Constraints on warped compactifications

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Abstract

We discuss a possible compactification of a higher-dimensional gravitational theory, which give rise to a de Sitter or an accelerating universe, to the extent that it can describe a low energy limit of string theory. Such analysis can be carried out by the usual integration over the internal space in the Einstein equations. The combined field equations in the higher-dimensional supergravity theory after integration give several new terms which may help to realize a de Sitter compactification. After developing the general framework, we describe some specific examples involving the dilaton and mass parameter.

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I. INTRODUCTION

This paper is devoted to investigate the realizability of non-singular de Sitter compactifications for higher-dimensional gravitational theories, in particular in string theory. Realization of cosmic acceleration is one of the requirements from observations [1–5] (see also the review [6] and references therein). At present, although we have not successfully explained it yet from the point of view of string theory, it would be desirable in future.

In that context, cosmological model is obtained by mainly two approaches. One is an extension to a classical solution to the Einstein equation that was introduced in [7, 8]. Another is the construction of a lower-dimensional effective theory after integrating over the volume of internal space [9–13]. The latter has a lot of works with respect to realizing a de Sitter universe. In the former case for type II supergravity and eleven-dimensional supergravity theory, there is the well-known obstacle about the NO-GO theorem to obtain de Sitter compactifications [14]. The NO-GO theorem forbids a de Sitter solution in any higher dimensional theory which does not contain higher curvature corrections and a nonpositive potential, involves massless degrees of freedom with positive kinetic terms and induces a finite effective gravitational constant. Then an explicit proof has been made under the assumptions that the warp factor is static, and that the internal space is smooth and compact. There are several examples e.g., in Ref. [15–17], where time dependence (without the warp factor) allows for accelerating universes. de Sitter compactification solutions also have been obtained by violating at least one of the above assumptions, for example, adding higher curvature corrections to the gravity action (see e.g., [18–20]) and introducing orbifold models with fluxes (see e.g., [14, 21–26]). However, it has not been explicitly clarified that de Sitter compactification solutions can be obtained by allowing the time dependence of the warp factor and without changing any other above assumption. The main purpose of this work is to see whether and how in this new class of compactifications the warped spacetime structure can realize de Sitter compactifications.

On the other hand, in the recent years, new development has been made in finding time dependent generalizations of \( p \)-brane solutions which are originally found by [7, 8]. Numerous number of time dependent solutions of D-branes and M-branes like \( p \)-brane in any dimension have been obtained, and their applications to cosmology have been widely studied in [27–29] (useful references are [30, 31], and physical explanations with some relevances to the
present paper are [27–29, 32–38]). A time dependent $p$-brane is explicitly calculable for any given link to a corresponding static solution. Indeed it was originally found with an explicit algebraic recipe (see [7, 8] for an accessible account). Unfortunately, no $p$-brane solution which leads to a de Sitter or an accelerating expansion of our universe has been found.

Let us review more on the properties of time dependent $p$-solutions and then deduce what should be revealed in this paper. In the case of the compactification in $p$-brane system discussed in [29], the problem is that of a decelerating expansion of the universe. When such a solution is found, the warp factor of the familiar static solutions is replaced by a function depending not only on the coordinates of the transverse space but on the linear function of time, studied in [27, 28]. This might suggest generalizing our discussion to the case of intersecting branes as done in [28, 38]. Hence there is a serious difficulty in obtaining an accelerating expansion from these solutions because of a choice of ansatz of fields with a given dilaton coupling parameter. Though these results are really natural in the viewpoint of the extension of the static solution, it prevents us from obtaining an accelerating expansion unless we consider additional matter. The previous approaches have involved the exact time dependent solutions with respect to the BPS $D$-brane solutions. For this purpose, we decide to investigate the realizability of a de Sitter or an accelerating expansion through a more general warped compactification. Thus, we will repeat the same argument of NO-GO theorem for warped compactification on the compact internal space, but allowing time dependence in the warp factor. This is important to study how to choose the ansatz of fields and make use of the assumptions to get realistic cosmological warped compactifications.

There are two issues to discuss cosmic acceleration in a warped compactification. As we mentioned above, one is NO-GO theorem of warped compactification which is analyzed after integrating over the internal space, which does not make the exact solution manifest. The other is an approach which makes contact with exact solutions of higher-dimensional field equations. The description in this paper contains both aspects: We discuss the possibility of a four-dimensional accelerating universe from loopholes of NO-GO theorem. We also obtain the exact solutions which describe de Sitter spacetime in warped compactifications. It would be highly desirable to discuss the de Sitter compactification from these two types of our calculation here because the condition of accelerating expansion as well as its computability are manifest.

We discuss the criterion for the possible de Sitter compactifications. We start with a
compactification because of a single form field strength, and then perform an integration over the compact manifold so that it remains convergent. This gives the minimal framework to analyze the possibilities of de Sitter compactification. An extension of our analyses to a more general theory including a scalar field and multi-form field strengths as in supergravity is straightforward. We will see that a warped compactification leads to a new possibility of an accelerating expansion.

The organization of this paper is as follows. In section III, we consider the time dependent, warped compactification that the internal space is smooth and compact. The Einstein equations are used to derive some necessary condition that de Sitter compactifications can be present, for examples which were considered in [14] where the no-go theorem of warped compactifications have been proposed. We also look for the de Sitter compactification with not only field strengths but also the scalar field. We will focus on the particular examples of supergravity theories, the ten-dimensional type IIA theory as a example from string theory. In the case of IIA supergravity, the Einstein equations imply that de Sitter compactifications may arise under certain conditions for the field strengths if the internal space is compact. Again, the results from static compactifications can be compared to results obtained in [14]. There seems to be some room where the no-go theorem may be broken by a sort of time dependence from the warp factor. The last section III is devoted to summary and discussion.

II. COMPACTIFICATIONS

A. Models with a single form field

We consider a gravitational theory with the metric $g_{MN}$ and an anti-symmetric tensor field strength of rank $n$. The action we consider is given by

$$S = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} \left( R - \frac{1}{2 \cdot n!} F^2 \right),$$

where $\kappa^2$ is the $D$-dimensional gravitational constant, $g$ is the determinant of the $D$-dimensional metric $g_{MN}$, and $F$ is an $n$-form field strength. After variations with respect to the metric, the $D$-dimensional Einstein equations are given by

$$R_{MN} = T_{MN} - \frac{1}{D-2} g_{MN} T,$$

$$= \frac{1}{2 \cdot n!} \left( n F_{ML2\cdots L_n} F_{NL2\cdots L_n} - \frac{n-1}{D-2} g_{MN} F^2 \right),$$
where $T$ is the trace of energy momentum tensor $T_{MN}$ which is defined by

$$T_{MN} = \frac{1}{2 \cdot n!} \left( n F_{ML \cdots L_n} F_{N}^{L_2 \cdots L_n} - \frac{1}{2} g_{MN} F^2 \right).$$  \hfill (3)

Now we assume the $D$-dimensional metric in the form

$$ds^2 = A^2(x, y) \left[ q_{\mu\nu}(X) dx^\mu dx^\nu + u_{ij}(Y) dy^i dy^j \right],$$  \hfill (4)

where $q_{\mu\nu}(X)$ is a $d$-dimensional Einstein space metric which depends only on the $d$-dimensional coordinates $x^\mu$, and $u_{ij}(Y)$ is the $(D - d)$-dimensional metric which depends only on the $(D - d)$-dimensional coordinates $y^i$. Under the assumption Eq. (4), the $(\mu, \nu)$ component of Ricci tensor in Eq. (2) gives

$$R_{\mu\nu} = R_{\mu\nu}(X) - (D - 2) (D_\mu D_\nu \ln A - \partial_\mu \ln A \partial_\nu \ln A) - q_{\mu\nu} \left[ \triangle_X \ln A + \triangle_Y \ln A + (D - 2) (q^{\rho\sigma} \partial_\rho \ln A \partial_\sigma \ln A + u^{ij} \partial_i \ln A \partial_j \ln A) \right],$$  \hfill (5)

where $R_{\mu\nu}(X)$ is the Ricci tensor of the metric $q_{\mu\nu}(X)$, and $\triangle_X$ and $\triangle_Y$ are the Laplace operators on the space of X and Y, respectively. The trace of the $(\mu, \nu)$ components of Einstein equations reduces to

$$\frac{(D + d - 2)}{\alpha} A^{-\alpha} \triangle_X A^\alpha + \frac{d}{(D - 2) A^{D-2}} \triangle_Y A^{D-2} = R(X) + A^2 \bar{T},$$  \hfill (6)

where $\alpha$ and $\bar{T}$ are defined by

$$\alpha = \frac{(D - 2)(d - 1)}{D - d},$$  \hfill (7)

$$\bar{T} = -T^\mu_{\mu} + \frac{d}{D - 2} T,$$  \hfill (8)

respectively. Substituting Eq. (3) into Eq. (8), we obtain

$$\bar{T} = \frac{1}{2 \cdot (n - 1)!} \left[ -F_{\mu M_1 \cdots M_{n-1}} F^{\mu M_1 \cdots M_{n-1}} + \frac{d(n - 1)}{(D - 2) n} F^2 \right].$$  \hfill (9)

The Eq. (9) implies $\bar{T} \geq 0$ if the field strength $F$ has only components along the $(D - d)$-dimensional space. In the case of the part of the field strength with components along the $d$-dimensional space, the trace over the index $\mu$ can be written by a particular order of contractions of the indices

$$F_{\mu M_1 \cdots M_{n-1}} F^{\mu M_1 \cdots M_{n-1}} = \frac{d}{n} F^2.$$  \hfill (10)
Substituting the Eq. (10) into Eq. (9), we get

$$\bar{T} = -\frac{d(D - n - 1)}{2(D - 2) \cdot n!} F^2 \geq 0,$$

(11)

where $F^2 \leq 0$ since the indices of the field strength are along the X space. Let us assume that the internal manifold should be compact. If the function $A$ depends only on the coordinate $y$, after multiplying Eq. (6) by $A^{D-2}$ and then integrating over the Y space, the $y$-derivative term in the left-hand side does not contribute since it becomes a total derivative term. Thus, we obtain the consistency condition

$$0 = \frac{d}{(D-2)} \int_Y d^{D-d}y \sqrt{u} \triangle_y A^{D-2} = \int_Y d^{D-d}y \sqrt{u} A^{D-2} \left[ R(X) + A^2 \bar{T} \right].$$

(12)

Nothing that $\bar{T} \geq 0$ for an $n$-form field strength, we obtain $R(X) \leq 0$. Therefore, de Sitter compactification can not be allowed [14].

We next consider the function $A$ which depends on the coordinate $x$ as well as on $y$. Then, we obtain

$$\frac{(D + d - 2)}{\alpha} \int_Y d^{D-d}y \sqrt{u} \triangle_x A^\alpha = \int_Y d^{D-d}y \sqrt{u} A^{D-2} \left[ R(X) + A^2 \bar{T} \right],$$

(13)

and thus, we can not abandon the possibility $R(X) \geq 0$ if the function $A$ satisfies

$$\frac{(D + d - 2)}{\alpha} A^{-\alpha + (D-2)} \triangle_x A^\alpha > 0.$$  

(14)

The Eq. (14) implies that there are de Sitter compactifications even if there is no higher derivative terms and is no scalar field which depends on time.

In principle we could have functions of scalar fields multiplying these expressions, as we have in some supergravity theories, and we could also have many types of $n$-form fields, as we will discuss in the next subsection. In fact, the action (1) is a straightforward generalization of the case of eleven-dimensional supergravity or ten-dimensional supergravity without dilaton coupling.

If there is a hypersurface where $A = 0$, a curvature singularity appears there. Even in such a case, we can always consider the region $A > \epsilon$ where the singularities are left out for small $\epsilon$, and then apply the same arguments for the integral (see [14] for details).

Let us go back to the Einstein equations. ($\mu, i$) and ($i, j$) components of Einstein equations...
are given by
\[ (D - 2) A \partial_\mu \partial_i A^{-1} = \frac{1}{2 \cdot (n - 1)!} F_{\mu A \cdots B} F_i^{A \cdots B}, \]  
(15a)
\[ (D - d) [A^{-1} \triangle_X A + (D - 3) q^\rho_\sigma \partial_\rho \ln A \partial_\sigma \ln A] + (2D - d - 2) A^{-1} \triangle_Y A 
+ [(D - d)(D - 3) - 2(D - 2)] u^{ij} \partial_i \ln A \partial_j \ln A = R(Y) + A^2 \tilde{T}, \]  
(15b)
where \( \tilde{T} \) is defined by
\[ \tilde{T} \equiv -T^i_i + \frac{(D - d)}{D - 2} T. \]  
(16)
Eq. (16) gives \( \tilde{T} < 0 \) if the field strength \( F \) has only components along the \( (D - d) \)-dimensional space. In the case of the part of the field strength with components along the \( d \)-dimensional space, we get again \( \tilde{T} < 0 \) due to Eq. (10).

Using Eqs. (6) and (15b), we get
\[ (D - 1) [(D - d - 2) \triangle_X \ln A - (D - 2) q^\rho_\sigma \partial_\rho \ln A \partial_\sigma \ln A - d \triangle_Y \ln A] 
= (D - d - 1) R(X) - dR(Y) + A^2 \left[ (D - d - 1) \tilde{T} - d\tilde{T} \right]. \]  
(17)
The second term of the left hand side of Eq. (17), i.e., the kinetic term of \( A \), contributes positively to it. Therefore, the de Sitter compactification may be allowed for the case of \( \triangle_X \ln A > 0 \) and \( R(Y) > 0 \) because of \( (D - d - 1) \tilde{T} - d\tilde{T} \geq 0 \) for \( D > d + 1 \).

Before closing this subsection, some remarks are in order. In getting the four-dimensional cosmology from Eq. (4), there would be two approaches. The first one is integrating over the \( Y \) space, giving rise to the four-dimensional effective theory with the de Sitter universe. The second one is assuming that we are living at some particular place of the internal space, like a braneworld picture. Then, the effective four dimensional metric \( A^2(x) q_{\mu \nu}(X) dx^\mu dx^\nu \) may be no longer de Sitter universe, even being decelerating universe.

Our previous argument is applied not only to the cases that the \( d \)-dimensional spacetime \( X \) is the Einstein space, but also to those of more general \( d \)-dimensional metric. For instance, let us consider the case that the \( d \)-dimensional metric is given by the Robertson-Walker form
\[ ds^2(X) = -dt^2 + t^{2\lambda} \delta_{ab} dx^a dx^b, \]  
(18)
where \( \lambda \) is the parameter, and \( \delta_{ab} \) denotes the \( (d - 1) \)-dimensional Euclidean space. The Ricci scalar \( R(X) \) is expressed as
\[ R(X) = d(d - 1) \lambda \left( \lambda - \frac{2}{d} \right) t^{-2}. \]  
(19)
Then, the Ricci scalar of the X space becomes $R(X) > 0$ only if the parameter $\lambda$ satisfies $\lambda > 2/d$. Hence, for $2/d < \lambda \leq 1$, the $d$-dimensional spacetime is not accelerating expansion while the Ricci scalar of the X space is positive. For $d = 4$, a positive $R(X)$ corresponds to an universe expanding faster than the radiation dominated universe with $\lambda = \frac{1}{2}$. The above thing will also be true also for the discussions in the rest of this paper, although we mainly focus on the case that X is an Einstein space.

B. Models with a scalar field and multi-form fields

We now discuss the de Sitter compactification in the theory with the multi-form field strengths as well as the scalar field, which is more relevant for the supergravity theories.

We consider a gravitational theory with the metric $g_{MN}$, dilaton $\phi$, the cosmological constant $\Lambda$, and anti-symmetric tensor fields of rank $n_I$. The action in the Einstein frame is given by

$$S = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} \left( R - 2e^{\phi} \Lambda - \frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - \sum_I \frac{1}{2 \cdot n_I!} e^{c_I \phi} F_{n_I}^2 \right),$$

where $\kappa^2$ is the $D$-dimensional gravitational constant, $g$ is the determinant of the $D$-dimensional metric $g_{MN}$, $F_{n_I}$ is $n_I$-form field strength, and $c_I$, $\chi$ are constants. The field equations are given by

$$R_{MN} = \frac{2}{D-2} e^{\phi} \Lambda g_{MN} + \frac{1}{2} \partial_M \phi \partial_N \phi + \sum_I \frac{1}{2 \cdot n_I!} e^{c_I \phi} \left( n F_{MA_2 \ldots A_{n_I}} F_{N A_2 \ldots A_{n_I}} - \frac{n-1}{D-2} g_{MN} F_{n_I}^2 \right),$$

$$\Delta \phi = \sum_I \frac{c_I}{2 \cdot n_I!} e^{c_I \phi} F_{n_I}^2 + 2\chi e^{\phi} \Lambda,$$

where $\Delta$ is the $D$-dimensional Laplace operator.

We assume the form of the $D$-dimensional metric as

$$ds^2 = A^2(x,y)q_{\mu\nu}(X)dx^\mu dx^\nu + B^2(x,y)u_{ij}(Y)dy^i dy^j,$$

where $q_{\mu\nu}(X)$ is a $d$-dimensional Einstein space metric which depends only on the $d$-dimensional coordinates $x^\mu$, and $u_{ij}(Y)$ is the $(D-d)$-dimensional metric which depends only on the $(D-d)$-dimensional coordinates $y^i$. With the same procedure as section IIA,
the scalar field equation and the trace of the \((\mu, \nu)\) components of the Einstein equation are given by

\[
A^{-d} B^{-(D-d)} D_\mu \left(A^{d-2} B^{D-d} \eta^{\mu\nu} \partial_{\nu} \phi \right) + A^{-d} B^{-(D-d)} D_i \left(A^d B^{D-d-2} u^{ij} \partial_j \phi \right)
= 2 \chi \epsilon^{ij} \Lambda + \frac{1}{2} \sum_l \frac{c_I}{n_I} \epsilon^{ij} \left[ \theta(n_I - d) F_{n_I \text{ex}}^2 + F_{n_I \text{in}}^2 \right], \tag{23a}
\]

\[
2(d - 1) A^{-1-d/2} D_\mu \left(A^{-2+d/2} \eta^{\mu\nu} \partial_{\nu} A \right) + (D - d) A^{-d} B^{-(d-1)} D_\mu \left(A^{d-2} B^{-1} \eta^{\mu\nu} \partial_{\nu} B \right)
+ d A^{-d} B^{-(D-d)} D_i \left(A^{d-1} B^{D-d-2} u^{ij} \partial_j A \right) = A^{-2} R(X) - \frac{2\Lambda d}{D-2} \epsilon^{ij} - \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi
- \frac{d}{2(D-2)} \sum_l \frac{1}{n_I} \epsilon^{ij} \left[(D - n_I - 1) \theta(n_I - d) F_{n_I \text{ex}}^2 - (n_I - 1) F_{n_I \text{in}}^2 \right], \tag{23b}
\]

where \(\theta(a)\) is defined by

\[
\theta(a) = \begin{cases} 
1 & \text{if } a \geq 0 \\
0 & \text{if } a < 0.
\end{cases} \tag{24}
\]

Here \(F_{n_I \text{ex}}^2\) is the square of a tensor with components purely in the internal dimensions and \(F_{n_I \text{in}}^2\) is that with components along the \(d\)-dimensional spacetime directions. \(D_\mu, D_i\) are the covariant derivatives with respect to the metric \(q_{\mu\nu}, u_{ij}\), respectively. After some algebra, these two equations can be combined into

\[
dD_\mu \left(A^{d-2} B^{D-d} \eta^{\mu\nu} \partial_{\nu} \phi \right) + dD_i \left(A^d B^{D-d-2} u^{ij} \partial_j \phi \right) + d(D - 2) D_\mu \left(A^{d-1} B^{D-d-2} u^{ij} \partial_j A \right)
+ 2(d - 1)(D - 2) A^{-1+d/2} B^{D-d} D_\mu \left(A^{-2+d/2} \eta^{\mu\nu} \partial_{\nu} A \right)
+ (D - d)(D - 2) B^{D-d-1} D_\mu \left(A^{d-2} B^{-1} \eta^{\mu\nu} \partial_{\nu} B \right)
= (D - 2) A^{d-2} B^{D-d} R(X) + 2\Lambda d(\chi - 1) \epsilon^{ij} - \frac{1}{2} (D - 2) A^d B^{D-d} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi
+ \frac{d}{2} A^d B^{D-d} \sum_l \frac{c_I}{n_I} \left[-(D - n_I - 1 - c_I) \theta(n_I - d) F_{n_I \text{ex}}^2 + (c_I + n_I - 1) F_{n_I \text{in}}^2 \right]. \tag{25}
\]

If the functions \(A\) and \(B\) depend only on \(y^i\), the right hand side of Eq. \((25)\) becomes positive because the square of a tensor with components in the internal dimensions is positive and the square of the tensor with components along the \(d\)-dimensional spacetime directions is negative. If we have a compact internal manifold, the left hand side in Eq. \((25)\) is zero since we have a total derivative after we integrate Eq. \((25)\) over the manifold. On the other hand, the right hand side in Eq. \((25)\) is non-zero unless \(R(X) = 0\) and \(F_{n_I} = 0\). Hence, for a compact internal manifold, there are no nonsingular de Sitter compactifications. However,
if $\Lambda(\chi - 1) < 0$ it may be possible to find de Sitter compactification even if the functions $A$ and $B$ depend only on $y^i$.

Furthermore, if the functions $A$ and $B$ depend on both $x^\mu$ and $y^i$, the left hand side in Eq. (25) is not zero due to the contribution of the terms $\partial_\mu A$ and $\partial_\mu B$. Then, we can find the nonsingular de Sitter compactifications.

C. de Sitter compactifications in the type IIA theory

In this subsection, we discuss the possible de Sitter compactifications of the ten-dimensional massive IIA supergravity. The action for the massive IIA supergravity in the Einstein frame can be written as [43]

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left( R - \frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - \frac{1}{2 \cdot 2!} e^{3\phi/2} F^2 - \frac{1}{2 \cdot 3!} e^{-\phi} H^2 
- \frac{1}{2 \cdot 4!} e^{3\phi/2} G^2 - \frac{1}{2} e^{5\phi/2} m^2 \right) + S_{CS},$$

(26)

where $S_{CS}$ denotes the action for Chern-Simons term, $R$ is the ten-dimensional Ricci scalar with respect to the ten-dimensional metric $g_{MN}$, $\kappa^2$ is the ten-dimensional gravitational constant, $m$ is constant, $g$ is the determinant of the ten-dimensional metric $g_{MN}$, and $F$, $H$, $G$ are 2-form, 3-form, 4-form field strengths, respectively. The expectation values of fermionic fields are assumed to be zero.

The ten-dimensional action (26) gives following field equations:

$$R_{MN} = \frac{1}{2} \partial_M \phi \partial_N \phi + \frac{1}{16} m^2 e^{5\phi/2} g_{MN} + \frac{1}{2 \cdot 2!} e^{3\phi/2} \left( 2 F_{MA} F_N^A - \frac{1}{8} g_{MN} F^2 \right)
+ \frac{1}{2 \cdot 3!} e^{-\phi} \left( 3 H_{MAB} H_{N}^{AB} - \frac{1}{4} g_{MN} H^2 \right)
+ \frac{1}{2 \cdot 4!} e^{\phi/2} \left( 4 G_{MABC} G_N^{ABC} - \frac{3}{8} g_{MN} G^2 \right),$$

(27a)

$$\Delta \phi = \frac{5}{4} e^{5\phi/2} m^2 + \frac{3}{4 \cdot 2!} e^{3\phi/2} F^2 - \frac{1}{2 \cdot 3!} e^{-\phi} H^2 + \frac{1}{4 \cdot 4!} e^{\phi/2} G^2,$$

(27b)

where $\Delta$ is the Laplace operator with respect to the metric $g_{MN}$.

Now the ten-dimensional metric is assumed to be Eq. (4). Under the same procedure as
in sec. [11] the trace of the $(\mu, \nu)$ components of the Einstein equations is given by
\[
\frac{(d + 8)}{\alpha} A^{-\alpha-2} \Delta_X A^\alpha + \frac{d}{8} A^{-10} \Delta_Y A^8 = A^{-2} R(X) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \\
+ \frac{d}{16} \left( \frac{1}{2} e^{3\phi/2} F_{in}^2 + \frac{2}{3!} e^{-\phi/2} H_{in}^2 + \frac{3}{4!} e^{\phi/2} G_{in}^2 \right) - \frac{d}{16} \partial^\phi/2 m^2 \\
- \frac{d}{16} \left[ \frac{7}{2} \theta(2 - d) e^{3\phi/2} F_{ex}^2 + \frac{6}{3!} \theta(3 - d) e^{-\phi} H_{ex}^2 + \frac{5}{4!} \theta(4 - d) e^{\phi/2} G_{ex}^2 \right], \quad (28)
\]
where $\Delta_X$ and $\Delta_Y$ are the Laplace operators on the space of $X$ and the space $Y$, and $R_{\mu\nu}(X)$ is the Ricci tensor of the metrics $q_{\mu\nu}$, and the function $\theta(a)$ is defined by Eq. (24), and $\alpha$ is defined by Eq. (8), and $F_{in}, H_{in}, G_{in}$ are the squares of a tensors with components purely in the internal dimensions and $F_{ex}, H_{ex}, G_{ex}$ are the squares of the tensors with components along the $d$-dimensional spacetime directions. The components $F_{ex}, H_{ex}, G_{ex}$ can only appear if the rank of the tensor is bigger or equal to $d$.

Substituting Eq. (11) into the equation of motion for the scalar field Eq. (27b), we find
\[
A^{-10} \left[ D_\mu \left( A^8 q^{\mu\nu} \partial_\nu \phi \right) + D_i \left( A^8 u^{ij} \partial_j \phi \right) \right] = \frac{5}{4} e^{\phi/2} m^2 + \frac{1}{2} \left( \frac{3}{2 \cdot 2!} e^{3\phi/2} F^2 - \frac{3}{3!} e^{-\phi} H^2 + \frac{1}{2 \cdot 4!} e^{\phi/2} G^2 \right), \quad (29)
\]
where $D_\mu, D_i$ are the covariant derivatives with respective to the metric $q_{\mu\nu}, u_{ij}$. In terms of Eq. (28) and Eq. (29), we get
\[
\frac{10(d + 8)}{\alpha} A^{-\alpha+8} \Delta_X A^\alpha + \frac{5d}{4} \Delta_Y A^8 + \frac{d}{2} D_\mu \left( A^8 q^{\mu\nu} \partial_\nu \phi \right) + \frac{d}{2} D_i \left( A^8 u^{ij} \partial_j \phi \right) \\
= 10 A^8 R(X) - 5 A^{10} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + d A^{10} \left( \frac{1}{2!} e^{3\phi/2} F_{in}^2 + \frac{1}{3!} e^{-\phi} H_{in}^2 + \frac{2}{4!} e^{\phi/2} G_{in}^2 \right) \\
- d A^{10} \left[ \frac{4}{2!} \theta(2 - d) e^{3\phi/2} F_{ex}^2 + \frac{4}{3!} \theta(3 - d) e^{-\phi} H_{ex}^2 + \frac{3}{4!} \theta(4 - d) e^{\phi/2} G_{ex}^2 \right]. \quad (30)
\]
If the function $A$ depends only on $y^i$, the right hand side of Eq. (30) is positive since the square of a tensor with components in the internal dimensions is positive and the square of the tensor with components along the $d$-dimensional spacetime directions is negative. For a compact internal manifold, the left hand side in Eq. (30) becomes zero since we have a total derivative after we integrate Eq. (30) over the internal space. On the other hand, the right hand side in Eq. (30) is non-zero unless $R(X) = 0$ and $F = H = G = 0$. Hence, there are no nonsingular de Sitter compactifications. However, if the function $A$ depends not only on $y^i$ but on $x^\mu$, the left hand side in Eq. (30) is not zero due to the contribution of $A^{-\alpha+8} \Delta_X A^\alpha$. Then, it is possible to find the de Sitter compactifications.
If we consider the regions where the function $A$ vanishes the curvature singularity will again emerge. Then we cannot analyze the construction of compactifications any further. If we consider again the region where the curvature singularities can be left out, we can find the de Sitter compactification by the same steps as in the section II.

III. DISCUSSIONS

In this work, we proposed a relatively simple and explicit class of de Sitter compactification in gravity theory coupled to form fields, including the supergravity theories. We showed how a few ingredients suffice to produce several new terms which exhibit de Sitter compactifications. In sections II we have noted that the condition of the de Sitter compactification can be interpreted formally as the condition that the warp factor determines a compactification. In the warped compactification, this interpretation is somewhat formal because the contribution of the warp factor can strictly determine the lower-dimensional geometry. Especially to have terms from the warp factor leads to the possibility of the de Sitter compactifications.

The time dependence of the warp factor may also support to obtain more general accelerating universes since our analysis can be applied to the cases that the four-dimensional geometry is not an Einstein space, as mentioned in Sec. II-A. It is also worth mentioned that the time dependence of the warp factor plays important roles in other dynamical systems, for example, it gives the dynamical black hole solutions in the asymptotically de Sitter spacetime in the Einstein-Maxwell theory [38, 45].

Clearly, an important direction for further work is fleshing out further the methods in section II for finding the cosmological model from the exact solution of Einstein equations. A convenient feature of the background is its simple ansatz for matter fields and metric in higher-dimensional spacetime, which make a controlled analysis of the warp factor and the internal space possible. In the analysis of the four-dimensional effective theory, the curvature of the internal and KK 5-brane configuration facilitates the de Sitter compactification by introducing useful competing forces [46].

One of the consideration is that the metric flux and KK 5-branes yield the de Sitter spacetime in the four-dimensional effective theory [46, 48]. Since these matter fields couple to the moduli of the internal space, the effects of fields modify the potential to realize the
four-dimensional de Sitter spacetime. It would be interesting to apply our construction to the problems of explicitly modeling inflation in string theory. One question is whether the time dependence of the warp factor could help to tune the inflationary scenario and accelerating expansion of our Universe. It might also be interesting to introduce higher derivative corrections or source term corresponding to the field strengths to these models, perhaps using an orientifold plane within the bulwark of D-branes and NS-branes to form brane constructions of the relevant field theories. Some rudimentary model-building observations based on this mechanism for obtaining the de Sitter compactification will appear elsewhere.

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