Multi-plaquette solutions for discretized Ashtekar gravity

Kiyoshi Ezawa

Department of Physics
Osaka University, Toyonaka, Osaka 560, Japan

Abstract

A discretized version of canonical quantum gravity proposed by Loll is investigated. After slightly modifying Loll’s discretized Hamiltonian constraint, we encode its action on the spin network states in terms of combinatorial topological manipulations of the lattice loops. Using this topological formulation we find new solutions to the discretized Wheeler-Dewitt equation. These solutions have their support on the connected set of plaquettes. We also show that these solutions are not normalizable with respect to the induced heat-kernel measure on $SL(2, \mathbb{C})$ gauge theories.

*Supported by JSPS. e-mail address: ezawa@funp.th.phys.sci.osaka-u.ac.jp
As an approach to quantize gravity nonperturbatively, canonical quantization of general relativity has been investigated for more than thirty years. In the conventional metric formulation [1], we have not yet found any solutions for its basic equation, i.e. the Wheeler-Dewitt (WD) equation [2] (or the Hamiltonian constraint). This had been one of the serious obstructions against smooth progress in this approach for a long time. The situation was drastically changed after the discovery of Ashtekar’s new canonical variables in 1986 [3]. Ashtekar’s variables consist of the complex-valued \( SU(2) \) connection \( A^i_a \) and the densitized triad \( \tilde{E}^{ia} \). Using these variables the Hamiltonian constraint takes the simple form:

\[
H = \epsilon^{ijk} F_{ab}^{i} \tilde{E}^{ja} \tilde{E}^{kb},
\]

where \( F_{ab}^{i} \) is the curvature of the connection \( A^i_a \). The WD equation in terms of new variables are therefore expected to have solutions. Indeed several types of solutions has been constructed using Wilson loops defined on smooth loops with or without intersections [4][5][6][7].

These solutions are, however, of little interest. Because they are already eliminated by the operator \( \epsilon^{ijk} \epsilon_{abc} \tilde{E}^{ja} \tilde{E}^{kb} \), naively we consider that they correspond to the states with degenerate metric. This suggests that we have to search for the solutions on which the action of the curvature plays an essential role. In terms of Wilson loops, multiplication by the curvature is encoded by the action of the area derivative [8]. It is therefore important to define the area derivative without any ambiguity. This seems to be a nontrivial problem in the continuum approach.

In the lattice formulation, we can in principle express the area derivative by the operation of inserting a plaquette to the lattice loop arguments. The lattice formulation therefore deserves studying as a heuristic model of the continuum approach.

A discretized version of Ashtekar’s formalism was proposed by Loll [9]. This model is defined on a 3 dimensional cubic lattice of size N. We will follow the notations in ref. [9] and label lattice sites by \( n \) and three positive directions of links by \( \hat{a} \). The connection \( A^i_a \) is replaced by the link variables \( V(n, \hat{a}) \) which takes the value in \( SL(2, \mathbb{C}) \) and the conjugate momenta \( \tilde{E}^{ia} \) is replaced by the left translation operator \( p_i(n, \hat{a}) \). Their commutation
relations are:

\[ [V(n, \hat{a}), V(m, \hat{b})] = 0, \quad [p_i(n, \hat{a}), V(m, \hat{b})] = -\frac{i}{2} \delta_{n,m} \delta_{\hat{a} \hat{b}} (\tau_i V(m, \hat{b})), \]
\[ [p_i(n, \hat{a}), p_j(m, \hat{b})] = i \delta_{n,m} \delta_{\hat{a} \hat{b}} \epsilon_{ijk} p_k(n, \hat{b}), \]

where \( \tau_i \) equals to \(-i\) times of the Pauli matrices.

Among the three constraints in Ashtekar’s formalism, the Gauss’ law constraint is solved by considering only the gauge-invariant functionals of link variables, namely, spin network states. The diffeomorphism constraint is formally solved by regarding our lattice to be a purely topological object (see also [13]). Thus we are left only with the Hamiltonian constraint, whose discretized form proposed by Loll is:

\[ H^C(n) = \sum_{\hat{a} < \hat{b}} \epsilon_{ijk} p_i(n, \hat{a}) p_j(n, \hat{b}) \text{Tr}(V(n, P_{\hat{a} \hat{b}}) \tau_k), \]  

(2)

where \( V(n, P_{\hat{a} \hat{b}}) \equiv V(n, \hat{a}) V(n + \hat{a}, \hat{b}) V(n + \hat{b}, \hat{a})^{-1} V(N, \hat{b})^{-1} \) denotes a plaquette loop.

This definition obviously lacks symmetry; only positive directions emanating from the site \( n \) are subject to the action of \( H^C(n) \). This is not desirable because there are privileged directions in the world.

In order to provide symmetric forms of discretized Hamiltonian constraint, we first introduce link variables in the negative direction \( V(n, -\hat{a}) \equiv V(n - \hat{a}, \hat{a})^{-1} \) and the right translation operator \( p_i(n, -\hat{a}) \):

\[ p_i(n, -\hat{a}) V(n, -\hat{a}) = -\frac{i}{2} \tau_i V(n, -\hat{a}), \quad p_i(n, -\hat{a}) V(n - \hat{a}, \hat{a}) = \frac{i}{2} V(n - \hat{a}, \hat{a}) \tau_i. \]  

(3)

Naively considering, we think of two candidates for the symmetric discretized Hamiltonian constraint. One is

\[ H^C_I(n) = \sum_{\hat{a} < \hat{b}} \epsilon_{ijk} \text{Tr}(\tilde{V}(n, \hat{a} \hat{b}) \tau_k) (p_i(n, \hat{a}) - p_i(n, -\hat{a}))(p_j(n, \hat{b}) - p_j(n, -\hat{b})), \]  

(4)

where \( \tilde{V}(n, \hat{a} \hat{b}) \equiv \frac{1}{2} (V(n, P_{\hat{a} \hat{b}}) + V(n, P_{\hat{b} \hat{a}}) + V(n, P_{\hat{a} \hat{b}}) + V(n, P_{\hat{b} \hat{a}})) \). The other is

\[ H^C_{II}(n) = \sum_{\hat{a} < \hat{b}} \sum_{m,n_2 = \pm} \epsilon_{ijk} \text{Tr}(V(n, P_{\eta_1 \eta_2 \hat{a} \hat{b}})) p_i(n, \eta_1 \hat{a}) p_j(n, \eta_2 \hat{b}). \]  

(5)

Probably \( H^C_I \) is more preferable than \( H^C_{II} \) because the area derivative in the former is uniformly expressed by the insertion of \( \frac{1}{2}(\tilde{V} - \tilde{V}^{-1}) \). Indeed the result of the action of \( H^C_I \)
is identical, up to the overall factor, to the action of the continuum Hamiltonian under the regularization used in ref.[7]. The WD equation using $H_C^I$ therefore has solutions which are the lattice analog of the solutions found in refs.[4][5][6][7], provided that the smooth loops are replaced by straight Polyakov loops.

Our purpose is, however, to find out “nontrivial solutions” which becomes the solution only after the area derivative is taken into account. To this end it is much easier to use $H_C^I$, because the terms appearing in $H_C^I$ is naively four times as many as those in $H_C^I$. While $H_C^I$ may be less suitable to regularize the continuum theory, we expect that it will provide some essential lessons concerning to the nontrivial solutions. For these reasons we will henceforth deal only with $H_C^I$.

The action of the discretized Hamiltonian constraint can be computed by using only the $SL(2, \mathbb{C})$ algebra. The two identities are particularly useful:

$$\tau_i \tau_j = -\delta_{ij} + \epsilon_{ijk} \tau_k, \quad (\tau_i)_A^B (\tau_i)_C^D = \delta_A^B \delta_C^D - 2 \delta_A^D \delta_C^B. \quad (6)$$

As in the continuum case, the nonvanishing contributions are obtained only from the series of link variables with kinks or intersections. Because the Hamiltonian constraint involves second order derivative, it is convenient to separate its action as follows:

$$H_C^I(n) = H_C^I(n)_1 + H_C^I(n)_2, \quad (7)$$

where $H_C^I(n)_1$ is the sum of the action on the single series and $H_C^I(n)_2$ is the sum of the action on the pairs of series. The problem of evaluating the action of the discretized Hamiltonian constraint $H_C^I$ is thus reduced to that of calculating the action on all possible types of kinks and intersections involving at most two series of links.

For example, the action on kinks is as follows ($\hat{a} \neq \hat{b}$):

$$H_C^I(n)_1 \cdot V(n - \hat{a}, \hat{a})V(n, \hat{b}) = -\frac{1}{2} V(n - \hat{a}, \hat{a})(V(n, P_{\hat{a}, \hat{b}}) - V(n, P_{\hat{b}, \hat{a}}))V(n, \hat{b}),$$

$$H_C^I(n)_2 \cdot (V(n - \hat{a}, \hat{a})V(n, \hat{b}) \otimes V(n - \hat{a}, \hat{a})V(n, \hat{b})) = 0. \quad (8)$$

Topologically, the former action can be interpreted as taking the difference of two operations of inserting plaquettes with the opposite orientations. The action on the other types of vertices can also be interpreted in terms of combinatorial topology. We will depict the action on some typical vertices in figure1, where the bold lines stand for the series of
link variables. Once we express the action of $H_{II}^C$ in terms of combinatorial topology, the orientation of the curve is irrelevant because of the symmetry of $H_{II}^C$.

A merit of this topological formulation is that we can visualize the action of the Hamiltonian constraint. It is expected that we can fully exploit this merit in finding solutions to the WD equation\textsuperscript{1}.

As an exercise we will provide a set of the simplest “nontrivial solutions” on which

\textsuperscript{1} A weakness of the topological formulation is that we have to work with the overcomplete basis of wavefunctions. In this respect, it would be better to describe the action of the Hamiltonian constraint in terms solely of spin network states, which are known to form a complete basis\textsuperscript{11,14}. In the spin network description, however, we have to deal with tedious linear combinations consisting of considerably many spin network states. This description is not considered to be suitable for the visual search of solutions, while it may be useful for computer analysis.
the contribution of area derivative is essential. We first consider the action on the trace of the \( l \)-th power of the plaquette \( \text{Tr}(V(n, P_\hat{ab})^l) \). This is calculated by using eq.(8)

\[
H_{II}^C(n) \cdot \text{Tr}(V(n, P_\hat{ab})^l) = \frac{l}{2} \left( \text{Tr}(V(n, P_\hat{ab})^{l+1}) - \text{Tr}(V(n, P_\hat{ab})^{l-1}) \right).
\]

(9)

This seems to imply the following equation

\[
H_{II}^C(n) \cdot \sum_{k=1}^{\infty} \frac{-2}{2k+1} \text{Tr}(V(n, P_\hat{ab})^{2k+1}) = 2.
\]

(10)

While the issue of the convergence remains, this can be resolved by exploiting the idea of analytic continuation. More explicitly we consider as follows. First we reinterpret eq.(9) as

\[
H_{II}^C(n) \cdot \text{Tr} F(V(n, P_\hat{ab})) = \text{Tr} \left[ \frac{(V(n, P_\hat{ab})^2 - 1}{2} \frac{d}{dV} F(V(n, P_\hat{ab})) \right],
\]

(11)

where \( F \) denotes an arbitrary polynomial. We can readily extend this equation to the case where \( F \) is a function which can be expressed by a Laurent series. Thus we find

\[
H_{II}^C(n) \cdot \text{Tr} \log \left( \frac{1 - V(n, P_\hat{ab})}{1 + V(n, P_\hat{ab})} \right) = 2.
\]

(12)

The power series expansion of the expression on the l.h.s yields the l.h.s of eq.(10). As for the action of \( H_{II}^C(m) \) with \( m \neq n \), the following can be said. When \( m \) coincides with one of the vertices of the plaquette \( P_\hat{ab} \), the result is identical to eq.(12) owing to the symmetry of \( H_{II}^C \). When \( m \) does not coincide, on the other hand, the action necessarily vanishes. Putting these results together, we find

\[
H_{II}^C(m) \cdot \text{Tr} \log \left( \frac{1 - V(n, P_\hat{ab})}{1 + V(n, P_\hat{ab})} \right) = \begin{cases} 2 & \text{for } m = n, n + \hat{a}, n + \hat{b}, n + \hat{a} + \hat{b}, \\ 0 & \text{for } m \neq n, n + \hat{a}, n + \hat{b}, n + \hat{a} + \hat{b}. \end{cases}
\]

(13)

Now we can provide the prescription for constructing “multi-plaquette solutions” on which the action of the area derivative is essential: i) prepare a connected set of plaquettes \( \{P\} \) in which each vertex belongs to at least two plaquettes; ii) assign to each plaquette \( P \) a weight factor \( w(P) \) so that the sum of weight factors of the plaquettes which meet at each vertex vanishes; iii) the following expression yields a solution

\[
< A|\{w(P)\} > \equiv \sum_{P \in \{P\}} w(P) \text{Tr} \log \left( \frac{1 - V(P)}{1 + V(P)} \right).
\]

(14)
The two simplest assignments of the weights \( \{w(P)\} \) are shown in figure 2. Because the product \( < A|\{w(P)\} > < A|\{w(P')\} > \) is also a solution if the sets \( \{P\} \) and \( \{P'\} \) are disconnected, we found a considerably large number of solutions to the discretized WD equation (5).

One may wonder whether or not these solutions are normalizable with respect to an appropriate inner product. To investigate this problem, it is convenient to translate the result into spin network states. In the present case, they are nothing but symmetrized traces

\[
\text{Tr} S(V(P)^k) \equiv V(P)_{A_1}^{B_1} \cdots V(P)_{A_k}^{B_k}.
\]

The action of \( H_{II}^C \) on these states are calculated as

\[
H_{II}^C(n) \text{Tr} S(V(n, P_{\hat{a}\hat{b}})^k) = \frac{k}{2} S(V(n, P_{\hat{a}\hat{b}})^{k+1}) - \frac{k + 2}{2} S(V(n, P_{\hat{a}\hat{b}})^{k-1})
\]

Thus, to obtain multi-plaquette solutions \( < A|\{w(P)\} >_S \) in the spin network representation, we have only to replace \( \log(\frac{1-V}{1+V}) \) in eq.(14) by

\[
< A|P >_S \equiv \sum_{m=0}^{\infty} \frac{1}{(2m + 3)(2m + 1)} \text{Tr} S(V(P)^{2m+1}). \tag{15}
\]

\footnote{We can construct a set of solutions to (3) by a similar procedure. However, the analogous prescription cannot apply to eq.(4). From this we expect that the continuum limit of \( < A|\{w(P)\} > \) are not solutions to the continuum WD equation, at least in a naive regularization.}
Let us now investigate the normalizability of multi-plaquette solutions. We first look into the induced Haar measure\[11\][15] regarding our model as a sort of $SU(2)$ gauge theory. Owing to the consistency property of the induced Haar measure and the orthogonality of the spin network states with different numbers of link variables, we have only to investigate the norm of a one-plaquette state (15)(which is not a solution)

$$∥ < A|P >_S ∥_H = \int dμ_H(V(P))| < A|P >_S |^2,$$

where $dμ_H$ is the Haar measure on $SU(2)$. We estimate this norm as follows. Using bi-$SU(2)$ invariance of $dμ_H$: $dμ_H(gVh) = dμ_H(V)$ with $V, g, h ∈ SU(2)$, and the fundamental identity $ε^{AC}V_A^BV_C^D = ε^{BD}$, the integration by the Haar measure is determined uniquely

$$\int dμ_H(V)\prod_{i=1}^{2n}V_{A_i}^{B_i} = \frac{1}{2^n(n+1)!n!} \sum_{σ ∈ P_{2n}} \prod_{k=1}^{n} ε_{A_2k-1}^A B_{σ_{2k-1}}^B ε_{B_{2k-1}}^{B_σ_{2k}},$$

where $P_{2n}$ is the group of permutations of $2n$ entries. The integration of $\prod_{i=1}^{2n+1}V_{A_i}^{B_i}$ vanishes identically. From this equation we find

$$\int dμ_H(V)\overline{TrS(V^n)}TrS(V^m) = δ_{nm},$$

where the bar denotes the complex conjugate. The desired norm is calculated from eqs. (15) (17). The result is

$$∥ < A|P >_S ∥_H = ∑_{m=0}^∞ \frac{1}{(2m+3)^2(2m+1)^2} < ∞,$$

i.e. the multi-plaquette solutions are normalizable w.r.t. the induced Haar measure.

Next we look into the induced heat-kernel measure $dν_t$ on $SL(2, C)$ gauge theories. Because $dν_t$ is also bi-$SU(2)$ invariant and possesses the consistency property, we can show that the spin network states with different numbers of link variables are orthonormal w.r.t. $dν_t$, in particular

$$\int dν_t(V)\overline{TrS(V^n)}TrS(V^m) = C(n)δ_{nm}. $$

There seems to be no algebraic principle which determines the constant factor $C(n)$. Using some facts on the coherent-state transform $C_t : L^2(SU(2), dμ_H) → L^2(SL(2, C), dν_t)^H$ (Theorem 2 and eq.(30) of ref.\[17\]), however, we can exactly estimate the constant factor

$$C(n) = e^{n(n+2)/4}t.$$
As a result, the one-plaquette states are not normalizable w.r.t. the induced heat-kernel measure $d\nu_t$:

$$
\int d\nu_t(V(P))| < A|P >^2_S \propto \sum_{m=0}^{\infty} e^{\frac{t(n+2)}{4}m} (2m+3)(2m+1)^2 \rightarrow \infty.
$$

(21)

We should note that the multi-plaquette solutions do not correspond to “geometrodynamical states” with nondegenerate metric. This is because the volume operators have vanishing eigenvalues on these solutions. In order to construct geometrodynamical solutions with nonvanishing volume, we have to consider the lattice-loop states which have three dimensional vertex of at least four-valent on almost every site. This is a highly non-trivial task and left to the future investigation. We saw, however, that the the action of the Hamiltonian constraint on a multi-plaquette solution completely cancels only when we consider an infinite number of terms. This seems to be the origin of non-normalizability of the multi-plaquette solutions w.r.t. the induced heat-kernel measure. We anticipate that this “cancelation of the action of the Hamiltonian constraint by an infinite number of terms” is a common feature of the nontrivial solutions involving geometrodynamical solutions. We therefore conjecture that the geometrodynamical solutions are, if any, not normalizable w.r.t. the induced heat-kernel measure on $SL(2, \mathbb{C})$ while they may be normalizable w.r.t. the induced Haar measure on $SU(2)$. But this should not be taken so seriously, because there still remains a gauge degree of freedom generated by the Hamiltonian constraint and because the induced heat kernel measure is thought to be a kind of “kinematical inner product” which does not take account of this gauge symmetry. It is often the case that the physical states are not normalisable w.r.t. these kinematical inner products. The problem is then to find a genuine physical inner product which makes the physical states normalizable and which implements the reality conditions.

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References

[1] R Arnowitt, S. Deser and C. W. Misner, in “Gravitation, An Introduction to Current Research”, ed. by L. Witten (John Willey and Sons, 1962) Chap. 7

[2] B. S. Dewitt, Phys. Lev. 160 (1967) 1113

[3] A. Ashtekar, Phys. Rev. Lett. 57 (1986) 2244 ; Phys. Rev. D36 (1987) 295

[4] T. Jacobson and L. Smolin, Nucl. Phys. B299 (1988) 295

[5] V. Husain, Nucl. Phys. B313 (1989) 711

[6] B. Brügmann and J. Pullin, Nucl. Phys. B363 (1991) 221

[7] K. Ezawa, OU-HET/217, gr-qc/9506043, to appear in Nucl. Phys. B

[8] S. Mandelstam, Phys. Rev. 175 (1968) 1580;
   Yu. M. Makeenko and A. A. Migdal, Nucl. Phys. B188 (1981) 269

[9] R. Loll, Nucl. Phys. B444 (1995) 619

[10] R. Penrose, in “Quantum Theory and Beyond”, ed. by. T. Bastin (Cambridge Univ. Press, Cambridge, 1971)

[11] J. C. Baez, “Spin networks in Nonperturbative Quantum Gravity”, gr-qc/9504036;
   “Spin Network States in Gauge Theory”, gr-qc/9411007

[12] C. Rovelli and L. Smolin, “Spin networks and quantum gravity”, gr-qc/9505006, to appear in Phys. Rev. D

[13] A. Ashtekar, J. Lewandowsli, D. Marolf, J. Mourão and T. Thiemann, “Quantization of diffeomorphism invariant theories of connections with local degrees of freedom” gr-qc/9504018

[14] W. Furmanski and A. Kolawa, Nucl. Phys. B291 (1987) 594;
   J. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395

[15] A. Ashtekar and J. Lewandowski, J. Math. Phys. 36 (1995) 2170
[16] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão and T. Thiemann, “Coherent State transform for Spaces of Connections”, gr-qc/9412014

[17] B. C. Hall, J. Func. Anal. 122 (1994) 103

[18] C. Rovelli and L. Smolin, Nucl. Phys. B442 (1995) 593;
    A. Ashtekar, C. Rovelli and L. Smolin, Phys. Rev. Lett. 69 (1992) 446

[19] R. Loll, “the Volume Operator in Discretized Quantum Gravity”, DFF 228/05/95, gr-qc/9506014
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