Composition operators acting on weighted Hilbert spaces of analytic functions

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Abstract. In this paper, we consider composition operators on weighted Hilbert spaces of analytic functions and observe that a formula for the essential norm, give a Hilbert-Schmidt characterization and characterize the membership in Schatten-class for these operators. Also, closed range composition operators are investigated.

1. Introduction

Let \( \mathbb{D} \) denotes the open unit disk \( \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \varphi \) be an analytic self map of \( \mathbb{D} \). The composition operator \( C_\varphi \) induced by \( \varphi \) is defined \( C_\varphi f = f \circ \varphi \), for any \( f \in H(\mathbb{D}) \), the space of all analytic functions on \( \mathbb{D} \). This operator can be generalized to the weighted composition operator \( uC_\varphi \), \( uC_\varphi f(z) = u(z)f(\varphi(z)) \), \( u \in H(\mathbb{D}) \). We consider a weight as a positive integrable function \( \omega \in C^2(0,1) \) which is radial, \( \omega(z) = \omega(|z|) \). The weighted Hilbert space of analytic functions \( \mathcal{H}_\omega \) consists of all analytic functions on \( \mathbb{D} \) such that

\[
||f'||^2_{\mathcal{H}_\omega} = \int_\mathbb{D} |f'(z)|^2 \omega(z) \, dA(z) < \infty,
\]

equipped with the norm \( ||f||^2_{\mathcal{H}_\omega} = |f(0)|^2 + ||f'||^2_{\mathcal{H}_\omega} \). Here \( dA \) is the normalized area measure on \( \mathbb{D} \). Also the weighted Bergman spaces defined by

\[
\mathcal{A}_\omega^p = \left\{ f \in H(\mathbb{D}) : ||f||^2_{\mathcal{H}_\omega} = \int_\mathbb{D} |f(z)|^2 \omega(z) \, dA(z) < \infty \right\}.
\]

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If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then \( f \in \mathcal{H}_\omega \) if and only if
\[
||f||^2_{\mathcal{H}_\omega} = \sum_{n=0}^{\infty} |a_n|^2 \omega_n < \infty,
\]
where \( \omega_0 = 1 \) and for \( n \geq 1 \)
\[
\omega_n = 2n^2 \int_0^1 r^{2n-1} \omega(r) dr,
\]
and \( f \in \mathcal{A}_\omega \) if and only if
\[
||f||^2_{\mathcal{A}_\omega} = \sum_{n=0}^{\infty} |a_n|^2 p_n < \infty,
\]
where
\[
p_n = 2 \int_0^1 r^{2n+1} \omega(r) dr, \quad n \geq 0.
\]
By letting \( \omega_\alpha(r) = (1 - r^2)^\alpha \) (standard weight), \( \alpha > -1 \), \( \mathcal{H}_{\omega_\alpha} = \mathcal{H}_\alpha \). If \( 0 \leq \alpha < 1 \), then \( \mathcal{H}_\alpha = \mathcal{D}_\alpha \), the weighted Dirichlet space, and \( \mathcal{H}_1 = \mathcal{H}^2 \), the Hardy space.

There are several papers that studied composition operators on various spaces of analytic functions. The best monographs for these operators are \([1, 7]\). In \([2]\), Kellay and Lefèvre studied composition operators on weighted Hilbert space of analytic functions by using generalized Nevanlinna counting function. They characterized boundedness and compactness of these operators. Pau and Pérez \([6]\) studied boundedness, essential norm, Schatten-class and closed range properties of these operators acting on weighted Dirichlet spaces.

Our aim in this paper is to generalize the results of \([6]\) to a large class of spaces. Throughout the remainder of this paper, \( c \) will denote a positive constant, the exact value of which will vary from one appearance to the next.

### 2. Preliminaries

In this section we give some notations and lemmas will be used in our work.

**Definition 2.1.** \([2]\) We assume that \( \omega \) is a weight function, with the following properties

\( W_1 \): \( \omega \) is non-increasing,
\( W_2 \): \( \omega(r)(1 - r)^{-(1+\delta)} \) is non-decreasing for some \( \delta > 0 \),
\( W_3 \): \( \lim_{r \to 1^-} \omega(r) = 0 \),
\( W_4 \): One of the two properties of convexity is fulfilled
\[
\begin{cases}
(W_4^{(I)}) & \text{\( \omega \) is convex and } \lim_{r \to 1^-} \omega'(r) = 0, \\
(W_4^{(II)}) & \text{\( \omega \) is concave}.
\end{cases}
\]

Such a weight \( \omega \) is called admissible.
If $\omega$ satisfies conditions $(W_1)$-$(W_3)$ and $(W_4^{(I)})$ (resp. $(W_4^{(II)})$), we shall say that $\omega$ is (I)-admissible (resp. (II)-admissible). Also we use weights satisfy (L1) condition (due to Lusky [5]):

$$(L1) \quad \inf_k \frac{\omega(1 - 2^{-k-1})}{\omega(1 - 2^{-k})} > 0.$$ 

This is equivalent to this condition (see[3]):

There are $0 < r < 1$ and $0 < c < \infty$ with $\frac{\omega(z)}{\omega(0)} \leq c$ for every $a, z \in \Delta(a, r)$, where $\Delta(a, r) = \{ z \in \mathbb{D} : |\sigma_a(z)| < r \}$ and $\sigma_a(z) = \frac{a - z}{1 - az}$ is the Mobius transformation on $\mathbb{D}$.

All characterizations in this paper are needed to the generalized counting Nevanlinna function. Let $\varphi$ be an analytic self map of $\mathbb{D}$ $(\varphi(\mathbb{D}) \subset \mathbb{D})$. The generalized counting Nevanlinna function associated to a weight $\omega$ defined as follows

$$N_{\varphi, \omega}(z) = \sum_{a : \varphi(a) = z} \omega(a), \quad z \in \mathbb{D}\{\varphi(0)\}.$$ 

By using the change of variables formula we have: If $f$ be a non-negative function on $\mathbb{D}$, then

$$\int_{\mathbb{D}} f(\varphi(z))|\varphi'(z)|^2 \omega(z) dA(z) = \int_{\mathbb{D}} f(z) N_{\varphi, \omega}(z) dA(z).$$

Also the generalized counting Nevanlinna function has the sub-mean value property (Lemmas 2.2 and 2.3 [2]). Let $\omega$ be an admissible weight. Then for every $r > 0$ and $z \in \mathbb{D}$ such that $D(z, r) \subset \mathbb{D}\{0, 1/2\}$

$$N_{\varphi, \omega}(z) \leq \frac{2}{r^2} \int_{\{z - |z| < r\}} N_{\varphi, \omega}(\zeta) dA(\zeta).$$

**Lemma 2.1.** [2] If $\omega$ is a weight satisfying $(W_1)$ and $(W_2)$, then there exists $c > 0$ such that

$$\frac{1}{c} \omega(z) \leq \omega(\sigma_{\varphi(0)}(z)) \leq c \omega(z), \quad z \in \mathbb{D}.$$

**Lemma 2.2.** [2] Let $\omega$ be a weight satisfying $(W_1)$ and $(W_2)$. Let $a \in \mathbb{D}$ and

$$f_a(z) = \frac{1}{\sqrt{\omega(a)}} \frac{(1 - |a|^2)^{1+\delta}}{(1 - az)^{1+\delta}}.$$

Then $||f_a||_{H_\omega} \asymp 1$.

### 3. Essential Norm

Recall that the essential norm $||T||_e$ of a bounded operator $T$ between Banach spaces $X$ and $Y$ is defined as the distance from $T$ to $K(X, Y)$, the space of all compact operators between $X$ and $Y$.

**Theorem 3.1.** Let $\omega$ be an admissible weight and $C_\varphi$ be a bounded operator on $H_\omega$. Then

$$||C_\varphi||_e = \limsup_{|z| \to 1^-} \frac{N_{\varphi, \omega}(z)}{\omega(z)}.$$
Proof. Consider the test function defined in Lemma 2.2. Then \( \{ f_a \}_{a \in \mathbb{D}} \) is bounded in \( \mathcal{H}_\omega \) and converges uniformly on compact subsets of \( \mathbb{D} \) to 0 as \( |a| \to 1^- \). Then for every compact operator \( K \) on \( \mathcal{H}_\omega \), \( \lim_{|a| \to 1^-} || K f_a ||_{\mathcal{H}_\omega} = 0 \). There exists a constant \( c > 0 \) such that
\[
|| C \varphi - K || \geq c \limsup_{|a| \to 1^-} || C \varphi f_a - K f_a ||_{\mathcal{H}_\omega}.
\]
Then for every compact operator \( K \) on \( \mathcal{H}_\omega \), \( \lim_{|a| \to 1^-} || K f_a ||_{\mathcal{H}_\omega} = 0 \). There exists a constant \( c > 0 \) such that
\[
|| C \varphi ||_{\mathcal{E}} \geq c \limsup_{|a| \to 1^-} || C \varphi f_a ||_{\mathcal{H}_\omega}.
\]
On the other hand
\[
|| C \varphi f_a ||^2_{\mathcal{H}_\omega} = || f_a(\varphi(0)) ||^2 + \int_{\mathcal{D}} |f'_a(\varphi(z))^2\varphi'(z)|^2 \omega(z) dA(z)
\]
\[
= || f_a(\varphi(0)) ||^2 + \int_{\mathcal{D}} |f'_a(z)|^2 N_{\varphi,\omega}(z) dA(z)
\]
\[
\geq c \frac{(1-|a|^2)^{2+2\delta}|a|^2}{\omega(a)} \int_{D(a, \frac{1-|a|}{2})} N_{\varphi,\omega}(z) dA(z).
\]
If \( |a| \) is close enough to 1, then \( \varphi(0) \notin D(a, \frac{1-|a|}{2}) \). So \( |1-\overline{z}| \approx (1-|a|) \) for \( z \in D(a, \frac{1-|a|}{2}) \). We have
\[
\limsup_{|a| \to 1^-} || C \varphi f_a ||^2_{\mathcal{H}_\omega} \geq \limsup_{|a| \to 1^-} \frac{|a|^2}{\omega(a)(1-|a|)^2} \int_{D(a, \frac{1-|a|}{2})} N_{\varphi,\omega}(z) dA(z).
\]
By the sub-mean value property of \( N_{\varphi,\omega} \), we get
\[
\limsup_{|a| \to 1^-} || C \varphi f_a ||^2_{\mathcal{H}_\omega} \geq \limsup_{|a| \to 1^-} \frac{N_{\varphi,\omega}(a)}{\omega(a)}.
\]
Now
\[
|| C \varphi ||^2_{\mathcal{E}} \geq c \limsup_{|a| \to 1^-} \frac{N_{\varphi,\omega}(a)}{\omega(a)},
\]
and the lower estimate is obtained. The upper estimate comes from p. 136 [1]. \( \square \)

Since the space of compact operators is a closed subspace of space of bounded operators, then a bounded operator \( T \) is compact if and only if \( || T ||_{\mathcal{E}} = 0 \). According to this fact we have the following corollary which is Theorem 1.4 [2].

**Corollary 3.1.** Let \( \omega \) be an admissible weight. Then \( C \varphi \) is compact on \( \mathcal{H}_\omega \) if and only if
\[
\lim_{|z| \to 1^-} \frac{N_{\varphi,\omega}(z)}{\omega(z)} = 0.
\]
4. Hilbert-Schmidt and Schatten-class

In this section we try to get a characterization of Hilbert-Schmidt composition operators. Another characterization we will have as a result of Schatten-class in the case $p = 2$.

**Theorem 4.1.** Let $\omega$ be a weight. Then $C_\varphi : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is Hilbert-Schmidt if and only if
\[
\int_{\mathbb{D}} ||R_z||^2 N_{\varphi,\omega}(z) dA(z) < \infty,
\]
where $R_z$ is the reproducing kernel of the weighted Bergman space $\mathcal{A}_2^\omega$.

**Proof.** Note that $\{z^n/||z^n||_{\mathcal{H}_\omega}\}$ is an orthonormal basis for $\mathcal{H}_\omega$. This implies that
\[
\sum_{n=1}^{\infty} \left|C_\varphi\left(\frac{z^n}{||z^n||_{\mathcal{H}_\omega}}\right)\right|^2 = \sum_{n=1}^{\infty} \int_{\mathbb{D}} \frac{n^2|\varphi(z)|^{2(n-1)}}{||z^n||_{\mathcal{H}_\omega}^2} |\varphi'(z)|^2 \omega(z) dA(z) = \sum_{n=1}^{\infty} \int_{\mathbb{D}} \frac{n^2|z|^{2(n-1)}}{||z^n||_{\mathcal{H}_\omega}^2} N_{\varphi,\omega}(z) dA(z) = \int_{\mathbb{D}} \sum_{n=1}^{\infty} \frac{n^2|z|^{2(n-1)}}{\omega_n} N_{\varphi,\omega}(z) dA(z) = \int_{\mathbb{D}} \langle ||R_z||^2 N_{\varphi,\omega}(z) dA(z) >
\]
This completes the proof. \(\square\)

For studying Schatten-class we need the Toeplitz operator. For more information about relation between Toeplitz operator and Schatten-class see [9]. Let $\psi$ be a positive function in $L^1(\mathbb{D}, dA)$ and $\omega$ be a weight. The Toeplitz operator associated to $\psi$ defined by
\[
T_\psi f(z) = \frac{1}{\omega(z)} \int_{\mathbb{D}} f(t) \psi(t) \omega(t) (1 - \overline{z}t)^2 dA(t).
\]
$T_\psi \in S_p(\mathcal{A}_2^\omega)$ if and only if the function
\[
\widehat{\psi}_r(z) = \frac{1}{(1 - |z|^2)^2 \omega(z)} \int_{\Delta(z,r)} \psi(t) \omega(t) dA(t)
\]
is in $L^p(\mathbb{D}, d\lambda)$, [4], where $d\lambda = (1 - |z|^2)^{-2} dA(z)$ is the hyperbolic measure on $\mathbb{D}$. According to the description of [6] pages 8 and 9, $C_\varphi \in S_p(\mathcal{H}_\omega)$ if and only if $\varphi' C_\varphi \in S_p(\mathcal{A}_2^\omega)$.

**Theorem 4.2.** Let $\omega$ be an admissible weight satisfy (L1) condition. Then $C_\varphi \in S_p(\mathcal{H}_\omega)$ if and only if
\[
\psi(z) = \frac{N_{\varphi,\omega}(z)}{\omega(z)} \in L^{p/2}(\mathbb{D}, d\lambda).
\]
PROOF. For any $f, g \in \mathcal{H}_\omega$ we have
\[
\langle (\varphi' C_\varphi)' (\varphi' C_\varphi) f, g \rangle = \langle (\varphi' C_\varphi) f, (\varphi' C_\varphi) g \rangle
\]
\[
= \int_D f(\varphi(z))g(\varphi(z))|\varphi'(z)|^2 \omega(z)dA(z)
\]
\[
= \int_D f(z)g(z)N_{\varphi, \omega}(z)dA(z).
\]
On the other hand, since $1/(1-|z|^2)$ is the reproducing kernel in $\mathcal{A}^2$,
\[
T_\psi f(z) = \frac{1}{\omega(z)} \int_D \frac{f(t)N_{\varphi, \omega}(t)}{(1 - |t|^2)^2} dA(t) = \frac{N_{\varphi, \omega}(z)f(z)}{\omega(z)}.
\]
Therefore
\[
(T_\psi f, g) = \int_D f(z)g(z)N_{\varphi, \omega}(z)dA(z).
\]
Thus $T_\psi = (\varphi' C_\varphi)' (\varphi' C_\varphi)$. Theorem 1.4.6 [11] implies that $\varphi' C_\varphi \in S_p(\mathcal{A}^2_p)$ if and only if $(\varphi' C_\varphi)' (\varphi' C_\varphi) \in S_{p/2}(\mathcal{A}^2_p)$. We get $\varphi' C_\varphi \in S_p(\mathcal{A}^2_p)$ if and only if $T_\psi \in S_{p/2}(\mathcal{A}^2_p)$ if and only if $\hat{\psi}_r(z) \in L^{p/2}(\mathbb{D}, d\lambda)$.

It is clear that $\Delta(z, r)$ contains an Euclidian disk centered at $z$ of radius $\eta(1 - |z|)$ with $\eta$ depending only on $r$. By the sub-mean value property of $N_{\varphi, \omega}$ we have
\[
\psi(z) = \frac{N_{\varphi, \omega}(z)}{\omega(z)} \leq \frac{2}{r^2 \omega(z)} \int_{\Delta(z, r)} N_{\varphi, \omega}(t)dA(t)
\]
\[
= \frac{2}{r^2} \hat{\psi}_r(z).
\]
So $\hat{\psi}_r(z) \in L^{p/2}(\mathbb{D}, d\lambda)$ implies $\psi(z) \in L^{p/2}(\mathbb{D}, d\lambda)$. Now, suppose that $\psi(z) \in L^{p/2}(\mathbb{D}, d\lambda)$. From the argument above, noting that $(1 - |t|^2) \asymp (1 - |z|^2) \asymp |1 - Tz|$ and $\omega(t)/\omega(z) \leq c$, for $t \in \Delta(z, r)$, we have
\[
\hat{\psi}_r(z)^{p/2} \leq \frac{c}{\omega(z)^{p/2}} \sup\{\psi(t)^{p/2}\omega(t)^{p/2} : t \in \Delta(z, r)\}
\]
\[
\leq \frac{c}{\omega(z)^{p/2}} \sup_{t \in \Delta(z, r)} \int_{\Delta(t, r)} \psi(s)^{p/2}\omega(s)^{p/2}dA(s)
\]
\[
\leq \frac{c}{\omega(z)^{p/2}} \sup_{t \in \Delta(z, r)} \int_{\Delta(t, r)} \frac{(1 - |z|^2)^2}{|1 - Tz|^4} \psi(s)^{p/2}\omega(s)^{p/2}dA(s).
\]
Since $t \in \Delta(z, r)$, we can choose $\Delta(t, r)$ so that $\Delta(t, r) \subset \Delta(z, r)$. Then
\[
\hat{\psi}_r(z)^{p/2} \leq \frac{c}{\omega(z)^{p/2}} \int_{\Delta(z, r)} \frac{(1 - |z|^2)^2}{|1 - Tz|^4} \psi(s)^{p/2}\omega(s)^{p/2}dA(s)
\]
\[
\leq c \int_D \frac{(1 - |z|^2)^2}{|1 - Tz|^4} \psi(s)^{p/2}dA(s).
\]
By Fubini’s Theorem and well known theorem (Theorem 1.12 [10]), we get
\[
\int_{\mathbb{D}} \frac{1}{\omega(z)(1 - |z|^2)^2} d\lambda(z) = \int_{\mathbb{D}} \frac{1}{\omega(z)} d\lambda(z) < \infty.
\]

If \( p = 2 \), then we have another characterization for Hilbert-Schmidt composition operators.

**Corollary 4.1.** Let \( \omega \) be an admissible weight satisfy (L1) condition. Then \( C_\varphi \) is Hilbert-Schmidt on \( H_\omega \) if and only if
\[
\int_{\mathbb{D}} N_{\varphi, \omega}(z) \omega(z)|N_{\varphi, \omega}(z)|^2 dA(z) = \int_{\mathbb{D}} N_{\varphi, \omega}(z) \omega(z) d\lambda(z) < \infty.
\]

5. Closed Range

It is well known that having the closed range for a bounded operator acting on a Hilbert space \( H \) is equivalent to existing a positive constant \( c \) such that for every \( f \in H \), \( \|Tf\|_H \geq c\|f\|_H \). Consider the function
\[
\tau_{\varphi, \omega}(z) = \frac{N_{\varphi, \omega}(z)}{\omega(z)}.
\]

**Proposition 5.1.** Let \( \omega \) be an admissible weight and \( C_\varphi \) be a bounded operator on \( H_\omega \). Then \( C_\varphi \) has closed range if and only if there exists a constant \( c > 0 \) such that for all \( f \in H_\omega \)
\[
\int_{\mathbb{D}} |f'(z)|^2 \tau_{\varphi, \omega}(z) \omega(z) dA(z) \geq c \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z).
\]

**Proof.** If \( \varphi(0) = 0 \), the we can consider \( C_\varphi \) acting on \( H_\omega \), the closed subspace of \( H_\omega \) consisting all functions with \( f(0) = 0 \). Note that \( C_\varphi \) has closed range if and only if there exists a constant \( c > 0 \) such that \( \|C_\varphi f\|_H \geq c\|f\|_H \). But
\[
\|C_\varphi f\|_H^2 = \int_{\mathbb{D}} |f'(\varphi(z))|^2 |\omega(\varphi(z))|^2 dA(z)
\]
\[
= \int_{\mathbb{D}} |f'(z)|^2 N_{\varphi, \omega} dA(z)
\]
\[
= \int_{\mathbb{D}} |f'(z)|^2 \tau_{\varphi, \omega}(z) \omega(z) dA(z).
\]

Thus, in this case the proposition is proved. If \( \varphi(0) = a \neq 0 \), define the function \( \psi = \sigma_a \circ \varphi \). Then \( C_\varphi = C_\psi C_{\sigma_a} \) and \( C_{\sigma_a} \) is invertible on \( H_\omega \). Therefore \( C_\varphi \) has closed range if and only if \( C_\psi \) has closed range. Since \( \psi(0) = 0 \), the argument above shows that \( C_\psi \) has closed range if and only if there exists a constant \( c > 0 \) such that
\[
\int_{\mathbb{D}} |f'(z)|^2 \tau_{\psi, \omega}(z) \omega(z) dA(z) \geq c \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z).
\]
We just prove that (5.1) and (5.2) are equivalent. If (5.1) holds, then
\[
\int_{D} |f'(z)|^2 \tau_{\psi, \omega}(z) \omega(z) \ dA(z) = \int_{D} |f'(z)|^2 N_{\psi, \omega} \ dA(z)
\]
\[
= \int_{D} |(f \circ \psi)'(z)|^2 \omega(z) \ dA(z)
\]
\[
= \int_{D} |(f \circ \sigma_a)'(\varphi(z))|^2 |\varphi'(z)|^2 \omega(z) \ dA(z)
\]
\[
= \int_{D} |(f \circ \sigma_a)'(z)|^2 \tau_{\varphi, \omega}(z) \omega(z) \ dA(z)
\]
\[
\geq c \int_{D} |(f \circ \sigma_a)'(z)|^2 \omega(z) \ dA(z)
\]
\[
= c \int_{D} |f'(z)|^2 \omega(\sigma_a(z)) \ dA(z)
\]
\[
= c \int_{D} |f'(z)|^2 \omega(z) \ dA(z).
\]
The last equation is due to Lemma 2.1. Hence (5.2) holds. Since \( \varphi = \sigma_a \circ \psi \), the proof of converse part is similar. \( \square \)

Fredholm composition operator is an example of composition operator with closed range property. Recall that a bonded operator \( T \) between two Banach spaces \( X, Y \) is called Fredholm if Kernel \( T \) and \( T^* \) are finite dimensional.

**Example 5.1.** Suppose that \( C_\varphi : \mathcal{H}_\omega \to \mathcal{H}_\omega \) be a Fredholm operator. By Theorem 3.29[1], \( \varphi \) is an authomorphism of \( \mathbb{D} \). Then \( N_{\varphi, \omega}(z) = \omega(\varphi^{-1}(z)) \). If \( \varphi(0) = 0 \), Schwarz Lemma implies that \( |\varphi^{-1}(z)| \leq |z| \). Since \( \omega \) is non-increasing, \( \omega(\varphi^{-1}(z)) = \omega(|\varphi^{-1}(z)|) \geq \omega(|z|) = \omega(z) \). Now (5.1) holds. If \( \varphi(0) \neq 0 \), then the same argument can be applied.

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