Monogamy relations for generalized W class states in terms of Tsallis entropy beyond qubits

Xian Shi

Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
University of Chinese Academy of Sciences, Beijing 100049, China
UTS-AMSS Joint Research Laboratory for Quantum Computation and Quantum Information Processing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Abstract

In this article, we consider the monogamy relations for the generalized W class states. Here we first present a monogamy inequality in terms of the squared TqEE for the reduced density matrix of the GW state, then we present a polygamy inequality in terms of TqEE for the reduced density matrix of the GW state. At last, we present a tighter polygamy inequality in terms of TqEEoA for the reduced density matrix of the superposition of generalized W class and the vacuum states.

PACS numbers: 03.67.Mn, 03.65.Ud
I. INTRODUCTION

Quantum entanglement [1] is an essential feature of quantum information theory, which distinguishes the quantum from classical theory. One of the fundamental differences between entanglement and classical relations is that there exists some restrictions on its distribution and sharability [2]. This property is known as monogamy of entanglement (MoE). Monogamy relations is valuable on the frustration effects observed in condensed matter physics [3]. MoE is also a key ingredient to make quantum cryptography secure as it quantifies how much information an eavesdropper could potentially obtain about the secret key to be extracted [4, 5].

Mathematically, MoE for a three-party system $\rho_{ABC}$ can be represented as in terms of some entanglement measure $E$,

$$E_{A|BC} \geq E_{AB} + E_{AC}. \quad (1)$$

This property was first shown by Coffman et al. [6] in terms of the squared concurrence for a three-qubit mixed state $\rho_{ABC}$, here we can denote this inequality as CKW inequality. It was generated for $n$-qubit systems in terms of the squared concurrence later [7]. Then this relation is generalized in terms of the TqEE [8, 9], the Renyi-$\alpha$ entropy [10], and the unified entropy [11] for multi-qubit systems. In 2014, Regula et al. proposed a stronger monogamy inequality which generalized the CKW inequality by conjecturing the nonnegativity of $n$-tangle when $n \geq 3$ [12]. However, the CKW inequality is invalid for higher dimensional systems in terms of the squared concurrence [13]. In 2016, Lancien et al. even showed any nontrivial monogamy relations cannot satisfy for a whole additive entanglement measures [14]. Up to date, it seems only one known entanglement measure, the squashed entanglement, is monogamous for arbitrary dimensional systems [15]. And there are results on states satisfying the monogamy relations in higher dimensional systems. In 2008, Kim and Sanders showed the generalized W class (GW) states satisfying the monogamy inequality in terms of the squared concurrence [16]. In 2015, Choi and Kim showed that the superposition of the generalized W-class states and the vacuum (GWV) states satisfy in terms of the squared convex roof extended negativity strong monogamy inequality [17]. In 2016, Kim showed that a partially coherent superposition (PCS) of a generalized W-class state and the vacuum saturates the strong monogamy inequality [18], this result is interesting, as it is the first kind of multiqudit mixed states that satisfy the strong monogamy inequality in terms
of the squared convex roof extended negativity.

As a generalization of von Neumann entropy, Tsallis-q entropy plays an important role in quantum information theory. It can be used to provide criterion for separability of compound quantum systems \[19, 20\], and it is used to generalize global quantum discord and provide a sufficient condition for an n-party quantum state to be monogamous \[21\]. Furthermore, M. Wajs et al. showed that the entropic bell inequalities in terms of the classical Tsallis-q entropy can be used to investigate the nonlocal corrections which is more suitable than the Shannon entropy \[22\].

In this article, we consider the monogamy relations in terms of the TqEE for the reduced density matrix of the GW state. In section II, we present some preliminary knowledge on this article. In section III, we present our main results. First we present a monogamy inequality for the GW state in terms of the squared TqEE, at last, we present a tighter polygamy inequality in terms of TqEEoA for the reduced density matrix of GWV states when q=2. In section IV, we end with a summary.

II. PRELIMINARY KNOWLEDGE

Given a bipartite pure state \(|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i} |ii\rangle\), the concurrence is defined as

\[
C(|\psi_{AB}\rangle) = \sqrt{2(1 - \text{Tr} \rho_A^2)} = \sqrt{2 \sum_{i \neq j} \lambda_i \lambda_j},
\]

where \(\rho_A = \text{Tr}_B \rho_{AB}\), when \(\rho_{AB}\) is a mixed state, its concurrence is defined as by the convex roof extended method,

\[
C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),
\]

where the minimum takes over all the decompositions of \(\rho_{AB}\). As a dual quantity to the concurrence, we can define the concurrence of assistance (CoA) as

\[
C(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),
\]

where the maximum takes over all the decompositions of \(\rho_{AB}\).

For a pure state \(|\psi_{AB}\rangle = \sum_i \sqrt{\lambda_i} |ii\rangle\), its TqEE is defined as

\[
T_q(|\psi_{AB}\rangle) = \frac{1 - \text{Tr} \rho_A^q}{q - 1} = \frac{1 - \sum_i \lambda_i^q}{q - 1},
\]
for any \( q > 0, q \neq 1 \), here we denote that \( \rho_A = \text{Tr}_B |\psi\rangle \langle \psi| \). Assume \( \rho_{AB} \) is a mixed state, its TqEE is defined as

\[
T_q(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i T_q(|\psi_i\rangle),
\]

(6)

where the minimum takes over all the decompositions of \( \rho_{AB} \). When \( q \to 1 \), \( T_q(\cdot) \) converges to the entanglement of formation \( E(\cdot) \). As a dual concept of TqEE, the Tsallis-q entanglement entropy of assistance (TqEEoA) was defined as

\[
T_q(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i T_q(|\psi_i\rangle),
\]

(7)

where the minimum takes over all the decompositions of \( \rho_{AB} \).

From the equalities (2) and (4), we see that when \( |\psi\rangle_{AB} = \sqrt{\lambda_0} |00\rangle + \sqrt{\lambda_1} |11\rangle \), we have that \( C^2(|\psi_{AB}\rangle) = 4\lambda_0 A_1, T_q(|\psi_{AB}\rangle) = \frac{1 - \lambda^q_0 - \lambda^q_1}{q - 1} \), from the above equalities, we have

\[
T_q(|\psi_{AB}\rangle) = f_q(C^2(|\psi_{AB}\rangle)),
\]

(8)

where the function \( f_q(x) = \frac{1}{q-1} [1 - (\frac{1 + \sqrt{q-1}}{2})^q - (\frac{1 - \sqrt{q-1}}{2})^q] \).

Now let us recall the definition of the GW states \( |W^d_n\rangle \) [17],

\[
|W^d_n\rangle_{A_1 \cdots A_n} = \sum_{i=1}^d (a_{1i}|i0\cdots0\rangle + \cdots + a_{ni}|00\cdots i\rangle),
\]

where we assume \( \sum_{i=1}^d \sum_{j=1}^n |a_{ji}|^2 = 1 \). Here Choi and Kim presented the following lemma,

**Lemma 1.** [17] Let \( |\psi\rangle_{A_1 \cdots A_n} \) be the superposition of the generalized W class states and vacuum (GWV), that is,

\[
|\psi\rangle_{A_1 \cdots A_n} = \sqrt{p}|W^d_n\rangle + \sqrt{1-p}|00\cdots0\rangle,
\]

(9)

for \( 0 \leq p \leq 1 \). Let \( \rho_{A_{j_1} \cdots A_{j_m}} \) be a reduced density matrix of \( |\psi_{A_1 \cdots A_n}\rangle \) onto \( m \)-qudit subsystems \( A_{j_1} \cdots A_{j_{m-1}} \) with \( 2 \leq m \leq n - 1 \). For any pure state decomposition of \( \rho_{A_{j_1} \cdots A_{j_m}} \) such that

\[
\rho_{A_{j_1} \cdots A_{j_m}} = \sum_k q_k |\phi_k\rangle_{A_{j_1} \cdots A_{j_m}} \langle \phi_k|_{A_{j_1} \cdots A_{j_m}},
\]

(10)

\( |\phi_k\rangle_{A_{j_1} \cdots A_{j_m}} \) is a GWV state.

As each GWV state \( |\psi_{A_{j_1} A_{j_2} \cdots A_{j_{i+1}} \cdots A_{j_m}}\rangle \) is a Schmidt rank 2 pure state by any partition and from the above lemma, we see that for any decomposition \( \{p_i, |\phi_i\rangle_{A_{j_1} A_{j_2} \cdots A_{j_{i}} A_{j_{i+1}} \cdots A_{j_{m}}}\} \) of a reduced density matrix of \( |\psi\rangle_{A_1 \cdots A_n}, |\phi_i\rangle_{A_{j_1} A_{j_2} \cdots A_{j_{i-1}} A_{j_{i+1}} \cdots A_{j_m}} \) is a Schmidt rank 2 pure state.
Before presenting our main results, we give some lemmas on the properties of the function $f_q$ in the equality (8) presented in [9].

**Lemma 2.** The function $f_q^2(C^2)$ is a monotonously increasing and convex function of the squared concurrence $C^2$ when $q \in \left[\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}\right]$.

**Lemma 3.** The function $f_q(C^2)$ is a monotonously increasing and concave function for the squared concurrence $C^2$ when $q \in \left[\frac{5-\sqrt{13}}{2}, 2\right] \cup \left[3, \frac{5+\sqrt{13}}{2}\right]$.

**Lemma 4.** The function $f_q(C^2)$ is a monotonic increasing function of the concurrence $C$ for any $q > 0$ and $0 < C < 1$, it is a convex function of the concurrence $C$ when $q \in \left[\frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}\right]$.

Then we have the following result by the similar method in [9].

**Theorem 1.** Assume $\rho_{A_1 \cdots A_m}$ is a reduced density matrix of a pure GW state, then we have

$$T_q(\rho_{A_1 | A_2 \cdots A_m}) = f_q(C^2(\rho_{A_1 | A_2 \cdots A_m})), \quad (11)$$

when $q \in \left[\frac{5-\sqrt{13}}{2}, 2\right] \cup \left[3, \frac{5+\sqrt{13}}{2}\right]$.

**Proof.** Here we denote $\rho_{A_1 | A_2 \cdots A_m}$ as $\rho_{AB}$ below. First we would prove $T_q(\rho_{AB}) \leq f_q(C^2(\rho_{AB}))$. Assume the decomposition $\{p_i, |\psi_i\rangle_{AB}\}$ is an optimal decomposition for the TqEE of $\rho_{AB}$, then we have

$$T_q(\rho_{AB}) = \sum_i p_i T_q(|\psi_i\rangle_{AB})$$

$$= \sum_i p_i f_q(C^2(|\psi_i\rangle_{AB}))$$

$$\leq \sum_k r_k f_q(C^2(|\phi_k\rangle_{AB}))$$

$$\leq f_q(\sum_k r_k C^2(|\phi_k\rangle_{AB})) = f_q(C^2(\rho_{AB})), \quad (12)$$

where in the first equality, we use the definition of TqEE, in the first inequality, we denote the decomposition $\{r_k, |\phi_k\rangle\}$ is optimal for concurrence $C^2(\rho_{AB}) = \min_{\{r_k, |\phi_k\rangle\}} \sum_k r_k C^2(|\phi_k\rangle)$.
The second inequality holds due to the concavity of the function \( f_q(C^2) \) for the squared concurrence \( C^2 \) for \( q \in \left[ \frac{5-\sqrt{13}}{2}, 2 \right] \cup \left[ 3, \frac{5+\sqrt{13}}{2} \right] \).

Then we will prove \( T_q(\rho_{AB}) \geq f_q(C^2(\rho_{AB})) \), we can obtain

\[
T_q(\rho_{AB}) = \sum_i p_i T_q(|\psi_i\rangle_{AB}) \\
= \sum_i p_i f_q(C(|\psi_i\rangle_{AB})) \\
\geq f_q(\sum_i p_i C(|\psi_i\rangle_{AB})) \\
\geq f_q(\sum_j s_j C(|\psi_j\rangle_{AB})) = f_q(C(\rho_{AB})). \tag{13}
\]

Here in the first inequality, we use the convexity of \( f_q(C) \) when \( q \in \left[ \frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2} \right] \) and the Cauchy-Schwartz inequality, in the second inequality, we denote that the decomposition \( \{ s_j, |\psi_j\rangle \} \) is the optimal decomposition for the concurrence \( C \). Combing the inequalities (12) and (13), we have

\[
f_q(C^2(\rho_{AB})) \geq T_q(\rho_{AB}) \geq f_q(C(\rho_{AB})), \tag{14}
\]

then as \( C(\rho_{AB}) = C_a(\rho_{AB}) \) \cite{16}, and by the method in \cite{23}, we could find a decomposition \( \{ p_m, |\theta_m\rangle \} \) of \( \rho_{AB} \) such that all of \( C(|\theta_m\rangle) \) are the same. Then we finish the proof. \( \Box \)

As in the second part of the proof, we have the following corollary,

**Corollary 1.** Assume \( |\psi_{AB}\rangle \) is a GW state, then when \( q \in \left[ \frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2} \right] \),

\[
T_q(\rho_{AB}) \geq f_q(C^2(\rho_{AB})). \tag{15}
\]

Next we will provide a monogamy relation in terms of the TqEE for the reduced density matrix of the GW states when \( q \in \left[ \frac{5-\sqrt{13}}{2}, 2 \right] \cup \left[ 3, \frac{5+\sqrt{13}}{2} \right] \).

**Theorem 2.** Assume \( \rho_{A_1A_2\cdots A_m} \) is the reduced density matrix of a GW state \( |\psi_{A_1\cdots A_n}\rangle \), and here we denote \( \{ P_1, P_2, \cdots, P_k \} \) is a partition of the set \( \{ A_{j_1}, A_{j_2}, \cdots, A_{j_m} \} \), when \( q \in \left[ \frac{5-\sqrt{13}}{2}, 2 \right] \cup \left[ 3, \frac{5+\sqrt{13}}{2} \right] \), we have the following monogamy inequality,

\[
T_q^2(\rho_{P_1|P_2\cdots P_k}) \geq \sum_{i=2}^{k} T_q^2(\rho_{P_1P_i}). \tag{16}
\]
Proof. When \( q \in [\frac{5-\sqrt{13}}{2}, 2] \cup [3, \frac{5+\sqrt{13}}{2}] \), we have

\[
T_q^2(\rho_{P_1|P_2\ldots P_k}) = f_q^2(C^2(\rho_{P_1|P_2\ldots P_k}))
= f_q^2(\sum_{i=2}^{k} C^2(\rho_{P_i}P_i))
\geq \sum_{i=2}^{k} f_q^2(C^2(\rho_{P_i}P_i))
= \sum_{i=2}^{k} T_q^2(\rho_{P_i}P_i).
\]

(17)

Here the second equality is due to the result \( \sum_{i=2}^{k} C^2(\rho_{P_i}P_i) = C^2(\rho_{P_1|P_2\ldots P_k}) \), the second inequality is due to the fact that \( f_q^2(C^2) \) is convex as a function of \( C^2 \) when \( q \in [\frac{5-\sqrt{13}}{2}, 2], [3, \frac{5+\sqrt{13}}{2}] \).

Trivially, we have the following monogamy relations in terms of the \( \alpha \)-th power of \( T_q\text{EE} \) for the GW states.

**Corollary 2.** Assume \( \rho_{A_{j_1}A_{j_2}\ldots A_{j_m}} \) is the reduced density matrix of a GW state \( |\psi_{A_1\ldots A_n}\rangle \), and here we denote \( \{P_1, P_2, \ldots, P_k\} \) is a partition of the set \( \{A_{j_1}, A_{j_2}, \ldots, A_{j_m}\} \), when \( q \in [\frac{5-\sqrt{13}}{2}, 2] \cup [3, \frac{5+\sqrt{13}}{2}] \), we have the following monogamy inequality,

\[
T_q^\alpha(\rho_{P_1|P_2\ldots P_k}) \geq \sum_{i=2}^{k} T_q^\alpha(\rho_{P_i}P_i).
\]

(18)

when \( \alpha \geq 2 \).

**Theorem 3.** Assume \( \rho_{A_{j_1}A_{j_2}\ldots A_{j_m}} \) is the reduced density matrix of a GW state \( |\psi_{A_1\ldots A_n}\rangle \), and here we denote \( \{P_1, P_2, \ldots, P_k\} \) is a partition of the set \( \{A_{j_1}, A_{j_2}, \ldots, A_{j_m}\} \), when \( q \in [\frac{5-\sqrt{13}}{2}, 2] \cup [3, \frac{5+\sqrt{13}}{2}] \), we have the following monogamy inequality,

\[
T_q(\rho_{P_1|P_2\ldots P_k}) \leq \sum_{i=2}^{k} T_q(\rho_{P_i}P_i).
\]

(19)
Proof. When \( q \in \left[ \frac{5 - \sqrt{13}}{2}, 2 \right] \cup \left[ 3, \frac{5 + \sqrt{13}}{2} \right] \), we have

\[
T_q(\rho_{P_1|P_2 \ldots P_k}) = f_q(C^2(\rho_{P_1|P_2 \ldots P_k}))
\]

\[
= f_q\left( \sum_{i=2}^{k} C^2(\rho_{P_i}) \right)
\]

\[
\leq \sum_{i=2}^{k} f_q(C^2(\rho_{P_i}))
\]

\[
= \sum_{i=2}^{k} T_q(\rho_{P_i}),
\] (20)

where the first inequality is due to the concavity of \( f_q(C^2) \) as a function of \( C^2 \).

From the proof of the monogamy inequalities for the GW states above, we see that the method can be generalized to derive monogamy inequalities for the GW states in terms of other entanglement measures, such as the squared convex roof extended negativity \[17, 18\], the Rényi \(-\alpha\) entropy for \( \alpha \) in some region \[24, 25\].

Then we consider the PCS states proposed by \[18\], which is defined as

\[
\rho_p = p|W_n^d\rangle\langle W_n^d| + (1 - p)|0\rangle^{\otimes n}\langle 0|,
\] (21)

where \( p \in [0, 1] \), then we consider the purification of \( \rho_p \) such that

\[
|\psi\rangle_p = \sqrt{p + (1 - p)\lambda^2}|\psi_1\rangle|0\rangle + \sqrt{(1 - \lambda^2)(1 - p)}|\psi_2\rangle|\phi\rangle,
\] (22)

where we denote that \( |\phi\rangle = \sum_{i=1}^d a_{n+1i}|i\rangle \) with \( \sum_i |a_{n+1i}|^2 = 1 \). Then we can write \( |\psi\rangle_p = \sum_{i=1}^d \sqrt{p(a_{1i}|00\ldots0\rangle + a_{2i}|0i0\ldots0\rangle + \cdots + a_{ni}|000\ldots i\rangle)|0\rangle + \sqrt{1-p}|00\ldots0\rangle|\phi\rangle \), it is easy to see that it is a GW state. Then we know that the above properties shown for the GW states are also valid for these mixed states.

Next we will present the polygamy relations for the GWV states in terms of \( T_q\text{EEoA} \) when \( q=2 \). Assume \( |\psi\rangle_{ABC} \) is a GWV state, then by the results in \[26\]

\[
T_2(|\psi\rangle_{A|BC}) \leq T_2(\rho_{A|B}) + T_2(\rho_{AC}).
\] (23)

And here we note that \( q = 2 \). Assume that \( \rho_{ABC} \) is a reduced density matrix of a GWV
state $|\psi\rangle_{ABCD}$, then we have

$$T_q^a(\rho_{A|BC}) = \max \sum_i p_i T_q(|\psi_i\rangle_{A|BC})$$

$$\leq \sum_i [p_i T_q(\rho_{A|B}) + p_i T_q(\rho_{A|C})]$$

$$\leq T_q(\rho_{A|B}) + T_q(\rho_{A|C}),$$ \hspace{1cm} (24)

where in the first equality, we use the definition of TqEEoA, in the first inequality, we use the theorem 3, in the second inequality is due to the linearity of the operation of the partial trace,

$$\sum_i p_i \rho_{AB} = \rho_{AB},$$ \hspace{1cm} (25)

and we also use the definition of the TqEEoA,

$$\sum_i p_i T_q(\rho_{A|B}^j) \leq T_q^a(\rho_{A|B}), \quad \sum_i p_i T_q(\rho_{A|C}^j) \leq T_q^a(\rho_{A|C}).$$ \hspace{1cm} (26)

Then we have the following theorem.

**Theorem 4.** Assume $\rho_{A_{j_1}...A_{j_m}}$ is the reduced density matrix of a GWV state $|\psi_{A_1A_2...A_n}\rangle$, here we denote that $\{P_1, P_2, \ldots, P_k\}$ is a partition of the set $\{j_1, j_2, \ldots, j_m\}$. When $q = 2$, we have the following polygamy inequality:

$$T_q^a(\rho_{P_1|P_2...P_k}) \leq \sum_{i=2}^k T_q^a(\rho_{P_i|P_k}).$$ \hspace{1cm} (27)

Recently, results on the tighter monogamy inequalities in terms of concurrence [27], negativity [27] for n-qubit systems and the entanglement of assistance for arbitrary dimensional systems [28] are proposed. However, the results on the study of high dimensional systems are less, next we present a tighter polygamy inequality for the GW states.

**Lemma 5.** Let $\beta \in [0, 1], x \in [0, 1]$, then we have

$$(1 + x)^\beta \leq 1 + (2^\beta - 1)x^\beta$$ \hspace{1cm} (28)

**Proof.** Let $t = \frac{1}{x}$, then the lemma is equivalent to get the maximum of $f(t)$ when $t \in [1, \infty)$,

$$f(t) = (1 + t)^\beta - t^\beta.$$ \hspace{1cm} (29)

As $t \in [1, \infty)$, and $f'(t) \leq 0$, that is, when $t = 1$, $f(t)$ get the maximum $2^t - 1$. At last, When we replace $t$ with $\frac{1}{x}$, we finish the proof. \hfill \square
We know that any number $j \in \mathbb{N}^+$ can be written as
\[ j = \sum_{i=0}^{n-1} j_i 2^i, \]
here we assume $\log_2 n, j_i \in [0,1]$. According to the equality (31), we have the following bijection:
\[ j \rightarrow \vec{j}, \]
\[ j \rightarrow (j_0, j_1, \ldots, j_{n-1}), \]
then we denote its Hamming distance $w_H(\vec{j})$ as the number 1 of the set $\{j_0, j_1, \ldots, j_{n-1}\}$.

Next we present the tighter polygamy inequality of the GW states in terms of $T_q$EEoA.

**Theorem 5.** Let $\beta \in [0,1]$, when $q = 2$, assume $\rho_{PP_{j_0} \ldots P_{j_{m-1}}}$ is a reduced density matrix of a GW state $|\psi_{AA_1 \ldots A_{n-1}}\rangle$, then there exists an appropriate order of $P_{j_0}, P_{j_1} \ldots P_{j_{m-1}}$ such that
\[ [T_q^a(\rho_{P|P_{j_0} \ldots P_{j_{m-1}}})]^\beta \leq \sum_{i=0}^{m-1} (2^\beta - 1)^{w_H(\vec{j})} [T_q^a(\rho_{P|P_{j_i}})]^\beta \]  

**Proof.** In the process of the proof, we always order the partite $P_{j_0}, P_{j_1} \ldots P_{j_{m-1}}$ such that
\[ T_q^a(\rho_{P|P_{j_i}}) \geq T_q^a(\rho_{P|P_{j_{i+1}}}), \quad i = 0, 1, \ldots, m-1. \]  

Here we denote that the set
\[ A = \{\rho_{PP_{j_0} \ldots P_{j_{m-1}}} | \rho_{PP_{j_0} \ldots P_{j_{m-1}}} \text{ is a reduced density matrix of a GW state}\}, \]
\[ B = \{\rho_{PP_{j_0} \ldots P_{j_{m-1}}} = \gamma_{PP_{j_0} \ldots P_{j_{m-1}}} \otimes |0_{m-k}\rangle \langle 0_{m-k}| \gamma_{PP_{j_0} \ldots P_{j_{m-1}}} \text{ is a reduced density matrix of a GW state}\} \]

Then we will prove the elements in the set $A \cup B$ is valid for the inequality (32).

Due to the theorem 4 and the definition of the set $B$, it is enough to prove
\[ \left[\sum_{i=0}^{m-1} T_q^a(\rho_{P|P_{j_i}})\right]^\beta \leq \sum_{i=0}^{m-1} (2^\beta - 1)^{w_H(\vec{j})} [T_q^a(\rho_{P|P_{j_i}})]^\beta \]

First we prove the theorem is correct when a tripartite mixed state $\rho_{ABC}$ is a reduced density matrix of a GWV state $|\psi_{AA_1 \ldots A_{n-1}}\rangle$, according to the inequality (33), we have
\[ (T_q^a(\rho_{P|P_{b}P_{a}}))^\beta \leq T_q^a(\rho_{P|P_{b}})^\beta + T_q^a(\rho_{P|P_{a}})^\beta \]
\[ = [T_q^a(\rho_{P|P_{b}})]^\beta \left[ 1 + \frac{T_q^a(\rho_{P|P_{a}})}{T_q^a(\rho_{P|P_{b}})} \right] \]
\[ \leq (T_q^a(\rho_{P|P_{b}}))^\beta + (2^\beta - 1)(T_q^a(\rho_{P|P_{a}}))^\beta, \]  

(33)
here the first inequality is due to the theorem 4, and when \( a > c > 0, b > 0, a^b > c^b \), and the second inequality is due to the lemma 5.

Then we use the mathematical induction. First let us assume when \( m < 2^n \), the theorem is correct. Then we have when \( m = 2^n \), from the inequality (33), we have

\[
\sum_{i=0}^{m/2-1} [T_q^a(\rho_{P|P_{j_i}})]^\beta + 2\sum_{i=m/2}^{m-1} [T_q^a(\rho_{P|P_{j_i}})]^\beta \leq \sum_{i=0}^{m-1} (2\beta - 1) \sum_{i=m/2}^{m-1} [T_q^a(\rho_{P|P_{j_i}})]^\beta \]

When \( m \) is an arbitrary number, we always can choose an \( n \in \mathbb{N}^+ \) such that \( 2^{n-1} \leq m \leq 2^n \). Then we choose a \( 2^n + 1 \) party quantum state in the set \( \mathcal{B} \),

\[
\gamma_{PP_j_0...P_{j_{m-1}}} = \rho_{PP_j_0...P_{j_{m-1}}} \otimes |0\rangle_{2^n-m} \langle 0|.
\]

Then due to the inequality (35), we have

\[
\sum_{i=0}^{m-1} [T_q^a(\gamma_{P|P_{j_i}})]^\beta \leq \sum_{i=0}^{m-1} (2\beta - 1)^{w_H}(T_q^a(\gamma_{P|P_{j_i}})]^\beta \leq \sum_{i=0}^{m-1} (2\beta - 1)^{w_H}(T_q^a(\gamma_{P|P_{j_i}})]^\beta
\]

From the definition of the state \( \gamma_{PP_j_0...P_{j_{2^n-1}}} \), we have

\[
T_q^a(\gamma_{P|P_{j_0...P_{j_{2^n-1}}}}) = T_q^a(\rho_{P|P_{j_0...P_{j_m-1}}})
\]

(37)

\[
T_q^a(\gamma_{P|P_{j_i}}) = T_q^a(\rho_{P|P_{j_i}}), i = 0, \ldots, m - 1,
\]

(38)

\[
T_q^a(\gamma_{P|P_{j_i}}) = 0, i = m, m + 1, \ldots, 2^n - 1,
\]

(39)
then we have

\[
[T_q^a(\gamma_{P|P_{j_0} \cdots P_{j_{2^n-1}}})]_{\beta}^\beta \\
= [T_q^a(\rho_{P|P_{j_0} \cdots P_{j_{m-1}}})]_{\beta}^\beta \\
\leq \sum_{i=0}^{2^n-1} (2^\beta - 1)^{w_H(\vec{j}_i)} [T_q^a(\gamma_{P|P_{j_i}})]_{\beta}^\beta \\
= \sum_{i=0}^{m-1} (2^\beta - 1)^{w_H(\vec{j}_i)} [T_q^a(\rho_{P|P_{j_i}})]_{\beta}^\beta
\]

\tag{40}

Proof. According to the lemma 5, we need to prove

\[
\sum_{i=0}^{m-1} (2^\beta - 1)^i [T_q^a(\rho_{P|P_{j_i}})]_{\beta}^\beta \\
\leq \sum_{i=0}^{m-1} (2^\beta - 1)^{w_H(\vec{j}_i)} [T_q^a(\gamma_{P|P_{j_i}})]_{\beta}^\beta,
\]

\tag{44}

Next we use the mathematical induction to prove the inequality (44). When \( m = 2 \), similar to the proof of the theorem 5, we see that the theorem is correct. When \( m \geq 2 \), due to the condition (43), we have

\[
0 \leq \sum_{k=1}^{m-1} \frac{T_q^a(\rho_{P|P_{j_k}})}{T_q^a(\rho_{P|P_{j_0}})} \leq 1,
\]

\tag{45}

In the proof of the theorem 5, as we assume that \( \beta \in [0, 1] \), then

**Corollary 3.** Let \( \beta \in [0, 1] \) and \( \rho_{PP_{j_0} \cdots P_{j_{m-1}}} \) is a reduced density matrix of a GW state \( |\psi_{PP_{1} \cdots P_{n-1}}\rangle \), when \( q = 2 \), then we have

\[
[T_q^a(\rho_{P|P_{j_0} \cdots P_{j_{m-1}}})]_{\beta}^\beta \leq \sum_{i=0}^{m-1} [T_q^a(\rho_{P|P_{j_i}})]_{\beta}^\beta
\]

\tag{41}

At last, we present a tighter polygamy relation in terms of TqEEoA for GW states under some conditions we present.

**Theorem 6.** When \( q = 2 \), let \( \beta \in [0, 1] \) and \( \rho_{PP_{j_0} \cdots P_{j_{m-1}}} \) is a reduced density matrix of a GWV state \( |\psi_{AA_{1} \cdots A_{n-1}}\rangle \), then

\[
T_q^a(\rho_{P|P_{j_i}}) \geq \sum_{k=i+1}^{m-1} T_q^a(\rho_{P|P_{j_k}}),
\]

\tag{42}

we have

\[
[T_q^a(\rho_{P|P_{j_0} \cdots P_{j_{m-1}}})]_{\beta}^\beta \leq \sum_{i=0}^{m-1} (2^\beta - 1)^i [T_q^a(\rho_{P|P_{j_i}})]_{\beta}^\beta.
\]

\tag{43}

**Proof.** According to the lemma 5, we need to prove

\[
\sum_{i=0}^{m-1} [T_q^a(\rho_{P|P_{j_i}})]_{\beta}^\beta \leq \sum_{i=0}^{m-1} (2^\beta - 1)^i [T_q^a(\rho_{P|P_{j_i}})]_{\beta}^\beta.
\]

\tag{44}

When \( m = 2 \), similar to the proof of the theorem 5, we see that the theorem is correct. When \( m \geq 2 \), due to the condition (43), we have

\[
0 \leq \sum_{k=1}^{m-1} \frac{T_q^a(\rho_{P|P_{j_k}})}{T_q^a(\rho_{P|P_{j_0}})} \leq 1,
\]

\tag{45}
then we have

\[
[T_q^a(\rho_{P|P_{j0}}\cdots P_{jm-1})]^\beta \\
\leq [T_q^a(\rho_{P|P_{j0}})]^\beta [1 + (2^\beta - 1)(\sum_{k=1}^{m-1} \frac{T_q^a(\rho_{P|P_{jk}})}{T_q^a(\rho_{P|P_{j0}})})]^\beta \\
=[T_q^a(\rho_{P|P_{j0}})]^\beta + (2^\beta - 1)(\sum_{k=1}^{m-1} T_q^a(\rho_{P|P_{jk}}))^\beta ,
\] (46)

At last, due to the mathematical induction, we have,

\[
\left( \sum_{k=1}^{m-1} T_q^a(\rho_{P|P_{jk}}) \right)^\beta \leq \sum_{k=1}^{m-1} (2^\beta - 1)^{k-1} [T_q^a(\rho_{P|P_{jk}})]^\beta ,
\] (47)

combing the inequality (47) and (48), we finish the proof. \hfill \Box

IV. CONCLUSION

In this article, we investigate the general monogamy inequalities for the GW states in terms of TqEE. First we present an analytical formula for the TqEE of the reduced density matrix of the GW state in terms of any partitions, then we present a monogamy inequality in terms of the squared TqEE for the reduced density matrices of the GW states, we also present a polygamy inequality in terms of the TqEE for the reduced density matrices of the GW states. At last, we present a tighter polygamy inequality in terms of TqEEoA for the reduced density matrix of GWV states. These results are meaningful as the GW states are in arbitrary n-qudit systems. Due to the importance of the study on the higher dimensional multipartite entanglement systems, and there are few results that the monogamy relations are valid for higher dimensional systems, our results can provide provide a reference for future work on the study of multiparty quantum entanglement.

[1] R. Horodecki, M. Horodecki, K. Horodecki. Rev. Mod. Phys. 81, 865 (2009).
[2] B. M. Terhal. IBM J. Res. Dev 48, 71 (2004).
[3] X. S. Ma, B. Dakic, W. Naylor, A. Zeilinger and P. Walther. Nat. Phys. 7, 399 (2011).
[4] L. Masanes. Phys. Rev. Lett. 102, 140501 (2009).
[5] J. M. Renes and M. Grassl. Phys. Rev. A 74, 022317 (2006).
[6] V. Coffman, J. Kundu, and W. K. Wootters. Phys. Rev. A. 61, 052306 (2000).
[7] T. J. Osborne and F. Verstraete. Phys. Rev. Lett. 96, 220503 (2006).
[8] J. S. Kim. Phys. Rev. A 81, 062328 (2010).
[9] Y. Luo, T Tian, L. H. Shao and Y. M. Li. Phys. Rev. A 93, 062340 (2016).
[10] J. S. Kim and B. C. Sanders. J. Phys. A: Math. Theor. 43, 442305 (2010).
[11] J. S. Kim and B. C. Sanders. J. Phys. A: Math. Theor. 44, 295303 (2011).
[12] B. Regula, S. Di Martino, S. Lee and G. Adesso. Phys. Rev. Lett. 113, 110501 (2014).
[13] Y. C. Ou. Phys. Rev. A 75, 034305 (2007).
[14] C. Lancien, S. D. Martino, M. Huber, M. Piani, G. Adesso and A. Winter. Phys. Rev. Lett. 117, 060501 (2016).
[15] M. Christandll and A. Winter. J. Math. Phys. (NY) 45, 829 (2004).
[16] J. S. Kim and B. C. Sanders. J. Phys. A: Math. Theor. 41, 495301 (2008).
[17] J. H. Choi and J. S. Kim. Phys. Rev. A 92, 042307 (2015).
[18] J. S. Kim. Phys. Rev. A 93, 032331 (2016).
[19] R. Rossignoli and N. Canosa. Phys. Rev. A 66, 042306 (2006).
[20] Anantha S Nayak, Sudha, A. R. Usha Devi, A. K. Rajagopal. Quantum Inf. Process 16, 51 (2017).
[21] C. D. Pyo, K. J. San and K. Lee. Phys. Rev. A. 87, 062339 (2013).
[22] M. Wajs, P. Kurzynski and D. Kaszlikowski. Phys. Rev. A. 91, 012114 (2014).
[23] W. K. Wootters. Phys. Rev. Lett. 80, 2245 (1998).
[24] J. S. Kim and B. C. Sanders. J. Phys. A: Math. Theor. 43, 445305 (2010).
[25] W. Song, Y.-K. Bai, M. Yang, M. Yang and Z.-L. Cao. Phys. Rev. A 93, 022306 (2016).
[26] Xian Shi. Int. J. Theo. Phys. 57, 3056 (2018).
[27] Z. X. Jin and S. M. Fei. Quantum Inf. Process. 16, 77 (2018).
[28] J. S. Kim. Phys. Rev. A 97, 042332 (2018).