OSCILLATOR TOPOLOGIES ON A PARATOPOLOGICAL GROUP AND RELATED NUMBER INVARIANTS

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Abstract. We introduce and study oscillator topologies on paratopological groups and define certain related number invariants. As an application we prove that a Hausdorff paratopological group $G$ admits a weaker Hausdorff group topology provided $G$ is 3-oscillating. A paratopological group $G$ is 3-oscillating (resp. 2-oscillating) provided for any neighborhood $U$ of the unity $e$ of $G$ there is a neighborhood $V \subset G$ of $e$ such that $V^{-1}VV^{-1} \subset UU^{-1}U$ (resp. $V^{-1}V \subset UU^{-1}$). The class of 2-oscillating paratopological groups includes all collapsing, all nilpotent paratopological groups, all paratopological groups satisfying a positive law, all paratopological SIN-group and all saturated paratopological groups (the latter means that for any nonempty open set $U \subset G$ the set $U^{-1}$ has nonempty interior). We prove that each totally bounded paratopological group $G$ is countably cellular; moreover, every cardinal of uncountable cofinality is a precaliber of $G$. Also we give an example of a saturated paratopological group which is not isomorphic to its mirror paratopological group as well as an example of a 2-oscillating paratopological group whose mirror paratopological group is not 2-oscillating.

This note was motivated by the following question of I. Guran [Gu]: Does every (regular) Hausdorff paratopological group $G$ admit a weaker Hausdorff group topology?

Under a paratopological group we understand a pair $(G, \tau)$ consisting of a group $G$ and a topology $\tau$ on $G$ making the group operation $\cdot : G \times G \to G$ of $G$ continuous. If, in addition, the operation $(\cdot)^{-1} : G \to G$ of taking the inverse is continuous with respect to the topology $\tau$, then $(G, \tau)$ is a topological group. A paratopological group $G$ is Lawson if $G$ possesses a neighborhood base at the unit, consisting of subsemigroups of $G$. Under the mirror paratopological group of a paratopological group $G = (G, \tau)$ we understand the paratopological group $G^{-} = (G, \tau^{-1})$ where $\tau^{-1} = \{U^{-1} : U \in \tau\}$. Let us mention that there are paratopological groups which are not isomorphic to their mirror paratopological groups, see Examples 2 and 4.

Given a paratopological group $G$ let $\tau_\circ$ be the strongest group topology on $G$, weaker than the topology of $G$. The topological group $G^\circ = (G, \tau_\circ)$, called the group reflexion of $G$, has the following characteristic property: the identity map $i : G \to G^\circ$ is continuous and for every continuous group homomorphism $h : G \to H$ from $G$ into a topological group $H$ the homomorphism $h \circ i^{-1} : G^\circ \to H$ is continuous. Our definition of the topology $\tau_\circ$ is categorical. An inner description of the topology $\tau_\circ$ can be given using the technique of $T$-filters, see [PZ, §3.1]. A subset $A$ of a paratopological group $G$ will be called $b$-closed (resp. $b$-open) if $A$ is closed (resp. open) in the topology $\tau_\circ$. A paratopological group $G$ is called $b$-separated provided its group reflexion $G^\circ$ is Hausdorff. We define a paratopological group $G$ to be $b$-regular if each neighborhood $U$ of the unit $e$ of $G$ contains a $b$-closed neighborhood of $e$. Observe that each Hausdorff $b$-regular paratopological group is regular and $b$-separated. The latter assertion follows from the fact that every point $x \neq e$ of $G^\circ$
can be separated from the unit \(e\) by a \(\flat\)-closed subset. This implies that the topological group \(G^\flat\) is separated and hence Hausdorff.

Observe that in terms of group reflexions the Guran question can be reformulated as follows: Is any (regular) Hausdorff paratopological group \(G\) \(\flat\)-separated?

The negative answer to this question was given by the second author in \([Ra_1]\) where he has constructed a non-commutative Hausdorff zero-dimensional paratopological group with non-Hausdorff group reflexion. In fact, any such a paratopological group necessarily is non-commutative: according to \([Ra_1]\) the group reflexion \(G^\flat\) of any abelian Hausdorff paratopological group \(G\) is Hausdorff. Moreover, in this case the topology of \(G^\flat\) has a very simple description: a base of neighborhoods at the unit in \(G^\flat\) consists of the sets \(UU^{-1}\) where \(U\) runs over neighborhoods of the unit in the group \(G\). A bit later it was realized that the same is true for any paratopological SIN-group, that is a paratopological group \(G\) possessing a neighborhood base \(B\) at the unit such that \(gUg^{-1} = U\) for any \(U \in B\) and \(g \in G\) (as expected, SIN is abbreviated from Small Invariant Neighborhoods). Unfortunately, Hausdorff paratopological SIN-groups do not exhaust all paratopological groups whose group reflexion is Hausdorff (for example any separated topological group has Hausdorff group reflexion but needs not be a paratopological SIN-group). In this situation it is natural to search for less restrictive conditions on a paratopological group \(G\) under which the group reflexion \(G^\flat\) of \(G\) is Hausdorff and admits a simple description of its topology. This is important since many results concerning paratopological groups require their \(\flat\)-separatedness, see \([KRS]\, [BR_1]\, [BR_3]\).

For each paratopological group \((G, \tau)\) we define a decreasing sequence \(\tau = \tau_1 \supset \tau_2 \supset \cdots \supset \tau_n\) of so-called oscillator topologies which are intermediate between the topology \(\tau\) of \(G\) and the topology \(\tau_1\) of its group reflexion. In some fortunate cases the topology \(\tau_n\) coincides with some oscillator topology \(\tau_n\) and thus admits a relatively simple description.

Given a subset \(U\) of a group \(G\) by induction define the sets \((\pm U)^n\) and \((\mp U)^n\), \(n \in \omega\), letting \((\pm U)^0 = (\mp U)^0 = \{e\}\) and \((\pm U)^{n+1} = U(\mp U)^n\), \((\mp U)^{n+1} = U^{-1}(\pm U)^n\) for \(n \geq 0\). Thus \((\pm U)^n = \bigcup_{k=0}^{n} U^{k}U^{-1} \cdots U^{(1)} \cdots U^{-1}\) and \((\mp U)^n = \bigcup_{k=0}^{n} U^{-1} \cdots U^{(1)} \cdots U^{k} \cdots U^{(1)}\). Note that \((\pm U)^n\) is even if \(n\) is even and \((\pm U)^n\) is odd if \(n\) is odd.

Under an \(n\)-oscillator (resp. a mirror \(n\)-oscillator) on a topological group \((G, \tau)\) we understand a set of the form \((\pm U)^n\) (resp. \((\mp U)^n\)) for some neighborhood \(U\) of the unit of \(G\). Observe that each \(n\)-oscillator in a paratopological group \((G, \tau)\) is a mirror \(n\)-oscillator in the mirror paratopological group \((G, \tau^{-1})\) and vice versa: each mirror \(n\)-oscillator in \((G, \tau)\) is an \(n\)-oscillator in \((G, \tau^{-1})\).

Under the \(n\)-oscillator topology on a paratopological group \((G, \tau)\) we understand the topology \(\tau_n\) consisting of sets \(U \subseteq G\) such that for each \(x \in U\) there is an \(n\)-oscillator \((\pm V)^n\) with \(x \in (\pm V)^n \subseteq U\).

Since \((\pm V)^{n+1} \supset (\pm V)^n \cup (\mp V)^n\) for each set \(V\) containing the unit of \(G\), we get \(\tau_{n+1} \supset \tau_n\) and \(\tau_{n+1} \supset (\tau^{-1})_n\) for every \(n \in \mathbb{N}\). Thus we obtain a decreasing sequence \(\tau = \tau_1 \supset \tau_2 \supset \cdots \supset \tau_n\) of oscillator topologies on the paratopological group \((G, \tau)\) and also a decreasing sequence \(\tau^{-1} = (\tau^{-1})_1 \supset (\tau^{-1})_2 \supset \cdots \supset (\tau^{-1})_n = \tau_n\) of oscillator topologies on the mirror paratopological group \((G, \tau^{-1})\). Observe that \((\tau_n)^{-1} = \tau_n\) if \(n\) is even and \((\tau_n)^{-1} = (\tau^{-1})_n\) if \(n\) is odd.

In general, \((G, \tau_n)\) is not a paratopological group but it is a semitopological group, that is a group endowed with a topology making the group operation separately continuous.
(equivalently, a group endowed with a shift-invariant topology). The following theorem detects the situation when the sequence of oscillator topologies eventually stabilize.

**Theorem 1.** For a paratopological group \((G, \tau)\) and a positive integer \(n\) the following conditions are equivalent:

1. \((G, \tau_n)\) is a topological group;
2. \(\tau_n = \tau\);
3. \(\tau_k = \tau_n = (\tau^{-1})_{k+1}\) for any \(k \geq n\);
4. \(\tau_n \subset (\tau^{-1})_n\) which means that for any neighborhood \(U\) of the unit \(e\) of \(G\) there is a neighborhood \(V \subset G\) of \(e\) such that \((\mp V)^n \subset (\pm U)^n\).

Moreover, if \(n\) is even, then the conditions (1)–(4) are equivalent to

5. \((G, \tau_n)\) is a paratopological group.

**Proof.** If \(n\) is even, then (1) \(\iff\) (5) because of the equality \((\tau_n)^{-1} = \tau_n\).

The implication (1) \(\Rightarrow\) (2) follows from the inclusions \(\tau \supset \tau_n \supset \tau\) and the fact that \(\tau\) is the strongest group topology weaker than \(\tau\).

The implication (2) \(\Rightarrow\) (3) follows from the inclusions \(\tau_0 \subset (\tau^{-1})_{k+1} \subset (\tau^{-1})_{n+1} \subset \tau_n \supset \tau_k \supset \tau\) holding for each \(k \geq n\).

The implication (3) \(\Rightarrow\) (4) follows from the inclusion \(\tau_0 \subset (\tau^{-1})_n\).

Finally, we show that (4) \(\Rightarrow\) (1). Let \(\mathcal{N}(e)\) be a base of open neighborhoods at the unit \(e\) of the paratopological group \(G\). Assume that \(\tau_n \subset (\tau^{-1})_n\) which means that for any \(U \in \mathcal{N}(e)\) there is \(V \in \mathcal{N}(e)\) with \((\mp V)^n \subset (\pm U)^n\).

To show that \((G, \tau_n)\) is a topological group we shall use the Pontriagin characterization [Po, §18] asserting that a group \(G\) endowed with a shift-invariant topology is a topological group if and only if the family \(\mathcal{B}\) of open neighborhoods of the unit \(e\) of \(G\) satisfies the following five Pontriagin conditions:

1. \((\forall U, V \in \mathcal{B}) (\exists W \in \mathcal{B}) \text{ with } W \subset U \cap V\);
2. \((\forall U \in \mathcal{B}) (\exists V \in \mathcal{B}) \text{ with } V^2 \subset U\);
3. \((\forall U \in \mathcal{B}) (\forall x \in U) (\exists V \in \mathcal{B}) \text{ with } xV \subset U\);
4. \((\forall U \in \mathcal{B}) (\forall x \in G) (\exists V \in \mathcal{B}) \text{ with } x^{-1}Vx \subset U\);
5. \((\forall U \in \mathcal{B}) (\exists V \in \mathcal{B}) \text{ with } V^{-1} \subset U\).

Thus to prove that \((G, \tau_n)\) is a topological group, it suffices to verify the Pontriagin conditions (P1)–(P5) for the family \(\mathcal{B}\) of all \(n\)-oscillators in \(G\).

The first condition (P1) is trivial.

To verify (P2), fix any open neighborhood \(U \in \mathcal{N}(e)\) and by finite induction find open neighborhoods \(U_0 \supset U_1 \supset \cdots \supset U_n\) of \(e\) in \(G\) such that \(U_0 = U\), \(U_k \cdot U_k \subset U_{k-1}\) if \(k\) is odd and \((\mp U_k)^n \subset (\pm U_{k-1})^n\) if \(k\) is even. It is easy to see that \((\pm U_n)^n \cdot (\pm U_n)^n \subset (\pm U)^n\) and thus the Pontriagin condition (P2) is satisfied too.

(P3) Fix any neighborhood \(U \in \mathcal{N}(e)\) and a point \(x = (\pm U)^n\). We have to find \(V \in \mathcal{N}(e)\) such that \(x(\pm V)^n \subset (\pm U)^n\). Write \(x = x_1x_2^{-1}x_3 \cdots x_n^{-1}x^{-1}\), where all \(x_i\) are in \(U\). By \(A\) denote the (finite) set of all products in the forms \(y_1 \cdots y_n\) where \(y_i \in \{e, x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}\) for every \(i\). Choose a neighborhood \(V \in \mathcal{N}(e)\) such that \((x a^{-1}V a) \cup (a^{-1}Vax) \subset U\) for every \(x \in \{x_1, \ldots, x_n\}\) and every \(a \in A\). Then

\[
x(\pm V)^n = x_1 x_2^{-1} \cdots x_n^{-1} (\pm V)^n = x_1 x_2^{-1} \cdots x_n^{-1} V x_n^{-1} \cdots x_3^{-1} x_2^{-1} \times \\
\times x_2^{-1} x_3 \cdots x_n^{-1} V^{-1} x_n^{-1} \cdots x_3^{-1} \cdots x_1^{-1} V^{-1} \subset U U^{-1} \cdots U^{-1} = (\pm U)^n\]
\]
To verify (P4), fix arbitrary $U \in \mathcal{N}(e)$ and $x \in G$. Choose a neighborhood $V \in \mathcal{N}(e)$ such that $x^{-1}Vx \subset U$. Then $x^{-1}(\pm V)^nx = (\pm x^{-1}Vx)^n \subset (\pm U)^n$.

To verify (P5), fix any $U \in \mathcal{N}(e)$. If $n$ is even, then $((\pm U)^n)^{-1} = (\pm U)^n$. If $n$ is odd, use the assumption $\tau_n \subset (\tau^{-1})_n$, to find $V \in \mathcal{N}(e)$ with $(\mp V)^n \subset (\pm U)^n$. Then $((\pm V)^n)^{-1} = (\mp V)^n \subset (\pm U)^n$. In any case the condition (P5) holds.

Hence the family $\mathcal{B}$ of $n$-oscillators in $G$ satisfies the Pontriagin conditions (P1)--(P5) and since $\mathcal{B}$ forms a neighborhood base of the topology $\tau_n$ at the unit of $G$, $(G, \tau_n)$ is a topological group.

Next, we consider some separation axioms for the oscillator topologies. We remind that a topology $\tau$ on a set $X$ is $T_1$ if for any distinct points $x, y \in X$ there is a neighborhood $U \in \tau$ of $x$ such that $y \notin U$; $\tau$ is $T_2$ if the topological space $(X, \tau)$ is Hausdorff.

**Theorem 2.** For a paratopological group $(G, \tau)$ and a positive integer $n$ the following conditions are equivalent:

1. the topology $\tau_n$ is $T_2$;
2. the topology $\tau_2$ is $T_1$;
3. the topology $\tau_{2n+1}$ is $T_1$;
4. the topology $(\tau^{-1})_{2n+1}$ is $T_1$;
5. the topology $(\tau^{-1})_{2n}$ is $T_1$;
6. the topology $(\tau^{-1})_n$ is $T_2$;

**Proof.** Let $\mathcal{N}(e)$ be a neighborhood base at the unit $e$ of $G$ and $n$ is a positive integer.

1) $\Rightarrow$ 2) Assume that the topology $\tau_n$ is Hausdorff. This means that for any $x \neq e$ there is a neighborhood $U \in \mathcal{N}(e)$ with $x(\pm U)^n \cap (\pm U)^n = \emptyset$. Then $x \notin (\pm U)^n \cdot ((\pm U)^n)^{-1} = (\pm U)^{2n}$ and thus $\bigcap_{U \in \mathcal{N}(e)}(\pm U)^{2n} = \{e\}$, i.e., the topology $\tau_2$ is $T_1$.

2) $\Rightarrow$ 3) Suppose the topology $\tau_2$ is $T_1$. Then $\bigcap_{U \in \mathcal{N}(e)}(\pm U)^{2n} = \{e\}$. To show that the topology $\tau_{2n+1}$ is $T_1$ we have to verify that $\bigcap_{U \in \mathcal{N}(e)}(\pm U)^{2n+1} = \{e\}$.

It suffices for each $x \neq e$ to find a neighborhood $W \in \mathcal{N}(e)$ with $x \notin (\pm W)^{2n+1}$. Since the topology $\tau_2$ is $T_1$, there is $U \in \mathcal{N}(e)$ such that $x \notin (\pm U)^{2n}$. Let $V, W \in \mathcal{N}(e)$ be such that $V \cdot V \subset U$ and $W \subset V$, $Wx^{-1} \subset x^{-1}V$. Then $xW^{-1} \subset Vx^{-1}$. We claim that $x \notin (\pm W)^{2n+1}$. Assuming the converse we would get $x \in (\pm W)^{2n+1} = (\pm W)^{2n}W$ and consequently, $xW^{-1} \cap (\pm W)^{2n} \neq \emptyset$. Since $xW^{-1} \subset V^{-1}$ and $(\pm W)^{2n} \neq \emptyset$ and thus $x \in V(\pm W)^{2n} \subset V(\pm V)^{2n} \subset (\pm U)^{2n}$, which contradicts to the choice of the neighborhood $U$.

3) $\Rightarrow$ 1) If the topology $\tau_{2n+1}$ is $T_1$, then so is the topology $\tau_2 \supset \tau_{2n+1}$. Consequently, for any distinct points $x, y \in G$ there is $U \in \mathcal{N}(e)$ such that $x^{-1}y \notin (\pm U)^{2n} = (\pm U)^n \cdot ((\pm U)^n)^{-1}$. Then $y \notin x(\pm U)^n \cdot ((\pm U)^n)^{-1}$ and consequently, $y(\pm U)^n \cap x(\pm U)^n = \emptyset$, i.e., the topology $\tau_n$ is Hausdorff.

The equivalence $(3) \iff (4)$ follows from the equality $(\tau_{2n+1})^{-1} = (\tau^{-1})_{2n+1}$.

Finally the equivalences $(4) \iff (5) \iff (6)$ follow from the equivalences $(3) \iff (2) \iff (1)$ applied to the mirror paratopological group $(G, \tau^{-1})$.

Theorem 2 allows us to introduce two number invariants of paratopological groups reflecting their separatedness properties. Given a paratopological group $(G, \tau)$ and $i = 1, 2$

$$T_i(G) = \sup \{n \in \mathbb{N} : \text{ the } n\text{-oscillator topology } \tau_n \text{ on } G \text{ is } T_i\} \in \mathbb{N} \cup \{\infty\}.$$  

We assume that $\sup \emptyset = 0$. Thus a paratopological group $G$ is $T_i$ for $i = 1, 2$ if and only if $T_i(G) > 0$.  

In terms of the invariants $T_1(G)$, $T_2(G)$, Theorem \cite{2} can be reformulated as follows.

**Corollary 1.** If $G$ is a paratopological group and $G^-$ is its mirror paratopological group, then $T_1(G) = T_1(G^-)$, $T_2(G) = T_2(G^-)$, and $T_1(G) = 2T_2(G) + 1$. In particular, $T_1(G) \geq 3$ for any Hausdorff paratopological group $G$.

In \cite{Ra}, the second author has constructed a regular zero-dimensional paratopological group $G$ with $T_1(G) = 3$ and $T_2(G) = 1$. This shows that the lower estimation in the above corollary cannot be improved. Below we use the idea of \cite{Ra} to construct a paratopological group $G$ with $T_2(G) = n$ for any given $n \in \mathbb{N}$.

Let $F$ be a free semigroup over a set $X$. A word $w = y_1 \cdots y_n \in F$, $y_i \in X$, is reduced if there is no pair $y_i, y_{i+1}$ such that $y_i^{-1} = y_{i+1}$. A reduced word is cyclic reduced if $y_i^{-1} \neq y_n$.

**Lemma 1.** \cite{LS} Theorem 5.5] Let $G$ be a group generated by an alphabet $A = \{a, b, c, \ldots\}$ with a relation $r^p = 1$ where $r$ is cyclic reduced and $p > 1$. Let $w$ be a nonempty reduced word in the alphabet $A$ such that $w$ equal to the unit of the group $G$. Then there exists a subword $s$ of the word $w$ which also is a subword of the word $r^p$ or $r^{-p}$ such that $l(s) > (p - 1)l(r^p)/p$, where $l(s)$ and $l(r^p)$ denote the lengths of the words $s$ and $r^p$ respectively.

Under the normal closure of a subset $A$ of a group $G$ we understand the smallest normal subgroup of $G$ containing the set $A$.

**Lemma 2.** Let $F^2$ be a free group over a two-point set $\{x, y\}$, $p > 1$ be integer and $N$ be the normal closure of the element $r^p = (xy^{-1})^p$ in $F^2$. Let $S \subset G$ be a semigroup generated by the elements $x$ and $y$. Then $(\pm S)^{2p-2} \cap N = \{e\}$.

**Proof.** Let $w \in (\pm S)^{2p-2} \cap N$ be a non trivial reduced word. Then Lemma \cite{1} implies that $w$ must contain a subword $s$ of length $> 2p - 2$ such that $s \notin (\pm S)^{2p-2}$, which is impossible. \qed

**Example 1.** For every positive integer $p$ there exists a Lawson regular countable paratopological group $G$ with $T_2(G) = p - 1$ and $T_1(G) = 2p - 1$.

**Proof.** Fix any positive integer $p$. For every $n \in \mathbb{N}$, let $F^2_n$ be a free group over a two-point set $\{x_n, y_n\}$. Denote by $H = \oplus_{n=1}^{\infty} F^2_n$ the direct sum of the groups $F^2_n$. Let $S_n \subset F^2_n$ be the semigroup generated by the elements $x_n$ and $y_n$. Denote the direct sum $\oplus_{m \geq n} S_m$ by $U_n$.

We show that the family $\{U_n : n \in \mathbb{N}\}$ satisfies the Pontriagin conditions (P1)--(P4) formulated in the proof of Theorem \cite{1}. The condition (P1) is satisfied because $U_n \cap U_m \supseteq U_{\max(m,n)}$. (P2) and (P3) hold since $U_n$ are semigroups. To show that (P4) holds fix arbitrary $n$ and $w \in G$. Find a number $m$ such that $w \in \oplus_{n=1}^{m} F^2_n$. Then $w^{-1} U_{\max(m,n)+1} w = U_{\max(m,n)+1} \subset U_n$ and thus the condition (P4) holds too. According to \cite{Ra}, $\{U_n : n \in \mathbb{N}\}$ is a neighborhood base at the unit of some (not necessarily Hausdorff) paratopology on the group $H$.

Let $F_n$ denote the quotient group of the group $F^2_n$ by the relation $r^p_n = (x_n y_n^{-1})^p$ and $\phi_n : F^2_n \to F_n$ be the canonical homomorphism with $N_n = \ker \phi_n$. Let $\psi_n : F^2_n \to \mathbb{Z}$ be a unique homomorphism such that $\psi_n(x_n) = 1$ and $\psi_n(y_n) = 0$. Define a map $\psi : H \to \mathbb{Z} \times \prod F_n$ as follows. Given $w = w_1 \cdots w_n \in H$ where $w_i \in F^2_i$ let $\psi(w) = (\sum \psi_i(w_i), \prod \phi_i(w_i))$. Let $G = \psi(H)$ and $\tau$ be the quotient paratopology on the group $G$, see \cite{Ra}. By definition, a base of this paratopology consists of the sets $\psi(U_n)$, $n \in \mathbb{N}$. 


We claim that the \((p - 1)\)-oscillator topology \(\tau_{p-1}\) on \(G\) is Hausdorff. According to Theorem\[\footnote{2}\] it suffices to show that the \((2p-2)\)-oscillator topology \(\tau_{2p-2}\) is \(T_1\). Observe that a neighborhood base of this topology consists of the sets \((\pm \psi(U_n))^{2p-2} = \psi((\pm U_n)^{2p-2})\), \(n \in \mathbb{N}\).

To show that the topology \(\tau_{2p-2}\) is \(T_1\) it suffices given an element \(w \in H \setminus \ker \psi\) to find \(n \in \mathbb{N}\) with \(\psi(w) \notin \psi((\pm U_n)^{2p-2})\). Since \(H = \bigoplus_{i=1}^{\infty} F_i^2\), there is a positive integer \(n\) with \(w \in \bigoplus_{i=1}^{n} F_i^2\).

If \(w \notin \bigoplus_{i=1}^{n} N_i\), then \(\psi(w) \notin \psi((\pm U_{n+1})^{2p-2})\). Next we consider the case \(w \in \bigoplus_{i=1}^{n} N_i\). We claim that \(\psi(w) \notin \psi((\pm U_1)^{2p-2})\). Assuming the converse we would find an element \(s \in (\pm U_1)^{2p-2}\) such that \(\psi(s) = \psi(w)\). Lemma\[\footnote{2}\] yields \(s = e\) and hence \(\psi(w) = \psi(e) = e \neq \psi(w)\), which is a contradiction. Hence the topology \(\tau_{2p-2}\) is \(T_1\) and the topology \(\tau_{p-1}\) is Hausdorff by Theorem\[\footnote{2}\].

Observe that the oscillator topology \(\tau_{2p}\) is not \(T_1\) since \((\psi(U_n)\psi(U_n)^{-1})^p \geq (\psi(x_n)\psi(y_n)^{-1})^p = (p, e)\) for every natural \(n\). It follows that \(T_1(G) = 2p - 1\) and \(T_2(G) = p\).

Finally we show that \(\psi(U_n)\) is a clopen subset of the group \(G\) for every \(n\) and hence \(\tau\) is a zero-dimensional regular topology. Let \(w \in H\) and \(\psi(w) \in \psi(U_n)\). Write \(w = w_1 \cdots w_m\), where \(m \geq n\) and \(w_i \in F_i^2\) for \(i \leq m\). There exist elements \(u \in U_{m+1}, v \in U_n\) such that \(ww^{-1} \in \ker \psi\). Write \(u = u_{m+1} \cdots u_k, v = v_n \cdots v_k\), where \(u_i, v_i \in F_i^2\). Then \(u_i v_i^{-1} \in N_i\) for \(i \geq m + 1\). Since \(u_i v_i^{-1} \in S_i S_i^{-1}\) for every \(i\), Lemma\[\footnote{2}\] implies that \(u_i = v_i\) for \(i \geq m + 1\). Therefore \(w \prod_{i=m}^{n} v_i^{-1} = w w^{-1} \in \ker \psi\) and \(\psi(w) = \psi(v_n \cdots v_m) \in \psi(U_n)\). \(\square\)

Another number invariant of paratopological groups is suggested by Theorem\[\footnote{1}\] which reflects some symmetry property of paratopological groups, which will be referred to as oscillation symmetry. We shall say that a paratopological group \(G\) has finite oscillation if there is a positive integer \(n\) such that any of the first four equivalent conditions of Theorem\[\footnote{1}\] holds. In particular, \(G\) has finite oscillation if there is a positive integer \(n\) such that for any neighborhood \(U\) of the unit in \(G\) the set \((\pm U)^n\) is a neighborhood of \(e\) in \(G^p\). We shall say that a paratopological group \(G\) has countable oscillation if for any neighborhood \(U \subset G\) of \(e\) there is a positive integer \(n\) such that \((\pm U)^n\) is neighborhood of the unit in \(G^p\).

Next, we define an invariant of paratopological groups related to the oscillation. This invariant takes value in the set \(\mathbb{N} \cup \{\omega, \infty\}\) linearly ordered so that \(n < m < \omega < \infty\) for each positive integers \(n < m\). For a paratopological group \(G\) with finite oscillation let \(\text{osc}(G)\) be the smallest positive integer \(n\) such that for any neighborhood \(U \subset G\) of \(e\) the set \((\pm U)^n\) is a neighborhood of \(e\) in \(G^p\). If \(G\) has countable oscillation but fails to have finite oscillation, then we put \(\text{osc}(G) = \omega\). If \(G\) fails to have countable oscillation we put \(\text{osc}(G) = \infty\). We shall say that a paratopological group \(G\) is \(n\)-oscillating if \(\text{osc}(G) \leq n\).

In particular, \(\text{osc}(G) \leq 2\) (resp. \(\text{osc}(G) \leq 3\)) means that the sets \(UU^{-1}\) (resp. \(UU^{-1}U\)) with \(U \in \mathcal{N}(e)\) form a neighborhood base at the unit of the topological group \(G^p\). The following Proposition is immediate.

**Proposition 1.** A paratopological group \(G\) is a topological group if and only if \(\text{osc}(G) = 1\).

Thus the oscillating number allows us to measure the distance from a paratopological group to the class of topological groups, i.e., paratopological groups with small oscillation in a sense are near to topological groups.

Next, we introduce a class of 2-oscillating paratopological groups which contains all topological groups and all paratopological SIN-groups. A paratopological group \(G\) is
defined to be a paratopological LSIN-group if for any neighborhood $U$ of the unit $e$ of $G$ there is a neighborhood $W \subset G$ of $e$ such that $g^{-1}Wg \subset U$ for any $g \in W$. It is clear that each topological group is a paratopological LSIN-group.

A paratopological group $G$ is totally bounded if for any neighborhood $U$ of the unit $e$ of $G$ there is a finite subset $F \subset G$ with $G = F \cdot U$. It is well-known that each totally bounded topological group is a SIN-group. It is interesting to remark that for paratopological groups it is not so, see Example 3.

**Proposition 2.** Each paratopological SIN-group is a paratopological LSIN-group. Conversely, each totally bounded paratopological LSIN-group is a paratopological SIN-group.

**Proof.** The first statement is trivial. To prove the second statement, suppose that $G$ is a totally bounded paratopological LSIN-group. Given a neighborhood $U$ of the unit $e$ in $G$, find a neighborhood $V \subset G$ of $e$ such that $x^{-1}Vx \subset U$ for all $x \in V$. By the total boundedness of $G$ find a finite subset $F \subset G$ such that $G = F \cdot V$. By the continuity of the group operation, find a neighborhood $W \subset G$ of $e$ such that $f^{-1}Wf \subset V$ for each $f \in F$. We claim that $g^{-1}Wg \subset U$ for each $g \in G$. Indeed, given arbitrary $g \in G$, find $f \in F$ and $x \in V$ such that $g = fx$. Then $g^{-1}Wg = x^{-1}f^{-1}Wfx \subset x^{-1}Vx \subset U$.

Therefore, for any neighborhood $U \subset G$ of $e$ we have found a neighborhood $W \subset G$ of $e$ such that $g^{-1}Wg \subset U$ for all $g \in G$. Hence $G$ is a paratopological SIN-group. □

Following I. Guran we say that a paratopological group $G$ is saturated if for any neighborhood $U \subset G$ of the unit the set $U^{-1}$ has nonempty interior in $G$. A standard example of a saturated paratopological group with discontinuous inverse is the Sorgenfrey line, i.e., the real line endowed with the Sorgenfrey topology generated by the base consisting of half-intervals $[a, b)$, $a < b$. Saturated paratopological groups seem to be very close to being a topological group (this vague thesis will be confirmed in the subsequent proposition). Let us mention that totally bounded paratopological groups are saturated, see Proposition 2.1 from [Ra3]. The following theorem shows that quite often we deal with 2-oscillating paratopological groups.

**Proposition 3.** The class of paratopological 2-oscillating groups contains all topological groups, all paratopological LSIN-groups and all saturated paratopological groups.

**Proof.** By Proposition 1 the class of paratopological 2-oscillating groups contains all topological groups. To see that each paratopological LSIN-group $G$ is 2-oscillating fix any neighborhood $U$ of the unit $e$ in $G$ and find a neighborhood $W \subset U$ of $e$ such that $g^{-1}Wg \subset U$ for all $g \in W$. Then $g^{-1}W \subset Ug^{-1}$ for all $g \in W$ and thus $W^{-1}W \subset UW^{-1} \subset UU^{-1}$. This means that the paratopological group $G$ is 2-oscillating.

Finally, let us show that each saturated paratopological group $(G, \tau)$ is 2-oscillating. Fix any neighborhood $U$ of the unit $e$ in $G$. We have to find a neighborhood $W \subset G$ of $e$ such that $W^{-1}W \subset UU^{-1}$. Find an open neighborhood $V \subset G$ of $e$ such that $V^2 \subset U$. Since $G$ is saturated, there are a point $x \in V$ and a neighborhood $W \subset G$ of $e$ such that $x^{-1}W \subset V^{-1}$. Then $W^{-1}x \subset V$ and $W^{-1} \subset Vx^{-1}$. We can assume that $W$ is so small that $x^{-1}W \subset Vx^{-1}$. In this case $W^{-1}W \subset Vx^{-1}W \subset VVx^{-1} \subset VV^{-1} \subset UU^{-1}$. Hence osc$(G) \leq 2$. □

Proposition 3 gives topological conditions under which a paratopological group is 2-oscillating. Next we consider some algebraic conditions yielding the same result.
A group $G$ is defined to be absolutely $n$-oscillating if any paratopological group algebraically isomorphic to $G$ is $n$-oscillating. In particular, each abelian group is absolutely 2-oscillating and each group of finite exponent is absolutely 1-oscillating.

We shall show that the absolute $n$-oscillation property follows from another algebraic property called the $n$-reversivity. A group $G$ is defined to be $(n, m)$-reversible where $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$ if $(\mp A)^n \subset (\pm A^m)^n$ for any subset $A \subset G$ containing the unit $e$ of $G$ (here $A^\infty = \bigcup_{n \in \mathbb{N}} A^m \subset G$ is the semigroup in $G$ generated by $A$). A group $G$ is called $n$-reversible if it is $(n, m)$-reversible for some $m \in \mathbb{N}$. Observe that each $n$-reversible group is $(n, \infty)$-reversible and $(n + 1)$-reversible. Note also that a group $G$ is $(1, \infty)$-reversible (resp. 1-reversible) if and only if $G$ is periodic ($G$ is of finite exponent).

Reversible groups were studied in [Ba] where it was shown that a group $G$ is 2-reversible if and only if $G$ is 3-reversible if and only if $G$ is collapsing in the sense of [SS], [S]. We remind that a group $G$ is collapsing if there are numbers $n, m \in \mathbb{N}$ such that $|A^n| < |A|^m$ for any $n$-element subset $A$ of $G$. Collapsing groups form a wide class of groups, containing all groups with positive laws, in particular all virtually nilpotent groups, see [SS], [S], and [Ma]. We remind that a group $G$ is virtually nilpotent if $G$ contains a nilpotent subgroup of finite index. According to the famous Gromov Theorem [Gr], a finitely generated group is virtually nilpotent if and only if its has polynomial growth. For finitely generated solvable groups a more precise characterization is true: such a group is virtually nilpotent if and only if it contains no free semigroup with two generators, see [Ro]. It is interesting to mention that a group $G$ contains no free semigroup with two generators if and only if $G$ is $(2, \infty)$-reversible if and only if $G$ is $(3, \infty)$-reversible, see [Ba]. Moreover, for any polycyclic group $G$ the following conditions are equivalent: (i) $G$ is virtually nilpotent, (ii) $G$ has polynomial growth, (iii) $G$ is collapsing, (iv) $G$ contains no free semigroup with two generators, (v) $G$ is $n$-reversible for some $n \in \mathbb{N}$, (vi) $G$ is $(n, \infty)$-reversible for some $n \in \mathbb{N}$, see [Ba].

For a group $G$ let $G_\omega^0 = \{(g_n) \in G^\omega : g_n$ be the unit of $G$ for almost all $n\}$ denote the direct sum of countably many copies of $G$. The following proposition describes the interplay between $n$-reversible and absolutely $n$-oscillating groups.

**Proposition 4.** Let $G$ be a group and $n$ be a positive integer.

1. If $G$ is $n$-reversible, then it is absolutely $n$-oscillating.
2. If $G$ is not $n$-reversible, then the group $G_\omega^0$ is not absolutely $n$-oscillating. More precisely, $G_\omega^0$ is isomorphic to a first-countable Hausdorff $b$-regular zero-dimensional paratopological group $H$ with osc$(H) > n$.
3. If $G$ is isomorphic to $G_\omega^0$, then $G$ is $n$-reversible if and only if $G$ is absolutely $n$-oscillating.

**Proof.** The last statement follows directly from the previous two statements.

Assume that a group $G$ is $n$-reversible and find $m \in \mathbb{N}$ such that $(\mp A)^n \subset (\pm A^m)^n$ for each subset $A \subset G$ containing the unit $e$ of $G$. To show that $G$ is absolutely $n$-oscillating, suppose that $\tau$ is a topology on $G$ making the group operation of $G$ continuous. Given any neighborhood $U$ of the unit $e$ in $(G, \tau)$, find a neighborhood $W \subset G$ of $e$ such that $W^m \subset U$. Then $(\mp W)^n \subset (\pm W^m)^n \subset (\pm U)^n$. By Theorem [1] the paratopological group $(G, \tau)$ is $n$-oscillating.

Next, assume that a group $G$ is not $n$-reversible. This means that for any $m \in \mathbb{N}$ there is a subset $A_m \subset G$ containing the unit of $G_n$ such that $(\mp A_m)^n \not\subset (\pm A^m)^n$. 

It is easy to find a countable family \( \mathcal{F} \) of non-decreasing maps \( f : \omega \to \mathbb{Z}_+ = \{0\} \cup \mathbb{N} \), satisfying the following conditions

1. \( \lim_{n \to \infty} \frac{f(n)}{n} = 0 \) and \( \lim_{n \to \infty} f(n) = \infty \) for any \( f \in \mathcal{F} \);
2. for any \( f, g \in \mathcal{F} \) there is \( h \in \mathcal{F} \) with \( 2h \leq \min\{f, g\} \).

For any \( f \in \mathcal{F} \) let \( U_f = \{(g_m)_{m \in \omega} \in G^\omega_0 : g_m \in A^f(m) \} \). Repeating the argument from the proof of Example 1, it can be shown that the family \( \mathcal{B} = \{U_f : f \in \mathcal{F} \} \) forms a neighborhood base at the unit of some zero-dimensional first-countable paratopology \( \tau \) on \( G^\omega_0 \). Since each set \( U_f, f \in \mathcal{F} \), is closed in the Tychonov product topology on \( G^\omega_0 \) which is weaker than \( \tau \), we get that \( U_f \) is \( b \)-closed. Hence the paratopological group \( (G^\omega_0, \tau) \) is Hausdorff and \( b \)-regular. We claim that it not \( n \)-oscillating.

Assuming that \( (G, \tau) \) is \( n \)-oscillating, we can find functions \( f, g \in \mathcal{F} \) such that \( (\exists U_g)^n \subset (\pm U_f)^n \) which means that \( (\exists A^g_m)^n \subset (\pm A^f_m)^n \) for all \( m \). Find \( m \in \omega \) such that \( 0 < g(m) \leq f(m) < m \). Then \( (\exists A^g_m)^n \subset (\pm A^g_m)^n \subset (\pm A^f_m)^n \subset (\pm A^m_m)^n \) which contradicts to the choice of the set \( A_m \).

\( \square \)

A similar statement holds for \((n, \infty)\)-reversible groups. We remind that a paratopological group \( G \) is \emph{Lawson} if it possesses a neighborhood base at the unit, consisting of subsemigroups of \( G \).

**Proposition 5.** Let \( G \) be a group and \( n \) be a positive integer.

1. If \( G \) is \((n, \infty)\)-reversible, then any Lawson paratopology on \( G \) is \( n \)-oscillating.
2. If \( G \) is not \((n, \infty)\)-reversible, then the group \( G^\omega_0 \) is isomorphic to a Lawson first-countable Hausdorff \( b \)-regular zero-dimensional paratopological group \( H \) with \( \text{osc}(H) > n \).

The first statement of this Proposition can be proven by analogy with the proof of the first statement of Proposition 4, while the second one follows from the next theorem whose proof repeats the argument of Example 1 and Proposition 4.

**Proposition 6.** Suppose \( G \) is a group and \( S \) is a subsemigroup of \( G \) containing the unit \( e \) of \( G \). Then the sets \( U_n = \{(g_i)_{i \in [n]} \in G^\omega_0 : g_i = e \text{ if } i \leq n \text{ and } g_i \in S \text{ if } i > n \} \), \( n \in \mathbb{N} \), form a neighborhood base of some Lawson paratopology \( \tau \) on \( G \) which has the following properties:

1. the paratopological group \( (G^\omega_0, \tau) \) is \( b \)-regular first-countable and zero-dimensional;
2. the paratopological group \( (G^\omega_0, \tau) \) is \( n \)-oscillating for some \( n \in \mathbb{N} \) if and only if \( (\mp S)^n \subset (\pm S)^n \).

We use the above Proposition to construct an example of a 2-oscillating paratopological group whose mirror paratopological group is not 2-oscillating. This example relies on a semigroup with is left reversible but not right reversible. We remind that a semigroup \( S \) is left (resp. right) \emph{reversible} if for any elements \( a, b \in S \) the intersection \( aS \cap bS \) (resp. \( Sa \cap Sb \)) is not empty, see [CP], §1.10. If \( S \) is a subsemigroup of a group, this is equivalent to saying that \( S^{-1}S \subset SS^{-1} \) (resp. \( SS^{-1} \subset S^{-1}S \)). The simplest example of a semigroup which is left reversible but not right reversible is the semigroup \( S \) generated by two transformations \( y = 2x \) and \( y = x + 1 \) in the group \( \text{Aff}(\mathbb{Q}) \) of affine transformations of the field \( \mathbb{Q} \) of rational numbers. This semigroup can be also defined in an abstract way as a semigroup generated by two elements \( a, b \) with the relation \( ab = b^2a \), see Example 1 to [CP], §1.10. The left and non-right reversivity of \( S \) implies that \( S^{-1}S \subset SS^{-1} \) but \( SS^{-1} \not\subset S^{-1}S \). Observe that the group \( \text{Aff}(\mathbb{Q}) \) is \emph{metabelian} in the sense that it contains a
normal abelian subgroup with abelian quotient. Applying Proposition 6 to the semigroup \( S \cup \{ e \} \subset \text{Aff}(Q) \), we get the following unexpected example showing that the oscillation number is powerful enough to distinguish between a paratopological group and its mirror group.

**Example 2.** There is a Lawson Hausdorff countable first-countable metabelian \( b \)-regular group \( G \) with \( \text{osc}(G) = 2 \) and \( \text{osc}(G^-) = 3 \).

In spite of the fact that the oscillation numbers \( \text{osc}(G) \) and \( \text{osc}^-(G) \) of a paratopological group \( G \) and its mirror paratopological group \( G^- \) need not be equal, they cannot differ very much. The following proposition can be easily derived from the definitions and the equality \((\tau_n)^{-1} = (\tau^{-1})_n\) holding for each odd \( n \).

**Proposition 7.** Suppose \( G \) is a topological group with finite oscillation and \( G^- \) is its mirror paratopological group. Then

1. \( \text{osc}(G) - 1 \leq \text{osc}(G^-) \leq \text{osc}(G) \) if the number \( \text{osc}(G) \) is odd;
2. \( \text{osc}(G) \leq \text{osc}(G^-) \leq \text{osc}(G) + 1 \) if \( \text{osc}(G) \) is even;

It is clear that each 2-oscillating paratopological group is 3-oscillating. We shall show that (regular) 3-oscillating paratopological groups are \( b \)-separated (and \( b \)-regular).

**Theorem 3.** Any (regular) Hausdorff 3-oscillating paratopological group \( G \) is \( b \)-separated (and \( b \)-regular).

*Proof.* Suppose \( G \) is a Hausdorff 3-oscillating paratopological group. This means that the 3-oscillator topology \( \tau_3 \) coincides with \( \tau_2 \). By Theorem 2 the topology \( \tau_3 \) is \( T_1 \). Consequently, the topological group \( G^0 = (G, \tau_3) \) is separated and hence is Hausdorff. This means that the group \( G \) is \( b \)-separated.

Next, let us verify that \( G \) is \( b \)-regular provided \( G \) is regular. Fix any neighborhood \( U \) of the unit \( e \) of \( G \). Since \( G \) is regular, we can assume that \( U \) is closed in \( G \). We have to find a neighborhood \( V \in \mathcal{B} \) such that the closure \( \overline{V}^\circ \) of \( V \) in the topology \( \tau_3 \) lies in \( U \). Let \( V \) be a neighborhood of \( e \) in \( G \) such that \( V^3 \subset U \).

To show that \( \overline{V}^\circ \subset U \), pick any point \( x \not\in U \). We have to find a 3-oscillator \( QO^{-1}Q \) such that \( xQO^{-1}Q \cap V = \emptyset \) (since \( G \) is 3-oscillating 3-oscillators form a neighborhood base at the unit of \( G^0 \)). Since \( U \) is closed in \( G \), there is a neighborhood \( W \) of \( e \) in \( G \) such that \( xW \cap U = \emptyset \). We can assume that \( W \) is so small that \( Wx^{-1} \subset x^{-1}V \). The group \( G \) is 3-oscillating and thus contains a neighborhood \( O \subset G \) such that \( O^{-1}QO^{-1} \subset WW^{-1}W \). We claim that \( xQO^{-1}Q \cap V = \emptyset \). Assuming the converse we would get \( x \in V \) \( V^{-1}OQ^{-1}Q \subset VV^{-1}W^{-1}W \) and thus \( xW^{-1} \cap VVW^{-1}W^{-1} \neq \emptyset \). Then \( Wx^{-1} \cap WW^{-1}V^{-1} \neq \emptyset \) and hence \( x^{-1}V \cap WW^{-1}V^{-1} \neq \emptyset \). After inversion we get \( V^{-1} \cap VVW^{-1}W^{-1} \neq \emptyset \) and \( x \in VVW^{-1}W^{-1} \subset V^3W^{-1} \). Then \( xW \cap U \subset xW \cap V^3 \neq \emptyset \) which contradicts to the choice of the neighborhood \( W \).

Theorem 3 and Propositions 3, 4 imply

**Corollary 2.** A Hausdorff (regular) paratopological group \( G \) is \( b \)-separated (and \( b \)-regular) provided \( G \) satisfies one of the following conditions:

1. \( G \) is a saturated paratopological group;
2. \( G \) is a paratopological LSIN-group;
3. \( G \) is absolutely 2-oscillating;
4. \( G \) is collapsing.
Recall that a topological space $X$ is Čech-complete if it is a $G_δ$-set in any its compactification, see [En, §3.9]. It is well known that each complete metric space is Čech-complete.

**Theorem 4.** A Hausdorff $b$-regular paratopological group $(G, \tau)$ has countable oscillation provided the group reflexion $G^b$ is a Lindelöf Čech-complete space.

**Proof.** Let $N(e)$ be a neighborhood base at the unit $e$ of the group $G$, consisting of $b$-closed sets. Then $B = \{\bigcup_{n \in \mathbb{N}} (\pm U)^n : U \in N(e)\}$ is a base at the unit of some (not necessary Hausdorff) group topology on $G$ weaker than $\tau$. It follows that for any neighborhood $U \in N(e)$ the set $\bigcup_{n \in \mathbb{N}} (\pm U)^n$, being an open subgroup of $G^b$, is closed in $G^b$ and thus is Lindelöf and Čech-complete.

Fix any neighborhood $U \in N(e)$. We have to find $m$ such that $(\pm U)^m$ is a neighborhood of $e$ in $G^b$. The group $H = \bigcup_{n \in \mathbb{N}} (\pm U)^n$, being Čech-complete, is Baire. Consequently, there is $n \in \mathbb{N}$ such that the set $A = (\pm U)^n$ is not meager in $H$. We claim that $A \cdot A^{-1}$ is a neighborhood of the unit in $H$.

We shall use Banach-Kuratowski-Pettis Theorem (see [Kei] p.279 or [Ke] 9.9) asserting that for any subset $B$ of a topological group the set $BB^{-1}$ is a neighborhood of the unit provided $B$ is non-meager and has the Baire Property in the group. We remind that a subset $B$ of a topological space $X$ has the Baire Property in $X$ if $B$ contains a $G_δ$-subset $C$ of $X$ such that $B \setminus C$ is meager in $X$. Thus to show that $AA^{-1}$ is a neighborhood of the unit in $H$ it suffices to verify that the set $A = (\pm U)^n$ has the Baire Property in $H$.

For this we shall use the well-known fact (see [RJ] or [Ha] 3.1) asserting that each $K$-analytic subspace $X$ of a Tychonoff topological space $Y$ has the Baire Property in $Y$. We remind that a topological space $X$ is $K$-analytic if $X$ is a continuous image of a Lindelöf Čech-complete space. It is known that the product of two $K$-analytic spaces is $K$-analytic and the continuous image of a $K$-analytic space is $K$-analytic, see [RJ]. Observe that the subspace $U \subset H$, being a closed subspace of the Lindelöf Čech-complete space $H$, is $K$-analytic. Then the space $A = (\pm U)^n \subset H$, being a continuous image of the product $U^n$, is $K$-analytic too. Hence $A$ has the Baire property in $H$ and by the Banach-Kuratowski-Pettis Theorem, $AA^{-1}$ is a neighborhood of the unit in $H$. Observing that $AA^{-1} \subset (\pm U)^{2n+2}$ we see that the set $(\pm U)^m$ is a neighborhood of the unit of $H$ for $m = 2n + 2$. Since the group $H$ is open in $G^b$, we get that $(\pm U)^m$ is a neighborhood of the unit in $G^b$. \[\square\]

Next, we give a $\pi$-base characterization of saturated paratopological groups. A collection $W$ of non-empty open subsets of a topological space $X$ is a $\pi$-base if any non-empty open set $U \subset X$ contains an element $W$ of $W$.

**Theorem 5.** A paratopological group $G$ is saturated if and only if the collection of nonempty $b$-open subsets forms a $\pi$-base in $G$.

**Proof.** To prove the "if" part, assume that the collection of nonempty $b$-open subsets forms a $\pi$-base for $G$ and fix any neighborhood $U$ of the unit $e$ in $G$. Find a $b$-open subset $V \subset U$. We can assume that $V = xW$ for some $x \in V$ and some $b$-open neighborhood $W$ of $e$ with $W = W^{-1}$. Then $W = W^{-1}$ is an open neighborhood of $e$ in $G$ such that $U \supset V = xW = xW^{-1}$ and thus $Wx^{-1} \subset U^{-1}$ which means that the paratopological group $G$ is saturated.

To prove the "only if" part, suppose that $G$ is a saturated paratopological group. Fix any neighborhood $U \subset G$ of the unit $e$ of $G$. Find a neighborhood $V \subset G$ of $e$ such that $V \cdot V \subset U$. Since $V^{-1}$ has nonempty interior in $G$, there is a point $x \in V$ and a
neighborhood \( W \subset V \) of the unit \( e \) such that \( x^{-1}W \subset V^{-1} \). Then \( W^{-1}x \subset V \) and thus \( WW^{-1}x \subset WV \subset U \). By Proposition 3 the set \( WW^{-1} \) is \( b \)-open. Hence the collection of nonempty \( b \)-open subsets forms a \( \pi \)-base for the space \( G \).

Theorem 3 implies that a saturated paratopological group \( G \) and its group reflexion \( G^p \) have many common properties (those that can be expressed via \( \pi \)-bases). In particular, the spaces \( G \) and \( G^p \) have the same Souslin number, the same calibers and precalibers, they simultaneously are (or are not) Baire or quasicomplete and simultaneously satisfy (or not) many chain conditions considered in \([CN]\) and \([AMN]\) (such as the properties ccc, productively-ccc, (\( \ast \)), (\( \ast \ast \)), (P), or (\( K_n \)) for \( n \geq 2 \)), see also \([BR_3]\).

We present here only one result of this sort, concerning precalibers of totally bounded paratopological groups. We remind that a cardinal \( \tau \) is a precaliber of a topological space \( X \) if any collection \( U \) of nonempty open subsets of \( X \) with \( |U| = \tau \) contains a centered subcollection \( V \) with \( |V| = \tau \) (a collection \( V \) being centered if \( \cap F \neq \emptyset \) for any finite subcollection \( F \) of \( V \)). It is easy to see that a topological space \( X \) is countably cellular if \( \aleph_1 \) is a precaliber of \( X \) (the converse is true under (MA+\( \neg \)CH) but is false under (CH), see [Ar, p.43] and [AMN]). It is well-known that each cardinal of uncountable cofinality is a precaliber of any totally bounded topological group (this follows from the dyadicity of compact topological groups). This fact and Theorem 3 imply the following useful result answering the “paratopological” version of Protasov’s Problem 6 from [BCGP].

**Corollary 3.** A totally bounded paratopological group \( G \) is countably cellular. Moreover, each cardinal of uncountable cofinality is a precaliber of \( G \).

It is interesting to mention that for any infinite cardinal \( \tau \) there is a zero-dimensional totally bounded left-topological group with Souslin number \( \tau \), see [PR].

Theorem 3 implies that any paratopological group \( G \) with \( 1 \leq T_2(G) < \infty \) satisfies \( \text{osc}(G) \geq 4 \). In particular, this concerns countable regular paratopological groups constructed in Example 1. Thus paratopological groups with large oscillation numbers exist. Moreover, such a group can be a subgroup of a paratopological group with small oscillation number. (In this context it is interesting to notice that the class of \( b \)-separated (\( b \)-regular) paratopological groups is closed with respect to taking subgroups and many other operations).

**Example 3.** There is a regular countable first-countable saturated paratopological group \( G \) with \( \text{osc}(G) = 2 \) containing a \( b \)-closed subgroup \( H \) with \( \text{osc}(G) = \infty \) and failing to be a paratopological LSIN-group.

**Proof.** We shall use the result of [BR_1] asserting that a countable first-countable paratopological group \( H \) is a \( b \)-closed subgroup of a \( b \)-regular countable first-countable saturated paratopological group provided \( H \) has a neighborhood base at the unit, consisting of subsets, closed in some weaker topology \( \sigma \) turning \( H \) into a first-countable topological SIN-group.

Thus to produce the required example it suffices to construct a countable first-countable group \( H \) with \( \text{osc}(H) = \infty \) possessing a neighborhood base at the unit, consisting of subsets closed in some weaker topology \( \sigma \) turning \( H \) into a first-countable topological SIN-group.

Consider the free group \( F_2 \) with two generators \( x, y \) and the unit \( e \) and let \( FS_2 \subset F_2 \) be the subsemigroup spanning the set \( \{e, x, y\} \). Let \( H = (F_2)_{0}^{\infty} \) and \( \tau \) be the paratopology
on $H$ generated by the semigroup $FS_2$ as indicated in Proposition 6 which implies that the paratopological group $(H, \tau)$ is countable, first countable and has infinite oscillation. Besides the topology $\tau$, the group $H$ carries the weaker topology $\sigma$, induced from the countable Tychonov power $(F_2)^\omega$ of the discrete group $F_2$. It is easy to see that $(H, \sigma)$ is a topological SIN-group and $(H, \tau)$ has a neighborhood base consisting of $\sigma$-closed neighborhoods.

Applying [BR], we conclude that $H$ is a $b$-closed subgroup of a first-countable countable $b$-regular saturated paratopological group $G$. The group $G$, being saturated, is 2-oscillating according to Proposition 3.

Assuming that $G$ is a paratopological LSIN-group, we would get that so is its subgroup $H$ which is not possible because $\text{osc}(H) = \infty$, see Proposition 3.

We saw in Example 2 that a paratopological group needs not be isomorphic to its mirror paratopological group. Below we construct a saturated example of this sort.

An automorphism $h : G \to G$ of a group is called an inner automorphism of $G$ if there is $g \in G$ such that $h(x) = g^{-1}xg$ for all $x \in G$.

**Proposition 8.** Suppose $G$ is a paratopological group such that any continuous automorphism $H : G^\circ \to G^\circ$ of its group reflexion is an inner automorphism. The paratopological group $G$ is isomorphic to its mirror paratopological group $G^-$ if and only if $G$ is a topological group.

**Proof.** The “if” part of the theorem is trivial. To prove the “only if” part, suppose that $h : G \to G^-$ is a topological isomorphism. It follows that $h$ is a continuous automorphism of the topological group $G^\circ$ and thus $h$ is an inner automorphism. Find $g \in G$ with $h(x) = g^{-1}xg$ for all $x \in G$. Then $x = gh(x)g^{-1}$ and hence the identity automorphism $id : G \to G^-$ is continuous. This means that for any neighborhood $U \subset G$ of the unit $e$ there is a neighborhood $V \subset G$ of $e$ such that $V \subset U^{-1}$, i.e., the inversion $(\cdot)^{-1} : G \to G$ is continuous and hence $G$ is a topological group.

As usual, under a character on a topological group $G$ we understand a continuous homomorphism $\chi : G \to \mathbb{T}$ of $G$ into the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ considered as a multiplicative subgroup of the complex plane $\mathbb{C}$. Each character $\chi : G \to \mathbb{T}$ induces a topology (called the Sorgenfrey paratopology) on $G$, whose neighborhood base at a point $g_0 \in G$ consists of the sets $U^+ = \{g \in U : \text{Arg}(\chi(g_0)) \leq \text{Arg}(\chi(g)) < \text{Arg}(\chi(g_0)) + \pi\}$ where $U$ runs over neighborhoods of $g_0$ in $G$ (as usual $\text{Arg}(z) \in [0, 2\pi)$ stands for the argument of a complex number $z \neq 0$). It is easy to see that $G$ endowed with the Sorgenfrey paratopology is a saturated paratopological group. If the subgroup $\text{Ker}(\chi) = \chi^{-1}(1)$ is not open in $G$, then this paratopological group fails to be a topological group. This observation together with Proposition 8 imply

**Corollary 4.** Let $G$ be a topological group such that each continuous automorphism of $G$ is inner and let $\chi : G \to \mathbb{T}$ be a character whose kernel $\text{Ker}(\chi)$ is not open in $G$. Suppose that $\tau$ is the Sorgenfrey paratopology on $G$ generated by the character $\chi$. Then the saturated paratopological group $(G, \tau)$ is not isomorphic to its mirror paratopological group $(G, \tau^{-1})$.

To construct a saturated paratopological group which is not isomorphic to its mirror paratopological group, it rests to find an example of a topological group satisfying the conditions of Corollary 4. Many such examples can be found using the theory of Lie groups and Lie algebras, see [GG], [VO].
Probably the simplest example is the Lie group \( \text{Aff}^+(\mathbb{R}) \) of all orientation-preserving affine transformations of the real line. This group can be represented by matrices of the form \( \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \) where \( a, b \in \mathbb{R}, a > 0 \). It is well-known that \( \text{Aff}^+(\mathbb{R}) \) endowed with the natural locally Euclidean topology is a metabelian Lie group which is not a SIN-group (see [Kel, p.279]). It follows from [GG, p.28] that any continuous automorphism of the Lie group \( \text{Aff}^+(\mathbb{R}) \) is inner. The group \( \text{Aff}^+(\mathbb{R}) \) admits a non-trivial character \( \chi : \text{Aff}^+(\mathbb{R}) \to \mathbb{T} \) assigning to each matrix \( A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \text{Aff}^+(\mathbb{R}) \) the complex number \( \chi(A) = e^{i \ln a} \in \mathbb{T} \). This character induces the Sorgenfrey topology \( \tau \) on \( \text{Aff}^+(\mathbb{R}) \) whose neighborhood base at the unit \( E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) consists of the sets \( U(\varepsilon) = \{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : 1 \leq a < 1 + \varepsilon, \ |b| < \varepsilon \} \) where \( \varepsilon > 0 \). Thus it is legal to apply Proposition 8 and Corollary 4 to get

**Example 4.** The paratopological group \( (\text{Aff}^+(\mathbb{R}), \tau) \) endowed with the Sorgenfrey topology \( \tau \) is not isomorphic to its mirror paratopological group \( (\text{Aff}^+(\mathbb{R}), \tau^{-1}) \). Yet, the paratopological groups \( (\text{Aff}^+(\mathbb{R}), \tau) \) and \( (\text{Aff}^+(\mathbb{R}), \tau^{-1}) \) are saturated paratopological LSIN-groups but are not paratopological SIN-groups.

**Remark 1.** In fact many other Lie groups have the properties of the group \( \text{Aff}^+(\mathbb{R}) \). In particular, each non-trivial solvable simply connected Lie group \( G \), being a semidirect product of a closed normal subgroup and a one-dimensional Lie group, admits a non-trivial character, [VO, p.59]. If, in addition, the Killing form of the Lie algebra of \( G \) is non-degenerated, then \( G \) all automorphisms of \( G \) are inner (see [GG §1.5]) and thus \( G \) admits a regular saturated paratopology \( \tau \) such that the paratopological group \( (G, \tau) \) is not isomorphic to its mirror paratopological group \( (G, \tau^{-1}) \).

Finally, let us state some open questions related to the introduced concepts.

**Problem 1.**  
1. Is every \((2n + 1)\)-reversible group \(2n\)-reversible? (The answer is “yes” for \( n = 1 \)).
2. Is there an absolutely 2-osculating group which is not 2-reversible?
3. Is every polycyclic group absolutely \( n \)-reversible for some \( n \in \mathbb{N} \)?
4. For which \( n \in \mathbb{N} \) there are a group \( G \) and a subsemigroup \( S \subset G \) such that \( (\pm S)^n \subset (\pm S)^n \) but \( (\pm S)^n \not\subset (\pm S)^n \)?
5. Is every regular \( b \)-separated paratopological group \( b \)-regular?
6. Suppose \( G \) is a paratopological LSIN-group. Is the mirror paratopological group \( G^\sim \) a paratopological LSIN-group?
7. Is it true that for every number \( n \geq 1 \) there is a \((b\text{-regular})\) paratopological group \( G \) with \( \text{osc}(G) = n \)? (The answer is “yes” for \( n \leq 3 \)).
8. Is there a \((\text{regular})\) Hausdorff paratopological group \( G \) such that the numbers \( T_2(G) \) and \( \text{osc}(G) \) are finite?
9. Is there a paratopological group \( G \) whose all oscillator topologies are Hausdorff, but the group reflexion \( G^\circ \) of \( G \) is not separated?
10. Is there a paratopological group \((G, \tau)\) such that \( \tau_b \neq \inf \tau_n \)?
11. Is there a paratopological group \( G \) whose group reflexion \( G^\circ \) carries the antidiscrete topology?
12. Has a \( b \)-regular paratopological group \( G \) finite oscillation if its group reflexion \( G^\circ \) is compact?

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