An exact master equation for the system-reservoir dynamics under the strong coupling regime and non-Markovian dynamics

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In this paper we present a method to derive an exact master equation for a bosonic system coupled to a set of other bosonic systems, which plays the role of the reservoir, under the strong coupling regime, i.e., without resorting to either the rotating-wave or secular approximations. Working with phase-space distribution functions, we verify that the dynamics are separated in the evolution of its center, which follows classical mechanics, and its shape, which becomes distorted. This is the generalization of a result by Glauber, who stated that coherent states remain coherent under certain circumstances, specifically when the rotating-wave approximation and a zero-temperature reservoir are used. We show that the counter-rotating terms generate fluctuations that distort the vacuum state, much the same as thermal fluctuations. Finally, we discuss conditions for non-Markovian dynamics.

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I. INTRODUCTION

The subject of open quantum systems has undergone substantial growth in the last three decades, starting with contributions to the field of fundamental quantum physics with the aim of understanding the process of decoherence. Based on the von Neumann approach to the reduction of the state vector\textsuperscript{[1]}, these contributions were mainly driven by the pioneering work of Zurek\textsuperscript{[2]}, Caldeira and Leggett\textsuperscript{[3]}, and Joos and Zeh\textsuperscript{[4]}. The repercussions of their work, together with the advent of the field of quantum information theory, led to
renewed interest in open quantum systems, the focus now shifting from fundamental issues to practical applications in circuits to implement quantum logic operations.

The master equation approach has long been used to derive system-reservoir dynamics, to account for energy loss under a weak coupling regime. Its effectiveness comes from the fact that the energy loss of most quantum mechanical systems, especially within quantum and atomic optics, can be handled by the single-pole Wigner-Weisskopf approximation, where a perturbative expansion is performed in the system-reservoir coupling. Following developments by Caldeira and Leggett, more sophisticated methods to deal with the system-reservoir strong coupling regime have been advanced, such as the Hu-Paz-Zhang master equation, with time-dependent coefficients, which allows for non-Markovian dynamics. Halliwell and Yu have published an alternative derivation of the Hu-Paz-Zhang equation, in which the dynamics is represented by the Wigner function, and an exact solution of this equation was given by Ford and O’Connell.

Recently, the non-Markovian dynamics of open quantum systems has been studied with renewed interest, especially in connection with quantum information theory, as in Refs. [10, 11]. However, in these studies, as well as in most of the derivations of master equations with time-dependent coefficients, the authors assume either the rotating-wave approximation (RWA) or the secular approximation (SA) for the system-reservoir coupling. Since non-Markovian behavior is sensitive to the counter-rotating terms in the interaction Hamiltonian, important features of the dynamics are missing under the RWA in the strong-coupling regime. It is worth mentioning that a study of the effect of the RWA and the SA on the non-Markovian behavior in the spin-boson model at zero temperature has already been advanced, without, however, deriving a master equation.

Our goal in this work is to derive and investigate the consequences of a master equation within the strong-coupling regime, which prevents us resorting to either the RWA or the SA in the system-reservoir coupling. Moreover, instead of the path integrals approach, we use the formalism of quasi-probability distributions, thus enabling us to cast the problem as the solution of a linear system of equations. Our results follow from the general treatment of a bosonic dissipative network we have previously presented in Ref. [14], where the network dynamics were investigated, and further used for quantum information purposes. However, differently from our previous developments, we first consider the general model for a network of bosonic non-dissipative oscillators and, subsequently, we focus on some of
these oscillators (or in just one of them) as our system of interest, and treat all the others as a (structured) reservoir. The exact dynamics of the network allows us to obtain an exact dynamics of the system-reservoir interaction. Moreover, we present a simple inequality to distinguish between Markovian and non-Markovian dynamics.

Finally, this development enables us to generalize an earlier result by Glauber [16]. When using the RWA and a zero-temperature reservoir, it was shown that the quasi-probability functions maintain their shape while they are displaced in phase space; in particular, coherent states remain coherent states. We find that, for a general Gaussian state, the center of its phase space distribution follows classical dynamics (as in Ref. [16]), but its shape is changed. Furthermore, this change can be derived from the evolution of the vacuum state, which is no longer stationary, because of the counter-rotating terms. The change in shape is affected by both quantum and thermal fluctuations, and these contributions can be distinguished, at least in theory. Our developments can be straightforwardly translated to the derivation of an exact master equation for fermionic systems, using the reasoning in Ref. [17].

II. UNITARY DYNAMICS OF THE UNIVERSE

The universe considered here consists of a set of \( M + N \) harmonic oscillators, which are linearly coupled to each other in an arbitrary network. We consider \( M \) of them to be part of our system of interest, and the remaining \( N \) to be part of a reservoir. However, at this stage, we are concerned with the full dynamics of the universe, and there is actually no difference between system and reservoir modes. The oscillators are described by mass \( m_k \) and natural, isolated frequencies \( \omega_k \); the coupling between modes \( k \) and \( j \), which occurs via their position coordinates, has strength \( \lambda_{kj} \) (which, without loss of generality, is symmetric in its indices). Before we write the Hamiltonian that describes such a universe, we note that it must be positive-definite, in order to be bounded from below and have a well-defined ground state. Then, the Hamiltonian which is compatible with this model is

\[
H = \frac{1}{2} \sum_{k=1}^{M+N} \left( \frac{1}{m_k} \dot{p}_k^2 + m_k \omega_k^2 \dot{q}_k^2 \right) + \frac{1}{4} \sum_{kj=1}^{M+N} \lambda_{kj} (\dot{q}_k - \dot{q}_j)^2 ,
\]

where the coefficients \( \lambda_{kj} \) form a real, symmetric matrix. We do not assume any particular form for them, so as to generate an arbitrary network, as depicted in Fig. II

The coupling
term induces a change in the natural frequency of each mode, that is now represented by

$$\omega_k = \sqrt{\omega_k^2 + \frac{1}{m_k} \sum_{j=1}^{N} \lambda_{kj}}.$$  

(2)

Using this renormalized frequency, we can define annihilation operators $a_k$ and rewrite the Hamiltonian as

$$H = \sum_{k=1}^{M+N} \omega_k a_k^\dagger a_k + \frac{1}{2} \sum_{kj=1}^{M+N} g_{kj} \left( a_k + a_k^\dagger \right) \left( a_j + a_j^\dagger \right),$$  

(3)

the coupling in this picture being given by

$$g_{kj} = \frac{\lambda_{kj}}{2\sqrt{m_k m_j \omega_k \omega_j}}.$$  

(4)

From here on, we will focus on $\omega_k$ and $g_{kj}$, the latter forming a real, symmetric matrix.

A. Characteristic function

The dynamics given by the Hamiltonian of Eq. (3) is best understood in terms of the characteristic function of a state, which is just the expected value of the multimode dis-
placement operator in the symmetric ordering,

$$\chi(\{\beta_k\}) = \left\langle \prod_{k=1}^{M+N} \exp\left(\beta_k a_k^\dagger - \beta_k^* a_k\right) \right\rangle,$$

(5)

where \(\{\beta_k\}\) represents all coordinates \(\beta_k\) with \(k = 1, \ldots, N\), as well as their complex conjugates.

The characteristic function carries the complete information about the state, and in particular information about moments of all orders; this is one of the reasons it is a better approach than using the Heisenberg equations of motion directly. The von Neumann equation in Hilbert space is mapped to a differential equation in dual phase space (where the characteristic function is defined):

$$\frac{\partial \chi}{\partial t} = i \sum_{k=1}^{M+N} \left(\omega_k \beta_k - \sum_{j=1}^N g_{kj} (\beta_j + \beta_j^*)\right) \frac{\partial \chi}{\partial \beta_k} + \text{H.c.}$$

(6)

Being linear and of first order, this equation admits a simple ansatz,

$$\chi(\{\beta_k\}, t) = \chi(\{\beta_k(t)\}, 0),$$

(7)

which implies that the characteristic function maintains its shape, but the underlying (dual) phase space undergoes a linear transformation, given by

$$\beta_k(t) = \sum_{j=1}^{M+N} \left(U_{j,k}(t) \beta_j - V_{j,k}(t) \beta_j^* \right).$$

(8)

This transformation is defined by the solution to a system of differential equations,

$$\frac{dU_{kj}}{dt} = i\omega_j U_{kj} - i \sum_{n=1}^{M+N} (U_{k,n} - V_{k,n}) g_{n,j},$$

(9a)

$$\frac{dV_{kj}}{dt} = -i\omega_j V_{kj} - i \sum_{n=1}^{M+N} (U_{k,n} - V_{k,n}) g_{n,j}.$$  

(9b)

The Heisenberg equations of motion for the first moments have a similar structure. However, since they refer only to first moments, they do not represent a complete solution of the problem, which can be obtained from the characteristic function with the same computational effort.
III. REDUCED DYNAMICS OF THE SYSTEM

From this point on, we shall be interested only in the behavior of a subset of \( M \) oscillators (the ones labeled 1 to \( M \)), which form our system of interest, while the oscillators labeled \( M + 1 \) to \( M + N \) play the role of a (structured) reservoir. The complete solution to the dynamics is given by Eq. (7); in order to eliminate the reservoir degrees of freedom, all we need to do is set \( \beta_k = 0 \) if \( k > M \) (i.e., evaluate the characteristic function at the origin of the phase space of the modes we want to eliminate from the description). Before continuing, we observe that although not strictly necessary in our method, for the sake of simplicity we assume the usual sudden-coupling hypothesis, i.e., that the states of system and reservoir are initially uncorrelated:

\[
\chi_{SR} (\{ \beta_k \}, 0) = \chi_S (\{ \beta_k \}_{k \leq M}, 0) \chi_R (\{ \beta_m \}_{m > M}).
\]  

(10)

Tracing out the reservoir degrees of freedom, following the procedure above, leads to

\[
\chi_S (\{ \beta_k \}, t) = \chi_S (\{ \beta_k (t) \}, 0) \chi_{in} (\{ \beta_k \}, t),
\]  

(11)

where the indices run only through the degrees of freedom of the system (i.e., \( k \) runs from 1 to \( M \)). Therefore, we must use Eq. (8) with \( \beta_k = 0 \) for \( k > M \), and it follows that we only need \( U_{kj} \) and \( V_{kj} \) for \( k \leq M \). Eqs. (9a,9b), although written as a matrix equation, are actually a set of \( N \) independent vector equations and we conclude that only a few of these need to be solved. In fact, if our system of interest were a single oscillator, we would reduce the problem of finding its exact dynamics to a single vector equation of dimension \( 2N \).

The two terms of Eq. (11) are called the homogeneous (because it depends on the initial state of the system) and inhomogeneous terms (because it is independent of it, depending only on the initial state of the reservoir). The homogeneous part of the solution is just the linear transformation of phase space induced only by the elements \( U_{kj} \) and \( V_{kj} \) for which both \( k, j \leq M \). These elements can be arranged in two general complex \( M \times M \) matrices, resulting in \( 4M^2 \) real parameters.

At this point, we make an additional assumption that the initial state of the reservoir is Gaussian [18], i.e., its characteristic function has the Gaussian form. Moreover, the reservoir is unbiased (i.e., \( \langle a_m \rangle = 0 \) for \( m > M \)). These are reasonable hypotheses, since the Gaussian states include the thermal states of quadratic Hamiltonians. The inhomogeneous
characteristic function is then also a Gaussian function:

\[
\chi_{\text{in}}(\{\beta_k\}, t) = \exp \left( -\frac{1}{2} \sum_{k_j=1}^{M} A_{kj} (t) \beta_k \beta_j^* \right) 
\times \exp \left( \sum_{k_j=1}^{M} B_{kj} (t) \beta_k \beta_j + \text{c.c.} \right). \tag{12}
\]

The time-dependent functions \(A_{kj}\) and \(B_{kj}\) may be divided into two terms, in the form \(A_{kj} = A_{kj}^{(0)} + A_{kj}^{(th)}\) (and similarly for \(B\)), the first of which is the solution for a zero-temperature reservoir,

\[
A_{kj}^{(0)} = \frac{1}{2} \sum_{m=M+1}^{M+N} \left( U_{km} U_{jm}^* + V_{km} V_{jm}^* \right) \tag{13a}
\]

\[
B_{kj}^{(0)} = \frac{1}{2} \sum_{m=M+1}^{M+N} \left( U_{km} V_{jm} + V_{km} U_{jm}^* \right), \tag{13b}
\]

while the second incorporates the effects of the reservoir initial state, which is completely characterized by the second-order moments \(\langle a_m^\dagger a_n \rangle_0\) and \(\langle a_m a_n \rangle_0\),

\[
A_{kj}^{(th)} = \sum_{m=M+1}^{M+N} \langle a_m^\dagger a_n \rangle_0 \left( U_{km} U_{jn}^* + V_{km} V_{jn}^* \right) + \sum_{m=M+1}^{M+N} \left( \langle a_m a_n \rangle_0 V_{km} U_{jn}^* + \text{c.c.} \right) \tag{14a}
\]

\[
B_{kj}^{(th)} = \sum_{m=M+1}^{M+N} \langle a_m^\dagger a_n \rangle_0 \left( U_{kn} V_{jm} + V_{kn} U_{jm}^* \right) + \sum_{m=M+1}^{M+N} \left( \langle a_m a_n \rangle_0 V_{km} V_{jn} + \text{c.c.} \right). \tag{14b}
\]

Both \(A\) and \(B\) form complex \(M \times M\) matrices; however, \(A\) must be Hermitian, while \(B\) is not. This represents an additional \(3M^2\) real parameters, giving a total of \(7M^2\) that completely specifies a given Gaussian evolution map (so called because, if the initial state of the system is Gaussian, it will remain Gaussian).

The functions \(A_{kj}^{(0)}\) and \(B_{kj}^{(0)}\) represent the solution for a zero-temperature reservoir; therefore, they represent the quantum, or zero-point fluctuations. The functions \(A_{kj}^{(th)}\) and \(B_{kj}^{(th)}\) represent the thermal fluctuations (when the reservoir is assumed to be in a thermal state), and other effects that may arise due to, e.g., squeezing in the reservoir modes.
IV. SINGLE-MODE DYNAMICS

The above result may be written in a simpler fashion for the case of a single oscillator taken as the system of interest:

\[ \chi(\beta, t) = \chi(U\beta - V\beta^*, 0) \]

\[ \times \exp \left( -A|\beta|^2 + \frac{1}{2}B\beta^2 + \frac{1}{2}B^*\beta^{*2} \right) , \]

where the indices 1, 1 are dropped. The single-mode Gaussian map is completely characterized by 7 real parameters (since \(A\) is real, and \(U, V\) and \(B\) are complex).

When a single mode is considered as the system of interest, we can perform a diagonalization of the reservoir part of the Hamiltonian, and consider the interaction of the system with each of the reservoir normal modes, as depicted in Fig. 2 (normal modes of the reservoir do not interact with each other, but interact with the system).

In order to get physical results in the limit \(N \to \infty\), it is essential to keep track of the oscillator masses \(m_k\) in Eq. (11). Essentially, the central oscillator must be much more massive than the reservoir modes. This is the case with Brownian motion, where the
observed particle, though mesoscopic, is still much larger than the bath of fluid molecules it interacts with. It is also the case in Quantum Optics, where the mode inside a cavity has a much smaller mode volume (i.e., it is concentrated in a small region) than the vacuum modes outside the cavity. We shall consider then that the central oscillator has mass $M$ and the reservoir modes have mass $\mu$, with $M \gg \mu$, and the renormalized frequencies and couplings are

$$\omega_1 = \sqrt{\frac{\omega_1^2}{M} + \frac{1}{M} \sum_{j=2}^{N+1} \lambda_{1j}}$$

$$\omega_j = \sqrt{\frac{\omega_j^2}{\mu} + \frac{1}{\mu} \lambda_{1j}} \quad (2 \leq j \leq N + 1)$$

$$g_j = \frac{1}{2\sqrt{\mu M}} \sqrt{\frac{\lambda_{1j}}{\omega_1 \omega_j}} \quad (2 \leq j \leq N + 1)$$

Dropping the first index, Eqs. (9a,9b) become

$$\frac{dU_1}{dt} = i\omega_1 U_1 - i \sum_{j=2}^{N} g_j (U_j - V_j)$$

$$\frac{dV_1}{dt} = -i\omega_1 V_1 - i \sum_{j=2}^{N} g_j (U_j - V_j)$$

$$\frac{dU_j}{dt} = i\omega_j U_j - ig_j (U_1 - V_1) \quad (j \neq 1)$$

$$\frac{dV_j}{dt} = -i\omega_j V_j - ig_j (U_1 - V_1) \quad (j \neq 1).$$

The bottom two equations can be solved by considering $U_1$ and $V_1$ as external parameters. Then, by substituting them into the top two equations, we get a pair of coupled integro-differential equations:

$$\frac{dU_1}{dt} = i\omega_1 U_1 + i \int_0^t d\tau h(t - \tau) (U_1(\tau) - V_1(\tau))$$

$$\frac{dV_1}{dt} = -i\omega_1 V_1 + i \int_0^t d\tau h(t - \tau) (U_1(\tau) - V_1(\tau)),$$

which depends on the reservoir topology only through the function

$$h(t) = \sum_{j=2}^{N+1} g_j^2 \sin(\omega_j t) = \frac{1}{4\mu M \omega_1} \sum_{j=2}^{N+1} \frac{\lambda_{1j}^2}{\omega_j} \sin(\omega_j t),$$

which in turn is related to the Fourier transform of the reservoir spectral density

$$J(\omega) = \sum_{j=2}^{N+1} g_j^2 \delta(\omega - \omega_j) = \frac{1}{4\mu M \omega_1} \sum_{j=2}^{N+1} \frac{\lambda_{1j}^2}{\omega_j} \delta(\omega - \omega_j).$$
This is the homogeneous part of the solution. To obtain the inhomogeneous one, we need to use the solution found previously for \( U_k \) and \( V_k \) in terms of the now known \( U_1 \) and \( V_1 \), and then use Eqs. (13) and (14).

V. MASTER EQUATION

The complete solution for single-mode dynamics is Eq. (15), with time-dependent functions \( U, V, A \) and \( B \). It was derived by assuming an explicit microscopic model for the reservoir as a set of other modes, which are coupled to the mode of interest, but over which the experimenter has little control (except for macroscopic parameters such as temperature). In this section, our goal is to find a dynamical equation (in fact, a master equation) whose solution is precisely Eq. (15), but which does not need to involve any other degrees of freedom, besides those of the system.

We start by differentiating Eq. (15) with respect to time, and then mapping it from phase space back to Hilbert space:

\[
\frac{d\rho}{dt} = -i \left[ H_S(t), \rho(t) \right] + \mathcal{D}_t(\rho(t)),
\]

where we have a time-dependent effective Hamiltonian

\[
H_S(t) = \omega(t) a\dagger a + \xi(t) a^{12} + \xi^*(t) a^2,
\]

and a time-dependent dissipation super-operator,

\[
\mathcal{D}_t(\rho) = \frac{\gamma_1(t) + \gamma_2(t)}{2} \left( [a\rho, a\dagger] + [a, \rho a\dagger] \right) \\
+ \frac{\gamma_2(t)}{2} \left( [a\dagger\rho, a] + [a\dagger, \rho a] \right) \\
- \frac{1}{2} \left( \eta(t) \left( [a\dagger\rho, a\dagger] + [a\dagger, \rho a\dagger] \right) + \text{H.c.} \right).
\]

This master equation depends on 7 real time-dependent parameters, which in turn depend on the 7 real parameters that define solution Eq. (15); the three real parameters

\[
\omega(t) = \frac{1}{|U|^2 - |V|^2} \Im \left( U^* \frac{dU}{dt} - V^* \frac{dV}{dt} \right), 
\]

\[
\gamma_1(t) = \frac{-2}{|U|^2 - |V|^2} \Re \left( U^* \frac{dU}{dt} - V^* \frac{dV}{dt} \right) \\
= -\frac{d}{dt} \log \left( |U|^2 - |V|^2 \right),
\]

(24b)
\[
\gamma_2(t) = \frac{dA}{dt} + \gamma_1 \left( A - \frac{1}{2} \right) + 2\Re(\xi^* B),
\]
(24c)

and the two complex parameters

\[
\xi(t) = \frac{-i}{|U|^2 - |V|^2} \left( U \frac{dV}{dt} - V \frac{dU}{dt} \right),
\]
(24d)

\[
\eta(t) = \frac{dB}{dt} + (\gamma_1 + 2i\omega) B + 2i\xi A.
\]
(24e)

The time-dependent functions \( \omega(t) \), \( \gamma_1(t) \) and \( \xi(t) \) are independent of the initial state of the reservoir, while \( \gamma_2(t) \) and \( \eta(t) \) depend on it.

The dissipator, Eq. (23), is not explicitly in Lindblad-like form, but can be put into it,

\[
\mathcal{D}_t(\rho) = \sum_{n=1}^2 \lambda_n(t) \left( [L_n(t) \rho, L_n^\dagger(t)] + [L_n(t), \rho L_n^\dagger(t)] \right)
\]
(25)

by defining the Lindblad operators

\[
L_1(t) = \cos \left( \frac{\theta}{2} \right) a - \sin \left( \frac{\theta}{2} \right) \frac{\eta}{|\eta|} a^\dagger
\]
(26a)

\[
L_2(t) = \cos \left( \frac{\theta}{2} \right) a^\dagger + \sin \left( \frac{\theta}{2} \right) \frac{\eta^*}{|\eta|} a,
\]
(26b)

and Lindblad rates

\[
\lambda_1(t) = \frac{\gamma_1}{2} + \frac{\gamma_1}{|\gamma_1|} \sqrt{\frac{\gamma_1^2}{4} + |\eta|^2 + \gamma_2}
\]
(27a)

\[
\lambda_2(t) = \frac{\gamma_1}{2} - \frac{\gamma_1}{|\gamma_1|} \sqrt{\frac{\gamma_1^2}{4} + |\eta|^2 + \gamma_2},
\]
(27b)

with the auxiliary definition

\[
\theta = \arctan \left( \frac{2|\eta|}{\gamma_1} \right) \quad \left( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right)
\]
(28)

The standard master equation derived with the Born-Markov approximation has the same form as equations Eq. (21)-(23), but with constant-in-time parameters. In it, each term has a physical meaning:

- The first term in Eq. (22), with \( \omega(t) = \omega_1 + \Delta \omega(t) \), accounts for the free dynamics of the system, modified by a frequency shift due to its interaction with the reservoir.

- The second term in Eq. (22) is a squeezing term, arising from an asymmetry between position and momentum variables in the coupling Hamiltonian. However, in the weak-coupling regime, this term is small (being exactly zero in the RWA), leading to a negligible squeezing effect.
• \( \gamma_1(t) \) is a decay rate, that drives the center of the system wave-packet towards its equilibrium at the origin of phase space.

• \( \gamma_2(t) \) is a diffusion coefficient, related to injection of extra noise into the system due to non-zero reservoir temperature and counter-rotating terms, which only spreads the wave-packet without affecting the trajectory of its center.

• \( \eta(t) \) is a coefficient of anomalous diffusion, which injects different levels of noise in position and momentum. From Eqs. (26a,26b), we see that, when \( \eta \neq 0 \), the Lindblad operators are not given by \( a \) and \( a^\dagger \), but by linear combinations of the two, giving rise to anomalous diffusion.

### A. Markovian and non-Markovian behavior

An interesting discussion in the current literature (see Ref. 19 and references therein) concerns non-Markovian behavior. The Born-Markov approximation always leads to a Lindblad equation with a dissipator written in the form of Eq.(25), with rates \( \lambda_n(t) \), which are positive but may vary in time (in which case it can be called a time-dependent Markovian process). If, at any given time, one of these rates assumes a negative value, then it is said to be a non-Markovian process, according to the divisibility criterion of Rivas-Huelga-Plenio 19, 20.

The model we have developed allows us to compute these rates exactly from the solution, obtained through the system-reservoir interaction Hamiltonian. We can thus describe the system as Markovian if the following conditions hold for all times \( t \):

\[
\begin{align}
\gamma_1(t) + 2\gamma_2(t) & \geq 0 \quad (29a) \\
\gamma_1(t) \gamma_2(t) + \gamma_2^2(t) - |\eta(t)|^2 & \geq 0, \quad (29b)
\end{align}
\]

where the functions are defined in Eq. (24b), Eq. (24c) and Eq. (24e).

### VI. ROTATING WAVE APPROXIMATION

In many physical systems described by the Hamiltonian of Eq. (3), the typical coupling intensity, \( |g_{kj}| \), is many orders of magnitude smaller than the frequencies \( \omega_k \), characterizing
the weak coupling regime. It is then a good approximation to drop the counter-rotating terms \((a_k a_j \text{ and } a_k^\dagger a_j^\dagger)\), a procedure which is known as the rotating wave approximation (RWA). Eqs. (9a, 9b) are greatly simplified, with \(V_{kj} = 0\) and \(U_{kj}\) obeying:

\[
\frac{dU_{kj}}{dt} = i\omega_j U_{kj} - i \sum_{n=1}^{N} U_{kn} g_{nj} .
\]

(30)

The condition \(V_{kj} = 0\) (for all \(kj\)) implies both \(\xi(t) = 0\) (no squeezing term in the effective system Hamiltonian) and \(B^{(0)} = 0\) and, unless the reservoir initial state has some degree of squeezing (i.e., \(\langle a_m a_n \rangle_0 \neq 0\) for some \(m, n\)), then also \(B^{(th)} = 0\). Together, this implies that \(\eta(t) = 0\). The condition \(\xi(t) = \eta(t) = 0\) is required to maintain the symmetry between position and momentum variables (the exchange \((\hat{q}, \hat{p}) \leftrightarrow (\hat{p}, -\hat{q})\) leaves the RWA Hamiltonian unchanged, while it changes the one in Eq. (1)). Therefore, in RWA, the squeezing term in Eq. (22) and the last term in Eq. (23) both vanish at all times, leading to the usual three terms (frequency shift, dissipation and diffusion) in the expression. The Markovianity condition is then simplified to

\[
\gamma_1(t) + 2\gamma_2(t) \geq 0 \quad (31a)
\]

\[
\gamma_2(t) \geq 0 \quad (31b)
\]

VII. NATURAL BASIS FOR SYSTEM EVOLUTION

It is a well known result [16] that a coherent state remains coherent when in contact with a reservoir at absolute zero, if one assumes RWA. This makes coherent states a natural basis to analyze system dynamics, ultimately motivating Glauber and Sudarshan to define the normal-order quasi-probability \(P\) function:

\[
\rho(t) = \int d^{2M} \{\alpha\} \ P(\{\alpha\} , t) \langle \{\alpha\} \rangle \langle \{\alpha\} \rangle .
\]

(32)

We have returned to the general case, where the system is composed of \(M\) modes. The coherent state follows a dynamics in phase space that can be written \(|\{\alpha\}\rangle \rightarrow |\{\alpha(t)\}\rangle\), where \(\{\alpha(t)\}\) is given by (compare with Eq. (8))

\[
\alpha_k(t) = \sum_{j=1}^{M} (U_{kj} a_j + V_{kj} a_j^\dagger) \quad (1 \leq k \leq M) .
\]

(33)
Combining these two equations, we have the familiar result

\[ \rho (t) = \int d^2 M \{ \alpha \} P (\{ \alpha \}, 0) |\{ \alpha (t) \}\rangle \langle \{ \alpha (t) \}|. \] (34)

The fact that coherent states remain coherent is intimately connected with the fact that the vacuum is a stationary state of this non-unitary evolution. However, for non-zero temperature, or when one includes the counter-rotating terms, this is no longer true: coherent states do not maintain their coherence, and we must resort to another basis, formed by Gaussian states. In the same way that the coherent states are generated by displacing the vacuum, the time-dependent Gaussian basis states are generated by displacing a squeezed thermal state:

\[ \rho_B (\{ \alpha \}, t) = D (\{ \alpha \}) \rho_o (t) D^\dagger (\{ \alpha \}), \] (35)

where \( \rho_o (t) \) is obtained by allowing an initial vacuum state to evolve in accordance with the solution presented in Eq. (15):

\[ |0\rangle \langle 0| \rightarrow \rho_o (t) = \int d^2 M \{ \alpha \} P_o (\{ \alpha \}, t) |\{ \alpha \}\rangle \langle \{ \alpha \}| \] (36)

Adopting then this natural Gaussian basis, we can write the evolution of any initial state as:

\[ \rho (t) = \int d^2 M \{ \alpha \} \int d^2 M \{ \eta \} P (\{ \alpha \}, 0) P_o (\{ \eta \}, t) \times |\{ \eta + \alpha (t)\}\rangle \langle \{ \eta + \alpha (t)\}|, \] (38)

where \( \{ \alpha (t) \} \) describe the evolution of the center of the wavepacket (which obeys a classical equation of motion, as required by the Ehrenfest theorem, and is independent of the state of the reservoir) and \( P_o (\{ \eta \}, t) \) describe the evolution of the shape of the wavepacket.

When the RWA and an absolute-zero reservoir are assumed, the wavepacket is not distorted, and \( P_o (\{ \eta \}, t) \) reduces to a delta function at the origin, making Eq. (38) identical to Eq. (15). Therefore, Eq. (38) is a generalization of Eq. (15) and we have obtained a generalization of the dynamics described in Ref. [16].
Another way to look at this result is that the displaced phase-space quasi-probability function is convoluted with another function, which accounts for the change in shape.

\[
P(\{\alpha\}, t) = \int d^2M \{\gamma\} P(\{\gamma\}, 0) P_o(\{\alpha - \gamma(t)\}, t)
\]

(39)

For a single mode, the center path follows \(\alpha(t) = U_1 \alpha + V_1 \alpha^*\), \(U_1\) and \(V_1\) being given by the solutions to Eqs. (18a) and (18b). The function \(P_o(\{\alpha\}, t)\) is just the solution when the initial state is the vacuum, i.e., it satisfies the initial condition \(P_o(\{\alpha\}, 0) = \delta(2)(\alpha)\). Under the RWA, this continues to be true at all times, \(P_{\text{RWA}}^{\alpha}(\{\alpha\}, t) = \delta(2)(\alpha)\).

**VIII. CONCLUSIONS**

We have presented a technique to derive an exact master equation for the system-reservoir dynamics under the strong coupling regime, where neither the rotating-wave-approximation nor the secular approximation apply. To this end, we adopted the strategy of considering a network of bosonic systems coupled to each other, picking out one of them as the system of interest and leaving the rest to play the role of the reservoir. Working with phase-space distribution functions and Gaussian states, we generalize an earlier result by Glauber, that a coherent state remains coherent despite dissipation when coupled to a zero temperature reservoir. We demonstrate that there is a class of Gaussian states which serves as a generalization of the coherent state basis of the Glauber-Sudarshan \(P\) representation. This class of Gaussian states follows from the distortion of the vacuum state which, in the strong-coupling regime, is no longer a stationary state, even for a zero temperature reservoir. We have also presented an investigation of the conditions that lead to a non-completely-divisible map, and thus non-Markovian dynamics. So far, conditions for non-Markovianity have been studied for finite Hilbert spaces under the rotating-wave and/or secular approximations. We remark that a master equation similar to the one derived here has been obtained using the Path Integrals approach \[7\]. The simplicity of our development, using phase-space distribution functions, offers the significant advantage of enabling us to cast the problem as the solution of a linear system of equations.
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[1] J. von Neumann, Mathematical Foundations of Quantum Mechanics (Princeton University Press, Princeton, New Jersey, 1955).

[2] W. H. Zurek, Phys. Rev. D 24, 1516 (1981); *ibid.* 26, 1862 (1982).

[3] A. O. Caldeira and A. J. Leggett, Physica 121A, 587 (1993), *ibid.*, Ann. Phys. (N.Y.) 149, 374 (1983), *ibid.*, Phys. Rev. A 31, 1059 (1985).

[4] E. Joos and H. D. Zeh, Z. Phys. B: Condens. Matter 59, 223 (1985).

[5] E. B. Davies, Quantum Theory of Open Systems (Academic Press, New York, 1976); D. Walls, G. Milburn, Quantum Optics (Springer-Verlag, Berlin, 1994); M. O. Scully, M. S. Zubairy, Quantum Optics (Cambridge Press, London, 1997).

[6] E. P. Wigner and V. F. Weisskopf, Z. Physik 63, 54 (1930).

[7] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D 45, 2843 (1992).

[8] J. J. Halliwell and T. Yu, Phys. Rev. D 53, 2012 (1996).

[9] G. W. Ford and R. F. O’Connell, Phys. Rev. D 64, 105020 (2001).

[10] W.-M. Zhang, P.-Y. Lo, H.-N. Xiong, M. W.-Y. Tu, and F. Nori, Phys. Rev. Lett. 109, 170402 (2012).

[11] H.-N. Xiong, W.-M, Zhang, X. Wang, and M.-H. Wu, Phys. Rev. A 82, 012105 (2010).

[12] H. Mäkelä and M. Möttönen, Phys. Rev. A 88, 052111 (2013).

[13] R. P. Feynman, A. R. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill, New York, 1965).

[14] M. A. de Ponte, S. S. Mizrahi, and M. H. Y. Moussa, Phys. Rev. A 76, 032101 (2007); M. A. de Ponte, M. C. de Oliveira, and M. H. Y. Moussa, Phys. Rev. A 70, 022324 (2004); *ibid.* Phys. Rev. A 70, 022325 (2004); *ibid.* Ann. Phys. (N.Y.) 317, 72 (2005).

[15] M. A. de Ponte, S. S. Mizrahi, and M. H. Y. Moussa, Ann. Phys 322, 2077 (2007); *ibid.*, Phys. Rev. A 84, 012331 (2011).

[16] R. Glauber, Quantum theory of optialoherene: seleted papers and letures (Berlin, Germany:
Wiley-VCH, 2007).

[17] K. E. Cahill and R. J. Glauber, Phys. Rev. A 59, 1538 (1999).

[18] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. 84, 621 (2012).

[19] C. Addis, B. Bylicka, D. Chruściński and S. Maniscalco, arXiv:1402.4975.

[20] A. Rivas, S. F. Huelga and M. B. Plenio, Phys. Rev. Lett. 105, 050403 (2010).