GENERALIZED HOMOTOPY THEORY IN CATEGORIES WITH A NATURAL CONE

FRANCISCO J. DÍAZ AND JOSÉ M. G. CALCINES

ABSTRACT. In proper homotopy theory, the original concept of point used in the classical homotopy theory of topological spaces is generalized in order to obtain homotopy groups that study the infinite of the spaces. This idea: “Using any arbitrary object as base point” and even “any morphism as zero morphism” can be developed in most of the algebraic homotopy theories. In particular, categories with a natural cone have a generalized homotopy theory obtained through the relative homotopy relation. Generalized homotopy groups and exact sequences of them are built so that respective classical pointed ones are a particular case of these.

1. Introduction

The main problem to build a homotopy theory based on the concept of cone object is to define the homotopy relation between morphisms through the notion of nullhomotopic morphism. In additive categories this problem can be solved using the addition of morphisms in the category, and generalizing the injective homotopy theory defined by Eckmann and Hilton [Hi] for \( R \)-modules. Algebraic theories in this sense were created by H. Kleisli [Kl] and J. A. Seebach [S]. Also, S. Rodríguez-Machín gives an algebraic homotopy theory based on a cone functor in additive categories, without using injective objects, that contains the injective homotopy theories mentioned above as particular cases [P]. A generalization to arbitrary categories has been given by S. Rodríguez-Machín and the first author of this paper in [D]: Identifying the notion of dual standard construction given by P. J. Huber [Hu] with the concept of cone, and adapting the axioms given by H. J. Baues about cofibrations in Categories with a Natural Cylinder [B] to a cone object obtained by collapsing the base of the cylinder to a single point. The algebraic homotopy theory in this way defined attains that classical homotopy theory of the topological spaces and pointed topological spaces can be developed using their respective cones. Also the proper homotopy theory of the topological spaces can be developed in this sense, through a cone functor.

Finally, in categories with a natural cylinder in the sense of Baues, the cone functor obtained by collapsing, using pushout diagrams, the base of the cylinder functor to a single point gives the same homotopy theory that the cylinder functor.

In order to study the infinite of topological spaces, the proper homotopy theory defines several homotopy groups using different spaces as base point. So, the Brown

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homotopy groups \([B\] can be defined using a sequence of points a base point, and the Steenrod homotopy groups \([C\) use spaces based on a base ray. H.J. Baues uses trees as base point of the spaces to create a proper homotopy theory \([?\) \([B\). These facts suggest a more general concept of base point in homotopy theory. Moreover, in many theories homotopy groups and exact sequences of them can be defined using arbitrary objects and morphisms as base point and zero morphism, respectively. These homotopy theories are called generalized.

A generalized homotopy theory can be defined in categories with a natural cone using relative homotopy: Generalized homotopy groups are homotopy groups relative to a cofibration based on a morphism. Also, generalized exact homotopy sequences of these groups are given. In this way, when the category is pointed, the classical homotopy theory is a generalized homotopy theory based on a point.

Finally, fixed an object as base point in a category with a natural cone, a generalization of the process used to obtain spheres beginning with a point in topological spaces allows one to define spheres beginning with the fixed object.

We point out that the main results of this paper were already announced in \([D-1\).

2. Notation and preliminaries

The following categorical notation will be useful in the interpretation of this paper.

Given functors \(B \to E \to C \to F \to G \to D \to H \to E\) and a transformation \(t : F \to G\), then the transformations \(t \circ x : FE \to GE\) and \(H \circ t : HF \to HG\) will be denoted by \(t_E\) and \(Ht\), respectively. When there is not a possibility of confusion, the morphism \(t_X : FX \to GX\) will be simply denoted by \(t\), for every object \(X\).

The pushout object of two morphisms \(f\) and \(g\) will be denoted by \(P\{f,g\}\). The induced morphisms will be denoted by \(\tilde{f} : \text{codom } g \to P\{f,g\}\) and \(\tilde{g} : \text{codom } f \to P\{f,g\}\). Given a morphism \(f\), if the notation \(\tilde{f}\) has been used, \(\tilde{f}\) will denote other induced morphism by \(f\) in a pushout. In particular, if \(f = g\) then \(\tilde{f} = \tilde{g}\) and \(\tilde{f}\) will denote the morphisms \(\tilde{f}\) and \(\tilde{g}\), respectively.

Given morphisms \(r\) and \(s\) verifying \(rf = sg\), the unique morphism \(h\) such that \(h\tilde{g} = r\) and \(h\tilde{f} = s\) will be denoted by \(\{r,s\}\). If \(codom f\) (resp. \(codom g\)) is a pushout object, the component \(r\) (resp. \(s\)) has an expression like \(\{r_1,r_2\}\) (resp. \(\{s_1,s_2\}\)). Frequently, in this case, the morphism \(\{r,s\} = \{r_1,r_2\}, s\) (resp. \(\{r,s_1,s_2\}\)) will be denoted by \(\{r_1,r_2,s\}\) (resp. \(\{r,s_1,s_2\}\)). In this way, expressions of the type \(\{h_0,h_1,...,h_n\}\) can appear.

Given two pushout objects \(P\{f,g\}\) and \(P\{f',g'\}\), and three morphisms \(r : codom f \to codom f'\), \(s : codom g \to codom g'\) and \(t : dom f = dom g \to dom f' = dom g'\) verifying \(rf = f'\tilde{t}\) and \(sg = g't\), we will denote the unique morphism \(\{g'r,\tilde{f}s\}\) by \(r \cup s\). If there is not possibility of confusion, expressions of the type \(h_0 \cup h_1 \cup ... \cup h_n\) will be used.

Finally, the set of extensions of a morphism \(u : B \to X\) relative to other morphism \(i : B \to A\) is defined by \(\text{Hom}(A,X)^{u(i)} = \{f : A \to X \mid fi = u\}\).

Next we recall some concepts given in \([D-1\) relative to a category with a natural cone.

Definition 2.1. A category with a natural cone, or \(C\)-category, is a category \(C\) together with a class “cof” of morphisms in \(C\), called cofibrations and denoted
by $\rightarrow$, a functor $C : C \rightarrow C$ which will be called the cone functor, and natural transformations $\kappa : 1 \rightarrow C$ and $\rho : CC \rightarrow C$, denominated inclusion and projection respectively, satisfying the following axioms:

C1. Cone axiom. $\rho \kappa = \rho = \rho \rho$.

C2. Pushout axiom. For any pair of morphisms $A \leftarrow B \rightarrow X$, where $i$ is a cofibration, there exists the pushout square

\[ \begin{array}{ccc} B & \xrightarrow{f} & X \\ i \downarrow & & \downarrow \tau \\ A & \xrightarrow{T} & P\{i, f\} \end{array} \]

and $T$ is also a cofibration. The cone functor carries this pushout diagram (called cofibrated pushout) into a pushout diagram, that is $C(P\{i, f\}) = P(Ci, Cf)$.

C3. Cofibration axiom. For each object $X$ the morphisms $1_X$ and $\kappa_X$ are cofibrations. The composition of two cofibrations is a cofibration. Moreover, there is a retraction for the cone of each cofibration. This last property is called nullhomotopy extension property (NEP).

C4. Relative cone axiom. Given a cofibration $i : B \rightarrow A$, the morphism $i_1 = \{Ci, \kappa\} : \Sigma^i = P\{\kappa, i\} \rightarrow CA$ is also a cofibration. The object $\Sigma^i$ is called relative cone of $i$.

Note that isomorphisms and cones of cofibrations are also cofibrations.

Given a cofibration $i$, for each non-negative integer $n$ one defines $i^n = (i^{n-1})_1$, with $i^n = i$.

Theorem 2.2. Given the commutative cubical diagram

\[ \begin{array}{ccc} X & \xrightarrow{g} & Z \\
\alpha & \Downarrow & \alpha \cup \beta \\
Y & \xleftarrow{f'} & \rightarrow \ P\{f, g\} \\
\gamma & \Downarrow & \beta \\
Y' & \xrightarrow{f'} & \rightarrow \ P\{f', g'\} \end{array} \]

where the top and bottom faces are pushouts and $\alpha, \beta, \gamma$ are cofibrations. If $\{g', \beta\} : P\{\gamma, g\} \rightarrow Z'$ or $\{f', \alpha\} : P\{\gamma, f\} \rightarrow Y'$ is a cofibration then so is $\alpha \cup \beta$.

Definition 2.3. A morphism $f : X \rightarrow Y$ is said to be nullhomotopic ($f \simeq 0$) if there exists an extension $F : CX \rightarrow Y$ of the morphism $f$ relative to the cofibration $\kappa_X$. $F$ is called a nullhomotopy for $f$ ($F : f \simeq 0$).

An object $X$ is said to be contractible ($X \simeq 0$) when $1_X \simeq 0$. 

So a morphism is nullhomotopic if and only if it may be factored through a contractible object. Hence the nullhomotopy extension property can be also stated in terms of nullhomotopy:

**Theorem 2.4 (NEP).** Given a morphism \( i : B \to A \), the following sentences are equivalent:

a) The morphism \( i \) verifies NEP.

b) Every nullhomotopic morphism \( f : B \to X \) has a nullhomotopic extension rel. \( i \).

c) Every nullhomotopic morphism \( f : B \to X \) has an extension rel. \( i \).

d) The inclusion \( \kappa : B \to CB \) has an extension rel. \( i \).

**Definition 2.5.** A cofibration \( i : B \to A \) is said to be contractible when \( B \) and \( A \) are contractible objects. These cofibrations will be the contractible objects in the category of pairs.

Observe that pushout objects of two contractible cofibrations are contractible objects. Hence if \( i \) is contractible then \( i_n \) and \( \Sigma^{n-1} i \) are, for each natural number \( n \).

**Definition 2.6.** Given a cofibration \( i : B \to CA \) and two morphisms \( f_0, f_1 : CA \to X \), \( f_0 \) is said to be homotopic to \( f_1 \) relative to the cofibration \( i \) \( (f_0 \simeq f_1 \text{ rel. } i) \) if there exists an extension \( F : C^2 A \to X \) of the morphism \( \{f_0 C i, f_1\} \) relative to the cofibration \( i_1 \). \( F \) is called homotopy from \( f_0 \) to \( f_1 \) relative to \( i \), in symbols \( F : f_0 \simeq f_1 \text{ rel. } i \).

**Remark 2.7.** The homotopy relation relative to a cofibration \( i \) is an equivalence relation compatible with the composition of morphisms in the following sense:

- If \( F : f_0 \simeq f_1 \text{ rel. } i \) then \( fF : f f_0 \simeq f f_1 \text{ rel. } i \).
- Given a commutative square \((Cf)i = jg\), if \( F : f_0 \simeq f_1 \text{ rel. } j \) then \( FC^2f : f_0 Cf \simeq f_1 Cf \text{ rel. } i \). If the commutative square is a pushout then \( f_0 \simeq f_1 \text{ rel. } j \) if and only if \( f_0 Cf \simeq f_1 Cf \text{ rel. } i \).

The quotient set \( \text{Hom}(CA, X) u(i)/\simeq \) will be denoted by \( [CA, X] u(i) \), where \( u = f_0 i = f_1 i \).

The following property is fundamental to obtain to obtain equalities among morphisms save homotopy:

**Theorem 2.8.** If \( X \) or \( i \) is contractible then \( f_0 \simeq f_1 \text{ rel. } i \) if and only if \( f_0 i = f_1 i \).

**Definition 2.9.** A \( C \)-category is said to be pointed if every object \( X \) is cofibrant (that is, the initial morphism \( \emptyset_X : \emptyset \to X \) is a cofibration) and \( C \emptyset = \emptyset \). In pointed categories the initial object is denoted by \( * \) and it is called point.

If \( X \) and \( Y \) are objects of a pointed \( C \)-category, then \( X \lor Y \) will denote the pushout object \( P\{*X, *Y\} \). Note that \( C(X \lor Y) = CX \lor CY \). If \( X \simeq 0 \) and \( Y \simeq 0 \) then \( X \lor Y \simeq 0 \). If \( i : B \to A \) and \( i' : B' \to A' \) are cofibrations then \( i \lor i' : B \lor B' \to A \lor A' \) so is.

**Remark 2.10.** \( [CA \lor CA', X] \{u, u'\}(i \lor i') \cong [CA, X] u(i) \times [CA', X] u'(i') \)
3. Generalized Homotopy Groups

In this section the homotopy groupoid relative to a cofibration \( i : B \to CA \) of an object \( X \), \( \mathbf{H}_i(X) \), is built in order to define the first homotopy group, \( \pi^1_i(X, h) \), relative to a cofibration \( i \) based on a morphism \( h : CA \to X \). Then higher homotopy groups are defined as first homotopy groups relative to iterated cofibrations: 
\[
\pi^n_i(X, h) = \pi^{n-1}_i(X, h\rho^{n-1}).
\]

Finally, main properties about the functorial character of these groups, and their relation with coproducts and contractible objects or cofibrations are studied.

The following commutative square is fundamental to obtain the groupoid \( \mathbf{H}_i(X) \):

\[
\begin{array}{ccc}
\Sigma^1 & \xrightarrow{\rho(Ci) \cup 1} & P\{i, i\} \\
\downarrow_{i_1} & & \downarrow_{\kappa} \\
C^2 A & \xrightarrow{\mu} & CP\{i, i\}
\end{array}
\]

\( \mu \) is an extension of the morphism \( \kappa \rho(Ci) \cup \kappa \) relative to the cofibration \( i_1 \).

Given \( f_0, f_1 \in \text{Hom}(CA, X) \), \( H_i(f_0, f_1) \) will denote the homotopy bracket \( [C^2 A, X]^{\{f_0, f_1\}(i_1)} \).

If \( F, G \in \text{Hom}(C^2 A, X) \cup \{f_0, f_1\}(i_1) \), then \( F \) and \( F * G \) will denote the morphisms \( \{F, f_0\}\mu \) and \( \{F, G\}\mu \), respectively.

**Lemma 3.1.** \( \mu^* : [P\{Ci, Ci\}, X]^{\{f_0, f_1\}(\kappa)} \to H_i(f_0, f_1) \) is a bijection.

**Proof.** \( (\mu^*)^{-1} : H_i(f_0, f_1) \to [P\{Ci, Ci\}, X]^{\{f_0, f_1\}(\kappa)} \) is defined by \( (\mu^*)^{-1}([F]) = \{f_0\rho, F\} \).

- \( \mu^* \) is well defined. If \( F : F_0 \simeq F_1 \) rel. \( \kappa \) then \( F\nu : F_0\mu \simeq F_1\mu \) rel. \( i_1 \), where \( \nu \) is an extension of the morphism \( \{(C\kappa)\mu \rho(Ci), i_1\} \) relative to \( i_2 \).
- \( (\mu^*)^{-1} \) is well defined. If \( F : F_0 \simeq F_1 \) rel. \( i_1 \), then \( \{f_0\rho\mu^2, F\} : \{f_0\rho, F_0\} \simeq \{f_0\rho, F_1\} \) rel. \( \kappa \).
- \( (\mu^*)^{-1} \mu^* = 1 \). \( \{FC\rho, \{F\rho, G\rho\}C\mu\} : \{F, G\} \simeq \{f_0\rho, \{F, G\}\mu\} \) rel. \( \kappa \).
- \( \mu^* (\mu^*)^{-1} = 1 \). \( \{f_0\rho^2, F\rho\}C\mu : F \simeq \{f_0\rho, F\}\mu \) rel. \( i_1 \).

\[\square\]

**Theorem 3.2.** \( \mathbf{H}_i(X) \) is a groupoid, with objects \( \text{Hom}(CA, X) \): morphisms from \( f_0 \) to \( f_1 \), \( H_i(f_0, f_1) \): identities \( 1 = [f_0] \); inverse morphisms \( [F]^{-1} = [F] \); and composite morphisms \( [F, G] = [F \star G] \).

**Proof.** Remark 1: If \( F : F_0 \simeq F_1 \) rel. \( i_1 \) and \( G : G_0 \simeq G_1 \) rel. \( i_1 \) then \( \{F, G\} : \{F_0, G_0\} \simeq \{F_1, G_1\} \) rel. \( \kappa \).

- Inverse morphisms are well defined: If \( [F_0] = [F_1] \in H_i(f_0, f_1) \), by Remark 1 \( \{F_0, f_0\rho\} \simeq \{F_1, f_0\rho\} \) rel. \( \kappa \). By Lemma 3.1 \( [F_0] = [F_1] \) in \( H_i(f_1, f_0) \).
- Composite morphisms are well defined: If \( [F_0] = [F_1] \in H_i(f_0, f_1) \) then \( [F_0] = [F_1] \). Moreover, if \( [G_0] = [G_1] \in H_i(f_1, f_2) \), by Remark 1 \( \{F_0, G_0\} \simeq \{F_1, G_1\} \) rel. \( \kappa \). By Lemma 3.1 \( [F_0 \star G_0] = [F_1 \star G_1] \in H_i(f_0, f_2) \).

Remark 2: By Lemma 3.1 and the definition of inverse morphism, \( [f_0\rho]^{-1} = [f_0\rho] \) for every \( f : CA \to X \).

- Left homotopy identity property: If \( [F] \in H_i(f_0, f_1) \), by Remarks 1, 2 and Lemma 3.1 \( [f_0\rho \star F] = [F] \).
- Right homotopy identity property: If \( [F] \in H_i(f_0, f_1) \), then \( \{(F, f_0\rho^2)C\mu, FC\rho\} : \{f_0\rho, F\} \simeq \{(F, f_0\rho)\mu, f_1\rho\} \) rel. \( \kappa \). By Lemma 3.1 \( [F] = [F \star f_1\rho] \).
Remark 3: \([F] = [F]\), for every \([F] \in H_i(f_0, f_1)\), since \(F = F \ast f_1 \rho\).

- Right homotopy inverse property: If \([F] \in H_i(f_0, f_1)\), then \(\{F \cup C \rho, F \cup C \rho\} : \{F, F\} \simeq \{f_0 \rho, f_0 \rho\}\) rel. \(\kappa\). By Remark 2 and Lemma 3.1 \([F \ast F] = [f_0 \rho]\).

- Left homotopy inverse property: Since the composition of morphisms is well defined, by Remark 3 and the right homotopy inverse property \([F \ast F] = [F], [F'] = [f_1 \rho]\).

Remark 4: If \([F] \in H_i(f_0, f_1)\) and \([G] \in H_i(f_1, f_2)\) then \(\{F \rho, G \rho\} C \mu, F \cup C \rho\} : \{G, F\} \simeq \{f_0 \rho, f_0 \rho\}\) rel. \(\kappa\). By Remarks 1, 3 and Lemma 3.1 \([G \ast F] = [F \ast G]\).

Remark 5: If \([F] \in H_i(f_0, f_1)\) and \([G] \in H_i(f_1, f_2)\) and \([H] \in H_i(f_1, f_3)\) then \(\{f_0 \rho, \rho \rho\} C \mu, \{H \rho, G \rho\} C \mu\} : \{F, G\} \simeq \{f_0 \rho, f_0 \rho\}\) rel. \(\kappa\). By Remarks 1, 3, 4 and Lemma 3.1 \([F \ast G] = [(F \ast H) \ast (G \ast H)]\).

- Homotopy associative property: If \([F] \in H_i(f_0, f_1)\), \([G] \in H_i(f_1, f_2)\) and \([H] \in H_i(f_1, f_3)\), by the left homotopy identity property and Remark 5 \(H = [f_0 \rho \ast H] = [(f_0 \rho \ast \rho) \ast (G \ast H)] = [G \ast (G \ast H)]\). So \([F \ast G] \ast [H] = [(F \ast G) \ast (G \ast H)] = [F \ast (G \ast H)]\) since the composition of morphisms is well defined and by Remark 5.

\[\square\]

Definition 3.3. The \(n\)-th homotopy group relative to a cofibration \(i : B \rightarrow CA\) of an object \(X\) based on a morphism \(h : CA \rightarrow X\) is

\[\pi_i^n(X, h) = H_{i-1} (h \rho^{n-1}, h \rho^n)\], \(n \in \mathbb{N}\)

Given a cofibration \(i : B \rightarrow A\) and a morphism \(h : CA \rightarrow X\), since \(\pi_i^n(X, h) = \pi_i^n(X, h^\ast)\) the \((n+1)\)-th homotopy group relative to a cofibration \(i : B \rightarrow A\) of an object \(X\) based on a morphism \(h : CA \rightarrow X\) is

\[\pi_{i+1}^n(X, h) = \pi_{i+1}^n(X, h)\]

Observe that \(\pi_i^n(X, h) = [C^n A, X]^{h \rho^n i_n}\).

By Theorem 2.8 if \(X\) or \(i\) are contractible then \(\pi_i^n(X, h) = \{0\}\) for all \(n\).

The compatibility of the homotopy relation with the composition of morphisms (Remark 2.7) gives functorial character to the homotopy groups:

Proposition 3.4. If \(f : X \rightarrow Y\) is a morphism then \(f_* : \pi_i^n(X, h) \rightarrow \pi_i^n(Y, f h)\) is a homomorphism of groups.

Proof. It is clearly seen that \(f_*([F], [G]) = f_*([F \ast G]) = ([f F] \ast (f G)]\).

\[\square\]

Proposition 3.5. If \(f i = g j\) is a is a commutative square relating cofibrations \(i\) and \(j\) then \((C^n f)^* \circ \pi_i^n(X, h(C f))\) is a homomorphism of groups.

Proof. \((C^n f)^*([F], [G]) = (C^n f)^*([F \ast G]) = [(F \ast G) C^n f] = [(F C^n f) \ast (G C^n f)]\) since \(\mu_n (C^n f) i_n = (C^n f \cup C^n f) \mu_i n, i_n\), and by Theorem 2.8 \(\mu_j (C^n f) \simeq (C^n f \cup C^n f) \mu_i n\) rel. \(i_n\).

\[\square\]

Corollary 3.6. If \(f i = g j\) is a pushout diagram then \((C^n f)^*\) is an isomorphism of groups.

In pointed categories with a natural cone, the relation between coproducts and products in homotopy groups is a consequence of Remark 2.10.
Proposition 3.7. \( \pi_{n+2}^{i,n+1}(X, \{h, h'\}) \cong \pi_n^i(X, h) \vee \pi_n^i(X, h') \).

Proof. Observe that \( P\{\kappa, i \lor i'\} = P\{\kappa \lor i, i'\} = P\{\kappa, i\lor i'\} \), where \( i \lor i' = \overline{i \lor i'} \) and \( \overline{\kappa} = \kappa \lor \overline{\kappa} = \overline{\overline{\kappa}} \). Hence \((i \lor i')_n = i_n \lor i'_n\), and so that \( \mu_{i_n} \lor \mu_{i'_n} \) is an extension of the type \( \mu_{(i \lor i')_n} \). Therefore \([\{F, F'\}, \{G, G'\}] = [{\{F, F'\} \ast \{G, G'\}}] = [{\{F \ast G, F' \ast G'\}}] \).

□

Classical homotopy groups in pointed categories with a natural cone are generalized homotopy groups in the following sense:

\( \pi_n^A(X) \equiv \pi_n^{A, \kappa}(X, 0) \) and \( \pi_n^A(X) \equiv \pi_n^A(X, 0) \).

In pointed categories \( \pi_n^{A, \kappa}(X, h) \) will be also denoted by \( \pi_n^A(X, h) \).

4. Generalized Exact Homotopy Sequences

In \( \square \) a pair \((X, Y)\) is defined as a cofibration \( f : Y \to X \). A morphism from \((X, Y)\) to \((X', Y')\) is a pair of morphisms \((g, h)\) such that \(gf = f'h\). Cofibrations of pairs are the morphisms \((u, v) : (X, Y) \to (X', Y')\) with \(v\) and \(\{f', u\} : P\{v, f\} \to X'\) cofibrations. In this way the category of pairs \( \text{cof} C \) of a category with a natural cone \( C \) is also a \( C \)-category. Therefore, concepts and results obtained in the previous section are also available in \( \text{cof} C \).

On the other hand, homotopy groupoids of \( \text{cof} C \) are related with respective ones of \( C \) in the following sense:

a) If \((f_0, g_0) \simeq (f_1, g_1)\) rel. \((u, v)\) then \(f_0 \simeq f_1\) rel. \(u\) and \(g_0 \simeq g_1\) rel. \(v\).

b) \([\{F_0, G_0\}, \{F_1, G_1\}] = [\{F_0, G_0\} \ast \{F_1, G_1\}] = [\{F_0 \ast F_1, G_0 \ast G_1\}] \).

In this way every generalized homotopy group in \( C \) can be also seen as a generalized homotopy group in \( \text{cof} C \):

Proposition 4.1. There is an isomorphism of groups

\[ \theta_n : \pi_{n+2}^i(X, h) \to \pi_{n+1}^{i,1}(X, (fh, h)) \]

for all pair \((X, Y)\).

Proof. If \([\{F, G\}] \in \pi_{n+1}^{i,1}(X, Y, (fh, h))\) then \((F,G)(i_{n+2}, 1, n+2) = (Fi_{n+2}, G) = (\{fh^{n+1}Ci_{n+1}, fh^{n+1}h\}, h^{n+1})\), so \(H = h^n\). Therefore \(F \leftrightarrow [\{F, h^n\}]\) is an isomorphism of groups.

□

Definition 4.2. The \((n+2)\)-th homotopy group relative to the cofibration \(i : B \to A\) of the pair \((X, Y)\) based on the morphism \(h : CA \to Y\) is

\[ \pi_{n+2}^i((X, Y), h) = \pi_{n+1}^{i,Ci}(X, Y, (fh, h)), \quad (n \in \mathbb{N}) \]

Observe that \((Ci, i) : (CB, B) \to (CA, A)\) is a cofibration since \(\{Ci, \kappa\} = i_1\).

Next the exact homotopy sequence relative to a cofibration \(i : B \to A\) associated to a pair \((X, Y)\) based on a morphism \(h : CA \to Y\) is given:

Theorem 4.3. The following sequence of groups is exact:

\[ \ldots \to \pi_3^i(Y, h) \xrightarrow{f_1} \pi_3^i(X, fh) \xrightarrow{\delta_1} \pi_3^i((X, Y), h) \xrightarrow{\delta_1} \pi_2^i(Y, h) \xrightarrow{f_2} \pi_2^i(X, fh) \]

where \(f_n\) is the homomorphism of groups used in Proposition 3.4, \(j_n = (1, 1)^* \theta_n\), with \(\theta_n\) the isomorphism of groups given in Proposition 4.1, \((1, 1)^*\) is defined using the proposition 3.4 and the commutative square \((1, 1)(Ci, i) = (i_1, 1)(i, i)\), where
Clearly $f_\star$, $j_\star$, and $\delta_n$ are homomorphisms of groups by their definition.

- $\delta_n(\langle F, h_\rho \rangle) = \delta_n(\langle [F, h_\rho^n] \rangle) = [h_\rho^n]$.
- $f_\star(\delta_n(\langle F, G \rangle)) = f_\star(\langle G \rangle) = \langle [f, h_\rho^n] \rangle$ since $HC^{n+2} \zeta \ast h_\rho^n \simeq fG$ rel. $i_{n+1}$, where $H$ is an extension of the morphism $\{h_\rho^{n+2}(C^2)_{i+1}, F\}$ relative to the cofibration $(Ci)_{n+2}$.
- $j_\star(\delta_n(\langle F \rangle)) = j_\star(\langle [F, h_\rho^n] \rangle) = \langle [f, h_\rho^{n+1}, h_\rho^n] \rangle$ since:

  If $n$ is odd then $(HC^{n+2}\kappa, F) : (f, h_\rho^{n+1}, h_\rho^n) \simeq (F, h_\rho^n)$ rel. $(Ci,n+1)$, where $H$ is an extension of the morphism $\{h_\rho^{n+3}(C^n+4)i, fF, H, h_\rho^{n+2}, \langle [F, h_\rho^n] \rangle \}$ relative to the cofibration $(Ci)_{n+2}$.

  If $n$ is even then $(HC^{n+2}\kappa, F) : (fF, h_\rho^n) \simeq (fF, h_\rho^{n+1}, h_\rho^n)$ rel. $(Ci,n+1)$, where $H$ is an extension of the morphism $\{h_\rho^{n+3}(C^n+4)i, fF, H, h_\rho^{n+2}, \langle [F, h_\rho^n] \rangle \}$ relative to the cofibration $(Ci)_{n+2}$.

- If $\delta(\langle F, G \rangle) = [G] = [h_\rho^n]$ then there is $H : h_\rho^n \simeq G$ rel. $i_{n+1}$. $j_n(\langle K\kappa \rangle) = [(K\kappa, h_\rho^n)] = [(F, G)]$ since $(K, H) : (K\kappa, h_\rho^n) \simeq (F, G)$ rel. $(Ci,n+1)$, where $K$ is an extension of the morphism $\{h_\rho^{n+2}(C^n+3)i, fF, H, h_\rho^{n+1}, ..., h_\rho^n, F\}$ relative to the cofibration $(Ci)_{n+1}$.

- If $f_\star(\langle F \rangle) = [fF] = [fh_\rho^n]$ then there is $G : fF \rho \simeq fF$ rel. $i_{n+1}$. Let $H$ be an extension of the morphism $\{h_\rho^{n+2}(C^n+3)i, fF, H, h_\rho^{n+1}, ..., h_\rho^n\}$ relative to the cofibration $(Ci)_{n+2}$, then $\delta(\langle H\kappa, F \rangle) = [F]$.

- If $j_n(\langle F \rangle) = \langle [F, h_\rho^n] \rangle = \langle [fh_\rho^{n+1}, h_\rho^n] \rangle$.

  When $n$ is even, taking $(H, G) : (F, h_\rho^n) \simeq (fh_\rho^{n+1}, h_\rho^n)$ rel. $(Ci,n+1)$, $(H_1(C^n+3)\kappa)i_{n+3} = \{fh_\rho^{n+2}(C^n+2)i_1, fG, fh_\rho^{n+1}, ..., fh_\rho^{n+1}, F\}$, where $H_1$ is an extension of the morphism $\{fh_\rho^{n+3}(C^n+4)i, fF, H, fh_\rho^{n+2}, ..., fh_\rho^{n+2}, FC^n, FC^{n+1}\}$ relative to the cofibration $(Ci)_{n+3}$.

  $(H_2(C^{n+2})\kappa)i_{n+3} = \{fh_\rho^{n+2}(C^n+2)i_2, fG, fh_\rho^{n+1}, ..., fh_\rho^{n+1}, F, fh_\rho^{n+1}\}$, where $H_2$ is an extension of the morphism $\{fh_\rho^{n+3}(C^n+3)i_1, fF, H, fh_\rho^{n+2}, ..., fh_\rho^{n+2}, FC^n, FC^{n+1}\}$ relative to the cofibration $(Ci)_{n+2}$.

  $(H_3(C^n)\kappa)i_{n+3} = \{fh_\rho^{n+2}(C^n+2)i_3, fG, fh_\rho^{n+1}, ..., fh_\rho^{n+1}, F\}$, where $H_3$ is an extension of the morphism $\{fh_\rho^{n+4}(C^n+2)i_2, fF, H, fh_\rho^{n+2}, ..., fh_\rho^{n+2}, FC^n, FC^{n+1}\}$ relative to the cofibration $(Ci)_{n+1}$.

  $(H_4(C^n)\kappa)i_{n+3} = \{fh_\rho^{n+2}(C^n+1)i_4, fG, fh_\rho^{n+1}, ..., fh_\rho^{n+1}, F, fh_\rho^{n+1}\}$, where $H_4$ is an extension of the morphism $\{fh_\rho^{n+3}(C^n+2)i_3, fF, H, fh_\rho^{n+2}, ..., fh_\rho^{n+2}, FC^n, FC^{n+1}\}$ relative to the cofibration $(Ci)_{n}$.

  This process can be iterated to obtain a homotopy $H_{n+1}C^3 \kappa : fG \simeq F$ rel. $i_{n+2}$.

  When $n$ is odd, the same process for a homotopy $(H, G) : (fh_\rho^{n+1}, h_\rho^n) \simeq (F, h_\rho^n)$ rel. $(Ci,n+1)$ gives $H_{n+1}C^3 \kappa : fG \simeq F$ rel. $i_{n+2}$. 

\end{proof}

In pointed categories with a natural cone $\mathbb{D}$ classical exact homotopy sequences associated to a pair are obtained as generalized exact homotopy sequences taking the morphism $h = 0$. 

\begin{flushright}
$\square$
\end{flushright}
5. Spherical Homotopy Groups

Every category with a natural cylinder and product is a category with a natural cone and the same cofibrations \([D^*]\). Therefore the category of topological spaces with the topological cone has a structure of natural cone. This topological cone is obtained as the pushout of the inclusion in the base of the cylinder with the projection on a point. In this way the cone of the empty topological space is a point. So the category of pointed topological spaces is a category under the cone of an object, that obtains homotopy groups using spheres.

The development described above can be generalized for any category with a natural cone:

The cone \(C\nabla\) of any object \(\nabla\) behave like a point. Fixed an object \(\nabla\), \(C^*\) is the full subcategory of \(C^{C\nabla}\) whose objects, \((X, x)\), are cofibrations \(x : C\nabla \rightarrow X\).

The cone functor \(C_* : C^* \rightarrow C^*\) is defined by \(C_*(X, x) = (C_*X, C_*x)\), where \(C_*X = P\{\rho, Cx\}\) and given \(f : (X, x) \rightarrow (Y, y)\), \(C_*f = 1 \cup Cf : P\{\rho, Cx\} \rightarrow P\{\rho, Cy\}\).

Every pointed object \((X, x)\) has associated the trivial pushout diagram of the object \(P\{1, x\}\). However other pushout diagrams can be also associated to the object \((X, x)\): Any cofibration pushout diagram with induced cofibration \(x\) will be considered a pushout diagram associated to \((X, x)\).

**Theorem 5.1.** \(C^*_n(X, x) = (P\{\rho^n, C^n x\}, C^n x)\), where \(\overline{C^n x} = x\) and \(\overline{C^n x} = C\ C^n x\) is obtained by iterated use of the functor \(C_*\).

**Proof.** It is enough to observe that the following square diagram is a pushout:

\[
\begin{array}{ccc}
C^{n+1} \nabla & \xrightarrow{\rho^n} & C\nabla \\
\downarrow \rho & & \downarrow \rho \\
C^n x & \xrightarrow{\rho C\ldots C^{n-1}\rho} & C^*_n x
\end{array}
\]

- **Commutativity:**

\[
\rho C\ldots C^{n-1}\rho C^n x = \rho C\ldots C^{n-1} \overline{C^n x} = C^{n-1} p =
\]

\[
= \rho C\ldots C^{n-2} C^{C^n x} (C^{n-2} \rho) (C^{n-1} \rho) = ...
\]

\[
= \overline{C^n x} \rho(C\rho)\ldots(C^{n-2} \rho)(C^{n-1} \rho) = \overline{C^n x} \rho^n
\]

- **Pushout property:**

Given \(F : C\nabla \rightarrow Y\) and \(G : C^n X \rightarrow Y\) with \(F\rho^n = GC^n x\), then \(F\rho^n\ C^n x = GC^n x\), and there is \(\{F\rho^n, G\} : C^n \rightarrow (C_* X) \rightarrow Y\). \(\{F\rho^n, G\} C^{C^n x} = \{F\rho^n, G\} C^{n-1} \overline{C^n x} = F\rho^{n-1} C^{n-1} \rho\), and there is \(\{F\rho^{n-1}, G\} : C^{n-2} C^n x \rightarrow Y\). This process can be iterated to obtain the unique \(\{F, F\rho, \ldots, F\rho^{n-1}, G\} = \{F, G\} : C^n X = P\{\rho^n, C^n x\} \rightarrow Y\).
Natural transformations $\kappa_\ast : 1 \to C_\ast$ and $\rho_\ast : C_\ast^2 \to C_\ast$ are defined by $\kappa_\ast(X,x) = 1 \cup \kappa_X : (X,x) = P\{1, x\} \to C_\ast(X,x) = P\{\rho, Cx\}$ and $\kappa_\ast(X,x) = 1 \cup \rho_X : C_\ast^2(X,x) = P\{p^2, C^2x\} \to C_\ast(X,x) = P\{\rho, Cx\}$.

**Proposition 5.2.** Given $f : (X,x) \to (Y,y)$, $C_n^nf = 1 \cup C^nf : C_n^\ast(X,x) = P\{\rho^n, C^nx\} \to C_n^\ast(Y,y) = P\{\rho^n, C^ny\}$; $\kappa_\ast C_\ast^2(X,x) = 1 \cup \kappa_\ast C_\ast^2(X,x) = C_n^\ast(X,x) = P\{\rho^n, C^nx\} \to C_n^\ast+1(X,x) = P\{\rho^{n+1}, C^{n+1}x\}$ and $\rho_\ast C_\ast^2(X,x) = 1 \cup \rho C_\ast^2(X,x) = C_n^\ast+2(X,x) = P\{\rho^{n+2}, C^{n+2}x\} \to C_n^\ast+1(X,x) = P\{\rho^{n+1}, C^{n+1}x\}$.

**Proof.** $1 \cup \kappa_\ast C_\ast^2 = \{\overline{C^nX}, \rho C_\ast^2 \ldots C_\ast^{n-1} \rho \kappa C_\ast^2\}$. So that $(1 \cup \kappa_\ast C_\ast^2)^{(n)} \overline{C^nX} = \bar{C^nX}$ and

$(1 \cup \kappa_\ast C_\ast^2)^{(n+1)} \overline{C^nX} = (1 \cup \kappa_\ast C_\ast^2)^{(n+1)} \overline{C^nX} \kappa C_\ast^2 \nu \kappa C_\ast^2 = \rho C_\ast^2 \nu \kappa C_\ast^2 = (1 \cup \kappa_\ast C_\ast^2)^{(n+1)} \overline{C^nX}$.

The morphisms $(1 \cup \kappa_\ast C_\ast^2)^{(n+1)} \overline{C^nX} = \rho C_\ast^2 \kappa C_\ast^2 = 1 \cup \rho C_\ast^2$ agree.

The other equalities are trivial by definition.

A morphism $i : (B,b) \to (A,a)$ is called a pointed cofibration when $i : B \to A$ is a cofibration in C.

**Theorem 5.3.** $C^\ast$, with the cone $C_\ast$, natural transformations $\kappa_\ast$, $\rho_\ast$ and the pointed cofibrations, is a pointed category with a natural cone.

**Proof.** Observe that $(C \nu, 1)$ is an initial object with initial morphisms, $x : (C \nu, 1) \to (X,x)$, pointed cofibrations; so every object is cofibrant. Clearly $C_\ast(C \nu, 1) = (P\{\rho, 1\}, \overline{T}) = (C \nu, 1)$.

**Cone axiom:** It is a simple verification.

**Pushout axiom:** Given a pointed cofibration $i : (B,b) \to (A,a)$ and a morphism $f : (B,b) \to (X,x)$, then $P\{f, i\} = P\{f, i\}, x \cup a$ with induced morphisms $\overline{f} : (A,a) \to P\{f, i\}, x \cup a$ and $\overline{f} : (X,x) \to P\{f, i\}, x \cup a$, where $x \cup a : C \nu = P\{1, b\} \to P\{f, i\}$ is a cofibration by Theorem 2.22 since $\{a, i\} = i : P\{1, b\} \to A$. Note that $\overline{f} = \overline{f} : (P\{1, x \cup a\})$ is a cofibration.

$P\{C_\ast f, C_\ast i\} = P\{1 \cup C_\ast f, 1 \cup C_\ast i\} = P\{\rho_\ast \cup \rho, C_\ast x \cup C_\ast a\} = C_\ast P\{f, i\}$, with induced morphisms $\overline{C_\ast f} = 1 \cup \overline{C_\ast f}$ and $\overline{C_\ast i} = 1 \cup \overline{C_\ast i}$.

**Cofibration axiom:** Clearly $1 \{x, x\}$ and the composition of pointed cofibrations are pointed cofibrations. $\kappa_\ast(X,x) = 1 \cup \kappa_X$ is a cofibration by Theorem 2.2 since $x \{x\}$ is so. Given a pointed cofibration $i : (B,b) \to (A,a)$, by NEP in C there is $r : CA \to CB$ such that $r(Ci) = 1$. Then $1 \cup r : C_\ast(A,a) = P\{\rho, Ca\} \to C_\ast(B,b) = P\{\rho, Cb\}$ is a retraction for $C_\ast i$.

**Relative cone axiom:** Given a pointed cofibration $i : (B,b) \to (A,a)$, $\{C_\ast i, \kappa_\ast \} : P\{\kappa_\ast, i\} \to C_\ast(A,a)$ will be denoted by $i_\ast : (\Sigma_\ast^\nu, Cb \cup a) \to C_\ast(A,a)$.

$\Sigma_\ast^i = P\{1 \cup \kappa, 1 \cup i\} = P\{\rho \cup 1, Ci \cup a\}$ with induced morphisms $\overline{\Sigma_\ast^\nu} = 1 \cup \overline{\kappa}$ and $\overline{\Sigma_\ast^i} = 1 \cup \overline{i}$, where $\overline{i}$ is the induced cofibration by $i$ in the pushout of $P\{\kappa, i\}$.
The concepts developed in pointed categories with a natural cone \([D -]\) are also available in \(C^*\). They can be related with the respective ones of the original category \(C\).

**Proposition 5.4.** Given \(f : (X, x) \to (Y, y)\), \(f \simeq 0\) in \(C^*\) if and only if \(f \simeq 0\) in \(C\).

**Proof.** If \(F : f \simeq 0\) in \(C^*\) then \(F\) is the induced morphism in the pushout of \(C, X\).

If \(F : f \simeq 0\) in \(C\) then \(\{FCxC\rho, F\} : \{y\rho, f\} \simeq 0\), and by NEP there is an extension \(H\) of the morphism \(\{y\rho, f\}\) relative to the cofibration \(x_1\). \(\{y, H\} : f \simeq 0\) in \(C^*\).

**Corollary 5.5.** \((X, x) \simeq 0\) if and only if \(X \simeq 0\).

Next pointed homotopy groups in \(C^*\) will be stated as generalized homotopy groups of \(C\). A study about pushout diagrams associated to pointed objects is necessary for it.

**Proposition 5.6.** If \(P\{s, j\}\) is a pushout diagram associated to \((X, x)\) then \(C^n(X, x) = P\{n^sC^n s, C^n j\}\).

**Proof.** It is enough to observe the following composition of pushouts:

\[
\begin{array}{c}
C^n S \\
\downarrow C^n j \\
C^n T
\end{array}
\quad \begin{array}{c}
C^n s \\
\downarrow C^n x
\end{array}
\quad \begin{array}{c}
\rho^n \\
\downarrow (n^s C^n x)
\end{array}
\quad \begin{array}{c}
C^n T
\end{array}
\quad \begin{array}{c}
\tilde{C}^n X
\end{array}
\]

**Corollary 5.7.** Given \(g = 1 \cup f : (X, x) = P\{s, j\} \to (X', x') = P\{s', j'\}\):

a) \(C^n g = 1 \cup C^n f : C^n(X, x) = P\{n^s C^n s, C^n j\} \to C^n(X', x') = P\{n^s C^n s', C^n j'\}\).

b) \(\kappa_{C^n(X, x)} = 1 \cup \kappa_{C^n T} : C^n(X, x) = P\{n^s C^n s, C^n j\} \to C^{n+1}(X, x) = P\{n^{s+1} C^{n+1} s, C^{n+1} j\}\).

c) \(\rho_{C^n(X, x)} = 1 \cup \rho_{C^n T} : C^{n+2}(X, x) = P\{n^{s+1} C^{n+1} s, C^{n+1} j\} \to C^{n+1}(X, x) = P\{n^{s+1} C^{n+1} s, C^{n+1} j\}\).

**Proof.** (a) \(1 \cup C^n f = (\tilde{C}^n x', \tilde{p}C^n \tilde{p}... \tilde{p}C^n s C^n f)\). So that:
Definition 5.8. A pointed cofibration \((B, b) \to (A, a)\) is said to have associated a pushout cofibration when \(u = 1 \cup i : (B, b) = P\{s, j\} \to (A, a) = P\{s, j\}'\), where \(i : T \to T'\) is a cofibration verifying \(j' = ij\).

Observe that every pointed cofibration \(i : (B, b) \to (A, a)\) has an associated pushout cofibration \(1 \cup i : (B, b) = P\{1, b\} \to (A, a) = P\{1, a\}\).

On the other hand, every object \(X\) can be considered as an iterated pushout diagram: \(X = P\{1_X, 1_X\} = P\{1_X, 1_X\} = P\{1_X, 1_X\} = \ldots = P\{1_X, 1_X\}\), where \(1_X = 1_X\) and \(1_X = 1_X : X \to P\{1_X^{-1}, 1_X\}\). In this way it is possible to define curly braces or unions with domain the object \(X\).

Theorem 5.9. Given a pushout cofibration \(1 \cup i : (B, b) = P\{s, j\} \to (A, a) = P\{s, j\}'\) associated to a pointed cofibration \(u\), the following square diagram is a pushout:
Theorem 5.10. Given a based pushout cofibration \( u = 1 \cup i : (B, b, \beta) = P\{\rho(Cs), j\} \rightarrow C_s(A, a, \alpha) = P\{\rho(Cs), Cj'\} \), then

\[
[(C, A, Ca), (X, x)]^{(u)} \cong [CT', X]^{x'y(i)}
\]
Definition 5.14. 

described in this paper, it is possible to obtain homotopy groups of its objects, as it occurs with topological spaces and pointed topological spaces.

\[ \theta \text{ is well defined since } G : g_0 \simeq g_1 \text{ rel. } i \text{ implies that } \{x, G\} : C^2_n(A, a) = P\{\rho^2C^2_s, C^2j'\} \to (X, x) \text{ verifies } \{x, G\} : \{x, g_0\} \simeq \{x, g_1\} \text{ rel. } u. \]

Clearly \( \theta \) is surjective.

\[ \theta \text{ is injective since if } G : \{x, g_0\} \simeq \{x, g_1\} \text{ rel. } u \text{ then } G\rho^2C^2s : g_0 \simeq g_1 \text{ rel. } i. \]

\[ \square \]

Corollary 5.11. Given a based pushout cofibration \( u = 1 \cup i : (B, b, \beta) = P\{s, j\} \to (A, a, \alpha) = P\{s, j'\} \), then

\[ \pi^0_n((X, x)) \cong \pi^i_n(X, x\rho(C\alpha)) \]

Proof.

\[ \pi^0_n((X, x)) = (C^i_n(A, a), (X, x))^{(u_n)} \cong \]

\[ \cong (C^i_nT', X)^{x(\rho^nC^n\beta, \rho^n-1C^n-1\alpha, \ldots, \rho^n-1C^n-1\alpha)(i_n)} = \]

\[ \cong \pi^1_n(X, x\rho(C\alpha)) \]

Given an extension \( \mu \) of the morphism \( \kappa\rho(Ci_{n-1}) \cup \kappa \) relative to the cofibration \( i_n \), then \( \mu^n\rho^nj' = (C^n\rho^nj' \cup C^n\rho^nj')\kappa\rho \) and \( \rho^nC^n\kappa s = (\rho^nC^n\kappa s)\kappa\rho \). Hence there is \( 1 \cup \mu : C^n_n(A, a) = P\{\rho^nC^n\kappa s, C^n\rho^n\kappa s\} \to (C, P\{u_{n-1s}, u_{n-1s}\}, u_{n-1s}C\alpha) = P\{\rho^nC^n\kappa s, C^n\rho^n\kappa s\} \) verifying

\[ (1 \cup \mu)u_n \approx (1 \cup \mu)(1 \cup i_n) = \]

\[ = 1 \cup (\kappa\rho(Ci_{n-1}) \cup \kappa) \approx \]

\[ \approx ((1 \cup \kappa)(1 \cup \rho)(1 \cup Ci_{n-1})) \cup (1 \cup \kappa) = \]

\[ = \kappa\rho Cu_{n-1s} \cup \kappa \]

Therefore the bijection above is an isomorphism of groups.

\[ \square \]

Corollary 5.12. Given a based pointed object \( (A, a, \alpha) \):

\[ \pi^0_n((X, x)) \cong \pi^i_n(X, x\rho(C\alpha)) \]

Corollary 5.13. The pointed exact sequence relative to the based pushout cofibration \( u = 1 \cup i \) of the pointed pair \( ((X, x), (Y, y)) \) is isomorphic to the exact sequence relative to the cofibration \( i \) of the pair \( (X, Y) \) based on the morphism \( x\rho(C\alpha) \)

The isomorphism of groups given in Corollary 5.11 let one to extend the definition of pointed homotopy groups to pointed objects \( (X, x) \) where \( x \) be not a cofibration:

\[ \pi^0_n((X, x)) = \pi^i_n(X, x\rho(C\alpha)) \]

In this way, although \( C^{CV} \) has not, in general, a natural cone in the sense described in this paper, it is possible to obtain homotopy groups of its objects, as it occurs with topological spaces and pointed topological spaces.

Next, taking as example topological spheres, spherical objects are defined.

Definition 5.14. The 0-sphere of \( C^{CV} \) is \( S^0 = P\{\kappa\nu, \kappa\nu\} \).
Observe that \((S^0, \pi, \{1, 1\})\) is a based pointed object.

**Definition 5.15.** The \(n\)-sphere of \(C^\wedge\) is \(S^n = S^n_\pi(S^0, \pi)\).

**Remark 5.16.** Note that, for \(n \geq 2\)
\[
\pi_{n^*}^S((X, x)) = \{S^n, (X, x)\} = (\text{Corollary 5.12}) = \\
\pi_n^\wedge(X, \{xp, xp\}) \cong (\text{pushout of definition of } S^0) \cong \\
\cong \pi_n^\wedge(X, x)
\]

Hence one can to extend the definition of \(\pi_{n^*}^S((X, x))\) for \(n = 1\) by:

\[
\pi_1^S((X, x)) = \pi_1^\wedge(X, x)
\]

**Definition 5.17.** \(\pi_{n^*}^S((X, x))\) are denominated Spherical Homotopy Groups of the pointed object \((X, x)\).

**Remark 5.18.** If \(C\) is a pointed category with a natural cone, then \(\pi_{n^*}^S((X, x)) = \pi_{n+1}^\wedge(X, x)\).

If \(C\) has initial object \(\emptyset\) and \(C\emptyset \neq \emptyset\), then \(\pi_{n^*}^S((X, x))\) are denominated Standard Spherical Homotopy Groups of the pointed object \((X, x)\). \(\pi_1^S((X, x)) = \pi_2^\emptyset(X, x)\) is also denominated Fundamental Group.

In the category of topological spaces, the standard spherical homotopy groups of a pointed topological space are the classical homotopy groups.

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Departamento de Matemática Fundamental, Universidad de La Laguna.  
38271 La Laguna.