Minimal Potentials with Very Many Minima

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Abstract

We demonstrate, by construction, that simple renormalizable matrix potentials with $S_N$, as opposed to $O(N)$, symmetry can exhibit an exponentially large number of inequivalent deep local minima.

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There are many situations where behavior of great complexity arises, or is thought to arise, from simple underlying equations. Extensively studied cases include chaos, turbulence, and spin glasses. Chaos and turbulence involve long-term dynamics and extended spatial structures, while spin glasses involve an element of randomness. Here we will analyze a much simpler case (the simplest known to us) involving a static, deterministic, and very symmetrical system, wherein simple equations exhibit quite a complicated space of solutions. In particular, we present a simple class of potentials in \( n \)-component order parameters, whose number of local minima unrelated by symmetry grows exponentially in \( n \). Our model is closely related to ones commonly used in studying large \( N \) limits of quantum field theory [1], differing only in that the assumption of some continuous symmetry among the fields (e.g., \( O(N) \)) is replaced by a discrete permutation symmetry (basically \( S_N \)). Of course, it is just such permutation symmetries which arise in studies of quenched disorder by the replica method, so there is a close connection to that branch of spin glass theory [2, 3]. In some circumstances the flexibility afforded by imposing less symmetry might allow better extrapolations than the traditional one, in the sense that \( 1/N \) corrections might be made smaller, and more complex behaviors captured.

To put the later results in perspective, and to highlight the minimal requirements for complexity in our framework, let us first consider an example that does not work. Suppose that we have \( N \) order parameters \( \phi_i \), for \( i = 1 \ldots N \). The most general renormalizable (i.e., no more than quartic) potential symmetric under the \( S_N \) permuting these parameters and under a change in all their signs simultaneously is

\[
V(\vec{\phi}) = \mu \sum_i \phi_i^2 + \alpha \sum_{i,j} \phi_i \phi_j + \beta_1 \sum_i \phi_i^4 + \beta_2 \sum_{i,j} \phi_i^3 \phi_j + \beta_3 \sum_{i,j,k} \phi_i^2 \phi_j \phi_k + \beta_4 \sum_{i,j,k,l} \phi_i \phi_j \phi_k \phi_l .
\]

(1)

Varying with respect to \( \phi_a \), we find that for an extremum \( \phi_a \) must obey a cubic equation. The equation can be written in a form that is the same for every value of \( a \). That is, for a particular fixed minimum, the equation can be written as a cubic equation in \( \phi_a \), so that the coefficients of the various terms (when evaluated as constant numbers for the particular minimum in question) are the same for all values of \( a \). This cubic equation has at most three real roots. Of these, at most two are local minima. Therefore, for any local minimum, the different components of the order parameter will take at most two distinct values. Thus for large values of \( N \) many of the components will be equal. Let us suppose there are \( n_1 \) components with value \( r_1 \) and \( n_2 \) components with value \( r_2 \), where \( n_1 + n_2 = N \), and \( n_2 \leq n_1 \). Then (for given \( n_1, n_2 \)) the conditions for an extremum will be two polynomial equations of degree 3 in the variables \( r_1 \), and \( r_2 \). In general, these will have at most 9 solutions. Taking into account that there are at most two solutions when \( n_2 = 0 \), for generic values of the parameters \( \mu, \alpha, \beta_1 \ldots \beta_5 \) in the potential we readily bound the number of distinct minima depths by \((9N+4)/2\) for \( N \) even, and \((9N-5)/2\) for \( N \) odd. We expect that with more care this number could be

1
further reduced. Non-generic values presumably correspond either to fine tuning of the parameters, which is not physically realistic, or to enhanced symmetry, which renders mathematically distinct solutions physically equivalent. In any case, one does not find here a straightforward possibility for the sort of exponential growth in the number of physically distinct minima that we will encounter shortly.

The next logical step after the vector order parameter is to consider matrices. For simplicity we impose that our matrices are symmetric, so that $M_{ij}$ and $M_{ji}$ are actually the same variable. Each index runs from 1 to $N$, implying that there are $N(N + 1)/2$ independent order parameters in the matrix $M$. We define the $a^{th}$ “row-column” of a matrix to be the union of the $a^{th}$ row, and the $a^{th}$ column of the matrix; it is the set of all $M_{ia}$’s and all $M_{ai}$’s for all $i$’s.

We assume the potential is symmetric under $M \rightarrow -M$, and under permutation of the values of the labels; none of the row-columns is to be singled out in any way. For example, one can take a matrix, and every time one sees index 3 in the matrix, replace it with index 7, and vice-versa. Thus entries $M_{37}$, and $M_{73}$ stay the same, while the entries $M_{33}$ and $M_{77}$ get interchanged, and for all other $i$’s, $M_{3i}$ swaps with $M_{7i}$, and $M_{3i}$ swaps with $M_{7i}$. We refer to this symmetry as the “row-column exchange symmetry,” or simply “exchange symmetry” of the potential.

Given these constraints, the allowed quadratic terms in a potential are:

$$M_{ii}M_{ii}, M_{ii}M_{ij}, M_{ii}M_{jj}, M_{ij}M_{ij}, M_{ii}M_{jk}, M_{ij}M_{ik}, \text{ and } M_{ij}M_{kl},$$

(2)

Here, and hereafter, summation over all indices, even if they are not repeated, is assumed unless explicitly stated otherwise.

There are many allowed quartic terms, and we will not write them all out here. But for future reference note that terms as highly structured as $M_{ij}M_{jk}M_{kl}M_{ii}$, and $M_{ij}M_{ij}M_{jk}M_{kl}$ are fair game now.

We will now demonstrate, by explicit construction, exponential proliferation of inequivalent local minima in this case. Our strategy will be to use a subset of the allowed terms to construct a very simple potential with many isolated local minima. These will be equivalent under a symmetry of the simplified potential, but not under the smaller symmetry of our full class of allowed potentials. Then we shall lift the degeneracy (and physical equivalence) of these minima in a controlled way by perturbing with additional allowed terms, in such a way that they remain local minima.

To begin, we form what we call a plastic-soda-bottle-bottom potential out of the allowed terms. (In contrast to the classic wine-bottle potential, the plastic-soda bottle potential in two variables has four symmetrically arranged dips.) This takes the form:

$$V(M) = a \left[ \sum_{i,j} (1 - M_{ij}^2) + \sum_i (1 - M_{ii}^2) \right]^2 + b \left[ \sum_{i,j} (1 - M_{ij}^2)^2 + \sum_i (1 - M_{ii}^2)^2 \right],$$

(3)

where $a, b > 0$ are arbitrary, and no summation over the indices is assumed within the
explicitly stated sums. All the local minima lie at:

\[
\begin{bmatrix}
\pm 1 & \pm 1 & \ldots & \pm 1 \\
\pm 1 & \pm 1 & & \pm 1 \\
& \ddots & \ddots & \\
& & \ddots & \\
\pm 1 & \pm 1 & \ldots & \pm 1
\end{bmatrix}
\]  

(4)

They are all related by the accidental symmetry of the plastic-soda-bottle potential, which allows both independent changes in the signs of individual components and interchange of any two components (not just row-columns).

Now we can experiment numerically by adding in more of the allowed terms. The positions of the minima, and their depths, will change as we vary the amounts of the various small terms we are adding. We will be careful that the terms we are adding are small enough so as not to destabilize any minimum, nor change the sign of any of the order parameters at the position of any of the minima. Let us add terms of the form \(M_{ij}M_{jk}M_{kl}M_{li}\) and \(M_{il}M_{ij}M_{jk}M_{kl}\) with small coefficients, with \(N = 2, 3, 4, 5, 6\), and track the depth of each minimum numerically. Then we can count the number of distinct numerical values for the potential at the perturbed minima. Of course, local minima with distinct energies must be physically inequivalent, i.e. unrelated by an underlying symmetry. The results are exhibited in Figure 1.

In Figure 1, it appears that the number of distinct minima classes grows exponentially in the number of the order parameters, in response to only these two particular terms for the perturbations. Now we shall discuss how this proliferation can be understood theoretically.

First, let us show that the number of minima which are not related by the exchange symmetry, or by the \(\mathbf{M} \leftrightarrow \mathbf{M}\) symmetry grows exponentially in the number of the order parameters. We focus our attention on one very particular subset of all minima, and prove that the logarithm of the number of minima in this subset that are not related by any of the allowed symmetries grows quadratically with \(N\). The subset in question consists of all minima that can be written in the following form:

\[
\begin{bmatrix}
\mathbf{B} & \mathbf{A} \\
\mathbf{A}^T & \mathbf{C}
\end{bmatrix}
\]

(5)

where when \(N\) is even, all matrices \(\mathbf{A}, \mathbf{B}, \) and \(\mathbf{C}\) have \(N/2\) rows and \(N/2\) columns; and when \(N\) is odd, \(\mathbf{B}\) has \((N + 1)/2\) rows and \((N + 1)/2\) columns, while \(\mathbf{C}\) has \((N - 1)/2\) rows and \((N - 1)/2\) columns; consequently, \(\mathbf{A}\) has \((N + 1)/2\) rows, and \((N - 1)/2\) columns. Furthermore, we require that the matrix \(\mathbf{B}\) has only positive values on the diagonal (from now on denoted by a +,) while the matrix \(\mathbf{C}\) has only negative values on the diagonal (from now on denoted by a −,) while any other entry of \(\mathbf{B}\), and \(\mathbf{C}\) is “free” to be either a +, or a −. Note that the logarithm of the number of such “free” entries grows quadratically in \(N\), or linearly in the number of order parameters, for
Figure 1: On a semi-log plot, the number of distinct minima classes versus the number of order parameters appears as a straight line. This is evidence that the number of distinct minima grows exponentially with the number of order parameters.

large $N$. Finally, all the entries of the matrix $A$ are fixed; if $N$ is even, all elements on or above the diagonal are $+$'s, while all the elements below the diagonal are $-$'s; if $N$ is odd, entry $A_{kl}$ is $+$ if $k \leq l$, and $-$ otherwise.

Concretely, if $N = 8$, an element of our subset looks like:

$$
\begin{bmatrix}
+ & ? & ? & ? & + & + & + & + \\
? & + & ? & ? & - & + & + & + \\
? & ? & + & ? & - & - & + & + \\
? & ? & ? & + & - & - & - & + \\
+ & - & - & - & ? & ? & ? & ? \\
+ & + & - & - & ? & ? & ? & ? \\
+ & + & + & - & ? & ? & ? & ? \\
+ & + & + & + & ? & ? & ? & - \\
\end{bmatrix},
$$

where ? can be either a $+$ or a $-$, as long as it is consistent with the requirement $M_{ij} = M_{ji}$; i.e. the elements below the diagonal are fixed once we pick the elements above the diagonal. If $N = 7$, an element of our subset looks like:
with same requirements as for the case \( N = 8 \).

The reason we focus our attention on this particular subset is that none of its elements are related by the symmetries of our class of potentials, as we now discuss. The proof proceeds in two steps. First, ignoring the existence of the \( \mathbf{M} \leftrightarrow \mathbf{M} \) symmetry, we prove that exchange symmetry alone cannot change one member of subset into another. Then we prove that the \( \mathbf{M} \leftrightarrow \mathbf{M} \) symmetry does not cause any further problems.

We propose a “painting scheme” to keep track where each entry of the matrix moves during the exchange process. This scheme also makes it easier to visualize what is going on. Paint each row-column with a different color. Consequently, each \( M_{ij} \) for \( i \neq j \) is covered with two layers of distinct paints; \( M_{ii} \) is covered with two layers of the same paint. Make sure to use “light colors” if \(+\) is on the diagonal entry of the row-column you are painting, and “dark colors” if \(−\) is on the diagonal entry. Each particular entry \( M_{ij} \) for \( i \geq j \) is now labeled uniquely by its two colors; of course, \( M_{ij} \) has the same colors as \( M_{ji} \), which suits us because they are the same variable anyway.

Say the 2\(^{nd}\) row-column is yellow, and the 5\(^{th}\) row-column is green. Exchanging indices 2 and 5 makes the 5\(^{th}\) row-column yellow, and the 2\(^{nd}\) row-column green. Using the coloring scheme, it is easy to keep track where each particular entry moved during the exchange. Say the 11\(^{th}\) row-column was blue initially, and we want to know where the entry \( M_{2,11} \) ended up after the exchange; we look for the square of the matrix that is covered precisely by the yellow, and the blue paint, and conclude that the entry in question is now at the position \( M_{5,11} \).

Note that every entry of the \( \mathbf{B} \) matrix initially contains only light colors, while the matrix \( \mathbf{C} \) contains only dark colors. In contrast, every entry of matrix \( \mathbf{A} \) is painted with precisely one light, and one dark color.

Now, we start with a matrix \( \mathbf{M}_1 \) and permute it into a matrix \( \mathbf{M}_2 \) so that both of these matrices are elements of our preferred subset. First, note that all rows of \( \mathbf{A}_2 \) and \( \mathbf{B}_2 \) are painted with light colors, while all columns of \( \mathbf{A}_2 \) and \( \mathbf{C}_2 \) are painted with darker colors; this is so because \( \mathbf{B}_2 \) has only +’s on the diagonal, while \( \mathbf{C}_2 \) has only −’s on the diagonal. Therefore, the set of all entries of \( \mathbf{A}_1 \) is exactly the same as the set of all entries of \( \mathbf{A}_2 \); only these entries are such as to have exactly one light, and one dark color. Suppose that the light colors we have are: yellow, orange, red and pink, and suppose \( N = 8 \). Furthermore, suppose that \( \mathbf{A}_1 \) has the 1\(^{st}\) row yellow, the 2\(^{nd}\) row orange, etc. Since two entries that were in the same row-column before the exchanges stay in the same row-column after the exchanges, the only way to get
exactly 4+’s in the 1st row of $A_2$ is to have the 1st row of $A_2$ yellow. This implies that the 1st row-column of $M_2$ is yellow. Furthermore, the only way to have exactly 3+’s in the 2nd row of $A_2$ is to have the 2nd row of $A_2$ orange, implying that the 2nd row-column of $M_2$ is orange, etc. This way we determine the position of all light colors, and thereby determine uniquely everything about the matrix $B_2$. In a similar manner, we determine everything about the matrix $C_2$. Therefore $A_1 = A_2$, $B_1 = B_2$, and $C_1 = C_2$, so $M_1 = M_2$, as we sought to prove, since this implies that there is no symmetry that relates any two elements of this particular subset.

The particular case $N = 8$ is just illustrative; everything we said generalizes immediately to any even $N$. Furthermore, everything we said applies with only minor modifications to the case where $N$ is odd.

Now we prove that during the whole process of transforming matrix $M_1$ into matrix $M_2$, one always has to multiply the matrix with $-1$ a total of an even number of times. The way to see this differs a bit in the case when $N$ is even, and when $N$ is odd. When $N$ is odd, we have to end up with less −’s than +’s on the diagonal of $M_2$, which is the same as for the diagonal we started with; however, none of the entries of the diagonal ever moves off the diagonal during the process. Similarly, in the case $N$ is even, we have to end up with less −’s than +’s in the matrix $A_2$, and we already proved during Step 1 of this proof that $A_1$ consists of the same set of elements as $A_2$. Therefore, the matrix has to be multiplied with $-1$ an even number of times during the process, both when $N$ is odd, and when $N$ is even. Since the operation of multiplication with $-1$ treats all the elements of the matrix indiscriminately, it does not matter at all when during the process we perform these operations; in particular, we could instead perform all of them before doing anything else; but then, we might as well not do them at all, since multiplying the matrix with $-1$ an even number of times leaves the matrix unchanged.

This concludes our proof that the number of local minima of the special potential that are unrelated by any symmetry of the general potential grows exponentially in the number of the order parameters for large $N$.

Physical intuition suggests that unless two minima have a very good reason to have the same depths (e.g. an underlying symmetry of the full potential), generically one would not expect them to have equal depths. Since the potentials of our class support an exponentially large number of minima unrelated by symmetry, we expect that such potentials will generally have a number of distinct depths at local minima that is exponential in the number of order parameters, unless the equations that determine them are insensitive to the symmetry-breaking structure. That of course is the behavior indicated by our numerical work, and it differs markedly from the earlier, vector case. The following consideration makes it plausible, though it does not prove, that the degeneracy among the physically distinct minima, which occurs for our initial plastic-soda-bottle potential, is lifted by perturbation with certain of the allowed potential terms. The point is that the derivative with respect to $M_{ij}$ of a term like $M_{ab} M_{bc} M_{cd} M_{da}$, that is $M_{ab} M_{bc} M_{cj}$, probes the whole structure of $M$ in a way
that is significantly different for each value of \(ij\). Thus, unlike in the vector case, here the response to the perturbation in principle knows enough about (contains enough independent measures of) the order parameter to encode its detailed structure. In the vector case, one would need to go to \(N^{th}\) order terms, of the type \(\phi_1\phi_2...\phi_N\), or higher to encounter similar sensitivity.

To illustrate this point further, we now examine the properties of some particular cases of our potentials, thus showing concretely how the various minima become physically inequivalent.

To keep things as simple as possible, we just add a tiny perturbation to the initial plastic-soda-bottle potential. Because the perturbations are tiny, we are justified in evaluating the changes in the potential only to the first order; we say that the depth of each minimum moves by whatever the perturbation we are adding evaluates to at the original position of the minimum in question; these positions are given in (4). To the first order, the degeneracy can not be broken into an exponentially large number of minima classes; for example, a quartic term that involves as many as 8 different indices can assume at most \(O(N^8)\) different values when evaluated at the positions given in (4), since it is a sum of \(N^8\) terms each of which can be either a +1, or a −1. Even if we add all the allowed terms, each multiplied by an arbitrary tiny coefficient, at lowest order we still have at best a power law breaking of the degeneracy.

Nevertheless, the number of distinct minima one can in principle get by analyzing only to the first order is quite large, especially if we include many allowed terms to create the perturbation. Furthermore, for small perturbations, the expectation values of different operators will typically not differ significantly if we evaluate the changes in the depths only to the first order, as opposed to evaluating them exactly. Moreover, in practice we sort the minima into energy bins of finite width in our plots. If our perturbation breaks the degeneracy to the first order into say \(O(N^8)\) distinct minima classes and \(N = 6\), we have in principle up to \(\sim 10^6\) distinct minima. Since our plots typically involve 200 bins, it does not matter for the plots that we evaluate the depth changes to the first order only instead of calculating them exactly.

Typical results are displayed in Figure 2. Plots A and B from that figure demonstrate that one can get quite a rich structure by using only a few of the allowed terms. Furthermore, the breaking of degeneracy is quite extensive even when we work to first order only. When we include more than one perturbative term, the degeneracy breaking is even bigger, producing quite a rich structure even at first order. This is visible in plots C and D of Figure 2, where we included 20 of the allowed terms, with random coefficients multiplying them. The plot D has a very high resolution of almost 40000 bins for the whole plot; both plots are for exactly the same potential. Note that in these plots we count the total number of minima, so that minima are counted as distinct even if they are related by a symmetry. Thus, much of the degeneracy is intrinsic, and will not be broken in any order of approximation.

An example of the general sort of structure described here arises in the analysis of QCD with many flavors of quarks at high density. For three flavors the color-flavor
Figure 2: Plots of bin occupation numbers versus the changes in minima depths, evaluated to the first order in the small perturbations. All plots are for $N = 6$. The $x$-axes are in arbitrary units. The width of the bins in plots A,B, and C is $10^{-2}$ energy units, and in plot D is $2.5 \times 10^{-4}$ energy units. The term added in plot A was $-M_{ii}M_{ij}M_{jk}M_{kl}$, while in plot B it was $-M_{ij}M_{jk}M_{kl}M_{li}$. To create plot C, we add 20 different terms, with random coefficients multiplying them. Plot D is exactly the same as plot C, except with much higher bin resolution.

locking condensate takes the form

$$
\langle q_\alpha^a q_\beta^b \rangle = U_\alpha^\gamma U_\beta^\delta (\kappa_1 \delta_\alpha^\gamma \delta_\beta^\delta + \kappa_2 \delta_\gamma^\gamma \delta_\delta^\gamma) ,
$$

where the Greek indices refer to color and the Latin to flavor. For present purposes we are suppressing various inessential complications (spin, chirality, momentum dependence), and emphasizing the existence of the matrix degree of freedom $U$, which parameterizes the degenerate vacua associated with the spontaneous symmetry breaking $SU(3)_{\text{color}} \times SU(3)_{\text{flavor}} \rightarrow SU(3)_{\text{color}+\text{flavor}}$.

It appears that for $3k$ flavors the favored condensation is repeated color-flavor locking. Thus we start with the ansatz

$$
\langle q_\alpha^a q_\beta^b \rangle = \sum_{i=1}^{k} U_{\gamma}^{(i)\alpha} U_{\delta}^{(i)\beta} (\kappa_1 \delta_{a-3i+3}^\gamma \delta_{b-3i+3}^\delta + \kappa_2 \delta_{b-3i+3}^\gamma \delta_{a-3i+3}^\delta) ,
$$

8
corresponding to the symmetry breaking \( SU(3)_{\text{color}} \times SU(3k)_{\text{flavor}} \rightarrow SU(3)_{\text{color+diagonal}} \times S_k \). The residual \( SU(3) \) acts on the flavor indices in blocks of 3, while the permutation symmetry \( S_k \) implements interchanges of the blocks.

Now the question arises how the energy depends on the relative alignment of the \( U^{(i)} \). Non-trivial relative alignments violate the permutation symmetry. We will not attempt here to determine whether this actually occurs in the ground state, or in other low-lying states, but we do want to point out that to analyze this question one would need to consider potentials resembling those discussed above, featuring permutation rather than rotation symmetry in internal space. This case is intermediate in complexity between the vector and matrix cases discussed above, in that the permutation acts on a single index (as in the vector case), but the objects being permuted are chosen from a complicated manifold, rather than being a simple choice of sign. Symmetry breaking correlations of the type \( \langle U^{(i)} U^{(j)} \rangle \sim M^{(ij)} \) could produce an effective matrix structure in the permutation index.

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