Reverse Test and Characterization of Quantum Relative Entropy

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Abstract

The aim of the present paper is to give axiomatic characterization of quantum relative entropy utilizing resource conversion scenario. We consider two sets of axioms: non-asymptotic and asymptotic. In the former setting, we prove that the upperbound and the lowerbound of $D^Q(\rho||\sigma)$ is $D^R(\rho||\sigma) := \text{tr} \, \rho \ln \sqrt{\sigma^{-1}} \sqrt{\rho}$ and $D(\rho||\sigma) := \text{tr} \, \rho (\ln \rho - \ln \sigma)$, respectively. In the latter setting, we prove uniqueness of quantum relative entropy, that is, $D^Q(\rho||\sigma)$ should equal a constant multiple of $D(\rho||\sigma)$. In the analysis, we define and use reverse test and asymptotic reverse test, which are natural inverse of hypothesis test.

1 Introduction

Many problems in quantum/classical information theory can be viewed as conversion between given resources and ‘standard’ resources, and such viewpoint had turned out to be very fruitful. This manuscript will exploit this scenario in asymptotic theory of quantum estimation theory (with some comments on classical estimation theory). Resource conversion scenario was first explored in axiomatic theory of entanglement measures. The optimal asymptotic conversion ratio from maximally entangled states (‘standard’ resource) to a given state is called entanglement cost, while the optimal ratio for inverse conversion is called distillable entanglement. It had been shown that all quantities which satisfies a set of reasonable axioms takes value between these two quantities. Similar argument had been applied to classical/quantum channels, and so on.

The aim of the present paper is to give axiomatic characterization of quantum relative entropy utilizing resource conversion scenario. We consider two sets of axioms: non-asymptotic and asymptotic. In both cases, we require a quantum relative entropy $D^Q(\rho||\sigma)$ is monotone decreasing by application of any CPTP map. In addition, in the former setting, we assume quantum relative entropy coincide with its classical counterpart for probability distributions $\{p, q\}$: $D^Q(p||q) = D(p||q)$. Then we can prove that the upperbound and the lowerbound of $D^Q(\rho||\sigma)$ is

$$D^R(\rho||\sigma) := \text{tr} \, \rho \ln \sqrt{\sigma^{-1}} \sqrt{\rho},$$

and

$$D(\rho||\sigma) := \text{tr} \, \rho (\ln \rho - \ln \sigma),$$

respectively. In the latter setting, in stead, $D^Q(\rho||\sigma)$ is supposed to satisfy some asymptotic properties, namely weak additivity and lower asymptotic continuity, which will be defined later. Under such assumptions, we prove uniqueness of quantum relative entropy, that is, $D^Q(\rho||\sigma)$ should equal a constant multiple of $D(\rho||\sigma)$.

In the analysis, newly defined reverse test and asymptotic reverse test play key role. The former is a conversion from a pair $\{p, q\}$ of probability distributions to a pair $\{\rho, \sigma\}$ of quantum states, and the latter is an approximate conversion from a pair $\{p^n, q^n\}$ of probability distributions over the binary set $\{0, 1\}$ to a pair $\{\rho^n, \sigma^n\}$ of quantum states. Each of them is natural inverse of optimal measurement for hypothesis and hypothesis test of Neyman-Pearson type, and optimal measurement for hypothesis test, respectively.

In the course of analyzing reverse test, we show operational meaning of RLD Fisher information. Also, we prove joint convexity of $D^R(\rho||\sigma)$. 

1
2 Main results

In the paper, the totality of density operators in the Hilbert space $\mathcal{H}$ is denoted by $S(\mathcal{H})$, and the totality rank $r$ elements is denoted by $S_r(\mathcal{H})$. Unless otherwise mentioned, we suppose $d := \dim \mathcal{H} < \infty$. We consider following conditions.

(M) (Monotonicity) For any CPTP map $\Lambda$,
$$D^Q(\rho||\sigma) \geq D^Q(\Lambda(\rho)||\Lambda(\sigma)).$$

(N) (Normalization) For any probability distributions $\{p, q\}$,
$$D^Q(p||q) = D(p||q) := \sum_x p(x) (\ln p(x) - \ln q(x)).$$

(A) (Weak additivity)
$$D^Q(\rho^\otimes n||\sigma^\otimes n) = n D^Q(\rho||\sigma).$$

(C) (Lower asymptotic continuity)
$$\lim_{n \to \infty} \left\| \tilde{\rho}^n - \rho^\otimes n \right\| = 0 \implies \lim_{n \to \infty} \frac{1}{n} \left\{ D^Q(\tilde{\rho}^n||\sigma^\otimes n) - D^Q(\rho^\otimes n||\sigma^\otimes n) \right\} \geq 0. \quad \text{(1)}$$

Define
$$D(\rho||\sigma) := \text{tr} \rho (\ln \rho - \ln \sigma),$$
$$D^R(\rho||\sigma) := \text{tr} \rho \ln \sqrt{\rho \sigma^{-1}} \sqrt{\rho},$$
and denote by $M(\rho)$ the probability distribution of the data from the application of the measurement $M$ to $\rho$.

Theorem 2.1 If (M) and (N) are satisfied,
$$\max_M D(M(\rho)||M(\sigma)) \leq D^Q(\rho||\sigma) \leq D^R(\rho||\sigma).$$

Theorem 2.2 If (M), (N) and (A) are satisfied,
$$D(\rho||\sigma) \leq D^Q(\rho||\sigma) \leq D^R(\rho||\sigma).$$

Theorem 2.3 If (M), (A), and (C) are satisfied,
$$D^Q(\rho||\sigma) = \text{const.} \times D(\rho||\sigma).$$

2.1 Proof of Main theorems

Below, reverse test of a pair of states $\{\rho, \sigma\}$ means the triplet $(\Phi, \{p, q\})$ of a CPTP map $\Phi$ and probability distributions $p, q$ with
$$\Phi(p) = \rho, \quad \Phi(q) = \sigma.$$

We use following theorems to prove main theorems:

Theorem 2.4
$$\min D(p||q) = D^R(\rho||\sigma),$$
where minimization is taken for over all reverse tests $(\Phi, \{p, q\})$ of $\{\rho, \sigma\}$.  

**Theorem 2.5** (Hiai-Petz [8]) For any states $\rho$ and $\sigma$, and constant $c > 0$, we can find a projective measurement $M^n := \{P^n, 1 - P^n\}$ such that:

$$\lim_{n \to \infty} \text{tr} P^n \rho \otimes^n = 1,$$

$$\lim_{n \to \infty} \frac{1}{n} \ln \text{tr} P^n \sigma \otimes^n \geq D(\rho||\sigma) - c,$$

$$\lim_{n \to \infty} \frac{1}{n} D(M^n (\rho \otimes^n) || M^n (\sigma \otimes^n)) \geq D(\rho||\sigma) - c.$$

**Proposition 2.6** $D(\rho||\sigma)$ satisfies the condition (C).

**Proof.** By Fannes’s inequality, when $n$ is very large,

$$\frac{1}{n} \left| D(\rho \otimes^n || \sigma \otimes^n) - D(\tilde{\rho} \otimes^n || \sigma \otimes^n) \right| \leq \frac{\left\| \rho \otimes^n - \tilde{\rho} \otimes^n \right\|_1 \ln d + \frac{1}{n}}{\ln \sigma} \to 0.$$



**Theorem 2.7** If $D(\rho_0||\sigma_0) > D(\rho||\sigma)$, there is a sequence $\{\Psi^n\}$ of TPCP map with

$$\lim_{n \to \infty} \left\| \Psi^n (\rho \otimes^n) - \rho \otimes^n \right\|_1 = 0, \quad \Psi^n (\sigma \otimes^n) = \sigma \otimes^n. \quad (2)$$

Conversely, if such $\{\Psi^n\}$ with (2) exists, $D(\rho_0||\sigma_0) \geq D(\rho||\sigma)$.

Proof of Theorem 2.1. That $D^R$ satisfies (M) is known [8], but here we give another proof. By Theorem 2.4,

$$D^R(\Lambda (\rho) || \Lambda (\sigma)) = \min \{ D(p||q) : \Phi (p) = \Lambda (\rho), \Phi (q) = \Lambda (\sigma) \}$$

$$\leq \min \{ D(p||q) : \Phi = \Psi \circ \Lambda, \text{ s.t.}, \Psi (p) = \rho, \Psi (q) = \sigma \}$$

$$= \min \{ D(p||q) : \Psi (p) = \rho, \Psi (q) = \sigma \}$$

$$= D^R(\rho||\sigma).$$

So $D^R$ satisfies (M). (N) is obviously satisfied. Also, that $\max_M D(M(\rho$$ || M(\sigma))$ satisfies (M) and (N) is trivial.

Letting $(\Phi, \{p, q\})$ be an optimal reverse test, due to (N) and (M),

$$D^R(\rho||\sigma) = D(p||q) \overset{(N)}{=} D^Q(p||q)$$

$$\geq \overset{(M)}{D^Q(\Phi(p)||\Phi(q))} = D^Q(\rho||\sigma).$$

Also,

$$D(M(\rho$$ || M(\sigma)) \overset{(N)}{=} D^Q(M(\rho$$ || M(\sigma)) \leq \overset{(M)}{D^Q(\rho||\sigma)}.$$

Proof of Theorem 2.2. Since $D^R(\rho||\sigma)$ is weakly additive, we have the upper bound. The lower bound is known [6], but also can be easily obtained by Theorem 2.1 and Theorem 2.5.

Proof of Theorem 2.3. Without loss of generality, we can suppose

$$D^Q(\rho_0||\sigma_0) = D(\rho_0||\sigma_0).$$
for some $\rho_0, \sigma_0$. For given $\rho$ and $\sigma$, let $l, l', m, m'$ be integers with
\[
\frac{l'}{m'}D (\rho_0||\sigma_0) < D (\rho||\sigma) < \frac{l}{m}D (\rho_0||\sigma_0).
\]
By Theorem 2.7 there is $\{\Psi^n\}$ with
\[
\lim_{n \to \infty} \|\Psi^n (\rho_0^{\otimes l n}) - \rho^{\otimes mn}\|_1 = 0,
\]
\[
\Psi^n (\sigma_0^{\otimes l n}) = \sigma^{\otimes mn}.
\]
Since $D^Q$ is satisfies (A), (C), and (M), we have
\[
mD^Q (\rho||\sigma) = \lim_{(A)} n \to \infty \frac{1}{n}D^Q (\rho^{\otimes mn}||\sigma^{\otimes mn})
\]
\[
\leq \lim_{(C)} n \to \infty \frac{1}{n}D^Q (\Psi^n (\rho_0^{\otimes l n}) ||\Psi^n (\sigma_0^{\otimes l n}))
\]
\[
\leq \lim_{(M)} n \to \infty \frac{1}{n}D^Q (\rho^{\otimes mn}||\sigma^{\otimes mn}) = lD^Q (\rho_0||\sigma_0),
\]
or
\[
D^Q (\rho||\sigma) \leq \frac{l}{m}D (\rho_0||\sigma_0) = \frac{l}{m}D (\rho||\sigma_0).
\]
Exchanging $\{\rho_0, \sigma_0\}$ and $\{\rho, \sigma\}$ in the above argument, we obtain
\[
D^Q (\rho||\sigma) \geq \frac{l'}{m'}D (\rho_0||\sigma_0) = \frac{l'}{m'}D (\rho||\sigma_0).
\]
Taking $\frac{l}{m}$ and $\frac{l'}{m'}$ arbitrarily close to $\frac{D (\rho||\sigma)}{D (\rho_0||\sigma_0)}$, we have
\[
D^Q (\rho||\sigma) = D (\rho||\sigma).
\]

3 Monotone metric

3.1 Classical Fisher Information as a monotone metric

Let us consider a family of probability distribution $\{ p_\theta; \theta \in \Theta \subset \mathbb{R}^m \}$ over the finite set $\mathcal{X}$, $|\mathcal{X}| < \infty$. A logarithmic derivative is defined by $l_{\theta,i} := \partial_i \ln p_\theta$, where $\partial_i := \frac{\partial}{\partial \theta_i}$. Fisher information $J_\theta$ (sometimes denoted as $J_{p_\theta}$) is defined by
\[
J_{\theta,i,j} := \sum_x p_\theta (x) l_{\theta,i} (x) l_{\theta,j} (x) = \sum_x l_{\theta,i} (x) \partial_j p_\theta (x).
\]
It is known that, with some regularity condition, the optimal asymptotic mean square error of an estimate of $\theta$ equals $J_\theta^{-1}$.

Being positive definite and covariant by the coordinate change of the parameter space, $J_\theta$ induces a Riemannian metric, or an inner product in the tangent space $T_\theta$ by
\[
J_\theta (\partial_i p_\theta, \partial_j p_\theta) := J_{\theta,i,j},
\]
where the representation of $T_\theta$ is chosen as span $\{\partial_i p_\theta; i = 1, \ldots, m\}$. This metric brings about the following intuitive picture: the precision of estimate is proportional to the distance between $p_\theta$ and $p_{\theta + \Delta \theta}$.

Hereafter, the differential map of affine map $\Lambda$ is also denoted by $\Lambda$, by abusing the notation. Cencov [2] had proven:

**Theorem 3.1** [2] Suppose a Riemannian metric $g_{p_\theta}$ is monotone decreasing by application of Markov maps,
\[
g_{p_\theta} (X, X) \geq g_{\Lambda (p_\theta)} (\Lambda (X), \Lambda (X)) .
\]
Then, $g_{p_\theta}$ is the one induced by Fisher information, up to a constant multiple.

In the proof, it is essential that the metric is Riemannian, i.e., the norm in the tangent space is defined via an inner product. This assumption can be replaced by weak additivity and asymptotic lower continuity [12].
3.2 SLD and RLD Fisher information

We consider a family \( \{ \rho_\theta : \theta \in \Theta \subset \mathbb{R}^m \} \) of density operators, and suppose the map \( \theta \to \rho_\theta \) is smooth enough, and \( \Theta \) is open. Define a symmetric logarithmic derivative (SLD) \( L^S_{\theta,i} \) and a right logarithmic derivative (RLD) \( L^R_{\theta,i} \) as a solution to the matrix equation,

\[
\partial_i \rho_\theta = \frac{1}{2} (L^S_{\theta,i} \rho_\theta + \rho_\theta L^S_{\theta,i} ) = L^R_{\theta,i} \rho_\theta.
\]

If \( \rho_\theta \) is strictly positive, \( L^S_{\theta,i} \) and \( L^R_{\theta,i} \) are uniquely defined in this way. If \( \rho_\theta \) has zero eigenvalues, \( L^S_{\theta,i} \) still can be defined, but not uniquely. \( L^R_{\theta,i} \) exists (and if exists, unique) if and only if \( \partial_i \rho_\theta \) has non-zero eigenvalues only in the support of \( \rho_\theta \). Observe they are quantum analogues of a classical logarithmic derivative, \( l_{\theta,i} = \partial_i \ln \rho_\theta(x) \).

SLD Fisher information matrix \( J^S_{\theta} \) and RLD Fisher information matrix \( J^R_{\theta} \) is defined as

\[
J^S_{\theta,i,j} = \Re \Tr \rho_\theta L^S_{\theta,i} L^S_{\theta,j}, \quad J^R_{\theta,i,j} = \Tr \rho_\theta L^R_{\theta,i} L^R_{\theta,j},
\]

respectively [9]. They are quantum analogues of classical Fisher information \( J_\theta \), and, being positive definite, each of them induces inner product to the tangent space \( \mathcal{T}_\theta \),

\[
J^S_{\theta}(\partial_\theta \rho_\theta, \partial_\theta \rho_\theta) := J^S_{\theta,i,j}, \quad J^R_{\theta}(\partial_\theta \rho_\theta, \partial_\theta \rho_\theta) := J^R_{\theta,i,j},
\]

where we represent \( \mathcal{T}_\theta \) by span \( \{ \partial_i \rho_\theta ; i = 1, \ldots, m \} \). We sometimes use notations such as \( J^S_{\rho_\theta} \) and \( J^R_{\rho_\theta} \) to indicate that the underlying family of states is \( \{ \rho_\theta \} \). Even if \( \rho_\theta \) is not full-rank, \( J^S_{\theta} \) is uniquely defined, regardless the indefiniteness of SLD.

An operational meaning of SLD Fisher metric is given through estimation of \( \theta \) in an asymptotic setting, just like its classical counterpart. For the detail, see, for example, [5]. Here, we point out relation of SLD Fisher information to classical Fisher information of the family \( \{ M(\rho_\theta) \} \).

**Theorem 3.2** [2][13][5]

\[
J^S_{M(\rho_\theta)} \leq J^S_{\rho_\theta}, \quad (3)
\]

Also, for any \( X \in \mathcal{T}_\theta \), there is a measurement \( M \) with

\[
J^S_{M(\rho_\theta)}(M(X), M(X)) = J^S_{\rho_\theta}(X, X).
\]

3.3 RLD and Reverse SLD

Denote by \( \mathcal{W} \) the totality of matrices \( W \) with \( \tr WW^\dagger = 1 \). The totality of \( d \times d' \) elements of \( \mathcal{W} \) is denoted by \( \mathcal{W}_{d,d'} \), where \( d' \geq r = \text{rank } \rho \). Consider a map form \( \mathcal{W} \) to \( \mathcal{S}(\mathcal{H}) \) such that

\[
W \to WW^\dagger.
\]

A meaning of this map is as follows. Let

\[
W = [\sqrt{p_1} |\phi_1\rangle, \ldots, \sqrt{p_{d'}} |\phi_{d'}\rangle],
\]

then,

\[
WW^\dagger = \sum_{i=1}^{d'} p_i |\phi_i\rangle \langle \phi_i|.
\]

**Proposition 3.3** There is a Hermitian matrix \( L \) with \( A = BL \) if and only if \( AB^\dagger = BA^\dagger \) and \( \text{Im } A \subset \text{Im } B \).

**Proof.** Since ‘only if’ is trivial, we show ‘if’. Consider the singular decomposition of \( B \):

\[
B = UXV,
\]

where \( U \) and \( V \) is \( d \times r \)- and \( r \times d' \)-matrix, with \( U^\dagger U = VV^\dagger = 1 \), respectively. Let

\[
L := V^\dagger X^{-1} U^\dagger AB^\dagger UX^{-1} V + \tilde{V}^\dagger \tilde{V} C^\dagger V + V^\dagger V C \tilde{V}^\dagger \tilde{V}.
\]
Here $\tilde{V}$ is $(d - r) \times d'$-matrix with $\tilde{V}\tilde{V}^\dagger = 1$ and $\tilde{V}V^\dagger = 0$, and $C$ is a matrix with $A = BC$ (existence of such $C$ is due to $\text{Im} A \subset \text{Im} B$). Since $AB^\dagger$ is Hermitian, $L$ is Hermitian. Also,

$$BL = U XV \left\{ V^\dagger X^{-1} U^\dagger A (UXV)^\dagger UX^{-1} V + \tilde{V}^\dagger \tilde{V} C^\dagger V^\dagger V + V^\dagger VC\tilde{V}^\dagger \tilde{V} \right\} = AV^\dagger V + UXV\tilde{V} C^\dagger V^\dagger V + UXVC\tilde{V}^\dagger \tilde{V} = A \left( V^\dagger V + \tilde{V}^\dagger \tilde{V} \right) = A.$$  

Hence, $L$ satisfies required condition, and the assertion is proved. ■

Since

$$(L_{\theta,i}^{R} W)^\dagger = L_{\theta,i}^{R,\theta} = \rho_{\theta} L_{\theta,i}^{R} = W \left( L_{\theta,i}^{R} \right)^\dagger,$$

due to Proposition 3.3

$$L_{\theta,i}^{R} W = WA_{\theta,i}^{R}, \quad \exists A_{\theta,i}^{R} = (A_{\theta,i}^{R})^\dagger.$$ $A_{\theta,i}^{R}$ is called the reverse SLD at $W$.

On the other hand, let $A$ be an arbitrary $d' \times d'$ Hermitian matrix. Observe that the image of $WAW^\dagger$ is a subspace of the image of $WW^\dagger$. Therefore, for an arbitrary reverse SLD, there is a RLD, i.e.,

$$\forall A = A^\dagger, \exists L^{R} L^{R} W W^\dagger = W A W^\dagger,$$

and, letting $Q$ be the projection onto $(\ker W)^\perp = \text{Im} W^\dagger,$

$$L^{R} W = W A Q.$$

(Especially, if $d' = r$, $L^{R} W = W A$.) Therefore, we have .

$$\text{Tr} \rho L^{R} L^{R} = \text{Tr} L^{R} W W^\dagger L^{R} = \text{Tr} W A W^\dagger \leq \text{Tr} W (A^2) W^\dagger. \quad (4)$$

Especially, if $d' = r$, the equality holds.

### 3.4 Reverse estimation of quantum state family and RLD

The heart of quantum statistics is optimization of a measurement, i.e., choice of a measurement which converts a family of quantum states to the most informative classical probability distribution family. In estimation of the parameter $\theta$ in asymptotic situation, we maximize the output Fisher information $J_{M(\rho_{\theta})}$ by modifying $M$.

Now, we consider the reverse of above, i.e., generation of the quantum state family $\{\rho_{\theta}\}$: a pair $(\Phi, \{\rho_{\theta}\})$ is said to be a reverse estimation of $\{\rho_{\theta}\}$ if

$$\Phi (\rho_{\theta}) = \rho_{\theta}, \quad \forall \theta \in \Theta.$$ 

Classical version of this is nothing but randomization criteria of deficiency, the concept which plays key role in statistical decision theory [18]. Let us introduce ‘local’ version of this condition. We say $(\Phi, \{\rho_{\theta}, \partial_{i} \rho_{\theta}; i = 1, \cdots, m\})$ is tangent reverse estimation of $\{\rho_{\theta}, \partial_{i} \rho_{\theta}; i = 1, \cdots, m\}$ if

$$\Phi (\rho_{\theta}) = \rho_{\theta}, \Phi (\partial_{i} \rho_{\theta}) = \partial_{i} \rho_{\theta},$$

hold at $\theta$. (In statistical decision theory, when this relation holds, we say $\{\rho_{\theta}, \partial_{i} \rho_{\theta}; i = 1, \cdots, m\}$ is locally deficient relative to $(\rho_{\theta}, \partial_{i} \rho_{\theta}; i = 1, \cdots, m)$ at $\theta$ [18].)

Now let us consider the $m = 1$-case, and optimize $(\Phi, \{\rho_{\theta}, d \rho_{\theta} / d \theta\})$ to minimize the Fisher information $J_{\rho_{\theta}}$. Let us denote by $\delta_{x}$, the delta-distribution at $x$. Suppose $\Phi (\delta_{x})$ is pure (this can be supposed without loss of generality) and let

$$|\phi_{x}\rangle \langle \phi_{x}| := \Phi (\delta_{x}).$$
Then (5) is rewritten as
\[
\rho_\theta = \sum_{x=1}^{d'} p_\theta(x) |\phi_x\rangle \langle \phi_x |,
\]
\[
\frac{dp_\theta}{d\theta} = \sum_{x=1}^{d'} \frac{dp_\theta(x)}{d\theta} |\phi_x\rangle \langle \phi_x |.
\]

If \( p_\theta(x) = 0 \) and \( \frac{dp_\theta}{d\theta} \neq 0 \), the input Fisher information is infinite. So let us suppose this is not the case, and let us define
\[
W = [\sqrt{p_\theta(1)} |\phi_1\rangle, \ldots, \sqrt{p_\theta(l)} |\phi_x\rangle]
\]
\[
A = \text{diag} \left( \frac{1}{p_\theta(1)} \frac{dp_\theta(1)}{d\theta}, \ldots, \frac{1}{p_\theta(d')} \frac{dp_\theta(d')}{d\theta} \right).
\]

Then, the input Fisher information is
\[
\sum_{x=1}^{d'} p_\theta(x) \left( \frac{1}{p_\theta(x)} \frac{dp_\theta(x)}{d\theta} \right)^2 = \text{tr} \, W^2 A^\dagger \geq J_{\rho_\theta}^R,
\]
where the inequality is due to (4).

On the other hand, let us suppose \( \text{rank} \, W = r \) and \( W \) satisfies \( WW^\dagger = \rho_\theta \), and let \( A \) be the reverse SLD at \( W \), \( WAW^\dagger = \frac{dp_\theta}{d\theta} \). Then this \( A \) achieves the equality of (4). Since \((WU)(WU)^\dagger = \rho_\theta \) for unitary matrix \( U \), we can suppose that \( A \) is diagonal, by choosing \( W \) properly. Therefore, one can define \( \Phi \) by tracking above process inverse way, which achieves identity of (4). Therefore, we have:

**Theorem 3.4** Suppose \( \text{dim } \Theta = 1 \). Then
\[
J_{\rho_\theta} \geq J_{\rho_\theta}^R
\]
holds for all the reverse estimation of \( \{\rho_\theta\} \), and there is a tangent reverse estimation \( (\Phi, \{p_\theta, \frac{dp_\theta}{d\theta}\}) \) with
\[
J_{\rho_\theta} = J_{\rho_\theta}^R.
\]

\( m > 1 \)-case is briefly discussed in Appendix [7].

### 3.5 Monotone metric

In this subsection, to avoid notational complexity, we let \( m = 1 \), and abbreviate \( J_{\rho_\theta}^S \left( \frac{dp_\theta}{d\theta}, \frac{dp_\theta}{d\theta} \right) \) as \( J_{\rho_\theta}^S \), and so on. Corresponding statement for \( m > 1 \)-case will be easily obtained by considering its appropriate one dimensional subfamily.

It is known that SLD Fisher metric and RLD Fisher metric are monotone decreasing by application of CPTP maps
\[
J_{\rho_\theta}^R \geq J_{\Lambda(\rho_\theta)}^R, \quad J_{\rho_\theta}^S \geq J_{\Lambda(\rho_\theta)}^S,
\]
and any monotone Riemannian metric \( g \), if a constant factor is properly chosen, takes values between SLD and RLD Fisher metric
\[
J_{\rho_\theta}^S \leq g_{\rho_\theta} \leq J_{\rho_\theta}^R.
\]

[17]. In this section, we show the operational proof of the slightly stronger version of these facts.

First, monotonicity of SLD is trivial because the optimization of measurement applied to the family \( \{\Lambda(\rho_\theta)\} \) is equivalent to the optimization of measurement applied to \( \{\rho_\theta\} \) over the restricted class of measurements of the form \( M \circ \Lambda \):
\[
J_{\Lambda(\rho_\theta)}^S = \max_M J_{M \circ \Lambda(\rho_\theta)} \leq \max_M J_{M(\rho_\theta)} = J_{\rho_\theta}^S
\]
The monotonicity of RLD Fisher metric is proven in the similar manner. Given a tangent reverse estimation \( \left( \Phi, \{ p_\theta, d\rho_\theta/d\theta \} \right) \) of \( \{ p_\theta, d\rho_\theta/d\theta \} \), \( \Lambda \circ \Phi \) is a tangent reverse estimation of \( \{ \Lambda (p_\theta), \Lambda (d\rho_\theta/d\theta) \} \). Since \( \{ \Lambda (p_\theta), \Lambda (d\rho_\theta/d\theta) \} \) may have a better tangent reverse estimation, we have
\[
J^R_{\Lambda (p_\theta)} = \min \{ J_{p_\theta}; \Phi (p_\theta) = \Lambda (p_\theta), \Phi (d\rho_\theta/d\theta) = \Lambda (d\rho_\theta/d\theta) \}
\leq \min \{ J_{p_\theta}; \Phi = \Psi \circ \Lambda, \ s.t. \ \Psi (p_\theta) = \rho_\theta, \ \Psi (d\rho_\theta/d\theta) = d\rho_\theta/d\theta \}
= J^R_{\rho_\theta}.
\]
Assume that a metric is not increasing by a quantum-classical (QC) channel, and coincides with classical Fisher information restricted to classical probability distributions. Then, this metric should be no smaller than SLD Fisher metric: an optimal tangent reverse estimation (\( \Phi \)) of classical Fisher information for probability distributions. Then, the metric should be no larger than RLD Fisher map \( \Phi \),
\[
\text{where the second identity is due to the assumption of normalization: therefore, due to monotonicity by the CQ Theorem 3.5}
\]
Assume that a (not necessarily Riemannian) metric \( g \) is not increasing by a quantum-classical (QC) channel, and coincides with classical Fisher information in the space of classical probability distributions. Then, if \( g \) is monotone decreasing by a QC map, \( g \) is no smaller than SLD Fisher metric. If \( g \) is monotone decreasing by a QC map, \( g \) is no larger than RLD Fisher metric.

Similarly, assume that a metric is not increasing by a classical-quantum (CQ) channel and coincides with classical Fisher information for probability distributions. Then, the metric should be no larger than RLD Fisher metric: an optimal tangent reverse estimation (\( \Phi \)) of \( \{ p_\theta, d\rho_\theta/d\theta \} \) satisfies
\[
J^R_{\rho_\theta} = J_{p_\theta} = g_{p_\theta},
\]
where the second identity is due to the assumption of normalization: therefore, due to monotonicity by the CQ map \( \Phi \),
\[
g_{p_\theta} \leq g_{p_\theta} = J^R_{\rho_\theta}.
\]

Here, we have not assumed that the metric is Riemannian, or induced from an inner product in the tangent space, different from the argument in [17]. Also, we have only assumed monotonicity by QC and CQ maps:

**Theorem 3.5** Assume that a (not necessarily Riemannian) metric \( g \) coincide with classical Fisher information in the space of classical probability distributions. Then, if \( g \) is monotone decreasing by a QC map, \( g \) is no smaller than SLD Fisher metric. If \( g \) is monotone decreasing by a QC map, \( g \) is no larger than RLD Fisher metric.

**Example 3.6** (Petz metrics) In [17], Petz had shown any monotone Riemannian metric can be written as
\[
g^f_{p_\theta} (\partial_i p_\theta, \partial_j p_\theta) := \text{tr} \partial_i p_\theta \left\{ R_{p_\theta} f (L_{p_\theta} R_{p_\theta}^{-1}) \right\}^{-1} \partial_j p_\theta,
\]
where \( L_{p_\theta} \) and \( R_{p_\theta} \) are map form \( \mathcal{B} (\mathcal{H}) \) to \( \mathcal{B} (\mathcal{H}) \) with
\[
L_{p_\theta} (A) = p_\theta A, \ R_{p_\theta} (A) = A p_\theta,
\]
and \( f \) is an operator monotone function with
\[
f (x) = xf (x^{-1}), \quad f (1) = 1.
\]
For RLD and SLD metric, \( f (x) = \frac{2x}{x^2 + 1} \) and \( f (x) = \frac{x+1}{x^2} \), respectively. If \( f (x) = \frac{x-1}{\ln x} \),
\[
g^f_{p_\theta} (\partial_i p_\theta, \partial_j p_\theta) = \text{tr} \partial_i p_\theta \partial_j \ln p_\theta := J^B_{\rho_\theta, i,j},
\]
which is called Bogoljubov-Kubo-Mori (BKM) metric. It had been known that
\[
J^S_{\rho_\theta} \leq J^B_{\rho_\theta} \leq J^R_{\rho_\theta}.
\]
Also, [7]
\[
f (x) = f_\alpha (x) = \left( 1 - \frac{\alpha^2}{4} \right) \frac{(x-1)^2}{(x^{\alpha/2} - 1)(x^{-\alpha/2} - 1)} \quad (|\alpha| \leq 3)
\]
is operator monotone for \( |\alpha| \leq 3 \), and corresponding metric will be denoted by \( J^\alpha_{\rho_\theta} \), hereafter. It holds that
\[
J^3_{\rho_\theta} = J^{-3}_{\rho_\theta} = J^R_{\rho_\theta}, \quad J^1_{\rho_\theta} = J^{-1}_{\rho_\theta} = J^B_{\rho_\theta}.
\]
Hence, \( J^\alpha_{\rho_\theta} \ (1 \leq \alpha \leq 3) \) ‘interpolates’ between \( J^B_{\rho_\theta} \) and \( J^R_{\rho_\theta} \). In addition, [7] shown
\[
J^\alpha_{\rho_\theta} \leq J^0_{\rho_\theta} \leq J^B_{\rho_\theta}.
\]
4 Non-asymptotic scenario

4.1 Parallel family of states

A family \( \{ \rho_\theta \} \) is said to be \( \text{RLD-parallel} \) if and only if:

\[
\rho_\theta = N M_\theta N^\dagger,
\]

where \( M_\theta = \text{diag} \left( p_\theta (1), \ldots, p_\theta (r) \right) \), \( N = \left[ |\phi_1 \rangle, |\phi_2 \rangle, \ldots, |\phi_r \rangle \right] \) is a linearly independent, normalized, but not necessarily orthogonal system of state vectors. This condition is equivalent to

\[
\rho_\theta = \sum_{x=1}^{r} p_\theta (x) |\phi_x \rangle \langle \phi_x |.
\]

Its operational meaning is as follows. Observe that \((\Phi, \{ p_\theta \})\) is the reverse estimation of \( \{ \rho_\theta \} \), with \( \Phi (\delta_x) = |\phi_x \rangle \langle \phi_x | \). The Fisher information \( J_\theta \) of \( \{ p_\theta \} \) is easily computed by observing

\[
L_{\theta, i} R = N \text{diag} \left( \partial_i \ln p_\theta (1), \ldots, \partial_i \ln p_\theta (r) \right) N^{-1},
\]

(10)

Hence this reverse estimation achieves the lower bound suggested by Theorem 3.4 at any \( \theta \).

Hereafter, let

\[
p_t^{(m)} := t p + (1 - t) q, \quad \rho_t^{(m)} := t \rho + (1 - t) \sigma.
\]

**Proposition 4.1** For any \( \rho \) and \( \sigma \) with \( \text{supp} \rho = \text{supp} \sigma \), there is an RLD-parallel manifold containing \( \rho \), \( \sigma \), and \( \rho_t^{(m)} \), for any \( 0 \leq t \leq 1 \).

**Proof.** Let \( P \) be the projection onto \( \text{supp} \rho = \text{supp} \sigma \). Let \( U \) be a unitary matrix such that \( PU (1 - P) = 0 \) and

\[
\sqrt{\rho} U \sqrt{\sigma} = \sqrt{\sigma} U^\dagger \sqrt{\rho},
\]

or equivalently

\[
\sqrt{\sigma^{-1}} \sqrt{\rho} U = U^\dagger \sqrt{\rho} \sqrt{\sigma^{-1}},
\]

where \( \rho^{-1} \) and \( \sigma^{-1} \) are generalized inverses. Such \( U \) is found out using the polar decomposition of \( \sqrt{\sigma^{-1}} \sqrt{\rho} \). By Proposition 3.3 there is \( X \) with

\[
\sqrt{\rho} U = \sqrt{\sigma} X, \quad X = X^\dagger.
\]

Let \( VD^\dagger \) be diagonalization of \( X \), and we obtain

\[
\sqrt{\rho} UV = \sqrt{\sigma} VD.
\]

Divide \( x \)th column vector of \( \sqrt{\sigma} V \) by its magnitude and denote the product by \( |\phi_x \rangle \). Then letting \( N := \left[ |\phi_1 \rangle, |\phi_2 \rangle, \ldots, |\phi_r \rangle \right] \), we have

\[
\rho = N \text{diag} \left( p (1), \ldots, p (r) \right) N^\dagger, \quad \sigma = N \text{diag} \left( q (1), \ldots, q (r) \right) N^\dagger,
\]

\[
\rho_t^{(m)} = N \text{diag} \left( p_t^{(m)} (1), \ldots, p_t^{(m)} (r) \right) N^\dagger,
\]

for some \( p (1), \ldots, p (r) \), and \( q (1), \ldots, q (r) \), and the assertion is proved. □
4.2 Reverse test

Consider test of the hypothesis ‘the given state is $\rho$’ against the alternative hypothesis ‘the given state is $\sigma$’.
(Hereafter, such test is referred to as “test ‘$\rho$ vs. $\sigma$’.”) Suppose we are given many copies of the unknown states, and the error $a_n$ of the first kind, or the probability of rejecting $\rho$ while $\rho$ is the true state, vanishes as $n \to \infty$. Then in maximizing the exponent of the error $\beta_n$ of the second kind, or the probability of rejecting $\sigma$ while $\sigma$ is the true state, the key step is optimization of QC map (measurement) $M$ to maximize the relative entropy $D(M(\rho^\otimes n)||M(\sigma^\otimes n))$.

We consider reverse test, or the inverse process of (the single copy version of) the above. Given a pair $\{\rho, \sigma\}$ of states, let $\Phi$ be CQ map with $\Phi(p) = \rho, \Phi(q) = \sigma$, where $\{p, q\}$ is a pair of probability distributions. A pair $(\Phi, \{p, q\})$ is called a reverse test of $\{\rho, \sigma\}$. (In the terminology of statistical decision theory, $\{\rho, \sigma\}$ is deficient relative to $\{p, q\}$.) Our task is to minimize $D(p||q)$ for all reverse tests.

To find the optimal reverse test, the following lemma plays a key role:

**Lemma 4.2** ([1], Chapter 3, Section 3.5)

$$D(p||q) = \int_0^1 \int_0^t J_{p_{12}^{(m)}} ds dt.$$  

Let $(\Phi, \{p, q\})$ be a reverse test of $\{\rho, \sigma\}$. Then

$$\Phi(p_t^{(m)}) = \rho_{t}^{(m)}$$

holds, and $(\Phi, \{p_t^{(m)}\})$ is a reverse estimation of $\{\rho_t^{(m)}\}$. Therefore, by Lemma 4.2 and Theorem 3.4, we have

$$D(p||q) \geq \int_0^1 \int_0^t R_{p_{12}^{(m)}} ds dt.$$

Also, by [11] there is a parallel family which contains $\{\rho_{t}^{(m)}\}$. Therefore,

$$\rho_u = N \text{diag} \left( p_t^{(m)}(1), \cdots, p_t^{(m)}(r) \right) N^\dagger,$$

holds for some $N = [|\phi_1\rangle, \cdots, |\phi_r\rangle]$. Hence, by [11], the reverse estimation $(\Phi_0, \{p_0, \sigma\})$, where $\Phi_0(\delta_x) = |\phi_x\rangle \langle \phi_x|$, achieves

$$R_{\rho_t^{(1)}} = R_{p_t^{(1)}}, \quad \forall t,$$

and $\Phi_0(p) = \rho, \Phi_0(q) = \sigma$. Therefore, the reverse test $(\Phi_0, \{p, q\})$ achieves

$$\int_0^1 \int_0^t R_{p_{12}^{(m)}} ds dt = \int_0^1 \int_0^t R_{p_t^{(m)}} ds dt = \sum_{x=1}^r p(x) \ln \frac{p(x)}{q(x)},$$

and hence is optimal. The right most side integral is computed in [6], although the detail is not described. Here we show a way to verify that the left most side equals $\mathcal{D}(\rho||\sigma)$, as in [11]. Observe that there is a $r \times r$ unitary matrix $U$ with

$$\rho_t^{1/2} = U D_0 N^\dagger = N D_0 U^\dagger,$$
$$\sigma_t^{1/2} = U D_1 N^\dagger = N D_1 U^\dagger,$$
$$D_t := \text{diag} \left( \sqrt{p_t^{(m)}(1)}, \cdots, \sqrt{p_t(r)} \right).$$
Therefore,
\[
\begin{align*}
\text{tr} \ln \rho \sigma^{-1} & = \text{tr} \left[ U D_0 N \U D_0 U^\dagger \ln \left( U D_0 N \U D_1 N \U^{-1} \left( N D_1 U \U^{-1} \right) - 1 \right) N D_0 U^\dagger \right] \\
& = \text{tr} \left( N \diag \left( p(1) \ln \frac{p(1)}{q(1)}, \cdots, p(r) \ln \frac{p(r)}{q(r)} \right) \right) \\
& = \sum_x \| \phi_x \|^2 \frac{p(x)}{q(x)} \ln \frac{p(x)}{q(x)} = \sum_x p(x) \ln \frac{p(x)}{q(x)}.
\end{align*}
\]

Thus we obtain:

**Theorem 2.4**

\[
\min D(p\|q) = D^R(\rho\|\sigma),
\]

where minimization is taken for over all reverse tests \((\Phi, \{p, q\})\) of \(\{\rho, \sigma\}\).

### 4.3 Monotone relative entropy

An example of the quantity satisfying (M) and (N) is \(D(\rho\|\sigma) = \text{Tr} \rho (\ln \rho - \ln \sigma)\). By Theorem 2.1, we obtain another proof of the inequality shown in [8],

\[
D(\rho\|\sigma) \leq D^R(\rho\|\sigma).
\]

Another example is

\[
D^g(\rho\|\sigma) := \int_0^1 \int_0^t g_{\rho_{t}^{(m)}} dsdt,
\]

where \(g\) is any properly normalized monotone metric. Note \(D^R = D^{JR}\). Also, it is known [14] [8] that

\[
D^{JR}(\rho\|\sigma) = D(\rho\|\sigma). \tag{11}
\]

Due to Lemma 4.2, \(D^g(p\|q) = D(p\|q)\) for all probability distributions \(p, q\). Also, since

\[
\Lambda \left( \rho_t^{(m)} \right) = (1 - t) \Lambda (\rho) + t \Lambda (\sigma),
\]

\(D^g(\rho\|\sigma)\) is monotone decreasing by application of CPTP maps:

\[
D^g(\Lambda (\rho) \| \Lambda (\sigma)) = \int_0^1 \int_0^t g_{\Lambda (\rho_{t}^{(m)})} dsdt \\
\leq \int_0^1 \int_0^t g_{\rho_{t}^{(m)}} dsdt = D^g(\rho\|\sigma).
\]

**Corollary 4.3**

\[
\lim_{n \rightarrow \infty} \frac{1}{n} D^{JS}(\rho^\otimes n \| \sigma^\otimes n) = D(\rho\|\sigma), \quad \lim_{n \rightarrow \infty} \frac{1}{n} D^R(\rho^\otimes n \| \sigma^\otimes n) = D(\rho\|\sigma).
\]

**Proof.** Since both of

\[
\lim_{n \rightarrow \infty} \frac{1}{n} D^{JS}(\rho^\otimes n \| \sigma^\otimes n), \quad \lim_{n \rightarrow \infty} \frac{1}{n} D^R(\rho^\otimes n \| \sigma^\otimes n)
\]

satisfy (M), (N), and (A), Theorem 2.2 implies

\[
\lim_{n \rightarrow \infty} \frac{1}{n} D^{JS}(\rho^\otimes n \| \sigma^\otimes n) \geq \lim_{n \rightarrow \infty} \frac{1}{n} D^R(\rho^\otimes n \| \sigma^\otimes n) \geq D(\rho\|\sigma).
\]
5.1 Asymptotic reverse test

The result of the test \( \rho^n \) vs. \( \sigma^n \) is binary, that is, accept \( \rho^n \) or \( \sigma^n \). Hence, a natural inverse problem would be generation of \( \{\rho^n, \sigma^n\} \) from the probability distributions \( \{p^n, q^n\} \) over binary set \( \{0, 1\} \). Let us define an asymptotic reverse test, or a pair \( (\Phi^n, \{p^n, q^n\}) \) with

\[
\lim_{n \to \infty} ||\Phi^n (p^n) - \rho^n||_1 = 0, \quad \Phi^n (q^n) = \sigma^n,
\]

\[
\lim_{n \to \infty} p^n (0) = \lim_{n \to \infty} q^n (1) = 1,
\]

and discuss the infimum of

\[
\lim_{n \to \infty} \frac{-1}{n} \ln q^n (0).
\]

To describe the infimum, we need the following object:

\[
D_{\max} \left( \{\rho^n\} || \{\sigma^n\} \right) := \inf \left\{ a; \; \bar{\rho}^n \leq e^{na} \sigma^n, \lim_{n \to \infty} \|\bar{\rho}^n - \rho^n\|_1 = 0 \right\}.
\]

The following proposition is trivial.
Proposition 5.1 \( D_{\text{max}}^\infty \) is monotone decreasing by application of a CPTP map \( \Lambda \),

\[
D_{\text{max}}^\infty (\{ \Lambda (\rho^n) \}||\{ \Lambda (\sigma^n) \}) \leq D_{\text{max}}^\infty (\{ \rho^n \}||\{ \sigma^n \}) , \tag{13}
\]

and asymptotically continuous about the first argument,

\[
\lim_{n \to \infty} \| \tilde{\rho}^n - \rho^n \| = 0 \implies D_{\text{max}}^\infty (\{ \tilde{\rho}^n \}||\{ \sigma^n \}) = D_{\text{max}}^\infty (\{ \rho^n \}||\{ \sigma^n \}) . \tag{14}
\]

Theorem 5.2

\[
\inf \lim_{n \to \infty} \frac{-1}{n} \ln q^n (0) = D_{\text{max}}^\infty (\{ \rho^n \}||\{ \sigma^n \})
\]

where \( \inf \) is taken over all the asymptotic reverse test.

Proof. First, we show \( \leq \). By definition of \( D_{\text{max}}^\infty \), for any \( c > 0 \), it is possible to define \( \Phi^n (\delta_0) \) so that

\[
\lim_{n \to \infty} \| \Phi^n (\delta_0) - \rho^n \|_1 = 0 , \quad \Phi^n (\delta_0) \leq \sigma^n \exp \{ n (D_{\text{max}}^\infty (\{ \rho^n \}||\{ \sigma^n \}) + c) \} .
\]

hold. Then, letting

\[
\lim_{n \to \infty} p^n (0) = 1 , \quad q^n (0) = \exp \{ -n (D_{\text{max}}^\infty (\{ \rho^n \}||\{ \sigma^n \}) + c) \} ,
\]

we have

\[
\lim_{n \to \infty} \| \Phi^n (p^n) - \rho^n \|_1 = \lim_{n \to \infty} \| \Phi^n (\delta_0) - \rho^n \|_1 = 0 , \quad \sigma^n - q^n (0) \Phi^n (\delta_0) \geq 0 .
\]

Therefore, it is possible to define \( \Phi^n (\delta_1) \) so that

\[
\Phi^n (q^n) = q^n (0) \Phi^n (\delta_0) + q^n (1) \Phi^n (\delta_1) = \sigma^n
\]

holds. To sum up, a sequence of reverse test \( (\Phi^n , (p^n , q^n)) \) satisfies the requirement \( 12 \), and satisfies

\[
\lim_{n \to \infty} \frac{-1}{n} \ln q^n (0) = D_{\text{max}}^\infty (\{ \rho^n \}||\{ \sigma^n \}) + c .
\]

Since this composition is possible for any \( c > 0 \), we have \( \leq \).

Second, we prove \( \geq \). Observe, due to \( 12 \),

\[
\inf \lim_{n \to \infty} \frac{-1}{n} \ln q^n (0) = D_{\text{max}}^\infty (\{ p^n \} || \{ q^n \}) .
\]

Therefore, by monotonicity \( 13 \) of \( D_{\text{max}}^\infty \),

\[
\inf \lim_{n \to \infty} \frac{-1}{n} \ln q^n (0) \geq D_{\text{max}}^\infty (\Phi^n (p^n) || \Phi^n (q^n)) = D_{\text{max}}^\infty (\Phi^n (p^n) || \sigma^n) .
\]

This, by asymptotic continuity \( 14 \) and \( 12 \), leads to \( \geq \). ■

A converse statement of Stein’s lemma can be made using \( D_{\text{max}}^\infty \), indicating asymptotic reverse test is a natural inverse problem of test.

Corollary 5.3 (converse statement of Stein’s lemma)

\[
D_{\text{max}}^\infty (\{ \rho^n \}||\{ \sigma^n \}) \geq \sup \left\{ \lim_{n \to \infty} \frac{-1}{n} \log \text{tr} P_n \sigma^n : \lim_{n \to 0} \text{tr} \rho^n P_n > 0 , 0 \leq P_n \leq 1 \right\} .
\]
Proof. Consider \((\Phi^n, \{p^n, q^n\})\) with (12), and let \(\hat{p}^n\) and \(\hat{q}^n\) be binary distributions with
\[
\hat{p}^n(0) := \text{tr} P^n \Phi^n(p^n), \quad \hat{q}^n(0) := \text{tr} P^n \Phi^n(q^n),
\]
and let \(P^n\) be a POVM element with \(\lim_{n \to 0} \text{tr} \rho^n P^n > 0\). Then,
\[
\lim_{n \to 0} \hat{p}^n(0) = \lim_{n \to 0} \text{tr} P^n \Phi^n(p^n) \geq \lim_{n \to 0} \{\text{tr} P^n \rho^n - \|\rho^n - \Phi^n(p^n)\|_1\} > 0. \tag{15}
\]

Observe that the composition of the map \(\Phi^n\) and the measurement \(\{P^n, 1 - P^n\}\) is a CPTP map, or a Markov map from binary distributions onto themselves. Hence, it should be written as
\[
\hat{p}^n(0) = a_{00}^n \rho^n(0) + a_{01}^n \rho^n(1), \quad \hat{q}^n(0) = a_{00}^n q^n(0) + a_{01}^n q^n(1).
\]
By (15), we should have \(\lim_{n \to \infty} a_{00}^n > 0\) or \(\lim_{n \to \infty} a_{01}^n > 0\). If the former is true,
\[
\lim_{n \to \infty} -\frac{1}{n} \log q^n(0) \geq \lim_{n \to \infty} -\frac{1}{n} \log \hat{q}^n(0) = \lim_{n \to \infty} -\frac{1}{n} \log \rho^n(0).
\]
If the latter is true, due to \(\lim_{n \to \infty} q^n(1) = 1\), we have
\[
\lim_{n \to \infty} -\frac{1}{n} \log \hat{q}^n(0) = 0.
\]
In either case, we have
\[
\lim_{n \to \infty} -\frac{1}{n} \log q^n(0) \geq \lim_{n \to \infty} -\frac{1}{n} \log \hat{q}^n(0) = \lim_{n \to \infty} -\frac{1}{n} \log \text{tr} P^n \sigma^n.
\]
Also, Theorem 5.2 implies that there is \((\Phi^n, \{p^n, q^n\})\) with
\[
\lim_{n \to \infty} -\frac{1}{n} \log q^n(0) \leq \mathcal{D}_{\text{max}}(\rho^n) + c,
\]
for any \(c > 0\). Therefore, we have to have the converse statement. 

5.2 Relations between \(\mathcal{D}_{\text{max}}^\infty, \mathcal{D}\) and \(\mathcal{D}\)

Nagaoka \cite{15} defined the following quantity to analyze quantum hypothesis test:
\[
\mathcal{D}(\rho^n) := \inf \left\{ a : \lim_{n \to \infty} \text{tr} \rho^n \{\rho^n - e^{-a} \sigma^n \leq 0\} = 1 \right\},
\]
where \(\{\rho^n - e^{-a} \sigma^n \leq 0\}\) is the projector onto the non-positive eigenspace of \(\rho^n - e^{-a} \sigma^n\).

Theorem 5.4 \cite{15, 16} \(\mathcal{D}(\rho^n)\) characterizes efficiency of the test \(\rho^n\) vs. \(\sigma^n\) as follows.
\[
\mathcal{D}(\rho^n) = \inf \left\{ a : \forall \{P^n\} \lim_{n \to \infty} -\frac{1}{n} \ln \text{tr} P^n \sigma^n \geq a \Rightarrow \lim_{n \to \infty} \text{tr} P^n \rho^n = 0 \right\}.
\]

Lemma 5.5 (Datta \cite{3}) If
\[
\varepsilon := 1 - \text{tr} \rho^n \{\rho^n - e^{-a} \sigma^n \leq 0\},
\]
there is a positive operator \(A^n\) with
\[
\|A^n - \rho^n\|_1 \leq \sqrt{8\varepsilon}, \quad A^n \leq e^{-a} \sigma^n.
\]
Let \( A^n \) as of Lemma 5.5 and \( \tilde{\rho}^n := \frac{1}{\text{tr} A^n} A^n \). Then,
\[
\| \tilde{\rho}^n - \rho^n \|_1 \leq \left\| \frac{1}{\text{tr} A^n} A^n - A^n \right\|_1 + \| A^n - \rho^n \|_1
\]
\[
= |1 - \text{tr} A^n| + \| A^n - \rho^n \|_1
\]
\[
= |\text{tr} \rho^n - \text{tr} A^n| + \| A^n - \rho^n \|_1
\]
\[
\leq 2 \| A^n - \rho^n \|_1
\]
\[
\leq 4 \sqrt{2} \varepsilon. \tag{16}
\]

Also,
\[
\tilde{\rho}^n \leq 2^{na} \sigma^n \leq 2^{na} \frac{1}{1 - \sqrt{8\varepsilon}} \sigma^n. \tag{17}
\]

**Theorem 5.6**
\[
\overline{D}(\{\rho^n\}||\{\sigma^n\}) = D_{\text{max}}(\{\rho^n\}||\{\sigma^n\})
\]

The proof is analogue of the proof of Theorem 2 of [3], and is given below with minor modification in accordance with the difference in their \( D_{\text{max}} \) and our \( D_{\text{max}}^\infty \).

**Proof.** First we show ‘\( \leq \)’. Let \( c > 0 \) and \( \{\tilde{\rho}^n\} \) be a sequence of states with
\[
\lim_{n \to \infty} \| \rho^n - \tilde{\rho}^n \|_1 = 0.
\]
Then, for any \( \{ P^n \} \) with
\[
\lim_{n \to \infty} \frac{1}{n} \ln \text{tr} P^n \sigma^n \geq D_{\text{max}}(\{\rho^n\}||\{\sigma^n\}) + 2c,
\]
we have
\[
\lim_{n \to \infty} \text{tr} P^n \rho^n = \lim_{n \to \infty} \text{tr} P^n \tilde{\rho}^n
\]
\[
\leq \lim_{n \to \infty} e^{n(D_{\text{max}}(\{\rho^n\}||\{\sigma^n\}) + c)} \text{tr} P^n \sigma^n = 0.
\]
Since \( c > 0 \) is arbitrary, by the second identity of Theorem 5.4, we have ‘\( \leq \)’.

Second, we show ‘\( \geq \)’. Let \( a = \overline{D}(\{\rho^n\}||\{\sigma^n\}) + c \) (\( c > 0 \)). Then,
\[
\lim_{n \to \infty} \text{tr} \rho^n \{\rho^n - e^{na} \sigma^n \leq 0\} = 1.
\]
Hence, by (16) and (17), we can compose \( \tilde{\rho}^n \) with
\[
\lim_{n \to \infty} \| \tilde{\rho}^n - \rho^n \|_1 = 0, \quad \tilde{\rho}^n \leq e^{n(a+c)} \sigma^n, \quad \forall c > 0 \exists n_0 \forall n \geq n_0,
\]
or
\[
\overline{D}(\{\rho^n\}||\{\sigma^n\}) + 2c \geq D_{\text{max}}(\{\rho^n\}||\{\sigma^n\}).
\]
Since \( c, c' > 0 \) are arbitrary, we have the assertion. ■

**Theorem 5.7** [15] [16]
\[
D(\rho||\sigma) = \overline{D}(\{\rho^\otimes n\}||\{\sigma^\otimes n\}).
\]

**Corollary 5.8**
\[
\overline{D}(\{\rho^\otimes n\}||\{\sigma^\otimes n\}) = D_{\text{max}}(\{\rho^\otimes n\}||\{\sigma^\otimes n\}) = D(\rho||\sigma).
\]
5.3 Asymptotically lower continuous and monotone relative entropy

**Theorem 2.7** If \( D(\rho_0||\sigma_0) > D(\rho||\sigma) \), there is a sequence \( \{\Psi^n\} \) of TPCP map with
\[
\lim_{n \to \infty} \|\Psi^n(\rho_0^{\otimes n}) - \rho^{\otimes n}\|_1 = 0, \quad \Psi^n(\sigma_0^{\otimes n}) = \sigma^{\otimes n}.
\] (18)

Conversely, if \( \{\Psi^n\} \) with (18) exists, \( D(\rho_0||\sigma_0) \geq D(\rho||\sigma) \).

**Proof.** Suppose \( D(\rho_0||\sigma_0) > D(\rho||\sigma) \) and let
\[
c := \frac{1}{2} \{D(\rho_0||\sigma_0) - D(\rho||\sigma)\}.
\]

By Theorem 2.5, there is a sequence of projector \( \{P^n\} \) with
\[
p^n(0) := \text{tr} P^n \rho_0^{\otimes n} \to 1 \quad (n \to \infty)
\]
\[
q^n(0) := \text{tr} P^n \sigma_0^{\otimes n} \leq e^{-n(D(\rho_0||\sigma_0) - c)} = e^{-n(D(\rho||\sigma) + c)}, \quad \exists n_0 \forall n \geq n_0.
\]

Let CPTP map \( \Phi^n \) as of 5.2. Then, due to \( D_{\max}(\rho^{\otimes n}||\sigma^{\otimes n}) = D(\rho||\sigma) \), the composition \( \Psi^n \) of the measurement \( \{P^n, 1 - P^n\} \) followed by \( \Phi^n \) satisfies (18). Thus we have the former half of the assertion.

In the sequel, we prove the latter half. Recall \( D(\rho||\sigma) \) satisfies (M), (A), and (C). Therefore,
\[
D(\rho_0||\sigma_0) = \lim_{n \to \infty} \frac{1}{n} D\left(\rho_0^{\otimes n}||\sigma_0^{\otimes n}\right)
\]
\[
\geq \lim_{n \to \infty} \frac{1}{n} D\left(\Psi^n(\rho_0^{\otimes n})||\Psi^n(\sigma_0^{\otimes n})\right)
\]
\[
= \lim_{n \to \infty} \frac{1}{n} D\left(\rho^{\otimes n}||\sigma^{\otimes n}\right) \geq \lim_{n \to \infty} \frac{1}{n} D\left(\rho^{\otimes n}||\sigma^{\otimes n}\right) = D(\rho||\sigma).
\]

**Corollary 5.9** \( D^F(\rho||\sigma) := \ln \|\sqrt[\lambda]{\sqrt[n]{\sigma}}\|_1 \) does not satisfy the condition (C).

**Proof.** \( D^F(\rho||\sigma) \) satisfies (M) and (A), but does not equal a constant multiple of \( D(\rho||\sigma) \). Therefore, we must have the assertion.

**Theorem 5.10** If \( g \) is a properly normalized monotone metric, then
\[
\inf \left\{ \lim_{n \to \infty} \frac{1}{n} D^g(\rho^{\otimes n}||\sigma^{\otimes n}) : \lim_{n \to \infty} \|\rho^n - \rho^{\otimes n}\|_1 = 0 \right\}
\]
\[
= D(\rho||\sigma).
\]

**Proof.** By Theorem 2.2, we have to prove the assertion only for \( g = J^R \). \( \geq \) is due to \( D^R \geq D \) and Proposition 2.4. To prove \( \leq \), let \( \left(\Phi^n, \{p^n, q^n\}\right) \) be an asymptotic reverse test with
\[
\lim_{n \to \infty} \frac{1}{n} \ln q^n(0) = D(\rho||\sigma) + c.
\]
Then by monotonicity of \( D^R \),
\[
\lim_{n \to \infty} \frac{1}{n} D^R(\Phi^n(p^n)||\sigma^{\otimes n}) \leq \lim_{n \to \infty} \frac{1}{n} D^R(p^n||q^n)
\]
\[
= \lim_{n \to \infty} \frac{1}{n} D(p^n||q^n)
\]
\[
= \lim_{n \to \infty} \frac{1}{n} \ln q^n(0) = D(\rho||\sigma) + c,
\]
which leads to the assertion.
6 Conclusions and Discussions

Using reverse test and asymptotic reverse test, we gave a characterization of quantum versions of relative entropy. Note that the uniqueness in the asymptotic scenario is valid also for classical relative entropy: any two-point functions over probability distribution with (A), (M) and (C) is constant multiple of relative entropy.

The condition (C) can be replaced by the following ‘weak monotonicity’ [10], which may be a bit more natural.

\[(WM) \text{ (weak monotonicity) If } \|\tilde{\rho}^{\otimes n} - \Lambda_n (\rho^{\otimes n})\| \to 0, \tilde{\sigma}^{\otimes n} = \Lambda_n (\sigma^{\otimes n}) \]

\[D_Q (\rho || \sigma) \geq D_Q (\tilde{\rho} || \tilde{\sigma}) .\]

It may be interesting to compare the asymptotic behavior of quantum relative entropy and corresponding quantum Fisher information (correspondence is made via Lemma 4.2). While it is known that \(J^R\) and \(J^S\) satisfies both of them [12], \(D_Q (\rho || \sigma)\) and \(D_{J^S} (\rho || \sigma)\) does not satisfy (C) and (A), respectively.

Some problems are left open. First, relaxing (C) in the following manner can be interesting:

\[(C') \text{ (Lower exponential asymptotic continuity) If } \lim_{n \to \infty} - \frac{1}{n} \ln \|\tilde{\rho}^n - \rho^{\otimes n}\| \geq a, \]

\[\lim_{n \to \infty} \frac{1}{n} \{D_Q (\tilde{\rho}^n || \sigma^{\otimes n}) - D_Q (\rho^{\otimes n} || \sigma^{\otimes n})\} \geq 0.\]

By relaxing (C) to (C’), quantities such as relative Renyi entropy may survive. Second, generalizing Theorem 5.2 and Theorem 2.7 (by increasing the numbers of states, changing constraint on error, etc.) may be also interesting.

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7 Reverse estimation for a multi-dimensional parameter family

By the argument parallel with the 1−dim-case, we have

\[ J^R_{p\theta}(X, X) = \min \{ J_{p\theta}(Y, Y) : \Psi(p\theta) = \rho_{\theta}, \Psi(Y) = X \} . \]

Therefore, for any reverse estimation \((\Phi, \{p\theta\})\),

\[ J_{p\theta} \leq J^R_{p\theta}, \]

which, for any real \(m \times m\)-matrix \(G > 0\), leads to

\[ \text{Tr} \, G \, J_{p\theta} \geq \min \{ \text{Tr} \, G \, J \geq J^R_{p\theta} \} = \text{Tr} \, G \, \Re J^R_{p\theta} + \text{Trabs} \, \Im J^R_{p\theta} \]

\[ = \text{Tr} \, G \, \Re J^R_{p\theta} + \text{Trabs} \, \Im J^R_{p\theta} \]

(19)

Note that

\[ \Im J^R_{\theta,ij} = \frac{1}{2} \left( \text{Tr} \rho_\theta L_{\theta,ij}^R L_{\theta,ji}^R - \text{Tr} L_{\theta,ij}^R L_{\theta,ji}^R \rho_\theta \right) = \frac{1}{2} \left( \text{Tr} L_{\theta,ij}^R \rho_\theta L_{\theta,ji}^R - \text{Tr} L_{\theta,ij}^R L_{\theta,ji}^R \rho_\theta \right) = -\frac{1}{2} \text{Tr}_{\rho_\theta} \left[ L_{\theta,ij}^R, L_{\theta,ji}^R \right] := \tilde{J}_{\theta, ij}, \]

This inequality is in many cases not achievable. However, if \(\{p\theta\}\) is RLD-parallel, \(\Im J^R_{\theta,ij} = 0\) and the inequality is written as

\[ \text{Tr} \, G^R_{p\theta} \geq \text{Tr} \, G \, \Re J^R_{p\theta} \]

which is achievable. Also:

**Example 7.1** Gaussian states are defined by its P-representation,

\[ \rho_\theta = \int \frac{dqdq}{2\pi N} \exp \left[ -\frac{(q - \theta^1)^2 + (p - \theta^2)^2}{2N} \right] \left| q + ip \right\rangle \left\langle q + ip \right|, \]

where \(|z\rangle\) is the coherent state with complex amplitude \(z\). Being infinite dimensional states, in strict sense, this example is out of the scope of our theory. However, the lower bound (19) can be explicitly computed as

\[ \text{Tr} \, \Re J^R_{p\theta} + \text{Trabs} \, \Im J^R_{p\theta} = \frac{2}{N}. \]
Also, using $P$-representation, one can compose a reverse estimation (generate coherent states according to the probability distribution defined by $P$-function), and

$$J = \begin{pmatrix} \frac{N}{N} & 0 \\ 0 & \frac{N}{N} \end{pmatrix}, \quad \text{Tr} \ J = \frac{2}{N}$$

achieving the lower bound.

Note that the optimal measurement for estimation of $\theta = (\theta^1, \theta^2)$ is

$$| \varphi_\theta \rangle := \sum_{n=0}^{\infty} \left( \frac{N}{N+1} \right)^{\frac{n}{2}} U_\theta | n \rangle \otimes U_\theta | n \rangle \in \mathcal{H} \otimes \mathcal{K},$$

where $U_\theta$ is the Weyl operator. Then $\rho_\theta = \text{tr}_\mathcal{K} | \varphi_\theta \rangle \langle \varphi_\theta | = \text{tr}_\mathcal{H} | \varphi_\theta \rangle \langle \varphi_\theta |$ and

$$\langle 2^{-\frac{1}{4}} \left( \hat{\theta}^1 + i \hat{\theta}^2 \right) | \varphi_\theta \rangle = \sqrt{\frac{1}{N+1}} \exp \left\{ - \frac{\left( \hat{\theta}^1 - \theta^1 \right)^2 + \left( \hat{\theta}^2 - \theta^2 \right)^2}{4 \left( \frac{N}{N+1} \right)} \right\} \sqrt{\frac{N}{N+1} \sqrt{\frac{1}{2}}}. $$

Therefore, if one measures $K$-part of $| \varphi_\theta \rangle$ by POVM \( \left\{ \left| \frac{\hat{\theta}^1 + i \hat{\theta}^2}{\sqrt{2}} \right\rangle \langle \frac{\hat{\theta}^1 + i \hat{\theta}^2}{\sqrt{2}} \right\} \), one obtains $\left| \tilde{\theta}_1 - i \tilde{\theta}_2 \sqrt{2} \right\rangle$ with probability \( \frac{1}{2 \pi N} \exp \left\{ - \frac{\left( \tilde{\theta}_1 - \theta^1 \right)^2 + \left( \tilde{\theta}_2 - \theta^2 \right)^2}{2N} \right\} \), or realizes the optimal reverse estimation.