On $\alpha$-excellent graphs

1Magda Dettlaff, 2Michael A. Henning* and 3Jerzy Topp

1Faculty of Mathematics, Physics and Informatics
University of Gdańsk
80-952 Gdańsk, Poland
Email: magda.dettlaff@pg.edu.pl

2Department of Mathematics and Applied Mathematics
University of Johannesburg
Auckland Park 2006, South Africa
mahenning@uj.ac.za

3Institute of Applied Informatics
University of Applied Sciences
82-300 Elbląg, Poland
j.topp@ans-elblag.pl

Abstract

A graph $G$ is $\alpha$-excellent if every vertex of $G$ is contained in some maximum independent set of $G$. In this paper, we characterize $\alpha$-excellent bipartite graphs, $\alpha$-excellent unicyclic graphs, $\alpha$-excellent simplicial graphs, $\alpha$-excellent chordal graphs, $\alpha$-excellent block graphs, and we show that every generalized Petersen graph is $\alpha$-excellent.

Keywords: independence number; independent domination number; excellent graph.
AMS subject classification: 05C69, 05C85

1 Introduction

A dominating set of a graph $G$ is a set $D$ of vertices of $G$ such that every vertex belonging to $V(G) \setminus D$ is adjacent to at least one vertex in $D$. A subset $I$ of $V(G)$ is independent if no two vertices belonging to $I$ are adjacent in $G$. An independent dominating set, abbreviated ID-set, of $G$ is a set that is both a dominating set and an independent set. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set in $G$, while the independent domination number $i(G)$ of $G$ is the minimum cardinality of an ID-set in $G$. The independence number $\alpha(G)$ of $G$ is the maximum cardinality of an independent set in $G$. An $\alpha$-set of $G$ is an independent set of $G$ of

*Research supported in part by the University of Johannesburg
maximum cardinality $\alpha(G)$. The study of independent sets in graphs was begun by Berge [11,2] and Ore [11]. We refer the reader to the survey [7] of results on independent domination in graphs by Goddard and Henning. For recent books on domination in graphs, we refer the reader to [8,9,10].

Recently another parameter concerning the existence and cardinality of independent sets in a graph, the common independence number of a graph, was introduced and studied by Dettlaff et al. [4]. Formally, the common independence number of a graph $G$, denoted by $\alpha_c(G)$, is defined as the greatest integer $r$ such that every vertex of $G$ belongs to some independent subset $X$ of $V(G)$ with $|X| \geq r$. Thus, the common independence number of $G$ refers to numbers of mutually independent vertices of $G$ and it emphasizes the notion of the individual independence of a vertex of $G$ from other vertices of $G$. It follows immediately from the above definitions that the common independence number is sandwiched between the independent domination number and the independence number. We state this formally as follows.

**Observation 1.1.** ([4]) For every graph $G$, $i(G) \leq \alpha_c(G) \leq \alpha(G)$.

A graph $G$ is said to be well-covered if $i(G) = \alpha(G)$. Equivalently, $G$ is well-covered if every maximal independent set of $G$ is a maximum independent set of $G$. The concept of well-covered graphs was introduced by Plummer [12] in 1970 and extensively studied in many papers. For a survey of well-covered graphs, see [13].

A graph $G$ is called an $\alpha$-excellent graph if $\alpha_c(G) = \alpha(G)$. Equivalently, $G$ is $\alpha$-excellent if every vertex is contained in some $\alpha$-set of $G$. The $\alpha$-excellent graphs, referred to as $B$-graphs in [1,2], have been studied in [1,2,3,4,5,6,15,16].

We are interested in characterizations of graphs $G$ for which $i(G) = \alpha_c(G)$, $i(G) = \alpha(G)$, and $\alpha_c(G) = \alpha(G)$, respectively. We remark that the family of well-covered graphs is properly contained in the family of $\alpha$-excellent graphs, and in the family of graphs $G$ for which $i(G) = \alpha_c(G)$, respectively. Trees $T$ for which $i(T) = \alpha_c(T)$, and block graphs $G$ such that $\alpha_c(G) = \alpha(G)$ have been characterized in [3]. The $\alpha$-excellent trees have also been studied in [5,6]. In this paper, we provide characterizations of $\alpha$-excellent bipartite graphs, unicyclic graphs, simplicial graphs, chordal graphs, block graphs, and we show that every generalized Petersen graph is $\alpha$-excellent.

### 1.1 Notation

For notation and graph theory terminology, we in general follow [10]. Specifically, let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. Two vertices are *neighbors* if they are adjacent. For a graph $G$ and a vertex $v \in V(G)$, the *open neighborhood* $N_G(v)$ of $v$ is the set of neighbors of $v$ in $G$, and the *closed neighborhood* $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the *open neighborhood* of $S$ is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$, and the *closed neighborhood* $N_G[S] = N_G(S) \cup S$.

An *end-edge* of a graph $G$ is an edge incident with at least one vertex of degree 1 in $G$. A *support vertex* is a vertex adjacent to a vertex of degree 1, that is, a support vertex is incident with an end-edge. A *strong support vertex* is a vertex with at least two neighbors of degree 1. We note that if two end-edges share a vertex, then such a vertex is a strong support vertex. An *interior vertex* of $G$ is a vertex of degree at least 2.

A *caterpillar-wheel* is a connected unicyclic graph $G$ of girth at least three in which a single cycle
C (the body) is incident to (or contains) every edge, that is, at least one end of every edge of \( G \) belongs to the cycle \( C \). An end-edge of a caterpillar-wheel is its leg.

A vertex \( v \) of a graph \( G \) is a simplicial vertex if every two vertices belonging to \( N_G(v) \) are adjacent in \( G \). Equivalently, a simplicial vertex is a vertex that appears in exactly one clique of a graph, where a clique of a graph \( G \) is a maximal complete subgraph of \( G \). A clique of a graph \( G \) containing at least one simplicial vertex of \( G \) is called a simplex (plural, simplexes or simplices) of \( G \). We note that if \( v \) is a simplicial vertex of \( G \), then \( G[N_G[v]] \) is the unique simplex of \( G \) containing \( v \).

A graph \( G \) is simplicial if every vertex of \( G \) is a simplicial vertex of \( G \) or is adjacent to a simplicial vertex of \( G \), that is, a graph \( G \) is simplicial if and only if every vertex of \( G \) belongs to at least one simplex of \( G \).

A graph \( G \) is said to be chordal (also called triangulated, rigid-circuit, perfect eliminated, or monotone sensitive in the literature) if every cycle of \( G \) of length four or more contains a chord, i.e., an edge joining two non-consecutive vertices of the cycle. We remark that chordal graphs have been also characterized in terms of the existence of simplicial vertices \([19]\): A graph \( G \) is chordal if and only if every induced subgraph of \( G \) has a simplicial vertex. Certainly, every induced subgraph of a chordal graph is chordal.

A block in a graph \( G \) is a maximal connected subgraph having the property that it contains no cut-vertex of its own. An end-block of \( G \) is a block containing exactly one cut-vertex, while an inner-block of \( G \) is a block containing at least two cut-vertices. A graph is a block graph if every block of \( G \) is a complete graph. It is known that a graph \( G \) is a block graph if and only if there exists a unique induced path between any two vertices of \( G \).

For a graph \( G \) and a family \( \mathcal{H} = \{H_v : v \in V(G)\} \) of nonempty graphs indexed by the vertices of \( G \), the corona \( G \circ \mathcal{H} \) of \( G \) and \( \mathcal{H} \) is the disjoint union of \( G \) and \( H_v \), \( v \in V(G) \), with additional edges joining each vertex \( v \) of \( G \) to all vertices of \( H_v \). Thus, to form the graph \( G \circ \mathcal{H} \) we add a copy of the graph \( H_v \) associated with \( v \) for each vertex \( v \) of \( G \), and join \( v \) to all vertices of \( H_v \). If all the graphs of the family \( \mathcal{H} \) are isomorphic to one and the same graph \( H \), then we shall write \( G \circ H \) instead of \( G \circ \mathcal{H} \). In particular, the corona \( G \circ K_1 \) is formed from \( G \) by adding for each vertex \( v \) in \( G \) a new vertex \( v' \) and the edge \( vv' \).

For \( k \geq 1 \) an integer, we use the standard notation \([k] = \{1, 2, \ldots, k\}\) and \([k]_0 = [k] \cup \{0\} = \{0, 1, \ldots, k\}\).

# 2 Preliminary results

In this section, we present some preliminary results that we will need when proving our main results. We have the following simple property of simplexes in \( \alpha \)-excellent graphs.

**Proposition 2.1.** The simplexes of an \( \alpha \)-excellent graph are pairwise vertex-disjoint. In particular, no \( \alpha \)-excellent graph contains a strong support vertex.

**Proof.** Let \( G \) be an \( \alpha \)-excellent graph, and so \( \alpha_e(G) = \alpha(G) \). Suppose, to the contrary, that a vertex \( v \) of a graph \( G \) belongs to two simplexes of \( G \), say to \( G[N_G[u]] \) and \( G[N_G[w]] \). Let \( I \) be a maximum independent set of \( G \) that contains \( v \), and so \( \alpha(G) \geq |I| \geq \alpha_e(G) = \alpha(G) \). Consequently, we must have equality throughout this inequality chain, implying that \( |I| = \alpha(G) \). However, \( (I \setminus \{v\}) \cup \{u, w\} \) is an independent set of \( G \) of cardinality \( |I| + 1 = \alpha(G) + 1 \), a contradiction. \( \square \)
Proposition 2.4. A connected graph $G$ of order at least 2 is well-covered if and only if every interior vertex of $G$ is a critical vertex of $G$. Let $H$ be a maximum independent set in $G$, and let $I_v$ be a maximum independent set in $G$ that contains the vertex $v$. If $H_v$ is not a complete graph, then replacing $v$ in $I_v$ with two non-adjacent vertices in $H_v$ produces an independent set of cardinality greater than $|I_v|$, implying that $G$ is not $\alpha$-excellent.

On the other hand, if $H_v$ is a complete graph for every vertex $v$ of $G$, then $\alpha(G) = |V(G)|$ and every vertex of $G$ belongs to some $\alpha$-set of $G$. This yields the following family of $\alpha$-excellent graphs.

Observation 2.2. Let $G$ be a graph, and let $H = \{H_v : v \in V(G)\}$ be a family of nonempty graphs indexed by the vertices of $G$. Then the corona $G \circ H$ is an $\alpha$-excellent graph if and only if $H$ consists of complete graphs. In particular, the corona $G \circ K_1$ is an $\alpha$-excellent graph for every graph $G$.

In [18], Ravindra proved that a connected bipartite graph $G$ of order at least 2 is well-covered if and only if $G$ has a perfect matching $M$ and for every $uw \in M$, the induced subgraph $G[N_G(\{u, v\})]$ is a complete bipartite graph. In particular, it was observed in [18] that a tree $G$ is well-covered if and only if every interior vertex of $G$ is adjacent to exactly one end vertex of $G$ or, equivalently, if and only if the end-edges of $G$ form a perfect matching of $G$, that is, if and only if $G$ is the corona graph of a tree. In Proposition 2.3 we have an analogue of these results for $\alpha$-excellent bipartite graphs. In fact, this analogue was proved by Berge [2] many years ago. Our proof given below is a modification of that given in [2]. We begin with one definition and two propositions.

A vertex $v$ is a critical vertex of $G$ if its removal from $G$ changes the independence number, that is, a vertex $v$ is a critical vertex of $G$ if any of the following equivalent conditions holds:

1. $\alpha(G - v) \neq \alpha(G)$.
2. $\alpha(G - v) = \alpha(G) - 1$.
3. Every maximum independent set of $G$ contains $v$.

Proposition 2.3. ([3]) If a graph $G$ contains no critical vertex, then every independent set $I$ of $G$ can be matched into $V(G) \setminus I$.

Proposition 2.4. A connected $\alpha$-excellent graph of order at least 2 has no critical vertex, and every independent set $I$ of $G$ can be matched into $V(G) \setminus I$.

Proof. Let $G$ be a connected $\alpha$-excellent graph of order at least two, and let $v$ be a vertex of $G$. To prove that $v$ is not a critical vertex, it suffices to show that $v$ does not belong to some $\alpha$-set of $G$. Let $u$ be any neighbor of $v$. From the fact that $G$ is an $\alpha$-excellent graph, it follows that $u$ belongs to some $\alpha$-set $I$ of $G$. Since $v$ is adjacent to $u$ and $u \in I$, the vertex $v$ does not belong to $I$, and this proves that $v$ is not a critical vertex of $G$. Consequently, $G$ has no critical vertex and, therefore, by Proposition 2.3 every independent set $I$ of $G$ can be matched into $V(G) \setminus I$.

Proposition 2.5. ([2]) If $G$ is a connected bipartite graph of order $n \geq 2$, then the following statements are equivalent:

(a) $G$ has a perfect matching.
(b) $\alpha(G) = n/2$.
(c) $G$ is an $\alpha$-excellent graph.
Proof. Assume that \( G \) is a connected bipartite graph of order \( n \geq 2 \), say \( G = (A, B, E) \), where \( A \) and \( B \) be the partite sets of \( G \), \( |A| \geq |B| \geq 1 \), and \( E = E(G) \). Since \( A \) and \( B \) are independent sets and \( |A| \geq |B| = n - |A| \), it follows that \( \alpha(G) \geq |A| \geq n/2 \geq |B| \).

Assume first that \( G \) has a perfect matching. Then, certainly, \( \alpha(G) \leq n/2 \) and, therefore, \( \alpha(G) = n/2 \). Consequently, \( |A| = |B| = n/2 = \alpha(G) \), and each of the sets \( A \) and \( B \) is an \( \alpha \)-set of \( G \). Thus every vertex of \( G \) belongs to an \( \alpha \)-set of \( G \) and \( G \) is an \( \alpha \)-excellent graph. This proves the implications (a) \( \Rightarrow \) (b) and (b) \( \Rightarrow \) (c).

Now assume that \( G \) is an \( \alpha \)-excellent graph. We shall prove that \( G \) has a perfect matching. By Proposition 2.4 \( G \) has no critical vertex and every independent set \( I \) of \( G \) can be matched into \( V(G) \setminus I \). In particular, \( A \) can be matched into \( B \) and \( B \) into \( A \). Thus, \( |A| = |B| \) and \( G \) has a perfect matching. This proves the implication (c) \( \Rightarrow \) (a).

The following simple characterization of \( \alpha \)-excellent trees follows immediately from Proposition 2.5 and it was already proved in [2], and independently in [4, 5, 6].

**Corollary 2.6.** A tree \( T \) of order at least 2 is an \( \alpha \)-excellent graph if and only if \( T \) has a perfect matching.

## 3 \( \alpha \)-Excellent unicyclic graphs

In this section, we characterize \( \alpha \)-excellent unicyclic graphs. Our characterization is a counterpart of characterizations of well-covered unicyclic graphs presented in [22, 23].

**Proposition 3.1.** Let \( G \) be a caterpillar-wheel of girth at least 3. Then \( G \) is an \( \alpha \)-excellent graph if and only if \( G \) is a cycle (that is, \( G \) is a caterpillar-wheel without any leg) or \( G \) is a caterpillar-wheel with a perfect matching and with at least two legs.

**Proof.** If \( G \) is a cycle, then \( G \) is a vertex transitive graph, and, certainly, \( G \) is an \( \alpha \)-excellent graph (as every two vertices of \( G \) have the same properties). Thus assume that \( G \) is a caterpillar-wheel with at least two legs. Let \( u_1, v_2, \ldots, v_n \) be the consecutive vertices of the only cycle \( C \) of \( G \), and let \( L = \{v_i, \pi_i, \ldots, v_{i_l}, \pi_{i_l}\} \) be the set of legs of \( G \), where \( 1 \leq i_1 < \cdots < i_l \leq n \), and \( \pi = \{\pi_{i_1}, \ldots, \pi_{i_l}\} \) be the set of leaves of \( G \). Let \( S \) be the set of support vertices in \( G \), that is, \( S = \{v_1, \ldots, v_{i_l}\} \). Let \( M \) be the (unique) perfect matching of \( G \). Certainly, \( L \) is a subset of \( M \).

If \( M = L \), then \( G = C_n \circ K_1 \) is a corona graph and it is an \( \alpha \)-excellent graph, see Observation 2.2. Thus assume that \( L \) is a proper subset of \( M \). Renaming the vertices on \( C \) if necessary, we may assume, without loss of generality, that \( i_1 = 1 < \cdots < i_l \leq n - 2 \). From the fact that \( M \) is a perfect matching of \( G \) and \( L \subset M \), it follows that every component of the subgraph \( G' = G - \{v_1, \pi_1, \ldots, v_{i_l}, \pi_{i_l}\} \) is a path of even order, say \( Q_1, \ldots, Q_k \) are the components of \( G' \), where \( V(Q_j) = \{u^j_1, u^j_2, \ldots, u^j_{2t_j}\} \) (for some positive integer \( t_j \)), and the order of vertices of \( Q_j \) is inherited from the order of the vertices \( u_1, v_2, \ldots, v_n \) on the cycle \( C \). It is evident that the perfect matching \( M \) is the set

\[
M = L \cup \bigcup_{j=1}^k \{u^j_1u^j_2, \ldots, u^j_{2t_j-1}u^j_{2t_j}\}
\]
and that \( \alpha(G) = |M|/2 \).

Now we shall prove that \( G \) is an \( \alpha \)-excellent graph. Let \( V^j_i \) denote the set \( \{u^j_1, \ldots, u^j_{2t_j-1}\} \) of “odd” vertices of the path \( Q_j \) (with odd subscripts), and, similarly, let \( V^j_j \) denote the set \( \{u^j_2, \ldots, u^j_{2t_j}\} \) of “even” vertices of \( Q_j \) (with even subscripts) for \( j \in [k] \). It is evident that each of the sets

\[
I_1 = \mathcal{L} \cup \bigcup_{j=1}^k V^j_1 \quad \text{and} \quad I_2 = \mathcal{L} \cup \bigcup_{j=1}^k V^j_2
\]

is an \( \alpha \)-set of \( G \). Consequently every vertex belonging to \( I_1 \cup I_2 = V(G) \setminus S \) belongs to an \( \alpha \)-set of \( G \). Assume that \( v_{i_a} \in S \). It remains to show that \( v_{i_a} \) belongs to some \( \alpha \)-set of \( G \). We now consider four cases. In what follows, all subscripts are taken modulo \( n \).

**Case 1.** \( v_{i_a-1} \in S \) and \( v_{i_a+1} \in S \). In this case, each of the sets \( (I_1 \setminus \{v_{i_a}\}) \cup \{v_{i_a}\} \) and \( (I_2 \setminus \{v_{i_a}\}) \cup \{v_{i_a}\} \) is an \( \alpha \)-set of \( G \) and \( v_{i_a} \) belongs to each of them.

**Case 2.** \( v_{i_a-1} \in S \) and \( v_{i_a+1} \notin S \). In this case, the set \( (I_2 \setminus \{v_{i_a}\}) \cup \{v_{i_a}\} \) is an \( \alpha \)-set of \( G \) that contains \( v_{i_a} \).

**Case 3.** \( v_{i_a-1} \notin S \) and \( v_{i_a+1} \in S \). In this case, the set \( (I_1 \setminus \{v_{i_a}\}) \cup \{v_{i_a}\} \) is an \( \alpha \)-set of \( G \) that contains \( v_{i_a} \).

**Case 4.** \( v_{i_a-1} \notin S \) and \( v_{i_a+1} \notin S \). In this case, without loss generality, assume that \( v_{i_a-1} \) and \( v_{i_a+1} \) belong to \( Q_k \) and \( Q_1 \), respectively. In this case, the set

\[
(\mathcal{L} \setminus \{v_{i_a}\}) \cup \{v_{i_a}\} \cup V^K_1 \cup \bigcup_{j=1}^{k-1} V^j_2
\]

is an \( \alpha \)-set of \( G \) that contains \( v_{i_a} \). This completes the proof of the first part of the theorem.

Assume now that \( G \) is an \( \alpha \)-excellent caterpillar-wheel which is neither a cycle nor the corona of a cycle. Thus at least one vertex on the cycle \( C \) has degree 2 in \( G \) or at least one vertex on the cycle \( C \) is a strong support vertex. By Proposition 2.1, the \( \alpha \)-excellent graph \( G \) contains no strong support vertex. Hence, at least one vertex on the cycle \( C \) is not a support vertex (and has degree 2 in \( G \)). We claim that \( G \) has at least two leaves. Suppose, to the contrary, that \( G \) has only one leaf. Let \( \tau \) be the only leaf of \( G \), and let \( v \) be the unique neighbor of \( \tau \). Adopting our earlier notation, let \( C \) denote the (unique) cycle of \( G \) and let \( C \) have length \( n \) where \( n \geq 3 \). Let \( I_v \) and \( I_\tau \) be a largest independent set of \( G \) containing the vertices \( v \) and \( \tau \), respectively. Since \( G \) is an \( \alpha \)-excellent graph, we have \( \alpha(G) = |I_v| = |I_\tau| \). However, \( |I_\tau| = 1 + \alpha(G - NG[\tau]) = 1 + \alpha(P_{n-1}) = 1 + [(n-1)/2] \) and \( |I_v| = 1 + \alpha(G - NG[v]) = 1 + \alpha(P_{n-3}) = 1 + [(n-3)/2] \), and so \( |I_v| > |I_\tau| \), a contradiction. Hence, \( G \) has at least two leaves.

Now we shall prove that \( G \) has a perfect matching. Let \( v \) be a support vertex of \( G \), let \( \tau \) be a leaf adjacent to \( v \), and let \( u \) be one of the two neighbors of \( v \) on the cycle \( C \). Let \( G' \) denote the graph \( G - uv \). Every independent set of \( G \) is an independent set of \( G - uv \), and so \( \alpha(G - uv) \geq \alpha(G) \). On the other hand, let \( J \) be a maximum independent set of \( G - uv \). At least one of the sets \( J \) or \( (J \setminus \{v\}) \cup \{\tau\} \) is an independent set of \( G \) and, therefore, \( \alpha(G - uv) = |J| = |(J \setminus \{v\}) \cup \{\tau\}| \leq \alpha(G) \). Consequently, \( \alpha(G - uv) = \alpha(G) \). From this and from the fact that \( G \) is an \( \alpha \)-excellent graph it is evident that \( G - uv \) is an \( \alpha \)-excellent graph. Thus, since \( G - uv \) is a tree, Corollary 2.6 implies that \( G - uv \) has a perfect matching. Therefore, \( G \) has a perfect matching. \( \square \)
Let $S$ be the family consisting of all cycles and of all caterpillar-wheels having a perfect matching and at least 2 legs. It follows from the proof of Proposition 3.1 that a caterpillar-wheel $G$ which is not a cycle belongs to the family $S$ if and only if $G$ has $\ell$ legs, where $\ell \geq 2$, and the distance between any two consecutive legs of $G$ is an odd integer.

Let $\mathcal{H}$ be the family of graphs defined recursively as follows:

1. The family $\mathcal{H}$ contains every graph belonging to the family $S$.
2. The family $\mathcal{H}$ is closed under the operation $O_1$ defined below:
   - **Operation $O_1$.** If a graph $H$ belongs to $\mathcal{H}$, then add a vertex disjoint copy of a complete graph $K_2$ of order 2 to $H$ and add an edge joining a vertex of $H$ with a vertex in the added copy of $K_2$.

**Proposition 3.2.** If $H$ is a connected graph of order at least 2, and $G$ is a graph obtained from $H$ by applying operation $O_1$, then the following properties hold:

1. $\alpha(G) = \alpha(H) + 1$.
2. $G$ is $\alpha$-excellent if and only if $H$ is $\alpha$-excellent.

**Proof.** Let $v$ and $w$ be the two vertices added to $H$ when constructing $G$ using operation $O_1$, and let $v$ be the vertex of $H$ that is joined to exactly one of $v$ and $w$, say $v$. Thus, $uvw$ is a path in $G$, where $v$ has degree 1 in $G$ and $v$ has degree 2 in $G$. Every $\alpha$-set in $H$ can be extended to an independent set of $G$ by adding to it the vertex $v$, implying that $\alpha(G) \geq \alpha(H) + 1$. On the other hand, every $\alpha$-set in $G$ is an independent set of $G - uv = H \cup K_2$, and therefore $\alpha(G) \leq \alpha(G - uv) = \alpha(H) + 1$. Consequently, $\alpha(G) = \alpha(H) + 1$. This proves part (1).

To prove part (2), assume first that $G$ is $\alpha$-excellent. Let $x$ be a vertex of $H$, and let $I_x$ be an $\alpha$-set of $G$ that contains $x$. We note that the set $I_x$ contains exactly one of $v$ and $w$, say $v$. If $v \in I_x$, then we can replace $v$ in $I_x$ with the vertex $w$. Hence, we can choose the set $I_x$ to contain the vertex $w$. The set $I_x \cup \{w\}$ is an independent set of $H$ of cardinality $|I_x| - 1 = \alpha(G) - 1 = \alpha(H)$, and is therefore an $\alpha$-set of $H$ that contains the vertex $x$. Since $x$ is an arbitrary vertex of $H$, the graph $H$ is, therefore, an $\alpha$-excellent graph.

Now assume that $H$ is an $\alpha$-excellent graph. Let $y$ be a vertex of $G$. If $y$ belongs to $V(H)$, and if $I_y$ is an $\alpha$-set of $H$ containing $y$, then $I_y \cup \{w\}$ is an $\alpha$-set of $G$ containing $y$ and $w$. If $t$ is a neighbor of $u$ in $H$ and if $I_t$ is an $\alpha$-set of $H$ containing $t$, then $I_t \cup \{v\}$ is an $\alpha$-set of $G$ containing $v$. This proves that $G$ is an $\alpha$-excellent graph, and completes the proof of part (2).

We define next the opposite operation to Operation $O_1$.

- **Operation $O_2$.** If $G$ is a graph of order at least 4 that contains a support vertex of degree 2, then remove this support vertex and its (unique) neighbor of degree 1 from the graph $G$.

We call Operation $O_2$ a **plucking operation**. The **plucked graph** of a graph $G$ of order at least 4, denoted by $P(G)$, is a graph obtained from $G$ by repeated applications of the plucking operation $O_2$ until no further plucking operation is possible. It follows from Corollary 2.6 and from the above definitions that a tree $T$ of order at least 2 is an $\alpha$-excellent graph if and only if the plucked graph $P(T)$ is $K_2$. It also follows from these definitions that if a graph $G$ belongs to the family $\mathcal{H}$, then $G$ and $P(G)$ are unicyclic graphs. Now from Propositions 3.1 and 3.2 we have the following characterization of the $\alpha$-excellent unicyclic graphs. We omit its straightforward inductive proof.
Theorem 3.3. Let $G$ be a connected unicyclic graph. Then, $G$ is an $\alpha$-excellent graph if and only if one of the following two statements holds:

1. The plucked graph $P(G)$ of $G$ is a cycle.
2. The plucked graph $P(G)$ of $G$ is a caterpillar-wheel having a perfect matching with $\ell$ legs where $\ell \geq 2$.

The subdivision graph of a graph $G$, denoted $S(G)$, is the graph obtained from $G$ by subdividing every edge of $G$ exactly once. As an immediate consequence of Theorem 3.3, we have the following result.

Corollary 3.4. If $G$ is a connected unicyclic graph, then its subdivision graph $S(G)$ is an $\alpha$-excellent graph.

Proof. The result is obvious if $G$ is a cycle. If $G$ is a connected unicyclic graph with at least one leg, then $S(G)$ is a unicyclic graph, and its plucked graph $P(S(G))$ is a cycle. Thus, by Theorem 3.3, $S(G)$ is an $\alpha$-excellent graph.

4 $\alpha$-Excellent simplicial, chordal, and block graphs

In this section we characterize simplicial graphs, chordal graphs, and block graphs which are $\alpha$-excellent graphs. We begin with the following characterization of well-covered simplicial graphs and well-covered chordal graphs which was proved in [14].

Proposition 4.1. ([14]) If $G$ is a simplicial graph or a chordal graph, then $G$ is well-covered if and only if every vertex of $G$ belongs to exactly one simplex of $G$.

The following theorem gives a simple characterization of the $\alpha$-excellent simplicial graphs. This theorem also shows that a simplicial graph is $\alpha$-excellent if and only if it is well-covered.

Theorem 4.2. If $G$ is a simplicial graph, then the following statements are equivalent:

(a) $G$ is a well-covered graph.
(b) $G$ is an $\alpha$-excellent graph.
(c) Every vertex of $G$ belongs to exactly one simplex of $G$.

Proof. The statements (a) and (c) are equivalent, by Proposition 4.1. The implication (a) \(\Rightarrow\) (b) is obvious, as if $i(G) = \alpha(G)$, then by Observation 1.1 we have $\alpha_c(G) = \alpha(G)$, and so $G$ is $\alpha$-excellent. Finally, to prove the implication (b) \(\Rightarrow\) (a), assume that $G$ is an $\alpha$-excellent graph and suppose, to the contrary, that $G$ is not well-covered. In this case the equivalence of (a) and (c) implies that some vertex of $G$ belongs to at least two simplexes in $G$, which is impossible in an $\alpha$-excellent graph, see Proposition 2.1.

The graph $G$ shown in Fig. 1 is an $\alpha$-excellent chordal graph which is not a well-covered graph, as it is easy to observe that $i(G) = 2 < 3 = \alpha_c(G) = \alpha(G)$. Thus, since every well-covered graph is an $\alpha$-excellent graph, the set of well-covered simplicial graphs is properly contained in the set of $\alpha$-excellent chordal graphs.

For our characterization of $\alpha$-excellent chordal graphs, we need the next two properties of simplicial vertices and the definition of a successive clique-cover of a graph.
Proposition 4.3. If \( v \) is a simplicial vertex of a graph \( G \), then \( \alpha(G - N_G[v]) = \alpha(G) - 1 \).

Proof. If \( I \) is a maximum independent set of \( G - N_G[v] \), then \( I \cup \{ v \} \) is an independent set of \( G \) and, therefore, \( \alpha(G) \geq |I \cup \{ v \}| = |I| + 1 = \alpha(G - N_G[v]) + 1 \). Now assume that \( J \) is a maximum independent set of \( G \). Since \( G[N_G[v]] \) is a complete graph, necessarily \( |J \cap N_G[v]| \leq 1 \) and \( J \setminus N_G[v] \) is an independent set of \( G - N_G[v] \), implying that \( \alpha(G - N_G[v]) \geq |J \setminus N_G[v]| \geq |J| - 1 = \alpha(G) - 1 \). Consequently, \( \alpha(G - N_G[v]) = \alpha(G) - 1 \).

Proposition 4.4. If \( v \) is a simplicial vertex of an \( \alpha \)-excellent graph \( G \), then \( G - N_G[v] \) is an \( \alpha \)-excellent graph.

Proof. Assume that \( v \) is a simplicial vertex of an \( \alpha \)-excellent graph \( G \). We shall show that \( G' = G - N_G[v] \) is an \( \alpha \)-excellent graph. Let \( u \) be an arbitrary vertex of \( G' \). By Proposition 4.3, \( \alpha(G') = \alpha(G) - 1 \). Since \( G \) is \( \alpha \)-excellent, in \( G \) there is an independent set, say \( J_u \), of cardinality \( \alpha(G) \) that contains \( u \). The set \( J_u \setminus N_G[v] \) is an independent set in \( G' \) of cardinality \( \alpha(G) - 1 = \alpha(G') \) that contains \( u \). Since \( u \) is an arbitrary vertex of \( G' \), every vertex of \( G' \) belongs to an \( \alpha \)-set of \( G' \), that is, \( G' \) is an \( \alpha \)-excellent graph.

An ordered sequence \((V_1, \ldots, V_n)\) of subsets of the vertex set \( V(G) \) of a graph \( G \) is said to be a successive clique-cover of \( G \) if the following three properties hold:

(a) The sets \( V_1, \ldots, V_n \) form a partition of \( V(G) \);
(b) \( G[V_i] \) is a simplex in \( G - \bigcup_{j<i} V_j \) for \( i \in [n] \);
(c) For every \( u_i \in V_i \) there are vertices \( u_{i+1} \in V_{i+1}, \ldots, u_n \in V_n \) such that \( \{u_i, u_{i+1}, \ldots, u_n\} \) is an independent set for \( i \in [n-1] \).

Proposition 4.5. If a graph \( G \) has a successive clique-cover, then \( G \) is an \( \alpha \)-excellent graph.

Proof. The result is obvious if \( G \) is a complete graph. Thus assume that \( G \) is a non-complete graph and \((V_1, \ldots, V_n)\) is a successive clique-cover of \( G \). Let \( v_i \) be a simplicial vertex of \( G - \bigcup_{j<i} V_j \) belonging to \( V_i \) where \( i \in [n] \). From the recursive definition of the sequence \((V_1, \ldots, V_n)\) it is evident that the set \( I = \{v_1, \ldots, v_n\} \) is independent in \( G \). Therefore, \( \alpha(G) \geq |I| = n \). On the other hand, let \( J \) be a maximum independent set of \( G \). Since \( G[V_i] \) is a complete graph, we note that \( |J \cap V_i| \leq 1 \) for \( i \in [n] \). Thus, since \( V_1, \ldots, V_n \) form a partition of \( V(G) \), we have \( \alpha(G) = |J| = |J \cap V(G)| = |J \cap \bigcup_{i=1}^n V_i| = \sum_{i=1}^n |J \cap V_i| \leq n \). Consequently, \( \alpha(G) = n \).

To prove that \( G \) is an \( \alpha \)-excellent graph, it suffices to show that every vertex \( v \) of \( G \) belongs to an independent set of cardinality \( n \). This is obvious if \( v \in \{v_1, \ldots, v_n\} \) (where, in fact, \( v_n \) can be any vertex belonging to \( V_n \), as \( G[V_n] \) is a complete graph), as \( \{v_1, \ldots, v_n\} \) is an independent set of cardinality \( n \) in \( G \). Thus assume that \( v \in V(G) \setminus \{(v_1, \ldots, v_{n-1}) \cup V_n\} \). If \( v \in V_1 \), then from the property (c) of the successive clique-cover \((V_1, \ldots, V_n)\) of \( G \) it follows that there are
vertices $u_2 \in V_2, \ldots, u_n \in V_n$ such that $\{v, u_2, \ldots, u_n\}$ is an independent set of cardinality $n$ that contains $v$. If $v \in V_i$, where $1 < i < n$, then again from the property (c) of the successive clique-cover $(V_1, \ldots, V_n)$ it follows that there are vertices $u_{i+1} \in V_{i+1}, \ldots, u_n \in V_n$ such that $\{v, u_{i+1}, \ldots, u_n\}$ is an independent set. In this case $\{v_1, \ldots, v_{i-1}, v, u_{i+1}, \ldots, u_n\}$ is an independent set of cardinality $n$ in $G$ that contains the vertex $v$. This proves that $G$ is an $\alpha$-excellent graph. 

The next theorem presents a characterization of the $\alpha$-excellent chordal graphs.

**Theorem 4.6.** A chordal graph $G$ is an $\alpha$-excellent graph if and only if $G$ has a successive clique-cover.

**Proof.** If a chordal graph $G$ has a successive clique-cover, then $G$ is an $\alpha$-excellent graph, by Proposition 4.4. We proceed by induction on the order $n$ to show that an $\alpha$-excellent chordal graph has a successive clique-cover. The result is immediate to check for small $n \leq 4$. Let $n \geq 5$ and assume that every $\alpha$-excellent chordal graph of order less than $n$ has a successive clique-cover. Let $G$ be an $\alpha$-excellent chordal graph of order $n$.

Let $v_0$ be a simplicial vertex of $G$ and let $G' = G - N_G[v_0]$. The graph $G'$ is a chordal graph of order less than $n$. In addition, by Proposition 4.4, the graph $G'$ is an $\alpha$-excellent graph and, by Proposition 4.3, $\alpha(G') = \alpha(G) - 1$. Now, by the induction hypothesis, $G'$ has a successive clique-cover, say $(V_1, \ldots, V_k)$ is a successive clique-cover of $G'$ (where $k = \alpha(G')$), that is, $(V_1, \ldots, V_k)$ is a sequence having the properties (a), (b) and (c) specified in the definition of a successive clique-cover of $G'$. It remains to prove that the sequence $(V_0, V_1, \ldots, V_k)$, where $V_0 = N_G[v_0]$, has the desired properties of a successive clique-cover of $G$.

Since $V_1, \ldots, V_k$ is a partition of $V(G')$ and $V_0 = V(G) \setminus V(G')$, we note that $V_0, V_1, \ldots, V_k$ is a partition of $V(G)$. Recall that $v_0$ is a simplicial vertex of $G$ and $V_0 = N_G[v_0]$. We note that $G[V_i] = G'[V_i]$ and $G'[V_i]$ is a simplex in $G' - \bigcup_{j<i} V_j = G - \bigcup_{j<i} V_j$ for $i \in [n]$. These observations imply that $G[V_0]$ is a simplex in $G = G - \bigcup_{j<i} V_j$. Finally, the property (c) of the sequence $(V_0, V_1, \ldots, V_k)$ is obvious, for if $u_i \in V_i$ and $I_{u_i}$ is a largest independent set containing $u_i$, then, since $G$ is an $\alpha$-excellent graph, we infer that $|I_{u_i}| = k + 1$. Moreover, since $G[V_j]$ is a complete graph and $V_0, V_1, \ldots, V_k$ is a partition of $V(G)$, we have $|I_{u_i} \cap V_j| = 1$ for $j \in [k]$. This proves that $(V_0, V_1, \ldots, V_k)$ is a successive clique-cover of $G$.

It was proved in [22] (see also [14, 17, 21]) that a block graph $G$ is well-covered if and only if every vertex of $G$ belongs to exactly one simplex of $G$. The $\alpha$-excellent block graphs have already been characterized in [4]. Here we present another characterization of such graphs. In order to state and prove our characterization, we need additional definitions and terminology.

A family $\mathcal{P}$ of vertex disjoint blocks of a block graph $G$ is a perfect block-cover of $G$ if every vertex of $G$ belongs to some block in $\mathcal{P}$. (It is easy to observe that every block graph has at most one perfect block-cover.) Let $K_n$ be a complete graph of order $n \geq 1$ with vertex set $\{v_1, \ldots, v_n\}$, and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of $n$ complete graphs. Recall $K_n \circ \mathcal{H}$ denotes a graph obtained from the disjoint union of the graphs $K_n, H_1, \ldots, H_n$ by joining the vertex $v_i$ of $K_n$ to every vertex of $H_i$ for $i \in [n]$. The graph $K_n \circ \mathcal{H}$ is said to be the general corona of $K_n$ and $\mathcal{H}$, and its complete subgraph induced by the vertices $v_1, \ldots, v_n$ is called the body of $K_n \circ \mathcal{H}$.

Let $\mathcal{F}$ be the family of graphs defined in [4] and such that:

1. contains every complete graph of order at least 2; and

10
(2) is closed under attaching general coronas of complete graphs

Thus, if a graph $G'$ belongs to the family $\mathcal{F}$ and $H = K_n \circ \mathcal{H}$ is a general corona of a complete graph $K_n$ and a family $\mathcal{H}$ of complete graphs, then to $\mathcal{F}$ belongs every graph $G'_\alpha(H)$ obtained from the disjoint union $G' \cup H$ by adding the edges joining the body of $H$ with the vertex $v$ of $G'$, see Fig. 2(a). It is clear from this definition that every graph belonging to the family $\mathcal{F}$ is a block graph.

We are now in a position to state and prove a characterization of the $\alpha$-excellent block graphs in terms of perfect block-covers.

**Theorem 4.7.** Let $G$ be a connected block graph of order at least 2. Then the following statements are equivalent:

(a) $G$ is an $\alpha$-excellent graph.

(b) $G \in \mathcal{F}$.

(c) $G$ has a perfect block-cover.

*Proof.* The equivalence between (a) and (b) was proved in [4]. By making a little modification of that proof, we shall prove the equivalence of the statements (b) and (c). Assume first that $G \in \mathcal{F}$. Thus, $G$ can be obtained from a sequence $G_1, \ldots, G_m$ of graphs belonging to $\mathcal{F}$, where $G_1$ is a complete graph of order at least 2 and $G = G_m$, and, if $m \geq 2$, $G_{i+1}$ can be obtained from $G_i$ by attaching a general corona (of a complete graph) for $i \in [m - 1]$. By induction on the number $m$ we shall prove that $G$ has a perfect block-cover.

If $m = 1$, then $G$ is a complete graph of order at least 2 and, certainly, the only block of $G$ forms a perfect block-cover of $G$. This establishes the base case of the induction. Assume, then, that the result holds for all graphs belonging to $\mathcal{F}$ that can be constructed from a sequence of fewer than $m$ graphs, where $m \geq 2$. Let $G$ be obtained from a sequence $G_1, \ldots, G_{m-1}$ by attaching a general corona $H = K_n \circ \{H_1, \ldots, H_n\}$ to a vertex $v$ of $G' = G_{m-1}$, that is, assume that $G = G'_\alpha(H)$. By the induction hypothesis, $G'$ has a perfect block-cover, say $\mathcal{P}'$. Now, it is obvious that $\mathcal{P}' \cup \{B_1, \ldots, B_n\}$ is a perfect block-cover of $G$, where $B_1, \ldots, B_n$ are the blocks of $G$ (and of $H$) induced by the vertex-sets $V(H_1) \cup \{v_1\}, \ldots, V(H_n) \cup \{v_n\}$, respectively, where $v_1, \ldots, v_n$ are the vertices of the body of $H$. This proves the implication (b) $\Rightarrow$ (c).

Finally, to prove the implication (c) $\Rightarrow$ (b), assume that $G$ is a block graph of order at least 2 and $G$ has a perfect block-cover, say $\mathcal{P} = \{B_1, \ldots, B_n\}$. We proceed by induction on the number $n$ of blocks in $\mathcal{P}$. If $n = 1$, then $G = B_1$ is a complete graph and, certainly, $G \in \mathcal{F}$. Assume, then, that $n \geq 2$ and that the statement holds for all block graphs having a perfect block-cover of cardinality less than $n$. From the fact that the blocks $B_1, \ldots, B_n$ are disjoint and $n \geq 2$, it follows that the diameter $d$ of $G$ is greater than 2. Let $P: u_0 u_1 \ldots u_d$ be a longest path without chords in $G$. Let $D_1$ and $D_2$ be the blocks in $G$ that contain the edges $u_0 u_1$ and $u_1 u_2$, respectively. From the choice of $P$ as the longest chordless path in $G$, it follows that $u_0$ is a simplicial vertex in $G$. Therefore, $D_1$ is the only block in $G$ that contains $u_0$. The block $D_1$, as does every other simplex of $G$, belongs to $\mathcal{P}$, and, consequently, every two simplexes of $G$ are pairwise vertex-disjoint. This also means that $D_1$ is the only simplex of $G$ that contains $u_1$. We now consider two cases depending on the order of $D_2$.

**Case 1.** $D_2$ is of order two. In this case, $\mathcal{P}' = \mathcal{P} - \{D_1\}$ is a perfect block-cover of the subgraph $G' = G - V(D_1)$ of $G$. Thus, by the induction hypothesis, $G' \in \mathcal{F}$. Consequently, $G$ belongs to
the family \( \mathcal{F} \) as \( G \) can be obtained from \( G' \) by attaching the general corona \( K_1 \circ \{D_1 - u_1\} = G[\{u_1\}] \circ \{D_1 - u_1\} \) to the vertex \( u_2 \) in \( G' \).

Case 2. \( D_2 \) is of order at least three. Assume that \( u_2, v_1 = u_1, v_2, \ldots, v_l \) are the vertices of \( D_2 \), where \( l \geq 2 \). Since \( D_1 \) and \( D_2 \) share a vertex, the block \( D_2 \) is not a simplex. Thus every vertex of \( D_2 \) is a cut-vertex in \( G \). From this and from the choice of \( P \) as the longest chordless path in \( G \), it follows that each vertex \( v_i \) belongs to some end-block of \( G \). In addition, since every end-block of \( G \) belongs to \( P \) and blocks in \( P \) are disjoint, every vertex \( v_i \) of \( D_2 \) belongs to exactly one end-block in \( G \), say \( B_i \) is an end-block of \( G \) that contains \( v_i \). Certainly, \( B_1 = D_1 \), and the subgraph \( H \) of \( G \) induced by the vertices belonging to the blocks \( B_1, B_2, \ldots, B_l \) is a general corona, \( H = K_1 \circ \mathcal{H} \), where \( K_l = G[\{v_1, \ldots, v_l\}] \) and \( \mathcal{H} = \{B_1 - v_1, \ldots, B_l - v_l\} \). Now, \( P \setminus \{B_1, \ldots, B_l\} \) is a perfect block-cover of the subgraph \( G' = G - V(H) \) of \( G \). Thus by the induction hypothesis, \( G' \in \mathcal{F} \). Consequently, \( G \) belongs to the family \( \mathcal{F} \) as \( G \) can be obtained from \( G' \) by attaching the general corona \( H \) to the vertex \( u_2 \) in \( G' \), see Fig. 2(b). This completes the proof of Theorem 4.7. □

\[
\begin{align*}
G' &
\end{align*}
\]

5 \( \alpha \)-Excellent generalized Petersen graphs

Let \( n \) and \( k \) be positive integers with \( n \geq 3 \) and \( k \in [n - 1] \). The \( \text{generalized Petersen graph} \ P_{n,k} \) is defined in the following way. It has \( 2n \) vertices, say \( v_0, v_1, \ldots, v_{n-1}, u_0, u_1, \ldots, u_{n-1} \), and edges \( v_iv_{i+1}, v_iu_i \), and \( u_iu_{i+k} \) for all \( i \) satisfying \( i \in [n - 1] \) with all subscripts taken modulo \( n \). By construction each vertex \( v_i \) is of degree 3 in \( P_{n,k} \). Similarly, each \( u_i \) is a vertex of degree 3 if \( k \neq n/2 \) but its degree is 2 if \( k = n/2 \). Since \( P_{n,k} \) is isomorphic to \( P_{n,n-k} \), we may therefore always assume that \( k \leq \lfloor n/2 \rfloor \). A simple analysis shows that \( P_{n,k} \) is a graph of girth 3 if and only if \( n = 3k \) for \( k \geq 1 \). Analogously, \( P_{n,k} \) is a graph of girth 4 if and only if \( k = 1 \) and \( n \geq 4 \), \( k = 2 \) and \( n = 4 \) or \( k \geq 1 \) and \( n = 4k \).

It was shown in [20] that \( P_{3,1}, P_{4,2}, P_{5,1}, P_{6,2}, \) and \( P_{7,2} \) (shown in Fig. 3) are the only well-covered generalized Petersen graphs. In contrast, all generalized Petersen graphs are \( \alpha \)-excellent graphs.

**Proposition 5.1.** Every generalized Petersen graph is an \( \alpha \)-excellent graph.
then at least one of the sets $I \in \{C_{n,1}, C_{n,2}, \ldots, C_{n,k}\}$ if and vice versa. It is known (see [20]) that a $C_{n,1}$ (that is, to exactly one simplex of $G$) does not know which $C_{n,1}$ we do not know which is an $\alpha$-excellent graph. Hence, $I_0 \cap U = \emptyset$. If $I_0 \cap U = \emptyset$, then $I_0 = V$, which is impossible as $V$ is not an independent set in $G$. Hence, $I_0 \cap V \neq \emptyset$. Now, if $I_l = \{v_{i+l}: v_i \in I_0 \cap V\} \cup \{u_{j+l}: u_j \in I_0 \cap U\}$, where $l \in [k-1]$ and all subscripts are taken modulo $n$, then each of the sets $I_0, I_1, \ldots, I_{k-1}$ is an $\alpha$-set of $G$, and, certainly, each vertex of $G$ belongs to at least one of the sets $I_0, I_1, \ldots, I_{k-1}$. This proves that $G$ is an $\alpha$-excellent graph. 

6 Concluding remarks

A graph $G$ is a $C_{(n)}$-tree if it can be constructed from a cycle of length $n$ by a finite number of applications of the following operation: add a new cycle of length $n$ and identify an edge of this cycle with an edge of the existing graph. Note that every 2-tree of order at least 3 is a $C_{(3)}$-tree and vice versa. It is known (see [20]) that a $C_{(n)}$-tree $G$ is a well-covered graph if and only if $G \in \{C_3, C_4, C_5, C_7\}$ or $G$ is a $C_{(3)}$-tree in which every vertex belongs to exactly one end cycle of $G$ (that is, to exactly one simplex of $G$). The way in which each $C_{(n)}$-tree is constructed implies that if $n$ is even, then every $C_{(n)}$-tree is a bipartite graph and has a perfect matching. From this and from Proposition 2.5 we conclude that every $C_{(n)}$-tree is an $\alpha$-excellent graph if $n$ is even. However, we do not know which $C_{(n)}$-trees are $\alpha$-excellent if $n$ is odd. The graphs $G_1$ and $G_2$ in Fig. 4 are $C_{(5)}$-trees, and only $G_2$ is an $\alpha$-excellent graph.

It would also be interesting to know which Cartesian products of graphs are $\alpha$-excellent graphs. The problem is not clear even for bipartite graphs. It is obvious that the Cartesian product $G \square H$ is a bipartite graph if and only if both $G$ and $H$ are bipartite and, as we know, a bipartite graph is an $\alpha$-excellent graph if and only if it has a perfect matching. Thus, if $G$ and $H$ are bipartite graphs and at least one of them is an $\alpha$-excellent graph, then the Cartesian product $G \square H$ is an $\alpha$-excellent graph. The opposite implication is not true, as, for example, the Cartesian product $S_{2,2} \square K_{1,2}$ in Fig. 4 is an $\alpha$-excellent graph but neither the double star $S_{2,2}$ nor the star $K_{1,2}$ is an $\alpha$-excellent graph.

We close this paper with the following two open problems that we have yet to settle.

1. Characterize the $\alpha$-excellent $C_{(n)}$-trees for odd $n \geq 3$.
2. Study $\alpha$-excellent Cartesian products of graphs.

\begin{figure}[h]
\centering
\begin{tabular}{ccccc}
\includegraphics[width=0.15\textwidth]{P3_1} & \includegraphics[width=0.15\textwidth]{P4_2} & \includegraphics[width=0.15\textwidth]{P5_1} & \includegraphics[width=0.15\textwidth]{P6_2} & \includegraphics[width=0.15\textwidth]{P7_2} \\
$P_{3,1}$ & $P_{4,2}$ & $P_{5,1}$ & $P_{6,2}$ & $P_{7,2}$ \\
\end{tabular}
\caption{The well-covered generalized Petersen graphs}
\end{figure}
References

[1] C. Berge, *Theory of Graphs and its Applications*. Methuen, London, 1962.

[2] C. Berge, Some common properties for regularisable graphs, edge-critical graphs and B-graphs. In: N. Saito, T. Nishizeki (eds), *Graph Theory and Algorithms*. Lecture Notes in Computer Science, 108, Springer, New York, 1981, 108–123.

[3] C. Berge, *Graphs*. North-Holland, Amsterdam, 1985.

[4] M. Dettlaff, M. Lemańska, and J. Topp, Common independence in graphs. *Symmetry* **13** (2021), 1411.

[5] G. S. Domke, J. H. Hattingh, and L. R. Markus, On weakly connected domination in graphs II. *Discrete Math.* **305** (2005), 112–122.

[6] G. H. Fricke, T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, and R. C. Laskar, Excellent trees. *Bull. Inst. Combin. Appl.* **34** (2002), 27–38.

[7] W. Goddard and M. A. Henning, Independent domination in graphs: A survey and recent results. *Discrete Math.* **313** (2013), 839–854.

[8] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning (eds), *Topics in Domination in Graphs*. Developments in Mathematics, 64, Springer, Cham, 2020.

[9] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning (eds), *Structures of Domination in Graphs*. Developments in Mathematics, 66, Springer, Cham, 2021.

[10] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning, *Domination in Graphs: Core Concepts*. Springer Monographs in Mathematics, Springer, Cham, 2022.

[11] O. Ore, *Theory of Graphs*. Amer. Math. Soc. Colloq. Publ. 38, Providence, RI, 1962.

[12] M. D. Plummer, Some covering concepts in graphs. *J. Combin. Theory* **8** (1970), 91–98.

[13] M. D. Plummer, Well-covered graphs: a survey. *Quaest. Math.* **16** (1993), 253–287.

[14] E. Prisner, J. Topp, and P. D. Vestergaard, Well covered simplicial, chordal and circular arc graphs. *J. Graph Theory* **21** (1996), 113–119.
[15] A. P. Pushpalatha, G. Jothilakshmi, S. Suganathi, and V. Swaminathan, $\beta_0$-excellent graphs. *Int. J. Contemp. Math. Sciences* **6** (2011), 1447–1451.

[16] A. P. Pushpalatha, G. Jothilakshmi, S. Suganathi, and V. Swaminathan, Very $\beta_0$-excellent graphs. *Taga J.* **14** (2018), 144–148.

[17] B. Randerath and L. Volkmann, A characterization of well covered block-cactus graphs. *Australas. J. Combin.* **91** (1994), 307–314.

[18] G. Ravindra, Well-covered graphs. *J. Comb. Inf. Syst. Sci.* **2** (1977), 20–21.

[19] D. J. Rose, On simple characterizations of $k$-trees. *Discrete Math.* **7** (1974), 317–322.

[20] J. Topp, Domination, independence and irredundance in graphs. *Dissertationes Math.* **342** (1995), 99 pp.

[21] J. Topp and L. Volkmann, On domination and independence numbers of graphs. *Result. Math.* **17** (1990), 333–341.

[22] J. Topp and L. Volkmann, Well covered and well dominated block graphs and unicyclic graphs. *Math. Pannon.* **1**/2 (1990), 55–66.

[23] J. Topp and L. Volkmann, Characterization of unicyclic graphs with equal domination and independence numbers. *Discuss. Math.* **11** (1991), 27–34.