Integrability, susy $SU(2)$ matter gauge theories and black holes

Davide Fioravanti*, Daniele Gregori* and Hongfei Shu††

* Sezione INFN di Bologna, Dipartimento di Fisica e Astronomia, Università di Bologna
Via Irnerio 46, 40126 Bologna, Italy
fioravanti .at. bo.infn.it , daniele.gregori6 .at. unibo.it

† Beijing Institute of Mathematical Sciences and Applications (BIMSA), Beijing, 101408, China
†† Yau Mathematical Sciences Center (YMSC), Tsinghua University, Beijing, 100084, China
shuphy124 .at. gmail.com

Abstract

We show that previous correspondence between some (integrable) statistical field theory quantities and periods of $SU(2)\, N = 2$ deformed gauge theory still holds if we add $N_f = 1, 2$ flavours of matter. Moreover, the correspondence entails a new non-perturbative solution to the theory. Eventually, we use this solution to give exact results on quasinormal modes of black branes and holes.
# Contents

1 Preliminaries and overview 2

2 ODE/IM correspondence for gauge theory 4
   2.1 Gauge/Integrability dictionary 4
   2.2 Integrability functional relations 5
   2.3 $Q$ function’s exact expressions and asymptotic expansion 8
   2.4 Integrability TBA 10

3 Integrability $Y$ function and dual gauge period 12
   3.1 Gauge TBA 12
   3.2 Seiberg-Witten gauge/integrability identification 15
   3.3 Exact quantum gauge/integrability identification for $Y$ 20

4 Integrability $T$ function and gauge period 24
   4.1 $T$ function and Floquet exponent 24
   4.2 Exact quantum gauge/integrability identification for $T$ 26

5 Applications of gauge-integrability correspondence 28
   5.1 Applications to gauge theory 28
   5.2 Applications to integrability 29

6 Gravitational correspondence and applications 30
   6.1 Gravitational correspondence $N_f = 2$ 30
   6.2 Gravitational correspondence $N_f = 1$ 34

7 Conclusions and perspectives 36

A Quantum Seiberg-Witten theory with fundamental matter 38

B $N_f = 1, 2$ Seiberg-Witten periods 39
   B.1 Massless $N_f = 1$ SW periods 39
      B.1.1 $Z_3$ R-symmetry 40
   B.2 Massive $N_f = 1, 2$ SW periods 40
   B.3 Relations between alternatively defined periods 42

C The $N_f = 0$ analytic proof of gauge-integrability relation for $Y$ function 43
   C.1 Asymptotic proof 43
   C.2 Exact analytic proof 45
   C.3 Limits of validity of the identification and wall crossing 45
   C.4 Relation with other gauge period 46

D Résumé of D3 brane quantization relations 47

E Floquet exponent through Hill determinant 48

F Doubly Confluent Heun equation 50
   F.1 Eigenvalue expansion 51
1 Preliminaries and overview

Since almost 30 years, many effective low energy $\mathcal{N} = 2$ supersymmetric gauge theories have been brilliantly solved by Seiberg and Witten (SW) [1, 2]. In fact, the advantage of their construction is that several quantities can be computed exactly. In particular, SW theory enjoys a weak-strong coupling duality which allows us to compute the full effective action for the light fields at any coupling. In practice, this theory prescribes to compute the effective correspondence [15, 16].

ODE/IM the solution to the ODEs, which both physical theories share, and formulate an extension of the so-called N the integral definition (1.2) (cf. computation methods of the quantum gauge periods. In this paper we give a new method to interpret and compute large $\Lambda$ can use the above converging series around $\epsilon$ then needs an exact resummation, difficult to perform in practice (cf. 13 and references therein). Moreover, one can use the above converging series around $\Lambda_0 = 0$, which however may be very difficult to use at relatively large $\Lambda_0$. The interested reader is invited to [14] and references therein for a thorough comparison of different computation methods of the quantum gauge periods. In this paper we give a new method to interpret and compute the integral definition (1.2) (cf. also the details in appendix C). In fact, the integral starting point is essential to us for linking quantum SW $\mathcal{N} = 2$ gauge theory to suitable integrable models (IMs). In a nutshell, we analyse the solution to the ODEs, which both physical theories share, and formulate an extension of the so-called ODE/IM correspondence [15, 16].

In this way, we found a novel correspondence between 4D (deformed) $\mathcal{N} = 2$ supersymmetric Yang-Mills (SYM) and 2D IMs for $N_f = 0$ flavours [17]. Importantly, the quantum gauge periods $a$ and $a_D$ are directly connected to the Baxter’s $Q$ and $T$ functions of the self-dual Liouville (central charge $c = 25$) vacuum momentum $p$ and rapidity $\theta$ by

$$Q(\theta, p) = \exp \left\{ \frac{2\pi i}{\hbar} a_D(h, u, \Lambda_0) \right\}, \quad T(\theta, p) = 2 \cos \left\{ \frac{2\pi}{\hbar} a(h, u, \Lambda_0) \right\},$$

with the gauge/integrability map

$$\frac{\hbar}{\Lambda_0} = e^{-\theta}, \quad \frac{u}{\Lambda_0^2} = \frac{1}{2} p^2 e^{-2\theta}.$$
Extending \cite{18}, the $Q, T$ functions above (and in the following) appear in the ODE/IM correspondence as connection coefficients of certain solutions of the ODEs, called sometimes radial and lateral, respectively. Then they are proven to satisfy (1.3) as explained in appendix C. Besides, they expand asymptotically at $\theta \rightarrow +\infty$ in terms of the eigenvalues of the local integrals of motion (LIMs) $\mathbb{I}_{2n-1}(p), n = 1, 2, \ldots$ of self-dual Liouville field theory: in this way the key objects of integrability express the different periods in the WKB expansion \cite{17}. Nonetheless, $Q$ and $T$ functions, as well as the $Y$ functions derived from them, satisfy certain exact functional relations, which differ if in the integrability or SYM side as a consequence of the map (1.4) \cite{17}. In particular, those for the $Y$’s may also be inverted in Thermodynamic Bethe Ansatz (TBA) (real non-linear integral) equations which can be concretely solved, at least numerically, both on the integrability \cite{19, 20} and on the gauge side \cite{17}. For the cases here there are similar differences and often we find convenient to deal with the gauge parametrisation and TBA as in \cite{17}. For some understanding on the physical origin of this apparently casual correspondence between ODEs and IMs, besides the SW geometrical motivation deepened here, we refer to our other previous works \cite{21, 22}. In these actually, we outline the way for deriving the ODE from the quantum system. Actually, without mention to the full integrable structure \cite{21}, some work on exact TBA equations for periods of other gauge theories has been developed from an arguably more conjectural framework in \cite{23, 24, 25}.\footnote{In particular, the latter contains some numerical work on particular cases of ours and is rather inspiring.}  

It was intuitively clear that the construction of \cite{17}, although realised in particular, hold much more in general and give a general correspondence. Indeed, in the next work \cite{26}, a similar correspondence has been found between the $SU(3)$ colour group gauge theory (with $N_f = 0$) and the $A_2$ Toda CFT with central charge $c = 98$. Here we want to show how integrability/gauge correspondence holds upon adding $N_f = 1$ and $N_f = 2$ matter multiplets to the $SU(2)$ colour group.

In the meantime, other extremely interesting developments occurred. Indeed, the very same NS-deformed $\mathcal{N} = 2$ $SU(2)$ gauge theories found new applications to black holes (BHs) physics, specifically in perturbation theory for the ringdown (final) phase of BHs merging \cite{27, 28, 29, 30}. It was found first that (Bohr-Sommerfeld like) quantisation conditions on quantum gauge periods $a_D, a$ provide a new analytic exact characterisation of quasinormal modes (QNMs)\footnote{QNMs are the characteristic frequencies of the gravitational wave signal in ringdown (after merging) phase.} and could be practically used to also compute them \cite{27}. Thank to this and exploiting the AGT duality \cite{31, 32} between four dimensional $\mathcal{N} = 2$ gauge theories and two dimensional Conformal Field Theories (CFTs), also the latter kind of theories found applications to BHs \cite{29}.\footnote{These CFTs are different from ours. In fact, we relate to $N_f = 0$ gauge theory the $c = 25$ self-dual Liouville, rather then the $c \rightarrow +\infty$ Liouville as AGT does for the NS limit \cite{31}. Further investigations on the relation between such two Liouville models would be interesting.} For instance thus were made new computations of other BHs observables such the greybody factor and Love numbers\footnote{The greybody factor, or absorption coefficient, is associated to Hawking radiation, while Love numbers describe tidal deformations of BHs.}, sometimes also more accurate \cite{29, 33, 34}. From these many other applications and new results followed, like for instance

- an isospectral simpler equation to the perturbation ODE \cite{35};
- improved theoretical proofs of BHs stability \cite{36};
- a simpler interpretation of Chandrasekhar transformation as exchange of gauge mass parameters \cite{37};
- precise determination of the conditions of invariance under (Couch-Torrence) transformations which exchange inner horizon and null infinity \cite{38};
- an exact formula for the thermal scalar two-point function in four-dimensional holographic conformal field theories \cite{39}.

Moreover, we emphasise that the BHs which can be studied through this approaches are also very ’real’ (for instance, the Schwarschild and Kerr BHs) and enter astrophysics and gravitation phenomenology \cite{27, 40}. For instance, if
real BHs possessed horizon-scale structure, forbidden by General Relativity (GR) but allowed by modified theories of gravity or String Theory, it would manifest itself as echoes in the gravitational wave signal in the later ringdown phase and would be accessible to future higher precision detectors [41, 28].

In [42], still using the ODE/IM correspondence, we related the mathematically precise definition of QNMs [43] to quantisation conditions (Bethe roots condition) on various Baxter’s-integrable functions. It turned out that QNMs $\omega_n$ are nothing but the zeros (Bethe roots) of the Baxter’s $Q$ function

$$Q(\omega_n) = 0 \quad \omega_n \sim e^{i\theta_n} \quad n \in \mathbb{N}$$

and can be computed very efficiently with a new method typical of integrability: the Thermodynamic Bethe Ansatz (TBA) nonlinear integral equation. We gave also an explanation and derivation of the basic result of [27], the Bohr-Sommerfeld quantisation condition on the gauge periods to compute QNMs, through the further connection we previously found of these gauge theories to quantum integrable models. In particular, we sketched this for the $N_f = 0$ and $N_f = 2$ NS-deformed $SU(2)$ gauge theories, in correspondence with the D3 brane and the intersection of four stacks of D3 branes, respectively (the latter are essentially a generalization of the extremal Reissner-Nordström (RN) BHs).

In this respect here, besides showing the extension of the integrability-gauge correspondence to the $SU(2)$ $N_f = 1, 2$ theory, we show the generalization of the integrability-gravity correspondence to also $N_f = 1$ theory. It corresponds physically to just the null entropy limit in the system of intersection of four stacks of D3 branes. Moreover, thanks to the extended integrability-gauge correspondence, here we clarify and explain also some conjectures we made on other uses of integrability structures in the gravitational physics. This paper is also considerably longer and with many more details than our previous [17, 42] and so hopefully more clearly understandable. It is structured as follows. In section 2 we derive the integrability structures we need for the $SU(2)$ $N_f = 1, 2$ theory. In sections 3 and 4 we connect the gauge periods $a, a_D$ to the integrability $Y$ and $T$ functions. In section 5 we show some applications of this connection as new results for both gauge theory and integrability. In section 6 we make explicit the gravity counterpart of the gauge and integrability theories involved and find other applications. Finally in section 7 we give some conclusions, point out present limitations of our methods and future possible developments.

2 ODE/IM correspondence for gauge theory

2.1 Gauge/Integrability dictionary

The quantum Seiberg-Witten curves for $SU(2)$ $N_f = 1, 2$ gauge theory, deformed in the Nekrasov Shatashvili limit $\epsilon_2 \to 0$, $\epsilon = \hbar \neq 0$ can be derived from the classical one as explained in appendix A and they are the following ODEs. For $N_f = 1$

$$-\hbar^2 \frac{d^2}{dy^2} \psi(y) + \left[ \frac{\Lambda_1^2}{4}(e^{2y} + e^{-y}) + \Lambda_1 me^y + u \right] \psi(y) = 0,$$

(2.1) ODEgau

for $N_f = 2$ (with the first realization $N_+ = 1$, see appendix A):

$$-\hbar^2 \frac{d^2}{dy^2} \psi(y) + \left[ \frac{\Lambda_1^2}{8} \cosh(2y) + \frac{1}{2} \Lambda_2 m_1 e^y + \frac{1}{2} \Lambda_2 m_2 e^{-y} + u \right] \psi(y) = 0,$$

(2.2) ODEgau

where $u$ is the moduli parameter, $\Lambda_1, \Lambda_2$ are the instanton coupling parameters, $m, m_1, m_2$ are masses of the flavour hypermultiplets [12]. We notice that both equations are of the Doubly Confluent Heun equations [44], with two irregular singularities at $y \to \pm \infty$, as shown in appendix F.

5For another arguably more physical explanations through AGT duality or (conjecturally) M-Theory see [29, 30]
The first physical observation we can make is that they can be mapped into the ODEs for the Integrable Perturbed Hairpin model (IPHM) in the ODE/IM correspondence approach [45] and its generalization:

\[
-d^2\frac{\psi(y)}{dy^2} + [e^{2\theta} (e^{2y} + e^{-y}) + 2e^\theta q e^y + p^2]\psi(y) = 0, \quad (2.3)
\]

\[
-d^2\frac{\psi(y)}{dy^2} + [2e^{2\theta} \cosh(2y) + 2e^\theta q_1 e^y + 2e^\theta q_2 e^{-y} + p^2]\psi(y) = 0, \quad (2.4)
\]

where \( \theta \) is the TBA rapidity, \( p, q \) parametrizes the Fock vacuum of the IPHM and \( q_1, q_2 \) their generalization. For \( q = 0 \), equation (2.3) can be related to the ODE (Generalized Mathieu equation) associated to the Integrable Liouville model with Liouville coupling \( b = \sqrt{2} \) [14, 17, 20]. In particular, the gauge/integrability parameter dictionary is the following

\[
\frac{\hbar}{\Lambda_1} = \frac{1}{2}e^{-\theta}, \quad \frac{u}{\Lambda_1} = \frac{1}{4}p^2 e^{-2\theta}, \quad \frac{m}{\Lambda_1} = \frac{1}{2}qe^{-\theta}, \quad (2.5)
\]

\[
\frac{\hbar}{\Lambda_2} = \frac{1}{4}e^{-\theta}, \quad \frac{u}{\Lambda_2} = \frac{1}{16}p^2 e^{-2\theta}, \quad \frac{m_{1,2}}{\Lambda_2} = \frac{1}{4}q_{1,2}e^{-\theta}, \quad (2.6)
\]

or also

\[
\frac{u}{\hbar^2} = p^2, \quad \frac{m}{\hbar} = q, \quad (2.7)
\]

\[
\frac{u}{\hbar^2} = p^2, \quad \frac{m_1}{\hbar} = q_1, \quad \frac{m_2}{\hbar} = q_2. \quad (2.8)
\]

In [45], \( P \) and \( q \) were considered fixed, on the other hand, in the gauge theory, it is natural to keep \( \Lambda, u \) and \( m \) fixed. The mixed dependence on \( \theta \) gives then a nontrivial map, producing for instance different integrable structures in different parameters.

### 2.2 Integrability functional relations

The integrability equations are invariant under the following discrete symmetries. For \( N_f = 1 \)

\[
\Omega_+ : y \rightarrow y + 2\pi i/3 \quad \theta \rightarrow \theta + i\pi/3 \quad q \rightarrow -q, \quad (2.9)
\]

\[
\Omega_- : y \rightarrow y - 2\pi i/3 \quad \theta \rightarrow \theta + 2i\pi/3 \quad q \rightarrow q, \quad (2.10)
\]

for \( N_f = 2 \)

\[
\Omega_+ : y \rightarrow y + i\pi/2, \quad \theta \rightarrow \theta + i\pi/2, \quad q_1 \rightarrow -q_1, \quad q_2 \rightarrow +q_2, \quad (2.11)
\]

\[
\Omega_- : y \rightarrow y - i\pi/2, \quad \theta \rightarrow \theta + i\pi/2, \quad q_1 \rightarrow q_1, \quad q_2 \rightarrow -q_2. \quad (2.12)
\]

This symmetry is spontaneously broken by the regular solutions for \( \text{Re}\ y \rightarrow \pm\infty \), defined by the asymptotics, for \( N_f = 1 \):

\[
\psi_{+,0}(y) \simeq 2^{-\frac{1}{2}}q e^{-(\frac{1}{2}+q)\theta-(\frac{1}{2}+q)y} e^{-\theta+y} \quad y \rightarrow +\infty, \quad (2.13)
\]

\[
\psi_{-,0}(y) \simeq 2^{-\frac{1}{2}}e^{-\theta+\frac{1}{2}y-2e^{-\theta-y}/2} \quad y \rightarrow -\infty
\]

and for \( N_f = 2 \):

\[
\psi_{+,0}(y) \simeq 2^{-\frac{1}{2}}q_1 e^{-(\frac{1}{2}+q_1)\theta-(\frac{1}{2}+q_1)y} e^{-\theta+y} \quad \text{Re} \ y \rightarrow +\infty \quad (2.14)
\]

\[
\psi_{-,0}(y) \simeq 2^{-\frac{1}{2}}q_2 e^{-(\frac{1}{2}+q_2)\theta+(\frac{1}{2}+q_2)y} e^{-\theta-y} \quad \text{Re} \ y \rightarrow -\infty
\]

The solutions \( (\psi_{+,0}, \psi_{-,0}) \) of course form a basis. However, we can generate other independent solutions by using the symmetries as

\[
\psi_{+,k} = \Omega_+^k \psi_+, \quad \psi_{-,k} = \Omega_-^k \psi_- \quad k \in \mathbb{Z}. \quad (2.15)
\]
For $k \neq 0$ such solutions are in general diverging for $y \to \pm \infty$. A basis of solutions is then given also, for instance, by $(\psi_{+,0}, \psi_{+,1})$. Importantly, the solutions $\psi_{\pm}$ are invariant under the symmetry $\Omega_\pm$ respectively:

$$\Omega_+ \psi_{-,k} = \psi_{-,k} \quad \Omega_- \psi_{+,k} = \psi_{+,k}. \quad (2.14)$$

The normalization so that we have the following wronskians for next neighbour $k$-$k+1$ solutions. For $N_f = 1$

$$W[\psi_{+,k+1}, \psi_{+,k}] = i e^{(-1)^k i \pi q} \quad W[\psi_{-,k+1}, \psi_{-,k}] = -i \quad (2.15)$$

for $N_f = 2$

$$W[\psi_{+,k+1}, \psi_{+,k}] = i e^{(-1)^k i \pi q_1} \quad W[\psi_{-,k+1}, \psi_{-,k}] = -i e^{(-1)^k i \pi q_2} \quad (2.16)$$

As is usual in ODE/IM correspondence, we can define the integrability Baxter’s $Q$ function as the wronskian of the regular solutions at different singular points $y \to \pm \infty$

$$Q = W[\psi_{+,0}, \psi_{-,0}] \quad (2.17)$$

Mathematically this quantity is called also the central connection coefficient, since it appears in the connection relations for solutions at different singular points $y \to \pm \infty$. To write such relations it is convenient to introduce the notation, for $N_f = 1$:

$$Q_{\pm}(\theta) = W[\psi_{+,0}, \psi_{-,0}](\theta, p, \pm q) \quad (2.18)$$

and for $N_f = 2$:

$$Q_{\pm,\pm}(\theta) = W[\psi_{+,0}, \psi_{-,0}](\theta, p, \pm q_1, \pm q_2) \quad Q_{\pm,\mp}(\theta) = W[\psi_{+,0}, \psi_{-,0}](\theta, p, \pm q_1, \mp q_2) \quad (2.19)$$

We have to expand the solutions $(\psi_{-,0}, \psi_{-,1})$ in terms of $(\psi_{+,0}, \psi_{+,1})$ with coefficients obtained very simply by taking the wronskians of both sides of the relations and using the symmetries $\Omega_\pm$ to change the parameters of $Q$. Thus we obtain, for $N_f = 1$

$$i e^{i \pi q} \psi_{-,0} = Q_- (\theta + \frac{i \pi}{3}) \psi_{+,0} - Q_+ (\theta) \psi_{+,1} \quad (2.20)$$

$$i e^{i \pi q} \psi_{-,1} = Q_- (\theta + i \pi) \psi_{+,0} - Q_+ (\theta + \frac{i 2 \pi}{3}) \psi_{+,1} \quad (2.21)$$

and for $N_f = 2$

$$i e^{i \pi q_1} \psi_{-,0} = Q_- (\theta + \frac{i \pi}{2}) \psi_{+,0} - Q_+ (\theta) \psi_{+,1} \quad (2.22)$$

$$i e^{i \pi q_1} \psi_{-,1} = Q_- (\theta + i \pi) \psi_{+,0} - Q_+ (\theta + \frac{i \pi}{2}) \psi_{+,1}. \quad (2.23)$$

By taking the wronskian of the first line with the second line (and also shifting $\theta$ and flipping the sign of $q$), we obtain the first integrability structure, that is the $QQ$ system. For $N_f = 1$

$$Q_+ (\theta + \frac{i \pi}{2}) Q_- (\theta - \frac{i \pi}{2}) = e^{-i \pi q} + Q_+ (\theta - \frac{i \pi}{6}) Q_- (\theta + i \frac{\pi}{6}). \quad (2.24)$$

and for $N_f = 2$

$$Q_{+,+} (\theta + \frac{i \pi}{2}) Q_{-,+} (\theta - \frac{i \pi}{2}) = e^{-i \pi (q_1 + q_2)} + Q_{-,+}(\theta) Q_{+,+}(\theta). \quad (2.25)$$

For this particular ODEs with two irregular singularities it is possible to define also an integrability $Y$ function and obtain a $Y$ system relation starting directly from the $Q$ function and $QQ$ system relation, rather than from the $T$ functions and $T$ system. So we define a function as, for $N_f = 1$

$$Y_{\pm}(\theta) = e^{\pm i \pi q} Q_{\pm}(\theta - \frac{i \pi}{6}) Q_+ (\theta + i \frac{\pi}{6}). \quad (2.26)$$
and for $N_f = 2$

$$Y_{\pm,\pm}(\theta) = e^{i\pi(q_1+q_2)}Q_{\pm,\pm}(\theta) Q_{\mp,\mp}(\theta)$$

$$Y_{-,\pm}(\theta) = e^{i\pi(-q_1+q_2)}Q_{-,\pm}(\theta) Q_{+,\mp}(\theta).$$  \hspace{1cm} (2.26) \text{Int}_Y \text{Def}

We notice that for $N_f = 1$ in the $Y$ function the $Q$ functions appear with different $\theta$ arguments and this will lead to several technical complications for this case, albeit corresponding to one hypermultiplet less. Equivalent definitions are obtained by the $QQ$ systems as, for $N_f = 1$:

$$e^{\pm i\pi q}Q_{\pm}(\theta + i\pi 2) Q_{\mp}(\theta - i\pi 2) = 1 + Y_{\pm}(\theta)$$  \hspace{1cm} (2.27) \text{QQY1}

and for $N_f = 2$:

$$e^{i\pi(q_1-q_2)}Q_{+,\mp}(\theta + i\pi 2) Q_{-,\mp}(\theta - i\pi 2) = 1 + Y_{+,\mp}(\theta).$$  \hspace{1cm} (2.28) \text{QQY2}

The $Y$ systems can be now obtained by taking a product of the $QQ$ system with itself with suitable parameters so to obtain a close relation in terms of $Y$ functions. For $N_f = 1$

$$Y_{\pm}(\theta + i\pi 2) Y_{\mp}(\theta - i\pi 2) = [1 + Y_{\pm}(\theta + i\pi 6)] [1 + Y_{\pm}(\theta - i\pi 6)]$$  \hspace{1cm} (2.29) \text{Int}_Y \text{Sys}

and for $N_f = 2$

$$Y_{+,\mp}(\theta + i\pi 2) Y_{-,\mp}(\theta - i\pi 2) = [1 + Y_{+,\mp}(\theta)] [1 + Y_{-,\mp}(\theta)].$$  \hspace{1cm} (2.30) \text{Y Sys2}

Now, the presence of the irregular singularities of ODEs (2.3)-(2.4) at $y \to +\infty$ (Stokes phenomenon) plays a rôle for defining the $T$ functions, for $N_f = 1$

$$T_+(\theta) = -iW[\psi_{-,1}, \psi_{+,1}], \quad \tilde{T}_+(\theta) = iW[\psi_{-,1}, \psi_{+,1}].$$  \hspace{1cm} (2.31) \text{T Def1}

and for $N_f = 2$

$$T_{+,\mp}(\theta) = -iW[\psi_{-,1}, \psi_{+,1}], \quad \tilde{T}_{+,\mp}(\theta) = iW[\psi_{-,1}, \psi_{+,1}].$$  \hspace{1cm} (2.32) \text{T Def2}

(with of course $T_-$ $T_{+,-}$ defined with the flipped masses as in (2.18) (2.19).) By expanding $\psi_{+,1}$ in terms of $\psi_{+,0}$,$\psi_{-,1}$, for $N_f = 1$

$$\psi_{+,1} = -e^{2i\pi q_1} \psi_{+,1} + e^{i\pi q_1} \tilde{T}_{+,\mp}(\theta) \psi_{+,0}$$

$$\psi_{-,1} = -e^{2i\pi q_2} \psi_{-,1} + T_{+,\mp}(\theta) e^{i\pi q_2} \psi_{-,0}$$  \hspace{1cm} (2.33)

or for $N_f = 2$

$$\psi_{+,1} = -e^{2i\pi q_1} \psi_{+,1} + e^{i\pi q_1} \tilde{T}_{+,\mp}(\theta) \psi_{+,0}$$

$$\psi_{-,1} = -e^{2i\pi q_2} \psi_{-,1} + T_{+,\mp}(\theta) e^{i\pi q_2} \psi_{-,0}$$  \hspace{1cm} (2.34)

we obtain the $TQ$ relations, for $N_f = 1$

$$T_{\pm}(\theta) Q_{\pm}(\theta) = Q_{\pm}(\theta - i\frac{2\pi}{3}) + Q_{\pm}(\theta + i\frac{2\pi}{3})$$

$$\tilde{T}_{\pm}(\theta) Q_{\pm}(\theta) = e^{\pm i\pi q_1} Q_{\pm}(\theta - i\frac{2\pi}{3}) + e^{\mp i\pi q_1} Q_{\pm}(\theta + i\frac{2\pi}{3})$$  \hspace{1cm} (2.35) \text{TQ1}

or for $N_f = 2$

$$T_{+,\mp}(\theta) Q_{+,\mp}(\theta) = e^{i\pi q_2} Q_{+,\mp}(\theta - i\frac{2\pi}{3}) + e^{-i\pi q_2} Q_{+,\mp}(\theta + i\frac{2\pi}{3})$$

$$\tilde{T}_{+,\mp}(\theta) Q_{+,\mp}(\theta) = e^{i\pi q_1} Q_{+,\mp}(\theta - i\frac{2\pi}{3}) + e^{-i\pi q_1} Q_{+,\mp}(\theta + i\frac{2\pi}{3})$$  \hspace{1cm} (2.36) \text{TQ2}

By applying the $\Omega_+$ and $\Omega_-$ symmetries to the $T$ and $\tilde{T}$ functions it is immediate to obtain also the periodicity relations, for $N_f = 1$

$$T_{\pm}(\theta + i\frac{\pi}{3}) = T_{\pm}(\theta)$$

$$\tilde{T}_{\pm}(\theta + i\frac{\pi}{3}) = \tilde{T}_{\pm}(\theta)$$  \hspace{1cm} (2.37) \text{T Per1}

and for $N_f = 2$

$$T_{+,\mp}(\theta + i\frac{\pi}{2}) = T_{+,\mp}(\theta)$$

$$\tilde{T}_{+,\mp}(\theta + i\frac{\pi}{2}) = \tilde{T}_{+,\mp}(\theta).$$  \hspace{1cm} (2.38) \text{T Per2}
2.3 Q function’s exact expressions and asymptotic expansion

From the ODE/IM analysis, cf. equations (2.20)-(2.22), we find a limit formula Baxter’s Q function as, for \( N_f = 1 \)

\[
Q_+(\theta) = -ie^{i\pi q} \lim_{y \to +\infty} \frac{\psi_{-0}(y, \theta)}{\psi_{-1}(y, \theta)},
\]

(2.39)_{defpsi} or for \( N_f = 2 \)

\[
Q_{+1}(\theta) = -ie^{i\pi q} \lim_{y \to +\infty} \frac{\psi_{-0}(y, \theta)}{\psi_{-1}(y, \theta)}.
\]

(2.40)

From this formula we can obtain another which concretely allows to compute \( Q \) as an integral. However, to do that, it is convenient first to transform the second order linear ODEs (2.3)-(2.4) for \( \psi \) into their equivalent first order nonlinear Riccati equations for the logarithmic derivative of \( \psi \). Besides, since we will need later to asymptotically expand the solution for \( y \to \pm \infty \) and \( \theta \to \infty \), it is convenient to change variable so to single out the leading order behaviour in \( y, \theta \) and simplify higher orders calculations. So we change variable as

\[
dw = \sqrt{\phi} dy \quad \phi = \begin{cases} 
-e^{2y} - e^{-y} & N_f = 1 \\
-2 \cosh(2y) & N_f = 2 
\end{cases}.
\]

(2.41)

To keep the ODE in normal form we have to let \( \psi \to \sqrt{\phi} \psi \). Then we take the logarithmic derivative of \( \psi \) in the new variable \( w \)

\[
\Pi = -i \frac{d}{dw} \ln(\sqrt{\phi} \psi)
\]

(2.42)

and we get for it the Riccati equation

\[
\Pi(y)^2 - i \frac{1}{\sqrt{\phi}} \frac{d}{dy} \Pi(y) = e^{2\theta} - e^{\theta} V(y) - U(y),
\]

(2.43)_{eqRiccati} with

\[
V(y) = \begin{cases} 
-\frac{2e^{2y}}{\cosh(2y)} & N_f = 1 \\
-\frac{e^{2y} + e^{2y}}{\cosh(2y)} & N_f = 2
\end{cases},
\]

\[
U(y) = \begin{cases} 
-\frac{p^2}{e^{-y} + e^{2y}} + \frac{e^{-y} - 40e^{4y} + 4e^{7y}}{16(e^{y+1})^3} & N_f = 1 \\
\frac{1}{2\cosh(2y)} [-p^2 - 1 + \frac{7}{4} \tanh^2(2y)] & N_f = 2
\end{cases}.
\]

(2.44)

The first asymptotic expansion we make is the one for \( y \to \pm \infty \), in the formal parameter \( e^{-y} \). The Riccati equation gets approximated, at the leading and subleading order as

\[
\Pi(y)^2 - i \frac{1}{\sqrt{\phi}} \frac{d}{dy} \Pi(y) \simeq \begin{cases} 
e^{2\theta} + 2e^{\theta} \delta_+ q e^{-y} & N_f = 1 \\
e^{2\theta} + 2e^{\theta} q_1 e^{+y} & N_f = 2 \end{cases} \quad y \to \pm \infty
\]

(2.45)

where for \( N_f = 1 \) \( \delta_+ = 1 \) for \( y \to +\infty \), \( \delta_+ = 0 \) for \( y \to -\infty \). Then the solution is asymptotic to

\[
\Pi(y) \simeq \begin{cases} e^{\theta} + \delta_+ q e^{-y} & N_f = 1 \\
e^{\theta} + q_1 e^{+y} & N_f = 2 \end{cases} \quad y \to \pm \infty
\]

(2.46)

This leading expansion for \( y \to \pm \infty \) allows us to fix the regularization in the integrals formulas we now write for the (logarithm) of \( \psi_{-0} \), for \( N_f = 1 \)

\[
\psi_{-0}(y) = \sqrt{\frac{2}{e^{2y} + e^{-y}}} \exp \left\{ -e^{\theta} (2e^{-y/2} - e^{y}) + 2q \ln(1 + e^{y/2}) \right\} \times \exp \left\{ \int_{-\infty}^{y} dy' \sqrt{e^{2y'} + e^{-y'} \Pi(y', \theta, p, q)} - e^{\theta} \left( e^{y'} + e^{-y'}/2 \right) - q \frac{1}{1 + e^{-y'/2}} \right\}
\]

(2.47)_{psi-0e}
and for $N_f = 2$

$$
\psi_{-,0}(y) = \frac{2^{1/2}q_2e^{-(1/2+q_2)\theta}}{\sqrt{e^{2y}+e^{-2y}}} \exp \left\{ -e^\theta (e^{-y} - e^y) + 2q_1 \ln(1 + e^{y/2}) - 2q_2 \ln(1 + e^{-y/2}) \right\} \times \\
\exp \left\{ \int_{-\infty}^{y} dy' \left[ \sqrt{e^{2y'} + e^{-2y'}} \Pi(y', \theta, p, q_1, q_2) - e^\theta (e^{y'} + e^{-y'}) - q_1 \frac{1}{1 + e^{-y'/2}} - q_2 \frac{1}{1 + e^{y'/2}} \right] \right\}.
$$

Then from the limit formula for $Q$ we get also an integral expression for it, for $N_f = 1$

$$
\ln Q_+(\theta) = \int_{-\infty}^{\infty} dy \left[ \sqrt{e^{2y} + e^{-2y}} \Pi(y, \theta, q, p) - e^\theta e^y - e^\theta e^{-y} - q_1 \frac{1}{1 + e^{-y/2}} \right] - (\theta + \ln 2) q.
$$

and for $N_f = 2$

$$
\ln Q_{+,+}(\theta) = \int_{-\infty}^{\infty} dy \left[ \sqrt{2 \cosh(2y)} \Pi(y, \theta, q_1, q_2, p) - 2e^\theta \cosh y - \left( \frac{q_1}{1 + e^{-y/2}} + \frac{q_2}{1 + e^{y/2}} \right) \right] - (\theta + \ln 2) (q_1 + q_2)
$$

To get the vacuum eigenvalues of the local integrals of motion (LIMs) we make instead the $\theta \to +\infty$ asymptotic expansion, at all orders

$$
\Pi(y, \theta) \approx e^\theta + \sum_{n=0}^{\infty} \Pi_n(y)e^{-n\theta} \quad \theta \to +\infty.
$$

Its coefficients $\Pi_n$ satisfy the recursion relation

$$
\Pi_{n+1} = \frac{1}{2} \left( \frac{i}{\sqrt{2}} \frac{d}{dy} \Pi_n - \sum_{m=0}^{n} \Pi_m \Pi_{n-m} \right) \quad n \geq 1
$$

with initial conditions

$$
\Pi_0 = -\frac{1}{2} V \\
\Pi_1 = \frac{1}{2} \left( \frac{i}{\sqrt{2}} \frac{d}{dy} \Pi_0 - \Pi_0^2 - U \right)
$$

The expansion of $\ln Q$ in terms of the LIMs is, for $N_f = 1$

$$
\ln Q_+(\theta) \approx -\frac{4\sqrt{3\pi^3}}{\Gamma(\frac{1}{6}) \Gamma(\frac{1}{3})} e^\theta - (\theta + \frac{1}{3} \ln 2) q - \sum_{n=1}^{\infty} e^{-n\theta} C_n \Pi_n \quad \theta \to +\infty
$$

and for $N_f = 2$

$$
\ln Q_{+,+}(\theta) \approx -\frac{4\sqrt{\pi^3}}{\Gamma(\frac{1}{4})^2} e^\theta - (\theta + \frac{1}{2} \ln 2) (q_1 + q_2) - \sum_{n=1}^{\infty} e^{-n\theta} C_n \Pi_n \quad \theta \to +\infty,
$$

with the local integrals of motion $\Pi_n$ times some normalization constants $C_n$ given by the integrals

$$
C_n \Pi_n(p, q) = -i \int_{-\infty}^{\infty} dy \sqrt{\phi(y)} \Pi_n(y, p, q) \quad n \geq 1.
$$
\( I_n(p, q) \) are in general polynomials in \( p, q \), where \( q \) of course here stands for either \( q \) for \( N_f = 1 \) or \((q_1, q_2)\) for \( N_f = 2 \). We have checked the first ones for \( N_f = 1 \) to match with those of IPHM given in [45],

\[
I_1(p, q) = \frac{1}{12} (4q^2 - 12p^2 - 1) \\
I_2(p, q) = \frac{1}{6\sqrt{3}} q \left( \frac{20}{3} q^2 - 12p^2 - 3 \right)
\]  

(2.57)

For \( N_f = 2 \) they were never given in the literature to our knowledge and they have the peculiar feature that the mixed \( q_1, q_2 \) terms have transcendental coefficients (Gamma functions). We notice also that the one step recursion very effective method of computation of LIMs explained in [17] does not directly generalize to this case where all \( e^{-i\theta} \) are present in the asymptotic expansion. Further investigations on such LIMs issues could be pursued.

### 2.4 Integrability TBA

Define as usual the pseudoenergy \( \varepsilon(\theta) = -\ln Y(\theta) \) and \( L = \ln[1 + \exp(-\varepsilon)] \) (with suitable subscripts omitted of course). Using the analytic properties of pseudoenergy \( \varepsilon \), we can transform the \( Y \) system (2.29) into the following 'integrability TBAs'. For \( N_f = 1 \) [45]

\[
\begin{align*}
\varepsilon_+(\theta) &= -\frac{12\sqrt{\pi^3} \theta}{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{3}{2} \right)} e^{\theta} - \frac{4}{3} i\pi q - (\varphi_{++} * L_+)(\theta) - (\varphi_{+-} * L_-)(\theta) \\
\varepsilon_-(\theta) &= -\frac{12\sqrt{\pi^3} \theta}{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{3}{2} \right)} e^{\theta} + \frac{4}{3} i\pi q - (\varphi_{++} * L_-)(\theta) - (\varphi_{+-} * L_+)(\theta),
\end{align*}
\]  

(2.58)

and for \( N_f = 2 \)

\[
\begin{align*}
\varepsilon_{+,+}(\theta) &= -\frac{8\sqrt{\pi^3} \theta}{\Gamma \left( \frac{1}{2} \right)^2} e^{\theta} - \frac{1}{3} i\pi (q_1 - q_2) - \varphi \ast (L_{++} + L_{--}) \\
\varepsilon_{+,+}(\theta) &= -\frac{8\sqrt{\pi^3} \theta}{\Gamma \left( \frac{1}{2} \right)^2} e^{\theta} - \frac{1}{3} i\pi (q_1 + q_2) - \varphi \ast (L_{++} - L_{--}) \\
\varepsilon_{-,+}(\theta) &= -\frac{8\sqrt{\pi^3} \theta}{\Gamma \left( \frac{1}{2} \right)^2} e^{\theta} + \frac{1}{3} i\pi (q_1 + q_2) - \varphi \ast (L_{--} + L_{+-}) \\
\varepsilon_{-,+}(\theta) &= -\frac{8\sqrt{\pi^3} \theta}{\Gamma \left( \frac{1}{2} \right)^2} e^{\theta} + \frac{1}{3} i\pi (q_1 - q_2) - \varphi \ast (L_{--} - L_{+-}).
\end{align*}
\]  

(2.59)

The leading (driving) term follows directly from the expansions (2.54)-(2.55) under the definitions for \( Y = \exp(-\varepsilon) \) (2.25)-(2.26). The symbol \( \ast \) stands for the \((-\infty, +\infty)\) convolution, which for general functions \( f, g \)

\[
(f \ast g)(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} f(\theta - \theta') g(\theta')
\]  

(2.60)

The kernel for \( N_f = 2 \) is the simple usual hyperbolic secant [46]

\[
\varphi(\theta) = \frac{1}{\cosh \theta},
\]  

(2.61)

while the one for \( N_f = 1 \) is slightly more involved because of the shifts in \( \theta \) also on the RHS of the \( Y \) system (2.29) but can be obtained by taking Fourier transform as explained in [47]

\[
\varphi_{+}(\theta) = \frac{\sqrt{3}}{\cosh \theta + 1}.
\]  

(2.62)
We notice that \( q, q_1, q_2 \) enter the integrability TBAs as chemical potentials \([48]\). In these TBAs the parameter \( p \) does not appear, but it enters in the boundary condition for the solution \( \varepsilon \) at \( \theta \to -\infty \), for \( N_f = 1 \)

\[
\varepsilon_{\pm}(\theta, p) \simeq 6p\theta \mp i\pi q + 2C(p, q) \quad \theta \to -\infty ,
\]

(2.63)

and for \( N_f = 2 \)

\[
\varepsilon_{+,+}(\theta, p) \simeq 4p\theta - i\pi(q_1 - q_2) + 2C(p, q_1, q_2) \quad \theta \to -\infty
\]

(2.64)

with

\[
C(p) = \left\{ \begin{array}{ll}
\ln \left[ \frac{2 - p\Gamma(2p)\Gamma(1 + 2p)}{\sqrt{2\pi^3} \Gamma(1 + p + q)\Gamma(1 + p - q)} \right] & \text{for } N_f = 1 \\
\ln \left[ \frac{1}{\sqrt{\Gamma(1 + 2p)\Gamma(1 - 2p)(p + q)(p - q)}} \right] & \text{for } N_f = 2
\end{array} \right.
\]

(2.65)

This asymptotic behaviour follows of course from the \( \theta \to -\infty \) perturbative expansion of the ODE (shifting \( y \) by \( \pm \theta \) in the ODE so to eliminate the leading terms at \( y \to \mp \infty \) and get the solution as confluent hypergeometric function, expanding it in \( e^{i\theta} \) and taking the wronskian, see also \([45]\) for \( N_f = 1 \)). We can solve therefore this TBA by adding and subtracting outside and inside the convolutions the boundary condition for \( \theta \to -\infty \) which depends on \( p \). For example, for \( N_f = 1 \) the numerically solvable integrability TBA reads

\[
\varepsilon_+(\theta) = \frac{12\sqrt{\pi}^3}{\Gamma(\frac{5}{6})\Gamma(\frac{1}{6})}\left[ e^{\theta} - \frac{4}{3}i\pi q - f_0(\theta) - f_1(\theta) - (\varphi_+ + (L_1 - L_0 - L_1))\theta - (\varphi_- + (L_1 - L_0 - L_1))\theta \right]
\]

\[
\varepsilon_-(\theta) = \frac{12\sqrt{\pi}^3}{\Gamma(\frac{5}{6})\Gamma(\frac{1}{6})}\left[ e^{\theta} + \frac{4}{3}i\pi q - f_0(\theta) - f_1(\theta) - (\varphi_+ + (L_1 - L_0 - L_1))\theta - (\varphi_- + (L_1 - L_0 - L_1))\theta \right]
\]

(2.66)

where the explicit terms can be derived in analogue way to \([20]\) as

\[
L_0(\theta) = 3p\ln \left[ 1 + e^{-2\theta} \right],
\]

\[
L_1(\theta) = C(p)(1 - \tanh \theta),
\]

(2.67)

\[
f_0(\theta) = \varphi_+ L_0 = 3p \left\{ \ln \left[ 1 + e^{-(\theta + i\pi/6)} \right] + \ln \left[ 1 + e^{-(\theta - i\pi/6)} \right] \right\},
\]

\[
f_1(\theta) = \varphi_+ L_1 = C(p) \left[ 1 - \frac{1}{2} \tanh \left( \frac{\theta}{2} + \frac{i\pi}{12} \right) - \frac{1}{2} \tanh \left( \frac{\theta}{2} - \frac{i\pi}{12} \right) \right].
\]

(2.68)

We notice that the constant term \( i\pi q \) in (2.63) is automatically produced by the contribution of the the complex convolution. We notice also that boundary condition (2.63) requires strictly \( p > 0 \), which in gauge theory will correspond to \( u/\Lambda_{1,2}^2 > 0 \) by (2.5). However, we shall see that we can solve the TBA in gauge variables for \( u/\Lambda_{1,2}^2 \in \mathbb{C} \) (small), thus providing an analytic continuation of the integrability TBA. For \( N_f = 2 \) instead the corresponding auxiliary functions are

\[
L_0(\theta) = 2p\ln \left[ 1 + e^{-2\theta} \right],
\]

\[
L_1(\theta) = C(p)(1 - \tanh \theta),
\]

(2.68)

\[
f_0(\theta) = \varphi_+ L_0 = 4p\ln \left[ 1 + e^{-\theta} \right],
\]

\[
f_1(\theta) = \varphi_+ L_1 = C(p) \left[ 1 - \tanh \left( \frac{\theta}{2} \right) \right].
\]

We notice that (2.59) generalizes the TBA found in \([45]\) for the Perturbed Hairpin IM and therefore we call the IM involved (with no much creativity admittedly) Generalized Perturbed Hairpin IM.
Now from the TBA solution we can obtain also $Q$ as follows. Writing from the $QQ$ system for $N_f = 1$ (2.27)

\[ [Q_+(\theta - i\pi/2)Q_-(\theta - i\pi/2)] = [1 + Y_+(\theta)][1 + Y_-(\theta)] \]

\[
\begin{bmatrix}
Q_+(\theta + i\pi/2) \\
Q_-(\theta + i\pi/2)
\end{bmatrix}
\begin{bmatrix}
Q_+(\theta - i\pi/2) \\
Q_-(\theta - i\pi/2)
\end{bmatrix}^{-1} = e^{-2\pi q} [1 + Y_+(\theta)] \]

\[
\begin{bmatrix}
Q_+(\theta + i\pi/2) \\
Q_-(\theta + i\pi/2)
\end{bmatrix}
\begin{bmatrix}
Q_+(\theta - i\pi/2) \\
Q_-(\theta - i\pi/2)
\end{bmatrix}^{-1} = \frac{1}{1 + Y_-(\theta)}
\]

we easily deduce the following integral expression for $Q$ for $N_f = 1$

\[
\ln Q_\pm(\theta) = -\frac{4\sqrt{3}\pi^3}{\Gamma\left(\frac{1}{3}\right)} e^\theta \mp \left(\theta + \frac{1}{3} \ln 2\right)q
\]

\[
+ \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \left\{ \ln[1 + \exp(-\epsilon_+(\theta'))][1 + \exp(-\epsilon_-(\theta'))] \mp i \frac{e^{\theta' - \theta}}{\cosh(\theta - \theta')}ight. \ln \left[1 + \exp(-\epsilon_+(\theta'))\right] \right. \]

\[
\left. \left. \left. \mp \epsilon \right. \ln \left[1 + \exp(-\epsilon_-(\theta'))\right] \right}\right\}
\]

Similarly for $N_f = 2$ it follows

\[
\ln Q_{\pm,\mp}(\theta) = -\frac{4\sqrt{\pi^3}}{\Gamma\left(\frac{1}{3}\right)^2} e^\theta \mp \left(\theta + \frac{1}{2} \ln 2\right)(q_1 - q_2)
\]

\[
+ \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \left\{ \ln[1 + \exp(-\epsilon_+(\theta'))][1 + \exp(-\epsilon_-(\theta'))] \mp i \frac{e^{\theta' - \theta}}{\cosh(\theta - \theta')}ight. \ln \left[1 + \exp(-\epsilon_+(\theta'))\right] \right. \]

\[
\left. \left. \left. \mp \epsilon \right. \ln \left[1 + \exp(-\epsilon_-(\theta'))\right] \right.\right\}
\]

3 Integrability $Y$ function and dual gauge period

3.1 Gauge TBA

To establish a connection between integrability and gauge theory, we need first of all to express all integrability definitions and relations in gauge variables through the parameter dictionaries (2.5)-(2.6). Thus for $N_f = 1$ we can introduce 6 gauge $Q$ and $Y$ functions defined, for $k = 0, 1, 2$, as

\[
Q_{\pm,k}(\theta) = Q(\theta, -u_k, \pm m_k, \Lambda_1), \quad Y_{\pm,k}(\theta) = Y(\theta, u_k, \pm i m_k, \Lambda_1).
\]

where for simplicity we denote

\[
u_k = e^{2\pi i/3}u_k \quad m_k = e^{-2\pi i/3}m_k \quad k = 0, 1, 2
\]

The explicit relation between $Q$ and $Y$ is for example, for $k = 0$ (from (2.25) and (2.5))

\[
Y_{\pm,0}(\theta) = Y(\theta, u, \pm im, \Lambda_1) = e^{\mp 2\pi i / \sqrt{3}} Q_{\pm,2}(\theta - i\pi/6)Q_{\pm,1}(\theta + i\pi/6).
\]

For $N_f = 2$ instead we have 8 $Q$ and $Y$ functions

\[
Y_{\pm,1}(\theta) = Y(\theta, u, m_1, \pm m_2, \Lambda_2) \quad Y_{\pm,2}(\theta) = Y(\theta, -u, \mp m_1, \pm im_2, \Lambda_2)
\]

It is convenient to write the gauge $Y$ system (2.29) explicitly as, for $N_f = 1$

\[
Y_{\pm,0}(\theta + i\pi/2)Y_{\mp,0}(\theta - i\pi/2) = [1 + Y_{\pm,1}(\theta + i\pi/6)][1 + Y_{\pm,2}(\theta - i\pi/6)]
\]

\[
Y_{\pm,1}(\theta + i\pi/2)Y_{\mp,1}(\theta - i\pi/2) = [1 + Y_{\pm,2}(\theta + i\pi/6)][1 + Y_{\pm,0}(\theta - i\pi/6)]
\]

\[
Y_{\pm,2}(\theta + i\pi/2)Y_{\mp,2}(\theta - i\pi/2) = [1 + Y_{\pm,0}(\theta + i\pi/6)][1 + Y_{\pm,1}(\theta - i\pi/6)]
\]
and for $N_f = 2$

$$
\dot{Y}_{\pm, \pm}(\theta + i\pi/2)\dot{Y}_{\pm, \pm}(\theta - i\pi/2) = [1 + Y_{\pm, \pm}(\theta)][1 + Y_{\pm, \pm}(\theta)]
$$

(3.6)

Notice that with respect to what happens in the integrability variables, in the gauge variables the number of $Q$, $Y$ functions increases (triples for $N_f = 1$, doubles for $N_f = 2$), as it happens for the $SU(2)$ $N_f = 0$ theory (where it doubles) [17]. Besides the $Q$ and $Y$ systems in gauge variables simplify their dependence on the flipped masses.

Again, as explained in [47], it straightforward to invert the $Y$-systems into the following 'gauge TBAs'. For $N_f = 1$:

$$
\varepsilon_{\pm, 0}(\theta) = \varepsilon_{\pm, 0}(\varepsilon^2 - (\varphi_+ * L_{\pm, 1})(\theta) - (\varphi_- * L_{\pm, 2})(\theta)
$$

(3.7)<sub>ga-TBA</sub>

$$
\varepsilon_{\pm, 1}(\theta) = \varepsilon_{\pm, 1}(\varepsilon^2 - (\varphi_+ * L_{\pm, 2})(\theta) - (\varphi_- * L_{\pm, 0})(\theta)
$$

$$
\varepsilon_{\pm, 2}(\theta) = \varepsilon_{\pm, 2}(\varepsilon^2 - (\varphi_+ * L_{\pm, 0})(\theta) - (\varphi_- * L_{\pm, 1})(\theta),
$$

and for $N_f = 2$

$$
\varepsilon_{\pm, \pm}(\theta) = \varepsilon_{\pm, \pm}(\varepsilon^2 - \varphi_*(\bar{L}_{\pm \pm} + \bar{L}_{\mp \mp})(\theta)
$$

(3.8)<sub>TBA2</sub>

$$
\bar{\varepsilon}_{\pm, \pm}(\theta) = \bar{\varepsilon}_{\pm, \pm}(\varepsilon^2 - \varphi_*(L_{\pm \pm} + L_{\mp \mp})(\theta).
$$

The new kernels for $N_f = 1$ $\varphi_\pm$ are defined as

$$
\varphi_\pm(\theta) = \frac{1}{\cosh(\theta \pm i\pi/6)}.
$$

(3.9)

The leading order coefficient for $N_f = 1$, for example for $k = 0$ writes explicitly as

$$
\varepsilon^{(0)}_{\pm, \pm} = -e^{-i\pi/6} \ln Q_0^{(0)}(-e^{-2\pi i/3} u_k, \pm e^{2\pi i/3} m_k, \Lambda_1) - e^{i\pi/6} \ln Q_0^{(0)}(-e^{-2\pi i/3} u_k, \pm e^{-2\pi i/3} m_k, \Lambda_1) \pm \frac{8\pi m_k}{3\Lambda_1},
$$

(3.10)<sub>eps0ga</sub>

where $\ln Q_0^{(0)}(u, m, \Lambda_1)$ is given by the integral

$$
\ln Q_0^{(0)}(u, m, \Lambda_1) = \int_{-\infty}^{\infty} \left[ \frac{1}{\lambda_1} e^{2y} - 2 \right] dy.
$$

(3.11)<sub>lnQ0ga</sub>

For $N_f = 2$ also

$$
\varepsilon^{(0)}_{\pm, \pm} = -\ln Q_0^{(0)}(u, m_1, m_2, \Lambda_2) - \ln Q_0^{(0)}(u, -m_1, -m_2, \Lambda_2) \mp \frac{4\pi i}{\Lambda_2}(m_1 - m_2)
$$

(3.12)

and

$$
\ln Q_0^{(0)}(u, m_1, m_2, \Lambda_2) = \int_{-\infty}^{\infty} \left[ \frac{1}{\lambda_2} e^{2y} + \frac{16u}{\lambda_2^2} - 2 \right] dy.
$$

(3.13)

We can simply compute concretely $\ln Q_0^{(0)}$ by expanding the square root integrand in multiple binomial series for small parameters and then getting simple Beta function integrals. In particular, for $N_f = 1$ we get

$$
\ln Q_0^{(0)}(u, m, \Lambda_1) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \binom{1/2}{n} \binom{1/2 - n}{l} B_1(n, l) \left( \frac{4m}{\lambda_1} \right)^n \left( \frac{4u}{\lambda_1^2} \right)^l
$$

(3.15)<sub>lnQ0ga</sub>
with
\[ B_1(n, l) = \frac{1}{3} B \left( \frac{1}{6} (2 l + 4 n - 1), \frac{1}{3} (2 l + n - 1) \right) \quad (n, l) \neq (1, 0) \]
\[ B_1(1, 0) = \frac{2 \ln(2)}{3} \]
and for \( N_f = 2 \) we obtain
\[ \ln Q^{(0)}(u, m_1, m_2, \Lambda_2) = \sum_{l, m, n=0} \left( \frac{1}{2} \right) \left( \frac{1}{l} \right) \left( \frac{1}{m} \right) \left( \frac{l - m + \frac{1}{2}}{n} \right) B_2(l, m, n) \left( \frac{8m_1}{\Lambda_2} \right)^n \left( \frac{16u}{\Lambda_2^2} \right)^m \left( \frac{8m_2}{\Lambda_2} \right)^l \]
\[ B_2(l, m, n) = \frac{\Gamma \left( \frac{1}{2} (3l + 2m + n - 1) \right) \Gamma \left( \frac{1}{2} (l + 2m + 3n - 1) \right)}{4 \Gamma \left( l + m + n - \frac{1}{2} \right)} \]
\[ B_2(1, 0, 0) = \frac{1}{2} (\ln 2 - 1) \quad B_2(0, 0, 1) = \frac{1}{2} \ln 2. \]

Of course, when \( u, m, \Lambda_1 \) \( (u, m_1, m_2, \Lambda_2) \) are such that the leading order (3.10) computed through (3.11) has a negative real part, the TBA (3.7) no longer converges. In general, we find the convergence region to correspond to \( u, m (u, m_1, m_2) \) finite but small with respect to \( \Lambda_1 (\Lambda_2) \). For instance in the \( N_f = 1 \) massless case, this region corresponds on the real axis of \( u \) precisely to the strong coupling region \( -3\Lambda_1^2/2^{3/2} < u < 3\Lambda_1^2/2^{3/2} \). For \( N_f = 2 \) instead it corresponds to the region \( -3\Lambda_2^2/8 < u < 3\Lambda_2^2/8 \) [49].

Following [20, 13], it is easy to find the boundary condition at \( \theta \rightarrow -\infty \) for the gauge TBA
\[ \epsilon_{\pm,k}(\theta) \simeq -2 \ln \left( -\frac{2}{\pi} \theta \right) \simeq \hat{f}(\theta), \quad \theta \rightarrow -\infty, \]
\[ \hat{f}(\theta) = -\ln \left( 1 + \frac{2}{\pi} \ln \left( 1 + e^{-\theta - \frac{\pi}{2}} \right) \right) - \ln \left( 1 + \frac{2}{\pi} \ln \left( 1 + e^{-\theta + \frac{3\pi}{2}} \right) \right). \]

Numerically, this condition is imposed by modified the TBA equations to be
\[ \epsilon_{\pm,k}(\theta) = \epsilon_{\pm,k}^{(0)} e^\theta + \hat{f}(\theta) - \left( \varphi_+ \ast (L_{\pm,(k+1) \mod 3} + \hat{L}) \right)(\theta) - \left( \varphi_- \ast (L_{\pm,(k+2) \mod 3} + \hat{L}) \right)(\theta), \]
where \( \hat{L} \) is fixed by \( \hat{f} = (\varphi_+ + \varphi_-) \ast \hat{L} \). Under this boundary condition (3.19), the dilogarithm trick leads to the “effective central charge” associated with the TBA equations (3.7)
\[ c_{\text{eff}} = \frac{6}{\pi^2} \int d\theta e^\theta \sum_{j=0}^2 \epsilon_{\pm,j}^{(0)} L_{\pm,j}(\theta) = 3, \]
which coincides the numeric test and thus tests the validity of our boundary condition. We notice that even if we had not added the boundary condition at \( \theta \rightarrow -\infty \), the solution of the gauge TBA (3.7) would have been fixed anyway, just giving a less precise numerical solution. We remark that the same thing would have not been true for the integrability TBA (2.58), since the boundary condition is strictly necessary to fix \( p \), which does not enter the forcing term [20].

Similarly for \( N_f = 2 \) the effective central charge of \( N_f = 2 \) case is found to be
\[ c_{\text{eff}} = \frac{6}{\pi^2} \int d\theta e^\theta \sum_{\pm} \left( \epsilon_{\pm,\pm}^{(0)} L_{\pm,\pm}(\theta) + \epsilon_{\pm,\pm}^{(0)} \hat{L}_{\pm,\pm}(\theta) \right) = 4, \]
and we find the consistent boundary condition at $\theta \to -\infty$:
\[
\varepsilon_{\pm,\pm}(\theta) \simeq -2 \ln \left( -\frac{2\theta}{\pi} \right) \simeq -2 \ln \left[ 1 + \frac{2}{\pi} \ln(1 + e^{-\theta}) \right] \quad \theta \to -\infty \tag{3.24}
\]
and so
\[
\hat{f}(\theta) = -2 \ln \left( 1 + \frac{2}{\pi} \ln \left( 1 + e^{-\theta} \right) \right) . \tag{3.25}
\]

### 3.2 Seiberg-Witten gauge/integrability identification

We can now begin to find, first at the leading $\hbar \to 0$ ($\theta \to +\infty$) order, a relation between the integrability quantity $\varepsilon^{(0)}$ and the gauge periods. We suggest to look also at appendix C for the much simpler and illuminating proof for the $SU(2)$ $N_f = 0$ gauge theory. We do now the proof for $N_f = 1$. In (3.10) for $k = 0$ we have the following integral contributions
\[
e^{-i\pi/6} \ln Q^{(0)}(-e^{-2\pi i/3}u, e^{2\pi i/3}m) = \int_{-\infty - 2\pi i/3}^{\infty - 2\pi i/3} \frac{1}{\Lambda_1} \left[ 1 + e^{-y/2 - \pi i/3} \right] dy \tag{3.26}
\]
and
\[
e^{i\pi/6} \ln Q^{(0)}(-e^{-2\pi i/3}u, e^{-2\pi i/3}m) = \int_{-\infty + 2\pi i/3}^{\infty + 2\pi i/3} \left[ \frac{1}{\Lambda_1} \left[ 1 + e^{y/2 + \pi i/3} \right] - \frac{1}{1 + e^{y/2 - \pi i/3}} \right] dy . \tag{3.27}
\]

We notice that the integrands in (3.26) and (3.28) are equal except for the mass regularizing term, which gives an integral difference
\[
2 \frac{im}{\Lambda_1} \int_{-\infty + 2\pi i/3}^{\infty + 2\pi i/3} \left[ \frac{1}{1 + e^{-y/2 + \pi i/3}} - \frac{1}{1 + e^{y/2 - \pi i/3}} \right] dy = \frac{2}{\Lambda_1} \left[ \frac{4\pi m}{3} \right] . \tag{3.30}
\]

We can then consider only the integrand of $\ln Q^{(0)}(-e^{-2\pi i/3}u, e^{2\pi i/3}m)$. We observe that such integral is nothing but the Seiberg-Witten differential $\lambda$, up to a total derivative
\[
e^{-i\pi/6} \ln Q^{(0)}(-e^{-2\pi i/3}u, e^{2\pi i/3}m) = i \int_{-\infty - 2\pi i/3}^{\infty - 2\pi i/3} dy \left[ \frac{e^{2y} + e^{-y} + \frac{4m}{\Lambda_1} e^{y} - \frac{4u}{\Lambda_1^2}}{\sqrt{e^{2y} + e^{-y} + \frac{4m}{\Lambda_1} e^{y} - \frac{4u}{\Lambda_1^2}}} \right] - \frac{d}{dy} \left[ \frac{e^{2y} + e^{-y} + \frac{4m}{\Lambda_1} e^{y} - \frac{4u}{\Lambda_1^2}}{\sqrt{e^{2y} + e^{-y} + \frac{4m}{\Lambda_1} e^{y} - \frac{4u}{\Lambda_1^2}}} \right] \tag{3.32}
\]
\[
= 4i \int_{-\infty - 2\pi i/3}^{\infty - 2\pi i/3} dy \left[ \frac{3}{8} e^{-y} + \frac{2m}{\Lambda_1} e^{y} - \frac{u}{\Lambda_1^2} - \text{reg} \right] = -\frac{4\sqrt{2\pi}}{\Lambda_1} \int_{-\infty - 2\pi i/3}^{\infty - 2\pi i/3} \lambda(y,-u,m,\Lambda_1) dy \tag{3.33}
\]
where the SW differential $\lambda$ [50] is defined as usual in the variable $x = -\frac{\Lambda_1^2}{4} e^{-y}$ as
\[
\lambda(x,-u) dx = \frac{1}{2\pi \sqrt{2}} \frac{-u - \frac{3}{8} \frac{x^3}{4} - \frac{\Lambda_1 m}{4} x - \frac{\Lambda_1^2}{64}}{\sqrt{x^3 + u x^2 + \frac{\Lambda_1 m}{4} x - \Lambda_1^2}} dx = -\frac{1}{2\pi \sqrt{2}} \frac{-u - \frac{3}{8} \frac{x^3}{4} - \frac{\Lambda_1 m}{4} x - \frac{\Lambda_1^2}{64}}{\sqrt{e^{2y} + e^{-y} + \frac{4m}{\Lambda_1} e^{y} - \frac{4u}{\Lambda_1^2}}} dy = \lambda(y,-u) dy \tag{3.34}
\]
Figure 3.1: A strip of the $y$ complex plane, where in yellow we show the contour of integration of SW differential for the $SU(2)$ $N_f = 1$ theory we use for the proof equality of the (alternative) SW period $a_1^{(0)}$ and the leading $\hbar \to 0$ order of the (minus the logarithm of the) integrability $Y$ function $\varepsilon^{(0)} = -\ln Y^{(0)}$. In red are shown the branch cuts of the SW differential.

Now we consider for $-i\lambda(y)$ the countour of integration as in figure 3.2. We have horizontal branch cuts for $\text{Im }y = \pm \frac{\pi}{2}$, $\text{Re }y < \text{Re }y_1$ and other two curved branch cuts $b_{\pm}$ from the branch points $y_2, y_3$ to their asymptotics at $\text{Im }y = \pm \frac{\pi}{2}$ for $\text{Re }y \to +\infty$. (This can be shown easily by considering the asymptotics of $e^{2y} + e^{-y} + \frac{4m}{\Lambda_1} - \frac{4u^2}{\Lambda_1}$ at $\text{Re }y \to \pm \infty$ and $\text{Im }y = \pm \frac{\pi}{2}, \pm \pi$, which are negative real). Now, the integral from the complex-conjugate branch points $y_2$ and $y_3$ is defined as the alternative gauge period $a_1^{(0)}$ (see for the definition appendix B)

$$a_1^{(0)}(-u, m, \Lambda_1) = 2 \int_{y_2}^{y_3} \lambda(y, -u, m, \Lambda_1) dy \quad (3.35)$$

We now find some symmetry properties of $\lambda(y)$ for $y \in \mathbb{C}$. Since for $y \in \mathbb{R}$ and $m, \Lambda > 0$ and $u > 0$ not large we have

$$i\lambda(y) \in \mathbb{R} \quad y \in \mathbb{R} \quad (3.36)$$

the analytic continuation is such that

$$i\lambda(y^*) = (i\lambda(y))^* \quad y \in \mathbb{C} \quad (3.37)$$

From this it follows that along the branch cuts upper $b_+^\pm$ and lower $b_-^\pm$ edge of the curved branch cuts $b_{\pm}$, where $i\lambda \in i\mathbb{R}$ we have the properties

$$i\lambda(y) \bigg|_{b_+^\pm} = -i\lambda(y) \bigg|_{b_-^\pm} = -i\lambda(y) \bigg|_{b_-^\pm} = +i\lambda(y) \bigg|_{b_+^\pm} \in i\mathbb{R} \quad (3.38)$$

where of course the change of sign between $b_+^\pm$ and $b_-^\pm$ is due to the fact these are branch cuts for a square root. Thus by considering the integration contour $C_2 = (y_2, y_3) \cup b_+^\cup b_-^\pm$ closed also at infinity (where thanks to the regularization there is no contribution) we have

$$0 = \oint_{C_2} i\lambda(y) dy = -\frac{i}{2} a_1^{(0)} + 2 \int_{b_+^\pm} i\lambda(y) dy, \quad (3.39)$$

$^6$We notice that the integrand of (3.30) has poles only at $y = \pm 4\pi i/3$ with periodicity of $4\pi i$. 

16
that means we can express the gauge period also as an integral along the branch cut \( b_+ \)
\[
a_1^{(0)} = 4 \int_{b_+} \lambda(y) \, dy. \tag{3.40}
\]

On the other hand by considering the integration contour \( C_1 = (-\infty + 2\pi i/3, +\infty + 2\pi i/3) \cup b_+ \cup b_- \cup b_+ \cup b_- \cup (\infty - 2\pi i/3, -\infty - 2\pi i/3) \) closed also at infinity we have
\[
0 = \oint_{C_1} i\lambda(y) \, dy = +\frac{1}{8\sqrt{2}\pi}\varepsilon^{(0)}(u, im) - 4 \int_{b_+} i\lambda(y) \, dy
\]
Hence
\[
\varepsilon^{(0)}(u, im, \Lambda_1) = \frac{4\sqrt{2}\pi}{\Lambda_1} a_1^{(0)}(-u, m, \Lambda_1) \tag{3.42}
\]
This result is also confirmed numerically. The change of basis of the periods is, at least for \( u > 0 \) (see for the derivation appendix B)
\[
a^{(0)}(-u, m) = -a_1^{(0)}(-u, m) + a_2^{(0)}(-u, m) + \frac{m}{\sqrt{2}} \tag{3.43}
\]
\[
a_D^{(0)}(-u, m) = -2a_1^{(0)}(-u, m) + a_2^{(0)}(-u, m) + \frac{3m}{2\sqrt{2}}
\]
Hence, we can write the gauge-integrability relation for all 3 gauge TBA’s forcing terms as
\[
\varepsilon^{(0)}(u, im) = 2\pi\sqrt{2} \left[ a^{(0)}(-u, m) - a_D^{(0)}(-u, m) + \frac{m}{2\sqrt{2}} \right] \frac{2\sqrt{2}}{\Lambda_1}
\]
\[
\varepsilon^{(0)}(e^{2\pi i/3}u, ie^{-2\pi i/3}m) = 2\pi\sqrt{2} \left[ a^{(0)}(-e^{2\pi i/3}u, e^{-2\pi i/3}m) - a_D^{(0)}(-e^{2\pi i/3}u, e^{-2\pi i/3}m) + \frac{e^{-2\pi i/3}m}{\sqrt{2}} \right] \frac{2\sqrt{2}}{\Lambda_1}
\]
\[
\varepsilon^{(0)}(e^{-2\pi i/3}u, ie^{2\pi i/3}m) = 2\pi\sqrt{2} \left[ -2a^{(0)}(-e^{-2\pi i/3}u, e^{2\pi i/3}m) + a_D^{(0)}(-e^{-2\pi i/3}u, e^{2\pi i/3}m) + \frac{e^{2\pi i/3}m}{2\sqrt{2}} \right] \frac{2\sqrt{2}}{\Lambda_1}. \tag{3.44}
\]
We notice that for all three pseudoenergies of the gauge TBA we find that the forcing term (leading order) is of the form of a central charge for the SW theory for \( SU(2) \) with \( N_f = 1 \) \( [2] \):
\[
Z = n_m a_D^{(0)} - n_e a^{(0)} + \frac{s \, m}{\sqrt{2}}, \tag{3.45}
\]
so that the mass of the BPS state is \( M_{BPS} = \sqrt{2}|Z| \) \( [50] \). We find a perfect match between the expected electric and magnetic charges \( n_e, n_m \) which multiply the periods \( a^{(0)} \) and \( a_D^{(0)} \) respectively (precisely, \( (-1, 0), (1, -1) \) and \( (0, 1) \) \( [49] \)). We notice that with this observation on the spectrum the same TBA equations can be derived
\[\text{We have checked this expression also numerically through the use of elliptic integrals of appendix B [50, 51] for the periods and the hypergeometric integral (77) to calculate \( \varepsilon^{(0)}_{\pm, \pm} \).}
\[\text{The mass constant term (physical flavour charge [52]) is ambiguous, but that it is just because the periods themselves are defined up to the well-known SW monodromy of exactly an integer multiple of \( \frac{2\pi i}{\sqrt{2}} \) [2, 51, 50]. We emphasize that that the central charge and mass of BPS states have no ambiguity. We notice also that, in integrability, there is no ambiguity since the wave functions and therefore the \( Q \) function in (2.39) cannot change. In other words, we are fixing through integrability what is in gauge theory is in general ambiguous.}
\[\text{The periods \( (a^{(0)}, a_D^{(0)}) \) are discontinuous on the moduli space, due to the singularities, and can be analytically continued to \( (\tilde{a}^{(0)}, \tilde{a}_D^{(0)}) \) by using the monodromy matrix around the singularity on the moduli space. Correspondingly, the charges \( (n_e, n_m) \) will also be transformed by the inverse of the monodromy matrix, because one needs to keep the physical mass and central charge invariant. Since the driving terms of TBA equations are given by the central charge, more precisely the BPS mass, the TBA equations are invariant under the monodromy transformation.} \]
Figure 3.2: A strip of the $y$ complex plane, where in yellow we show the contour of integration of SW differential for the $SU(2) \ N_f = 2 = (1,1)$ theory we use for the proof equality of the (alternative) SW period $a^{(0)}_2$ and the leading $\hbar \to 0$ order of the (minus the logarithm of the) integrability $Y$ function $\varepsilon^{(0)} = - \ln Y^{(0)}$. In red are shown the branch cuts of the SW differential.

formally by taking the conformal limit of the integral equations in the framework of Gaiotto, Moore and Neitzke in[23, 24, 53, 25, 13]. However, we remark that though their framework is used very generally, it is for that very reason arguably more conjectural than our bottom-up approach from the precise four dimensional gauge theory ODEs.

Similarly for $N_f = 2$ we find At the $\hbar \to 0$ leading SW order we have the relations, for $u, m_1, m_2 > 0$

$$\varepsilon^{(0)}(-u, im_1, -im_2, \Lambda_2) = \frac{8\sqrt{2\pi}}{\Lambda_2} a^{(0)}_D(u, m_1, m_2, \Lambda_2)$$

$$\varepsilon^{(0)}(-u, -im_1, im_2, \Lambda_2) = \frac{8\sqrt{2\pi}}{\Lambda_2} a^{(0)}_D(u, m_1, m_2, \Lambda_2) + \frac{8\pi}{\Lambda_2} (m_1 + m_2)$$

$$\varepsilon^{(0)}(u, m_1, m_2, \Lambda_2) = \frac{8\sqrt{2\pi}}{\Lambda_2} a^{(0)}_D(-u, -im_1, im_2, \Lambda_2) - \frac{4\pi i}{\Lambda_2} (m_1 - m_2)$$

$$\varepsilon^{(0)}(u, -m_1, -m_2, \Lambda_2) = \frac{8\sqrt{2\pi}}{\Lambda_2} a^{(0)}_D(-u, -im_1, im_2, \Lambda_2) + \frac{4\pi i}{\Lambda_2} (m_1 - m_2)$$

We give an analytic proof also of this result. The leading order of $\varepsilon$ as $\hbar \to 0$ (that is, $\theta \to \infty$) is

$$\varepsilon(\theta, u, m_1, m_2, \Lambda_2) \simeq e^\theta \varepsilon^{(0)}(u, m_1, m_2, \Lambda_2)$$

$$= e^\theta \left[ -\ln Q^{(0)}(u, m_1, m_2, \Lambda_2) - \ln Q^{(0)}(u, -m_1, -m_2, \Lambda_2) + \frac{4\pi i}{\Lambda_2} (m_1 - m_2) \right]$$

(3.47)
with
\[
\ln Q^{(0)}(u, m_1, m_2, \Lambda_2) = \int_{-\infty}^{\infty} \left[ \sqrt{e^{2y} + e^{-2y} + \frac{8m_1}{\Lambda_2} e^y + \frac{8m_2}{\Lambda_2} e^{-y} + \frac{16u}{\Lambda^2}} 
- 2 \cosh y \frac{4m_1}{\Lambda_2} \frac{1}{1 + e^{-y/2}} - \frac{4m_2}{\Lambda_2} \frac{1}{1 + e^{y/2}} \right] dy
\]

\[
\ln Q^{(0)}(u, -m_1, -m_2, \Lambda_2) = \int_{-\infty}^{\infty} \left[ \sqrt{e^{2y} + e^{-2y} - \frac{8m_1}{\Lambda_2} e^y - \frac{8m_2}{\Lambda_2} e^{-y} + \frac{16u}{\Lambda^2}} 
- 2 \cosh y \frac{4m_1}{\Lambda_2} \frac{1}{1 + e^{-y/2}} + \frac{4m_2}{\Lambda_2} \frac{1}{1 + e^{y/2}} \right] dy
\]  (3.48)

Now we can trade the change of sign in the masses as a shift in \( y \) by \( i\pi \)

\[
\ln Q^{(0)}(u, -m_1, -m_2, \Lambda_2) = \int_{-\infty + i\pi}^{\infty + i\pi} \left[ \sqrt{e^{2y} + e^{-2y} + \frac{8m_1}{\Lambda_2} e^y + \frac{8m_2}{\Lambda_2} e^{-y} + \frac{16u}{\Lambda^2}} 
+ 2 \cosh y \frac{4m_1}{\Lambda_2} \frac{1}{1 + ie^{-y/2}} + \frac{4m_2}{\Lambda_2} \frac{1}{1 - ie^{y/2}} \right] dy
\]  (3.49)

We can use the same integrand and integrate it in the countour of figure 3.2 if we separate and add outside the term coming from the regularizing part

\[
\int_{-\infty}^{\infty} \left[ \frac{4(m_1 e^{y/2} + m_2)}{\Lambda_2 (e^{y/2} + 1)} - \frac{4(m_1 e^{y/2} + im_2)}{\Lambda_2 (e^{y/2} + i)} \right] dy = \frac{4i\pi(m_1 - m_2)}{\Lambda_2}
\]  (3.50)

Now the SW differential as defined in appendix A from the quartic SW curve (A.5) gives

\[
\int \lambda(x, -u, -im_1, im_2, \Lambda_2) dx = \int \left[ \sqrt{e^{2y} + e^{-2y} + \frac{8m_1}{\Lambda_2} e^y + \frac{8m_2}{\Lambda_2} e^{-y} + \frac{16u}{\Lambda^2}} 
- 2 \cosh y \frac{4m_1}{\Lambda_2} \frac{1}{1 + e^{-y/2}} - \frac{4m_2}{\Lambda_2} \frac{1}{1 + e^{y/2}} + d \frac{d}{dy} (...) \right] dy
\]  (3.51)

Therefore

\[
\varepsilon^{(0)}(u, m_1, m_2, \Lambda_2) = \int \lambda(y) dy - \frac{4\pi i}{\Lambda_2} (m_1 - m_2)
\]  (3.52)

We notice that for \( y = t + is \) along the (almost) horizontal branch cuts we have

\[
\text{Re} \: \mathcal{P}^{(0)}(y) = 0
\]
\[
\text{Im} \: \mathcal{P}^{(0)}(t + is) = -\text{Im} \: \mathcal{P}^{(0)}(-t + is)
\]  (3.53)

so that the only contribution is from the vertical branch cuts. That is

\[
\varepsilon^{(0)}(u, m_1, m_2, \Lambda_2) = \frac{8\sqrt{2\pi}}{\Lambda_2} a_2^{(0)}(-u, -im_1, im_2, \Lambda_2) - \frac{4\pi i}{\Lambda_2} (m_1 - m_2)
\]  (3.54)

For \( u \to \infty, \Lambda_2 \) we have \( a_2^{(0)}(-u, m_1, m_2, \Lambda_2) \sim a_D(-u, m_1, m_2, \Lambda_2) \sim \frac{i}{2\pi} \sqrt{2u} \ln \frac{u}{\Lambda^2} \) and then (3.46) follows.

In this way TBA (6.5) constitute a generalization of that found in [54] \( N_f = 2 \) gauge theory with equal masses \( m_1 = m_2 \) respectively (see a numerical test for different masses below in table 3.2).
3.3 Exact quantum gauge/integrability identification for $Y$

We can use the following differential operators [12] to get higher $k \to 0$ ($\theta \to +\infty$) orders of either the periods $a_k$ or $\ln Q$

$$a_k(\theta, u, m, \Lambda_1) \simeq \sum_{n=0}^{\infty} e^{-2n\theta} a_k^{(n)}(u, m, \Lambda_1) \quad \theta \to +\infty$$

$$\ln Q(\theta, u, m, \Lambda_1) \simeq \sum_{n=0}^{\infty} e^{\theta(1-2n)} \ln Q^{(n)}(u, m, \Lambda_1) \quad \theta \to +\infty$$  (3.55)

For $N_f = 1$ they are

$$a_k^{(1)}(u, m, \Lambda_1) = \left( \frac{\Lambda_1}{2} \right)^2 \frac{1}{12} \left[ \frac{\partial}{\partial u} + 2m \frac{\partial}{\partial m} \frac{\partial}{\partial u} + 2u \frac{\partial^2}{\partial u^2} \right] a_k^{(0)}(u, m, \Lambda_1)$$

$$a_k^{(2)}(u, m, \Lambda_1) = \left( \frac{\Lambda_1}{2} \right)^4 \frac{1}{1440} \left[ 28m^2 \frac{\partial^2}{\partial u^2} \frac{\partial^2}{\partial m^2} + 28u^2 \frac{\partial^4}{\partial u^4} + 132m \frac{\partial^2}{\partial u^2} \frac{\partial}{\partial m} + 56mu \frac{\partial^3}{\partial u^3} \frac{\partial}{\partial m} \right] a_k^{(0)}(u, m, \Lambda_1)$$

$$+ \frac{1}{4} \frac{\partial^2}{\partial u^2} + \frac{1}{2} \frac{\partial^3}{\partial u^3} \right] a_k^{(0)}(u, m, \Lambda_1)$$

(3.56) [oper-h]

and for $N_f = 2$

$$a_k^{(1)}(u, m_1, m_2, \Lambda_2) = \left( \frac{\Lambda_2}{4} \right)^2 \frac{1}{6} \left[ 2u \frac{\partial^2}{\partial u^2} + \frac{3}{2} \left( m_1 \frac{\partial}{\partial m_1} \frac{\partial}{\partial u} + m_2 \frac{\partial}{\partial m_2} \frac{\partial}{\partial u} \right) + \frac{\partial}{\partial u} \right] a_k^{(0)}(u, m_1, m_2, \Lambda_2)$$

$$a_k^{(2)}(u, m_1, m_2, \Lambda_2) = \left( \frac{\Lambda_2}{4} \right)^4 \frac{1}{360} \left[ 28u^2 \frac{\partial^4}{\partial u^4} + 120u \frac{\partial^3}{\partial u^3} + 75 \frac{\partial^2}{\partial u^2} + 42 \left( m_1 \frac{\partial}{\partial m_1} \frac{\partial^3}{\partial u^3} + m_2 \frac{\partial}{\partial m_2} \frac{\partial^3}{\partial u^3} \right) \right]$$

$$+ \left( m_1 \frac{\partial}{\partial m_1} \frac{\partial^2}{\partial u^2} + m_2 \frac{\partial}{\partial m_2} \frac{\partial^2}{\partial u^2} \right) + \frac{63}{4} \left( m_1 \frac{\partial}{\partial m_1} \frac{\partial^2}{\partial u^2} + m_2 \frac{\partial}{\partial m_2} \frac{\partial^2}{\partial u^2} \right)$$

(3.57) [diffop]

The same operators can be used also to obtain $\ln Q^{(n)}$ of course.

Remarkably, we find the same higher orders of $a_k$ to be given by the asymptotic expansion of the gauge TBA.

For $N_f = 1$

$$\varepsilon^{(1)}_{+,0} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta e^{\theta} \left\{ -2(-1)^{5/6} L_{+,1}(\theta') + 2(-1)^{1/6} L_{+,2}(\theta') \right\}$$

$$= -e^{i\pi/6} \ln Q^{(1)}(-e^{-2\pi i/3} u, e^{2\pi i/3} m, \Lambda_1) - e^{-i\pi/6} \ln Q^{(1)}(-e^{2\pi i/3} u, e^{-2\pi i/3} m, \Lambda_1)$$

$$= -\frac{4\pi \sqrt{2}}{\Lambda_1} a_1^{(1)}(-u, m, \Lambda_1)$$

(3.58) [higher]

$$\varepsilon^{(2)}_{+,0} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta e^{3\theta} \left\{ 2i L_{+,1}(\theta') - 2i L_{+,2}(\theta') \right\}$$

$$= -e^{i\pi/2} \ln Q^{(2)}(-e^{-2\pi i/3} u, e^{2\pi i/3} m, \Lambda_1) - e^{-i\pi/2} \ln Q^{(2)}(-e^{2\pi i/3} u, e^{-2\pi i/3} m, \Lambda_1)$$

$$= \frac{4\pi \sqrt{2}}{\Lambda_1} a_1^{(2)}(-u, m, \Lambda_1)$$
Table 3.1: Comparison between the higher $\hbar \to 0$ asymptotic expansion modes for the $N_f = 1$ gauge theory and Perturbed Hairpin IM. The first line is the result from the $\theta \to \infty$ expansion of the gauge TBA (3.7). The second line is the result from the differential operators (3.56) acting on the leading order hypergeometric functions (3.10), (3.15). The third line are the higher periods computed through the same differential operators acting on the elliptic integral of the SW order, as in appendix B, with $c_k = \frac{i\pi\sqrt{2}}{\Lambda^k} \left( \frac{\Lambda^2}{\pi} \right)^k$ and $d_k = \left( \frac{\Lambda^2}{\pi} \right)^k$. Here the parameters are $u = 0.1, \Lambda_1 = 1, m = \frac{1}{20\sqrt{2}}$ and of course $u_k = e^{i\pi k/3} u, m_k = e^{-i\pi k/3} m$.

| $\varepsilon(k)$ $(u, m_1, m_2, \Lambda_2)$ $c_k a_1^{(k)} (-u_1, m_0)$ | $k=1$ | $k=2$ |
|---|---|---|
| $\varepsilon_{+1}$ $d_k \Delta \ln Q^{(k)} + 0$ | $-0.140549 + 0.00193600i$ | $-0.00142739 + 0.00311155i$ |
| $(-1)^k c_k a_1^{(k)} (-u_1, m_1)$ | $-0.140552 + 0.00193603i$ | $-0.00142740 + 0.00311157i$ |

Table 3.2: Comparison of higher orders $\varepsilon^{(k)}$ from gauge TBA (6.5) and $a_2^{(k)}$ from elliptic integrals (through differential operators (3.57), with $c_k = \frac{8\pi i\sqrt{2}}{\pi^2} \Lambda_2^{2k-1}$. In the second and third column $m_1 = m_2 = \frac{1}{8}, \Lambda_2 = 4, u = 1$. In the fourth and fifth column $m_1 = \frac{1}{16}, m_2 = \frac{1}{8}, \Lambda_2 = 4, u = 1$.

| $\varepsilon^{(k)}(u, m_1, m_2, \Lambda_2)$ $c_k a_2^{(k)} (-u, -im_1, im_2, \Lambda_2)$ | $k=1, m_1 = m_2$ | $k=2, m_1 = m_2$ | $k = 1, m_1 \neq m_2$ | $k = 2, m_1 \neq m_2$ |
|---|---|---|---|---|
| $\varepsilon^{(k)}(u, m_1, m_2, \Lambda_2)$ $c_k a_2^{(k)} (-u, -im_1, im_2, \Lambda_2)$ | $-0.5025004$ | $0.3120101$ | $-0.5000211$ | $0.29418949$ |

The numerical check is shown in table 3.1. Thus we have the asymptotic expansion, for $N_f = 1$

$$\varepsilon(\theta, u, \Lambda_1) \equiv \sum_{n=0}^{\infty} e^{\theta(1-2n)} \varepsilon(n)(u, \im \Lambda_1) = \frac{4\sqrt{2} \pi}{\Lambda_1} \sum_{n=0}^{\infty} e^{\theta(1-2n)} (-1)^n a_1^{(n)}(-u, \im \Lambda_1) \quad \theta \to +\infty$$

For $N_f = 2$ we have similarly

$$\varepsilon(n)^{(n)}_{++} = \left[ -\frac{1}{\pi} \int_{-\infty}^{\infty} d\theta e^{\theta(2n-1)} \left[ L_{++}(\theta') + L_{-\im}(\theta') \right] \right]$$

$$= - \ln Q^{(n)}(u, m_1, m_2, \Lambda_2) - \ln Q^{(n)}(u, -m_1, -m_2, \Lambda_2)$$

$$= (-1)^n \frac{8\pi i\sqrt{2}}{\Lambda_2} a_2^{(n)}(-u, -im_1, im_2, \Lambda_2)$$

$$\varepsilon(n)^{(n)}_{++} = \left[ -\frac{1}{\pi} \int_{-\infty}^{\infty} d\theta e^{\theta(2n-1)} \left[ L_{++}(\theta') + L_{-\im}(\theta') \right] \right]$$

$$= - \ln Q^{(n)}(-u, -im_1, im_2, \Lambda_2) - \ln Q^{(n)}(-u, im_1, -im_2, \Lambda_2)$$

$$= (-1)^n \frac{8\pi i\sqrt{2}}{\Lambda_2} a_2^{(n)}(u, m_1, m_2, \Lambda_2).$$
Figure 3.3: Plots (at low and high magnification) of the matching between the $N_f = 1$ gauge and Perturbed Hairpin IM pseudoenergies $\varepsilon(\theta, u, im)$ and $\varepsilon(\theta, P, iq)$ for $u = 0.1, q = 0.1, \Lambda = 1$, for $\theta_0 = 0$.

Table 3.3: Table which shows the very good match between $N_f = 1$ gauge and integrability pseudoenergies at $\theta = \theta_0 = 0$ with parameters $u = 0.1, m = \frac{1}{20}, \Lambda = 1$. In the third line we show also a match with the result from direct numerical integration of the Riccati equation (2.43).

| $(\theta_0, q_1, q_2, P)$ | $(0, \frac{1}{10}, \frac{1}{20}, 1)$ | $(0, \frac{1}{10}, \frac{1}{20}, 1)$ |
|--------------------------|-------------------------------|-------------------------------|
| $\varepsilon^{\text{INT}}_{\pm \pm}$ | 1.428378 | 1.416945047 ± 0.19634954i |
| $\varepsilon^G_{\pm \pm}$ | 1.428383 | 1.416939137 ± 0.19634954i |
| $\varepsilon^{\text{INT}}_{\pm \pm}$ | 1.4133849 ± 0.78539816i | 1.40946127 ± 0.58904862i |
| $\varepsilon^G_{\pm \pm}$ | 1.4133714 ± 0.78539816i | 1.40944721 ± 0.58904862i |

Table 3.4: Comparison of $N_f = 2$ gauge and generalized Perturbed Hairpin IM TBA for different values of parameters.

The numerical check is shown in table 3.2. Thus we have the asymptotic expansion, for $N_f = 2$

$$\varepsilon(\theta, u, \pm m_1, \pm m_2, \Lambda_2) = \sum_{n=0}^{\infty} e^{\theta(1-2n)} \varepsilon(n)(u, \pm m_1, \pm m_2, \Lambda_2)$$

$$= 8\sqrt{2\pi} \frac{\Lambda}{\Lambda_2} \left[ e^\theta a_2^{(0)} (-u, -im, +im_2, \Lambda_2) + \frac{1}{2\sqrt{2}}(im_1 - im_2) + \sum_{n=1}^{\infty} e^{\theta(1-2n)} a_2^{(n)} (-u, -im, +im_2, \Lambda_2) \right] \theta \to +\infty$$

(3.61)

Therefore we can identify the exact gauge pseudoenergy $\varepsilon$ as defining the exact periods $a_k$. Moreover, we can numerically prove that the exact gauge pseudoenergy is equivalent, under change of variable, to the exact integrability pseudoenergy.

$$\varepsilon(\theta, p, q) = \varepsilon(\theta, u, m, \Lambda)$$

$$\varepsilon(\theta, p, q_1, q_2) = \varepsilon(\theta, u, m_1, m_2, \Lambda_2)$$

(3.62)

This check is shown in tables 3.3-3.4 and figure 3.3.

We have defined the exact gauge periods as cycle integrals of the solution of the Riccati equation $P(y)$, the Seiberg-Witten quantum differential (see appendix C). However, in gauge theory they are properly defined from the
\{\theta, \Lambda_1, p, q\} \sim \{5, 0.1, 5, 0.1\} \quad \sim \{-2.5, 0.1, 10, 0.1\} \quad \sim \{0, 0.1, 5, 0.1\}

\{h, u, m, \frac{\Lambda_1^2}{u}\} \sim (5, 10^3, 0.5, 10^{-6}) \quad \sim \{0.5, 10^2, 10^{-1}, 10^{-4}\} \quad \sim \{10^{-2}, 10^{-1}, 10^{-3}, 10^{-1}\}

\varepsilon(\theta, p, iq) = -267.1186026 \quad \sim -381.1795517 \quad \sim -54.9818090

\frac{2\pi \sqrt{2^a - aD + \frac{\Lambda_1^2}{i\hbar}}}{\hbar(\theta)} \sim -267.1186297 \quad \sim -381.1797573 \quad \sim -54.9949700

Table 3.5: A table which shows the match between the integrability pseudoenergy for positive mass \(i\) in the \(\theta\)-non-perturbative region and the instanton expansion for the right combination of the \(N_f = 1\) gauge periods which we have analytically proven to be equal to it.

This instanton expansion (around \(\Lambda_1 = 0\)), which is, for also small \(h\), for \(N_f = 1\)

\begin{align*}
a(\theta, u, m, \Lambda_1) &= \sqrt{\frac{u}{2}} \left[ \frac{\Lambda_1^3}{2^4\sqrt{2}} \right] + \frac{3\Lambda_1^6}{2^6\sqrt{2}} + \ldots + \hbar(\theta)^2 \left[ \frac{\Lambda_1^3}{2^6\sqrt{2}} \right] + \frac{15\Lambda_1^6}{2^8\sqrt{2}} + \ldots + \hbar(\theta)^4 \left[ \frac{\Lambda_1^3}{2^8\sqrt{2}} \right] + \frac{63\Lambda_1^6}{2^{10}\sqrt{2}} + \ldots + \hbar(\theta)^n \left[ \frac{\Lambda_1^3}{2^{n+2}\sqrt{2}} \right] + \ldots,
\end{align*}

\begin{align*}
a_D(\theta, u, m, \Lambda_1) &= \frac{i}{2\sqrt{2\pi}} \left[ \sqrt{\frac{u}{2}} \left[ h(\theta)^{2k}(u, m, \Lambda_1) \right] \left( i\pi - 3\ln \frac{16u}{\Lambda_1^2} \right) \right] \left( \frac{6\sqrt{u} + \frac{m^2}{\sqrt{u}} + \frac{m^4}{6\Lambda_1^2}}{\Lambda_1^2} \right) + \ldots + \hbar(\theta)^2 \left[ \frac{1}{4\sqrt{u}} \right] + \frac{m^2}{12\sqrt{u}} + \frac{m^4}{64\Lambda_1^2} + \ldots + \hbar(\theta)^4 \left[ \frac{1}{160\sqrt{u}} \right] + \frac{7m^2}{240\sqrt{u}} + \frac{7m^4}{960} + \frac{127\Lambda_1^2}{2560} + \ldots + \ldots
\end{align*}

The results are shown in table 3.5. Notice that through formulas (3.63) and (3.64) we can reach even the non-perturbative (non-WKB) large \(h\) regime, the important thing to be necessarily small being the ratio \(\Lambda_1^2/\sqrt{u}\).

Hence we find a first identification between an integrability quantity, the \(Y\) function, and the exact gauge periods.

For \(N_f = 1\), for \(u, m, \Lambda_1 > 0\)

\begin{align*}
\varepsilon(\theta, p, iq) &= \frac{2\pi \sqrt{2}}{h(\theta)} \left[ a(\theta - i\pi/2, -u, m) - a_D(\theta - i\pi/2, -u, m) + \frac{m}{\sqrt{2}} \right] \quad u, m, \Lambda_1 > 0
\end{align*}

or more generally for \(u, m \in \mathbb{C}\), with \(\arg u = -\arg m\)

\begin{align*}
\varepsilon(\theta, p, iq) &= \frac{2\pi \sqrt{2}}{h(\theta)} a_1(\theta - i\pi/2, -u, m) - \frac{2\pi \sqrt{2} i}{h(\theta - i\pi/2)} a_1(\theta - i\pi/2, -u, m) \quad u, m \in \mathbb{C} \quad \arg u = -\arg m \quad \Lambda_1 > 0
\end{align*}
Similarly for \( N_f = 2 \) and \( u, m, \Lambda_2 > 0 \)\(^{10} \)

\[
\varepsilon(\theta, ip, iq_1, -iq_2) = \frac{2\sqrt{2\pi}}{h(\theta)} a_D(\theta, u, m_1, m_2, \Lambda_2)
\]

\[
\varepsilon(\theta, ip, -iq_1, iq_2) = \frac{2\sqrt{2\pi}}{h(\theta)} \left[ a_D(\theta, u, m_1, m_2) + \frac{1}{\sqrt{2}}(m_1 + m_2) \right]
\]

\[
\varepsilon(\theta, p, q_1, q_2) = \frac{2\sqrt{2\pi}}{h(\theta)} \left[ a_D(\theta, -u, -im_1, im_2, \Lambda_2) - i \frac{1}{2\sqrt{2}}(m_1 - m_2) \right]
\]

\[
\varepsilon(\theta, p, -q_1, -q_2) = \frac{2\sqrt{2\pi}}{h(\theta)} \left[ a_D(\theta, -u, -im_1, im_2, \Lambda_2) + i \frac{1}{2\sqrt{2}}(m_1 - m_2) \right]
\]  

Relations (3.65)-(3.67) show a new connection between the \( SU(2) \) \( N_f = 1, 2 \) gauge periods and the \( Y \) function (Generalized) Perturbed Hairpin integrable model. This generalizes to the case of massive hypermultiplets matter the integrability-gauge correspondence already developed for the \( SU(2) \) \( N_f = 0 \) and the self-dual Liouville model (cf. the first (1.3), with \( Q = \sqrt{Y} \)) [17]. (3.65) and (3.67) are in some sense expressions for a \( N_f = 1, 2 \) SW exact central charge. As explained in appendix C by considering different particles in the spectrum or definition of gauge periods other than the integral one, different relations could be found like those for the \( N_f = 0 \) and \( N_f = 1 \) theory in [13, 14].

Besides, we remark that these gauge-integrability identifications holds as they are written only in a restricted strip of of the complex \( \theta \) plane: \( \text{Im} \theta < \pi / 3 \) and \( \text{Im} \theta < \pi / 2 \) for the \( N_f = 1 \) and \( N_f = 2 \) theory. Beyond such strips the gauge TBAs (3.7) (6.5) needs analytic continuation (of its solution) since poles of the kernels are found on the \( \theta' \) integrating axis. A modification of TBAs equation as usually done in integrability by adding the residue is possible, but then the \( Y \)s no longer identifies with the gauge periods: in fact the former are entire functions while the latter are not [55, 56, 57]. A similar singular behaviour is found using the \( y \) integral definition of gauge periods as explained in appendix C. This is a manifestation of the so-called wall-crossing phenomenon, whereby the spectrum of SW theory changes and therefore a fundamental change in its relation to integrability is to be expected. We hope to investigate further and write more on this issue in the future.

\section{Integrability \( T \) function and gauge period}

\subsection{\( T \) function and Floquet exponent}

In this subsection we follow and adapt the monograph on Doubly Confluent Heun equation in [44]. Define the periodicity operator

\[
\Upsilon \psi(y) = \psi(y + 2\pi i)
\]  

(4.1)

We can express \( \Upsilon \) in terms of the \( \Omega_\pm \) symmetry operators, for \( N_f = 1 \) as

\[
\Upsilon = \Omega_+^2 \Omega_-^{-1}
\]  

(4.2)

and for \( N_f = 2 \) as

\[
\Upsilon = \Omega_+^2 \Omega_-^{-2}
\]  

(4.3)

Then we write, for \( N_f = 1 \)

\[
\psi_{+,1}(y + 2\pi i) = e^{2\pi i q} \psi_{+,1} = -e^{2\pi i q} \psi_{+,1} + i e^{i\pi q} \hat{T}_+(\theta) \psi_{+,0}
\]

(4.4)

\[
\psi_{+,0}(y + 2\pi i) = -e^{i\pi q} \hat{T}_-(\theta + i\pi / 3) \psi_{+,1} + [-e^{-2\pi i q} + \hat{T}_-(\theta + i\pi / 3) \hat{T}_+(\theta)] \psi_{+,1}
\]

\(^{10}\)We remark that the first two relation with imaginary \( p \) parameters are not directly implemented in the integrability variables (since the integrability TBA does not converge), but they will in the gravity variables in section 6 (in (6.11) precisely this range of parameters is involved).
and for $N_f = 2$

$$
\psi_{+,1}(y + 2\pi i) = \psi_{+,1} = -e^{2\pi i q_1} \psi_{+,1} + i e^{\pi i q_1} \tilde{T}_{+,+}(\theta) \psi_{+,0},
$$

$$
\psi_{+,0}(y + 2\pi i) = \psi_{+,2} = -e^{i\pi q_1} \tilde{T}_{-,-}(\theta + i\pi/2) \psi_{+,1} + [-e^{-2\pi i q_1} + \tilde{T}_{-,-}(\theta + i\pi/2) \tilde{T}_{+,+}(\theta)] \psi_{+,1}.
$$

(4.5)

We can write these relations also in matrix form

$$
\mathcal{T} \psi_+ = \mathcal{T}_+ \psi_+
$$

(4.6)

where we defined $\psi = (\psi_{+,1}, \psi_{+,0})$ and, for $N_f = 1$

$$
\mathcal{T}_+ = \begin{pmatrix}
-e^{2\pi i q} & -e^{i\pi q_1} \tilde{T}_{+,+}(\theta) \\
e^{i\pi q_1} \tilde{T}_{-,-}(\theta + i\pi/3) & -e^{-2\pi i q_1} + \tilde{T}_{-,-}(\theta + i\pi/3) \tilde{T}_{+,+}(\theta)
\end{pmatrix}
$$

(4.7)

and for $N_f = 2$

$$
\mathcal{T}_+ = \begin{pmatrix}
-e^{2\pi i q_1} & -e^{i\pi q_1} \tilde{T}_{+,+}(\theta) \\
e^{i\pi q_1} \tilde{T}_{-,-}(\theta + i\pi/2) & -e^{-2\pi i q_1} + \tilde{T}_{-,-}(\theta + i\pi/2) \tilde{T}_{+,+}(\theta)
\end{pmatrix}.
$$

(4.8)

Now we can say that $\nu$ is a characteristic exponent of the Doubly confluent Heun equation (2.4) if and only if $e^{\pm 2\pi i \nu}$ are eigenvalues of $\mathcal{T}_+$. It then follows that $\nu$ is determined from

$$
2 \cos 2\pi \nu = \text{tr} \mathcal{T}_+
$$

(4.9)

or more explicitly, for $N_f = 1$

$$
2 \cos 2\pi \nu + 2 \cos 2\pi q = 4 \cos \pi (q + \nu) \cos \pi (q - \nu) = \tilde{T}_+(\theta) \tilde{T}_-(\theta + i\pi/3)
$$

(4.10)

and for $N_f = 2$

$$
2 \cos 2\pi \nu + 2 \cos 2\pi q_1 = 4 \cos \pi (q_1 + \nu) \cos \pi (q_1 - \nu) = \tilde{T}_{+,+}(\theta) \tilde{T}_{-,-}(\theta + i\pi/2)
$$

(4.11)

Similarly we can prove relations for $T$, for $N_f = 1$

$$
2 \cos 2\pi \nu = 4 \cos^2 \pi \nu = T_+(\theta) T_-(\theta + i\pi/3) = T_+^2(\theta)
$$

(4.12)

and for $N_f = 2$

$$
2 \cos 2\pi \nu + 2 \cos 2\pi q_2 = 4 \cos \pi (q_2 + \nu) \cos \pi (q_2 - \nu) = T_{+,+}(\theta) T_{-,-}(\theta + i\pi/2)
$$

(4.13)

These relations between $T$ and $\nu$ generalize both what found numerically by Zamolodchikov and us [20, 17] for the self-dual Liouville model ($N_f = 0$) and also that found by D.F. and R. Poghossian and H. Poghosyan for SU(3) $N_f = 0$ [26].

For $N_f = 1$, from the $T$ periodicity $T_+(\theta + i\pi/3) = T_-(\theta)$ it follows the Floquet (anti)-periodicity

$$
\nu(\theta + i\pi/3, -q) = \nu(\theta, q) = \pm \nu(\theta - i\pi/3, -q) \mod(n) \in \mathbb{Z}
$$

(4.14)

Thus for $N_f = 1$ we prove the following conjecture by Fateev and Lukyanov [45].

$$
T_+(\theta) = T(\theta, p, q) = \exp\{-i\pi \nu(\theta + i\pi/3, p, q)\} + \exp\{i\pi \nu(\theta - i\pi/3, p, -q)\},
$$

(4.15)

which follows immediately from (4.12) and (4.14). We show also its numerical proof in the massless case in table 4.1, where $\nu$ is computed in practice through the well-known method of the Hill determinant [58] (see appendix E).
\[ T(\theta, p, 0) \text{ TBA, TQ} \exp[-2\pi\nu(\theta + i\pi/3, p, 0)] + \exp[2\pi\nu(\theta - i\pi/3, p, 0)] \text{ Hill} \]

| \( \theta \) | \( -10. \) | \( -0.409791 \) | \( -0.409791 \) |
|---|---|---|---|
| \( -8. \) | \( -0.409791 \) | \( -0.409791 \) |
| \( -6. \) | \( -0.40979 \) | \( -0.40979 \) |
| \( -4. \) | \( -0.409786 \) | \( -0.409791 \) |
| \( -2. \) | \( -0.412355 \) | \( -0.412353 \) |
| \( -1. \) | \( -1.44334 \) | \( -1.44332 \) |
| 0. | \( -371.911 \) | \( -371.912 \) |
| 1. | \( -3.99263 \cdot 10^6 \) | \( -3.99263 \cdot 10^6 \) |
| 2. | \( -1.02835 \cdot 10^{17} \) | \( -1.02835 \cdot 10^{17} \) |
| 3. | \( 1.00886 \cdot 10^{48} \) | \( 1.00886 \cdot 10^{48} \) |
| 4. | \( -2.63656 \cdot 10^{130} \) | \( -2.63656 \cdot 10^{130} \) |
| 5. | \( 6.00739 \cdot 10^{353} \) | \( 6.00739 \cdot 10^{353} \) |

Table 4.1: Here we make a table, with \( p = 0.2 \) and several \( \theta \) in the lines, of three quantities: \( T(\theta, p, q = 0) \) from the TBA and \( TQ \) system (\( Q \) function), \( \exp[-2\pi\nu(\theta + i\pi/3, p, 0)] + \exp[2\pi\nu(\theta - i\pi/3, p, 0)] \), were \( \nu \) is Hill’s Floquet (see appendix E). (Here in \( \theta \) we discretize the interval \((-50, 50)\) in \( 2^8 \) parts, which is no big effort, but we go up to \( 2^{13} \) iterations for the TBA or \( 2^{14} \) as the Hill matrix’s width.)

### 4.2 Exact quantum gauge/integrability identification for \( T \)

The gauge period is defined from the \( \Lambda_1 \) (\( \Lambda_2 \)) derivative of the instanton part of the gauge prepotential \( F_{NS} \) through the Matone’s relation, for \( N_f = 1 \)

\[
2u = a^2 - \frac{\Lambda_1}{3} \frac{\partial F_{NS}^{\text{inst}}}{\partial \Lambda_1}
\]

and for \( N_f = 2 \)

\[
2u = a^2 - \frac{\Lambda_2}{2} \frac{\partial F_{NS}^{\text{inst}}}{\partial \Lambda_2}.
\]

where the instanton prepotential \( F_{NS}^{\text{inst}} \) is given by, for \( N_f = 1 \)

\[
F_{NS}^{\text{inst}} = \sum_{n=0}^{\infty} \Lambda_1^3 F_{NS}^{(n)}
\]

with first terms

\[
F_{NS}^{(1)} = -\frac{2m_1}{4(a^2 - 2h^2)}
\]

\[
F_{NS}^{(2)} = -\frac{4m_1^2(20a^2 + 14h^2) - 3(4a^2 - 2h^2)^2}{256(a^2 - 2h^2)(4a^2 - 2h^2)^3}
\]

\[
F_{NS}^{(3)} = -\frac{4m_1^3(144a^4 + 464a^2h^2 + 116h^4) - m_1(28a^2 + 34h^2)(4a^2 - 2h^2)^2}{192(4a^2 - 2h^2)^5(4a^4 - 26a^2h^2 + 36h^4)}
\]

and for \( N_f = 2 \)

\[
F_{NS}^{\text{inst}} = \sum_{n=0}^{\infty} \Lambda_2^2 F_{NS}^{(n)}
\]
Table 4.2: Comparison of \( \nu \) as computed by the Hill determinant and \( a \) for \( N_f = 1 \) as computed from the instanton series (with \( \hbar = 1 \)).

| \( \Lambda_1 \) | \( u \) | \( m \) | \( \nu \) | \( \frac{a}{\pi} \) | \( \Lambda_2 \) | \( u \) | \( m \) | \( \nu \) | \( \frac{a}{\pi} \) |
|----------------|------|------|------|--------|----------------|------|------|------|--------|
| 0.04           | 1.1  | 0    | 0.0488088 | 1 + 0.0488088 | 0.04 | 1.1  | 0.3  | 0.0488075 | 1 + 0.0488085 |
| 0.08           | 1.1  | 0    | 0.0488089 | 1 + 0.0488088 | 0.08 | 1.1  | 0.3  | 0.0487981 | 1 + 0.0488062 |
| 0.12           | 1.1  | 0    | 0.0488089 | 1 + 0.0488089 | 0.12 | 1.1  | 0.3  | 0.047726  | 1 + 0.0487998 |
| 0.16           | 1.1  | 0    | 0.0488094 | 1 + 0.0488089 | 0.16 | 1.1  | 0.3  | 0.0487231 | 1 + 0.0487874 |

Table 4.3: Comparison of \( \nu \) as computed by the Hill determinant and \( a \) for \( N_f = 2 \) as computed from the instanton series (with \( \hbar = 1 \)).

| \( \Lambda_2 \) | \( u \) | \( m_1 \) | \( m_2 \) | \( \nu \) | \( \frac{a}{\pi} \) |
|----------------|------|--------|--------|------|--------|
| 0.04           | 1.1  | 0      | 0      | 0.0488088 | 1 + 0.0488088 |
| 0.08           | 1.1  | 0      | 0      | 0.0488085 | 1 + 0.0488088 |
| 0.12           | 1.1  | 0      | 0      | 0.0488069 | 1 + 0.0488084 |
| 0.16           | 1.1  | 0      | 0      | 0.0488027 | 1 + 0.0488073 |
| 0.04           | 1.1  | 0.2    | 0.2    | 0.0488043 | 1 + 0.0488077 |
| 0.08           | 1.1  | 0.2    | 0.2    | 0.0487906 | 1 + 0.0488043 |
| 0.12           | 1.1  | 0.2    | 0.2    | 0.048767  | 1 + 0.0487982 |
| 0.16           | 1.1  | 0.2    | 0.2    | 0.0487325 | 1 + 0.0487892 |

with

\[
\mathcal{F}_{NS}^{(1)} = -\frac{1}{8} + \left[ \frac{1}{8} - \frac{4m_1m_2}{8(a^2 - 2\hbar^2)} \right]
\]

\[
\mathcal{F}_{NS}^{(2)} = -\frac{64a^2(a^4 + 3a^2(m_1^2 + m_2^2) + 5m_1^2m_2^2) - 8\hbar^4 + 48\hbar^2(a^2 + m_1^2 + m_2^2) - 32\hbar^2[3a^4 + 6a^2(m_1^2 + m_2^2) - 7m_1^2m_2^2]}{1024(a^2 - 2\hbar^2)(a^2 - 2\hbar^2)^3}
\]  

(4.21)

In tables 4.2 and 4.3 we check the equality to this order of approximation

\[
\nu = \frac{1}{\sqrt{2}\hbar} a \mod(n), \quad n \in \mathbb{Z}
\]  

(4.22)

We notice that (for \( N_f = 2 \)) the first instanton series coeffient match the general mathematical analytical result (from continued fractions technique) for the expansion of the eigenvalue of Doubly Confluent Heun equation in \( \Lambda \) as computed by the Hill determinant and as computed from the instanton period-Floquet identification (4.22) and the Floquet-\( T \) function identifications (4.10)-(4.13) follow new gauge-integrability basic connection formulas for the \( T \) function and \( a \) period. For \( N_f = 1 \)

\[
T^2_+(\theta) = 2 \cos \frac{\sqrt{2}a}{\hbar} \theta
\]

(4.23)

\[
\dot{T}_+^{(1)}(\theta)(\theta + i\frac{\pi}{3}) = 2 \cos \frac{\sqrt{2}a}{\hbar} \theta + 2 \cos \frac{2\pi m}{\hbar}
\]

and for \( N_f = 2 \)

\[
T_{+,+}(\theta)T_{+,+}(\theta + i\frac{\pi}{2}) = 2 \cos \frac{\sqrt{2}a}{\hbar} \theta + 2 \cos \frac{2\pi m_2}{\hbar}
\]

(4.24)

\[
\dot{T}_+^{(1)}(\theta)(\theta + i\frac{\pi}{2}) = 2 \cos \frac{\sqrt{2}a}{\hbar} \theta + 2 \cos \frac{2\pi m_1}{\hbar}
\]
5 Applications of gauge-integrability correspondence

We now show some applications of the gauge-integrability correspondence as new results on both sides. In particular, for gauge theory we find a gauge interpretation of integrability’s functional relations, namely as exact $R$-symmetry relations never found before to our knowledge. For integrability instead we find new formulas for the local integrals of motions in terms of the asymptotic gauge periods, which may sometimes be convenient.

5.1 Applications to gauge theory

Consider first $N_f = 2$. We have the relation (4.13) which considering that $a = \nu$ (cf. (4.22)) becomes

$$ T_{++}(\theta)T_{--}(\theta + i\pi/2) = 4 \cos(a - q_2) \cos(a + q_2) \tag{5.1} $$

Now using the $T$ periodicity relation (2.38) and the $TQ$ relation (2.36) becomes

$$ T_{++}(\theta)T_{--}(\theta) = \frac{1}{Q_{++}(\theta)Q_{--}(\theta)} \left[ Q_{+-}(\theta + i\pi/2)Q_{++}(\theta + i\pi/2) + Q_{+-}(\theta - i\pi/2)Q_{++}(\theta - i\pi/2) + e^{2i\pi q_2}Q_{+-}(\theta + i\pi/2)Q_{++}(\theta + i\pi/2) \right] \tag{5.2} $$

Now we claim that thanks to our connection of $T$ function and $Q/Y$ function to gauge periods $a$ and $a_D$, this $TQ$ relation becomes an $\mathbb{Z}_2$ $R$-symmetry relation for the exact gauge periods $a, a_D$. Indeed, such relations where already known in the $SU(2)$ $N_f = 0$ case for the $h \to 0$ asymptotic expansion modes $a^{(n)}, a^{(n)}_D$ [10]. For the massless $SU(2)$ $N_f = 2$ case the periods are the same, up to a factor $2$ [49]. If $u > 0$ they are

$$ a^{(0)}(u, 0, 0) = -ia^{(0)}(u, 0, 0) \tag{5.3} $$

$$ a^{(0)}_D(u, 0, 0) = -ia^{(0)}_D(u, 0, 0) - a^{(0)}(u, 0, 0) $$

Indeed, expressing (5.2) in terms of gauge periods through (5.1) and (3.46) we get

$$ a^{(0)}(-u, 0, 0) = -a^{(0)}(-u) - ia^{(0)}(u) \tag{5.4} $$

which is consistent with the same relations (5.3). Actually, relations (5.3) can be considered to be derived from the $TQ$ relation when coupled with the $T$ periodicity relation (2.38)

$$ T_{++}(\theta + i\pi/2) = T_{++}(\theta) \tag{5.5} $$

which inside (5.1) reads

$$ T_{++}(\theta)T_{--}(\theta) = T_{++}(\theta + i\pi/2)T_{--}(\theta + i\pi/2) \tag{5.6} $$

and is then another $\mathbb{Z}_2$ $R$-symmetry relation for the exact gauge periods $a$. Indeed, in the massless $N_f = 2$ case reduces precisely to the first of (5.3). Thus we conclude that $\mathbb{Z}_2$ $R$-symmetry for exact gauge theory periods is encoded in the integrability $TQ$ and $T$ periodicity functional relations.

Similarly for $N_f = 1$ the $T$ periodicity is easily shown to be interpreted in gauge theory in the same way. If $u > 0$ and $m = 0$ the other exact relation from the $T$ periodicity (2.37) reduces to the $\mathbb{Z}_3$ symmetry in the asymptotic $h \to 0$ (cf. (B.9))

$$ a^{(0)}(e^{-2\pi i/3}u, 0) = -e^{2\pi i/3}a^{(0)}(u, 0) \tag{5.7} $$

$$ a^{(n)}(e^{-2\pi i/3}u, 0) = -e^{2\pi i/3(1-n)}a^{(n)}(u, 0) \tag{5.8} $$

We avoid though for the moment considering the $N_f = 1 TQ$ relation since it requires some non-trivial analytic continuation of gauge-integrability relations beyond the complex strip $\text{Im} \theta < \pi/3$ in which the TBA holds without analytic continuation.

We see that the new exact relations following from the integrability functional relations are a $\mathbb{Z}_2, \mathbb{Z}_3 N_f = 2, 1$ $R$-symmetry relations. They were never found previously in the literature, to our knowledge. We knew only the $h \to 0$ perturbative relations, also in the massless case in [49].
5.2 Applications to integrability

We now find a new ways to compute either the local integrals of motions for the Perturbed Hairpin IM or the asymptotic expansion modes of the $N_f = 1$ quantum gauge periods.

Consider the large energy asymptotic expansion (2.54) of $Q$ in terms of the LIMs. We set first $q = 0$ so to recover the LIMs of Liouville $b = \sqrt{2}$. For this particular case the expansion simplifies as

$$\ln Q(\theta, p) = -C_0 e^\theta - \sum_{n=1}^\infty e^{\theta(1-2n)} C_n \theta^{2n-1}, \quad \theta \to +\infty, \quad p \text{ finite}.$$  \hfill (5.9)

The normalization constants are given (cf. [17] with $b = \sqrt{2}$)

$$C_n = \frac{\Gamma\left(\frac{2n}{3} - \frac{1}{2}\right) \Gamma\left(\frac{n}{3} - \frac{1}{6}\right)}{3\sqrt{2} \pi n!}. \hfill (5.10)$$

We can also expand the LIMs $I_{2n-1}$, as polynomials in $p^2$ with coefficients $\Upsilon_{n,k}$

$$I_{2n-1} = \sum_{k=0}^n \Upsilon_{n,k} p^{2k}. \hfill (5.11)$$

The leading and subleading coefficients are found to be [17]

$$\Upsilon_{n,n} = (-1)^n, \quad \Upsilon_{n,n-1} = \frac{1}{24} (-1)^n n(2n-1). \hfill (5.12)$$

Now, since in Seiberg-Witten theory $u$ is finite as $\theta \to +\infty$, to connect the IM $\theta \to +\infty$ asymptotic expansion, it is necessary to take the further limit

$$p^2(\theta) = 4 \frac{u}{\Lambda_1^2} e^{2\theta} \to +\infty. \hfill (5.13)$$

In this double limit, an infinite number of LIMs $\Upsilon_{n,k}$, through their coefficients $\Upsilon_{n,k}$, are re-summed into a quantum gauge period asymptotic mode (a sort of LIM on its way). For instance the leading order is obtained from the resummation of all $\Upsilon_{n,n} = (-1)^n$ terms as

$$\ln Q^{(0)}(u, 0, \Lambda_1) = -\sum_{n=0}^\infty \frac{\Gamma\left(\frac{2n}{3} - \frac{1}{2}\right) \Gamma\left(\frac{n}{3} - \frac{1}{6}\right)}{3\sqrt{2} \pi n!} \left(-\frac{4u}{\Lambda_1^2}\right)^n \hfill (5.14)$$

and from it we can derive the higher orders as usual through differential operators (3.56). In particular, in the massless case the first simplify as

$$\ln Q^{(1)}(u, 0, \Lambda_1) = \left(\frac{\Lambda_1}{2}\right)^2 \left[\frac{u}{6} \frac{\partial^2}{\partial u^2} + \frac{1}{12} \frac{\partial}{\partial u}\right] \ln Q^{(0)}(u, 0, \Lambda_1)$$

$$\ln Q^{(2)}(u, 0, \Lambda_1) = \left(\frac{\Lambda_1}{2}\right)^4 \left[\frac{7}{360} \frac{u^2}{\partial u^2} \frac{\partial^4}{\partial u^4} + \frac{31}{360} \frac{u}{\partial u^3} \frac{\partial^3}{\partial u^3} + \frac{9}{160} \frac{\partial^2}{\partial u^2}\right] \ln Q^{(0)}(u, 0, \Lambda_1)$$

$$\ln Q^{(3)}(u, 0, \Lambda_1) = \left(\frac{\Lambda_1}{2}\right)^6 \left[\frac{31}{15120} \frac{u^3}{\partial u^6} + \frac{443}{18144} \frac{u^2}{\partial u^5} + \frac{43}{576} \frac{u}{\partial u^4} + \frac{557}{10368} \frac{\partial^3}{\partial u^3}\right] \ln Q^{(0)}(u, 0, \Lambda_1). \hfill (5.15)$$
Indeed these expression match with the resummation of LIMs at higher orders:

\[
\begin{align*}
\ln Q^{(1)}(u, 0, \Lambda_1) &= \left( \frac{\Lambda_1}{2} \right)^2 \sum_{n=0}^{\infty} \frac{\left( \frac{n}{12} + \frac{1}{24} \right) \Gamma \left( \frac{2n}{3} + \frac{1}{3} \right) \Gamma \left( \frac{n}{3} + \frac{1}{6} \right)}{3\sqrt{2\pi} n!} \left( \frac{4u}{\Lambda_1^2} \right)^n \\
\ln Q^{(2)}(u, 0, \Lambda_1) &= -\left( \frac{\Lambda_1}{2} \right)^4 \sum_{n=0}^{\infty} \frac{(14n + 27)(2n + 3)}{5760} \frac{\Gamma \left( \frac{2n}{3} + 1 \right) \Gamma \left( \frac{2}{3} + \frac{1}{2} \right)}{3\sqrt{2\pi} n!} \left( \frac{4u}{\Lambda_1^2} \right)^n \\
\ln Q^{(3)}(u, 0, \Lambda_1) &= \left( \frac{\Lambda_1}{2} \right)^6 \sum_{n=0}^{\infty} \frac{\left[ 1 + 4n(93n + 596) + 3899 \right](2n + 5)}{362880} \frac{\Gamma \left( \frac{2n}{3} + \frac{5}{6} \right) \Gamma \left( \frac{2}{3} + \frac{5}{6} \right)}{3\sqrt{2\pi} n!} \left( \frac{4u}{\Lambda_1^2} \right)^n 
\end{align*}
\]

(5.16) (5.17) (5.18)

So in general we find the relation

\[
\ln Q^{(k)}(u, 0, \Lambda_1) = (-1)^{k+1} \left( \frac{\Lambda_1}{2} \right)^{2k} \sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{k+n}{3} - \frac{1}{6} \right) \Gamma \left( \frac{2(k+n)}{3} - \frac{1}{3} \right)}{3\sqrt{2\pi} (k + n)!} \left( \frac{4u}{\Lambda_1^2} \right)^n .
\]

(5.19)

Thus this procedure can actually be a convenient way to compute the LIMs coefficients \( \Upsilon_{n+k, n} \) for general \( n \) at each successive \( k \) order. Alternatively and equivalently, we can use it to compute the \( k \)-th mode of the (alternative dual) quantum period \( a_1 \)

\[
\frac{4\sqrt{2\pi}}{\Lambda_1} a_1^{(k)}(u, 0, \Lambda_1) = -\sum_{k=0}^{\infty} \frac{\Gamma \left( \frac{k+n}{3} - \frac{1}{6} \right) \Gamma \left( \frac{2(k+n)}{3} - \frac{1}{3} \right)}{3\sqrt{2\pi} (k + n)!} \sin \left( \frac{1}{3} \pi (k + n + 1) \right) \left( \frac{4u}{\Lambda_1^2} \right)^n .
\]

(5.20)

6 Gravitational correspondence and applications

6.1 Gravitational correspondence \( N_f = 2 \)

Our two-fold integrability-gauge correspondence actually is three-fold method as black hole’s perturbation theory involves the same ODEs we use. In particular the Doubly Confluent Heun equation (see appendix F) we have for the \( SU(2) \) \( N_f = 0, 1, 2 \) gauge theory and Generalized Perturbed Hairpin integrable model is typically associated to extremal black holes. In particular, for the \( N_f = 2 \) we consider now the gravitational background given by the intersection of four stacks of D3-branes in type IIB supergravity. This geometry is characterised by four different charges \( Q_i \), which, if all equal, lead to an extremal RN BH, that is maximally charged. In isotropic coordinates the line element writes \([59, 30]\)

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)] ,
\]

(6.1) linein

with \( f(r) = \prod_{i=1}^{4} (1 + Q_i/r)^{-\frac{1}{2}} \). The ODE describing the scalar perturbation is, with \( \Sigma_k = \sum_{i_1 < \cdots < i_k} Q_{i_1} \cdots Q_{i_k} \)

\[
\frac{d^2 \phi}{dr^2} + \left[ -\frac{1}{r^2} + \omega^2 \sum_{k=0}^{4} \frac{\Sigma_k}{r^k} \right] \phi = 0 .
\]

(6.2) ODEgra

Changing variables as \( r = \sqrt[3]{\sum_4} e^{\psi} \) and

\[
\omega \sqrt{\sum_4} = -ie^\theta \quad q_j = \frac{1}{2} \Sigma_{2j-1} e^{\theta} \quad p^2 = (l + \frac{1}{2})^2 - \omega^2 \Sigma_2 ,
\]

(6.3) DictIn

\((j = 1, 2)\) the ODE takes precisely the form of the Generalized Perturbed Hairpin IM (2.4).
Setting up ODE/IM in gravity variables (6.3), we notice that the discrete symmetries (2.10) are consistent with the brane dictionary (6.3), as the brane parameters vary as $\Sigma_1 \rightarrow \pm i\Sigma_1$, $\Sigma_2 \rightarrow -\Sigma_2$, $\Sigma_3 \rightarrow \mp i\Sigma_3$, $\Sigma_4 \rightarrow \Sigma_4^{11}$. So in gravity variables the $Y$ system reads

$$Y(\theta + \frac{i\pi}{2}, -i\Sigma_1, -\Sigma_2, i\Sigma_3)Y(\theta - \frac{i\pi}{2}, -i\Sigma_1, -\Sigma_2, i\Sigma_3) = [1 + Y(\theta, \Sigma_1, \Sigma_2, \Sigma_3)][1 + Y(\theta, -\Sigma_1, \Sigma_2, -\Sigma_3)],$$

(6.4) \text{Y syst 2 grav}

(with $\Sigma_4$ omitted since it is fixed). We remark we shall pay particular attention to the change of variables from gravity or gauge to integrability: this results in different TBA equations as first noted in [17]. Indeed, $Y$ system (6.4) can be inverted into the TBA in gravitational variables

$$\varepsilon_{\pm, \pm}(\theta) = [f_{0, +} \pm \frac{i\pi}{2} (\Sigma_1 \mp \frac{\Sigma_3}{\Sigma_4})] e^\theta - \varphi^*(L_{\pm\pm} + L_{\mp\mp})(\theta)$$

(6.5) \text{TBA 2}

where we defined $\varepsilon(\theta) = -\ln Y(\theta, \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$, $\varepsilon(\theta) = \varepsilon(\theta, i\Sigma_1, -\Sigma_2, -i\Sigma_3, \Sigma_4)$, $L = \ln[1 + \exp(-\varepsilon)]$, $\varphi(\theta) = (\cosh(\theta))^{-1}$ and

$$f_{0, \pm} = c_{0, +, \pm} + c_{0, -, \mp} \quad c_{0, +, \pm} = c_0(\Sigma_1, \Sigma_2, \pm \Sigma_3, \Sigma_4)$$

(6.6)

with

$$c_0(\Sigma) = \int_{-\infty}^{\infty} \left[ \frac{2 \cosh(2y) + \frac{\Sigma_1}{\sqrt{\Sigma_4}} e^y + \frac{\Sigma_3}{\sqrt{\Sigma_4}} e^{-y} + \frac{\Sigma_2}{\sqrt{\Sigma_4}}}{2 \cosh y - 1} - \frac{1}{2} \frac{\Sigma_1}{\sqrt{\Sigma_4}} \frac{1}{1 + e^{-y/2}} \frac{1}{2} \frac{\Sigma_3}{\sqrt{\Sigma_4}} \right] dy$$

(6.7)

which in turn can be expressed either through a triple power series for small parameters or as an elliptic integral as

$$c_0(\Sigma) = \sum_{l,m,n=0}^{\infty} \left(\frac{1}{2}\right)^l \left(\frac{1}{2} - \frac{l}{m}\right) \left(-l - m + \frac{1}{2}\right) \frac{n}{\Sigma_4} \left(\frac{\Sigma_1}{\sqrt{\Sigma_4}}\right)^n \left(\frac{\Sigma_2}{\sqrt{\Sigma_4}}\right)^m \left(\frac{\Sigma_3}{\sqrt{\Sigma_4}}\right)^l$$

(6.8)

$$B_2(l, m, n) = \frac{\Gamma \left(\frac{1}{4} \left(3l + 2m + n - 1\right)\right)}{4\Gamma \left(l + m + n - \frac{1}{2}\right)} \Gamma \left(\frac{1}{4} \left(3l + 2m + 3n - 1\right)\right)$$

$$B_2(1, 0, 0) = \frac{1}{2} (\ln 2 - 1), \quad B_2(0, 0, 1) = \frac{1}{2} \ln 2$$

We have to numerically input $l$ in the TBA with the boundary condition at $\theta \rightarrow -\infty$:

$$\varepsilon_{\pm, \pm}(\theta) \simeq 4p\theta \simeq 4(l + 1/2) + 2C(p)\theta \quad \theta \rightarrow -\infty,$$

(6.9)

$$C(p) = \ln \left(\frac{2^{1-2p}\Gamma(2p)^2}{\Gamma \left(p + \frac{1}{2}\right)^2}\right)$$

(6.10)

also following from the asymptotic of the ODE (2.4) (the precision improves by adding also the constant at the subleading order, as explained in [45]). Through this TBA we find again the QNMs to be given by the Bethe roots condition

$$\varepsilon_{+, +}(\theta_n - i\pi/2) = -i\pi(2n' + 1), \quad Q_{+, +}(\theta_n) = 0 \quad n' \in \mathbb{Z}$$

(6.11) \text{quant}

and we show in tables 6.2 their agreement with continued fraction (Leaver) method and WKB approximation ($l \rightarrow \infty$) [60]. We notice that for $\Sigma_1 \neq \Sigma_3$ and $\Sigma_4 \neq 1$ the Leaver method is not applicable, at least in its original version since the recursion produced by the ODE involves more than 3 terms (compare [28, 60]) and thus also for
Table 6.1: Comparison of QNMs obtained from TBA (6.5), Leaver method (through (6.11) with $n' = 0$) and WKB approximation ($\Sigma_1 = \Sigma_3 = 0.2$, $\Sigma_2 = 0.4$, $\Sigma_4 = 1$).

| $n$ | $l$ | TBA         | Leaver      | WKB          |
|-----|-----|-------------|-------------|--------------|
| 0   | 1   | $0.869623 - 0.372022i$ | $0.868932 - 0.372859i$ | $0.89642 - 0.36596i$ |
| 0   | 2   | $1.477900 - 0.368144i$ | $1.477888 - 0.368240i$ | $1.49100 - 0.36596i$ |
| 0   | 3   | $2.080200 - 0.367076i$ | $2.080168 - 0.367097i$ | $2.0916 - 0.36596i$ |
| 0   | 4   | $2.680363 - 0.366637i$ | $2.680350 - 0.366642i$ | $2.6893 - 0.36596i$ |

Table 6.2: Comparison of QNMs obtained from TBA (6.5), (through (6.11) with $n' = 0$) and WKB approximation ($\Sigma_1 = 0.1$, $\Sigma_2 = 0.2$, $\Sigma_3 = 0.3$, $\Sigma_4 = 1$). Since $\Sigma_1 \neq \Sigma_3$ the Leaver method seems not applicable, at least in its original version (N.A.).

\[
\frac{2\sqrt{2}\pi}{\hbar(\theta)}a_D(\theta, u, m_1, m_2, \Lambda_2) = -i\pi(2n' + 1) \tag{6.12}
\]

This constitutes a (mathematical?) proof of the essential finding of [27] and the following literature (see the introduction).

A note of caution, though. Literature following [27] uses another definition of gauge period which we denote by $A_D$ which derives from the instanton expansion of the prepotential. As we explain in appendix D the two definitions can be actually related by formulas like (C.19) for the $N_f = 0$ theory. Generalizations of formula (C.19), already exist for the subcase of the $N_f = 1$ gauge theory [14] (see next subsection) and so we expect them to exist also for the whole $N_f = 2$ theory and even more generally. In this way we expect that in general the integrable Bethe roots condition, which we have shown to follow straightforwardly from BHs physics, in gauge theory indeed corresponds to the quantization of the gauge $A_D$ period as stated in [27].

By making considerations on these $TQ$ systems and the $QQ$ system (2.24) like done in [42] and reported briefly in appendix D, we are not in general able to conclude any quantization condition on the $T$ function, except in the case of equal masses $q_1 = q_2 \equiv q$ where we find

\[
T_{+,+}(\theta_n)T_{-,+}(\theta_n) = 4. \tag{6.13}
\]

that generalizes (D.8) for $N_f = 0$. We now prove (6.13). From the $QQ$ system (2.24) we can write, for general

\footnote{This observation does not mean that a dictionary not consistent with the discrete symmetry would imply ODE/IM cannot be used: in that case we should just do ODE/IM in the suitable variables and then afterwards change to the variables of interest.}
\[ e^{i\pi q_1} Q_{-+}(\theta - i\pi/2) = c_0 \left[ 1 \pm ie^{i\pi\frac{q_1-q_2}{2}} \sqrt{Q_{+-}(\theta)Q_{--}(\theta)} \right] \]  \hspace{1cm} (6.14)  
\[ e^{-i\pi q_2} Q_{++}(\theta + i\pi/2) = \frac{1}{c_0} \left[ 1 \mp ie^{i\pi\frac{q_1-q_2}{2}} \sqrt{Q_{++}(\theta)Q_{--}(\theta)} \right] \]  \hspace{1cm} (6.15)  
\[ e^{i\pi q_2} Q_{+-}(\theta - i\pi/2) = \frac{1}{c_0} \left[ 1 \mp ie^{-i\pi\frac{q_1-q_2}{2}} \sqrt{Q_{+-}(\theta)Q_{--}(\theta)} \right] \]  \hspace{1cm} (6.16)  
\[ e^{-i\pi q_1} Q_{++}(\theta + i\pi/2) = c_0' \left[ 1 \mp ie^{-i\pi\frac{q_1-q_2}{2}} \sqrt{Q_{++}(\theta)Q_{--}(\theta)} \right]. \]  \hspace{1cm} (6.17)

From the 2 \( TQ \) system \((2.36)\) at the Bethe roots we get the same relation

\[ c_0(-q_1, q_2) = -c_0'(-q_1, q_2). \]  \hspace{1cm} (6.18)

We can also exchange the masses in \((6.14)\) and \((6.16)\) to obtain the relation

\[ c_0(-q_1, q_2)c_0(-q_2, q_1) = -1. \]  \hspace{1cm} (6.19)

In addition, considering real parameters, we have

\[ c_0 = -c_0'. \]  \hspace{1cm} (6.20)

However, we cannot fix \( c_0 \) completely in general, only when \( q_1 = q_2 = q \) we can say

\[ c_0(q_1, q_2 = q_1) = \pm i. \]  \hspace{1cm} (6.21)

We notice also that

\[ Q_{+-} = Q_{--}, \quad q_1 = q_2 = q. \]  \hspace{1cm} (6.22)

We can generalize the \( N_f = 0 \) procedure by considering the \( Y \) system instead of the \( Q \) system.

\[ T_{+,+}(\theta)T_{-,-}(\theta)Y_{+,+}(\theta) = [e^{i\pi q}Q_{+-}(\theta - i\pi/2) + e^{-i\pi q}Q_{+-}(\theta + i\pi/2)][e^{i\pi q}Q_{-+}(\theta - i\pi/2) + e^{-i\pi q}Q_{-+}(\theta + i\pi/2)] \]
\[ = Y_{++,}(\theta - i\pi/2) + Y_{--}(\theta + i\pi/2) + 2 + 2Y_{+,+}(\theta). \]  \hspace{1cm} (6.23)

Notice that we can write shifted \( Y \) as

\[ Y_{++,}(\theta - i\pi/2) = e^{2i\pi q}Q_{++,}(\theta - i\pi/2)Q_{--,}(\theta - i\pi/2) \]
\[ = -1 \mp 2i \sqrt{Q_{++,}(\theta)Q_{--,}(\theta) + Q_{++,}(\theta)Q_{++,}(\theta)Q_{--,}(\theta)} \] \hspace{1cm} (6.24)

and

\[ Y_{--}(\theta + i\pi/2) = -1 \pm 2i \sqrt{Y_{++,}(\theta) + Y_{++,}(\theta)} \] \hspace{1cm} (6.25)

Inserting these shifted-\( Y \) expressions in what we could call the \( TY \) relation \((6.23)\) we find

\[ T_{+,+}(\theta)T_{-,-}(\theta)Y_{++,}(\theta) = +4Y_{++,}(\theta), \]  \hspace{1cm} (6.26)

that is nothing but quantization relation on \( T \) \((6.13)\).
Now, on plugging the \( T \) periodicity relations (2.38) \( T_{+,-}(\theta + \frac{\pi}{2}) = T_{+,+}(\theta) \), \( \tilde{T}_{+,-}(\theta + \frac{\pi}{2}) = \tilde{T}_{+,+}(\theta) \) in the relations between \( T, \tilde{T} \) and \( \nu \) (4.13), (4.11) we get the simplification to only one \( T \)

\[
\pm \sqrt{2 \cos 2\pi\nu + 2 \cos 2\pi q_2} = T_{+,+}(\theta) \\
\pm \sqrt{2 \cos 2\pi\nu + 2 \cos 2\pi q_1} = \tilde{T}_{+,+}(\theta) 
\]

Now we notice from the \( \nu = a \) instanton series terms (4.21) that

\[
\nu(q_1, q_2) = \nu(-q_1, -q_2) 
\]

so we can write the same relations for also opposite masses

\[
\pm \sqrt{2 \cos 2\pi\nu + 2 \cos 2\pi q_2} = T_{-,-}(\theta) \\
\pm \sqrt{2 \cos 2\pi\nu + 2 \cos 2\pi q_1} = \tilde{T}_{-,-}(\theta). 
\]

Now from \( T \) quantization for \( q_1 = q_2 = q \) (6.13)

\[
T_{+,+}(\theta)T_{-,-}(\theta) = \pm [2 \cos 2\pi\nu + 2 \cos 2\pi q] = 4 
\]

it follows a quantization condition on the combination of \( \nu \) and \( q \)

\[
[\cos 2\pi\nu + \cos 2\pi q]_{\theta=\theta_n} = \pm 2. 
\]

In conclusion, from this derivation we do not expect that the alternative QNMs quantization condition on the gauge a period found in [28] for \( N_f = 0 \) generalizes to other gauge theories, both because the integrability \( T \) function is not quantized generally (for different masses \( m_1, m_2, q_1, q_2 \)) and because even when it is, it implies a quantization on only the combination of a \( \nu \) period and masses.

Now we can find also an integrability interpretation of the symmetry under Couch-Torrence transformation found for this gravitational background in [63], thanks to identifications of certain scattering angles with the SW a period. It refers to the symmetry that exchange infinity \( (y \rightarrow +\infty) \) and the (analogue) horizon \( (y \rightarrow -\infty) \), leaving the photon sphere \( (y = 0) \) fixed. In our ODE approach, it correspondence to the following wave function properties

\[
\psi_{+,0}(y) = \psi_{-,0}(-y), \quad (q_1 = q_2) 
\]

which we notice holds only for equal masses. In this respect, under (6.32) we have the \( T \) and \( \tilde{T} \) identity

\[
\tilde{T}_{+,+}(\theta) = T_{+,+}(\theta) \quad (q_1 = q_2), 
\]

as can be understood by looking to their very definitions (2.32).

All the considerations of this subsection show how integrability structures give valuable insights in several gauge-gravity correspondence mathematical physics issues.

### 6.2 Gravitational correspondence \( N_f = 1 \)

Now, to get a gravitation counterpart of the \( N_f = 1 \) gauge theory, we can simply take the limit from the \( N_f = 2 \) theory, as explained in appendix G. In gravity variables such limit corresponds to

\[
\Sigma_4 \rightarrow 0 
\]

and in terms of charges can be realised for instance with \( Q_4 \rightarrow 0. \) Upon this limit, get the following gravity-integrability parameters dictionary

\[
\omega \sqrt{\Sigma_3} = -ie^\theta, \quad \frac{\Sigma_4}{\sqrt{\Sigma_3}} = 2q_1e^{-\theta}, \quad p^2 = (l + \frac{1}{2})^2 - \omega^2 \Sigma_2 
\]
The \( N_f = 1 \) \( Y \) system in gravitational variables reads
\[
Y(\theta + i\pi/2, -i\Sigma_1, -\Sigma_2) Y(\theta - i\pi/2, -i\Sigma_1, -\Sigma_2) = [1 + Y(\theta + i\pi/6, -ie^{-2\pi i/3}\Sigma_1, -e^{2\pi i/3}\Sigma_2)] [1 + Y(\theta - i\pi/6, -ie^{2\pi i/3}\Sigma_1, -e^{-2\pi i/3}\Sigma_2)],
\]
(6.36)
from which it appears convenient to define
\[
Y_{0, \pm}(\theta) = Y(\theta, i\Sigma_1, -\Sigma_2) \quad Y_{1, +}(\theta) = Y(\theta, ie^{2\pi i/3}\Sigma_1, -e^{-2\pi i/3}\Sigma_2) \quad Y_{2, +}(\theta) = Y(\theta, ie^{-2\pi i/3}\Sigma_1, -e^{2\pi i/3}\Sigma_2)
\]
Y_{0, -}(\theta) = Y(\theta, -i\Sigma_1, -\Sigma_2) \quad Y_{1, -}(\theta) = Y(\theta, -ie^{2\pi i/3}\Sigma_1, -e^{-2\pi i/3}\Sigma_2) \quad Y_{2, -}(\theta) = Y(\theta, -ie^{-2\pi i/3}\Sigma_1, -e^{2\pi i/3}\Sigma_2).
\]
(6.37)
The \( Y \) system can be inverted in a TBA made of 6 coupled equations as
\[
\varepsilon_{0, \pm}(\theta) = \left( f_{0, \pm} \pm \frac{4}{3} \pi \Sigma_1 \right) e^{\varphi} - (\varphi_± L_{1, \pm})(\theta) - (\varphi_± L_{2, \pm})(\theta)
\]
(6.38)
\[
\varepsilon_{1, \pm}(\theta) = \left( f_{1, \pm} \pm \frac{4}{3} \pi e^{2\pi i/3}\Sigma_1 \right) e^{\varphi} - (\varphi_± L_{2, \pm})(\theta) - (\varphi_± L_{0, \pm})(\theta)
\]
(6.39)
\[
\varepsilon_{2, \pm}(\theta) = \left( f_{2, \pm} \pm \frac{4}{3} \pi e^{-2\pi i/3}\Sigma_1 \right) e^{\varphi} - (\varphi_± L_{0, \pm})(\theta) - (\varphi_± L_{1, \pm})(\theta)
\]
(6.40)
with of course \( L_{k, \pm} = \ln[1 + \exp\{-\varepsilon_{k, \pm}\}] \) and the kernels
\[
\varphi_{\pm}(\theta) = \frac{1}{2\pi \cosh(\theta \pm i\pi/6)}
\]
(6.41)\(\text{kernel}\)
Under change to gravity variables \( q(\theta) = \frac{1}{2} \Sigma_1 \frac{\Sigma_2}{\sqrt{\Sigma_3}} e^{\varphi} \) and so the leading order is given by
\[
f_{k, \pm} = -e^{-i\pi/6} c_0(\mp ie^{2\pi i/3}\Sigma_1, -e^{-2\pi i/3}\Sigma_2) - e^{i\pi/6} c_0(\mp ie^{-2\pi i/3}\Sigma_1, -e^{2\pi i/3}\Sigma_2)
\]
(6.42)
\[
c_0(\Sigma_1; \Sigma_2, \Sigma_3) = \int_{-\infty}^{\infty} \left( e^{2y} + e^{-y} + \frac{\Sigma_1}{\sqrt{\Sigma_3}} e^y + \frac{\Sigma_2}{\sqrt{\Sigma_3}} e^{-y} - e^{-y/2} - \frac{1}{2} \frac{\Sigma_1}{\sqrt{\Sigma_3}} \frac{1}{1 + e^{-y/2}} \right) dy.
\]
(6.43)
We can compute this integral analytically as usual by expanding it in double binomial series for small \( \Sigma_1, \Sigma_2 \)
\[
c_0(\Sigma_1; \Sigma_2, \Sigma_3) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{\Sigma_1}{\sqrt{\Sigma_3}} \right)^n \left( \frac{\Sigma_2}{\sqrt{\Sigma_3}} \right)^l \left( 1/2 - l \right) \left( 1/2 - l \right) B(n, l)
\]
(6.44)
\[
B(n, l) = \frac{1}{3} B \left( \frac{1}{6} (2l + 4n - 1), \frac{1}{3} (2l + n - 1) \right) \quad (n, l) \neq (1, 0)
\]
(6.45)
\[
B(1, 0) = \frac{2 \log(2)}{3}
\]
As in the Liouville model, also in the Hairpin model the TBA does not contain explicitly \( p \), so that is has to be solved through the boundary condition
\[
\varepsilon_{\pm}(\theta) \simeq 6p \theta \simeq 6(l + \frac{1}{2}) \theta_1 + 2C(p), \quad \theta \to -\infty
\]
(6.46)
\[
C(p) = \log \left( \frac{2^{-p^2/2} \Gamma(\sqrt{2p}) \Gamma(2\sqrt{2p})}{\pi} \right)
\]
(6.47)
\begin{align*}
\{n, t, \Sigma_1, \Sigma_2, \Sigma_3\} & \quad \text{TBA} & \quad \text{WKB} \\
\{0, 1, 0, 1, 0, 2, 1, 1\} & \quad 0.996031 - 0.308972i & \quad 1.018635 - 0.317055i \\
\{0, 2, 0, 1, 0, 2, 1\} & \quad 1.6945 - 0.301444i & \quad 1.69772 - 0.31706i \\
\{0, 3, 0, 1, 0, 2, 1\} & \quad 2.39612 - 0.294969i & \quad 2.37681 - 0.31706i \\
\{0, 1, 0, 2, 0, 4, 1\} & \quad 0.943852 - 0.263758i & \quad 0.959219 - 0.28132i \\
\{0, 2, 0, 2, 0, 4, 1\} & \quad 1.59951 - 0.250208i & \quad 1.59870 - 0.28132i \\
\{0, 3, 0, 2, 0, 4, 1\} & \quad 2.25939 - 0.237859i & \quad 2.23818 - 0.28132i \\
\{0, 1, 0, 4, 0, 1, 1\} & \quad 0.966828 - 0.337457i & \quad 0.990202 - 0.300483i \\
\{0, 2, 0, 4, 0, 1, 1\} & \quad 1.64269 - 0.357236i & \quad 1.65034 - 0.30048i \\
\{0, 3, 0, 4, 0, 1, 1\} & \quad 2.32242 - 0.37745i & \quad 2.31047 - 0.30048i \\
\end{align*}

Table 6.3: QNMs for \( N_f = 1 \). Since the Leaver method is not applicable to this case, at least in its original version, we were able to compare only with the WKB approximation, by which however the match is necessarily very rough.

From the general analysis of [42] we can safely affirm that the QNMs are given by zeros of \( Q_+ \)

\[ Q_+(\theta_n) = 0 \]  \hspace{1cm} (6.48)

or the equivalent condition on \( Y \)

\[ Y_{0,+}(\theta_n - i\pi/2) = -1. \]  \hspace{1cm} (6.49)

or \( \varepsilon \)

\[ \varepsilon_{0,+}(\theta_n - i\pi/2) = -i\pi(2n' + 1) \quad n' \in \mathbb{Z} \]  \hspace{1cm} (6.50)

With the last relation we can actually compute the QNMs as usual\(^{12}\). We report their values obtained in table 6.2. Again, we find the Leaver method is not applicable to this case, at least in its original version \([60]\), so we are able to compare only with the WKB approximation, which gives however necessarily a very rough match. Now from our gauge-integrability identification (3.66) we can prove a quantization on the (alternative) gauge period \( a_1 \)

\[ \frac{2\pi\sqrt{2}}{\hbar} a_1(\theta - i\pi/2, u, m) = -i\pi(2n' + 1) \quad n' \in \mathbb{Z} \]  \hspace{1cm} (6.51)

and as discussed in the previous subsection we surely expect a similar quantization condition on the other differently defined \( A_D \) period actually used in the literature on the new gauge-gravity correspondence following [27]. In particular, we can now compare directly with the work \([14]\) in which eq. 8.12 (in the first arXiv version) shows that zeros of \( Q \) correspond to quantization conditions on the gauge periods, thus again recovering the characterization of QNMs of [27].

Applying the \( N_f = 1 \) TQ system (2.35) to also this background, we find the same limitations as for \( N_f = 2 \) in finding quantization conditions for \( T \) and \( a \) as in (6.13) and (6.31).

### 7 Conclusions and perspectives

In conclusion, we have shown how 2D integrable models when studied in the ODE/IM correspondence approach can find a natural connection to (NS-deformed) \( \mathcal{N} = 2 \) supersymmetric gauge theory, as well as to black hole perturbation theory and shed light on the relation recently found between the last two. This triple new correspondence, besides being interesting in itself, allows also to derive new results on all three sides and at the non-perturbative exact level. Moreover, since the ODEs in the \( N_f = 0, 1, 2 \) \( SU(2) \) gauge theories possess two irregular singularities\(^{13}\), the connection to IMs constitute a development of ODE/IM correspondence itself.

\(^{12}\)We notice that to implement this condition through TBA it is NOT necessary to analytically continue (since \( Y \) functions are analytic) beyond the poles of the kernels (6.41) at the points \( \theta - \theta' = i\pi \) by adding their residue.

\(^{13}\)Rather than just one irregular and one regular as it is often found in ODE/IM literature.

36
On these new directions, much extension work in either breadth and depth can still be done. Indeed, the $SU(2)$ $N_f = 0, 1$ and $N_f = 2 = (1, 1)$ gauge theories are in correspondence with BH perturbation theory with the gravitational background given by the intersection of four stacks of D3 branes (generalization of extremally charged BHs). However, the other $SU(2)$ $N_f = 2 = (2, 0)$ (asymmetrical) and $N_f = 3, 4$ as well as $SU(2)$ quivers have been found correspond to still many other gravitational backgrounds [27, 30]. We have not yet related them to IMs, but from the generality of the ODE/IM construction it appears to us manifest that our method should still apply. We notice indeed that much of the BH theory seems to go in parallel to the ODE/IM correspondence construction and its 2D integrable field theory interpretation, beyond the determination of QNMs. For instance also the greybody factor that parametrizes the Hawking radiation seems to be ratio of $Q$s\textsuperscript{14}. Similarly it is very intriguing to investigate still other applications of integrability to BHs\textsuperscript{15}.

Also in-depth developments of the gauge-integrability correspondence are possible, among which especially interesting would be the extension of the basic identifications under gauge theory wall crossing, with possibly other useful applications.

Perhaps most importantly, our work as part of the larger new field of application of $\mathcal{N} = 2$ gauge theories to BHs perturbation theory, constitutes a unexpected fruitful application of extended supersymmetry. Thus, even though particle physics has not yet found supersymmetric new elementary particles, the new field of gravitational phenomenology in some sense compensates and gives some physical substance of the sophisticated mathematics of extended supersymmetric gauge theories.

In conclusion, the new application of $\mathcal{N} = 2$ supersymmetry gauge theory and quantum integrability to black holes physics allows us to deal with non-perturbative effects and so illuminates aspects of classical and quantum gravitational theories difficult to access through standard methods.

Acknowledgements We thank M. Bianchi, D. Consoli, A. Grassi, A. Grillo, F. Morales, H. Poghossian, K. Zarembo for discussions and suggestions. This work has been partially supported by the grants: GAST (INFN), the MPNS-COST Action MP1210, the EC Network Gatis and the MIUR-PRIN contract 2017CC72MK_003. DG and HS thank NORDITA for warm hospitality.

\textsuperscript{14}This can be understood by considering its absorption coefficient role as viewed in 1D quantum mechanics. We hope to write more details on this in the future.

\textsuperscript{15}Another present technical limitation which should be overcome in the future is that analytic continuations of the TBAs in rapidity $\theta$ and the moduli are necessary to obtain overtones $\omega_n, n \geq 1$ and also some particular gravitational systems.
A Quantum Seiberg-Witten theory with fundamental matter

The Seiberg-Witten (SW) curve for $N = 2$ $SU(2)$ with $N_f$ fundamental matter flavour hypermultiplets is given by

$$K(p) - \frac{\Lambda}{2} (K_+(p)e^{ix} + K_-(p)e^{-ix}) = 0$$ (A.1)

where

$$\Lambda = \begin{cases} 
\Lambda_0^2 & N_f = 0 \\
\Lambda_1^{3/2} & N_f = 1 \\
\Lambda_2^1 & N_f = 2 
\end{cases}$$ (A.2)

$$K(p) = \begin{cases} 
p^2 - u & N_f = 0 \\
p^2 - u & N_f = 1 \\
p^2 - u + \frac{\Lambda^2}{8} & N_f = 2
\end{cases}$$ (A.3)

$$K_+(p) = \prod_{j=1}^{N_+} (p + m_j), \quad K_-(p) = \prod_{j=N_++1}^{N_f} (p + m_j).$$ (A.4)

$u$ is the Coulomb moduli parameter and $m_i$ are the masses $1 \leq N_+ \leq N_f$. By introducing $y_{SW} = \bar{\Lambda}K_+(p)e^{ix} - K(p)$ we get the SW curve in standard form

$$y_{SW}^2 = K(p)^2 - \bar{\Lambda}^2 K_+(p)K_-(p)$$ (A.5)

The SW differential is then defined to be

$$\lambda = pd\ln \frac{K_+}{K_-} - 2\pi ip dx$$ (A.6)

and defines a symplectic form $d\lambda = dp \wedge dx$, which doubly integrated gives the SW periods [12]

$$a = \oint_A p(x) dx \quad a_D = \oint_B p(x) dx.$$ (A.7)

The quantum SW curve is obtained by letting $p$ become the differential operator $-i\hbar \frac{d}{dx}$ [12]:

$$(K(-i\hbar \partial_x)) - \frac{\Lambda}{2} (e^{ix/2}K_+(e^{ix/2} + e^{-ix/2}K_-(e^{-ix/2}) \psi(x) = 0.$$ (A.8)

Let $N_f = 0$ and $x = -iy$. We get

$$-\hbar^2 \frac{d^2}{dy^2} \psi + (\Lambda_0^2 \cosh y + u) \psi = 0$$ (A.9)

Let $N_f = 1$ and $x = -iy$. We get

$$-\hbar^2 \frac{d^2}{dy^2} \psi + \left[ \frac{1}{16}\Lambda_1^3 e^{2y} + \frac{1}{2}\Lambda_1^{3/2} e^{-y} + \frac{1}{2}\Lambda_2^{3/2} m_1 e^y + u \right] \psi = 0$$ (A.10)

Let $N_f = 1$ and $x = -iy$, $y \rightarrow y - \frac{1}{2} \ln \Lambda_1 + \ln 2$. We get

$$-\hbar^2 \frac{d^2}{dy^2} \psi + \left[ \frac{1}{4}\Lambda_1^2 e^{2y} + e^{-y} \right] \psi = 0$$ (A.11)
Let $N_f = 2$, $N_+ = 1$ and $x = -iy$. We get

$$-\hbar^2 \frac{d^2}{dy^2} \psi + \left[ \frac{1}{16} \Lambda_2^2 (e^{2y} + e^{-2y}) + \frac{1}{2} \Lambda_2 m_1 e^y + \frac{1}{2} \Lambda_2 m_2 e^{-y} + u \right] \psi = 0 $$  \hspace{1cm} (A.12)

Let $N_f = 2$, $N_+ = 2$ and $x = -iy$. We get

$$-\hbar^2 \frac{d^2}{dy^2} \psi + \frac{e^{2y} \Lambda_2^2 (m_1 - m_2)^2 + e^y (\Lambda_2^2 - 2 \Lambda_2 \hbar^2 + 8 \Lambda_2 m_1 m_2 - 8 \Lambda_2 u) + 16u - 6 \Lambda_2^2 + 8 \Lambda_2 e^{-y}}{4 (\Lambda_2 e^y - 2)^2} \psi = 0 \hspace{1cm} (A.13)$$

In this paper to relate to BHs and IMs we need to consider only the first realization $N_+ = 1$ for $N_f = 2$. We notice also that the second realization $N_+ = 2$ has a rather different singular structure: two regular and one irregular singularities instead of two irregular singularities. Therefore it is a Confluent Heun equation rather than a Doubly Confluent Heun equation as all the others considered in this paper. We refer though to [30] for a dictionary with BHs also for this second realization.

**B $N_f = 1, 2$ Seiberg-Witten periods**

In this appendix we define and give some relations for the Seiberg-Witten periods for the $SU(2) \ N_f = 1, 2$ theories, that is, the leading $\hbar \to 0$ of the quantum (or deformed) exact periods which we prove are connected to integrability exact $Y$ and $T$ functions.

**B.1 Massless $N_f = 1$ SW periods**

The massless $N_f = 2$ gauge periods are just the $N_f = 0$ gauge periods already dealt with in [17]. Hence we treat here the (much more complex) $N_f = 1$ massless $m = 0$ case, following and extending [49]. In that case the low energy effective action has three finite $\mathbb{Z}_3$ symmetric singularities, corresponding to dyon BPS particles becoming massless. If we set $\Lambda_1 = \Lambda_1^*$ with

$\Lambda_1^* = \sqrt[3]{\frac{256}{27}}, \hspace{1cm} (B.1)$

those singularities are situated at

$$u_0 = -1 \hspace{1cm} u_1 = -e^{2\pi i/3} \hspace{1cm} u_2 = -e^{-2\pi i/3}.\hspace{1cm} (B.2)$$

The massless $m = 0$ $N_f = 1$ SW curve is

$$y_{SW}^2(u, \Lambda_1) = x^3 - ux^2 - \frac{\Lambda_1^6}{64}, \hspace{1cm} (B.3)$$

and it gives the SW periods through the integrals

$$\left( \begin{array}{c} a^{(0)}(u, \Lambda_1) \\ a_D^{(0)}(u, \Lambda_1) \end{array} \right) = \frac{\sqrt{2}}{8\pi} \int_{A,B} dx \frac{2u - 3x}{\sqrt{x^3 - ux^2 - \frac{\Lambda_1^6}{64}}}. \hspace{1cm} (B.4)$$

It can be shown then that $\Pi^{(0)} = a^{(0)}, a_D^{(0)}$ satisfy the SW Picard-Fuchs equation

$$\left( \frac{27 \Lambda_1^6}{256} + u^3 \right) \frac{\partial^2 \Pi^{(0)}(u)}{\partial u^2} + \frac{u}{4} \Pi^{(0)}(u) = 0, \hspace{1cm} (B.5)$$

39
with boundary condition as \( u \to \infty \) as

\[
a^{(0)}(u, \Lambda_1) \simeq \frac{u}{2} \quad u \to \infty
\]

\[
a^{(0)}_D(u, \Lambda_1) \simeq -i \left[ \frac{1}{2\pi} a^{(0)}(u, 0, \Lambda_1) \left( -i\pi - 3\ln \frac{16u}{\Lambda_1^2} \right) + \frac{3}{\pi} \sqrt{\frac{u}{2}} \right] \quad u \to \infty.
\]

(B.6)

The massless SW Picard-Fuchs equation can be mapped into an hypergeometric equation and then explicit formulas for \( a^{(0)}, a^{(0)}_D \) follow:

\[
a^{(0)}(u, \Lambda_1) = \sqrt{\frac{u}{2}} \ _2F_1 \left( \frac{1}{6}, \frac{1}{6}; 1; -\frac{27\Lambda_1^6}{256u^3} \right)
\]

\[
a^{(0)}_D(u, \Lambda_1) = \begin{cases} 
- a^{(0)}(u, \Lambda_1) + e^{-i\pi/3} f_D(u, \Lambda_1) & 0 < \arg(u) \leq \frac{2\pi}{3} \\
 f_D(u, \Lambda_1) - 2 a^{(0)}(u, \Lambda_1) & \frac{2\pi}{3} < \arg(u) \leq \pi \\
 a^{(0)}(u, \Lambda_1) - f_D(u, \Lambda_1) & -\pi < \arg(u) < -\frac{2\pi}{3}
\end{cases}
\]

(B.7)

(sectors given assuming \( \Lambda_1 > 0 \)) where

\[
f_D(u, \Lambda_1) = \frac{\Lambda_1 \left( \frac{256u^3}{27\Lambda_1^6} + 1 \right) \ _2F_1 \left( \frac{5}{6}, \frac{5}{6}; 2; \frac{256u^3}{27\Lambda_1^6} + 1 \right)}{4\sqrt{2}/\sqrt{\sqrt{3}}}
\]

(B.8)

So defined, \( a^{(0)} \) has a branch cut for \( u < 0 \) (due to the square root and three other cuts from the origin \( u = 0 \) to \( u_0, u_1 \) and \( u_2 \) (due to the hypergeometric function). Instead, \( a^{(0)}_D \) so defined has a branch cut for \( u < 0 \) and from \( u = 0 \) to \( u_2 \).

### B.1.1 \( \mathbb{Z}_3 \) R-symmetry

We find the following \( \mathbb{Z}_3 \) R-symmetry relations

\[
\begin{align*}
& a^{(0)}(e^{2\pi i/3} u) = -e^{-2\pi i/3} a^{(0)}(u) & -\pi < \arg(u) \leq \pi/3 \\
& a^{(0)}(e^{-2\pi i/3} u) = e^{2\pi i/3} a^{(0)}(u) & \pi/3 < \arg(u) \leq \pi \\
& a^{(0)}(e^{-2\pi i/3} u) = -e^{2\pi i/3} a^{(0)}(u) & -\pi/3 < \arg(u) \leq \pi \\
& a^{(0)}_D(e^{2\pi i/3} u) = e^{2\pi i/3} a^{(0)}_D(u) & -\pi < \arg(u) \leq -\pi/3 \\
& a^{(0)}_D(e^{-2\pi i/3} u) = -e^{2\pi i/3} \left[ a^{(0)}_D(u) - a^{(0)}(u) \right] & -\pi < \arg(u) \leq -\pi/3 \\
& a^{(0)}_D(e^{-2\pi i/3} u) = -e^{2\pi i/3} \left[ a^{(0)}_D(u) + a^{(0)}(u) \right] & -\pi/3 < \arg(u) \leq \pi
\end{align*}
\]

(B.9)

### B.2 Massive \( N_f = 1, 2 \) SW periods

The massive \( N_f = 1 \) SW curve is \([50]\)

\[
y_{SW}^2 = x^3 - ux^2 + \frac{A_1^3}{4} m_1 x - \frac{A_1^6}{64}
\]

(B.10)

The SW differential is

\[
\lambda = \frac{\sqrt{2}}{4\pi} \left[ - \left( 3x - 2u + \frac{A_1^3}{4} m_1 x \right) \right] \frac{dx}{2y_{SW}}
\]

(B.11)
The SW periods $a_1^{(0)}$, $a_2^{(0)}$ are given by the integrals
\[
\int_{\gamma_1} \lambda = \frac{\sqrt{2}}{4\pi} \left[ u I_1^{(i)} - 3 I_2^{(i)} - \frac{\Lambda_1^3}{4} m I_3^{(i)} \left( - \frac{u}{3} \right) \right] \tag{B.12}
\]
Define $e_k$ as the roots of the Seiberg-Witten curve in canonical form
\[
y_{SW}(x = \xi + \frac{u}{3}) = (\xi - e_1)(\xi - e_2)(\xi - e_3)
= -\frac{\Lambda_1^6}{64} + \xi \left( \frac{\Lambda_1^3 m}{4} - \frac{u^2}{3} \right) + \frac{1}{12} \Lambda_1^3 m u + \xi^3 - \frac{2u^3}{27}, \tag{B.13}
\]
Basic integrals over the cycle $\gamma_1$
\[
I_1^{(1)} = 2 \int_{e_3}^{e_2} \frac{d\xi}{\eta} = \frac{2}{(e_1 - e_3)^{1/2}} K(k)
I_2^{(1)} = 2 \int_{e_3}^{e_2} \frac{\xi d\xi}{\eta} = \frac{2}{(e_1 - e_3)^{1/2}} \left[ e_1 K(k) + (e_3 - e_1) E(k) \right] \tag{B.14}
I_3^{(1)} = 2 \int_{e_3}^{e_2} \frac{d\xi}{\eta (\xi - c)} = \frac{2}{(e_1 - e_3)^{3/2}} \left[ \frac{1}{1 - \tilde{c} + k^2} K(k) + \frac{4k'}{1 + k'} \left( \Pi_1 \left( \nu(c), \frac{1 - k'}{1 + k'} \right) \right) \right]
\]
\[
k^2 = \frac{e_2 - e_3}{e_1 - e_3}, k^2 = 1 - k^2 \tag{B.15}
\]
Elliptic integrals of the first, second and third kind:
\[
K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}
E(k) = \int_0^1 \frac{dx}{1 - k^2 x^2}
\Pi_1(\nu, k) = \int_0^1 \frac{dx}{(1 - x^2)(1 - k^2 x^2)^{1/2}(1 + \nu x^2)} \tag{B.16}
\]
The corresponding integrals $I_1^{(2)}$ over the cycle $\gamma_2$ are obtained by exchanging in $I_1^{(1)} e_1$ and $e_3$.

The massive $N_f = 1$ SW periods also satisfy the Picard-Fuchs equation [64]
\[
\frac{\partial^2 \Pi^{(0)}(u, m)}{\partial u^2} + \frac{81 \Lambda_1^6 - 2048 m^4 u - 384 \Lambda_1^3 m^3 + 3840 m^2 u^2 - 1536 u^3}{(4 m^2 - 3 u) \left( 27 \Lambda_1^6 + 256 u^2 (u - m^2) + 32 \Lambda_1^3 m (8 m^2 - 9 u) \right)} \frac{\partial \Pi^{(0)}(u, m)}{\partial u} = 0 \tag{B.17}
\]
with boundary conditions
\[
a^{(0)}(u, m, \Lambda_1) \simeq \left[ \sqrt{\frac{u}{2}} - \frac{m}{2 \sqrt{2} \Lambda_1^{3/2}} \right] \quad u \to \infty
\]
\[
a_D^{(0)}(u, m, \Lambda_1) \simeq -i \left[ \sqrt{2} a^{(0)}(u, m, \Lambda_1) \left( -i\pi - 3 \ln \frac{16u}{\Lambda_1^2} \right) + 6\sqrt{u} + \frac{m^2}{\sqrt{u}} + \frac{m^4}{u^{3/2}} \right] \quad u \to \infty \tag{B.18}
\]
Notice however that the periods $a^{(0)}$ and $a^{(0)}_D$ so defined are in principle different from the periods $a_1^{(0)}$ and $a_2^{(0)}$ defined as integrals. They are in fact linear combinations of each other, which also possible separate mass term contribution.

For $N_f = 2$ we have similarly (in the cubic SW curve conventions [50])

\[
y_{SW}^2 = x^3 - ux^2 - \frac{\Lambda_1^2}{64}(x-u) + \frac{\Lambda_2^2}{4}m_1m_2x - \frac{\Lambda_1^2}{64}(m_1^2 + m_2^2).
\]

(B.19)

\[
\lambda = -\frac{\sqrt{2}}{4\pi} \frac{dx}{y_{SW}} \left[ x - u - \frac{\Lambda_2^2}{16} \frac{(m_1 - m_2)^2}{x - \frac{\Lambda_2^2}{8}} + \frac{\Lambda_2^2}{16} \frac{(m_1 + m_2)^2}{x + \frac{\Lambda_2^2}{8}} \right]
\]

(B.20)

\[
\int \lambda = \frac{\sqrt{2}}{4\pi} \left[ \frac{4}{3} u I_1 - 2I_2 + \frac{\Lambda_2^2}{8}(m_1 - m_2)^2 I_3 \left( \frac{\Lambda_2^2}{8} - \frac{u}{3} \right) - \frac{\Lambda_2^2}{8}(m_1 + m_2)^2 I_3 \left( -\frac{\Lambda_2^2}{8} - \frac{u}{3} \right) \right].
\]

(B.21)

### B.3 Relations between alternatively defined periods

We show now the relation between $a^{(0)}$, $a^{(0)}_D$ and $a_1^{(0)}$, $a_2^{(0)}$ in the massless case. Assuming $u > 0$ and with small $|u|$ we have

\[
a^{(0)}(u) = a^{(0)}_1(u) \quad \text{Re } a^{(0)}(u) > 0
\]

\[
a^{(0)}_D(u) = -a^{(0)}_2(u) \quad \text{Re } a^{(0)}_D(u) < 0
\]

With their inverses

\[
a^{(0)}_1(u) = a^{(0)}(u) \quad \text{Re } a^{(0)}_1(u) > 0
\]

\[
a^{(0)}_2(u) = -a^{(0)}_D(u) \quad \text{Re } a^{(0)}_2(u) > 0
\]

\[
a^{(0)}_1(e^{2\pi i/3}u) = a^{(0)}_D(e^{2\pi i/3}u) + 2a^{(0)}(e^{2\pi i/3}u) \quad \text{Re } e^{2\pi i/3}a^{(0)}_1(e^{2\pi i/3}u) < 0
\]

\[
a^{(0)}_2(e^{2\pi i/3}u) = a^{(0)}_D(e^{2\pi i/3}u) - a^{(0)}(e^{2\pi i/3}u) \quad \text{Re } e^{2\pi i/3}a^{(0)}_2(e^{2\pi i/3}u) > 0
\]

\[
a^{(0)}_1(e^{-2\pi i/3}u) = a^{(0)}(e^{-2\pi i/3}u) - a^{(0)}_D(e^{-2\pi i/3}u) \quad \text{Re } e^{-2\pi i/3}a^{(0)}_1(e^{-2\pi i/3}u) < 0
\]

\[
a^{(0)}_2(e^{-2\pi i/3}u) = -a^{(0)}_D(e^{-2\pi i/3}u) \quad \text{Re } e^{-2\pi i/3}a^{(0)}_2(e^{-2\pi i/3}u) > 0
\]

Also

\[
a^{(0)}(-u) = -a^{(0)}_1(-u) + a^{(0)}_2(-u)
\]

\[
a^{(0)}_D(-u) = 3a^{(0)}_1(-u) - 2a^{(0)}_2(-u)
\]

\[
a^{(0)}(e^{2\pi i/3}u) = a^{(0)}_2(-e^{2\pi i/3}u)
\]

\[
a^{(0)}_D(e^{2\pi i/3}u) = -a^{(0)}_1(-e^{2\pi i/3}u) + a^{(0)}(-e^{2\pi i/3}u)
\]

\[
a^{(0)}(e^{-2\pi i/3}u) = a^{(0)}_2(-e^{-2\pi i/3}u)
\]

\[
a^{(0)}_D(e^{-2\pi i/3}u) = -a^{(0)}_1(-e^{-2\pi i/3}u) - 2a^{(0)}(-e^{-2\pi i/3}u)
\]

(B.24)
Figure C.1: A region of the $y$ complex plane, where in yellow we show the contour of integration of SW differential for the $SU(2) N_f = 0$ theory we use for the proof equality of the dual SW period $a_D^{(0)}$ and the leading $h \to 0$ order of the logarithm of the Baxter’s $Q$ function $\ln Q^{(0)}$. In red are shown the branch cuts of the SW differential.

In the massive case, similar relations can be found by looking at the large $u$ asymptotics (B.18) and, if the small $u$ region is of interest, also to the continuous behaviour of the functions involved.

C The $N_f = 0$ analytic proof of gauge-integrability relation for $Y$ function

C.1 Asymptotic proof

We report here for clarity the proof of the gauge integrability equivalence between $Y = Q^2$ and $a_D$ in the $SU(2) N_f = 0$ gauge theory case. In that case, the SW order gauge periods are

\begin{align*}
  a^{(0)}(u, \Lambda) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sqrt{2u - 2\Lambda^2 \cos z} \, dz = \Lambda \sqrt{2(u/\Lambda^2 + 1)} \, 2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{2}{1 + u/\Lambda^2}\right), \\
  a_D^{(0)}(u, \Lambda) &= \frac{1}{2\pi} \int_{\arccos(u/\Lambda^2) - \pi}^{\arccos(u/\Lambda^2) - \pi} \sqrt{2u - 2\Lambda^2 \cos z} \, dz = -i\Lambda \left(\frac{u/\Lambda^2 - 1}{2}\right) \, 2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; 1 - u/\Lambda^2\right).
\end{align*}

Let us consider the integral for $\ln Q$ at the leading $h$ (Seiberg-Witten) order. For the modified Mathieu equation (C.3)

\[-\frac{h^2}{2} \frac{d^2}{dy^2} \psi(y) + [\Lambda^2 \cosh y + u] \psi(y) = 0.\]

Then, the leading order of the quantum momentum is

\[P_{-1} = -i\Lambda \sqrt{2 \cosh y' + 2 \frac{u}{\Lambda^2}}.\] (C.4)

Since, in the limits $y \to \pm \infty$, we have $P_{-1} = -i\Lambda e^{\pm y/2} + O(e^{\mp y/2})$, it follows that the Seiberg-Witten regularized momentum is

\[P_{reg,-1}(y) = P_{-1}(y) + 2i\Lambda \cosh \frac{y}{2} = -i\Lambda \left[\sqrt{2 \cosh y' + 2 \frac{u}{\Lambda^2} - 2 \cosh \frac{y'}{2}}\right].\] (C.5)
The leading order of $\ln Q$ is then \[\ln Q^{(0)}(u, \Lambda) = \int_{-\infty}^{\infty} i\mathcal{P}_{reg,-1}(y) \, dy = \Lambda \int_{-\infty}^{\infty} \left[ \sqrt{2 \cosh y + 2 \frac{u}{\Lambda^2} - 2 \cosh \frac{y}{2}} \right] \, dy. \tag{C.6}\]

We assume $u < \Lambda^2$. Let us consider the integral of $i\mathcal{P}_{reg,-1}(y)$ on the (oriented) closed curve which runs along the real axis, slightly below the cut and closes laterally. Mathematically, it is $\gamma = \gamma_1 \cup \gamma_{lat,R} \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \gamma_{lat,L}$, with $\gamma_1 = (-\infty, +\infty)$, $\gamma_2 = (+\infty + i\pi - i0, 0^+ + i\pi - i0)$, $\gamma_3 = (0^+ + i\pi - i0, 0^+ + i\pi - i \arccos(u/\Lambda^2))$, $\gamma_4 = (0^- + i\pi - i \arccos(u/\Lambda^2), 0^- + i\pi - i0)$, $\gamma_5 = (0^- + i\pi - i0, -\infty + i\pi - i0)$, and $\gamma_{lat,L}$ $\gamma_{lat,R}$ are the lateral contours which close the curve (see figure C.1). We expect the integral of $\mathcal{P}_{reg,-1}(y)$ on $\gamma$ to be zero, since the branch cuts are avoided and no singularities are inside the curve. By expanding the square root for $Re y \to \pm \infty$, $|Im y| < \pi$, we get the asymptotic behaviour:

\[\frac{\hbar}{\Lambda} i\mathcal{P}_{reg,-1}(y) = -\left( \frac{u}{\Lambda^2} + 1 \right) e^{-y/2} + o(e^{-y/2}) \quad Re y \to +\infty \tag{C.7}\]

\[\frac{\hbar}{\Lambda} i\mathcal{P}_{reg,-1}(y) = -\left( \frac{u}{\Lambda^2} + 1 \right) e^{y/2} + o(e^{y/2}) \quad Re y \to -\infty, \tag{C.8}\]

from which, we deduce that the integrals on the lateral contours $\gamma_{lat,L/R}$ are exponentially suppressed. For $\gamma_2$ and $\gamma_5$, we consider $\mathcal{P}_{reg,-1}(t + i\pi - i0)$ for $t \in \mathbb{R}$:

\[\frac{\hbar}{\Lambda} i\mathcal{P}_{reg,-1}(t + i\pi - i0) = \sqrt{-2 \cosh t + 2 \frac{u}{\Lambda^2} - 2i \sinh \frac{t}{2}}. \tag{C.9}\]

Since for $t = 0$ it is necessary to cross a cut, we find the oddness property $\mathcal{P}_{-1}(t + i\pi - i0) = -\mathcal{P}_{-1}(-t + i\pi - i0)$. Besides also the regularizing part is odd and therefore, for $t \in \mathbb{R}$ we have

\[\mathcal{P}_{reg,-1}(t + i\pi - i0) = -\mathcal{P}_{reg,-1}(-t + i\pi - i0) \tag{C.10}\]

As a consequence, the integrals on $\gamma_2$ and $\gamma_5$ cancel each other. The integrals on $\gamma_3$ and $\gamma_4$, around the cut, can be better taken into account in the variable $z = -iy - \pi$. There is no contribution from the regularizing part, which has no cut. Instead $\mathcal{P}_{-1}$, which is

\[\mathcal{P}_{-1}(z - i0) = \Lambda \sqrt{-2 \cos (z - i0) + 2 \frac{u}{\Lambda^2}}. \tag{C.11}\]

has the oddness property

\[\mathcal{P}_{-1}(-z + i0) = -\mathcal{P}_{-1}(z - i0) \quad z \in \mathbb{R} \tag{C.12}\]

It follows that the integrals on $\gamma_3$ and $\gamma_4$ add to each other

\[\int_{-\arccos(u/\Lambda^2)}^{0} \mathcal{P}_{-1}(z - i0) \, dz + \int_{0}^{+\arccos(u/\Lambda^2)} \mathcal{P}_{-1}(z + i0) \, dz = \int_{-\arccos(u/\Lambda^2)}^{+\arccos(u/\Lambda^2) - i0} \mathcal{P}_{-1}(z) \, dz. \tag{C.13}\]

In conclusion, we find a relation between the integrals on $\gamma_1$ and on $\gamma_3$ and $\gamma_4$:

\[\int_{-\infty}^{+\infty} i\mathcal{P}_{reg,-1}(y) \, dy = \int_{-\arccos(u/\Lambda^2) - i0}^{+\arccos(u/\Lambda^2) - i0} i\mathcal{P}_{-1}(z) \, dz, \tag{C.14}\]

which in terms of physical quantities is

\[\ln Q^{(0)}(u, \Lambda) = 2\pi i a_D^{(0)}(u, \Lambda). \tag{C.15}\]
θ = 0 + i 0.π

Figure C.2: Poles for the quantum SW differential \( P(h, u, \Lambda_0) \) for the \( SU(2) \) \( N_f = 0 \) theory. The set of poles in the periodicity strip \( |\text{Im } y| < \pi \) we denote by \( B \).

### C.2 Exact analytic proof

We can also imagine here an \( \hbar \)-exact analytic proof of the relation between the Baxter’s \( Q \) function and \( a_D \) period.

\[
Q(\theta, P) = \exp \frac{2\pi i a_D(h, u, \Lambda_0)}{\hbar} \tag{C.16}
\]

following on the lines of the \( \hbar \to 0 \) (classical SW) proof, by using Cauchy theorem to relate the exact integral for the Baxter’s \( Q \) function and \( a_D \) period. Since \( \ln Q \) is \( i \) times the integral over \( (-\infty, +\infty) \) of the regularised NS momentum (as \( b = 1 \)) (as in (2.49), but see also [17])

\[
P_{\text{reg}}(y) = P(y) + 2ie^{\theta} \cosh \frac{y}{2} - \frac{i}{4} \tanh y \tag{C.17}
\]

let us consider the integral of \( iP_{\text{reg}}(y) \) on the (oriented) closed curve with the actual numerically computed poles in figure C.2. We can define the exact dual periods as the exact integrals of \( P(y) = -i \frac{d}{dy} \ln \psi(y) \) written as sum over residues at the poles which as \( \hbar \to 0 \) reduce to the classical cycles (branch cuts), as shown in figure C.2.

\[
\frac{1}{\hbar} a_D(h, u, \Lambda_0) \equiv \oint_{\mathbb{B}} P(y, h, u, \Lambda_0) \, dy = 2\pi i \sum_{n} \text{Res} P(y) \bigg|_{y=n}\tag{C.18}
\]

One may argue that the choice of poles for the two cycles is not well defined. However, on one hand we numerically find that the period \( a \) is given precisely as the integral from \(-i\pi \) to \( i\pi \) as required by the equality \( a = \nu \). On the other hand the choice of poles for the period \( a_D \) is unambiguous because it includes all of them. Along this lines we should be able to prove analytically precisely (C.16).

### C.3 Limits of validity of the identification and wall crossing

The behaviour of the poles of the exact SW differential for complex \( \hbar \sim e^{-\theta} \) is shown in figure C.3. We notice that at \( \text{Im } \theta = \pi/2 \) we have an horizontal line of poles accumulation which crosses the real axis, where we would have to integrate to get \( \ln Q \). Therefore our proof breaks down at such point. A similar singular behaviour can be deduced by the TBA equations [17] where similarly we find a singularity of the kernels at such \( \text{Im } \theta = \pi/2 \).

Now, it can be easily shown that at the SW level crossing such point in \( \theta \) is equivalent to crossing the so-called \textit{curve of marginal stability} in the \( u \) complex plane, which separate the strong-coupling from the weak coupling...
Figure C.3: Poles for the quantum SW differential $\mathcal{P}(\hbar, u, \Lambda_0)$ in the complex $y$ plane for varying $\text{Im} \theta$. We show also an horizontal line of accumulation at $\text{Im} y = -\pi + 2\text{Im} \theta$.

Table C.1: Numerical check of formula (C.19). We used only two instanton contribution and so to have a good match we have to restrict to small $\Lambda_0$.

| $\{\Lambda_0, p, \hbar\}$ | $-\frac{1}{2} \epsilon(\theta, p)$ | $\ln i \sinh A_D/\sinh(2\pi a/\hbar)$ |
|-----------------------------|-------------------------------|----------------------------------|
| $\{\frac{1}{16\sqrt{\pi}}, 2, -i\}$ | 9.27325 | 9.273204 |
| $\{\frac{1}{16\sqrt{\pi}}, 3, -i\}$ | 18.7522 | 18.752173 |
| $\{e^{-1} \frac{1}{16\sqrt{\pi}}, 2, -i\}$ | 17.2829 | 17.282910 |
| $\{e^{1} \frac{1}{16\sqrt{\pi}}, 2, -i\}$ | 1.04849 | 1.04235 |

region. Since different particles are present in the two regions, we do expect some fundamental change in our relations to take place. Also, the $Q, Y$ function in integrability are defined to be entire in $\theta$, while the gauge periods not. Since the strong coupling spectrum involves only the dual gauge period $A_D^{16}$, while the weak coupling spectrum involves necessarily also $a^{17}$, we conjecture that the integrability structure involved in wall crossing is also the $TQ$ system (because $T$ is associated to $a$, cf. (1.3)).

C.4 Relation with other gauge period

It was found in [13] a relation between the $Q$ function and the gauge periods $A_D, a$ (in our conventions)

$$Q(\hbar, a, \Lambda_0) = i \frac{\sinh \frac{1}{2} A_D(\hbar, a, \Lambda_0)}{\sinh \frac{2\pi a}{\hbar}}$$  \hspace{1cm} (C.19)

Actually, we could easily check numerically this relation by computing the l.h.s. by the Liouville TBA (D.4) for $b = 1$ and the r.h.s. relies on the expansion of the prepotential $\mathcal{F}$ in $\Lambda_0$ (number of instantons) [66, 6]: the period $a$ is related to the moduli parameter $u$ (or $P$) through the Matone’s relation [67, 68] and the dual one is given by $A_D = \partial \mathcal{F}/\partial a$. In this respect we noticed that only the first instanton contributions are easily accessible and

---

16The particles at strong coupling being only the magnetic monopole associated to $a_D(u)$ and the dyon associated to $a_D(-u)$ [65, 17].

17At weak coupling there are infinite dyonic BPS particles differing by units of electric charge, associated to $a$ [65, 13].
summing them up (naively) is accurate as long as $|\Lambda_0|/\hbar \ll 1$. The gauge period is defined as

$$
\frac{A_D}{\hbar} = \frac{4a^2}{\hbar} \ln \Lambda_0 + \ln \frac{\Gamma(1 + \frac{2a}{\hbar})}{\Gamma(1 - \frac{2a}{\hbar})} + \frac{8a}{\hbar (4a^2 - \hbar^2)^2} \Lambda_0^4 + O(\Lambda_0^8)
$$

(C.20)

$$
2u = a^2 - \frac{\Lambda_0}{4} \frac{\partial F}{\partial \Lambda_0} = a^2 + \frac{\Lambda_0^4}{2(4a^2 - \hbar^2)} + O(\Lambda_0^8)
$$

(C.21)

The instanton prepotential is given by

$$
F_{NS}^{\text{inst}} = \sum_{n=0}^\infty \Lambda_0^{4n} F_{NS}^{(n)}
$$

(C.22)

with

$$
F_{NS}^{(1)} = -\frac{2}{4a^2 - \hbar^2}
$$

$$
F_{NS}^{(2)} = -\frac{20a^2 + 7\hbar^2}{4(4a^2 - \hbar^2)^3}
$$

$$
F_{NS}^{(3)} = -\frac{4(144a^4 + 232a^2\hbar^2 + 29\hbar^4)}{3(4a^2 - \hbar^2)^5(4a^4 - 13a^2\hbar^2 + 9\hbar^4)}
$$

Thus, thanks to (C.16), relation (C.19) of Grassi, Gu and Marino becomes a relation between the two definition of dual cycles

$$
i \sinh \frac{1}{\hbar} A_D(h, a, \Lambda_0) = \exp \frac{2\pi i a_D(h, u)}{\hbar}.
$$

(C.25)

This relation means that the two cycles $a_D$ and $A_D$ differ by non-perturbative terms in $\hbar$. From the gauge theory point of view, they are precisely respectively the dyon and monopole period in the strong coupling region [13].

**D** Résumé of D3 brane quantization relations

As shown in [42] the physical QNMs condition translates into

$$
Q(\theta_n) = 0,
$$

(D.1)

namely the zeros of the Baxter’s $Q$ function which are the Bethe roots [69]. Now, we prove that condition (D.1) is equivalent to the quantization condition of the dual gauge period

$$
\frac{1}{\hbar} A_D(a, \Lambda_{0,n}, \hbar) = in , \quad n \in \mathbb{Z}.
$$

(D.2)

as conjectured by [27]. For it was already proposed in [13] on a numerical basis the relevant relation (C.19).

Eventually, the $QQ$ system (D.6) characterizes the QNMs as $Y(\theta_n - i\pi/2) = -1$, *i.e.* the TBA quantization condition

$$
\varepsilon(\theta_n' - i\pi/2) = -i\pi(2n' + 1), \quad n' \in \mathbb{Z}
$$

(D.3)

\(^{18}\) Beware that for the $N_f = 0$ theory with respect to the $N_f = 1, 2$ theories we rescale $\hbar \rightarrow \hbar/\sqrt{2}$. This explains the differences with the formulas in subsection 4.2.
which can be easily implemented by using the TBA

\[ \varepsilon(\theta) = \frac{16\sqrt{\pi}}{\Gamma(\frac{1}{4})^2} e^\theta - 2 \int_{-\infty}^{\infty} \frac{\ln[1 + \exp\{-\varepsilon(\theta')\}]}{\cosh(\theta - \theta')} \, d\theta'. \] (D.4)

The \( TQ \) system

\[ T(\theta)Q(\theta) = Q(\theta - i\pi/2) + Q(\theta + i\pi/2) \] (D.5)

and the \( QQ \) relation

\[ Q(\theta + i\pi/2)Q(\theta - i\pi/2) = 1 + Q(\theta)^2, \] (D.6)

impose

\[ Q(\theta_n \pm i\pi/2) = \pm i \] (D.7)

Again (D.6) around \( \theta_n \) forces \( Q(\theta + i\pi/2) = i \pm Q(\theta) + \ldots \) and \( Q(\theta - i\pi/2) = -i \pm Q(\theta) + \ldots \) up to smaller corrections (dots). Therefore, the \( TQ \) system imposes

\[ T(\theta_n) = \pm 2. \] (D.8)

We can derive also the relation [17]

\[ T(\theta) = 2 \cos \left\{ \frac{2\pi}{\hbar} a \right\}, \] (D.9)

from which it follows that the period \( a \) is also quantised (setting \( \hbar = 1 \) as in the literature [27])

\[ a(\theta_n) = \frac{n}{2}, \quad n \in \mathbb{Z}. \] (D.10)

This is exactly the condition used by [28]. Yet, here we have fixed the general limits of its validity as relying on specific forms of the \( TQ \) and \( QQ \) systems (D.5) and (D.6) respectively: it does not work in general, but we will see in the next section the specific conditions for its validity.

**E Floquet exponent through Hill determinant**

Consider the more general \( N_f = 2 \) equation and change variable as \( z = iy \). We get

\[ \frac{d^2}{dz^2} \psi + [\theta_0 + \theta_2 e^{2iz} + \theta_{-2} e^{-2iz} + \theta_1 e^{iz} + \theta_{-1} e^{-iz}] \psi = 0 \] (E.1)

with

\[ \theta_0 = p^2 \quad \theta_{\pm 2} = e^{2\theta} \quad \theta_{\pm 1} = 2e^\theta q_{1,2} \] (E.2)

We search for Floquet solutions, such that

\[ \psi_+(z + 2\pi) = e^{2\pi \nu} \psi_+(z) \quad \psi_-(z + 2\pi) = e^{-2\pi \nu} \psi_-(z) \] (E.3)

that implies they can be expanded in Fourier series as

\[ \psi(z) = e^{\nu z} \sum_{n=-\infty}^{\infty} b_n e^{inz} \] (E.4)

From the equation we get the recursion

\[ (\nu + in)^2 b_n + \sum_{m=-2}^{2} \theta_n b_{n-m} = 0 \] (E.5)
Dividing by $\theta_0 - n^2$ we get the matrix with convergent determinant

$$
\begin{pmatrix}
\vdots & \vdots & \vdots \\
\xi_{n,n-1} & 1 & \xi_{n,n+1} & \xi_{n,n+2} & 0 & \cdots \\
\xi_{n+1,n-1} & \xi_{n+1,n} & 1 & \xi_{n+1,n+2} & \xi_{n+1,n+3} & \cdots \\
0 & \xi_{n+2,n} & \xi_{n+2,n+1} & 1 & \xi_{n+2,n+3} & \cdots \\
0 & 0 & \xi_{n+3,n+1} & \xi_{n+3,n+2} & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\begin{pmatrix}
b_{n-1} \\
b_n \\
b_{n+1} \\
b_{n+2} \\
\vdots \\
\end{pmatrix} = 0.
$$

(E.6)

with

$$
\xi_{mn} = \frac{-\theta_{m-n}}{(m - ivn)^2 - \theta_0} \quad \xi_{m,m} = 1
$$

(E.7)

Defining $A_n$ as the finite $2n + 1 \times 2n + 1$ submatrix We also introduce a $(2n + 1) \times (2n + 1)$ matrix

$$
A_n =
\begin{pmatrix}
\varepsilon_{1-n} & \varepsilon_{0-n+1} & \varepsilon_{1-n+2} & \cdots & 0 \\
\varepsilon_{0-n} & \varepsilon_{0} & 1 & \varepsilon_{0+1} & 0 & \cdots \\
\varepsilon_{0+2} & \varepsilon_{1+1} & 1 & \varepsilon_{1+2} & \varepsilon_{1+3} & \cdots \\
& & & \ddots & & \\
0 & \varepsilon_{n-n-2} & \varepsilon_{n-n-1} & 1 & \varepsilon_{n-n} \\
\end{pmatrix}
$$

(E.8)

and

$$
\Delta(iv) = \lim_{n \to \infty} \det A_n
$$

(E.9)

by ordinary methods [58] we arrive at this relation

$$
\Delta(iv) = \Delta(0) - \frac{\sin^2(\pi iv)}{\sin^2 \pi \sqrt{\theta_0}}
$$

(E.10)

The Floquet exponent is then given by the roots of the equation

$$
\sin^2(\pi iv) = \Delta(0) \sin^2 \pi \sqrt{\theta_0}
$$

(E.11)

or

$$
\cosh(2\pi \nu) = 1 - 2\Delta(0) \sin^2 \pi \nu
$$

(E.12)

In particular for $N_f = 2 \xi_{m,n}$ are given by

$$
\xi_{m,m+2}^{(2)} = -\frac{e^{2\theta}}{(m - iv)^2 - p^2} \quad \xi_{m,m+1}^{(2)} = -\frac{2e^{2\theta}q_{1,2}}{(m - iv)^2 - p^2}
$$

(E.13)

while for $N_f = 1$

$$
\xi_{m,m-2}^{(1)} = -\frac{e^{2\theta}}{(m - iv)^2 - p^2} \quad \xi_{m,m+1}^{(1)} = -\frac{e^{2\theta}}{(m - iv)^2 - p^2} \quad \xi_{m,m-1}^{(1)} = -\frac{2e^{2\theta}q_{1}}{(m - iv)^2 - p^2}
$$

(E.14)
F Doubly Confluent Heun equation

Let us now show that the equations for \( N_f = 0, 1, 2 \) are just particular cases of the doubly confluent Heun equation\(^{19}\):

\[
\frac{d^2w}{dz^2} + \left( \frac{\gamma}{z^2} + \frac{\delta}{z} + \epsilon \right) \frac{dw}{dz} + \frac{\alpha z - \bar{q}}{z^2} w = 0
\]  

(\text{F.1})

It’s general solution is given by Mathematica as

\[
w = c_1 \text{HeunD}[\bar{q}, \alpha, \gamma, \delta, \epsilon, z] + c_2 z^2 - \delta e^{\gamma z} - z \epsilon \text{HeunD}[\delta + \bar{q} - 2, \alpha - 2\epsilon, -\gamma, 4 - \delta, -\epsilon, z]
\]  

(\text{F.2})

It is enough to just change variable as \( z = e^y \)

\[
\frac{d^2w}{dy^2} + (\delta + \gamma e^{-y} + e^y \epsilon - 1) \frac{dw}{dy} + (\alpha e^y - \bar{q})w = 0
\]  

(\text{F.3})

and transforming the solution as

\[
\psi(y) = \exp \left\{ \frac{1}{2} \left( \gamma e^{-y} + (1 - \delta)y - \epsilon e^y \right) \right\} w(y)
\]  

(\text{F.4})

to get

\[
\frac{d^2\psi}{dy^2} - \frac{1}{4} \left[ \gamma^2 e^{-2y} + 2\gamma(\delta - 2)e^{-y} + (2\gamma \epsilon + (\delta - 1)^2 + 4\bar{q}) + e^y(2\delta \epsilon - 4\alpha) + \epsilon^2 e^{2y} \right] \psi(y) = 0
\]  

(\text{F.5})

By comparing with the quantum SW curve for \( N_f = 2 \)

\[
-\hbar^2 \frac{d^2}{dy^2} \psi + \left( \frac{\Lambda_2^2}{16} e^{2y_2} + \frac{\Lambda_2 m_3}{2} e^{y_2} + \frac{\Lambda_2 m_2}{2} e^{-y_2} + \frac{\Lambda_2^2}{16} e^{-2y_2} + u \right) \psi = 0
\]  

(\text{F.6})

we get the parameter dictionary

\[
\begin{align*}
\gamma &= \pm \frac{\Lambda_2}{2\hbar} & \epsilon &= \frac{\Lambda_2}{2\hbar} \\
\delta &= \frac{2(1 \pm m_2)}{\hbar} \\
\alpha &= \frac{1}{2\hbar^2}(\Lambda_2 \hbar - m_1 \Lambda_2 \mp m_2 \Lambda_2) \\
\bar{q} &= \frac{1}{8\hbar^2}[-2\hbar^2 + 8u - 8m_2^2 \mp 8m_2 \hbar \mp \Lambda_2^2]
\end{align*}
\]  

(\text{F.7})

or

\[
\begin{align*}
\gamma &= \pm \frac{\Lambda_2}{2\hbar} & \epsilon &= -\frac{\Lambda_2}{2\hbar} \\
\delta &= \frac{2(1 \pm m_2)}{\hbar} \\
\alpha &= \frac{1}{2\hbar^2}(\Lambda_2 \hbar - m_1 \Lambda_2 \mp m_2 \Lambda_2) \\
\bar{q} &= \frac{1}{8\hbar^2}[-2\hbar^2 + 8u - 8m_2^2 \mp 8m_2 \hbar \mp \Lambda_2^2]
\end{align*}
\]  

(\text{F.8})

\(^{19}\) in the Mathematica’s notation, let \( \delta \leftrightarrow \gamma \) and set \( \epsilon = 1 \)
By comparing with the quantum SW curve for $N_f = 1$ with $y \to -y_1$

$$-\hbar^2 \frac{d^2}{dy_1^2} \psi + \left( \frac{\Lambda_1^2}{4} e^{-y_1} + \Lambda_1 m_1 e^{y_1} + \frac{\Lambda_1^2}{4} e^{2y_1} + u \right) \psi = 0 \quad (F.9)$$

we get the parameter dictionary

$$\begin{align*}
\gamma &= \pm \frac{\Lambda_1}{\hbar} \\
\epsilon &= 0 \\
\delta &= \frac{2(h \pm m_1)}{\hbar} \\
\alpha &= -\frac{\Lambda_1^2}{4} \\
\bar{q} &= \frac{1}{4\hbar^2} [-h^2 + 4u - 4m_1^2 \mp 4m_1\hbar] \\
\end{align*} \quad (F.10)$$

By comparing with the quantum SW curve for $N_f = 0$, after also change of variable $y \to y_0/2^{20}$

$$-\hbar^2 \frac{d^2}{dy_0^2} \psi + \left( \frac{\Lambda_0^2}{2} e^{y_0} + \frac{\Lambda_0^2}{2} e^{-y_0} + u \right) \psi = 0 \quad (F.11)$$

$$\begin{align*}
\gamma &= \pm \frac{2\sqrt{2}\Lambda_0}{\hbar} \\
\epsilon &= \frac{2\sqrt{2}\Lambda_0}{\hbar} \\
\delta &= \frac{2\sqrt{2}\Lambda_0}{\hbar} \\
\alpha &= \frac{2\sqrt{2}\Lambda_0}{\hbar} \\
q &= \frac{1}{4\hbar^2} [-h^2 \mp 16\Lambda_0^2 + 16u] \\
\end{align*} \quad (F.12)$$

or

$$\begin{align*}
\gamma &= \pm \frac{2\sqrt{2}\Lambda_0}{\hbar} \\
\epsilon &= -\frac{2\sqrt{2}\Lambda_0}{\hbar} \\
\delta &= \frac{2\sqrt{2}\Lambda_0}{\hbar} \\
\alpha &= -\frac{2\sqrt{2}\Lambda_0}{\hbar} \\
\bar{q} &= \frac{1}{4\hbar^2} [-h^2 \pm 16\Lambda_0^2 + 16u] \\
\end{align*} \quad (F.13)$$

F.1 Eigenvalue expansion

In the book on Heun equations [44] it is given another form for the doubly confluent Heun equation, namely

$$z \frac{d}{dz} z \frac{d}{dz} w + \alpha \left( z + \frac{1}{z} \right) z \frac{d}{dz} w + \left( \frac{\beta_1 + 1}{2} \alpha z + \left( \frac{\alpha^2}{2} - \gamma \right) + (\beta_{-1} - \frac{1}{2}) \frac{\alpha}{z} \right) w = 0 \quad (F.14)$$

Transforming in normal form, then changing variable as $z = e^y$ and transforming again into normal form we get

$$-\frac{d^2}{dy^2} \psi + \left( \gamma + \frac{1}{4} \alpha^2 e^{-2y} + \frac{1}{4} \alpha^2 e^{2y} - \alpha \beta_{-1} e^{-y} - \alpha \beta_1 e^y \right) \psi = 0 \quad (F.15)$$

We have

$$w(z) = e^{-\frac{z}{2}} (z^{-\frac{1}{2}}) \psi(y) \quad (F.16)$$

We get the parameters map for $N_f = 2$

$$\begin{align*}
\alpha &= \pm \frac{\Lambda_2}{2\hbar} = \pm 2e^\theta \\
\beta_1 &= \mp \frac{m_1}{\hbar} = \mp q_1 \\
\beta_{-1} &= \mp \frac{m_2}{\hbar} = \mp q_1 \\
\gamma &= \frac{u}{\hbar^2} = p^2 \\
\end{align*} \quad (F.17)$$

\(^{20}\)Notice though that as for the $N_f = 1, 2$ theories in this paper, with respect to $N_f = 0$ in [17] we use make the rescaling $\hbar \to \sqrt{2}\hbar$. 

51
The authors [44] in particular have solutions corresponding to the lower sign convention

\[ w_{\infty,1}(y) \simeq (-2e^{\theta+y})^{-\left(\frac{1}{2}+q_1\right)} e^{e^{\theta+y} \psi_{+,0}(y)} \quad y \to +\infty \]  
(F.18)  

\[ w_{\infty,2}(y) \simeq e^{2e^{\theta+y}q_1^{-\frac{1}{2}}} e^{e^{\theta+y} \psi_{+,1}} \quad y \to +\infty \]  
(F.19)  

with

\[ W[w_{\infty,2}, w_{\infty,1}] = 1 \]  
(F.20)  

Define

\[ \lambda = \gamma - \alpha^2/2 \]  
(F.21)  

The DCHE has a countable number of eigenvalues, denoted \( \lambda_{\mu}(\alpha, \beta) \) with

\[ \mu \in \nu + \mathbb{Z} \]  
(F.22)  

where \( \nu \) is the Floquet characteristic exponent. The eigenvalues have expansion

\[ \lambda_{\mu}(\alpha, \beta) = \mu^2 + \sum_{m=1}^{\infty} \lambda_{\mu,m}(\beta) \alpha^{2m}. \]  
(F.23)  

The first coefficient is [44]

\[ \lambda_{\mu,1}(\beta) = -\frac{1}{2} + \frac{2\beta-1}{4\mu^2 - 4}. \]  
(F.24)  

### G Limit to lower flavours gauge theories

#### G.1 Limit from \( N_f = 1 \) to \( N_f = 0 \)

The Seiberg-Witten curve for \( N_f = 1 \)

\[ y_{SW,1}^2 = x^2(x-u) + \frac{\Lambda_1^4}{4} m_1 x - \frac{\Lambda_1^6}{64} \]  
(G.1)  

in the limit

\[ \Lambda_1 \to 0, \quad m_1 \to \infty, \quad \text{with } \Lambda_1^4 m_1 = \Lambda_0^4. \]  
(G.2)  

flows to the Seiberg-Witten curve for \( N_f = 0 \)

\[ y_{SW,0}^2 = x^2(x-u) + \frac{\Lambda_0^4}{4} x. \]  
(G.3)  

Similarly the \( N_f = 1 \) quantum Seiberg-Witten curve:

\[- \hbar^2 \frac{d^2}{dy_1^2} \psi + \left[ \frac{1}{16} \Lambda_1^3 e^{2y_1} + \frac{1}{2} \Lambda_1^{3/2} e^{-y_1} + \frac{1}{2} \Lambda_1^{3/2} m_1 e^{y_1} + u \right] \psi = 0. \]  
(G.4)  

if we let

\[ y_1 = y_0 - \frac{1}{2} \ln m_1 \to -\infty \]  
(G.5)  

becomes

\[- \hbar^2 \frac{d^2}{dy_0^2} \psi + \left[ \frac{1}{16} \frac{\Lambda_1^3}{m_1} e^{2y_0} + \frac{1}{2} \Lambda_1^{3/2} m_1^{1/2} e^{-y_0} + \frac{1}{2} \Lambda_1^{3/2} m_1^{1/2} e^{y_0} + u \right] \psi = 0 \]  
(G.6)
that is precisely reduce to the $N_f = 0$ equation:
\begin{equation}
- \hbar^2 \frac{d^2}{dy_0^2} \psi + (\Lambda_0^2 \cosh y_0 + u) \psi = 0 .
\end{equation}

We can also consider the limit on the integrability equation as follows. The Perturbed Hairpin IM ODE/IM equation is
\begin{equation}
- \frac{d^2}{dy_1^2} \psi(y_1) + \left[ e^{2\theta_1} (e^{2y_1} + e^{-y_1}) + 2q e^{\theta_1} e^{y_1} + p_0^2 \right] \psi(y_1) = 0 .
\end{equation}

and it must reduce to the ODE/IM equation for the Liouville model studied in [17]
\begin{equation}
- \frac{d^2}{dy_0^2} \psi(y_0) + \left\{ e^{2\theta_0} [e^{3y_0} + e^{-3y_0}] + p_0^2 \right\} \psi(y_0) = 0 ,
\end{equation}

In order for (G.8) to go into (G.9) we need to impose
\begin{equation}
e^{2\theta_1 - y_1} = e^{2\theta_0 - y_0} \quad e^{2\theta_1 + y_1} = e^{2\theta_0 + y_0} \quad p_1 = p_0
\end{equation}

or
\begin{equation}q = \frac{1}{2} e^{6\theta_0}, \quad y_1 = y_0 - 2\theta_0 + 2\theta_1
\end{equation}

Now the limit requires $\theta_1 + y_1 \to -\infty$, that is
\begin{equation}\theta_1 \to -\infty
\end{equation}

and as a consequence
\begin{equation}q \sim e^{-3\theta_1} \to \infty \quad \theta_1 \to -\infty
\end{equation}

We now consider also the limit on gauge periods. We numerically find, for $u, m_1, \Lambda_1 > 0, \Lambda_1 \to 0, m_1 \to \infty, \Lambda_1^3 m_1 = \Lambda_0^4$
\begin{equation}a_{1,1}^{(0)}(u, m_1, \Lambda_1) \to -a_{0,D}^{(0)}(u, \Lambda_0)
\end{equation}
\begin{equation}a_{1,1}^{(0)}(-u, m_1, \Lambda_1) \to -a_{0,D}^{(0)}(-u, \Lambda_0) + a_{0}^{(0)}(-u + i0, \Lambda_0)
\end{equation}
\begin{equation}=-ia_{0,D}^{(0)}(u, \Lambda_0)
\end{equation}
\begin{equation}a_{1,2}^{(0)}(\pm u, m_1, \Lambda_1) + \frac{m_1}{\sqrt{2}} \to \frac{1}{2} a_{0}^{(0)}(\pm u, \Lambda_0)
\end{equation}
\begin{equation}a_{1,1}^{(0)}(e^{\pm 2\pi i/3} u, e^{\mp 2\pi i/3} m_1, \Lambda_1) - \frac{e^{\mp 2\pi i/3} m_1}{\sqrt{2}} \to \frac{1}{2} a_{0}^{(0)}(u, e^{\mp i\pi/6} \Lambda_0)
\end{equation}
\begin{equation}a_{1,1}^{(0)}(-e^{2\pi i/3} u, e^{-2\pi i/3} m_1, \Lambda_1) - \frac{e^{-2\pi i/3} m_1}{\sqrt{2}} \to e^{-2\pi i/3} [a_{0,D}^{(0)}(u, \Lambda_0) - \frac{1}{2} a_{0}^{(0)}(-u + i0, \Lambda_0)]
\end{equation}
\begin{equation}a_{1,1}^{(0)}(-e^{-2\pi i/3} u, e^{2\pi i/3} m_1, \Lambda_1) - \frac{e^{2\pi i/3} m_1}{\sqrt{2}} \to e^{2\pi i/3} [-\frac{1}{2} a_{0}^{(0)}(-u + i0, \Lambda_0)]
\end{equation}
G.2 Limit from $N_f = 2$ to $N_f = 1$

Staring from the $N_f = 2$ quantum Seiberg Witten curve
\[ -\hbar^2 \frac{d^2 \psi}{dy_2^2} + \left[ \frac{1}{16} \Lambda_1^2 (e^{2y_2} + e^{-2y_2}) + \frac{1}{2} \Lambda_2 m_1 e^{y_2} + \frac{1}{2} \Lambda_2 m_2 e^{-y_2} + u \right] \psi = 0, \]  
(G.21)
since we have
\[ \Lambda_2^2 m_2 = \Lambda_1^3 \quad m_2 \to \infty \quad \Lambda_2 \to 0 \]  
(G.22)
we can set
\[ y_2 = y_1 + \frac{1}{2} \ln m_2 \to +\infty \]  
(G.23)
so the equation becomes
\[ -\hbar^2 \frac{d^2 \psi}{dy_1^2} + \left[ \frac{1}{16} \Lambda_1^2 (m_2 e^{2y_1} + m_2 e^{-2y_1}) + \frac{1}{2} \Lambda_2 \sqrt{m_2 m_1} e^{y_1} + \frac{1}{2} \Lambda_2 \sqrt{m_2} e^{-y_1} + u \right] \psi = 0 \]  
(G.24)
which in the limit reduces to the $N_f = 1$ quantum Seiberg-Witten curve equation:
\[ -\hbar^2 \frac{d^2 \psi}{dy_1^2} + \left[ \frac{1}{16} \Lambda_1^3 e^{2y_1} + \frac{1}{2} \Lambda_1^{3/2} e^{-y_1} + \frac{1}{2} \Lambda_1^{3/2} m_1 e^{y_1} + u \right] \psi = 0. \]  
(G.25)

In integrability variables, we impose the conditions that allow the limit of the differential equations
\[ e^{2\theta_2+2y_2} = e^{2\theta_1+2y_1}, \quad e^{\theta_2+y_2} q_1 = e^{\theta_1+y_1} q_1, \quad 2 e^{\theta_2-y_2} q_2 = e^{2\theta_1-y_1}, \quad e^{2\theta_2-2y_2} \to 0, \quad p_2^2 = p_1^2. \]  
(G.26)
from which we deduce that we have to take the limit
\[ y_2 = -\theta_2 + \theta_1 + y_1 \quad \theta_2 \to -\infty \quad M_2 = \frac{1}{2} e^{3\theta_1-2\theta_2} \to \infty \]  
(G.27)

References

[1] N. Seiberg and E. Witten, “Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory,” Nucl. Phys. B 426 (1994) 19–52, arXiv:hep-th/9407087. [Erratum: Nucl.Phys.B 430, 485–486 (1994)].

[2] N. Seiberg and E. Witten, “Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD,” Nucl. Phys. B 431 (1994) 484–550, arXiv:hep-th/9408099.

[3] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, and Y. I. Manin, “Construction of Instantons,” Phys. Lett. A 65 (1978) 185–187.

[4] R. Flume and R. Poghossian, “An Algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential,” Int. J. Mod. Phys. A 18 (2003) 2541, arXiv:hep-th/0208176.

[5] N. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” Adv. Theor. Math. Phys. 7 (2004) 831, arXiv:hep-th/0306211.

[6] N. Nekrasov and A. Okounkov, “Seiberg-Witten theory and random partitions,” Prog. Math. 244 (2006) 525–596, arXiv:hep-th/0306238.
[7] N. Nekrasov and A. Okounkov, “Seiberg-Witten theory and random partitions,” *Prog. Math.* **244** (2006) 525–596, arXiv:hep-th/0306238 [hep-th].

[8] N. A. Nekrasov and S. L. Shatashvili, “Quantization of Integrable Systems and Four Dimensional Gauge Theories,” in *16th International Congress on Mathematical Physics*. 8, 2009, arXiv:0908.4052 [hep-th].

[9] A. Mironov and A. Morozov, “Nekrasov Functions and Exact Bohr-Sommerfeld Integrals,” *JHEP* **04** (2010) 040, arXiv:0910.5670 [hep-th].

[10] G. Basar and G. V. Dunne, “Resurgence and the Nekrasov-Shatashvili limit: connecting weak and strong coupling in the Mathieu and Lamâ© systems,” *JHEP* **02** (2015) 160, arXiv:1501.05671 [hep-th].

[11] A.-K. Kashani-Poor and J. Troost, “Pure $\mathcal{N} = 2$ super Yang-Mills and exact WKB,” *JHEP* **08** (2015) 160, arXiv:1504.08324 [hep-th].

[12] K. Ito, S. Kanno, and T. Okubo, “Quantum periods and prepotential in $\mathcal{N} = 2$ SU(2) SQCD,” *JHEP* **08** (2017) 065, arXiv:1705.09120 [hep-th].

[13] A. Grassi, J. Gu, and M. Mariño, “Non-perturbative approaches to the quantum Seiberg-Witten curve,” *JHEP* **07** (2020) 106, arXiv:1908.07065 [hep-th].

[14] A. Grassi, Q. Hao, and A. Neitzke, “Exact WKB methods in $SU(2)N_f = 1$,” arXiv:2105.03777 [hep-th].

[15] P. Dorey and R. Tateo, “Anharmonic oscillators, the thermodynamic Bethe ansatz, and nonlinear integral equations,” *J. Phys.* **A32** (1999) L419–L425, arXiv:hep-th/9812211 [hep-th].

[16] S. L. V. Bazhanov and A. Zamolodchikov, “Spectral determinants for Schrödinger equation and Q-operators of Conformal Field Theory,” *J.Stat.Phys.* **102** (1999) 567–576, arXiv:9812247 [hep-th].

[17] D. Fioravanti and D. Gregori, “Integrability and cycles of deformed $\mathcal{N} = 2$ gauge theory,” *Phys. Lett. B* **804** (2020) 135376, arXiv:1908.08030 [hep-th].

[18] P. Dorey and R. Tateo, “On the relation between Stokes multipliers and the T-Q systems of conformal field theory,” *Nucl.Phys. B* **563** (1999) 573–602, arXiv:9906219 [hep-th].

[19] A. B. Zamolodchikov, “On the thermodynamic Bethe ansatz equation in sinh-Gordon model,” *J. Phys. A* **39** (2006) 12863–12887, arXiv:hep-th/0005181 [hep-th].

[20] A. Zamolodchikov, *Quantum Field Theories in two dimensions: Collected works of Alexei Zamolodchikov - Generalized Mathieu Equation and Liouville TBA*, vol. 2. World Scientific, 2012.

[21] D. Fioravanti, M. Rossi, and H. Shu, “$QQ$-system and non-linear integral equations for scattering amplitudes at strong coupling,” *JHEP* **12** (2020) 086, arXiv:2004.10722 [hep-th].

[22] D. Fioravanti and M. Rossi, “On the origin of the correspondence between classical and quantum integrable theories,” arXiv:2106.07600 [hep-th].

[23] D. Gaiotto, G. W. Moore, and A. Neitzke, “Four-dimensional wall-crossing via three-dimensional field theory,” *Commun. Math. Phys.* **299** (2010) 163–224, arXiv:0807.4723 [hep-th].

[24] D. Gaiotto, G. W. Moore, and A. Neitzke, “Wall-crossing, Hitchin Systems, and the WKB Approximation,” arXiv:0907.3987 [hep-th].
[25] D. Gaiotto, “Opers and TBA,” arXiv:1403.6137 [hep-th].
[26] D. Fioravanti, H. Poghosyan, and R. Poghossian, “$T$, $Q$ and periods in SU(3) $\mathcal{N} = 2$ SYM,” JHEP 03 (2020) 049, arXiv:1909.11100 [hep-th].
[27] G. Aminov, A. Grassi, and Y. Hatsuda, “Black Hole Quasinormal Modes and Seiberg-Witten Theory,” arXiv:2006.06111 [hep-th].
[28] M. Bianchi, D. Consoli, A. Grillo, and J. F. Morales, “QNMs of branes, BHs and fuzzballs from Quantum SW geometries,” arXiv:2105.04245 [hep-th].
[29] G. Bonelli, C. Iossa, D. P. Lichtig, and A. Tanzini, “Exact solution of Kerr black hole perturbations via CFT$_2$ and instanton counting,” arXiv:2105.04483 [hep-th].
[30] M. Bianchi, D. Consoli, A. Grillo, and J. F. Morales, “More on the SW-QNM correspondence,” arXiv:2109.09804 [hep-th].
[31] L. F. Alday, D. Gaiotto, and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” Lett. Math. Phys. 91 (2010) 167–197, arXiv:0906.3219 [hep-th].
[32] D. Gaiotto, “Asymptotically free $\mathcal{N} = 2$ theories and irregular conformal blocks,” J. Phys. Conf. Ser. 462 no. 1, (2013) 012014, arXiv:0908.0307 [hep-th].
[33] G. Bonelli, C. Iossa, D. P. Lichtig, and A. Tanzini, “Irregular Liouville correlators and connection formulae for Heun functions,” arXiv:2201.04491 [hep-th].
[34] D. Consoli, F. Fucito, J. F. Morales, and R. Poghossian, “CFT description of BH’s and ECO’s: QNMs, superradiance, echoes and tidal responses,” arXiv:2206.09437 [hep-th].
[35] Y. Hatsuda, “An alternative to the Teukolsky equation,” arXiv:2007.07906 [gr-qc].
[36] M. Casals and R. T. da Costa, “Hidden spectral symmetries and mode stability of subextremal Kerr(-dS) black holes,” arXiv:2105.13329 [gr-qc].
[37] H. Nakajima and W. Lin, “New Chandrasekhar transformation in Kerr spacetime,” Phys. Rev. D 105 no. 6, (2022) 064036, arXiv:2111.05857 [gr-qc].
[38] M. Bianchi and G. Di Russo, “Turning rotating D-branes and BHs inside out their photon-halo,” arXiv:2203.14900 [hep-th].
[39] M. Dodelson, A. Grassi, C. Iossa, D. Panea Lichtig, and A. Zhiboedov, “Holographic thermal correlators from supersymmetric instantons,” arXiv:2206.07720 [hep-th].
[40] D. R. Mayerson, “Fuzzballs and Observations,” Gen. Rel. Grav. 52 no. 12, (2020) 115, arXiv:2010.09736 [hep-th].
[41] V. Cardoso and P. Pani, “Tests for the existence of black holes through gravitational wave echoes,” Nature Astron. 1 no. 9, (2017) 586–591, arXiv:1709.01525 [gr-qc].
[42] D. Fioravanti and D. Gregori, “A new method for exact results on Quasinormal Modes of Black Holes,” arXiv:2112.11434 [hep-th].
[43] H.-P. Nollert, “Quasinormal modes: the characteristic sound of black holes and neutron stars,” Classical and Quantum Gravity 16 no. 12, (Nov, 1999) R159–R216. https://doi.org/10.1088/0264-9381/16/12/201.
[44] A. Ronveaux, *Heun's Differential Equations*. Oxford University Press, 1995.

[45] V. A. Fateev and S. L. Lukyanov, “Boundary RG flow associated with the AKNS soliton hierarchy,” *J. Phys. A* **39** (2006) 12889–12926, arXiv:hep-th/0510271.

[46] A. B. Zamolodchikov, “On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories,” *Phys. Lett. B* **253** (1991) 391–394.

[47] A. Fabbri, D. Fioravanti, S. Piscaglia, and R. Tateo, “Exact results for the low energy $AdS_4 \times \mathbb{CP}^3$ string theory,” *JHEP* **11** (2013) 073, arXiv:1308.1861 [hep-th].

[48] T. R. Klassen and E. Melzer, “The Thermodynamics of purely elastic scattering theories and conformal perturbation theory,” *Nucl. Phys. B* **350** (1991) 635–689.

[49] A. Bilal and F. Ferrari, “Curves of marginal stability, and weak and strong coupling BPS spectra in N=2 supersymmetric QCD,” *Nucl. Phys. B* **480** (1996) 589–622, arXiv:hep-th/9605101.

[50] A. Bilal and F. Ferrari, “The BPS spectra and superconformal points in massive N=2 supersymmetric QCD,” *Nucl. Phys. B* **516** (1998) 175–228, arXiv:hep-th/9706145.

[51] L. Alvarez-Gaume, M. Marino, and F. Zamora, “Softly broken N=2 QCD with massive quark hypermultiplets. I.,” *Int. J. Mod. Phys. A* **13** (1998) 403–430, arXiv:hep-th/9703072.

[52] F. Ferrari, “Charge fractionization in N=2 supersymmetric QCD,” *Phys. Rev. Lett.* **78** (1997) 795–798, arXiv:hep-th/9609101.

[53] H.-Y. Chen and K. Petunin, “Notes on Wall Crossing and Instanton in Compactified Gauge Theory with Matter,” *JHEP* **10** (2010) 106, arXiv:1006.5957 [hep-th].

[54] K. Imaizumi, “Quantum periods and TBA equations for $\mathcal{N}=2 \ SU(2) \ N_f=2$ SQCD with flavor symmetry,” *Phys. Lett. B* **816** (2021) 136270, arXiv:2103.02248 [hep-th].

[55] C. Destri and H. J. de Vega, “Nonlinear integral equation and excited states scaling functions in the sine-Gordon model,” *Nucl. Phys. B* **504** (1997) 621–664, arXiv:hep-th/9701107.

[56] S. Cecotti and M. Del Zotto, “$Y$ systems, $Q$ systems, and 4D $\mathcal{N}=2$ supersymmetric QFT,” *J. Phys. A* **47** no. 47, (2014) 474001, arXiv:1403.7613 [hep-th].

[57] K. Ito, M. Marino, and H. Shu, “TBA equations and resurgent Quantum Mechanics,” *JHEP* **01** (2019) 228, arXiv:1811.04812 [hep-th].

[58] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*. Cambridge Mathematical Library. Cambridge University Press, 4 ed., 1996.

[59] T. Ikeda, M. Bianchi, D. Consoli, A. Grillo, J. F. Morales, P. Pani, and G. Raposo, “Black-hole microstate spectroscopy: Ringdown, quasinormal modes, and echoes,” *Phys. Rev. D* **104** no. 6, (2021) 066021, arXiv:2103.10960 [gr-qc].

[60] E. W. Leaver, “An Analytic representation for the quasi normal modes of Kerr black holes,” *Proc. Roy. Soc. Lond. A* **402** (1985) 285–298.

[61] E. W. Leaver, “Quasinormal modes of reissner-nordström black holes,” *Phys. Rev. D* **41** (May, 1990) 2986–2997. https://link.aps.org/doi/10.1103/PhysRevD.41.2986.
[62] S. Prem Kumar, A. O’Bannon, A. Pribytok, and R. Rodgers, “Holographic Coulomb branch solitons, quasinormal modes, and black holes,” *JHEP* **05** (2021) 109, arXiv:2011.13859 [hep-th].

[63] M. Bianchi and G. Di Russo, “Turning black-holes and D-branes inside out their photon-spheres,” arXiv:2110.09579 [hep-th].

[64] Y. Ohta, “Prepotential of N=2 SU(2) Yang-Mills gauge theory coupled with a massive matter multiplet,” *J. Math. Phys.* **37** (1996) 6074–6085, arXiv:hep-th/9604051.

[65] A. Bilal and F. Ferrari, “The Strong-Coupling Spectrum of the Seiberg-Witten Theory,” *Nucl.Phys. B* **469** (1996) 387–402, arXiv:9602082 [hep-th].

[66] N. A. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” *Adv. Theor. Math. Phys.* **7** no. 5, (2003) 831–864, arXiv:hep-th/0206161.

[67] M. Matone, “Instantons and recursion relations in N=2 SUSY gauge theory,” *Phys. Lett. B357* (1995) 342–348, arXiv:hep-th/9506102 [hep-th].

[68] R. Flume, F. Fucito, J. F. Morales, and R. Poghossian, “Matone’s relation in the presence of gravitational couplings,” *JHEP* **04** (2004) 008, arXiv:hep-th/0403057 [hep-th].

[69] S. Lukyanov, V. Bazhanov and A. Zamolodchikov, “Integrable Structure of Conformal Field Theory II. Q-operator and DDV equation,” *Commun.Math.Phys.* **190** (1997) 247–278, arXiv:9604044 [hep-th].

58