On Weak Separation Property for Affine Fractal Functions.

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1. Introduction. Weak separation property (WSP), which was developed since 90-ies in papers of C.Bandt[2], K.-S.Lau and S.-M.Ngai [6] and M.Zerner[10] remains one of the main tools of analyzing dimension problems. In recent years it proved to be useful for the study of geometric structure of self-similar sets [8] and rigidity of self-similar structures [9].

In this short note we apply this notion to the theory of affine fractal interpolation functions.

Standard definition (see [3], [4], [7]) of affine fractal function $f : [a, b] \rightarrow \mathbb{R}$ deals with a partition $a = x_0 < x_1 < \ldots < x_m = b$ of the interval $[a, b]$ and a system $S = \{ S_1, \ldots, S_m \}$ of affine transformations

$$S_i(x, y) = \begin{pmatrix} p_i & 0 \\ r_i & q_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_i \\ s_i \end{pmatrix}, |p_i| < 1, |q_i| < 1,$$

which send vertical strip $a \leq x \leq b$ to vertical strips $L_i = \{(x, y) : x_{i-1} \leq x \leq x_i\}$. These strips divide the graph $\Gamma(f)$ to non-overlapping pieces $\Gamma_i = S_i(\Gamma(f)) = \Gamma(f) \cup L_i$ whose union is $\Gamma(f)$.

But a more general approach must take into account the possibility of overlaps:

For example, a system $S$ consisting of 4 maps

$$S_1 : \begin{pmatrix} 1/5 & 0 \\ 1/5 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad S_2 : \begin{pmatrix} 1/3 & 0 \\ -1/5 & -1/5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/5 \\ 1/5 \end{pmatrix},$$

$$S_3 : \begin{pmatrix} 1/3 & 0 \\ 1/5 & -1/5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 7/15 \\ 0 \end{pmatrix}, \quad S_4 : \begin{pmatrix} 1/5 & 0 \\ -1/5 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix}$$

defines a self-affine function whose graph passes through the points

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(0, 0), (1/5, 1/5), (7/15, 0), (8/15, 0), (4/5, 1/5), (1, 0) and has overlapping pieces

\[ S_2(\Gamma) \cap S_3(\Gamma) = S_2S_4(\Gamma) = S_3S_1(\Gamma) = \Gamma \left( f\big|_{\frac{7}{15} \leq x \leq \frac{8}{15}} \right) \]

Figure 1: The graph of \( f(x) \): overlapping pieces are blue and red.

In view of the above argument, we use the following definition which allows the overlaps:

**Definition 1.** Let \( S = \{S_1, \ldots, S_m\} \) be a system of affine maps

\[ S_i(x, y) = (p_i x + h_i, q_i y + r_i x + s_i); |p_i|, |q_i| < 1 \]

A function \( f(x) \) is an affine fractal function on \( [a, b] \) defined by the system \( S \), if its graph \( \Gamma(f) = \{(x, f(x)), x \in [a, b]\} \) is the attractor of the system \( S \).

To formulate the main Theorem we recall some definitions and notation:

We denote the projections of \( S_i \) to \( \mathbb{R} \) by \( S_i^{\diamond} \left( x \right) = p_i x + h_i \) and we denote \( S^\circ = \{S_1^\circ, \ldots, S_m^\circ\} \).

\( G \) denotes the semigroup generated by \( S \) and \( G^\circ \) denotes the semigroup generated by \( S^\circ \).

Observe that each element \( g_i = S_{i_1}S_{i_2} \ldots S_{i_k} \) of the semigroup \( G \) is a map of the form \( g_i(x, y) = (p_i x + h_i, q_i y + r_i y + s_i); |p_i|, |q_i| < 1 \) where \( p_i = p_{i_1}p_{i_2} \ldots p_k \), \( g_i = g_1 g_2 \ldots g_k \), while \( g_i^\circ(x) = S_{i_1}^\circ S_{i_2}^\circ \ldots S_{i_k}^\circ(x) = (p_i x + h_i) \).

We define associated families \( \mathcal{F} = G^{-1} \circ G \) and \( \mathcal{F}^\circ = G^\circ^{-1} \circ G^\circ \) for the system \( S \) (resp. \( S^\circ \)). Each element of the family \( \mathcal{F} \) is a composition \( g = g_j^{-1} g_i \) and also has the form \( g(x, y) = (px + h, qy + rx + s) \), while its projection \( g^\circ = g_1^\circ \ldots g_k^\circ \) satisfies \( g^\circ(x) = px + h \).

**Definition 2.** *The system \( S \) satisfies weak separation property (WSP) if \( \Id \) is an isolated point in the associated family \( \mathcal{F} \).*
So, if the system \( S \) does not satisfy the weak separation property, there is a sequence \( g_n \in \mathcal{F} \) which converges to \( \text{Id} \).

2. The main Theorem.

In this paper we prove the following Theorem:

**Theorem 3.** Let \( f(x) \) be the affine fractal function defined by a system \( S \) on the segment \([a, b]\). If \( S^\circ \) does not satisfy weak separation property, then the graph \( \Gamma(f) \) is a segment of a parabola.

First of all, it follows from the definition that each fractal affine function is continuous, because its graph \( \Gamma(f) \) is a compact set.

Second, a remarkable property of the maps \( g \in \mathcal{F} \) is that these maps move the points of \( \Gamma(f) \) along \( \Gamma(f) \):

**Lemma 4.** If \( g \in \mathcal{F} \) and for some \( x \in [a, b] \), \( g^\circ(x) \in [a, b] \), then \( g(x, f(x)) = (g^\circ(x), f(g^\circ(x))) \).

**Proof.** Let \( g \in \mathcal{F} \), so \( g = g_1^{-1}g_2 \). If \( (x, y) \in \Gamma(f) \), then \( g(x, y) \in \Gamma(f) \). Suppose \((u, v) \in \Gamma(f) \) and \( g_2^r(u) = g_1^r(x) \). Since \( g_1(u, v) \in \Gamma(f) \), \( g_1(u, v) = g_1(x, y) \), therefore \( g_1^{-1}g_1(x, y) = (u, v) \), so \( g(x, f(x)) = (g^\circ(x), f(g^\circ(x))) \).

These facts imply that weak separation property holds for both systems \( S^\circ \) and \( S \) simultaneously:

**Lemma 5.** Let \( f(x) \) be the affine fractal function defined by a system \( S \) on the segment \([a, b]\) whose graph is not a straight line segment. The system \( S \) satisfies WSP iff \( S^\circ \) satisfies WSP.

**Proof.** Suppose that WSP does not hold for \( \mathcal{F}^\circ \). Take three points \((x_i, y_i)\), \( i = 1, 2, 3 \) on \( \Gamma(f) \) which do not lie on a line. If \( g_n^r \to \text{Id} \) then for each \( i \), \( g_n^r(x_i) \to x_i \). Since \( f \) is continuous, \( g_n(x_i, y_i) \to (x_i, y_i) \). This means that \( g_n \) converges to \( \text{Id} \) and WSP does not hold for \( \mathcal{F} \).

Suppose now that WSP does not hold for \( \mathcal{F} \) and there is a sequence \( g_n \in \mathcal{F} \) which converges to \( \text{Id} \). Consider the coefficients of \( g_n(x, y) = (p_n x + h_n, q_n y + r_n x + s_n) \); \( p_n \) and \( q_n \) converge to 1, while \( h_n, r_n \) and \( s_n \) converge to 0. Therefore \( g_n^r(x) = p_n x + h_n \) also converges to \( \text{Id} \).

**Lemma 6.** Suppose \( U \) is a family of functions \( \varphi(x) \in C^3[a, b] \), satisfying inequality \( |\varphi(x)| \leq M \). If for any \( \varphi(x) \in U \), \( \varphi''(x) \) and \( \varphi'''(x) \) are monotonous and do not change the sign on \([a, b]\), then for any segment \([a', b'] \subset (a, b)\), the family \( U' = \{\varphi|_{[a', b']} ; \varphi \in U\} \) is bounded in \( C^3([a', b']) \)

**Proof.** Without loss of generality, we suppose \([a, b] = [0, 1]\) and \( \varphi''(x) > 0 \) on \([0, 1]\). Take some \( \lambda \in (2^{1/3}, 1) \).
Since \( \varphi(1) \leq M \) and \( \varphi(\lambda) \geq -M, \varphi'(\lambda) < \frac{2M}{1-\lambda} \). Similarly, we get \( \varphi'(1-\lambda) > \frac{-2M}{1-\lambda} \). So \( \varphi'(x) < \left| \frac{2M}{1-\lambda} \right| \) on \([1-\lambda, \lambda]\).

Repeating the same step for \( \varphi' \) we get \( 0 < \varphi''(\lambda^2) < \frac{4M}{\lambda(1-\lambda)^2} \) if \( \varphi'' \) increases and \( \varphi''(1-\lambda^2) > \frac{4M}{\lambda(1-\lambda)^2} \) if \( \varphi'' \) decreases, so \( \varphi''(x) < \frac{8M}{\lambda^3(1-\lambda)^3} \) on \([1-\lambda^3, \lambda^3]\). Taking such \( \lambda \), that \([a', b'] \subset [1-\lambda^3, \lambda^3]\), we obtain the statement for the segment \([a', b']\). ■

**Lemma 7.** Let \( g \in \mathcal{F} \) and fix \((g^0) \notin [a, b] \). Suppose that
(i) if \( x_1, x_2 \in [a, b] \) and \(|x_1 - x_2| < \delta \), then \(|(x_1, f(x_1)) - (x_2, f(x_2))| < \varepsilon; \)
(ii) \(|g(x, y) - (x_2, f(x_2))| < \delta \) for any point \((x, y) \in \Gamma(f)\).
Then for some \( M \in \mathbb{N} \) either \((g^n(a, f(a)), n = 0, \ldots, M) \) or \((g^n(b, f(b)), n = 0, \ldots, M) \) is an \( \varepsilon \)-net in \( \Gamma(f) \).

**Proof.** The condition (i) implies that if \( \{x_1, \ldots, x_k\} \) is a \( \delta \)-net in \([a, b] \), then \( \{(x_1, f(x_1)), \ldots, (x_k, f(x_k))\} \) is an \( \varepsilon \)-net in \( \Gamma(f) \). So we have to show that \( g^{3n}(a) \) or \( g^{3n}(b) \) form a \( \delta \)-net in \([a, b]\).

Since \( \text{fix}(g^0) \notin [a, b] \), we have either \( g^0(x) > x \) for any \( x \in [a, b] \) or \( g^0(x) < x \) for any \( x \in [a, b] \).

Suppose \( g^0(a) > a \). Then for any point \( x \in [a, b] \), \( g^0(x) > x \) and \( g^0(x) - x < \delta \).
Since the limit point of the sequence \( g^{3n}(a) \) is outside \([a, b]\), there is such \( M \in \mathbb{N} \) for which \( g^{3M}(a) < b < g^{3M+1}(a) \), so for any \( n = 1, \ldots, M, g^{3n}(a) - g^{3n-1}(a) < \delta \) and \( b - g^{3n}(a) < \delta \). Therefore \( \{g^n(a, f(a)), n = 0, 1, \ldots, M\} \) is an \( \varepsilon \)-net in \( \Gamma(f) \). The proof in the case \( g^0(b) < b \) is similar. ■

**Lemma 8.** Suppose \( g(x, y) \in \mathcal{F}, \text{fix}(g^0) \notin [a, b] \) and \( g^0(x) > x \) on \([a, b]\).
Let \( g^0(a) = b \). Then the set \( \{g^t(a, f(a)), t \in [0, T]\} \) is a graph of one of the following functions on \([a, b]\):
1. \( y = Ax^2 + Bx + C; \)
2. \( y = Ax + Be^{Kx} + C; \)
3. \( y = Ax + B(\log(x - C)) + D, C \notin [a, b]. \)
4. \( y = Ax + B(x - C)^K + D, C \notin [a, b]; \)
5. \( y = A(x - C) \log(x - C) + Dx + E, C \notin [a, b]. \)

**Proof.** It is sufficient to check the statement in the case \( a = 0, b = 1, f(0) = 0 \) and \( p > 1 \). Since \( g \) is close to \( Id, p \) and \( q \) are close to 1 and therefore positive.

The five types of functions arise from direct solution of recurrence equations:
1. If \( g(x, y) = (x + h, y + rx + s) \), then the points \( g^n(0, 0) \) lie on a parabola \( y = Ax^2 + Bx \), where \( A = \frac{r}{2h} \) and \( B = \frac{2s - hr}{2h} \).

2. If \( g(x, y) = (x + h, qy + rx + s) \), \( q \neq 1 \), then the points \( g^n(0, 0) \) lie on a graph of a function \( y = Ax + B(e^{Kx} - 1) \), where \( K = \frac{\log q}{h} \), \( A = \frac{r}{q - 1} \).

\[
B = \frac{hr + (q - 1)s}{(q - 1)^2};
\]

3. If \( g(x, y) = (px + h, y + rx + s) \), then the points \( g^n(0, 0) \) lie on a graph of \( y = Ax + B(\log(1 + x/C)) \), where \( C = \frac{h}{p - 1} \), \( A = \frac{r}{p - 1} \), \( B = \frac{hr + (1 - p)s}{(1 - p)\log p} \).

4. If \( g(x, y) = (px + h, qy + rx + s) \), then the points \( g^n(0, 0) \) lie on a graph of a function \( y = Ax + B(x/C + 1)^K - B \), where \( A = \frac{r}{p - q} \), \( C = \frac{h}{p - 1} \).

\[
B = \frac{hr + s(q - p)}{(q - 1)(q - p)} \quad \text{and} \quad K = \frac{\log q}{\log p};
\]

5. If \( g(x, y) = (px + h, py + rx + s) \), then the points \( g^n(0, 0) \) lie on a graph of a function \( y = A(x/C + 1)\log(x/C + 1) + Bx \), where \( C = \frac{h}{p - 1} \), \( A = \frac{rC}{p \log p} \).

\[
B = \frac{Cr - s}{C - Cp}.
\]

Applying to x coordinate a linear transformation which sends \([a, b]\) to \([0, 1]\), we get the formulas 1-5 of the statement.

**Proof of the Theorem 1.** Take such sequence \( g_n \to \text{Id} \), \( g_n \in \mathcal{F} \) and such segment \([a_1, b_1] \subset (a, b)\), that for any \( n \), \( \text{fix}(g^n_{a_1}) \notin [a_1, b_1] \).

Since \( g^{-1}_n \) also converge to Id, we may suppose that for any \( n \), \( p_n \geq 1 \).

Without loss of generality we may suppose that for any \( n \), \( g^n_{a_1}(1) > a_1 \). Let \( T_n \) be such number, that \( g^n_{a_1}(1) = b_1 \). Each curve \( \{g^n_{a_1}(t, f(a_1)), t \in [0, T_n]\} \) is a graph of a function \( \varphi_n(x) \) on the segment \([a_1, b_1]\).

It follows from Lemma 7 that \( \varphi_n(x) \) uniformly converges to \( f(x) \) on \([a_1, b_1]\).

By Lemma 8, each of these functions is of one of 5 types, indicated by the Lemma. Therefore the functions \( \varphi_n(x) \) have monotone derivatives \( \varphi''_n(x) \), \( \varphi'''_n(x) \), which do not change their sign on \([a_1, b_1]\). By Lemma 6, for any \([a_2, b_2] \subset (a_1, b_1)\), the family \( \{\varphi_n(x)|_{a_2, b_2}\} \) is a bounded subset of \( C^3([a_2, b_2]) \). Therefore some subsequence of \( \varphi_n(x) \) converges in \( C^2([a_2, b_2]) \), which implies that \( f(x) \) is twice differentiable on \([a_2, b_2]\). This means that \( f(x) \in C^2([a, b]) \). As it was proved in [1] Theorem 3 this implies that \( \Gamma(f) \) is a parabolic segment.

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