Asymptotic Behavior of Colored Jones polynomial and
Turaev-Viro Invariant of figure eight knot

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Abstract

In this paper we investigate the asymptotic behavior of the colored Jones polynomial and the Turaev-Viro invariant for the figure eight knot. More precisely, we consider the $M$-th colored Jones polynomial evaluated at $(N + 1/2)$-th root of unity with a fixed limiting ratio, $s$, of $M$ and $(N + 1/2)$. We find out the asymptotic expansion formula (AEF) of the colored Jones polynomial of figure eight knot with $s$ close to 1. An upper bound for the asymptotic expansion formula of the colored Jones polynomial of figure eight knot with $s$ close to 1/2 is also obtained. It is known that the Turaev-Viro invariant of figure eight knot can be expressed in terms of a sum of its colored Jones polynomials. Our results show that this sum is asymptotically equal to the sum of the terms with $s$ close to 1/2 or 1. As an application of the asymptotic behavior of the colored Jones polynomials, we obtain the asymptotic expansion for the Turaev-Viro invariant of the figure eight knot. Finally, we suggest a possible generalization of our approach so as to relate the AEF for the colored Jones polynomials and the AEF for the Turaev-Viro invariants for general hyperbolic knots.

1 Introduction

This paper aims to find out the asymptotic expansion formula (AEF) for the $M$-th colored Jones polynomial of figure eight knot at $(M + a)$-th root of unity, with $a$ and $M$ satisfying some limiting relation. The method is motivated by the work in [23] in which an asymptotic expansion of an $SU(n)$-invariant of the figure eight knot is given. In particular, we are interested in the case where $a = (N - M + 1/2)$ with $N > M$, where $M \in \mathbb{N}$ is a sequence of integers in $N$ with limiting ratio $s = \lim_{N \to \infty} \frac{M}{N + 1/2}$ close to 1/2 or 1. From the AEF of the colored Jones polynomial of figure eight knot, we find out explicitly the large $r$ behavior of the Turaev-Viro invariant $TV_r(SS^3 \setminus \mathcal{A}_1)$. This tells us what kinds of topological information can be extracted from the AEF of the TV invariant.

1.1 Overview of the volume conjecture

The main theme of this paper is to establish the AEF for the Turaev-Viro invariant of the figure eight knot complement. The study of the volume conjecture of the Turaev-Viro invariant started from [3], in which Q.Chen and T.Yang discovered a version of volume conjecture of Turaev-Viro invariant at the $2r$-th root of unity with an odd integer $r$. The conjecture can be stated as follows.

Conjecture 1. For every hyperbolic 3-manifold $M$, we have

$$\lim_{r \to \infty} \frac{2\pi}{r} \log \left( \text{TV}_r(M, e^{\frac{2\pi i}{r}}) \right) = \text{Vol}(M)$$

where $r$ is odd positive integer.

This result is surprising since according to the Witten’s Asymptotic Expansion conjecture, the Reshetikhin-Turaev invariant and Turaev-Viro invariant should grow polynomially in $r$. In particular, when $K$ is the figure eight knot 41, and $M$ is the complement of $K$ in $SS^3$, numerical evidence shows that Conjecture [3] is true. Furthermore, Chen and Yang find that $\frac{2\pi}{r} \log \left( \text{TV}_r(M, e^{\frac{2\pi i}{r}}) \right)$ goes faster to the hyperbolic volume than $\frac{2\pi}{r} \log \left| J_r(K; e^{\frac{2\pi i}{r}}) \right|$. 

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To explain the gap between asymptotic behavior of these invariants, we need to relate the Turaev-Viro invariant with the colored Jones polynomial such that a comparison can be done. The following relationship between the two invariants is given by Theorem 1.1 in [3].

**Theorem 1.** Let $L$ be a link in $SS^3$ with $n$ components. Then given an odd integer $r = 2N + 1 \geq 3$, we have

$$TV_r \left( SS^3 \setminus L, e^{\frac{2\pi i}{N}} \right) = 2^{n-1} (\eta'/r)^2 \sum_{1 \leq M \leq r-1} \left| J_M \left( L, e^{\frac{2\pi i}{N}} \right) \right|^2,$$

where

$$\eta' = \frac{2 \sin \left( \frac{2\pi}{r} \right)}{\sqrt{r}}.$$

From this relationship between the Turaev-Viro invariant and the colored Jones polynomial, Conjecture 1 has been proved for the case of figure eight knot complement (Theorem 1.6 in [6]). Furthermore, in order to find out the AEF of the Turaev-Viro invariant, it is natural to consider the AEF of the $M$-th colored Jones polynomials, where $M = 1, 2, \ldots, N$.

The asymptotic behavior of the colored Jones polynomial has been investigated for a very long time. The classical volume conjecture (Conjecture 2 below) states that the evaluation of the $N$-th colored Jones polynomial of a knot $K$ at an $N$-th root of unity captures the simplicial volume of the knot complement $SS^3 \setminus K$.

**Conjecture 2.** (Classical volume conjecture [7, 13]) Let $K$ be a knot and $J_N(K; q)$ be the $N$-th colored Jones polynomial of $K$ evaluated at $q$. We have

$$\lim_{N \to \infty} \frac{\log |J_N(K; e^{2\pi i/N})|}{N \pi} = \frac{\text{Vol}(SS^3 \setminus K)}{2\pi},$$

where $\text{Vol}(SS^3 \setminus K)$ is the simplicial volume of the knot complement.

In [1] Anderson and Hansen used saddle point approximation to find out the AEF for the $N$-th colored Jones polynomial of figure eight knot evaluated at $N$-th root of unity.

**Theorem 2.** The AEF for the the $N$-th colored Jones polynomial of figure eight knot evaluated at $N$-th root of unity is given by

$$J_N \left( 4_1; \exp \left( \frac{2\pi i}{N} \right) \right) \sim N^{3/2} \exp \left( \frac{N \text{Vol}(SS^3 \setminus 4_1)}{2\pi} \right) \left( \frac{2}{\sqrt{3}} \right)^{1/2} \left( \frac{N}{2\pi i} \right)^{3/2} \exp \left( \frac{N}{2\pi i} \times i \text{Vol}(SS^3 \setminus 4_1) \right).$$

As a generalization of Theorem 2 in [16] H.Murakami obtained the asymptotic expansion formula of the colored Jones polynomial, which captures the Chern-Simons invariant together with the Reidemeister torsion of the knot. (A related result on colored HOMFLY polynomial is obtained in [23].) To introduce the theorem, for any $0 < u < \log((3 + \sqrt{5})/2) = 0.9624\ldots$ we define

$$S(u) = \Li_2 \left( e^{u^2} \right) - \Li_2 \left( e^{u^2} \right) - u \varphi(u)$$

and

$$T(u) = \frac{2}{\sqrt{(e^u + e^{-u} + 1)(e^u + e^{-u} - 3)}}.$$

Here $\varphi(u) = \arccosh(\cosh(u) - 1/2)$ and

$$\Li_2(z) = - \int_0^z \frac{\log(1 - x)}{x} dx$$

is the dilogarithm function.

The functions $S(u)$ and $T(u)$ are the Chern-Simons invariant and the cohomological twisted Reidemeister torsion respectively, both of which are associated with an irreducible representation of $\pi_1(SS^3 \setminus 4_1)$ into $SL(2; \mathbb{C})$ sending the meridian to an element with eigenvalues $\exp(u/2)$ and $\exp(-u/2)$ [15].

H.Murakami proved the following asymptotic equivalence.
Theorem 3. (Asymptotic expansion formula for the colored Jones polynomial of figure eight knot)
Let $u$ be a real number with $0 < u < \log((3 + \sqrt{5})/2)$ and put $\xi = 2\pi i + u$. Then we have the following asymptotic equivalence for the colored Jones polynomial of the figure-eight knot $A_1$:

$$J_N(A_1; \exp(\frac{\xi}{N})) \sim_{N \to \infty} \frac{\sqrt{\pi}}{2\sinh(u/2)} T(u)^{1/2} \left(\frac{N}{\xi}\right)^{1/2} \exp \left( \frac{N}{\xi} S(u) \right).$$

1.2 AEF of the $M$-th colored Jones polynomial of figure eight knot at $(N + \frac{1}{2})$-th root of unity

The main results of this paper are summarized as follows. First of all we consider the following discussion we only focus on the other situations.

1.3 Main Results

The main results of this paper are summarized as follows. First of all we consider the $M$-th colored Jones polynomial around $(M + a)$-th root of unity $q = \exp \left( \frac{2\pi i u}{M + a} \right)$ with some fixed non-negative real number $a$. One can easily see that when $a \in \mathbb{N}$ and $u = 0$, we have $\lim_{M \to \infty} J_M(A_1, q) = 1$. So in the following discussion we only focus on the other situations.
Theorem 5. For \( q = \exp \left( \frac{2\pi i + u}{M + u} \right) \), if \( a \not\in \mathbb{N} \) or \( u \neq 0 \), we have

\[
J_M(41, q) \sim \frac{\sin \alpha \pi}{a \pi} 2^{3/2} \left( \frac{2}{\sqrt{3}} \right)^{1/2} \left( \frac{M + a}{2\pi i} \right)^{3/2} \exp \left( \frac{M + a}{2\pi i} \right) \times \text{Vol}(\mathbb{S}^3 \setminus 4_1) \]

where \( \frac{\sin \alpha \pi}{a \pi} = 1 \) when \( a = 0 \).

Next we consider the case where \( a \) and \( M \) satisfies some limiting constraints. Theorem 7 below corresponds to the case where \( s \sim 1 \).

Theorem 6. When \( u = 0 \) and \( a \not\in \mathbb{N} \), we have

\[
J_M(41, q) \sim \frac{\sin \alpha \pi}{a \pi} 2^{1/2} \left( \frac{M + a}{2\pi i} \right)^{1/2} \exp \left( \frac{M + a}{2\pi i} \right) \times \text{Vol}(\mathbb{S}^3 \setminus 4_1) \]

where \( \frac{\sin \alpha \pi}{a \pi} = 1 \) when \( a = 0 \).

Theorem 7. Let \( q = \exp \left( \frac{2\pi i}{N + \frac{1}{2}} \right) \), i.e. \( a = N - M + \frac{1}{2} \). Let \( s = \lim_{N \to \infty} \frac{M}{N + 1/2} \). Then there exists some \( \delta > 0 \) such that for any \( 1 - \delta < s < 1 \), we have

\[
J_M(41, q) \sim \frac{1}{(M + s)^{1/2}} \exp \left( \left( N + \frac{1}{2} \right) \Phi_M^{(s)}(z_M) \right) \frac{\text{Vol}(\mathbb{S}^3 \setminus 4_1)}{\Phi_M^{(s)}(z_M)}
\]

where

\[
\Phi_M^{(s)}(z) = \frac{1}{2\pi i} \left[ \text{Li}_2 \left( e^{-2\pi i z} \right) - \text{Li}_2 \left( e^{2\pi i z} \right) \right] + 2\pi i \left( 1 - \frac{M}{N + 1/2} \right) z
\]

and \( z_M^{(s)} \) satisfies the equation

\[
\beta_M \omega^2 - (\beta_M^2 + 1 - \beta_M) \omega + \beta_M = 0,
\]

where \( \beta_M = e^{2\pi i (N + \frac{1}{2})} \) and \( \omega_M = e^{2\pi i z_M^{(s)}} \).

For \( s \sim \frac{1}{2} \), we find an upper bound for the AEF for the colored Jones polynomial. The following theorem gives an upper bound for the asymptotic behavior of \( |J_M(41; q)| \) with \( s \sim \frac{1}{2} \).

Theorem 8. Let \( q = \exp \left( \frac{2\pi i}{N + \frac{1}{2}} \right) \), i.e. \( a = N - M + \frac{1}{2} \). Let \( s = \lim_{N \to \infty} \frac{M}{N + 1/2} \). Then there exists some \( \zeta > 0 \) such that for any \( \frac{1}{2} - \zeta < s < \frac{1}{2} + \zeta \), we have

\[
|J_M(41; q)| = O \left( \frac{1}{1 + e^{-2\pi i (s - 1/2)}} \right) \left( N + \frac{1}{2} \right)^{1/2} \sqrt{\frac{2\pi}{|\Phi_M^{(s)}(z_M)|}} \exp \left( \left( N + \frac{1}{2} \right) \chi_M^{(s)}(z_M^{(s)}) \right)
\]

where

\[
\chi_M^{(s)}(x) = \text{Re} \left[ \frac{1}{2\pi i} \left( \text{Li}_2 \left( e^{-2\pi i (x + \frac{1}{2}) + 2\pi i \left( \frac{\phi}{N + \frac{1}{2}} - \frac{1}{2} \right)} \right) - \text{Li}_2 \left( e^{2\pi i (x + \frac{1}{2}) + 2\pi i \left( \frac{\phi}{N + \frac{1}{2}} - \frac{1}{2} \right)} \right) \right) \right]
\]
and $x_M^{(s)}$ is the solution of the following equation
\[
\sin(A + B)\sin(-A + B) = \frac{1}{4}
\]
where $A = \pi(x_M^{(s)} + \frac{1}{2})$ and $B = \pi(\frac{M}{N+1/2} - \frac{1}{2})$ respectively.

Using Theorem 7 and Theorem 8 in Section 2 we will show that the sum of the colored Jones polynomials with $s \sim 1$ dominates that with $s \sim 1/2$. Furthermore, the AEF for the former sum can be found out by the Laplace’s method. As a result, we obtain the AEF for the Turaev-Viro invariant for the figure eight knot stated as follows.

**Theorem 9.** For any $r = 2N + 1 > 3$, the AEF of the Turaev-Viro invariant of the figure eight knot complement is given by
\[
\frac{1}{2}\pi\frac{r^{1/2}}{\sqrt{2\pi}} \left(\frac{1}{2\pi\sqrt{3}}\right)^{3/2} \exp\left(\frac{r}{2\pi} \text{Vol}(SS^3\setminus S^1)\right)
\]
Furthermore, the evaluation of the function $\tilde{\Phi}_0^{(1)}$ at the point $z = \frac{5}{6}$ gives the hyperbolic volume of figure eight knot:
\[
\tilde{\Phi}_0^{(1)}\left(\frac{5}{6}\right) = \frac{1}{2\pi^2} \left[\text{Li}_2\left(e^{-2\pi i}\right) - \text{Li}_2\left(e^{2\pi i}\right)\right]
\]

This observation is consistent with our expectation that the growth rate of the colored Jones polynomial with $s \sim 1$ is close to the hyperbolic volume of figure eight knot.
Moreover, the growth rate of the colored Jones polynomial with \( s \sim 1 \) is given by

\[
\Phi_M^{(s)}(z_M^{(s)}) = \frac{1}{2\pi i} \left[ \text{Li}_2 \left( e^{-2\pi i z_M^{(s)} + 2\pi i \left( \frac{M}{N+1/2} \right)} \right) - \text{Li}_2 \left( e^{2\pi i z_M^{(s)} + 2\pi i \left( \frac{M}{N+1/2} \right)} \right) \right] + 2\pi i \left( 1 - \frac{M}{N + 1/2} \right) z_M^{(s)}
\]  

(9)

with \( z_M^{(s)} \) satisfies the equation

\[
\beta_M \omega_M^2 - (\beta_M^2 + 1) \omega_M + \beta_M = 0,
\]

(10)

where \( \beta_M = e^{2\pi i \left( \frac{M}{N+1/2} \right)} \) and \( \omega_M = e^{2\pi i z_M^{(s)}} \).

Note that Equation (10) is equivalent to the equation

\[
\omega_M + \omega_M^{-1} = \beta_M + \beta_M^{-1} - 1
\]

If we write \( \omega = \omega_M \) and \( B = \beta_M^{-1} \), then the equation can be written as

\[
\omega + \omega^{-1} = B + B^{-1} - 1
\]

(11)

and the value \( \Phi_M^{(s)}(z_M^{(s)}) \) can be expressed in the form

\[
\Phi_M^{(s)}(z_M^{(s)}) = \frac{1}{2\pi i} \left[ \text{Li}_2 \left( \omega^{-1} B^{-1} \right) - \text{Li}_2 \left( \omega B^{-1} \right) \right] + 2\pi i \left( 1 - \frac{M}{N + 1/2} \right) z_M^{(s)}
\]

(12)

\[= \frac{1}{2\pi i} \left[ \text{Li}_2 \left( \omega^{-1} B^{-1} \right) - \text{Li}_2 \left( \omega B^{-1} \right) \right] + 2\pi i (z_M^{(s)} - 1) \left( 1 - \frac{M}{N + 1/2} \right) + 2\pi i \left( 1 - \frac{M}{N + 1/2} \right)
\]

\[= \frac{1}{2\pi i} \left[ \text{Li}_2 \left( \omega^{-1} B^{-1} \right) - \text{Li}_2 \left( \omega B^{-1} \right) + (2\pi i (z_M^{(s)} - 1)) \left( 1 - \frac{M}{N + 1/2} \right) \right] + 2\pi i \left( 1 - \frac{M}{N + 1/2} \right)
\]

\[= \frac{1}{2\pi i} \left[ \text{Li}_2 \left( \omega^{-1} B^{-1} \right) - \text{Li}_2 \left( \omega B^{-1} \right) + \log \omega \log B \right] + 2\pi i \left( 1 - \frac{M}{N + 1/2} \right)
\]

(13)

where \( H(x, y) \) is the function appear in [13] given by

\[
H(x, y) = \text{Li}_2(x^{-1} y^{-1}) - \text{Li}_2(xy^{-1}) + \log x \log y
\]

and \( \log \) is the principal logarithm. In particular, we have

\[
\exp \left( \left( N + \frac{1}{2} \right) \Phi_M^{(s)}(z_M^{(s)}) \right) = \exp \left( \frac{N + 1/2}{2\pi i} H(\omega, B) + \pi i \right) = - \exp \left( \frac{N + 1/2}{2\pi i} H(\omega, B) \right)
\]

As a result, the growth rate can be interpreted as the hyperbolic volume of the cone manifold with singularity the figure eight knot [Theorem 1.2 and Remark 1.4 in [12]].

Finally we compare our result with [12]. In [12], the growth rates of the colored Jones polynomial evaluated at \( \exp(2\pi i r/N) \) is computed. It depends on whether \( r \) is irrational. The difference between rational and irrational \( r \) is that for rational \( r \), the equation \( g(j) = 0 \), where

\[
g(j) = q^{(N+j)/2} - q^{-(N+j)/2} (q^{(N-j)/2} - q^{-(N-j)/2}) = 4 \sin(\pi r j/N + \pi r) \sin(\pi r j/N - \pi r),
\]

has an integer solution \( B = N(1-r)/r \) for certain choices of \( N \). In particular we have

\[
\prod_{j=1}^{N-1} g(j) = 0 \times \text{something that may have exponential growth} = 0
\]
It is natural to compare our Theorem 4 to Murakami’s result with \( r = \frac{N}{4 \pi i}. \) Nevertheless, the evaluation of \( g(j) \) at \( \exp(2\pi i/(N+1/2)) \) never vanish. Precisely, the analogue of the equation \( g(j) = 0 \) is given by

\[
-4\sin(\pi(N+j)/(N+1/2)) \sin(\pi(N-j)/(N+1/2)) = 0
\]

Since \( j < N \), one can show that such integer solution does not exist.

We suggest that this kind of vanishing phenomenon is the reason why the Turaev-Viro invariant and the Reshetikhin-Turaev invariants grow exponentially at \( 2r \)-root of unity but grow polynomially at \( 4r \)-root of unity. Similar phenomenon can also be found in the evaluation of \( M \)-th colored Jones polynomial at \((M+\text{integer})\)-th root of unity.

### 1.5 From volume conjecture of colored Jones polynomial to volume conjecture of Turaev-Viro invariant

Finally we summarized the techniques used in this paper and try to relate the AEF of colored Jones polynomials and that of Turaev-Viro invariant from an analytical perspective.

It is suggested by H. Murakami [15] that the AEF of the colored Jones polynomial can be expressed in the form of a contour integral of a function with the form \( e^{N\Phi(z)} \) along some suitable contour \( C \). Here the function \( \Phi(z) \) is called the potential function of the knot \( K \) with the property that the evaluation of the potential function at some saddle point is equal to the complex volume of the knot \( K \).

In this paper, the authors generalize the above approach by using the one-parameter family of saddle point approximation. This gives a family of potential functions \( \Phi_M(z) \) for \( s = \lim_{N \to \infty} \frac{M}{N+1/2} \approx 1 \). From this we obtained the AEF stated as Theorem 5, 6 and 7.

This idea can be summarized by the following conjecture:

**Conjecture 3.** For any hyperbolic knot \( K \) and two integers \( M \) and \( N \), we denote \( s = \lim_{N \to \infty} \frac{M}{N+1/2} \). Then there exists a small neighborhood \( U \subset \mathbb{R} \) of \( s = 1 \) such that for any \( M, N \) with \( s \in U \), we have a holomorphic function \( \Phi_M(z) \) satisfying the following properties:

1. the holomorphic function \( \Phi_M(z) \) gives the potential function of the knot \( K \) by taking limit \( N \to \infty \), i.e.

   \[
   \lim_{N \to \infty} \Phi_M(z) = \Phi_0(z).
   \]

   Furthermore, denote \( z_0 \) to be the non-degenerate saddle point of the potential function \( \Phi_0(z) \) that gives the complex volume of the knot \( K \), i.e.

   \[
   \frac{d}{dz} \Phi_0(z) = 0 \quad \text{and} \quad \frac{d^2}{dz^2} \Phi_0(z) \neq 0.
   \]

   Then there exist a smooth choices of saddle point \( z_M \) of the family of potential functions such that

   (a) the points \( z_M \) satisfy the saddle point equations and they are non-degenerate, i.e.

   \[
   \frac{d}{dz} \Phi_M(z_M) = 0 \quad \text{and} \quad \frac{d^2}{dz^2} \Phi_M(z_M) \neq 0;
   \]

   (b) as \( M \to N \), we have

   \[
   z_M \to z_0 \quad \text{and} \quad \Phi_M(z_M) \to \Phi_0(z_0) = \text{Vol}(K) + i \text{CS}(K).
   \]

2. the family of potential functions determine the AEF of the colored Jones polynomial in the following way:

   \[
   J_M(K, e^{\frac{2\pi i}{M}}) \sim_{M \to \infty} (\text{constant}) \times \left( \frac{N}{2\pi i} \right)^{3/2} \exp\left( \frac{N+1/2}{2\pi} \times \Phi_M(z_M) \right). \]
In particular Conjecture 3 is true for $K = 4_1$.

Similar idea can be applied to the study of AEF of the Turaev-Viro invariant. Naively the TV invariant can be thought of a double integral over a suitable surface, with one integral corresponding to the sum inside each colored Jones polynomials and the other integral corresponding to the sum over all the colored Jones polynomials. Note that this idea can be found in [1] where the AEF for RT invariant is studied. So it is natural to think that the volume conjecture of TV invariant is equivalent to the 2-dimensional saddle point approximation over a suitable surface.

In this paper our approach is different from that in [1]. We break the 2-dimensional saddle point approximation into iterated 1-dimensional saddle point approximation, by first finding the AEF of colored Jones polynomial with $s \sim 1$ parametrized by the ratio $\frac{M}{N+1/2}$ and then apply the saddle point approximation again along the parameter $s$.

Besides, from the development of the volume conjecture, we expect that the AEF of the colored Jones polynomial should be related to the character variety of the knot complement. By Mostow rigidity, there exists a unique point on the character variety which corresponds to the complete hyperbolic structure. The classical volume conjecture is about the topology (Reidemeister torsion) and geometry (hyperbolic volume) at this point. From an analytical perspective, the volume conjecture corresponds to the classical saddle point approximation.

In this paper we study the AEF of the $M$-th colored Jones polynomial evaluated at $(N+1/2)$-th root of unity. By introducing the limiting ratio $s = \lim_{N \to \infty} \frac{M}{N+1/2}$ the AEF of $J_M(4_1, \exp(2\pi i/(N+1/2)))$ has been found out for $s \sim 1$. Moreover, the real part of the exponential growth rate coincides with the volume of the cone manifold. Therefore, the number $s$ can be thought of a parametrization of the points on the variety. Using the idea of the classical case, the AEF of the colored Jones polynomial with limiting ratio $s$ should also capture the same types of topological and geometrical information. From an analytical perspective, this kinds of ‘volume conjecture’ corresponds to the one-parameter family of saddle point approximation.

In our study about figure eight knot, although we cannot find out the explicit AEF for the case where $s \sim 1/2$, we are able to show that it has an upper bound which is dominated by the contribution of the colored Jones polynomial with $s \sim 1$. This can be explained as follows. By the work of Thurston [20], we know that the hyperbolic volume of the manifold with complete hyperbolic structure is strictly greater than that with incomplete hyperbolic structure. Hence, if $s$ is not close to 1 (that means the point is away from the point with complete structure), then the exponential growth rate (volume of the manifold at that point) is strictly smaller and hence can be ignored.

From above discussion, we expect that this kind of phenomenon is true for any hyperbolic knot. More precisely the conjecture can be stated as follows.

**Conjecture 4.** In the content of Conjecture 3 for any hyperbolic knot $K$, the sum of colored Jones polynomials with $s \in U$ dominates the ones with $s \notin U$, i.e.

$$\sum_{M : s \notin U} \left| J_M \left( K, e^{\frac{2\pi i}{N+1/2}} \right) \right|^2 = o \left( \sum_{M : s \in U} \left| J_M \left( K, e^{\frac{2\pi i}{N+1/2}} \right) \right|^2 \right),$$

where $a_N = o(b_N)$ if and only if $\lim_{N \to \infty} \frac{a_N}{b_N} = 0$.

With Conjecture 4 our approach of finding the AEF of the TV invariant can be formulated as the following conjecture:
Conjecture 5. In the content of Conjecture 3 and 4, for any hyperbolic knot $K$, we can find a function $\Phi(s, z)$: $D \subset U \times \mathbb{C} \rightarrow \mathbb{C}$ holomorphic in $z$ such that

1. the holomorphic function $\Phi(s, z)$ recovers the potential functions stated in Conjecture 3, i.e.
$$\Phi\left(\frac{M}{N + 1/2}, z\right) = \Phi_M^0(z) \quad \text{and} \quad \Phi(1, z) = \Phi(z);$$

2. there exists a smooth choice of non-degenerate saddle points $z(s)$ such that for each $s \in D$,
$$\frac{d}{dz} \Phi(s, z(s)) = 0 \quad \text{and} \quad \frac{d^2}{dz^2} \Phi(s, z(s)) \neq 0;$$

3. the AEF of the Turaev-Viro invariant is given by
$$TV_r(K) = TV_r(SS^3\setminus K, e^{\frac{2\pi i}{N+1/2}})$$
$$= (\eta_+')^2 \sum_{1 \leq M \leq N} |J_M\left(K, e^{\frac{2\pi i}{N+1/2}}\right)|^2$$
$$\sim r^{-\infty} \left(\frac{\pi^{5/2}}{4}\right) \left(\frac{r}{2\pi}\right)^{1/2} [T(K)]^{1/2} \exp\left(\frac{r}{2\pi} \times (\text{Vol}(SS^3\setminus K))\right),$$

where $T(K)$ and $\text{Vol}(K)$ are the twisted Reidemeister torsion and the hyperbolic volume associated with the unique complete hyperbolic structure of $SS^3\setminus K$ respectively.

1.6 Organization

In Section 2 we will outline the proof of the main theorems. In order to focus on the key ideas, the proofs of the technical statements will be collected in Section 3.

2 Proof Outline of the Main Theorem

This section is divided into three parts. The first part aims to prove Theorem 5 and illustrate the techniques used in [23] and [10]. We will show the AEF for the colored Jones polynomial at $(M + a)$-th root of unity with fixed $a \geq 0$. The AEF will then be generalized to the case where $a > 0$ satisfies some limiting relation with $M$. This gives the proof of Theorem 7 and Theorem 8. Finally, we apply the AEF’s obtained in part two to prove Theorem 9.

2.1 AEF for the colored Jones polynomial around $(M + a)$-th root of unity with fixed $a \geq 0$

Fixed $a \geq 0$. We are going to consider the asymptotic behavior of the $M$-th colored Jones polynomial around $(M + a)$-th root of unity, i.e. $q = \exp\left(\frac{2\pi i u}{M + a}\right)$ with $0 \leq u < \log((3 + \sqrt{5})/2)$.

Recall that the formula of colored Jones polynomial, the definition of quantum dilogarithm and its functional equation are given as follows:

1. $J_M(41; q) = \sum_{k=0}^{M-1} q^{-kM} \prod_{l=1}^{k} (1 - q^{M-l}) (1 - q^{M+l})$

2. Fix $\gamma \in \mathbb{C}$ with $\text{Re}(\gamma) > 0$. Then for any $|\text{Re}(z)| < \pi + \text{Re}(\gamma)$, the quantum dilogarithm function is defined to be
$$S_\gamma(z) = \exp\left(\frac{1}{4} \int_{C_R} \frac{e^{zt}}{\sinh(\pi t) \sinh(\gamma t)} \frac{dt}{t}\right),$$

where $C_R = (-\infty, -R] \cup \Omega_R \cup [R, \infty)$ with $\Omega_R = \{ Re^{i(\pi - t)} \mid 0 \leq s \leq \pi \}$ for $0 < R < \min\{\pi/|\gamma|, 1\}$. 

3. For \( |\text{Re}(z)| < \pi \), the quantum dilogarithm satisfies the functional equation:

\[
(1 + e^{iz}) S_r(z + \gamma) = S_r(z - \gamma)
\]

Using the functional equation of the quantum dilogarithm, one may extend the definition of quantum dilogarithm to any complex number \( z \) with

\[
\text{Re}(z) \neq \pi + 2m\text{Re}(\gamma) \quad \text{and} \quad \text{Re}(z) \neq -\pi - 2m'\text{Re}(\gamma)
\]

for any \( m, m' \in \mathbb{N} \).

Now we are going to obtain the AEF of the colored Jones polynomials at \((M + a)\)th root of unity with \( a > 0, a \notin \mathbb{N} \). In fact we are going to find out the AEF around the root of unity, i.e. \( q = \exp(\frac{2\pi i + u}{M + a}) \). Then by taking \( u = 0 \) we can get our desired result.

Applying the functional equation of the quantum dilogarithm with the values

\[
\gamma = \frac{2\pi - iu}{2(M + a)}, \quad \xi = 2\pi i + u \quad \text{and} \quad z = \pi - iu - 2(l + a)\gamma
\]

and observing that \( \frac{\xi}{M + a} = 2i\gamma \), we have

\[
\prod_{l=1}^{k} \left(1 - e^{\frac{2\pi i + l\gamma}{M + a}}\right) = \frac{S_r(\pi - iu - (2(k + a) + 1)\gamma)}{S_r(\pi - iu - (2a + 1)\gamma)} \tag{14}
\]

Similarly, putting \( z = -\pi - iu + 2(l - a)\gamma \), we have

\[
\prod_{l=1}^{k} \left(1 - e^{\frac{2\pi i + l\gamma}{M + a}}\right) = \frac{S_r(-\pi - iu + (1 - 2a)\gamma)}{S_r(-\pi - iu + (2(k - a) + 1)\gamma)} \tag{15}
\]

Furthermore, we split the colored Jones polynomial into two parts:

\[
J_M(4_1; q) = \sum_{k=0}^{M-1} q^{-kM} \prod_{l=1}^{k} (1 - q^{M-1}) (1 - q^{M+1})
\]

\[
= \left[1 + \sum_{k=1}^{[a]-1} q^{-kM} \prod_{l=1}^{k} (1 - q^{M-1}) (1 - q^{M+1})\right]
\]

\[
+ \sum_{k=\lceil a \rceil}^{M-1} q^{-kM} \prod_{l=1}^{k} (1 - q^{M-1}) (1 - q^{M+1})
\]

Overall from (14) and (15) we have

\[
J_M(4_1; q) = \left[1 + \sum_{k=1}^{[a]-1} q^{-kM} \prod_{l=1}^{k} (1 - q^{M-1}) (1 - q^{M+1})\right]
\]

\[
+ \frac{S_r(-\pi - iu - (2a - 1)\gamma)}{S_r(\pi - iu - (2a + 1)\gamma)} \times \sum_{k=\lceil a \rceil}^{M-1} e^{-\frac{2\pi ik}{M+1}} \frac{S_r(\pi - iu - (2k + 2a + 1)\gamma)}{S_r(-\pi - iu + (2k - 2a + 1)\gamma)} \tag{16}
\]

Define

\[
g_M(z) = \exp \left(- (M + a)(u - \frac{a\xi}{M + a})z\right) \frac{S_r(\pi - iu + i\xi z + i\xi(\frac{a\xi}{M + a}))}{S_r(-\pi - iu - i\xi z + i\xi(\frac{a\xi}{M + a}))}
\]
Now we want to find an integral expression for $J_M(41;q)$ by using residue theorem. Since $S_\gamma(z)$ is defined for $|\text{Re}(z)| < \pi + \text{Re}(\gamma)$ and $\text{Re}(\gamma) > 0$, one may check that $g$ is well-defined and analytic on the domain (i.e. open, connected) $D$ where

$$D = \left\{ x + iy \in \mathbb{C} \mid -2\pi \left( x + \frac{\alpha}{M + \alpha} \right) < \text{Re}(\gamma) < 2\pi \left( 1 - \left( x + \frac{\alpha}{M + \alpha} \right) \right) \right\}$$

$$= \left\{ x + iy \in \mathbb{C} \mid -2\pi \left( x - \frac{\alpha}{M + \alpha} \right) < \text{Re}(\gamma) < 2\pi \left( 1 - \left( x - \frac{\alpha}{M + \alpha} \right) \right) \right\}$$

Next, let $\epsilon = \frac{2a + \frac{1}{2}}{2(M + a)}$. Consider the contour $C(\epsilon) = C_+ (\epsilon) \cup C_- (\epsilon)$ with the polygonal lines $C_\pm (\epsilon)$ defined by

$C_+(\epsilon) : \quad 1 - \epsilon \rightarrow 1 - \frac{\epsilon}{2\pi} - \epsilon + i \rightarrow 0 + \epsilon + i \rightarrow \epsilon$

$C_-(\epsilon) : \quad \epsilon \rightarrow 0 - \epsilon + \frac{\epsilon}{2\pi} - i \rightarrow -1 - \epsilon + \frac{\epsilon}{2\pi} - i \rightarrow 1 - \epsilon$

Note that for $k = [a], [a] + 1, [a - 1] + 2, \ldots, M - 1$, the singularities $\frac{2k + 1}{2(M + a)}$ of the function $z \mapsto \tan((M + a)\pi z)$ lie in $D$. This is the reason why we need to split $J_M(41;q)$ into two parts. Using Residue Theorem, we may express the colored Jones polynomial as

$$J_M(41;q) = \left[ 1 + \sum_{k=1}^{[a]-1} q^{-kM} \prod_{l=1}^{k} (1 - q^{M-l}) (1 - q^{M+l}) \right] +$$

$$\frac{S_\gamma(-\pi - iu - (2a - 1)\gamma)}{S_\gamma(\pi - iu - (2a + 1)\gamma)} \cdot \frac{(M + a)i \exp(i \frac{\pi}{2} - \frac{a\epsilon}{2(M + a)})}{2} \int_{C(\epsilon)} \tan((M + a)\pi z) g_M(z) dz \quad (17)$$

Note that as $M$ goes to infinity, the first part of $J_M(41;q)$ grows at most polynomially. So it suffices to consider the large $M$ behavior of the second part. In order to estimate the integral, let

$$G_{\pm}(M, \epsilon) = \int_{C_{\pm}(\epsilon)} \tan((M + a)\pi z) g_M(z) dz .$$

Then one may rewrite

$$J_M(41;q) = \left[ 1 + \sum_{k=1}^{[a]-1} q^{-kM} \prod_{l=1}^{k} (1 - q^{M-l}) (1 - q^{M+l}) \right] +$$

$$\frac{S_\gamma(-\pi - iu - (2a - 1)\gamma)}{S_\gamma(\pi - iu - (2a + 1)\gamma)} \times \frac{(M + a)i \exp(i \frac{\pi}{2} - \frac{a\epsilon}{2(M + a)})}{2} (G_+ (M, \epsilon) + G_- (M, \epsilon)) \quad (18)$$

The integral in $G_{\pm}$ may be split by adding and subtracting the same term as follows,

$$G_{\pm}(M, \epsilon) = \pm i \int_{C_{\pm}(\epsilon)} g_M(z) dz + \int_{C_{\pm}(\epsilon)} \tan((M + a)\pi z) \mp i) g_M(z) dz$$

The second term can be controlled by the following analogue of Proposition 2.2 in $[23]$.

**Proposition 1.** There exists a constant $K_{\gamma, \pm}$ independent of $M$ and $\epsilon$ such that

$$\left| \int_{C_{\pm}(\epsilon)} \tan((M + a)\pi z) \pm i) g_M(z) dz \right| < \frac{K_{\gamma, \pm}}{M + a} .$$
To approximate $g_M$, define a function

$$
\tilde{\Phi}_M(z) = \frac{1}{\xi} \left[ \text{Li}_2 \left( e^{u-(z+\frac{\pi i}{\xi})^\epsilon} \right) - \text{Li}_2 \left( e^{u+(z-\frac{\pi i}{\xi})^\epsilon} \right) \right] - \left( u - \frac{a \xi}{M + a} \right) z
$$

Since $\text{Li}_2$ is analytic in $\mathbb{C} \setminus [1, \infty)$, by considering the region where

$$
\text{Im} \left( u - \left( z + \frac{a}{M + a} \right) \xi \right), \text{Im} \left( u + \left( z - \frac{a}{M + a} \right) \xi \right) \in (-2\pi, 0),
$$

one may verify that the function $\Phi_M(z)$ is analytic in the region

$$
D' = \left\{ x + iy \in \mathbb{C} \, | \, -2\pi < x - \frac{a}{M + a} < 2\pi \left( 1 - \frac{a}{M + a} \right) \right\} \subset D
$$

Note that the contour $C(\epsilon)$ and the poles of $\tan((M + a)\pi z)$ lie inside $D'$. The following result is an analogue of Proposition 2.3 in [23].

**Proposition 2.** Let $p(\epsilon)$ be any contour in the parallelogram bounded by $C(\epsilon)$ connecting from $\epsilon$ to $1 - \epsilon$, then there exists a constant $K_2 > 0$ independent of $M$ and $\epsilon$ such that

$$
\left| \int_{p(\epsilon)} g_M(z) dz - \int_{p(\epsilon)} \exp((M + a)\tilde{\Phi}_M(z)) dz \right| \leq \frac{K_2 \log(M + a)}{M + a} \max_{\omega \in p(\epsilon)} \left\{ \exp((M + a)\text{Re} \tilde{\Phi}_M(z)) \right\}
$$

Since $\tilde{\Phi}_M(z)$ is analytic on $D'$, by Cauchy’s theorem

$$
\int_{C_+(\epsilon)} \exp \left( (M + a)\tilde{\Phi}_M(z) \right) dz = - \int_{C_-(\epsilon)} \exp \left( (M + a)\tilde{\Phi}_M(z) \right) dz
$$

Define a new function $\Phi(z)$ by

$$
\Phi_M(z) = \frac{1}{\xi} \left[ \text{Li}_2 \left( e^{u-(z+\frac{\pi i}{\xi})^\epsilon} \right) - \text{Li}_2 \left( e^{u+(z-\frac{\pi i}{\xi})^\epsilon} \right) \right] - u z
$$

The contour integral can be further expressed as

$$
\int_{C_-(\epsilon)} \exp \left( (M + a)\tilde{\Phi}_M(z) \right) dz = \int_{C_-(\epsilon)} \exp(a \xi z) \exp \left( (M + a)\Phi_M(z) \right) dz
$$

To approximate the above two integrals, we need the following generalized saddle point approximation. The proof is similar to that of Theorem 2.4 at [23].

**Theorem 10.** (One-parameter family version for saddle point approximation) Let $\{ \Phi_y(z) \}_{y \in [0, 1]}$ be a family of holomorphic functions smoothly depending on $y \in [0, 1]$. Let $C(y, t) : [0, 1]^2 \rightarrow \mathbb{C}$ be a continuous family of contours with length uniformly bounded above by a fixed constant $L$, such that for each $y \in [0, 1]$, $C(y, t)$ lies inside the domain of $\Phi_y(z)$, for which $z_y$ is the only saddle point along the $\alpha$ of contour $C_y$ and $\max \text{Re} \Phi_y(z)$ is attained at $z_y$. Further assume that $\left| \text{arg} \left( \sqrt{-\frac{d^2\Phi_y}{dz^2}(z_y)} \right) \right| < \pi/4$. Suppose we have an analytic function $f(z)$ along the contour such that $f(z_y) \neq 0$ for an $y \in [0, 1]$. Then for any sequence $\{ y_M \}_{M \in \mathbb{N}}$ with $y_M \rightarrow 0$ as $M \rightarrow \infty$, we have the following generalized saddle point approximation:

$$
\int_{C(y_M, t)} f(z) \exp \left( (M + a)\Phi_{y_M}(z) \right) dz = \frac{2\pi}{(M + a) \left( -\frac{d^2\Phi_{y_M}}{dz^2}(z_{y_M}) \right)} f(z_{y_M}) \exp \left( (M + a)\Phi_{y_M}(z_{y_M}) \right) \left( 1 + O \left( \frac{1}{M + a} \right) \right)
$$
In our case, we have
\[ \Phi_M(z) = \frac{1}{\xi} \left[ \text{Li}_2 \left( e^{z+i\frac{\pi}{2}} \xi \right) - \text{Li}_2 \left( e^{z-i\frac{\pi}{2}} \xi \right) \right] - uz \]
\[ \Phi(z) = \frac{1}{\xi} \left[ \text{Li}_2 \left( e^{z-\xi} \right) - \text{Li}_2 \left( e^{z+\xi} \right) \right] - uz \]

**Lemma 2.** The curves described in Theorem 10 exist.

By Theorem 10 and Lemma 2, we have

**Theorem 11.** (Large \( M \) behavior of \( \int_{C_z(\epsilon)} \exp(a\xi z) \exp \left( (M + a)\Phi_M(z) \right) dz \)) Let \( z_M \) be the saddle point of \( \Phi_M \) inside the contour \( C_z(\epsilon) \). Then
\[
\int_{C_z(\epsilon)} \exp(a\xi z) \exp \left( (M + a)\Phi_M(z) \right) dz \xrightarrow{M \to \infty} \sqrt{2\pi} \frac{\exp \left( (M + a)\Phi_M(z_M) \right)}{\sqrt{M + a} \sqrt{-\frac{d^2\Phi_M}{dz^2}(z_M)}} \]

Together with the following proposition, which provides a control on the right-hand side, the integral in Theorem 11 is ensured to have exponentially growth.

**Proposition 3.** \( \text{Re} \Phi_M(z_M) \) is positive for \( 0 \leq u < \log((3 + \sqrt{5})/2) \) and \( M \) is large.

Combining the controls in Propositions 1 and 2 and Theorem 11, we are able to estimate \( G_{\pm}(M, \epsilon) \), namely,
\[
\lim_{M \to \infty} \left| \frac{G_{\pm}(M, \epsilon)}{-i \int_{C_z(\epsilon)} \exp(a\xi z) \exp \left( (M + a)\Phi_M(z) \right) dz} - 1 \right| \leq K_{1,\pm} + \frac{K_2 \log(M + a)}{M + a} \frac{\exp \left( (M + a)\text{Re} \Phi_M(z_M) \right)}{\int_{C_z(\epsilon)} \exp(a\xi z) \exp \left( (M + a)\Phi_M(z) \right) dz} \]

Thus, up to this point, we can asymptotically express \( J_M \) in terms of quantum dilogarithm and a contour integral involving exponential of \((M + a)\Phi_M\). That is,
\[
J_M \left( a_1, e^{\xi/(M+a)} \right) \xrightarrow{M \to \infty} \frac{S_3(-\pi - iu - (2a - 1)\gamma)}{S_3(\pi - iu - (2a + 1)\gamma)} \times (M + a) \exp \left( \frac{u}{2} - \frac{a\xi}{2(M + a)} \right) \int_{C_z(\epsilon)} \exp(a\xi z) \exp \left( (M + a)\Phi_M(z) \right) dz
\]

Moreover, we also have the fact that (see p.200 of [16])
\[
\lim_{M \to \infty} \frac{d^2\Phi_M}{dz^2}(z_M) = \frac{d^2\Phi}{dz^2}(z) = \xi \sqrt{(2\cosh u + 1)(2\cosh u - 3)}
\]

The asymptotic behavior of the ratio of the quantum dilogarithm is given by the following lemma.
Lemma 3. For $\gamma = \frac{2\pi - i u}{2(M + a)}$ with $u > 0$, let $a = b + c$ for some $b \in \mathbb{N} \cup \{0\}$, $c \in (0,1)$.

1. If $b \neq 0$ and $u \neq 0$, we have

$$\frac{S_r(-\pi - i u - (2a - 1)\gamma)}{S_r(\pi - i u - (2a + 1)\gamma)} = \frac{\exp(u - 2c\gamma i)}{\exp(u - 2\alpha i)} \times \frac{S_r(-\pi - i u - (2c - 1)\gamma)}{S_r(\pi - i u - (2c + 1)\gamma)} = \frac{\exp(u - 2c\gamma i)}{\exp(u - 2\alpha i)} \times \frac{\exp(u - 2\gamma i)}{\exp(u - 2\gamma i)} \sim \frac{\exp(2\pi i u (M + a)/\xi - 2\alpha i)}{\exp(u - 1)}.$$  

2. If $u = 0$, we have

$$\frac{S_r(-\pi - (2a - 1)\gamma)}{S_r(\pi - (2a + 1)\gamma)} = \frac{\exp(-2\alpha i)}{\exp(-2\alpha i)} = \frac{\exp(-a\pi i)}{\exp(-a\gamma i)} \sim \frac{\exp(-a\pi i + a\gamma i)}{a\pi} (M + a).$$

Since $b$ is a non-negative integer, we also have $\exp(2\pi i a i) = \exp(2\pi i c i)$. By Theorem [11] [19], [20] and Lemma 3, we have

$$J_M (A, q) \sim \frac{e^{2\pi i u (M + a)/\xi - 2\alpha i}}{e^u - 1} (M + a)^{1/2} \left(e^{u/2 - \alpha i/2(M + a)}\right) \exp(a\xi z_M) \times \sqrt{2\pi} \exp((M + a)\Phi_M (z_M))$$

$$\sim \frac{e^{2\pi i u (M + a)/\xi - 2\alpha i}}{e^u - 1} \sqrt{\left(\frac{(2\cosh(u) + 1)(2\cosh(u) - 3)}{2\cosh(u)}\right)} (M + a)^{1/2} \times \sqrt{\pi e^{u/2}} \exp(a\xi z_M) \exp((M + a)\Phi_M (z_M))$$

$$\sim \frac{e^{2\pi i u (M + a)/\xi - 2\alpha i}}{e^u - 1} T(u)^{1/2} \left(\frac{M + a}{\xi}\right)^{1/2} \times \sqrt{-\pi e^{u/2}} \exp(a\xi z_M) \exp((M + a)\Phi_M (z_M))$$

In order to apply the saddle point approximation, we have to solve the equation

$$\frac{d\Phi_M}{dz}(z) = 0.$$  

Recall that

$$\Phi_M(z) = \frac{1}{\xi} \left[\text{Li}_2\left(e^{u-(z+\frac{2\pi i}{\xi})}\xi\right) - \text{Li}_2\left(e^{u+(z+\frac{2\pi i}{\xi})}\xi\right)\right] - u z$$

$$\frac{d}{dq}\text{Li}_2(e^q) = \text{Li}_1(e^q) = -\log(1 - e^q)$$

The desired saddle point equation (22) can be rewritten as below,

$$\log(1 - e^{u-(z+\frac{2\pi i}{\xi})}) (1 - e^{u+(z+\frac{2\pi i}{\xi})}) - u = 0,$$
which in turns becomes,

\[(1 - e^{u-(z + a)\xi})(1 - e^{u+(z - a)\xi}) = e^u.\]  

(23)

With \(a = e^u, b = e^{\frac{\pi}{4}a\xi}\) and \(w = e^{z\xi}\), the above equation is equivalent to

\[ab\omega^2 - (a^2 + b^2 - ab)\omega + ab = 0\]  

(24)

**Remark 1.** By putting \(b = 1\) we obtained the quadratic equation appeared in p.200 of [16].

Let \(\omega_M\) be the solution for \(\omega\) inside the domain \(C(\varepsilon)\) and \(e^{z\xi_M} = \omega_M\). Furthermore, let \(z_0\) be the solution of the saddle point equation of \(\Phi_0(z)\), where \(\Phi_0(z)\) is defined to be the limit of \(\Phi_M(z)\):

\[\Phi_0(z) = \frac{1}{\xi} \left[ Li_2 \left(e^{u-z\xi}\right) - Li_2 \left(e^{u+z\xi}\right) \right] - uz\]

Note that we have \(z_M \to z_0\) as \(M \to \infty\). The last step to establish Theorem 5 is to change \(\Phi_M\) into \(\Phi_0\). The estimation between them is given by the following lemma, which is direct consequence of L’Hospital rule.

**Lemma 4.** For any \(z \in D\),

\[\lim_{M \to \infty} (M+a)(\Phi_M(z) - \Phi_0(z)) = a[\log(1 - e^{u-z\xi}) - \log(1 - e^{u+z\xi})]\]

From Equation (3.1) in [16] we know that \(z_0 = \frac{\phi(u) + 2\pi i}{\xi}\). That means

\[\exp ((M+a)\Phi_0(z_M)) \sim \exp [(M+a)(\Phi_0(z_M))]\]

(25)

Using \(24\), one can show that \(z_M - z_0 = O\left(\frac{1}{M+a}\right)\). Together with the fact that \(z_0\) satisfies the equation \(\frac{d\Phi_0}{dz} \bigg|_{z_0} = 0\), we have

**Lemma 5.** \(\lim_{M \to \infty} (M+a)(\Phi_0(z_M) - \Phi_0(z_0)) = 0\)

As a result, \(25\) becomes

\[\exp ((M+a)\Phi_M(z_M)) \sim \frac{1 - e^{u-\phi(u)}a}{1 - e^{u+\phi(u)}a} \exp ((M+a)(\Phi_0(z)))\]  

(26)

Altogether, by \(21\) and \(26\), we have
Proposition 2.

Now, we try to apply the arguments in previous subsection to the case where $a\leq \frac{N}{2}$ is close to $1$. Then we can split the colored Jones polynomial as before. Note that the following arguments also work if we replace $\frac{1}{2}$ by any other number $c\in (0,1)$ under suitable modification.

We repeat the trick as in previous subsection. Take $\epsilon = \frac{2a + 1}{2(M + a)}$ and define the contour $C(\epsilon) = C_+(\epsilon) \cup C_-(\epsilon)$ with the polygonal lines $C_\pm(\epsilon)$ defined by

$C_+ (\epsilon) : \quad 1 - \epsilon \rightarrow 1 - \frac{u}{2\pi} - \epsilon \rightarrow -\frac{u}{2\pi} + \epsilon \rightarrow \epsilon$

$C_- (\epsilon) : \quad \epsilon \rightarrow \epsilon + \frac{u}{2\pi} - i \rightarrow 1 - \epsilon + \frac{u}{2\pi} - i \rightarrow 1 - \epsilon$

Define $g_M(\epsilon)$ and $G(M, \pm \epsilon)$ as before. We have the following analogues of the Proposition \[ and Proposition \[.
Proposition 4. There exists some $\eta_1 > 0$ such that for any $s \in (1 - \eta_1, 1]$, there exists a constant $K_{1, \pm}$ independent of $M, N, s$ and $\epsilon$ such that

$$
\left| \int_{C_{\pm}(\epsilon)} (\tan((N + 1/2)\pi z) \pm i)g_M(z)dz \right| < \frac{K_{1, \pm}}{N + 1/2}.
$$

Proposition 5. Let $p(\epsilon)$ be any contour in the parallelogram bounded by $C(\epsilon)$ connecting from $\epsilon$ to $1 - \epsilon$. Then there exists some $\eta_2 > 0$ such that for any $s \in (1 - \eta_2, 1]$, there exists a constant $K_2 > 0$ independent of $M, N, s$ and $\epsilon$ such that

$$
\left| \int_{p(\epsilon)} g_M(z)dz - \int_{p(\epsilon)} \exp((N + \frac{1}{2})i\Phi_M(z))dz \right| \leq \frac{K_2 \log(N + 1/2)}{N + 1/2} \max_{w \in p(\epsilon)} \left\{ \exp((N + \frac{1}{2}) \text{Re} \Phi_M(z)) \right\}
$$

When $u = 0$, our function $\Phi_M(z)$ is given by

$$
\Phi_M(z) = \frac{1}{2\pi i} \left[ \text{Li}_2 \left( e^{-2\pi i s + 2\pi i(\frac{M}{N} + \frac{z}{N})} \right) - \text{Li}_2 \left( e^{2\pi i s + 2\pi i(\frac{M}{N} + \frac{z}{N})} \right) \right] + 2\pi i \left( 1 - \frac{M}{N + 1/2} \right) \sum_{k=1}^{[a]} \frac{\xi_k z}{M + a}
$$

Now since $a = a_M$ is no longer fixed, the limiting function of $\Phi_M(z)$’s are different for each $s$. What we have discussed in previous subsection can be considered as a special case where $s = 1$. In general we define the function $\Phi_M^{(s)}$ and the limiting function $\Phi_0^{(s)}$ by

$$
\Phi_M^{(s)}(z) = \frac{1}{2\pi i} \left[ \text{Li}_2 \left( e^{-2\pi i s + 2\pi i(\frac{M}{N} + \frac{z}{N})} \right) - \text{Li}_2 \left( e^{2\pi i s + 2\pi i(\frac{M}{N} + \frac{z}{N})} \right) \right] + 2\pi i \left( 1 - \frac{M}{N + 1/2} \right) \sum_{k=1}^{[a]} \frac{\xi_k z}{M + a}
$$

$$
\Phi_0^{(s)}(z) = \frac{1}{2\pi i} \left[ \text{Li}_2 \left( e^{-2\pi i s + 2\pi i s} \right) - \text{Li}_2 \left( e^{2\pi i s + 2\pi i s} \right) \right] + 2\pi i \left( 1 - s \right) \sum_{k=1}^{[a]} \frac{\xi_k z}{M + a}
$$

Let $z_0^{(s)}$ be the solution of the saddle point equation

$$
\frac{d}{dz} \Phi_0^{(s)}(z) = 0
$$

Since $z_0^{(s)}$ and the contour depend continuously on $s$, there exists a positive real number $\zeta < \min\{\eta_1, \eta_2\}$ such that for any $1 - \zeta < s \leq 1$, the saddle point $z_0^{(s)}$ lies inside the contour $C(\epsilon)$. From now on we consider those $M$ satisfied $1 - \zeta < s < 1$.

As a result, from [19], Lemma [3] and Theorem [10] we have

$$
J_M(4_1, q, g) \sim \left[ 1 + \sum_{k=1}^{[a]} \frac{1}{(1 - q^{M - k})(1 - q^{M + k})} \right]
$$

$$
+ \frac{S_\gamma(-\pi - (2a - 1)\gamma)}{S_\gamma(\pi - (2a + 1)\gamma)} \frac{\sqrt{2\pi \exp((N + \frac{1}{2})\Phi_M^{(s)}(z_0^{(s)}))}}{\sqrt{(N + \frac{1}{2})\Phi_0^{(s)}(z_0^{(s)})}}
$$

(27)

The following proposition ensures that the second term grows exponentially.
**Proposition 6.** We may choose $\zeta > 0$ such that for every $1 - \zeta < s \leq 1$, $\text{Re} \tilde{\Phi}_M^{(s)}(z_M^{(s)})$ is positive when $M$ is sufficiently large.

Note that since now $a$ depends on $M$, the first term is not a finite sum. To deal with this term, we need to following lemma, which follows easily from the arguments in Theorem 4.1 of [6].

**Lemma 6.** Let $g_M(k) = \prod_{i=1}^{k} \left| \left( q^{M+i} - q^{-M+i} \right) \left( q^{M+i} - q^{-M+i} \right) \right|$. For each $M$, let $k_M \in \{1, \ldots, M-1\}$ such that $g_M(k_M)$ achieves the maximum among all $g_M(k)$. Assume that $\frac{M}{r} = \frac{M}{2N+1} \rightarrow d \in [0, \frac{1}{2}]$ and $k_M \rightarrow k_d$ as $r \rightarrow \infty$. Then we have

$$
\lim_{r \rightarrow \infty} \frac{1}{r} \log(g_M(k_M)) = -\frac{1}{2\pi} (\Lambda(2\pi(k_d - d)) + \Lambda(2\pi(k_d + d))) \leq \frac{\text{Vol}(SS^3 \backslash 41)}{4\pi}.
$$

Furthermore, the equality holds if and only if $(s = 2d = 1)$ and $2k_d = \frac{4}{3}$ or $(s = 2d = \frac{1}{3}$ and $2k_d = \frac{4}{3}$).

By Lemma 6 since $j_d$ depends continuously on $d$, there exists a small neighborhood of $\frac{1}{2}$ such that for any $d$ in that neighborhood, $\frac{1}{2}$ is close to $\frac{1}{2}$. In other word, we may choose a small $\zeta$ to ensures that for any $1 - \zeta < s \leq 1$, the maximum terms among all $g_M(j)$ appears in the second summation. Furthermore, the growth rate of the first summation will then be strictly less than a multiple (a number in $(0,1)$) of the growth rate of the second one. As a result, the first summation decays exponentially when it is compared to the second one.

To conclude, from Proposition 4, Proposition 3 and Theorem 3 we have

$$
J_M(41, q) \sim \frac{S_z(-\pi - (2a - 1)\gamma)}{S_z(\pi - (2a + 1)\gamma)} (N + \frac{1}{2})^{1/2} e^{-a\gamma i} \times \sqrt{2\pi} \exp \left( (N + \frac{1}{2}) \tilde{\Phi}_M^{(s)}(z_M^{(s)}) \right) \frac{\tilde{\Phi}_M^{(s)\prime\prime}(z_M^{(s)})}{\tilde{\Phi}_M^{(s)}(z_M^{(s)})}.
$$

By Lemma 3 we have

$$
\frac{S_z(-\pi - (2a - 1)\gamma)}{S_z(\pi - (2a + 1)\gamma)} = -\frac{2}{e^{-2a\gamma i} - 1} = \frac{1}{ie^{-a\gamma i} \sin(a\gamma)} \sim M \rightarrow \frac{1}{ie^{-a\gamma i} \sin(s\gamma)}.
$$

Altogether we have

$$
J_M(41, q) \sim \frac{1}{M \rightarrow \frac{1}{2}} (N + \frac{1}{2})^{1/2} \sqrt{2\pi} \exp \left( (N + \frac{1}{2}) \tilde{\Phi}_M^{(s)}(z_M^{(s)}) \right) \frac{\tilde{\Phi}_M^{(s)\prime\prime}(z_M^{(s)})}{\tilde{\Phi}_M^{(s)}(z_M^{(s)})}.
$$

Now we explore to the saddle point equation in more detail. By direct computation one can see that the saddle point equation is given by

$$
\beta_M \omega_M^2 - (\beta_M^2 + 1 - \beta_M) \omega_M + \beta_M = 0,
$$

where $\beta_M = e^{2\pi i(\frac{a}{M+1})}$ and $\omega_M = e^{2\pi i(\frac{s}{M})}$. This is exactly Equation (10) and we complete the proof of Theorem 7.

Here we give a remark on (29) that will be used to find the AEF for the TV invariant later. Note that the suitable solution of the saddle point equation is given by

$$
\omega_M = \frac{\beta_M^2 + 1 - \beta_M - \sqrt{(-\beta_M^2 + 1 - \beta_M)(3\beta_M^2 + 1 - \beta_M)}}{2\beta_M}
$$

Let $\omega$ be the solution of the following equation

$$
\omega = \frac{\beta^2 + 1 - \beta - \sqrt{(-\beta^2 + 1 - \beta)(3\beta^2 + 1 - \beta)}}{2\beta}
$$
with $\beta = e^{2\pi i s}$ and $\omega = e^{2\pi i z(s)}$. Define the function $\Theta(s)$ by

$$
\Theta(s) = \delta^{(s)}(z(s)) = \frac{1}{2\pi i} \left[ \text{Li}_2 \left( e^{-2\pi i z(s)} + 2\pi i s \right) - \text{Li}_2 \left( e^{-2\pi i z(s)} + 2\pi i s \right) \right] + 2\pi i (1 - s) z(s).
$$

Then we can see that $\Theta(s)$ depends smoothly on $s$ with $\Theta(\frac{M}{N + 1/2}) = \hat{\Phi}^s_M(z_M^{(s)})$. Similarly, for the evaluation of the second derivative at the saddle point, define the function $\Xi(s)$ by

$$
\Xi(s) = 2\pi i e^{2\pi i s} (e^{-2\pi i z(s)} - e^{2\pi i z(s)}).
$$

Then by using the property that $\frac{d}{dz} \hat{\Phi}^s_M(z_M) = 0$, one can verify that $\Xi(\frac{M}{N + 1/2}) = \hat{\Phi}^{(s)}_M(z_M^{(s)})$. Note that since $z(1) = \frac{5}{3}$, we also have $\Xi(1) = 2\pi \sqrt{3}$.

### 2.3 An upper bound for the AEF of the $M$-th colored Jones polynomial of figure eight knot at $(N + \frac{1}{2})$-th root of unity with $s$ closes to $\frac{1}{2}$

In this subsection we are going to find an upper bound for the AEF of the $M$-th colored Jones polynomial at $(N + \frac{1}{2})$-th root of unity with the condition that

$$
s = 2d = \lim_{N \to \infty} \frac{M}{N + 1/2} \in \left( \frac{1}{2} - \delta, \frac{1}{2} + \delta \right),
$$

where $\delta > 0$ is a small number that will be clarified later.

First of all we split the $J_M(4_1; q)$ into two parts

$$
J_M(4_1; q) = \left[ 1 + \sum_{k=1}^{M-1} \sum_{\frac{N+1/2}{2} \leq \frac{M}{N+1/2} \leq \frac{M}{N+1/2} + \frac{1}{2}} q^{-kM} \prod_{l=1}^{k} (1 - q^{M-l}) (1 - q^{M+l}) \right] + \sum_{k=1}^{M-1} \sum_{\frac{M}{N+1/2} + \frac{1}{2} \leq \frac{M}{N+1/2} \leq \frac{M}{N+1/2} + \frac{1}{2}} q^{-kM} \prod_{l=1}^{k} (1 - q^{M-l}) (1 - q^{M+l})
$$

Basically we follow the same method as before. Applying the functional equation of the quantum dilogarithm with the values

$$
\gamma = \frac{2\pi}{2N + 1}, \quad \xi = 2\pi i \quad \text{and} \quad z = 2 \left( M - \frac{N + \frac{1}{2}}{2} - l \right) \gamma
$$

and observing that $\frac{\xi}{N + \frac{1}{2}} = 2i\gamma$, we have

$$
\prod_{l=1}^{k} \left( 1 - e^{\frac{M-l}{2} \gamma} \right) = \frac{S_{\gamma} \left( 2 \left( M - \frac{N + 1/2}{2} \right) \gamma - (2k + 1)\gamma \right)}{S_{\gamma} \left( 2 \left( M - \frac{N + 1/2}{2} \right) \gamma - \gamma \right)}
$$

(30)

Similarly, putting $z = 2 \left( M - \frac{N + 1}{2} + l \right) i\gamma$, we have

$$
\prod_{l=1}^{k} \left( 1 - e^{\frac{M+l}{2} \gamma} \right) = \frac{S_{\gamma} \left( 2 \left( M - \frac{N + 1/2}{2} \right) \gamma + \gamma \right)}{S_{\gamma} \left( 2 \left( M - \frac{N + 1/2}{2} \right) \gamma + (2k + 1)\gamma \right)}
$$

(31)

From (30) and (31), by using triangle inequality, we have
\[ |J_M(A; q)| \leq \left[ 1 + \sum_{k=1, \frac{N+1/2}{2} < \frac{k}{\pi} \leq \frac{k}{\pi} \leq 2} q^{-kM} \prod_{l=1}^{k} (1 - q^{M-l}) (1 - q^{M+l}) \right] \\
+ \sum_{k=1, \frac{N+1/2}{2} > \frac{k}{\pi} \geq 2} q^{-kM} \prod_{l=1}^{k} (1 - q^{M-l}) (1 - q^{M+l}) \\
= \left[ 1 + \left| \frac{S_\gamma\left(2 \left(M - \frac{N+1/2}{2}\right) \gamma + \gamma\right)}{S_\gamma\left(2 \left(M - \frac{N+1/2}{2}\right) \gamma - \gamma\right)} \right| \right] \\
\times \sum_{k=1, \frac{N+1/2}{2} \leq \frac{k}{\pi} \leq \frac{k}{\pi}} \left| \frac{S_\gamma\left(2 \left(M - \frac{N+1/2}{2}\right) \gamma - (2k+1)\gamma\right)}{S_\gamma\left(2 \left(M - \frac{N+1/2}{2}\right) \gamma + (2k+1)\gamma\right)} \right| \\
+ \sum_{k=1, \frac{N+1/2}{2} > \frac{k}{\pi} \geq 2} q^{-kM} \prod_{l=1}^{k} (1 - q^{M-l}) (1 - q^{M+l}) \right] (32) \]

To show that the second summation can be ignored when it is compared to the first one, by Lemma 6 there exists a small \( \delta \) such that for any \( s = 2d \in \left( \frac{1}{2} - \delta, \frac{1}{2} + \delta \right) \), we have \( 2jd < \frac{1}{12} \). The choice of \( \delta \) ensures that the maximum terms among all \( qM(j) \) appears in the first summation. Furthermore, the growth rate of the second summation will then be strictly less than some multiple (a number in \((0, 1)\)) of the growth rate of the first one. As a result, the second summation decays exponentially when it is compared to the first one.

Recall that for \(|\text{Re}(z)| < \pi\), the quantum dilogarithm can be expressed by [Equation (4.2) in [1]]

\[ S_\gamma(z) = \exp \left( \frac{1}{2\gamma} \text{Li}_2(-e^{iz}) + I_\gamma(z) \right) = \exp \left( \frac{N + \frac{1}{2}}{2\pi i} \text{Li}_2(-e^{iz}) + I_\gamma(z) \right), \]

where

\[ I_\gamma(z) = \frac{1}{4} \int_{C_R} \frac{e^{zt}}{t \sinh(\pi t)} \left( \frac{1}{\sinh(\gamma t)} - \frac{1}{\gamma t} \right) dt. \]

Recall the following lemma [Lemma 3 in [1]], which gives an estimate of \(|I_\gamma(z)|\).

**Lemma 7.** There exist \( A, B > 0 \) depending only on \( R \) such that if \(|\text{Re}(z)| < \pi\), then we have

\[ |I_\gamma(z)| \leq A \left( \frac{1}{\pi - \text{Re}(z)} + \frac{1}{\pi + \text{Re}(z)} \right) |\gamma| + B(1 + e^{-\text{Im}(z)R})|\gamma| \]

From this we can see that for \(|\text{Re}(z)| < \pi\), \( I_\gamma(z) \) goes to zero when \( N \) goes to infinity. In particular, when \( N \) is sufficiently large, we can find a constant \( K \) such that

\[ \exp \left( I_\gamma \left( 2 \left(M - \frac{N + \frac{1}{2}}{2}\right) \gamma - (2k+1)\gamma \right) \right) \leq K \]

and

\[ \exp \left( I_\gamma \left( 2 \left(M - \frac{N + \frac{1}{2}}{2}\right) \gamma + (2k+1)\gamma \right) \right) \leq K \]

for any \( k \) with \( \frac{k}{N+1/2} \leq \frac{5}{12} \).

Define the analytic functions \( \chi_M^{(s)}(x) \) by

\[ \chi_M^{(s)}(x) = \frac{1}{2\pi i} \left( \text{Li}_2 \left( e^{-2\pi i(x+\frac{1}{2})+2\pi i\left(\frac{N+1/2}{2}\right)} \right) - \text{Li}_2 \left( e^{2\pi i(x+\frac{1}{2})+2\pi i\left(\frac{N+1/2}{2}\right)} \right) \right). \]
Then when $N$ is sufficiently large, we have the following upper bound for $|J_M(4; q)|$:

$$
|J_M(4; q)| \leq K \left| \frac{S_c \left( \frac{2(N+1/2)}{2M} \gamma + \gamma \right)}{S_c \left( \frac{2(N+1/2)}{2M} \gamma - \gamma \right)} \right| \times (N + \frac{1}{2}) \times \sum_{k=1, \frac{1}{M} \leq \frac{1}{2}}^{M-1} \frac{1}{N + 1/2} \left| \exp \left( \left( \frac{N + 1}{2} \right) \chi_M^{(s)} \left( \frac{2k+1}{2(N+1)} \right) \right) \right| \tag{33}
$$

Before we proceed, consider the limiting function $\chi_0^{(s)}(x)$ defined by

$$
\chi_0^{(s)}(x) = \frac{1}{2\pi i} \left( \text{Li}_2(e^{-2\pi i(x+\frac{1}{2})}) - \text{Li}_2(e^{-2\pi i(s-\frac{1}{2})}) \right)
$$

Note that for $s = 2d = \frac{1}{3}$, we have $\chi_0^{(s)}(x) = \Phi_0^{(1)}(x + \frac{1}{3})$. Therefore, $x = 2j_d = \frac{1}{3}$ is a solution of the saddle point equation $\frac{d}{dx} \chi_0^{(s)}(x) = 0$ with

$$
\chi_0^{(s)}(\frac{1}{3}) = \Phi_0^{(1)}(\frac{5}{6}) = \frac{1}{2\pi i} \left( \text{Li}_2(e^{-\frac{2\pi i}{3}}) - \text{Li}_2(e^{\frac{2\pi i}{3}}) \right) = \frac{\text{Vol}(SS^3\setminus 41)}{2\pi}
$$

Moreover, we have

$$
\frac{d^2}{dx^2} \chi_0^{(s)}(\frac{1}{3}) = \frac{d^2}{dx^2} \Phi_0^{(1)}(\frac{5}{6}) = 2\pi i\sqrt{-3} = 2\pi \sqrt{3}
$$

More generally, let $x_M^{(s)}$ be the solution of the equation

$$
\frac{d}{dx} \text{Re} \chi_M^{(s)}(x) = 0.
$$

Recall that for any $\theta \in \mathbb{R}$, we have

$$
\text{Im} \text{Li}_2(e^{i\theta}) = 2\Lambda(\theta).
$$

By using this formula, one can verify that the equation (34) is given by

$$
4 \left| \sin \left( -\pi \left( x_M^{(s)} + \frac{1}{2} + \left( \frac{M}{N+1/2} - \frac{1}{2} \right) \right) \right) \right| \times \left| \sin \left( \pi \left( x_M^{(s)} + \frac{1}{2} + \left( \frac{M}{N+1/2} - \frac{1}{2} \right) \right) \right) \right| = 1
$$

Now we continue the discussion on the upper bound. Note that the Riemann sum in (33) can be further expressed in an integral form. This is guaranteed by the following proposition.

**Proposition 7.** Let $f(z)$ be an analytic function defined on a domain $D$ containing $[a, b]$. Assume that

1. $x_{\text{crit}} \in [a, b]$ is the only critical point of $\text{Re} f$ along $[a, b]$ on which $\text{Re} f(z)$ attains its maximum;
2. $x_{\text{crit}}$ is non-degenerate with $(\text{Re} f)'(x_{\text{crit}}) < 0$.

Then for any positive $C^1$ function $h(x)$ on $[a, b]$, we have the following asymptotic equivalence:

$$
\int_a^b h(x) \left| \exp \left( \left( \frac{N + 1}{2} \right) f(x) \right) \right| \, dx = \sum_{k=1, \frac{1}{M} \leq \frac{1}{2}}^{M-1} \left( \frac{1}{N+1/2} h \left( \frac{2k+1}{2N+1} \right) \right) \left| \exp \left( \left( \frac{N + 1}{2} \right) f \left( \frac{2k+1}{2N+1} \right) \right) \right| \times \left( 1 + O \left( \frac{1}{(N+1/2)^{1/3}} \right) \right).
$$
By Proposition 7, we have

$$\sum_{k=1}^{M-1} \frac{1}{N+1/2} \exp \left( \left( N + \frac{1}{2} \right) \chi^{(s)} (M) \left( \frac{2k + 1}{2N + 1} \right) \right)$$

$$= \left( \int_{0}^{\infty} \exp \left( \left( N + \frac{1}{2} \right) \frac{\chi^{(s)} (x)}{x} \right) \, dx \right) \left( 1 + O \left( \frac{1}{(N + 1/2)^{1/3}} \right) \right)$$

(35)

By one-parameter family version of Laplace’s method (the proof is similar to that of Theorem 10), the AEF for the upper bound is given by

$$K \left| \frac{S_{\gamma} \left( 2 \left( M - \frac{N+1/2}{2} \right) \gamma + \gamma \right)}{S_{\gamma} \left( 2 \left( M - \frac{N+1/2}{2} \right) \gamma - \gamma \right)} \right| \left( N + \frac{1}{2} \right)^{1/2} \sqrt{\frac{2\pi}{\left| \chi^{(s)} (x^{*}_{M}) \right|}} \exp \left( \left( N + \frac{1}{2} \right) \chi^{(s)} (x^{*}_{M}) \right)$$

(36)

Finally, the ratio of the quantum dilogarithms in (33) is given by the following lemma.

**Lemma 8.** We have the following formula

$$\frac{S_{\gamma} \left( 2 \left( M - \frac{N+1/2}{2} \right) \gamma + \gamma \right)}{S_{\gamma} \left( 2 \left( M - \frac{N+1/2}{2} \right) \gamma - \gamma \right)} = \frac{1}{1 + e^{-2\pi(N+1/2)i}} \xrightarrow{M \to \infty} \frac{1}{1 - e^{-2\pi(s-1/2)i}}$$

Overall, we have the following estimation:

$$|J_{M}(41; q)| = O \left( \frac{1}{1 + e^{-2\pi(s-1/2)i}} \left( N + \frac{1}{2} \right)^{1/2} \sqrt{\frac{2\pi}{\left| \chi^{(s)} (x^{*}_{M}) \right|}} \exp \left( \left( N + \frac{1}{2} \right) \chi^{(s)} (x^{*}_{M}) \right) \right)$$

This completes the proof of Theorem 8.

To end this subsection, let $x(s)$ be the solution of the equation

$$\sin \left( -\pi(x(s) + \frac{1}{2}) + \pi(s - \frac{1}{2}) \right) \sin \left( -\pi(x(s) + \frac{1}{2}) \right) = \frac{1}{4}$$

Define the function $\Psi(s)$ by

$$\Psi(s) = \text{Re} \left( \frac{1}{2\pi i} \left( \text{Li}_{2}(e^{-2\pi i(x(s) + \frac{1}{2}) + 2\pi(s - \frac{1}{2}))} - \text{Li}_{2}(e^{2\pi i(x(s) + \frac{1}{2}) + 2\pi(s - \frac{1}{2}))} \right) \right)$$

$$= \frac{1}{2\pi} \left( \Lambda(-\pi x(s) + \pi s) + \Lambda(-\pi x(s) - \pi s) \right)$$

Then we have the following equation which will be used in the next subsection.

$$\Psi \left( \frac{M}{N + 1/2} \right) = \chi^{(s)} (x^{(s)}_{M})$$

### 2.4 AEF for the Turaev-Viro invariant of the figure eight knot complement

As an application of AEF’s obtained in previous subsections, we are going to find out the AEF for the Turaev-Viro invariant of the figure eight knot complement as follows.

Recall from Theorem 1 that the TV invariants and the colored Jones polynomials of a link $L$ are related by

$$TV_{q} \left( SS^{3} \right) = 2^{n-1} \left( q_{t}^{2} \right)^{2} \sum_{1 \leq M \leq \frac{c-1}{2}} \left| J_{M} \left( L, e^{2\pi i \frac{M}{N+1/2}} \right) \right|^{2}$$
where \( r = 2N + 1 \) and \( \eta'_n = \frac{2 \sin(\frac{n\pi}{2N})}{\sqrt{r}} \).

For the figure eight knot \( L = 4_1 \), we can split the TV invariant into three parts.

\[
TV_r \left( SS^3 \setminus A_1, e^{\frac{2\pi i}{2N}} \right) = (\eta'_n)^2 \left( \sum_{M : s \in (1, 1]} \left| J_M \left( L, e^{\frac{2\pi i}{N + 1}} \right) \right|^2 + \sum_{M : s \in \left( \frac{1}{2} - \delta, \frac{1}{2} + \delta \right)} \left| J_M \left( L, e^{\frac{2\pi i}{N + 1}} \right) \right|^2 \right) + (\eta'_n)^2 \sum_{1 \leq M \leq N, s \in \left( \frac{1}{2} - \delta, \frac{1}{2} + \delta \right)} \left| J_M \left( L, e^{\frac{2\pi i}{N + 1}} \right) \right|^2
\]

The last summation can be estimated by using Lemma 9. Using the same arguments as in the proof of Theorem 7 and Theorem 8, we can see that the growth rate of the last term is strictly less than some multiple (a number in (0, 1)) of that of the first and the second summations. As a result,

\[
TV_r \left( SS^3 \setminus A_1, e^{\frac{2\pi i}{2N}} \right) \sim_{N \to \infty} (\eta'_n)^2 \left( \sum_{M : s \in (1, 1]} \left| J_M \left( L, e^{\frac{2\pi i}{N + 1}} \right) \right|^2 + \sum_{M : s \in \left( \frac{1}{2} - \delta, \frac{1}{2} + \delta \right)} \left| J_M \left( L, e^{\frac{2\pi i}{N + 1}} \right) \right|^2 \right)
\]

Note that each summands satisfy the condition in the previous subsections. To apply the formulas obtained in previous section, we need the following lemma.

**Lemma 9.** For each \( M \in \mathbb{N} \), let \( a^M_N \) and \( b^M_N \) be two sequence of positive real numbers such that \( |a^M_N - b^M_N| \leq b^M_N \log\left( \frac{\log N}{N} \right) \), where \( K(N) \) is a sequence of positive real numbers independent on \( M \) such that \( \lim_{N \to \infty} K(N) = 0 \). Then we have

\[
\sum_{M = 1}^{N} a^M_N \sim_{N \to \infty} \sum_{M = 1}^{N} b^M_N.
\]

In (28), the error term \( O\left( \frac{\log N}{N} \right) \) comes from Proposition 5 and Theorem 10. From the proof of Proposition 5 and Theorem 10, we can see that the error depends continuously on the functions \( \tilde{\Phi}_M(z) \). Since our functions \( \Phi^{(s)}_M(z) \) converges uniformly to analytic functions \( \tilde{\Phi}^{(s)}_0(z) \), the error terms can be controlled uniformly.

Therefore, for the first summation, by Lemma 9, we have

\[
\sum_{M : s \in (1, 1]} \left| J_M \left( L, e^{\frac{2\pi i}{N + 1}} \right) \right|^2 = (2N + 1) \pi \sum_{M : s \in (1, 1]} \left| \frac{S_s \left( -\pi - 2 \left( \frac{N - M + \frac{1}{2}}{N + 1} \right) \pi \right)}{S_s \left( \pi - 2 \left( \frac{N - M + \frac{1}{2}}{N + 1} \right) \pi \right)} \right|^2 \times \left| \exp \left( (2N + 1) \Phi^{(s)}_M (z_M) \right) \right| \left( 1 + O \left( \frac{\log N}{N} \right) \right)
\]

Again we use Proposition 7 and the Laplace’s method to deal with this kind of summation. Note that the sum can be expressed in the form
Therefore, from direct calculation, one can show that
\[ z \]

where the functions \( \Theta(z) \) and \( \Xi(s) \) are defined in previous subsections by

\[ \Theta(s) = \tilde{\Phi}_0(z(s)) = \frac{1}{2\pi i} \left[ \text{Li}_2 \left( e^{-2\pi is} + 2\pi is \right) \right] - \text{Li}_2 \left( e^{2\pi i(s) + 2\pi is} \right) + 2\pi i (1 - z(s) z(s) = 2\pi i e^{2\pi i(s)} (e^{-2\pi i(s)} - e^{2\pi i(s)}).

By Proposition\[7\] we have

\[ \sum_{M:s \in (1, \zeta], 1]} \left| S_\gamma \left( -\pi - 2 \left( 1 - \frac{M}{N+1/2} \right) \pi + \gamma \right) \right|^2 \frac{\exp \left( (2N + 1) \theta \left( \frac{M}{N+1/2} \right) \right)}{\Xi \left( \frac{M}{N+1/2} \right)} \]

\[ N \rightarrow \infty \quad \sum_{M:s \in (1, \zeta], 1]} \left| S_\gamma \left( -\pi - 2 \left( 1 - \frac{M}{N+1/2} \right) \pi + \gamma \right) \right|^2 \frac{\exp \left( (2N + 1) \theta \left( \frac{M}{N+1/2} \right) \right)}{\Xi \left( \frac{M}{N+1/2} \right)} \]

To find out the value of \( \text{Re} \Theta''(1), \) recall that \( z(s) \) satisfies the equation

\[ \beta \omega^2 - (\beta^2 + 1 - \beta) \omega + \beta = 0, \]

where \( \beta = e^{2\pi i s} \) and \( \omega = e^{2\pi i z(s)}. \) Differentiate both sides with respect to \( s \) and put \( s = 1, \) we can check that \( z'(1) = 0. \) Furthermore, when \( s = 1, \) we have

\[ z(1) = \frac{5}{6} \quad \text{and} \quad \log(1 - e^{-2\pi i z(s)} + 2\pi is) + \log(1 - e^{2\pi i z(s)} + 2\pi is) = 0 \]

Therefore, from direct calculation, one can show that

\[ \text{Re} \Theta'(1) = 0 \quad \text{and} \quad \text{Re} \Theta''(1) = -2\sqrt{3} \pi < 0 \]

As a result, by Laplace’s method we have

\[ \sum_{M:s \in (1, \zeta], 1]} \left| J_M \left( L, e^{\frac{2\pi i}{N + 1/2}} \right) \right|^2 \]

\[ \sim \quad (2N + 1) \frac{\sqrt{2}}{\pi} \frac{1}{\Xi(1)} \sqrt{\frac{2\pi}{(2N + 1) \text{Re} \Theta''(1)}} \exp \left( (2N + 1) \theta(1) \right) \]

\[ \frac{(2N + 1)^{3/2} \pi^{3/2}}{2 \sqrt{2} (2\pi \sqrt{3})^{3/2}} \left| S_\gamma \left( -\pi + \gamma \right) \right|^2 \exp \left( (2N + 1) \theta(1) \right) \]

\[ \frac{2\pi}{(2N + 1) \text{Re} \Theta''(1)} \exp \left( (2N + 1) \theta(1) \right) \]

(37)
Lemma 10. We have \( \frac{S_\gamma(-\pi + \gamma)}{S_\gamma(\pi - \gamma)} = N + \frac{1}{2} \).

Altogether, we have

\[
\sum_{M : s \in (1 - \zeta, 1]} \left| J_M \left( L, e^{\frac{s\pi i}{N + \frac{1}{2}}} \right) \right|^2 \sim \frac{(2N + 1)^{3/2}}{2^{3/2}} \frac{\pi^{3/2}}{(2\pi \sqrt{3})^{3/2}} \frac{1}{N + \frac{1}{2}} \exp \left( (2N + 1) \frac{\Vol(SS^3\backslash A_1)}{2\pi} \right)
\]

For the second summation, similarly the sum can be expressed in an integral form

\[
\sum_{M : s \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta]} \left| J_M \left( L, e^{\frac{s\pi i}{N + \frac{1}{2}}} \right) \right|^2 \sim \frac{(2N + 1)^2}{2^{3/2}} \frac{\pi^{3/2}}{(2\pi \sqrt{3})^{3/2}} S_\gamma(2\pi(s - \frac{\theta}{2}) + \gamma) \left| \frac{S_\gamma(2\pi(s - \frac{\theta}{2}) - \gamma)}{S_\gamma(2\pi(s - \frac{\theta}{2}) + \gamma)} \right|^2 \exp \left( (2N + 1) \frac{\Theta(s)}{\Psi(s)} \right) \frac{d\Psi(s)}{s},
\]

where the functions \( \Psi(s) \) and \( \Theta(s) \) are defined by

\[
\Psi(s) = \Re \left( \frac{1}{2\pi i} \left( \Li_2(e^{-2\pi i(x(s) + \frac{\theta}{2}) + 2\pi i(s - \frac{\theta}{2})i}) - \Li_2(e^{2\pi i(x(s) + \frac{\theta}{2}) + 2\pi i(s - \frac{\theta}{2})i}) \right) \right)
\]

\[
\Theta(s) = 2\pi i e^{2\pi i(s - \frac{\theta}{2})}(e^{-2\pi i x(s)} - e^{2\pi i x(s)}).
\]

Note that \( \Psi(s) \) attains its maximum at \( s = \frac{1}{2} \). Furthermore, we have

\[
\Psi(\frac{1}{2}) = \chi(\frac{1}{2}) = \chi_0(\frac{1}{2}) \frac{\Vol(SS^3\backslash A_1)}{2\pi} \text{ and } \Theta(\frac{1}{2}) = \Phi''(\frac{1}{2}) = 2\pi \sqrt{3}
\]

Therefore we have

\[
\sum_{M : s \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta]} \left| J_M \left( L, e^{\frac{s\pi i}{N + \frac{1}{2}}} \right) \right|^2 \sim \frac{(2N + 1)^2}{2^{3/2}} \frac{\pi^{3/2}}{(2\pi \sqrt{3})^{3/2}} S_\gamma(\gamma) \left( \frac{1}{\Psi'(\frac{1}{2})} \right)^\frac{2\pi}{\sqrt{3}} \exp \left( (2N + 1) \frac{\Vol(SS^3\backslash A_1)}{2\pi} \right)
\]

The ratio of the quantum dilogarithm is given by the lemma below.

Lemma 11. We have \( \frac{S_\gamma(\gamma)}{S_\gamma(-\gamma)} = \frac{1}{2} \).

One can also check that

\[
\Re \Phi''(\frac{1}{2}) = \Re \Theta''(1) = 2\pi \sqrt{3}
\]

Moreover we have

\[
\eta'' = \frac{2 \sin \frac{2\pi}{\sqrt{3}}}{\sqrt{3}} \sim \frac{4\pi}{r^{3/2}}.
\]

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Hence,

\[
\sum_{M: s \in \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right)} |J_M \left(L, e^{\frac{2\pi i}{3}}\right)|^2 \\
N \sim (2N + 1)^{3/2} \frac{\pi^{3/2}}{\sqrt{2}} \frac{1}{(2\pi \sqrt{3})^{3/2}} \left(\frac{1}{4}\right) \exp \left((2N + 1) \frac{\text{Vol}(SS^3 \setminus \{4\})}{2\pi}\right)
\]

From (38) and (40), due to the difference between the ratios of quantum dilogarithm, we can see that the contribution of the colored Jones polynomials with \( s \sim 1 \) dominates that with \( s \sim \frac{1}{2} \).

Overall, the AEF of the Tureav-Viro invariant of figure eight knot is given by

\[
TV_r \left(SS^3 \setminus \{4\}, e^{\frac{2\pi i}{3}}\right) \sim (\eta_r^3)^{3/2} \frac{\pi^{3/2}}{\sqrt{2}} \frac{1}{(2\pi \sqrt{3})^{3/2}} \left(\frac{1}{4}\right) \exp \left(\frac{r}{2\pi} \text{Vol}(SS^3 \setminus \{4\})\right)
\]

This complete the proof of Theorem 9.

### 3 Proof of Results listed in Section 2

**Proof of Proposition 2** and Proposition 4 We follow the line of the proof in [23] and [16] with suitable modification. First of all, recall that for \(|\text{Re}(z)| < \pi\), or \(|\text{Re}(z)| = \pi\) and \(|\text{Im}(z)| > 0\),

\[
\frac{1}{2i} \text{Li}_2(-ez) = \frac{1}{4} \int_{Cn} \frac{e^{zt}}{t^2 \sinh(\pi t)} dt
\]

which implies

\[
S_\gamma(z) = \exp \left(\frac{1}{2\gamma} \text{Li}_2(-e^{iz}) + I_\gamma(z)\right)
\]

\[
= \exp \left(\frac{M + a}{\xi} \text{Li}_2(-e^{iz}) + I_\gamma(z)\right),
\]

where

\[
I_\gamma(z) = \frac{1}{4} \int_{Cn} \frac{e^{zt}}{t \sinh(\pi t)} \left(\frac{1}{\sinh(\gamma t)} - \frac{1}{\gamma t}\right) dt.
\]

Recall that our function \(g_M\) is given by

\[
g_M(z) = \exp \left( -(M + a) \left( u - \frac{a \xi}{M + a}\right) z \right) \frac{S_\gamma(\pi - iu + i\xi z + i\xi(\frac{a}{M + a}))}{S_\gamma(-\pi - iu - i\xi z + i\xi(\frac{a}{M + a}))}
\]

Substituting the above equation for \(S_\gamma\) into the definition of \(g_M\) leads to

\[
g_M(z) = \exp \left[ -(M + a) \left( u - \frac{a \xi}{M + a}\right) z \right]
\]

\[
\exp \left[ \frac{M + a}{\xi} \left( \text{Li}_2(e^{u - z - \frac{a \xi}{M + a}}) - \text{Li}_2(e^{u + z - \frac{a \xi}{M + a}})\right) \right] \times
\]

\[
\exp \left[ I_\gamma(\pi - iu + i\xi z + i\xi(\frac{a}{M + a})) - I_\gamma(-\pi - iu - i\xi z + i\xi(\frac{a}{M + a}))\right]
\]

Let

\[
\Phi_M(z) = \frac{1}{\xi} \left[ \text{Li}_2 \left( e^{u - z - \frac{a \xi}{M + a}}\xi\right) - \text{Li}_2 \left( e^{u + z - \frac{a \xi}{M + a}}\xi\right) \right] - \left( u - \frac{a \xi}{M + a}\right) z
\]
We have

\[
g_M(z) = \exp \left( (M + a) \Phi_M^{(s)}(z) \right) \times \exp \left[ I_\gamma((\pi - iu + i\xi z + i\xi) - \frac{a}{M + a}) - I_\gamma(-\pi - iu - i\xi z + i\xi) - \frac{a}{M + a}) \right]
\]

Decompose \( C_+ \) as \( C_{+,1}, C_{+,2} \) and \( C_{+,3} \) by \( \epsilon \to \epsilon - \frac{u}{2\pi} + i \to (1 - \epsilon - \frac{u}{2\pi} + i) \to 1 - \epsilon \) and \( C_- \) as \( C_{-,1}, C_{-,2} \) and \( C_{-,3} \) by \( \epsilon \to \epsilon + \frac{u}{2\pi} - i \to (1 - \epsilon + \frac{u}{2\pi} - i) \to 1 - \epsilon \).

Write \( I_{+,i}(N) \) be the integral along \( C_{+,i} \) respectively. We are going to show the following controls on the integrals:

\[
|I_{+1}(N)| < \frac{K_{+,1}}{M + a} \tag{41}
\]

\[
|I_{+2}(N)| < \frac{K_{+,2}}{M + a} \tag{42}
\]

\[
|I_{+3}(N)| < \frac{K_{+,3}}{M + a} \tag{43}
\]

\[
|I_{-,1}(N)| < \frac{K_{-,1}}{M + a} \tag{44}
\]

\[
|I_{-,2}(N)| < \frac{K_{-,2}}{M + a} \tag{45}
\]

\[
|I_{-,3}(N)| < \frac{K_{-,3}}{M + a} \tag{46}
\]

Let us observe the comparison between (i) \( \tilde{\Phi}_M^{(s)} \), (ii) its limiting function \( \tilde{\Phi}_0^{(s)} \) and (iii) the function \( \Phi \) (which is \( \Phi_0^{(1)} \) in our notation) in [16].

\[
\Phi(z) = \frac{1}{\xi} \left(\text{Li}_2(e^{u-\xi z}) - \text{Li}_2(e^{u+\xi z})\right) - uz
\]

\[
\tilde{\Phi}_M(z) = \frac{1}{\xi} \left(\text{Li}_2(e^{u-(z+\frac{u}{2\pi})}) - \text{Li}_2(e^{u+(z-\frac{u}{2\pi})})\right) - \left( u - \frac{a\xi}{M + a} \right) z
\]

\[
\tilde{\Phi}_0^{(s)}(z) = \frac{1}{\xi} \left(\text{Li}_2(e^{u-z\xi-(1-s)z}) - \text{Li}_2(e^{u+z\xi-(1-s)z})\right) - (u - (1-s)) z
\]

The proof of the above estimates for the contour integrals is basically the same as the one of Proposition 3.1 in [16].

To prove (11), first we estimate \( |\tan((M + a)\pi((-u/2\pi + i)t + \epsilon)) - i| \). By using (6.8) in [16], we have

\[
|\tan((M + a)\pi((-u/2\pi + i)t + \epsilon)) - i| \leq \frac{2e^{-2(M+a)\pi t}}{1 - e^{-2/\pi u}}
\]

So we have

\[
|I_{+,1}(N + n - 2)| \leq \frac{2}{1 - e^{-2/\pi u}} \int_0^1 e^{-2(M+a)\pi t} \left| g_N((-u/2\pi + i)t + \epsilon) \right| dt
\]

Recall the Lemma 6.1 in [11] that for \( |\text{Re}(z)| \leq \pi \) we have

\[
|I_\gamma(z)| \leq 2A + B|\gamma| \left(1 + e^{-|\text{Im}(z)||R}}\right)
\]

That means \( \exp(1 \text{ part}) \) is bounded above by some constant \( K > 0 \) and

\[
\left| g_N((-u/2\pi + i)t + \epsilon) \right| \leq Ke^{(M+a)\text{Re}(\gamma((M+a)(i)t + \epsilon)}
\]

From the proof of (6.2) in [16], we know that \( \text{Re}(\Phi((M+a)(i)t + \epsilon)) < 0 \) for sufficiently small \( \epsilon > 0 \). We have two cases:
1. if \( a \) is fixed and \( u \neq 0 \), since we have
\[
\hat{\phi}_M^{(s)} \xrightarrow{M \to \infty} \Phi_0.
\]
for \( M \) large enough we have \( \text{Re} \, \Phi_M^{(s)}((\frac{w}{2\pi} + it) + \epsilon) < 0 \).

2. if \( a = N - M + \frac{t}{2} \) and \( u = 0 \), since we have
\[
\hat{\phi}_M^{(s)} \xrightarrow{M \to \infty} \hat{\phi}_0^{(s)} \quad \text{and} \quad \hat{\phi}_0^{(s)} \xrightarrow{s \to 0} \hat{\phi}_0,
\]
there exists a small \( \zeta_1 > 0 \) such that whenever \( 1 - \zeta_1 < s \leq 1 \) and \( M \) is large enough, we have
\[
\text{Re} \, \Phi_M^{(s)}((\frac{w}{2\pi} + it) + \epsilon) \leq 0.
\]

Hence we have
\[
|I_{1.1}(N)| \leq \frac{2}{1 - e^{-\pi M u}} K \int_0^1 e^{-(M+a)\pi t} dt \leq \frac{K_{+1}}{M + a}
\]
This establishes the inequality (11). The proof of the other inequalities (12, 14) are basically the same. \( \square \)

**Proof of Proposition 2 and Proposition 3**

Write
\[
g_M(z) = \exp((M + a)\Phi_M^{(s)}(z)) \times \exp(\text{I part})
\]
First, note that
\[
|\int_{p(\epsilon)} g_M(\omega) d\omega - \int_{p(\epsilon)} \exp((M + a)\hat{\Phi}_M^{(s)}(z)) d\omega| = |\int_{p(\epsilon)} \exp((M + a)\hat{\Phi}_M^{(s)})[\exp(\text{I part}) - 1] d\omega|
\]
\[
\leq \max_{\omega \in p(\epsilon)} \{\exp((M + a)\text{Re} \, \hat{\Phi}_M^{(s)}(\omega))\} \int_{p(\epsilon)} |\exp(\text{I part}) - 1| d\omega
\]
\[
= \max_{\omega \in p(\epsilon)} \{\exp((M + a)\text{Re} \, \hat{\Phi}_M^{(s)}(\omega))\} \int_{\epsilon}^{1-\epsilon} |h_\gamma(\omega)| d\omega
\]
where
\[
h_\gamma(\omega) = \sum_{n=1}^\infty \frac{1}{n!} [I_\gamma(\pi - iu + i\xi t + i\xi(\frac{a}{M + a})) - I_\gamma(-\pi - iu - i\xi t + i\xi(\frac{a}{M + a}))]^n
\]
In the above we use the analyticity of \( h_\gamma(\omega) \) to change the contour to straight line parametrized by \( t, t \in (\epsilon, 1-\epsilon) \).

Recall the lemma 3 in [1] that there exist \( A, B > 0 \) dependent only on \( R \) such that if \( |\text{Re}(z)| < \pi \), we have
\[
|I_\gamma(z)| \leq A(\frac{1}{\pi - \text{Re}(z)} + \frac{1}{\pi + \text{Re}(z)})|\gamma| + B(1 + e^{-\text{Im}(z)R})|\gamma|
\]
So we can find a positive constant \( B' \) such that
\[
|I_1| = |I_\gamma(\pi - iu + i\xi t + i\xi(\frac{a}{M + a}))| \leq A|\gamma| \left( \frac{1}{2\pi(t + \frac{u}{2\pi})} + \frac{1}{2\pi - 2\pi(t + \frac{u}{2\pi})} \right) + B|\gamma|(1 + e^{u-u(t+\frac{u}{2\pi})R})
\]
\[
\leq A|\gamma| \left( \frac{1}{2\pi(t + \frac{u}{2\pi})} + \frac{1}{2\pi - 2\pi(t + \frac{u}{2\pi})} \right) + B'|\gamma| \quad (47)
\]

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and
\[ |I_2| = |I_1(\pi - iu - i\xi t + i\xi(\frac{a}{M + a})| \]
\leq A|\gamma| \left( \frac{1}{2\pi - 2\pi(t - \frac{a}{M + a})} + \frac{1}{2\pi(t - \frac{a}{M + a})} \right) + B|\gamma|(1 + e^{(a-(u(t-\pi\xi))}R)\)
\leq A|\gamma| \left( \frac{1}{2\pi - 2\pi(t - \frac{a}{M + a})} + \frac{1}{2\pi(t - \frac{a}{M + a})} \right) + B'|\gamma| \quad (48)\]

Let \( f(t) = \frac{1}{t} + \frac{1}{t} \). Note that \( f(t) \geq 4 \) for \( t \in (\epsilon, 1 - \epsilon) \).

From (47) and (48), we have
\[ |I_1 - I_2| \leq |\gamma| \left( \frac{A}{2\pi} \left( f \left( t + \frac{a}{M + a} \right) + f \left( t - \frac{a}{M + a} \right) \right) + B'f(t) \right) \]

Note that for \( t \in (\epsilon, 1 - \epsilon) \) and \( N \) large,
\[ \frac{f(t)}{f(t + \frac{a}{M + a})} = t + \frac{a}{M + a} \times \frac{1 - (t + \frac{a}{M + a})}{1 - t} \geq 1 - \frac{a}{M + a} = 1 - \frac{2a}{2a + 1} \geq \frac{1}{3} \]
Similarly,
\[ \frac{f(t)}{f(t - \frac{a}{M + a})} = t - \frac{a}{M + a} \times \frac{1 - (t - \frac{a}{M + a})}{1 - t} \geq 1 - \frac{a}{M + a} = 1 - \frac{2a}{2a + 1} \geq \frac{1}{3} \]
Thus, there exists some positive constant \( A'' \) such that
\[ |I_1 - I_2| \leq |\gamma|A''f(t) \]

As a result,
\[ \int_{\epsilon}^{1-\epsilon} |h_n(t)| \, dt \leq \sum_{n=1}^{\infty} \frac{A''^n |\gamma|^n}{n!} \int_{\epsilon}^{1-\epsilon} f(t)^n \, dt \leq \sum_{n=1}^{\infty} \frac{A''^n |\gamma|^n}{n!} \int_{|\gamma|}^{1-|\gamma|} f(t)^n \, dt \]

Follow the argument in [1], p.537, for \( n \geq 1 \) we have
\[ \int_{|\gamma|}^{1-|\gamma|} f(t)^n \, dt \leq 2^{2n+1} \int_{|\gamma|}^{1-|\gamma|} \frac{dt}{t^n} \]

Also,
\[ \int_{|\gamma|}^{1-|\gamma|} \frac{dt}{t^n} = \log(M + a) - \log(1) = \log(M + a) \quad \text{and} \]
\[ \int_{|\gamma|}^{1-|\gamma|} \frac{dt}{t^n} = \frac{1}{n - 1} \left( \frac{1}{|\gamma|^{n-1}} - 2^{n-1} \right) = \frac{1}{(n - 1)|\gamma|^{n-1}} \quad \text{for} \quad n \geq 2 \]
Therefore we have
\[ \int_{\epsilon}^{1-\epsilon} |h_n(t)| \, dt \leq \sum_{n=1}^{\infty} \frac{1}{n!} (A''^n |\gamma|^n) \int_{|\gamma|}^{1-|\gamma|} f(t)^n \, dt \]
\[ \leq 2^{2n+1} \left( 4A''^n \log(M + a) + \sum_{n=2}^{\infty} \frac{(4A'')^n}{(n - 1)n!} \right) \leq \frac{K}{M + a} \left( 4A''^n \log(M + a) + e^{4A''^n} - 4A''^n - 1 \right) \]
\[ \leq \frac{K \log(M + a)}{M + a} \]
\[ \square \]
Proof of Lemma 3. We only prove the formula for $c = \frac{1}{2}$. The general case can be proved similarly. Note that

$$\frac{S_r(-\pi - iu)}{S_r(\pi - iu - 2\gamma)} = \exp \left( \frac{1}{4} \int_{C_R} \frac{e^{iut}e^{-\gamma t}}{\sinh(\pi t) \sinh(\gamma t)}(e^{-\pi t + \gamma t} - e^{\pi t - \gamma t}) dt \right)$$

$$= \exp \left( \frac{1}{4} \int_{C_R} \frac{e^{-iut}e^{-\gamma t}}{\sinh(\pi t) \sinh(\gamma t)}(e^{-\pi t + \gamma t} - e^{\pi t - \gamma t}) dt \right)$$

$$= \exp \left( \frac{1}{2} \int_{C_R} \frac{e^{-iut}e^{-\gamma t} \coth(\pi t)}{l} - \frac{e^{-iut}e^{-\gamma t} \coth(\gamma t)}{l} dt \right)$$

Now we modify the proof in [16]. For $r > 0$, let $U_i, i = 1, 2, 3$ be the segments defined by $r \overset{U_i}{\rightarrow} r - r' \overset{U_2}{\rightarrow} -r - r' \overset{U_3}{\rightarrow} -r$ with $r' = \frac{3\pi}{4}$. Since the zeros of $\sinh(\pi t)$ and $\sinh(\gamma t)$ are discrete, for generic $r'$, $U_2$ does not pass through those singular points.

Now we want to show that for $i = 1, 2, 3$,

$$\lim_{r \to \infty} \int_{U_i} \frac{e^{-iut}}{\sinh(\pi t) \sinh(\gamma t)}(e^{-\pi t} - e^{\pi t - 2\gamma t}) dt = 0$$

We will show the convergence on (i) $U_1$, (ii) $U_3$, (iii) $U_2$.

First of all we define

$$r = \frac{(2l + \frac{3}{2})\pi}{4\pi^2/(N + \frac{1}{2})u + u/(N + \frac{1}{2})} \quad \text{where} \quad l \in \mathbb{N}$$

Clearly $r \to \infty$ if and only if $l \to \infty$. The choice of $r$ helps us to avoid the pole of $\sinh(\gamma t)$ and get a good estimation of the integrals. More precisely, for $s \in [0, r']$ we consider the four functions

$$p(s) = |1 - e^{-2\pi(r - si)}|, \quad q(s) = |e^{2\pi(r - si)} - 1|$$

$$g(s) = |e^{-2\pi(r - si)} - 1| \quad \text{and} \quad k(s) = |2\sinh(\gamma(r - si))| = |e^{\gamma(r - si)} - e^{-\gamma(r - si)}|$$

In the above $g(s)$ is the distance between $e^{-2\gamma(r - si)}$ and 1. These functions correspond to the terms appear in the integrals as shown later. Now we are going to construct lower bound for these functions. When $r$ is large,

$$p(s) = |1 - e^{-2\pi(r - si)}| \geq 1 - e^{-2\pi r} \geq 1/2;$$

$$q(s) = |e^{2\pi(r - si)} - 1| \geq e^{2\pi r} - 1 \geq 1$$

Also, one can check that

$$g(s) = |e^{-2\gamma(r - si)} - 1| \geq |e^{R(s)}e^{i\theta(s)} - 1|,$$

where $R(s) = \frac{us}{N + \frac{1}{2}} - \frac{2\pi e}{N + \frac{1}{2}}$ and $\theta(s) = \frac{u}{N + \frac{1}{2}} + \frac{2\pi s}{N + \frac{1}{2}}$. Moreover, due to the choice of $r$,

- when $s = \frac{3u}{r}$, we have $R(s) = 0$, $\theta(s) = 2l\pi + \frac{3\pi}{4}$;

- when $s = \frac{2u}{r} - \frac{N + \frac{1}{2}}{8}$, we have $R(s) = -\frac{u}{8}$, $\theta(s) = 2l\pi + \frac{\pi}{2}$;

- when $s = \frac{2u}{r} + \frac{3(N + \frac{1}{2})}{8}$, we have $R(s) = \frac{3\pi}{8}$, $\theta(s) = 2l\pi + \frac{3\pi}{2}$.

Since $R(s)$ and $\theta(s)$ are strictly increasing in $s$ and $g(s)$ is the distance between $e^{R(s)}e^{i\theta(s)}$ and 1,

- for $0 \leq s \leq \frac{2u}{r} - \frac{N + \frac{1}{2}}{8}$, $g(s) \geq \min_{|z| \leq e^{-u/s}} |z - 1| = 1 - e^{-u/s}$;

- for $\frac{2u}{r} - \frac{N + \frac{1}{2}}{8} \leq s \leq \frac{2u}{r} + \frac{3(N + \frac{1}{2})}{8}$, with $\theta(s) = |(2l + 1)\pi + \frac{\pi}{4}, (2l + 1)\pi + \frac{3\pi}{4}|$, we must have $g(s) \geq 1$. 

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for $\frac{2\pi}{u} r + \frac{3(N + \frac{1}{2})}{8} \leq s \leq \frac{2\pi}{u} r, \ g(s) \geq \min_{|z| \geq e^{3u/8}} |z - 1| = e^{3u/8} - 1$.

Finally for $k(s)$, note that the function

$$K(s) = |k(s)|^2 = 4|\sinh(z)|^2 = 4(\sinh^2(R(s) + \sin^2\theta(s))$$

has derivative

$$K'(s) = 8(\sinh R(s) - \frac{u}{N + 1/2}) + \sin(\theta(s))\frac{2\pi}{N + 1/2}) \geq 0$$

for $\theta(s) \in [2\pi + \frac{\pi}{2}, (2l + 1)\pi], k(s)$ is increasing on $[2\pi + \frac{\pi}{2}, (2l + 1)\pi]$ with

$$k(s) \geq k(2\pi + \frac{\pi}{2}) = |\sinh(-u/8)| = \sinh(u/8)$$

For $\theta(s) \notin [2\pi + \frac{\pi}{2}, (2l + 1)\pi]$, we have

$$k(s) = |e^{\gamma(r-si)} - e^{-\gamma(r-si)}| \geq ||e^{\gamma(r-si)}| - |e^{-\gamma(r-si)}|| \geq e^{u/8} - e^{-u/8}$$

The last inequality above is due to the fact that the function is strictly increasing on $R(s)$ and $R(s) > u/8$ when $\theta(s) \notin [2\pi + \frac{\pi}{2}, (2l + 1)\pi]$.

To conclude, we can find positive constants $M_1, M_2, M_3$ and $M_4$ independent on $r$ such that

$$\frac{1}{p(s)} \leq M_1, \quad \frac{1}{q(s)} \leq M_2, \quad \frac{1}{g(s)} \leq M_3 \quad \text{and} \quad \frac{2}{k(s)} \leq M_4$$

Now we can get a good control of the integrals.

(i) On $U_1$,

$$\left| \int_{U_1} e^{-iu t} \frac{e^{-it}}{\sinh(\pi t) \sinh(\gamma t)} (e^{-\pi t}) dt \right| \leq 4 \int_0^{r'} \left| \frac{e^{-iu(r-si)}}{r - si} \right| \left| \frac{e^{-\gamma(r-si)}}{\sinh(\pi(r-si)) \sinh(\gamma(r-si))} \right| ds$$

$$\leq \frac{4M_2M_4}{r} \int_0^{r'} e^{-us} ds$$

$$= \frac{4M_2M_4}{ur}(1 - e^{-ur'}) \xrightarrow{r \to \infty} 0.$$

Similarly,

$$\left| \int_{U_1} e^{-iu t} \frac{e^{-it}}{\sinh(\pi t) \sinh(\gamma t)} (e^{-\pi t-2\gamma t}) dt \right| \leq 4 \int_0^{r'} \left| \frac{e^{-iu(r-si)}}{r - si} \right| \left| \frac{e^{-\gamma(r-si)}}{\sinh(\pi(r-si)) \sinh(\gamma(r-si))} \right| ds$$

$$\leq \frac{4}{r} \int_0^{r'} e^{-us}|e^{-\gamma(r-si)}| \frac{1}{p(s) g(s)} ds$$

$$\leq \frac{4M_1M_3}{r} \int_0^{r'} e\left(-1 + \frac{1}{s} + \frac{1}{\gamma^2}\right) \frac{us - \gamma^2 s}{ur} ds$$

Hence

$$\left| \int_{U_1} e^{-iu t} \frac{e^{-it}}{\sinh(\pi t) \sinh(\gamma t)} (e^{-\pi t-2\gamma t}) dt \right| \leq \frac{4M_1M_4}{r} \int_0^{r'} e\left(-1 + \frac{1}{s} + \frac{1}{\gamma^2}\right) \frac{us - \gamma^2 s}{ur} - e^{-\gamma^2} ds$$

$$\leq \frac{4M_1M_4}{ur} (e\left(-1 + \frac{1}{s} + \frac{1}{\gamma^2}\right) us - \frac{\gamma^2 s}{ur} - e^{-\gamma^2}) \xrightarrow{r \to \infty} 0.$$

(ii) On $U_3$,

$$\left| \int_{U_3} e^{-iu t} \frac{e^{-it}}{\sinh(\pi t) \sinh(\gamma t)} (e^{-\pi t}) dt \right|$$

$$\leq 4 \int_0^{r'} \left| \frac{e^{-iu(r-si)}}{r - si} \right| \left| \frac{e^{-\gamma(r-si)}}{\sinh(\pi(r-si)) \sinh(\gamma(r-si))} \right| dt$$

$$\leq \frac{4}{r} \int_0^{r'} e^{-us} \frac{1}{e(2\pi + \gamma)(r-si) + e^{-\gamma(r-si)} - e(\gamma(r-si) - e(2\pi - \gamma)(r-si))} dt$$
Note that the modulus of the terms in the denominator are \( e^{-2\pi r - \frac{2\pi r + 2\pi}{2N + 1}}, e^{\frac{2\pi r + 2\pi}{2N + 1}} \) and \( e^{\frac{2\pi r + 2\pi}{2N + 1}} \) respectively. For large \( r \), the dominant term is given by \( e^{\frac{2\pi r + 2\pi}{2N + 1}} \to \infty \). This show that the denominator is bounded below. So we can find some constant \( M_5 \) such that

\[
\left| \int_{U_3} \frac{e^{-iut}}{\sinh(\pi t) \sinh(\gamma t)} (e^{-\pi t} dt) \right| \leq \frac{M_5}{r} \int_0^{r'} e^{-us} dt \leq \frac{M_5}{ur} (1 - e^{-ur'}) \to \infty \to 0.
\]

Similarly,

\[
\left| \int_{U_3} \frac{e^{-iut}}{\sinh(\pi t) \sinh(\gamma t)} (e^{\pi t - 2\pi t} dt) \right| \\
\leq 4 \int_0^{r'} \left| \frac{e^{-iut}}{r - si} \right| \left| \frac{e^{\pi t - 2\gamma t}}{e^{\pi t - 2\gamma t} - e^{-\pi t - 2\gamma t}} \right| dt \\
\leq 4 \frac{M_6}{r} \int_0^{r'} e^{-us} dt \leq \frac{M_6}{ur} (1 - e^{-ur'}) \to \infty \to 0.
\]

where \( q_1, q_2, q_3 \) and \( q_4 \) are given by

\[
q_1 = 3\gamma(-r - si), \quad q_2 = (-2\pi + \gamma)(-r - si), \\
q_3 = -\gamma(-r - si), \quad q_4 = (-2\pi + 3\gamma)(-r - si).
\]

Note that the modulus of the terms in the denominator are \( e^{\frac{3\pi u + 2\pi}{2N + 1}}, e^{\frac{2\pi r - 3\pi u + 2\pi}{2N + 1}}, e^{\frac{2\pi r}{2N + 1}} \) and \( e^{\frac{2\pi r - 3\pi u + 2\pi}{2N + 1}} \) respectively. For large \( r \), the dominant term is given by \( e^{\frac{2\pi r - 3\pi u + 2\pi}{2N + 1}} \to \infty \). This show that the denominator is bounded below. Again we can find some constant \( M_6 \) such that

\[
\left| \int_{U_3} \frac{e^{-iut}}{\sinh(\pi t) \sinh(\gamma t)} (e^{\pi t - 2\gamma t} dt) \right| \leq \frac{M_6}{r} \int_0^{r'} e^{-us} dt \\
\leq \frac{M_6}{ur} (1 - e^{-ur'}) \to \infty \to 0.
\]

(iii) On \( U_2 \), we consider the expression

\[
\frac{S_\gamma(-\pi + iu)}{S_\gamma(-\pi - iu - 2\gamma)} = \exp \left( \frac{1}{2} \int_{C_R} \frac{e^{-iut} e^{-\gamma t} \coth(\pi t)}{t} - \frac{e^{-iut} e^{-\gamma t} \coth(\gamma t)}{t} \ dt \right)
\]

Note that for \( t = s - r' i, s \in [-r, r] \),

\[
|e^{-\gamma t}| = e^{-\frac{2\pi r'}{2N + 1}} \leq e^{\frac{2\pi r'}{2N + 1}} \leq e^{\frac{2\pi r'}{2N + 1}}.
\]

Write \( \kappa = \alpha - \beta i \), where \( \kappa = \pi \) or \( \gamma \),

\[
\left| \int_{U_2} \frac{e^{-iut} \coth(\kappa t)}{t} e^{-\gamma t} dt \right| \leq \int_{U_2} \left| \frac{e^{-iut} \coth(\kappa t)}{t} \right| |e^{-\gamma t}| dt \leq \frac{e^{-ur'(1 - \frac{1}{2N + 1})}}{r'} \int_{U_2} |\coth(\kappa t)| dt
\]

By the similar trick in [10], put \( \delta = \max_{-1 \leq \kappa \leq 1} |\coth(\kappa s)| > 0 \). This helps us to get away from the singularity of \( \coth(s\pi) \) in the proof shown below. Now we have

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\[
\int_{U_2} |\coth(\alpha t)| dt = \int_{-r}^{r} |\coth(\alpha t - r' \beta) - (\alpha r' \beta)| dt
\]

\[
\leq 2\delta + \int_{-r}^{r} \left( e^{\alpha r' \beta - (\alpha r' \beta)} - e^{-(\alpha r' \beta)} \right) dt
\]

\[
= 2\delta + \int_{-r}^{r} \left( e^{\alpha r' \beta} - e^{-(\alpha r' \beta)} \right) dt
\]

\[
= 2\delta + \int_{-r}^{r} \left( e^{\alpha r' \beta} - e^{-(\alpha r' \beta)} \right) dt
\]

\[
\int_{U_2} |\coth(\alpha t)| dt = \int_{-r}^{r} |\coth(\alpha t - r' \beta) - (\alpha r' \beta)| dt
\]

\[
\leq 2\delta + \int_{-r}^{r} \left( e^{\alpha r' \beta} - e^{-(\alpha r' \beta)} \right) dt
\]

\[
= 2\delta + \int_{-r}^{r} \left( e^{\alpha r' \beta} - e^{-(\alpha r' \beta)} \right) dt
\]

Hence

\[
\int_{U_2} \frac{e^{-i\alpha t} \coth(\alpha t)}{t} e^{-i\gamma t} dt
\]

\[
\leq \frac{e^{-i\alpha t}(1 - \frac{\pi}{r^2 \alpha})}{r^2 \alpha} \left[ 2\delta + \frac{\log(\sinh(\alpha - r' \beta)) - \log(\sinh(\alpha - r' \beta))}{\alpha} \right.
\]

\[
+ \frac{\log(\sinh(-\alpha - r' \beta)) - \log(\sinh(-\alpha - r' \beta))}{\alpha}
\]

\[
\rightarrow \infty 0
\]

Let \( C_r = [-r, -R] \cup \Omega_R \cup [R, r] \). Denote \( U_1 \cup U_2 \cup U_3 \) by \( U_{123} \). By (i)-(iii) we get

\[
\int_{C_r} \frac{e^{-i\alpha t} e^{-i\gamma t}}{\sinh(\alpha t) \sinh(\gamma t)} dt
\]

\[
= \lim_{r \rightarrow \infty} \int_{C_r} \frac{e^{-i\alpha t} e^{-i\gamma t}}{\sinh(\alpha t) \sinh(\gamma t)} dt
\]

\[
= \lim_{r \rightarrow \infty} \int_{U_{123}} \frac{e^{-i\alpha t} \coth(\alpha t)}{t} e^{-i\gamma t} dt
\]

\[
- 2\pi i \text{Res} \left( \frac{e^{-i\alpha t} \coth(\alpha t)}{t} e^{-i\gamma t}, t = -li \right)
\]

\[
- 2\pi i \text{Res} \left( \frac{e^{-i\alpha t} \coth(\alpha t)}{t} e^{-i\gamma t}, t = \frac{-l\pi i}{\gamma} \right)
\]

\[
- 2\pi i \sum_{l=0}^{\infty} \left[ \text{Res} \left( \frac{e^{-i\alpha t} \coth(\alpha t)}{t} e^{-i\gamma t}, t = -li \right)
\]

\[
- \text{Res} \left( \frac{e^{-i\alpha t} \coth(\alpha t)}{t} e^{-i\gamma t}, t = \frac{-l\pi i}{\gamma} \right) \right]
\]

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\[=-2\pi i \sum_{l=0}^{\infty} \left[ \text{Res} \left( \frac{e^{-i \gamma t}}{t} e^{-\gamma t}, t = -li \right) - \text{Res} \left( \frac{e^{-\gamma t}}{t} e^{-\gamma t}, t = -l\pi i \right) \right] \]

In the above, the negative sign before the residue term is due to the negative orientation of the contour. Moreover, the term \(l = 0\) corresponds to the residue at zero. To find out the residue at \(z = 0\), we consider the following series expansions:

\[e^{-iu z} = 1 - iuz + \frac{(-iu z)^2}{2} + \ldots, \quad \coth(\kappa z) = \frac{1}{\kappa z} + \frac{\kappa z}{3} + \frac{(\kappa z)^3}{45} + \ldots \]

\[e^{-\gamma z} = 1 - \gamma z + \frac{(-\gamma z)^2}{2} + \ldots \]

Hence the residue (coefficient of \(z^{-1}\)) is given by \(-iu - \gamma \kappa\) where \(\kappa = \pi\) or \(\gamma\). From this we can find that

\[
\int_{C_r} \frac{e^{-iut \coth(\pi t)}}{t} e^{-\gamma t} dt = -2\pi i \sum_{l=0}^{\infty} \left[ \text{Res} \left( \frac{e^{-iut \coth(\pi t)}}{t} e^{-\gamma t}, t = -li \right) - \text{Res} \left( \frac{e^{-iut \coth(\gamma t)}}{t} e^{-\gamma t}, t = -\frac{l\pi i}{\gamma} \right) \right]
\]

Similarly,

\[
\int_{C_r} \frac{e^{-iut \coth(\gamma t)}}{t} e^{-\gamma t} dt = -2\pi i \sum_{l=0}^{\infty} \left[ \text{Res} \left( \frac{e^{-iut \coth(\gamma t)}}{t} e^{-\gamma t}, t = -\frac{l\pi i}{\gamma} \right) \right]
\]

Overall we have

\[
\frac{S_\gamma(-\pi - iu)}{S_\gamma(\pi - iu - 2\gamma)} = \frac{e^{u\pi/\gamma} - pi(1 + e^{-u\pi/\gamma})}{e^{u-\gamma i}(1 - e^{-u+\gamma i})} \approx -\frac{e^{u\pi/\gamma} + 1}{e^{u-\gamma i} - 1} \sim \frac{e^{2\pi i u (N+\frac{1}{2})/\xi}}{e^{u} - 1}
\]

In particular, when \(u = 0\),

\[
\frac{S_\gamma(-\pi)}{S_\gamma(\pi - 2\gamma)} = \lim_{u \to 0} \frac{S_\gamma(-\pi - iu)}{S_\gamma(\pi - iu - 2\gamma)} = \frac{2}{1 - e^{-\gamma i}} \sim \frac{2N + 1}{\pi i}
\]
Proof of Lemma 8. For any positive real number \( u \), note that
\[
\frac{S_0(2(M - \frac{N+1/2}{2})\gamma + iu + \gamma)}{S_0(2(M - \frac{N+1/2}{2})\gamma + iu - \gamma)} = \exp \left( \frac{1}{2} \int_{C_R} \frac{e^{2(M - \frac{N+1/2}{2})\gamma + iu + \gamma} e^{-\gamma t}}{t \sinh(\pi t)} dt \right)
\]
\[
= \exp \left( \frac{1}{2} \int_{C_R} \frac{e^{2(M - \frac{N+1/2}{2})\gamma + iu + \gamma} e^{-\gamma t}}{t \sinh(\pi t)} dt \right).
\]

For any \( l \in \mathbb{N} \), let \( U_i, i = 1, 2, 3 \) be the segments defined by
\[
l \xrightarrow{U_1} l + \left( l + \frac{1}{2} \right) i \xrightarrow{U_2} -l + \left( l + \frac{1}{2} \right) i \xrightarrow{U_3} -l.
\]

We are going to estimate the number
\[
T(z) = \left| \frac{e^{2(M - \frac{N+1/2}{2})\gamma + iu + \gamma}}{z \sinh(\pi z)} \right|^2 = \frac{1}{|z|^2} \frac{e^{2(M - \frac{N+1/2}{2})\gamma + iu + \gamma}}{\sinh^2(\pi z)} \quad \text{for line segments } U_1 \text{ and } U_3.
\]

For the line segments \( U_1 \) and \( U_3 \), one can show that
\[
\lim_{l \to \infty} T(z) = 0
\]

For the line segment \( U_2 \), using the fact that
\[
\lim_{x \to \infty} \frac{e^{2(M - \frac{N+1/2}{2})\gamma + iu + \gamma}}{\sinh^2(\pi z)} = 0,
\]
we can find a constant \( K > 0 \) such that for any \( x \geq K \),
\[
\frac{e^{2(M - \frac{N+1/2}{2})\gamma + iu + \gamma}}{\sinh^2(\pi z)} \leq 1.
\]

For \( x \leq K \), it is clear that the term \( \frac{e^{2(M - \frac{N+1/2}{2})\gamma + iu + \gamma}}{\sinh^2(\pi z)} \) is bounded.

Altogether, along the line segment \( U_1 \) we also have
\[
\lim_{l \to \infty} \frac{1}{|z|^2} \frac{e^{2(M - \frac{N+1/2}{2})\gamma + iu + \gamma}}{\sinh^2(\pi z)} = 0.
\]

Therefore, by Residue theorem,
\[
\int_{C_R} \frac{e^{2(M - \frac{N+1/2}{2})\gamma + iu + \gamma}}{t \sinh(\pi t)} dt = 2\pi i \sum_{l=1}^{\infty} \text{Res} \left( \left( e^{2(M - \frac{N+1/2}{2})\gamma + iu + \gamma} \right) \frac{1}{t \sinh(\pi t)}, li \right)
\]
\[
= 2\pi i \sum_{l=1}^{\infty} e^{2(M - \frac{N+1/2}{2})\gamma + iu + \gamma} \frac{1}{l^2 \pi i} (-1)^l \pi i
\]
\[
= -2 \log(1 + e^{-2(M - \frac{N+1/2}{2})\gamma + iu + \gamma})
\]

As a result,
\[
\frac{S_0(2(M - \frac{N+1/2}{2})\gamma + iu + \gamma)}{S_0(2(M - \frac{N+1/2}{2})\gamma + iu - \gamma)} = \exp \left( \frac{1}{2} \int_{C_R} \frac{e^{2(M - \frac{N+1/2}{2})\gamma + iu + \gamma}}{t \sinh(\pi t)} dt \right) = \frac{1}{1 + e^{-2(M - \frac{N+1/2}{2})\gamma + iu + \gamma}}
\]

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Take $u \to 0$, we get

$$
\frac{S_\gamma(2(M - \frac{N+1/2}{2}) \gamma + \gamma)}{S_\gamma(2(M - \frac{N+1/2}{2}) \gamma - \gamma)} = \lim_{u \to 0} \frac{S_\gamma(2(M - \frac{N+1/2}{2}) \gamma + iu + \gamma)}{S_\gamma(2(M - \frac{N+1/2}{2}) \gamma + iu - \gamma)}
$$

$$
= \frac{1}{1 + e^{-2(u+\gamma - \frac{1}{2})\pi i}}
$$

$$
\sim \frac{1}{1 + e^{-2(s-\frac{1}{2})\pi i}}
$$

Proof of Lemma 10. Follow the proof of Lemma 3, we have

$$
\frac{S_\gamma(-\pi + \gamma)}{S_\gamma(\pi - \gamma)} = \lim_{u \to 0} \frac{S_\gamma(-\pi - iu + \gamma)}{S_\gamma(\pi - iu - \gamma)}
$$

$$
= \lim_{u \to 0} \frac{e^{i\pi/\gamma} - 1}{e^u - 1}
$$

$$
= \lim_{u \to 0} \frac{\pi}{\gamma} \times \frac{e^{i\pi/\gamma}}{e^u}
$$

$$
= N + \frac{1}{2}
$$

Proof of Lemma 11. Follow the proof of Lemma 8, we have

$$
\frac{S_\gamma(\gamma)}{S_\gamma(-\gamma)} = \exp\left(\frac{1}{2} \int_{C_R} \frac{1}{t \sinh(\pi t)} dt\right)
$$

Since

$$
\int_{C_R} \frac{1}{t \sinh(\pi t)} dt = 2\pi i \sum_{l=1}^{\infty} \text{Res} \left(\frac{1}{t \sinh(\pi t)}, li\right)
$$

$$
= 2\pi i \sum_{l=1}^{\infty} \frac{1}{(-1)^l l \pi i}
$$

$$
= -2 \ln 2,
$$

we have

$$
\frac{S_\gamma(\gamma)}{S_\gamma(-\gamma)} = \exp\left(\frac{1}{2} \int_{C_R} \frac{1}{t \sinh(\pi t)} dt\right) = \frac{1}{2}
$$

Proof of Proposition 3 and Proposition 6. From Lemma 3.5 in [16] we know that $\text{Re} \Phi(0)(z_0^{(1)}) > 0$ for $0 \leq u < \log((3 + \sqrt{5})/2)$. Since $\Phi_M(z_0^{(1)}) \to \Phi(0)(z_0^{(1)})$ as $M \to \infty$ and $s \to 1$, we get the result.

Proof of Lemma 4. Recall that

$$
\Phi(z) = \frac{1}{\xi} (\text{Li}_2(e^{-\xi z}) - \text{Li}_2(e^{\xi z})) - uz
$$

$$
\Phi_M(z) = \frac{1}{\xi} \left[\text{Li}_2\left(e^{-\frac{z+\frac{1}{M+a}}{\xi}}\right) - \text{Li}_2\left(e^{\frac{z-\frac{1}{M+a}}{\xi}}\right)\right] - uz
$$

Put $y = \frac{a}{M + a}$, we have

$$
\Phi_M(z) - \Phi(z) = \frac{1}{\xi} \left[\left(\text{Li}_2(e^{u-\xi z - \xi y}) - \text{Li}_2(e^{u-\xi z})\right) - (\text{Li}_2(e^{u+\xi z - \xi y}) - \text{Li}_2(e^{u+\xi z}))\right]
$$
As a result, by L’Hospital’s rule

\[
\lim_{M \to \infty} (M + a)(\Phi_M(z) - \Phi(z)) = \left( \frac{a}{\xi} \lim_{y \to 0} \frac{[\text{Li}_2(e^{u-\xi-y}) - \text{Li}_2(e^{u-\xi})] - (\text{Li}_2(e^{u+\xi-y}) - \text{Li}_2(e^{u+\xi}))}{y} \right)
\]

\[
= \frac{a}{\xi} \lim_{y \to 0} \frac{d}{dy} \left[ (\text{Li}_2(e^{u-\xi-y}) - \text{Li}_2(e^{u-\xi})) - (\text{Li}_2(e^{u+\xi-y}) - \text{Li}_2(e^{u+\xi})) \right]
\]

\[
= a[\log(1 - e^{u-\xi}) - \log(1 - e^{u+\xi})]
\]

\[\square\]

**Proof of Lemma 5.** To remove the \(N\) dependence of \(z_N\), recall that from (24) that

\[ab\omega_N^2 = (a^2 + b^2 - ab^2)\omega_N + ab = 0\]

where \(a = e^u\), \(b = e^{-\frac{\omega}{\omega+2}}\) and \(\omega_N = e^{z_N\xi}\). When \(b = 1\), we have the equation

\[a\omega_0^2 = (a^2 + 1)\omega_0 + a = 0\]

By subtracting two equations we get

\[a(\omega_N^2 - \omega_0^2) - (a^2 - a + 1)(\omega_N - \omega_0) = -a(b^2 - 1)\omega_N^2 + (a^2(b - 1) + (b + 1)(b - 1)) - a(b - 1)
\]

This implies

\[\omega_N - \omega_0 = (b - 1)\frac{-a(b + 1)\omega_N^2 + (a^2 + b + 1) - a}{a(\omega_N^2 + 1) - (a^2 - a + 1)} \]

For simplicity, we denote the right hand side by \((b - 1)K_N\). Note that \(K_N \xrightarrow{N \to \infty} K \neq 0\). On the other hand,

\[\omega_N - \omega_0 = e^{z_N\xi} - e^{z_0\xi}\]

\[= e^{z_0\xi} \left[ e(z_N - z_0)\xi - 1 \right] = e^{z_0\xi} (z_N - z_0)\xi \sum_{k=1}^{\infty} \frac{((z_N - z_0)\xi)^{k-1}}{k!}
\]

As a result,

\[z_N - z_0 = (b - 1) \frac{K_N}{\xi e^{z_0\xi} \left(\sum_{k=1}^{\infty} \frac{((z_N - z_0)\xi)^{k-1}}{k!}\right)}
\]

\[= \frac{1}{N + \frac{1}{2}} \sum_{k=1}^{\infty} \frac{\left[\xi/(N + 1/2)\right]^{k-1}}{k!} e^{z_0\xi} \left(\sum_{k=1}^{\infty} \frac{((z_N - z_0)\xi)^{k-1}}{k!}\right)
\]

\[= \frac{M_N}{N + \frac{1}{2}},
\]

where \(M_N \xrightarrow{N \to \infty} M < \infty\). Therefore, we have

\[\lim_{N \to \infty} (N + \frac{1}{2})(\Phi(z_N) - \Phi(z_0)) = \lim_{N \to \infty} \frac{\Phi(z_0 + \frac{M_N}{N + 1/2}) - \Phi(z_0)}{\Phi(z_0 + \frac{M_N}{N + 1/2}) - \Phi(z_0)} M_N = 0,
\]

where in the last equality we use the fact that \(z_0\) is the solution of the saddle point equation

\[
\frac{d\Phi(z)}{dz} = 0
\]

\[\square\]
Proof of Proposition \[\text{For the second and the third sum of } E \text{ can be expressed in the form}
\]
\[E = \int_a^b h(x) \left| \exp \left( \left( N + \frac{1}{2} \right) f(x) \right) \right| dx
\]
\[- \sum_{k=1, \substack{a \leq \frac{2k+1}{2N+1} \leq b}} \frac{1}{N+1/2} \frac{1}{h} \left( \frac{2k+1}{2N+1} \right) \left| \exp \left( \left( N + \frac{1}{2} \right) f \left( \frac{2k+1}{2N+1} \right) \right) \right| \]
\[= \sum_{k=1, \substack{a \leq \frac{2k+1}{2N+1} \leq b}} E(k), \]
\]
where \(E(k)\) is defined by
\[E(k) = \int_{a + \frac{2k+1}{2N+1}}^{a + \frac{2k+1}{2N+1}} h(x) \left| \exp \left( \left( N + \frac{1}{2} \right) f(x) \right) \right| dx
\]
\[-h \left( \frac{2k+1}{2N+1} \right) \left| \exp \left( \left( N + \frac{1}{2} \right) f \left( \frac{2k+1}{2N+1} \right) \right) \right| \]
\]
For each \(N\), let \(a_N\) and \(b_N\) be the least and largest integers such that
\[x_{\text{crit}} = \frac{1}{(N+1/2)^{1/3}} \leq a + \frac{2a_N + 1}{2N+1} < a + \frac{2b_N + 1}{2N+1} \leq x_{\text{crit}} + \frac{1}{(N+1/2)^{1/3}}
\]
Note that
\[(a + \frac{2a_N + 1}{2N+1}) - (a + \frac{2b_N + 1}{2N+1}) \leq \frac{2}{(N+1/2)^{1/3}}
\]
Then the error term can be split into two parts:
\[E = \sum_{k=a_N}^{b_N} E(k) + \sum_{k<a_N} E(k) + \sum_{k>b_N} E(k)
\]
By Laplace’s method, we know that
\[\int_a^b h(x) \left| \exp \left( \left( N + \frac{1}{2} \right) f(x) \right) \right| dx = \int_a^b h(x) \exp \left( \left( N + \frac{1}{2} \right) \text{Re} f(x) \right) dx
\]
\[\sim \sqrt{2\pi h(x_{\text{crit}})} \exp((N + 1/2)\text{Re} f(x_{\text{crit}})) \frac{1}{(N+1/2)^{1/3}}]
\]
For the second and the third sum of \(E\), note that we have
\[x_{\text{crit}} = \frac{2k+1}{2N+1} > \frac{1}{(N+1/2)^{1/3}}
\]
So for each such \(k\),
\[\left| \exp \left( \left( N + \frac{1}{2} \right) f \left( \frac{2k+1}{2N+1} \right) \right) \right| / \left| \exp \left( \left( N + \frac{1}{2} \right) f(x_{\text{crit}}) \right) \right|
\[= \left| \exp \left( \left( N + \frac{1}{2} \right) \left( f(x_{\text{crit}}) + f''(x_{\text{crit}}) \frac{(x_{\text{crit}} - \frac{2k+1}{2N+1})^2}{2} + \ldots \right) - N f(x_{\text{crit}}) \right) \right|
\[\leq \left| \exp \left( \left( N + \frac{1}{2} \right)^{1/3} \frac{f''(x_{\text{crit}})}{2} + \text{lower order terms} \right) \right| \xrightarrow{N \to \infty} 0
\]
Therefore, the second and third sums decay exponentially when they are compared with the integral
\[\int_a^b h(x) \left| \exp \left( \left( N + \frac{1}{2} \right) f(x) \right) \right| dx.\]
For the first sum, by the Mean-Value Theorem, for each \( i \) there exists some \( \xi_k \in \left( a + \frac{2k - 1}{2N + 1}, a + \frac{2k + 1}{2N + 1} \right) \)

such that

\[
\left| \exp \left( \left( N + \frac{1}{2} \right) f(x) \right) - \exp \left( \left( N + \frac{1}{2} \right) f \left( \frac{2k + 1}{2N + 1} \right) \right) \right| \\
\leq \left| \left( N + \frac{1}{2} \right) (\Re f)'(\xi_i) \right| \exp \left( \left( N + \frac{1}{2} \right) \Re (\xi_i) \right) \left( \frac{1}{N + 1/2} \right) \\
= |(\Re f)'(\xi_i)| \exp \left( \left( N + \frac{1}{2} \right) \Re (\xi_i) \right)
\]

For \( x \in [\frac{2aN + 1}{2N + 1}, \frac{2bN + 1}{2N + 1}] \), define a function \( g(x) \) by

\[
g(x) = |(\Re f)'(x)| \exp \left( \left( N + \frac{1}{2} \right) \Re (x) \right).
\]

Note that \( g(x_{\text{crit}}) = 0 \). By assumption, since \( x_{\text{crit}} \) is the only critical point for \( \Re f \) and \( \Re f \) attains its maximum at \( x_{\text{crit}} \), we have

\[
g(x) = \begin{cases} 
(\Re f)'(x) \exp \left( \left( N + \frac{1}{2} \right) \Re (x) \right) & \text{if } x \in [\frac{2aN + 1}{2N + 1}, x_{\text{crit}}] \\
-(\Re f)'(x) \exp \left( \left( N + \frac{1}{2} \right) \Re (x) \right) & \text{if } x \in [x_{\text{crit}}, \frac{2bN + 1}{2N + 1}]
\end{cases}
\]

For \( x \in [\frac{2aN + 1}{2N + 1}, x_{\text{crit}}] \), \( g'(x) \) is given by

\[
g'(x) = \left( N + \frac{1}{2} \right) ((\Re f)'(x))^2 \exp \left( \left( N + \frac{1}{2} \right) \Re (x) \right) \\
+ (\Re f)''(x) \exp \left( \left( N + \frac{1}{2} \right) \Re (x) \right)
\]

Let \( x_{\text{max}} \) be the maximum point of \( g(x) \) on \([\frac{2aN}{N}, x_{\text{crit}}]\). Note that \( g'(x_{\text{crit}}) < 0 \). We have two possibilities:

1. \( x_{\text{max}} \in [\frac{2aN}{N}, x_{\text{crit}}] \) and \( x_{\text{max}} \notin [\frac{2aN}{N}, x_{\text{crit}}] \).

(1) if \( x_{\text{max}} \in [\frac{2aN}{N}, x_{\text{crit}}] \), we have \( g'(x_{\text{max}}) = 0 \), i.e.

\[
(\Re f)'(x_{\text{max}}) = \sqrt{\left( N + \frac{1}{2} \right)^{-1} (\Re f)''(x_{\text{max}})}
\]

Thus,

\[
g(x_{\text{max}}) = \sqrt{\left( N + \frac{1}{2} \right)^{-1} (\Re f)''(x_{\text{max}}) \exp(N(\Re f)(x_{\text{max}}))} \\
\leq \sqrt{\left( N + \frac{1}{2} \right)^{-1} (\Re f)''(x_{\text{max}}) \exp(N(\Re f)(x_{\text{crit}}))}
\]

and

\[
\left| \exp \left( \left( N + \frac{1}{2} \right) f(x) \right) - \exp \left( \left( N + \frac{1}{2} \right) f \left( \frac{2k + 1}{2N + 1} \right) \right) \right| \\
\leq \sqrt{\left( N + \frac{1}{2} \right)^{-1} (\Re f)''(x_{\text{max}}) \exp(N(\Re f)(x_{\text{crit}}))}
\]

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On the other hand, by Mean Value Theorem, for each \( k \) there exists some
\[
d_k \in \left( a + \frac{2k - 1}{2N + 1}, a + \frac{2k + 1}{2N + 1} \right)
\]
such that
\[
|h(x) - h\left( \frac{2k + 1}{2N + 1} \right)| = \left| h'(d_k) \right| \left| x - \frac{2k + 1}{2N + 1} \right|
\]
Let \( y_1 \) and \( y_2 \) be the maximum points of \( h(x) \) and \( h'(x) \) on \([a, b]\) respectively, and let \( c_N \) be the largest integer such that \( a + \frac{2N+1}{2N+1} \leq x_{\text{crit}} \). Altogether,

\[
\sum_{k=a_N}^{c_N} E(k) \leq \sum_{k=a_N}^{c_N} \int_{a + \frac{2k + 1}{2N + 1}}^{a + \frac{2k + 1}{2N + 1}} h\left( \frac{2k + 1}{2N + 1} \right) \times \left( \left| \exp\left( N + \frac{1}{2} \right) f(x) \right| - \left| \exp\left( N + \frac{1}{2} \right) f\left( \frac{2k + 1}{2N + 1} \right) \right| \right) dx
\]

\[
+ \sum_{k=a_N}^{c_N} \int_{a + \frac{2k + 1}{2N + 1}}^{a + \frac{2k + 1}{2N + 1}} \left| h(x) - h\left( \frac{2k + 1}{2N + 1} \right) \right| \left| \exp\left( N + \frac{1}{2} \right) f(x) \right| dx
\]

\[
\leq h(y_1) \sum_{k=a_N}^{c_N} \int_{a + \frac{2k + 1}{2N + 1}}^{a + \frac{2k + 1}{2N + 1}} \left( \left| \exp\left( N + \frac{1}{2} \right) f(x) \right| \right) dx
\]

\[
+ h'(y_2) \sum_{k=a_N}^{c_N} \int_{a + \frac{2k + 1}{2N + 1}}^{a + \frac{2k + 1}{2N + 1}} \left| \exp\left( N + \frac{1}{2} \right) f(x) \right| dx
\]

\[
\leq \sqrt{-\Re f''(x_{\text{max}})} \exp\left( N + \frac{1}{2} \right) \Re f(x_{\text{crit}}) \left( \frac{2}{(N + 1/2)^{5/6}} \right)
\]

\[
+ \frac{h'(y_2)}{N + 1/2} \sum_{k=a_N}^{c_N} \int_{a + \frac{2k + 1}{2N + 1}}^{a + \frac{2k + 1}{2N + 1}} \left| \exp\left( N + \frac{1}{2} \right) f(x) \right| dx
\]

Since
\[
\frac{\sqrt{-\Re f''(x_{\text{max}})} \exp\left( N + \frac{1}{2} \right) \Re f(x_{\text{crit}}) \left( \frac{2}{(N + 1/2)^{5/6}} \right)}{\sqrt{2 \pi h(x_{\text{crit}}) \exp((N + 1/2) \Re f(x_{\text{crit}}))}} \xrightarrow{N \to \infty} 0
\]

and
\[
\frac{\frac{h'(y_2)}{N + 1/2} \int_a^b \left| \exp\left( N + \frac{1}{2} \right) f(x) \right| dx}{\sqrt{2 \pi h(x_{\text{crit}}) \exp((N + 1/2) \Re f(x_{\text{crit}}))}} \xrightarrow{N \to \infty} 0,
\]

we have the desired result.

(2) if \( x_{\text{max}} \notin \left( \frac{a}{N}, x_{\text{crit}} \right) \), since \( g'(x_{\text{crit}}) < 0 \), we have \( x_{\text{max}} = a + \frac{2a+1}{2N+1} \).
In particular, since \(|x_{\text{crit}} - x_{\text{max}}| \geq \frac{1}{2N^{1/3}}\), we have

\[
g(x_{\text{max}})/|\exp(Nf(x_{\text{crit}}))| \\
\leq |N(\text{Re}f)'(x_{\text{max}})| \left| \exp \left( \left( N + \frac{1}{2} \frac{f''(x_{\text{crit}})}{2} \right) + \text{lower order terms} \right) \right| \\
\xrightarrow{N \to \infty} 0
\]

Hence in this case the error term decay exponentially compared with the integral \(\int_{x_{\text{crit}}}^{b} |\exp(Nf(x))| dx\).

Similar method can be applied to the interval \([x_{\text{crit}}, a+\frac{2\pi}{2N+1}]\). This completes the proof.

**Proof of Lemma 3.** We are going to construct the contour using the same idea as in the proof of Lemma 3.4 of [16]. To do so, we only need to check that the conditions in the construction are also satisfied in our case.

Let \(q_M(t) = z_M^{(s)} t \) for \(0 < t < \text{Re}(1/z_M^{(s)})\). Since \(\lim_{s \to 1} \lim_{M \to \infty} z_M^{(s)} = z_0^{(1)} < 1\) (see the proof of Lemma 3.4 of [16]), \(\text{Re}(1/z_M^{(s)}) > 1\) for \(s\) sufficiently close to 1 and \(M\) sufficiently large.

Also, since \(d^2 \Phi_0^{(s)}(z_0^{(1)})/dz^2 \neq 0\) and \(\lim_{s \to 1} \lim_{M \to \infty} \Phi_0^{(s)} \to \Phi_0^{(1)}\), we have

\[
d^2 \Phi_M^{(s)}(z_M^{(s)})/dz^2 \neq 0
\]
for \(s\) sufficiently close to 1 and \(M\) sufficiently large.

By definition we have \(d \Phi_M^{(s)}(z_M^{(s)})/dz = 0\). This implies

\[
\text{Re} d \Phi_M^{(s)}(q_M(1))/dt = 0
\]
for any \(M\). Since \(\max\{\text{Re} \Phi_0^{(1)}(z)\}\) takes place at \(z = z_0^{(1)}\), we must have

\[
\max\{\text{Re} \Phi_M^{(s)}(z)\} = \text{Re} \Phi_M^{(s)}(z_M^{(s)})
\]
along the line \(q_N(t)\) for \(s\) sufficiently close to 1 and \(M\) sufficiently large.

Moreover, from the proof of Lemma 3.4 of [16] that the difference between the argument of \(z_0^{(1)}\) and \(1/\sqrt{-d^2 \Phi_0^{(1)}(z_0^{(1)})/dz^2}\) is strictly smaller than \(\pi/4\). Hence the difference between the argument of \(z_M^{(s)}\) and \(1/\sqrt{-d^2 \Phi_M^{(s)}(z_M^{(s)})/dz^2}\) is also strictly smaller than \(\pi/4\) for \(s\) sufficiently close to 1 and \(M\) sufficiently large. As a result the same construction of the path \(Q\) in the proof of Lemma 3.4 of [16] still applies.

Finally we connect \(z_M^{(s)}(\text{Re}1/z_M^{(s)})\) and 1 by a line segment \(L\). Since from the proof of Lemma 3.4 in [16] that \(\text{Re} \Phi_0^{(1)}(z) < 0\) on the segment connecting \(2\pi i / \xi\) and 1, we also have \(\text{Re} \Phi_M^{(s)}(\omega) \leq 0\) on the segment \(L\) for \(s\) sufficiently close to 1 and \(M\) sufficiently large. This finishes the construction of the paths.

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