Multiscaling in Infinite Dimensional Collision Processes

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We study relaxation properties of two-body collisions in infinite spatial dimension. We show that this process exhibits multiscaling asymptotic behavior as the underlying distribution is characterized by an infinite set of nontrivial exponents. These nonequilibrium relaxation characteristics are found to be closely related to the steady state properties of the system.

PACS numbers: 05.40.+j, 05.20.Dd, 02.50.Ey

Our understanding of the statistical mechanics of nonequilibrium systems remains incomplete, in sharp contrast with their equilibrium counterpart. The rich phenomenology associated with dynamics of far from equilibrium interacting particle systems exposes the lack of a unifying theoretical framework. Simple tractable microscopic models can therefore help us gain insight and better the description of nonequilibrium dynamics.

In this study, we focus on the nonequilibrium relaxation of an infinite particle system interacting via two body collisions. We find that a hierarchy of scales underlies the relaxation. In particular, we devise an extremely simple system which exhibits multiscaling in infinite dimension, while in finite dimensions simple scaling behavior is restored. Furthermore, we show that this behavior extends to a broader class of collision processes.

We are interested in modeling collision processes in a structureless, infinite dimensional space. Therefore, we place an infinite number of identical particles on the nodes of a completely connected graph. Particles are characterized by a single parameter, their velocity \( v \).

Two-body collisions are realized by choosing two particles at random and changing their velocities according to \((u_1, u_2) \rightarrow (v_1, v_2)\) with

\[
\begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix} = \begin{pmatrix}
  \gamma & 1 - \gamma \\
  1 - \gamma & \gamma
\end{pmatrix} \begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}
\]

and \(0 \leq \gamma \leq 1\). In other words, the post-collision velocities are given by a linear combination of the pre-collision velocities. Both the total momentum \((u_1 + u_2 = v_1 + v_2)\) and the total number of particles are conserved by this process. In fact, the collision rule \((1)\) is the most general linear combination which obeys

momentum conservation and Galilean invariance, i.e., invariance under velocity translation \( v \rightarrow v - v_0 \).

Our motivation for studying this problem is inelastic collisions in one-dimensional granular gases \((2)\). While the two problems involve different collision rates, they share the same trivial final state where all velocities vanish, \(P(v, t) \rightarrow \delta(v)\) when \(t \rightarrow \infty\) (without loss of generality, the average velocity was set to zero by invoking the transformation \( v \rightarrow v - \langle v \rangle \)). We chose to describe this work in slightly more general terms since closely related dynamics were used in different contexts including voting systems \((2)\), asset exchange processes \((3)\), combinatorial processes \((3)\), traffic flows \((3)\), and force fluctuations in bead packs \((3)\). We will show that multiscaling characterizes fluctuations in some of these problems as well.

Velocity fluctuations may be obtained via the probability distribution function \(P(v, t)\) which evolves according to the following master equation

\[
\frac{\partial P(v, t)}{\partial t} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du_1 du_2 P(u_1, t)P(u_2, t) 
\times \left[ \delta(v - \gamma u_1 - (1 - \gamma) u_2) - \delta(v - u_2) \right].
\]

The \(\delta\)-functions on the right-hand side reflect the collision rule \((1)\) and guarantee conservation of the number of particles, \(\int dvP(v, t) = 1\), and the total momentum \(\int dv vP(v, t) = 0\). Eq. \((2)\) can be simplified by eliminating one of the integrations

\[
\frac{\partial P(v, t)}{\partial t} + P(v, t) = \frac{1}{1 - \gamma} \int_{-\infty}^{\infty} du P(u, t)P\left(\frac{v - \gamma u}{1 - \gamma}, t\right).
\]

Further simplification may be achieved via the Fourier transform \(\hat{P}(k, t) = \int dv e^{ikv} P(v, t)\) which obeys

\[
\frac{\partial}{\partial t} \hat{P}(k, t) + \hat{P}(k, t) = \hat{P}[(1 - \gamma)k, t].
\]

Although the integration is eliminated, this compact equation is still challenging as the nonlinear term becomes nonlocal.

Velocity fluctuations can be quantified using the moments of the velocity distribution, \(M_n(t) = \int dv v^n P(v, t)\). The moments obey a closed and recursive set of the ordinary differential equations. The corresponding equations can be derived by inserting the expansion \(\hat{P}(k, t) = \sum_n \frac{(ik)^n}{n!} M_n(t)\) into Eq. \((4)\) or directly from Eq. \((4)\).

The first few moments evolve according to \(M_0 = M_1 = 0\), and

\[
\begin{align*}
M_2 &= -a_2 M_2, \\
M_3 &= -a_3 M_3, \\
M_4 &= -a_4 M_4 + a_{24} M_2^2,
\end{align*}
\]

with the coefficients

\[
a_n \equiv a_n(\gamma) = 1 - (1 - \gamma)^n - \gamma^n,\]

1
and $a_{24} = 6\gamma^2 (1 - \gamma)^2$. Integrating these rate equations yields $M_0 = 1$, $M_1 = 0$ and

$$
M_2(t) = M_2(0) e^{-a_2 t} \\
M_3(t) = M_3(0) e^{-a_2 t} \\
M_4(t) = [M_4(0) + 3M_2^2(0)] e^{-a_2 t} - 3M_2^2(t).
$$

The asymptotic behavior of the first few moments suggests that knowledge of the RMS fluctuation $v^* \equiv M_2^{1/2}$ is not sufficient to characterize higher order moments since $M_3^{1/3} / v^*, M_4^{1/4} / v^* \to \infty$, as $t \to \infty$.

This observation extends to higher order moments as well. In general, the moments evolve according to

$$
M_n + a_n M_n = \sum_{m=2}^{n-2} \binom{n}{m} \gamma^m (1 - \gamma)^{n-m} M_m M_{n-m}.
$$

Note that for $0 < \gamma < 1$, the coefficients $a_n$ satisfy $a_n < a_m + a_{n-m}$ when $1 < m < n - 1$. This inequality can be shown by introducing $G(\gamma) = a_n(\gamma) + a_{n-m}(\gamma) - a_n(\gamma)$ which satisfies $G(0) = 0$ and $G(\gamma) = G(1 - \gamma)$. Therefore, one needs to show that $G'((\gamma) = m(b_m - b_n) + (n - m)[b_{n-m} - b_n] > 0$ for $0 < \gamma < 1/2$ with $b_n \equiv b_n(\gamma) = (1 - \gamma)^{n-1} - \gamma^{n-1}$. One can verify that the $b_n$'s decrease monotonically with increasing $n$, $b_n \geq b_{n+1}$ for $n \geq 2$, therefore proving the desired inequality. Since moments decay exponentially, this inequality shows that the right hand side in the above equation is negligible asymptotically. Thus, the leading asymptotic behavior for all $n > 0$ is $M_n \sim \exp(-a_n t)$. Since the $a_n$'s increase monotonically, $a_n < a_{n+1}$, the moments decrease monotonically in the long time limit, $M_n > M_{n+1}$. Furthermore, in terms of the second moment one has

$$
M_n \propto M_2^{\alpha_n}, \quad \alpha_n = \frac{1 - (1 - \gamma)^n - \gamma^n}{1 - (1 - \gamma)^2 - \gamma^2}.
$$

While the prefactors depend on the details of the initial distribution, the scaling exponents are universal. Therefore, the velocity distribution does not follow a naive scaling form $P(v, t) = \frac{1}{t} P(\frac{v}{t})$. Such a distribution would imply the linear exponents $\alpha_n = \alpha_n^* = n/2$. Instead, the actual behavior is given by Eq. (9) with the exponents $\alpha_n$ reflecting a multiscaling asymptotic behavior with a nontrivial (non-linear) dependence on the index $n$. For instance, the high order exponents saturate, $\alpha_n \to a_2^{-1}$ for $n \to \infty$, instead of diverging. One may quantify the deviation from ordinary scaling via a properly normalized set of indices $\beta_n = \alpha_n / \alpha_n^*$ defined from $M_n^{1/n} \sim (v^*)^2$. By evaluating the $\gamma = 1/2$ case where multiscaling is most pronounced, a bound can be obtained for these indices: $7/8, 31/48 \leq \beta_n \leq 1$ for $n = 4, 6$ respectively. Furthermore, $\beta_2 \to 1 - \frac{\alpha_2}{2} \gamma$ when $\gamma \to 0$ indicating that the deviation from ordinary scaling vanishes for weakly inelastic collisions. Thus, the multiscaling behavior can be quite subtle.

The above shows that a hierarchy of scales underlies fluctuations in the velocity. In parallel, a hierarchy of diverging time scales characterizes velocity fluctuations

$$
M_n^{1/n} \sim \exp(-t/\tau_n), \quad \tau_n = \frac{n}{\alpha_n}.
$$

These time scales diverge for large $n$ according to $\tau_n \approx n$. Large moments reflect the large velocity tail of a distribution. Indeed, the distribution of extremely large velocities is dominated by persistent particles which experienced no collisions up time $t$. The probability for such events decays exponentially with time $P(v, t) \sim P(v, 0) \exp(-t)$ for $v \gg 1$ (alternatively, this behavior emerges from Eq. (9) since the gain term is negligible for the tail and hence $\dot{P} + P = 0$). This decay is consistent with the large order moment decay $M_n \sim \exp(-t)$ when $n \to \infty$.

**FIG. 1.** Development of a singularity for a compact initial distribution. Shown is the probability distribution obtained by simulating the collision process of Eq. (9) with $\gamma = 1/2$. The data represents an average over 200 independent realizations in a system with $10^7$ particles, starting from a uniform distribution in the range $[-1 : 1]$.

Although the leading asymptotic behavior of the moments was established, understanding the entire distribution $P(v, t)$ remains a challenge. Simulations of the $\gamma = 1/2$ process reveal an interesting structure for compact distributions. Starting from a uniform velocity distribution, $P_0(v) = 1/2$ for $-1 < v < 1$, the distribution loses analyticity at $v = \pm1/2$. Our analysis of Eq. (9) shows that such a singularity should indeed develop at $v = \pm1/2$ and it additionally implies the appearance of (progressively weaker and weaker) singularities at $v = \pm1/4$, etc. More generally, for an arbitrary compact initial distribution and an arbitrary $\gamma$, the distribution $P(v, t)$ loses analyticity for $t > 0$ and develops an infinite (countable) set of singularities whose locations depend on the arithmetic nature of $\gamma$ (e.g., it is very different for rational and irrational $\gamma$'s). On the other hand, un-
bounded distributions do not develop such singularities, and therefore, the loss of analyticity is not necessarily responsible for the multiscaling behavior.

Asymptotically, our system reaches a trivial steady state \( P(v, t = \infty) = \delta(v) \). To examine the relation between dynamics and statics, a non-trivial steady state can be generated by considering the driven version of our model [11]. External forcing balances dissipation due to collisions and therefore results in a non-trivial nonequilibrium steady state. Specifically, we assume that in addition to changes due to collisions, velocities may also change due to an external forcing: 

\[
\frac{dv}{dt}_{\text{heat}} = \xi. 
\]

We assume standard uncorrelated white noise \( \langle \xi(t)\xi(t') \rangle = 2D\delta(t-t') \) with zero average \( \langle \xi \rangle = 0 \). The left hand side of the master equation (3) should therefore be modified by the diffusion term

\[
\frac{\partial P(v,t)}{\partial t} \rightarrow \frac{\partial P(v,t)}{\partial t} - D \frac{\partial^2 P(v,t)}{\partial v^2}. 
\] (11)

Of course, the addition of the diffusive term does not alter conservation of the total particle number and the total momentum, and one can safely work in a reference frame moving with the center of mass velocity.

We restrict our attention to the steady state, obtained by setting the time derivative to zero. The corresponding Fourier transform \( \hat{P}_\infty(k) \equiv \hat{P}(k, t = \infty) \) satisfies

\[
(1 + Dk^2)\hat{P}_\infty(k) = \hat{P}_\infty[|\kappa|k]\hat{P}_\infty[(1 - \gamma)k]. 
\] (12)

The solution to this functional equation which obeys the conservation laws \( \hat{P}_\infty(0) = 1 \) and \( \langle v \rangle = \hat{P}_\infty(0) = 0 \) is found recursively

\[
\hat{P}_\infty(k) = \prod_{i=0}^{\infty} \prod_{j=0}^{i} \left[ 1 + \gamma^2i(1 - \gamma)^2(i-j)Dk^2 \right]^{-i}. 
\] (13)

To simplify this double product we take the logarithm and transform it as follows

\[
\ln \hat{P}_\infty(k) = -\sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{i}{j} \ln \left[ 1 + \gamma^2j(1 - \gamma)^2(i-j)Dk^2 \right] 
\]

\[
= \sum_{n=1}^{\infty} \sum_{i=0}^{n} \binom{i}{j} \frac{(-Dk^2)^n \gamma^{2jn} (1 - \gamma)^{2(i-j)n}}{n} 
\]

\[
= \sum_{n=1}^{\infty} \frac{(-Dk^2)^n \gamma^{2jn} (1 - \gamma)^{2(n-i)}}{n} 
\]

\[
= \sum_{n=1}^{\infty} \frac{(-Dk^2)^n \gamma^{2n + (1 - \gamma)^{2n}}}{n}. 
\] (14)

The second identity follows from the series expansion

\[
\ln(1 + q) = -\sum_{n \geq 1} n^{-1} (-q)^n, \quad \text{and the forth from the binomial identity } \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} = (p + q)^n. \]

Finally, using the geometric series \( (1-x)^{-1} = \sum_{n \geq 0} x^n \), the Fourier transform at the steady state is found

\[
\hat{P}_\infty(k) = \exp \left\{ \sum_{n=1}^{\infty} \frac{(-Dk^2)^n}{n a_2n(\gamma)} \right\}, 
\] (15)

with \( a_2(\gamma) \) given by Eq. (1). The \( n \)th cumulant of the steady state distribution \( \kappa_n \) can be readily found from

\[
\ln \hat{P}_\infty(k) = \sum_{n=2}^{\infty} \frac{(-Dk^2)^n}{n \kappa_n}. 
\]

Therefore, the odd cumulants vanish while the even cumulants are simply proportional to the time scales characterizing the exponential relaxation of the corresponding moments:

\[
\kappa_{2n} = \frac{(2n-1)!}{n} D^n \gamma_{2n}. 
\] (16)

Of course, the moments can be constructed from these cumulants. Interestingly, a direct correspondence between the steady state characteristics and the nonequilibrium relaxation time scales is established via the cumulants of the probability distribution.

None of the (even) cumulants vanish, thereby reflecting significant deviations from a Gaussian distribution. Nevertheless, for sufficiently large velocities, one may concentrate on the small wave number behavior. Using the inverse Fourier transform of \( \hat{P}_\infty(k) \) one finds the tail of the distribution

\[
P_\infty(v) \simeq \sqrt{\frac{a_2}{4\pi D}} \exp \left\{ -\frac{a_2 v^2}{4D} \right\}, \quad v \gg \sqrt{D/a_2}. 
\] (17)

This in particular implies the large moment behavior \( M_{2n} \to (2n-1)!/(4D/a_2)^n \) as \( n \to \infty \).

To examine how general is the above behavior, we briefly discuss a few generalizations and extensions of the basic model. Relaxing Galilean invariance, the most general momentum conserving collision rule is

\[
\left( v_1 \gamma_1 \begin{array}{c} 1 \\gamma_1 \end{array} \begin{array}{c} 1 - \gamma_1 \gamma_2 \end{array} \left( u_1 \begin{array}{c} u_2 \end{array} \right) \right). 
\] (18)

Following the same steps that led to (1) shows that when \( \gamma_1, \gamma_2 \neq 0, 1 \) and when \( M_1 = 0 \) this process also exhibits multiscaling with the exponents \( a_n = a_n/\sqrt{D} \), where \( a_n(\gamma_1, \gamma_2) = \frac{1}{\sqrt{2}} \left[ a_n(\gamma_1) + a_n(\gamma_2) \right] \). When \( \gamma_1 = 1 - \gamma_2 = \gamma \) one recovers the model introduced by Melzak [4], and when \( \gamma_1 = \gamma_2 = \gamma \) one recovers inelastic collisions. Since \( a_n(\gamma) = a_n(1-\gamma) \) both models have identical multiscaling exponents. Furthermore, a multiscaling behavior with the very same exponents \( a_n(\gamma) \) is also found for the following process \( (u_1, u_2) \to (u_1 - \gamma_1 v_1, v_1 + \gamma_2 u_2) \) investigated in the context of asset distributions [7] and headway distributions in traffic flows [8].

One can also consider stochastic rather than deterministic collision processes by assuming that the collision (5) occurs with probability density \( \sigma_1(\gamma_1, \gamma_2) \). Our findings extend to this model as well and the multiscaling exponents are given by the same general expression \( a_n = a_n/\sqrt{D} \) with \( a_n = \int d\gamma_1 \int d\gamma_2 \sigma(\gamma_1, \gamma_2) a_n(\gamma_1, \gamma_2) \). In particular, for completely random inelastic collisions, i.e.,
\( \sigma \equiv 1 \) and \( \gamma_1 = \gamma_2 = \gamma \), one finds \( a_n = \frac{n!}{n^{n+1}} \) and hence \( \alpha_n = \frac{3n}{n+1} \).

So far, we discussed only two-body interactions. We therefore consider \( N \)-body interactions where a collision is symbolized by \( (u_1, \ldots, u_N) \rightarrow (v_1, \ldots, v_N) \). We consider a generalization of the \( \gamma = \frac{1}{2} \) two-body case where the post-collision velocities are all equal. Momentum conservation implies \( v_1 = \bar{u} = N^{-1} \sum u_i \). The master equation is a straightforward generalization of the two-body case and we merely quote the moment equations

\[
M_n + a_n M_n = N^{-n} \sum_{n_i \neq 1} \binom{n}{n_1 \ldots n_N} M_{n_1} \cdots M_{n_N} \quad (19)
\]

with \( a_n = 1 - N^{1-n} \). Using the inequality \( a_n < a_n + a_{n-m} \) for all \( 1 < m < n-1 \), and its kin like \( a_n < a_{n_1} + a_{m_2} + a_{n-m_1-m_2} \) for all \( 1 < m_1, m_2 < n-1 \), etc., we find that the right-hand side of the above equation remains asymptotically negligible. Therefore, \( M_n \sim e^{-\alpha_n t} \) and

\[
M_n \sim M_2^{\alpha_n}, \quad \alpha_n = \frac{1 - N^{1-n}}{1 - N^{-1}}. \quad (20)
\]

Thus, this \( N \)-body “averaging” process exhibits multiscale asymptotic behavior as well.

Thus far, we considered the behavior on a mean field level, i.e., in an infinite dimensional space. It is natural to consider the finite-dimensional counterpart. Specifically, we assume that particles reside on a \( d \) dimensional lattice and that only nearest neighbors interact. Here, the above dynamics is essentially equivalent to a diffusion process \([3]\). As a result, the underlying correlation length is diffusive, \( L(t) \sim t^{1/2} \). Within this correlation length the velocities are “well mixed” and momentum conservation therefore implies that \( v \sim L^{-d/2} \sim t^{-d/4} \).

Indeed, the infinite dimension limit is consistent with the above exponential decay. Furthermore, an exact solution for moments of arbitrary order is possible \([3]\). We do not detail it here and simply quote that ordinary scaling is restored \( M_n \sim t^{-n/4} \), i.e. \( \alpha_n = \alpha_n^* = n/2 \). Thus, spatial correlations counter the mechanism responsible for multiscale.

In summary, we have investigated inelastic collision processes in infinite dimension. We have shown that such systems are characterized by multiscale, or equivalently by an infinite hierarchy of diverging time scales. Multiscale holds for several generalizations of the basic model including stochastic collision models and even processes which do not obey Galilean invariance. In this latter case, however, multiscale is restricted to situations with zero total momentum. This perhaps explains why multiscale asymptotic behavior was overlooked in previous studies \([\text{[1]}]\). Another explanation is that this behavior may be difficult to detect from numerical simulations. Indeed, in other problems such as multidimensional fragmentation \([\text{[1]}]\), and in fluid turbulence, low order moments deviate only slightly from the normal scaling expectation.

There are a number of extensions of this work which are worth pursuing. We have started with a simplified model of a 1D granular gas with a velocity independent collision rate. One possibility is to approximate the collision rate with the RMS velocity fluctuation. This leads to the algebraic decay \( M_n \sim t^{-2\alpha_n} \) with \( \alpha_n \) given by Eq. \( (1) \) and in particular, Haff’s cooling law \( T = M_2 \sim t^{-2} \) is recovered \([\text{[1]}]\). Our numerical studies indicate that when velocity dependent collision rates are implemented, ordinary scaling behavior is restored. One may also use this model as an approximation for inelastic collisions in higher dimensions as well, following the Maxwell approximation in kinetic theory \([\text{[3]}][\text{[5]}]\).

This research was supported by the DOE (W-7405-ENG-36), NSF (DMR9632059), and ARO (DAAH04-96-1-0114).

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