AN ALGEBRA OF SKEW PRIMITIVE ELEMENTS

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We study the various term operations on the set of skew primitive elements of Hopf algebras, generated by skew primitive semi-invariants of an Abelian group of grouplike elements. All 1-linear binary operations are described and trilinear and quadrilinear operations are given a detailed treatment. Necessary and sufficient conditions for the existence of multilinear operations are specified in terms of the property of particular noncommutative polynomials being linearly dependent and of one arithmetic condition. We dub the conjecture that this condition implies, in fact, the linear dependence of the polynomials in question and so is itself sufficient (a proof of this conjecture see in "An existence condition for multilinear quantum operations," Journal of Algebra, 217, 1999, 188 – 228).

INTRODUCTION

Skew primitive elements in quantum group theory play roughly the same part as the primitive elements play in the theory of classical continuous groups. The significance of primitive elements is determined by the fact that, in any Hopf algebra $H$, the set $L_1$ of primitive elements form a Lie algebra under the bilinear term operation $[xy] = xy - yx$, and the subalgebra generated by $L_1$ in $H$ is isomorphic to a universal enveloping (or $p$-enveloping, if the characteristic $p$ of a ground field is positive) algebra of the Lie algebra $L_1$.

The study of skew derivations of associative rings, we observe, is also tightly linked with research on skew primitive elements of Hopf algebras, since skew primitive elements, indeed, always act as skew derivations; see [1].

The goal of the present article is to elucidate an algebraic structure of the set of skew primitive elements of some Hopf algebras. To take off the ground, it is worth noting that even the linear structure of that set exhibits itself in a more complex manner as compared to the way the structure of the set of primitive elements does. That set forms a comb (see [2-4]) or, in other terms, its linear span $L$ is an Yetter–Drinfeld module (cf. [5, 6]) over a group algebra of the group $G$ of grouplike elements in a given Hopf algebra. The multiplicative structure (more precisely, the Lie structure), too, suffers a sufficient distortion and turns into a partial operation of variable (quantum) arity; in other words, it splits into the set of closely related partial operations of different arities. These operations are the subject matter of our research.

We focus on the Hopf algebras generated by semi-invariants w.r.t. a commutative group $G$ of grouplike elements acting by conjugations. We call such Hopf algebras character. Among them, for instance, are quantum enveloping Drinfeld–Jimbo algebras; $G$-universal enveloping algebras of Lie color superalgebras; a quantum plane;
and any Hopf algebra generated by skew primitive elements, provided that it has exactly \( n \) grouplike elements which commute pairwise, and the ground field contains a primitive \( n \)th root of unity.

In quantum group theory spaces of primitive elements of braided Hopf algebras (cf. [7] and [8]) are currently often treated as “quantum” Lie algebras (see, e.g., [9]). Our present results apply in research of such “quantum” Lie algebras provided that braiding is defined via the bigrading by a commutative group and by its character group using the formula

\[
(a \otimes b)(c \otimes d) = \chi^c(g_b)^{-1} \cdot ac \otimes bd,
\]

because term operations on those “quantum” Lie algebras are defined by the same terms (polynomials) as are the quantum operations dealt with in the article (see Prop. 4.2 below or Radford’s theorem in [10] on embedding braided Hopf algebras in ordinary ones via biproduct).

In this article, we give a description of all unary “quantum” operations (Thm 5.1) and of all binary operations linear in one of the variables (Thm. 6.1), and specify a necessary and sufficient condition for a nonzero \( n \)-linear operation to exist (Thm. 7.5). This criterion is then used in Sec. 8 to study in detail trilinear and quadrilinear operations. Also, we introduce the notion of a partial main operation of variable arity in terms of which all quantum operations of degree \( \leq 4 \) are expressible in the case of a ground field of characteristic 0.

1. BASIC NOTIONS AND EXAMPLES

Let \( H \) be an arbitrary Hopf algebra with comultiplication \( \Delta \), counity \( \varepsilon \), and antipode \( S \). Denote by \( G \) the set of all grouplike elements

\[
G = \{g \in H \mid \Delta(g) = g \otimes g, \ \varepsilon(g) = 1\}.
\]

It is well known that \( G \) is a group and \( S(g) = g^{-1} \), in which case the linear space generated by \( G \) in \( H \) is a group algebra of \( G \), that is, distinct grouplike elements are linearly independent in \( H \). For \( g \in G \), put

\[
L_g = \{h \in H \mid \Delta(h) = h \otimes 1 + g \otimes h\}.
\]

This set forms a linear space over the ground field, and we call its elements \( g \)-primitive, or skew primitive if \( g \) is not specified. The action of \( G \) on \( H \) is defined by conjugations \( h^g = g^{-1}hg \). It is easy to see that linear spaces \( L_g \) are independent, that is, their linear span \( L \) is the direct sum

\[
L = \sum_{g \in G} \oplus L_g,
\]

in which case the \( L \) is invariant under the above-specified action, and \( L_g^2 = L_g^{-1}sl_g \). In other words, \( L \) is an Yetter–Drinfeld module (cf. [5, 6]) over a Hopf subalgebra \( k[G] \), which is a group algebra of the group \( G \).
Definition 1.1. We say that $h \in H$ is a character element, or call it a semi-invariant, if there exists a character $\chi : G \to k^*$ such that, for all $g \in G$,

$$g^{-1}hg = \chi(g)h.$$ 

If $h$ is a nonzero semi-invariant, then the character $\chi$ is uniquely determined by (2), and we call $\chi$ a weight of $h$ and denote it by $\chi^h$.

Definition 1.2. A Hopf algebra $H$ is called character if the group $G$ is commutative and $H$ is generated as an algebra with unity by character skew primitive elements.

The product of two semi-invariants is again a semi-invariant, and $\chi^{ab} = \chi^a \chi^b$ (if $ab \neq 0$). Therefore, semi-invariants generate a character Hopf algebra also as a linear space. Moreover, using (2), we can easily show that nonzero semi-invariants of different weights are linearly independent. This means that any character Hopf algebra is graded by the character group $G^*$ of $G$:

$$H = \sum_{\chi \in G^*} \oplus H^\chi.$$ 

Definition 1.3. Throughout the article, we refer to the Yetter-Drinfeld module over a group algebra of the Abelian group $G$ as a quantized space (in view of the fact that in a character Hopf algebra, that module plays the role of a generating space). In this way the quantized space is a linear space, graded by an Abelian group, on which the action of the group is defined in such a way as to leave homogeneous components invariant.

Consider some examples of character Hopf algebras.

Example 1.4. Quantum enveloping algebra $\mathcal{KM}$. Let $A = |a_{ij}|$ be an arbitrary $n \times n$-matrix, for which there exist elements $d_1, \ldots, d_n$ such that $d_i a_{ij} = d_j a_{ji}$ (any Cartan matrix, for instance, has this property). The Hopf algebra $\mathcal{KM}$ is generated as an algebra with unity by elements $E_i, F_i, K_i, K_i^{-1}, 1 \leq i \leq n$, and is defined by the following relations:

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$K_i^{-1} E_j K_i = q^{-d_i a_{ij}} E_j, \quad K_i^{-1} F_j K_i = q^{d_i a_{ij}} F_j,$$

where $q$ is some fixed parameter — normally, a formal variable which is freely adjoined to the ground field. Comultiplication is obtained via the formulas

$$\Delta(E_i) = E_i \otimes K_i^{-1} + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + K_i \otimes F_i,$$

by which the counity and antipode are uniquely determined thus:

$$\varepsilon(K_i) = 1, \quad \varepsilon(K_i^{-1}) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0;$$

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -q^{-d_i a_{ii}} E_i, \quad S(F_i) = -q^{d_i a_{ii}} F_i.$$
In the present example, the group $G$ is generated by elements $K_i$; its skew primitive generating elements are the semi-invariants
\[ e_i = E_i K_i, \quad f_i = F_i K_i, \quad 1 - K_i, \]
whose weights are given by the formulas:
\[ \chi^{e_i}(K_j) = q^{-d_{ij}} a_{ji}, \quad \chi^{f_i}(K_j) = q^{d_{ij}}, \quad \chi^{1 - K_i} = id. \]

**Example 1.5.** Quantum Drinfeld–Jimbo enveloping algebra $U_q(g)$. This can be exemplified by a quotient Hopf algebra of $\mathcal{K}\mathcal{M}$ for the case where $A$ is a Cartan matrix (in particular, $a_{ii} = 2$, $a_{ij} \leq 0$; $|a_{ij}|, d_i \in \{1, 2, 3\}$ for $i \neq j$), defined via
\[ [E_i, F_j] = \delta_{ij} \left( \frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}} \right) \]
(3)

$\delta_{ij}$ is a Kronecker symbol), and by the Serre quantum relations
\[ \sum_{\xi=0}^{1-a_{ij}} (-1)^{\xi} \left( \frac{1 - a_{ij}}{\xi} \right) q^{2d_i} E_i^{1-a_{ij}-\xi} E_j E_i^\xi = 0 \quad (i \neq j), \]
(4)
\[ \sum_{\xi=0}^{1-a_{ij}} (-1)^{\xi} \left( \frac{1 - a_{ij}}{\xi} \right) q^{2d_i} F_i^{1-a_{ij}-\xi} F_j F_i^\xi = 0 \quad (i \neq j), \]
(5)

where the parentheses with indices (binomial coefficients) are given explicitly as values of the polynomials
\[ \binom{m}{n}_t = \frac{(t^m - t^{-m})(t^{m-1} - t^{-(m-1)}) \cdots (t^{m-n+1} - t^{-(m-n+1)})}{(t - t^{-1})(t^2 - t^{-2}) \cdots (t^n - t^{-n})}. \]

Relations (3) for skew primitive generators take up the form
\[ e_i f_j - q^{-2d_i a_{ij}} f_j e_i = \delta_{ij} \left( \frac{K_i^4 - 1}{q^{4d_i} - 1} \right), \]
(6)
whereas Serre $q$-relations (4) and (5) are left fixed (with $E_i$ and $F_i$ replaced by $e_i$ and $f_i$, respectively).

**Example 1.6.** Quantum analog for a Lie–Heisenberg algebra. This is a Hopf subalgebra of $U_q(sl(3))$:
\[ U_q(H) = k\langle E_1, E_2, K_1, K_2, K_1^{-1}, K_2^{-1} \rangle \subseteq U_q(sl(3)). \]

Since the Cartan matrix of the algebra $sl(3)$ has the form $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, the Serre $q$-relations take up the form
\[ E_i^2 E_j + E_j E_i^2 = (q^2 + q^{-2}) E_i E_j E_i. \]
for \(i, j \in \{1, 2\}, i \neq j\).

**Example 1.7.** Quantum plane. We have
\[
A_{q}^{(2|0)} = k\langle g, x \mid xg = qgx \rangle
\]
\[
G = \langle g \rangle, \Delta(x) = x \otimes 1 + g \otimes x.
\]

**Example 1.8.** Assume that the primitive \(n\)th root of unity belongs in the ground field. Then any Hopf algebra with exactly \(n\) grouplike elements which commute pairwise will be character provided that it is generated by skew primitive elements.

Indeed, \(k[G]\), in this case, is a commutative and completely reducible algebra of dimension \(n\), whose irreducible modules all have dimension 1. In particular, the invariant spaces \(L_g\) split into direct sums of one-dimensional invariant subspaces, which consist of character elements generating \(H\).

**Example 1.9.** Universal \(G\)-enveloping algebras of Lie color superalgebras. The concept of a Lie color superalgebra is related to some fixed Abelian group \(G\) and symmetric bicharacter \(\lambda : G \times G \to k^*\), defined thus:
\[
\lambda(fh, g) = \lambda(f, g)\lambda(h, g), \quad \lambda(f, hg) = \lambda(f, h)\lambda(f, g), \quad \lambda(f, g)\lambda(g, f) = 1.
\]
The linear \(G\)-graded space \(\Lambda = \sum_{g \in G} \oplus \Lambda_g\) is called a **Lie color superalgebra** if it is augmented with a bilinear operation satisfying the following properties:
\[
[a, b] = -\lambda(f, g)[b, a], \quad a \in \Lambda_f, \quad b \in \Lambda_g, \quad c \in \Lambda_h,
\]
\[
\lambda(f, h)[a, [b, c]] + \lambda(h, g)[c, [a, b]] + \lambda(g, h)[b, [c, a]] = 0.
\]

Any Lie \((G, \lambda)\)-color superalgebra \(\Lambda\) has a universal associative enveloping algebra \(U\). That is, there exists a \(G\)-graded associative algebra \(U = \sum \oplus U_g\), which contains \(\Lambda\) as a generating graded subspace \(\Lambda_g \subseteq U_g\), and the operation on homogeneous elements \(a \in \Lambda_f, b \in \Lambda_g\) and \(\Lambda\) is expressed via multiplication in \(U\) by the formula
\[
[a, b] = ab - \lambda(f, g)ba.
\]
Moreover, \(U\) satisfies the categorical universality condition (for details, see [11-14]). On \(U\), we can define the action of the group \(G\) by setting \(a^g = \lambda(f, g)a, a \in \Lambda_f\), and consider a skew group ring \(H^{\text{col}} = G \ast U\), which has the structure of a Hopf algebra with comultiplication defined on \(G\) and \(\Lambda\) via \(\Delta(g) = g \otimes g\) and \(\Delta(a) = a \otimes 1 + f \otimes a\), \(a \in \Lambda_f\) (and this is exactly the Radford biproduct \(U(\Lambda) \ast k[G]\); see [15]).

Now it is easy to see that \(H^{\text{col}}\) is a character Hopf algebra, for which
\[
\chi^a = \lambda(f, ), \quad a \in \Lambda_f; \quad L_g = \Lambda_g \oplus (1 - g)k.
\]
2. QUANTUM VARIABLES AND QUANTUM OPERATIONS

In what follows, we fix an Abelian group \( G \) and assume that, in the Hopf algebras under examination, the \( G \) is interpreted by grouplike elements. In other words, we consider the category of Hopf algebras \( H \) with distinguished homomorphisms \( \varphi_H : k[G] \to H \).

**Definition 2.1.** A quantum variable is one to which an element \( g \in G \) and a character \( \chi \in G^* \) are associated. In a Hopf algebra, accordingly, the quantum variable \( x = x^\chi_g \), can assume only \( g \)-primitive semiinvariants of weight \( \chi \) values only. A character corresponding to \( x \) is denoted by \( \chi^x \), and a grouplike element — by \( g_x \).

**Definition 2.2.** A quantum operation in quantum variables \( x_1, \ldots, x_n \) refers to an associative polynomial in \( x_1, \ldots, x_n \) which yields a skew primitive element given any values of the quantum variables \( x_1, \ldots, x_n \) in Hopf algebras.

In particular, a homogeneous quantum operation has this form:

\[
[x_1, \ldots, x_n] = \sum_{\pi \in S_n} \alpha_{\pi} x_{\pi(1)} \cdots x_{\pi(n)},
\]

where \( x_1, \ldots, x_n \) are not necessarily distinct quantum variables. If those variables are mutually distinct (but not necessarily of different types), the operation is called multilinear.

We give some examples of quantum operations.

1. **Commutator.** If \( G \) is a trivial group, then the usual commutator \( xy - yx \) is a quantum operation. If the ground field has a positive characteristic \( p > 0 \), there exists a nonmultilinear operation \( x^p \). We can show that all other operations (if, of course, \( G = \text{id} \)) will be superpositions of these two (this in essence exhausts the content of the known theorem due to Friedrichs; see, e.g., [16, Ch. V, Sec. 4]).

2. **Skew commutator.** Let \( x \) and \( y \) be quantum variables. Write \( p_{12} = \chi^x(g_y) \) and \( p_{21} = \chi^y(g_x) \), assuming that these parameters are related via \( p_{12}p_{21} = 1 \). Then the skew commutator

\[
[x, y]_{p_{12}} = xy - p_{12}yx
\]

(7)
is a quantum operation.

In Example 1.5, we have \( \chi^f_i(g_{c_{ij}})\chi^{c_{ij}}(g_{f_i}) = q^{-2a_{i}a_{ij}}q^{2d_{i}a_{ij}} = 1 \), and so the left sides of (6) are values of the quantum operations and the right ones are skew primitive constants.

Likewise, the Lie operation in a Lie color superalgebra \( \Lambda \) (if \( \Lambda \) is assumed embedded in the Hopf algebra \( H^{col} \)) is a quantum operation since \( \chi^a(g_b)\chi^b(g_a) = \lambda(f, g)\lambda(g, f) = 1. \)

It is easy to see that skew commutators essentially exhaust all the bilinear quantum operations; see Thm 6.1 for \( n = 1 \).
3. Pareigis quantum operation. Let \( \zeta \) be a primitive \( n \)th root of unity and \( x_1, x_2, \ldots, x_n \) be quantum variables such that \( \chi^{x_1}(g_{x_j}) \chi^{x_j}(g_{x_1}) = \zeta^2 \). Then

\[
P_n(x_1, \ldots, x_n) = \sum_{\pi \in S_n} \left( \prod_{i<j \& \pi(i) > \pi(j)} (\zeta^{-1} \chi^{x_{\pi(i)}}(g_{x_{\pi(j)}})) \right) x_{\pi(1)} \cdots x_{\pi(n)}
\]

is a quantum operation (see [9, Thm. 3.1, p. 147] and the remarks under Sec. 4 below).

4. Serre quantum operation. Let \( x \) and \( y \) be such that

\[
\chi^x(g_y) = q^{d_i} a_{ij}, \quad \chi^y(g_x) = q^{d_i},
\]

where the parameters \( d_i \) and \( a_{ij} \) are the same as in Example 1.5. We can show, then, that the left parts of Serre quantum relations are values of the following quantum operations:

\[
S_{ij}(x, y) = \sum_{\xi=0}^{1-a_{ij}} (-1)^\xi \left( \frac{1-a_{ij}}{\xi} \right) q^{2d_i} y^{1-a_{ij}-\xi} x y^\xi
\]

for \( x = e_j \) and \( y = e_i \) or for \( x = f_j \) and \( y = f_i \).

These are examples of homogeneous binary quantum operations, linear in one of the variables. A complete description of such operations will be given later, under Sec. 6. Now we present the construction of a “tensor algebra” for a quantized space, which makes it possible to formally operate with polynomials in quantum variables as if with elements of Hopf algebras.

### 3. FREE ENVELOPING ALGEBRA OF A QUANTIZED SPACE

Let \( L = \sum L_g \) be some quantized space. Denote by \( \mathbf{k}\langle L \rangle \) the tensor algebra of a linear space \( L \). If we distinguish some basis \( X \) in \( L \), consisting of character elements, then \( \mathbf{k}\langle L \rangle \) will be a free associative algebra of \( X \) — in particular, it has a basis consisting of all words on \( X \) (including the empty word equal to unity). The action of \( G \) is uniquely extended to \( \mathbf{k}\langle L \rangle \), and so we can define the skew group algebra \( G * \mathbf{k}\langle L \rangle \), on which the structure of a Hopf algebra arises naturally. We have

\[
\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}, \quad g \in G;
\]

\[
\Delta(l) = l \otimes 1 + g \otimes l, \quad \varepsilon(l) = 0, \quad S(l) = -g^{-1}l, \quad l \in L_g.
\]

**Definition 3.1.** A free enveloping algebra of a quantized space \( L \) is the Hopf algebra specified above, and we denote it by \( H\langle L \rangle \), or by \( H\langle X \rangle \) if \( L \) has some distinguished basis \( X \) consisting of homogeneous character elements.

Denote the quantized space of skew primitive elements of the free enveloping algebra \( H\langle L \rangle \) by \( \mathbf{k}_{q-tie}\langle L \rangle \) or by \( \mathbf{k}_{q-tie}\langle X \rangle \). It is then obvious that \( L \) is a quantum subspace in \( \mathbf{k}_{q-tie}\langle L \rangle \).
Now suppose that some set $X$ of quantum variables is given. Consider linear spaces $L_x = kx$ spanned by the variables $x$, and denote by $L_g$ the direct sum of all $L_x$ such that $g_x = g$. Define on $L_g$ the action of $G$ by setting $x^h = \chi^h(x)x$. In this way the direct sum $L$ of spaces $L_g$ turns into a quantized space. It is easy to see that quantum operations are elements of the quantized space $q_{-\text{tie}}(X)$, in the representations of which there are no grouplike elements (i.e., they lie in $k(X)$). Of course, it is not pointless to treat all elements $k_{q_{-\text{tie}}}(X)$ as quantum operations (with constants). This might seem to be even natural since we have fixed the constants. However, the reader will see that this does not in fact give way to any new operations (but for the constants $1-g$ proper).

A free enveloping algebra is the algebra $KM$ from Example 1.4. In this case $G$ is an Abelian group, freely generated by elements $K_i$, $1 \leq i \leq n$; linear spaces $L_g$ are either two-dimensional $L_{K_i^2} = ke_i \oplus kf_i$ or zero; the characters are determined from columns of the matrix $[a_{ij}]$ by setting $\chi^j(K_i) = q^{a_{ij}}$; and the action of the group is defined so that $f_j$ turns into a semi-invariant of weight $\chi^j$ and $e_j$ turns into a semi-invariant of weight $(\chi^j)^{-1}$. Curiously, in order to construct that quantized space, we need not impose any restrictions on the matrix $[a_{ij}]$; moreover, if $[a_{ij}]$ is freed of zero columns, and $q \neq \pm 1$, then, for any character $\chi \in G^*$ and for every element of $g \in G$, the space $L_g^x$ is not more than one-dimensional or, in other words, $KM$ does not contain distinct quantum variables of the same type.

We make some trivial but important remarks. First, we bring out the general form into which a word in quantum variables is expanded under comultiplication. Let $w = x_1x_2\cdots x_n$; then

$$\Delta(w) = \Delta(x_1)\Delta(x_2)\cdots \Delta(x_n) = (x_1 \otimes 1 + g_{x_1} \otimes x_1)(x_2 \otimes 1 + g_{x_2} \otimes x_2)\cdots (x_n \otimes 1 + g_{x_n} \otimes x_n).$$

Removing the parentheses gives

$$\Delta(w) = \sum_{v \in B(w)} w\mid v \otimes v, \quad (8)$$

where $B(w)$ denotes the set of all subwords of $w$ including the empty word (a subword of $w$ is a word obtained from $w$ by deleting the letters); $w\mid v$ is a word obtained from $w$ by replacing all variables $x_i$ in $v$ by respective $g_{x_i}$. Put $g_w = g_{x_1}g_{x_2}\cdots g_{x_n}$. Taking into account that grouplike elements $g_x$ commute pairwise and that $xg = \chi_x(g)\cdot gx$, we can reduce (8) to the form

$$\Delta(w) = \sum_{v \in B(w)} \alpha_v g_v[w-v] \otimes v, \quad (9)$$

where $[w-v]$ is a word obtained from $w$ by deleting $v$, and $\alpha_v$ is a product of all elements of the form $\chi^x(g_y)$ for all pairs of variables $x,y$ which occur in $w$ and are such that $x$ is in $[w-v]$ and $y$ in $v$, and the quantum variable $x$ occurs in $w$ to the left of $y$. 

8
The above formula can be presented in another form, by writing the grouplike elements \( g_v \) to the right of \([w - v]\) in the left components of tensors:

\[
\Delta(w) = \sum_{v \in B(w)} \alpha'_v [w - v] g_v \otimes v,
\]

where \( \alpha'_v \) is a product of all elements of the form \((\chi^x(g_y))^{-1}\) for all pairs of variables \(y, x\) which occur in \(w\) and are such that \(x\) is in \([w - v]\) and \(y\) in \(v\), and the quantum variable \(x\) occurs in \(w\) to the right of \(y\).

Further, on the free enveloping algebra \(H \langle X \rangle\) we can define a degree function \(d\) by setting \(d(g) = 0, g \in G\); \(d(x) = 1, x \in X\). On the tensor product, that function induces two degree functions:

\[
d_l(w \otimes 1) = d(w) \quad \text{and} \quad d_l(1 \otimes w) = 0
\]

and

\[
d_r(w \otimes 1) = 0 \quad \text{and} \quad d_r(1 \otimes w) = d(w).
\]

Also, we can define the degree \(d_+ = d_l + d_r\). The tensor square of a free enveloping algebra has gradings relative to each one of the degrees. It is worth mentioning that comultiplication will be homogeneous in view of (9) once we have assumed that \(H \langle X \rangle\) is graded by \(d\), and \(H \langle X \rangle \otimes H \langle X \rangle\) — by \(d_+\). In particular, it follows that all \(d\)-homogeneous components of quantum operations are quantum operations themselves, and we can therefore limit our treatment to \(d\)-homogeneous quantum operations. It is not hard to see that the filtration, defined by the degree function \(d\),

\[
k[G] \subseteq k[G]L \subseteq k[G]L^2 \subseteq \cdots \subseteq k[G]L^n \subseteq \cdots
\]

is contained in the coradical filtration; see [17, p. 60].

We will need yet another degree function which is related to some distinguished variable \(x \in X\) and is defined similarly as follows: \(d^{(x)}(x) = 1\) and \(d^{(x)}(y) = 0\), for \(y \in X\) and \(y \neq x\), and \(d^{(x)}_l(w \otimes v) = d^{(x)}(w)\), \(d^{(x)}_r(w \otimes v) = d^{(x)}(v)\). Clearly, comultiplication will be homogeneous if we consider the degree \(d^{(x)}\) on \(H \langle X \rangle\) and consider \(d^{(x)}_+ = d^{(x)}_l + d^{(x)}_r\) on \(H \langle X \rangle \otimes H \langle X \rangle\).

Now note that no new grouplike elements arise from a free enveloping algebra.

**Lemma 3.2.** Every grouplike element of a free enveloping algebra of a quantized space belongs to \(G\).

**Proof.** By construction, the basis of a free enveloping algebra consists of words of the form \(gw\), where \(g \in G\) and \(w\) is a word in some set \(X\) of quantum variables. Then the basis for a tensor product \(H \langle X \rangle \otimes H \langle X \rangle\) consists of tensors of the form \(gw \otimes hv\) — in particular, those tensors are linearly independent. If \(f = \sum \alpha_{gw} gw\) is a grouplike element, then \(\Delta f = f \otimes f\); therefore,

\[
\sum_{g, w} \alpha_{gw} (g \otimes g) \Delta(w) = \sum_{g, w} \alpha_{gw} gw \otimes \sum_{g, w} \alpha_{gw} gw.
\]
If \( w \) is some nonempty word of the greatest length possible, occurring in the expansion of \( f \) with nonzero \( \alpha_{gw} \), then the right-hand side of (11) has the term \( \alpha_{gw}^2 gw \otimes gw \), which cannot arise from the left by (9) [or else in view of the property of comultiplication being \((d, d_+)-\)homogeneous]. Thus, all words in the expansion of \( f \) are empty, that is, \( f \in k[G] \).

It remains to appeal to the trite fact that all grouplike elements of a group algebra belong to the initial group. The lemma is proved.

Further, we note that the concept of a quantum operation with constants brings about nothing new.

**Proposition 3.3.** Every quantum operation with constants of positive \( d \)-degree lies in \( k\langle X \rangle \).

**Proof.** Let \( f = \sum_{h,w} \beta_{hw} hw \) be a skew primitive element of a free enveloping algebra. Then

\[
 f \otimes 1 + g_f \otimes f = \sum_{h,w} \beta_{hw} hw \otimes 1 + g_f \otimes \sum_{h,w} \beta_{hw} hw = \Delta(\sum_{h,w} \beta_{hw} hw) =
\]

\[
 \sum_{h,w} \beta_{hw} (h \otimes h) \sum_{v \in B(w)} \alpha_v g_v [w - v] \otimes v = \sum_{h,w,v} \beta_{hw} \alpha_v h g_v [w - v] \otimes hv.
\]

Since the linear spaces \( 1k, hL, h \in G \), form a direct sum in the free enveloping algebra, all terms of the form \( ... \otimes 1 \) should be cancelable, that is,

\[
 \sum_{h,w} \beta_{hw} hw \otimes 1 = \sum_{w} \beta_{1w} w \otimes 1.
\]

Because \( hw, h \in G \), are linearly independent, we conclude that \( \beta_{hw} = 0 \) for \( h \neq 1 \).

The proposition is proved.

### 4. BIGRADED HOPF ALGEBRAS

In quantum group theory, spaces of primitive elements of braided Hopf algebras are sometimes treated as quantum analogs of Lie algebras. A braided Hopf algebra is defined in essentially the same way as is an ordinary Hopf algebra, the difference being that, instead of the usual tensor product of algebras, in which the left and right components commute so that

\[
(1 \otimes a)(b \otimes 1) = b \otimes a = (b \otimes 1)(1 \otimes a),
\]

we take another product \( \cdot \), for which the commutation rule is given by a Yang–Baxter operator. In this event the notion of a character Hopf algebra is translated into a concept of a \( G \times G^* \)-graded braided Hopf algebra. Namely, let \( \mathcal{H} \) be an associative algebra graded by the group \( G \times G^* \):

\[
 \mathcal{H} = \sum_{g \in G, \chi \in G^*} \oplus \mathcal{H}_g^\chi.
\]
Redefine multiplication on the tensor product $\mathcal{H} \otimes \mathcal{H}$ of linear spaces by setting

$$(a \otimes b) \cdot (c \otimes d) = (\chi^c(g_b))^{-1}(ac \otimes bd).$$

The result is an associative algebra, denoted by $\mathcal{H} \otimes \mathcal{H}$. Now if, in the definition of a Hopf algebra, we change the sign $\otimes$ by $\otimes$ and assume that the coproduct, counity, and antipode are homogeneous, we arrive at a definition of the braided bigraded Hopf algebra. It will be natural to take the primitive element $\Delta(a) = a \otimes 1 + 1 \otimes a$, lying in the component $\mathcal{H}^1$, to be the value of the quantum variable $x = x^\chi$ in the braided bigraded Hopf algebra. Accordingly, the definition of a quantum operation undergoes a slight change, not in content, but in form.

**Definition 4.1.** A braided operation in quantum variables $x_1, x_2, \ldots, x_n$ is a polynomial in $x_1, x_2, \ldots, x_n$, which turns into a primitive element given any values of quantum variables in braided bigraded Hopf algebras.

**Proposition 4.2.** A homogeneous polynomial $F$ is a braided operation if and only if it is a quantum operation.

**Proof.** We need only compare the formulas used to compute the coproducts of monomials in nonbraided [formula (10)] vs. braided Hopf algebras:

$$\Delta(w) = \Delta(x_1, x_2, \ldots, x_n) = (x_1 \otimes 1 + 1 \otimes x_1) \cdot (x_2 \otimes 1 + 1 \otimes x_2) \cdot \ldots \cdot (x_n \otimes 1 + 1 \otimes x_n) = \sum_{v \in B(w)} \beta_v [w - v] \otimes v,$$

where by

$$(1 \otimes x)(y \otimes 1) = (\chi^y(g_x))^{-1}y \otimes x,$$

the coefficients $\beta_v$ are defined in exactly the same way the $\alpha'_v$ are defined in (10), that is, $\beta_v = \alpha'_v$. The proposition is proved.

For a set $X$ of quantum variables, the structure of a bigraded braided Hopf algebra is naturally defined on the free algebra $\langle X \rangle$ in a way that quantum variables become primitive elements. By the above proposition, then, the set of primitive elements of that algebra is exactly the set of all quantum operations in $X$.

The above definition of a bigraded braided Hopf algebra is somewhat different in form from the conventional concept of a $(G, \lambda)$-graded Hopf algebra (see, e.g., [15]), in which one grading $A = \sum A_g$ and a bicharacter $\lambda : G \times G \to k^*$ are specified and the commutation rule is defined via

$$(a \otimes b)(c \otimes d) = \lambda(g_b, g_c)(ac \otimes bd),$$

where $b \in A_{g_b}$ and $c \in A_{g_c}$. If we put $\chi^c = \lambda(\cdot, g_c)^{-1}$, the $A$ then turns into the bigraded braided Hopf algebra defined above. Conversely, if, on the group $G = G^* \times G$, the bicharacter is defined via $\lambda(\chi \times g, \chi' \times g') = \chi'(g)^{-1}$, then the $H$ will turn into a $(G, \lambda)$-graded Hopf algebra.

Besides, the Radford biproduct $A \ast kG$ is an (ordinary) Hopf algebra (see [15, Cor. 3.5]), and if $A$ is generated by primitive elements, then $A \ast kG$ is a character
Hopf algebra whose $g$-primitive elements all have equal weights. Yet, in an arbitrary (ordinary) Hopf algebra, $g$-primitive elements do not necessarily all have equal weights — this is the reason why we opt for the present definitions.

5. Unary Operations

**Theorem 5.1.** For a quantum variable $x$, there exists a quantum operation $x^n$, $n > 1$, if and only if $p = \chi^x(g_x)$ is a primitive $m$th root of unity, and $n = mlr$ where $l = 1$, if the characteristic of the ground field $k$ is zero, and $l = \text{char } k$ if it is positive.

**Proof.** We have
$$\Delta(x^n) = (x \otimes 1 + g_x \otimes x)^n,$$
in which case $(x \otimes 1)(g_x \otimes x) = p(g_x \otimes x)(x \otimes 1)$. Therefore, we can use the so-called quantum binomial formula, which says that if $XY = pYX$, then
$$(X + Y)^n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_{t=p} Y^k X^{n-k},$$
where $\left[ \begin{array}{c} n \\ k \end{array} \right]_t$ is a polynomial in $t$, having the following rational representation
$$\left[ \begin{array}{c} n \\ k \end{array} \right]_t = \frac{t^{[n]}t^{[n-1]}\cdots t^{[n-k+1]}}{t^{[1]}t^{[2]}\cdots t^{[k]}},$$
in which by definition, $t^{[s]} = 1 + t + \cdots + t^{s-1}$, $t^{[0]} = 0$. Using this formula, we obtain
$$\Delta(x^n) = x^n \otimes 1 + g_x^m \otimes x^n + \sum_{k=1}^{n-1} \left[ \begin{array}{c} n \\ k \end{array} \right]_{t=p} g_x^k x^{n-k} \otimes x^k.$$
In a free enveloping algebra, the elements $1, x, \ldots, x^n$ are linearly independent; therefore, the $x^n$ will be primitive if and only if all polynomials $\left[ \begin{array}{c} n \\ k \end{array} \right]_t$, $1 \leq k \leq n-1$ vanish at $t = p$. In particular, $p^{[n]} = \left[ \begin{array}{c} n \\ 1 \end{array} \right]_{t=p} = 0$, that is, $p^n = 1$.

Thus, if $x^n$ is an operation, then $p$ is a primitive $m$th root of unity for some $m$, and $n$ is divisible by $m$, $n = mq$ (if $p = 1$, we put $m = 1$). Consider the coefficient at $\ldots \otimes x^m$. Note that the following general formula is valid:
$$t^{[ms+d]} = (t^m)^{[s]}t^{[m]} + t^{ms}t^{[d]},$$
with $0 \leq d < m$. Using it yields
$$\left[ \begin{array}{c} n \\ m \end{array} \right]_{t=p} = \frac{t^{[mq]}t^{[mq-1]}\cdots t^{[mq-k+1]}}{t^{[1]}t^{[2]}\cdots t^{[m]}} = \frac{(t^m)^{[q]}t^{[m]}}{t^{[m]}} \frac{p^{[m-1]}p^{[m-2]}\cdots p^{[1]}}{t^{[1]}t^{[2]}\cdots t^{[m]}} = (t^m)^{[q]}|_{t=p} = q.$
For the case of characteristic zero, we obtain $n \left[ \frac{n}{m} \right] \neq 0$, which is possible only whenever $m = n$. If $l = \text{char } k$ is positive, then $q$ is divisible by $l$, and so $n = mlq_1$.

In a similar way, we can compute the coefficient at $\cdots \otimes x^{ml}$, which in fact equals $q_1$, that is, either $ml = n$, or $q_1$ is divisible by $l$. Carrying these computations over and over, we obtain $n = mlr$. Such routines can be done away with if we use a simple induction argument along the following lines.

First, $x^m$ is a quantum operation since all coefficients $\left[ \frac{m}{k} \right]_{t=p}$ are such that each has $p^{[m]} = 0$ in the numerator and all of their denominators do not vanish. In addition, $g_x^m = g_x^m$ and $\chi^{x^m} = (\chi^x)^m$; in particular, $\chi^{x^m}(g_x^m) = (p^m)^m = 1$. Consider a new quantum variable $y$ with parameters $\chi^y = (\chi^x)^m$ and $g_y = (g_x)^m$. The (one-dimensional) quantized space generated by $x^m$ is then isomorphic to a quantized space $k_y$. Therefore, free enveloping algebras of these spaces are also isomorphic under $y \leftrightarrow x^m$. And $y^a$ is a primitive element since it corresponds to the primitive element $x^n$. Now, if $m > 1$, we can apply the induction hypothesis to $y$, to eventually obtain $q = l^s$, because $p_y = \chi^y(y) = 1$. But if $m = 1$, that is, $p = 1$, then $x^l$ is also a quantum operation, and a similar inductive step can be taken to treat the quantum variable $y$ with parameters $\chi^y = (\chi^x)^l$ and $g_y = g_x^l$. We have thus proved the necessity of the condition $n = mlr$.

Conversely, if $p$ is a primitive $m$th root of unity and if $n = mlr$, then $x^n = (\cdots ((x^n)^l) \cdots)^l$, that is, $x^n$, being a superposition of the quantum operations $x^m$ and $y^l$, is also a quantum operation. The theorem is proved.

The present theorem shows that there exist two basic types of unary quantum operations: $x^m$ and $y^l$, where the quantum variables $x$ and $y$ are such that $\chi^y(y) = 1$ and $\chi^x(x)$ is a primitive $m$th root of unity, and all other operations are superpositions of these two operations. In fact, the two types of operations do not differ in a crucial respect, since in either case the existence of an operation $x^n$ follows from the fact that $n$ is a least number such that $\chi^x(g_x)^n = 0$. This allows us to introduce one main unary (partial) operation $[\ ]$ on an arbitrary Hopf algebra, defined thus:

$$[a] \overset{\text{def}}{=} \begin{cases} a^m, & \text{a is a skew primitive semi-invariant and} \\ m \text{ a minimal number with the property } \chi^a(g_a)^m = 0; & \text{undefined otherwise.} \end{cases}$$

The main unary operation on an arbitrary braided bigraded Hopf algebra is defined in exactly the same way: $[a] = a^m$ if $a$ is a homogeneous element and $\chi^a(g_a)^m = 0$, $m$ is minimal.

Clearly, the set of all unary quantum operations form a unary algebra relative to $[\ ]$, and that algebra is generated by quantum variables.
6. BINARY OPERATIONS LINEAR IN ONE OF THE VARIABLES

Let $x$ and $y$ be quantum variables. Denote $p_{12} = \chi^x(g_y)$, $p_{21} = \chi^y(g_x)$, and $p_{22} = \chi^y(g_y)$.

**THEOREM 6.1.** For quantum variables $x$ and $y$, there exists a nonzero, quantum, linear in $x$ operation:

$$W(x, y) = \sum_{k=0}^{n} \alpha_k y^k x y^{n-k}$$

if and only if either

$$p_{12} p_{21} = p_{22}^{1-n},$$

or $p_{22}$ is a primitive $m > 1$th root of unity, $m | n$, and

$$p_{12}^{m} p_{21}^{m} = 1.$$ (16)

If one of these conditions is satisfied, or both, then there exists a unique, up to multiplication by a scalar, nonzero quantum operation of degree $n$ w.r.t. $y$, which is linear in $x$.

If condition (15) holds, then the coefficients $\alpha_k$ are equal to coefficients of the polynomial

$$\prod_{s=0}^{n-1} (t - p_{12} p_{22}^{s}) = \sum_{k=0}^{n} \alpha_k t^k.$$ (17)

In this event the operation has the following $q$-commutator representations:

$$W(x, y) = [\ldots [[xy]_{p_{12} p_{22}} y]_{p_{12} p_{22}^{n-1}}],$$

$$W(x, y) = [\ldots [[xy]_{p_{21} p_{22}} y]_{p_{21} p_{22}^{n-1}}].$$ (19)

If (16) holds, then

$$W(x, y) = [\ldots [[xy]^m]_{p_{12}^{m} y^m}]_{p_{12}^{m}}.$$ (20)

**Proof.** First we show that if (15) is satisfied, then the right-hand sides of equalities (18) and (19) are equal operations whose coefficients are specified by equality (17).

We consider the sequence of elements $v_0 = x$, $v_{k+1} = [w_k y]_{p_{12} p_{22}^{k}}$ and use induction to show that

$$d_{i}^{(x)} (\Delta(v_k) - v_k \otimes 1) = 0.$$ (21)

Let

$$\Delta(v_k) - v_k \otimes 1 = \sum_{i=0}^{k} g_x g_y^{n-i} y^i \otimes u_i.$$ Then

$$\Delta(v_{k+1}) = \Delta(v_k) \Delta(y) - p_{12} p_{22}^{k} \Delta(y) \Delta(v_k) = \ldots$$
\[(v_k \otimes 1 + \sum_{i=0}^{k} g_x y^{n-i} y^i \otimes u_i)(y \otimes 1 + g_y \otimes y) - p_{12} p_{22}^k (y \otimes 1 + g_y \otimes y)(v_k \otimes 1 + \sum_{i=0}^{k} g_x y^{n-i} y^i \otimes u_i).\]

Removing the parentheses and neglecting terms of the form \(g y^j \otimes \ldots, j \geq 0\), (which we can do since their \(d_i^{(x)}\)-degrees equal zero), in view of \(v_k g_y = p_{12} p_{22}^k g_y v_k\) we obtain the equality

\[\Delta(v_{k+1}) \equiv v_k y \otimes 1 + v_k g_y \otimes y - p_{12} p_{22}^k (y v_k \otimes 1 + g_y v_k \otimes y) \equiv (v_k y - p_{12} p_{22}^k y v_k) \otimes 1 \equiv v_{k+1} \otimes 1,\]
as required.

Likewise we consider the sequence of elements \(w_0 = x, w_{k+1} = [w_k y]_{p_{21}^{-1} p_{22}^{k}}\) and show that

\[d_r^{(x)}(\Delta(w_k) - g_x g_y^k \otimes w_k) = 0.\]  
(22)

If

\[\Delta(w_k) - g_x g_y^k \otimes w_k = \sum_{i=0}^{k} u_i \otimes y^i,\]

then

\[\Delta(w_{k+1}) = \Delta(w_k) \Delta(y) - p_{21}^{-1} p_{22}^{-k} \Delta(y) \Delta(w_k) =
\]

\[(g_x g_y^k \otimes w_k + \sum_{i=0}^{k} u_i \otimes y^i)(y \otimes 1 + g_y \otimes y) - p_{21}^{-1} p_{22}^{-k} (y \otimes 1 + g_y \otimes y)(g_x g_y^k \otimes w_k + \sum_{i=0}^{k} u_i \otimes y^i).\]

Removing the parentheses and neglecting terms whose \(d_r^{(x)}\)-degrees are zero, in view of \(y g_x g_y^k = p_{21} p_{22}^k g_x g_y^k y\) we obtain

\[\Delta(w_{k+1}) \equiv g_x g_y^k y \otimes w_k + g_x g_y^{k+1} \otimes w_k y - p_{21}^{-1} p_{22}^{-k} (y g_x g_y^k \otimes w_k + g_x g_y^{k+1} \otimes y w_k) \equiv g_x g_y^{k+1} \otimes w_{k+1},\]
as required.

Now we show that \(v_n = w_n\). To do this, consider an operator representation of the skew commutators \([w y]_p = w \cdot (R_y - p L_y)\), where \(L_y\) is an operator of left multiplication by \(y\) and \(R_y\) an operator of right multiplication by \(y\). We have

\[v_n = x \cdot (R_y - p_{12} L_y)(R_y - p_{12} p_{22} L_y) \ldots (R_y - p_{12} p_{22}^{n-1} L_y),\]  
(23)

\[w_n = x \cdot (R_y - p_{21}^{-1} L_y)(R_y - p_{21}^{-1} p_{22}^{-1} L_y) \ldots (R_y - p_{21}^{-1} p_{22}^{-n-1} L_y).\]  
(24)

At this point we note that all operators occurring in the two representations are pairwise commuting, and if condition (15) is satisfied, then \(p_{12} p_{22}^k = p_{21}^{-1} p_{22}^{1-(n-k)}\).

Therefore, the left parts of those two equalities have equal operators. Thus, \(v_n = w_n = W\), and we have

\[d_r^{(x)}(\Delta(W) - W \otimes 1 - g_x g_y^n \otimes W) =
\]

\[d_i^{(x)}(\Delta(v_n) - v_n \otimes 1 - g_x g_y^n \otimes v_n) + d_r^{(x)}(\Delta(w_n) - w_n \otimes 1 - g_x g_y^n \otimes w_n) = 0,\]
which means that $W$ is a skew primitive element. Now, if we remove the parentheses in (23) and (17) we see that the coefficients in (14) and in (17) are equal (we can simply replace $t$ with $\frac{R_w}{R_y}$, assuming that $R_y$ and $L_y$ are formal commuting symbols).

The sufficiency of condition (15) is thereby established.

If the second condition of the theorem is satisfied, then $z = y^m$ is a primitive element, and $p'_{21} = \chi^z(g_z) = p'_{21}$, $p'_{12} = \chi^z(g_z) = p'_{12}$, and $p'_{22} = \chi^z(g_z) = p'_{22}$. For $x$ and $z$, then, condition (15) holds with $n = 1$, that is, $z_1 = [xy^m]_{p'_{22}}$ is a primitive element by the above. Following up this argument, we will see that the right-hand side of (20) is a quantum operation.

We argue for the way back. Assume $W(x, y) = \sum_{k=0}^{n} \alpha_k y^k x y^{n-k}$ is a quantum operation. Then by (8),

$$
\sum_{k=0}^{n} \sum_{\substack{v \in B(y^k x y^{n-k}) \\ v \neq 0, v \neq y^k x y^{n-k}}} \alpha_k y^k x y^{n-k} |v \otimes v = 0. \tag{25}
$$

First we argue for uniqueness. To do this, it suffices to show that $\alpha_0 = 0$ implies $W = 0$. Suppose, to the contrary, that $\alpha_0 = \ldots = \alpha_{r-1} = 0$, $\alpha_r \neq 0$, $r \geq 1$. Consider all terms of the form $\ldots \otimes xy^{n-r}$ in (25). The word $xy^{n-r}$ is a subword of just one word occurring with nonzero coefficient in $W$; therefore, (25) will have only one term in this form:

$$
\alpha_r y^r g_x g_y^{n-r} \otimes xy^{n-r}.
$$

Consequently, the whole sum cannot vanish. This is a contradiction, which proves the uniqueness.

Now assume that $\alpha_0 \neq 0$ and consider all terms of the form $\ldots \otimes y$ in (25). The word $y^k x y^{n-k}$ has $n$ different entries of the subword $y$; therefore, the second sum in (25) has the form

$$
(\sum_{k=0}^{n} \alpha_k y^k g_y y^{k-s-1} x y^{n-s} + \sum_{s=0}^{n-k} \alpha_k y^k x y^s g_y y^{n-k-s-1}) \otimes y =
$$

$$
(\alpha_k p_{22}^{[k]} y^{k-1} x y^{n-k} + \alpha_k p_{22}^{[n-k]} p_{12} p_{22}^{k} x y^{n-k-1}) \otimes y.
$$

Summing all terms of this form over all $k$ and keeping in mind that different words are linearly independent, we obtain the system of equalities

$$
\alpha_k p_{22}^{[k]} + \alpha_{k-1} p_{22}^{[n-k+1]} p_{12} p_{22}^{k-1} = 0, \quad k = 1, \ldots, n. \tag{26}
$$

Consider all terms of the form $\ldots \otimes xy^{n-1}$ in (25). The word $xy^{n-1}$ is a subword of just two words: $xy^{n-1}$ and $xy^n$. In (25), therefore, there are only two terms in the desired form: $\alpha_1 y g_x g_y^{n-1} \otimes xy^{n-1}$ and $\alpha_0 \sum_{s=0}^{n-1} g_x g_y s g_y^{n-s-1} \otimes xy^{n-1}$. It follows that
\[ \alpha_1 p_{21} p_{22}^{n-1} + \alpha_0 [n] p_{22} = 0. \]
Comparing this equality with (26) for \( k = 1 \), \( \alpha_1 + \alpha_0 [n] p_{22} = 0 \),
we obtain
\[ \alpha_0 [n] (-p_{12} p_{21} p_{22}^{n-1} + 1) = 0. \] (27)

Therefore, if \( [n] p_{22} \neq 0 \), then the first condition of the theorem [see (15)] holds, and \( W \) has the desired form by uniqueness.

To prove the remaining part, we use induction on \( n \), that is, assume that for all lesser values of the parameter \( n \), Theorem 6.1 is satisfied in full measure. The basis of induction is the case \( n = 1 \), which is proved since \( p_{22}^{[1]} = 1 \neq 0 \).

Let \( m \) be a least number such that \( p_{22}^{[m]} = 0 \), \( m > 1 \). Then \( n \) is divisible by \( m \), \( n = mq \). By (13), we obtain \( p_{22}^{[ms]} = 0 \) and \( p_{22}^{[nms]} \neq 0 \) for \( 1 \leq d \leq m - 1 \). This allows us to solve the system of equations (26). The result is \( \alpha_{ms+d} = 0 \) for \( 1 \leq d \leq m - 1 \), and \( \alpha_{ms} \) are arbitrary parameters, that is, the \( W \) takes up the form
\[ W(x, y) = \sum_{s=0}^{q} \alpha_{ms} y^{ms} x y^{m(q-s)}. \] (28)

By Theorem 5.1, the element \( y^m \) is primitive. It is also obvious that \( x \) and \( y^m \) generate a free subalgebra in the free enveloping algebra \( H(X) \). This means that \( x \) and \( y^m \), together with the group \( G \), generate in \( H(X) \) a free enveloping algebra of the quantized space \( kx + ky^m \). Now consider a new quantum variable \( z \) with parameters \( \chi^z = (\chi^y)^m \) and \( g_z = g_y^m \). Then the quantized spaces \( kx + ky^m \) and \( kx + kz \) will be isomorphic, and so are their free enveloping algebras under \( x \leftrightarrow x \), \( y^m \leftrightarrow z \). Since (28) is a primitive element in \( H(kx + ky^m) \), we conclude that
\[ N(x, y) = \sum_{s=0}^{q} \alpha_{ms} z^s x z^{n-s} \]
is a quantum operation of lesser degree, and so the theorem applies. We have \( \chi^z(g_z) = p_{12}^m, \chi^z(g_x) = p_{21}^m, \) and \( \chi^z(g_z) = (p_{22}^m)^m = 1 \). In particular, the latter equality shows that the second condition cannot be satisfied for \( N \), and so the first will hold: \( p_{12}^m p_{21}^n = 1^{1-q} = 1 \), and
\[ N(x, z) = [\ldots [[[x z] p_{12}^m z] p_{22}^m \ldots z] p_{22}^m. \]

The theorem is complete.

We make some useful remarks. First, it is interesting that the theorem just proved is — in form — not sensitive to the characteristic of the ground field. This is associated with the fact that the condition of there being an operation \( [xy^{ml}]_{p_{12}^{ml}} \) is given by the equality \( p_{12}^{ml} p_{21}^{ml} = 1 \), which in the case of characteristic \( l > 0 \) is equivalent to \( p_{12}^{ml} p_{21}^{ml} = 1 \) and ensures the existence of a commutator operation \( [xy]_{p_{22}^m}. \)

If the variables \( x \) and \( y \) are such that \( p_{12} = p_{21} = q^{2d_{aij}} \) and \( p_{22} = q^{ld_{ij}} \), as in the Drinfeld–Jimbo algebra, and \( q \) is not a root of unity, then the second condition of the theorem fails. Therefore, the condition of there being an operation has the form \( q^{id_{aij}} = q^{ld_{ij}(1-n)} \), whence \( n = 1 - a_{ij} \), that is, only Serre \( q \)-operations obtain. If \( q \) is a root of unity, and \( (q^{id_{ij}})^m_i = 1, (q^{id_{aij}})^{s_i} = 1, m_i, s_i \) are minimal, then \( \varepsilon = p_{12}^l = \pm 1, \)

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\[ \delta = p_{12}^{m_i} = \pm 1, \] and the Serre operations are complemented by \[ \ldots [[xy^a] \varepsilon y^a] \varepsilon \ldots y^a] \varepsilon \] and by \[ \ldots [[S(x,y)y^{m_i}] \delta y^{m_i}] \delta \ldots y^{m_i}] \delta. \] Moreover, by Theorem 6.1, an operation of degree \( n \) relative to \( y \) exists only if \( a_{ij} + n - 1 \) is divisible by \( m_i \) or \( n \) is divisible by \( s_i \), that is, there are no other (binary 1-linear) operations by uniqueness.

Furthermore, it is notable that if both conditions are satisfied together, that is, \( p_{12}p_{21} = p_{22}, p_{22} \) is a primitive \( m \)th root of unity, \( n = mq \), the uniqueness yields some identity
\[ [xy^m]_{p_{12}} = \ldots [[xy]_{p_{12}}y]_{p_{12}p_{22}} \ldots y]_{p_{12}p_{22}}^{m-1}. \] (29)

It is also curious that the quantum operations involved in the second case are superpositions of the quantum operations in a lesser degree. Therefore, it seems natural that the specialization of the multilinear main operation \([a, b, \ldots, b]\) should be specified by conditions \( p_{12}p_{21} = p_{22}^{-n} \) and \( p_{22}^{[n]} \neq 0 \). Yet, for the moment we define only a bilinear main operation and observe that braided bigraded Hopf algebras over a field of characteristic 0, for which that operation is defined on the whole quantized space of primitive elements, are universal enveloping algebras of Lie color superalgebras.

**Definition 6.2.** Main bilinear operation:
\[
[a, b] \overset{\text{def}}{=} \begin{cases} 
abla p_{12}ba & \text{if } a \text{ and } b \text{ are skew primitive character elements and } 
p_{12}p_{21} = 1, \text{ where } p_{12} = \chi^a(g_b) \text{ and } p_{21} = \chi^b(g_a); \\
\text{undefined} & \text{otherwise.}
\end{cases}
\]

On braided bigraded Hopf algebras, \([a, b]\) is defined in a similar way.

It is easy to see that the above operation satisfies the identities
\[ [a, b] = -\chi^a(g_b)[b, a], \]
\[ \chi^a(g_c)[a, [b, c]] + \chi^c(g_b)[c, [a, b]] + \chi^b(g_a)[b, [c, a]] = 0, \] (31)
subject to the condition that all values \([ \ ]\) involved in the representation are determined.

Now let \( \mathcal{H} \) be a braided bigraded Hopf algebra and assume that the main operation is defined on all pairs of homogeneous primitive elements. By linearity, then, \([ \ ]\) is uniquely determined on the space \( \Lambda \) of all primitive elements, and the last two formulas show that \((\Lambda, [\ ]\)) is a Lie \((\mathcal{G}, \lambda)-color \) superalgebra, where \( \mathcal{G} \) is a subgroup of \( G^* \times G \), generated by elements of the form \( \chi^a \times g_a \), and the bicharacter is defined by \( \lambda(\chi \times g, \chi' \times g') = \chi(g') \), while the symmetry of that bicharacter is implied by the fact that the main operation is total. A standard and well-known argument will show that in the case of characteristic 0, \( \mathcal{H} \) is itself a universal enveloping algebra of \( \Lambda \); see, e.g., [13].
7. MULTILINEAR OPERATIONS

Fix some set of quantum variables $x_1, \ldots, x_n$ and distinguish one variable $x = x_1$ in it. In what follows, we use the following notation:

$$
    \chi^i = \chi^{x_i}; \quad g_i = g_{x_i}; \quad p_{ij} = \chi^i(g_j); \quad q_k = \prod_{i=1}^{k-1} p_{ik}.
$$

Denote by $S_n$ the permutation group on the set $\{1, 2, \ldots, n\}$, and by $S_n^1$ its subgroup consisting of all permutations leaving the unity fixed. For our goals, both a functional and an exponential notation for the action of $S_n$ on the index set might seem convenient; yet, here we opt for the second, that is, assume that $i^{(\pi\nu)} = (i^{\pi})^{\nu} = \nu(\pi(i))$.

Write $\tau$ to denote the permutation

$$
    \tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ n & n-1 & n-2 & \cdots & 1 \end{pmatrix}.
$$

For brevity, we make the convention to write $\pi(A)$ or $A^\pi$ for the permutation $\pi$ and for an arbitrary expression $A$, meaning that $\pi(A)$ is obtained from $A$ by applying $\pi$ to each index occurring in $A$ at letters $p_{ij}$ or at variables $x_i$ — for instance, $p_{ij}^\pi = p_{\pi(i)\pi(j)}$ or $\pi(q_k) = \prod_{i=1}^{k-1} p_{\pi(i)\pi(k)}$, but not $\pi(q_k) = q_{\pi(k)}$. In so doing, we do not require that $S_n$ act on the ground field. For instance, it might be the case that, in $k$, $p_{12} = p_{23}$ is satisfied but $p_{123}^{(123)} = p_{23}^{(123)}$ is not, that is, this is merely a notational convention, which is used only unless it leads to confusion. An arbitrary multilinear polynomial, in accordance with the above conventions, can be written in the form

$$
    W(x_1, \ldots, x_n) = \sum_{\pi \in S_n} \alpha_{\pi}(x_1 \cdots x_n).
$$

**Definition 7.1.** An element $W$ of the free enveloping algebra $H\langle X \rangle$ of the set $X$ is called left primitive w.r.t. $x \in X$ if

$$
    d^{(x)}_l(\Delta(W) - W \otimes 1) = 0.
$$

Similarly, if we write $H\langle X \rangle_g$ for the linear span of words $hw$ such that $hg_v = g$, then we say that $W \in H\langle X \rangle_g$ is right primitive w.r.t. $x$ if

$$
    d^{(x)}_r(\Delta(W) - g \otimes W) = 0.
$$

**Lemma 7.2.** If a multilinear polynomial $W$ depending on $x$ is left and right primitive w.r.t. $x$, then the $W$ defines a quantum operation.

**Proof.** Since $d^{(x)}_l(g_W \otimes W) = d^{(x)}_l(W \otimes 1) = 0$, we have

$$
    d^{(x)}_+(\Delta(W) - W \otimes 1 - g_W \otimes W) =
$$
\[ d^{(x)}_i(\Delta(W) - W \otimes 1 - g_W \otimes W) + d^{(x)}_r(\Delta(W) - W \otimes 1 - g_W \otimes W) = 0. \]

The lemma is proved.

**Theorem 7.3.** Polynomial (34) is left primitive w.r.t. \( x \) if and only if it has the following representation:

\[ W = \sum_{\nu \in S^1} \beta_{\nu}(\ldots[[x_1x_2]q_2x_3]q_3 \ldots x_n]q_n). \] (35)

**Proof.** First we prove that all summands on the right of (35) are left primitive w.r.t. \( x \). By symmetry, it suffices to consider only the case \( \nu = 1 \). Put \( v_1 = x_1 \), \( v_{k+1} = [v_kx_{k+1}]q_{k+1} \) and use induction to show that all \( v_k \) are left primitive w.r.t. \( x \).

Let \( \Delta(v_k) = v_k \otimes 1 + \sum u_i \otimes d_i \), where the words \( u_i \) are independent of \( x \). Then, if we take into account that \( v_k g_{k+1} = \chi^{v_k}(g_{k+1})g_{k+1}v_k = q_{k+1}g_{k+1}v_k \) and neglect terms whose \( d_i \)-degrees equal 0, we obtain

\[ \Delta(v_{k+1}) = \Delta(v_k)\Delta(x_{k+1}) - q_{k+1}\Delta(x_{k+1})\Delta(v_k) = \]

\[ (v_k \otimes 1 + \sum u_i \otimes d_i)(x_{k+1} \otimes 1 + g_{k+1} \otimes x_{k+1}) - q_{k+1}(x_{k+1} \otimes 1 + g_{k+1} \otimes x_{k+1})(v_k \otimes 1 + \sum u_i \otimes d_i) \equiv \]

\[ v_k x_{k+1} \otimes 1 + v_k g_{k+1} \otimes x_{k+1} - q_{k+1}(x_{k+1}v_k \otimes 1 + g_{k+1} v_k \otimes x_{k+1}) \equiv v_{k+1} \otimes 1. \]

Conversely, assume that \( W \) is left primitive w.r.t. \( x \). Consider the element

\[ W' = \sum_{\nu \in S^1} (-1)^{n-1} \alpha_{\nu}(\prod_{i=2}^n q_i)^{-1}([[x_1x_2]q_2x_3]q_3 \ldots x_n]q_n). \]

In the latter formula, we note, each long skew commutator has exactly one word ending in \( x_1 = x \), and that word equals \( x_\nu(n) \ldots x_\nu(2)x_1 \) and occurs with coefficient \((-1)^{n-1}\nu(\prod_{i=2}^n q_i)\). This means that all words ending in \( x \) have equal coefficients in \( W \) and in \( W' \), that is, the difference \( W - W' \) has no words ending in \( x \). It remains to show that an element with this property, left primitive w.r.t. \( x \) equals zero.

Thus, let \( \alpha_{\pi} = 0 \) for \( \pi(n) = 1 \). Denote by \( w = uvx \) a word which occurs with nonzero coefficient \( \alpha_{\pi} \) in the representation of \( W \) and is such that the subword \( v \), which succeeds \( x \), has the least length possible. Then the word \( ux \) is not a subword of the type \([w' - v]\) of any word \( w' \), except \( w \), occurring in the representation of \( W \) with nonzero coefficient. Therefore, the expansion of \( \Delta(W) \) via (8) will show that \( \Delta(W) - W \otimes 1 \) has a single tensor of the form \( g_{ux} \otimes v \) with nonzero coefficient. Thus, \( W \) cannot be left primitive in the present case. The theorem is proved.

**Theorem 7.4.** Polynomial (34) is left primitive w.r.t. \( x \) if and only if it has the representation

\[ W = \sum_{\nu \in S^1} \beta_{\nu}(\ldots[[x_1x_2]q_2x_3]q_3 \ldots x_n]q_n), \] (36)

where \( q_k = \prod_{i=1}^{k-1} p_{ki}^{-1} \).
The \textbf{proof} follows the same line of argument as in the previous theorem using relations \( g_1 g_2 \cdots g_k x_{k+1} = q_{k+1} x_{k+1} g_1 g_2 \cdots g_k \) and treating words beginning with \( x \).

**THEOREM 7.5.** For quantum variables \( x_1, \ldots, x_n \), there exists a nonzero quantum multilinear operations if and only if \( \prod_{1 \leq i \neq j \leq n} p_{ij} = 1 \), and the polynomials

\[
D_\nu \overset{\text{def}}{=} \nu([\ldots [[x_1 x_2]_{q_2} x_3]_{q_3} \ldots x_n]_{q_n}),
\]

where \( q_k^* = \prod_{i=1}^{k-1} p_{ki}^{-1} \) and \( \nu \) runs through \( S_n^1 \), are linearly dependent in a free associative algebra. In addition, associated to each linear dependence \( \sum \beta_\nu D_\nu = 0 \) is the quantum operation

\[
W(x_1, \ldots, x_n) = \sum_{\nu \in S_n^1} \beta_\nu \nu([\ldots [[x_1 x_2]_{q_2} x_3]_{q_3} \ldots x_n]_{q_n}).
\]

Conversely, every multilinear quantum operation has a presentation by (38), in which the coefficients \( \beta_\nu \) determine the linear dependence of \( D_\nu \).

**Proof.** Put

\[
D_\nu^+ = \nu([\ldots [[x_1 x_2]_{q_2} x_3]_{q_3} \ldots x_n]_{q_n}),
\]

\[
D_\nu^- = \nu([\ldots [[x_1 x_2]_{q_2} x_3]_{q_3} \ldots x_n]_{q_n^*}).
\]

Then \( D_\nu = D_\nu^+ - D_\nu^- \). Therefore, if \( \sum \beta_\nu D_\nu = 0 \), then \( \sum \beta_\nu D_\nu^+ = \sum \beta_\nu D_\nu^- \). By Theorems 7.3 and 7.4, (38) is both left and right primitive w.r.t. \( x \), that is, \( W \) is a quantum operation by Lemma 7.1. Once we have noted that the polynomials \( D_\nu^+ \) are linearly independent in a free algebra, we see that different linear dependences among \( D_\nu \) correspond to different operations.

Conversely, if \( W \) is a quantum operation, we have representation (38) by Theorem 7.3 and have

\[
W = \sum \beta_\nu \nu([\ldots [[x_1 x_2]_{q_2} x_3]_{q_3} \ldots x_n]_{q_n})
\]

by Theorem 7.4. For each permutation \( \nu \in S_n^1 \), among all terms on the right of (38), there is only one containing the word \( x_1 x_{\nu(2)} \cdots x_{\nu(n)} \) — this is \( \beta_\nu D_\nu^+ \). In addition, the coefficient at that word equals \( \beta_\nu \). By a similar argument, the coefficient at the same word in (41) equals \( \beta_\nu \), that is, \( \beta_\nu = \beta_\nu' \), and hence \( \sum \beta_\nu D_\nu = 0 \).

We follow the same line to treat words ending in \( x \). Comparing coefficients at \( x_{\nu(n)} \cdots x_{\nu(2)} x_1 \) on the right of (38) and of (41), we arrive at a system of \( (n - 1)! \) equalities

\[
\beta_\nu (-1)^{n-1} \prod_{k=2}^{n} \left( \prod_{i=1}^{k-1} p_{\nu(i) \nu(k)} \right) = \beta_\nu (-1)^{n-1} \prod_{k=2}^{n} \left( \prod_{i=1}^{k-1} p_{\nu(k) \nu(i)} \right), \quad \nu \in S_n^1.
\]

Clearly, all the equalities are equivalent to one:

\[
\prod_{1 \leq i \neq j \leq n} p_{ij} = 1.
\]
The theorem is proved.

Our further objective is to prove that condition (42) guarantees that $D_\nu$ are linearly dependent, and hence it is a necessary and sufficient condition for a nonzero quantum operation to exist. We set the general solution of this problem aside for a separate Article (see [18]); here, only the cases $n = 3$ and $n = 4$ will be discussed in detail.

8. TRILINEAR AND QUADRILINEAR QUANTUM OPERATIONS

THEOREM 8.1. For quantum variables $x_1$, $x_2$, and $x_3$, a nonzero trilinear quantum operation exists if and only if

$$p_{12}p_{21}p_{13}p_{31}p_{23}p_{32} = 1. \quad (43)$$

If one of the inequalities

$$p_{12}p_{21} \neq 1, \quad p_{13}p_{31} \neq 1, \quad p_{23}p_{32} \neq 1 \quad (44)$$

holds, then there exists exactly one (up to multiplication by a scalar) such operation. If no one of them holds, then all trilinear operations are linearly expressed in terms of $[x_1, [x_2, x_3]]$ and $[x_2, [x_3, x_1]]$ via (30) and (31).

Proof. For $n = 3$, the group $S_3^3$ consists of two elements, $id$ and (23), for which we have

$$D_{id} = (p_{21}^{-1} - p_{12})x_2x_1x_3 + (p_{31}^{-1}p_{32}^{-1} - p_{13}p_{23})x_3x_1x_2,$$

$$D_{(23)} = (p_{21}^{-1}p_{23}^{-1} - p_{12}p_{32})x_2x_1x_3 + (p_{31}^{-1} - p_{13})x_3x_1x_2$$

[see (37)]. If (43) is met and one of the inequalities (44) holds (let it be $p_{13}p_{31} \neq 1$ for definiteness), then

$$(p_{21}^{-1} - p_{12})(p_{31}^{-1} - p_{13}) = (p_{31}^{-1}p_{32}^{-1} - p_{13}p_{23})(p_{21}^{-1}p_{23}^{-1} - p_{12}p_{32}),$$

and hence

$$D_{id} - \frac{p_{31}^{-1}p_{32}^{-1} - p_{13}p_{23}}{p_{31}^{-1} - p_{13}} D_{(23)} = 0.$$ 

Here, $D_{(23)} \neq 0$, that is, the space generated by $D_{id}$ and $D_{(23)}$ is one-dimensional, and by Theorem 7.5, there exists the unique trilinear operation

$$[[x_1x_2]_{p_{12}}x_3]_{p_{13}p_{23}} = \frac{p_{31}^{-1}p_{32}^{-1} - p_{13}p_{23}}{p_{31}^{-1} - p_{13}} [[x_1x_3]_{p_{13}}x_2]_{p_{12}p_{32}}. \quad (45)$$

But, if all products $p_{ij}p_{ji}, \ i \neq j$, are equal to unity, then $D_{id} = D_{(23)} = 0$, that is, there exist exactly two linear dependences between $D_{id}$ and $D_{(23)}$; hence, there are exactly two linearly independent operations. In this case, on the other hand, all the three values, $[x_1, x_2]$, $[x_1, x_3]$, and $[x_2, x_3]$, of the main bilinear operation are defined. Moreover, since $g_{[x_i, x_j]} = g_i g_j$ and $\chi^{[x_i, x_j]} = \chi^i \chi^j$, we see that $\chi^{[x_i, x_j]}(g_k)\chi^k(g_{[x_i, x_j]}) = \chi^{[x_i, x_j]}(g_k)\chi^j$. 

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The theorem is proved.

Note that if exactly one of the inequalities (44) fails, say, \( p_{12}p_{21} = 1 \), then the superposition \([x_3, [x_1, x_2]]\) will be defined; hence, the unique (by Thm. 8.1) quantum operation will equal that superposition. This circumstance allows us to define the main trilinear operation in this way:

**Definition 8.2.** Main trilinear operation:

\[
[a, b, c] \overset{\text{def}}{=} \begin{cases} 
[[ab]_{p_{12}}c]_{p_{13}p_{23}} - \frac{p_{13}^{-1} - p_{21}^{-1}}{p_{21}^{-1} - p_{13}^{-1}}[[ac]_{p_{13}}b]_{p_{12}p_{23}} & \text{if } \prod_{i \neq j} p_{ij} = 1 \text{ and } p_{ij}p_{ji} \neq 1 \\
\text{undefined} & \text{for } i \neq j, \text{ where } a, b, c \\
\text{are character skew primitive elements and } p_{12} = \chi^a(g_0), \ p_{13} = \chi^a(g_c), \text{ etc.; otherwise.}
\end{cases}
\]

On braided bigraded Hopf algebras, the main trilinear operation is defined in exactly the same way.

The operation being unique has an implication that if we rename the variables \( x_i \to x_{\pi(i)} \), then the value of the main operation on \( x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)} \) (of course, it is defined on that sequence since (43) is invariant under such substitutions) should be linearly expressed via its value on \( x_1, x_2, x_3 \), that is,

\[
[x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}] = \alpha_\pi [x_1, x_2, x_3].
\]

If we compare the coefficients at \( x_{\pi(1)}x_{\pi(2)}x_{\pi(3)} \) on the right- and left-hand sides of (46) we see that \( \alpha_\pi = \gamma_\pi^{-1} = \gamma_\pi^{-1} \) where \( \gamma_\pi \) are precisely coefficients in the expansion

\[
[x_1, x_2, x_3] = \sum \gamma_\pi x_{\pi(1)}x_{\pi(2)}x_{\pi(3)};
\]

or, again, by routine computations,

\[
\alpha_{id} = 1, \ \alpha_{(123)} = \frac{p_{31} - p_{13}^{-1}}{p_{21}^{-1} - p_{21}^{-1}}, \ \alpha_{(132)} = \frac{p_{31} - p_{13}^{-1}}{p_{23} - p_{32}^{-1}}, \ \alpha_{(13)} = p_{21}p_{32}p_{31},
\]

\[
\alpha_{(12)} = p_{21}p_{32}p_{31} \frac{p_{31} - p_{13}^{-1}}{p_{23} - p_{32}^{-1}}, \ \alpha_{(23)} = p_{12}p_{32}p_{13} \frac{p_{31} - p_{13}^{-1}}{p_{12} - p_{21}^{-1}}.
\]

We pass to the case \( n = 4 \). To the conventions and notation fixed at the beginning of Sec. 7, we add the following:

\[
\{p_{ij}p_{kl} \cdots p_{rs}\} \overset{\text{def}}{=} p_{ij}p_{kl} \cdots p_{rs} - p_{ij}^{-1}p_{ik}^{-1} \cdots p_{sr}^{-1},
\]

and for the word \( A \) depending on \( p_{ij} \), denote by \( \overline{A} \) a word obtained from \( A \) by replacing all letters \( p_{ij} \) with \( p_{ji} \). These are again merely notational conventions since we by no means mean that the equality of words in \( k \) has any bearing on those operators.
LEMMA 8.3. Let $C$, $D$, and $E$ be some words in $p_{ij}$. Then

$$\{CE\}{DE} - \{C\}{D} = \{CD\}{E}. \tag{48}$$

**Proof.** Using (47), we rewrite the left- and right-hand sides of (48) in this way:

$$\{CE\}{DE} - \{C\}{D} = (CE - CE^{-1})(DE - (DE)^{-1} - (C - C^{-1})(D - D^{-1}) = CDE - C^{-1}E^{-1}DE - CED^{-1}E^{-1} + C^{-1}E^{-1}D^{-1}E^{-1} - CD + C^{-1}D + CD^{-1} - C^{-1}D^{-1} = CED + CDE^{-1}E^{-1} - CD - C^{-1}D^{-1},$$

$$\{CDE\}{E} = (CDE - CDE^{-1}E^{-1})(E - E^{-1}) = CDEE - CDE^{-1}E^{-1} - CD + CDE^{-1}E^{-1}.$$  

The lemma is proved.

Note that (43) can be written via braces thus: $\{p_{12}p_{13}p_{23}\} = 0$. Therefore, it might be useful to point out the following trivial properties of the braces:

$$\{C\} = 0 \rightarrow \{CD\} = C\{D\}, \tag{49}$$

$$\{C\} = 0 \& \{CD\} = 0 \rightarrow \{D\} = 0. \tag{50}$$

THEOREM 8.4. For quantum variables $x_1, x_2, x_3, x_4$, a nonzero quadrilinear quantum operation exists if and only if

$$p_{12}p_{21}p_{13}p_{31}p_{14}p_{41}p_{23}p_{32}p_{24}p_{42}p_{34}p_{43} = 1. \tag{51}$$

If this equality holds, and there is a pair of indices $i, j$ such that

$$\Gamma_{4}^{(ij)} \Leftrightarrow \{p_{ij}\} \neq 0 \& \{p_{ij}p_{ik}p_{kj}\} \neq 0 \& \{p_{ij}p_{ik}p_{kj}\} \neq 0, \tag{52}$$

where $i, j, k, s$ are distinct indices, then there exist exactly two linearly independent quadrilinear operations.

If condition $\Gamma_{4}^{(ij)}$ fails for all $i \neq j$, then all quadrilinear operations are expressed via the main operation of ranks 2 and 3.

**Proof.** We seek an element $D_{id}$ in an explicit form. Expanding the skew commutators in (37) yields

$$D_{id} = -\{p_{12}\}x_2x_1x_3x_4 - \{p_{13}p_{23}\}x_3x_1x_2x_4 - \{p_{14}p_{24}p_{34}\}x_4x_1x_2x_3 + \{p_{12}p_{13}p_{23}\}x_3x_2x_1x_4 + \{p_{12}p_{14}p_{24}p_{34}\}x_4x_2x_1x_3 + \{p_{13}p_{23}p_{14}p_{24}p_{34}\}x_4x_3x_1x_2. \tag{53}$$

Now assume that $\beta_{\nu}$ are unknown parameters. Consider the linear combination $\sum \beta_{\nu}(D_{id})$ and the coefficients at its distinct words. Setting that combination equal to zero, we obtain a homogeneous system of twelve equations (equal to the number
of distinct words not beginning with and not ending in $x_1$) with six unknowns. We show that, under conditions (51) and $\Gamma_4^{(1)}$, that system has exactly two linearly independent solutions.

Consider the coefficient at $x_2x_1x_3x_4$. If we apply $\nu \in S_n^1$, the element $x_1$ will be left fixed; therefore, the word $x_2x_1x_3x_4$ arises in $\nu(D_{id})$ only from the first three summands of (53). If it arises from the second, then $\nu(3) = 2$, $\nu(2) = 3$, and $\nu(4) = 4$, that is, $\nu = (23)$. If it arises from the third, then $\nu(4) = 2$, $\nu(2) = 3$, and $\nu(3) = 4$, that is, $\nu = (234)$. Therefore, the whole coefficient at $x_2x_1x_3x_4$ is equal to

$$-\{p_{12}\} \beta_{id} - \{p_{12}p_{32}\} \beta_{(23)} - \{p_{12}p_{32}p_{42}\} \beta_{(234)}.$$

In a similar way, if we compute coefficients at other six words $\nu(x_2x_1x_3x_4)$, $\nu \in S_n^1$, with $x_1$ holding second place, we obtain the first group of six equations

$$[-\{p_{12}\} \beta_{id} - \{p_{12}p_{32}\} \beta_{(23)} - \{p_{12}p_{32}p_{42}\} \beta_{(234)}]^\mu = 0, \mu \in S_4^1. \quad (54)$$

At this point we use the conventions made at the beginning of Sec. 7, assuming in addition that permutations $\mu$ act on the indices at $\beta$ by right multiplications: $[\ldots \beta_{\nu} \ldots]^\mu = \ldots \beta_{\nu^\mu} \ldots$.

In exactly the same way we consider the coefficient at $x_4x_3x_1x_2$. This word arises in $\nu(D_{id})$ from the last three terms only. If it arises from the last but one term, then $\nu(4) = 4$, $\nu(2) = 3$, and $\nu(3) = 2$, that is, $\nu = (23)$. And, if it arises from the fourth, then $\nu(3) = 4$, $\nu(2) = 3$, and $\nu(4) = 2$, that is, $\nu = (234)$. Therefore, the coefficient is equal to

$$\{p_{13}p_{23}p_{14}p_{24}p_{34}\} \beta_{id} + \{p_{13}p_{14}p_{34}p_{24}\} \beta_{(23)} + \{p_{13}p_{14}p_{34}\} \beta_{(234)}.$$

and we obtain yet other six equations

$$\{p_{13}p_{23}p_{14}p_{24}p_{34}\} \beta_{id} + \{p_{13}p_{14}p_{34}p_{24}\} \beta_{(23)} + \{p_{13}p_{14}p_{34}\} \beta_{(234)}] = 0. \quad (55)$$

Now compare two equations that correspond to one permutation $\mu$ in (54) and (55). Using Lemma 8.3 to compute all the three minors in that system of two equations, we make it sure that, under condition (51), they all are equal to zero, for example,

$$\{p_{12}\} \{p_{13}p_{14}p_{34}\} - \{p_{12}p_{32}p_{42}\} \{p_{13}p_{23}p_{14}p_{24}p_{34}\} = \{p_{12}p_{13}p_{14}p_{34}p_{23}p_{24}\} \{p_{32}p_{42}\} = 0.$$

Besides, if some coefficient in (54) equals zero, then by (50), the corresponding coefficient in (55), too, will be equal to zero, that is, the whole system of twelve equations is equivalent to the six in (54).

We order elements of the group $S_4^1$ in this way: $id$, (23), (234), (34), (24), (243).
The matrix of the system then has the form

\[
\begin{pmatrix}
\{p_{12}\} & \{p_{12}p_{32}\} & \{p_{12}p_{32}p_{42}\} & 0 & 0 & 0 \\
\{p_{13}p_{23}\} & \{p_{13}\} & 0 & \{p_{13}p_{23}p_{43}\} & 0 & 0 \\
0 & 0 & \{p_{13}\} & 0 & \{p_{13}p_{43}\} & \{p_{13}p_{43}p_{23}\} \\
0 & 0 & 0 & \{p_{12}\} & \{p_{12}p_{42}p_{32}\} & \{p_{12}p_{42}\} \\
0 & \{p_{14}p_{34}p_{24}\} & \{p_{14}p_{34}\} & 0 & \{p_{14}\} & 0 \\
\{p_{14}p_{24}p_{34}\} & 0 & 0 & \{p_{14}p_{24}\} & 0 & \{p_{14}\}
\end{pmatrix}
\]

If, in this matrix, we delete the first two columns and the third and fourth rows, we obtain a triangular submatrix, the leading diagonal of which has elements \(\{p_{12}p_{32}p_{42}\}, \{p_{13}p_{23}p_{43}\}, \{p_{14}\}\). That is, by condition \(\Gamma^{(14)}\) and remark (50), the corresponding minor is distinct from zero and the whole system has not more than two linearly independent solutions. Put \(\beta_{id} = 1\) and \(\beta_{(23)} = 0\), and find one solution for the system of the first two and last two equations:

\[
\beta_{id} = 1, \beta_{(23)} = 0, \beta_{(234)} = \frac{\{p_{12}\}}{\{p_{12}p_{32}p_{42}\}}, \beta_{(34)} = \frac{\{p_{13}p_{23}\}}{\{p_{13}p_{23}p_{43}\}},
\]

\[
\beta_{(24)} = \frac{\{p_{12}\}\{p_{14}p_{34}\}}{\{p_{14}\}\{p_{12}p_{32}p_{42}\}}, \beta_{(243)} = \frac{\{p_{13}p_{24}p_{34}p_{13}p_{23}\}}{\{p_{14}\}\{p_{13}p_{23}p_{43}\}}.
\]

Using Lemma 8.3, we verify whether these values are solutions for the third equation:

\[
-\{p_{13}\} \frac{\{p_{12}\}}{\{p_{12}p_{32}p_{42}\}} + \{p_{13}p_{43}\} \frac{\{p_{12}\}\{p_{14}p_{34}\}}{\{p_{14}\}\{p_{12}p_{32}p_{42}\}} - \frac{\{p_{13}p_{23}p_{43}\}\{p_{43}\}\{p_{14}p_{24}p_{34}p_{13}p_{23}\}}{\{p_{14}\}\{p_{12}p_{32}p_{42}\}\{p_{14}\}} = 0.
\]

Likewise for the fourth equation (with the “−” sign):

\[
\{p_{12}\} \frac{\{p_{13}p_{23}\}}{\{p_{13}p_{23}p_{43}\}} - \{p_{12}p_{32}p_{42}\} \frac{\{p_{12}\}\{p_{14}p_{34}\}}{\{p_{12}p_{32}p_{42}\}\{p_{14}\}} +
\]

26
\[
\{p_{12}p_{42}\} \frac{\{p_{43}\}\{p_{14}p_{24}p_{34}p_{13}p_{23}\}}{\{p_{14}\}\{p_{13}p_{23}p_{43}\}} = \frac{\{p_{12}\} \{p_{14}\}\{p_{13}p_{23}\}\{p_{14}\}}{\{p_{14}\}\{p_{13}p_{23}p_{43}\}} - \\
\{p_{13}p_{23}p_{43}\}\{p_{14}p_{34}\}\} + \ldots = -\frac{\{p_{12}\} \{p_{13}p_{23}p_{14}p_{34}\}\{p_{43}\}}{\{p_{14}\}\{p_{13}p_{23}p_{43}\}} + \\
\frac{\{p_{12}p_{42}\}\{p_{43}\}\{p_{14}p_{24}p_{34}p_{13}p_{23}\}}{\{p_{14}\}\{p_{13}p_{23}p_{43}\}} = \frac{\{p_{43}\} \{p_{14}\}\{p_{13}p_{23}p_{14}p_{34}\}\{p_{42}\}}{\{p_{14}\}\{p_{13}p_{23}p_{43}\}} = 0.
\]

Thus, by Theorem 7.5, the computed values of \(\beta_\nu\) determine the quadrilinear operation
\[
[x_1, x_2, x_3, x_4] = \sum \beta_\nu D_\nu^+. \tag{57}
\]

Since \(\beta_{id} = 1, \beta_{(23)} = 0\), and the word \(\nu(x_1x_2x_3x_4)\) in (57) occurs only in the summand \(D_\nu^+\), we see that the coefficient at \(x_1x_2x_3x_4\) in the expansion (34) of the polynomial \([x_1, x_2, x_3, x_4]\) equals 1, and the coefficient at \(x_1x_3x_2x_4\) is zero.

Consider a sequence of quantum variables \(y_1 = x_1, y_2 = x_3, y_3 = x_3, y_4 = x_4\). This sequence satisfies both conditions (51) and \(\Gamma_{4}^{(14)}\); hence, by the above, there exists a quantum operation \([y_1, y_2, y_3, y_4] = [x_1, x_3, x_3, x_4]\) such that the coefficient at \(x_1x_3x_2x_4\) equals 1 and the one at \(x_1x_2x_3x_4\) equals 0. In this way \([x_1, x_3, x_2, x_4]\) supplies the second solution for the system under consideration, which proves the first part of the theorem.

Now assume that condition \(\Gamma_{4}^{(ij)}\) is not satisfied for any pair \(i \neq j\). We call the set of quantum variables \(Y\) conforming if condition (42) is satisfied for it. The failure of condition \(\Gamma_{4}^{(ij)}\) will mean, then, that the variables \(x_i\) and \(x_j\) enter some two- or three-element conforming subset. If the pair \(x_i, x_j\) is itself conforming, then the value \([x_i, x_j]\) is defined, and the set \([x_i, x_j], x_k, x_l\) too is conforming. Therefore, one of the superpositions \([[x_i, x_j], x_k, x_l]]\) or \([[x_i, x_j], x_k], x_l]]\) is determined. Similarly, if the triple \(x_i, x_k, x_j\) is conforming, then either \([[x_i, x_j], x_k], x_l]]\) or \([[x_i, x_j], x_k, x_l]]\) is defined.

We turn on to consider the possible cases where the six conditions \(\Gamma_{4}^{(ij)}\) are all fallible.

1. All two-element subsets are conforming. The system (54) has only zero coefficients, and by Theorem 7.5, we then find six linearly independent quantum operations \(D_\nu^+, \nu \in S_{4}^1\),
\[
D_\nu^+ = \nu([[x_1, x_2], x_3], x_4]]).
\]

2. All four three-element subsets are conforming. In view of the above, we can assume that one of the two-element subsets is not conforming. Suppose \(\{p_{12}\} \neq 0\). Then the system (54) splits into three pairs of equations: \(id, (23); (243), (24); (34), (243)\). Here, the first and third pairs have rank 1 and the second has rank \(\leq 1\). Thus, if at least one of the inequalities \(\{p_{14}\} \neq 0, \{p_{13}\} \neq 0, \text{or} \{p_{43}\} \neq 0\) holds, then the
whole system has exactly three solutions, and these, in accordance with Theorem 7.5, yield the following three operations:

\[ [[[x_1, x_2, x_3]', x_4]]; [[[x_1, x_2, x_4]', x_2]]; [[[x_1, x_3, x_4]', x_3]]. \]

Here, \( [\cdot'] \) denotes the ternary operation whose uniqueness is asserted by Theorem 8.1, that is, it is either the main operation or a superposition of the form \( [[\cdot, \cdot], \cdot] \). There then exists one more superposition \( [[[x_2, x_3, x_4], x_1]] \), which should be linearly expressed in terms of the solutions that we have found. Consequently, using the fact that the bilinear (30) and trilinear (46) operations are symmetric, we arrive at an analog of the Jacobi identity

\[
\sum_{k=0}^{3} \xi_k \sigma^k ([[x_1, x_2, x_3], x_4]) = 0,
\]

(58)

where \( \sigma = (1234) \) is a cyclic permutation, the coefficients \( \xi_k \) are uniquely determined up to multiplication by a common scalar, and all values of the main operation are assumed determined.

If \( \{p_{14}\} = \{p_{13}\} = \{p_{43}\} = 0 \), then the second pair of equations disappears, and instead of \( [[[x_1, x_2, x_4]', x_2]] \), there appear two operations: \( D_{(234)}^+ = [[[x_1, x_3] x_4, x_2]] \) and \( D_{(24)}^+ = [[[x_1, x_4] x_3, x_2]] \).

3. Three three-element subsets are conforming. Condition (51) then implies that the fourth subset is also conforming.

4. Exactly two three-element subsets are conforming. To be specific, let \( \{p_{12}p_{14}p_{24}\} = \{p_{13}p_{14}p_{34}\} = 0 \), \( \{p_{12}p_{23}p_{13}\} \neq 0 \), \( \{p_{23}p_{24}p_{34}\} \neq 0 \). Since condition \( \Gamma_{(23)}^+ \) fails and the two triples involved are not conforming, we have \( \{p_{23}\} = 0 \). If we write (51) in the form \( \{p_{12}p_{14}p_{24}p_{13}p_{23}p_{43}\} = 0 \), by formulas (49) and (50), we obtain \( 0 = \{p_{13}p_{23}p_{43}\} = p_{23}\{p_{13}p_{43}\} \), and similarly \( 0 = \{p_{12}p_{32}p_{42}\} = p_{32}\{p_{12}p_{42}\} \). In other words, \( \{p_{13}p_{43}\} = \{p_{12}p_{42}\} = \{p_{23}\} = 0 \), and again condition (51) yields \( \{p_{14}\} = 0 \). In the matrix of (54), in particular, the last two columns will disappear, and the minor corresponding to the first four rows and columns will equal \( \{p_{12}p_{23}p_{13}\}\{p_{13}\}\{p_{12}\} \).

Now if \( \{p_{13}\} \{p_{12}\} \neq 0 \), then the whole system has rank 4 and its solutions are determined by arbitrary values of \( \beta_{(24)} \) and \( \beta_{(243)} \), that is, we obtain two operations, \( D_{(24)}^+ \) and \( D_{(243)}^+ \):

\[
[[[x_1, x_4] x_3, x_2]]; [[[x_1, x_4] x_2, x_3]],
\]

(59)
in terms of which all other operations defined in the present case are expressible:

\[
[[[x_1, x_2, x_4], x_3]]; [[[x_1, x_3, x_4], x_2]]; [[[x_1, x_3, x_4], x_2]],
\]

\[
[[[x_1, x_2, x_3], x_4]]; [[[x_1, x_2, x_3], x_4]].
\]

If \( \{p_{13}\} \{p_{12}\} = 0 \), in view of the initial conditions being symmetric under the permutation \( 2 \leftrightarrow 3 \), it suffices to consider the case \( \{p_{13}\} = 0 \). We have \( \{p_{12}p_{24}\} = \{p_{13}\} = \{p_{14}\} = \{p_{23}\} = \{p_{34}\} = 0 \), \( \{p_{12}\} \neq 0 \), \( \{p_{24}\} \neq 0 \) (if not all pairs are
conforming. And we face only one additional solution $D^+_{(234)} = [[[x_1, x_3]x_4], [x_2]]$ since the minor corresponding to the first, fourth, and sixth rows and to the first, second, and fourth columns is not equal to zero.

5. Only one three-element subset is conforming. Let it be $x_2, x_3, x_4$. Then the failure of conditions $\Gamma^{(12)}, \Gamma^{(13)},$ and $\Gamma^{(14)}$ implies that $\{p_{12}\} = \{p_{13}\} = \{p_{14}\} = 0$. In this case $\{p_{34}\} \neq 0$, since otherwise the triple $x_1, x_3, x_4$ would be conforming. Similarly, $\{p_{24}\} \neq 0$ and $\{p_{23}\} \neq 0$. These imply $\{p_{23}p_{24}\} \neq 0$, $\{p_{23}p_{34}\} \neq 0$, and $\{p_{24}p_{34}\} \neq 0$ since, for instance, condition $\{p_{23}p_{24}\} = 0$, combined with $\{p_{11}\} = 0$, $i = 2, 3, 4,$ and (51), yields $\{p_{34}\} = 0$. Under these conditions, the system splits into three pairs of rank 1 equations: $(23), (243); id, (24); (234), (34)$. The first pair agrees with the operation

$$D^+_{id} = \frac{\{p_{23}p_{43}\}}{\{p_{23}\}} D^+_{(34)} = [[[x_1, x_2], x_3, x_4]],$$

and the other two operations result from substitutions $(23)$ and $(24)$. All other superpositions defined in the present case are linearly expressed via these three. Specifically, we have an identity of the form

$$[x_1, [x_2, x_3, x_4]] = \xi_1 \left([[x_1, x_2], x_3, x_4] + \xi_2 [x_2, [x_1, x_3], x_4] + \xi_3 [x_2, x_3, [x_1, x_4]]\right].$$

6. No one of the three-element subsets is conforming. Then two-element subsets cannot all be conforming; therefore, one of the conditions $\Gamma^{(ij)}_4$ is satisfied.

The theorem is proved.

**Definition 8.5.** Under condition $\Gamma^{(14)}_4$, the main quadrilinear operation is defined by

$$[a_1, a_2, a_3, a_4] = \sum_{\nu(1)=1} \beta_\nu \left([[a_1a_{\nu(2)}], [a_{\nu(3)}a_{\nu(4)}]], [a_{\nu(3)}a_{\nu(4)}], [a_{\nu(3)}a_{\nu(4)}]\right),$$

where $a_i$ are skew primitive character elements, $p_{ij} = \chi^a(g_{a_i})$, $\nu(q_i) = \prod_{k=1}^{i-1} p_{\nu(k)}(q_i)$, and the coefficients $\beta_\nu$ are given as in (56).

If no proper subset of the set $x_1, x_2, x_3, x_4$ is conforming, then all conditions $\Gamma^{(ij)}_4$ are satisfied. Therefore, all possible $4! = 24$ permutation variants $[x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}]$, $\pi \in S_4$, are determined, and by Theorem 8.4, they all are expressible via any pair of them. In order to find that representation, we write the main operation in the form

$$[x_1, x_2, x_3, x_4] = \sum \alpha_\pi x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}x_{\pi(4)}, \quad (60)$$

where $\alpha_\pi$ are particular rational functions in $p_{ij}$, obtained by expanding the skew commutators in Definition 8.5. We have already mentioned that $\alpha_{id} = 1$ and $\alpha_{(23)} = 0$. Given an arbitrary replacement $x_i \rightarrow x_{\mu(i)}$, $\mu \in S_4$, we obtain

$$[x_{\mu(1)}, x_{\mu(2)}, x_{\mu(3)}, x_{\mu(4)}] = \sum (\alpha_\pi)^\mu x_{\mu(\pi(1))}x_{\mu(\pi(2))}x_{\mu(\pi(3))}x_{\mu(\pi(4))}.$$
On the right-hand side of the latter equality, the coefficient at $x_1x_2x_3x_4$ equals $\alpha_{\mu-1}^\mu$ and the one at $x_1x_3x_2x_4$ equals $\alpha_{(23)\mu-1}^\mu$. Therefore, we have a formula that replaces the twisted symmetry in (46):

$$
\begin{bmatrix}
x_{\mu(1)}, x_{\mu(2)}, x_{\mu(3)}, x_{\mu(4)}
\end{bmatrix}
= \alpha_{\mu-1}^\mu \begin{bmatrix}
x_1, x_2, x_3, x_4
\end{bmatrix}
+ \alpha_{(23)\mu-1}^\mu \begin{bmatrix}
x_1, x_3, x_2, x_4
\end{bmatrix}.
$$

(61)

Clearly, the same trick will help us find an expression for any quadrilinear operation in terms of the main operation since coefficients in the expansion are equal to those at $x_1x_2x_3x_4$ and $x_1x_3x_2x_4$. For the Pareigis operation $P_4$, for instance, we have

$$
P_4 = \begin{bmatrix}
x_1, x_2, x_3, x_4
\end{bmatrix}
+ \zeta^{-1}p_{23}\begin{bmatrix}
x_1, x_3, x_2, x_4
\end{bmatrix}.
$$

(62)

By definition, then, $p_{ij}p_{ji} = -1$; therefore, $\{A\} = 2A$ for words of odd length and $\{A\} = 0$ for words of even length. That is, the main operation has the following representation:

$$
\begin{bmatrix}
x_1, x_2, x_3, x_4
\end{bmatrix}
= D_{id}^+ - p_{23}p_{24}D_{(234)}^+ - p_{24}p_{34}D_{(243)}^+.
$$

In conclusion we note that any (not multilinear) operation admits a full and partial linearizations. On identifying variables in the linearized operation, we obtain the initial operation multiplied by an integer dividing $n!$. Therefore, if the ground field has characteristic zero, all operations will be expressed via multilinear ones. If the characteristic of $k$ is distinct from 2, 3, then all operations of degree $\leq 4$ are expressed via the main operation of variable arity, defined in the article. The picture changes if the characteristic equals 2 or 3. Assume, for instance, that it equals 2. Then $\begin{bmatrix}x, y, y\end{bmatrix} = 0$ if the left-hand side is determined, and by Theorem 6.1, there still exists a nonzero operation $xy^2 + (p_{12} + p_{12}p_{22})yxy + y^2x$.

Acknowledgement. I am extremely indebted to the participants of Shirshov Seminar on Ring Theory of the Institute of Mathematics RAS held in July, 1997, particularly L.A. Bokut’, I. P. Shestakov, V. T. Filippov, A. N. Koryukin, V. N. Zhelyabin, K. N. Ponomarev, V. N. Gerasimov, and O. N. Smirnov, for giving careful considerations to my results notwithstanding the vacation time. Thanks also are due to Drs. Jaime Torres Keller and Suemi Rodríguez-Romo for the beautiful facilities for my research work in the Center of Theoretical Research (FES-C UNAM) and to Prof. Zbigniew Oziewicz for interesting comments on the subject matter.
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