Recent Leibniz scholarship has sought to gauge which foundational framework provides the most successful account of the procedures of the Leibnizian calculus (LC). While many scholars (e.g. Ishiguro, Levey) opt for a default Weierstrassian framework, Arthur compares LC to a non-Archimedean framework SIA (Smooth Infinitesimal Analysis) of Lawvere–Kock–Bell. We analyze Arthur’s comparison and find it rife with equivocations and misunderstandings on issues including the non-punctiform nature of the continuum, infinite-sided polygons, and the fictionality of infinitesimals. Rabouin and Arthur claim that Leibniz considers infinites as contradictory, and that Leibniz’ definition of incomparables should be understood as nominal rather than as semantic. However, such claims hinge upon a conflation of Leibnizian notions of bounded infinity and unbounded infinity, a distinction emphasized by early Knobloch. The most faithful account of LC is arguably provided by Robinson’s framework for infinitesimal analysis. We exploit an axiomatic framework for infinitesimal analysis SPOT to formalize LC.

We find here [a] very important feature of Leibniz’s strategy when dealing with infinite and infinitely small quantities: nowhere does he express his conviction that they are contradictory notions. – Rabouin & Arthur, 2020

The most popular grounds for denying the existence of infinitesimals, besides a lingering sense of their simply being conceptually ‘repugnant,’ are (1) the lack of any useful mathematical application, and (2) their incompatibility with the so-called axiom of Archimedes. – Levey, 1998

This article has been corrected with minor changes. These changes do not impact the academic content of the article.
Leibniz] has defined … what it is for one quantity (say, \(dx\)) to be incomparable in relation to another (say, \(x\)). Formally, Leibniz’s definition gives:

For a given \(x\) and \(dx\), \(dx \ \text{INC} \ x\) if \(\neg (\exists n)(ndx > x)\). – Rabouin & Arthur, 2020

1. Introduction

Recent Leibniz scholarship has sought to gauge which foundational framework furnishes the most successful account of the procedures of the Leibnizian calculus (LC). While many scholars (e.g. Ishiguro, Levey) opt for a default Weierstrassian framework, some recent work explores additional possibilities, such as the non-Archimedean systems provided by Lawvere’s framework and by Robinson’s framework. Siegmund Probst observes:

For us, the question is whether Leibniz based his metaphysical foundations of the calculus wholly on the concept of the syncategorematic infinite or whether he pursued also an alternative approach of accepting infinitesimals similar to what is done in modern non-standard analysis. (Probst 2018, 221)

We will extend the scope of Probst’s question to include the additional possibility of Lawvere’s framework. We will give summaries of all three accounts in this introduction.

1.1. Grafting of the Epsilontik on the calculus of Leibniz

The title of this section is inspired by that of Cajori (1923). Commenting on the role of the theory of limits, Cajori noted: ‘[Leibniz] never actually founded his calculus upon that theory’ (Cajori 1923, 223). Cajori saw this aspect of LC as a shortcoming:

At that time both [Leibniz and Newton] used infinitely small quantities which were dropped when comparatively small. It is one of the curiosities in the history of mathematics that this rough procedure was adopted, even though before this time theories of limits had been worked out in geometry by Giovanni Benedetti, S. Stevin, L. Valerius, Gregory St. Vincent and Tacquet, and in higher arithmetic by John Wallis. (ibid.; emphasis added).

Cajori went on to summarize his view concerning the role of limits as follows: ‘[W]e see on the European continent the grafting of the Newtonian and pre-Newtonian concepts of limits upon the calculus of Leibniz’ (Cajori 1923, 234). Cajori was writing a number of years before the following mathematical developments took place:

1. Skolem developed proper extensions of the naturals satisfying the axioms of Peano Arithmetic in (1933 and 1934);
2. Hewitt introduced hyper-real ideals in (1948);
3. Łoś proved his theorem in (1955);
4. Robinson pioneered his framework for infinitesimal analysis in (1961);

1See note 3 for Boyer’s summary of the procedure of dropping negligible terms, and Section 4.5 for a formalization.
Lawvere pioneered his category-theoretic framework in the 1960s and 70s (see e.g. Lawvere 1980).

These developments are not to be construed as breakthroughs in Leibniz scholarship, but rather as mathematical developments that enabled modern theories of infinitesimals, one of which was compared to LC by Arthur (see Section 1.8).

1.2. From Berkeley’s ghosts to Guicciardini’s limits

In 1734, George Berkeley claimed that infinitesimals were logically contradictory. Namely, he claimed that they are simultaneously zero and nonzero. There was a bit of a flurry in England at the time around Berkeley’s pamphlet; see e.g. Andersen (2011). However, it was mostly ignored until the end of the nineteenth century. In the closing decades of the nineteenth century, the trend-setters in the mathematics community ruled against infinitesimals, and Berkeley’s criticism became popular again. This development was due to the fact that some working mathematicians (though by no means all) agreed with its conclusion (‘infinitesimals are contradictory’), not because they actually analyzed his argument. Since then, many historians have quoted Berkeley’s claims without analyzing them properly.

Robinson’s work (1966) cast doubt upon Berkeley’s conclusion, leading scholars like David Sherry (1987) to take a fresh look at Berkeley’s criticism to see if it actually made sense, and they were not impressed. Contrary to Berkeley’s claim, Leibniz never asserted that an infinitesimal literally equals zero. On the contrary, Leibniz emphasized repeatedly that he worked with a generalized relation of equality ‘up to’ a negligible term (see Section 2.6). In this way, Leibniz justified moves such as the passage from $\frac{2x + \delta x}{a}$ to $\frac{2x}{a}$ in the calculation of $\frac{dy}{dx}$ when $ay = x^2$ (see Section 6.2). Even such a generalized relation had antecedents. Thus, Fermat’s method of adequality involved discarding the remaining terms in $E$ (rather than setting them equal to zero); see Strømholm (1968, 51). Somewhat paradoxically, Robinson’s insights appear to have led to an intensification of the efforts by some historians to demonstrate that (contrary to Cajori’s claim) Leibniz did found his calculus on limits. Thus, Guicciardini writes:

In several of his mature writings, [Leibniz] stated that his differentials are ‘well founded,’ since they are symbolic abbreviations for limit procedures. (Guicciardini 2018, 90; emphasis added)

In a similar vein, Guicciardini claims that

2Possibly Johann Bernoulli did, but there may have been significant differences between Leibniz and Bernoulli; see Nagel (2008).

3As observed by Boyer, ‘The general principle that in an equation involving infinitesimals those of higher order are to be discarded inasmuch they have no effect on the final result is sometimes regarded as the basic principle of the differential calculus. … this type of doctrine constituted the central theme in the developments leading to the calculus of Newton and Leibniz … It was, indeed, precisely upon this general premise that Leibniz sought to establish the method of differentials’ (Boyer 1941, 79). For a possible formalization of the procedure of discarding terms see Section 4.5.
Leibniz distinguished between the search for rigor and the effectiveness of algorithms. The former was the purview of metaphysicians, because it ultimately rests on metaphysical principles, most notably on the ‘principles of continuity,’ on which Leibniz ‘well founded,’ as he used to say, his method of *infinitesimals as limits*. (*ibid*.; emphasis added)

Here Guicciardini alludes to Leibnizian references to infinitesimals as *well-founded fictions*. However, Leibniz never wrote anything about any ‘method of infinitesimals as limits.’ If, as per Guicciardini, Leibniz viewed infinitesimals as limits, then Leibniz’s vision of the infinitesimal calculus was significantly different from l’Hospital’s. Unlike Guicciardini, editors Bradley et al. of an English edition of l’Hospital’s *Analyse des infiniment petits* take for granted a *continuity* between Leibniz’s and l’Hospital’s visions of the calculus:

Leibniz’ differential calculus tells us how to find the relations among infinitely small increments $dx$, $dy$, etc., among the variables $x$, $y$, etc., in an equation. L’Hospital gives the definition on 2: ‘The infinitely small portion by which a variable quantity continually increases or decreases is called the *Differential.*’ (Bradley et al. 2015, xvii; emphasis in the original)

While there were differences between the views of Leibniz and l’Hospital (notably with regard to the question of the reality of infinitesimals), the assumption of a *discontinuity* between them is a thesis that needs to be argued rather than postulated (for a discussion of the issue of historical continuity see Katz 2020). The assumption that Leibniz based his calculus on limits is akin to Ishiguro’s interpretation; see Section 1.3.

We will examine several modern interpretations of LC, with special attention to Arthur’s comparison of LC to Smooth Infinitesimal Analysis (SIA); see Section 1.8. An introduction to SIA appears in Section 5.

1.3. *Modern interpretations*

Interpretations of LC by Bos (1974) and Mancosu (1989) contrast with interpretations by Ishiguro (1990, Chapter 5) and Arthur. These interpretations were explored by Katz and Sherry (2012, 2013), Bascelli et al. (2016), Blaszczyk et al. (2017), Bair et al. (2017b, 2018).

*The view of *infinitesimals as *limits* is no late stratagem for Guicciardini. Nearly two decades earlier, he already claimed that ‘Leibniz carefully defines the infinitesimal and the infinite in terms of limit procedures’ (Guicciardini 2000). The claim occurs in a review of (Knobloch 1999). Note that the article under review does not mention the word ‘limit’ in connection with the Leibnizian calculus.

*Similar remarks apply to the question of continuity between Leibniz and other Leibnizians such as Jacob Hermann; see note 16. Jesseph analyzes the use of infinitesimals by Torricelli and Roberval, and concludes: ‘By taking indivisibles as infinitely small magnitudes of the same dimension as the lines, figures, or solids they compose, [Roberval] could avoid the paradoxes that seemed to threaten Cavalieri’s methods, just as Torricelli had done’ (Jesseph 2021, 117). Thus Torricelli and Roberval viewed infinitesimals as homogeneous with ordinary quantities but incomparable with them. In Section 1.5 we argue that this was Leibniz’s view, as well. Guicciardini’s limit-centered interpretation of Leibnizian infinitesimals anachronistically postulates a discontinuity between Leibniz with both the preceding and the following generations of mathematicians, and is symptomatic of butterfly-model thinking (see Section 1.7).*
Ishiguro interprets Leibnizian infinitesimals as follows:

It seems that when we make reference to infinitesimals in a proposition, we are not designating a fixed magnitude incomparably smaller than our ordinary magnitudes. Leibniz is saying that whatever small magnitude an opponent may present, one can assert the existence of a smaller magnitude. In other words, we can paraphrase the proposition with a universal proposition with an embedded existential claim. (Ishiguro 1990, 87; emphasis added)

Ishiguro posits that when Leibniz wrote that his inassignable \( dx \), or alternatively \( \varepsilon \), was smaller than every given quantity \( Q \), what he really meant was a quantifier statement to the effect that for each given \( Q > 0 \) there exists an \( \varepsilon > 0 \) such that \( \varepsilon < Q \) (or that the error is smaller than \( Q \) if the increment is smaller than \( \varepsilon \)). This is the standard syncategorematic interpretation, via alternating logical quantifiers, of terms involving infinity. According to such interpretations, Leibnizian infinitesimals are logical fictions. Sherry and Katz (2012) contrasted such interpretations with a pure fictionalist interpretation. Every occurrence of a logical fiction is eliminable by a quantifier paraphrase, whereas pure fictions are not.

Arthur has endorsed Ishiguro’s reading in a series of articles and books (see e.g. Arthur 2007, 2014, 2015, 2018; Rabouin and Arthur 2020). The so-called ‘Leibniz’s syncategorematic infinitesimals’ currently come in two installments: (Arthur 2013), communicated by Guicciardini and (Rabouin and Arthur 2020), communicated by Jeremy Gray. Even earlier, Arthur produced ‘Leibniz’s Archimedean infinitesimals’ (Arthur 2007).

However, Leibniz made it clear that the infinitesimal method is an alternative to, rather than being merely a façon de parler for, Archimedean methods involving exhaustion. Thus, Leibniz wrote in his De Quadratura Arithmetica (DQA) already in 1675/76:

> What we have said up to this point about infinite and infinitely small quantities will appear obscure to certain people, as does everything new—although we have said nothing that cannot be easily understood by each of them after a little reflection: indeed, whoever has understood it will recognize its fecundity. (Leibniz as translated by Rabouin and Arthur 2020, 419)

How can one overcome such apprehensions concerning the novelty of the infinite and the infinitely small? Leibniz offers a way forward by dispensing with ontological preoccupations as to their existence ‘in nature,’ and treating them as fictional:

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6Thus Ishiguro’s Chapter 5 fits in a modern tradition of scholarship in search of, to paraphrase Berkeley, the ghosts of departed quantifiers that targets not only Leibniz but also Cauchy (see e.g. Bair et al. 2017a).

7Among the authors who have pursued such an Archimedean line is Levey. Levey is of the opinion that ‘Leibniz has abandoned any ontology of actual infinitesimals and adopted the syncategorematic view of both the infinite and the infinitely small … The interpretation is worth stating in some detail, both for propaganda purposes and for the clarity it lends to some questions that should be raised concerning Leibniz’s fictionalism’ (Levey 2008, 107; emphasis on ‘syncategorematic’ in the original; emphasis on ‘propaganda’ added). In conclusion, ‘Leibniz will emerge at key points to be something of an Archimedean’ (ibid.). Levey does not elaborate what ‘propaganda purposes’ he may have in mind here, but apparently they don’t include considering Robinson’s interpretation as a possibility. Most recently, Levey claims that ‘Particularly critical for grasping the conceptual foundation of his method … is the fact that for Leibniz the idea of “infinitely small” is itself understood in terms of a variable finite quantity that can be made arbitrarily small’ (Levey 2021, 142; emphasis in the original), but provides no evidence.
It does not matter whether there are such quantities in nature, for it suffices that they be introduced by a fiction, since they provide abbreviations of speaking and thinking, and thereby of discovery as well as of demonstration, … (ibid.)

We deliberately interrupted the passage at a comma for reasons that will become clear shortly. Leibniz’s rejection of the assumption that infinitesimals exist in nature occurs in the same sentence as their description as fictions, indicating that their fictionality is the contrary of being instantiated in nature. Leibniz continues:

… so that it is not always necessary to use inscribed or circumscribed figures and to infer ad absurdum, and to show that the error is smaller than any assignable. (ibid.)

Leibniz asserts that the infinitesimal method renders it unnecessary to use inscribed or circumscribed figures. He thus makes it clear that the infinitesimal method (freed of any ontological assumptions as to infinitesimals being instantiated in nature) provides an alternative to, rather than being a façon de parler for – as per the syncategorematic reading – Archimedean methods (involving inscribed and circumscribed figures mentioned in the passage). Clearly at odds with the Leibnizian intention in this passage, the syncategorematic reading specifically makes it necessary to use inscribed and circumscribed figures.8 The fictionalist view as expressed in DQA is consistent with what Leibniz wrote decades later concerning the interpretation of the infinitely small. Thus, in a letter to Varignon, Leibniz commented on the ideal status of infinitesimals in a way similar to that in DQA:

D’où il s’ensuit, que si quelcun9 n’admet point des lignes infinies et infiniment petites à la rigueur metaphysique et comme des choses reelles, il peut s’en servir seurement comme des notions ideales qui abregent le raisonnement, semblables à ce qu’on nomme racines imaginaires dans l’analyse commune…10 (Leibniz to Varignon, 2 February 1702, in Gerhardt 1850–63, IV, 92; emphasis added).

Here Leibniz describes infinitesimals as ideal notions, and expresses a reluctance to get into metaphysical debates as to their reality.

1.4. Was the fictionalist interpretation a late development?

Rabouin and Arthur (RA) allege the following:

(1) Some authors claim that when Leibniz called them ‘fictions’ in response to the criticisms of the calculus by Rolle and others at the turn of the century, he had

8Levey quotes this passage in (2021, 146) and claims that ‘Leibniz’s view in DQA of the infinitely small and its correlative idea of the infinitely large as useful fictions, whose underlying truth is understood in terms of relations among variable finite quantities that could be spelled out painstakingly if necessary, will be his longstanding position’ (ibid.), without however providing any evidence. As we argued, rather than support the syncategorematic reading, the 1676 passage actually furnishes evidence against it.

9Here and in the following we preserve the spelling as found in Gerhardt.

10Loemker’s English translation: ‘It follows from this that even if someone refuses to admit infinite and infinitesimal lines in a rigorous metaphysical sense and as real things, he can still use them with confidence as ideal concepts which shorten his reasoning, similar to what we call imaginary roots in the ordinary algebra’ (Leibniz 1989c, 543).
in mind a different meaning of ‘fiction’ than in his earlier work, involving a commitment to their existence as non-Archimedean elements of the continuum. (Rabouin and Arthur 2020, abstract)

(2) It has been objected [by such authors] that although Leibniz characterized infinitesimals as non-existent fictions in DQA and other writings of the mid-1670s, it cannot be assumed that he continued to hold the same view of fictions after he had fully developed the differential calculus. (Rabouin and Arthur 2020, 403–404)

In these passages, RA attribute to their opponents the view that Leibniz may have had a different meaning of fiction when he fully developed his calculus as compared to his DQA. However, RA provide no evidence to back up the claim that their opponents ‘persist in holding that Leibniz’s fictionalist interpretation was a late development, prompted by the criticisms of Rolle and others in the 1690s.’\footnote{In their note 8, RA attribute such a view to Jesseph. However, Jesseph observes that ‘there are certainly traces of [Leibniz’s fictionalist position] as early as the 1670s’ (Jesseph 2015, 195). RA give an erroneous page for Jesseph’s observation.} Did Leibniz change his mind about the approach developed in DQA? It is true that, already in the late 1670s, Leibniz felt that the new infinitesimal calculus may have superseded the techniques of DQA. Thus, in a 1679 letter to Huygens, Leibniz wrote:

> I have left my manuscript on arithmetical quadratures at Paris so that it may some day be printed there.\footnote{Knobloch quotes only the first sentence ‘I have left my manuscript on arithmetical quadratures at Paris so that it may some day be printed there’ and infers that ‘Leibniz did want to publish it’ (Knobloch 2017, 282). The rest of the quotation makes such an inference less certain.} But I have advanced far beyond studies of this kind and believe that we can get to the bottom of most problems which now seem to lie beyond our calculation; for example, quadratures, the inverse method of tangents, the irrational roots of equations, and the arithmetic of Diophantus. (Leibniz in 1679 as translated by Loemker (Leibniz 1989c, 282)).

Thus already by 1679, Leibniz ‘advanced far beyond’ his DQA. However, DQA being possibly superseded does not mean that Leibniz changed his mind about the meaning of infinitesimals as fictions. And if Leibniz did change his mind, then RA would be contradicting Leibniz himself (rather than contradicting their opponents), as they do when they claim to account for Leibnizian infinitesimals ‘in keeping with the Archimedean axiom’ (Rabouin and Arthur 2020, abstract), whereas Leibniz made it clear that his incomparables violate Euclid’s Definition V.4 in a pair of 1695 texts (Leibniz 1965a, 1695b) (see Section 1.5).

Similarly, Leibnizian bounded infinities are greater than any assignable quantity (see Section 4.3 for a modern formalization). Leibniz clarified the meaning of the term incomparable as follows:

> C’est ce qui m’a fait parler autres fois des incomparables, par ce que ce que j’en dis a lieu soi qu’on entende des grandeurs infiniment petites ou qu’on employe des grandeurs d’une petitesse inconsiderable et suffissante pour faire l’erreur moindre que celle qui est donnée. (Leibniz quoted by Pasini (1988, 708); emphasis added)
Leibniz’ sentence exploits the structure ‘soit ... ou ...’ to present a pair of distinct meanings of the term *incomparable*: (A) as a practical tool enabling one to make an error smaller than a value given in advance, and (B) as an infinitesimal properly speaking. The 1695 texts used the term in the sense (B). RA quote this Leibnizian passage twice yet fail to notice that it challenges their main thesis concerning Leibnizian infinitesimals (see Section 3 for an analysis of the difficulties with RA’s reading).

1.5. *Euclid’s Definition V.4*

A pair of Leibnizian texts (1965a; 1695b) from 1695 mention Euclid’s definition V.4 (referred to as V.5 by Leibniz): a letter to l’Hospital and a published response to Nieuwentijt. Breger notes the violation of Euclid’s definition by Leibnizian incomparables:

> In a letter to L’Hôpital of 1695, Leibniz gives an explicit definition of incomparable magnitudes: two magnitudes are called incomparable if the one cannot exceed the other by means of multiplication with an arbitrary (finite) number, and he expressly points to Definition 5 of the fifth book of Euclid quoted above. (Breger 2017, 73–4)

See (Bair et al. 2018, Sections 3.2–3.4) for a detailed analysis of these two mentions of V.4.

Levey claims that ‘in a 1695 response to criticisms of the calculus by Bernard Nieuwentijt ... Leibniz explains that his infinitely small differential (say, $dx$ in $x+dx$) is not to be taken to be a fixed small quantity’ (Levey 2021, 147). Levey goes on to translate a Leibnizian passage as follows:

> Such an increment cannot be exhibited by construction. Certainly, I agree with Euclid bk. 5, defin. 5, that only those homogeneous quantities are comparable of which one when multiplied by a number, that is, a finite number, can exceed the other. And I hold that any entities whose difference is not such a quantity are equal. (…) This is precisely what is meant by saying that the difference is smaller than any given. (GM 5.322) (Leibniz as translated by Levey in (2021, 147))

Arthur makes similar claims about this passage in (2013, 562). However, such claims concerning this Leibnizian passage are problematic, as signalled in the following seven points.

1. Both Arthur and Levey skip Leibniz’s crucial introductory sentences (quoted in Section 2.6 below) where Leibniz discusses both his generalized relation of equality up to negligible terms, and his incomparables.

2. Arthur and Levey overlook the fact that Leibniz is not claiming that homogeneous quantities are always comparable. Rather, Leibniz merely gives the definition of comparability in terms of V.4. Levey seeks to create a spurious impression that

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13The *incomparables* in this sense are sometimes described by Leibniz as *common*. Thus, they are described as *incomparables communs* in the 1702 letter to Varignon quoted in Section 3.2. The distinction has often been overlooked by commentators; see note 34.

14Leibnizian conciliatory methodology (see Mercer 2006) could account for his desire to include a mention of the A-method (favored by some contemporaries on traditionalist grounds) alongside the B-method.
Leibniz endorses the Archimedean property, whereas Leibniz merely gives a definition of what it means for quantities to be comparable.

3. The promoters of the syncategorematic reading are in disarray when it comes to interpreting this Leibnizian passage. RA (Rabouin and Arthur 2020, 432) acknowledge the violation of Euclid V.4 at least as a nominal definition of incomparable quantities that may be used as if they exist under certain specified conditions (see Section 3.4). In a comment appearing in our third epigraph, RA even summarize the Leibnizian definition of incomparability in modern notation as follows. They denote by ‘INC’ the relation of being incomparable, and write:

[Leibniz] has defined … what it is for one quantity (say, \(dx\)) to be incomparable in relation to another (say, \(x\)). Formally, Leibniz’s definition gives: For a given \(x\) and \(dx\), \(dx \perp INC x\) iff \(¬(\exists n)(ndx > x)\). (Rabouin and Arthur 2020, 433; emphasis in the original)

By contrast, Levey (2021, 147) and Arthur (2013) interpret the passage directly in an Archimedean fashion involving understanding the clause ‘not a comparable difference’ as no difference rather than possibly an incomparable difference.

4. The contention that Leibniz had in mind a generalized relation of equality up to negligible terms is corroborated by the fact that Jacob Bernoulli’s student, Jacob Hermann, gave a similar definition in his 1700 rebuttal of Nieuwentijt’s criticisms:

According to Hermann the whole difficulty is based on an ambiguous use of the two notions ‘aequalis’ and ‘incomparabilis’. With regard to equality, he reiterates Leibniz’s definition, formulating it like this: ‘quaecunque data quavis minore differentia differunt, aequalia esse.’ (‘Whatever differs by a difference smaller than any given quantity is equal’). (Nagel 2008, 206)

Hermann also gave a definition of infinitesimals similar to Leibniz’s. While the idea of Leibniz being ahead of his time possesses certain a priori philosophical plausibility, historically speaking the idea of Leibniz as a harbinger of the Epsilontik must postulate a problematic discontinuity with both his predecessors Torricelli and Roberval and his successors l’Hospital, Hermann, the Bernoullis and others (see also note 5).

5. Levey’s opening claim that ‘Leibniz explains that his infinitely small differential ... is not to be taken to be a fixed small quantity’ is not supported by the Leibnizian passage at all, since the passage mentions neither constant nor variable quantities.

6. In view of Breger’s remark quoted at the beginning of the current Section 1.5, Levey appears to attribute to Leibniz a rather paradoxical stance of claiming that his incomparables violate V.4 in one 1695 text (letter to l’Hospital), and satisfy V.4 in another text (response to Nieuwentijt) from the same year. The analysis presented

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15 As mentioned earlier, this interpretation is untenable since in the preceding sentences – not quoted by Levey – Leibniz defines his generalized equality where the difference may be incomparably small rather than necessarily absolutely zero.

16 Hermann defines infinitesimals as follows: ‘Quantitas vero infinite parva est, quae omni assignabili minor est: & talis Infinitesima vel Differentiale vocatur’ (Hermann 1700, 56). Thus an infinitesimal is smaller than every assignable quantity, just as in Leibniz.
in (Bair et al. 2018) indicates that in both texts, Leibniz made it clear that his incomparables violate V.4.

7. Levey follows Ishiguro’s narrative in the following sense. He first presents what he describes as the ‘popular history’ of LC, and then proposes an alternative. According to the popular history as described by Levey, Leibniz used unrigorous infinitesimals to advance geometry without worrying about the foundations of his method. Levey’s alternative account holds that Leibniz was in fact rigorous (by the standards of his day), but his concept of infinitesimal was misunderstood: what he really meant was Archimedean variable quantities. Thus, Levey writes:

Leibniz’s calculus was developed with scrupulous care for foundational matters and was never wedded to the idea of infinitesimals as fixed quantities greater than zero but less than any finite value. (Levey 2021, 140)

We can agree that LC was developed with scrupulous care for foundational matters. However, Levey’s alternative account is rooted in a Weierstrassian concept of rigor, and overlooks another possibility—namely, that Lebnizian infinitesimals were rigorous (by the standards of his time), but to appreciate it one needs to overcome the Weierstrassian limitations and be open to taking Leibniz at face value when he makes it clear that his infinitesimals violate Definition V.4. The coherence of LC with its incomparables and bounded infinities is underscored by the existence of modern formalizations (see Section 4) faithful to the Leibnizian procedures.

1.6. Questions for dialog

While the syncategorematic reading of LC is not the main focus of our analysis in the present text, we would like to present a few questions for a possible dialog concerning the Ishiguro–Arthur reading. In the recent literature, one detects oblique references of the following type:

- ‘Some recent theories of infinitesimals as non-Archimedean entities’ (Arthur 2013, 553).
- ‘Advocates of nonstandard analysis routinely refuse to acknowledge this’ (Spalt 2015, 121).
- ‘Recently there have been attempts to argue that Leibniz, Euler, and even Cauchy could have been thinking in some informal version of rigorous modern non-standard analysis, in which infinite and infinitesimal quantities do exist. However, a historical interpretation such as the one sketched above that aims to understand Leibniz on his own terms, and that confers upon him both insight and consistency, has a lot to recommend it over an interpretation that has only been possible to defend in the last few decades’ (Gray 2015, 11).

17Not to be confused with common incomparables; see notes 13 and 34.
18Additional difficulties with Levey’s reading are signalled in notes 7, 8, 37, 40 and 51.
19Apart from its failure to either cite or name scholars being criticized, Gray’s position endorsing historical authenticity seems reasonable—until one examines Gray’s own historical work, where Euler’s foundations are claimed to be ‘dreadfully weak’ (Gray 2008a, 6) and ‘cannot be said to be more than a gesture’ (Gray 2015, 3), whereas Cauchy’s continuity is claimed to be among concepts Cauchy defined using ‘limiting arguments’ (Gray 2008b, 62). Such claims are arguably symptomatic of butterfly-model thinking (see...
Certain scholars of the calculus have denied that the interpretation of infinitesimals as syncategorematic was Leibniz’s mature view, and have seen them as fictions in a different sense (Arthur 2018, 156).

‘on se gardera néanmoins de forcer un parallèle que Leibniz ne place pas là où l’interprétation formaliste le place’ (Rabouin in Leibniz 2018, 96).

Alas, such references come unequipped with either quotation or citation. These authors apparently feel that the answer to Probst’s question (see Section 1) is evident. It is to be hoped that a more meaningful dialog with proponents of the Ishiguro–Arthur reading and its variants can be engaged in the future. Such a dialog could use the following questions as a starting point:

1. **Were there two separate methods in the Leibnizian calculus or only one?** Bos claims that there were two (one involving exhaustion in an Archimedean context, and a separate one relying on the law of continuity and exploiting inassignable infinitesimals), and provides textual evidence. A number of authors disagree with Bos, including Arthur (2013, 561), Breger (2008, 196–7), and Spalt (2015, 118).

2. **Are there hidden quantifiers lurking behind Leibnizian calculus?** Ishiguro said there were (see Section 1.3). Rabouin seeks to distance himself from hidden quantifiers in (2015, 362) but goes on to endorse Ishiguro in note 25 on the same page. Breger endorses ‘absorbed’ quantifiers in (2008, 194).

3. **Is it or is it not appropriate to use Robinson’s framework in order to formalize Leibnizian analysis?** Here there seems to be a uniform agreement among scholars including Arthur, Bassler, Breger, Gray, Ishiguro, Levey, Rabouin (Leibniz 2018, note 2, 95–96), and Spalt that a Robinsonian interpretation is not possible. Yet Arthur (2013) compares LC to Smooth Infinitesimal Analysis where infinitesimals similarly exhibit non-Archimedean behavior. Granted, he finds some differences in addition to similarities, but he does not seem to accord a status of ‘comparison is possible’ to Robinson’s framework. The reasons for this need to be understood.

4. **Do scholars following Ishiguro tend to apply a modern framework to interpreting Leibniz, namely the Weierstrassian one?** In some of his later writings, Knoebloch explicitly describes Leibnizian infinitesimals in terms of Weierstrass and the Epsilontik; see e.g. Knoebloch (2021, 2, 7).

5. **Must Leibnizian infinitesimals be interpreted as limits?** Guicciardini (2018, 90) replies in the affirmative.

Philosophical partis pris among historians of mathematics have been analyzed by Hacking; see Section 1.7.

Section 1.7). For a more balanced approach to Euler and Cauchy see respectively (Bair et al. 2017b) and (Bair et al. 2020).

It is commendable that in a recent text, Rabouin and Arthur (2020) engage their opponents in a more open fashion than has been the case until now, even though we raise many questions concerning the quality of their engagement in Sections 1.4 and 3.

Rabouin and Arthur (2020) seem to acknowledge the presence of two methods, but then go on to claim that method B exploiting infinitesimals \(dx \) and \(dy \) is easily transcribed in terms of assignable quantities \((d)x\) and \((d)y\). The mathematical coherence (or otherwise) of such a claim is analyzed in Section 6.2.
1.7.  **Hacking’s butterfly / Latin models; Carnap’s ‘empty words’**

Some historians may be conditioned by their undergraduate mathematical training to interpret Leibniz in anachronistic ways. Thus, in one of his later articles, Knobloch writes:

> The set of all finite cardinal numbers 1, 2, 3, … is a transfinite set. Its cardinal number is \( \aleph_0 \). This is the least cardinal number being larger than any finite cardinal number. Leibniz’s terminology implies actual infinity though he rejects the existence of an infinite number, etc. (Knobloch 2018, 13–14)

The assumption that the only way to formalize infinity is by means of Cantorian infinite cardinalities creates tensions with what Leibniz wrote. Specifically, *infinita terminata* can be understood without Cantorian infinite cardinalities; see Section 4.4.

Some Leibniz scholars work under the related assumption that ‘Archimedean’ is the only natural way of thinking of the continuum. Thus Arthur and Rabouin endeavours to clear Leibniz of any suspicion of entertaining unnatural thoughts such as infinitesimals. They would rather declare Leibniz to be mired in contradictions (see Section 3.1) than admit that he dealt with non-Archimedean phenomena.

Such ‘conceptually repugnant’ attitudes are illustrated in our second epigraph taken from Levey (1998, 55, note 9). Such scholars fail to take into account the contingency of the historical development of mathematics emphasized by Hacking (2014, 72–75, 119).

Hacking contrasts a model of a deterministic (genetically determined) biological development of animals like butterflies (the egg–larva–cocoon–butterfly sequence), with a model of a contingent historical evolution of languages like Latin. Emphasizing determinism over contingency in the historical evolution of mathematics can easily lead to anachronism.²² With reference to infinitesimal calculus, the Latin model challenges the assumption that the so-called search for rigor inevitably led to the arithmetisation of analysis, accompanied by the banning of infinitesimals and the establishment of Weierstrassian foundations ultimately formalized in Zermelo–Fraenkel set theory (ZF) in the \( \varepsilon \)-language (formalizing the membership relation). The development towards the Weierstrassian limit concept seemed inevitable to Carnap who wrote:

> Leibniz and Newton … thought that they had a definition which allowed them to have a conceptual understanding of ‘derivative’. However, their formulations for this definition used such expressions as ‘infinitesimally small magnitudes’ and quotions [sic] of such, which, upon more precise analysis, turn out to be pseudo concepts (empty words). It took more than a century before an unobjectionable definition of the general concept of a limit and thus of a derivative was given. Only then all those mathematical results which had long since been used in mathematics were given their actual meaning. (Carnap 2003, 306–307)

Carnap’s assumption is that the ‘actual meaning’ of a mathematical concept like the derivative could not possibly involve ‘empty words’ like infinitesimals. In 2020 one finds a historian who claims that ‘Euler was not entirely successful in achieving

²²For a case study in Cauchy scholarship see Bair et al. (2019). Opposition to Marburg neo-Kantians’ interest in infinitesimals is documented in Mormann and Katz (2013).
his aim since he introduced infinitesimal considerations in various proofs’ (Ferraro 2020, 11). Such assumptions obscure the possibility that the so-called arithmetisation of analysis could have occurred while retaining infinitesimals. Hacking’s Latin model enables scholars to contemplate alternative possibilities of historical development of mathematics. Pace Carnap, set theories in the more versatile st-∈-language (formalizing both the Leibnizian assignable/inassignable distinction and the membership relation) provide more faithful conceptualisations of the procedures of LC than the theory ZF; see Section 4 for further details.

1.8. **Grafting of SIA on the calculus of Leibniz**

Richard Arthur compared LC to a modern theory of infinitesimals, namely Smooth Infinitesimal Analysis (SIA). Arthur claims to find ‘many points in common’ (in addition to differences) between LC and SIA:

I then turn to a comparison of Leibniz’s approach with the recent theory of infinitesimals championed by John Bell, Smooth Infinitesimal Analysis (SIA), of which I give a brief synopsis in Section 3. As we shall see, this has many points in common with Leibniz’s approach: the non-punctiform nature of infinitesimals, their acting as parts of the continuum, the dependence on variables (as opposed to the static quantities of both Standard and Non-standard Analysis), the resolution of curves into *infinite-sided polygons*, and the finessing of a commitment to the existence of infinitesimals.23 (Arthur 2013, 554; emphasis added)

The claimed similarities including the following issues:

- (AR1) non-punctiform nature of the continuum;
- (AR2) variables;
- (AR3) infinite-sided polygons;
- (AR4) fictionality of infinitesimals.

We will argue that issue (AR1) is largely irrelevant to interpreting the procedures of LC, while Arthur’s case for (AR2) is based on equivocation. Furthermore, Robinson’s framework is more successful than SIA on issues (AR3) and (AR4).

We present a detailed analysis of Arthur’s comparison of LC to SIA in Section 2. The position of Rabouin and Arthur is analyzed in Section 3. We propose a formalization of the procedures of the Leibnizian calculus in modern mathematics in Section 4. Some technical details on SIA are reviewed in Section 5. In Section 6 we focus on interpretations of the seminal text ‘Cum Prodiisset’.

2. **Leibniz and smooth infinitesimal analysis**

Arthur developed a comparison of LC with SIA in (2013). In this section we will analyze Arthur’s comparison. Any discussion of SIA should mention the foundational work of Lawvere starting in 1967 (see e.g. Lawvere 1980) and the books by Moerdijk and Reyes (1991) and by Kock (2006), among other authors.

23Arthur’s finessing comment is analyzed in Section 5.1.
2.1. **Knobloch on Arthur and SIA**

Knobloch analyzed Arthur’s comparison of LC and SIA in a *Zentralblatt* review:

[Richard Arthur] compares Leibniz’s infinitesimals with those of Bell’s smooth infinitesimal analysis (SIA) published in 1998. Hence, in Section 3, he gives a brief synopsis of SIA. Both notions of infinitesimals use non-punctiform infinitesimals and the *resolution of curves into infinite-sided polygons*. The Leibnizian polygonal representation of curves is closely related to Bell’s principle of microstraightness. Yet, there are crucial differences. (Knobloch 2013; emphasis added)

Note that SIA depends on a category-theoretic foundational framework and on intuitionistic logic to enable nilpotency of infinitesimals (see Section 5.1). Thus Arthur seeks to rely on the resources of modern mathematics, including non-classical logics, in a comparison with LC. While we welcome such a display of pluralism on Arthur’s part, we also agree with both John L Bell and Arthur that SIA is closer to Nieuwenhuis’s approach to calculus than to Leibniz’s (see further in Section 2.3). All SIA nilpotent infinitesimals \( \varepsilon \) have the property that both \( \varepsilon \) and \(-\varepsilon\) are smaller than \( \frac{1}{n} \):

\[
\text{for all } \varepsilon \text{ in } \Delta,^{24} \text{ one has } \varepsilon < \frac{1}{n},
\]

signifying non-Archimedean behavior; see (Bell 2008, Exercise 8.2, p 110).

Leibniz endorsed what is known today as the law of excluded middle in a passage written around 1680 in the following terms:

Every judgment is either true or false. No judgment is simultaneously true and false. Either the affirmation or the negation is true. Either the affirmation or the negation is false. For every truth a reason can be provided, excepting those first truths in which the same thing is affirmed of the thing itself or is denied of its opposite. \( A \) is \( A \). \( A \) is not \( \neg A \). (Leibniz as translated by Arthur in Arthur (2001), 237–9)

The assumption that proposition \( A \) is equivalent to \( \neg \neg A \) is the law of excluded middle, rejected by intuitionists. Arthur translated and edited this Leibnizian passage in 2001 but did not report it in his text (Arthur 2013) comparing LC to the theory SIA based on intuitionistic logic. The following question therefore arises.

**Question.** Why does Arthur’s voluminous output on the Leibnizian calculus systematically eschew readings based on Robinson’s framework (Robinson 1966) or the foundational approaches of Hárbacek (1978) and Nelson (1977), based as they are on classical logic, and enabling straightforward transcription (see Section 4) of both Leibniz’s assignable/inassignable dichotomy and his infinitely many orders of infinitesimal and infinite numbers?

---

24The definition of the part \( \Delta \) appears in Section 5.3.
An attempt by Rabouin and Arthur to address this question is analyzed in Section 3. Meanwhile, Arthur appears to avoid Robinsonian infinitesimals as zealously as Leibniz avoided atoms and material indivisibles (for an analysis of Leibniz on atoms and indivisibles see Bair et al. (2018, Section 2.6)).

2.2. Infinita terminata

A key Leibnizian distinction between bounded and unbounded infinity is sometimes insufficiently appreciated by commentators (including Rabouin and Arthur). In DQA, Leibniz contrasts bounded infinity and unbounded infinity in the following passages:

But as far as the activity of the mind with which we measure infinite areas is concerned, it contains nothing unusual because it is based on a certain fiction and proceeds effortlessly on the assumption of a certain, though bounded, but infinite line; therefore it has no greater difficulty than if we were to measure an area that is finite in length. (Leibniz, DQA (Leibniz 2004), Scholium following Propositio XI; translation ours; emphasis added)

Leibniz elaborates as follows:

Just as points, even of infinite numbers, are unsuccessfully added to and subtracted from a bounded line, so a bounded line can neither form nor exhaust an unbounded one, however many times it has been repeated. This is different with a bounded but infinite line thought to be created by some multitude of infinite lines, although this multitude exceeds any number. And just as a bounded infinite line is made up of finite ones, so a finite line is made up of infinitely small ones, yet divisible.25 (ibid.)

The importance of the distinction is stressed by Mancosu (1996, 144). RA appear to be aware of the distinction and even mention the term terminata three times in their article, but don’t fully appreciate its significance.26 We propose a possible formalization of the bounded/unbounded distinction in modern mathematics in Section 4.4.

2.3. Infinite-sided polygons

Arthur claims that a common point between LC and SIA consists in viewing a curve as a polygon with infinitely many sides (see Section 1.8). Knobloch takes note of such alleged similarity in his review of Arthur quoted in Section 2.1. Are infinite-sided polygons a point in common between LC and SIA, as Arthur claims? A comparison of LC and SIA based on infinite-sided polygons would apparently require both sides of the comparison to envision such polygons. Arthur goes on to mention

25It is worth noting that bounded/unbounded is not the same distinction as potential/actual infinity. Bounded infinity is a term Leibniz reserves mainly to characterize the quantities used in his infinitesimal calculus, namely the infinitely small and infinitely large.

26Furthermore, RA misinterpret Leibnizian bounded infinities when they compare them to compactifications in modern mathematics; see Section 3.8.
the quotation Bertoloni Meli gives from Leibniz’s letter to Claude Perrault … written in 1676: ‘I take it as certain that everything moving along a curved line endeavours to escape along the tangent of this curve; the true cause of this is that curves are polygons with an infinite number of sides, and these sides are portions of the tangents…’ (quoted from Bertoloni Meli 1993, 75). (Arthur 2013, 568–9, note 17; emphasis added).

Thus far, we have sourced Leibnizian infinite-sided polygons. What about the SIA side of Arthur’s comparison? Arthur’s source Bell does mention *infinilateral polygons* in his article ‘Continuity and infinitesimals’ (Bell 2005–2013). However, Bell attributes such polygons to Leibniz and Nieuwentijt rather than to SIA. We give three examples:

1. ‘The idea of considering a curve as an infinilateral polygon was employed by a number of … thinkers, for instance, Kepler, Galileo and Leibniz.’ (Bell 2005–13)
2. ‘Leibniz’s definition of tangent employs both infinitely small distances and the conception of a curve as an infinilateral polygon.’ *(ibid.)*
3. ‘Nieuwentijdt’s infinitesimals have the property that the product of any pair of them vanishes; in particular each infinitesimal is ‘nilsquare’ in that its square and all higher powers are zero. This fact enables Nieuwentijdt to show that, for any curve given by an algebraic equation, the hypotenuse of the differential triangle generated by an infinitesimal abscissal increment \( e \) coincides with the segment of the curve between \( x \) and \( x+e \). That is, a curve truly *is* an infinilateral polygon.’ *(ibid.; emphasis in the original)*

While Bell speaks of Leibniz and Nieuwentijt as using infinilateral polygons, Bell’s article is silent on infinilateral polygons in connection with SIA.

Similarly, Bell’s book on SIA (Bell 2008) makes no mention of infinilateral polygons. There is a good reason for Bell’s silence. Nilsquare infinitesimals, say \( \varepsilon \), are by definition not invertible. Therefore in SIA one cannot view a curve as made up of, say, \( \frac{\varepsilon}{2} \) sides or something of that order. Rather, in SIA, a curve microlocally ‘looks like’ a straight infinitesimal segment in the sense that, microlocally, a function coincides with its first Taylor polynomial; see formula (2) in Section 5.3. Therefore infinite-sided polygons can hardly be said to be a point in common between LC and SIA. Arthur’s claim to the contrary is dubious.

### 2.4. Bell’s microquantities

In his more recent book (Bell 2019) published six years after Arthur’s article (2013), Bell mentions infinilateral polygons in the context of his discussion of Stevin, Kepler, Galileo, Barrow, Leibniz, l’Hôpital, and Nieuwentijt in his historical Chapter 2.

In Chapter 10, Bell gives his definition of *microstraightness* in terms of the set (part) of microquantities \( \Delta \text{ }^27 \) as follows:

If we think of a function \( y = f(x) \) as defining a curve, then, for any \( a \), the image under \( f \) of the ‘microinterval’ \( \Delta + a \) obtained by translating \( \Delta \) to \( a \) is straight

\[^27\text{The definition of the part } \Delta \text{ appears in Section 5.3 below.}\]
and coincides with the tangent to the curve at \( x = a \) … In this sense … each curve is ‘microstraight’. (Bell 2019, 241; emphasis in the original)

Bell goes on to claim that

closed curves can be treated as infinitesimal polygons, as they were by Galileo and Leibniz (ibid., note 18, p. 241)

However, he provides no further elaboration. The implied assumption that Leibnizian infinitesimal polygons were akin to SIA’s seems dubious; to justify such an assumption one would have to stretch the meaning of the term ‘infinitesimal’ beyond its apparent meaning of ‘possessing infinitely many sides.’ This is because the microquantities in question are not invertible, as already mentioned in Section 2.3. The matter is analyzed further in Section 2.5.

2.5. Infinitesimal or infinitangular?
In his seminal 1684 text *Nova Methodus*, Leibniz refers to a curve with infinitely many **angles**, rather than infinitely many sides.

While curves in SIA are microstraight, enabling many elegant calculations, they don’t fit the Leibnizian description. Having an angle requires having a vertex, but from the SIA viewpoint, curves not only do not have infinitely many vertices but they have no vertices (where one could speak of an **angle** not not different from \( \pi \)) at all. Perhaps it is for this reason that Bell can only speak of infinitesimal polygons rather than infinitangular polygons. How does Bell handle the issue? Bell writes:

Leibniz’s definition of tangent employs both infinitely small distances and the conception of a curve as an infinitesimal polygon:

> We have to keep in mind that to find a tangent means to draw a line that connects two points of a curve at an infinitely small distance, or the continued side of a polygon with an infinite number of angles, which for us takes the place of a curve.

In thinking of a curve as an infinitesimal polygon … the abscissae \( x_0, x_1, \ldots \) and the ordinates \( y_0, y_1 \), are to be regarded as lying infinitesimally close to one another; … (Bell 2019, 70; emphasis added)

The indented Leibnizian passage is from (Leibniz 1684, 223). Leibniz speaks of infinitangular polygons, whereas Bell speaks of infinitesimal polygons both before and following the Leibnizian passage, without commenting on the discrepancy. Leibnizian infinitangular polygons are arguably modeled less faithfully in SIA than in Robinson’s framework, as we argue in Section 2.6.

2.6. Infinitangular and infinitesimal polygons in Robinson
We saw in Section 2.4 that infinite-sided polygons are not a feature of the SIA framework in the context of nilsquare microquantities. Meanwhile, in Robinson’s framework, an
infinite-sided polygon\textsuperscript{28} can easily be chosen to approximate a given curve. This enables one to determine the usual geometric entities, such as the tangent line from a pair of adjacent vertices or the curvature from a triple of adjacent vertices of the polygon.

To provide a modern interpretation, consider the problem of determining the tangent line to a curve. Take a pair of infinitely close distinct points $D,E$ on the curve, and consider the line $DE$. One can normalize the equation of the line $DE$ as $ax + by = c$ where $a^2 + b^2 = 1$. Then one can take the standard part (see Section 4.5) of the coefficients of the equation of the line $DE$ to obtain the equation $a_0x + b_0y = c_0$ of a true tangent line to the curve at the standard point $D_0 = E_0$, where $r_0$ is the standard part of $r$ for each of $r = a, b, c, D, E$. In this sense, the line $DE$ is an approximation to the true tangent line, meaning that they coincide up to negligible terms. For more advanced applications see e.g. Albeverio et al. (1986).

Approximation procedures are not unusual for Leibniz. Leibniz often mentioned his use of a generalized notion of equality. Thus, in his response to Nieuwentijt Leibniz wrote:

Furthermore I think that not only those things are equal whose difference is absolutely zero, but also those whose difference is incomparably small. And although this [difference] need not absolutely be called Nothing, neither is it a quantity comparable to those whose difference it is. (Leibniz 1695b, 322)

Bell makes the following claims:

In SIA, [1] all curves are microstraight, and [2] closed curves [are] infinilateral polygons. [3] Nothing resembling this is present in NSA [Nonstandard Analysis]. (Bell 2019, 259; numerals [1], [2], [3] added)

Bell’s claim [1] is beyond dispute. However, both of Bell’s claims [2] and [3] are dubious. Claim [2] was already analyzed in Section 2.4. Claim [3] may be literally true in the sense that in Robinson’s framework, the infinite-sided polygon does not literally coincide with the closed curve but is only an approximation to the curve. However, the approximation is good enough to compute all the usual entities such as tangent line and curvature, as noted above. Leibniz defined the curvature via (the reciprocal of) the radius of the osculating circle (see Bos 1974, 36). The osculating circle can be defined via the circle passing through a triple of infinitely close points on the curve (alternatively, the center of the osculating circle to the curve at a point $p$ can be defined via the intersection of two ‘consecutive’ normals to the curve near $p$). Johann Bernoulli’s approach to osculating circles is analyzed by Blåsjö (2017, 89–93). The calculation of curvature from a triple of infinitely close points on the curve is straightforward in Robinson’s framework (or its axiomatic formulations; see Section 4).

2.7. Leibniz to Wallis on Archimedes

In connection with Leibniz’s reference to Archimedes in a letter to Wallis, Arthur comments as follows:

\textsuperscript{28}More precisely, an internal polygon with an infinite hyperinteger number of vertices. In Nelson’s framework or in the theory SPOT one considers a polygon with $\mu$ sides for a nonstandard $\mu \in \mathbb{N}$; see Section 4.6.
The strict proof operating only with assignable quantities justifies proceeding by simply appealing to the fact that \( dv \) is incomparable with respect to \( v \): in keeping with the Archimedean axiom, it can be made so small as to render any error in neglecting it smaller than any given. (Arthur 2013, 567–568; emphasis added)

Here Arthur is assuming the \( dv \) is a common incomparable and therefore assignable.\(^{29}\) Is the Leibnizian \( dv \) an assignable quantity, as Arthur claims? Arthur goes on to quote Leibniz as follows:

Thus, in a letter to Wallis in 1699, Leibniz justifies the rule for \( d(xy) = xdy + ydx \) as follows:\(^{30}\)

\[
\ldots \text{there remains } xdy + ydx + dx\text{~}dy. \text{ But this [term] } dx\text{~}dy \text{ should be rejected, as it is incomparably smaller than } xdy + ydx, \text{ and this becomes } d(xy) = xdy + ydx, \text{ inasmuch as, if someone wished to translate the calculation into the style of Archimedes, it is evident that, when the thing is done using assignable qualities, the error that could accrue from this would always be smaller than any given. (ibid.; emphasis on ‘translate’ added)}.
\]

When Leibniz speaks of the possibility of using assignable quantities only, he explicitly refers to the possibility of a translation of the calculation into the style of Archimedes, rather than to the original infinitesimal calculation itself.\(^{30}\) Arthur’s inference of the alleged Archimedean nature of Leibnizian \( dv \) is dubious.\(^{31}\)

In a letter to Wallis two years earlier, Leibniz already emphasized the distinction between Archimedean and non-Archimedean techniques:

Leibniz’s response [to Wallis] was first to distinguish between two kinds of tetragonistic methods, both of which could be traced back to Archimedes, and then to distinguish these from recent infinitesimal techniques. Of the older established methods one, he writes, considers geometrical figures and bodies to be collections of an infinite number of quantities each of which is incomparably smaller than the whole, while in the other,\(^{32}\) quantities remain comparable to the whole and are taken successively in infinite number so as eventually to exhaust that whole. … Effectively, the concept of analysis with the infinite as its object became the distinguishing factor between the old and the new. (Beeley 2013, 57; emphasis added)

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\(^{29}\)See Section 1.4 and note 34 on the distinction between incomparables and common incomparables.

\(^{30}\)The Leibnizian passage is also quoted by RA (Rabouin and Arthur 2020, 437) who similarly fail to account for the fact that talking about a translation into the style of Archimedes entails the existence of a separate method exploiting infinitesimals à la rigueur.

\(^{31}\)Furthermore, Arthur’s attempt to account for the Leibnizian derivation of the law \( d(xy) = xdy + ydx \) in Archimedean terms, by means of quantified variables representing \( dx, dy, d(xy) \) taking assignable values and tending to zero, would run into technical difficulties. Since all three tend to zero, the statement to the effect that ‘the error that could accrue from this would always be smaller than any given’ understood literally is true but vacuous: 0 = 0 + 0. To assign non-vacuous meaning to such an Archimedean translation of Leibniz, one would have to rewrite the formula, for instance, in the form \( \frac{d(xy)}{dx} = x \frac{dy}{dx} + y \). While such a paraphrase works in the relatively simple case of the product rule, it becomes more problematic for calculations involving, e.g. transcendental functions.

\(^{32}\)We add a comma for clarity.
3. Rabouin–Arthur indictment of infinitesimals

Rabouin and Arthur (RA) address the Question posed at the end of Section 2.1 by purporting to identify a factor that ‘distinguishes Leibniz’s position from Robinson’s’ (Rabouin and Arthur 2020, 407):

For what Robinson demonstrated with his non-standard analysis was precisely that one can introduce infinite numbers and infinitesimals without contradiction. The whole point of any ‘non-standard’ approach is the building of a non-standard model in which what is said about entities such as ‘infinite numbers’ or ‘infinitely small quantities’ has to be literally and rigorously true. By contrast, Leibniz repeatedly claims that what is said about infinitesimals is not true ‘à la rigueur’ and entails a paradoxical way of speaking.18 (ibid.; emphasis added)

(More on RA’s footnote 18 in Section 3.3 below.) Here RA seek to identify a discrepancy between the approaches of Leibniz and Robinson. They claim to find one in terms of the presence of a contradiction (in the notion of infinitesimal) in Leibniz but not in Robinson. Such an indictment of Leibnizian infinitesimals entails the claim that allegedly only Robinson ‘can introduce infinite numbers and infinitesimals without contradiction’ according to RA. Note, however, the following eight items.

3.1. Do infinitesimals contradict the part-whole axiom?

RA’s underlying assumption is that Leibniz views infinitesimals as being contradictory, just as unbounded infinite collections (taken as a whole) are. In fact, RA’s purported discrepancy between Leibnizian and Robinsonian infinitesimals depends crucially on such an assumption (Leibnizian infinitesimals are contradictory; Robinson’s aren’t; QED). However, what Leibniz repeatedly described as contradictory is only unbounded infinity seen as a whole (and contradicting the part-whole axiom), not bounded infinity (infinita terminata, such as the reciprocal of an infinitesimal), which he treated as a fiction because it does not exist in nature (see Section 2.2). Since the issue is not the existence or otherwise of Leibnizian infinitesimals but rather their being contradictory or not, RA have failed to show that Leibniz viewed infinitesimals as contradicting the part-whole axiom.

3.2. Did Leibniz envision infinity à la rigueur?

RA allege that Leibniz claimed that ‘what is said about infinitesimals is not true “à la rigueur.”’ However, Leibniz did allow for the possibility of infinities à la rigueur as for instance in the following passage:

‘Et c’est pour cet effet que j’ay33 donné un jour des lemmes des incomparables dans les Actes de Leipzic, qu’on peut entendre comme on vent [i.e., ‘veut’], soit des infinis à la rigueur, soit des grandeurs seulement, qui n’entrent point en ligne de compte les unes au prix des autres. Mais il faut considérer en même

33Here and in the following we preserve the original spelling.
temps, que ces incomparables communs\textsuperscript{34} mèmes n’estant nullement fixes ou
determinés, et pouvant estre pris aussi petits qu’on veut dans nos raisonnnemens
Geometriques, font l’effet des infiniment petits rigoureux’ (Leibniz 1702, 92;
emphasis added).

Leibniz’ ‘soit … soit …’ construction clearly indicates that infinity à la rigueur is
one of the possibilities. The precise meaning Leibniz attached to such infinis à la
rigueur is fictional quantities (lacking instantiation in nature) greater than any assign-
ables (see Section 1.4).

3.3. Leibniz to Wolff on fictions

Concerning RA’s footnote 18 (attached to the passage reproduced at the beginning
of the current Section 3), we note the following. Here RA seek to source their claim that
when Leibniz referred to infinitesimals as fictional, he meant that they involve a con-
tradiction. To this end, they quote a lengthy passage from a 1713 letter from Leibniz
to Wolff. Here Leibniz writes: ‘[E]ven though in my opinion [the infinitely small]
ensash something of the fictive and imaginary, this can nevertheless be rectified
by a reduction to ordinary expressions so readily that no error can intervene’ (GM,
V, 385) (Leibniz as translated by RA in (Rabouin and Arthur 2020, note 18)). In
the passage quoted, Leibniz makes no mention of contradictions, and merely
describes infinitesimals as fictive and imaginary. If RA wish to demonstrate by this
that Leibniz viewed fictions (such as infinitesimals) as contradictory then their dem-
onstration is circular. Thus, the passage does not substantiate RA’s claim.

3.4. What kind of definition?

RA claim that the Leibnizian definition of incomparables is nominal rather than
semantic. While they acknowledge that it is ‘clear and distinct’ (Rabouin and
Arthur 2020, 406), they claim that it does not assert existence or even possible exist-
ence. According to RA, Leibniz’ theory of knowledge enables one to derive truths
using contradictory concepts,\textsuperscript{35} but only non-contradictory concepts can refer to
possibly existing things; i.e. only non-contradictory concepts can be regarded as
defined semantically.

\footnotesize
\textsuperscript{34}Such common incomparables are not to be confused with the inassignable ones; see Section 1.4. Loemker
gives the following translation of this sentence: ‘[T]hese incomparable magnitudes themselves, as commonly
understood, are not at all fixed or determined but can be taken to be as small as we wish in our geometrical
reasoning and so have the effect of the infinitely small in the rigorous sense’ (Leibniz 1989c, 543). The trans-
lation is not successful as it obscures the fact that communs modifies incomparables. Leibniz is not speaking
of common understanding of incomparable quantities, but rather of common incomparables as opposed to
bona fide ones. Horváth in (1986, 66) provides both the original French and an English translation, but
omits the adjective communs in his translation.

\textsuperscript{35}To buttress such a claim, RA offer what they describe as ‘a short digression on Leibniz’s theory of knowl-
dge’ (Rabouin and Arthur 2020, 406). The evidence they provide is the following Leibnizian passage: ‘For
we often understand the individual words in one way or another, or remember having understood them
before, but since we are content with this blind thought and do not pursue the resolution of notions far
enough, it happens that a contradiction involved in a very complex notion is concealed from us’
(Leibniz as translated by RA in Rabouin and Arthur 2020, note 13). Readers may judge for themselves
how compelling this evidence is for RA’s claim.
RA claim that Leibniz considers infinitely large or infinitely small quantities as contradictory. They write:

But, as we have shown, Leibniz always claimed that infinite entities, be they infinitely large or infinitely small, could not be considered as genuine quantities without violating a constitutive property of quantities given by the part-whole axiom. Hence, they cannot be introduced into the system without contradiction. (Rabouin and Arthur 2020, 433–434)

RA go on to conclude that Leibniz could not have understood incomparables in non-Archimedean terms.

However, as we noted in connection with the infinita terminata, RA provide no evidence that Leibniz considered all infinities (including bounded ones) as contradictory. Therefore, RA’s conclusion is based on an erroneous premise and a conflation of the notions of bounded and unbounded infinity; see Section 2.2 for details on the distinction. Section 4 proposes an interpretation of the distinction in modern mathematics.

While RA base their claims concerning the nature of Leibnizian infinities on their perception of alleged contradictions therein, it is important to note that they furnish no corroborating evidence whatsoever based on any textual analysis of Leibniz’s actual definitions of infinities and infinite magnitudes; instead, they rely on their claim of alleged contradictions. In a passage we quoted in our epigraph, RA even seem to admit that no such textual evidence exists.

RA assert that Leibniz did not provide a semantic definition of infinities. Could one expect him to provide such a definition? Modern mathematical logic makes a clear distinction between syntactic and semantic notions and between theory and model. Modern philosophy makes a distinction between procedures and ontology. Such insights are not easily attributable to Leibniz, though he did emphasize that applying infinities in geometry and physics should be independent of metaphysical investigations. Leibniz only specified the properties he expected his inassignables to have, rather than providing models that would allow him to ‘have’ them in any semantic sense.

As far as his ‘theory of knowledge’ is concerned, Leibniz specifically stated that one needn’t get involved in metaphysical questions as to the reality of such entities, and that the geometer’s task is limited to exploring the consequences of assuming the existence of such mental constructs. When he asserts that infinities ‘have their proof with them’ what he seems to mean is that their fruitful application justifies their use, but not that they are metaphysically real (in a few instances Leibniz specifically denies that material infinities exist, as in a letter to des Bosses). The way Leibniz employs his inassignables is our only aid in interpreting LC. The evidence that he views them as consistent comes from the fact that he exploits them in his mathematics. Leibniz insisted many times on the rigorousness of his calculus, and couldn’t afford to exploit inconsistent notions.

3.5. **RA vs Knobloch**

RA repeatedly cite Knobloch’s article (2002) (without providing either quotation or page number), as when they claim that ‘the nub of the proof is an exploitation of the Archimedean property to prove that quantities whose difference can be reduced to a quantity smaller than any given quantity are equal’ (Rabouin and Arthur...
In his earlier papers, Knobloch set out his position clearly as follows (see e.g. Knobloch 1990, 42; Knobloch 1994, 266–267). Leibniz contrasts bounded infinity and unbounded infinity. Bounded infinity is fictional. Unbounded infinity exists in the physical world (and is even described by Leibniz as ‘actual’), but contradicts the part-whole axiom if taken as a whole. No claim concerning the presence of contradictions is made with regard to bounded infinity, of which infinitesimals are one of the manifestations. Knobloch’s position in his early papers is in stark contrast with what RA claim already in their abstract, namely that both (i) infinitesimals and (ii) unbounded infinity taken as a whole, lead to contradictions.

3.6. RA’s case against Bernoullian continua

RA criticize the use of the term Bernoullian continuum by their opponents. RA quote Leibniz as follows:

‘We can conceive an infinite series consisting merely of finite terms, or terms ordered in a decreasing geometric progression. I concede the infinite multiplicity of terms, but this multiplicity forms neither a number nor one whole’ (To Johann Bernoulli, Feb. 24, 1699, GM III 575; A III 8, 66). (Leibniz as translated by RA in Rabouin and Arthur 2020)

Here Leibniz clearly rejects infinite wholes. RA then claim the following:

So we can see that the idea that Leibniz subscribed to a ‘Bernoullian continuum’, containing actual infinitesimals as its elements, is emphatically rejected by Leibniz himself, writing to the same Bernoulli. (Rabouin and Arthur 2020, 411)

However, RA provide no evidence that their opponents asserted that Leibniz believed in infinite wholes. Furthermore, the articles where the Archimedean vs Bernoullian distinction was introduced state explicitly that they use Bernoulli’s name merely because Bernoulli exploited only B-track methods whereas Leibniz used both A-track and B-track methods. RA provide no evidence that infinitesimals violate the part-whole axiom.

3.7. What about Archimedean continua?

The choice of target of RA’s criticism is significant. RA criticize the use only of the term Bernoullian continuum but not of the term Archimedean continuum. Such an attitude may be related to their undergraduate mathematical training resulting in a presentist belief that ‘Archimedean’ is the only natural way of thinking about the continuum, and is symptomatic of butterfly-model commitments (see Section 1.7). Meanwhile, Leibniz himself viewed any type of infinite whole as contradicting the part-whole axiom.

36Similarly, Jesseph notes that ‘a central task for Leibniz’s fictionalism about infinitesimal magnitudes is to show that their introduction is essentially harmless in the sense that it does not yield a contradiction’ (Jesseph 2015, 196; emphasis added).

37Levey claims that ‘For Leibniz there is no genuine infinite magnitude, great or small: no infinite line, no infinite quantity, no infinite number’ (Levey 2021, 147). He seeks support for his claim in the following Leibnizian passage from (Leibniz 1921): ‘It is perfectly correct to say that there is an infinity of things,
3.8. Compactifications

RA misinterpret Leibnizian bounded infinities when they compare them to compactifications (Rabouin and Arthur 2020, 421) in modern mathematics (such as adding a point at infinity to an open infinite space). Leibnizian bounded infinities are an arithmetical extension rather than a topological compactification. Just as infinitesimals can be further divided, according to Leibniz, *infinita terminata* can be expanded further. For an infinite line from 0 to μ there is also a line from 0 to 2μ, and so on. While this would not make any sense with compactifications, it fits well with the arithmetic employing a nonstandard μ (cf. Section 4). Leibnizian comparisons of infinitesimals to imaginaries and to ideal points of intersection of parallel lines only refer to their status as fictions possessing no instantiation in *rerum natura*.

3.9. RA on Kunen and paradoxes

RA pursue a puzzling venture into modern set theory and the paradox of ‘the set of all sets’. They provide a lengthy quotation from Kunen’s textbook (1980), and propose the following comparison:

> The standard set theorist can … claim at the same time that proper classes do not exist (in her axiomatic system), and that her surface language remains neutral as regard this question of existence, since there are other ways of interpreting it … As we will see, this is *exactly the position endorsed by Leibniz* in his public declarations regarding infinitesimals. (Rabouin and Arthur 2020, note 19; emphasis added)

Here RA appear to be comparing the fictionality of Leibnizian infinitesimals with the fictionality of the set of all sets in modern set theories. However, such set-theoretic paradoxes involve entities more similar to Leibnizian unbounded infinities than to his infinitesimals (see Section 2.2). We therefore question the exactness of RA’s claim that ‘this is exactly the position endorsed by Leibniz’ regarding infinitesimals.

See also Blåsjö’s analysis of the RA text in (2020).

4. Leibnizian calculus via modern infinitesimals

Recall that the canonical Zermelo–Fraenkel set theory (ZF) is a set theory in the ∈-language. Here ∈ is the two-place membership relation. Since the emergence of axiomatic nonstandard set theories in the work of Hrbacek (1978) and Nelson (1977), it has become clear that it is possible to develop analysis in set theories i.e. that there are always more of them than one can specify. But it is easy to demonstrate that there is no infinite number, nor any infinite line or other infinite quantity, if these are taken to be genuine wholes’ (emphasis added). Levey appears not to have noticed that the quoted claim is *conditional* upon taking infinity to be a whole. Unbounded infinity taken as a whole contradicts the part-whole axiom; bounded infinity doesn’t (see Section 2.2).

38 Arthur’s conjured-up Leibniz specifically compares the violation of the part-whole axiom by unbounded infinities, to the set-theoretic paradox of the set of all ordinals, in Arthur (2019, 104).
exploiting the more versatile \( \text{st}\-\in\)-language. Here \( \text{st} \) is a one-place predicate; \( \text{st}(x) \) means ‘\( x \) is standard.’ A recently developed theory SPOT (Hrbacek and Katz 2020) in the \( \text{st}\-\in\)-language is a conservative extension of ZF. The theory SPOT is a subtheory of theories developed by Hrbacek and Nelson. For a comprehensive treatment see Kanovei and Reeken (2004) and the references therein, as well as the survey by Fletcher et al. (2017).

Objections have been raised in the literature based on the assumption that modern infinitesimal analysis allegedly depends on non-effective foundational material such as the Axiom of Choice. Thus, Ehrlich comments:

The aforementioned ultrapower construction of hyperreal number systems is another source of nonuniqueness since the construction depends on an arbitrary choice of a nonprincipal ultrafilter. (Ehrlich 2021, note 24, 523)

In a similar vein, Henle comments:

There was hope, when Abraham Robinson developed nonstandard analysis …, that intuition and rigor had at last joined hands. His work indeed gave infinitesimals a foundation as members of the set of hyperreal numbers. But it was an awkward foundation, dependent on the Axiom of Choice. … Nonstandard analysis requires a substantial investment (mathematical logic and the Axiom of Choice) but pays great dividends. (Henle 1999, 67, 72)

There are many such comments in the literature.\(^{39}\) Such an assumption turns out to be incorrect. Infinitesimal analysis can be done in an axiomatic framework conservative over ZF, as shown in Hrbacek and Katz (2020). In particular, it can be developed without assuming either the existence of nonprincipal ultrafilters or the axiom of choice.

In this section, we will detail a formalization of the procedures of the Leibnizian calculus in SPOT.

4.1. Assignable vs inassignable

The predicate \( \text{st} \) provides a formalization of the Leibnizian distinction between assignable and inassignable quantities, as used in e.g. Leibniz (1701) (for an analysis see Bos (1974) as well as Section 6 below). Here assignable corresponds to standard whereas inassignable corresponds to nonstandard. Leibniz distinguished notationally between the inassignable \( dx, dy \) and the assignable \( (d)x, (d)y \); see Section 6.2.

\(^{39}\)Thus, Väth comments: ‘Without the existence of \( \delta\)-free ultrafilters … we were not able to construct non-standard embeddings. … [I]n the author’s opinion … nonstandard analysis is not a good model for ‘real-world’ phenomena’ (Väth 2007, 85). Easwaran and Towsner claim to ‘point out serious problems for the use of the hyperreals (and other entities whose existence is proven only using the Axiom of Choice) in describing the physical world in a real way’ (Easwaran and Towsner 2019, 1). Sanders catalogue many such comments in (2020, 459). A rebuttal of the claims by Easwaran and Towsner and Väth appears in Bottazzi et al. (2019). A rebuttal of related criticisms by Pruss (2018) appears in Bottazzi and Katz (2021a, 2021b). The theories developed in Hrbacek and Katz (2020) expose as factually incorrect a claim by Alain Connes to the effect that ‘as soon as you have a non-standard number, you get a non-measurable set’ (Connes 2007, 26).
4.2. Law of continuity and transfer

By the transfer principle, if $\phi$ is an $\in$-formula with standard parameters, then the following entailment holds:

$$\forall^{st}x \; \phi(x) \rightarrow \forall x \phi(x).$$

Here $\forall^{st}$ denotes quantification over standard entities only. SPOT’s transfer principle provides a formalization of the Leibnizian law of continuity, to the effect that ‘the rules of the finite are found to succeed in the infinite’ (Robinson 1966, 266). Namely, the rules of arithmetic for standard natural numbers (finite numbers) extend to the rules of arithmetic of (all) natural numbers; here the nonstandard ones correspond to the Leibnizian infinita terminata (see Section 2.2). The transfer principle can be thought of as a (belated) response to Rolle’s critique of the Leibnizian calculus and the law of continuity; see (Bair et al. 2018, Sections 2.7–2.11) for details.40

4.3. Infinitesimals and Euclid V.4

A real number $\epsilon$ is infinitesimal if it satisfies the formula

$$\forall n \in \mathbb{N} \left[ \text{st}(n) \rightarrow |\epsilon| < \frac{1}{n} \right].$$

Such a number necessarily satisfies $\neg \text{st}(\epsilon)$ if $\epsilon = 0$. The formula above formalizes the violation of Euclid Definition V.4 (see Section 1.5).

A real number smaller in absolute value than some standard real number is called finite, and otherwise infinite. Every nonstandard natural number is infinite.

4.4. Bounded vs unbounded infinity: a formalization

The theory SPOT enables a formalization of the Leibnizian distinction between bounded infinity and unbounded infinity (see Section 2.2) as follows. Unbounded infinity is exemplified by the natural numbers $\mathbb{N}$, whereas bounded infinity is exemplified by $\mu$ for a nonstandard (in Leibnizian terminology, inassignable) element $\mu \in \mathbb{N}$ (thus we have $\neg \text{st}(\mu)$).41 In modern mathematics, the existence of $\mathbb{N}$ as a whole depends on infinitary axioms of set theory, which certainly contradict the part-

40In connection with the law of continuity, Levey observes: ‘It is not hard to think of counterexamples to the law [‘the rules of the finite are found to succeed in the infinite’] and at a minimum the law of continuity requires additional sharpening and guidance to be used correctly in specific cases’ (Levey 2021, 153). However, Levey fails to point out that such sharpening and guidance are indeed provided in Robinson’s classic framework for infinitesimal analysis and its modern axiomatic versions (see the beginning of the current Section 4). Unlike Probst (2018), Levey provides no hint of the existence of an alternative to the syncategorematic reading.

41Bassler claims the following: ‘while it is difficult if not impossible to imagine a line starting at a particular point, going on forever, and then terminating at another point, the idea that two points on a line could be infinitely close to each other seems considerably more palatable. … At any rate, this is much different from Leibniz’s conception that the model for the mathematical infinite is the sequence of natural numbers, which has a beginning but no end’ (Bassler 2008, 145). Arguably the two conceptions are not ‘much different’ since the interval $[0, \mu] \subseteq \mathbb{R}$ would formalize Leibniz’s bounded infinite line. It is therefore not difficult to imagine a line starting at a particular point, terminating at another point, and having infinite length.
whole principle (viewed as axiomatic by Leibniz). Meanwhile, using $\mu$ itself in the context of Skolem’s model (Skolem 1933, 1934) of Peano Arithmetic entails no such foundational woes, since Peano Arithmetic and the extended Zermelo-like theory of finite sets are definitionally equivalent (Tarski and Givant 1987, 225).42

4.5. Discarding negligible terms
Leibniz often exploited the technique of discarding negligible terms (see note 3). His transcendental law of homogeneity in (1710) is a codification of the procedure; see Katz and Sherry (2012) for an analysis. The technique is formalized in terms of the standard part principle, to the effect that if $|x| < r$ for some standard $r$, then there exists a standard $x_0$ such that $x - x_0$ is infinitesimal.

4.6. Curves as infinitesimal polygons
A curve $\alpha(t), \ t \in [0, 1]$ can be approximated by a polygon with a nonstandard (inassignable) number $\mu$ of sides, with vertices at the points $\alpha(\frac{i}{\mu})$ as $i$ runs from 0 to $\mu$. Such approximations are sufficient for calculating the usual geometric entities such as the tangent line and the osculating circle; see Section 2.6. For the corresponding treatment in SIA, see Sections 2.3 and 5.1.

4.7. Parabola as status transitus of ellipses
Leibniz notes that a parabola can be obtained as status transitus from a family of ellipses (see Section 6.1). Consider a family of ellipses $(E_t)$ such that every ellipse passes through the origin and has one focus at $(0, 1)$ and the other at $(0, t + 1)$. The family is parametrized by the distance $t$ between the foci. Then the family includes an internal ellipse $E_m$ for an infinite value $t = \mu$. Taking standard part of the coefficients of a normalized equation of $E_m$, we obtain the equation of a standard parabola.

Other ways of formalizing Leibnizian principles are explored by Forti (2018).

5. Infinitesimals of smooth infinitesimal analysis
In this section we will analyze more closely the details of the foundational approach adopted in SIA, and their ramifications for Arthur’s comparison of Leibnizian calculus with SIA.

5.1. Fictions and equivocations
The intuitionistic logic relied upon in SIA enables an infinitesimal $\varepsilon$ to satisfy

$$\neg\neg(\varepsilon = 0),$$

42Fragments of nonstandard arithmetic are studied by Avigad (2005), Sommer and Suppes (1996), Nelson (1987), Sanders (2020), van den Berg and Sanders (2019), Yokoyama (2010), and others.
or in words: ‘$e$ is not nonzero’ (see e.g. Bell 2008, formula (8.1), 105). On the other hand, one can prove neither ($e = 0$) nor $\neg(e = 0)$.

In this sense it could perhaps be said that the nilpotent infinitesimals of SIA hover at the boundary of existence. Such a mode of existence could be loosely described as fictional (or ‘finessing of a commitment to the existence of infinitesimals’ as Arthur puts it; see Section 2.3). Even if one does use the term fiction to describe them, are such SIA fictions related to the logical fictions (see Section 1.3) as posited by the Ishiguro–Arthur (IA) syncategorematic reading of Leibnizian infinitesimals?

On the IA reading, an infinitesimal $e$ is understood as generated by ordinary real values, say $e_n$, chosen small enough to defeat a pre-assigned error bound. But then the quantity $H$ generated by $H_n = \frac{1}{e_n}$ will surely represent the inverse of the infinitesimal quantity $e$, so that we have a nonzero product $eH = 1$. Thus such an $e$ would necessarily be invertible and therefore provably nonzero: $\neg(e = 0)$, contradicting the SIA assumption (1) that $\neg\neg(e = 0)$; cf. (Bell 2008, formula (8.2), 105). Similar reasoning applies if Leibnizian $e$ is taken to be a variable quantity taking ordinary real values, inevitably leading to invertibility.

Thus Leibnizian infinitesimals, especially on the IA reading, are unlike SIA infinitesimals. Arthur’s claim of similarity between them (see Section 2.3) is largely rhetorical, and amounts to thinly veiled equivocation on the meaning of the qualifier fictional.

5.2. The punctiform issue

An important issue in the history of analysis concerns the nature of the continuum as being punctiform or nonpunctiform; Leibniz endorsed the latter view. In an interesting twist, Arthur (following Bell 2008, 3) speaks of infinitesimals as being nonpunctiform, as noted by Knobloch; see Section 2.3. This tends to obscure the possibility that the exact nature of any continuum Leibniz may have envisioned may be irrelevant to the actual procedures of his infinitesimal calculus. Whether points merely mark locations on the continuum (as was the case for Leibniz) or are more fundamental to the actual make-up of the continuum (as in modern set-theoretic approaches whether of Archimedean or non-Archimedean type) is an issue mainly of foundational ontology that is transverse to Leibniz’s mathematical practice.44 Leibniz may never have envisioned a punctiform continuum, but infinitesimals that are not nonzero but not provably zero (see Section 5.1) have no known source in Leibniz’s writings, either.45

5.3. Variables and static quantities

One of Arthur’s claimed similarities between LC and SIA is ‘the dependence on variables (as opposed to the static quantities of both Standard and Non-standard Analysis)’ (see Section 1.8 for the full quotation). Let us examine Arthur’s static claim.

43Thus, speaking of Leibnizian infinite-sided polygons, Arthur claims that ‘this means that a curve can be construed as an ideal limit of a sequence of such polygons, so that its length $L$ will be the limit of a sequence of sums $ns$ of their sides $s$ as their number $n \to \infty$’ (Arthur 2001, note 4, 393).

44For an analysis of the procedures/ontology distinction see Błaszczyk et al. (2017).

45See Section 2.1 for Arthur’s translation of Leibniz’s endorsement of the law of excluded middle.
The analog of the real line $\mathbb{R}$ of traditional analysis is the SIA line $\mathbb{R}$. Bell defines a part $\Delta$ of the SIA line $\mathbb{R}$ by the formula $\Delta = \{x: x^2 = 0\};$ see Bell (2008, 20). The part $\Delta$ does not reduce to $\{0\}$ as discussed in our Section 5.1. The principle of microaffineness asserts the following:

$$\forall g: \Delta \rightarrow R \exists! b \in R \forall \epsilon \in \Delta, \quad g(\epsilon) = g(0) + b \cdot \epsilon$$

(see Bell 2008, 21). The fact that the for all $\epsilon$ in $\Delta$ is crucial for the uniqueness of $b$.

Arthur notes that ‘[t]he letter $\epsilon$ then denotes a variable ranging over $\Delta’$ (Arthur 2013, 537). This is the only mention of variables in Arthur’s discussion of the SIA approach to the derivative. Concerning SIA infinitesimals, Arthur writes:

The sense in which they are fictions in SIA, however, is that although it is denied that an infinitesimal neighbourhood of a given point, such as 0, reduces to zero, it cannot be inferred from this that there exists any point in the infinitesimal neighbourhood distinct from 0. (Arthur 2013, 572–3)

Apparently the claimed similarity between LC and SIA based on ‘variables’ amounts to the fact that the existence of a unique $b$ as in formula (2) depends crucially on universal quantification $\forall \epsilon$ over the part $\Delta$.

What are we to make of Arthur’s claimed contrast between such variables and what he describes as ‘the static quantities of both Standard and Non-standard Analysis’? It is true that in the classical setting, if a function $f$ is differentiable, the derivative $L = f'(x)$ can be computed from a single infinitesimal $\epsilon = 0$ by taking the standard part. Namely, $L$ is the standard part of $\frac{f(x+\epsilon) - f(x)}{\epsilon}$, see Section 2.6 and Keisler (1986). Here universal quantification over $\epsilon$ is not needed. Such an $\epsilon$ could possibly be described as ‘static’ (or more precisely fixed).

On the other hand, defining differentiability in the classical setting does require universal quantification over the infinitesimal $\epsilon$. Namely, denoting by $\forall^\text{un}$ universal quantification over nonzero infinitesimals and by $\exists^\text{in}_0$ existential quantification over (possibly zero) infinitesimals, we have the following:

$$f'(x) = L \text{ if and only if } \forall^\text{un}_0 \exists^\text{in}_0 \ell \frac{f(x + \epsilon) - f(x)}{\epsilon} = L + \ell.$$  

In this sense, Robinsonian infinitesimals are no more ‘static’ (in an Arthurian sense) than the Lawvere–Kock–Bell ones. Arthur’s rhetorical claim of similarity between Leibniz and SIA on account of ‘variables’ and statics hinges on equivocation on the meaning of the term variable.

6. Law of continuity and status transitus in ‘Cum Prodiisset’

Leibniz’s unpublished text ‘Cum Prodiisset’ dates from around 1701. The most adequate translation of Leibniz’s law of continuity as it appeared in ‘Cum Prodiisset’ was given by Child:
In any supposed [continuous] transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.46 (Leibniz as translated in Child 2005, 147)

Child used the noncommittal term terminus. Meanwhile, both Bos (1974, 56) and Arthur (2013, 562) use the term limit (or limiting case) in their translations. Such a translation risks being presentist in that it is suggestive of the modern notion of limit, confirming Cajori’s grafting diagnosis (see Section 1.1).47

6.1. Status transitus

In the formulation cited above, Leibniz used the expression status terminus. However, what Leibniz is really getting at here is clear from his development that follows the formulation of the law in ‘Cum Prodiisset’. Namely, the key issue is actually what Leibniz refers to as the status transitus:

… a state of transition [status transitus in the original] may be imagined, or one of evanescence, in which indeed there has not yet arisen exact equality or rest or parallelism, but in which it is passing into such a state, that the difference is less than any assignable quantity; (Leibniz as translated by Child in (2005, 149); emphasis on ‘evanescence’ added)

Leibniz goes on to provide some examples:

… also that in this state there will still remain some difference, some velocity, some angle, but in each case one that is infinitely small; and the distance of the point of intersection, or the variable focus, from the fixed focus will be infinitely great, and the parabola may be included under the heading of an ellipse, etc.48 (ibid.)

The status transitus is essentially the evanescent stage, as when, for example, one evaluates an equation of an ellipse at an infinite value of the parameter to produce an ellipse with infinite distance between the foci. The law of continuity asserts the legitimacy of such a procedure. The law as stated here is closely related to Leibniz’s formulation of the law of continuity in Leibniz (1702, 93–4) as quoted by Robinson: ‘The rules of the finite are found to succeed in the infinite’. (Robinson 1966, 266) (see Section 4.2 for a modern formalization). Namely, substituting an infinite value of the parameter produces a status transitus given by a legitimate conic/
ellipse. Then the finite part of the conic/ellipse is indistinguishable from a parabola (see Section 4.7).

6.2. Which things are taken to be equal?
How does Arthur handle the Leibnizian law of continuity? Immediately following his translation of Leibniz’s law of continuity using the presentist ‘limiting case,’ Arthur quotes ‘Cum Prodiisset’ as follows:

Hence it may be seen that in all our differential calculus there is no need to call equal those things that have an infinitely small difference, but those things are taken as equal that have no difference at all, … (Leibniz as translated by Arthur in Arthur 2013, 563)

Here we deliberately interrupted the Leibnizian passage at a comma, for reasons that will become clear presently. At first glance, it may appear that Leibniz is contradicting what he wrote in many texts including his published response to Nieuwentijt (Leibniz 1695b) as well as (Leibniz 1710), to the effect that he is working with a generalized notion of equality up to a negligible term (see e.g. Section 2.6). Note however that the unusually long sentence in ‘Cum Prodiisset’ does not stop with the apparent endorsement of exact equality, but rather continues to list a number of conditions and qualifications. One of the qualifications is that exact equality results only after one performs suitable algebraic simplifications and omissions (such as omitting $dx$ where appropriate):

… provided that the calculation is supposed to have been rendered general, applying equally to the case where the difference is something and to where it is zero; and only when the calculation has been purged as far as possible through legitimate omissions and ratios of non-vanishing quantities until at last application is made to the ultimate case, is the difference assumed to be zero.49 (Leibniz as translated by Arthur in (Arthur 2013), 563–564; emphasis added)

Thus, while the simplification of the differential ratio in the case of the parabola $x^2 = ay$ results in $\frac{2x + dx}{a}$, subsequently discarding the term $dx$ does result in an exact equality $\frac{dy}{dx} = \frac{2x}{a}$.50 The Leibnizian $(d)x$ and $(d)y$ were rendered $dx$ and $dy$ in Bos (1974). The notation $(d)x$ and $(d)y$ refers to assignable quantities whose ratio is the modern derivative.51 RA quote the following Leibnizian passage:

49Arthur’s translation is rather unsatisfactory. Child translated this passage as follows: ‘Hence, it may be seen that there is no need in the whole of our differential calculus to say that those things are equal which have a difference that is infinitely small, but that those things can be taken as equal that have not any difference at all, provided that the calculation is supposed to be general, including both the cases in which there is a difference and in which the difference is zero; and provided that the difference is not assumed to be zero until the calculation is purged as far as is possible by legitimate omissions, and reduced to ratios of non-evanescent quantities, and we finally come to the point where we apply our result to the ultimate case’ (Leibniz as translated by Child in (2005, 151–152).

50See note 3 on discarding terms.

51Levey’s page-and-a-half discussion of this Leibnizian derivation manages to avoid mentioning the crucial Leibnizian distinction between $dx$ and $(d)x$. He concludes that ‘in fact $dx$ stands for a variable finite
But if we want to retain \( dx \) and \( dy \) in the calculation in such a way that they denote non-vanishing quantities even in the ultimate case, let \((d)x\) be assumed to be any assignable straight line whatever; and let the straight line which is to \((d)x\) as \(y\) or \(1X1Y\) is to \(1XT\) be called \((d)y\)…

What has Leibniz accomplished here? Denoting the assignable value of the differential quotient by \(L\), we note that Leibniz chooses an assignable \((d)x\) and then defines a new quantity \((d)y\) to be the product \(L(dx)\). RA claim that the method using \((d)x\) and \((d)y\) ‘differs from the first one in the sense that it does not rely on vanishing quantities’ (Rabouin and Arthur 2020, 439). However, the method most decidedly does depend on vanishing quantities. The Leibnizian \((d)x\) and \((d)y\) may all be assignable but that is not where the substantive part of infinitesimal calculus is. The nontrivial work goes into the evaluation of \(L\) using the appropriate ‘legitimate omissions’, whereas introducing a new quantity by setting \((d)y = L(dx)\) is of lesser mathematical import. RA’s claim to the contrary stems from paying exaggerated attention to the rhetorical content of the surface language of the Leibnizian text, at the expense of the mathematical content. Their claim that ‘the introduction of the Law of Continuity as a postulate … is fully compatible with the ‘syncategorematic’ view’ (Rabouin and Arthur 2020, 404) remains unsubstantiated.

6.3. Are inassignable quantities dispensable?

RA claim to provide a ‘new reading’ (Rabouin and Arthur 2020, 404) of the mature texts by Leibniz on the foundations of the calculus, in particular the famous text ‘Cum Prodiisset’. They claim that Leibniz presents his strategy based on the Law of Continuity as being provably rigorous according to the accepted standards in keeping with the Archimedean axiom and that a recourse to inassignable quantities is therefore avoidable. They base their claim on the Leibnizian statement that one can switch to assignable quantities \((d)x, (d)y\) that keep the same ratio as \(dx, dy\).

However, as noted in Section 6.2, the nontrivial part of the calculus consists in determining the assignable value of the differential ratio, involving omission of negligible terms. Therefore we cannot agree with RA’s assertion that the Leibnizian strategy based on the Law of Continuity is independent of inassignable quantities.

Both in his discussion of the example of the parabola and in the passage quoted by Arthur, Leibniz speaks of omitting terms. To elaborate, Leibniz specifically speaks of a stage where ‘the calculation has been purged as far as possible through legitimate omissions.’ Namely, these are the stages in the calculation where negligible terms are omitted. Only afterwards does one attain a stage where ‘those things are taken as equal that have no difference at all.’

This important feature of the calculations found in Leibniz does not occur in a modern epsilon-delta definition of the concept of limit, where

\[
\lim_{x \to c} f(x) = L \quad \text{is defined by the quantifier formula}
\]

\[
(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)[0 < |x - c| < \delta \implies |f(x) - f(c)| < \epsilon].
\]

quantity, and its behavior reflects precisely the fact – ensured by the continuity of the curve \(AY\) – that the difference between the abscissas can be taken as small as one wishes, all the way to zero’ (Levey 2021, 152). However, the conclusion applies only to the \((d)x\), not the \(dx\) as per Levey.
No omission ever takes place here. A freshman who attempts, Leibniz-style, to omit terms on a test in a course following the Epsilontik approach will surely lose part of the credit. Discarding terms is not only not a feature of the modern definition in terms of limits, but is on the contrary seen as a typical freshman calculus error. On the other hand, from the viewpoint of Robinson’s framework, Leibniz’s procedure admits of straightforward formalization in terms of taking the standard part (see Sections 2.6 and 4.5). Thus we cannot agree with Arthur’s appreciation:

It can be appreciated, I think, how close [Leibniz’s approach] is to a modern justification of differentiation in terms of limits. (Arthur 2013, 564; emphasis added)

Arthur’s appreciation is yet another case of the grafting of the modern theory of limits, in a mandatory Archimedean context (symptomatic of butterfly-model thinking; see Section 1.7), on the calculus of Leibniz.

7. Conclusion

We have examined three interpretations of the procedures of the Leibnizian calculus in the current literature, related respectively to frameworks developed by Weierstrass, Lawvere, and Robinson. Arthur pursued comparisons of the Leibnizian procedures to the first two frameworks. We have argued that both of Arthur’s readings of the procedures of the Leibnizian calculus are less successful than the interpretation in terms of the procedures of Robinson’s framework for analysis with infinitesimals.

Rabouin and Arthur claim to ‘show that by 1676 Leibniz had already developed an interpretation from which he never wavered, according to which infinitesimals, like infinite wholes, cannot be regarded as existing because their concepts entail contradictions.’ Their position is at odds with that elaborated by Eberhard Knobloch in a number of early texts, concerning the distinction between bounded infinity and unbounded infinity in Leibniz. While it is true, as RA claim, that Leibniz’s fictionalist view as elaborated in De Quadratura Arithmetica is consistent with later views (e.g. those expressed in the February 1702 letter to Varignon), not only did RA not show that Leibniz held infinitesimals to be contradictory but in fact RA appear to admit (in the passage quoted in our first epigraph) that Leibniz never expressed such an alleged conviction.

Leibniz held that infinitesimals are unlike infinite wholes, in that infinitesimals pertain to bounded infinity (infinita terminata) whereas infinite wholes pertain to unbounded infinity and contradict the part-whole principle.

RA appear to base their conclusions on an analysis of the Leibnizian texts that dwells excessively on the rhetorical content of the surface language, and takes insufficient note of the mathematical content. This is apparent in their claim that the Leibnizian method exploiting the law of continuity was independent of the use of infinitesimals. As we have shown, the claim does not hold up mathematically. The same is true of Arthur’s comparison of Leibnizian infinitangular polygons with curves in Smooth Infinitesimal Analysis. To answer Probst’s question (see Section 1), the reading based on Robinson’s framework (or its axiomatic conceptualisations such as SPOT (Hrbacek and Katz 2020)) is arguably more successful on a number of issues, including infinitangular polygons and fictionality of infinitesimals.

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ORCID
Piotr Błaszczyk http://orcid.org/0000-0002-3501-3480
Robert Ely http://orcid.org/0000-0002-2477-3484
Mikhail G. Katz http://orcid.org/0000-0002-3489-0158
Karl Kuhlemann http://orcid.org/0000-0002-7713-4782

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