New representations of Poincaré group for consistent Relativistic Particle Theories

Giuseppe Nisticò
Dipartimento di Matematica e Informatica, Università della Calabria, Italy and INFN – gruppo collegato di Cosenza, Italy
E-mail: giuseppe.nistico@unical.it

Abstract. The class of representations of the Poincaré group for the formulation of relativistic quantum theories of single particle is consistently extended. In so doing new species of such representations are singled out. By making use of the enlarged class quantum theories of Klein-Gordon free particle are formulated without the problems suffered by the early theory. Furthermore, new species of consistent theories can be developed.

1. Introduction

The symmetry character of Poincaré transformations for a free elementary particle implies that the Hilbert space $\mathcal{H}$ of its quantum theory must admit [1] an irreducible representation $U$ of the Poincaré group $\mathcal{P}$ such that the quantum transformation $A \rightarrow S_{\mathbf{q}}[A]$ of each observable $A$, operated by a Poincaré transformation $\mathbf{q}$, is obtained according to the relation $S_{\mathbf{q}}[A] = U_{\mathbf{q}}AU_{\mathbf{q}}^{-1}$. Therefore, the identification of the irreducible representations of $\mathcal{P}$ lays the groundwork for the formulation of relativistic quantum theories of one free particle. Elsewhere [2] we have shown that the collection of the irreducible representations of $\mathcal{P}$ considered in the literature does not exhaust the irreducible representations of $\mathcal{P}$ a quantum theory of an elementary particle can be coherently based on. In particular, the irreducible representations with anti-unitary space inversion operator are currently discarded [3], [4] but in fact quantum theories of a particle characterized by anti-unitary space inversion operator can be consistently developed [2]; in particular, these further representations turn out to be successful for formulating quantum theories of a Klein Gordon particle without the inconsistencies that plagued the early theory [4].

For this reason it is important to re-determine the possible irreducible representations of $\mathcal{P}$ without a priori preclusions. In section 2 we outline such a re-determination, limited to positive mass and spin 0 irreducible representations just for reason of space. In so doing, for every value of the of mass four irreducible representations are identified besides those considered by the current literature. Furthermore we show a further possibility of forming irreducible representations of $\mathcal{P}$ from reducible representations of the subgroup $\mathcal{P}^{+}_{\uparrow}$, completely neglected so far.

In section 3 we show that making use of this re-determination, it is possible to formulate particle theories that do not suffer the problems of early theories, and also new species of theories.
2. Positive mass and spin 0 irreducible representations of \( \mathcal{P} \)

2.1. Notation and mathematical prerequisites

2.1.1. Poincaré group. Given any vector \( \vec{x} = (x_0, \mathbf{x}) \in \mathbb{R}^4 \), we call \( x_0 \) the time component of \( \vec{x} \) and \( \mathbf{x} = (x_1, x_2, x_3) \) the spatial component of \( \vec{x} \). The proper orthochronous Poincaré group \( \mathcal{P}_+^\uparrow \) is the separable locally compact group of all transformations of \( \mathbb{R}^4 \) generated by the ten one-parameter sub-groups \( T_0, T_j, R_j, B_j, j = 1, 2, 3 \), of, respectively, time translations, spatial translation, proper spatial rotations and Lorentz boosts relative to axis \( x_j \). The Euclidean group \( \mathcal{E} \) is the sub-group generated by all \( T_j \) and \( R_j \). The sub-group generated by \( R_j, B_j \) is the proper orthochronous Lorentz group \( \mathcal{L}_+^\uparrow \) [5]. It does not include time reversal \( \mathcal{t} \) and space inversion \( \mathcal{s} \). Time reversal \( \mathcal{t} \) transforms \( \vec{x} = (x_0, \mathbf{x}) \) into \((-x_0, \mathbf{x}) \); space inversion \( \mathcal{s} \) transforms \( \vec{x} = (x_0, \mathbf{x}) \) into \((x_0, -\mathbf{x}) \). The group generated by \{\( \mathcal{P}_+^\uparrow, \mathcal{t}, \mathcal{s} \)\} is the separable and locally compact Poincaré group \( \mathcal{P} \). By \( \mathcal{L}_+^\uparrow \) we denote the subgroup generated by \( \mathcal{L}_+^\uparrow \) and \( \mathcal{t} \), while \( \mathcal{L}_+^\uparrow \) denotes the subgroup generated by \( \mathcal{L}_+^\uparrow \) and \( \mathcal{s} \); analogously, \( \mathcal{P}_+^\uparrow \) denotes the subgroup generated by \( \mathcal{P}_+^\uparrow \) and \( \mathcal{t} \), while \( \mathcal{P}_+^\uparrow \) is the subgroup generated by \( \mathcal{P}_+^\uparrow \) and \( \mathcal{s} \).

2.1.2. Mathematical structures. The following mathematical structures, based on a complex and separable Hilbert space \( \mathcal{H} \), are of general interest in quantum theory.

- The set \( \Omega(\mathcal{H}) \) of all self-adjoint operators of \( \mathcal{H} \); in a quantum theory these operators represent quantum observables.
- The lattice \( \Pi(\mathcal{H}) \) of all projections operators of \( \mathcal{H} \); in a quantum theory they represent observables with spectrum \( \{0, 1\} \).
- The set \( \Pi_1(\mathcal{H}) \) of all rank one orthogonal projections of \( \mathcal{H} \).
- The set \( \mathcal{S}(\mathcal{H}) \) of all density operators of \( \mathcal{H} \); in a quantum theory these operators represent quantum states.
- The set \( \mathcal{V}(\mathcal{H}) \) of all unitary or anti-unitary operators of the Hilbert space \( \mathcal{H} \).
- The set \( \mathcal{U}(\mathcal{H}) \) of all unitary operators of \( \mathcal{H} \); trivially, \( \mathcal{U}(\mathcal{H}) \subseteq \mathcal{V}(\mathcal{H}) \) holds.

2.1.3. Generalized representations of groups. The following definition introduces generalized notions of group representation.

**Definition 2.1.** Let \( G \) be a separable, locally compact group with identity element \( e \). A correspondence \( U : G \rightarrow \mathcal{V}(\mathcal{H}), g \rightarrow U_g \), with \( U_e = \mathbb{I} \), is a generalized projective representation (shortly gprep) of \( G \) if the following conditions are satisfied.

i) A complex function \( \sigma : G \times G \rightarrow \mathcal{C} \), called multiplier, exists such that \( U_{g_1g_2} = \sigma(g_1, g_2)U_{g_1}U_{g_2} \); the modulus \( |\sigma(g_1, g_2)| \) is always 1, of course;

ii) for all \( \phi, \psi \in \mathcal{H} \), the mapping \( g \rightarrow (U_g\phi | \psi) \) is a Borel function in \( g \).

Whenever \( U_g \) is unitary for all \( g \in G \), \( U \) is called projective representation (shortly \( \text{prep} \)), or \( \sigma \)-representation.

A generalized projective representation is said to be continuous if for any fixed \( \psi \in \mathcal{H} \) the mapping \( g \rightarrow U_g\psi \) from \( G \) to \( \mathcal{H} \) is continuous with respect to \( g \).

If \( g \rightarrow U_g \) is a gprep and \( \theta(g) \in \mathbb{R} \), then \( g \rightarrow \tilde{U}_g = e^{\theta(g)}U_g \) is a gprep equivalent [6] to \( g \rightarrow U_g \).

In [7] we have proved that the following statement holds.

**Proposition 2.1.** If \( G \) is a connected group, then every continuous generalized projective representation of \( G \) is a projective representation, i.e. \( U_g \in \mathcal{U}(\mathcal{H}) \), for all \( g \in G \).
2.2. Generalized representations of the Poincaré group \( \mathcal{P} \)

All sub-groups \( \mathcal{T}_0, \mathcal{T}_j, \mathcal{R}_j, \mathcal{B}_j \) of \( \mathcal{P}_+^1 \) of Poincaré group \( \mathcal{P} \) are additive; in fact, \( \mathcal{B}_j \) is not additive with respect to the parameter relative velocity \( u \), but it is additive with respect to the parameter \( \varphi(u) = \frac{1}{2} \ln \frac{1+u}{1-u} \). Then, according to Stone’s theorem \([9]\), for every continuous prep of \( \mathcal{P}_+^1 \), an equivalent prep \( U \) exists for which there are ten self-adjoint generators \( P_0, P_j, J_j, K_j, j = 1, 2, 3 \), of the ten one-parameter unitary subgroups \( \{e^{iP_j}, a \in \mathbb{R}\}, \{e^{-ij\varphi(a)}, \varphi \in \mathbb{R}\}, \{e^{-ijK_j\varphi(u)}, u_j \in \mathbb{R}\} \) of \( U(\mathcal{H}) \) that represent the one-parameter sub-groups \( \mathcal{T}_0, \mathcal{T}_j, \mathcal{R}_j, \mathcal{B}_j \) according to the projective representation \( g \rightarrow U_g \) of the Poincaré group \( \mathcal{P}_+^1 \).

2.2.1. Commutation relations. The structural properties of \( \mathcal{P}_+^1 \) as a Lie group imply that every continuous prep of \( \mathcal{P}_+^1 \) admits an equivalent prep \( U \) such that the following commutation relations \([10]\) hold for its generators.

\[
\begin{align*}
(i) \quad [P_j, P_k] &= 0, & (ii) \quad [J_j, P_k] &= i\epsilon_{jkl}P_l, & (iii) \quad [J_j, J_k] &= i\epsilon_{jkl}J_l, \\
(iv) \quad [J_j, K_k] &= i\epsilon_{jkl}K_l, & (v) \quad [K_j, K_k] &= -i\epsilon_{jkl}J_l, & (vi) \quad [K_j, P_k] &= i\epsilon_{jkl}P_l, \\
(vii) \quad [P_j, P_0] &= 0, & (viii) \quad [J_j, P_0] &= 0, & (ix) \quad [K_j, P_0] &= 0,
\end{align*}
\]

where \( \epsilon_{jkl} \) is the Levi-Civita symbol \( \epsilon_{jkl} \) restricted by the condition \( j \neq l \neq k \).

Let \( U : \mathcal{P} \rightarrow \mathcal{U}(\mathcal{H}) \) be a ggrep of \( \mathcal{P} \) whose restriction to \( \mathcal{P}_+^1 \) is continuous. Since time reversal \( \mathcal{T} = U_\mathcal{T} \) and space inversion \( \mathcal{S} = U_\mathcal{S} \) can be unitary or anti-unitary; The phase factor \( e^{i\theta(a)} \) can be chosen in such a way that the following statements hold in the equivalent ggrep \([10]\).

If \( \mathcal{S} \) is unitary, then \( \{\mathcal{S}, P_0\} = 0, \quad \mathcal{S}P_j = -P_j\mathcal{S}, \quad \mathcal{S}J_j = -J_j\mathcal{S}, \quad \mathcal{S}K_j = -K_j\mathcal{S}, \quad \mathcal{S}^2 = \mathbb{I}; \) \( (2) \)

If \( \mathcal{S} \) is anti-unitary, then \( \mathcal{S}P_0 = -P_0\mathcal{S}, \quad \mathcal{S}P_j = \mathcal{S}P_j, \quad \mathcal{S}J_j = -J_j\mathcal{S}, \quad \mathcal{S}K_j = K_j\mathcal{S}; \) \( (3) \)

\( \mathcal{S}^2 = c\mathbb{I}, \) so that \( \mathcal{S}^{-1} = c\mathcal{S} \), where \( c = 1 \) or \( c = -1 \).

If \( \mathcal{T} \) is unitary, then \( \mathcal{T}P_0 = -P_0\mathcal{T}, \quad \mathcal{T}P_j = \mathcal{T}P_j, \quad \mathcal{T}J_j = \mathcal{T}J_j, \quad \mathcal{T}K_j = -K_j\mathcal{T}, \quad \mathcal{T}^2 = \mathbb{I}; \) \( (4) \)

If \( \mathcal{T} \) is anti-unitary, then \( \mathcal{T}P_0 = P_0\mathcal{T}, \quad \mathcal{T}P_j = -P_j\mathcal{T}, \quad \mathcal{T}J_j = K_j\mathcal{T}, \quad \mathcal{T}K_j = K_j\mathcal{T}; \) \( (5) \)

\( \mathcal{T}^2 = c\mathbb{I}, \) so that \( \mathcal{T}^{-1} = c\mathcal{T} \), either \( c = 1 \) or \( c = -1 \) must hold.

The commutation condition for the pair \( \mathcal{S}, \mathcal{T} \), is \( \mathcal{S}\mathcal{T} = \omega \mathcal{T}\mathcal{S}, \) with \( \omega \in \mathbb{C} \) and \( |\omega| = 1 \). \( (6) \)

2.3. Classifying and identifying irreducible representations of \( \mathcal{P} \)

From now on we specialize to irreducible generalized projective representations of \( \mathcal{P} \), with \( U \mid_{\mathcal{P}_+^1} \) continuous, where \((1)-(6)\) hold. The following proposition holds \([8]\).

**Proposition 2.2.** Properties \((1)-(6)\) imply that if a ggrep \( U \) of \( \mathcal{P} \) is irreducible, then two real numbers \( \eta, \varpi \) exist such that the following equalities hold.

\[
\begin{align*}
(i) \quad P_0^2 - P^2 &= \eta \mathbb{I} \quad \text{and} \quad (ii) \quad W^2 &= W_0^2 - (W_1^2 + W_2^2 + W_3^2) = \varpi \mathbb{I}
\end{align*}
\]

where \( W_0 = \mathbf{P} \cdot \mathbf{J} \) and \( W_j = P_0 J_j - (\mathbf{P} \times \mathbf{K})_j \).

Therefore every irreducible ggrep of \( \mathcal{P} \) is characterized by the real constant \( \eta, \varpi \). We restrict our investigation to those irreducible ggrep of \( \mathcal{P} \) for which \( \eta > 0 \), so that \( \eta = \mu^2 \), with \( \mu > 0 \); with this restriction it can be proved that \( s \in \frac{1}{2}\mathbb{N} \) exists such that \( \varpi = \mu s(s + 1) \).

The following proposition \([2]\) establishes that the spectrum \( \sigma(P_0) \) of \( P_0 \) must be one of three definite subsets \( I_2^+, I_2^-, I_2^+ \cup I_2^- \) of \( \mathbb{R} \), where \( I_2^+ = [\mu, \infty) \) and \( I_2^- = (-\infty, -\mu] \). The different possibilities are ruled over by the following proposition.
Proposition 2.3. If $U : \mathcal{P} \to \mathcal{V}(\mathcal{H})$ is an irreducible generalized projective representation, then there are only the following mutually exclusive possibilities for the spectrum $\sigma(P_0)$ of $P_0$.

$(u)$ $\sigma(P_0) = I^+_\mu$ and $\sigma(P_0) = [\mu, \infty]$, up spectrum;

$(d)$ $\sigma(P_0) = I^-_\mu$ and $\sigma(P_0) = (-\infty, -\mu]$, down spectrum;

$(s)$ $\sigma(P_0) = I^+_\mu \cup I^-_\mu$, and $\sigma(P_0) = [\mu, \infty) \cup (-\infty, -\mu]$, symmetrical spectrum.

If $^\mathcal{T}$ is anti-unitary and $^\mathcal{S}$ is unitary, then either $\sigma(P_0) = I^+_\mu$ or $\sigma(P_0) = I^-_\mu$, and hence $\sigma(P_0) = I^+_\mu \cup I^-_\mu$ cannot occur.

If $^\mathcal{S}$ is anti unitary then $\sigma(P_0) = I^+_\mu \cup I^-_\mu$, independently of $^\mathcal{T}$.

If $^\mathcal{T}$ is anti unitary then $\sigma(P_0) = I^+_\mu \cup I^-_\mu$, independently of $^\mathcal{T}$.

Given an irreducible gprep $U$ of $\mathcal{P}$, we define the projection operators $E^\pm = \int_{P_0} E^{(0)}(\psi)$ where $E^{(0)}$ is the resolution of the identity of $P_0$. In [2] we prove the following proposition.

Proposition 2.4. In an irreducible gprep $U$ of $\mathcal{P}$ the relation $[E^\pm, U_g] = \Phi$ holds for all $g \in \mathcal{P}_+^\uparrow$.

According to Prop. 2.4, in the case of symmetrical spectrum $\sigma(P_0) = I^+_\mu \cup I^-_\mu$, the restriction $U \mid_{P_+^\uparrow}$ is always reduced by $E^+$ into $U^+ \mid_{P_+^\uparrow} = E^+ U \mid_{P_+^\uparrow}$ and by $U^- \mid_{P_+^\downarrow} = E^- U \mid_{P_+^\downarrow}$. If $\sigma(P_0) = I^+_\mu$ (resp., $\sigma(P_0) = I^-_\mu$), then $U \mid_{P_+^\uparrow} = U^+ \mid_{P_+^\uparrow}$ (resp., $U \mid_{P_+^\downarrow} = U^- \mid_{P_+^\downarrow}$).

In any case, given an irreducible gprep $U : \mathcal{P} \to \mathcal{U}(\mathcal{H})$, once established which of the three conditions $(u)$, $(d)$, $(s)$ is satisfied, $U^+ \mid_{P_+^\uparrow}$ or $U^- \mid_{P_+^\downarrow}$ can be reducible or not, though the “mother” representation $U$ is irreducible. In sections 2.3.1 and 2.3.2 we completely identify the possible irreducible gpreps $U$ of $\mathcal{P}$ for which $U^+ \mid_{P_+^\uparrow}$ and $U^- \mid_{P_+^\downarrow}$ are irreducible. For reasons of space, we restrict ourselves to the case $s = \varpi = 0$; nevertheless, the identification is extendable to all values of $s$ [2].

In section 2.4 we show that also irreducible gpreps of $\mathcal{P}$ exist with $U^+ \mid_{P_+^\uparrow}$ (or $U^- \mid_{P_+^\downarrow}$) reducible exist, though they are never considered by the literature.

2.3.1. The case $\sigma(P_0) = I^+_\mu$ with $U^\pm \mid_{P_+^\uparrow}$ irreducible. For each value $\mu = \sqrt{\eta} > 0$, modulo unitary isomorphisms there is only one irreducible projective representation of $\mathcal{P}$ with $\sigma(P_0) = I^+_\mu$ and only one with $\sigma(P_0) = I^-_\mu$, that we briefly present. Once fixed $\mu$, the Hilbert space of the projective representation is the space $L_2(\mathbb{R}^3, dv)$ of all complex functions $\psi : \mathbb{R}^3 \to \mathbb{C}$, $p \rightarrow \psi(p)$, square integrable with respect to the measure $dv(p) = \frac{dp_1 dp_2 dp_3}{\sqrt{\mu^2 + p^2}}$.

For the well known [10] irreducible preps with $\sigma(P_0) = I^+_\mu$ the following statements hold.

- The generators $P_j$ are the multiplication operators defined by $(P_j \psi)(p) = p_j \psi(p)$;
- $(P_0 \psi)(p) = p_0 \psi(p)$ where $p_0 = +\sqrt{\mu^2 + p^2}$; this $P_0$ has a positive spectrum;
- $J_k = i \left( p_k \frac{\partial}{\partial p_0} - p_j \frac{\partial}{\partial p_l} \right)$, $(k, l, j)$ being a cyclic permutation of $(1, 2, 3)$;
- the generators $K_j$ are given by $K_j = ip_0 \frac{\partial}{\partial p_j}$;
- the space inversion operator is $^\mathcal{S} = \mathcal{T}$ and the time reversal operator is $^\mathcal{T} = K^\mathcal{T}$, where $^\mathcal{T}$ is the unitary operator defined by $(^\mathcal{T} \psi)(p) = \psi(-p)$,
- $K$ is the anti-unitary complex conjugation operator defined by $K \psi(p) = \overline{\psi(p)}$.

The irreducible projective representation with $\sigma(P_0) = I^-_\mu$ differs only for $P_0$ and $K$: $(P_0 \psi)(p) = -p_0 \psi(p)$, and $K_j = -ip_0 \frac{\partial}{\partial p_j}$.
2.3.2. The case \( \sigma(P_0) = I_\mu^+ \cup I_\mu^- \) with \( U^+_{|\mathcal{P}_\perp} \) irreducible. For every value \( \mu > 0 \) there are six inequivalent irreducible generalized projective representations \( U^{(l)}, l = 1, 2, \ldots, 6 \) of \( \mathcal{P} \); modulo unitary isomorphisms, the Hilbert space of each \( U^{(l)} \) is \( L_2(\mathbb{R}^3, d\nu) \oplus L_2(\mathbb{R}^3, d\nu) \). If each vector \( \psi \in \mathcal{H} \) is represented as the column vector \( \psi = \begin{bmatrix} \psi^+ \\ \psi^- \end{bmatrix} \), with \( \psi^\pm \in L_2(\mathbb{R}^3, d\nu) \), then \( E^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \), \( E^- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \), and the generators of \( U^{(l)}_{|\mathcal{P}_\perp} \) are represented by the following matrices, known as the canonical form.

\[
P_j = \begin{bmatrix} p_j & 0 \\ 0 & -p_j \end{bmatrix}, \quad P_0 = \begin{bmatrix} p_0 & 0 \\ 0 & -p_0 \end{bmatrix}, \quad J_k = \begin{bmatrix} j_k & 0 \\ 0 & -j_k \end{bmatrix}, \quad K_j = \begin{bmatrix} k_j & 0 \\ 0 & -k_j \end{bmatrix},
\]

(8)

where \( j_k = i \left( p_k \frac{\partial}{\partial p_k} - p_j \frac{\partial}{\partial p_j} \right) \) and \( k_j = ip_0 \frac{\partial}{\partial p_0} \).

The six representations \( U^{(l)}, l = 1, 2, \ldots, 6 \) differ just for the different combinations of time reversal and space inversion operators, according to the following list.

- \( U^{(1)} \) has unitary \( T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and unitary \( S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \);
- \( U^{(2)} \) has unitary \( T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and unitary \( S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \);
- \( U^{(3)} \) has unitary \( T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and anti-unitary \( S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} K \);
- \( U^{(4)} \) has unitary \( T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and anti-unitary \( S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} K \);
- \( U^{(5)} \) has anti-unitary \( T = 1K^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and anti-unitary \( S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} K \);
- \( U^{(6)} \) has anti-unitary \( T = 1K^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and anti-unitary \( S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} K \).

Thus we have identified, besides the well known \([10]\) irreducible \( g^{preps} U^{(1)}, U^{(2)}, \) the \( g^{preps} U^{(3)}, U^{(4)} \) with \( T \) anti-unitary and \( S \) unitary, \( U^{(5)}, U^{(6)} \) or with \( T \) anti-unitary and \( S \) anti-unitary, which are neglected by the literature about elementary particles theory, at the best of our knowledge.

2.4. New species of \( g^{preps} \): irreducible \( U : \mathcal{P} \rightarrow \mathcal{U}(\mathcal{H}) \) with \( U^\pm_{|\mathcal{P}_\perp} \) reducible

Hence, for each \( \mu > 0 \) there are eight inequivalent irreducible \( g^{preps} \) of \( \mathcal{P} \). The class of all such octets, however, does not exhaust the class \( \mathcal{I}_U \) of all irreducible \( g^{preps} \) of \( \mathcal{P} \) with \( s = 0 \), because the class \( \mathcal{I}_U(U^\pm\text{red}.) \) of the irreducible representations with \( U^+_{|\mathcal{P}_\perp} \) or \( U^-_{|\mathcal{P}_\perp} \) reducible is not empty.

2.4.1. The case \( \sigma(P_0) = I_\mu^\pm \). Fixed any \( \mu > 0 \), let us consider the Hilbert space \( \mathcal{H} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \), where \( \mathcal{H}^{(1)} = \mathcal{H}^{(2)} = L_2(\mathbb{R}^3, d\nu) \). We represent every vector \( \psi = \psi_1 + \psi_2 \in \mathcal{H} \), with \( \psi_n \in \mathcal{H}^{(n)} \), as the column vector \( \psi \equiv \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \), so that every linear (resp., anti-linear) operator \( A \) of \( \mathcal{H} \) can be represented by a matrix \( A_{11} \begin{bmatrix} A_{12} \\ A_{21} \end{bmatrix} A_{22} \), where \( A_{nm} \) is a linear (resp., anti-linear) operator from \( \mathcal{H}^{(m)} \) to \( \mathcal{H}^{(n)} \), and \( A\psi = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} A_{11}\psi_1 + A_{12}\psi_2 \\ A_{21}\psi_1 + A_{22}\psi_2 \end{bmatrix} \).

Let us define \( P_0 = \begin{bmatrix} p_0 & 0 \\ 0 & p_0 \end{bmatrix}, P_j = \begin{bmatrix} p_j & 0 \\ 0 & -p_j \end{bmatrix}, J_k = \begin{bmatrix} j_k & 0 \\ 0 & -j_k \end{bmatrix} \), \( J_j = \begin{bmatrix} k_j & 0 \\ 0 & -k_j \end{bmatrix} \), they satisfy (1).
In such a way, a reducible prep $U : \mathcal{P}_+^1 \rightarrow L_2(\mathbb{R}^3, dv) \oplus L_2(\mathbb{R}^3, dv)$ is completely determined, where $E^+ = \mathbb{I}$, so that $U^+ = U$. The possible extensions to the whole $\mathcal{P}$ are obtained by adding a time reversal operator $\mathcal{T}$ and a space inversion operator $\mathcal{S}$ that satisfy (2)-(6). Now we show that some of these extensions are irreducible gpreps of $\mathcal{P}$. Indeed, the operators

$$
\mathcal{S} = \mathcal{T} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{T} = \mathcal{K} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
$$

(10)

satisfy all conditions (2)-(6). Therefore, by defining $U_\mathcal{S} = \mathcal{T} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad U_\mathcal{T} = \mathcal{K} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}

the projective representation $U$ of $\mathcal{P}_+^1$ is extended to a generalized projective representation of $\mathcal{P}$. Moreover, if $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is any self-adjoint operator of $\mathcal{H} = L_2(\mathbb{R}^3, dv) \oplus L_2(\mathbb{R}^3, dv)$, then the conditions $[A, P_0] = [A, P_j] = [A, J_k] = [A, K_j] = [A, \mathcal{S}] = [A, \mathcal{T}] = \mathcal{O}$ imply $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \equiv a\mathbb{I}$. Thus, this generalized projective representation $U$ of $\mathcal{P}$ is irreducible, while $U^+ \mid_{\mathcal{P}_+^1}$ is reducible.

2.4.2. The case $\sigma(P_0) = I_\mu^+ \cup I_\mu^-$. Let us introduce the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3, dv) \oplus L_2(\mathbb{R}^3, dv) \oplus L_2(\mathbb{R}^3, dv) \oplus L_2(\mathbb{R}^3, dv)$. Every vector $\psi \in \mathcal{H}$ is represented as a column vector

$$
\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}, \quad \text{with} \quad \psi \in L_2(\mathbb{R}^3, dv).
$$

To identify a gprep $U$ of $\mathcal{P}$ with $U^+ \mid_{\mathcal{P}_+^1}$ reducible, we first define the self-adjoint generators of a prep $U : \mathcal{P}_+^1 \rightarrow \mathcal{U}(\mathcal{H})$ as

$$
P_0 = \begin{bmatrix} p_0 & 0 & 0 & 0 \\ 0 & -p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & -p_0 \end{bmatrix}, \quad P_j = \begin{bmatrix} p_j & 0 & 0 & 0 \\ 0 & p_j & 0 & 0 \\ 0 & 0 & p_j & 0 \\ 0 & 0 & 0 & p_j \end{bmatrix},
$$

$$
J_k = \begin{bmatrix} j_k & 0 & 0 & 0 \\ 0 & j_k & 0 & 0 \\ 0 & 0 & j_k & 0 \\ 0 & 0 & 0 & j_k \end{bmatrix}, \quad K_j = \begin{bmatrix} k_j & 0 & 0 & 0 \\ 0 & -k_j & 0 & 0 \\ 0 & 0 & k_j & 0 \\ 0 & 0 & 0 & -k_j \end{bmatrix}.
$$

So, a reducible projective representation of $\mathcal{P}_+^1$ is defined. To extend it to an irreducible gprep of $\mathcal{P}$ we introduce $\mathcal{K} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (unitary) and $\mathcal{S} = \mathcal{K} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$ (anti-unitary), that satisfy (2)-(6).

Now, if $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}$ is any self-adjoint operator of $\mathcal{H}$, then the conditions $[A, P_0] = [A, P_j] = [A, J_k] = [A, K_j] = [A, \mathcal{S}] = [A, \mathcal{T}] = \mathcal{O}$ are satisfied if and only if

$$
A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \equiv a\mathbb{I} \quad \text{with} \quad a \in \mathbb{R}.
$$

Thus, a new species of irreducible gprep of $\mathcal{P}$ is identified.
3. Consistent relativistic quantum theories of elementary spin-0 particle

By localizable free particle, shortly free particle, we mean an isolated system whose quantum theory is endowed with a unique triple \((Q_1, Q_2, Q_3) \equiv Q\) of quantum observables, called position operator, such that

\[
(Q.1) \quad [Q_j, Q_k] = 0, \text{ for all } j, k \in \{1, 2, 3\}.
\]

This condition requires that a measurement of position yields all three values of the coordinates of the particle position.

\[(Q.2)\] The triple \((Q_1, Q_2, Q_3) \equiv Q\) is characterized by the specific properties of transformation of position with respect to the group \(\mathcal{P}\); hence, in particular,

\[
(Q.2.a) \quad S^T[g][Q] = Q \quad \text{and} \quad S[g][Q] = -Q, \quad \text{equivalent to} \quad Q \rightarrow Q^T = Q \quad \text{and} \quad S(Q) = -Q.
\]

\[(Q.2.b)\] If \(g \in \mathcal{E}\) then \(S_g[Q] = U_g QU_g^{-1} = g(Q)\), where \(x \rightarrow g(x)\) is the function that realizes \(g\).

A free particle is said elementary if the group \(U\) for which \(S_g[A] = U_g AU_g^{-1}\) is irreducible. Accordingly, by selecting the irreducible group \(U\) of \(\mathcal{P}\), satisfying (1)-(6), that admit such a triple \(Q\) satisfying \((Q.1)\) and \((Q.2)\) we identify the possible theories of an elementary free particle.

3.1. Elementary particle theories with \(s = 0\), and \(U^\pm |p^\pm\) irreducible

For the irreducible groups with \(\sigma(P_0) = I_P^+\) of section 2.3.1, conditions \((Q.1)\) and \((Q.2.a,b)\) are sufficient [2] to univocally determine \(Q\) as \(Q_j = F_j\), where \(F_j = i \frac{\partial}{\partial \phi_j} - \frac{1}{2m} p_j\) are the Newton and Wigner operators [11].

If \(\sigma(P_0) = I_P^+ \cup I_P^-\), \((Q.1)\) and \((Q.2.a,b)\) are sufficient [2] to univocally determine \(Q\) only for \(U^{(5)}\) and \(U^{(6)}\) in section 2.3.1, where \(Q = \tilde{F} = \begin{bmatrix} F_j & 0 \\ 0 & F_j \end{bmatrix}\). So we have four complete theories. Thus, our approach proves that anti-unitary space inversion operators, discarded in the literature, are consistent in a particle theory with \(U^\pm |p^\pm\) irreducible and \(\sigma(P_0)\) symmetrical.

The early theory for such a kind of particle is Klein-Gordon theory [12]-[14], that suffered serious problems.

A first problem is that the wave equation of Klein-Gordon theory is second order in time, while according to the general laws of quantum theory it should be first order.

Furthermore, in Klein-Gordon theory \(\tilde{\rho}(t, x) = i \frac{\partial}{\partial \psi} \tilde{\Psi}_t = \tilde{\psi}_t \frac{\partial}{\partial \tilde{\psi}_t}\) was interpreted as the probability of position density, and \(\tilde{j}(t, x) = i \frac{\partial}{\partial \psi} \tilde{\psi}_t = \tilde{\psi}_t \frac{\partial}{\partial \tilde{\psi}_t}\) as its current density. This interpretation is at the basis of the Dirac concern that, due to the time derivative of \(\psi\), such position probability density can be negative. A way to overcome the difficulty was proposed by Feshbach and Villars [15]. They derive an equivalent form of Klein-Gordon equation as a first order equation \(i \frac{\partial}{\partial t} \psi_t = H \Psi_t\) for the state vector \(\Psi_t = \begin{bmatrix} \phi_t \\ \chi_t \end{bmatrix}\), where \(\phi_t = \frac{1}{\sqrt{2}} (\psi_t + \frac{1}{m} \frac{\partial}{\partial \psi} \psi_t)\), \(\chi_t = \frac{1}{\sqrt{2}} (\psi_t - \frac{1}{m} \frac{\partial}{\partial \psi} \psi_t)\), and \(H = (\sigma_3 + \sigma_2) \frac{1}{2m}(\nabla + m \sigma_3)\), \(\psi_t\) being the Klein-Gordon wave function and the \(\sigma_j\)’s are the Pauli matrices; in this representation \(\tilde{\rho} = \frac{1}{2} (|\phi_t|^2 - |\chi_t|^2)\), without time derivatives. The minus sign in \(\tilde{\rho}\) forbids to interpret it as probability density of position; Feshbach and Villars proposed to interpret it as density probability of charge. Nevertheless, according to Barut and Malin [16], covariance with respect to boosts should imply that \(\tilde{\rho}\) must be the time component of a four-vector. Barut and Malin proved that is not the case!

In order to check our theories with respect to these problems, the theories based on \(U^{(5)}\) and \(U^{(6)}\) are reformulated in equivalent forms, obtained by means of unitary transformations operated by the unitary operator \(Z = Z_1 Z_2\), where \(Z_2 = \frac{1}{\sqrt{m}} I\) and \(Z_1\) is the inverse of the Fourier-Plancherel operator, that transforms \(\psi(p)\) into \((Z \psi)(x) \equiv (\hat{\psi})(x)\). In the so reformulated theories
the Hilbert space for both turns out to be \( \mathcal{H} = Z \left( L_2(\mathbb{R}^3, d\nu) \oplus L_2(\mathbb{R}^3, d\nu) \right) \equiv L_2(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3) \); the new self-adjoint generators are
\[
\hat{P}_j = \left[ -i \frac{\partial}{\partial x_j}, 0 \right], \quad \hat{P}_0 = \sqrt{\mu^2 - \nabla^2} \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right], \quad \hat{J}_k = -i \left( x_l \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_l} \right) \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \quad \hat{K}_j = \frac{1}{2} \left( x_j \sqrt{\mu^2 - \nabla^2} + \sqrt{\mu^2 - \nabla^2} x_j \right) \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right].
\]
The wave equation trivially is
\[
i \frac{\partial}{\partial t} \psi_t = \hat{P}_0 \psi_t,
\]
that is first order.

The Hilbert space for both turns out to be \( \mathcal{H} = Z \left( L_2(\mathbb{R}^3, d\nu) \oplus L_2(\mathbb{R}^3, d\nu) \right) \equiv L_2(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3) \); the new self-adjoint generators are
\[
\hat{P}_j = \left[ -i \frac{\partial}{\partial x_j}, 0 \right], \quad \hat{P}_0 = \sqrt{\mu^2 - \nabla^2} \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right], \quad \hat{J}_k = -i \left( x_l \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_l} \right) \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \quad \hat{K}_j = \frac{1}{2} \left( x_j \sqrt{\mu^2 - \nabla^2} + \sqrt{\mu^2 - \nabla^2} x_j \right) \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right].
\]
The wave equation trivially is
\[
i \frac{\partial}{\partial t} \psi_t = \hat{P}_0 \psi_t,
\]
that is first order.

The position operator is \( \hat{Q}_j = \left[ \begin{array}{cc} x_j & 0 \\ 0 & x_j \end{array} \right] \), so that also the other problems disappear. Indeed, the position is represented by the multiplication operator; therefore, the probability density of position must necessarily be given by the non negative function
\[
\rho(t, x) = |\psi_t(x)|^2 + |\psi_t(x)|^2.
\]
On the other hand, the covariance properties with respect to boosts, according to \((Q.2)\), are explicitly expressed by
\[
S_g[Q] = e^{iK_j \varphi(u)} Q e^{-iK_j \varphi(u)},
\]
being \( K_j \) and \( Q \) explicitly known, and there is no need of a four-density concept.

### 3.2. New species of particle theories

Conditions \((Q.1), (Q.2.a,b)\) univocally determine \([2]\) the position operator also for the irreducible gprep of section 2.3.3 with \( \sigma(P_0) = I_\mu \) \([2]\) where \( \mathcal{H} = L_2(\mathbb{R}^3, d\nu) \oplus L_2(\mathbb{R}^3, d\nu) \), and again
\[
\hat{Q}_j = \left[ \begin{array}{cc} F_j & 0 \\ 0 & F_j \end{array} \right].
\]
Thus, complete theories of an elementary free particle turn out to be identified, which corresponds to none of the early theories. Only Nature establishes whether the zoo of all existing particles also hosts also particles described by these theories.

---

[1] Wigner EP. 1959 Group Theory and its Applications to the Quantum Theory of Atomic Spectra (Boston: Academic Press)
[2] Nisticò G. 2018 Arxiv:1811.01546
[3] Costa G, Fogli G. 2012 Lecture Notes in Physics 823 (New York: Springer)
[4] Weinberg S. 1995 The quantum theory of fields. Vol.I (Cambridge: Cambridge University press)
[5] Barut AO, Racza R. 1986 Theory of group representations and applications (Singapore: World Scientific)
[6] Mackey GW. 1968 Induced Representations of Group and Quantum Mechanics (New York: Benjamin Inc.)
[7] Nisticò G. 2017 Group theoretical derivation of the minimum coupling principle. Proc. R. Soc. A473 20160629
[8] Bargmann V, Wigner EP. 1948 Nat.Ac.Sc. 34 211
[9] Simon B. 1976 in Studies in Mathematical Physics: Essays in Honor of Valentine Bargmann ed Lieb E H, Simon B and Wightman A S (Princeton: Princeton Un. Press) p. 327
[10] Foldy LL. 1956 Phys. Rev. 102 568
[11] Newton TD, Wigner EP. 1949 Rev. Mod. Phys. 21 400
[12] Klein O. 1926, Z. Physik 37 895
[13] Fock V. 1926 Z. Physik 37 242
[14] Gordon W. 1926 Z. Physik 40 117
[15] Felsbach H, Villars F. 1958 Rev. Mod. Phys. 30 24
[16] Barut AO, Malin S. 1968 Rev. Mod. Phys. 40 632