Compactness and existence results of the prescribing fractional $Q$-curvatures problem on $\mathbb{S}^n$

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Abstract
This paper is devoted to establishing the compactness and existence results of the solutions to the prescribing fractional $Q$-curvatures problem of order $2\sigma$ on $n$-dimensional standard sphere when $n - 2\sigma = 2$, $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$. The compactness results are novel and optimal. In addition, we prove a degree-counting formula of all solutions to achieve the existence. From our results, we can know where blow up occur. Furthermore, the sequence of solutions that blow up precisely at any finite distinct location can be constructed. It is worth noting that our results include the case of multiple harmonic.

Mathematics Subject Classification (2020): 35R09,35B44,35J35

1 Introduction
The study of the prescribing scalar curvature problem on Riemannian manifolds, which dates back to [34, 35, 36], has received a lot of attention. In the case of $n$-dimensional standard sphere $(\mathbb{S}^n, g_0)$, this is known as Nirenberg problem. The classical Nirenberg problem is as follows: which function $K$ on $(\mathbb{S}^n, g_0)$ is the scalar curvature (Gauss curvature in dimension $n = 2$) of a metric $g$ that is conformal to $g_0$? If we denote $g = e^{2v}g_0$ in the two dimensional case and $g = v^{\frac{n-2}{2}}g_0$ in the $n$ ($n \geq 3$) dimensional case, this problem is equivalent to solving the following nonlinear elliptic equations:

$$-\Delta_{g_0} v + 1 = Ke^{2v} \quad \text{on } \mathbb{S}^2, \quad (1.1)$$
and

$$-\Delta_{g_0} v + c(n)R_0 v = c(n)Kv^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{S}^n, \quad n \geq 3, \quad (1.2)$$

*The research was supported by National Science Foundation of China(12071036,12126306)
where $\Delta_{g_0}$ is the Laplace-Beltrami operator, $c(n) = \frac{n-2}{4(n-1)}$, $R_0 = n(n-1)$ is the scalar curvature associated to $g_0$.

A first answer to the Nirenberg problem was given by Koutroufiotis [38], which established the existence of the solutions to (1.1) by assuming that $K$ is an antipodally symmetric function which close to 1. Morse [44] proved the existence of antipodally symmetric solutions to (1.1) for all antipodally symmetric functions $K$ which are positive somewhere. Chang and Yang [11] further extended this existence result to the case of $K$ without making any symmetry assumptions. Moreover, Bahri and Coron [6] presented a sufficient condition for the existence of solutions to (1.2) in dimension $n = 3$. As for the compactness of all solutions in dimensions $n = 2, 3$, Chang et al. [12], Han [25], and Schoen and Zhang [49] proved that a sequence of solutions cannot blow up at more than one point. Li [39, 40] established the compactness and existence results for (1.2). In these two papers, the compactness result is very different from the previous low-dimensional case. In fact, when $n = 2$ or $n = 3$, a sequence of solutions cannot blow up at more than one point. However, if $n > 3$, there could be blow up at many points, which considerably complicates the study of the problem. There have been many papers on the problem and related ones, see e.g., [9, 13, 20, 26, 28, 47, 48, 50].

The linear operators defined on left-hand side of (1.1) and (1.2) are called the conformal Laplacian associated to the metric $g_0$ and are denoted as $P_{1}^{g_0}$. For any Riemannian manifold $(M, g)$, let $R_g$ be the scalar curvature of $(M, g)$, and the conformal Laplacian be defined as $P_{1}^{g} = -\Delta_g + \frac{n-2}{4(n-1)}R_g$. The Paneitz operator $P_{2}^{g}$ is another conformal invariant operator, which was discovered by Paneitz [45]. Graham et al. [23] constructed a sequence of conformally covariant elliptic operators $\{P_{k}^{g}\}$ on Riemannian manifolds for all positive integers $k$ if $n$ is odd, and for $k \in \{1, \cdots, n/2\}$ if $n$ is even, which are called GJMS operators. Juhl [32, 33] found an explicit formula and a recursive formula for GJMS operators and $Q$-curvatures (see also Fefferman and Graham [22]). Graham and Zworski [24] presented a family of fractional order conformally invariant operators $P_{\sigma}^{g}$ of non-integer order $\sigma \in (0, n/2)$ on the conformal infinity of asymptotically hyperbolic manifolds. In addition, Chang and González [10] showed that the operator $P_{\sigma}^{g}$, $\sigma \in (0, n/2)$ can be defined as a Dirichlet-to-Neumann operator of a conformally compact Einstein manifold by using localization method in [8], they also provided some new interpretations and properties of those fractional operators and their associated fractional $Q$-curvatures.

Regarded as a generalization of Nirenberg problem, the prescribing fractional $Q$-curvature problem of order $2\sigma$ on $\mathbb{S}^n$ can be described as: which function $K$ on $\mathbb{S}^n$ is the fractional $Q$-curvature of a metric $g$ on $\mathbb{S}^n$ conformally equivalent to $g_0$? If we denote $g = v^{4/(n-2\sigma)}g_0$, this problem can be represented as finding the solution of the following nonlinear equation with critical exponent:

$$P_{\sigma}^{g_0}(v) = c(n, \sigma)K v^{\frac{n+2\sigma}{n-2\sigma}} \text{ on } \mathbb{S}^n, \quad (1.3)$$

where $n \geq 2$, $0 < \sigma < n/2$, $c(n, \sigma) = \Gamma(\frac{n}{2} + \sigma)/\Gamma(\frac{n}{2} - \sigma)$, $\Gamma$ is the Gamma function,
$K$ is a function defined on $S^n$, $P^{g_0}_\sigma$ is an intertwining operator of $2\sigma$-order:

$$P^{g_0}_\sigma = \frac{\Gamma(B + \frac{1}{2} + \sigma)}{\Gamma(B + \frac{1}{2} - \sigma)}, \quad B = \sqrt{-\Delta_{g_0} + \left(\frac{n-1}{2}\right)^2}.$$  

In what follows, $P^{g_0}_\sigma$ is simply written as $P_\sigma$. It can be viewed as the pull back operator of the $\sigma$ power of the Laplacian $(-\Delta)^\sigma$ on $\mathbb{R}^n$ via the stereographic projection:

$$(P_\sigma(v)) \circ F = |J_F|^{\frac{n+2\sigma}{2n}} (-\Delta)^\sigma (|J_F|^{\frac{n-2\sigma}{2n}} (v \circ F)) \quad \text{for } v \in C^{2\sigma}(S^n),$$

where $F$ is the inverse of the stereographic projection and $|J_F|$ is the determinant of the Jacobian of $F$. In addition, the Green function of $P_\sigma$ is the spherical Riesz potential, i.e.,

$$P^{-1}_\sigma f(\xi) = c_{n,\sigma} \int_{S^n} \frac{f(\zeta)}{|\xi - \zeta|^{n-2\sigma}} d\text{vol}_{g_{S^n}}(\zeta) \quad \text{for } f \in L^p(S^n), \quad (1.4)$$

where $c_{n,\sigma} = \frac{\Gamma(\frac{n-2\sigma}{2})}{2\sigma \pi^{\frac{n}{2}} \Gamma(\sigma)}$, $p > 1$, and $|\cdot|$ is the Euclidean distance in $\mathbb{R}^{n+1}$.

Many research have been conducted on the fractional operators $P^{g_0}_\sigma$ and their associated fractional $Q$-curvature, for instance, see [2, 3, 15, 16, 17, 18, 19, 21, 29, 30, 31, 43]. The flatness of the prescribing fractional $Q$-curvature function $K$ plays a crucial role in the study of this problem. We begin with the definition of the $\beta$-flatness condition that characterizes flatness.

**$\beta$-flatness condition:** Let $K \in C^1(S^n)$ ($K \in C^{1,1}(S^n)$ if $0 < \sigma \leq 1/2$) be a positive function and $\beta$ is a positive constant, we say that $K$ satisfies the $\beta$-flatness condition if for every critical point $\xi_0$ of $K$, in some geodesic normal coordinates $\{y_1, \cdots, y_n\}$ centered at $\xi_0$, there exists a small neighborhood $\mathcal{O}$ of $0$ and $a_j(\xi_0) \neq 0$, $\sum_{j=1}^n a_j(\xi_0) \neq 0$, such that

$$K(y) = K(0) + \sum_{j=1}^n a_j(\xi_0)|y_j|^{\beta} + R(y) \quad \text{in } \mathcal{O},$$

where $\sum_{s=0}^{[\beta]} |\nabla^s R(y)||y|^{-\beta+s} \to 0$ as $y \to 0$, here $\nabla^s$ denotes all possible derivatives of order $s$ and $[\beta]$ is the integer part of $\beta$. We call $\beta$ the flatness order.

For $\sigma \in (0,1)$ and $\beta \in (n-2\sigma, n)$, Jin et al. [29, 30] proved the existence of the solutions to (1.3) and derived some compactness properties when $K$ satisfies the $\beta$-flatness condition by using the approach based on approximation of the solutions to (1.3) by a blow up subcritical method. For $\sigma \in (0, n/2)$ and $\beta \in (n-2\sigma, n)$, Jin et al. [31] developed a unified approach to establish blow up profiles, compactness and existence of positive solutions to (1.3) when $K$ satisfies $\beta$-flatness condition by making use of integral representations. Since their conclusions are valid only when the flatness order $n - 2\sigma < \beta < n$, some very interesting functions $K$ are excluded. In fact, note that an important class of functions, which is worth including in the
results of existence and compactness for (1.3), are the Morse functions with only non-degenerate critical points. Such functions satisfy the 2-flatness condition.

Existence results of the solutions to (1.3) were given when \( \beta \in (1, n - 2\sigma] \) by Abdelhedi et al. [3], and when \( \beta \in [n - 2\sigma, n) \) by Chtioui and Abdelhedi [16]. Under a so-called “non-degenerate condition”, Khadijah and Chtioui [37] studied the lack of compactness and provided the existence results for (1.3) when \( \beta = n - 2\sigma = 2 \), \( \sigma \in (0, n/2) \).

However, under the assumption of the flatness order \( \beta = n - 2\sigma \) of prescribing curvature function \( K \), the precise compactness results of the solutions to (1.3) are unknown. When \( \sigma = 1 \) and \( n = 2\sigma + 2 = 4 \), Li [40] obtained the optimal compactness and a degree-counting formula of the solutions to (1.2) when \( K \) is some special class of functions satisfying condition \( 2 = n - 2\sigma \)-flatness condition. Therefore, a quite natural question arises: can we establish the optimal compactness results and provide a degree-counting formula of the solutions to (1.3) when the curvature function \( K \) is specified as a special function satisfying the \( \beta \)-flatness condition with \( \beta = n - 2\sigma = 2 \)? The main target of this article is to give an affirmative answer to this question.

In the present paper, we are interested to the prescribing fractional \( Q \)-curvature problem (1.3), \( n = 2\sigma + 2 \), \( \sigma = 1 + m/2 \), \( m \in \mathbb{N}_+ \). Our aim is to establish the optimal compactness and existence results of the solutions, when the prescribing curvature function \( K \) is some special function satisfying \( 2 = n - 2\sigma \)-flatness condition. In order to obtain an existence result, we will prove a degree-counting formula of the solutions to (1.3). This counting formula, together with the compactness results completely describes where blow up occur. Especially, from our results, we can construct a sequence of solutions to (1.3) that blow up precisely at these points for any finite distinct points on \( S^n \).

First of all, Eq. (1.3) is not always solvable. Indeed, we have the Kazdan-Warner type obstruction: for any conformal Killing vector field \( X \) on \( S^n \), there holds

\[
\int_{S^n} (\nabla_X K) v^{2n/(n-2\sigma)} \, dvol_{g_{S^n}} = 0
\]

for any solution \( v \) of (1.3), see [7, 51].

Before state our results, we introduce some definitions and notations.

For \( K \in C^2(S^n) \), we introduce the following notation:

\[
\mathcal{K} = \{ q \in S^n : \nabla_{g_0} K(q) = 0 \}, \\
\mathcal{K}^+ = \{ q \in S^n : \nabla_{g_0} K(q) = 0, \Delta_{g_0} K(q) > 0 \}, \\
\mathcal{K}^- = \{ q \in S^n : \nabla_{g_0} K(q) = 0, \Delta_{g_0} K(q) < 0 \}, \\
\mathcal{M}_K = \{ v \in C^{2\sigma}(S^n) : v \text{ satisfies (1.3)} \}.
\] (1.5)

For any \( k (k \geq 1) \) distinct points \( q^{(1)}, \ldots, q^{(k)} \in \mathcal{K} \setminus \mathcal{K}^+ \), the \( k \times k \) symmetric
matrix \( M = (M(q^{(1)}, \cdots, q^{(k)})) \) is defined by
\[
M_{ii} = -\frac{\Delta_{g_0} K(q^{(i)})}{K(q^{(i)})^{n/2}},
\]
\[
M_{ij} = -n(n-1)\frac{G_{q^{(i)}}(q^{(j)})}{(K(q^{(i)})K(q^{(j)}))^{1/2}}, \quad i \neq j,
\]
where
\[
G_{q^{(i)}}(q^{(j)}) = \frac{1}{1 - \cos d(q^{(i)}, q^{(j)})}
\]
is the Green’s function of \( P_{\sigma} \) on \( S^n \), and \( d(\cdot, \cdot) \) denotes the geodesic distance. Let \( \mu(M) \) denote the smallest eigenvalue of \( M \), and when \( k = 1 \),
\[
\mu(M) = M = -\frac{\Delta_{g_0} K(q^{(1)})}{K(q^{(1)})^{n/2}}.
\]

In what follows, we define
\[
C^2(S^n)^* := \{ K \in C^2(S^n) : K > 0 \text{ on } S^n, \quad K \text{ has only non-degenerate critical points} \},
\]
and
\[
\mathcal{A} = \{ K \in C^2(S^n)^* : \Delta_{g_0} K \neq 0 \text{ on } \mathcal{K}, \quad \mu(M(q^{(1)}, \cdots, q^{(k)})) \neq 0, \forall q^{(1)}, \cdots, q^{(k)} \in \mathcal{K}^{-}, k \geq 2 \}.
\]
We can observe that \( \mathcal{A} \) is open in \( C^2(S^n) \) and \( \mathcal{A} \) is dense in \( C^2(S^n)^* \) with respect to the \( C^2(S^n) \) norm.

**Remark 1.1.** In this paper, we mainly establish the compactness and existence results for (1.3) when \( K \in \mathcal{A} \). It is worth noting that the sign of the smallest eigenvalue of \( M(q^{(1)}, \cdots, q^{(k)}) \) plays a key role in counting formula of all solutions and compactness results.

We will introduce an integer-valued continuous function \( \text{Index} : \mathcal{A} \rightarrow \mathbb{Z} \), which has an explicit formula for \( K \in \mathcal{A} \) being a Morse function.

**Definition 1.1.** We define \( \text{Index} : \mathcal{A} \rightarrow \mathbb{Z} \) by the following properties:

(i) For any Morse function \( K \in \mathcal{A} \) with \( \mathcal{K}^{-} = \{q^{(1)}, \cdots, q^{(s)}\} \), we define
\[
\text{Index}(K) = -1 + \sum_{k=1}^{s} \sum_{\mu(M(q^{(1)}), \cdots, q^{(k)}) > 0, j_1 < \cdots < j_k < s} (-1)^{k-1 + \sum_{j=1}^{k} i(q^{(j)})},
\]
where \( i(q^{(i)}) \) denotes the Morse index of \( K \) at \( q^{(i)} \).
(ii) Index : \( \mathcal{A} \to \mathbb{Z} \) is continuous with respect to the \( C^2(\mathbb{S}^n) \) norm of \( \mathcal{A} \) and hence is locally constant.

**Remark 1.2.** The existence and uniqueness of the Index mapping follows from Theorem 1.1 and the proof of Theorem 1.2 below.

Our first result is about the compactness of the solutions when \( K \in \mathcal{A} \), which is:

**Theorem 1.1.** Let \( \sigma = 1 + m/2 \), \( m \in \mathbb{N}_+ \) and \( n = 2\sigma + 2 \). Let \( \mathcal{A} \) be as in (1.9) and \( K \in \mathcal{A} \). Then for any \( \alpha \in (0, 1) \), there exists constants \( \delta = \delta(K) > 0 \) and \( C = C(K) > 0 \), such that for any \( K \in C^2(\mathbb{S}^n) \) satisfying \( \|K - K\|_{C^2(\mathbb{S}^n)} < \delta \), and any \( v \in \mathcal{M}_K \), we have

\[
v \in C^{2\sigma,\alpha}(\mathbb{S}^n) : 1/C < v < C, \|v\|_{C^{2\sigma,\alpha}(\mathbb{S}^n)} < C, \quad (1.11)
\]

where \( \mathcal{M}_K \) is as in (1.5).

For any given \( \sigma = \frac{n-2}{2}, 0 < \alpha < 1, R > 0 \), we define

\[
\mathcal{O}_R := \{v \in C^{2\sigma,\alpha}(\mathbb{S}^n) : 1/R < v < R, \|v\|_{C^{2\sigma,\alpha}(\mathbb{S}^n)} < R \}. \quad (1.12)
\]

Our second result is about degree-counting formula and the existence of the solutions to (1.3), which is:

**Theorem 1.2.** Let \( \sigma = 1 + m/2 \), \( m \in \mathbb{N}_+ \) and \( n = 2\sigma + 2 \). Let \( \mathcal{A} \) be as in (1.9), \( K \in \mathcal{A} \) and \( \text{Index}(K) \) be as in Definition 1.1. Then for any \( \alpha \in (0, 1) \), there exists a constant \( R_0 = R_0(K, \alpha) \), such that for all \( R > R_0 \), we have

\[
\deg_{C^{2\sigma,\alpha}}(v - P^{-1}_\sigma(c(n, \sigma)K_v^{\frac{2\sigma}{n-2}}), \mathcal{O}_R, 0) = \text{Index}(K), \quad (1.13)
\]

where \( \deg_{C^{2\sigma,\alpha}} \) denotes the Leray-Schauder degree in \( C^{2\sigma,\alpha}(\mathbb{S}^n) \).

Furthermore, if \( \text{Index}(K) \neq 0 \), then (1.3) has at least one solution.

**Remark 1.3.** It follows from Theorem 2.1 that when \( K \in \mathcal{A} \), the solutions to (1.3) belong to \( \mathcal{O}_R \) for some \( R > 0 \). We call the left-hand side of (1.13) the total degree of the solutions to the conformally invariant equation. From Theorem 1.2, the total degree is \( \text{Index}(K) \).

For any finite subset \( \mathcal{R} \subset \mathbb{S}^n \), we use \( \sharp \mathcal{R} \) to denote the number of elements in the set \( \mathcal{R} \). Let us now state a corollary of Theorem 1.2, which is:

**Corollary 1.1.** Let \( \sigma = 1 + m/2 \), \( m \in \mathbb{N}_+ \) and \( n = 2\sigma + 2 \). Let \( \mathcal{A} \) be as in (1.9) and \( K \in \mathcal{A} \) be a Morse function satisfying \( \sharp \mathcal{K}^- \leq 1 \) or for any distinct \( P, Q \in \mathcal{K}^- \),

\[
\Delta_{g_0}K(P)\Delta_{g_0}K(Q) < \frac{n^2(n-1)^2}{4}K(P)K(Q). \quad (1.14)
\]
Then for any $\alpha \in (0, 1)$, there exists a constant $C = C(K, \alpha) > 0$, such that for all solutions $v$ to (1.3), we have $v \in \mathcal{O}_C$, and for all $R \geq C$,

$$\deg_{C^{2\alpha, \alpha}}(v - P_\sigma^{-1}(c(n, \sigma)K v^{\frac{n+2\sigma}{n-2\sigma}}), \mathcal{O}_R, 0) = -1 + \sum_{\nabla_{q_0} K(q_0) = 0, \Delta_{q_0} K(q_0) < 0} (-1)^{i(q_0)},$$

where $\mathcal{O}_C$ is as in (1.12) and $i(q_0)$ denotes the Morse index of $K$ at $q_0$.

Furthermore, if

$$\sum_{\nabla_{q_0} K(q_0) = 0, \Delta_{q_0} K(q_0) < 0} (-1)^{i(q_0)} \neq 1,$$

then (1.3) has at least one solution.

Our third result is about the blow up behavior of the solutions when the prescribing fractional $Q$-curvature function $K \in C^2(\mathbb{S}^n)^* \setminus \mathcal{A} = \partial \mathcal{A}$, which is:

**Theorem 1.3.** Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$ and $n = 2\sigma + 2$. Let $\mathcal{A}$ be as in (1.9) and $C^2(\mathbb{S}^n)^*$ be as in (1.8). Then for any $K \in C^2(\mathbb{S}^n)^* \setminus \mathcal{A}$, there exists $K_i \to K$ in $C^2(\mathbb{S}^n)$ and $v_i \in \mathcal{M}_{K_i}$, such that

$$\lim_{i \to \infty} (\max_{\mathbb{S}^n} v_i) = \infty, \quad \lim_{i \to \infty} (\min_{\mathbb{S}^n} v_i) = 0, \quad (1.15)$$

where $\mathcal{M}_{K_i}$ is as in (1.5).

From Theorems 1.1, 1.2, and 1.3, we can know that the total degree of solutions to (1.3) strongly depend on the sign of the smallest eigenvalue of $M(q^{(1)}, \ldots, q^{(k)})$. In fact, the points $q^{(1)}, \ldots, q^{(k)}$ for which $\mu(M(q^{(1)}, \ldots, q^{(k)}))$ is positive characterize the so-called asymptotic in the theory of critical points at infinity developed by Bahri [4, 6]. For instance, considering a continuous family of functions $K_t (0 \leq t \leq 1)$, the total degree changes when the smallest eigenvalue of $M(K_t; (q^{(1)}, \ldots, q^{(k)}))$ crosses zero while it remains unchanged when other eigenvalues cross zero.

It follows from Theorem 1.3 that when $K \in C^2(\mathbb{S}^n)^* \setminus \mathcal{A}$, the solutions to (1.3) may blow up. A natural question is where the blow up occur? The following results present the accurate location of the blow up.

For any $K \in C^2(\mathbb{S}^n)$, we first define

$$\mathcal{K}(K) = \{(q^{(1)}, \ldots, q^{(k)}): k \geq 1, q^{(j)} \in \mathcal{K} \setminus \mathcal{K}^+, \forall j: 1 \leq j \leq k, q^{(j)} \neq q^{(\ell)}, \forall j \neq \ell, \mu(M(q^{(1)}, \ldots, q^{(k)})) = 0\}.$$  (1.16)

Combined with Theorem 2.1, we give the fourth result in this paper, which is about the location of blowing up when $K \in C^2(\mathbb{S}^n)^* \setminus \mathcal{A}$:

**Theorem 1.4.** Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$ and $n = 2\sigma + 2$. Let $\mathcal{A}$ be as in (1.9) and $C^2(\mathbb{S}^n)^*$ be as in (1.8). For a given function $K \in C^2(\mathbb{S}^n)^* \setminus \mathcal{A}$, we have the following results:
(i) For any $K_i \to K$ in $C^2(\mathbb{S}^n)$, and $v_i \in \mathcal{M}_{K_i}$ with $\max_{\mathbb{S}^n} v_i \to \infty$, then for some $(q^{(1)}, \ldots, q^{(k)}) \in \mathcal{H}(K)$, \{v_i\} (after passing to a subsequence) blows up at precisely the $k$ points.

(ii) For any $(q^{(1)}, \ldots, q^{(k)}) \in \mathcal{H}(K)$, there exists $K_i \to K$ in $C^2(\mathbb{S}^n)$, $v_i \in \mathcal{M}_{K_i}$, such that \{v_i\} blows up at precisely the $k$ points.

**Corollary 1.2.** For any $k \in \mathbb{N}_+$ distinct points $q^{(1)}, \ldots, q^{(k)} \in \mathbb{S}^n$, there exists a sequence of Morse functions $\{K_i\} \subset \mathcal{A}$, such that for some $v_i \in \mathcal{M}_{K_i}$, \{v_i\} blows up at precisely the $k$ points.

In order to obtain the compactness results, we need to further characterize the behavior of the blow up point of the solutions to (1.3) (see Theorem 2.1 below). More precisely, we will use the Pohozaev type identity (see Proposition A.3 below) to judge the sign of the Laplacian of the prescribing curvature function at these isolated simple blow up point (see Definition 2.3 below). Due to the limit of the form of the Pohozaev type identity, the proof method is only effective for the case $n - 2\sigma = 2$. In addition, when proving the existence results, we transform the conclusion to be proved into solving the Brouwer degree of the operator on finite dimensional manifolds through the homotopy invariance of the Leray-Schauder degree. In the process of solving, we need to get a strictly convex function according to the form of the operator, and the condition “$n - 2\sigma = 2$” just ensures the existence of the form of strictly convex function. For $n = 2\sigma + 2$, $0 < \sigma < 1$, we obtain the corresponding compactness and existence results with $n = 3, \sigma = 1/2$, see [42].

The paper is organized as follows:

In Section 2, our main task is to prove Theorem 1.1. Before that, we should further characterizes the behavior of blow up points for solutions to (1.3) (see Theorem 2.1 below), we mainly consider the subcritical equation with $\tau > 0$ small:

$$P_\tau v_i = c(n, \sigma) K v_i^{n-1-\tau}, \quad v_i > 0 \quad \text{on} \quad \mathbb{S}^n. \quad (1.17)$$

In proving Theorem 2.1, we first use the Green’s representation (1.4) to transform (1.17) into

$$v_i(\xi) = \frac{\Gamma(n+2\sigma/2)}{2^{2\sigma} \pi^{n/2} \Gamma(\sigma)} \int_{\mathbb{S}^n} \frac{K_i(\eta)v_i(\eta)^{n-1-\tau}}{|\xi - \eta|^2} \, d\eta \quad \text{on} \quad \mathbb{S}^n,$$

and then use some results of blow up analysis given in Appendix A to complete the proof. By using Theorem 2.1, integral representation, Harnack inequality and Schauder type estimates, we have completed the proof of Theorem 1.1.

Section 3 is devoted to proving the Theorems 1.2, 1.3, and 1.4. Firstly, recall the classification of solutions for integral equation ([14]) and optimal representation in small tubular neighborhood ([6]), we give the definition of $\Sigma_\tau = \Sigma_\tau(P_1, \ldots, P_k)$ for $P_1, \ldots, P_k \in \mathcal{A}$ with $\mu(M(P_1, \ldots, P_k)) > 0$. Then by using Theorem 2.1 and some results in [31], we obtain that for $\tau > 0$ very small, the solutions to (1.17) either stay
bounded or stay in one of the $\Sigma_{\tau}$ (see Proposition 3.1 below). Furthermore, we obtain the $H^\sigma$ topological degree of the solutions to (1.17) on $\Sigma_{\tau}$ (see Theorem 3.1 below). It follows from the above results that for all $0 < \tau < 2$, the $H^\sigma$ total degree of the solutions to (1.17) is equal to $-1$ (see Proposition A.7 below). Then we can conclude that $H^\sigma$ topological degree of those solutions to (1.17) which remain bounded as $\tau$ tends to zero is equal to Index($K$). Some well-known results in degree theory imply that the $H^\sigma$ degree contribution above is equal to the $C^{2,\alpha}$ topological degree of those bounded solutions to (1.17). Thus, we proved Theorem 1.2. Furthermore, we complete the proof of Theorem 1.3 by using the degree-counting formula and perturbing the function $K$ near its critical point. In the end, using Theorem 2.1 and the idea of the proof of Theorem 1.3, we prove Theorem 1.4.

In Appendix A, since the fact that by the Green’s representation (1.4) and the stereographic projection, we can write Eq. (1.3) as the form

$$u(x) = \int_{\mathbb{R}^n} \frac{K(y)u(y)^p_i}{|x-y|^{n-2\sigma}} \, dy \quad \text{on } \mathbb{R}^n, \quad (1.18)$$

we first review the Hölder estimates, Schauder type estimates, blow up profile for nonlinear integral equations (1.18) established by Jin-Li-Xiong [31].

In Appendix B, we provide some useful technical results and elementary estimates.

2 The characterization of blow up behavior and compactness result

In this section, our main task is to prove Theorem 1.1. Before that, we need further characterizes the blow up points for solutions to (1.3) by using integral representation and some estimates in the Appendix A (see Theorem 2.1 below), which plays a key role in proving main result concerning compactness and existence. We first review some definitions of blow up points.

Let $\Omega$ be a domain in $\mathbb{R}^n$ and $K_i$ are nonnegative bounded functions in $\mathbb{R}^n$. Let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of nonnegative constants satisfying $\lim_{i \to \infty} \tau_i = 0$, and set

$$p_i = \frac{n + 2\sigma}{n - 2\sigma} - \tau_i.$$

Suppose that $0 \leq u_i \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ satisfies the nonlinear integral equation

$$u_i(x) = \int_{\mathbb{R}^n} \frac{K_i(y)u_i(y)^{p_i}}{|x-y|^{n-2\sigma}} \, dy \quad \text{in } \Omega. \quad (2.1)$$

We assume that $K_i \in C^1(\Omega)$ ($K_i \in C^{1,1}(\mathbb{S}^n)$ if $\sigma \leq 1/2$) and, for some positive constants $A_1$ and $A_2$,

$$1/A_1 \leq K_i, \quad \text{and} \quad \|K_i\|_{C^1(\Omega)} \leq A_2, \quad (\|K_i\|_{C^{1,1}(\Omega)} \leq A_2 \text{ if } \sigma \leq \frac{1}{2}). \quad (2.2)$$
Definition 2.1. Suppose that \( \{K_i\} \) satisfies (2.2) and \( \{u_i\} \) satisfies (2.1). A point \( \overline{y} \in \Omega \) is called a blow up point of \( \{u_i\} \) if there exists a sequence \( y_i \) tending to \( \overline{y} \) such that \( u_i(y_i) \to \infty \).

Definition 2.2. A blow up point \( \overline{y} \in \Omega \) is called an isolated blow up point of \( \{u_i\} \) if there exists \( 0 < r < \text{dist}(\overline{y}, \Omega), \) \( C > 0, \) and a sequence \( y_i \) tending to \( \overline{y} \), such that \( y_i \) is a local maximum point of \( u_i, u_i(y_i) \to \infty \) and

\[
u_i(y) \leq C|y - y_i|^{-2\sigma/(p_i-1)} \quad \text{for all} \quad y \in B_{\overline{y}}(y_i).
\]

Let \( y_i \to \overline{y} \) be an isolated blow up point of \( \{u_i\} \), and define, for \( r > 0, \)

\[
\overline{u}_i(r) := \frac{1}{|\partial B_r(y_i)|} \int_{\partial B_r(y_i)} u_i \quad \text{and} \quad \overline{w}_i(r) := r^{2\sigma/(p_i-1)} \overline{u}_i(r).
\]

Definition 2.3. A point \( y_i \to \overline{y} \in \Omega \) is called an isolated simple blow up point if \( y_i \to \overline{y} \) is an isolated blow up point such that for some \( \rho > 0 \) (independent of \( i \)), \( \overline{w}_i \) has precisely one critical point in \((0, \rho)\) for large \( i \).

2.1 Characterization of blow up behavior

Recall the definitions of the matrix \( M \) given in (1.6) and its smallest eigenvalue \( \mu(M) \). The result about characterization of blow up behavior of the solutions to (1.3) is:

**Theorem 2.1.** Let \( \sigma = 1 + m/2, m \in \mathbb{N}_+ \) and \( n = 2\sigma + 2 \). Let \( K \in C^2(\mathbb{S}^n) \) be a positive function and \( \mathcal{K}, \mathcal{K}^-, \mathcal{K}^+ \) be as in (1.5). Let \( p_i \leq \frac{n+2\sigma}{n-2\sigma} = \frac{n+2\sigma}{2} = n - 1, \)

\( p_i \to n - 1, \) \( K_i \in C^2(\mathbb{S}^n) \) satisfy \( K_i \to K \) in \( C^2(\mathbb{S}^n) \), and \( v_i \in C^{2\sigma}(\mathbb{S}^n) \) satisfy

\[
P_\sigma v_i = c(n, \sigma) K_i v_i^{p_i}
\]

and

\[
\lim_{i \to \infty} \max_{\mathbb{S}^n} v_i = \infty.
\]

Then there exists a constant \( \delta^* > 0 \) depending only on \( \min_\mathbb{S}^n K, \|K\|_{C^2(\mathbb{S}^n)} \), and the modulus of the continuity of \( \nabla_\mathcal{g}_0 K \) if \( \sigma > 1/2 \) such that after passing to a subsequence, we have:

(i) \( \{v_i\} \) (still denote the subsequence by \( \{v_i\} \)) has only isolated simple blow up points \( q^{(1)}, \ldots, q^{(k)} \in \mathcal{K} \setminus \mathcal{K}^+ \) \((k \geq 1)\) with \( q^{(j)} - q^{(\ell)} \geq \delta^*, \forall j \neq \ell, \) and \( \mu(M(q^{(1)}, \ldots, q^{(k)})) \geq 0. \) Furthermore, \( q^{(1)}, \ldots, q^{(k)} \in \mathcal{K}^- \) if \( k \geq 2. \)

(ii) Let \( q^{(1)}, \ldots, q^{(k)} \) be as in (i), and \( q^{(j)}_i \) be the local maximum of \( v_i \) with \( q^{(j)}_i \to q^{(j)} \), we have

\[
\lambda_j := K(q^{(j)})^{-1/2\sigma} \lim_{i \to \infty} v_i(q^{(1)}_i)(v_i(q^{(j)}_i))^{-1} \in (0, \infty),
\]

\[
\mu^{(j)} := \lim_{i \to \infty} \tau_i v_i(q^{(j)}_i)^2 \in [0, \infty).
\]
(iii) Let \( \lambda_j, \mu^{(j)}, j = 1, \ldots, k \) be as in (ii), then when \( k = 1 \),

\[
\mu^{(1)} = -\frac{2 \Delta_{g_0} K(q^{(1)})}{\sigma K(q^{(1)})^{n/2\sigma}},
\]

(2.7)

when \( k \geq 2 \),

\[
\sum_{\ell=1}^k M_{\ell j}(q^{(1)}\ldots q^{(k)})\lambda_{\ell} = \frac{\sigma}{2} \lambda_j \mu^{(j)}, \quad \forall j : 1 \leq j \leq k.
\]

(2.8)

(iv) \( \mu^{(j)} \in (0, \infty), \forall j = 1, \ldots, k \), if and only if \( \mu(M(q^{(1)}\ldots q^{(k)})) > 0 \).

We first give the following proposition:

**Proposition 2.1.** Let \( K \in C^2(S^n) \), \( n \geq 2 \), be a positive function and \( \mathcal{K}, \mathcal{K}^-, \mathcal{K}^+ \) be as in (1.5). Let \( p_i \) satisfy \( p_i \leq \frac{n + 2\sigma}{n - 2\sigma} p_i \rightarrow \frac{n + 2\sigma}{n - 2\sigma} \), \( K_i \in C^2(S^n) \) satisfy \( K_i \rightarrow K \) in \( C^2(S^n) \), and \( v_i \) satisfy

\[
P_{\sigma} v_i = c(n, \sigma) K_i^p_i.
\]

Then exists a constant \( \delta^* > 0 \) depending only on \( \min_{S^n} K, \|K\|_{C^2(S^n)} \), and the modulus of the continuity of \( \nabla_{g_0} K \) if \( \sigma > 1/2 \) such that, after passing to a subsequence, either \( \{v_i\} \) stays bounded in \( L^\infty(S^n) \) or \( \{v_i\} \) has only isolated simple blow up points and the distance between any two blow up points is bounded blow by \( \delta^* \).

**Proof.** The proof follows from the same arguments used to prove Theorem 3.3 in [31], so we omit it. \( \square \)

**Proof of Theorem 2.1.** From Proposition 2.1 and \( \lim_{i \rightarrow \infty} \max_{S^n} v_i = \infty \), there exists a constant \( \delta^* > 0 \) depending only on \( \min_{S^n} K, \|K\|_{C^2(S^n)} \), and the modulus of the continuity of \( \nabla_{g_0} K \) if \( \sigma > 1/2 \) such that \( \{v_i\} \) has only isolated simple blow up points \( q^{(1)}\ldots q^{(k)} \in \mathcal{K} \) \( (k \geq 1) \) with \( |q^{(j)} - q^{(\ell)}| \geq \delta^* (j \neq \ell) \).

By (1.4), (2.4) is equivalent to

\[
v_i(\xi) = \frac{\Gamma(\frac{n + 2\sigma}{2})}{2^n \pi^{n/2} \Gamma(\sigma)} \int_{\mathbb{S}^n} K_i(\eta)v_i(\eta)^{p_i} \frac{d\eta}{|\xi - \eta|^2} \quad \text{on} \quad S^n.
\]

(2.9)

Let \( F \) be the stereographic projection with with \( q^{(j)} \) being the south pole:

\[
F : \mathbb{R}^n \rightarrow S^n \setminus \{-q^{(j)}\},
\]

\[
x \mapsto \left( \frac{2x}{1 + |x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1} \right),
\]

Let \( \tau_i = n - 1 - p_i \), via the stereographic projection, the equation (2.9) is translated to

\[
u_i(x) = \frac{\Gamma(\frac{n + 2\sigma}{2})}{2^n \pi^{n/2} \Gamma(\sigma)} \int_{\mathbb{R}^n} \tilde{K}_i(y) H(y)^{\tau_i} u_i(y)^{p_i} \frac{dy}{|x - y|^2} \quad \text{on} \quad \mathbb{R}^n,
\]

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where
\[ H(x) = \frac{2}{1 + |x|^2}, \quad u_i(x) = H(x)v_i(F(x)), \quad \tilde{K}_i(x) = K_i(F(x)). \tag{2.10} \]

Let \( x_i^{(j)} \) be the local maximum of \( u_i \) and \( x_i^{(j)} \to 0 \). It follows from Propositions A.7 and A.8 that
\[ u_i(x_i^{(j)})u_i(x) \to h^{(j)}(x) := aK(q^{(j)})^{-1/\sigma}|x|^{-2} + b^{(j)}(x) \]
\[ \text{in } C^2_{\text{loc}}(\mathbb{R}^n \setminus \{\cup_{\ell=1}^k x_i^{(\ell)}\}), \tag{2.11} \]
where
\[ a = 4c_{n,\sigma}c(n,\sigma) \int_{\mathbb{R}^n} \left( \frac{1}{1 + |y|^2} \right)^{n-1} \, dy = 2c_{n,\sigma}c(n,\sigma)|\mathbb{S}^{n-1}|B(\sigma,n/2), \tag{2.12} \]
\( B(\sigma,n/2) \) is the Beta function, and \( c_{n,\sigma} \) is as in (1.4). From the maximum principle, \( b^{(j)}(x) \) satisfies
\[ b^{(j)}(x) \equiv 0 \quad \text{if } k = 1, \quad b^{(j)}(x) > 0 \quad \text{if } k \geq 2. \tag{2.13} \]

By (2.37) and \( y_i^{(j)} \to 0 \) as \( i \to \infty \), we have
\[ \lim_{i \to \infty} v_i(q_i^{(j)})v_i(q) = \frac{1}{4} \lim_{i \to \infty} (1 + |x|^2)u_i(x_i^{(j)})u_i(x), \]
combining with (2.11), it is easy to see that for \( q \neq q^{(j)} \) and close to \( q^{(j)} \),
\[ \lim_{i \to \infty} v_i(q_i^{(j)})v_i(q) = \frac{aG_{q^{(j)}}(q)}{2K(q^{(j)})^{1/\sigma}} + \tilde{b}^{(j)}(q) \quad \text{in } C^2_{\text{loc}}(\mathbb{S}^n \setminus \{\cup_{\ell=1}^k q^{(\ell)}\}), \tag{2.14} \]
where \( a \) is as in (2.12), and \( \tilde{b}^{(j)}(q) \) is some regular function on \( \mathbb{S}^n \setminus \cup_{\ell \neq j} q^{(\ell)} \) satisfying \( P_{q} \tilde{b}^{(j)} = 0 \), and \( G_{q^{(j)}}(q) \) is the Green function defined as in (1.7).

When \( k \geq 2 \), taking into account the contribution of all the poles, we deduce
\[ \lim_{i \to \infty} v_i(q_i^{(j)})v_i(q) = \frac{aG_{q^{(j)}}(q)}{2K(q^{(j)})^{1/\sigma}} + \frac{a}{2} \sum_{\ell \neq j} \lim_{i \to \infty} v_i(q_i^{(j)}) \frac{G_{q^{(j)}}(q)}{v_i(q_i^{(j)}) K(q^{(j)})^{1/\sigma}} \]
\[ \quad \text{in } C^2_{\text{loc}}(\mathbb{S}^n \setminus \{\cup_{\ell=1}^k q^{(\ell)}\}). \tag{2.15} \]

In fact, subtracting all the poles from the limit function, we obtain a regular function \( \tilde{b}_0 : \mathbb{S}^n \to \mathbb{R} \) such that \( P_{q} \tilde{b}_0 = 0 \) on \( \mathbb{S}^n \), so it must be \( \tilde{b}_0 \equiv 0 \). Using (2.15), we have, for \( |y| > 0 \) small,
\[ h^{(j)}(y) = \frac{a}{K(q^{(j)})^{1/\sigma}|y|^2} + 2a \sum_{\ell \neq j} \lim_{i \to \infty} v_i(q_i^{(j)}) \frac{G_{q^{(j)}}(q^{(j)})}{v_i(q_i^{(j)}) K(q^{(j)})^{1/\sigma}} + O(|y|), \tag{2.16} \]
where \( a \) is as in (2.12). The conclusion obtained from the above is easy to see that (2.5) is true.

Before stating the result to be proved, we give the following estimates (2.17) and (2.18). Using Proposition A.10, we obtain

\[
|\nabla K_i(\gamma^{(j)}_i)| = O(u_i(\gamma^{(j)}_i)^{-1}), \quad \tau_i = O(u_i(\gamma^{(j)}_i)^{-2}).
\]  

(2.17)

It is obvious that (2.6) can be proved by (2.17). We have proved Part (ii).

Let \( y = (y_1, \cdots, y_n) \in \mathbb{R}^n \). It follows from Propositions A.5, A.7, and A.9, that for sufficiently small \( \delta > 0 \),

\[
\sum_{j=1}^n \left| \int_{B_\delta} y_{(j)} u_i(y + x^{(j)}_i) \right| = o(u_i(x^{(j)}_i)^{-1}),
\]

\[
\sum_{j \neq \ell} \left| \int_{B_\delta} y_{(j)} y_{(\ell)} u_i(y + x^{(j)}_i) \right| = o(u_i(x^{(j)}_i)^{-2}),
\]

\[
\int_{\partial B_\delta} u_i(y + x^{(j)}_i) \right| = O(u_i(x^{(j)}_i)^{-p_i - 1}),
\]

\[
\lim_{t \to 0} u_i(x^{(j)}_i) \int_{B_t} |y|^{2} u_i(y + x^{(j)}_i) = \frac{n2^{1+n}S^{n-1}}{n + 2\sigma} \frac{B(\sigma, n/2)}{K(q^{(j)})^{1+2/\sigma}}.
\]  

(2.18)

In fact, the first three formulas in (2.18) can be easily obtained from Proposition A.9. For the last formula in (2.18), let \( R_i \) be as in Proposition A.5 and

\[
m_{ij} := u_i(x^{(j)}_i), \quad r_{ij} := R_i m_{ij}^{-p_i - 1/2\sigma}, \quad k_{ij} := 2^{-2} \tilde{K}_i(x^{(j)}_i)^{1/\sigma}.
\]  

(2.19)

Using Proposition A.5 again, we have

\[
m_{ij}^2 \int_{|y| \leq r_{ij}} |y|^{2} u_i(y + x^{(j)}_i) \right| dy = \frac{n2^{1+n}S^{n-1}}{n + 2\sigma} \frac{B(\sigma, n/2)}{K(q^{(j)})^{1+2/\sigma}}.
\]

We have completed the proof of (2.18).

By \( n - 2\sigma = 2 \) and \( \tau_i = (n + 2\sigma)/(n - 2\sigma) - p_i \), it is easy to see that

\[
\frac{1}{p_i + 1} = \frac{1}{2\sigma + 2 - \tau_i} = \frac{1}{n} \left( 1 + \frac{\tau_i}{n} + O(\tau_i^2) \right).
\]  

(2.20)
For sufficiently small $\delta > 0$, $u_i$ satisfy

$$u_i(x) = \frac{c_{n,\sigma}c(n, \sigma)}{2^{2\sigma}} \int_{B_\delta(x_i^{(j)})} \frac{\tilde{K}_i(y)H(y)^{n}u_i(y)^{p_i}}{|x-y|^2} \, dy + h_\delta(x),$$

where

$$h_\delta(x) = \frac{c_{n,\sigma}c(n, \sigma)}{2^{2\sigma}} \int_{\mathbb{R}^n \setminus B_\delta(x_i^{(j)})} \frac{\tilde{K}_i(y)H(y)^{n}u_i(y)^{p_i}}{|x-y|^2} \, dy. \quad (2.21)$$

By Proposition A.3, we have

$$\left(\frac{n-2\sigma}{2} - \frac{n}{p_i+1}\right) \int_{B_\delta(x_i^{(j)})} \tilde{K}_i(x)H(x)^{n}u_i(x)^{p_i+1} \, dx$$

$$= \frac{n-2\sigma}{2} \int_{B_\delta(x_i^{(j)})} \tilde{K}_i(x)H(x)^{n}u_i(x)^{p_i} h_\delta(x) \, dx \quad (2.22)$$

$$+ \int_{B_\delta(x_i^{(j)})} (x - x_i^{(j)}) \cdot \nabla h_\delta(x) \tilde{K}_i(x)H(x)^{n}u_i(x)^{p_i} \, dx$$

$$- \frac{\delta}{p_i+1} \int_{\partial B_\delta(x_i^{(j)})} \tilde{K}_i(x)H(x)^{n}u_i(x)^{p_i+1} \, ds.$$

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, by (2.20) we have

$$- \frac{1}{p_i+1} \int_{B_\delta} x \cdot \nabla (\tilde{K}_i(x + x_i^{(j)})H(x + x_i^{(j)})^{n}u_i(x + x_i^{(j)})^{p_i+1} \, dx$$

$$= - \frac{1}{n} \sum_{\ell=1}^{n} \int_{B_\delta} x(\ell) \frac{\partial \tilde{K}_i}{\partial x(\ell)} (x + x_i^{(j)})u_i(x + x_i^{(j)})^{p_i+1} \, dx + o(m_{ij}^{-2})$$

$$= - \frac{1}{n} \int_{B_\delta} x \cdot \nabla \tilde{K}(x_i^{(j)})u_i(x + x_i^{(j)})^{p_i+1} \, dx$$

$$- \frac{1}{n} \sum_{\ell,m} \int_{B_\delta} x(\ell)x(m) \frac{\partial^2 \tilde{K}}{\partial x(\ell)\partial x(m)} (x_i^{(j)})u_i(x + x_i^{(j)})^{p_i+1} \, dx + o(m_{ij}^{-2})$$

$$= - \frac{1}{n^2} \Delta \tilde{K}(0) \int_{B_\delta} |x|^2 u_i(x + x_i^{(j)})^{p_i+1} \, dx + o(m_{ij}^{-2})$$

$$= - \frac{4}{n^2} \Delta_{g_0} K(q^{(j)}) \int_{B_\delta} |x|^2 u_i(x + x_i^{(j)})^{p_i+1} \, dx + o(m_{ij}^{-2}). \quad (2.23)$$

Then, by (2.18) and (2.23),

$$\lim_{i \to \infty} - \frac{m_{ij}^2}{p_i+1} \int_{B_\delta} x \cdot \nabla \tilde{K}_i(x + x_i^{(j)})u_i(x + x_i^{(j)})^{p_i+1} \, dx$$

$$= - \frac{2^{3+n}|S^{n-1}|B(\sigma, n/2)}{n(n+2\sigma)} \frac{\Delta_{g_0} K(q^{(j)})}{K(q^{(j)})^{1+2/\sigma}}. \quad (2.24)$$
Let $r_i$ be as in (2.19) and by (2.20), we have
\[
\left(\frac{n - 2\sigma}{2} - \frac{n}{p_i + 1}\right) \int_{B_{\delta}} \tilde{K}_i(x + x_i^{(j)})H(x + x_i^{(j)}) \tau_i u_i(x + x_i^{(j)})^{p_i + 1} \, dx
\]
\[= -\frac{\tau_i}{n} \int_{B_{\delta}} \tilde{K}_i(x + x_i^{(j)}) u_i(x + x_i^{(j)})^{p_i + 1} \, dx + o(m_{ij}^{-2})
\]
\[= -\frac{\tau_i}{n} \int_{B_{\delta}} \tilde{K}_i(x_i^{(j)}) u_i(x + x_i^{(j)})^{p_i + 1} \, dx
\]
\[+ O\left(\left| \int_{B_{\delta}} x \cdot \nabla \tilde{K}_i(x_i^{(j)}) u_i(x + x_i^{(j)})^{p_i + 1} \, dx \right| \right)
\]
\[+ O\left(\int_{B_{\delta}} |x|^2 u_i(x + x_i^{(j)})^{p_i + 1} \, dx \right) + o(m_{ij}^{-2})
\]
\[= -\frac{\tau_i}{n} \int_{|x| < r_i} u_i(x + x_i^{(j)})^{p_i + 1} \, dx + o(m_{ij}^{-2})
\]
\[= -\frac{\tau_i^2 2n}{n} K(q^{(j)})^{-1/\sigma} \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^n} \, dx + o(m_{ij}^{-2})
\]
\[= -\frac{\tau_i^2 2n|S^{n-1}|}{n} \left(\frac{\sigma}{n + 2\sigma}\right) B(n/2, \sigma) K(q^{(j)})^{-1/\sigma} + o(m_{ij}^{-2}).
\]  
(2.25)

It follows from (2.6) and (2.25) that
\[
\lim_{i \to \infty} m_{ij}^2 \left(1 - \frac{n}{p_i + 1}\right) \int_{B_{\delta}} \tilde{K}_i(x + x_i^{(j)})H(x + x_i^{(j)}) \tau_i u_i(x + x_i^{(j)})^{p_i + 1} \, dx
\]
\[= -\frac{2n^2|S^{n-1}|}{n(n + 2\sigma)} \frac{\mu^{(j)}}{B(n/2, \sigma) K(q^{(j)})^{1/\sigma}}.
\]  
(2.26)

In view of (2.18), we obtain
\[
\lim_{i \to \infty} m_{ij}^2 \delta \frac{\tau_i}{p_i + 1} \int_{B_{\delta}} \tilde{K}_i(x + x_i^{(j)})H(x + x_i^{(j)}) \tau_i u_i(x + x_i^{(j)})^{p_i + 1} \, dx
\]
\[= \lim_{i \to \infty} m_{ij}^2 \delta \frac{\tau_i}{n} \int_{B_{\delta}} \tilde{K}_i(x + x_i^{(j)}) u_i(x + x_i^{(j)})^{p_i + 1} \, dx
\]
\[= 0.
\]  
(2.27)

Using (2.21) and Proposition A.5, we have
\[
m_{ij}^2 \left\{\frac{n - 2\sigma}{2}\right\} \int_{B_{\delta}(x_i^{(j)})} \tilde{K}_i(x) u_i(x)^{p_i} H(x)^{\tau_i} \, dx
\]
\[= m_{ij}^2 \int_{B_{\delta}(x_i^{(j)})} \left(\tilde{K}_i(x_i^{(j)}) + (x - x_i^{(j)}) \cdot \nabla \tilde{K}_i(x_i^{(j)}) + O(|x - x_i^{(j)}|^2)\right) u_i(x)^{p_i} h_\delta(x) \, dx
\]
\[= m_{ij}^2 \left\{\frac{n(p_i - 1)}{2\sigma} + \frac{p_i}{2\sigma}\right\} \tilde{K}_i(x_i^{(j)}) \int_{|y| < R_i} (m_{ij}^2 u_i(m_{ij}^2 y + x_i^{(j)}))^{p_i} h_\delta(m_{ij}^2 y + x_i^{(j)}) \, dy + o(1)
\]
it follows that

$$\lim_{i \to \infty} m_{ij}^{-2} \frac{n - 2\sigma}{2} \int \limits_{B_\delta(x_i^{(j)})} \tilde{K}_i(x) u_i(x) p_\delta H(x) \gamma_{ij} h_\delta(x) \, dx$$

$$= 2^{n-1} |S^{n-1}| B(\sigma, n/2) b^{(j)}(0) + o(1),$$

(2.28)

When $|x - x_i^{(j)}| < \delta$, a direct calculation gives

$$|\nabla h_\delta(x)| \leq \begin{cases} C |\delta^{2\sigma - 1} - (\delta - |x - x_i^{(j)}|)2\sigma - 1| m_{ij}^{-1} & \text{if } \sigma \neq 1/2, \\ C |\log \delta - \log(\delta - |x - x_i^{(j)}|)| m_{ij}^{-1} & \text{if } \sigma = 1/2. \end{cases}$$

(2.29)

The detailed proof of (2.29) can refer to [31]. Using Proposition A.5 and (2.29), we can obtain

\[ \int_{|x - x_i^{(j)}| < \delta} (x - x_i^{(j)}) \nabla h_\delta(x) \tilde{K}_i(x) H(x)^{\gamma_{ij}} u_i(x) p_\delta \, dx \leq C m_{ij}^{-1} \int_{|x - x_i^{(j)}| < \delta} |x - x_i^{(j)}| u_i(x) p_\delta \, dx \]

\[ \leq C m_{ij}^{-1 - \frac{(n+1)(p_\delta - 1)}{2\sigma}} \int_{|y| < R_i} |y| (m_{ij}^{-1} u_i(m_{ij}^{-\frac{p_\delta - 1}{2\sigma}} y + x_i^{(j)})) p_\delta \, dy \]

\[ = o(m_{ij}^{-2}). \quad (2.30) \]

By (2.22), (2.24), (2.26), (2.27), (2.28), and (2.30), we have

$$\frac{8\sigma \mu^{(j)}_i}{n(n + 2\sigma)} \frac{1}{K(q^{(j)})^{1/\sigma}} + \frac{16}{n(n + 2\sigma)} \Delta_{\sigma_i} K(q^{(j)}) = - \frac{1}{K(q^{(j)})^{1/\sigma}} b^{(j)}(0).$$

(2.31)

Consequently, $q^{(j)} \in \mathcal{K} \setminus \mathcal{K}^+, 1 \leq j \leq k$, and when $k \geq 2$, $q^{(j)} \in \mathcal{K}^-, 1 \leq j \leq k$. It is easy to see that (2.7) follows from (2.13) and (2.31) when $k = 1$.

From (2.16), (2.12), (2.37), and (2.5), we can obtain

$$b^{(j)}(0) = 2^2 \Gamma(n/2) |S^{n-1}| \frac{\lambda_\ell}{\pi^{n/2}} \sum_{\ell \neq j} \frac{G_{q^{(j)}}(q^{(j)})}{\lambda_j (K(q^{(j)}) K(q^{(\ell)}))^{1/2\sigma}}$$

$$= 8 \sum_{\ell \neq j} \frac{\lambda_\ell}{\lambda_j (K(q^{(j)}) K(q^{(\ell)}))^{1/2\sigma}} G_{q^{(j)}}(q^{(j)}).$$

(2.32)
Substituting (2.32) into (2.31) to get

$$-n(n-1) \sum_{\ell \neq j} \frac{G_{q(\ell)}(q(j))}{(K(q(j))K(q(\ell)))^{1/2}} \lambda_\ell - \frac{\Delta_{q(\ell)} K(q(j))}{K(q(j))n^{1/2}} \lambda_j = \frac{\sigma}{2} \lambda_j \mu^{(j)}.$$ 

We have established (2.8) and thus verified Part (iii).

We claim that there exists some

$$\eta = (\eta_1, \cdots, \eta_k) \neq 0$$

such that

$$\sum_{\ell=1}^k M_{\ell j}(q^{(1)}, \cdots, q^{(k)}) \eta_\ell = \mu(M) \eta_j, \quad \forall \ j = 1, \cdots, k.$$ 

Indeed, choose \( \Lambda > \max_i M_{ii} \), then the matrix \( \Lambda I - M \) is a positive matrix (see [27] for the definition), where \( I \) denotes the unit matrix. The claim can follows from [27, Theorem 8.2.2].

Multiplying (2.8) by \( \eta_j \) and summing over \( j \), then using Part (ii) and (2.33), we have

$$\mu(M) \sum_j \lambda_j \eta_j = \sum_{\ell, j} M_{\ell j} \lambda_\ell \eta_j = \frac{1}{4} \sum_j \lambda_j \eta_j \mu^{(j)} \geq 0.$$ 

It follows that \( \mu(M) \geq 0 \). We have verified part (i) of Theorem 2.1. Part (iv) follows from (i)–(iii). The proof is completed. \( \square \)

### 2.2 Proof of Theorem 1.1

Using some results of blow up analysis in Appendix A and Theorem 2.1, we are going to prove Theorem 1.1.

**Proof of Theorem 1.1.** We first prove the existence of upper bounds. Suppose the assertion of the theorem is false. Then we can find that there exists \( K_i \to K \) in \( C^2(S^n) \) such that \( \max_{S^n} v_i \to \infty \) for some \( v_i \in M_{K_i} \). Theorem 2.1 shows that \( \{v_i\} \) has only isolated simple blow up points \( \{q^{(1)}, \cdots, q^{(k)}\} \subset \mathcal{K} \setminus \mathcal{K}^+ \).

Next, we prove that \( k > 1 \). Let \( q_0 \) be the isolated simple blow up point of \( v_i \). It follows from Proposition A.10 and \( K \in \mathcal{K}^+ \) that \( q_0 \) is a non-degenerate critical point of \( K \). Let \( F \) be the stereographic projection with \( q_0 \) being the south pole, and \( \tilde{K} := K(F(y)) \).

We assert that for any \( \tilde{y} \in \mathbb{R}^n \),

$$f_{\mathbb{R}^n} \nabla^2 \tilde{K}(0)(y + \tilde{y})(1 + |y|^2)^{-n}$$

$$\left( \int_{\mathbb{R}^n} \frac{1}{2} ((y + \tilde{y}), \nabla^2 \tilde{K}(0)(y + \tilde{y}))(1 + |y|^2)^{-n} \right) \neq 0.$$ 

(2.35)
In fact, if there exists some \( \hat{y} \in \mathbb{R}^n \) such that

\[
\int_{\mathbb{R}^n} \nabla^2 \tilde{K}(0)(y + \hat{y})(1 + |y|^2)^{-n} = 0,
\]

then by the property of odd function, the non degeneracy of \( \nabla^2 \tilde{K}(0) \), and \( \Delta \tilde{K}(0) \neq 0 \), we can obtain that

\[
\int_{\mathbb{R}^n} \frac{1}{2} \langle (y + \hat{y}), \nabla^2 \tilde{K}(0)(y + \hat{y}) \rangle (1 + |y|^2)^{-n} \neq 0.
\]

Thus (2.35) is proved.

Suppose the contrary that \( q_0 \) is the only blow up of \( v_i \). We are going to find some \( \hat{y} \) such that (2.35) fails. By (1.4), we know that (1.3) is equivalent to

\[
v_i(\xi) = \frac{\Gamma\left(\frac{n+2\sigma}{2}\right)}{2^{2\sigma} \pi^{n/2} \Gamma(\sigma)} \int_{\mathbb{S}^n} \frac{K_i(\eta)v_i(\eta)^{n-1}}{|\xi - \eta|^2} \, d\eta \quad \text{on } \mathbb{S}^n.
\]

Under the stereographic projection \( F \), the equation (2.36) is transformed to

\[
u_i(x) = \frac{\Gamma\left(\frac{n+2\sigma}{2}\right)}{2^{2\sigma} \pi^{n/2} \Gamma(\sigma)} \int_{\mathbb{R}^n} \frac{\tilde{K}_i(y)u_i(y)^{n-1}}{|x - y|^2} \, dy \quad \text{on } \mathbb{R}^n,
\]

where

\[H(x) = \frac{2}{1 + |x|^2}, \quad u_i(x) = H(x)v_i(F(x)), \quad \tilde{K}_i(x) = K_i(F(x)).\]

Let \( y_i \) be the local maximum point of \( u_i(y) \) and \( m_i =: u_i(y_i) \). First, we establish

\[|y_i| = O(m_i^{-1}).\]

Since we have assumed that \( v_i \) has no blow up point other than \( q_0 \), it follows from Proposition A.7 and the Harnack inequality that \( u_i(y) \leq C(\varepsilon)|y|^{-2m_i^{-1}} \) for \( |y| \geq \varepsilon > 0 \).

By the Kazdan-Warner condition, we have

\[
\int_{\mathbb{R}^n} \nabla \tilde{K}_i u_i^n = 0.
\]

It follows that for \( \varepsilon > 0 \) small we have

\[
\left| \int_{B_{\varepsilon}} \nabla \tilde{K}_i(y + y_i)u_i(y + y_i)^n \right| \leq C(\varepsilon)m_i^{-n}.
\]

For \( |y| \leq \varepsilon \),

\[
\tilde{K}_i(y) = \tilde{K}_i(0) + \frac{1}{2} \langle y, \nabla^2 \tilde{K}_i(0)y \rangle + o(|y|^2),
\]

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it follows that
\[
\lim_{|y| \to 0} \nabla \left( \tilde{K}_i(y) - \langle y, \nabla^2 \tilde{K}_i(0)y \rangle \right) |y|^{-1} = 0, \tag{2.42}
\]
where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$. Since $\det(\nabla^2 \tilde{K}(0)) \neq 0$ and $\tilde{K}_i \to \tilde{K}$, there exists a constant $C > 0$ such that
\[
\left| \frac{1}{2} \nabla \langle y, \nabla^2 \tilde{K}_i(0)y \rangle \right| = |\nabla^2 \tilde{K}_i(0)y| \geq C|y|, \quad \forall |y| \leq \varepsilon. \tag{2.43}
\]
By (2.40), (2.42) and (2.43), we can obtain
\[
\left| \int_{B_{\varepsilon}} (1 + o_{\varepsilon}(1)) \nabla^2 \tilde{K}_i(0)(y + y_i)u_i(y + y_i)^n \right| \leq C(\varepsilon)m_i^{-n}.
\]
Multiplying the above by $m_i$, and let $\tilde{y}_i := m_iy_i$, we have
\[
\left| \int_{B_{\varepsilon}} (1 + o_{\varepsilon}(1)) \nabla^2 \tilde{K}_i(0)(m_iy + \tilde{y}_i)u_i(y + y_i)^n \right| \leq C(\varepsilon)m_i^{-n}. \tag{2.43}
\]
Suppose (2.38) is false, namely $\tilde{y}_i \to \infty$ along a subsequence. From Proposition A.5, we can choose $R_i \leq |\tilde{y}_i|/4$ such that
\[
\left| \int_{|y| \leq R_i m_i^{-1}} (1 + o_{\varepsilon}(1)) \nabla^2 \tilde{K}_i(0)(m_iy + \tilde{y}_i)u_i(y + y_i)^n \right|
= \left| \int_{|z| \leq R_i} (1 + o_{\varepsilon}(1)) \nabla^2 \tilde{K}_i(0)(z + \tilde{y}_i)(m_i^{-1}u_i(m_i^{-1}z + y_i))^n \right| \sim |\tilde{y}_i|.
\]
On the other hand, it follows from Proposition A.9 that
\[
\left| \int_{R_i m_i^{-1} \leq |y| \leq \varepsilon} (1 + o_{\varepsilon}(1)) \nabla^2 \tilde{K}_i(0)(m_iy + \tilde{y}_i)u_i(y + y_i)^n \right|
\leq C \left| \int_{R_i m_i^{-1} \leq |y| \leq \varepsilon} (|m_iy| + |\tilde{y}_i|)u_i(y + y_i)^n \right| \leq o(1)|\tilde{y}_i|.
\]
It follows that $|\tilde{y}_i| \leq C(\varepsilon)m_i^{-n}$. This contradicts to $\tilde{y}_i \to \infty$. Thus (2.38) is proved.
It follows from the Kazdan-Warner condition that
\[
\int_{\mathbb{R}^n} \langle y, \nabla \tilde{K}_i(y + y_i) \rangle u_i(y + y_i)^n = 0.
\]
Similar to (2.40), we have for any $\varepsilon > 0$,\[
\left| \int_{B_{\varepsilon}} \langle y, \nabla \tilde{K}_i(y + y_i) \rangle u_i(y + y_i)^n \right| \leq C(\varepsilon)m_i^{-n}.
\]
By (2.41), (2.42), (2.43), and Proposition A.9, we have

\[
\left| \int_{B_{R_0}} \langle y, \nabla^2 \tilde{K}_i(0)(y + y_i) \rangle u_i(y + y_i)^n \right|
\leq C(\varepsilon)m_i^{-n} + o_\varepsilon(1) \int_{B_{R_0}} (|y|^2 + |y||y_i|) u_i(y + y_i)^n
\leq C(\varepsilon)m_i^{-n} + o_\varepsilon(1)m_i^{-2}.
\]

Multiplying the above by \(m_i^2\, due to \(n - 2 = 2\sigma\), we have

\[
\lim_{i \to \infty} m_i^2 \int_{B_{R_0}} \langle y, \nabla^2 \tilde{K}_i(0)(y + y_i) \rangle u_i(y + y_i)^n = o_\varepsilon(1).
\]

Let \(R_i \to \infty\) as \(i \to \infty\), and \(r_i := R_i m_i^{-1}\). By Proposition A.9, we have

\[
m_i^2 \left| \int_{r_i \leq |y| \leq \varepsilon} \langle y, \nabla^2 \tilde{K}_i(y + y_i) \rangle u_i(y + y_i)^n \right|
\leq C m_i^2 \left| \int_{r_i \leq |y| \leq \varepsilon} (|y|^2 + |y||y_i|) u_i(y + y_i)^n \right| \to 0 \quad \text{as } i \to \infty.
\]

Using Proposition A.5, making a change of variable \(z = m_i y\), and then letting \(\varepsilon \to 0\), we have,

\[
\int_{\mathbb{R}^n} \langle z, \nabla^2 \tilde{K}(0)(z + z_0) \rangle (1 + k|z|^2)^{-n} = 0, \quad (2.44)
\]

where \(z_0 = \lim_{i \to \infty} m_i y_i\) and \(k = \lim_{i \to \infty} \tilde{K}_i(y_i)^{1/\sigma}/4\).

It follows from (2.39) that

\[
\int_{\mathbb{R}^n} \nabla \tilde{K}_i(y + y_i) u_i(y + y_i)^n = 0.
\]

Using the same method above, we obtain

\[
\int_{\mathbb{R}^n} \nabla^2 \tilde{K}(0)(z + z_0)(1 + k|z|^2)^{-n} = 0. \quad (2.45)
\]

It follows from (2.44) and (2.45) that

\[
\int_{\mathbb{R}^n} \frac{1}{2} \langle z + z_0, \nabla^2 \tilde{K}(0)(z + z_0) \rangle (1 + k|z|^2)^{-n} = 0. \quad (2.46)
\]

From (2.45) and (2.46) we can see that (2.35) does not hold for \(\tilde{\gamma} = k^{1/2} z_0\). Therefore, we proved that \(k > 1\).

By Part (i) of Theorem 2.1, we have \(\{q^{(1)}, \ldots, q^{(k)}\} \subset \mathcal{H}^-\) and \(\mu(M(q^{(1)}, \ldots, q^{(k)}) \geq 0\). It follows from \(v_i \in \mathcal{M}_K\) that \(\tau_i = 0\). Applying Part (iv) of Theorem 2.1, we deduce that \(\mu(M(q^{(1)}, \ldots, q^{(k)})) = 0\). This leads to a contradiction with \(K \in \mathcal{A}\). From the Harnack inequality and Schauder type estimates, we complete the proof of Theorem 1.1. □
3 The degree-counting formula and existence results

This section is devoted to the proof of Theorems 1.2, 1.3, and 1.4. It is worth noting that due to Theorem 1.1, homotopy invariance of Leray-Schauder degree and the properties of “Index”, we only need to prove Theorem 1.2 for $K \in \mathcal{A}$ being a Morse function. Once this is achieved, we also prove that the Index as in Definition 1.1 is well defined on $\mathcal{A}$. Therefore, we always assume that $K \in \mathcal{A}$ is a Morse function in this section.

3.1 On the case of subcritical equations

Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$, and $n = 2\sigma + 2$. In this subsection, we consider the following subcritical equation:

$$ P_{\sigma}v = c(n, \sigma)Kv^{n-1-\tau} \quad \text{on } S^n, \quad (3.1) $$

where $c(n, \sigma) = \Gamma(n - 1)$, $K \in C^2(S^n)$, and $\tau > 0$.

We will soon prove that when $K \in \mathcal{A}$, the solutions to $(3.1)$ either stay bounded and converge to the solutions to critical equations $(1.3)$ in $C^2_\sigma$ norm or become unbounded and blow up at finite points as $\tau \to 0^+$.

Denote the $H^\sigma(S^n)$ inner product and norm by

$$ \langle u, v \rangle = \int_{S^n} (P_{\sigma}u)v, \quad \|u\|_{\sigma} = \sqrt{\langle u, u \rangle}. \quad (3.2) $$

The Euler-Lagrange functional associated with $(3.1)$ is

$$ I_\tau(u) = \frac{1}{2} \int_{S^n} (P_{\sigma}u)u - \frac{\Gamma(n - 1)}{n - \tau} \int_{S^n} K|u|^{n-\tau}, \quad \forall u \in H^\sigma(S^n). \quad (3.2) $$

**Definition 3.1.** Let $K \in C^2(S^n)$, $\mathcal{K}^-$ be as in $(1.5)$ and $k \in \mathbb{N}_+$. Let $\mathcal{T}_1, \ldots, \mathcal{T}_k \in \mathcal{K}^-$ be the critical points of $K$ with $\mu(M(\mathcal{T}_1, \ldots, \mathcal{T}_k)) > 0$, and $\varepsilon_0 > 0$ be sufficiently small. Define

$$ \Omega_{\varepsilon_0} = \Omega_{\varepsilon_0}(\mathcal{T}_1, \ldots, \mathcal{T}_k) $$

$$ =\{(\alpha, t, P) \in \mathbb{R}^k_+ \times \mathbb{R}_+^k \times (S^n)^k : |\alpha_i - (K(P_i))^{-1/2\sigma}| < \varepsilon_0, \quad t_i > 1/\varepsilon_0, \quad |P_i - \mathcal{T}_i| < \varepsilon_0, \quad 1 \leq i \leq k\}. $$

For $P \in S^n$ and $t > 0$,

$$ \delta_{P,t}(x) = \frac{t}{1 + t^2 \left(1 - \cos d(x, P)\right)}, \quad x \in S^n \quad (3.3) $$
is the family of the solutions for
\[ P_\sigma v = \Gamma(n-1)v^{n-1}, \quad v > 0 \quad \text{on } \mathbb{S}^n. \] (3.4)

We have the following lemma based on the ideas provided by Bahri in [5]:

**Lemma 3.1.** Let \( \varepsilon_0 \) be sufficiently small and \( \Omega_{\varepsilon_0} = \Omega_{\varepsilon_0}(\overline{P}, \cdots, \overline{P}_k) \) be as in Definition 3.1. For any \( u \in H^\sigma(\mathbb{S}^n) \) satisfying
\[
\left\| u - \sum_{i=1}^{k} \alpha_i \delta_{P_i, P} \right\|_\sigma < \frac{\varepsilon_0}{2}
\]
for some \((\alpha, \tilde{t}, \tilde{P}) \in \Omega_{\varepsilon_0/2}, \) then there exists a unique \((\alpha, t, P) \in \Omega_{\varepsilon_0} \) such that
\[
u = \sum_{i=1}^{k} \alpha_i \delta_{P_i, t_i} + v,
\]
with \( v \) satisfies
\[
\langle v, \delta_{P_i, t_i} \rangle = \langle v, \frac{\partial \delta_{P_i, t_i}}{\partial \overline{P}_i} \rangle = \langle v, \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \rangle = 0,
\] (3.5)
where \( \frac{\partial}{\partial \overline{P}_i} \) denotes the corresponding derivatives.

In what follows, we say that \( v \in E_{P,t} \) if \( v \) satisfies (3.5) and we work in some orthonormal basis near \( \{\overline{P}_1, \cdots, \overline{P}_k\} \).

**Definition 3.2.** Let \( \tau, \varepsilon_0, \nu_0 > 0 \) be sufficiently small, \( A > 0 \) be sufficiently large, and \( \Omega_{\varepsilon_0/2} = \Omega_{\varepsilon_0}(\overline{P}, \cdots, \overline{P}_k) \) be as in Definition 3.1. Define
\[
\Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k) = \{ (\alpha, t, P, v) \in \Omega_{\varepsilon_0/2} \times H^\sigma(\mathbb{S}^n) : \\
|P - \overline{P}| < \tau^{1/2} \log \tau, \quad A^{-1} \tau^{-1/2} < t_i < A \tau^{-1/2}, \quad \nu \in E_{P,t}, \|v\|_\sigma < \nu_0 \}. \] (3.6)
Without confusion we use the same notation for
\[
\Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k) = \left\{ u = \sum_{i=1}^{k} \alpha_i \delta_{P_i, t_i} + v : (\alpha, t, P, v) \in \Sigma_\tau \right\} \subset H^\sigma(\mathbb{S}^n).
\]

Combined with Theorem 2.1, we can obtain the necessary conditions on blowing up solutions to (3.1) when \( K \in \mathcal{A} \) as \( \tau \) tends to \( 0^+ \).

**Proposition 3.1.** Let \( \sigma = 1 + m/2, \) \( m \in \mathbb{N}_+, \) and \( n = 2\sigma + 2 \). Let \( K \in \mathcal{A} \) be a Morse function and \( \mathcal{K}^- \) be as in (1.5). Then for any \( \alpha \in (0, 1) \), there exists some positive constants \( \varepsilon_0, \nu_0 \ll 1, \) and \( A, R \gg 1 \) depending only on \( K \), such that when \( \tau > 0 \) is sufficiently small, for all \( u \) satisfying \( u \in H^\sigma(\mathbb{S}^n), \) \( u > 0, \ I'_\tau(u) = 0, \) we have
\[
u \in \Theta_{R} \cup \left\{ \cup_{k \geq 1} \cup_{\overline{P}_k \in \mathcal{K}^-, \mu(M(\overline{P}_1, \cdots, \overline{P}_k)) > 0} \Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k) \right\},
\]
where \( I'_\tau(u) \) is as in (3.1), \( \Theta_{R} \) is as in (1.12) and \( \Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k) \) is as in (3.6).
Proof. For any \( \tau > 0 \) sufficiently small, let \( u_\tau \in H^\sigma(S^n) \), \( u_\tau > 0 \) be a critical point of \( I_\tau(u) \). If \( u_\tau \) is uniformly bounded, then by the Schauder type estimates we know that there exists a \( R > 0 \) such that \( u_\tau \in \mathcal{O}_R \). The proof is now completed. If not, there exists \( \tau_i \to 0 \) such that \( \max_{S^n} u_{\tau_i} \to \infty \). It follows from Theorem 2.1 and \( K \in \mathcal{A} \) that there exists a constant \( \delta^* > 0 \) such that \( \{u_{\tau_i}\} \) has only isolated simple blow up points \( q^{(1)}, \ldots, q^{(k)} \in \mathcal{X}^- \), with \( |q^{(j)} - q^{(\ell)}| \geq \delta^* \), \( \forall j \neq \ell \), and \( \mu(M(q^{(1)}, \ldots, q^{(k)})) > 0 \). Then Proposition 3.1 can be deduced from Propositions A.6, A.7, A.8, and Lemma 3.1.

Now we are going to show that if \( K \in \mathcal{A} \) is a Morse function, one can construct solutions highly concentrating at arbitrary points \( q^{(1)}, \ldots, q^{(k)} \in \mathcal{X}^- \) provided \( \mu(M(q^{(1)}, \ldots, q^{(k)})) > 0 \).

**Theorem 3.1.** Let \( \sigma = 1 + m/2 \), \( m \in \mathbb{N}_+ \), and \( n = 2\sigma + 2 \). Let \( K \in \mathcal{A} \) be a Morse function and \( \mathcal{X}^- \) be as in (1.5). Let \( \tau, \varepsilon_0, \nu_0 > 0 \) be sufficiently small, \( A > 0 \) be sufficiently large and \( k \in \mathbb{N}_+ \). Then for any \( P_1, \ldots, P_k \in \mathcal{X}^- \) satisfying \( \mu(M(P_1, \ldots, P_k)) > 0 \), we have

\[
\deg_{H^\sigma}(u - P_\sigma^{-1}(c(n, \sigma)K|u|^{2\sigma-\tau}u), \Sigma_\tau(P_1, \ldots, P_k), 0) = (-1)^{k+\sum_{j=1}^k i(P_j)}, \quad (3.7)
\]

where \( \deg_{H^\sigma} \) denotes the Leray-Schauder degree in \( H^\sigma(S^n) \), and \( i(P_j) \) is the Morse index of \( K \) at \( P_j \).

In order to prove Theorem 3.1, we need the following Lemmas 3.2, 3.3 and Propositions 3.2, 3.3, 3.4, 3.5, whose proofs mainly uses the estimates in the appendix.

**Lemma 3.2.** Under the hypotheses of Theorem 3.1, in addition that \( \Sigma_\tau = \Sigma_\tau(P_1, \ldots, P_k) \) is as in Definition 3.2 for the given \( \tau, \varepsilon_0, \nu_0, A \), and \( P_1, \ldots, P_k \in \mathcal{X}^- \). Then for any \( (\alpha, t, P, v) \in \Sigma_\tau \), we have:

\[
I_\tau \left( \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} + v \right) = \frac{\Gamma(n-1)}{2} \left( \sum_{i=1}^k \alpha_i^2 \int_{S^n} \delta_{P_i, t_i} + \sum_{i \neq j} \alpha_i \alpha_j \int_{S^n} \delta_{P_i, t_i} \delta_{P_j, t_j} \right) - \frac{\Gamma(n-1)}{n-\tau} \int_{S^n} K \left( \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} \right)^{n-\tau} + f_\tau(v) + Q_\tau(v, v) + V(\tau, \alpha, t, P, v),
\]

where

\[
f_\tau(v) := -\Gamma(n-1) \int_{S^n} K \left( \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} \right)^{n-1-\tau} v, \quad (3.8)
\]

\[
Q_\tau(v, v) := \frac{1}{2} \int_{S^n} (P_\sigma v) v - (n-1-\tau) \frac{\Gamma(n-1)}{2} \int_{S^n} K \left( \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} \right)^{2\sigma-\tau} v^2, \quad (3.9)
\]

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and there exists a constant \( C > 0 \) depends only on \( K, \nu_0, \) and \( A \) such that
\[
|V(\tau, \alpha, t, P, v)| \leq C\|v\|_\sigma^3.
\]

**Proof.** By (3.2) and (3.5), we have
\[
I_\tau \left( \sum_{i=1}^{k} \alpha_i \delta_{P,t_i} + v \right) = \frac{\Gamma(n-1)}{2} \left( \sum_{i=1}^{k} \alpha_i^2 \delta_{P,t_i}^n + \sum_{j \neq i} \alpha_i \alpha_j \int_{\mathbb{S}^n} \delta_{P,t_i}^{n-1} \delta_{P,t_j} \right) + \frac{1}{2} \int_{\mathbb{S}^n} (P_{\sigma}v) v 
\]
(3.10)

Then, it follows from Lemma B.1 and (B.6) that Lemma 3.2 holds.

**Lemma 3.3.** Under the hypotheses of Lemma 3.2, in addition that \( E_{P,t} \) is as in (3.5). Then for any \( (\alpha, t, P, v) \in \Sigma_{\tau} \), there exists some function \( V_v \) and a constant \( C > 0 \) depending only on \( K, \nu_0, \) and \( A \) such that
\[
I_\tau' \left( \sum_{i=1}^{k} \alpha_i \delta_{P,t_i} + v \right) \varphi = f_\tau(\varphi) + 2Q_\tau(v, \varphi) + \langle V_v(\tau, \alpha, t, P, v), \varphi \rangle,
\]
and
\[
\|V_v(\tau, \alpha, t, P, v)\|_\sigma \leq \|v\|_\sigma^2,
\]
where \( f_\tau(v) \) is as in (3.8) and \( Q_\tau(v, \varphi) \) is as in (3.9).

**Proof.** For any \( \varphi \in E_{P,t} \), by using (3.10), Lemma B.1, and (3.5), we have
\[
I_\tau' \left( \sum_{i=1}^{k} \alpha_i \delta_{P,t_i} + v \right) \varphi
\]
\[
= \int_{\mathbb{S}^n} P_{\sigma}(v) \varphi - \Gamma(n-1) \int_{\mathbb{S}^n} K \left| \sum_{i=1}^{k} \alpha_i \delta_{P,t_i} + v \right|^{2\sigma-\tau} \left( \sum_{i=1}^{k} \alpha_i \delta_{P,t_i} + v \right) \varphi
\]
\[
= \int_{\mathbb{S}^n} P_{\sigma}(v) \varphi - \Gamma(n-1) \int_{\mathbb{S}^n} K \left( \sum_{i=1}^{k} \alpha_i \delta_{P,t_i} \right)^{n-1-\tau} \varphi
\]
\[
- \Gamma(n-1)(n-1-\tau) \int_{\mathbb{S}^n} K \left( \sum_{i=1}^{k} \alpha_i \delta_{P,t_i} \right)^{2\sigma-\tau} v \varphi
\]
\[
+ \langle V_v(\tau, \alpha, t, P, v), \varphi \rangle.
\]
Then, the estimates of \( V_v(\tau, \alpha, t, P, v) \) can be obtained by Sobolev imbedding and (B.6).

\[\square\]
Proposition 3.2. Under the hypotheses of the Theorem 3.1, in addition that $\Sigma_\tau(P_1, \cdots, P_k)$ is as in (3.7) and $E_{P,t}$ is as in (3.5) for the given $(\alpha, t, P)$. Then there exists a unique minimizer $v = v_\tau(\alpha, t, P) \in E_{P,t}$ of $I_\tau(\sum_{i=1}^k \alpha_i P_i, t_i + v)$ with respect to $\{v \in E_{P,t} : \|v\|_\sigma < \nu_0\}$. Furthermore, there exists a constant $C$ independent of $\tau$ such that

$$\|v\|_\sigma \leq C \sum_{i=1}^k \|\nabla K(P_i)\|_{1/2} + C\tau\|\log \tau\| \leq C\tau\|\log \tau\|. \quad (3.11)$$

Proof. From Lemma 3.3, we have, for all $\varphi \in E_{P,t}$,

$$I'_\tau\left(\sum_{i=1}^k \alpha_i P_i, t_i + v\right)\varphi = f_\tau(\varphi) + 2Q_\tau(v, \varphi) + \langle V_v(\tau, \alpha, t, P, v), \varphi \rangle, \quad (3.12)$$

where

$$f_\tau(\varphi) := -\Gamma(n-1) \int_{S^n} K\left(\sum_{i=1}^k \alpha_i P_i, t_i\right)^{n-1-\tau} \varphi,$$

and

$$Q_\tau(v, \varphi) := \frac{1}{2} \int_{S^n} (P_\sigma v)\varphi - (n-1-\tau) \frac{\Gamma(n-1)}{2} \int_{S^n} K\left(\sum_{i=1}^k \alpha_i P_i, t_i\right)^{2\sigma-\tau} v \varphi.$$

It is obviously that $f_\tau$ is a continuous linear functional over $E_{P,t}$, there exists a unique $\tilde{f}_\tau \in E_{P,t}$ such that

$$f_\tau(\varphi) = \langle \tilde{f}_\tau, \varphi \rangle, \quad \forall \varphi \in E_{P,t}. \quad (3.13)$$

By the same method of proving the coercivity of the quadratic form $Q_\tau$ in [1, 15], it follows that there exists a constant $\delta_0 > 0$ (independent of $\tau$) such that

$$Q_\tau(v, v) \geq \frac{\delta_0}{2} \|v\|_{\sigma}^2, \quad \forall (\alpha, t, P, v) \in \Sigma_\tau, \quad (3.14)$$

thus, there exists a unique symmetric continuous and coercive operator $\tilde{Q}_\tau$ from $E_{P,t}$ onto itself such that

$$Q_\tau(v, \varphi) = \langle \tilde{Q}_\tau v, \varphi \rangle, \quad \forall \varphi \in E_{P,t}. \quad (3.15)$$

Using these notations, (3.12), (3.13), and (3.15), we have

$$I'_\tau\left(\sum_{i=1}^k \alpha_i P_i, t_i + v\right) = \tilde{f}_\tau + 2\tilde{Q}_\tau v + V_v(\tau, \alpha, t, P, v). \quad (3.16)$$

There is an equivalence between the existence of minimizer $\bar{v}_\tau$ and

$$\tilde{f}_\tau + 2\tilde{Q}_\tau v + V_v(\tau, \alpha, t, P, v) = 0, \quad v \in E_{P,t}. \quad (3.17)$$
As in [43, 46], by the implicit function theorem, there exist a $C^1$-map $\varphi : (\alpha, t, P) \mapsto E_{P, t}$ satisfying (3.17) and
\[
\|\varphi\|_\sigma \leq C\|\tilde{f}_\tau\|_\sigma.
\] (3.18)

Therefore, in order to prove (3.11), we only need to estimate $\|\tilde{f}_\tau\|_\sigma$.

Applying Lemma B.2, (B.10), (B.11), (3.6), and (B.13), we can obtain
\[
f_\tau(v) = -\Gamma(n - 1) \int_{\mathbb{S}^n} K\left(\sum_{i=1}^{k} (\alpha_i \delta_{P_i})^{n-1}\right) v + O\left(\sum_{i \neq j}^{k} \int_{\mathbb{S}^n} \delta_{P_i}^{n-2-\tau} \delta_{P_j} \mid v \right)
\]
\[
= -\Gamma(n - 1) \int_{\mathbb{S}^n} (K - K(P_i)) \sum_{i=1}^{k} \alpha_i^{n-1-\tau} \delta_{P_i} v
\]
\[
+ O\left(\sum_{i=1}^{k} \int_{\mathbb{S}^n} |\delta_{P_i}^{n-1} - \delta_{P_j}^{n-1}| \mid v \right) + O\left(\sum_{i \neq j}^{k} \|\delta_{P_i}^{n-2-\tau} \delta_{P_j} \|_{L^n/(n-1)(\mathbb{S}^n)} \| v \|_\sigma \right)
\]
\[
= O\left(\sum_{i=1}^{k} |\nabla g_{i} K(P_i)| \int_{\mathbb{S}^n} |P - P_i|^{\delta_{P_i}^{n-1}} \mid v \right) + O\left(\sum_{i=1}^{k} \int_{\mathbb{S}^n} |P - P_i|^{2\delta_{P_i}^{n-1}} \mid v \right)
\]
\[
+ O(\tau \log \tau \| v \|_\sigma),
\]
where $|P - P_i|$ represents the distance between two points $P$ and $P_i$ after through a stereographic projection with $P_i$ as the south pole of $\mathbb{S}^n$.

From (3.6) and (B.13), we have, for all $(\alpha, t, P, v) \in \Sigma_{\tau}(\overline{P}_1, \cdots, \overline{P}_k)$,
\[
|f_\tau(v)| \leq C\left\{\tau^{1/2} \sum_{i=1}^{k} |\nabla K(P_i)| + \tau + \tau \log \tau \right\} \| v \|_\sigma
\]
\[
\leq C\tau \| v \|_\sigma,
\] (3.19)
this, combining (3.13) and (3.18), we obtain (3.11). \hfill \Box

**Proposition 3.3.** Under the hypotheses of Theorem 3.1, then for any $(\alpha, t, P, v) \in \Sigma_{\tau}(\overline{P}_1, \cdots, \overline{P}_k)$, we have
\[
\frac{\partial}{\partial \alpha_i} I_\tau\left(\sum_{j=1}^{k} \alpha_j \delta_{P_j} + v \right) = -2\sigma\|\delta_{P_i} \|_\sigma^2 \beta_i + V_\alpha(\tau, \alpha, t, P, v),
\]
where $\beta = (\beta_1, \cdots, \beta_k)$, $\beta_i := \alpha_i - K(P_i)^{-1/2\sigma}$, $i = 1, \cdots, k$, and
\[
V_\alpha(\tau, \alpha, t, P, v) = O(|\beta|^2) + O(\tau \log \tau) + O(\| v \|^2_\sigma).
\]
Furthermore, let $\overline{v}$ be as in Proposition 3.2, then we have
\[
\frac{\partial}{\partial \alpha_i} I_\tau\left(\sum_{j=1}^{k} \alpha_j \delta_{P_j} + v \right) = -2\sigma\|\delta_{P_i} \|_\sigma^2 \beta_i + O(\| \beta \|^2 + \tau \log \tau).
\]
Proof. Using Lemma B.1, (B.7), (B.10), and Lemma B.2 we have

$$\frac{\partial}{\partial \alpha_i} I_\tau \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right)$$

$$= \Gamma(n-1) \left( \alpha_i \int_{S^n} \delta_{P_i,t_i} + \sum_{j \neq i} \alpha_j \int_{S^n} \delta_{P_j,t_j}^{n-1} \delta_{P_j,t_j} \right)$$

$$- \Gamma(n-1) \int_{S^n} K \left| \sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i} + v \right|^{n-2-\tau} \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right) \delta_{P_i,t_i}$$

$$= \Gamma(n-1) \left( \alpha_i \int_{S^n} \delta_{P_i,t_i} - \int_{S^n} K \left| \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} \right|^{n-1-\tau} \delta_{P_i,t_i} \right)$$

$$- \Gamma(n-1)(n-\tau-1) \int_{S^n} K \left| \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} \right|^{n-2-\tau} v \delta_{P_i,t_i} + O(\tau) + O(\|v\|_\sigma^2)$$

$$= \Gamma(n-1) \left( \alpha_i \int_{S^n} \delta_{P_i,t_i} - \int_{S^n} K \left( \sum_{j=1}^{k} (\alpha_j \delta_{P_j,t_j})^{n-1-\tau} \right) \delta_{P_i,t_i} \right)$$

$$- \Gamma(n-1) \int_{S^n} K \left( \sum_{j=1}^{k} (\alpha_j \delta_{P_j,t_j})^{n-2-\tau} \right) v \delta_{P_i,t_i} + O(\tau) + O(\|v\|_\sigma^2).$$

It follows from (B.13) and (3.6) that

$$\int_{S^n} K \alpha_i^{n-1} \delta_{P_i,t_i}^{n-\tau} = \int_{S^n} K(P_i) \alpha_i^{n-1} \delta_{P_i,t_i}^{n-\tau} - \int_{S^n} (K(P) - K(P_i)) \alpha_i^{n-1} \delta_{P_i,t_i}^{n-\tau}$$

$$= \int_{S^n} K(P_i) \alpha_i^{n-1} \delta_{P_i,t_i}^{n-\tau} + O(\tau). \quad (3.20)$$

Similarly, by (3.5), (B.11), (3.6), and (B.13), we have

$$\int_{S^n} K \alpha_i^{n-2} \delta_{P_i,t_i}^{n-1-\tau} v$$

$$= \int_{S^n} K(P_i) \alpha_i^{n-2} \delta_{P_i,t_i}^{n-1-\tau} v + \int_{S^n} (K(P) - K(P_i)) \alpha_i^{n-2} \delta_{P_i,t_i}^{n-1-\tau} v + O(\tau \log \|v\|_\sigma)$$

$$= O(\tau \log \|v\|_\sigma) + O(\|v\|_\sigma^2). \quad (3.21)$$

By using the fact $|\alpha_i^{n-1-\tau} - \alpha_i^{n-1}| = O(\tau)$, (3.20), (3.21), (B.2), and (B.12) that

$$\frac{\partial}{\partial \alpha_i} I_\tau \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right)$$

$$= \Gamma(n-1) \left( \alpha_i \int_{S^n} \delta_{P_i,t_i} - K(P_i) \int_{S^n} \alpha_i^{n-1-\tau} \delta_{P_i,t_i}^{n-\tau} - \int_{S^n} K \alpha_i^{n-2} \delta_{P_i,t_i}^{n-1-\tau} v \right)$$

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follows from the above.

\[ P(B.1) \]

where \( \Theta \) are positive constants, \( G_{\alpha}(\tau) \) is as in (1.7), and

\[ \forall \alpha, (\alpha, t, P, v) = O(\|\beta\|^2) + O(\|v\|^2) + O(\|v\|^2). \]

Combining with (3.19), we obtain

\[
\frac{\partial}{\partial t_i} I_\tau \left( \sum_{j=1}^{k} \alpha_j \delta_{P_{j,t}} + v \right) = -2\sigma \|\delta_{P_{i,t}}\|_{\sigma}^2 \beta_i + V_{\alpha}(\tau, \alpha, t, P, v),
\]

where

\[
V_{\alpha}(\tau, \alpha, t, P, v) = O(\|\beta\|^2) + O(\|v\|^2) + O(\|v\|^2).
\]

Proposition 3.3 follows from the above.

**Proposition 3.4.** Under the hypotheses of Proposition 3.3, then for any \((\alpha, t, P, v) \in \Sigma_{\tau}(\overline{P}_1, \ldots, \overline{P}_k)\), we have

\[
\frac{\partial}{\partial t_i} I_\tau \left( \sum_{j=1}^{k} \alpha_j \delta_{P_{j,t}} + v \right) = \Theta_1 \frac{1}{K(P_i)^{1/\sigma} t_i} + \Theta_2 \frac{\Delta g_{\sigma} K(P_i) 1}{K(P_i)^{n/2\sigma} t_i^2} + \Theta_3 \sum_{j \neq i} \frac{G_{\alpha}(\tau)}{(K(P_i) K(P_j))^{1/2\sigma} t_i^{2} t_j} + V_{\alpha}(\tau, \alpha, t, P, v),
\]

where \( \Theta_1, \Theta_2, \Theta_3 \) are positive constants, \( G_{\alpha}(\tau) \) is as in (1.7), and

\[
V_{\alpha}(\tau, \alpha, t, P, v) = O(\|v\|_{\sigma}) + O(\|v\|^2) + O(\|v\|^2) + o(\|v\|^2).
\]

**Proof.** By (3.10), Lemma B.1, Hölder inequality, and Sobolev embedding, we have

\[
\frac{\partial}{\partial t_i} I_\tau \left( \sum_{j=1}^{k} \alpha_j \delta_{P_{j,t}} + v \right)
\]

\[= \Gamma(n-1) \left( \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_{j,t}} - \int_{S^n} K \left( \sum_{j=1}^{k} \alpha_j \delta_{P_{j,t}} \right)^{n-1-\tau} \alpha_i \frac{\partial \delta_{P_{j,t}}}{\partial t_i} \right)
\]

\[= \Gamma(n-1)(n-1-\tau) \int_{S^n} K \left( \sum_{j=1}^{k} \alpha_j \delta_{P_{j,t}} \right)^{n-2-\tau} \alpha_i \frac{\partial \delta_{P_{j,t}}}{\partial t_i} + O \left( \|v\|^2 \left\| \frac{\partial \delta_{P_{j,t}}}{\partial t_i} \right\|_{\sigma} \right).
\]

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From (3.5), we can obtain

\[
\int_{\mathbb{S}^n} \delta_{P_i,t_i}^{\tau} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} = \frac{2}{n+2\sigma} \int_{\mathbb{S}^n} v \frac{\partial}{\partial t_i} (\delta_{P_i,t_i}^{\tau -1})
\]

\[
= \frac{2}{(n+2\sigma)c(n,\sigma)} \frac{\partial}{\partial t_i} \langle v, \delta_{P_i,t_i} \rangle
\]

\[
= \frac{2}{(n+2\sigma)c(n,\sigma)} \langle v, \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \rangle = 0.
\]

It follows from (3.24), (3.6), (B.11), (B.8), and (B.14) that

\[
\left| \int_{\mathbb{S}^n} K \delta_{P_i,t_i}^{n-2-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \right|
\]

\[
= \left| \int_{\mathbb{S}^n} (K - K(P_i)) \delta_{P_i,t_i}^{n-2} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} + \int_{\mathbb{S}^n} K \delta_{P_i,t_i}^{n-2-\tau} - \delta_{P_i,t_i}^{n-2} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \right|
\]

\[
\leq C(\tau^{1/2} |\log \tau|) \left( P - P_i \right) \| \delta_{P_i,t_i}^{n-2} \| - \delta_{P_i,t_i}^{n-2} \| \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \|_\sigma \| v \|_\sigma
\]

\[
+ O \left( \| \delta_{P_i,t_i}^{n-2-\tau} - \delta_{P_i,t_i}^{n-2} \|_{L^\infty(n)} \| \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \|_\sigma \| v \|_\sigma \right)
\]

\[
\leq (\tau^{1/2} |\log \tau|) O \left( \| \delta_{P_i,t_i}^{n-2} \|_\sigma \| \delta_{P_i,t_i}^{n-2} \|_\sigma \| \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \|_\sigma \| v \|_\sigma \right) + O \left( \tau^{3/2} |\log \tau| \| v \|_\sigma \right)
\]

\[
\leq C \tau^{3/2} |\log \tau| \| v \|_\sigma,
\]

this, Lemma B.2, (B.16), and (B.17) yields,

\[
\left| \int_{\mathbb{S}^n} K \left( \sum_{j=1}^k \alpha_j \delta_{P_j,t_j}^{n-2-\tau} \frac{\partial \delta_{P_j,t_j}}{\partial t_j} \right) \right|
\]

\[
\leq \left| \int_{\mathbb{S}^n} K \alpha_i \delta_{P_i,t_i}^{n-2-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \right| \| v \|
\]

\[
+ C \sum_{j \neq i} \left| \int_{\mathbb{S}^n} \delta_{P_j,t_j}^{n-2-\tau} \left| \frac{\partial \delta_{P_j,t_j}}{\partial t_j} \right| \right| | v |
\]

\[
\leq C \left( \tau^{3/2} |\log \tau| \| v \|_\sigma + \tau^{3/2} \| v \|_\sigma \right)
\]

\[
\leq C \tau^{3/2} |\log \tau| \| v \|_\sigma.
\]

Using (3.23), (3.25), Lemma B.2, and (B.15), we obtain

\[
\frac{\partial}{\partial t_i} I_x \left( \sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v \right)
\]

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\[ \Gamma(n-1) \left( \sum_{j \neq i} \alpha_i \partial \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_j, t_j} \delta_{P_i, t_i}^{n-1} - \int_{S^n} K(\alpha_i \delta_{P_i, t_i})^{n-1-\tau} \alpha_i \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \right) \]
\[ - \Gamma(n-1) \int_{S^n} K \left( \sum_{j \neq i} \alpha_j \delta_{P_j, t_j} \right)^{n-1-\tau} \alpha_i \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \]
\[ - \Gamma(n-1)(n-1-\tau) \int_{S^n} K(\alpha_i \delta_{P_i, t_i})^{n-2-\tau} \left( \sum_{j \neq i} \alpha_j \delta_{P_j, t_j} \right) \alpha_i \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \]
\[ + O(\tau^{3/2}) + O(\tau^{3/2} \log \tau \|v\|_\sigma) + O(\tau^{1/2} \|v\|_\sigma^2) \]
\[ = \Gamma(n-1) \left( \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_j, t_j} \delta_{P_i, t_i}^{n-1} - \int_{S^n} K \alpha_i \delta_{P_i, t_i}^{n-1-\tau} \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \right) \]
\[ - \Gamma(n-1) \int_{S^n} \alpha_i K \left( \sum_{j \neq i} \alpha_j \delta_{P_j, t_j} \right)^{n-1-\tau} \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \]
\[ - \Gamma(n-1)(n-1) \int_{S^n} \alpha_i^{-1} K \left( \sum_{j \neq i} \alpha_j \delta_{P_j, t_j} \right) \delta_{P_i, t_i}^{n-2-\tau} \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \]
\[ + O(\tau^{3/2}) + O(\tau^{3/2} \log \tau \|v\|_\sigma) + O(\tau^{1/2} \|v\|_\sigma^2). \]

(3.26)

It follows from Lemma B.2 that
\[ \left( \sum_{j \neq i} \alpha_j \delta_{P_j, t_j} \right)^{n-1-\tau} = \sum_{j \neq i} (\alpha_j \delta_{P_j, t_j})^{n-1-\tau} + O \left( \sum_{j \neq i, \ell \neq i, j \neq \ell} \delta_{P_j, t_j}^{n-2-\tau} \delta_{P_i, t_i} \right). \]  

(3.27)

By (3.26), (3.27), (B.8), (B.10), and (B.15), we can obtain

\[ \frac{\partial}{\partial t_i} I_\tau \left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} + v \right) \]
\[ = \Gamma(n-1) \left( \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_j, t_j} \delta_{P_i, t_i}^{n-1} - \int_{S^n} K \alpha_i \delta_{P_i, t_i}^{n-1-\tau} \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \right) \]
\[ - \Gamma(n-1) \int_{S^n} \alpha_i K \left( \sum_{j \neq i} \alpha_j \delta_{P_j, t_j} \right)^{n-1-\tau} \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \]
\[ - \Gamma(n-1) \alpha_i^{-1} \sum_{j \neq i} \int_{S^n} K \alpha_j \delta_{P_j, t_j} \frac{\partial}{\partial t_i} (\delta_{P_i, t_i})^{n-1-\tau} \]
\[ + O(\tau^{3/2}) + O(\tau^{3/2} \log \tau \|v\|_\sigma) + O(\tau^{1/2} \|v\|_\sigma^2). \]

By (B.18), we have
\[ \int_{S^n} K \delta_{P_j, t_j} \frac{\partial}{\partial t_i} (\delta_{P_i, t_i})^{n-1-\tau} \]
\[ = \frac{\partial}{\partial t_i} \int_{S^n} K \delta_{P_j, t_j} \delta_{P_i, t_i}^{n-1-\tau} \]

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\[ = K(P_i) \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_i,t_j} \delta_{P_i,t_i}^{n-1-\tau} + \frac{\partial}{\partial t_i} \int_{S^n} (K - K(P_i)) \delta_{P_i,t_j} \delta_{P_i,t_i}^{n-1-\tau} \]

\[ = K(P_i) \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_i,t_j} \delta_{P_i,t_i}^{n-1-\tau} + O(\tau^2), \quad (3.28) \]

and by (B.19),

\[ \int_{S^n} K \delta_{P_j,t_j}^{n-1-\tau} \frac{\partial \delta_{P_j,t_i}}{\partial t_i} = \frac{\partial}{\partial t_i} \int_{S^n} K \delta_{P_i,t_i} \delta_{P_j,t_j}^{n-1-\tau} \]

\[ = K(P_j) \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_i,t_i} \delta_{P_j,t_j}^{n-1-\tau} + \frac{\partial}{\partial t_i} \int_{S^n} (K - K(P_j)) \delta_{P_i,t_i} \delta_{P_j,t_j}^{n-1-\tau} \]

\[ = K(P_j) \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_i,t_i} \delta_{P_j,t_j}^{n-1-\tau} + O(\tau^2). \quad (3.29) \]

Thus, from (3.28), (3.29), (B.3), (B.4), and (B.5), we get

\[ \frac{\partial}{\partial t_i} I_r \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right) \]

\[ = \Gamma(n - 1) \left( \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P,j,t_j} \delta_{P_i,t_i}^{n-1} - \int_{S^n} K \alpha^a_i \delta_{P_i,t_i}^{n-1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \right) \]

\[ - \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_i,t_i}^{n-1-\tau} \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_j,t_j} \delta_{P_i,t_i}^{n-1} \]

\[ - \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_i,t_i} \delta_{P_j,t_j}^{n-1-\tau} + o(\tau^{3/2}) + O(\tau^{3/2} \log \tau \| v \|_\sigma) + O(\tau^{1/2} \| v \|_\sigma^2) \]

\[ = \Gamma(n - 1) \left( \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P,j,t_j} \delta_{P_i,t_i}^{n-1} - \frac{1}{n - \tau} \int_{S^n} K(P_i) \alpha_i^n \frac{\partial \delta_{P_i,t_i}^{n-\tau}}{\partial t_i} \right) \]

\[ - \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_i,t_i}^{n-\tau} \frac{\partial}{\partial t_i} \int_{S^n} |P - P_i|^2 \alpha^n_i \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \]

\[ - \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_i,t_i} \delta_{P_j,t_j}^{n-1-\tau} + o(\tau^{3/2}) + O(\tau^{3/2} \log \tau \| v \|_\sigma) + O(\tau^{1/2} \| v \|_\sigma^2) \]

\[ = \Gamma(n - 1) \sum_{j \neq i} \{ \alpha_i \alpha_j - \alpha_i \alpha_j K(P_i) - \alpha_i \alpha_j K(P_j) \} \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P,j,t_j} \delta_{P_i,t_i}^{n-1} \]

\[ - \frac{\Gamma(n - 1)}{n - \tau} \alpha^n_i K(P_i) \frac{\partial}{\partial t_i} \int_{S^n} \delta_{P_i,t_i}^{n-\tau} \]
\[-2\Gamma(n-1)\frac{n(n-\tau)}{\Delta_{\gamma_0}K(P_i)\alpha_i^2}\frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} |P - P_i|^2 \delta_{P_i,t_i}^{n-\tau} + o(\tau^{3/2}) + O(\tau^{3/2} |\log \tau||v||_\sigma) + O(\tau^{1/2}||v||_\sigma^2)\]

\[= -\Gamma(n-1) \sum_{j \neq i} \frac{1}{(K(P_i)K(P_j))^{1/2\sigma}} \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} \delta_{P_j,t_j} \delta_{P_i,t_i}^{n-1}\]

\[-\Gamma(n-1) \frac{1}{n} \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} \delta_{P_i,t_i}^{n-\tau}\]

\[-\frac{2\Gamma(n-1)}{n^2} \frac{\Delta_{\gamma_0}K(P_i)}{K(P_i)^{n/2\sigma}} \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} |P - P_i|^2 \delta_{P_i,t_i}^{n-\tau} + o(\tau^{3/2}) + O(\tau^{3/2} |\log \tau||v||_\sigma) + O(\tau^{1/2}||v||_\sigma^2) + O(\tau^{1/2}||v||_\sigma^2) + O(\tau^{1/2}||v||_\sigma^2),\]

where |P - P_i| represents the distance between two points P and P_i after through a stereographic projection with P_1 as the south pole of S^n.

It follows that

\[\frac{\partial}{\partial t_i} I_\tau \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right) \]

\[= \Theta_1 \frac{1}{K(P_i)^{1/\sigma}} t_i + \Theta_2 \frac{\Delta_{\gamma_0}K(P_i)}{K(P_i)^{n/2\sigma}} \frac{1}{t_i^2}, \]

\[+ \Theta_3 \sum_{j \neq i} \frac{G_{P_i}(P_j)}{(K(P_i)K(P_j))^{1/2\sigma}} \frac{1}{t_i^2 t_j} + V_{t_i}(\tau, \alpha, t, P, v), \] (3.30)

where

\[\Theta_1 = 2^{n-2}\Gamma(n-1)|\mathbb{S}^{n-1}| \frac{n-2}{n(n-1)} B\left(\frac{n}{2}, \frac{n}{2} - 1\right),\]

\[\Theta_2 = 2^n\Gamma(n-1)|\mathbb{S}^{n-1}| \frac{1}{n(n-1)} B\left(\frac{n}{2}, \frac{n}{2} - 1\right),\]

\[\Theta_3 = 2^n\Gamma(n-1)|\mathbb{S}^{n-1}| B\left(\frac{n}{2}, \frac{n}{2} - 1\right),\]

and

\[V_{t_i}(\tau, \alpha, t, P, v) = o(\tau^{3/2}) + O(\tau||v||_\sigma) + O(\tau^{1/2}||v||^2_\sigma) + O(\tau^{1/2}||v||^2_\sigma) + O(\tau^{1/2}||v||^2_\sigma).\]

Proposition 3.4 follows from the above. \(\square\)

**Proposition 3.5.** Under the hypotheses of Proposition 3.3, then for any \((\alpha, t, P, v) \in \Sigma_{\tau}(\overline{P_1}, \ldots, \overline{P_k})\), we have

\[\frac{\partial}{\partial P_i} I_\tau \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right) = -\Theta_4 \nabla_{\gamma_0}K(P_i) + V_{P_i}(\tau, \alpha, t, P, v),\]

where \(\Theta_4 \geq \nu_1 > 0\) is a constant, and

\[V_{P_i}(\tau, \alpha, t, P, v) = O(\tau^{1/2}) + O(||v||_\sigma) + O(\tau^{-1/2}||v||^2_\sigma).\]
Proof. Using (3.10) and Lemma B.1, we have

\[
\frac{\partial}{\partial P_i} I_\tau \left( \sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v \right)
= \Gamma(n-1) \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial P_i} \int_{S^n} \delta_{P_j,t_j}^{n-1} \delta_{P_i,t_i}
- \Gamma(n-1) \int_{S^n} K \left[ \sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v \right]^{n-\tau} \left( \sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v \right) \alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i}
= \Gamma(n-1) \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial P_i} \int_{S^n} \delta_{P_j,t_j}^{n-1} \delta_{P_i,t_i}
- \Gamma(n-1) \int_{S^n} K \left( \sum_{j=1}^k \alpha_j \delta_{P_j,t_j} \right)^{n-\tau} \alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i}
- \Gamma(n-1) \frac{1}{n-1} \int_{S^n} K \left( \sum_{j=1}^k \alpha_j \delta_{P_j,t_j} \right)^{n-\tau} v \alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i}
+ O \left( \|v\|_\sigma^2 \left\| \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \right\|_\sigma \right).
\]

(3.31)

It follows from Lemma B.2 that

\[
\left( \sum_{j=1}^k \alpha_j \delta_{P_j,t_j} \right)^{n-\tau} = \left( \alpha_i \delta_{P_i,t_i} + \sum_{j \neq i} \alpha_j \delta_{P_j,t_j} \right)^{n-\tau}
= \left( \alpha_i \delta_{P_i,t_i} \right)^{n-\tau} + O \left( \sum_{j \neq i} \delta_{P_i,t_i}^{n-\tau} \delta_{P_j,t_j} + \sum_{j \neq i} \delta_{P_j,t_j}^{n-\tau} \right).
\]

By (B.22), (B.9), (B.10), (B.14), and (B.11), we have

\[
\int_{S^n} K \left( \sum_{j=1}^k \alpha_j \delta_{P_j,t_j} \right)^{n-\tau} v \alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i}
= \int_{S^n} K \left( \alpha_j \delta_{P_j,t_j} \right)^{n-\tau} v \alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} + O \left( \sum_{j \neq i} \int_{S^n} \delta_{P_i,t_i}^{n-\tau} \delta_{P_j,t_j} \left| \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \right| \|v\| \right)
+ O \left( \sum_{j \neq i} \int_{S^n} \delta_{P_j,t_j}^{n-\tau} \left| \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \right| \|v\| \right)
= \int_{S^n} K(P_i) \left( \alpha_j \delta_{P_j,t_j} \right)^{n-\tau} v \alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} + O \left( \int_{S^n} |P - P_i| \delta_{P_i,t_i}^{n-\tau} \left| \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \right| \|v\| \right)
+ O(\tau^{1/2}\|v\|_\sigma) + O(\tau\|v\|_\sigma)
= O \left( \int_{S^n} \left| \delta_{P_i,t_i}^{n-\tau} - \delta_{P_i,t_i}^{n-\tau} \right| \left| \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \right| \|v\| \right) + O(\|v\|_\sigma)
\]
\[ = O(\|v\|_\sigma). \]

From Lemma B.2, we have
\[
\left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} \right)^{n-1-\tau} = \left( \alpha_i \delta_{P_i,t_i} + \sum_{j \neq i} \alpha_j \delta_{P_j,t_j} \right)^{n-1-\tau} = (\alpha_i \delta_{P_i,t_i})^{n-1-\tau} + \left( \sum_{j \neq i} \alpha_j \delta_{P_j,t_j} \right)^{n-1-\tau} + (n-1-\tau) \alpha_i \delta_{P_i,t_i}^{n-2-\tau} \left( \sum_{j \neq i} \alpha_j \delta_{P_j,t_j} \right) + O \left( \sum_{j \neq i} \delta_{P_i,t_i}^{n-3-\tau} \delta_{P_j,t_j}^{2} \right),
\]

then, by using (B.9) and (B.23), (B.21), (B.24), we can obtain
\[
\frac{\partial}{\partial P_i} I_{\tau} \left( \sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v \right) = \Gamma(n-1) \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial P_i} \int_{\mathbb{S}^n} \delta_{P_j,t_j}^{n-1} \delta_{P_i,t_i}^{n-1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
- \Gamma(n-1) \alpha_i \int_{\mathbb{S}^n} K(\alpha_i \delta_{P_i,t_i})^{n-1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
- \Gamma(n-1) \alpha_i (n-1-\tau) \int_{\mathbb{S}^n} K \left( \sum_{j \neq i} \alpha_j \delta_{P_j,t_j} \right) \left( \alpha_i \delta_{P_i,t_i} \right)^{n-2-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
- \Gamma(n-1) \alpha_i \int_{\mathbb{S}^n} \left( \sum_{j \neq i} \alpha_j \delta_{P_j,t_j} \right)^{n-1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial P_i} + O(\|v\|_\sigma) + O(\tau^{-1/2} \|v\|_\sigma^2) + O(\tau^{3/2}) \\
= - \Gamma(n-1) \alpha_i \int_{\mathbb{S}^n} K(\delta_{P_i,t_i})^{n-1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial P_i} + O(\tau^{1/2}) + O(\|v\|_\sigma) + O(\tau^{-1/2} \|v\|_\sigma^2) \\
= - \Theta_4(\tau, \alpha, t, P, v) \nabla K(P_i) + V_{P_i}(\tau, \alpha, t, P, v),
\]

where
\[
\Theta_4(\tau, \alpha, t, P, v) \geq \nu_1 > 0 \quad \text{with } \nu_1 \text{ independent of } \tau,
\]

and
\[
V_{P_i}(\tau, \alpha, t, P, v) = O(\tau^{1/2}) + O(\|v\|_\sigma) + O(\tau^{-1/2} \|v\|_\sigma^2). \tag{3.32}
\]

We now prove that the existence of \( \nu_1 \). Let \( P_i \) be the south pole and make a stereographic projection \( F \) to the equatorial plane of \( \mathbb{S}^n \) with \( y = (y(1), \cdots, y(n)) \) as
the stereographic projection coordinates, let \( \tilde{K} = K(F(y)) \) and \( |J_F| := (2/(1+|y|^2))^n \). Then we have \( F(0) = P_i \) and
\[
\int_{S^n} K \delta_{P_i,t_i} (\partial \delta_{P_i,t_i} / \partial P_i) = \int_{[0,t_i]} \omega_{\tilde{y},t_i} (\nabla \tilde{K}) \cdot y + O(|y|^2) g_t(y) \frac{\partial \omega_{\tilde{y},t_i}}{\partial y_i} =: \mathcal{L} = (\mathcal{L}^{(1)}, \cdots, \mathcal{L}^{(n)}),
\]
where \( \omega_{\tilde{y},t_i}(y) = \frac{2t_i}{1 + t_i^2 |y|^2} \), and \( g_t(y) := (\omega_{\tilde{y},t_i}^{-1} |J_F|^{1/n})^\tau \). For \( \ell = 1, \cdots, n \), we have
\[
\mathcal{L}^{(\ell)} = \int_{[0,t_i]} \omega_{\tilde{y},t_i} (\nabla \tilde{K}) \cdot y + O(|y|^2) g_t(y) \frac{\partial \omega_{\tilde{y},t_i}}{\partial y_i} = \int_{[0,t_i]} t_i \omega_{\tilde{y},t_i} (\nabla \tilde{K}) + O(|y|^2) g_t(y) \frac{\partial \omega_{\tilde{y},t_i}}{\partial y_i} = \frac{1}{n} \frac{\partial \tilde{K}}{\partial y(\ell)} (0) = \int_{[0,t_i]} t_i |y|^2 \omega_{\tilde{y},t_i} g_t(y) + O(\tau^{1/2}),
\]
thus,
\[
\mathcal{L} = \nabla g_t K(P_i) + \frac{2}{n} \int_{S^n} t_i |y|^2 \omega_{\tilde{y},t_i} g_t(y) + O(\tau^{1/2}).
\]
It follows from \( t_i^{-\tau} \leq g_t(y) \leq t_i^\tau \) that
\[
\int_{S^n} t_i |y|^2 \omega_{\tilde{y},t_i} g_t(y) \geq \int_{S^n} t_i |y|^2 \omega_{\tilde{y},t_i} \rightarrow \int_{[0,t_i]} 2^{n+1} \left( \frac{2^{n+1}}{1 + |y|^2} \right)^n
\]
as \( \tau \to 0 \). This ensures the existence of \( \nu_1 \). We have proved Proposition 3.5. \( \square \)

By using Propositions 3.2, 3.3, 3.4, 3.5, and constructing a family of homotopy Id+compact operators, we will obtain the degree-counting formula of the solutions to the subcritical equation \( (I + \text{compact operators}) \) on \( \Sigma_{\tau}(\overline{P}_1, \cdots, \overline{P}_k) \).

**Proof of Theorem 3.1.** The \( \mathcal{K}^- \) be as in \( (1.5) \) for the given \( K \) and \( \Sigma_{\tau}(\overline{P}_1, \cdots, \overline{P}_k) \) be as in \( (3.6) \) for the given \( \overline{P}_1, \cdots, \overline{P}_k \in \mathcal{K}^- \).

For \( u = \sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v \in \Sigma_{\tau}(\overline{P}_1, \cdots, \overline{P}_k) \), we have
\[
T_u H^\sigma(S^n) = E_{P,t} \bigoplus \text{span} \{ \delta_{P_i,t_i}, \partial \delta_{P_i,t_i} / \partial t_i, \partial \delta_{P_i,t_i} / \partial P_i \}.
\]
Since \( I_{\tau}'(u) \in T_u H^\sigma(S^n) \), there exist \( \xi \in E_{P,t}, \eta \in \text{span} \{ \delta_{P_i,t_i}, \partial \delta_{P_i,t_i} / \partial t_i, \partial \delta_{P_i,t_i} / \partial P_i \} \) such that
\[
I_{\tau}'(u) = \xi + \eta.
\]
By Lemma 3.3, we have
\[ \langle \xi, \varphi \rangle = I'_\tau(u)\varphi = f_\tau(\varphi) + 2Q_\tau(v, \varphi) + \langle V_\nu(\tau, \alpha, t, P, v), \varphi \rangle, \quad \forall \varphi \in E_{P,t}, \]  
where \( \| V_\nu(\tau, \alpha, t, P, v) \|_\sigma \leq C\| v \|_\sigma^2 \). Replacing \( \varphi \) by \( v \) in (3.33) and using (3.14), we have
\[ \| \xi \|_\sigma \geq \delta_0\| v \|_\sigma - \| f_\tau \| - O(\| v \|_\sigma^2) \geq \frac{\delta_0}{2}\| v \|_\sigma - \| f_\tau \|, \]  
where \( \delta_0 \) is as in (3.14). Let \( \beta = (\beta_1, \cdots, \beta_k) \), \( \beta_i = \alpha_i - K(P_i)^{-1/2} \) be as in Proposition 3.3, we define
\[ \hat{\Sigma}_\tau = \left\{ u = \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} + v \in \Sigma_\tau(P_1, \cdots, P_k) : \| v \|_\sigma < \tau |\log \tau|^3, |\beta| < \tau |\log \tau| \right\}. \]
It follows from Proposition 3.2 and (3.22) that
\[ I'_\tau(u) \neq 0, \quad \forall u \in \Sigma_\tau(P_1, \cdots, P_k) \setminus \hat{\Sigma}_\tau. \]
For any \( u = \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} + v \in \hat{\Sigma}_\tau \), by (3.10) and Proposition 3.3, we have
\[ \langle \eta, \delta_{P_i, t_i} \rangle = I'_\tau(u)\delta_{P_i, t_i} \]
\[ = \Gamma(n - 1) \left( \alpha_i \int_{\mathbb{S}^n} \delta_{P_i, t_i}^n + \sum_{j=1}^k \alpha_j \int_{\mathbb{S}^n} \delta_{P_i, t_i}^{n-1} \delta_{P_j, t_j} \right) \]
\[ - \Gamma(n - 1) \int_{\mathbb{S}^n} K \left| \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} + v \right|^{n-2}\left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} + v \right) \delta_{P_i, t_i} \]
\[ = \frac{\partial}{\partial \alpha_i} I_\tau \left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} + v \right) \]
\[ = -2\sigma \| \delta_{P_i, t_i} \|_2^2 \beta_i + V_{\alpha_i}(\tau, \alpha, t, P, v), \]
and
\[ V_{\alpha_i}(\tau, \alpha, t, P, v) = O(\tau |\log \tau|). \]  
It follows from (3.23) and (3.30) that
\[ \langle \eta, \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \rangle = I'_\tau(u) \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \]
\[ = \frac{1}{\alpha_i} \frac{\partial}{\partial t_i} I_\tau \left( \sum_{j=1}^k \alpha_j \delta_{P_j, t_j} + v \right) \]
\[ = \frac{1}{\alpha_i} \left\{ \Theta_1 \frac{1}{K(P_i)^{1/2}} \frac{\tau}{t_i} + \Theta_2 \frac{\Delta_{\nu_i} K(P_i) 1}{K(P_i)^{n/2}} \frac{1}{t_i} \right\} \]
\[ + \Theta_3 \sum_{j \neq i} \frac{G_{P_j}(P_j)}{(K(P_i) K(P_j))^{1/2}} \frac{1}{t_j} + V_{t_i}(\tau, \alpha, t, P, v) \} \].
where
\[ |V_t^i(\tau, \alpha, t, P, v)| = o(\tau^{3/2}). \tag{3.36} \]

By (3.31) and (3.32), we obtain
\[
\langle \eta, \frac{\partial \delta_{P,t_i}}{\partial P_i} \rangle = I'_t(u) \frac{\partial \delta_{P,t_i}}{\partial P_i} \\
= \frac{1}{\alpha_i} \frac{\partial}{\partial P_i} I_t \left( \sum_{i=1}^k \alpha_i \delta_{P,t_i} + v \right) \\
= \frac{1}{\alpha_i} \left\{ -\Theta_4 \nabla_{g_0} K(P_i) + V_{P_i}(\tau, \alpha, t, P, v) \right\},
\]
with \( V_{P_i} \) satisfying
\[ |V_{P_i}(\tau, \alpha, t, P, v)| = O(\tau^{1/2}). \tag{3.37} \]

Using the estimates stated above, we define a family of operators on \( \widehat{\Sigma}_\tau \) as follows: for any \( u = \sum_{i=1}^k \alpha_i \delta_{P,t_i} + v \in \widehat{\Sigma}_\tau \),
\[ X_\theta(u) := \xi(u) + \eta_\theta(u), \quad 0 \leq \theta \leq 1, \]
where, for any \( \varphi \in E_{P,t} \),
\[ \langle \xi_\theta, \varphi \rangle := \theta f_\tau(\varphi) + (1 - \theta)\langle \varphi, \phi \rangle + 2\theta Q_\tau(\varphi, v) + \theta \langle V_v(\tau, \alpha, t, P, v), \varphi \rangle, \tag{3.38} \]
and
\[
\langle \eta_\theta, \delta_{P,t_i} \rangle := -2 \sigma \| \delta_{P,t_i} \|^2 \left\{ \alpha_i - \frac{\theta}{K(P_i)^{1/2}} - \frac{1 - \theta}{K(P_i)^{1/2}} \right\} \\
+ \theta V_{\alpha_i}(\tau, \alpha, t, P, v),
\]
\[
\langle \eta_\theta, \frac{\partial \delta_{P,t_i}}{\partial t_i} \rangle := \left\{ \frac{\tau}{\alpha_i} + (1 - \theta) \right\} \left\{ \frac{\Theta_1}{K(P_i)^{1/2} t_i} + \frac{\Theta_2 \Delta_{g_0} K(P_i(\theta))}{K(P_i(\theta))^{n/2} t_i^3} \right\} \\
+ \sum_{j \neq i} \frac{\Theta_3 G_{P_i(\theta)}(P_j(\theta))}{K(P_i(\theta))^{1/2} t_i^{1/2} t_j^3} + \frac{\theta}{\alpha_i} V_{t_i}(\tau, \alpha, t, P, v),
\]
\[
\langle \eta_\theta, \frac{\partial \delta_{P,t_i}}{\partial P_i} \rangle := - \left\{ (1 - \theta) + \frac{\tau}{\alpha_i} \Theta_4 \right\} \nabla_{g_0} K(P_i) + \frac{\theta}{\alpha_i} V_{P_i}(\tau, \alpha, t, P, v),
\]
where \( P_i(\theta) \) is the short geodesic trajectory on \( S^n \) with \( P_i(0) = \overline{P}_i, P_i(1) = P_i \).

Obviously, \( X_1 = I'_t(u) = \xi + \eta \). It is well known from (3.2) that \( I'_t(u) \) is of the form \( \text{Id} + \text{compact} \) on \( H^s(S^n) \). From Sobolev compact imbedding theorem, the explicit forms of \( V_v, V_{\alpha_i}, V_{t_i}, V_{P_i}, A^{-2} < t_i^2 \tau < A^2 \), (3.35), (3.36), (3.37), and \( \Omega_{t_0/2} \) in the definition of \( \widehat{\Sigma}_\tau \) is a finite dimensional submanifold of \( H^s(S^n) \), we can conclude that \( X_\theta \) (0 \leq \theta \leq 1) is the form \( \text{Id} + \text{compact} \). Furthermore, we have \( X_\theta \neq 0 \) on \( \partial \widehat{\Sigma}_\tau \), \( \forall 0 \leq \theta \leq 1 \). In fact, for a given \( u = \sum_{i=1}^k \alpha_i \delta_{P,t_i} + v \in \partial \widehat{\Sigma}_\tau \), we obtain \( \xi \neq 0 \) by
using (3.34) and (3.19). When \( \theta = 0, \xi_0 = v \neq 0 \). It follows from (3.38) that \( \xi_\theta \neq 0, \forall 0 < \theta < 1 \). By the homotopy invariance of the Leray-Schauder degree, we have

\[
\deg_{H^s}(X_1, \hat{\Sigma}_r, 0) = \deg_{H^s}(X_0, \hat{\Sigma}_r, 0). \tag{3.40}
\]

It is easily seen from (3.38) and (3.39) that for any \( u = \sum_{i=1}^{k} \alpha_i \delta_{P_i, t_i} + v \in \hat{\Sigma}_r, \)

\[
X_0(u) = \xi_0(u) + \eta_0(u),
\]

where \( \xi_0 \in E_{P,t}, \eta_0 \in \text{span}\{\delta_{P_i, t_i}, \frac{\partial \delta_{P_i, t_i}}{\partial t_i}, \frac{\partial \delta_{P_i, t_i}}{\partial P_i}\} \)

satisfy

\[
\langle \xi_0, \varphi \rangle = \langle v, \varphi \rangle,
\]

\[
\langle \eta_0, \delta_{P_i, t_i} \rangle = -\beta_2 \gamma \|\delta_{P_i, t_i}\|_2^2 (\alpha_i - K(\hat{\mathcal{P}}_i)^{-1/2\sigma}),
\]

\[
\left\langle \eta_0, \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \right\rangle = \frac{\Theta_1}{K(\hat{\mathcal{P}}_i)^{1/\sigma} t_i} + \frac{\Theta_2 \Delta_{g_0} K(\hat{\mathcal{P}}_i) 1}{K(\hat{\mathcal{P}}_i)^{n/2\sigma} t_i^2} + \sum_{j \neq i} \frac{\Theta_3 G_{\hat{\mathcal{P}}_i}(\hat{\mathcal{P}}_j)}{(K(\hat{\mathcal{P}}_i) K(\hat{\mathcal{P}}_j))^{1/2\sigma} t_i^2 t_j}, \tag{3.41}
\]

\[
\left\langle \eta_0, \frac{\partial \delta_{P_i, t_i}}{\partial P_i} \right\rangle = -\nabla_{g_0} K(\hat{\mathcal{P}}_i).
\]

Recalling the definition of \( M(\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_k) \). From the above, we can easily get

\[
X_0(u) = 0 \quad \text{on } \hat{\Sigma}_r,
\]

if and only if

\[
\alpha_i = K(\hat{\mathcal{P}}_i)^{-1/2\sigma}, \quad P_i = \hat{\mathcal{P}}_i, \quad v = 0, \tag{3.42}
\]

\[
\frac{\sigma}{4K(\hat{\mathcal{P}}_i)^{1/\sigma}} t_i - \sum_{j=1}^{k} M_{ij}(\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_k) \frac{1}{t_i^2 t_j} = 0.
\]

For any \((s_1, \ldots, s_k) \in \mathbb{R}^k, s_i > 0, i = 1, \ldots, k\), we define

\[
F(s_1, \ldots, s_k) := -\frac{\sigma \tau}{4} \sum_{j=1}^{k} \frac{1}{K(\hat{\mathcal{P}}_j)^{1/\sigma}} \log s_j + \frac{1}{2} \sum_{i,j=1}^{k} M_{ij}(\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_k) s_i s_j,
\]

and for \( t_i = s_i^{-1} \),

\[
\hat{F}(t_1, \ldots, t_k) := F(s_1, \ldots, s_k).
\]

The derivative with respect to \( t_i \) is

\[
\frac{\partial \hat{F}}{\partial t_i}(t_1, \ldots, t_k) = \frac{\sigma \tau}{4K(\hat{\mathcal{P}}_i)^{1/\sigma} t_i} - \sum_{j=1}^{k} M_{ij}(\hat{\mathcal{P}}_1, \ldots, \hat{\mathcal{P}}_k) \frac{1}{t_i^2 t_j},
\]

combining this and (3.41), we have

\[
\left\langle \eta_0, \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \right\rangle = \frac{\partial \hat{F}}{\partial t_i}(t_1, \ldots, t_k).
\]
It is obvious that $\nabla \tilde{F}(t_1, \cdots, t_k) = 0$ if and only if $\nabla F(s_1, \cdots, s_k) = 0$. Since
\[ \mu(M(\bar{P}_1, \cdots, \bar{P}_k)) > 0, \]
a trivial verification shows that $F(s_1, \cdots, s_k)$ is a strictly convex function, and having a unique critical point in the first quadrant. It follows that $\tilde{F}(t_1, \cdots, t_k)$ has unique critical point in the first quadrant with Morse index zero. Hence $X_0$ has precisely one non-degenerate zero in $\tilde{S}_\tau$. Furthermore, by (3.42) we can easily obtain
\begin{equation}
\deg_{H^\sigma}(X_0, \tilde{S}_\tau, 0) = (-1)^{k + \sum_{i=1}^{k} i(\bar{P}_i)}.
\end{equation}
(3.43)
Combining (3.43) and (3.40), we complete the proof of Theorem 3.1.

Recall the definition of $\mathcal{O}_R$ in (1.12). For $\delta > 0$ suitably small, define
\begin{equation}
\mathcal{O}_{R, \delta} := \{ u \in H^\sigma(S^n) : \inf_{\omega \in \mathcal{O}_R} \| u - \omega \|_\sigma < \delta \}.
\end{equation}
(3.44)

**Proposition 3.6.** Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$, and $n = 2\sigma + 2$. Let $K \in \mathcal{A}$ be a Morse function and $0 < \tau_0 \leq \tau \leq 4/(n - 2\sigma) - \tau_0$. Then there exists some constants $C_0 > 0$, $\delta_0 > 0$ depending only on $\tau_0$ and $K$, such that
\begin{equation}
\{ u \in H^\sigma(S^n) : u > 0 \text{ a.e., } I'_\tau(u) = 0 \} \subset \mathcal{O}_{C_0, \delta_0}.
\end{equation}
(3.45)
Furthermore, we have $I'_\tau(u) \neq 0$ on $\partial \mathcal{O}_{C_0, \delta_0}$ and
\begin{equation}
\deg_{H^\sigma}(u - P_\sigma^{-1}(\Gamma(n - 1)K|u|^{\frac{4\sigma}{n - 2\sigma} - \tau} u), \mathcal{O}_{C_0, \delta_0}, 0) = -1.
\end{equation}
(3.46)

**Proof.** From Proposition 3.1, we know that for $\tau > 0$ small there exists some suitable value of $\nu_0$, $A, R$ such that $u$ satisfying $u \in H^\sigma(S^n)$, $u > 0$, a.e., $I'_\tau(u) = 0$ are either in $\mathcal{O}_R$ or in some $\Sigma_\tau(q^{(1)}, \cdots, q^{(k)})$. Combining (3.6), (3.5), (B.1), and (B.6), we conclude that there exists some positive constants $C_0$ and $\delta_0$ such that (3.45) holds.

For $K^*(x) = x_{(n+1)} + 2$, $x = (x_{(1)}, \cdots, x_{(n+1)}) \in S^n \subset \mathbb{R}^{n+1}$ and $t \in (0, 1)$, we consider $K_t = tK + (1 - t)K^*$. By the homotopy invariance of the Leray-Schauder degree, we only need to establish (3.46) for $K^*$ and $\tau$ very small. It is easy to see that $K^* \in \mathcal{A}$ is a Morse function. The proof of (3.46) is straightforward by the Kazdan-Warner condition, Theorem 3.1, and a homotopy argument.

**3.2 Proof of Theorems 1.2, 1.3 and 1.4**

Using Theorem 3.1 and Proposition 3.6, we next prove Theorem 1.2.

**Proof of Theorem 1.2.** The existence of $R_0$ can be easily obtained from Theorem 1.1. For all $R \geq R_0$, using Theorem 1.1, Proposition 3.1, and By the homotopy invariance of the Leray-Schauder degree, we have
\begin{equation}
\deg_{C^{2, \alpha}, \alpha}(u - P_\sigma^{-1}(\Gamma(n - 1)Ku^{n-1}), \mathcal{O}_R, 0) = \deg_{C^{2, \alpha}, \alpha}(u - P_\sigma^{-1}(\Gamma(n - 1)K|u|^{2\sigma - \tau} u), \mathcal{O}_R, 0)
\end{equation}
(3.47)
for \( \tau > 0 \) sufficiently small.

Let \( C_0 \gg R, 0 < \delta_1 \ll \delta_0 \), and \( \tau_0 \) be given by Proposition 3.6. Using (3.46), Proposition 3.1, (3.7), (1.10), and the excision property of the degree, we have

\[
\deg_{H^s}(u - P_{\sigma}^{-1}(\Gamma(n - 1)K|u|^{2\sigma - \tau}u), \partial R, \delta_1, 0) = \text{Index}(K). \tag{3.48}
\]

As in the proof of Proposition 3.6, one can check that there are no critical points of \( I_{\tau} \) in \( \overline{\partial R, \delta_1} \). Using the same proof idea as Li [39, Theorem B.2] and [31, Theorems 2.4 and 2.5], we can easily get

\[
\deg_{C_{2\sigma}, \sigma}(u - P_{\sigma}^{-1}(\Gamma(n - 1)K|u|^{2\sigma - \tau}u), \partial R, 0) = \deg_{H^s}(u - P_{\sigma}^{-1}(\Gamma(n - 1)K|u|^{2\sigma - \tau}u), \partial R, \delta_1, 0). \tag{3.49}
\]

It follows from (3.47)–(3.49) that for \( R \geq R_0 \), (1.13) is proved. Theorem 1.2 follows from the above. \( \Box \)

Using Theorem 1.2 and perturbing the prescribing function near its critical point, we can know exactly where the blow up occur when \( K \notin \mathcal{A} \).

**Proof of the Theorem 1.3.** Since the Morse functions in \( C^2(\mathbb{S}^n)^* \setminus \mathcal{A} = \partial \mathcal{A} \) are dense in \( \partial \mathcal{A} \), without loss of generality we consider the case that \( K \in \partial \mathcal{A} \) is a Morse function. First recall the definition of \( \mathcal{K} \) and \( \mathcal{K}^+ \), we can assume here \( \mathcal{K} \setminus \mathcal{K}^+ = \{q^{(1)}, \ldots, q^{(m)}\}, m \in \mathbb{N}_+ \). From the definition of \( \mathcal{A} \) and \( K \in \partial \mathcal{A} \), we know that there exists \( 1 \leq i_1 < \cdots < i_k \leq m, k \geq 1 \), such that

\[
\mu(M(q^{(i_1)}, \ldots, q^{(i_k)}))) = 0. \tag{3.50}
\]

**Case 1:** There is only one such \( \{q^{(i_1)}, \ldots, q^{(i_k)}\} \) satisfying (3.50). Using the same \( C^2 \) perturbation method as in Li [40, 41], we can obtain a smooth, one-parameter family of Morse functions \( \{K_t\} \) \((-1 \leq t \leq 1\) with the following properties:

(a) \( K_t \) \((-1 \leq t \leq 1\) are identically the same as \( K \) except in some small balls around \( q^{(i_1)}, \ldots, q^{(i_k)} \) and \( K_0 = K \). \( K_t \) have the same critical points with the same Morse index for any \(-1 \leq t \leq 1\).

(b) \( \mu(M(K_t; q^{(i_1)}, \ldots, q^{(j_s)})) \) have the same sign for \(-1 < t < 1\) for any \( 1 \leq j_1 < \cdots < j_s \leq m, (j_1, \ldots, j_s) \neq (i_1, \ldots, i_k) \). Furthermore,

\[
\mu(M(K_t; q^{(i_1)}, \ldots, q^{(i_k)})) \begin{cases} < 0, & \text{if } -1 < t < 0, \\ = 0, & \text{if } t = 0, \\ > 0, & \text{if } 0 < t < 1. \end{cases}
\]

It is easily seen that \( K_t \in \mathcal{A} \) when \( t \neq 0 \). From the definition of Index, we have

\[
\text{Index}(K_1) = \text{Index}(K_{-1}) + (-1)^{k - 1 + \sum_{j=1}^{k} i(j)}.
\]
evidently, Index($K_1$) $\neq$ Index($K_{-1}$).

By the homotopy invariance of the Leray-Schauder degree and Theorem 1.2, there exists $t_i$ and $v_i \in M_{K_{t_i}}$ such that
\[ \lim_{i \to \infty} \|v_i\|_{C^{2,\alpha}(\mathbb{S}^n)} = \infty \] or \[ \lim_{i \to \infty} (\min_{\mathbb{S}^n} v_i) = 0. \]

In fact, we can prove that if \( \lim_{i \to \infty} (\min_{\mathbb{S}^n} v_i) = 0 \), then \( \lim_{i \to \infty} \|v\|_{C^{2,\alpha}(\mathbb{S}^n)} = \infty \). If not, it means that \( \{v_i\} \) has no blow up point, then \( v_i \equiv 0 \) on \( \mathbb{S}^n \) can be obtained from \( \lim_{i \to \infty} (\min_{\mathbb{S}^n} v_i) = 0 \) and Hanarck inequality. This leads to contradictions and we deduce that (1.15) holds.

It follows from \( K_t \in \mathscr{A} (t \neq 0) \) and Theorem 1.1 that \( t_i \to 0 \), namely, \( K_{t_i} \to K \). Then by Theorem 2.1, we can know that \( \{v_i\} \) blows up exactly at \( k \) points \( q^{(i_1)}, \ldots, q^{(i_k)} \).

Case 2: If \( \{q^{(i_1)}, \ldots, q^{(i_k)}\} \) satisfying (3.50) is not unique, we can perturb as described above the function \( K \) near its some critical points to change the Hessian matrix of \( K \) at these points, such that there exists a sequence of Morse functions \( K_\ell \) satisfying: \( K_\ell \to K \), \( K_\ell \) are identically the same as \( K \) except in some small balls and have the same critical points with the same Morse index; there is only one such \( (i_1, \ldots, i_k) \) such that (3.50) is true for any \( \ell \). From Case 1, we know that there exists a sequence of \( K_i \to K \) in \( C^2(\mathbb{S}^n) \), \( v_i \in M_{K_i} \) such that \( \{v_i\} \) blows up at precisely the \( k \) points \( q^{(i_1)}, \ldots, q^{(i_k)} \). We have thus proved Theorem 1.3.

Using Theorem 2.1 and the proof method of Theorem 1.3, we show Theorem 1.4 holds.

Proof of Theorem 1.4. By using Theorem 2.1 we can prove the Part (i) of Theorem 1.4. The Part (ii) of Theorem 1.4 is similar to the proof of Theorem 1.3, we omit it here.

A Appendix

In this section, we review some results about the local analysis and blow up profiles for nonlinear integral equations obtained in Jin-Li-Xiong [31]. For any \( x \in \mathbb{R}^n \) and \( r > 0 \), the symbol \( B_r(x) \) denotes the ball in \( \mathbb{R}^n \) with radius \( r \) and center \( x \), and \( B_r := B_r(0) \).

A.1 Hölder estimates and Schauder type estimates

Consider nonnegative solutions of the integral equation
\[ u(x) = \int_{\mathbb{R}^n} \frac{V(y)u(y)}{|x-y|^{n-2\sigma}} dy \quad \text{a.e in } B_3, \] \hspace{1cm} (A.1)
where \( 0 < \sigma < n/2 \).

The Hölder estimates for solutions to (A.1) is following:
Proposition A.1. For \( n \geq 1, 0 < \sigma < n/2, r > n/(n - 2\sigma) \) and \( p > n/2\sigma \), let \( 0 \leq V \in L^p(B_3), 0 \leq u \in L^r(B_3) \) and \( 0 \leq Vu \in L^1_{\text{loc}}(\mathbb{R}^n) \). If \( u \) satisfies (A.1), then \( u \in C^\alpha(B_1) \),

\[
\|u\|_{C^\alpha(B_1)} \leq C\|u\|_{L^r(B_3)},
\]

and \( u \) satisfies the Harnack inequality

\[
\max_{B_1} u \leq C\min_{B_1} u,
\]

where \( C > 0 \) and \( \alpha \in (0,1) \) depend only on \( n, \sigma, p, \) and an upper bound of \( \|V\|_{L^p(B_3)} \).

The Schauder type estimates for solutions \( u \) to (A.1) is following:

Proposition A.2. In addition to the assumptions in Proposition A.1, we assume that \( V \in C^\alpha(B_3) \) for some \( \alpha > 0 \) but not an integer, then \( u \in C^{2\sigma + \alpha'}(B_1) \) and

\[
\|u\|_{C^{2\sigma + \alpha'}(B_1)} \leq C\|u\|_{L^r(B_3)},
\]

where \( \alpha' = \alpha \) if \( 2\sigma + \alpha \notin \mathbb{N}_+ \), otherwise \( \alpha' \) can be any positive constant less than \( \alpha \). Here \( C > 0 \) depends only on \( n, \sigma, \alpha \) and an upper bound of \( \|V\|_{C^\alpha(B_1)} \).

A.2 Blow up profiles for nonlinear integral equations

Proposition A.3 (Pohozaev type identity). Let \( u \geq 0 \) in \( \mathbb{R}^n \), and \( u \in C(\overline{B}_R) \) be a solution of

\[
u(x) = \int_{B_R} \frac{K(y)u(y)^p}{|x - y|^{n-2\sigma}} \, dy + h_R(x),
\]

where \( 1 < p \leq \frac{n+2\sigma}{n-2\sigma} \), and \( h_R(x) \in C^1(B_R), \nabla h_R \in L^1(B_R) \). Then

\[
\begin{align*}
&\left(\frac{n-2\sigma}{2} - \frac{n}{p+1}\right) \int_{B_R} K(x)u(x)^{p+1} \, dx - \frac{1}{p+1} \int_{B_R} x\nabla K(x)u(x)^{p+1} \, dx \\
= &\frac{n-2\sigma}{2} \int_{B_R} K(x)u(x)^p h_R(x) \, dx + \int_{B_R} x\nabla h_R(x)K(x)u(x)^p \, dx \\
&- \frac{R}{p+1} \int_{\partial B_R} K(x)u(x)^{p+1} \, ds.
\end{align*}
\]

Proposition A.4. Suppose that \( 0 \leq u_i \in L^\infty_{\text{loc}}(\mathbb{R}^n) \) satisfies (2.1) with \( K_i \) satisfying (2.2). Suppose that \( x_i \to 0 \) is an isolated blow up point of \( \{u_i\} \), i.e., for some positive constants \( A_3 \) and \( \bar{r} \) independent of \( i \),

\[
|x - x_i|^{2\sigma/(p_i-1)}u_i(x) \leq A_3 \quad \text{for all } x \in B_{\bar{r}} \subset \Omega.
\]

Then for any \( 0 < r < \bar{r}/3 \), we have the following Harnack inequality

\[
\sup_{B_{2r}(x_i) \setminus B_{r/2}(x_i)} u_i \leq C \inf_{B_{2r}(x_i) \setminus B_{r/2}(x_i)} u_i,
\]

where \( C \) is a positive constant depending only on \( \sup_i \|K_i\|_{L^\infty(B_{\bar{r}}(x_i))}, n, \sigma, \bar{r} \) and \( A_3 \).
Proposition A.5. Assume the hypotheses in Proposition A.4. Then for every \( R_i \to \infty, \varepsilon_i \to 0^+ \), we have, after passing to a subsequence (still denoted as \( \{u_i\}, \{x_i\}, \) etc.), that
\[
\left\| m_i^{-1} u_i (m_i^{-1/2 \sigma} \cdot + x_i) - (1 + k_i \cdot |2|^{2 \sigma - n}/2) \right\|_{C^2(B_2 R_i(0))} \leq \varepsilon_i,
\]
where \( m_i := u_i(x_i) \) and \( k_i := (K_i(x_i) \pi^{n/2} \Gamma(\sigma) / \Gamma(\frac{n}{2} + \sigma))^{1/\sigma} \).

Proposition A.6. Under the hypotheses of Proposition A.5, there exists a positive constant \( C = C(n, \sigma, A_1, A_2, A_3) \) such that,
\[
u_i(x) \geq C^{-1} m_i(1 + k_i m_i^{(p_i - 1)/\sigma} |x - x_i|^2)^{2 \sigma - n}/2 \quad \text{for all} \quad |x - x_i| \leq 1.
\]
in particular, for any \( e \in \mathbb{R}^n, |e| = 1 \), we have
\[
u_i(x_i + e) \geq C^{-1} m_i^{-1 + ((n-2\sigma)/2) \tau_i}
\]
where \( \tau_i = (n + 2\sigma)/(n - 2\sigma) - p_i \).

Proposition A.7. Under the hypotheses of Proposition A.4 with \( \bar{r} = 2 \), and in addition that \( x_i \to 0 \) is only an isolated simple blow up point with constant \( \rho \), we have
\[
\tau_i = O(u_i(x_i)^{-c_1 + o(1)}) \quad \text{and} \quad u_i(x_i)^{\tau_i} = 1 + o(1),
\]
where \( c_1 = \min \{2, 2/(n-2\sigma)\} \). Moreover,
\[
\nu_i(x) \leq C u_i^{-1}(x_i) |x - x_i|^{2\sigma - n} \quad \text{for all} \quad |x - x_i| \leq 1.
\]

Proposition A.8. Under the hypotheses of Proposition A.7, let
\[
T_i(x) := u_i(x_i) \int_{B_1(x_i)} K_i(y) u_i(y)^{p_i} dy + u_i(x_i) \int_{\mathbb{R}^n \setminus B_1(x_i)} K_i(y) u_i(y)^{p_i} / |x - y|^{n-2\sigma} dy
\]
\[
= : T_i'(x) + T_i''(x).
\]
Then, after passing a subsequence,
\[
T_i'(x) \to a|x|^{2\sigma - n} \quad \text{in} \quad C^2_{\text{loc}}(B_1 \setminus \{0\})
\]
and
\[
T_i''(x) \to h(x) \quad \text{in} \quad C^2_{\text{loc}}(B_1)
\]
for some \( h(x) \in C^2(B_2) \), where
\[
a = \left( \frac{\pi^{n/2} \Gamma(\sigma)}{\Gamma(\frac{n}{2} + \sigma)} \right)^{-\frac{\sigma}{2}} \int_{\mathbb{R}^n} \left( \frac{1}{1 + |y|^2} \right)^{n/2 + \sigma} dy \lim_{i \to \infty} K_i(0)^{2\sigma - n}.
\]
Consequently, we have
\[
u_i(x_i) u_i(x) \to a|x|^{2\sigma - n} + h(x) \quad \text{in} \quad C^2_{\text{loc}}(B_1 \setminus \{0\}).
\]
Proposition A.9. Under the hypotheses of Proposition A.7, we have

\[
\int_{|x-x_i| \leq r_i} |x-x_i|^su_i(x)^{p_i+1} \, dx = \begin{cases} 
O(u_i(x_i)^{-2s/(n-2\sigma)}), & -n < s < n, \\
O(u_i(x_i)^{-2n/(n-2\sigma)} \log u_i(x_i)), & s = n, \\
O(u_i(x_i)^{-2n/(n-2\sigma)}), & s > n,
\end{cases}
\]
and

\[
\int_{r_i < |x-x_i| \leq 1} |x-x_i|^su_i(x)^{p_i+1} \, dx = \begin{cases} 
O(u_i(x_i)^{-2s/(n-2\sigma)}), & -n < s < n, \\
O(u_i(x_i)^{-2n/(n-2\sigma)} \log u_i(x_i)), & s = n, \\
O(u_i(x_i)^{-2n/(n-2\sigma)}), & s > n,
\end{cases}
\]

where \( r_i \) is as in Proposition A.5.

Proposition A.10. Let \( \sigma = 1 + m/2, m \in \mathbb{N}_+, n = 2\sigma + 2, \) and \( K_i \to K \) in \( C^2(B_3) \). Let \( p_i \leq \frac{n+2\sigma}{n-2\sigma} = n-1, p_i \to n-1, \) and \( \tau_i = n-1-p_i \). Let \( u_i(x) \) satisfy

\[
u_i(x) = \int_{\mathbb{R}^n} \frac{K_i(y)H(y)^{\tau_i}(y)u_i(y)^{p_i}}{|x-y|^2} \, dy \quad \text{for} \quad x \in B_3,
\]

where \( H(y) = 2/(1+|y|^2) \). Let \( x_i \to 0 \) is an isolated simple blow up point of \( \{u_i\} \) with constant \( A_3 \) and \( \rho \), i.e., \( |x-x_i|^{(p_i-1)/2\sigma}u_i(x) \leq A_3 \), and \( r_i^{\frac{p_i}{2\sigma}}\bar{u}_i(r) \) has precisely one critical point in \( (0, \rho) \) for large \( i \), where \( \bar{u}_i(r) = \int_{B_{r_i}(x_i)} u_i \, ds \).

Then there exists some constants \( C_1, C_2 \) depending only on \( n, A_3, \|K\|_{C^2(B_3)}, \rho \), such that

\[|\nabla K_i(x_i)| \leq C_1u_i(x_i)^{-1}, \quad \tau_i \leq C_2u_i(x_i)^{-2}.
\]

B Appendix

In this appendix, we provide some estimates that can be verified by elementary calculations which have been used in the proof of Theorem 1.2.

For \( P \in \mathbb{S}^n \) and \( t > 0 \), let

\[
\delta_{P,t}(x) = \frac{t}{1 + \frac{t^2-1}{2}(1 - \cos d(x, P))}, \quad x \in \mathbb{S}^n,
\]

where \( d(\cdot, \cdot) \) is the distance induced by the standard metric of \( \mathbb{S}^n \). Let \( P \) be the south pole of \( \mathbb{S}^n \) and make a stereographic projection with respect to the equatorial plane, we then have

\[
\delta_{P,t}(y) = \frac{t(1+|y|^2)}{1 + t^2|y|^2}, \quad \forall y \in \mathbb{R}^n.
\]
Lemma B.1. Let $2 \leq \alpha \leq \beta$, there exists a positive constant $C$ depending only on $\beta$ such that, for any $a \geq 0$, $b \in \mathbb{R}$,
\[
\left| |a + b|^{\alpha - 1}(a + b) - a^\alpha - \alpha a^{\alpha - 1}b - \frac{\alpha(\alpha - 1)}{2}a^{\alpha - 2}b^2 \right| \leq C(|b|^\alpha + a\gamma |b|^{\alpha - \gamma}),
\]
where $\gamma = \max\{0, \alpha - 3\}$.

Lemma B.2. For any $2 \leq \alpha \leq 3$ and any $a, b \geq 0$, there exists some universal constant $C > 0$ such that for any $a, b \geq 0$, we have
\[
| (a + b)^\alpha - a^\alpha - b^\alpha - \alpha a^{\alpha - 1}b | \leq C a^{\alpha - 2}b^2,
\]
\[
| (a + b)^\alpha - a^\alpha - b^\alpha | \leq C a^{\alpha - 1}b + ab^{\alpha - 1}.
\]
For any $1 \leq \alpha \leq 2$, there exists some universal constant $C > 0$ such that for any $a, b \geq 0$, we have
\[
| (a + b)^\alpha - a^\alpha | \leq C (a^{\alpha - 1}b + b^\alpha).
\]

Lemma B.3. We have
\[
\int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^n} \delta_{\frac{n-1}{2}} (\mathbf{S}^n - x) \bigg| B\left(\frac{n}{2}, \frac{n}{2} - 1\right),
\]
\[
\int_{\mathbb{R}^n} \frac{|x|^2}{(1 + |x|^2)^n} = \frac{n|\mathbb{S}^{n-1}|}{4(n - 1)} B\left(\frac{n}{2}, \frac{n}{2} - 1\right),
\]
\[
\int_{\mathbb{R}^n} \frac{|x|^2 - 1}{(1 + |x|^2)^n} = \frac{|\mathbb{S}^{n-1}|}{2(n - 1)} B\left(\frac{n}{2}, \frac{n}{2} - 1\right),
\]
where $B\left(\frac{n}{2}, \frac{n}{2} - 1\right)$ is the Beta function.

Lemma B.4. Let $\varepsilon_0, \tau > 0$ be suitably small and $A > 0$ be suitably large. Let $A^{-1} \tau^{-1/2} < t_1, t_2 < A \tau^{-1/2}$, $P_1, P_2 \in \mathbb{S}^n$, $|P_1 - P_2| \geq \varepsilon_0$, $\delta_{P_1, t_1}$ be as in (3.3) and $G_{P_1}(P_2)$ be as in (1.7), where $|P_1 - P_2|$ represents the distance between two points $P_1$ and $P_2$ after through a stereographic projection. Then, we have,
\[
\int_{\mathbb{S}^n} \delta_{\frac{n-1}{2}}^{\frac{n-1}{2}} \delta_{P_1, t_1} \delta_{P_2, t_2} = 2^n \left( \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^n-1} G_{P_1}(P_2) \right) t_1 t_2 + O(\tau^2), \tag{B.1}
\]
\[
\int_{\mathbb{S}^n} \delta_{\frac{n-1}{2}}^{\frac{n-1}{2}} \delta_{P_1, t_1} \delta_{P_2, t_2} = O(\tau), \tag{B.2}
\]
\[
\frac{\partial}{\partial t_1} \int_{\mathbb{S}^n} \delta_{\frac{n-1}{2}}^{\frac{n-1}{2}} \delta_{P_1, t_1} \delta_{P_2, t_2} = -(n - 1)2^n \left( \int_{\mathbb{R}^n} \frac{|x|^2 - 1}{(1 + |x|^2)^n} \right) t_1 t_2 + O(\tau^2), \tag{B.3}
\]
\[
\frac{\partial}{\partial t_1} \int_{\mathbb{S}^n} \delta_{\frac{n-1}{2}}^{\frac{n-1}{2}} = -\tau \int_{\mathbb{R}^n} \left(2^n \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^n} + O(\tau^\frac{n-1}{2} \log \tau)\right), \tag{B.4}
\]
\[
\frac{\partial}{\partial t_1} \int_{\mathbb{S}^n} |P - P_1|^2 \delta_{\frac{n-1}{2}}^{\frac{n-1}{2}} = -\frac{2^n}{t_1^3} \int_{\mathbb{R}^n} \frac{|x|^2}{(1 + |x|^2)^n} + O(\tau^\frac{n-1}{2} \log \tau). \tag{B.5}
\]
Lemma B.5. Under the hypotheses of Lemma B.4, in addition that $\Theta_5, \Theta_6$ are positive constants independent of $\tau$. Then, we have,

$$\langle \delta P_{1,t_1}, \delta P_{1,t_2} \rangle = 2^{n-1}|S^{n-1}|B\left(\frac{n}{2}, \frac{n}{2}\right), \quad (B.6)$$

$$\langle \delta P_{1,t_1}, \delta P_{2,t_2} \rangle = O(\tau), \quad (B.7)$$

$$\left\langle \frac{\partial \delta P_{1,t_1}}{\partial t_1}, \frac{\partial \delta P_{1,t_1}}{\partial t_1} \right\rangle = \Theta_5 t_1^{-2} = O(\tau), \quad (B.8)$$

$$\left\langle \frac{\partial \delta P_{1,t_1}}{\partial P_1^{(1)}}, \frac{\partial \delta P_{1,t_1}}{\partial P_1^{(1)}} \right\rangle = \Theta_6 t_1^2, \quad \left\langle \frac{\partial \delta P_{1,t_1}}{\partial P_1^{(1)}}, \frac{\partial \delta P_{1,t_1}}{\partial P_1^{(m)}} \right\rangle = 0, \forall \ell \neq m, \quad (B.9)$$

$$\|\delta_{P,1,t_1}^{n-1-\tau} \delta_{P,2,t_2}^{n/(n-1)}(S^n) = O(\tau), \quad (B.10)$$

$$\|\delta_{P,1,t_1}^{n-3-\tau} \delta_{P,2,t_2}^{n/(n-1)}(S^n) = O(\tau), \quad (B.11)$$

$$\|\delta_{P,1,t_1}^{n-1-\tau} - \delta_{P,1,t_1}^{n-1} \|_{L^{n/(n-1)}(S^n)} = O(\tau |\log \tau|),$$

$$\|\delta_{P,1,t_1}^{n-2-\tau} - \delta_{P,1,t_1}^{n-2} \|_{L^{n/(n-2)}(S^n)} = O(\tau |\log \tau|), \quad (B.12)$$

$$\|\delta_{P,1,t_1}^{n-\tau} - \delta_{P,1,t_1}^{n-1} \|_{L^1(S^n)} = O(\tau |\log \tau|),$$

$$\|\delta_{P,1,t_1}^{n-1} \|_{L^{n/(n-1)}(S^n)} = O(\tau^{1/2}), \quad (B.13)$$

$$\|\delta_{P,1,t_1}^{n-2} \|_{L^{n/(n-1)}(S^n)} = O(\tau),$$

$$\|\delta_{P,1,t_1}^{n-2} \|_{L^{n/(n-2)}(S^n)} = O(\tau^{1/2}), \quad (B.14)$$

$$\|\delta_{P,1,t_1}^{n-3-\tau} \frac{\partial \delta P_{1,t_1}}{\partial t_1} \|_{L^1(S^n)} = o(\tau^{3/2}), \quad (B.15)$$

$$\|\delta_{P,1,t_1}^{n-2-\tau} \frac{\partial \delta P_{1,t_1}}{\partial t_1} \|_{L^{n/(n-1)}(S^n)} = O(\tau^{3/2}), \quad (B.16)$$

$$\|\delta_{P,1,t_1}^{n-3-\tau} \frac{\partial \delta P_{1,t_1}}{\partial t_1} \|_{L^{n/(n-1)}(S^n)} = O(\tau^{3/2}). \quad (B.17)$$
Lemma B.6. In addition to the hypotheses of Lemma B.4, we assume that $K \in C^1(\mathbb{S}^n)$. Then

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^n} (K - K(P_1)) \delta_{P_2,t_2} \delta_{P_1,t_1}^{n-1-\tau} = O(\tau^2), \quad (B.18)$$

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^n} (K - K(P_2)) \delta_{P_1,t_1} \delta_{P_2,t_2}^{n-1-\tau} = O(\tau^2). \quad (B.19)$$

Lemma B.7. Let $\varepsilon_0, \tau, A$ be as in Lemma B.4, $P_1, P_2, P_3 \in \mathbb{S}^n$ satisfy $|P_i - P_j| \geq \varepsilon_0$, $i \neq j$, and $A^{-1}\tau^{-1/2} < t_1, t_2, t_3 \leq A\tau^{-1/2}$. Then, we have,

$$\left\| \delta_{P_2,t_2}^{n-2-\tau} \delta_{P_3,t_3} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\|_{L^1(\mathbb{S}^n)} = o(\tau^{3/2}), \quad (B.20)$$

$$\int_{\mathbb{S}^n} \delta_{P_1,t_1}^{n-2-\tau} \delta_{P_2,t_2} |\frac{\partial \delta_{P_1,t_1}}{\partial P_1}| = O(\tau^{1/2}), \quad (B.21)$$

$$\left\| \delta_{P_1,t_1}^{n-3-\tau} \delta_{P_2,t_2} \left| \frac{\partial \delta_{P_1,t_1}}{\partial P_1} \right| \right\|_{L^{n/(n-1)}(\mathbb{S}^n)} = O(\tau^{1/2}), \quad (B.22)$$

$$\int_{\mathbb{S}^n} \delta_{P_1,t_1}^{n-3-\tau} \delta_{P_2,t_2}^2 \left| \frac{\partial \delta_{P_1,t_1}}{\partial P_1} \right| = O(\tau^{3/2}), \quad (B.23)$$

$$\left| \frac{\partial}{\partial P_1} \int_{\mathbb{S}^n} \delta_{P_2,t_2}^{n-1-\tau} \delta_{P_1,t_1} \right| = O(\tau). \quad (B.24)$$

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