Accelerated Method for Stochastic Composition Optimization with Nonsmooth Regularization

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Abstract

Stochastic composition optimization draws much attention recently and has been successful in many emerging applications of machine learning, statistical analysis, and reinforcement learning. In this paper, we focus on the composition problem with nonsmooth regularization penalty. Previous works either have slow convergence rate, or do not provide complete convergence analysis for the general problem. In this paper, we tackle these two issues by proposing a new stochastic composition optimization method for composition problem with nonsmooth regularization penalty. In our method, we apply variance reduction technique to accelerate the speed of convergence. To the best of our knowledge, our method admits the fastest convergence rate for stochastic composition optimization: for strongly convex composition problem, our algorithm is proved to admit linear convergence; for general composition problem, our algorithm significantly improves the state-of-the-art convergence rate from \( O(T^{-1/2}) \) to \( O((n_1 + n_2)^{2/3}T^{-1}) \). Finally, we apply our proposed algorithm to portfolio management and policy evaluation in reinforcement learning. Experimental results verify our theoretical analysis.

1 Introduction

Stochastic composition optimization draws much attention recently and has been successful in addressing many emerging applications of different areas, such as reinforcement learning [2, 14], statistical learning [13] and risk management [4]. The authors in [13, 14] proposed composition problem, which is the composition of two expected-value functions:

\[
\min_{x \in \mathbb{R}^N} \underbrace{\mathbb{E}_i F_i(\mathbb{E}_j G_j(x)) + h(x)}_{f(x)},
\]

(1)

where \( G_j(x) : \mathbb{R}^N \rightarrow \mathbb{R}^M \) are inner component functions, \( F_i(y) : \mathbb{R}^M \rightarrow \mathbb{R} \) are outer component functions. The regularization penalty \( h(x) \) is a closed convex function but not necessarily smooth. In reality, we usually
solve the finite-sum scenario for composition problem (1), and it can be represented as follows:

$$
\min_{x \in \mathbb{R}^N} H(x) = \min_{x \in \mathbb{R}^N} \frac{1}{n_1} \sum_{i=1}^{n_1} F_i \left( \frac{1}{n_2} \sum_{j=1}^{n_2} G_j(x) \right) + h(x),
$$

(2)

where it is defined that $F(y) = \frac{1}{n_1} \sum_{i=1}^{n_1} F_i(y)$ and $G(x) = \frac{1}{n_2} \sum_{j=1}^{n_2} G_j(x)$. Throughout this paper, we mainly focus on the case that $F_i$ and $G_j$ are smooth. However, we do not require that $F_i$ and $G_j$ have to be convex.

Minimizing the composition of expected-value functions (1) or finite-sum functions (2) is challenging. Classical stochastic gradient method (SGD) and its variants are well suited for minimizing traditional finite-sum functions (1). However, they are not directly applicable to the composition problem. To apply SGD, we need to compute the unbiased sampling gradient ($\nabla G_j(x))^T \nabla F_i(G(x))$ of problem (2), which is time-consuming when $G(x)$ is unknown. Evaluating $G(x)$ requires traversing all inner component functions, which is unacceptable to compute in each iteration if $n_2$ is a large number.

Table 1: The table shows the comparisons of SCGD, Accelerated SCGD, ASC-PG, Compositional-SVRG-1, Compositional-SVRG-2, com-SVR-ADMM and our VRSC-PG. For fair comparison, we consider query complexity in the convergence rate. We define that one query of Sampling Oracle (SO) has three cases: (1) Given $x \in \mathbb{R}^N$ and $i \in \{1, 2, ..., n_2\}, SO$ returns $G_j(x) \in \mathbb{R}^M$; (2) Given $x \in \mathbb{R}^N$ and $j \in \{1, 2, ..., n_2\}$, SO returns $\nabla G_j(x) \in \mathbb{R}^{M \times N}$; (3) Given $y \in \mathbb{R}^M$ and $i \in \{1, 2, ..., n_1\}$, SO returns $\nabla F_i(y) \in \mathbb{R}^M$. $T$ denotes the total number of iterations, $\kappa$ is condition number and $0 < \rho < 1$.

| Algorithm                        | $h(x) \neq 0$ | Strongly Convex | General Problem         |
|----------------------------------|---------------|-----------------|------------------------|
| SCGD [13]                        | ×             | $O(T^{-2/3})$   | $O(T^{-1/4})$          |
| Accelerated SCGD [13]            | ×             | $O(T^{-4/5})$   | $O(T^{-2/7})$          |
| Compositional-SVRG-1 [8]         | ×             | $O(\rho^{n_1+n_2+\kappa^2})$ | -                      |
| Compositional-SVRG-2 [8]         | ×             | $O(\rho^{n_1+n_2+\kappa^2})$ | -                      |
| ASC-PG [14]                      | ✓             | $O(T^{-4/5})$   | $O(T^{-4/9})$          |
| ASC-PG (if $G_j(x)$ are linear)  [14] | ✓             | $O(T^{-1})$     | $O(T^{-1/2})$          |
| com-SVR-ADMM [16]                | ✓             | $O(\rho^{n_1+n_2+\kappa^2})^{1/4}$ | -                      |
| VRSC-PG (Our)                    | ✓             | $O(\rho^{n_1+n_2+\kappa^2})$ | $O((n_1 + n_2)^{2/3}T^{-1})$ |

In [13], the authors considered the problem (1) with $h(x) = 0$ and proposed stochastic compositional gradient descent algorithm (SCGD) which is the first stochastic method for composition problem. In their paper, they proved that the convergence rate of SCGD for strongly convex composition problem is $O(T^{-2/3})$, and for general problem is $O(T^{-1/4})$. They also proposed accelerated SCGD by using Nesterov smoothing technique [9] which is proved to admit faster convergence rate. SCGD has constant query complexity per iteration, however, their convergence rate is far worse than full gradient method because of the noise induced by sampling gradients. Recently, variance reduction technique [7] was applied to accelerate the convergence of stochastic composition optimization. [8] first utilized the variance reduction technique and proposed two variance reduced stochastic compositional gradient descent methods (Compositional-SVRG-1 and Compositional-SVRG-2). Both methods are proved to admit linear convergence rate. However, the methods proposed in [13] and [8] are not applicable to composition problem with nonsmooth regularization penalty.
Composition problem with nonsmooth regularization was then considered in \cite{14, 16}. In \cite{14}, the authors proposed accelerated stochastic compositional proximal gradient algorithm (ASC-PG). They proved that the optimal convergence rate of ASC-PG for strongly convex problem and general problem is $O(T^{-1})$ and $O(T^{-1/2})$ respectively. However, ASC-PG suffers from slow convergence because of the noise of the sampling gradients. \cite{16} proposed com-SVR-ADMM using variance reduction. Although com-SVR-ADMM admits linear convergence for strongly convex composition problem, it is not optimal. Besides, they did not analyze the convergence for general (nonconvex) composition problem either. We review the convergence rate of stochastic composition optimization in Table 1.

In this paper, we propose variance reduced stochastic compositional proximal gradient method (VRSC-PG) for composition problem with nonsmooth regularization penalty. Applying the variance reduction technique to composition problem is nontrivial because the optimization procedure and convergence analysis are essentially different. We investigate the convergence rate of our method: for strongly convex problem, we prove that VRSC-PG has linear convergence rate $O\left(\frac{T}{\rho^{n_1+n_2+\kappa^2}}\right)$, which is faster than com-SVR-ADMM; For general problem, sometimes nonconvex, VRSC-PG significantly improves the state-of-the-art convergence rate of ASC-PG from $O(T^{-1/2})$ to $O((n_1+n_2)^{2/3}T^{-1})$. To the best of our knowledge, our result is the new benchmark for stochastic composition optimization. We further evaluate our method by applying it to portfolio management and reinforcement learning. Experimental results verify our theoretical analysis.

2 Preliminary

In this section, we briefly review stochastic composition optimization and proximal stochastic variance reduced gradient.

2.1 Stochastic Composition Optimization

The objective function of the stochastic composition optimization is the composition of expected-value \cite{1} or finite-sum \cite{2} functions, which is much more complicated than traditional finite-sum problem. The full gradient of composition problem using chain rule is $\nabla f(x) = (\nabla G(x))^T \nabla F(G(x))$. Given $x$, applying the classical stochastic gradient descent method in constant queries to compute the unbiased sampling gradient $(\nabla G(x))^T \nabla F_i(G(x))$ is not available, when $G(x)$ is unknown yet. In problem \cite{2}, evaluating $G(x)$ is time-consuming which requires $n_2$ queries in each iteration. Therefore, classical SGD is not applicable to composition optimization. In \cite{13}, the authors proposed the first stochastic compositional gradient descent (SCGD) for minimizing the stochastic composition problem \cite{1} with $h(x) = 0$. In their paper, they proposed to use an auxiliary variable $y$ to approximate $G(x)$. In each iteration $t$, we store $x_t$ and $y_t$ in memory. SCGD are briefly described in Algorithm \cite{1}.

In the algorithm, $\alpha_t$ and $\beta_t$ are learning rate. Both of them are decreasing to guarantee convergence because of the noise induced by sampling gradients. In their paper, they supposed that $x \in \mathcal{X}$. In each iteration, $x$ is projected to $\mathcal{X}$ after step 4. Furthermore, the authors proposed Accelerated SCGD by applying Nesterov smoothing \cite{9}, which is proved to converge faster than basic SCGD.

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\footnote{1 In \cite{16}, their result is $O(\rho^{n_1+n_2+\kappa^2})$. We prove that to get linear convergence, it must be satisfied that $A$ and $m$ are proportional to $\kappa^2$, which is not included in their paper. Check Remark \cite{2} in supplementary material.}
Algorithm 1 SCGD

1: Initialize $x_0 \in \mathbb{R}^N, y_0 \in \mathbb{R}^M$;
2: for $t = 0, 1, 2, \ldots, T - 1$ do
3: Uniformly sample $j$ from $\{1, 2, \ldots, n_2\}$ with replacement and query $G_j(x_t)$ and $\nabla_j G(x_t)$; $\triangleright$ 2 queries
4: Update $y_{t+1}$ using:

$$y_{t+1} \leftarrow (1 - \beta_t)y_t + \beta_t G_j(x_t);$$  (3)

5: Uniformly sample $i$ from $\{1, 2, \ldots, n_1\}$ with replacement and query $\nabla F_i(y_{t+1})$; $\triangleright$ 1 query
6: Update $x_{t+1}$ using:

$$x_{t+1} \leftarrow x_t - \alpha_t (\nabla_G j(x_t))^T \nabla F_i(y_{t+1});$$  (4)

7: end for

2.2 Proximal Stochastic Variance Reduced Gradient

Stochastic variance reduced gradient (SVRG) \cite{Johnson13} was proposed to minimize finite-sum functions:

$$\min_{x \in \mathbb{R}^N} \frac{1}{n_1} \sum_{i=1}^{n_1} f_i(x),$$  (5)

where component functions $f_i(x) : \mathbb{R}^N \to \mathbb{R}$. In large-scale optimization, SGD and its variants use unbiased sampling gradient $\nabla f_i(x)$ as the approximation of the full gradient, which only requires one query in each iteration. However, the variance induced by sampling gradients forces us to decrease learning rate to make the algorithm converge. Suppose $x^*$ is the optimal solution to problem (5), full gradient $\frac{1}{n_1} \sum_{i=1}^{n_1} \nabla f_i(x^*) = 0$, while sampling gradient $\nabla f_i(x^*) \neq 0$. We should decrease learning rate, otherwise the convergence of the objective function value cannot be guaranteed. However, the decreasing learning rate makes SGD converge very slow at the same time. For example, if problem (5) is strongly convex, gradient descent method (GD) converges with linear rate, while SGD converges with a learning rate at $O(T^{-1})$. Reducing the variance is one of the most important ways to accelerate SGD, and it has been widely applied to large-scale optimization \cite{Shakeel15, Johnson13, Zhao16}. In \cite{Johnson13}, the authors proposed to use the aggregation of old sampling gradients to regulate the current sampling gradient. In \cite{Shakeel15}, the authors considered the nonsmooth regularization penalty $h(x) \neq 0$ and proposed proximal stochastic variance reduced gradient (Proximal SVRG). Proximal SVRG is briefly described in Algorithm 2. In their paper, they used $v_t$ as the approximation of full gradient, where $E v_t = 0$.

It was also proved that the variance of $v_t$ converges to zero: $\lim_{t \to \infty} \mathbb{E}\|v_t - \frac{1}{n_1} \sum_{i=1}^{n_1} \nabla f_i(x_t)\|^2 \to 0$. Therefore, we can keep learning rate $\eta$ constant in the procedure. In step 7, $\text{Prox}_{\eta h_{i}()}(x)$ denotes proximal operator. With the definition of proximal mapping, we have:

$$\text{Prox}_{\eta h_{i}()}(x) = \arg \min_{x'} (h(x') + \frac{1}{\eta}\|x' - x\|^2),$$  (6)

Convergence analysis and experimental results confirmed that Proximal SVRG admits linear convergence in expectation for strongly convex optimization. In \cite{Azevedo16}, the authors proved that Proximal SVRG has sublinear convergence rate of $O(n_1^{2/3}T^{-1})$ when $f_i(x)$ is nonconvex.
Algorithm 2 Proximal SVRG

1: Initialize $\tilde{x}^0 \in \mathbb{R}^N$;
2: for $s = 0, 1, 2, \ldots, S - 1$ do
3: \hspace{1em} $x_s^{s+1} \leftarrow \tilde{x}^s$;
4: \hspace{1em} $f' \leftarrow \frac{1}{n_1} \sum_{i=1}^{n_1} \nabla f_i(\tilde{x}^s)$; ▷ $n_1$ queries
5: \hspace{1em} for $t = 0, 1, 2, \ldots, m - 1$ do
6: \hspace{2em} Uniformly sample $i$ from $\{1, 2, \ldots, n_1\}$ with replacement and query $\nabla f_i(x_t^{s+1})$ and $\nabla f_i(\tilde{x}^s)$; ▷ 2 queries
7: \hspace{2em} Update $\nu_t^{s+1}$ using:
8: \hspace{2em} $\nu_t^{s+1} \leftarrow \nabla f_i(x_t^{s+1}) - \nabla f_i(\tilde{x}^s) + f'$;
9: \hspace{1em} end for
10: \hspace{1em} $x_t^{s+1} \leftarrow \text{Prox}_{\eta h(\cdot)}(x_t^{s+1} - \eta \nu_t^{s+1})$;
11: end for

3 Variance Reduced Stochastic Compositional Proximal Gradient

In this section, we propose variance reduced stochastic compositional proximal gradient method (VRSC-PG) for solving the finite-sum composition problem with nonsmooth regularization penalty (2).

The description of VRSC-PG is presented in Algorithm 3. Similar to the framework of Proximal SVRG \cite{proximal-svrg}, our VRSC-PG also has two-layer loops. At the beginning of the outer loop $s$, we keep a snapshot of the current model $\tilde{x}^s$ in memory and compute the full gradient:

$$\nabla f(\tilde{x}^s) = \frac{1}{n_2} \sum_{j=1}^{n_2} (\nabla G_j(\tilde{x}^s))^T \frac{1}{n_1} \sum_{i=1}^{n_1} \nabla F_i(G^s),$$

(9)

where $G^s = \frac{1}{n_2} \sum_{j=1}^{n_2} G_j(\tilde{x}^s)$ denotes the value of the inner functions and $\nabla G(\tilde{x}^s) = \frac{1}{n_2} \sum_{j=1}^{n_2} \nabla G_j(\tilde{x}^s)$ denotes the gradient of inner functions. Computing the full gradient of $f(x)$ in problem (2) requires $(n_1 + 2n_2)$ queries.

To make the number of queries in each inner iteration irrelevant to $n_2$, we need to keep $\hat{G}_t^{s+1}$ and $\nabla \hat{G}_t^{s+1}$ in memory to work as the estimates of $G(x_t^{s+1})$ and $\nabla G(x_t^{s+1})$ respectively. In our algorithm, we query $G_{A_t}(x_t^{s+1})$ and $G_{A_t}(\tilde{x}^s)$, then $\hat{G}_t^{s+1}$ is evaluated as follows:

$$\hat{G}_t^{s+1} = G^s - \frac{1}{A} \sum_{1 \leq j \leq A} (G_{A_t[j]}(\tilde{x}^s) - G_{A_t[j]}(x_t^{s+1})),$$

(10)

where $A_t[j]$ denotes element $j$ in the set $A_t$ and $|A_t| = A$. The elements of $A_t$ are uniformly sampled from $\{1, 2, \ldots, n_2\}$ with replacement. In \cite{variance-reduced}, we reduce the variance of $G_{A_t}(x_t^{s+1})$ by using $G^s$ and $G_{A_t}(\tilde{x}^s)$. Similarly, we sample $B_t$ with size $B$ from $\{1, 2, \ldots, n_2\}$ uniformly with replacement, and query $\nabla G_{B_t}(x_t^{s+1})$ and $\nabla G_{B_t}(\tilde{x}^s)$. The estimation of $\nabla G(x_t^{s+1})$ is evaluated as follows:

$$\nabla \hat{G}_t^{s+1} = \nabla G(\tilde{x}^s) - \frac{1}{B} \sum_{1 \leq j \leq B} (\nabla G_{B_t[j]}(\tilde{x}^s) - \nabla G_{B_t[j]}(x_t^{s+1}))$$

(11)
where $B_t[j]$ denotes element $j$ in the set $B_t$ and $|B_t| = B$. It is important to note that $A_t$ and $B_t$ are independent. Computing $\tilde{G}_t^{s+1}$ and $\nabla \tilde{G}_t^{s+1}$ requires $(2A + 2B)$ queries in each inner iteration.

Now, we are able to compute the estimate of $\nabla f(\hat{x}^{s+1})$ in inner iteration $t$ as follows:

\[
v_t^{s+1} = \frac{1}{b_1} \sum_{i_t \in I_t} \left( \left( \nabla \tilde{G}_t^{s+1} \right)^T \nabla F_{i_t}(\tilde{G}_t^{s+1}) \right) - \left( \nabla G(\hat{x}^s) \right)^T \nabla F_{i_t}(G^s) + \nabla f(\hat{x}^s),
\]

where $I_t$ is a set of indexes uniformly sampled from $\{1, 2, ..., n_1\}$ and $|I_t| = b_1$. As per (12), we need to query $\nabla F_{i_t}(\tilde{G}_t^{s+1})$ and $\nabla F_{i_t}(G^s)$, and it requires $2b_1$ queries. Finally, we update the model with proximal operator:

\[
x_t^{s+1} = \text{Prox}_{\eta v_t^{s+1}} \left( x_t^{s+1} - \eta v_t^{s+1} \right),
\]

where $\eta$ is the learning rate.

**Algorithm 3 VRSC-PG**

**Input:** The total number of iterations in the inner loop $m$, the total number of iterations in the outer loop $S$, the size of the mini-batch sets $A, B$ and $b_1$, learning rate $\eta$.

1. Initialize $\tilde{x}^0 \in \mathbb{R}^N$;
2. for $s = 0, 1, 2, \cdots, S - 1$ do
3. 
4. $x_0^{s+1} \leftarrow \tilde{x}^s$;
5. $G^s \leftarrow \frac{1}{n_2} \sum_{j=1}^{n_2} G_j(\tilde{x}^s)$;   \hspace{1cm} $\triangleright$ $n_2$ queries
6. $\hat{v}_t^{s+1} \leftarrow \frac{1}{n_2} \sum_{j=1}^{n_2} \nabla G_j(\tilde{x}^s)$; \hspace{1cm} $\triangleright$ $n_2$ queries
7. Compute the full gradient $\nabla f(\hat{x}^s)$ using (9); \hspace{1cm} $\triangleright$ $n_1$ queries
8. for $t = 0, 1, 2, \cdots, m - 1$ do
9. 
10. $\tilde{v}_t^{s+1} \leftarrow \tilde{v}_t^{s+1}$ using (10); \hspace{1cm} $\triangleright$ $2A$ queries
11. Uniformly sample $A_t$ from $\{1, 2, ..., n_2\}$ with replacement and $|A_t| = A$;
12. Uniformly sample $B_t$ from $\{1, 2, ..., n_2\}$ with replacement and $|B_t| = B$;
13. Update $\tilde{G}_t^{s+1}$ using (11); \hspace{1cm} $\triangleright$ $2B$ queries
14. Update $\tilde{v}_t^{s+1}$ using (12); \hspace{1cm} $\triangleright$ $2b_1$ queries
15. Update model $x_t^{s+1}$ using:
16. $x_t^{s+1} \leftarrow \text{Prox}_{\eta v_t^{s+1}} \left( x_t^{s+1} - \eta v_t^{s+1} \right)$ \hspace{1cm} (14)
17. end for
18. $\hat{x}^{s+1} \leftarrow x_{m+1}^{s+1}$;
19. end for

4 Convergence Analysis

In this section, we prove that (1) VRSC-PG admits linear convergence rate for the strongly convex problem; (2) VRSC-PG admits sublinear convergence rate $O((n_1 + n_2)^{2/3}T^{-1})$ for the general problem. To the best of our knowledge, both of them are the best results so far. Following are the assumptions commonly used for stochastic composition optimization [13,14,8].
Strongly convex: To analyze the convergence of VRSC-PG for the strongly convex composition problem, we assume that the function $f$ is $\mu$-strongly convex.

**Assumption 1** The function $f(x)$ is $\mu$-strongly convex. Therefore $\forall x$ and $\forall y$, we have:

$$\|\nabla f(x) - \nabla f(y)\| \geq \mu\|x - y\|. \hspace{1cm} (15)$$

Equivalently, $\mu$-strongly convexity can also be written as follows:

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2}\|x - y\|^2. \hspace{1cm} (16)$$

Lipschitz Gradient: We assume that there exist Lipschitz constants $L_F$, $L_G$ and $L_f$ for $\nabla F_i(x)$, $\nabla G_j(x)$ and $\nabla f(x)$ respectively.

**Assumption 2** There exist constants $L_F$, $L_G$ and $L_f$ for $\nabla F_i(x)$, $\nabla G_j(x)$ and $\nabla f(x)$ satisfying that $\forall x$, $\forall y$, $\forall i \in \{1, \cdots, n_1\}$, $\forall j \in \{1, \cdots, n_2\}$:

$$\|\nabla F_i(x) - \nabla F_i(y)\| \leq L_F\|x - y\|, \hspace{1cm} (17)$$

$$\|\nabla G_j(x) - \nabla G_j(y)\| \leq L_G\|x - y\|, \hspace{1cm} (18)$$

$$\| (\nabla G_j(x))^T \nabla F_i(G(x)) - (\nabla G_j(y))^T \nabla F_i(G(y))\| \leq L_f\|x - y\|. \hspace{1cm} (19)$$

As proved in [8], according to (19), we have:

$$\|\nabla f(x) - \nabla f(y)\| \leq L_f\|x - y\|, \forall x, \forall y. \hspace{1cm} (20)$$

Equivalently, (20) can also be written as follows:

$$f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle + \frac{L_f}{2}\|x - y\|^2, \forall x, \forall y \hspace{1cm} (21)$$

Bounded gradients: We assume that the gradients $\nabla F_i(x)$ and $\nabla G_j(x)$ are upper bounded.

**Assumption 3** The gradients $\nabla F_i(x)$ and $\nabla G_j(x)$ have upper bounds $B_F$ and $B_G$ respectively.

$$\|\nabla F_i(x)\| \leq B_F, \forall x, \forall i \in \{1, \cdots, n_1\} \hspace{1cm} (22)$$

$$\|\nabla G_j(x)\| \leq B_G, \forall x, \forall j \in \{1, \cdots, n_2\} \hspace{1cm} (23)$$

Note that we do not need the strong convexity assumption when we analyze the convergence of VRSC-PG for the general problem.

### 4.1 Strongly Convex Problem

In this section, we prove that our VRSC-PG admits linear convergence rate for strongly convex finite-sum composition problem with nonsmooth penalty regularization [4]. We need Assumptions 1, 2 and 3 in this section. Unlike Prox-SVRG in [15], the estimated $v_t^{s+1}$ is biased, i.e., $E_{r_t, A_t, B_t}[v_t^{s+1}] \neq \nabla f(x_t^{s+1})$. It makes the theoretical analysis for proving the convergence rate of VRSC-PG more challenging than the analysis in [15]. In spite of this, we can demonstrate that $E\|v_t^{s+1} - \nabla f(x_t^{s+1})\|^2$ is upper bounded as well.
Lemma 1 Let $x^*$ be the optimal solution to problem (2) $H(x)$ such that $x^* = \arg \min_{x \in \mathbb{R}^n} H(x)$. We define $\gamma = \left( \frac{64}{\mu} \left( \frac{B_1^2 L_2^2}{B_2^2} + \frac{B_2^2 L_3^2}{A^2} \right) + 8 L_f \right)$. Supposing Assumptions 1, 2 and 3 hold, from the definition of $v_t^{s+1}$ in (12), the following inequality holds that:

$$
\mathbb{E}\|v_t^{s+1} - \nabla f(x_t^{s+1})\|^2 \leq \gamma \left[ H(x_t^{s+1}) - H(x^*) + H(\hat{x}^s) - H(x^*) \right]
$$

(24)

Therefore, when $x_t^{s+1}$ and $\hat{x}^s$ converges to $x^*$, $\mathbb{E}\|v_t - \nabla f(x_t^{s+1})\|^2$ also converges to zero. Thus, we can keep learning rate constant, and obtain faster convergence.

Theorem 1 Suppose Assumptions 1, 2, and 3 hold. We let $x^* = \arg \min_{x \in \mathbb{R}^n} H(x)$, if $m, A, B$ and $\eta$ are selected properly so that $\rho < 1$, where $\rho$ is defined as follows:

$$
\rho = \frac{\frac{2}{\mu} + 2\eta \left( \frac{6\eta L_f + \left( \frac{B_1^2 L_2^2}{B_2^2} + \frac{B_2^2 L_3^2}{A^2} \right) \frac{32}{\mu} \left( \frac{B_1^2 L_2^2}{B_2^2} + \frac{B_2^2 L_3^2}{A^2} \right) \right)}{2\eta \left( \frac{B_1^2 L_2^2}{B_2^2} + \frac{B_2^2 L_3^2}{A^2} \right)} (m + 1)}
$$

(25)

we can prove that our VRSC-PG admits linear convergence rate:

$$
\mathbb{E}H(\hat{x}^s) - H(x^*) \leq \rho^S \left( \mathbb{E}H(x^0) - H(x^*) \right)
$$

(26)

As per Theorem 1, we need to choose $\eta, m, A$ and $B$ properly to make $\rho < 1$. We provide an example to show how to select these parameters.

Corollary 1 According to Theorem 1, we set $\eta, m, A$ and $B$ as follows:

$$
\eta = \frac{1}{96 L_f}
$$

(27)

$$
m = 16 \left( 1 + \frac{96 L_f}{\mu} \right)
$$

(28)

$$
A = \frac{2048 B_1^4 L_2^2}{\mu^2}
$$

(29)

$$
B = \frac{2048 B_1^2 L_2^2}{\mu^2}
$$

(30)

we have the following linear convergence rate for VRSC-PG:

$$
\mathbb{E}H(\hat{x}^s) - H(x^*) \leq \left( \frac{2}{3} \right)^S \left( \mathbb{E}H(x^0) - H(x^*) \right)
$$

(31)

Remark 1 According to Theorem 1, in order to obtain $\mathbb{E}H(\hat{x}^s) - H(x^*) \leq \varepsilon$, the number of stages $S$ is required to satisfy:

$$
S \geq \log \frac{\mathbb{E}H(x^0) - H(x^*)}{\varepsilon} / \log \frac{1}{\rho}
$$

(32)

As per Algorithm 3 and the definition of Sampling Oracle in [14], to make the objective value gap $\mathbb{E}H(\hat{x}^s) - H(x^*) \leq \varepsilon$, the total query complexity we need to take is $O \left( (n_1 + n_2 + m(A + B + b_1)) \log \left( \frac{1}{\varepsilon} \right) \right) = O \left( (n_1 + n_2 + \kappa^3) \log \left( \frac{1}{\varepsilon} \right) \right)$, where we let $\kappa = \max \left\{ \frac{L_1}{\mu}, \frac{L_2}{\mu}, \frac{L_3}{\mu} \right\}$ and $b_1$ can be smaller than or proportional to $\kappa^2$. It is better than com-SVR-ADMM[16] whose total query complexity is $O \left( (n_1 + n_2 + \kappa^4) \log \left( \frac{1}{\varepsilon} \right) \right)$.
4.2 General Problem

In this section, we prove that VRSC-PG admits a sublinear convergence rate $O(T^{-1})$ for the general finite-sum composition problem with nonsmooth regularization penalty. It is much better than the state-of-the-art method ASC-PG [14] whose optimal convergence rate is $O(T^{-1/2})$. In this section, we only need Assumption 2 and 3. The unbiased $v_{t+1}$ makes our analysis nontrivial and it is much different from previous analysis for finite-sum problem [11]. In our proof, we define $G_\eta(x) = \frac{1}{\eta} (x - \text{Prox}_{\eta h}(x - \nabla f(x)))$.

**Theorem 2** Suppose Assumptions 2 and 3 hold. Let $x^*$ be the optimal solution to problem (2), we have $x^* = \arg\min_{x \in \mathbb{R}^N} H(x)$. If $m, A, B, b_1$ and $\eta$ are selected properly such that:

$$
4\left( \frac{\eta m^2 L_f^2}{b_1} + \frac{2\eta m^2 B_G^4 L_f^2}{A} + \frac{2\eta m^2 B_G^2 L_f}{B} \right) + \frac{L_f}{2} \leq \frac{1}{2\eta},
$$

then the following inequality holds that:

$$
\mathbb{E}\|G_\eta(x_a)\|^2 \leq \frac{2}{(1 - 2\eta L_f)\eta} \frac{H(\tilde{x}^0) - H(x^*)}{T}
$$

where $x_a$ is uniformly selected from $\{\{x_{t+1}^m\}_{t=0}^{m-1}\}_{t=0}^{S-1}$ and $T$ is a multiple of $m$. As per Theorem 2, we need to choose $m, A, B, b_1$ and $\eta$ appropriately to make condition (33) satisfied. We provide an example to show how to select these parameters.

**Corollary 2** According to Theorem 2 we let $m = \left\lfloor \left( n_1 + n_2 \right)^{\frac{1}{2}} \right\rfloor$, $\eta = \frac{1}{4L_f}$, $b_1 = (n_1 + n_2)^{\frac{2}{3}}$ and $T$ be a multiple of $m$, it is easy to know that if $A$ and $B$ are lower bounded:

$$
A \geq \frac{8m^2 B_G^4 L_f^2}{L_f}
$$

$$
B \geq \frac{8m^2 B_G^2 L_f^2}{L_f}
$$

we can obtain sublinear convergence rate for VRSC-PG:

$$
\mathbb{E}\|G_\eta(x_a)\|^2 \leq 16L_f \frac{H(\tilde{x}^0) - H(x^*)}{T}
$$

As per Algorithm 3 and the definition of Sampling Oracle in [14], to obtain $\varepsilon$-accurate solution, $\mathbb{E}\|G_\eta(x_a)\|^2 \leq \varepsilon$, the total query complexity we need to take is $O(n_1 + n_2 + A + B + b_1) = O\left( n_1 + n_2 + \frac{(n_1 + n_2)^{2/3}}{\varepsilon} \right)$, where $A, B$ and $b_1$ are proportional to $(n_1 + n_2)^{2/3}$. Therefore, our method improves the state-of-the-art convergence rate of stochastic composition optimization for general problem from $O(T^{-1/2})$ (Optimal convergence rate for ASC-PG) to $O\left( (n_1 + n_2)^{2/3}T^{-1} \right)$.

5 Experimental Results

We conduct two experiments to evaluate our proposed method: (1) application to portfolio management; (2) application to policy evaluation in reinforcement learning.

In the experiments, we compare our proposed VRSC-PG with two other related methods:
Figure 1: Experimental results for meaning-variance portfolio management on synthetic data. \(\kappa_{\text{cov}}\) is the conditional number of the covariance matrix of the corresponding Gaussian distribution which is used to generate reward. We use time as \(x\) axis, and it is proportional to the query complexity. In \(y\) axis, the objective value gap is defined as \(H(x) - H(x^*)\), where \(x^*\) is obtained by running our methods for enough iterations until convergence. \(\|G(x)\|^2\) denotes the \(\ell_2\)-norm of the full gradient, where \(G(x) = \nabla f(x) + \partial h(x)\).

- Accelerated stochastic compositional proximal gradient (ASC-PG) [14];
- Stochastic variance reduced ADMM for Stochastic composition optimization (com-SVR-ADMM) [16];

In our experiments, learning rate \(\eta\) is tuned from \(\{1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}\). We keep the learning rate constant for com-SVR-ADMM and VRSC-PG in the optimization. For ASC-PG, in order to guarantee convergence, learning rate is decreased as per \(\eta_{t+1} = \eta_t / \left(1 + \frac{t}{T}\right)\), where \(t\) denotes the number of iterations.

5.1 Application to Portfolio Management

Suppose there are \(N\) assets we can invest, \(r_t \in \mathbb{R}^N\) denotes the rewards of \(N\) assets at time \(t\). Our goal is to maximize the return of the investment and to minimize the risk of the investment at the same time. Portfolio
management problem can be formulated as the mean-variance optimization as follows:

\[
\min_{x \in \mathbb{R}^N} \frac{1}{n} \sum_{t=1}^{n} \langle r_t, x \rangle + \frac{1}{n} \sum_{t=1}^{n} \left( \langle r_t, x \rangle - \frac{1}{n} \sum_{j=1}^{n} \langle r_j, x \rangle \right)^2
\]

(38)

where \(x \in \mathbb{R}^N\) denotes the investment quantity vector in \(N\) assets. According to [8], problem (38) can also be viewed as the composition problem as (2). In our experiment, we also add a nonsmooth regularization penalty \(h(x) = \lambda |x|\) in the mean-variance optimization problem (38).

Similar to the experimental settings in [8], we let \(n = 2000\) and \(N = 200\). Rewards \(r_t\) are generated in two steps: (1) Generate a Gaussian distribution on \(\mathbb{R}^N\), where we define the condition number of its covariance matrix as \(\kappa_{\text{cov}}\). Because \(\kappa_{\text{cov}}\) is proportional to \(\kappa\), in our experiment, we will control \(\kappa_{\text{cov}}\) to change the value of \(\kappa\); (2) Sample rewards \(r_t\) from the Gaussian distribution and make all elements positive to guarantee that this problem has a solution. In the experiment, we compared three methods on two synthetic datasets, which are generated through Gaussian distributions with \(\kappa_{\text{cov}} = 2\) and \(\kappa_{\text{cov}} = 10\) separately. We set \(\lambda = 10^{-3}\) and \(A = B = b_1 = 5\). We just select the values of \(A, B, b_1\) casually, it is probable that we can get better results as long as we tune them carefully.

Figure 1 shows the convergence of compared methods regarding time. We suppose that the elapsed time is proportional to the query complexity. Objective value gap means \(H(x_t) - H(x^*)\), where \(x^*\) is the optimal solution to \(H(x)\). We compute \(H(x^*)\) by running our method until convergence. Firstly, by observing the \(x\) and \(y\) axises in Figure 1, we can know that when \(\kappa_{\text{cov}} = 10\), all compared methods need more time to minimize problem (38), which is consistent with our analysis. Increasing \(\kappa\) will increase the total query complexity. Secondly, we can also find out that com-SVR-ADMM and VRSC-PG admit linear convergence rate. ASC-PG runs faster at the beginning, because of their low query complexity in each iteration. However, their convergence slows down when the learning rate gets small. In four figures, our SVRC-PG always has the best performance compared to other compared methods.

5.2 Application to Reinforcement Learning

We then apply stochastic composition optimization to reinforcement learning and evaluate three compared methods in the task of policy evaluation. In reinforcement learning, let \(V^\pi(s)\) be the value of state \(s\) under
policy $\pi$. The value function $V^\pi(s)$ can be evaluated through Bellman equation as follows:

$$V^\pi(s_1) = \mathbb{E}[r_{s_1, s_2} + \gamma V^\pi(s_2) | s_1]$$

(39)

for all $s_1, s_2 \in \{1, 2, ..., S\}$, where $S$ represents the number of total states. According to [14], the Bellman equation (39) can also be written as a composition problem. In our experiment, we also add sparsity regularization $h(x) = \lambda |x|$ in the objective function.

Following [3], we generate a Markov decision process (MDP). There are 400 states and 10 actions at each state. The transition probability is generated randomly from the uniform distribution in the range of $[0, 1]$. We then add $10^{-5}$ to each element of transition matrix to ensure the ergodicity of our MDP. The rewards $r(s, s')$ from state $s$ to state $s'$ are also sampled uniformly in the range of $[0, 1]$. In our experiment, we set $\lambda = 10^{-3}$ and $A = B = b_1 = 5$. We also select these values casually, better results can be obtained if we tune them carefully.

In Figure 2, we plot the convergence of the objective value and $\|G(x)\|^2$ in terms of time. We can observe that VRSC-PG is much faster than ASC-PG, which has been reflected in the analysis of convergence rate already. It is also obvious that our VRSC-PG converges faster than com-SVR-ADMM. Experimental results on policy evaluation also verify our theoretical analysis.

6 Conclusion

In this paper, we propose variance reduced stochastic compositional proximal gradient method (VRSC-PG) for composition problem with nonsmooth regularization penalty. We also analyze the convergence rate of our method: (1) for strongly convex composition problem, VRSC-PG is proved to admit linear convergence; (2) for general composition problem, VRSC-PG significantly improves the state-of-the-art convergence rate from $O(T^{-1/2})$ to $O((n_1 + n_2)^{2/3}T^{-1})$. Both of our theoretical analysis, to the best of our knowledge, are the state-of-the-art results for stochastic composition optimization. Finally, we apply our method to two different applications, portfolio management and reinforcement learning. Experimental results show that our method always has the best performance in different cases and verify the conclusions of theoretical analysis.

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A Strongly Convex Problem

Proof to Lemma[1]

Proof: Following [15], we define function $\phi_i(x)$:

$$
\phi_i(x) = F_i(G(x)) - F_i(G(x^*)) - \langle (\nabla G(x^*))^T \nabla F_i(G(x^*)), (x - x^*) \rangle
$$

(40)

where $x^* = \arg \min_x H(x)$ denotes the optimal solution of $H(x)$. It is easy to know that $\nabla \phi_i(x^*) = 0$ and $\min_x \phi_i(x) = \phi_i(x^*) = 0$. For any $x, y \in \mathbb{R}^N$, we have:

$$
\| \nabla \phi_i(x) - \nabla \phi_i(y) \|
\leq \frac{1}{n_2} \sum_{j=1}^{n_2} \| (\nabla G_j(x))^T F_i(G(x)) - (\nabla G_j(y))^T F_i(G(y)) \|
\leq \frac{1}{n_2} \sum_{j=1}^{n_2} L_f \| x - y \|
\leq L_f \| x - y \|,
$$

(41)
where the first inequality follows from the triangle inequality, and the second inequality follows from (19) in Assumption $d$. Therefore, $\nabla \phi_i(x)$ is $L_f$-Lipschitz continuous. According to Theorem 2.1.5 in [10], we have:

$$
\frac{1}{2L_f} \| \nabla \phi_i(x) - \nabla \phi_i(x^*) \|^2 \leq \phi_i(x) - \phi_i(x^*) - \langle \nabla \phi_i(x^*), x - x^* \rangle
$$

(42)

Because $\phi_i(x^*) = 0$ and $\nabla \phi_i(x^*) = 0$, it follows that:

$$
\frac{1}{2L_f} \| (\nabla G(x))^T \nabla F_i(G(x)) - (\nabla G(x^*))^T \nabla F_i(G(x^*)) \|^2 \
\leq F_i(G(x)) - F_i(G(x^*)) - \langle (\nabla G(x^*))^T \nabla F_i(G(x^*)), (x - x^*) \rangle
$$

(43)

Because $x^* = \arg \min_x f(x) + h(x)$, there exists $\xi^* \in \partial h(x^*)$ so that $(\nabla G(x^*))^T \nabla F(G(x^*)) + \xi^* = 0$. Then it holds that:

$$
\frac{1}{m_1} \sum_{i=1}^{m_1} \| (\nabla G(x))^T \nabla F_i(G(x)) - (\nabla G(x^*))^T \nabla F_i(G(x^*)) \|^2 \
\leq 2L_f \left[ F(G(x)) - F(G(x^*)) - \langle (\nabla G(x^*))^T \nabla F(G(x^*)), (x - x^*) \rangle \right] \
= 2L_f \left[ F(G(x)) - F(G(x^*)) + \langle \xi^*, (x - x^*) \rangle \right] \
\leq 2L_f \left[ F(G(x)) - F(G(x^*)) + h(x) - h(x^*) \right] \
\leq 2L_f \left[ H(x) - H(x^*) \right]
$$

(44)

where the third inequality is from the convexity of $h(x)$. We let $u_t$ be the unbiased estimate of $\nabla f(x_t^{s+1})$:

$$
uu_s \xrightarrow{} 1 \sum_{i \in I_t} \left( (\nabla G(x_t^{s+1}))^T \nabla F_i(G(x_t^{s+1})) - (\nabla G(\tilde{x}^s))^T \nabla F_i(\tilde{G}^s) \right) + \nabla f(\tilde{x}^s)
$$

(45)

where $E u_t = \nabla f(x_t^{s+1})$. We can get the upper bound of the variance of $u_t$:

$$
E \| u_t - \nabla f(x_t^{s+1}) \|^2 \
= E \| \frac{1}{b_1} \sum_{i \in I_t} \left( (\nabla G(x_t^{s+1}))^T \nabla F_i(G(x_t^{s+1})) - (\nabla G(\tilde{x}^s))^T \nabla F_i(\tilde{G}^s) + \nabla f(\tilde{x}^s) - \nabla f(x_t^{s+1}) \right) \|^2 \
= \frac{1}{b_1} \sum_{i \in I_t} E \| (\nabla G(x_t^{s+1}))^T \nabla F_i(G(x_t^{s+1})) - (\nabla G(\tilde{x}^s))^T \nabla F_i(\tilde{G}^s) + \nabla f(\tilde{x}^s) - \nabla f(x_t^{s+1}) \|^2 \
\leq \frac{1}{b_1} \sum_{i \in I_t} E \| (\nabla G(x_t^{s+1}))^T \nabla F_i(G(x_t^{s+1})) - (\nabla G(\tilde{x}^s))^T \nabla F_i(\tilde{G}^s) \|^2 \
\leq \frac{2}{b_1} \sum_{i \in I_t} \left( E \| (\nabla G(x_t^{s+1}))^T \nabla F_i(G(x_t^{s+1})) - (\nabla G(x^*))^T \nabla F_i(G(x^*)) \|^2 \
+ E \| (\nabla G(\tilde{x}^s))^T \nabla F_i(\tilde{G}^s) - (\nabla G(x^*))^T \nabla F_i(G(x^*)) \|^2 \right) \
\leq 4L_f \left[ H(x_t^{s+1}) - H(x^*) + H(\tilde{x}^s) - H(x^*) \right]
$$

(46)
where the second equality follows from Lemma 4, the last inequality follows from (44). Then, we compute the upper bound of $\mathbb{E}\|u_t^{s+1} - v_t^{s+1}\|^2$:

$$
\mathbb{E}\|u_t^{s+1} - v_t^{s+1}\|^2 = \mathbb{E}\left\| \frac{1}{b_t} \sum_{i \in I_t} \left( (\nabla G(x_t^{s+1}))^T \nabla F_{i_t}(G(x_t^{s+1})) - (\nabla \hat{G}_t^{s+1})^T \nabla F_{i_t}(\hat{G}_t^{s+1}) \right) \right\|^2
$$

$$
\leq \frac{1}{b_t} \sum_{i \in I_t} \mathbb{E}\left\| (\nabla G(x_t^{s+1}))^T \nabla F_{i_t}(G(x_t^{s+1})) - (\nabla \hat{G}_t^{s+1})^T \nabla F_{i_t}(\hat{G}_t^{s+1}) \right\|^2
$$

$$
\leq 2B_F^2 \mathbb{E}\left\| (\nabla G(x_t^{s+1}))^T - (\nabla \hat{G}_t^{s+1})^T \right\|^2 + 2B_F^2 \mathbb{E}\| \nabla F_{i_t}(G(x_t^{s+1})) - \nabla F_{i_t}(\hat{G}_t^{s+1}) \|^2
$$

$$
\leq 2B_F^2 \mathbb{E}\left\| (\nabla G(x_t^{s+1}))^T - (\nabla \hat{G}_t^{s+1})^T \right\|^2 + 2B_F^2 \mathbb{E}\| \nabla F_{i_t}(G(x_t^{s+1})) - \nabla F_{i_t}(\hat{G}_t^{s+1}) \|^2
$$

(47)

where the inequality follows from Lemma 4. Then we can get the upper bound of $T_3$ as follows:

$$
T_1 \leq 2B_F^2 \mathbb{E}\left\| (\nabla G(x_t^{s+1}))^T - (\nabla \hat{G}_t^{s+1})^T \right\|^2 + 2B_F^2 \mathbb{E}\| \nabla F_{i_t}(G(x_t^{s+1})) - \nabla F_{i_t}(\hat{G}_t^{s+1}) \|^2
$$

$$
\leq 2B_F^2 \mathbb{E}\left\| (\nabla G(x_t^{s+1}))^T - (\nabla \hat{G}_t^{s+1})^T \right\|^2 + 2B_F^2 \mathbb{E}\| \nabla F_{i_t}(G(x_t^{s+1})) - \nabla F_{i_t}(\hat{G}_t^{s+1}) \|^2
$$

$$
\leq 2B_F^2 \mathbb{E}\left\| (\nabla G(x_t^{s+1}))^T - (\nabla \hat{G}_t^{s+1})^T \right\|^2 + 2B_F^2 \mathbb{E}\| \nabla F_{i_t}(G(x_t^{s+1})) - \nabla F_{i_t}(\hat{G}_t^{s+1}) \|^2
$$

(48)
where the first inequality follows from Lemma 4, the second, third, fourth inequality follows from Assumption 3, the third equality follows from Lemma 3, the last inequality follows from the strong convexity of $H(x)$, such that $\|x - x^*\|^2 \leq \frac{2}{\mu}(H(x) - H(x))$. Combine (46) and (47), then we have:

$$
\begin{align*}
\mathbb{E} \|v_t - \nabla f(x_t^{s+1})\|^2 & \leq 2\mathbb{E} \|u_t^{s+1} - v_t^{s+1}\|^2 + 2\mathbb{E} \|u_t - \nabla f(x_t^{s+1})\| \\
& \leq \left( \frac{64}{\mu} \left( \frac{B_2^2L_G^2}{B} + \frac{B_4^4L_F^2}{A} \right) + 8L_f \right) \left( H(x_t^{s+1}) - H(x^*) + H(\tilde{x}^s) - H(x^*) \right)
\end{align*}
$$

(49)

where the first inequality follows from Lemma 4.

**Lemma 2** Suppose all assumption hold, given $x^*$ is the optimal solution to problem $H(x)$, there exists a constant $\alpha > 0$ such that:

$$
\begin{align*}
\mathbb{E} \left( v_t^{s+1} - \nabla f(x_t^{s+1}), x^* - x_t^{s+1} \right) & = \mathbb{E} \left( u_t^{s+1} - \nabla f(x_t^{s+1}), x^* - x_t^{s+1} \right) + \mathbb{E} \left( v_t^{s+1} - u_t^{s+1}, x^* - x_t^{s+1} \right) \\
& = \mathbb{E} \left( u_t^{s+1} - \nabla f(x_t^{s+1}), x^* - \tilde{x}_t^{s+1} \right) + \mathbb{E} \left( u_t^{s+1} - \nabla f(x_t^{s+1}), \tilde{x}_t^{s+1} - x_t^{s+1} \right)
\end{align*}
$$

(50)

**Proof:** Define $\tilde{x}_t^{s+1} = \text{Prox}_{\eta h(\cdot)} \left( x_t^{s+1} - \eta \nabla f(x_t^{s+1}) \right)$, we have that

$$
\begin{align*}
\mathbb{E} \left( v_t^{s+1} - \nabla f(x_t^{s+1}), x^* - x_t^{s+1} \right) & = \mathbb{E} \left( u_t^{s+1} - \nabla f(x_t^{s+1}), x^* - x_t^{s+1} \right) + \mathbb{E} \left( v_t^{s+1} - u_t^{s+1}, x^* - x_t^{s+1} \right) \\
& = \mathbb{E} \left( u_t^{s+1} - \nabla f(x_t^{s+1}), x^* - \tilde{x}_t^{s+1} \right) + \mathbb{E} \left( u_t^{s+1} - \nabla f(x_t^{s+1}), \tilde{x}_t^{s+1} - x_t^{s+1} \right)
\end{align*}
$$

(51)

where the third inequality follows from triangle inequality and Young’s inequality, the last inequality follows from Lemma 4. This completes the proof.
Proof to Theorem 1

Proof: We define the virtual gradient $g_t^{s+1}$ as follows:

$$g_t^{s+1} = \frac{1}{\eta} \left( x_t^{s+1} - \text{Prox}_{\eta h(\cdot)} \left( x_t^{s+1} - \eta v_t^{s+1} \right) \right)$$  \hfill (52)

Because $x_{t+1}^{s+1} = \arg \min_{x} \frac{1}{2} \| x - (x_t^{s+1} + \eta \nu_t^{s+1}) \|^2 + h(x)$, based on the optimality condition, we have that $\frac{1}{\eta} \left( -x_{t+1}^{s+1} + (x_t^{s+1} - \eta \nu_t^{s+1}) \right) \in \partial h(x_{t+1}^{s+1})$. Thus, we have:

$$g_t^{s+1} = \frac{1}{\eta} \left( x_t^{s+1} - x_{t+1}^{s+1} \right) = v_t^{s+1} + \xi_t^{s+1}$$  \hfill (53)

where $\xi_t^{s+1} \in \partial h(x_{t+1}^{s+1})$. Note that the virtual gradients $g_t^{s+1}$ are only used for the analysis, not computed in the implementation. Because $x^*$ is the optimal solution, we have:

\[
H(x^*) = f(x^*) + h(x^*) \\
\geq f(x_t^{s+1}) + \langle \nabla f(x_t^{s+1}), x^* - x_t^{s+1} \rangle + h(x_t^{s+1}) + \langle \xi_t^{s+1}, x^* - x_t^{s+1} \rangle \\
\geq f(x_t^{s+1}) - \langle \nabla f(x_t^{s+1}), x_t^{s+1} - x_t^{s+1} \rangle - \frac{L_f}{2} \| x_t^{s+1} - x_t^{s+1} \|^2 \\
+ \langle \nabla f(x_t^{s+1}), x^* - x_t^{s+1} \rangle + h(x_t^{s+1}) + \langle \xi_t^{s+1}, x^* - x_t^{s+1} \rangle \\
= H(x_t^{s+1}) + (v_t^{s+1} + \xi_t^{s+1}, x^* - x_t^{s+1}) + (v_t^{s+1} + \xi_t^{s+1}, x_t^{s+1} - x_t^{s+1}) \\
+ \langle \nabla f(x_t^{s+1}) - v_t^{s+1}, x^* - x_t^{s+1} \rangle - \frac{L_f}{2} \| x_t^{s+1} - x_t^{s+1} \|^2 \\
= H(x_t^{s+1}) + (g_t^{s+1}, x^* - x_t^{s+1}) + \eta \| g_t^{s+1} \|^2 + \langle \nabla f(x_t^{s+1}) - v_t^{s+1}, x^* - x_t^{s+1} \rangle - \frac{L_f\eta}{2} \| g_t^{s+1} \|^2
\]

where the first inequality uses the convexity of $F(x)$ and $R(x)$, the second inequality follows from that $f(x)$ is $L_f$-smooth. Thus, we have:

$$\langle g_t^{s+1}, x^* - x_t^{s+1} \rangle$$

\[
\leq H(x^*) - H(x_t^{s+1}) - \eta \| g_t^{s+1} \|^2 - \langle \nabla f(x_t^{s+1}) - v_t^{s+1}, x^* - x_t^{s+1} \rangle + \frac{L_f\eta}{2} \| g_t^{s+1} \|^2
\]

As per the iteration of $x_t^{s+1}$, we have:

\[
\begin{align*}
\mathbb{E} \| x_{t+1}^{s+1} - x^* \|^2 & = \mathbb{E} \| x_t^{s+1} - \eta g_t^{s+1} - x^* \|^2 \\
& = \mathbb{E} \| x_t^{s+1} - x^* \|^2 + \eta^2 \mathbb{E} \| g_t^{s+1} \|^2 - 2 \eta \mathbb{E} \langle g_t^{s+1}, x_t^{s+1} - x^* \rangle \\
& \leq \mathbb{E} \| x_t^{s+1} - x^* \|^2 + \eta^2 \mathbb{E} \| g_t^{s+1} \|^2 - 2 \eta \mathbb{E} (\mathbb{E} H(x_t^{s+1}) - H(x^*)) + \\
& \quad 2 \eta \mathbb{E} \langle \nabla f(x_t^{s+1}) - v_t^{s+1}, x_t^{s+1} - x^* \rangle - 2 \eta^2 \mathbb{E} \| g_t^{s+1} \|^2 + L_f \eta^3 \| g_t^{s+1} \|^2 \\
& \leq \mathbb{E} \| x_t^{s+1} - x^* \|^2 - 2 \eta \mathbb{E} (\mathbb{E} H(x_t^{s+1}) - H(x^*)) + 2 \eta \mathbb{E} \langle \nabla f(x_t^{s+1}) - v_t^{s+1}, x_t^{s+1} - x^* \rangle \\
& \leq \mathbb{E} \| x_t^{s+1} - x^* \|^2 - 2 \eta \mathbb{E} (\mathbb{E} H(x_t^{s+1}) - H(x^*)) + 2 \eta \frac{\alpha}{\mu} \mathbb{E} (H(x_t^{s+1}) - H(x^*)) \\
& \quad + 2 \eta \left( 6 \eta L_f + \frac{\eta}{2} + \frac{1}{2 \alpha} \right) \frac{32}{\mu} \left( \frac{B_f^2 L_g^2}{B} + \frac{B_g^4 L_f^2}{A} \right) \mathbb{E} (H(x_t^{s+1}) - H(x^*) + H(\tilde{x}^s) - H(x^*)) \\
& = \mathbb{E} \| x_t^{s+1} - x^* \|^2 - 2 \eta \left( 1 - \frac{\alpha}{\mu} \right) \mathbb{E} (\mathbb{E} H(x_t^{s+1}) - H(x^*)) \\
& \quad + 2 \eta \left( 6 \eta L_f + \frac{\eta}{2} + \frac{1}{2 \alpha} \right) \frac{32}{\mu} \left( \frac{B_f^2 L_g^2}{B} + \frac{B_g^4 L_f^2}{A} \right) \mathbb{E} (H(x_t^{s+1}) - H(x^*) + H(\tilde{x}^s) - H(x^*))
\end{align*}
\]

(56)
where the first inequality follows from (55), the second inequality follows from $\eta L_f \leq 1$, the third inequality follows from Lemma 2. By summing the inequality (56) over $t = 0, \cdots, m - 1$, because $\bar{x}^s = x^s$ and $\bar{x}^s = x_0^{s+1}$, we obtain the following inequality:

$$
\mathbb{E} \left\| x^s_m - x^* \right\|^2 + 2\eta \left( 1 - \frac{\alpha}{\mu} \right) \left( \mathbb{E} H(x^s_m) - H(x^*) \right) + 2\eta \left( 1 - \frac{\alpha}{\mu} \right) \sum_{t=1}^{m-1} \left( \mathbb{E} H(x^s_t) - H(x^*) \right)
$$

$$
\leq \mathbb{E} \left\| x^s - x^* \right\|^2 + 2\eta \left( \frac{\eta}{2} + 1 \right) \sum_{t=1}^{m-1} \left( \frac{B_f^2 L_f^2}{B} \frac{B_G^2 L_G^2}{A} \right) \left( m + 1 \right) \sum_{t=1}^{m-1} \mathbb{E} \left( H(x^s_t) - H(x^*) \right)
$$

According to (57) and $\| x^s - x^* \|^2 \leq \frac{2}{m} (H(\bar{x}) - H(x^*))$, the following inequality holds:

$$
2\eta \left( 1 - \frac{\alpha}{\mu} \right) \left( \frac{\eta}{2} + 1 \right) \sum_{t=1}^{m-1} \left( \frac{B_f^2 L_f^2}{B} \frac{B_G^2 L_G^2}{A} \right) \left( m + 1 \right) \sum_{t=1}^{m-1} \mathbb{E} \left( H(x^s_t) - H(x^*) \right)
$$

As per the convexity of $H(x)$ and the definition of $\bar{x}^s$, we have $H(\bar{x}^s) \leq \frac{1}{m} \sum_{t=1}^{m} H(x^s_t)$. According to (58), and let $\alpha = \frac{6}{8}$, we have that:

$$
\mathbb{E} H(\bar{x}^{s+1}) - H(x^*)
$$

$$
\leq \frac{2}{\mu} + 2\eta \left( \frac{\eta}{2} + 1 \right) \sum_{t=1}^{m-1} \left( \frac{B_f^2 L_f^2}{B} \frac{B_G^2 L_G^2}{A} \right) \left( m + 1 \right) \mathbb{E} \left( H(\bar{x}) - H(x^*) \right)
$$

Define $\rho = \frac{\eta}{8} \left( 6\eta L_f + \left( \frac{\eta}{2} + 1 \right) \sum_{t=1}^{m-1} \left( \frac{B_f^2 L_f^2}{B} \frac{B_G^2 L_G^2}{A} \right) \left( m + 1 \right) \mathbb{E} \left( H(x^s_t) - H(x^*) \right) \right)$, where $\rho < 1$. Applying (59) recursively, we have:

$$
\mathbb{E} H(\bar{x}^S) - H(x^*) \leq \rho^S \left( H(x^0) - H(x^*) \right)
$$

This completes the proof.

**Proof to Corollary 1**

**Proof:** Appropriately choosing $\eta$, $m$ and $A$ in Algorithm 3, we can get the linear convergence rate for our algorithm. We choose $A$ and $B$ to satisfy:

$$
\frac{\eta}{2} \frac{B_f^2 L_f^2}{B} \mu \leq \frac{1}{16}
$$

$$
\frac{\eta}{2} \frac{B_G^2 L_G^2}{A} \mu \leq \frac{1}{16}
$$

$$
\frac{4}{\mu} \frac{B_f^2 L_f^2}{B} \mu \leq \frac{1}{16}
$$

$$
\frac{4}{\mu} \frac{B_G^2 L_G^2}{A} \mu \leq \frac{1}{16}
$$
The above inequalities are equivalent with the following conditions:

\[
A \geq \max \left\{ \frac{256\eta B^2 L_F^2}{\mu}, \frac{2048 B^2 L_G^2}{\mu^2} \right\} \tag{65}
\]

\[
B \geq \max \left\{ \frac{256\eta B^2 L_F^2}{\mu}, \frac{2048 B^2 L_G^2}{\mu^2} \right\} \tag{66}
\]

We choose \( \eta \) satisfying \( 6\eta L_f \leq \frac{1}{16} \), such that \( \eta \leq \frac{1}{96L_f} \). It follows that:

\[
\frac{2}{\mu} + 2\eta \left( 6\eta L_f + \left( \frac{\eta}{2} + \frac{4}{\mu} \right) \frac{32}{\mu} \left( \frac{B^2 L_F^2}{B} + \frac{B^2 L_G^2}{A} \right) \right) (m + 1)
\]

\[
= \frac{2\mu m + 2\eta \frac{5}{16} (1 + \frac{1}{m})}{2\eta \frac{9}{16}}
\]

\[
\leq \frac{2\mu m + 2\eta \frac{5}{16} (1 + \frac{1}{m})}{2\eta \frac{9}{16}}
\]

\[
= \frac{5}{9} + \frac{2\mu m + 2\eta \frac{1}{m}}{2\eta \frac{9}{16}} \tag{67}
\]

We then set \( m \) satisfying \( \frac{2\mu m + 2\eta \frac{1}{m}}{2\eta \frac{9}{16}} \leq \frac{1}{3} \), which is equivalent to \( m \geq 16 \left( 1 + \frac{1}{\mu\eta} \right) \). Thus, by setting \( \eta, m, A \) and \( B \) appropriately:

\[
\eta = \frac{1}{96L_f} \tag{68}
\]

\[
m = 16 \left( 1 + \frac{96L_f}{\mu} \right) \tag{69}
\]

\[
A = \max \left\{ \frac{8B^2 L_F^2}{3\mu L_f}, \frac{2048 B^2 L_G^2}{\mu^2} \right\} \tag{70}
\]

\[
B = \max \left\{ \frac{8B^2 L_F^2}{3\mu L_f}, \frac{2048 B^2 L_G^2}{\mu^2} \right\} \tag{71}
\]

we can obtain a linear convergence rate \( \frac{2}{3} \). This completes the proof.

**B General Problem**

**Proof to Theorem 2**

Proof: We define \( u^{s+1}_t \) to be an unbiased estimate for \( \nabla f(x^{s+1}_t) \):

\[
u_t^{s+1} = \frac{1}{b_1} \sum_{i \in I_t} \left( \nabla G(x^{s+1}_t) \right)^T \nabla F_i (G(x^{s+1}_t)) - \left( \nabla G(x^s_t) \right)^T \nabla F_i (G^s) + \nabla f(x^s) \tag{72}
\]

where \( I_t \) is uniformly sampled from \{1, 2, ..., \( n_1 \) \} such that \(|I_t| = b_1\). We also define:

\[
\bar{x}^{s+1}_{t+1} = \text{Prox}_{\eta h} (x^s_t + 1 - \eta \nabla f(x^{s+1}_t)) \tag{73}
\]

According to Lemma 2 in [12], if \( y = \text{prox}_{\eta h} (x - \eta d') \), \( d' \in \mathbb{R}^d \) and \( \forall z \in \mathbb{R}^d \), following equality holds that:

\[
H(y) \leq H(z) + \langle y - z, \nabla f(x) - d' \rangle + \left[ \frac{L_f}{2} - \frac{1}{2\eta} \right] ||y - x||^2 + \left[ \frac{L_f}{2} + 1 \frac{1}{2\eta} \right] ||z - x||^2 - \frac{1}{2\eta} ||y - z||^2 \tag{74}
\]
Thus if we let $y = \bar{x}_{t+1}^{s+1}$, $x = z = x_t^{s+1}$ and $d' = \nabla f(x_t^{s+1})$, we have:

$$
E[H(\bar{x}_{t+1}^{s+1})] \leq E \left[ H(x_t^{s+1}) + \left[ \frac{L_f}{2} - \frac{1}{2\eta} \right] \|x_{t+1}^{s+1} - x_t^{s+1}\|^2 - \frac{1}{2\eta} \|\bar{x}_{t+1}^{s+1} - x_t^{s+1}\|^2 \right].
$$

(75)

If we let $y = x_{t+1}^{s+1}$, $x = x_t^{s+1}$, $z = x_{t+1}^{s+1}$ and $d' = u_t^{s+1}$, then we have:

$$
E[H(x_t^{s+1})] \leq E \left[ H(x_t^{s+1}) + \left[ \frac{L_f}{2} - \frac{1}{2\eta} \right] \|x_{t+1}^{s+1} - x_t^{s+1}\|^2 + \left[ \frac{L_f}{2} - \frac{1}{2\eta} \right] \|x_{t+1}^{s+1} - x_t^{s+1}\|^2 
- \frac{1}{2\eta} \|x_{t+1}^{s+1} - x_t^{s+1}\|^2 + \langle x_{t+1}^{s+1} - \bar{x}_{t+1}^{s+1}, \nabla f(x_t^{s+1}) - u_t^{s+1} \rangle \right].
$$

(76)

Combine (75) and (76), we have:

$$
E[H(x_t^{s+1})] \leq E \left[ H(x_t^{s+1}) + \left[ \frac{L_f}{2} - \frac{1}{2\eta} \right] \|x_{t+1}^{s+1} - x_t^{s+1}\|^2 + \left[ \frac{L_f}{2} - \frac{1}{2\eta} \right] \|x_{t+1}^{s+1} - x_t^{s+1}\|^2 
- \frac{1}{2\eta} \|x_{t+1}^{s+1} - x_t^{s+1}\|^2 + \langle x_{t+1}^{s+1} - \bar{x}_{t+1}^{s+1}, \nabla f(x_t^{s+1}) - u_t^{s+1} \rangle \right].
$$

(77)

Then we can have an upper bound of $T_1$:

$$
E[T_1] \leq \frac{1}{2\eta} E\|x_{t+1}^{s+1} - x_t^{s+1}\|^2 + \left( \frac{1}{2\eta} \right) \|\nabla f(x_t^{s+1}) - u_t^{s+1}\|^2
\leq \frac{1}{2\eta} E\|x_{t+1}^{s+1} - x_t^{s+1}\|^2 + \eta E\|\nabla f(x_t^{s+1}) - u_t^{s+1}\|^2 + \eta E\|u_t^{s+1} - u_t^{s+1}\|^2
$$

(78)

where the first inequality follows from Cauchy-Schwarz and Young’s inequality. As per the definition of $u_t^{s+1}$, we have:

$$
E\|\nabla f(x_t^{s+1}) - u_t^{s+1}\|^2
= E\left[ \frac{1}{b_1} \sum_{i_t \in t} \left( \nabla f(x_t^{s+1}) - \nabla G(x_t^{s+1}) \right)^T \nabla F_{i_t} (G(x_t^{s+1})) \right] 
+ \left( \nabla G(x^{s}) \right)^T \nabla F_{i_t} (G^{s}) - \nabla f(x^{s}) \right] \|^2
= \frac{1}{b_1^2} E\| \sum_{i_t \in t} \left( \nabla f(x_t^{s+1}) - \nabla G(x_t^{s+1}) \right)^T \nabla F_{i_t} (G(x_t^{s+1})) 
+ \left( \nabla G(x^{s}) \right)^T \nabla F_{i_t} (G^{s}) - \nabla f(x^{s}) \|^2
= \frac{1}{b_1^2} E\| \sum_{i_t \in t} \left( \nabla f(x_t^{s+1}) - \nabla G(x_t^{s+1}) \right)^T \nabla F_{i_t} (G(x_t^{s+1})) 
+ \left( \nabla G(x^{s}) \right)^T \nabla F_{i_t} (G^{s}) - \nabla f(x^{s}) \|^2
\leq \frac{L_f^2}{b_1^2} E\|x_t^{s+1} - \bar{x}\|^2
$$

(79)
where the third equality follows from Lemma\textsuperscript{3}, the first inequality follows from:

$$\mathbb{E}\|\xi - \mathbb{E}[\xi]\|^2 = \mathbb{E}\|\xi\|^2 - \|\mathbb{E}[\xi]\|^2,$$

and the last inequality follows from Assumption\textsuperscript{2}. We can bound $\mathbb{E}\|v_t^{s+1} - u_t^{s+1}\|^2$ as follows:

$$
\begin{align*}
\mathbb{E}\|v_t^{s+1} - u_t^{s+1}\|^2 & = \mathbb{E}\left|\frac{1}{b_1} \sum_{i_t \in I_t} \left| \left( \nabla \hat{G}_t^{s+1} \right)^T \nabla F_{i_t}(\hat{G}_t^{s+1}) - \left( \nabla G(x_t^{s+1}) \right)^T \nabla F_{i_t}(G(x_t^{s+1})) \right| \right|^2 \\
& \leq \frac{1}{b_1} \mathbb{E} \sum_{i_t \in I_t} \left( \left( \nabla \hat{G}_t^{s+1} \right)^T \nabla F_{i_t}(\hat{G}_t^{s+1}) - \left( \nabla G(x_t^{s+1}) \right)^T \nabla F_{i_t}(G(x_t^{s+1})) \right)^2 \\
& \leq \frac{1}{b_1} \mathbb{E} \sum_{i_t \in I_t} \left( \left( \nabla \hat{G}_t^{s+1} \right)^T \nabla F_{i_t}(\hat{G}_t^{s+1}) - \left( \nabla G(x_t^{s+1}) \right)^T \nabla F_{i_t}(G(x_t^{s+1})) \right)^2 \\
& \leq \frac{1}{b_1} \mathbb{E} \sum_{i_t \in I_t} \left( \left( \nabla \hat{G}_t^{s+1} \right)^T \nabla F_{i_t}(\hat{G}_t^{s+1}) - \left( \nabla G(x_t^{s+1}) \right)^T \nabla F_{i_t}(G(x_t^{s+1})) \right)^2 \\
& \leq \frac{2B_2^2 L_g^2}{B^2} \sum_{1 \leq j \leq B} \mathbb{E} \left\| \nabla G_{B_t[j]}(x_t^{s+1}) - \nabla G_{B_t[j]}(\bar{x}) + \nabla G(\bar{x}) - \nabla G(x_t^{s+1}) \right\|^2 \\
& \leq \frac{2B_2^2 L_g^2}{B^2} \sum_{1 \leq j \leq B} \mathbb{E} \left\| \nabla G_{B_t[j]}(x_t^{s+1}) - \nabla G_{B_t[j]}(\bar{x}) \right\|^2 \\
& \leq \left( \frac{2B_4^4 L_g^2}{A} + \frac{2B_2^2 L_g^2}{B} \right) \mathbb{E} \|x_t^{s+1} - \bar{x}\|^2 \tag{81}
\end{align*}
$$

where the first inequality follows from Lemma\textsuperscript{4}, the second inequality follows from \textup{(}80\textup{)}, the third inequality follows from \textup{(}80\textup{)}, and the last inequality follows from the Assumption of bounded gradients. From \textup{(}79\textup{)} and \textup{(}81\textup{)}, we get the upper bound of $T_1$:

$$
\mathbb{E}[T_1] \leq \frac{1}{2\eta} \mathbb{E}\|x_t^{s+1} - \bar{x}\|^2 + \frac{\eta L_f^2}{b_1} \mathbb{E}\|x_t^{s+1} - \bar{x}\|^2 + \left( \frac{2B_4^4 L_g^2}{A} + \frac{2B_2^2 L_g^2}{B} \right) \mathbb{E}\|x_t^{s+1} - \bar{x}\|^2. \tag{82}
$$

Substituting the upper bound of $T_1$ in \textup{(}77\textup{)}, we have:

$$
\begin{align*}
\mathbb{E}\left[H(x_t^{s+1})\right] & \leq \mathbb{E}\left[H(x_t^{s+1}) + \left[ L_f - \frac{1}{2\eta} \right] \|x_t^{s+1} - x_t^{s+1}\|^2 + \left[ \frac{L_f}{2} - \frac{1}{2\eta} \right] \|x_t^{s+1} - x_t^{s+1}\|^2 \\
& \quad + \left( \frac{\eta L_f^2}{b_1} + \frac{2B_4^4 L_g^2}{A} + \frac{2B_2^2 L_g^2}{B} \right) \mathbb{E}\|x_t^{s+1} - \bar{x}\|^2 \right]. \tag{83}
\end{align*}
$$

Following the analysis in \textup{(}11\textup{)}, we define the Lyapunov function $R_t^{s+1} = \mathbb{E}[H(x_t^{s+1}) + c_t \|x_t^{s+1} - \bar{x}\|^2]$. 
Then we have the upper bound of $R_{t+1}^{s+1}$:

\[
R_{t+1}^{s+1} = \mathbb{E} \left[ H(x^{s+1}_{t+1}) + c_{t+1} \|x^{s+1}_{t+1} - x^{s+1}_t - \bar{x}^s\|^2 \right]
\]

\[
\leq \mathbb{E} \left[ H(x^{s+1}_{t+1}) + c_{t+1} (1 + \frac{1}{\beta}) \|x^{s+1}_{t+1} - x^{s+1}_t\|^2 + c_{t+1} (1 + \beta) \|x^{s+1}_t - \bar{x}^s\|^2 \right]
\]

\[
\leq \mathbb{E} \left[ H(x^{s+1}_t) + \left[ L_f - \frac{1}{2\eta} \right] \|x^{s+1}_{t+1} - x^{s+1}_t\|^2 + \left[ c_{t+1} (1 + \frac{1}{\beta}) + \frac{L_f}{2} - \frac{1}{2\eta} \right] \|x^{s+1}_t - x^{s+1}_t\|^2 
+ \left( c_{t+1} (1 + \beta) + \frac{\eta L_f^2}{b_1} + \frac{2\eta B_G^4 L_F^2}{A} + \frac{2\eta B_F^2 L_G^2}{B} \right) \mathbb{E} \|x^{s+1}_t - \bar{x}^s\|^2 \right]
\]

\[
\leq \sum_{t=0}^{m-1} \left[ L_f - \frac{1}{2\eta} \right] \mathbb{E} \|x^{s+1}_{t+1} - x^{s+1}_t\|^2
\]

(84)

where the last inequality follows from that:

\[
c_t = c_{t+1} (1 + \frac{1}{\beta}) + \frac{\eta L_f^2}{b_1} + \frac{2\eta B_G^4 L_F^2}{A} + \frac{2\eta B_F^2 L_G^2}{B}
\]

\[
c_{t+1} (1 + \frac{1}{\beta}) + \frac{L_f}{2} \leq \frac{1}{2\eta}
\]

(85)

(86)

It is easy to know that $c_t$ is decreasing, if we let $c_m = 0$ and $\beta = \frac{1}{m}$, then we have:

\[
c_0 = \left( \frac{\eta m L_f^2}{b_1} + \frac{2\eta m B_G^4 L_F^2}{A} + \frac{2\eta m B_F^2 L_G^2}{B} \right) ((1 + \frac{1}{m})^m - 1)
\]

\[
\leq \left( \frac{\eta m L_f^2}{b_1} + \frac{2\eta m B_G^4 L_F^2}{A} + \frac{2\eta m B_F^2 L_G^2}{B} \right) (e - 1)
\]

(87)

where the inequality follows from that $(1 + \frac{1}{2})^l$ is increasing if $l > 0$ and $\lim_{l \to \infty} (1 + \frac{1}{2})^l = e$, where $e$ is Euler’s number. Therefore, in order to have (84), it should be satisfied that:

\[
c_0 (1 + \frac{1}{\beta}) + \frac{L_f}{2} \leq 2 \left( \frac{\eta m L_f^2}{b_1} + \frac{2\eta m B_G^4 L_F^2}{A} + \frac{2\eta m B_F^2 L_G^2}{B} \right) (1 + m) + \frac{L_f}{2}
\]

\[
\leq 4 \left( \frac{\eta m^2 L_f^2}{b_1} + \frac{2\eta m^2 B_G^4 L_F^2}{A} + \frac{2\eta m^2 B_F^2 L_G^2}{B} \right) + \frac{L_f}{2}
\]

\[
\leq \frac{1}{2\eta}
\]

(88)

where the first and second inequalities follow from that $e - 1 < 2$ and $m \geq 1$. Therefore, the condition can be written as:

\[
4 \left( \frac{\eta m^2 L_f^2}{b_1} + \frac{2\eta m^2 B_G^4 L_F^2}{A} + \frac{2\eta m^2 B_F^2 L_G^2}{B} \right) + \frac{L_f}{2} \leq \frac{1}{2\eta}
\]

(89)

Summing over (84) from $t = 0$ to $m - 1$, we have:

\[
R_{m+1}^{s+1} \leq R_0^{s+1} + \sum_{t=0}^{m-1} \left[ L_f - \frac{1}{2\eta} \right] \mathbb{E} \|x^{s+1}_{t+1} - x^{s+1}_t\|^2
\]

(90)

22
Because $R_s^{s+1} = \mathbb{E}[H(x_m^{s+1})] = \mathbb{E}[H(\bar{x})]$ and $R_0^{s+1} = \mathbb{E}[H(x_0^{s+1})] = \mathbb{E}[H(\bar{x})]$. Then adding up (90) from $s = 0$ to $S - 1$, we get:

$$
\sum_{s=0}^{S-1} \sum_{t=0}^{m-1} \left[ \frac{1}{2\eta} - L_f \right] \mathbb{E}\|\bar{x}_{t+1}^s - x_{t+1}^s\|^2 \leq H(\bar{x}) - H(\bar{x}_S) \leq H(\bar{x}) - H(x^*)
$$

(91)

where $x^*$ is the optimal solution. We define $\mathcal{G}_\eta(x_t^{s+1}) = \frac{1}{\eta}(x_t^{s+1} - \bar{x}_t^{s+1})$ and $x_a$ is uniformly selected from $\{x_t^{s+1}\}_{t=0}^{m-1} S_{t=0}^{S-1}$, we have:

$$
\mathbb{E}\|\mathcal{G}_\eta(x_a)\|^2 \leq \frac{2}{(1 - 2\eta L_f)\eta} \frac{H(\bar{x}) - H(x^*)}{T}
$$

(92)

## C Other Lemmas

**Lemma 3 (11)** For random variables $z_1, ..., z_r$ are independent and mean 0, we have:

$$
\mathbb{E}[\|z_1 + ... + z_r\|^2] = \mathbb{E}[\|z_1\|^2 + ... + \|z_r\|^2]
$$

(93)

**Lemma 4** For any $z_1, ..., z_r$, it holds that:

$$
\|z_1 + ... + z_r\|^2 \leq r(\|z_1\|^2 + ... + \|z_r\|^2)
$$

(94)

**Remark 2** In this remark, we follow the notations in [16], please check their paper for details. As per Theorem 1 in [16], the overall query complexity for com-SVR-ADMM is $O\left((m + n + \kappa^4) \log(1/\varepsilon)\right)$ and its convergence rate in terms of query is $O\left(\rho^{m+n+\kappa^4}\right)$. (Note: Corresponding notations in our paper, $n_1 \leftarrow n$, $n_2 \leftarrow m$, $m \leftarrow K$, $A \leftarrow N$, therefore, their query complexity is represented as $O\left(\rho^{n_1+n_2+\kappa^4}\right)$ in Table 1.)

**Proof:** As per Theorem 1 in [16], they prove that:

$$
\gamma_1 \mathbb{E}G(\bar{u}) \leq \gamma_2 G(\bar{u}^{s-1})
$$

(95)

where $\gamma_1$ and $\gamma_2$ is defined as follows:

$$
\gamma_1 = (2\eta - \frac{32\eta^2 C_G^4 L_f^2}{\mu_F N} - \frac{48\eta^2 L_f^2}{\mu_F} + 8\eta DC_G L_f L_G N^{-0.5}) K
$$

(96)

$$
\gamma_2 = (K + 1)(\frac{32\eta^2 C_G^4 L_f^2}{\mu_F N} + \frac{48\eta^2 L_f^2}{\mu_F} + 8\eta DC_G L_f L_G N^{-0.5}) + \frac{2}{\mu_F} + \frac{2\eta \rho \|A^T A\|}{\mu_F} + \frac{2L_f \eta}{\rho \sigma_{\min}(AA^T)}
$$

(97)
Their conclusion is true under two conditions: (1) $\gamma_1 > 0$; (2) $\frac{\gamma_2}{\gamma_1} < 1$. To make $\gamma_1 > 0$, following inequality should be satisfied that:

$$\eta \left( 2 - \frac{32\eta C_G L_f^2}{\mu_F N} - \frac{48\eta L_f^2}{\mu_F} - \frac{8DC_G L_f L_G N^{-0.5}}{\mu_F} \right) > 0 \quad (98)$$

Suppose that:

$$\frac{32\eta C_G L_f^2}{\mu_F N} < \frac{1}{4} \quad (99)$$

$$\frac{48\eta L_f^2}{\mu_F} < \frac{1}{4} \quad (100)$$

$$\frac{8DC_G L_f L_G N^{-0.5}}{\mu_F} < \frac{1}{4} \quad (101)$$

we have $\gamma_1 > \frac{5\eta K}{4}$. Therefore, following inequalities should be hold that:

$$N > \max \left\{ \frac{128\eta C_G L_f^2}{\mu_F}, \frac{32D_L^2 C_G^2 L_f^2 L_G^2}{\mu_F^2} \right\} \quad (102)$$

$$\eta < \frac{\mu_F}{192L_f^2} \quad (103)$$

Follow the definition of $\kappa$ in the paper, we have $\kappa = \max \{ \frac{L_f}{\mu_F}, \frac{L_G}{\mu_F}, \frac{L_f^2}{\mu_F} \}$, thus $N$ should be proportional to $\kappa^2$. Given $\gamma_1 = \frac{5\eta K}{4}$, in order to make $\frac{\gamma_2}{\gamma_1} < 1$, following inequality should be satisfied that:

$$\frac{3(K + 1)}{5K} + \frac{8}{5\eta \mu_F K} + \frac{8\rho \|A^T A\|}{5\mu_F K} + \frac{8L_f}{5\rho \sigma_{\min}(AA^T) K} < 1 \quad (104)$$

To make the above inequality true, at least, it should holds that:

$$\frac{8}{5\eta \mu_F K} < 1 \quad (105)$$

Therefore, we can also obtain that:

$$\frac{8 \times 192L_f^2}{5\mu_F^2} < K \quad (106)$$

So far, we know that $K$ is also proportional to $\kappa^2$. In [16], the authors claimed that their overall query complexity is $O \left( (m + n + KN) \log(1/\epsilon) \right)$. Because $K$ and $N$ are proportional to $\kappa^2$, their query complexity can also be represented as $O \left( (m + n + \kappa^4) \log(1/\epsilon) \right)$ and convergence rate in terms of query is $O \left( \frac{\epsilon}{\rho^{m+n+\kappa^4}} \right)$. 

24