 1, 2, \ldots \), and therefore \( \bigcap_{i=1}^{\infty} S_{\delta_1} \subset S_i \), and in the sixth step we used the assumption that \( \eta \notin S_i \).

The contradiction between (51) and (53) shows that the assumption that led to (53) must be false, and therefore (49) must be true. \( \Box \)

**Proof of Theorem 5**

Thm.H.26.3 implies that for any projection \( p \),

\[
\text{Range}(p) = \{ \varphi \in H : ||p\varphi|| = ||\varphi|| \}. \tag{54}
\]

This and the definition of \( M_\varphi(A) \) when \( A \in \mathcal{A} \), (39), show that we can define

\[
\mathcal{H}_A = \{ \varphi \in H : M_\varphi(A) = ||\varphi||^2 \}, \tag{55}
\]

for all \( A \in \mathcal{M} \), consistently with the previous definition in case \( A \in \mathcal{A} \) (in the second paragraph of this section). Once we show that for each \( A \in \mathcal{M} \), \( \mathcal{H}_A \) is a subspace, we can, also consistently, define \( p_A \) as the projection on \( \mathcal{H}_A \). For later use, note that \( \mathcal{M}(\text{PVM7}) \) is satisfied, since for any \( A \in \mathcal{M} \), \( M_\varphi(A) \leq M_\varphi(\Omega) = ||\varphi||^2 \).

We will show next that for each \( A \in \mathcal{M} \), \( \mathcal{H}_A \) is indeed a subspace and

\[
M_\varphi(A) = ||p_A \varphi||^2. \tag{56}
\]

This will be done in two steps. First we consider \( A \in \mathcal{A}_\sigma \). In this case we can write \( A = \bigcup_{i=1}^{\infty} A_i \), where \( A_i \in \mathcal{A}, \ i = 1, 2, \ldots \) are disjoint sets. Hence

\[
M_\varphi(A) = \sum_{i=1}^{\infty} M_\varphi(A_i) = \sum_{i=1}^{\infty} ||p_{A_i} \varphi||^2 = \left| \left| \sum_{i=1}^{\infty} p_{A_i} \varphi \right| \right|^2, \tag{57}
\]

where in the second equality we used the definition of \( M_\varphi \), and in the third equality we used the orthogonality stated in \( \mathcal{A}(\text{PVM6}) \). Thm.H.28.2 states that a sum of orthogonal projections is a projection. Therefore \( \sum_{i=1}^{\infty} p_{A_i} \) is a projection and from (55), (57) and (54),

\[
\mathcal{H}_A = \left\{ \varphi \in H : \left| \left| \sum_{i=1}^{\infty} p_{A_i} \varphi \right| \right|^2 = ||\varphi||^2 \right\} = \text{Range} \left( \sum_{i=1}^{\infty} p_{A_i} \right), \tag{58}
\]

implying that \( \mathcal{H}_A \) is indeed a subspace and that

\[
p_A = \sum_{i=1}^{\infty} p_{A_i}. \tag{59}
\]

Feeding (59) back into (57), we obtain (56) in case \( A \in \mathcal{A}_\sigma \).

We turn now to general \( A \in \mathcal{M} \). Define \( \mathcal{A}_\sigma(A) = \{ B \in \mathcal{A}_\sigma : A \subset B \} \). Since \( M_\varphi(B) \leq M_\varphi(\Omega) = ||\varphi||^2 \) for any \( B \in \mathcal{M} \), identity (42), in conjunction with (55), implies that

\[
\mathcal{H}_A = \{ \varphi \in H : M_\varphi(B) = ||\varphi||^2 \text{ for all } B \in \mathcal{A}_\sigma(A) \} = \bigcap \{ \mathcal{H}_B : B \in \mathcal{A}_\sigma(A) \}. \tag{60}
\]

Since intersections of subspaces are subspaces, this implies that \( \mathcal{H}_A \) is a subspace.

We will now apply Theorem 6 with \( \Lambda = \mathcal{A}_\sigma(A) \), and for \( B \in \Lambda, \ S_B = \mathcal{H}_B \). To verify the condition in that theorem, given \( B', B'' \in \Lambda \), take \( B = B' \cap B'' \), which does belong to \( \Lambda \), since \( A \subset B \) and intersections of finitely many elements of \( \mathcal{A}_\sigma \) are also in \( \mathcal{A}_\sigma \). And since we already know that \( \mathcal{M}(\text{PVM7}) \) holds (as observed after (55)), we have \( \mathcal{H}_B \subset \mathcal{H}_{B'} \cap \mathcal{H}_{B''} \), as required. Theorem 6 and (60) give us then

\[
||p_B \varphi||^2 = \inf \{ ||p_B \varphi||^2 : B \in \mathcal{A}_\sigma(A) \} = \inf \{ M_\varphi(B) : B \in \mathcal{A}_\sigma(A) \} = M_\varphi(A),
\]

where in the second step we used the fact that (56) has already been proved for sets in \( \mathcal{A}_\sigma \), and in the last step we used (42). This concludes the proof that (56) holds for every \( A \in \mathcal{M} \).

Our next task is to show that (56) implies \( \mathcal{M}(\text{PVM2}) \). Computations and arguments identical to the ones involving (57), (58) and (59) show that this task will be fulfilled if we prove that \( \mathcal{M}(\text{PVM6}) \) holds. To do it, first we observe that, using (56), we obtain, for each \( A \in \mathcal{M} \),

\[
\mathcal{H}_{A^c} = \{ \varphi \in H : M_\varphi(A^c) = ||\varphi||^2 \} = \{ \varphi \in H : M_\varphi(A) = 0 \} = \{ \varphi \in H : ||p_A \varphi||^2 = 0 \} = \mathcal{H}_A^+. \tag{61}
\]
Since we already know that $\mathcal{M}$-(PVM7) holds (as observed after (55)), we have that if $A \cap B = \emptyset$, then $B \subset A^\sigma$, and hence $\mathcal{H}_B \subset \mathcal{H}_{A^\sigma}$. Therefore (61) implies $\mathcal{H}_A \perp \mathcal{H}_B$. This establishes $\mathcal{M}$-(PVM6) and completes the proof of $\mathcal{M}$-(PVM2).

Since $\mathcal{M}$-(PVM1) is the same as $\mathcal{A}$-(PVM1), it is already assumed to be true, and we have completed the proof that \{p_A : A \in \mathcal{M}\} is a p.v.m..

This proof also provided us with the claims (43), (44) and (45), which appeared above as (56), (55) and (60), respectively.

To show the uniqueness of the extension to $\Sigma$, suppose that \{p'_A : A \in \Sigma\} is a p.v.m. that extends \{p_A : A \in \mathcal{A}\}. For $\varphi \in \mathcal{H}$ and $A \in \Sigma$, define $M'_\varphi(A) = ||p'_A\varphi||^2$. Then \{M'_\varphi(A) : A \in \Sigma\} is a measure on $\Sigma$ that agrees with $M_\varphi(A)$ when $A \in \mathcal{A}$. By uniqueness of extension of finite measures from the algebra $\mathcal{A}$ to the sigma-algebra $\Sigma$ that it generates, we must also have $M'_\varphi(A) = M_\varphi(A)$, for all $A \in \Sigma$.

Using now (54), we have, when $A \in \Sigma$,

$$\text{Range}(p'_A) = \{\varphi \in \mathcal{H} : M'_\varphi(A) = ||\varphi||^2\} = \{\varphi \in \mathcal{H} : M_\varphi(A) = ||\varphi||^2\} = \text{Range}(p_A),$$

showing that $p'_A = p_A$. $\square$

**Remark on alternative proof:** The proof of the existence part given above goes in steps, from $\mathcal{A}$ to $\mathcal{A}_{\sigma}$ to $\mathcal{M}$. There is a more direct approach, at the cost of more abstraction, that is worth pointing out. We will only indicate the ideas, leaving the details to the interested reader.

The key tool is again Theorem 6. But now, given $A \in \mathcal{M}$, we take $\Lambda = \Lambda(A)$ given by

$$\Lambda = \{\{A_i\}_{i=1,...} : A_i \in \mathcal{A}, A_i \cap A_j = \emptyset \text{ if } i \neq j, A \subset \cup_i A_i\}.$$ 

And for $\alpha = \{A_i\}_{i=1,...} \in \Lambda$, we define

$$S_\alpha = \bigoplus_{i=1}^\infty \mathcal{H}_{A_i}.$$ 

Given two elements of $\Lambda$: $\alpha = \{A_i\}_{i=1,...}$ and $\beta = \{B_j\}_{j=1,...}$, one can check that if $\gamma = \{A_i \cap B_j\}_{i,j=1,...}$, then $\gamma \in \Lambda$ and $S_\gamma \subset S_\alpha \cap S_\beta$. Therefore we can apply Theorem 6. This gives us, for any $\varphi \in \mathcal{H}$,

$$||p_\Lambda \varphi||^2 = \inf_{\alpha \in \Lambda} ||p_\alpha \varphi||^2 = \inf \left\{ \sum_{i} ||p_{A_i} \varphi||^2 : \{A_i\}_{i=1,...} \in \Lambda \right\} = \inf \left\{ \sum_{i} M_\varphi(A_i) : \{A_i\}_{i=1,...} \in \Lambda \right\} = M_\varphi(A), \quad (62)$$

where in the third equality we used (59) and in the fourth equality we used (60), the observation that the infimum is not altered by taking only disjoint sets $A_i$, and the definition of $M_\varphi(A)$, for $A \in \mathcal{M}_\varphi$, given by (61). By its definition, $p_\Lambda$ is the projection on $\cap_{\alpha \in \Lambda} S_\alpha$. From this and the properties of a p.v.m., one can readily verify that in case $A \in \mathcal{A}$,

$$p_A = p_\Lambda.$$ 

Therefore, we can consistently extend this equation as the definition of $p_A$, for $A \in \mathcal{M}$. Equation (43) follows then from (62). Equation (44) follows from (43) and (54). The proofs of (45) and of the claim that \{p_A : A \in \mathcal{M}\} is a p.v.m. follow from (44) and (43) by the arguments in the proof above of the Theorem.

6 A partial converse to part (b) of Theorem 1

In this section we return to the setting introduced in Section 2, but we will suppose that the set $S$ is totally ordered. In case $S \subset \mathbb{R}$, which is the case in applications to quantum mechanics, in which elements of $S$ are moments in time, we can think that $S$ inherits the order from $\mathbb{R}$. Using $\leq$ for the order relation, we will, as usual, write $s < t$ in case $s \leq t$ and $s \neq t$.

In the statement of the theorem below we refer to a subset $V$ of $\mathcal{H}$ that is dense in $\mathcal{H}$ (i.e., $\overline{V} = \mathcal{H}$). Important examples of such sets are the domains of self-adjoint operators, including the Hamiltonian (see Subsection 8.1). The relevance of stating the theorem in terms of such a subset of $\mathcal{H}$ rather than $\mathcal{H}$ itself relates to its applicability to typical pilot-wave theories, including Bohmian mechanics, in Subsection 8.4.
Theorem 7 Suppose $S$ is a totally ordered set and $V \subset H$ is dense in $H$. If for every $\varphi \in V$ there is a probability measure $\mathbb{P}_\varphi$ on $(\Omega, \Sigma)$ such that

$$\mathbb{P}_\varphi(X_i = a_i, i = 1, \ldots, k) = ||p_{a_k}^t \cdots p_{a_1}^t \varphi||^2,$$

(63)

for every $t_1 < \ldots < t_k$ and $a_1, \ldots, a_k$, then $H_x = H$, i.e., $p_a^t$ and $p_b^t$ commute, for every $s, t \in S$, $a \in \Gamma(s), b \in \Gamma(t)$.

Proof: If $s = t$, then $p_a^s p_a^t = p_b^s p_a^t = 0$, if $a \neq b$ and $p_a^s p_a^t = p_b^s p_a^t = p_a^s$, if $a = b$. So, with no loss, we only need to consider the case in which $s < t$, to which we turn now.

For $\varphi \in V \setminus \{0\}$ and $b \in \Gamma(t)$, we have from (63),

$$\sum_a ||p_a^t \varphi||^2 = \sum_a \mathbb{P}_\varphi(X_s = a, X_t = b) = \mathbb{P}_\varphi(X_t = b) = ||p_b^t \varphi||^2.$$

This implies that for every $\varphi \in V$,

$$\sum_a ||p_a^t \varphi||^2 = ||p_b^t \varphi||^2.$$

(64)

On the other hand, for any $\varphi \in H$, we have

$$||p_b^t \varphi||^2 = \langle p_b^t \varphi, p_b^t \varphi \rangle = \left( \sum_a ||p_b^t p_a^t \varphi||^2 \right) = \sum_a \langle p_b^t p_a^t \varphi, p_b^t \varphi \rangle = \sum_a \left( \langle p_b^t p_a^t \varphi, p_b^t (I - p_a^t) \varphi \rangle + \langle p_b^t p_a^t \varphi, p_b^t p_a^t \varphi \rangle \right)$$

$$= \sum_a \left( ||p_b^t \varphi||^2 ||p_a^t \varphi||^2 \right),$$

(65)

where in the second equality we used (1) and Thm.H.28.1 (as in the proof of part (e) of Proposition 2), and the third equality is justified by Thm.H.7.3, since $\sum_a p_b^t p_a^t = p_b^t$ is well defined.

Combining (64) with (65), we now have, for every $\varphi \in V$,

$$\sum_a \langle \varphi, Q_a \varphi \rangle = 0, \quad \text{where} \quad Q_a = p_a^s p_b^t (I - p_a^t).$$

(66)

Next we will show that $\sum_a Q_a \varphi$ converges for every $\varphi \in H$, defining a bounded operator $\sum_a Q_a$. Since the projections $p_a^t$, $a \in \Gamma(s)$ are orthogonal to each other, so are also the vectors $Q_a \varphi$. Hence the claimed convergence is equivalent to the statement that $\sum_a ||Q_a \varphi||^2 < \infty$, which we easily verify:

$$\sum_a ||Q_a \varphi||^2 \leq 2 \sum_a \left( ||p_b^t p_a^t \varphi||^2 + ||p_a^s p_b^t p_a^t \varphi||^2 \right) \leq 2 \sum_a \left( ||p_b^t \varphi||^2 + ||p_a^t \varphi||^2 \right)$$

$$= 2 \left( \sum_a ||p_a^t \varphi||^2 + \sum_a ||p_a^t \varphi||^2 \right) = 2 \left( ||p_b^t \varphi||^2 + ||\varphi||^2 \right) \leq 4 ||\varphi||^2,$$

where in the second and in the last steps we used the fact that projections cannot increase the norm of a vector, and in the third and fourth steps we used the orthogonality of the projections $p_a^t$, $a \in \Gamma(s)$, and (1), respectively. The norm of $\sum_a Q_a$ can be estimated from $||\sum_a Q_a \varphi||^2 = \sum_a ||Q_a \varphi||^2 \leq 4 ||\varphi||^2$, as being at most 2.

The convergence of $\sum_a Q_a \varphi$ allows the application of Thm.H.7.3 to (66), to obtain

$$\left\langle \varphi, \sum_a Q_a \varphi \right\rangle = 0,$$

(67)

for any $\varphi \in V$. But since $\sum_a Q_a$ is a bounded (and hence continuous) operator and inner products are jointly continuous in their two arguments, (67) extends by continuity, from the dense $V$, to all $\varphi \in H$. And Theorem 12.7 of [28] tells us that (since our Hilbert space is over the Complex field) this implies

$$\sum_a Q_a \varphi = 0,$$

(68)
for all \( \varphi \in \mathcal{H} \). Recall that since the projections \( p^a_s \), \( a \in \Gamma(s) \) are orthogonal to each other, so are also the vectors \( Q_a \varphi \). And since a sum of orthogonal vectors can only be 0 if each one of them is, (68) yields, for each \( a \in \Gamma(s) \) and \( \varphi \in \mathcal{H} 
abla \varepsilon \)

\[
Q_a \varphi = 0.
\]

So we have proved that

\[
p^a_s p^b_t = p^a_s p^b_t p^a_s,
\]

for each \( a \in \Gamma(s) \) and \( b \in \Gamma(t) \). Since the right-hand side of this equation is a self-adjoint operator, so has to be the left-hand side. But the adjoint of \( p^a_s p^b_t \) is \( p^b_t p^a_s \). So we have learned that \( p^a_s p^b_t = p^b_t p^a_s \), completing the proof. \( \square \)

It is natural to ask if when \( \mathcal{H}_\pi \neq \mathcal{H} \), there could still be some exceptional \( \varphi \notin \mathcal{H}_\pi \) for which (63) holds. An example with \( S = \{ s, t \} \), \( s \prec t \), shows that this is possible. By (1) we have \( \mathcal{H}_\pi \neq \mathcal{H} \). Therefore, if \( \mathcal{H}_\pi \neq \mathcal{H} \), there must exist some \( c \in \Gamma(s) \) for which there is some \( \varphi \in \mathcal{H}_c^\pi \), with \( \varphi \notin \mathcal{H}_\pi \). Obviously \( \varphi \neq 0 \), so that we can compute

\[
\|p^a_s p^a_c \varphi\|^2 = \|p^b_t \varphi\|^2 \delta_{a,c},
\]

(69)

where \( \delta_{a,c} = 1 \) if \( a = c \) and \( \delta_{a,c} = 0 \) if \( a \neq c \). The numbers in the right-hand side of (69) are non-negative and satisfy

\[
\sum_{a, b} \|p^a_s \varphi\|^2 \delta_{a, c} = \sum_{b} \|p^b_t \varphi\|^2 = \left\| \sum_{b} p^b_t \varphi \right\|^2 = \|\varphi\|^2 = 1,
\]

where we used (2) and (1). Therefore (69) defines a probability measure \( \mathbb{P}_\varphi \) on \( (\Omega, \Sigma) \), that satisfies

\[
\mathbb{P}_\varphi(X_s = a, X_t = b) = \|p^b_t p^a_s \varphi\|^2 = \|p^b_t \varphi\|^2 \delta_{a,c}.
\]

This probability measure satisfies also

\[
\mathbb{P}_\varphi(X_s = a) = \sum_{b} \mathbb{P}_\varphi(X_s = a, X_t = b) = \sum_{b} \|p^b_t \varphi\|^2 \delta_{a,c} = \delta_{a,c} = \|p^a_s \varphi\|^2.
\]

And

\[
\mathbb{P}_\varphi(X_t = b) = \sum_{a} \mathbb{P}_\varphi(X_s = a, X_t = b) = \sum_{a} \|p^b_t \varphi\|^2 \delta_{a,c} = \|p^b_t \varphi\|^2.
\]

The last three displays show that (63) is satisfied by \( \mathbb{P}_\varphi \).

7 Refinements and coarsenings

The concepts of refinement and coarsening discussed in this section are the same as those in the consistent, or decoherent, approach to quantum mechanics (see, e.g., 25, 15, 17).

Suppose that \( \pi \) is as defined in Section 2. A refinement of \( \pi \) is another set of projections in \( \mathcal{H} \),

\[
\pi' = \{ p^b_t : t \in S', b \in \Gamma'(t) \},
\]

where \( S \subset S' \) and for each \( t \in S \), \( \Gamma'(t) \) is the disjoint union of some sets \( \Gamma'_a(t), a \in \Gamma(t) \), with the property that

\[
p^a_t = \sum_{b \in \Gamma'_a(t)} p^b_t.
\]

(70)

This implies that the condition

\[
\sum_{b \in \Gamma'(t)} p^b_t = I
\]

is satisfied for every \( t \in S \), and we assume that it is satisfied for every \( t \in S' \). Informally, we are increasing the set of times from \( S \) to \( S' \) and, for each \( t \in S \), breaking each \( p^a_t \) into a sum of smaller orthogonal projections, according to (70). (By smaller projections we mean as usual that their ranges are smaller subspaces. And the orthogonality of the ranges of the \( p^b_t \) in (70) is a consequence of Thm.H.28.2 according to which a sum of projections can only be a projection if their ranges are orthogonal to each other.)

From the definition above, it is clear that

\[
\mathcal{H}_{\pi'} \subset \mathcal{H}_\pi.
\]

(71)
We will use primes to denote, in a self-explanatory fashion, the following objects associated to \( \pi' \): \( \Omega' \), \( \Sigma' \), \( A' \), \( X'_s \), \( s \in S' \).

Define now, for each \( A \subset \Omega \),

\[
A' = \left\{ \omega' \in \Omega' : \text{for some } \omega \in A, \omega'(t) \in \Gamma'_\omega(t), \text{ for all } t \in S \right\}.
\]

(72)

The following properties of \( A' \) are immediate:

- The notation \( \Omega' \) was defined twice above, but consistently.
- If \( A \text{ and } B \) are disjoint subsets of \( \Omega \), then \( A' \) and \( B' \) are disjoint subsets of \( \Omega' \).
- If \( \{A_\alpha\} \) is an arbitrary family of subsets of \( \Omega \) and \( A = \cup_\alpha A_\alpha \), then \( A' = \cup_\alpha A'_\alpha \).
- For any \( A \subset \Omega \), \( (A')^c = (A^c)' \).

When \( \pi' \) is a refinement of \( \pi \), we say that \( \pi \) is a coarsening of \( \pi' \). A simple example, from Section 4, is \( \pi(D) \), defined by (32), as a coarsening of \( \pi \). Lemma 8 in that section is an instance of one of the statements in Theorem 8 below. A number of interesting examples will appear in Subsection 8.12.

**Theorem 8** If \( A \subset \Sigma \), then \( A' \subset \Sigma' \) and

\[
p_{A'} = p_{A \pi} = p_{\pi'}p_A = p_A \land p_{\pi'}.
\]

(73)

So that in particular \( p_A \) and \( p_{\pi'} \) commute and \( \mathcal{H}_A = \mathcal{H}_A \cap \mathcal{H}_{\pi'} \).

**Proof:** We will use twice the \( \pi-\lambda \) Theorem (see, e.g., Theorem 3.2 of [6]).

Consider the following class of subsets of \( \Omega \)

\[
\mathcal{P} = \{ \{X_{t_1} = a_1, \ldots, X_{t_k} = a_k\} : t_1, \ldots, t_k \in S, a_1 \in \Gamma(t_1), \ldots, a_k \in \Gamma(t_k) \} \cup \{\emptyset\}.
\]

This class is clearly closed with respect to finite intersections, which means that it is a \( \pi \)-system.

Consider now the class

\[
\mathcal{L}_1 = \{ A \subset \Omega : A' \subset \Sigma' \}.
\]

The properties of the mapping from \( A \) to \( A' \) listed above imply that \( \mathcal{L}_1 \) has the three properties that define a \( \lambda \)-system:

- \( \Omega \in \mathcal{L}_1 \).
- \( \mathcal{L}_1 \) is closed with respect to complements.
- \( \mathcal{L}_1 \) is closed with respect to countable disjoint unions.

We claim that for any \( A \in \mathcal{P} \), we have \( A' \subset \mathcal{A}' \). Indeed, if \( A = \emptyset \), then \( A' = \emptyset \), and for \( A = \{X_{t_1} = a_1, \ldots, X_{t_k} = a_k\} \in \mathcal{P} \), we have

\[
A' = \left\{ X'_{t_1} \in \Gamma_\alpha(t_1), \ldots, X'_{t_k} \in \Gamma_\alpha(t_k) \right\} \in \mathcal{A}'.
\]

(74)

This means that \( \mathcal{P} \subset \mathcal{L}_1 \), and, since the smallest sigma-algebra that contains \( \mathcal{P} \) is \( \Sigma \), we learn from the \( \pi-\lambda \) Theorem that \( \Sigma \subset \mathcal{L}_1 \), completing the proof that \( \mathcal{A}' \subset \Sigma' \) whenever \( A \subset \Sigma \).

Thm.H.29.1 states that a product of projections is a projection if and only if they commute, and in this case their product in any order is equal to their meet. Therefore we only need to prove the first equality in (73), and the others follow.

If \( \varphi \downarrow \mathcal{H}_{\pi'} \), then \( p_{A'} \varphi = 0 = p_{A' \pi'} \varphi \), so it is sufficient to prove that if \( \varphi \in \mathcal{H}_{\pi'} \), then \( p_{A' \pi'} \varphi = p_{A' \pi} \varphi \). and this is the same as the statement that

\[
\text{if } \varphi \in \mathcal{H}_{\pi'} \text{, then } p_{A'} \varphi = p_{A' \pi} \varphi.
\]

Set

\[
\mathcal{L}_2 = \{ A \subset \Sigma : p_{A' \pi} \varphi = p_{A' \pi} \varphi \text{ for every } \varphi \in \mathcal{H}_{\pi'} \}.
\]

The class \( \mathcal{L}_2 \) is a \( \lambda \)-system, since for every \( \varphi \in \mathcal{H}_{\pi'} \) we have:

- For \( A = \Omega \), \( p_{A' \pi} \varphi = p_{\pi'} \varphi = p_{\pi'} \varphi = p_{\pi'} \varphi = p_{A'} \varphi \), where we used (71), in the third equality. Therefore \( \Omega \in \mathcal{L}_2 \).
- If \( A \in \mathcal{L}_2 \), then

\[
p_{A' \gamma} \varphi = (p_{A' \gamma} - p_A) \varphi = (p_{A' \gamma} - p_A) \varphi = p_{A' \gamma} \varphi,
\]

where we used the already proved facts that \( A' \) and \( (A')' \) are in \( \Sigma' \), part (c) of Theorem 1 for \( \pi \) and for \( \pi' \) and specifically property (PVM5) of a p.v.m. (see Section 5), as well as the fact from the previous item. Therefore \( A' \in \mathcal{L}_2 \).
• If $A_1, A_2, \ldots$ are disjoint sets in $L_2$ and $A = \cup_{i=1}^{\infty} A_i$, then, from the properties of the mapping from $A$ to $A'$ listed before this theorem, we have that also $A_1', A_2', \ldots$ are disjoint sets and $A' = \cup_{i=1}^{\infty} A'_i$. And since we already know that $A'_i \in \mathcal{S}'$, we can use again part (c) of Theorem 1 for $\pi$ and for $\pi'$ and property (PVM2) of a p.v.m. (see Section 5), to write: $p_{A'}\varphi = \sum_{i=1}^{\infty} p_{A_i'}\varphi = \sum_{i=1}^{\infty} p_{A_i}\varphi = p_A\varphi$. Therefore $A \in \mathcal{L}_2$.

Our next task is to show that
\[ \mathcal{P} \subset \mathcal{L}_2. \] (76)

Clearly $\emptyset \in \mathcal{L}_2$, and for $A = \{X_{t_1} = a_1, \ldots, X_{t_k} = a_k\} \in \mathcal{P}$, we have (74), so that, for $\varphi \in \mathcal{H}_\pi'$,
\[ p_{A'}\varphi = \sum_{b_1 \in \Gamma_{a_1}(t_1), \ldots, b_k \in \Gamma_{a_k}(t_k)} p_{b_1, \ldots, b_k}^{t_1, \ldots, t_k} \varphi = \sum_{b_1 \in \Gamma_{a_1}(t_1), \ldots, b_k \in \Gamma_{a_k}(t_k)} p_{b_1}^{t_1} \cdots p_{b_k}^{t_k} \varphi = \varphi. \]

In the first and in the last equalities, we used part (c) of Theorem 1 first for $\pi'$, then for $\pi$, and in the former we also used property (PVM2) of a p.v.m. (see Section 5). In the second and in the next-to-last equalities, we used part (d) of Proposition 2 first for $\pi'$ (fine since $\varphi \in \mathcal{H}_\pi'$) then for $\pi$ (fine since, thanks to (71), also $\varphi \in \mathcal{H}_\pi$). And in the third and fourth equalities, we used Thm.H.28.1 (as in the proof of part (e) of Proposition 2 and (70)).

Since $\mathcal{P}$ is a $\pi$-system that generates $\mathcal{S}$ and $\mathcal{L}_2$ is a $\lambda$-system, the $\pi$-$\lambda$ Theorem tells us that (76) implies the stronger statement
\[ \mathcal{S} \subset \mathcal{L}_2, \]
which means that (75) holds for every $A \in \mathcal{S}$, completing the proof of the theorem. □

If $\pi'$ is a refinement of $\pi$, we write $\pi' \leq \pi$, as this is a partial order in the set of possible $\pi$.

In the set of possible $\pi$ with a given fixed $S$, the minimal element is the one in which, for each $t \in S$, $\Gamma(t)$ has a single element $a_t$ and $p_{a_t}^t = 1$. (This is unique modulo the choice of the labels $a_t$.) This set of $\pi$ also has maximal elements, those being characterized by the sets $\mathcal{H}_\pi'$ having dimension 1, for all $t \in S, a \in \Gamma(t)$.

8 Examples, Remarks and Applications

This section combines mathematical issues with issues of interpretation. It includes some applications that illustrate the use of the theorems proved in the previous sections to evaluate proposed interpretations, or to propose different ones. The first application described in the abstract of the paper appears in Subsection 8.14 and is further elaborated in Subsection 8.16. The second one appears in Subsection 8.11 and is further elaborated in the following three subsections. The third one appears in Subsection 8.15.

8.1 Basic examples

In the standard quantum mechanics setting, in addition to the Hilbert space $\mathcal{H}$, there is a strongly continuous group of unitary operators $(U_t)_{t \in \mathbb{R}}$, that provide the evolution of the state in the Schrödinger picture, or of the operators in the Heisenberg picture. In the Schrödinger picture the state at time $t$ is given by $\Psi_t = U_t \Psi$, if at time 0 it is $\Psi \in \mathcal{H}$. In the Heisenberg picture $\Psi$ does not change with time, but each operator $Q$ evolves to $Q_t = U_{-t} Q U_t$ at time $t$.

The assumptions on $(U_t)_{t \in \mathbb{R}}$ are expressed as
\[ U_0 = I, \quad U_{t+s} = U_t U_s, \quad U_{-t}^* = U_t^{-1}, \] (77)
for each $t, s \in \mathbb{R}$, where the star denotes the adjoint. And
\[ \lim_{s \to t} U_s \varphi = U_t \varphi, \] (78)
for each $\varphi \in \mathcal{H}, t \in \mathbb{R}$.

Note that (77) implies that
\[ U_{-t} = U_t^{-1} = U_t^*. \] (79)

Stone’s Theorem and its converse (see Theorems VIII.7 and VIII.8 of 26) state that the conditions above on $(U_t)_{t \in \mathbb{R}}$ are equivalent to the existence of a self-adjoint operator $H$, in this context called the Hamiltonian, such that
\[ U_t = \exp(-itH), \]
We turn now to "particle models", that are important examples of the setting in Subsection 8.1. For simplicity, we suppose that \( \sum_{a} p_{a} = I \).

In other words, this set of projections is a partition of the identity.

Define now, for each \( t \in \mathbb{R} \),
\[
p_{a}^{t} = U_{-t} p_{a} U_{t},
\]
then, using (79),
\[
\sum_{a} p_{a}^{t} = \sum_{a} U_{-t} p_{a} U_{t} = U_{-t} \left( \sum_{a} p_{a} \right) U_{t} = U_{-t} I U_{t} = I,
\]
so that (1) is satisfied. If \( S \subset \mathbb{R} \), then
\[
\pi = \{ p_{a}^{t} : t \in S, a \in \Gamma \}
\]
is an example of the sort of family of projections studied in this paper, with the particular feature that \( \Gamma(t) = \Gamma \), for all \( t \in S \).

Every such \( \pi \) has a natural refinement
\[
\pi' = \{ p_{a}^{t} : t \in \mathbb{R}, a \in \Gamma \}.
\]
And in case \( S \) is dense in \( \mathbb{R} \), i.e., \( \overline{S} = \mathbb{R} \), (78) implies that
\[
\mathcal{H}_{\pi} = \mathcal{H}_{\pi'}.
\]
The group property in (77) yields, for each \( t \in \mathbb{R} \),
\[
\begin{align*}
p_{a_{1}}^{t_{1}} \cdots p_{a_{k}}^{t_{k}} U_{t} \varphi &= U_{-t_{1}} p_{a_{1}} U_{t_{1}} \cdots U_{-t_{k}} p_{a_{k}} U_{t_{k}} U_{t} \varphi \\
&= U_{t} \left( U_{-(t_{1}+t_{2})} p_{a_{1}} U_{t_{1}+t_{2}} \cdots U_{-(t_{k}+t_{k+1})} p_{a_{k}} U_{t_{k}+t_{k+1}} \right) \varphi \\
&= U_{t} p_{a_{1}+t}^{t_{1}} \cdots p_{a_{k}+t}^{t_{k}} \varphi.
\end{align*}
\]
And this implies that \( \mathcal{H}_{\pi} \) is invariant under \( U_{t} \), i.e.,
\[
U_{t} \mathcal{H}_{\pi} \subset \mathcal{H}_{\pi}.
\]
This conclusion applied to \(-t\), in conjunction with (79) implies that \( \mathcal{H}_{\pi} \) is also invariant under \( U_{t}^{*} \). And Thm.H.23.2 implies then that \( \mathcal{H}_{\pi}^{\perp} \) is invariant under \( U_{t}^{*} \):
\[
U_{t} \mathcal{H}_{\pi}^{\perp} \subset \mathcal{H}_{\pi}^{\perp}.
\]
Together, (83) and (84) are expressed by saying that \( \mathcal{H}_{\pi} \) reduces \( U_{t} \). And Thm.H.26.2 then tells us that \( U_{t} \) commutes with \( p_{a} \), for every \( t \in \mathbb{R} \).

8.2 Particle models

We turn now to "particle models", that are important examples of the setting in Subsection 8.1. For simplicity, we consider first a universe with a single type of particle, and no creation or annihilation of particles. Suppose that the dimension of the physical space is 3 and that there are n particles. In this case \( \mathcal{H} = L^{2}(\mathbb{R}^{3n}) \), and \( C = \mathbb{R}^{3n} \) is called the configuration space, and is endowed with its Borel sigma-algebra and Lebesgue measure. For each measurable \( R \subset \mathbb{R}^{3n} \), let \( I_{R} \) denote its indicator function, i.e., \( I_{R}(x) = 1 \), if \( x \in R \), and \( I_{R}(x) = 0 \), if \( x \not\in R \).

Let \( \Gamma \) be a countable set and let \( \{ R_{a} : a \in \Gamma \} \) be a partition of \( \mathbb{R}^{3n} \) into measurable disjoint sets \( R_{a} \) that have boundaries of Lebesgue measure 0. Define now the projections \( p_{a} \) by
\[
(p_{a} \varphi)(x) = I_{R_{a}}(x) \varphi(x),
\]
x \( \in \mathbb{R}^{3n} \). It is clear that (80) is satisfied, and hence we have an example of the setting discussed in Subsection 8.1. Clearly also, the range of \( p_{a} \) is
\[
\mathcal{H}_{a} = \{ \varphi \in \mathcal{H} : \text{supp} \varphi \subset \overline{R_{a}} \},
\]
where \( \text{supp} \varphi \) denotes the essential support of \( \varphi \), i.e., the smallest closed subset of \( \mathbb{R}^{3n} \) such that \( \varphi = 0 \) almost everywhere on the complement of this set. If we use the notation, \( \varphi_{t} = U_{t} \varphi \), then it follows that, for each \( t \in S \) and \( a \in \Gamma(t) \),
\[
\mathcal{H}_{a}^{t} = \{ \varphi \in \mathcal{H} : p_{a}^{t} \varphi = \varphi \} = \{ \varphi \in \mathcal{H} : U_{-t} p_{a} U_{t} \varphi = \varphi \} = \{ \varphi \in \mathcal{H} : p_{a} \varphi_{t} = \varphi_{t} \}
\]
= \{ \varphi \in \mathcal{H} : \varphi_{t} \in \mathcal{H}_{a} \} = \{ \varphi \in \mathcal{H} : \text{supp} \varphi_{t} \subset \overline{R_{a}} \}.\]
where in the third equality we used (79). It follows that for each $t_1, \ldots, t_k$ and $a_1, \ldots, a_k$,

$$\mathcal{H}^{t_1, \ldots, t_k}_{a_1, \ldots, a_k} = \{ \varphi \in \mathcal{H} : \text{supp } \varphi_i \subset \mathcal{T}_{a_i}, i = 1, \ldots, k \}.$$ 

In words, this subspace is the set of wave functions at time 0 that evolve with time in the Schrödinger picture in such a way that at each time $t_i$, $i = 1, \ldots, k$ their essential support is contained in $\mathcal{T}_{a_i}$, and hence they are almost everywhere identically 0 outside of $\mathcal{T}_{a_i}$ (recall that the boundary of each $\mathcal{T}_{a_i}$ has Lebesgue measure 0). In applications, the sets $\mathcal{T}_{a_i}$ may correspond to physically meaningful macroscopic descriptions. For instance, in one of these sets some of the particles may form a healthy cat, or a measuring device with a pointer indicating some outcome to an experiment, or a computer in a certain computational state, or human beings with brains in configurations that correspond to certain mental states. The subspace $\mathcal{H}^{t_1, \ldots, t_k}_{a_1, \ldots, a_k}$ should then be understood as the set of time-0 wave functions with the property that at the times $t_1, \ldots, t_k$ the respective physical descriptions indexed by $a_1, \ldots, a_k$ correspond to our unique macroscopic reality.

There is no difficulty in modifying the example above to allow for creation and annihilation of particles. In this case the configuration space should be taken as the disjoint union $\mathcal{C} = \bigcup_{n=0}^{\infty} \mathbb{R}^{3n}$, where $\mathbb{R}^{0}$ is a set with a single element, called the “vacuum configuration”, and the Hilbert space will be the Fock space

$$\mathcal{H} = C(0) = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^{3n}) = \left\{ (\varphi_0, \varphi_1, \ldots) : \varphi_n \in L^2(\mathbb{R}^{3n}), n = 0, 1, \ldots, \sum_{n=0}^{\infty} ||\varphi_n||^2 < \infty \right\},$$

where $L^2(\mathbb{R}^0) = \mathbb{C}$, the set of complex numbers.

Let $\Gamma$ be a countable set and, for each $n = 0, 1, \ldots$, let $\{ R_{a,n} : a \in \Gamma \}$ be a partition of the corresponding $\mathbb{R}^{3n}$ into measurable disjoint sets $R_{a,n}$ that have boundaries of Lebesgue measure 0. Define now the projections $p_a$ by

$$p_a \varphi = p_a(\varphi_0, \varphi_1, \ldots) = (p_{a,0} \varphi_0, p_{a,1} \varphi_1, \ldots),$$

where

$$p_{a,n} \varphi_n(x) = \hat{I}_{R_{a,n}}(x) \varphi_n(x),$$

$x \in \mathbb{R}^{3n}$. Then, similarly to the previous example,

$$\mathcal{H}_{a,n} = \{(\varphi_0, \varphi_1, \ldots) \in \mathcal{H} : \text{supp } \varphi_n \subset \mathcal{T}_{a,n}, n = 0, 1, \ldots \}.$$

And, if we use the notation $U_\varepsilon \varphi = U_\varepsilon(\varphi_0, \varphi_1, \ldots) = (\varphi_{t,0}, \varphi_{t,1}, \ldots)$, then for any $t_1, \ldots, t_k$ and $a_1, \ldots, a_k$, we have

$$\mathcal{H}_{a_1, \ldots, a_k} = \{(\varphi_0, \varphi_1, \ldots) \in \mathcal{H} : \text{supp } \varphi_{t,i} \subset \mathcal{T}_{a_i,n}, i = 1, \ldots, k, n = 0, 1, \ldots \}.$$

And as in the previous example, this subspace admits the same sort of interpretation that that one has. Suppose that we take the partitions $\{ R_{a,n} : a \in \Gamma \}$ in such a way that configurations in any of the sets $R_{a,n}, n = 0, 1, \ldots$ correspond to the same macroscopic description indexed by $a \in \Gamma$. Then the subspace $\mathcal{H}_{a_1, \ldots, a_k}$ should be understood as the set of time-0 wave functions with the property that at the times $t_1, \ldots, t_k$ the respective physical descriptions indexed by $a_1, \ldots, a_k$ correspond to our unique macroscopic reality.

We can also include different kinds of particles, possibly with different spins, without any further difficulty, by replacing in the Fock space $L^2(\mathbb{R}^{3n})$ with the appropriate tensor products (see, e.g., Section II.4 of (26)). The configuration space becomes then a disjoint union $\mathcal{C} = \bigcup_{n_1=0}^{\infty} \bigcup_{n_2=0}^{\infty} \mathbb{R}^{3n_1} \times \cdots \times \mathbb{R}^{3n_l}$, where the indices $1, \ldots, l$ correspond to the different types of particles.

### 8.3 Remarks on the expression $||p_{a_1}^{t_1} \ldots p_{a_l}^{t_l} \varphi||^2$

This expression, that appears in the right-hand side of (6), in part (b) of Theorem 1 can be rewritten in ways that more explicitly show its relation with Born’s rule and (apparent) collapse of the wave function at “observation” times. Here we are assuming that $S \subset \mathbb{R}$ and $t_1 < \ldots < t_k$. We are also supposing that at each time $t_i, i = 1, \ldots, k$ an “observation” is being made which has possible outcomes in $\Gamma(t_i)$ and that if $a_i$ is “observed”, standard quantum mechanics with collapse postulates collapse of the (Heisenberg-picture) wave function into its projection on the subspace $\mathcal{H}_{a_i}$. In the usual jargon, and assuming that the indexes $a_i$ are identified with real numbers, at time $t_i$ the observable corresponding to the self-adjoint operator $\sum_{a_i \in \Gamma(t_i)} a_i p_{a_i}^{t_i}$ is being measured.

In the case $k = 1$, we have $||p_{a_1}^{t_1} \varphi||^2 = ||p_{a_1}^{t_1} \hat{\varphi}||^2$, which indeed is the probability given by Born’s rule, for an “observation” of $a_1$ at time $t_1$, when the state (in the Heisenberg picture) is $\varphi$.

Set $\varphi_0 = \varphi$, and, for $i = 1, \ldots, k$, recursively define $\varphi_i = p_{a_i}^{t_i} \hat{\varphi}_{i-1}$, if $\varphi_{i-1} \neq 0$, and $\varphi_i = 0$, if $\varphi_{i-1} = 0$. If $\varphi_k = 0$, let

$$i_0 = \min\{i \in \{1, \ldots, k\} : \varphi_i = 0\}.$$
Then, when $\varphi_k \neq 0$, we have
\[
||p_{a_k}^{t_k} \cdots p_{a_2}^{t_2} \hat{\varphi}||^2 = ||p_{a_k}^{t_k} \cdots p_{a_2}^{t_2} \varphi_1||^2 = \ldots = ||\varphi_k||^2 \cdots ||\varphi_1||^2 = ||p_{a_k}^{t_k} \hat{\varphi}_{k-1}||^2 \cdots ||p_{a_2}^{t_2} \hat{\varphi}_{0}||^2.
\]
This is precisely the probability that standard quantum mechanics with collapse gives to the successive "observations" of $a_1$ at $t_1$, ..., $a_k$ at $t_k$, with collapse of the wave function at each "observation".

When $\varphi_k = 0$,
\[
||p_{a_k}^{t_k} \cdots p_{a_2}^{t_2} \hat{\varphi}||^2 = \ldots = ||p_{a_k}^{t_k} \cdots p_{a_2}^{t_2} \varphi_{i_0-1}||^2 = ||\varphi_{i_0-1}||^2 \cdots ||\varphi_1||^2 = 0.
\]
This also agrees with standard quantum mechanics with collapse, since then the $(i_0 - 1)$-th "observation" would have collapsed the wave function to $\varphi_{i_0-1}$ which is incompatible with the observation of $a_{i_0}$ at time $t_{i_0}$, because $p_{a_{i_0}}^{t_{i_0}} \varphi_{i_0-1} = \varphi_{i_0} = 0$.

8.4 Pilot-wave theories in configuration space and physical space that are fully equivalent to standard quantum mechanics in a path-wise sense

Part (b) of Theorem 1 can be seen as stating the existence of a pilot-wave theory in "$\Gamma$-space" that is in full agreement with standard quantum mechanics. This can be used to build pilot-wave theories in configuration space, and hence also in physical space, for the particle models discussed in Subsection 8.2.

In these models the configuration space $C$ is partitioned into sets $R_{a,i}$, where $a \in \Gamma$ and $i$ specifies the number of particles of each kind present. The interpretation being that all points in each $R_a = \cup_i R_{a,i}$ correspond to the same physically meaningful macroscopic description, labeled by $a \in \Gamma$.

Suppose now that $x$ is a function from $\Gamma$ to $C$, with the property that, $x(a) \in R_a$, for each $a \in \Gamma$. For $\omega \in \Omega$ and $t \in \mathbb{R}$, set $x_t(\omega) = x(\omega)$. Then, for each $\varphi \in \mathcal{H}_{\pi}$, $(x_t)_{t \in \mathbb{R}}$ is a stochastic process on the probability space $(\Omega, \Sigma, P_{\varphi})$ (measurability issues are automatically satisfied because $\Gamma$ is a discrete space). And from (6) we obtain
\[
P_{\varphi}(x_{t_1} \in R_{a_1}, \ldots, x_{t_k} \in R_{a_k}) = ||p_{a_k}^{t_k} \cdots p_{a_2}^{t_2} \hat{\varphi}||^2,
\]
(85) for every $t_1 < \ldots < t_k$, and $a_1, \ldots, a_k$. (Actually we obtain (85) under these assumptions, for every $t_1, \ldots, t_k$. But for our purposes in this subsection and subsequent ones, when we quote (85) we mean it with the times in the stated order.)

Since a point in $C$ specifies how many particles of each kind are present, and where they are located, one can see $(x_t)_{t \in \mathbb{R}}$ as describing particles in physical space moving and being created and annihilated. This all happening in fashions that, through $P_{\varphi}$, are guided by the wave function $\varphi$ and the unitary group $(U_t)_{t \in \mathbb{R}}$.

The argument above is one of existence of processes $(x_t)_{t \in \mathbb{R}}$ with the described properties. From that construction, it is clear that uniqueness is not at all true. And unfortunately, it is not clear what properties, including Markovianity, smoothness properties of the paths, etc, a process $(x_t)_{t \in \mathbb{R}}$ that satisfies (85) may, or may not have.

Compare the construction above with the more standard pilot-wave theories, including the paradigmatic Bohmian mechanics. Those are usually Markovian and have continuous paths, except when particles are created or annihilated. But in those, one usually is satisfied with a weaker condition than (85), namely:
\[
P_{\varphi}(x_t \in R_a) = ||p_a^t \hat{\varphi}||^2,
\]
(86) for every $t$ and $a$. To prove (86), one usually shows a property called "equivariance", which states that if (86) holds at one time, then it holds at any other time. One then assumes that it holds at one given time, sometimes with the support of some plausibility argument. There is also a competing idea, that (86) was not always true in our universe, but that it is a sort of equilibrium condition, that resulted from good mixing properties of the underlying pilot-wave process. The first of these two approaches to (86) appears in most of the papers on pilot-wave theories listed in the introduction. The second view is defended in [25]. For appraisals of both approaches, see [8] and [24].

One should point out that in typical pilot-wave theories, the partition of the configuration space into the sets $R_a$ so that (86) holds can be fairly arbitrary, with only measurability requirements being necessary. And $\varphi$ can then typically be chosen arbitrarily from a dense, linearly closed, subset of the full Hilbert space $\mathcal{H}$, not just $\mathcal{H}_{\pi}$.

The expression "fully equivalent to quantum mechanics" in the title of this section refers to pilot-wave theories that satisfy (85), rather than simply satisfying (86).

There are arguments, related to the idea of an "effective collapse of the wave function", that suggest that Bohmian mechanics may satisfy the full (85), at least approximately (see, e.g., Section 9.2 of [13], Section 5.1.6 of [7], and...
Section 8 of [16]. But it seems that whether exact agreement with this equation holds for Bohmian mechanics is an open question.

Lack of full agreement with quantum mechanics in the sense discussed here was one of the criticisms of stochastic mechanics (another well known pilot-wave theory) by its own first developer, in Section 10.2 of [22], and Section 5 of [23].

As pointed out in [2], Section 5, it is easy to produce stochastic processes that satisfy (86), but do not satisfy (85). For instance one can take a point $x_t \in C$ at each time $t \in \mathbb{R}$ independently of anything else, with probability $\mathbb{P}_\phi(x_t \in R_a) = ||p_t^{a}_\phi||^2$.

An important philosophical question that arises is if (86) should be considered sufficient to make a pilot-wave theory plausible. The point, made in [2], [22] and [23], is that if we had (86) we would not be able to perceive that we do not have the full (85), based on experiments. We could nevertheless have incorrect records (including those in our brains) of our true history. Think of the example in the last paragraph, for a dramatic case of complete lack of correlation across time, and in particular between memories and true pasts. Similarly, the models introduced in [9] are diffusions with arbitrarily large diffusion coefficients, and will show very low correlation between memories and true pasts when this coefficient is large, despite the paths being continuous. If one is not bothered by this, then one can simply propose the independent choices of $x_t$ at different times as a satisfactory interpretation of quantum mechanics. But if one finds this possibility unacceptable, as emphasized in [21], [22] and [23] then one should ask which pilot-wave theories satisfy (85) (We should observe that, due to tunneling, quantum mechanics may produce false records of the past. What (85) entails is that the correlations between the records and the true past are given correctly by the quantum dynamics, and not modified by additional phenomena pertaining to the pilot-wave theory, as in the examples given in this paragraph.)

A second question is whether approximately satisfying (85), which may turn out to be the case for Bohmian mechanics, should be considered philosophically satisfactory. And what one then means by a satisfactory approximate fulfillment of this condition.

A most interesting mathematical question stressed and left open here is whether Bohmian mechanics satisfies (85) exactly, for the kind of $\pi$ discussed in Subsection 8.2 with the corresponding sets $R_a$ corresponding to certain macroscopic descriptions labeled by $a \in \Gamma$. Note that since Bohmian mechanics can be defined for all $\phi$ in the domain of the Hamiltonian, which is a dense subset of $L^2(\mathbb{R}^{3n})$, Theorem 7 would imply, if the answer is positive, that $\mathcal{H}_\pi = \mathcal{H} = L^2(\mathbb{R}^{3n})$ in this case. And from part (b) of Theorem 1 we would then learn that there is a pilot-wave theory that satisfies (85) for all $\phi \in \mathcal{H} = L^2(\mathbb{R}^{3n})$.

A related important open problem is how regular the paths of pilot-wave theories that satisfy (85) can be. Can they be continuous in the case in which particles are not created or annihilated? Can they be continuous from one side, with limits from the other when particles can be created and annihilated?

Especially in view of Theorem 7, one can ask what is the value of having (85), that applies to $\phi \in \mathcal{H}_\pi$, if it turns out that we live in a universe that is in a Heisenberg-picture state $\Psi \notin \mathcal{H}_\pi$, for the relevant $\pi$. We will answer this question in Subsection 8.16, where, building on previous subsections, we will propose that a pilot-wave theory that satisfies (85) with $\phi = \Psi_\pi$ should be a good candidate for an interpretation of quantum mechanics.

### 8.5 When $S$ is finite

When $S = \{t_1, \ldots , t_K\}$ is a finite set, there are major simplifications to many of the proofs in this paper.

In this case $\Omega = \Gamma(t_1) \times \ldots \times \Gamma(t_K)$ is countable, and $\Sigma = \mathcal{A}$ contain all the subsets of $\Omega$. Also

$$\mathcal{H}''_\pi = \mathcal{H}^{t_1 \ldots t_K} = \bigoplus_{a_1, \ldots , a_K} \mathcal{H}^{t_1 \ldots t_K}_{a_1, \ldots , a_K} = \text{Range} \left\{ \sum_{a_1, \ldots , a_K} p_{a_1, \ldots , a_K}^{t_1 \ldots t_K} \right\},$$

and, using (3), (2) and the fact that a sum of orthogonal vectors can only be 0 if all these vectors are 0,

$$N = \{ \phi \in \mathcal{H} : p_{a_1, \ldots , a_K}^{t_1 \ldots t_K} \phi = 0 \text{ for all } a_1, \ldots , a_K \}$$

$$= \left\{ \phi \in \mathcal{H} : \sum_{a_1, \ldots , a_K} p_{a_1, \ldots , a_K}^{t_1 \ldots t_K} \phi = 0 \right\} = \text{Kernel} \left\{ \sum_{a_1, \ldots , a_K} p_{a_1, \ldots , a_K}^{t_1 \ldots t_K} \right\}.$$

Therefore it is immediate that $N$ is a vector space that, in this case, is topologically closed, i.e., $N = \overline{N}$, and that we have $\mathcal{H}''_\pi = N^\perp$.  

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I am not aware of any other simplification in the proof of part (a) of Theorem 1. But it is worth pointing out that once one has proved these statements in case \( S \) is finite, the general case follows simply by taking intersections over \( \{t_1, \ldots, t_k\} \).

Parts (b), (c), (d) and (f) of Theorem 1 as well as parts (a) and (c) of Theorem 2 are greatly simplified. And, as with \( N \), also the sets \( F_A \) are topologically closed.

One can start by defining, for \( A \in \Sigma \),

\[
\mathcal{H}_A = \bigoplus_{(a_1, \ldots, a_K) \in A} \mathcal{H}_{a_1, \ldots, a_K} = \text{Range} \left\{ \sum_{(a_1, \ldots, a_K) \in A} p_{a_1, \ldots, a_K}^{t_1, \ldots, t_K} \right\}.
\]

And noting, again using (3), (2) and the fact that a sum of orthogonal vectors can only be 0 if all these vectors are 0, that for all \( A \in \Sigma \),

\[
F_A = \{ \varphi \in \mathcal{H} : p_{a_1, \ldots, a_K}^{t_1, \ldots, t_K} \varphi = 0 \text{ for all } (a_1, \ldots, a_K) \in A \}
= \left\{ \varphi \in \mathcal{H} : \sum_{(a_1, \ldots, a_K) \in A} p_{a_1, \ldots, a_K}^{t_1, \ldots, t_K} \varphi = 0 \right\} = \text{Kernel} \left\{ \sum_{(a_1, \ldots, a_K) \in A} p_{a_1, \ldots, a_K}^{t_1, \ldots, t_K} \right\}.
\]

It is then easy to check that \( \{ \mathcal{H}_A : A \in \Sigma \} \) has the properties claimed in part (c) of Theorem 1. And it is immediate that \( F_A = \overline{F_A} \) and that \( F_A = \mathcal{H}_A \), as stated in part (a) of Theorem 2. Because \( F_A \) is closed, part (c) of Theorem 2 now reads \( \mathcal{H}_A = F_A \cap \mathcal{H}_\pi \), and it follows immediately from part (b) of that theorem and (8) in part (c) of Theorem 1.

If one now recalls that \( p_A \) is the projection on \( \mathcal{H}_A \) and defines, for \( \varphi \in \mathcal{H}_\pi \setminus \{0\} \) and \( A \in \Sigma \),

\[
P_\varphi(A) = ||p_A \hat{\varphi}||^2,
\]

then it is easy to check that \( P_\varphi \) has the properties claimed in part (b) of Theorem 1 and that the second claim in part (d) of that theorem also holds.

Finally, for part (f) of Theorem 1 given \( f : \Omega \rightarrow \mathbb{R} \), we can simply set

\[
Q_f = \sum_{\omega \in \Omega} f(\omega)p(\omega),
\]

with domain \( D_f \) as defined there, and check easily the required properties.

### 8.6 Do we need to consider infinite \( S \)? Uncountable \( S \)? Infinite \( \Gamma(t) \)?

In light of the remarks in Subsection 8.5, it is natural to ask what is gained, as far as applications to foundations of quantum mechanics are at stake, from considering infinite \( S \). One important reason for considering countably infinite sets \( S \) is to be able to use the limit theorems of probability theory, like the strong law of large numbers, that apply to idealized settings with infinitely many random variables. Those would correspond, for instance, to idealized sequences of experiments.

Less clear is if, for the sake of physics, there is a need for considering uncountably large sets \( S \). An argument in favor is in the fact that we usually consider physical time to be a real number, so that we should consider \( S = \mathbb{R} \) as our fundamental setting. But is there really a reason for thinking that physical time is not limited to rational values? And that the real line comes in simply as a mathematical tool, providing completeness in the mathematical sense as a convenience, but not an additional physical reality? This is an interesting philosophical issue that will not affect the applicability of the results in this paper in situations in which the sets \( \Gamma(t), t \in S \), of interest are all finite, thanks to the results in Section 4.

And this raises the question whether there is any reason for considering infinite \( \Gamma(t) \) in applications to foundations of quantum mechanics. In applications of the kind proposed in Subsection 8.2 when the number of particles in the universe is fixed (so that the configuration space is \( \mathcal{C} = \mathbb{R}^{3n} \)), there should be only a finite number of sets \( R_n \) that are macroscopically distinguishable from each other and meaningful to us. After all, in such a universe, there can only be a finite number of computational devices (including human brains), each one capable of holding some finite number of distinct computational states. Even if particles can be created, energy considerations may limit the number of particles and hence the number of bits that all the computers (including our brains) can hold.

In any case, we will see in Subsection 8.15 that the partial results obtained in case of infinite \( \Gamma(t) \) and uncountable \( S \) in Section 4 are sufficient to draw the conclusion that, if we accept certain intuitive assumptions, then events for
which we compute Born-probability 0 should not happen, even if $S$ is uncountable and $\Gamma$ is infinite. What is currently missing in this case is the converse. So we have not ruled out that, in this case, there could be events of positive probability that will not happen.

8.7 Can $F_A$ in Section 3 be replaced with a set $N_A$ that provides more uniformity in time?

For each $A \subset \Omega$, define

$$N_A = \left\{ \varphi \in \mathcal{H} : \text{for some } t_1, \ldots, t_k, \ p_{t_1, \ldots, t_k}^1 \varphi = 0 \text{ for all } \omega \in A \right\}. \quad (87)$$

Note that $N_\Omega = N$, and $N_A \subset F_A$. The difference between $N_A$ and $F_A$ is the extra uniformity, with respect to $\omega \in A$, in the choice of $t_1, \ldots, t_k$ in $N_A$. It is natural to ask if in Section 3 we could have used $N_A$ instead of $F_A$, and in particular whether in part (b) of Theorem 2, which is directly related to interpretation, we could replace $F_A$ with $N_A$.

The answer is that in some parts of Section 3 we can make this replacement, but not in others that include Theorem 2. This discussion highlights some of the technical details of the proofs in that section.

The following example shows that the problem is not only with the proof, but with the conclusion in this lemma, which is a special case of part (a) of Theorem 2.

Suppose that $S = \{t_1, t_2, \ldots\}$. For $i = 1, 2, \ldots$, choose $G_i \subset \Gamma(t_i)$, $G_i \neq \emptyset$, and set $A_i = \{X_{t_i} \in G_i\}$, $A = \bigcup_{i=1}^{\infty} A_i$. It is clear that for each $i, A_i \subset A$, and hence $A \in \mathcal{A}_\pi$. Now, using (3) (as in (25)),

$$N_A = \bigcup_{k=1}^{\infty} \left\{ \varphi \in \mathcal{H} : p_{t^k_1, \ldots, t^k_k}^1 \varphi = 0, \text{ for all } (a_1, a_2, \ldots) \in A \right\} = \bigcup_{k=1}^{\infty} \left\{ \varphi \in \mathcal{H} : p_{t^k_1, \ldots, t^k_k}^1 \varphi = 0, \text{ for all } (a_1, \ldots, a_k) \right\} = N.$$

In the second equality, we used the fact that for any $(a_1, \ldots, a_k)$ there is some $a_{k+1} \in G_{k+1} \subset \Gamma(t_{k+1})$, such that $(a_1, a_k, a_{k+1}, \ldots) \in A_{k+1} \subset A$.

If we could replace $F_A$ with $N_A$ in the statement of Lemma 4 or part (a) of Theorem 2, we would then have

$$\mathcal{H}_A = N_A^{1} = N^{1} = \mathcal{H}_\pi,$$

where the last equality is from part (a) of Theorem 2. In particular, for every $\varphi \in \mathcal{H}_\pi \setminus \{0\}$, we would have $\mathbb{P}_\varphi(A) = ||p_A \varphi||^2 = 1$. This is certainly absurd in many applications, since the sets $G_i$ can be very small subsets of the corresponding $\Gamma(t_i)$, only assumed to be non-empty above.

For a counter-example, let $\mathcal{H} = L^2([0,1])$, $S = \{1, 2, \ldots\}$, $\Gamma(t) = \{1, 2\}$ and $p^1_\varphi$ be defined by $(p^1_\varphi)(x) = I_{[0.3,1)}(x) \varphi(x)$, where (as before in this paper) $I_R(x) = 1$, if $x \in R$ and 0 otherwise. By necessity, $p^1_\varphi = I - p^1_\varphi$. In this setting, $\mathcal{H}_\pi \subset \mathcal{H}_\pi$, whenever $s < t$, and this implies $p_\varphi(1) = p_\varphi^1 p_\varphi^1 = p_\varphi^1$. It follows that $\mathcal{H}_\pi = \mathcal{H}$.

Take $G = \{1\}$ for each $i$. Then, for $\varphi$ defined by $\varphi(x) = 1$, we have

$$\mathbb{P}_\varphi(A) \leq \sum_{i=1}^{\infty} \mathbb{P}_\varphi(A_i) = \sum_{i=1}^{\infty} \mathbb{P}_\varphi(X_i = 1) = \sum_{i=1}^{\infty} ||p_i^1 \varphi||^2 = \sum_{i=1}^{\infty} \left( \frac{1}{3} \right)^i < 1.$$

It is worth pointing out that the counter-example above has $\Gamma(t)$ finite for all $t \in S$, so that this extra assumption (as made in parts of Section 4) would not change the conclusion here.
8.8 Can we eliminate the topological closure of $F_A$ in part (b) of Theorem\ref{thm:7} or part (d) of Theorem\ref{thm:9}?

It is natural to ask if we can replace $F_A$ with $F_A$ in part (b) of Theorem\ref{thm:7} or part (d) of Theorem\ref{thm:9}. This is important for interpretations of quantum mechanics, as the condition $\Psi \in F_A$ can naturally be proposed to imply that if the state of the universe is $\Psi$, then the event $A$ should not be part of our experiences. But that the condition $\Psi \in F_A$ should also have this implication is a more delicate philosophical issue. In \cite{31} and \cite{32} this lead to the consideration of a version of the superposition principle to reach this conclusion. (See Subsection 8.9 below.)

A simple argument, though, shows that in relevant situations the closure of $F_A$ is needed to make part (b) of Theorem\ref{thm:7} and part (d) of Theorem\ref{thm:9} true. As observed in Proposition\ref{prop:10} for any family of events $A_\alpha \in \Sigma$, if $\varphi \in F_{A_\alpha}$, for each $\alpha$, then $\varphi \in F_{\cup \alpha A_\alpha}$. But $\mathbb{P}_\varphi(A_\alpha) = 0$, for all $\alpha$ does not imply that $\mathbb{P}_\varphi(\cup \alpha A_\alpha) = 0$, unless this family of events is countable.

8.9 $\overline{F_A}$, $N_A$ and superposition of states

We recall now how in \cite{31} and \cite{32} the statement $\Psi \in \overline{F_A}$ was translated into the statement that $\Psi$ is a superposition of states in $F_A$. $\Psi \in F_A$ means that there are $\zeta_1, \zeta_2, \ldots$ such that $\zeta_i \in F_A$ and $\zeta_i \rightarrow \Psi$, as $i \rightarrow \infty$. Equivalently, $\zeta_1 + (\zeta_2 - \zeta_1) + (\zeta_3 - \zeta_2) + \ldots$ converges to $\Psi$. We can apply the Gram-Schmidt orthonormalization procedure (see p.46 of \cite{26}, or p.167 of \cite{14}) to the vectors $\zeta_1, \zeta_2 - \zeta_1, \zeta_3 - \zeta_2, \ldots$ to produce a sequence of orthonormalized vectors $\eta_1, \eta_2, \ldots$ that have the same closed span, to which $\Psi$ belongs. Since $F_A$ is a vector space and $\zeta_i \in F_A$, this procedure (which only involves linear operations) gives us that also $\eta_i \in F_A$. Set $\Psi_i = \langle \Psi, \eta_i \rangle \eta_i$, $i = 1, 2, \ldots$. Then the vectors $\Psi_i$ are orthogonal to each other, $\Psi_i \in F_A$ for each $i = 1, 2, \ldots$ and $\sum_{i=1}^{\infty} \Psi_i = \Psi$. A converse statement is trivial, any convergent series of vectors in $F_A$ converges to a vector in $\overline{F_A}$.

Referring to $N_A$, as defined by \cite{87}, since those are also vector spaces, a similar derivation applies to $N_A$.

In words, using common quantum-mechanics jargon: Belonging to $\overline{F_A}$ (resp. $N_A$) is the same as being a superposition of orthogonal states in $F_A$ (resp. $N_A$).

The version of the superposition principle proposed in \cite{31} and \cite{32} can be rephrased, replacing prediction with ontology, in the following fashion. Here all the mentioned universes are supposed to be described by the same Hilbert space $\mathcal{H}$ and group of unitary evolution operators $\{U_t\}$, and their state is given in the Heisenberg picture by an element of $\mathcal{H}$.

One-Sided Superposition Principle: If an event $A$ is not realized in universes that are in states $\Psi_i$, $i = 1, 2, \ldots$, then it is also not realized in a universe in state $\Psi = \sum_{i=1}^{\infty} \Psi_i$.

The reason for the title of “one-sided superposition principle” is that if we remove the word “not” in the two places that it appears, we obtain a statement that is certainly false, due to interference. Superpositions cannot create new realities, but they can eliminate realities by interference.

If we accept the idea that in a universe in a state $\Phi \in F_A$ (resp. $\Phi \in N_A$), the event $A$ is not realized and accept also the one-sided superposition principle, then we conclude that the same is the case in a universe in a state $\Psi \in F_A$ (resp. $\Psi \in N_A$).

In particular, if we accept the one-sided superposition principle and the idea that in a universe in a state $\Phi \in N$ no event in $\Sigma$ is realized, then we conclude that no event in $\Sigma$ is realized in a universe in a state $\Psi \in N = \mathcal{H}_+^\perp$.

8.10 Should we believe that in our universe $\mathcal{H}_\pi \neq \{0\}$, and $\Psi_\pi \neq 0$? The role of decoherence, the “we-are-here” argument, and the ordinary nature of the present time on a cosmological scale

In this subsection we are considering one of the particle models of Subsection 8.2 as a model for our universe. And we are considering the sort of $\pi$ discussed there, associated to a partition of the configuration space according to sets with macroscopically meaningful descriptions. But we should make one modification in how $\pi$ is chosen, because we are only interested in times that, on a cosmological scale, are not too early nor too late. For this reason we will assume $S = (t_-, t_+)$, where $-\infty < t_- < s < t_+ < \infty$, with $s$ being the present moment, and the differences $s - t_-$ and $t_+ - s$ being of the order of cosmological times.

It is natural to ask whether, for some values of $t_-$ and $t_+$ as above, we should believe that $\mathcal{H}_\pi \neq \{0\}$, and more specifically $\Psi_\pi \neq 0$, where $\Psi$ is the Heisenberg-picture state of our model universe.

There are three complementary ideas to discuss.
What would it mean if \( \Psi \) the mathematical results in this paper support a minimalist ontology for non-collapse quantum mechanics, that would not be there to realize that we were wrong.

In this scenario, whatever event in the universe after time \( \pi \) will indeed not happen. In this scenario, as our own existence would not go beyond time \( \pi \), we would have predicted not to happen after time \( \pi \) will actually not happen. In this scenario, as our own existence would not go beyond time \( \pi \), we would have erred in the opposite direction, incorrectly predicting events to happen that will actually not happen. In this scenario, as our own existence would not go beyond time \( \pi \), we would not be there to realize that we were wrong.

### 8.11 A minimalistic ontology for non-collapse quantum mechanics

The mathematical results in this paper support a minimalistic ontology for non-collapse quantum mechanics, that conforms with our experiences, including our perceptions of apparent collapses of the wave function according to Born’s rule. This ontology is also compatible with the one-sided superposition principle of Subsection 8.9.
In this subsection we will build the theory based only on the mathematical results in Section 2. In the next two, we will further elaborate on this theory, using also notions from Section 7. And in Subsection 8.14 we will see how this ontology relates to the results in Sections 8 and 4.

In this ontology, the primary physical reality is limited to a vector $\Psi$ which belongs to a Hilbert space $\mathcal{H}$ and a strongly continuous group of unitary operators on $\mathcal{H}$, $(U_t)_{t \in T}$, where $T = \mathbb{R}$, or $T = \mathbb{Q}$. In the latter case we are assuming that only rational times have physical meaning, as discussed in Subsection 8.6. We could also entertain the idea of assuming $T = \epsilon \mathbb{N}$, where $\epsilon$ is a time interval shorter than anything that we can (presently) measure.

At this point it is important to explain what we mean by “primary reality” and how it differs from the broader use of “reality” below. For a good illustration of the distinction consider the concept of cellular automata, as the well known Game of Life, [20]. The primary reality is limited to a grid, a deterministic updating rule in discrete time and an initial configuration of alive and dead cells of the grid. (Those are analogous, respectively, to our $\pi$-patterns of $\Psi$.)

In addition to this primary reality, there are patterns of alive and dead cells that develop and propagate in time. And this is actually the reason for the interest in the model. In particular because these propagating patterns can produce computations as a Turing machine. In our terminology, such patterns are elements of the “derived reality”, or simply “reality” of the system. If the grid and the deterministic updating rule are fixed, we may regard the patterns that develop and propagate in time as features of the initial configuration.

We need to propose a theory about the nature of our experiences, compatible with the primary quantum ontology proposed above, and with the fact that these experiences are well described by textbook quantum mechanics (with collapse according to Born’s rule). Theorem 11 and the analogy above suggest an answer: Our experiences are in one-to-one correspondence with a class of patterns in $\Psi$.

We should think of each possible $\pi$, with $S \subset T$, as a tool for analysing the features of $\Psi$. For this reason, we will call each such $\pi$ an “analysers”. Given such an analyser, we have its associated sets $\Omega$ and $\Sigma$. And Theorem 11 provides us with $\{p_A : A \in \Sigma\}$, which is a projection valued measure on $\mathcal{H}_\pi$. Given $A \in \Sigma$, we say that $A$ is a $\pi$-pattern in $\Psi$ if $p_A \Psi \neq 0$.

The proposal is to regard any $\pi$-pattern $A$ for any analyser $\pi$ as part of the reality defined by (or derived from) $\Psi$. The corresponding postulate is:

**Ontological Postulate:** For any analyser $\pi$ and any $A \in \Sigma$,

\[ A \text{ is part of reality } \iff p_A \Psi \neq 0, \quad (88) \]

And the idea is that our experiences are in one-to-one correspondence with the $\pi$-patterns of $\Psi$ for an appropriate $\pi$.

In short: that our experiences are $\pi$-patterns of $\Psi$, for a certain $\pi$.

There are several interesting aspects of such a theory.

First, it satisfies the one-sided superposition principle of Subsection 8.9:

If $p_A \Psi_i = 0$, for $i = 1, 2, \ldots$, then $p_A (\sum_i \Psi_i) = 0$.

Second,

\[ p_A \Psi \neq 0 \iff p_A \Psi_\pi \neq 0, \quad (89) \]

so that, for the relevant $\pi$, our experiences are only affected by $\Psi_\pi$, not by what $\Psi - \Psi_\pi$ may be. As a consequence, we have no information, through our experiences, of what $\Psi - \Psi_\pi$ is. For us, it is as if the Heisenberg-picture state of the universe were $\Psi_\pi$, rather than $\Psi$. And since we have experiences, it must be the case that $\Psi_\pi \neq 0$.

Third, Theorem 11 implies that, if $\Psi_\pi \neq 0$, then (88) is equivalent to

\[ A \text{ is part of reality } \iff \mathbb{P}_{\Psi_\pi}(A) \neq 0. \quad (90) \]

Now, which $\pi$ is relevant in describing our human experiences? We will start with a broad proposal, then scrutinize it and settle for a very precise instance of that proposal as our $\pi$.

The natural starting point is to assume that the setting is one of the particle models of Subsection 8.2. And that $\pi$ is as defined there, with each $p_{i\alpha}$ associated to a subset $R_{i\alpha} = \bigcup_i R_{\alpha,i}$ of the configuration space $C$, which admits a microscopically meaningful description to us, labeled by $\alpha \in \Gamma$, and where $i$ gives the number of particles of each type present in each component $R_{\alpha,i}$ of $R_{\alpha}$. To assure that $\Psi_\pi \neq 0$, the set $S$ may need to be limited to an interval $(t_-, t_+)$ of $T$, for some $t_-$ that is finite and significantly smaller than the present time on a cosmological scale, and some $t_+$ that is finite and significantly larger than the present time on a cosmological scale, as explained in Subsection 8.10.
And, as emphasized in that subsection, the physical phenomenon responsible for $\Psi_\pi \neq 0$ is decoherence. And the meaning of “macroscopic” in the definition of $\pi$ relates to our sets $R_\pi$ being defined by the positions of large numbers of particles that allow for environmental decoherence to happen.

We can now theorize that our experiences are the events $A$ that are $\pi$-patterns of $\Psi$, for a $\pi$ as defined in the last paragraph.

This works well in that (90) implies that human experiences are precisely the ones that have positive probability according to standard Born-collapse quantum mechanics. Here it is important to stress that we perceive the Heisenberg-picture state of the universe as $\Psi_\pi$, rather than $\Psi$, as $\Psi - \Psi_\pi$ does not affect us. And that the “effective state of the universe for us”, $\Psi_\pi$, is what we use in computing the Born probabilities. It is also important to understand that if $\mathbb{P}_{\Psi_\pi}(A) > 0$ and also $\mathbb{P}_{\Psi_\pi}(A^c) > 0$, then both are human experiences, but in the Everettian sense that humans branch, and along each branch only perceive one of these two events. What is accomplished here is to eliminate the naive, but important, criticism of non-collapse quantum mechanics, according to which if no collapse happens, then every event $A \in \Sigma$ would happen. Events with $\mathbb{P}_{\Psi_\pi}(A) = 0$ have $p(A)\Psi = 0$ and, according to the postulate above, do not happen in any branch!

There is a way to explain the last statement above that may be helpful in convincing skeptical readers, who would insist that without collapses every $A \in \Sigma$ would happen. Consider a fictitious model universe, in which nature picks an infinite collection of independent realizations of one of the stochastic processes $(x_t)_{t \in \mathbb{R}}$, defined in Subsection 8.3, that satisfy (89) with $\varphi = \Psi_\pi$. In this model universe, we have infinitely many trajectories $x_t$ in configuration space, and we can think of this ensemble of independent trajectories as an ensemble of different worlds that do not interact with each other. In each one there are human experiences that are identical to those predicted by Born-collapse quantum mechanics, in a universe in state $\Psi_\pi$. And since there are infinitely many of these worlds, each event $A \in \Sigma$ that has $\mathbb{P}_{\Psi_\pi}(A) > 0$ is experienced in some of them (actually in infinitely many of them). But no event $A \in \Sigma$ that has $\mathbb{P}_{\Psi_\pi}(A) = 0$ is experiences in any of them. The assumption that our experiences correspond to events that are $\pi$-patterns of $\Psi$ is equivalent, thanks to Theorem 1, to the statement that our experiences are identical to those of the humans in this model-infinite-ensemble universe. Now, imagine that in this model-infinite-ensemble universe a group of scientists is performing a sequence of identical independent quantum experiments that may each time result in outcome 1 with Born-probability 0.9, or outcome 2 with Born-probability 0.1. In each experiment there are worlds in which outcome 1 happens and worlds in which outcome 2 happens. But in no world does the frequency of outcomes 1 converge to a number different from 0.9. Having every possible outcome in each single experiment in this universe does not imply having every possible outcome in infinite series of experiments.

Another issue that may be raised is whether (90) captures all the ways in which Born’s-rule-collapse probabilities are used in standard collapse quantum mechanics. Argument answering this question in the affirmative is presented in Sections 3 and 6 of [31].

Now to some essential criticism of the kind of choice above of $\pi$. There is a substantial amount of subjectivity involved. What is “meaningful”? Would we all agree that a certain family of sets $R_\alpha$ correspond to each label $\alpha \in \Gamma$ that we describe in a certain way? How fine can the partition of the configuration space be to still allow decoherence to happen? Fortunately, there is a good way to solve these issues, if we accept the (currently standard) view that our perceptions are encoded in the physical state of our brains, and that our mental processes are in one-to-one correspondence with computational processes produced as our brains behave according to the same physical laws that apply to everything else.

Before returning to humans, it is helpful to consider a computer of the kind that we build with silicon. In this context, we can introduce the relevant analyser $\pi$ by partitioning the configuration space into the sets $R_\alpha$ labeled by the computational states of the computer, including a set in which the computer is not present in the universe and a set in which the computational state of the computer may not be well defined. Instead of “experiences that it has”, we should talk of “computation that the computer performs”. If occasionally the computer receives bits of input that correspond to outcomes of quantum experiments, the computer will branch in an Everettian sense, with each branch continuing to compute separately from the others. If we accept (89), and suppose that, due to decoherence (expected, since each bit of information is encoded by the state of a very large number of particles) $\Psi_\pi \neq 0$, then, by (90), events $A \in \Sigma$ (which now pertain to the computational processes of this computer) will be part of the collective reality of the branching versions of the computer when and only when $\mathbb{P}_{\Psi_\pi}(A) > 0$. The important point that we want to emphasize is that here the partition of the configuration space into the parts $R_\alpha$ is objective!

If the story above involving a computer is understood and we accept the hypothesis that our mental processes are manifestations of computational processes in our brains, then there is no relevant difficulty in replacing the computer by the family of humans. We should partition the configuration space $\mathcal{C}$ according to the computational states of our brains, including a set in which there are no humans present and (possibly) sets in which some humans have undefined
When we consider a $\pi$ and all our perceptions are the same that would happen if the state of our universe were $\Phi = \Psi$, computers, books (including all the letters and digits printed on each page), the moon (even if no one looks at it), ... Using Theorem 8 and the fact that $H_A = 0$, which from (89) and what we saw in the last paragraph.

One can make the case that in interpreting quantum mechanics all we have to accomplish is to produce a coherent, logically consistent, theory and predict our human perceptions correctly from it. (See, e.g., Chapter 9 of [30] and references therein.) The theory presented here, with $\pi_H$ as the relevant analyser from the human perspective, fulfils these requirements!

One should not misunderstand the statement that $\pi_H$ is the relevant analyser for the purpose of predicting human perceptions, with the incorrect idea that in the theory only $\pi_H$-patterns are part of reality. The postulate above applies to every analyser $\pi$, as stated. And there are good philosophical reasons for thinking about what we can learn from considering other related analysers. This is the subject of the next subsection.

8.12 Refinements of $\pi_H$ and coarsenings of refinements of $\pi_H$. The realm of classical physics $\pi_C$.

Among the partitions of the configuration space into sets with “macroscopically meaningful descriptions”, an important class is that of refinements of $\pi_H$. It is true that we may sometimes disagree on the precise borders of the sets $R_a$, but often we all agree with several of the relevant macroscopic descriptions, as for instance with the description of a measuring device pointing to a certain result, or with the letters that are printed on a piece of paper. And this allows us to consider various interesting refinements of $\pi_H$. It is worth looking at this in some more detail. The analyser $\pi_H$ corresponds to a partition of the configuration space into sets $R_a$, where each $a$ corresponds to a given state of mind for each human present. And this means that in producing this partition, we are concerned with the location of the particles in the brains of the humans. To refine $\pi_H$, we break each $R_a$ into parts, according to the location of many other particles that form all sorts of other things we may include and that have non-controversial macroscopic descriptions for us. This could include all the other particles forming the bodies of these humans, forming other animals, like cats (alive and dead), computers, books (including all the letters and digits printed on each page), the moon (even if no one is looking at it), ...

The effective state of the universe based on our perceptions is the vector $\Phi = \Psi_{\pi_H}$. And when we compute Born-rule probabilities we are using the best information we have to approximate $\Phi$ (or at least how $\Phi$ looks inside our lab). When we consider a $\pi$ that refines $\pi_H$, as in the previous paragraph, we may wonder if $\Psi_{\pi}$ would not be substantially different from $\Phi$. But we should not worry about it when the refinement of $\pi_H$ is, as above, based on “macroscopic descriptions”, once again because of decoherence. The point is that we are aware of the phenomenon of environmental decoherence, and have proposed ways in which it happens in our universe, based on our knowledge about the state of the universe that we perceive, namely $\Phi$. From this knowledge, we see that $\Phi$ includes a rich enough environment to assure enough decoherence affecting cats, books, the moon, etc, that $\Phi \in \mathcal{H}' = \mathcal{H}_\pi$, as explained in Subsection 8.10.

Therefore, since $\mathcal{H}_\pi \subset \mathcal{H}_{\pi_H}$, we have $\Psi_{\pi} = p_{\pi} \Psi = p_{\pi} p_{\pi_H} \Psi = p_{\pi} \Psi_{\pi_H} = p_{\pi} \Phi$. $\Phi$

It is interesting to compare a typical event $A \in \mathcal{H}_{\pi}$ with its refinement $A' \in \Sigma$, corresponding to a $\pi$ that refines $\pi_H$ in the fashion described above. (For the definition of $A'$ see Section 7.) The intuitive meaning of $A'$ is that it provides all the ways in which $A$ could happen in terms of descriptions based on $\pi$.) $A$ could, for instance, be the event that “at time $t_1$ Jane saw three moons of Jupiter, and between times $t_2$ and $t_3$ Hui heard a meow sound”. In comparison, depending on what $\pi$ is, $A'$ could also include a description of the telescopes that Jane could have used, the hats she possibly had on, the possible expressions on her face, ..., and the cat that produced the sound that Hui heard, or the person who was imitating the sound of a cat, or .... Using Theorem 8 and the fact that $p_{\pi} \Phi = \Phi$, we have

$$P_\Phi(A') = ||p_{A'} \Phi||^2 = ||p_{A'} p_\Phi \Phi||^2 = ||p_{A'} \Phi||^2 = P_\Phi(A).$$

So that $A'$ is part of reality if and only if $A$ is. We can also consider $B \subset A'$ that specifies a certain telescope, a certain hat, and a certain cat. In the view presented here, these are parts of the reality derived from $\Psi$, provided that $p_B \Psi \neq 0$, which from [39] and what we saw in the last paragraph amounts to $p_B \Phi \neq 0$, or equivalently $P_\Phi(B) > 0$. Now, if in $C \subset A'$ Jane was looking at Jupiter, at time $t_1$, with naked eyes, then the laws of physics would entail $p_C \Psi = p_C \Phi = 0$ and, in particular, we would have $P_\Phi(C) = 0$. 

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And as much as Jane had branched in her life, this event would not be part of her reality along any branch. The point of all of this is that human perceptions and their causes “out there in the world” are, in the view presented here, possible parts of reality, and physics laws restrict them and relate them in the proper way. (In this paragraph we abused notation in the following way. Theorem 1 provides a probability measure $\mathbb{P}_\Phi$, on $(\Omega_H, \Sigma_H)$ associated to $\pi_H$ and a different one, on $(\Omega, \Sigma)$, associated to its refinement $\pi$. We should have distinguished them in the notation used, but did not do it, since when we write $\mathbb{P}_\Phi(A)$, with $A \in \Sigma_H$, or $\mathbb{P}_\Phi(C)$, with $C \in \Sigma$, it should be clear which probability measure we mean. We will continue to abuse notation in this way, when no confusion is possible.)

Of course, there should also be refinements of $\pi_H$ that specify the mental state of other animals. The only reason we did not include this feature in $\pi_H$ itself was lack of necessity, when our task was to account for human experience.

It is also natural to consider coarsenings of $\pi_H$, or of the refinements of $\pi_H$ discussed above. Those can naturally be obtained by focusing on a subset of the humans and then lumping together the $R_a$ according to the state of mind of the humans in this subset, regardless of the state of mind of the other humans. For instance, at the risk of being called solipsistic, an individual may consider only the states of his or her mind in defining $\pi$, and perhaps refine it to include only certain aspects of the world of his or her interest. In another example, as we make predictions for our foreseeable future, we may cap the number of humans that we distinguish in $\pi$ to a large but finite upper bound, lumping together in one $R_a$ all the configurations with a larger number of humans. Assuming that each human brain can only encode finitely many different computational states, we see that the corresponding $\Gamma$ is now finite.

Again we should wonder if $\Psi_x$ could be substantially different from $\Phi$ in case, say, that only the state of mind of some of the humans is used in the definition of $\pi$. Here the equality $\Psi_x = \Phi$ seems to be justified by invoking the assumption of “homogeneity of the scales on which decoherence operates”. As we understand it, decoherence affects our brains by means of phenomena that operate in homogeneous fashion in a very large scale, including the electromagnetic background radiation that fills the universe. It is true that the phenomena that we are aware of as producing decoherence are accounted for by $\Phi$, and we do not know anything about $\Psi - \Phi$. But it seems reasonable to think that that component of $\Psi$ shares this sort of large scale homogeneity feature, since it is part of the same natural phenomenon, namely our universe. This assumption implies that the component of $\Psi$ on which the brain of each one of us decoheres should be the same one. And therefore it should be the common $\Phi = \Psi_{\pi_H}$ that we infer from our perceptions.

We can now summarize what we propose for our universe, based on the ideas presented above. We suppose that it is well modeled by a particle model of the kind described in Subsection 8.2 with the appropriate particles. Those, as far as we currently understand it, are the photons, the quarks, the leptons, etc, from what is called the “standard model”. The corresponding Hilbert space $\mathcal{H}$ is the appropriate Fock space. The nature of the Hamiltonian and of the Heisenberg-picture state vector $\Psi \in \mathcal{H}$ are such that they allow for an analyser $\pi_C$, which is a substantial refinement of $\pi_H$ based on the objects that we describe in classical physics, and has the properties that yield what we call our “classical world(s)”, or the “realm of classical physics” (hence the “C” in $\pi_C$). These properties include the assumption that $\Psi_x = \Phi$ is the same for a wide range of coarsenings $\pi$ of $\pi_C$. The idea is that $\Phi$ provides enough decoherence, and $\Psi - \Phi$ does not add to it across the relevant scales, to assure this constancy of $\Psi_x$. Here $\pi$ could be as fine as $\pi_C$ itself, or as coarse as only describing a piece of dust, say (anything that is still well described by classical physics would fit here). Other examples of allowed $\pi$ would be $\pi_H$ and the other analysers mentioned above in this subsection, or the $\pi$ associated to the computational states of a computer, from the previous subsection. How fine can the partition corresponding to $\pi_C$ be? This is the question of understanding how small something can be to still be the subject of enough decoherence in $\Phi$ to fit in the definition of $\pi_C$ (and hence be called “macroscopic”). Reflecting on this question makes clear that there is a certain fuzziness in the definition of $\pi_C$, and the answer may depend on the cosmological time scale considered, for the reasons presented in Subsection 8.10. This question is nothing but the question of determining the limits of the realm of classical physics, which certainly has a fuzzy boundary, which can and should be investigated experimentally. Another question is what we can say about $\Psi - \Phi$. The answer suggested by all the considerations so far is nothing. This vector could be 0, it could be comparable to $\Phi$ in norm, or it could be much larger than $\Phi$. In any case it would not affect our experiences in the theory proposed here, based on the Ontological Postulate from Subsection 8.11 and the idea that our mental processes correspond to $\pi_H$-patterns of $\Psi$, or equivalently, $\pi_C$-patterns of $\Psi$. (To see this last equivalence, let $A \in \Sigma_H$ and $A'$ be its refinement to $\pi_C$. Then Theorem 8 tells us that $\mathbb{P}_A\Psi = \mathbb{P}_A\Phi = \mathbb{P}_A\Psi$, so that $A$ is a $\pi_H$-pattern of $\Psi$ iff $A'$ is a $\pi_C$-pattern of $\Psi$. Note that, as in the example involving Jane and Hui, here we also have $\mathbb{P}_A(A) = \mathbb{P}_A(A')$. Since $\Psi - \Phi$ has no effect on our experiences, and is therefore inaccessible to us via experiments, it is tempting to assume that $\Psi = \Phi$, i.e., that $\Psi \in \mathcal{H}_{\pi_C}$. This assumption is nevertheless not needed and may be criticized as being anthropocentric. For this reason we take an agnostic position on whether $\Psi \in \mathcal{H}_{\pi_1}$, or not. The analyser $\pi_H$ and the other ones discussed so far in this subsection involve “macroscopic” objects (we are supposing that each computational bit in our brains is also encoded by the state of a large number of particles). The relevance of
this assumption was emphasized repeatedly above, in connection to the need of decoherence. But in a refinement of \( \pi_H \), could we include microscopic phenomena too? For instance, if we are performing a double slit experiment with one electron, can we refine \( \pi_C \), by partitioning the sets \( R_n \) according to the location of this electron while in flight? Nothing prevents us from doing it. We lose the assurance that \( \Psi_\pi \neq 0 \). But in case we still have \( \Psi_\pi \neq 0 \), we can even associate probability \( P_{\pi_\phi}(A) = ||p_A||^2/||\Psi_\pi||^2 \) to an event \( A \) involving the location of the electron. And this probability will be positive if and only if the event \( A \) is part of reality. If \( \Psi_\pi = 0 \), then also \( p_A \Psi = 0 \), and \( A \) is not part of reality. Now, it should be pointed out that these considerations have no operational meaning for us humans. When for an event \( B \in \Sigma_H \) we compute \( P_{\pi_\phi}(B) \), the meaning is that \( B \) is part of our perceptual reality if and only if this probability is positive. So that we can predict that if this probability is 0, \( B \) will not happen. But in the case of the event \( A \) mentioned above, whether it is or is not part of reality has no implication for our perceptions, unless those are also accounted for by an event \( B \in \Sigma_H \). From a pragmatic point of view, this is in agreement with the quantum recipe of the textbooks, that tell us to only associate probabilities to outcomes of experiments, not to events like the location of the electron in the double slit experiment, before it hits the screen. In the view presented here, it is not that such probabilities cannot be defined and related to the ontology of the theory. They can, when \( \Psi_\pi \neq 0 \). The issue is that they are related to aspects of that ontology that are not amenable to experimental scrutiny by us.

We end this subsection with a further discussion and clarification of the role of some of the different probability measures provided by Theorem 11 part (b). Strictly speaking, for each analyser \( \pi \) we have a distinct measurable space \( (\Omega, \Sigma) \), and then, for each \( \varphi \in \mathcal{H}_\pi \setminus \{0\} \) we have a distinct probability measure \( P_{\pi_\phi} \) on this space. This is a very large set of probability measures! Now, the proposal in the previous subsection and in this one is that the most relevant analysers, from our perspective, are \( \pi_H \) and its refinement \( \pi_C \), that both share \( \Phi = \Psi_\pi \) and that the relevant probabilities are \( P_{\phi}(A) \), either for \( A \in \Sigma_H \), or \( A \in \Sigma_C \) (abusing notation as explained before). The probabilities \( P_{\phi}(A) \), for \( A \in \Sigma_C \), are the ones that the quantum mechanics textbooks tell us to compute, to make predictions. (In doing it, they sometimes state that a microscopic system must interact with a macroscopic measuring device, for a “potentiality” to become a “reality”. The role of decoherence removes the need for such mysterious statements.) For instance, in a double slit experiment with a single electron, in which we are using an old fashioned photographic plate as the screen, \( A_i \) may be the event in \( \Sigma_C \) that a pixel \( i \) is sensitized. \( B_i \) may be the event in \( \Sigma_H \) that Carla sees the pixel \( i \) as a white dot, as she looks at the plate after developing it. And \( B'_i \) will denote the refinement of \( B_i \) to \( \Sigma_C \). The textbooks tell us how to compute \( P_{\phi}(A_i) \) (precisely in the same way that Theorem 11 does), and our understanding of what happens: if one looks at a developed photographic plate tells us that \( B'_i = A_i \), and therefore \( P_{\pi}(A_i) = P_{\phi}(B'_i) = P_{\phi}(B_i) \). (For the justification of the last equality, one can use Theorem 8 as in the example involving Jane and Hui.) Suppose now that \( \pi \) is a refinement of \( \pi_C \) based on the position of the electron at a certain time \( t \), before it hits the screen. And let \( A''_i \) denote the corresponding refinement of \( A_i \in \Sigma_C \) to the sigma-algebra \( \Sigma \) associated to \( \pi \). Lack of decoherence of the electron’s location in \( \Phi \), i.e., lack of recording of the electron’s position at time \( t \) in the environment provided by \( \Phi \), means that \( \Phi \notin \mathcal{H}_\pi^{''} = \mathcal{H}_\pi \) (see Subsection 8.11). And this implies that \( \Psi_\pi = p_C \Psi = p_\pi \pi_C \Psi = p_\pi \Phi = \Phi_\pi \neq \Phi \). In case \( \Phi_\pi \neq 0 \), the probability \( P_{\Psi_\pi}(A''_i) = P_{\Psi_\pi}(A''_i) \) is well defined, but is not related to the relevant probability \( P_{\phi}(A_i) = P_{\phi}(B_i) \) in any simple way.

8.13 The relevance of \( \{ p_A : A \in \Sigma \} \) being a projection valued measure for the validity of our logical reasoning about our perceptions, and for the computational aspect of life

In the theory of Subsection 8.11 our perceptions are events \( A \in \Sigma_H \) that satisfy \( p_A \Psi \neq 0 \). The fact that \( \{ p_A : A \in \Sigma_H \} \) is a p.v.m. explains then the validity of our use of some basic rules of logic in thinking about these perceptions. Below go some basic instances:

If \( A \subseteq B \) and we believe that \( B \) will not be one of our perceptions (along any branch of our existence), then we reason that also \( A \) will not be one of our perceptions. And indeed, (PVM7) implies that if \( p_B \Psi = 0 \), then \( p_A \Psi = 0 \).

If we believe that each one of the disjoint \( A_1, \ldots, A_n \) will not be one of our perceptions, then we reason that also \( A = \bigcup_{i=1}^n A_i \) will not be one of our perceptions. And indeed, (PVM4) implies that if \( p_{A_i} \Psi = 0 \), \( i = 1, \ldots, n \), then \( p_A \Psi = 0 \).

If we believe that one among \( A_1, \ldots, A_n \) will not be one of our perceptions, then we reason that also \( A = \bigcap_{i=1}^n A_i \) will not be one of our perceptions. And indeed, (PVM8) implies that if \( p_{A_i} \Psi = 0 \), for some \( i = 1, \ldots, n \), then \( p_A \Psi = 0 \).

If we believe that each one of the disjoint \( A_1, A_2, \ldots \) will not be one of our perceptions, then we reason that also \( A = \bigcup_{i=1}^\infty A_i \) will not be one of our perceptions. And indeed, (PVM2) implies that if \( p_{A_i} \Psi = 0 \), \( i = 1, 2, \ldots \), then \( p_A \Psi = 0 \).

More formally, the examples above and others can be derived from the observation that the set \( \{ A \in \Sigma_H : p_A \Psi = 0 \} \) is a sigma-ideal, i.e., a family of elements of \( \Sigma_H \) that has the following three properties: It contains \( \emptyset \), is closed with respect to taking subsets and with respect to taking countable unions.
But we do make mistakes: If we believe that each one of the disjoint, but possibly uncountably many, \( A_\alpha \), will not be one of our perceptions, then we (sometimes) reason that also \( A = \cup_\alpha A_\alpha \) will not be one of our perceptions. This is not justified, and counter-examples are not hard to find. For instance, consider an infinite sequence of independent identical experiments that may result in one of two outcomes, each with a Born probability that is positive. Each possible sequence of outcomes has Born probability 0, and therefore will not be one of our perceptions (along any branch). But the union of all the individual outcomes is \( \Omega \), which has \( p_\Omega \Psi = p_\pi \Psi = \Psi_\pi \neq 0 \).

Such mistakes are of little consequence for the survival of a species that usually only needs to deal with finite sets of \( A_\alpha \) at a time. So the persistence of such mistakes, not having been eliminated by natural selection is not a surprise.

It is very interesting to observe that the structure of quantum mechanics, in a rich enough universe like ours, provides for the existence of patterns in the state vector that embed rules of logic and therefore can instantiate classical computations. Computations performed by DNA-based, or RNA-based wetware that are essential for life as we know it, animal brains and silicon based computers are possible thanks to this structure and this richness. (See also the concept of IGUS in \([15]\).)

### 8.14 Reducing the Ontological Postulate to more intuitive statements

The Ontological Postulate can be justified by its success. It produces a theory that makes the same predictions of Born’s-rule-collapse quantum mechanics, without the collapses. But one may wonder if it can be reduced to more intuitive statements.

This can be done as we relate the theory proposed in Subsection 8.11 to the results in Sections 3 and 4 and the ideas from \([31]\) and \([32]\) presented in Subsection 8.9. Suppose that we accept the following two premises:

1. If \( \Psi \in F_A \), then \( A \) is not part of reality. 
2. The one-sided superposition principle.

Then, as explained in Subsection 8.9, we conclude that when \( \Psi \in \overline{F_A} \), \( A \) should not be part of reality. Combining this with part (a) of Theorem \([8]\) we have

\[
p_A \Psi = 0 \implies \Psi \in \overline{F_A} \implies A \text{ is not part of reality.} \tag{91}
\]

This reduces half of our postulate to (P1) and (P2) above.

Now, suppose we go further and also accept a third premise:

3. If \( A \) is not excluded from reality by (P1) and (P2) above, then \( A \) is part of reality.

Since \( F_A \subset \overline{F_A} \) and \( \overline{F_A} \) is closed with respect to taking superpositions, we have then

\[
p_A \Psi = 0 \iff \Psi \in \overline{F_A} \iff A \text{ is not part of reality.} \tag{92}
\]

where the leftmost implication to the left can currently only be justified by Theorem \([2]\) if \( S \) is countable (e.g., \( S \subset \mathbb{Q} \), or \( S \subset \mathbb{N} \)), or by Theorem \([3]\) if \( \Gamma \) is finite. This reduces our postulate to (P1), (P2) and (P3), in these cases.

### 8.15 A close relationship between Born’s rule and the one-sided superposition principle

From \([11]\) in part (d) of Theorem \([1]\) we know that

\[
p_A \Psi = 0 \iff \Psi_\pi = 0 \text{ or } \mathbb{P}_{\Psi_\pi} (A) = 0.
\]

Therefore, if we accept the assumptions (P1) and (P2), from Subsection 8.14, then (91) implies

\[
\mathbb{P}_{\Psi_\pi} (A) = 0 \implies A \text{ is not part of reality.}
\]

And if we also accept (P3) and assume \( S \) countable or \( \Gamma \) finite, then (92) implies

\[
\mathbb{P}_{\Psi_\pi} (A) > 0 \iff A \text{ is part of reality.}
\]

(In the left-hand side, the assumption \( \Psi_\pi \neq 0 \) is implicit, since otherwise \( \mathbb{P}_{\Psi_\pi} (A) \) would not be defined.) These considerations show how closely related Born’s rule is, in the context of non-collapse quantum mechanics, to the one-sided superposition principle, expanding on the thesis of \([31]\) and \([32]\).

It is natural to ask if one can have a version of non-collapse quantum mechanics that does not satisfy Born’s rule, but instead satisfies some other probability rule. This is the case in the context of the particle models of Subsection 8.2 if nature chooses at each time \( t \in \mathbb{R} \), independently of anything else, a point \( x_t \) from the configuration space \( C \) with probability \( \mathbb{P}_{x_t} (x_t \in \cup \mathbb{R}_{\alpha t}) \) proportional to \( ||\phi_\alpha||^2 \), \( \phi = \Psi_\pi \), with some \( \alpha \neq 2 \), where \( \Psi \) is the Heisenberg-picture state of the universe and \( \pi \) is some special analyser, that satisfies \( \Psi_\pi \neq 0 \).
8.16 Choosing between pilot-wave theories and the minimalistic ontology of Subsection 8.11

As we observed in Subsection 8.11 the minimalistic ontology proposed there cannot be distinguished through experiments from the alternative proposal that we live in an infinite-ensemble universe, in which each world in the ensemble is an independent realization of a process \((x_t)_{t \in S}\) that satisfies (85) with \(\varphi = \psi_\pi\), where the partition of the configuration space into the sets \(R_a, a \in \Gamma\) and the set \(S\) correspond to the analyser \(\pi = \pi_H\). The same is true if \(\pi\) is an appropriate refinement of \(\pi_H\), as described in Subsection 8.12 including \(\pi_C\).

Experiments would also not distinguish our experiences under these proposals from those in a universe with a single, or any finite number of independent realizations of such a process \((x_t)_{t \in S}\).

Choosing among these theories seems to be a pure matter of personal taste. If the issue is only experimental adequacy, also pilot-wave theories that satisfy (86), but do not satisfy (85), are alternatives. But here there are already serious manifestations of discontent in the literature: in Section 10.2 of [22] and Section 5 of [23], and in Section 5 of [2], because in such theories our memories and records do not have to correspond to our true past.

One should try to show that (under appropriate conditions on the Hamiltonian, perhaps) a process \((x_t)_{t \in S}\) that satisfies (85) can have very nice properties, including continuous paths interrupted by jumps at creation and annihilation of particles, Markovianity, being described by a differential equation, etc.

If this turns out to be the case, such a pilot-wave theory would probably be very attractive. On the other hand, for people who prefer less baggage in a metaphysical theory and see abstraction as no obstacle (perhaps even an advantage) the minimalistic ontology of Subsection 8.11 will probably still be a better choice.

In the discussion above the choice between the minimalistic ontology of Subsection 8.11 and the alternatives is only about what is considered to be in the primary ontology of the theory. Even in the minimalistic ontology, the processes \((x_t)_{t \in S}\) that satisfy (85) exist in the derived ontology (as mathematical constructs). And an infinite ensemble of independent versions of such processes exists as well in the same sense. And as we saw in Subsection 8.11 such an infinite ensemble helps us understand the meaning of the Ontological Postulate introduced there. This postulate is equivalent to the statement that the only events in \(\Sigma\) that are realized are those that are realized in this infinite-ensemble of independent processes. Therefore one can think of these processes as aids to the visualization of the realities encoded in the primary ontology given simply by \(\mathcal{H}, (U_t)_{t \in \mathbb{T}}\) and \(\Psi\).

A person choosing a metaphysics in which a realization of a process \((x_t)_{t \in S}\) is also part of the primary ontology will be faced with the question of why to prefer one single realization of such a process with this status rather than infinitely many independent ones. In other words, between a pilot-wave theory with a single realization and one with an infinite ensemble, why prefer one to the other? (The remaining choice of a finite number, larger than 1, of realizations will probably be discarded in comparison with those, for lack of motivation, or even on aesthetic grounds.)

In favor of a single realization one can argue that it is more economic and at the same time sufficient. (But then, the minimalistic ontology is even more so.) Perhaps one would add a strong preference for having a theory without human branching. (But humans would still be branching in the wave function, even if only one branch would be considered to be real. And this raises the question of what to make of the humans in the branches of the wave function that evolve and behave like you and me, without being “real”.)

In favor of an infinite ensemble one can argue that it preserves the symmetry among the branches of the wave function. For instance, at the end of the infamous experiment involving a cat, there will be two branches that evolve quite differently (for the cat at least). Regardless of the way a single process \((x_t)_{t \in S}\) goes at the end of this experiment, both branches evolve according to \((U_t)_{t \in \mathbb{T}}\) in ways that encode coherent subsequent stories. Why would nature produce all of this and only realize one of these stories?

The debate above, between one or an infinite ensemble of realizations of \((x_t)_{t \in S}\) may be taken as an argument in favor of the minimalistic ontology. That ontology avoids the choice, by considering any number of such processes with the same status of derived realities. All very useful for our understanding of how the universe evolves, of our place in it, and why we can use textbook quantum mechanics to predict our future and describe our past.

One should not think that by being placed in the derived ontology the pilot-wave processes \((x_t)_{t \in S}\) become less important. Their existence, as mathematical objects, shows that it is possible to have the particles of the universe move in physical space, between their creation and annihilation, in ways that are compatible with our experiences and our records and memories of these experiences. This negates the very common statements according to which nothing like this could be done. See for instance Chapters 1, 2 and 7 of [7], for an extensive criticism of such statements.
The reader should have inferred from the discussion in this subsection and previous ones what my own preferences for interpretations of quantum mechanics are. Among those discussed here, they are ordered as follows. The minimalistic ontology first, followed by an infinite-ensemble-pilot-wave model that satisfies (85), followed by a single-pilot-wave model that satisfies (85). The option of a pilot-wave model that satisfies (86), but does not satisfy (85) does not seem plausible to me, for the reasons (admitting incorrect records and memories) presented before.

Readers are invited to come to their own conclusions and to possibly apply the theorems in this paper in different ways that may further shed light on issues in the interpretation of quantum mechanics. They are also invited to expand and elaborate on the proposals in this paper, and to possibly settle mathematical issues left open here, as the conjecture raised after Theorem 3 and the questions posed in Subsection 8.4.

Acknowledgements: It is a pleasure to thank Marek Biskup, Christopher de Firmian and Jim Ralston for enlightening conversations. Jim Ralston also deserves many thanks for carefully reading the paper and making several constructive suggestions. Thanks are also given to Jean Bricmont and Shelly Goldstein for enlightening conversations on pilot-wave theories, especially Bohmian mechanics.

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