Examination of Wandzura-Wilczek Relation 
for $g_2(x, Q^2)$ in pQCD

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(March 14, 1997, Revised May 20, 1997)

Abstract

In order to examine the validity of Wandzura-Wilczek relation for the polarized DIS structure function $g_2(x, Q^2)$, we use the light-front time-ordering pQCD to calculate $g_2(x, Q^2)$ at order $\alpha_s$ on a quark target. We find that the study of the transverse polarized structure function in pQCD is only meaningful if we begin with massive quarks. The result shows that the Wandzura-Wilczek relation for $g_2(x, Q^2)$ is strongly violated by the quark mass in pQCD.

PACS: 11.10Ef; 11.40.-q; 12.38.Bx

Keywords: polarized structure function; light-front time-ordering perturbative QCD; Wandzura-Wilczek relation

1. Introduction

The transverse polarized structure function $g_2(x, Q^2)$ is perhaps the least well-known structure function in deep inelastic lepton-nucleon scatterings (DIS). In the early naive parton model, Feynman claimed that $g_2$, just like $g_1$, has a simple parton interpretation:

$$g_1(x) + g_2(x) = \frac{1}{2} \sum_q e_q^2 \Delta q_T(x),$$

where $\Delta q_T(x)$ is the difference between the number density of quarks polarized along the same direction of the transverse polarization of nucleon and those polarized along the opposite direction $[1]$. He also pointed out that one would permit a measurement of the transverse quark polarization if one keeps terms proportional to quark masses. This implies that the measurement of transverse polarization in DIS is a “higher twist” effect, based on some common understanding of the concept of twist. Some years later, Wandzura and Wilczek studied the properties of $g_2$ in DIS by the use of the operator product expansion (OPE) analysis $[2]$. They claimed that except for a twist three contribution $\overline{g}_2(x)$ which may be negligible in their model calculation, $g_2(x)$ can be related to an integral over $g_1(x)$:

$$g_2(x) = g_2^{WW}(x) + \overline{g}_2(x), \quad g_2^{WW}(x) = -g_1(x) + \int_x^1 \frac{dy}{y} g_1(y). \quad (1)$$

The relation between $g_2^{WW}$ with $g_1(x)$ is called Wandzura-Wilczek (WW) relation. In their work, the quark mass effect is neglected. After a couple of years, Altarelli and Muzzetto explicitly computed $g_2(x, Q^2)$ in perturbative QCD and found that quark masses play an important and nontrivial role in determining $g_2$ but unfortunately the work was not published.
The OPE analysis with the quark mass effect included was given by Kodaira et al. at the same time which results in the same solution as in but the picture of $g_2$ is not very clear in OPE analysis. Later on, Shuryak and Vainshtein pointed out that the twist-three contribution $g_2$ is a direct quark-gluon interaction effect which is important in $g_2(x, Q^2)$. Thus, the measurement of $g_2$ may also be very sensitive to the interaction dependent higher twist effects in QCD.

Since then, much work on the subject is concentrated on the questions whether $g_2$ can be described approximately by Feynman’s parton picture, and whether it is a relatively good approximation to predict $g_2$ from the longitudinal polarized structure function $g_1$ via the WW relation, or whether the quark-gluon coupling (a twist-3 operator) can provide a significant contribution to $g_2$. Different authors used different methods and obtained different results, and these are not all compatible with each other. However, because of the lack of experimental data, the understanding of $g_2$ is very limited in the investigations of past twenty years. Very recently, $g_2(x, Q^2)$ has been preliminarily measured by the SMC experiment in CERN and the E143 experiment in SLAC. Although it seems that the existing data can be fit quite well by the WW relation, whether this simple relation can reasonably describe the transverse polarization physics of hadrons in DIS and whether one may also see the effect of direct quark-gluon interaction from $g_2$ are still in question.

Roberts and Ross attempted to show from their parton model picture that the WW relation should be a good approximation in the description of $g_2$ even if quark masses and quark transverse momentum are important in determining $g_2$. While, Jaffe and Ji have argued from OPE that the twist-3 contribution $g_2$ may be very significant. Some bag model calculations have demonstrated that the twist-3 contribution is significantly large in comparison with $g_{WW}$. The preliminary data is not sufficient to justify these arguments.

Physically, if the WW relation would be a good description of $g_2$, then $g_2$ would provide no new information about QCD dynamics in transversely polarized hadrons since $g_2$ can be determined from $g_1$. This seems to be highly unlikely. On the other hand, if the twist-3 contribution in $g_2$ would be significantly large and, as the bag model calculations claimed there should be a huge cancellation between the twist-3 and the twist-2 contributions for $g_2$, then the classification in terms of twist may not be useful. This certainly contradicts the current understanding of DIS phenomena based on the twist expansion in $1/Q$.

Indeed, to get an intuitive picture of deep inelastic scattering in field theory, it is extremely helpful to keep close contact with parton ideas. However, partons were originally introduced as collinear, massless, on-mass shell objects which do not interact with each other. The question, then, is if one can generalize this concept and introduce field theoretic partons as non-collinear and massive (in the case of quarks) but still on-mass shell objects in interacting field theory? Light-front Hamiltonian description of composite systems which utilizes many body wavefunctions for the constituents allows us to precisely achieve this goal.

Within the light-front Hamiltonian description, we recently find that the pure transverse polarized structure function in DIS,

$$g_T(x, Q^2) = g_1(x, Q^2) + g_2(x, Q^2),$$

constitutes indeed a direct measurement of the QCD dynamics of chiral symmetry break-
ing [17]. The major contributions from all the sources (the quark mass effect, the quark transverse momentum contribution and the direct quark-gluon coupling dynamics) to $g_T$ are proportional to the simple transverse polarized parton distribution [1] due to the dynamical chiral symmetry breaking. Based on the hadronic bound state structure analysis, the so-called truly twist-3 contributions from the direct quark-gluon coupling and transverse quark momentum dynamics which have no simple parton interpretation should be rather unimportant. Our result indicates that the WW relation is not essential in determining $g_2$.

In this paper we shall compute the transverse polarized structure function in light-front Hamiltonian (namely time-ordering) perturbative QCD (pQCD) up to the order of $\alpha_s$ on a quark target. The result explicitly shows that the WW relation is strongly violated in pQCD by the quark mass effect (or chiral symmetry breaking dynamics).

2. Light-front time-ordered pQCD calculation of $g_2(x, Q^2)$

Recently, several authors have used different methods to calculate $g_2$ on the quark target to examine if it obeys the so-called Burkhardt-Cottingham (BC) sum rule [19,20] in pQCD. We find that BC sum rule is a trivial result in our formulation to all orders in pQCD (see later discussion). But so far, none has explicitly paid attention on the WW relation in pQCD. Of course, the result on quark target cannot be tested from experiments, but it can provide some basis for the theoretical understanding of the polarized structure functions themselves. This is not only because the structure functions on quark target can be exactly calculated in pQCD, but with the use of the factorization theorem, all we have understood from QCD about the hard processes extracted from DIS are also indeed based on these quark and gluon target structure function calculations [21]. Here we shall use the light-front Hamiltonian analysis of DIS structure functions [22] to directly calculate $g_1(x, Q^2)$ and $g_T(x, Q^2)$ in pQCD. In light-front Hamiltonian approach (utilizing old-fashioned perturbation theory) the physical picture is very transparent at each stage of the calculation since all the calculations within this framework are performed in the physical space. In addition, compared to the other approaches [19,20], the light-front formulation calculations are far more simple and straightforward.

Note that both in experiment and theory, with the target being polarized either in the longitudinal or transverse direction, one can only measure and calculate either the helicity structure function $g_1$ or the transverse polarized structure function $g_T$. The $g_2$ structure function which is well-defined from the hadronic tensor based on the Lorentz structure and the current conservation, however, cannot be measured and computed directly. It can only be extracted from the definition eq.(2). This is very much different from the calculation in the OPE framework [19,20]. Therefore, we shall first calculate $g_1$ and $g_T$ in the quark helicity state and the quark state polarized in transverse direction, respectively, up to $\alpha_s$ in pQCD. Then we can obtain the $g_2$ unambiguously from eq.(2).

As we have recently shown [17], in the large $q^-$ (BJL) limit [23], the leading contributions to the polarized structure functions in DIS are given by

$$g_1(x, Q^2) = \frac{1}{8\pi S^+} \int_{-\infty}^{\infty} d\eta e^{-i\eta x} \langle PS | \overline{\psi}(\xi^-) Q^2 \gamma^+ \gamma_5 \psi(0) + h.c. | PS \rangle,$$

$$g_T(x, Q^2) = \frac{1}{8\pi (S_+ - E_+ S^+)} \int_{-\infty}^{\infty} d\eta e^{-i\eta x} \langle PS | \overline{\psi}(\xi^-) Q^2$$
where $P$ and $S$ are the target four-momentum and polarization vector, respectively ($P^2 = M^2, S^2 = -M^2, S \cdot P = 0$), and $q$ is the virtual-photon four momentum ($Q^2 = -q^2, \nu = P \cdot q, x = Q^2/2\nu$). The parameter $\eta \equiv \frac{1}{2}P^+\xi^-$ with $\xi^-$ being the light-front longitudinal coordinate, and $Q$ the quark charge operator. This is a general expression for the DIS polarized structure functions in which the target has not been assigned to any specific Lorentz frame. If we let the target be in the rest frame, eqs. (3-4) are reduced to the expressions previously derived by Jaffe and Ji in the impulse approximation [12] and also by Efremov et al. in QCD field theoretic model [24].

From eqs. (3) and (4), one can easily obtain

$$g_2(x, Q^2) = \frac{1}{8\pi(S_- - P_-)} \int_{-\infty}^{\infty} d\eta e^{-i\eta x} \langle PS| \bar{\psi}(\xi^-) Q^2 \left( \gamma_\perp - \frac{S_+}{S_+} \gamma^+ \right) \gamma_5 \psi(0) + h.c.|PS \rangle,$$

This is again the most general matrix element expression for $g_2(x, Q^2)$. Note that if there is no anomalous contribution,

$$\langle PS| \bar{\psi} \gamma^\mu \gamma_5 \psi |PS \rangle \sim \Delta q_i S^\mu.$$

Then it is straightforward to show that

$$\int_0^1 dx g_2(x, Q^2) = \frac{1}{2(S_- - P_-)} \langle PS| \bar{\psi}(0) Q^2 \left( \gamma_\perp - \frac{S_+}{S_+} \gamma^+ \right) \gamma_5 \psi(0)|PS \rangle = 0$$

which is just the BC sum rule and it is valid to all orders in pQCD and in the full theory of hadron dynamics. But anomalous contribution may spoil this sum rule which we will not discuss in this paper.

Since we will focus on the WW relation in $g_2(x, Q^2)$ in this paper, we must individually calculate the contributions of $g_2(x, Q^2)$ from various sources: the quark mass contribution, the quark transverse momentum contribution and quark-gluon coupling effect, etc. It is perhaps the most convenient to perform such calculation in the framework of light-front field theory. As we can see, the matrix elements in eqs. (3-4) are expressed on the equal light-front time surface. On the light-front, $\psi$ is decoupled into $\psi = \psi_+ + \psi_\perp, \psi_\perp = \frac{1}{2}\gamma^0\gamma^+ \psi$, and the component $\psi_\perp$ is not physically independent: $\psi_\perp = \frac{1}{2\gamma}(iD_\perp + m)\psi_+$ in QCD, where $D_\perp = \partial_\perp - igA_\perp$ is the transverse component of the covariant derivative [23]. Hence the precise expression of eqs. (3-4) should be [17]

$$g_1(x, Q^2) = \frac{1}{4\pi S^+} \int_{-\infty}^{\infty} d\eta e^{-i\eta x} \langle PS|\psi_+^{1}(\xi^-) Q^2 \gamma_5 \psi_+(0) + h.c.|PS \rangle,$$

$$g_T(x, Q^2) = \frac{1}{8\pi(S_- - P_- S^+)} \int_{-\infty}^{\infty} d\eta e^{-i\eta x} \langle PS|\left( O_m + O_{k_\perp} + O_g \right) + h.c.|PS \rangle$$

$$= g_T^m(x, Q^2) + g_T^{k_\perp}(x, Q^2) + g_T^g(x, Q^2),$$

4
where

$$O_m = m_q \psi_+^\dagger(\xi^-) \mathcal{Q}^2 \gamma_\perp \left( \frac{1}{i \partial^+} - \frac{1}{i \partial^+} \right) \gamma_5 \psi_+(0),$$

$$O_{k_\perp} = -\psi_+^\dagger(\xi^-) \mathcal{Q}^2 \left( \frac{1}{i \partial^+} \bar{\Phi}_\perp + \bar{\Phi}_\perp \frac{1}{i \partial^+} \gamma_\perp + 2 \frac{P^+}{P^+} \right) \gamma_5 \psi_+(0),$$

$$O_g = g \psi_+^\dagger(\xi^-) \mathcal{Q}^2 \left( A_\perp(\xi^-) \frac{1}{i \partial^+} \gamma_\perp - \frac{1}{i \partial^+} A_\perp(0) \right) \gamma_5 \psi_+(0),$$

(10)

and $m_q$ and $g$ are the quark mass and quark-gluon coupling constant in QCD and $A_\perp = \sum_a T^a A_{a\perp}$ the transverse gauge field. The light-front expression of $g_T$ makes the physical picture clear: it explicitly shows the contributions associated with the quark mass, quark transverse momentum and quark-gluon coupling operators.

Now we calculate $g_1$ and $g_T$ in single quark states. Here the calculation is performed in the light-front time-ordering perturbation theory, in which the single quark state with fixed helicity can be expressed as

$$|k^+, k_\perp, \lambda\rangle = \mathcal{N} \left\{ b_\lambda^\dagger(k)|0\rangle + \sum_{\lambda_1, \lambda_2} \frac{dk^+_1 d^2 k_{1\perp}}{2(2\pi)^3} \frac{dk^+_2 d^2 k_{2\perp}}{2(2\pi)^3} 2(2\pi)^3 \delta^3(k - k_1 - k_2) \right.$$ 

$$\times \Phi^\lambda_{\lambda_1, \lambda_2}(x, k_\perp) b_{\lambda_1}^\dagger(k_1) a_{\lambda_2}^\dagger(k_2)|0\rangle + \cdots \right\},$$

(11)

where $\mathcal{N}$ is the normalization constant determined by

$$\langle k^+, k'_\perp, \lambda'|k^+, k_\perp, \lambda\rangle = 2(2\pi)^3 k^+ \delta_{\lambda, \lambda'} \delta(k^+ - k'^+) \delta^2(k_\perp - k'_\perp),$$

(12)

$b_\lambda^\dagger(k)$ and $a_\lambda^\dagger(k)$ the creation operators of quarks and gluons on the light-front which are defined by [23]:

$$\psi_+(x) = \sum_{\lambda} \chi_\lambda \int \frac{dk^+ d^2 k_{\perp}}{2(2\pi)^3} (b_\lambda(k) e^{-ikx} + d_\lambda^\dagger(k) e^{ikx}),$$

(13)

$$A_{a\perp}(x) = \sum_{\lambda} \int \frac{dk^+ d^2 k_{\perp}}{2(2\pi)^3} (\xi_\lambda^a(\lambda) a_\lambda(k) e^{-ikx} + h.c)$$

(14)

with

$$\{ b_\lambda(k), b_{\lambda'}^\dagger(k') \} = \{ d_\lambda(k), d_{\lambda'}^\dagger(k') \} = 2(2\pi)^3 \delta(k^+ - k'^+) \delta^2(k_\perp - k'_\perp),$$

(15)

$$\{ a_\lambda(k), a_{\lambda'}^\dagger(k') \} = 2(2\pi)^3 k^+ \delta(k^+ - k'^+) \delta^2(k_\perp - k'_\perp),$$

(16)

and $\chi_\lambda$ is the eigenstate of $\sigma_z$ in the two-component spinor of $\psi_+$ by the use of the following light-front $\gamma$ matrix representation [26]:

$$\gamma^0 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \gamma^3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} -i\bar{\sigma}^i & 0 \\ 0 & i\bar{\sigma}^i \end{bmatrix}$$

(17)

with $\bar{\sigma}^1 = \sigma^2, \bar{\sigma}^2 = -\sigma^1$ and $\xi^a_\lambda(\lambda)$ the polarization vector of transverse gauge field. The amplitude $\Phi_{\lambda_1, \lambda_2}^\lambda(x, k_\perp)$ in eq.(11) is
\[
\Phi_{\lambda_1\lambda_2}(x, \kappa) = -gT^a \frac{x(1-x)}{\kappa^2 + m_q^2(1-x)^2} \chi^\dagger \lambda \left\{ 2 - \frac{\kappa^i}{1-x} \right. \\
+ \frac{1}{x} (\sigma_{\perp} \cdot \kappa) \sigma^i - im_q \sigma^i \frac{1-x}{x} \left. \right\} \chi \lambda \varepsilon^{i*}(\lambda). \tag{18}
\]

Note that the \( m_q \) dependence in the above wave function has arisen from the helicity flip part of light-front QCD Hamiltonian. This is an essential term in the determination of transverse polarization dynamics. The transverse polarized quark target in the \( x \)-direction can then be expressed by

\[
|k^+, k, S^1\rangle = \frac{1}{\sqrt{2}} (|k^+, k, \uparrow\rangle \pm |k^+, k, \downarrow\rangle) \tag{19}
\]

with \( S^1 = \pm m^R_q \), and \( m^R_q \) is the renormalized quark mass.

Without the QCD correction (i.e., for the free quark state), it is easy to show that

\[
g_T(x) = g^m_T(x) = \frac{e^2_q}{2} \frac{m_q}{S^1} \delta(1-x) = \frac{e^2_q}{2} \delta(1-x), \quad g^{k^+}_T(x) = 0 = g^T_T(x). \tag{20}
\]

Here \( m_q/S^1 = 1 \) since the renormalized mass is the same as the bare mass at the tree level of QCD. We see that only the quark mass term contributes to \( g_T \) in eq.(19). The quark transverse momentum term alone does not contribute to \( g_T \) since it cannot cause helicity flip in the free theory. This result indicates that physically the dominant contributions to \( g_T \) is not controlled by the twist classification. A direct calculation of \( g_1(x) \) from eq.(18) in the free quark helicity state will immediately lead to the well-known solution:

\[
g_1(x) = \frac{e^2_q}{2} \delta(1-x). \tag{21}
\]

Thus, for the free quark, we have

\[
g_2(x) = g_T(x) - g_1(x) = 0. \tag{22}
\]

It is obvious that for free theory, the Burkhardt-Cottingham (BC) sum rule is trivially obeyed. But as we can see (as it has also been previously noticed) the Wandzura-Wilczek relation,

\[
g_2(x) = -g_1(x) + \int_x^1 dy g_1(y), \tag{23}
\]

is not satisfied in free theory. Jaffe and Ji argued that the WW relation is violated in free theory because the quark mass contribution in \( g_2(x) \) which is thought to be a twist-3 contribution \([12]\) cancels the \( g_2^{ww} \) contribution (a twist-2 contribution). This has become a common understanding why WW relation is not satisfied in free theory. However, this may not be a proper explanation because if one ignored the quark mass contribution (naively started with a massless quark theory), then there would be no transverse polarization. In other words, one cannot predefine \( g_T(x) \) in a massless quark theory. Thus the investigation of \( g_2 \) in terms of a massless quark theory is totally ambiguous in this way. On the other
hand, if one started with a massive quark theory, the resulting \( g_T \) would be independent of quark mass (the mass dependence does not occur for a quark target). One can then take the chiral limit (\( m_q = 0 \)) at the end of calculation to obtain the unambiguous \( g_T \) in the massless theory which is the same as that in the massive theory. Thus, there is no so-called cancellation between the twist-two contribution and the quark mass contribution in the determination of \( g_2 \) in the free theory. Eq.\( \text{(22)} \) just indicates that there does not exist the WW relation, at least in free theory. Such a picture is not manifest in OPE approach.

Next, we consider the QCD corrections up to order \( \alpha_s \), where the quark-gluon interaction is explicitly included. We find that all the three terms in eq.\( \text{(9)} \) have nonzero contribution to \( g_T \),

\[
\begin{align}
g_T^m(x, Q^2) &= \frac{e_q^2 m_q}{2 S^1} \left\{ \delta(1-x) + \frac{\alpha_s}{2\pi} C_f \ln \frac{Q^2}{\mu^2} \left[ \frac{2}{1-x} - \delta(1-x) \int_0^1 dx' \frac{1+x'^2}{1-x'} \right] \right\}, \\
g_T^{k\perp}(x, Q^2) &= -\frac{e_q^2 m_q \alpha_s}{2 S^1 2\pi} C_f \ln \frac{Q^2}{\mu^2} (1-x), \\
g_T^q(x, Q^2) &= \frac{e_q^2 m_q \alpha_s}{2 S^1 2\pi} C_f \ln \frac{Q^2 \delta(1-x)}{\mu^2} 2,
\end{align}
\]

where we have set a hadronic scale such that \( |k\perp|^2 >> \mu^2 >> (m_q)^2 \) and \( \mu \) is the hadronic factorization scale for separating the “hard” and “soft” dynamics of QCD. As a matter of fact, the above result represents purely the pQCD dynamics. The integral term in eq.\( \text{(24)} \) comes from the dressed quark wavefunction normalization constant \( \mathcal{N} \) in eq.\( \text{(11)} \) (corresponds to the virtual contribution in the standard Feynman diagrammatic approach). It shows that up to the order \( \alpha_s \), the matrix elements from \( O_{k\perp} \) (quark transverse momentum effect) and \( O_q \) (quark-gluon interaction effect) in eq.\( \text{(9)} \) are also proportional to quark mass. In other words, the transverse quark momentum and quark-gluon coupling contributions to \( g_T(x, Q^2) \) arise from quark mass effect. Explicitly, these contributions arise from the interference of the \( m_q \) term with the non-\( m_q \) dependent terms in the wave function of eq.\( \text{(18)} \) through the quark transverse momentum operator and the quark-gluon coupling operator in the \( g_T \) expression. This result is not surprising since, as we have pointed out, the pure transverse polarized structure function measures the dynamical effect of chiral symmetry breaking \( \text{[17]} \). Physically only these interferences related to quark mass can result in the helicity flip (i.e., chiral symmetry breaking) in pQCD so that they can contribute to \( g_T(x, Q^2) \). From this result, we may see that only the operators themselves or their twist structures may not give us useful information about their importance in the determination of structure functions.

Combining the above results together, we obtain

\[
\begin{align}
g_T(x, Q^2) &= \frac{e_q^2 m_q}{2 S^1} \left\{ \delta(1-x) + \frac{\alpha_s}{2\pi} C_f \ln \frac{Q^2}{\mu^2} \left[ \frac{1+2x-x^2}{(1-x)_+} + 2\delta(1-x) \right] \right\}.
\end{align}
\]

Note that in the above solution, \( m_q \) is the bare quark mass, while the dressed quark polarization \( S^1 = m_q^R \), and up to order \( \alpha_s \),
\[ m^R_q = m_q \left( 1 + \frac{3}{4\pi} \alpha_s C_f \ln \frac{Q^2}{\mu^2} \right). \]  

(28)

We must emphasize that on the light-front there are two mass scales in the QCD Hamiltonian, one is proportional to \( m^2_q \) which does not violate chiral symmetry, and the other is proportional to \( m_q \) which we discuss here and is associated with explicit chiral symmetry breaking in QCD [27]. An important feature of light-front QCD is that the above two mass scales are renormalized in different ways even in the perturbative region. The renormalization of \( m^2_q \) in pQCD is different from the above result, the details of which can be found in our previous work [25]. With this consideration, we have

\[ g_T(x, Q^2) = \frac{e_q^2}{2} \left\{ \delta(1 - x) + \frac{\alpha_s}{2\pi} C_f \ln \frac{Q^2}{\mu^2} \left[ \frac{1}{(1 - x)_+} + \frac{1}{2} \delta(1 - x) \right] \right\}. \]  

(29)

The final result is independent quark mass but we have to emphasize again that one must start with massive quark theory. Otherwise, there is no definition for \( g_T \) at the beginning.

By a similar calculation for \( g_1 \), we have

\[ g_1(x, Q^2) = \frac{e_q^2}{2} \left\{ \delta(1 - x) + \frac{\alpha_s}{2\pi} C_f \ln \frac{Q^2}{\mu^2} \left[ \frac{1 + x^2}{(1 - x)_+} + \frac{3}{2} \delta(1 - x) \right] \right\}, \]  

(30)

which is independent of quark mass in any step of calculation. Thus, up to order \( \alpha_s \), we find \( g_2 \) for a quark target

\[ g_2(x, Q^2) = \frac{e_q^2}{2} \frac{\alpha_s}{2\pi} C_f \ln \frac{Q^2}{\mu^2} \left[ 2x - \delta(1 - x) \right]. \]  

(31)

It is easy to check that the above result of \( g_2(x, Q^2) \) obeys the BC sum rule,

\[ \int_0^1 dx g_2(x, Q^2) = 0, \]  

(32)

as is expected. However, it also shows that the WW relation is violated. It is not possible to let \( g_2 \) satisfy the WW relation by ignoring quark mass effect and/or quark-gluon interaction contributions in pQCD. In the physical space, one cannot separate quark mass contribution and quark-gluon coupling contribution and ignore the quark mass effect in addressing the transverse polarization processes.

3. Comparison with the OPE formulation

As is well-known, the WW relation was originally derived from OPE method by ignoring the twist-3 contribution to \( g_2(x, Q^2) \). The OPE method has been thought as a fundamental approach in the investigation of DIS structure functions. Using the OPE, one can show that the moments of the polarized structure functions \( g_1 \) and \( g_2 \) can be expressed by

\[ \int_0^1 dx x^{n-1} g_1(x, Q^2) = \frac{1}{2} a_n, \quad n = 1, 3, 5, \ldots, \]  

(33)

\[ \int_0^1 dx x^{n-1} g_2(x, Q^2) = -\frac{1}{2} n \frac{a_n - d_n}{n}, \quad n = 3, 5, \ldots, \]  

(34)
where $a_n$ and $d_n$ are determined by the twist-2 and twist-3 hadronic matrix elements. Note that OPE is incomplete in describing $g_2$ since it cannot define the first moment of $g_2$. In fact, the OPE only defines the odd moments of the polarized structure functions.

From eqs.(33-34),

$$
\int_0^1 dx x^{n-1} \left\{ g_1(x, Q^2) + g_2(x, Q^2) \right\} = \frac{1}{2n} a_n + \frac{1}{2} \frac{n-1}{n} d_n, \quad n = 3, 5, \ldots,
$$

(35)

If one assumed that eq.(35) is valid for all integers $n$, then using the convolution theorem for Mellin transformation, one could find

$$
g_2(x, Q^2) = -g_1(x, Q^2) + \int_x^1 \frac{dy}{y} g_1(y, Q^2) + \bar{g}_2(x, Q^2),
$$

(36)

where $\bar{g}_2(x, Q^2)$ is the so-called twist-3 contribution that is related to $d_n$. If one further assumed that the twist-3 contribution $\bar{g}_2$ can be ignored, then eq.(36) would be reduced to $g_2^{WW}(x, Q^2)$ that obeys the famous WW relation. Obviously, to obtain the WW relation, one has to make some unjustified assumptions.

Now we shall examine whether one can assume the validity of eq.(35) for all integers $n$ and how big the twist-3 contribution can be in pQCD.

In our approach, instead of the moments, we directly calculate the structure functions $g_1$ and $g_2$. Then using the identity,

$$
\int_0^1 dx x^{n-1} \frac{1}{(1-x)^+} = -\sum_{j=1}^{n-1} \frac{1}{j},
$$

(37)

we calculate from eqs.(30-29) all the moments for $g_1(x, Q^2)$ and $g_T(x, Q^2)$ up to $\alpha_s$ in pQCD. The result is

$$
\int_0^1 dx x^{n-1} g_1(x, Q^2) = \frac{e_q^2}{2} \left\{ 1 + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2} \left[ -\frac{1}{2} + \frac{1}{n(n+1)} - 2 \sum_{j=2}^{n} \frac{1}{j} \right] \right\},
$$

(38)

$$
\int_0^1 dx x^{n-1} g_T(x, Q^2) = \frac{e_q^2}{2} \left\{ 1 + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2} \left[ -\frac{3}{2} + \frac{1}{n} + \frac{1}{n+1} - 2 \sum_{j=2}^{n} \frac{1}{j} \right] \right\},
$$

(39)

Since the above result is directly obtained from pQCD without using the OPE, it is valid for all the integers $n$. It is easy to see that the first moment of $g_1$ is the same as that of $g_T$ for quark target,

$$
\int_0^1 dx g_1(x, Q^2) = \int_0^1 dx g_T(x, Q^2) = \frac{e_q^2}{2},
$$

(40)

and is independent on $Q^2$, as is expected in the leading order pQCD. Again from the above result, we see that the BC sum rule is satisfied in pQCD.

Comparing eqs.(38-39) with eqs.(33-34), we obtain

$$
a_n = e_q^2 \left\{ 1 + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2} \left[ -\frac{1}{2} + \frac{1}{n(n+1)} - 2 \sum_{j=2}^{n} \frac{1}{j} \right] \right\},
$$

(41)

$$
d_n = e_q^2 \left\{ 1 + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2} \left[ -\frac{3}{2} + \frac{1}{n} - 2 \sum_{j=2}^{n} \frac{1}{j} \right] \right\}.
$$

(42)
The above result is valid for all the integers \(n\). In other words, when \(g_2(x, Q^2)\) is written by eq.\((36)\), \(a_n\) and \(d_n\) are the same order for all the \(n,\)

\[
a_n - d_n = e^2 \alpha_s \frac{Q^2}{2\pi} \ln \left(\frac{Q^2}{\mu^2} \left[1 - \frac{1}{n + 1}\right]\right) \tag{43}
\]

This shows that \(\bar{g}_2(x, Q^2)\) is comparable with \(g_{2W}(x, Q^2)\) so that the WW relation is badly violated not only in free theory but also in pQCD by the quark mass effects. It is easy to check that in terms of moments, the result presented here is consistent with the result by Altarelli et al.\(^{[20]}\) except for a finite part which we did not include here. This finite contribution comes from the region \(0 \leq |k_\perp|^2 \leq \mu^2\) [see the discussion after eq.\((26)\)] which is the soft parton domain, and is not part of pQCD contribution in the standard physical picture. But it is straightforward to include this contribution and the result does not change the conclusion in this paper.

It is worth noticing that here \(d_n\) only represents a part of twist-3 contribution in the real hadronic process that is associated with quark mass effect (including a part of contributions from quark-gluon coupling and quark transverse momentum operators). The possible contributions purely related to the quark-gluon coupling and intrinsic transverse quark momentum dynamics do not occur in the case of using quark target. Is there any possibility that these possible twist-3 contributions in nucleon targets may cancel the quark mass effect so \(a_n\) can become dominant in hadronic processes? None can provide a definite answer for this possibility at the current status of QCD. However, as we have pointed out recently\(^{[17]}\), based on the hadronic bound state structure, the pure twist-3 contribution related to the direct quark-gluon coupling dynamics and the quark transverse momentum effect is rather small since it is proportional to the off-diagonal hadronic matrix elements in different Fock states. Meanwhile, the pure mass effect in pQCD may also be suppressed by the factor \(\frac{m_q}{S_z} \rightarrow \frac{m_q}{M}\) in the real hadronic process, where \(M\) is the hadronic mass. However, we find that the physical picture of the transverse polarized structure function is that it essentially describes the chiral symmetry breaking dynamics in DIS. Accompanying with the above mass effect, there is a nonperturbative QCD effect at the factorization scale that is generated by the dynamical chiral symmetry breaking. This effect cannot be suppressed by the factor \(\frac{1}{M}\). The physics of the structure function \(g_2\) is not described by the WW relation, nor it measures the higher-twist dynamics. It has a simple parton picture and is characterized by the chiral symmetry breaking dynamics that is not described by the WW relation\(^{[17]}\).

In conclusion, since it was proposed twenty years ago, the WW relation has not been seriously examined in pQCD. In this paper we explicitly show how the WW relation is badly violated even in pQCD to the order \(\alpha_s\). It is hard to think how it can be a good representation of \(g_2\) if it cannot be satisfied in free theory and in pQCD.

A.H would like to thank Prakash Mathews and K. Sridhar, and W.M.Z would like to thank H. Y. Cheng, C. Y. Cheung and H. L. Yu for useful discussions. This work is partially supported by NSC86-2112-M-001-020 and NSC86-2816-M001-009R-L(WMZ).
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