Technical Report: Observability of a Linear System under Sparsity Constraints
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Abstract—Consider an $n$-dimensional linear system where it is known that there are at most $k < n$ non-zero components in the initial state. The observability problem, that is the recovery of the initial state for such a system is considered. We obtain sufficient conditions on the number of the available observations to be able to recover the initial state exactly for such a system. Both deterministic and stochastic setups are considered for system dynamics. In the former setting, the system matrices are known deterministically, whereas in the latter setting, all of the matrices are picked from a randomized class of matrices. The main message is that, one does not need to obtain full $n$ observations to be able to uniquely identify the initial state of the linear system, even when the observations are picked randomly, when the initial condition is known to be sparse.

I. INTRODUCTION

A linear system of dimension $n$ is said to be observable if an ensemble of at most $n$ successive observations guarantee the recovery of the initial state. Observability is an essential notion in control theory as, with the sister notion of controllability, these form the essence of modern linear control theory.

In this paper, we consider the observability problem when the number of non-zeros in the initial state in a linear system is strictly less than the dimension of the system. This might arise in systems where natural or external forces give rise to a certain subset of components of a linear system to be activated or excited, for example an external force may give rise to a subset of locally unstable states while keeping certain other states intact.

Furthermore, with the increasing emphasis on networked control systems, it has been realized that the controllability and observability concepts for linear systems with controllers having full access to sensory information is not practical. Many research efforts have focused on both stochastic settings, as well as information theoretic settings to adapt the observability notion to control of linear systems with limited information. One direction in this general field is the case when the observations available at a controller comes at random intervals. In this context, in both the information theory literature as well as automatic control literature, a rich collection of papers have studied the recursive estimation problem and its applications in remote control [1], [2], [3], [4].

In the following, we describe the system model. In Section III preliminaries on compressive sensing theory are presented. It follows a formal discussion of observability of linear systems: since the analytical tools and results are significantly different for different cases, we first treat a deterministic setup in Section IV and then study a stochastic setup in Section V. Detailed proofs are given in Section VI. Concluding remarks are discussed in Section VII.

II. PROBLEM FORMULATION

For the purpose of observability analysis, we consider the following discrete-time linear time-invariant system (with zero control input): $x_{t+1} = Ax_t$, $y_t = \eta_tCx_t$, where $t \in \mathbb{Z}_+$ denotes the discrete time instant, $x_t \in \mathbb{R}^n$ and $y_t \in \mathbb{R}^d$ are the state of the system and the observation of the system respectively, the matrices $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{d \times n}$ denote the state transfer matrix and the observation matrix respectively, and $\eta_t$ takes value either 0 or 1 ($\eta_t = 1$ means an observation at time $t$ is available, and $\eta_t = 0$ otherwise).

The problem we are interested in is the observability of a system with a sparse initial state: Given $m < n$ observations ($m$ instances where $\eta_t = 1$), can we reconstruct the initial state $x_0 \in \mathbb{R}^n$ exactly? Suppose that the receiver observes the output of the system $y_t$ at the (stopping) time instances $t_1, t_2, \ldots, t_m$. Let the overall observation matrix be the stacked observation matrices $O_{T_m} = [(CA^{t_1})^T, (CA^{t_2})^T, \ldots, (CA^{t_m})^T]^T$ and the overall observation be $y_{T_m} = [y_{t_1}^T, y_{t_2}^T, \ldots, y_{t_m}^T]^T$, where the subscript $T_m$ emphasizes that only the observations at time instants $T_m := \{t_1, t_2, \ldots, t_m\}$ are available. Then $y_{T_m} = O_{T_m}x_0$.

In order to infer the initial state $x_0$ from $y_{T_m}$, the columns of $O_{T_m}$ have to be linearly independent, or equivalently, the null-space of the matrix $O_{T_m}$ must be trivial.

While the general setup has been well understood, the problem of our particular interest is the observability when the initial state $x_0$ is sparse. The definition of a sparse vector is given as follows.

Definition 1. Let $B \in \mathbb{R}^{n \times n}$ be an orthonormal basis, i.e., $B$ contains $n$ orthonormal columns. A vector $x \in \mathbb{R}^n$ is $K$-sparse under $B \in \mathbb{R}^{n \times n}$ if $x = Bs$ for some $s \in \mathbb{R}^m$ with $\|s\|_0 \leq K$, where $\|s\|_0$ gives the number of non-zero components in the vector $s$ ($\|s\|_0$ is often referred to as the $\ell_0$-norm, even though it is not a well-defined norm).

Our formulation appears to be new in the control theory literature, except for a paper [5] which considers a similar setting for observability properties of a stochastic model to be considered later in the paper. The differences between the

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approaches in the stochastic setup are presented in Section V.

Another related work is [6] which designs control algorithms based on sparsity in the state, where compressive sensing tools are used to reconstruct the state for control purposes.

III. PRELIMINARIES AND COMPRESSIVE SENSING

Compressive sensing is a signal processing technique that encodes a signal $x$ of dimension $n$ by computing a measurement vector $y$ of dimension $m \ll n$ via linear projections, i.e., $y = \Phi x$, where $\Phi \in \mathbb{R}^{m \times n}$ is referred to as the measurement matrix. In general, it is not possible to uniquely recover the unknown signal $x$ using measurements $y$ with reduced-dimensionality. Nevertheless, if the input signal is sufficiently sparse, exact reconstruction is possible. In this context, suppose that the unknown signal $x \in \mathbb{R}^n$ is at most $K$-sparse, i.e., that there are at most $K$ nonzero entries in $x$. A naive reconstruction method is to search among all possible signals and find the sparsest one which is consistent with the linear measurements. This method requires only $m = 2K$ random linear measurements, but finding the sparsest signal representation is an NP-hard problem. On the other hand, Donoho and Candès et al. [7], [8] demonstrated that reconstruction of $x$ from $y$ is a polynomial time problem if more measurements are taken. This is achieved by casting the reconstruction problem as an $\ell_1$-minimization problem, i.e.,

$$\min \|x\|_1 \text{ subject to } y = \Phi x,$$

where $\|x\|_1 = \sum_{i=1}^n |x_i|$ denotes the $\ell_1$-norm of the vector $x$. It is a convex optimization problem and can be solved efficiently by linear programming (LP) techniques. The reconstruction complexity equals $O(m^2n^{3/2})$ if the convex optimization problem is solved using interior point methods [9]. More recently, an iterative algorithm, termed subspace pursuit (SP), was proposed independently in [10] and [11]. The corresponding computational complexity is $O(Km(n+K^2))$, which is significantly smaller than that of $\ell_1$-minimization when $K \ll n$.

A sufficient and necessary condition for $\ell_1$-minimization to perform exact reconstruction is the so called the null-space condition [12].

**Theorem 2.** If and only if for all $w \in \mathbb{R}^n$ such that $\Phi w = 0$, and for all sets $T \subset \{1, 2, \ldots, n\}$ such that $|T| = K$, there exists a constant $c > 1$ such that

$$c \sum_{t \in T} |w_t| \leq \sum_{j \in \mathbb{T}^c} |w_j|,$$

where $T^c = \{1, 2, \ldots, n\} - T$, then $\ell_1$-minimization reconstructs $x$ exactly.

A sufficient condition for both the $\ell_1$-minimization and SP algorithms to perform exact reconstruction is based on the so called restricted isometry property (RIP) [8]. A matrix $\Phi \in \mathbb{R}^{m \times n}$ is said to satisfy the Restricted Isometry Property (RIP) with coefficients $(K, \delta)$ for $K \leq m$, $0 \leq \delta \leq 1$, if for all index sets $I \subset \{1, \ldots, n\}$ such that $|I| \leq K$ and for all $q \in \mathbb{R}^{|I|}$, one has

$$(1 - \delta) \|q\|^2 \leq \|\Phi_I q\|^2 \leq (1 + \delta) \|q\|^2,$$

where $\Phi_I$ denotes the matrix formed by the columns of $\Phi$ with indices in $I$. The RIP parameter $\delta_K$ is defined as the infimum of all parameters $\delta$ for which the RIP holds. It was shown in [8], [13], [10] that both $\ell_1$-minimization and SP algorithms lead to exact reconstructions of $K$-sparse signals if the matrix $\Phi$ satisfies the RIP with a constant parameter, i.e., $\delta_{kK} \leq \delta_0$ where $\delta_0 \in (0, 1)$ and $K \in \mathbb{R}_+$ are independent of $K$. We note that different algorithms may have different parameter values for $\delta_0$ and $k$. Examples of random and deterministic RIP matrices can be found in [12], [8], [13], [16].

For later use, we also consider a particular class of the measurement matrices $\Phi$. We will assume that $\Phi^T \in S_{n,m} (\mathbb{R})$ (that is, the rows of $\Phi \in \mathbb{R}^{m \times n}$ are orthonormal) is isotropically distributed (the definition of $S_{n,m} (\mathbb{R})$ and the isotropic distribution on $S_{n,m} (\mathbb{R})$ will be introduced in Section V-A). Under this assumption, it has been shown in [17] that if the number of measurements satisfies $m \geq C \cdot K \log(n/K)$ for some positive constant $C$, then with high probability ($\geq 1 - \epsilon = \epsilon_{nc}$ for some positive constant $c$) the $\ell_1$-minimization perfectly reconstructs the input unknown signal $x$.

IV. THE DETERMINISTIC MODEL

This section characterizes the number of measurements needed for observability for different scenarios. We assume that $x_0$ is $K$-sparse under a basis $B \in S_{n,n} (\mathbb{R})$ and $B$ is known in advance. Recall that observability generally requires that the observability matrix $O_{T_m}$ has full rank, i.e., at least $n$ measurements should be collected. When $x_0$ is sparse, the number of observations required for observability can be significantly reduced.

We start with a special case where particular structures are imposed on $A$, $B$ and $C$ to reduce the number of required observations to $2K + 1$.

**Proposition 3.** Suppose that $x_0$ is $K$-sparse under the natural basis $B = I$. Assume that $A \in \mathbb{R}^{n \times n}$ is diagonal, and that all diagonal entries are nonzero and distinct. Let all of the entries of $C \in \mathbb{R}^{1 \times n}$ ($d_y = 1$) be non-zero. Then $x_0$ can be exactly reconstructed after exactly $2K + 1$ measurements by algorithms with polynomial complexity in $n$.

**Proof:** See Section VI-A.

**Remark 4.** The reconstruction relies on the Reed-Solomon decoding method presented in [18]. Note that the reconstruction is not robust to noise and hence not very useful in practice.

The following proposition considers the case where $\ell_1$-minimization is used for reconstruction. We have further restrictions on the initial state and observation time.

**Proposition 5.** Let all of the entries of $C \in \mathbb{R}^{1 \times n}$ ($d_y = 1$) be non-zero. Suppose $c_i x_{0,i} \geq 0$ for all $i$, where $C = [c_1, \ldots, c_n]$. Further assume that $A \in \mathbb{R}^{n \times n}$ is diagonal, and that all diagonal entries are nonzero. If the decoder receives $2K + 1$ successive observations at times $t = 0, \ldots, 2K$, the decoder can reconstruct the initial state perfectly and the unique solution can be obtained by the solution of the linear program $\min \|x\|_1 \text{ s.t. } O_t x = y$, where $O_t = \left[ C^T, (CA)^T, \ldots, (CA^{2K})^T \right]^T$.

**Proof:** See Section VI-B.
We note that, one can relax the above to the case when the observations are periodic such that \( t_2 - t_1 = t_3 - t_2 = \ldots = t_m - t_{m-1} \), where 1, 2, \ldots, \( m \) are the observation times.

In the following, we consider more general settings.

**Proposition 6.** Suppose that \( A \in \mathbb{R}^{n \times n} \) is of Jordan canonical form, all diagonal entries are nonzero, and the eigenvalues corresponding to different Jordan blocks are distinct. Let the entries of \( C \in \mathbb{R}^{n \times n} \) (\( d_y = 1 \)) be non-zero for all the leading components of Jordan blocks (that is, for the first entry corresponding to a Jordan block). If the decoder receives \( m \) random observations, at random times \( T_m = \{t_1, t_2, \ldots, t_m\} \), let \( O_{T_m} = \left[ (CA^{t_1})^T, (CA^{t_2})^T, \ldots, (CA^{t_m})^T \right]^T \). Let \( O_{T_m}(i) \) denote the \( i^{th} \) column of \( O_{T_m} \) for \( 1 \leq i \leq n \). Define

\[
M(T_m) = \sup_{i \neq j} \frac{1}{||O_{T_m}(i)||_2} ||O_{T_m}(i)||_2 \frac{1}{||O_{T_m}(j)||_2} ||O_{T_m}(j)||_2 < 1.
\]

Then \( x_0 \) can be exactly reconstructed after \( m \) measurements if:

\[
\|x_0\|_0 \leq \frac{1}{2} \left( 1 + \frac{1}{M(T_m)} \right)
\]

by algorithms with polynomial complexity in \( n \). In particular, a linear program (LP) can be used to recover the initial state.

**Proof:** See Section IV-C.

**Remark 7.** We recall that the observability of a linear system described by the pair \( (A, C) \) can be verified by the following criterion, known as the Hautus-Rosenbrock test: The pair is observable if and only if for all \( \lambda \in \mathbb{C} \), the matrix \( \left[ (\lambda I - A)^T, C^T \right]^T \) is full rank. Clearly, one needs to check the rank condition only for the eigenvalues of \( A \). It is a consequence of the above that, if the component of \( C \) corresponding to the first entry of a Jordan block is zero, then the corresponding component cannot be recovered even with \( n \) successive observations, since this is a necessary condition for observability.

A more general case is studied in the next proposition.

**Proposition 8.** Given \( A \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{d_y \times n} \) and \( T_m = \{t_1, \ldots, t_m\} \), if \( \Phi = O_{T_m}B \) satisfies the null-space condition \( \mathcal{H} \), then \( \ell_1 \)-minimization \( \min \|s\|_1 \) s.t. \( y_t = O_{T_m}Bs \) reconstructs \( s \) and \( x_0 = Bs \) exactly. Suppose that \( \Phi \) satisfies the RIP with proper parameters, both \( \ell_1 \)-minimization and SP algorithm leads to exact reconstruction of the initial state \( x_0 \).

This proposition is a direct application of the results presented in Section III. This result implies a protocol in which one keeps collecting available observations \( y_{t_1}, y_{t_2}, \ldots \) until the null-space or RIP condition is satisfied. However, the computation complexity of verifying either of them generally increases exponentially with \( n \). There are two approaches to avoid this extremely expensive computational cost. The first approach is reconstruction on the fly by trying to reconstruct the unknown initial state \( x_0 \) every time when certain number of new observations are received; and continue this process until the reconstruction is good enough. In the second approach, certain suboptimal but computationally more efficient conditions, for example, the incoherence condition, are employed to judge whether current observations are sufficient for reconstruction.

## V. The Stochastic Model

In this section, we discuss a stochastic model for the system matrices. One advantage of the stochastic model is that it helps in understanding more general cases that are difficult to analyze using the deterministic model. Examples include Theorem 12 and Corollary 14. Our analysis is based on the concept of rotational invariance, defined in Subsection V-A. The intuition is that rotational invariance provides a rich structure to “mix” the non-zeros in the initial state and this “mixing” ensures an observability with significantly reduced number of measurements.

During the preparation of this paper, we noticed that the stochastic model was also discussed in an independent work [5]. The major differences between our approach and that in [5] are as follows. First, in [5], the observation matrix \( C_k \)'s are assumed to be random Gaussian matrices. In contrast, our model relies on rotationally invariant random matrices, which are much more general. Second, though the work [5] is targeted for general state transition matrix \( A \), the analysis and results best suit for the \( A \) matrices with concentrated spectrum, for example, unitary matrices. As a comparison, in our stochastic model, we separate the rotational invariance and the spectral property and hence the spectral property can be very much relaxed.

### A. The Isotropy of Random Matrices

To define rotational invariance, we need to define the set of rotational matrices, often referred to as the Stiefel manifold. Formally, the Stiefel manifold \( S_{n,k}(\mathbb{R}) \) is defined as \( S_{n,k}(\mathbb{R}) = \{U \in \mathbb{R}^{n \times k} : U^T U = I_k\} \), where \( I_k \) is the \( k \times k \) identity matrix. When \( n = k \), a matrix in \( S_{n,n}(\mathbb{R}) \) is an orthonormal matrix and represents a rotation. A left rotation of a measurable set \( H \subset \mathbb{R}^{m \times n} \) under a given rotation represented by \( A \in S_{m,m} \) is given by the set \( AH = \{AH : H \in H\} \subset \mathbb{R}^{m \times n} \). Similarly defines the right rotation of \( H \) given by \( BH \) for a given \( B \in S_{n,n} \). An invariant isotropic probability measure \( \mu_1 \) [19], [20, Sections 2 and 3] is defined by the property that for any measurable set \( M \subset \mathbb{R}^{m \times n} \) and rotation matrices \( A \in S_{n,n}(\mathbb{R}) \) and \( B \in S_{n,k}(\mathbb{R}) \), \( \mu_1 (AM) = \mu_1 (BM) \). The invariant probability on the Stiefel manifold is essentially the uniform probability measure, i.e., \( \mu_1 \{\{A \in S_{n,n}(\mathbb{R}) : \|A - U\|_F \leq \epsilon\} \} \) is independent of the choice of \( U \in S_{n,k}(\mathbb{R}) \).

The main results in this subsection are Lemmas 9 and 10 which show that an rotationally invariant random matrix admits rotationally invariant matrix products and decompositions. These results are the key for proving results regarding observability in Subsection V-B.

**Lemma 9.** Let \( A \in S_{n,k}(\mathbb{R}) \) be isotropically distributed. Let \( B \in S_{n,n}(\mathbb{R}) \) be random. Let \( C = B \cdot A \). Then \( C \in S_{n,k}(\mathbb{R}) \) is isotropically distributed and independent of \( B \).

**Proof:** In order to show that \( C \) is independent of \( B \), it is sufficient to show that for given arbitrary \( B \in S_{n,n}(\mathbb{R}) \)
and arbitrary measurable set $M \subset S_{n,k}(\mathbb{R})$, the conditional probability $\Pr(C \in M | B)$ is independent of $B$. This can be verified by observing

$$\Pr(C \in M | B) = \Pr(A \in B^{-1}M | B) \overset{(a)}{=} \Pr(A \in B^{-1}M)$$

where $(a)$ follows from the fact that $A$ is independent of $B$, and $(b)$ comes from the fact that $A$ is isotropically distributed and that $B \in S_{n,n}(\mathbb{R})$ and hence $B^{-1} = BT \in S_{n,n}(\mathbb{R})$. This proves the lemma.

Let $H \in \mathbb{R}^{n \times n}$ be a standard Gaussian random matrix, i.e., the entries of $H$ are independent and identically distributed Gaussian random variables with zero mean and unit variance. Consider the Jordan matrix decomposition $H = PJP^{-1}$, where $J$ is often referred to as the Jordan normal form of $H$. Let $P = U \Lambda P V_T$ be the singular value decomposition of $P$, where $\Lambda$ is the diagonal matrix composed of singular values of $P$. Then $P^{-1} = V_T \Lambda^{-1} U_T$. The following lemma states that the orthogonal matrix $U_T$ is isotropically distributed.

**Lemma 10.** Let $H \in \mathbb{R}^{n \times n}$ be a standard Gaussian random matrix, let $H = PJP^{-1}$ be the corresponding Jordan matrix decomposition, and let $P = U \Lambda P V_T$ be the singular value decomposition of $P$. Then $P \in S_{n,n}(\mathbb{R})$ is isotropically distributed and independent of $J, \Lambda_P$ and $V_P$.

**Proof:** According to the statement of this lemma, $H$ is a standard Gaussian random matrix. Hence, the distribution of $H$ is left and right rotationally invariant. That is, for measurable sets $H \subset \mathbb{R}^{n \times n}$ and arbitrary $Q \in S_{n,n}(\mathbb{R})$, $\Pr(H \in H) = \Pr(H \in HQH)$, and therefore, $\Pr(H \in H) = \Pr(H \in HQH^T)$. To simplify the notation, let $H = U \Lambda P V_T$, where $B = \Lambda_P V_T J V_T \Lambda^{-1}$. Let $U_T \subset S_{n,n}(\mathbb{R})$ be an arbitrary measurable set of $U_T$. Let $\Pr(U_T)$ be the probability measure of $U_T$ induced from the probability measure of $H$.

The isotropics of $U_T$ means that $\Pr(U_T \in U_T)$ = $\Pr(U_T \in Q)$ for an arbitrarily given $Q \in S_{n,n}(\mathbb{R})$. To reach this end, note that $\Pr(U_T \in U_T) = \Pr(H : \exists U_T \in U_T)$, and

$$\Pr(U_T \in Q) = \Pr(H : \exists U_T \in Q) = \Pr(H : \exists U_T \in QU_T)$$

In other words, for any $H$ that induces a $U_T \in U_T$, $QHQT$ induces a $U_T \in QU_T$, and vice versa. Because we have shown $\Pr(H \in H) = \Pr(H \in HQH^T)$, we conclude that $U_T$ is isotropically distributed. Furthermore, the above argument also suggests that $U_T$ is independent of the matrix $B$, therefore independent of $J, \Lambda_P$ and $V_P$. This lemma is proved.

**Remark 11.** Although Lemma 10 only treats standard Gaussian random matrices, the same result holds for general random matrix ensembles, whose distributions are left and right rotationally invariant: The proof of Lemma 10 can be carried over.

**B. Results for Stochastic Models**

Recall that a general linear system is observable if and only if the observability matrix $O_{T_m}$ has full row rank. One may expect that the row rank of $O_{T_m}$ still indicates the observability of a linear system with sparse initial state and partial observations. The next theorem confirms the intimate relation between the row rank and the observability. The difference between our results and the standard results is that the required minimum rank is much smaller than the signal dimension $n$ in our setting.

**Theorem 12.** Suppose that $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{d_u \times n}$ are independent drawn from a random matrix ensemble whose distribution is left and right rotationally invariant. Let $r$ be the row rank of the overall observation matrix $O_{T_m}$. If $r \geq O(K \log n)$, then the $\ell_1$-minimization method perfectly reconstructs $x_0$ from $y_t = O_t x_0$ (where we write $t = T_m$ for notational convenience) with high probability (at least $1 - e^{-nc}$ for some positive constant $c$ independent of $n$ and $r$).

The proof of Theorem 12 rests on the following Lemma.

**Lemma 13.** Assume the same set-ups as in Theorem 12 and let $t = T_m$ for notational convenience. Let $O_t = U_t \Lambda_t V_t^T$ be the corresponding singular value decomposition, where $U_t \in S_{n,d_u,md_u}(\mathbb{R})$, $V_t \in S_{n,n}(\mathbb{R})$ are the left and right singular vector matrices respectively. Then $V_t$ is isotropically distributed and independent of $U_t$ and $\Lambda_t$.

While Lemma 13 is proved in Section VI-D, the detailed proof of Theorem 12 is presented in Section VI-D. The detailed reconstruction procedure using $\ell_1$-minimization is explicitly presented in the proof.

The next corollary presents a special case where the diagonal form is involved.

**Corollary 14.** Suppose that $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{1 \times n}$ are independent drawn from random matrix ensembles whose distribution is left and right rotationally invariant. Suppose that the Jordan normal form $J = P^{-1}AP$ is diagonal with distinct diagonal entries with probability one. Then after $m \geq O(K \log n)$ measurements, the $\ell_1$-minimization method perfectly reconstructs $x_0$ with high probability (at least $1 - e^{-nc}$ for some positive constant $c$).

**Proof:** See Section VI-D.

Acute readers may ask whether there exists a random matrix ensemble such that the random sample $A$ satisfies the required conditions in Corollary 14. In fact, if $A = HH^T$ where $H \in \mathbb{R}^{n \times n}$ is a standard Gaussian random matrix, then all the conditions required for $A$ hold. This corollary guarantees that blindly collecting $m \geq O(K \log n)$ observations is sufficient for perfect reconstruction with high probability.

**VI. PROOFS**

**A. Proof of Proposition 3**

Let $A = \text{diag}(\lambda)$ where $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]^T$ is the vector containing the diagonal entries of $A$. Let $c_i$ denote the $i^{th}$ entry of the row vector $C$. Then $CA_i = [c_1 \lambda_i, c_2 \lambda_i, \ldots, c_n \lambda_i] = [\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i]$ diag($C$), where diag($C$) is the diagonal matrix whose $i^{th}$ diagonal
Since all the entries of $C$ are non-zero, $\text{diag}(C)x_0$ is $K$-sparse under the natural basis. On the other hand, since $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all distinct, the matrix $\Lambda_k$ is a truncation of the full rank Vandermonde matrix $\left[ \begin{array}{c} 1 \\ \lambda_k \\ \lambda_k^2 \\ \vdots \\ \lambda_k^{m} \end{array} \right]$. Now according to the Reed-Solomon decoding method presented in [18] and the corresponding proof, as long as $m \geq 2K + 1$, one can exactly reconstruct $\text{diag}(C)x_0$ and therefore $x_0$ from $y_t$ with the number of algebraic operations polynomial in $n$. This proposition is therefore proved.

\section*{B. Proof of Proposition 3}

We first consider the case when $A$ is diagonal. Since $A$ is diagonal, it is of the form $A = \text{diag}(\{\lambda_1, \ldots, \lambda_n\})$. Furthermore, assume that $C = [c_1, \ldots, c_n]$ is a row vector.

With $m$ many successive observations, we have a linear system described by

$$y_t = \left[ \begin{array}{cccc} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_n^{m-1} \end{array} \right] \text{diag} ([c_1, \ldots, c_n]) x_0 = \text{diag}(C)x_0.$$ 

Define $z \in \mathbb{R}^n$ such that $z_i = c_i x_0_{i} \geq 0$. Then the corresponding $\ell_1$-minimization problem becomes

$$\min \|z\|_1 \text{ subject to } y_t = Mz.$$ 

Once we solve the above optimization problem, it is clear that $x_0_{i} = z_i / (\lambda_i^t c_i)$ where $t_1 = 0$.

For this case, we first show that the $\ell_1$-minimization has a unique solution. Via duality theory, for a constrained minimization problem of a convex function with an equality constraint, the minimization has a unique solution if one can find a Lagrange multiplier (in the dual space) for which the inequality holds since $\lambda_i$’s are distinct. Let $f \in \mathbb{R}^m, f := [\alpha_{2K}, \alpha_{2K-1}, \ldots, \alpha_1, \alpha_0, 0, \ldots, 0]^T$. It can be verified that the inner product $\langle f, [1, \lambda_1, \ldots, \lambda_i^{m-1}]^T \rangle = \prod_{k=1}^{n} (\lambda_k - \lambda_i)^2 = P(\lambda_i)$.

If $c_1$ is non-zero, and the entries corresponding to leading entries of Jordan blocks are non-zero, the columns of the matrix become linearly independent. By multiplying the initial condition with a diagonal matrix, we can normalize the columns such that the $l_2$ norm of each column is equal to 1.

The rest of the proof now follows from Theorem 3 of [25].

\section*{C. Proof of Proposition 6}

We now discuss the result for a Jordan matrix $A$. Observe that

$$J = \left[ \begin{array}{ccc} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{array} \right] \Rightarrow J^n = \left[ \begin{array}{ccc} \lambda_1^n & (\lambda_1^t)^{n-1} & (\lambda_1^t)^{n-2} \\ 0 & \lambda_1^n & (\lambda_1^t)^{n-1} \\ 0 & 0 & \lambda_1^n \end{array} \right].$$

Thus, it follows that if $A$ is of the diagonal form: $\text{diag} (\lambda_1, \ldots, \lambda_n)$, the random observation matrix writes as:

$$M = \left[ \begin{array}{cccc} c_1 \lambda_1^t & c_1 t_1 \lambda_1^t - 1 & c_2 \lambda_1^t & \cdots & c_n \lambda_1^t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 \lambda_1^{m_t} & c_1 t_m \lambda_1^{m_t} - 1 & c_2 \lambda_1^{m_t} & \cdots & c_n \lambda_1^{m_t} \end{array} \right].$$

If $c_1$ is non-zero, and the entries corresponding to leading entries of Jordan blocks are non-zero, the columns of the matrix become linearly independent. By multiplying the initial condition with a diagonal matrix, we can normalize the columns such that the $l_2$ norm of each column is equal to 1.

The rest of the proof now follows from Theorem 3 of [25].

\section*{D. Proof of Lemma 7}

Consider the Jordan decomposition $A = PJP^{-1}$ and the singular value decomposition $P = U_P \Lambda_P V_P^T$. It is clear that $P^{-1} = V_P \Lambda_P^{-1} U_P^T$. For notational compactness, let $A = \Lambda_P V_P^T$, $JVP^{-1}$ so that $A = U_P \check{A} U_P^T$. It is elementary to verify that $A^t = U_P \check{A}^t U_P^T$. Hence,

$$O_t = \left[ \begin{array}{c} CA^{t_1} \\ \vdots \\ CA^{t_m} \end{array} \right] = \left[ \begin{array}{c} C U_P \check{A}^{t_1} U_P^T \\ \vdots \\ C U_P \check{A}^{t_m} U_P^T \end{array} \right].$$

We shall show that $U_P$ is independent of both $\check{A}$ and $C U_P$. Since $A$ is left and right rotation-invariantly distributed, according to Remark 11 $U_P$ is isotropically distributed and independent of $\check{A}$. In order to show that $U_P$
is independent of $CU_P$, we resort to the singular value decomposition $C = UC\Lambda CV_T^T$. Since $C$ is right rotation-invariantly distributed, $V_C$ is isotropically distributed. Thus $V_T^T := V_C^T U_P$ is isotropically distributed and independent of $U_P$ according to Lemma 9. As a result, $CU_P = UC\Lambda CV_T^T$ is independent of $U_P$. Write $O_t = O_t U_P^T$, where $O_t = \left[ \left( CU_P \tilde{A}_t \right)^T, \ldots, \left( CU_P \tilde{A}_m \right)^T \right]^T$. Since $U_P$ is independent of both $\tilde{A}$ and $CU_P$, $U_P$ is independent of $O_t$. Write the singular value decompositions of $O_t$ and $O_t$ as $O_t = U_t \Lambda_t V_t^T$ and $O_t = U_t \Lambda_t V_t^T$. Clearly $V_t = U_P V_t$. Since $U_P$ is isotropically distributed and independent of $O_t$, $V_t = U_P V_t$ is isotropically distributed and independent of both $\Lambda_t$ and $U_t$ according to Lemma 9. This completes the proof.

E. Proof of Theorem 12

We transfer the considered reconstruction problem to the standard compressive sensing reconstruction. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the $r$ non-zero singular values of $O_t$ and $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r]^T$. The singular value decomposition of $O_t$ can be written in the form

$$O_t = U_t \left[ \begin{array}{cc} \text{diag} (\lambda) & 0 \\ 0 & 0 \end{array} \right] V_t^T,$$

where $\text{diag} (\lambda)$ is the diagonal matrix generated from $\lambda$. Note that

$$U_t^T y_t = \left[ \begin{array}{cc} \text{diag} (\lambda) & 0 \\ 0 & 0 \end{array} \right] V_t^T x_0.$$

The $r + 1, r + 2, \ldots, m$ entries of $U_t^T y_t$ are zeros: they do not carry any information about $x_0$. Define $\tilde{y}_t$ to be the vector containing the first $r$ entries of $U_t^T y_t$. We have

$$\tilde{y}_t = \left[ \begin{array}{c} 0 \\ \text{diag} (\lambda) \end{array} \right] V_t^T x_0 = \left[ \begin{array}{cc} I_r & 0 \end{array} \right] V_t^T x_0 = \left[ \begin{array}{cc} I_r & 0 \end{array} \right] V_t^T B s,$$

where $I_r$ is the $r \times r$ identity matrix.

The unknown $s$ ($K$-sparse) can be reconstructed by $\ell_1$-minimization with high probability. Since $V_t$ is isotropically distributed and independent of $B$, the matrix $V_t^T B$ is isotropically distributed. The matrix $\left[ \begin{array}{cc} I_r & 0 \end{array} \right] V_t^T B \in S_{n,r}(\mathbb{R})$, containing the first $r$ rows of $V_t^T B$ as columns, is therefore isotropically distributed. Provided that $r \geq O(K \log (n/K))$, the unknown signal $s$ can be exactly reconstructed from $\text{diag} (\lambda)^{-1} \tilde{y}_t$ via $\ell_1$-minimization [17]. Theorem 12 is proved.

Remark 15. The reconstruction procedure involves singular value decomposition, matrix production, and $\ell_1$-minimization. The numbers of algebraic operations required for all these steps are polynomial in $n$. Hence, the complexity of the whole reconstruction process is polynomial in $n$.

F. Proof of Corollary 14

Since both $A$ and $C$ are left and right rotation-invariantly distributed, Theorem 12 can be applied. Let $A = PJP^{-1}$ be a Jordan decomposition. Corollary 14 holds if

$$O_t = \begin{bmatrix} CA_1 & & \\ & \ddots & \\ & & CA_m \end{bmatrix} = \begin{bmatrix} CPJ_1 & & \\ & \ddots & \\ & & CPJ_m \end{bmatrix} P^{-1},$$

is full row ranked with probability one, i.e., $\text{rank} (O_t) = m \geq O(K \log (n/K))$ with probability one.

Suppose that the Jordan normal form $J = P^{-1}AP$ is diagonal. Denote the $j$th diagonal entry of $J$ by $J_j$. Note that

$$CPJ^t = \left[ (CP)_1 J_1^t, (CP)_2 J_2^t, \ldots, (CP)_n J_n^t \right]$$

$$= \left[ J_1^t, J_2^t, \ldots, J_n^t \right] \text{diag} (CP),$$

where $\text{diag} (CP)$ is the diagonal matrix generated from the row vector $CP$. Define

$$J_{V,t} = \begin{bmatrix} J_1^t & J_2^t & \cdots & J_n^t \\ J_1 & J_2 & \cdots & J_n \end{bmatrix}.$$ 

Then $O_t = J_{V,t} \text{diag} (CP) P^{-1}$. Note that $J_V$ is composed of $m$ rows of the Vandermonde matrix

$$J_V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ J_1 & J_2 & \cdots & J_n \end{bmatrix}.$$ 

The matrix $J_{V,t}$ has full row rank. By definition of $P$, $P^{-1}$ has full rank as well. Therefore, $O_t$ has full row rank if and only if $CP$ does not contain any zero entries.

The fact that the row vector $CP$ does not contain any zero entries holds with probability one. This fact will be established by the isotropy of $C$. Let $P_{:,j}$ denote the $j$th column of $P$. Since $P$ is full rank, $P_{:,j} \neq 0$ for all $j = 1, 2, \ldots, n$. By assumption, $C$ is isotropically distributed. This implies that $CP_{:,j} \neq 0$ with probability one [20]. $CP$ is composed of finite columns. It follows that with probability one, no entry of $CP$ is zero.

So far, we have proved that $O_t$ has full row rank with probability one if the Jordan normal form $J = P^{-1}AP$ is diagonal. Note that by assumption, the Jordan normal form is diagonal with probability one. We have $\text{rank} (O_t) = m \geq O(K \log (n/K))$ with probability one. This proves this corollary.

VII. CONCLUDING REMARKS

In this paper we obtained sufficiency conditions for the observability of a linear system where the number of non-zeros in the initial states is known to be less than the dimensionality of the system. The discussion also applies to the case if certain elements have known values and we wish to reconstruct the unknown values.

Two models were included; one is for a deterministic model and the other for a stochastic model. We observed that a much lower number of observations (even when the observations are randomly picked) can be used to recover the initial condition. Furthermore, this can be done by a linear or quadratic program.
An interesting extension of this problem is for the case when there are some non-zero terms but terms which are known to have small magnitude, that is a robust formulation of initial condition recovery when the disturbance is an \( l_2 \) ball of small radius.

Compressive sensing offers new directions for design of information structures in networked control systems. Recent work \([6]\) lays out designs based on compressive sensing principles for such systems. We believe there will be further results specific to control systems, in particular on the inherent interaction between estimation and control in decentralized control systems.

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