CONVERGENCE ANALYSIS OF THE DISCRETE DUALITY
FINITE VOLUME SCHEME FOR THE REGULARISED
HESTON MODEL

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ABSTRACT. The aim of the paper is to study problem of financial derivatives
pricing based on the idea of the Heston model introduced in [9]. Following
the approach stated in [6] and in [7] we construct the regularised version of
the Heston model and the discrete duality finite volume (DDFV) scheme for
this model. The numerical analysis is performed for this scheme and stability
estimates on the discrete solution and the discrete gradient are obtained. In
addition the convergence of the DDFV scheme to the weak solution of the
regularised Heston model is proven. The numerical experiments are provided
in the end of the paper to test the regularisation parameter impact.

1. Studied problem introduction. Heston model, see [9], is known as popular
and useful generalization of the linear Black-Scholes model, see [2]. Primary it deals
with the constant volatility of the underlying asset assumption of the Black-Scholes
model. In the Heston’s case is the underlying asset price equation considered in the form

\[ dS_t = \mu S_t dt + \sqrt{v_t} S_t \, dw_t, \]

where \( \{w_t\}_{t \geq 0} \) is the Wiener process and \( \{v_t\}_{t \geq 0} \) is the variance of the underlying
process assumed to be modelled by the equation:

\[ dv_t = \kappa(\theta - v_t)dt + \sigma \sqrt{v_t} \, dz_t, \quad (1) \]

where \( \{z_t\}_{t \geq 0} \) is the Wiener process correlated with \( w_t \) by \( E[\,dw_t \, dz_t \,] = \rho \, dt \).

Therefore is the financial derivative price \( V \) assumed to be a function of time \( t \) and
two stochastic variables - underlying asset price \( S \) and underlying asset volatility \( v \).

For more information regarding Heston model principle see for instance [10].

As soon as we establish the transformations

\[ x = \ln \left( \frac{S}{E} \right), \quad y = v, \quad \tau = T - t, \quad u(x, y, \tau) = \frac{V(S, v, t)}{E}, \quad (2) \]

we are able to get the compact form of the Heston’s PDE:

\[ \frac{\partial u}{\partial \tau} + \vec{A} \cdot \nabla u = \nabla \cdot (B \nabla u) - ru, \quad (3) \]

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where

\[ B = \frac{1}{2} \sigma \left( \frac{1}{\rho \sigma} \rho \sigma \right), \quad \vec{A} = -\left( r - \frac{1}{2} y - \frac{1}{2} \rho \sigma \right) \cdot \vec{n}. \]

It is necessary to define initial and boundary conditions to complete the Heston model for the purposes of the numerical solution of the problem. They are derived from the usual Heston model conditions with respect to the substitution (2):

\[ u(x, y, 0) = \max(0, e^x - 1), \quad u(x, y, \tau) = 0, \]
\[ u(x, -\infty, y, \tau) = \frac{1}{E}(Ee^{\tau} - Ee^{-\tau}) \rightarrow \infty, \quad \frac{\partial u(x, y, -\infty, \tau)}{\partial \vec{n}} = 0, \]

where \( \vec{n} \) is the outward normal to the boundary \( \partial \Omega \). Analogly of the condition for the boundary \( y = 0 \), is not necessary as it is consequence of Fichera condition (cf. [5]). As one can find in [11] the Fichera condition states that if the advection velocity vector \( \vec{A} \) in (3) projected onto the unit normal vector is greater than zero then only the outflow is realized in this point. This can be, for \( y = 0 \), written in the form:

\[-\left( \begin{array}{c} r - \frac{1}{2} \rho \sigma \\ \kappa \theta - \frac{1}{2} \sigma^2 \end{array} \right) \cdot \left( \begin{array}{c} 0 \\ -1 \end{array} \right) = \kappa \theta - \frac{1}{2} \sigma^2 \geq 0.\]

In other words, no information can enter the domain from region where \( y < 0 \) and no boundary condition has to be prescribed.

2. Studied model and its parameters. The regularisation of the Heston model is done based on the ideas presented in [13], it means we consider following problem: find an unknown function \( v = v(x, y, \tau) \) - approximate solution to the following equation:

\[ \frac{\partial v}{\partial \tau} + \vec{A} \cdot \nabla v = \epsilon \Delta v + \nabla \cdot (B \nabla v) - rv, \quad (x, y, \tau) \in \Omega \times [t_1, t_2], \]

where \( \epsilon \) is the regularisation parameter of the problem and where \( \Omega \) is rectangular 2D domain, \( \Omega = (X_a, X_b) \times (0, Y) \), such that \( X_a < 0, X_b > 0 \) and \( Y > 0 \) and \( I = [t_1, t_2] \) is time interval, such that \( 0 < t_1 < t_2 < \infty \). Constants \( X_a, X_b, Y \) and \( t_1, t_2 \) are chosen by us and they should imitate the space domain \((-\infty, \infty) \times (0, \infty)\) and the time interval \((0, \infty)\) of the original model.

The initial and boundary conditions of the problem are derived from the original Heston model with the respect to the now finite domain and for the European call option are as follow:

\[ v(x, y, 0) = \max\{0, e^x - 1\}, \quad v(X_a, y, \tau) = 0, \quad v(X_b, y, \tau) = e^{X_b} - e^{-\tau y}, \]
\[ \frac{\partial v(x, 0, \tau)}{\partial \vec{n}} = 0, \quad \frac{\partial v(x, Y, \tau)}{\partial \vec{n}} = 0, \]

where \( \frac{\partial v}{\partial \vec{n}} \) means the co normal derivation of the \( v \) to the boundary \( \partial \Omega \) in the sense of Ladyzhenskaya, see [12], which means

\[ \frac{\partial v}{\partial \vec{N}} = ((B + \epsilon) \nabla v) \cdot \vec{n} = ((b_{11} + \epsilon)v_x + b_{12}v_y)_{n_1} + (b_{21}v_x + (b_{22} + \epsilon)v_y)_{n_2} \]

and where the boundary condition \( \frac{\partial v(x, 0, \tau)}{\partial \vec{n}} = 0 \) replaces the Fichera condition for \( y = 0 \) as the Fichera condition holds no more for the regularised case.

Based on the boundary condition we divide \( \Gamma \), boundary of the domain \( \Omega \), into two parts such that

\[ \Gamma = \Gamma_D \cup \Gamma_N, \]
where $\Gamma_D = \{(x, y) \in \Gamma : x = X_a \lor x = X_b\}$, part of the $\Gamma$ where Dirichlet boundary conditions are prescribed and $\Gamma_N = \{(x, y) \in \Gamma : y = 0 \lor y = Y\}$, part of the $\Gamma$ where Neumann boundary conditions are prescribed.

Explanation of the model parameters was discussed in detail in [10]:

- $\epsilon > 0$ is the regularisation parameter;
- $\mu \in \mathbb{R}$ is the shift of the process describing the underlying asset price;
- $\rho \in \langle -1, 1 \rangle$ is the correlation parameter between underlying asset price and the volatility of the financial derivative;
- $\sigma > 0$ is the volatility variance, which is taken to be a stochastic variable as defined in (1);
- $\theta > 0$ is the long term variance, around which the financial derivative volatility oscillate;
- $\kappa > 0$ is the reversion speed of the underlying asset volatility return to the long term variance;
- $\lambda > 0$ is the interest rate;
- $\rho$ is the market price of risk, which models the risk impact.

To simplify the problem we are studying we will find the function $v = v(x, y, \tau)$ in the form $v(x, y, \tau) = u(x, y, \tau) + w(x, y, \tau)$, where

$$w(x, y, \tau) = \max\{0, e^x - e^{-r\tau}\}.$$  

Function $w$ naturally fulfils the initial condition and allboundary conditions (5) except the one for $x = X_a$. To secure this condition we have to prescribe the restriction $X_a < -rT$. In reality this is no substantial restriction as we consider $X_a << 0$ to imitate $-\infty$ and it is only up to us how we define $X_a$.

Following problem holds then for the function $u$:

$$\frac{\partial u}{\partial \tau} + \vec{A} \cdot \nabla u = \epsilon \Delta u + \nabla \cdot (B \nabla u) - ru + f(x, y, \tau), \quad (x, y, \tau) \in \Omega \times [t_1, t_2],$$

where

$$f(x, y, \tau) = \begin{cases} 0 & \text{for } x < -r\tau, \\ \epsilon e^x & \text{for } x \geq -r\tau \end{cases}$$

and $u$ fulfils homogeneous initial and boundary conditions.

**Definition 2.1. (Weak solution of (6))** We say that $u$ is a weak solution of (6) if following holds, for all $I = (t_1, t_2)$, $0 < t_1 < t_2 < \infty$:

1. $u \in L^2(I; V(\Omega))$, where $V(\Omega) := \{u \in H^1(\Omega) : u|_{\Gamma_D} = 0\}.$

2. 

$$\int_0^1 \int_\Omega -u(x, y, \tau) \frac{\partial \psi}{\partial \tau}(x, y, \tau) + \vec{A} \cdot \nabla u(x, y, \tau) \psi(x, y, \tau) + \epsilon \nabla u(x, y, \tau) \nabla \psi(x, y, \tau) + B \nabla u(x, y, \tau) \nabla \psi(x, y, \tau) + ru(x, y, \tau) \psi(x, y, \tau) + f(x, y, \tau) \psi(x, y, \tau) dx dy d\tau =$$

$$\int_0^1 \int_\Omega f(x, y, \tau) \psi(x, y, \tau) dx dy d\tau,$$

$\forall \psi \in A := \{\psi \in C^1(I; C^1(\Omega)) : \psi(t_2, \cdot) = 0 \land \psi|_{\Gamma_D} = 0\}.$

3. **Discretisation and DDFV numerical scheme.** The DDFV scheme is based on the Finite Volume Method approach. FVM was applied for the Heston model before for instance in [11]. The derivation of the DDFV numerical scheme we are presenting here follows the notation from [7] mostly and from [6] and [8] as well.

We construct the fully implicit scheme. For the time discretisation we use uniform time step $k = \frac{t_n - t_0}{N}$ and $t_n = nk$ for $n = 0, 1, ..., N$. The time derivative is
approximated using the backward difference. On each time interval $[t_{n-1}, t_n]$ by $u^n$ we denote the piecewise constant function and by $f^n$ the average value of function $f$, which is piecewise constant as well. Using this notation for (6), where equality

$$\vec{A} \cdot \nabla u = \nabla \cdot (\vec{A}u) - (\nabla \cdot \vec{A})u$$

was used, we have:

$$\frac{u^n - u^{n-1}}{k} - \epsilon \Delta u^n - \nabla \cdot (B \nabla u^n) + \nabla \cdot (A u^n) - (\nabla \cdot \vec{A}) u^n + r u^n = f^n.$$  \tag{9}

For the space discretisation the method based on the finite volumes is used. By $u^n_{ij}$ and $f^n_{ij}$ we denote the piecewise constant function on each control volume $V_{ij}$ and each time interval $[t_{n-1}, t_n]$ as an average function on each finite volume and time interval.

Here we consider the rectangular domain in 2D. The finite volume mesh consists of cells $V_{ij} \in T_h$ with the measure $m(V_{ij})$, each associated with the point $x_{ij} = (x_{i,j}, y_{i,j}) \in V_{ij}$ for $i = 1, \ldots, N_1$ and $j = 0, 1, \ldots, N_2$, such that $\Omega = \bigcup_{V_{ij} \in T_h} V_{ij}$.

In addition by $\sigma^n_{ij}$ we denote the edges of the control volume $V_{ij}$ (their measure is then denoted by $m(\sigma^n_{ij})$) and by $d^n_{ij} = |x_{ij} - x_{i+1,j}|$ we denote the distance between the neighbouring representative points, $|p| + |q| = 1$. Especially, due to homogeneous Dirichlet boundary conditions, we denote $d_{11} = \text{dist}(x_{11}, \partial \Omega)$ and $d_{N_21} = \text{dist}(x_{N_21}, \partial \Omega)$. Unit outward normal vector to the edge $\sigma^n_{ij}$ we denote by $n_{ij}^\sigma$, $|p| + |q| = 1$.

Using standard FVM approach and this notation we conclude

$$\frac{u^n - u^{n-1}}{k} m(V_{ij}) - \epsilon \sum_{|p| + |q| = 1} \int_{\sigma^0_{ij}} \nabla u^n \vec{n}_{ij}^\sigma ds - \sum_{|p| + |q| = 1} \int_{\sigma^1_{ij}} B \nabla u^n \vec{n}_{ij}^\sigma ds + \sum_{|p| + |q| = 1} \int_{\sigma^1_{ij}} \vec{A} u^n \vec{n}_{ij}^\sigma ds - u^n_{ij} \int_{V_{ij}} (\nabla \cdot \vec{A}) dx + ru^n_{ij} m(V_{ij}) = f^n_{ij} m(V_{ij}).$$  \tag{10}

For each finite volume $V_{ij}$ and each edge $\sigma^n_{ij}$ we denote the coefficients for the tensor and advection term in the form:

$$B^n_{ij} = \begin{pmatrix} b_{11}^{ij} \\ b_{12}^{ij} \\ b_{21}^{ij} \\ b_{22}^{ij} \end{pmatrix}, A^n_{ij} = \begin{pmatrix} a_{11}^{ij} \\ a_{12}^{ij} \\ a_{21}^{ij} \\ a_{22}^{ij} \end{pmatrix}.$$ 

For our numerical approximation we can express all coefficients as a constant function on the whole edge for example by a value at the central point of an edge.

We construct two meshes as usual for the DDFV approach. The first one is described above and the second one, called dual mesh, is shifted to the north-east direction.

The dual mesh is created and denoted in analogy to the primal one, for instance the dual mesh consists of control volumes $\tilde{V}_{ij} \in \tilde{T}_h$ with measure $m(\tilde{V}_{ij})$ associated with points $\tilde{x}_{ij}$ for $i = 1, \ldots, N_1$ and $j = 0, 1, \ldots, N_2$ in such a way that $\tilde{x}_{ij}$ is the right top corner for the volume $V_{ij}$ of the original mesh. Again, all inner dual finite volumes are rectangles and boundary volumes are created in such a way that $\tilde{\Omega} = \bigcup_{\tilde{V}_{ij} \in \tilde{T}_h}$.

All other entities are denoted in similar way as defined above for primal mesh but "barred".

We can easily define constant gradients on diamond cells using this approach. It is an union of $\mathcal{D}_h$ and $\mathcal{D}_h$, where

$$\mathcal{D}_h = \bigcup_{(i,j) = (0,0), \ldots, (N_1, N_2)} D_{ij},$$
where $D_{ij}$ has vertices $\{x_{ij}, \bar{x}_{i,j-1}, x_{i+1,j}, \bar{x}_{i,j}\}$ with degenerated triangle diamonds on the boundaries (for $i = 0 \lor N_1$), and where

$$
\mathcal{D}_h = \bigcup_{(i,j)=(0,0), \ldots, (N_1, N_2)} D_{ij},
$$

where $\mathcal{D}_{ij}$ has vertices $\{x_{ij}, \bar{x}_{i,j}, x_{i,j+1}, \bar{x}_{i-1,j}\}$ with degenerated triangle diamonds on the boundaries (for $j = 0 \lor N_2$).

For each time interval $[t_{n-1}, t_n]$ we can define the gradient in the form:

$$
\nabla u^n_{ij} = \left( \frac{u^n_{i+1,j} - u^n_{ij}}{d^n_{ij}}, \frac{u^n_{i,j+1} - u^n_{ij}}{m^n(\sigma^n_{ij})} \right) = (u^n_{x_{ij}}, u^n_{y_{ij}}) \text{ on } D_{ij},
$$

$$
\nabla u^n_{ij} = \left( \frac{\bar{u}^n_{ij} - \bar{u}^{n-1}_{ij}}{d^n_{ij}}, \frac{u^n_{ij+1} - u^n_{ij}}{m^n(\sigma^n_{ij})} \right) = (u^n_{x_{ij}}, \bar{u}^n_{y_{ij}}) \text{ on } \mathcal{D}_{ij}.
$$

Following definition is taken from [7] and was stated in [8] before:

**Definition 3.1.** If we denote by $(k, h)$ the time-space discretisation of $(t_1, t_2) \times \Omega$, then the function $u_{k,h}$ is a piecewise function in space and time defined as follows

$$
u^n_{k,h}(x, y, \tau) = \frac{1}{2} \left( \nu^n_{k,h,V}(x, y, \tau) + \nu^n_{k,h,V}(x, y, \tau) \right),
$$

where

$$
u^n_{k,h,V}(x, y, \tau) = \nu^n_{ij} \text{ for } (x, y) \in V_{ij}, \tau \in ((n-1)k, nk],
$$

$$
u^n_{k,h,V}(x, y, \tau) = \bar{u}^n_{ij} \text{ for } (x, y) \in \bar{V}_{ij}, \tau \in ((n-1)k, nk].
$$

Especially for the $n$-th time step we have

$$
u^n_{k,h}(x, y, \tau) = \frac{1}{2} \left( \nu^n_{k,h,V}(x, y, \tau) + \nu^n_{k,h,V}(x, y, \tau) \right),
$$

where

$$
u^n_{k,h,V}(x, y, \tau) = \nu^n_{ij} \text{ for } (x, y) \in V_{ij},
$$

$$
u^n_{k,h,V}(x, y, \tau) = \bar{u}^n_{ij} \text{ for } (x, y) \in \bar{V}_{ij}.
$$

Approximative time derivation and approximative gradient we define as

$$
\delta u^n_{k,h}(x, y, \tau) = \frac{1}{2} \left( \delta u^n_{k,h,V}(x, y, \tau) + \delta u^n_{k,h,V}(x, y, \tau) \right),
$$

$$
\delta u^n_{k,h,V}(x, y, \tau) = \frac{u^n_{ij} - u^n_{ij-1}}{k} \text{ for } (x, y) \in V_{ij}, \tau \in ((n-1)k, nk],
$$

$$
\delta u^n_{k,h,V}(x, y, \tau) = \frac{\bar{u}^n_{ij} - \bar{u}^{n-1}_{ij}}{k} \text{ for } (x, y) \in \bar{V}_{ij}, \tau \in ((n-1)k, nk],
$$

$$
\nabla u^n_{k,h}(x, y, \tau) = \begin{cases} 
\nabla u^n_{ij} \text{ for } (x, y) \in D_{ij}, \tau \in ((n-1)k, nk],
\n\nabla \bar{u}^n_{ij} \text{ for } (x, y) \in \mathcal{D}_{ij}, \tau \in ((n-1)k, nk].
\end{cases}
$$

We construct both meshes as super admissible mesh consisting of rectangles only with edges $h_x$ in $x$-direction and $h_y$ in $y$-direction. This means that $m(V_{ij}) = m(\bar{V}_{ij}) = h_x h_y$, $m(\sigma_{ij}) = d_{ij} = h_y$, $m(\bar{\sigma}_{ij}) = d_{ij} = h_x$.

The DDFV numerical scheme for the Heston model was derived in [6] and stated in [7] as well. Here we are stating the DDFV numerical scheme for the regularised Heston model (6). The (10) equation terms we approximate in analogy to the [6] and [7]. In addition the regularisation term is approximated as follows:

$$
\epsilon \sum_{|p|+|q|=1} \int_{\sigma_{ij}} \nabla u^n_{ij} \tilde{r}^{pq}_{ij} ds \approx \epsilon(h_y[u^n_{ij} - u^{i-1,j,n}] + h_x[u^n_{ij} - \bar{u}^{i,j-1,n}]),
$$
For an unknown value $u_{ij}$ (primal mesh) from (9) we get

$$
\frac{u^n_{ij} - u^{n-1}_{ij}}{k} h_x h_y - \epsilon (h_y [u^{i+1,j,n} - u^{i,j,n}] + h_x [u^{i,j+1,n} - u^{i,j,n}]) - h_y [b_{i,j+1,10}^{11} u^{i+1,j,n} + b_{i,j+1,10}^{12} u^{i,j+1,n}] - h_x [b_{i+1,j,10}^{21} u^{i,j+1,n} + b_{i+1,j,10}^{22} u^{i+1,j,n}] + h_y [b_{i,j,10}^{11} u^{i-1,j,n} + b_{i,j,10}^{12} u^{i,j-1,n}] + h_x [b_{i,j,10}^{21} u^{i,j-1,n} + b_{i,j,10}^{22} u^{i+1,j,n}] + h_y a_{i+1,j,10} [u^{i+1,j+1,n} - u^{i+1,j,n}] + h_x a_{i,j+1,10} [u^{i,j+1,n} - u^{i,j,n}] - h_y a_{i+1,j,10} [u^{i+1,j,n} - u^{i,j,n}] - h_x a_{i,j+1,10} [u^{i,j+1,n} - u^{i,j,n} - \frac{k}{2} y_{ij} + \frac{1}{2} \rho \sigma - r] + a_{ij}^2 = \frac{1}{2} \sigma^2 - \kappa \theta + \kappa y_{ij} + \lambda y_{ij} \text{ for the numerical analysis of the model (6).}
$$

**Lemma 4.1.** Let the discretization has the properties described in the Section 3 and let $k$, $h_x$ and $h_y$ are the same as defined there. Let it hold

$$
\sigma \geq |\rho|, \quad 1 \geq |\rho| \sigma, \quad Y(\lambda + \kappa) \geq \kappa \theta - \frac{1}{2} \sigma^2.
$$

In addition let $\epsilon$ be fixed regularisation parameter from (4) and (6). Then for the numerical solution of the scheme (15) - (16) the following stability estimates hold:

$$
\|u_{k,h}\|_{L_\infty (I; L_2(\Omega))} \leq C, \quad \|\nabla u_{k,h}\|_{L_2(I; L_2(\Omega))} \leq C(\epsilon),
$$

where $C(\epsilon)$ is generic constant depending only on the data of the problem and the regularisation parameter $\epsilon$, not on parameters $k$, $h_x$ and $h_y$.

**Proof.** Before we begin the proof one should notice that due to the page limitation we do not repeat parts of the proof published in [6] before and we pay attention on the new oginal additions as the original proof was done for the Heston model and we are dealing with the regularised version.

$h_y$ is the vertical diagonal size of each $D_{ij}$ and $h_x$ is the horizontal diagonal size of each $D_{ij}$. The tensor and convection terms coefficients evaluated at the barycentre of $D_{ij}$ we denote by $b_{ij}$ and $a_{ij}$ respectively. Analogously $h_y$ is the vertical diagonal size of each $D_{ij}$, $h_x$ is the horizontal diagonal size of each $D_{ij}$ and $b_{ij}$ and $a_{ij}$ are the tensor and convection terms coefficients evaluated at the barycentre of $D_{ij}$. Finally we denote by $D_{int}$ the set of all diamonds $D_{ij}$ for $i = 1, ..., N_1 - 1$ and for $j = 1, ..., N_2$ and by $D_{ext}$ the set of all diamonds $D_{ij}$ for $i = 2, ..., N_1 - 1$ and for $j = 1, ..., N_2 - 1$. These represent the diamonds that have no point belonging to the Dirichlet boundary condition.
Now as a first step we multiply the scheme represented by the equation (15) by \(2k u_{ij}^n\) and sum over primal mesh finite volumes.

For the terms with regularisation parameter \(\epsilon\) it holds:

\[
-2k(h_x \sum_{V_{ij} \in \mathcal{T}} u_{ij}^n [u_{ij}^{n-1} - u_{ij}^{n-1,0}] + h_y \sum_{V_{ij} \in \mathcal{T}} u_{ij}^n [\bar{u}_{ij}^{n-1} - \bar{u}_{ij}^{n-1,0}]) = \epsilon 2k \left( \sum_{D_{ij} \in \mathcal{D}} h_x h_y (u_{ij}^{n-1,0})^2 + \sum_{D_{ij} \in \mathcal{D}} h_x h_y (\bar{u}_{ij}^{n-1,0})^2 \right).
\]

(19)

Using the results from [6] and together with (19) and its analogy for the dual scheme and summing for \(n = 1, \ldots, m\) gives us:

\[
\sum_{V_{ij} \in \mathcal{T}} (u_{ij}^m)^2 h_x h_y + \sum_{V_{ij} \in \mathcal{T}} (\bar{u}_{ij}^m)^2 h_x h_y + 2kr \sum_{n=1}^m \left( \sum_{V_{ij} \in \mathcal{T}} (u_{ij}^n)^2 h_x h_y + \sum_{V_{ij} \in \mathcal{T}} (\bar{u}_{ij}^n)^2 h_x h_y \right) + \epsilon 2k \sum_{n=1}^m \left( \sum_{D_{ij} \in \mathcal{D}} (u_{ij}^{n-1,0})^2 + (u_{ij}^{n-1})^2 + \sum_{V_{ij} \in \mathcal{T}} (\bar{u}_{ij}^{n-1,0})^2 + (\bar{u}_{ij}^{n-1})^2 \right) h_x h_y + 2k \sum_{n=1}^m \sum_{D_{ij} \in \mathcal{D}} \left( b_{ij}^{11} (u_{ij}^{n-1,0})^2 + b_{ij}^{22} (u_{ij}^{n-1})^2 + 2b_{ij}^{12} u_{ij}^{n-1,0} u_{ij}^{n-1} \right) h_x h_y + 2k \sum_{n=1}^m \sum_{D_{ij} \in \mathcal{D}} \left( \bar{b}_{ij}^{11} (\bar{u}_{ij}^{n-1,0})^2 + \bar{b}_{ij}^{22} (\bar{u}_{ij}^{n-1})^2 + 2\bar{b}_{ij}^{12} \bar{u}_{ij}^{n-1,0} \bar{u}_{ij}^{n-1} \right) h_x h_y + A_1 + A_2 = \sum_{V_{ij} \in \mathcal{T}} (u_{ij}^0)^2 h_x h_y + \sum_{V_{ij} \in \mathcal{T}} (\bar{u}_{ij}^0)^2 h_x h_y + 2k \sum_{V_{ij} \in \mathcal{T}} f_{ij}^0 u_{ij}^0 h_x h_y + 2k \sum_{V_{ij} \in \mathcal{T}} f_{ij}^0 \bar{u}_{ij}^0 h_x h_y,
\]

(20)

where we used the symmetry of the matrix \(B\) and where

\[
A_1 = 2k \sum_{n=1}^m \sum_{D_{ij} \in \mathcal{D}} \left( h_x a_{ij}^1 ((u_{ij}^{n+1})^2 - (u_{ij}^n)^2) + h_y a_{ij}^2 ((\bar{u}_{ij}^{n})^2 - (\bar{u}_{ij-1}^n)^2) \right),
\]

\[
A_2 = 2k \sum_{n=1}^m \sum_{D_{ij} \in \mathcal{D}} \left( h_y \bar{a}_{ij}^1 ((\bar{u}_{ij}^n)^2 - (\bar{u}_{ij-1}^n)^2) + h_x a_{ij}^2 ((u_{ij+1}^n)^2 - (u_{ij}^n)^2) \right).
\]

The results from [6], where (17) was used, guarantee that:

\[
A_1 + A_2 \geq -(\lambda + \kappa) \sum_{n=1}^m k \left( \sum_{V_{ij} \in \mathcal{T}} (u_{ij}^n)^2 h_x h_y + \sum_{V_{ij} \in \mathcal{T}} (\bar{u}_{ij}^n)^2 h_x h_y \right).
\]

(21)

We substitute (21) to (20), realize non-negativity of the second row of (20), take into account the homogeneous initial condition for \(u\) and therefore get:

\[
\sum_{V_{ij} \in \mathcal{T}} (u_{ij}^n)^2 h_x h_y + \sum_{V_{ij} \in \mathcal{T}} (\bar{u}_{ij}^n)^2 h_x h_y + 2kr \sum_{n=1}^m \left( \sum_{V_{ij} \in \mathcal{T}} (u_{ij}^n)^2 h_x h_y + \sum_{V_{ij} \in \mathcal{T}} (\bar{u}_{ij}^n)^2 h_x h_y \right) + \]

(22)
\[
2k \sum_{n=1}^{m} \sum_{D_{ij} \in \mathcal{D}} \left( \sum_{D_{ij} \in \mathcal{D}} (u_{x}^{ij,n})^2 + (u_{y}^{ij,n})^2 + \sum_{D_{ij} \in \mathcal{D}} (\bar{u}_{x}^{ij,n})^2 + (\bar{u}_{y}^{ij,n})^2 \right) \|h_x h_y +
\]
\[
2k \sum_{n=1}^{m} \sum_{D_{ij} \in \mathcal{D}} \left( \bar{b}_{ij}^{11} (u_{x}^{ij,n})^2 + \bar{b}_{ij}^{22} (u_{y}^{ij,n})^2 + 2\bar{b}_{ij}^{12} u_{x}^{ij,n} u_{y}^{ij,n} \right) h_x h_y +
\]
\[
2k \sum_{n=1}^{m} \sum_{D_{ij} \in \mathcal{D}} \left( \bar{b}_{ij}^{11} (\bar{u}_{x}^{ij,n})^2 + \bar{b}_{ij}^{22} (\bar{u}_{y}^{ij,n})^2 + 2\bar{b}_{ij}^{12} \bar{u}_{x}^{ij,n} \bar{u}_{y}^{ij,n} \right) h_x h_y \leq
\]
\[
k(2 + (\lambda + \kappa)) \sum_{n=1}^{m} \left( \sum_{V_{ij} \in \mathcal{T}} (u_{ij}^{n})^2 h_x h_y + \sum_{V_{ij} \in \mathcal{T}} (\bar{u}_{ij}^{n})^2 h_x h_y \right) + 2\|f_{k,h}\|^{2}_{L_{2}(I; L_{2}(\Omega))},
\]
where the Cauchy-Schwarz inequality was used in the last step.

From the previous results published in [6] using (17) we know that
\[
k c \sum_{n=1}^{m} \sum_{D_{ij} \in \mathcal{D}} \bar{y}_{ij} (\nabla \bar{u}_{ij})^2 \leq 2k \sum_{n=1}^{m} \sum_{D_{ij} \in \mathcal{D}} \left( \bar{b}_{ij}^{11} (u_{x}^{ij,n})^2 + \bar{b}_{ij}^{22} (u_{y}^{ij,n})^2 + 2\bar{b}_{ij}^{12} u_{x}^{ij,n} u_{y}^{ij,n} \right).
\]

Analogical estimate holds for the “overlined” coefficients:
\[
k c \sum_{n=1}^{m} \sum_{D_{ij} \in \mathcal{D}} \bar{y}_{ij} (\nabla \bar{u}_{ij})^2 \leq 2k \sum_{n=1}^{m} \sum_{D_{ij} \in \mathcal{D}} \left( \bar{b}_{ij}^{11} (\bar{u}_{x}^{ij,n})^2 + \bar{b}_{ij}^{22} (\bar{u}_{y}^{ij,n})^2 + 2\bar{b}_{ij}^{12} \bar{u}_{x}^{ij,n} \bar{u}_{y}^{ij,n} \right).
\]

Using (22), (23) and (24) and realizing non-negativity of few terms in the scheme we get:
\[
\sum_{V_{ij} \in \mathcal{T}} (u_{ij}^{n})^2 h_x h_y + \sum_{V_{ij} \in \mathcal{T}} (\bar{u}_{ij}^{n})^2 h_x h_y \leq 2\|f_{k,h}\|^{2}_{L_{2}(I; L_{2}(\Omega))} +
\]
\[
(2 + (\lambda + \kappa)) \sum_{n=1}^{m} k \left( \sum_{V_{ij} \in \mathcal{T}} (u_{ij}^{n})^2 h_x h_y + \sum_{V_{ij} \in \mathcal{T}} (\bar{u}_{ij}^{n})^2 h_x h_y \right).
\]

From (7) we can derive that \(\|f_{k,h}\|^{2}_{L_{2}(I; L_{2}(\Omega))} < \infty\), so Gronwall inequality could be used:
\[
\|u_{m,h}\|^{2}_{L_{2}(I; L_{2}(\Omega))} + 2r ||u_{k,h}\|^{2}_{L_{2}(I; L_{2}(\Omega))} + 2r ||\nabla u_{k,h}\|^{2}_{L_{2}(I; L_{2}(\Omega))} + 2r ||\nabla u_{k,h}\|^{2}_{L_{2}(I; L_{2}(\Omega))} \leq
\]
\[
(2 + (\lambda + \kappa)) T \left( \frac{2}{\alpha} ||f_{k,h}\|^{2}_{L_{2}(I; L_{2}(\Omega))} + \frac{2r}{\alpha} ||\nabla u_{k,h}\|^{2}_{L_{2}(I; L_{2}(\Omega))} \right) + 2||f_{k,h}\|^{2}_{L_{2}(I; L_{2}(\Omega))}.
\]

As (25) holds for every \(m\) we have \(\|u_{k,h}\|_{L_{\infty}(I; L_{2}(\Omega))} \leq C\). In addition it is clear that \(\|\nabla u_{k,h}\|^{2}_{L_{2}(I; L_{2}(\Omega))} \leq \frac{C}{2r} \), which concludes proof of the lemma.

5. Convergence of the DDFV scheme. We use stability estimates (18) proven in the Lemma 4.1 to obtain the convergence of the numerical scheme (15) - (16) to the weak solution of the problem (6).

Lemma 5.1 (Convergence properties). Let \(\Omega\) be rectangular 2D domain and \([t_1, t_2]\) be the time interval, \(0 < t_1 < t_2 < \infty\). Let \(u_{k,h}\) be defined by (11) - (12) and by (13) - (14) for the n-th time step. Let \((k_m, h_m)\) denotes a sequence of space-time discretisations such that \(k_m \to 0\) and \(h_m \to 0\) as \(m \to \infty\). Then there exists \(\bar{u} \in L^{2}(I; H^{1}(\Omega))\) such that \(u_{k_m,h_m} \rightharpoonup \bar{u}\) in \(L^{2}(I; H^{1}(\Omega))\) as \(m \to \infty\).
Proof. From (18) it is clear that \( u_{k_m,h_m} \) is bounded in \( L^\infty(I; L^2(\Omega)) \) for all \( m = 1, \ldots, N_T \). Furthermore we know that \( \nabla u_{k_m,h_m} \) is bounded in \( L^2(I; L^2(\Omega))^2 \) for all \( m = 1, \ldots, N_T \).

As our mesh is the admissible one we can use the Lemma 3.6 from [1], which is the DDFV analogy of the Lemma 4.6 from [3], stating the time translation estimate on the numerical solution of the scheme (15) - (16) and the Theorem 6.1 from [4], which is a generalization of Ascoli’s theorem, to show the convergence \( u_{k_m,h_m}(\cdot, t) \rightarrow \tilde{u} \in L^2(I; L^2(\Omega)) \).

Moreover, thanks to the Lemma 3.6 from [1], we have that \( \tilde{u} \in L^2(I; H^1(\Omega)) \) and that \( \nabla u_{k_m,h_m} \rightharpoonup \nabla \tilde{u} \in L^2(I \times \Omega)^2 \).

Now it is necessary to show that there is a unique numerical solution \( u_{k,h} \) of the scheme (15) - (16) and there is a unique solution \( \tilde{u} \) of the regularised Heston model (6).

Uniqueness of the \( u_{k,h} \) was proven in [6] for the original Heston model DDFV scheme and the proof can be easily modified for our case. Uniqueness of the solutions of types of problems Heston model belongs to was proven by Ladyzhenskaya, see [12] and references therein, for the Dirichlet and Neumann boundary conditions. As in our problem we are dealing with the mixture of both types of boundary conditions the original proofs have to be used together. Uniqueness of \( u_{k,h} \) and \( \tilde{u} \) conclude the proof of the lemma.

\[ \square \]

**Theorem 5.2.** Let \( \Omega \) be rectangular 2D domain and \( I = [t_1, t_2] \) be the time interval, \( 0 \leq t_1 < t_2 < \infty \). Let \( u_{k,h} \) be defined by (11) - (12) and by (13) - (14) for the \( n \)-th time step. Let \( (k_m, h_m) \) be a sequence of space-time discretisations such that \( k_m \rightarrow 0 \) and \( h_m \rightarrow 0 \) as \( m \rightarrow \infty \). Then the function \( \tilde{u} \in L^2(I; H^1(\Omega)) \) such that \( u_{k_m,h_m} \rightharpoonup \tilde{u} \) in \( L^2(I; H^1(\Omega)) \) as \( m \rightarrow \infty \) is the weak solution of (6) in the sense of the Definition 2.1.

Proof. From the Lemma 5.1 we know that there exists \( \tilde{u} \in L^2(I; H^1(\Omega)) \) such that \( u_{k_m,h_m} \rightharpoonup \tilde{u} \in L^2(I; H^1(\Omega)) \). We rewrite (8) to the form

\[
- \int_I \int_{\Omega} \frac{\partial \tilde{\psi}}{\partial t} dxdy \tau + \int_I \int_{\Omega} \tilde{\phi} \cdot \nabla \tilde{u} dxdy \tau + \int_I \int_{\Omega} ((B + \epsilon) \nabla \tilde{u} \nabla \psi dxdy \tau + \int_I \int_{\Omega} ru \psi dxdy \tau = \int_I \int_{\Omega} f \psi dxdy \tau,
\]

\( \forall \psi \in A := \{ \varphi \in C^1(I; C^1(\Omega)) : \varphi(t_2, \cdot) = 0 \wedge \varphi|_{\Gamma_D} = 0 \} \).

Let choose an arbitrary \( \phi \in A \) such that \( \phi \in C^1(I; C^2(\Omega)) \) and define \( \phi_{ij}^n = \phi(x, y, \tau) \) for \( (x, y) \in V_{ij}, \tau \in ((n - 1)k, nk] \) and \( \delta \phi_{ij} = \phi(x, y, \tau) \) for \( (x, y) \in V_{ij}, \tau \in ((n - 1)k, nk] \). In analogy to the Definition 3.1 we define symbols \( \delta \phi_{ij}^n, \phi_{ij,n}, \delta \phi_{ij,n}^n, \phi_{ij,n}^n \) and \( \delta \phi_{k,h} \).

Now we multiply the scheme (15) by \( \frac{1}{2} k \phi_{ij}^n \) and sum it over all \( V_{ij} \) and \( n \) to get

\[
\frac{1}{2} \sum_{n=1}^{N} \sum_{V_{ij} \in \mathcal{I}} k \sum_{n=1}^{N} \frac{u_{ij}^n - u_{ij}^{n-1}}{k} \phi_{ij}^n h_x h_y + \frac{1}{2} \sum_{n=1}^{N} \sum_{D_{ij} \in \mathcal{D}} h_x h_y (b_{ij}^{11} \phi_{ij}^n + b_{ij}^{12} u_{ij}^n \phi_{ij}^n + b_{ij}^{13} u_{ij}^{n+1} \phi_{ij}^n) + \sum_{n=1}^{N} \sum_{D_{ij} \in \mathcal{D}} h_y a_{ij}^1 (u_{i+1,j}^n - u_{ij}^n) (\phi_{ij}^n + \phi_{ij}^{n+1}) + (26)
\]
$\sum_{n=1}^{N} \sum_{D_{ij} \in \mathcal{D}} h_x \varphi_y (\tilde{b}_{ij}^{22} + \epsilon) \tilde{a}^{ij,n}_{xy} \tilde{a}^{ij,n}_{x} + \tilde{b}_{ij}^{12} u^{ij,n}_{x} \tilde{b}^{ij,n}_{x} + \tilde{b}_{ij}^{21} u^{ij,n}_{x} \tilde{b}^{ij,n}_{x} + \tilde{b}_{ij}^{22} + \epsilon u^{ij,n}_{x} + \phi^{ij,n}_{y})$.

Similarly we multiply the scheme (16) by $\frac{1}{2}k \tilde{b}_{ij}^{ij,n}$ and sum it over all $\tilde{V}_{ij}$ and $n$. After summing (26) and its dual version we obtain the equation

$$T_1 + T_2 + T_3 + T_4 = T_5,$$

where

$$T_1 = \frac{1}{2} \left( \sum_{n=1}^{N} \sum_{V_{ij} \in T} \frac{u^{n}_{ij} - u^{n-1}_{ij}}{k} \phi^{n}_{ij} h_x h_y + \sum_{n=1}^{N} \sum_{V_{ij} \in T} \frac{\tilde{u}^{n}_{ij} - \tilde{u}^{n-1}_{ij}}{k} \tilde{\phi}^{n}_{ij} h_x h_y \right),$$

$$T_2 = \frac{1}{2} \sum_{n=1}^{N} \left( \sum_{D_{ij} \in \mathcal{D}} \frac{1}{2} \left( h_y a_{ij}^{n} (u^{n}_{ij} + u^{n}_{ij}) (\phi^{n}_{ij} + \phi^{n}_{ij}) + h_x \alpha_{ij}^{n} (u^{n}_{ij} - u^{n}_{ij}) (\phi^{n}_{ij} + \phi^{n}_{ij}) \right) \right),$$

$$T_3 = \frac{1}{2} \sum_{n=1}^{N} \left( \sum_{D_{ij} \in \mathcal{D}} \left( h_x h_y (b_{ij}^{11} + \epsilon) u^{ij,n}_{x} \phi^{n}_{y} + b_{ij}^{12} u^{ij,n}_{x} \phi^{n}_{x} + b_{ij}^{21} u^{ij,n}_{x} \phi^{n}_{x} + b_{ij}^{22} + \epsilon u^{ij,n}_{x} + \phi^{ij,n}_{y} \right) \right),$$

$$T_4 = \frac{1}{2} \left( \sum_{n=1}^{N} \sum_{V_{ij} \in T} r u^{n}_{ij} \phi^{n}_{ij} h_x h_y + \sum_{n=1}^{N} \sum_{V_{ij} \in T} \tilde{r} \tilde{u}^{n}_{ij} \tilde{\phi}^{n}_{ij} h_x h_y \right),$$

$$T_5 = \frac{1}{2} \left( \sum_{n=1}^{N} \sum_{V_{ij} \in T} f^{n}_{ij} \phi^{n}_{ij} h_x h_y + \sum_{n=1}^{N} \sum_{V_{ij} \in T} \tilde{f}^{n}_{ij} \tilde{\phi}^{n}_{ij} h_x h_y \right).$$

From this point going forward we are dividing the proof into 5 steps as we are showing above mentioned convergence term by term.

**Step 1.** If we take a look at the first part of $T_1$ and reorder terms there we can write

$$\sum_{n=1}^{N} \sum_{V_{ij} \in T} \frac{u^{n}_{ij} - u^{n-1}_{ij}}{k} \phi^{n}_{ij} h_x h_y = \sum_{n=1}^{N} \sum_{V_{ij} \in T} \frac{(u^{n}_{ij} - u^{n-1}_{ij}) \phi^{n}_{ij} h_x h_y = \sum_{n=1}^{N} \sum_{V_{ij} \in T} u^{n}_{ij} (\phi^{n-1}_{ij} - \phi^{n}_{ij}) h_x h_y + u_{n}^{N} \phi^{N}_{ij} - u_{0}^{0} \phi^{0}_{ij} = \sum_{n=1}^{N} \sum_{V_{ij} \in T} \frac{u^{n}_{ij} \phi^{n}_{ij} - \phi^{n-1}_{ij}}{k} h_x h_y$$

thanks to the homogeneous initial condition and construction of the space $\mathcal{A}$. 

As the similar equation holds for the “overlined” coefficients we obtain
\[
T_1 = -\frac{1}{2} \left( \sum_{n=1}^{N} \sum_{i,j \in T} u^n_{ij} \phi^n_{ij} - \phi^n_{ij} h_x h_y + \sum_{n=1}^{N} \sum_{i,j \in T} \bar{u}^n_{ij} \phi^n_{ij} - \phi^n_{ij} h_x h_y \right),
\]
which, with the definition of \( \delta \phi_{k,h} \), concludes that
\[
T_1 = -\int_{\Omega} u_{k,h} \delta \phi_{k,h} dx dy dt.
\]
In addition (strong) convergence of the \( \delta \phi_{k,h} \) to \( \frac{\partial \psi}{\partial t} \) in \( L^2(I; L^2(\Omega)) \) and weak convergence of \( u_{k,h} \) to \( \bar{u} \) in \( L^2(I; L^2(\Omega)) \) guarantees that
\[
T_1 \to -\int_{\Omega} \bar{u} \frac{\partial \psi}{\partial t} dx dy dt.
\]

Step 2. We define
\[
\bar{T}_2 = \int_{\Omega} \hat{\nabla} u_{k,h} \hat{\phi} dx dy dt,
\]
where \( \hat{\phi}(x,y,\tau) := \phi(x,y,t_n) \) for \( \tau \in (t_{n-1},t_n) \) (\( \hat{\phi} \) is piecewise constant in time) and write down \( T_2 = \bar{T}_2 + (T_2 - \bar{T}_2) \). Thanks to the strong convergence of the \( \phi \) to \( \hat{\phi} \) in \( L^2(I; L^2(\Omega)) \) and weak convergence of \( \nabla u_{k,h} \) to \( \nabla \bar{u} \) in \( L^2(I; L^2(\Omega)) \) we are able to conclude
\[
\bar{T}_2 \to \int_{\Omega} \hat{\nabla} \bar{u} \hat{\phi} dx dy dt.
\]
Now we want to show the convergence of the residual \( T_2 - \bar{T}_2 \) to zero. We define following symbols
\[
\phi^n_{i+\frac{1}{2},j} = \frac{\phi^n_{i+1,j} + \phi^n_{i,j}}{2}, \phi^n_{i,j+\frac{1}{2}} = \frac{\phi^n_{i,j+1} + \phi^n_{i,j}}{2}, \phi^n_{i,j-\frac{1}{2}} = \frac{\phi^n_{i,j} + \phi^n_{i-1,j}}{2}, \phi^n_{i-\frac{1}{2},j} = \frac{\phi^n_{i,j} + \phi^n_{i,j-1}}{2},
\]
which, thanks to the smoothness of the \( \phi \), could be used for the \( T_2 \) term approximation
\[
T_2 = \frac{1}{2} \sum_{n=1}^{N} k \left( \sum_{D_{i,j} \in D} h_x h_y \left( a^1_{i,j} u_x^{ij,n} \phi^n_{i+\frac{1}{2},j} + a^2_{i,j} u_y^{ij,n} \phi^n_{i,j+\frac{1}{2}} \right) + \sum_{D_{i,j} \in D} h_x h_y \left( a^1_{i,j} u_x^{ij,n} \phi^n_{i-\frac{1}{2},j} + a^2_{i,j} u_y^{ij,n} \phi^n_{i,j-\frac{1}{2}} \right) \right).
\]
(27)
Turning back to the \( \bar{T}_2 \) study using the \( \nabla u_{k,h} \) definition stated in the Definition 3.1 we are able to develop it as follows
\[
\bar{T}_2 = \sum_{n=1}^{N} k \left( \sum_{D_{i,j} \in D} \left( a^1_{i,j} \int_{D_{i,j}} a_1 \hat{\phi} dx dy + u_y^{ij,n} \int_{D_{i,j}} a_2 \hat{\phi} dx dy \right) + \sum_{D_{i,j} \in D} \left( a^1_{i,j} \int_{D_{i,j}} \hat{a}_1 \hat{\phi} dx dy + a^2_{i,j} \int_{D_{i,j}} \hat{a}_2 \hat{\phi} dx dy \right) \right).
\]
(28)
Comparing (27) with (28) we can see that we have to estimate four terms, where the first one is defined as
\[
Q^{ij}_{1,j} := \left( a^1_{i,j} \phi_{i+\frac{1}{2},j} h_x h_y - \int_{D_{i,j}} a_1 \hat{\phi} dx dy \right) \int_{D_{i,j}} \left( a^1_{i,j} \phi_{i+\frac{1}{2},j} - a_1 \hat{\phi} \right) dx dy.
\]
(29)
Using smoothness of the function $\phi$ and fact that the coefficient $a_1$ does not depend on the variable $x$ we are able to develop (29) to get
\[
|Q^1_{ij}| = \left| \int_{D_{ij}} \left( \frac{a_{ij}^n}{h_x y} \frac{h_x h_y}{2} - \frac{2}{h_x y} \frac{h_x h_y}{2} \right) dxdy \right| \leq c_1 \max \{h_x, h_y\} \frac{h_x h_y}{2},
\]
where $c_1 := c_{11}|a_{ij}^n| + c_{12} \max_{D_{ij}} |\hat{\phi}|$. Similar considerations lead to following estimates
\[
|Q^2_{ij}| := \left| \frac{a_{ij}^n}{h_x y} \frac{h_x h_y}{2} - \frac{2}{h_x y} \frac{h_x h_y}{2} \right| \leq c_2 \max \{h_x, h_y\} \frac{h_x h_y}{2},
\]
\[
|\tilde{Q}^1_{ij}| := \left| \frac{a_{ij}^n}{h_x y} \frac{h_x h_y}{2} - \frac{2}{h_x y} \frac{h_x h_y}{2} \right| \leq c_3 \max \{h_x, h_y\} \frac{h_x h_y}{2},
\]
\[
|\tilde{Q}^2_{ij}| := \left| \frac{a_{ij}^n}{h_x y} \frac{h_x h_y}{2} - \frac{2}{h_x y} \frac{h_x h_y}{2} \right| \leq c_4 \max \{h_x, h_y\} \frac{h_x h_y}{2}.
\]
For the sake of simplicity we define $\tilde{c} := \max \{c_1, c_2, c_3, c_4\} \max \{h_x, h_y\}$ and we estimate the $T_2 - \hat{T}_2$ differential. We use the Cauchy-Schwarz inequality and inequality $\sqrt{A^2 + B^2} \leq \sqrt{2(A^2 + B^2)}$:
\[
|T_2 - \hat{T}_2| \leq \tilde{c} \sum_{n=1}^{N} k \left( \sum_{D_{ij} \in \mathcal{D}} \left( u_{ij}^n + u_{ij}^m \right) \frac{h_x h_y}{2} \right) \sum_{D_{ij} \in \mathcal{D}} \frac{h_x h_y}{2} \leq 2 \tilde{c} \sqrt{2m(\Omega)} \sum_{n=1}^{N} k \left( \sum_{D_{ij} \in \mathcal{D}} \left( (\nabla u_{ij}^n)^2 \right) \frac{h_x h_y}{2} \right) \sum_{D_{ij} \in \mathcal{D}} \frac{h_x h_y}{2} \leq (30)
\]
\[
2 \tilde{c} \sqrt{t_2 - t_1} \sqrt{m(\Omega)} \|
\]
for $h_x, h_y \to 0$ as we know that $\nabla u_{k,h}$ is bounded in $L_2(I; L_2(\Omega))^2$ from the (18).

**Step 3.** In analogy to the previous step we define
\[
\hat{T}_3 = - \int_I \int_{\Omega} \nabla \cdot ((B + \epsilon) \nabla \phi) u_{k,h} dxdy d\tau
\]
and divide $T_3$ as follows $T_3 = \hat{T}_3 + (T_3 - \hat{T}_3)$. Thanks to the smoothness of the $\phi$ and weak convergence of $u_{k,h}$ to $\tilde{u}$ in $L^2(I; L^2(\Omega))$ we get
\[
\hat{T}_3 = - \int_I \int_{\Omega} \nabla \cdot ((B + \epsilon) \nabla \phi) \tilde{u} dxdy d\tau = \int_I \int_{\Omega} ((B + \epsilon) \nabla \phi) \tilde{u} dxdy d\tau.
\]
Let us turn to the study of the $T_3 - \hat{T}_3$. Under the symbol $e_{ij}$ we understand the vertical diagonal of the diamond cell $D_{ij}$ and $e_{ji}$ is the horizontal diagonal and vice versa we define the $\tilde{e}_{ij}$ as the vertical diagonal of the diamond cell $\hat{D}_{ij}$ and as $\tilde{e}_{ji}$ the horizontal diagonal.

\[
\hat{T}_3 = \frac{1}{2} \sum_{n=1}^{N} k \left( \sum_{V_{ij} \in \mathcal{V}} \int_{V_{ij}} \nabla \cdot ((B + \epsilon) \nabla \phi) u_{ij}^n dxdy + \sum_{V_{ij} \in \mathcal{V}} \int_{V_{ij}} \nabla \cdot ((B + \epsilon) \nabla \phi) \tilde{u}_{ij}^n dxdy \right)
\]
\[
= \frac{1}{2} \sum_{n=1}^{N} k \left( \sum_{V_{ij} \in \mathcal{V}} u_{ij}^n \int_{V_{ij}} (B + \epsilon) \nabla \phi \cdot \tilde{n} d\sigma + \sum_{V_{ij} \in \mathcal{V}} \tilde{u}_{ij}^n \int_{V_{ij}} (B + \epsilon) \nabla \phi \cdot \tilde{n} d\sigma \right) =
\]
\[
\frac{1}{2} \sum_{n=1}^{N} k \left( \sum_{D_{ij} \in \mathcal{D}} h_x u_{ij}^{x,n} \int_{e_{ij}} (b_{11} + \epsilon) \phi_x + b_{21} \phi_y ds + h_y u_{ij}^{y,n} \int_{e_{ij}} (b_{12} + \epsilon) \phi_x + b_{22} \phi_y ds \right.
\]
\[
+ \sum_{D_{ij} \in \mathcal{D}} h_x u_{ij}^{x,n} \int_{e_{ij}} (b_{11} + \epsilon) \phi_x + b_{21} \phi_y ds + h_y u_{ij}^{y,n} \int_{e_{ij}} (b_{12} + \epsilon) \phi_x + b_{22} \phi_y ds \right).
\]
Looking back on the $T_3$ definition we can see that actually we have to estimate four terms. The first one is in the form

$$|R_{ij}^1| := \left| \left( \tilde{b}_{ij}^1 \varphi_x^{ij,n} + \tilde{b}_{ij}^2 \varphi_y^{ij,n} \right) h_y - \int_{t_{ij}}^{t_{ij+1}} ((b_{11} + \epsilon) \phi_x + b_{21} \phi_y) \cdot \tilde{\mathbf{n}} ds \right| \leq C_4(h_x + h_y)h_y$$

thanks to the smoothness of the diffusion tensor $\mathbf{B}$ coefficients and function $\phi$ appropriate second order Taylor expansion. Similarly for the other terms we get

$$|R_{ij}^2| := \left| \left( \tilde{b}_{ij}^{11} \varphi_x^{ij,n} + (\tilde{b}_{ij}^{12} + \epsilon) \tilde{\varphi}_x^{ij,n} \right) h_y - \int_{t_{ij}}^{t_{ij+1}} ((b_{11} + \epsilon) \phi_x + b_{21} \phi_y) ds \right| \leq C_5(h_x + h_y)h_y,$$

$$|R_{ij}^3| := \left| \left( \tilde{b}_{ij}^{21} \varphi_y^{ij,n} + (\tilde{b}_{ij}^{22} + \epsilon) \tilde{\varphi}_y^{ij,n} \right) h_y - \int_{t_{ij}}^{t_{ij+1}} ((b_{11} + \epsilon) \phi_x + b_{21} \phi_y) ds \right| \leq C_6(h_x + h_y)h_y.$$

Following the reasoning given in (30) we get

$$|T_3 - T_3| = \frac{1}{2} \sum_{n=1}^{N} \left( \sum_{D_{ij} \in \mathcal{D}} h_x |u_x^{ij,n} R_{ij}^1| + h_y |u_y^{ij,n} R_{ij}^2| \right) \leq \max\{C_1, C_2, C_3, C_4\}(h_x + h_y)\sqrt{T_3 - t_1}\sqrt{m(\Omega)} \to 0$$

for $h_x, h_y \to 0$ thanks to the $\nabla u_{k,h}$ boundedness in $L^2(I; L^2(\Omega))^2$ proven in (18).

**Step 4.** Definitions of $u_{k,h}$ and $\phi_{k,h}$ provide that

$$T_4 = \int_I \int_{\Omega} r u_{k,h} \phi_{k,h} dxdy\tau.$$

As $r$ is a constant, $\phi_{k,h}$ is strongly convergent in $L^2(I; L^2(\Omega))$ to $\phi$ and $u_{k,h} \rightharpoonup \tilde{u}$ in $L^2(I; L^2(\Omega))$ we get

$$T_4 \to \int_I \int_{\Omega} r \tilde{u} \phi dxdy\tau.$$

**Step 5.** From the $f_{k,h}$ and the $\phi_{k,h}$ definitions it is clear that

$$T_5 = \int_I \int_{\Omega} f_{k,h} \phi_{k,h} dxdy\tau.$$

From the strong convergences of the $\phi_{k,h}$ to $\phi$ and $f_{k,h}$ to $f$ in $L^2(I; L^2(\Omega))$ we conclude

$$T_5 \to \int_I \int_{\Omega} f \phi dxdy\tau.$$

Putting all five steps together finalizes the proof. \hfill \Box

6. **Regularisation parameter importance experiment.** We are extending the experiment presented in [7] originaly stated in [10], so we are using the same computation domain

$$\Omega := \{(x, y) \in \mathbb{R}^2 : -7 \leq x \leq 3, 0 \leq y \leq 1\},$$

the time interval $< 0, 0.05 >$, the strike price $E = 100$, initial condition $u(x, y, 0) = \max\{0, e^x - 1\}$ and boundary conditions

$$u(3, y, \tau) = e^3 - e^{-\tau}, u(-7, y, \tau) = 0 = \frac{\partial u}{\partial y}(x, 0, \tau) = \frac{\partial u}{\partial y}(x, 1, \tau).$$

There is an exact solution known for this case, so we are able to compare the results gained by the regularised scheme against the one without any regularisation. We
use the $L^2$ error definition from [10]. That means we are computing the error only on sub domain $<-1,1> \times <0,1>$. Taking into account used transformation $x = \ln \frac{S}{S_0}$ we obtain the interval $<36,272>$ for the underlying asset price $S$, which represents its usual interval, see [10].

Under the symbol $N_x$ we understand the number of the primal mesh finite volumes along the horizontal boundary and under the symbol $N_y$ along the vertical boundary. $N_{ts}$ is the number of computed time steps. By $L^2_D$ we denote the errors for the DDFV scheme for the classic Heston model and by $L^2_R$ the errors for the regularised Heston model DDFV scheme. As $L^2_R$ depends on $\epsilon$ in every table there are more options of the $L^2_R$ for specified $\epsilon$ stated.

**Experiment 1.** In the first experiment we take the parameters as given in [7]:

$$\rho = -0.5, \, \sigma = 0.5, \, r = 0.1, \, \kappa = 5, \, \theta = 0.07, \, \lambda = 0.$$ 

In the Table 1 we can see the results for various options of the parameter $\epsilon$ choice:

| $N_x$ | $N_y$ | $N_{ts}$ | $L^2_D$ | $L^2_R, \epsilon = 10^{-2}$ | $L^2_R, \epsilon = 10^{-4}$ | $L^2_R, \epsilon = 10^{-6}$ |
|-------|-------|----------|---------|---------------------|---------------------|---------------------|
| 20    | 10    | 1        | 0.00318557 | 0.00329745         | 0.00318659         | 0.00318559         |
| 40    | 20    | 4        | 0.00206132 | 0.00211980         | 0.00206182         | 0.00206133         |
| 80    | 40    | 16       | 0.00151241 | 0.00156704         | 0.00151286         | 0.00151242         |
| 160   | 80    | 64       | 0.00125001 | 0.00130976         | 0.00125050         | 0.00125002         |

**Table 1.** Results for the regularised and the original DDFV scheme comparison, Experiment Nr. 1.

One can observe that errors of both models decrease with increasing number of the space and time steps. In addition it is clear that $L^2_R(\epsilon) \to L^2_D$ as $\epsilon \to 0$ for all listed meshes and for $\epsilon$ sufficiently small are the results for the regularised model almost the same as for the non-regularised case.

Both mentioned observations are requisite and in line with the expectations.

**Experiment 2.** For the second presented experiment is the set of the model parameters as follows:

$$\rho = 0.9, \, \sigma = 0.35, \, r = 0.1, \, \kappa = 5, \, \theta = 0.07, \, \lambda = 0.$$ 

This setting imitates the situation when the assumptions of the Lemma 4.1, the stability estimate, are not fulfilled. Second motivation of this experiment is to study the case when there is a positive correlation between processes $\{w_t\}_{t \geq 0}$ and $\{z_t\}_{t \geq 0}$.

One can take a look on the Table 2 to observe that $L^2_R(\epsilon) \to L^2_D$ as $\epsilon \to 0$ still holds, but the most interesting observation shown in the Table 2 is that the smallest error occurs with the biggest $\epsilon$ value for the sparse meshes:

| $N_x$ | $N_y$ | $N_{ts}$ | $L^2_D$ | $L^2_R, \epsilon = 10^{-2}$ | $L^2_R, \epsilon = 10^{-4}$ | $L^2_R, \epsilon = 10^{-6}$ |
|-------|-------|----------|---------|---------------------|---------------------|---------------------|
| 20    | 10    | 1        | 0.00377821 | 0.00371450         | 0.00377742         | 0.00377822         |
| 40    | 20    | 4        | 0.00269958 | 0.00264958         | 0.00269896         | 0.00269957         |
| 80    | 40    | 16       | 0.00199309 | 0.00197965         | 0.00199286         | 0.00199309         |
| 160   | 80    | 64       | 0.00155891 | 0.00157838         | 0.00155904         | 0.00155891         |

**Table 2.** Results for the regularised and the original DDFV scheme comparison, Experiment Nr. 2.
This insight needs deeper future research to identify its cause and to conclude whether the regularisation with the bigger value of $\epsilon$ could be numerically useful for problems with special parameters sets. Another option is to define the regularisation parameter dependent on the mesh properties, parameters or variables.

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