THE BOUNDARY OF A FIBERED FACE OF THE MAGIC 3-MANIFOLD
AND THE ASYMPTOTIC BEHAVIOR OF THE MINIMAL PSEUDO-ANOSOV DILATATIONS

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Abstract. Let $\delta_{g,n}$ be the minimal dilatation of pseudo-Anosovs defined on an orientable surface of genus $g$ with $n$ punctures. Tsai proved that for any fixed $g \geq 2$, the logarithm of the minimal dilatation $\log \delta_{g,n}$ is on the order of $\log n/n$. We prove that if $2g+1$ is relatively prime to $s$ or $s+1$ for each $0 \leq s \leq g$, then

$$\limsup_{n \to \infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2.$$ 

Our examples of pseudo-Anosov $\phi$'s which provide the upper bound above have the following property: The mapping torus $M\phi$ of $\phi$ is a single hyperbolic 3-manifold $N$ called the magic manifold, or the fibration of $M\phi$ comes from a fibration of $N$ by Dehn filling cusps along the boundary slopes of a fiber. The main tool in this paper is the boundary of a fibered face of $N$.

1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus $g$ with $n$ punctures and Mod($\Sigma$) the mapping class group of $\Sigma$. According to the work of Nielsen and Thurston, elements of Mod($\Sigma$) are classified into three types: periodic, reducible, pseudo-Anosov, see [19]. Pseudo-Anosov mapping classes have rich dynamical and geometric properties. The hyperbolization theorem by Thurston [20] relates the dynamics of pseudo-Anosovs and the geometry of hyperbolic fibered 3-manifolds. The theorem asserts that $\phi \in$ Mod($\Sigma$) is pseudo-Anosov if and only if the mapping torus $M\phi$ of $\phi$ is a hyperbolic 3-manifold with finite volume.

Each pseudo-Anosov element $\phi \in$ Mod($\Sigma$) has a representative $\Phi : \Sigma \to \Sigma$ called a pseudo-Anosov homeomorphism. Such a homeomorphism is equipped with a constant $\lambda = \lambda(\Phi) > 1$ called the dilatation of $\Phi$. If we let $\text{ent}(\Phi)$ be the topological entropy of $\Phi$, then the identity $\text{ent}(\Phi) = \log \lambda(\Phi)$ holds, see [3] Exposé 10]. The dilatation $\lambda$ of $\Phi$ does not depend on the choice of a pseudo-Anosov homeomorphism $\Phi \in \phi$, and hence the dilatation $\lambda(\phi)$ of $\phi$ is defined to be $\lambda(\Phi)$. We call the quantities $\text{ent}(\phi) = \log \lambda(\phi)$ and $\text{Ent}(\phi) = |\chi(\Sigma)| \log \lambda(\phi)$ the entropy and normalized entropy of $\phi$ respectively, where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$.

If we fix $\Sigma$, the set of entropies of pseudo-Anosovs defined on $\Sigma$ is a closed discrete subset of $\mathbb{R}$, see [7]. In particular there exists a minimal entropy, and hence there exists a minimal dilatation. We denote by $\delta(\Sigma) > 1$, the minimal dilatation of pseudo-Anosov elements in Mod($\Sigma$). The minimal dilatations are determined in only a few cases, see [2].
Let us set $\delta_{g,n} = \delta(\Sigma_{g,n})$ and $\delta_{g} = \delta_{g,0}$. Penner proved in \cite{16} that $\log \delta_{g} \asymp \frac{1}{g}$. This work by Penner was a starting point on the study of the asymptotic behavior of the minimal dilatations on surfaces varying topology. Later it was proved by Hironaka-Kin\cite{6} that $\log \delta_{0,n} \asymp \frac{1}{n}$, and by Tsai\cite{21} that $\log \delta_{1,n} \asymp \frac{1}{n}$. See also Valdivia\cite{22}. The following theorem, due to Tsai, is in contrast with the cases of genus 0 or 1.

**Theorem 1.1** (\cite{21}). For any fixed $g \geq 2$, we have

$$\log \delta_{g,n} \asymp \frac{\log n}{n}.$$  

We ask the following question which is motivated by Theorem 1.1.

**Question 1.2.** Given $g \geq 2$, does $\lim_{n \to \infty} \frac{n \log \delta_{g,n}}{\log n}$ exist? What is its value?

This is an analogous question, posed by McMullen, which is asking whether $\lim_{g \to \infty} g \log \delta_{g}$ exists or not, see \cite{14}.

**Theorem 1.3.** Given $g \geq 2$, there exists a sequence $\{n_{i}\}_{i=0}^{\infty}$ with $n_{i} \to \infty$ such that

$$\limsup_{i \to \infty} \frac{n_{i} \log \delta_{g,n_{i}}}{\log n_{i}} \leq 2.$$  

We note that for any $g \geq 2$, Tsai’s examples in \cite{21} yield the upper bound $\limsup_{n \to \infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2(2g + 1)$, which is proved by a similar computation as in the proof of Theorem 1.3.

We define the polynomial $B_{(g,p)}(t)$ for nonnegative integers $g$ and $p$:

$$B_{(g,p)}(t) = t^{2p+1}(t^{2g+1} - 1) - 1 - 2t^{p+1} - t^{2g}.$$  

We shall see that there exists a unique real root $r_{(g,p)}$ greater than 1 of $B_{(g,p)}(t)$ such that

$$\lim_{p \to \infty} \frac{p \log r_{(g,p)}}{\log p} = 1$$  

(Lemma 4.1). The root $r_{(g,p)}$ gives the following upper bound of the minimal dilatations.

**Theorem 1.4.** For $g \geq 2$ and $p \geq 0$, suppose that $\gcd(2g+1, p + g + 1) = 1$. Then

$$\delta_{g,2p+i} \leq r_{(g,p)}$$  

for each $i \in \{1, 2, 3, 4\}$.

If $g$ enjoys (*) in the next theorem then one can take a subsequence $\{n_{i}\}_{i=0}^{\infty}$ in Theorem 1.3 to be the sequence $\{n\}_{n=1}^{\infty}$ of natural numbers.

**Theorem 1.5.** Suppose that $g \geq 2$ satisfies

(*) $\gcd(2g+1,s) = 1$ or $\gcd(2g+1,s+1) = 1$ for each $0 \leq s \leq g$.

Then

$$\limsup_{n \to \infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2.$$  

For example, (*) holds for $g = 4$ since 9 is relatively prime to 1, 2, 4 and 5; (*) does not hold for $g = 7$ because $\gcd(15, 5) = 5$ and $\gcd(15, 6) = 3$. We point out that infinitely many $g$’s satisfy (*). In fact if $2g+1$ is prime, then $2g+1$ is relatively prime to $s'$ for each $1 \leq s' \leq g + 1$. Such a $g$ enjoys (*), and this leads to

\footnote{Let $A_{g}$ and $B_{g}$ be functions on $g$. We write $A_{g} \asymp B_{g}$ if there exists a constant $c$, independent of $g$, such that $A_{g}/c < B_{g} < cA_{g}$.}
Corollary 1.6. If $2g + 1$ is prime for $g \geq 2$, then
$$\limsup_{n \to \infty} \frac{n \log \delta_{g,n}}{\log n} \leq 2.$$ 

Remark 1.7. One can simplify $(*)$ in Theorem 1.6, since $2g + 1$ is relative prime to $1, 2$ and $g$. In the case $g \geq 5$, $(*)$ is equivalent to
$$\text{(**) } \gcd(2g + 1, s) = 1 \text{ or } \gcd(2g + 1, s + 1) = 1 \text{ for each } 3 \leq s \leq g - 2.$$ 

Our results are proved by using the theory on fibered faces of hyperbolic fibered 3-manifolds $M$, developed by Thurston [18], Fried [4], Matsumoto [13] and McMullen [14]. (See Section 3) Let $\| \cdot \| : H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}$ be the Thurston norm, and let $\Omega$ be a fibered face of $M$. The work of Thurston tells us that if $M$ has the second Betti number more than 1, then it admits a family of fibrations on $M$ dominated by $int(C_0)$, where $C_0$ is the core over $\Omega$ with the origin and $int(C_0)$ is its interior. In other words, such a fibered 3-manifold provides infinitely many pseudo-Anosovs defined on surfaces with variable topology. By work of Fried, the entropy function defined on these fibrations admits a unique continuous extension $\text{ent} : int(C_0) \to \mathbb{R}$. By the continuity of $\| \cdot \|$ and ent, we have the continuous function
$$\text{Ent} = \| \cdot \| \cdot \text{ent}(\cdot) : int(C_0) \to \mathbb{R}.$$ 

The normalized entropy function Ent is constant on each ray in $int(C_0)$ through the origin. It is shown by Fried that the restriction $\text{ent}_{|int(\Omega)} (= \text{ent}_{|int(\Omega)} : int(\Omega) \to \mathbb{R}$ has the property such that $\text{ent}(a)$ goes to $\infty$ as $a \in int(\Omega)$ goes to a point on the boundary of $\Omega$. 

These properties give us the following observation: Fix a manifold $M$ as above. For any compact set $\Delta \subset int(\Omega)$, there exists a constant $C = C_D > 0$ satisfying the following. Let $a \in int(C_0)$ be any integral class of $H_2(M, \partial M; \mathbb{Z})$ and let $\Phi_a$ be the monodromy of the fibration associated to $a$. Then the normalized entropy $\text{Ent}(\Phi_a)$ is bounded by $C$ from above whenever $\pi \in \Delta$, where $\pi$ is the projective class of $a$.

This observation enables us to investigate the asymptotic behavior of the minimal dilatations. The following asymptotic inequalities (which are the best known upper bounds) are proved by using a similar technique.

1. $\limsup_{n \to \infty} n \log \delta_{0,n} \leq 2 \log(2 + \sqrt{3})$, see [3] [10].

2. $\limsup_{n \to \infty} n \log \delta_{1,n} \leq 2 \log \lambda_0$, where $\lambda_0 \approx 2.2966$ is the largest real root of $t^4 - 2t^3 - 2t + 1$, see [9].

3. $\limsup_{g \to \infty} g \log \delta_g \leq \log \left(\frac{4 + \sqrt{7}}{2}\right)$, see [5] [11] [3].

However for any fixed $g \geq 2$, the observation as above doesn’t work to investigate the asymptotic behavior $\delta_{g,n}$ varying $n$ because of Theorem 1.4. Theorem 1.4 implies that there exists no constant $C > 0$, independent of $g$ so that $|\chi(\Sigma_{g,n})| \log \delta_{g,n} < C$. Thus if there exists a sequence of integral classes $\{a_i\}$ with $a_i \in int(C_0)$ such that the fiber of the fibration associated to $a_i$ is a surface of genus $g$ having $n_i$ boundary components, then the accumulation points of the sequence of projective classes $\{\pi_i\}$ must lie on the boundary of $\Omega$. (This is because there exists no constant $C > 0$, independent of $i$, such that $\text{Ent}(\Phi_{a_i}) (= |\chi(\Sigma_{g,n_i})| \log(\Phi_{a_i})) \leq C$.)

Nevertheless we focus on a fibered face of a particular hyperbolic fibered 3-manifold, called the magic manifold $N$. This manifold is the exterior of the 3 chain link $C_3$, see Figure 1. Our examples of pseudo-Anosov $\phi$’s which provide the upper bounds in Theorems 1.3 [4] and [5] have the following property: The mapping torus $M_0$ of $\phi$ is homeomorphic to $N$, or the fibration of $M_0$ comes from a fibration of $N$ by Dehn filling cusps along the boundary slopes of a fiber. We also
point out that a family of the integral classes of $H_2(N, \partial N; \mathbb{Z})$ is a main ingredient to prove the asymptotic inequalities (1)–(3) above, see [9].

We turn to the hyperbolic volume of hyperbolic 3-manifolds. The set of volumes of hyperbolic 3-manifolds is a well-ordered closed subset in $\mathbb{R}$ of order type $\omega^\omega$, see [17]. In particular if we fix a surface $\Sigma$, then there exists a minimum among volumes of hyperbolic $\Sigma$-bundles over the circle.

The proofs of Theorems 1.3, 1.5 imply the following.

\textbf{Proposition 1.8.} Given $g \geq 2$, there exists a sequence $\{n_i\}_{i=0}^\infty$ with $n_i \to \infty$ such that the minimal volume of $\Sigma_{g,n_i}$-bundles over the circle is less than or equal to $\text{vol}(N) \approx 5.3334$, the volume of the magic manifold $N$. Furthermore if $g \geq 2$ satisfies $(\ast)$, then for any $n \geq 3$, the minimal volume of $\Sigma_{g,n}$-bundles over the circle is less than or equal to $\text{vol}(N)$.

We close the introduction by asking

\textbf{Question 1.9 (cf. Theorems 1.3 and 1.5).} Does $\limsup_{n \to \infty} \frac{n \log f_{g,n}}{\log n} \leq 2$ hold for any $g \geq 2$?

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2. \textsc{Roots of Polynomials}

This section concerns the asymptotic behavior of roots of families of polynomials. Let

$$g(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

be a polynomial with real coefficients $a_0, a_1, \cdots, a_n$ $(a_1, a_2, \cdots, a_n \neq 0)$, where $g(t)$ is arranged in the order of descending powers of $t$. Let $\mathfrak{D}(g)$ be the number of variations in signs of the coefficients $a_n, a_{n-1}, \cdots, a_0$. For example if $g(t) = +t^4 + t^3 - 2t^2 + t - 1$, then $\mathfrak{D}(g) = 3$; if $h(t) = +t^4 + t^3 - 2t^2 + t + 1$, then $\mathfrak{D}(h) = 2$. Descartes’s rule of signs (see [23]) says that the number of positive real roots of $g(t)$ (counted with multiplicities) is equal to either $\mathfrak{D}(g)$ or less than $\mathfrak{D}(g)$ by an even integer.

\textbf{Lemma 2.1.} Let $r \geq 0$, $s > 0$ and $u > 0$ be integers. Let

$$P_m(t) = t^{2m+r} (t^s - 1) + 1 - Q(t) t^m - t^u$$

be a polynomial for each $m \in \mathbb{N}$, where $Q(t)$ is a polynomial whose coefficients are positive integers. ($Q(t)$ could be a positive constant.)
Suppose that $t^{2m+r+s}$ is the leading term of $P_m(t)$. Then $P_m(t)$ has a unique real root $\lambda_m$ greater than 1.

Given $0 < c_1 < 1$ and $c_2 > 1$, we have

$$m^{\frac{c_1}{m}} < \lambda_m < m^{\frac{c_2}{m}} \text{ for } m \text{ large}.$$\n
In particular

$$\lim_{m \to \infty} \frac{m \log \lambda_m}{\log m} = 1.$$\n
(3) For any real numbers $q \neq 0$ and $v$, we have

$$\lim_{m \to \infty} \frac{(qm + v) \log \lambda_m}{\log(qm + v)} = q.$$\n
Proof. (1) Under the assumption on $P_m(t)$, we have $D(P_m) = 2$. By Descartes’s rule of signs, the number of positive real roots of $P_m(t)$ is either 2 or 0. Since $P_m(0) = 1$ and $P_m(1) = -Q(1) < 0$, the number of positive real roots of $P_m(t)$ is exactly 2. Because $P_m(t)$ goes to $\infty$ as $t$ does, $P_m(t)$ has a unique real root $\lambda_m > 1$.

(2) We have

$$P_m(t)t^{-(2m+r)} = t^s - 1 + t^{-(2m+r)} - Q(t)t^{-(m+r)} - t^{-(2m+r-u)}.$$\n
We define $f_m(t)$ and $g_m(t)$ such that $P_m(t)t^{-(2m+r)} = f_m(t) + g_m(t)$ as follows.

$$f_m(t) = t^s - 1 + t^{-(2m+r)}, \text{ and}$$
$$g_m(t) = Q(t)t^{-(m+r)} + t^{-(2m+r-u)}.$$\n
We let $t = m^{\frac{s}{m}}$ for $c > 0$. Then

$$f_m(m^{\frac{s}{m}}) = (m^{\frac{s}{m}})^s - 1 + (m^{\frac{s}{m}})^{-(2m+r)}$$
$$= (e^{\frac{sc}{m}})^s - 1 + m^{-c(2+s/m)}$$
$$= e^{\frac{sc \log m}{m}} - 1 + m^{-c(2+s/m)}.$$\n
By Maclaurin expansion of $e^{\frac{sc \log m}{m}}$, we have

$$e^{\frac{sc \log m}{m}} = 1 + \frac{sc \log m}{m} + R_2,$$\n
where

$$R_2 = \frac{e^w}{2} \left( \frac{sc \log m}{m} \right)^2 \text{ for some } 0 < w < \frac{sc \log m}{m}.$$\n
Since $\frac{sc \log m}{m}$ goes to 0 as $m$ goes to $\infty$, we may assume that $\frac{e^w}{2} < B$ for some constant $B > 0$. Then
\[ f_m(m^{\frac{c}{m}}) = \frac{sc \log m}{m} + R_2 + \frac{m^{1-c(2+\frac{r}{m})}}{m} \]

\[
< \frac{sc \log m}{m} + B\left(\frac{sc \log m}{m}\right)^2 + \frac{m^{1-c(2+\frac{r}{m})}}{m} \\
= \frac{sc \log m}{m} + Bs^2c^2\left(\frac{\log m}{m}\right)^2 + \frac{m^{1-c(2+\frac{r}{m})}}{m} \\
< \frac{sc \log m}{m} + Bs^2c^2\left(\frac{\log m}{m}\right) + \frac{m^{1-c(2+\frac{r}{m})}}{m} \\
= \frac{(sc + Bs^2c^2) \log m + m^{1-c(2+\frac{r}{m})}}{m}.
\]

(The last inequality comes from \(0 < \frac{\log m}{m} < 1\) for \(m\) large.) Thus

\[
(2.1) \quad f_m(m^{\frac{c}{m}}) < \frac{(sc + Bs^2c^2) \log m + m^{1-c(2+\frac{r}{m})}}{m}.
\]

The first equality \(f_m(m^{\frac{c}{m}}) = \frac{sc \log m}{m} + R_2 + \frac{m^{1-c(2+\frac{r}{m})}}{m}\) above together with \(R_2 > 0\) and \(\frac{m^{1-c(2+\frac{r}{m})}}{m} > 0\) tells us that

\[
(2.2) \quad f_m(m^{\frac{c}{m}}) > \frac{sc \log m}{m}.
\]

Recall that all coefficients of \(Q(t)\) (appeared in \(P_m(t)\)) are positive integers. If we write \(Q(t) = \sum_{j=0}^{\ell} a_j t^j\), where \(a_j \geq 0\), then

\[
g_m(m^{\frac{c}{m}}) = Q(m^{\frac{c}{m}})m^{-c(1+\frac{r}{m})} + m^{-c(2+\frac{r}{m} - \frac{u}{m})} \\
= \left(\sum_{j=0}^{\ell} a_j m^{-c(1+\frac{r}{m})} \right) + m^{-c(2+\frac{r}{m} - \frac{u}{m})}.
\]

Thus we obtain

\[
(2.3) \quad g_m(m^{\frac{c}{m}}) = \left(\sum_{j=0}^{\ell} a_j m^{-c(1+\frac{r}{m})} \right) + m^{1-c(2+\frac{r}{m} - \frac{u}{m})}.
\]

For the proof of the claim (1), it is enough to prove that for \(0 < c_1 < 1\) and \(c_2 > 1\), we have \(f_m(m^{\frac{c_1}{m}}) < g_m(m^{\frac{c_2}{m}})\) and \(f_m(m^{\frac{c_2}{m}}) > g_m(m^{\frac{c_1}{m}})\) for \(m\) large.

First, suppose that \(0 < c < \frac{1}{2}\). Let us consider how the following four terms grow.

\[
(2.4) \quad \log m, \quad m^{1-c(2+\frac{r}{m})}, \quad m^{1-c(1+\frac{r}{m} - \frac{u}{m})}, \quad \text{and} \quad m^{1-c(2+\frac{r}{m} - \frac{u}{m})}.
\]

The first two terms are appeared in (2.1), and the last two are coming from (2.3). All four terms go to \(\infty\) as \(m\) does, since the last three terms have the positive powers of \(m\). Note that for any \(C > 0\), we have \(\log m < m^C\) for \(m\) large. Keeping in mind of this, we observe that among the four terms in (2.4), \(m^{1-c(1+\frac{r}{m} - \frac{u}{m})}\) is dominant. This is because

\[1 - c(1 + \frac{r}{m} - \frac{j}{m}) > 1 - c(2 + \frac{r}{m} - \frac{u}{m}) \geq 1 - c(2 + \frac{r}{m})\]

for \(m\) large. These imply that \(f_m(m^{\frac{c_1}{m}}) < g_m(m^{\frac{c_2}{m}})\) holds for \(m\) large, since \(m^{1-c(1+\frac{r}{m} - \frac{u}{m})}\) is appeared in the numerator of \(g_m(m^{\frac{c_1}{m}})\), see (2.3).
Next, we suppose that \( \frac{1}{2} \leq c < 1 \). We can check that \( m^{1-c(1+\frac{r}{m} - \frac{1}{c})} \) is still dominant among the four in (2.4). (The second and fourth terms are bounded as \( m \) goes to \( \infty \).) Therefore we still have \( f_m(m^n) < g_m(m^n) \) for \( m \) large.

Finally we suppose that \( c > 1 \). Then the last three terms in (2.4) go to 0 as \( m \) goes to \( \infty \), because they have the negative powers of \( m \) for \( m \) large. Thus the numerator of \( g_m(m^n) \), see (2.3), goes to 0 as \( m \) tends to \( \infty \). On the other hand, \( f_m(m^n) > \frac{sc \log m}{m} \) holds (see (2.2)), and hence the numerator of

\[
\frac{sc \log m + mR_2 + m^{1-c(2+\frac{r}{m})}}{m} (= f_m(m^n))
\]

goes to \( \infty \) as \( m \) does. Thus \( f_m(m^n) > g_m(m^n) \) for \( m \) large. This completes the proof of the first part of the claim (2).

Taking the logarithm of the both sides of

\[
m^c < \lambda_m < m^c
\]

yields

\[
c < \frac{m \log \lambda_m}{\log m} < c
\]

for \( m \) large.

Since \( 0 < c_1 < 1 \) and \( c_2 > 1 \) are any numbers, we have the desired limit. This completes the proof of the second half of the claim (2).

(3) By the claim (2),

\[
\frac{c_1 \log m}{m} < \log \lambda_m < \frac{c_2 \log m}{m}
\]

for \( m \) large.

Let us set \( n = qm + v \). We substitute \( m = \frac{n-v}{q} \) for the inequality above:

\[
\frac{c_1 \log \left( \frac{n-v}{q} \right)}{n-v} < \log \lambda_m < \frac{c_2 \log \left( \frac{n-v}{q} \right)}{n-v}.
\]

Hence

\[
\frac{qc_1 \left( \log(n-v) - \log q \right)}{n-v} < \log \lambda_m < \frac{qc_2 \left( \log(n-v) - \log q \right)}{n-v}.
\]

We multiply all sides above by \( \frac{n}{\log n} > 0 \) (for \( n \) large). Then

\[
\frac{qc_1 n \left( \log(n-v) - \log q \right)}{(n-v) \log n} < \frac{n \log \lambda_m}{\log n} < \frac{qc_2 n \left( \log(n-v) - \log q \right)}{(n-v) \log n}.
\]

Note that \( \frac{n \left( \log(n-v) - \log q \right)}{(n-v) \log n} \) goes to 1 as \( n \) (and hence \( m \)) goes to \( \infty \). Since \( 0 < c_1 < 1 \) and \( c_2 > 1 \) are any numbers, it follows that

\[
\lim_{m \to \infty} \frac{n \log \lambda_m}{\log n} = \lim_{m \to \infty} \frac{(qm+v) \log \lambda_m}{\log(qm+v)} = q.
\]

\[\square\]

3. Thurston norm and fibered 3-manifolds

Let \( M \) be an oriented hyperbolic 3-manifold with boundary \( \partial M \) (possibly \( \partial M = \emptyset \)). The Thurston norm \( \| \cdot \| \) is defined on an integral class \( a \in H_2(M, \partial M; \mathbb{Z}) \) as follows.

\[
\| a \| = \min_F \{-\chi(F)\},
\]

where the minimum is taken over all oriented surface \( F \) embedded in \( M \), satisfying \( a = |F| \), with no components of non-negative Euler characteristic. The surface \( F \) which realizes this minimum is
called the \textit{minimal representative} of \(a\), denoted by \(F_a\). The norm \(\| \cdot \|\) defined on integral classes admits a unique continuous extension \(\| \cdot \|: H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}\) which is linear on the ray through the origin. The unit ball \(U_M\) with respect to the Thurston norm is a compact, convex polyhedron. See [18] for more details.

Suppose that \(M\) is a surface bundle over the circle and let \(F\) be its fiber. The fibration determines a cohomology class \(a^* \in H^1(M; \mathbb{Z})\), and hence a homology class \(a \in H_2(M, \partial M; \mathbb{Z})\) by Poincaré duality. Thurston proved in [18] that there exists a top dimensional face \(\Omega\) on \(\partial U_M\) such that \(a = [F]\) is an integral class of \(\text{int}(C_{\Omega})\). On the other hand, the minimal representative \(F_a\) for any integral class \(a\) in \(\text{int}(C_{\Omega})\) becomes a fiber of the fibration associated to \(a\). Such a face \(\Omega\) is called a \textit{fibered face}, and an integral class \(a \in \text{int}(C_{\Omega})\) is called a \textit{fibered class}.

The set of integral and rational classes of \(\text{int}(C_{\Omega})\) are denoted by \(\text{int}(C_{\Omega}(\mathbb{Z}))\) and \(\text{int}(C_{\Omega}(\mathbb{Q}))\) respectively. When \(a \in \text{int}(C_{\Omega}(\mathbb{Z}))\) is primitive, the associated fibration on \(M\) has a connected fiber represented by \(F_a\). Let \(\Phi_a: F_a \to F_a\) be the monodromy. Since \(M\) is hyperbolic, \(\phi_a = [\Phi_a]\) is pseudo-Anosov. The \textit{dilatation} \(\lambda(a)\) and \textit{entropy} \(\text{ent}(a) = \log \lambda(a)\) are defined as the dilatation and entropy of \(\phi_a\) respectively.

The entropy defined on primitive fibered classes is extended to rational classes as follows: For a rational number \(r\) and a primitive fibered class \(a\), the entropy \(\text{ent}(ra)\) is defined by \(\frac{1}{|r|}\text{ent}(a)\). It is shown by Fried in [2] that \(\frac{1}{\text{ent}}: \text{int}(C_{\Omega}(\mathbb{Q})) \to \mathbb{R}\) is concave, and in particular \(\text{ent}: \text{int}(C_{\Omega}(\mathbb{Q})) \to \mathbb{R}\) admits a unique continuous extension

\[
\text{ent}: \text{int}(C_{\Omega}) \to \mathbb{R}.
\]

Moreover Fried proved that the restriction of ent to the open fibered face \(\text{int}(\Omega)\) has the property such that \(\text{ent}(a)\) goes to \(\infty\) as \(a \in \text{int}(\Omega)\) goes to a point on \(\partial \Omega\). Thus we have a continuous function

\[
\text{Ent} = \| \cdot \| \text{ent}(\cdot): \text{int}(C_{\Omega}) \to \mathbb{R}
\]

which is constant on each ray in \(\text{int}(C_{\Omega})\) through the origin. Thus the function

\[
\text{Ent}|_{\text{int}(\Omega)}(= \text{ent}|_{\text{int}(\Omega)}): \text{int}(\Omega) \to \mathbb{R}
\]

has a minimum, denoted by \(\text{min Ent}(M, \Omega)\). Matsumoto [13] refined the result by Fried. (See also McMullen [14].) He proved that \(\frac{1}{\text{ent}}|_{\text{int}(\Omega)}: \text{int}(\Omega) \to \mathbb{R}\) is strictly concave. This implies that \(\text{min Ent}(M, \Omega)\) is achieved by a unique point in \(\text{int}(\Omega)\). The quantity \(\text{min Ent}(M, \Omega)\) is a significant invariant on the pairs \((M, \Omega)\), but we do not discuss this invariant in the present paper.

Teichmüller polynomial \(P_\Omega\), developed by McMullen [14] organizes the dilatations \(\lambda(a)\) for all \(a \in \text{int}(C_{\Omega})\). Once one computes \(P_\Omega\), the largest real root of the polynomial determined by \(P_\Omega\) and a given fibered class \(a \in \text{int}(C_{\Omega})\) gives us the dilatation \(\lambda(a)\).

4. The Magic 3-Manifold \(N\)

Monodromies of fibrations on \(N\) have been studied in [9][10][11]. (See also a survey [8].) In Sections 4.1 and 4.2 we recall some results which tell us that the topology of fibered classes \(a\) and the actual value of \(\lambda(a)\). In Section 4.3 we define a family of fibered classes \(a_{(g,p)}\) of \(N\) with two variables \(g\) and \(p\), and we shall prove that it is a suitable family to prove theorems in Section 1 (cf. Remark 4.4).

Recall that \(\Sigma_{g,n}\) is an orientable surface of genus \(g\) with \(n\) punctures. Abusing the notation, we sometimes denote by \(\Sigma_{g,n}\), an orientable surface of genus \(g\) with \(n\) boundary components.
4.1. Fibered face $\Delta$. Let $K_\alpha$, $K_\beta$ and $K_\gamma$ be the components of the 3 chain link $C_3$. They bound the oriented disks $F_\alpha$, $F_\beta$ and $F_\gamma$ with 2 holes, see Figure 1. Let $\alpha = [F_\alpha]$, $\beta = [F_\beta]$, $\gamma = [F_\gamma] \in H_2(N, \partial N; \mathbb{Z})$. The set $\{\alpha, \beta, \gamma\}$ is a basis of $H_2(N, \partial N; \mathbb{Z})$. Figure 1 illustrates the Thurston norm ball $U_N$ for $N$ which is the parallelepiped with vertices $\pm \alpha, \pm \beta, \pm \gamma, \pm (\alpha + \beta + \gamma)$ (Example 3 in Section 2). Because of the symmetry of $C_3$, every top dimensional face of $U_N$ is a fibered face.

We denote a class $\alpha x + y \beta + z \gamma \in H_2(N, \partial N)$ by $(x, y, z)$. We pick a fibered face $\Delta$ with vertices $\alpha = (1, 0, 0)$, $\alpha + \beta + \gamma = (1, 1, 1)$, $\beta = (0, 1, 0)$ and $-\gamma = (0, 0, -1)$, see Figure 1. The open face $int(\Delta)$ is written by

$$int(\Delta) = \{(X, Y, Z) \mid X + Y - Z = 1, \ X > 0, \ Y > 0, \ X > Z, \ Y > Z\}.$$  

A class $a = (x, y, z) \in H_2(N, \partial N)$ is an element of $int(C_\Delta)$ if and only if $x > 0, y > 0, x > z$ and $y > z$. In this case, we have $|a| = x + y - z$.

Let $a = (x, y, z)$ be a fibered class in $int(C_\Delta)$. The minimal representative of this class is denoted by $F_a$ or $F_{(x, y, z)}$. We recall the formula which tells us that the number of the boundary components of $F_a$. We denote the tori $\partial N(K_\alpha), \partial N(K_\beta), \partial N(K_\gamma)$ by $T_\alpha, T_\beta, T_\gamma$, respectively, where $N(K)$ be a regular neighborhood of a knot $K$ in $S^3$. Let us set $\partial_a F_{(x, y, z)} = \partial_F \cap T_a$ which consists of the parallel simple closed curves on $T_a$. We define the subsets $\partial_b F_{(x, y, z)}, \partial_c F_{(x, y, z)} \subset \partial F_{(x, y, z)}$ in the same manner. By Lemma 3.1, the number of the boundary components $\sharp(\partial_F_{(x, y, z)}) = \sharp(\partial_a F_{(x, y, z)}) + \sharp(\partial_b F_{(x, y, z)}) + \sharp(\partial_c F_{(x, y, z)})$ is given by

$$\sharp(\partial_a F_{(x, y, z)}) = \gcd(x, y + z), \ \sharp(\partial_b F_{(x, y, z)}) = \gcd(y, z + x), \ \sharp(\partial_c F_{(x, y, z)}) = \gcd(z, x + y),$$

where $\gcd(0, w)$ is defined by $|w|$.

4.2. Dilatations $\lambda(a)$ and the stable foliation $F_a$ of fibered classes $a$. Teichmüller polynomial $P_\lambda$ on the fibered face $\Delta$ is computed in Section 3.2, and it tells us that the dilatation $\lambda(x, y, z)$ of a fibered class $(x, y, z) \in int(C_\Delta)$ is the largest real root of

$$f_{(x, y, z)}(t) = t^{x+y+z} - t^x - t^y - t^{x+z} - t^{y+z} + 1,$$

see Theorem 1.1. (In fact, $\lambda(x, y, z)$ is a unique real root greater than 1 of $f_{(x, y, z)}(t)$ by Descartes’s rule of signs.)

Let $\Phi_{(x, y, z)} : F_{(x, y, z)} \rightarrow F_{(x, y, z)}$ be the monodromy of the fibration associated to a primitive class $(x, y, z) \in int(C_\Delta)$. Let $\mathcal{F}_{(x, y, z)}$ be the stable foliation of the pseudo-Anosov $\Phi_{(x, y, z)}$. The components of $\partial_a F_{(x, y, z)}$ (resp. $\partial_b F_{(x, y, z)}$, $\partial_c F_{(x, y, z)}$) are permuted cyclically by $\Phi_{(x, y, z)}$. In particular the number of prongs of $\mathcal{F}_{(x, y, z)}$ at a component of $\partial_a F_{(x, y, z)}$ (resp. $\partial_b F_{(x, y, z)}$, $\partial_c F_{(x, y, z)}$) is independent of the choice of the component. By Proposition 3.3, the stable foliation $\mathcal{F}_{(x, y, z)}$ has the property such that:

- each component of $\partial_a F_{(x, y, z)}$ has $x/\gcd(x, y + z)$ prongs,
- each component of $\partial_b F_{(x, y, z)}$ has $y/\gcd(y, x + z)$ prongs, and
- each component of $\partial_c F_{(x, y, z)}$ has $(x + y - 2z)/\gcd(z, x + y)$ prongs.

$\mathcal{F}_{(x, y, z)}$ does not have singularities in the interior of $F_{(x, y, z)}$.

4.3. Proofs of theorems. For $g \geq 0$ and $p \geq 0$, define a fibered class $a_{(g, p)}$ as follows.

$$a_{(g, p)} = (p + g + 1)a + (p - g)b = (p + g + 1, 2p + 1, p - g) \in int(C_\Delta).$$

The class $a_{(g, p)}$ is primitive if and only if $2g + 1$ and $p + g + 1$ are relatively prime. One can check the identity

$$B_{(g, p)}(t) = f_{(p + g + 1, 2p + 1, p - g)}(t).$$
(see Section 1 for the definition of \( B_{(g,p)}(t) \)). We denote by \( r_{(g,p)} \), the dilatation \( \lambda(a_{(g,p)}) \) of the fibered class \( a_{(g,p)} \). (Thus the dilatation \( r_{(g,p)} = \lambda(a_{(g,p)}) \) of \( a_{(g,p)} \) is a unique real root greater than 1 of \( B_{(g,p)}(t) \), see Section 4.2)

**Lemma 4.1.** We fix \( g \geq 0 \). Given \( 0 < c_1 < 1 \) and \( c_2 > 1 \), we have

\[
\lim_{p \to \infty} \frac{p \log r_{(g,p)}}{\log p} = 1.
\]

In particular

\[
p^{-c_2} < r_{(g,p)} < p^{-c_1} \quad \text{for } p \text{ large}.
\]

**Proof.** Apply Lemma 2.1 to the polynomial \( B_{(g,p)}(t) \).

**Lemma 4.2.** Suppose that \( a_{(g,p)} \) is primitive. Then the minimal representative \( F_{a_{(g,p)}} \) is a surface of genus \( g \) with \( 2p + 4 \) boundary components, and the stable foliation \( F_{a_{(g,p)}} \) has the following properties. If \( p + g \) is odd (resp. even), then \( \sharp(\partial_a F_{a_{(g,p)}}) = 2 \) (resp. 1) and \( \sharp(\partial_\gamma F_{a_{(g,p)}}) = 1 \) (resp. 2). A component of \( \partial_a F_{a_{(g,p)}} \) has \( \frac{p+g-1}{2} \) prongs (resp. \( p+g+1 \) prongs), and a component of \( \partial_\gamma F_{a_{(g,p)}} \) has \( p+3g+2 \) prongs (resp. \( \frac{p+3g+2}{2} \) prongs).

**Proof.** By (1.1), we have that \( \sharp(\partial_\beta F_{a_{(g,p)}}) = 2p + 1 \). We have

\[
\sharp(\partial_a F_{a_{(g,p)}}) = \gcd(p + g + 1, 3p - g + 1) = \gcd(p + g + 1, 2(2g + 1)).
\]

Since \( a_{(g,p)} \) is primitive, \( p + g + 1 \) and \( 2g + 1 \) must be relatively prime. Hence \( \sharp(\partial_a F_{a_{(g,p)}}) = 1 \) (resp. 2) if \( p + g \) is even (resp. odd). Let us compute \( \sharp(\partial_\gamma F_{a_{(g,p)}}) \). We have

\[
\sharp(\partial_\gamma F_{a_{(g,p)}}) = \gcd(3p + g + 2, p - g) = \gcd(2(2g + 1), p - g).
\]

Since \( \gcd(2g + 1, p - g) = \gcd(2g + 1, p + g + 1) = 1 \), we have that \( \sharp(\partial_\gamma F_{a_{(g,p)}}) = 2 \) (resp. 1) if \( p - g \) is even (resp. odd), equivalently \( p + g \) is even (resp. odd). The genus of \( F_{a_{(g,p)}} \) is computed from the identities \( |a_{(g,p)}| = |\chi(F_{a_{(g,p)}})| = 2p + 2g + 2 \) and \( \sharp(\partial F_{a_{(g,p)}}) = 2p + 4 \).

The singularity data of \( F_{a_{(g,p)}} \) is obtained from the formula in Section 4.2.

By Lemma 4.2, it is straightforward to prove

**Lemma 4.3.** Suppose that \( a_{(g,p)} \) is primitive. Then \( (g,p) \notin \{(0,0), (0,1), (1,0)\} \) and only if \( F_{a_{(g,p)}} \) has the property such that each component of \( \partial_a F_{a_{(g,p)}} \cup \partial_\gamma F_{a_{(g,p)}} \) does not have 1 prong. In particular if \( g \geq 2 \) and \( p \geq 0 \), then each component of \( \partial_a F_{a_{(g,p)}} \cup \partial_\gamma F_{a_{(g,p)}} \) does not have 1 prong.

We are now ready to prove theorems in Section 1.

**Proof of Theorem 1.3.** There exists a sequence of primitive fibered classes \( \{a_{(g,p_i)}\}_{i=0}^\infty \) with \( p_i \to \infty \). (In fact, if we take \( p_i = (g + 1) + (2g + 1)i \), then \( 2g + 1 \) and \( p_i + g + 1 \) are relatively prime. Hence \( a_{(g,p_i)} \) is primitive.) Then \( N \) is a \( \Sigma g_{2p_i+4} \)-bundle over the circle whose monodromy of the fibration has the dilatation \( r_{(g,p_i)} \). Therefore \( \delta g_{2p_i+4} \leq r_{(g,p_i)} \). If we set \( n_i = 2p_i + 4 \), then

\[
\frac{n_i \log \delta_{g,n_i}}{\log n_i} \leq \frac{n_i \log r_{(g,p_i)}}{\log n_i} = \frac{(2p_i + 4)r_{(g,p_i)}}{\log(2p_i + 4)}.
\]

The right hand side goes to 2 as \( i \) goes to \( \infty \), see Lemmas 2.1 and 4.4. This completes the proof.

□
Proof of Theorem 1.4. The monodromy $\Phi_{a_{(g,p)}}$ of the fibration associated to the primitive fibered class $a_{(g,p)}$ is defined on the surface of genus $g$ with $2p + 4$ boundary components. It has the dilatation $r_{(g,p)}$, and hence $\delta_{g,2p+4} \leq r_{(g,p)}$. Now let us prove $\delta_{g,2p+1} \leq r_{(g,p)}$. The fibration associated to $a_{(g,p)}$ extends naturally to a fibration on the manifold obtained from $N$ by Dehn filling two cusps specified by the tori $T_\alpha$ and $T_\gamma$ along the boundary slopes of the fiber. Then $\Phi_{a_{(g,p)}} : F_{a_{(g,p)}} \to F_{a_{(g,p)}}$ extends to the monodromy $\hat{\Phi} : \hat{F} \to \hat{F}$ of the extended fibration, where the extended fiber $\hat{F}$ is obtained from $F_{a_{(g,p)}}$ by filling each disk bounded by each component of $\partial_{\alpha}F_{a_{(g,p)}} \cup \partial_{\gamma}F_{a_{(g,p)}}$. Thus $\hat{F}$ has the genus $g$ with $2p + 1$ boundary components, see Lemma 4.2. By Lemma 4.3, $F_{a_{(g,p)}}$ does not have 1 prong at each component of $\partial_{\alpha}F_{a_{(g,p)}} \cup \partial_{\gamma}F_{a_{(g,p)}}$. Hence $F_{a_{(g,p)}}$ extends canonically to the stable foliation $\hat{F}$ of $\hat{\Phi}$. Therefore $\hat{\Phi} = [\Phi]$ is pseudo-Anosov with the same dilatation as $\Phi_{a_{(g,p)}}$. This implies that $\delta_{g,2p+1} \leq r_{(g,p)}$.

The proofs of the rest of bounds $\delta_{g,2p+2} \leq r_{(g,p)}$ and $\delta_{g,2p+3} \leq r_{(g,p)}$ are similar. In fact, the extended fiber of the fibration on the manifold obtained from $N$ by Dehn filling a cusp specified by $T_\alpha$ or $T_\gamma$ along the boundary slope of the fiber has the genus $g$ with $2p + 2$ or $2p + 3$ boundary components, see Lemma 1.2. Lemma 1.3 ensures that the extended monodromy is pseudo-Anosov with the same dilatation as $\Phi_{a_{(g,p)}}$. □

Proof of Theorem 1.5. By Theorem 1.4 together with the assumption (*) in Theorem 1.5, we have that for any $p \geq 0$ and for $j \in \{3, 4\}$,

$$\delta_{g,2p+j} \leq r_{(g,p)} \text{ or } \delta_{g,2p+j} \leq r_{(g,p+1)}.$$  

Thus

$$(4.2) \quad \frac{(2p+j) \log \delta_{g,2p+j}}{\log(2p+j)} \leq \frac{(2p+j) \log r_{(g,p)}}{\log(2p+j)} \leq \frac{(2p+j) \log r_{(g,p+1)}}{\log(2p+j)}.$$  

By Lemma 2.1 it is easy to see that the both right hand sides in (4.2) go to 2 as $p$ goes to $\infty$. Thus

$$\limsup_{p \to \infty} \frac{(2p+j) \log \delta_{g,2p+j}}{\log(2p+j)} \leq 2.$$  

Since this holds for $j \in \{3, 4\}$, the proof is done. □

Proof of Proposition 1.8. We prove the claim in the second half. (The proof in the first half is similar.) If $g \geq 2$ satisfies (*), then for any $p \geq 0$ there exist a $\Sigma_{g,2p+3}$-bundle and a $\Sigma_{g,2p+4}$-bundle over the circle obtained from $N$, see proof of Theorem 1.5. More precisely such a bundle is homeomorphic to $N$ or it is obtained from $N$ by Dehn filling cusps along the boundary slopes of the fiber. Thus Proposition 1.8 holds from the result which says that the hyperbolic volume decreases after Dehn filling, see [15] [17]. □

Remark 4.4. To address Question 1.22, we had explored fibered classes of the magic manifold whose dilatations have a suitable asymptotic behavior. We found a family of primitive fibered classes $a_{(g,p)}$ by computer. By Lemma 4.3, most of the components of $\partial F_{a_{(g,p)}}$ lie on the torus $T_\beta$. The pseudo-Anosov stable foliation associated to $a_{(g,p)}$ has the property such that each component of $\partial_\beta F_{a_{(g,p)}}$ has 1 prong. The striking property of $a_{(g,p)}$ is that the slope of the components of $\partial_\beta F_{a_{(g,p)}}$ is exactly equal to $-1$. Moreover, for any fixed $g$, the projective class $\bar{a}_{(g,p)}$ goes to a single point $(\frac{1}{2}, 1, \frac{1}{2}) \in \partial \Delta$ as $p$ goes to $\infty$. On the other hand, it is proved by Martelli and Petronio[12] that the manifold $N(-1)$ obtained from $N$ by Dehn filling a cusp along the boundary slope $-1$ is not hyperbolic. The
property such that each component of $\partial_\beta F_{\alpha(s,p)}$ has 1 prong can also be seen from the fact that $\mathbb{N}(-1)$ is a non-hyperbolic manifold.

References

[1] J. W. Aaber and N. M. Dunfield, Closed surface bundles of least volume, Algebraic and Geometric Topology 10 (2010), 2315-2342.
[2] J. H. Cho and J. Y. Ham, The minimal dilatation of a genus-two surface, Experimental Mathematics 17 (2008), 257-267.
[3] A. Fathi, L. Laudenbach and V. Poenaru, Travaux de Thurston sur les surfaces, Astérisque, 66-67, Société Mathématique de France, Paris (1979).
[4] D. Fried, Flow equivalence, hyperbolic systems and a new zeta function for flows, Commentarii Mathematici Helvetici 57 (1982), 237-259.
[5] E. Hironaka, Small dilatation mapping classes coming from the simplest hyperbolic braid, Algebraic and Geometric Topology 10 (2010), 2041-2060.
[6] E. Hironaka and E. Kin, A family of pseudo-Anosov braids with small dilatation, Algebraic and Geometric Topology 6 (2006), 699-738.
[7] N. V. Ivanov, Stretching factors of pseudo-Anosov homeomorphisms, Journal of Soviet Mathematics, 52 (1990), 2819–2822, which is translated from Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 167 (1988), 111–116.
[8] E. Kin, Notes on pseudo-Anosovs with small dilatations coming from the magic 3-manifold, Representation spaces, twisted topological invariants and geometric structures of 3-manifolds, RIMS Kokyuroku 1836 (2013) 45-64.
[9] E. Kin, S. Kojima and M. Takasawa, Minimal dilatations of pseudo-Anosovs generated by the magic 3-manifold and their asymptotic behavior, preprint (2011), arXiv:1104.3939v3, to appear in “Algebraic and geometric topology”.
[10] E. Kin and M. Takasawa, Pseudo-Anosov braids with small entropy and the magic 3-manifold, Communications in Analysis and Geometry 19 (4) (2011), 705-758.
[11] E. Kin and M. Takasawa, Pseudo-Anosovs on closed surfaces having small entropy and the Whitehead sister link exterior, Journal of the Mathematical Society of Japan 65 (2) (2013), 411-446.
[12] B. Martelli and C. Petronio, Dehn filling of the “magic” 3-manifold, Communications in Analysis and Geometry 14 (2006), 969-1026.
[13] S. Matsumoto, Topological entropy and Thurston’s norm of atoroidal surface bundles over the circle, Journal of the Faculty of Science, University of Tokyo, Section IA. Mathematics 34 (1987), 763-778.
[14] C. McMullen, Polynomial invariants for fibered 3-manifolds and Teichmüller geodesic foliations, Annales Scientifiques de l’École Normale Supérieure. Quatrième Série 33 (2000), 519-560.
[15] W. D. Neumann and D. Zagier, Volumes of hyperbolic three-manifolds, Topology 24 (3) (1985), 307-332.
[16] R. C. Penner, Bounds on least dilatations, Proceedings of the American Mathematical Society 113 (1991), 443-450.
[17] W. Thurston, The geometry and topology of 3-manifolds, Lecture Notes, Princeton University (1979).
[18] W. Thurston, A norm of the homology of 3-manifolds, Memoirs of the American Mathematical Society 339 (1986), 99-130.
[19] W. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bulletin of the American Mathematical Society 19 (1988), 417-431.
[20] W. Thurston, Hyperbolic structures on 3-manifolds II: Surface groups and 3-manifolds which fiber over the circle, preprint, arXiv:math/9801045
[21] C. Y. Tsai, The asymptotic behavior of least pseudo-Anosov dilatations, Geometry and Topology 13 (2009), 2253-2278.
[22] A. D. Valdivia, Sequences of pseudo-Anosov mapping classes and their asymptotic behavior, New York Journal of Mathematics 18 (2012), 609-620.
[23] X. Wang, A simple proof of Descartes’s rule of signs, American Mathematical Monthly 111 (6) (2004), 525-526.
THE BOUNDARY OF A FIBERED FACE OF THE MAGIC 3-MANIFOLD

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