Effects of singular external fields and boundary condition on the vacuum of massless fermions in QFT

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Abstract

Effects of the configuration of an external static magnetic field in the form of a singular vortex on the vacuum of a quantized massless spinor field are studied. The most general boundary conditions at the punctured singular point which make the twodimensional Dirac Hamiltonian to be self-adjoint are employed.

1 Introduction

A study of effects of singular external fields (zero-range potentials) in quantum mechanics has a long history and has been comprehensively conducted (see [1] and references therein). Contrary to this, effects of singular external fields in quantum field theory are at the initial stage of consideration, and much has to be elucidated. Singular background can act on the vacuum of a second-quantized spinor field in a rather unusual manner: a leak of quantum numbers from the singularity point occurs. This is due, apparently, to the fact that a solution to the Dirac equation, unlike that to the Schrodinger one, does not obey a condition of regularity at the singularity point. It is necessary then to specify a boundary condition at this point, and the least restrictive, but physically acceptable, condition is such that guarantees self-adjointness of the Dirac Hamiltonian. Thus, effects of polarization of the vacuum by a singular background appear to depend on the choice of the boundary condition at the singularity point, and the set of permissible boundary conditions is labelled, most generally, by a self-adjoint extension parameter.

In this talk we shall consider some effects of polarization of the massless fermionic vacuum in the background of a pointlike magnetic vortex in 2+1-dimensional space-time (see also [2, 3]).
2 Boundary Condition at the Location of the Vortex

The operator of the second-quantized spinor field is presented in the form

\[ \Psi(x, t) = \sum_{E_\lambda > 0} e^{-iE_\lambda t} |x|\lambda a_\lambda + \sum_{E_\lambda < 0} e^{-iE_\lambda t} |x|\lambda b_\lambda^\dagger, \]

where \( a_\lambda^\dagger \) and \( a_\lambda \) (\( b_\lambda^\dagger \) and \( b_\lambda \)) are the spinor particle (antiparticle) creation and annihilation operators satisfying the anticommutation relations, and \( |x|\lambda \) is the solution to the stationary Dirac equation

\[ H <x|\lambda >= E_\lambda <x|\lambda >, \]

\( H \) is the Dirac Hamiltonian, \( \lambda \) is the set of parameters (quantum numbers) specifying a state, \( E_\lambda \) is the energy of a state; symbol \( \sum \int \) means the summation over discrete and the integration (with a certain measure) over continuous values of \( \lambda \). Ground state \( |\text{vac} > \) is defined conventionally as

\[ a_\lambda|\text{vac} >= b_\lambda|\text{vac} >= 0. \]

In the case of quantization of a massless spinor field in the background of a static vector field \( V(x) \), the Dirac Hamiltonian takes form

\[ H = -i\alpha[\partial - iV(x)], \]

where \( \alpha = \gamma^0\gamma \) and \( \beta = \gamma^0 \) (\( \gamma \) and \( \gamma^0 \) are the Dirac \( \gamma \) matrices). In 2+1-dimensional space-time \( (x, t) = (x_1, x_2, t) \), the Clifford algebra has two inequivalent irreducible representations which can be differed in the following way:

\[ i\gamma^0\gamma^1\gamma^2 = s, \quad s = \pm 1. \]

Choosing the \( \gamma^0 \) matrix in the diagonal form, one gets

\[ \gamma^0 = \sigma_3, \quad \gamma^1 = e^{\pm i\sigma_3\chi_1}i\sigma_1e^{-\frac{i}{2}\sigma_3\chi_3}, \quad \gamma^2 = e^{\pm i\sigma_3\chi_3}i\sigma_2e^{-\frac{i}{2}\sigma_3\chi_3}, \]

where \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are the Pauli matrices, and \( \chi_1 \) and \( \chi_{-1} \) are the parameters that are varied in interval \( 0 \leq \chi_1 < 2\pi \) to go over to the equivalent representations.
The configuration of external field $\mathbf{V}(x) = (V_1(x), V_2(x))$ is chosen as

\[
V_1(x) = -\Phi(0) \frac{x^2}{(x^1)^2 + (x^2)^2}, \quad V_2(x) = \Phi(0) \frac{x^1}{(x^1)^2 + (x^2)^2},
\]

which corresponds to the magnetic field strength in the form of a singular vortex

\[
\partial \times \mathbf{V}(x) = 2\pi \Phi(0) \delta(x),
\]

where $\Phi(0)$ is the total flux (in $2\pi$ units) of the vortex – i.e. of the thread that pierces the plane $(x^1, x^2)$ at the origin.

A solution to the Dirac equation (2) with Hamiltonian (4) in background (7) can be presented as

\[
\begin{pmatrix}
    f_n(r, E) e^{im\varphi} \\
    g_n(r, E) e^{i(n+s)\varphi}
\end{pmatrix}, \quad n \in \mathbb{Z},
\]

where $\mathbb{Z}$ is the set of integer numbers, $r = \sqrt{(x^1)^2 + (x^2)^2}$ and $\varphi = \arctan(x^2/x^1)$ are the polar coordinates, and the radial functions $f_n$ and $g_n$ satisfy the system of equations

\[
e^{-i\chi s}[-\partial_r + s(n - \Phi(0)) r^{-1}] f_n(r, E) = E g_n(r, E),
\]

\[
e^{i\chi s} [\partial_r + s(n - \Phi(0) + s) r^{-1}] g_n(r, E) = E f_n(r, E).
\]

When vortex flux $\Phi(0)$ is integer, the requirement of square integrability for wave function (9) provides its regularity everywhere on plane $(x^1, x^2)$, rendering partial Dirac Hamiltonians for every value of $n$ to be essentially self-adjoint. When $\Phi(0)$ is fractional, the same is valid only for $n \neq n_0$, where

\[
n_0 = \lfloor \Phi(0) \rfloor + \frac{1}{2} - \frac{1}{2} s,
\]

$\lfloor u \rfloor$ is the integer part of quantity $u$ (i.e., the greatest integer that is less than or equal to $u$). For $n = n_0$, each of the two linearly independent solutions to system (10) meets the requirement of square integrability. Any particular solution in this case is characterized by at least one (at most both) of the radial functions being divergent as $r^{-p}$ ($p < 1$) for $r \to 0$. Therefore, contrary to the case of $n \neq n_0$, the partial Dirac Hamiltonian in the case of
\[ n = n_0 \text{ is not essentially self-adjoint. The Weyl-von Neumann theory of self-adjoint operators (see, e.g., Ref. [4]) has to be employed in order to consider the possibility of a self-adjoint extension in the case of } n = n_0. \text{ It can be shown that the self-adjoint extension exists indeed and is parametrized by one continuous real variable denoted in the following by } \Theta. \text{ Thus, the partial Dirac Hamiltonian in the case of } n = n_0 \text{ is defined on the domain of functions obeying the condition}

\[
\cos\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) \lim_{r \to 0} (\mu r)^F f_{n_0} = -e^{ixs} \sin\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) \lim_{r \to 0} (\mu r)^{1-F} g_{n_0},
\]

where \( \mu > 0 \) is the parameter of the dimension of inverse length and

\[
F = s\|\Phi^{(0)}\| + \frac{1}{2} - \frac{1}{2}s,
\]

\( \|u\| = u - [u] \) is the fractional part of quantity \( u \), \( 0 \leq \|u\| < 1 \); note here that Eq.(12) implies that \( 0 < F < 1 \), since, for \( F = \frac{1}{2} - \frac{1}{2}s \), both \( f_{n_0} \) and \( g_{n_0} \) obey the condition of regularity at \( r = 0 \). Note also that Eq.(12) is periodic in \( \Theta \) with the period of \( 2\pi \); therefore, without a loss of generality, all permissible values of \( \Theta \) will be restricted in the following to the range \( -\pi \leq \Theta \leq \pi \).

All solutions to the Dirac equation in the background of a singular magnetic vortex correspond to the continuous spectrum and, therefore, obey the orthonormality condition

\[
\int d^2x < E, n | x > < x | E', n' > = \frac{\delta(E - E')}{\sqrt{|EE'|}} \delta_{nn'}.
\]

In the case of \( 0 < F < 1 \) one gets the following expressions corresponding to the regular solutions with \( sn > sn_0 \):

\[
\begin{pmatrix}
    f_n \\
g_n
\end{pmatrix}
= \frac{1}{2\sqrt{\pi}} \begin{pmatrix}
    J_{l-F}(kr) e^{i\chi s} \\
    \text{sgn}(E) J_{l+1-F}(kr)
\end{pmatrix}, \quad l = s(n - n_0),
\]

the regular solutions with \( sn < sn_0 \):

\[
\begin{pmatrix}
f_n \\
g_n
\end{pmatrix}
= \frac{1}{2\sqrt{\pi}} \begin{pmatrix}
    J_{l'+F}(kr) e^{i\chi s} \\
    -\text{sgn}(E) J_{l'-1+F}(kr)
\end{pmatrix}, \quad l' = s(n_0 - n),
\]
and the irregular solution:

\[
\begin{pmatrix}
  f_{n_0} \\
  g_{n_0}
\end{pmatrix} = \frac{1}{2\sqrt{\pi[1 + \sin(2\nu_E)\cos(F\pi)]}} \times \\
\times \left(\begin{array}{c}
  \sin(\nu_E)J_{-F}(kr) + \cos(\nu_E)J_F(kr) e^{ix} \\
  \text{sgn}(E)[\sin(\nu_E)J_{1-F}(kr) - \cos(\nu_E)J_{1+F}(kr)]
\end{array}\right);
\]

(17)

Here \( k = |E| \), \( J_\rho(u) \) is the Bessel function of order \( \rho \) and

\[
\text{sgn}(u) = \begin{cases} 
1, & u > 0 \\
-1, & u < 0
\end{cases}
\]

Substituting the asymptotic form of Eq.(17) at \( r \to 0 \) into Eq.(12), one arrives at the relation between the parameters \( \nu_E \) and \( \Theta \):

\[
\tan(\nu_E) = \text{sgn}(E) \left( k^2 \frac{2F-1}{\Gamma(1-F)} \frac{\Gamma(1-F)}{\Gamma(F)} \tan(\frac{s\Theta}{2} + \frac{\pi}{4}) \right),
\]

(18)

where \( \Gamma(u) \) is the Euler gamma function.

## 3 Fermion Number

In the second-quantized theory the operator of the fermion number is given by the expression

\[
\hat{N} = \int d^2x \frac{1}{2}[\Psi^+(x,t),\Psi(x,t)] = \sum \left[a_\lambda^+ a_\lambda - b_\lambda^+ b_\lambda - \frac{1}{2}\text{sgn}(E_\lambda)\right],
\]

(19)

and, consequently, its vacuum expectation value takes form

\[
\mathcal{N} \equiv \langle \text{vac}|\hat{N}|\text{vac}>= -\frac{1}{2} \sum \text{sgn}(E_\lambda) = -\frac{1}{2} \int d^2x \text{ tr } <x|\text{sgn}(H)|x>.
\]

(20)

From general arguments, one could expect that the last quantity vanishes due to cancellation between the contributions of positive and negative energy solutions to the Dirac equation (2). Namely this happens in a lot of cases. That is why every case of a nonvanishing value of \( \mathcal{N} \) deserves a special attention.
Considering the case of the background in the form of singular magnetic vortex (7) – (8), one can notice that the contribution of regular solutions (15) and (16) is cancelled upon summation over the sign of energy, whereas irregular solution (17) yields a nonvanishing contribution to $N$ (20). Defining the vacuum fermion number density

$$N_x = -\frac{1}{2} \text{tr} < x | \text{sgn}(H) | x >,$$  

we get

$$N_x = -\frac{1}{8\pi} \int_0^\infty dkk \left\{ A \left( \frac{k}{\mu} \right)^{2F-1} \left[ L_+ + L_- \right] \left[ J_{2F}(kr) + 

+ J_{1-F}(kr) \right] + 2\left[ L_+ - L_- \right] \left[ J_- (kr) J_F(kr) - J_{1-F}(kr) J_{1+F}(kr) \right] + 

+ A^{-1} \left( \frac{k}{\mu} \right)^{1-2F} \left[ L_+ + L_- \right] \left[ J_F^2(kr) + J_{1+F}^2(kr) \right] \right\},$$  

(22)

where

$$A = 2^{1-2F} \frac{\Gamma(1-F)}{\Gamma(F)} \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} \right),$$  

(23)

$$L_{(\pm)} = 2^{-1} \left\{ \cos(F\pi) \pm \cosh[(2F-1)\ln(\frac{k}{\mu}) + \ln A] \right\}^{-1}.$$  

(24)

Transforming the integral in Eq.(22), we get the final expression

$$N_x = -\frac{\sin(F\pi)}{2\pi^3 r^2} \int_0^\infty dw \frac{K_\rho^2(w) - K_\rho^2(1-F)(w)}{\cosh[(2F-1)\ln(\frac{w}{\mu r}) + \ln A]},$$  

(25)

where $K_\rho(w)$ is the Macdonald function of order $\rho$. Vacuum fermion number density (25) vanishes at half integer values of the vortex flux ($F = \frac{1}{2}$) as well as at $\cos \Theta = 0$. Otherwise, at large distances from the vortex we get

$$N_{x_{r \to \infty}} = -(F - \frac{1}{2}) \frac{\sin(F\pi)}{2\pi^2 r^2} \left\{ \begin{array}{ll}
(\mu r)^{2F-1} A^{-1} \frac{\Gamma(\frac{3}{2} - F) \Gamma(\frac{3}{2} - 2F)}{\Gamma(2 - F)}, & 0 < F < \frac{1}{2} \\
(\mu r)^{1-2F} A \frac{\Gamma(F + \frac{1}{2}) \Gamma(2F - \frac{1}{2})}{\Gamma(1 + F)}, & \frac{1}{2} < F < 1
\end{array} \right.$$

(26)
Integrating Eq.(25) over plane \((x^1, x^2)\), we obtain the total vacuum fermion number
\[
\mathcal{N} = -\frac{1}{2} \text{sgn}_0 \left[ (F - \frac{1}{2}) \cos \Theta \right],
\] (27)
where
\[
\text{sgn}_0(u) = \begin{cases} 
\text{sgn}(u), & u \neq 0 \\
0, & u = 0 
\end{cases}
\].

### 4 Parity Breaking Condensate

Since two-dimensional massless Dirac Hamiltonian (4) anticommutes with the \(\beta\) matrix
\[
[H, \beta]_+ = 0,
\] (28)
Dirac equation (2) is invariant under the parity transformation
\[
E_\lambda \rightarrow -E_\lambda, \quad <x|\lambda > \rightarrow \beta <x|\lambda >.
\] (29)
However, this invariance is violated by the boundary condition (12), unless \(\cos \Theta = 0\). Consequently, the parity breaking condensate emerges in the vacuum:
\[
C_x = \langle \text{vac} | \frac{1}{2} [\psi^+(x,t), \beta \psi(x,t)]_- | \text{vac} >= -\frac{1}{2} \text{tr} <x|\beta \text{sgn}(H)|x >. 
\] (30)

The contribution of regular solutions (15) and (16) to Eq.(30) is cancelled upon summation over the energy sign. Thus, only the contribution of irregular solution (17) to Eq.(30) survives:
\[
C_x = -\frac{1}{8\pi} \int_0^\infty dk k \left\{ A \left( \frac{k}{\mu} \right)^{2F-1} [L_+(+) + L_(-)] [J^2_+F(kr) - J^2_1-F(kr)] + 
+ 2[L_+(+) - L_(-)][J_1-F(kr)J_F(kr) + J_{1-F}(kr)J_{1+F}(kr)] + 
+ A^{-1} \left( \frac{k}{\mu} \right)^{1-2F} [L_+(+) + L_(-)][J^2_F(kr) - J^2_{1+F}(kr)] \right\},
\] (31)
where \(A\) and \(L_\pm\) are given by Eqs.(23) and (24), respectively. Transforming the integral in Eq.(31), we get
\[
C_x = -\frac{\sin(F\pi)}{2\pi^3 r^2} \int_0^\infty dw w \frac{K^2_F(w) + K^2_{1-F}(w)}{\cosh[(2F - 1) \ln\left(\frac{x}{\mu r}\right) + \ln A]},
\] (32)
Evidently, Eq. (32) vanishes if \( \cos \Theta = 0 \). At half integer values of the vortex flux \( (F = \frac{1}{2}) \), we get

\[
C_x|_{F=\frac{1}{2}} = -\frac{\cos \Theta}{4\pi^2 r^2}.
\]

(33)

At large distances from the vortex we get

\[
C_x \approx -\frac{\sin(F\pi)}{2\pi^2 r^2} \left\{ \begin{array}{ll}
(\mu r)^{2F-1} A^{-1} \frac{\Gamma\left(\frac{3}{2} - F\right)\Gamma\left(\frac{3}{2} - 2F\right)}{\Gamma\left(1 - F\right)}, & 0 < F < \frac{1}{2} \\
(\mu r)^{1-2F} A \frac{\Gamma(F + \frac{1}{2})\Gamma(2F - \frac{1}{2})}{\Gamma(3F - 1)}, & \frac{1}{2} < F < 1
\end{array} \right. .
\]

(34)

Integrating Eq. (32) over plane \((x^1, x^2)\), we obtain the total vacuum condensate

\[
C \equiv \int d^2 x C_x = -\frac{\text{sgn}_0(\cos \Theta)}{4|F - \frac{1}{2}|}.
\]

(35)

Thus, the total vacuum condensate is infinite at \( F = \frac{1}{2} \) if \( \cos \Theta \neq 0 \).

5 Angular Momentum

Let \( \hat{M} \) be an operator in the first-quantized theory, which commutes with the Dirac Hamiltonian

\[
[\hat{M}, H]_- = 0.
\]

(36)

Then, in the second-quantized theory, the vacuum expectation value of the dynamical variable corresponding to \( \hat{M} \) is presented in the form

\[
\mathcal{M} = \int d^2 x \mathcal{M}_x,
\]

(37)

where

\[
\mathcal{M}_x = \langle \text{vac} | \hat{M} | \Psi^+(x, t), \hat{M} \Psi(x, t) | \text{vac} \rangle = -\frac{1}{2} \text{tr} < x | \hat{M} \text{sgn}(H) | x > .
\]

(38)

The commutation relation (36) is the evidence of invariance of the theory with \( \hat{M} \) being the generator of the symmetry transformations. Since, in background (7) – (8), there is invariance with respect to rotations around
the location of the vortex, one can take \( \hat{M} \) as the generator of rotations — the operator of angular momentum in the first-quantized theory (see [5] for more details):

\[
\hat{M} = -i \mathbf{x} \times \mathbf{\partial} + \frac{1}{2} s \beta.
\]  

(39)

Note that the eigenvalues of operator \( \hat{M} \) (39) on spinor functions are half integer.

Decomposing Eq.(39) into the orbital and spin parts, we get in the second-quantized theory

\[
\mathcal{M}_x = \mathcal{L}_x + \mathcal{S}_x,
\]

(40)

where

\[
\mathcal{L}_x = \frac{1}{2} \text{tr} < \mathbf{x}|(i \mathbf{x} \times \mathbf{\partial}) \text{sgn}(H)|\mathbf{x} >
\]

(41)

and

\[
\mathcal{S}_x = -\frac{1}{4} s \text{tr} < \mathbf{x}|\beta \text{sgn}(H)|\mathbf{x} >.
\]

(42)

Since vacuum spin density (42) is related to vacuum condensate (30),

\[
\mathcal{S}_x = \frac{1}{2} s C_x,
\]

(43)

there remains only vacuum orbital angular momentum density (41) to be considered.

The contribution of regular solutions (15) and (16) is cancelled upon summation over the energy sign, whereas the contribution of irregular solution (17) yields:

\[
\mathcal{L}_x = -\frac{1}{8\pi} \int_0^\infty dk \left\{ A \left( \frac{k}{\mu} \right)^{2F-1} [L_{(+)} + L_{(-)}][n_0 J_{-F}^2(kr) + (n_0 + s) J_{1-F}^2(kr)] + 
+ 2[L_{(+)} - L_{(-)}][n_0 J_{-F}(kr) J_F(kr) - (n_0 + s) J_{1-F}(kr) J_{-1+F}(kr)] + 
+ A^{-1} \left( \frac{k}{\mu} \right)^{1-2F} [L_{(+)} + L_{(-)}][n_0 J_F^2(kr) + (n_0 + s) J_{-1+F}^2(kr)] \right\}.
\]

(44)

Transforming the integral in Eq.(44), we get

\[
\mathcal{L}_x = -\frac{\sin(F\pi)}{2\pi^3 r^2} \int_0^\infty dw \frac{n_0 K_F^2(w) - (n_0 + s) K_{1-F}^2(w)}{\cosh[(2F-1) \ln(w) + \ln A]}.
\]

(45)

9
Summing Eqs.(45) and (46), taking into account Eqs.(32) and (11), we obtain the following expression for the vacuum angular momentum density in background (7) – (8):

$$\mathcal{M}_x = \left(\left\lfloor \Phi^{(0)}(0) \right\rfloor + \frac{1}{2} \right) \mathcal{N}_x,$$

(46)

where vacuum fermion number density $\mathcal{N}_x$ is given by Eq.(25). Consequently, the total vacuum angular momentum takes form (see Eq.(27))

$$\mathcal{M} = -\frac{1}{2} \left(\left\lfloor \Phi^{(0)}(0) \right\rfloor + \frac{1}{2} \right) \text{sgn}[\left(\frac{1}{2} \cos \Theta\right)].$$

(47)

Thus, if the vacuum angular momentum could become somehow detectable, then this would provide us with a unique explicit evidence in favour of physical effects that depend essentially both on integer and fractional parts of the flux of a singular magnetic vortex. 

Acknowledgements. The work was supported by the U.S. Civilian Research and Development Foundation (CRDF grant UP1-2115).

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