Galactic tide

J. Klačka

Abstract Equation of motion for the galactic tide is derived under the assumption of cylindrically symmetric gravitational potential of the Galaxy. The paper considers galactic tide both for the galactic plane $x-$ and $y-$ components and also for the normal $z-$ component. The $x-$ and $y-$ components of the acceleration come not only from the $x-$ and $y-$ components of the position of a body, but also from its $z-$component of the position vector.

Values of the Oort constants are $A = (14.2 \pm 0.5) \ km \ s^{-1} \ kpc^{-1}$ and $B = (-12.4 \pm 0.5) \ km \ s^{-1} \ kpc^{-1}$. Mass density in the solar neighborhood, 30 pc above the galactic equatorial plane, equals to $(0.117 \pm 0.005) \ M_\odot \ pc^{-3}$. The result for the acceleration is written in the form easily applicable to Solar System studies, to the evolution of comets in the Oort cloud.

Keywords Galaxy · Oort constants · Oort cloud · Equation of motion

1 Introduction

Global galactic gravitational field influences relative motion of two close bodies in the form of galactic tide. This relative motion is important in Solar System studies on evolution of comets within the Oort cloud of comets. This paper derives the equation of motion for galactic tide under the assumption of cylindrically symmetric gravitational potential of the Galaxy. The effect of the normal $z-$component is well understood in the case when motion of one of the bodies is fixed to galactic equatorial plane (see, e.g., Mihalas and McRae Routly 1968, pp. 221-222; analytic approach to the secular evolution of orbital elements in the gravitational field of the Sun is presented in Klačka and Gajdošík 1999). Acceleration in the galactic plane comes not only from the $x-$ and $y-$ components of the galactic field, but also from the $z-$component of the galactic components. Similarly, acceleration in the $z-$component comes also from the $x-$ and $y-$ components of the galactic field acceleration/tide. The acceleration in the plane parallel to the galactic equatorial plane and the acceleration normal to the galactic equatorial plane are presented in the relevant general form for two sets of
galactic components: one case corresponds to Dauphole et al. (1996) potential model for galactic bulge, disk and halo, the other case considers potential galactic bulge of Dauphole et al. (1996), galactic disk is represented by mass density function (Maoz 2007) and galactic halo is given by flat rotation curve.

2 Values of the Oort constants

In order to have a realistic model for galactic gravitational force in disposal, one needs to know also the values of Oort constants. They refer for the region of the Sun, in galactocentric distance about 8 kpc.

There is relatively great uncertainty in the values of the Oort constants. The values with errors are given in Table 1, where also sources of the values are given.

| A [km s^{-1} kpc^{-1}] | B [km s^{-1} kpc^{-1}] | source |
|-------------------------|-------------------------|--------|
| 15.0 ± 0.8              | −10.0 ± 1.2             | Kulikovskij (1985, p. 96) |
| 14.4 ± 1.2              | −12.0 ± 2.8             | Kerr and Lynden-Bell (1986) |
| 11.3 ± 1.1              | −13.9 ± 0.9             | Hanson (1987) |
| 14.8 ± 0.8              | −12.4 ± 0.6             | Feast and Whitelock (1997) |
| 14.5 ± 1.5              | −12 ± 3                 | Barbieri (2007, p. 167) |

Table 1 Values of the Oort constants taken from various sources.

The values presented by Hanson (1987) are given also by Olling and Merrifield (1998), the values presented by Feast and Whitelock (1997) can be found also in Sparke and Gallagher (2007, pp. 92-93) and Carroll and Ostlie (2007, p. 913).

The scatter of the values of A and B is large. The best estimates of the quantities A and B are the weighted averages

\[ A_{wav} = \frac{\sum_{i=1}^{N} w_A i \ A_i}{\sum_{i=1}^{N} w_A i}, \]
\[ B_{wav} = \frac{\sum_{i=1}^{N} w_B i \ B_i}{\sum_{i=1}^{N} w_B i}, \]

(1)

where the weight \(w_A i\) (\(w_B i\)) of each measurement is the reciprocal square of the corresponding uncertainty

\[ w_A i = \frac{1}{\sigma_A i^2}, \]
\[ w_B i = \frac{1}{\sigma_B i^2}, \quad i = 1, 2, ..., N \]

(2)

(Taylor 1997, pp. 175-176) and the uncertainties in \(A_{wav}\) and \(B_{wav}\) are

\[ \sigma_{A_{wav}} = \frac{1}{\sqrt{\sum_{i=1}^{N} w_A i}}, \]
\[ \sigma_{B_{wav}} = \frac{1}{\sqrt{\sum_{i=1}^{N} w_B i}}. \]

(3)
Using the above presented values, we can calculate, on the basis of Eqs. (1)-(3), the most probable values of $A$ and $B$ (we omit the index “wav”):

$$A = (14.2 \pm 0.5) \text{km s}^{-1} \text{kpc}^{-1},$$
$$B = (-12.4 \pm 0.5) \text{km s}^{-1} \text{kpc}^{-1}.$$  

(4)

The values are consistent with the statements that “$A$ has a value of about 14 km s$^{-1}$ kpc$^{-1}$” and “$B$ takes a value around $-12$ km s$^{-1}$ kpc$^{-1}$” (Phillipps 2005, p. 101). However, these values significantly differ from the IAU recommended values $A = 15$ km s$^{-1}$ kpc$^{-1}$, $B = -10$ km s$^{-1}$ kpc$^{-1}$ (Karttunen et al. 2007, p. 358). Also the results obtained from the model of Dauphole et al. (1996) are not fully consistent with the results presented in Eq. (4). The model of Dauphole et al. (1996) yields $A = 14.2$ km s$^{-1}$ kpc$^{-1}$, $B = -13.9$ km s$^{-1}$ kpc$^{-1}$. Similarly, the cepheids data of Karimova and Pavlovskaja (1974) (see also Kulikovskij 1985, p. 98) yield rotation curve $v(R) = v_0 = (243 \pm 1) \text{km s}^{-1}$, which leads to $A = -B = 15.2$ km s$^{-1}$ kpc$^{-1}$ for $R = 8$ kpc [the angular velocity is $\omega(R) = v_0/R$, $A(R) \equiv (v(R)/R - dv(R)/dR) / 2 = \omega(R) / 2$, in this special case when $v(R) = v_0$; if $\omega(R)$ is a convex function, then $A(R)$ is the convex function $\omega(R) / 2$].

3 Models of Galaxy

Galaxy is standardly considered to consist of three components, galactic bulge, galactic disk and galactic halo (see, e.g., Sec. 6 in Maoz 2007). If one does not take into account spiral structure of the Galaxy, then simple models can be created.

The model of Dauphole et al. (1996) considers spherical symmetry for galactic bulge and halo and cylindrical symmetry for galactic disk. Disadvantage of the model is that it does not yield values of Oort constants corresponding to Eqs. (4). Moreover, the model cannot be used as a realistic model of Galaxy for galactocentric distances larger than about 40 kpc, since the model produces a decreasing rotation curve for these distances.

We can create a new simple model, better consistent with Eqs. (4) and with flat rotation curve of the Galaxy. The new model considers galactic bulge of Dauphole et al. (1996) and its gravitational potential is

$$\Phi_{\text{bulge}}(r) = -\frac{GM_b}{\sqrt{r^2 + b_b^2}},$$

$$M_b = 1.3955 \times 10^{10} \text{M}_\odot,$$
$$b_b = 0.35 \text{kpc},$$  

(5)

where $G$ is the gravitational constant.

Galactic disk of our new model is represented by mass density function. We can take relevant information from, e.g., Maoz (2007, Sec. 6.1.1.1): “The Galactic disk has a mass distribution that falls exponentially with both distance $R$ from the center and height $z$ above or below the plane of the disk:

$$\rho_{\text{disk}}(R, z) = \rho_0 \left[ \exp \left( -\frac{R}{R_d} \right) \right] \left[ \exp \left( -\frac{|z|}{h_d} \right) \right].$$  

(6)
The scale length of the disk, \( R_d = (3.5 \pm 0.5) \) kpc, and hence at \( R = 8 \) kpc the Sun is in the outer regions of the Galaxy. The characteristic scale height is \( h_d = 330 \) pc for the lower-mass (older) stars in the disk and \( h_d = 160 \) pc for the gas-and-dust disk. The Sun is located at \( z = 30 \) pc above the midplane of the disk. The mass of the disk within one scale radius is \( M_{\text{disk}} \approx 10^{10} M_\odot \), most of it is in stars (and about 10 % in gas)." Eq. (3) is consistent with the statement presented by Bertin (2000, p. 33): "The luminosity profiles of galaxy disks are approximately exponential, being reasonably well fitted by the law \( I(R) = I_0 \exp(-R/h) \), which has two scale parameters, the central brightness \( I_0 \) and the exponential length \( h \)."

Finally, the simple model of galactic halo is given by a flat rotation curve. The rotation curve is given by circular speed \( v_{\text{halo}}(r) \) for spherical halo:

\[
[v_{\text{halo}}(r)]^2 = v_H^2 \left\{ 1 - \alpha \frac{a_H}{r} \arctan \left( \frac{r}{a_H} \right) - (1 - \alpha) \exp \left[ - \left( \frac{r}{b_H} \right)^2 \right] \right\},
\]

\[ v_H = 220 \text{ km s}^{-1}, \]
\[ \alpha = 0.174, \]
\[ a_H = 0.04383 \text{ kpc}, \]
\[ b_H = 37.3760 \text{ kpc}, \]

(7)

(compare with Sparke and Gallagher 2007, p. 95). The numerical values of \( \alpha, a_H \) and \( b_H \) are calculated on the basis of Eqs. (4)-(7) for the given value of \( v_H \) (although a different value can be used), \([v_{\text{halo}}(R)]^2 = [v_{\text{Galaxy}}(R)]^2 - [v_{\text{bulge}}(R)]^2 - [v_{\text{disk}}(R)]^2\), \( v_{\text{Galaxy}}(R) = (A - B)R \), \( dv_{\text{Galaxy}}(R)/dR = -(A + B) \); moreover, the values yield minimum potential energy of the halo and also minimum radius of the Galaxy for a given mass of the Galaxy.

4 Motion in Galaxy – general description

Let us consider an approximation when global galactic gravitational field is described by cylindrically symmetric potential \( \Phi(R, z) \), \( R \) being distance from the axis of rotation and \( z \) the coordinate of a body above/below the galactic equatorial plane (\( z = 0 \) corresponds to the galactic equatorial plane; right-handed system \( x - y - z \) has its origin at the center of the Galaxy, \( z \) is positively oriented toward the north pole of the Galaxy; \( R = \sqrt{x^2 + y^2} \)). The galactic gravitational potential is generated by mass distribution within the Galaxy. We can write

\[ \Phi(R, z) = \Phi_{\text{bulge}}(R, z) + \Phi_{\text{halo}}(R, z) + \Phi_{\text{disk}}(R, z). \]

Acceleration of the body is given by the following equations in cartesian coordinates:

\[
\frac{d^2X}{dt^2} = - \frac{\partial \Phi(R, z)}{\partial X} = - \frac{\partial \Phi(R, z)}{\partial R} \frac{X}{R},
\]
\[
\frac{d^2Y}{dt^2} = - \frac{\partial \Phi(R, z)}{\partial Y} = - \frac{\partial \Phi(R, z)}{\partial R} \frac{Y}{R},
\]
\[
\frac{d^2Z}{dt^2} = - \left[ \frac{\partial \Phi(R, z)}{\partial z} \right]_{z=Z},
\]

\[ R = \sqrt{X^2 + Y^2}. \]
For the case represented by Eqs. (5)-(7) we have
\[ \Phi_{\text{bulge}}(R, z) = \Phi_{\text{bulge}}(r), \]
\[ r = \sqrt{R^2 + z^2}, \]
(10)
\[ \Phi_{\text{disk}}(R, z) \] is found from the solution of Poisson’s equation
\[ \Delta \Phi_{\text{disk}}(R, z) = 4 \pi G \varrho_{\text{disk}}(R, z), \]
(11)
and, finally,
\[ \Phi_{\text{halo}}(R, z) = \Phi_{\text{halo}}(r), \]
\[ r = \sqrt{R^2 + z^2}, \]
\[ \frac{d \Phi_{\text{halo}}(r)}{dr} = \frac{[v_{\text{halo}}(r)]^2}{r}. \]
(12)

5 Motion near the galactic equator

On the basis of Eqs. (5)-(7) and (10)-(12) we can write
\[ \left[ \frac{\partial \Phi_i}{\partial R} (R, z) \right]_{R_0} = \left[ \frac{v_i(R_0)}{R_0} \right]^2 \left( 1 + \alpha_i z^2 + \beta_i z^4 \right), \]
\[ i = \text{bulge, disk, halo}, \]
(13)
and,
\[ \left[ \frac{\partial \Phi_i}{\partial R} (R, z) \right]_{R_0} = \sum_i \left[ \frac{\partial \Phi_i}{\partial R} (R, z) \right]_{R_0} \]
\[ = \frac{v_G^2}{R_0} \left\{ 1 + \sum_i \frac{v_i^2}{v_G^2} \alpha_i z^2 + \sum_i \frac{v_i^2}{v_G^2} \beta_i z^4 \right\}, \]
\[ v_G^2 = \sum_i v_i^2, \]
\[ v_G \equiv v_0 = v(R_0) = (A - B) R_0. \]
(14)
Using numerical values for the model, we can write
\[ \left[ \frac{\partial \Phi_i}{\partial R} (R, z) \right]_{R_0} = (A - B)^2 R_0 \left( 1 - \Gamma_1 z^2 + \frac{1}{2} \Gamma_2 z^4 \right), \]
\[ \Gamma_1 = 0.124 \text{ kpc}^{-2}, \]
\[ \Gamma_2 = 1.586 \text{ kpc}^{-4}, \]
\[ R_0 = 8.0 \text{ kpc}. \]
(15)
We have added also the value of \( R_0 \), the galactocentric distance of the Sun. The values of \( A \) and \( B \) are given in Eqs. (4).

The rotation curve is given by the circular velocity \( v(R) = \sqrt{R \frac{\partial \Phi(R, 0)}{\partial R}} \). Eqs. (9) can be written, then: \( d^2X/dt^2 = -(v^2/R^2)X \) and \( d^2Y/dt^2 = -(v^2/R^2)Y \) for \( x \)- and \( y \)-components in the plane \( z = 0 \). Using two close points, \((X_0, Y_0, Z_0)\) and \((X, Y, Z)\), \( X = X_0 + \xi, Y = Y_0 + \eta, Z = Z_0 + \zeta \), one can write \((X_0 \xi + Y_0 \eta + Z_0 \zeta)/\sqrt{R_0^4 + Z_0^2} \approx \).
\[
(X_0 \xi + Y_0 \eta + Z_0 \zeta)/R_0 \text{ for the difference between magnitudes of their position vectors, if higher orders in } Z_0/R_0, \xi, \eta \text{ and } \zeta \text{ are neglected. For the galactic plane } [v(R)]^2 = v_0^2 \{1 + 2(v_0/R_0)(X_0 \xi + Y_0 \eta)/R_0\}, \text{ where the prime denotes differentiation with respect to } R \text{ (} v_0 \equiv v(R_0), v_0' \equiv [dv(R)/dR(R_0)] \text{ and, again, higher orders in } \xi \text{ and } \eta \text{ are neglected. We are dealing only with } |z| \ll 1 \text{ kpc; } Z_0 = 0.03 \text{ kpc, at present (Maoz 2007), } R_0 = 8 \text{ kpc.}
\]

The total action of all galactic components can be summarized (see Eqs. 9):

\[
\frac{d^2 X}{dt^2} = - \frac{v_0^2}{R_0^3} \left\{ X_0 + \xi + 2 \left( R_0 v_0'/v_0 - 1 \right) \left[ \left( \frac{X_0}{R_0} \right)^2 \xi + \frac{X_0 Y_0}{R_0^2} \eta \right] - X_0 \left[ \Gamma_1 (Z_0^2 + 2 Z_0 \xi) - \frac{1}{2} \Gamma_2 \left( Z_0^4 + 4 Z_0^3 \xi \right) \right] \right\},
\]

\[
\frac{d^2 Y}{dt^2} = - \frac{v_0^2}{R_0^3} \left\{ Y_0 + \eta + 2 \left( R_0 v_0'/v_0 - 1 \right) \left[ \frac{X_0 Y_0}{R_0^2} \xi + \left( \frac{Y_0}{R_0} \right)^2 \eta \right] - Y_0 \left[ \Gamma_1 (Z_0^2 + 2 Z_0 \zeta) - \frac{1}{2} \Gamma_2 \left( Z_0^4 + 4 Z_0^3 \xi \right) \right] \right\},
\]

if higher orders in \( \xi, \eta, \zeta \) are neglected.

6 Relative acceleration

The relative acceleration of the body with respect to the Sun, if \( (X_0, Y_0, Z_0) \) represents position vector of the Sun, is:

\[
d^2 \xi/dt^2 \equiv -GM_\odot \xi/r^3 + d^2 X/dt^2 - d^2 X_0/dt^2, \quad d^2 \eta/dt^2 \equiv -GM_\odot \eta/r^3 + d^2 Y/dt^2 - d^2 Y_0/dt^2,
\]

or

\[
\frac{d^2 \xi}{dt^2} = - \frac{GM_\odot}{r^3} \xi
- \frac{v_0^2}{R_0^3} \left\{ \left[ 1 - 2 \left( \frac{Y_0}{R_0} \right)^2 \left( 1 - R_0 \frac{v_0'}{v_0} \right) \right] \xi - 2 \frac{X_0 Y_0}{R_0^2} \left( 1 - R_0 \frac{v_0'}{v_0} \right) \eta \right.
- 2 X_0 Z_0 \left( \Gamma_1 - \Gamma_2 Z_0^2 \right) \zeta \bigr],
\]

\[
\frac{d^2 \eta}{dt^2} = - \frac{GM_\odot}{r^3} \eta
- \frac{v_0^2}{R_0^3} \left\{ \left[ 1 - 2 \left( \frac{Y_0}{R_0} \right)^2 \left( 1 - R_0 \frac{v_0'}{v_0} \right) \right] \eta - 2 \frac{X_0 Y_0}{R_0^2} \left( 1 - R_0 \frac{v_0'}{v_0} \right) \xi \right.
- 2 Y_0 Z_0 \left( \Gamma_1 - \Gamma_2 Z_0^2 \right) \zeta \bigr],
\]

\[
r = \sqrt{\xi^2 + \eta^2 + \zeta^2}.
\]

Using Oort constants \( A \) and \( B \), fulfilling the relations \( A - B = \omega \equiv v_0/R_0, \quad A + B = -v_0' \), Eq. (17) yields

\[
\frac{d^2 \xi}{dt^2} = - \frac{GM_\odot}{r^3} \xi + (A - B) [A + B + 2A \cos (-2 \omega t)] \xi
+ 2A(A - B) \sin (-2 \omega t) \eta
+ 2 (A - B)^2 \left( \Gamma_1 - \Gamma_2 Z_0^2 \right) R_0 Z_0 \cos (-\omega_0 t) \zeta,
\]

\[
\frac{d^2 \eta}{dt^2} = - \frac{GM_\odot}{r^3} \eta + 2A(A - B) \sin (-2 \omega_0 t) \xi.
\]
Eqs. (18) represent $z$-component of the acceleration. In order to be the system of differential equations complete, we need also the $z$-component of the acceleration. It can be obtained from the Poisson’s equation

$$ \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Phi}{\partial R} \right) + \frac{\partial^2 \Phi}{\partial z^2} = 4 \pi G \varrho(R, z). $$

(19)

Taking into account that $R \Phi / \partial R = [v(R)]^2$ and $[v(R)]R_0 = \omega_0 = A - B$, $[d v(R)/d R]R_0 = -(A + B)$, where $A$ and $B$ are the Oort constants, and, the $z$-component of the acceleration is $a_z = - \partial \Phi / \partial z$, Eq. (19) yields

$$ \frac{\partial a_z}{\partial z} = -4 \pi G \varrho(R_0, z) - 2 (A^2 - B^2)
- 4 \pi G \left[ \frac{\partial \varrho(R, z)}{\partial R} \right]_{R_0} \left\{ \frac{X_0}{R_0} (x - X_0) + \frac{Y_0}{R_0} (y - Y_0) \right\}. $$

(20)

The first term in Eq. (14) is dominant for the region around the Sun. Taking $\varrho(R_0, z) = \varrho(R_0, z_0)$ in the right-hand side of Eq. (20), one finally obtains

$$ a_z = - \left[ 4 \pi G \varrho(R_0, z_0) + 2 (A^2 - B^2) \right] z
- 4 \pi G \left[ \frac{\partial \varrho(R_0, z_0)}{\partial R} \right]_{R_0} \left\{ \frac{X_0}{R_0} (x - X_0) + \frac{Y_0}{R_0} (y - Y_0) \right\} z. $$

(21)

We can take $\varrho(R_0, z_0)$ as the sum of densities coming from the disk, halo and bulge. The density coming from the bulge is of negligible importance (see, e.g., the bulge model of Dauphole et al. 1996). The contribution of the disk from the Maoz model yields $\varrho_{disk} = (0.126 \pm 0.005) M_\odot pc^{-3}$ for $z_0 = 0$. Considering spherically symmetric halo, its contribution to mass density is

$$ \varrho_{halo}(r) = (4\pi G)^{-1} r^{-2} d(r v_{halo}^2)/dr, $$

where $v_{halo}$ is circular velocity of an object due to the gravity of the halo. One can use Eq. (7). We show another possibility, now. We have $[v_{Galaxy}(R)]^2 = [v_{disk}(R)]^2 + [v_{halo}(R)]^2 + [v_{bulge}(R)]^2, v_{Galaxy}(R) = [A(R) - B(R)] R_0, v_{disk}(R) = [A_{disk}(R) - B_{disk}(R) R_0, v_{halo} = (4\pi G)^{-1} R_0^{-1} \{ v_{halo}^2 - 2(A^2 - B^2) \} + 2(A^2_{disk} - 2B_{disk}) R_0 - 2v_{bulge} [dv_{bulge}(r)/dr] R_0, \}$ if the relations $[d v_{Galaxy}(R)/d R] R_0 = -(A + B), [d v_{disk}(R)/d R] R_0 = -(A_{disk} + B_{disk})$ are used. We have $\varrho_{halo} = \varrho_{halo}(R_0)$, $3.94 \times 10^{-3} M_\odot pc^{-3}$. (As a comparison we can mention that the halo mass density
according to the halo model of Dauphole et al. (1996) is 2.35-times greater than our result: \(\varphi_{\text{halo}}(R_0)\) (Dauphole et al.) = \(9.26 \times 10^{-3} \, M_\odot \, \text{pc}^{-3}\). We can summarize

\[
\begin{align*}
\lVert v_{\text{Galaxy}}(R) \rVert^2 &= \lVert v_{\text{disk}}(R) \rVert^2 + \lVert v_{\text{halo}}(R) \rVert^2 + \lVert v_{\text{bulge}}(R) \rVert^2, \\
v_{\text{Galaxy}}(R) &= [A(R) - B(R)] R, \\
v_{\text{disk}}(R) &= [A_{\text{disk}}(R) - B_{\text{disk}}(R)] R, \\
v_{\text{bulge}}(r) &= \sqrt{r \frac{\partial \Phi_{\text{bulge}}(r)}{\partial r}}, \\
\Phi_{\text{bulge}}(r) &= - \frac{GM_b}{\sqrt{r^2 + b_b^2}}, \\
\varphi_{\text{halo}}(R) &= (4\pi G)^{-1} \lVert X(\text{Galaxy}) - X(\text{disk}) - X(\text{bulge}) \rVert, \\
X(\text{Galaxy}) &= - [A(R) - B(R)] \times [A(R) + 3B(R)], \\
X(\text{disk}) &= - [A_{\text{disk}}(R) - B_{\text{disk}}(R)] \times [A_{\text{disk}}(R) + 3B_{\text{disk}}(R)], \\
X(\text{bulge}) &= \frac{v_{\text{bulge}}(R)}{R} \left( \frac{v_{\text{bulge}}(R)}{R} + 2 \frac{dv_{\text{bulge}}(R)}{dR} \right), \\
\sigma_{\varphi_{\text{halo}}} &= \frac{2}{4\pi G} \times \sqrt{Y}, \\
Y &= (A + B)^2 \sigma_A^2 + (A - 3B)^2 \sigma_B^2 + (A_{\text{disk}} + B_{\text{disk}})^2 \sigma_{A_{\text{disk}}}^2 + (A_{\text{disk}} - 3B_{\text{disk}})^2 \sigma_{B_{\text{disk}}}^2, \\
M_b &= 1.3955 \times 10^{10} \, M_\odot, \\
b_b &= 0.35 \, \text{kpc}, \\
A_{\text{disk}} &= A_{\text{disk}}(R_0) = (10.81 \pm 1.70) \, \text{km s}^{-1} \, \text{kpc}^{-1}, \\
B_{\text{disk}} &= B_{\text{disk}}(R_0) = (-9.97 \pm 0.34) \, \text{km s}^{-1} \, \text{kpc}^{-1}, \\
\varphi_{\text{disk}} &= (0.126 \pm 0.005) \, M_\odot \, \text{pc}^{-3}, \\
\varphi_{\text{halo}} &= (0.004 \pm 0.001) \, M_\odot \, \text{pc}^{-3}, \\
\varphi &= \varphi_{\text{disk}} + \varphi_{\text{halo}},
\end{align*}
\]

(22)

if also the bulge model of Dauphole et al. (1996) is inserted. The contribution of the bulge to the total mass density is \(\varphi_{\text{bulge}} / (\varphi_{\text{disk}} + \varphi_{\text{halo}}) = 1 \times 10^{-4}\). The main part of the bulge is concentrated inside a sphere of small radius and the value of \(\varphi_{\text{halo}}\) is not very sensitive to the form of the bulge potential, \(v_{\text{bulge}} \approx \sqrt{GM_b/r}\).

The total value of the mass density, coming from the disk and the halo, is \(\varphi = 0.130 \, M_\odot \, \text{pc}^{-3}\). The value obtained from the model of Dauphole et al. (1996), 0.143 \(M_\odot \, \text{pc}^{-3}\), is in 10% greater than our result. The value of \(\varphi\) seems to be in a good agreement with the values (0.120 \(\pm 0.008\)) \(M_\odot \, \text{pc}^{-3}\) and (0.138 \(\pm 0.009\)) \(M_\odot \, \text{pc}^{-3}\) obtained from the results of Agekjan in 1962 and Agekjan and Ogorodnikov in 1974 (see Kulikovskij 1985, pp. 158-159) under the assumption that average mass of the stars in the solar neighborhood equals to 1 \(M_\odot\) (it does not significantly differ from 1 \(M_\odot\)).

As for \(\partial \varphi / \partial R\) at \(R_0\), the model of Maoz (2007) yields \(\partial \varphi(R, z = 0)/\partial R\) at \(R_0\) \(= (-0.0360 \pm 0.0037) \, M_\odot \, \text{pc}^{-3} \, \text{kpc}^{-1}\) for \(z_0 = 0\). Our model yields \(\partial \varphi(r)/\partial r \big|_{R_0 \, \text{halo}} = -0.62 \times 10^{-3} \, M_\odot \, \text{pc}^{-3} \, \text{kpc}^{-1}\).

Perhaps, the result represented by Eq. (21) can be a little improved using the fact that

\[\varphi(R_0, z) = \varphi_{\text{disk}} \left(1 - u \vert z \vert \right) + \varphi_{\text{halo}},\]
\[
\left[ \frac{\partial \theta(R,z)}{\partial R} \right]_R = \theta'_{\text{disk}} (1 - u |z|) + \theta'_{\text{halo}} ,
\]
where the value of \( u \) follows from the model of the disk.

Eqs. (20) and (23) are applied to both bodies, the one with the coordinates \((X_0, Y_0, Z_0)\) and the second with the coordinates \((X, Y, Z)\). In application to the Solar System and the Oort cloud, we have the Sun and the comet. We have

\[
\begin{align*}
\frac{d^2 Z_0}{dt^2} &= - \left\{ 4 \pi G \left[ \theta_{\text{disk}} \left( 1 - \frac{1}{2} u |Z_0| \right) + \theta_{\text{halo}} \right] + 2 \left( A^2 - B^2 \right) \right\} Z_0 , \\
\frac{d^2 Z}{dt^2} &= - \left\{ 4 \pi G \left[ \theta_{\text{disk}} \left( 1 - \frac{1}{2} u |Z_0| \right) + \theta_{\text{halo}} \right] + 2 \left( A^2 - B^2 \right) \right\} Z \\
&- 4 \pi G \left\{ \frac{X_0}{R_0} (X - X_0) + \frac{Y_0}{R_0} (Y - Y_0) \right\} \times \\
&\quad \times \left\{ \theta'_{\text{disk}} \left( 1 - \frac{1}{2} u |Z| \right) + \theta'_{\text{halo}} \right\} Z .
\end{align*}
\]

Initial conditions for the case of the Sun are: \( Z_0 = 30 \text{ pc} \), \( dZ_0/dt = 6 \text{ km s}^{-1} \) (information on the basic solar motion is used, see, e.g., Mihalas and McRae Routly 1968, p. 101). Using \( X = X_0 + \xi \), \( Y = Y_0 + \eta \) and \( Z = Z_0 + \zeta \), Eq. (24) yields

\[
\frac{d^2 \zeta}{dt^2} = - \frac{GM_\odot}{r^3} \zeta \\
- 4 \pi G \left\{ \varrho \zeta - \varrho_{\text{disk}} \frac{1}{2} u |Z_0 + \zeta| (Z_0 + \zeta) - |Z_0| \right\} \\
- 2 \left( A^2 - B^2 \right) \zeta \\
- 4 \pi G \left( \varrho' - \varrho'_{\text{disk}} \frac{1}{2} u |Z_0 + \zeta| \right) (Z_0 + \zeta) \left( \frac{X_0}{R_0} \xi + \frac{Y_0}{R_0} \eta \right) ,
\]

if also acceleration from the body with the coordinates \((X_0, Y_0, Z_0)\) (Sun) is taken into account. The last term of Eq. (25) assures that motion in a plane parallel to the plane of the galactic equator does not exist: the inclination of the comet relative to the galactic equatorial plane does not fulfill the relation \( i = 0 \), since \( Z_0 = 0 \) does not hold. The statement that “plane-parallel motions do not exist” is equivalent to the fact that initial conditions \( \zeta = 0 \) and \( d\zeta/dt = 0 \) do not admit \( \zeta = \text{constant} = 0 \) during the motion. Thus, our approach differs from the approach of Bottlinger in 1924-1925 reducing to Oort equations for the galactic equatorial plane (see Kulikovskij 1985, pp. 90-91).

6.2 Summary

Summarizing the most relevant equations and results presented above, we obtain

\[
\frac{d^2 \xi}{dt^2} = - \frac{GM_\odot}{r^3} \xi + (A - B) \left[ A + B + 2A \cos(2 \omega_0 t) \right] \xi
\]
10

\[ -2A(A - B)\sin(2\omega_0 t)\eta \\
+ 2(A - B)^2(\Gamma_1 - \Gamma_2 Z_0^2)R_0 Z_0 \cos(\omega_0 t)\zeta , \]

\[
\frac{d^2\eta}{dt^2} = -GM_\odot \frac{\eta}{r^3} - 2A(A - B)\sin(2\omega_0 t)\xi \\
+ (A - B)\left[A + B - 2A\cos(2\omega_0 t)\right]\eta \\
- 2(A - B)^2(\Gamma_1 - \Gamma_2 Z_0^2)R_0 Z_0 \sin(\omega_0 t)\zeta ,
\]

\[
\frac{d^2\zeta}{dt^2} = -GM_\odot \frac{\zeta}{r^3} - [4\pi G \varrho + 2(A^2 - B^2)]\zeta \\
- 4\pi G \varrho' Z_0 \left[\cos(\omega_0 t)\xi - \sin(\omega_0 t)\eta\right],
\]

\[
\frac{d^2Z_0}{dt^2} = -\left[4\pi G \varrho + 2(A^2 - B^2)\right]Z_0 ,
\]

\[
r = \sqrt{\xi^2 + \eta^2 + \zeta^2} ,
\]

\[
\omega_0 = A - B ,
\]

(26)

if the terms containing the quantity \(u\) are neglected and the relations \(X_0 = R_0 \cos(-\omega_0 t)\) and \(Y_0 = R_0 \sin(-\omega_0 t)\) are used. The numerical values of the relevant quantities are:

\[
A = (14.2 \pm 0.5) \text{ km s}^{-1} \text{ kpc}^{-1} ,
\]

\[
B = (-12.4 \pm 0.5) \text{ km s}^{-1} \text{ kpc}^{-1} ,
\]

\[
\Gamma_1 = 0.124 \text{ kpc}^{-2} ,
\]

\[
\Gamma_2 = 1.586 \text{ kpc}^{-4} ,
\]

\[
\varrho = (0.130 \pm 0.005) M_\odot \text{ pc}^{-3} ,
\]

\[
\varrho' = (-0.037 \pm 0.004) M_\odot \text{ pc}^{-3} \text{ kpc}^{-1} .
\]

(27)

We stress that \(\varrho \equiv \varrho(R_0, z = 0) = 0.130 M_\odot \text{ pc}^{-3}, R_0 = 8.0 \text{ kpc}\). The mass density in the solar neighborhood, 30 pc above the galactic equatorial plane, is \(\varrho \equiv \varrho(R_0,z = 0.03 \text{ kpc}) = (0.117 \pm 0.005) M_\odot \text{ pc}^{-3}\), see also Eq. (23).

One must bear in mind that the values of \(A\) and \(B\) significantly differ from the IAU recommended values \(A = 15 \text{ km s}^{-1} \text{ kpc}^{-1}, B = -10 \text{ km s}^{-1} \text{ kpc}^{-1}\). The following results should be for certain:

\[
A \in (10.0, 15.0) \text{ km s}^{-1} \text{ kpc}^{-1} ,
\]

\[
B \in (-15.0, -10.0) \text{ km s}^{-1} \text{ kpc}^{-1} ,
\]

\[
\varrho \in (0.07, 0.15) M_\odot \text{ pc}^{-3} ,
\]

\[
\varrho' \in (-0.043, -0.035) M_\odot \text{ pc}^{-3} \text{ kpc}^{-1} .
\]

(28)

if also the result for the mass density of Crézé et al. (1998) and the IAU recommended values are considered. Unfortunately, there exist results which do not fit Eq. (28) (e.g., there are values of \(A\) and \(B\) which are out of the above presented intervals, see Clemens (1985)).
7 Galactic tide for Dauphole et al. (1996) model of Galaxy

Taking into account Dauphole et al. (1996) model of Galaxy, the relevant equations for galactic tide are:

\[
\frac{d^2 \xi}{dt^2} = - \frac{GM_{\odot}}{r^3} \xi + (A - B) [A + B + 2A \cos(2 \omega_0 t)] \xi \\
- 2A (A - B) \sin(2 \omega_0 t) \eta \\
+ (A - B)^2 (\Gamma_{1D}/\sqrt{b_d^2 + Z_0^2} + \Gamma_{2D}) R_0 Z_0 \cos(\omega_0 t) \zeta ,
\]

\[
\frac{d^2 \eta}{dt^2} = - \frac{GM_{\odot}}{r^3} \eta - 2A (A - B) \sin(2 \omega_0 t) \xi \\
+ (A - B) [A + B - 2A \cos(2 \omega_0 t)] \eta \\
- (A - B)^2 (\Gamma_{1D}/\sqrt{b_d^2 + Z_0^2} + \Gamma_{2D}) R_0 Z_0 \sin(\omega_0 t) \zeta ,
\]

\[
\frac{d^2 \zeta}{dt^2} = - \frac{GM_{\odot}}{r^3} \zeta - \left[ 4 \pi G \varrho + 2(A^2 - B^2) \right] \zeta \\
- 4 \pi G \varrho' Z_0 \cos(\omega_0 t) \xi - \sin(\omega_0 t) \eta ,
\]

\[
\frac{d^2 Z_0}{dt^2} = - \left[ 4 \pi G \varrho + 2(A^2 - B^2) \right] Z_0 ,
\]

\[
r = \sqrt{\xi^2 + \eta^2 + \zeta^2} ,
\]

\[
\omega_0 = A - B ,
\]

\[
A = 14.25 \text{ km s}^{-1} \text{kpc}^{-1} ,
\]

\[
B = -13.89 \text{ km s}^{-1} \text{kpc}^{-1} ,
\]

\[
\Gamma_{1D} = 0.084 \text{kpc}^{-1} ,
\]

\[
\Gamma_{2D} = 0.008 \text{kpc}^{-2} ,
\]

\[
\varrho = 0.143 \text{M}_{\odot} \text{pc}^{-3} ,
\]

\[
\varrho' = -0.0425 \text{M}_{\odot} \text{ kpc}^{-1} ,
\]

\[
b_d = 0.25 \text{ kpc} .
\]
Acknowledgement

This work was supported by the Scientific Grant Agency VEGA, Slovak Republic, grant No. 2/0016/09.

References

1. Barbieri, C.: Fundamentals of Astronomy. Taylor & Francis Group, LLC, Boca Raton, 366pp. (2007)
2. Bertin, G.: Dynamics of Galaxies. Cambridge University Press, Cambridge, 414pp. (2000)
3. Carroll, B.W., Ostlie, D.A.: An Introduction to Modern Astrophysics. Pearson Education, Inc., publishing as Addison-Wesley, San Francisco (2nd ed.), 1278pp. (2007)
4. Clemens, D.P.: Massachusetts - Stony Brook galactic plane CO survey: the galactic disk rotation curve. Astrophys. J. 295, 422-436 (1985)
5. Crézé, M., Chereul, É., Bienaymé, O., Pichon, C.: The distribution of nearby stars in phase space mapped by Hipparcos. I. The potential well and local dynamical mass. Astron. Astrophys. 329, 929-936 (1998)
6. Dauphole, B., Colin, J., Geffert, M., Odenkirchen, M., Tucholke, H.-J.: The mass distribution of the Milky Way deduced from globular cluster dynamics. In: Blitz, L. and Teuben, P. (eds.) Unsolved Problems of the Milky Way, pp. 697-702. IAU Symposium 169 (1996)
7. Feast, M., Whitelock, P.: Galactic kinematics of Cepheids from Hipparcos proper motions. Mon. Not. R. Astron. Soc. 291, 683-693 (1997)
8. Hanson, R.B.: Lick northern proper motion program. II. Solar motion and Galactic rotation. Astron. J. 94, 409-415 (1986)
9. Karimova, D.K., Pavlovskaja, E.D.: On galactic rotation of centroids of various objects. Astronomicheskij Zhurnal 50, 737-746 (1974) (in Russian)
10. Karttunen, H., Kroger, P., Oja, H., Poutanen, M., Donner, K.J.: Fundamental Astronomy. Springer-Verlag, Berlin (5th ed.), 510pp. (2007)
11. Kerr, F.J., Lynden-Bell, B.: Review of galactic constants. Mon. Not. R. Astron. Soc. 221, 1023-1038 (1986)
12. Klacka, J., Gajkošik, M.: Orbital motion in outer Solar System. In: Pretka-Zionek, H., Wnuk, E., Seidelmann, P.K., Richardson, D. (eds.) Dynamics of Natural and Artificial Celestial Bodies, pp. 347-349. Kluwer Academic Press, Dordrecht, arXiv:astro-ph/9910041 (2001)
13. Kulikovskij, P.G.: Stellar Astronomy. Nauka, Moscow (2nd ed.), 272pp. (1985) (in Russian)
14. Marz, D.: Astrophysics in a Nutshell. Princeton University Press, Princeton, 249pp. (2007)
15. Mihalas, D., McRaе Routly, P.: Galactic Astronomy. W. H. Freeman and Company, San Francisco, 257pp. (1968)
16. Olling, R.P., Merrifield, M.R.: Refining the Oort and Galactic constants. Mon. Not. R. Astron. Soc. 297, 943-952 (1998)
17. Phillipps, S.: The Structure and Evolution of Galaxies. John Wiley & Sons, Ltd., Chichester, 305pp. (2005)
18. Sparke, L. S., Gallagher, J.S.: Galaxies in the Universe: An Introduction. Cambridge University Press, Cambridge (2nd ed.), 431pp. (2007)
19. Taylor, J.R.: An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements. University Science Books, Sausalito (2nd ed.), 327pp. (1997)