The Taylor expansion of Ruelle L-function at the origin and the Borel regulator

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April 17, 2008

Abstract

We will prove that Ruelle L-function for a cuspidal unitary local system on an odd dimensional complete hyperbolic manifold with finite volume satisfies a functional equation and an analogue of Riemann hypothesis. We will also compute its Laurent expansion at the origin and will prove that the second coefficient coincides with a rational multiple of the volume up to a certain contribution from cusps. Moreover if the dimension is three we will identify the leading coefficient. Both of them will be interpreted as a period of a certain element of K-group of C. Also a relation with the $L^2$-torsion will be discussed.

2000 Mathematics Subject Classification : 11M36,11G55, 18F25, 19Bxx,19Dxx,57Q10.

Key words : Ruelle L-function, Selberg trace formula, the Franz-Reidemeister torsion, Cheeger-Müller’s theorem.

1 Introduction

Researches of an L-functions may be roughly classified in the following three subjects:

1. a functional equation,
2. (Riemann hypothesis) a distribution of zeros and poles,
3. an arithmetic or a geometric meaning of special values.

For example let us consider the zeta function for a number field $F$:

$$\zeta_F(s) = \prod_{\mathfrak{p}} (1 - e^{-s \log N(\mathfrak{p})})^{-1}.$$

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Here \( \mathfrak{P} \) runs through all prime ideals and \( N(\mathfrak{P}) \) is the norm. Then \( \zeta_F(s) \) satisfies a functional equation and the Riemann hypothesis is still a far reaching problem. It has a zero at \( s = 0 \) of order \( r_1 + r_2 - 1 \) where \( r_1 \) (resp. \( r_2 \)) is the number of real (resp. complex) places and the leading coefficient of the Taylor expansion is given by

\[
\lim_{s \to 0} s^{-(r_1 + r_2 - 1)} \zeta_F(s) = -\frac{2\text{Pic}(\mathcal{O}_F)}{\mathfrak{z}(\mathcal{O}_F)_{\text{tors}}} \cdot R.
\]

\( R \) is the covolume of the image of \( \mathcal{O}_F^\times \oplus \mathbb{Z} \) by the classical regulator which is defined by the logarithmic function:

\[
\mathcal{O}_F^\times \oplus \mathbb{Z} \xrightarrow{\gamma \mapsto \mathbb{R}^{r_1+r_2}} \mathbb{R}^{r_1+r_2}.
\]

The observation that \( K_1(\mathcal{O}_F) \) is isomorphic to \( \mathcal{O}_F^\times \) and the fact that the order of \( \zeta_F(s) \) at \( s = 1 - l \) is equal to the dimension \( d_l \) of \( K_{2l-1}(F) \otimes \mathbb{Q} \) for \( l \geq 2 \) lead Lichtenbaum to a conjecture; There should be a map

\[
K_{2l-1}(F) \xrightarrow{\gamma \mapsto \mathbb{R}^{d_l}} \mathbb{R}^{d_l}.
\]

such that

\[
\lim_{s \to 1 - l} (s + 1 - l)^{-d_l} \zeta_F(s) = \text{vol}(\mathbb{R}^{d_l}/r_{l,F}(K_{2l-1}(F))).
\]

This was solved by Borel. He has also constructed a map

\[
K_{2l-1}(\mathbb{C}) \xrightarrow{\gamma \mapsto \mathbb{R}},
\]

and each of \( r_{l,F} \) and \( r_l \) is referred as the Borel regulator. In this paper, under a certain condition, we will show that Ruelle L-function for a unitary local system on an odd dimensional hyperbolic manifold (especially a threefold) with finite volume carries similar properties.

Let \( X \) be a complete hyperbolic \( d \)-fold \( (d = 2n + 1, \ n \geq 1) \) of finite volume. Thus it is a quotient of the Poincaré upper half space \( \mathbb{H}^d \) by a torsion free discrete subgroup \( \Gamma \) in \( \text{SO}(d, 1) \), the connected component of the isometry groups \( \text{SO}(d, 1) \) of \( \mathbb{H}^d \). Notice that there is the natural bijection between a set of hyperbolic conjugacy classes \( \Gamma_{\text{hyp}} \) of \( \Gamma \) and a set of closed geodesics of \( X \). By this the length \( l(\gamma) \) of a hyperbolic conjugacy class \( \gamma \) is defined to be one of the corresponding closed geodesic. A closed geodesic will be referred as \textit{prime} if it is not a positive multiple of an another one. Using the bijection we define a subset \( \Gamma_{\text{prim}} \) of hyperbolic conjugacy classes which consists of elements corresponding to prime closed geodesics. Let \( \rho \) be a unitary representation of \( \Gamma \) with degree \( r \), i.e. the dimension of the representation space \( V_\rho \) is \( r \). Now Ruelle L-function is defined to be

\[
R_X(z, \rho) = \prod_{\gamma \in \Gamma_{\text{prim}}} \det[1 - \rho(\gamma)e^{-z(l(\gamma))}].
\]
It absolutely converges if $\text{Res} > 2n$ and is meromorphically continued to the whole plane ([14], see also §2.4). Hereafter, otherwise mentioned, we will assume that $\rho$ is cuspidal (see §2).

We will show that $R_X(z, \rho)$ satisfies a functional equation

$$R_X(z, \rho) \cdot R_X(-z, \rho)^{-1} = \exp\left[\frac{\text{vol}(X)}{\pi} Y(z) + 4 \sum_{j=0}^{n} (-1)^j \delta(X, \rho) z^j\right],$$

where $Y(z)$ is a polynomial of rational coefficients which vanishes at $z = 0$ and $\delta(X, \rho)$ is a certain constant determined by special values of Epstein L-functions of the fundamental groups at cusps (see §2.3). Notice that here and hereafter if $X$ is closed, since it has no cusp, $\delta(X, \rho)$ does not appear. It will be also shown that its zeros and poles are located on

$$\{z \in \mathbb{C} \mid \text{Re} z = -n, -(n-1), \ldots, n-1, n\},$$

except for finitely many of them. For example if $d = 3$, i.e. $n = 1$,

$$R_X(z, \rho) \cdot R_X(-z, \rho)^{-1} = \exp\left[\frac{2r}{\pi} \text{vol}(X)(\frac{z^3}{3} - 3z)\right].$$

Its logarithmic derivative

$$r_X(z, \rho) + r_X(-z, \rho) = \frac{2r}{\pi} \text{vol}(X)(z^2 - 3), \quad r_X(z, \rho) = \frac{d}{dz} \log R_X(z, \rho),$$

may be compared to the functional equation of Weil conjecture. In fact Hasse-Weil’s congruent zeta function of a smooth projective variety $M$ with dimension $m$ over a finite field $\mathbb{F}_q$ is defined to be

$$\zeta_M(z) = \prod_{P \in |M|} (1 - q^{-\text{deg}(x)})^{-1},$$

where $|M|$ is the set of closed points of $M$ and $\text{deg}(x)$ is the extension degree of the residue field of $x$ over $\mathbb{F}_q$. Weil conjecture implies that

$$\sigma_M(z) = \frac{d}{dz} \log_q \zeta_M(z + \frac{m}{2}),$$

satisfies

$$\sigma_M(z) + \sigma_M(-z) = \chi(M),$$

where $\chi(M)$ is the Euler characterisite of $M$. Thus replacing $\chi(M)$ by $2r \text{vol}(X)(z^2 - 3)/\pi$ (which may be not so absurd if we think about Gauss-Bonnet’s formula), we find that the logarithmic derivative of Ruelle L-function for a cuspidal unitary local system on a hyperbolic threefold satisfies a functional equation similar to $\sigma_M(s)$.  

3
Let us expand $R_X(z, \rho)$ at the origin which is a symmetric point of the functional equation:

$$R_X(z, \rho) = c_0 z^h(1 + c_1 z + \cdots), \quad c_0 \neq 0.$$  

We are interested in the coefficients $c_0$ and $c_1$.

**Theorem 1.1.** $c_1 - 2 \sum_{j=0}^{n} (-1)^j \delta(X, \rho)$ is a rational multiple of $\text{vol}(X)/\pi$.

For example if $d = 3$, we will show

$$c_1 = -3r \frac{\text{vol}(X)}{\pi}.$$  

Combining with the results of Goncharov ([6]) this yields

**Corollary 1.1.** There is an element $\gamma_X \in K_{2n+1}(\mathbb{C})$ called the Borel element so that $c_1 - 2 \sum_{j=0}^{n} (-1)^j \delta(X, \rho)$ is a rational multiple of $r_{n+1}(\gamma_X)/\pi$.

In §4 we will recall a construction of $\gamma_X$ for a closed hyperbolic threefold and will show

$$c_1 = -3r \frac{\text{vol}(X)}{\pi}.$$  

It is natural to expect that the leading coefficient $c_0$ has a similar interpretation. In fact it is true at least if $d = 3$.

**Theorem 1.2.** Let $h^p(X, \rho)$ be the dimension of $H^p(X, \rho)$. Suppose that $d = 3$. Then $R_X(z, \rho)$ has a zero at the origin of order $2 h^1(X, \rho)$ and

$$c_0 = (\tau^*(X, \rho) \cdot \text{Per}(X, \rho))^2.$$  

Here $\tau^*(X, \rho)$ and $\text{Per}(X, \rho)$ are the modified Franz-Reidemeister torsion and the period of $(X, \rho)$, respectively. (See §3.4.) (If $h^1(X, \rho)$ is zero $\tau^*(X, \rho)$ is the usual Franz-Reidemeister torsion $\tau(X, \rho)$. If $X$ is closed the theorem has been already proved by Fried ([5]). In fact he has proved it for a closed odd dimensional hyperbolic manifold.

Suppose $d = 3$ and that $h^1(X, \rho)$ vanishes. Let us fix a triangulation of $X$ and a unitary basis $\{e_1, \cdots, e_r\}$ of $V_\rho$. By Poincaré duality we know that all $H^p(X, \rho)$ vanishes for all $p$ and therefore the cochain complex $C^*(X, \rho)$ is acyclic. Then Milnor has constructed an element $\tau(X, \rho, e)$ in $K_1(\mathbb{C}) \simeq \mathbb{C}^\times$ which is referred as the Milnor element and has shown ([12]):

$$\log \tau(X, \rho) = 2\pi r_1(\tau(X, \rho, e)).$$

Summarizing there are rational numbers $\alpha$ and $\beta$ such that

$$\log R_X(0, \rho) = \alpha \pi r_1(\tau(X, \rho, e)), \quad \frac{d}{dz} \log R_X(z, \rho)|_{z=0} = \frac{\beta}{\pi} \cdot r_2(\gamma_X).$$
Thus replacing the logarithmic derivative by a shift:
\[ f = f(z) \rightarrow f^{[k]}(z) = f(z - k), \]
our formula will correspond to Lichtenbaum conjecture.

In [14], Park has obtained

**Fact 1.1.** Let \( X \) be an odd dimensional complete hyperbolic manifold with finite volume and \( \rho \) a unitary local system on \( X \) which may not satisfy the cuspidal condition. Then the leading coefficient of the Laurent expansion at the origin is \( \exp(-\zeta_X'(0, \rho)) \) where \( \zeta_X(s, \rho) \) is the spectral zeta function (see §3.4).

By Hodge theory \( H^p(X, \rho) \) is isomorphic to \( \text{Ker} \Delta^p_X \), the kernel of Hodge Laplacian \( \Delta^p_X \). They are subspaces of \( C'(X, \rho) \) and \( L^2(X, \Omega^p(\rho)) \), the space of square integrable sections of \( p \)-forms twisted by \( \rho \) on \( X \), respectively. (Here notice that \( H^p(X, \rho) \) is isomorphic to the kernel of combinatoric Laplacian acting on \( C^p(X, \rho) \), see §3.4.) Using this two metrics will be defined on the determinant line bundle \( \text{det} H^*(X, \rho) \). One is Franz-Reidemeister metric which is defined in terms of the combinatoric \( L^2 \)-norm \( \cdot \cdot_{L^2, X} \) induced from the natural metric on \( C'(X, \rho) \) and the modified Franz-Reidemeister torsion:
\[ || \cdot ||_{FR} = \cdot \cdot_{L^2, X} \cdot \tau^*(X, \rho)^{1/2}. \]

The other is Ray-Singer metric:
\[ || \cdot ||_{RS} = \cdot \cdot_{L^2, X} \cdot \exp\left(-\frac{1}{2} \zeta_X(0, \rho)\right), \]
where \( \cdot \cdot_{L^2, X} \) is the analytic \( L^2 \)-norm derived from the inner product on \( L^2(X, \Omega^p(\rho)) \). Since by definition Per\( (X, \rho) \) is \( \cdot \cdot_{L^2, X}/\cdot \cdot_{L^2, X} \), Theorem 2.1 is reduced to show the following Cheeger-Müller type theorem.

**Theorem 1.3.**
\[ || \cdot ||_{FR} = || \cdot ||_{RS}. \]

For a convenience we will give a proof of Fact 1.1 in Appendix under an assumption that \( d = 3 \) and \( \rho \) is cuspidal. In particular the last assumption implies that it is not necessary to take care of the scattering term in Selberg trace formula and one can prove the desired result just following Fried’s argument ([5]).

**Acknowledgment.** The author express heartly gratitude to Professor J. Park who kindly show him a preprint [14] which is indispensab le to finish this work.

## 2 The second coefficient

Let
\[ \mathbb{H}^d = \{(x_1, \cdots, x_{d+1}) \mid x_1^2 + \cdots + x_d^2 - x_{d+1}^2 = -1, x_{d+1} > 0\} \]
be the hyperbolic space form \((d = 2n + 1, \; n \geq 1)\). We will choose its origin to be \(o = (0, \cdots, 0, 1)\). The connected component \(G = \text{SO}(d, 1)\) of its isometry group \(\text{SO}(d, 1)\) transitively acts on \(\mathbb{H}^d\) and the isotropy subgroup at \(o\) is a maximal compact subgroup \(K = \text{SO}(d)\). Thus we have a surjective map

\[ G \xrightarrow{\pi} \mathbb{H}^d, \quad \pi(g) = g \cdot o, \]

which induces a diffeomorphism

\[ G/K \simeq \mathbb{H}^d. \tag{1} \]

Let \(\mathfrak{g}\) be the Lie algebra of \(G\) and \(\theta\) the Cartan involution. We define the normalized Cartan-Killing form to be

\[ (X, Y) = -\frac{1}{4\pi} \text{Tr}(\text{ad}X \circ \text{ad}(\theta Y)), \quad X, Y \in \mathfrak{g}. \]

The Cartan involution provides a decomposition of the Lie algebra \(\mathfrak{g}\) of \(G\):

\[ \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}, \]

where \(\mathfrak{k}\) and \(\mathfrak{h}\) are the +1- and −1-eigenspaces, respectively. \(\mathfrak{h}\) may be identified with the tangent space of \(\mathbb{H}^n\) at the origin and the normalized Cartan-Killing form defines a Riemannian metric on \(\mathbb{H}^n\) with constant curvature −1. Let \(\mathfrak{a}\) be the maximal abelian subalgebra of \(\mathfrak{h}\) and \(\beta\) the positive restricted root of \((\mathfrak{g}, \mathfrak{a})\). Then \(\mathfrak{a}\) is one dimensional and let us choose \(H \in \mathfrak{a}\) satisfying \(\beta(H) = 1\). Then the Lie subgroup \(A\) of \(\mathfrak{a}\) is isomorphic to \(\mathbb{R}\) by a map:

\[ \mathbb{R} \simeq A, \quad t \mapsto \exp(tH). \tag{2} \]

Let \(\mathfrak{n}\) be the positive root space of \(\beta\) and \(N = \exp(\mathfrak{n})\) the associated Lie subgroup. Then using the Iwasawa decomposition

\[ G = KAN, \]

we introduce a Haar measure on \(G\) by

\[ dg = a^{2\rho}dk \cdot da \cdot dn. \]

Here \(\rho = n\beta\) is the half sum of positive roots of \((\mathfrak{g}, \mathfrak{a})\) and \(a^{2\rho} = \exp(2\rho(\log a))\). \(dk\) is the Haar measure on \(K\) whose total mass is one and \(da\) is the push forward of the Lebesgue measure on \(\mathbb{R}\) by (2). The volume form \(dn\) of \(N\) is induced by the normalized Cartan-Killing form. Let \(M \simeq \text{SO}(d-1)\) be the centralizer of \(A\) in \(K\) and \(P = MAN\) a proper parabolic subgroup.

Let \(X\) be a complete hyperbolic \(d\)-fold with finite volume and \(\{\infty_1, \cdots, \infty_h\}\) be the cusps. Thus it is a quotient of \(\mathbb{H}^d\) by a torsion free discrete subgroup \(\Gamma\) in \(G\). A conjugate \(P_\nu = g_\nu Pg_\nu^{-1} (g_\nu \in G)\) corresponds to a cusp \(\infty_\nu\) and the fundamental group \(\Gamma_\nu\) at \(\infty_\nu\) is defined to be

\[ \Gamma_\nu = \Gamma \cap P_\nu. \]
We will normalize so that $\infty_1 = \infty$ and $g_1$ is the identity. Since $\Gamma$ is torsion free $\Gamma_\nu$ is equal to $\Gamma \cap N_\nu$ ($N_\nu = g_\nu N g_\nu^{-1}$) which is a lattice in $\mathbb{R}^{2n}$. Let $\rho$ be a unitary representation of $\Gamma$ of degree $r$ and $\rho_\nu$ its restriction to $\Gamma_\nu$. Since $\Gamma_\nu$ is abelian $\rho_\nu$ is decomposed into a direct sum of characters:

$$\rho_\nu = \bigoplus_{i=1}^{r} \chi_{\nu,i}.$$ 

Throughout the paper we will assume that $\rho$ is cuspidal, i.e. none of $\{\chi_{\nu,i}\}_{1 \leq i \leq r, 1 \leq \nu \leq h}$ is trivial. This terminology will be justified in Lemma 3.1.

Let $\Omega^j_X$ be the vector bundle of $j$-forms on $X$ and $\Omega^j_X(\rho)$ its twist by $\rho$. Then the pullback $\Omega^j$ of $\Omega^j_X$ on $\mathbb{H}^d$ is a homogeneous vector bundle. In fact let $\xi$ be the standard action of $SO(d)$ on $\mathbb{R}^d$ and $SO(d) \xleftarrow{\xi} GL(\wedge^j \mathbb{R}^d)$ its exterior product. Then $\Omega^j$ is isomorphic to $SO(d,1) \times_{SO(d),\xi} \wedge^j \mathbb{R}^d$. By an inclusion:

$$SO(d-1) = SO(2n) \to SO(d), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

the restriction of $\xi$ to $SO(2n)$ is decomposed into a direct sum of the standard representation $\sigma$ of $SO(2n)$ on $\mathbb{R}^{2n}$ and the trivial module $1$. Therefore we have

$$\xi_j|_{SO(2n)} \simeq \sigma_j \oplus \sigma_j^{-1},$$

where $\sigma_j$ is the $j$-th exterior product of $\sigma$. Let us observe that $\sigma_0$ and $\sigma_{2n}$ are trivial and $\sigma_j$ is isomorphic to $\sigma_{2n-j}$. $\sigma_j \otimes \mathbb{C}$ is irreducible for $j \neq n$ whereas $\sigma_n \otimes \mathbb{C}$ splits into a direct sum of two irreducible representations, $\sigma_n^+$ and $\sigma_n^-$. We prepare notation. Let $\gamma \in \Gamma$ be a hyperbolic element. Then it is conjugate an element of $MA$. $m_\gamma \exp[l(\gamma)H] (m_\gamma \in M, \exp[l(\gamma)H] \in A)$ where $l(\gamma)$ is the length of the conjugacy class of $\gamma$. There is a $\gamma_0 \in \Gamma$ which determines a prime conjugacy class and that $\gamma = \gamma_0^{\mu(\gamma)}$, where $\mu(\gamma)$ is a positive integer. For $0 \leq j \leq 2n$ we put

$$\alpha_j(\gamma) = \frac{\text{Tr} \rho(\gamma) \cdot \text{Tr} \sigma_j(m_\gamma) \cdot l(\gamma_0)}{\Delta(\gamma)}, \quad e_j = |n-j|$$

and

$$S_j(z) = \exp\left[ - \sum_{\gamma \in \Gamma_{hyp}} \alpha_j(\gamma) \frac{l(\gamma)}{l(\gamma_0)} e^{-zl(\gamma)} \right],$$

where

$$\Delta(\gamma) = \text{det}(I_d - e^{-l(\gamma)} m_\gamma).$$

Let $s_j$ be the logarithmic derivative of $S_j$:

$$s_j(z) = \frac{d}{dz} \log S_j(z) = \sum_{\gamma \in \Gamma_{hyp}} \alpha_j(\gamma) e^{-zl(\gamma)}.$$ 

If $\text{Re} z$ is sufficiently large $S_j(z)$ absolutely converges.
Fact 2.1. ([5], (RS))

\[ R_X(z, \rho) = \prod_{j=0}^{2n} S_j(z + j)^{(-1)^{j+1}}. \]

An isomorphism \( \sigma_j \simeq \sigma_{2n-j} \) induces

\[ \alpha_j(\gamma) = \alpha_{2n-j}(\gamma) \quad \text{and} \quad S_j(z) = S_{2n-j}(z), \]

and therefore

Lemma 2.1.

\[ r_X(z, \rho) = \sum_{j=0}^{2n} (-1)^{j+1} s_j(z + j) \]

\[ = \sum_{j=0}^{n-1} (-1)^{j+1} \{ s_j(z + j) + s_j(z + 2n - j) \} + (-1)^{n+1} s_n(z + n). \]

Lemma 2.2. Let \( f \) be a meromorphic function defined on a neighborhood of the origin and

\[ f(z) = a_0 z^h(1 + a_1 z + \cdots), \quad a_0 \neq 0 \]

its Laurent expansion. Then

\[ a_1 = \frac{1}{2} \lim_{z \to 0} \{ \frac{f'(z)}{f(z)} + \frac{f'(-z)}{f(-z)} \}. \]

Proof. Set \( f(z) = a_0 z^h g(z) \), where \( g(z) = 1 + a_1 z + \cdots \). Then

\[ \frac{f'}{f}(z) = \frac{h}{z} + \frac{g'}{g}(z). \]

and therefore

\[ \lim_{z \to 0} \{ \frac{f'}{f}(z) + \frac{f'(-z)}{f(-z)} \} = \frac{2g'(0)}{g(0)} = 2a_1. \]

Let us regard a meromorphic continuation of \( R_X(z, \rho) \) for a moment and

\[ R_X(z, \rho) = c_0 z^h(1 + c_1 z + \cdots), \quad c_0 \neq 0 \]

the Taylor expansion at the origin. By Lemma 2.2 we obtain

\[ c_1 = \frac{1}{2} \lim_{z \to 0} \{ r_X(z, \rho) + r_X(-z, \rho) \}. \]

Using Selberg trace formula we will compute RHS. Let \( \Delta_X^j \) be Hodge Laplacian acting on the space of smooth sections of \( \Omega^j_X(\rho) \) and its selfadjoint extension to
$L^2(X, \Omega_X^1(\rho))$ will be denoted by the same character. Since $\rho$ is cuspidal $\Delta_X^j$ has only discrete spectrum which do not accumulate and Selberg trace formula for the heat kernel becomes

$$\text{Tr}[e^{-t\Delta_X^j}] = H_j(t) + I_j(t) + U_j(t), \quad t > 0.$$  

Here $H_j(t)$, $I_j(t)$ and $U_j(t)$ are the hyperbolic, the identical and the unipotent orbital integral, respectively ([17], Theorem 2). In this section we will compute the derivative of Laplace transform of each of them:

$$L(f)(z) = 2z \int_0^\infty e^{-lz^2} f(t) dt, \quad f = H_j, I_j, U_j.$$

### 2.1 The hyperbolic orbital integral

For $0 \leq j \leq 2n$, let us put

$$h_j(t) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in \Gamma_{hyp}} \alpha_j(\gamma) \exp\left\{-\frac{(\gamma e_j)^2}{4t} + tc_j^2 + nl(\gamma)\right\}.$$  

Then a hyperbolic orbital integral is given by ([5], Theorem 2)

$$H_j(t) = h_j(t) + h_{j-1}(t),$$

where $h_{-1}(t)$ is understood to be 0. Notice that

$$h_j(t) = h_{2n-j}(t), \quad 0 \leq j \leq n$$

Although the following lemma seems to be well known, we will give a proof for a completeness.

**Lemma 2.3.** Let $l$ and $z$ be positive numbers. Then

$$\int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-z^2t - \frac{l^2}{4t}} dt = \frac{e^{-lz}}{z}.$$  

**Proof.** Let us remember the well known formula:

$$\int_0^\infty e^{-t^2 - \frac{x^2}{t}} dt = \frac{\sqrt{\pi}}{2} e^{-2x}, \quad x > 0.$$  

If we differentiate it with respect to $x$, we obtain

$$x \int_0^\infty \frac{1}{t^2} e^{-t^2 - \frac{x^2}{t^2}} dt = \frac{\sqrt{\pi}}{2} e^{-2x}.$$  

A change of variables, $t = z\sqrt{y}$, $x = \frac{t^2}{2}$ will yield

$$\frac{1}{\sqrt{4\pi}} \int_0^\infty y^{-\frac{3}{2}} e^{-y^2 - \frac{2}{y}} dy = \frac{e^{-lz}}{l}.$$  

Take a derivative of this equation with respect to $z$, the desired formula will be proved.
Therefore if $z$ is a sufficiently large positive number,

\[
L(e^{tc^2}h_j)(z) = 2z \int_0^\infty \sum_{\gamma \in \Gamma_{hyp}} \alpha_j(\gamma) e^{-n\ell(\gamma)} \frac{1}{\sqrt{4\pi t}} e^{-z^2 t - \frac{\ell^2(\gamma)^2}{4t}} dt
\]

and we have proved the following proposition.

**Proposition 2.1.** For a sufficiently large positive number $z$,

\[
L(e^{tc^2}h_j)(z) = s_j(z + n). \tag{3}
\]

Park has obtained the following proposition even though $\rho$ is not cuspidal ([14]). For the sake of a convenience, we will give a proof in §2.4 under our assumption.

**Proposition 2.2.** $s_j(z)$ is continued to the entire plane as a meromorphic function whose singularities are at most only simple poles with integral residues.

This implies a meromorphic continuation of $S_j$ to the whole plane. Therefore by Fact 2.1 $R_X(z, \rho)$ is also meromorphically continued.

### 2.2 The identical orbital integral

We put

\[
i_j(t) = i_{2n-j}(t) = \frac{r}{4\pi} \operatorname{vol}(\Gamma \setminus G) \int_{-\infty}^\infty e^{-t(\lambda^2 + c^2)} P_j(\lambda) d\lambda, \quad 0 \leq j \leq n - 1,
\]

and

\[
i_n(t) = \frac{r}{2\pi} \operatorname{vol}(\Gamma \setminus G) \int_{-\infty}^\infty e^{-t\lambda^2} P_n(\lambda) d\lambda.
\]

Here $\operatorname{vol}(\Gamma \setminus G)$ is the volume of $\Gamma \setminus G$ and $P_j$ is the Plancherel measure for $\sigma_j$ ([10]):

\[
P_j(\lambda) = \frac{4^{1-n}}{(2n-1)!!^2 \pi} \binom{2n}{j} q_j(\lambda),
\]

where

\[
q_j(\lambda) = \prod_{k=1}^j \{\lambda^2 + (n - k + 1)^2\} \prod_{k=j+1}^n \{\lambda^2 + (n - k)^2\}
\]
Then the identical orbital integral is given by

\[ I_j(t) = i_j(t) + i_{j-1}(t). \]

Since \( \Gamma \) is torsin free its intersection with \( K \) is the only identity element. Remember that we have normalized the Haar measure so that \( \text{vol}(K) \) to be one and thus

\[ \text{vol}(\Gamma \setminus G) = \text{vol}(\Gamma \setminus \mathbb{H}^d) \cdot \text{vol}(K) = \text{vol}(X). \]

Hence we have obtained

\[ i_j(t) = i_{2n-j}(t) = \frac{r}{4\pi} \text{vol}(X) \int_{-\infty}^{\infty} e^{-t(\lambda^2 + c_j^2)} P_j(\lambda) d\lambda, \quad 0 \leq j \leq n - 1, \]

and

\[ i_n(t) = \frac{r}{2\pi} \text{vol}(X) \int_{-\infty}^{\infty} e^{-t\lambda^2} P_n(\lambda) d\lambda. \]

For example if \( d = 3 \), i.e. \( n = 1 \), using

\[ \int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda = \sqrt{\pi t^{-\frac{1}{2}}}, \quad (4) \]

one can see

\[ i_0(t) = i_2(t) = \frac{r}{4\pi^2} \text{vol}(X) \int_{-\infty}^{\infty} e^{-t(\lambda^2 + 1)\lambda^2} d\lambda = \frac{r \cdot \text{vol}(X)}{8\pi \sqrt{\pi}} e^{-t - \frac{3}{2}}, \quad (5) \]

\[ i_1(t) = \frac{r}{\pi^2} \text{vol}(X) \int_{-\infty}^{\infty} e^{-t\lambda^2}(\lambda^2 + 1)d\lambda = \frac{r \cdot \text{vol}(X)}{2\pi \sqrt{\pi}} (2t^{-\frac{3}{2}} + t^{-\frac{5}{2}}). \quad (6) \]

In order to compute \( L(e^{t\lambda^2} i_j)(z) \) let us expand \( q_j(\lambda) \) as

\[ q_j(\lambda) = \sum_{k=0}^{n} \gamma_{j,k} \lambda^{2k}, \]

where \( \gamma_{j,k} \) is an integer and \( \gamma_{j,n} = 1 \).

**Lemma 2.4.** Let \( z \) be a positive number. Then for a nonnegative integer \( k \),

\[ L(\int_{-\infty}^{\infty} e^{-t\lambda^2} \lambda^{2k} d\lambda)(z) = (-1)^k 2\pi z^{2k}. \]

In particular \( L(\int_{-\infty}^{\infty} e^{-t\lambda^2} \lambda^{2k} d\lambda)(z) \) is entirely continued to the whole plane.

**Proof.** Let us take a \( k \)-times derivative of (4) with respect to \( t \). Then we obtain

\[ \int_{-\infty}^{\infty} e^{-t\lambda^2} \lambda^{2k} d\lambda = 2^{-k} (2k - 1)!! \sqrt{\pi \pi} t^{-\frac{1}{2} - k}, \]

and

\[ L(\int_{-\infty}^{\infty} e^{-t\lambda^2} \lambda^{2k} d\lambda)(z) = 2^{-k} (2k - 1)!! \sqrt{\pi} L(t^{-\frac{1}{2} - k})(z). \]
Now since
\[
L(t^{\frac{1}{2}-k})(z) = 2z \int_0^{\infty} e^{-tz^2} t^{\frac{1}{2}-k} dt
= 2\Gamma\left(\frac{1}{2} - k\right) z^{2k}
= \frac{(-1)^k 2^{k+1} \sqrt{\pi}}{(2k-1)!!} z^{2k},
\]
the desired equation has been proved.

\[\square\]

**Proposition 2.3.** We have
\[
L(e^{t^2 i_{2j}})(z) = L(e^{t^2 i_{2n-j} i_{2n-j}})(z) = \frac{4^{1-n}r}{2(2n-1)!!2\pi} \binom{2n}{j} vol(X) \sum_{k=0}^{n} (-1)^k \gamma_{j,k} z^{2k}.
\]
for \(0 \leq j \leq n-1\) and
\[
L(i_{n})(z) = \frac{4^{1-n}r}{(2n-1)!!2\pi} \binom{2n}{n} vol(X) \sum_{k=0}^{n} (-1)^k \gamma_{n,k} z^{2k}.
\]

**Corollary 2.1.** Suppose \(n = 1\). Then
\[
L(e^{t^0 i_0})(z) = L(e^{t^2 i_2})(z) = -\frac{r}{2\pi} vol(X) z^2
\]
and
\[
L(i_{1})(z) = \frac{2r}{\pi} vol(X)(1 - z^2).
\]

### 2.3 The unipotent orbital integral

Let \(\zeta_{\nu}(s, \chi_{\nu,i})\) be the Epstein L-function:
\[
\zeta_{\nu}(s, \chi_{\nu,i}) = \sum_{0 \neq \eta \in \Gamma_{\nu}} \chi_{\nu,i}(\eta) |X_\eta|^{-2n(s+1)},
\]
where \(X_\eta\) is an element of the Lie algebra of \(N_\nu\) such that \(\exp(X_\eta) = \eta\). The norm is taken with respect to the normalized Cartan-Killing form. It absolutely converges if \(\text{Re } s\) is sufficiently large and is meromorphically continued to the whole plane. Since \(\chi_{\nu,i}\) is nontrivial it is regular at the origin and we put
\[
\tau_{\nu} = \sum_{i=1}^{r} \zeta_{\nu}(0, \chi_{\nu,i}).
\]
Let \(A(n)\) be the volume of the unit sphere in \(n\). By [13] we find the unipotent orbital integral is given by
\[
U_j(t) = u_j(t) + u_{j-1}(t),
\]
where

\[ u_j(t) = u_{2n-j}(t) = \frac{1}{2\pi A(n)} \sum_{\nu=1}^{h} \text{vol}(\Gamma_{\nu} \setminus N_{\nu}) \tau_{\nu} \int_{-\infty}^{\infty} e^{-t(\lambda^2 + c^2_j)} d\lambda \]  

(7)

\[ = \frac{1}{2\sqrt{\pi} A(n)} \sum_{\nu=1}^{h} \text{vol}(\Gamma_{\nu} \setminus N_{\nu}) \tau_{\nu} e^{-tc_j^2 t^{-\frac{1}{2}}} \]  

(8)

for \( 0 \leq j \leq n-1 \) and

\[ u_n(t) = \frac{1}{\pi A(n)} \sum_{\nu=1}^{h} \text{vol}(\Gamma_{\nu} \setminus N_{\nu}) \tau_{\nu} \int_{-\infty}^{\infty} e^{-t \lambda^2} d\lambda \]  

(9)

\[ = \frac{1}{\sqrt{\pi} A(n)} \sum_{\nu=1}^{h} \text{vol}(\Gamma_{\nu} \setminus N_{\nu}) \tau_{\nu} t^{-\frac{1}{2}}. \]  

(10)

Here we have used (4).

**Lemma 2.5.** Let \( z \) be a positive number. Then

\[ L(\int_{-\infty}^{\infty} e^{-t \lambda^2} d\lambda)(z) = 2\pi, \]

and \( L(\int_{-\infty}^{\infty} e^{-t \lambda^2} d\lambda)(z) \) is entirely continued to the whole plane as \( 2\pi \).

**Proof.** We compute,

\[ L(\int_{-\infty}^{\infty} e^{-t \lambda^2} d\lambda)(z) = 2z \int_{0}^{\infty} dt e^{-tz^2} \int_{-\infty}^{\infty} e^{-t \lambda^2} d\lambda \]

\[ = 2z \int_{-\infty}^{\infty} d\lambda \int_{0}^{\infty} e^{-t(\lambda^2 + z^2)} dt \]

\[ = 2z \int_{-\infty}^{\infty} d\lambda \int_{0}^{\infty} e^{-t(\lambda^2 + z^2)} dt \]

The desired formula will be obtained by the contour integration. \( \Box \)

Thus putting

\[ \delta(X, \rho) = \frac{1}{A(n)} \sum_{\nu=1}^{h} \text{vol}(\Gamma_{\nu} \setminus N_{\nu}) \tau_{\nu}, \]

we have proved the following proposition.

**Proposition 2.4.** For \( 0 \leq j \leq n-1 \), both \( L(e^{tc_j^2 u_j})(z) \) and \( L(e^{tc_{2n-j}^2 u_{2n-j}})(z) \) are analytically continued to the entire plane as a constant \( \delta(X, \rho) \), whereas \( L(u_n)(z) \) is continued as \( 2\delta(X, \rho) \).
2.4 An application of Selberg trace formula

Let $0 \leq j \leq n$. Then by definition we know

$$h_j(t) = \sum_{k=0}^{j} (-1)^{j-k} H_k(t), \quad i_j(t) = \sum_{k=0}^{j} (-1)^{j-k} I_k(t)$$

and

$$u_j(t) = \sum_{k=0}^{j} (-1)^{j-k} U_k(t).$$

If we put

$$\delta_j(t) = \sum_{k=0}^{j} (-1)^{j-k} \text{Tr}[e^{-t\Delta_k}],$$

Selberg trace formula implies

$$\delta_j(t) = h_j(t) + i_j(t) + u_j(t). \quad (11)$$

The following lemma will directly follow from the definition of $L$.

**Lemma 2.6.**

$$L(e^{tc^2_j}\delta_j)(-z) = -L(e^{tc^2_j}\delta_j)(z).$$

Now we will prove **Proposition 2.2**. Since $\rho$ is cuspidal, $\Delta^k_X$ has only discrete spectrum $\{\sigma_k(l)\}$ which do not accumulate and are nonnegative. Let us fix $z \in \mathbb{C}$ so that

$$\text{Re} z^2 > n^2.$$

Then

$$L(e^{tc^2_j}\delta_j)(z) = \sum_{k=0}^{j} (-1)^{j-k} \sum_l 2z \int_0^{\infty} e^{-t(z^2 - c^2_j + \sigma_k(l))} dt$$

and

$$L(e^{tc^2_j}\delta_j)(z-c_j) = \sum_{k=0}^{j} (-1)^{j-k} \sum_l \left\{ \frac{1}{z - c_j + \sqrt{c^2_j - \sigma_k(l)}} + \frac{1}{z - c_j - \sqrt{c^2_j - \sigma_k(l)}} \right\}. \quad (12)$$

Thus $L(e^{tc^2_j}\delta_j)(z)$ is meromorphically continued to the whole plane and has only simple poles with integral residues. Using **Proposition 2.1**, **Proposition 2.3** and **Proposition 2.4** Selberg trace formula imply

$$s_j(z + n) = L(e^{tc^2_j}\delta_j)(z)$$

$$- \frac{4^{1-n}c_j + \delta_j(n)}{2(2n-1)!t^2 \pi} \left( \begin{array}{c} 2n \\ j \end{array} \right) \text{vol}(X) \sum_{k=0}^{n} (-1)^k \gamma_{j,k} z^{2k}$$

$$- (1 + \delta_j(n)) \delta(X, \rho), \quad (13)$$

14
where \( \delta_{j,n} \) is the Kronecker’s delta. This proves Proposition 2.2.

\[ \square \]

Using Lemma 2.1 and (13), the above computation shows

\[
\begin{align*}
r_X(z, \rho) &= \sum_{j=0}^{n-1} (-1)^{j+1} \sum_{k=0}^{j} (-1)^{j-k} \sum_{l} \left\{ \frac{1}{z - c_j + \sqrt{c_j^2 - \sigma_k(l)}} + \frac{1}{z - c_j - \sqrt{c_j^2 - \sigma_k(l)}} \right\} \\
&\quad + \sum_{j=0}^{n-1} (-1)^{j+1} \sum_{k=0}^{j} (-1)^{j-k} \sum_{l} \left\{ \frac{1}{z + c_j + \sqrt{c_j^2 - \sigma_k(l)}} + \frac{1}{z + c_j - \sqrt{c_j^2 - \sigma_k(l)}} \right\} \\
&\quad + (-1)^{n+1} \sum_{k=0}^{n} (-1)^{n-k} \sum_{l} \left\{ \frac{1}{z + \sigma_k(l)\sqrt{-1}} + \frac{1}{z - \sigma_k(l)\sqrt{-1}} \right\} \\
&\quad + E(z),
\end{align*}
\]

where \( E(z) \) is an entire function. Thus remembering \( h^0(X, \rho) = 0 \), we have

\[
\text{Res}_{z=0} r_X(z, \rho) = 2 \sum_{l=0}^{n-1} (-1)^l (n - l) h^{l+1}(X, \rho).
\]

It is obvious that the zeros and poles of \( R_X(z, \rho) \) are located on

\[ \Xi = \{ z \in \mathbb{C} | \text{Re}z = -n, -(n-1), \ldots, n-1, n \}, \]

except for finitely many of them. For example if \( d = 3 \) (i.e. \( n = 1 \)), we conclude \( R_X(z, \rho) \) has a zero at the origin of order \( 2h^1(X, \rho) \). If the minimum of spectrum of \( \Delta_0^X \) is greater than or equal to 1, all zeros and poles are located on \( \Xi \). The following theorem is a consequence of Lemma 2.1, Lemma 2.6 and (13).

**Theorem 2.1.**

\[
r_X(z, \rho) + r_X(-z, \rho) = \frac{4^{1-n_r} \cdot vol(X)}{(2n-1)!! \cdot 2\pi} \chi(z) + 4 \sum_{j=0}^{n} (-1)^j \delta(X, \rho),
\]

where

\[
\chi(z) = \sum_{j=0}^{n} (-1)^j \left( \begin{pmatrix} 2n \\ j \end{pmatrix} \right) \sum_{k=0}^{n} (-1)^k \gamma_{j,k} \{ (z + j - n)^{2k} + (z - j + n)^{2k} \}.
\]

**Corollary 2.2.** Suppose \( d = 3 \). Then

\[
r_X(z, \rho) + r_X(-z, \rho) = \frac{2r}{\pi} \text{vol}(X)(z^2 - 3).
\]

Noting the order of \( R_X(z, \rho) \) is even, an easy computation will show

\[
\lim_{z \to 0} R_X(z, \rho) R_X(-z, \rho)^{-1} = 1.
\]
By Theorem 2.1,
\[
\frac{d}{dz} \log(R_X(z, \rho)R_X(-z, \rho)^{-1}) = r_X(z, \rho) + r_X(-z, \rho)
\]
\[
= \frac{4^{1-n} r \cdot \text{vol}(X)}{(2n-1)!! \pi} \chi(z) + 4 \sum_{j=0}^{n} (-1)^n \delta(X, \rho),
\]
and therefore $R_X(z, \rho)$ satisfies a functional equation:
\[
R_X(z, \rho) \cdot R_X(-z, \rho)^{-1} = \exp[4^{1-n} r \cdot \text{vol}(X) \chi(z) + 4 \sum_{j=0}^{n} (-1)^n \delta(X, \rho) z],
\]
where $X(z)$ is the primitive function of $\chi(z)$ so that $X(0) = 0$. For example if $d = 3$,
\[
R_X(z, \rho) \cdot R_X(-z, \rho)^{-1} = \exp[\frac{2r}{\pi} \text{vol}(X)(\frac{z^3}{3} - 3z)].
\]

Now the second coefficient of the Taylor expansion is obtained by Lemma 2.2 and Theorem 2.1.

Theorem 2.2. Let
\[
R_X(z, \rho) = c_0 z^h (1 + c_1 z + \cdots), \quad c_0 \neq 0,
\]
be the Laurent expansion. Then $c_1 - 2 \sum_{j=0}^{n} (-1)^n \delta(X, \rho)$ is a rational multiple of $\text{vol}(X)/\pi$.

Corollary 2.3. Suppose $d = 3$. Then
\[
c_1 = -\frac{3r}{\pi} \text{vol}(X).
\]

3 The leading coefficient

Throughout this section we assume that $X$ is a hyperbolic threefold with finite volume. We will compute the leading coefficient of the Taylor expansion of $R_X(z, \rho)$ at the origin. In §2.4 we have seen $\text{ord}_{z=0} R_X(z, \rho) = 2h^1(X, \rho)$. The following fact is a special case of [14].

Fact 3.1.
\[
\lim_{z \to 0} z^{-2h^1(X, \rho)} R_X(z, \rho) = \exp(-\zeta_X(0, \rho)).
\]
Here
\[
\zeta_X(z, \rho) = \sum_{p=0}^{3} (-1)^p p \cdot \zeta_X^{(p)}(z, \rho),
\]
where
\[
\zeta_X^{(p)}(z, \rho) = \frac{1}{\Gamma(z)} \int_{0}^{\infty} \{ \text{Tr} [e^{-tA_X}] - h^p(X, \rho) \} t^{z-1} dt.
\]
ζ^{(p)}(z, \rho) absolutely converges if Rez is sufficiently large and is meromorphically continued to the whole plane. In fact let us put

\[ \theta_p(t) = \text{Tr}[e^{-t\Delta^p}] - h^p(X, \rho). \]

Then the computation of orbital integrals in §2 and Selberg trace formula show that it has an asymptotic expansion on \((0, 1]\) such that

\[ \theta_p(t) \sim t^{-\frac{3}{2}} \sum_{l=0}^{N} c_l t^l + O(t^{N-\frac{3}{2}}). \] (14)

Therefore if Rez > \(N - 3/2\),

\[ \int_0^1 \theta_p(t)t^{z-1}dt = \sum_{l=0}^{N} \frac{c_l}{z+l-\frac{3}{2}} + R_N(z), \]

where \(R_N(z)\) is a regular function on \(\{z \in \mathbb{C} | \text{Rez} > \frac{3}{2} - N\}\) which is meromorphically continued to the whole plane. Since \(\theta_p(t)\) exponentially decays as \(t \to \infty\), \(\int_1^\infty \theta_p(t)t^{z-1}dt\) is an entire function. Thus writing

\[ \int_0^\infty \theta_p(t)t^{z-1}dt = \int_0^1 \theta_p(t)t^{z-1}dt + \int_1^\infty \theta_p(t)t^{z-1}dt, \]

we know that \(\zeta^{(p)}(z, \rho)\) is meromorphically continued to the whole plane and that it vanishes at the origin. Since we assume that \(\rho\) is cuspidal it is possible to prove Fact 3.1 just following the arguments of [5]. For a convenience we will give a proof in Appendix. Thus the leading coefficient is \(\exp(-\zeta_X^{(0, \omega)})\) but we want to express this by a more geometric term.

### 3.1 Boundary conditions

We will use the Poincaré upper half space model:

\[ \mathbb{H}^3 = \{(x, y, r) \in \mathbb{R}^3 | r > 0\}, \quad g = \frac{dx^2 + dy^2 + dr^2}{r^2}. \]

For \(a \in \mathbb{R}\) we put

\[ \mathbb{H}^3_a = \cap_{\nu=1}^a g_{\nu}, \quad \mathbb{H}^3_{a, \infty} = \{(x, y, r) \in \mathbb{H}^3 | r \leq e^a\}. \]

(Remember that \(g_{\nu} \in \text{PSL}_2(\mathbb{C})\) is chosen to satisfy

\[ N_{\nu} = g_{\nu} N g_{\nu}^{-1}, \quad N = \left( \begin{array}{cc} 1 & z \\ 0 & 1 \end{array} \right) \quad | z \in \mathbb{C}. \])

Let \(X_a\) be the image of \(\mathbb{H}^3_a\) by the natural projection

\[ \mathbb{H}^3 \to X, \]

17
and $Y_a$ the closure of $X \setminus X_a$. If $a$ is sufficiently large $Y_a$ is a disjoint union of $Y_{a, \nu}$ $(1 \leq \nu \leq h)$ which is a warped product of a flat 2-torus $\mathbb{T}_\nu = N_\nu / \Gamma_\nu (= \mathbb{R}^2 / \Gamma_\nu)$ and an interval $[e^a, \infty)$ with the metric,
\[ g = du^2 + e^{-2u}(dx^2 + dy^2). \]
(Here we have made a change of variables: $r = e^u$.) According to the decomposition
\[ \rho_\nu = \bigoplus_{i=1}^r \chi_{\nu,i}, \]
a smooth section $\varphi$ of $\Omega^p_X(\rho)$ over $Y_{a, \nu}$ is written as
\[ \varphi = \sum_{i=1}^r \varphi_i, \quad \varphi_i = \sum_{|\alpha|=p} \varphi_{i,\alpha} dx^\alpha \in C^\infty(Y_{a, \nu}, \Omega^p_X(\chi_{\nu,i})), \]
where we have put
\[ x_0 = u, \quad x_1 = x, \quad x_2 = y. \]

**Lemma 3.1.**
\[ \int_{\mathbb{T}_\nu} \varphi_{i,\alpha} dxdy = 0. \]

**Proof.** Let us choose $\gamma \in \Gamma_\nu$ so that
\[ \chi_{\nu,i}(\gamma) \neq 1. \]
By definition we have
\[ \gamma^* \varphi_{i,\alpha} = \chi_{\nu,i}(\gamma) \varphi_{i,\alpha}. \]
and the desired result will follow from
\[ \int_{\mathbb{T}_\nu} \varphi_{i,\alpha} dxdy = \int_{\mathbb{T}_\nu} \gamma^* \varphi_{i,\alpha} dxdy = \chi_{\nu,i}(\gamma) \int_{\mathbb{T}_\nu} \varphi_{i,\alpha} dxdy. \]
\[ \square \]

We will consider an eigenvalue problem of Hodge Laplacian $\Delta^p_X$ on spaces $L^2(X_a, \Omega^p_X(\rho))$ or $L^2(Y_{a, \nu}, \Omega^p_X(\rho))$ under a certain boundary condition. Hereafter for simplicity we omit the subscript $X$ of $\Delta^p_X$. The restriction $\Omega^p_X(\rho)$ to the boundary $\mathbb{T}_\nu$ of $Y_{a, \nu}$ is decomposed into
\[ \Omega^p_X(\rho)|_{\mathbb{T}_\nu} = \Omega^p_{\mathbb{T}_\nu}(\rho) \oplus du \wedge \Omega^{p-1}_{\mathbb{T}_\nu}(\rho), \]
and according to this a section $\omega$ of $\Omega^p_X(\rho)|_{\mathbb{T}_\nu}$ is expressed by
\[ \omega = \omega_{\text{tan}} + \omega_{\text{norm}}, \]
where $\omega_{\text{tan}}$ (resp. $\omega_{\text{norm}}$) is a section of $\Omega^p_{\mathbb{T}_\nu}(\rho)$ (resp. $du \wedge \Omega^{p-1}_{\mathbb{T}_\nu}(\rho)$).
**Definition 3.1.** Let $\omega$ be a smooth section of $\Omega^p_X(\rho)$ on $X_a$ or $Y_{a,\nu}$. We call it satisfies the absolute boundary condition if both $\omega_{\text{norm}}$ and $(d\omega)_{\text{norm}}$ vanish on the every boundary. If the Hodge dual $\ast \omega$ satisfies the absolute boundary condition we will refer that it satisfies the relative boundary condition. If $\omega$ satisfies both of the absolute and the relative boundary condition, we call it satisfies the Dirichlet boundary condition.

It is easy to see that $\omega$ satisfies the relative boundary condition if and only if both $\omega_{\text{tan}}$ and $(d\omega)_{\text{tan}}$ vanish on every $T_\nu$. Thus $\omega$ satisfies the Dirichlet boundary condition if and only if the restrictions of both $\omega$ and $d\omega$ become the 0-section of $\Omega^p_X(\rho)|_{T_\nu}$ and $\Omega^{p+1}_X(\rho)|_{T_\nu}$ for every $1 \leq \nu \leq h$. More concretely the latter condition means that if we write

$$
\omega = \sum_{|\alpha|=p} f_\alpha \, dx^\alpha, \quad d\omega = \sum_{|\beta|=p+1} g_\beta \, dx^\beta,
$$

all $f_\alpha$ and $g_\beta$ vanish along $T_\nu$ for every $\nu$. Notice that $\ast$ interchanges the absolute and relative boundary conditions and preserves the Dirichlet one. Since $\rho$ is unitary the associated local system possesses a fiberwise hermitian inner product $Tr_\rho$. For $\omega, \eta \in \Omega^p_X(\rho)$ we put

$$
(\omega, \eta) = \frac{\text{Tr}_\rho(\omega \wedge \ast \eta)}{dv_g},
$$

which defines a hermitian inner product on $\Omega^p_X(\rho)$. Here $dv_g$ is the volume form of $g$. Let $M$ be $X_a$ or $Y_{a,\nu}$ and $\nabla$ the covariant derivative. If both $\omega$ and $\eta$ satisfy one of the boundary conditions,

$$
\int_M (\Delta^p \omega, \eta) dv_g = \int_M (\nabla \omega, \nabla \eta) dv_g = \int_M (\omega, \Delta^p \eta) dv_g,
$$

by Stokes theorem. Therefore $\Delta^p$ has a selfadjoint extension $\Delta^p_{\text{abs}}$, $\Delta^p_{\text{rel}}$ or $\Delta^p_{\text{dir}}$ according to a boundary condition. If $\ast$ is $\text{abs}$ (resp. $\text{rel}$ or $\text{dir}$) its dual $\ast$ is defined to be $\text{rel}$ (resp. $\text{abs}$ or $\text{dir}$). Since the Hodge $\ast$-operator intertwines the action of $\Delta^p$ on $L^2(M, \Omega^p_X(\rho))$ and one of $\Delta_{\ast}^{3-p}$ on $L^2(M, \Omega^{3-p}_X(\rho))$, we will only consider the case of $p = 0$ or 1.

For a later purpose we will introduce one more boundary condition. Let $\alpha$ be a real number greater than one. For a sufficiently large $a$, $Y_{a,\nu} \cap X_{aa}$ is diffeomorphic to $T_\nu \times \{e^a, e^{\alpha a}\}$. If $\omega \in C^\infty(Y_{a,\nu} \cap X_{aa}, \Omega^p_X(\rho))$ satisfies the Dirichlet condition on $T_\nu \times \{e^a\}$ and a condition $\ast = \text{abs, rel or dir}$ on $T_\nu \times \{e^{\alpha a}\}$ we call it satisfies Dirichlet/$\ast$-condition. If $\omega \in C^\infty(Y_{a} \cap X_{aa}, \Omega^p_X(\rho))$ satisfies Dirichlet/$\ast$-condition on every connected component ($\ast$ does not depend on a component) we will refer that it satisfies Dirichlet/$\ast$-condition.

### 3.2 Spectrum of Hodge Laplacian at cusps

Since $\Delta^p$ commutes with the action of $\Gamma$ it preserves the decomposition,

$$
C^\infty(Y_{a,\nu}, \Omega^p_X(\rho)) = \oplus_{i=1}^r C^\infty(Y_{a,\nu}, \Omega^p_X(\chi_{\nu,i})).
$$
Thus the spectral problem of Hodge Laplacian on $L^2(Y_{a,\nu}, \Omega_X^\rho(\chi_{\nu,i}))$ is reduced to one on $L^2(Y_{a,\nu}, \Omega_X^\rho(\chi_{\nu,i}))$. We will give an explicit formula of $\Delta^\rho$ on $Y_{a,\nu}$. A straightforward computation will show the following lemma.

**Lemma 3.2.** Let $\Delta_T$ be the positive Laplacian on a flat torus,

$$\Delta_T = -(\partial_x^2 + \partial_y^2).$$

1. For $f \in C^\infty(Y_{a,\nu}, \Omega_X^0(\chi_{\nu,i}))$,

$$\Delta_0 f = e^{2u} \Delta_T f - \partial_x^2 f + 2\partial_x f.$$

2. For $\omega = f dx + g dy + h du \in C^\infty(Y_{a,\nu}, \Omega_X^1(\chi_{\nu,i}))$,

$$\Delta^1 \omega = (e^{2u} \Delta_T f - \partial_x^2 f + 2\partial_x h) dx$$
$$+ (e^{2u} \Delta_T g - \partial_y^2 g + 2\partial_y h) dy$$
$$+ (e^{2u} \Delta_T h - \partial_x^2 h + 2\partial_x h - 2e^{2u}(\partial_x f + \partial_y g)) du.$$

**Fact 3.2.** ([16] Theorem XIII.1, The min-max principle) Let $A$ be a selfadjoint operator with domain $D(A)$, which is bounded below. Define

$$\mu_n(A) = \sup_{\varphi_1, \cdots, \varphi_{n-1}} U_A(\varphi_1, \cdots, \varphi_{n-1}),$$

where

$$U_A(\varphi_1, \cdots, \varphi_{n-1}) = \inf_{\psi \in D(A), \|\psi\| = 1, \psi \in <\varphi_1, \cdots, \varphi_{n-1}>} (\psi, A\psi),$$

and $<\varphi_1, \cdots, \varphi_{n-1}>^\perp$ is the orthogonal complement of a vector space $<\varphi_1, \cdots, \varphi_{n-1}>$ spanned by $\{\varphi_1, \cdots, \varphi_{n-1}\}$. Then either of the followings holds:

1. there are $n$ eigenvalues below the bottom of the essential spectrum and $\mu_n(A)$ is the $n$-th eigenvalue counting with multiplicity,
2. $\mu_n(A)$ is the bottom of the essential spectrum.

Later on we will need a variant of this.

**Lemma 3.3.** Let $A$ be a selfadjoint operator bounded below such that $(A - \lambda)^{-1}$ is compact for a certain $\lambda \in \rho(A)$, where $\rho(A)$ is the resolvent set. Then the $n$-th eigenvalue $\mu_n(A)$ is obtained by

$$\mu_n(A) = \inf_{M \in \text{Gr}_n D(A)} \sup_{0 \neq v \in M} \frac{(Av, v)}{\|v\|^2}.$$
Proof. Let $\mu_n'(A)$ be the RHS of the above equation. By the assumption there is a complete orthonormal basis $\{\varphi_n\}_n$ in $D(A)$ such that $A\varphi_n = \mu_n(A)\varphi_n$ with $\mu_1(A) \leq \mu_2(A) \leq \cdots$ and $\mu_n(A) \to \infty$. Let $\mathcal{M}$ be an $n$-dimensional space spanned by $\{\varphi_1, \cdots, \varphi_n\}$. Thus

$$\mu_n(A) = \sup_{0 \neq v \in \mathcal{M}} \frac{(Av, v)}{||v||^2},$$

and $\mu_n'(A) \leq \mu_n(A)$ by definition. Suppose $\mu_n'(A)$ is strictly less than $\mu_n(A)$. Then there is an $n$-dimensional subspace $\mathcal{M}$ of $D(A)$ so that

$$\mu_n'(A) \leq \sup_{0 \neq v \in \mathcal{M}} \frac{(Av, v)}{||v||^2} < \mu_n(A).$$

But by the equation (2a) in pp.77 of [16], the dimension of $\mathcal{M}$ should be less than $n$, which is a contradiction.

Let $A$ and $B$ are selfadjoint operators bounded below which act on a Hilbert space $H$. Suppose that they have the same domain $D$ and that $A \geq B$, i.e. $(Av, v) \geq (Bv, v)$ for any $v \in D$. Then Fact 3.1 implies

Lemma 3.4.

$$\mu_n(A) \geq \mu_n(B)$$

Let $a$ and $a'$ be positive numbers so that $a' \geq a$. Extending as 0-map on the outside $L^2(X, \Omega^p_X(\rho))$ is embedded into $L^2(X_{a'}, \Omega^p_X(\rho))$. Thus $D(\Delta^p_{\text{dir}}|_{X_{a'}})$ is a subspace of $D(\Delta^p_{\text{dir}}|_{X_{a'}})$. In particular $\text{Gr}_n(D(\Delta^p_{\text{dir}}|_{X_{a'}}))$ is a subset of $\text{Gr}_n(D(\Delta^p_{\text{dir}}|_{X_{a'}}))$. Since $\Delta^p_{\text{dir}}|_{X_{a'}}$ satisfies the assumption of Lemma 3.3,

$$\mu_n(\Delta^p_{\text{dir}}|_{X_{a'}}) \leq \mu_n(\Delta^p_{\text{dir}}|_{X_{a'}}).$$

Since $\rho$ is cuspidal $\Delta^p$ also satisfies the assumption of Lemma 3.3. The same argument will yield the following lemma.

Lemma 3.5. 1. Let $a$ and $a'$ be positive numbers so that $a' \geq a$. Then,

$$\mu_n(\Delta^p_{\text{dir}}|_{X_{a'}}) \leq \mu_n(\Delta^p_{\text{dir}}|_{X_{a'}}).$$

2. For a positive $a$,

$$\mu_n(\Delta^p) \leq \mu_n(\Delta^p_{\text{dir}}|_{X_{a'}}).$$

and

$$\mu_n(\Delta^p|_{X_{a'}}) \leq \mu_n(\Delta^p_{\text{dir}}|_{X_{a'}}),$$

where $*$ is abs or rel.

Remark 3.1. The above lemma also follows from the Rayleigh-Ritz technique.([16] Theorem XIII.3)
Let $\Gamma^*_\nu$ be the dual lattice of $\Gamma_\nu$. We will define its norm to be

$$||\Gamma^*_\nu|| = \text{Min}\{\gamma| |0 \neq \gamma \in \Gamma^*_\nu\}.$$ 

Here the modulus $| \cdot |$ is taken with respect to the standard Euclidean metric $dx^2 + dy^2$ on $\mathbb{R}^2$.

**Proposition 3.1.**

$$\mu_1(\Delta^0_{\text{dir}}|_{Y_\alpha,\nu}) \geq e^{2a}||\Gamma^*_\nu||^2.$$

**Proof.** Let us consider a nonnegative selfadjoint operator

$$P_a = e^{2a}\Delta_T - \partial_u^2 + 2\partial_u$$

on $L^2(Y_\alpha,\nu,\Omega^0(\chi_{\nu,i}))$ under Dirichlet condition at the boundary. Since

$$\Delta^0 - P_a = (e^{2u} - e^{2a})\Delta_T$$

is a nonnegative operator **Lemma 3.4** implies

$$\mu_1(\Delta^0_{\text{dir}}|_{Y_\alpha,\nu}) \geq \mu_1(P_a).$$

For $f \in C^\infty_c(Y_\alpha,\nu,\Omega^0(\chi_{\nu,i}))$,

$$\int_{Y_\alpha,\nu} (P_a f, f) dv_g = e^{2a} \int_{Y_\alpha,\nu} \Delta_T f \cdot \bar{f} e^{-2u} dx dy du + \int_{Y_\alpha,\nu} |\partial_u f|^2 e^{-2u} dx dy du$$

$$\geq e^{2a} \int_{Y_\alpha,\nu} \Delta_T f \cdot \bar{f} e^{-2u} dx dy du$$

$$= e^{2a} \int_a^\infty du e^{-2u} \int_{T_\nu} \Delta_T f \cdot \bar{f} dx dy.$$

Let

$$f = \sum_{\gamma \in \Gamma^*_\nu} \{f_{\gamma}(u)e_{\gamma}(z) + f^*_{\gamma}(u)e_{\gamma}(\bar{z})\}, \quad e_{\gamma}(z) = \exp(2\pi i \gamma z)$$

be a Fourier expansion with respect to $T_\nu$-direction. Here notice that by **Lemma 3.1** $\gamma$ runs through nonzero elements of $\Gamma^*_\nu$. Then

$$\int_{T_\nu} \Delta_T f \cdot \bar{f} = \text{vol}(T_\nu) \sum_{0 \neq \gamma \in \Gamma^*_\nu} |\gamma|^2 \{ |f_{\gamma}(u)|^2 + |f^*_{\gamma}(u)|^2 \}$$

$$\geq ||\Gamma^*_\nu||^2 \text{vol}(T_\nu) \sum_{0 \neq \gamma \in \Gamma^*_\nu} \{ |f_{\gamma}(u)|^2 + |f^*_{\gamma}(u)|^2 \}$$

$$= ||\Gamma^*_\nu||^2 \int_{T_\nu} |f|^2 dx dy,$$

and therefore we have obtained

$$\int_{Y_\alpha,\nu} (P_a f, f) dv_g \geq e^{2a}||\Gamma^*_\nu||^2 \int_{Y_\alpha,\nu} (f, f) dv_g.$$ 

Now **Fact 3.3** implies $\mu_1(P_a) \geq e^{2a}||\Gamma^*_\nu||^2$. 

22
The same argument will prove

**Proposition 3.2.** For \( \alpha > 1 \) and \( * = \text{abs or rel} \),

\[
\mu_1(\Delta_{\text{dir},a}^0|_{X_{aa} \cap Y_{a,v}}) \geq e^{2a||\Gamma_v^*||^2}.
\]

Next we will estimate \( \mu_1(\Delta_{\text{dir}}^1|_{Y_{a,v}}) \) from below. Before doing this we will give some remarks. Let us fix a positive number \( \alpha \) less than \( \alpha \) and we make a change of variables,

\[
u = v + \alpha.
\]

Then \( Y_{a,v} \) is isometric to a warped product,

\[
[a', \infty) \times T'_{a}, \quad a' = a - \alpha,
\]

with metric

\[
dg = dv^2 + e^{-2v}(dx^2 + dy^2).
\]

Here the boundary \( T'_{a} \) is a quotient of \( \mathbb{R}^2 \) with the standard Euclidean metric \( dx^2 + dy^2 \) by a lattice \( e^{-\alpha} \Gamma_v \). Thus replacing \( \Gamma_v \) (resp. \( T_v \)) by \( e^{-\alpha} \Gamma_v \) (resp. \( T'_v \)) for a sufficiently large \( \alpha \), we may initially assume that \( ||\Gamma_v|| < 1 \), or equivalently \( ||\Gamma_v^*|| > 1 \). Taking \( \alpha \) sufficiently large we also assume that \( e^{2a} > 32 \). Let \( \omega = f dx + g dy + h du \) be an element of \( C_c(Y_{a,v}, \Omega^1(x_{\nu,i})) \). Then a computation in **Proposition 3.1** implies

\[
\int_{Y_{a,v}} \Delta_T f \cdot \bar{f} dxdyu \geq ||\Gamma_v^*||^2 \int_{Y_{a,v}} |f|^2 dxdyu \geq \int_{Y_{a,v}} |f|^2 dxdyu, \quad (15)
\]

\[
\int_{Y_{a,v}} \Delta_T g \cdot \bar{g} dxdyu \geq ||\Gamma_v^*||^2 \int_{Y_{a,v}} |g|^2 dxdyu \geq \int_{Y_{a,v}} |g|^2 dxdyu, \quad (16)
\]

and

\[
\int_{Y_{a,v}} \Delta_T h \cdot \bar{h} e^{-2u} dxdyu \geq ||\Gamma_v^*||^2 \int_{Y_{a,v}} |h|^2 e^{-2u} dxdyu. \quad (17)
\]

Using

\[
||dx|| = ||dy|| = e^u, \quad ||du|| = 1
\]

and **Lemma 3.2**, an integration by parts shows

\[
\int_{Y_{a,v}} (\Delta^1 \omega, \omega) dv_g = \int_{Y_{a,v}} e^{2u}(\Delta_T f \cdot \bar{f} + \Delta_T g \cdot \bar{g}) dxdyu \\
+ \int_{Y_{a,v}} |\nabla_T h|^2 dxdyu \\
+ \int_{Y_{a,v}} (|\partial_u f|^2 + |\partial_u g|^2 + |\partial_u h|^2 e^{-2u}) dxdyu \\
+ 2 \int_{Y_{a,v}} \{(\partial_x h \cdot \bar{f} + \partial_x \bar{h} \cdot f) + (\partial_y h \cdot \bar{g} + \partial_y \bar{h} \cdot g)\} dxdyu
\]

23
we obtain

$$\int_{Y_{\alpha,\nu}} (e^{2u} - 16) (\Delta_T f \cdot \tilde{f} + \Delta_T g \cdot \tilde{g}) dx dy du$$

By (15) and (16), (19) is nonnegative. Moreover

$$\int_{Y_{\alpha,\nu}} (e^{2u} - 16) (\Delta_T f \cdot \tilde{f} + \Delta_T g \cdot \tilde{g}) dx dy du = \int_{Y_{\alpha,\nu}} (e^{2u} - 16) (|\nabla_T f|^2 + |\nabla_T g|^2) dx dy du$$

we obtain

$$\int_{Y_{\alpha,\nu}} (\Delta^1 \omega, \omega) dv_g \geq (e^{2a} - 16) \int_{Y_{\alpha,\nu}} (\Delta_T f \cdot \tilde{f} + \Delta_T g \cdot \tilde{g}) dx dy du$$

and

$$\int_{Y_{\alpha,\nu}} (\Delta^1 \omega, \omega) dv_g \geq (e^{2a} - 16) \int_{Y_{\alpha,\nu}} (|\nabla_T f|^2 + |\nabla_T g|^2) dx dy du$$

Here we have used the fact $e^{2a}$ is greater than 32. Thus by (15), (16) and (17) we see

$$\int_{Y_{\alpha,\nu}} (\Delta^1 \omega, \omega) dv_g \geq \frac{1}{2} e^{2a} \int_{Y_{\alpha,\nu}} ||\Gamma^*||^2 ||\omega||^2 dv_g.$$ 

Now Fact 3.1 yields
Proposition 3.3. For a sufficiently large $a$, 
\[ \mu_1(\Delta^1_{\text{dir}}|_{Y_{a,\nu}}) \geq \frac{1}{2}e^{2a||\Gamma^*_\nu||^2}. \]

By the same argument we will find

Proposition 3.4. Suppose $\alpha > 1$. Then for a sufficiently large $a$ and $\ast = \text{abs or rel}$, 
\[ \mu_1(\Delta^1_{\ast/\ast}|_{X_{a,\ast}}) \geq \frac{1}{2}e^{2a||\Gamma^*_\ast||^2}. \]

3.3 A convergence of spectrum

In Lemma 3.5 we have shown that $\mu_n(\Delta^p_{\text{dir}}|X_a)$ is a monotone decreasing function of $a$ which is bounded below by $\mu_n(\Delta^p)$. In this section we will show the following theorems.

Theorem 3.1. 
\[ \lim_{a \to \infty} \mu_n(\Delta^p_{\text{dir}}|X_a) = \mu_n(\Delta^p). \]

Theorem 3.2. 
\[ \lim_{a \to \infty} \mu_n(\Delta^p|X_a) = \mu_n(\Delta^p) \quad \ast = \text{abs, rel}. \]

Corollary 3.1. Let $t$ be a positive number. Then 
\[ \text{Tr}[e^{-t\Delta^p}] = \lim_{a \to \infty} \text{Tr}[e^{-t\Delta^p_{\ast/\ast}|X_a}] = \lim_{a \to \infty} \text{Tr}[e^{-t\Delta^p_{\ast/\ast}|X_a}] = \mu_n(\Delta^p) \quad \ast = \text{abs, rel}. \]

Let us fix a positive $a_0$ so that $Y_{a_0}$ is a disjoint union: 
\[ Y_{a_0} = \Pi_{\nu=1}^k T_{\nu} \times [a_0, \infty). \]

We may assume that $e^{2a_0} > 32$ and $||\Gamma^*_\nu|| > 1$ for every $\nu$. Thus Proposition 3.1 and Proposition 3.3 are available for $a > a_0$. Let us fix such an $a$ and let $\chi$ be a smooth function on $X$ satisfying

1. $0 \leq \chi \leq 1$.
2. $\chi|_{X_a} = 1$ and $\chi|_{Y_{2a}} = 0$.
3. $|\nabla \chi| \leq a^{-1}$.

Let $\varphi_i$ be its eigenform of $\Delta^p$ whose eigenvalue is $\mu_i(\Delta^p)$ and $\mathcal{M}_n$ an element of $\text{Gr}_n D(\Delta^p)$ spanned by $\{\varphi_1, \ldots, \varphi_n\}$. Then for an arbitrary $\varphi \in \mathcal{M}_n$ we have 
\[ \int_X ||\nabla \varphi||^2 dv_g = \int_X (\Delta^p \varphi, \varphi) dv_g \leq \mu_n(\Delta^p) \int_X ||\varphi||^2 dv_g. \quad (24) \]
The LHS is
\[ \int_X ||\nabla \varphi||^2 dv_g = \int_X ||\nabla(\chi \varphi) + \nabla((1 - \chi)\varphi)||^2 dv_g \]
\[ = \int_X ||\nabla(\chi \varphi)||^2 dv_g + \int_X ||\nabla((1 - \chi)\varphi)||^2 dv_g + 2\text{Re} \int_X (\nabla(\chi \varphi), \nabla((1 - \chi)\varphi)) dv_g. \]

Since
\[ \chi (1 - \chi) \leq \frac{1}{4} \quad \text{and} \quad |\nabla \chi| \leq \frac{1}{a} \]
and by Schwartz inequality,
\[ |(\nabla(\chi \varphi), \nabla((1 - \chi)\varphi))| \leq \left( \frac{1}{a} + \frac{1}{a^2} \right) ||\varphi||^2 + \left( \frac{1}{a} + \frac{1}{4} \right) ||\nabla \varphi||^2. \]

Therefore (24) implies
\[ \mu_n(\Delta^p) \int_X ||\varphi||^2 dv_g \geq \int_X ||\nabla((1 - \chi)\varphi)||^2 dv_g + \int_X ||\nabla(\chi \varphi)||^2 dv_g \]
\[ - 2\left( \frac{1}{a} + \frac{1}{a^2} \right) \int_X ||\varphi||^2 dv_g - 2\left( \frac{1}{a} + \frac{1}{4} \right) \int_X ||\nabla \varphi||^2 dv_g \]
\[ \geq \int_X ||\nabla((1 - \chi)\varphi)||^2 dv_g \]
\[ - 2\left( \frac{1}{a} + \frac{1}{a^2} + \mu_n(\Delta^p)\left( \frac{1}{a} + \frac{1}{4} \right) \right) \int_X ||\varphi||^2 dv_g. \]

Notice that \((1 - \chi)\varphi\) is contained in the domain of \(\Delta^p_{\text{dir}}|_{Y_a}\). By Fact 3.2 and Proposition 3.1 (if \(p = 0\)), or Proposition 3.3 (if \(p = 1\)),
\[ \int_X ||\nabla((1 - \chi)\varphi)||^2 dv_g \geq \mu_1(\Delta^p_{\text{dir}}|_{Y_a}) \int_X ||(1 - \chi)\varphi||^2 dv_g \]
\[ \geq \mu_1(\Delta^p_{\text{dir}}|_{Y_a}) \int_{Y_{2a}} ||\varphi||^2 dv_g \]
\[ \geq C e^{2a} \int_{Y_{2a}} ||\varphi||^2 dv_g, \]
where \(C\) is a positive constant independent of \(a\). So we have obtained
\[ \{ \mu_n(\Delta^p) + 2\left( \frac{1}{a} + \frac{1}{a^2} + \mu_n(\Delta^p)\left( \frac{1}{a} + \frac{1}{4} \right) \right) \} \int_X ||\varphi||^2 dv_g \geq C e^{2a} \int_{Y_{2a}} ||\varphi||^2 dv_g. \]

Now put
\[ \rho_n(a) = 2C^{-1} e^{-2a} \left\{ \left( \frac{1}{a} + \frac{1}{a^2} \right) + \mu_n(\Delta^p)\left( \frac{1}{a} + \frac{3}{4} \right) \right\}, \]
then we have proved the following proposition.
Proposition 3.5. For $\varphi \in \mathcal{M}_n$

$$\int_{Y_{2a}} \|\varphi\|^2 dv_g \leq \rho_n(a) \int_X \|\varphi\|^2 dv_g.$$ 

Let $\alpha$ be greater than two and $\phi_i$ an eigenform of $\Delta_p\big|_{X_{\alpha a}}$ whose eigenvalue is $\mu_i(\Delta_p\big|_{X_{\alpha a}})$ and $\mathcal{M}_n(\alpha a)$ an element of $\text{Gr}(\Delta_p\big|_{X_{\alpha a}})$ spanned by $\{\phi_1, \cdots, \phi_n\}$.

Proposition 3.6. For $\phi \in \mathcal{M}_n(\alpha a)$,

$$\int_{Y_{2a} \cap X_{\alpha a}} \|\phi\|^2 dv_g \leq \rho^0_n(a) \int_{X_{\alpha a}} \|\phi\|^2 dv_g,$$

where

$$\rho^0_n(a) = 2C^{-1} e^{-2a} \left\{ \left( \frac{1}{a} + \frac{1}{a^2} \right) + \mu_n(\Delta_p\big|_{X_{\alpha a}}) \left( \frac{1}{a} + \frac{3}{4} \right) \right\}.$$

Proof. The argument is almost same as one of Proposition 3.5. For $\phi \in \mathcal{M}_n(\alpha a)$,

$$\int_{X_{\alpha a}} \|\nabla \phi\|^2 dv_g = \int_{X_{\alpha a}} (\Delta_p \phi, \phi) dv_g \leq \mu_n(\Delta_p\big|_{X_{\alpha a}}) \int_{X_{\alpha a}} \|\phi\|^2 dv_g.$$  \hspace{1cm} (25)

Using Proposition 3.2 and Proposition 3.4 instead Proposition 3.1 and Proposition 3.3, respectively the previous computation will show

$$\mu_n(\Delta_p\big|_{X_{\alpha a}}) \int_{X_{\alpha a}} \|\phi\|^2 dv_g \geq \int_{X_{\alpha a}} \|\nabla \phi\|^2 dv_g \geq Ce^{2a} \int_{Y_{2a} \cap X_{\alpha a}} \|\phi\|^2 dv_g - 2 \left\{ 1 + \frac{1}{a^2} + \mu_n(\Delta_p\big|_{X_{\alpha a}}) \left( \frac{1}{a} + \frac{1}{4} \right) \right\} \int_{X_{\alpha a}} \|\phi\|^2 dv_g,$$

which yields

$$\int_{Y_{2a} \cap X_{\alpha a}} \|\phi\|^2 dv_g \leq 2C^{-1} e^{-2a} \left\{ \left( \frac{1}{a} + \frac{1}{a^2} \right) + \mu_n(\Delta_p\big|_{X_{\alpha a}}) \left( \frac{1}{a} + \frac{3}{4} \right) \right\} \int_{X_{\alpha a}} \|\phi\|^2 dv_g.$$ 

By Lemma 3.5,

$$\mu_n(\Delta_p\big|_{X_{\alpha a}}) \leq \mu_n(\Delta_{dir}\big|_{X_{\alpha 0}})$$

and we have proved the proposition.

\hspace{1cm} $\square$

A proof of Theorem 3.1
As before let \( \varphi_i \) be an eigenvector of \( \Delta^p \) whose eigenvalue is \( \mu_i(\Delta^p) \) and \( M_{n, \chi} \) an \( n \)-dimensional subspace of \( D(\Delta^p_{\text{dir}}|_{X_{2a}}) \) spanned by \( \{\chi \varphi_1, \cdots, \chi \varphi_n\} \). Let us choose \( \varphi \in M_n \) to be
\[
\frac{\int_X ||\nabla(\varphi)||^2 dv_g}{\int_X ||\chi \varphi||^2 dv_g} = \sup_{f \in M_{n, \chi}} \frac{\int_X ||\nabla f||^2 dv_g}{\int_X ||f||^2 dv_g}.
\]
By Lemma 3.3 the RHS is greater than or equal to \( \mu_n(\Delta^p_{\text{dir}}|_{X_{2a}}) \) and therefore
\[
\int_X ||\nabla(\varphi)||^2 dv_g \geq \mu_n(\Delta^p_{\text{dir}}|_{X_{2a}}) \int_X ||\chi \varphi||^2 dv_g.
\]
On the other hand by a choice of \( \chi \),
\[
||\nabla(\varphi)||^2 \leq ||\nabla \chi \cdot \varphi||^2 + 2|\text{Re}(\nabla \chi \cdot \varphi, \chi \nabla \varphi)| + ||\chi \nabla \varphi||^2
\leq \frac{1}{a^2}||\varphi||^2 + \frac{2}{a}|\text{Re}(\varphi, \nabla \varphi)| + ||\nabla \varphi||^2
\leq \left( \frac{1}{a^2} + \frac{1}{a} \right)||\varphi||^2 + \left( \frac{1}{a} + 1 \right)||\nabla \varphi||^2
\]
hence
\[
\left( \frac{1}{a^2} + \frac{1}{a} \right) \int_X ||\varphi||^2 dv_g + \left( \frac{1}{a} + 1 \right) \int_X ||\nabla \varphi||^2 dv_g
\geq \mu_n(\Delta^p_{\text{dir}}|_{X_{2a}}) \int_X ||\chi \varphi||^2 dv_g
\geq \mu_n(\Delta^p_{\text{dir}}|_{X_{2a}}) \int_{X_{\chi}} ||\varphi||^2 dv_g
= \mu_n(\Delta^p_{\text{dir}}|_{X_{2a}})(\int_X ||\varphi||^2 dv_g - \int_{Y_\chi} ||\varphi||^2 dv_g).
\]
Here by (24) the most left hand side is less than or equal to
\[
\left\{ \frac{1}{a} + 1 \right\} \mu_n(\Delta^p) + \left( \frac{1}{a^2} + \frac{1}{a} \right) \int_X ||\varphi||^2 dv_g,
\]
and by Proposition 3.5 the most right hand side is greater than or equal to
\[
\mu_n(\Delta^p_{\text{dir}}|_{X_{2a}})(1 - \rho_n(\frac{a}{2})) \int_X ||\varphi||^2 dv_g.
\]
Thus we have obtained
\[
\left( \frac{1}{a} + 1 \right) \mu_n(\Delta^p) \geq \mu_n(\Delta^p_{\text{dir}}|_{X_{2a}})(1 - \rho_n(\frac{a}{2})) - \left( \frac{1}{a^2} + \frac{1}{a} \right).
\]
Now since
\[
\lim_{a \to \infty} \rho_n\left(\frac{a}{2}\right) = 0,
\]
and Lemma 3.5 implies the desired result.
A proof of Theorem 3.2.

Since a proof is almost same as one of Theorem 3.1 we will only indicate where a modification is necessary. As before let $\phi_i$ be an eigenform of $\Delta^p|_{X_{3a}}$ whose eigenvalue is $\mu_i(\Delta^p|_{X_{3a}})$ and $\mathcal{M}_n(3a)$ a $n$-dimensional subspace of $D(\Delta^p_{dir}|_{X_{2a}})$ spanned by $\{\chi_{\phi_1}, \cdots, \chi_{\phi_n}\}$. We choose $\phi \in \mathcal{M}_n(3a)$ so that

$$\int_{X_{3a}} ||\nabla(\chi_{\phi})||^2 dv_g \geq \mu_n(\Delta^p_{dir}|_{X_{2a}}) \int_{X_{3a}} ||\chi_{\phi}||^2 dv_g.$$  

Then Lemma 3.3 implies

$$\int_{X_{3a}} ||\nabla(\chi_{\phi})||^2 dv_g \geq \mu_n(\Delta^p_{dir}|_{X_{2a}}) \int_{X_{3a}} ||\chi_{\phi}||^2 dv_g.$$  

Using (25) and Proposition 3.6 instead (24) and Proposition 3.5, respectively the same computation as in Theorem 3.1 will yield

$$\left(\frac{1}{a} + 1\right) \mu_n(\Delta^p|_{X_{3a}}) \geq \mu_n(\Delta^p_{dir}|_{X_{2a}})(1 - \rho_n(\frac{a}{2}) - \left(\frac{1}{a^2} + \frac{1}{a}\right).$$

By Lemma 3.5 $\mu_n(\Delta^p|_{X_{3a}})$ is bounded by $\mu_n(\Delta^p_{dir}|_{X_{3a}})$ from above. Therefore

$$\lim_{a \to \infty} \mu_n(\Delta^p|_{X_{3a}}) = \lim_{a \to \infty} \mu_n(\Delta^p_{dir}|_{X_{3a}}),$$

and this is equal to $\mu_n(\Delta^p)$ by Theorem 3.1.

3.4 A theorem of Cheeger-M{"u}ller type

Since both $\Delta^p$ and $\Delta^p_{abs}|_{X_a}$ have only discrete spectrum the $p$-th cohomology groups $H^p(X_a, \rho)$ and $H^p(X, \rho)$ is isomorphic to $\operatorname{Ker} \Delta^p_{abs}|_{X_a}$ and $\operatorname{Ker} \Delta^p_X$ by Hodge theory, respectively. If $a$ is sufficiently large, $H^p(X_a, \rho)$ is isomorphic to $H^p(X, \rho)$ by the restriction and therefore $\operatorname{Ker} \Delta^p_X$ is also isomorphic to $\operatorname{Ker} \Delta^p_{abs}|_{X_a}$. Since $\rho$ is cuspidal $H^0(T_\nu, \rho) = 0$ for every $\nu$ and

$$H^2(T_\nu, \rho) = 0,$$

by Poincaré duality. Because $\rho$ is a local system on a flat torus $T_\nu$ we see

$$H^1(T_\nu, \rho) = 0,$$

by the index theorem. Therefore the exact sequence

$$H^{p-1}(\partial X_a, \rho) = \oplus_{\nu=1}^h H^{p-1}(T_\nu, \rho) \to H^p(X_a, \partial X_a, \rho) \to H^p(X_a, \rho) \to H^p(\partial X_a, \rho) = \oplus_{\nu=1}^h H^p(T_\nu, \rho)$$

29
shows $H^p(X_a, \partial X_a, \rho)$ is isomorphic to $H^p(X_a, \rho)$. Thus we find

$$h^p(X, \rho) = h^{3-p}(X, \rho),$$

by Poincaré duality. (Recall $h^p(X, \rho)$ is the dimension of $H^p(X_a, \rho)$.) In particular since $h^0(X, \rho)$ vanishes so does $h^3(X, \rho)$. Moreover Hodge $*$ operator yields an isomorphism

$$\text{Ker } \Delta^p_{abs}|_{X_a} \cong \text{Ker } \Delta^{3-p}_{rel}|_{X_a},$$

and therefore

$$h^p(X, \rho) = \dim \text{Ker } \Delta^p_{abs}|_{X_a} = \dim \text{Ker } \Delta^{3-p}_{rel}|_{X_a}.$$

A partial spectral zeta function of $\Delta^p|_{X_a}$ ($* = \text{abs or rel}$) is defined to be

$$\zeta_{X_a,*}^{(p)}(z, \rho) = \frac{1}{\Gamma(z)} \int_0^\infty \theta_p(t, a) t^{z-1} dt, \quad \theta_p(t, a) = \text{Tr}[e^{-t\Delta^p_{|X_a}}] - h^p(X, \rho).$$

(Here $a$ is assumed to be sufficiently large.) If $\Re z$ is sufficiently large it absolutely converges. Since $\theta_p(t, a)$ has an asymptotic expansion

$$\theta_p(t, a) \sim t^{-\frac{3}{2}} \sum_{l=0}^{2N} c_l(a) t^{l/2} + O(t^{N-\frac{3}{2}}), \quad \text{as } t \to 0. \quad (26)$$

the same argument as the beginning of this section will show that $\zeta_{X_a,*}^{(p)}(z, \rho)$ is meromorphically continued to the whole plane and that it is regular at the origin. We define the spectral zeta function of $X_a$ as

$$\zeta_{X_a}(z, \rho) = \sum_{p=0}^{3} (-1)^p \zeta_{X_a,\text{abs}}^{(p)}(z, \rho).$$

Since Hodge $*$ operator commutes with Hodge Laplacian and since it interchanges two boundary conditions,

$$\zeta_{X}^{(p)}(z, \rho) = \zeta_{X,a}^{(3-p)}(z, \rho), \quad \zeta_{X,a,\text{abs}}^{(p)}(z, \rho) = \zeta_{X,a,\text{rel}}^{(3-p)}(z, \rho).$$

Therefore

$$\zeta_{X_a}(z, \rho) = 2\zeta_{X,a,\text{rel}}^{(1)}(z, \rho) - \zeta_{X,a,\text{abs}}^{(1)}(z, \rho) - 3\zeta_{X,a,\text{rel}}^{(0)}(z, \rho)$$

and

$$\zeta_{X}(z, \rho) = \zeta_{X,a}^{(1)}(z, \rho) - 3\zeta_{X}^{(0)}(z, \rho).$$

**Theorem 3.3.**

$$\lim_{a \to \infty} \zeta_{X_a}(z, \rho) = \zeta_{X}(z, \rho).$$

**Corollary 3.2.**

$$\lim_{a \to \infty} \zeta_{X_a}'(0, \rho) = \zeta_{X}'(0, \rho).$$
Proof of Theorem 3.3. Let us write
\[ \int_0^\infty \theta_p(t) t^{z-1} dt = \int_0^1 \theta_p(t) t^{z-1} dt + \int_1^\infty \theta_p(t) t^{z-1} dt, \]
\[ \int_0^\infty \theta_p(t,a) t^{z-1} dt = \int_0^1 \theta_p(t,a) t^{z-1} dt + \int_1^\infty \theta_p(t,a) t^{z-1} dt. \]
We will investigate convergence of corresponding terms in RHS. For the second term Theorem 3.2 implies
\[ \lim_{a \to \infty} \int_1^\infty \theta_p(t,a) t^{z-1} dt = \int_1^\infty \theta_p(t) t^{z-1} dt. \]
As we have seen at the beginning of this section \( \theta_p(t) \) has an asymptotic expansion (see (14)),
\[ \theta_p(t) \sim t^{-\frac{3}{2}} \sum_{l=0}^{2N} c_l t^{l/2} + O(t^{N-\frac{3}{2}}), \quad c_{2l+1} = 0, \]
around \( t = 0 \). We put
\[ \theta_p(t,a)^{(N)} = \theta_p(t,a) - t^{-\frac{3}{2}} \sum_{l=0}^{2N} c_l(a) t^{l/2}, \quad \theta_p(t)^{(N)} = \theta_p(t) - t^{-\frac{3}{2}} \sum_{l=0}^{2N} c_l t^{l/2}. \]
Then \( t^{\frac{3}{2}-N} \theta_p(t,a)^{(N)} \) and \( t^{\frac{3}{2}-N} \theta_p(t)^{(N)} \) are bounded on \((0,1]\). By Theorem 3.2 we see
\[ \lim_{a \to \infty} c_l(a) = c_l, \quad 0 \leq l \leq 2N, \]
which implies in turn
\[ \lim_{a \to \infty} \theta_p(t,a)^{(N)} = \theta_p(t)^{(N)} \quad \text{on } (0,1] \]
Therefore if \( \text{Re} z > \frac{3}{2} - N, \)
\[ \lim_{a \to \infty} \int_0^1 \theta_p(t,a)^{(N)} t^{z-1} dt = \int_0^1 \theta_p(t)^{(N)} t^{z-1} dt. \]
and
\[ \int_0^1 \theta_p(t,a) t^{z-1} dt = \sum_{l=0}^{2N} \frac{c_l(a)}{z + \frac{l-3}{2}} + \int_0^1 \theta_p(t,a)^{(N)} t^{z-1} dt, \]
converges to
\[ \int_0^1 \theta_p(t) t^{z-1} dt = \sum_{l=0}^{2N} \frac{c_l}{z + \frac{l-3}{2}} + \int_0^1 \theta_p(t)^{(N)} t^{z-1} dt, \]
as \( a \to \infty \).
For a finite dimensional vector space $V$ we set
\[
\det V = \wedge^{\dim V} V.
\]

The determinant of a bounded complex of finite dimensional vector spaces $(C, \partial)$ is defined to be
\[
\det(C, \partial) = \otimes_i (\det C^i)^{(-1)^i}.
\]

Here for a one dimensional complex vector space $L$, $L^{-1}$ is its dual. Due to Knudsen and Mumford, there is a canonical isomorphism
\[
\det(C, \partial) \simeq \otimes_i \det \mathcal{H}^i(C, \partial)^{(-1)^i}.
\]

Let $\Sigma = \{\Sigma_p\}_p$ be a triangulation of $X_a$ where $\Sigma_p$ is the set of $p$-simplices and $e = \{e_1, \cdots, e_r\}$ a unitary basis of $\rho$. We define a Hermitian inner product on the group of $p$-cochains:
\[
\mathcal{C}^p(\Sigma, \rho) = \mathcal{C}^p(\Sigma) \otimes \rho,
\]
so that $\{[\sigma]^* \otimes e_i\}$ form its unitary basis, where $[\sigma]^*$ is the dual vector of $[\sigma]$. Now the Knudsen and Mumford isomorphism induces a metric $\| \cdot \|_{FR,a}$ on $\det \mathcal{H}^i(X_a, \rho) = \otimes_i \det \mathcal{H}^i(C^i(\Sigma, \rho))^{(-1)^i}$, which is referred as Franz-Reidemeister metric. Via the isomorphism
\[
\mathcal{H}^i(X_a, \rho) \simeq \mathcal{H}^i(X, \rho),
\]

it yields a metric $\| \cdot \|_{FR,a}$ on $\det \mathcal{H}^i(X, \rho)$. Notice that they are independent of $a$ as far as it is sufficiently large since we can use the same triangulations to define them. Therefore the limit
\[
\| \cdot \|_{FR} = \lim_{a \to \infty} \| \cdot \|_{FR,a}
\]
is well-defined. For a later purpose we will describe it in terms of a combinatorial zeta function.

A triangulation $\Sigma$ of $X_a$ induces a simplicial decomposition $\tilde{\Sigma}$ of the universal covering $\tilde{X}_a$. Let $\{\sigma_1^{(p)}, \cdots, \sigma_p^{(p)}\}$ the set of $p$-simplices of $\tilde{\Sigma}$ which are a lift of $\Sigma_p$. Then $\mathcal{C}_p(\tilde{\Sigma})$ is a free $\mathbb{C}[\Gamma]$-module generated by these elements. A twisted chain complex is defined to be
\[
\mathcal{C}(\Sigma, \rho) = \mathcal{C}(\tilde{\Sigma}) \otimes_{\mathbb{C}[\Gamma]} \rho,
\]

which is a bounded complex of finite dimensional vector spaces. We will introduce a Hermitian inner product so that $\{\sigma_i^{(p)} \otimes e_j\}$ is a unitary basis. Here is
an explicit description of the boundary map: Let \( \sum_k (-1)^k \gamma_k \sigma_k \in C^p(\Sigma) \) be the boundary of \( [\sigma_i^{(p)}] \in C^p(\tilde{\Sigma}) \). Then
\[
\partial([\sigma_i^{(p)}] \otimes e_j) = \sum_k (-1)^k \gamma_k \sigma_k \otimes \rho(\gamma_k) e_j.
\]
Let \((C^*(\Sigma, \rho), \delta)\) be the dual complex. By the inner product we may identify \( C^*(\Sigma, \rho) \) with \( C^*(\Sigma, \rho) \) and in particular the dual vector of \([\sigma_i^{(p)}] \otimes e_j\) will be identified with itself. Thus \((C^*(\Sigma, \rho), \delta)\) is a complex such that each \( C^p(\Sigma, \rho) \) is nothing but \( C_p(\Sigma, \rho) \) as a vector space but the differential \( \delta \) is the Hermitian dual of \( \partial \). Let us define a (positive) combinatorial Laplacian \( \Delta^p_{\text{comb}} \) on \( C^p(\Sigma, \rho)(= C_p(\Sigma, \rho)) \) to be
\[
\Delta^p_{\text{comb}} = \partial \delta + \delta \partial.
\]
Then
\[
H^p(X_a, \rho)(= H_p(X_a, \rho)) = \text{Ker}[\Delta^p_{\text{comb}}],
\]
has the inner product \((\cdot, \cdot)_{2, X_a}\) induced by the metric on \( C_p(\Sigma, \rho) \) (Here we have identified \( H^p(X_a, \rho) \) with \( \text{Ker}[\Delta^p_{\text{comb}}] \) which is a subspace of \( C_p(\Sigma, \rho) \)). It induces a norm \(|\cdot|_{2, X_a}\) on the determinant \( \otimes \rho \text{det}H^p(X_a, \rho \cdot (-1)^p \). Let us define the combinatorial zeta function to be
\[
\zeta_{\text{comb}}(s, X_a) = \sum_p (-1)^p \cdot \zeta_{\text{comb}}^{(p)}(s, X_a),
\]
where
\[
\zeta_{\text{comb}}^{(p)}(s, X_a) = \sum \lambda^{-s}.
\]
Here \( \lambda \) runs through positive eigenvalues of \( \Delta^p_{\text{comb}} \) on \( C^p(\Sigma, \rho) \). The modified Franz-Reidemeister torsion \( \tau^*(X_a, \rho) \) is defined as
\[
\tau^*(X_a, \rho) = \exp(-\frac{1}{2} \zeta_{\text{comb}}'(0, \rho)).
\]
If \( H^1(X, \rho) \) vanishes so does every \( H^p(X, \rho) \) and \( \tau^*(X_a, \rho) \) is the usual Franz-Reidemeister torsion \( \tau(X_a, \rho)([15]) \). It is known that \(|\cdot|_{FR,a} \) is equal to \((2)[15]\).

By construction since both \(|\cdot|_{2, X_a} \) and \( \tau^*(X_a, \rho) \) depend only on a triangulation \( \Sigma \), they are independent of sufficiently large \( a \) as before. Thus the limit
\[
|\cdot|_{2, X} = \lim_{a \to \infty} |\cdot|_{2, X_a}, \quad \tau^*(X, \rho) = \lim_{a \to \infty} \tau^*(X_a, \rho),
\]
is well-defined and we set
\[
||\cdot||_{FR} = |\cdot|_{2, X} \cdot \tau^*(X, \rho).
\]
On the other hand since \( H^p(X_a, \rho) \) is isomorphic to 
\[
\text{Ker}\Delta^p_{\text{abs}}|_{X_a} \subset L^2(X_a, \Omega^p(\rho))
\]
the inner product on \( L^2(X_a, \Omega^p(\rho)) \) induces a metric on \( H^p(X_a, \rho) \). Thus by the
isomorphism \( H^p(X, \rho) \simeq H^p(X_a, \rho) \) we have a norm \( \cdot |_{L^2, X_a} \) on \( \det H^p(X, \rho) \).
The Ray-Singer metric \( || \cdot ||_{RS,a} \) is defined to be
\[
|| \cdot ||_{RS,a} = | \cdot |_{L^2, X_a} \cdot \exp\left(-\frac{1}{2} \zeta_X'(0, \rho)\right).
\]
Similary using the canonical isomorphism
\[
H^p(X, \rho) \simeq \text{Ker}\Delta^p \subset L^2(X, \Omega^p(\rho))
\]
Ray-Singer metric \( || \cdot ||_{RS} \) on \( \det H^p(X, \rho) \) is defined as
\[
|| \cdot ||_{RS} = | \cdot |_{L^2, X} \cdot \exp\left(-\frac{1}{2} \zeta_X'(0, \rho)\right).
\]
Then we will show
\[\text{Theorem 3.4.} \ || \cdot ||_{FR} \text{ and } || \cdot ||_{RS} \text{ coincide. In particular}\]
\[
\exp(-\zeta_X'(0, \rho)) = \left(\frac{| \cdot |_{L^2, X}^2}{| \cdot |_{L^2, X}^2}\right)^2 \tau^*(X, \rho)^2.
\]
\[\text{Proposition 3.7.} \ \lim_{a \to \infty} | \cdot |_{L^2, X_a} = | \cdot |_{L^2, X}.
\]
In fact for a sufficiently large \( a \) let \( \{\xi_{a,i}\}_i \) be an orthonormal base of \( \text{Ker}\Delta^p_{\text{abs}}|_{X_a} \) and we define a map
\[
\text{Ker}\Delta^p \xrightarrow{P_a} \text{Ker}\Delta^p_{\text{abs}}|_{X_a}
\]
as
\[
P_a \psi = \sum_i \int_{X_a} (\psi, \xi_{a,i}) dv_g \cdot \xi_{a,i}.
\]
Then we claim the following.
\[\text{Lemma 3.6.} \ \lim_{a \to \infty} \int_{X_a} || \psi - P_a \psi ||^2 dv_g = 0.
\]
The following corollary will imply \textbf{Proposition 3.7}.
\[\text{Corollary 3.3.} \ \text{For } \psi \in \text{Ker}\Delta^p, \]
\[
\lim_{a \to \infty} \int_{X_a} || P_a \psi ||^2 dv_g = \int_X || \psi ||^2 dv_g.
\]
\[\text{Proof.} \ \text{Immediately from Proposition 3.5 and Lemma 3.6.}\]
Proof of Lemma 3.6. In the following arguments all $C$ are positive constants independent of $a$. Let $\phi_\lambda$ be an eigenform of $\Delta^p_{abs}|_{X_a}$ whose eigenvalue is $\lambda$ and that

$$\int_{X_a} ||\phi_\lambda||^2 dv_g = 1.$$ 

Let us expand $\psi$ as

$$\psi = \sum_{\lambda} \int_{X_a} (\psi, \phi_\lambda) dv_g \cdot \phi_\lambda.$$ 

Since

$$\int_{X_a} ||\psi - P_a \psi||^2 dv_g = \sum_{\lambda > 0} |\int_{X_a} (\psi, \phi_\lambda) dv_g|^2,$$

it is sufficient to show that for $\phi = \phi_\lambda$ ($\lambda > 0$),

$$|\int_{X_a} (\psi, \phi) dv_g| \leq Ce^{-a} (\int_{X_a} ||\psi||^2 dv_g + C).$$

Let us choose $\chi \in C^\infty_c(X_a)$ so that

1. $0 \leq \chi \leq 1$.
2. $|\nabla \chi|, |\Delta \chi|$ are bounded by 1.
3. $\chi|_{X_a/2} = 1$.

By Stokes theorem,

$$\int_{X_a} (\Delta^p(\chi \psi), \phi) dv_g = \int_{X_a} (\chi \psi, \Delta^p \phi) dv_g = \lambda \int_{X_a} \chi (\psi, \phi) dv_g \quad (28)$$

Since $\Delta^p \psi = 0$ and by the property 3 of $\chi$, LHS of (28) becomes

$$\int_{X_a} (\Delta^p(\chi \psi), \phi) dv_g = \int_{X_a} (\Delta \chi \cdot \psi, \phi) dv_g + 2 \int_{X_a} (\nabla \chi \cdot \nabla \psi, \phi) dv_g$$

$$= \int_{Y_a/2 \cap X_a} (\Delta \chi \cdot \psi, \phi) dv_g + 2 \int_{Y_a/2 \cap X_a} (\nabla \chi \cdot \nabla \psi, \phi) dv_g.$$

Let us consider the first term. Using the property 2 of $\chi$ the Schwartz inequality implies

$$|\int_{Y_a/2 \cap X_a} (\Delta \chi \cdot \psi, \phi) dv_g| \leq \frac{1}{2} \left( \int_{Y_a/2 \cap X_a} ||\psi||^2 dv_g + \int_{Y_a/2 \cap X_a} ||\phi||^2 dv_g \right)$$

$$\leq \frac{1}{2} \left( \int_{Y_{a/2}} ||\psi||^2 dv_g + \int_{Y_{a/2} \cap X_a} ||\phi||^2 dv_g \right).$$
By Proposition 3.5,
\[
\int_{Y_a/2} ||\psi||^2 dv_g \leq Ce^{-a/2} \int_X ||\psi||^2 dv_g
\]
\[
= Ce^{-a/2} \left( \int_{X_a} ||\psi||^2 dv_g + \int_{Y_a} ||\psi||^2 dv_g \right)
\]
\[
\leq Ce^{-a/2} \int_{X_a} ||\psi||^2 dv_g + Ce^{-a/2} \int_{Y_a/2} ||\psi||^2 dv_g,
\]
and therefore changing $C$ we obtain
\[
\int_{Y_a/2} ||\psi||^2 dv_g \leq Ce^{-a/2} \int_{X_a} ||\psi||^2 dv_g.
\]

Using Proposition 3.6 instead Proposition 3.5 the same computation will show
\[
\int_{Y_a/2 \cap X_a} ||\phi||^2 dv_g \leq Ce^{-a/2} \int_{X_a} ||\phi||^2 dv_g = Ce^{-a/2}
\]
and thus
\[
|\int_{Y_a/2 \cap X_a} (\Delta \chi \cdot \psi, \phi) dv_g| \leq Ce^{-a/2} \left( \int_{X_a} ||\psi||^2 dv_g + C \right).
\]

Next we will estimate the second term. Using the property 2 of $\chi$, the Schwartz inequality yields,
\[
|2 \int_X (\nabla \chi \cdot \nabla \psi, \phi) dv_g| \leq \int_{Y_a/2} ||\psi||^2 dv_g + \int_{Y_a/2 \cap X_a} ||\phi||^2 dv_g.
\]
Since
\[
\int_{Y_a/2} ||\psi||^2 dv_g \leq \int_X ||\nabla \psi||^2 dv_g = \int_X (\psi, \Delta \psi) dv_g = 0,
\]
and by (29), we obtain
\[
|2 \int_X (\nabla \chi \cdot \nabla \psi, \phi) dv_g| \leq Ce^{-a/2}.
\]

Thus LHS of (28) is bounded by $Ce^{-a/2}(\int_{X_a} ||\psi||^2 dv_g + C)$. Let us consider RHS of (28). The property 3 implies
\[
\int_{X_a} \chi(\psi, \phi) dv_g = \int_{X_a} (\psi, \phi) dv_g + \int_{Y_a/2 \cap X_a} \chi(\psi, \phi) dv_g.
\]
But by the previous arguments
\[
|\int_{Y_a/2 \cap X_a} \chi(\psi, \phi) dv_g | \leq \int_{Y_a/2 \cap X_a} |(\psi, \phi)| dv_g
\]
\[
\leq \frac{1}{2} \left\{ \int_{X_a} ||\psi||^2 dv_g + \int_{Y_a/2 \cap X_a} ||\phi||^2 dv_g \right\}
\]
\[
\leq Ce^{-a/2}(\int_{X_a} ||\psi||^2 dv_g + C),
\]

and we will obtain
\[ |\int_{X_{a/2}} (\psi, \phi) dv_g| \leq Ce^{-a/2}(\int_X ||\psi||^2 dv_g + C). \]

Notice that
\[ |\int_X (\psi, \phi) dv_g - \int_{X_{a/2}} (\psi, \phi) dv_g| = |\int_{X_{a/2}} (\psi, \phi) dv_g| \leq \int_{X_{a/2}} (\psi, \phi) dv_g \leq Ce^{-a/2}(\int_X ||\psi||^2 dv_g + C), \]
and the desired result has been obtained since
\[ |\int_X (\psi, \phi) dv_g| \leq |\int_X (\psi, \phi) dv_g - \int_{X_{a/2}} (\psi, \phi) dv_g| + |\int_{X_{a/2}} (\psi, \phi) dv_g| \leq Ce^{-a/2}(\int_X ||\psi||^2 dv_g + C). \]

□

Let us choose a sufficiently large number \( a \) and small positive number \( \delta \). Let \( g_0 \) be a Riemannian metric on \( X \) such that
\[ g_0(x) = \begin{cases} g(x) & \text{if } x \in X_{a-\delta} \\ du^2 + e^{-2a}(dx^2 + dy^2) & \text{if } x \in Y_a \end{cases} \]

We will consider a one parameter family of metrics:
\[ g_q = (1 - q)g_0 + qg, \quad 0 \leq q \leq 1. \]

Let \( \{e^0, e^1, e^2\} \) be an orthonormal frame of \( \Omega^1|_{\partial X_a} \) with respect to \( g(q)|_{\partial X_a} = du^2 + e^{-2a}(dx^2 + dy^2) \) so that \( e^0 = du \). Let \( h(q) \) and \( R(q) \) be the second fundamental of \( \partial X_a \) with respect to \( g(q) \) and the curvature tensor of \( g(q) \), respectively. Then we define elements
\[ \hat{h}(q) = \sum_{1 \leq a, b \leq 2} h(q)_{ab} e^a \otimes e^b \]
and
\[ \hat{R}_0(q) = \frac{1}{4} \sum_{j,k,l} R(q)_{ijkl} e^j \otimes (e^k \wedge e^l) \]
of \( \Omega^1|_{\partial X_a} \otimes \Omega^1|_{\partial X_a} \). Using Berezin integral \( \int_B \), we put
\[ \phi_a = \int_0^1 dq \int_B \hat{h}(q) \hat{R}_0(q) \in \Omega_{\partial X_a}. \]
Fact 3.3. ([3])

$$\log \left( \frac{|| \cdot ||_{RS,a}}{|| \cdot ||_{FR,a}} \right) = \chi(\partial X_a, \rho) \log 2 + \gamma \cdot r \int_{\partial X_a} \phi_a,$$

where $\gamma$ is an absolute constant.

Notice that the term $\hat{c}(g_0, g_a)$ in the original formula vanishes because the dimension of $X$ is three. A straightforward computation will show that the norm of $\phi_a$ is bounded by a constant $C$ which is independent of $a$. Thus,

$$\left| \int_{\partial X_a} \phi_a \right| \leq C \cdot \text{vol}(\partial X_a) \leq C' \cdot e^{-2a},$$

where $C'$ is also independent of $a$. Since $\rho$ is a unitary local system on $\partial X_a$ which is a disjoint union of flat tori, $\chi(\partial X_a, \rho)$ vanishes by the index theorem. Therefore we have shown

Proposition 3.8.

$$\lim_{a \to \infty} || \cdot ||_{RS,a} = || \cdot ||_{FR}.$$  

Proof of Theorem 3.4. By Proposition 3.8

$$|| \cdot ||_{FR} = \lim_{a \to \infty} \{ | \cdot |_{L^2,X_a} \cdot \exp\left( -\frac{1}{2} \xi'_{X_a}(0, \rho) \right) \}.$$

But by Proposition 3.7 and Corollary 3.2 this is equal to $|| \cdot ||_{RS}$. □

3.5 A computation of the leading coefficient

We will interprete the ratio $| \cdot |_{L^2} / | \cdot |_{L^2,X}$ as a period. Hereafter we will identify $H^p(X, \rho)$ and $\text{Ker} \Delta_X^p$ by Hodge theory. Let $\phi(p) = \{ \phi_1^{(p)}, \cdots, \phi_{h^p(X, \rho)}^{(p)} \}$ and $\psi(p) = \{ \psi_1^{(p)}, \cdots, \psi_{h^p(X, \rho)}^{(p)} \}$ be its unitary basis with respect to $(\cdot, \cdot)_{L^2,X}$ and $(\cdot, \cdot)_{L^2,X}$, respectively. Then $\phi(p)$ determines the dual basis $\phi(p) = \{ \phi(p,i) \}_{1 \leq i \leq h^p(X, \rho)}$ of $H^p(X, \rho)$ and we write

$$\psi_1^{(p)} = \sum_{j=1}^{h^p(X, \rho)} \int_{\phi(p),j} \psi_1^{(p)} \cdot \phi_j^{(p)}.$$

A period matrix of twisted $p$-forms is defined to be

$$P(X, \rho)_p = (\int_{\phi(p),j} \psi_1^{(p)})_{ij}.$$

We call an alternating product $\prod_p | \det P(X, \rho)_p |^{-1/p}$ a period of $(X, \rho)$ and will denote it by $\text{Per}(X, \rho)$. A simple computation shows

$$\psi_1^{(p)} \wedge \cdots \wedge \psi_{h^p(X, \rho)}^{(p)} = \det P(X, \rho)_p \cdot \phi_1^{(p)} \wedge \cdots \wedge \phi_{h^p(X, \rho)}^{(p)}.$$
and by definition we have
\[ | \otimes_p (\psi^{(p)}_1 \wedge \cdots \wedge \psi^{(p)}_{h^p(X,\rho)}) (\psi^{(p)}_1 \wedge \cdots \wedge \psi^{(p)}_{h^p(X,\rho)}) (\psi^{(p)}_1 \wedge \cdots \wedge \psi^{(p)}_{h^p(X,\rho)}) (-1)^p |_{L^2, X} = | \otimes_p (\phi^{(p)}_1 \wedge \cdots \wedge \phi^{(p)}_{h^p(X,\rho)}) (\phi^{(p)}_1 \wedge \cdots \wedge \phi^{(p)}_{h^p(X,\rho)}) (\phi^{(p)}_1 \wedge \cdots \wedge \phi^{(p)}_{h^p(X,\rho)}) (-1)^p |_{L^2, X} = 1 \]

Therefore
\[ | \otimes_p (\psi^{(p)}_1 \wedge \cdots \wedge \psi^{(p)}_{h^p(X,\rho)}) (\psi^{(p)}_1 \wedge \cdots \wedge \psi^{(p)}_{h^p(X,\rho)}) (\psi^{(p)}_1 \wedge \cdots \wedge \psi^{(p)}_{h^p(X,\rho)}) (-1)^p |_{L^2, X} = \prod_p | \det P(X,\rho)p| (-1)^p \]
\[ = \text{Per}(X,\rho). \]

Now using Fact 3.1, Theorem 3.4 is reformulated as the following.

**Theorem 3.5.**
\[ \lim_{z \to 0} z^{-2h^1(X,\rho)} R_X(z,\rho) = (\tau^*(X,\rho) \cdot \text{Per}(X))^2. \]

**Corollary 3.4.** Suppose that \( h^1(X,\rho) \) vanishes. Then
\[ R_X(0,\rho) = \tau(X,\rho)^2, \]
where \( \tau(X,\rho) \) is the usual Franz-Reidemeister torsion.

### 3.6 An example

Let \( K \) be a knot in \( S^3 \) whose complement \( X_K \) admits a complete hyperbolic structure of finite volume and \( \rho \) a cuspidal unitary local system of rank \( r \) on \( X_K \). Here the representation of \( \pi_1(X_K) \) associated to \( \rho \) is denoted by the same character. \( X_K \) is obtained by attaching 3-cells to a two dimensional CW-complex \( L \) which is a deformation retract of \( X_K \). The argument of Lemma 7.2 will show the following.

**Lemma 3.7.**
\[ \tau(X_K,\rho) = \tau(L,\rho). \]

We will compute \( \tau(L,\rho) \). Let
\[ \pi_1(X_K) = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_{n-1} \rangle \]
be the Wirtinger presentation. Here \( \{ x_i \}_i \) (resp. \( \{ r_j \}_j \)) is generators (resp. relators). \( H_1(X_K,\mathbb{Z}) \) is an infinite cyclic group and we fix a generator \( t \). We choose \( x_i \) so that Hurewicz map
\[ \pi_1(X_K) \xrightarrow{\tau} H_1(X_K,\mathbb{Z}) \]
sends \( x_i \) to \( t \). Then a group ring \( \mathbb{C}[H_1(X_K,\mathbb{Z})] \) is isomorphic to Laurent polynomial ring \( \Lambda = \mathbb{C}[t,t^{-1}] \) and a ring homomorphism:
\[ \mathbb{C}[\pi_1(X_K)] \to \Lambda. \]
induced by Hurewicz map will be also denoted by $\epsilon$. Also the representation $\rho$ yields a homomorphism
$$C[\pi_1(X_K)] \xrightarrow{\rho} M_r(\mathbb{C})$$
and let
$$C[\pi_1(X_K)] \xrightarrow{\epsilon \otimes \rho} M_r(\Lambda).$$
be the tensor product of them. Composing this with the homomorphism induced by the natural projection from the free group $F_n$ of $n$-generators to $\pi_1(X_K)$ we obtain,
$$C[\pi_1(X_K)] \xrightarrow{\Phi} M_r(\Lambda).$$

The set of 0-cells of $L$ consists of only one point $P_0$ and one of 1-cells is
$$\{x_1, \cdots, x_n\}.$$  
In order to obtain the relation it is necessary to attach 2-cells
$$\{y_1, \cdots, y_{n-1}\},$$  
where $y_j$ realizes the relator $r_j$. Let $\tilde{L}$ be the universal covering of $L$ and $L\infty$ an infinite cyclic covering which corresponds to $\ker \epsilon$. According to $p = 0$ (resp. $p = 1$ or $p = 2$), the $p$-th chain group $C_p(L, \mathbb{C})$ is a free right $C[\pi_1(X_K)]$ module generated by $P_0$ (resp. $\{x_1, \cdots, x_n\}$ or $\{y_1, \cdots, y_{n-1}\}$) and the chain complex $C(L, \rho)$ is defined to be
$$C_p(L, \rho) = C_p(\tilde{L}, \mathbb{C}) \otimes_{C[\ker \epsilon]} \rho.$$  
Thus the chain complex
$$C_2(L, \rho) \xrightarrow{\partial_2} C_1(L, \rho) \xrightarrow{\partial_1} C_0(L, \rho),$$  
becomes
$$(\Lambda \oplus \mathbb{r})^{n-1} \xrightarrow{\partial_2} (\Lambda \oplus \mathbb{r})^n \xrightarrow{\partial_1} \Lambda \oplus \mathbb{r},$$  
and the differentials are described by Fox’s free differential calculus. In fact it is known (12):
$$\partial_1 = \begin{pmatrix} 
\Phi(x_1 - 1) \\
\vdots \\
\Phi(x_n - 1) 
\end{pmatrix} = \begin{pmatrix} 
\rho(x_1)t - I_r \\
\vdots \\
\rho(x_n)t - I_r 
\end{pmatrix},$$  
and
$$\partial_2 = \begin{pmatrix} 
\Phi(\frac{\partial r_1}{\partial x_1}) & \cdots & \Phi(\frac{\partial r_1}{\partial x_n}) \\
\vdots & \ddots & \vdots \\
\Phi(\frac{\partial r_n}{\partial x_1}) & \cdots & \Phi(\frac{\partial r_n}{\partial x_n}) 
\end{pmatrix}.$$  
Here an each entry is an element of $M_r(\Lambda)$. $C_p(L, \rho)$ is considered as a space of row vectors and differentials act from the right. It is known that the determinant
of a certain entry of $\partial$ is not zero\([19]\). Therefore rearranging indices we may assume that $\det(\rho(x_n)t - I_r)$ is not zero and will denote it by $\Delta_0(t)$. Let us put $\Delta_1(t) = \det \begin{pmatrix} \Phi(\frac{\partial}{\partial x_1}) & \cdots & \Phi(\frac{\partial}{\partial x_{n-1}}) \\ \vdots & \ddots & \vdots \\ \Phi(\frac{\partial}{\partial x_1}) & \cdots & \Phi(\frac{\partial}{\partial x_{n-1}}) \end{pmatrix}$. Then the twisted Alexander function is defined to be \([7][8][19]\),

$$\Delta_{K,\rho}(t) = \frac{\Delta_1(t)}{\Delta_0(t)}.$$ 

In the following we will assume $\Delta_1(t)$ is not zero. This implies that after tensored with $\mathbb{C}(t)$ \((30)\) becomes acyclic. Thus $H_L(L,\rho)$ are torsion $\Lambda$-modules and in particular they are finite dimensional vector spaces. Let $\tau_i$ be the representation matrix of $t$. Remember that $C(L,\rho)$ is quasi-isomorphic to $C(X_K,\rho)$ and that

$$C(L,\rho) = C(L,\rho) \otimes_\Lambda \mathbb{C}.$$ 

Here $\mathbb{C}$ is regarded as a $\Lambda$-module by $\mathbb{C} \simeq \Lambda/(t - 1)$. Thus the exact sequence of complexes

$$0 \rightarrow C(L,\rho) \rightarrow C(L,\rho) \rightarrow C(L,\rho) \rightarrow 0,$$

induces

$$
\begin{array}{c}
0 & \rightarrow & H_2(L,\rho) & \xrightarrow{\tau_2 - \text{id}} & H_2(L,\rho) & \rightarrow & H_2(X_K,\rho) \\
\rightarrow & H_1(L,\rho) & \xrightarrow{\tau_1 - \text{id}} & H_1(L,\rho) & \rightarrow & H_1(X_K,\rho) \\
\rightarrow & H_0(L,\rho) & \xrightarrow{\tau_0 - \text{id}} & H_0(L,\rho) & \rightarrow & H_0(X_K,\rho)
\end{array}
$$

Since $h^0(X_K,\rho) = 0$, $H_0(X_K,\rho)$ vanishes by the universal coefficient theorem. Thus $\tau_0 - \text{id}$ is an isomorphism. Using the fact $\Delta_i(t)$ is a multiplication of the characteristic polynomial of $\tau_i$ and a certain unit of $\Lambda$, the following lemma is an easy consequence of \((31)\).

**Lemma 3.8.** The following are equivalent:

1. $\Delta_{K,\rho}(1) \neq 0$.

2. $h^1(X_K,\rho) = 0$.

3. $h^1(X_K,\rho) = h^2(X_K,\rho) = 0$.

In [13] we have proved that the fact $h^i(X_K,\rho) = 0$ for all $i$ implies

$$\tau(X_K,\rho) = |\Delta_{K,\rho}(1)|.$$ 

\((32)\)

Thus **Corollary 3.4** and \((32)\) prove the following.
Theorem 3.6. Suppose $\Delta_{K,\rho}(1) \neq 0$. Then

$$R_{X_K}(0, \rho) = |\Delta_{K,\rho}(1)|^2.$$ 

Here is an example of $\rho$ such that a special value of Ruelle L-function at the origin can be computed explicitly. Let $\xi$ be a complex number of modulus one. We define a morphism

$$H_1(X_K, \mathbb{Z}) \overset{\rho}{\to} U(1)$$

to be

$$\rho(t) = \xi.$$ 

Composing it with Hurewicz map we obtain a unitary character

$$\pi_1(X_K) \overset{\rho}{\to} U(1),$$

which yields a unitary local system of rank one on $X_K$. Notice that $t$ represents a meridian of the boundary of a tubular neighborhood of $K$. Thus if $\xi \neq 1$, $\rho$ is cuspidal. Moreover it is known ([7] §3.3):

$$\Delta_0(t) = 1 - \xi t, \quad \Delta_1(t) = A_K(\xi t),$$

where $A_K(t)$ is the Alexander polynomial. Now let us choose $\xi$ so that $\xi \neq 1$ and that $A_K(\xi) \neq 0$. By Theorem 3.6 we have the following.

Corollary 3.5.

$$R_{X_K}(0, \rho) = \left| \frac{A_K(\xi)}{1 - \xi} \right|^2.$$

4 A geometric meaning of coefficients

4.1 The K-group and regulators

Fact 4.1. ([6]) Let

$$K_{2n+1}(\mathbb{C}) \overset{r_{n+1}}{\to} \mathbb{R}$$

be the Borel regulator map. Then there is a natural element $\gamma_X$ in $K_{2n+1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}$ such that

$$\text{vol}(X) = r_{n+1}(\gamma_X).$$

Thus Theorem 2.2 shows if $n$ is odd the ratio of the first and the second coefficient of the Taylor expansion at the origin is interpreted as an evaluation of the Borel regulator against a certain element of the algebraic K-group whereas if $n$ is even a correction from cusps is necessary. It seems natural to expect the first coefficient also has such an interpretation. In fact it is true at least for a hyperbolic threefold, which will be explained below. Following [4] we will also explain how to construct $\gamma_X$ for a closed hyperbolic threefold.
Let $M$ be a smooth manifold and $P \to M$ a principal $\mathbb{C}^\times$-bundle with a flat connection. By Chern-Weil theory the image of the first Chern class $c_1(P) \in H^2(M, \mathbb{Z})$ in $H^2(M, \mathbb{C})$ vanishes. Thus the exact sequence

$$H^1(M, \mathbb{C}/\mathbb{Z}) \xrightarrow{\beta} H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{C})$$

shows there is an element $\hat{c}_1(P) \in H^1(M, \mathbb{C}/\mathbb{Z})$ which maps to $c_1(P)$ by $\beta$. Let $GL_1(\mathbb{C})$ be the multiplicative group $\mathbb{C}^\times$ with the discrete topology and $BGL_1(\mathbb{C})$ the classifying space. If we apply the previous construction to the universal flat $\mathbb{C}^\times$-bundle, we obtain an element $\hat{c}_1$ of $H^1(BGL_1(\mathbb{C}), \mathbb{C}/\mathbb{Z})$. Since

$$H^1(BGL_1(\mathbb{C}), \mathbb{C}/\mathbb{Z}) \simeq H^1(GL_1(\mathbb{C}), \mathbb{C}/\mathbb{Z}) \simeq \text{Hom}(\mathbb{C}^\times, \mathbb{C}/\mathbb{Z}),$$

$\hat{c}_1$ may be regarded as a homomorphism from $K_1(\mathbb{C}) \simeq \mathbb{C}^\times$ to $\mathbb{C}/\mathbb{Z}$. By definition $r_1$ is $\text{Im}\hat{c}_1$ and it is known

$$\hat{c}_1(g) = \frac{i}{2\pi} \log |g|, \quad g \in \mathbb{C}^\times,$$

and thus $r_1(g) = \log |g|/2\pi$. Let us choose a unitary basis $e = \{e_1, \ldots, e_r\}$ of $\rho$. Suppose that $H^1(X, \rho)$ vanishes. Then as we have seen at the beginning of §3.4, $C'(X, \rho)$ is acyclic. Following [12] a certain $\tau(X, \rho, e) \in K_1(\mathbb{C})$ is defined which will be referred as the Milnor element. The Franz-Reidemeister torsion is nothing but its image in $K_1(\mathbb{C})/U(1) \simeq \mathbb{R}$, i.e. its modulus. Thus

$$\log R_X(0, \rho) = 2 \log \tau(X, \rho)$$

$$= 2 \log |\tau(X, \rho, e)|$$

$$= 4\pi r_1(\tau(X, \rho, e)).$$

and we have found that $\log R_X(0, \rho)$ is interpreted as a period of the Milnor element by the first Borel regulator.

Taking account of an exceptional isomorphism

$$\text{Spin}(3, 1) \simeq SL_2(\mathbb{C}),$$

let us apply the previous construction to the universal flat bundle $SL_2(\mathbb{C})$-bundle on $BSL_2(\mathbb{C})^\delta$. Then we will obtain the Chern-Simon class

$$\hat{c}_2 \in H^3(BSL_2(\mathbb{C})^\delta, \mathbb{C}/\mathbb{Z}) \simeq H^3(SL_2(\mathbb{C}), \mathbb{C}/\mathbb{Z})$$

$$\simeq \text{Hom}(H_3(SL_2(\mathbb{C}), \mathbb{Z}), \mathbb{C}/\mathbb{Z})$$

which is a lift of the second Chern class $c_2 \in H^4(BSL_2(\mathbb{C})^\delta, \mathbb{Z})$. In particular its imaginary part $\text{Im}\hat{c}_2$ yields a $\mathbb{Q}$-linear map:

$$H_3(SL_2(\mathbb{C}), \mathbb{Q}) \xrightarrow{\text{Im}\hat{c}_2} \mathbb{R},$$

and it is known ([4])

$$\text{Im}\hat{c}_2(g_1|g_2|g_3) = \frac{1}{4\pi^2} \text{vol}(T(\infty, g_1\infty, g_1g_2\infty, g_1g_2g_3\infty)), \quad g_i \in SL_2(\mathbb{C}).$$
Here $T(z_1, z_2, z_3, z_4)$ is a tetrahedron in $\mathbb{H}^3$ whose vertices are $\{z_1, z_2, z_3, z_4\}$ and edges are geodesics. Remember that the volume of an ideal tetrahedron is computed by the Bloch-Wigner function:

$$D(z) = \text{arg}(1 - z)\text{arg}(z) + \text{ImLi}_2(z),$$

where $\text{Li}_2(z)$ is the dilogarithm:

$$\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

More precisely it is known (3)

$$\text{vol}(T(\infty, g_1\infty, g_1g_2\infty, g_1g_2g_3\infty)) = \text{vol}(T(\infty, 0, 1, z(g_1, g_2, g_3))) = D(z(g_1, g_2, g_3)),$$

where $z(g_1, g_2, g_3)$ is the cross ratio of $\{\infty, g_1\infty, g_1g_2\infty, g_1g_2g_3\infty\}$. Suppose that $X$ is closed. Then since $H_3(B\text{PSL}_2(\mathbb{C})^\delta, \mathbb{Q})$ is isomorphic to $H_3(B\text{SL}_2(\mathbb{C})^\delta, \mathbb{Q})$ the natural inclusion $\Gamma \hookrightarrow \text{SO}(3, 1) \simeq \text{PSL}_2(\mathbb{C})$ induces

$$H_3(X, \mathbb{Q}) \simeq H_3(B\Gamma, \mathbb{Q}) \to H_3(B\text{SL}_2(\mathbb{C})^\delta, \mathbb{Q}) \simeq H_3(\text{SL}_2(\mathbb{C}), \mathbb{Q}).$$

Here recall that $H_3(\text{SL}_2(\mathbb{C}), \mathbb{Z})$ is a direct summand of the Quillen’s K-group $K_3(\mathbb{C})$. In fact (3, (9.6)),

$$K_3(\mathbb{C}) \simeq H_3(\text{SL}_2(\mathbb{C}), \mathbb{Z}) \oplus K_3^{\text{Mil}}(\mathbb{C}),$$

where $K_3^{\text{Mil}}(\mathbb{C})$ is the Milnor K-group. Thus the fundamental class $[X]$ of $X$ defines an element $\gamma_X$ in $K_3(\mathbb{C}) \otimes \mathbb{Q}$. According to Weil’s rigidity $\Gamma$ is conjugate a subgroup of $\text{PSL}_2(\mathbb{Q})$ and $\gamma_X$ is contained in $K_3^{\text{Mil}}(\mathbb{Q}) \otimes \mathbb{Q}$. We define the second Borel regulator

$$K_3(\mathbb{C}) \otimes \mathbb{Q} \xrightarrow{\tau_2} \mathbb{R}$$

to be a composition of $4\pi^2\text{Im} \hat{c}_2$ and the natural projection:

$$K_3(\mathbb{C}) \otimes \mathbb{Q} \to H_3(\text{SL}_2(\mathbb{C}), \mathbb{Q}).$$

Then Corollary 2.3 yields

$$\frac{d}{dz} \log R_X(z, \rho)|_{z=0} = -\frac{3r}{\pi} \cdot r_2(\gamma_X).$$

If $X$ is not closed, $H_3(X)$ vanishes. So we have to use the relative homology group to define $\gamma_X$. (See 3 for details.) Thus we have found that the leading and the second coefficient of Taylor expansion of $R_X(z, \rho)$ at the origin are expressed by the logarithm and the dilogarithm, respectively.

44
4.2 The $L^2$-torsion

The constant term of the logarithmic derivative of Ruelle L-function at the origin is also related to $L^2$-analytic torsion ([1][9]). Following [1], we remember the von Neumann trace. Let $\omega_p$ the action of $\Gamma$ on $L^2(\mathbb{H}^d, \Omega^p)$. Then $L^2(\mathbb{H}^d, \Omega^p \otimes \mathbb{C}^r)$ is a $\Gamma$-module by $\omega_p \otimes \rho$ and there is an isomorphism of $\Gamma$-modules:

$$L^2(\mathbb{H}^d, \Omega^p \otimes \mathbb{C}^r) \simeq L^2(\Gamma) \otimes L^2(\mathbb{H}^d, \Omega^p). \quad (33)$$

Here $\Gamma$ acts on $L^2(\Gamma)$ by the left regular representation and we regard $L^2(\mathbb{H}^d, \Omega^p \otimes \mathbb{C}^r)$ is the trivial module. Since Hodge Laplacian $\Delta^p$ on $L^2(\mathbb{H}^d, \Omega^p \otimes \mathbb{C}^r)$ commutes with $\Gamma$ so does $e^{-t\Delta^p}$. Let $U$ be the fundamental domain of $\Gamma$ and $\psi$ its characteristic function. Using (14) von Neumann trace of $e^{-t\Delta^p}$ is given by ([1])

$$\tau(e^{-t\Delta^p}) = \text{Tr}(\psi \cdot e^{-t\Delta^p} \cdot \psi),$$

which is equal to $I_p(t)$. Let us put

$$\zeta_2(s) = \sum_p (-1)^p \zeta_2^{(p)}(s), \quad \zeta_2^{(p)}(s) = \frac{1}{\Gamma(s)} M(\tau(e^{-t\Delta^p}))(s).$$

Then the analytic $L^2$-torsion of $(X, \rho)$ is defined to be

$$\tau_{an}^{(2)}(X, \rho) = \exp(-\frac{1}{2}\zeta_2'(0)).$$

[9] Lemma 6.4 (see also Appendix, Lemma 5.1 below) and the computation in §2.2 show that $M(I_p)(0)$ is a rational multiple of $\text{vol}(X)/\pi$ and that

$$\frac{d}{ds}\zeta_2^{(p)}(s)|_{s=0} = M(I_p)(0).$$

Now Theorem 2.2 implies

**Theorem 4.1.** Let $h$ be the order of $R_X(\rho, z)$ at the origin. Then there is a rational number $\alpha$ such that

$$\lim_{z \to 0} \left\{ \frac{d}{dz} \log R_X(\rho, z) - \frac{h}{z} \right\} - 2 \sum_{j=0}^{n} (-1)^j \delta(X, \rho) = \alpha \cdot \log \tau_{an}^{(2)}(X, \rho).$$

For example suppose $d = 3$. Since ([9] Corollary 6.7)

$$\log \tau_{an}^{(2)}(X, \rho) = M(I_0)(0) = \frac{r}{6\pi} \text{vol}(X),$$

we obtain

$$\lim_{z \to 0} \left\{ \frac{d}{dz} \log R_X(\rho, z) - \frac{2h^1(X, \rho)}{z} \right\} = -18 \cdot \log \tau_{an}^{(2)}(X, \rho).$$
5 Appendix

We will show Fact 3.1 under the assumption that \( \rho \) is cuspidal and \( d = 3 \). In the following the Mellin transform of a function \( f \) will be denoted by \( M(f) \):

\[
M(f)(s) = \int_0^\infty f(t)t^{s-1}dt.
\]

**Lemma 5.1.** Let \( t \) be a positive number.

1. \[
\int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda = \sqrt{\pi} t^{-\frac{1}{2}}.
\]

2. For a positive integer \( k \),

\[
\int_{-\infty}^{\infty} e^{-t\lambda^2} \lambda^{2k} d\lambda = \frac{\sqrt{\pi}(2k-1)!!}{2^k} t^{-\frac{k}{2}}.
\]

3. Let \( c \) be a positive number and \( P \) an even polynomial. Then the Mellin transform of \( \int_{-\infty}^{\infty} e^{-t(\lambda^2+c^2)} P(\lambda)d\lambda \) is meromorphically continued to \( \mathbb{C} \). It is regular at \( s = 0 \) and

\[
M(\int_{-\infty}^{\infty} e^{-t(\lambda^2+c^2)} P(\lambda)d\lambda)(0) = -2\pi \int_0^c P(iy)dy.
\]

**Proof.** See [[5]|Lemma 3].

We put

\[
\theta_p(t) = \text{Tr}[e^{-\Delta_p^X}] - h_p(X, \rho).
\]

Since \( H^0(X, \rho) \) vanishes Selberg trace formula shows

\[
\theta_0(t) = \delta_0(t) = h_0(t) + e_0(t), \tag{34}
\]

where

\[
e_0(t) = i_0(t) + u_0(t) = \frac{r \cdot \text{vol}(X)}{4\pi^2} \int_{-\infty}^{\infty} e^{-t(\lambda^2+1)} X^2 d\lambda + \frac{\delta(X, \rho)}{2\pi} \int_{-\infty}^{\infty} e^{-t(\lambda^2+1)} d\lambda
\]

\[
= \frac{r \cdot \text{vol}(X)}{8\pi \sqrt{\pi}} e^{-t} t^{-\frac{1}{2}} + \frac{\delta(X, \rho)}{2\sqrt{\pi}} e^{-t} t^{-\frac{1}{2}}.
\]

By Lemma 5.1, \( M(e_0)(s) \) is regular at the origin and

\[
M(e_0)(0) = \frac{r}{6\pi} \text{vol}(X) - \delta(X, \rho).
\]
If \( t > 0 \) is sufficiently small,
\[
h_0(t) \sim \frac{a}{\sqrt{4\pi t}} e^{-\frac{c_X^2}{4t}},
\]
where \( c_X \) is the length of minimal closed geodesic. Since \( \theta_0(t) \) exponentially decays as \( a \to \infty \) so does \( h_0(t) \). Therefore \( M(h_0)(s) \) is an entire function and \( M(\theta_0)(s) \) is a meromorphic function on the whole plane regular at the origin. Notice that \( \Gamma(s) \) has simple pole with residue 1 at \( s = 0 \).

**Proposition 5.1.** \( \zeta_X^{(0)}(s, \rho) = M(\theta_0)(s)/\Gamma(s) \) satisfies the following properties.

1. It is a meromorphic function on \( \mathbb{C} \) and vanishes at the origin.
2. \[ \frac{d}{ds} \zeta_X^{(0)}(s, \rho)|_{s=0} = M(\theta_0)(0) - \frac{r}{6\pi} \text{vol}(X) - \delta(X, \rho). \]

By definition the functional determinant is
\[ \det \Delta_X^p = \exp\left(-\frac{d}{ds} \zeta_X^{(0)}(s, \rho)|_{s=0}\right). \]

Hence
\[ -\log \det \Delta_X^0 = M(\theta_0)(0) = M(h_0)(0) + \frac{r}{6\pi} \text{vol}(X) - \delta(X, \rho). \]  
(35)

We put
\[ \eta_1(t) = h_1(t) + e_1(t) - h^1(X, \rho), \]
where
\[
e_1(t) = i_1(t) + u_1(t) = \frac{r \cdot \text{vol}(X)}{2\pi \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t\lambda^2} (\lambda^2 + 1) d\lambda + \frac{\delta(X, \rho)}{\pi} \int_{-\infty}^{\infty} e^{-t\lambda^2} d\lambda = \frac{r \cdot \text{vol}(X)}{2\pi \sqrt{\pi}} (2t^{-\frac{1}{2}} + t^{-\frac{3}{2}}) + \frac{\delta(X, \rho)}{\sqrt{\pi}} t^{-\frac{1}{2}}.
\]

Then
\[ \theta_1(t) = \eta_1(t) + \theta_0(t). \]

In order to investigate the Mellin transform of \( \eta_1 \) we consider
\[
\mu_1(t) = \eta_1(t) + (h^1(X, \rho) - e_1(t)) \cdot \chi_{[0,1)} \]
(36)
\[ = h_1(t) - (h^1(X, \rho) - e_1(t)) \cdot \chi_{(1, \infty)}, \]  
(37)
where \( \chi \) is a characteristic function. For a sufficiently small positive number \( t \), (37) shows
\[
\mu_1(t) \sim h_1(t) \sim \frac{a}{\sqrt{4\pi t}} e^{-\frac{c_X^2}{4t}}. 
\]
and (36) implies for sufficiently large $t$
\[ \mu_1(t) \sim \eta_1(t) \sim e^{-\gamma t}, \quad \gamma > 0. \]
Thus $M(\mu_1)(s)$ is an entire function. If $\text{Res} > \frac{3}{2}$,
\[
M(\eta_1)(s) = M(\mu_1)(s) + \int_0^1 (e_1(t) - h^1(X, \rho)) t^{s-1} dt
\]
\[
= M(\mu_1)(s) - \frac{h^1(X, \rho)}{s} + \left( \frac{r \cdot \text{vol}(X)}{\pi \sqrt{\pi}} + \frac{\delta(X, \rho)}{\sqrt{\pi}} \right) \frac{1}{s - 1/2} + \frac{r \cdot \text{vol}(X)}{2\pi \sqrt{\pi}} \frac{1}{s - 3/2}.
\]
and if $\text{Res} < \frac{1}{2}$,
\[
M(h_1)(s) = M(\mu_1)(s) - \int_1^\infty (e_1(t) - h^1(X, \rho)) t^{s-1} dt
\]
\[
= M(\mu_1)(s) - \frac{h^1(X, \rho)}{s} + \left( \frac{r \cdot \text{vol}(X)}{\pi \sqrt{\pi}} + \frac{\delta(X, \rho)}{\sqrt{\pi}} \right) \frac{1}{s - 1/2} + \frac{r \cdot \text{vol}(X)}{2\pi \sqrt{\pi}} \frac{1}{s - 3/2}.
\]
we see that both $M(\eta_1)(s)$ and $M(h_1)(s)$ are meromorphically continued to the whole plane as the same function. Moreover we find that $M(\eta_1)(s) + h^1(X, \rho)/s$ is regular at the origin. Together with Proposition 5.1 this shows

**Proposition 5.2.** $\zeta_X^{(1)}(s, \rho) = M(\theta_1)(s)/\Gamma(s)$ satisfies the following properties.

1. It is a meromorphic function on $\mathbb{C}$ and is regular at the origin. Moreover
\[
\zeta_X^{(1)}(0, \rho) = -h^1(X, \rho).
\]

2. \[
\frac{d}{ds} \zeta_X^{(1)}(s, \rho)|_{s=0} = M(\theta_0)(0) + \lim_{s \to 0} \{ \Gamma(s) h^1(X, \rho) + M(\eta_1)(s) \}.
\]

By Proposition 2.1, Corollary 2.1, Proposition 2.4 and (34) we obtain
\[
\begin{align*}
\quad s_0(z + 1) &= L(e^t h_0)(z) \\
&= L(e^t \delta_0)(z) - L(e^t i_0)(z) - L(e^t u_0)(z) \\
&= L(e^t \delta_0)(z) + \frac{r \cdot \text{vol}(X)}{2\pi} z^2 - \delta(X, \rho).
\end{align*}
\]
By Lemma 2.6 $L(e^t \delta_0)(z)$ is an odd function and thus
\[
\begin{align*}
\quad s_0(1 - z) + s_0(1 + z) &= \frac{r}{\pi} \text{vol}(X) z^2 - 2\delta(X, \rho).
\end{align*}
\]
Since $s_j(z)$ is the logarithmic derivative of $S_j(z)$, (35) yields
\[
\log S_0(2) - \log S_0(0) = \int_0^1 (s_0(1 + z) + s_0(1 - z)) dz
\]
\[
= \frac{r \cdot \text{vol}(X)}{3\pi} - 2\delta(X, \rho)
\]
\[
= -2 \log \det \Delta_X^0 - 2M(h_0)(0).
\]
The equation ([5], pp535 (13)):
\[ M(h_0)(0) = -\log S_0(2). \]

and (35) implies

**Proposition 5.3.**

\[ \log(S_0(0)S_0(2)) = 2\log \det \Delta_X^0 = -2M(\theta_0)(0). \]

Now we will compute the Ray-Singer torsion. By **Proposition 5.1**, **Proposition 5.2** and **Proposition 5.3**

\[ \zeta_X'(0, \rho) = \frac{d}{ds} \zeta_X^{(0)}(s, \rho)|_{s=0} - 3\frac{d}{ds} \zeta_X^{(0)}(s, \rho)|_{s=0} \]
\[ = \lim_{s \to 0} \{ \Gamma(s)h_1(X, \rho) + M(\eta_1)(s) \} + \log(S_0(0)S_0(2)). \]

The arguments of pp.536 of [5](especially (25)) shows that the leading term of the Taylor expansion of \( S_1(z + 1) \) at the origin is \( \delta z^{2h_1(X, \rho)} \). Here \( \delta \) is given by

\[ -\log \delta = \lim_{s \to 0} \{ \Gamma(s)h_1(X, \rho) + M(\eta_1)(s) \}. \]

Thus

\[ \lim_{s \to 0} \{ \Gamma(s)h_1(X, \rho) + M(\eta_1)(s) \} = -\log(\lim_{z \to 0} z^{2h_1(X, \rho)}S_1(z + 1)), \]

and **Fact 2.1** shows

**Theorem 5.1.**

\[ \lim_{z \to 0} z^{-2h_1(X, \rho)}R_X(z, \rho) = \exp(-\zeta_X'(0, \rho)). \]

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