On the Steinberg character of an orthogonal group over a finite field

A. E. Zalesski

Dedicated to Roderick Gow on the occasion of his 65th birthday

Abstract We determine the irreducible constituents of the Steinberg character of an orthogonal group over a finite field restricted to the orthogonal group of one less dimension.

1 Introduction

The Steinberg character plays an outstanding role in general theory of characters of Chevalley groups. It has many interesting features studied in numerous papers. In particular, experts are interested in branching rules for restrictions of the Steinberg character to natural subgroups. In this paper we study the branching rule for the special case described above.

Let $F_q$ denote the field of $q$ elements, and let $V'$ be a vector space over $F_q$, endowed with a non-degenerate quadratic form. Let $H = SO(V')$ be the special orthogonal group. For a non-degenerate subspace $V$ of $V'$ of codimension 1, set $G = SO(V)$. Let $St_H$ denote the Steinberg character of $H$. Theorems 1.1 and 1.2 are stated for $q$ odd, but the results remain valid for $q$ even if one takes for $G$ the group $\Omega(V)$ formed by the elements of spinor norm 1.

Theorem 1.1

1. Suppose that $\dim V$ is odd. Then the restriction of $St_H$ to $G$ is a multiplicity free character.

2. Suppose that $\dim V$ is even. Then the multiplicities of the irreducible constituents of $St_H|_G$ does not exceed 2.

We also determine the irreducible constituents of the restriction $St_H|_G$ of $St_H$ to $G$. The nature of the irreducible constituents of $St_H|_G$ is described as follows (the meaning of the term "regular character" is as in the Deligne-Lusztig theory):

Theorem 1.2 Let $\rho$ be an irreducible constituent of $St_H|_G$. Then $\rho$ is either a regular character, or a constituent of the induced character $\sigma^G$, where $\sigma$ is an irreducible character of a maximal parabolic subgroup $P$ of $G$.

More precisely, if $L$ is a Levi subgroup of $P$ then $\sigma$ is trivial on the unipotent radical of $P$ and $\sigma|_L$ is a regular character of $L$. Furthermore, $L = GL_m(q) \times SO(V_1)$ for some $m > 0$ and non-degenerate subspace $V_1$ of $V$ (or $V_1 = 0$). Then $\sigma|_L = St_{GL_m(q)} \otimes \rho'$ for some regular character $\rho'$ of $SO(V_1)$.

If $G = SO_{2n+1}(q)$ then every regular character of $G$ is a constituent of $St_H|_G$ (Proposition 6.2). Note that the group $G$ is not quasi-simple if $q$ is odd, and the above results are not valid for the derived subgroup of $G$.

The key ingredient of our consideration is the analysis of the Curtis dual $\omega_G$ of $St_H|_G$ (see [11, §8], [8, §71] for the duality of generalized characters of finite reductive groups).
In our situation $\omega_G$ is a generalized character, whose restrictions to the maximal tori of $G$ determine the irreducible constituents of $St_{H|G}$.

The method used here for proving Theorems 1.1 and 1.2 is analogous to that developed in [19]. There for $G = Sp_{2n}(q)$ and $U_n(q)$, the authors determined the irreducible constituents of the character $\omega \cdot St$, where $\omega$ is the Weil character of $G$. Note that, if $q$ is even then $SO_{2n+1}(q) \cong Sp_{2n}(q)$, and this case has been already treated in [19].

A priori, the method in [19] does not seem to work for $G$ orthogonal, as an orthogonal group has no proper analog of the Weil character. The main idea of this paper is the suggestion of the Curtis dual of $St_{H|G}$ for the role played in [19] by the Weil character.

This deserves to be explained in more detail. First, for the purpose of [19] the values of the Weil character at the non-semisimple elements are irrelevant, and hence the Weil character can be replaced by any other generalized character which coincides with the Weil character at the semisimple elements. This could be, for instance, the Brauer lift of the Weil character. The following "multiplication theorem" is one of the most useful properties of the Weil character. Let $\omega$ be the Curtis dual of the Weil character. A priori, the Curtis dual of the Weil character does not seem to work for $G$ orthogonal, as an orthogonal group has no proper analog of the Weil character. (The character in question was introduced in [16, p. 413] for a different purpose.) I have found out that the Gow character coincides with the Curtis dual of the Weil character at the semisimple elements with the Curtis dual of $St_{H|G}$.

Technically, the reasoning in [19] uses quite heavily certain very specific properties of the Weil character. A priori, the Curtis dual $\omega_G$ of $St_{H|G}$ may not enjoy such properties. However, rather surprisingly, it does have a lot of properties analogous to those of the Weil character. The following "multiplication theorem" is one of the most useful properties of $\omega_G$.

**Theorem 1.3** Let $V$ be an orthogonal space, and $V = V_1 \oplus V_2$, where $V_1, V_2$ are non-degenerate. Let $G = SO(V), G_i = SO(V_i)$ for $i = 1, 2$, and let $\omega_G, \omega_{G_i}$ be the generalized characters as defined above.

(i) Suppose that at least one of $\dim V_1, \dim V_2$ is even. Then $\omega_G(g) = \omega_{G_1}(g) \cdot \omega_{G_2}(g)$ for every $g \in G_1 G_2$.

(ii) Suppose that both $\dim V_1, \dim V_2$ are odd. Then $\omega_G(g) = q \cdot \omega_{G_1}(g) \cdot \omega_{G_2}(g)$ for every $g \in G_1 G_2$.

The branching rule for the restriction of the cuspidal regular irreducible representations $\phi$ of $H = SO(V')$ to $G = SO(V)$, where $q$ and $\dim V$ are odd, is obtained in Reeder [26]. The case where $\phi = St_H$ is somehow at the opposite extreme, as $St_H$ is cuspidal only if $H$ is abelian. It is stated in [26, p. 573] that one can deduce from unpublished work of Bernstein and Rallis that the restriction $\phi|G$ is multiplicity free for every irreducible representation $\phi$ of $H$ provided $\dim V$ is odd.

**Notation.** $F_q$ is the field of $q$ elements, where $q$ is a power of a prime $p$. The algebraic closure of $F_q$ is denoted by $\overline{F_q}$. All vector spaces under consideration are over $F_q$, unless otherwise is said explicitly. Let $V$ be a vector space over $F_q$. We set $\overline{V} = V \otimes \overline{F_q}$. By $End V$ we denote the ring of all $F_q$-homomorphisms $V \rightarrow V$. If $g \in End V$ then $V^g$ is the subspace of vectors fixed by $g$ (unless otherwise is stated). The general and special linear groups are...
denoted by \( GL(V) \) and \( SL(V) \), respectively. If \( n = \dim V \) then we also use the notation \( GL_n(q) \) for \( GL(V) \) and \( SL_n(q) \) for \( SL(V) \). If \( A, B, C \) are square matrices, we denote by \( \text{diag}(A, B, C) \) the block-diagonal matrix with consecutive blocks \( A, B, C \) (similarly, for any number of blocks).

We say that \( V \) is an orthogonal space if \( V \) is endowed with a quadratic form \( Q \), say, which is non-degenerate unless \( q \) is even and \( \dim V \) is odd; in the latter case the form \( Q \) is assumed to be non-defective [9]. If \( q \) is odd then the quadratic form can be replaced by the associated bilinear form. It is well known that \( V \) can be endowed with a quadratic form \( \overline{Q} \) such that \( Q \) is the restriction of \( \overline{Q} \) to \( V \).

Let \( V, V' \) be orthogonal spaces over \( F_q \) defined by quadratic forms \( Q, Q' \), respectively. An embedding \( e : V \to V' \) is called natural if \( Q \) is the restriction of \( Q' \) to \( e(V) \).

A subspace \( U \) of \( V \) is called totally singular if \( Q \) vanishes on \( U \), and non-degenerate if the restriction of \( Q \) to \( U \) yields a non-degenerate associated bilinear form.

Let \( V \) be an orthogonal space. We say that \( V \) is of Witt defect \( d \) if \( \dim V = 2n \) is even and maximal totally singular subspaces of \( V \) are of dimension \( n - d \). Note that \( n - d \) is called the Witt index of \( V \).

The isometry group of an orthogonal space \( V \) is denoted by \( O(V) \), and we set \( SO(V) = O(V) \cap SL(V) \). The subgroup of \( SO(V) \) of elements of spinor norm 1 is denoted by \( \Omega(V) \) [1] [15]. Note that \( |SO(V) : \Omega(V)| = 2 \). If \( d = \dim V \) is odd, the group \( SO(V) \) is also denoted by \( SO_d(q) \). If \( d \) is even, there are two non-isomorphic non-degenerate orthogonal spaces. If \( V \) is of Witt defect 0, we often use \( SO^+_d(q) \) for \( SO(V) \), and if the Witt defect equals 1 then we write \( SO^-_d(q) \) for \( SO(V) \).

Groups \( G = SO(V) \) for \( n > 2 \) are groups with a split BN-pair, for which one defines an irreducible character called the Steinberg character, see [8]; this is denoted here by \( St_G \). For uniformity, if \( \dim V \leq 2 \), we define \( St_G = 1_G \). The Steinberg-plus character \( St^+_G \) is defined in Section 4.

If \( X \) is a finite group, we denote by \( \text{Irr} \ X \) the set of irreducible characters, by \( p_{X}^{reg} \) the character of the regular representation of \( X \) and by \( 1_X \) the trivial character. For a subgroup \( Y \) of \( X \) and a character (or a representation) \( \mu \) of \( X \) the symbol \( \mu|_Y \) denotes the restriction of \( \mu \) to \( Y \). If \( \lambda \) is a character of \( Y \), we write \( \lambda^G \) for the induced character. For the notion of Harish-Chandra induction a reader may consult [8] §70]. Note that we use cross to express a irreducible character of a direct product of two groups in terms of characters of the multiple.

Let \( G \) be a reductive connected algebraic group over an algebraically closed field of characteristic \( p > 0 \). An algebraic group endomorphism \( Fr : G \to G \) is called Frobenius if its fixed point group \( G := G^{Fr} \) is finite. A group \( G \) is called a finite reductive group or a group of Lie type in characteristic \( p \) if there exists a reductive connected algebraic group \( G \) over a field of characteristic \( p \) and a Frobenius endomorphism \( Fr : G \to G \) such that \( G = G^{Fr} \). A subgroup \( T \) of \( G \) is called a maximal torus if there exists an \( Fr \)-stable maximal torus \( T \) of \( G \) such that \( T = T^{Fr} \). Note that saying that two maximal tori \( T, T' \) of \( G \) are \( G \)-conjugate means that the respective \( Fr \)-stable maximal tori in \( G \) are \( G \)-conjugate. For a maximal torus \( T = T^{Fr} \) we set \( W(T) = N_G(T)/T \).

For simple algebraic groups \( G \) Frobenius endomorphisms and the groups \( G = G^{Fr} \) have been classified, see Carter [4] §1.19]. Let \( G \) be the simply connected simple algebraic group of type \( D_n \) or \( B_n \). We identify \( G \) with the \( F_q \)-form of \( G \). Then there is an algebraic group homomorphism \( \eta : G \to SO(\overline{V}) \) with finite kernel. If \( q \) is odd, \( \eta \) is surjective, \( SO(\overline{V}) = \Omega(\overline{V}) \) and the group \( SO(\overline{V}) \) is connected [1] p. 258]. If \( q \) is even then \( \eta(G) = \Omega(\overline{V}) \), so \( \Omega(\overline{V}) \) is connected. Therefore, for \( q \) even \( SO(\overline{V}) \) is not connected as \( \Omega(\overline{V}) \) has index 2 in
\( SO(\mathbb{V}) \). Thus, if \( q \) is odd then \( SO(V) \) is a finite reductive group, if \( q \) is even then so is \( \Omega(V) \).

For a connected algebraic group \( G \) with Frobenius endomorphism \( Fr \) one defines the relative rank \( \text{rel.rk} \) of \( G \) and the function \( \varepsilon_G = (-1)^{\text{rel.rk}(G)} \) called the sign of \( G \), see Carter [8] p. 197-199]. If \( G \) is not connected, we define the sign to be that of \( G^0 \), the connected component of \( G \).

For the notions of dual groups \( G^* \) of \( G \) and \( G^*_r \) of \( G \), see [8 Ch. 4] and [11] 13.10]. (Note that \( G^*_r = (G^*)^{Fr^*} \), where \( Fr^* \) is a suitable Frobenius endomorphism of \( G^* \). To simplify the notation, we shall use \( Fr \) for \( Fr^* \), which should not lead to any confusion.)

Notation for Deligne-Lusztig characters is introduced in Section 6.

### 2 Some properties of orthogonal groups

**Lemma 2.1** Let \( G = SO(V), \) \( q \) odd, or \( \Omega(V), \) \( q \) even. Suppose that \( \dim V \) is even. Let \( s \in G \) be a semisimple element. Then \( C_{O(V)}(s) \subset G \) if and only if neither 1 nor \(-1\) is an eigenvalue of \( s \).

Proof. Let \( \mathbb{V} \) the natural module for \( O(2n, \mathbb{F}_q) \). We can assume that the form defining \( G \) is the restriction to \( V \) of that defining \( O(2n, \mathbb{F}_q) \). As \( SO(V) = SO(\mathbb{V}) \cap O(V) \) and \( \Omega(V) = \Omega(\mathbb{V}) \cap O(V) \), respectively, if \( q \) is odd or even, it follows that it suffices to prove that \( C_{O(\mathbb{V})}(s) \subseteq SO(\mathbb{V}) \) and \( C_{O(\mathbb{V})}(s) \subseteq \Omega(\mathbb{V}) \), respectively.

The "if" part. Let \( \mathbb{V} = V_1 \oplus \cdots \oplus V_t \) be the decomposition of \( \mathbb{V} \) as a direct sum of the homogeneous components of \( s \) on \( \mathbb{V} \). Then the restriction of \( s \) to \( V_i \) \((i = 1, \ldots, t)\) is scalar \( \alpha_i \cdot \text{Id} \), say. Suppose that \( \pm 1 \) is not an eigenvalue of \( s \). Then each \( V_i \) is totally singular, \( t \) is even and \( V_1, \ldots, V_t \) can be reordered so that \( V_{2i-1} \) and \( V_{2i} \) were dual. (That is, the Gram matrix of the bilinear form on \( V_{2i-1} \oplus V_{2i} \) under a certain basis is \( \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix} \)). Let \( x \in C_{O(\mathbb{V})}(s) \). Then \( xV_i = V_i \) for every \( i = 1, \ldots, t \). Let \( x_i \) denote the restriction of \( x \) to \( V_i \). Then \( x_{2i} = x_{2i-1}^{-1} \cdot x_{2i-1} \), where \( x_{2i-1}^{-1} \) means the transpose of \( x_{2i-1} \). It follows that \( \det x = 1 \), and hence, if \( q \) is odd, then \( x \in SO(\mathbb{V}) \) as claimed. If \( q \) is even then \( |O(V) : \Omega(V)| = 2 \), whereas \( |GL(V_i) : SL(V_i)| \) is odd. So the claim follows.

The "only if" part. Suppose the contrary. Let \( Y \neq 0 \) be the 1- or \(-1\)-eigenspace of \( s \) on \( V \). Then \( Y \) is non-degenerate. Obviously, \( C_{O(V)}(s) \) contains a subgroup isomorphic to \( O(Y) \), and hence \( C_{O(V)}(s) \) contains a matrix of determinant \(-1\) if \( q \) is odd, or of spinor norm \(-1\) if \( q \) is even. So \( C_{O(V)}(s) \) is not contained in \( G \).

**Lemma 2.2** (1) Let \( G = Sp_{2n}(\mathbb{F}_q) \). Then \( C_G(s) \) is connected for any semisimple element \( s \in G \).

(2) Let \( G = SO_{2n}(\mathbb{F}_q) \) and \( s \in G \) a semisimple element. Then \( C_G(s) \) is connected if and only if either 1 or \(-1\) is not an eigenvalue of \( s \). In particular, if \( q \) is even then \( C_G(s) \) is always connected. In addition, if \( C_G(s) \) is connected then \( C_L(s) \) is connected for every Levi subgroup \( L \) of \( G \) containing \( s \).

Proof. (1) As \( Sp_{2n}(\mathbb{F}_q) \) is simply connected, the statement is a special case of [27] Part E, Ch.2, 3.9].

(2) Let \( \mathbb{V} \) be the natural module for \( G \). If \( \lambda \) is an eigenvalue of \( s \), let \( V(\lambda) \) denote the \( \lambda \)-eigenspace of \( V \). If \( \lambda \neq \pm 1 \) then \( V(\lambda) \) is totally singular, and hence \( V(\lambda^{-1}) \neq 0 \). Set \( V_\lambda = V(\lambda) + V(\lambda^{-1}) \). Then \( V_\lambda \) is non-degenerate. Moreover, it is well known that the
common stabilizer of $V(\lambda), V(\lambda^{-1})$ is isomorphic to $GL(V(\lambda))$. Therefore, if 1, −1 are not eigenvalues of $s$ then $C_{O(V)}(s)$ is the direct product of $GL(V(\lambda))$, and hence connected.

Furthermore, suppose that 1 or −1 (but not each of them) is an eigenvalue of $s$, and let $Y$ be the 1- or −1-eigenspace of $s$. Then $C_{O(V)}(s)$ is the direct product of $GL(V(\lambda))$ with $\lambda \neq \pm 1$ and $O(Y)$, so $C_{SO(V)}(s)$ is the direct product of $GL(V(\lambda))$ with $\lambda \neq \pm 1$ and $SO(Y)$. Again, this group is connected.

Finally, suppose that both the 1- and −1-eigenspaces of $s$ are non-zero (so $g$ is odd), and let $Y, Z$ be the 1- or −1-eigenspaces of $s$, respectively. Then $C_{SO(V)}(s)$ contains a subgroup isomorphic to the direct product of $GL(V(\lambda))$ with $\lambda \neq \pm 1$, $SO(Y)$ and $SO(Z)$. This group is connected. However, $C_{SO(V)}(s)$ additionally contains all elements $g_1 g_2$, where $g_1$ (respectively, $g_2$) acts trivially on $Y^\perp$ (respectively, $Z^\perp$), and $\det g_1 = \det g_2 = -1$. It follows that $C_{SO(V)}(s)$ contains a connected subgroup of index 2, and hence is not connected.

For the additional claim, observe that $L$ coincides with the stabilizer in $G$ of the direct sum of subspace $(U_1 + U_1') + \cdots + (U_t + U_t') + U'$ of $\overline{V}$, where $U'$ is non-degenerate or $\{0\}$, and $U_1, U_1'$ are totally singular and dual to each other ($i = 1, \ldots, t$). Therefore, $L$ is isomorphic to the direct product of $SO(U_i)$ with $GL(U_1) \times \cdots \times GL(U_t)$. As $s \in L$, all $U_i, U_i'$ are stabilized by $s$. Let $s_i$ be the restriction of $s$ to $U_i$, and $s'$ the restriction of $s$ to $U'$. Then $C_L(s)$ is isomorphic to the direct product of $C_{SO(U_i)}(s_i)$ with $C_{GL(U)}(s_1) \times \cdots \times C_{GL(U)}(s_t)$. Each group $C_{GL(U)}(s_i)$ is well known to be connected. As $C_G(s)$ is connected, either 1 or −1 is not an eigenvalue of $s$ on $\overline{V}$, and this remains true for $s'$. So $C_{SO(U)}(s')$ is connected, whence the result.

**Lemma 2.3** Let $G = SO(V), q$ odd, and let $t \in G$ be a semisimple element. Let $V^t$ be the 1-eigenspace of $t$ on $V$, and $W$ the unique $t$-stable complement of $V^t$. Let $t_1$ be the restriction of $t$ to $W$. Then $SO(V^t) \times C_{SO(W)}(t_1)$ is a subgroup of $C_G(t)$ of index at most 2, and the index equals 2 if and only if $V^t \neq 0$ and −1 is an eigenvalue of $t$.

Proof. Let $Y$ be the −1-eigenspace of $t$ and $M$ the unique $t$-stable complement of $V^t + Y$ in $V$. Let $t_1$ be the projection of $t$ to $M$. Obviously, $C_G(s) \subseteq O(V^t) \times O(Y) \times C_{O(M)}(t_1)$. By Lemma 2.1 $C_{O(M)}(t_1) = C_{SO(M)}(t_1)$. This implies the lemma if $V^t$ or $Y$ is the zero space. Otherwise, $C_G(t) = \langle O(V^t) \times O(Y) \times C_{O(M)}(t_1), g \rangle$, where $g \in O(V^t) \times O(Y) \times O(M)$ is such that the projection of $g$ to $M$ is the identity, whereas the projections to $V^t$ and to $Y$ are not in $SO(V^t), SO(Y)$, respectively. So the lemma follows.

Let $\alpha \in \{1, -1\}$. In many cases it is convenient to use uniform notation for $SO_{2n}^+(q)$ and $SO_{2n}^-\alpha(q)$ by writing $SO_{2n}^\alpha(q)$ and interpreting $\alpha$ as the plus sign if $\alpha = 1$ and the minus sign if $\alpha = -1$. The following is well known:

**Lemma 2.4** Let $G = SO_{2n}^\alpha(q) = SO(V)$ if $q$ is odd, otherwise let $G = SO_{2n}^\alpha(q)$. Then $G$ contains a cyclic subgroup $T$ of order $q^{n} - \alpha$. Moreover, if $\alpha = -1$ then $T$ is irreducible, otherwise $T$ stabilizes a maximal totally singular subspace of $V$ and acts on it irreducibly. In addition, $T$ is a maximal torus of $G$.

Proof. Suppose first that $\alpha = 1$. Then $V = U + U'$, where $U, U'$ are totally singular subspaces of $V$ of dimension $n$. Moreover, there are bases in $U, U'$ such that the matrix $\text{diag}(g, g^{-1})$ belongs to $O(V)$ for every $g \in GL(U)$. (Here $^t g$ means the transpose of $g$.) Note that $GL(U) \cong GL_n(q)$ contains an irreducible element $h$ of order $q^n - 1$. We set $T = \langle \text{diag}(h, h^{-1}) \rangle$. Furthermore, $(h)$ is self-centralizing in $GL(U)$.

Let $\alpha = -1$. By Huppert [22] Satz 3], $O(V)$ contains a cyclic irreducible subgroup $T$ of order $q^{n} + 1$. In fact, $t \in G$. This is trivial if $q$ is even as $|O(V) : \Omega(V)| = 2$ and $q^n + 1$ is
odd. Let \( q \) be odd. View \( t \) as an element of the group \( O(V) \). Then there is a basis \( B \), say, of \( V \), under which the matrix of \( t \) is diagonal. Moreover, \( t \) does not have eigenvalues \( \pm 1 \). (Indeed, \( t \) acts irreducibly on \( V \), and hence no eigenvalue of \( t \) on \( V \) belongs to \( F_q \).) This implies that every vector of \( B \) is singular, and hence \( B \) can be assumed to be a hyperbolic basis. (That is, one can replace every element of \( B \) by a suitable multiple so that the Gram matrix of the associated bilinear form under the new basis becomes block-diagonal with \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).) It easily follows from this that \( \det t = 1 \).

We show that \( T \) is self-centralizing. Set \( Y = G \) if \( q \) is odd, otherwise denote by \( Y \) the symplectic group of the associated bilinear form on \( V \). It is well known that there exists an involutory anti-automorphism \( \sigma \) of \( \text{End} \ V \) such that \( Y = \{ x \in \text{End} \ V : x\sigma(x) = \text{Id} \} \). Let \( K \) be the centralizer of \( T \) in \( \text{End} \ V \). By Schur’s lemma, \( K \) is a field of order \( q^{2n} \). Set \( T_1 = \{ x \in K : x\sigma(x) = \text{Id} \} \). Then \( T_1 \subset Y \) and \( C_Y(T_1) = T_1 \). As \( \sigma \) is an automorphism of \( K \) of order 2, an easy Galois argument implies that \( |T_1| = q^n + 1 \). As \( T \subset T_1 \), we have \( T = T_1 \). As \( C_Y(T) = T \), it follows that \( C_G(T) = T \).

Finally, let \( D \cong GL(U) \) be the group \( \{ \text{diag}(g, t^{-1}) : g \in GL(U) \} \) and \( D \cong GL(U) \). As \( T \) is a cyclic group, \( T \) is contained in a maximal torus \( T' \) of \( G \) (respectively, \( D \)) if \( \alpha = -1 \) (respectively, \( \alpha = 1 \)), see [27, Ch. II, 1.1.1]. Let \( T' = T \text{primeFr} \). As \( C_G(T) = T \) (respectively, \( C_D(T) = T \) unless \( n = 2, q \leq 3 \)), we have \( T = T' \). If \( n = 2, q \leq 3 \) then it is easy to observe that \( T' = T \). As the rank of the algebraic group \( D \) equals the rank of \( G = SO(F_q) \), a maximal torus of \( D \) remains maximal in \( G \).

**Lemma 2.5** Let \( \dim V \) be odd and let \( T \subset G \subset O(V) \) be a maximal torus. Then \( T \) stabilizes a non-degenerate subspace \( V' \), say, of \( V \), and hence \( T \subset O(V') \), when \( O(V') \) is viewed as a subgroup of \( O(V) \). In addition, \( T \subset SO(V') \) if \( q \) is odd, otherwise \( T \subset \Omega(V') \).

Proof. Observe first that there is a vector \( v \in V \) fixed by \( T \). Indeed, it is well known that such a vector exists in \( V \otimes F_q \). Let \( b_1, \ldots, b_n \) be a basis in \( V \). Consider linear equations \( t \sum x_i b_i = \sum x_i b_i \) for every \( t \in T \) with respect to indeterminates \( x_1, \ldots, x_n \). By the above, these equations have a common solution with \( x_i \in F_q \). Therefore, there exists a solution with \( x_i \in F_q \), as claimed. Set \( V^T = \{ y \in V : ty = y \text{ for all } t \in T \} \). If \( q \) is odd then \( V^T \) is non-degenerate, and hence there is an anisotropic vector \( v \in V^T \), so we take \( V' = v^\perp \).

Suppose that \( q \) is even. By Maschke’s theorem, \( T \) is completely reducible on \( V \), and hence there a \( T \)-stable complement \( V' \) of \( \langle v \rangle \) in \( V \). One observes that \( V' \) is non-degenerate, so \( T \subset O(V') \). The additional statement is well known.

**Lemma 2.6** Let \( V \) be an orthogonal space, and let \( U_1, U_2 \) be totally singular subspaces of \( V \) of equal dimension. Then \( gU_1 = U_2 \) for some \( g \in G \), unless when \( \dim U_1 = \dim U_2 = \dim V/2 \) (and hence \( V \) is of Witt defect 0). In the exceptional case there are two \( G \)-orbits of the subspaces in question.

Proof. By Witt’s theorem, \( hU_1 = U_2 \) for some \( g \in O(V) \). Suppose that \( \dim U_1 < \dim V/2 \). Let \( W \) be a complement of \( U_1 \) in \( U_1^\perp \). Then \( W \neq 0 \) is non-degenerate and \( U_1 \subset W^\perp \). Furthermore, there is an element \( x \in O(V) \) such that \( xv = v \) for all \( v \in W^\perp \) and (a) \( \det x = -1 \) if \( q \) is odd and (b) the projection of \( x \) into \( O(W) \) is not in \( \Omega(W) \) if \( q \) is even. Then, if \( h \notin G \), we have \( h_x U_1 = U_2 \) and \( h x \in G \) in (a). So the first claim follows. Suppose that \( \dim U_1 = \dim V/2 \). Then the result follows by [23, Lemma 2.5.8(ii)].
Lemma 2.7 Let $G$ be a reductive algebraic group, $Z$ a finite central subgroup of $G$, $G_1 := G/Z$ and let $\eta : G \to G_1$ be the natural homomorphism. Let $Fr$ be a Frobenius endomorphism of $G$, $G := G^{Fr}$ and $G_1 := G_1^{Fr}$. Let $T$ be a maximal $Fr$-stable torus of $G$, $T_1 = \eta(T)$, $T = T^{Fr}$, $T_1 = T_1^{Fr}$ and $Z = Z^{Fr}$. Then $|G_1 : \eta(G)| = |Z|$ and $G_1 = \eta(G) \cdot T_1$.

Proof. Obviously, $\eta(G) \subset G_1$, $\eta(T) \subset T_1$. Furthermore, $\eta(G)$ is a normal subgroup of $G_1$, $|G_1 | = |G_1 |$ and $|T_1 | = |T_1 |$, see [12] 4.2.3. Obviously, $Z = G \cap Z = G \cap \ker \eta$. So $|G_1 | = |G_1 |$ implies $|G_1 | = |\eta(G)| \cdot |Z|$.

Observe that $\eta(T) = \eta(G) \cap T_1$. (Indeed, let $x \in \eta(G) \cap T_1$. As $T_1 = \eta(T)$, there is $t \in T$ such that $\eta(t) = x$. Let $g \in G$ with $\eta(g) = x$. Then $\eta(g) = \eta(t)$, or $\eta(t^{-1}g) = 1$. Therefore, $z := t^{-1}g \in Z$. As $T$ is a maximal torus of $G$, it follows that $Z \subset T$, and hence $z \in T$. Let $t' = tz$. Then $g = tz = t' \in T$. This means that $g \in T$, and hence $g \in T \cap G = T$. So $x \in \eta(T)$, as claimed.)

Set $m = |T_1 / (\eta(G) \cap T_1 )| = |T_1 / \eta(T)|$. As $Z \subset T$, we have $Z \subseteq T$, and hence $Z = T \cap \ker \eta$. As above, $|T_1 |$ implies $|Z| = |T_1 / \eta(T)| = m$. Therefore, $|\eta(G) \cdot T_1 |$ is of order $|\eta(G)| \cdot m = |G_1 |$, whence the result.

In the classical group theory an important role is played by the notion of spinor norm $[2, 18]$. This is a homomorphism $\nu : O(V) \to K$, where $K$ is an abelian group of exponent 2 and of order at most 4. It is well known that $\Omega(V)$ is a subgroup of $SO(V)$ of index 2 unless $q$ is even and dim $V$ is odd, or dim $V = 1$, see [9] or [23]. Furthermore, if $W$ is a non-degenerate subspace of $V$ then $\Omega(W) = O(W) \cap \Omega(V)$, when $O(W)$ is viewed as a subgroup of $O(V)$.

If $q$ is even and dim $V$ is odd then $O(V) \cong Sp(U)$, where $U$ is a symplectic space of dimension dim $V - 1$, see [18 Theorem 14.2]; therefore in this case $O(V)$ is simple unless $(n, q) = (3, 2)$ or $(5, 2)$.

Note that $G = SO(V)$ has a unique subgroup of index 2, unless dim $V = 1$, or $q$ is even, dim $V$ is odd, or $G = SO_2^\perp (2)$, see [23 2.5.7]. With this exceptions, $\Omega(V)$ can be defined as the subgroup of index 2 in $SO(V)$.

Lemma 2.8 Assume that $q$ is odd and dim $V > 1$. Let $T$ be a maximal torus in $SO(V)$. Then $T$ contains an element of spinor norm $-1$.

Proof. If $n = 2$, the statement is trivial as $SO(V) = T$. Otherwise, this follows from Lemma 2.7. (Indeed, in notation of Lemma 2.7, we have $|Z| = 2$ and hence $|G_1 : \eta(G)| = 2$. Therefore, $\eta(G) = \Omega(V)$ as $\Omega(V)$ is the only subgroup of index 2 in $SO(V)$).

Lemma 2.9 Let dim $V = 2n$, $q$ odd. Suppose that the Witt defect of $V$ equals 1. Then $SO(V) \setminus \Omega(V)$ contains an element $t$ of order $q^n + 1$ such that $C_G(t) = C_G(t^2) = \langle t \rangle$.

Proof. If $n = 1$ then $SO(V)$ is abelian, and the statement is trivial. Let $n > 1$. By Lemma 2.4 $SO(V)$ contains a cyclic irreducible subgroup $T$ of order $q^n + 1$, which is a maximal torus in $SO(V)$. Let $T = \langle t \rangle$.

By Lemma 2.8 $T$ is not contained in $\Omega(V)$, and hence $t \notin \Omega(V)$. We have shown in Lemma 2.7 that $T$ is self-centralizing, so $C_G(t) = T$. We show that $T = C_G(t^2)$. Let $K$ be as in the proof of Lemma 2.4. If $t$ is irreducible in $V$ then the claim follows by Schur’s lemma. Suppose that $T$ is reducible. Then $t$ belongs to a proper subfield $K_1$, say, of $K$. Let $K^x, K_1^x$ be the multiplicative groups of $K, K_1$, respectively. Then $t^2 \in K_1$ implies $|K^x / K_1^x | = 2$, which is false. So the lemma follows.

If $G$ is an algebraic group then $G^0$ denotes the connected component of $G$. 

7
Lemma 2.10 Assume that $q$ is odd. Then there exists $t \in G \setminus \Omega(V)$ such that $|C_G(t)|_p = |C_G(t^2)|_p$ and $V^t = V^{t^2}$. In addition, $C_G(t)^0 = C_G(t^2)^0$.

Proof. Set $G = SO(V)$ and $\dim V = n$. If $n = 2$ then $G$ is abelian, so the statement is trivial. Let $n > 2$. In view of Lemma 2.3, we can assume that the Witt defect of $V$ equals 0. Let $W$ be a non-degenerate subspace of $V$ of Witt defect 1 and $2k = \dim W$. We choose $W$ so that $2k = n - 2$ if $n$ is even, otherwise $2k = n - 1$. By Lemma 2.3, there is an element $t \in SO(W) \setminus \Omega(W)$ of order $q^k + 1$ acting on $W$ irreducibly. Let us view $t$ as an element of $G$. Then $t \notin \Omega(V)$.

If $G = SO^{+}_4(3)$ or $SO_3(3)$ then $|t| = 4$, and the $-1$-eigenspace of $t^2$ on $V$ is of dimension 2. So $C_G(t^2)$ is a subgroup of $O(W) \times O(W^\perp)$, which has no element of order 3. The equality $V^t = V^{t^2}$ is here obvious. In addition, the connected component of $C_G(t^2)$ is $SO(W) \times SO(W^\perp)$, so the additional claim of the lemma is true in this case.

With exception of the above two groups, $t^2$ is irreducible on $W$. Indeed, let $K$ be the enveloping algebra of $t$ in $End W \cong \text{Mat}(2k, q)$. By Schur’s lemma, $K$ is a field of order $q^{2k}$. If $t^2$ is reducible then $t^2$ belongs to a proper subfield $L$ of $K$, in fact $K/L$ is a quadratic extension. Let $\gamma$ be the Galois automorphism of $K/L$. Then the group $\{x \in K : x\gamma(x) = 1\}$ is of order $q^{k+1}$ and $t$ is a generator of this group. Therefore, $t^2 \in L$ if and only if $q^{k+1} = 2$ or 4. This implies $k = 1, q = 3$, and hence $G \in \{SO^{+}_4(3), SO_3(3)\}$.

Thus, $t^2$ is irreducible on $W$. It follows that $C_G(t) = C_G(t^2)$, as claimed. As all eigenvalues of $t^2$ on $W$ are distinct, it easily follows that $C_G(t)^0 = C_G(t^2)^0$.

Remark. One observes that if $G \notin \{SO^{+}_4(3), SO_3(3)\}$ then $|C_G(t)|_p = |C_G(t^2)|_p = 1$.

Lemma 2.11 Let $V$ be a non-degenerate subspace of an orthogonal space $U$, let $G = SO(V) \subset D = SO(U)$ be a natural embedding, and let $t \in G$ be as in Lemma 2.10. Then $|C_D(t)|_p = |C_D(t^2)|_p$. In addition, if $D := SO(\overline{V})$ then $C_{D}(t)^0 = C_{D}(t^2)^0$.

Proof. Let $W = V^\perp$ so $U = V \oplus W$. Then $W$ is a trivial $F_q G$-module. Therefore, $U^t = V^t + W = V^{t^2} + W = U^{t^2}$. Set $V' = (V^t)^\perp, U' = (U^t)^\perp$. Then $V' = U'$. Let $t'$ be the projection of $t$ to $V'$. As $V^t = V^{t^2}$, it follows that $t$ does not have eigenvalue $-1$. By Lemma 2.3, $C_G(t) = SO(V^t) \times C_{SO(V')}(t')$. Furthermore, $|C_G(t)|_p = |C_G(t^2)|_p$, so $|C_{O(V')}(t')|_p = |C_{O(V')}(t^2)|_p$, and hence $|C_{O(U')}(t)|_p = |C_{O(U')}(t^2)|_p$ as $U' = V'$. We have $C_D(t) = SO(U^t) \times C_{O(U')}(t')$ and $C_D(t^2) = SO(U^{t^2}) \times C_{O(U')}(t^2)$. So the first statement of the lemma follows. Similarly, $C_{D}(t) = SO(V^t) \times C_{SO(V')}(t')$, so $C_{D}(t)^0 = SO(V^t) \times C_{SO(V')}(t')^0$. By Lemma 2.10, $C_{SO(V')}(t')^0 = C_{SO(V')}(t^2)^0$. As $C_D(t)^0 = SO(V^t) \times C_{SO(V')}(t^2)^0$, the second statement follows.

The following result by Gow and Szechman [17 Theorem 4.1] considerably simplifies the computational aspect of our reasoning below. This is called the character comparison theorem in [17] and generalizes an earlier result by Knörr [24 Proposition 1.1].

Theorem 2.12 (The comparison theorem) Let $X$ be a finite group, and $p$ a prime divisor of $|X|$. Let $\phi, \psi$ be generalized characters of $X$. Suppose that $\phi(g) = \pm \psi(g) = \pm p^{m(g)}$ for every $p'$-element $g$ of $G$, where $m(g) \geq 0$ is an integer. Suppose also that $\phi(1) = \psi(1)$. Then there exists a linear character $\lambda$ of $X$ such that $\lambda^2 = 1$ and $\phi(g) = \lambda(g)\psi(g)$ for all $p'$-elements $g \in X$. 


3 Orthogonal decompositions and maximal tori

Let $V$ be the natural module for $G$. We need to compute the restriction $\omega_{G|T}$ for every maximal torus $T$ of $G$. In order to express the result in a convenient and uniform way, we introduce a so-called $T$-decomposition of $V$, see [19]. If $q$ is even, we assume $\dim V$ to be even. Let

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_k \oplus V_{k+1} \oplus \cdots \oplus V_{k+l},$$

where

(a) $k, l \geq 0$ and $\dim V_i$ is even for $i > 0$;

(b) $V_1, \ldots, V_{k+l}$ are non-degenerate subspaces of $V$ orthogonal to each other, $V_0 = (V_1 \oplus \cdots \oplus V_{k+l})^{\perp}$; $V_0 = 0$ if $n$ is even and $\dim V_0 = 1$ otherwise;

(c) $V_1, \ldots, V_k$ are of Witt defect 0;

(d) $V_{k+1}, \ldots, V_{k+l}$ are of Witt defect 1;

(e) $\dim V_1 \leq \cdots \leq \dim V_k$ and $\dim V_{k+1} \leq \cdots \leq \dim V_{k+l}$.

We call this an orthogonal decomposition of $V$. Note that the part $V_1 \oplus \cdots \oplus V_k$ or $V_{k+1} \oplus \cdots \oplus V_{k+l}$ may be absent. We express this by writing $k = 0$ or $l = 0$. Additionally, we set

$$V' = V_1 \oplus \cdots \oplus V_k, \quad \text{and} \quad V'' = V_{k+1} \oplus \cdots \oplus V_{k+l}.$$  

Obviously, the Witt defect of $V'$ is 0.

The list of the dimensions $\dim V_1, \ldots, \dim V_k$ and $\dim V_{k+1}, \ldots, \dim V_{k+l}$ is an essential invariant of an orthogonal decomposition. There is a convenient way (for our purposes) to encode the list of these dimensions in terms of two functions $i \to d_i, j \to e_j$, where $d_i, e_j \geq 0$ are integers. We often record these functions in the form $[1^{d_1}, 2^{d_2}, \ldots]$ and $[1^{e_1}, 2^{e_2}, \ldots]$. These are interpreted as follows. The entry $i^{d_i}$ tells us that in the list $V_1, \ldots, V_k$ there are $d_i$ subspaces of dimension $2i$. Similarly, the entry $j^{e_j}$ tells us that in the list $V_{k+1}, \ldots, V_{k+l}$ there are $e_j$ terms of dimension $2j$. Observe that some $d_i, e_j$ may be zeros. Obviously, $k = d_1 + d_2 + \cdots$ and $l = e_1 + e_2 + \cdots$. Note that

$$\dim V = \sum_i 2id_i + \sum_i \sum 2je_j \text{ if } n \text{ is even, and } \dim V = 1 + \sum_i 2id_i + \sum_i \sum 2je_j \text{ if } n \text{ is odd.}$$

Computing the Witt defect of $V$ in terms of $V_i$ [23, 2.5.11], one observes that $l$ is even (respectively, odd) if $G = SO^{+}_{2n}(q)$ (respectively, $G = SO^{-}_{2n}(q)$).

Lemma 3.1 For every orthogonal decomposition (1) there is a maximal torus $T$ of $G$ of order $\Pi_{i,j}(q^i - 1)^{d_i} \cdot \Pi_{j}(q^j + 1)^{e_j}$. In fact,

$$T = T_1 \times \cdots \times T_k \times T_{k+1} \times \cdots \times T_{k+l},$$

where $TV_i = V_i$ and $T_i$ is the restriction of $T$ to $V_i$ for $i = 1, \ldots, k+l$.

Proof. To every decomposition (1) one can correspond an abelian subgroup $T$ of $SO(V)$, which is the direct product $T_1 \times \cdots \times T_k \times T_{k+1} \times \cdots \times T_{k+l}$ as follows. Let $T_i \subset SO(V_i)$ be a maximal torus of $SO(V_i)$ of order $q^{\dim V_i/2} - 1$ for $1 \leq i \leq k$ and of order $q^{\dim V_i/2} + 1$ for $k+1 \leq i \leq k+l$ according with Lemma 27.4. Let $G = G^{Fr}$. Set $Y = SO(V_1) \times \cdots \times SO(V_{k+l})$ and $Y = SO(V_1) \times \cdots \times SO(V_{k+l})$. Then $Y$ and $Y$ are subgroups of $G = SO(V)$ and of $G$, respectively (in the obvious sense). Then $Y = Y \cap G$. It follows that $Fr(Y) = Y$ and $Y = Y^{Fr}$. Saying that $T_i$ is a maximal torus of $G_i$ means that $T_i = T_i^{Fr}$ for some
Fr-stable maximal torus $T_i$ of $SO(V_i)$. Therefore, $T := T_1 \times \cdots \times T_{k+l}$ is a maximal Fr-stable torus of $Y$ and $T = T^{Fr}$. By dimension reason, $T$ is a maximal Fr-stable torus of $G$. (One can also mimic the reasoning in [19 §2.4].)

Observe that every maximal torus $T$ of $G$ determines an orthogonal decomposition with properties described in Lemma 3.1. This means (in more precise terms) that if $T = T^{Fr}$ for an Fr-stable maximal torus $T$ of the algebraic group $G$, then $T$ determines an orthogonal decomposition in question, which we call the $T$-decomposition of $V$ in this paper. For symplectic groups, which appear below as the dual groups of $SO_{2n+1}(q)$, this notion was introduced in [19].

**Lemma 3.2** If dim $V$ is odd then there is a bijection between the $G$-conjugacy classes of maximal Fr-stable tori in $G$ and the pairs of functions $[1^{d_1}, 2^{d_2}, \ldots]$ and $[1^{e_1}, 2^{e_2}, \ldots]$ satisfying (2).

Proof. This is well known (the case with $q$ even is not excluded), see for instance [19]. (Note that in this case there is a bijection between the $G$-conjugacy classes of maximal Fr-stable tori in $G$ and the conjugacy classes of the Weyl group $W$ of $G$.)

Thus, if dim $V$ is odd then the maximal tori of $G$ are parametrized by the above functions. Note that this parametrization differs a bit from the parametrization by bipartitions which are pairs $(\lambda_1, \ldots, \mu_1, \ldots)$, where $2\lambda_i = \dim V_i$ for $i = 1, \ldots, k$, and $2\mu_j = \dim V_{k+j}$ for $j = 1, \ldots, l$.

The situation is more complex if $n$ is even. However, if one wishes to consider the maximal tori of $G$ up to conjugacy in the full orthogonal groups, then we have a similar statement:

**Lemma 3.3** Suppose that dim $V = 2n$ is even. Then there is a bijection between $O(V)$-conjugacy classes of maximal Fr-stable tori in $G$ and the pairs of functions $[1^{d_1}, 2^{d_2}, \ldots]$ and $[1^{e_1}, 2^{e_2}, \ldots]$ satisfying (2).

Proof. Note that $O(V)$ is a subgroup of $O(V)$ and $SO(V) = SO(V) \cap O(V)$. Consider $G$ as a subgroup of $H = SO(W)$, where dim $W = 2n + 1$. Then a maximal torus $T$ of $G$ remains maximal in $H$. Set $X = \{\text{diag}(\det g^{-1}, g) : g \in O(V)\}$. Obviously, $X \cong O(V)$ and the restriction of $X$ to $V$ is exactly $O(V)$. Then two maximal tori $T, T'$ of $G$ are conjugate in $H$ if and only if they are conjugate in $X$. Indeed, tori $T, T'$ of $G$ stabilizes a complement $\mathcal{U}$ of $V$ in $W$, and $\mathcal{U}$ is the fixed point space of each $T, T'$ on $W$. Therefore, if $hT h^{-1} = T'$ for $h \in H$ then $h\mathcal{U} = \mathcal{U}$. Similarly, $hV = V$. Therefore, $hV = V$ and the restriction of $h$ to $V$ belongs to $O(V)$. As $h \in SO(W)$, it follows that $h \in X$. Now the lemma follows from Lemma 3.2.

We need some observations on the Weyl groups of the BN-pairs of type $B_n$ and $D_n$. We denote them here by $W(B_n)$ and $W(D_n)$, respectively. Recall that $W(B_n)$ is a semidirect product of a normal subgroup $A$ and of the symmetric group $S_n$; in addition, $A$ is exponent 2 and rank $n$. In the natural realization as a group of $(n \times n)$-matrices over the rationals, $W(B_n)$ is exactly the group of monomial matrices with non-zero entries $\pm 1$. So there is a basis $b_1, \ldots, b_n$ of the underlying space such that the elements of $W(B_n)$ permute the set $\pm b_1, \ldots, \pm b_n$. If $w \in W(B_n)$, we denote by $\tilde{w}$ the projection of $w$ into $S_n$. Let $A_0$ be the subgroup of $A$ formed by matrices of determinant 1. Then $|A : A_0| = 2$, and $W(D_n)$ is a semidirect product of $A_0$ and $S_n$.

The conjugacy classes of $W(B_n)$ and $W(D_n)$ are determined in [7]. Let $w \in W(B_n)$. The conjugacy classes of $B = W(B_n)$ are determined by pairs of functions $i \rightarrow d_i, j \rightarrow e_j$,
and a representative \( w \) of the respective conjugacy class is the permutation of the set \( \{ \pm b_1, \ldots, \pm b_n \} \) obtained as follows. Let \( k = \sum_i d_i \) and \( l = \sum_j e_j \) (so \( k + l = n \)), and let \( \pi \in S_k \times S_l \subset S_n \) be the permutation in which the cycle of length \( i \) occurs with multiplicity \( d_i \), and the cycle of length \( j \) occurs with multiplicity \( e_j \). If \( m \leq k \) then \( w^t(b_m) \in \{ b_1, \ldots, b_k \} \) for any \( t \), that is, \( w \) acts by permuting the basis elements \( b_1, \ldots, b_k \). If \( m > k \) and \( \tilde{w} \, r \cdot m \) is a cycle of length \( t \) then \( w \) can be chosen so that \( w^r b_m \in \{ b_1, \ldots, b_k \} \) for \( r < t \) and \( w^r b_m = -b_m \). In addition, \( w \in W(D_n) \) if and only if \( \pi \) is even.

**Lemma 3.4** Set \( D = W(D_n) \) and \( B = W(B_n) \). For \( w \in B \) denote by \( w^D \) (respectively, \( w^B \)) the \( D \)- (respectively, \( B \)-) conjugacy class of \( w \). Let \( i \to d_i, j \to e_j \) be the functions which determine \( w^B \).

1. \( C_D(w) = C_B(w) \) if and only if \( w \in D, l = 0 \) and all \( i \) with \( d_i > 0 \) are even. In the exceptional case \( |C_B(w) : C_D(w)| = 2 \).
2. Either \( w^D = w^B \), or \( w \in D, l = 0 \), each cycle of \( \tilde{w} \) has even size and \( w^B \) is a union of two \( D \)-conjugacy classes of equal size.

Proof. Clearly, (1) and (2) are equivalent. For (2), if \( w \in D \), the result is contained in Carter [7, Proposition 25]. Let \( w \notin W(D_n) \). As \( |B : D| = 2 \), it follows that \( C_D(w) \neq C_B(w) \) if and only if \( C_B(w) \) contains an element \( x \notin C_D(w) \). As \( w \notin D \), we can take \( x = w \).

Our next aim is to obtain a formula for \( W(T) \). Recall that \( W(T) \) is defined to be \( (N_G(T)/T)^{Fr} = N_G(T)/T \). It is well known that there is a bijection between the conjugacy classes of \( Fr \)-stable maximal tori of \( G \) and the so called \( Fr \)-conjugacy classes in \( W(G) \), the Weyl group of \( G \) [11, 3.23]. If \( G = B_n(q), D_n^+(q) \) then \( W(G) = W(B_n) \) and \( W(D_n) \), respectively, and the \( Fr \)-conjugacy classes of \( W(G) \) are ordinary conjugacy classes.

The following definition plays an essential role in what follows.

**Definition 3.5** Let \( T \) be a maximal torus in \( G \), and let \( i \to d_i, j \to e_j \) be the corresponding functions. We say that \( T \) is neutral if \( l = 0 \), and exceptional if \( l = 0 \) and all \( i \) with \( d_i > 0 \) are even.

**Lemma 3.6** (1) Let \( H = SO_{2n+1}(q) \) and \( T \) a maximal torus of \( H \) corresponding to the functions \( i \to d_i, j \to e_j \). Then \( |W(T)| = \Pi_i(2i)^{d_i}! \cdot \Pi_j(2j)^{e_j}! \cdot 2^k \cdot m^l \).

(2) Let \( G = SO_{2n}^+(q) \) and \( T \) a maximal torus of \( G \) corresponding to a function \( i \to d_i, j \to e_j \). Then \( |W(T)| = \Pi_i(2i)^{d_i}! \cdot \Pi_j(2j)^{e_j}! \cdot 2^k \cdot m^l \).

(3) Let \( G = SO_{2n}^-(q) \) and \( T \) a maximal torus of \( G \) corresponding to a function \( i \to d_i, j \to e_j \). Then \( |W(T)| = \Pi_i(2i)^{d_i}! \cdot \Pi_j(2j)^{e_j}! \cdot 2^k \cdot m^l \).

(4) Let \( G = SO(V) \), where \( \dim V = 2n \), and \( G = SO_{2n}(T_q) \). Let \( Fr \) be a Frobenius endomorphism of \( G \) such that \( G = G^{Fr} \). Let \( T \) be a maximal \( Fr \)-stable torus of \( G \) and \( T = T^{Fr} \). Then \( N_G(T) \neq N_{O(V)}(T) \), except for the case where \( G = SO^+(2n, q) \) and \( T \) is exceptional.

Proof. (1) See [11, Lemma 2.3].

(2) Let \( T = T_w \), where \( w \in W(G) \cong W(D_n) \). Set \( B = W(B_n), D = W(D_n) \) as in Lemma 3.3.4. Recall that \( W(T) \cong C_{W(G)}(w) = C_D(w) \) [6, 3.3.6]. We view \( G \) as a natural subgroup of \( H \), and also consider \( T \) as a maximal torus of \( H \). Then \( T = T_w \), where \( w \) is viewed as an element of \( W(H) \cong W(B_n) = B \). Then \( W_H(T) \cong C_H(w) \), where we use the
subscript $H$ in $W_H(T)$ to indicate that $T$ is viewed as a torus in $H$. As above, for $w \in D$ denote by $w^D$ (respectively, $w^B$) the conjugacy class of $w$ in $D$ (respectively, $B$). Recall that $|B : D| = 2$. So $w^B \subset D$, and we have $|C_D(w)| = \frac{|D|}{|w^B|}$ as well as $|C_B(w)| = \frac{|B|}{|w^B|}$. By Lemma 3.4, either $w^D = w^B$ or $w^B$ is a union of two $D$-conjugacy classes of equal size. In the former case $|C_D(w)| = \frac{|D|}{|w^B|} = \frac{|D|}{2|w^B|} = \frac{1}{2}|C_B(w)|$, in the latter case, we have $|C_D(w)| = |C_B(w)|$. So (2) follows from (1) and the fact that $w^D \neq w^B$ if and only if all $i$ with $d_i > 0$ are even.

(3) Now $G = SO_{2n}^-(q)$. The $G$-conjugacy classes of maximal tori are in bijection with the $Fr$-conjugacy classes in $W(G) \cong W(D_n) = D$. These are defined in terms of the action of $Fr$ on $W(G) = D$. This is known to be realized as the conjugation by some element $a \in B \setminus D$, see [23] §11 where the action in question is explicitly described. Furthermore, an $Fr$-conjugacy class of $D$ is of shape $\{x^{-1}wax^{-1} : x \in D\}$, which is the orbit of $w$ under the action $w \to x^{-1}wax^{-1}$, $x \in D$. By [6, 3.3.6], $W(T_w)$ is isomorphic to the point stabilizer of $w$ in $D$ under this action. The mapping $w \to wa$ transforms the action in $D$-conjugacy action on $B \setminus D$. Therefore, $W(T_w) \cong C_B(wa)$. By Lemma 3.4, for every $w \in Da$ the $D$-conjugacy class coincides with $B$-conjugacy class. This implies the equality $2 \cdot |C_D(w)| = |C_B(w)|$ as $|B : D| = 2$. So the result follows.

(4) Set $\overline{G} = O(V)$. The natural embedding $G \to H = SO_{2n+1}(q)$ extends to the embedding $\overline{G} = O_{2n}^+(q) \to H$ in an obvious way. Let $H = SO_{2n+1}(\overline{F}_q)$. Then the above embedding extends to an embedding $e : O_{2n}(\overline{F}_q) \to H$. Observe that $e(N_{O_{2n}(\overline{F}_q)}(T)) = N_H(T)$. Indeed, $e(T)$ fixes a unique a singular line $U$ on $\overline{V}$ (the natural $H$-module), and hence $N_H(T)$ stabilizes $U$. As the stabilizer of $U$ in $H$ coincides with $e(O_{2n}(\overline{F}_q))$, the claim follows. Recall that $W(T) = (N_G(T)/T)^{Fr}$ and $W_H(T) = (N_H(T)/T)^{Fr}$. Clearly, $(N_H(T)/T)^{Fr} \cong N_G(T)/T$. If $G = SO_{2n}^+(q)$ then (4) follows in this case by comparison of the items (1) and (2) above. If $G = SO_{2n}^-(q)$ then (3) implies that $N_G(T) \not\subseteq N_G(T)$, which yields the result.

**Lemma 3.7** (1) Let $G = SO(V) \cong SO_{2n}^+(q)$. Then the functions $i \to d_i, j \to e_j$, where $l = \sum_j e_j$ is odd, determine a maximal torus in $G$ up to conjugacy. In particular, $G$ has no neutral maximal torus.

(2) Let $G = SO(V) \cong SO_{2n}^-(q)$. Then the functions $i \to d_i, j \to e_j$, where $l = \sum_j e_j$ is even, determine a maximal torus in $G$ up to conjugacy, unless this is exceptional. In the latter case there are two non-conjugate maximal tori corresponding to the function $i \to d_i$.

**Proof.** Let $T, T'$ be maximal tori in $G$ corresponding to the same pair of functions $i \to d_i, j \to e_j$. Let $T, T'$ be maximal $Fr$-stable tori of $G$ such that $T = T^{Fr}, T' = T'^{Fr}$. By Lemmas 3.2 and 3.3, $gTg^{-1} = T'$ for some $g \in O(V)$. If $T$ is not exceptional, then $N_{O(V)}(T)$ is not contained in $SO(V)$ (Lemma 3.6(4)), so there is $x \in O(V) \setminus SO(V)$ such that $xT^{-1} = T$. If $g \not\in SO(V)$ then $gx \in SO(V)$, and $gxT(gx)^{-1} = T'$. So in this case the lemma follows, in particular, (1) holds (see Definition 3.5). Suppose that $T$ is exceptional. Then $V$ is of Witt defect 0, and $SO(V) \cong SO_{2n}^-(q)$. Let $S$ denote the $O(V)$-orbit of $T$. If $S$ coincides with the $G$-orbit of $T$ then $N_{O(V)}(T) \neq N_G(T)$. This is not the case by Lemma 3.6(4).

Remark. Lemma 3.7 is equivalent to saying that an $O(V)$-conjugacy class of maximal tori in $SO(V)$ consists of a single $SO(V)$-conjugacy class if the tori of $S$ are non-exceptional, otherwise it consists of two $SO(V)$-conjugacy classes.
4 The Steinberg-plus character and its Curtis dual

For a finite reductive group $H$ one can define an irreducible character $St_H$ called the Steinberg character. We use for $St_H$ the definition in [11, 9.3], which simultaneously gives the values of $St_H$. Specifically, $St_H$ is defined to vanish at all non-semisimple elements, whereas for the semisimple elements $h \in H$ one defines

$$St_H(h) = \varepsilon_h \varepsilon_{C_H(h)} \cdot |C_H(h)|_p,$$

where $C_H(h)$ is the centralizer of $h$ in $H$, $|C_H(h)|_p$ is the $p$-part of the order $|C_H(h)|$ of $C_H(h)$ and $p$ is the defining characteristic of $H$. For the meaning of $\varepsilon_h, \varepsilon_{C_H(h)}$, see Notation.

Let $G = SO_{2n-1}(q)$. A priori, $St_{H|G}$ may depend on the choice of $SO^+_{2n}(q)$ or $SO^-_{2n}(q)$ for $H$. We show that, in fact, $St_{H|G}$ does not depend on the choice of $H \in \{SO^+_2(q)\}$. Furthermore, if $q$ is odd, there are two non-equivalent embeddings $G \rightarrow H^+$ (as well as $G \rightarrow H^-$) obtained as follows. Let $H = SO(V)$ and let $Q$ be the quadratic form defining $H$. Let $v, v' \in V$ be non-singular vectors such that $Q(v)/Q(v')$ is a non-square in $F_q$. Then $G \cong Stab_H(v) \cong Stab_H(v')$, but the groups $Stab_H(v)$, $Stab_H(v')$ are not conjugate in $H$. Therefore, $St_{H|G}$ may depend on the choice $Stab_H(v)$ or $Stab_H(v')$ for $G$. In fact, this is not the case. This means that the character $St_{H|G}$ depends on $G$ only, and we shall call it the Steinberg-plus character and denote by $St^+_G$. We keep this term for the restriction $St_{H|G}$ when $G = SO^\pm_{2n}(q)$ and $H = SO_{2n+1}(q)$ (in this case there is no ambiguity in the definition).

**Proposition 4.1** Set $G = SO_{2n-1}(q), H^+ = SO^+_{2n}(q)$ and $H^- = SO^-_{2n}(q)$. Then $St_{H^+|G} = St_{H^-|G}$. In addition, both the left and the right hand sides do not depend on the choice of a natural embedding $G \rightarrow H^+$ and $G \rightarrow H^-$. Proof. It suffices to show that the equality holds at the semisimple elements of $G$ as both the characters vanish at all non-semisimple elements.

Recall that the Steinber character of any finite reductive group $X$ is of defect 0, and hence remains irreducible under reduction modulo $p$. Furthermore, the representation obtained is the restriction to $X$ of a representation $\tau$, say, of the algebraic group $X$ [28, Theorem 43]. We apply this to $X = H^+$ and $X = H^-$, and observe that $X$ is the same in both the cases. (One can assume that $X = SO_{2n}(\mathbb{F}_q)$ and $G = SO_{2n}(\mathbb{F}_q)$). In addition, the following diagrams are commutative:

$$
\begin{array}{ccc}
G \rightarrow H^+ & & G \rightarrow H^- \\
\downarrow & & \downarrow \\
G \rightarrow X & & G \rightarrow X
\end{array}
$$

Therefore, $G \rightarrow H^+ \rightarrow X$ and $G \rightarrow H^- \rightarrow X$ yield the same embedding $G \rightarrow X$. Furthermore, the groups $Stab_H(v)$ and $Stab_H(v')$ (where $v, v' \in V$ are chosen as in the paragraph prior the lemma) are conjugate in $SO_{2n}(\mathbb{F}_q)$. Therefore, $\tau|_G$ is independent from the choice of the embedding $G \rightarrow H$, and lemmas follows.

We denote by $\omega_G$ the Curtis dual of the character $St^+_G = St_{H|G}$ [11, 8.8] or [6, Ch.8, §2]. So $\omega_G$ is a generalized character of $G$.

**Lemma 4.2** Let $\Phi$ be a class function on $G$ vanishing at all non-semisimple elements, and let $\phi$ be the Curtis dual of $\Phi$. Let $g \in G$. Then $\phi(g) = \Phi(g)/St_G(g)$ if $g$ is semisimple. In general, if $g = su$, where $s$ is semisimple, $u$ is unipotent and $su = us$ then $\phi(g) = \phi(s)$.
Proof. This follows from Broué [3, Theorem 1 and §[2]], see also [20][ Proof of Lemma 4.1], where this result is deduced from Carter [6][ 7.6.4].

As $St_H|G$ vanishes at all non-semisimple elements of $G$, it follows that $\omega_G(g) = \frac{St_H(g)}{St_G(g)}$ for all semisimple elements $g \in G$ (see [3] or [20] Lemma 4.1). Furthermore, if $g = su$, where $s$ is semisimple, $u$ is unipotent and $su = us$ then $\omega_G(g) = \omega_G(s)$.

Lemma 4.3 Let $V$ be the natural $F_q G$-module. Then $\omega_G(g) = \pm \omega_{SO(V^g)}(1) = \pm q^{[\dim V^g/2]}$. In particular, $\omega_G(1) = q^{[\dim V^g/2]}$.

Proof. Suppose first that $g = 1$. If $\dim V = 2n$ then $St_G(1) = |G|_p = q^{n(n-1)}$ whereas $St_H(1) = |H|_p = q^{n^2}$. If $\dim V = 2n+1$ then $St_G(1) = |G|_p = q^{n^2}$ whereas $St_H(1) = |H|_p = q^{n(n+1)}$. In both the cases $\omega_G(1) = q^n$ as required. In general, let $V \subset W$, where $W$ is an orthogonal space of dimension $1 + \dim V$ and $H = SO(W)$. We can write $V = V^q \oplus (1-g)W$ and $W = W^q \oplus (1-g)W$. Note that $(1-g)V = (1-g)W$. Set $g' = g_{(1-g)W}$. As $g$ fixes no non-zero vector in $(1-g)V$ and $(1-g)W$, we have $|C_G(g)|_p = |SO(V^q)|_p \cdot |C_{SO(1-g)W}(g')|_p$ (at least for $p > 2$), and $|C_H(g)|_p = |SO(W^q)|_p \cdot |C_{SO(1-g)W}(g')|_p$. Therefore, $\omega_G(g) = \frac{St_H(g)}{St_G(g)} = \pm \frac{|SO(V^q)|}{|SO(V^q)|} = \omega_{SO(V^q)}(1)$, and the lemma follows.

Examples. Here we consider some degenerate cases where $n = 1, 2, 3$.

1) $n = 1$. Then $G = \{1\}$ and $H = SO^+_{2}(q)$. Therefore, $\omega_G = St_H(1) \cdot 1_G$. As $St_H(1)$ is the identity, we have $St_H(1) = 1$, so $\omega_G = 1_G$.

2) $n = 2$. The group $G = SO^\pm_{2}(q)$ is abelian of order $q + 1$. In particular, $G$ has no non-trivial $p$-element, so $|C_G(g)|_p = 1$ for $1 \neq g \in G$. It follows that $St_G = 1_G$. Therefore, $\omega_G = St_G$. Recall that $H = SO_3(q)$ and $St_H(1) = q$. One observes that $C_H(g)$ is a $p'$-group for every $1 \neq g \in G$, so $St_H(g) = \pm 1$. Then the $F_q$-rank of $SO^\pm_{2}(q)$ equals 1, while the $F_q$-rank of $SO^-_{2}(q)$ equals 0. As $G$ is abelian, $C^n_H(g) \cong G$ for every $1 \neq g \in G$. So for $1 \neq g \in G$ we have

$$\omega_G(g) = St_H(g) = \begin{cases} 1 & \text{if } G = SO^+_{2}(q) \\ -1 & \text{if } G = SO^-_{2}(q). \end{cases}$$

One easily deduces from this, that $\omega_G = \rho_G^\text{reg} + 1_G$ if $G = SO^+_{2}(q)$, and $\rho_G^\text{reg} - 1_G$ if $G = SO^-_{2}(q)$.

3) $n = 3$. Here $G = SO_3(q) \cong PGL_2(q)$, and $H = SO^+_{4}(q)$.

Group $G$ has two maximal tori $T_1, T_2$ of order $q - 1$ and $q + 1$, respectively. Let $V$ be the natural module for $H$. One observes that if $1 \neq t \in G$ is a semisimple then $(t - 1)V$ and $V^t$ are 2-dimensional non-degenerate subspaces of $V$ orthogonal to each other, and $V = (t - 1)V \oplus V^t$. It follows from this that $|C_G(t)|_p = 1$. Furthermore, it is easy to observe that $St_H(t) = 1$, respectively, $-1$ if $t \in T_1$, respectively, $t \in T_2$. In addition, $C_G(t) = 1$, respectively, $-1$ if $t \in T_1$, respectively, $t \in T_2$. In fact $\omega_G(g) = 1$, respectively, $-1$ if $t \in T_1$, respectively, $t \in T_2$. (Note that $\omega_G(u) = q$ for every unipotent element $u \in G$; in particular, $\omega_G(1) = q$.)

4.1 Multiplication theorem

Lemma 4.4 Let $V$ be an orthogonal space, and $V = V_1 \oplus V_2$, where $V_1, V_2$ are non-degenerate subspaces of $V$ orthogonal to each other. Set $G_1 = SO(V_1)$, $G_2 = SO(V_2)$. Let $g \in G$ be a semisimple element such that $gV_i = V_i$, and set $g_i = g|_{V_i}$ for $i = 1, 2$. 

Lemma 4.5
Let $V$ be an orthogonal space, and $V = V_1 \oplus V_2$, where $V_1, V_2$ are non-degenerate.

(i) Suppose that at least one of $\dim V_1, \dim V_2$ is even. Then there is a linear character $\lambda$ of $G_1G_2$ such that $\lambda^2 = 1$ and $\omega_G(g) = \lambda(g) \cdot \omega_{G_1}(g) \cdot \omega_{G_2}(g)$ for every semisimple element $g \in G_1G_2$.

(ii) Suppose that both $\dim V_1, \dim V_2$ are odd. Then there is a linear character $\lambda$ of $G_1G_2$ such that $\lambda^2 = 1$ and $\omega_G(g) = \lambda(g) \cdot q \cdot \omega_{G_1}(g) \cdot \omega_{G_2}(g)$ for every semisimple element $g \in G_1G_2$.

In addition, $\lambda(u) = 1$ for all unipotent elements $u \in G_1G_2$.

Proof. Applying Theorem 2.12 to $X = G_1G_2$, $\phi = \omega_{G_1}|_{G_1G_2}$ and to $\psi = \omega_{G_1} \cdot \omega_{G_2}$, we have the equality required at the semisimple elements $g$. Lemma 4.4 guarantees that the hypothesis of Theorem 2.12 holds.

Furthermore, let $\lambda'$ be the linear character of $G_1G_2$ such that $\lambda(s) = \lambda'(s)$ at the semisimple elements and $\lambda'(1) = 1$ for all unipotent elements of $G_1G_2$. Let $x = gu$, where $g$ is semisimple, $u$ is unipotent and $gu = ug$. We know that $\omega_G(x) = \omega_G(g)$ and $\omega_{G_i}(x) = \omega_{G_i}(g)$ for $i = 1, 2$. So in (1) we have $\omega_G(x) = \omega_G(g) = \lambda(g) \cdot \omega_{G_1}(g) \cdot \omega_{G_2}(g) = \lambda(g) \cdot \omega_{G_1}(x) \cdot \omega_{G_2}(x)$, and similarly in (2).

Remarks. (1) If $\dim V_2 = 1$ then $\omega_{G_2} = 1_{G_2}$, so $\omega_G(g) = \lambda(g) \cdot \omega_{G_1}(g)$. If $\dim V_2 = 2$ then $G_2$ is a torus, but $\omega_{G_2} \neq 1_{G_2}$. (2) If one replaces $G, G_1, G_2$ by the subgroup $G', G'_1, G'_2$ of index 2 in $G, G_1, G_2$, respectively, then $\lambda|_{G'_1G'_2} = 1_{G'_1G'_2}$.

Lemma 4.6
The character $\lambda$ in Lemma 4.5 is trivial.

Proof. If $q$ is even then semisimple elements of $G$ are of odd order. As $\lambda^2 = 1$ and $\lambda$ is trivial at every unipotent element, we conclude that $\lambda$ is trivial.

Let $q$ be odd. Suppose the contrary. Then $\lambda_i \neq 1$ for $i \in \{1, 2\}$. Fix any $i$ with $\lambda_i \neq 1$.

Suppose first that $\dim V_i$ is even for $i = 1, 2$. Let $G = G_1 \times G_2$, and let $\lambda$ be a character of $G_1G_2$ of order 2. Then $\lambda = \lambda_1\lambda_2$, where $\lambda_i = \lambda|_{G_i}$.

Note that $G_i$ has a subgroup of index 2 [23, 2.5.7]. As $\Omega(V_i)$ has index 2 in $G_i$, it follows that if $\lambda_i(G_i) \neq 1$ then $\lambda_i(x) = 1$ (x in $G_i$) if and only if $x \in \Omega(V_i)$. By Lemma 2.10 there is $t \in SO(V_i) \setminus \Omega(V_i)$ such that $|C_{G_i}(t)|_p = |C_{G_i}(t^2)|_p$ and $V_i^t = V_i^t$.

View $t$ as an element of $G$. Then $\omega_G(t) = \lambda_i(t)\omega_{G_i}(t) = -\omega_{G_i}(t)$ as $\lambda_i^2 = 1$. By Lemma 2.11 $|C_G(t)|_p = |C_G(t^2)|_p$ and $|C_H(t)|_p = |C_H(t^2)|_p$; in addition $C_G(t)^0 = C_G(t^2)^0$ and $C_H(t)^0 = C_H(t^2)^0$. Therefore, $St_H(t) = St_H(t^2)$ and $St_G(t) = St_G(t^2)$. This yields $\omega_G(t) = St(t)/St_G(t) = \omega_G(t^2)$. Then $-\omega_{G_i}(t) = \omega_G(t) = \omega_G(t^2) = \lambda_i(t^2)\omega_{G_i}(t^2) = \omega_{G_i}(t^2) = \omega_G(t)$, which is a contradiction.
**Corollary 4.7** Let $V$ be an orthogonal space of odd dimension $n$ over $F_q$, and let $V'$ be a non-degenerate subspace of dimension $n - 1$ in $V$. Set $G = SO(V)$ and $X = SO(V')$ if $q$ is odd, otherwise set $G = \Omega(V)$ and $X = \Omega(V')$. View $X$ as a subgroup of $G$. Then $\omega_G|_X = \omega_X$.

Proof. If $q$ is odd then the corollary straightforwardly follows from Lemmas 4.5 and 4.6. Suppose that $q$ is even. In this case we have $V = V_1 + V'$, where $\dim V_1 = 1$ however, $V_1$ is not non-degenerate. Nonetheless, the formula $\omega_G(g) = \pm q^{(1+\dim V')/2} = \pm q^{\dim V'/2}$ remains true as $\dim V_2^{q}$ is even. This implies $\omega_G(g) = \pm \omega_X(g)$ for $g \in X$. The reasoning in the proofs of Lemmas 4.5 and 4.6 remain valid, whence the result.

**Proof of Theorem 1.3** The result follows from Lemmas 4.5 and 4.6.

The standard embedding $GL_n(q) \rightarrow SO_2n^+(q)$ is well known. This comes from fixing a maximal totally singular subspace $W$ of $V$. Every basis $b_1, \ldots, b_n$ of $W$ can be extended to a basis $b_1, \ldots, b_{2n}$ of $V$ such that the Gram matrix of this basis is

$$
\begin{pmatrix}
0 & \text{Id}_n \\
\text{Id}_n & 0
\end{pmatrix}
$$

Then the matrix $\text{diag}(g, g^{-1})$ belongs to $SO_2n^+(q)$ for every $g \in GL_n(q)$. (Here $t^g$ denotes the transpose of $g$.)

If $n$ is even (respectively, odd) then there is an embedding $U(n, q) \rightarrow O^+_{2n}(q)$ (respectively, $O^-_{2n}(q)$). These can be obtained as follows. Let $W$ be the natural $F_q^r$-module for $H = U_n(q)$. There is a surjective embedding of $h : W \rightarrow V$, where $V$ is an orthogonal $F_q$-space of dimension $2n$, such that every non-degenerate one-dimensional subspace of $W$ goes to a two-dimensional anisotropic subspace. Moreover, $h$ regards the orthogonality relation, that is, if subspaces $X, Y$ of $W$ are orthogonal then so are $h(X), h(Y)$. Therefore, $V$ is an orthogonal sum of two-dimensional anisotropic subspaces. It follows from [23, 2.5.11] that the Witt defect of $V$ equals 0 if $n$ is even, otherwise equals 1. Therefore, $h$ yields an embeddings $e : U_n(q) \rightarrow O^+_{2n}(q)$ if $n$ is even, and $U_n(q) \rightarrow O^-_{2n}(q)$ if $n$ is odd. In addition, $V^e(g) = h(W^g)$ and $2\dim W^g = \dim h(W^g) = \dim V^e(g)$ for $g \in U_n(q)$. In fact, $e(U_n(q))$ is contained in $G$. This is trivial if $q$ even, moreover, in this case $e(U_n(q))$ is contained in $\Omega_{2n}^+(q)$ as $U_n(q) : SU_n(q)$ is odd. Let $q$ be odd. Note that $U_n(q)$ contains a central element $z$, say, of order greater than 2. Then $e(z)$ does not have eigenvalue $\pm 1$, and hence the claim follows from Lemma 2.1.

Furthermore, if $t, n$ are odd then there is an embedding $U(n, q^t) \rightarrow U_{2n}(q)$ [22]. In particular, $U_1(q^t)$ embeds into $U_1(q)$ and hence into $SO_{2n}^+(q)$.

We use Gerardin’s definition of the Weil representation of a unitary group. This is the one afforded by the character, which is defined for $g \in U_m(q)$ by $\chi(g) = (-1)^m(-q)^{\dim W^g}$.

**Lemma 4.8** (1) Let $G = SO_{2n}^+(q)$ and $X \subset G$ a standard subgroup isomorphic to $GL_n(q)$. Then there is a linear character $\lambda$ of $X$ such that $\omega_G(g) = \lambda(g)\pi_X(g)$ for all semisimple elements $g \in X$, where $\pi_X$ is the character of the permutation representation of $X$ associated with the action of $X$ on the vectors of $F_q^m$.

(2) Let $G = SO_{2n}^+(q)$ or $SO_{2n}^-(q)$. Let $Y \subset G$ be a subgroup isomorphic to $U_n(q)$. Then there is a linear character $\lambda$ of $Y$ such that $\omega_G(g) = \lambda(g)\phi(g)$ for all semisimple elements $g \in Y$, where $\phi$ is the character of the Weil representation of $Y$.

Proof. (1) By the formula for $g \in G$ in by Lemma 4.3 we have $\omega_G(g) = \pm q^{\dim V^g/2}$. As the Witt index of $V$ is equal to $n$, there is a basis $b_1, \ldots, b_{2n}$ in $V$ with Gram matrix

$$
\begin{pmatrix}
0 & \text{Id}_n \\
\text{Id}_n & 0
\end{pmatrix}
$$

Then the elements $g := \text{diag}(x, x^{-1})$ with $x \in GL_n(q)$ preserve the Gram
matrix so \( g \in SO_{2n}^\pm(q) \). Let \( U_1 = (b_1, \ldots, b_n), U_2 = (b_{n+1}, \ldots, b_{2n}) \). Then \( V^g = U_1^g + U_2^g \). Therefore, \( \omega_G(g) = \pm q^{\dim V^g} \). Note that \( \pi(g) = q^{\dim U_1^g} \). So (1) follows from the comparison theorem.

(2) Let \( g \in U_n(q) \) be a semisimple element. As the Weil character value at \( g \) is \((-1)^n(-q)^{\dim W^g} \) and \( \omega_G(e(g)) = \pm q^{\dim V(e(g))/2} \) (Lemma 4.3), the result again follows by the comparison theorem.

**Lemma 4.9** The character \( \lambda \) in Lemma [1.8] is trivial.

Proof. If \( n > 2 \) then the derived subgroup \( G' \) of \( G = SO_n^\pm(q) \) has index 2, so \( \lambda^2 = 1_G \). We show that \( \lambda = 1_G \). Suppose the contrary.

(a) \( X = GL_n(q) \). Let \( t \in G \) be of order \( q^n - 1 \). Then it is known that \( t \notin G' \), and hence \( \lambda(t) = -1 \) and \( \lambda(t^2) = 1 \). As both \( t \) and \( t^2 \) are regular semisimple in \( G \) and in \( H \), we have \( C_H(t) = C_H(t^2) \) and \( C_G(t) = C_G(t^2) \). Therefore, we have

\[
\omega_G(t) = St_H(t)/St_G(t) = \frac{\varepsilon_H \varepsilon_{C_H(t)}|C_H(t)|_p}{\varepsilon_G \varepsilon_{C_G(t)}|C_G(t)|_p} = \omega_G(t^2).
\]

Now \( \lambda(t) = \omega_G(t) = \omega_G(t^2) = \pi(t^2) = \pi(t) \), a contradiction.

(b) \( X = U_1(q^n) \). Let \( t \in G \) be of order \( q^n + 1 \). Then it is known that \( t \notin G' \), and hence \( \lambda(t) = -1 \) and \( \lambda(t^2) = 1 \). Let \( \phi \) denote the Weil representation of \( U(1, q) \). Then \( \phi(t) = (-q^n)^0 = \phi(t^2) = -1 \). As both \( t \) and \( t^2 \) are regular semisimple in \( G \), \( H \), we have \( C_H(t) = C_H(t^2) \) and \( C_G(t) = C_G(t^2) \). Therefore, we have

\[
\omega_G(t) = St_H(t)/St_G(t) = \frac{\varepsilon_H \varepsilon_{C_H(t)}|C_H(t)|_p}{\varepsilon_G \varepsilon_{C_G(t)}|C_G(t)|_p} = \omega_G(t^2).
\]

Now \( \lambda(t) = \omega_G(t) = \omega_G(t^2) = \phi(t^2) = \phi(t) \), a contradiction.

5 Maximal tori and Curtis dual

5.1 Character formula

In this section we give a formula that describes in a convenient way the restriction \( \omega_G|_T \). Recall that maximal tori in \( G \) are determined up to \( O(V) \)-conjugation by the orthogonal decompositions \( \square \), which in turn yields an "orthogonal" decomposition \( \square \) for the corresponding torus. The formula depends on the \( O(V) \)-conjugacy class of a torus in question, rather than on the \( SO(V) \)-conjugacy class. This is not surprising as \( SO_{2n+1}(q) \) contains \( O_{2n}^\pm(q) \), and hence \( St_G^\pm \) is invariant under \( O(V) \) for \( G = SO(V) = SO_{2n}^\pm(q) \).

First, we deal with two special cases:

**Lemma 5.1** (1) Let \( G = SO_{2n}^+(q) \) and \( T \) a maximal torus of order \( q^n - 1 \) (that is, \( T \) corresponds to an orthogonal decomposition with \( k = 1, l = 0 \)). Then \( \omega_G|_T = \rho_T^{reg} + 1_T \).

(2) Let \( G = SO_{2n}^-(q) \) and \( T \) a maximal torus of order \( q^n + 1 \) (that is, \( T \) corresponds to an orthogonal decomposition with \( k = 0, l = 1 \)). Then \( \omega_G|_T = \rho_T^{reg} - 1_T \).
Proof. This follows from Lemma 4.8.

Next, as in [19], Theorem 4.3 allows us to express the character of \( \omega_G|_T \) in terms of the characters \( \omega_{G_i}|_{T_i} \). In turn, \( \omega_{G_i}(T_i) \) can be expressed in terms of the regular character \( \rho_{T_i}^{\text{reg}} \) and the trivial character \( 1_{T_i} \). (For a group \( H \) we denote by \( \rho_H^{\text{reg}} \) the character of the regular representation of \( H \)).

**Theorem 5.2** Let \( G = SO(V) \) if \( q \) is odd, and \( G = \Omega(V) \) if \( q \) is even. Let \( T = T_1 \times \cdots \times T_k \) be a maximal torus of \( G \) as in (1). Then
\[
\omega_G|_T = (\rho_{T_1}^{\text{reg}} + 1_{T_1}) \otimes \cdots \otimes (\rho_{T_k}^{\text{reg}} + 1_{T_k}) \otimes (\rho_{T_{k+1}}^{\text{reg}} - 1_{T_{k+1}}) \otimes \cdots \otimes (\rho_{T_{k+l}}^{\text{reg}} - 1_{T_{k+l}}).
\]

Proof. If \( \dim V \) is even, the result follows from Theorem 1.3 and Lemma 5.1. Suppose that \( \dim V \) is odd. Then, by Lemma 2.5, \( T \) stabilizes a non-degenerate subspace \( V' \), say, of \( V \). So \( T \subset X \), where \( X \cong SO(V) \) if \( q \) is odd and \( X \cong \Omega(V') \) if \( q \) is even. So the result follows from that for the case where \( \dim V \) is even.

**Corollary 5.3** The restriction of \( \omega_G \) to every maximal torus \( T \) of \( G \) is a proper character of \( T \).

Examples. (1) \( G = SO_3(q) \). We have two maximal tori \( M_1, M_2 \) of order \( q - 1, q + 1 \), respectively. We have \( \omega_G|_{M_1} = \rho_{M_1}^{\text{reg}} + 1_{M_1} \) and \( \omega_G|_{M_2} = \rho_{M_2}^{\text{reg}} - 1_{M_2} \).

(2) \( G = SO_4^\pm(q) \), \( q \) odd. Up to conjugacy, there are two maximal tori \( M_1, M_2 \) of order \( q^2 - 1, q^2 + 1 \), respectively. They correspond to the decompositions with \( k = l = 1 \) and \( k = 0, l = 1 \), respectively. (That is, \( \dim V_1 = \dim V_2 = 1 \) in (1) in the former case, and \( \dim V_1 = 4 \) with \( l = 1 \) in the latter case. Note that \( M_1 \) is not a cyclic group.) So \( M_1 = T_1 \times T_2 \), where \( |T_1| = q - 1, |T_2| = q + 1 \). We have \( \omega_G|_{M_1} = (\rho_{T_1}^{\text{reg}} + 1_{T_1}) \otimes (\rho_{T_2}^{\text{reg}} - 1_{T_2}) \) and \( \omega_G|_{M_2} = \rho_{M_2}^{\text{reg}} - 1_{M_2} \). (Note that \( G \cong PSL_2(q^2) \) if \( q \) is even, otherwise \( G \cong PSL_2(q^2) \times (\text{Id}) \). Indeed, \( \Omega_4^\pm(q^2) \cong PSL_2(q^2) \) [23, 29.1], and \( -\text{Id} \in SO_4^\mp(q^2) \setminus \Omega_4^\pm(q^2) \). As \( |SO_4^\pm(q^2) : \Omega_4^\pm(q^2)| = 2 \), the claim follows.)

(3) \( G = SO_4(q) \). We have four maximal tori \( M_1, M_2, M_3, M_4 \) of orders \( (q - 1)^2, (q + 1)^2, q^2 - 1, q^2 + 1 \), respectively. They correspond to the decompositions with \( (k, l) = (2, 0), (0, 2) \), and remaining two tori correspond to \( (k, l) = (1, 0) \), that is, \( \dim V_k = 4 \) for \( k = 1 \). We have \( \omega_G|_{M_1} = (\rho_{T_1}^{\text{reg}} + 1_{T_1})(\rho_{T_2}^{\text{reg}} + 1_{T_2}), \omega_G|_{M_2} = (\rho_{T_1}^{\text{reg}} - 1_{T_1})(\rho_{T_2}^{\text{reg}} - 1_{T_2}) \), and \( \omega_G|_{M_3} = \omega_G|_{M_4} = \rho_{T_1}^{\text{reg}} - 1_{T_1} \) in notation of Theorem 5.2. (Note that \( G \cong SL_2(q) \times SL_2(q) \) if \( q \) is even, otherwise \( G \cong (SL_2(q) \circ SL_2(q)) \cdot 2 \).)

(4) Let \( G = SO_2(q) \), \( q \) odd. Then \( \omega_G(-\text{Id}) = \alpha \). Indeed, if \( \alpha = 1 \) then \( -\text{Id} \) belongs to a maximal torus with \( k = n = 0 \), and the claim follows from Theorem 5.2 as \( l = 0 \). Alternatively, one can use Lemma 5.4. If \( \alpha = -1 \), then \( -\text{Id} \) belongs to a maximal torus with \( k = n - 1 = l = 1 \), and again apply Theorem 5.2.

**Lemma 5.4** Let \( G \) and \( \nu \) be as in Theorem 5.2. Let \( \theta = \theta_1 \otimes \cdots \otimes \theta_{k+l} \) be an irreducible character of \( T = T_1 \times \cdots \times T_{k+l} \).

1. If \( \theta_{k+j} = 1_{T_{k+j}} \) for some \( j > 0 \) then \( \theta \) does not occur as an irreducible constituent of \( \omega_G|_T \) (that is, \( \langle \omega_G|_T, \theta \rangle = 0 \)).

2. Suppose that \( \theta_{k+j} \neq 1_{T_{k+j}} \) for every \( j = k + 1, \ldots, l \). Let \( k(\theta) \) be the number of \( 0 \leq i \leq k \) such that \( \theta_i = 1_{T_i} \). Then \( \langle \omega_G|_T, \theta \rangle = 2^{k(\theta)} \).

3. Suppose that \( \theta_i \neq 1_{T_i} \) for every \( 1 \leq i \leq k + l \). Then \( \langle \omega_G|_T, \theta \rangle = 1 \).
Proof. This follows from Theorem 5.2.

Lemma 5.5 can be restated in terms of the dual group $G^*$. Note that the dual of $SO_{2n}^+(q)$ (respectively, $SO_{2n}^-(q)$, respectively, $SO_{2n+1}(q)$) is isomorphic to $SO^*_{2n}(q)$ (respectively, $SO_{2n}^-(q)$, respectively, $Sp_{2n}(q)$), see [3, p. 120]. Let $V^*$ denote the natural module for $G^*$, and we assume that $V^*$ endows a bilinear or quadratic form defining $G^*$.

A torus $T^*$ of $G^*$ dual to $T$ has the same structure as $T$, and in fact there is an orthogonal decomposition $V^* = V^*_1 \oplus \cdots \oplus V^*_k \oplus V^*_{k+1} \oplus \cdots \oplus V^*_{k+l}$ with the same properties for $T^*$ as those for $T$ on $V$. The only difference is that we do not have $V^*_0$ anymore. In particular, $\dim V^*_i = \dim V_i$ and $|T^*_i| = |T_i|$ for $i = 1, \ldots, k+l$. In addition, $T^* = T_1^* \times \cdots \times T_{k+l}^*$, and $T^*_i$ can be viewed as a maximal torus of $SO(V^*_i)$. This also tells us that if $\theta = \theta_1 \otimes \cdots \otimes \theta_{k+l}$ is a linear character of $T$ then the corresponding element $s \in T^*$ can be expressed as $(s_1, \ldots, s_{k+l})$, where $s_i \in T_i^*$ for $i = 1, \ldots, k+l$. In addition, one observes that $\theta_i = 1_{T_i}$ if and only if $s_i = 1$.

Now, we can restate Lemma 5.5 as follows:

**Lemma 5.5** Let $s \in T^*$ be a semisimple element and a maximal torus in $G^*$, which correspond to a pair $(T, \theta)$ under the duality $G \to G^*$. Let $s = \text{diag}(s_1, \ldots, s_{k+l})$, where $s_i$ means the restriction of $s$ to $V^*_i$.

1. If $s_{k+j} = 1$ for some $j > 0$ then $(\omega_G|_T, \theta) = 0$ (that is, $\theta$ does not occur as an irreducible constituent of $\omega_G|_T$).

2. Suppose that $s_{k+j} \neq 1$ for every $j = k+1, \ldots, l$. Let $m(\theta)$ be the number of $0 \leq i \leq k$ such that $s_i = 1$. Then $(\omega_G|_T, \theta) = 2^{m(\theta)}$.

3. Suppose that $s_i \neq 1$ for every $1 \leq i \leq k+l$. Then $(\omega_G|_T, \theta) = 1$.

**Corollary 5.6** $(\omega_G|_T, \theta) = 0$ if and only if the 1-eigenspace of $s$ on $V^*$ is non-zero and not contained in $V^*_1 \oplus \cdots \oplus V^*_k$.

**Corollary 5.7** Let $s \in G^*$, and let $T^*$ be a maximal torus of $G^*$ containing $s$. Let $\theta \in \text{Irr}T$ corresponds to $s \in T^*$. Suppose that $s$ does not have eigenvalue 1 on the natural module for $G^*$. Then $(\omega_G|_T, \theta) = 1$.

**Corollary 5.8** Let $s \in G^*$, and let $T^*$ be a maximal torus of $G^*$ containing $s$. Let $\theta \in \text{Irr}T$ corresponds to $s \in T^*$. Let $V^*$ be the natural module for $G^*$, let $V^*_s$ be the 1-eigenspace of $s$ on $V^*_s$ and let $V^* = V^*_1 \oplus \cdots \oplus V^*_k \oplus V^*_{k+1} \oplus \cdots \oplus V^*_k \oplus V^*_{k+l}$ a $T^*$-decomposition of $V^*$. Suppose that $(\omega_G|_T, \theta) > 1$. Then $V^*_s \neq 0$ and $V^*_s \subseteq V^*_1 \oplus \cdots \oplus V^*_k$. In particular, $V^*_s$ is of Witt defect 0.

### 5.2 The characteristic 2 case

Let $G$ be a simple algebraic group of rank $r$ and let $\lambda_1, \ldots, \lambda_r$ denote the fundamental weights of $G$. The irreducible representations of $G$ are parametrized by the dominant weights $\lambda$ of $G$, which are linear combinations $a_1 \lambda_1 + \cdots + a_r \lambda_r$ with non-negative coefficients $a_1, \ldots, a_r$. We write $\phi_\lambda$ for the irreducible representation of $G$ corresponding to $\lambda$. A dominant weight is called $q$-restricted if all $a_1, \ldots, a_r$ do not exceed $q - 1$. Let $Fr$ be a Frobenius endomorphism of $G$ and $G = G^{Fr}$. If $G$ is not of type $^2B_2(q), ^2G_2(q), ^2F_4(q)$ then the restriction $\phi_\lambda$ to $G$ is irreducible whenever $\lambda$ is $q$-restricted. Moreover, the irreducible representations of $G$ are parametrized by the $q$-restricted dominant weights. So we denoted by $\beta_\lambda$ the Brauer character of $\phi_\lambda|_G$. 

---

19
Proposition 5.9 Let \( q \) be even and \( G = Sp_{2n}(q) \cong \Omega_{2n+1}(q) \). Then \( St^+_G = \beta_{(q-1)\lambda_n} \cdot St_G \).

Proof. It suffices to observe that \( \omega_G(g) \) coincides with \( \beta_{(q-1)\lambda_n}(g) \) for every semisimple element \( g \in G \). This follows by comparison of Theorem 5.2 with the formula in \([19, Proposition 4.12]\).

For \( G_1 = SO_{2n}(\mathbb{F}_2) \) with \( n \geq 4 \), let \( \mu_1, \ldots, \mu_n \) denote the fundamental weights of \( G_1 \). Denote by \( \Delta t \) the set of weights \( \nu_i + 2\nu_2 + \cdots + 2^{t-1}\nu_{t-1} \), where \( \nu_i \in \{ \mu_{n-1}, \mu_n \} \) for \( i = 1, \ldots, t \). Let \( \lambda_1, \ldots, \lambda_n \) and \( \mu_1, \ldots, \mu_n \) denote the fundamental weights of \( G \cong SO_{2n+1}(\mathbb{F}_2) \).

Lemma 5.10 Let \( G_1 \subset G \cong SO_{2n+1}(\mathbb{F}_2) \) be the natural embedding. Then the restriction to \( G_1 \) of the irreducible representation \( \phi_{(q-1)\lambda_n} \) of \( G \) is the direct sum of the representations \( \phi_\nu \) when \( \nu \) runs over \( \Delta t \).

Proof. Let \( \varepsilon_1, \ldots, \varepsilon_n \) be the weights of \( G \) introduced in \([2, Planchee III]\). The weights of \( \phi_{\lambda_n} \) are well known to be \( \pm \varepsilon_1, \ldots, \pm \varepsilon_n \) in terms of \( \varepsilon_1, \ldots, \varepsilon_n \). They form a single \( W(G) \)-orbit. Viewing \( W(G_1) \) as a subgroup of \( W(G) \), these weights form two \( W(G_1) \)-orbits. Note that a maximal torus of \( G_1 \) remains a maximal torus in \( G \). It follows that the restriction \( \phi_{\lambda_n} |_{G_1} \) consists of at most two composition factors. The weights of \( \phi_\mu \) and \( \phi_\mu_{n-1} \) are also known, and one easily deduces from this that the composition factors are exactly \( \phi_\mu \) and \( \phi_\mu_{n-1} \). Observe that the restriction \( \phi_{\lambda_n} |_{G_1} \) can be obtained from the restriction \( \phi_{\lambda_n} |_{O_{2n}(\mathbb{F}_2)} \).

As \( \phi_\mu \) is not invariant under \( O_{2n}(\mathbb{F}_2) \), it follows that \( \phi_{\lambda_n} |_{O_{2n}(\mathbb{F}_2)} \) is irreducible, and, by Clifford’s theorem, \( \phi_{\lambda_n} |_{G_1} \) is the direct sum of two irreducible constituents. This implies the lemma for \( q = 2 \).

In general, let \( q = 2^t > 2 \). Then \( q - 1 = 1 + 2 + \cdots + 2^{t-1} \). It follows that \( \phi_{(q-1)\lambda_n} = \phi_{\lambda_n} \otimes \phi_{2\lambda_n} \otimes \cdots \otimes \phi_{2^{t-1}\lambda_n} \). Therefore, the restriction of \( \phi_{(q-1)\lambda_n} \) to \( G_1 \) is the tensor product of the restrictions of \( \phi_{2^i\lambda_n} \) to \( G_1 \) for \( i = 0, \ldots, t - 1 \). As \( \phi_{2^i\lambda_n} |_{G_1} = \phi_{2^i\mu_{n-1}} \oplus \phi_{2^i\mu_n} \), the lemma follows. (Note that the representations \( \phi_\nu \) for \( \nu \in \Delta t \) are irreducible by \([28, Theorem 43]\).)

Let \( n \geq 4 \) and \( G_1 = SO_{2n}(q) \) or \( SO_{2n}^+(q) \), so \( G_1 = G^{Fr} \) for a suitable Frobenius endomorphism \( Fr \) of \( G = SO_{2n}(\mathbb{F}_2) \). Denote by \( st^+_{G_1} \) and \( st_G \) the projective modules with the characters \( St^+_{G_1} \) and \( St_G \), respectively.

Proposition 5.11 \( st^+_{G_1} = \sum_{\nu \in \Delta t} \phi_\nu \otimes st_{G_1} \).

Proof. It is well known that \( \phi_\nu \otimes st_{G_1} \) is a projective module (possibly decomposable). By Corollary 4.7, the Curtis dual \( \omega_G |_{G_1} \) of \( st^+_{G_1} \) coincides with \( \omega_G |_{G_1} \). By Lemma 5.10 and the above remarks, \( \beta_{(q-1)\lambda_n} |_{G_1} \) coincides with the sum of \( \beta_\nu \) for all \( \nu \in \Delta t \). Therefore, \( St^+_{G_1} \) is the direct sum of \( \beta_\nu \cdot St_{G_1} \) \( \nu \in \Delta t \), where we keep \( \beta_\nu \) for the Brauer lift of the Brauer character \( \beta_\nu \). It follows that \( st^+_{G_1} \) and \( \sum_{\nu \in \Delta t} \phi_\nu \otimes st_{G_1} \) have the same Brauer characters. It is well known that projective modules with the same Brauer characters are isomorphic. So the result follows.

6 \( s \)-components of the Steinberg-plus character

In this section we recall some general facts of character theory of finite groups of Lie type and results from the paper \([20]\) which contains some approach to the analysis of characters of \( G \) vanishing on the non-semisimple elements.
Our main references for the character theory of groups of Lie type are [6] and [11]. One of the principal notions of the theory is that of the dual groups $G^*$ of $G$ and $G^*$ of $G$, see [6] Ch. 4 and [11] 13.10. (Recall that to simplify notation, we keep $Fr$ for $Fr^*$ (the Frobenius endomorphism of $G^*$ defining $G^*$).) The duality yields a bijection between the maximal tori $T$ of $G$ and maximal tori $T^*$ of $G^*$ such that $T^*$ is naturally identified with $\text{Irr} T$, the set of irreducible characters of $T$. The group $G$ acts on the set of maximal $Fr^*$-stable tori of $G$ by conjugation, and this induces the action of $G$ on the set of pairs $(T, \theta)$, where $T = T^{Fr^*}$. By [11] 13.13, $G$-orbits of the pairs $(T, \theta)$ are in bijection with the $G^*$-orbits of the pairs $(s, T^*)$, where $s \in T^{s Fr^*}$.

To every pair $(T, \theta)$ the theory corresponds a generalized character $R_{T, \theta}$ of $G$, called a Deligne-Lusztig character of $G$. If $(T', \theta')$ is another pair then $R_{T, \theta} = R_{T', \theta'}$ if $(T, \theta)$ and $(T', \theta')$ are $G$-conjugate, otherwise $(R_{T, \theta}, R_{T', \theta'}) = 0$, where $(\cdot, \cdot)$ means the usual inner product of functions on $G$ [11] 11.15. The duality allows us to parametrize $R_{T, \theta}$ by the $G^*$-orbits of the pairs $(s \in T^*)$. We denote by $R_s$ the set of pairs $(T, \theta)$, where $s \in G^*$ is a fixed semisimple element, whereas $T^*$ vary within $C_{G^*}(s)$ such that $C_{G^*}(s)$-conjugate tori are counted once. In other words, $R_s$ consists of the pairs $(T, \theta)$ such that the dual $T^*$ of $T$ contains $s$ and $\theta$ corresponds to $s$ under the isomorphism $\text{Irr} T = T^*$. In addition, $R_s$ contains at most one representative of the $G$-orbit of every pairs $(T, \theta)$. The irreducible constituents of the Deligne-Lusztig characters that belong to $R_s$ form the Lustzig rational series usually denoted by $E_s$. This yields a partition $\text{Irr} G = \cup_s E_s$, when $s$ runs over representatives of the semisimple conjugacy classes in $G^*$ [11] 14.41.

Recall that $W(T) := N_G(T)/T$. The group $N_G(T)$ acts on $T$ by conjugation and stabilizes $T = G \cap T$. This yields an action of $W(T)$ on $\text{Irr} T$. If $\theta$ is an irreducible character of $T$ we set $W(T) = C_{W(T)}(\theta)$, that is, $W(T) \theta$ is the stabilizer of $\theta$ in $W(T)$. In addition, recall that $\varepsilon_G := (-1)^r$, where $r$ is the relative rank of a connected algebraic group $G$, which therefore is meaningful for $T$ as well (consult [11] pp. 64, 66).

**Proposition 6.1** Let $\Phi$ be a class function on $G$ vanishing at all non-semisimple elements. Let $\Phi$ be the Curtis dual of $\Phi$. Then $\Phi = \sum_{(T, \theta)} (\Phi(T, \theta))/W(T) \varepsilon_G \varepsilon_T R_{T, \theta}$, where the sum is over representatives of the $G$-orbits of $(T, \theta)$.

**Proof.** This is [20] Lemma 2.1.

Applying Proposition 6.1 to the Steinberg-plus character $St^+_G$ of $G = SO(V)$ and the Curtis dual $\omega_G$ of $St^+_G$, we have

$$St^+_G = \sum_{(T, \theta)} \frac{(\omega_G|T, \theta)}{|W(T)|} \varepsilon_G \varepsilon_T R_{T, \theta},$$

(5)

where the sum is over representatives of the $G$-orbits of $(T, \theta)$. For every fixed semisimple element $s \in G^*$ we denote by $St^+_s$ the partial sum consisting from the terms with $(T, \theta) \in R_s$. Thus, $St^+_G = \sum_s St^+_s$, with the sum over representatives of the $G^*$-conjugacy classes of semisimple elements of $G^*$. By the above comments, if $t$ is a representative of another semisimple conjugate class of $G^*$ then $St^+_t$ and $St^+_s$ has no common irreducible constituents. It follows that $St^+_s$ is multiplicity free if and only if so are $St^+_s$ for all semisimple elements $s \in G^*$. The argument depends on certain properties of $s$. For convenience of further references, we record:

$$St^+_s = \sum_{(T, \theta) \in R_s} \frac{(\omega_G|T, \theta)}{|W(T)|} \varepsilon_G \varepsilon_T R_{T, \theta}.$$
Obviously, if $St^+_s \neq 0$ then $St^+_s$ is the sum of characters that belong to $\mathcal{E}_s$. Our goal is to determine these characters. However, we first identify the cases where $St^+_s = 0$.

From now on we specify $G$ to be $SO(V)$. Let $V^*$ be the natural module for $G^*$, $T^*$ is a maximal torus of $G^*$ and let $V^*_1 \oplus \cdots \oplus V^*_k$ be the respective $T^*$-decomposition of $V^*$. If $s \in G^*$ is semisimple, we denote by $V^*_s$ the 1-eigenspace $V^*_s$ of $s$ on $V^*$. If $s \in T^*$ then we denote by $T^*_s$ the projection of $T^*$ to $V^*$. So $T^*_s$ is a maximal torus in $SO(V^*_s)$. Recall that, if $\dim V = 2n + 1$ is odd then $G^* = Sp_{2n}(q)$, otherwise $G^* \cong G$. It follows that $\dim V^*_s$ is even. (Indeed, if $G^* = Sp_{2n}(q)$ then $V^*_s$ is non-degenerate, and hence of even dimension. Let $G^* = SO(V^*)$ so $\dim V^*$ is even. Viewing $s$ as an element of $V^*$, one can choose a basis $b_1, \ldots, b_{2n}$ of $V^*$ such that $sb_i = \nu_i b_i$ for $i = 1, \ldots, 2n$ and some $\nu_i \in \mathbb{F}_q$. If $\nu_i \neq \pm 1$ then $\nu_i^{-1}$ is an eigenvalue of $s$, and one easily observes that the multiplicities of $\nu_i, \nu_i^{-1}$ as eigenvalues of $s$ are equal. As $\det s = 1$, it follows that the multiplicity of $-1$ is even (or equals zero). Therefore, the eigenvalue 1 must be of even multiplicity, as claimed.)

**Proposition 6.2** $St^+_s = 0$ if and only if $V^*_s$ has Witt defect 1. In particular, if $\dim V$ is odd then $St^+_s \neq 0$ for every semisimple element $s \in G^*$.

Proof. As distinct characters $R_{T, \theta}$ with $(T, \theta) \in R_s$ are linearly independent, $St^+_s = 0$ if and only if every term in (1) equals 0, that is, $(\omega_G|T, \theta) = 0$ for every $(T, \theta) \in R_s$. By Corollary 5.6 $(\omega_G|T, \theta) = 0$ if and only if $V^*_s$ is contained in $V^*_1 \oplus \cdots \oplus V^*_k$, see (1). This is equivalent to saying that $T^*_s$ is a neutral torus in $SO(V^*_s)$ (Definition 3.5). As $\dim V^*$ is even, $SO(V^*_s)$ has no neutral torus if and only if $V^*_s$ is of Witt defect 1.

Remark. The second statement of Proposition 6.2 can be proven in a more conceptual way. This follows from the fact that $St^+ - \Gamma_G$ is a proper character of $G$, where $\Gamma_G$ denotes the Gelfand-Graev character of $G = SO(V)$. Indeed, $\Gamma_G$ is multiplicity free and consists of $q^n$ irreducible constituents called the regular characters, see [11 14.42,14.39]. In addition, as $\Gamma_G$ vanishes at the non-identity semisimple elements of $G$ and $St^+_G$ vanishes at the non-semisimple ones, it follows that $(St^+_G, \Gamma_G) = St^+_G(1)/\Gamma_G(1) = q^n$. So every irreducible constituents of $\Gamma_G$ must occur in $St^+_G$.

The following result from [20, Theorem 1.3] plays an essential role here:

**Theorem 6.3** Let $s \in G^*$ be semisimple and let $\phi$ be a generalized character of $G$. Suppose that $\langle \phi|T, \theta \rangle \in \{0, 1\}$ for all $(T, \theta) \in R_s$ corresponding to the conjugacy class of $s$. Then $\phi \cdot St$ has at most one constituent from $\mathcal{E}_s$, and this is a unique regular character.

We shall apply this theorem to the situation where $\phi = \omega_G$ and $\phi \cdot St_G = St^+_G$.

**Proposition 6.4** If $s$ does not have eigenvalue 1 on $V^*$ then $St^+_s$ is a regular character in $\mathcal{E}_s$.

Proof. By Lemma 5.3 if $(T, \theta) \in R_s$ and $s$ does not have eigenvalue 1 on $V^*$, then $(\omega_G|T, \theta) = 1$. Then the result follows from Theorem 6.3.

It is convenient to state here the following technical fact:

**Lemma 6.5** Let $V^*_s$ be the 1-eigenspace of $s$ on $V^*$. Suppose that $V^*_s \neq 0$ and $St^+_s \neq 0$. Then $V^*_s$ has Witt defect 0.
Proof. Let $T^*$ be a maximal torus containing $s$. Obviously, $T^*V^*_s = V^*_s$. Therefore, a $T^*$-decomposition of $V^*_s$ can be extended to that of $V^*$. In other words, there is a $T^*$-decomposition $V^*_s = V^*_1 \oplus \cdots \oplus V^*_k$ of $V^*$ such that $V^*_1$ is the sum of some $V^*_i$ $(1 \leq i \leq k + l)$. If $i > k$ for some $V^*_i \subseteq V^*_s$ then $S_{1_s}^+ = 0$ by Proposition 6.2 This contradicts the assumption. Therefore, $V^*_s$ is contained in $V^*_1 \oplus \cdots \oplus V^*_k$, and hence $V^*_s$ is the direct sum of non-degenerate subspaces of Witt defect 0. So $V^*_s$ is of Witt defect 0.

7 The constituents of the Steinberg-plus character

Recall that $G = SO(V)$ if $q$ is odd, otherwise $G = \Omega(V)$, and $G^*$ the dual group. As above, $V^*$ denote the natural module for $G^*$. Observe that $G^* = Sp(V^*)$ if dim $V$ is odd, otherwise $G^* = SO(V^*)$. For a semisimple element $s \in G^*$ we keep notation $V^*_s$ for the 1-eigenspace of $s$ on $V^*$. Then $V^*_s$ is non-degenerate of even dimension. In this section we deal with the case, where $V^*_s \neq 0$ and $S^+_s \neq 0$. By Lemma 6.1 $V^*_s$ contains a totally singular (or totally isotropic) subspace $U$ of dimension $(\dim V)/2$. Let $P_U$ be the stabiliser of $U$ in $G^*$, and $L$ a parabolic subgroup of $P_U$. Then the stabilizer of $Y$ in the reasonong below. Observe that there is a bijection between parabolic subgroups of $G$ and $G^*$, and that the one corresponding to $P_U$ can also be defined as the stabilizer of a totally singular subspace $U'$, say, of $V$. Similarly, $L$ is the stabilizer of the direct sum of two totally singular subspaces of $V$, each of dimension equal to dim $U$. Let $V'$ be a complement of $U'$ in $U'^\perp$. Then $L \cong GL(U') \times SO(V')$ if $q$ is odd, and $L \cong GL(U) \times \Omega(V')$ if $q$ is even. If dim $V$ is odd then $L \cong GL(U') \times Sp(V')$. This observation is frequently used below without reference. Furthermore, we usually write $L = G_1 \times G_2$, where $G_1 \cong GL(U')$ and $G_2 \cong SO(V')$.

Note that $V^* = V^*_s \oplus V^*_s^\perp$. We write $s = \text{diag}(s_1, s_2)$, where $s_1$, respectively, $s_2$, is the restriction of $s$ to $V^*_s$, respectively, $V^*_s^\perp$.

We define a subgroup $X^* = X^*_1 \times X^*_2$ as follows. If $G^* = Sp(V^*)$ then we set $X^*_1 = Sp(V^*_s)$, $X^*_2 = Sp(V^*_s^\perp)$, otherwise, $X^*_1 = SO(V^*_s)$, $X^*_2 = SO(V^*_s^\perp)$ for $q$ odd, and $X^* = X^*_1 \oplus \Omega(V^*_s)$, $X^*_2 = \Omega(V^*_s^\perp)$ for $q$ even.

Observe that we also have $\nabla^* = \nabla^*_s \oplus \nabla^*_s^\perp$. Then the stabilizer of $\nabla_s^*$ in $O(\nabla^*)$ is $O(\nabla^*_s) \times O(\nabla^*_s)$ if $O(\nabla^*)$ is orthogonal. The connected component of this algebraic group is $X^* = SO(\nabla^*_s) \times SO(\nabla^*_s)$ if $q$ is odd, and hence $X^* = X^*F_r$. Similarly, in the other cases.

As above, $X^*$ is the dual group of $X \cong SO(\nabla^*_s) \times SO(\nabla^*_s)$. It is clear that $G_2 = X_2$.

Similarly, $L^* = G_1^* \times G_2^*$, where $G_2^* = X_2^*$.

Lemma 7.1 Let $s \in G^*$ be a semisimple element such that $V^*_s \neq 0$. Then $C_{G^*}(s) \subseteq X^*$ if and only if either dim $V$ is odd or $-1$ is not an eigenvalue of $s$. In particular, if $q$ is even then $C_{G^*}(s) \subseteq X^*$ for any $s$.

Proof. Set $Y = C_{G^*}(s)$. Then $YV^*_s = V^*_s$ and $YV^*_s^\perp = V^*_s^\perp$. If dim $V$ is odd then $G^* = Sp(V^*)$ and $X^* = Sp(V^*_s) \times Sp(V^*_s^\perp)$. So the lemma follows in this case.

Suppose that dim $V$ is even. Let $s = \text{diag}(s_1, s_2)$ as above. Then $Y \subseteq O(V^*_s) \times C_{O(V^*_s^\perp)}(s_2)$. As 1 is not an eigenvalue of $s_2$, by Lemma 2.1, $C_{O(V^*_s^\perp)}(s_2) \subseteq SO(V^*_s^\perp)$ if and only if $-1$ is not an eigenvalue of $s_2$. Let $y \in Y$, and $y = \text{diag}(y_1, y_2)$ for $y_1 \in O(V^*_s)$, $y_2 \in O(V^*_s^\perp)$. If $q$ is even then $y_2 \in X^*_s$, and hence $y_1 \in X^*_1$ as the spinor norm of $y$ is the product of the spinor norms of $y_1$ and $y_2$. So $y \in X^*$ is this case.

Suppose first that $q$ is odd. Then $y \in SO(V^*)$ if and only if det $y_1 = \text{det} y_2$. Therefore, $y \not\in X^*$ if and only if det $y_1 = \text{det} y_2 = -1$. As 1 is not an eigenvalue of $s_2$, the lemma follows in this case too.
Next suppose that $s$ has eigenvalue $1$ on $V^*$ and $St_s^+ \neq 0$. Let $U$ be a maximal totally singular (or totally isotropic) subspace of $V^*_s$. Let $P_U$ be the stabilizer of $U$ in $G^*$. If $u := \dim U < \dim V/2$ then $P_U$ is determined by $u$ up to $G^*$-conjugacy.

**Lemma 7.2** Let $s \in G^*$ be a semisimple element such that $V^*_s \neq 0$. Suppose that $St_s^+ \neq 0$.

1. Let $U_1, U_2$ be maximal totally singular subspaces of $V^*_s$. Then $gU_1 = U_2$ for some $g \in C_{G^*}(s)$, unless $\dim V$ is even and $-1$ is not an eigenvalue of $s$. In the exceptional case there are two $C_{G^*}(s)$-orbits of maximal totally singular subspaces of $V^*_s$.

2. Let $P_i$ $(i = 1, 2)$ be the stabilizer of $U_i$ in $G^*$. Then there is a Levi subgroup of $P_i$ $(i = 1, 2)$ stabilizing $V^*_s$.

3. Let $L_i^s$ be a Levi subgroup of $P_i$ stabilizing $V^*_s$. Then $gL_i^s = L_i^s$ for some $g \in C_{G^*}(s)$, unless $\dim V$ and $\dim U_i$ are even and $s$ does not have eigenvalue $-1$. In the exceptional case $L_1^s$, $L_2^s$ are $C_{G^*}(s)$-conjugate if and only if $gU_1 = U_2$ for some $g \in C_{G^*}(s)$.

Proof. By Lemma 6.3, $V^*_s$ is of Witt defect $0$.

1. Let $U$ be the set of all maximal totally singular subspaces of $V^*_s$. If $C_{G^*}(s)$ is not contained in $X^*$, then the restriction of $C_{G^*}(s)$ to $V^*_s$ coincides with $O(V^*_s)$. As $O(V^*_s)$ is transitive on $U$, in this case $C_{G^*}(s)$ is transitive on $U$, and the claim follows. Suppose that $C_{G^*}(s) \subseteq X^*$. By Lemma 7.1, this happens if and only if either $\dim V$ is odd or $-1$ is not an eigenvalue of $s$. In the latter case the restriction of $C_{G^*}(s)$ to $V^*_s$ coincides with $SO(V^*_s)$. As $\dim V^*_s$ is even, the claim now follows from Lemma 2.6.

Suppose that $\dim V$ is even and $-1$ is an eigenvalue of $s$. Then, by Lemma 7.1, $C_{G^*}(s) \not\subset X^*$, and hence the restriction of $C_{G^*}(s)$ to $V^*_s$ coincides with $O(V^*_s)$. So $C_{G^*}(s)$ is transitive on $U$.

2. Let $V_s^* = U_i \oplus U'_i$, where $U'_i$ is an arbitrary totally singular subspace of $V^*_s$. Then the stabilizer $L_i^s$ of $U_i$ and $U'_i$ in $G^*$ is known to be a Levi subgroup of $P_i$. Obviously, $L_i^s$ stabilizes $V^*_s$, whence (3). (Note that all Levi subgroups of $P_i$ stabilizing $V^*_s$ can be obtained in this way.)

3. Let $P_i^*$ denote the stabilizer of $U_i$ in $X_i^*$, and let $L_i^{\prime*}$ be the projection of $L_i^s$ to $X_1^*$ (so $L_i^{\prime*} \cong G_1$). Then $P_i^*$ is a parabolic subgroup of $X_1^*$, and $L_i^{\prime*}$ is a Levi subgroup of $P_i^*$.

Suppose first that $\dim V$ is odd. Then $C_{G^*}(s)$ is contained in $X^*$ by Lemma 7.1. Note that here $X^* = X_1^* \times X_2^*$, where $X_1^* = Sp(V^*_s)$ and $G_2 = Sp(V^*_s)$ is odd. In addition, $L_i^s = G_1^i \times G_2$, where $G_1^i \cong GL(U) \subset X_1^*$ is the Levi subgroup of $P_i$. Therefore, $gU_1 = U_2$ for some $g' \in C_{G^*}(s)$. So we can assume that $P_1 = P_2$. Then $L_1^s$, $L_2^s$ are $C_{G^*}(s)$-conjugate if and only if $L_1^{\prime s}$, $L_2^{\prime s}$ are $X_1^*$-conjugate. This is the case as Levi subgroups of any parabolic subgroup are conjugate.

Suppose that $\dim V$ is even. Recall that $L_i^s = G_1^i \times G_2$, where $G_1^i \cong GL(U) \subset X_1^*$ is the Levi subgroup of $P_i$. Observe that $C_{G^*}(s)$ coincides either with $X_1 \cdot C_{X_2^*}(s)$ or, for $q$ odd, with $\langle (g_1g_2)(X_1 \cdot C_{X_2^*}(s)) \rangle$, where $g_1 \in O(V^*_s)$, $g_2 \in C_{O(V^*_s)}(s)$ and $g_1 = g_2 = -1$. So in the latter case the restriction of $C_{G^*}(s)$ to $V^*_s$ coincides with $O(V^*_s)$.

Clearly, this happens if and only if $C_{G^*}(s)$ is not contained in $X^*$.

As $X_2^*$ is normal in $O(V^*_s)$, it follows that $L_1^s$, $L_2^s$ are $C_{G^*}(s)$-conjugate if and only if $L_i^{\prime s} = hL_i^s$ for some $h$ from the projection of $C_{G^*}(s)$ to $V^*_s$. As Levi subgroups of any parabolic subgroup are conjugate, it follows that $hL_i^s = L_i^s$ for some $h \in X_1^*$, unless $U_1$, $U_2$ are not in the same $X_1^*$-orbit. The latter happens if and only if $U_1$ is even (Lemma 2.6).

Note that $g'U_1 = U_2$ for some $g' \in O(V^*_s)$. It follows that, if $C_{G^*}(s) \not\subset X^*$ then $U_1 = U_2$ for some $g' \in C_{G^*}(s)$. So in this case (3) follows. Thus, we are left with the case where $C_{G^*}(s) \subset X^*$ and $\dim U_1$ is even. In this case (3) follows by Lemma 7.1.
The following lemma is one of the key points of our argument.

**Lemma 7.3** Let \( s \in G^* \) be a semisimple element, and let \( T^* \) be a maximal torus in \( G^* \) containing \( s \). Suppose that \( V^*_s \neq 0 \). Then the following are equivalent:

1. \((\omega_G(T), \theta) \neq 0; \)
2. \( T^* \) is \( C_{G^*}(s) \)-conjugate to a subgroup of \( P_U \), where \( U \) is some totally singular subspace of \( V^*_s \) of dimension \( \dim V^*_s/2 \).
3. \( T^* \) is \( C_{G^*}(s) \)-conjugate to a Levi subgroup \( L^* \) of \( P_U \) such that \( L^*V^*_s = V^*_s \).

In addition, suppose that \(-1 \) is not an eigenvalue of \( s \). Let \( T^*, T'^* \) be two maximal tori of \( L^* \). If \( T^*, T'^* \) are \( C_{G^*}(s) \)-conjugate then they are \( C_{L^*}(s) \)-conjugate.

**Proof.** \((1) \to (3) \) By Lemma 7.2 (see the proof of item \(1) \)), \( T^*V^*_s = V^*_s \) and any \( T^* \)-decomposition of \( V^*_s \) can be extended to a \( T^* \)-decomposition \( V^*_1 \oplus \cdots \oplus V^*_k \) of \( V^* \). Moreover, \( V^*_s \) is contained in \( V^*_1 \oplus \cdots \oplus V^*_k \) (Corollary 5.8). Each \( V^*_i \) with \( i \leq k \) is the direct sum of two totally singular \( T^* \)-stable subspaces of equal dimension. Let \( U \) be the sum of any such subspaces, chosen by one from every \( V^*_i \subset V^*_s \). Then \( U \) is totally singular, \( T^* \)-stable and of dimension \( \dim V^*_s/2 \). Moreover, it is easy to observe that there is a \( T^* \)-stable totally singular (or totallyisotropic) subspace \( U' \subset V^*_s \) such that \( V^*_s = U \oplus U' \). Then the stabilizer of both \( U, U' \) in \( G \) is a Levi subgroup of \( P_U \), as required.

\((2) \to (1) \) follows from Lemma 5.5 and \((3) \to (2) \) is trivial.

For the additional statement, let \( L^* = G_1^* \times G_2^* \), where \( G_1^* \) is the projection of \( L^* \) to \( V^*_s \) and \( G_2^* \) is the projection of \( L^* \) to \( V^*_s^\perp \). Then let \( T^* = T_1^* \times T_2^* \) with \( T_1^* \subset G_1^*, T_2^* \subset G_2^* \), and similarly, let \( T'^* = T_1'^* \times T_2'^* \). As \(-1 \) is not an eigenvalue of \( s \), we have \( C_{G^*}(s) \subseteq X^* = X_1^* \times X_2^* \) (Lemma 7.1). Since \( X_2^* = G_2^* \), we have \( C_{L^*}(s) = G_1^* \times C_{G_2^*}(s_2) \) and \( C_{G^*}(s) = X_1^* \times C_{G_2^*}(s_2) \). It follows that \( T^*, T'^* \) are \( C_{L^*}(s) \)-conjugate if and only if \( T_1^*, T_1'^* \) are \( G_1^* \)-conjugate. As \( T^*, T'^* \) are \( C_{G^*}(s) \)-conjugate, it follows that \( T_1^*, T_1'^* \) has the same partition function as maximal tori of \( X_1^* \), and hence as those in \( G_1^* \cong GL(U) \). Therefore, \( T_1^*, T_1'^* \) are conjugate in \( G_1^* \), as claimed.

Remark. Strictly speaking, we have to prove that if \( T^*, T'^* \) are \( Fr \)-stable tori in the algebraic groups \( C_{L^*}(s) \) such that \( T^* = T^*_{Fr} \) and \( T'^* = (T'^*)_{Fr} \), and if \( T^* \) is \( C_{G^*}(s) \)-conjugate to \( T'^* \) then \( T^* \) is \( C_{L^*}(s) \)-conjugate to \( T'^* \). Clearly, this only requires routine changes of the above reasoning.

Let \( L \) be the set of all Levi subgroups \( L^* \) of \( P_U \), when \( U \) runs over all maximal totally singular subspaces of \( V^*_s \), satisfying the condition \( L^*V^*_s = V^*_s \). By Lemma 7.2 \( L \) forms a single \( C_{G^*}(s) \)-orbit, unless \( \dim V \) and \( \dim U \) are even and \(-1 \) is not an eigenvalue of \( s \). In the latter case \( L \) consists of two \( C_{G^*}(s) \)-orbits. With this notation we have the following refinement of Lemma 7.3.

**Lemma 7.4** Suppose \( G^* = SO^{\pm}(2n, q) \), \( n \) even and let \( s = Id \). Let \( L_1^*, L_2^* \in L \) be two non-conjugate subgroups. Let \( T_i^* \) be an maximal torus of \( L_i^* \) \( (i = 1, 2) \) corresponding to the same function \( i \to d_i \). Then \( T_i^* \) is not \( C_{G^*}(s) \)-conjugate to \( T_2^* \) if and only if \( T_i^* \) are exceptional tori in \( G^* \).

**Proof.** By Lemma 3.7, non-exceptional neutral maximal tori of \( G^* \) with the same function \( i \to d_i \) are conjugate in \( G^* \) as stated, whereas exceptional neutral maximal tori of \( G^* \) with the same function \( i \to d_i \) form two \( G^* \)-conjugacy classes. Suppose the contrary, that \( T_1^* \) and \( T_2^* \) are exceptional but \( G^* \)-conjugate. As every neutral maximal torus of \( G^* \) is conjugate to that in \( L_1^* \) or \( L_2^* \), and maximal tori in each \( L_1^*, L_2^* \) with the same function \( i \to d_i \) are conjugate, it follows that all neutral maximal tori in \( G^* \) with the same function \( i \to d_i \) are conjugate in \( G^* \), which is a contradiction.
Lemma 7.5  Suppose that $(\omega_G|_T, \theta) \neq 0$ and, for $s \in T^*$ corresponding to $\theta$, let $V^*_s \neq 0$.

(1) Suppose that $\mathcal{L}$ consists of a single $G_{C*}(s)$-orbit, and let $L^* \in \mathcal{L}$. Then maximal tori of $G_{C*}(s)$ are $G_{C*}(s)$-conjugate to tori in $L^*$.

(2) Suppose that $\mathcal{L}$ consists of two $G_{C*}(s)$-orbits, and let $L^*_1, L^*_2 \in \mathcal{L}$ be their representatives. Then every maximal torus $T^*$ of $G_{C*}(s)$ is $G_{C*}(s)$-conjugate to a torus of $L^*_1$ or $L^*_2$. Moreover, $T^*$ is $G_{C*}(s)$-conjugate to a torus of each $L^*_1$ and $L^*_2$ if and only if $T^*_1$ is non-exceptional.

Proof. Let $T^*$ be maximal torus of $X^* \subset C_{G*}(s)$. By Lemma 7.3, $T^*$ is $G_{C*}(s)$-conjugate to a torus in a Levi subgroup $L^* \in \mathcal{L}$. So (1) is immediate. Consider (2). As in the case (1), it follows that every maximal torus of $X^*$ such that $(\omega_G|_T, \theta) \neq 0$ is $X^*$-conjugate to a torus in $L^*_1$ or $L^*_2$, whence the first claim of (2).

Furthermore, $T^* \subset X^* = X^*_1X^*_2$. Let $T^* = T^*_1T^*_2$, where $T^*_1 \subset X^*_1, T^*_2 \subset X^*_2$. By Lemma 7.2, in this case $\dim V^*/2$ are even and $-1$ is not an eigenvalue of $s$. So $G_{C*}(s) \subseteq X^*$ by Lemma 7.1. Therefore, maximal torus $T^*, T^*_2$ of $X^*$ are $G_{C*}(s)$-conjugate if and only if they are $X^*$-conjugate, and hence if and only if $T^*_1, T^*_2$ are $X^*_1$-conjugate. If $T^*_1$ is non-exceptional in $X^*_1$, then $T^*_1, T^*_2$ are conjugate in $X^*_1$ (Lemma 7.3), and the statement is true in this case.

Suppose that $T^*_1$ is exceptional. Denote by $L^*_1, L^*_2$ the projections of $L^*_1, L^*_2$ to $X^*_1$. Then $L^*_1, L^*_2$ are $G_{C*}(s)$-conjugate if and only if $L^*_1, L^*_2$ are $X^*_1$-conjugate. Therefore, the second statement in (2) follows from Lemma 7.4 applied to $X^*_1 \cong SO(V^*_s)$.

Lemma 7.6  Let $G = SO(V)$, $L^* \in \mathcal{L}$ and let $s \in T^* \subset L^*$. Suppose that $\dim V$ is odd, or $\dim V$ is even and $-1$ is not an eigenvalue of $s$. Then $W(T)_\theta = W_X(T)_\theta$.

Proof. Recall that $G^* = Sp(V^*)$ if $\dim V$ is odd. Note that $G_{C*}(s)$ is contained in $X^*_1 \cdot X^*_2$, where $X^*_1 = Sp(V^*_s)$ and $X^*_2 = Sp(V^*_s \perp)$. (Indeed, $X^*_1 \cdot X^*_2$ coincides with the stabilizer of $V^*_s$ in $G^*$.) This implies the claim, as $W(T)_\theta$ is canonically isomorphic to $W_{C_{G*}(s)}(T^*) = C_{W(T)}(s)$, which is exactly $N_{C_{G*}(s)}(T^*)/T^*$. Next, suppose $\dim V$ is even. Observe that $W(T)_\theta \cong C_{W(T)}(s) = N_{C_{G*}(s)}(T)/T$ and $W_X(T)_\theta \cong C_{W_X(T)}(s) = N_{C_{X*}(s)}(T)/T$. As $C_{G*}(s) \subset X^*$ by Lemma 7.1, we have $C_{G*}(s) = C_{X*}(s)$, and the lemma follows.

Recall that we write $L^* = G^*_1 \times G^*_2$, where $G^*_1 \cong GL(U)$ and $G^*_2 \cong X^*_2$. If $T^*$ is a maximal torus of $L^*$, we write $T^* = T^*_1 \times T^*_2$, where $T^*_r$ is a maximal torus in $G^*_r$ for $r = 1, 2$. Similarly, if $T$ is a dual torus in $L$, we write $T = T_1 \times T_2$, where $T_r$ is a maximal torus in $G^*_r$.

We denote by $R_L$ the set of representatives of the $L$-conjugacy classes of pairs $(T, \theta)$ with $T \subset L$. They are in bijection with $C_{L*}(s)$-conjugacy classes of maximal tori $T^* \subset C_{L*}(s)$.

Lemma 7.7  Let $G = SO(V)$ with $\dim V$ odd, and let $s \in G^* \cong Sp(V^*)$ be a semisimple element. Let $V^*_s$ be the 1-eigenspace of $s$ on $V^*$. Suppose that $V^*_s \neq 0$, and let $U$ be a maximal totally isotropic subspace of $V^*_s$, $P_U$ the stabilizer of $U$ in $G$, and let $L^*$ be a Levi subgroup of $P_U$. Then

$$St^+_s = \sum_{(T, \theta) \in R_s} (\omega_G|_T, \theta) \frac{1}{[W(T)_\theta]} [\epsilon G \in T R_{T, \theta} = \sum_{(T, \theta) \in R_L} \frac{1}{[W_L(T)_\theta]} [\epsilon G \in T R_{T, \theta}. \quad (7)$$

Proof. Recall that $(\omega_G|_T, \theta) = 0$, if $T$ is not conjugate to a torus in $L$ (Lemma 7.3). So, by Lemma 7.6, the elements of $R_s$ corresponding to the non-zero terms of (7) can be chosen
in $\mathbb{R}^k$. As above, let $s = \text{diag}(s_1, s_2)$, where $s_1$ is the projection of $s$ to $V^*_s$ and $s_2$ is the projection of $s$ to $V^*_s$.

Let $x \in X$, and let $x = x_1x_2 \in X$ with $x_i \in X_i$, $i = 1, 2$. By Lemmas 4.5 and 4.9, $\omega_G(x) = \omega_{X_1}(x_1)\omega_{X_2}(x_2)$, and hence $(\omega_G|T, \theta) = (\omega_{X_1}|T_1, \theta_1) \cdot (\omega_{X_2}|T_2, \theta_2)$, where $\theta_r \in \text{Irr}T_r$ corresponds to $s_r$, $r = 1, 2$. As $s_1 = \text{Id}$, we have $\theta_1 = 1_{T_1}$. As $s_2$ does not have eigenvalue 1 on $V^*_s$, by Lemma 5.2, we have $(\omega_{X_2}|T_2, \theta_2) = 1$. Furthermore, the torus $T_1$ is neutral in $X_1$. By Lemma 5.3, $|W_{X_1}(T_1)| = 2^{\text{dim}(T_1)}$, where $m(T_1)$ is the number of the parts in the partition that determines $T_1$ (because $\theta_1 = 1_{T_1}$). Therefore, $|W_{\omega_G(T_1)}| = (\omega_{X_1}|T_1, \theta_1)^{-1} = 2^{\text{dim}(T_1)}$.

Let $T_1$ correspond to a function $i \rightarrow d_i$ (with zero function $j \rightarrow e_j$). Then $\sum_i d_i = m(T_1)$ and $|W_{X_1}(T_1)| = \Pi_i(2i)^{d_i}d_i!$ by Lemma 5.4 whereas $|W_{G_1}(T_1)| = \Pi_i i^{d_i}d_i!$. So $|W_{G_1}(T_1)| = 2^{m(T_1)} \cdot |W_{G_1}(T_1)|$.

Finally, by Lemma 7.6, $W(T_1) = W(X_1)\theta = W(X_1)\theta$. We recall that $W_X(T_1) \cong C_{W(T_1)}(s) = W_{X_1}(T_1) \times C_{W_{X_2}(T_2)}(s_2)$ as $s_1 = 1$. By the above $|W_{G_1}(T_1)| = 2^{m(T_1)} \cdot |W_{G_1}(T_1)|$. As $L^* = G_1^* \times G_2^*$ and $G_2^* = G_2$, the lemma follows.

Let $G = SO(V)$, where $\text{dim} V$ is even. In Lemma 7.8 below, we assume that $-1$ is an eigenvalue of $s$, and hence, by Lemma 7.2, $L$ forms a single $C_{G^*}(s)$-orbit. Statement (A) of Lemma 7.8 refines further Lemma 7.3 for the case where $-1$ is not an eigenvalue of $s$.

Lemma 7.8 Let $G = SO(V)$, where $\text{dim} V$ is even, $L \in L$, and $s \in T^* \subset L^*$. Let $S$ be the set of maximal $Fr$-stable tori $T^* \subset C_{L^*}(s)$ that are $C_{G^*}(s)$-conjugate to $T^*$. Let $Y$ be the $-1$-eigenspace of $s$ on $V^*$, and $T^*_3$ the restriction of $T^*$ to $Y$. Suppose $Y \neq 0$.

(A) The set $S$ consists of two $C_{L^*}(s)$-orbits if and only if $T^*_3$ is exceptional but $T^*_1$ is not exceptional.

(B) The following statements are equivalent:

1. $N_{C_{G^*}}(s)(T^*)$ is contained in $X^*$.
2. either $T_1$ or $T_3$ is exceptional;
3. The subgroup $W_X(T^*) = W(T_1) \times W_{X_2}(T_2)\theta_2$ coincides with $W(T_1)$.

(C) The following statements are equivalent:

4. $N_{C_{G^*}}(s)(T^*)$ is not contained in $X^*$.
5. $T_1$ and $T_3$ are non-exceptional;
6. The subgroup $W_X(T^*) = W(T_1) \times W_{X_2}(T_2)\theta_2$ has index 2 in $W(T_1)$.

Proof. (A) Set $M := (V^*_s + Y)^{-1}$. Let $s_2 = \text{diag}(-\text{Id}, s_4)$, where $s_4$ is the restriction of $s$ to $M$. Then $C_{X^*}(s) = X_1^* \times X_3^* \times X_4^*$ where $X_3^* = SO(Y)$ and $X_4^* = C_{SO(M)}(s_4)$, whereas $C_{G^*}(s) = (C_{X^*}(s), g)$, where $g = \text{diag}(g_1, g_2, \text{Id})$, det $g_1 = \det g_2 = -1$ and $g_1, g_2$ are the projections of $g$ to $U, Y$, respectively (see Lemma 20). Also $C_{L^*}(s) = G_1^* \times X_3^* \times X_4^*$. Furthermore, $T^* = T_1^* \times T_3^* \times T_4^*$, where $T_3^*, T_4^*$ are projections of $T^*$ to $Y, M$, respectively. Similarly, let $T^* = T_1^* \times T_3^* \times T_4^*$, and $T_1^*, T_3^*$ are conjugate to $T^*_3$ in $O(V^*_s)$, so they correspond to the same partition function as tori in $T^*$, and hence in $G_1^*$. We know that the maximal tori corresponding to the same partition are conjugate in $G_1^* \cong GL(U)$. As $X_1^*$ is common for $C_{L^*}(s)$ and $C_{G^*}(s)$, it suffices to look at $T^*_3$. Again, $T_3^*$ and $T^*_3$ are conjugate in $X_3^*$ unless $T_3^*$ is exceptional, see Lemma 7.4.

So we are left with the case where $T_3^*$ is exceptional. As $T_1^*$ and $T^*_3$ are conjugate in $G_1^* \cong GL(U)$, we can assume $T_1^* = T^*_3$. If $T^*_1$ is non-exceptional then $N_{O(V^*_s)}(T^*_1)$
contains an element $g_1$ with $\det g_1 = -1$ (Lemma 3.6). Therefore, $C_{G^*}(s)$ contains an
element $g = \text{diag}(g_1, g_2, \text{Id})$ with $\det g_2 = -1$. It follows that the restriction of $C_{G^*}(s)$ to $Y$
coincides with $O(Y)$, and hence all maximal tori of $X^*_3$ whose partition function is the same
as that of $T^*_3$ are in the same $O(Y)$-orbit (Lemma 3.3). This implies that, given a maximal
torus $D$, say, in this orbit, there is a torus in $S$ whose restriction to $Y$ is $D$. However,
these tori $D$ form two $SO(Y)$-orbits (Lemma 3.3). It follows that that $S$ consists of two
$C_{L^*}(s)$-orbits.

Finally, suppose that $T^*_1$ is exceptional. Let $h \in C_{G^*}(s)$ be an element such that
$h T^* h^{-1} = T^*$. Then $h = \text{diag}(h_1, h_3, h_4)$, where $h_1 \in O(V^*_s)$, $h_3 \in O(Y)$ and $h_4 \in O(M)$.
As above, we can assume that $T^*_1 = T^*_1'$ and $T^*_4 = T^*_4'$. Then $N_{O(V^*_s)}(T^*_1) \subset SO(V^*_s)$
and $N_{O(M)}(T^*_1) \subset SO(M)$. Therefore, $\det h_1 = 1$ and $\det h_4 = 1$. As $h \in G^* = SO(V^*_s)$, it
follows that $\det h_3 = 1$ and hence $h_3 \in SO(Y) = X^*_3$. As $X^*_3$ is a multiple of $L^*$, we can
assume that $T^*_3 = T^*_3'$. So (A) follows.

(B) Observe first that (1) and (2) are equivalent. Let $T^*$ be a maximal $Fr$-stable torus
of $L^*$ such that $T^* = T^* \cdot Fr$. Note that $(N_{C_{G^*}(s)}(T^*))_{Fr} = N_{C_{G^*}(s)}(T^*)$. Observe that
$N_{C_{G^*}(s)}(T^*) \subset C_{G^*}(s) \subset O(V^*_s) \times O(V^*_s)$. Furthermore, $N_{C_{G^*}(s)}(T^*)$ is not contained
in $X^* = X^*_1 X^*_2 = SO(V^*_s) \times SO(V^*_s)$ if and only if there are $g_1 \in O(V^*_s)$ and $g_2 \in C_{O(V^*_s)}(s_2)$
such that $\det g_1 = -1$ and $g_1$ normalizes $T^*_i$, $i = 1, 2$. Equivalently, if and only if $N_X(T^*_1) \neq N_{O(V^*_s)}(T^*_1)$ and $N_{X^*_2}(s_2) \neq N_{C_{O(V^*_s)}(s_2)}(T^*_2)$. By Lemma 3.6(4),
$N_X(T^*_1) \neq N_{O(V^*_s)}(T^*_1)$ if and only if $T^*_1$ is non-exceptional. Let $g_3$ be the projection of
$g$ to $Y$. Then $g_2 \in X^*_2$ and only if $g_3 \in SO(Y)$. By Lemma 3.6(4), $N_{X^*_2}(s_2) \neq N_{C_{O(V^*_s)}(s_2)}(T^*_2)$
if and only if $T^*_3$ is non-exceptional. This implies the equivalence of (1) and (2).

The equivalence of (1) and (3) is obvious. (Recall that $W_X(T)_{\theta} \cong N_{C_{X^*}^*}(s)(T^*) / T^*$.) So (B) follows.

(C) The statements (3), (4) and (5) are the negations of (1), (2) and (3), respectively,
except for the additional statement on the index $|W(T)_{\theta} : W_X(T)_{\theta}|$. This coincides with the index
$|N_{C_{G^*}}(T^*) : N_{C_{G^*}}(s)(T^*)|$. As $|C_{G^*}(s) : C_{X^*}(s)| \leq 2$, it suffices to observe that
$N_{C_{G^*}}(T^*) \neq N_{C_{X^*}^*}(s)(T^*)$. This is stated in (4).

\textbf{Lemma 7.9} Let $T = T_1 \times T_2$, let $i \to d_i$ be the function defining $T_1$ and $m(T_1) = \sum d_i$.
Set $t = 2^{m(T_1)}$ if $T_1$ is exceptional, otherwise $t = 2^{m(T_1)} - 1$.

If both the tori $T^*_1, T^*_3$ are non-exceptional then $|W(T)_{\theta}| = 2 \cdot |W_X(T)_{\theta}| = 2^{m(T_1)} \cdot
|W_L(T)_{\theta}|$, otherwise $|W(T)_{\theta}| = |W_X(T)_{\theta}| = t \cdot |W_L(T)_{\theta}|$.

Proof. The first equality in both the cases follows from Lemmas 7.6 and 7.8. Furthermore,
$W_X(T)_{\theta} = W_{X_1}(T_1) \times W_{X_2}(T_2)_{\theta_2}$ and $W_L(T)_{\theta} = W_{G_1}(T_1) \times W_{X_2}(T_2)_{\theta_2}$,
as $G_2 = X_2$ and $s_1 = 1$. In addition, $|W_{G_1}(T_1)| = \Pi_i d_i i!$, whereas $|W_{X_1}(T_1)| = \Pi_i (2i)^{d_i} i! / 2$, unless all parts
of the partition defining $T_1$ are even, in which case $|W_{X_1}(T_1)| = \Pi_i (2i)^{d_i} i!$ (Lemma 3.6).
Therefore, $W_X(T)_{\theta} = t \cdot W_L(T)_{\theta}$.

\textbf{Lemma 7.10} Keep the notation of Lemma 7.8. Denote by $\nu(s, T^*)$ the number of $C_{L^*}(s)$-
orbits of tori $T^*_s \subset C_{L^*}(s)$ that are $C_{G^*}(s)$-conjugate to $T^*$.

(1) Suppose that $T^*_3$ is exceptional and either $Y = 0$ or $T^*_3$ is exceptional. Then
$\nu(s, T^*) = 1$ and $\frac{|\omega|_{\phi, \theta}}{|W(T)_{\theta}|} = \frac{1}{|W_L(T)_{\theta}|}$.

(2) Suppose that $T^*_1$ is exceptional but $T^*_3$ is not. Then $\nu(s, T^*) = 1$ and $\frac{|\omega|_{\phi, \theta}}{|W(T)_{\theta}|} = \frac{1}{|W_L(T)_{\theta}|}$.
(3) Suppose that \( Y \neq 0 \) and \( T_3^* \) is exceptional but \( T_1^* \) is not exceptional. Then \( \nu(s, T^*) = 2 \) and
\[
\frac{\langle \omega_G|T, \theta \rangle}{|W(T)_\theta|} = \frac{2}{|W_L(T)_\theta|}.
\]

(4) Suppose that both \( T_1^* \) and \( T_3^* \) are non-exceptional. Then \( \nu(s, T^*) = 1 \) and
\[
\frac{\langle \omega_G|T, \theta \rangle}{|W(T)_\theta|} = \frac{1}{|W_L(T)_\theta|}.
\]

(5) Suppose that \( Y = 0 \) and \( T_1 \) is non-exceptional. Then \( \nu(s, T^*) = 1 \) and
\[
\frac{\langle \omega_G|T, \theta \rangle}{|W(T)_\theta|} = \frac{2}{|W_L(T)_\theta|}.
\]

Proof. In the notation of Lemma 7.8 \( \nu(s, T^*) \) is the number of \( C_L-(s) \) orbits in \( \mathcal{S} \). So the formulae for \( \nu(s, T^*) \) follow from that lemma. The formulae for \( \langle \omega_G|T, \theta \rangle \) follow from Lemma 7.9 and the fact that \( (\omega_G|T, \theta) = 2^{m(T)} \) (Lemma 5.5).

We shall use Lemma 7.10 to transform formula (6).

**Proposition 7.11** Let \( G = SO(V) \), where \( \dim V \) is even, and let \( s \in G^* \) be a semisimple element such that \( V_s^* \neq 0 \).

1. Suppose that \(-1\) is not an eigenvalue of \( s \) and \( \dim U \) is odd. Then
\[
St_s^+ = \sum_{(T, \theta) \in \mathcal{R}_s} \frac{\langle \omega_G|T, \theta \rangle}{|W(T)_\theta|} \varepsilon_G \varepsilon_T R_{T, \theta} = 2 \sum_{(T, \theta) \in \mathcal{R}_s} \frac{1}{|W_L(T)_\theta|} \varepsilon_L \varepsilon_T R_{T, \theta}.
\]

2. Suppose that \(-1\) is not an eigenvalue of \( s \) and \( \dim U \) is even. Then
\[
St_s^+ = \sum_{(T, \theta) \in \mathcal{R}_s} \frac{\langle \omega_G|T, \theta \rangle}{|W(T)_\theta|} \varepsilon_G \varepsilon_T R_{T, \theta} =
\]
\[
= \sum_{(T, \theta) \in \mathcal{R}_s} \frac{1}{|W_L(T)_\theta|} \varepsilon_L \varepsilon_T R_{T, \theta} + \sum_{(T, \theta) \in \mathcal{R}_s} \frac{1}{|W_L(T)_\theta|} \varepsilon_L \varepsilon_T R_{T, \theta}.
\]

(The above expression is understood so that if \( T \) is not \( C_{G^*}(s) \)-conjugate to \( L_i \) for \( 1 \leq i \leq 2 \) then there is no term in the corresponding sum.)

3. Suppose that \(-1\) is an eigenvalue of \( s \). Then
\[
St_s^+ = \sum_{(T, \theta) \in \mathcal{R}_s} \frac{\langle \omega_G|T, \theta \rangle}{|W(T)_\theta|} \varepsilon_G \varepsilon_T R_{T, \theta} = \sum_{(T, \theta) \in \mathcal{R}_s} \frac{1}{|W_L(T)_\theta|} \varepsilon_L \varepsilon_T R_{T, \theta}.
\]

Proof. The terms in (6) with \( \langle \omega_G|T, \theta \rangle = 0 \) can be dropped. If \( \langle \omega_G|T, \theta \rangle \neq 0 \) then, by Lemma 7.8 (and the definition of \( \mathcal{L} \) prior Lemma 7.1), \( T^* \) is \( C_{G^*}(s) \)-conjugate to a torus in some \( L \in \mathcal{L} \). If (a) \( \mathcal{L} \) consists of a single \( C_{G^*}(s) \)-orbit then we can fix \( L \in \mathcal{L} \). Furthermore, if (b) \( \mathcal{L} \) consists of a single \( C_{G^*}(s) \)-orbit then the elements of \( (T, \theta) \in \mathcal{R}_s \) are in bijection with \( \mathcal{R}_s \). Note that \( \varepsilon_G = \varepsilon_L \) for every Levi subgroup \( \varepsilon_L \) of \( \varepsilon_G \).

Suppose that (1) holds. As \( \dim U \) is odd, (a) holds by Lemma 7.12. As \(-1\) is not an eigenvalue of \( s \), (b) holds too. So the result follows by Lemma 7.10(5).

(2) In this case we have two Levi subgroups \( L_1^*, L_2^* \in \mathcal{L} \) which are not conjugate in \( C_{G^*}(s) \), see Lemma 7.5. So a maximal torus \( T^* \) of \( C_{G^*}(s) \) is \( C_{G^*}(s) \)-conjugate either (i) to a torus of \( L_1^* \) but not \( L_2^* \), or to a torus of \( L_2^* \) but not \( L_1^* \), or (ii) to a torus of each \( L_1^*, L_2^* \). The option (i) happens if and only if \( T_1^* \) is exceptional (Lemma 7.4).
If $T_2^*$ is not exceptional then we are in the case (5) of Lemma 7.10. As $|W_{L_2}(T)\theta| = |W_{L_2}(T)\theta|$, when $T$ is viewed as a torus in $L_1^*$ or $L_2^*$, we have:

$$\frac{(\omega_G|T,\theta)}{|W(T)\theta|} \varepsilon_{G^T} r_{T,\theta} = \frac{1}{|W_{L_1}(T)\theta|} \varepsilon_{L_1} \varepsilon_{G^T} r_{T,\theta} + \frac{1}{|W_{L_2}(T)\theta|} \varepsilon_{L_2} \varepsilon_{G^T} r_{T,\theta}.$$

If $T_1^*$ is exceptional then $T^*$ is $C_{G^*}(s)$-conjugate to a torus either in $L_1^*$ or in $L_2^*$, but not in both of them. As we are in case (1) of Lemma 7.10, the term with this $T$ occurs only in one of the sums in (7.11). So again (by the convention in the parentheses of (2)) we have:

$$\frac{(\omega_G|T,\theta)}{|W(T)\theta|} \varepsilon_{G^T} r_{T,\theta} = \frac{1}{|W_{L_1}(T)\theta|} \varepsilon_{L_1} \varepsilon_{G^T} r_{T,\theta} + \frac{1}{|W_{L_2}(T)\theta|} \varepsilon_{L_2} \varepsilon_{G^T} r_{T,\theta}.$$  

This implies the second equality of (7.11) as $\varepsilon_{L_1} \varepsilon_{L_2} = \varepsilon_G$.

(3) As $-1$ is an eigenvalue of $s$, all subgroups of $L$ are $C_{G^*}(s)$-conjugate by Lemma 7.2. So $L \in L$ can be fixed. If dim $U$ is odd, then either (3) or (4) of Lemma 7.10 holds. If (3) of Lemma 7.10 holds then $L$ consists of two $C_{L^*}(s)$-orbits, and hence the element $(T, \theta) \in R_s$ corresponds to two elements $(T, \theta), (T', \theta')$ in $R_s^L$. Therefore,

$$\frac{(\omega_G|T,\theta)}{|W(T)\theta|} \varepsilon_{G^T} r_{T,\theta} = \frac{1}{|W_{L^*}(T)\theta|} \varepsilon_{L^*} \varepsilon_{G^T} r_{T,\theta} + \frac{1}{|W_{L^*}(T)\theta|} \varepsilon_{L^*} \varepsilon_{G^T} r_{T,\theta}.$$  

If (4) of Lemma 7.10 holds then $L$ consists of a single $C_{L^*}(s)$-orbit. Therefore, we have

$$\frac{(\omega_G|T,\theta)}{|W(T)\theta|} \varepsilon_{G^T} r_{T,\theta} = \frac{1}{|W_{L^*}(T)\theta|} \varepsilon_{G^T} r_{T,\theta}.$$  

So (11) holds.

Finally suppose that dim $U$ is even. We show that (12) holds.

Let $S$ be as in Lemma 7.8. By Lemma 7.8, $S$ consists of two $C_{L^*}(s)$-orbits if and only if $T_3^*$ is non-exceptional and $T_3^*$ is exceptional. In this case (3) of Lemma 7.10 holds. Then there is a single class of $C_{G^*}(s)$-conjugacy class of tori $T \subset C_{L^*}(s)$, which splits in two $C_{L^*}(s)$-conjugacy classes. As $\frac{(\omega_G|T,\theta)}{|W(T)\theta|} = \frac{2}{|W_{L^*}(T)\theta|}$, we get (11).

So we are left with the situation where $S$ consists of a single $C_{L^*}(s)$-orbit. Then we have one of the cases (1), (2) or (4) of Lemma 7.10.

If (1) of Lemma 7.10 holds then there are two $C_{G^*}(s)$-conjugacy classes of tori $T \subset C_{L^*}(s)$, and each of them form a single class of $C_{L^*}(s)$-conjugate tori. So (12) holds.

Suppose that (2) of Lemma 7.10 holds. Then there is a single class of $C_{G^*}(s)$-conjugacy class of tori $T \subset C_{L^*}(s)$, and all these tori are $C_{L^*}(s)$-conjugate. So again (12) holds.

Suppose that (4) of Lemma 7.10 holds. Then there is a single class of $C_{G^*}(s)$-conjugacy class of tori $T \subset C_{L^*}(s)$, and all these tori are $C_{L^*}(s)$-conjugate. So (12) holds.

This implies the statement (3) of the proposition.

Next we show that the right hand side of the equalities in Lemmas 7.7 and 7.11 can be expressed in terms of regular characters of $L$. Recall that if $L$ is a Levi subgroup of a parabolic subgroup $P$ of $G$ and $\lambda$ is a character of $L$ then $\lambda^{\#G}$ denotes the Harish-Chandra induced character. This is exactly the induced character $\lambda^P_\theta$, where $\lambda_\theta$ is the inflation of $\lambda$ to $P$ via the projection $P \to L$. Note that $\lambda^{\#G}$ does not depend on the choice of $P$ [8, 70.10]. Note that if $L$ is abelian then every character of $L$ is regular (by convention).
Similarly to the usage of the notation \(R_s\), for \(s \in L^*\) we denote by \(R_s^L\) the set of representatives of the \(L\)-conjugacy classes of pairs \((T, \theta)\) with \(T \subset L\). They are in bijection with \(C_{L^*}(s)\)-conjugacy classes of maximal tori \(T^* \subset C_{L^*}(s)\). The argument is based on the fact that \(T \subset L\) then \(R_{T, \theta} = (R_{T, \theta}^L)^{\#G}\), see [6, 7.4.4].

**Proposition 7.12** Let \(G = SO(V)\) and let \(s \in G^*\) be a semisimple element such that \(St^+_s \neq 0\) and \(V^*_s \neq 0\). Let \(U\) be a maximal totally singular (or totally isotropic) subspace of \(V^*_s\), \(P_U\) the stabilizer of \(U\) in \(G^*\) and \(L^*\) a Levi subgroup of \(P_U\) such that \(L^*V^*_s = V^*_s\). Then

\[
\sum_{(T, \theta) \in R_s^L} \frac{1}{|W_L(T)|} \varepsilon_L \varepsilon_T R_{T, \theta} = \left( \sum_{(T, \theta) \in R_s^L} \frac{1}{|W_L(T)|} \varepsilon_G \varepsilon_T R_{T, \theta}^L \right)^{\#G}
\]

and

\[
\lambda_s := \sum_{(T, \theta) \in R_s^L} \frac{1}{|W_L(T)|} \varepsilon_L \varepsilon_T R_{T, \theta}^L
\]

is a regular irreducible character of \(L\). In addition, \(\lambda_s\) is a unique regular character of \(E_s^L\).

Proof. Let \(u = \dim U\). Observe that the group \(C_{L^*}(s)\) is connected. Indeed, if \(\dim V\) is odd then \(L^* \cong GL_u(F_q) \times Sp_{2(n-u)}(F_q)\), and hence \(C_{L^*}(s) \cong GL_u(F_q) \times C_{Sp_{2(n-u)}(F_q)}(s_2)\). So the claim is true as the latter group is simply connected (Lemma 2.2). If \(\dim V\) is even then \(L^* = G_1^* \times G_2^*\), where \(G_1^* \cong GL_u(F_q)\) and \(G_2^* \cong SO_{2(n-u)}(F_q)\). Therefore, \(C_{L^*}(s)\) is connected if and only if \(CG_2^*(s_2)\) is connected. Again, this is the case by Lemma 2.2.

The expression for \(\lambda_s\) coincides with the class function on \(L\) defined in [11, 14.40]. As \(C_{L^*}(s)\) is connected, it follows, by [11, 14.43], that \(\lambda_s\) is the unique regular character of \(E_s^L\).

By [6, 7.4.4], if \(T \subset L\) then \(R_{T, \theta} = (R_{T, \theta}^L)^{\#G}\), where \(R_{T, \theta}^L\) is a Deligne-Lusztig character of \(L\). Note that \(\varepsilon_L = \varepsilon_G\) and \(\varepsilon_T\) is unchange when we view \(T\) as a torus of \(L\). So the proposition follows.

The result of Proposition 7.12 is not sufficient to prove that \(\lambda_s^{\#G}\) is multiplicity free. For this we shall additionally show that \(\lambda_s = St_{G_1} \otimes \rho_{s_2}\), where \(\rho_{s_2}\) is a regular character of \(G_2\).

**Proposition 7.13** Let \(\lambda_s\) be as in Proposition 7.12 Then \(\lambda_s = St_{G_1} \otimes \rho_{s_2}\), where \(\rho_{s_2}\) is a regular character of \(G_2\) from the Lusztig series \(E_{s_2}\).

Proof. Recall that \(L = G_1 \times G_2\), where \(G_1 \cong GL_1(F_q)\) and \(G_2 \cong SO_{\dim V - 2r}(F_q)\). Then \(T = T_1 \times T_2\) and \(\theta = \theta_1 \otimes \theta_2\), where \(T_i\) is a maximal torus of \(G_i\) and \(\theta_i\) is a linear character of \(T_i\), \(i = 1, 2\).

It is well known that \(R_{T, \theta}^L = R_{T_1, \theta_1}^G \otimes R_{T_2, \theta_2}^G\), \(\varepsilon_L = \varepsilon_{G_1} \cdot \varepsilon_{G_2}\) and \(\varepsilon_T = \varepsilon_{T_1} \cdot \varepsilon_{T_2}\). Furthermore, \(W_L(T, \theta) = W_{G_1}(T_1)_{\theta_1} \times W_{G_2}(T_2)_{\theta_2} = W_{G_1}(T_1) \times W_{G_2}(T_2)_{\theta_2}\) as \(\theta_1 = 1_{T_1}\) and \(s_2\) does not have eigenvalue \(1\). It follows that

\[
\sum_{(T, \theta) \in R_s^L} \frac{\varepsilon_G \varepsilon_T}{|W_G(T)|} R_{T, \theta}^L = \left( \sum_{(T_1, \theta_1) \in R_{G_1}^L} \frac{\varepsilon_{G_1} \varepsilon_{T_1}}{|W_{G_1}(T_1)|} R_{T_1, \theta_1}^G \right) \otimes \left( \sum_{(T_2, \theta_2) \in R_{G_2}^L} \frac{\varepsilon_{G_2} \varepsilon_{T_2}}{|W_{G_2}(T_2)|} R_{T_2, \theta_2}^G \right).
\]

The expression in the first parentheses yields the Steinberg character of \(G_1\), see [6, 7.6]. Therefore, \(\lambda' = St_{G_1}\) and \(\lambda''\) equals the expression in the second parentheses. The latter coincides with the class function on \(G_2\) defined in [11, 14.10]. As \(\lambda''\) is irreducible, it is
exactly the unique regular character of $G_2$ that belongs to the Lusztig series $\mathcal{E}^{G_2}$, see [11, 14.49]. (Alternatively, as $s = \text{diag}(s_1, s_2)$, the regular character of the Lusztig series $\mathcal{E}_s$ is the product of the regular characters in $\mathcal{E}^{G_1}$ and $\mathcal{E}^{G_2}$. As $s_1 = \text{Id}$, the regular characters in $\mathcal{E}^{G_1}$ is $St_{G_1}$, whence the result.)

Recall that if $-1$ is not an eigenvalue of $s$, dim $V$ is even and dim $V_s* \equiv 0 \pmod{4}$ (that is, dim $U$ is even), then $\mathcal{L}$ consists of two $C_{G^*}(s)$-orbits (Lemma 7.2). We choose $L_1^s, L_2^s$ from distinct orbits, and write $L_1 = G_1^s \times G_2^s$ and $L_2 = G_1^{s*} \times G_2$. (Note that $\rho_{s_2}$, as defined in Proposition 7.13 is the same in both the cases, as $\rho_{s_2}$ depends only on $s_2$.) In the other cases $\mathcal{L}$ forms a single $C_{G^*}(s)$-orbit.

**Proposition 7.14** Let $G = SO(V)$, and let $s \in G^*$ be a semisimple element. Let $\text{Spec} \, s$ denote the set of eigenvalues of $s$ on $V^*$. Suppose that $V_s* \neq 0$ and $St_{s^*}^+ \neq 0$.

1. If $\dim V$ is odd then $St_s^+ = (St_G \otimes \rho_{s_2})^G$.
2. If $\dim V$ is even then

$$St_s^+ = \begin{cases} (St_G \otimes \rho_{s_2})^G & \text{if } -1 \in \text{Spec} \, s, \\ 2 \cdot (St_G \otimes \rho_{s_2})^G & \text{if } -1 \notin \text{Spec} \, s \text{ and } \dim V_s* \equiv 2 \pmod{4}, \\ ((St_G^1 \oplus St_G^{s*}) \otimes \rho_{s_2})^G & \text{if } -1 \notin \text{Spec} \, s \text{ and } \dim V_s* \equiv 0 \pmod{4}. \end{cases}$$

Proof. This follows from Lemma 7.11 and Proposition 7.13. (Note that $\rho_{s_2}$ has to be omitted if $s = 1$.)

If $q$ is odd, we denote by $1_G^-$ the only non-trivial one-dimensional character of $G$, and set $St_G^- = St_G \otimes 1_G^-$. It is well known that $St_G$ and $St_G^-$ are the only irreducible characters of $G$ of defect 0.

**Corollary 7.15** (1) Let $G = SO_{2n+1}(q)$, $n > 0$. Then $(St_G^+, St_G) = 1 = (St_G^-, St_G)$, and $(St_G^+, 1_G) \neq 0$ if and only if $n = 1$. In addition, $(St_G^+, 1_G^-) = 0$.

(2) Let $G = SO_{2n}(q)$, $n > 1$. Then $(St_G^+, St_G) = 1 + \alpha$, and $(St_G^+, 1_G) = 0 = (St_G^+, 1_G^-)$. In addition, $(St_G^+, St_G^-) = 1$.

Proof. It is well known that $1_G, St_G \in \mathcal{E}_1$, that is, $s = 1$. If $q$ is odd then $1_G^-, St_G^- \in \mathcal{E}_s$ for $s = -\text{Id}$. So $1_G^-$ is a constituent of $St_G^+$ if and only if $St_G^-$ is one-dimensional, and hence $G$ is abelian. As $n > 1$, this is not the case. So the claims about $1_G^-$ follow.

By Proposition 6.4, $St_{-\text{Id}}^+$ is the regular character in $\mathcal{E}_{-\text{Id}}$, which is exactly $St_G^-$. It is also well known that the Harish-Chandra restriction of $St_G$ to any Levi subgroup $L$ of $G$ is $St_L$, see [11, p. 72]. If $s = 1$ then $L = G_1$.

1. In this case $St_1^+ = St_{G_1}^G$, where $G_1 \cong GL_n(q)$. By Harish-Chandra reciprocity,

$$(St_{G_1}^G, St_G) = (St_G(1), St_{G_1}(1)) = 1.$$ In addition, $(St_{G_1}^G, 1_G) = (St_{G_1}(1), 1_G)$, which is non-zero if and only if $St_G(1) = 1_{G_1}$, that is, when $G_1$ is abelian. This implies $n = 3$.

2. Let $\alpha = -1$. Then $St_1^+ = 0$ by Lemma 5.2. So $1_G, St_G$ are not constituents of $St_G^+$.

Let $\alpha = 1$. Then $St_1^+ = 2 \cdot St_{G_1}^G$ if dim $V^*/2$ is odd, and $St_{G_1}^G + St_{G_1}^{s*}$ otherwise.

Here $G_1 \cong GL_n(q)$, as well as $G'_1$ and $G''_1$. As above, $St_G$ is a constituent of each $St_{G_1}^G$ and $St_{G_1}^{s*}$ by Harish-Chandra reciprocity, whereas $1_G$ is a constituent of neither $St_{G_1}^G$ nor $St_{G_1}^{s*}$, unless $G'_1 \cong G''_1$ is a torus. The letter implies $n = 1$. 

32
8 The decomposition of \((St_{G_1} \otimes \rho_{s_2})^G\)

Let \(U \subset V^*\) be a totally singular (or totally isotropic) subspace, \(P_U\) its stabilizer in \(G^*\) and \(L^*\) a Levi subgroup of \(P_U\). Then \(L^* = G_1^* \times G_2^*\), where \(G_1^* \cong GL(U), G_2^* \cong SO(V^*)\) and \(V^*\) be a complement of \(U\) in \(U^\perp\). These correspond to respective objects in \(G\): there is a totally singular subspace \(R\) of \(V\) of dimension \(\dim U\) whose stabilizer in \(G\) is a parabolic subgroup and its Levi subgroup \(L\) is dual to \(L^*\). If we set \(V' = R^\perp/R\) then \(SO(V')\) is dual to \(G_2^*\).

Let \(\rho\) be a regular character of \(SO(V')\). In this section we show that \((St_{G_1} \otimes \rho)^G\) is a multiplicity free character of \(G\). We denote by \(S_n\) the symmetric group of permutations of \(n\) objects.

We start with a special case where \(G^* = SO_{2n}^+(q), \ n\) even, and \(s = 1\). Let \(U_1, U_2\) be two totally singular subspaces of \(V^*\) of dimension \(n\) such that \(gU_1 \neq U_2\) for any \(g \in G^*\) (see Lemma 2.6). Let \(P_i\) be the stabilizer of \(U_i\) for \(i = 1, 2\). Then \(P_1, P_2\) are non-conjugate parabolic subgroups of \(G\). Let \(L_1^*, L_2^*\) be Levi subgroups of \(P_1, P_2\), respectively. By Lemma 7.2 for \(s = 1\), the groups \(L_1^*, L_2^*\) are not conjugate in \(G^*\). Let \(W\) be the Weyl group of type \(D_n\). Then the Weyl groups \(W_1, W_2\) of \(L_1^*, L_2^*\), respectively, can be viewed as subgroups of \(W\), each isomorphic to \(S_n\), the symmetric group. Groups \(W_1, W_2\) are known to be conjugate in \(W(B_n)\), but not in \(W\) if \(n\) is even.

**Lemma 8.1** Let \(W\) be the Weyl group of type \(D_n\) for \(n > 2\) even.

(1) \(W_1, W_2\) are not conjugate in \(W\);
(2) \((1^W_{W_1}, 1^W_{W_2}) = n/2\) and \((1^W_{W_1}, 1^W_{W_1}) = (n + 2)/2\).

**Proof.** View \(W\) as a group of monomial matrices with non-zero entries \(\pm 1\), see comments prior Lemma 6.6. Let \(D\) be the subgroup of diagonal matrices in \(W\). Then \(D\) is of exponent 2, of order \(2^{n-1}\), and \(\det d = 1\) for every \(d \in D\). In addition, \(W\) contains a subgroup \(W_1\), consisting of monomial matrices with entries 0, 1 and \(W = DW_1 = W_1 D\). Let \(t = \text{diag}(-1, 1, \ldots, 1)\). Then \(t \notin W\), but \(tWt^{-1} = W\). We set \(W_2 = tW_1t^{-1}\). We first observe that \(W_1, W_2\) are not conjugate in \(W\). Indeed, if \(W_2 = xW_1x^{-1}\) for \(x \in W\) then \(tx\) normalizes \(W_1\). Let \(x = ds\), where \(d \in D, s \in W_1\). So \(td\) normalizes \(W_1\). One easily observes that a diagonal matrix normalizing \(W_1\) must be scalar. Therefore, \(t = \pm d\), which is false as \(\det d = \det(-d) = 1\) whereas \(\det t = -1\).

Thus, \(W_1, W_2\) are not conjugate in \(W\). We show that \((1^W_{W_1}, 1^W_{W_1}) = n/2\). The fact that \((1^W_{W_1}, 1^W_{W_1}) = (n + 2)/2\) is known, and also follows from our reasoning below. It is well known that \((1^W_{W_1}, 1^W_{W_1})\) equals the number of the orbits of \(W_2\) on the cosets \(gW_1\). So we proceed with computing these orbits.

As \(W = DW_1\), it follows that representatives of cosets \(gW_1\) can be chosen in \(D\). Obviously, all cosets \(dW_1\) are distinct when \(d\) runs over \(D\). This implies that \(W_1dW_1 = W_1dW_1\) if and only if the ranks of the matrices \(d - Id\) and \(d' - Id\) coincide. The ranks may be equal \(2i\) for \(i = 0, \ldots, n/2\). So \((1^W_{W_1}, 1^W_{W_1}) = (n + 2)/2\).

Let \(Y\) be the subgroup of \(W_1\) consisting of matrices with 1 at the (1, 1)-position. Then \(Y \cong S_{n-1}\) and \([t, Y] = 1\). Note that \(W_1 = \cup_{i=1}^{n} a_i Y\), where \(a_i \in W_1\) is the matrix whose non-zero non-diagonal entries are at the \((1, i)\) and \((i, 1)\)-positions. Clearly, \(Y\) is also contained in \(W_2\). It is easy to observe that representatives of the double cosets \(YxW_1\) \((x \in W)\) can be chosen to be \(\text{diag}(\pm 1, 1, \ldots, 1, -1, \ldots, -1)\), where the number of entries \(-1\) may be 0. As \(W_2\) contains the matrix whose entries at the positions \((1, i)\) and \((i, 1)\) are \(-1\) and 0 or 1 elsewhere, it follows that representatives of the double cosets \(YxW_1\) \((x \in W)\) can be chosen to be of shape \(d_i = \text{diag}(1, 1, \ldots, 1, -1, \ldots, -1)\), where the number of 1-entries is even.
and ranges between 1 and \( n/2 \). Moreover, one observes that all double cosets \( W_2 d_i W_1 \) are distinct, and hence \( d_1, \ldots, d_{n/2} \) are representatives of all double cosets \( W_1 x W_1 \) (\( x \in W \)). Therefore, \((1^W_{W_1}, 1^W_{W_2}) = n/2\), as claimed.

As both \( 1^W_{W_1} \) and \( 1^W_{W_2} \) are multiplicity free, it follows that \( 1^W_{W_1} \) and \( 1^W_{W_2} \) have exactly \( n/2 \) common irreducible constituents.

Remark. One can identify the characters of \( W(D_n) \) occurring in both \( 1^W_{W_1} \) and \( 1^W_{W_2} \). In the bijection between the conjugacy classes and irreducible characters of \( W(D_n) \) described in [14 Section 5] they correspond to the partitions \([k], [l]\), where \( k + l = n \) and \( k \neq l \). (This follows from [14 Proposition 6.1.5], which is more precise than our elementary lemma 8.1.) So \( k \) runs over the numbers \( 0, 2, \ldots, n \) except \( n/2 \) (so \( l = n - k \)), one obtains exactly \( n/2 \) characters.

**Lemma 8.2** With above notation, let \( St_i \) be the Steinberg character of \( L_i \). Then \( St^{#G}_{i} \) and \( St^{#G}_{2} \) are multiplicity free characters, having \( n/2 \) common irreducible constituents. In addition, \( St_G \) is a constituent of both \( St^{#G}_{i} \) and \( St^{#G}_{2} \).

Proof. It is known [8, 70.24] that the irreducible constituents of \( St^{#G}_{i} \) for \( i = 1, 2 \) are in bijection with the irreducible constituents of \( \varepsilon^W_i \), where \( W = W(D_n) \), \( W_1, W_2 \) are as in Lemma 8.1 and \( \varepsilon_i \) is the sign character of \( W_i \). Moreover, the bijection regards the multiplicities of the irreducible constituents. Note that \( \varepsilon^W_i = (1^W_{W_i}) \otimes \nu_i \), where \( \nu_i \) is a linear character of \( W \) such that \( \nu_i|_{W_i} = \varepsilon_i \) (there are two such characters \( \nu_i \) for every \( i = 1, 2 \)). So the first statement of the lemma follows from Lemma 8.1. The proof of Corollary 7.15 contains a proof of the additional statement.

Let \( G \) be a group with BN-pair, which later will be specified to be \( SO(V) \). Let \( W(G) \) be the Weyl group of the BN-pair. In terms of algebraic groups \( W(T) \) as defined above for a maximal torus \( T \) of a Borel subgroup of \( G \). By Harish-Chandra theory, every irreducible character \( \lambda \) of \( L \) is a constituent of \( \delta^{#L} \), where \( \delta \) is a cuspidal character of a Levi subgroup \( M \), say, of some parabolic subgroup \( Q \) of \( L \). Furthermore, the decomposition of \( \delta^{#L} \) as a sum of irreducible constituents is described in terms of the group \( W(\delta) := \{ w \in W(G) : w(M) = M \text{ and } w(\delta) = \delta \} \), see [21]. This can be also applied to \( \text{Irr} \ L \), so we define \( W_L(\delta) = \{ w \in W(L) : w(M) = M \text{ and } w(\delta) = \delta \} \). The standard embedding of \( L \) into \( G \) yields an embedding \( W_L(\delta) \) into \( W_G \), and hence \( W_L(\delta) = W(\delta) \cap W_L \).

Let \( \zeta \in \text{Irr} \ G \) be a constituent of \( \lambda^{#G} \). Then \( \zeta \) is a constituent of \( \delta^{#G} \) by transitivity of Harish-Chandra induction. By the above we can label \( \zeta \) and \( \rho \) by elements of \( \text{Irr} \ W(L) \) and \( \text{Irr} \ W_L(\delta) \), respectively. So let \( \zeta = \zeta_\mu \) and \( \lambda = \lambda_\nu \) for some \( \mu \in \text{Irr} \ W(\delta) \) and \( \nu \in \text{Irr} \ W_L(\delta) \). Then \( (\zeta, \lambda^{#G}) = (\mu, \nu^{W(\delta)}) \) by Howlett and Lehrer [21 Theorem 5.9] and Geck [13, Corollary 2].

If \( \lambda \) is a regular character of \( L \) then \( \delta \) is a regular character of \( M \), see for instance [25 Proposition 2.5]. Note that \( \delta \) is not unique in general, however, if \( \delta \in \mathcal{E}^L_\mu \) then \( \delta \) can be chosen in \( \mathcal{E}^M_\mu \). (Indeed, let \( \Gamma_L \) be a Gelfand-Graev character containing \( \rho \) as a constituent. By [10 Theorem 2.9], the Harish-Chandra restriction \( \Gamma_M \) of \( \Gamma_L \) to \( M \) is a Gelfand-Graev character of \( M \). Let \( \lambda_M \) denote the Harish-Chandra restriction of \( \lambda \) to \( M \). (In general, \( \rho_M \) is reducible.) By Harish-Chandra reciprocity, \( \delta \) is a constituent of \( \lambda_M \). Let \( \mathcal{E}^M_\mu \) be the rational Lusztig series of \( \text{Irr} \ M \) containing \( \lambda_M \). Then the irreducible constituents of \( \delta^{#L} \) belongs to \( \mathcal{E}^L_\mu \) (see, for instance, [5 Theorem 8.25]). In particular, \( \lambda \in \mathcal{E}^L_\mu \), and hence \( \mathcal{E}^L_{\lambda_\nu} = \mathcal{E}^L_{\mu} \). It follows that \( s, s' \) are conjugate in \( L \). Therefore, we can take \( s' \) for \( s \).

Let now \( G = SO(V) \) and let \( L \) be a Levi subgroup of \( G \) defined in the beginning of this section. We shall prove that \( \nu^{W(\delta)} \) is multiplicity free for \( \lambda = \lambda_s = St_{G_1} \otimes \rho_{s_2} \). As \( L \) is the
direct product of $G_1$ and $G_2$, it follows that $M = M_1 \times M_2$ and $\delta = \delta_1 \otimes \delta_2$, where $M_i$ (for $i = 1, 2$) is a Levi subgroup of some parabolic subgroup of $G_i$ and $\delta_i$ is a cuspidal character of $M_i$. We have to show that $(St_{G_1} \otimes \rho_{s_2})^{\# G}$ is a multiplicity free character of $G$. We shall do this by analysis of $(\delta_1 \otimes \delta_2)^{\# G}$ (this is meaningful as $M$ is a quotient group of a parabolic subgroup of $G$ by a unipotent normal subgroup). We first obtain some information on the decomposition of $(\delta_1 \otimes \delta_2)^{\# G}$ (specified to our situation where $\lambda = \lambda_s = St_{G_1} \otimes \rho_{s_2}$).

An advantage of our situation is that $W(\delta) \subset W = W_{X_1} \times W_{X_2}$. As $St_{G_1}$ is well known to be a constituent of $1^{\# G_1}_T$, we have $\delta = 1_{T_0} \otimes \delta_2$, where $\delta_2$ is a cuspidal character of $M_2$.

**Lemma 8.3** $W(\delta) \subset W(X_1) \times W(X_2)$.

Proof. By the definition of $W(\delta)$ above, if $w \in W(\delta)$ then $w(M) = M$. By the comments prior the lemma, we can assume that $\delta$ is a regular character of $M$ and $\delta \in \mathcal{E}^M$. Recall that $C_L(s)$ is connected (see the proof of Proposition (7.12)), and hence so is $C_M(s)$ by Lemma (2.2). It follows that that $\delta$ is the only regular character in $\mathcal{E}^M$, see [11, 14.40, 14.43], and

$$
\delta = \sum_{(T, \theta) \in \mathcal{R}^M} a_{T, \theta} \cdot R_{T, \theta}^M,
$$

where $R_{T, \theta}^M$ are Deligne-Lusztig characters of $M$ and $a_{T, \theta}$ are rational numbers, specified in [11 14.40].

Let $w \in W(\delta)$. Then $w$ acts on $M$ as an automorphism, and this induces an action of $w$ on the characters of $M$. Therefore,

$$
\delta = w(\delta) = \sum_{(T, \theta) \in \mathcal{R}^M} a_{T, \theta} \cdot w(R_{T, \theta}^M).
$$

Observe that $w(R_{T, \theta}^M) = R_{w(T), w(\theta)}^M$. Therefore, $w(R_{T, \theta}^M)$ is a Deligne-Lusztig character of $M$. Furthermore, distinct Deligne-Lusztig characters of $M$ are orthogonal, and if $R_{T, \theta}^M = R_{T', \theta'}^M$ then $(T', \theta') \in \mathcal{R}^M$, see [11 11.15]. So $\delta = w(\delta)$ implies $(w(T), w(\theta)) \in \mathcal{R}^M$.

Let $T^*$ be a torus in $M^*$ dual to $T$. (Note that $T^*$ is also dual to $T$ under the duality $G \to G^*$ as $M$ is a Levi subgroup of $G$.) Recall that $(T, \theta) \in \mathcal{R}^M$ is equivalent to saying that $s \in T^*$ up to conjugacy in $M^*$. Therefore, we can choose $T^* \subset M^*$ so that $s \in T^*$. According with Section 3, let $V^* = V_1^* \oplus \cdots \oplus V_k^* \oplus V_{k+1}^* \oplus \cdots \oplus V_{k+l}^*$ be the $T^*$-decomposition of $V^*$, and $T^* = T_1^* \times \cdots \times T_k^* \times T_{k+1}^* \times \cdots \times T_{k+l}^*$ the respective decomposition of $T^*$. Correspondingly, let $s = (s_1, \ldots, s_{k+l})$, where $s_i \in T_i^*$ for $i = 1, \ldots, k+l$. Recall that $V_s^*$ denotes the 1-eigenspace of $s$ on $V^*$. So $V_s^*$ is the sum of the $V_i^*$ that are contained in $V_s^*$. Set $J = \{i : V_i \subset V_s^*\}$. Then $s_i = 1$ if and only if $i \in J$. Taking the dual decomposition $T = T_1 \times \cdots \times T_{k+l}$ and $\theta = (\theta_1, \ldots, \theta_{k+l})$, where $\theta_i$ is the linear character of $T_i$ corresponding to $s_i$ $(i = 1, \ldots, k+l)$, we observe that $\theta_i(T_i) = 1$ if and only if $i \in J$.

As $(w(T), w(\theta)) \in \mathcal{R}^M$, we can apply the previous paragraph reasoning to $(w(T), w(\theta))$. We conclude that $w(\theta_i)(w(T_i)) = 1$ implies that $w(T_i)^*$ acts trivially on $V_s^*$, and hence $w(T_i)^* \subset X_i^*$. (We may view $T_i$ as a subtorus of $T$, and $T_i^*$ as a subtorus of $T^*$.) In turn, this implies $w(T_i) \subset X_1$ for $i \leq k$, and the lemma follows.

**Corollary 8.4** $W(\delta) = W(X_1) \times W_2(\delta_2)$, where $W_2(\delta_2) = \{w \in W(X_2) : w(M_2) = M_2, w(\delta_2) = \delta_2\}$. Similarly, $W_L(\delta) = W(G_1) \times W_2(\delta_2)$. 
Proof. By Lemma 8.3, \( W(\delta) \subset W(X_1) \times W(X_2) \), and hence \( W(\delta) = W_1(\delta_1) \times W_2(\delta_2) \). It is well known that \( \text{St}_{G_1} \) is a constituent of \( (1_{T_0})^\#G_1 \), where \( T_0 \) is a maximal torus of a Borel subgroup of \( G_1 \) and \( 1_{T_0} \) is the trivial character of \( T_0 \). Thus, here \( M_1 = T_0 \) and \( \delta_1 = 1_{T_0} = 1_{M_1} \). It follows that \( W_1(\delta_1) \) coincides \( W(X_1) \). This implies the lemma.

Now we come back to proving that \( \lambda^\#G \) is multiplicity free. As explained prior Lemma 8.3, it suffices to show that \( \nu^{W(\delta)} \) is multiplicity free. As \( \nu \in \text{Irr}(W(G_1) \times W_2(\delta_2)) \), we can write \( \nu = \nu_1 \otimes \nu_2 \), where \( \nu_1 \in \text{Irr}(W(G_1)) \), \( \nu_2 \in \text{Irr}(W_2(\delta_2)) \).

Lemma 8.5 \( \nu^{W(\delta)} = \nu_1^{W(X_1)} \otimes \nu_2 \) is a multiplicity free character of \( W(\delta) \).

Proof. We have \( \nu^{W(\delta)} = \nu_1^{W(X_1)} \otimes \nu_2^{W(\delta_2)} \). As \( \nu_2 \in \text{Irr}(W_2(\delta_2)) \), \( \nu_2^{W(\delta_2)} \) is simply \( \nu_2 \) itself. So it suffices to prove that \( \nu_1^{W(X_1)} \) is multiplicity free. Recall that \( \nu_1 \) is the Harish-Chandra correspondent of \( \text{St}_{G_1} \) and \( G_1 = GL_m(q) \) for some \( m \). It is well known that \( W(GL_m(q)) \cong S_m \) and \( \nu_1 \) is the sign character of \( S_m \). Furthermore, \( X_1 \cong SO^+_{2m}(q) \), so \( W(X_1) \) is of type \( D_m \). The fact that \( \nu_1^{W(X_1)} \) is multiplicity free is well known and has been already discussed in the proof of Lemma 8.1.

Proposition 8.6 Let \( \lambda_s \) be as in Lemma 7.13. Then the character \( \lambda^\#G \) is multiplicity free.

Proofs of Theorems 1.1 and 1.2. Recall that \( \text{St}_{H|G} \) is denoted by \( \text{St}_G^+ \) throughout the paper. Furthermore, it follows from Deligne-Lusztig theory that \( \text{St}_G^+ = \sum_s \text{St}_s^+ \), where \( s \) runs over semisimple elements of \( G^* \), and \( \text{St}_s^+ \) is either zero or the sum of all irreducible constituents of \( \text{St}_G^+ \) that belong to the Lusztig (rational) series \( \mathcal{E}_s \). Let \( V^* \) be the natural module for \( G^* \) and \( V_s^* \) the 1-eigenspace of \( s \) on \( V^* \). Then \( \text{St}_s^+ = 0 \) if and only if \( V_s^* \) is non-zero and of Witt defect 1 (Proposition 6.2). If \( V_s^* = 0 \) then \( \text{St}_s^+ \) is the regular character of \( \mathcal{E}_s \) (Proposition 6.4). Suppose that \( \text{St}_s^+ \neq 0 \) and \( V_s^* \neq 0 \). Then the character \( \text{St}_s^+ \) is described by Proposition 7.14. If \( \dim V \) is odd then \( \text{St}_s^+ \) is Harish-Chandra induced from a regular character of a suitable Levi subgroup of \( G \), and it is multiplicity free by Proposition 8.6. This implies Theorem 1.1 and Theorem 1.2 in this case. Suppose that \( \dim V \) is even. Then \( \text{St}_s^+ \) is the sum of at most two characters, each is Harish-Chandra induced from a regular character of a suitable Levi subgroup of \( G \). So again the results follow from Proposition 8.6. The nature of the regular characters of the Levi subgroups in question is explained in Proposition 7.13.

Examples. (1) Let \( G = SO_3(q) \). Then \( \text{St}_G^+ = 1_G + \Gamma_G \), where \( \Gamma_G \) is the Gelfand-Graev character of \( G \). Moreover, the linear character of \( G \) of order 2 is the only irreducible character of \( G \) that is not a constituent of \( \text{St}_G^+ \).

Indeed, in this case \( G^* = Sp_2(q) = SL_2(q) \), and \( \text{Id} \) is the only semisimple element of \( G^* \) which has eigenvalue 1. If \( s = \text{Id} \) then \( L^* = T^* \), where \( T^* \) is a split maximal torus of \( G^* \). Then \( \text{St}_L = 1_L + 1_T \) so \( \text{St}_G^+ = \text{St}_T^+ \), where \( B \) is a Borel subgroup of \( G \) containing \( T \). It is well known that \( 1_B^G = 1_G + \text{St}_G \). If \( s \neq \text{Id} \) then \( s \) does not have eigenvalue 1, and hence \( \text{St}_s^+ \) is a regular character in \( \mathcal{E}_s \) by Proposition 6.4. Let \( \Gamma_G \) be the sum of all regular characters of \( G \) (as the center of \( G \) is trivial and hence connected, \( \Gamma_G \) is the Gelfand-Graev character of \( G \)). Recall that \( \text{St}_G \) is the regular character of \( \mathcal{E}_1 \). This implies the first claim. We omit the proof of the second claim.

Let \( D \) denote the Curtis duality operation. Then \( \omega_G = D(1_G) + D(\Gamma_G) = \text{St}_G + D(\Gamma_G) \). It is known that \( D(\Gamma_G) \) vanishes at all semisimple elements of \( G \). Therefore, \( w_G \) coincides with \( \text{St}_G \) at the semisimple elements of \( G \). It follows that \( \text{St}_G^+ = \omega_G \cdot \text{St}_G = \text{St}_G^2 \).
(2) \( G = SO_4^-(q), \) \( q \) odd. Then \( G \cong PSL_2(q^2) \times \langle -\text{Id} \rangle \) (as \( \Omega_4^{-}(q) \cong PSL_2(q^2) \)).

By Proposition 6.2, \( St_s^+ \neq 0 \) unless \( s = 1 \) or \( V_s^* \) is an anisotropic subspace of \( V^* \). Suppose that \( St_s^+ \neq 0 \). If \( V_s^* = 0 \) then \( St_s^+ \) is an irreducible regular character of \( G \) (Proposition 6.4). If \( V_s^* \neq 0 \) then \( V_s^* \) is of Witt defect 0 and \( \dim V_s^* = 2 \). We can assume that \( V_s^* \) is the same for all such \( s \). By Lemma 7.2 \( L \) consists of a single \( C_G^*(s) \)-orbit. If \( L \in L \), then \( L \cong L^* \cong GL_1(q) \times SO_2^-(q) \) is a maximal torus of \( G \) of order \( q^2 - 1 \). As the Steinberg character of \( GL_1(q) \) is \( 1_{GL_1(q)} \), by Lemma 7.11 we have \( St_s^+ = 2 \cdot (1_{GL_1(q)} \otimes \rho_{s_2})^{\#G} \) if \( s_2 \neq -\text{Id} \), otherwise \( St_s^+ = (1_{GL_1(q)} \otimes \rho_{s_2})^{\#G} \). (Here \( s_2 \neq 1_{SO_2^-(q)} \).

The irreducible constituents of \( (1_{GL_1(q)} \otimes \rho_{s_2})^{\#G} \) are in bijection with those of \( 1_{SO_2^-(q)} \), and this character is trivial as \( SO_2^+(q) = GL_1(q) \). So each character \( \chi_{s_2} := (1_{GL_1(q)} \otimes \rho_{s_2})^{\#G} \) is irreducible of degree \( q^2 + 1 \). One observes that there are \( (q - 1)/2 \) \( G^* \)-conjugacy classes of \( s \) such that \( V_s^* \) is of Witt defect 0, \( \dim V_s^* = 2 \) and \( s_2 \neq -\text{Id} \) and a single class with \( s_2 = -\text{Id} \). Let \( \Gamma_G \) be the sum of all regular characters of \( G \). (As \( G \) is not with connected center, there are two Gelfand-Graev characters \( \Gamma_1, \Gamma_2 \) of \( G \), and each regular character of \( G \) is a constituent of \( \Gamma_1 \) or \( \Gamma_2 \).) Therefore, \( St_G^+ = \Gamma_G + \sum_{s_2 \neq \pm \text{Id}} \chi_{s_2} - \sum \chi_s \), where \( \chi_s \) runs over the regular characters of \( G \) that does not occur in \( St_G^+ \). This means that \( s \) runs over representatives of the conjugacy classes \( s \) of \( G^* \) such that either \( s = 1 \) or \( V_s^* \) is anisotropic. The number of classes with \( V_s^* \) anisotropic equals \( (q - 1)/2 \). So \( St_G^+ = \Gamma_G + \sum_{s_2 \neq \pm \text{Id}} \chi_{s_2} - St_G - \sum \chi_s \), where \( s \) is such that \( \dim V_s^* = 2 \) and \( V_s^* \) is anisotropic (chosen single in each semisimple conjugacy class).

Acknowledgement. This work was partially supported by the Hausdorff Research Institute for Mathematics (Bonn University, Germany) during the Trimester program ”Universality and Homogeneity” (Autumn 2013).

References

[1] A. Borel, Linear algebraic groups, 2nd edition, Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991

[2] N. Bourbaki, Groupes et algébras de Lie, Ch. IV - VI, Hermann, Paris, 1968.

[3] M. Broué, Dualité de Curtis et caractères de Brauer, Comp. Rend. Acad. Sc. Paris, ser. I-559, 295(1982).

[4] O. Brunat, Reassignment the Steinberg character in finite linear and unitary group, Repres. theory 13(2009), 485 - 459.

[5] M. Cabanes and M. Enguehard, Representation theory of finite reductive groups, Cambridge Univ. Press, Cambridge, 2004.

[6] R. Carter, Finite groups of Lie type: Conjugacy classes and complex characters, Wiley, 1985.

[7] R. Carter, Conjugacy classes in the Weyl groups, Compositio Math. 25(1972), 1 - 59.

[8] C. W. Curtis, I. Reiner, Methods of representation theory with applications to finite groups and orders, Vol. 2, Wiley, New York e.a., 1987.
[9] J. Dieudonné. *La géométrie des groupes classiques*, Troisième édition, Springer-Verlag, Berlin, 1971.

[10] F. Digne, G. Lehrer and J. Michel, The characters of groups of rational points of a reductive group with non-connected center, J. Reine Angew. Math. 425(1992), 155 - 192.

[11] F. Digne and J. Michel, *Representations of finite groups of Lie type*, London Math. Soc. Student Texts 21, Cambridge University Press, 1991.

[12] M. Geck, *An introduction to algebraic geometry and algebraic groups*, Oxford Univ. Press, Oxford, 2003.

[13] M. Geck, A note on Harish-Chandra induction, Manuscripta Math. 80(1993), 393 - 401.

[14] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, Clarendon press, Oxford, 2000.

[15] R. Gow, A generalized character of the orthogonal group and its relation to the Weil representation, Preprint, Dublin, 1993.

[16] R. Gow, An approach to the Steinberg character of a finite group of Lie type via the orthogonal geometry of its adjoint module, J. Algebra 165(1994), 410 - 436.

[17] R. Gow and F. Szechtman, The Weil character of unitary group associated to a finite local ring, Canad. J. Math. 54(2002), 1229 - 1253.

[18] L. Grove, *Classical groups and geometric algebra*, Amer. Math. Soc. Providence, 2002.

[19] G. Hiss and A. Zalesski, The Weil-Steinberg character of finite classical groups, Representation thory 13(2009), 427 - 459.

[20] G. Hiss and A. Zalesski, Tensoring generalized characters with the Steinberg character, Proc. Amer. Math. Soc. 138(2010), 1907 - 1921.

[21] R. Howlett and G. Lehrer, Representations of generic algebras and finite groups of Lie type, Trans. Amer. Math. Soc. 280(1983), 753 - 779.

[22] B. Huppert, Singer-Zyklen in klassischen Gruppen, Math. Z. 117(1970), 141-150.

[23] P. Kleidman and M.Liebeck, *Subgroup structure of finite classical groups*, LMS Lecture Notes 129, Cambridge University Press 1990.

[24] R. Knörr, On the number of characters in a $p$-block of a $p$-solvable group, Illinois J. Math. 28(1984), 181 - 210.

[25] D. Kotlar, On the irreducible components of degenerate Gelfand-Graev characters, J. Algebra 173(1995), 348 - 360.

[26] M. Reeder, On the restriction of Deligne-Lusztig characters, J. Amer. Math. Soc. 20(2007), 573 - 602.
[27] T. Springer and R. Steinberg, Conjugacy classes, In: "Seminar on algebraic groups and related finite groups, Lect. Notes in Math, Springer-Verlag, Berlin, 1970", Part E.

[28] R. Steinberg, Lectures on Chevalley groups. Mimeographic notes, Yale univ, 1969.

National Academy of Sciences, Minsk, Belarus and University of East Anglia, Norwich, UK
e-mail: alexandre.zalesski@gmail.com