The spooky paradox of Einstein, Podolsky and Rosen, challenging the completeness of the mathematical framework of quantum theory, represents one of the deepest foundational puzzle of quantum theory. Quantum spookiness was first time discovered [1] in 1935 and it took thirty years until this paradox has been resolved by John Bell in Ref. [2]. He has realized that theories compatible with the concept of local realism (used to create the paradox) necessarily satisfy certain inequalities, nowadays known as Bell inequalities. He has shown that quantum theory violates them and the phenomenon of quantum (Bell) non-locality [3] has been discovered. Since then the mathematical origin of the EPR paradox was clarified and non-locality-enabled quantum technologies have been developed, however, its intuitive spookiness remains (see for instance Refs. [4, 5]).

In general, the non-locality requires the existence of spatially separated independent experimental facilities. It is common to name these local experimental stations Alice and Bob. It is assumed that both Alice and Bob can choose independently between particular (local) experimental setups. In the simplest case [6] each of them is free to switch between a pair of experimental setups \((A, A')\) for Alice and \(B, B'\) for Bob) leading to binary outcomes \(a, a', b, b' = \pm 1\), respectively. The whole setting is described by conditional probabilities \(P(x, y|X, Y)\), i.e. by 16 numbers satisfying the elementary probability constraints \(0 \leq P(x, y|X, Y) \leq 1\) and \(\sum_{x,y} P(x, y|X, Y) = 1\) for all \(x, y = \pm 1\), \(X \in \{A, A'\}\) and \(Y \in \{B, B'\}\). In summary, the whole experiment is schematically illustrated as a non-local box (see Fig. 1) understood as a device with (binary) inputs \(X, Y\) and (binary) outputs \(x, y\).

In a special case, when \(P(x, y|X, Y)\) is compatible with local hidden variable model, i.e. \(P(x, y|X, Y) = \sum_{\lambda} \pi(\lambda) P(x|X, \lambda) P(y|Y, \lambda)\) for some probability distribution \(\pi(\lambda)\) over the (local hidden) parameter space \(\lambda\), we say the box is local. By definition, the local boxes do not exhibit any non-locality and satisfy all Bell inequalities [7–9]. In particular, let us denote by \(\langle X \otimes Y \rangle = \sum_{x,y = \pm 1} xy P(x, y|X, Y)\) the expectation value for joint measurement of \(X\) and \(Y\) performed by Alice and Bob, respectively. Since for all choices of \(a, a', b, b' = \pm 1\) the identity \(a(b+b') + a'(b - b') = \pm 2\) holds, it follows that for any local box \(|\langle A \otimes B \rangle + \langle A' \otimes B' \rangle + \langle A' \otimes B' \rangle - \langle A \otimes B' \rangle| \leq 2\). This inequality is known as CHSH Bell inequality [6].

For quantum (state-based) non-local boxes the conditional probabilities are given by Born formula \(P(x, y|X, Y) = \text{tr} [\rho (A_x \otimes B_y)]\), where \(\rho\) is some density operator (representing a joint state of a pair of quantum systems) and \(A_x, B_y\) are effects (positive operators smaller than identity operator) forming the used measurements (describing the outcomes observed by Alice and Bob, respectively). It is well known that such quantum non-local boxes are violating CHSH inequality and that for any choice of the state and the measurements \(|\langle A \otimes B \rangle + \langle A' \otimes B' \rangle + \langle A' \otimes B' \rangle - \langle A \otimes B' \rangle| \leq 2\sqrt{2}\). This quantum limitation is known as Tsirelson bound [10].

**Popescu-Rohrlich box.** In their seminal work [11, 12] Popescu and Rohrlch questioned the conceptual and operational origin of quantum correlations and, especially, of the existence of Tsirelson bound. It is straightforward to verify that the algebraic maximum of CHSH formula (being a sum of four terms bounded by 1) is four. Popescu and Rohrlich were wondering why this value of non-locality is not achieved by quantum states. Extending CHSH framework they have identified the concept of non-local boxes and motivated the study of so-called general probabilistic theories (GPT) [13, 14]. In particular, they have introduced a family of no-signaling (non-local) boxes characterized by the conditions \(P(x|X) = \sum_{x'} P(x, y|X, Y) = \sum_{y'} P(x, y'|X, Y')\) for any \(X, Y, Y'\) and \(P(y|Y) = \sum_{x} P(x, y|X, Y) = \sum_{x'} P(x', y|X, Y)\) for any \(X, X', Y\). In other words, the marginal distributions

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$P(x|X), P(y|Y)$ are independent of the choices of $Y$ and $X$, respectively. Such no-signaling restriction ensures that measurements performed in the local laboratories are not influencing each other.

Popescu and Rohrlich gave an example of no-signaling non-local box (coined as PR box) violating Tsirelson bound for CHSH inequality and achieving the algebraic maximum. Relabeling the outcome values to $\{0, 1\}$ and denoting non-dashed and dashed observables as choices 0 and 1, respectively, we set for all variables the same value $x, y, X, Y \in \{0, 1\}$. Using such notation the conditional probabilities for Popescu-Rohrlich box (PR box) can be compactly expressed as $P_{PR}(x, y|X, Y) = 1/2$ if $x \oplus y = XY$ and vanishes otherwise.

Recently, it was realized that PR box would represent a resource both qualitatively and quantitatively stronger than any quantum non-local box [3, 15–17]. For example [18–20], PR box would make any communication trivial (from the complexity point of view). However, such device is purely hypothetical and currently there are no known (in practise) existing realizations. Several proposals and attempts have been made to realize PR box in laboratories [21–23]. All these constructions are employing post-processing techniques to achieve the desired goal. In this work we will present qualitatively different PR box implementations. In particular, we will show that a general probabilistic theory in which channels are playing the role of states allows for maximal violation of CHSH inequality.

**General probabilistic theory of quantum processes.** When designing a general probabilistic theory we can follow the following algorithm. First, we specify a convex set playing the role of states (density operators in quantum theory), i.e. mathematical representation identified with the set of preparations of the object of experiments. Outcomes of measurements are then naturally identified with affine (thus convex structure preserving) functionals, i.e. linear maps from the convex sets into an interval $[0, 1]$. These numbers are interpreted as probabilities of observing the outcome (effects in quantum theory) given the measured object is in some given state (element of the convex set).

Further, we need to give some mathematical meaning to experiments composed of more than one object (being the tensor product for quantum theory). There is a freedom in the definition of a suitable tensor product and particular choice determines the features of the theory. However, the discussion of these consequences is not relevant for the purposes of this work.

The transformations of the object are identified with affine maps transforming the state space into itself including all the object’s extensions (channels in quantum theory). In this work the concept of dynamics does not play any role, thus, we will skip the discussion of dynamical features of general probabilistic theories.

Our object of interest is the set of quantum processes (quantum channels), i.e. the set of completely positive tracepreserving linear maps defined on the set of density operators (embedded in the positive cone of trace-preserving linear maps defined on the set of density operators (quantum channels), i.e. the set of completely positive trace-preserving linear maps defined on the set of density operators (embedded in the positive cone of trace-preserving linear maps defined on the set of density operators (quantum channels), i.e. the set of completely positive trace-preserving linear maps defined on the set of density operators (positive operators on $H$). The role of process effects is then represented by functionals $f(\rho) = \text{tr}[F\rho]$, where $F$ is a positive operator describing the outcome of an experiment addressing the properties of quantum channels. In general, such experiment consists of the preparation of the input state $\rho$, application of the channel $(\mathcal{E} \otimes \mathcal{I})[\rho]$ and the measurement of the (transformed) output state recording as the outcome some effect $O \leq E \leq I$. The process effect $E$ captures the relevant characteristics of $\rho$ and $E$. In particular, $E = (\mathcal{I} \otimes \mathcal{R}_\rho)[E]$, where $\mathcal{R}_\rho$ is defined via the identity $(\mathcal{I} \otimes \mathcal{R}_\rho)[\rho] = \rho$ and $\mathcal{R}_\rho$ denotes the associated dual map [25, 26].

In summary, the general probabilistic theory of quantum processes (GPTQP) identifies the state space with the set of all Choi-Jamiolkowski states (being a subset of state space $\mathcal{S}(H \otimes H)$). The role of process effects is played by positive operators $O \leq F \leq I \otimes \rho$ (for some density operator $\rho$) containing (up to a constant factor $d = \dim H$) the set of effects $\mathcal{E}(H \otimes H)$. Fortunately, for the statement and the derivation of the result itself we do no need to employ (explicitly) the introduced mathematical formalism of GPTQP. Therefore, we omit any further details of GPTQP.

It is known that incompatibility is intimately related with the phenomenon of non-locality [27–32]. Indeed, both the entangled states and the incompatible measurements are necessary for the demonstration of quantum non-locality (violation of Bell inequality). The question of incompatibility of process observables was addressed recently in Ref. [33], where it was show that incompatibility of process observables is both qualitatively and quantitatively different from incompatibility of observables. In particular, unlike for quantum (state) observables for GPTQP the maximum incompatibility degree is attained already for smallest GPTQP systems (being the qubit processes). It was exactly this observation that has driven our curiosity to investigate the CHSH settings within the GPTQP framework.

**Results.** Denote by $|0\rangle, |1\rangle$ the vectors forming the computational basis of qubit Hilbert space $\mathcal{H}_2$. Consider a two-qubit measure-and-prepare channel in which, each of the qubits is (independently) measured in the computational basis and subsequently (based on the observed outcomes $j, k \in \{0, 1\}$) the two-qubits are prepared in a fixed two-qubit state. In particular, $\xi_{\text{cos}} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$ if $jk = 0$ and $\xi_{\text{acor}} = \frac{1}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|)$ if $jk = 1$. These output states describe the (classically) correlated and anticoerlated state of two qubits, respect-
tively. In summary, the total action of such channel leads to the following state transformation

$$\Phi[\rho] = (1 - \kappa)\xi_{\text{cor}} + \kappa\xi_{\text{acor}}$$

where $\kappa = \theta_{11,11} = (11|\theta_{11})$.

In order to perform Bell like experiment we need to specify local measurements of such non-local process. Let us denote by $Z_0$ an experiment, in which the quantum process acts on the (single qubit) test state $|\psi\rangle$ and the output system is measured in $\sigma_z$ basis. Suppose on both sides one may perform either $Z_0$, or $Z_1$ measurement, thus, the test states are $|0\rangle$, or $|1\rangle$, respectively. In the settings of Bell inequalities $X,Y \in \{Z_0, Z_1\}$ and $x,y = \{\pm 1\}$. We have everything we need to evaluate the conditional probabilities $P(x,y|X,Y)$ and, subsequently, evaluate the CHSH expression.

Let us denote by $\Phi(jk) = \Phi(|j\rangle\langle j| \otimes |k\rangle\langle k|)$. For measurement choices $Z_j, Z_k$ with $j = 0$ we obtain

$$P(+, +|Z_j, Z_k) = \langle 00|\Phi(jk)|00\rangle = \langle 00|\xi_{\text{cor}}|00\rangle = 1/2,$$
$$P(+, -|Z_j, Z_k) = \langle 01|\Phi(jk)|01\rangle = \langle 01|\xi_{\text{cor}}|01\rangle = 0,$$
$$P(-, +|Z_j, Z_k) = \langle 10|\Phi(jk)|10\rangle = \langle 10|\xi_{\text{cor}}|10\rangle = 0,$$
$$P(-, -|Z_j, Z_k) = \langle 11|\Phi(jk)|11\rangle = \langle 11|\xi_{\text{cor}}|11\rangle = 1/2.$$ 

Consequently,

$$\langle Z_0 \otimes Z_0\rangle = \langle Z_0 \otimes Z_1\rangle = \langle Z_1 \otimes Z_0\rangle = \langle Z_1 \otimes Z_1\rangle = 1.$$ 

And for the remaining choice $Z_1, Z_1$ one finds

$$P(+, +|Z_1, Z_1) = \langle 00|\Phi(11)|00\rangle = \langle 00|\xi_{\text{acor}}|00\rangle = 0,$$
$$P(+, -|Z_1, Z_1) = \langle 01|\Phi(11)|01\rangle = \langle 01|\xi_{\text{acor}}|01\rangle = 1/2,$$
$$P(-, +|Z_1, Z_1) = \langle 10|\Phi(11)|10\rangle = \langle 10|\xi_{\text{acor}}|10\rangle = 1/2,$$
$$P(-, -|Z_1, Z_1) = \langle 11|\Phi(11)|11\rangle = \langle 11|\xi_{\text{acor}}|11\rangle = 0,$$

and $\langle Z_1 \otimes Z_1\rangle = -1$. Therefore,

$$\langle (Z_0 + Z_1) \otimes Z_0 + (Z_0 - Z_1) \otimes Z_1\rangle = 4.$$ 

In conclusion, this proves that the designed experiment achieves (algebraic) maximum of CHSH quantity, thus, overcoming the Tsirelson bound valid for quantum states and demonstrating the existence of Popescu-Rohrlich non-local box. Let us stress that the induced probabilities $P(x,y|X,Y)$ are indeed no-signaling.

Discussion. It is well known that Popescu-Rohrlich non-local boxes do exist as mathematical objects in (artificial) general probabilistic theories. In this work we have shown that PR boxes are accommodated within an existing formalism of quantum theory - the general probabilistic theory of quantum processes, thus, PR boxes are not anymore purely conceptual objects appearing in thought experiments and applications. Indeed, the above equations for conditional probabilities show explicitly that the introduced channel $\Phi$ is a mathematically faithful realization of Popescu-Rohrlich non-local box.

A question that emerges is related to the interpretation of non-locality. In particular, unlike for the case of states, it is not possible to separate parts of the process and distribute them to distant places (in space). In current physics, we are lacking long-distance interactions and therefore it is essentially forbidden to implement the PR box channel $\Phi$ in a spatially separated manner. However, the mathematical formalism of non-local boxes does not really rely on the concept of space. The concept of locality can be understood purely in terms of subsystems, i.e. in our ability to address subsystems individually. Adopting such perspective we do not affect any mathematical statement regarding the non-locality, thus, the existence of such non-locality with respect to subsystems is of foundational relevance and interest. On the other side in such case the interpretation of non-local boxes as non-local objects in space is not necessarily applicable and this fact have potential consequences for communication-based applications, where the spatial separation between the participants matters.

Consequently, for sake of clarification it turns out be important to distinguish conceptually between system non-locality and spatial non-locality. Having such clarification in mind we may say that the introduced channel $\Phi$ is a faithful implementation of Popescu-Rohrlich non-local box not reflecting the spatial non-locality. Following this interpretation, one may wonder to which extent the spatial non-locality of $\Phi$ can be simulated. Similar question we are facing in entanglement swapping protocols [34, 35], where we are aiming to implement the swap gate between spatially separated systems. Again, such transformation is forbidden in universes without non-local interactions. However, having access to entanglement and communication such gate can be simulated (essentially by implementing the quantum teleportation protocol [36]). The drawback is that the gate implementation is not instantaneous. In fact, its realization is limited by the speed of communication, thus, by the mutual space distance between the systems. Can we do something similar for the proposed channel-based realization of PR box?

Further, we will propose simulation of spatially separated implementation of $\Phi$, however, it is important to stress that this simulation does not really simulate the spatial non-locality phenomenon, because the implementation of $\Phi$ is not instantaneous. In fact, it will be based on exchange of classical information that is known to be sufficient to simulate the action of PR box. By definition the PR box channel $\Phi$ can be simulated by a pair of spatially separated observers (Alice and Bob) in the following way. Suppose Alice and Bob share a classical key $k$ representing the shared (uniform) randomness and they want to act by $\Phi$ on the input two-qubit state $\rho$. Next both of them perform $\sigma_z$ measurement and record outcomes $a$ and $b$, respectively. Alice sends her outcome $a$ to Bob and outputs qubit in the state $|k\rangle$. Bob compares $a$ and $b$ and outputs qubit in the state $|k \oplus (ab)\rangle$ (alternatively, Bob applies NOT gate on the qubit $|k\rangle$ if $ab = 1$). To summarize, the simulation of $\Phi$ assumes the existence of the pre-shared classical randomness and its performance is limited by the communication cost.
Testing CHSH expression itself requires very specific factorized input states $\rho$ depending on the choice of the measurement setups.

In practice, the proposed CHSH experiment with PR box can be seen as follows. Both Alice and Bob choose one random bit $j$ and $k$ (determining the input test state). If $jk = 0$, then (after the action of the channel $\Phi$) they observe perfectly correlated outcomes and if $jk = 1$ they found their results perfectly anticorrelated. Seeing the whole experiment from this perspective it is natural to ask what is quantum in this experiment? The answer may sound rather surprising. The channel $\Phi$ can be seen as purely classical channel acting on a pair of classical bits and all the results derived for $\Phi$ remain valid, including the lack of spatial non-locality feature. This is clearly very unexpected result that PR-type non-locality emerges already in the mathematical formalism of classical structures, namely for general probabilistic theory of classical processes.

In conclusion, we found that Popescu-Rohrlich non-local box can be faithfully implemented within general probabilistic theory of quantum and classical processes. This implies that superquantum (non-signalling) correlations are not forbidden in our experiments, however, also that the spatial non-locality (being the true phenomenon) should be distinguished from the system non-locality (being the presented phenomenon in general probabilistic theories for processes). Our observation induces a plethora of open questions and exciting foundational research directions. For example, the presented construction is based on measure-and-prepare channel that is in a sense classical. One may wonder whether there are also some intrinsically “non-classical” channels maximizing the CHSH expression, probably exhibiting even stronger correlations for different types of Bell inequalities. More general characterization of superquantum non-local boxes will be addressed in [37]. The information-theoretic interpretation and applications of our findings represent another interesting research directions with a strong foundational potential. In particular, the presented PR box can be understood as a logical AND gate. It would be interesting to understand whether such non-local gate is in some sense universal for communication, or whether there is something more to discover in the space of process-based non-local boxes.

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