Self-similar groups and the zig-zag and replacement products of graphs

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Abstract

Every finitely generated self-similar group naturally produces an infinite sequence of finite $d$-regular graphs $\Gamma_n$. We construct self-similar groups, whose graphs $\Gamma_n$ can be represented as an iterated zig-zag product and graph powering: $\Gamma_{n+1} = \Gamma_n^k \circ \Gamma$ ($k \geq 1$). Also we construct self-similar groups, whose graphs $\Gamma_n$ can be represented as an iterated replacement product and graph powering: $\Gamma_{n+1} = \Gamma_n^k \otimes \Gamma$ ($k \geq 1$). This gives simple explicit examples of self-similar groups, whose graphs $\Gamma_n$ form an expanding family, and examples of automaton groups, whose graphs $\Gamma_n$ have linear diameters $\text{diam}(\Gamma_n) = O(n)$ and bounded girth.

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1 Introduction

A sequence of finite $d$-regular graphs $(\Gamma_n)_{n \geq 1}$ is an expanding family if there exists $\varepsilon > 0$ such that $\lambda(\Gamma_n) < 1 - \varepsilon$ for all $n \in \mathbb{N}$, where $\lambda(\Gamma)$ is the second largest (in absolute value) eigenvalue of the normalized adjacency matrix of $\Gamma$. Expanding graphs have many interesting applications in different areas of mathematics and computer science (see [11] and the references therein). That is why many constructions of expanding families were proposed for the last decades, most of which have algebraic nature.

In [16], Reingold, Vadhan, and Wigderson discovered a simple combinatorial construction of expanding graphs. Their construction is based on the new operation on regular graphs — the zig-zag product $\circ$. The estimates on the second eigenvalue of the zig-zag product of graphs proved in [16] lead to the construction of expanders as an iterated zig-zag product and graph squaring: the sequence $\Gamma_{n+1} = \Gamma_n^2 \circ \Gamma$, $\Gamma_1 = \Gamma^2$ is an expanding family if $\lambda(\Gamma)$ is small enough. Later, the zig-zag product showed its effectiveness in constructing graphs with other exceptional properties, various codes, in computational complexity theory, etc.
The zig-zag product is directly related to the simpler replacement product $\circlearrowright_r$, which replace every vertex of one graph by a copy of another graph. This product was widely used in various contexts. For example, the replacement product of the graph of the $d$-dimensional cube and the cycle on $d$ vertices is the so-called cube-connected cycle, which is used in the network architecture for parallel computations. Gromov [10] considered the graphs of $d$-dimensional cubes for different dimensions and estimated the second eigenvalue of their iterated replacement product (iterated cubical graphs). Previte [15] studied the convergence of iterated replacement product $\Gamma_{n+1} = \Gamma_n \circlearrowright_r \Gamma$, normalized to have diameter one, in the Gromov-Hausdorff metric and their limit spaces. The estimate on the second eigenvalue of the replacement product of graphs proved in [16] leads to the expanding family $\Gamma_{n+1} = \Gamma_n \circlearrowright_r \Gamma$ when $\lambda(\Gamma)$ and $\lambda(\Gamma_1)$ are small enough [14].

In this paper we establish a connection between the zig-zag and replacement products of graphs and self-similar groups. The theory of self-similar groups [13] was developed from several examples of groups (mainly the Grigorchuk group) that enjoy many extreme properties (intermediate growth, finite width, non-uniformly exponential growth, periodic groups, amenable but not elementary amenable groups, just-infinite groups, etc.) Self-similar groups are specific groups of transformations on the space of all finite words over an alphabet that preserve the length of words. Every self-similar group can be easily defined by a finite system of wreath recursions, while properties of the group remain mysterious.

By fixing a generating set of a self-similar group, we get a sequence of $d$-regular graphs $\Gamma_n$ associated to the action of generators on words of length $n$. A natural question arises whether we can produce an expanding family in this way. However, the graphs $\Gamma_n$ were studied mostly for the opposite case of contracting self-similar groups. In this case, the graphs $\Gamma_n$ converge in certain sense to a compact fractal space, which lead to the notion of a limit space of a contracting self-similar group and further developed into the beautiful theory of iterated monodromy groups [13]. The diameter of graphs $\Gamma_n$ for contracting groups has exponential growth in terms of $n$ (polynomial in the number of vertices), what makes them opposite to expanding graphs and the zig-zag product.

An important class of self-similar groups is the class of automaton groups. These groups are given by finite-state transducers (Mealy automata) with the same input and output alphabets. Every state of such an automaton $A$ produces a transformation of words over an alphabet. If all these transformations are invertible, they generate a self-similar group under composition of functions called the automaton group $G_A$ generated by $A$. For example, the Grigorchuk group is generated by a 5-state automaton over a 2-letter alphabet. The graphs $\Gamma_n$ for an automaton group $G_A$ can be expressed through the standard operation of composition of automata, namely $\Gamma_n = \hat{A} \circ \ldots \circ \hat{A}$ ($n$ times), where $\hat{A}$ is the dual automaton. However, expanding properties of automata composition are unknown. The complete spectrum of graphs $\Gamma_n$ was computed only for a few automaton groups [3, 8, 9], and the general case remains widely open. Nevertheless, Glasner and Mozes [6] realized certain groups with property (T) as automaton groups, what implies that the associated graphs $\Gamma_n$ form an expanding family. The corresponding automata are large and were not described explicitly. At the same time, there are two specific 3-state automata over a 2-letter alphabet, the Aleshin and Bellaterra automata, whose graphs $\Gamma_n$ form asymptotic
expander [17, Section 10], and the question is raised [17, Problem 10.1] whether actually these graphs are expanders. This problem remains open. Even the asymptotic of diameters of $\Gamma_n$ for these two automata is unknown; the best known upper bound is $O(n^2)$ [12].

In this paper, given $k \geq 1$ and a graph $\Gamma$ with certain restrictions, we construct self-similar groups, whose graphs $\Gamma_n$ can be represented as iterated zig-zag or replacement products and graph powering: $\Gamma_{n+1} = \Gamma_k \circ \Gamma$ for all $n \geq 1$ or $\Gamma_{n+1} = \Gamma_k \circ \Gamma$ for all $n \geq 1$. This gives explicit examples of self-similar groups whose graphs $\Gamma_n$ form a family of expanders. The established connection between self-similar groups and the zig-zag product is not surprising—the zig-zag product of graphs is closely related to the semidirect product of groups [1], while self-similar groups to the wreath product of groups. We also note that our construction modeling iterated zig-zag product is a self-similar analog of the construction from [17]. In the case $k = 1$, the constructed groups are automaton groups. This gives simple explicit examples of automaton groups whose graphs $\Gamma_n$ have linear diameters $O(n)$ (logarithmic in the number of vertices) and bounded girth. Interestingly, some of the automaton groups modeling iterated replacement product belong to the class of GGS groups [2, 4]. In particular, these groups are not finitely presented and have intermediate growth.

2 The zig-zag and replacement products of graphs

All graphs in this paper are regular, undirected, and may have loops and multiple edges.

Let $\mathcal{G}$ be a $D$-regular graph on $N$ vertices and let $\Gamma$ be a $d$-regular graph on $D$ vertices. We label the edges near every vertex of $\mathcal{G}$ by the vertices of $\Gamma$ in one-to-one fashion; for $v \in V(\mathcal{G})$ and $x \in V(\Gamma)$, let $v[x]$ be the $x$-neighbor of $v$. If an edge is labeled by $x$ near $v$ and by $y$ near $u$, i.e., $v[x] = u$ and $u[y] = v$, we write $v \xrightarrow{x} y u$. The zig-zag and replacement products depend on the chosen labeling.

The zig-zag product. The zig-zag product $\mathcal{G} \circ \Gamma$ is a $d^2$-regular graph on $ND$ vertices $V(\Gamma) \times V(\mathcal{G})$. The edges of $\mathcal{G} \circ \Gamma$ are formed by “zig-zag” paths of length three:

1. for every edge $x \rightarrow x'$ in $\Gamma$ (the zig-step),
2. the edge $v \xrightarrow{x} y \xrightarrow{y'} u$ in $\mathcal{G}$,
3. and every edge $y' \rightarrow y$ in $\Gamma$ (the zag-step),

there is an edge between $(x, v)$ and $(y, u)$ in $\mathcal{G} \circ \Gamma$. (Classically, the vertices of the zig-zag product are written as pairs $(v, x)$. We switched the order to show a similarity with action graphs of self-similar groups. As usual, by switching from right to left, we get a connection between two object studied in different contexts.)

The next basic properties easily follow from the definition. The zig-zag product of any two graphs has girth $\leq 4$ and diameter $\text{diam}(\mathcal{G} \circ \Gamma) \leq \text{diam}(\mathcal{G}) + 2\text{diam}(\Gamma)$. The zig-zag product of connected graphs is not always connected; one easy sufficient condition is the following: If any two vertices of $\Gamma$ can be connected by a path of even length, then the graph $\mathcal{G} \circ \Gamma$ is connected for any connected graph $\mathcal{G}$.
Many applications of the zig-zag product are based on the following spectral property proved in [16]:

$$\lambda(G \odot \Gamma) \leq \lambda(G) + \lambda(\Gamma) + \lambda(\Gamma)^2,$$  \hspace{1cm} (1)

where $\lambda(\Gamma)$ is the second largest (in absolute value) eigenvalue of the normalized adjacency matrix of $\Gamma$.

**The replacement product.** The replacement product $G \circ \circ \Gamma$ is a $(d+1)$-regular graph on $ND$ vertices $V(\Gamma) \times V(G)$ with the following edges:

1. for every edge $x - y$ in $\Gamma$ and $v \in V(G)$ there is an edge $(x, v) - (y, v)$ in $G \circ \circ \Gamma$;

2. for every edge $v - x - y - u$ in $G$ there is an edge $(x, v) - (y, u)$ in $G \circ \circ \Gamma$.

In other words, we replace each vertex $v$ of $G$ with a copy of $\Gamma$ (keeping all the edges of $\Gamma$ in all the copies), and adjoin edges adjacent to $v$ in $G$ to the corresponding vertices of $\Gamma$ using the chosen one-to-one correspondence between these edges and vertices of $\Gamma$.

The next properties easily follow from the definition. The replacement product of connected graphs is connected, the diameter satisfies $\text{diam}(G \circ \circ \Gamma) \leq \text{diam}(G) \cdot \text{diam}(\Gamma)$, and the girth of $G \circ \circ \Gamma$ is not greater than the girth of $\Gamma$.

In [16], the expansion property of the replacement product is estimated as

$$\lambda(G \circ \circ \Gamma) \leq (p + (1 - p)(\lambda(G) + \lambda(\Gamma) + \lambda(\Gamma)^2))^{1/3},$$  \hspace{1cm} (2)

where $p = d^2/(d + 1)^3$.

**Iterative construction of expanders.** Let us describe the construction of expanding families using the zig-zag product and graph powering presented in [16]. Take a $d$-regular graph $\Gamma$ on $d^4$ vertices such that $\lambda(\Gamma) \leq 1/5$ (such graphs exist by probabilistic arguments). Define the sequence of graphs $(\Gamma_n)_{n \geq 1}$ as follows:

$$\Gamma_1 = \Gamma^2, \quad \Gamma_{n+1} = \Gamma_n \circ \circ \Gamma, \quad n \geq 1.$$  \hspace{1cm} (3)

(The $k$-th power $\Gamma^k$ of a graph $\Gamma$ is the graph on the vertices of $\Gamma$, where each edge corresponds to a path of length $k$ in $\Gamma$. Note that $\lambda(\Gamma^k) = \lambda(\Gamma)^k$. Then the estimate (1) implies that the graphs $\Gamma_n$ are $d^2$-regular graphs with $\lambda(\Gamma_n) \leq 2/5$.

Analogous construction works with the replacement product as well [14]. Take a $(d+1)$-regular graph $\Gamma_1$ and a $d$-regular graph $\Gamma$ on $(d + 1)^4$ vertices such that $\lambda(\Gamma_1) \leq 1/5$, $\lambda(\Gamma) \leq 1/5$. Define the sequence of graphs $(\Gamma_n)_{n \geq 1}$ as follows:

$$\Gamma_{n+1} = \Gamma_n \circ \circ \Gamma, \quad n \geq 1.$$  \hspace{1cm} (4)

The estimate (2) implies that the graphs $\Gamma_n$ are $(d + 1)$-regular graphs with $\lambda(\Gamma_n) \leq 1/10$. 

4
3 Self-similar groups and their action graphs

Every finitely generated self-similar group can be given by a finite system (wreath recursion)

\[
\begin{align*}
  s_1 &= \pi_1(w_{11}, w_{12}, \ldots, w_{1d}) \\
  s_2 &= \pi_2(w_{21}, w_{22}, \ldots, w_{2d}) \\
  &\quad \vdots \nonumber \\
  s_k &= \pi_k(w_{k1}, w_{k2}, \ldots, w_{kd}) \nonumber
\end{align*}
\]

(5)

where \(\pi_i\) is a permutation on \(X = \{1, 2, \ldots, d\}\) and \(w_{ij}\) is a word over \(S \cup S^{-1}\), \(S = \{s_1, s_2, \ldots, s_k\}\). The system defines the action of \(S\) on the set \(X^*\) of all finite words over \(X\) (we use left actions). Each \(s_i\) acts on \(X\) by the permutation \(\pi_i\), and the action on words over \(X\) is defined by the recursive rule

\[s_i(xv) = \pi_i(x)w_{ix}(v), \quad x \in X, v \in X^*,\]

where \(w_{ix}\) acts by composition. These transformations are invertible, and the group generated by them under composition is called the self-similar group \(G = \langle S \rangle\) associated to the system \((5)\).

When all words \(w_{ix}\) in \((5)\) are letters, i.e., \(w_{ix} = s_{ix} \in S\), the system \((5)\) can be represented by a finite-state automaton-transducer \(A\) over the alphabet \(X\) with states \(S\). The automaton \(A\) is represented by a finite directed graph with vertices \(S\) and arrows \(s_i \rightarrow s_{ix}\) labeled by \(x|\pi_i(x)\) for all \(x \in X\) and \(s_i \in S\). The action of \(s_i\) on \(X^*\) can be described using the automaton \(A\) as follows. Given a word \(v = x_1x_2\ldots x_n \in X^*\), there exists a unique directed path in the automaton \(A\) starting at the state \(s_i\) and labeled by \(x_1|y_1, x_2|y_2, \ldots, x_n|y_n\) for some \(y_i \in X\). Then the word \(y_1y_2\ldots y_n\) is the image of \(x_1x_2\ldots x_n\) under \(s_i\). In this case, the group \(G = \langle S \rangle\) is called the automaton group given by the automaton \(A\).

Self-similar groups preserve the length of words under the action on \(X^*\), and we can restrict the action to \(X^n\), words of length \(n\), for each \(n \in \mathbb{N}\). By choosing a finite symmetric generating set \(S\) of a self-similar group \(G\), we get a sequence of \(|S|\)-regular graphs \((\Gamma_n)_{n \geq 1}\) of the action on \(X^n, n \geq 1\). The vertex set of \(\Gamma_n\) is \(X^n\), and for every \(s \in S\) and \(v \in X^n\) there is an edge between the vertices \(v\) and \(s(v)\). The graph \(\Gamma_n\) is a Schreier coset graph of \(G\) if the group acts transitively on \(X^n\).

If the generating set \(S\) is given by the system \((5)\), the graphs \(\Gamma_n = \Gamma_n(S)\) can be constructed iteratively, very similar to \((3)\). Every \(s_i \in S\) produces an edge \(xv - \pi_i(x)w_{ix}(v)\) in \(\Gamma_n\), which can be interpreted as a zig-step \(x - \pi_i(x)\) in the graph \(\Gamma_1\), and a walk \(v - w_{ix}(v)\) in the graph \(\Gamma_{n-1}\). In contrast to the zig-zag product, we are missing the zag-step (and the walk \(w_{ix}\) is not agreed with \(\pi_i(x)\)), but we will see in the next section that one can make the edge \(x - \pi_i(x)\) to be already the combination of the zig and zag steps.
4 Modeling iterated zig-zag and replacement products by automaton groups

In this section we construct automaton groups whose action graphs satisfy $\Gamma_{n+1} = \Gamma_n \odot \Gamma$ for all $n \geq 1$ or $\Gamma_{n+1} = \Gamma_n \odot \Gamma$ for all $n \geq 1$, where $\Gamma$ is a fixed graph.

**Modeling iterated zig-zag product.** Let $X = \{1, 2, \ldots, d\}$ and $P$ be a symmetric set of permutations on $X$ such that $d = |P|^2$. We introduce formal symbols $s_{(\pi, \tau)}$ for $\pi, \tau \in P$ and define wreath recursion (5) for the set $S_P = \{s_{(\pi, \tau)} : \pi, \tau \in P\}$, $|S_P| = d$ as follows. Choose an order on $S_P$: let $S_P = \{s_1, \ldots, s_d\}$. Let $\gamma$ be the permutation on $X$ given by the rule: if $s_x = s_{(\pi, \tau)}$ then $s_\gamma(x) = s_{(\tau^{-1}, \pi^{-1})}$. Notice that $\gamma = \gamma^{-1}$. Define wreath recursion by

$$s_{(\pi, \tau)} = \tau \gamma (s_1, s_2, \ldots, s_d) \pi = \tau \gamma \pi (s_{\pi(1)}, s_{\pi(2)}, \ldots, s_{\pi(d)}), \pi, \tau \in P,$$

(here $\pi$ and $\tau$ will play a role of the zig and zag steps respectively). Let $G_P$ be the self-similar group defined by this recursion. It is important to note that $S_P$ defines a symmetric generating set of $G_P$, where $s_{(\pi, \tau)}^{-1} = s_{(\tau^{-1}, \pi^{-1})}$ (in other notations, $s_x^{-1} = s_{\gamma(x)}$). This follows inductively from the recursions

$$s_{(\pi, \tau)}^{-1} (xv) = \pi^{-1} \gamma^{-1} \tau^{-1} (x) s_{\gamma^{-1} \tau^{-1} (x)} (v),$$

$$s_{(\tau^{-1}, \pi^{-1})}^{-1} (xv) = \pi^{-1} \gamma^{-1} \tau^{-1} (x) s_{\tau^{-1} \pi^{-1} (x)} (v),$$

$x \in X, v \in X^*$ (use $\gamma = \gamma^{-1}$).

**Theorem 1.** The action graphs $\Gamma_n$ of the group $G_P = \langle S_P \rangle$ satisfy $\Gamma_{n+1} = \Gamma_n \odot \Gamma$, $n \geq 1$, where $\Gamma$ is the graph of the action of $P$ on $X$.

**Proof.** The graph $\Gamma$ is a $|P|$-regular graph on $d$ vertices, while $\Gamma_n$ are $d$-regular graphs. In order to define the zig-zag product $\Gamma_n \odot \Gamma$, we should label the edges of $\Gamma_n$ by the vertices of $\Gamma$. For $x \in X$, define the $x$-neighbor of a vertex $v \in X^n$ as $v[x] := s_x(v)$. In this way we get the labeling of edges $v \xrightarrow{x \cdot \gamma(x)} s_x(v)$. Now we can consider the zig-zag product $\Gamma_n \odot \Gamma$. The vertex set of $\Gamma_n \odot \Gamma$ can be naturally identified with the vertex set $X^{n+1}$ of $\Gamma_{n+1}$ via $(x, v) \leftrightarrow xv$. For every zig-zag path

$$x \xrightarrow{\pi} x' = \pi(x) \text{ in } \Gamma, \quad v \xrightarrow{\pi' \gamma'} v[x'] \text{ in } \Gamma_n, \quad y' \xrightarrow{\tau} y = \tau(y') \text{ in } \Gamma$$

there is an edge $xv - yv[x']$ in $\Gamma_n \odot \Gamma$. Here $y' = \gamma(x') = \gamma(\pi(x))$ and $v[x'] = s_{\pi(x)}(v)$. Therefore this edge is precisely the edge of $\Gamma_{n+1}$ given by $s_{(\pi, \tau)}$:

$$xv - \tau(\gamma(\pi(x))) s_{\pi(x)}(v).$$

The next statement immediately follows from the properties of the zig-zag product.
Figure 1: The generating automata for two examples of groups $G_P$ and $G_Q$

**Corollary 1.1.** The action graphs $\Gamma_n$ of the group $G_P = \langle S_P \rangle$ have bounded girth and linear diameters $\text{diam}(\Gamma_n) = O(n)$. If $\Gamma_1$ is connected ($P \gamma P$ acts transitively on $X$) and there is a path of even length between any two vertices of $\Gamma$, then all graphs $\Gamma_n$ are connected (the group $G_P$ acts transitively on $X^n$).

The wreath recursion for the group $G_P$ defines a finite automaton over $X$ with $d$ states. Therefore every $G_P$ is an automaton group.

**Example 1.** Let $d = 4$, $X = \{1, 2, 3, 4\}$ and $P = \{(1 2), (1 4)(2 3)\}$. Then $\gamma = (2 3)$ and the group $G_P$ is generated by $s_1, s_2, s_3, s_4$ given by the wreath recursion:

\[
\begin{align*}
  s_1 &= (1 2)(2 3)(s_1, s_2, s_3, s_4)(1 2) = (1 3)(s_2, s_1, s_3, s_4) \\
  s_2 &= (1 4)(2 3)(2 3)(s_1, s_2, s_3, s_4)(1 2) = (1 2 4)(s_2, s_1, s_3, s_4) \\
  s_3 &= (1 2)(2 3)(s_1, s_2, s_3, s_4)(1 4)(2 3) = (1 4 2)(s_4, s_3, s_2, s_1) \\
  s_4 &= (1 4)(2 3)(2 3)(s_1, s_2, s_3, s_4)(1 4)(2 3) = (2 3)(s_4, s_3, s_2, s_1)
\end{align*}
\]

The generating automaton is shown on the left-hand side of Figure 1.

The construction can be modified for the case $d > |P|^2$. We add $d - |P|^2$ empty words $e$ to the wreath recursion:

\[ s_{(\pi, \tau)} = \tau \gamma(s_1, s_2, \ldots, s_{|P|^2}, e, \ldots, e)\pi, \]

where $e$ acts trivially on $X^*$. Then the action graphs satisfy $\Gamma_{n+1} = \Gamma_n \odot \Gamma$, where $\Gamma_n^e$ is obtained from $\Gamma_n$ by adding $d - |P|^2$ loops to every vertex.

**Modeling iterated replacement product.** Let $Q = \{\pi_1, \ldots, \pi_d\}$ be a symmetric set of permutations on $X = \{1, 2, \ldots, d + 1\}$. Let $\gamma$ be the involution on $X$ given by the rule:
\[ \pi_{\gamma(x)} = \pi_x^{-1} \] and \[ \gamma(d+1) = d+1. \] We define wreath recursion for the set \( S_Q = \{s_1, \ldots, s_{d+1}\} \) by
\[
\begin{align*}
s_i &= \pi_i(e, e, \ldots, e), \quad i = 1, 2, \ldots, d; \\
s_{d+1} &= \gamma(s_1, s_2, \ldots, s_{d+1}).
\end{align*}
\]

Let \( G_Q \) be the self-similar group defined by this recursion. Notice that the generating set \( S_Q \) is symmetric, because \( s_i^{-1} = s_{\gamma(i)} \) for \( i = 1, 2, \ldots, d \) and \( s_{d+1}^2 = e \).

Every \( s_i \) for \( i = 1, \ldots, d \) changes only the first letter in any word over \( X \). In this case one usually identifies \( s_i \) and \( \pi_i \); then we can write \( G_Q = \langle \pi_1, \ldots, \pi_d, s \rangle \), where \( s = s_{d+1} \) is given by the recursion \( s = \gamma(\pi_1, \ldots, \pi_d, s) \).

**Theorem 2.** The action graphs \( \Gamma_n \) of the group \( G_Q = \langle S_Q \rangle \) satisfy \( \Gamma_{n+1} = \Gamma_n \overline{\circ} \Gamma, \ n \geq 1 \), where \( \Gamma \) is the graph of the action of \( Q \) on \( X \). In particular, if \( \Gamma \) is connected, then all \( \Gamma_n \) are connected.

**Proof.** The graph \( \Gamma \) is a \( d \)-regular graph on \( d+1 \) vertices, while \( \Gamma_n \) are \( (d+1) \)-regular graphs. In order to define the replacement product \( \Gamma_n \overline{\circ} \Gamma \), we label the edges of \( \Gamma_n \) by the vertices of \( \Gamma \) as follows. For \( x \in X \), define the \( x \)-neighbor of a vertex \( v \in X^n \) as \( v[x] := s_x(v) \). In this way we get the labeling of edges \( v \xrightarrow{x \gamma(x)} s_x(v) \). Now we can consider the replacement product \( \Gamma_n \overline{\circ} \Gamma \). The vertex set of \( \Gamma_n \overline{\circ} \Gamma \) can be naturally identified with the vertex set \( X^{n+1} \) of \( \Gamma_{n+1} \) via \( (x, v) \leftrightarrow xv \). For every edge \( x \xrightarrow{\pi_i} y = \pi_i(x) \) in \( \Gamma \), the edges \( xv - yv \) in \( \Gamma_n \overline{\circ} \Gamma \) coincide with edges in \( \Gamma_{n+1} \) given by \( s_i \). For every edge \( v \xrightarrow{x \gamma(x)} s_x(v) \) in \( \Gamma_n \), the edge \( xv - \gamma(x)s_x(v) \) in \( \Gamma_n \overline{\circ} \Gamma \) coincide with the edge in \( \Gamma_{n+1} \) given by \( s_{d+1} \).

The wreath recursion for the group \( G_Q \) defines a finite automaton over \( X \) with \( d+2 \) states. All these automata belong to the important class of bounded automata. In particular, the groups \( G_Q \) belong to the class of contracting self-similar groups (see [13]). The action graphs \( \Gamma_n \) of groups generated by bounded automata were studied in [3]. In particular, the diameters of graphs \( \Gamma_n \) have exponential growth in terms of \( n \), and there is an algorithmic method to find the exponent of growth as the Perron-Frobenius eigenvalue of certain non-negative integer matrix.

Some of the groups \( G_Q \) were studied before as interesting examples of automaton groups. To see this, assume that all permutations in \( Q \) are involutions (then \( \gamma \) is trivial), \( d \geq 2 \), and the graph \( \Gamma \) is connected. Then the group \( G_Q \) is a GGS group studied in [2, 3]. In particular, in this case \( G_Q \) is not finitely presented and has intermediate growth. Is it true that all groups \( G_Q \) have subexponential growth?

**Example 2.** Let \( d = 2 \) and \( Q = \{\sigma, \sigma^{-1}\} \), \( \sigma = (1\ 2\ 3) \). The group \( G_Q \) is generated by \( \sigma \) and \( s = (1\ 2)(\sigma, \sigma^{-1}, s) \). The generating automaton is shown on the right-hand side of Figure [4].
In this section we construct wreath recursions that model the iterations (3) and (4).

Let $G$ be the self-similar group generated by this recursion. Notice that $S_P$ defines a symmetric generating set of $G_{P,1}$ from the previous section.

Let us consider the associated action graphs $\Gamma_n$. Note that each word $w_x$ represents a path of length $k$ in $\Gamma_n$, which is an edge in the graph $\Gamma_n^k$. For $x \in X$, define the $x$-neighbor of a vertex $v \in X^n$ in $\Gamma_n^k$ as $v[x] := w_x(v)$. Then each edge $s_{(\pi,\tau)}(\pi^x(x))w_{\pi(x)}(v)$ in $\Gamma_{k+1}$ is precisely the zig-zag path in $\Gamma_n \boxtimes \Gamma$. We get the following statement.

**Theorem 3.** The action graphs $\Gamma_n$ of the group $G_{P,k} = \langle S_P \rangle$ satisfy $\Gamma_{n+1} = \Gamma_n^k \boxtimes \Gamma$, $n \geq 1$, where $\Gamma$ is the graph of the action of $P$ on $X$.

If $\lambda(\Gamma)$ and $\lambda(\Gamma_1)$ are small enough (for example, less than $1/5$), then we get a sequence of expanders. Therefore this construction gives simple explicit examples of self-similar groups whose graphs $\Gamma_n$ form an expanding family.

As above the construction can be modified for the case $d > |P|^k$ by adding empty words to the wreath recursion.

Similarly we model the iteration (4). Fix $k \geq 1$. Let $Q = \{\pi_1, \ldots, \pi_d\}$ be a symmetric set of permutations on $X = \{1, 2, \ldots, (d + 1)^k\}$. Let $s$ be a formal symbol and $S_Q = \{\pi_1, \ldots, \pi_d, s\}$, where $\pi_i$ is considered as transformation of $X^*$ that changes the first letter of words. Take all words of length $k$ over $S_Q$, there are $(d + 1)^k$ such words, and fix an order on them: $w_1, w_2, \ldots, w_{(d+1)^k}$. Let $\gamma$ be the involution on $X$ such that if $w_x = s_1s_2\ldots s_k$ then $w_{\gamma(x)} = s_k^{-1}\ldots s_2^{-1}s_1^{-1}$, where for the symbol $s_i$ we put $s_i^{-1} := s$. We define the self-similar group $G_{Q,k} = \langle \pi_1, \ldots, \pi_d, s \rangle$, where $s$ is given by the wreath recursion

$$s = \gamma(w_1, w_2, \ldots, w_{(d+1)^k}).$$

The set $S_Q$ defines a symmetric generating set of $G_{Q,k}$, because $Q$ is symmetric and $s^2 = e$.

As above we get the following statement.

**Theorem 4.** The action graphs $\Gamma_n$ of the group $G_{Q,k} = \langle S_Q \rangle$ satisfy $\Gamma_{n+1} = \Gamma_n^k \boxtimes \Gamma$, $n \geq 1$, where $\Gamma$ is the graph of the action of $Q$ on $X$. 

9
This construction produces other examples of self-similar groups whose graphs $\Gamma_n$ form an expanding family when $k \geq 4$ and $\lambda(\Gamma)$ and $\lambda(\Gamma_1)$ are small enough.

**Questions.** It is interesting what are algebraic and geometric properties of the groups $G_{P,k}$ and $G_{Q,k}$. Are these groups finitely presented? have property (T)? What are their profinite completions? What are the properties of their action on the boundary of the space $X^*$, i.e., infinite sequences $x_1x_2\ldots$ over $X$?

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