ON THE GREY BAKER-THOMPSON RULE

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Abstract. Cost sharing problems can arise from situations in which some service is provided to a variety of different customers who differ in the amount or type of service they need. One can think of and airports computers, telephones. This paper studies an airport problem which is concerned with the cost sharing of an airstrip between airplanes assuming that one airstrip is sufficient to serve all airplanes. Each airplane needs an airstrip whose length can be different across airplanes. Also, it is important how should the cost of each airstrip be shared among airplanes. The purpose of the present paper is to give an axiomatic characterization of the Baker-Thompson rule by using grey calculus. Further, it is shown that each of our main axioms (population fairness, smallest-cost consistency and balanced population impact) together with various combinations of our minor axioms characterizes the best-known rule for the problem, namely the Baker-Thompson rule. Finally, it is demonstrated that the grey Shapley value of airport game and the grey Baker-Thompson rule coincides.

1. Introduction. The grey systems theory is a new methodology that focuses on the study of problems involving small samples and poor information [9]. It deals with uncertain systems with partially known information through generating, excavating, and extracting useful information from what is available. So, systems’ operational behaviors and their laws of evolution can be correctly described and effectively monitored. In the natural world, uncertain systems with small samples and poor information exist commonly. That fact determines the wide range of applicability of grey systems theory. Moreover, grey systems and information are

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a powerful theory capable of bringing forward practically beneficial impacts to the advancement of the human society [10].

In these years, “uncertainty” is entering more and more in all areas of natural sciences, engineering, Operations Research (OR), economics and finance, as subject of consideration, of modeling, of control and optimization [4, 7, 8, 18, 12, 13, 20, 22, 25, 27, 30]. [1] gives an axiomatic characterization of the Baker Thompson rule with interval uncertainty. [14] considers dynamical gene-environment networks under ellipsoidal uncertainty which is a kind of uncertainty. [29] surveys and mathematically expands recent advances in modelling and prediction the environment and aspects of errors by using uncertainty data. Finally, [3] presents and identify the interval Baker-Thompson rule for solving the aircraft fee problem of an airport with one runway when there is uncertainty regarding the costs of the pieces of the runway. What in the past was regarded as a matter left alone to the “soft” human and social sciences, now enters core areas of “hard” research, of computation and calibration. This has been transforming the view on uncertainty, supported by approaches such as uncertainty quantization, interval uncertainty, grey uncertainty, ellipsoidal uncertainty, robust counterparts of optimization and of stochastic optimal control, e.g. The fast dynamics and growth has just really started.

Airport situations have paid much attention in the literature ([16]). In these situations, we focus on the appealing rule which is Baker-Thompson rule introduced by [6] and [28]. This rule provides a fair and easy share for the costs of the landings. In [15] showed that the Baker-Thompson rule coincides with the Shapley value.

The main contribution of this paper to give another characterization of the grey Baker-Thompson rule. Our intuition is from [11] who study the axiomatic characterization of the classical Baker-Thompson rule and [24] which introduce the grey Baker-Thompson rule. In this study, we show that each of our major axioms together with various combinations of our minor axioms characterizes the best-known rule for the problem, namely the Baker-Thompson rule. This rule requires that all airplanes using a given section of the airstrip contribute equally to the cost of this section. Each airplane’s contribution is the sum of terms, one for each section of the airstrip that it uses.

The rest of the paper is organized as follows. We recall in Section 2 basic notions and facts from airport situations and the theory of cooperative grey games. Section 3 is devoted to the grey Baker-Thompson rule and the airport grey games. In Section 4, we give the properties of the grey Baker-Thompson rule. In Section 5, we give an axiomatic characterization of this rule by using minor and major axioms. We conclude in Section 6 with some final remarks.

The flowchart of this study is shown in Figure 1.

2. Preliminaries. In this section, we give some preliminaries from cooperative game theory and grey calculus.

A cooperative cost game in coalitional form is an ordered pair \(< N, c >\), where \(N = \{1, 2, ..., n\}\) is the set of players, and \(c : 2^N \rightarrow \mathbb{R}\) is a map, assigning to each coalition \(S \in 2^N\) a real number, such that \(c(\emptyset) = 0\). This function \(c\) is called the characteristic function of the game and \(c(S)\) is called the worth (or value) of coalition \(S\). We identify cooperative cost game \(< N, c >\) with its characteristic function \(c\). The set of coalitional games with player set \(N\) is denoted \(G^N\) [26].

The set \(G^N\) of coalitional games with player set \(N\) forms with the usual operators of addition and scalar multiplication of functions a \((2^{(N-1)})\)-dimensional linear
space; a basis of this space is supplied by the dual unanimity games based \( u^*_T \) on \( T \).

\[
    u^*_T = \begin{cases} 
    1, & i \in T \\
    0, & i \in N \setminus T.
    \end{cases}
\]

We recall the definition of the cooperative interval game and the length game [2]. A cooperative interval game is an ordered pair \( <N, w> \) where \( N = \{1, \ldots, n\} \) is the set of players, and \( w : 2^N \to I(\mathbb{R}) \) is the characteristic function such that \( w(\emptyset) = [0, 0] \), where \( I(\mathbb{R}) \) is the set of all nonempty, compact intervals in \( \mathbb{R} \). For each \( S \in 2^N \), the worth set (or worth interval) \( w(S) \) of the coalition \( S \) in the interval game \( <N, w> \) is of the form \([w(S), \overline{w}(S)]\), where \( w(S) \) is the minimal reward which coalition \( S \) could receive on its own and \( \overline{w}(S) \) is the maximal reward which coalition \( S \) could get. The family of all interval games with player set \( N \) is denoted by \( IG^N \).
A grey number is such a number whose exact value is known but a range within that the value lies is known. In applications, a grey number in general is an interval or a general set of numbers.

In this paper, we consider the interval grey numbers.

A grey number with both a lower limit \( a \) and an upper limit \( \bar{a} \) is called an interval grey number, denoted as \( \otimes \in [a, \bar{a}] \).

For example, the weight of a polar bear is between 800 and 900 kg. A plane-tree’s height is between 30 and 50 meters. These two grey numbers can be respectively written as

\[ \otimes_1 \in [800, 900] \quad \text{and} \quad \otimes_2 \in [30, 50]. \]

Now, we discuss various operations on interval grey numbers.

Let

\[ \otimes_1 \in [a, b], \quad a < b \quad \text{and} \quad \otimes_2 \in [c, d], \quad c < d. \]

The sum of \( \otimes_1 \) and \( \otimes_2 \), written \( \otimes_1 + \otimes_2 \), is defined as follows:

\[ \otimes_1 + \otimes_2 \in [a + c, b + d]. \]

For example, let \( \otimes_1 \in [3, 4] \) and \( \otimes_2 \in [5, 8] \), then,

\[ \otimes_1 + \otimes_2 \in [8, 12]. \]

Assume that \( \otimes \in [a, b] \), \( a < b \), and \( k \) is a positive real number. The scalar multiplication of \( k \) and \( \otimes \) is defined as follows:

\[ k \otimes \in [ka, kb]. \]

We denote by \( G(\mathbb{R}) \) the set of interval grey numbers in \( \mathbb{R} \). Let \( \otimes_1, \otimes_2 \in G(\mathbb{R}) \), \( |\otimes_1| = b - a \) and \( \alpha \in \mathbb{R}_+ \). Then,

(i) \( \otimes_1 + \otimes_2 \in [a + c, b + d] \);  
(ii) \( \alpha \otimes = [\alpha a, \alpha b] \).

By (i) and (ii) we see that \( G(\mathbb{R}) \) has a cone structure \( [9, 17] \).

A cooperative grey game is an ordered pair \( < N, c' > \) where \( N = \{1, \ldots, n\} \) is the set of players, and \( c' : 2^N \rightarrow G(\mathbb{R}) \) is the characteristic function such that \( c'() \in [0,0] \), grey payoff function \( c'(S) \in [A_S, \overline{A}_S] \) refers to the value of the grey expectation benefit belonging to a coalition \( S \in 2^N \), where \( \overline{A}_S \) and \( \overline{A}_S \) represent the maximum and minimum possible profits of the coalition \( S \). So, a cooperative grey game can be considered as a classical cooperative game with grey profits \( c' \). Grey solutions are useful to solve reward/cost sharing problems with grey data using cooperative grey games as a tool. Building blocks for grey solutions are grey payoff vectors, i.e., vectors whose components belong to \( G(\mathbb{R}) \). We denote by \( G(\mathbb{R}) \) the set of all such grey payoff vectors. We denote by \( G^N \) the family of all cooperative grey games.

We recall that the definition of the grey Shapley value and the properties of the grey Shapley value.

For \( c, c_1, c_2 \in IG^N \) and \( c_1', c_2' \in GG^N \) we say that \( c_1' \in c_1 \leq c_2' \in c_2 \) if \( c_1'(S) \leq c_2'(S) \) where \( c_1'(S) \in c_1(S) \) and \( c_2'(S) \in c_2(S) \) and, for each \( S \in 2^N \). For \( c_1', c_2' \in GG^N \) and \( \lambda \in \mathbb{R}_+ \) we define \( < N, c_1' + c_2' > \) and \( < N, \lambda c' > = (c_1' + c_2')(S) = c_1'(S) + c_2'(S) = (\lambda c')(S) = \lambda c'(S) \) for each \( S \in 2^N \). So, we conclude that \( G^N \) endowed with \( \leq \) is a partially ordered set and has a cone structure with respect to addition and multiplication with non-negative scalars described above. For \( c_1', c_2' \in GG^N \) where \( c_1' \in c_1 \), \( c_2' \in c_2 \) with \( |c_1(S)| \geq |c_2(S)| \) for each \( S \in 2^N \), \( < N, c_1' - c_2' > \) is defined by \( (c_1' - c_2')(S) = c_1'(S) - c_2'(S) \in c_1(S) - c_2(S) \).
We call a game \( < N, c' > \) grey size monotonic if \( < N, |c| > \) is monotonic, i.e., \(|c|(S) \leq |c|(T)\) for all \( S, T \in 2^N \) with \( S \subset T \). For further use we denote by \( SMGG^N \) the class of all grey size monotonic games with player set \( N \).

The grey marginal operators and the grey Shapley value are defined on \( SMGG^N \). Denote by \( \Pi(N) \) the set of permutations \( \sigma : N \rightarrow N \) of \( N \). The grey marginal operator \( m^\sigma : SMGG^N \rightarrow G(\mathbb{R})^N \) corresponding to \( \sigma \), associates with each \( c' \in SMGG^N \) the grey marginal vector \( m^\sigma(c') \) of \( c' \) with respect to \( \sigma \) defined by
\[
m^\sigma_i(c') = c'(P^\sigma(i) \cup \{i\}) - c'(P^\sigma(i)) \in [A^\sigma_{P^\sigma(i) \cup \{i\}}, A^\sigma_{P^\sigma(i)} - A^\sigma_{P^\sigma(i) \cup \{i\}}]
\]
for each \( i \in N \), where
\[
P^\sigma(i) = \{ r \in N \mid \sigma^{-1}(r) < \sigma^{-1}(i) \}
\]
and \( \sigma^{-1}(i) \) denotes the entrance number of player \( i \). For grey size monotonic games \( < N, c' > \), \( c'(T) - c'(S) \in c(T) - c(S) \) is defined for all \( S, T \in 2^N \) with \( S \subset T \) since \(|c(T)| = |c|(T) \geq |c|(S) = |c|(S)|. Now, we notice that for each \( c' \in SMGG^N \) the grey marginal vectors \( m^\sigma(c') \) are defined for each \( \sigma \in \Pi(N) \), because the monotonicity of \(|c| \) implies \( A^\sigma_{S \cup \{i\}} - A^\sigma_{S \cup \{i\}} \geq A^\sigma_S - A^\sigma_S \), which can be rewritten as \( A^\sigma_{S \cup \{i\}} - A^\sigma_S \geq A^\sigma_{S \cup \{i\}} - A^\sigma_S \). So, \( c'(S \cup \{i\}) - c'(S) \in c(S \cup \{i\}) - c(S) \) is defined for each \( S \subset N \) and \( i \notin S \). We notice that all the grey marginal vectors of a grey size monotonic game are efficient grey payoff vectors.

The grey Shapley value \( \Phi' : SMGG^N \rightarrow G(\mathbb{R})^N \) is defined by
\[
\Phi'(c') := \frac{1}{m!} \sum_{\sigma \in \Pi(N)} m^\sigma(c') \in [\frac{1}{m!} \sum_{\sigma \in \Pi(N)} m^\sigma(A), \frac{1}{m!} \sum_{\sigma \in \Pi(N)} m^\sigma(\overline{A})],
\]
for each \( c' \in SMGG^N \) \([21]\).

3. Airport grey situations and their Baker-Thompson rule. This section is based on airport grey games and their Baker-Thompson rule.

In the sequel we first recall the classical airport situations. We consider the aircraft fee problem of an airport with one runway and suppose that the planes which are to land are classified into \( m \) types. For each \( 1 \leq j \leq m \), denote the set of landings of planes of type \( j \) by \( N_j \) and its cardinality by \( n_j \). Then \( N = \bigcup_{j=1}^m N_j \) represents the set of all landings. Let \( c_j \) represent the cost of a runway adequate for planes of type \( j \). We assume that the types are ordered such that \( 0 = c_0 < c_1 < \ldots < c_m \). We consider the runway divided into \( m \) consecutive pieces \( P_j, 1 \leq j \leq m \), where \( P_1 \) is adequate for landings of planes of type \( 1 \); \( P_1 \) and \( P_2 \) together for landings of planes of type \( 2 \), and so on. The cost of piece \( P_j, 1 \leq j \leq m \), is the marginal cost \( c_j - c_{j-1} \). According to this the Baker-Thompson rule is given by
\[
\beta_i = \frac{\sum_{k=1}^j (\sum_{r=k}^m n_r)^{-1} (c_k - c_{k-1})}{\sum_{k=1}^m (\sum_{r=k}^m n_r)^{-1} (c_k - c_{k-1})} \text{ whenever } i \in N_j.
\]
That is, every landing of planes of type \( j \) contributes to the cost of the pieces \( P_k, 1 \leq k \leq j \), equally allocated among its users \( \bigcup_{r=k}^m N_r \).

In this section, we consider airport situations where cost of pieces of the runway are interval grey numbers. Then, we associate as in the classical case to such a situation a grey game and extend to airport grey games the results presented above.

Let \( G \in G(\mathbb{R}_+), T \setminus \{\emptyset\} \), and let \( u_+^G : 2^N \rightarrow \mathbb{R} \) be the classical dual unanimity game based on \( T \). Here, the grey game \( < N, Gu^r_+ > \) defined by \( (Gu^r_+)(S) := u^r_+(S)G \) for each \( S \in 2^N \) will play an important rule. We notice that \( \Phi'(Gu^r_+) \) for the grey
game \( \langle N, G^T \rangle \) is related with the Shapley value \( \Phi(u^*_T) \) of the classical game \( \langle N, G^T \rangle \) as follows:

\[
\Phi'(G^T) = \Phi_i(u^*_T) = \Phi_i(u^*_T) \quad < T \quad G \in \left\{ \begin{array}{ll} \mathbb{G}/|T|, & i \in T \\ [0,0], & i \in N \setminus T \end{array} \right. 
\]  

(2)

Consider the aircraft fee problem of an airport with one runway. Suppose that the planes which are to land are classified into \( m \) types. For each \( 1 \leq j \leq m \), denote the set of landings of planes of type \( j \) by \( N_j \) and its cardinality by \( n_j \). Then \( N = \bigcup_{j=1}^m N_j \) represents the set of all landings. Consider the runway divided into \( m \) consecutive pieces \( P_j, 1 \leq j \leq m \), where \( P_1 \) is adequate for landings of planes of type 1; \( P_1 \) and \( P_2 \) together for landings of planes of type 2, and so on. Let the grey \( T_j \) with non-negative finite bounds represent the grey cost of piece \( P_j \), \( 1 \leq j \leq m \).

Next we propose a grey allocation rule \( \beta \), which we call the grey Baker-Thompson rule. For a given airport grey situation \( \langle N, (T_k)_{k=1,...,m} \rangle \) the Baker-Thompson allocation for each player \( i \in N_j \) is given by:

\[
\beta_i = \sum_{k=1}^j \sum_{r=k}^m n_r^{-1} T_k
\]  

(3)

Note that for the piece of \( P_k \) of the runway the users are \( \bigcup_{r=k}^m N_r \). So, \( \sum_{r=k}^m n_r^{-1} T_k \) is the equal grey cost share of each user of the piece \( P_k \). This means that a player \( i \in N_j \) contributes to the cost of the pieces \( P_1, ..., P_j \).

The characteristic cost function \( c' \) of the airport grey game \( \langle N, c' \rangle \) is given by \( c'(\emptyset) \in [0,0] \) and \( c'(S) \in \sum_{k=1}^j T_k \) for all coalitions \( S \subset N \) satisifying \( S \cap N_j \neq \emptyset \) and \( S \cap N_k \neq \emptyset \) for all \( j + 1 \leq k \leq m \). Now, we give the definition of the airport grey game as follows:

\[
c' \in \sum_{k=1}^m T_k u^*_N.
\]  

(4)

Here, \( N^* = \bigcup_{r=k}^m N_r \). In the following theorem, we show that the grey Baker-Thompson allocation for the airport situation with grey data coincides with grey Shapley value of the corresponding airport grey game.

**Theorem 3.1.** Let \( \langle N, c' \rangle \) be an airport grey game. Then, the grey allocation \( \beta \) agrees with the grey Shapley value \( \Phi'(c') \).

**Proof.** For \( i \in N_j \) we have

\[
\Phi_i(c') = \Phi_i(\sum_{k=1}^m T_k u^*_N) = \sum_{k=1}^m \Phi_i(T_k u^*_N) \quad \text{by additivity of } \Phi'
\]  

(5)

\[
= \sum_{k=1}^j (\sum_{r=k}^m n_r^{-1} T_k = \beta_i
\]  

(6)

This is what we wanted to show. \( \square \)

Now, we reconsider on the grey Baker-Thompson rule. Consider the aircraft fee problem of an airport with one runway. There is a universe of “potential” airplanes, denoted by \( I \subseteq \mathbb{N} \) where \( \mathbb{N} \) is the set of natural numbers. Let \( \mathcal{N} \) be the class of
non-empty and finite subsets of \( I \). Given \( N \in \mathcal{N} \) and \( i \in N \), let \( T_i \in \mathcal{G}(\mathbb{R}_+) \) be airplane \( i \)'s grey cost, and \( \mathcal{T} = (T_i)_{i \in N} \) the grey costs vector. An airport grey problem for \( N \) is a list \( \mathcal{T} \in \mathcal{G}(\mathbb{R}_+^N) \). Let \( \mathcal{C}^N \) be the class of all problems for \( N \). A grey contributions vector for \( \mathcal{T} \in \mathcal{C}^N \) is a vector \( \mathcal{G} \in \mathcal{G}(\mathbb{R}_+^N) \). Let \( \mathcal{G}(\mathcal{T}) \) be the set of all grey contributions vectors for \( \mathcal{T} \in \mathcal{C}^N \). A grey rule is a function defined on \( \bigcup_{N \in \mathcal{N}} \mathcal{C}^N \) that associates with each \( N \in \mathcal{N} \) and each \( \mathcal{T} \in \mathcal{C}^N \) a vector in \( \mathcal{G}(\mathcal{T}) \).

For all \( N \in \mathcal{N} \) and \( T \in \mathcal{C}^N \), the grey Shapley value by using Theorem 3.1.

Let \( (N, (T_k)_{k=1,...,m}) \) be an airport grey situation \( (N, (T_k)_{k=1,...,m}) \) be an airport grey situation and grey Baker-Thompson rule by using the upper bounds of the grey costs. These grey Baker-Thompson rule by using the lower bounds of the grey costs; and the upper bound of the grey Baker-Thompson rule by using the upper bounds of the grey costs. These calculations can be done easily by using the classical Baker-Thompson rule.

**Example 3.3.** Let \( (N = \{1, 2, 3\}, (T_k)_{k=1,2,3}) \) be an airport grey situation with the grey costs \( T_1 \in [30, 36] \), \( T_2 \in [44, 54] \), \( T_3 \in [60, 80] \). Then,
\[
\begin{align*}
\beta_1 &= \frac{T_1}{3} = 10 \\
\beta_2 &= \frac{T_1}{3} + \frac{T_2 - T_1}{3 - 1} = 17 \\
\beta_3 &= \frac{T_1}{3} + \frac{T_2 - T_1}{3 - 1} + \frac{T_3 - T_2}{3 - 2} = 33 \\
\end{align*}
\]

\[\bar{\beta} = (10, 17, 33)\] and

\[
\begin{align*}
\bar{\beta}_1 &= \frac{T_1}{3} = 12 \\
\bar{\beta}_2 &= \frac{T_1}{3} + \frac{T_2 - T_1}{3 - 1} = 21 \\
\bar{\beta}_3 &= \frac{T_1}{3} + \frac{T_2 - T_1}{3 - 1} + \frac{T_3 - T_2}{3 - 2} = 47 \\
\end{align*}
\]

\[\bar{\beta} = (12, 21, 47)\] and by Theorem 3.1, \(\beta_i \in ([10, 12], [17, 21], [33, 47]).\)

Let \(\langle N, c' \rangle\) be a three-person airport grey game corresponding to the airport grey situation. The grey costs of the pieces are given by \(T_1 \in [30, 36], T_2 \in [44, 54], T_3 \in [60, 80].\) Then,

\[
\begin{align*}
&c'(\emptyset) \in [0, 0], \\
&c'(1) \in [30, 36], \\
&c'(2) = c'(12) \in [44, 54], \\
&c'(3) = c'(13) = c'(23) = c'(N) \in [60, 80].
\end{align*}
\]

Then the grey marginal vectors are given in the following table, where \(\sigma : N \to N\) is identified with \((\sigma(1), \sigma(2), \sigma(3))\). Firstly, for \(\sigma_1 = (1, 2, 3)\), we calculate the grey marginal vectors. Then,

\[
\begin{align*}
&m_1^2(\sigma)(c') = c'(1) \in [30, 36], \\
&m_2^2(\sigma)(c') = c'(12) - c'(1) \in [44, 54] - [30, 36] = [14, 18], \\
&m_3^2(\sigma)(c') = c'(123) - c'(12) \in [60, 80] - [44, 54] = [16, 26].
\end{align*}
\]

The others can be calculated similarly, which is shown in Table 1.

\[
\begin{array}{|c|c|c|c|}
\hline
\sigma & m_1^2(\sigma)(c') & m_2^2(\sigma)(c') & m_3^2(\sigma)(c') \\
\hline
\sigma_1 = (1, 2, 3) & [30, 36] & [14, 18] & [16, 26] \\
\sigma_2 = (1, 3, 2) & [30, 36] & [0, 0] & [30, 44] \\
\sigma_3 = (2, 1, 3) & [0, 0] & [44, 54] & [16, 26] \\
\sigma_4 = (2, 3, 1) & [0, 0] & [44, 54] & [16, 26] \\
\sigma_5 = (3, 1, 2) & [0, 0] & [0, 0] & [60, 80] \\
\sigma_6 = (3, 2, 1) & [0, 0] & [0, 0] & [60, 80] \\
\hline
\end{array}
\]
Table 1 illustrates the grey marginal vectors of the cooperative grey game in Example 3.3. The average of the six grey marginal vectors is the grey Shapley value of this game which can be shown as:

$$\Phi'(c') \in ([10, 12], [17, 21], [33, 47]).$$

It is easy to show that the grey Baker-Thompson rule coincides with the grey Shapley value. That is;

$$\beta_i (c') = \Phi'(c') \in ([10, 12], [17, 21], [33, 47]).$$

4. The properties of the grey Baker-Thompson rule. We define a grey allocation rule for a given airport grey situation $$(N, (T_k)_{k=1,...,m})$$ as a map $$\mathcal{F}$$ associating each allocation situation $$(N, (T_k)_{k=1,...,m})$$ to a unique rule $\mathcal{F}(N, (T_k)_{k=1,...,m}) = \mathcal{F}(N, ([T_k, \bar{T}_k])_{k=1,...,m}) \in \mathbb{R}^N$$ with

$$\sum_{i \in N} \mathcal{F}_i (N, (T_k)_{k=1,...,m}) = \sum_{i=1}^m [T_i, \bar{T}_i] = \sum_{i=1}^m T_i.$$ 

In this context, we give some properties the major and minor axioms on the airport interval problems. The major axioms are:

**Grey population fairness (GPF):** For all $$N \in \mathcal{N},$$ all $$\mathcal{T} \in \mathcal{C}^N,$$ all $$i \in I \setminus N,$$ and all $$j, k \in N$$ such that $$\min \{ T_j, T_k \} \geq T_i,$$ we have $$\varphi_j (T_{N \cup \{i\}}) - \varphi_j (T) = \varphi_k (T_{N \cup \{i\}}) - \varphi_k (T).$$

Grey population fairness requires that upon the arrival of a new airplane, the grey contributions of all airplanes whose grey costs are not less than the grey cost of the new airplane should be affected by equal amounts.

**Grey smallest-cost consistency (GSCC):** For all $$N \in \mathcal{N},$$ all $$\mathcal{T} \in \mathcal{C}^N,$$ all $$N' \subset N$$ and all $$G \in \mathcal{G}(\mathbb{R}),$$ if $$G = \varphi (T),$$ then $$r_{N'} (T) \in \mathcal{C}^N$$ and $$x_{N'} = \varphi (r_{N'} (T)).$$

Grey smallest-cost consistency requires that upon the departure of an airplane with the smallest grey cost, if the problem is reevaluated from the viewpoint of the remaining airplanes, then all the remaining airplanes should contribute the same amounts as they did initially.

The minor axioms are:

**Grey Reasonableness (GR):** For all $$N \in \mathcal{N},$$ all $$\mathcal{T} \in \mathcal{C}^N,$$ and all $$i \in N,$$

$$[0, 0] \preceq \varphi_i (T) \preceq \bar{T}_i.$$ 

Grey reasonableness requires that each airplane should contribute a non-negative grey amount, but no more than its individual grey cost.

**Grey Efficiency (GE):** For all $$N \in \mathcal{N},$$ all $$\mathcal{T} \in \mathcal{C}^N,$$ $$\varphi_i (T) = T^{\max}.$$ 

Grey efficiency requires that the sum of all grey contributions should be equal to the largest grey cost.

**Grey Equal share lower bound (GESLB):** For all $$N \in \mathcal{N},$$ all $$\mathcal{T} \in \mathcal{C}^N,$$ and all $$i \in N,$$

$$\varphi_i (T) \geq \frac{T_i}{|N|}.$$ 

Grey equal share lower bound requires that each airplane should contribute at least as much as an equal division of its individual grey cost.

**Grey Cost monotonicity (GCM):** For all $$N \in \mathcal{N},$$ all $$\mathcal{T} \in \mathcal{C}^N,$$ all $$\mathcal{T}' \in \mathcal{C}^N,$$ and all $$i \in N,$$ if $$\mathcal{T}' \succ T_j$$ and for all $$j \in N \setminus \{i\}, T'_j = T_j,$$ then for all $$j \in N \setminus \{i\},$$

$$\varphi_j (T') \preceq \varphi_j (T).$$
Grey cost monotonicity requires that if an airplane’s grey cost increases, then all other airplanes should contribute at most as much as they did initially.

In [5] it is shown that the Baker-Thompson rule satisfies the classical properties above and they characterize the Baker-Thompson rule by using these classical properties. Our aim is to extend these results to the grey setting.

5. An axiomatic characterization of the grey Baker-Thompson rule. We give the axiomatic characterizations of the grey Baker-Thompson rule by using the major and the minor axioms. The first characterization is given by using \( \mathcal{G}_{PF} \), \( \mathcal{G}_{E} \), \( \mathcal{G}_{ESLB} \), \( \mathcal{G}_{CM} \). The second characterization is given by using \( \mathcal{G}_{SCC} \), \( \mathcal{G}_{R} \), \( \mathcal{G}_{ESLB} \), \( \mathcal{G}_{CM} \). Finally, the third characterization is given by using \( \mathcal{G}_{BPI} \), \( \mathcal{G}_{E} \).

The following lemma is required for the first characterization.

**Lemma 5.1.** For all \( N \in \mathcal{N} \), all \( T \in \mathcal{C}_N \), if a grey rule \( \varphi \) satisfies the grey efficiency, the grey equal share lower bound, and grey cost monotonicity, then

\[
\varphi_{\eta^{-1}(1)}(T) \in \left[ \frac{T_{\eta^{-1}(1)}}{n}, \frac{T_{\eta^{-1}(1)}}{n} \right].
\]

**Proof.** The proof is a straightforward generalization from the classical case and can be obtained by following the steps of Lemma 1 in [5].

Now, we give our first characterization.

**Theorem 5.2.** The grey Baker-Thompson rule is the only rule satisfying \( \mathcal{G}_{E} \), \( \mathcal{G}_{ESLB} \), \( \mathcal{G}_{CM} \) and \( \mathcal{G}_{PF} \).

**Proof.** It is obvious that the grey Baker-Thompson rule satisfies the four axioms. We only need to show the uniqueness. For uniqueness, it is clear from [5] that \( \beta_i \) and \( \overline{\beta}_i \) for each \( i \in N \) are the unique allocations satisfying the four properties \( \mathcal{G}_{E} \), \( \mathcal{G}_{ESLB} \), \( \mathcal{G}_{CM} \) and \( \mathcal{G}_{PF} \). Finally, by Theorem 2.1 we conclude that \( \beta_i = [\beta_i, \overline{\beta}_i] \) for each \( i \in N \) is unique. Hence \( \beta_i \) is the unique grey allocation satisfying \( \mathcal{G}_{E} \), \( \mathcal{G}_{ESLB} \), \( \mathcal{G}_{CM} \) and \( \mathcal{G}_{PF} \).

Next we present our second characterization of the grey Baker-Thompson rule:

**Theorem 5.3.** The grey Baker-Thompson rule is the only rule satisfying \( \mathcal{G}_{R} \), \( \mathcal{G}_{ESLB} \), \( \mathcal{G}_{CM} \) and \( \mathcal{G}_{SCC} \).

**Proof.** It is obvious that the grey Baker-Thompson rule satisfies the four axioms. We only need to show the uniqueness. For uniqueness, it is clear from [5] that \( \beta_i \) and \( \overline{\beta}_i \) for each \( i \in N \) are the unique allocations satisfying the four properties \( \mathcal{G}_{R} \), \( \mathcal{G}_{ESLB} \), \( \mathcal{G}_{CM} \) and \( \mathcal{G}_{SCC} \). Finally, by Theorem 2.1 we conclude that \( \beta_i = [\beta_i, \overline{\beta}_i] \) for each \( i \in N \) is unique. Hence \( \beta_i \) is the unique grey allocation satisfying \( \mathcal{G}_{R} \), \( \mathcal{G}_{ESLB} \), \( \mathcal{G}_{CM} \) and \( \mathcal{G}_{SCC} \).

Finally, we give another characterization of grey Baker-Thompson rule. If an airplane leaves an airport problem, this will typically affect the contributions of other remaining airplanes. Next, we require that the effect of airplane \( i \) leaving on the contribution of another airplane \( j = i \) should be equal to the effect of airplane \( j \) leaving on the contribution of airplane \( i \).

**Grey balanced population impact (\( \mathcal{G}_{BPI} \)):** For all \( N \in \mathcal{N} \), all \( T \in \mathcal{C}_N \), all \( i, j \in N \),

\[
\varphi_i(T) - \varphi_i(T_{N\setminus\{i\}}) = \varphi_j(T) - \varphi_j(T_{N\setminus\{i\}}).
\]
This axiom was introduced for TU games by [19] under the name of “balanced contributions.” For airport problems, the sequential contributions rule satisfies it. Moreover, it can be characterized by efficiency and balanced population impact, as in the Myerson’s characterization of the Shapley value.

**Theorem 5.4.** The grey Baker-Thompson rule is the only rule satisfying $GE$ and $GBPI$.

**Proof.** The proof can be done by using a similar argument.

6. **Conclusion.** In real life situations, airport operational costs related aircraft fees are not known exactly. Therefore, we address grey airport games aspect of major and minor axioms characterizations on the grey Baker-Thompson rule. Particularly, this paper shows that cooperative grey game theory can help us to provide a fair and effective share for the costs of the landings by using Baker-Thompson rule and its axioantic characterizations.

In this paper, we study on airport situations and related games under grey uncertainty. Grey uncertainty is a natural type of uncertainty which may influence cooperation between players. We axiomatically characterized the grey Baker-Thompson rule and show that one of our major axioms (population fairness, smallest-cost consistency and balanced population impact) together with various combinations of our minor axioms characterizes the best-known rule for the problem, namely the Baker-Thompson rule.

Moreover, we show that the grey Shapley value and the grey Baker-Thompson rule coincides. Other characterizations can be applied to different situations by using grey uncertainty. Other OR situations and combinatorial optimization problems with grey data among which flow situations, linear production situations and inventory situations can give rise to interesting grey games. We notice that the grey Baker-Thompson rule is useful to informs users about what they can expect to pay for the construction of the runway.

We aim to associate our study with further forms of uncertainty quantification and their dynamics, and, within Operations Research, combine our contributions to game theory with emerging modeling and decision making methods of optimization theory. It should be noted that, optimization techniques embodies a vast and significant area of science that interfaces many related subjects. Included among these are linear programming, Operations Research and game theory. The scientific community is invited to future research in compromised decision making, as represented by collaborative game theory, under uncertainty and in time dependence. Here, optimization theory will be a grey technology in the needed stages of data mining, modeling, decision making and optimal control.

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