Dynamics of state-wise prospective reserves in the presence of non-monotone information

Marcus C. Christiansen\textsuperscript{1} and Christian Furrer\textsuperscript{2,3,}\textsuperscript{*}

\textsuperscript{1}Institut für Mathematik, Carl von Ossietzky Universität Oldenburg, Carl-von-Ossietzky-Straße 9–11, DE-26129 Oldenburg, Germany.
\textsuperscript{2}PFA Pension, Sundkrogsgade 4, DK-2100 Copenhagen Ø, Denmark.
\textsuperscript{3}Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark.
\textsuperscript{*Corresponding author. E-mail: furrer@math.ku.dk.

In the presence of monotone information, stochastic Thiele equations describing the dynamics of state-wise prospective reserves are closely related to the classic martingale representation theorem. When the information utilized by the insurer is non-monotone, classic martingale theory does not apply. By taking an infinitesimal approach, we derive generalized stochastic Thiele equations that allow for information discarding. The results and their implication in practice are illustrated via examples where information is discarded upon and after stochastic retirement.

Keywords: Life insurance; Stochastic Thiele equations; Infinitesimal martingales; Marked point processes; Stochastic retirement

JEL Classification: G22; C02

1. Introduction

Life insurers frequently employ reduced information in the valuation of liabilities due to e.g. legal constraints and data privacy considerations or to achieve model simplifications. The possibility of information discarding leads to potentially decreasing flows of information for which classic martingale theory does not apply. Based on the novel \textit{infinitesimal approach} proposed and developed in Christiansen (2020), we study the dynamics of so-called state-wise prospective reserves in the presence of non-monotone information. Our main contribution is a generalization of the stochastic Thiele equations of Norberg (1992, 1996) to allow for non-monotone information. Secondary contributions include a careful study of the concept of state-wise prospective reserves and a discussion of current actuarial practices regarding valuation in relation to information discarding upon and after stochastic retirement.

In this paper, the only source of randomness consists of the state of the insured, which is modeled as a non-explosive pure jump process on a finite state space. This places our work within the field of multi-state life insurance mathematics. The definitions of retrospective and prospective reserves in Norberg (1991) encompass non-monotone information, and under (semi-)Markovian
assumptions specific instances of non-monotone information appear in the study of retrospective reserves and bonus prognosis, see Norberg (1991, 1999, 2001) and Helwich (2008). But to our knowledge, the literature contains no attempts at the development of a unifying theory for non-Markovian models under non-monotone information. Our contribution constitutes the first step towards this goal, since we impose no restrictions on the intertemporal dependency structure and allow for general information discarding occurring at stopping times w.r.t. the state of the insured.

The multi-state approach to life insurance dates back at least to Hoem (1969), where Thiele equations describing the dynamics of the state-wise prospective reserves are derived under the assumption that the process governing the state of the insured is Markovian. These differential/integral equations were revisited by Norberg in his seminal paper Norberg (1991) and have since been generalized in various directions. This includes relaxing the assumption of Markovianity to allow for duration dependency (semi-Markovianity), taking market risks into account, and the study of higher order moments of prospective reserves, see e.g. Møller (1993), Steffensen (2000), Helwich (2008), Adékambi and Christiansen (2017), and Bladt et al. (2020). We should mention that while the approach of Steffensen (2000) is very general, the results are only established under strict smoothness conditions that might not be satisfied in practice.

The ordinary Thiele equations are essentially Feynman-Kac type results. In contrast, the stochastic Thiele equations of Norberg (1992, 1996) are stochastic differential equations that apply irregardless of the intertemporal dependency structure and reveal the universality of Thiele’s original equation. Furthermore, under Markovian assumptions, stochastic Thiele equations can be used to elegantly derive Feynman-Kac formulas for the prospective reserve.

In the presence of monotone information, the dynamics of prospective reserves are characterized by identifying integrands in the classic martingale representation theorem for random counting measures (Norberg, 1992, 1996; Christiansen and Djehiche, 2019). In similar fashion, our approach relies on the infinitesimal martingale representation theorem of Christiansen (2020), which extends the classic martingale representation theorem for random counting measures to allow for non-monotone information. Essentially, our methodology and results accompany Christiansen (2020); while Christiansen (2020) contains the general theory for so-called infinitesimal compensators and infinitesimal martingales, this theory is here applied to multi-state life insurance.

Although we focus on state-wise prospective reserves and their dynamics, we expect the setting and mathematical techniques presented here to be applicable beyond this specific application, e.g. in relation to estimation and efficient computation of expected cash flows and reserves in the presence of non-monotone information. Broadly speaking, with this paper we initialize a program that aims at the development of general mathematical methodology for multi-state life insurance in the presence of non-monotone information.

The paper is structured as follows. In Section 2, we present the probabilistic setup and the main examples concerning information discarding upon and after retirement. In Section 3, we develop a mathematically sound concept of state-wise prospective reserves in the presence of potentially non-monotone information. Section 4 contains our main result, namely a generalization of the stochastic Thiele equations to allow for non-monotone information, and its application to information discarding upon and after retirement. In particular, we illustrate the pertinence and usefulness of the generalized stochastic Thiele equations by deriving Feynman-Kac formulas beyond the (semi-)Markovian case.
2. Monotone and non-monotone information structures

In this section, we introduce a general modeling framework for the random pattern of states of the insured in the presence of non-monotone information. The framework is strongly related to the general theory of non-monotone information for jump processes introduced by Christiansen (2020). To clarify the theoretical as well as practical relevance of an approach allowing for non-monotone information and general intertemporal dependency structures, we further discuss a motivating example concerning stochastic retirement. This leads to the specification of some explicit cases of non-monotone information that serve as the main examples in the ensuing investigation.

2.1. General setting

Let \((\Omega, \mathcal{A}, P)\) be a complete probability space with null sets \(\mathcal{N}\), and let \(Z = (Z_t)_{t \geq 0}\) be a random pattern of states (pure jump process) on the finite state space \(S = \{1, \ldots, J + 1, J + 2\}\), \(J \in \mathbb{N}_0\), with initial state \(Z_0 \equiv z_0 \in S\), giving at each time \(t\) the state of the insured in \(S\).

The total information available is denoted \(F = (F_t)_{t \geq 0}\); it is the right-continuous and complete filtration given by

\[
F_t = \sigma(Z_s : s \leq t) \vee \mathcal{N}.
\]

Since \(F\) is a filtration, it represents monotone (increasing) information.

We relate to the random pattern of states \(Z\) a multivariate counting process \(N = (N(t))_{t \geq 0}\) with components \(N_{jk} = (N_{jk}(t))_{t \geq 0}\), \(j, k \in S\), \(j \neq k\), giving the number of jumps of \(Z\) from state \(j\) to state \(k\):

\[
N_{jk}(t) = \# \{s \in (0, t] : Z_s^- = j, Z_s = k\}, \quad t \geq 0.
\]

We impose the following technical condition. It ensures that \(Z\) is non-explosive and that compensated counting processes are true martingales.

**Assumption 2.1** (No explosions and true martingales). We assume that

\[
E \left[ \sum_{j, k \in S, j \neq k} N_{jk}(t) \right] < \infty
\]

for all \(t \geq 0\).

If we denote by \(T(t)\) the next jump after time \(t\),

\[
T(t) = \inf \{s \in (t, \infty) : Z_s \neq Z_t\},
\]

\[
T(\infty) = \infty,
\]

and employ the convention \(\inf \emptyset = \infty\), we can also define a marked point process \((\tau_i, Z_{\tau_i})_{i \in \mathbb{N}_0}\) by

\[
\tau_0 = 0, \quad \tau_i = T(\tau_{i-1}), \quad i \in \mathbb{N},
\]

with \(Z_\infty = \nabla\) for some arbitrary cemetery state \(\nabla\). The marked point process, multivariate counting process, and random pattern of states formulations of the setup are equivalent in the sense that the information generated by these processes agree.

A life insurance contract between the insured and the insurer is stipulated by the specification of a payment process \(B = (B(t))_{t \geq 0}\) representing the accumulated benefits minus premiums. In general, we suppose that \(B\) is an \(F\)-adapted process that has càdlàg sample paths, finite expected variation on compacts (in particular, it has sample paths of finite variation on compacts), and a deterministic initial value \(B(0) \in \mathbb{R}\).
2.2. Non-monotone information

Due to e.g. legal constraints, privacy considerations, or to achieve model and/or computational simplifications, the insurer might not have access to or desire to utilize all information available to it. Examples include the newly introduced General Data Protection Regulation 2016/679 of the European Union, where Article 17 describes a so-called right to erasure, and the restriction to a Markovian type of information even when the Markov property is not satisfied. Representing the resulting utilized information as a subsequence of $\sigma$-algebras, one typically finds that the sequence is non-monotone because certain pieces of information are discarded en route.

To describe the information reductions, we introduce a subsequence of $\sigma$-algebras as follows. Let $(T_i)_{i \in \mathbb{N}}$ and $(S_i)_{i \in \mathbb{N}}$ be sequences of $\mathcal{F}$-stopping times with $S_i \geq T_i$, $i \in \mathbb{N}$. Further, let $(\zeta_i)_{i \in \mathbb{N}}$ be a sequence of random variables with values in a separable complete metric space $E$ and corresponding Borel $\sigma$-algebra $\mathcal{E} := \mathcal{B}(E)$, and suppose that each $\zeta_i$ is $\mathcal{F}_{T_i}$-measurable. For the sake of a convenient notation, without loss of generality we assume that $0 \notin E$. The information $\zeta_i$ is recorded at time $T_i$ and then discarded at a later time $S_i$; here $S_i = \infty$ signifies no discarding. Thus the admissible information at time $t \geq 0$ is given by the $\sigma$-algebra $\mathcal{G}_t \subseteq \mathcal{F}_t$ defined by

$$\mathcal{G}_t = \sigma(\{T_i \leq t < S_i\} \cap \{\zeta_i \in A\} : i \in \mathbb{N}, A \in \mathcal{E}) \vee \mathcal{N},$$

while the information available immediately before time $t > 0$ is given by the $\sigma$-algebra $\mathcal{G}_{t-} \subseteq \mathcal{F}_{t-}$ defined by

$$\mathcal{G}_{t-} = \sigma(\{T_i < t \leq S_i\} \cap \{\zeta_i \in A\} : i \in \mathbb{N}, A \in \mathcal{E}) \vee \mathcal{N}.$$  \hfill (2.2)

We introduce the notation $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ and $\mathcal{G}_- = (\mathcal{G}_{t-})_{t > 0}$.

The subsequence of $\sigma$-algebras $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ is in general non-monotone and the random times $T_i$ and $S_i$ are not necessarily stopping times w.r.t. $\mathcal{G}$. We do not assume the random times $(T_i)_{i \in \mathbb{N}}$ and $(S_i)_{i \in \mathbb{N}}$ to take a specific order in time other than $T_i \leq S_i$, and we even allow for simultaneous events. We can recover $\mathcal{F}$ by taking $S_i = \infty$, $T_i = \tau_i$, and $\zeta_i = (\tau_i, Z_{\tau_i})$ for all $i \in \mathbb{N}$, and from this point and onward, that representation is always assumed whenever $\mathcal{G} = \mathcal{F}$.

Let $\mathcal{S} := \{x \in \mathbb{N} : |x| < \infty\}$ be the finite subsets of the natural numbers. Note that $\mathcal{S}$ is countable. For each $x \in \mathcal{S}$ we define the indicator processes

$$I_x(t) := \left\{ \begin{array}{ll} 1 & : \cap_{t \in x} \{T_i \leq t < S_i\} \cap \cap_{i \notin x} (\Omega \setminus \{T_i \leq t < S_i\}), \\ 0 & : \text{else,} \end{array} \right.$$  \hfill (2.3)

so that $I_x(t)$ is $\mathcal{G}_t$-measurable for each $t \geq 0$ and $x \in \mathcal{S}$. We assume in continuation of Assumption [2.1] that

$$E\left[ \sum_{i=1}^{\infty} 1_{\{T_i \leq t\}} \right] < \infty, \quad t \geq 0,$$

which implies that on each compact interval we can almost surely find at most finitely many random times $T_i, S_i$, $i \in \mathbb{N}$. As a result, the indicator processes $I_x$ have càdlàg paths of finite variation on compacts. The family of indicator processes $I := (I_x)_{x \in \mathcal{S}}$ corresponds to the $\mathcal{G}$-adapted non-explosive random pattern of states

$$Z_t = \sum_{x \in \mathcal{S}} x I_x(t), \quad t \geq 0.$$
This random pattern of states describes the state of information: \( Z_t = x \) if and only if exactly the information \((\zeta_i)_{i \in x}\) is available at time \( t \); in particular, the information \((\zeta_i)_{i \not\in x}\) has either been recorded and already discarded or is yet to be recorded.

We generally suppose that
\[
\sigma(Z_t) \subseteq \mathcal{G}_t, \quad t \geq 0.
\]  
(2.4)
Since we assumed that \( 0 \not\in E \), the information at time \( t \) and at time \( t^- \) can be alternatively represented as
\[
\mathcal{G}_t = \sigma(\zeta_x I_x(t) : x \in S) \vee \mathcal{N}, \quad t \geq 0,
\]
\[
\mathcal{G}_{t^-} = \sigma(\zeta_x I_x(t^-) : x \in S) \vee \mathcal{N}, \quad t > 0,
\]  
(2.5)
where \( \zeta_x := (\zeta_i)_{i \in x}, x \in S \). Let
\[
T_{xy} := \inf\{t \geq 0 : I_x(t^-) I_y(t) = 1\},
\]
using the convention \( \inf \emptyset := \infty \). We see that \( T_{xy} \) is the exact point in time where the state of information changes from state \( x \) to state \( y \) by discarding information \( \zeta_{x \setminus y} \) and recording information \( \zeta_{y \setminus x} \); here we ignore information that is recorded and immediately discarded. The total information either discarded or recorded at time \( T_{xy} \) is thus \( \zeta_{xy} := (\zeta_i)_{i \in x \Delta y} \), where \( x \Delta y = (x \setminus y) \cup (y \setminus x) \).

The extended marked point process \((T_i, S_i, \zeta_i)_{i \in N}\) corresponds to the random counting measures \( \nu_{xy}, x, y \in S, y \neq x \), defined as the unique completions of
\[
\nu_{xy}([0,t] \times A) := 1_{\{T_{xy} \leq t\}} 1_{\{\zeta_{xy} \in A\}}, \quad t \geq 0, A \in \mathfrak{B}(E_{xy}),
\]
where \( E_{xy} := E^{[x \Delta y]} \).

If \( T_i = \tau_i \) for all \( i \in N \), the \( \sigma \)-algebra \( \mathcal{G}_t \) reveals in particular the indices \( i \) of the admissible observations and thus gives a lower bound on the number of past discards, cf. Remark 3.1 in Christiansen (2020), which might be an unwanted feature. As further discussed in Christiansen (2020), by considering suitable permutations it is often possible to obtain non-informative indices; in that case, the number of past discards becomes non-admissible information. In the next subsection, we introduce some specific instances of non-monotone information concerning stochastic retirement and embed them into the framework above. In particular, we exemplify how to obtain non-informative indices using suitable permutations.

### 2.3. Stochastic retirement

Suppose \( J \geq 1 \) and \( z_0 \notin \{J + 1, J + 2\} \), and let \( \delta \) and \( \eta \) be the first hitting times of \( \{J + 2\} \) and \( \{J + 1\} \), respectively, by \( Z \):
\[
\delta = \inf\{t \geq 0 : Z_t = J + 2\},
\]
\[
\eta = \inf\{t \geq 0 : Z_t = J + 1\}.
\]
We think of \( \delta \) as the time of death and \( \eta \) as the time of retirement. Accordingly, the states \( \{1, \ldots, J\} \) describe the health state of the insured up until retirement or death. In this subsection, we assume a decrement structure such that retirement occurs at most once and death is a terminal event:
Assumption 2.2 (Decrement structure concerning retirement and death). We assume that

\[
[0, \infty) \ni t \mapsto \sum_{j \in S} \sum_{k \leq J, k \leq J} N_{jk}(t) = 0,
\]

\[
[0, \infty) \ni t \mapsto \sum_{j \in S} \sum_{k \leq J, k \leq J} N_{jk}(t) = 0,
\]

almost surely.

Note that the structure of the state space entails that the insurer is not updating its information concerning the health state of the insured upon or after retirement. In Figure 2.1 we have exemplified this setup for the case \(J = 2\) corresponding to a disability model allowing for recovery before retirement.

Figure 2.1: Extension of classic disability model with recovery to allow for stochastic retirement.

In actuarial practice, it is common to impose some Markovian structure by assuming the random pattern of states \(Z\) to be e.g. Markovian or semi-Markovian. In the following, we illustrate why such assumptions might be insufficient and, as an alternative, how to represent similar assumptions as non-monotone information substructures. This motivates the general non-Markovian framework with non-monotone information introduced in Subsections 2.1–2.2.

It is natural to imagine the random pattern of states \(Z\) as embedded into a larger framework. Let \(\tilde{Z}\) be a random pattern of states on an extended state space \(\tilde{S} = \{1, \ldots, J+1, J+2, \ldots, 2J+1\}\) with initial state \(\tilde{Z}_0 = \tilde{z}_0 \in \{1, \ldots, J\}\). Denote the corresponding multivariate counting process by \(\tilde{N}\). Suppose that

\[
\mathbb{E}\left[ \sum_{j,k \in \tilde{S}} \tilde{N}_{jk}(t) \right] < \infty \quad (2.6)
\]

for all \(t \geq 0\), and that

\[
[0, \infty) \ni t \mapsto \sum_{j \in \tilde{S}} \sum_{k \leq J, k \leq J} \tilde{N}_{jk}(t) = 0,
\]

\[
[0, \infty) \ni t \mapsto \sum_{k \in \tilde{S}} \sum_{J < k \leq 2J} \tilde{N}_{(2J+1)k}(t) = 0,
\]

almost surely. We think of the states \\{\(J + 1, \ldots, 2J\)\} as providing information concerning the health state of the insured upon or after retirement. In Figure 2.2 we have exemplified this
setup for the case $J = 2$ corresponding to a disability model allowing for recovery and stochastic retirement. In general, we can now redefine $Z$ by

$$Z_t = \begin{cases} 
\tilde{Z}_t & \text{if } \tilde{Z}_t \in \{1, \ldots, J\}, \\
J + 1 & \text{if } Z_t \in \{J + 1, \ldots, 2J\}, \\
J + 2 & \text{if } \tilde{Z}_t = 2J + 1
\end{cases}$$

for all $t \geq 0$, when we find that $z_0 \notin \{J + 1, J + 2\}$ and that Assumptions 2.1–2.2 remain satisfied.

---

**Figure 2.2**: Extension of the disability model with retirement of Figure 2.1 where the health status of the insured remains observed upon and after retirement.

The information available to the insured is represented by the filtration $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0}$ given by

$$\tilde{\mathcal{F}}_t = \sigma(\tilde{Z}_s : s \leq t) \vee \mathcal{N}.$$ 

In many cases, the information $\tilde{\mathcal{F}}$ is not available to the insurer, and then the insurer must resort to the information given by $\mathcal{F}$; this can e.g. be the case if upon retirement, disability coverage ceases.

It appears consistent with actuarial practice to propose that the underlying random pattern of states $\tilde{Z}$ is Markovian or semi-Markovian. We now study the resulting implications on $Z$, which is the natural modeling object given information $\mathcal{F}$. Let $U = (U_t)_{t \geq 0}$ and $\tilde{U} = (\tilde{U}_t)_{t \geq 0}$ be the duration processes associated with $Z$ and $\tilde{Z}$, respectively, given by

$$U_t = t - \sup\{s \in [0, t] : Z_s \neq Z_t\},$$

$$\tilde{U}_t = t - \sup\{s \in [0, t] : \tilde{Z}_s \neq \tilde{Z}_t\}.$$

Note that $1_{\{t \leq \eta\}} U_t = 1_{\{t \leq \eta\}} \tilde{U}_t$. Let $U^r = (U^r_t)_{t \geq 0}$ be the time since retirement given by

$$U^r_t = \begin{cases} 
0 & \text{if } t < \eta, \\
\eta - t & \text{if } t \geq \eta,
\end{cases}$$

let $H = (H_t)_{t \geq 0}$ be the state of the insured just before retirement given by

$$H_t = \begin{cases} 
Z_t & \text{if } t < \eta, \\
Z_{\eta-} & \text{if } t \geq \eta,
\end{cases}$$
Lemma 2.4. The implies (semi-)Markovianity of $Z$

Proof. (2.2)

Further that $\tilde{Z}$ is Markovian. Then $(Z,U^h, U^r, H)$ is Markovian. Suppose further that $\tilde{Z}$ is Markovian. Then $(Z,U^r, H)$ is Markovian.

Proof. See Appendix A

Example 2.5. Let

$$T_1 = \eta, \quad S_1 = \infty, \quad \zeta_1 = (T_1, Z_{T_1}),$$

$$T_2 = \delta, \quad S_2 = \infty, \quad \zeta_2 = (T_2, Z_{T_2}),$$

$$S_{2+i} = T_1 \wedge T_2, \quad \zeta_{2+i} = (T_{2+i}, Z_{T_{2+i}}), \quad i \in \mathbb{N},$$

and let $U^h = (U^h_t)_{t \geq 0}$ be the duration of the latest sojourn before retirement given by

$$U^h_t = \begin{cases} U_t & \text{if } t < \eta, \\ U_{\eta} & \text{if } t \geq \eta. \end{cases}$$

Proposition 2.3. Suppose $(\tilde{Z}, \tilde{U})$ is Markovian. Then $(Z,U^h, U^r, H)$ is Markovian. Suppose further that $\tilde{Z}$ is Markovian. Then $(Z,U^r, H)$ is Markovian.

Proof. See Appendix A

It is possible to derive necessary and sufficient conditions for which (semi-)Markovianity of $\tilde{Z}$ implies (semi-)Markovianity of $Z$, see e.g. Serfozo [1971]. In general, these conditions are very restrictive and do not apply to models of actuarial relevance: in this sense, the complex intertemporal dependency structure implied by Proposition 2.3 must be taken into account. This serves as a motivation for the general non-Markovian framework presented in Subsection 2.1.

Although Proposition 2.3 indicates that the mortality as retiree might depend on the past through e.g. the time since retirement and the last health state before retirement, it is common in actuarial practice to rely on a standard mortality table – an example is the longevity benchmark of the Danish financial supervisory authority, cf. Jørgen and Møller (2013). This in a sense corresponds to imposing an ‘as if’ Markovian assumption or, alternatively, to only utilize information corresponding to a specific subsequence of $\sigma$-algebras rather than $F$ itself. Therefore, we introduce two subsequences of $\sigma$-algebras $G^1 = (\mathcal{G}^1_t)_{t \geq 0}$ and $G^2 = (\mathcal{G}^2_t)_{t \geq 0}$ given by

$$\mathcal{G}^1_t = \sigma(Z_t \mathbf{1}_{\{Z_t \in \{1, \ldots, J\}\}}, \mathbf{1}_{\{\eta \leq s\}}, \mathbf{1}_{\{\delta \leq s\}} : s \leq t) \vee \mathcal{N},$$

$$\mathcal{G}^2_t = \sigma(Z_t \mathbf{1}_{\{Z_t \in \{1, \ldots, J\}\}}, \mathbf{1}_{\{\eta < t\}}, \mathbf{1}_{\{\delta < s\}} : s \leq t) \vee \mathcal{N}.$$

The information $\mathcal{G}^1$ corresponds to the case where upon retirement or death the insurer discards the previous health records of the insured. The sub-information $\mathcal{G}^2 \subset \mathcal{G}^1$ even keeps no record on the time of retirement. For most if not all practical purposes, the discarding of previous information upon death is of no importance.

Further, for describing the admissible information immediately before time $t \geq 0$ we define sequences of $\sigma$-algebras $\mathcal{G}^1 = (\mathcal{G}^1_t)_{t \geq 0}$ and $\mathcal{G}^2 = (\mathcal{G}^2_t)_{t \geq 0}$ by

$$\mathcal{G}^1_t = \sigma(Z_t \mathbf{1}_{\{Z_t \in \{1, \ldots, J\}\}}, \mathbf{1}_{\{\eta < s\}}, \mathbf{1}_{\{\delta < s\}} : s \leq t) \vee \mathcal{N},$$

$$\mathcal{G}^2_t = \sigma(Z_t \mathbf{1}_{\{Z_t \in \{1, \ldots, J\}\}}, \mathbf{1}_{\{\eta < t\}}, \mathbf{1}_{\{\delta < s\}} : s \leq t) \vee \mathcal{N}.$$

Lemma 2.4. The $\sigma$-algebras $\mathcal{G}^1_t$, $\mathcal{G}^1_-, \mathcal{G}^2_t$, and $\mathcal{G}^2_-$, $t \geq 0$, can be brought on the form of (2.1)–(2.2).

Proof. See Appendix A

Lemma 2.4 gives a link to the general setting; note that the condition (2.3) is satisfied. From this point onward, for $\mathcal{G}^1$ and $\mathcal{G}^2$ the respective extended marked point process $(T_i, S_i, \zeta_i)_{i \in \mathbb{N}}$ is always taken to be that from the proof of Lemma 2.4; see also Example 2.5 below.

Example 2.5.
and let $T_{2+i}, i \in \mathbb{N}$, be the jump times of the process counting the number of jumps of $Z$ except retirement and death. Then according to the proof of Lemma 2.4, cf. Appendix A,

$$G_1^1 = \sigma(\{T_i \leq t < S_i \} \cap \{\zeta_i \in A \} : i \in \mathbb{N}, A \in \mathcal{E}) \vee \mathcal{N},$$

$$G_1^{1-} = \sigma(\{T_i < t \leq S_i \} \cap \{\zeta_i \in A \} : i \in \mathbb{N}, A \in \mathcal{E}) \vee \mathcal{N},$$

for $t \geq 0$. The jump times have been permuted so that retirement and death have indices one and two, respectively. Consequently, the index of the jump time corresponding to retirement does not carry information concerning the total number of previous jumps.

In the following, we develop a mathematically sound concept of state-wise prospective reserves in the case of non-monotone information, and we derive so-called stochastic Thiele equations describing the dynamics of state-wise prospective reserves in the presence of non-monotone information. The results are exemplified with non-monotone information given by $G^1$ and $G^2$, respectively, allowing us to discuss current actuarial practice regarding valuation of insurance liabilities in the presence of (possibly stochastic) retirement.

3. Prospective reserves in the presence of non-monotone information

In the case of monotone information, prospective reserves are so-called optional projections of accumulated future payments, suitably discounted. To our knowledge, there appears to be no unifying definition of general state-wise prospective reserves in the actuarial literature; in Norberg (1992), state-wise prospective reserves are given implicitly as prospective reserves evaluated on the relevant event, while Norberg (1996) in principle casts them based on somewhat arbitrary functional representations of prospective reserves. The properties of the state-wise prospective reserves as stochastic processes, including the existence and uniqueness of suitably regular versions, are not investigated. Furthermore, it is unclear from these proposals how to define state-wise prospective reserves in the presence of non-monotone information.

In this section, we present a sound and fruitful definition of state-wise prospective reserves in the presence of monotone as well as non-monotone information. In the presence of non-monotone information, the main idea is to take as underlying state process not $Z$ giving the state of the insured but rather $\mathcal{Z}$ giving the state of information. The section is structured as follows. In Subsection 3.1, we introduce so-called state-wise counterparts and reveal the non-triviality of developing the concept of state-wise prospective reserves. In Subsection 3.2, we follow Christiansen (2020) on optional projections in the presence of non-monotone information, which turns out to be a fruitful Ansatz for a mathematically sound definition of state-wise quantities. Definitions of state-wise prospective reserves are introduced and discussed in Subsection 3.3.

3.1. State-wise counterparts

Suppose that $\mathcal{C} = (\mathcal{C}_t)_{t \geq 0}$ is a sequence of $\sigma$-algebras such that

$$\sigma(Z_t) \vee \mathcal{N} \subseteq \mathcal{C}_t \subseteq \mathcal{F}_t, \quad t \geq 0.$$ 

Examples include $\mathcal{C} = \mathcal{G}$. We define sequences of families of sets $\mathcal{C}_j = (\mathcal{C}_{t,j})_{t \geq 0}$, $j \in S$, by

$$\mathcal{C}_{t,j} = \{A \in \mathcal{F}_t^- : A \cap \{Z_t = j\} \in \mathcal{G}_t\}.$$
Lemma 3.1. For each $(t, j) \in [0, \infty) \times \mathcal{Z}$ the family of sets $\mathcal{C}_{t,j}$ is a $\sigma$-algebra. Moreover,

$$\mathcal{C}_t = \sigma(A \cap \{Z_t = j\} : A \in \mathcal{C}_{t,j}, j \in S)$$

for any $t \geq 0$.

Proof. Follows by standard set-theoretic calculations. □

Example 3.2. Consider monotone information $\mathcal{F}$. Then $\mathcal{F}_{t,j} = \mathcal{F}_{t-}$ since $\mathcal{F}_{t-} \vee \sigma(Z_t) \subseteq \mathcal{F}_t$. o

Example 3.3. Consider the setting of Subsection 2.3. By defining

$$\psi^1_j(s) := Z_s1_{\{t,\ldots,j\}}(j) + 1_{\{\eta < \delta\}}1_{\{j+1\}}(j) + (1_{\{\eta < \delta\}}, 1_{\{\delta \leq \eta\}})1_{\{j+2\}}(j)$$

$$\psi^2_j(s) := Z_s1_{\{t,\ldots,j\}}(j) + 1_{\{\delta < \eta\}}1_{\{j+2\}}(j)$$

for $s \geq 0$ and $j \in S$ we find that

$$\mathcal{G}^1_{t,j} = \sigma(\psi^1_j(s) : s < t) \vee \mathcal{N}$$

$$\mathcal{G}^2_{t,j} = \sigma(\psi^2_j(s) : s < t) \vee \mathcal{N}.$$ o

Let $Y = (Y(t))_{t \geq 0}$ be a real-valued stochastic process, and suppose that $Y(t)$ is $\mathcal{C}_t$-measurable for each $t \geq 0$. We now define the state-wise counterparts as follows:

Definition 3.4. A family of real-valued stochastic processes $(Y_j)_{j \in S} = (Y_j(t))_{t \geq 0, j \in S}$ is said to be state-wise counterparts to $Y$ if for each $(t, j) \in [0, \infty) \times S$:

- $Y_j(t)$ is $\mathcal{C}_{t,j}$-measurable,
- $1_{\{Z_t = j\}}Y_j(t) \overset{a.s.}{=} 1_{\{Z_t = j\}}Y(t).$ △

In general, we suppress the dependency of state-wise counterparts on the specific sequence of $\sigma$-algebras $\mathcal{C}$.

Suppose for the moment that $Y^\mathcal{G} = (Y^\mathcal{G}(t))_{t \geq 0}$ is the prospective reserve under information $\mathcal{G}$ (to be formally defined later on). Then it is intuitively appealing to base the definition of the state-wise prospective reserves on the state-wise counterparts $(Y^\mathcal{G}_j)_{j \in S}$ to $Y^\mathcal{G}$: they satisfy the key identity $1_{\{Z_t = j\}}Y^\mathcal{G}_j(t) \overset{a.s.}{=} 1_{\{Z_t = j\}}Y^\mathcal{G}(t)$ and only rely on the information $\mathcal{G}_{t,j}$, which is the information available at time $t$—that remains available at time $t$ if $Z_t = j$.

For each $t \geq 0$ let $m_t$ be the sub-probability measure that is uniquely defined on $\sigma(A \times \{j\} : A \in \mathcal{C}_{t,j}, j \in S)$ by

$$m_t(A \times \{j\}) = m_{t,j}(A) := P(A \cap \{Z_t = j\}), \quad A \in \mathcal{C}_{t,j}, j \in S.$$

Proposition 3.5. Let $Y = (Y(t))_{t \geq 0}$ be a real-valued stochastic process such that $Y(t)$ is integrable and $\mathcal{C}_t$-measurable for each $t \geq 0$. Then the state-wise counterparts $(Y_j)_{j \in S}$ to $Y$ exist and for each $t \geq 0$ the mapping $\Omega \times S \ni (\omega, j) \mapsto Y_j(t)(\omega)$ is $m_t$-almost everywhere unique.

Proof. See Appendix A. □

The uniqueness of the state-wise counterparts does not extend beyond $m_t$-almost everywhere for fixed $t \geq 0$. In other words, viewed as processes the state-wise counterparts are not almost surely unique and thus not well-defined. Consequently, the definition of state-wise counterparts is
mathematically flawed and it might therefore be unfortunate to base the definition of state-wise prospective reserves thereon.

Before we turn the attention to an alternative foundation based on an explicit representation of optional projections, we first present some results for the state-wise counterparts that are useful later.

Define a class of functionals $L_1(\Omega, \mathcal{A}, P) \ni X \mapsto E_{t,j}[X|C_{t,j}]$ by

$$E_{t,j}[X|C_{t,j}] := E[X\mathbf{1}_{\{Z_t=j\}}|C_{t,j}],$$

where we impose the convention $0/0 := 0$. If $P(Z_t = j) > 0$, it holds that $E_{t,j}[X|C_{t,j}]$ are versions of the conditional expectations of $Y(t)$ given $C_{t,j}$ w.r.t. the probability measure $P_{t,j}$ given by

$$P_{t,j}(A) = \frac{P(A \cap \{Z_t = j\})}{P(Z_t = j)}, \quad A \in \mathcal{A},$$

cf. Exercise 34.4(a) of Billingsley (1994).

Based on similar techniques as in the proof of Proposition 3.5, one can then show that

$$Y_j(t) \overset{a.s.}{=} E_{t,j}[Y(t)|C_{t,j}], \quad (3.1)$$

This provides an explicit representation of the state-wise counterparts.

We are now ready to derive the following law of iterated expectations:

**Lemma 3.6.** Let $X \in L_1(\Omega, \mathcal{A}, P)$. Then for each $(t, j) \in [0, \infty) \times S$:

$$E_{t,j}[E[X|C_t]|C_{t,j}] \overset{a.s.}{=} E_{t,j}[X|C_{t,j}].$$

**Proof.** See Appendix A. \hfill \Box

When $C_{t,j}$ is generated by $F_{t,j} = f_{t,j}((Z_s)_{0 \leq s < t})$ added null sets $\mathcal{N}$ with $f_{t,j}$ some measurable function, it can be shown that

$$E_{t,j}[Y(t)|C_{t,j}] \overset{a.s.}{=} E[Y(t)|F_{t,j}, Z_t = j], \quad (3.2)$$

where the latter refers to path-wise integration w.r.t. the conditional distribution of $Y(t)$ given $(F_{t,j}, Z_t)$ and, further, evaluated in $\{F_{t,j}(\omega), j\}$. This provides an alternative explicit representation of the state-wise counterparts. Rewrites in the spirit of (3.2) are typical and occur frequently and opaquely in the remainder of the paper.

### 3.2. Optional projections and state-wise quantities

Let $Y = (Y(t))_{t \geq 0}$ be a real-valued stochastic process such that $Y(t)$ is integrable for each $t \geq 0$. If there exists an almost surely unique process $X = (X(t))_{t \geq 0}$ such that for each $t \geq 0$,

$$X(t) = E[Y(t)|\mathcal{G}_t]$$

almost surely, then we denote $Y^\mathcal{G} := X$ as the optional projection of $Y$ with respect to $\mathcal{G}$.

In the following we calculate conditional expectations given $(\zeta_x, T_{xy}, \zeta_{xy})$, $x, y \in \mathcal{S}, x \neq y$. We throughout assume that they are defined as path-wise integrals with respect to arbitrary but fixed
regular conditional distributions $P(\cdot \mid \zeta_x, T_{xy}, \zeta_{xy})$. For a càdlàg or càglàd process $Y = (Y(t))_{t \geq 0}$ with finite expected variation on compacts, let

\[
\begin{align*}
\mathcal{Y}^G_x(t) &= \frac{E[I_x(t)Y(t) \mid \zeta_x]}{E[I_x(t) \mid \zeta_x]}, \quad t \geq 0, \\
\mathcal{Y}^G_{x-}(t) &= \frac{E[I_x(t-1)Y(t) \mid \zeta_x]}{E[I_x(t-1) \mid \zeta_x]}, \quad t > 0, \\
\mathcal{Y}^G_{xx}(t) &= \frac{E[I_x(t-1)I_x(t)Y(t) \mid \zeta_x]}{E[I_x(t-1)I_x(t) \mid \zeta_x]}, \quad t > 0, \\
\mathcal{Y}^G_{xy}(t, e) &= \frac{E[I_x(t-1)Y(t) \mid \zeta_x, T_{xy} = t, \zeta_{xy} = e]}{E[I_x(t-1) \mid \zeta_x]}, \quad x \neq y, e \in E_{xy}, t > 0, \\
\mathcal{Y}^G_y(t, e) &= \frac{E[I_y(t)Y(t) \mid \zeta_y, T_{xy} = t, \zeta_{xy} = e]}{E[I_y(t) \mid \zeta_y]}, \quad x \neq y, e \in E_{xy}, t > 0, \\
\end{align*}
\]  

(3.3)

which are almost surely unique processes, cf. the discussion between Theorem 4.2 and Proposition 4.3 of Christiansen (2020). The above state-wise quantities refer to the state of information and changes in the state of information, rather than the state of the insured. In Subsection 3.3 we interpret these state-wise quantities when $Y$ describes the accumulated future payments. The following proposition helps us in this regard.

**Proposition 3.7.** Let $Y$ be a càdlàg or càglàd process with finite expected variation on compacts. For each $t > 0$ we almost surely have

\[
I_x(t)\mathcal{Y}^G_x(t) = I_x(t)E[Y(t) \mid \mathcal{G}_t],
\]

\[
I_x(t-1)\mathcal{Y}^G_{x-}(t) = I_x(t-1)E[Y(t) \mid \mathcal{G}_{t-}],
\]

\[
I_x(t-1)\mathcal{Y}^G_{xx}(t) = I_x(t-1)E[Y(t) \mid \mathcal{G}_{t-}, \mathcal{Z}_t = x],
\]

\[
I_x(t)\mathcal{Y}^G_{xy}(t, e) = I_x(t)E[Y(t) \mid \mathcal{G}_t, \mathcal{T}_{xy} = t, \zeta_{xy} = e],
\]

\[
I_y(t)\mathcal{Y}^G_y(t, e) = I_y(t)E[Y(t) \mid \mathcal{G}_t, \mathcal{T}_{xy} = t, \zeta_{xy} = e].
\]

**Proof.** See Proposition 4.3 and the proof of Theorem 4.2 in Christiansen (2020). \(\square\)

Note that for all $t > 0$ and $x \in S$, by Proposition 3.7 and (2.3), the latter which essentially allows us to replace $\mathcal{G}_{t-}$ by $\zeta_x$ on $\{\mathcal{Z}_{t-} = x\}$,

\[
I_x(t-1)\mathcal{Y}^G_{xx}(t) = I_x(t-1)E[Y(t) \mid \mathcal{G}_{t-}, \mathcal{Z}_t = x] = I_x(t-1)\frac{E[I_x(t)Y(t) \mid \zeta_x]}{E[I_x(t) \mid \zeta_x]} = I_x(t-1)\mathcal{Y}^G_x(t)
\]

(3.4)

almost surely.

The state-wise quantities $\mathcal{Y}^G_x$ allow for a rather explicit characterization of the optional projection $Y^G$:

**Proposition 3.8.** Let $Y = (Y(t))_{t \geq 0}$ be a càdlàg process with with finite expected variation on compacts. Then the optional projection $Y^G$ of $Y$ exists and has the almost surely unique representation

\[
Y^G(t) = \sum_{x \in S} I_x(t)\mathcal{Y}^G_x(t), \quad t \geq 0.
\]

For each $x \in S$ the process $t \mapsto I_x(t)\mathcal{Y}^G_x(t)$ has a càdlàg modification with paths of finite variation on compacts.
Proof. See Theorem 4.2 in Christiansen (2021).

The following corollary shows that the path properties of the optional projection $Y^G$ are passed on to the state-wise quantities of interest. It ensures all later applications of e.g. integration by parts to be feasible.

**Corollary 3.9.** Let $Y = (Y(t))_{t \geq 0}$ be a càdlàg process with finite expected variation on compacts. Then the processes

$$[0, \infty) \ni t \mapsto 1_{\{Z_i = j\}} Y^G_j(t), \quad (0, \infty) \ni t \mapsto 1_{\{Z_i = j\}} Y^G_j(t),$$

and the process

$$(0, \infty) \ni t \mapsto I_x(t-)Y^G_x(t)$$

almost surely have càdlàg paths of finite variation on compacts.

**Proof.** See Appendix A.

In the special case of monotone information, we now establish a more direct relation between the different concepts of state-wise quantities. Setting $(T_i, S_i, \zeta_i)_{i \in \mathbb{N}} := (\tau_i, \infty, (Z_{\tau_i}, \tau_i))_{i \in \mathbb{N}}$ we recover the filtration $\mathcal{F} = \mathcal{G}$. In this case, let

$$Y^F_{jk}(t) := 1_{\{Z_i = j\}} \sum_{x,y \in S, x \neq y} I_x(t-)Y^F_{xy}(t, k, t),$$

$$Y^F_{jj}(t) := 1_{\{Z_i = j\}} \sum_{x \in S} I_x(t-)Y^F_{xx}(t)$$

for $j, k \in S, j \neq k$, and $t > 0$.

**Remark 3.10.** In case of $(T_i, S_i, \zeta_i)_{i \in \mathbb{N}} := (\tau_i, \infty, (Z_{\tau_i}, \tau_i))_{i \in \mathbb{N}}$, only those indicator processes $I_x$ are different from constantly zero that have an $x$ of the form $x = \{1, \ldots, n\} \in S$ for some $n \in \mathbb{N}_0$; here we define $\{1, \ldots, n\}$ as the empty set in case of $n = 0$. In particular, we have

$$I_x(t-) = 1_{\{\tau_n < t \leq \tau_{n+1}\}} \quad \text{if } x = \{1, \ldots, n\}$$

for $t > 0$ and with $\tau_0 := 0$. Moreover, the stopping times $T_{xy}$ are only then different from constantly infinity if $x$ and $y$ are of the form $x = \{1, \ldots, n\}$, $y = \{1, \ldots, n+1\}$, $n \in \mathbb{N}_0$. In particular, for each $t > 0$ we almost surely have

$$Y^F_{jk}(t) = 1_{\{Z_i = j\}} \sum_{n=0}^{\infty} 1_{\{\tau_n < t \leq \tau_{n+1}\}} E[Y(t) | (Z_{\tau_1}, \tau_1), \ldots, (Z_{\tau_n}, \tau_n), (Z_{\tau_n+1}, \tau_{n+1}) = (k, t)],$$

$$Y^F_{jj}(t) = 1_{\{Z_i = j\}} \sum_{n=0}^{\infty} 1_{\{\tau_n < t \leq \tau_{n+1}\}} \frac{E[1_{\{Z_i = j\}} Y(t) | (Z_{\tau_1}, \tau_1), \ldots, (Z_{\tau_n}, \tau_n)]}{E[1_{\{Z_i = j\}} | (Z_{\tau_1}, \tau_1), \ldots, (Z_{\tau_n}, \tau_n)]}$$

for $j, k \in S, j \neq k$.

In the presence of monotone information, the following result relates the state-wise counterparts to the state-wise quantities introduced by (3.3).
**Proposition 3.11.** Let \( Y = (Y(t))_{t \geq 0} \) be a càdlàg process with finite expected variation on compacts. Denote by \( Y^F \) the corresponding optional projection and by \( (Y^F_j)_{j \in S} \) the state-wise counterparts to \( Y^F \). At each time \( t > 0 \) it almost surely holds that
\[
Y^F_j(t) = Y^F_{jj}(t) + \sum_{k \in S, j \neq k} Y^F_{kj}(t)
\]
for \( j \in S \), where \( Y^F_{jj} \) and \( Y^F_{kj}, k \neq j \) are almost surely unique predictable processes defined by \((3.5)\).

**Proof.** See Appendix A.

In the following, the notation \( Y^F_j \) always refers to the modification given by Proposition 3.11. Insisting on this essentially solves the issue of well-definedness of the state-wise counterparts in the presence of monotone information. In the general case, where we allow for non-monotone information, the issue persists.

**Example 3.12.** Consider the accumulated payments \( B \), which is an \( F \)-adapted càdlàg process with finite expected variation on compacts; in particular, \( B^F = B \). Proposition 3.11 yields
\[
B(t) = \sum_{j \in S} 1_{\{Z_t = j\}} B^F_j(t)
\]
\[
= \sum_{j \in S} 1_{\{Z_t = j\}} B^F_{jj}(t) + \sum_{j,k \in S, j \neq k} 1_{\{Z_t = j\}} B^F_{kj}(t)
\]
almost surely for all \( t > 0 \). Recall that \( B^F_{jj}(t) = 1_{\{Z_t = j\}} B^F_{jj}(t) \) and \( B^F_{jk}(t) = 1_{\{Z_t = j\}} B^F_{jk}(t) \) for all \( j, k \in S, j \neq k \). By applying integration by parts and rearranging the terms, one then finds
\[
B(dt) = \sum_{j \in S} 1_{\{Z_t = j\}} B^F_{jj}(dt) + \sum_{j,k \in S, j \neq k} (B^F_{jk}(t) - B^F_{jj}(t)) N_{jk}(dt)
\]  
almost surely. This recovers the classic decomposition into sojourn payments and transition payments in the following sense. Suppose the accumulated payments \( B \) are defined as
\[
B(dt) = \sum_{j \in S} 1_{\{Z_t = j\}} B_j(dt) + \sum_{j,k \in S, j \neq k} b_{jk}(t) N_{jk}(dt),
\]
where the cumulative sojourn payments \( B_j \) shall be \( F \)-predictable càdlàg processes with finite expected variation on compacts and the transition payments \( b_{jk} \) shall be bounded and \( F \)-predictable processes. By calculating \( B^F_{jj} \) and \( B^F_{jk} \) in \((3.7)\) explicitly and comparing the results with the definition of \( B \), one can show that
\[
1_{\{Z_t = j\}} B_j(dt) = 1_{\{Z_t = j\}} B^F_{jj}(dt)
\]
almost surely for \( j \in S \) and for each \( t > 0 \),
\[
1_{\{Z_t = j\}} b_{jk}(t) = 1_{\{Z_t = j\}} (B^F_{jk}(t) - B^F_{jj}(t))
\]
almost surely for \( j, k \in S, j \neq k \). By defining the process \( \beta = (\beta(t))_{t \geq 0} \) via
\[
\beta(t) = \sum_{j, k \in S, j \neq k} b_{jk}(t) \Delta N_{jk}(t), \quad t \geq 0,
\]
which equals the difference of a càdlàg and a càglàd process, we can alternatively recover the transition payments via the representation
\[
1\{Z_{t-} = j\} b_{jk}(t) = 1\{Z_{t-} = j\} \beta_{jk}^{\mathcal{G}}(t),
\]
which holds almost surely for all \( t > 0 \) and \( j, k \in S, j \neq k \).

### 3.3. State-wise prospective reserves

In the previous two subsections, we have introduced a range of state-wise concepts and quantities, including the state-wise counterparts, and we have studied their interrelation – in particular in the presence of monotone information. Building on this, we now turn our attention to mathematical sound definitions of state-wise prospective reserves. In the presence of monotone information, the definition bases on the concept of state-wise counterparts and refers to the state of the insured, while in the presence of non-monotone information, we rely on the state-wise quantities appearing in the explicit characterization of optional projections; these quantities refer to the state of information rather than the state of the insured.

Consider a deterministic bank account \( \kappa : [0, \infty) \mapsto (0, \infty) \) assumed measurable, càdlàg, and of finite variation on compacts, with initial value \( \kappa(0) = 1 \). Denote with \( v \) the corresponding discount function given by
\[
[0, \infty) \ni t \mapsto v(t) = \frac{1}{\kappa(t)}.
\]
Denote from this point on by \( Y = (Y(t))_{t \geq 0} \) the accumulated future payments, suitably discounted, given by
\[
Y(t) = \int_{(t, \infty)} \frac{\kappa(t)}{\kappa(s)} B(ds).
\]
Note that \( Y \) has càdlàg sample paths of finite variation on compacts. We further suppose that \( Y(t) \) has finite expected variation on compacts. This is for example the case if \( \kappa \) is bounded away from zero.

The prospective reserve under possibly non-monotone information is the almost surely unique optional projection \( Y^{\mathcal{G}} = (Y^{\mathcal{G}}(t))_{t \geq 0} \) of \( Y \) w.r.t. \( \mathcal{G} \) satisfying for each \( t \geq 0 \)
\[
Y^{\mathcal{G}}(t) = \mathbb{E}[Y(t) \mid \mathcal{G}_t] = \mathbb{E}\left[ \int_{(t, \infty)} \frac{\kappa(t)}{\kappa(s)} B(ds) \mid \mathcal{G}_t \right], \quad t \geq 0, \tag{3.8}
\]
almost surely. This definition is consistent with the one proposed in [Norberg (1991)]. State-wise prospective reserves are now defined as follows:

**Definition 3.13.** For \( j \in S \) the **classic state-wise prospective reserve in insured state** \( j \) is the not necessarily unique process \( Y^{\mathcal{G}}_j = (Y^{\mathcal{G}}_j(t))_{t \geq 0} \), where \( (Y^{\mathcal{G}}_j)_{j \in S} \) are the state-wise counterparts to the
prospective reserve $Y^G$. For $x \in S$, the non-classic state-wise prospective reserve in information state $x$ is the almost surely unique process $Y^G_x = (Y^G_x(t))_{t \geq 0}$ given by

$$Y^G_x(t) = \frac{E[I_x(t)Y(t) | \zeta_x]}{E[I_x(t) | \zeta_x]}$$

for $t \geq 0$.

In the following we shall follow the conventions of the literature and write $(V^j)_{j \in S}$ for the classic state-wise prospective reserves in the presence of monotone information $G = F$. Similarly, we write $V$ for the prospective reserve in the presence of monotone information.

Note that for each $t \geq 0$, $j \in S$, and $x \in S$, it almost surely holds that

$$1\{Z_t=j\}Y^G_j(t) = 1\{Z_t=j\}Y^G_x(t),
I_x(t)Y^G_x(t) = I_x(t)Y^G_j(t),$$

cf. Definition 3.4 and Proposition 3.7. The proposed explicit definitions are therefore consistent with the implicit definition in the presence of monotone information put forward by Norberg (1992).

By an application of the law of iterated expectations, cf. Remark 3.6 and the identity (3.1), we can for each $t \geq 0$ cast the classic state-wise prospective reserves as

$$Y^G_j(t) \overset{a.s.}= E \left[ \int_{(t, \infty)} \frac{\kappa(t)}{\kappa(s)} B(ds) \left| G_{t,j}, Z_t = j \right. \right], \quad j \in S. \quad (3.9)$$

**Example 3.14.** Consider the case of monotone information $F$, when by Example 3.2 we have $F_{t,j} = F_{t-}$. It follows that for each $t \geq 0$ and $j \in S$,

$$V_j(t) \overset{a.s.}= E \left[ \int_{(t, \infty)} \frac{\kappa(t)}{\kappa(s)} B(ds) \left| (Z_s)_{0 \leq s < t}, Z_t = j \right. \right],$$

cf. (3.9) and (3.2).

**Example 3.15.** Consider the framework of Subsection 2.3 with non-monotone information $G^i$, $i \in \{1, 2\}$, when by Example 3.3 we have $G^i_{t,j} = \sigma(\psi^i_j(t)) \vee N$. In the presence of non-monotone information $G^i$, $i \in \{1, 2\}$, we then for each $t \geq 0$ and $j \in S$ have

$$Y^G_j(t) \overset{a.s.}= E \left[ \int_{(t, \infty)} \frac{\kappa(t)}{\kappa(s)} B(ds) \left| (\psi^i_j(s))_{0 \leq s < t}, Z_t = j \right. \right],$$

cf. (3.9). For example,

$$Y^G_j(t) \overset{a.s.}= E \left[ \int_{(t, \infty)} \frac{\kappa(t)}{\kappa(s)} B(ds) \left| U_t, Z_t = J + 1 \right. \right],$$

where $U = (U_t)_{t \geq 0}$ is the duration process associated with $Z$.

Note that for each $t \geq 0$,

$$Y^G_j(t) \overset{a.s.}= V_j(t)$$

16
for \( j \in \{1, \ldots, J\} \), while applying (3.2), (3.3), and the constructions of \( \mathcal{G}^1 \) and \( \mathcal{G}^2 \) according to the proof of Lemma 2.4, yields

\[
Y_{G^i_j+1}(t) \overset{\text{a.s.}}{=} Y_{G^i_{(1)}}(t), \\
Y_{G^i_j+2}(t) \overset{\text{a.s.}}{=} 1_{\{\eta \leq t\}} Y_{G^i_{(1,2)}}(t) + 1_{\{\eta > t\}} Y_{G^i_{(2)}}(t).
\]

In the following, \((Y_{G^i_j})_{j \in S}\) always refers to the modifications given by the above identities. Insisting on this ensures the classic state-wise prospective reserves to be well-defined in the presence of non-monotone information \( \mathcal{G}^i \).

As already discussed in Subsections 3.1-3.2, the state-wise counterparts are as a rule not well-defined as stochastic processes, since they are defined up to null-sets for an uncountable number of time points. In the presence of monotone information, \( \mathcal{G} = \mathcal{F} \), we insist on taking the modification given by Proposition 3.11 which solves the problem of well-definedness, and in the following section we show how the concept of classic state-wise prospective reserves is sufficient to study dynamics of state-wise prospective reserves under monotone information. In the presence of non-monotone information, the classic state-wise prospective reserves are not well-defined as stochastic processes. Furthermore, as we show in the following section, the concept of non-classic state-wise prospective reserves, as well as the additional state-wise quantities given by (3.3), is necessary to study the dynamics of state-wise prospective reserves under non-monotone information. To develop the general theory of stochastic Thiele equations, we thus focus on the non-classic state-wise prospective reserves, which refer to the state of information. Still, when meaningful and relevant for specific instances of information, cf. Example 3.15, we cast the results in terms of the more intuitively appealing classic state-wise prospective reserves, which refer to the state of the insured.

In addition to the classic and non-classic state-wise prospective reserves, the additional state-wise quantities given by (3.3) prove useful. Based on Proposition 3.8 and Proposition 3.7 for each \( x, y \in S, x \neq y \), we interpret the state-wise quantities \( \mathcal{Y}_{xx}^G, \mathcal{Y}_{xy}^G, \) and \( \mathcal{Y}_{-xy}^G \) as follows:

- \( \mathcal{Y}_{xx}^G(t) \) is the prospective reserve for staying in information state \( x \) at time \( t \): if in information state \( x \) at time \( t- \) or time \( t \), what one would set aside in case no change in information state occurs at time \( t \),
- \( \mathcal{Y}_{xy}^G(t, e) \) is the backward prospective reserve at transition from information state \( x \) to information state \( y \) with information change \( e \): if in information state \( y \) at time \( t \), what one would set aside in case a change \( \underline{\text{from}} \) information state \( x \) occurred with change in information \( e \) at exactly time \( t \),
- \( \mathcal{Y}_{-xy}^G(t, e) \) is the forward prospective reserve at transition from information state \( x \) to \( y \) with information change \( e \): if in information state \( x \) at time \( t- \), what one would set aside in case a change \( \underline{\text{to}} \) information state \( y \) occurs with change in information \( e \) at exactly time \( t \).

### 4. Dynamics of state-wise prospective reserves

In this section, we present the main results of the paper by deriving so-called stochastic Thiele equations describing the dynamics of state-wise prospective reserves in the presence of non-monotone information. In principle, our method is based on the infinitesimal approach introduced and developed by Christiansen (2020) and relies on the explicit infinitesimal martingale representation theorem (see Theorem 6.1 and Theorem 7.1 in Christiansen, 2020). In comparison,
stochastic Thiele equations in the presence of monotone information are closely related to the classic martingale representation theorem, see e.g. Norberg (1992) and Christiansen and Djehiche (2019).

In Subsection 4.1 we present and derive so-called infinitesimal forward/backward compensators describing the systematic part of the development of the state of information and the payments. Generalized stochastic Thiele equations are derived and interpreted in Subsection 4.2. Finally, in Subsection 4.3 we impose the specific framework of Subsection 2.3 with non-monotone information to information discarding upon and after stochastic retirement and derive stochastic Thiele equations in the presence of monotone information are closely related to the classic martingale representation theorem, see e.g. Norberg (1992) and Christiansen and Djehiche (2019).

In the remainder of the paper, we generally suppose that

\[ B(dt) = \sum_{j \in S} 1\{Z_{t-} = j\}b_j(t) \mu(dt) + \sum_{j,k \in S, j \neq k} b_{jk}(t) N_{jk}(dt), \]

where \( b_j \) and \( b_{jk} \) are \( \mathcal{F} \)-predictable bounded Dirac processes and the measure \( \mu \) is a sum of the Lebesgue-measure \( m \) and a countable number of Dirac-measures \((\epsilon_n)_{n \in \mathbb{N}}:\)

\[ \mu(A) = m(A) + \sum_{n=1}^{\infty} \epsilon_n(A), \quad A \in \mathcal{B}([0, \infty)), \]

for deterministic time points \( 0 \leq t_1 < t_2 < ... \) that are increasing to infinity (i.e. there are at most a finite number of such time points on each compact interval).

### 4.1. Infinitesimal compensators

The so-called compensator \( \lambda_{xy} \) of the random counting measure \( \nu_{xy} \) is the unique \( \mathcal{F} \)-predictable random measure such that the difference \([0, \infty) \ni t \mapsto \nu_{xy}([0, t] \times A) - \lambda_{xy}([0, t] \times A)\) is an \( \mathcal{F} \)-martingale for each \( A \in \mathcal{B}(E_{xy}) \). In particular, we have

\[ \lambda_{xy}([0, t] \times A) = \lim_{n \to \infty} \sum_{S_n} \mathbb{E}[\nu_{xy}((t_k, t_{k+1}] \times A) | \mathcal{F}_{t_k}] \quad (4.1) \]

almost surely for each \( t > 0 \), where \((S_n)_{n \in \mathbb{N}}\) is any increasing sequence (i.e. \( S_n \subseteq S_{n+1} \) for all \( n \)) of partitions \( 0 = t_0 < \cdots < t_n = t \) of the interval \([0, t]\) such that \( |S_n| := \max\{t_k - t_{k-1} : k = 1, \ldots, n\} \to 0 \) for \( n \to \infty \). Christiansen (2020) expands this property to the non-monotone information \( \mathcal{G} \) and denotes the random measures \( \gamma_{xy}^G \) and \( \gamma_{xy}^F \) defined by

\[ \gamma_{xy}^{\mathcal{G}}([0, t] \times A) = \lim_{n \to \infty} \sum_{S_n} \mathbb{E}[\nu_{xy}((t_k, t_{k+1}] \times A) | \mathcal{G}_{t_k}], \quad t > 0, A \in \mathcal{B}(E_{xy}), \]

\[ \gamma_{xy}^{\mathcal{F}}([0, t] \times A) = \lim_{n \to \infty} \sum_{S_n} \mathbb{E}[\nu_{xy}((t_k, t_{k+1}] \times A) | \mathcal{G}_{t_{k+1}}], \quad t > 0, A \in \mathcal{B}(E_{xy}), \]

as infinitesimal forward compensator (IF-compensator) and infinitesimal backward compensator (IB-compensator) of \( \nu_{xy} \) with respect to \( \mathcal{G} \), given that the limits exist for all \( t > 0 \) almost surely.

In the special case of monotone information \( \mathcal{G} = \mathcal{F} \) the IF-compensator equals the classic compensator and the IB-compensator equals the counting measure itself, i.e. \( \gamma_{xy}^{\mathcal{F}} = \lambda_{xy} \) and \( \gamma_{xy}^{\mathcal{F}} = \nu_{xy} \) almost surely.
Proposition 4.1. For each \( x, y \in S, x \neq y \), the IF-compensator \( \gamma_{xy}^G \) and the IB-compensator \( \gamma_{xy}^G \) of \( \nu_{xy} \) exist and satisfy for each \( A \in \mathcal{B}(E_{xy}) \) and all \( t > 0 \)

\[
\gamma_{xy}^G((0,t] \times A) = \int_{(0,t] \times A} \frac{I_x(s-)}{E[I_x(s-)|\zeta_x]} P((T_{xy}, \zeta_{xy}) \in ds \times de | \zeta_x),
\]

\[
\gamma_{xy}^G((0,t] \times A) = \int_{(0,t] \times A} \frac{I_y(s)}{E[I_y(s)|\zeta_y]} P((T_{xy}, \zeta_{xy}) \in ds \times de | \zeta_y)
\]

almost surely.

Proof. See Proposition 5.1 and Theorem 5.2 in Christiansen (2020).

Denote by \( b \) the sojourn payment rate given by

\[
b(t) = \sum_{j \in S} 1 \{ Z_t - j \} b_j(t), \quad t > 0,
\]

and denote by \( \beta \) the transition payments given by

\[
\beta(t) = \sum_{j,k \in S} b_{jk}(t) \Delta N_{jk}(t), \quad t > 0.
\]

Proposition 4.2. The payment process \( B \) has an IF-compensator \( C_B^G \) with respect to \( G \) of the form

\[
C_B^G(dt) \stackrel{as.}{=} \sum_{x \in S} I_x(t-) b_x^G(t) \mu(dt) + \sum_{x,y \in S, y \neq x} \int_{E_{xy}} \beta_{xy}^G(t,e) \gamma_{xy}^G(dt \times de),
\]

where \( b_x^G \) and \( \beta_{xy}^G \) are the processes defined from \( b \) and \( \beta \) by the second and fourth line in (3.3), respectively.

Proof. See Theorem 5.2 and Example 7.2 in Christiansen (2020). Note that (2.4) holds and that \( \beta \) can be decomposed into a sum of a càdlàg and a càgàl process both with finite expected variation on compacts.

Applying similar techniques as in the proof of Proposition 4.1 and the proof of Proposition 4.2, one can show that if for all \( t > 0 \) each \( b_j(t) \) and \( b_{jk}(t) \) is \( G \)-measurable, then

\[
C_B^G(dt) \stackrel{as.}{=} \sum_{j \in S} 1 \{ Z_t - j \} b_j(t) \mu(dt) + \sum_{j,k \in S, j \neq k} b_{jk}(t) \Gamma_{jk}^G(dt),
\]

where \( \Gamma^G \) are the IF-compensators of the multivariate counting process \( N \) (associated with \( Z \)) w.r.t. \( G \).

In general, we thus interpret \( b_x^G \) as the (\( G \)-averaged) sojourn payments in information state \( x \in S \) and \( \beta_{xy}^G(\cdot,e) \) as the (\( G \)-averaged) transition payment for a change in information \( e \) from information state \( x \) to information state \( y \).
4.2. Stochastic Thiele equations

We are now ready to present stochastic differential equations describing the dynamics of the non-classic state-wise prospective reserves \((Y^G_x)_{x \in S}\) in the presence of general non-monotone information \(G\):

**Theorem 4.3** (Generalized stochastic Thiele equation). The non-classic state-wise prospective reserves \((Y^G_x)_{x \in S}\) almost surely satisfy the stochastic differential equation

\[
0 = \sum_{x \in S} I_x(t-) \left( Y^G_x(dt) - Y^G_x(t-) \frac{\kappa(dt)}{\kappa(t-)} + b^G_x(t) \mu(dt) + \sum_{y: y \neq x} \int_{E_{xy}} R^G_{-}(t, x, y, e) \gamma^G_{xy}(dt \times de) \right. \\
- \sum_{y: y \neq x} \int_{E_{xy}} R^G(t, y, x, e) \gamma^G_{yx}(dt \times de),
\]

where for \(x, y \in S, x \neq y\),

\[
R^G_{-}(t, x, y, e) := \beta^G_{xy}(t, e) + Y^G_{xy}(t, e) - Y^G_{xx}(t), \\
R^G(t, y, x, e) := Y^G_{yx}(t, e) - Y^G_{xx}(t).
\]

**Remark 4.4.** Due to (3.4), we could equally likely cast \(R^G_{-}\) and \(R^G\) via

\[
R^G_{-}(t, x, y, e) := \beta^G_{xy}(t, e) + Y^G_{xy}(t, e) - Y^G_{xx}(t), \\
R^G(t, y, x, e) := Y^G_{yx}(t, e) - Y^G_{xx}(t).
\]

In the following, we prefer this representation.

In the presence of monotone information \(G = F\), starting from Theorem 4.3 one can derive the following stochastic differential equations describing the dynamics of the classic state-wise prospective reserves \((V_j)_{j \in S}\):

**Corollary 4.5** (Classic stochastic Thiele equation). The classic state-wise prospective reserves \((V_j)_{j \in S}\) almost surely satisfy the stochastic differential equation

\[
0 = \sum_{j \in S} \mathbf{1}_{\{Z_t = j\}} \left( V_j(dt) - V_j(t-) \frac{\kappa(dt)}{\kappa(t-)} + b_j(t) \mu(dt) + \sum_{k: k \neq j} R_{jk}(t) \Lambda_{jk}(dt) \right),
\]

where \(R_{jk}(t) := b_{jk}(t) + V_k(t) - V_j(t)\) are the classic sum at risks and where \(\Lambda_{jk} := \Gamma_{jk}^F\) are the classic \(F\)-compensators of the multivariate counting process \(N\).

Before we present the proofs of Theorem 4.3 and Corollary 4.5, we first provide an interpretation of the results. In the presence of monotone information, Corollary 4.5 yields stochastic differential equations that are directly comparable to the stochastic Thiele equations of Norberg (1992, 1996). In Norberg (1992, 1996), the \(F\)-compensators \(\Lambda\) of \(N\) are assumed to admit densities w.r.t. the Lebesgue-measure, and the result is derived by suitably applying the martingale representation theorem and identifying the integrands. The method of the present paper, while extended to also cover the non-monotone case, is based on a suitable application of the explicit infinitesimal martingale representation theorem. In particular, Corollary 4.5 can also be derived directly from
the classic martingale representation theorem following Christiansen and Djehiche (2019); in this case, the restriction to slightly less general payments, cf. beginning of Section 4, is not necessary.

The stochastic differential equation of Theorem 4.3 is in a twofold manner fundamentally different from the stochastic Thiele equation in the presence of monotone information. Firstly, the sum at risks appearing in the term involving the IF-compensators, which correspond to ordinary compensators in the presence of monotone information, take a different form. Rather than being the difference of two state-wise prospective reserves added the relevant transition payment, it involves the difference of the forward state-wise prospective reserve and the current ordinary state-wise prospective reserve added relevant transition payment. In the presence of monotone information, we can show that the forward state-wise prospective reserve can be replaced by a relevant ordinary state-wise prospective reserve, but this is not necessarily the case in the presence of non-monotone information. Here the possibility of information discarding entails a possible ordinary state-wise prospective reserve, but this is not necessarily the case in the presence of non-monotone information. Here the possibility of information discarding entails a possible improvement in the accuracy of the reserving by utilizing the information available at time $t$ and time $t$—rather than utilizing only the information available at time $t$.

Secondly, the stochastic differential equation of Theorem 4.3 contains an additional term involving the IB-compensators. In the presence of monotone information, we can show that this term is zero. It is the backward looking equivalent of the term involving the IF-compensators. In the presence of monotone information, we can show that this term is zero. It is the backward looking equivalent of the term involving the IF-compensators.

In Subsection 4.3, we derive and interpret stochastic Thiele equations in the presence of specific examples of non-monotone information related to stochastic retirement. We refer to this subsection for further interpretation and discussion of the general results.

Proof of Theorem 4.3. Analogously to Proposition 4.2, one can show that the discounted payment process $\bar{B}$ given by $\bar{B}(0) = B(0)$ and $\bar{B}(dt) := v(t)B(dt)$ admits the IF-compensator

$$C^G_{\bar{B}}(dt) = \sum_{x \in S} I_x(t-):v(t) b^+_x(t) \mu(dt) + \sum_{x,y \in S \atop x \neq y} \int_{E_{xy}} v(t) \beta^G_{xy}(t,e) \gamma^G_{xy}(dt \times de).$$

According to Theorem 7.1 in Christiansen (2020), the process $[0, \infty) \ni t \mapsto \bar{Y}(t) = v(t)Y(t)$ almost surely satisfies the equation

$$\bar{Y}^G(dt) = -C^G_{\bar{B}}(dt) + \sum_{x,y \in S \atop x \neq y} \int_{E_{xy}} v(t) \left( \gamma^G_{xy}(t,e) - \gamma^G_{yx}(t) \right)(dt \times de) - \sum_{x,y \in S \atop x \neq y} \int_{E_{xy}} v(t) \left( \gamma^G_{xy}(t,e) - \gamma^G_{yx}(t) \right)(dt \times de).$$

On the other hand, by applying integration by parts on $\bar{Y}(t) \overset{a.s.}{=} \sum_{x \in S} I_x(t)\overline{Y}_x^G(t)$ and using $\overline{Y}^G \overset{a.s.}{=} v(t)Y^G$, we can show that

$$\bar{Y}^G(dt) \overset{a.s.}{=} \sum_{x \in S} I_x(t-):\overline{Y}_x^G(dt) + \sum_{x,y \in S \atop x \neq y} v(t) \left( \gamma^G_{xy}(t) - \gamma^G_{yx}(t) \right) \nu_{xy}(dt \times E_{xy}).$$

Thus, by equating the latter two equations and rearranging the terms, while using (3.4), the fact that $\gamma^G_{xy}(dt \times de) = I_x(t-):\gamma^G_{xy}(dt \times de)$ and $\gamma^G_{yx}(dt \times de) = I_x(t)\gamma^G_{yx}(dt \times de)$ almost surely, and
the equation \( I_x(t) = I_x(t^-)I_x(t) + 1_{\{z_{-} \neq x\}}I_x(t) \), we obtain

\[
0 \overset{a.s.}{=} \sum_{x \in S} I_x(t^-) \left( \mathcal{Y}_x^G(dt) + v(t)\mathcal{R}^G_x(t) \mu(dt) + \sum_{y:y \neq x} \int_{E_{xy}} v(t)\mathcal{R}^G_x(t, y, e) \gamma_{xy}^G(dt \times de) \right. \\
- \left. \sum_{y:y \neq x} \int_{E_{yx}} v(t)\mathcal{R}^G_x(t, y, x) \gamma_{yx}^G(dt \times de) \right),
\]

\[
- \sum_{x,y \in S \atop x \neq y} v(t) \left( \mathcal{Y}_{xy}^G(t, e) - \mathcal{Y}_{xy}^G(t, e) \right) \nu_{xy}(dt \times E_{xy})
\]

\[
+ \sum_{x,y \in S \atop x \neq y} v(t) \left( \mathcal{Y}_{xy}^G(t) - \mathcal{Y}_{xy}^G(t) \right) \nu_{xy}(dt \times E_{xy})
\]

The third line equals zero because of (6.7) in Christiansen (2020). The forth, fifth and sixth line together equal

\[
\sum_{x,y \in S \atop x \neq y} v(t) \left( \mathcal{Y}_{xy}^G(t) - \mathcal{Y}_{xy}^G(t) \right) \nu_{xy}(dt \times E_{xy})
\]

almost surely, because \( \sum_{x:x \neq x \neq x \neq x} \nu_{xx}(\{t\} \times E_{xx}) \) is almost surely non-zero only at finitely many time points. The last line equals zero since \( I_x(t^-)(\mathcal{Y}_{xx}^G(t) - \mathcal{Y}_{xx}^G(t)) = 0 \) according to (3.4). The remain-
ing two lines also add up to zero since, by applying Proposition 5.7, 2.5, and Proposition 4.1,
\[ I_x(t)Y_{x}^{G}(t) = I_x(t)E[Y(t) | \zeta_{x}, \mathcal{Z}_{t} = x] \]
\[ = I_x(t)E[Y(t)I_x(t-) | \zeta_{x}, \mathcal{Z}_{t} = x] + I_x(t)E[Y(t) \sum_{x:x \neq x} I_x(t-) | \zeta_{x}, \mathcal{Z}_{t} = x] \]
\[ = I_x(t)E[Y(t) | \zeta_{x}, \mathcal{Z}_{t} = x, \mathcal{Z}_{t-} = x]E[I_x(t-) | \zeta_{x}, \mathcal{Z}_{t} = x] \]
\[ + \sum_{x:x \neq x} \int_{E_{xx}} E[Y(t) | \zeta_{x}, \mathcal{Z}_{t} = x, T_{xx} = t, \zeta_{xx} = e] \gamma_{xx}^{G}(\{t\} \times de) \]
\[ = I_x(t)Y_{xx}^{G}(t) \left(1 - \sum_{x:x \neq x} \gamma_{xx}^{G}(\{t\} \times E_{xx})\right) + \sum_{x:x \neq x} \int_{E_{xx}} Y_{xx}^{G}(t,e) \gamma_{xx}^{G}(\{t\} \times de) \]
almost surely. All in all, we have
\[ 0 = \sum_{x \in S} I_x(t) \left( \int_{E_{xx}} \gamma_{xx}^{G}(\{t\} \times dt) + \sum_{y: y \neq x} \int_{E_{xy}} v(t)R_{xy}^{G}(t,x,y,e) \gamma_{yx}^{G}(\{t\} \times dt \times de) \right) \]
\[ - \sum_{y: y \neq x} \int_{E_{xy}} v(t)R_{xy}^{G}(t,x,y,e) \gamma_{yx}^{G}(\{t\} \times dt \times de) \].

Now apply integration by parts on \( \tilde{Y}_{x}^{G}(t) = v(t)Y_{x}^{G}(t) \) and rearrange the terms in order to end up with the statement of the theorem.

Proof of Corollary 4.1.\[\] By setting \((T_{i}, S_{i}, \zeta_{i})_{i \in \mathbb{N}} = (\tau_{i}, \infty, (\tau_{i}, Z_{\tau_{i}}))_{i \in \mathbb{N}} \) we obtain \( \mathcal{G} = \mathcal{F} \) such that \((Y_{x}^{F})_{x \in S} \) satisfy \( 4.2 \) almost surely. Recall that \( \gamma_{yx}^{F} = \nu_{yx} \), when
\[ I_x(t-) \sum_{y: y \neq x} \int_{E_{yx}} R_{xy}^{F}(t,x,y,e) \gamma_{yx}^{F}(\{t\} \times dt \times de) = 0 \]
almost surely. By Remark 3.10 and starting from \( 4.2 \), similar arguments as in the proof of Proposition 3.11 yield the following stochastic differential equations:
\[ \sum_{n=0}^{\infty} 1_{\{Z_{\tau_{n}} = j\}} 1_{\{\tau_{n} \leq \tau_{n+1}\}} Y_{\{1, \ldots, n\}}^{F}(dt) \]
\[ = a_{n} 1_{\{Z_{\tau_{n}} = j\}} \left( V_{j}(t-) - \frac{\kappa_{j}(dt)}{\kappa(t-)} - b_{j}(t) \mu(dt) - \sum_{k:k \neq j} (b_{jk}(t) + V_{k}(t) - V_{j}(t)) \Gamma_{jk}^{F}(dt) \right) \]
for \( j \in S \). By tedious yet straightforward calculations, it is possible to show that
\[ \sum_{n=0}^{\infty} (V_{j}(t) - Y_{\{1, \ldots, n\}}^{F}) d(1_{\{Z_{\tau_{n}} = j\}} 1_{\{\tau_{n} \leq \tau_{n+1}\}} ) \overset{a.s.}{=} 0, \quad j \in S, \]
which implies
\[ \sum_{n=0}^{\infty} 1_{\{Z_{\tau_{n}} = j\}} 1_{\{\tau_{n} \leq \tau_{n+1}\}} Y_{\{1, \ldots, n\}}^{F}(dt) \overset{a.s.}{=} 1_{\{Z_{\tau_{n}} = j\}} V_{j}(dt), \quad j \in S, \]
by an application of integration by parts. Collecting results establishes the desired result. \[\]
In the case where the payments \( B \) themselves depend on the prospective reserve \( V \), the (stochastic) Thiele equations rather than \((3.3)\) might serve as definition for the prospective reserve \( V \), see e.g. [Djehiche and Löfdahl (2016) and Christiansen and Djehiche (2019)]. In the presence of monotone information, this point of view is encapsulated by the following result.

**Proposition 4.6.** Let there be a maximal contract time \( n < \infty \), i.e. each \( b_j \) and \( b_{jk} \) is constantly zero on the interval \((n, \infty)\). Suppose that \( W_j, j \in S \), are \( \mathcal{F} \)-predictable bounded processes such that \([0, \infty) \ni t \mapsto 1_{\{Z_i=j\}} W_j(t) \) almost surely has càdlàg paths for all \( j \in S \). If \( W_j, j \in S \), satisfy the stochastic differential equations

\[
0 = 1_{\{Z_i=j\}} \left( W_j(dt) - W_j(t-) \frac{\kappa(dt)}{\kappa(t-)} + b_j(t) \mu(dt) + \sum_{k \neq j \in S} (b_{jk}(t) + W_k(t) - W_j(t)) \Delta_{jk}(dt) \right) \tag{4.4}
\]

with terminal condition \( W_j(n) = 0, j \in S \), then \( W_{Z_i}(t) = V(t) \) almost surely for all \( t \in [0, n] \).

**Proof.** Since \([0, \infty) \ni t \mapsto 1_{\{Z_i=j\}} W_j(t) \) almost surely has càdlàg paths for all \( j \in S \), the stochastic differential equations imply that \([0, \infty) \ni t \mapsto 1_{\{Z_i=j\}} W_j(t) \) and \((0, \infty) \ni t \mapsto 1_{\{Z_i=j\}} W_j(t) \) almost surely have càdlàg paths of finite variation for all \( j \in S \), cf. proof of Corollary 3.9. By applying integration by parts and the stochastic differential equations for the processes \( W_j, j \in S \), we obtain

\[
\begin{align*}
\frac{d}{dt}(v(t)W_{Z_i}(t)) &= \sum_{j \in S} 1_{\{Z_i=j\}} \left( v(t)W_j(dt) - W_j(t-)v(t) \frac{\kappa(dt)}{\kappa(t-)} \right) + \sum_{j_1, j_2 \in S, j_1 \neq j_2} v(t)(W_{j_1}(t) - W_{j_2}(t)) N_{j_1,j_2}(dt) \\
&= -v(t) B(dt) + \sum_{j_1, j_2 \in S, j_1 \neq j_2} v(t)(b_{j_1}(t) + W_{j_2}(t) - W_{j_1}(t)) (N_{j_1,j_2} - \Delta_{j_1,j_2})(dt)
\end{align*}
\]

almost surely. Since each \([0, \infty) \ni t \mapsto b_{j_1}(t) + W_{j_2}(t) - W_{j_1}(t) \) is \( \mathcal{F} \)-predictable and bounded, the last term is an \( \mathcal{F} \)-martingale. Thus, we obtain

\[
v(t)W_{Z_i}(t) = \mathbb{E} \left[ v(t) \sum_{j \in S} 1_{\{Z_i=j\}} W_j(t) \bigg| \mathcal{F}_t \right] \\
= \mathbb{E} \left[ v(t) \int_{(t,n]} \frac{k(t)}{k(s)} B(ds) \bigg| \mathcal{F}_t \right] = v(t)V(t)
\]

almost surely for all \( t \in [0, n] \). Noting \( v > 0 \) completes the proof. \( \Box \)

### 4.3. Examples

In this subsection, we consider the framework of stochastic retirement from Subsection 2.3 and the non-monotone information given by \( G_1 \) and \( G_2 \). The time of retirement and death are given by the hitting times \( \eta \) and \( \delta \), respectively. Recall that \( G_1 \) corresponds to the case where upon retirement or death the insurer discards the previous health records of the insured, while \( G_2 \) even keeps no record on the time of retirement.

In Subsection 4.3.1 we present some auxiliary results characterizing the relevant IF- and IB-compensators and state-wise quantities. Stochastic Thiele equations are then derived in Subsection 4.3.2 using the general theory developed in Subsection 4.2. Finally, in Subsection 4.3.3 we specialize the inter-temporal dependency structure, derive Feynman-Kac formulas, and relate the results to actuarial practice.
4.3.1. Preliminaries

Denote for \( j, k \in S, \ j \neq k \), by \( \Gamma^i_{jk} \) and \( \Gamma^j_{jk} \) the IF- and IB-compensator of \( N_{jk} \), respectively, w.r.t. \( \mathcal{G}^i \), \( i = 1, 2 \). Recall that \( \Lambda \) denotes the classic \( F \)-compensators of \( N \). The following result gives an explicit characterization of the relevant IF- and IB-compensators of \( N \) w.r.t. \( \mathcal{G}^1 \) and \( \mathcal{G}^2 \).

**Proposition 4.7.** For all \( t > 0 \) we almost surely have

\[
\begin{align*}
\Gamma^1_{jk}(t) &= \Gamma^2_{jk}(t) = \Lambda_{jk}(t), \quad j \in \{1, \ldots, J\}, k \in S \setminus \{j\}, \\
\Gamma^1_{jk}(t) &= \Gamma^2_{jk}(t) = N_{jk}(t), \quad \text{either } j \in S, k \in \{1, \ldots, J\} \setminus \{j\} \text{ or } j = J + 1, k = J + 2, \\
\Gamma^1_{jk+1}(t) &= \int_{(0,t]} \frac{1_{\{\eta < s \leq \delta\}}}{\mathbb{P}(\delta \geq s | \eta)} \mathbb{P}(\delta \in ds | \eta), \quad j = J + 1, k = J + 2, \\
\Gamma^2_{jk+1}(t) &= \int_{(0,t]} \frac{1_{\{\eta < s \leq \delta\}}}{\mathbb{P}(\eta < s \leq \delta)} \mathbb{P}(\delta \in ds | \eta), \quad j = J + 1, k = J + 2, \\
\Gamma^1_{jk}(t) &= \int_{(0,t]} \mathbb{P}(Z_{\eta} = j | \eta = s) \sum_{\ell=1}^{J} N_{\ell k}(ds), \quad j \in \{1, \ldots, J\}, k = J + 1, \\
\Gamma^2_{jk}(t) &= \int_{(0,t]} \mathbb{P}(Z_{\eta} = j | \eta = s) \sum_{\ell=1}^{J} N_{\ell k}(ds), \quad j \in \{1, \ldots, J\}, k = J + 1, \\
\Gamma^1_{jk+1}(t) &= \int_{(0,t]} \frac{1_{\{\eta < \delta \leq \beta\}}}{\mathbb{P}(\beta \geq \delta | \eta)} \mathbb{P}(\beta \in d\beta | \eta), \quad j \in \{1, \ldots, J\}, k = J + 2.
\end{align*}
\]

All remaining IF- and IB-compensators of \( N \) equal zero almost surely.

**Sketch of proof.** Calculate the IF-compensator \( \gamma^i_{xy} \) and the IB-compensator \( \gamma^{i}_{xy} \) of \( \nu_{xy} \) from Proposition 4.4 and use the construction of \( \mathcal{G}^1 \) and \( \mathcal{G}^2 \) according to the proof of Lemma 2.4. \( \square \)

In the following, \((Y^i_{\mathcal{G}})_{i \in S}\) refers to the modification of the classic state-wise prospective reserves w.r.t. \( \mathcal{G}^i \) presented in Example 3.1. The next result provides a characterization of the remaining key terms appearing in the stochastic Thiele equations w.r.t. \( \mathcal{G}^1 \) and \( \mathcal{G}^2 \).

**Proposition 4.8.** For each \( i \in \{1, 2\} \) and \( t > 0 \) we have

\[
\begin{align*}
b^i_{J+1}(t) := b^i_1(t) &= \mathbb{E}[b_{J+1}(t) | \mathcal{G}^i_t], \\
\beta^i_{J+1}(t) &= \beta^i_{11}(t, t, J + 2) = \mathbb{E}[b_{J+1}(t) | \mathcal{G}^i_t, \delta = t] \\
b^i_{J+2}(t) := 1_{\{\eta < t\}} b^i_{12}(t) + 1_{\{\eta \geq t\}} b^i_{22}(t) &= \mathbb{E}[b_{J+2}(t) | \mathcal{G}^i_t]
\end{align*}
\]

almost surely on \( \{Z_{t-} = J + 1\} \),

\[
\begin{align*}
b^i_{J+2}(t) := 1_{\{\eta < t\}} b^i_{12}(t) + 1_{\{\eta \geq t\}} b^i_{22}(t) &= \mathbb{E}[b_{J+2}(t) | \mathcal{G}^i_t]
\end{align*}
\]

almost surely on \( \{Z_{t-} = J + 2\} \),

\[
\begin{align*}
R^i_{J+1}(t) := \beta^i_{J+1}(t) + Y^i_{J+2}(t) - Y^i_{J+1}(t) &= E[\beta(t) + Y(t) | \mathcal{G}^i_t, t, Z_{t-} = J + 1] \\
L^2_{J+1}(t) := E[Y(t) | \eta = t, Z_{\eta-} = j] - \mathbb{E}[Y(t) | \mathcal{G}^2_t, Z_{t-} = J + 1] &= E[Y(t) | \mathcal{G}^2_t, t, Z_{t-} = J + 1]
\end{align*}
\]

almost surely on \( \{Z_t = J + 1\} \) and for each \( t \geq 0 \) and \( j \in \{1, \ldots, J\} \) we have

\[
\begin{align*}
R^i_{J+1}(t) := \beta^i_{J+1}(t) + Y^i_{J+2}(t) - Y^i_{J+1}(t) &= E[\beta(t) + Y(t) | \mathcal{G}^i_t, t, Z_{t-} = J + 1] \\
L^2_{J+1}(t) := E[Y(t) | \eta = t, Z_{\eta-} = j] - \mathbb{E}[Y(t) | \mathcal{G}^2_t, Z_{t-} = J + 1] &= E[Y(t) | \mathcal{G}^2_t, t, Z_{t-} = J + 1]
\end{align*}
\]

almost surely on \( \{Z_t = J + 1\} \).
Sketch of proof. Combine suitably the contents of Example 3.15, Proposition 4.7, the constructions of \( G^1 \) and \( G^2 \) according to the proof of Lemma 2.4 and (3.3).

4.3.2. Stochastic Thiele equations

Based on the characterization of relevant IF- and IB-compensators and state-wise quantities from Subsection 4.3.1, the following two theorems yield stochastic Thiele equations for the classic state-wise prospective reserves w.r.t. non-monotone information \( G^1 \) and \( G^2 \).

**Theorem 4.9.** The classic state-wise prospective reserves \((Y_j^{G^1})_{j \in S}\) almost surely satisfy \(Y_j^{G^1} = V_j\) for \(j \in \{1, \ldots, J\}\) and
\[
0 = 1_{\{Z_{t-} = J + 1\}} \left( Y_{J+1}^{G^1}(dt) - Y_{J+1}^{G^1}(t-) \frac{\kappa(dt)}{\kappa(t-)} + b_{J+1}^1(t) \mu(dt) + R_{J+1(J+2)}^1(t) \Gamma_{(J+1)(J+2)}^{G^1}(dt) \right),
\]
\[
0 = 1_{\{Z_{t-} = J + 2\}} \left( Y_{J+2}^{G^1}(dt) - Y_{J+2}^{G^1}(t-) \frac{\kappa(dt)}{\kappa(t-)} + b_{J+2}^1(t) \mu(dt) \right).
\]

**Theorem 4.10.** The classic state-wise prospective reserves \((Y_j^{G^2})_{j \in S}\) almost surely satisfy \(Y_j^{G^2} = V_j\) for \(j \in \{1, \ldots, J\}\) and
\[
0 = 1_{\{Z_{t-} = J + 1\}} \left( Y_{J+1}^{G^2}(dt) - Y_{J+1}^{G^2}(t-) \frac{\kappa(dt)}{\kappa(t-)} + b_{J+1}^2(t) \mu(dt) + R_{J+1(J+2)}^2(t) \Gamma_{(J+1)(J+2)}^{G^2}(dt) \right)
\]
\[
- \sum_{k=1}^{J} L_{k(J+1)}^2(t) \Gamma_{k(J+1)}^{G^2}(dt),
\]
\[
0 = 1_{\{Z_{t-} = J + 2\}} \left( Y_{J+2}^{G^2}(dt) - Y_{J+2}^{G^2}(t-) \frac{\kappa(dt)}{\kappa(t-)} + b_{J+2}^2(t) \mu(dt) \right).
\]

Sketch of proof of Theorem 4.9 and Theorem 4.10. Since \(\{Z_t = J + 1\} = \{\eta \leq t < \delta\} = \{Z_t = \{1\}\}\), \(\{Z_t = J + 2, \eta \leq t\} = \{Z_t = \{1, 2\}\}\), and \(\{Z_t = J + 2, \eta > t\} = \{Z_t = \{2\}\}\) for all \(t \geq 0\), following along the lines of the proof of Corollary 4.5 and pointing to Example 3.15 yields
\[
1_{\{Z_{t-} = J + 1\}} Y_{J+1}^{G^1}(dt) = I_{\{1\}}(t-) Y_{\{1\}}^{G^1}(dt),
\]
\[
1_{\{Z_{t-} = J + 2\}} Y_{J+2}^{G^1}(dt) = I_{\{1, 2\}}(t-) Y_{\{1, 2\}}^{G^1}(dt) + I_{\{2\}}(t-) Y_{\{2\}}^{G^1}(dt)
\]
almost surely. Now apply Theorem 4.2, calculate the terms explicitly, collect them, and apply Proposition 4.7 and Proposition 4.8.

**Remark 4.11.** Note that the term
\[
1_{\{Z_{t-} = J + 1\}} \sum_{k=1}^{J} L_{k(J+1)}^2(t) \Gamma_{k(J+1)}^{G^2}(dt)
\]
can be replaced by
\[
1_{\{Z_{t-} = J + 1\}} L_{\bullet(J+1)}^2(t) \Gamma_{\bullet(J+1)}^{G^2}(dt),
\]
where
\[
L_{\bullet(J+1)}^2(t) := E[Y(t)|\eta = t] - Y_{J+1}^{G^2}(t),
\]
\[
\Gamma_{\bullet(J+1)}^{G^2}(dt) := \frac{1_{\{\eta \leq t < \delta\}}}{P(\eta \leq t < \delta)} P(\eta \in dt).
\]
To see this, apply Proposition 4.7, Proposition 4.8, and (3.3).
The stochastic differential equations that follow from Theorem 4.9 and Theorem 4.10 are fundamentally different from the stochastic differential equations appearing in the presence of monotone information. Since \( Y^G_j \) almost surely equals \( V_j \) for \( j \in \{1, \ldots, J\} \), Corollary 4.13 yields the stochastic differential equations

\[
0 \overset{a.s.}{=} 1_{\{\zeta_{i-1} = j\}} \left( Y^G_j (dt) - Y^G_j (t-) \frac{\kappa(dt)}{\kappa(t-)} + b_j(t) \mu(dt) + \sum_{k: k \neq j} (b_{jk}(t) + V_k(t) - Y^G_j) \Lambda_{jk}(dt) \right)
\]

for \( j \in \{1, \ldots, J\} \). The sum at risks for \( k \in \{J + 1, J + 2\} \) take an unusual form as they involve \( V_{J+1} \) and \( V_{J+2} \) rather than \( Y^G_{J+1} \) and \( Y^G_{J+2} \). Since information discarding occurs upon or after retirement and death, this just reflects full utilization of all available information (before retirement and death).

Another fundamental difference is evident in Theorem 4.10. Recall that \( G^2 \) does not have the time since retirement as admissible information. Referring to Remark 4.11, the stochastic differential equation for \( Y^G_{J+1} \) includes the term

\[
\left( E[Y(t)|\eta = t] - Y^G_{J+1}(t) \right) \frac{1_{\{\eta \leq t < \delta\}}}{P(\eta \leq t < \delta)} P(\eta \in dt).
\]

It adjusts the dynamics to take into account the possibility that retirement has just occurred. In this case, at time \( t \) one would reserve \( E[Y(t)|\eta = t] \) rather than the averaged \( Y^G_{J+1}(t) \). This constitutes the first part of the product, while the second part is exactly the infinitesimal probability of retirement having just occurred, conditionally on the insured presently being retired.

4.3.3. Feynman-Kac formulas

We now specialize and simplify the setting to provide a more straightforward and less technical discussion of the general results and their relation to actuarial practice.

Suppose that \( \bar{Z} \) is semi-Markovian such that the \( F \)-compensators \( \Lambda \) of \( N \) admit densities w.r.t. the Lebesgue measure and such that \((\eta, \delta)\) is a continuous random variable. Denote by \( f_{(\eta, \delta)} \) the joint density function of \((\eta, \delta)\), by \( f_{\eta|\delta} \) the conditional density function of \( \eta \) given \( \delta \), and by \( f_\eta \) and \( f_\delta \) the marginal density functions of \( \eta \) and \( \delta \). Further, suppose that \( b_j \) and \( b_{jk} \) are deterministic for all \( j, k \in S, j \neq k \), and let there be a maximal contract time \( n < \infty \), i.e. each \( b_j \) and \( b_{jk} \) is constantly zero on the interval \((n, \infty)\).

Because of Proposition 2.3, the compensators \( \Lambda \) have representations of the form

\[
\Lambda_{jk}(dt) = 1_{\{\zeta_{i-1} = j\}} \alpha_{jk}(t, t - U_{i-1}) dt, \quad j \in \{1, \ldots, J\}, \quad k \in \{1, \ldots, J + 2\} \setminus \{j\},
\]

\[
\Lambda_{(J+1)(J+2)}(dt) = 1_{\{\zeta_{i-1} = J+1\}} \alpha_{(J+1)(J+2)}(t, t - U^h_{i-1}, t - U^r_{i-1}, H_{i-1}) dt
\]

for deterministic functions \( \alpha_{jk} \) and \( \alpha_{(J+1)(J+2)} \), so-called transition rates.

The next results provide Feynman-Kac formulas that can serve as the starting point for the development of numerical schemes for the classic state-wise prospective reserves \((V_j)_{j \in S}\).

**Proposition 4.12.** Suppose the assumptions from the beginning of this subsection hold. If the function \( W_{J+2}(\cdot) \) is a bounded càdlàg solution of

\[
W_{J+2}(dt) = W_{J+2}(t- \frac{\kappa(dt)}{\kappa(t-)} - b_{J+2}(t) \mu(dt), \quad t > 0,
\]

(4.5)
with terminal condition $W_{J+2}(n) = 0$, and the function $W_{J+1}(\cdot, \cdot, \cdot)$ is a bounded and càdlàg solution of

$$W_{J+1}(dt, s, r, k) = W_{J+1}(t-, s, r, k) \frac{K(dt)}{K(t-)} - b_{J+1}(t) \mu(dt)$$

$$- \left( b_{(J+1)(J+2)}(t) + W_{J+2}(t) - W_{J+1}(t, s, r, k) \right) \alpha_{(J+1)(J+2)}(t, s, r, k) dt,$$  \hspace{1cm} (4.6)

with terminal conditions $W_{J+1}(n, s, r, k) = 0$ for $0 \leq s \leq r \leq n$ and $k \in \{1, \ldots, J \}$, and the functions $W_{j}(\cdot, \cdot)$, $j \in \{1, \ldots, J \}$, are bounded and càdlàg solutions of

$$W_{j}(dt, s) = W_{j}(t-, s) \frac{K(dt)}{K(t-)} - b_{j}(t) \mu(dt) +$$

$$- \sum_{k \leq j; k \neq j} \left( b_{jk}(t) + W_{k}(t, t) - W_{j}(t, s) \right) \alpha_{jk}(t, s) dt$$  \hspace{1cm} (4.7)

with terminal conditions $W_{j}(n, s) = 0$ for $0 \leq s \leq n$, then for all $t \geq 0$ and $j \in \{1, \ldots, J \}$,

$$1_{\{Z_{i} = j\}} W_{j}(t, t - U_{i}) = 1_{\{Z_{i} = j\}} V_{j}(t) = 1_{\{Z_{i} = j\}} V(t)$$

almost surely, and for all $t \geq 0$,

$$1_{\{Z_{i} = J+1\}} W_{J+1}(t, t - U_{i}^{n}, t - U_{i}^{r}, H_{i}) = 1_{\{Z_{i} = J+1\}} V_{J+1}(t) = 1_{\{Z_{i} = J+1\}} V(t),$$

$$1_{\{Z_{i} = J+2\}} W_{J+2}(t) = 1_{\{Z_{i} = J+2\}} V_{J+2}(t) = 1_{\{Z_{i} = J+2\}} V(t)$$

almost surely.

**Proof.** Note that the right-continuity of the solutions of the differential/integral equations allows us to uniquely expand the domains of the solutions to $t \geq s \geq 0$, $t \geq r \geq s \geq 0$ and $t \geq 0$. That means that $W_{j}(t, t)$ and $W_{j+1}(t, s, r, k)$ are indeed given by the solutions.

Since the bounded and càdlàg solution $W_{J+2}$ of (4.3) is deterministic, it is also $F$-predictable and by multiplying (4.3) with $1_{\{Z_{i} = J+2\}}$ we obtain (4.4) for $J = J + 2$. By multiplying equation (4.4) with $1_{\{Z_{i} = J+1\}}$ and replacing $s$, $r$ and $k$ by $t - U_{i}^{h}$, $t - U_{i}^{r}$, and $H_{i}$, we obtain that $W_{J+1}(t, t - U_{i}^{h}, t - U_{i}^{r}, H_{i})$ is an $F$-predictable, bounded, and càdlàg solution of (4.4) for $J = J + 1$.

Multiplying equation (4.7) with $1_{\{Z_{i} = j\}} 1_{\{\tau_{i} < t \leq \tau_{i+1}\}}$ and replacing $s$ by $\tau_{i} 1_{\{Z_{i} = j\}} + t 1_{\{Z_{i} \neq j\}}$, we obtain that $W_{j}(t, \tau_{i} 1_{\{Z_{i} = j\}} + t 1_{\{Z_{i} \neq j\}}, j \in \{1, \ldots, J \}$, is a solution of (4.4) on the interval $(\tau_{i}, \tau_{i+1})$. This follows from the almost sure identities

$$1_{\{Z_{i} = j\}} W_{k}(t, \tau_{i} 1_{\{Z_{i} = k\}} + t 1_{\{Z_{i} \neq k\}}) = 1_{\{Z_{i} = j\}} W_{k}(t, t),$$

$$1_{\{Z_{i} = j\}} W_{j+1}(t, \tau_{i} 1_{\{Z_{i} = j\}} + t 1_{\{Z_{i} \neq j\}}, 0, j) = 1_{\{Z_{i} = j\}} W_{J+1}(t, t - U_{i}^{h}, t - U_{i}^{r}, H_{i})$$

for all $t \in (\tau_{i}, \tau_{i+1})$ and $j, k \in \{1, \ldots, J \}$, $j \neq k$. Summing over $i \in \mathbb{N}$ yields that the bounded and càdlàg $F$-predictable processes

$$W_{j}(t, t - U_{i} 1_{\{Z_{i} = j\}}) = \sum_{i=0}^{\infty} 1_{\{\tau_{i} < t \leq \tau_{i+1}\}} W_{j}(t, \tau_{i} 1_{\{Z_{i} = j\}} + t 1_{\{Z_{i} \neq j\}}), \quad j \in \{1, \ldots, J \},$$

28
are solutions of (4.4) for \( j \in \{1, \ldots, J\} \) due to the fact that
\[
1_{\{Z_t = j\}} W_k(t, t - U_t - 1_{\{Z_t = k\}}) = 1_{\{Z_t = j\}} W_k(t, t)
\]
almost surely for all \( t > 0 \) and \( j, k \in \{1, \ldots, J\} \), \( j \neq k \).

All in all, we conclude that the processes \( W_j(t, t - U_{t-} 1_{\{Z_{t-} = j\}}), j \in \{1, \ldots, J\}, W_{j+1}(t, t - U_{t-} - t - U_{t-} - H_{t-}) \), and \( W_{j+2}(t) \) form an \( \mathcal{F} \)-predictable bounded and càdlàg solution of the equation system (4.4), which implies that, according to Proposition 4.3, they equal the classic state-wise prospective reserves \( V_j(t) \) on \( \{Z_t = j\} \) for \( j \in \{1, \ldots, J + 2\} \). Since \( Z_t = J + 1 \) implies \( \eta \leq t \), we may replace \( 1_{\{Z_t = J + 1\}} W_{j+1}(t, t - U_{t-}, t - U_{t-}, H_{t-}) \) by \( 1_{\{Z_t = J + 1\}} W_{j+1}(t, t - U_{t-}, t - U_{t-}, H_{t-}) \). Moreover, we have
\[
1_{\{Z_t = j\}} V_j(t, t - U_t) = 1_{\{Z_t = j\}} W_j(t, t - U_t - 1_{\{Z_t = j\}}), \quad j \in \{1, \ldots, J\},
\]
almost surely for all \( t \geq 0 \) under the conventions \( U_{0-} := 0 \) and \( Z_{0-} := Z_0 \). This implies the statement of the Proposition.

The numerical schemes that can be developed based on Proposition 4.12 are significantly more complex than in the classic (semi-)Markovian case, see e.g. Adekambi and Christiansen (2017). The sum at risks involve \( W_{j+1}(t, s, t, j) \), which must be computed based on (4.6) for all \( 0 \leq s < t \) using e.g. the method of lines.

Recall that \( Y^j_{\{\cdot\}} = V_j \) almost surely for \( j \in \{1, \ldots, J\} \), cf. Theorem 4.9 and Theorem 4.10 and due to the assumptions given at the beginning of this subsection, we also have \( Y^j_{\{\cdot\}} = V_j \) almost surely. The next results provide Feynman-Kac formulas for the resdiuary classic state-wise prospective reserve in the presence of non-monotone information \( \mathcal{G}^1 \) and \( \mathcal{G}^2 \). Proofs are given at the end of the subsection.

**Proposition 4.13.** Suppose the assumptions from the beginning of this subsection hold. If \( W^1_{\{\cdot\}}(\cdot, \cdot) \) is a bounded and càdlàg solution of
\[
W^1_{j+1}(dt, r) = W^1_{j+1}(t-, r) \frac{\kappa(dt)}{\kappa(t-)} - b^1_{j+1}(t) \mu(dt)
- (b^1_{(J+1)(J+2)}(t) + W^1_{j+2}(t) - W^1_{j+1}(t, r)) \alpha^1_{(J+1)(J+2)}(t, r) \; dt
\]
for \( 0 < r < t \) with terminal conditions \( W^1_{j+1}(n, r) = 0 \) for \( 0 \leq r \leq n \), and where \( W^1_{j+2}(\cdot) \) solves (4.5) while
\[
\alpha^1_{(J+1)(J+2)}(t, r) := \frac{f_{\delta_0}(t|\eta = r)}{P(\delta \geq t | \eta = r)}, \quad 0 \leq r \leq t,
\]
then \( 1_{\{Z_t = J + 1\}} W^1_{j+1}(t, \eta) = 1_{\{Z_t = J + 1\}} Y^1_{\{J+1\}}(t) = 1_{\{Z_t = J + 1\}} Y^{1\{\cdot\}}(t) \) almost surely for all \( t \geq 0 \).

**Proposition 4.14.** Suppose the assumptions from the beginning of this subsection hold. If \( W^2_{j+1}(\cdot, \cdot) \) is a bounded and càdlàg solution of
\[
W^2_{j+1}(dt) = W^2_{j+1}(t-) \frac{\kappa(dt)}{\kappa(t-)} - b^1_{j+1}(t) \mu(dt)
- (b^1_{(J+1)(J+2)}(t) + W^2_{j+2}(t) - W^2_{j+1}(t)) \alpha^2_{(J+1)(J+2)}(t) \; dt
+ (W^2_{j+1}(t, t) - W^2_{j+1}(t)) \xi_{J+1}(t) \; dt
\]
(4.10)
for $0 < t$ with terminal condition $W_{j+1}^2(n) = 0$, and where $W_{j+2}^1(\cdot)$ and $W_{j+1}^1(\cdot, \cdot)$ solve (4.3) and (4.8), while

$$
\alpha_{(j+1)(j+2)}^2(t) := \frac{\int_{0}^{t} f_{\eta(t)}(s, t) \, ds}{P(\eta < t \leq \delta)},
$$

(4.11)

$$
\xi_{j+1}(t) := \frac{f_{\eta}(t)}{P(\eta \leq t \leq \delta)},
$$

(4.12)

then $1_{\{Z_t = J+1\}} W_{j+1}^2(t) = 1_{\{Z_t = J+1\}} Y_{j+1}^G(t) = 1_{\{Z_t = J+1\}} Y_{j+1}^G(t)$ almost surely for all $t \geq 0$.

In order to reduce the computation time and simplify actuarial modeling and statistical estimation, practitioners, when computing the prospective reserve for non-retirees based on $W_j$, $j \in \{1, \ldots, J\}$, often approximate $W_{j+1}(t, s, t, j)$ by a less complex quantity such as $W_{j+1}^1(t, t)$, which discards information concerning previous health records, or $W_{j+2}^1(t)$, which additionally discards information concerning the time of retirement. Replacing $W_{j+1}$ by $W_{j+1}^1$ produces approximation errors on the individual level (and redistribution of wealth on the portfolio level for non-retirees).

Proposition 4.13 and Proposition 4.14 can be used to develop computational schemes for $W_{j+1}^1$ and $W_{j+1}^2$, respectively. Focusing on $W_{j+1}^2$, this involves the transition rate $\alpha_{(j+1)(j+2)}^2$, which by (4.11) is the hazard rate corresponding to a classic mortality table for retirees. It also involves the adjustment term

$$
(W_{j+1}^1(t, t) - W_{j+1}^2(t)) \xi_{j+1}(t) \, dt,
$$

where according to (4.12), $\xi_{j+1}(t) \, dt$ is the infinitesimal probability of retirement having just occurred (at time $t$), conditionally on the insured presently being retired.

If the mortality does not depend on the time since retirement, i.e. if $\alpha_{(j+1)(j+2)}^1(t, r) \equiv \alpha_{(j+1)(j+2)}^2(t)$, we end up with the differential/integral equations

$$
W_{j+1}(dt) = W_{j+1}(t-) \frac{\kappa(dt)}{\kappa(t-)} - b_{j+1}(t) \mu(dt) - (b_{(j+1)(j+2)}(t) + W_{j+2}(t) - W_{j+1}(t)) \alpha_{(j+1)(j+2)}^2(t) \, dt.
$$

(4.13)

Even though the mortality of retirees might depend on the time since retirement, practitioners often still utilize (4.13) directly. This produces additional approximation errors on the individual level (and redistribution of wealth on the portfolio level for retirees as well as non-retirees).

**Proof of Proposition 4.13** Note that (4.8) implies that $W_{j+1}^1(\cdot, \eta)$ has paths of finite variation on compacts. By applying integration by parts, we obtain

$$
1_{\{\eta < t\}} \, d\left(1_{\{Z_t = J+1\}} v(t) W_{j+1}^1(t, \eta)\right)
$$

$$
= 1_{\{Z_t = J+1\}} \left(v(t) W_{j+1}^1(t, \eta) - v(t) W_{j+1}^1(t-, \eta) \frac{\kappa(dt)}{\kappa(t-)}\right) - v(t) W_{j+1}^1(t, \eta) N_{(j+1)(j+2)}(dt).
$$

almost surely. Inserting (4.8) into the latter term leads to

$$
1_{\{\eta < t\}} \, d\left(1_{\{Z_t = J+1\}} v(t) W_{j+1}^1(t, \eta)\right)
$$

$$
= -1_{\{Z_t = J+1\}} v(t) B(dt) - v(t) W_{j+2}(t) N_{(j+1)(j+2)}(dt)
$$

$$
+ v(t) R_{(j+1)(j+2)}^1(t) M_{(j+1)(j+2)}^1(dt)
$$

almost surely. Inserting (4.8) into the latter term leads to

30
almost surely; here \( R^1_{(J+1)(J+2)}(t) := b_{(J+1)(J+2)}(t)+W_{J+2}(t)-W_{J+1}(t) \) and \( M^1_{(J+1)(J+2)}(dt) := N_{(J+1)(J+2)}(dt) - \mathbb{1}_{\{\eta < t\}} \alpha^2_{(J+1)(J+2)}(t, \eta) \) dt. Thus, since \( \{ \eta < t \} \subseteq \{ \eta < s \} \) for \( s \geq t \geq 0 \), we find that almost surely for all \( t \geq 0 \),

\[
1_{\{\eta < t\}} 1_{\{Z_t = J+1\}} v(t) W^1_{J+1}(t, \eta) = E[1_{\{\eta < t\}} 1_{\{Z_t = J+1\}} v(t) W^1_{J+1}(t, \eta) | G^1_t] = 1_{\{\eta < t\}} E \left[ v(t) \int_{[t,n]} 1_{\{Z_s = J+1\}} \frac{\kappa(t)}{\kappa(s)} B(ds) \left| G^1_t \right. \right] + 1_{\{\eta < t\}} E \left[ \int_{[t,n]} v(s) W^1_{J+2}(s) N_{(J+1)(J+2)}(ds) \left| G^1_t \right. \right] - 1_{\{\eta < t\}} E \left[ \int_{[t,n]} v(s) R^1_{(J+1)(J+2)}(s) M^1_{(J+1)(J+2)}(ds) \left| G^1_t \right. \right] = 1_{\{\eta < t\}} 1_{\{Z_t = J+1\}} v(t) Y^{G^1}_{J+1}(t) - 1_{\{\eta < t\}} 1_{\{Z_t = J+1\}} E \left[ \int_{[t,n]} v(s) R^1_{(J+1)(J+2)}(s) M^1_{(J+1)(J+2)}(ds) \left| G^1_t \right. \right]
\]

Recall that \( \{Z_t = J + 1\} = \{Z \in \{1\}\} \). Pointing to Proposition 3.7, the constructions of \( G^1 \) according to the proof of Lemma 2.2.5 and 3.3.6, straightforward calculations then yield that the last line equals zero. All in all, we conclude that

\[
1_{\{\eta < t\}} 1_{\{Z_t = J+1\}} v(t) W^1_{J+1}(t, \eta) = 1_{\{\eta < t\}} 1_{\{Z_t = J+1\}} v(t) Y^{G^1}_{J+1}(t) = 1_{\{\eta < t\}} 1_{\{Z_t = J+1\}} v(t) Y^{G^1}_{J+1}(t)
\]

almost surely for all \( t \geq 0 \). Since \( v \) and \( Y^{G^1} \) almost surely have càdlàg sample paths, cf. Proposition 3.8, we may replace \( 1_{\{\eta < t\}} 1_{\{Z_t = J+1\}} \) by \( 1_{\{\eta < t\}} 1_{\{Z_t = J+1\}} = 1_{\{Z_t = J+1\}} \). Using \( v > 0 \) completes the proof.

**Proof of Proposition 4.13** Note (4.10) implies that \( W^2_{J+1}(\cdot) \) has paths of finite variation on compacts. By applying integration by parts, inserting (4.10), applying Theorem 4.11 and referring to Remark 4.11 straightforward calculations yield

\[
d \left( 1_{\{Z_t = J+1\}} v(t) W^2_{J+1}(t) - 1_{\{Z_t = J+1\}} v(t) Y^{G^2}_{J+1}(t) \right)^2 = v(t)^2 1_{\{Z_t = J+1\}} (W^2_{J+1}(t) - Y^{G^2}_{J+1}(t))^2 \left( \alpha^2_{(J+1)(J+2)}(t) - \xi_{J+1}(t) \right) dt - v(t)^2 (W^2_{J+1}(t) - Y^{G^2}_{J+1}(t))^2 \left( N_{(J+1)(J+2)}(dt) - 1_{\{Z_t = J+1\}} \alpha^2_{(J+1)(J+2)}(t) dt \right) + v(t)^2 (W^2_{J+1}(t) - Y^{G^2}_{J+1}(t))^2 \left( -1_{\{Z_t = J+1\}} \xi_{J+1}(t) dt + \sum_{k=1}^J N_{k(J+1)}(dt) \right)
\]

almost surely. Following along the lines of the proof of Proposition 4.13, we find that

\[
v(t)^2 P(Z_t = J + 1) (W^2_{J+1}(t) - Y^{G^2}_{J+1}(t))^2 = E \left[ v(t)^2 1_{\{Z_t = J+1\}} (W^2_{J+1}(t) - Y^{G^2}_{J+1}(t))^2 \right] = -E \left[ \int_t^n v(s)^2 1_{\{Z_s = J+1\}} (W^2_{J+1}(t) - Y^{G^2}_{J+1}(t))^2 \left( \alpha^2_{(J+1)(J+2)}(s) - \xi_{J+1}(s) \right) ds \right] = -\int_t^n v(s)^2 P(Z_s = J + 1) (W^2_{J+1}(t) - Y^{G^2}_{J+1}(t))^2 \left( \alpha^2_{(J+1)(J+2)}(s) - \xi_{J+1}(s) \right) ds
\]
almost surely. This means that the function
\[ f(t) := v(t)^2 P(Z_t = J + 1) (W_j^2(t) - Y_j^{2^2}(t))^2 \]
aalmost surely satisfies the integral equation
\[ f(t) = -\int_t^n f(s) \left( \alpha_{(J+1)}^2(s) - \xi_{J+1}(s) \right) ds \]
for all \( t \in [0, n] \) under the convention \( (n, n] = \emptyset \). Note that
\[ |f(t)| \leq \int_t^n |f(s)| \left| \alpha_{(J+1)}^2(s) - \xi_{J+1}(s) \right| ds \]
almost surely for all \( t \in [0, n] \). According to the backward Grönwall inequality (see Lemma 4.7 in Cohen and Elliott 2012), \( f(t) = 0 \) almost surely for all \( t \in [0, n] \). Since \( v > 0 \), for each \( t \geq 0 \) it then holds that \( 1_{\{Z_t = J + 1\}} W_j^2(t) = 1_{\{Z_t = J + 1\}} Y_j^{2^2}(t) \) almost surely. Since the implicated processes almost surely have càdlàg sample paths, cf. Corollary 3.9, there exists a joint \( P \)-null set. Thus \( 1_{\{Z_t = J + 1\}} W_j^2(t) = 1_{\{Z_t = J + 1\}} Y_j^{2^2}(t) \) almost surely for all \( t \geq 0 \) as desired.

\[ \square \]

Acknowledgments and declarations of interest

Christian Furrer’s research is partly funded by the Innovation Fund Denmark (IFD) under File No. 7038-00007. The authors declare no competing interests.

A. Proofs

Proof of Proposition 2.3 As a consequence of Assumption 2.2, the only non-trivial statement of the proposition relates to intertemporal dependency structure of \( Z \) after retirement, so it suffices to study the quantities
\[ P(Z_s = J + 1 \mid \mathcal{F}_t) \]
on the event \( \{Z_t = J + 1\} \) for \( 0 \leq t < s < \infty \). To this end, consider sets \( A^n_p := \{Z_t = J + 1, \tilde{N}(t) = n\}, n \in \mathbb{N} \), where \( \tilde{N} = (\tilde{N}(t))_{t \geq 0} \) is the process counting the total number of jumps of \( Z \) given by
\[ \tilde{N}(t) = \sum_{j,k \in S, j \neq k} N_{jk}(t), \quad t \geq 0, \]
and denote with \( \tau = (\tau_i)_{i \in \mathbb{N}} \) and \( \tilde{\tau} = (\tilde{\tau}_i)_{i \in \mathbb{N}} \) the point processes corresponding to the jump times of \( Z \) and \( \tilde{Z} \), respectively. Fix \( 0 \leq t < s < \infty \), and fix \( n \in \mathbb{N} \). On \( A^n_p \) it then almost surely holds that
\[ \tau_n = \tilde{\tau}_n = \eta, \quad \tilde{Z}_{\tilde{\tau}_n} \in \{J + 1, \ldots, 2J\}, \]
\[ \tau_i = \tilde{\tau}_i, \quad Z_{\tau_i} = \tilde{Z}_{\tilde{\tau}_i} \in \{1, \ldots, J\}, \quad \forall i = 1, \ldots, n - 1. \]
In particular,
\[ P(Z_s = J + 1 \mid \mathcal{F}_t) 1_{A^n_p} \overset{a.s.}{=} P \left( \tilde{Z}_s \in \{J + 1, \ldots, 2J\} \mid \tilde{\tau}_n, Z_{\tilde{\tau}_n}, Z_{\tilde{\tau}_{n-1}}, \tilde{Z}_{\tilde{\tau}_{n-1}}, \ldots, \tilde{\tau}_1, Z_{\tilde{\tau}_1} \right) 1_{A^n_p}. \]
Suppose $(\bar{Z}, \bar{U})$ is Markovian such that $\bar{Z}$ is semi-Markovian. By the law of iterated expectations and the strong Markov property, cf. Theorem 7.5.1 in [Jacobsen (2000)], it follows that

$$P(\bar{Z}_s \in \{J + 1, \ldots, 2J\} \mid \bar{\tau}_n, Z_{\bar{\tau}_n}, \bar{\tau}_{n-1}, \ldots, \bar{\tau}_1, \bar{Z}_{\bar{\tau}_n}) 1_{A^n_t} = \mathbb{E} \left[ P(\bar{Z}_s \in \{J + 1, \ldots, 2J\} \mid \bar{\tau}_n, Z_{\bar{\tau}_n}, \bar{\tau}_{n-1}, \ldots, \bar{\tau}_1, \bar{Z}_{\bar{\tau}_n}) 1_{A^n_t} \right] = \mathbb{E} \left( \mathbb{E} \left( P(\bar{Z}_s \in \{J + 1, \ldots, 2J\} \mid Z_{\bar{\tau}_n}, \bar{\tau}_n - \bar{\tau}_n - 1, \bar{Z}_{\bar{\tau}_n-1}) 1_{A^n_t} \right) \mid \bar{\tau}_n, Z_{\bar{\tau}_n}, \bar{\tau}_n - \bar{\tau}_n - 1, \bar{Z}_{\bar{\tau}_n-1}, \bar{Z}_{\bar{\tau}_n} \right).$$

Thus on $A^n_t = \{Z_t = J + 1, \bar{N}(t) = n\}$ it almost surely holds that

$$P(Z_s = J + 1 \mid F_t) = P(Z_s = J + 1 \mid \bar{\tau}_n, Z_{\bar{\tau}_n}, \bar{\tau}_{n-1}, \bar{Z}_{\bar{\tau}_n-1}) = P(Z_s = J + 1 \mid t - \bar{\tau}_n, Z_{\bar{\tau}_n}, t - \bar{\tau}_{n-1}, \bar{Z}_{\bar{\tau}_n-1}) = P(Z_s = J + 1 \mid U^r_s, Z_t, U^h_t, H_t),$$

which does not depend on $n$. We conclude that if $(\bar{Z}, \bar{U})$ is Markovian, then on $\{Z_t = J + 1\}$,

$$P(Z_s = J + 1 \mid F_t) \overset{a.s.}{=} P(Z_s = J + 1 \mid U^r_s, Z_t, U^h_t, H_t)$$

proving the first part of the proposition. The proof of the second and final part follows by similar arguments.

**Proof of Lemma 2.7.** Let $N^-(t) = (N^-(t))_{t \geq 0}$ be the process counting the number of jumps of $Z$ except retirement and death given by

$$N^-(t) = \sum_{j,k \in \mathbb{N}, k \not\in \{j, j+1, j+2\}} N_{jk}(t), \quad t \geq 0,$$

and denote by $(\tau_i^-)_{i \in \mathbb{N}}$ the point process corresponding to the jumps of $N^-$. The $\sigma$-algebras $(2.1)$ and $(2.2)$ are equivalent to $G^1_t$ and $G^2_{T^-}$, respectively, if we set

$$T_1 = \eta, \quad S_1 = \infty, \quad \zeta_1 = (T_1, Z_{T_1}),$$
$$T_2 = \delta, \quad S_2 = \infty, \quad \zeta_2 = (T_2, Z_{T_2}),$$
$$T_{2+i} = \tau_i^-, \quad S_{2+i} = T_1 \wedge T_2, \quad \zeta_{2+i} = (T_{2+i}, Z_{T_{2+i}}), \quad i \in \mathbb{N}.$$ 

If we replace $\zeta_1 = (T_1, Z_{T_1})$ by the constant $\zeta_1 = (0, Z_{\tau_1}) = (0, J + 1)$, then $(2.1)$ and $(2.2)$ are equivalent to $G^1_T$ and $G^2_{T^-}$, respectively.

**Proof of Proposition 3.5.** Since $Y$ is integrable, for each $j \in S$ and $t \geq 0$ the mapping

$$C_{t,j} \ni A \mapsto \nu_{t,j}(A) := \int_A Y(t) \, dm_{t,j}$$

is a finite signed measure on $C_{t,j}$ which is absolutely continuous with respect to the sub-probability measure $m_{t,j}$ given by

$$C_{t,j} \ni A \mapsto m_{t,j}(A) = P(A \cap \{Z_t = j\}).$$
According to the Radon-Nikodym theorem there exist mappings $\omega \mapsto Y_j(t)(\omega)$ that are $C_{t,j}$-measurable and satisfy

$$\nu_{t,j}(A) = \int_A Y_j(t) \, dm_{t,j}, \quad A \in C_{t,j}. \quad (A.1)$$

In particular

$$\int_{A \cap \{Z_t = j\}} Y(t) \, dP = \int_{A \cap \{Z_t = j\}} Y_j(t) \, dP, \quad j \in S, A \in C_{t,j},$$

which by Lemma 3.11 yields

$$\int_A Y(t) \mathbf{1}_{\{Z_t = j\}} \, dP = \int_A Y_j(t) \mathbf{1}_{\{Z_t = j\}} \, dP, \quad j \in S, A \in C_t.$$

We conclude that $\mathbf{1}_{\{Z_t = j\}}Y_j(t) \overset{a.s.}{=} \mathbf{1}_{\{Z_t = j\}}Y(t)$ for each $j \in S$ and $t \geq 0$. This establishes existence of the state-wise counterparts. Furthermore, if there is another real-valued random variable $\tilde{Y}_j(t)$ that has the properties of $Y_j(t)$, we necessarily have

$$0 = \int_{A \cap \{Z_t = j\}} (Y_j(t) - \tilde{Y}_j(t)) \, dP = \int_{A \times \{j\}} (Y_j(t)(\omega) - \tilde{Y}_j(t)(\omega)) \, dm_t(\omega, j),$$

for $A \in C_{t,j}$, which means that the mapping $(\omega, j) \mapsto Y_j(t)(\omega) - \tilde{Y}_j(t)(\omega)$ is $m_t$-almost everywhere zero. This establishes the desired uniqueness of the state-wise counterparts. \hfill $\square$

**Proof of Lemma 3.6.** If $P(Z_t = j) = 0$, the result is trivial. Thus suppose $P(Z_t = j) > 0$. Since $E_{t,j}[E[X \mid C_t] \mid C_{t,j}]$ is the conditional expectation of $E[X \mid C_t]$ given $C_{t,j}$ w.r.t. $P_{t,j}$, we find that

$$\int_A E_{t,j}[E[X \mid C_t] \mid C_{t,j}] \, dP_{t,j} = \int_A E[X \mid C_t] \, dP_{t,j}.$$ 

Note that by definition of $C_{t,j}$, we have $A \cap \{Z_t = j\} \in C_t$. It follows that

$$\int_A E_{t,j}[E[X \mid C_t] \mid C_{t,j}] \, dP_{t,j} = \frac{1}{P(Z_t = j)} \int_{A \cap \{Z_t = j\}} E[X \mid C_{t,j}] \, dP = \frac{1}{P(Z_t = j)} \int_A X \mathbf{1}_{\{Z_t = j\}} \, dP = \int_A X \, dP_{t,j},$$

where we have used that $E[X \mid C_t]$ is the conditional expectation of $X$ given $C_t$ w.r.t. $P$. In conclusion, $E_{t,j}[E[X \mid C_{t,j}] \mid C_{t,j}]$ is a version of the conditional expectation of $X$ given $C_{t,j}$ w.r.t. $P_{t,j}$ which completes the proof. \hfill $\square$

**Proof of Corollary 3.9.** From Proposition 3.8 we know that $Y^G$ almost surely has càdlàg paths of finite variation on compacts. Since $\mathbf{1}_{\{Z_t = j\}}Y^G_j(t) \overset{a.s.}{=} \mathbf{1}_{\{Z_t = j\}}Y^G(t)$, the path properties of $Y^G$ also hold almost surely for $t \mapsto \mathbf{1}_{\{Z_t = j\}}Y^G_j(t)$. Proposition 3.8 directly states the path properties for $t \mapsto I_x(t)Y^G_x(t)$. Since Assumption 2.1 and 2.2 imply that the paths of $Z$ and $Z$ have at most a finite number of jumps on $[0,t]$, the càdlàg and finite variation property are still true for $t \mapsto \mathbf{1}_{\{Z_t = j\}}Y^G_j(t)$ and $t \mapsto I_x(t-)Y^G_x(t)$.
Proof of Proposition 3.11. Suppose that \((T_i, S_i, \zeta_i)_{i \in \mathbb{N}} := (\tau_i, \infty, (Z_{\tau_i}, \tau_\i))_{i \in \mathbb{N}}\), such that \(\mathcal{F} = \mathcal{G}\). Fix \(t > 0\) and \(j \in S\). By (3.6) we almost surely find
\[
\sum_{k \in S} Y_{kj}^{F-}(t) = 1\{Z_{t-} \neq j\} \sum_{n=0}^{\infty} 1\{\tau_n < t \leq \tau_{n+1}\} \frac{E[1\{Z_t = j\} Y(t) | (Z_{\tau_1}, \tau_1), \ldots, (Z_{\tau_n}, \tau_n)]}{E[1\{Z_t = j\} | (Z_{\tau_1}, \tau_1), \ldots, (Z_{\tau_n}, \tau_n)]}.
\]
Since \(\{Z_{t-} \neq j, Z_t = j, \tau_n < t \leq \tau_{n+1}\} = \{Z_{t-} \neq j, \tau_n < t \leq \tau_{n+1}, \tau_{n+1} = t, Z_{\tau_{n+1}} = j\}\) for any \(n \in \mathbb{N}\), we further conclude on the basis of Example 3.2 and (3.2) that
\[
\sum_{k \in S} Y_{kj}^{F-}(t) = 1\{Z_{t-} \neq j\} \sum_{n=0}^{\infty} 1\{\tau_n < t \leq \tau_{n+1}\} \frac{E[1\{Z_t = j\} Y(t) | (Z_{\tau_1}, \tau_1), \ldots, (Z_{\tau_n}, \tau_n)]}{E[1\{Z_t = j\} | (Z_{\tau_1}, \tau_1), \ldots, (Z_{\tau_n}, \tau_n)]} = 1\{Z_{t-} \neq j\} Y_j^{F-}(t)
\]
almost surely. Similarly, \(Y_{jj}^{F-}(t) \overset{a.s.}{=} 1\{Z_{t-} = j\} Y_j^{F-}(t)\). Writing
\[
Y_j^{F-}(t) = Y_j^{F-}(t) 1\{Z_{t-} = j\} + Y_j^{F-}(t) 1\{Z_{t-} \neq j\}
\]
and collecting terms completes the proof. \(\square\)

References

F. Adékambi and M.C. Christiansen. Integral and differential equations for the moments of multistate models in health insurance. *Scandinavian Actuarial Journal*, 2017:29–50, 2017. doi: 10.1080/03461238.2015.1058854.

M. Bladt, S. Asmussen, and M. Steffensen. Matrix representations of life insurance payments. *European Actuarial Journal*, 2020. doi: 10.1007/s13385-019-00222-0.

M.C. Christiansen. A martingale concept for non-monotone information in a jump process framework, 2020. arXiv: 1811.00952v3.

M.C. Christiansen and B. Djehiche. Nonlinear reserving and multiple contract modifications in life insurance, 2019. arXiv: 1911.06159.

S.N. Cohen and R.J. Elliott. Existence, uniqueness and comparisons for BSDEs in general spaces. *The Annals of Probability*, 40(5):2264–2297, 2012. doi: 10.1214/11-AOP679.

B. Djehiche and B. Löfdahl. Nonlinear reserving in life insurance: Aggregation and mean-field approximation. *Insurance: Mathematics and Economics*, 69(3):1–13, 2016. doi: 10.1016/j.insmatheco.2016.04.002.

M. Helwich. *Durational effects and non-smooth semi-Markov models in life insurance*. PhD thesis, University of Rostock, 2008.
J.M. Hoem. Markov Chain Models in Life Insurance. *Blätter der DGVFM*, 9:91–107, 1969. doi: 10.1007/BF02810082.

M. Jacobsen. *Point process theory and applications: Marked point and piecewise deterministic processes*. Probability and its Applications. Birkhäuser, 2006. doi: 10.1007/0-8176-4463-6.

S.F. Jarner and T. Møller. A partial internal model for longevity risk. *Scandinavian Actuarial Journal*, 2015(4):352–382, 2015. doi: 10.1080/03461238.2013.836561.

C.M. Møller. A stochastic version of Thiele’s differential equation. *Scandinavian Actuarial Journal*, 1993:1–16, 1993. doi: 10.1080/03461238.1993.10413910.

R. Norberg. Reserves in Life and Pension Insurance. *Scandinavian Actuarial Journal*, 1991:3–24, 1991. doi: 10.1080/03461238.1991.10557357.

R. Norberg. Hattendorff’s theorem and Thiele’s differential equation generalized. *Scandinavian Actuarial Journal*, 1992:2–14, 1992. doi: 10.1080/03461238.1992.10413894.

R. Norberg. Addendum to Hattendorff’s Theorem and Thiele’s Differential Equation Generalized, SAJ 1992, 214. *Scandinavian Actuarial Journal*, 1996:50–53, 1996. doi: 10.1080/03461238.1996.10413962.

R. Norberg. A theory of bonus in life insurance. *Finance and Stochastics*, 3:373–390, 1999. doi: 10.1007/s007800050067.

R. Norberg. On bonus and bonus prognoses in Life Insurance. *Scandinavian Actuarial Journal*, 2001(2):126–147, 2001. doi: 10.1080/03461230152592773.

R.F. Serfozo. Functions of Semi-Markov Processes. *SIAM Journal on Applied Mathematics*, 20(3):530–535, 1971. doi: 10.1137/0120055.

M. Steffensen. A no arbitrage approach to Thiele’s differential equation. *Insurance: Mathematics and Economics*, 27(2):201–214, 2000. doi: 10.1016/S0167-6687(00)00048-2.