FAST TRACK COMMUNICATION

Translation-invariant models for non-commutative gauge fields

Daniel N Blaschke, François Gieres, Erwin Kronberger, Manfred Schweda and Michael Wohlgenannt

1 Institute for Theoretical Physics, Vienna University of Technology, Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria
2 Université de Lyon, Institut de Physique Nucléaire, Université Lyon 1 and CNRS/IN2P3, Bat. P. Dirac, 4 rue Enrico Fermi, F-69622 Villeurbanne, France
3 Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria

E-mail: blaschke@hep.itp.tuwien.ac.at, gieres@ipnl.in2p3.fr, kronberger@hep.itp.tuwien.ac.at, mkschweda@tpb.tuwien.ac.at and michael.wohlgenannt@univie.ac.at

Received 16 April 2008
Published 4 June 2008
Online at stacks.iop.org/JPhysA/41/252002

Abstract
Motivated by the recent construction of a translation-invariant renormalizable non-commutative model for a scalar field [1], we introduce models for non-commutative U(1) gauge fields along the same lines. More precisely, we include some extra terms into the action with the aim of getting rid of the UV/IR mixing.

PACS numbers: 11.10.Nx, 11.15.−q, 11.10.Gh

1. Introduction

Non-commuting spacetime coordinates naturally appear in various approaches to quantum gravity, e.g. see reviews [2]. Field theories on non-commutative space generally suffer from a new class of problematic infrared divergences which have the same degree as the usual ultraviolet divergences at the perturbative level. This phenomenon is commonly referred to as UV/IR mixing, see [2] and references therein. Recently, this problem was overcome within certain models of scalar field theories. The first of these models, which was introduced by Grosse and Wulkenhaar [3], is the $\phi^4$ theory supplemented by an oscillator term in the Euclidean $x$-space action: this model has been proved to be renormalizable to all orders of perturbation theory by different methods [4]. Since the oscillator term breaks the translational invariance, Gurau et al [1] recently introduced another renormalizable model in which the oscillator term in $x$ space is replaced in the Euclidean momentum space action by a $1/\tilde{k}^2$ term (with $\tilde{k}^2 = \tilde{k}_\mu \tilde{k}_\mu$ and $\tilde{k}_\mu = \theta_{\mu\nu} k^\nu$, where $\theta_{\mu\nu}$ are the non-commutativity parameters for the Euclidean spacetime coordinates.) This term is motivated by the fact that the 1-loop
self-energy of the standard non-commutative $\phi^4$ model has a quadratic IR divergence which is proportional to $1/k^2$: the new term in the momentum space action yields a dressed propagator at 1-loop level involving a similar contribution. (As a matter of fact, such a term had already been considered earlier in connection with a resummation procedure [5, 6].)

The deformation matrix $(\theta_{\mu\nu})$ can be (and is) assumed to have the simple form

$$ (\theta_{\mu\nu}) = \theta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \text{with } \theta \in \mathbb{R}. $$

The action of Gurau et al [1] is given in Euclidean momentum space by

$$ S = \int d^4k \left[ \frac{1}{2} k_{\mu} \phi k^{\mu} \phi + \frac{1}{2} m^2 \phi \phi + \frac{a}{2} \theta k^2 \phi \phi + \frac{\lambda}{4!} \phi \phi \phi \phi \right], \quad (1) $$

or, more explicitly [7],

$$ S[\hat{\phi}] = \int d^4k \left[ \frac{1}{2} \hat{\phi}(-k) \left( k^2 + m^2 + \frac{a}{\theta k^2} \right) \hat{\phi}(k) + \frac{\lambda}{4!} F(\phi \phi \phi \phi) (k) \right], $$

where $a > 0$ and where $\hat{\phi} \equiv F\phi$ denotes the Fourier transform of $\phi$. This leads to the improved propagator

$$ G^{\phi\phi}(k) = \frac{1}{k^2 + m^2 + \frac{a}{\theta k^2}}, \quad (2) $$

which has a ‘damping’ behaviour for vanishing momentum:

$$ \lim_{k \to 0} G^{\phi\phi}(k) = 0. $$

As expected, matters are more complicated in gauge field theories. Although there have been several suggestions as to how to handle the UV/IR mixing [8, 9], the corresponding models have some drawbacks. The one introduced by Slavnov [8] relies on a constraint which reduces the degrees of freedom of the gauge field whereas those involving an oscillator-type term [9] (in analogy to the scalar field model of Grosse and Wulkenhaar) break the translational invariance. Accordingly, the goal of the present communication is to put forward some ideas for generalizing the procedure of Gurau et al in view of constructing a renormalizable and translation-invariant model for $U(1)$ gauge fields in four-dimensional non-commutative Euclidean space.
2. New gauge field model I

The quadratic IR divergence of a non-commutative $U(1)$ gauge theory is known to be of the form
\[ \Pi^\text{IR}_{\mu\nu} \propto \tilde{k}_\mu \tilde{k}_\nu \left( \frac{k^2}{2} \right)^2, \tag{5} \]
and to be independent of the chosen gauge fixing (see [10]). This expression motivated the authors of [6] to introduce the following gauge invariant term into their action (in connection with a resummation procedure):
\[ \int d^4x \tilde{F} \ast \frac{1}{(D^2)^2} \ast \tilde{F}. \tag{6} \]
Here,
\[ \tilde{F} = \theta^{\mu\nu} F_{\mu\nu}, \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu \ast A_\nu], \]
\[ \tilde{D}^2 = \tilde{D}^\mu \ast \tilde{D}_\mu, \quad \text{with} \quad \tilde{D}_\mu = \theta^{\mu\nu} D_\nu. \tag{7} \]
hence \( \frac{1}{D^2} \ast \tilde{F} = \frac{1}{D^2} \ast \tilde{F} \). The expression \( \frac{1}{D^2} \ast \tilde{F} \equiv Y \) is to be understood as a formal power series in the gauge field \( A_\mu \) which may be determined recursively as follows. First, note that
\[ \tilde{F} = D^2 \ast \frac{1}{D^2} \ast \tilde{F} = D^2 Y = \partial^\mu (D_\mu Y) - ig[A_\mu \ast D_\mu Y] \]
\[ = \Box Y - ig\partial^\mu [A_\mu \ast Y] - ig[A_\mu \ast \partial_\mu Y] + (ig)^2 [A_\mu \ast [A_\mu \ast Y]]. \tag{8} \]
By applying \( \Box^{-1} \equiv \Box^{-1} \) (i.e. the Green function of the operator \( \Box \), see equation (4)) to this relation, we find
\[ Y = \frac{1}{\Box} \tilde{F} + ig\partial^\mu [A_\mu \ast Y] + ig[A_\mu \ast \partial_\mu Y] - (ig)^2 [A_\mu \ast [A_\mu \ast Y]]. \tag{9} \]
The quantity \( Y \) can be determined from this equation up to an arbitrary order:
\[ Y^{(0)} = \frac{1}{\Box} \tilde{F}, \]
\[ Y^{(1)} = \frac{1}{\Box} \tilde{F} + ig\partial^\mu [A_\mu \ast \frac{1}{\Box} \tilde{F}] + ig[A_\mu \ast \partial_\mu \frac{1}{\Box} \tilde{F}] - (ig)^2 [A_\mu \ast [A_\mu \ast \frac{1}{\Box} \tilde{F}]], \tag{10} \]
and so on.

Next, we define the BRST transformations of the gauge field \( A_\mu \), the ghost \( c \), the anti-ghost \( \bar{c} \) and the Lagrange multiplier \( B \) as usual:
\[ s A_\mu = D_\mu c \equiv \partial_\mu c - ig[A_\mu \ast c], \quad s \bar{c} = B, \]
\[ sc = ig c \ast c, \quad sB = 0, \tag{11} \]
\[ s^2 \varphi = 0 \quad \text{for} \quad \varphi \in \{ A_\mu, c, \bar{c}, B \}. \]
The \( s \) variation of \( A_\mu \) implies \( s \tilde{F} = ig[c \ast \tilde{F}] \), from which it follows (as we will now show) that
\[ s \left( \frac{1}{D^2} \ast \tilde{F} \right) = ig \left[ c \ast \frac{1}{D^2} \ast \tilde{F} \right]. \tag{12} \]
Indeed, for a field \( \Phi \) transforming as \( \tilde{F} \), i.e.
\[ s \Phi = ig[c \ast \Phi], \tag{13} \]
the field \( D^2 \Phi \) also transforms covariantly: \( s (D^2 \Phi) = ig[c \ast D^2 \Phi] \). From \( s(D^2 \Phi) = (sD^2)\Phi + D^2(s\Phi) \)
and the previous transformation law, we obtain the operatorial relation

\[
(s D^2)\Phi = -i g [D^2 c \Phi] - 2i g [D^\mu c \cdot \Phi].
\]  

(14)

By applying the \(s\)-operator to \(\Phi = D^2 \frac{1}{D^2} \Phi\),

\[
s\Phi = (s D^2) \Phi + D^2 \frac{1}{D^2} s\Phi,
\]

we can deduce the transformation law of \(\frac{1}{D^2} \Phi\): \(s\Phi = (s D^2) \Phi + D^2 \frac{1}{D^2} s\Phi\). Substitution of (13) and (14) into this relation leads to the conclusion that \(\frac{1}{D^2} \Phi\) transforms in the same manner as \(\Phi\),

\[
s\left(\frac{1}{D^2} \Phi\right) = \frac{1}{D^2} \Phi - 1\frac{1}{D^2} (s D^2) \Phi + D^2 \Phi - \frac{1}{D^2} s D^2 \Phi - B, \tag{15}
\]

whence result (12).

Consider now the following action for the \(U(1)\) gauge field \(A_\mu\) in four-dimensional non-commutative Euclidean space:

\[
\Gamma^{(0)} = S_{\text{inv}} + S_{\text{gf}},
\]

\[
S_{\text{inv}} = \int d^4x \left[ \frac{1}{4} F^{\mu\nu} \cdot F_{\mu\nu} + \beta \frac{1}{4} \left( \frac{1}{D^2} \cdot F\right) \left( \frac{1}{D^2} \cdot F\right) \right],
\]

\[
S_{\text{gf}} = s \int d^4x \bar{c} \left[ \left( 1 + \gamma \frac{\square}{\tilde{\square}} \right) \partial_{\mu} A_\mu - \frac{1}{2} B \right] - \bar{c} \left( 1 + \gamma \frac{\square}{\tilde{\square}} \right) \square B
\]

\[
= \int d^4x \left[ B \left( 1 + \gamma \frac{\square}{\tilde{\square}} \right) \partial_{\mu} A_\mu - \frac{1}{2} B B - \bar{c} \left( 1 + \gamma \frac{\square}{\tilde{\square}} \right) \partial_{\mu} D_\mu c \right]. \tag{16}
\]

Here, \(\beta\) and \(\gamma\) are constants, and the term parametrized by \(\gamma\) has been introduced in order to improve the IR behaviour in the ghost sector. (For \(\gamma \to 0\), one recovers the Feynman gauge expression.) Furthermore, \(\square = \partial^\mu \partial_\mu\) and \(\partial_\mu = \partial_\mu c\). The action \(\Gamma^{(0)}\) is invariant under the BRST transformations (11), (12). Its bilinear part \(S_{\text{bil}}\) yields the following equations of motion for the free fields:

\[
0 = \delta S_{\text{bil}} \delta A^\mu = -\left( \square\delta_\mu - \partial_\mu \partial_\eta A^\eta + \frac{\beta}{\tilde{\square}} \tilde{\square}\delta_\mu A^\mu - \left( 1 + \gamma \frac{\square}{\tilde{\square}} \right) \partial_\mu B, \tag{17}
\]

This leads to the following propagators in momentum space:

\[
G^A_{\mu\nu}(k) = \frac{1}{k^2} \left( \delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} - \frac{k_\mu k_\nu}{k^2 (1 + \gamma \frac{\square}{\tilde{\square}})} - \beta \frac{k_\mu k_\nu}{(k^2)^2 (\gamma \frac{\square}{\tilde{\square}} + \frac{\gamma}{\tilde{\square}})} \right),
\]

\[
G^{\bar{c}c}(k) = \frac{1}{k^2 + \frac{\gamma}{\tilde{\square}}}. \tag{18}
\]

Since the gauge field propagator \(G^A_{\mu\nu}\) involves an overall factor \(\frac{1}{k^2}\), it is not damped for \(k \to 0\) and one may argue that it does not sufficiently mix long and short scales. If one takes this issue of 'mixing' more seriously, one is led to the alternative model presented in the following section.
3. New gauge field model II

Considering that the scaling behaviour of propagator (2) of Gurau et al [1] ensures the IR finiteness of their model, we look for a BRST invariant action leading to a similar propagator for the $U(1)$ gauge field $A_{\mu}$. Accordingly, we introduce the following action in four-dimensional non-commutative Euclidean space:

\[ \Gamma^{(0)} = S_{\text{inv}} + S_{\text{gf}}, \]

\[ S_{\text{inv}} = \int d^{4}x \left[ \frac{1}{4} F^{\mu\nu} \star F_{\mu\nu} + \frac{1}{4} F^{\mu\nu} \star \frac{1}{D^{2}D^{2}} \star F_{\mu\nu} \right], \]

\[ S_{\text{gf}} = \int d^{4}x \bar{c} \star \left[ \left( 1 + \frac{1}{\Box} \right) \partial^{\mu} A_{\mu} - \frac{\alpha}{2} B - \bar{c} \left( 1 + \frac{1}{\Box} \right) \partial^{\mu} D_{\mu} c \right]. \]  

Here, $\alpha$ is a real parameter and \( \frac{1}{D^{2}D^{2}} \star F_{\mu\nu} \) is again to be understood as a formal power series in the gauge field $A_{\mu}$. The functional $\Gamma^{(0)}$ is invariant under the BRST transformations (11) which imply

\[ s \left( \frac{1}{D^{2}D^{2}} \star F_{\mu\nu} \right) = ig \left[ c \star \frac{1}{D^{2}D^{2}} \star F_{\mu\nu} \right]. \]  

The bilinear part of the action now leads to the following equations of motion for the free fields:

\[ 0 = \frac{\delta S_{\text{bil}}}{\delta A^{\nu}} = - \left( 1 + \frac{1}{\Box} \right) \left( \Box \delta_{\nu\mu} - \partial_{\nu} \partial_{\mu} \right) A^{\mu} - \left( 1 + \frac{1}{\Box} \right) \partial_{\nu} B, \]

\[ 0 = \frac{\delta S_{\text{bil}}}{\delta B} = \left( 1 + \frac{1}{\Box} \right) \Box A_{\mu} - \alpha B, \]

\[ 0 = \frac{\delta S_{\text{bil}}}{\delta \bar{c}} = - \left( 1 + \frac{1}{\Box} \right) \Box c. \]  

Hence, we get the following propagators in momentum space:

\[ G_{\mu\nu}^{A}(k) = \frac{1}{k^{2} + \frac{1}{\xi^{2}}} \left( -\delta_{\mu\nu} + \frac{k_{\mu}k_{\nu}}{k^{2}} - \frac{\alpha}{k^{2} + \frac{1}{\xi^{2}}} \right), \]

\[ G^{cc}(k) = \frac{1}{k^{2} + \frac{1}{\xi^{2}}}. \]

If one chooses the Landau gauge $\alpha = 0$ for the gauge parameter, then the gauge field propagator simplifies to

\[ G_{\mu\nu}^{A}(k) = \frac{1}{k^{2} + \frac{1}{\xi^{2}}} \left( -\delta_{\mu\nu} + \frac{k_{\mu}k_{\nu}}{k^{2}} \right). \]  

4. Concluding remarks

In the preceding sections, we introduced two natural models for non-commutative $U(1)$ gauge fields. These models are both BRST invariant and translation invariant, and they are devised for curing the UV/IR mixing problem. The second model has the advantage that the gauge field propagator has an improved ‘damping’ behaviour for vanishing momentum. The question
of whether this property is sufficient for ensuring the renormalizability of the model obviously requires further and more involved investigations (work in progress).

Acknowledgments

It is a great pleasure to thank Jean-Christophe Wallet for his valuable comments on a preliminary version of the present communication. The work of D N Blaschke, E Kronberger and M Wohlgenannt was supported by the 'Fonds zur Förderung der Wissenschaftlichen Forschung' (FWF) under contracts P20507-N16, P19513-N16 and P20017-N16, respectively.

References

[1] Gurau R, Magnen J, Rivasseau V and Tanasa A 2008 A translation-invariant renormalizable non-commutative scalar model (Preprint arXiv:0802.0791)
[2] Douglas M R and Nekrasov N A 2001 Noncommutative field theory Rev. Mod. Phys. 73 977–1029 (Preprint hep-th/0106048)
[3] Grosse H and Wulkenhaar R 2003 Renormalisation of phi**4 theory on noncommutative R**2 in the matrix model J. High Energy Phys. JHEP12(2003)019 (Preprint hep-th/0307017)
[4] Grosse H and Wulkenhaar R 2005 Renormalisation of phi**4 theory on noncommutative R**4 in the matrix model Comm. Math. Phys. 256 305–74 (Preprint hep-th/0401128)
[5] Grosse H and Vignes-Tourneret F 2008 Minimalist translation-invariant non-commutative scalar field theory (Preprint arXiv:0803.1035)
[6] Slavnov A A 2003 Consistent noncommutative quantum gauge theories? Phys. Lett. B 565 246–52 (Preprint hep-th/0304141)
[7] Slavnov A A 2004 Gauge-invariant U(1) model in the axial gauge on the noncommutative plane Teor. Mat. Fiz. 140 388–95
[8] Blaschke D N, Hohenegger S and Schweda M 2005 Divergences in non-commutative gauge theories with the Slavnov term J. High Energy Phys. JHEP11(2005)041 (Preprint hep-th/0510100)
[9] Blaschke D N, Gieres F, Piguet O and Schweda M 2006 A vector supersymmetry in noncommutative U(1) gauge theory with the Slavnov term J. High Energy Phys. JHEP05(2006)059 (Preprint hep-th/0604154)
[10] de Goursac A, Wallet J-C and Wulkenhaar R 2007 A generalization of Slavnov-extended non-commutative gauge theories J. High Energy Phys. JHEP08(2007)032 (Preprint arXiv:0705.3007)
[11] Hayakawa M 2000 Perturbative analysis on infrared aspects of noncommutative QED on R^4 Phys. Lett. B 478 394–400 (Preprint hep-th/9912094)
Ruiz F R 2001 Gauge-fixing independence of IR divergences in non-commutative U(1), perturbative tachyonic instabilities and supersymmetry Phys. Lett. B 502 274–8 (Preprint hep-th/0012171)
Van Raamsdonk M 2001 The meaning of infrared singularities in noncommutative gauge theories J. High Energy Phys. JHEP11(2001)006 (Preprint hep-th/0110093)
Attems M, Blaschke D N, Ortner M, Schweda M, Stricker S and Weiretma H M 2005 Gauge independence of IR singularities in non-commutative QFT—and interpolating gauges J. High Energy Phys. JHEP07(2005)071 (Preprint hep-th/0506117)