Subgroups of the additive group of real line

Jitender Singh
Department of Mathematics, Guru Nanak Dev University,
Amritsar-143005, Punjab, INDIA
sonumaths@gmail.com

Abstract
Without assuming the field structure on the additive group of real numbers \( \mathbb{R} \) with the usual order \(<\), we explore the fact that, every proper subgroup of \( \mathbb{R} \) is either closed or dense. This property of the subgroups of the additive-group of reals is special and well known (see Abels and Monoussos [4]). However, by revisiting it, we provide another direct proof. We also generalize this result to arbitrary topological groups in the sense that, any topological group having this property of the subgroups, in a given topology, is either connected or totally-disconnected.

By a topological group, we mean, an abstract group \( G \) which is also a topological space where the two maps \( G \times G \to G \) and \( G \to G \) defined by \( (x, y) \mapsto xy \) and \( x \mapsto x^{-1} \), are continuous. An example is, the additive group of real numbers \( (\mathbb{R}, +) \) with the standard topology of \( \mathbb{R} \). A subgroup of a topological group is also a topological group in the subspace topology and, for any fixed elements \( a \) and \( b \) of \( G \), the map \( x \mapsto axb \) is a homeomorphism of \( G \) onto itself. Consequently for any open subset \( U \) of \( G \), the subsets \( U^{-1} := \{x^{-1} | x \in U\} \), \( xU := \{xu | u \in U\} \), and \( WU := \cup_{w \in W} wU \) with the similar definitions for \( Ux \) and \( UW \), are all open in \( G \), where \( W \subset G \). Similarly, for any closed subset \( V \) of \( G \), the subsets \( xV \), \( Vx \), \( V^{-1} \) are closed in \( G \) but each of the sets \( WV \) and \( VW \), being arbitrary union of closed subsets of \( G \), need not be closed in \( G \). It is easy to see that any open subgroup \( H \) of \( G \) is closed in \( G \) because its compliment \( G \sim H \) is open as being union of left co-sets of \( H \) in \( G \) each of which is an open subset of \( G \). However, a closed subgroup of \( G \), of finite index, is open in \( G \) Also if \( H \) is a subgroup of \( G \) then so is its closure \( \overline{H} \). To see this, note that, for any \( x, y \in \overline{H} \) and the basis elements \( B_x \ni x \) and \( B_y \ni y \); the open sets \( xB_y^{-1}, B_x y^{-1} \) contain \( xy^{-1} \). Since \( B_x B_y^{-1} \cap H \neq \emptyset \) as \( B_x \) and \( B_y \) intersect \( H \) nontrivially, it follows that \( xy^{-1} \in \overline{H} \). It is also easy to observe that any subgroup \( H \) of \( G \) that contains an open subset \( U \ni 1_G \) is always open, as then, it would be union of the open sets obtained by the co-sets of \( U \) in \( H \). An interesting feature of a topological group \( G \) is that, it is homogeneous, i.e., for any pair of points \( x \) and \( y \) in \( G \), there is a homeomorphism...
of $G$ sending $x \mapsto y$ (e.g. $t \mapsto tx^{-1}y$) (see [1, pp. 95-119], [2, p.219], [3, p.16]). The following
Theorem 1 is an easy consequence of these basic notions about the algebraic structure of the
real line with respect to the operation of usual addition (see for a ‘semigroup version proof’
of this result in [4] namely the lemma 2.2-2.3).

**Theorem 1.** Any proper subgroup of the additive group $\mathbb{R}$ is either its closed subset or its
dense subset.

**Proof.** Let $H$ be a subgroup of $\mathbb{R}$. Assume w.l.o.g that $H$ is not a closed subset of $\mathbb{R}$. If
possible, let us suppose that there is a basis element $(a, b)$, which does not intersect $H$. Then
there is a limit point $x \notin H$ of $H$ in $\mathbb{R}$. So, every basis element $(c, d) \ni x$ intersects $H$.
For each integer $n$ and $t \in H$, $nx$ and $x + t$ are also limit points of $H$, since, the
maps defined by $x \mapsto nx$ and $x \mapsto x + t$ are homeomorphisms of $\mathbb{R}$. Assume w.l.o.g. that
$(a, b) \subset (mx, (m+1)x)$ for some integer $m$. Then for every positive integer $q$, satisfying
$q(b-a) > x$, there is an integer $p < q$ and a real number $z$ such that $qz = px \in (qa, qb)$
or $z \in (a, b)$. The point $z$ is a limit point of $H$ since so is $qz$ and the map $z \mapsto qz$, is a
homeomorphism. But, then $H \cap (a, b) \neq \emptyset$ which is a contradiction. This completes the
proof.

**Remark 2.** Here is more simplified version of the proof of Theorem 1 based on author’s
personal communication with Dr. Keerti Vardhan Madahar. Let $x \in \mathbb{R} - H$ be a limit point of
$H$. Choose two points (say $x_1$ and $x_2$) of $H$ which belong to the open set $U = (x - \epsilon/2, x + \epsilon/2)$
of $x$ for some $\epsilon > 0$. Now $y = x_1 - x_2$ and all its integral multiples lie in $H$. Also notice that
if $(a, b)$ is an arbitrary basis element of length at least $\epsilon$ then some multiple of $y$ must lie in
$(a, b)$ (otherwise there is a positive integer $N$ such that $Ny \leq a < b \leq (N + 1)y$ which gives
$\epsilon \leq (b-a) \leq y < \epsilon$ a contradiction). That shows $(a, b)$ has a non-trivial intersection with
$H$, so $H$ is dense in $\mathbb{R}$.

The hypothesis of Theorem 1 does not hold incase we consider the topological group $\mathbb{R}^n$, $n \geq 2$
either is a closed subset of $\mathbb{R}^n$ nor it is a dense subgroup (i.e. its closure is homeomorphic to $\mathbb{R}^{n-1}$). It is also easy
to see that the above result is not true for the multiplicative group $(\mathbb{R}^\times, \cdot, <)$, because, the
subgroup $\{e^r \mid r \in \mathbb{Q}\}$ is neither dense nor a closed subset of $\mathbb{R}^\times$.

**Remark 3.** Let $H \neq \{0\}$ be a proper subgroup of the additive group $\mathbb{R}$. Define $A := \{|x| : x \in H - \{0\}\}$ and $\alpha := \inf \{A\}$.

If $H$ is closed in $\mathbb{R}$ then so is $H - \{0\}$ and $\alpha \in \bar{A} \subset \overline{H - \{0\}} = H - \{0\}$; it follows that
$\alpha \neq 0$ and $\alpha \in H$. For any $x \in H$ by division algorithm, $|x| = n\alpha + r$, $0 \leq r < \alpha$ for some
positive integer \( n \). This means that \( r = (|x| - n\alpha) \in H \) such that \( r < \alpha \) which is possible only when \( r = 0 \). Hence \( H = \alpha \mathbb{Z} \). We have proved that \( H \) is cyclic.

Conversely, if \( \alpha \neq 0 \) then for any \( x \in H, |x| = n\alpha + r, 0 \leq r < \alpha \) for some positive integer \( n \). We claim that \( r = 0 \). If possible let \( r > 0 \). Using definition of inf, choose for every \( \epsilon > 0 \) a \( y \in H \) s.t. \( -n\alpha \in (y, y+\epsilon) \). Take \( \epsilon = \alpha + r \), s.t. \( |x| + y \in H \) and \( -\alpha = r - \epsilon < |x| + y < r < \alpha \) or \( ||x| + y| < \alpha \). This is a contradiction since \( \alpha = \inf \{A\} \). Thus \( r = 0 \) and \( H = \alpha \mathbb{Z} \) which is closed and cyclic subgroup of \( \mathbb{R} \).

It is clear that if \( \hat{H} = \mathbb{R} \) then \( \alpha := \inf \{A\} = 0 \). Conversely, if \( \alpha = 0 \) then \( H \) is not closed subset of \( \mathbb{R} \). Therefore by Theorem 1, \( H \) is dense in \( \mathbb{R} \).

We have proved the following necessary and sufficient conditions for the subgroups of the additive group of reals.

**Theorem 4.** Let \( H \neq \{0\} \) be a proper subgroup of the additive group \( \mathbb{R} \) and \( \alpha := \inf \{|x| : x \in H - \{0\}\} \). Then

(a) \( H \) is closed in \( \mathbb{R} \) if and only if \( \alpha \neq 0 \). Moreover such a subgroup is cyclic.

(b) \( H \) is dense in \( \mathbb{R} \) if and only if \( \alpha = 0 \).

As an application of the preceding result, we have following special case of Kronecker’s (1884) approximation theorem.

**Theorem 5.** (Kronecker) For an irrational \( \alpha \), the subgroup \( \alpha \mathbb{Z} + \mathbb{Z} \) is dense in \( \mathbb{R} \).

**Proof.** First note that \( \alpha \mathbb{Z} + \mathbb{Z} \leq \mathbb{R} \) and so is its closure. If possible let \( \alpha \mathbb{Z} + \mathbb{Z} \) be closed in \( \mathbb{R} \). Then by preceding theorem, \( \alpha \mathbb{Z} + \mathbb{Z} = \beta \mathbb{Z} \) for some nonzero real \( \beta \). But then \( \mathbb{Z} \subseteq \beta \mathbb{Z} \) and \( \alpha \mathbb{Z} \subseteq \beta \mathbb{Z} \). This gives \( 1 = \beta n \) and \( \alpha = \beta m \) for some \( 0 \neq m, n \in \mathbb{Z} \) from which we obtain \( \beta = \frac{1}{n} \) and \( \alpha = \beta m = \frac{m}{n} \in \mathbb{Q} \). This contradicts the fact that \( \alpha \in \mathbb{R} - \mathbb{Q} \). \( \square \)

**Example 6.** If we let \( S := \{\sin n \mid n \in \mathbb{Z}\} \) then closure of \( S \) in \( \mathbb{R} \) is the interval \([-1, 1]\).

We prove this well known fact (see for detail [5]) using the preceding result. For any real \( r \in [-1, 1] \), and \( \alpha = 2\pi \) there exist \( s \in [-\pi/2, \pi/2] \) s.t. \( \sin s = r \) and integer-sequences \( \{p_n\} \) and \( \{q_n\} \) such that \( 2\pi p_n + q_n \to s \). By continuity of \( \sin \) function, we have \( \sin(2\pi p_n + q_n) \to \sin s = r \) or \( \sin q_n \to r \). Thus every point of the interval \([-1, 1]\) is limit of some subsequence of the sequence \( \{\sin n\} \).

We now establish a general result concerning the topological groups having the nice property of Theorem 1.
Theorem 7. Let $G$ be a topological group such that
(a) $G$ has a proper dense subgroup $H$
(b) any proper subgroup $K$ of $G$ is such that either $\bar{K} = K$ or $\bar{K} = G$.
Then $G$ is either connected or totally disconnected.

Proof. Let $G$ is not connected and $C \neq G$ be the connected component of $G$ containing the identity element. If there is a connected open subset $U \neq G$ of $G$ containing a point $x$ of $G$ then so is the open set $x^{-1}U \ni 1_G$ such that $x^{-1}U \cap C \neq \emptyset$; consequently, $x^{-1}U \cup C$ is connected and hence $C \supset x^{-1}U$. It follows that $C$ is open in $G$; as $C \leq G$ and $\bar{H} = G$ we see using the hypothesis (b) that $\bar{H} \cap C = \bar{C} = C$ is closed subset of $G$ which gives $\bar{H} \cap \bar{C} = H \cap C = C$ or $C \subset H$. But then $H$ is open subgroup of $G$ and since every open subgroup of a topological group is also a closed subset, it follows that $\bar{H} = H$ contradicting the fact that $H$ was assumed to be dense subgroup of $G$. We have proved that there does not exist any connected open subset of $G$ and by (a) none of the singleton set in $G$ can be open. It follows that every proper open subset of $G$ is totally disconnected. Finally, since $C$ is closed subset of $G$ and if $C \neq G$, it follows that $G - C$ is totally disconnected. But then for every $y \in G - C$, $yC = \{y\}$ which is possible only if $C = \{1_G\}$. This proves that $G$ is totally disconnected. \hfill \Box

In view of the hypothesis of Theorem 7 through Theorem 1, we see that $\mathbb{R}$ is connected in ‘usual topology’, ‘indiscrete topology’, and ‘finite-closed topology’ while $\mathbb{R}$ is totally disconnected in ‘the lower limit topology’. In the latter topology we see that every interval in $\mathbb{R}$ is totally disconnected!

Acknowledgement

The author is indebted to Dr. Keerti Vardhan Madahar for useful discussion on proof of theorem 1.

References:

[1] L. S. Pontryagin. Selected Works: Topological Groups Vol 2. (English translation by A. Brown), Gordon and Breach Science Publishers, New York, 1986.

[2] N. Bourbaki. Elements of Mathematics: Part I-General Topology, Addison Wesley, 1966.
[3] P. J. Higgins. *Introduction to Topological Groups, Lecture notes 15*, London Mathematical Society. Cambridge University Press, 1974.

[4] H. Abels and A. Manoussos. Topological generators of abelian Lie groups and hypercyclic finitely generated abelian semigroups of matrices. (2010) pp 1-14.

[5] J. H. Staib and M. S. Demos. On the limit points of the sequence $\sin n$. *Mathematics Magazine*, 40:4 (1967) pp 210-213.