Surface modes and critical velocity in trapped Bose-condensates.

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We propose the novel mechanism leading to superfluidity breakdown in dilute Bose-condensed gases. We discuss the properties and highlight the role played by surface excitations in trapped condensates. We show that the critical velocity measured in recent experiments is nothing else but the critical velocity associated with the spontaneous creation of surface modes. The latter turns out to be larger than that found by the analysis of a superflow stability with respect to vortex nucleation.

Superfluidity is one of the most striking manifestations of quantum statistics in many-body interacting systems. It is usually defined as “the property of a liquid to move without friction inside capillaries or around obstacles”. Superfluidity is confined to small flow velocities. There is a certain critical value of the fluid speed v_c beyond which the dissipation settles in. The value of the critical velocity is directly related to excitations properties (1,2). The analysis of the energy conservation in a moving liquid shows (3) that the flow becomes thermodynamically unstable when the liquid velocity exceeds

\[ v_c = \min_p \frac{\epsilon(p)}{p}. \]  

Here \( \epsilon(p) \) is the energy of an excitation with the momentum \( p \) in the laboratory frame.

When applied to liquid helium, the criterion (1) gives the critical velocity coinciding with the sound velocity, which is too high to explain the experimentally measured value. Moreover, the experimental results appear to be non-universal and depend on the geometry of the liquid sample (3). This contradiction was resolved by R. Feynman (4) who suggested, that the dissipation is related to creation of quantized vortices. If this is taken into account, the critical velocity turns out to be both sufficiently small and depends on the external factors. The combination of the vortex hypothesis and the Landau criterion (1) very well explains the experiments in liquid helium (3).

These fundamental theories are now being tested in the new experiments with ultra-cold Bose-condensed gases (see (5) for a review of experimental and theoretical activity in the field of Bose-Einstein condensation (BEC) in weakly interacting gases). It is now possible to study dynamical properties of superfluidity in systems, which are both exceptionally well controlled experimentally and understood theoretically. The closest analogy for the critical velocity measurement in long capillaries is the experiment done at MIT (6). There the onset of the dissipation caused by a moving laser beam was studied as a function of the beam velocity. The reported value of the critical velocity is well below the speed of sound. This can be explained either by the vortex nucleation (7), or by the modification of the excitation spectrum in inhomogeneous condensates (8).

Another way to measure critical velocity is to rotate a superfluid and find the minimum (critical) angular velocity \( \Omega_c \) at which an excitation shows up in the system. The straightforward generalization of Eq.(1) for the case of a rotating superfluid gives (2):

\[ \Omega_c = \min_M \frac{\epsilon(M)}{M}, \]  

where \( \epsilon(M) \) is the energy of an excitation as a function of its angular momentum \( M \). In a Bose-condensed gas, the lowest value of \( \Omega_c \) is obtained when \( \epsilon(M) \) is the energy of a single vortex state. Remarkably, the critical velocity (2) is smaller than that observed at ENS (9).

In this Letter we perform a simple analysis of critical velocity issues in a trapped Bose-condensed gas. We point out that the universal criterion (1) gives no information about the dissipation time scales. Actually, Eqs. (1) and (2) merely reflect the fact that dissipation is not forbidden by the energy conservation law as soon as the (angular) velocity of the liquid reaches a certain value. In spite of the fact that vortex excitations may have the lowest values of the critical velocity, in a dilute gas they can hardly be formed suddenly out of nothing. Instead, as it is prompted both by the experiment (6) and numerical simulations (10), vortices are nucleated at the condensate border from the condensate shape oscillations. This can only happen if the velocity of the superflow exceeds the critical velocity corresponding to the formation of the necessary surface modes, which later evolve into the vortices in the course of non-linear condensate dynamics. To support this view we analyze the elementary excitations of a confined Bose-condensed gas. We show that, in a sufficiently dense condensate, the lowest possible values of the critical velocity correspond to the excitations localized at the low density region close to the edge of the condensate. Remarkably, the surface modes turn out to be responsible for the previously reported decrease of \( v_c \) compared to the velocity of sound in inhomogeneous condensates (8). We note that this effect is a novel feature characteristic of dilute Bose-condensed gases and is absent in “conventional” superfluids, such as liquid helium. In the latter case the superfluid is incompressible and the surface excitations of the mentioned type cannot exist. Using a simple model of the surface modes, we calculate both \( v_c \) (for an MIT-type experiment) and \( \Omega_c \) (for
an ENS-type experiment), assuming that the generation of the surface modes is the necessary precondition for the vortex nucleation.

Consider a Bose-condensate confined in an infinitely long cylindrical harmonic trap characterized by the transverse frequency \( \omega \) (hereafter we assume \( h = m = 1 \), where \( m \) is the mass of the condensate atom). The condensate is characterized by its chemical potential \( \mu \) and is assumed to be in the Thomas-Fermi(TF) regime: \( \mu \gg \omega \).

In this case, the condensate density has a simple form:

\[
\rho_0(\rho) = n_0(1 - \rho^2/R_c^2),
\]

where \( \rho \) is the radial coordinate and \( R_c = (2\mu/\omega^2)^{1/2} \) is the TF size of the condensate. The condensate density on the axis of the trap \( n_0 \) is related to the chemical potential: \( \mu = 4\pi n_0a \), where \( a \) is the scattering length \( [3] \).

The character of the excitations in this geometry is pretty well understood by now. The lowest elementary excitations are phonons. The velocity of sound is found to be \( \sqrt{2} \) times smaller than that in a homogeneous condensate of the same density \( n_0 \): \( c_g = \sqrt{g n_0}/2 \) \([2]\). The radial profile of the phonon modes turns out to be very similar to the profile of the condensate wavefunction. Hence, the phonons are delocalized inside the condensate and are analogous to the “bulk” phonons in homogeneous Bose-condensates or in liquid helium. As the axial wave-vector of the excitations increases (i.e. when \( pR_c \sim 1 \)), the radial dependence of the phonon modes becomes more and more involved, and the spectrum ceases to be linear \([2]\). The behavior of the excitations for arbitrary values of \( p \) was studied numerically in \([8]\). The critical velocity of sound obtained with the help of Eq.(1) was shown to decrease together with the TF parameter \( \eta = \omega/\mu \) can be much smaller than the velocity of sound \( c_g \).

As we will see below, the excitations with the lowest critical velocity are in fact the surface modes, i.e. the excitations localized at the TF border of the condensate. In this region the condensate wavefunction \( \psi_0 \) has a universal form and can be analyzed with the help of the following universal equation \([3][4]\):

\[
-\frac{1}{2}\Delta \chi_0 + \chi_0^\prime + x\chi_0 = 0.
\]

Here \( x = (\rho - R_c)/d \), \( d = R_c(\omega^2/2\mu^2)^{1/3} \ll R_c \) is the “condensate border thickness”, and \( \chi_0 = (d^2\bar{g})^{1/2}/\psi_0 \) is the dimensionless condensate wavefunction. Deeply inside the condensate spatial region ( \( x \to -\infty \)), the solution \( \chi_0 \) matches the TF density profile: \( \chi_0 \to \sqrt{-x} \). Far away from the condensate border ( \( x \to \infty \) ) the condensate density quickly decreases: \( \chi_0 \to 0.19 \exp(-2x^{3/2}/3)/x^{1/2} \) \([3]\).

The spectrum of elementary excitations can be found by studying the behavior of quantized linear fluctuations of the order parameter \( \delta \tilde{\psi} = \tilde{\psi} - \psi_0 \), where \( \tilde{\psi} \) is the annihilation operator of the atomic field \([4]\), and

\[
\delta \tilde{\psi} = \sum_{pm} (\hat{a}_{pm} u_{pm} e^{im\phi + ipz} - \hat{a}^\dagger_{pm} u_{pm} e^{-im\phi - ipz}).
\]

Here \( \hat{a}_{pm}, (\hat{a}^\dagger_{pm}) \) are the annihilation(creation) operators of the Bogolyubov excitations characterized by the longitudinal and angular momentums \( p \) and \( m \), respectively. One can show that the functions \( f_{pm}^- = u_{pm} \mp v_{pm} \) satisfy the following Bogolyubov-de Gennes equations \([5]\):

\[
\epsilon_{pm} f_{pm}^- = (-\frac{\Delta_p}{2} + \frac{m^2}{2p^2} + \frac{p^2}{2} + \frac{\Delta\psi_0}{2\psi_0}) f_{pm}^-,
\]

\[
\epsilon_{pm} f_{pm}^+ = (-\frac{\Delta_p}{2} + \frac{m^2}{2p^2} + \frac{p^2}{2} + \frac{\Delta\psi_0}{2\psi_0} + 2g\psi_0^2) f_{pm}^+.
\]

The whole spectrum of the elementary excitations can be recovered by solving Eqs.(3),(4) numerically together with the exact Gross-Pitaevskii equation for the condensate wave function \( \psi_0 \). Below we will take advantage of the representation (3) and analyze only the modes confined to the vicinity of the condensate border \( |p - R_c| \sim d \). This can be done by rewriting Eqs.(3),(4) in the same units as Eq.(3). Since the excitations are radially localized inside a shell with the thickness \( d \ll R_c \), the Laplacian \( \Delta \) can be substituted by the second derivative in \( x \), and the centrifugal term can be approximated by its value at \( p = R_c \). Thus, Eqs.(3),(4) take the simple universal form

\[
\tilde{\epsilon}_{pm} f_{pm}^- = (-\frac{d^2}{2dx^2} + \frac{k^2}{2} + \frac{\Delta\chi_0}{2\chi_0}) f_{pm}^-,
\]

\[
\tilde{\epsilon}_{pm} f_{pm}^+ = (-\frac{d^2}{2dx^2} + \frac{k^2}{2} + \frac{\Delta\chi_0}{2\chi_0} + 2\chi_0^2) f_{pm}^-,
\]

where \( \tilde{\epsilon} = ed^2 \), and

\[
k^2 = (p^2d^2 + \frac{m^2d^2}{R_c^2}).
\]

Since Eqs.(3),(7) contain only the single parameter \( k^2 \), the spectrum of the excitations is given by the universal relation: \( \epsilon_{pm} = d^{-2} F(k^2) \). Here \( F \) is a dimensionless spectral function found numerically (see Fig. 1).

![FIG. 1. Dimensionless spectral function F for the lowest surface excitations obtained by numerical solution of Eqs.(6) and (7).](image-url)
To make sense of the numerical results we resort to a simple model, based on WKB approximation to Eqs. (1) and (3). Quasiclassical radial momentum \( p_r \) has the following Bogolyubov-type form

\[
\frac{p_r^2}{2} + \frac{k^2}{2} = \sqrt{\frac{\epsilon_{pm}^2}{C_0} + \chi_0^2 - \frac{\lambda_0^2}{2\chi_0}}. \tag{8}
\]

To make the things even more simple, we ignore the exact behavior of the condensate wave function close to the condensate border \( |x| \lesssim 1 \) and substitute \( \chi_0 \) with its asymptotic values at \( |x| \gg 1 \). For sufficiently high momentums \( k^2 \gg 1 \), we find, that the classical turning points are close to the condensate border \( x = 0 \). The dimensionless energy of the excitations is much larger than the mean field interaction \( \chi_0 \). Hence, the dispersion relation (8) can be written as

\[
\frac{p_r^2}{2} + \frac{k^2}{2} = \epsilon_{pm} + |x|.
\]

Born-Sommerfeld quantization rule gives essentially the single-particle spectrum

\[
\epsilon_{pm}(n) = \frac{C_n}{d^2} + \frac{p_r^2}{2} + \frac{m^2}{2R_c^2}, \tag{9}
\]

where \( n = 0, 1, \ldots \) is the radial quantum number, and \( C_n = (3\pi/8\sqrt{2})^{2/3}(4n + 1)^{2/3} \) is the dimensionless constant found from the boundary condition at the classical turning points (16)(19). The quasiclassical value \( C_0 = 0.885 \) found in our model calculation is remarkably close to that obtained by exact numerical solution of Eqs. (1) and (3): \( C_0 \approx 0.9 \). This coincidence is somewhat surprising, since our analysis fails to recognize the complicated character of the condensate wavefunction close to the condensate border. At the same time the numerical solution shows that the mode functions are localized at \( |x| \gtrsim 1 \), where our approximations are better justified. Moreover, WKB approximation is known to provide very reasonable estimations for excitation energies even if the quantum numbers are small (7). The exact solution of Eqs. (1) and (3) also shows that the approximation (8) holds fairly well even for \( pd, md/R_c \sim 1 \), in spite of the fact that the excitations characterized by small momentums \( k \lesssim 1 \) spread more inside the condensate spatial region and, hence, are not single particles (see Fig. 4).

We note that, although the surface modes (4) arise from the analysis of the Bogolyubov dispersion relation, they are quite different from the Bogolyubov excitations in an homogeneous condensate. At smaller momentums \( (pd \lesssim 1) \) the localization size of the solutions to Eqs. (1), (3) increases and gradually approaches the size of the condensate. In this limit, the dispersion relation recovers its phonon limit. The surface modes (4) exist in a thin layer around \( \rho = R_c \) and which is still thicker than the healing length \( l_0 = 1/\sqrt{8\pi a_n a_0} \) in the condensate: \( d \gg l_0 \) in the TF limit. This may happen only in a compressible liquid and thus the dispersion relation similar to (4) can hardly be found in “traditional” superfluids, such as liquid helium. In the latter case the condensate density drops on a distance scale of order \( l_0 \) at the wall of the confining vessel.

The validity of Eq. (4) is restricted by a number of assumptions. First, we require that the radial localization length of the surface excitations be much smaller than the size of the condensate. Since we are mostly interested in \( \epsilon_{pm} \sim 1/d^2 \), the radial size of the modes is \( \sim d \ll R_c \) in the TF regime. In the same way we can neglect the spatial variations of the centrifugal potential provided that the momentums of the involved excitations are sufficiently small. Indeed, a simple WKB calculation shows that the first correction to Eq. (1) arising from the variations of the centrifugal potential close to \( \rho \approx R_c \) results in a slight change of the ground state energy:

\[
\Delta C_0(C_0) \sim 2m^2/\mu R_c^2.
\]

The solution of Eq. (3) corresponds to \( m \approx R_c/d \ll R_c/\mu^{1/2} \), so that for our purposes, the finite-\( m \) corrections to \( C_0 \) can be safely neglected.

The dispersion relation (4) is the central result of our work. It allows us to study the critical velocities (4) and (6) in the deep TF regime \( (\eta \ll 1) \). First we turn to an MIT-type experiment. The application of criterion (4) shows that the lowest value of critical velocity corresponds to the instability of the excitations with \( m = 0 \):

\[
v_c = \sqrt{2C_0/d^2} \sim 1/d \sim c_Z \eta^{-1/3} \ll c_Z. \tag{10}
\]

This result has to be compared with the stability analysis of the vortex excitations. Consider a superfluid flow of velocity \( v_s \) without vortices. The energy of the liquid scales as \( \sim \rho_s v_s^2 R_c^2 L \), where \( \rho_s \approx n_0 \) is the super-fluid density, and \( L \) is the length of the trap. Let us consider instead the vortex configuration similar to that discussed in relation to the experiment [13], i.e. a sequence of vortices at the distance \( \pi/v_s \) from each other along the trap axis. The average velocity of the superfluid would be the same \( v_s \). The energy of this vortex chain is roughly the sum of the vortex energies and thus can be estimated as \( \rho_s R_c \log(1/\eta) v_s L \). Comparing the energy of the superflow with and without the vortices, we find that the vortex configuration has a lower energy as soon as \( v_s > v_c^{(v)} \sim \eta \log(\eta^{-1}) c_Z \). This means that in a TF condensate, vortices form the true ground state already at \( \eta > v_c^{(v)} \). At the same time, since the vortices are nucleated only at the edge of the condensate, they originate from the surface excitations, which only become unstable if \( v > v_c^{(v)} \). Therefore, in the TF regime no vortices can be nucleated unless the superfluid velocity reaches \( v_c \). This statement does not contradict the conclusions derived in [13], since the vortex states possess very small energies compared to the irrotational superflow of velocity \( v_s \), and therefore further relaxation leads to the creation of vortices.

The dispersion relation for the surface excitations also gives us a way to discuss the critical angular velocity in an ENS-type rotating superfluid experiment. Again, applying the criterion (4) to the dispersion relation (3) we
immediately find that the surface modes in a cylindrical trap first become unstable if \( p = 0 \) and
\[
\Omega > \Omega_c = \frac{v_c}{R_c} \sim \omega \eta^{1/3}.
\]

This value has to be compared with the lowest angular velocity at which a vortex can be formed in the condensate. It can be derived using the same kind of simple arguments as above: \( \Omega_{c}^{(s)} \sim \omega \eta \log(\eta^{-1}) \). As in the discussion above, in the deep TF limit \( \Omega_{c}^{(s)} < \Omega \) and, we conclude that, in order to make up a vortex on the surface of a rotating superfluid, one has to first reach the angular velocity (11). Only then the surface modes are nucleated and, finally, a vortex appears in the superfluid in the course of the turbulent liquid dynamics. This point of view can be supported by earlier work [10], where the authors conclude that \( \Omega_c \) coincides with the frequencies of the phonon-like excitations with \( m = 2, 3 \) (depending on the symmetry of the rotating perturbation). We argue that the TF parameter in the ENS experiment is not sufficiently small. Since the critical velocities (11) and (11) change very slowly as \( \eta \) decreases, the distinction between the bulk and the surface modes is not very sharp at moderate values of the TF parameter.

Finally, we would like to comment on the physical nature of criteria (1) and (2). Consider an obstacle (probe) of a size \( l \) moving inside a superfluid with the velocity \( v \). In a weakly interacting Bose-condensed gas the dissipation arises pair creation of excitations with radial momentums \( p \) and \( p' \). The energy conservation law (1) requires that
\[
\epsilon_p + \epsilon_{p'} - v(p + p') = 0.
\]

If the fluid velocity slightly exceeds \( v_c \) (1), both solutions of Eq. (12) \( p, p' \approx p_s \), where \( p_s \) is found from the equation \( \epsilon(p_s) = v_c p_s \). At the same time, the uncertainty principle requires \( (p + p')^2 \lesssim 1/l \). Therefore, \( p_t \lesssim 1 \) and thus the dissipation at \( v \approx v_c \) is only possible if the probe is sufficiently small: \( l \lesssim p_s^{-1} \). In our case we find that \( p_s \sim d^{-1} \) and thus the probe dissipates only if \( l \lesssim d \). In this sense the criteria (1) and (3) give the values of the critical velocities measured by an ideal, point size probe. We note that this requirement is not at all automatically satisfied in the experiments. This, in turn, may lead to additional complications in ascribing a particular mechanism to the reported critical velocity values. For example, a finite size probe may nucleate a vortex ring on its surface. This instability leads to a sufficiently small value of the critical velocity: \( v_c \sim 1/l \) (see [4] and refs. therein). In the latter case, the mechanism of the critical velocity decrease is somewhat similar to that reported here: the increase of the superfluid velocity around a moving body leads to the decrease of the condensate density, so that the moving body-superfluid interface can host an unstable “surface” excitation.

In conclusion, we develop a simple theory of critical velocity in trapped BEC. We highlight the role played by the surface excitations in large TF condensates. We suggest that, since the vortex excitations are formed at the condensate border and thus are initially surface excitations, the critical velocity measured in the recent experiments cannot be smaller than the critical velocity associated with the generation of the surface excitations. The latter turns out to be larger than that found by the analysis of thermodynamic stability of superflow with respect to vortex nucleation both in a long trap with axially moving superfluid and in a rotating trapped Bose-condensate. We note that this novel feature is a direct consequence of compressibility of trapped Bose-condensed gases and hence cannot be related to traditional superfluids, such as liquid helium.

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