Characteristic-dependent linear rank inequalities via complementary vector spaces

Victor Peña-Macias *
Humberto Sarria†
Departamento de Matemáticas
Universidad Nacional de Colombia
Bogotá
Colombia

Abstract

A characteristic-dependent linear rank inequality is a linear inequality that holds by ranks of subspaces of a vector space over a finite field of determined characteristic, and does not in general hold over other characteristics. In this paper, we produce new characteristic-dependent linear rank inequalities by an alternative technique to the usual Dougherty’s inverse function method [9]. We take up some ideas of Blasiak [4], applied to certain complementary vector spaces, in order to produce them. Also, we present some applications to network coding. In particular, for each finite or co-finite set of primes \(P\), we show that there exists a sequence of networks \(\mathcal{N}(k)\) in which each member is linearly solvable over a field if and only if the characteristic of the field is in \(P\), and the linear capacity, over fields whose characteristic is not in \(P\), \(\to 0\) as \(k \to \infty\).

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Introduction

Network Coding is a branch of Information Theory introduced by Ahlswede, Cai, Li and Yeung in 2000 that studies the problem of information flow through a network [1]. It has been proven that network
coding is a great tool for improving information management in contrast to the usual way routing. It is known that an algorithm exists to calculate the routing capacity of a network [5] but it is unknown if there is one for the linear capacity of a network, much less for the non-linear capacity [6]. Information inequalities play an important role in the calculation of these capacities because upper bounds have been found by treating the messages involved in the network as random variables. Any advance in the understanding of the regions determined by entropies of random variables implies an advance in network coding [8, 6, 9, 19].

There are networks whose linear capacity is smaller than the non-linear capacity [9]. Therefore, in order to understand the linear capacity of a network, it is necessary to study inequalities that are valid for random variables induced by finite dimensional vector spaces. It is well known that the entropy of these random variables is completely determined by the dimension (usually referred to as rank) of the associated vector spaces. The mentioned inequalities are called linear rank inequalities. Formally, a linear rank inequality is a linear inequality that is always satisfied by ranks of subspaces of a vector space. All information inequalities are linear rank inequalities but not all linear rank inequalities are information inequalities [18]. The first example of a linear rank inequality that is not an information inequality was found by Ingleton in [12]. This inequality was useful to calculate the linear capacity (over any field) of the Vámos network [7]. Other inequalities have been presented in [8, 10, 13].

The linear capacity of a network depends on the characteristic of the scalar field associated to the vector space of the network codes. In other words, it is possible to achieve a higher rate of linear communication by choosing one characteristic over another, an example is the Fano network [6, 7]. Therefore, when we study linear capacities over specific fields, it is also convenient to work with “linear rank inequalities” that depend on the characteristic of the scalar field associated to vector space. A characteristic-dependent linear rank inequality is a linear inequality that is always satisfied by ranks of subspaces of a vector space over fields of certain characteristic and does not in general hold over other characteristics. These are the appropriate inequalities to calculate capacities over specific fields. It is worth noting that all linear rank inequalities for up to and including five variables are known and are all characteristic-independent [8]. The first two characteristic-dependent linear rank inequalities (over seven variables) were presented by Blasiak, Kleinberg and Lubetzky in 2011. Specifically, the first inequality holds for all fields whose characteristic is not two and does not in general hold over characteristic two. The second
inequality holds for all fields whose characteristic is two and does not in general hold over characteristics other than two [4]. Their applications used linear programming problems whose constraints express information inequalities (and their new inequalities) to produce separation between linear and non-linear network coding. Using lexicographic products, the separation is amplified, yielding a sequence of networks in which the difference in linear and non-linear capacity is bigger in each network.

In 2013, Dougherty, Freiling and Zeger presented two new characteristic-dependent linear rank inequalities; again, one inequality is valid for characteristic two and the other inequality is valid for every characteristic except for two [9]. The technique used to produce these inequalities is called The inverse function method and is different from the technique used by Blasiak et al. in their inequalities. These inequalities are then used to provide upper bounds for the linear capacity of the Fano network and non-Fano network. In 2014, E. Freiling in [11, Ph.D. thesis], for each finite or co-finite set of prime numbers, constructed a characteristic-dependent linear rank inequality that is valid only for vector spaces over fields whose characteristic is in the aforementioned set. The technique that Freiling used is a generalization of the inverse function method. He also showed that for each finite or co-finite set of primes, there exists a network that is linearly solvable over a field if and only if the characteristic of the field is in the set. In this thesis appears the natural question: Are there other techniques to tighten these inequalities?

Organization of the work and contributions. This work is organized into two sections. In section 1, we introduce the basic definitions related to Linear Algebra and Information Theory. Then, we produce new characteristic-dependent linear rank inequalities by taking the central ideas of Blasiak et al. [4] but modifying some of their arguments: We take a matrix which is a generalization matrix of the representation matrix of the Fano and non-Fano matroids. Some vectorial matroids associated to this matrix are known in [14]. This matrix is used as a guide to extract some properties of vector spaces and obtain certain conditional inequalities. Then, we turn these inequalities into characteristic-dependent linear rank inequalities. We also present some cases when the desired inequalities are indeed true over any field. In section 2, we review some concepts of Network Coding and Index Coding, as well as some results of Blasiak [4] in order to define our linear programming problems which are useful for our application theorem to network coding: For each finite or co-finite set of primes \( P \), we show that there exists a sequence of networks \( \mathcal{N}(k) \) in which each member is linearly solvable over a field if and only if
the characteristic of the field is in $P$, and the linear capacity, over fields whose characteristic is not in $P$, $\rightarrow 0$ as $k \rightarrow \infty$. This means that we have a sequence of solvable networks in which we can achieve a higher rate of linear communication by choosing one characteristic in $P$ over another in the complement set of $P$, and the rate of linear communication on this last set can be as bad as we want. We remark that these networks are associated to index coding instances from vector matroids whose matrix is used in section 1. Also, we remark that the gap in capacities is obtained via lexicograph product and improves the above mentioned result of Freiling [11, Theorem 3.3.1 and 3.3.2]. Additionally, as a corollary we present many sequences of networks which the rate of (non-linear) communication is better than the rate of linear communication. It is notable that one of these sequences is a modified version of the sequence that was presented by Blasiak et al. [4, Theorem 1.2]. By last, we show that our sequences of networks have a good coding gain [15].

1. Characteristic-dependent linear rank inequalities

Let $A, A_1, \ldots, A_n, B$ be vector subspaces of a finite dimensional vector space $V$. There is a correspondence between linear rank inequalities and information inequalities associated to certain class of random variables induced by vector spaces, see [18, Theorem 2]. So, we can use notation of information theory to refer dimension of vector spaces. Let $A_i := \sum_{i \in I} A_i$ denote the span or sum of $A_i$, $i \in I \subseteq [n] := \{1, 2, \ldots, n\}$, the entropy of $A_i$ is the dimension, $H(A_i) = \dim(A_i, i \in I)$. The mutual information of $A$ and $B$ is $I(A; B) = \dim(A \cap B)$. If $B$ is a subspace of a subspace $A$, then we denote the codimension of $B$ in $A$ by $\text{codim}_A(B) := H(A) - H(B)$. For $A$ and $B$ vector subspaces, $H(A | B) = \text{codim}_A(A \cap B)$.

The sum $A + B$ is a direct sum if and only if $A \cap B = O$, the notation for such a sum is $A \oplus B$. Subspaces $A_1, \ldots, A_n$ are called mutually complementary subspaces in $V$ if every vector of $V$ has an unique representation as a sum of elements of $A_1, \ldots, A_n$. Equivalently, they are mutually complementary subspaces in $V$ if and only if $V = A_1 \oplus \cdots \oplus A_n$. In this case, $\pi_s$ denotes the canonical projection function $V \rightarrow \oplus_{i \in S} A_i$.

In the principal proof of this section we will need to calculate the difference in dimension between vector spaces, so inequalities associated to codimension given by the following two lemmas are important.
**Lemma 1** : For any subspaces $A_1, \ldots, A_m, A'_1, \ldots, A'_m$ of finite dimensional vector space $V$ such that $A'_i \leq A_i$,

$$\operatorname{codim}_{[A_m]} A'_m \leq \sum_{i=1}^{m} \operatorname{codim}_{A_i} A'_i$$

with equality if and only if $A_{k+1} \cap A_{[k]} = A'_{k+1} \cap A'_{[k]}$ for all $k$.

**Lemma 2** : For any subspaces $A, B, C$ of finite dimensional vector space $V$ such that $B \leq A$,

$$\operatorname{codim}_{(A \cap C)} (B \cap C) \leq \operatorname{codim}_{A} B$$

with equality if and only if there exists a subspace of $C$ which is complementary to $B$ in $A$.

**Inequalities using a suitable matrix as a guide.** For $n \geq 2$, $L_n$ denotes the $(n + 1) \times (2n + 3)$-matrix

$$
\begin{pmatrix}
A_1 & \cdots & A_n & A_{n+1} & B_1 & \cdots & B_n & B_{n+1} & C \\
1 & \cdots & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\
0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 & 1 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 1 & 0 & 1 & \cdots & 0 & 1 & 1 \\
0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & 1 
\end{pmatrix}
$$

The rank of the submatrix $B_{[n+1]}$ depends on the field where its inputs are defined: If the characteristic of the field divides $n$, the rank is $n$; and if the characteristic of the field does not divide $n$, the rank is maximum. Lemmas 5 and 6 (with the help of Lemma 3) present a general version of this. Specifically, these lemmas abstract the properties of linear independence between the vector spaces (over a field with certain characteristic) generated by the columns of $L_n$ to obtain inequalities associated to the rank of the vector space generated by the columns of the submatrix $B_{[n+1]}$ and the rank of the vector space generated by the column $C$.

**Lemma 3** : Let $A_{1}, A_{2}, \ldots, A_{n+1}$ be mutually complementary vector subspaces of a vector space $V$ over a field $\mathbb{F}$, and $C$ a subspace of $V$ such that the sum of $\bigoplus_{i=1,i \neq k}^{n+1} A_i$ and $C$ is a direct sum for all $k$. Then

$$H\left(\left\{\pi_{[n+1]-i}(C)\right\}_{j=1}^{n+1}\right) = \begin{cases} nH(C) & \text{if } \operatorname{char}(\mathbb{F}) \mid n \\ (n+1)H(C) & \text{if } \operatorname{char}(\mathbb{F}) \nmid n. \end{cases}$$
**Proof**: We have the following claim: A non-zero element of $C$ has $n + 1$ non-zero coordinates. Moreover, for all $i$, $H(\pi_{[n+1]-i}(C)) = H(C)$. Proof of claim.

Let $v \in C$, we can write $v = \sum_{i=1}^{n+1} v_i$, where $v_i \in A$, for $i = 1, \ldots, n + 1$. If $v_k = 0$ for some $1 \leq k \leq n + 1$, then $v \in \bigoplus_{i=1, i \neq k} A_i$, but $C$ is complementary to this space. It follows $v = 0$. Now, we consider the case when $\text{char}(\mathbb{F})$ divides $n$. For any $v = \sum_{i=1}^{n+1} v_i \in V$, taking into account that $n = 0$ and $n - 1$ is invertible in $\mathbb{F}$, we get

$$\frac{1}{n-1} \sum_{i=1}^{n} \pi_{[n+1]-i}(v) = \frac{1}{n-1} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n+1} v_j$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n+1} v_j + O \frac{n}{n-1} v_{n+1}$$

$$= \sum_{i=1}^{n} v_i$$

$$= \pi_{[n+1)-(n+1)}(v).$$

Hence, $\pi_{[n+1)-(n+1)}(C) \leq \sum_{i=1}^{n} \pi_{[n+1]-i}(C)$. Furthermore, the subspaces $\pi_{[n+1]-i}(C)$ with $i \in [n]$ form a direct sum. In effect, let $v_i = \sum_{j=1}^{n+1} v_j \in C$, $i \in [n]$ such that $\sum_{i=1}^{n} \pi_{[n+1]-i}(v_i) = 0$. Then for every $1 \leq k \leq n$, we get $\sum_{i=1}^{n} v_{ik}$ and $\sum_{i=1}^{n} v_{i}^{n+1}$ are equal to zero. Then, applying claim to each $1 \leq k \leq n$, we get $\sum_{i=1, i \neq k}^{n} v_i$ and $\sum_{i=1}^{n} v_i$ are equal to zero vector. Thus, for every $1 \leq k \leq n$, we get $v_k = \sum_{i=1}^{n} v_i - \sum_{i=1, i \neq k}^{n} v_i = 0$. Consequently, the subspaces $\pi_{[n+1]-i}(C)$, with $i \in [n]$, are mutually complementary. Applying claim to this fact, we get $H(\pi_{[n+1]-i}(C), i \in [n+1]) = H(\pi_{[n+1]-i}(C), i \in [n]) = nH(C)$. Now, we consider the case when $\text{char}(\mathbb{F})$ does not divide $n$. It is enough to prove that $\sum_{i=1}^{n+1} \pi_{[n+1]-i}(C)$ is a direct sum. In effect, for each $i = 1, \ldots, n + 1$ take $v_{i} = \sum_{j=1}^{n+1} v_{ij}$ in $C$ such that $\sum_{i=1}^{n+1} \pi_{[n+1]-i}(v_{i}) = 0$. Then for every $1 \leq k \leq n + 1$,
we get ∑_{i=1,i≠k}^{n+1} v_i^k = 0, which for claim implies, ∑_{i=1,i≠k}^{n+1} v_i = O for all k. Fixed j, add member to member all these inequalities except the inequality corresponding to k = j, we get

\[ O = \sum_{k=1,k≠j}^{n+1} \left( \sum_{i=1,i≠k}^{n+1} v_i^k \right) \]

\[ = n v_j + O(n-1) \sum_{i=1,i≠j}^{n+1} v_i. \]

Since char(\mathbb{F}) does not divide n, v_j = O. □

**Remark 4**: We remark that a subspace C as described in previous lemma holds H(C) ≤ 1.

**Lemma 5**: Let A_1, A_2, ..., A_{n+1}, B_1, B_2, ..., B_{n+1}, C be subspaces of a finite-dimensional vector space V over a scalar field \mathbb{F} whose field characteristic divides n and

(i) A_1, ..., A_{n+1} are mutually complementary in V, and subspaces C and A_{[n+1]-k} form a direct sum for all k.

(ii) B_k ≤ A_{[n+1]-k} ∩ (A_k + C) for all k.

Then H(B_{[n+1]}) ≤ nH(C).

**Proof**: By hypotheses (i) and the condition on the characteristic, we apply lemma 3 to get

\[ H(\pi_{[n+1]-k}(C), i \in [n+1]) = n H(C). \]

Furthermore, \[ \pi_{[n+1]-k}(C) = (C + A_k) \cap A_{[n+1]-k}, \] for all k. In effect, let \( v \in C \) such that \( v = \sum_{i=1}^{n+1} v_i \), where \( v_i \in A_i, i \in [n+1] \) and fixed \( k \in [n+1] \).

Noting that \[ \pi_{[n+1]-k}(v) = \sum_{i=1,i≠k}^{n+1} v_i, \]

\[ H(\pi_{[n+1]-i}(C) : i \in [n+1]) = (n+1)H(C). \]

Furthermore, \[ \pi_{[n+1]-k}(C) ≤ B_k \] for all k. In effect, fixed \( k \in [n+1] \) and let \( v = \sum_{i=1}^{n+1} v_i \in C \), where \( v_i \in A_i \). By hypothesis (iii), there exist \( a_k \in A_k \) and \( b_k \in B_k \) such that \( v = a_k + b_k \). By hypothesis (ii), there exist \( a_i \in A_i \) for
\[ j \in [n + 1] - k, \text{ such that } b_k = \sum_{j=1, j \neq k}^{n+1} a_j. \] Then \[ v = \sum_{i=1}^{n+1} v_i = a_k + \sum_{j=1, j \neq k}^{n+1} a_j \] but \( v \) has unique writing in terms of \( A_j, i \in [n + 1], \) in particular, \( a_k \in v. \) We get \( \pi_{n+1-k}(v) = v - v_k = b_k \in B_k. \) In other words, \( \pi_{n+1-k}(C) \leq B_k. \) Hence, \( \sum_{k=1}^{n+1} \pi_{n+1-k}(C) \leq \sum_{k=1}^{n+1} B_k. \) Therefore, using equation (1.2) we get, \( H(B_{n+1}) \geq (n + 1) H(C). \) 

Inequalities imply by Lemmas 5 and 6 are conditional characteristic-dependent linear rank inequalities, in the sense that they are true only for vector spaces with certain relations of linear dependency. Theorems 7 and 9 will use these inequalities to obtain characteristic-dependent linear rank inequalities. The demonstrations consists of finding vector subspaces of the original vector subspaces that satisfy the conditions of these lemmas. Then, we find an upper bounds and a lower bounds of the inequalities imply by these lemmas in terms of the original subspaces. To accomplish this, we introduce the following construction: First, we build mutually complementary subspaces \( A'_1, \ldots, A'_{n+1} \) in \( A_{n+1} \) from \( A_1, \ldots, A_{n+1} : \) Define \( A'_1 := A_1, \) and for \( k = 2, \ldots, n + 1 \) denote by \( A'_k \) a subspace of \( A_k \) which is a complementary subspace to \( A_{n+1} \) in \( A_k. \) Then \( A'_1, \ldots, A'_{n+1} \) are mutually complementary and the following equations hold:

\[
\text{codim}_{A'_k}(A'_k) = I(A_{n+1-k}; A_k), \quad (1.3)
\]

where \( A_0 = O. \) Second, we built a subspace \( \overline{C} \) of \( C \cap A'_{n+1} \) such that \( \overline{C} \) and \( A'_{n+1-k} \) form a direct sum for all \( k. \) Let \( C^{(0)} := C \cap A_{n+1}. \) Recursively, for \( k = 1, \ldots, n + 1 \) denote by \( C^{(k)} \) a subspace of \( C^{(k-1)} \) which is a complementary subspace to \( A'_{n+1-k} \) in \( C^{(k-1)} + A'_{n+1-k}. \) We denote \( \overline{C} := C^{(n+1)}, \) this space satisfies the required condition and the following equation:

\[
\text{codim}_{C}(\overline{C}) \leq H(C | A_{n+1}) + \sum_{i=1}^{n+1} I(A_{n+1-i}; C). \quad (1.4)
\]

Summarizing, from \( V, A_1, \ldots, A_{n+1} \) and \( C, \) we built a tuple of vector subspaces

\[
A'_1, \ldots, A'_{n+1}, \overline{C} \quad (1.5)
\]
in which the sum of any members is a direct sum. We remark that this tuple is not unique but in the proofs of the following two theorems we will fix one of these.
**Theorem 7:** For any \( n \geq 2 \). Let \( A_1, A_2, \ldots, A_{n+1}, B_1, B_2, \ldots, B_{n+1}, C \) be subspaces of a finite-dimensional vector space \( V \) over a scalar field \( \mathbb{F} \) whose field characteristic divides \( n \),

\[
H(B_{[n+1]}) \leq nI(A_{[n+1]}; C) + \sum_{i=1}^{n+1} H(B_i | A_{[n+1]-i}) + \sum_{i=1}^{n+1} H(B_i | A_i, C) + n \sum_{i=2}^{n} I(A_{[n+1]-i}; A_i) \\
+(n+1) \left[ I(A_{[n]}; A_{n+1}) + H(C | A_{[n+1]}) + \sum_{i=1}^{n+1} I(A_{[n+1]-i}; C) \right].
\]

**Proof:** The tuple (1.5) obtained from the given vector spaces satisfies the condition (i) of the lemma in the space \( V' = A_{[n+1]} \). To meet condition (ii), we define for \( k = 1 \) to \( k = n + 1 \), \( B_k := B_k \cap (A_{[n+1]-k}) \cap (A'_k + C) \). Subspaces \( A'_1, \ldots, A'_{n+1}, B'_1, \ldots, B'_{n+1}, C \) of \( V' \) satisfy all hypothesis of lemma 5 over a scalar field \( \mathbb{F} \) whose field characteristic divides \( n \), we get

\[
H(B'_{[n+1]}) \leq nH(C). \tag{1.6}
\]

An upper bound of this inequality (1.6) is given by

\[
H(C) \leq I(A_{[n+1]}; C) \text{ [from } \overline{C} \leq C^{(0)} \text{].} \tag{1.7}
\]

We look for an upper bound on \( \text{codim}_{B'_{[n+1]}} B'_{[n+1]} \) in order to get a lower bound on \( H(B'_{[n+1]}) \).

\[\text{codim}_{B'_{[n+1]}} B'_{[n+1]} \leq \sum_{i=1}^{n+1} \text{codim}_{B_i} B'_i \text{ [from Lemma 1].}\]

For \( k \in [n] \), we have

\[
\text{codim}_{B_k} B'_k \leq H(B_k | A'_{[n+1]-k}) + H(B_k | A'_k, C)
\]

\[
= \text{codim}_{B_k} (A'_{[n+1]-k} \cap B_k) + \text{codim}_{B_k} ([A'_k + C] \cap B_k)
\]

\[
= \text{codim}_{B_k} (A_{[n+1]-k} \cap B_k) + \text{codim}_{B_k} ([A_k + C] \cap B_k)
\]

\[
+ \text{codim}_{A_{[n+1]-k} \cap B_k} (A'_{[n+1]-k} \cap B_k) + \text{codim}_{[A_k + C] \cap B_k} ([A'_k + C] \cap B_k)
\]

\[
\leq \text{codim}_{B_k} (A_{[n+1]-k} \cap B_k) + \text{codim}_{B_k} ([A_k + C] \cap B_k)
\]

\[
+ \text{codim}_{A_k + C} (A'_k + C) \text{ [from Lemma 2].}
\]
= H(B_k \mid A_{[n+1]-k}) + H(B_k \mid A_k, C) + \text{codim}_{A_{[n+1]-k}}(A'_{[n+1]-k}) \\
+ \text{codim}_{A_k+C}(A' + C') \\
\leq H(B_k \mid A_{[n+1]-k}) + H(B_k \mid A_k, C) + \sum_{i=1}^{n+1} \text{codim}_{A_i}(A'_i) \\
+ \text{codim}_{C}(C) \quad \text{[from Lemma 1].} \\
\leq H(B_k \mid A_{[n+1]-k}) + H(B_k \mid A_k, C) + \sum_{i=2}^{n+1} I(A_{[i-1]}; A_i) + H(C \mid A_{[n+1]}) \\
+ \sum_{i=1}^{n+1} I(A_{[i+1]-1}; C) \quad \text{[from 1.3].} \\

For k = n + 1, noting that \text{codim}_{A_{[n]}}(A'_{[n]}) = 0, we get \\
\text{codim}_{B'_{[n+1]}} \leq H(B_{[n+1]} \mid A_{[n]}) + H(B_{[n+1]} \mid A_{[n+1]}, C) + I(A_{[n]}; A_{[n+1]}) \\
+ H(C \mid A_{[n+1]}) + \sum_{i=1}^{n+1} I(A_{[n+1]-1}; C). \\

Then, we find that \\
\text{codim}_{B'_{[n+1]}} \leq \sum_{i=1}^{n+1} H(B_i \mid A_{[n+1]-i}) + \sum_{i=1}^{n+1} H(B_i \mid A_i, C) + \sum_{i=1}^{n} I(A_{[i-1]}; A_{[n+1]-i}) \\
+ \sum_{i=2}^{n+1} I(A_{[i-1]}; A_i) + (n+1) \left[ H(C \mid A_{[n+1]}) + \sum_{i=1}^{n+1} I(A_{[i+1]-1}; C) \right]. \\
(1.8) \\

From (1.6), (1.7) and (1.8), we get the desired inequality. The inequality does not hold in general over vector spaces whose characteristic does not divide n. A counter example would be: In \( V = GF(p)^{n+1}, p \nmid n \), take the vector space \( A_1, \ldots, A_{n+1}, B_1, \ldots, B_{n+1} \) and \( C \) generated by the columns of the matrix \( L_n \). Then, all information measures are zero but \( H(B_{[n+1]}) = n + 1 \) and \( I(A_{[n+1]}; C) = 1 \). We get \( n \geq n + 1 \) which is a contradiction.

**Theorem 8:** If the dimension of vector space \( V \) is at most \( n \), then inequality implicated by Theorem 7 is true over any field.

**Proof:** We supose that there exist vector subspaces \( A_1, A_2, \ldots, A_{n+1}, B_1, B_2, \ldots, B_{n+1}, C \) of a vector space \( V \) of dimension at most \( n \) that do not hold the desired inequality i.e.
\[ H(B_{[n+1]}) > nI(A_{[n+1]}; C) + \sum_{i=1}^{n+1} H(B_i \mid A_{[n+1]-i}) + \sum_{i=1}^{n+1} H(B_i \mid A_i, C) \]
\[
+ n \sum_{i=2}^{n+1} I(A_{[i-1]}; A_i) + (n+1) \left[ I(A_{[n]}; A_{n+1}) + H(C \mid A_{[n+1]}) + \sum_{i=1}^{n+1} I(A_{[i+1]-i}; C) \right],
\]
and find a contradiction. Since \( H(B_{[n+1]}) \leq n \), the right side of the inequality is at most \( n - 1 \). Hence, \( I(A_{[i-1]}; A_i) = I(A_{[n+1]}; C) = H(C \mid A_{[n+1]}) = 0 \) for all \( i \). So, we get \( \emptyset A_i \), is a direct sum and \( C \) is the zero space. Then, the inequality becomes \( H(B_{[n+1]}) > \sum_{i=1}^{n+1} \left[ H(B_i \mid A_{[n+1]-i}) + H(B_i \mid A_i) \right] \). We note that if \( H(B_i \mid A_{[n+1]-i}) = 0 \) then \( H(B_i \mid A_i) = H(B_i) \); if \( H(B_i \mid A_i) = 0 \) then \( H(B_i \mid A_{[n+1]-i}) = H(B_i) \), and at least \( n + 3 \) summands are zeros in the right side of the inequality. With this in mind, we get an inequality of the form \( H(B_S) > \sum_{i \in S} H(B_i) \), where \( B_i \neq \emptyset \) for \( i \in S \) which is a contradiction.

We want to remark that the characteristic-dependent linear rank inequalities which is valid for fields whose characteristic is different from two in [4, Theorem 6.2] is wrong. The error is due to a failure to determine upper bounds on the ranks of some vector spaces in the proof. A counter example for that inequality would be: Let \( V_{100} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, V_{010} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, V_{001} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, V_{101} = V_{110} = V_{111} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) be vector subspaces of \( \text{GF}(p)^3 \) with \( p \neq 2 \). Then we get \(-3 \geq 0\) which is a contradiction. So, in the case \( n = 2 \), the following inequality corrects this error.

**Theorem 9:** For any \( n \geq 2 \). Let \( A_1, A_2, \ldots, A_{n+1}, B_1, B_2, \ldots, B_{n+1}, C \) be subspaces of a finite-dimensional vector space \( V \) over a scalar field \( \mathbb{F} \) whose field characteristic does not divide \( n \),

\[
H(C) \leq \frac{1}{n+1} H(B_{[n+1]}) + H(C \mid A_{[n+1]}) + \sum_{i=1}^{n+1} I(A_{[n+1]-i}; C) + \sum_{i=1}^{n+1} H(C \mid A_i, B_i) \]
\[
+ n \sum_{i=2}^{n} I(A_{[i-1]}; A_i) + (n+1) I(A_{[n]}; A_{n+1}) + \sum_{i=1}^{n+1} H(B_i \mid A_{[n+1]-i}).
\]
Proof: The tuple $(1.5)$ obtained from the given vector spaces satisfies the condition (i) of the lemma 6 in the space $V' = A'_{[n+1]}$. To meet condition (ii), we define for $k = 1$ to $k = n + 1$, $B'_k := B_k \cap (A'_{[n+1]} - k)$. We get

$$\text{codim}_{B_k} (B'_k) \leq H(B_k | A_{[n+1]-k}) + \sum_{i=2, i \neq k}^{n+1} I(A_{[i-1]}; A_i), k \in [n] \quad (1.9)$$

$$\text{codim}_{B_{n+1}} (B'_{n+1}) = H(B_{n+1} | A_{[n]}). \quad (1.10)$$

By last, to meet condition (iii), we obtain a new subspace of $\bar{C}$ that also satisfies (i) by following way. Let $\bar{C}^{(0)} := \bar{C}$, for $k = 1$ to $k = n + 1$, denote by $\bar{C}^{(k)} := \bar{C}^{(k-1)} \cap (A'_k + B'_k)$. Define $\hat{C} = \bar{C}^{(n+1)}$. The subspaces $A'_1, \ldots, A'_{n+1}, B'_1, \ldots, B'_{n+1}$, $\hat{C}$ of $V'$ satisfy all hypothesis of lemma 6, we get

$$(n+1)H(\hat{C}) \leq H(B'_{[n+1]}) \quad (1.11)$$

We have to get an upper bound and a lower bound using (1.11). Obviously,

$$H(B'_{[n+1]}) \leq H(B_{[n+1]}). \quad (1.12)$$

We look for an upper bound on codim$_{\bar{C}} \hat{C}$ in order to get a lower bound on $H(\hat{C})$,

$$\text{codim}_{\bar{C}} \hat{C} = \text{codim}_{\bar{C}} \bar{C} + \text{codim}_{\bar{C}} \hat{C}$$

$$\leq H(C | A_{[n+1]} - n + 1) + \sum_{i=1}^{n+1} I(A_{[i+1]-n+1}; C) + \sum_{k=1}^{n+1} H(C | A'_{k} + B'_k) \quad \text{[from (1.4) and definition of $\hat{C}$]}

= H(C | A_{[n+1]} - n + 1) + \sum_{i=1}^{n+1} I(A_{[i+1]-n+1}; C) + \sum_{k=1}^{n+1} \text{codim}_{\bar{C}} (C \cap [A'_{k} + B'_k])

= H(C | A_{[n+1]} - n + 1) + \sum_{i=1}^{n+1} I(A_{[i+1]-n+1}; C) + \sum_{k=1}^{n+1} \text{codim}_{\bar{C}} (C \cap [A'_{k} + B'_k])

+ \sum_{k=1}^{n+1} \text{codim}_{(C \cap [A'_{k} + B'_k])} (C \cap [A'_{k} + B'_k])

\leq H(C | A_{[n+1]} - n + 1) + \sum_{i=1}^{n+1} I(A_{[i+1]-n+1}; C) + \sum_{k=1}^{n+1} H(C | A'_{k} + B'_k)
+\sum_{k=1}^{n+1} \text{codim}_{A_k \oplus B_k} (A'_k + B'_k) \quad \text{[from Lemmas 2 and (1)]}

\leq H(C \mid A_{[n+1]}) + \sum_{i=1}^{n+1} I(A_{[n+1]-i}; C) + \sum_{k=1}^{n+1} H(C \mid A_k, B_k) + \sum_{i=1}^{n+1} \text{codim}_{A'_i} (A'_i)

\leq H(C \mid A_{[n+1]}) + \sum_{i=1}^{n+1} I(A_{[n+1]-i}; C) + \sum_{k=1}^{n+1} H(C \mid A_k, B_k) + \sum_{i=1}^{n+1} I(A_{[n+1]-i}; A_i) + \sum_{j=1}^{n+1} H(B_j \mid A_{[n+1]-i})

\leq H(C \mid A_{[n+1]}) + \sum_{i=1}^{n+1} I(A_{[n+1]-i}; C) + \sum_{k=1}^{n+1} H(C \mid A_k, B_k) + n \sum_{i=2}^{n} I(A_{[n+1]-i}; A_i) + (n+1)I(A_{[n+1]}; A_{n+1}) + \sum_{j=1}^{n+1} H(B_j \mid A_{[n+1]-i})

\text{[from (1.9)]}

From (1.11), (1.12) and last inequality, we get the desired inequality. The inequality does not hold in general over vector spaces whose characteristic divides \( n \). A counter example would be: In \( V = GP(p)^{n+1} \), \( p \mid n \), take the vector space \( A_1, A_2, \ldots, A_{n+1}, B_1, B_2, \ldots, B_{n+1} \) and \( C \) generated by the columns of the matrix \( L_n \). Then, all information measures are zero but \( H(B_{[n+1]}) = n \) and \( H(C) = 1 \). We get \( n + 1 \leq n \) which is a contradiction.

**Theorem 10**: If the dimension of vector space \( V \) is at most \( n \), then inequality implicated by Theorem 9 is true over any field.

**Proof**: We suppose that there exist vector subspaces \( A_1, A_2, \ldots, A_{n+1}, B_1, B_2, \ldots, B_{n+1}, C \) of a vector space \( V \) of dimension at most \( n \) that do not hold the desired inequality i.e.

\[
H(C) > \frac{1}{n+1} H(B_{[n+1]}) + H(C \mid A_{[n+1]}) + \sum_{i=1}^{n+1} I(A_{[n+1]-i}; C) + \sum_{i=1}^{n+1} H(C \mid A_i, B_i) + n \sum_{i=2}^{n} I(A_{[n+1]-i}; A_i) + (n+1)I(A_{[n+1]}; A_{n+1}) + \sum_{j=1}^{n+1} H(B_j \mid A_{[n+1]-i}).
\]
and find a contradiction. Since $H(A_{[n+1]}) \leq n$, there exists at least one $A_k$ such that $A_k \leq A_{[n+1]-k}$. Then, the summing $H(C | A_{[n+1]}) + \sum_{i=1}^{n+1} I(A_{[n+1]-i};C)$ on the right side of the desired inequality can be write as $H(C) + \sum_{i=1, i \neq k}^{n+1} I(A_{[n+1]-i};C)$. Hence, the right side of the inequality has negative information measures which is a contradiction.

An alternative demonstration of the above theorem and Theorem 8 can be obtained by noting that Lemma 3 is trivial when the dimension of $V$ is at most $n$.

2. Network Coding

We will first briefly review some concepts of network coding in order to fix some index coding terms. We study network coding with networks in representation circuit, see [17], so each node represents a coding function and hence the same message flows every edge coming out of the same node. We emphasize that this approach loses no generality and can be modified to coincide with other network models such as the one used by Dougherty et al. [5, 6]. Formally, a network $N = (V, E)$ is an acyclic multidirected-graph. There exist source and receiver nodes and a (demand) function $\tau$ from collection of receivers $T$ onto collection of sources $S$. There exist an alphabet $A$, and a finite collection of $k$-tuples of $A$ called messages. Each source node has a message. A $(k, n)$-network code specifies a alphabet $A$, two natural numbers $k$ and $n$, and a collection of functions, one for each node of the network ($f_v$), such that

- If $v$ is a source, $f_v = \text{id}_{A^k}$ (these functions are generally omitted).
- If $v$ is not neither source or receiver, $f_v$ is a function from $\text{Im} f_{v^-}$ to $A^v$, where $\text{Im} f_{v^-} := \prod_{w \in v^-} \text{Im} f_w$.
- If $v$ is a receiver, $f_v$ is a function called decoding function from $\text{Im} f_{v^-}$ to $A^v$.

A network code is linear if all their functions are linear fictions over the same finite field.

To capture the idea of transmit information through the network, there is another collection of functions $(f_v^*)_{v \in V}$ on $A^{|\mathcal{A}|}$, specified by the network code, defined by
\[ f_v^* := \pi_v, \text{ if } v \text{ is a source.} \]

\[ f_v^*(x) := f_v(f_v^*(x)) = f_v((f_v^*(x)) \text{ over } v^* \text{ for all } x \in \mathcal{A}^{|\mathcal{H}|}, \text{ if } v \in V - S. \]

The value \( f_v^*(x) \) gives the message that is carried on the node for a given tuple of messages \( x \). A network code is a solution if for all tuple of messages \( x \) and \( t \in T, f_t^*(x) = x_{t(i)} \) (i.e. the demand of each receiver is satisfied).

The network coding problem of \( \mathcal{N} \) is to find some alphabet, and efficient solution over this alphabet. The efficiency is measured by the ratio \( \frac{k}{n} \). The capacity of \( \mathcal{N} \) respect to a class of functions \( \mathcal{D} \) over \( A \) is

\[
C_D^A(\mathcal{N}) := \sup \left\{ \frac{k}{n} : \exists (k,n) - \text{solution in } \mathcal{D} \text{ over } A \right\}.
\]

\( \mathcal{D} \) is usually though as the collection of all network codes, in this case the capacity is usually refereed as non-linear coding capacity. Also \( \mathcal{D} \) can be taken as the collection of linear codes over determined finite fields (or over any finite field).

A network is defined to be [6, 7]:

- **Solvable over** \( A \) if there exists a \((1, 1)\)-solution over \( A \), and solvable if the network is solvable over some \( A \).
- **Scalar linearly solvable over** \( F \) if there exists a \((1, 1)\)-linear solution over \( F \), and scalar linearly solvable if the network is scalar linearly solvable over some \( F \).
- **(Vector) Linearly solvable over** \( F \) if there exists a \((k, k)\)-linear solution over \( F \), for some \( k \geq 1 \), and linearly solvable if the network is (vector) linearly solvable over some \( F \).
- **Asymptotically solvable over** \( A \) if for any \( \epsilon > 0 \), there exists a \((k, n)\)-solution over \( A \) such that \( \frac{k}{n} > 1 - \epsilon \), and the network is asymptotically solvable if the network is asymptotically solvable over some \( A \).
- **Asymptotically linearly solvable over** \( F \) if for any \( \epsilon > 0 \), there exists a \((k, n)\)-linear solution over \( F \) such that \( \frac{k}{n} > 1 - \epsilon \), and the network is asymptotically linearly solvable if the network is asymptotically linearly solvable over some \( F \).

In this paper, we will use the following class of networks.
**Definition 11**: Let $m$ be a natural number. A $m$-index coding-network is a network with sources $S$ and receivers $T$ and a collection $[m]$ of $m$-intermediate nodes called $m$-block such that $S \times [m], [m] \times T \subseteq E$.

The network in case $m = 1$ is simply called index coding-network and corresponds to the index coding instance studied in [2, 3]. In this case, the set of messages indexed by nodes of $t \cap S$ is known as the additional information of $t$. The message carried on intermediate node is called broadcast message. Also, the network is completely determined by $(S, E^*)$, where $E^* := \{((\pi(t), t \cap S) \in E : t \in T)\}$. To refer to these networks, we write $N = (S, E)$. From this, it is easy to obtain other $m$-index coding network $N[m] = (S, E)$, letting $E = (S \times [m]) \cup ([m] \times T) \cup E^*$. The relationship between $N$ and $N[m]$ is established by the following lemma.

**Lemma 12**: Let $m \in \mathbb{N}$. A $(k, n)$-solution of index coding-network $N$, implies a $(mk, n)$-solution of $N[m]$. Indeed, $C \_D(N[m]) = mC \_D(N)$, where $D$ can be the collection of all the codes or linear codes.

From parameter of index coding instances to network coding parameters. The broadcast rate for an index coding instance is defined in [2]. This parameter coincides with the inverse multiplicative of the capacity of the index coding network associated to the instance. In the following we show some results from [4] in our network coding context.

We use the following class of linear programming problems [4]: The $(LP)$ linear programming problem with constraint matrix $A$ for an index coding-network $N$ is to determine $\min(z)_{Y \subseteq S}$ for tuples of non-negative real numbers $(z_Y)_{Y \subseteq S}$ such that

(i) $z_Y = |S|$

(ii) $\forall Z \subseteq Y$ $z_Y - z_Z \leq |Y - \text{cl}(Z)|$, where $\text{cl}(Z) := Z \cup \{s \in S : \exists (s, Y) \in E^*, Y \subseteq Z\}$.

(iii) $Az \geq 0$.

The optimal solution is denoted by $b_\_A(N)$. The inverse multiplicative of this value is denoted\footnote{in case $b = 0$, $B = \infty$.} by $B_\_A(N)$. We remark that conditions (i) and (ii) are associated to information flow of $N$, and condition (iii) enumerates a list $A$ of constraints correspond to information inequalities or (characteristic-dependent) linear rank inequalities. When $A$ enumerates the constraints correspond to information inequalities, $B_\_A$ is an upper bound on the
capacity of \(N\); when \(A\) enumerates the constraints correspond to (characteristic-dependent) linear rank inequalities, \(B_\lambda\) is an upper bound on the linear capacity of \(N\) over the alphabets in which the linear rank inequalities are valid. This is easy to see, consider a \((k, n)\)-solution of \(N\) over \(A\). Let \(X_1, \ldots, X_{|S|}\) be independent uniformly distributed random variables (associated to messages) over \(A^k\) and \(P\) be a random variable (associated to broadcast message) over \(A^n\). Take the base of the entropy function as \(|A|^k\).

Let \(z = H(X_Y \cup P)\), we can verify that \((z_Y)_{Y \subseteq S}\) is a feasible primal solution of linear programming problem. Thus, \(z = H(P) \leq \frac{n}{k}\), yielding \(C(N) \leq B_\lambda(N)\). The upper bound on the linear capacity is obtained in a similar way. The subscript in \(b_\lambda(N)\) or \(B_\lambda(N)\) is omitted when \(A\) corresponds to the constraints of the submodular inequality.

The lexicographic product of index coding networks \(N_1\) and \(N_2\), denoted by \(N_1 \cdot N_2\), is a index coding network whose source set is \(S_1 \times S_2\). Each receiver \(t\) is indexed by a pair \((t_1, t_2)\) of receivers of \(N_1\) and \(N_2\) such that \(t(t) = (\pi(t_1), \pi(t_2))\) and \(\mathcal{F} \cap (S_1 \times S_2) = [(\pi(t_1) \cap S_1) \times S_2] \cup [\pi(t_1) \times (\pi(t_2) \cap S_2)]\). The \(k\)-fold lexicographic power of \(N\) is denoted by \(N^\ast_k\). Since the broadcast rate is sub-multiplicative and \(b\) is super-multiplicative under the lexicographic products [4], the capacity of index coding-networks is super-multiplicative and \(B\) is sub-multiplicative under the lexicographic products i.e. \(C(N_1) \leq C(N_1 \cdot N_2)\) and \(B(N_1 \cdot N_2) \leq B(N_1)B(N_2)\).

We want to define linear programming problems, using our inequalities, whose solutions behave super-multiplicatively under lexicographic products, we make this by the following argument: In [4, Theorem 6.3], it is presented a matrix \(B\) whose transpose matrix has the property that if \(\beta\) is the associated vector of a linear rank inequality over \(\mathbb{F}\), then \(\beta = B\) is the associated vector of a tight linear rank inequality over \(\mathbb{F}\). We can take the associated vectors of the inequalities of the Theorems 7 and 9. Then, we apply this matrix to get two tight characteristic-dependent linear rank inequalities: For any \(A_1, A_2, \ldots, A_{n+1}, B_1, B_2, \ldots, B_{n+1}, C\) and \(P\) vector subspaces of \(V\), we get

\[
H(B_{[n+1]} | P) + \sum_{i=1}^{n+1} H(B_i \mid A_{[n+1]} \setminus B_{[n+1]-i}, C, P) + (n+1)H(C \mid A_{[n+1]} \setminus B_{[n+1]}, P) \\
\leq (n+1)\sum_{i=1}^{n+1} I(A_{[n+1]-i} \mid C \mid P) + nI(A_{[n+1]} \mid C \mid P) + \sum_{i=1}^{n+1} H(B_i \mid A_{[n+1]-i}, P)
\]

\(2\) A linear inequality \(\alpha \bullet v \geq 0\) is called tight if it is balanced and \(\Sigma \alpha_i = 1\).
\[ + \sum_{i=1}^{n+1} H(B_i | A_i, C, P) + n \sum_{i=2}^{n} I(A_{[i-1]} ; A_i | P) + (n+1)I(A_{[n]} ; A_{n+1} | P) + (n+1)H(C | A_{[n+1]} , P), \]  

(2.1)

when char($\mathbb{F}$) divides $n$;

\[
\begin{align*}
H(C | P) + (n+1)H(C | A_{[n+1]} , B_{[n+1]} , P) + \frac{n+2}{n+1} \sum_{i=1}^{n+1} H(B_i | A_{[n+1]} , B_{[n+1]-i} , C, P) \\
\leq \frac{1}{n+1} H(B_{[n+1]} | P) + H(C | A_{[n+1]} , P) + \sum_{i=1}^{n+1} I(A_{[n+1]-i} ; C | P) \\
+ \sum_{i=1}^{n+1} H(C | A_i , B_i , P) + n \sum_{i=2}^{n} I(A_{[i-1]} ; A_i | P) + (n+1)I(A_{[n]} ; A_{n+1} | P) \\
+ \sum_{i=1}^{n+1} H(B_i | A_{[n+1]-i} , P),
\end{align*}
\]

(2.2)

when char($\mathbb{F}$) does not divide $n$. We use these inequalities to define two new linear programming problems adding the constraints imply by each one of theses inequalities to the matrix $A$ of LP with constraint matrix given by submodular inequality. The linear programming problem which use the first inequality, we shall call LP-$A_n$; and the linear programming problem which use the second inequality, we shall call LP-$B_n$. The optimal solutions are denoted by $b_A^n$ and $b_B^n$. The following inequality is a constraint which is satisfied by LP-$A_n$, this is obtained from inequality 2.1 and [4, Lemma 6.4],

\[
(2n^2 + 3n + 1)z_\varnothing + 2(n+1)z_{A_{[n+1]} , B_{[n+1]} , C} + z_{\varnothing_{[n+1]} , C} + \sum_{i=1}^{n+1} \left( z_{A_i , C} + (n+1)z_{A_{[n+1]-i} , C} \right) \\
+ (n+2)z_{A_{[n+1]} , B_{[n+1]} , C} \\
\leq + z_{\varnothing_{[n+1]} , C} + n \sum_{i=1}^{n} \left( z_{A_i} + z_{A_{[n+1]-i}} \right) + (n^2 + 3n + 1)z_C \\
+ (n+1)(z_{A_{[n]} , A_{[n]} , B_{[n+1]}} + z_{A_{[n]} , A_{[n]} , B_{[n+1]}}) \\
+ \sum_{i=1}^{n+1} (z_{A_{[n+1]-i} , B_i , C} + z_{A_i , B_i , C} + z_{A_{[n+1]-i} , B_i , C}); \tag{2.3}
\]

in analogous way, the following inequality is a constraint which is satisfied by LP-$B_n$, this is obtained from inequality 2.2 and [4, Lemma 6.4],

\[
+ (2n+3)z_{A_{[n+1]} , B_{[n+1]} , C} + \sum_{i=1}^{n+1} (z_{A_{[n+1]-i} , B_i , C} + z_{A_i , B_i , C}) + (n+2)z_{A_{[n+1]} , B_{[n+1]} , C} \\
+ \frac{n^3 + 2n^2 + 2n + 2}{n+1} z_\varnothing \leq \frac{1}{n+1} z_{\varnothing_{[n+1]} , C} + z_{C , A_{[n+1]} , B_{[n+1]} , C} + (n+1)z_{A_{[n+1]} , B_{[n+1]} , C};
\]
\[ + \frac{n+2}{n+1} \sum_{i=1}^{n} z_{\beta_{i+1}}^2 + z_{\alpha_{i+1}}^2 + z_{\alpha_{i}}^2 + (n+1)z_{\alpha_{n+1}}^2 \\
\]
\[ +nz_c + \sum_{i=1}^{n+1} \left( z_{\alpha_{i}, \beta_{i}}^2 + z_{\alpha_{i+1}, \beta_{i}}^2 \right). \] (2.4)

By last, from [4, Theorem 3.4], we get that the optimal solutions of our LP-problems are super-multiplicative under lexicographic products.

**Index coding from matroids.** A matroid is an abstract structure that captures the notion of independence in linear algebra [16]. Let \( \mathcal{M} = (S, r) \) be a matroid and let \( J \) be the set of coloops of \( \mathcal{M} \) (each element is in no circuit). Consider the matroid obtained by deletion of \( J \), \( \mathcal{M} | J = (S - J, r|_J) \). Define the index coding network associated to \( \mathcal{M} \) by an index coding-network, denoted by \( \mathcal{N}_\mathcal{M} \), with source set \( S - J \) and \( E_M^* := \{(s, C - s) : C \text{ is a circuit in } \mathcal{M} | J, s \in C \} \). This construction is a modification of the construction given by Blasiak et al. [4, Definition 5.1]. Our network has a smaller number of sources and receivers because it is completely determined by the circuits of the matroid. We introduce the following definition in order to study the properties of this network.

**Definition 13:** An index coding network \( \mathcal{N}' = (S, E_{\mathcal{N}'}^*) \) is called an index coding-subnetwork of \( \mathcal{N} \) if \( E_{\mathcal{N}'}^* \subseteq E_{\mathcal{N}}^* \) and there exists a collection \( \{(s, S_s)\}_{s \in S} \) of elements of \( E_{\mathcal{N}}^* \) such that \( T := \bigcup_{s \in S} S_s \) is a minimum subset of \( S \), with the property that for all \( s \in S \), \((s, T_s) \in E_{\mathcal{N}'}^* \), for some \( T_s \subseteq T \). This is equivalent to \( cl_{\mathcal{N}'} \leq cl_{\mathcal{N}} \) and \( r_{d_{\mathcal{N}'}} = r_{d_{\mathcal{N}}^*} \), where \( r_{cl} := \min \{|T| : cl(T) = S\} \).

The definition of subnetwork guarantees that the network flow of a subnetwork behaves like the network flow of the network. Specifically, a solution of \( \mathcal{N} \) is a solution of \( \mathcal{N}' \) and \( b(\mathcal{N}') \leq b(\mathcal{N}) \). Furthermore, the index coding network of a matroid \( \mathcal{M} \) is an index coding-subnetwork of the index coding-network obtained from the index coding instance associated to the matroid \( \mathcal{M} | J \) of Blasiak et al. With this in mind, the following proposition (and proof) is a rewriting of [4, Proposition 5.2 and Theorem 5.4].

**Proposition 14:** Let \( \mathcal{M} = (S, r) \) be a matroid. For any index coding-subnetwork \( \mathcal{N} \) of the index coding network \( \mathcal{N}_\mathcal{M} \)

\[ B(\mathcal{N}) = \frac{1}{|S| - r_{\mathcal{M}}}. \]

Also, if some \( \mathcal{M} \) is representable over \( \mathbb{F} \), then
\[ C(\mathcal{N}) = C_{\text{linear}}^\mathcal{F}(\mathcal{N}) = \frac{1}{|S|-r_M} \]

and this capacity is achieved by a \((1, |S| - r_M)\)-linear solution over \(\mathcal{F}\).

**Applications**: We use index coding-networks from matroids for our theorem. Fixed \(n\). For a field \(\mathcal{F}\), matrix \(L_n\) over \(\mathcal{F}\) induces a vector matroid \(\mathcal{M}(L_n)\) with ground set \(S := \{A_1, \ldots, A_{n+1}, B_1, \ldots, B_{n+1}, C\}\), some of these are known in [14] for \(n\) prime. If we change the field, it is possible that the vector matroid changed. However, these matroids have some properties in common. Specifically, certain subsets of the ground set of \(\mathcal{M}(L_n)\) are always circuits according to the characteristic of \(\mathcal{F}\) divides or does not divide \(n\). We classify them in two types\(^3\): The collection \(A_n := \{A_{i+1}, C, A_{i+1}, B_i, A_i B C, B_{i+1} : i \in [n+1]\}\) is a subclass of circuits in any \(\mathcal{M}(L_n)\) over \(\mathcal{F}\), when \(\text{char}(\mathcal{F})\) divides \(n\); and the collection \(B_n := \{A_{i+1}, C, A_{i+1}, B_i, A_i B C, B_{i+1} C : i \in [n+1]\}\) is a subclass of circuits in any \(\mathcal{M}(L_n)\) over \(\mathcal{F}\), when \(\text{char}(\mathcal{F})\) does not divide \(n\). We define \(\mathcal{N}_{A_n}\) as the index coding with the source set \(S\) and \(\mathcal{E}_{A_n}^\mathcal{C} := \{(s, C - s) : C \in A_n, s \in \mathcal{C}\}\); and \(\mathcal{N}_{B_n}\) as the index coding with the source set \(S\) and \(\mathcal{E}_{B_n}^\mathcal{C} := \{(s, C - s) : C \in B_n, s \in \mathcal{C}\}\). Before continuing, the following statements are useful.

**Lemma 15**: For any \(\mathcal{N}_1\) and \(\mathcal{N}_2\). If \(\mathcal{N}_1\) has a \((n, m)\)-linear solution and \(\mathcal{N}_2\) has a \((k, n)\)-linear solution both over the same field, then \(\mathcal{N}_1 \cdot \mathcal{N}_2\) has a \((k, m)\)-linear solution.

**Proof**: Let \(f\) be the function on the intermediate node and \(f_{t_1}\) be the decoding function on a receiver \(t_1\) of the desired \((n, m)\)-linear solution of \(\mathcal{N}_1\), and let \(g\) be the function on the intermediate node and \(g_{t_2}\) be the decoding function on a receiver \(t_2\) of the desired \((k, n)\)-linear solution of \(\mathcal{N}_2\). Define \(g'(x) := (g(x_{s_x} s_2))_{x \in \mathcal{E}^dI, s_2} \in \mathcal{F}^m\), and let \(h = f g'\) be the function on the intermediate node in \(\mathcal{N}_1 \cdot \mathcal{N}_2\). We obtain the broadcast message \(h(x) \in \mathcal{F}^m\). Let \(t\) be a receiver in \(\mathcal{N}_1 \cdot \mathcal{N}_2\) such that \(\tau(t) = (\tau(t_1), \tau(t_2))\) and \(t \cap (S_1 \times S_2) = [(t'_1 \cap S_1) \times S_2] \cup [(t'_2 \cap S_2) \times (t'_2 \cap S_2)]\). We have \(f_{t_1}(h(x), (g(x_{s_x} s_2))_{x \in t'_1 \cap S_1}) = f_{t_1}(f((g(x_{s_x} s_2))_{x \in t'_1 \cap S_1}), (g(x_{s_x} s_2))_{x \in t'_2 \cap S_2}) = g(x_{\tau(t_1) \times S_2})\). Then,

\(^3\) Here we use the notation \(A_i := \{A_i : i \in I\}\).
These equations and \( h \) clearly define a \((k, m)\)-linear solution of \( N_1 \cdot N_2 \).

**Lemma 16**: For \( k \in \mathbb{N} \). If \( N \) has a \((1, n)\)-linear solution, then \( N^{*^k} \) has a \((1, n^k)\)-linear solution.

**Proof**: By induction, case \( k = 2 \), take \( N_1 = N_2 = N \) in Lemma 15 and note that \( N_2 \) has a \((n, n^2)\)-linear solution by repetition of the given solution of \( N \). We get a \((1, n^2)\)-linear solution of \( N_1 \cdot N_2 \). Now, we suppose that case \( k - 1 \) holds i.e. \( N^{*^{k-1}} \) has a \((1, n^{k-1})\)-linear solution. Take \( N_1 = N, N_2 = N^{*^{k-1}} \) in Lemma 15 and note that \( N_1 \) has a \((n^{k-1}, n^n)\)-linear solution by repetition of the given solution of \( N \). Then, \( N^{*^k} \) has a \((1, n^n)\)-linear solution.

**Theorem 17**: For any \( k, n \in \mathbb{N}, n \geq 2 \). We have,

(i) \( N^{*^k}_A[(n + 2)^k] \) is linearly solvable over a field \( \mathbb{F} \) if, and only if, \( \text{char}(\mathbb{F}) \) divides \( n \). Also, when \( \text{char}(\mathbb{F}) \nmid n \),

\[
\left( \frac{n + 2}{n + 3} \right)^k \leq C_{\text{linear}}^{\mathbb{F}} \left( N^{*^k}_A \left[ (n + 2)^k \right] \right) \leq \left( \frac{5n^3 + 22n^2 + 31n + 14}{5n^3 + 22n^2 + 31n + 15} \right)^k.
\]

(ii) \( N^{*^k}_B[(n + 2)^k] \) is linearly solvable over a field \( \mathbb{F} \) if, and only if, \( \text{char}(\mathbb{F}) \) does not divide \( n \). Also, when \( \text{char}(\mathbb{F}) \mid n \),

\[
\left( \frac{n + 2}{n + 3} \right)^k \leq C_{\text{linear}}^{\mathbb{F}} \left( N^{*^k}_B \left[ (n + 2)^k \right] \right) \leq \left( \frac{n^3 + 8n^2 + 19n + 14}{n^3 + 8n^2 + 19n + 15} \right)^k.
\]

**Proof**: For (i), we have that \( N_{A_n} \) is an index coding-subnetwork of any \( N_{A_n^{(l_{un})}} \) when \( \text{char}(\mathbb{F}) \) divides \( n \). Using Lemma 14, we have \( C(N_{A_n}) = C^{\mathbb{F}}_{\text{linear}}(N_{A_n}) = \frac{1}{n+2} \) when \( \text{char}(\mathbb{F}) \) divides \( n \) and this capacity is achieved by a \((1, n + 2)\)-linear solution over \( \mathbb{F} \). By Lemma 16 with \( N = N_{A_n}, N^{*^k}_A \) has a \((1, (n + 2)^k)\)-linear solution over \( \mathbb{F} \). Finally, by Lemma 12, \( N^{*^k}_A[(n + 2)^k] \) has a \((n + 2)^k, (n + 2)^k\)-linear solution over \( \mathbb{F} \) which implies that \( N^{*^k}_A[(n + 2)^k] \) is linearly solvable over a field \( \mathbb{F} \) whose \( \text{char}(\mathbb{F}) \) divides \( n \). We estimate an upper bound on \( C_{\text{linear}}^{\mathbb{F}}(N_{A_n}) \) when \( \text{char}(\mathbb{F}) \) does not divide \( n \), using the LP-B_n: Let \( (\zeta_S)_{S \subseteq V} \) be a solution of LP-B_n for \( N_{A_n} \). From definition of \( N_{A_n} \), we have:
(a) If $Y$ is a dependent set in each $\mathcal{M}(L_n)$ (char ($F$) divides $n$), then
\[ z_Y \leq z_{\emptyset} + r_{\mathcal{M}(L_n)}(Y). \]

(b) If $Y$ is an independent set in each $\mathcal{M}(L_n)$ (char ($F$) divides $n$), then
\[ |Y| + n + 2 \leq z_Y \leq |Y| + z_{\emptyset}. \]

We can use constraints implied by these conditions along with the constraint 2.4 to get
\[ z_{\emptyset} \geq \frac{5n^2 + 22n^2 + 31n + 15}{5n^2 + 12n + 7} \]
which implies that $b_{\mathcal{B}_n}(\mathcal{N}_{A_n}^*) \geq \frac{5n^2 + 22n^2 + 31n + 15}{5n^2 + 12n + 7}$. By super-multiplicative of $b_{\mathcal{B}_n}$ under lexicographic products,
\[ b_{\mathcal{B}_n}(\mathcal{N}_{A_n}^*) \geq \left( \frac{5n^2 + 22n^2 + 31n + 15}{5n^2 + 12n + 7} \right)^k. \]
Then, $C_{\text{linear}}^F(\mathcal{N}_{A_n}^*) \leq \frac{5n^2 + 12n + 7}{5n^2 + 22n^2 + 31n + 15}$. Hence, using Lemma 12 with $m = (n + 2)^k$, $C_{\text{linear}}^F(\mathcal{N}_{A_n}^*((n + 2)^k)) \leq \frac{5n^2 + 12n + 7}{5n^2 + 22n^2 + 31n + 15} < 1$, when char($F$) does not divide $n$.

For (ii), we have that $\mathcal{N}_{A_n}$ is an index coding-subnetwork of any $\mathcal{N}_{\mathcal{M}(L_n)}$ when char($F$) does not divide $n$. Using Lemma 14, we have $C(\mathcal{N}_{\mathcal{B}_n}^*) = C_{\text{linear}}^F(\mathcal{N}_{\mathcal{B}_n}^*) = \frac{1}{n + 2}$ when char($F$) does not divide $n$ and this capacity is achieved by a $(1, n + 2)$-linear solution over $F$. Then, we apply an argument as in (i) to get the required linear solution of $\mathcal{N}_{\mathcal{B}_n}^*((n + 2)^k)$. We estimate an upper bound on $C_{\text{linear}}^F(\mathcal{N}_{\mathcal{B}_n}^*)$ when char($F$) divides $n$ using the LP-$\mathcal{A}_n$: Let $(z_s)_{s \subseteq V}$ be a solution of LP-$\mathcal{A}_n$ for $\mathcal{N}_{\mathcal{B}_n}^*$. From definition of $\mathcal{N}_{\mathcal{B}_n}^*$, we have that this network satisfies conditions (a)-(b) of part (i) when the matroid $\mathcal{M}(L_n)$ is taken over a field $F$ whose char($F$) does not divide $n$. We can use constraints implied by these conditions along with the constraint
\[ 2.3 \text{ to get } z_{\emptyset} \geq \frac{n^2 + 8n^2 + 19n + 15}{n^2 + 6n + 7} \]
which implies that $b_{\mathcal{A}_n}(\mathcal{N}_{\mathcal{B}_n}^*) \geq \frac{n^2 + 8n^2 + 19n + 15}{n^2 + 6n + 7}$. Then, by super-multiplicative of $b_{\mathcal{A}_n}$ under lexicographic products,
\[ b_{\mathcal{A}_n}(\mathcal{N}_{\mathcal{B}_n}^*) \geq \left( \frac{n^2 + 8n^2 + 19n + 15}{n^2 + 6n + 7} \right)^k. \]
Thus, $C_{\text{linear}}^F(\mathcal{N}_{\mathcal{B}_n}^*) \leq \frac{n^2 + 6n + 7}{n^2 + 8n^2 + 19n + 15}$. When char($F$) divide $n$. Hence, using Lemma 12 with $m = (n + 2)^k$, $C_{\text{linear}}^F(\mathcal{N}_{\mathcal{B}_n}^*((n + 2)^k)) \leq \frac{n^2 + 8n^2 + 19n + 14}{n^2 + 8n^2 + 19n + 15} < 1$, when char($F$) divides $n$.

For the remaining lower bounds on the linear capacities over fields in which the networks are not linearly solvable, we use the network topology in common of $\mathcal{N}_{A_n}$ and $\mathcal{N}_{B_n}$: We add the message of $C$ to the broadcast...
message of the $(1, n + 2)$-linear solution of $\mathcal{N}_{\mathcal{B}_n}$ over $\mathbb{F}$ when $\text{char}(\mathbb{F})$ does not divide $n$ to obtain a $(1, n + 3)$-linear code which is a linear solution of $\mathcal{N}_{\mathcal{A}_n}$ over this field. Then, the solution is extended to a $(n + 2)^k, (n + 3)^k)$-linear solution of $\mathcal{N}_{\mathcal{A}_n}^k [(n + 2)^k]$ yielding $\left( \frac{n+2}{n+3} \right)^k \leq C_{\text{linear}}^F (\mathcal{N}_{\mathcal{A}_n}^k [(n + 2)^k])$. In an analogous way, we get the respective lower bound on $C_{\text{linear}}^F (\mathcal{N}_{\mathcal{B}_n}^k [(n + 2)^k])$, when $\text{char}(\mathbb{F})$ divides $n$.

Corollary 18: Let $P$ be a finite or co-finite set of primes. There exists a sequence of networks $(\mathcal{N}_p^k)_k$ in which each member is linearly solvable over a field $\mathbb{F}$ if and only if the characteristic of $\mathbb{F}$ is in $P$. Furthermore, when $\text{char}(\mathbb{F})$ is not in $P$, $C_{\text{linear}}^F (\mathcal{N}_p^k) \rightarrow 0$ as $k \rightarrow \infty$.

Proof: In the previous theorem, take $n = \prod_{p \in P} P$ if $P$ is finite and $n = \prod_{p \in P} P$ if $P$ is co-finite.

The following corollary is a straightforward consequence of the theorem 17, and it is a generalization of [4, Theorem 1.2]. The proof is followed taking: $\mathcal{N}_n^k = \mathcal{N}_{\mathcal{A}_n} \cdot \mathcal{N}_{\mathcal{B}_n}$, and for all $k \in \mathbb{N}$, $\mathcal{N}_{\mathcal{A}_n}^k = \mathcal{N}_{\mathcal{A}_n}^k [(n + 2)^k]$. Then, we apply an argument as the previous theorem.

Corollary 19: There exists a infinite collection of sequences of networks $\{(\mathcal{N}_n^k)_k : n \in \mathbb{N}, n \geq 2\}$ in which each member of each sequence is asymptotically solvable but is not asymptotically linearly solvable and the linear capacity $\rightarrow 0$ as $k \rightarrow \infty$ in each sequence.

The network coding gain is equal to the coding capacity divided by the routing capacity. In [11, 15], there are two sequences of networks $\mathcal{N}(k)$ ($i = 1, 2$) such that the coding gain $\rightarrow \infty$ as $k \rightarrow \infty$. The routing capacities of $\mathcal{N}_p^k$ and $\mathcal{N}_n^k$ are $\left( \frac{n+2}{2n+3} \right)^k$ and $\left( \frac{n^2+2n+4}{4n^2+12n+9} \right)^k$, respectively. Hence, any sequence of networks presented previously satisfies this property.

Corollary 20: The network coding gain of the sequences $(\mathcal{N}_p^k)_k$ and $(\mathcal{N}_n^k)_k$ $\rightarrow \infty$ as $k \rightarrow \infty$.

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