A new sequence convergent to Euler–Mascheroni constant

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Abstract
In this paper, we provide a new sequence converging to the Euler–Mascheroni constant. Finally, we establish some inequalities for the Euler–Mascheroni constant by the new sequence.

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1 Introduction
The Euler–Mascheroni constant was first introduced by Leonhard Euler (1707–1783) in 1734 as the limit of the sequence

$$\gamma(n) := \sum_{m=1}^{n} \frac{1}{m} - \ln n. \quad (1.1)$$

There are many famous unsolved problems about the nature of this constant (see, e.g., the survey papers or books of Brent and Zimmermann [1], Dence and Dence [2], Havil [3], and Lagarias [4]). For example, it is a long-standing open problem if the Euler–Mascheroni constant is a rational number. A good part of its mystery comes from the fact that the known algorithms converging to $\gamma$ are not very fast, at least when they are compared to similar algorithms for $\pi$ and $e$.

The sequence $(\gamma(n))_{n\in\mathbb{N}}$ converges very slowly toward $\gamma$, like $(2n)^{-1}$. Up to now, many authors are preoccupied to improve its rate of convergence; see, for example, [2, 5–14] and references therein. We list some main results:

$$\sum_{m=1}^{n} \frac{1}{m} - \ln \left( n + \frac{1}{2} \right) = \gamma + O(n^{-2}) \quad \text{(DeTemple [6])},$$

$$\sum_{m=1}^{n} \frac{1}{m} - \ln \left( \frac{n^3 + \frac{3}{2} n^2 + \frac{247}{240} n + \frac{107}{480}}{n^2 + n + \frac{27}{240}} \right) = \gamma + O(n^{-6}) \quad \text{(Mortici [13])},$$

$$\sum_{m=1}^{n} \frac{1}{m} - \ln \left( 1 + \frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \frac{23}{5760n^4} \right) = \gamma + O(n^{-5}) \quad \text{(Chen and Mortici [5])}.$$
Recently, Mortici and Chen [14] provided a very interesting sequence

\[ v(n) = \sum_{m=1}^{n} \frac{1}{m} - \frac{1}{2} \ln \left( n^2 + n + \frac{1}{3} \right) - \left( \frac{1}{180} \frac{8}{2835} \frac{5}{1512} \frac{592}{93,555} \right) \]

and proved that

\[ \lim_{n \to \infty} n^{12} (v(n) - \gamma) = -\frac{796,801}{43,783,740}. \] (1.2)

Hence the rate of the convergence of the sequence \((v(n))_{n\in\mathbb{N}}\) is \(n^{-12}\).

Very recently, by inserting the continued fraction term into (1.1), Lu [9] introduced a class of sequences \((r_k(n))_{n\in\mathbb{N}}\) (see Theorem 1) and showed that

\[ \frac{1}{72(n+1)^3} < \gamma - r_2(n) < \frac{1}{72n^3}, \] (1.3)

\[ \frac{1}{120(n+1)^4} < r_3(n) - \gamma < \frac{1}{120(n-1)^4}. \] (1.4)

In fact, Lu [9] also found \(a_4\) without proof, and his works motivate our study. In this paper, starting from the well-known sequence \(\gamma_n\), based on the early works of Mortici, DeTemple, and Lu, we provide some new classes of convergent sequences for the Euler–Mascheroni constant.

**Theorem 1** For the Euler–Mascheroni constant, we have the following convergent sequence:

\[ r(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n - \ln \left( 1 + \frac{a_1}{n} \right) - \ln \left( 1 + \frac{a_2}{n^2} \right) - \cdots, \]

where

\[ a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{24}, \quad a_3 = -\frac{1}{24}, \quad a_4 = \frac{143}{5760}, \]

\[ a_5 = -\frac{1}{160}, \quad a_6 = -\frac{151}{290,304}, \quad a_7 = -\frac{1}{896}, \quad \ldots \]

Let

\[ r_k(n) := \sum_{m=1}^{n} \frac{1}{m} - \ln n - \sum_{m=1}^{k} \ln \left( 1 + \frac{a_m}{n^m} \right). \]

For \(1 \leq k \leq 7\), we have

\[ \lim_{n \to \infty} n^{k+2} (r_k(n) - \gamma) = C_k, \] (1.5)
where
\[
\begin{align*}
C_1 &= \frac{1}{24}, & C_2 &= -\frac{1}{24}, & C_3 &= \frac{143}{5760}, & C_4 &= -\frac{1}{160}, \\
C_5 &= -\frac{151}{290,304}, & C_6 &= -\frac{1}{896}, & C_7 &= \frac{109,793}{22,118,400}, & \ldots
\end{align*}
\]

Furthermore, for \( r_2(n) \) and \( r_3(n) \), we also have the following inequalities.

**Theorem 2** Let \( r_2(n) \) and \( r_3(n) \) be as in Theorem 1. Then
\[
\frac{1}{24}\frac{1}{(n+1)^3} < \gamma - r_2(n) < \frac{1}{24}\frac{1}{n^3}, \tag{1.6}
\]
\[
\frac{143}{5760}\frac{1}{(n+1)^4} < r_3(n) - \gamma < \frac{143}{5760}\frac{1}{n^4}. \tag{1.7}
\]

**Remark 1** Certainly, there are similar inequalities for \( r_k(n) \) \((1 \leq k \leq 7)\); we omit the details.

### 2 Proof of Theorem 1

The following lemma gives a method for measuring the rate of convergence. This lemma was first used by Mortici [15, 16] for constructing asymptotic expansions or accelerating some convergences. For a proof and other details, see, for example, [16].

**Lemma 1** If the sequence \( (x_n)_{n \in \mathbb{N}} \) converges to zero and there exists the limit
\[
\lim_{n \to +\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty] \tag{2.1}
\]
with \( s > 1 \), then there exists the limit
\[
\lim_{n \to +\infty} n^{s-1}x_n = \frac{l}{s - 1}. \tag{2.2}
\]

We need to find the value \( a_1 \in \mathbb{R} \) that produces the most accurate approximation of the form
\[
r_1(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \ln \left(1 + \frac{a_1}{n}\right). \tag{2.3}
\]

To measure the accuracy of this approximation, we usually say that an approximation \(2.3\) is better as \( r_1(n) - \gamma \) faster converges to zero. Clearly,
\[
r_1(n) - r_1(n + 1) = \ln \left(1 + \frac{1}{n}\right) - \frac{1}{n+1} + \ln \left(1 + \frac{a_1}{n+1}\right) - \ln \left(1 + \frac{a_1}{n}\right). \tag{2.4}
\]

Developing expression \(2.4\) into a power series expansion in \(1/n\), we obtain
\[
r_1(n) - r_1(n + 1) = \left(\frac{1}{2} - a_1\right)\frac{1}{n^2} + \left(-\frac{2}{3} + a_1 + a_1^2\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right). \tag{2.5}
\]

From Lemma 1 we see that the rate of convergence of the sequence \( (r_1(n) - \gamma)_{n \in \mathbb{N}} \) is even higher as the value \( s \) satisfies \(2.1\). By Lemma 1 we have
(i) If $a_1 \neq 1/2$, then the rate of convergence of the $(r_1(n) - \gamma)_{n \in \mathbb{N}}$ is $n^{-2}$, since

$$\lim_{n \to \infty} n(r_1(n) - \gamma) = \frac{1}{2} - a_1 \neq 0.$$ 

(ii) If $a_1 = 1/2$, then from (2.5) we have

$$r_1(n) - r_1(n + 1) = \frac{1}{12} n^3 + O\left(\frac{1}{n^4}\right).$$

Hence the rate of convergence of the $(r_1(n) - \gamma)_{n \in \mathbb{N}}$ is $n^{-3}$, since

$$\lim_{n \to \infty} n^3(r_1(n) - \gamma) = \frac{1}{24} = C_1.$$ 

We also observe that the fastest possible sequence $(r_1(n))_{n \in \mathbb{N}}$ is obtained only for $a_1 = 1/2$.

We repeat our approach to determine $a_1$ to $a_7$ step by step. In fact, we can easily compute $a_k$, $k \leq 15$, by the Mathematica software. In this paper, we use the Mathematica software to manipulate symbolic computations.

Let

$$r_k(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \sum_{m=1}^{k} \ln \left(1 + \frac{a_m}{n^m}\right). \quad (2.6)$$

Then

$$r_k(n) - r_k(n + 1)$$

$$= \ln \left(1 + \frac{1}{n}\right) - \frac{1}{n + 1} + \sum_{m=1}^{k} \ln \left(1 + \frac{a_m}{(n + 1)^m}\right) - \sum_{m=1}^{k} \ln \left(1 + \frac{a_m}{n^m}\right). \quad (2.7)$$

Hence the key step is to expand $r_k(n) - r_k(n + 1)$ into power series in $1/n$. Here we use some examples to explain our method.

**Step 1:** For example, given $a_1$ to $a_4$, find $a_5$. Define

$$r_5(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - 5 \sum_{m=1}^{5} \ln \left(1 + \frac{a_m}{n^m}\right).$$

By using the Mathematica software (the Mathematica Program is very similar to that given further in Remark 2; however, it has the parameter $a_8$) we obtain

$$r_5(n) - r_5(n + 1) = \left(-\frac{1}{32} - 5a_5\right) \frac{1}{n^6} + \left(\frac{4385}{48,384} + 15a_5\right) \frac{1}{n^7} + O\left(\frac{1}{n^8}\right). \quad (2.8)$$

The fastest possible sequence $(r_5(n))_{n \in \mathbb{N}}$ is obtained only for $a_5 = -\frac{1}{160}$. At the same time, it follows from (2.8) that

$$r_5(n) - r_5(n + 1) = -\frac{151}{48,384} \frac{1}{n^7} + O\left(\frac{1}{n^8}\right). \quad (2.9)$$
The rate of convergence of \((r_8(n) - \gamma)_{n \in \mathbb{N}}\) is \(n^{-7}\), since

\[
\lim_{n \to \infty} n^7 (r_8(n) - \gamma) = -\frac{151}{290,304} := C_5.
\]

We can use this approach to find \(a_k\) (\(1 \leq k \leq 15\)). From the computations we may the

Step 2: Check \(a_6 = -\frac{151}{290,304}\) to \(a_7 = -\frac{1}{896}\).

Let \(a_1, \ldots, a_6, \) and \(r_6(n)\) be defined as in Theorem 1. Applying the Mathematica software, we obtain

\[
r_6(n) - r_6(n + 1) = -\frac{1}{128} \frac{1}{n^8} + O\left(\frac{1}{n^7}\right).
\]

The rate of convergence of \((r_6(n) - \gamma)_{n \in \mathbb{N}}\) is \(n^{-8}\), since

\[
\lim_{n \to \infty} n^8 (r_6(n) - \gamma) = -\frac{1}{896} := C_6.
\]

Finally, we check that \(a_7 = -\frac{1}{896}\):

\[
r_7(n) - r_7(n + 1) = \left(-\frac{1}{128} - 7a_7\right) \frac{1}{n^8} + \left(\frac{196,193}{2,764,800} + 28a_7\right) \frac{1}{n^9} + O\left(\frac{1}{n^{10}}\right).
\]

Since \(a_7 = -\frac{1}{896}\) and

\[
\lim_{n \to \infty} n^8 (r_7(n) - \gamma) = \frac{109,793}{22,118,400} := C_7,
\]

the rate of convergence of the \((r_7(n) - \gamma)_{n \in \mathbb{N}}\) is \(n^{-9}\).

This completes the proof of Theorem 1.

**Remark 2** From the computations we can guess that \(a_{n+1} = C_n, n \geq 1\). It is a very interesting problem to prove this. However, it seems impossible by the provided method.

### 3 Proof of Theorem 2

Before we prove Theorem 2, let us give a simple inequality, which follows from the Hermite–Hadamard inequality and plays an important role in the proof.

**Lemma 2** *Let \(f\) be a twice continuously differentiable function. If \(f''(x) > 0\), then*

\[
\int_a^{a+1} f(x) \, dx > f(a + 1/2).
\]

*By \(P_k(x)\) we denote polynomials of degree \(k\) in \(x\) such that all its nonzero coefficients are positive; it may be different at each occurrence.*

Let us prove Theorem 2. Noting that \(r_2(\infty) = 0\), we easily see that

\[
\gamma - r_2(n) = \sum_{m=n}^{\infty} (r_2(m + 1) - r_2(m)) = \sum_{m=n}^{\infty} f(m),
\]
where

\[ f(m) = \frac{1}{m+1} - \ln\left(1 + \frac{1}{m}\right) - \ln\left(1 + \frac{a_1}{m+1}\right) - \ln\left(1 + \frac{a_2}{(m+1)^2}\right) \]

\[ + \ln\left(1 + \frac{a_1}{m}\right) + \ln\left(1 + \frac{a_2}{m^2}\right). \]

Let \( D_1 = 1/2 \). By using the Mathematica software we have

\[-f'(x) - D_1 \frac{1}{(x+1)^5} = \frac{300 + 2739x + 11,434x^2 + 24,870x^3 + 28,314x^4 + 15,936x^5 + 3472x^6}{2x(1+x)^3(1+2x)(3+2x)(1+24x^2)(25 + 48x + 24x^2)} > 0 \]

and

\[-f'(x) - D_1 \frac{1}{(x + \frac{1}{2})^5} = -\frac{P(x)(x-1) + 151,085}{2x^3(1+x)^3(1+2x)(3+2x)(1+24x^2)(25 + 48x + 24x^2)} < 0. \]

Hence, we get the following inequalities for \( x \geq 1 \):

\[ D_1 \frac{1}{(x+1)^5} < -f'(x) < D_1 \frac{1}{(x + \frac{1}{2})^5}. \] (3.3)

Since \( f(\infty) = 0 \), from the right-hand side of (3.3) and Lemma 2 we get

\[ f(m) = -\int_m^\infty f'(x) \, dx \leq D_1 \int_m^\infty \left(1 + \frac{1}{2}\right)^{-5} \, dx \]

\[ = D_1 \left(1 + \frac{1}{2}\right)^{-4} \leq D_1 \int_m^{m+1} x^{-4} \, dx. \] (3.4)

From (3.1) and (3.4) we obtain

\[ \gamma - r_2(n) \leq \sum_{m=n}^\infty D_1 \int_m^{m+1} x^{-4} \, dx \]

\[ = D_1 \int_n^{\infty} x^{-4} \, dx = D_1 \frac{1}{12 n^3}. \] (3.5)

Similarly, we also have

\[ f(m) = -\int_m^\infty f'(x) \, dx \geq D_1 \int_m^\infty (x+1)^{-5} \, dx \]

\[ = D_1 (m+1)^{-4} \geq D_1 \frac{1}{4} \int_{m+1}^{m+2} x^{-4} \, dx \]
\[ \gamma - r_2(n) \geq \sum_{m=n}^{\infty} \frac{D_1}{4} \int_{m+1}^{m+2} x^{-4} \, dx \]
\[ = \frac{D_1}{4} \int_{n+1}^{\infty} x^{-4} \, dx = \frac{D_1}{12} \frac{1}{(n+1)^3}. \]  

(3.6)

Combining (3.5) and (3.6) completes the proof of (1.6).

Noting that \( r_3(\infty) = 0 \), we easily deduce

\[ r_3(n) - \gamma = \sum_{m=n}^{\infty} (r_3(m) - r_3(m+1)) = \sum_{m=n}^{\infty} g(m), \]  

(3.7)

where

\[ g(m) = r_3(m) - r_3(m+1). \]

Let \( D_2 = \frac{143}{288} \). By using the Mathematica software we have

\[ -g'(x) - D_2 \frac{1}{(x+1)^6} = \frac{P_{12}(x)}{288n(1+n)^6(1+2n)(3+2n)(1+24n^2)(25+48n+24n^2)(-1+24n^3)P_3(x)} > 0 \]

and

\[ -g'(x) - D_2 \frac{1}{(x+\frac{1}{2})^6} = \frac{P_{12}(x)(x-4) + 2,052,948,001,087,775}{9x(1+x)^2(1+2x)^6(3+2x)(1+24x^2)(25+48x+24x^2)(-1+24x^3)P_3(x)} < 0. \]

Hence, for \( x \geq 4 \),

\[ D_2 \frac{1}{(n+1)^6} < -g'(x) < D_2 \frac{1}{(x+\frac{1}{2})^6}. \]  

(3.8)

Since \( g(\infty) = 0 \), by (3.8) we get

\[ g(m) = -\int_{m}^{\infty} g'(x) \, dx \leq D_2 \int_{m}^{\infty} \left( x + \frac{1}{2} \right)^{-6} \, dx \]
\[ = \frac{D_2}{5} \left( m + \frac{1}{2} \right)^{-5} \]
\[ \leq \frac{D_2}{5} \int_{m}^{m+1} x^{-5} \, dx. \]  

(3.9)

It follows from (3.7), (3.9), and Lemma 2 that

\[ r_3(n) - \gamma \leq \sum_{m=n}^{\infty} \frac{D_2}{5} \int_{m}^{m+1} x^{-5} \, dx \]
\[ = \frac{D_2}{5} \int_{n}^{\infty} x^{-5} \, dx = \frac{D_2}{20} \frac{1}{n^4}. \]  

(3.10)
Finally,
\[ g(m) = - \int_m^\infty g'(x) \, dx \geq D_2 \int_m^\infty (x+1)^{-6} \, dx \]
\[ = \frac{D_2}{5} (m+1)^{-5} \geq \frac{D_2}{5} \int_{m+1}^{m+2} x^{-5} \, dx \]

and
\[ r_5(n) - \gamma \geq \sum_{m=n}^{\infty} \frac{D_2}{5} \int_{m+1}^{m+2} x^{-5} \, dx \]
\[ = \frac{D_2}{5} \int_n^{\infty} x^{-5} \, dx = \frac{D_2}{20} \frac{1}{(n+1)^4}. \] (3.11)

Combining (3.10) and (3.11) completes the proof of (1.7).

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The authors declare that they have no competing interests.

Authors’ contributions
The authors read and approved the final manuscript.

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