SMOOTHNESS OF HEAT KERNEL MEASURES ON INFINITE-DIMENSIONAL HEISENBERG-LIKE GROUPS

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Abstract. We study measures associated to Brownian motions on infinite-dimensional Heisenberg-like groups. In particular, we prove that the associated path space measure and heat kernel measure satisfy a strong definition of smoothness.

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1. Introduction

Recall that a measure $\mu$ on $\mathbb{R}^n$ is smooth if $\mu$ is absolutely continuous with respect to Lebesgue measure and the associated density is a smooth function on $\mathbb{R}^n$. If one wishes to generalize this notion of smoothness of measure to an infinite-dimensional space, one immediately encounters complications due to the lack of an infinite-dimensional Lebesgue measure. Thus, we consider the following more intrinsic definition of smoothness for a measure on $\mathbb{R}^n$: for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0,1,2,\ldots\}^n$, there exists a function $z_\alpha \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mu)$ such that

$$\int_{\mathbb{R}^n} \partial^\alpha f \, d\mu = \int_{\mathbb{R}^n} f z_\alpha \, d\mu,$$

for all $f \in C^\infty_c(\mathbb{R}^n)$, where $L^\infty := \cap_{p \geq 1} L^p$ and $\partial^\alpha = \prod_{i=1}^n \partial_i^{\alpha_i}$. This definition of smoothness is in fact equivalent to our first understanding (see for example [6]), and it is obviously better suited to adapt to infinite dimensions and the absence of a canonical reference measure.

In the present paper we adapt the above definition to give a direct proof of the smoothness of elliptic heat kernel measures on infinite-dimensional Heisenberg-like groups. Typically, it is not possible to verify that a measure on an infinite-dimensional space is smooth in this way and much weaker interpretations must be made; see for example [3,11,12].

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Let $G$ be an infinite-dimensional Heisenberg-like group, $\mathfrak{g}_{CM}$ be its Cameron-Martin Lie subalgebra, and $\{\xi_t\}_{t \geq 0}$ be a Brownian motion on $G$ (see Section 2 for definitions). Then we have the following theorem.

**Theorem 1.1.** Fix $T > 0$, and let $m \in \mathbb{N}$ and $h_1, \ldots, h_m \in \mathfrak{g}_{CM}$. Then there exist $\tilde{z}, \hat{z} \in L^\infty$ depending on $h_1, \ldots, h_m$ such that, for any suitably nice function $f$ on $G$,

$$E\left[ (\tilde{h}_1 \cdots \tilde{h}_m f)(\xi_T) \right] = E[f(\xi_T)\tilde{z}]$$

and

$$E\left[ (\hat{h}_1 \cdots \hat{h}_m f)(\xi_T) \right] = E[f(\xi_T)\hat{z}],$$

where $\tilde{h}$ and $\hat{h}$ are the left and right invariant vector fields, respectively, associated to $h \in \mathfrak{g}_{CM}$.

This result is proved by first establishing smoothness results for the induced measure on the associated path space. In particular, let $W_T(G)$ denote continuous path space on $G$ and $\mathcal{H}_T(\mathfrak{g}_{CM})$ denote the space of absolutely continuous paths on $\mathfrak{g}_{CM}$ with finite energy (see Notation 3.1). Then we prove the following theorem.

**Theorem 1.2.** Let $m \in \mathbb{N}$ and $h_1, \ldots, h_m \in \mathcal{H}_T(\mathfrak{g}_{CM})$. Then there exists $\hat{Z} \in L^\infty$ depending on $h_1, \ldots, h_m$ such that, for any suitably nice function $F$ on $W_T(G)$,

$$E \left[ (\hat{h}_1 \cdots \hat{h}_m F)(\xi) \right] = E[F(\xi)\hat{Z}],$$

where $\hat{h}$ is the right invariant vector field associated to $h \in \mathcal{H}_T(\mathfrak{g}_{CM})$.

Theorem 1.2 is stated more precisely and proved in Theorem 3.15; Theorem 1.1 is the content of Theorem 4.2 and Corollary 4.4. Note that these theorems give a strong satisfaction of smoothness for measures in infinite dimensions.

The organization of the paper is as follows. Section 2 recalls the definitions of infinite-dimensional Heisenberg-like groups and Brownian motions on these groups, first studied in [7]. In Section 3, we recall the quasi-invariance and first-order integration by parts results proved in [7] for the path space measure, and, building on these results, give the integration by parts formulae that prove Theorem 1.2. In Section 4, we show how these path space results immediately give integration by parts formulæ for heat kernel measures on the group.

Finally, let us here mention some references to other quasi-invariance and integration by parts results for measures in infinite-dimensional curved settings; see [1, 2, 4, 5, 8, 9] and their references.

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## 2. Brownian Motion on Infinite-Dimensional Heisenberg-Like Groups

In this section, we recall the definitions of infinite-dimensional Heisenberg-like groups and Brownian motion on these spaces. For more details on this construction, see Sections 2 and 4 of [7]. One may also consult this reference for motivating examples, including the finite-dimensional Heisenberg groups as well as the Heisenberg group of a symplectic vector space.

Let $(W, H, \mu)$ denote an abstract Wiener space; that is, $W$ is a real separable Banach space equipped with Gaussian measure $\mu$ and $H$ is the associated Cameron-Martin subspace. Let $C$ be a real vector space with inner product $\langle \cdot, \cdot \rangle_C$ and...
\[ \text{dim}(C) = N < \infty. \] Let \( \omega : W \times W \to C \) be a continuous skew-symmetric bilinear form on \( W \).

**Definition 2.1.** Let \( g \) denote \( W \times C \) when thought of as a Lie algebra with the Lie bracket given by
\[
[[X_1, V_1], [X_2, V_2]] := (0, \omega(X_1, X_2)).
\]

We may also equip \( W \times C \) with the group multiplication given by
\[
(w_1, c_1) \cdot (w_2, c_2) = \left( w_1 + w_2, c_1 + c_2 + \frac{1}{2} \omega(w_1, w_2) \right).
\]

We will denote \( W \times C \) by \( G \) when thought of as a group, and we will call \( G \) constructed in this way a Heisenberg-like group.

It is easy to verify that, given this bracket and multiplication, \( g \) is indeed a Lie algebra and \( G \) is a group with \( g^{-1} = -g \) and identity \( e = (0, 0) \).

**Notation 2.2.** Let \( g_{CM} \) denote \( H \times C \) when thought of as a Lie subalgebra of \( g \), and we will refer to \( g_{CM} \) as the Cameron-Martin subalgebra of \( g \). The space \( g = G = W \times C \) is a Banach space with the norm
\[
\|(w, c)\|_g := \|w\|_W + \|c\|_C,
\]
and \( g_{CM} = H \times C \) is a Hilbert space with respect to the inner product
\[
\langle (A, a), (B, b) \rangle_{g_{CM}} := \langle A, B \rangle_H + \langle a, b \rangle_C.
\]

The associated Hilbertian norm on \( g_{CM} \) is given by
\[
\|(A, a)\|_{g_{CM}} := \sqrt{\|A\|_H^2 + \|a\|_C^2}.
\]

Let \( i : H \to W \) denote the inclusion map, \( i^* : W^* \to H^* \) denote its transpose, and \( H_* := \{ h \in H : \langle \cdot, h \rangle_H \in \text{Range}(i^*) \} \). Let \( \{B_t, B_t^0\}_{t \geq 0} \) be a Brownian motion on \( g \) with variance determined by
\[
\mathbb{E}[(B_s, B_s^0), (A, a)]_{g_{CM}} = \langle (A, a), (C, c) \rangle_{g_{CM}} \min(s, t),
\]
for all \( s, t \geq 0 \), \( A, C \in H_* \), and \( a, c \in C \).

**Definition 2.3.** The continuous \( G \)-valued process given by
\[
\xi_t = (B_t, B_t^0) + \frac{1}{2} \int_0^t \omega(B_s, dB_s)
\]
is a Brownian motion on \( G \). For \( T > 0 \), let \( \nu_T = \text{Law}(\xi_T) \) denote the heat kernel measure at time \( T \) on \( G \).

Proposition 4.1 of [7] gives details on how the above stochastic integral is defined, and more generally that reference proves many properties of the process \( \xi_t \) and its distribution. In particular, in Corollary 4.9 of that reference it is proved that \( \nu_T \) is invariant under the inversion map \( g \mapsto g^{-1} \); that is, for any \( T > 0 \),
\[
\mathbb{E}[f(\xi_T)] = \int_G f(g) \, d\nu_T(g) = \int_G f(g^{-1}) \, d\nu_T(g) = \mathbb{E}[f(\xi_T^{-1})].
\]
3. The path space measure

In this section, we prove that $\nu = \text{Law}(\xi)$ satisfies its own strong smoothness properties.

**Notation 3.1.** Fix $T > 0$. For a Banach space $X$, let

$$W_T(X) := \{ x : [0, T] \to X : x \text{ continuous and } x(0) = 0 \}$$

equipped with the sup norm topology, and, for a Hilbert space $K$, let $\mathcal{H}_T(K)$ denote the absolutely continuous paths in $W_T(K)$ with finite energy. In particular, for $X = G$

$$\|g\|_{W_T(G)} := \sup_{0 \leq t \leq T} \|g(t)\|_G = \sup_{0 \leq t \leq T} (\|w(t)\|_W + \|c(t)\|_C)$$

for all $g = (w, c) \in W_T(G)$, and for $K = g_{CM}$

$$\|h\|^2_{\mathcal{H}_T(g_{CM})} := \int_0^T \|\dot{h}(t)\|^2_{g_{CM}} dt = \int_0^T \left( \|\dot{A}(t)\|^2_H + \|\dot{a}(t)\|^2_C \right) dt$$

for all $h = (A, a) \in \mathcal{H}_T(g_{CM})$.

**Remark 3.2.** Recall that, for $\{B_t\}_{t \geq 0}$ Brownian motion on $W$, $\text{Law}(B)$ is a Gaussian measure on the separable Banach space $W_T(W)$. Thus, by Fernique's theorem (see for example Theorem 3.1 of [10]), there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$

$$\mathbb{E} \left[ \exp(\delta \|B\|_{W_T(W)}) \right] < \infty.$$

Additionally, in Proposition 4.1 of [7], it is proved that for any $p \in [1, \infty)$

$$\mathbb{E} \left\| \int_0^T \omega(B_s, dB_s) \right\|^p_{W_T(C)} < \infty.$$

The following theorem is a slight generalization of Theorem 5.2 in [7], and the proof is analogous.

**Theorem 3.3.** Let $h = (A, a) \in \mathcal{H}_T(g_{CM})$. If $F, Z : W_T(G) \to [0, \infty]$ are measurable functions, then

$$\mathbb{E}[F(h \cdot \xi)Z(B, B^0)] = \mathbb{E}[F(\xi)Z(B - A, B^0 - a - u_A)J_h],$$

where

$$u_A(t) := \frac{1}{2} \int_0^t \omega(A(s) - 2B_s, \dot{A}(s)) ds \in \mathcal{H}_T(C)$$

and $J_h = J_h(B, B^0)$ is given by

$$J_h := \exp \left\{ \int_0^T \langle \dot{A}(t), dB_t \rangle_H + \left\langle \dot{a}(t) + \frac{1}{2} \omega(A(t) - 2B_t, \dot{A}(t)), dB^0_t \right\rangle_C \
- \frac{1}{2} \int_0^T \left( \|\dot{A}(t)\|^2_H + \|\dot{a}(t) + \frac{1}{2} \omega(A(t) - 2B_t, \dot{A}(t))\|^2_C \right) dt \right\}.$$

Moreover, equation (3.1) holds for all measurable $F, Z : W_T(G) \to \mathbb{R}$ such that

$$\mathbb{E}[F(h \cdot \xi)Z(B, B^0)] = \mathbb{E}[F(\xi)Z(B - A, B^0 - a - u_A)J_h] < \infty.$$
Proof. First combining (2.2) and (2.3) gives
\[
\mathbb{E}[F(h \cdot \xi)Z(B, B^0)] = \mathbb{E}
\left[
F\left(B + A, B^0 + a + \frac{1}{2} \int_0^1 \omega(B_s, dB_s) + \frac{1}{2} \omega(A, B)\right) Z(B, B^0)\right].
\]
Now translating \((B, B^0) \mapsto (B - A, B^0 - a)\) and applying the standard Cameron-Martin theorem (see for example Theorem 1.2 of Chapter II of [10]) implies that
\[
\mathbb{E}[F(h \cdot \xi)Z(B, B^0)] = \mathbb{E}
\left[
F\left(B, B^0 + \frac{1}{2} \int_0^1 \omega(B_s - A(s), d(B_s - A(s))) + \frac{1}{2} \omega(A, B - A)\right)
\times Z(B - A, B^0 - a) \tilde{J}_h(B, B^0)\right].
\]
where \(\tilde{J}_h = \tilde{J}_h(B, B^0)\) is given by
\[
\tilde{J}_h := \exp\left(\int_0^T \langle \dot{A}(t), dB_t \rangle_H - \frac{1}{2} \int_0^T \|\dot{A}(t)\|^2_H dt\right)
\times \exp\left(\int_0^T \langle \dot{a}(t), dB^0_t \rangle_C - \frac{1}{2} \int_0^T \|\dot{a}(t)\|^2_C dt\right).
\]
This may be rewritten as
\[
\mathbb{E}[F(h \cdot \xi)Z(B, B^0)] = \mathbb{E}
\left[
F\left(B, B^0 + \frac{1}{2} \int_0^1 \omega(B_s, dB_s) + \frac{1}{2} \int_0^1 \omega(A(s) - 2B_s, \dot{A}(s)) ds\right)
\times Z(B - A, B^0 - a) \tilde{J}_h(B, B^0)\right].
\]
Freezing integration over \(B\) (that is, using Fubini) and translating again, this time \(B_0 \mapsto B_0 - u_A\) with \(u_A\) as defined in (3.2), we may again apply the Cameron-Martin theorem to get that
\[
\mathbb{E}[F(h \cdot \xi)Z(B, B^0)] = \mathbb{E}
\left[
F(\xi)Z(B - A, B^0 - a - u_A) \tilde{J}_h(B, B^0 - u_A) \tilde{J}(0, u_A)\right].
\]
Now one may simplify to show that
\[
\tilde{J}_h(B, B^0 - u_A) \tilde{J}(0, u_A) = J_h,
\]
where \(J_h\) is as defined in (3.3). □

Remark 3.4. If we take \(Z \equiv 1\) in the previous theorem, this is the statement that \(\nu = \text{Law}(\xi)\) is quasi-invariant under left translation by elements of \(\mathcal{H}_T(g_{CM})\). It is worth recalling that the above proof fails for right translation, as the requisite translating element in that case is not absolutely continuous and thus the Cameron-Martin theorem is no longer available; see Remark 5.3 of [7] for details.

We now have a few technical estimates and notations that will allow us to prove the desired integration by parts formulae in Theorem 3.15. The following result is a restatement of Proposition 5.4 of [7]. We include the proof here for completeness.
Proposition 3.5. Let $p \in [1, \infty)$. Then there exists $\kappa = \kappa(p) > 0$ such that, for all $h \in \mathcal{H}_T(g_{CM})$ such that $\|h\|_{\mathcal{H}_T(g_{CM})} < \kappa$,

$$\mathbb{E}[J_h(B, B^0)^p] < \infty.$$  

Proof. For the purpose of this proof, let $\mathbb{E}_{B^0}$ and $\mathbb{E}_B$ denote expectation relative to $B^0$ and $B$, respectively. We may write

$$J_h(B, B^0)^p = \exp \left\{ p \int_0^T \left\langle \dot{a}(t) + \frac{1}{2} \omega(A(t) - 2B_t, \dot{A}(t)), dB^0_t \right\rangle \right\}$$

$$\times \exp \left\{ p \int_0^T \langle \dot{A}(t), dB_t \rangle_H - \frac{1}{2} p \int_0^T \|\dot{A}(t)\|_H^2 \, dt \right\}$$

$$\times \exp \left\{ \frac{1}{2} p \int_0^T \left\| \dot{a}(t) + \frac{1}{2} \omega(A(t) - 2B_t, \dot{A}(t)) \right\|_C^2 \, dt \right\}.$$  

Since

$$\mathbb{E}_{B^0} \left[ \exp \left\{ p \int_0^T \left\langle \dot{a}(t) + \frac{1}{2} \omega(A(t) - 2B_t, \dot{A}(t)), dB^0_t \right\rangle \right\} \right] = \exp \left\{ \frac{1}{2} p^2 \int_0^T \left\| \dot{a}(t) + \frac{1}{2} \omega(A(t) - 2B_t, \dot{A}(t)) \right\|_C^2 \, dt \right\},$$  

we may write $\mathbb{E}_{B^0}[J_h(B, B^0)^p] = UV$, where

$$U := \exp \left\{ p \int_0^T \langle \dot{A}(t), dB_t \rangle_H - \frac{1}{2} p \int_0^T \|\dot{A}(t)\|_H^2 \, dt \right\}$$

and

$$V := \exp \left\{ \frac{1}{2} (p^2 - p) \int_0^T \left\| \dot{a}(t) + \frac{1}{2} \omega(A(t) - 2B_t, \dot{A}(t)) \right\|_C^2 \, dt \right\}.$$  

In particular, when $p = 1$, this and Tonelli’s theorem imply that

$$\mathbb{E}[J_h(B, B^0)] = \mathbb{E}_B \mathbb{E}_{B^0}[J_h(B, B^0)] = \mathbb{E}_B[U] = 1.$$  

When $p > 1$, applying Tonelli again and the Cauchy-Schwarz inequality gives

$$\mathbb{E}[J_h(B, B^0)^p] = \mathbb{E}_B[UV] \leq (\mathbb{E}_B[U^2])^{1/2} (\mathbb{E}_B[V^2])^{1/2}.$$  

For the first factor, we have that

$$\mathbb{E}_B[U^2] = \exp \left( \frac{1}{2} (p^2 - p) \int_0^T \|\dot{A}(t)\|_H^2 \, dt \right) \leq \exp \left( \frac{1}{2} (p^2 - p) \|h\|_{\mathcal{H}_T(g_{CM})}^2 \right) < \infty.$$  

For the second factor, first note that

$$\left\| \dot{a}(t) + \frac{1}{2} \omega(A(t) - 2B_t, \dot{A}(t)) \right\|_C^2 \leq 2\|\dot{a}(t)\|_C^2 + 2 \cdot \frac{1}{4} \|\omega(A(t) - 2B_t, \dot{A}(t))\|_C^2.$$  

$$\leq 2\|\dot{a}(t)\|_C^2 + \frac{1}{2} \|\omega\|_0^2 \|A(t) - 2B_t\|_W^2 \|\dot{A}(t)\|_W^2.$$  

$$\leq 2\|\dot{a}(t)\|_C^2 + \|\omega\|_0^2 \left( \|A(t)\|_W^2 + 4\|B\|_{\mathcal{H}_T(W)}^2 \right) \|\dot{A}(t)\|_W^2.$$
Recall that $\| \cdot \|_W \leq C \| \cdot \|_H$ for some $C < \infty$ (see for example Theorem A.1 of [7]). Combining this with the fact that

$$\| A(t) \|_H \leq \int_0^T \| \dot{A}(s) \|_H ds \leq \sqrt{T} \left( \int_0^T \| \dot{A}(s) \|_H^2 ds \right)^{1/2} \leq \sqrt{T} \| h \|_{\mathcal{H}_T(g_{CM})},$$

implies that

$$V^2 \leq \exp \left\{ (p^2 - p) \left( 2 \| h \|^2_{\mathcal{H}_T(g_{CM})} + C^2 \| \omega \|^2_{\mathcal{H}_T(g_{CM})} \right) \right\} \times \exp \left\{ 4(p^2 - p)C^2 \| \omega \|^2_{\mathcal{H}_T(g_{CM})} \| B \|^2_{\mathcal{W}_T(W)} \right\} .$$

So letting $\delta_0$ be as in Remark 3.2 $\mathbb{E}_B[V^2] < \infty$ as long as

$$4(p^2 - p)C^2 \| \omega \|^2_{\mathcal{H}_T(g_{CM})} < \delta_0,$$

that is, for all $\| h \|_{\mathcal{H}_T(g_{CM})} < \kappa := \sqrt{\delta_0/4(p^2 - p)C^2 \| \omega \|^2_{\mathcal{H}_T(g_{CM})}}$. \hfill $\square$

In a similar way we may prove the following proposition.

**Proposition 3.6.** Let $p \in [1, \infty)$ and $h \in \mathcal{H}_T(g_{CM})$. Then there exists $\varepsilon_0 = \varepsilon_0(p) > 0$ such that

$$\mathbb{E} \left[ \sup_{|\varepsilon| \leq \varepsilon_0} \left| \frac{d}{d\varepsilon} J_{ch}(B, B^0) \right|^p \right] < \infty.$$

**Proof.** Note that

$$J_{ch} = \exp \left( \varepsilon \alpha_1 + \varepsilon^2 \alpha_2 + \varepsilon^3 \alpha_3 + \varepsilon^4 \alpha_4 \right)$$

where

$$\alpha_1 = \alpha_1(h) = \int_0^T (\dot{A}(t), dB_t)_H + (\dot{A}(t) - \omega(B_t, \dot{A}(t)), dB^0_t)_C$$

$$\alpha_2 = \alpha_2(h) = -\frac{1}{2} \int_0^T \| \dot{A}(t) \|^2_H dt + \frac{1}{2} \int_0^T \langle \omega(A(t), \dot{A}(t)), dB^0_t \rangle_C$$

$$- \frac{1}{2} \int_0^T \| \dot{A}(t) - \omega(B_t, \dot{A}(t)) \|^2_C dt$$

$$\alpha_3 = \alpha_3(h) = -\frac{1}{2} \int_0^T (\dot{A}(t) - \omega(B_t, \dot{A}(t)), \omega(A(t), \dot{A}(t)))_C dt,$$

and

$$\alpha_4 = \alpha_4(h) = -\frac{1}{8} \int_0^T \| \omega(A(t), \dot{A}(t)) \|^2_C dt.$$ 

Thus,

$$\frac{d}{d\varepsilon} J_{ch} = J_{ch} \cdot (\alpha_1 + 2\varepsilon \alpha_2 + 3\varepsilon^2 \alpha_3 + 4\varepsilon^3 \alpha_4).$$

For fixed $p \in [1, \infty)$, we may choose $\varepsilon_0 = \varepsilon_0(p)$ sufficiently small that $\varepsilon < \varepsilon_0$ implies $\varepsilon \| h \|_{\mathcal{H}_T(g_{CM})} < \kappa$, where $\kappa$ is as given in Proposition 3.5 and so $\mathbb{E}[J_{ch}^p] < \infty.$
For the \(\alpha_i\)'s, note that \(\int_0^T \langle \dot{A}, dB \rangle_H\) and \(\int_0^T \langle \omega(A, \dot{A}), dB^0 \rangle_C\) are Gaussian and hence have finite moments of all orders. Also,

\[
\int_0^T \|\dot{a}(t) - \omega(B_t, \dot{A}(t))\|_C^2 \, dt \leq 2 \int_0^T \left( \|\dot{a}(t)\|_C^2 + \|\omega(B_t, \dot{A}(t))\|_C^2 \right) \, dt
\]

\[
\leq 2 \int_0^T \left( \|\dot{a}(t)\|_C^2 + \|\omega\|^2 \|B\|_{W_T(W)}^2 \|\dot{A}(t)\|_H^2 \right) \, dt
\]

\[
\leq 2 \left( \|h\|^2_{\mathcal{H}_T(G,M)} + \|\omega\|^2 \|B\|^2_{W_T(W)} \|h\|^2_{\mathcal{H}_T(G,M)} \right)
\]

\[
\leq C \left( 1 + \|B\|^2_{W_T(W)} \right),
\]

So by Fernique’s Theorem (see Remark 3.2) this term is in \(L^p\) for all \(p \in [1, \infty)\).

Now if \(N_t := \int_0^t (\dot{a} - \omega(B, \dot{A}), dB^0)\), then \(N\) is a martingale and \(\langle N \rangle_T = \int_0^T \|\dot{a} - \omega(B, \dot{A})\|_C^2 \, dt\). So by the previous estimate, \(E[\langle N \rangle_T^p] < \infty\) for all \(p \in [1, \infty)\) and hence by the Burkholder-Davis-Gundy inequalities, \(E[N_T]^p < \infty\). Finally, applying the Cauchy-Schwarz inequality and again the previous estimate implies that

\[
\int_0^T \|\langle \dot{a}(t) - \omega(B_t, \dot{A}(t)), \omega(A(t), \dot{A}(t)) \rangle_C \|_C \, dt \leq C \left( 1 + \|B\|^2_{W_T(W)} \right)
\]

which is again finite by Fernique’s theorem. The remaining terms are deterministic and clearly finite. \(\square\)

**Notation 3.7.** For \(h_i = (A_i, a_i) \in \mathcal{H}_T(g_{CM})\), define

\[
Z_i := Z_{h_i}(B, B^0) := \int_0^T \langle \dot{A}_i(t), dB_i \rangle_H + \langle \dot{a}_i(t) - \omega(B_t, \dot{A}_i(t)), dB^0_t \rangle_C,
\]

\[
Z_{ij} := Z_{h_i,h_j}(B, B^0) := \int_0^T \langle \omega(A_j(t), \dot{A}_i(t)) \rangle_C\]

\[
- \int_0^T \left[ \langle \dot{A}_i(t), \dot{A}_j(t) \rangle_H + \langle \dot{a}_i(t) - \omega(B_t, \dot{A}_i(t)), \dot{a}_j(t) - \omega(B_t, \dot{A}_j(t)) \rangle_C \right] \, dt,
\]

\[
Z_{ijk} := Z_{h_i,h_j,h_k}(B, B^0) := - \int_0^T \left[ \langle \dot{a}_i(t) + \omega(B_t, \dot{A}_i(t)), \omega(A_k(t), \dot{A}_j(t)) \rangle_C\right.
\]

\[
+ \langle \dot{a}_j(t) + \omega(B_t, \dot{A}_j(t)), \omega(A_k(t), \dot{A}_i(t)) \rangle_C
\]

\[
+ \langle \dot{a}_k(t) + \omega(B_t, \dot{A}_k(t)), \omega(A_j(t), \dot{A}_i(t)) \rangle_C \right] \, dt,
\]

and

\[
Z_{ijkl} := Z_{h_i,...,h_l} := - \int_0^T \left[ \langle \omega(A_i(t), \dot{A}_i(t)), \omega(A_k(t), \dot{A}_j(t)) \rangle_C\right.
\]

\[
+ \langle \omega(A_k(t), \dot{A}_i(t)), \omega(A_i(t), \dot{A}_j(t)) \rangle_C
\]

\[
+ \langle \omega(A_j(t), \dot{A}_i(t)), \omega(A_i(t), \dot{A}_k(t)) \rangle_C \right] \, dt.
\]

The following lemma provides some motivation for Notation 3.7. In particular, these functions will comprise the factors appearing in the integration by parts formulæ.
Lemma 3.8. Let $J_h$ be as given in equation (3.3) and $Z_i$, $Z_{ij}$, $Z_{ijk}$, and $Z_{ijkl}$ be as in Notation 3.7. Then
\[
\begin{align*}
(i) & \quad Z_i = \frac{d}{d\varepsilon} J_{\varepsilon h}, \\
(ii) & \quad Z_{ij} = \frac{d}{d\varepsilon} Z_i(B - \varepsilon A_j, B^0 - \varepsilon a_j - u \varepsilon A_j) \\
(iii) & \quad Z_{ijk} = \frac{d}{d\varepsilon} Z_{ij}(B - \varepsilon A_k, B^0 - \varepsilon a_k - u \varepsilon A_k) \\
(iv) & \quad Z_{ijkl} = \frac{d}{d\varepsilon} Z_{ij}(B - \varepsilon A_l, B^0 - \varepsilon a_l - u \varepsilon A_l).
\end{align*}
\]

Proof. The lemma follows from simple computations. For example, recall from equation (3.5) that
\[
\left(\frac{d}{d\varepsilon} J_{\varepsilon h}\right)_{\varepsilon=0} = (J_{\varepsilon h} \cdot (\alpha_1 + 2\varepsilon\alpha_2 + 3\varepsilon^2\alpha_3 + 4\varepsilon^3\alpha_4))_{\varepsilon=0} = \alpha_1,
\]
where $\alpha_1 = \alpha_1(h)$ is given in (3.4). Taking $h = h_i$ and noting that $\alpha_1(h_i) = Z_{h_i} = Z_i$ completes the proof of (3.3).

Similarly, it may be checked that
\[
Z_i(B - \varepsilon A_j, B^0 - \varepsilon a_j - u \varepsilon A_j) = Z_i + \varepsilon Z_{ij} + \varepsilon^2 \beta_2 + \varepsilon^3 \beta_3,
\]
where
\[
\beta_2 = -\int_0^T \left\{ \frac{1}{2} \langle \dot{a}_i(t) - \omega(B_t, \dot{A}_i(t)), \omega(A_j(t), \dot{A}_j(t)) \rangle_C \\
+ \langle \dot{a}_j(t) - \omega(B_t, \dot{A}_j(t)), \omega(A_j(t), \dot{A}_i(t)) \rangle_C \right\} dt
\]
and
\[
\beta_3 = -\frac{1}{2} \int_0^T \langle \omega(A_j(t), \dot{A}_i(t)), \omega(A_j(t), \dot{A}_j(t)) \rangle_C dt,
\]
thus satisfying (3.4). The computations for (3.5) and (3.6) are analogous. \qed

Proposition 3.9. For all $p \in [1, \infty)$, $E|Z|^p < \infty$, where $Z$ represents any element from $\{Z_i, Z_{ij}, Z_{ijk}, Z_{ijkl} : h_i, h_j, h_k, h_l \in \mathcal{H}_T(g_{CM})\}$.

Proof. The integrability of $Z_i = \alpha_1(h_i)$ was already verified in the proof of Proposition 3.6. The terms in $Z_{ij}$ and $Z_{ijk}$ can be handled similarly as in that proof, and $Z_{ijkl}$ is deterministic and clearly finite. \qed

In a similar way to Propositions 3.5 and 3.6 we may prove the following.

Proposition 3.10. For any $p \in [1, \infty)$ and $h = (A, a) \in \mathcal{H}_T(g_{CM})$,
\[
E \left[ \sup_{|\varepsilon| \leq 1} \left| Z(B - \varepsilon A, B^0 - \varepsilon a - u \varepsilon A) \right|^p \right] < \infty
\]
and
\[
E \left[ \sup_{|\varepsilon| \leq 1} \left| \frac{d}{d\varepsilon} Z(B - \varepsilon A, B^0 - \varepsilon a - u \varepsilon A) \right|^p \right] < \infty,
\]
where $Z$ represents any element from $\{Z_i, Z_{ij}, Z_{ijk} : h_i, h_j, h_k, h_l \in \mathcal{H}_T(g_{CM})\}$. 
Proof. Recall from equation (3.7) that
\[ Z_i(B - \varepsilon A_j, B^0 - \varepsilon a_j - u \varepsilon A_j) = Z_i + Z_{ij} \varepsilon + \beta_2 \varepsilon^2 + \beta_3 \varepsilon^3, \]
where \( \beta_2 \) and \( \beta_3 \) are as given in (3.8) and (3.9). The integrability of \( Z_i \) and \( Z_{ij} \) follows from Proposition 3.8 and thus one need only justify the integrability of \( \beta_2 \) (as \( \beta_3 \) is deterministic). This is easily done using the polynomial integrability of \( \|B\|_{\mathcal{W}_T(W)} \) (compare with (3.6)). Similar arguments work for \( Z_{ij} \) and \( Z_{ijk} \). \( \square \)

**Notation 3.11.** For \( m \in \mathbb{N} \), let
\[ \Lambda_m := \{ \text{partitions } \theta \text{ of } \{1, \ldots, m\} : \theta = \{\gamma_1^0, \ldots, \gamma_{k_0}^0\} \text{ with } \#\gamma_r^0 \leq 4 \text{ for } r = 1, \ldots, k_0 \}. \]
For \( \gamma = \{\ell_1, \ldots, \ell_n\} \in \theta \in \Lambda_m \), we will always assume that elements are listed in increasing order \( \ell_1 < \cdots < \ell_n \). (Note that \( 1 \leq n \leq 4 \).)

**Notation 3.12.** For any \( m \in \mathbb{N} \), \( \gamma = \{\ell_1, \ldots, \ell_n\} \in \theta \in \Lambda_m \), and \( h_1, \ldots, h_m \in \mathcal{H}_T(g_{CM}) \) with \( h_k = (A_k, a_k) \), let \( Z_{\gamma} := Z_{\ell_1, \ldots, \ell_n} \) where the right hand side is as defined in Notation 3.7. Also let \( \Phi_{h_1, \ldots, h_m} := \Phi_{h_1, \ldots, h_m}(B, B^0) \) be defined by
\[ \Phi_{h_1, \ldots, h_m} := \sum_{\theta \in \Lambda_m} Z_{\gamma}^{\theta} \cdot \cdots \cdot Z_{\gamma}^{\theta}. \]
Further, for \( h_{m+1} \in \mathcal{H}_T(g_{CM}) \), let
\[ Z_{\gamma}^{h_{m+1}} := Z_{\gamma}^{\theta}(B - \varepsilon A_{m+1}, B^0 - \varepsilon a_{m+1} - u \varepsilon A_{m+1}), \]
where \( u_A \) is as defined in (3.2), and
\[ \Phi_{h_1, \ldots, h_m}^{\varepsilon h_{m+1}} := \Phi_{h_1, \ldots, h_m}(B - \varepsilon A_{m+1}, B^0 - \varepsilon a_{m+1} - u \varepsilon A_{m+1}) \]
\[ = \sum_{\theta \in \Lambda_m} Z_{\gamma}^{h_{m+1}} \cdot \cdots \cdot Z_{\gamma}^{h_{m+1}}. \]

**Definition 3.13.** Given a normed space \( X \) and a function \( F : X \to \mathbb{R} \), we say \( F \) is polynomially bounded if there exist constants \( K, M < \infty \) such that
\[ |F(x)| \leq K (1 + \|x\|_X)^M \]
for all \( x \in X \).

**Definition 3.14.** Given \( h \in \mathcal{H}_T(g_{CM}) \), we say a function \( F : \mathcal{W}_T(G) \to \mathbb{R} \) is right \( h \)-differentiable if
\[ (\hat{h}F)(g) := \frac{d}{d\varepsilon} \bigg|_{0} F(\varepsilon h \cdot g) \]
exists for all \( g \in \mathcal{W}_T(G) \). We will say that \( F \) is smooth if \((\hat{h}_1 \cdots \hat{h}_m F)(g) \) exists for all \( m \in \mathbb{N} \), \( h_1, \ldots, h_m \in \mathcal{H}_T(g_{CM}) \), and \( g \in \mathcal{W}_T(G) \).

**Theorem 3.15.** Let \( m \in \mathbb{N} \) and \( h_1, \ldots, h_m \in \mathcal{H}_T(g_{CM}) \), and suppose that \( F : \mathcal{W}_T(G) \to \mathbb{R} \) is a smooth function such that \( F \) and its right derivatives of all orders are polynomially bounded. Then
\[ E \left[ (\hat{h}_1 \cdots \hat{h}_m F)(\xi) \right] = E \left[ F(\xi) \Phi_{h_1, \ldots, h_m} \right] \]
and \( E |\Phi_{h_1, \ldots, h_m}|^p < \infty \) for all \( p \in [1, \infty) \).
Proof: That \( \Phi_{h_1, \ldots, h_m} \in L^p \) for all \( p \in [1, \infty) \) follows from the definition of \( \Phi \) and Proposition 3.9, since \( L^{\infty} \) is closed under products. Given the integrability results of Propositions 3.5, 3.6, 3.9, and 3.10, verifying the integration by parts is now straightforward. First note that, if \( \hat{h}F \) is polynomially bounded, then there exist \( K, M < \infty \) such that

\[
\sup_{|\varepsilon| \leq 1} \left| \frac{d}{d\varepsilon} F(\varepsilon h \cdot \xi) \right| = \sup_{|\varepsilon| \leq 1} \left| (\hat{h}F)(\varepsilon h \cdot \xi) \right| 
\leq \sup_{|\varepsilon| \leq 1} K \left( 1 + \| \varepsilon h \cdot \xi \|_{W^p(\mathbb{G})} \right)^M \leq C(h) \left( 1 + \| \xi \|_{W^p(\mathbb{G})} \right)^M,
\]

where this last expression is integrable by Remark 3.2.

Now consider the \( m = 1 \) case. This is the content of Corollary 5.6 of \([7]\), but we include it here for completeness. By Theorem 3.3 we have that

\[
E\left[ (h_1 F)(\xi) \right] = E \left[ \frac{d}{d\varepsilon} F(\varepsilon h_1 \cdot \xi) \right] = \frac{d}{d\varepsilon} E \left[ F(\varepsilon h_1 \cdot \xi) \right]
= \frac{d}{d\varepsilon} E \left[ F(\xi) J_{h_1} \right] = E \left[ F(\xi) \frac{d}{d\varepsilon} J_{h_1} \right],
\]

where the two interchanges of differentiation and integration are justified by (3.10) and Proposition 3.9 respectively. Then Lemma 3.8 implies that

\[
\frac{d}{d\varepsilon} J_{h_1} = Z_{h_1} = \Phi_{h_1},
\]

completing the proof for \( m = 1 \).

Now, assuming the formula for general \( m \), we have that

\[
E\left[ (\hat{h}_1 \cdots \hat{h}_{m+1} F)(\xi) \right] = E \left[ (\hat{h}_{m+1} F)(\xi) \Phi_{h_1, \ldots, h_m} (B, B^0) \right]
= E \left[ \frac{d}{d\varepsilon} F(\varepsilon h_{m+1} \cdot \xi) \Phi_{h_1, \ldots, h_m} (B, B^0) \right]
= \frac{d}{d\varepsilon} E \left[ F(\varepsilon h_{m+1} \cdot \xi) \Phi_{h_1, \ldots, h_m} (B, B^0) \right]
\]

where again we justify the interchange of differentiation and integration by the estimate in (3.10) above. Now by Theorem 3.3

\[
E[F(\varepsilon h_{m+1} \cdot \xi) \Phi_{h_1, \ldots, h_m} (B, B^0)]
= E \left[ F(\xi) \Phi_{h_1, \ldots, h_m} (B - \varepsilon a_{m+1}, B^0 - \varepsilon a_{m+1} - u_{\varepsilon a_{m+1}}) J_{\varepsilon h_{m+1}} \right]
= E \left[ F(\xi) \Phi_{h_1, \ldots, h_m} J_{\varepsilon h_{m+1}} \right].
\]

Since

\[
\frac{d}{d\varepsilon} \Phi_{h_1, \ldots, h_m} J_{\varepsilon h_{m+1}} = \sum_{\theta \in \Lambda_m} \sum_{j=1}^{k_\theta} \left( \frac{d}{d\varepsilon} Z_{\gamma_j} \right) \prod_{l \neq j}^{\theta} Z_{\gamma_l}^{h_{m+1}} J_{\varepsilon h_{m+1}}
+ \left( \frac{d}{d\varepsilon} J_{\varepsilon h_{m+1}} \right) \left( \sum_{\theta \in \Lambda_m} \prod_{j=1}^{k_\theta} Z_{\gamma_j}^{h_{m+1}} \right),
\]

we have

\[
\frac{d}{d\varepsilon} \Phi_{h_1, \ldots, h_m} J_{\varepsilon h_{m+1}} = \sum_{\theta \in \Lambda_m} \sum_{j=1}^{k_\theta} \left( \frac{d}{d\varepsilon} Z_{\gamma_j} \right) \prod_{l \neq j}^{\theta} Z_{\gamma_l}^{h_{m+1}} J_{\varepsilon h_{m+1}}
+ \left( \frac{d}{d\varepsilon} J_{\varepsilon h_{m+1}} \right) \left( \sum_{\theta \in \Lambda_m} \prod_{j=1}^{k_\theta} Z_{\gamma_j}^{h_{m+1}} \right).
\]
Propositions 3.5, 3.6, and 3.10 imply that, for all $p \in [1, \infty)$, there exists $\varepsilon_0 > 0$ such that

$$
E \left[ \sup_{|\varepsilon| \leq \varepsilon_0} \left| \frac{d}{d\varepsilon} \Phi^{ch_{m+1}}_{h_1, \ldots, h_m} J\varepsilon h_{m+1} \right|^p \right] < \infty.
$$

Thus,

$$
\frac{d}{d\varepsilon} \left| 0 \right| E \left[ F(\varepsilon) \Phi^{ch_{m+1}}_{h_1, \ldots, h_m} J\varepsilon h_{m+1} \right] = E \left[ F(\varepsilon) \frac{d}{d\varepsilon} \Phi^{ch_{m+1}}_{h_1, \ldots, h_m} J\varepsilon h_{m+1} \right].
$$

By Lemma 3.8,

$$
\frac{d}{d\varepsilon} \Phi^{ch_{m+1}}_{h_1, \ldots, h_m} = \sum_{\theta \in \Lambda_m} \frac{d}{d\varepsilon} Z^{\gamma \theta}_{\gamma_1 \gamma_2} \cdots Z^{\gamma \theta}_{\gamma_k} = \sum_{\theta \in \Lambda_m} \sum_{j=1}^{k_\theta} Z^{\gamma_{\theta_j} m+1}_{\gamma_j} \prod_{l \neq j} Z^{\gamma_l},
$$

where, for $\gamma = \{\ell_1, \ldots, \ell_n\}$,

$$
Z^{\gamma, m+1}_{\gamma_j} := \begin{cases} 
Z^{\gamma_j} & \text{for } \gamma' = \{\ell_1, \ldots, \ell_n, m+1\} \\
0 & \text{if } n = 1, 2, 3 \\
0 & \text{if } n = 4
\end{cases}
$$

Thus, we have that

$$
\frac{d}{d\varepsilon} \Phi^{ch_{m+1}}_{h_1, \ldots, h_m} J\varepsilon h_{m+1} = \frac{d}{d\varepsilon} \Phi^{ch_{m+1}}_{h_1, \ldots, h_m} + \Phi^{ch_{m+1}}_{h_1, \ldots, h_m} \frac{d}{d\varepsilon} J\varepsilon h_{m+1}
$$

$$
= \sum_{\theta \in \Lambda_m} \sum_{j=1}^{k_\theta} \left( Z^{\gamma_{\theta_j} m+1}_{\gamma_j} \prod_{l \neq j} Z^{\gamma_l} \right) + \Phi^{ch_{m+1}}_{h_1, \ldots, h_m} J\varepsilon h_{m+1},
$$

and notice that each term in this sum is a partition of $\{1, \ldots, m, m+1\}$. In particular, one may see that the final sum is over all of $\Lambda_{m+1}$, thus yielding the desired expression $\Phi^{ch_{m+1}}_{h_1, \ldots, h_m, h_{m+1}}$. \qed

We conclude this section with the following remark, which gives the reader some comparison between the integration by parts formula of Theorem 3.15 (and indeed the formulae to come in Theorem 4.12 and Corollary 4.11) and the usual “flat” integration by parts for Gaussian measures. In particular, one should think of the functions $\Phi$ as akin to Hermite functions for the measure $\nu$.

**Remark 3.16.** Let us recall the integration by parts formula for an abstract Wiener space $(W, H, \mu)$ following from the standard Cameron-Martin theorem. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis of $H$, and let $\partial_i$ denote the derivative in the direction $e_i$. Then, for any $k \in \mathbb{N}$, distinct indices $i_1, \ldots, i_k$, and multi-index $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$, we have

$$
\int_W (\partial_{i_1}^{\alpha_1} \cdots \partial_{i_k}^{\alpha_k} f)(w) d\mu(w) = \int_W f(w) H^{\alpha}_{i_1, \ldots, i_k}(w) d\mu(w)
$$

for $H^{\alpha}_{i_1, \ldots, i_k}(w) := \prod_{j=1}^k H_{\alpha_j}(\langle e_{i_j}, w \rangle_H)$, where $H_{\alpha}$ are the usual Hermite polynomials and $\langle e_i, w \rangle_H$ is the Paley-Wiener integral.
On the other hand, Theorem 3.13 implies that, for all \( h_1, \ldots, h_m \in \mathcal{H}_T(\mathfrak{g}_{CM}) \), there exists \( \hat{\Phi}_{h_1, \ldots, h_m} \in L^\infty \) such that

\[
\int_{\mathcal{W}_T(G)} (\mathbf{h}_1 \cdots \mathbf{h}_m) F(\omega) \, d\nu(\omega) = \mathbb{E} \left[ (\mathbf{h}_1 \cdots \mathbf{h}_m) F(\xi) \right] = \mathbb{E} \left[ F(\xi) \hat{\Phi}_{h_1, \ldots, h_m}(\xi) \right] = \int_{\mathcal{W}_T(G)} F(\omega) \hat{\Phi}_{h_1, \ldots, h_m}(\omega) \, d\nu(\omega).
\]

In particular, \( \hat{\Phi}_{h_1, \ldots, h_m}(\xi) = \mathbb{E}[\Phi_{h_1, \ldots, h_m} | \sigma(\xi, t \in [0, T])] \) a.s., and comparing this with the above flat case leads one to think of \( \Phi \) as a polynomial of order \( m \) in

\[
(h_1, (B, B^0))_{\mathcal{H}_T(\mathfrak{g}_{CM})} := \int_0^T (\mathbf{h}_1(t), d(B_t, B^0_t))_{\mathfrak{g}_{CM}} = \int_0^T (\dot{A}_t(t), dB_t)_H + \int_0^T (\dot{A}_t(t), dB^0_t)_C
\]

as well as additional terms like \( \int_0^T (\omega(B, \dot{A}_t), dB^0)_C \). The presence of these additional terms of course follows from the non-commutativity of the setting. That is, our formula coincides with the flat case in the event that \( \omega \equiv 0 \).

4. Smooth heat kernel measures on \( G \)

The smoothness results for the path space measure in the previous section now allow us to prove smoothness results for the heat kernel measure on \( G \). For example, in [7] the path space quasi-invariance was used to show quasi-invariance for \( \nu_T \) under left and right translations by elements of the Cameron-Martin subspace; see Theorem 6.1, Corollary 6.2, and Proposition 6.3 of that reference.

For \( g \in G \), let \( r_g, \ell_g : G \rightarrow G \) denote right and left multiplication by \( g \), respectively. As \( G \) is a vector space, to each \( g \in G \) we can associate the tangent space \( T_gG \) to \( G \) at \( g \), which is naturally isomorphic to \( G \). For \( h \in \mathfrak{g} \), we define the right and left invariant vector fields associated to \( h \):

\[
\hat{h}(g) := r_g h = \frac{d}{dz} \bigg|_0 \varepsilon h \cdot g \quad \text{and} \quad \bar{h}(g) := \ell_g h = \frac{d}{dz} \bigg|_0 g \cdot \varepsilon h, \quad \text{for all} \quad g \in G.
\]

The vector fields \( \hat{h} \) and \( \bar{h} \) act on smooth functions in the standard way; for example, for \( f : G \rightarrow \mathbb{R} \) a Fréchet smooth function on \( G \),

\[
(\hat{h}f)(g) = \frac{d}{dz} \bigg|_0 f(\varepsilon h \cdot g).
\]

**Notation 4.1.** Fix \( T > 0 \). For \( m \in \mathbb{N} \), and \( h_1, \ldots, h_m \in \mathfrak{g}_{CM} \), let \( \Psi_{h_1, \ldots, h_m} := \Phi_{h_1, \ldots, h_m} \in \mathcal{H}_T(\mathfrak{g}_{CM}) \) and define \( \hat{\Psi}_{h_1, \ldots, h_m} := \Phi_{h_1, \ldots, h_m} \), where \( \Phi \) is as in Notation 4.1.

**Theorem 4.2.** Fix \( T > 0 \). Let \( m \in \mathbb{N} \), and \( h_1, \ldots, h_m \in \mathfrak{g}_{CM} \), and suppose that \( f : G \rightarrow \mathbb{R} \) is a smooth function such that \( f \) and its right derivatives of all orders are polynomially bounded. Then

\[
\mathbb{E} \left[ (\hat{h}_1 \cdots \hat{h}_m) f(\xi_T) \right] = \mathbb{E} [f(\xi_T) \Psi_{h_1, \ldots, h_m}] \quad \text{where} \quad \mathbb{E} \vert \Psi_{h_1, \ldots, h_m} \vert^p < \infty \quad \text{for all} \quad p \in [1, \infty).
\]
Proof. Clearly, the integrability of $\Phi$ proved in Theorem 3.15 and the definition of $\hat{\Psi}$ imply that $\hat{\Psi}_{h_1, \ldots, h_m} \in L^p$ for all $p \in [1, \infty)$. The integration by parts also follows from Theorem 3.15. To see this, let $F : W_T(G) \to \mathbb{R}$ be given by $F(g) = f(g(T))$ and $h(t) = \frac{d}{dt} h_i \in H_T(g_{CM})$. Now note that

$$\mathbb{E} \left[ \hat{\Psi}_{h_1, \ldots, h_m}(\xi_T) \right] = \mathbb{E} \left[ \frac{d}{d\xi_1} \cdots \frac{d}{d\xi_m} f(\xi_m, \cdots (\xi_1 \cdot \xi)) \right]$$

$$= \mathbb{E} \left[ \frac{d}{d\xi_1} \cdots \frac{d}{d\xi_m} F(\xi_m, \cdots (\xi_1 \cdot \xi)) \right]$$

$$= \mathbb{E} \left[ \hat{h}_1 \cdots \hat{h}_m\xi_T \right] = \mathbb{E}[f(\xi)\Phi_{h_1, \ldots, h_m}]$$

$$= \mathbb{E}[f(\xi)\hat{\Psi}_{h_1, \ldots, h_m}].$$

\[\square\]

Remark 4.3. As in the path measure case (see Remark 3.10), Theorem 4.2 implies that, for all $h_1, \ldots, h_m \in g_{CM}$, there exists $\hat{\Psi}_{h_1, \ldots, h_m} \in L^\infty(\nu_T)$ such that

$$\int_G (\hat{h}_1 \cdots \hat{h}_m) f(g) \, d\nu_T(g) = \int_G f(g) \hat{\Psi}_{h_1, \ldots, h_m}(g) \, d\nu_T(g),$$

where $\hat{\Psi}_{h_1, \ldots, h_m}(\xi_T) = \mathbb{E}[\Psi_{h_1, \ldots, h_m} \mid \sigma(\xi_T)]$ a.s.

Corollary 4.4. Under the hypotheses of Theorem 4.2,

$$\mathbb{E}[(\hat{h}_1 \cdots \hat{h}_m)\xi_T] = \mathbb{E}[f(\xi)\hat{\Psi}_{h_1, \ldots, h_m}](\xi_T)],$$

where

$$\hat{\Psi}_{h_1, \ldots, h_m}(g) := (-1)^m \hat{\Psi}_{h_1, \ldots, h_m}(g^{-1}).$$

and $\hat{\Psi}$ is as in Remark 4.3.

Proof. Take $u(g) := f(g^{-1}) = f(-g)$. We proceed by induction. The $m = 1$ case is proved in Corollary 6.5 of [7], but we include the proof here for completeness. Note first that, for any $g \in G$ and $h \in g_{CM}$,

$$(\hat{h} f)(g) = \frac{d}{d\xi} f(\xi) = \frac{d}{d\xi} u(-\xi h \cdot g^{-1}) = -(\hat{h}u)(g^{-1}).$$

Thus, making repeated use of equation (2.4), we have that

$$\mathbb{E}[(\hat{h}f)(\xi_T)] = -\mathbb{E}[(\hat{h}u)(\xi_T^{-1})] = -\mathbb{E}[(\hat{h}u)(\xi_T)]$$

$$= -\mathbb{E}[u(\xi_T)\hat{\Psi}_h(\xi_T)] = -\mathbb{E}[f(\xi_T^{-1})\hat{\Psi}_h(\xi_T)]$$

$$= -\mathbb{E}[f(\xi_T)\hat{\Psi}_h(\xi_T^{-1})].$$
where we have applied Theorem 4.2 in the third equality. Now assuming the formula for $m$ and again using equations (4.1) and (2.4) and Theorem 4.2 gives

$$
\mathbb{E}\left[ (\tilde{h}_{1} \cdots \tilde{h}_{m+1} f)(\xi_T) \right] = (-1)^{m+1} \mathbb{E}\left[ (\tilde{h}_{m+1} f)(\xi_T) \hat{\Psi}_{h_{1}, \ldots, h_{m}}(\xi_T^{-1}) \right] \\
= (-1)^{m+1} \mathbb{E}\left[ (\hat{h}_{m+1} u)(\xi_T^{-1}) \hat{\Psi}_{h_{1}, \ldots, h_{m}}(\xi_T^{-1}) \right] \\
= (-1)^{m+1} \mathbb{E}\left[ (\hat{h}_{m+1} u)(\xi_T) \hat{\Psi}_{h_{1}, \ldots, h_{m}}(\xi_T) \right] = (-1)^{m+1} \mathbb{E}\left[ (\hat{h}_{m+1} u)(\xi_T) \hat{\Psi}_{h_{1}, \ldots, h_{m}}(\xi_T) \right] \\
= (-1)^{m+1} \mathbb{E}\left[ u(\xi_T) \hat{\Psi}_{h_{1}, \ldots, h_{m+1}}(\xi_T) \right] = (-1)^{m+1} \mathbb{E}\left[ u(\xi_T) \hat{\Psi}_{h_{1}, \ldots, h_{m+1}}(\xi_T) \right] \\
= (-1)^{m+1} \mathbb{E}\left[ f(\xi_T^{-1}) \hat{\Psi}_{h_{1}, \ldots, h_{m+1}}(\xi_T) \right] = (-1)^{m+1} \mathbb{E}\left[ f(\xi_T) \hat{\Psi}_{h_{1}, \ldots, h_{m+1}}(\xi_T^{-1}) \right] .
$$

$\square$

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