On $K_5$ and $K_{3,3}$-minors of graphs and regular matroids

João Paulo Costalonga
joaocostalonga@yahoo.com.br
Universidade Estadual de Maringá
Av. Colombo, 5790, Dpto. de Matemática, Bl. 67, sl. 221
Maringá-PR, 87020-900, Brazil
Tel. +55 44 3011 5356

Abstract

In this paper we prove two main results about obstruction to graph planarity. One is that, if $G$ is a 3-connected graph with a $K_5$-minor and $T$ is a triangle of $G$, then $G$ has a $K_5$-minor $H$, such that $E(T) \subseteq E(H)$.

Other is that if $G$ is a 3-connected simple non-planar graph not isomorphic to $K_5$ and $e, f \in E(G)$, then $G$ has a minor $H$ such that $e, f \in E(H)$ and, up to isomorphisms, $H$ is one of the four non-isomorphic simple graphs obtained from $K_{3,3}$ by the addition of 0, 1 or 2 edges. We generalize this second result to the class of the regular matroids.

Keywords: graph minors, graph planarity, regular matroid, Kuratowski Theorem, Wagner Theorem

1. Introduction

We use the terminology set by Oxley [4]. Our graphs are allowed to have loops and multiple edges. When there is no ambiguity we denote by $uv$ the edge linking the vertices $u$ and $v$. We use the notation $si(G/e)$ for a simplification of $G$ (a graph obtained from $G$ by removing all loops, and all, but one, edges in each parallel class). Usually we choose the edge-set of $si(G)$ satisfying our purposes with no mentions. It is a consequence of Whitney’s 2-Isomorphism Theorem (Theorem 5.3.1 of [4]) that, for each 3-connected graphic matroid $M$, there is, up to isomorphisms, a unique graph whose $M$ is the cycle matroid. We also use this result without mention, so as Kuratowski and Wagner Theorems about graph planarity. When talking about a triangle in a graph we may be referring both to the subgraph corresponding to the triangle as to its edge-set. We say that a set of vertices in a graph is stable if such set has no pair of vertices linked by an edge.

Preprint submitted to Elsevier  May 11, 2014
Let $U$ and $V$ be different maximal stable sets of vertices in $K_{3,3}$. We define $K_{3,3}^{i,j}$ to be the simple graph obtained from $K_{3,3}$ by adding $i$ edges linking pairs of vertices of $U$ and $j$ edges linking pairs of vertices of $V$. By default, we label the vertices of $K_{3,3}^{i,j}$ like in Figure 1.

A family $\mathcal{F}$ of matroids (graphs, resp.) is said to be $k$-rounded in a minor-closed class of matroids (graphs, resp.) $\mathcal{N}$ if each member of $\mathcal{F}$ is $(k+1)$-connected and, for each $(k+1)$-connected matroid (graph, resp.) $M$ of $\mathcal{N}$ with an $\mathcal{F}$-minor and, for each $k$-subset $X \subseteq E(M)$, $M$ has an $\mathcal{F}$-minor with $X$ in its ground set (edge set, resp.). When $\mathcal{N}$ is omitted we consider it as the class of all matroids (graphs, resp.). By Whitney’s 2-isomorphism Theorem, the concepts of $k$-roundedness for graphs and matroids agree, for $k \geq 2$. Such definition is a slight generalization of that one made by Seymour [7]. For more information about $k$-roundedness we refer the reader to Section 12.3 of [4].

The second main result stated in the abstract is Corollary 1.2, that follows from the next Theorem we establish here:

**Theorem 1.1.** The following families of graphs are 2-rounded:

(a) $\{K_{3,3}, K_{0,1}^{0,1}_{3,3}, K_{0,2}^{0,2}_{3,3}, K_{1,1}^{1,1}_{3,3}\}$ and
(b) $\{K_{3,3}, K_{0,1}^{1,1}_{3,3}, K_{0,2}^{2,2}_{3,3}, K_{1,1}^{3,3}, K_{5}\}$.

Moreover, the following families of matroids are 2-rounded in the class of the regular matroids.

(c) $\{M(K_{3,3}), M(K_{0,1}^{0,1}_{3,3}), M(K_{0,2}^{0,2}_{3,3}), M(K_{1,1}^{1,1}_{3,3})\}$ and
(d) $\{M(K_{3,3}), M(K_{0,1}^{1,1}_{3,3}), M(K_{0,2}^{2,2}_{3,3}), M(K_{1,1}^{3,3}), M(K_{5})\}$.

Seymour [6, (7.5)] proved that each 3-connected simple non-planar graph not isomorphic to $K_5$ has a $K_{3,3}$-minor. So, as consequence of Theorem 1.1 we have:

**Corollary 1.2.** If $G$ is a 3-connected simple non-planar graph and $e, f \in E(G)$, then either $G \cong K_5$ or $G$ has a minor $H$ isomorphic to $K_{3,3}, K_{0,1}^{0,1}_{3,3}, K_{0,2}^{0,2}_{3,3}$ or $K_{1,1}^{1,1}_{3,3}$ such that $e, f \in E(H)$. 

2
The next Corollary follows from Theorem 1.1 combined with Bixby’s Theorem about decomposition of connected matroids into 2-sums ([4, Theorem 8.3.1]). To derive the next corollary, instead of Theorem 1.1, we also may use a result of Seymour [7], which states that \(\{U_{2,4}, M(K_{3,3}), M(K_{0,1}^{0,1}), M(K_5)\}\) is 1-rounded.

**Corollary 1.3.** If \(G\) is a non-planar 2-connected graph and \(e \in E(G)\), then \(G\) has a minor \(H\) isomorphic to \(K_5, K_{3,3}\) or \(K_{0,1}^{0,1}\) such that \(e \in E(H)\).

The first result we state at the abstract is Corollary 1.5 that follows from the following theorem:

**Theorem 1.4.** If \(G\) is a 3-connected simple graph with a \(K_{3,3}^{1,1}\)-minor and \(T\) is a triangle of \(G\), then \(G\) has a \(K_{3,3}^{1,1}\)-minor with \(E(T)\) as edge-set of a triangle.

**Corollary 1.5.** If \(G\) is a 3-connected simple graph with a \(K_5\)-minor and \(T\) is a triangle of \(G\), then \(G\) has a \(K_5\)-minor with \(E(T)\) as edge-set of a triangle.

Other results about getting minors preserving a triangle were proved by Asano, Nishizeki and Seymour [1]. Truemper [8] proved that if \(G\) has a \(K_{3,3}\)-minor, and \(e, f\) and \(g\) are the edges of \(G\) adjacent to a degree-3 vertex, then \(G\) has a \(K_{3,3}\)-minor using \(e, f\) and \(g\).

We define a class \(\mathcal{F}\) of 3-connected matroids to be \((3, k, l)\)-rounded in \(\mathcal{N}\) provided the following property holds: if \(M\) is a 3-connected matroid in \(\mathcal{N}\) with an \(\mathcal{F}\)-minor, \(X \subseteq E(M), |X| = k\) and \(r(X) \leq l\), then \(M\) has an \(\mathcal{F}\)-minor \(N\) such that \(X \subseteq E(N)\) and \(N[X] = M[X]\).

Another formulation for Theorem 1.4 and Corollary 1.5 is that \(\{M(K_{3,3}^{1,1})\}\) and \(\{M(K_5)\}\) are \((3, 3, 2)\)-rounded in the class of graphic matroids. Costalona [3] (in the last comments of the introduction) proved:

**Proposition 1.6.** Let \(2 \leq l \leq k \leq 3\). Let \(\mathcal{F}\) be a finite family of matroids and \(\mathcal{N}\) a class of matroids closed under minors. Then, there is a \((3, k, l)\)-rounded family of matroids \(\mathcal{F}'\) such that each \(M \in \mathcal{F}'\) has an \(\mathcal{F}\)-minor \(N\) satisfying \(r(M) - r(N) \leq k + \lfloor \frac{k-1}{2}\rfloor\).

In [3] there are more results of such nature. Although a minimal \((3, 3, 3)\)-rounded family of graphs containing \(\{K_5, K_{3,3}\}\) exists and even has a size that allows a computer approach, it has shown to be complicated. Such family must at least include the graphs \(K_{3,3}^{i,j}\), for \(i + j \leq 3, K_5\) and the following two graphs in Figure 2, obtained, respectively, from \(K_{3,3}\) and \(K_5\) by the same kind of vertex expansion, which shall occur in such kind of families.
2. Proofs for the Theorems

The proof of Theorem 1.1 is based on the following theorem:

**Theorem 2.1.** (Seymour [7], see also [4, Theorem 12.3.9]) Let \( \mathcal{N} \) be a class of matroids closed under minors, and \( \mathcal{F} \) be a family of 3-connected matroids. If, for each matroid \( M \), for each \( e \in E(M) \) such that \( M/e \in \mathcal{F} \) or \( M\setminus e \in \mathcal{F} \) and for each \( f \in E(M) - e \) there is an \( \mathcal{F} \)-minor using \( e \) and \( f \), then \( \mathcal{F} \) is 2-rounded in \( \mathcal{N} \).

Seymour proved Theorem 2.1 when \( \mathcal{N} \) is the class of all matroids. But the same proof holds for this more general version. By Whitney’s 2-isomorphism Theorem, the analogous for graphs of Theorem 2.1 holds.

**Proof of Theorem 1.1:** For items (a) and (b) we will consider \( \mathcal{N} \) as the class of graphic matroids and for items (c) and (d) we will consider \( \mathcal{N} \) as the class of regular matroids. In each item we will verify the criterion given by Theorem 2.1.

First we prove item (a). We begin looking at the 3-connected simple graphs \( G \) such that \( G/e \in \mathcal{F}_a := \{ K_{3,3}, K_{3,3}^{0,1}, K_{3,3}^{0,2}, K_{3,3}^{1,1} \} \). We may assume that \( G \notin \mathcal{F}_a \). So, up to isomorphisms, \( G = K_{3,3}^{0,3} \) or \( G = K_{3,3}^{1,k} \) for some \( k \in \{1, 2, 3\} \). Thus \( e \notin E(K_{3,3}) \).

Define \( H := G[E(K_{3,3}) \cup \{ e, f \}] \). If \( f \in E(K_{3,3}) \), then \( H \cong K_{3,3}^{0,1} \), otherwise \( H \cong K_{3,3}^{0,2} \) or \( H \cong K_{3,3}^{1,1} \). Thus \( H \) is an \( \mathcal{F}_a \)-minor of \( G \) and we may suppose that \( G/e \in \mathcal{F}_a \).

We have that \( G \) is 3-connected and simple, in particular, \( G \) has no degree-2 vertices, hence \( G \) must be obtained from \( G/e \) by the expansion of a vertex with degree at least 4. This implies that \( G \cong K_{3,3} \). Thus, we may assume that \( G/e \) is one of graphs \( K_{3,3}^{0,1}, K_{3,3}^{1,1} \) or \( K_{3,3}^{0,2} \). We denote \( e := w_1 w_2 \).

If \( G/e = K_{3,3}^{0,1} \), then \( G \) is obtained from \( G/e \) by the expansion of a degree-4 vertex. In this case we may assume without losing generality that \( G \) is the graph \( G_1 \), defined in Figure 3. Note that, in this case, \( G_1/u_3 w_2 \cong K_{3,3} \) and that \( G_1/u_3 v_1 \cong K_{3,3}^{0,1} \) (with \( \{ u_1, u_2, w_2 \} \) stable). So, one of \( G_1/u_3 w_2 \) or \( G_1/u_3 v_1 \) is an \( \mathcal{F}_a \)-minor we are looking for. So we may assume that \( G \neq K_{3,3}^{0,1} \).

If \( G \cong K_{3,3}^{1,1} \), then \( G \cong G_1 + u_2 u_3 \) and the result follows as in the preceding case. Hence we may assume that \( G/e \cong K_{3,3}^{0,2} \).
If \( G \) is obtained from \( G/e \) by the expansion of a degree-4 vertex, then \( G \cong G_2 \cong G_1 + v_1 w_1 \). In this case we may proceed as in the first case again.

Thus, if \( G/e = K_{3,3}^{0,2} \), we can assume that \( G \) is obtained from \( G/e \) by the expansion of the degree-5 vertex. If \( \{v_1 w_1, v_2 w_2\} \) or \( \{v_1 w_2, v_3 w_1\} \) is contained in \( E(G) \), then \( G \) is again isomorphic to \( G_2 \) and we are reduced to the first case again. Without loss of generality, say that \( v_1 w_2, v_2 w_2 \in E(G) \). Then \( G \) is one of the graphs \( G_3 \) or \( G_4 \) in Figure 3. If \( G = G_3 \), then one of \( G_3/v_1 w_2 \) or \( G_3/w_2 v_3 \), both isomorphic to \( K_{3,3}^{0,2} \) is the \( F_a \)-minor we are looking for. If \( G = G_4 \), then one of \( si(G_4/u_3 w_2) \) or \( si(G_4/u_2 v_1) \) (with \( \{u_1, u_2, w_2\} \) stable) is such an \( F_a \)-minor. This proves item (a).

Now we prove item (b). We just have to examine the 3-connected simple single-element extensions and coextensions of \( K_5 \), since other verifications were made in the proof of item (a). The unique graph \( G \) with an edge \( e \) such that \( G/e \cong K_5 \) or \( G/e \cong K_5 \) is \( K_{3,3}^{1,1} \) (up to isomorphisms). So, we have item (b).

Now we prove item (c). By the proof of item (a), it is just left to examine the 3-connected extensions and coextensions of the matroids in \( F_c := \{M(K_{3,3}), M(K_{3,3}^{0,1}), M(K_{3,3}^{0,2}), M(K_{3,3}^{1,1})\} \) which are not graphic. By [4, Theorem 13.1.2 and Proposition 12.2.8], each 3-connected regular matroid is graphic, cographic, isomorphic to \( R_{10} \) or has a \( R_{12} \)-minor. But no cographic matroid has a minor in \( F_c \). Moreover, by cardinality, \( R_{10} \) also has no \( F_c \)-minor. So, the unique non-graphic matroids \( M \) such that \( M/e \) or \( M/e \) is possibly in \( F_c \) are those with \( R_{12} \)-minors. Specifically, by cardinality and rank the unique non-graphic matroid that possibly have a single element deletion or contraction in \( F_c \) is \( R_{12} \), up to isomorphisms. Usually \( R_{12} \) is defined as the matroid represented over \( GF(2) \) by

\[
F_c := \{M(K_{3,3}), M(K_{3,3}^{0,1}), M(K_{3,3}^{0,2}), M(K_{3,3}^{1,1})\}
\]
the following matrix:

\[
B := \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}
\]

Now, we build a representation of \( s_i(R_{12}/1) \) as follows. First, we eliminate the first row and column of \( B \) and eliminate column 9, that became equal column 5, after that, we add row \( z_5 \) to row \( z_6 \) and, finally, we add an extra row \( z_7 \) equal to the sum of the other rows. So we get the matrix \( A \), defined next:

\[
A := \begin{pmatrix}
2 & 3 & 4 & 5 & 6 & 7 & 8 & 10 & 11 & 12 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Note that \( R_{12}/1 \sim s_i(R_{12}/1) \sim R_{12}/1 \sim R_{12}/1 \sim R_{12}/6 \sim s_i(R_{12}/7) \sim M(K_{3,3}^0) \) and the ground set of one of these matroids can be chosen containing \( \{e, f\} \). Therefore, for each pair of elements of \( R_{12} \), there is an \( \mathcal{F}_c \)-minor containing both. This proves item (c).

To prove item (d) we observe that if \( M/e = M(K_5) \) or \( M \cong (K_5) \), then \( |E(M)| = 11 \), so \( M \) is not isomorphic to \( R_{10} \) neither has an \( R_{12} \)-minor. Moreover \( M \) is not cographic in this case. So, all matroids we have to deal are graphic, and the proof of item (d) is reduced to item (b). \( \square \)

**Lemma 2.2.** Let \( G \) be a 3-connected simple graph not isomorphic to \( K_5 \). Then \( G \) has a \( K_5 \)-minor if and only if \( G \) has a \( K_{3,3}^{1,1} \)-minor.
If $G$ has a $K_{3,3}^{1,1}$-minor, then $G$ has a $K_5$-minor, because $K_5 \cong K_{3,3}^{1,1} / u_1 v_1$. In other hand, suppose that $G$ has a $K_5$-minor. By the Splitter Theorem (Theorem 12.1.2 of [4]), $G$ has a 3-connected simple minor $H$ with an edge $e$ such that $H\setminus e \cong K_5$ or $H\setminus e \cong K_3$. But no simple graph $H$ has an edge $e$ such that $H\setminus e \cong K_5$. So $H\setminus e \cong K_5$. Now, it is easy to verify that $H \cong K_{3,3}^{1,1}$ and conclude the lemma. □

The next result is Corollary 1.8 of [2].

**Corollary 2.3.** Let $G$ be a simple 3-connected graph with a simple 3-connected minor $H$ such that $|V(G)| - |V(H)| \geq 3$. Then there is a 3-subset $\{x, y, z\}$ of $E(G)$, which is not the edge-set of a triangle of $G$, such that $G\setminus x, G\setminus y, G\setminus z$ and $G\setminus x, y$ are all 3-connected graphs having $H$-minors.

**Proof of Theorem [14]** Suppose that $G$ and $T$ is a counter-example to the theorem minimizing $|V(G)|$. If $|V(G)| \geq 8$, by Corollary 2.3 applied to $G$ and $K_5$, $G$ has an edge $e$ such that, $e \notin cl_{(G)}(T)$ and $G\setminus e$ is 3-connected and have a $K_5$-minor. Thus $si(G\setminus e)$ is a 3-connected simple graph having $T$ as triangle. By Lemma 2.2, $si(G\setminus e)$ has a $K_{3,3}^{1,1}$-minor, contradicting the minimality of $G$. Thus $|V(G)| \leq 7$. If $|V(G)| = 6$, then $G \cong K_{3,3}^{1,1}$ for some $1 \leq i \leq j \leq 3$. In this case, the Theorem can be verified directly. Thus $|V(G)| = 7$.

So, there is $e \in E(G)$ and $X \subseteq E(G)$ such that $G\setminus X/e \cong K_{3,3}^{1,1}$. If $e \notin T$, $si(G\setminus e)$ contradicts the minimality of $T$, so $e \in T$. We split the proof into two cases now.

The first case is when $e$ is adjacent to a degree-2 vertex $v$ of $G\setminus X$. Let $f$ be the other edge adjacent to $v$ in $G\setminus X$. So $e, f \in T$, otherwise, $si(G/f)$ would contradict the minimality of $G$.

Up to isomorphisms, $G\setminus X$ can be obtained from $K_{3,3}^{1,1} \cong G\setminus X/e$ by adding the vertex $v$ in the middle of some edge $e'$. By symmetry, we may assume that $e' = [u_1 v_2, v_2 v_3, u_1 v_1]$. So, there are, up to isomorphisms, three possibilities for $G\setminus (X - T)$, those in Figure 5. Since $G$ is simple, $G$ has a third edge $g$ adjacent to $v$. For any of the graphs in Figure 5, it verifies that $si(G\setminus (X - T)/g)$ contradicts the minimality of $G$. So the proof is done in the first case.

In the second case, $e$ is an edge of $G\setminus X$ whose adjacent vertices has degree at least 3. We may suppose that the end-vertices $w_1$ and $w_2$ of $e$ collapses into $v_2$ when contracting $e$ in $G\setminus X$. Let $S$ be the set of edges incident to $v_2$ in $G\setminus X/e$. We also may assume that $w_2$ is adjacent to $v_3$ in $G\setminus X$. With this assumptions
$G \setminus (X \cup S)$ is the graph $G_4$ of Figure 6. Note also that $G \setminus X$ is obtained from $G_4$ adding 3 edges, each incident to a different vertex in $\{u_1, u_2, u_3\}$, two of then incident to $w_1$ and one incident to $w_2$. Since switching $u_2$ and $u_3$ in $G_4$ induces an automorphism, we may suppose that $u_2 w_1 \in E(G \setminus X)$. Then, without losing generality, $G \setminus X$ is one of the graphs $G_5$ or $G_6$ in Figure 6.

In the case that $G = G_5$, in Figure 7, in the first row, for each possibility for $T$ we draw $G \setminus (X - T)$. The bold edges are those of $T$. In each graph of the first row, the double edge $g$ has the property that the graph $s_i(G \setminus (X - T)/g)$, draw in the second row in the respective column, contradicts the minimality of $G$. The vertex obtained in the contraction is labelled by $z$. In the third and fourth rows of Figure 7, we have the same for the case in which $G = G_6$. This proves the theorem.

Proof of Theorem 1.5 Suppose that $G$ is a 3-connected simple graph with a $K_5$-minor and $T$ is a triangle of $G$. We may suppose that $G \not\cong K_5$. By Lemma 2.2 $G$ has a 3-connected simple minor $H \cong K_{3,3}^1$. By Theorem 1.4 we choose $H$ having the edges of $T$ in a triangle. Let $e \in H$ be the edge such that $H/e \cong K_5$. Note that $e$ is in no triangle of $H$. So $H/e$ is the $K_5$-minor we are looking for.

References

[1] T. Asano, T. Nishizeki and P.D. Seymour, A note on non-graphic matroids, J. Combin. Theory Ser. B 37 (1980), 290-293.

[2] J. P. Costalonga, On 3-connected minors of 3-connected matroids an graphs, European J. Combin., 33 (2012), 72-81.

[3] J.P. Costalonga., Vertically N-contractible elements in 3-connected ma-
troids, arXiv:1210.0023 (2012).

[4] J.G. Oxley, Matroid Theory, Second Edition, Oxford University Press, New York, 2011.
Figure 7:

[5] P.D. Seymour, *Adjacency in Binary Matroids*, European J. Combin., 7 (1986), 171-176.

[6] P.D. Seymour, *Decomposition of regular matroids*, J. Combin. Theory Ser. B 28 (1980), 305-359.

[7] P.D. Seymour, *Minors of 3-connected matroids*, European J. Combin. 6 (1985), 375-382.

[8] K. Truemper, *A decomposition theory for matroids III. Decomposition conditions*, J. Combin. Theory Ser. B 41 (1986), 275-305.