WELL-POSEDNESS AND ENERGY DECAY OF SOLUTIONS TO
A BRESSE SYSTEM WITH A BOUNDARY DISSIPATION OF
FRACTIONAL DERIVATIVE TYPE

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ABSTRACT. We consider the Bresse system with three control boundary conditions of fractional derivative type. We prove the polynomial decay result with an estimation of the decay rates. Our result is established using the semigroup theory of linear operators and a result obtained by Borichev and Tomilov.

1. Introduction. In this paper we discuss the existence and energy decay rate of solutions for the initial boundary value problem of the linear Bresse system of the type

\[
\begin{aligned}
\rho_1\varphi_{tt} - Gh(\varphi_x + \psi + lw)_x - lEh(\omega_x - l\varphi) &= 0 \\
\rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + lw) &= 0 \\
\rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x + \psi + lw) &= 0
\end{aligned}
\]

where \((x,t) \in (0,L) \times (0, +\infty)\). This system is subject to the boundary conditions of the form

\[
\begin{aligned}
\varphi(0,t) &= 0, \quad \psi(0,t) = 0, \quad \omega(0,t) = 0 \quad \text{in } (0, +\infty) \\
Gh(\varphi_x + \psi + lw)(L,t) &= -\gamma_1\partial_t^{\alpha,\eta}\varphi(L,t) \quad \text{in } (0, +\infty) \\
EI\psi_x(L,t) &= -\gamma_2\partial_t^{\alpha,\eta}\psi(L,t) \quad \text{in } (0, +\infty) \\
Eh(\omega_x - lw)(L,t) &= -\gamma_3\partial_t^{\alpha,\eta}\omega(L,t) \quad \text{in } (0, +\infty)
\end{aligned}
\]

where \(\gamma_i > 0, i = 1, 2, 3\). The notation \(\partial_t^{\alpha,\eta}\) stands for the generalized Caputo’s fractional derivative of order \(\alpha\) with respect to the time variable. It is defined as follows

\[
\partial_t^{\alpha,\eta}w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s) \, ds, \quad \eta \geq 0.
\]

These exponentially modified fractional integro-differential operators were first proposed in Choi and MacCamy [10]. In other words, we investigate three dissipative effects at the boundary. The system is finally completed with initial conditions

\[
\begin{aligned}
\varphi(x,0) &= \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\
\psi(x,0) &= \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\
\omega(x,0) &= \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), 
\end{aligned} \quad x \in (0, L)
\]
where the initial data \((\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1)\) belong to a suitable function space. By \(\omega, \varphi\) and \(\psi\) we are denoting the longitudinal, vertical and shear angle displacements. The original Bresse system is given by the following equations (see [7]):

\[
\begin{align*}
\rho_1 \varphi_{tt} &= Q_x + lN + F_1, \\
\rho_2 \psi_{tt} &= M_x - Q + F_2, \\
\rho_1 \omega_{tt} &= N_x - lQ + F_3,
\end{align*}
\]

where we use \(N, Q\) and \(M\) to denote the axial force, the shear force and the bending moment respectively. These forces are stress-strain relations for elastic behavior and given by

\[
\begin{align*}
N &= Eh(\omega_x - l\varphi), \\
Q &= Gh(\varphi_x + \psi + l\omega), \\
M &= EI\psi_x,
\end{align*}
\]

where \(G, E, I\) and \(h\) are positive constants. Finally, by the terms \(F_i\) we are denoting external forces.

The Bresse system is more general than the well-known Timoshenko system where the longitudinal displacement \(\omega\) is not considered \((l = 0)\). There are a number of publications concerning the stabilization of the Timoshenko system with different kinds of damping (see [12], [17], [18], [19] and [20]). Raposo et al. [20] proved the exponential decay of the solution for the following linear system of Timoshenko-type beam equations with linear frictional dissipative terms:

\[
\begin{align*}
\rho_1 \varphi_{tt} - Gh(\varphi_x + \psi)_x + \mu_1 \varphi_t &= 0, \\
\rho_2 \psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi) + \mu_1 \psi_t &= 0.
\end{align*}
\]

Messaoudi and Mustafa [17] (see also [19]) considered the stabilization for the following Timoshenko system with nonlinear internal feedbacks:

\[
\begin{align*}
\rho_1 \varphi_{tt} - Gh(\varphi_x + \psi)_x + g_1(\psi_t) &= 0, \\
\rho_2 \psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi) + g_2(\psi_t) &= 0.
\end{align*}
\]

Recently, Park and Kang [19] considered the stabilization of the Timoshenko system with weakly nonlinear internal feedbacks.

In [24], Soriano, Wenden Charles, Rodrigo Schulz considered a Bresse system with three internal feedbacks. They proved the exponential decay of the solution.

In [21], Liu and Rao considered a Bresse system coupled with two heat equations. The two wave equations for the longitudinal displacement and the shear angle displacement are effectively globally damped by the dissipation from the two heat equations. The wave equation about the vertical displacement is subject to a weak thermal damping and indirectly damped through the coupling. They establish exponential energy decay rate when the vertical and the longitudinal waves have the same speed of propagation. Otherwise, a polynomial-type decay is established.

However, for a Bresse system with the dissipative effect taking place at the boundary very little is known in the literature, more general and recent results in this direction were obtained in [1]. In this paper the authors established a result of exponential stability for a Bresse system with three dissipative effects concentrated at the boundary that is

\[
\begin{align*}
\varphi(0, t) &= 0, & \psi(0, t) &= 0, & \omega(0, t) &= 0 & \text{in} & (0, +\infty), \\
Gh(\varphi_x + \psi + l\omega)(L, t) &= -\gamma_1 \partial_t \varphi(L, t) & \text{in} & (0, +\infty), \\
EI\psi_x(L, t) &= -\gamma_2 \partial_t \psi(L, t) & \text{in} & (0, +\infty), \\
Eh(\omega_x - l\varphi)(L, t) &= -\gamma_3 \partial_t \omega(L, t) & \text{in} & (0, +\infty).
\end{align*}
\]
The boundary feedback under the consideration here are of fractional type and are described by the fractional derivatives
\[ \partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s) ds, \quad \eta \geq 0. \]

The order of our derivatives is between 0 and 1. These convolutions with locally integrable kernels are not simple to treat: analytically, the singular character of kernel \( t^{-\alpha} \) (with \( 0 < \alpha < 1 \)) problematizes the use of methods and techniques developed for convolution terms with regular and/or integrable kernels.

It has been shown (see [16]) that, as \( \partial_t \), the fractional derivative \( \partial_t^{\alpha} \) forces the system to become dissipative and the solution to converge to the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations.

Boundary fractional dissipations are not only important from the theoretical point of view but also for applications. They naturally arise in hereditary processes and fractal media to describe memory effects and anomalous phenomena (see [22]). Indeed, it has been observed by experiments that many concepts cannot be described in Newtonian terms. In other words, in many fields, phenomena with strange kinetics cannot be described within the framework of classical theory using integer-order derivatives. It could lead to a more adequate modeling and more robust control performance. For example, in viscoelasticity, due to the nature of the material microstructure, both elastic solid and viscous fluid like response qualities are involved. Using Boltzmann assumptions, we end up with a stress-strain relationship defined by a time convolution. More precisely, the stress at each point and at each instant does not depend only on the present value of the strain but also on the entire temporal prehistory of the motion from 0 up to time \( t \). This is interpreted by a time convolution with a “relaxation function” as kernel. Viscoelastic response occurs in a variety of materials, such as soils, concrete, rubber, cartilage, biological tissue, glasses, and polymers (see [3], [4], [5] and [14]). In our case, the fractional dissipations may simply describe an active boundary viscoelastic damper designed to reduce unwanted vibrations.

Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem \((P)\) for linear damping. To obtain global solutions to the problem \((P)\), we use the argument combining the semigroup theory ([8]) with the energy estimate method. To prove decay estimates, we use a frequency domain approach and a Theorem of A. Borichev and Y. Tomilov.

2. Augmented model. This section is concerned with the reformulation of the model \((P)\) into an augmented system. For that, we need the following claims.

**Theorem 2.1** (see [15]). Let \( \mu \) be the function:
\[ \mu(\xi) = |\xi|^{2\alpha-1/2}, \quad -\infty < \xi < +\infty, \quad 0 < \alpha < 1. \]  
Then the relationship between the ‘input’ \( U \) and the ‘output’ \( O \) of the system
\[ \partial_t \phi(\xi,t) + \xi^2 \phi(\xi,t) + \eta \phi(\xi,t) - U(t) \mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \]  
\[ \phi(\xi,0) = 0, \]  
\[ O(t) = (\pi)^{-1} \sin(\alpha \pi) \int_{-\infty}^{+\infty} \mu(\xi) \phi(\xi,t) d\xi \]
is given by
\[ O = I^{1-\alpha,\eta}U = D^\alpha,\eta U. \] (5)

where
\[ [I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} e^{-\eta(t-\tau)} f(\tau) \, d\tau \]

Lemma 2.2. If \( \lambda \in D = \{ \lambda \in \mathbb{C} : \text{Re}\lambda + \eta > 0 \} \cup \{ \lambda \in \mathbb{C} : \text{Im}\lambda \neq 0 \} \) then
\[ g(\lambda) = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} \, d\xi = \frac{\pi}{\sin \alpha \pi} (\lambda + \eta)^{\alpha-1}. \]

Proof. Let us set
\[ f_\lambda(\xi) = \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2}. \]

We have
\[ \left| \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} \right| \leq \frac{\mu^2(\xi)}{\eta_0 + \eta + \xi^2}. \]

Then the function \( f_\lambda \) is integrable. Moreover
\[ \left| \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} \right| \leq \begin{cases} \frac{\mu^2(\xi)}{\eta_0 + \eta + \xi^2} & \text{for all } \text{Re}\lambda \geq \eta_0 > -\eta \\ \frac{\mu^2(\xi)}{\eta_0 + \xi^2} & \text{for all } |\text{Im}\lambda| \geq \eta_0 > 0 \end{cases} \]

From Theorem 1.16.1 in [25], the function
\[ g : D \to \mathbb{C} \text{ is holomorphic.} \]

For a real number \( \lambda > -\eta \), we have
\[
\int_{-\infty}^{\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} \, d\xi = \int_{-\infty}^{+\infty} \frac{|\xi|^{2\alpha-1}}{\lambda + \eta + \xi^2} \, d\xi = \int_{0}^{+\infty} \frac{x^{\alpha-1}}{\lambda + \eta + x} \, dx \text{ (with } \xi^2 = x) \\
= (\lambda + \eta)^{\alpha-1} \int_{-\infty}^{+\infty} y^{-1}(y - 1)^{\alpha-1} \, dy \text{ (with } y = x/(\lambda + \eta) + 1) \\
= (\lambda + \eta)^{\alpha-1} \int_{0}^{1} z^{-\alpha} (1 - z)^{\alpha-1} \, dz \text{ (with } z = 1/y) \\
= (\lambda + \eta)^{\alpha-1} B(1 - \alpha, \alpha) = (\lambda + \eta)^{\alpha-1} \Gamma(1 - \alpha)\Gamma(\alpha) = (\lambda + \eta)^{\alpha-1} \frac{\pi}{\sin \pi \alpha}.
\]

Both holomorphic functions \( g \) and \( \lambda \mapsto (\lambda + \eta)^{\alpha-1} \frac{\pi}{\sin \pi \alpha} \) coincide on the half line \([-\eta, +\infty[\), hence on \( D \) following the principle of isolated zeroes. \( \square \)

We are now in a position to reformulate system \( (P) \). Indeed, by using Theorem 2.1, system \( (P) \) may be recast into the augmented model:
Lemma 2.3. Let $(\varphi, \phi_1, \psi, \phi_2, \omega, \phi_3)$ be a solution of $(P')$. Then, the energy functional defined by (6) satisfies

$$E(t) = \frac{\rho_1}{2} \|\varphi_t\|^2 + \frac{\rho_2}{2} \|\psi_t\|^2 + \frac{\rho_1}{2} \|\omega_t\|^2 + \frac{EI}{2} \|\psi_x\|^2 + \frac{Gh}{2} \|\varphi_x + \psi + l\omega\|^2$$

$$+ \frac{Eh}{2} \|\omega_x - l\varphi\|^2 + (\pi)^{-1} \sin(\alpha\pi) \sum_{i=1}^{3} \gamma_i \int_{-\infty}^{+\infty} (\phi_i(\xi, t))^2 d\xi.$$  

(6)

Lemma 2.3. Let $(\varphi, \phi_1, \psi, \phi_2, \omega, \phi_3)$ be a solution of $(P')$. Then, the energy functional defined by (6) satisfies

$$E'(t) = -(\pi)^{-1} \sin(\alpha\pi) \sum_{i=1}^{3} \gamma_i \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi_i(\xi, t))^2 d\xi \leq 0.$$  

(7)

Proof. Multiplying the first equation in $(P')$ by $\varphi_t$, the third equation by $\psi_t$, the five equation by $\omega_t$, integrating over $(0, L)$ and using integration by parts, we get

$$\frac{1}{2} \rho_1 \frac{d}{dt} \|\varphi_t\|^2 = \frac{1}{2} \rho_2 \frac{d}{dt} \|\psi_t\|^2 - \frac{1}{2} \rho_1 \frac{d}{dt} \|\omega_t\|^2 + \frac{EI}{2} \|\psi_x\|^2 + \frac{Gh}{2} \|\varphi_x + \psi + l\omega\|^2$$

$$\frac{1}{2} \rho_1 \frac{d}{dt} \|\omega_t\|^2 = \frac{1}{2} \rho_2 \frac{d}{dt} \|\psi_t\|^2 - EH \int_0^L (\varphi_x + \psi + l\omega) \varphi_t dx - lEh \int_0^L (\omega_x - l\varphi) \varphi_t dx = 0$$

Then

$$\frac{d}{dt} \left( \frac{\rho_1}{2} \|\varphi_t\|^2 + \frac{\rho_2}{2} \|\psi_t\|^2 + \frac{\rho_1}{2} \|\omega_t\|^2 + \frac{EI}{2} \|\psi_x\|^2 + \frac{Gh}{2} \|\varphi_x + \psi + l\omega\|^2 + \frac{Eh}{2} \|\omega_x - l\varphi\|^2 \right) + \phi_1(\xi, t) \int_{-\infty}^{+\infty} \mu(\xi) \phi_1(\xi, t) d\xi$$

$$+ \phi_2(\xi, t) \int_{-\infty}^{+\infty} \mu(\xi) \phi_2(\xi, t) d\xi + \phi_3(\xi, t) \int_{-\infty}^{+\infty} \mu(\xi) \phi_3(\xi, t) d\xi = 0,$$  

(8)
where \( \zeta = (\pi)^{-1} \sin(\alpha \pi) \gamma_i \). Multiplying second, fourth and sixth equations in \((P')\) by \( \zeta \phi_i \) respectively and integrating over \((-\infty, +\infty)\), to obtain:

\[
\begin{align*}
\frac{\zeta_1}{2} \frac{d}{dt} \| \phi_1 \|_2^2 + \zeta_1 \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi_i(\xi, t))^2 d\xi - \zeta_1 \psi_i(L, t) \int_{-\infty}^{+\infty} \mu(\xi) \phi_1(\xi, t) d\xi &= 0, \\
\frac{\zeta_2}{2} \frac{d}{dt} \| \phi_2 \|_2^2 + \zeta_2 \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi_i(\xi, t))^2 d\xi - \zeta_2 \psi_i(L, t) \int_{-\infty}^{+\infty} \mu(\xi) \phi_2(\xi, t) d\xi &= 0, \\
\frac{\zeta_3}{2} \frac{d}{dt} \| \phi_3 \|_2^2 + \zeta_3 \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi_i(\xi, t))^2 d\xi - \zeta_3 \omega_i(L, t) \int_{-\infty}^{+\infty} \mu(\xi) \phi_3(\xi, t) d\xi &= 0.
\end{align*}
\]

From (6), (8) and (9) we get

\[
E'(t) = -\sum_{i=1}^{3} \zeta_i \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi_i(\xi, t))^2 d\xi.
\]

This completes the proof of the lemma. \(\square\)

3. Global existence. In this section we will give well-posedness results for problem \((P')\) using semigroup theory. Let us introduce the semigroup representation of the Bresse system \((P')\). Let \( U = (\varphi, \psi, \phi_1, \psi_1, \phi_2, \omega, \omega_1, \omega_3)^T \) and rewrite \((P')\) as

\[
\begin{align*}
U' &= AU, \\
U(0) &= (\varphi_0, \varphi_1, \phi_{01}, \psi_0, \psi_1, \phi_{02}, \omega_0, \omega_1, \omega_3),
\end{align*}
\]

where the operator \( A \) is defined by

\[
A = \begin{pmatrix}
\varphi \\
u \\
\phi_1 \\
\psi \\
v \\
\phi_2 \\
\omega \\
\phi_3 \\
\end{pmatrix} = \begin{pmatrix}
Gh(\varphi_x + \psi + l\omega)_x + \frac{lEh}{\rho_1}(\omega_x - l\varphi) \\
-\frac{lEh}{\rho_1}(\omega_x - l\varphi) \\
\frac{EI}{\rho_2} \psi_{xx} - \frac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) \\
-\frac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) \\
\frac{Eh}{\rho_1}(\omega_x - l\varphi)_x - \frac{lEh}{\rho_1}(\omega_x - l\varphi) \\
-\frac{lEh}{\rho_1}(\varphi_x + \psi + l\omega) \\
\end{pmatrix}
\]

with domain

\[
D(A) = \left\{(\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \omega_1, \omega_3)^T \in \mathcal{H} : \varphi, \psi, \omega \in H^2(0, L) \cap H_0^1(0, L), \right. \\
\left. \begin{array}{l}
u, \phi_2, \omega \in H^2_0(0, L), (-\xi^2 + \eta) \phi_1 + u(L) \mu(\xi) \in L^2(-\infty, +\infty), \\
-(\xi^2 + \eta) \phi_2 + v(L) \mu(\xi) \in L^2(-\infty, +\infty), \\
Gh(\varphi_x + \psi + l\omega)(L) + \zeta_1 \int_{-\infty}^{+\infty} \mu(\xi) \phi_1(\xi) d\xi = 0, \\
EI \psi_x(L) + \zeta_2 \int_{-\infty}^{+\infty} \mu(\xi) \phi_2(\xi) d\xi = 0, \\
Eh(\omega_x - l\varphi)(L) + \zeta_3 \int_{-\infty}^{+\infty} \mu(\xi) \phi_3(\xi) d\xi = 0, \\
|\xi| \phi_1, |\xi| \phi_2, |\xi| \phi_3 \in L^2(-\infty, +\infty) \end{array} \right\}
\]

where, the energy space \( \mathcal{H} \) is defined as

\[
\mathcal{H} = (H_0^1(0, L) \times L^2(0, L) \times L^2(-\infty, +\infty))^3
\]
where
\[ H^1_L(0, L) = \{ \varphi \in H^1(0, L) : \varphi(0) = 0 \} . \]

For \( U = (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \dot{\omega}, \phi_3)^T, \overline{U} = (\overline{\varphi}, \overline{\psi}, \overline{\phi_1}, \overline{\psi}, \overline{\phi_2}, \overline{\psi}, \overline{\omega}, \overline{\phi_3})^T, \) we define the following inner product in \( \mathcal{H} \)
\[
\langle U, \overline{U} \rangle_{\mathcal{H}} = \int_0^L \left( \rho_1 u\overline{u} + \rho_2 v\overline{v} + \rho_1 \omega\overline{\omega} + EI\psi_x\overline{\psi_x} + Gh(\varphi_x + \psi + l\omega)(\overline{\varphi_x} + \overline{\psi} + \overline{l\omega}) \\
+ Eh(\omega_x - l\varphi)(\overline{\omega_x} - l\overline{\varphi}) \right) dx + \sum_{i=1}^3 \zeta_i \int_{-\infty}^{+\infty} \phi_i \overline{\phi}_i \, d\xi .
\]

We show that the operator \( \mathcal{A} \) generates a \( C_0 \)-semigroup in \( \mathcal{H} \). In this step, we prove that the operator \( \mathcal{A} \) is dissipative. Let \( U = (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \dot{\omega}, \phi_3)^T \). Using (10), (7) and the fact that
\[
\mathcal{E}(t) = \frac{1}{2} \| U \|^2_{\mathcal{H}} , \quad (13)
\]
we get
\[
\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\sum_{i=1}^3 \zeta_i \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi_i(\xi))^2 \, d\xi . \quad (14)
\]

Consequently, the operator \( \mathcal{A} \) is dissipative. Now, we will prove that the operator \( \lambda I - \mathcal{A} \) is surjective for \( \lambda > 0 \). For this purpose, let \( (f_1, f_2, f_3, f_5, f_6, f_7, f_8, f_9)^T \in \mathcal{H}, \) we seek \( U = (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \dot{\omega}, \phi_3)^T \in D(\mathcal{A}) \) solution of the following system of equations
\[
\begin{aligned}
&\lambda \varphi - u = f_1, \\
&\lambda u - \frac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \frac{Eh}{l\rho_1}(\omega_x - l\varphi) = f_2, \\
&\lambda \phi_1 + (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = f_3, \\
&\lambda \psi - v = f_4, \\
&\lambda v - \frac{EI}{\rho_2}\psi_{xx} + \frac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) = f_5, \\
&\lambda \phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = f_6, \\
&\lambda \omega - \dot{\omega} = f_7, \\
&\lambda \dot{\omega} - \frac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \frac{Gh}{\rho_1}(\varphi_x + \psi + l\omega) = f_8, \\
&\lambda \phi_3 + (\xi^2 + \eta)\phi_3 - \dot{\omega}(L)\mu(\xi) = f_9.
\end{aligned}
\]
Suppose that we have found \( \varphi, \psi \) and \( \omega \). Therefore, the first, the fourth and the seventh equation in (15) give
\[
\begin{aligned}
u &= \lambda \varphi - f_1, \\
\omega &= \lambda \psi - f_4, \\
\dot{\omega} &= \lambda \omega - f_7.
\end{aligned}
\]
It is clear that \( u \in H^1_L(0, L), v \in H^1_L(0, L) \) and \( \omega \in H^1_L(0, L) \). Furthermore, by (15) we can find \( \phi_i (i = 1, 2, 3) \) as
\[
\begin{align}
\phi_1 &= \frac{f_3(\xi) + \mu(\xi)u(L)}{\xi^2 + \eta + \lambda} \\
\phi_2 &= \frac{f_6(\xi) + \mu(\xi)v(L)}{\xi^2 + \eta + \lambda} \\
\phi_3 &= \frac{f_9(\xi) + \mu(\xi)\dot{\omega}(L)}{\xi^2 + \eta + \lambda}
\end{align}
\]
By using (15) and (16) the functions $\varphi, \psi$ and $\omega$ satisfying the following system

$$
\begin{align*}
\lambda^2 \varphi - \frac{Gh}{\rho_1} (\varphi_x + \psi + l\omega)_x - \frac{1}{\rho_1} (\omega_x - l\varphi) &= f_2 + \lambda f_7, \\
\lambda^2 \psi - \frac{EI}{\rho_2} \psi_{xx} + \frac{Gh}{\rho_2} (\varphi_x + \psi + l\omega) &= f_5 + \lambda f_7, \\
\lambda^2 \omega - \frac{\rho_1}{\rho_1} (\omega_x - l\varphi)_x + \frac{lGh}{\rho_1} (\varphi_x + \psi + l\omega) &= f_8 + \lambda f_7.
\end{align*}
$$

(18)

Solving system (18) is equivalent to finding $(\varphi, \psi, \omega)$ such that

$$
\begin{align*}
\int_0^L (\rho_1 \lambda^2 \varphi w - Gh(\varphi_x + \psi + l\omega)_x w - lEh(\omega_x - l\varphi)w) \, dx &= \int_0^L \rho_1 (f_2 + \lambda f_7)w \, dx, \\
\int_0^L (\rho_2 \lambda^2 \psi \chi - EI\psi_{xx} \chi + Gh(\varphi_x + \psi + l\omega) \chi) \, dx &= \int_0^L \rho_2 (f_5 + \lambda f_7) \chi \, dx, \\
\int_0^L (\rho_1 \lambda^2 \omega \zeta - Eh(\omega_x - l\varphi)_x \zeta + lGh(\varphi_x + \psi + l\omega) \zeta) \, dx &= \int_0^L \rho_1 (f_8 + \lambda f_7) \zeta \, dx
\end{align*}
$$

for all $(w, \chi, \zeta) \in H^1_\infty(0, L) \times H^1_\infty(0, L) \times H^1_\infty(0, L)$. By using (19) and (17) the functions $\varphi, \psi$ and $\omega$ satisfying the following system

$$
\begin{align*}
\int_0^L (\rho_1 \lambda^2 \varphi w + Gh(\varphi_x + \psi + l\omega)_x w - lEh(\omega_x - l\varphi)w) \, dx &= \int_0^L + \tilde{\zeta}_1 u(L)w(L) \\
&= \int_0^L \rho_1 (f_2 + \lambda f_7)w \, dx - \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi) \, d\xi \, w(L), \\
\int_0^L (\rho_2 \lambda^2 \psi \chi + EI\psi_{xx} \chi + Gh(\varphi_x + \psi + l\omega) \chi) \, dx &= \int_0^L \rho_2 (f_5 + \lambda f_7) \chi \, dx - \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_6(\xi) \, d\xi \, \chi(L), \\
\int_0^L (\rho_1 \lambda^2 \omega \zeta + Eh(\omega_x - l\varphi)_x \zeta + lGh(\varphi_x + \psi + l\omega) \zeta) \, dx &= \int_0^L \rho_1 (f_8 + \lambda f_7) \zeta \, dx - \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_7(\xi) \, d\xi \, \zeta(L)
\end{align*}
$$

(20)

where $\tilde{\zeta}_i = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \eta + \lambda} \, d\xi$. Using again (16), we deduce that

$$
\begin{align*}
u(L) &= \lambda \varphi(L) - f_1(L), \\
\omega(L) &= \lambda \psi(L) - f_4(L), \\
\tilde{\omega}(L) &= \lambda \omega(L) - f_7(L).
\end{align*}
$$

(21)

Inserting (21) into (20), we get
Applying the Lax-Milgram theorem, we deduce that for all $(\varphi, \psi, \omega) \in L^2(0, L) \times L^2(0, L) \times L^2(0, L)$, it is easy to verify that

$$
\lambda > 0.
$$

(1) If $U_0 \in D(\mathcal{A})$, then system (10) has a unique strong solution

$$
U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).
$$

(1) If $U_0 \in \mathcal{H}$, then system (10) has a unique weak solution

$$
U \in C^0(\mathbb{R}_+, \mathcal{H}).
$$
4. Lack of exponential stability. We first state three well-known theorems.

**Theorem 4.1 ([23]).** Let \( S(t) = e^{At} \) be a \( C_0 \)-semigroup of contractions on Hilbert space \( \mathcal{H} \). Then \( S(t) \) is exponentially stable if and only if

\[
\rho(A) \supset \{ i\beta : \beta \in \mathbb{R} \} \equiv i\mathbb{R}
\]

and

\[
\lim_{|\beta| \to \infty} \| (i\beta I - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} < \infty
\]

**Theorem 4.2 ([6]).** Let \( S(t) = e^{At} \) be a \( C_0 \)-semigroup on a Hilbert space \( \mathcal{H} \). If

\[
i\mathbb{R} \subset \rho(A) \quad \text{and} \quad \sup_{|\beta| \geq 1} \frac{1}{|\beta|^l} \| (i\beta I - A)^{-1} \|_{\mathcal{L}(\mathcal{H})} < M
\]

for some \( l \), then there exist \( c \) such that

\[
\| e^{At} U_0 \|^2 \leq \frac{c}{t^l} \| U_0 \|^2_{D(A)}
\]

**Theorem 4.3 ([2]).** Let \( A \) be the generator of a uniformly bounded \( C_0 \)-semigroup \( \{ S(t) \}_{t \geq 0} \) on a Hilbert space \( \mathcal{H} \). If:

(i) \( A \) does not have eigenvalues on \( i\mathbb{R} \).

(ii) The intersection of the spectrum \( \sigma(A) \) with \( i\mathbb{R} \) is at most a countable set.

Then the semigroup \( \{ S(t) \}_{t \geq 0} \) is asymptotically stable, i.e., \( \| S(t) z \|_{\mathcal{H}} \to 0 \) as \( t \to \infty \) for any \( z \in \mathcal{H} \).

Our main result is the following

**Theorem 4.4.** The semigroup generated by the operator \( A \) is not exponentially stable.

We state and prove a proposition that will be needed later. We consider the case when \( l \to 0 \) i.e., when \( (P) \) takes the following form

\[
(P_0) \begin{cases}
\rho_1 \varphi_{tt} - Gh(\varphi_x + \psi)_x = 0 & \text{in } (0, L) \times (0, \infty) \\
\rho_2 \psi_{tt} - EI \psi_{xx} + Gh(\varphi_x + \psi) = 0 & \text{in } (0, L) \times (0, \infty) \\
\rho_1 \omega_{tt} - Eh(\omega_x)_x = 0 & \text{in } (0, L) \times (0, \infty) \\
\varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), & \text{in } (0, L) \times (0, \infty) \\
\psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), & \text{in } (0, L) \times (0, \infty) \\
\omega(x, 0) = \omega_0(x), \quad \omega_t(x, 0) = \omega_1(x), & \text{in } (0, L) \times (0, \infty) \\
\varphi(0, t) = 0, \quad \psi(0, t) = 0, \quad \omega(0, t) = 0 & \text{in } (0, +\infty) \\
Gh(\varphi_x + \psi)(L, t) = -\gamma_1 \partial_t^\alpha \varphi(L, t) & \text{in } (0, +\infty) \\
EI \psi_x(L, t) = -\gamma_2 \partial_t^\alpha \psi(L, t) & \text{in } (0, +\infty) \\
Eh \omega_x(L, t) = -\gamma_3 \partial_t^\alpha \omega(L, t) & \text{in } (0, +\infty) \\
\end{cases}
\]

System \( (P_0) \) can be reduced to the Timoshenko system and an independent wave equation:

\[
(PT) \begin{cases}
\rho_1 \varphi_{tt} - Gh(\varphi_x + \psi)_x = 0 & \text{in } (0, L) \times (0, \infty) \\
\rho_2 \psi_{tt} - EI \psi_{xx} + Gh(\varphi_x + \psi) = 0 & \text{in } (0, L) \times (0, \infty) \\
\varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), & \text{in } (0, L) \times (0, \infty) \\
\psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), & \text{in } (0, L) \times (0, \infty) \\
\varphi(0, t) = 0, \quad \psi(0, t) = 0, & \text{in } (0, +\infty) \\
Gh(\varphi_x + \psi)(L, t) = -\gamma_1 \partial_t^\alpha \varphi(L, t) & \text{in } (0, +\infty) \\
EI \psi_x(L, t) = -\gamma_2 \partial_t^\alpha \psi(L, t) & \text{in } (0, +\infty)
\end{cases}
\]
\( \rho_1 \omega_{tt} - Eh \omega_{xx} = 0 \) in \( ]0, L[ \times ]0, \infty[ \)
\( \omega(x, 0) = \omega_0(x), \quad \omega_t(x, 0) = \omega_1(x), \)
\( \omega(0, t) = 0, \quad \text{in } (0, +\infty) \)
\( Eh \omega_x(L, t) = -\gamma_3 \partial_x^\alpha \omega(L, t) \) in \( (0, +\infty) \).

The abstract formulation of \( (P_0) \) is:

\[
\begin{pmatrix}
\varphi \\
u \\
\phi_1 \\
\psi \\
v \\
\phi_2 \\
\omega \\
\tilde{\omega} \\
\phi_3
\end{pmatrix} =
\begin{pmatrix}
u \\
\frac{Gh}{\rho_1} (\varphi_x + \psi) \\
-(\xi^2 + \eta) \phi_1 + u(L) \mu(\xi) \\
E I \psi_{xx} - \frac{Gh}{\rho_2} (\varphi_x + \psi) \\
-(\xi^2 + \eta) \phi_2 + v(L) \mu(\xi) \\
E h \omega_{xx} \\
-(\xi^2 + \eta) \phi_3 + \tilde{\omega}(L) \mu(\xi)
\end{pmatrix}
\]

with domain

\[
D(A_0) = \begin{cases}
(\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T \text{ in } H : \varphi, \psi, \omega \in H^2(0, L) \cap H^1_1(0, L), \\
u, \psi, \tilde{\omega} \in H^1_1(0, L), -((\xi^2 + \eta) \phi_1 + u(L) \mu(\xi)) \in L^2(-\infty, +\infty), \\
-(\xi^2 + \eta) \phi_2 + v(L) \mu(\xi), -(\xi^2 + \eta) \phi_3 + \tilde{\omega}(L) \mu(\xi) \in L^2(-\infty, +\infty) \\
G h(\varphi_x + \psi)(L) + \zeta_1 \int_{-\infty}^{+\infty} \mu(\xi) \phi_1(\xi) \, d\xi = 0 \\
E I \psi_x(L) + \zeta_2 \int_{-\infty}^{+\infty} \mu(\xi) \phi_2(\xi) \, d\xi = 0 \\
E h \omega_x(L) + \zeta_3 \int_{-\infty}^{+\infty} \mu(\xi) \phi_3(\xi) \, d\xi = 0,
\end{cases}
\]

\[
|\xi| \phi_1, |\xi| \phi_2, |\xi| \phi_3 \in L^2(-\infty, +\infty)
\]

Proposition 1. The semigroup generated by operator \( A_0 \) is not exponentially stable.

Proof. This result is due to the fact that a subsequence of eigenvalues of \( A_0 \) is close to the imaginary axis.

Let \( H_1 \) be the subspaces of \( H \) defined by

\[ H_1 = \{ U \in H : U = (0, 0, 0, 0, 0, \omega, \tilde{\omega}, \phi_3) \} \]

and

\[ A_1 = A_0|_{H_1}. \]

Observe that the generator \( A_0 \) becomes the operator \( A_1 \) defined by

\[
D(A_1) = \begin{cases}
(0, 0, 0, 0, 0, 0, \omega, \tilde{\omega}, \phi_3)^T \text{ in } H_1 : \omega \in H^2(0, L) \cap H^1_1(0, L), \\
\tilde{\omega} \in H^1_1(0, L), -((\xi^2 + \eta) \phi_3 + \tilde{\omega}(L) \mu(\xi)) \in L^2(-\infty, +\infty), \\
E h \omega_x(L) + \zeta_3 \int_{-\infty}^{+\infty} \mu(\xi) \phi(\xi) \, d\xi = 0 \\
|\xi| \phi_3 \in L^2(-\infty, +\infty)
\end{cases}
\]
and

\[
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\omega \\
\tilde{\omega} \\
\phi_3
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\tilde{\omega} \\
\frac{E_1}{\rho_1} \omega_{xx} \\
-(\xi^2 + \eta) \phi_3 + \tilde{\omega}(L) \mu(\xi)
\end{pmatrix}
\]

for all \( U = (0, 0, 0, 0, 0, \omega, \tilde{\omega}, \phi_3)^T \in D(A_1) \).

We first compute the characteristic equation that gives the eigenvalues of \( A_1 \). Let \( \lambda \) be an eigenvalue of \( A_1 \) with associated eigenvector \( U = (0, 0, 0, 0, 0, \omega, \tilde{\omega}, \phi_3)^T \).

Then \( A_1 U = \lambda U \) is equivalent to

\[
\begin{align*}
\lambda \omega - \tilde{\omega} &= 0, \\
\lambda \tilde{\omega} - \frac{E_1}{\rho_1} \omega_{xx} &= 0, \\
\lambda \phi_3 + (\xi^2 + \eta) \phi - \tilde{\omega}(L) \mu(\xi) &= 0.
\end{align*}
\]

(27)

From (27)_1, we have

\[
\tilde{\omega} = \lambda \omega.
\]

(28)

Inserting (28) in (27)_2, we get

\[
\lambda^2 \omega - \frac{E_1}{\rho_1} \omega_{xx} = 0
\]

(29)

with the following conditions

\[
\begin{align*}
\omega(0) &= 0, \\
E_1 \omega_x(L) &= -\gamma_3 \lambda(\lambda + \eta) \phi_3 - \tilde{\omega}(L) \mu(\xi).
\end{align*}
\]

(30)

The matrix of the system determining is not singular. Set \( X = (\omega, \omega_x)^T \)

\[
\frac{d}{dx} X = \tilde{\mathcal{B}} X,
\]

(31)

where

\[
\tilde{\mathcal{B}} = \begin{pmatrix}
0 & 1 \\
\frac{\rho_1}{E_1} \lambda^2 & 0
\end{pmatrix}.
\]

The characteristic polynomial of \( \tilde{\mathcal{B}} \) is

\[
s^2 - \frac{\rho_1}{E_1} \lambda^2 = 0.
\]

We find the roots

\[
t_1(\lambda) = \sqrt{\frac{\rho_1}{E_1}} \lambda, \quad t_2(\lambda) = -\sqrt{\frac{\rho_1}{E_1}} \lambda.
\]

Here and below, for simplicity we denote \( t_i(\lambda) \) by \( t_i \). The solution \( \omega \) is given by

\[
\omega(x) = \sum_{i=1}^{2} c_i e^{t_i x}
\]

(32)

Thus the boundary conditions may be written as the following system:

\[
\tilde{M}(\lambda) C(\lambda) = \begin{pmatrix}
1 \\
\frac{1}{h(t_1)} e^{t_1 L} 1 \\
h(t_2) e^{t_2 L}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

(33)
where we have set
\[ h(r) = K r + \gamma_3 \lambda (\lambda + \eta)^{\alpha - 1} \quad (\text{we set } Eh = K). \]
Hence a non-trivial solution \( \omega \) exists if and only if the determinant of \( \tilde{M}(\lambda) \) vanishes. Set \( f(\lambda) = \text{det} \tilde{M}(\lambda), \) thus the characteristic equation is \( f(\lambda) = 0. \)

Our purpose in the sequel is to prove, thanks to Rouché’s Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since \( A_1 \) is dissipative, we study the asymptotic behavior of the large eigenvalues \( \lambda \) of \( A_1 \) in the strip \(-\alpha_0 \leq R(\lambda) \leq 0\), for some \( \alpha_0 > 0 \) large enough and for such \( \lambda \), we remark that \( e^{\rho_i}, i = 1, 2 \) remains bounded.

**Lemma 4.5.** There exists \( N \in \mathbb{N} \) such that
\[ \{\lambda_k\}_{k \in \mathbb{Z}, |k| \geq N} \subset \sigma(A_1), \quad (34) \]
where
\[ \lambda_k = i \frac{1}{rL} \left( k + \frac{1}{2} \right) \pi + \frac{\tilde{\alpha}}{K^{1-\alpha}} + \frac{\beta}{|k|^{1-\alpha}} + o \left( \frac{1}{|k|^{1-\alpha}} \right), \]
\[ r = \sqrt{\frac{\rho_1}{K}} k \geq N, \tilde{\alpha} \in i \mathbb{R}, \beta \in \mathbb{R}, \beta < 0. \]
\[ \lambda_k = \overline{\lambda_{-k}} \text{ if } k \leq -N. \]
Moreover for all \( |k| \geq N \), the eigenvalues \( \lambda_k \) are simple.

**Proof.** The proof is decomposed in three steps:

**Step 1.**
\[
\begin{align*}
  f(\lambda) &= e^{Lt_2}h(t_2) - e^{Lt_1}h(t_1) \\
  &= -e^{-\sqrt{\frac{\pi}{K}} \lambda L} h(\sqrt{\frac{\pi}{K}} \lambda) \left( e^{2 \sqrt{\frac{\pi}{K}} \lambda L} + \frac{K(\lambda + \eta)^{1-\alpha} - \gamma_3}{K(\lambda + \eta)^{1-\alpha} + \gamma_3} \right) \\
  &= -e^{-\sqrt{\frac{\pi}{K}} \lambda L} h(\lambda) \left( e^{2 \sqrt{\frac{\pi}{K}} \lambda L} + 1 - \frac{2 \gamma_3}{\gamma_3 + K(\lambda + \eta)^{1-\alpha}} \right) \\
  \end{align*}
\]
(35)

We set
\[
\begin{align*}
  \tilde{f}(\lambda) &= e^{2 \sqrt{\frac{\pi}{K}} \lambda L} + 1 - \frac{2 \gamma_3}{\gamma_3 + K(\lambda + \eta)^{1-\alpha}} \\
  &= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{1-\alpha}} + \frac{f_2(\lambda)}{\lambda^{2-2\alpha}} + o \left( \frac{1}{\lambda^{2-2\alpha}} \right) \\
  f_0(\lambda) &= e^{2 \sqrt{\frac{\pi}{K}} \lambda L} + 1 \\
  f_1(\lambda) &= -2 \gamma_3/K \\
  f_2(\lambda) &= 2 \gamma_3^2/K^2 \\
  \end{align*}
\]
(36) (37) (38) (39)

Note that \( f_0, f_1, f_2 \) remain bounded in the strip \(-\alpha_0 \leq R(\lambda) \leq 0\).

**Step 2.** We look at the roots of \( f_0 \). From (37), \( f_0 \) has one familie of roots that we denote \( \lambda^0_k \).
\[
  f_0(\lambda) = 0 \Leftrightarrow e^{2 \sqrt{\pi \lambda L}} = -1.
\]
Hence
\[
  2r \lambda L = i(2k+1)\pi, \quad k \in \mathbb{Z}, r = \sqrt{\frac{\rho_1}{K}}
\]
i.e.,
\[
  \lambda^0_k = \frac{i(2k+1)\pi}{2rL}, \quad k \in \mathbb{Z}.
\]
Now with the help of Rouché’s Theorem, we will show that the roots of $\tilde{f}$ are close to those of $f_0$. Let us start with the first family. Changing in (36) the unknown $\lambda$ by $u = 2\sqrt{\pi k}L$ then (36) becomes

$$\tilde{f}(u) = (e^u + 1) + O\left(\frac{1}{u^{(1-\alpha)}}\right) = f_0(u) + O\left(\frac{1}{u^{(1-\alpha)}}\right)$$

The roots of $f_0$ are $u_k = \frac{i(k+\frac{1}{2})}{rL}\pi, k \in \mathbb{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of $k$ such that $|e^u + 1| \geq Cr$ for $r$ small enough. This allows to apply Rouché’s Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots $u_k$ of $f_0$. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k| \geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $\frac{i(k+\frac{1}{2})}{rL} \pi$ of $f_0$. Finally for $|k| \geq N, \lambda_k$ is simple since $\lambda_k^0$ is.

**Step 3.** From Step 2, we can write

$$\lambda_k = \frac{1}{rL} \left( k + \frac{1}{2} \right) \pi + \varepsilon_k. \quad (40)$$

Using (40), we get

$$e^{2r\lambda_k L} = -1 - 2rL\varepsilon_k - 2rL^2\varepsilon_k^2 + o(\varepsilon_k^2). \quad (41)$$

Substituting (41) into (36), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = -2rL\varepsilon_k - \frac{2\gamma_3}{K(\lambda_k^0)^{1-\alpha}} + o(\varepsilon_k) = 0 \quad (42)$$

and hence

$$\varepsilon_k = -\frac{\gamma_3}{Kr^\alpha L^\alpha((k+\frac{1}{2})\pi)^{1-\alpha}} + o\left(\frac{1}{k^{1-\alpha}}\right)$$

$$= \begin{cases} 
-\frac{\gamma_3}{K r^\alpha L^\alpha((k+\frac{1}{2})\pi)^{1-\alpha}} \left(\cos(1-\alpha)\frac{\pi}{2} - i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{1-\alpha}}\right) \\
-\frac{\gamma_3}{K r^\alpha L^\alpha(-\frac{1}{2}\pi)^{1-\alpha}} \left(\cos(1-\alpha)\frac{\pi}{2} + i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{1-\alpha}}\right) 
\end{cases}$$

for $k \geq 0$

and

for $k \leq 0$

(43)

From (43) we have in that case $|k|^{1-\alpha}R\lambda_k \sim \beta$, with

$$\beta = -\frac{\gamma_3}{Kr^\alpha L^\alpha \pi^{1-\alpha}} \cos(1-\alpha)\frac{\pi}{2}.$$ 

The operator $A_1$ has a non exponential decaying branche of eigenvalues. Thus the proof of Proposition 1 is complete.

**Proof of Theorem 4.4.** We will examine two cases.

**Case 1** $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $A$. Indeed, noting that $(\sin x, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \in \mathcal{H}$, and denoting by $(\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T$ the image of $(\sin x, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ by $A^{-1}$, we see that $\phi_1(\xi) = [\xi]\frac{\sin k}{\pi} \sin L$. But, then $\phi_1 \notin L^2(-\infty, +\infty)$, since $\alpha \in ]0, 1[$. And so $(\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T \notin D(A)$.

**Case 2** $\eta \neq 0$: We aim to show that an infinite number of eigenvalues of $A$ approach the imaginary axis which prevents the Bresse system $(P)$ from being exponentially
Hence from (44), we have

We set

From the boundary conditions, we get

Then

From (47), after derivation, we have

From (46), after derivation, we have

Inserting (49) into (50), we find

From (46), after derivation, we have

stable. Indeed We first compute the characteristic equation that gives the eigenvalues of $A$.

Let $\lambda$ be an eigenvalue of $A$ with associated eigenvector $U = (\phi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T$. We set $\rho_1 = \rho_2 = 1, Gh = 1, EI = a, Eh = k$. To solve $AU = \lambda U$ is enough to solve

$$
\begin{align*}
-\lambda^2 \phi + (\phi_x + \psi + l\omega)_x + lk(\omega_x - l\phi) &= 0, \\
-\lambda^2 \psi &= a \psi_{xx} - (\phi_x + \psi + l\omega), \\
-\lambda^2 \omega + k(\omega_x - l\phi)_x - l(\phi_x + \psi + l\omega) &= 0,
\end{align*}
$$

From the boundary conditions, we get

$$
\begin{align*}
\lambda \phi_1 + (\xi^2 + \eta) \phi_1 - u(L)\mu(\xi) &= 0, \\
\lambda \phi_2 + (\xi^2 + \eta) \phi_2 - v(L)\mu(\xi) &= 0, \\
\lambda \phi_3 + (\xi^2 + \eta) \phi_3 - \tilde{\omega}(L)\mu(\xi) &= 0.
\end{align*}
$$

We set

$$
\tilde{\phi} = (\phi_x + \psi + l\omega), \quad \tilde{\psi} = \psi_x, \quad \tilde{\omega} = (\omega_x - l\phi).
$$

Hence from (44), we have

$$
\begin{align*}
-\lambda^2 \phi + \tilde{\phi}_x +lk\tilde{\omega} &= 0, \\
-\lambda^2 \tilde{\psi} + a\tilde{\psi}_x - \tilde{\phi} &= 0, \\
-\lambda^2 \omega + k\tilde{\omega}_x - l\tilde{\phi} &= 0.
\end{align*}
$$

Then

$$
\begin{align*}
-(\lambda^2 + l^2 + 1)\tilde{\phi} + \tilde{\phi}_{xx} + a\tilde{\psi}_x + 2lk\tilde{\omega}_x &= 0, \\
-\lambda^2 \tilde{\psi} + a\tilde{\psi}_xx - \tilde{\phi}_x &= 0, \\
-(\lambda^2 + l^2k)\tilde{\omega} + k\tilde{\omega}_{xx} + 2l\tilde{\phi}_x &= 0.
\end{align*}
$$

From (47), after derivation, we have

$$
-(\lambda^2 + l^2k)\tilde{\omega}_x + k\tilde{\omega}_{xxx} - 2l\tilde{\phi}_{xx} = 0.
$$

From (46), after derivation, we have

$$
\begin{align*}
\tilde{\omega}_x &= \frac{1}{2lk}((\lambda^2 + l^2 + 1)\tilde{\phi} - \tilde{\phi}_{xx} - a\tilde{\psi}_x), \\
\tilde{\omega}_{xxx} &= \frac{1}{2lk}((\lambda^2 + l^2 + 1)\tilde{\phi}_{xx} - \tilde{\phi}_{xxxx} - a\tilde{\psi}_{xxx}).
\end{align*}
$$

Inserting (49) into (50), we find

$$
\begin{align*}
-(\lambda^2 + l^2k) \left( (\lambda^2 + l^2 + 1)\tilde{\phi} - \tilde{\phi}_{xx} - a\tilde{\psi}_x \right) \\
+ \frac{1}{2l} (\lambda^2 + l^2 + 1)\tilde{\phi}_{xx} - \tilde{\phi}_{xxxx} - a\tilde{\psi}_{xxx} - 2l\tilde{\phi}_{xx} = 0
\end{align*}
$$

From (46), after derivation, we have

$$
\begin{align*}
\tilde{\phi} &= \lambda^2 \psi + a\tilde{\psi}_x, \\
\tilde{\phi}_{xx} &= \lambda^2 \psi_{xx} + a\tilde{\psi}_{xx}, \\
\tilde{\phi}_{xxxx} &= \lambda^2 \psi_{xxxx} + a\tilde{\psi}_{xxxx}.
\end{align*}
$$
Inserting (51)\textsubscript{1}, (51)\textsubscript{2}, (51)\textsubscript{3} into (50), we find
\[\psi_{xxxxx} + \left(-\frac{1}{k} + \frac{1}{\lambda} + 1\right) \lambda^2 + 2t^2 \right) \psi_{xxxx} + \left(\frac{1}{a} + \frac{1}{a} + \frac{1}{\lambda} \right) \lambda^4 + \left(\frac{1}{a} + (-\frac{2}{a} + \frac{1}{k} + 1)l^2 \right) \lambda^2 + l^4 \right) \psi_{xx} \]
\[- \left(\frac{1}{ak} \lambda^6 + \left(\frac{1}{ak} l^2 + \frac{1}{a} l^2 + \frac{1}{ak} \right) \lambda^4 + \left(\frac{1}{ak} l^4 + \frac{1}{a} l^2 \right) \lambda^2 \right) \psi = 0,\]  
(52)
whose general solutions depend on the roots of the polynomial
\[p(s) = s^6 + \left(-\frac{1}{k} + \frac{1}{\lambda} + 1\right) \lambda^2 + 2t^2 \right) s^4 + \left(\frac{1}{a} + \frac{1}{a} + \frac{1}{\lambda} \right) \lambda^4 + \left(\frac{1}{a} + (-\frac{2}{a} + \frac{1}{k} + 1)l^2 \right) \lambda^2 + l^4 \right) s^2 \]
\[- \left(\frac{1}{ak} \lambda^6 + \left(\frac{1}{ak} l^2 + \frac{1}{a} l^2 + \frac{1}{ak} \right) \lambda^4 + \left(\frac{1}{ak} l^4 + \frac{1}{a} l^2 \right) \lambda^2 \right) \cdot\]  
(53)
If we put \(s^2 = S\), we can write (53) as
\[\tilde{p}(S) = S^3 + \left(-\frac{1}{k} + \frac{1}{\lambda} + 1\right) \lambda^2 + 2t^2 \right) S^2 + \left(\frac{1}{a} + \frac{1}{a} + \frac{1}{\lambda} \right) \lambda^4 + \left(\frac{1}{a} + (-\frac{2}{a} + \frac{1}{k} + 1)l^2 \right) \lambda^2 + l^4 \right) S \]
\[- \left(\frac{1}{ak} \lambda^6 + \left(\frac{1}{ak} l^2 + \frac{1}{a} l^2 + \frac{1}{ak} \right) \lambda^4 + \left(\frac{1}{ak} l^4 + \frac{1}{a} l^2 \right) \lambda^2 \right) \cdot\]  
(54)
The polynomial \(\tilde{p}(S)\), given in (54), has one real root and two complex conjugate roots.

The general solution of (52) must be of the form
\[\psi(x) = \sum_{i=1}^{6} c_i e^{i t_i x},\]  
(55)
where \(t_i(\lambda) (i = 1, \ldots, 6)\) are the roots of (53) such that \(t_2(\lambda) = -t_1(\lambda), t_4(\lambda) = -t_3(\lambda), t_6(\lambda) = -t_5(\lambda)\). \(t_3(\lambda)\) and \(t_5(\lambda)\) are complex conjugate. Then, we write (45) uniquely in function of \(\psi\). From (46)\textsubscript{1} we have
\[\varphi = \frac{1}{\lambda^2} (\tilde{\varphi} + l k \tilde{\omega}).\]  

From (47)\textsubscript{3} we have
\[\tilde{\omega} = - \frac{2l}{\lambda^2 + l^2 k} \tilde{\varphi}_x + \frac{k}{\lambda^2 + l^2 k} \tilde{\omega}_{xx}.\]
From (47)\textsubscript{1}, after derivation, we get
\[\tilde{\omega}_{xx} = \frac{\lambda^2 + l^2 + 1}{2l k} \tilde{\varphi}_x - \frac{1}{2l k} \tilde{\varphi}_{xxx} - \frac{a}{2l k} \tilde{\varphi}_{xxx}.\]
Thus we find
\[\tilde{\omega} = \frac{2l}{\lambda^2 + l^2 k} \tilde{\varphi}_x + \frac{\lambda^2 + l^2 + 1}{2l(\lambda^2 + l^2 k)} \tilde{\varphi}_x - \frac{1}{2l(\lambda^2 + l^2 k)} \tilde{\varphi}_{xxx} - \frac{a}{2l(\lambda^2 + l^2 k)} \tilde{\varphi}_{xxx} - \frac{1}{2l(\lambda^2 + l^2 k)} \tilde{\varphi}_{xxx} - \frac{a}{2l(\lambda^2 + l^2 k)} \tilde{\varphi}_{xxx}.\]
We deduce that
\[\varphi = - \frac{(k + 2) \lambda^2 + k(1 - l^2)}{2(\lambda^2 + kl^2)} \psi_x + \frac{((k + 2) a + k) \lambda^2 - ak l^2}{2\lambda^2(\lambda^2 + kl^2)} \psi_{xxx} - \frac{ka}{2\lambda^2(\lambda^2 + kl^2)} \psi_{xxxxx}.\]
From (46)_3 and (47)_1 we have
\[
\omega = \frac{1}{\lambda^2}(k\hat{\omega}_x - l\hat{\xi})
\]
\[
\hat{\omega}_x = \frac{\lambda^2 + l^2 + 1}{2l}\hat{\xi} - \frac{1}{2l\lambda^2}\hat{\psi}_{xx} - \frac{a}{2l\lambda^2}\hat{\psi}_x.
\]
Then from (46)_2 and (47)_2 and (56), we get
\[
\omega = -\frac{\lambda^2 - l^2 + 1}{2l}\psi + \frac{(a + 1)\lambda^2 - al^2}{2l\lambda^2}\psi_{xx} - \frac{a}{2l\lambda^2}\psi_{xxx}.
\]
Thus the boundary conditions (45) may be written as the following system:
\[
\psi(0) = 0 \implies \sum_{i=1}^{6} c_i = 0,
\]
\[
\varphi(0) = 0 \implies \sum_{i=1}^{6} (\alpha_1 t_i + \alpha_3 t_i^3 + \alpha_5 t_i^5) c_i = 0,
\]
\[
\omega(0) = 0 \implies \sum_{i=1}^{6} (\alpha_0 + \alpha_2 t_i^2 + \alpha_4 t_i^4) c_i = 0,
\]
\[
a\psi(L) = -\gamma_2 \lambda(\lambda + \eta)^{\alpha-1}\psi(L) \implies \sum_{i=1}^{6} (at_i + \gamma_2 \lambda(\lambda + \eta)^{\alpha-1}) e^{t_i L} c_i = 0,
\]
\[
\hat{\varphi}(L) = -\gamma_1 \lambda(\lambda + \eta)^{\alpha-1}\varphi(L)
\]
\[
\implies \sum_{i=1}^{6} (-\lambda^2 + at_i^2 + \gamma_1 \lambda(\lambda + \eta)^{\alpha-1}(\alpha_1 t_i + \alpha_3 t_i^3 + \alpha_5 t_i^5)) e^{t_i L} c_i = 0,
\]
\[
k\hat{\omega}(L) = -\gamma_3 \lambda(\lambda + \eta)^{\alpha-1}\omega(L)
\]
\[
\implies \sum_{i=1}^{6} (k(\delta_1 t_i + \delta_3 t_i^3 + \delta_5 t_i^5) + \gamma_3 \lambda(\lambda + \eta)^{\alpha-1}(\alpha_0 + \alpha_2 t_i^2 + \alpha_4 t_i^4)) e^{t_i L} c_i = 0.
\]
Thus the boundary conditions may be written as the following system:
\[
\mathcal{M}_t C(\lambda) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]
where \(C(\lambda) = (c_1, c_2, c_3, c_4, c_5, c_6)^T\) and
\[
\mathcal{M}_t =
\begin{pmatrix}
1 & h_1(t_1) & h_1(t_2) & h_1(t_3) & h_1(t_4) & h_1(t_5) & h_1(t_6) \\
1 & h_2(t_1) & h_2(t_2) & h_2(t_3) & h_2(t_4) & h_2(t_5) & h_2(t_6) \\
1 & h_3(t_1) e^{t_1 L} & h_3(t_2) e^{t_2 L} & h_3(t_3) e^{t_3 L} & h_3(t_4) e^{t_4 L} & h_3(t_5) e^{t_5 L} & h_3(t_6) e^{t_6 L} \\
1 & h_4(t_1) e^{t_1 L} & h_4(t_2) e^{t_2 L} & h_4(t_3) e^{t_3 L} & h_4(t_4) e^{t_4 L} & h_4(t_5) e^{t_5 L} & h_4(t_6) e^{t_6 L} \\
1 & h_5(t_1) e^{t_1 L} & h_5(t_2) e^{t_2 L} & h_5(t_3) e^{t_3 L} & h_5(t_4) e^{t_4 L} & h_5(t_5) e^{t_5 L} & h_5(t_6) e^{t_6 L} \\
1 & h_6(t_1) & h_6(t_2) & h_6(t_3) & h_6(t_4) & h_6(t_5) & h_6(t_6) 
\end{pmatrix}.
\]
where
\[ h_1(r) = \alpha_1 r + \alpha_3 r^3 + \alpha_5 r^5, \]
\[ h_2(r) = \alpha_0 + \alpha_2 r^2 + \alpha_4 r^4, \]
\[ h_3(r) = a r + \gamma_2 \lambda(\lambda + \eta)^{\alpha - 1}, \]
\[ h_4(r) = -\lambda^2 + a r^2 + \gamma_1 \lambda(\lambda + \eta)^{\alpha - 1}(\alpha_1 r + \alpha_3 r^3 + \alpha_5 r^5), \]
\[ h_5(r) = k(\delta_1 r + \delta_3 r^3 + \delta_5 r^5) + \gamma_3 \lambda(\lambda + \eta)^{\alpha - 1}(\alpha_0 + \alpha_2 r^2 + \alpha_4 r^4), \]

where
\[ \alpha_1 = -\frac{(k + 2)\lambda^2 + k(1 - t^2)}{2(\lambda^2 + k t^2)}, \alpha_3 = \frac{(k + 2)a + k \lambda^2 - a k l^2}{2\lambda^2(\lambda^2 + k l^2)}, \alpha_5 = -\frac{ka}{2\lambda^2(\lambda^2 + k l^2)}, \]
\[ \alpha_0 = -\lambda^2 - t^2 + 1, \alpha_2 = \frac{(a + 1)\lambda^2 - a l^2}{\lambda^2}, \alpha_4 = -\frac{a}{\lambda^2}, \]
\[ \delta_1 = -\frac{\lambda^2 - 3t^2 + 1}{(\lambda^2 + t^2 k)}, \delta_3 = \frac{(a + 1)\lambda^2 - 3a l^2}{(\lambda^2 + t^2 k)}, \delta_5 = -\frac{a}{(\lambda^2 + t^2 k)}. \]

Hence a non-trivial solution \( \psi \) exists if and only if the determinant of \( M_t \) vanishes. Set \( f_1(\lambda) = \det M_t \), thus the characteristic equation is \( f_1(\lambda) = 0 \).

We remark that \( f_0(\lambda) = \det M_0 \) is a smooth function with the parameter \( l \). In the expansion of \( t_i(\lambda)(i = 1, \ldots, 6) \) and \( h_i(t_j)(i = 1, \ldots, 5, j = 1, \ldots, 6) \), the parameter \( l \) appears only in lower terms. Hence, in the development of \( \det M_t \) in power series following \( \lambda \), we obtain same development as \( \det M_0 \) modulo lower terms depending on \( l \). Hence \( \mathcal{A}_t \) (if we note \( \mathcal{A} \) by \( \mathcal{A}_t \)) and \( \mathcal{A}_0 \) have same branches of eigenvalues modulo lower terms depending on \( l \).

From Proposition 1 and the fact that \( f_0(\lambda) = \det M_0 = 0 \) give the eigenvalues of Bresse system when \( l = 0 \) we conclude our result.

**Remark 1.** We can also show the lack of exponential stability by proving that the second condition in Theorem 4.1 does not hold. In particular, it can be shown that there is a sequence \( \lambda_n \in \mathbb{R} \) diverging to \( \infty \), and a bounded sequence \( F_n \in \mathcal{H} \) such that
\[ \| (i\lambda_n - \mathcal{A})^{-1} F_n \| \to \infty \] for all \( n \) large enough.
We give an idea of the proof in the Appendix.

5. **Asymptotic stability.** In this section, we use a general criteria of Arendt-Batty in [2] to show the strong stability of the \( C_0 \)-semigroup \( e^{t\mathcal{A}} \) associated to the system \( (P) \) in the absence of the compactness of the resolvent of \( \mathcal{A} \). Our main result is the following theorem:

**Theorem 5.1.** Then, the \( C_0 \)-semigroup \( e^{t\mathcal{A}} \) is strongly stable in \( \mathcal{H} \), i.e, for all \( U^0 \in \mathcal{H} \), the solution of (10) satisfies
\[ \lim_{t \to +\infty} \| e^{t\mathcal{A}} U^0 \|_\mathcal{H} = 0. \]

**Lemma 5.2.** We have
\[ \sigma(\mathcal{A}) \cap \{ i\lambda, \lambda \in \mathbb{R}, \lambda \neq 0 \} = \emptyset. \]

**Lemma 5.3.** \( \lambda = 0 \) is not an eigenvalue of \( \mathcal{A} \).

Let us first prove Lemma 5.3.
Proof. From (11) we get that \((\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T \in Ker(A) \subset D(A)\) if and only if

\[
\left\{
\begin{array}{l}
-u = 0, \\
-Gh (\varphi_x + \psi + l\omega)_x - \frac{lEh}{\rho_1} (\omega_x - l\varphi) = 0, \\
(\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = 0, \\
-v = 0, \\
-EI \frac{\psi_{xx}}{\rho_2} + \frac{Gh}{\rho_2} (\varphi_x + \psi + l\omega) = 0, \\
(\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = 0, \\
-\tilde{\omega} = 0, \\
-Eh (\omega_x - l\varphi)_x + \frac{lGh}{\rho_1} (\varphi_x + \psi + l\omega) = 0, \\
(\xi^2 + \eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = 0.
\end{array}
\right. 
\]  

(57)

This implies that \(u = v = \tilde{\omega} = 0, \phi_1 = \phi_2 = \phi_3 = 0\) and

\[
\left\{
\begin{array}{l}
-Gh (\varphi_x + \psi + l\omega)_x - \frac{lEh}{\rho_1} (\omega_x - l\varphi) = 0, \\
-\frac{\rho_1}{EI} \psi_{xx} + \frac{Gh}{\rho_2} (\varphi_x + \psi + l\omega) = 0, \\
-\frac{\rho_2}{Eh} (\omega_x - l\varphi)_x + \frac{lGh}{\rho_1} (\varphi_x + \psi + l\omega) = 0,
\end{array}
\right. 
\]  

(58)

with

\[
\left\{
\begin{array}{l}
\varphi(0) = \psi(0) = \omega(0) = 0, \\
\varphi_x(L) + \psi(L) + l\omega(L) = 0, \quad \psi_x(L) = 0, \quad \omega_x(L) - l\varphi(L) = 0.
\end{array}
\right.
\]

This implies that

\[
Gh \|\varphi_x + \psi + l\omega\|^2 + EI \|\psi_x\|^2 + Eh \|\omega_x - l\varphi\|^2 = 0. 
\]  

(59)

(59) implies that \(\psi\) is a constant function and

\[
\varphi_x + \psi + l\omega = 0, \quad \omega_x - l\varphi = 0.
\]

As \(\psi(0) = 0\), we deduce that \(\psi \equiv 0\). Hence

\[
\varphi_x + l\omega = 0, \quad \omega_x - l\varphi = 0.
\]

Then, we have

\[
\varphi = c \sin lx, \quad \omega = -c \cos lx.
\]

Hence \(\omega(0) = 0\) imply that \(c = 0\). Thus \((\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \tilde{\omega}, \phi_3)^T = 0\). This concludes the proof of Lemma 5.3. 

\[\square\]

Now, we prove Lemma 5.2.
Let us suppose that there is $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $U \neq 0$, such that $AU = i\lambda U$. Then, we get

$$
\begin{align*}
&i\lambda \varphi - u = 0, \\
i\lambda u - \frac{Gh}{\rho_1} (\varphi_x + \psi + l\omega)_x - \frac{1}{\rho_1} (\omega_x - l\varphi) = 0, \\
i\lambda \phi_1 + (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = 0, \\
i\lambda \psi - v = 0, \\
i\lambda v - \frac{EI}{\rho_2} \psi_{xx} + \frac{Gh}{\rho_2} (\varphi_x + \psi + l\omega) = 0, \\
i\lambda \phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = 0, \\
i\lambda \omega - \bar{\omega} = 0, \\
i\lambda \bar{\omega} - \frac{Eh}{\rho_1} (\omega_x - l\varphi)_x + \frac{1}{\rho_1}Gh (\varphi_x + \psi + l\omega) = 0, \\
i\lambda \phi_3 + (\xi^2 + \eta)\phi_3 - \omega(L)\mu(\xi) = 0.
\end{align*}
$$

Then, from (14) we have

$$
\phi_i \equiv 0, \quad i = 1, 2, 3. \quad (61)
$$

From (60)$_3$, (60)$_6$ and (60)$_9$, we have

$$
u(L) = v(L) = \bar{\omega}(L) = 0. \quad (62)
$$

Hence, from (60) and (P') we obtain

$$
\varphi(L) = \psi(L) = \omega(L) = 0 \quad \text{and} \quad \varphi_x(L) = \psi_x(L) = \omega_x(L) = 0. \quad (63)
$$

From (60), we have

$$
\begin{align*}
-\lambda^2 \rho_1 \varphi - Gh (\varphi_x + \psi + l\omega)_x - 1Eh (\omega_x - l\varphi) = 0, \\
-\lambda^2 \rho_2 \psi - EI \psi_{xx} + Gh (\varphi_x + \psi + l\omega) = 0, \\
-\lambda^2 \rho_1 \omega - Eh (\omega_x - l\varphi)_x + 1Gh (\varphi_x + \psi + l\omega) = 0.
\end{align*}
$$

Consider $X = (\varphi, \psi, \omega, \varphi_x, \psi_x, \omega_x)$. Then we can rewrite (63) and (64) as the initial value problem

$$
\frac{d}{dx} X = AX \quad (65)
$$

where

$$
\mathcal{A} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-\frac{\lambda^2 \rho_1 + \lambda^2 E}{G} & 0 & 0 & 0 & 0 & 1 \\
0 & -\frac{\rho_1 \lambda^2 + Gh}{EI} & \frac{Gh}{EI} & -1 & \frac{(E+G)\xi}{E} & 0 \\
0 & \frac{Gh}{EI} & -\frac{\rho_1 \lambda^2 + Gh^2}{E} & 0 & 0 & 0
\end{pmatrix}
$$

By the Picard Theorem for ordinary differential equations the system (65) has a unique solution $X = 0$. Therefore $\varphi = 0, \psi = 0, \omega = 0$. It follows from (60), that $u = 0, v = 0, \bar{\omega} = 0$, i.e., $U = 0$.

The condition (ii) of Theorem 4.3 will be satisfied if we show that any point $\sigma(\mathcal{A}) \cap \{i\mathbb{R}\}$ is at most a countable set.

We will prove that the operator $i\lambda - \mathcal{A}$ is surjective for $\lambda \neq 0$. For this purpose, let $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathcal{H}$, we seek $U = (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \bar{\omega}, \phi_3)^T \in D(\mathcal{A})$ solution of solution of the following equation

$$(i\lambda - \mathcal{A})U = F.$$
Equivalently, we have the following system

\[
\begin{align*}
  i\lambda\phi - u &= f_1, \\
  i\lambda u - \frac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \frac{lEh}{\rho_1}(\omega_x - l\varphi) &= f_2, \\
  i\lambda\phi_1 + (\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) &= f_3, \\
  i\lambda\psi - v &= f_4, \\
  i\lambda\psi_x + \frac{E\psi}{\rho_2}\psi_{xx} + \frac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) &= f_5, \\
  i\lambda\phi_2 + (\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) &= f_6, \\
  i\lambda\omega - \tilde{\omega} &= f_7, \\
  i\lambda\omega_x - \frac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \frac{1Gh}{\rho_1}(\varphi_x + \psi + l\omega) &= f_8, \\
  i\lambda\phi_3 + (\xi^2 + \eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) &= f_9
\end{align*}
\]

with the following conditions

\[
\begin{align*}
  &\begin{cases}
    Gh(\varphi_x + \psi + l\omega)(L) = -\gamma_1(\pi)^{-1}\sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi, t) \, d\xi \\
    E\psi_x(L) = -\gamma_2(\pi)^{-1}\sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi, t) \, d\xi \\
    Eh(\omega_x - l\varphi)(L) = -\gamma_3(\pi)^{-1}\sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi_3(\xi, t) \, d\xi.
  \end{cases}
\end{align*}
\]

We get

\[
\begin{align*}
  -\lambda^2\varphi - \frac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \frac{lEh}{\rho_1}(\omega_x - l\varphi) &= f_2 + i\lambda f_1, \\
  -\lambda^2\psi - \frac{E\psi}{\rho_2}\psi_{xx} + \frac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) &= f_5 + i\lambda f_4, \\
  -\lambda^2\omega - \frac{E\omega}{\rho_1}(\omega_x - l\varphi)_x + \frac{1Gh}{\rho_1}(\varphi_x + \psi + l\omega) &= f_8 + i\lambda f_7.
\end{align*}
\]

Solving system (68) is equivalent to finding \((\varphi, \psi, \omega) \in (H^2 \cap H^1_1(0, L))^3\) such that

\[
\begin{align*}
  &\begin{cases}
    \int_0^L (-\rho_1\lambda^2\varphi w - Gh(\varphi_x + \psi + l\omega)_w - lEh(\omega_x - l\varphi)w) \, dx \\
    = \int_0^L \rho_1(f_2 + i\lambda f_1)w \, dx, \\
    \int_0^L (-\rho_2\lambda^2\psi \chi - E\psi_{xx}\chi + Gh(\varphi_x + \psi + l\omega)\chi) \, dx \\
    = \int_0^L \rho_2(f_5 + i\lambda f_4)\chi \, dx, \\
    \int_0^L (-\rho_1\lambda^2\omega \zeta - Eh(\omega_x - l\varphi)_x\zeta + lGh(\varphi_x + \psi + l\omega)\zeta) \, dx \\
    = \int_0^L \rho_1(f_8 + i\lambda f_7)\zeta \, dx.
  \end{cases}
\end{align*}
\]
for all \((w, \chi, \zeta) \in H^1_1(0, L) \times H^1_1(0, L) \times H^1_1(0, L)\). By using (19) and (17) the functions \(\varphi, \psi\) and \(\omega\) satisfying the following system

\[
\begin{cases}
\int_0^L (-\rho_1 \lambda^2 \varphi w + G h(\varphi x + \psi + l \omega) w_x - l E h(\omega x - l \varphi) w) \, dx + \zeta_1 u(L) w(L) \\
= \int_0^L \rho_1 (f_2 + i \lambda f_1) \, dx - \zeta_1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + i \lambda^2} f_3(\xi) \, d\xi \, w(L),
\end{cases}
\]

\[
\begin{cases}
\int_0^L (-\rho_2 \lambda^2 \psi \chi + E I \psi x \chi_x + G h(\varphi x + \psi + l \omega) \chi) \, dx + \zeta_2 v(L) \chi(L) \\
= \int_0^L \rho_2 (f_5 + i \lambda f_4) \, dx - \zeta_2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + i \lambda^2} f_6(\xi) \, d\xi \, \chi(L),
\end{cases}
\]

\[
\begin{cases}
\int_0^L (-\rho_1 \lambda^2 \omega \zeta + E h(\omega x - l \varphi) \zeta_x + l G h(\varphi x + \psi + l \omega) \zeta) \, dx + \zeta_3 \tilde{\omega}(L) w(L) \\
= \int_0^L \rho_1 (f_8 + i \lambda f_7) \, dx - \zeta_3 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + i \lambda^2} f_9(\xi) \, d\xi \, \zeta(L),
\end{cases}
\]

where \(\zeta_i = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2 + \eta + i \lambda} \, d\xi\). Using again (16), we deduce that

\[
\begin{cases}
u(L) = i \lambda \varphi(L) - f_3(L), \\
\tilde{\omega}(L) = i \lambda \omega(L) - f_7(L).
\end{cases}
\]

Inserting (71) into (70), we get

\[
\begin{cases}
\int_0^L (-\rho_1 \lambda^2 \varphi w + G h(\varphi x + \psi + l \omega) w_x - l E h(\omega x - l \varphi) w) \, dx + i \lambda \tilde{\zeta}_1 \varphi(L) w(L) \\
= \int_0^L \rho_1 (f_2 + i \lambda f_1) \, dx - \zeta_1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + i \lambda} f_3(\xi) \, d\xi \, w(L),
\end{cases}
\]

\[
\begin{cases}
\int_0^L (-\rho_2 \lambda^2 \psi \chi + E I \psi x \chi_x + G h(\varphi x + \psi + l \omega) \chi) \, dx + i \lambda \tilde{\zeta}_2 \psi(L) \chi(L) \\
= \int_0^L \rho_2 (f_5 + i \lambda f_4) \, dx - \zeta_2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + i \lambda} f_6(\xi) \, d\xi \, \chi(L),
\end{cases}
\]

\[
\begin{cases}
\int_0^L (-\rho_1 \lambda^2 \omega \zeta + E h(\omega x - l \varphi) \zeta_x + l G h(\varphi x + \psi + l \omega) \zeta) \, dx + i \lambda \tilde{\zeta}_3 \omega(L) w(L) \\
= \int_0^L \rho_1 (f_8 + i \lambda f_7) \, dx - \zeta_3 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + i \lambda} f_9(\xi) \, d\xi \, \zeta(L).
\end{cases}
\]

We can rewrite (72) as

\[-(L \lambda U, V)_{H^1_1} + (U, V)_{H^1_1} = l(V).\]
Using the compactness embedding from $L^2(0, L)$ into $H^{-1}(0, L)$ and from $H^1_0(0, L)$ into $L^2(0, L)$ we deduce that the operator $L_\lambda$ is compact from $(L^2(0, L))^3$ into $(L^2(0, L))^3$. Consequently, by Fredholm alternative, proving the existence of $U$ solution of (73) reduces to proving that 1 is not an eigenvalue of $L_\lambda$. Indeed if 1 is an eigenvalue, then there exists $U \neq 0$, such that
\[ (L_\lambda U, V)_{H^1_R} = (U, V)_{H^1_R} \quad \forall V \in H^1_R. \] (74)
In particular for $V = U$, it follows that
\[
\lambda^2 \left[ \rho_1 \norm{\varphi}_{L^2(0, L)}^2 + \rho_2 \norm{\psi}_{L^2(0, L)}^2 + \rho_1 \norm{\omega}_{L^2(0, L)}^2 \right] \\
- i\lambda (\tilde{\zeta}_1 \varphi(L) + \tilde{\zeta}_2 \psi(L) + \tilde{\zeta}_3 \omega(L)) = 0.
\]
From (74), we obtain
\[ \varphi(L) = \psi(L) = \omega(L) = 0. \] (75)
Hence, we have
\[ \varphi_x(L) = \psi_x(L) = \omega_x(L) = 0 \] (76)
and
\[
\begin{cases}
-\lambda^2 \varphi' - \frac{Gh}{\rho_1} (\varphi_x + \psi + l\omega)_x - \frac{Leh}{\rho_1} (\omega_x - l\varphi) = 0, \\
-\lambda^2 \psi' - \frac{Gh}{\rho_2} \psi_{xx} + \frac{Leh}{\rho_2} (\varphi_x + \psi + l\omega) = 0, \\
-\lambda^2 \omega' - \frac{Gh}{\rho_1} (\omega_x - l\varphi)_x + \frac{Leh}{\rho_1} (\varphi_x + \psi + l\omega) = 0.
\end{cases}
\] (77)
Consider $X = (\varphi, \psi, \omega, \varphi_x, \psi_x, \omega_x)$. Then we can rewrite (77), (75) and (76) as the initial value problem
\[
\frac{d}{dx} X = AX \\
X(0) = 0
\] (78)
where
\[
A = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-\lambda^2 \rho_1 t^2 Eh & 0 & 0 & 0 & 0 & 0 \\
0 & -\rho_2 \lambda^2 + Gh & 0 & 0 & 0 & -1 & \frac{Gh (E+G) t}{E} \\
0 & 0 & -\rho_1 \lambda^2 + Gh^2 & 0 & 0 & 0 & \frac{Gh^2 (E+G) t}{E} \\
0 & 0 & 0 & -\rho_1 \lambda^2 + Gh^2 & 0 & 0 & 0
\end{pmatrix}
\]
By the Picard Theorem for ordinary differential equations the system (78) has a unique solution $X = 0$. Therefore $\varphi = 0, \psi = 0, \omega = 0$. It follows from (60), that $u = 0, v = 0, \dot{\omega} = 0$, i.e., $U = 0$.

**Lemma 5.4.** If $\eta \neq 0$, we have
\[ 0 \in \rho(A). \]
Proof. From (66)

\[
\begin{cases}
-u = f_1, \\
\frac{Gh}{\rho_1}(\varphi_x + \psi + l \omega)_x - \frac{iEh}{\rho_1}(\omega_x - l \varphi) = f_2, \\
(\xi^2 + \eta)\phi_1 - u(L)\mu(\xi) = f_3, \\
-v = f_4, \\
-\frac{E\psi_{xx}}{\rho_2} + \frac{Gh}{\rho_2}(\varphi_x + \psi + l \omega) = f_5, \\
(\xi^2 + \eta)\phi_2 - v(L)\mu(\xi) = f_6, \\
-\tilde{\omega} = f_7, \\
\frac{Eh}{\rho_1}(\omega_x - l \varphi)_x + \frac{lGh}{\rho_1}(\varphi_x + \psi + l \omega) = f_8, \\
(\xi^2 + \eta)\phi_3 - \tilde{\omega}(L)\mu(\xi) = f_9.
\end{cases}
\]

with the following conditions

\[
\begin{cases}
Gh(\varphi_x + \psi + l \omega)(L) = -\gamma_1(\pi)^{-1}\sin(\alpha \pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi_1(\xi, t) d\xi \\
E\psi_x(L) = -\gamma_2(\pi)^{-1}\sin(\alpha \pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi_2(\xi, t) d\xi \\
Eh(\omega_x - l \varphi)(L) = -\gamma_3(\pi)^{-1}\sin(\alpha \pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi_3(\xi, t) d\xi
\end{cases}
\]

Consequently, problem (22) is equivalent to the problem

\[
a_\eta((\varphi, \psi, \omega), (w, \chi, \zeta)) = L_\eta(w, \chi, \zeta)
\]

where the bilinear form \( a_\eta : [H^1_0(0, L) \times H^1_0(0, L) \times H^1_0(0, L)]^2 \to \mathbb{R} \) and the linear form \( L_\eta : H^1_0(0, L) \times H^1_0(0, L) \times H^1_0(0, L) \to \mathbb{R} \) are defined by

\[
a_\eta((\varphi, \psi, \omega), (w, \chi, \zeta)) = \int_0^L Gh(\varphi_x + \psi + l \omega)(w_x + \chi + l \zeta) dx \\
+ \int_0^L E\psi_x \chi_x dx + \int_0^L Eh(\omega_x - l \varphi)(\zeta_x - lw) dx
\]
and

\[
L_\eta(w, \chi, \zeta) = \int_0^L \rho_1 f_3 w \, dx - \zeta_3 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta} f_3(\xi) \, d\xi 
\]

\[
+ \int_0^L \rho_2 f_5 \chi \, dx - \zeta_2 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta} f_5(\xi) \, d\xi \chi(\xi) + \tilde{\zeta}_2 f_4(\chi(L))
\]

\[
+ \int_0^L \rho_1 f_5 \zeta \, dx - \zeta_1 \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta} f_5(\xi) \, d\xi \zeta(L) + \tilde{\zeta}_1 f_7(\zeta(L)).
\]

It is easy to verify that \(a_\eta\) is continuous and coercive, and \(L_\eta\) is continuous. So applying the Lax-Milgram theorem, we deduce that for all \((w, \chi, \zeta) \in H^1_1(0, L) \times H^1_2(0, L) \times H^1_1(0, L)\) problem (23) admits a unique solution \((\varphi, \psi, \omega) \in H^1_1(0, L) \times H^1_2(0, L) \times H^1_1(0, L)\). Applying the classical elliptic regularity, it follows from (22) that \((\varphi, \psi, \omega) \in H^2(0, L) \times H^2(0, L) \times H^2(0, L)\). Therefore, the operator \(A\) is surjective.

\[\Box\]

**Lemma 5.5.** Let \(A\) be defined by (11). Then

\[
A^* = \begin{pmatrix}
\varphi \\
u \\
\phi_1 \\
\psi \\
v \\
\phi_2 \\
\omega \\
\phi_3
\end{pmatrix} = \begin{pmatrix}
-G_h \frac{(\varphi_x + \psi + l\omega)_x}{\rho_1} - \frac{IE_h}{\rho_1} (\omega_x - l\varphi) \\
-\frac{E I}{\rho_2} \psi_x + \frac{G_h}{\rho_2} (\varphi_x + \psi + l\omega) \\
-\chi^2 \varphi_1 - \chi \psi \\
\frac{E h}{\rho_1} (\omega_x - l\varphi)_x + \frac{IE_h}{\rho_1} (\varphi_x + \psi + l\omega) \\
-\chi^2 \varphi_3 - \chi \omega
\end{pmatrix}
\]

with domain

\[
D(A^*) = \left\{(\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \varphi, \phi_3)^T \in H : \varphi, \psi, \omega \in H^2(0, L) \cap H^1_1(0, L), \right. \\
\left. \begin{array}{l}
\frac{E I}{\rho_1} \psi(L) + \frac{E h}{\rho_1} (\omega_x - l\varphi)(L) + \zeta_1 \int_{-\infty}^{+\infty} \mu(\xi) \phi_1(\xi) \, d\xi = 0 \\
\int_{-\infty}^{+\infty} \mu(\xi) \phi_2(\xi) \, d\xi = 0 \\
E h(\omega_x - l\varphi)(L) + \zeta_3 \int_{-\infty}^{+\infty} \mu(\xi) \phi_3(\xi) \, d\xi = 0,
\end{array} \right\}
\]

\[(83)\]

**Proof.** Let \(U = (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \varphi, \phi_3)^T\) and \(V = (\tilde{\varphi}, \tilde{u}, \tilde{\phi}_1, \tilde{\psi}, \tilde{v}, \tilde{\phi}_2, \tilde{\omega}, \tilde{\varphi}, \tilde{\phi}_3)^T\).

We have
\[<AU,V>_{H} = <U,A^*V>_{H}.\]

\[<AU,V>_{H} = Gh \int_{0}^{L} \tilde{u}(\varphi_x + \tilde{\psi} + l\omega)x dx + lEh \int_{0}^{L} \tilde{u}(\omega_x - l\varphi) dx + EI \int_{0}^{L} \tilde{v}\psi_x dx - Gh \int_{0}^{L} \tilde{v}(\varphi_x + \psi + l\omega) dx + Eh \int_{0}^{L} \tilde{v}(\omega_x - l\varphi) dx - lGh \int_{0}^{L} \tilde{w}(\varphi_x + \psi + l\omega) dx + Gh \int_{0}^{L} (\tilde{\varphi}_x + \tilde{\psi} + l\tilde{\omega})(u_x + v + l\varpi) dx + Eh \int_{0}^{L} (\tilde{\omega}_x - l\tilde{\varphi})(\varpi_x - lu) dx + EI \int_{0}^{L} \tilde{\psi}_x \psi_x dx + \zeta_1 \int_{-\infty}^{+\infty} -[(\xi^2 + \eta)\phi_1 + u(L)\mu(\xi)] \tilde{\psi}_1 d\xi + \zeta_2 \int_{-\infty}^{+\infty} -[(\xi^2 + \eta)\phi_2 + v(L)\mu(\xi)] \tilde{\psi}_2 d\xi + \zeta_3 \int_{-\infty}^{1} -[(\xi^2 + \eta)\phi_3 + \varpi(L)\mu(\xi)] \tilde{\psi}_3 d\xi = -Gh \int_{0}^{L} (\tilde{u}_x + \tilde{v} + \tilde{\omega}r)(\varphi_x + \psi + l\omega) dx + Eh \int_{0}^{L} (\tilde{\varpi}_x - l\tilde{u})(\omega_x - l\varphi) dx - EI \int_{0}^{L} \tilde{\psi}_x \psi_x dx + Gh(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{\omega})(L)u(L) + \zeta_1 u(L) \int_{-\infty}^{+\infty} \mu(\xi) \tilde{\psi}_1 d\xi - \int_{0}^{L} u[Gh(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{\omega})x + lEh(\tilde{\omega}_x - l\tilde{\varphi})] dx - \int_{0}^{L} v[EI \tilde{\psi}_xx - Gh(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{\omega})] dx - \int_{0}^{L} \varpi[Eh(\tilde{\omega}_x - l\tilde{\varphi})x - lGh(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{\omega})] dx \]

If we set

\[Gh(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{\omega})(L)u(L) + \zeta_1 u(L) \int_{-\infty}^{+\infty} \mu(\xi) \tilde{\psi}_1 d\xi = 0,\]

\[EI \tilde{\psi}_x(L)v(L) + \zeta_2 u(L) \int_{-\infty}^{+\infty} \mu(\xi) \tilde{\psi}_2 d\xi = 0,\]

\[Eh(\tilde{\omega}_x - l\tilde{\varphi})(L)\varpi(L) + \zeta_3 \varpi(L) \int_{-\infty}^{+\infty} \mu(\xi) \tilde{\psi}_3 d\xi = 0,\]

we get

\[<AU,V>_{H} = - \int_{0}^{L} u[Gh(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{\omega})x + lEh(\tilde{\omega}_x - l\tilde{\varphi})] dx - \int_{0}^{L} v[EI \tilde{\psi}_xx - Gh(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{\omega})] dx - \int_{0}^{L} \varpi[Eh(\tilde{\omega}_x - l\tilde{\varphi})x - lGh(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{\omega})] dx + Eh \int_{0}^{L} (\tilde{\varpi}_x - l\tilde{u})(\omega_x - l\varphi) dx - \int_{0}^{L} (\tilde{\omega}_x - l\tilde{\varphi})(\omega_x - l\varphi) dx \]
Theorem 5.6. $\sigma_r(\mathcal{A}) = \emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.

Proof. Since $\lambda \in \sigma_r(\mathcal{A})$, $\bar{\lambda} \in \sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. This is because obviously the eigenvalues of $\mathcal{A}$ are symmetric on the real axis. From (83), the eigenvalue problem $\mathcal{A}^* Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (\varphi, u, \phi_1, \psi, v, \phi_2, \omega, \pi, \phi_3) \in D(\mathcal{A}^*)$ we have

$$
\begin{align*}
\lambda \varphi + u &= 0, \\
\lambda u + \frac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x + \frac{lEh}{\rho_1}(\omega_x - l\varphi) &= 0, \\
\lambda \phi_1 + (\xi^2 + \eta)\phi_1 + u(L)\mu(\xi) &= 0, \\
\lambda \psi + v &= 0, \\
\lambda v + \frac{EI}{\rho_2}\psi_{xx} - \frac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) &= 0, \\
\lambda \phi_2 + (\xi^2 + \eta)\phi_2 + v(L)\mu(\xi) &= 0, \\
\lambda \omega + \pi &= 0, \\
\lambda \pi + \frac{Eh}{\rho_1}(\omega_x - l\varphi)_x - \frac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) &= 0, \\
\lambda \phi_3 + (\xi^2 + \eta)\phi_3 + \pi(L)\mu(\xi) &= 0.
\end{align*}
$$

From (85) and (85)\textsubscript{2}, (85)\textsubscript{4} and (85)\textsubscript{5}, (85)\textsubscript{8} and (85)\textsubscript{9}, we get

$$
\begin{align*}
-\lambda^2 \varphi + \frac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x + \frac{lEh}{\rho_1}(\omega_x - l\varphi) &= 0, \\
-\lambda^2 \psi + \frac{EI}{\rho_2}\psi_{xx} - \frac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) &= 0, \\
-\lambda^2 \omega + \frac{Eh}{\rho_1}(\omega_x - l\varphi)_x - \frac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) &= 0.
\end{align*}
$$

From (85)\textsubscript{3}, (85)\textsubscript{6}, (85)\textsubscript{9} and the boundary conditions, we get

$$
\begin{align*}
Gh(\varphi_x + \psi + l\omega)(L) &= -\gamma_1\lambda(\lambda + \eta)^{\alpha - 1}\varphi(L), \\
EI\psi_x(L) &= -\gamma_2\lambda(\lambda + \eta)^{\alpha - 1}\psi(L), \\
Eh(\omega_x - l\varphi)(L) &= -\gamma_3\lambda(\lambda + \eta)^{\alpha - 1}\omega(L), \\
\varphi(0) &= \psi(0) = \omega(0) = 0.
\end{align*}
$$

System (86)-(87) is exactly the eigenvalue problem of $\mathcal{A}$. Hence $\mathcal{A}^*$ has the same eigenvalues of $\mathcal{A}$. The proof is complete.

Case $\eta \neq 0$

Theorem 5.7. The semigroup $S_{\mathcal{A}}(t)_{t \geq 0}$ is polynomially stable and

$$
\|S_{\mathcal{A}}(t)U_0\|_H \leq \frac{1}{t^{2(1-\alpha)}}\|U_0\|_{D(\mathcal{A})}
$$

Proof. We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$
\begin{align*}
i\lambda \varphi - u &= f_1, \\
i\lambda \rho_1 u - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) &= \rho_1 f_2, \\
i\lambda \phi_1 + (\xi^2 + \eta)\phi_1 - u(1)\mu(\xi) &= f_3, \\
i\lambda \psi - v &= f_4, \\
i\lambda \rho_2 v - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) &= \rho_2 f_5, \\
i\lambda \phi_2 + (\xi^2 + \eta)\phi_2 - v(1)\mu(\xi) &= f_6, \\
i\lambda \omega - \dot{\pi} &= f_7, \\
i\lambda \rho_3 \dot{\pi} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x + \psi + l\omega) &= \rho_3 f_8, \\
i\lambda \phi_3 + (\xi^2 + \eta)\phi_3 - \ddot{\omega}(1)\mu(\xi) &= f_9.
\end{align*}
$$
where $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T$. Taking inner product in $\mathcal{H}$ with $U$ and using (14) we get
\[ |\text{Re}(AU, U)| \leq ||U||_{\mathcal{H}} ||F||_{\mathcal{H}}. \] (89)
This implies that
\[ \sum_{i=1}^{3} \zeta_i \int_{-\infty}^{+\infty} (\xi^2 + \eta)(\phi_i(\xi, t))^2 d\xi \leq ||U||_{\mathcal{H}} ||F||_{\mathcal{H}}. \] (90)
and, applying (88), we obtain
\[
\begin{align*}
|\lambda| |\varphi(L)| - |f_1(L)|^2 & \leq |u(L)|^2 \\
|\lambda| |\psi(L)| - |f_4(L)|^2 & \leq |v(L)|^2 \\
|\lambda| |\omega(L)| - |f_7(L)|^2 & \leq |\tilde{\omega}(L)|^2.
\end{align*}
\]
We deduce that
\[
\begin{align*}
|\lambda|^2 |\varphi(L)|^2 & \leq c |f_1(L)|^2 + c |u(L)|^2 \\
|\lambda|^2 |\psi(L)|^2 & \leq c |f_4(L)|^2 + c |v(L)|^2 \\
|\lambda|^2 |\omega(L)|^2 & \leq c |f_7(L)|^2 + c |\tilde{\omega}(L)|^2.
\end{align*}
\]
Moreover, since
\[
\begin{align*}
Gh|\varphi_x + \psi + l\omega)(L)|^2 + EI|\psi_x(L)|^2 + Eh|\omega_x - l\varphi)(L)|^2 \\
\leq \sum_{i=1}^{3} \zeta_i^2 \int_{-\infty}^{+\infty} \mu(\xi)\phi_i(\xi) d\xi \\
\leq \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)^{-1} |\mu(\xi)|^2 d\xi\right) \sum_{i=1}^{3} \zeta_i^2 \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi_i(\xi)|^2 d\xi \\
\leq c ||U||_{\mathcal{H}} ||F||_{\mathcal{H}}.
\end{align*}
\]
From (88), we obtain
\[
\begin{align*}
u(L)|\mu(\xi)| = (i\lambda + \xi^2 + \eta)\phi_1 - f_3(\xi) \\
\psi(L)|\mu(\xi)| = (i\lambda + \xi^2 + \eta)\phi_2 - f_6(\xi) \\
\omega(L)|\mu(\xi)| = (i\lambda + \xi^2 + \eta)\phi_3 - f_9(\xi).
\end{align*}
(91)
By multiplying (91) by $(i\lambda + \xi^2 + \eta)^{-1}|\mu(\xi)|$, we get
\[
(i\lambda + \xi^2 + \eta)^{-1} u(L)|\mu^2(\xi) = \mu(\xi)|\phi_1 - ((i\lambda + \xi^2 + \eta)^{-1}|\mu(\xi)| f_3(\xi). \] (92)
Hence, by taking absolute values of both sides of (92), integrating over the interval $]-\infty, +\infty[$ with respect to the variable $\xi$ and applying Cauchy-Schwartz inequality, we obtain
\[
\begin{align*}
\mathcal{S}|u(L)| & \leq \mathcal{U} \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi_1|^2 d\xi\right)^{\frac{1}{2}} + \mathcal{V} \left(\int_{-\infty}^{+\infty} |f_3(\xi)|^2 d\xi\right)^{\frac{1}{2}}
\end{align*}
\] (93)
where
\[
\begin{align*}
\mathcal{S} & = \int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-1} |\mu(\xi)|^2 d\xi \\
\mathcal{U} & = \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)^{-1} |\mu(\xi)|^2 d\xi\right)^{\frac{1}{2}} \\
\mathcal{V} & = \left(\int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-2} |\mu(\xi)|^2 d\xi\right)^{\frac{1}{2}}.
\end{align*}
\]
Thus, by using again the inequality $2PQ \leq P^2 + Q^2$, $P \geq 0, Q \geq 0$, we get
\[ S^2|u(L)|^2 \leq 2U^2 \left( \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi_1|^2 d\xi \right) + 2V^2 \left( \int_{-\infty}^{+\infty} |f_3(\xi)|^2 d\xi \right). \] (94)

We deduce that
\[ |u(L)|^2 + |v(L)|^2 + |\omega(L)|^2 \leq c|\lambda|^{2-2\alpha} \|U\|_{\mathcal{H}} F\|_{\mathcal{H}} + cF^2. \] (95)

It follows that
\[ |\varphi_x(L)|^2 + |\psi_x(L)|^2 + |\omega_x(L)|^2 \leq c \left( \frac{1}{|\lambda|^2} + 1 \right) \|U\|_{\mathcal{H}} F\|_{\mathcal{H}} + c \frac{1}{|\lambda|^2} F^2. \] (96)

Let us introduce the following notation
\[
\begin{align*}
I_{\varphi} (\alpha) &= \rho_1|u(\alpha)|^2 + G\lambda|\varphi_{x}(\alpha)|^2, \\
I_{\psi} (\alpha) &= \rho_2|v(\alpha)|^2 + EI|\psi_{x}(\alpha)|^2, \\
I_{\omega} (\alpha) &= \rho_1|\omega(\alpha)|^2 + E\lambda|\omega_{x}(\alpha)|^2, \\
I (\alpha) &= I_{\varphi}(\alpha) + I_{\psi}(\alpha) + I_{\omega}(\alpha),
\end{align*}
\]
\[
\mathcal{E}_{\varphi}(L) = \int_0^L I_{\varphi}(s) \, ds, \quad \mathcal{E}_{\psi}(L) = \int_0^L I_{\psi}(s) \, ds, \quad \mathcal{E}_{\omega}(L) = \int_0^L I_{\omega}(s) \, ds.
\]

**Lemma 5.8.** Let $q \in H^1(0, L)$. We have that
\[
\begin{align*}
\mathcal{E}_{\varphi}(L) &= \left[q I_{\varphi}\right]_0^L - Ehl^2 [q|\varphi|^2]_0^L + 2G h Re \int_0^L q\varphi_x \overline{\varphi}_x \, dx + Eh\lambda^2 \int_0^L q'(x)|\varphi|^2 \, dx \\
&\quad + 2(G + E)hlRe \int_0^L q\omega_x \overline{\psi}_x \, dx + R_1
\end{align*}
\] (97)
\[
\begin{align*}
\mathcal{E}_{\psi}(L) &= \left[q I_{\psi}\right]_0^L - Gh [q|\psi|^2]_0^L - 2Gh Re \int_0^L q\varphi_x \overline{\psi}_x \, dx + Gh \int_0^L q'(x)|\psi|^2 \, dx \\
&\quad - 2Gh Re \int_0^L q\omega \overline{\psi}_x \, dx + R_2
\end{align*}
\] (98)
and
\[
\begin{align*}
\mathcal{E}_{\omega}(L) &= \left[q I_{\omega}\right]_0^L - Ghl^2 [q|\omega|^2]_0^L - 2Ghl Re \int_0^L q\psi_x \overline{x}_x \, dx + Ghl^2 \int_0^L q'(x)|\omega|^2 \, dx \\
&\quad - 2(G + E)hlRe \int_0^L q\varphi_x \overline{\omega}_x \, dx + R_3
\end{align*}
\] (99)

where $R_i$ satisfies
\[ |R_i| \leq C \|U\|_{\mathcal{H}} F\|_{\mathcal{H}}, \quad i = 1, 2, 3. \]

for a positive constant $C$.

**Proof.** To get (97), let us multiply the equation (88) by $q\overline{\varphi}_x$. Integrating on $(0, L)$ we obtain
\[
\begin{align*}
i\lambda \rho_1 \int_0^L uq\overline{\varphi}_x \, dx - Gh \int_0^L (\varphi_x + \psi + l\omega)_x q\overline{\varphi}_x \, dx - lEh \int_0^L (\omega_x + l\varphi)_x q\overline{\varphi}_x \, dx \\
= \rho_1 \int_0^L f_2 q\overline{\varphi}_x \, dx
\end{align*}
\]
or
\[-\rho_1 \int_0^L uq(\overline{\lambda \varphi_x}) \, dx - Gh \int_0^L q \varphi_{xx} \varphi_x \, dx - Gh \int_0^L q \psi_x \overline{\varphi_x} \, dx
\]
\[-(G + E) l h \int_0^L q \omega_x \overline{\varphi_x} \, dx + l^2 Eh \int_0^L q \varphi \overline{\varphi_x} \, dx = \rho_1 \int_0^L f_2 q \overline{\varphi_x} \, dx.\]

Since \(i \lambda \varphi_x = u_x + f_{1x}\) taking the real part in the above equality results in
\[-\rho_1 \int_0^L q \frac{d}{dx}|u|^2 \, dx - \frac{Gh}{2} \int_0^L q \frac{d}{dx} |\varphi_x|^2 \, dx\]

\[= \rho_1 \text{Re} \int_0^L f_2 q \overline{\varphi_x} \, dx + \rho_1 \text{Re} \int_0^L uq \overline{f_{1x}} \, dx + \text{Gh} \text{Re} \int_0^L q \psi_x \overline{\varphi_x} \, dx + (G + E) l h \text{Re} \int_0^L q \omega_x \overline{\varphi_x} \, dx - \frac{l^2 Eh}{2} \int_0^L q \frac{d}{dx} |\varphi|^2 \, dx.\]

Performing an integration by parts we get
\[
\int_0^L q'(s) |\rho_1 |u(s)|^2 + \text{Gh} |\varphi_x(s)|^2 | \, ds
\]
\[= [q \mathcal{L}_f]_0^L - l^2 Eh[q|\varphi|^2]_0^L + 2 \text{Gh} \text{Re} \int_0^L q \psi_x \overline{\varphi_x} \, dx
\]
\[+ l^2 Eh \int_0^L q'(s) |\varphi(s)|^2 \, ds + 2 (G + E) l h \text{Re} \int_0^L q \omega_x \overline{\varphi_x} \, dx + R_1\]

where
\[R_1 = 2 \rho_1 \text{Re} \int_0^L f_2 q \overline{\varphi_x} \, dx + 2 \rho_1 \text{Re} \int_0^L uq \overline{f_{1x}} \, dx.\]

Similarly, multiplying equation (88) by \(q \overline{\varphi_x}\), integrating on \((0, L)\) and taking the real part we obtain
\[i \lambda \rho_2 \int_0^L v q \overline{\psi_x} \, dx - EI \int_0^L \psi_{xx} q \overline{\varphi_x} \, dx + \text{Gh} \int_0^L (\varphi_x + \psi + l \omega) q \overline{\psi_x} \, dx = \rho_2 \int_0^L f_5 q \overline{\psi_x} \, dx\]
or
\[-\rho_2 \int_0^L v q(\overline{\lambda \psi_x}) \, dx - EI \int_0^L q \psi_{xx} \overline{\psi_x} \, dx + \text{Gh} \int_0^L q \varphi_x \overline{\psi_x} \, dx
\]
\[+ \text{Gh} \int_0^L q \psi \overline{\psi_x} \, dx + \text{Gh} l \int_0^L q \omega \overline{\psi} \, dx
\]
\[= \rho_2 \int_0^L f_5 q \overline{\psi_x} \, dx.\]

Since \(i \lambda \psi_x = v_x + f_{1x}\) taking the real part in the above equality results in
\[-\frac{\rho_2}{2} \int_0^L q \frac{d}{dx} |\psi|^2 \, dx - \frac{EI}{2} \int_0^L q \frac{d}{dx} |\psi_x|^2 \, dx = \rho_2 \text{Re} \int_0^L f_5 q \overline{\psi_x} \, dx
\]
\[+ \rho_2 \text{Re} \int_0^L q \overline{f_{1x}} \, dx - \text{Gh} \text{Re} \int_0^L q \varphi_x \overline{\psi_x} \, dx - \text{Gh} l \text{Re} \int_0^L q \omega \overline{\psi_x} \, dx
\]
\[- \frac{\text{Gh}}{2} \int_0^L q \frac{d}{dx} |\psi|^2 \, dx.\]
Performing an integration by parts we get
\[ \int_0^L q'(s)|\rho_2|v(s)|^2 + EI|\psi_x(s)|^2 \, ds \]
\[ = \left[ qI \right]_0^L - Gh|q|^2|_0^L - 2Gh \Re \int_0^L q\phi_x\bar{\psi}_x \, dx \]
\[ -2Ghl \Re \int_0^L q\bar{q}x \, dx + G\int_0^L q|\psi|^2 \, dx + R_2 \]
where
\[ R_2 = 2\rho_2 \Re \int_0^L f_5 q\bar{q}x \, dx + 2\rho_2 \Re \int_0^L q\bar{q}I_4x \, dx. \]

Finally, multiplying equation (88) by \( q\bar{\omega}_x \), integrating on \( (0,L) \) and taking the real part, after some algebraic manipulations we obtain (99) for
\[ \int_0^L \bar{\omega}q\bar{\omega}_x \, dx - Eh \int_0^L (\omega_x - l\phi)xq\bar{\omega}_x \, dx + lGh \int_0^L (\phi_x + \psi + l\omega)q\bar{\omega}_x \, dx \]
\[ = \rho_1 \int_0^L f_8 q\bar{\omega}_x \, dx \]
or
\[ -\rho_1 \int_0^L \bar{\omega}q(\bar{i}\omega_x) \, dx - Eh \int_0^L q\omega_x\bar{\omega}_x \, dx + lGh \int_0^L q\bar{\omega}_x \, dx 
+ (G + E)lh \int_0^L q\phi_x\bar{\omega}_x \, dx + \rho_1 \int_0^L f_8 q\bar{\omega}_x \, dx. \]

Since \( i\lambda \omega_x = \bar{\omega}_x + f_7x \) taking the real part in the above equality results in
\[ -\rho_1 \int_0^L \bar{\omega}q(\bar{i}\omega_x) \, dx - \frac{Eh}{2} \int_0^L \frac{d}{dx} \bar{\omega}_x |\omega_x|^2 \, dx \]
\[ = \rho_1 \Re \int_0^L f_8 q\bar{\omega}_x \, dx + \rho_1 \Re \int_0^L \bar{T}_7xq\bar{\omega} \, dx - lGh \Re \int_0^L q\bar{\omega}_x \, dx 
- (G + E)lh \Re \int_0^L q\phi_x\bar{\omega}_x \, dx - \frac{l^2Gh}{2} \int_0^L \frac{d}{dx} |\omega|^2 \, dx. \]

Performing an integration by parts we get
\[ \int_0^L q'(s)|\rho_1|\bar{\omega}(s)|s|^2 + Eh|\omega_x(s)|^2 \, ds \]
\[ = \left[ qI \right]_0^L - l^2Gh|q|^2|_0^L - 2lGh \Re \int_0^L q\psi\bar{\omega}_x \, dx - 2(G + E)lh \Re \int_0^L q\phi_x\bar{\omega}_x \, dx 
+l^2Gh \int_0^L q|\omega|^2 \, dx + R_3 \]
where
\[ R_3 = 2\rho_1 \Re \int_0^L f_8 q\bar{\omega}_x \, dx + 2\rho_1 \Re \int_0^L \bar{T}_7xq\bar{\omega} \, dx. \]
If we take \( q(x) = x \) in Lemma 5.8 and if we add (97)-(99) we arrive at
\[
\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) + \mathcal{E}_\omega(L)
= LL\varphi(L) - EhlL|\varphi(L)|^2 + EhL^2 \int_0^L |\varphi|^2 \, dx
+ L\mathcal{I}_\psi(L) - Gl|\psi(L)|^2 + G\int_0^L |\psi|^2 \, dx + L\mathcal{I}_\omega(L) - GhlL|\omega(L)|^2
+ Ghl^2 \int_0^L |\omega|^2 \, dx + R_1 + R_2 + R_3
- 2Ghl \int_0^L x\omega \bar{\varphi}_x \, dx - 2Ghl \int_0^L x\omega \bar{\varphi}_x \, dx.
\]
Since
\[
- 2Ghl \int_0^L x\omega \bar{\varphi}_x \, dx - 2Ghl \int_0^L x\omega \bar{\varphi}_x \, dx = -2GhlLRe(\omega)\bar{\psi}(L) + 2Ghl \int_0^L \psi \bar{\omega} \, dx.
\]
Using Lemma 5.8 and the Young inequality we get
\[
\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) + \mathcal{E}_\omega(L)
= LL\varphi(L) - EhlL|\varphi(L)|^2 + EhL^2 \int_0^L |\varphi|^2 \, dx
+ L\mathcal{I}_\psi(L) + Gl|\psi(L)|^2 + 2G\int_0^L |\psi|^2 \, dx + L\mathcal{I}_\omega(L) + GhlL|\omega(L)|^2
+ 2Ghl^2 \int_0^L |\omega|^2 \, dx + c\|U\|_\mathcal{H}\|F\|_\mathcal{H}
\]
for a positive constant \( C \). It results by (95) and (96) that we can find a positive constant \( C \) such that
\[
\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) + \mathcal{E}_\omega(L)
= EhL^2 \int_0^L |\varphi|^2 \, dx + 2GhL^2 \int_0^L |\psi|^2 \, dx + c\|U\|_\mathcal{H}\|F\|_\mathcal{H} + c\|F\|_\mathcal{H}^2
\]
for \( \lambda \neq 0 \). Since that \( \varphi = \frac{u + f_1}{2\lambda} \), \( \psi = \frac{v + f_2}{2\lambda} \) and \( \omega = \frac{\omega + f_3}{2\lambda} \) we obtain
\[
\mathcal{E}_\varphi(L) + \mathcal{E}_\psi(L) + \mathcal{E}_\omega(L)
= c\|\lambda|^{2\alpha-2}\|U\|_\mathcal{H}\|F\|_\mathcal{H} + c\|U\|_\mathcal{H}\|F\|_\mathcal{H} + c\|F\|_\mathcal{H}^2
+ \frac{c}{|\lambda|^2}\|U\|_\mathcal{H}\|F\|_\mathcal{H}.
\]
Since that
\[
\int_{-\infty}^{\infty} (\phi_i(\xi))^2 \, d\xi \leq C \int_{-\infty}^{\infty} (\xi^2 + \eta)\phi_i(\xi)^2 \, d\xi
\]
for \( \lambda \neq 0 \). If \( |\lambda| > 1 \) we get
\[
\|U\|_\mathcal{H}^2 \leq |\lambda|^{4(1-\alpha)}\|F\|_\mathcal{H}^2.
\]
It follows that
\[
\frac{1}{|\lambda|^{2-2\alpha}}\|(i\lambda I - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \quad \forall \lambda \in \mathbb{R},
\]
for a positive constant \( C \). The conclusion then follows by applying the Theorem 4.2.
Appendix. We will show the lack of exponential stability by frequency domain method.

We show the existence of a sequence \( (\lambda_{\mu}) \subset \mathbb{R} \) with \( \lim_{\mu \to \infty} |\lambda_{\mu}| = \infty \) and \( (U_{\mu}) \subset D(A) \) to \( F_{\mu} \subset \mathcal{H} \) such that \( (\lambda_{\mu}I-A)U_{\mu} = F_{\mu} \) is bounded in \( \mathcal{H} \) and \( \lim_{\mu \to \infty} \|U_{\mu}\|_{\mathcal{H}} = \infty \). Let \( F = F_{\mu} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \) with \( U_{\mu} = (\varphi_{\mu}, u_{\mu}, \phi_{1\mu}, \psi_{\mu}, v_{\mu}, \phi_{2\mu}, \omega_{\mu}, \hat{\omega}_{\mu}, \phi_{3\mu})^T \).

Equivalently, we have

\[
\begin{aligned}
\lambda^2 \varphi + i\lambda \varphi_x + lk \bar{\omega} &= -(f_2 + i\lambda f_1), \\
\lambda^2 \psi + a\tilde{\psi}_x - \tilde{\psi} &= -(f_5 + i\lambda f_4), \\
\lambda^2 \omega + k\bar{\omega}_x - i\tilde{\psi} &= -(f_8 + i\lambda f_7).
\end{aligned}
\]  

Then

\[
\begin{aligned}
(\lambda^2 - l^2 - 1)\tilde{\varphi} + \tilde{\varphi}_{xx} + a\tilde{\psi}_x + 2lk\bar{\omega}_x &= \lambda^2 \psi + a\tilde{\psi}_x - \tilde{\varphi} = -(f_5 + i\lambda f_4), \\
(\lambda^2 - l^2)\bar{\omega} + k\bar{\omega}_x - 2l\tilde{\varphi} &= -(f_8 + i\lambda f_7). \\
\end{aligned}
\] 

From (101)_3, after derivation, we have

\[
(\lambda^2 - l^2)\bar{\omega}_x + k\bar{\omega}_{xxx} - 2l\tilde{\varphi}_{xx} = -L_0
\] 

where

\[
L_0 = (f_{8xx} + i\lambda f_{7xx}) + l(f_{2x} + i\lambda f_{1x}).
\] 

From (101)_1, after derivation, we have

\[
\begin{aligned}
\bar{\omega}_x &= -\frac{1}{2lk}(\lambda^2 - l^2 - 1)\tilde{\varphi} + \tilde{\varphi}_{xx} + a\tilde{\psi}_x + L_1, \\
\bar{\omega}_{xxx} &= -\frac{1}{2lk}(\lambda^2 - l^2 - 1)\tilde{\varphi}_{xx} + \tilde{\varphi}_{xxxx} + a\tilde{\psi}_{xxx} + L_{1xx}.
\end{aligned}
\] 

where

\[
L_1 = [(f_{2x} + i\lambda f_{1x}) + (f_5 + i\lambda f_4) + l(f_8 + i\lambda f_7)].
\] 

Now, we replace in (102), we get

\[
-\frac{(\lambda^2 - l^2)k}{2lk} \left((\lambda^2 - l^2 - 1)\tilde{\varphi} + \tilde{\varphi}_{xx} + a\tilde{\psi}_x + L_1\right) \\
-\frac{1}{2l} \left((\lambda^2 - l^2 - 1)\tilde{\varphi}_{xx} + \tilde{\varphi}_{xxxx} + a\tilde{\psi}_{xxx} + L_{1xx}\right) - 2l\tilde{\varphi}_{xx} = -L_0
\]

Then

\[
\begin{aligned}
\frac{(\lambda^2 - l^2)(\lambda^2 - l^2 - 1)}{2lk} \tilde{\varphi} + \frac{(k + 1)\lambda^2 + k(2l^2 - 1)}{2lk} \tilde{\varphi}_{xx} \\
+ \frac{1}{2l} \tilde{\varphi}_{xxxx} + \frac{(\lambda^2 - l^2)k}{2lk} a\tilde{\psi}_x + \frac{a}{2l} \tilde{\varphi}_{xxx}
\end{aligned}
\]

\[
= -\frac{(\lambda^2 - l^2)k}{2lk} L_1 - \frac{1}{2l} L_{1xx} + L_0.
\]

From (101)_2, after derivation, we have

\[
\begin{aligned}
\tilde{\varphi} &= \lambda^2 \psi + a\tilde{\psi}_x + (f_5 + i\lambda f_4), \\
\tilde{\varphi}_{xx} &= \lambda^2 \psi_{xx} + a\tilde{\psi}_{xx} + (f_{5xx} + i\lambda f_{4xx}), \\
\tilde{\varphi}_{xxxx} &= \lambda^2 \psi_{xxxx} + a\tilde{\psi}_{xxxx} + (f_{5xxxx} + i\lambda f_{4xxxx}).
\end{aligned}
\]
Then, we deduce that
\[
\frac{(\lambda^2 - l^2 k)(\lambda^2 - l^2 - 1)}{2lk} (\lambda^2 \psi + a \lambda \psi_x + L_2) + \frac{(k + 1)\lambda^2 + k(2l^2 - 1)}{2lk}
\]
\[
(\lambda^2 \psi_{xx} + a \psi_{xxx} + L_{2xxx}) + \frac{1}{2l}(\lambda^2 \psi_{xxxx} + a \psi_{xxxxx} + L_{2xxxx})
\]
\[
+ \frac{(\lambda^2 - l^2 k)a \lambda \psi_x + \frac{a}{2l} \psi_{xxx}}{2lk} = -\frac{1}{2l} L_{1xx} + L_0.
\]

Hence
\[
\psi_{xxxxxx} + \left( \left( \frac{1}{k} + \frac{1}{a} + 1 \right) \lambda^2 + 2l^2 \right) \psi_{xxxxx}
\]
\[
+ \left( \left( \frac{1}{k} + \frac{1}{a} + 1 \right) \lambda^4 + \left( \frac{5}{a} + \frac{1}{a} - 1 \right) l^2 \right) \psi_{xx}
\]
\[
+ \frac{\lambda^6}{2\pi} \left( \left( \frac{5}{a} + \frac{1}{a} - 1 \right) l^2 + \frac{2}{a} \right) \lambda^4 + \left( \frac{5}{a} l^4 + \frac{2}{a} l^2 \right) \lambda^2 \psi
\]
\[
= - \frac{1}{2l} L_{1xx} + L_0 = \frac{(\lambda^2 - l^2 k)(\lambda^2 - l^2 - 1)}{2lk} L_2
\]
\[
- \frac{(k + 1)\lambda^2 + k(2l^2 - 1)}{2lk} L_{2xxx} - \frac{1}{2l} L_{1xxx} + L_0.
\]

The general solution of the homogeneous differential equation is of the form
\[
C_1 e^{r_1 x} + C_2 e^{-r_1 x} + C_3 \cos r_2 x e^{r_2 x} + C_4 \sin r_2 x e^{r_2 x} + C_5 \cos r_2 x e^{-r_2 x} + C_6 \sin r_2 x e^{-r_2 x}
\]

Using the variation of constants method with boundary conditions and choosing
\[
f_2 = f_3 = f_4 = f_5 = f_6 = f_7 = f_8 = f_9 = 0 \text{ and } f_1 \in H_L(0, L),\]

we find an explicit solution. Thus, we calculate \( \|\psi_x\|_2 \) and choosing \( \lambda = \lambda_n = \frac{\pi}{T} (n + \frac{1}{2}) \) (if \( k = a = 1 \)),

we can prove
\[
\|\psi_x\|_2 \geq C \lambda_n^{\lambda - \alpha} \|F\|_H.
\]

This implies that there exists some constant \( M > 0 \) independent of \( \lambda \) such that
\[
\|(i \lambda_n - A)^{-1}\|_H \geq M \lambda_n^{(1-\alpha)}.
\]

The proof is completed.

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