Research Article

On Coupled Systems for Hilfer Fractional Differential Equations with Nonlocal Integral Boundary Conditions

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In this paper, we study a coupled system involving Hilfer fractional derivatives with nonlocal integral boundary conditions. Existence and uniqueness results are obtained by applying Leray-Schauder alternative, Krasnoselskii’s fixed point theorem, and Banach’s contraction mapping principle. Examples illustrating our results are also presented.

1. Introduction

The theory of fractional differential equations has been widely used in pure mathematics and applications in the fields of physics, biology, and engineering. There are many interesting results for qualitative analysis and applications. We refer the interested reader, in fractional calculus, to the classical reference texts such as [1–7]. In the literature, there exist several different definitions of fractional integrals and derivatives, and the most popular of them are Riemann–Liouville, Caputo, and other less-known such as Hadamard fractional derivative and the Erdély–Kober fractional derivative. A generalization of derivatives of both Riemann–Liouville and Caputo was given by Hilfer in [8] as

\[ ^{H}D^x_{a}u(t) = \Gamma(\beta)(n-a)^{\beta-1} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-1-a} \beta \frac{d^\beta}{dt^\beta} u(s) \, ds, \]

where \( n-1 < a < n, \ 0 \leq \beta \leq 1, \ t > a \geq 0, \) and \( D^\beta = (d^\beta/dt^\beta). \)

He named it as generalized fractional derivative of order \( a \) and a type \( \beta. \) Many authors call it the Hilfer fractional derivative. We notice that when \( \beta = 0, \) the Hilfer fractional derivative corresponds to the Riemann–Liouville fractional derivative:

\[ ^{H}D^\alpha_{a}u(t) = D^\alpha_{a}u(t). \]  \( (2) \)

When \( \beta = 1, \) the Hilfer fractional derivative corresponds to the Caputo fractional derivative:

\[ ^{H}D^\alpha_{a}u(t) = D^\alpha_{a}u(t). \]  \( (3) \)

Such derivative interpolates between the Riemann–Liouville and Caputo derivative. Some properties and applications of the Hilfer derivative are given in [9, 10] and the references cited therein.

Initial value problems involving Hilfer fractional derivatives were studied by several authors, see, for example [11–16] and references therein. Nonlocal boundary value problems for Hilfer fractional derivative were studied in [17].

To the best of our knowledge, there is no work carried out on systems of boundary value problems with Hilfer fractional derivative in the literature. This paper come to fill
this gap. Thus, the objective of the present work is to introduce a new class of coupled systems of Hilfer-type fractional differential equations with nonlocal integral boundary conditions and develop the existence and uniqueness of solutions. In precise terms, we consider the following coupled system:

\[
\begin{align*}
H D^{\alpha,\beta} x (t) &= f (t, x (t), y (t)), & t & \in [a, b], \\
H D^{\alpha,\beta} y (t) &= g (t, x (t), y (t)), & t & \in [a, b], \\
x (a) &= 0, x (b) = \sum_{i=1}^{m} \theta_i I^{\nu_i} y (\xi_i), \\
y (a) &= 0, y (b) = \sum_{j=1}^{n} \zeta_j I^{\nu_j} x (z_j),
\end{align*}
\] (4)

where \( H D^{\alpha,\beta} \) and \( H D^{\alpha,\beta} \) are the Hilfer fractional derivatives of orders \( \alpha \) and \( \alpha_i, \, 1 < \alpha, \alpha_i < 2 \), and parameters \( \beta \) and \( \beta_i \), respectively, \( 0 \leq \beta, \beta_i \leq 1 \), and \( I^{\nu}, \, I^{\nu} \) are the Riemann–Liouville fractional integrals of order \( \nu \), \( \nu > 0 \), respectively, the points \( \xi_i, \, z_j \in [a, b], \, i \geq 0 \), \( f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions and \( \theta_i, \, \zeta_j, \, \xi_i \in \mathbb{R}, \, i = 1, 2, \ldots, m, \, j = 1, 2, \ldots, n \) are given real constants.

The paper is organized as follows. We present our main results in Section 3, by applying Leray–Schauder alternative, Krasnoselski’s fixed point theorem, and Banach’s contraction mapping principle, while Section 2 contains some preliminary concepts related to our problem. Examples are constructed to illustrate the main results.

## 2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later [2, 5].

**Definition 1.** The Riemann–Liouville fractional integral of order \( \alpha > 0 \) of a continuous function \( u : [a, \infty) \rightarrow \mathbb{R} \) is defined by

\[
I^{\alpha} u (t) = \frac{1}{\Gamma (\alpha)} \int_{a}^{t} (t - s)^{\alpha-1} u (s) ds,
\] (5)

provided the right-hand side exists on \( (a, \infty) \).

**Definition 2.** The Riemann–Liouville fractional derivative of order \( \alpha > 0 \) of a continuous function is defined by

\[
\begin{align*}
\text{RL} D^{\alpha} u (t) &= D^{\nu} I^{\nu-\alpha} u (t) = \frac{1}{\Gamma (n-\alpha)} \frac{d^n}{dt^n} \int_{a}^{t} (t-s)^{n-\alpha-1} u (s) ds,
\end{align*}
\] (6)

where \( n = [\alpha] + 1 \), \( [\alpha] \) denotes the integer part of real number \( \alpha \), provided the right-hand side is pointwise defined on \( (a, \infty) \).

**Definition 3.** The Caputo fractional derivative of order \( \alpha > 0 \) of a continuous function is defined by

\[
C D^{\nu} u (t) := I^{n-\alpha} D^{n} u (t) = \frac{1}{\Gamma (n-\alpha)} \int_{a}^{t} (t-s)^{n-\alpha-1} \left( \frac{d^n}{ds^n} u (s) \right) ds,
\] (7)

provided the right-hand side is pointwise defined on \( (a, \infty) \).

**Lemma 1** (see [18]). If \( g \in C^n [a, b], \, n-1 < \alpha < n, \) and \( 0 \leq \beta \leq 1 \), then

\[
\begin{align*}
(1) \quad &I^{\alpha} D^{\nu} I^{\beta} g (t) = g (t) - \sum_{k=1}^{n} \binom{\nu}{k} (t-a)^{\nu-k} \Gamma (\nu-k+1) \left( \frac{d^{k-\beta}}{dt^{k-\beta}} g (a) \right), \\
(2) \quad &H D^{\nu} I^{\beta} g (t) = g (t)
\end{align*}
\]

The following lemma deals with a linear variant of problem (4).

**Lemma 2.** Let \( \varphi_i, \, \psi_j > 0, \, \xi_i, \, z_j \in [a, b], \, a \geq 0, \, \theta_i, \, \zeta_j, \, \xi_i \in \mathbb{R}, \, i = 1, 2, \ldots, m, \, j = 1, 2, \ldots, n, \, 1 < \alpha, \alpha_i < 2, \, 0 \leq \beta, \beta_i \leq 1, \)

\[
\begin{align*}
\gamma = &\frac{(b-a)^{\nu+\gamma-2}}{\Gamma (\nu+\gamma)} - \left( \sum_{i=1}^{m} \theta_i (\xi_i-a)^{\nu+\gamma-1} \Gamma (\nu+\gamma) - \sum_{j=1}^{n} \zeta_j (z_j-a)^{\nu+\gamma-1} \Gamma (\nu+\gamma) \right) \\
\neq &0.
\end{align*}
\] (8)

Then, the system

\[
\begin{align*}
H D^{\alpha,\beta} x (t) &= h (t), & t & \in [a, b], \\
H D^{\alpha,\beta} y (t) &= h_1 (t), & t & \in [a, b], \\
x (a) &= 0, \\
x (b) &= \sum_{i=1}^{m} \theta_i I^{\nu_i} y (\xi_i), \\
y (a) &= 0, \\
y (b) &= \sum_{j=1}^{n} \zeta_j I^{\nu_j} x (z_j),
\end{align*}
\] (9)

is equivalent to the following integral equations:


\[
x(t) = I^a h(t) + \frac{(t-a)^{y-1}}{\Gamma(y)} \left[ \sum_{i=1}^{m} \theta_i I_t^{\alpha + \gamma_i} h_1(\xi_i) - I^a h(b) \right] + \left( \sum_{i=1}^{m} \theta_i (\xi_i - a)^{\gamma_i - 1} \right) \left( \sum_{j=1}^{n} \zeta_j I_t^{\alpha + \gamma_j} h(z_j) - I^a h(b) \right) \tag{10}
\]

\[
y(t) = I^a h_1(t) + \frac{(t-a)^{y-1}}{\Gamma(y)} \left[ \sum_{i=1}^{m} \theta_i I_t^{\alpha + \gamma_i} h_1(\xi_i) - I^a h_1(b) \right] + \left( \sum_{i=1}^{m} \theta_i (\xi_i - a)^{\gamma_i - 1} \right) \left( \sum_{j=1}^{n} \zeta_j I_t^{\alpha + \gamma_j} h(z_j) - I^a h_1(b) \right) \tag{11}
\]

**Proof.** Operating fractional integral \(I^a\) on both sides of the first equation in (9) and using Lemma 1, we obtain

\[
x(t) = \frac{2}{k!} \frac{(t-a)^{y-1}}{\Gamma(y)} \left( \frac{d}{dt} \right)^{2-k} I^{(1-\beta)(2-a)} x(a) = I^a h(t). \tag{12}
\]

Then, we have, since \((1-\beta)(2-a) = 2-y, \)

\[
x(t) = \frac{(t-a)^{y-1}}{\Gamma(y)} \left( \frac{d}{dt} \right)^2 x(t)|_{t=a} + \frac{(t-a)^{y-2}}{\Gamma(y-1)} I^a x(t)|_{t=a} + I^a h(t).
\]

where

\[
c_1 = H D^{\gamma - 1} x(t)|_{t=a},
\]

\[
c_2 = I^a x(t)|_{t=a}, \tag{15}
\]

By a similar way, we obtain

\[
y(t) = \frac{d_1}{\Gamma(y_1)} (t-a)^{y_1-1} + \frac{d_2}{\Gamma(y_1 - 1)} (t-a)^{y_1-2} + I^a h_1(t). \tag{16}
\]

By setting

\[
d_1 = H D^{\gamma - 1} y(t)|_{t=a},
\]

\[
d_2 = I^a y(t)|_{t=a}, \tag{17}
\]

from the boundary conditions \(x(a) = 0\) and \(y(a) = 0, \)

we obtain \(c_2 = 0\) and \(d_2 = 0, \)

Then, we obtain

\[
x(t) = \frac{c_1}{\Gamma(y)} (t-a)^{y-1} + I^a h(t), \tag{18}
\]

\[
y(t) = \frac{d_1}{\Gamma(y_1)} (t-a)^{y_1-1} + I^a h_1(t), \tag{19}
\]

From \(x(b) = \sum_{j=1}^{m} \theta_j I^\gamma y_j(\xi_j) \quad \text{and} \quad y(b) = \sum_{j=1}^{n} \zeta_j I^\gamma z_j, \)

we have

\[
c_1 = \frac{1}{\Lambda} \left[ \frac{(b-a)^{y_1-1}}{\Gamma(y_1)} \left( \sum_{j=1}^{m} \theta_j I_t^{\alpha + \gamma_j} h_1(\xi_j) - I^a h(b) \right) \right] + \left( \sum_{i=1}^{m} \theta_i (\xi_i - a)^{\gamma_i - 1} \right) \left( \sum_{j=1}^{n} \zeta_j I_t^{\alpha + \gamma_j} h(z_j) - I^a h_1(b) \right) \tag{20}
\]

Substituting the values of \(c_1\) and \(d_1\) in (18), we obtain solutions (10) and (11). The converse follows by direct computation. This completes the proof.

## 3. Main Results

Let \(C = C([a, b], \mathbb{R}), a \geq 0, \) denote the Banach space of all continuous functions from \([a, b]\) to \(\mathbb{R}.\) The space \(X = \{ x : x(t) \in C^1([a, b], \mathbb{R}) \} \) endowed with the norm \(\| x \| = \sup \{|x(t)|, \ t \in [a, b]\}\) is a Banach space. Let \(Y = \{ y : y(t) \in C^1([a, b], \mathbb{R}) \} \) with the norm \(\| y \| = \sup \{|y(t)|, \ t \in [a, b]\}. \) It is obvious that the product
space \((X \times Y, \| (x, y) \|)\) is Banach space with the norm \(\| (x, y) \| = \| x \| + \| y \|\).

In view of Lemma 2, we define two operators \(\mathcal{K}: X \times Y \rightarrow X \times Y\) by
\[
\mathcal{K}(x, y)(t) = \begin{pmatrix} \mathcal{K}_1(x, y)(t) \\ \mathcal{K}_2(x, y)(t) \end{pmatrix},
\]
where
\[
\mathcal{K}_1(x, y)(t) = \begin{pmatrix} I^a f_{x,y}(t) + \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \left[ (b-a)^{\alpha-1} \left( \sum_{i=1}^{n} \theta_i \Gamma(i) g_{x,y}(\xi_i) - I^a f_{x,y}(b) \right) \\
+ \left( \sum_{i=1}^{n} \theta_i \xi_i^{\alpha+\varphi-1} \Gamma(1+\varphi) \left( \sum_{j=1}^{n} \xi_j^{\alpha+\psi} f_{x,y}(z_j) - I^a g_{x,y}(b) \right) \right) \end{pmatrix},
\]
\[
\mathcal{K}_2(x, y)(t) = \begin{pmatrix} I^a g_{x,y}(t) + \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \left[ (b-a)^{\alpha-1} \left( \sum_{i=1}^{n} \theta_i \Gamma(i) f_{x,y}(\xi_i) - I^a f_{x,y}(b) \right) \\
+ \left( \sum_{i=1}^{n} \theta_i \xi_i^{\alpha+\varphi-1} \Gamma(1+\varphi) \left( \sum_{j=1}^{n} \xi_j^{\alpha+\psi} g_{x,y}(z_j) - I^a g_{x,y}(b) \right) \right) \end{pmatrix},
\]
\[
\text{where}
\]
\[
f_{x,y}(t) = f(t, x(t), y(t)),
g_{x,y}(t) = g(t, x(t), y(t)), \quad t \in [a, b].
\]

For computational convenience, we set
\[
M_1 = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \left[ (b-a)^{\alpha+\varphi-1} \left( \sum_{i=1}^{n} \theta_i \Gamma(i+1) \left( \xi_i^{\alpha+\psi} \right) - I^a f_{x,y}(b) \right) \\
+ \left( \sum_{i=1}^{n} \theta_i \xi_i^{\alpha+\varphi-1} \Gamma(1+\varphi) \left( \sum_{j=1}^{n} \xi_j^{\alpha+\psi} f_{x,y}(z_j) - I^a g_{x,y}(b) \right) \right) \right],
\]
\[
M_2 = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \left[ (b-a)^{\alpha+\varphi-1} \left( \sum_{i=1}^{n} \theta_i \Gamma(i+1) \left( \xi_i^{\alpha+\psi} \right) - I^a f_{x,y}(b) \right) \\
+ \left( \sum_{i=1}^{n} \theta_i \xi_i^{\alpha+\varphi-1} \Gamma(1+\varphi) \left( \sum_{j=1}^{n} \xi_j^{\alpha+\psi} g_{x,y}(z_j) - I^a g_{x,y}(b) \right) \right) \right],
\]
\[
M_3 = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \left[ (b-a)^{\alpha+\varphi-1} \left( \sum_{i=1}^{n} \theta_i \Gamma(i+1) \left( \xi_i^{\alpha+\psi} \right) - I^a g_{x,y}(b) \right) \\
+ \left( \sum_{i=1}^{n} \theta_i \xi_i^{\alpha+\varphi-1} \Gamma(1+\varphi) \left( \sum_{j=1}^{n} \xi_j^{\alpha+\psi} f_{x,y}(z_j) - I^a g_{x,y}(b) \right) \right) \right],
\]
\[
M_4 = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \left[ (b-a)^{\alpha+\varphi-1} \left( \sum_{i=1}^{n} \theta_i \Gamma(i+1) \left( \xi_i^{\alpha+\psi} \right) - I^a g_{x,y}(b) \right) \\
+ \left( \sum_{i=1}^{n} \theta_i \xi_i^{\alpha+\varphi-1} \Gamma(1+\varphi) \left( \sum_{j=1}^{n} \xi_j^{\alpha+\psi} g_{x,y}(z_j) - I^a f_{x,y}(b) \right) \right) \right].
\]

Then, system (4) has a unique solution on \([a, b]\), if
\[(M_1 + M_3)(\ell_1 + \ell_2) + (M_2 + M_4)(n_1 + n_2) < 1.\]

Proof. Define \(sup_{t \in [a, b]} f(t, 0, 0) = N_1 < \infty\) and \(sup_{t \in [a, b]} g(t, 0, 0) = N_2 < \infty\) such that
\[
r > \frac{(M_1 + M_3)N_1 + (M_2 + M_4)N_2 \Gamma(1+\varphi)}{1 - (M_1 + M_3)(\ell_1 + \ell_2) + (M_2 + M_4)(n_1 + n_2)}.\]

Now, we will show that the set \(\mathcal{K}B \subset B_r,\) where \(B_r = \{(x, y) \in X \times Y : \| (x, y) \| \leq r \}.\) For any \((x, y) \in B_r,\) \(t \in [a, b],\) we find that
\[
|f(t, x(t), y(t)) - f(t, x(t), y(t))| \leq |f(t, x(t), y(t)) - f(t, 0, 0) + f(t, 0, 0)| \leq |f(t, x(t), y(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \leq \ell_1 \| x \| + \ell_2 \| y \| + N_1,
\]
\[
|g(t, x(t), y(t))| \leq n_1 \| x \| + n_2 \| y \| + N_2.
\]
Similarly, we have

\[ \| \mathcal{K}_1(x, y) \| \leq M_1 (\ell_1 + \ell_2) + M_2 (n_1 + n_2) r + M_1 N_1 + M_2 N_2. \]

Hence,

\[ \| \mathcal{K}_1(x, y) \| \leq M_1 (\ell_1 + \ell_2) + M_2 (n_1 + n_2) + \frac{1}{r} \left( M_1 N_1 + M_2 N_2 \right). \]

Similarly, we have

\[ \| \mathcal{K}_2(x, y) \| \leq M_3 (\ell_1 \| x \| + \ell_2 \| y \| + N_1) + M_4 (n_1 \| x \| + n_2 \| y \| + N_2), \]

and hence

\[ \| \mathcal{K}_2(x, y) \| \leq M_3 (\ell_1 + \ell_2) + M_4 (n_1 + n_2) + \frac{1}{r} \left( M_1 N_1 + M_2 N_2 \right). \]

Consequently, it follows that

\[ \| \mathcal{K}(x, y) \| \leq \| \mathcal{K}_1(x, y) \| + \| \mathcal{K}_2(x, y) \| \leq M_1 \ell_1 + M_2 \ell_2 + M_3 n_1 + M_4 n_2 r + M_1 N_1 + M_2 N_2. \]

which implies \( \mathcal{K} \subset B_r \). Next, we will show that the operator \( \mathcal{K} \) is a contraction mapping. For any \( (x_1, y_1), (x_2, y_2) \in X \times Y \), we obtain
\[ |\mathcal{X}_1(x_1, y_1)(t) - \mathcal{X}_1(x_2, y_2)(t)| \]
\[ \leq I^n f_{x_1, y_1} - f_{x_2, y_2} (b) \]
\[ + \frac{(b-a)^{n-1}}{|A| \Gamma (\gamma)} \left[ \frac{(b-a)^{\gamma-1}}{\Gamma (\gamma)} \left( \sum_{i=1}^{m} \left| g_{x_1, y_1} - g_{x_2, y_2} \right| (\xi_i) + I^q f_{x_1, y_1} - f_{x_2, y_2} (b) \right) \right] \]
\[ + \left( \sum_{j=1}^{n} \left| j \right|^{\gamma+q} \left( f_{x_1, y_1} - f_{x_2, y_2} (z_j) \right) \right) \]
\[ \leq (\ell_1 \| x_1 - x_2 \| + \ell_2 \| y_1 - y_2 \|) I^n (1) (b) \]
\[ + \frac{(b-a)^{n-1}}{|A| \Gamma (\gamma)} \left[ \frac{(b-a)^{\gamma-1}}{\Gamma (\gamma)} \left( \sum_{i=1}^{m} \left| g_{x_1, y_1} - g_{x_2, y_2} \right| (\xi_i) \right) \right] \]
\[ + \left( \sum_{j=1}^{n} \left| j \right|^{\gamma+q} \left( f_{x_1, y_1} - f_{x_2, y_2} (z_j) \right) \right) \]
\[ \leq M_1 (\ell_1 \| x_1 - x_2 \| + \ell_2 \| y_1 - y_2 \|) + M_2 (n_1 \| x_1 - x_2 \| + n_2 \| y_1 - y_2 \|) \]
\[ = (M_1 \ell_1 + M_2 n_1) \| x_1 - x_2 \| + (M_1 \ell_2 + M_2 n_2) \| y_1 - y_2 \|. \]

Therefore, we obtain the following inequality:
\[ \| \mathcal{X}_1(x_1, y_1) - \mathcal{X}_1(x_2, y_2) \| \leq [M_1 (\ell_1 + \ell_2) + M_2 (n_1 + n_2)] \]
\[ \cdot (\| x_1 - x_2 \| + \| y_1 - y_2 \|). \]

In addition, we also obtain
\[ \| \mathcal{X}_2(x_1, y_1) - \mathcal{X}_2(x_2, y_2) \| \leq [M_1 (\ell_1 + \ell_2) + M_4 (n_1 + n_2)] \]
\[ \cdot (\| x_1 - x_2 \| + \| y_1 - y_2 \|). \]

From (39) and (40), it yields
\[ \| \mathcal{X}(x_1, y_1) - \mathcal{X}(x_2, y_2) \| \leq [(M_4 + M_4) (n_1 + n_2)] \]
\[ \cdot (\| x_1 - x_2 \| + \| y_1 - y_2 \|). \]

As \((M_4 + M_4) (n_1 + n_2) < 1\), therefore, \(\mathcal{X}\) is a contraction operator. By Banach’s fixed point theorem, the operator \(\mathcal{X}\) has a unique fixed point, which is the unique solution of (4) on \([a, b]\). The proof is completed.

Now, we prove our second existence result via Leray–Schauder alternative.

**Lemma 3** (Leray–Schauder alternative, see [19]). Let \(F: E \rightarrow E\) be a completely continuous operator. Let \(\xi (F) = \{ x \in E: x = \lambda F(x) \text{ for some } 0 < \lambda < 1 \}. \)

Then, either the set \(\xi (F)\) is unbounded, or \(F\) has at least one fixed point.

**Theorem 2.** Assume that there exist real constants \(u_i, v_i \geq 0\) for \(i = 1, 2\) and \(u_0, v_0 > 0\) such that, for any \(x_i \in \mathbb{R} \ (i = 1, 2)\), we have
\[ f(t, x_1, x_2) \leq u_0 + u_1 |x_1| + u_2 |x_2|, \]
\[ g(t, x_1, x_2) \leq v_0 + v_1 |x_1| + v_2 |x_2|. \]

If \((M_4 + M_4) u_1 + (M_3 + M_3) v_1 < 1\) and \((M_1 + M_1) u_2 + (M_2 + M_2) v_2 < 1\), where \(M_1, M_2, M_3, \) and \(M_4\) are given in (24)–(27), then (4) has at least one solution on \([a, b]\).

**Proof.** By continuity of the functions \(f\) and \(g\) on \([a, b] \times \mathbb{R} \times \mathbb{R}\), the operator \(\mathcal{X}\) is continuous. We will show that the operator \(\mathcal{X}: \{ x, y \} \rightarrow \{ x, y \}\) is completely continuous. Let \(\Phi \subset \{ x, y \}\) be bounded. Then, there exist positive constants \(L_1\) and \(L_2\) such that
\[ |f(t, x, y)| \leq L_1, \quad |g(t, x, y)| \leq L_2, \quad \forall (x, y) \in \Phi. \]

Then, for any \((x, y) \in \Phi\), we have
\[
|\mathcal{K}_1(x, y)(t)| \leq L_1^a|f_{x,y}|(b) \\
+ \frac{(b-a)^{\gamma-1}}{|\Gamma(y)|} \left[ (b-a)\gamma^{-1} \right] \left( \sum_{i=1}^{m} \left| \Theta_i \right| \Gamma^{|\alpha+i|} \right) \left| g_{x,y} \left( \xi_i \right) + L_i^a \right| |f_{x,y}|(b) \\
+ \left( \sum_{i=1}^{m} \left| \Theta_i \right| \right) \left( \sum_{j=1}^{n} \left| \zeta_j \right| \Gamma^{|\alpha+i|} \right) \left| f_{x,y} \left( \zeta_j \right) + L_i^a \right| |g_{x,y}|(b) \right] \leq L_1 L_1^a (1) (b) \\
+ \frac{(b-a)^{\gamma-1}}{|\Gamma(y)|} \left( \sum_{i=1}^{m} \left| \Theta_i \right| \right) \left( \sum_{j=1}^{n} \left| \zeta_j \right| \Gamma^{|\alpha+i|} \right) \left( L_1 \left( \sum_{j=1}^{n} \left| \zeta_j \right| \Gamma^{|\alpha+i|} \right) + L_2 L_2^a \right) \\
\leq L_1 M_1 + L_2 M_2,
\]

which yields
\[
\|\mathcal{K}_1(x, y)\| \leq L_1 M_1 + L_2 M_2.
\]

Hence, from the above inequalities, we obtain that the set \( \mathcal{K} \Phi \) is uniformly bounded. Next, we are going to prove that the set \( \mathcal{K} \Phi \) is equicontinuous. For any \((x, y) \in \Phi\) and \(r_1, r_2 \in [a, b]\) such that \(r_1 < r_2\), we have

\[
\|\mathcal{K}_1(x, y)(r_2) - \mathcal{K}_1(x, y)(r_1)\| \leq L_1^a|f_{x,y}|(r_2) - L_1^a|f_{x,y}|(r_1) \\
+ \frac{(r_2 - a)^{\gamma-1} - (r_1 - a)^{\gamma-1}}{|\Gamma(y)|} \left[ (b-a)^{\gamma-1} \right] \left( \sum_{i=1}^{m} \left| \Theta_i \right| \Gamma^{|\alpha+i|} \right) \left| g_{x,y} \left( \xi_i \right) + L_i^a \right| |f_{x,y}|(b) \\
+ \left( \sum_{i=1}^{m} \left| \Theta_i \right| \right) \left( \sum_{j=1}^{n} \left| \zeta_j \right| \Gamma^{|\alpha+i|} \right) \left| f_{x,y} \left( \zeta_j \right) + L_i^a \right| |g_{x,y}|(b) \right] \leq L_1 \left( \int_a^b \frac{(r_2 - s)^{a-1} - (r_1 - a)^{a-1}}{\Gamma(a)} ds + \int_{r_1}^{r_2} \frac{(r_2 - s)^{a-1}}{\Gamma(a)} ds \right) \\
+ \frac{(r_2 - a)^{\gamma-1} - (r_1 - a)^{\gamma-1}}{|\Gamma(y)|} \left( \sum_{i=1}^{m} \left| \Theta_i \right| \right) \left( \sum_{j=1}^{n} \left| \zeta_j \right| \Gamma^{|\alpha+i|} \right) \left( L_1 \left( \sum_{j=1}^{n} \left| \zeta_j \right| \Gamma^{|\alpha+i|} \right) + L_2 L_2^a \right) \\
\leq \frac{L_1}{\Gamma(a + 1)} \left( 2(r_2 - r_1)^{a} + (r_2 - a)^{a} - (r_1 - a)^{a} \right) \\
+ \frac{(r_2 - a)^{\gamma-1} - (r_1 - a)^{\gamma-1}}{|\Gamma(y)|} \left[ (b-a)^{\gamma-1} \right] \left( \sum_{i=1}^{m} \left| \Theta_i \right| \right) \left( \sum_{j=1}^{n} \left| \zeta_j \right| \Gamma^{|\alpha+i|} \right) \left( \int_a^{\alpha + \phi_i + 1} + L_1 \frac{(b-a)^{a}}{\Gamma(a + 1)} \right) \\
\leq \frac{L_1}{\Gamma(a + 1)} \left[ \frac{2(r_2 - r_1)^{a} + (r_2 - a)^{a} - (r_1 - a)^{a}}{\Gamma(a + 1)} \right].
\]
Therefore, we obtain
\[
|\mathcal{K}_1(x, y)(\tau_2) - \mathcal{K}_1(x, y)(\tau_1)| \rightarrow 0, \quad \text{as } \tau_1 \rightarrow \tau_2.
\]
(49)

Analogously, we can obtain the following inequality:
\[
|\mathcal{K}_2(x, y)(\tau_2) - \mathcal{K}_2(x, y)(\tau_1)| \rightarrow 0, \quad \text{as } \tau_1 \rightarrow \tau_2.
\]
(50)

Hence, the set \( \mathcal{K} \Phi \) is equicontinuous. By applying the Arzelà–Ascoli theorem, the set \( \mathcal{K} \Phi \) is relatively compact which implies that the operator \( \mathcal{K} \) is completely continuous. Lastly, we shall show that the set \( \xi = \{(x, y) \in X \times Y : (x, y) = \lambda \mathcal{K}(x, y), \ 0 \leq \lambda \leq 1\} \) is bounded. Let any \((x, y) \in \xi\), then \((x, y) = \lambda \mathcal{K}(x, y)\). For any \( t \in [a, b] \), we have
\[
x(t) = \lambda \mathcal{K}_1(x, y)(t), \\
y(t) = \lambda \mathcal{K}_2(x, y)(t).
\]
(51)

Then, we obtain
\[
\|x\| \leq (u_0 + u_1\|x\| + u_2\|y\|)M_1 + (v_0 + v_1\|x\| + v_2\|y\|)M_2, \\
\|y\| \leq (u_0 + u_1\|x\| + u_2\|y\|)M_3 + (v_0 + v_1\|x\| + v_2\|y\|)M_4,
\]
(52)

which imply that
\[
\|x\| + \|y\| \leq (M_1 + M_3)u_0 + (M_2 + M_4)v_0 \\
\quad + [(M_1 + M_3)u_1 + (M_2 + M_4)v_1]\|x\| \\
\quad + [(M_1 + M_3)u_2 + (M_2 + M_4)v_2]\|y\|.
\]
(53)

Thus, we obtain
\[
\|(x, y)\| \leq \frac{(M_1 + M_3)u_0 + (M_2 + M_4)v_0}{M^*},
\]
(54)

where \( M^* = \min\{1 - (M_1 + M_3)u_1 - (M_2 + M_4)v_1, \ 1 - (M_1 + M_3)u_2 - (M_2 + M_4)v_2\} \), which shows that the set \( \xi \) is bounded. Therefore, by applying Lemma 3, the operator \( \mathcal{K} \) has at least one fixed point. Therefore, we deduce that problem (4) has at least one solution on \([a, b]\). The proof is complete.

The last existence theorem is based on Krasnoselskii’s fixed point theorem.

**Lemma 4** (Krasnoselskii’s fixed point theorem, see [20]). Let \( M \) be a closed, bounded, convex, and nonempty subset of a Banach space \( X \). Let \( A \) and \( B \) be operators such that (i) \( Ax + By \in M \), where \( x, y \in M \), (ii) \( A \) is compact and continuous, and (iii) \( B \) is a contraction mapping. Then, there exists \( z \in M \) such that \( z = Az + Bz \).

**Theorem 3.** Assume that \( f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions satisfying assumption \((H_1)\) in Theorem 1. In addition, we suppose and there exist two positive constants \( P, Q \) such that, for all \( t \in [a, b] \) and \( x_i, y_i \in \mathbb{R}, i = 1, 2, \)
\[
|f(t, x_1, x_2)| \leq P, \\
|g(t, x_1, x_2)| \leq Q.
\]
(55)

If
\[
\frac{(b - a)\alpha}{\Gamma(\alpha + 1)}(\ell_1 + \ell_2) + \frac{(b - a)\alpha^2}{\Gamma(\alpha + 1)}(n_1 + n_2) < 1,
\]
(56)

then problem (4) has at least one solution on \([a, b]\).

**Proof.** To apply Lemma 4, we decompose the operator \( \mathcal{K} \) into four operators \( \mathcal{K}_{1,1}, \mathcal{K}_{1,2}, \mathcal{K}_{2,1}, \) and \( \mathcal{K}_{2,2} \) as

\[
\mathcal{K}_{1,1}(x, y)(t) = \frac{(t - a)^{n-1}}{\Gamma(n)} \left[ \frac{(b - a)^{n-1}}{\Gamma(n)} \left( \sum_{i=1}^{m} \theta_i t^{\alpha+\psi} g_{x,y}(\xi_i) - I^{\alpha} f_{x,y}(b) \right) \right] \\
+ \left( \sum_{i=1}^{m} \theta_i (\xi_i - a)^{\alpha+\psi-1} \right) \left( \sum_{j=1}^{n} \xi_j t^{\alpha+\psi} f_{x,y}(z_j) - I^{\alpha} g_{x,y}(b) \right),
\]
(57)

\[
\mathcal{K}_{1,2}(x, y)(t) = I^{\alpha} f_{x,y}(t),
\]

\[
\mathcal{K}_{2,1}(x, y)(t) = \frac{(t - a)^{n-1}}{\Gamma(n)} \left[ \frac{(b - a)^{n-1}}{\Gamma(n)} \left( \sum_{j=1}^{n} \xi_j t^{\alpha+\psi} f_{x,y}(z_j) - I^{\alpha} g_{x,y}(b) \right) \right] \\
+ \left( \sum_{j=1}^{n} \xi_j (z_j - a)^{\alpha+\psi-1} \right) \left( \sum_{i=1}^{m} \theta_i t^{\alpha+\psi} g_{x,y}(\xi_i) - I^{\alpha} f_{x,y}(b) \right),
\]

\[
\mathcal{K}_{2,2}(x, y)(t) = I^{\alpha} g_{x,y}(t).
\]
Note that $\mathcal{H}_1(x, y)(t) = \mathcal{H}_{1,1}(x, y)(t) + \mathcal{H}_{1,2}(x, y)(t)$ and $\mathcal{H}_2(x, y)(t) = \mathcal{H}_{2,1}(x, y)(t) + \mathcal{H}_{2,2}(x, y)(t)$. Also, observe that the ball $B_\delta$ is a closed, bounded, and convex subset of the Banach space $\mathcal{C}$. Let $B_\delta = \{(x, y) \in X \times Y: \|x, y\| \leq \delta\}$ be a ball, where a constant $\delta \geq \max\{M_1 P + M_2 Q, M_3 P + M_4 Q\}$. Now, we will show that $\mathcal{H}_B \subset B_\delta$ for satisfying condition (i) of Lemma 4. Setting $x = (x_1, x_2)$ and $y = (y_1, y_2) \in B_\delta$, and using condition (55), then we have, as in Theorem 2 that

$$\|\mathcal{H}_{1,1}(x_1, x_2)(t) + \mathcal{H}_{1,2}(y_1, y_2)(t)\| \leq M_1 P + M_2 Q \leq \delta.$$ (58)

Similarly, we can find that

$$\|\mathcal{H}_{2,1}(x_1, x_2)(t) + \mathcal{H}_{2,2}(y_1, y_2)(t)\| \leq M_3 P + M_4 Q \leq \delta.$$ (59)

That yields $\mathcal{H}_1 x + \mathcal{H}_2 y \in B_\delta$. To show that the operator $(\mathcal{H}_{1,2}, \mathcal{H}_{2,2})$ is a contraction mapping satisfying condition (iii) of Lemma 4, for $(x_1, y_1), (x_2, y_2) \in B_\delta$, we have

$$\|\mathcal{H}_{1,2}(x_1, y_1)(t) - \mathcal{H}_{1,2}(x_2, y_2)(t)\| \leq \|f_{x_1, y_1} - f_{x_2, y_2}\| \leq (b - a)^x \|x_1 - x_2\| + (b - a)^y \|y_1 - y_2\| \leq \frac{(b - a)^x}{\Gamma(a + 1)} (\|x_1 - x_2\| + \|y_1 - y_2\|)$$ (60)

$$\|\mathcal{H}_{2,2}(x_1, y_1)(t) - \mathcal{H}_{2,2}(x_2, y_2)(t)\| \leq \|g_{x_1, y_1} - g_{x_2, y_2}\| \leq \frac{(b - a)^y}{\Gamma(a + 1)} (\|x_1 - x_2\| + \|y_1 - y_2\|).$$ (61)

It follows from (60) and (61) that

$$\|\mathcal{H}_{1,2}, \mathcal{H}_{2,2})(x_1, y_1) - (\mathcal{H}_{1,2}, \mathcal{H}_{2,2})(x_2, y_2)\| \leq \left( \frac{(b - a)^x}{\Gamma(a + 1)} \ell_1 + \ell_2 + \frac{(b - a)^y}{\Gamma(a + 1)} (n_1 + n_2) \right) \cdot (\|x_1 - x_2\| + \|y_1 - y_2\|),$$ (62)

which is a contraction by inequality in (56). Therefore, condition (iii) of Lemma 4 is satisfied. Next, we will show that the operator $(\mathcal{H}_{1,1}, \mathcal{H}_{2,1})$ satisfies condition (ii) of Lemma 4. By applying the continuity of the functions $f, g$ on $[a, b] \times \mathbb{R} \times \mathbb{R}$, we can conclude that the operator $(\mathcal{H}_{1,1}, \mathcal{H}_{2,1})$ is continuous. For each $(x, y) \in B_\delta$, one has
and similarly
\[|\mathcal{K}_{2,1}(x, y)(t)| \leq Q'. \quad (64)\]
Then, we obtain the following fact:
\[\left\| (\mathcal{K}_{1,1}, \mathcal{K}_{2,1})(x, y) \right\| \leq P' + Q'. \quad (65)\]

which implies that the set \((\mathcal{K}_{1,1}, \mathcal{K}_{2,1})B_{\delta} \) is uniformly bounded. In the next step, we will show that the set \((\mathcal{K}_{1,1}, \mathcal{K}_{2,1})B_{\delta} \) is equicontinuous. For \(\tau_1, \tau_2 \in [a, b] \) such that \(\tau_1 < \tau_2\) and for any \((x, y) \in B_{\delta}\), we can prove that of Lemma 4, we have that problem (4) has at least one solution on \([a, b] \). This completes the proof.

**Example 1.** Consider the coupled system of Hilfer fractional differential equations with nonlocal integral boundary conditions of the form

\[
\begin{align*}
H^{D(3/2),(1/2)} & x(t) = f(t, x(t), y(t)), \quad t \in \left[ \frac{1}{3}, \frac{10}{3} \right], \\
H^{D(5/4),(2/3)} & y(t) = g(t, x(t), y(t)), \quad t \in \left[ \frac{1}{3}, \frac{10}{3} \right], \\
x \left( \frac{1}{3} \right) = 0, \quad x \left( \frac{10}{3} \right) = \frac{1}{2} l^{1/3} y \left( \frac{2}{3} \right) + 2^{1/2} y(1) + \frac{3}{4} l^{3/5} y \left( \frac{4}{3} \right) + \frac{4}{5} l^{2/3} y \left( \frac{5}{3} \right) \\
y \left( \frac{1}{3} \right) = 0, \quad y \left( \frac{10}{3} \right) = \frac{3}{2} l^{3/2} x(2) + 4^{1/2} x \left( \frac{2}{3} \right) + \frac{5}{4} l^{7/4} y \left( \frac{8}{3} \right) + \frac{6}{5} l^{5/3} x(3).
\end{align*}
\]  

Here, \(\alpha = 3/2, \quad \alpha' = 5/4, \quad \beta = 1/2, \quad \beta' = 2/3, \quad \alpha = 1/3, \quad b = 10/3, \quad m = n = 4, \quad \beta_i = (ii + 1)), \quad \xi_i = ((2i + 1)/(1 + i)), \quad \varphi_i = (ii((i + 1))/((i + 1))), \quad i = 1, 2, 3, 4, \quad \xi_j = (j/3), \quad j = 2, 3, 4, 5, \quad \alpha_{i,j} = (r/3), \quad r = 6, 7, 8, 9. \) Then, we can compute constants as \(\gamma = 7/4, \quad \gamma_1 = 41/21, \quad \Lambda = 6.371398411, \quad M_1 = 12.5680951, \quad M_2 = 3.2535588460, \quad M_3 = 48.72536839, \quad M_4 = 22.05071608.\)

(i) Let the nonlinear functions \(f\) and \(g\) be defined by
Observe that condition \((H_1)\) in Theorem 1 is satisfied for nonlinear functions \(f\) and \(g\) with Lipschitz constants \(\ell_1 = 1/10, \ell_2 = 1/15, n_1 = 1/20,\) and \(n_2 = 1/25.\) Next, we can find that \((M_1 + M_3)(\ell_1 + \ell_2) + (M_2 + M_4) (n_1 + n_2) = 12.4926389 > 1.\) Then, Theorem 1 cannot be used to obtain the existence criteria for the investigated problem. However, we calculate that \(|f(t,x,y)| \leq 5/6, |g(t,x,y)| \leq (108 + 5\pi)/200,\) and

\[
\frac{(b-a)^{n_1}}{\Gamma(a_1 + 1)} (\ell_1 + \ell_2) + \frac{(b-a)^{n_2}}{\Gamma(a_1 + 1)} (n_1 + n_2) > 0.9650966816 < 1.
\]

Hence, all assumptions of Theorem 3 hold. Therefore, by Theorem 3, problem (67) with (74)-(75) has at least one solution \((x,y)\) on \([1/3,10/3].\)

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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