Steady-state analysis of the Join the Shortest Queue model in the Halfin-Whitt regime

Anton Braverman

1Kellogg School of Management at Northwestern University

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Abstract

This paper studies the steady-state properties of the Join the Shortest Queue model in the Halfin-Whitt regime. We focus on the process tracking the number of idle servers, and the number of servers with non-empty buffers. Recently, [9] proved that a scaled version of this process converges, over finite time intervals, to a two-dimensional diffusion limit as the number of servers goes to infinity. In this paper we prove that the diffusion limit is exponentially ergodic, and that the diffusion scaled sequence of the steady-state number of idle servers and non-empty buffers is tight. Our results mean that the process-level convergence proved in [9] implies convergence of steady-state distributions. The methodology used is the generator expansion framework based on Stein’s method, also referred to as the drift-based fluid limit Lyapunov function approach in [35]. One technical contribution to the framework is to show that it can be used as a general tool to establish exponential ergodicity.

1 Introduction

We consider a system with \( n \) identical servers, where customers arrive according to a Poisson process with rate \( n\lambda \), and service times are i.i.d. exponentially distributed with rate 1. Each server maintains an individual buffer of infinite length. When a customer arrives, he will either enter service immediately if an idle server is available, or be routed to the server with the smallest number of customers in its buffer; ties are broken arbitrarily. Once a customer is routed to a server, he cannot switch to a different server. This model is known as the Join the Shortest Queue (JSQ) model.

To describe the system, let \( Q_i(t) \) be the number of servers with \( i \) or more customers at time \( t \geq 0 \), and let \( Q(t) = (Q_i(t))_{i=1}^\infty \). Then \( \{Q(t)\}_{t \geq 0} \) is a continuous time Markov chain (CTMC), and it is positive recurrent provided \( n\lambda < 1 \) [3]. Let \( Q_i \) be the random variables having the stationary distributions of \( \{Q_i(t)\}_t \).

In this paper we work in the Halfin-Whitt regime [23], which assumes that

\[
\lambda = 1 - \beta / \sqrt{n},
\]

(1.1)
for some fixed $\beta > 0$. The first paper to study the JSQ model in this regime is [9], which shows that the scaled process

$$\left\{ \left( \frac{Q_1(t) - n}{\sqrt{n}}, \frac{Q_2(t)}{\sqrt{n}}, \frac{Q_3(t)}{\sqrt{n}}, \ldots \right) \right\}_{t \geq 0}$$

(1.2)

converges to a diffusion limit as $n \to \infty$. The diffusion limit of (1.2) is essentially two dimensional, because $Q_i(t)/\sqrt{n}$ becomes negligible for $i \geq 3$. The results of [9] are restricted to the transient behavior of the JSQ model and steady-state convergence is not considered, i.e. convergence to the diffusion limit is proved only for finite time intervals.

In the present paper, we study the steady-state properties of the JSQ system. Specifically, we prove the existence of an explicitly known constant $C(\beta) > 0$ depending only on $\beta$ such that

$$n - EQ_1 = n(1 - \lambda),$$
$$EQ_2 \leq C(\beta)\sqrt{n},$$
$$EQ_i \leq C(\beta), \quad i \geq 3, \quad n \geq 1.$$  

(1.3)

In other words, the expected number of idle servers is known, the expected number of non-empty buffers is of order $\sqrt{n}$, and the expected number of buffers with two or more waiting customers is bounded by a constant independent of $n$. A consequence of (1.3) is tightness of the sequence of diffusion-scaled stationary distributions.

In addition to (1.3), we also prove that the two-dimensional diffusion limit of the JSQ model is exponentially ergodic. Stability of this diffusion limit remained an open question until the present paper. Combining the process-level convergence of [9], tightness of the prelimit stationary distributions in (1.3), and stability of the diffusion limit, we are able to justify convergence of the stationary distributions via a standard limit-interchange argument.

To prove our results, we use the generator expansion framework, which is a manifestation of Stein’s method [33] in queueing theory and was recently introduced to the stochastic systems literature in [4, 19]; see [5] for an accessible introduction. The idea is to perform Taylor expansion on the generator of a CTMC, and by looking at the second-order terms, to identify a diffusion model approximating the CTMC. One then proves bounds on the steady-state approximation error of the diffusion, which commonly results in convergence rates to the diffusion approximation [4, 5, 10, 19, 26]. In this paper, we use only the first-order terms of the generator expansion, which correspond to the generator of a related fluid model. We then carry out the machinery of Stein’s method to prove convergence rates to the fluid model equilibrium. The bounds in (1.3) are then simply an alternative interpretation of these convergence rates. For other examples of Stein’s method for fluid, or mean-field models, see [15, 16, 38, 39]. Specifically, [38] was the first to make the connection between Stein’s method and convergence rates to the mean-field equilibrium.

Our approach can also be tied to the drift-based fluid limit (DFL) Lyapunov functions used in [36], which appeared a few years before [38]. As we will explain in more detail in Section 3.1, the DFL approach and Stein’s method for mean-field approximations are essentially one and the same.
This paper contributes another example of the successful application of the generator expansion method to the queuing literature. Although the general framework has already been laid out in previous work, examples of applying the framework to non-trivial systems are the only way to display the power of the framework and promote its adoption in the research community. Furthermore, tractable examples help showcase and expand the versatility of the framework and the type of results it can prove. The present paper contributes from this angle in two ways. First, the JSQ model is an example where the dimension of the CTMC is greater than that of the diffusion approximation. To justify the approximation, one needs a way to show that the additional dimensions of the CTMC are asymptotically negligible; this is known as state space collapse (SSC). Our way of dealing with SSC in Section 3.2 differs from the typical solution of bounding the magnitude of the SSC terms [4, 8, 29, 30] (only [4] of the aforementioned papers uses the generator expansion framework, but the rest still deal with steady-state SSC in a conceptually similar way). Second, this paper presents the first working example of the generator expansion framework being used to prove exponential ergodicity of the diffusion approximation. The insight used is simple, but can be easily generalized to prove exponential ergodicity for other models.

1.1 Literature Review and Contributions

Early work on the JSQ model appeared in the late 50’s and early 60’s [21, 28], followed by a number of papers in the 70’s–90’s [11, 12, 22, 24, 40]. This body of literature first studied the JSQ model with two servers, and later considered heavy-traffic asymptotics in the setting where the number of servers $n$ is fixed, and $\lambda \to 1$; see [9] for an itemized description of the aforementioned works. A more recent paper [8] considers the steady-state behavior of the JSQ model, but again in the setting where $n$ is fixed, and $\lambda \to 1$.

The asymptotic regime where $n \to \infty$ has been untouched until very recently. In [34], the author studies a variant of the JSQ model where the routing policy is to join an idle server if one is available, and otherwise join any buffer uniformly: this is known as the Join the Idle Queue (JIQ) policy. In that paper, the arrival rate is $n\lambda$ where $\lambda < 1$ is fixed, and $n \to \infty$. The author shows that in this underloaded asymptotic regime, JIQ is asymptotically optimal on the fluid scale, and therefore asymptotically equivalent to the JSQ policy. We have already described [9], which is the first paper to study a non-underloaded regime. In [32], the authors work in the Halfin-Whitt regime and show that JIQ is asymptotically optimal, and therefore asymptotically equivalent to JSQ, on the diffusion scale. Most recently, [18] studies the JSQ model in the non-degenerate slowdown (NDS) regime introduced in [1]. In this regime, $\lambda = 1 - \beta/n$ for some fixed $\beta > 0$, i.e. NDS is even more heavily loaded than the Halfin-Whitt regime. The authors of [18] establish a diffusion limit for the total customer count process.

In the asymptotic regime where $n \to \infty$, all previous considerations of the diffusion-scaled model [9, 18, 32] have been in the transient setting. In particular, convergence to the diffusion limit is only proved over finite time intervals. In contrast, the present paper deals with steady-state distributions. Since the seminal work of [14], justifying convergence of steady-state distributions has become the standard in heavy-traffic approximations, and is recognized as being a non-trivial step beyond convergence over finite-time intervals [4, 5, 6, 13, 19, 20, 26, 27, 35, 36, 37, 41].
The methodology used in this paper can be discussed in terms of [15, 35, 38, 39]. The main technical driver of our results are bounds on the derivatives of the solution to a certain first order partial differential equation (PDE) related to the fluid model of the JSQ system. In the language of [35], we need to bound the derivatives of the DFL Lyapunov function. These derivative bounds are a standard requirement to apply Stein’s method, and [15, 38, 39] provide sufficient conditions to bound these derivatives for a large class of PDEs. The bounds in [15, 38, 39] require continuity of the vector field defining the fluid model, but the JSQ fluid model does not satisfy this continuity due to a reflecting condition at the boundary. To circumvent this, we leverage knowledge of how the fluid model behaves to give us an explicit expression for the PDE solution, and we bound its derivatives directly using this expression. Using the behavior of the fluid model is similar to what was done in [35]. However, bounding the derivatives in this way requires detailed understanding of the fluid model, and as such this is a case-specific approach that varies significantly from one model to another. Furthermore, unlike [35] where the dimension of the CTMC equals the dimension of the diffusion approximation, our CTMC is infinite-dimensional whereas the diffusion process is two-dimensional. These additional dimensions in the CTMC create additional technical difficulties which we handle in Section 3.2.

Regarding our proof of exponential ergodicity. The idea of using a fluid model Lyapunov function to establish exponential ergodicity of the diffusion model was initially suggested in Lemma 3.1 of [19]. However, the discussion in [19] is at a conceptual level, and it is only after the working example of the present paper that we have a simple and general implementation of the idea. Indeed, our Lyapunov function in (3.12) of Section 3.3 violates the condition in Lemma 3.1 of [19].

1.2 Notation

We use \( \Rightarrow \) to denote weak convergence, or convergence in distribution. We use \( 1(A) \) to denote the indicator of a set \( A \). We use \( D = D([0, \infty), \mathbb{R}) \) to denote the space of right continuous functions with left limits mapping \([0, \infty)\) to \( \mathbb{R} \). For any integer \( k \geq 2 \), we let \( D^k = D([0, \infty), \mathbb{R}^k) \) be the product space \( D \times \ldots \times D \).

The rest of the paper is structured as follows. We state our main results in Section 2, and prove them in Section 3. Section 4 is devoted to understanding the JSQ fluid model, and using this to prove the derivative bounds that drive the proof of our main results.

2 Model and Main Results

Consider the CTMC \( \{Q(t)\}_{t \geq 0} \) introduced in Section 1. The state space of the CTMC is

\[
S = \{ q \in \{0, 1, 2, \ldots, n\}^\infty \mid q_i \geq q_{i+1} \text{ for } i \geq 1 \text{ and } \sum_{i=0}^{\infty} q_i < \infty \}.
\]

The requirement that \( \sum_{i=0}^{\infty} q_i < \infty \) if \( q \in S \) means that we only consider states with a finite number of customers. Recall that \( Q_i \) are random variables having the stationary
distributions of \( \{Q_i(t)\}_t \), and let \( Q = (Q_i) \) be the corresponding vector. The generator of the CTMC \( G_Q \) acts on function \( f : S \rightarrow \mathbb{R} \), and satisfies

\[
G_Q f(q) = n \lambda 1(q_1 < n) \left( f(q + e^{(1)}) - f(q) \right) \\
+ \sum_{i=2}^{\infty} n \lambda 1(q_1 = \ldots = q_{i-1} = n, q_i < n) \left( f(q + e^{(i)}) - f(q) \right) \\
+ \sum_{i=1}^{\infty} (q_i - q_{i+1}) \left( f(q - e^{(i)}) - f(q) \right),
\]

where \( e^{(i)} \) is the infinite dimensional vector where the \( i \)th element equals one, and the rest equal zero. The generator of the CTMC encodes the stationary behavior of the chain. Exploiting the relationship between the generator and the stationary distribution can be done via the following lemma, which is proved in Section A.1.

**Lemma 1.** For any function \( f : S \rightarrow \mathbb{R} \) such that \( \mathbb{E} |f(Q)| < \infty \),

\[
\mathbb{E} G_Q f(Q) = 0. \quad (2.1)
\]

By choosing different test functions \( f(q) \), we can use (2.1) to obtain stationary performance measures of our CTMC. For example, we are able to prove the following using rather simple test functions.

**Lemma 2.**

\[
\begin{align*}
\mathbb{E} Q_1 &= n \lambda, \\
\mathbb{E} Q_i &= n \lambda \mathbb{P}(Q_1 = \ldots = Q_{i-1} = n), \quad i > 1.
\end{align*}
\]

*Proof.* Fix \( M > 0 \) and let \( f(q) = \min(M, \sum_{i=1}^{\infty} q_i) \). Then

\[
G_Q f(q) = n \lambda 1(\sum_{i=1}^{\infty} q_i < M) - q_1 1(\sum_{i=1}^{\infty} q_i \leq M).
\]

Using (2.1),

\[
n \lambda \mathbb{P}(T < M) = \mathbb{E}(Q_1 1(T \leq M)),
\]

where \( T = \sum_{i=1}^{\infty} Q_i \) is the total customer count. Although the infinite series in the definition of \( T \) may seem worrying at first, stability of the JSQ model in fact implies that \( T < \infty \) almost surely. To see why this is true, observe that an alternative way to describe the JSQ model is via the CTMC \( \{(S_1(t), \ldots, S_n(t))\}_{t \geq 0} \), where \( S_i(t) \) be the number of customers assigned to server \( i \) at time \( t \); we can view \( Q(t) \) as a deterministic function of \( (S_1(t), \ldots, S_n(t)) \). This new CTMC is also positive recurrent, but now the total number of customers in the system at time \( t \) is the finite sum \( \sum_{i=1}^{n} S_i(t) \). Therefore, \( T < \infty \) almost surely, and we can take \( M \to \infty \) and apply the monotone convergence theorem to conclude that

\[
\mathbb{E} Q_1 = n \lambda.
\]
Repeating the argument above with \( f(q) = \min \left( M, \sum_{j=1}^{\infty} q_j \right) \) gives us
\[
n\lambda \mathbb{P}(Q_1 = \ldots = Q_{i-1} = n) = \mathbb{E}Q_i.
\]
\[\square\]

Although Lemma 2 does characterize quantities like \( \mathbb{E}Q_i \), its results are of little use to us unless we can control \( \mathbb{P}(Q_1 = \ldots = Q_{i-1} = n) \). One may continue to experiment by applying \( GQ_i \) to various test functions in the hope of getting more insightful results from (2.1). In general, the more complicated the Markov chain, the less likely this strategy will be productive. In this paper we take a more systematic approach to selecting test functions.

To state our main results, define
\[
X_1(t) = \frac{Q_1(t)}{n}, \quad X_i(t) = \frac{Q_i(t)}{n}, \quad i \geq 2,
\]
and let \( X(t) = (X_i(t))_{i=1}^{\infty} \) be the fluid-scaled CTMC. Also let \( X_i \) be the random variables having the stationary distributions of \( \{X_i(t)\}_t \) and set \( X = (X_i)_{i=1}^{\infty} \). Our first result is about bounding the expected value of \( X_2 \). The main ingredients needed for the proof are presented in Section 3.1, and are followed by a proof in Section 3.2.

**Theorem 1.** For all \( n \geq 1, \beta > 0, \) and \( \kappa > \beta, \)
\[
\mathbb{E}\left((X_2 - \kappa/\sqrt{n})1(X_2 \geq \kappa/\sqrt{n})\right) \leq \frac{1}{\beta \sqrt{n}} \left(12 + \frac{6\kappa}{\kappa - \beta}\right) \mathbb{P}(X_2 \geq \kappa/\sqrt{n} - 1/n),
\]
which implies that
\[
\mathbb{E}\sqrt{n}X_2 \leq 2\kappa + \frac{1}{\beta} \left(12 + \frac{6\kappa}{\kappa - \beta}\right).
\]  
(2.3)

To parse the bound in (2.3) into a friendlier form, let us choose \( \kappa = \beta + \varepsilon \) to get
\[
\mathbb{E}\sqrt{n}X_2 \leq 2(\beta + \varepsilon) + \frac{1}{\beta} \left(12 + \frac{6(\beta + \varepsilon)}{\varepsilon}\right), \quad \varepsilon > 0,
\]
and to see that the right hand side above can be bounded by a constant independent of \( n \). A consequence of Theorem 1 is the following result, which tells us that for \( i \geq 3 \), we can bound \( \mathbb{E}Q_i \) by a constant that is independent of \( n \).

**Theorem 2.** For all \( i \geq 3, \beta > 0, \) \( \kappa > \beta, \) \( n \geq 1 \) such that \( \max(\beta/\sqrt{n}, 1/n) < 1, \) and \( \hat{\kappa} \in (\max(\beta/\sqrt{n}, 1/n), 1) , \)
\[
\mathbb{E}Q_i \leq \frac{1}{\beta(1 - \hat{\kappa})} \left(12 + \frac{6\hat{\kappa}}{\kappa - \beta/\sqrt{n}}\right) \frac{1}{\hat{\kappa} - 1/n} \left(2\kappa  + \frac{1}{\beta} \left(12 + \frac{6\kappa}{\kappa - \beta}\right)\right).
\]  
(2.4)

**Remark 1.** The bound in (2.4) is intended to be used with \( \kappa = \beta + \varepsilon \) and \( \hat{\kappa} \) being some constant, say \( \hat{\kappa} = 1/2 \). Then for \( n \) large enough so that \( 1/2 > \max(\beta/\sqrt{n}, 1/n) \), the bound implies that
\[
\mathbb{E}Q_i \leq \frac{2}{\beta} \left(12 + \frac{3}{0.5 - \beta/\sqrt{n}}\right) \frac{1}{0.5 - 1/n} \left(2(\beta + \varepsilon) + \frac{1}{\beta} \left(12 + \frac{6(\beta + \varepsilon)}{\varepsilon}\right)\right), \quad \varepsilon > 0.
\]
Since there only finitely many values of \( n \) violate \( 1/2 > \max(\beta/\sqrt{n}, 1/n) \), the above bound means that \( \mathbb{E}Q_i \) can be bounded by a constant independent of \( n \).
Proof of Theorem 2. Since $E_Q \leq E_{Q_i}$ for $i \geq 3$, it suffices to prove (2.4) for $i = 3$. Fix $\hat{\kappa} \in (\max(\beta/\sqrt{n}, 1/n), 1)$ and invoke Theorem 1 with $\sqrt{n}\hat{\kappa}$ in place of $\kappa$ there to see that

$$
E\left((X_2 - \hat{\kappa})1(X_2 \geq \hat{\kappa})\right) \leq \frac{1}{\beta \sqrt{n}} \left(12 + \frac{6\hat{\kappa}}{\hat{\kappa} - \beta/\sqrt{n}}\right) \mathbb{P}(X_2 \geq \hat{\kappa} - 1/n)
= \frac{1}{\beta n} \left(12 + \frac{6\hat{\kappa}}{\hat{\kappa} - \beta/\sqrt{n}}\right) \sqrt{n} \mathbb{E}\left(\frac{X_2}{X_2}1(X_2 \geq \hat{\kappa} - 1/n)\right)
\leq \frac{1}{\beta n} \left(12 + \frac{6\hat{\kappa}}{\hat{\kappa} - \beta/\sqrt{n}}\right) \frac{1}{\hat{\kappa} - 1/n} \mathbb{E}\sqrt{n}X_2
\leq \frac{1}{\beta n} \left(12 + \frac{6\hat{\kappa}}{\hat{\kappa} - \beta/\sqrt{n}}\right) \frac{1}{\hat{\kappa} - 1/n} \left(2\kappa + \frac{1}{\beta} \left(12 + \frac{6\kappa}{\kappa - \beta}\right)\right),
$$

where in the last inequality we used (2.3). Therefore,

$$
\frac{1}{\beta n} \left(12 + \frac{6\hat{\kappa}}{\hat{\kappa} - \beta/\sqrt{n}}\right) \frac{1}{\hat{\kappa} - 1/n} \left(2\kappa + \frac{1}{\beta} \left(12 + \frac{6\kappa}{\kappa - \beta}\right)\right) \geq E\left((X_2 - \hat{\kappa})1(X_2 \geq \hat{\kappa})\right)
\geq (1 - \hat{\kappa}) \mathbb{P}(X_2 = 1)
= (1 - \hat{\kappa}) \mathbb{P}(Q_2 = n)
\geq (1 - \hat{\kappa}) \frac{1}{n} E_Q^3,
$$

where in the second inequality we used the fact that $\hat{\kappa} < 1$, and in the last inequality we used Lemma 2.

Remark 2. The bound in (2.4) will be sufficient for our purposes, but it is unlikely to be tight. The argument in (2.5) can be modified by observing that for any integer $m > 0$,

$$
\mathbb{P}(X_2 \geq \hat{\kappa} - 1/n) = \frac{n^m}{n^m \mathbb{E}(\frac{X_2}{X_2}^m1(X_2 \geq \hat{\kappa} - 1/n))} \leq \frac{1}{n^m(\hat{\kappa} - 1/n)^2} \mathbb{E}(\sqrt{n}X_2)^{2m}.
$$

Provided we have a bound on $\mathbb{E}(\sqrt{n}X_2)^{2m}$ that is independent of $n$, it follows that $E_Q^3 \leq C(\beta)/n^{m-1/2}$. Although we have not done so, we believe the arguments used in Theorem 1 can be extended to provide the necessary bounds on $\mathbb{E}(\sqrt{n}X_2)^{2m}$.

2.1 The Diffusion Limit: Exponential Ergodicity

Let us now consider the diffusion limit of the JSQ model. The following result is copied from [32] (but it was first proved in [9]).

Theorem 3 (Theorem 1 of [32]). Suppose $Y(0) = (Y_1(0), Y_2(0)) \in \mathbb{R}^2$ is a random vector such that $\sqrt{n}X_i(0) \Rightarrow Y_i(0)$ for $i = 1, 2$ as $n \to \infty$ and $\sqrt{n}X_i(0) \Rightarrow 0$ for $i \geq 3$ as $n \to \infty$. Then the process $\{\sqrt{n}(X_1(t), X_2(t))\}_{t \geq 0}$ converges uniformly over bounded
It follows from Ito’s lemma that

\[ Y_1(t) = Y_1(0) + \sqrt{2}W(t) - \beta t + \int_0^t (-Y_1(s) + Y_2(s))ds - U(t), \]
\[ Y_2(t) = Y_2(0) + U(t) - \int_0^t Y_2(s)ds, \]

where \( \{W(t)\}_{t \geq 0} \) is standard Brownian motion and \( \{U(t)\}_{t \geq 0} \) is the unique non-decreasing, non-negative process in \( D \) satisfying \( \int_0^\infty 1(Y_1(t) < 0)dU(t) = 0. \)

Theorem 3 proves that \( \{\sqrt{n}(X_1(t), X_2(t))\}_{t \geq 0} \) converges to a diffusion limit. Convergence was established only over finite time intervals, but convergence of steady-state distributions was not justified. In fact, it has not been shown that the process in (2.6) is even positive recurrent. We show that not only is this process positive recurrent, but it is also exponentially ergodic. We first need to introduce the generator of the diffusion process \( \{(Y_1(t), Y_2(t))\}_{t \geq 0}. \)

Let

\[ \Omega = (-\infty, 0] \times [0, \infty). \]

Going forward, we adopt the convention that for any function \( f : \Omega \to \mathbb{R}, \) partial derivatives are understood to be one-sided derivatives for those values \( x \in \partial \Omega \) where the derivative is not defined. For example, the partial derivative with respect to \( x_1 \) is not defined on the set \{\( x_1 = 0, \ x_2 \geq 0 \). In particular, for any integer \( k > 0 \) we let \( C^k(\Omega) \) be the set of \( k \)-times continuously differentiable functions \( f : \Omega \to \mathbb{R} \) obeying the notion of one-sided differentiability just described. We use \( f_i(x) \) to denote \( \frac{df(x)}{dx_i}. \)

For any function \( f(x) \in C^2(\Omega) \) such that \( f_1(0, x_2) = f_2(0, x_2), \) define \( G_Y \) as

\[ G_Y f(x) = (-x_1 + x_2 - \beta)f_1(x) - x_2f_2(x) + f_{11}(x), \quad x \in \Omega. \]

It follows from Ito’s lemma that

\[ f(Y_1(t), Y_2(t)) - f(Y_1(0), Y_2(0)) - \int_0^t G_Y f(Y_1(s), Y_2(s))ds \]

is a martingale, and \( G_Y \) is the generator of this diffusion process. The condition \( f_1(0, x_2) = f_2(0, x_2) \) is essential due to the reflecting term \( U(t) \) in (2.6); otherwise (2.8) is not a martingale. The following theorem proves the existence of a function satisfying the Foster-Lyapunov condition [31] that is needed for exponential ergodicity. The proof of the theorem is given in Section 3.3, where one can also obtain more insight about the form of the Lyapunov function.

**Theorem 4.** For any \( \alpha, \kappa_1, \kappa_2 > 0 \) with \( \beta < \kappa_1 < \kappa_2, \) there exists a function \( V^{(\kappa_1, \kappa_2)} : \Omega \to [1, \infty) \) with \( V^{(\kappa_1, \kappa_2)}(x) \to \infty \) as \( |x| \to \infty \) such that

\[ G_Y V^{(\kappa_1, \kappa_2)}(x) \leq - \alpha V^{(\kappa_1, \kappa_2)}(x) + \alpha d, \]

(2.9)
where
\[ c = 1 - \frac{12}{(\kappa_2 - \kappa_1)^2} \log \left( \frac{\kappa_2 - \beta}{\kappa_1 - \beta} \right) - \frac{1}{\beta(\kappa_1 - \beta)} \left( 1 + \frac{\kappa_1}{\kappa_1 - \beta} \frac{4(\kappa_1 - \beta)}{\kappa_2 - \kappa_1} \right) \]
\[ - \frac{4}{\kappa_2 - \kappa_1} \log \left( \frac{\kappa_2 - \beta}{\kappa_1 - \beta} + 1/\beta \right)^2, \]
\[ d = \frac{(\kappa_2 - \beta \kappa_1)}{(\kappa_1 - \beta \kappa_1)} e^\alpha e^{2 \frac{\alpha}{\kappa_1}}. \]

We can choose \( \kappa_1 = \beta + \varepsilon \) and \( \kappa_2 = \beta + 2\varepsilon \) in Theorem 4 so that
\[ c = 1 - \frac{12}{(\varepsilon)^2} - \frac{1 + 4(1 + \beta/\varepsilon)}{\beta \varepsilon} - \alpha (4/\varepsilon + 1/\beta)^2, \quad d = 4^\alpha e^{\alpha(\varepsilon/\beta)}. \] (2.10)

We can make \( c < 0 \) by choosing \( \varepsilon \) and \( \alpha \) appropriately. A consequence of Theorem 4 is exponential ergodicity of the diffusion in (2.6).

**Corollary 1.** The diffusion process \( \{(Y_1(t), Y_2(t))\}_{t \geq 0} \) defined in (2.6) is positive recurrent. Furthermore, if \( Y = (Y_1, Y_2) \) is the vector having its stationary distribution, then for any \( \kappa_1, \kappa_2 > 0 \) with \( \beta < \kappa_1 < \kappa_2 \) such that the constant \( c \) in (2.10) is negative, there exist constants \( b < 1 \) and \( B < \infty \) such that
\[ \sup_{|f| \leq V(\kappa_1, \kappa_2)} |\mathbb{E}_x f(Y(t)) - \mathbb{E} f(Y)| \leq B V(\kappa_1, \kappa_2)(x) b^t. \]

**Proof.** The proof is an immediate consequence of Theorem 6.1 of [31]. \( \square \)

### 2.2 Convergence of Stationary Distributions

In this section we leverage the results of Theorems 1–3 together with positive recurrence of the diffusion limit to verify steady-state convergence.

**Theorem 5.** Let \( Y = (Y_1, Y_2) \) have the stationary distribution of the diffusion process defined in (2.6). Then
\[ \sqrt{n}(X_1, X_2) \Rightarrow Y \quad \text{as} \quad n \to \infty. \] (2.11)

**Proof of Theorem 5.** Lemma 2 and Theorem 1 imply that the sequence \( \{\sqrt{n}(X_1, X_2)\}_n \) is tight. It follows by Prohorov’s Theorem [2] that the sequence is also relatively compact. We will now show that any subsequence of \( \{\sqrt{n}(X_1, X_2)\}_n \) has a further subsequence that converges weakly to \( Y \).

Fix \( n > 0 \) and initialize the process \( \{X(t)\}_{t \geq 0} \) by letting \( \sqrt{n}X(0) \) have the same distribution as \( \sqrt{n}X \). Prohorov’s Theorem implies that for any subsequence
\[ \{\sqrt{n'}X(0)\}_{n'} \subset \{\sqrt{n}X(0)\}_n, \]
there exists a further subsequence
\[ \{\sqrt{n''}X(0)\}_{n''} \subset \{\sqrt{n'}X(0)\}_{n'} \]

that converges weakly to some random vector $Y^{(0)} = (Y_1^{(0)}, Y_2^{(0)}, \ldots)$. Theorem 2 implies that $Y_i^{(0)} = 0$ for $i \geq 3$. Now for any $t \geq 0$, let $(Y_1(t), Y_2(t))$ solve the integral equation in (2.6) with initial condition $(Y_1(0), Y_2(0)) = (Y_1^{(0)}, Y_2^{(0)})$. Then

$$\sqrt{n'}(X_1(0), X_2(0)) \overset{d}{=} \sqrt{n'}(X_1(t), X_2(t)) \Rightarrow (Y_1(t), Y_2(t)), \text{ as } n \to \infty,$$

where the weak convergence follows from Theorem 3. We conclude that

$$\lim_{n \to \infty} \sqrt{n'}(X_1(0), X_2(0)) = \lim_{t \to \infty} \lim_{n \to \infty} \sqrt{n'}(X_1(0), X_2(0)) \overset{d}{=} \lim_{t \to \infty} (Y_1(t), Y_2(t)) \overset{d}{=} (Y_1, Y_2).$$

3 Proving the Main Results

In this section we prove our main results. Section 3.1 outlines the main components needed for the proofs, and then Sections 3.2 and 3.3 follow up with proofs of Theorems 1 and 4, respectively.

3.1 Proof Ingredients: Generator Expansion

For any differentiable function $f : \Omega \to \mathbb{R}$, let

$$Lf(x) = (-x_1 + x_2 - \beta/\sqrt{n})f_1(x) - x_2f_2(x), \quad x \in \mathbb{R}^2,$$

where $f_i(x) = \frac{df(x)}{dx_i}$. For any function $h : \Omega \to \mathbb{R}$, consider the partial differential equation (PDE)

$$Lf(x) = -h(x), \quad x \in \Omega,$n

$$f_1(0, x_2) = f_2(0, x_2), \quad x_2 \geq 0. \quad (3.1)$$

The PDE above is related to the fluid model corresponding to $\{(X_1(t), X_2(t))\}_{t \geq 0}$. This connection will be expanded upon in Section 4. Assume for now that a solution to (3.1) exists and denote it by $f^{(h)}(x)$. Let $G_X$ be the generator of the CTMC $\{X(t)\}_{t \geq 0}$. For a function $f : \mathbb{R} \to \mathbb{R}$, define the lifted version $Af : S \to \mathbb{R}$ by

$$(Af)(q) = f(x_1, x_2) = f(x), \quad q \in S,$$

where $x_1 = (q_1 - n)/n$, and $x_2 = q_2/n$. We know that $\mathbb{E}|f(X_1, X_2)| < \infty$ for any $f : \mathbb{R} \to \mathbb{R}$ because $(X_1, X_2)$ can only take finitely many values. A variant of Lemma 1 then tells us that

$$\mathbb{E}G_XAf^{(h)}(X) = 0.$$

We can therefore take expected values in (3.1) to conclude that

$$\mathbb{E}h(X) = \mathbb{E}(G_XAf^{(h)}(X) - Lf^{(h)}(X)). \quad (3.2)$$
Lemma 3. For any \( q_i \in \mathbb{Z}_+ \), let \( x_1 = (q_1 - n)/n \) and \( x_i = q_i/n \) for \( i \geq 2 \). Suppose \( f(x_1, x_2) \) is defined on \( \Omega \), and has absolutely continuous first-order partial derivatives. Then for all \( q \in S \),

\[
G_X Af(q) - Lf(x) = (f_2(x) - f_1(x)) \lambda 1(q_1 = n) - f_2(x) \lambda 1(q_1 = q_2 = n) \\
+ q_3 \int_{x_2}^{x_2 - 1/n} f_2(x_1, u)du \\
+ n \lambda 1(q_1 > n) \int_{x_1}^{x_1 + 1/n} (x_1 + 1/n - u) f_{11}(u, x_2)du \\
+ n \lambda 1(q_1 = n, q_2 < n) \int_{x_2}^{x_2 + 1/n} (x_2 + 1/n - u) f_{22}(x_1, u)du \\
+ (q_1 - q_2) \int_{x_1 - 1/n}^{x_1} (u - (x_1 - 1/n)) f_{11}(u, x_2)du \\
+ q_2 \int_{x_2}^{x_2 - 1/n} (u - (x_2 - 1/n)) f_{22}(x_1, u)du.
\]

Lemma 3 tells us that to bound the error in (3.2), we need to know more about the solution to (3.1) and its derivatives. Section 4 is devoted to proving the following lemma.

Lemma 4. Fix \( \kappa > \beta \) and consider the PDE (3.1) with \( h(x) = \left( (x_2 - \kappa/\sqrt{n}) \vee 0 \right) \). There exists a solution \( f^{(h)} : \Omega \to \mathbb{R} \) with absolutely continuous first-order partial derivatives, such that the second-order weak derivatives satisfy

\[
f^{(h)}_{11}(x), f^{(h)}_{22}(x) \geq 0, \quad x \in \Omega, \quad (3.3) \\
f^{(h)}_{11}(x) = f^{(h)}_{22}(x) = 0, \quad x_2 \in [0, \kappa/\sqrt{n}], \quad (3.4) \\
f^{(h)}_{11}(x) \leq \frac{\sqrt{n}}{\beta} \left( \frac{\kappa}{\kappa - \beta} + 1 \right), \quad f^{(h)}_{22}(x) \leq \frac{\sqrt{n}}{\beta} \left( 5 + \frac{2\kappa}{\kappa - \beta} \right), \quad x_2 \geq \kappa/\sqrt{n}. \quad (3.5)
\]
3.2 Proof of Theorem 1

Proof of Theorem 1. Let \( f(x) = f^{(b)}(x) \) from Lemma 4. By Lemma 3,

\[
G_X Af(q) - Lf(x) = (f_2(x) - f_1(x))\lambda_1(q_1 = n) \tag{3.6}
\]

\[
- f_2(x)\lambda_1(q_1 = q_2 = n) + q_3 \int_{x_{2-1/n}}^{x_2} f_2(x_1, u)du \tag{3.7}
\]

\[
+ n\lambda_1(q_1 < n) \int_{x_1}^{x_{1+1/n}} (x_1 + 1/n - u)f_{11}(u, x_2)du \tag{3.8}
\]

\[
+ n\lambda_1(q_1 = n, q_2 < n) \int_{x_2}^{x_{2+1/n}} (x_2 + 1/n - u)f_{22}(x_1, u)du \tag{3.9}
\]

\[
+ (q_1 - q_2) \int_{x_{1-1/n}}^{x_1} (u - (x_1 - 1/n))f_{11}(u, x_2)du \tag{3.10}
\]

\[
+ q_2 \int_{x_{2-1/n}}^{x_2} (u - (x_2 - 1/n))f_{22}(x_1, u)du. \tag{3.11}
\]

We know (3.6) equals zero because of the derivative condition in (3.1). Assume for now that lines (3.8)-(3.11) are all non-negative, and their sum is upper bounded by \( \frac{1}{\beta \sqrt{n}} (12 + \frac{6\kappa}{\kappa - \beta}) 1(x_2 \geq \kappa/\sqrt{n} - 1/n) \). Then (3.2) implies

\[
0 \leq E((X_2 - \kappa/\sqrt{n}) \vee 0) = E(G_X Af(X) - Lf(X)) \leq \frac{1}{\beta \sqrt{n}} (12 + \frac{6\kappa}{\kappa - \beta}) \mathbb{P}(X_2 \geq \kappa/\sqrt{n} - 1/n)
\]

\[
- f_2(0, 1)\lambda \mathbb{P}(Q_1 = Q_2 = n) + E \left[ Q_3 \int_{X_{2-1/n}}^{X_2} f_2(X_1, u)du \right].
\]

The term containing \( Q_3 \) above is present because our CTMC is infinite dimensional, but the PDE in (3.1) is two-dimensional. To deal with this error term, we invoke Lemma 2,

\[
- f_2(0, 1)\lambda \mathbb{P}(Q_1 = Q_2 = n) + E \left[ Q_3 \int_{X_{2-1/n}}^{X_2} f_2(X_1, u)du \right]
\]

\[
= - f_2(0, 1) \frac{1}{n}E Q_3 + E \left[ Q_3 \int_{X_{2-1/n}}^{X_2} f_2(X_1, u)du \right]
\]

\[
= E \left[ Q_3 \int_{X_{2-1/n}}^{X_2} (f_2(X_1, u) - f_2(0, 1))du \right]
\]

\[
\leq 0,
\]

where in the last inequality we used \( f_{21}(x), f_{22}(x) \geq 0 \) from (3.3). To conclude the proof, it remains to verify the bound on (3.8)-(3.11). By (3.3) we know that (3.8)-(3.11) all equal zero when \( x_2 < \kappa/\sqrt{n} - 1/n \). Now suppose \( x_2 \geq \kappa/\sqrt{n} - 1/n \). From (3.5) and the fact that \( q_1 \geq q_2 \) if \( q \in S \), we can see that each of (3.8) and (3.10) is non-negative

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and bounded by $\frac{1}{\beta\sqrt{n}} \left( \frac{\kappa}{\kappa-\beta} + 1 \right)$. Similarly, (3.5) tells us that each of (3.9) and (3.11) is non-negative and bounded by $\frac{1}{\beta\sqrt{n}} \left( 5 + \frac{2\kappa}{\kappa-\beta} \right)$.

### 3.3 Proving Theorem 4

Given $h(x)$ and $\alpha > 0$, let $f^{(h)}(x)$ solve the PDE (3.1) and set

$$
g(x) = e^{\alpha f^{(h)}(x/\sqrt{n})}. \tag{3.12}$$

Observe that $g_1(0, x_2) = g_2(0, x_2)$, which means that the diffusion generator applied to $g(x)$ is

$$
G_Y g(x) = (-x_1 + x_2 - \beta)g_1(x) - x_2g_2(y) + g_{11}(x)
= (-x_1/\sqrt{n} + x_2/\sqrt{n} - \beta/\sqrt{n})f_1^{(h)}(x/\sqrt{n})g(x) - \frac{x_2}{\sqrt{n}} f_2^{(h)}(x/\sqrt{n})g(x)
+ g_{11}(x/\sqrt{n})
= -h(x/\sqrt{n})g(x) + \frac{1}{n} \left( \alpha f_1^{(h)}(x/\sqrt{n}) + \alpha^2 (f_1^{(h)}(x/\sqrt{n}))^2 \right) g(x), \tag{3.13}
$$

where the last equality comes from (3.1). The $h(x)$ we will use is a smoothed version of the indicator function. Namely, for any $\ell < u$, let

$$
\phi^{(\ell,u)}(x) = \begin{cases} 
0, & x \leq \ell, \\
(x - \ell)^2 \left( \frac{2}{((u+\ell)/2-u)^2} - \frac{2}{((u+\ell)/2-u)^2(u-\ell)} \right), & x \in [\ell, (u+\ell)/2], \\
1 - (x - u)^2 \left( \frac{2}{((u+\ell)/2-u)^2(u-\ell)} - \frac{2}{((u+\ell)/2-u)^2(u-\ell)} \right), & x \in [(u+\ell)/2, u], \\
1, & x \geq u.
\end{cases} \tag{3.14}
$$

It is straightforward to check that $\phi^{(\ell,u)}(x)$ has an absolutely continuous first derivative, and that

$$
\left| (\phi^{(\ell,u)})'(x) \right| \leq \frac{4}{u - \ell}, \quad \text{and} \quad \left| (\phi^{(\ell,u)})''(x) \right| \leq \frac{12}{(u - \ell)^2}. \tag{3.15}
$$

**Lemma 5.** Fix $\kappa_1 < \kappa_2$ such that $\kappa_1 > \beta$. There exists functions $f^{(1)}(x)$ and $f^{(2)}(x)$ satisfying

$$
Lf^{(1)}(x) = -\phi^{(\kappa_1,\kappa_2)}(-x_1), \quad x \in \Omega,
$$

$$
f^{(1)}_1(0, x_2) = f^{(1)}_2(0, x_2), \quad x_2 \geq 0, \tag{3.16}
$$

and

$$
Lf^{(2)}(x) = -\phi^{(\kappa_1,\kappa_2)}(x_2), \quad x \in \Omega,
$$

$$
f^{(2)}_1(0, x_2) = f^{(2)}_2(0, x_2), \quad x_2 \geq 0, \tag{3.17}
$$

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Such that both \( f^{(1)}(x) \) and \( f^{(2)}(x) \) belong to \( C^2(\Omega) \),

\[
\begin{align*}
f^{(1)}(x) &\leq \log \left( \frac{\kappa_2 - \beta}{\kappa_1 - \beta} \right), \\
f^{(2)}(x) &\leq \log(\kappa_2 / \kappa_1) + \frac{\kappa_2 - \kappa_1}{\beta},
\end{align*}
\]  
(3.18)  

and for all \( x \in \Omega \),

\[
\begin{align*}
|f^{(1)}_1(x)| &\leq \frac{4\sqrt{n}}{\kappa_2 - \kappa_1} \log \left( \frac{\kappa_2 - \beta}{\kappa_1 - \beta} \right), \\
|f^{(2)}_1(x)| &\leq \frac{n}{\kappa_2 - \kappa_1} \log \left( \frac{\kappa_2 - \beta}{\kappa_1 - \beta} \right),
\end{align*}
\]  
(3.20)  

Proof of Theorem 4. Fix \( \kappa_1 < \kappa_2 \) with \( \kappa_1 > \beta \) and \( \alpha > 0 \), let \( f^{(1)}(x) \) and \( f^{(2)}(x) \) be as in Lemma 5, and let \( V(\kappa_1, \kappa_2)(x) = e^{\alpha(f^{(1)}(x / \sqrt{n}) + f^{(2)}(x / \sqrt{n}))} \). It follows from (3.13) and (3.20)-(3.21) that

\[
\begin{align*}
\frac{G_Y V(\kappa_1, \kappa_2)(x)}{\alpha V(\kappa_1, \kappa_2)(x)}
&= -\phi^{(\kappa_1, \kappa_2)}(-x_1 / \sqrt{n}) - \phi^{(\kappa_1, \kappa_2)}(x_2 / \sqrt{n}) + \frac{1}{n} (f^{(1)}_1(x / \sqrt{n}) + f^{(2)}_1(x / \sqrt{n})) \\
&\quad + \frac{\alpha}{n} (f^{(1)}_1(x / \sqrt{n}) + f^{(2)}_1(x / \sqrt{n}))^2 \\
&\leq -\phi^{(\kappa_1, \kappa_2)}(-x_1 / \sqrt{n}) - \phi^{(\kappa_1, \kappa_2)}(x_2 / \sqrt{n}) + \frac{12}{(\kappa_2 - \kappa_1)^2} \log \left( \frac{\kappa_2 - \beta}{\kappa_1 - \beta} \right) \\
&\quad + \frac{1}{\beta(\kappa_1 - \beta)} \left( 1 + \frac{\kappa_1}{\kappa_2 - \kappa_1} + \frac{4(\kappa_1 - \beta)}{\kappa_2 - \kappa_1} \right) + \frac{\alpha}{\beta(\kappa_1 - \beta)} \left( \frac{4}{\kappa_2 - \kappa_1} \log \left( \frac{\kappa_2 - \beta}{\kappa_1 - \beta} \right) - 1/\beta \right)^2.
\end{align*}
\]

If \( x_1 \leq -\kappa_2 \) or \( x_2 \geq \kappa_2 \), then \( -\phi^{(\kappa_1, \kappa_2)}(-x_1 / \sqrt{n}) - \phi^{(\kappa_1, \kappa_2)}(x_2 / \sqrt{n}) \leq -1 \), and (2.9) is satisfied. If \( x \in [-\kappa_2 / \sqrt{n}, 0 \times [0, \kappa_2 / \sqrt{n}] \), then

\[
\begin{align*}
\frac{G_Y V(\kappa_1, \kappa_2)(x)}{\alpha V(\kappa_1, \kappa_2)(x)}
&\leq 1 - 1 + \frac{12}{(\kappa_2 - \kappa_1)^2} \log \left( \frac{\kappa_2 - \beta}{\kappa_1 - \beta} \right) + \frac{1}{\beta(\kappa_1 - \beta)} \left( 1 + \frac{\kappa_1}{\kappa_2 - \kappa_1} \right) + \frac{\alpha}{\beta(\kappa_1 - \beta)} \left( \frac{4}{\kappa_2 - \kappa_1} \log \left( \frac{\kappa_2 - \beta}{\kappa_1 - \beta} \right) - 1/\beta \right)^2,
\end{align*}
\]

multiplying both sides by \( \alpha V(\kappa_1, \kappa_2)(x) \) and applying the bound in (3.18)-(3.19) verifies (2.9) and concludes the proof. 

Remark 3. In the proof of Theorem 4 we compare the generator of the diffusion process \( G_Y \) to \( L \), which can be thought of as the generator of the associated fluid model. One may wonder why we do not use a similar argument to compare \( L \) to \( G_X \), and prove that the CTMC is also exponentially ergodic. The answer is that the CTMC is infinite dimensional, while the operator \( L \) acts on functions of only two variables. As a result, comparing \( G_X \) to \( L \) leads to excess error terms that \( L \) does not account for, e.g. \( q_3 \) in (3.7). Although we were able to get around this issue in the proof of
Theorem 1 by taking expected values, the same trick will not work now because (2.9) has to hold for every state. To prove exponential ergodicity, one needs to replace the operator $L$ and the PDE (3.1) by infinite-dimensional counterparts corresponding to the infinite-dimensional fluid model of $\{(X_1(t), X_2(t), X_3(t), \ldots)\}_{t \geq 0}$. This is left as an open problem to the interested reader, as Theorem 4 is sufficient for the purposes of illustrating the proof technique.

4 Derivative Bounds

The focus of this section is to prove Lemma 4. The following informal discussion provides a roadmap of the procedure. Given $x \in \Omega$, consider the system of integral equations

$$
\begin{align*}
v_1(t) &= x_1 - \frac{\beta}{\sqrt{n}} t - \int_0^t (v_1(s) - v_2(s))ds - U_1(t), \\
v_2(t) &= x_2 - \int_0^t v_2(s)ds + U_1(t), \\
\int_0^\infty v_1(s)dU_1(s) &= 0, \quad U_1(t) \geq 0, \quad t \geq 0,
\end{align*}
$$

and let $v^\pi(t)$ denote the solution; existence and uniqueness was proved in [9, Lemma 1]. The dynamical system above is the fluid model of $\{(X_1(t), X_2(t))\}_{t \geq 0}$. The key idea is that

$$
f^{(h)}(x) = \int_0^\infty h(v^\pi(t))dt \tag{4.2}
$$

solves the PDE (3.1). Our plan is to a) better understand the behavior of the fluid model and b) use this understanding to obtain a closed form representation of the integral in (4.2) to bound its derivatives. Section 4.1 takes care of a), while b) and the proof of Lemma 4 can be found in Section 4.2. The function in (4.2) is precisely what [35] calls the drift-based fluid limit Lyapunov function. Derivative bounds play the same role in that paper as they do in the present one.

We wish to point out that in the arguments that follow, we never actually have to prove existence and uniqueness of $v^\pi(t)$, or that (4.2) does indeed solve the PDE, or that the fluid model behaves the way we will describe it in Section 4.1. All of this is merely guiding intuition for our final product Lemma 8 in Section 4.2, where we describe a well-defined function that happens to solve the PDE.

4.1 Understanding the Fluid Model

Let us (heuristically) examine the fluid model in (4.1). We refer to $U_1(t)$ as the regulator, because it prevents $v_1^\pi(t)$ from becoming positive. In the absence of this regulator, i.e. $U_1(t) \equiv 0$, the system would have been a linear dynamical system

$$
\dot{v} = F(v), \quad \text{where} \quad F(v) = (-v_1 + v_2 - \beta/\sqrt{n}, -v_2).
$$
Figure 1: Dynamics of the fluid model. Any trajectory starting below the dashed curve will not hit the vertical axis, and anything starting above the curve will hit the axis and travel down until reaching the point (0, \(\beta/\sqrt{n}\)).

However, due to the presence of the regulator, for values in the set \(\{v_1 = 0, v_2 \geq \beta/\sqrt{n}\}\) it is as if the vector field becomes

\[
F(v) = (0, -\beta/\sqrt{n}).
\]

The dynamics of the fluid model are further illustrated in Figure 1. The key to characterizing \(v^*(t)\) is the quantity

\[
\inf\{t \geq 0 : v_1^*(t) = 0\}
\]

which is the first hitting time of the vertical axis given initial condition \(x\). The following lemma characterizes a curve \(\Gamma^{(\kappa)} \subset \Omega\), such that for any point \(x \in \Gamma^{(\kappa)}\), the fluid path \(v^*(t)\) first hits the vertical axis at the point \((0, \kappa/\sqrt{n})\). It is proved in Section B.1.1.

**Lemma 6.** Fix \(\kappa \geq \beta\) and \(x_1 \leq 0\). The nonlinear system

\[
-\beta/\sqrt{n} + (x_1 + \beta/\sqrt{n})e^{-\eta} + \eta ve^{-\eta} = 0, \\
\nu e^{-\eta} = \kappa/\sqrt{n}, \\
\nu \geq \kappa/\sqrt{n}, \quad \eta \geq 0.
\]

(4.3)

has exactly one solution \((\nu^*(x_1), \eta^*(x_1))\). Furthermore, for every \(x_1 \leq 0\), let us define the curve

\[
\gamma^{(\kappa)}(x_1) = \left\{ (t - \beta/\sqrt{n} + (x_1 + \beta/\sqrt{n})e^{-t} + t\nu^*(x_1)e^{-t}, \nu^*(x_1)e^{-t}) \mid t \in [0, \eta^*(x_1)] \right\}
\]

and let

\[
\Gamma^{(\kappa)} = \{ x \in \Omega \mid x_2 = \nu^*(x_1) \}.
\]

Then \(\gamma^{(\kappa)}(x_1) \subset \Gamma^{(\kappa)}\) for every \(x_1 \leq 0\).
Given \( \kappa \geq \beta \) and \( x \in \Omega \), let \( \Gamma^{(\kappa)} \) and \( \nu^*(x_1) \) be as in Lemma 6. Let us adopt the convention of writing
\[
x > \Gamma^{(\kappa)} \text{ if } x_2 > \nu^*(x_1),
\]
(4.4)
and define \( x \geq \Gamma^{(\kappa)} \), \( x < \Gamma^{(\kappa)} \), and \( x \leq \Gamma^{(\kappa)} \) similarly. Observe that the sets
\[
\{ x \in \Omega \mid x > \Gamma^{(\kappa)} \}, \quad \{ x \in \Omega \mid x < \Gamma^{(\kappa)} \}, \quad \text{and} \quad \{ x \in \Omega \mid x \in \Gamma^{(\kappa)} \}
\]
are disjoint, and that their union equals \( \Omega \). Furthermore,
\[
\{ x \in \Omega \mid x \geq \Gamma^{(\kappa)} \} \cap \{ x \in \Omega \mid x \leq \Gamma^{(\kappa)} \} = \{ x \in \Omega \mid x \in \Gamma^{(\kappa)} \}.
\]
The next lemma characterizes the first hitting time of the vertical axis given initial condition \( x \), and shows that this hitting time is differentiable in \( x \). It is proved in Section B.1.2.

**Lemma 7.** Fix \( \kappa \geq \beta \) and \( x \in (-\infty, 0] \times [\kappa/\sqrt{n}, \infty) \). Provided it exists, define \( \tau(x) \) to be the smallest solution to
\[
\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-\eta} - \eta x_2 e^{-\eta} = 0, \quad \eta \geq 0,
\]
and define \( \tau(x) = \infty \) if no solution exists. Let \( \Gamma^{(\kappa)} \) be as in Lemma 6.

1. If \( x > \Gamma^{(\kappa)} \), then \( \tau(x) < \infty \) and
\[
x_2 e^{-\tau(x)} > \kappa/\sqrt{n},
\]
and if \( x \in \Gamma^{(\kappa)} \), then \( \tau(x) < \infty \) and \( x_2 e^{-\tau(x)} = \kappa/\sqrt{n} \).
2. If \( \kappa > \beta \), then the function \( \tau(x) \) is differentiable at all points \( x \geq \Gamma^{(\kappa)} \) with
\[
\tau_1(x) = -\frac{e^{-\tau(x)}}{x_2 e^{-\tau(x)} - \beta/\sqrt{n}} \leq 0, \quad \tau_2(x) = \tau_1(x) \tau(x) \leq 0, \quad x \geq \Gamma^{(\kappa)},
\]
where \( \tau_1(x) \) is understood to be the left derivative when \( x_1 = 0 \).
3. For any \( \kappa_1, \kappa_2 \) with \( \beta < \kappa_1 < \kappa_2 \),
\[
x \geq \Gamma^{(\kappa_2)} \text{ implies } x > \Gamma^{(\kappa_1)},
\]
i.e. the curve \( \Gamma^{(\kappa_2)} \) lies strictly above \( \Gamma^{(\kappa_1)} \).

Armed with Lemmas 6 and 7, we are now in a position to describe the solution to (3.1) and prove Lemma 4.

### 4.2 Proving Lemma 4

Fix \( \kappa > \beta \) and partition the set \( \Omega \) into three subdomains
\[
\{ x_2 \in [0, \kappa/\sqrt{n}] \}, \quad \{ x \leq \Gamma^{(\kappa)}, \ x_2 \geq \kappa/\sqrt{n} \}, \quad \text{and} \quad \{ x \geq \Gamma^{(\kappa)} \},
\]
where \( \Gamma^{(\kappa)} \) is as in Lemma 6. From (4.5) we know that \( x \geq \Gamma^{(\kappa)} \) implies \( x_2 \geq \kappa/\sqrt{n} \), and therefore any point in \( \Omega \) must indeed lie in one of the three subdomains.
Lemma 8. Fix $\kappa > \beta$ and let $\Gamma^{(k)}$ be as in Lemma 6, and $\tau(x)$ be as in Lemma 7. For $x \in \Omega$, let

$$f(x) = \begin{cases} 0, & x \in [0, \kappa/\sqrt{n}], \\ x_2 - \frac{\kappa}{\sqrt{n}} - \frac{\kappa}{\sqrt{n}} \log(\sqrt{n}x_2/\kappa), & x \leq \Gamma^{(k)} \text{ and } x_2 \geq \kappa/\sqrt{n}, \\ x_2(1 - e^{-\tau(x)}) - \frac{1}{\sqrt{n}} \tau(x) + \frac{1}{2} \frac{\sqrt{n}}{\kappa} (x_2 e^{-\tau(x)} - \kappa/\sqrt{n})^2, & x \geq \Gamma^{(k)}. \end{cases} \tag{4.8}$$

This function is well-defined and has absolutely continuous first-order partial derivatives. When $x \leq \Gamma^{(k)}$ and $x_2 \geq \kappa/\sqrt{n}$,

$$f_1(x) = 0, \quad f_2(x) = 1 - \frac{\kappa}{x_2 \sqrt{n}}, \quad f_{22}(x) = \frac{1}{x_2^2 \sqrt{n}}, \tag{4.9}$$

and when $x \geq \Gamma^{(k)}$,

$$f_1(x) = \frac{\sqrt{n}}{\beta} e^{-\tau(x)} (x_2 e^{-\tau(x)} - \kappa/\sqrt{n}), \tag{4.10}$$

$$f_2(x) = 1 - \frac{1}{x_2 \sqrt{n}} + \frac{\sqrt{n}}{\beta} (x_2 e^{-\tau(x)} - \kappa/\sqrt{n}) \left( \frac{x_2 e^{-\tau(x)} - \beta/\sqrt{n}}{x_2} + \tau(x) e^{-\tau(x)} \right). \tag{4.11}$$

A full proof of the lemma is postponed to Section B.2, but the following is an outline containing the main bits of intuition. We write down (4.8) motivated by (4.2) and the behavior of the fluid model. At this point, we can think of (4.8) as a standalone mathematical object that we simply guess to be a solution to (3.1). Deriving the forms of the derivatives in (4.9)-(4.11) is purely an algebraic exercise that relies on the characterization of $\tau(x)$ from Lemma 7.

Proof of Lemma 4. Let $f(x)$ be as in Lemma 8. It is straightforward to verify that our candidate $f(x)$ solves the PDE (3.1) with $h(x) = ((x_2 - \kappa/\sqrt{n}) \vee 0)$ there; for the reader wishing to verify this, recall that $\tau(x) = 0$ when $x_1 = 0$. We can now prove Lemma 4.

We now prove that (3.3)–(3.5) hold when $x \geq \Gamma^{(k)}$. We omit the proof when $x \in \{x_2 \in [0, \kappa/\sqrt{n}]\}$ and $x \in \{x \leq \Gamma^{(k)}, \ x_2 \geq \kappa/\sqrt{n}\}$ because those cases are simple. Differentiating (4.10) and using (4.6) one arrives at

$$f_{11}(x) = \frac{\sqrt{n}}{\beta} e^{-2\tau(x)} \frac{x_2 e^{-\tau(x)} + (x_2 e^{-\tau(x)} - \kappa/\sqrt{n})}{(x_2 e^{-\tau(x)} - \beta/\sqrt{n})},$$

from which we conclude that

$$0 \leq f_{11}(x) \leq \frac{\sqrt{n}}{\beta} \left( \frac{x_2 e^{-\tau(x)}}{x_2 e^{-\tau(x)} - \beta/\sqrt{n}} + 1 \right) = \frac{\sqrt{n}}{\beta} \left( \frac{1}{\frac{1}{\beta} - \frac{\beta}{x_2 e^{-\tau(x)}}} + 1 \right) \leq \frac{\sqrt{n}}{\beta} \left( \frac{1}{\frac{1}{\beta} - \frac{\beta}{\kappa}} + 1 \right),$$

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where all three inequalities above follow from the fact that $x_2 e^{-\tau(x)} \geq \kappa/\sqrt{n} > \beta/\sqrt{n}$; c.f. (4.5) in Lemma 7. Taking the derivative in (4.10) with respect to $x_2$, we see that
\[
f_{12}(x) = \frac{\sqrt{n}}{\beta} (-\tau_2(x)e^{-\tau(x)})(x_2 e^{-\tau(x)} - \kappa/\sqrt{n}) + \frac{\sqrt{n}}{\beta} e^{-\tau(x)}(e^{-\tau(x)} - \tau_2(x)x_2 e^{-\tau(x)}).
\]
The quantity above is non-negative because $x_2 e^{-\tau(x)} \geq \kappa/\sqrt{n}$ and $-\tau_2(x) \geq 0$; the latter follows from (4.6). Lastly, we can differentiate (4.11) and use $\tau_2(x) = \tau_1(x)\tau(x)$ from (4.6) to see that
\[
f_{22}(x) = \frac{1}{x_2^2} \frac{\kappa}{\sqrt{n}} + \frac{\sqrt{n}}{\beta} \left(e^{-\tau(x)} - \tau_1(x)\tau(x)x_2 e^{-\tau(x)}\right)\left(\frac{x_2 e^{-\tau(x)} - \beta/\sqrt{n}}{x_2} + \tau(x)e^{-\tau(x)}\right)
+ \frac{\sqrt{n}}{\beta} \left(x_2 e^{-\tau(x)} - \kappa/\sqrt{n}\right)\left(\frac{\beta/\sqrt{n}}{x_2^2} - \tau_1(x)\tau^2(x)e^{-\tau(x)}\right).
\]
Again, $f_{22}(x) \geq 0$ because $x_2 e^{-\tau(x)} \geq \kappa/\sqrt{n}$ and $-\tau_1(x) \geq 0$. Let us now bound $f_{22}(x)$. The first term on the right hand side above is bounded by $\sqrt{n}/\kappa$, because $x \geq \Gamma^{(\kappa)}$ implies $x_2 \geq \kappa/\sqrt{n}$. For the second term, note that
\[
x_2 e^{-\tau(x)} - \beta/\sqrt{n} + \tau(x)e^{-\tau(x)} \leq 1 + e^{-1} \leq 2,
\]
and using the form of $\tau_1(x)$ from (4.6),
\[
e^{-\tau(x)} - \tau_1(x)\tau(x)x_2 e^{-\tau(x)} = e^{-\tau(x)} + \frac{e^{-\tau(x)}}{x_2 e^{-\tau(x)} - \beta/\sqrt{n}} \tau x_2 e^{-\tau(x)}
= e^{-\tau(x)} + \frac{\tau e^{-\tau(x)}}{1 - \frac{\beta/\sqrt{n}}{x_2 e^{-\tau(x)}}} \leq 1 + \frac{e^{-1}}{1 - \frac{\beta}{\kappa}} \leq 1 + \frac{\kappa}{\kappa - \beta}.
\]
For the third term, observe that
\[
\begin{align*}
&\left(x_2 e^{-\tau(x)} - \kappa/\sqrt{n}\right)\left(\frac{\beta/\sqrt{n}}{x_2^2} - \tau_1(x)\tau^2(x)e^{-\tau(x)}\right)
\leq \left(x_2 e^{-\tau(x)} - \kappa/\sqrt{n}\right)\left(\frac{1}{x_2 \kappa} - \tau_1(x)\tau^2(x)e^{-\tau(x)}\right)
= x_2 e^{-\tau(x)} - \kappa/\sqrt{n} \beta x_2^2 \kappa + x_2 e^{-\tau(x)} - \kappa/\sqrt{n} \frac{\beta}{\kappa} \tau^2(x)e^{-2\tau(x)}
\leq \frac{\beta}{\kappa} + 1 \leq 2,
\end{align*}
\]
where we use $x_2 \geq \kappa/\sqrt{n}$ in the first inequality, the form of $\tau_1(x)$ in (4.6) in the first equation, and the fact that $\beta < \kappa$ in the last two inequalities. Combining the bounds on all three terms, we conclude that
\[
f_{22}(x) \leq \frac{\sqrt{n}}{\kappa} + \frac{\sqrt{n}}{\beta} 2\left(1 + \frac{\kappa}{\beta - \kappa}\right) + 2\frac{\sqrt{n}}{\beta} \leq \frac{\sqrt{n}}{\beta} \left(5 + \frac{2\kappa}{\beta - \kappa}\right),
\]
where in the last inequality we used the fact that $\sqrt{n}/\kappa < \sqrt{n}/\beta$. \hfill \qedsymbol
Acknowledgements

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References

[1] ATAR, R. (2012). A diffusion regime with nondegenerate slowdown. Operations Research, 60 490–500. URL http://dx.doi.org/10.1287/opre.1110.1030.

[2] BILLINGSLEY, P. (1999). Convergence of probability measures. 2nd ed. Wiley, New York.

[3] BRAMSON, M. (2011). Stability of join the shortest queue networks. Ann. Appl. Probab., 21 1568–1625. URL https://doi.org/10.1214/10-AAP726.

[4] BRAVERMAN, A. and DAI, J. G. (2017). Stein’s method for steady-state diffusion approximations of $M/Ph/n + M$ systems. Annals of Applied Probability, 27 550–581. URL http://dx.doi.org/10.1214/16-AAP1211.

[5] BRAVERMAN, A., DAI, J. G. and FENG, J. (2016). Stein’s method for steady-state diffusion approximations: an introduction through the Erlang-A and Erlang-C models. Stochastic Systems, 6 301–366. URL http://www.i-journals.org/ssy/viewarticle.php?id=212&layout=abstract.

[6] BUDHIRAJA, A. and LEE, C. (2009). Stationary distribution convergence for generalized Jackson networks in heavy traffic. Math. Oper. Res., 34 45–56. URL http://dx.doi.org/10.1287/moor.1080.0353.

[7] CORLESS, R. M., GONNET, G. H., HARE, D. E. G., JEFFREY, D. J. and KNUTH, D. E. (1996). On the Lambert W function. Advances in Computational Mathematics, 5 329–359. URL https://doi.org/10.1007/BF02124750.

[8] ERYILMAZ, A. and SRIKANT, R. (2012). Asymptotically tight steady-state queue length bounds implied by drift conditions. Queueing Systems, 72 311–359. URL http://dx.doi.org/10.1007/s11134-012-9305-y.

[9] ESCHENFELDT, P. and GAMARNIK, D. (2015). Join the shortest queue with many servers. the heavy traffic asymptotics. URL https://arxiv.org/abs/1502.00999.

[10] FENG, J. and SHI, P. (2017). Steady-state diffusion approximations for discrete-time queue in hospital inpatient flow management. URL https://arxiv.org/abs/1612.00790.

[11] FLATTO, L. and MCKEAN, H. P. (1977). Two queues in parallel. Communications on Pure and Applied Mathematics, 30 255–263. URL https://dx.doi.org/10.1002/cpa.3160300206.
[12] Foschini, G. J. and Salz, J. (1978). A basic dynamic routing problem and diffusion. *IEEE Transactions on Communications*, **26** 320–327. URL https://dx.doi.org/10.1109/TCOM.1978.1094075.

[13] Gamarnik, D. and Stolyar, A. L. (2012). Multiclass multiserver queueing system in the Halfin-Whitt heavy traffic regime: asymptotics of the stationary distribution. *Queueing Systems*, **71** 25–51. URL http://dl.acm.org/citation.cfm?id=2339029.

[14] Gamarnik, D. and Zeevi, A. (2006). Validity of heavy traffic steady-state approximation in generalized Jackson networks. *Ann. Appl. Probab.*, **16** 56–90. URL http://dx.doi.org/10.1214/105051605000000638.

[15] Gast, N. (2017). Expected values estimated via mean-field approximation are 1/n-accurate. *Proc. ACM Meas. Anal. Comput. Syst.*, **1** 17:1–17:26. URL http://doi.acm.org/10.1145/3084454.

[16] Gast, N. and Van Houdt, B. (2017). A Refined Mean Field Approximation. *Proceedings of the ACM on Measurement and Analysis of Computing Systems*, **1**. URL https://hal.inria.fr/hal-01622054.

[17] Glynn, P. W. and Zeevi, A. (2008). Bounding stationary expectations of Markov processes. In *Markov processes and related topics: a Festschrift for Thomas G. Kurtz*, vol. 4 of *Inst. Math. Stat. Collect.* Inst. Math. Statist., Beachwood, OH, 195–214. URL http://dx.doi.org/10.1214/074921708000000381.

[18] Gupta, V. and Walton, N. (2017). Load balancing in the non-degenerate slowdown regime. URL https://arxiv.org/abs/1707.01969.

[19] Gurvich, I. (2014). Diffusion models and steady-state approximations for exponentially ergodic Markovian queues. *The Annals of Applied Probability*, **24** 2527–2559. URL http://dx.doi.org/10.1214/13-AAP984.

[20] Gurvich, I. (2014). Validity of heavy-traffic steady-state approximations in multiclass queueing networks: the case of queue-ratio disciplines. *Mathematics of Operations Research*, **39** 121–162. URL http://dx.doi.org/10.1287/moor.2013.0593.

[21] Haight, F. A. (1958). Two queues in parallel. *Biometrika*, **45** 401–410. URL http://dx.doi.org/10.1093/biomet/45.3-4.401.

[22] Halfin, S. (1985). The shortest queue problem. *Journal of Applied Probability*, **22** 865–878. URL http://www.jstor.org/stable/3213954.

[23] Halfin, S. and Whitt, W. (1981). Heavy-traffic limits for queues with many exponential servers. *Oper. Res.*, **29** 567–588.

[24] Hanqin, Z. and Rongxin, W. (1989). Heavy traffic limit theorems for a queueing system in which customers join the shortest line. *Advances in Applied Probability*, **21** 451–469. URL http://www.jstor.org/stable/1427169.
[25] Henderson, S. G. (1997). Variance reduction via an approximating Markov process. Ph.D. thesis, Department of Operations Research, Stanford University. http://people.orie.cornell.edu/shane/pubs/thesis.pdf.

[26] Huang, J. and Gurvich, I. (2016). Beyond heavy-traffic regimes: universal bounds and controls for the single-server queue. Submitted for publication, URL http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2784752.

[27] Katsuda, T. (2010). State-space collapse in stationarity and its application to a multiclass single-server queue in heavy traffic. Queueing Syst., 65 237–273. URL http://dx.doi.org/10.1007/s11134-010-9178-x.

[28] Kingman, J. F. C. (1961). Two similar queues in parallel. The Annals of Mathematical Statistics, 32 1314–1323.

[29] Maguluri, S. T., Burle, S. K. and Srikant, R. (2016). Optimal heavy-traffic queue length scaling in an incompletely saturated switch. SIGMETRICS Perform. Eval. Rev., 44 13–24. URL http://doi.acm.org/10.1145/2964791.2901466.

[30] Maguluri, S. T. and Srikant, R. (2016). Heavy traffic queue length behavior in a switch under the maxweight algorithm. Stochastic Systems, 6 211–250. URL http://dx.doi.org/10.1214/15-SSY193.

[31] Meyn, S. P. and Tweedie, R. L. (1993). Stability of Markovian processes III: Foster-Lyapunov criteria for continuous time processes. Adv. Appl. Probab., 25 518–548.

[32] Mukherjee, D., Borst, S. C., van Leeuwaarden, J. S. H. and Whiting, P. A. (2016). Universality of load balancing schemes on the diffusion scale. J. Appl. Probab., 53 1111–1124. URL https://projecteuclid.org:443/euclid.jap/1481132840.

[33] Stein, C. (1986). Approximate computation of expectations. Lecture Notes-Monograph Series, 7. URL http://www.jstor.org/stable/4355512.

[34] Stolyar, A. L. (2015). Pull-based load distribution in large-scale heterogeneous service systems. Queueing Systems, 80 341–361. URL https://doi.org/10.1007/s11134-015-9448-8.

[35] Stolyar, A. L. (2015). Tightness of stationary distributions of a flexible-server system in the Halfin-Whitt asymptotic regime. Stoch. Syst., 5 239–267. URL http://dx.doi.org/10.1214/14-SSY139.

[36] Tezcan, T. (2008). Optimal control of distributed parallel server systems under the Halfin and Whitt regime. Mathematics of Operations Research, 33 51–90. URL http://search.proquest.com/docview/212618995?accountid=10267.

[37] Ye, H.-Q. and Yao, D. D. (2012). A stochastic network under proportional fair resource control—diffusion limit with multiple bottlenecks. Operations Research, 60 716–738. URL http://dx.doi.org/10.1287/opre.1120.1047.
A Miscellaneous Proofs

This appendix contains proofs to a few miscellaneous lemmas used in the paper.

A.1 Lemma 1

Proof of Lemma 1. A sufficient condition to ensure that

\[ \mathbb{E}[G_Q f(Q)] = 0 \]

is given by [25, Proposition 1.1] (alternatively, see [17, Proposition 3]). Namely, we require that

\[ \mathbb{E}[|G_Q(Q, Q)f(Q)|] < \infty, \tag{A.1} \]

where \( G_Q(q, q) \) is the diagonal entry of the generator matrix \( G_Q \) corresponding to state \( q \in S \). It is not hard to check that in the JSQ system, \( |G_Q(q, q)| < n\lambda + n \) for all states \( q \in S \). Our assumption that \( \mathbb{E}[f(Q)] < \infty \), is enough to ensure (A.1) is satisfied. \( \square \)

A.2 Lemma 3 (generator expansion)

Proof of Lemma 3. The CTMC generator satisfies

\[ G_X Af(x) = n\lambda_1(q_1 < n)(f(x_1 + 1/n, x_2) - f(x_1, x_2)) \]
\[ + n\lambda_1(q_1 = n, q_2 < n)(f(x_1, x_2 + 1/n) - f(x_1, x_2)) \]
\[ + (q_1 - q_2)(f(x_1 - 1/n, x_2) - f(x_1, x_2)) \]
\[ + (q_2 - q_1)(f(x_1, x_2 - 1/n) - f(x_1, x_2)). \tag{A.2} \]
It is straightforward to verify that

\begin{align*}
  f(x + e^{(1)}/n) - f(x) &= \frac{1}{n} f_1(x) + \int_{x_1}^{x_1 + 1/n} (x_1 + 1/n - u)f_{11}(u,x_2)du, \\
  f(x - e^{(1)}/n) - f(x) &= -\frac{1}{n} f_1(x) + \int_{x_1 - 1/n}^{x_1} (u - (x_1 - 1/n))f_{11}(u)du, \quad \text{(A.3)}
\end{align*}

and that a similar expansion holds for \( f(x + e^{(2)}/n) \pm f(x) \). Applying (A.3) to (A.2) (but leaving the \( q_3 \) term untouched), we see that

\begin{align*}
  G_X Af(x) &= f_1(x)\frac{1}{n} (n\lambda 1(q_1 < n) - (q_1 - q_2)) + f_2(x)\frac{1}{n} (n\lambda 1(q_1 = n, q_2 < n) - q_2) \\
  &+ n\lambda 1(q_1 < n) \int_{x_1}^{x_1 + 1/n} (x_1 + 1/n - u)f_{11}(u)du \\
  &+ n\lambda 1(q_1 = n, q_2 < n) \int_{x_2}^{x_2 + 1/n} (x_2 + 1/n - u)f_{22}(u)du \\
  &+ (q_1 - q_2) \int_{x_1 - 1/n}^{x_1} (u - (x_1 - 1/n))f_{11}(u)du \\
  &+ q_2 \int_{x_2 - 1/n}^{x_2} (u - (x_2 - 1/n))f_{22}(u)du \\
  &- q_3 (f(x_1, x_2 - 1/n) - f(x_1, x_2)). \quad \text{(A.4)}
\end{align*}

To conclude, we rewrite the first line of (A.4) as

\begin{align*}
  f_1(x)\frac{1}{n} (n\lambda - (q_1 - q_2)) - f_2(x)\frac{1}{n} q_2 \\
  &+ (f_2(x) - f_1(x))\lambda 1(q_1 = n) - f_2(x)\lambda 1(q_1 = q_2 = n) \\
  &= f_1(x)(-\beta/\sqrt{n} - x_1 + x_2 - x_2 f_2(x) \\
  &+ (f_2(x) - f_1(x))\lambda 1(q_1 = n) - f_2(x)\lambda 1(q_1 = q_2 = n) \\
  &= Lf(x) + (f_2(x) - f_1(x))\lambda 1(q_1 = n) - f_2(x)\lambda 1(q_1 = q_2 = n).
\end{align*}

\[ \square \]

\section{Technical Lemmas: Section 4}

In this appendix we prove the key technical lemmas from Section 4. Section B.1 has the proofs for Lemmas 6 and 7 and Section B.2 has the proof for Lemma 8.

\subsection{Lemmas in Section 4.1}

A function known as the Lambert W function will play a central role here; the following discussion is based on [7]. Define \( W(x) \) as the solution to

\[ x = W(x)e^{W(x)}, \quad x \in [-e^{-1}, \infty). \quad \text{(B.1)} \]
Figure 2: A plot of $W(x)$ taken from [7]. For $x \leq 0$, the dashed line represents $W_{-1}(x)$ and the solid line represents $W_0(x)$.

The function $W(x)$ exists and is known as the Lambert W function. Taking logarithms on both sides of (B.1),

$$W(x) = \log x - \log W(x).$$  \hspace{1cm} (B.2)

As is depicted in the plot of $W(x)$ in Figure 2, $W(-e^{-1}) = -1$, $W(0) = 0$, and $W(x) \to \infty$ as $x \to \infty$. Furthermore, $W(x)$ is multi-valued for $x \in (-e^{-1}, 0)$, where it is separated into two ‘branches’ $W_0(x)$ and $W_{-1}(x)$; the former is commonly called the principal branch. We will also need to use the fact that $W(x)$ and $W_0(x)$ are differentiable for $x > 0$ and $x \in (-e^{-1}, 0)$, respectively, and that

$$W'(x) = \frac{W(x)}{x(1 + W(x))} > 0, \quad x \in (-e^{-1}, 0) \cup (0, \infty);$$  \hspace{1cm} (B.3)

c.f. section 3 of [7]. Going forward, we adopt the convention of using $W(x)$ to mean $W_0(x)$ for negative values of $x$. A useful property of $W(x)$ is that

$$x = W(xe^x), \quad x \geq -1.$$  \hspace{1cm} (B.4)

This can be seen by applying $W(x)$ to both sides of (B.1) and using the fact that the range of $W(x)$ is $[-1, \infty]$ (again, we are using the convention $W(x) = W_0(x)$ for $x \in (-e^{-1}, 0)$). Furthermore, $W(x)$ is invertible; indeed, $W^{-1}(x) = xe^x$ due to (B.4).
B.1.1 Proving Lemma 6

We first prove a technical result about $W(x)$, and then prove Lemma 6.

**Lemma 9.** Fix $\kappa \geq \beta$ and $x_1 \leq 0$. The equation

$$W\left(\frac{-\beta}{\sqrt{n}} e^{-\frac{(x_1 + \beta/\sqrt{n})}{\nu}}\right) = -\frac{\beta}{\kappa}$$

has a unique solution $\nu^* \geq \kappa/\sqrt{n}$. Furthermore,

$$\frac{d}{d\nu}\left(\frac{-\beta}{\sqrt{n}} e^{-\frac{(x_1 + \beta/\sqrt{n})}{\nu}}\right) > 0, \quad \nu \geq \nu^*.$$  

**Proof of Lemma 9.** Let

$$f(\nu) = \frac{-\beta}{\sqrt{n}} e^{-\frac{(x_1 + \beta/\sqrt{n})}{\nu}}.$$ 

Since $\kappa \geq \beta$ and $W(x)$ is one-to-one (recall our convention that $W(x) = W_0(x)$ for $x \leq 0$), (B.4) implies that (B.5) is satisfied if and only if

$$f(\nu) = \frac{-\beta}{\kappa} e^{-\beta/\kappa}.$$ 

If $x_1 = 0$, then $\nu = \kappa/\sqrt{n}$ is the unique solution, and so we assume that $x_1 < 0$ and argue that (B.7) has a unique solution. Since the domain of $W(x)$ is $[-e^{-1}, \infty)$, we can only consider $\nu$ large enough such that $f(\nu) \leq -1$. Observe that

$$f(\kappa/\sqrt{n}) = \frac{-\beta}{\kappa} e^{-\beta/\kappa}$$ 

Differentiating,

$$f'(\nu) = \frac{\beta}{\nu^2} e^{-\frac{(x_1 + \beta/\sqrt{n})}{\nu^2}} - \frac{\beta}{\nu} e^{-\frac{(x_1 + \beta/\sqrt{n})}{\nu}} \left(\frac{x_1 + \beta/\sqrt{n}}{\nu^2}\right)$$

We know that $f'(\nu) \to 0$ as $\nu \to \infty$.

**Case 1:** $x_1 + \beta/\sqrt{n} \leq 0$. In this case $f'(\nu) > 0$ for all $\nu > 0$, which implies that there exists a unique $\nu^*$ such that (B.7) is satisfied.

**Case 2:** $x_1 + \beta/\sqrt{n} \geq 0$. The form of $f'(\nu)$ tells us that $f(\nu)$ is decreasing on $(0, x_1 + \beta/\sqrt{n})$, but starts increasing after that. This again implies that a unique $\nu^*$ exists, and that $\nu^* \geq x_1 + \beta/\sqrt{n}$, implying $f'(\nu) > 0$ for all $\nu \geq \nu^*$.

**Proof of Lemma 6.** Fix $\kappa \geq \beta$ and $x_1 \leq 0$. We begin by showing that the system (4.3) has a unique solution. The first step is to write $\eta$ in terms of $\nu$. We rearrange

$$-\frac{\beta}{\sqrt{n}} + (x_1 + \beta/\sqrt{n})e^{-\eta} + \eta e^{-\eta} = 0$$

(B.8)
\[
\frac{-\beta}{\sqrt{n}} \nu = -(x_1 + \beta / \sqrt{n}) e^{-\eta} - \eta e^{-\eta} = \left( -\frac{(x_1 + \beta / \sqrt{n})}{\nu} - \eta \right) e^{-\eta},
\]
or
\[
\left( -\frac{(x_1 + \beta / \sqrt{n})}{\nu} - \eta \right) e^{-\frac{(x_1 + \beta / \sqrt{n})}{\nu} - \eta} = -\frac{\beta}{\sqrt{n}} e^{-\frac{(x_1 + \beta / \sqrt{n})}{\nu}}. \tag{B.9}
\]
Observe that the left hand side of (B.9) is in the form \(-xe^{-x}\), and so must lie in \([-e^{-1}, \infty)\). Therefore, existence of a solution to (4.3) imposes a natural constraint on \(\nu\) that the right hand side above must lie in \([-e^{-1}, \infty)\). Assuming this is the case, we apply \(W(x)\) to both sides of (B.9) and using (B.4), we arrive at
\[
\eta = -\frac{(x_1 + \beta / \sqrt{n})}{\nu} - W\left( -\frac{\beta}{\sqrt{n}} e^{-\frac{(x_1 + \beta / \sqrt{n})}{\nu}} \right). \tag{B.10}
\]
Since the Lambert W function is multivalued for \(x \leq 0\), the above equation tells us that given \(\nu\), there can be two potential choices for \(\eta\). Plugging the above form of \(\eta\) back into (B.8), we see that
\[
-\frac{\beta}{\sqrt{n}} + -W\left( -\frac{\beta}{\sqrt{n}} e^{-\frac{(x_1 + \beta / \sqrt{n})}{\nu}} \right) \nu e^{-\eta} = 0,
\]
which, after using the fact that \(\nu e^{-\eta} = \kappa / \sqrt{n}\), becomes
\[
W\left( -\frac{\beta}{\sqrt{n}} e^{-\frac{(x_1 + \beta / \sqrt{n})}{\nu}} \right) = -\frac{\beta}{\kappa}. \tag{B.11}
\]
Lemma 9 tells us that (B.11) does indeed have a unique solution \(\nu^*(x_1)\). From (B.10) we know \(\eta\) can have up to two values, but we narrow this number down to one using the fact that \(\nu e^{-\eta} = \kappa / \sqrt{n}\). As an aside, it can be verified that \(\nu^*(0) = \kappa / \sqrt{n}\), and \(\eta^*(0) = 0\).

We now prove the second claim in the lemma that \(\gamma^{(\kappa)}(x_1) \subset \Gamma^{(\kappa)}\) for every \(x_1 \leq 0\). Recall that
\[
\gamma^{(\kappa)}(x_1) = \left\{ \left( -\frac{\beta}{\sqrt{n}} + (x_1 + \beta / \sqrt{n}) e^{-t} + t \nu^*(x_1) e^{-t}, \nu^*(x_1) e^{-t} \right) \bigg| t \in [0, \eta^*(x_1)) \right\}.
\]
Given \(x_1 \leq 0\) and \(t \in [0, \eta^*(x_1)]\), define
\[
\bar{x}_1 = -\beta / \sqrt{n} + (x_1 + \beta / \sqrt{n}) e^{-t} + t \nu^*(x_1) e^{-t}.
\]
By uniqueness of \(\nu^*(\bar{x}_1)\) and \(\eta^*(\bar{x}_1)\), it suffices to show that the pair
\[
\nu = \nu^*(x_1) e^{-t}, \quad \eta = \eta^*(x_1) - t
\]
solves (4.3) with \(\bar{x}_1\) replacing \(x_1\) there. Indeed,
\[
\nu^*(x_1) e^{-t} e^{-(\eta^*(x_1) - t)} = \nu^*(x_1) e^{-\eta^*(x_1)} = \kappa / \sqrt{n},
\]
which completes the proof of the lemma.
and
\[-\beta/\sqrt{n} + (x_1 + \beta/\sqrt{n})e^{-(\eta^*(x_1) - t)} + (\eta^*(x_1) - t)\nu^*(x_1)e^{-t}e^{-(\eta^*(x_1) - t)}\]
\[= -\beta/\sqrt{n} + (x_1 + \beta/\sqrt{n})e^{-(\eta^*(x_1) - t)} + (\eta^*(x_1) - t)\nu^*(x_1)e^{-\eta^*(x_1)}\]
\[= -\beta/\sqrt{n} + ((x_1 + \beta/\sqrt{n})e^{-t} + t\nu^*(x_1)e^{-t})e^{-(\eta^*(x_1) - t)} + (\eta^*(x_1) - t)\nu^*(x_1)e^{-\eta^*(x_1)}\]
\[= -\beta/\sqrt{n} + (x_1 + \beta/\sqrt{n})e^{-\eta^*(x_1)} + \eta^*(x_1)\nu^*(x_1)e^{-\eta^*(x_1)} = 0.\]

\(\blacksquare\)

**B.1.2 Proving Lemma 7**

*Proof of Lemma 7.* Fix \(\kappa \geq \beta\) and \(x \in \Omega\). Assume that \(x \geq \Gamma(\kappa)\), which by definition in (4.4) implies that there exists some \(\delta \geq 0\) such that
\[(x_1, x_2 - \delta) \in \Gamma(\kappa).\]

Note that if \(x > \Gamma(\kappa)\), then \(\delta > 0\).

Let us prove (4.5). Consider the equation
\[\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-\eta} - \eta x_2 e^{-\eta} = 0.\]  
(B.12)

We first argue that
\[\eta = -\frac{(x_1 + \beta/\sqrt{n})}{x_2} - W\left(-\frac{\beta/\sqrt{n}}{x_2} e^{-(x_1 + \beta/\sqrt{n})}\right).\]  
(B.13)

Starting with (B.12), we can replicate the steps used to get (B.9) to see that (B.12) is equivalent to
\[\left(-\frac{(x_1 + \beta/\sqrt{n})}{x_2} - \eta\right)e^{-(x_1 + \beta/\sqrt{n})} - \eta = -\frac{\beta/\sqrt{n}}{x_2} e^{-(x_1 + \beta/\sqrt{n})}.\]  
(B.14)

Let us assume that \(x \geq \Gamma(\kappa)\) implies that the right hand side of (B.14) is in the interval \([-e^{-1}, 0]\); we postpone the verification of this claim for now. We can apply \(W(\cdot)\) to both sides of (B.14) and use (B.4) to conclude (B.13). Plugging (B.13) back into (B.12),
\[-\beta/\sqrt{n} - W\left(-\frac{\beta/\sqrt{n}}{x_2} e^{-(x_1 + \beta/\sqrt{n})}\right)x_2 e^{-\eta} = 0,\]  
(B.15)
or
\[x_2 e^{-\eta} = \frac{-\beta/\sqrt{n}}{W\left(-\frac{\beta/\sqrt{n}}{x_2} e^{-(x_1 + \beta/\sqrt{n})}\right)} \geq \frac{-\beta/\sqrt{n}}{W\left(-\frac{\beta/\sqrt{n}}{x_2 - \delta} e^{-(x_1 + \beta/\sqrt{n})}\right)} = \kappa/\sqrt{n}.\]

Observe that the inequality above is strict if \(x > \Gamma(\kappa)\), and that it becomes an equality if \(\delta = 0\) (which means that \(x \in \Gamma(\kappa)\)).
To conclude the proof of (4.5), it remains verify our assumption that \( x \geq \Gamma^{(\kappa)} \) implies that the right hand side of (B.14) is in the interval \([-e^{-1}, 0)\). Equation (B.11) in the proof of Lemma 6 tells us that \((x_1, x_2 - \delta) \in \Gamma^{(\kappa)} \) implies
\[
W\left( \frac{-\beta/\sqrt{n}}{x_2 - \delta} \frac{-(x_1 + \beta/\sqrt{n})}{x_2 - \delta} \right) = -\frac{\beta}{\kappa},
\]
or that
\[
-\frac{\beta}{\sqrt{n}} \frac{-(x_1 + \beta/\sqrt{n})}{x_2 - \delta} = W^{-1}(-\beta/\kappa) \geq W^{-1}(-1) = -e^{-1},
\]
where in the inequality above we used the fact that \( \kappa \geq \beta \) and that \( W^{-1}(\cdot) \) is an increasing function. Now from (B.6) we know that
\[
d\frac{d}{d\nu} \left( -\frac{\beta}{\sqrt{n}} e^{-\frac{(x_1 + \beta/\sqrt{n})}{\nu}} \right) > 0, \quad \nu \geq x_2 - \delta,
\]
which implies
\[
-\frac{\beta}{\sqrt{n}} \frac{-(x_1 + \beta/\sqrt{n})}{x_2 - \delta} \geq -\frac{\beta}{\sqrt{n}} \frac{-(x_1 + \beta/\sqrt{n})}{x_2 - \delta} = W^{-1}(-\beta/\kappa) \geq -e^{-1}.
\]
This concludes the proof of (4.5).

We now address the differentiability of \( \tau(x) \) to prove (4.6). Fixing \( \kappa > \beta \) and \( x \geq \Gamma^{(\kappa)} \), we see from (B.13) that
\[
\tau(x) = -\frac{(x_1 + \beta/\sqrt{n})}{x_2} - W\left( \frac{-\beta/\sqrt{n}}{x_2} \frac{-(x_1 + \beta/\sqrt{n})}{x_2} \right).
\]
We know that \( W'(u) \) exists for \( u \in (-e^{-1}, 0) \), and that
\[
-\frac{\beta}{\sqrt{n}} \frac{-(x_1 + \beta/\sqrt{n})}{x_2} > -e^{-1}, \quad x \geq \Gamma^{(\kappa)},
\]
which can be derived from (B.16). Therefore, \( \tau(x) \) is differentiable at all points \( x \geq \Gamma^{(\kappa)} \) with \( x_1 < 0 \). Only the one-sided derivative exists for \( x \in \{x_1 = 0, \ x \geq \Gamma^{(\kappa)} \} \), i.e. those \( x \) that are on the vertical axis. To characterize the derivatives of \( \tau(x) \), let us use the form
\[
\tau(x) = -\frac{(x_1 + \beta/\sqrt{n})}{x_2} + \frac{\beta/\sqrt{n}}{x_2 e^{-\tau(x)}},
\]
which is implied by (B.13) and (B.15). Differentiating gives us
\[
\tau_1(x) = \frac{1}{x_2} \left( \frac{\beta}{\sqrt{n}} \right)^{-1} = \frac{1}{x_2} x_2 e^{-\tau(x)} = \frac{e^{-\tau(x)}}{x_2 e^{-\tau(x)} - \beta/\sqrt{n}}.
\]
where $\tau_1(x)$ is understood to be the left derivative when $x_1 = 0$. Note that $x \geq \Gamma(\kappa)$ means the denominator in $\tau_1(x)$ is strictly positive due to our recently proved (4.5). Furthermore,

$$
\tau_2(x) = \frac{x_1 + \beta/\sqrt{n}}{x_2^2} - \frac{\beta/\sqrt{n}}{x_2^2 e^{-\tau(x)}} \tau_2(x) + \frac{\beta/\sqrt{n}}{x_2 e^{-\tau(x)}} = -\frac{1}{x_2} \tau(x) + \tau_2(x) \frac{\beta/\sqrt{n}}{x_2 e^{-\tau(x)}},
$$

and so

$$
\tau_2(x) = -\frac{1}{x_2} \tau(x) \left(1 - \frac{\beta/\sqrt{n}}{x_2 e^{-\tau(x)}}\right)^{-1} = \tau_1(x) \tau(x).
$$

This proves (4.6), and now prove the last claim of the lemma. Fix $x = (x_1, x_2)$ and assume that $x \geq \Gamma(\kappa)$. By (4.5), we know that $x_2 e^{-\tau(x)} \geq \kappa_2/\sqrt{n} > \kappa_1/\sqrt{n}$. Now

$$
\frac{d}{dx_2} x_2 e^{-\tau(x)} = e^{-\tau(x)} - \tau_2(x) x_2 e^{-\tau(x)} > 0, \quad x \geq \Gamma(\kappa),
$$

where the inequality follows from the form of $\tau_2(x)$ in (4.6). Therefore,

$$(x_2 + \varepsilon) e^{-\tau(x_1, x_2 + \varepsilon)} \geq \kappa_2/\sqrt{n} > \kappa_1/\sqrt{n}, \quad \varepsilon \geq 0.
$$

In other words, $(x_1, x_2 + \varepsilon) \not\in \Gamma(\kappa)$ for all $\varepsilon \geq 0$ by definition of $\Gamma(\kappa)$ in Lemma 6. However, also by Lemma 6, there must exist some $\tilde{x}_2 \geq 0$ such that $(x_1, \tilde{x}_2) \in \Gamma(\kappa_1)$, which means that $\tilde{x}_2 = x_2 - \tilde{\varepsilon}$ for some $\tilde{\varepsilon} > 0$, or that $x \not\in \Gamma(\kappa_1)$.

**B.2 Lemmas in Section 4.2**

**Proof of Lemma 8.** We first prove (4.9)-(4.11), and then make sure the first-order partial derivatives of $f(x)$ are absolutely continuous. Verification of (4.9) is omitted because it is straightforward from (4.8). Let us prove (4.10). Recalling that

$$
f(x) = x_2 (1 - e^{-\tau(x)}) - \frac{\kappa}{\sqrt{n}} \tau_1(x) + \frac{1}{2} \frac{\sqrt{n}}{\beta} (x_2 e^{-\tau(x)} - \kappa/\sqrt{n})^2, \quad x \geq \Gamma(\kappa),
$$

we differentiate to see that

$$
f_1(x) = \tau_1(x) x_2 e^{-\tau(x)} - \frac{\kappa}{\sqrt{n}} \tau_1(x) + \frac{\sqrt{n}}{\beta} (x_2 e^{-\tau(x)} - \kappa/\sqrt{n}) (-x_2 \tau_1(x) e^{-\tau(x)})
$$

$$
= -\tau_1(x) \left(- x_2 e^{-\tau(x)} + \frac{\kappa}{\sqrt{n}} + x_2^2 e^{-2\tau(x)} \frac{\sqrt{n}}{\beta} - x_2 e^{-\tau(x)} \frac{\kappa}{\beta}\right)
$$

$$
= -\tau_1(x) \frac{\sqrt{n}}{\beta} \left(x_2 e^{-2\tau(x)} - \frac{\beta}{\sqrt{n}} x_2 e^{-\tau(x)} - x_2 e^{-\tau(x)} \frac{\kappa}{\sqrt{n}} + \frac{\kappa \beta}{n}\right)
$$

$$
= -\tau_1(x) \frac{\sqrt{n}}{\beta} (x_2 e^{-\tau(x)} - \beta/\sqrt{n}) (x_2 e^{-\tau(x)} - \kappa/\sqrt{n})
$$

$$
= \frac{\sqrt{n}}{\beta} e^{-\tau(x)} (x_2 e^{-\tau(x)} - \kappa/\sqrt{n}),
$$
where in the last equality we used (4.6). To prove (4.11) we differentiate again:

\[
f_2(x) = (1 - e^{-\tau(x)}) + \tau_2(x)x_2e^{-\tau(x)} - \frac{\kappa}{\sqrt{n}}\tau_2(x)
\]

\[
+ \frac{\sqrt{n}}{\beta} (x_2e^{-\tau(x)} - \kappa/\sqrt{n})(e^{-\tau(x)} - \tau_2(x)x_2e^{-\tau(x)})
\]

\[
= (1 - e^{-\tau(x)}) + \tau_2(x)x_2e^{-\tau(x)} - \frac{\kappa}{\sqrt{n}}\tau_2(x)
\]

\[
+ \frac{\sqrt{n}}{\beta} (x_2e^{-2\tau(x)} - \tau_2(x)x_2e^{-2\tau(x)} - e^{-\tau(x)}\kappa/\sqrt{n} + \tau_2(x)x_2e^{-\tau(x)}\kappa/\sqrt{n}),
\]

which equals

\[
= (1 - e^{-\tau(x)}) + \frac{\sqrt{n}}{\beta} x_2e^{-2\tau(x)} - e^{-\tau(x)}\frac{\kappa}{\beta} - \tau_2(x)\frac{\sqrt{n}}{\beta} (x_2e^{-2\tau(x)} - \beta/\sqrt{n} x_2e^{-\tau(x)} - \frac{\kappa}{\sqrt{n}}x_2e^{-\tau(x)} + \frac{\kappa\beta}{n}).
\]

We focus on (B.19), which equals

\[
1 + \frac{\sqrt{n}}{\beta} \frac{1}{x_2} \left( x_2^2 e^{-2\tau(x)} - x_2e^{-\tau(x)} \frac{\kappa}{\sqrt{n}} - x_2e^{-\tau(x)} \frac{\beta}{\sqrt{n}} \right)
\]

\[
= 1 - \frac{1}{x_2} \frac{x_2e^{-\tau(x)} - \beta/\sqrt{n}}{x_2e^{-\tau(x)} - \kappa/\sqrt{n}} (x_2e^{-\tau(x)} - \beta/\sqrt{n}).
\]

With the help of (4.6), we see that (B.20) equals

\[- \tau_2(x)\frac{\sqrt{n}}{\beta} (x_2e^{-\tau(x)} - \beta/\sqrt{n})(x_2e^{-\tau(x)} - \kappa/\sqrt{n}) = \frac{\sqrt{n}}{\beta} \tau e^{-\tau(x)} (x_2e^{-\tau(x)} - \kappa/\sqrt{n}).
\]

Therefore, for all \( x \geq \Gamma^{(\kappa)} \),

\[
f_2(x) = 1 - \frac{1}{x_2} \frac{x_2e^{-\tau(x)} - \kappa/\sqrt{n}}{x_2e^{-\tau(x)} - \beta/\sqrt{n}} \left( x_2e^{-\tau(x)} - \beta/\sqrt{n} + \tau e^{-\tau(x)} \right).
\]

Recall that the support of \( f(x) \) is naturally partitioned into three subdomains:

\[
\{ x_2 \in [0, \kappa/\sqrt{n}] \}, \quad \{ x_2 \leq \Gamma^{(\kappa)}, \ x_2 \geq \kappa/\sqrt{n} \}, \quad \text{and} \quad \{ x_2 \geq \Gamma^{(\kappa)} \}.
\]

Continuity of the partial derivatives on the interiors of these subdomains is evident, and it remains to verify continuity on the intersections, which are

\[
\{ x_2 \in [0, \kappa/\sqrt{n}] \} \cap \{ x \leq \Gamma^{(\kappa)}, \ x_2 \geq \kappa/\sqrt{n} \} = \{ x_2 = \kappa/\sqrt{n} \},
\]

\[
\{ x \leq \Gamma^{(\kappa)}, \ x_2 \geq \kappa/\sqrt{n} \} \cap \{ x \geq \Gamma^{(\kappa)} \} = \{ x \in \Gamma^{(\kappa)} \},
\]

\[
\{ x_2 \in [0, \kappa/\sqrt{n}] \} \cap \{ x \geq \Gamma^{(\kappa)} \} = \{ (0, \kappa/\sqrt{n}) \} \subset \Gamma^{(\kappa)}.
\]

The fact that \( \{ (0, \kappa/\sqrt{n}) \} \subset \Gamma^{(\kappa)} \) follows from the definition of \( \Gamma^{(\kappa)} \) in Lemma 6. When \( x_2 = \kappa/\sqrt{n} \), we see from (4.9) that \( f_1(x) = f_2(x) = 0 \), which confirms continuity on \( \{ x_2 = \kappa/\sqrt{n} \} \). Now by definition, \( x \in \Gamma^{(\kappa)} \) implies that \( x_2e^{-\tau(x)} = \kappa/\sqrt{n} \), from which we see that (4.9) coincides with (4.10)-(4.11). \( \square \)
C Proving Lemma 5

In this section we prove Lemma 5 by constructing solutions to (3.16) and (3.17). The intuition behind the forms of these solutions is the same as in Section 4. Namely, that

\[ f(h(x)) = \int_0^\infty h(v^*(t))dt \]

solves the PDE (3.1). For the remainder of this section, we fix \( \kappa_1 < \kappa_2 \) with \( \kappa_1 > \beta \), and let us write \( \phi(x) \) instead of \( \phi(\kappa_1, \kappa_2)(x) \) to simplify notation.

C.1 Solving the first PDE

Recall the definition of \( W(x) \) from Section B.1, and for any \( \kappa > \beta \), define

\[ \tilde{\tau}^{(\kappa)}(x) = \frac{-(x_1 + \beta/\sqrt{n})}{x_2} - W\left( \frac{(\kappa - \beta)/\sqrt{n} e^{-(x_1 + \beta/\sqrt{n})/x_2}}{x_2} \right), \quad x_1 \leq -\kappa/\sqrt{n}, \ x_2 > 0. \]  

(C.1)

The quantity in (C.1) is well defined because the argument of \( W(\cdot) \) is positive. Furthermore, differentiability of \( W(\cdot) \) implies differentiability of \( \tilde{\tau}^{(\kappa)}(x) \). By repeating the arguments used in the proof of Lemma 6 in Section B.1.1, one can check that

\[ -\beta/\sqrt{n} + (x_1 + \beta/\sqrt{n})e^{-\tilde{\tau}^{(\kappa)}(x)} + x_2\tilde{\tau}^{(\kappa)}(x)e^{-\tilde{\tau}^{(\kappa)}(x)} = -\kappa/\sqrt{n} \]  

(C.2)

for those \( x \) where \( \tilde{\tau}^{(\kappa)}(x) \) is defined. Intuitively, \( \tilde{\tau}^{(\kappa)}(x) \) is the time the fluid model hits the set \( \{x_1 = -\kappa/\sqrt{n}\} \). We prove the following lemma in Section C.1.1.

Lemma 10. For any \( \kappa > \beta \) and \( x_1 \leq -\kappa/\sqrt{n} \),

\[ \lim_{x_2 \downarrow 0} \tilde{\tau}^{(\kappa)}(x) = \log \left( -\sqrt{n}x_1 - \beta \right), \]  

(C.3)

meaning that the function in (C.1) can be extended to \( x_2 \geq 0 \). Furthermore,

\[ \tilde{\tau}^{(\kappa)}(-\kappa/\sqrt{n}, x_2) = 0, \quad x_2 \geq 0 \]  

(C.4)

and

\[ -\tilde{\tau}_1^{(\kappa)}(x) = \frac{e^{-\tilde{\tau}^{(\kappa)}(x)}}{x_2 e^{-\tilde{\tau}^{(\kappa)}(x)} + (\kappa - \beta)/\sqrt{n}}, \quad \tilde{\tau}_2^{(\kappa)}(x) = -\tilde{\tau}_1^{(\kappa)}(x)\tilde{\tau}^{(\kappa)}(x) \]  

(C.5)

for \( x_1 \leq -\kappa/\sqrt{n} \) and \( x_2 \geq 0 \), where the derivatives at \( \{x_1 = -\kappa/\sqrt{n}\} \) and \( \{x_2 = 0\} \) are interpreted as the one-sided derivatives.

The following lemma presents the candidate solution to (3.16); it is proved in Section C.1.2.
Lemma 11. The function $f^{(1)} : \Omega \to \mathbb{R}_+$ defined as

$$f^{(1)}(x) = \begin{cases} f^{(\kappa_1)}(x) + f_0^{(\kappa_1)}(x) \phi(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t})dt, & x_1 \leq -\kappa_2/\sqrt{n}, \\ f_0^{(\kappa_1)}(x) \phi(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t})dt, & x_1 \in \left[ -\frac{\kappa_2}{\sqrt{n}}, -\frac{\kappa_1}{\sqrt{n}} \right], \\ 0, & x_1 \in \left[ -\kappa_1/\sqrt{n}, 0 \right]. \end{cases}$$

belongs to $C^2(\Omega)$. Furthermore,

$$f^{(1)}_{11}(x) = \begin{cases} -f^{(\kappa_1)}(x) e^{-t} \phi'\left(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t}\right)dt, & x_1 \leq -\kappa_2/\sqrt{n}, \\ -f_0^{(\kappa_1)}(x) e^{-t} \phi'\left(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t}\right)dt, & x_1 \in \left[ -\frac{\kappa_2}{\sqrt{n}}, -\frac{\kappa_1}{\sqrt{n}} \right], \\ 0, & x_1 \in \left[ -\kappa_1/\sqrt{n}, 0 \right]. \end{cases}$$

and

$$f^{(1)}_{22}(x) = \begin{cases} -f^{(\kappa_1)}(x) t e^{-t} \phi'\left(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t}\right)dt, & x_1 \leq -\kappa_2/\sqrt{n}, \\ -f_0^{(\kappa_1)}(x) t e^{-t} \phi'\left(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t}\right)dt, & x_1 \in \left[ -\frac{\kappa_2}{\sqrt{n}}, -\frac{\kappa_1}{\sqrt{n}} \right], \\ 0, & x_1 \in \left[ -\kappa_1/\sqrt{n}, 0 \right]. \end{cases}$$

Let us first verify that $f^{(1)}(x)$ from Lemma 11 satisfies (3.16). The boundary condition $f^{(1)}_{11}(0, x_2) = f^{(1)}_{22}(0, x_2)$ is trivially satisfied. For $x_1 \leq -\kappa_2/\sqrt{n}$,

$$(-x_1 + x_2 - \beta/\sqrt{n})f^{(1)}_{11}(x) - x_2f^{(1)}_{22}(x)$$

$$= \int_{\frac{\kappa_1}{\sqrt{n}}}^{\frac{\kappa_2}{\sqrt{n}}} \left( (x_1 - x_2 + \beta/\sqrt{n})e^{-u} + x_2e^{-u} \right) \phi'(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2e^{-t})dt$$

$$= \int_{\frac{\kappa_1}{\sqrt{n}}}^{\frac{\kappa_2}{\sqrt{n}}} \phi'(u)du = \phi(\kappa_1/\sqrt{n}) - \phi(\kappa_2/\sqrt{n}) = -1 = -\phi(-x_1),$$

and a similar argument works when $x_1 \in \left[ -\frac{\kappa_2}{\sqrt{n}}, -\frac{\kappa_1}{\sqrt{n}} \right]$. Therefore, $f^{(1)}(x)$ solves the (3.16).

We now bound $f^{(1)}_{11}(x)$ and $f^{(1)}_{11}(x)$ to prove (3.20). Since $\phi'(-\kappa_2/\sqrt{n}) = \phi'(\kappa_1/\sqrt{n}) = 0$, differentiating $f^{(1)}_{11}(x)$ gives us

$$f^{(1)}_{11}(x) = \begin{cases} -f^{(\kappa_1)}(x) e^{-2t} \phi''\left(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t}\right)dt, & x_1 \leq -\kappa_2/\sqrt{n}, \\ -f_0^{(\kappa_1)}(x) e^{-2t} \phi''\left(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t}\right)dt, & x_1 \in \left[ -\frac{\kappa_2}{\sqrt{n}}, -\frac{\kappa_1}{\sqrt{n}} \right], \\ 0, & x_1 \in \left[ -\kappa_1/\sqrt{n}, 0 \right]. \end{cases}$$
Using (3.15),
\[
\begin{align*}
|f_1^{(1)}(x)| &\leq 4\sqrt{n} \frac{\tilde{\tau}^{(\kappa_1)}(x) - \tilde{\tau}^{(\kappa_2)}(x)}{\kappa_2 - \kappa_1}, & x_1 \leq -\kappa_2/\sqrt{n}, \\
|f_{11}^{(1)}(x)| &\leq \frac{12n}{(\kappa_2 - \kappa_1)^2} |\tilde{\tau}^{(\kappa_1)}(x) - \tilde{\tau}^{(\kappa_2)}(x)|, & x_1 \leq -\kappa_2/\sqrt{n}, \\
|f_1^{(1)}(x)| &\leq 4\sqrt{n} \frac{|\tilde{\tau}^{(\kappa_1)}(x)|}{\kappa_2 - \kappa_1}, & x_1 \in \left[-\frac{\kappa_2}{\sqrt{n}}, -\frac{\kappa_1}{\sqrt{n}}\right].
\end{align*}
\]

Now when \(x_1 \leq -\kappa_2/\sqrt{n},\)
\[
\begin{align*}
\tilde{\tau}^{(\kappa_1)}(x) - \tilde{\tau}^{(\kappa_2)}(x) &= W\left(\frac{(\kappa_2 - \beta)/\sqrt{n}}{x_2} e^{-\beta/\sqrt{n}}\right) - W\left(\frac{(\kappa_1 - \beta)/\sqrt{n}}{x_2} e^{-\beta/\sqrt{n}}\right) \\
&= \log\left(\frac{\kappa_2 - \beta}{\kappa_1 - \beta}\right) - \log\left(\frac{W\left(\frac{(\kappa_2 - \beta)/\sqrt{n}}{x_2} e^{-\beta/\sqrt{n}}\right)}{W\left(\frac{(\kappa_1 - \beta)/\sqrt{n}}{x_2} e^{-\beta/\sqrt{n}}\right)}\right) \\
&\leq \log\left(\frac{\kappa_2 - \beta}{\kappa_1 - \beta}\right), & x_1 \leq -\kappa_2/\sqrt{n},
\end{align*}
\] (C.6)

where the second equation follows from (B.2) and the inequality follows from the fact that \(W(\cdot)\) is an increasing function and \(\kappa_2 > \kappa_1\). The first equation and monotonicity of \(W(\cdot)\) means that \(\tilde{\tau}^{(\kappa_1)}(x) - \tilde{\tau}^{(\kappa_2)}(x) > 0\). When \(x_1 \in \left[-\frac{\kappa_2}{\sqrt{n}}, -\frac{\kappa_1}{\sqrt{n}}\right],\)
\[
0 = \tilde{\tau}^{(\kappa_1)}(-\kappa_1/\sqrt{n}, x_2) \leq \tilde{\tau}^{(\kappa_1)}(x) \\
\leq \tilde{\tau}^{(\kappa_1)}(-\kappa_2/\sqrt{n}, x_2) \\
= \tilde{\tau}^{(\kappa_1)}(-\kappa_2/\sqrt{n}, x_2) - \tau^{(\kappa_2)}(-\alpha_2/\sqrt{n}, x_2) \\
\leq \log\left(\frac{\kappa_2 - \beta}{\kappa_2 + \beta}\right).
\]

The two equalities above are due to (C.4), the first two inequalities follow from the fact that \(\tilde{\tau}_1^{(\kappa_1)}(x) \leq 0\) in (C.5), and the last inequality comes from (C.6). Therefore,
\[
|f_1(x)| \leq \frac{4\sqrt{n}}{\kappa_2 - \kappa_1} \log\left(\frac{\kappa_2 - \beta}{\kappa_2 + \beta}\right), & |f_{11}(x)| \leq \frac{12n}{(\kappa_2 - \kappa_1)^2} \log\left(\frac{\kappa_2 - \beta}{\kappa_2 + \beta}\right),
\]
which proves (3.20). The above argument can be repeated to prove (3.18).

### C.1.1 Proof of Lemma 10

**Proof of Lemma 10.** Let
\[
u = \frac{(\kappa - \beta)/\sqrt{n}}{\kappa_2 - \kappa_1} e^{-\beta/\sqrt{n}},
\]

Using (B.2),
\[ \dot{\tau}(\kappa)(x) = -\frac{(x + \beta/\sqrt{n})}{x^2} - W(u) = \log \left( \frac{\sqrt{n}}{\kappa - \beta} x_2 W(u) \right), \quad x_1 \leq -\kappa/\sqrt{n}, \ x_2 > 0. \]
Therefore, it remains to evaluate
\[ \lim_{x_2 \downarrow 0} \frac{W\left(\frac{(\kappa - \beta)/\sqrt{n}}{x_2} e^{-\frac{(x + \beta/\sqrt{n})}{x_2}}\right)}{1/x_2}, \]
which we do using L'Hopital's rule. The derivative of the numerator with respect to \(x_2\) is
\[ W'(u)\left(-\frac{1}{x_2^2} + \frac{x_1 + \beta/\sqrt{n}}{x_2^2} \right) = \frac{W(u)}{1 + W(u)} \frac{1}{x_2^2} (-x_2 + x_1 + \beta/\sqrt{n}), \]
where we used (B.3) to get the equality above. Therefore
\[ \lim_{x_2 \downarrow 0} \frac{W\left(\frac{(\kappa - \beta)/\sqrt{n}}{x_2} e^{-\frac{(x + \beta/\sqrt{n})}{x_2}}\right)}{1/x_2} = -x_1 - \beta/\sqrt{n}, \]
and (C.3) follows. We now prove (C.4), or that \(\dot{\tau}(\kappa)(-\kappa/\sqrt{n}, x_2) = 0\) for \(x_2 > 0\). The claim is true when \(x_2 = 0\) by (C.3). For \(x_2 > 0\),
\[ \dot{\tau}(\kappa)(-\kappa/\sqrt{n}, x_2) = -\frac{\sqrt{n}}{x_2} - W\left(\frac{(\kappa - \beta)/\sqrt{n}}{x_2} e^{-\frac{(x + \beta/\sqrt{n})}{x_2}}\right) = 0, \]
where the second equality comes from the fact that \(W(xe^\kappa) = x\); c.f. (B.4). That Differentiability of \(\dot{\tau}(\kappa)(x)\) follows from the differentiability of \(W(\cdot)\). To verify (C.5), plug in (C.1) into (C.2) to see that
\[ W\left(\frac{(\kappa - \beta)/\sqrt{n}}{x_2} e^{-\frac{(x + \beta/\sqrt{n})}{x_2}}\right) = \frac{(\kappa - \beta)/\sqrt{n}}{x_2^2 e^{-\dot{\tau}(\kappa)(x)}}, \]
and therefore
\[ \dot{\tau}(\kappa)(x) = -\frac{(x + \beta/\sqrt{n})}{x_2} - \frac{(\kappa - \beta)/\sqrt{n}}{x_2^2 e^{-\dot{\tau}(\kappa)(x)}}. \]
The proof of (C.5) is then identical to the arguments used to prove (B.17)-(B.18).

\[ \square \]

\section*{C.1.2 Proof of Lemma 11}

\textbf{Proof of Lemma 11}. For convenience, we recall that
\[ f^{(1)}(x) \]
\[ = \begin{cases} \dot{\tau}(x_2)(x) + \int_{\dot{\tau}(x_2)(x)}^{\dot{\tau}(x_2)(x)} \phi(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t})dt, & x_1 \leq -\kappa_2/\sqrt{n}, \\ \int_{0}^{\dot{\tau}(x_2)(x)} \phi(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t})dt, & x_1 \in \left[ -\frac{\kappa_2}{\sqrt{n}}, -\frac{\kappa_1}{\sqrt{n}} \right], \\ 0, & x_1 \in \left[ -\kappa_1/\sqrt{n}, 0 \right]. \end{cases} \]
Continuity of $f^{(1)}(x)$ on the sets $\{x_1 = -\kappa_2/\sqrt{n}\}$ and $\{x_1 = -\kappa_1/\sqrt{n}\}$ follows from (C.4). We can differentiate $f^{(1)}(x)$ using the Leibniz rule and the fact that
\[
\phi(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-\tilde{\tau}(\kappa_2)(x)} - x_2\tilde{\tau}(\kappa_2)(x)e^{-\tilde{\tau}(\kappa_2)(x)}) = \phi(\kappa_2/\sqrt{n}) = 1,
\]
\[
\phi(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-\tilde{\tau}(\kappa_1)(x)} - x_2\tilde{\tau}(\kappa_1)(x)e^{-\tilde{\tau}(\kappa_1)(x)}) = \phi(\kappa_1/\sqrt{n}) = 0,
\]
to see that
\[
f^{(1)}_1(x) = \begin{cases} 
- f^{\tilde{\tau}(\kappa_1)} e^{-t}\phi'(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t})dt, & x_1 \leq -\kappa_2/\sqrt{n}, \\
- f^{\tilde{\tau}(\kappa_1)} e^{-t}\phi'(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t})dt, & x_1 \in \left[-\frac{\kappa_2}{\sqrt{n}}, -\frac{\kappa_1}{\sqrt{n}}\right], \\
0, & x_1 \in [-\kappa_1/\sqrt{n}, 0].
\end{cases}
\]
and
\[
f^{(1)}_2(x) = \begin{cases} 
- f^{\tilde{\tau}(\kappa_1)} te^{-t}\phi'(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t})dt, & x_1 \leq -\kappa_2/\sqrt{n}, \\
- f^{\tilde{\tau}(\kappa_1)} te^{-t}\phi'(\beta/\sqrt{n} - (x_1 + \beta/\sqrt{n})e^{-t} - x_2te^{-t})dt, & x_1 \in \left[-\frac{\kappa_2}{\sqrt{n}}, -\frac{\kappa_1}{\sqrt{n}}\right], \\
0, & x_1 \in [-\kappa_1/\sqrt{n}, 0].
\end{cases}
\]
Continuity of the first and second order derivatives can be verified using (C.4). \(\square\)

### C.2 Solving the Second PDE

Recall the definitions of $\Gamma^{(\kappa)}$ and $\tau(x)$ from Lemmas 6 and 7, respectively. Partition $\Omega$ into four subdomains:

- $S_0 = \{x \in \Omega \mid x_2 \leq \kappa_1/\sqrt{n}\}$,
- $S_1 = \{x \in \Omega \mid x_2 \geq \kappa_1/\sqrt{n}, x \leq \Gamma^{(\kappa_1)}\}$,
- $S_2 = \{x \in \Omega \mid \Gamma^{(\kappa_1)} \leq x \leq \Gamma^{(\kappa_2)}\}$,
- $S_3 = \{x \in \Omega \mid x \geq \Gamma^{(\kappa_2)}\}$.

Figure 3 helps to visualize the four sets. Let us verify that $S_0 \cup S_1 \cup S_2 \cup S_3$ does indeed equal $\Omega$. Fix $x_1 \leq 0$ and $x_2 \geq 0$. Then $x$ must lie in one of

- $\{x_2 \leq \kappa_1/\sqrt{n}\}$,
- $\{x_2 \geq \kappa_1/\sqrt{n}, x \leq \Gamma^{(\kappa_1)}\}$,
- $\{x_2 \geq \kappa_1/\sqrt{n}, \Gamma^{(\kappa_1)} \leq x \leq \Gamma^{(\kappa_2)}\}$,
- $\{x_2 \geq \kappa_1/\sqrt{n}, x \geq \Gamma^{(\kappa_2)}\}$.

or

$\{x_2 \geq \kappa_1/\sqrt{n}, \Gamma^{(\kappa_1)} \leq x \leq \Gamma^{(\kappa_2)}\}$,

From (4.5) we see that $x \geq \Gamma^{(\kappa_2)}$ implies $x_2 \geq \kappa_1/\sqrt{n}$, or

- $\{x_2 \geq \kappa_1/\sqrt{n}, \Gamma^{(\kappa_1)} \leq x \leq \Gamma^{(\kappa_2)}\} = S_2$, and
- $\{x_2 \geq \kappa_1/\sqrt{n}, x \geq \Gamma^{(\kappa_1)}\} = \{x \geq \Gamma^{(\kappa_1)}, x \geq \Gamma^{(\kappa_2)}\}$.  

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From (4.7) we know that \( x \geq \Gamma(\kappa_1) \) implies \( x > \Gamma(\kappa_2) \), or
\[
\{ x \geq \Gamma(\kappa_1), \ x \geq \Gamma(\kappa_2) \} = S_3.
\]
Therefore \( S_0 \cup S_1 \cup S_2 \cup S_3 \) equals \( \Omega \). The following lemma is proved at the end of this section.

**Lemma 12.** The function

\[
f^{(2)}(x) = \begin{cases} 
0, & x \in S_0, \\
\int_0^{\log(\pi x_2/\kappa_1)} \phi(x_2 e^{-t}) dt, & x_2 \leq \kappa_2 / \sqrt{n}, \ x \in S_1, \\
\log(\pi x_2 / \kappa_2) + \int_0^{\log(\kappa_2/\kappa_1)} \phi(\frac{x_2 e^{-t}}{\sqrt{n}}) dt, & x_2 > \kappa_2 / \sqrt{n}, \ x \in S_1, \\
\int_0^{\tau(x)} \phi(x_2 e^{-t}) dt + \sqrt{n} \int_{\kappa_1 / \sqrt{n}}^{x_2 e^{-\tau(x)}} \phi(t) dt, & x_2 \leq \kappa_2 / \sqrt{n}, \ x \in S_2, \\
\log(\pi x_2 / \kappa_2) + \int_0^{\tau(x)} \phi(x_2 e^{-t}) dt + \frac{\sqrt{n} \int_{\kappa_1 / \sqrt{n}}^{x_2 e^{-\tau(x)}} \phi(t) dt}{\beta / \sqrt{n}}, & x_2 > \kappa_2 / \sqrt{n}, \ x \in S_2, \\
\tau(x) + \frac{x_2 e^{-\tau(x)} - \kappa_2 / \sqrt{n}}{\beta / \sqrt{n}} + \frac{\sqrt{n} \int_{\kappa_1 / \sqrt{n}}^{x_2 e^{-\tau(x)}} \phi(t) dt}{\beta / \sqrt{n}}, & x \in S_3,
\end{cases}
\]

Figure 3: A aide to visualize \( S_0, S_1, S_2, S_3 \).
is well-defined for \( x \in \Omega \) and belongs to \( C^2(\Omega) \). Its derivatives are

\[
f_1^{(2)}(x) = \begin{cases} 
0, & x \in S_0, \\
0, & x \in S_1, \\
\frac{\sqrt{n}}{\beta} \phi(x_2 e^{-\tau(x)}) e^{-\tau(x)}, & x \in S_2, \\
\frac{\sqrt{n}}{\beta} e^{-\tau(x)}, & x \in S_3,
\end{cases}
\]

and

\[
f_2^{(2)}(x) = \begin{cases} 
0, & x \in S_0, \\
\frac{1}{x_2^2} \phi(x_2), & x \in S_1, \\
\frac{1}{x_2^2} (\phi(x_2) - \phi(x_2 e^{-\tau(x)})) + \phi(x_2 e^{-\tau(x)}) \frac{\sqrt{n}}{\beta} e^{-\tau(x)} (\tau(x) + 1), & x \in S_2, \\
\frac{\sqrt{n}}{\beta} e^{-\tau(x)} (\tau(x) + 1), & x \in S_3.
\end{cases}
\]

We now verify that \( f^{(2)}(x) \) solves (3.17). Using the fact that

\[-(x_1 + \beta/\sqrt{n}) e^{-\tau(x)} - x_2 \tau(x) e^{-\tau(x)} = -\beta/\sqrt{n},
\]

one can check that

\[-(x_1 + x_2 - \beta/\sqrt{n}) f_1(x) - x_2 f_2(x) = -\phi(x), \quad x \in \Omega.
\]

Furthermore, \( \tau(0, x_2) = 0 \) suggests that

\[f_1^{(2)}(0, x_2) = f_2^{(2)}(0, x_2), \quad x \in S_2 \cup S_3. \quad \text{(C.7)}
\]

Furthermore, the only point in \( S_1 \) with \( x_1 = 0 \) is the point \( (0, \kappa_1/\sqrt{n}) \), which means that (C.7) holds for \( x \in S_1 \) as well. Therefore, \( f^{(2)}(x) \) solves (3.17). We now bound \( f_1^{(2)}(x) \) and \( f_{11}^{(2)}(x) \) to prove (3.21). The bound on \( f_1^{(2)}(x) \) is straightforward. Differentiating \( f_1^{(2)}(x) \), we see that

\[
f_{11}^{(2)}(x) = \begin{cases} 
0, & x \in S_0, \\
0, & x \in S_1, \\
-\tau_1(x) \frac{\sqrt{n}}{\beta} \phi(x_2 e^{-\tau(x)}) e^{-\tau(x)} - \tau_1(x) x_2 e^{-2\tau(x)} \frac{\sqrt{n}}{\beta} \phi'(x_2 e^{-\tau(x)}), & x \in S_2, \\
-\tau_1(x) \frac{\sqrt{n}}{\beta} e^{-\tau(x)}, & x \in S_3.
\end{cases}
\]

Recall from (4.6) that

\[-\tau_1(x) = \frac{e^{-\tau(x)}}{x_2 e^{-\tau(x)} - \beta/\sqrt{n}}.
\]

From (4.5), we know that \( x_2 e^{-\tau(x)} \geq \kappa_2/\sqrt{n} \) for \( x \geq \Gamma(\kappa_2) \), which means that

\[
\left| f_{11}^{(2)}(x) \right| \leq \frac{\sqrt{n}}{\beta} \frac{1}{\kappa_2/\sqrt{n} - \beta/\sqrt{n}} = \frac{n}{\beta (\kappa_2 - \beta)}, \quad x \in S_3.
\]
Similarly,
\[
|f^{(2)}_{11}(x)| \leq \frac{n}{\beta(k_1 - \beta)} + \frac{e^{-\tau(x)}}{x_2e^{-\tau(x)} - \beta/\sqrt{n}} x_2e^{-2\tau(x)} \sqrt{n} \frac{\phi'(x_2e^{-\tau(x)})}{}\]
\[
\leq \frac{n}{\beta(k_1 - \beta)} + \frac{x_2e^{-\tau(x)}}{\sqrt{n}} \frac{4\sqrt{n}}{\beta \kappa_2 - \kappa_1}, \quad x \in S_2,
\]
where in the second inequality we used (3.15) and in the last inequality we used the fact that \(x_2e^{-\tau(x)} \geq \kappa_1/\sqrt{n}\) for \(x \geq \Gamma^2\). This proves (3.21) and we now prove the bound on \(f^{(2)}(x)\) in (3.19). From the form of \(f^{(2)}(x)\), we see that
\[
f^{(2)}(x) \leq \log(\sqrt{n}x_2/\kappa_1) \leq \log(\kappa_2/\kappa_1), \quad x_2 \leq \kappa_2/\sqrt{n}, \quad x \in S_1,
\]
\[
f^{(2)}(x) \leq \tau(x) + \frac{1}{\beta}(x_2e^{-\tau(x)} - \kappa_1/\sqrt{n}) \leq \tau(x) + \frac{\kappa_2 - \kappa_1}{\beta}, \quad x_2 \leq \kappa_2/\sqrt{n}, \quad x \in S_2.
\]
We do not need to bound \(f^{(2)}(x)\) for \(x \in S_3\), because from (4.5) we know that \(x \in S_3\) implies \(x_2 \geq \kappa_2/\sqrt{n}\). To bound \(\tau(x)\) in the second line above, note from (4.5) that
\[
x_2e^{-\tau(x)} \geq \kappa_1/\sqrt{n}, \quad x_2 \leq \kappa_2/\sqrt{n}, \quad x \in S_2,
\]
which means that
\[
\tau(x) \leq \log(\kappa_2/\kappa_1), \quad x_2 \leq \kappa_2/\sqrt{n}, \quad x \in S_2.
\]
This proves (3.19).

**Proof of Lemma 12.** Recall for convenience that
\[
f^{(2)}(x) = \begin{cases} 0, & x \in S_0, \\
\log(\sqrt{n}x_2/\kappa_2) + \frac{1}{\beta}(\phi(x_2e^{-\tau(x)} - \kappa_1/\sqrt{n}) dt, \quad x_2 \leq \kappa_2/\sqrt{n}, \quad x \in S_1, \\
\log(\sqrt{n}x_2/\kappa_2) + \frac{1}{\beta}(\phi(x_2e^{-\tau(x)}) dt + \frac{\sqrt{n}}{\beta} \int_{\kappa_1/\sqrt{n}}^{x_2e^{-\tau(x)}} \phi(t) dt, \quad x_2 \leq \kappa_2/\sqrt{n}, \quad x \in S_2, \\
\log(\sqrt{n}x_2/\kappa_2) + \frac{1}{\beta}(\phi(x_2e^{-\tau(x)}) dt + \frac{\sqrt{n}}{\beta} \int_{\kappa_1/\sqrt{n}}^{x_2e^{-\tau(x)}} \phi(t) dt, \quad x_2 \geq \kappa_2/\sqrt{n}, \quad x \in S_2, \\
\tau(x) + \frac{x_2e^{-\tau(x)} - \kappa_2/\sqrt{n}}{\beta/\sqrt{n}} + \frac{\sqrt{n}}{\beta} \int_{\kappa_1/\sqrt{n}}^{\kappa_2/\sqrt{n}} \phi(t) dt, \quad x \in S_3, \end{cases}
\]
and
\[
S_0 = \{x \in \Omega \mid x_2 \leq \kappa_1/\sqrt{n}\},
\]
\[
S_1 = \{x \in \Omega \mid x_2 \geq \kappa_1/\sqrt{n}, \quad x \leq \Gamma^2\},
\]
\[
S_2 = \{x \in \Omega \mid \Gamma^2 \leq x \leq \Gamma^2\},
\]
\[
S_3 = \{x \in \Omega \mid x \geq \Gamma^2\}.
\]

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Let us first verify the continuity of \( f^{(2)}(x) \) on each of \( S_i \cap S_j \) for \( i, j \in \{0, 1, 2, 3\} \); the only non-empty intersections are \( S_0 \cap S_1, S_1 \cap S_2, \) and \( S_2 \cap S_3 \). Continuity on \( S_0 \cap S_1 \) is straightforward because

\[
S_0 \cap S_1 = \{ x_2 = \kappa_1 / \sqrt{n} \}.
\]

Continuity of this set is straightforward. Now \( S_1 \cap S_2 = \{ x \in \Gamma(\kappa_1) \} \), and recall from (4.5) that

\[
x_2 e^{-\tau(x)} = \kappa / \sqrt{n} \quad \text{or} \quad \tau(x) = \log(\sqrt{n} x_2 / \kappa), \quad x \in \Gamma(\kappa).
\]

Therefore, we can compare the definitions of \( f^{(2)}(x) \) on

\[
x_2 \leq \kappa_2 / \sqrt{n}, \quad x \in S_1 \quad \text{vs.} \quad x_2 \leq \kappa_2 / \sqrt{n}, \quad x \in S_2
\]

and

\[
x_2 \geq \kappa_2 / \sqrt{n}, \quad x \in S_1 \quad \text{vs.} \quad x_2 \geq \kappa_2 / \sqrt{n}, \quad x \in S_2
\]

to see that they coincide. Lastly, \( S_2 \cap S_3 = \{ x \in \Gamma(\kappa_2) \} \), and recall that \( x \in \Gamma(\kappa_2) \) implies \( x_2 \geq \kappa_2 / \sqrt{n} \). Therefore, we need only to compare the definitions of \( f^{(2)}(x) \) on

\[
x_2 \geq \kappa_2 / \sqrt{n}, \quad x \in S_2 \quad \text{vs.} \quad x \in S_3
\]

and use (C.8) to conclude that \( f^{(2)}(x) \) is continuous.

Differentiating \( f^{(2)}(x) \) and using the Leibniz integration rule, we get

\[
f_1^{(2)}(x) = \begin{cases} 
0, & x \in S_0, \\
0, & x \in S_1, \\
\tau_1(x) \phi(x_2 e^{-\tau(x)}) \left( 1 - \frac{\sqrt{n}}{\beta} x_2 e^{-\tau(x)} \right), & x \in S_2, \\
\tau_1(x) \left( 1 - \frac{\sqrt{n}}{\beta} x_2 e^{-\tau(x)} \right), & x \in S_3,
\end{cases}
\]

and

\[
f_2^{(2)}(x) = \begin{cases} 
\frac{1}{\sqrt{\tau_2}} \phi(x_2), & x \in S_0, \\
\frac{1}{\sqrt{\tau_2}} \left( \phi(x_2) - \phi(x_2 e^{-\tau(x)}) \right) + \tau_2(x) \phi(x_2 e^{-\tau(x)}) \\
+ \frac{\sqrt{n}}{\beta} \phi(x_2 e^{-\tau(x)}) \left( e^{-\tau(x)} - \tau_2(x) x_2 e^{-\tau(x)} \right), & x \in S_2,
\end{cases}
\]

\[
\tau_2(x) + \frac{\sqrt{n}}{\beta} \left( e^{-\tau(x)} - \tau_2(x) x_2 e^{-\tau(x)} \right), & x \in S_3.
\]

From (4.6), we know that

\[
-\tau_1(x) = \frac{e^{-\tau(x)}}{x_2 e^{-\tau(x)} - \beta / \sqrt{n}}, \quad \text{and} \quad \tau_2(x) = \tau_1(x) \tau(x),
\]

which proves the forms of \( f_1^{(2)}(x) \) and \( f_2^{(2)}(x) \) as stated in the Lemma. Continuity of the first and second order derivatives can be checked using the same logic used in the continuity of \( f^{(2)}(x) \).