A counterexample to monotonicity of relative mass in random walks

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Abstract

For a finite undirected graph \( G = (V, E) \), let \( p_{u,v}(t) \) denote the probability that a continuous-time random walk starting at vertex \( u \) is in \( v \) at time \( t \). In this note we give an example of a Cayley graph \( G \) and two vertices \( u, v \in G \) for which the function

\[
r_{u,v}(t) = \frac{p_{u,v}(t)}{p_{u,u}(t)} \quad t \geq 0
\]

is not monotonically non-decreasing. This answers a question asked by Peres in 2013.

1 Introduction

Let \( G = (V, E) \) be a finite undirected regular graph. Let \( p_{u,v}(t) \) denote the probability that a continuous-time random walk starting at vertex \( u \) is in \( v \) at time \( t \). In this note we are interested in the function

\[
r_{u,v}(t) = \frac{p_{u,v}(t)}{p_{u,u}(t)} \quad t \geq 0.
\]

Clearly, in regular connected graphs for any \( u \neq v \), we have \( r_{u,v}(0) = 0 \) and \( \lim_{t \to \infty} r_{u,v}(t) = 1 \). One might wonder if the function is monotonically non-decreasing. It is not difficult to see that there are regular graphs for which this is not the case. In fact, there are regular graphs such that \( r_{u,v}(t) > 1 \) for some vertices \( u, v \) and time \( t \); in particular, \( r_{u,v}(t) \) is not monotonically non-decreasing. We give an example of such a graph in Appendix A. We thank Jeff Cheeger [Che15] for pointing this out to us.

For vertex-transitive graphs, however, it holds that \( r_{u,v}(t) \leq 1 \) for all vertices \( u, v \) and all \( t \geq 0 \). Indeed, using Cauchy-Schwarz and the reversibility of the walk,

\[
p_{u,v}(t) = \sum_{w \in V} p_{u,w}(t/2)p_{w,v}(t/2)
\]

\[
\leq \left( \sum_{w} p_{u,w}(t/2)^2 \right)^{1/2} \cdot \left( \sum_{w} p_{w,v}(t/2)^2 \right)^{1/2}
\]

\[
= p_{u,u}(t)^{1/2} \cdot p_{v,v}(t)^{1/2} = p_{u,u}(t).
\]
This motivates the following question, asked in 2013 by Peres [Per13]:

Is the function \( r_{u,v} \) monotonically non-decreasing in \( t \) for all vertex-transitive graphs and all vertices \( u, v \)?

More recently, a special case of that question was asked independently by Price [Pri14]. Namely, Price asked whether for Brownian motion on flat tori (i.e., on \( \mathbb{R}^n \) modulo a lattice), it holds that for any point \( x \), the density at \( x \) divided by the density at the starting point \( x_0 \) is monotonically non-decreasing in time. This would follow from a positive answer to Peres’s question through a limit argument. Price gave a positive answer to his question for the case of a cycle \( (n = 1) \) and recently, a positive answer for arbitrary flat tori was found [RSD15]. This can be seen as further evidence for a positive answer to Peres’s question.

In this note we give a negative answer to Peres’s question. In fact, we do so through a Cayley graph.

**Theorem 1.1.** There exists a Cayley graph \( G = (V, E) \) and two vertices \( u, v \in V \) such that the function \( r_{u,v} \) is not monotonically non-decreasing.

One remaining open question is whether \( r_{u,v} \) is monotonically non-decreasing for *Abelian* Cayley graphs. The positive result of [RSD15] is a special case of that.

### 1.1 Some basic facts about continuous-time random walks

Given a weighted finite graph \( G = (V, E) \) with weight function \( w : E \to \mathbb{R}_+ \), a continuous-time random walk \( X = (X_t)_{t \geq 0} \) on \( G \) is defined by its heat kernel \( H_t \), that at time \( t > 0 \) is equal to

\[
H_t = e^{-tL},
\]

where \( L \) is the Laplacian matrix of \( G \) given by \( L_{u,v} = -w(u,v) \) for \( u \neq v \), and \( L_{u,u} = \sum_v w(u,v) \).

As a result, for a random walk \( X \) starting at a vertex \( u \) the probability that \( X \) is in \( v \) at time \( t \) is equal to \( p_{u,v}(t) := H_t(u,v) \). When \( G \) is a \( d \)-regular unweighted simple graph, we think of the edges as all having weight \( 1/d \), in which case the Laplacian of \( G \) is given by

\[
L_{u,v} = \begin{cases} 
-1/d & \text{if } (u,v) \in E \\
1 & \text{if } u = v \\
0 & \text{otherwise}.
\end{cases}
\]

In this note we consider only vertex-transitive graphs, for which the sum \( \sum_v w(u,v) \) is the same for all vertices \( u \) of the graph. Note that we do not insist that this sum is equal to 1, though this can be achieved by normalizing \( L \), which corresponds to changing the speed of the random walk.

For basic facts about continuous-time random walks see, e.g., [LPW09].

If \( G \) is a weighted Cayley graph with a generating set \( S \) and a weight function \( w : S \to \mathbb{R}_+ \), then a continuous-time random walk \( X = (X_t)_{t \geq 0} \) on \( G \) is described by mutually independent Poisson processes of rate \( w(g) \) for each group generator \( g \in S \), where each process indicates the times when \( X \) jumps along the corresponding edge.
2 Non-monotonicity of time spent at the origin in the hypercube graph

For an integer \( d \geq 1 \) denote by \( Q_d \) the \( d \)-dimensional hypercube graph. The vertices of \( Q_d \) are \( \{0, 1\}^d \) and there is an edge between two vertices \( u \) and \( v \) if and only if they differ in exactly one coordinate. Let \( X = (X_t)_{t>0} \) be a continuous-time random walk on \( Q_d \) starting at the origin, denoted by \( 0 = (0, \ldots, 0) \in \{0, 1\}^d \). Denote by \( C_d(t) \) the expected time spent at the origin until time \( t \), conditioned on the event that \( X_t = 0 \). That is,

\[
C_d(t) = \int_0^t \Pr[X_s = 0|X_t = 0] \, ds.
\]

In this section we show that for \( d \) sufficiently large \( C_d(t) \) is not monotonically non-decreasing.

Lemma 2.1. Let \( d \in \mathbb{N} \) be sufficiently large. Then, there are some \( t_1 < t_2 \) such that \( C_d(t_1) > C_d(t_2) \), and in particular, the function \( C_d \) is not monotonically non-decreasing in \( t \).

Remark. Numerically, one can see that the function \( C_d \) is not monotone for \( d \geq 5 \). See Figure 1. Since \( C_d \) has a closed form expression (as can be seen from the calculations below), one can probably show non-monotonicity directly for \( C_5 \) by analyzing the function, though doing so would likely be messy and not too illuminating.

Before proving Lemma 2.1 we prove the following claim.

Claim 2.2. Let \( d \geq 1 \), and let \( Q_d \) be the \( d \)-dimensional hypercube graph. Let \( X = (X_t)_{t>0} \) be a continuous-time random walk on \( Q_d \) starting at \( 0 \). Then,

\[
\Pr[X_t = 0] = \left( \frac{1 + e^{-2t/d}}{2} \right)^d.
\]

Proof. Since \( X \) moves in each coordinate with rate \( 1/d \), it follows that for each \( i \in [d] \) the number of steps in direction \( i \) up to time \( t \) is distributed like \( \text{Pois}(t/d) \). Therefore,

\[
\Pr[(X_t)_i = 0] = \Pr[\text{Pois}(t/d) \text{ is even}] = (1 + e^{-2t/d})/2,
\]

Figure 1: \( C_d(t) \) for \( d = 4, 5, 6, 7 \) (from top right to bottom right).
where we used that the probability that \( \text{Pois}(\lambda) \) is even is
\[
\Pr[\text{Pois}(\lambda) \text{ is even}] = e^{-\lambda} \cdot \sum_{j \text{ even}} \frac{\lambda^j}{j!} = e^{-\lambda} \cdot \frac{1}{2} \left( \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} + \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \right) = e^{-\lambda} \cdot \frac{1}{2} (e^\lambda + e^{-\lambda}).
\]
Since the coordinates of \( X \) move independently the result follows.

We now prove Lemma 2.1.

**Proof of Lemma 2.1**. We show below that for all \( d \geq 1 \), it holds that

1. \( C_d(\sqrt{d}) \geq e^{-1}\sqrt{d} \),
2. \( C_d(d) \leq 6 \).

This clearly proves the lemma for \( d \) sufficiently large.

To prove Item 1, we show that if a walk starting from the origin is at the origin at time \( \sqrt{d} \), then with constant probability it stayed at the origin throughout that time interval. Intuitively, this is because the probability of a coordinate flipping twice during that time is of order only \( 1/d \) and so with constant probability none of the \( d \) coordinates flips. In more detail, by Claim 2.2
\[
\Pr[X_{\sqrt{d}} = 0] = \left(1 + e^{-2/\sqrt{d}}\right)^d \leq \left(1 - \frac{1}{\sqrt{d}} + \frac{1}{d} \right)^d = \left(1 - \frac{\sqrt{d} - 1}{d}\right)^d \leq e^{-\sqrt{d}+1},
\]
where we used the inequality \( e^{-x} \leq 1 - x + x^2/2 \) valid for all \( x \geq 0 \). On the other hand, by definition of a continuous-time random walk the probability that \( X \) stays in 0 during the entire time interval \([0, \sqrt{d}]\) is equal to \( \Pr[X_{[0, \sqrt{d}]} \equiv 0] = e^{-\sqrt{d}} \). Therefore,
\[
\Pr[X_{[0, \sqrt{d}]} \equiv 0 | X_{\sqrt{d}} = 0] \geq e^{-1},
\]
and hence the expected time spent at the origin conditioned on \( X_{\sqrt{d}} = 0 \) is as claimed in Item 1.

We next prove Item 2. Intuitively, here there is enough time for coordinates to flip twice, and only a very small part of the time will be spent at the origin. By definition of \( C_d \) and Claim 2.2 we have
\[
C_d(t) = \int_0^t \Pr[X_s = 0] \cdot \Pr[X_{t-s} = 0] \frac{d}{\Pr[X_t = 0]} ds = \int_0^t (h_d(t,s))^d ds,
\]
where
\[
h_d(t,s) = \frac{(1 + e^{-2s/d}) (1 + e^{-2(t-s)/d})}{2(1 + e^{-2t/d})} = \frac{1 + e^{-2s/d} + e^{-2(t-s)/d} + e^{-2t/d}}{2(1 + e^{-2t/d})}.
\]
Since \( h_d(t,s) \) is convex as a function of \( s \), for all \( 0 \leq s \leq t/2 \) we have \( h_d(t,s) \leq \ell(s) \) where \( \ell \) is the unique linear function satisfying \( \ell(0) = h_d(t,0) \) and \( \ell(t/2) = h_d(t,t/2) \). Therefore, taking \( t = d \), we get
\[
h_d(d,s) \leq \ell(s/d) = 1 - \frac{cs}{d},
\]
where \( c = (1 - e^{-1})^2/(1 + e^{-2}) \). Noting that \( h_d(t, s) = h_d(t, t - s) \), we get

\[
C_d(d) = \int_0^d (h_d(d, s))^d ds = 2 \int_0^{d/2} (h_d(d, s))^d ds < 2 \int_0^{d/2} (1 - \frac{cs}{d})^d ds < 2 \int_0^{d/2} e^{-cs} ds \leq \frac{2}{c}.
\]

This completes the proof of Lemma 2.1.

### 3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We first give a proof for a weighted graph, and then remark on how to convert it into an unweighted graph. For \( d \in \mathbb{N} \) sufficiently large we define the weighted graph \( G \) to be the lamplighter graph on \( Q_d \), whose edges corresponding to steps on \( Q_d \) are of weight \( 1/d \), and edges corresponding to toggling a lamp are of weight \( \varepsilon \), for some \( \varepsilon > 0 \) sufficiently small that depends on \( d \) and \( t_1, t_2 \) from Lemma 2.1.

In more detail, the weighted lamplighter graph \( G \) is described by placing a lamp at each vertex of \( Q_d \) and a lamplighter walking on \( Q_d \). A vertex of \( G \) is described by the location \( x \in \{0, 1\}^d \) of the lamplighter, and a configuration \( f : \{0, 1\}^d \rightarrow \{0, 1\} \) indicating which lamps are currently on. In each step the lamplighter either makes a step in the graph \( Q_d \) or toggles the state of the lamp in the current vertex. More formally, we have an edge between \((x, f)\) and \((y, g)\) if and only if either

1. \((x, y) \in E_d \) and \( f = g \) (this corresponds to a step in \( Q_d \)) or
2. \( x = y \) and \( f \) and \( g \) differ on the input \( x \) and are equal on all other inputs (this corresponds to toggling a lamp at \( x \)).

The weights of the edges of the first type are \( 1/d \), and the edges of the second type are of weight \( \varepsilon \). Thus, in a random walk on \( G \), the steps of the lamplighter are distributed as in a random walk on \( Q_d \), and the number of times the lamps are toggled in a time interval of length \( T \) is distributed like \( \text{Pois}(\varepsilon T) \) independently of the the lamplighter’s walk. It is well known that the lamplighter graph is a Cayley graph (see, e.g., [PR04]).

Let \( u \) be the vertex in \( G \) corresponding to the lamplighter being at the origin with all lights off. Let \( v \) be the vertex in \( G \) corresponding to the lamplighter being at the origin with the light at the origin being on, and all other lights off. We show below that \( r_u, v \) is not monotonically non-decreasing. More specifically, we show that \( r_{u, v}(t_1) > r_{u, v}(t_2) \), where \( t_1 < t_2 \) are from Lemma 2.1.

Let \( X = (X_t)_{t \geq 0} \) be a continuous-time random walk on \( G \) starting at \( X_0 = u \). Denote by \( Y_t \) the number of times a toggle occurred during the time interval \( [0, t] \). Denote by \( Z = (Z_t)_{t \geq 0} \) the trajectory of the lamplighter, i.e., the projection of \( X \) to the first coordinate. Note that by definition \( Z \) is a continuous-time random walk on \( Q_d \), and that \( Z \) is independent of \( Y_t \).

**Claim 3.1.** Let \( u, v \in V \) be as above. Then, for all \( t > 0 \) it holds that

\[
0 \leq p_{u, v}(t) - e^{-\varepsilon t} \cdot \Pr[Z_t = 0] \leq \varepsilon^2 t^2,
\]

and

\[
0 \leq p_{u, v}(t) - \varepsilon e^{-\varepsilon t} \cdot C_d(t) \cdot \Pr[Z_t = 0] \leq \varepsilon^2 t^2.
\]
Using the claim, 
\[ r_{u,v}(t) = \frac{p_{u,v}(t)}{p_{u,u}(t)} = \varepsilon \cdot C_d(t) \pm O(\varepsilon^2), \]
where \( O(\cdot) \) hides a constant that depends on \( d \) and \( t \). In particular, for \( t_1 < t_2 \) from Lemma 2.1 and \( \varepsilon > 0 \) sufficiently small we get that \( r_{u,v}(t_1) > r_{u,v}(t_2) \), which proves Theorem 1.1.

Intuitively, \( (\mathbb{I}) \) holds because the probability of toggling a lamp twice is very small, and hence \( p_{u,u}(t) \) is approximately equal to the probability that no lamp has changed its state multiplied by the probability that a random walk on \( Q_d \) will be at the origin at time \( t \). The intuition for \( (\mathbb{II}) \) is that in order to get from \( u \) to \( v \), in addition to getting back to the origin, the lamplighter must toggle the switch while being at the origin, and the probability of that is roughly \( \varepsilon \cdot C_d(t) \).

**Proof of Claim 3.1** For \( p_{u,u} \) we have
\[ p_{u,u} = \Pr[X_t = u \land Y_t = 0] + \Pr[X_t = u \land Y_t \geq 2]. \]
Since \( Y_t \) is distributed like \( \text{Pois}(\varepsilon t) \), the second term satisfies
\[ 0 \leq \Pr[X_t = u \land Y_t \geq 2] \leq \Pr[Y_t \geq 2] \leq \varepsilon^2 t^2, \]
and for the first term, by independence between \( Y_t \) and \( Z_t \) we have
\[ \Pr[X_t = u \land Y_t = 0] = \Pr[Z_t = 0 \land Y_t = 0] = e^{-\varepsilon t} \cdot \Pr[Z_t = 0], \]
proving \( (\mathbb{I}) \).

For \( p_{u,v} \) we similarly have
\[ p_{u,v}(t) = \Pr[X_t = v \land Y_t = 1] + \Pr[X_t = v \land Y_t \geq 2]. \]
As above, the second term is at most \( \varepsilon^2 t^2 \). For the first term, let \( E_t \) be the event that \( Y_t = 1 \), and the unique lamp that is on at time \( t \) is the lamp at the origin. Denote by \( T_0 \) the time spent by \( Z \) at the origin in the time interval \( [0, t] \). Then, conditioning on \( Z \), the event \( E_t \) holds if and only if there is a unique switch happened during \( T_0 \) time, and zero switches in the remaining time. Therefore, by independence of a Poisson process in disjoint intervals
\[ \Pr[E_t|Z] = \Pr[\text{Pois}(\varepsilon T_0) = 1|Z] \cdot \Pr[\text{Pois}(\varepsilon(t-T_0)) = 0|Z] = \varepsilon T_0 \cdot e^{-\varepsilon T_0} \cdot e^{-\varepsilon(t-T_0)} = e^{-\varepsilon t} \cdot T_0. \]
This implies that
\[ \Pr[X_t = v \land Y_t = 1] = \Pr[E_t|Z_t = 0] \cdot \Pr[Z_t = 0] = \varepsilon e^{-\varepsilon t} \cdot \Pr[T_0|Z_t = 0] \cdot \Pr[Z_t = 0]. \]
Therefore, since \( C_d(t) = \mathbb{E}[T_0|Z_t = 0] \) we get \( (\mathbb{II}) \), and the claim follows.

**Converting \( G \) into an unweighted graph.** Below we show how to convert a weighted Cayley graph \( G \) into an unweighted one, while preserving the property in Theorem 1.1. Let \( (G, S_G) \) be a weighted Cayley graph with the generating set \( S_G = \{g_1, \ldots, g_k\} \), and suppose that the weights \( w : S_G \to \mathbb{R}_+ \) are integers for all \( g \in S_G \). For \( N \in \mathbb{N} \) sufficiently large define the graph \( H \) by replacing each vertex \( v \in G \) with an \( N \)-clique \( \{(v,i) : i \in Z_N\} \), and replacing each edge \((u,ug)\) in \( G \) of weight \( w(g) \) with \( w(g) \) perfect matchings \( \{(u,i),(ug,i+j) : i \in Z_N\} \).
Formally, the vertices of the graph $H$ are $G \times \mathbb{Z}_N = \{(v,i) : v \in G, i \in \mathbb{Z}_N\}$, and the set of generators $S_H$ for the Cayley graph on $H$ is given by

$$S_H = \{(0,i) : i \in \mathbb{Z}_N \setminus \{0\} \} \cup \cup_{g \in S_G} \{(g,j) : j \in \{1, \ldots, w(g)\}\}.$$ 

Note that the projection of a continuous-time random walk on $H$ to the first coordinate is a random walk on $G$, slowed down by $\deg(H)$. Moreover, assuming $N$ is larger than, say, $\sum_{g \in S_G} w(g)$, after constant time the two coordinates become close to independent with the second coordinate being uniform. Therefore, if $u, v$ are vertices in $G$, and $x = (u,0), y = (v,0)$ are the corresponding vertices in $H$, then for any time $t > 0$ and $t' = \deg(H) \cdot t$ it holds that $p_{x,y}(t') = \frac{1}{N}(p_{u,v}(t) + o_N(1))$ and hence $r_{x,y}(t') = r_{u,v}(t') + o_N(1)$.

For the graph $G$ given in the proof of Theorem 1.1 above, we may assume that $1/\epsilon$ is an integer, and so, by multiplying all weights by $d/\epsilon$ we get a Cayley graph with integer weights. Hence, by applying the foregoing transformation we get a simple unweighted Cayley graph $H$ for which $r_{u,v}$ is not monotonically non-decreasing for some $u, v \in H$.

**A Appendix: A counterexample in a regular non-transitive graph**

Below we give a simple example of a regular non-transitive graph such that $r_{u,v}(t) > 1$ for some vertices $u, v$ and some time $t$; in particular, $r_{u,v}(t)$ is not monotonically non-decreasing, since $r_{u,v}(t) \to 1$ as $t \to \infty$. We thank Jeff Cheeger [Che15] for pointing this out to us.

**Proposition A.1.** Let $L$ be the Laplacian of a regular graph on vertex set $V$. Denote its eigenvalues by $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|V|}$ and by $f_i \in \mathbb{R}^V$ the corresponding normalized eigenvectors. Suppose that $0 < \lambda_2 < \lambda_3$, and that $f_2$ is such that $f_2(v) > f_2(u) > 0$ for some vertices $u, v$. Then, there is some $t > 0$ such that $r_{u,v}(t) > 1$.

**Proof.** Let $\pi_u \in \mathbb{R}^V$ be the vector with $\pi_u(u) = 1$ and $\pi_u(u') = 0$ for all $u' \neq u$. Writing $\pi_u = \sum \alpha_i f_i$ for $\alpha_i = \langle \pi_u, f_i \rangle = f_i(u)$, for all $w \in V$ we have

$$e^{-tL}\pi_u(w) = \sum_{i=1}^{\mid V \mid} e^{-t\lambda_i} \alpha_i \cdot f_i(w) = c + e^{-\lambda_3 t} f_2(u) f_2(w) + O(e^{-\lambda_3 t}),$$

where $O()$ hides some constants that may depend on the graph, but not on $t$, and $c = \alpha_1 \cdot f_1(w)$ is independent of $w$ since $f_1$ is a constant function. Using the facts that $f_2(v) > f_2(u) > 0$ and $\lambda_3 > \lambda_2$, it follows that for sufficiently large $t$,

$$r_{u,v}(t) = \frac{e^{-tL}\pi_u(v)}{e^{-tL}\pi_u(u)} > 1,$$

as desired. \qed

Graphs satisfying the constraints in Proposition A.1 are in abundance. As a concrete example, consider the 4-regular graph on 10 vertices shown in Figure 2. Using Mathematica, we see that the second eigenvalue of the Laplacian of this graph is $\lambda_2 = \frac{1}{2}(7 - \sqrt{17}) \approx 0.36$, and it is a simple eigenvalue. The corresponding (non-normalized) eigenvector with vertices ordered from left to right is $(c, 1, 1, 1, 1, -1, -1, -1, -1, -1, -c)$, where $c = 3 - \frac{1}{2}(7 - \sqrt{17}) \approx 1.56$. In particular, Proposition A.1 is applicable to this graph.
Figure 2: A cube with two square pyramids attached.

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