Decomposition of Spaces of Periodic Functions into Subspaces of Periodic Functions and Subspaces of Antiperiodic Functions

Hailu Bikila Yadeta
email: haybik@gmail.com

Salale University, College of Natural Sciences, Department of Mathematics,
Fiche, Oromia, Ethiopia

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Abstract
In this paper, we prove that the space \( P_p \) of all periodic function of fundamental period \( p \) is a direct sum of the space \( P_{p/2} \) of all periodic functions of fundamental period \( p/2 \) and the space \( AP_{p/2} \) of all antiperiodic functions of fundamental antiperiod \( p/2 \). The decomposition can be continued by applying the decomposition process to the successively raising periodic subspaces. It is shown that, under certain condition, a periodic function can be written as a convergent infinite series of antiperiodic functions of distinct fundamental antiperiods. In addition, we characterize the space of all periodic functions of period \( p \in \mathbb{N} \) in terms of all its periodic and antiperiodic subspaces of integer periods (or antiperiods). We show that the elements of a subspace of such a space of periodic functions take a specific form (not arbitrary) of linear combinations of the shifts of the elements of the given space. Lastly, we introduce a lattice diagram named periodicity diagram for a space of periodic function of a fixed period \( p \in \mathbb{N} \). As a particular example, the periodicity diagram of \( P_{12} \) is shown.

Keywords: periodic function, antiperiodic function, direct sum, decomposition, difference equation, periodicity diagram, \( n \)-th periodic generation, \( n \)-th antiperiodic generation

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1 Introduction and preliminaries

It is clear that any 1-periodic function is also a 2-periodic function. Therefore, the set of all 1-periodic functions is a subset of the set of 2-periodic functions. The main question is "what functions other than the 1-periodic ones are contained in the set of all 2-periodic functions?". This gives rise to the study of the decomposition of spaces of periodic functions into subspaces. The answers to such and similar questions will be answered in this paper. However, before introducing the main result of the current paper, we give examples from classical results in mathematics, where a vector space is decomposed as a direct sum of its subspaces. Let \( F \) represent the space of all real-valued function of real domain. That is

\[
F := \{ f : \mathbb{R} \to \mathbb{R} \}.
\]
Let us denote by \( E \) the subspace of all even functions in \( F \), where
\[
E = \{ f \in F : f(-x) = f(x), \forall x \in \mathbb{R} \}. \tag{1.2}
\]
Let us denote by \( O \) the subspace of all odd functions in \( F \), where
\[
O = \{ f \in F : f(-x) = -f(x), \forall x \in \mathbb{R} \}. \tag{1.3}
\]
Then we have the decomposition
\[
F = E \oplus O. \tag{1.4}
\]
According to (1.4), each element \( f \in F \) can be written as
\[
f = f_e + f_o, \quad f_e \in E, \quad f_o \in O, \tag{1.5}
\]
where
\[
f_e(x) = \frac{f(x) + f(-x)}{2}, \quad f_o(x) = \frac{f(x) - f(-x)}{2}.
\]
The only element common to both \( E \) and \( O \) is the constant function \( f(x) = 0 \). A good example of such decomposition of the form (1.5)
\[
e^x = \cosh x + \sinh x.
\]
Such decomposition is applicable in the Fourier series expansion of periodic functions. The even component is represented by Fourier cosine series, whereas the odd component is represented by Fourier sine series. See, for example, \([3],[5]\). The second example is the decomposition of space \( M_{n \times n}(\mathbb{R}) \) of square matrices with real entries into the space \( S_{n \times n}(\mathbb{R}) \) of symmetric matrices, and the space \( SS_{n \times n}(\mathbb{R}) \) of skew-symmetric matrices of real entries, where
\[
S_{n \times n}(\mathbb{R}) = \{ A \in M_{n \times n}(\mathbb{R}) : A^T = A \}, \quad SS_{n \times n}(\mathbb{R}) = \{ A \in M_{n \times n}(\mathbb{R}) : A^T = -A \}
\]
In fact,
\[
M_{n \times n}(\mathbb{R}) = S_{n \times n}(\mathbb{R}) \oplus SS_{n \times n}(\mathbb{R}),
\]
see \([12]\), pp. 151, Theorem 9.1.4. There, \( M^{m \times n}(\mathbb{F}) \) should be corrected as \( M^{n \times m}(\mathbb{F}) \). Also see \([11]\). If \( H_1 \) is a closed subspace of a Hilbert space \( H \), then \( H \) is a direct sum of \( H_1 \) and \( H_1^\perp \). That is
\[
H = H_1 \oplus H_1^\perp, \tag{1.6}
\]
where
\[
H_1^\perp = \{ h \in H : \langle h, g \rangle = 0 \quad \forall g \in H_1 \}.
\]
The representation of \( H \) as in (1.6) is called orthogonal decomposition of \( H \). See \([13]\). Let us denote by \( \mathbb{P}_p \), the space of all periodic functions of period \( p \),
\[
\mathbb{P}_p = \{ f \in F : f(x + p) = f(x) \}, \tag{1.7}
\]
and by $\mathbb{AP}_p$ the space of all antiperiodic functions of antiperiod $p$.

$$\mathbb{AP}_p = \{ f \in \mathcal{F} : f(x + p) = -f(x) \}. \tag{1.8}$$

The spaces $\mathbb{P}_p$, and $\mathbb{AP}_p$ form subspaces of $\mathcal{F}$. In this paper, we show that any periodic function $f$ with period $p$ can be decomposed in a unique way into a periodic function of period $p/2$ and an antiperiodic function of antiperiod $p/2$. To the best of the authors understanding and knowledge, this type of periodic-antiperiodic decomposition is not available in common literature and is new. Periodic and antiperiodic functions play important role in the solution of linear difference equations. In that, the general solutions are linear combinations of independent solutions over arbitrary periodic functions (antiperiodic) functions of some period. See [4], [8]. The motivation for the current work is the study of the difference equation with continuous argument

$$y(x + 2) - y(x) = 0, \tag{1.9}$$

whose characteristic equation is $\lambda^2 - 1 = 0$, and whose general solution

$$y(x) = f(x) + g(x), \tag{1.10}$$

where $f$ is an arbitrary 1-periodic function, and $g$ is an arbitrary 1-antiperiodic function. See [4], [6], [7]. On the other hand, the general solution of (1.9) is

$$y(x) = h(x), \tag{1.11}$$

where $h$ is an arbitrary periodic function of period 2. Comparing the general solutions of the difference equation (1.9), given in (1.10), and (1.11) in different forms, we come to the assertion that any arbitrary periodic function $h$ of period 2 can be decomposed into the sum of a periodic function $f$ of period 1 and an antiperiodic function $g$ of antiperiod 1. Consequently,

$$\mathbb{P}_2 = \mathbb{P}_1 \oplus \mathbb{AP}_1.$$

This assertion is true for an arbitrary periodic function of period $p$ and we prove it in this paper. We shall also prove that some periodic functions can be written as an infinite series of terms of antiperiodic functions of different antiperiods. We also study all periodic (and antiperiodic) of period (antiperiod) $d \in \mathbb{N}$ subspaces of a periodic space of functions with specific period $p \in \mathbb{N}$, where $d$ is a divisor of $p$. The decomposition of periodic functions into spaces of periodic and antiperiodic function is connected to difference equations, both in discrete argument or continuous argument (see [2], [3], in the study of some classes of operators defined on spaces of periodic functions.

### 1.1 The shift operator and periodicity

For $h \in \mathbb{R}$, we define the shift operator $E^h$ and the identity operator $I$ as

$$E^h y(x) := y(x + h), \quad Iy(x) := y(x).$$
For $h = 1$, we write $E^h$ only as $E$ than $E^1$. We agree that $E^0 = I$. We define the forward difference operator $\Delta$ and the back ward difference operators $\nabla$ as follows

$$\Delta y(x) := (E - I)y(x) = y(x + 1) - y(x), \quad \nabla y(x) = (I - E^{-1})y(x) = y(x) - y(x - 1).$$

**Definition 1.1.** A function $f$ is said to be $p$-periodic if there exists a $p > 0$ such that $f(x) = f(x + p)$, $x \in \mathbb{R}$. The least such $p$ is called the *period* of $f$. In terms of shift operator we write this as $E^p f(x) = f(x)$.

**Definition 1.2.** [10], [9] A function $f$ is said to be $p$-antiperiodic if there exists a $p > 0$ such that $f(x + p) = -f(x)$, $x \in \mathbb{R}$. The least such $p$ is called the *antiperiod* of $f$. In terms of shift operator we write this as $E^p f(x) = -f(x)$.

**Example 1.3.** The following functions are 1-periodic or 1-antiperiodic functions:

- The functions $f_n(x) = \cos 2n\pi x$, $n \in \mathbb{N}$ are 1-periodic.
- The functions $g_n(x) = \cos(2n + 1)\pi x$, $n \in \mathbb{N}$ are 1-antiperiodic.
- The function $f(x) = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer not greater than $x$, is a 1-periodic function.

**Remark 1.4.** Every $p$-antiperiodic function is $2p$-periodic. However not every $2p$-periodic functions is $p$-antiperiodic function. Further properties of $p$-antiperiodic function are available in literatures. For example, finite linear combinations, or convergent infinite series each of whose terms are $p$-periodic (p-antiperiodic) function is a $p$-periodic(p-antiperiodic) function. For example

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(2n + 1)x}{n^2}$$

is $\pi$-antiperiodic function defined by a uniformly convergent series each of its terms is $\pi$-antiperiodic. See [10].

**Remark 1.5.** The constant function $f(x) = 0$ is the only function that is both periodic and antiperiodic with any period and antiperiod.

**Theorem 1.6.** The composition of periodic function with even or odd function is given as follows:

- If $f \in \mathcal{O}$, $g \in \mathcal{A} \mathcal{P}_p$, then $f \circ g \in \mathcal{A} \mathcal{P}_p$.
- If $f \in \mathcal{E}$, $g \in \mathcal{A} \mathcal{P}_p$, then $f \circ g \in \mathcal{P}_p$.
- If $f \in \mathcal{F}$, $g \in \mathcal{P}_p$, then $f \circ g \in \mathcal{P}_p$.

**Theorem 1.7.** Let $\omega > 0$, and $f \in \mathcal{F}$. Define $g(x) = f(\omega x)$. If $f \in \mathcal{A} \mathcal{P}_p$ then $g \in \mathcal{A} \mathcal{P}_{\frac{p}{\omega}}$. If $f \in \mathcal{P}_p$ then $g \in \mathcal{P}_{\frac{p}{\omega}}$.

**Proof.** If $f \in \mathcal{A} \mathcal{P}_p$ then

$$g(x + \frac{p}{\omega}) = f(\omega(x + \frac{p}{\omega})) = f(\omega x + p) = -f(\omega x) = -g(x).$$
The proof for \( f \in \mathbb{P}_p \) is similar. \( \square \)

# 2 Main Results

## 2.1 Decomposition of spaces of periodic functions

**Theorem 2.1.** The space \( \mathbb{P}_p \) of all \( p \)-periodic functions is the direct sum of the space \( \mathbb{P}_{p/2} \) of all \( p/2 \)-periodic function and the space \( \mathbb{AP}_{p/2} \) of all \( p/2 \)-antiperiodic function.

**Proof.** Let \( h \in \mathbb{P}_p \). Suppose that \( h(x) = f(x) + g(x) \),

\[
(2.1)
\]

for some \( f \in \mathbb{P}_{p/2} \), and \( g \in \mathbb{AP}_{p/2} \). Then

\[
h(x + p/2) = f(x + p/2) + g(x + p/2) = f(x) - g(x). \tag{2.2}
\]

Then solving (2.1) and (2.2) simultaneously we get

\[
(2.3)
\]

Then \( f \) and \( g \) defined as in (2.3) satisfy the required condition. It remains to show that the representation is unique. Suppose that \( f_1, f_2 \in \mathbb{P}_p \) and \( g_1, g_2 \in \mathbb{AP}_{p/2} \) such that \( h = f_1 + g_1 = f_2 + g_2 \). Then we have \( f_1 - f_2 = g_2 - g_1 \) Hence \( f_1 - f_2 \in \mathbb{P}_p \) and \( g_1 - g_2 \in \mathbb{AP}_{p/2} \) we have \( f_1 - f_2 = g_2 - g_1 = 0 \). \( \square \)

**Example 2.2.** Let \( f(x) = x - \lfloor x \rfloor \). Then \( f \) is a 1-period function. The decomposition of \( f \) yields

\[
f_1(x) = \frac{1}{2}(2x + 1/2 - \lfloor x \rfloor - \lfloor x + 1/2 \rfloor),
\]

\[
\tilde{f}_1(x) = \frac{1}{2}(\lfloor x + 1/2 \rfloor - \lfloor x \rfloor - 1/2).
\]

We have proved that a periodic function of period \( p \) can be decomposed into a periodic function of period \( p/2 \) and an antiperiodic function of antiperiodic \( p/2 \). We may continue this decomposition process by taking the new period function of period \( p/2 \) by decomposing it into a periodic function of period \( p/4 \) and an antiperiodic function of antiperiod \( p/4 \) and so on.

**Definition 2.3.** Given a periodic function \( f \) period \( p \) the \( n \)-th periodic generation of \( f \), denoted by \( f_n \), is a periodic function of period \( p/2^n \) derived from \( f \) after \( n \) decompositions. The \( n \)-th antiperiodic generation of \( f \), denoted by \( \tilde{f}_n \), is an antiperiodic function of antiperiod \( p/2^n \) derived from \( f \) after \( n \) decompositions.

**Remark 2.4.** If \( f \) is a \( p \)-periodic function that is also a \( p/2 \)-antiperiodic function, then the decomposition in Theorem 2.1 yields \( f = \tilde{f}_1 + 0 \). That is, the first periodic generation \( f_1 \) of \( f \) is 0, and the first antiperiodic generation \( \tilde{f}_1 \) of \( f \) is itself. Consequently, all subsequent periodic and antiperiodic generations of \( f \) are all 0 s.

**Theorem 2.5.** Given a periodic function \( f \) of period \( p \) the \( n \)-th periodic generation of \( f \) is given by

\[
f_n = \frac{1}{2^n} \prod_{i=1}^{n} (1 + E^{\frac{2\pi}{p}i}) f. \tag{2.4}
\]
Theorem 2.6. Given a periodic function $f$ of period $p$, the $n$-th antiperiodic generation of $f$ is given by

$$\tilde{f}_n = \frac{1}{2^n}(I - E^p) \prod_{i=1}^{n-1} (1 + E^p) f.$$  (2.5)

Proof. The $(n-1)$-th periodic generation $f_{n-1}$ is decomposed into the $n$-th periodic generation $\tilde{f}_n$ and the $n$-th antiperiodic generation $\tilde{f}_n$. That is $f_{n-1} = f_n + \tilde{f}_n$. Consequently by (2.4)

$$\tilde{f}_n = f_{n-1} - f_n = \frac{1}{2^{n-1}} \prod_{i=1}^{n-1} (1 + E^p) f - \frac{1}{2^n} \prod_{i=1}^{n} (1 + E^p) f,$$

which, upon simplification, gives the desired result. \qed

Theorem 2.7. For each $n \in \mathbb{N}$,

$$f = f_n + \sum_{k=1}^{n} \tilde{f}_k.$$

Theorem 2.8. Let $f$ be the $p$-periodic function with $n$th periodic generation $f_n$. If

$$\lim_{n \to \infty} \sup_{x_0 \leq x \leq x_0 + p} |f_n(x)| = 0,$$

then

$$f(x) = \sum_{n=0}^{\infty} \tilde{f}_n(x).$$  (2.6)

2.2 Periodic functions of integer periods

Theorem 2.9. Let $\mathbb{P}_p$ denote the set of all $p$-periodic functions. Let

$$LC\mathbb{P}_p := \left\{ \sum_{i=0}^{p-1} c_i E^i f, \quad f \in \mathbb{P}_p, \quad c_i \in \mathbb{R} \right\}. $$  (2.7)

Then

$$LC\mathbb{P}_p = \mathbb{P}_p$$  (2.8)

Proof. If $f \in LC\mathbb{P}_p$ then $f \in \mathbb{P}_p$. For a space of all periodic functions of period $p$ are invariant under translations (shift operators), and invariant under scalar multiplication. Conversely, if $f \in \mathbb{P}_p$ then $f = 1 f$, with all other coefficients equal to zero. So $\mathbb{P}_p \subseteq LC\mathbb{P}_p$. \qed

Remark 2.10. For any $n, p \in \mathbb{N}$, there exists integers $m, r$ such that $n = mp + r, 0 \leq r < p$, so that

$$E^n f = E^{mp+r} f = E^r E^{mp} f = E^r f, \forall f \in \mathbb{P}_p.$$ 

Therefore, only the powers $E^i$ with $0 \leq i < p$ are considered in the definition of $LC\mathbb{P}_p$.

Theorem 2.11. Let $p = md$, where $m, d \in \mathbb{N}$. Then any element of the form

$$f_d = (I + E^d + E^{2d} + \ldots + E^{(m-1)d})g, \quad g \in \mathbb{P}_p$$  (2.9)

is an element of $\mathbb{P}_d$. 

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Proof. Since \( g \in P_p \), \( E^{md}g = E^{p}g = g \). Consequently,

\[
E^{d}f_d = E^{d}(I + E^{d} + E^{2d} + \ldots + E^{(m-1)d})g \\
= (E^{d} + E^{2d} + \ldots + E^{(m-1)d} + E^{md})g \\
= (I + E^{d} + E^{2d} + \ldots + E^{(m-1)d})f = f_d 
\]

Therefore \( f_d \in P_d \).

\[ \Box \]

**Theorem 2.12.** Let \( f_d \in P_d \). Then there exists \( g \in P_p \) (not necessarily unique) such that \( f_d \) can be written in the form \( 2.9 \).

Proof. Since \( f_d \in P_d \), we have \( \frac{1}{m}f_d \in P_d \subset P_p \). Take \( g = \frac{1}{m}f_d \) so that

\[
\left(I + E^{d} + E^{2d} + \ldots + E^{(m-1)d}\right) \frac{1}{m}f_d \\
= \frac{1}{m} \left(I + E^{d} + E^{2d} + \ldots + E^{(m-1)d}\right) f_d = \frac{1}{m}(mf_d) = f_d. 
\]

\[ \Box \]

**Corollary 2.13.** Let \( p \in \mathbb{N} \), and \( f \in P_p \). Then any element of the form

\[
(1 + E + E^{2} + \ldots + E^{p-1})f 
\]  

is an element of \( P_1 \). Conversely, any element \( f_1 \in P_1 \) can be written, not necessarily uniquely, in the form \( 2.10 \) for some \( f \in P_p \).

**Remark 2.14.** Regards to the non uniqueness of the element \( g \in P_p \) in Theorem 2.12, assume that there are elements \( g, \tilde{g} \in P_p \). Then we have

\[
(I + E^{d} + E^{2d} + \ldots + E^{(m-1)d})(g - \tilde{g}) = 0. 
\]

Therefore, \( \tilde{g} = g + h \) where \( h \) is any element in the null space of \( I + E^{d} + E^{2d} + \ldots + E^{(m-1)d} \).

**Theorem 2.15.** Let \( p, m, d \in \mathbb{N} \), such that \( p = md \) and that \( m \) is odd. Then \( AP_{d} \) is a subspace of \( AP_{p} \) and that every element \( \tilde{f}_d \in AP_{d} \) can be written as

\[
\tilde{f}_d = (I - E^{d} + E^{2d} - E^{3d} + \ldots + E^{(m-1)d}) \tilde{f}, 
\]

where \( \tilde{f} \in AP_{p} \).

**Lemma 2.16** (Bezout’s identity [1]). If \( a \) and \( b \) are integers not both zero then there exists integers \( u \) and \( v \) such that

\[
gcd(a, b) = au + bv 
\]

**Theorem 2.17.** Let \( d = \gcd(m, n) \). Then \( P_d = P_m \cap P_n \).

Proof. \( d|m \Rightarrow P_d \subseteq P_m \), and \( d|n \Rightarrow P_d \subseteq P_n \). Consequently

\[
P_d \subseteq P_m \cap P_n. 
\]  

(2.11)
By Bezout’s identity, \( d = \gcd(m, n) \Rightarrow \exists \alpha, \beta \in \mathbb{Z}, \alpha m + \beta n = d \). If \( f \in \mathbb{P}_m \cap \mathbb{P}_n \), we have \( E^m f = f, \ E^n f = f \).

Consequently
\[
E^d f = E^{\alpha m + \beta n} f = E^{\alpha m} E^{\beta n} f = E^{\alpha m} f = f.
\]

This shows that \( f \) is \( d \)-periodic. Therefore,
\[
\mathbb{P}_m \cap \mathbb{P}_n \subseteq \mathbb{P}_d. \tag{2.12}
\]

By (2.11) and (2.12) it follows that \( \mathbb{P}_m \cap \mathbb{P}_n = \mathbb{P}_d \).

**Corollary 2.18.** If \( m \) and \( n \) are relatively prime, then \( \mathbb{P}_m \cap \mathbb{P}_n = \mathbb{P}_1 \).

## 3 Practical Examples

### 3.1 Decomposition of the spaces \( \mathbb{P}_3, \mathbb{P}_6, \) and \( \mathbb{P}_{12} \)

We know that \( \mathbb{P}_1 \subset \mathbb{P}_3 \). Therefore \( f \in \mathbb{P}_1 \) then \( f \in \mathbb{P}_3 \). The important question is: What is the set of elements of \( \mathbb{P}_3 \) that are not in \( \mathbb{P}_1 \)?

**Theorem 3.1.** Let
\[
S = \{ f \in \mathcal{F} : E^2 f + Ef + f = 0 \}. \tag{3.1}
\]

- \( S \subset \mathbb{P}_3 \),
- \( \mathbb{P}_3 = \mathbb{P}_1 \oplus S \).

**Proof.** Let \( f \in S \). Then \( E^2 f = -Ef - f \). Consequently,
\[
E^3 f = -E^2 f - Ef = Ef + f - Ef = f.
\]

This shows that \( f \in \mathbb{P}_3 \). If \( f \in S \cap \mathbb{P}_1 \), then
\[
0 = E^2 f + Ef + f = f + f + f = 3f.
\]

So \( f = 0 \). Let \( f \in \mathbb{P}_3 \) is given. Suppose that
\[
f = g + h, \quad g \in \mathbb{P}_1, \ h \in S. \tag{3.2}
\]

Applying shift operator \( E \) to (3.2), we get
\[
Ef = g + Eh \tag{3.3}
\]

Subtracting (3.3) from (3.2), we get
\[
f - Ef = h - Eh = h + h + E^2 h = 2h + E^2 h.
\]

Consequently,
\[
h = \frac{I - E}{2I + E^2} f, \quad g = \frac{I + E + E^2}{2I + E^2} f.
\]

\[\square\]
Example 3.2. Let
\[ f(x) = \cos \frac{2\pi x}{3}, \quad g(x) = \sin \frac{2\pi x}{3}, \quad h(x) = x - \lfloor x \rfloor. \]
Then \( f, g \in S \subset P_3, \) \( f \notin P_1, \) \( g \notin P_1, \) \( h \in P_1, \) \( h \notin S. \)

By definition of \( S \) in (3.1), we see that \( S \) is the kernel of the operator \( E^2 + E + I := L, \) and it is clear that \( \ker \triangle = P_1. \) Next we want to determine the images of the operators \( \triangle \) and \( L. \)

Lemma 3.3.
\[ \{ \triangle f : f \in S \} = S \]

Proof. Let \( f \in S. \) Then
\[ E^2(\triangle f) + E(\triangle f)f + \triangle f = \triangle(E^2f + Ef + f) = 0. \]
Therefore, \( \{ \triangle f : f \in S \} \subset S. \) On the other hand, if \( s \in S \) then \( E^2s + Es + s = 0. \) Rearrangement yields,
\[ s = -E^2s - Es = (E - I)(-Es - 2s) - 2s \]
so that
\[ s = -\frac{1}{3}(Es + 2s) \in \{ \triangle f : f \in S \}. \]
Hence the Lemma is proved.

Theorem 3.4. Let \( L := E^2 + E + I, \) and \( \triangle := E - I \) the forward difference operator.
\[ L : P_3 \to P_3, \quad \triangle : P_3 \to P_3 \]

Then
\[ \ker \triangle = \Im L = P_1, \quad \Im \triangle = \ker L = S, \]
so that, by Theorem 3.1
\[ \ker L \oplus \Im L = P_3 = \ker \triangle \oplus \Im \triangle. \]

Proof. \( \Im L = \{ Lf : f \in P_3 \}, \) and \( \{ \triangle Lf : f \in P_3 \} = \{ (E^3 - I)f : f \in P_3 \} = \{ 0 \}. \) Consequently, \( \Im L \subset P_1 = \ker \triangle. \) Let \( f \in P_1 \subset P_3, \) then \( Lf = f. \) This implies that \( f \in \Im L. \) Therefore \( \Im L \subset \Im L. \) Using Lemma 3.3 and Theorem 3.1
\[ \Im \triangle = \{ \triangle f : f \in P_3 \} = \{ \triangle f : f \in P_1 \} \cup \{ \triangle f : f \in S \} = \{ 0 \} \cup S = S. \]

We have seen that the space of all 3-periodic functions, \( P_3, \) can be decomposed into the space of all 1-periodic function \( P_1, \) and the space \( S \) of all 3-periodic functions that satisfy the second order difference equation \( E^2f + Ef + f = 0. \) However according to Theorem 2.8
\[ P_3 = \{ \alpha E^2f + \beta Ef + \gamma f : \alpha, \beta, \gamma, \in \mathbb{R}, f \in P_3 \} \]
We show that \( 0 \in S \cap P_1 \) can take only two forms \( \{ f \in P_3 | E^2f + Ef + f = 0 \} \) or \( \{ f \in P_3 | Ef - f = 0 \}. \)
**Theorem 3.5.** $0 \in \mathbb{P}_3$ can be written either as $E^2 f + Ef + f = 0$, in which case $f \in \mathbb{S} \subset \mathbb{P}_3$ or $Ef - f = 0$, in which case $f \in \mathbb{P}_1 \subset \mathbb{P}_3$.

**Proof.** Let

$$\alpha E^2 f + \beta Ef + \gamma f = 0 \quad (3.4)$$

Applying the shift operator $E$ we get,

$$\alpha f + \beta E^2 f + \gamma Ef = 0 \quad (3.5)$$

Applying shift operator to (3.5) we get

$$\alpha Ef + \beta f + \gamma E^2 f = 0 \quad (3.6)$$

Writing (3.4) (3.5) (3.6) as a homogeneous system we

$$\begin{bmatrix} \gamma & \beta & \alpha \\ \alpha & \gamma & \beta \\ \beta & \alpha & \gamma \end{bmatrix} \begin{bmatrix} f \\ Ef \\ E^2 f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.7)$$

This homogeneous system has non trivial solution $[f \quad Ef \quad E^2 f]^T$ only if the determinant of the coefficient matrix is zero. This can happen when $\alpha + \beta + \gamma = 0$, or $\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \alpha\gamma - \beta\gamma = 0$. For the first case we have

$$0 = \alpha E^2 f + \beta Ef - \alpha f - \beta f$$

$$= \alpha(E^2 - I)f + \beta(E - I)f$$

$$= (E - I)(\alpha E + (\alpha + \beta)I)f$$

$$= (E - I)g, \quad (3.8)$$

where $g := (\alpha E + (\alpha + \beta)I)f \in \mathbb{P}_3$. It is easy to show that $\mathbb{P}_3 = \{(\alpha E + (\alpha + \beta)I) f, f \in \mathbb{P}_3\}$. For the second case, since

$$\alpha^2 + \beta^2 + \gamma^2 \geq 2(|\alpha\beta| + |\alpha\gamma| + |\beta\gamma|) \geq \alpha\beta + \alpha\gamma + \beta\gamma,$$

and equality holds if $\alpha = \beta = \gamma$, we have

$$0 = \alpha E^2 f + \beta Ef + \gamma f = (E^2 + E + f)g,$$

where, $g := \alpha f$. Since $\alpha$ is arbitrary, $\mathbb{P}_3 = \{\alpha f, f \in \mathbb{P}_3\}$, we have the desired result.

According to Theorem 2.1, we have $\mathbb{P}_6 = \mathbb{P}_3 \oplus \mathbb{A}\mathbb{P}_3$. In Theorem 3.1, we have seen the decomposition $\mathbb{P}_3 = \mathbb{P}_1 \oplus \mathbb{S}$. Now we see that the component $\mathbb{A}\mathbb{P}_3$ can be decomposed into some direct sum of its subspaces.

**Theorem 3.6.** Let

$$T = \{ f \in \mathcal{F} | E^2 f - Ef + f = 0 \}, \quad (3.9)$$

Then

- $T \subset \mathbb{A}\mathbb{P}_3$, 

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\[ A_P^3 = A_P^1 \oplus T. \]

**Proof.** Clearly \( A_P^1 \subset A_P^3 \). Let \( f \in T \), then \( E^2 f = Ef - f \). Consequently

\[ E^3 f = E^2 f - Ef = Ef - f - Ef = -f. \]

This shows that \( f \in A_P^3 \). Suppose that \( f \in A_P^1 \cap T \). Then

\[ 0 = E^2 f - Ef + f = f + f = 3f = 0 \]

For any \( f \in A_P^3 \), \( f = g + h \), where \( g \in A_P^1 \) and \( h \in T \) are given by

\[ g = \frac{E^2 - E + I}{2I + E^2} f, \quad h = \frac{I + E}{2I + E^2} f. \]

**Example 3.7.** Let \( f(x) = \cos \frac{\pi x}{3} \), \( g(x) = \sin \frac{\pi x}{3} \). Then \( f, g \in T \).

**Theorem 3.8.** Let \( f_1 \in P_1, f_2 \in P_2, f_3 \in P_3, \) and \( \tilde{f}_1 \in P_1, \tilde{f}_2 \in A_P^2, \tilde{f}_3 \in A_P^3 \). Then for appropriate \( f, g, h, \tilde{f}, \tilde{g}, \tilde{h} \in P_6 \),

\[ f_1 = (I + E + E^2 + E^3 + E^4 + E^5) f, \]
\[ f_2 = (I + E^2 + E^4) g, \]
\[ f_3 = (I + E^3) h, \]
\[ \tilde{f}_1 = (I - E + E^2 - E^3 + E^4 - E^5) \tilde{f}, \]
\[ \tilde{f}_2 = (I - E^2 + E^4) \tilde{g}, \]
\[ \tilde{f}_3 = (I - E^3) \tilde{h}. \]

**Remark 3.9.** We have seen than a 1- periodic element \( f_1 \) in \( P_3 \) has the form \( (I + E + E^2) f, f \in P_3 \), and the same element being in \( P_6 \), has the form \( (I + E + E^2 + E^3 + E^4 + E^5) g, g \in P_6 \). Now observe that

\[ f_1 = (I + E + E^2 + E^3 + E^4 + E^5) g \]
\[ = (I + E + E^2)(I + E^3) g \]
\[ = (I + E + E^2) f, \]

where \( f = (I + E^3) g, g \in P_6 \), is a 3-periodic element in \( P_6 \) and hence in \( P_3 \). This agrees with the representation of \( f_1 \) in \( P_3 \).

**Example 3.10.** We show that \( f \in P_2 \) can be written uniquely as \( f = x + y \), where \( x \in P_3, y \in A_P^3 \). Since \( f \in P_2 \subset P_6 \) and that \( P_6 = P_3 \oplus A_P^3 \), let

\[ f = x + y. \] (3.10)

Then applying the operator \( E^3 \), we get

\[ E^3 f = E^3 x + E^3 y \Rightarrow Ef = x - y. \] (3.11)
Solving equations (3.10) and (3.11) simultaneously, we get
\[ x = \frac{1}{2}(f + Ef), \quad y = \frac{1}{2}(f - Ef). \]

**Theorem 3.11.** Let
\[ U = \{ f \in \mathcal{F} \mid E^4f - E^2f + f = 0 \}. \] (3.12)

then
\[ \bullet U \subset P_6 \]
\[ \bullet AP_6 = AP_2 \oplus U \]

**Proof.** Clearly \( AP_2 \subset AP_6 \). If \( f \in U \), then \( E^4f = E^2f - f \). Consequently
\[ E^6f = E^4f - E^2f = E^2f - f - E^2f = -f. \]
This shows that \( f \in AP_6 \). Suppose that \( f \in AP_2 \cap U \). Then
\[ 0 = E^4f - E^2f + f = f + f + f = 3f = 0 \]
For any \( f \in AP_6 \), \( f = g + h \), where \( g \in AP_2 \), and \( h \in U \) are given by
\[ g = \frac{E^4 - E^2 + I}{2I + E^4}f, \quad h = \frac{I + E^2}{2I + E^4}f. \]

**Theorem 3.12.** For appropriate \( f \in \mathbb{P}_{12} \), the elements \( f_i, \tilde{f}_i \in \mathbb{A}P_i, i = 1, 2, 3, 4, 6, 12 \) take the form
\[ f_1 = (I + E + E^2 + E^3 + E^4 + E^5 + E^6 + E^7 + E^8 + E^9 + E^{10} + E^{11})f, \]
\[ f_2 = (I + E^2 + E^4 + E^6 + E^8 + E^{10})f, \]
\[ f_3 = (I + E^3 + E^6 + E^{9})f, \]
\[ f_4 = (I + E^4 + E^8) f, \]
\[ f_6 = (I + E^6)f, \]
\[ \tilde{f}_1 = (I + E + E^2 + E^3 + E^4 - E^5 + E^6 + E^7 + E^8 - E^9 + E^{10} - E^{11})f, \]
\[ \tilde{f}_2 = (I + E^2 + E^4 - E^6 + E^8 - E^{10})f, \]
\[ \tilde{f}_3 = (I + E^3 + E^6 - E^{9})f, \]
\[ \tilde{f}_6 = (I - E^6)f. \]

**Theorem 3.13.** we have the following decomposition of the space \( \mathbb{P}_{12} \) into its subspaces:
\[ \mathbb{P}_{12} = S \oplus \mathbb{P}_1 \oplus AP_1 \oplus T \oplus AP_2 \oplus U. \]

**Proof.** The proof follows from Theorem 2.1 and Theorem 3.1.
**Definition 3.14.** A *periodicity diagram* is a lattice graph depicting how a periodic space of an integer period, and its periodic (or antiperiodic) subspaces of integer period (or antiperiod) are related.

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**4 Conclusions and possible further Works**

In this paper, we have discussed the decomposition of a periodic function of period \( p \) into a periodic function of \( p/2 \) and an antiperiodic function of antiperiodic \( p/2 \). The author named the thus newly raised periodic function as first periodic generation. Continuing the process with the first periodic generation, we get the second periodic generation. If the magnitude of the \( n \)-th periodic generation of a given periodic function tends to zero uniformly on an initial interval \([x_0, x_0 + p]\) of length \( p \), which is equal to the fundamental period of \( f \), then we can write a periodic function \( f \) as an infinite sequence of antiperiodic functions of different fundamental antiperiods. In addition to that, we have discussed the possible forms of elements in periodic subspace of a space of periodic functions of period equal to \( p \in \mathbb{N} \). Such subspaces are composed of linear combinations of the shifts elements \( \mathbb{P}_p \).

The elements of subspaces of the main periodic space \( \mathbb{P}_p \) satisfy certain difference equations (linear combination of shifts of some function equated to zero) according to the subspace they belong to. For such decomposition of periodic spaces into direct sums of subspaces, we have depicted the diagram called periodicity diagram of space periodic functions. Periodicity diagram shows only subspace of spaces of periodic (or antiperiodic) functions and how they are related to each other. The author’s work is based on the consideration of solutions of difference equations with the nature of the roots of the characteristic equations. This work has open rooms for further developments and applications in the generation of periodic series whose terms are not trigonometric series as we have been accounted to.
Conflict of interests

The author declare that there is no conflict of interests regarding the publication of this paper.

Data availability

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