SPECTRUM OF P-ADIC LINEAR DIFFERENTIAL EQUATIONS I

THE SHAPE OF THE SPECTRUM

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ABSTRACT. This paper extends our previous works [Azz20; Azz22] on determining the spectrum, in the Berkovich sense, of ultrametric linear differential equations. Our previous works focused on equations with constant coefficients or over a field of formal power series. In this paper, we investigate the spectrum of p-adic differential equations at a generic point on a quasi-smooth curve. This analysis allows us to establish a significant connection between the spectrum and the spectral radii of convergence of a differential equation when considering the affine line. Furthermore, the spectrum offers a more detailed decomposition compared to Robba’s decomposition based on spectral radii [Rob75].

1. Introduction

In the ultrametric setting, linear differential equations present phenomena that do not appear over the complex field. Indeed, the solutions of such equations may fail to converge everywhere, even without the presence of poles. This leads to a non-trivial notion of radius of convergence, and its knowledge allows to obtain several interesting information about the equation. Notably, it controls the finite dimensionality of the de Rham cohomology. In practice, the radius of convergence is really hard to compute and it represents one of the most complicate features in the theory of p-adic differential equations [Pul15; PP15; PP13a; PP13b]. The radius of convergence can be expressed as the spectral norm of a certain operator. A natural notion refining it is the entire spectrum of this operator, in the sense of Berkovich.

In our previous works [Azz20; Azz22], we determine the spectrum of differential equations over a field of power series, and in the ultrametric case, differential equation with constant coefficients. The general case in the p-adic situation is the subject of this paper, and it is much more difficult, since it requires a fine improvement of some of the fundamental theorems in the theory of p-adic differential equations.

Let \((k, |\cdot|)\) be an algebraically closed complete ultrametric field of characteristic zero, and let \(\mathbb{A}_k^{1,\text{an}}\) be the Berkovich affine line. We fix \(T\) to be a coordinate function on \(\mathbb{A}_k^{1,\text{an}}\). For any positive real number \(r\) and \(c \in k\), let \(x_{c,r}\) be the point of \(\mathbb{A}_k^{1,\text{an}}\) associated to the multiplicative norm \(k[T] \to \mathbb{R}_+, \sum_i a_i (T - c)^i \mapsto \max_i |a_i|^r\). For a point \(x \in \mathbb{A}_k^{1,\text{an}}\), we denote by \(\mathcal{H}(x)\) the associated complete residue field, and by \(r_k(x)\) its radius (cf. (2.5)).

This work is in line with the work on differential modules defined over a generic point of a quasi-smooth\(^1\) curve, initiated by Dwork in [Dwo73]. We will restrict ourselves to points \(x\) of \(\mathbb{A}_k^{1,\text{an}} \setminus k\), and consider differential module \((M, \nabla)\) of finite rank over \((\mathcal{H}(x), d))\), where \(d\) is a “reasonable” \(k\)-linear bounded derivation defined over \(\mathcal{H}(x)\). We will explain later why we do not lose generality by such a restriction. For a point \(x \in \mathbb{A}_k^{1,\text{an}} \setminus k\) and a differential module \((M, \nabla)\) over \((\mathcal{H}(x), d)\), we set \(\mathcal{R}_i^{(M, \nabla), \text{Sp}}(x) \leq \cdots \leq \mathcal{R}_n^{(M, \nabla), \text{Sp}}(x)\) to be the spectral radii of convergence, as originally given in [CD94] for \(\mathcal{R}_1^{(M, \nabla), \text{Sp}}(x)\), and generalized for \(i \geq 2\) in [Ked10].

In this setting, the spectrum of \((M, \nabla)\) is the spectrum in the sense of Berkovich (see [Ber90, Chapter 7]) \(\Sigma_{\nabla, k}(\mathcal{L}_k(M))\) of \(\nabla\) as an element of \(\mathcal{L}_k(M)\), the \(k\)-Banach algebra of \(k\)-linear bounded endomorphisms of \(M\). This spectrum is a subset of \(\mathbb{A}_k^{1,\text{an}}\) instead of just \(k\), and also enjoys the same nice properties as in

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\(^1\)Quasi-smooth means that its sheaf of differential forms \(\Omega_X\) is locally free of rank 1, see [Duc, Definition 3.1.11].
the complex case, like non-emptiness and compactness. To avoid any confusion, we fix another coordinate function $S$ on $\mathbb{A}^{1,\text{an}}_k$ for differential modules.

Now let us comeback to the relation between the spectrum and the radius of convergence, consider a point $x \in \mathbb{A}^{1,\text{an}}_k \setminus k$ and a differential module $(M, \nabla)$ over $(\mathcal{H}(x), \frac{d}{dz})$. On the one hand we have $\|\nabla\|_{\text{sp}} = \lim_{n \to \infty} \|\nabla^n\|^\frac{1}{n} = \frac{\omega}{\mathcal{R}_{i(M,\nabla),\text{sp}}(x)}$, where $\omega := \lim |n|! \cdot n^{-\frac{1}{n}}$ (cf. [CD94, p. 676] or [Ked10, Definition 9.4.4]). On the other hand, the spectral norm $\|\nabla\|_{\text{sp}}$ is also equal to the radius of the smallest closed disk centered at zero and containing $\Sigma_{\nabla,k}(L_k(M))$ (cf. [Ber90, Theorem 7.1.2]). In our first computation of the spectrum in [Azz20], we prove that the spectrum of a linear differential equation with constant coefficients $(M, \nabla)$ is a finite union of closed disks $\cup D_i$. Furthermore, we observe that for each $\mathcal{R}_{i(M,\nabla),\text{sp}}(x)$ there exists $D_j$ such that the smallest closed disk centered at zero containing $D_j$ has radius equal to $\frac{\omega}{\mathcal{R}_{i(M,\nabla),\text{sp}}(x)}$, and conversely for each $D_i$. We may ask ourselves, how far can this relations between the spectrum and the radii of convergence be generalized? In our work [Azz22], we observe another phenomenon, the spectrum is intrinsic to the choice of the derivation. This means that given a differential module $(M, \nabla)$ over $(\mathcal{H}(x), d)$ for some $x \in \mathbb{A}^{1,\text{an}}_k \setminus k$ and $g \in \mathcal{H}(x) \setminus \{0\}$, then even if the kernel of $(M, \nabla)$ and $(M, g\nabla)$ define the same differential system, the respective spectra of these differential modules may differ. For example, suppose that $k$ is trivially valued and let $x_{0,r} \in \mathbb{A}^{1,\text{an}}_k \setminus k$ with $r < 1$, then the spectrum of $(\mathcal{H}(x), S_{\frac{d}{dz}})$ is equal to $\mathbb{Z} \cup \{0, 1\}$ (cf. [Azz22, Proposition 3.7]), this computation is generalized in Section 3.1, however the spectrum of $(\mathcal{H}(x), \frac{d}{dz})$ is equal to the closed disk $D^+(0, \frac{1}{2})$ (cf. [Azz20]). So, the following questions arise:

- What is the most suitable choice of derivations?
- If we choose a derivation different from $\frac{d}{dz}$, does the relation between the spectrum and the radii of convergence remain?

The derivation $\frac{d}{dz}$ appears to be the most natural, however we quickly realize that the determination of the spectrum becomes very hard in this case. Indeed, the usual technique used for the computation of the radii, like the ramification of the indeterminate or the push-forward by the Frobenius map, cannot be used for the determination of the spectrum. That is why we privilege the use of the derivation $S_{\frac{d}{dz}}$ in [Azz22] to determine the spectrum of a differential module over a field of formal power series. The interesting part is that by choosing $S_{\frac{d}{dz}}$, not only the determination of the spectrum becomes affordable, but in this situation we can also recover all the data of radii of convergence and others information in the spectrum. For these reasons, in this paper, for a point $x_{c,r} \in \mathbb{A}^{1,\text{an}}_k \setminus k$, we claim that $(S - c)_{\frac{d}{dz}}$ is the most convenient choice.

In this paper we focus mainly on the case where $\text{char}(\mathbb{k}) = p > 0$, and provide an algorithm to determine the spectrum of any differential module $(M, \nabla)$ over $(\mathcal{H}(x), (S - c)_{\frac{d}{dz}})$, with $x = x_{c,r}$ in $\mathbb{A}^{1,\text{an}}_k$. Moreover, we establish a link between the spectrum and all radii of convergence, which is the purpose of the main result of the paper. For $(\Omega, |.|)$ an extension of $(\mathbb{k}, |.|)$, we define the following map

\begin{equation}
\delta_{(\Omega,|.|)} : \Omega \rightarrow \mathbb{R}_+ \hspace{1cm} z \mapsto \inf_{n \in \mathbb{Z}} |z - n|.
\end{equation}

We may denote it only by $\delta$ instead of $\delta_{(\Omega,|.|)}$. We denote by $\pi_{\Omega/k} : \mathbb{A}^{1,\text{an}}_\Omega \rightarrow \mathbb{A}^{1,\text{an}}_k$ the canonical projection. The main result of the paper is the following.

**Theorem 1.1 (Theorem 5.45).** Assume that $\text{char}(\mathbb{k}) = p > 0$ and $x := x_{0,r} \in \mathbb{A}^{1,\text{an}}_k \setminus k$. Let $(M, \nabla)$ be a differential module $(\mathcal{H}(x), S_{\frac{d}{dz}})$. Let $\text{Frob}_p : \mathbb{A}^{1,\text{an}}_k \rightarrow \mathbb{A}^{1,\text{an}}_k$, $S \mapsto S^p$. We denote by $\mathcal{R}_a(x)$ the spectral radii of $(\mathcal{H}(x), S_{\frac{d}{dz}} - a)$ and $\text{Frob}_p^a(x)$ by $x^{\text{Frob}}_a^i$.

- There exist $z_1, \ldots, z_r \in \mathbb{A}^{1,\text{an}}_k \setminus k$ and $a_1, \ldots, a_r \in k$, such that $\Sigma_{\nabla,k}(L_k(M)) = \{z_1, \ldots, z_r, a_1, \ldots, a_r\} + \mathbb{Z}_p$, where $z_i$ has the same type as $x$, and $(r, \mu)$ is not equal to $(0, 0)$.
- We can choose $z_i$ and $a_j$ such that the set $\{z_1, \ldots, z_r, a_1, \ldots, a_r\}$ has minimal cardinality. Indeed it is enough to keep only $z_i$ and $a_j$, for which we have $\{z_i\} + \mathbb{Z}_p \cap \{z_j\} + \mathbb{Z}_p = \emptyset$ and $\{a_j\} + \mathbb{Z}_p \cap \{a_j\} + \mathbb{Z}_p = \emptyset$ for $i \neq i'$ and $j \neq j'$.
• We choose \( \{z_1, \ldots, z_n, a_1, \ldots, a_n\} \) to be minimal. Then we have a unique (up to an isomorphism) decomposition

\[
(M, \nabla) = \bigoplus_{i=1}^{\nu} (M_{z_i}, \nabla_{z_i}) \oplus \bigoplus_{j=1}^{\mu} (M_{a_j}, \nabla_{a_j}),
\]

such that, \( \Sigma_{\nabla_{a_j}}(L_k(M_{a_j})) = \{z_i\} + \mathbb{Z}_p \) and \( \Sigma_{\nabla_{a_j}}(L_k(M_{a_j})) = \{a_j\} + \mathbb{Z}_p \).

• Let \( c_i \in k \) and \( r_i > 0 \) such that \( z_i = x_{c_i,r_i} \). If \( |p|^l \leq r_i < |p|^{l-1} \), with \( l \in \mathbb{N} \setminus \{0\} \), then \( \text{Card}\{\{z_i\} + \mathbb{Z}_p\} = p^l \) and \( \{z_i\} + \mathbb{Z}_p = \{x_{c_i,r_i}, x_{c_i+1,r_i}, \ldots, x_{c_i+p^l-1,r_i}\} \). If \( r_i \geq 1 \) then we have \( \text{Card}\{\{z_i\} + \mathbb{Z}_p\} = 1 \) and \( \{z_i\} + \mathbb{Z}_p = \{x_{c_i,r_i}\} \).

• If \( r_i > 1 \), let \( P_{z_i}(S_{d_{r_i}}^a) \) be a differential polynomial associated to \((M_{z_i}, \nabla_{z_i})\). Then the image by \( \pi_{x_{d_{r_i}/k}} \) of all roots of \( P_{z_i}(T) \) (the commutative polynomial associated to \( P_{z_i}(S_{d_{r_i}}^a) \)) is equal to \( z_i \).

• If \( |p|^l < r_i \leq |p|^{l-1} \), let \( P_{z_i}(p^l S_{d_{r_i}}^a) \) be a differential polynomial associated to \((\text{Prob}_{p^l})_a(M, \nabla)\) (as a differential module over \((\mathcal{H}(x^p), p^l S_{d_{r_i}}^a)\)). Then the image by \( \pi_{x_{d_{r_i}/k}} \) of all roots of \( P_{z_i}(T) \) (the commutative polynomial associated to \( P_{z_i}(p^l S_{d_{r_i}}^a) \)) is equal to \( \{x_{c_i,r_i}, x_{c_i+1,r_i}, \ldots, x_{c_i+p^l-1,r_i}\} \). In the special case where \( r_i = |p|^{l-1} \) we have \( \{x_{c_i,r_i}, x_{c_i+1,r_i}, \ldots, x_{c_i+p^l-1,r_i}\} = \{x_{c_i,r_i}, x_{c_i+1,r_i}, \ldots, x_{c_i+p^l-1-1,r_i}\} \).

• If \( r_i \geq 1 \). For all \( a \in k \), the differential module \((M_{z_i}, \nabla_{z_i} - a)\) is pure (all its spectral radii are equal). For \( a \in D^+(c_i, r_i) \cap k \) we have

\[
R_{1}^{(M_{z_i}, \nabla_{z_i} - a)}(x) = \frac{\omega}{r_i},
\]

and for all \( a \in k \setminus D^+(c_i, r_i) \)

\[
R_{1}^{(M_{z_i}, \nabla_{z_i} - a)}(x) = \frac{\omega}{|a - c_i|}. \tag{1.3}
\]

• If \( |p|^l \leq r_i < |p|^{l-1} \). For all \( a \in k \), the differential module \((M_{z_i}, \nabla_{z_i} - a)\) is pure. We have for all \( a \in \bigcup_{j=0}^{p^l-1} D^+(c_i + j, r_i) \cap k \)

\[
R_{1}^{(M_{z_i}, \nabla_{z_i} - a)}(x) = \left( \frac{|p|^l \omega}{r_i} \right)^{\frac{1}{p^l}} r_i, \tag{1.4}
\]

and for all \( a \in k \setminus \bigcup_{j=0}^{p^l-1} D^+(c_i + j, r_i) \)

\[
R_{1}^{(M_{z_i}, \nabla_{z_i} - a)}(x) = R_{a-c_i}(x). \tag{1.5}
\]

• For all \( a \in k \), the differential module \((M_{a_i}, \nabla_{a_i} - a)\) is pure. More precisely, for all \( a \in \{a_i\} + \mathbb{Z}_p \), \((M_{a_i}, \nabla_{a_i} - a)\) is solvable (all its radii are equal to \( r_i \)), and for all \( a \in k \setminus \{a_i\} + \mathbb{Z}_p \), we have

\[
R_{1}^{(M_{a_i}, \nabla_{a_i} - a)}(x) = R_{a-c_i}(x). \tag{1.6}
\]

Let \((M, \nabla)\) be as in Proposition 1.1 and \( \Sigma_{\nabla} = \{\omega_1, \ldots, \omega_\mu\} + \mathbb{Z}_p \) be the spectrum of \( \nabla \), where \( \{\omega_1, \ldots, \omega_\mu\} \) has minimal cardinality and \( \omega_i \in A_k^{1,\text{an}} \). We emphasize that if \((M, \nabla) = \bigoplus_{i=1}^{\mu}(M_{\omega_i}, \nabla_{\omega_i})\) is as in (1.2), then

\[
\dim_{M_{\omega_1}} \text{ times } R_{1}^{(M_{\omega_1}, \nabla_{\omega_1})}(x), \ldots, R_{1}^{(M_{\omega_1}, \nabla_{\omega_1})}(x) \quad \cdots \quad \dim_{M_{\omega_\mu}} \text{ times } R_{1}^{(M_{\omega_\mu}, \nabla_{\omega_\mu})}(x), \ldots, R_{1}^{(M_{\omega_\mu}, \nabla_{\omega_\mu})}(x),
\]

after a suitable permutation corresponding exactly to \( R_{n}^{(M, \nabla)}(x), \ldots, R_{n}^{(M, \nabla)}(x) \).

This theorem gives a kind of geometrical incarnation of Robba’s decomposition by spectral radii [Rob75], that stating a differential module \((M, \nabla)\) over \((\mathcal{H}(x), d)\) decomposes into a direct sum of pure differential modules, and each component has all its spectral radii equal to one of the value of the spectral radii of convergence. However, the decomposition provided by the spectrum is finer than the one provided by radii. Moreover, when we vary \( a \in k \), the radii of \((M, \nabla - a)\) are well controlled by the points of the spectrum of
\( \nabla \). Let us give an example where we see concretely how the decomposition provided by the spectrum is that finer.

**Example 1.2.** Let \((M, \nabla)\) be a differential module over \( \mathcal{H}(x_{0,1}), S_{d}^{d} \), associated to the differential polynomial \( P(d) = d^{2} - [(a+b)S + b]d + bS(aS + b) \) with \( a, b \in k \) and \( 1 < |a| < |b| \). By Young’s theorem, \((M, \nabla)\) is a pure differential module with radius equal to \( \frac{1}{ \min |a, b|} \). However, by Theorem 1.4, we have \( \Sigma_{\nabla} = \{ x_{0,|b|}, x_{b,|a|} \} \).

This means that we can decompose \((M, \nabla)\) with respect to the spectrum.

More generally we have the following results.

**Theorem 1.3** (Theorem 5.49). Assume that \( \text{char}(\tilde{k}) = p > 0 \). Let \( C \) be a quasi-smooth curve and \( x \in C \) of type \((2)\) or \((3)\). Let \((M, \nabla)\) be a differential module over \( \mathcal{H}(x), d \), where \( d = \psi^{*}(S_{d}^{d}) \), \( \psi \) is a finite étale morphism from a neighbourhood of \( x \) to \( \mathbb{P}_{k}^{1,\mathrm{an}} \), with \( \psi(x) = x_{0,r} \). Then there exist \( z_{1}, \cdots, z_{\mu} \in \mathbb{A}_{k}^{1,\mathrm{an}} \), with \((\{ z_{i} \} + \mathbb{Z}_{p}) \cap (\{ z_{j} \} + \mathbb{Z}_{p}) = \emptyset \) for \( i \neq j \), such that:

\[
\Sigma_{\nabla,k}(\mathcal{L}_{k}(M)) = \{ z_{1}, \cdots, z_{\mu} \} + \mathbb{Z}_{p}.
\]

Note that for any point \( x \) in a quasi-smooth curve \( C \), there exists a finite étale morphism \( \psi : Y \to W \), where \( Y \) is an affinoid neighbourhood of \( x \) and \( W \) an affinoid domain of the projective Berkovich line \( \mathbb{P}_{k}^{1,\mathrm{an}} \) (cf. [Duc, Theorem 4.5.4], [PP15, Theorem 3.12]). If \( x \) has a neighbourhood isomorphic to an open disk or annulus, then the link between the radii and the spectrum is similar to the case of the affine line. More generally, for the case of a quasi-smooth curve, to establish the link between the spectral radii of convergence and the spectrum we need extra material, notably the continuity of the spectrum on branches out of \( x \). This is the object of our work [Azz23].

The proof of the main result requires to develope some key techniques. The most important one is a kind of spectral version of Young’s theorem, that states the following:

**Theorem 1.4** (Theorem 4.23). Let \((\Omega, |.|)\) be a complete extension of \((k, |.|)\) and \( d : \Omega \to \Omega \) be a \( k \)-linear bounded derivation. We set \( \pi_{\Omega^{\text{alg}}/k} : \mathbb{A}_{k,\text{an}}^{1,\text{an}} \to \mathbb{A}_{k}^{1,\text{an}} \) to be the canonical projection. Let \((M, \nabla)\) be a differential module over \( \Omega, d \), 

\( P(T) = \sum_{i=0}^{n} a_{i}T^{i} + T^{n} \in \Omega[T] \) and \( \{ z_{1}, \cdots, z_{n} \} \subset \Omega^{\text{alg}} \) be the multiset of the roots of \( P(T) \).

Suppose that in some basis the associated matrix of \((M, \nabla)\) is

\[
G = \begin{pmatrix}
0 & -a_{0} & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -a_{n-1}
\end{pmatrix}.
\]

If \( \min_{i} r_{k}(\pi_{\Omega^{\text{alg}}/k}(z_{i})) > \| d \| \) (\( \| d \| \) is the operator norm of \( d \)), then

\[
\Sigma_{\nabla,k}(\mathcal{L}_{k}(M)) = \pi_{\Omega^{\text{alg}}/k}(\{ z_{1}, \cdots, z_{n} \}).
\]

The other key technique is to provide the relation between the spectrum of a differential module and the spectrum of its push-forward by an étale morphisms (mainly by the Frobenius map).

The paper is organized as follows. Section 2 recalls all the necessary material from Berkovich geometry. In particular, we recall some important tools developed in our previous work [Azz20].

Section 3 is divided into two parts. In the first one, we determine the spectrum of \( S_{d}^{d} : \mathcal{H}(x) \to \mathcal{H}(x) \), where \( x \in \mathbb{A}_{k}^{1,\text{an}} \setminus k \). We treat separately the case where \( \text{char} \tilde{k} = p > 0 \) from the case where \( \text{char} \tilde{k} = 0 \), and the case where \( x \in (0, \infty) \) form the case where \( x \not\in (0, \infty) \cup k \). Indeed, for each situation we use different methods. Assume \( \text{char} \tilde{k} = p > 0 \), if \( x \in (0, \infty) \), we use the push-forward by Frobenius map, and if \( x \not\in (0, \infty) \cup k \), we use the push-forward by logarithm map. In the case where \( \text{char} \tilde{k} = 0 \), if \( x \in (0, \infty) \), we compute the spectrum without extra method, and if \( x \not\in (0, \infty) \cup k \) we also use the push-forward by logarithm map. In the second part, the spectrum of \( S_{d}^{d} \) allows us to deduce easily the spectrum of any differential
module with regular singularities. On the other hand, in the $p$-adic case, we prove that the variation of the spectrum satisfies a continuity property. However, if $\text{char}(\tilde{k}) = 0$, the variation is surprisingly not continuous at all.

In Section 4, we provide the spectral version of Young’s theorem. Since in $\text{char}(\tilde{k}) = 0$, the radii are either solvable or small, this spectral version permits to determine the spectrum of any differential module with non solvable radii. In particular, we can recover the result of our paper [Azz22] easily and without using Turrittin’s theorem.

Section 5 is devoted to prove the main result of the paper, i.e. the determination of the spectrum of any differential module $(M, \nabla)$ over $(\mathcal{H}(x_0, r), S_{dS})$, when $\text{char}(\tilde{k}) = p > 0$. For that we proceed as follows: we start by establishing, when the radii are small, the link between the spectrum and the spectral radii of convergence, especially when we choose $p'(S - c) \frac{da}{ds}$ as derivation. Then we use the Frobenius map (pull-back and push-forward) to compute the spectrum and establish the link with the spectral radii of convergence. In the last part of the section, we explain how we can deduce from the main result the shape of the spectrum of a differential module $(M, \nabla)$ over $(\mathcal{H}(x), d)$, where $x$ is a point of a quasi-smooth curve of type (2) or (3), and $d$ is well chosen $k$-linear bounded derivation.

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\section*{Contents}
\begin{itemize}
\item Introduction \hfill 1
\item Preliminaries \hfill 5
\item 2. Definitions and notations \hfill 5
\item 2.2 Push-forward and pull-back of a differential module and their spectra \hfill 10
\item 3. Spectrum of a differential module with regular singularities \hfill 11
\item 3.1 Spectrum of the derivation $S_{dS}$ \hfill 12
\item 3.2 Spectrum of a regular singular differential module \hfill 17
\item 4. Spectral version of Young’s theorem \hfill 20
\item 5. Spectrum of differential module \hfill 27
\item 5.1 Link between the radius of convergence and the spectrum when the radii are small \hfill 27
\item 5.2 Frobenius and spectrum \hfill 31
\item 5.3 The main result \hfill 34
\item 5.4 Spectrum of a differential equation at a point of a quasi-smooth curve \hfill 41
\item Appendix \hfill 41
\item 6. Analytic spectrum of a Banach ring \hfill 41
\item 6.2 Description of some étale morphisms \hfill 43
\end{itemize}

\section{2. Preliminaries}

2.1. Definitions and notations. All rings are supposed to have a unit element. We will denote by $\mathbb{R}$ the field of real numbers, by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}$ the set of nonnegative integers. We set $\mathbb{R}_+$ for $\{ r \in \mathbb{R}; r \geq 0 \}$ and $\mathbb{R}_0^*$ for $\mathbb{R}_+ \setminus \{0\}$.

In all the paper, we fix $(k, |.|)$ to be an ultrametric complete field of characteristic 0. Let $|k|$ be $\{|a|; a \in k\}$. Let $E(k)$ be the category whose objects are $(\Omega, |.|_{\Omega})$, where $\Omega$ is a field extension of $k$, complete with respect to the valuation $|.|_{\Omega}$, and whose isomorphisms are isometric rings morphisms. For $(\Omega, |.|_{\Omega}) \in E(k)$, let $\Omega^{\text{alg}}$ be an algebraic closure of $\Omega$, the absolute value extends uniquely to an absolute value defined on $\Omega^{\text{alg}}$. We denote by $\Omega^{\text{alg}}$ the completion of $\Omega^{\text{alg}}$ with respect to this absolute value.
2.1.1. Differential modules. Let $\Omega \in E(k)$, in all the paper any $k$-linear derivation $d : \Omega \rightarrow \Omega$ is supposed to be bounded. Recall that a differential module $(M, \nabla)$ over $(\Omega, d)$ is an $\Omega$-vector space with $\dim_{\Omega}(M) < +\infty$ and equipped with a $k$-linear map $\nabla : M \rightarrow M$, called connection of $M$, satisfying $\nabla(fm) = df.m + f.\nabla(m)$ for all $f \in \Omega$ and $m \in M$.

Notation 2.1. Let $(\Omega, d)$ be a differential field. We denote by $d - \text{Mod}(\Omega)$ the category of differential modules over $(\Omega, d)$ whose arrows are morphisms of differential modules.

Notation 2.2. Let $(\Omega, d)$ be a differential field. Let $\mathcal{D}_{\Omega,d} := \bigoplus_{n \in \mathbb{N}} \Omega \cdot d^n$ be the ring of differential polynomials on $d$ with coefficients in $\Omega$, where the multiplication is non-commutative and defined as follows: $d \cdot f = df + f \cdot d$ for all $f \in \Omega$. Let $P(d) = g_0 + \cdots + g_{n-1}d^{n-1} + d^n$ be a monic differential polynomial. The quotient $\mathcal{D}_{\Omega,d}/\mathcal{D}_{\Omega,d} \cdot P(d)$ is an $\Omega$-vector space of dimension $n$, the multiplication by $d$ induces a structure of differential module over $(\Omega, d)$. If there is no confusion about the derivation we will simply denote by $\mathcal{D}_\Omega$.

We will denote by $\text{Hom}_{\mathcal{D}}(M, N)$ the set of arrows between the two objects $M$ and $N$ of $d - \text{Mod}(\Omega)$.

2.1.2. Analytic spaces. In this paper we will consider $k$-analytic spaces in the sense of Berkovich (see [Ber90]). We denote by $\mathbb{A}_k^{1,\text{an}}$ (resp. $\mathbb{P}_k^{1,\text{an}}$) the analytic affine (resp. projective) line over the ground field, with coordinate $T$. Let $(X, \mathcal{O}_X)$ be an analytic space. For any $x \in X$, the residue field of the local ring $\mathcal{O}_{X,x}$ is naturally valued and we denote by $\mathcal{H}^r(x)$ its completion.

Let $\Omega \in E(k)$ and $c \in \Omega$. For $r \in \mathbb{R}_+^*$ we set

\begin{align}
(2.1) & \quad D^+_\Omega(c, r) := \{ x \in \mathbb{A}_\Omega^{1,\text{an}} \mid |T(x) - c| \leq r \} \\
(2.2) & \quad D^-_\Omega(c, r) := \{ x \in \mathbb{A}_\Omega^{1,\text{an}} \mid |T(x) - c| < r \}
\end{align}

Denote by $x_{c,r}$ the unique point in the Shilov boundary of $D^+_\Omega(c, r)$. For $r_1, r_2 \in \mathbb{R}_+$, such $0 < r_1 \leq r_2$ we set

\begin{align}
(2.3) & \quad C^+_\Omega(c, r_1, r_2) := \{ x \in \mathbb{A}_\Omega^{1,\text{an}} \mid r_1 \leq |T(x) - c| \leq r_2 \}
\end{align}

and for $r_1 < r_2$ we set:

\begin{align}
(2.4) & \quad C^+_\Omega(c, r_1, r_2) := \{ x \in \mathbb{A}_\Omega^{1,\text{an}} \mid r_1 < |T(x) - c| < r_2 \}
\end{align}

We will drop the subscript $\Omega$ when no confusion is possible.

Definition 2.3. Let $x \in \mathbb{A}_k^{1,\text{an}}$ and $y \in \pi_k^{-1}(x)$. We define the radius of $x$ to be the value:

\begin{align}
(2.5) & \quad r_k(x) = \inf_{c \in k^{alg}} |T(y) - c|.
\end{align}

It does not depend on the choice of $y$. We will drop $k$ when no confusion is possible.

Remark 2.4. Assume that $k$ is algebraically closed. For a point $x_{c,r}$ of type (1), (2) or (3), we have $r_k(x_{c,r}) = r$.

Remark 2.5. Let $\Omega \in E(k)$ and $\pi_{\Omega/k} : \mathbb{A}_\Omega^{1,\text{an}} \rightarrow \mathbb{A}_k^{1,\text{an}}$ be the canonical projection. Let $x \in \mathbb{A}_k^{1,\text{an}}$ and $y \in \pi_{\Omega/k}^{-1}(x)$. In general, we have

\begin{align}
(2.6) & \quad r_k(x) \neq r_\Omega(y).
\end{align}

We will show further when the equality holds.

Assume that $k$ is algebraically closed. Let $c \in k$. The following map

\begin{align}
(2.7) & \quad [0, +\infty) \rightarrow \mathbb{A}_k^{1,\text{an}} \\
& \quad r \mapsto x_{c,r}
\end{align}

induces a homeomorphism between $[0, +\infty)$ and its image.

Notation 2.6. We will denote by $[x_{c,r}, \infty)$ (resp. $(x_{c,r}, +\infty)$) the image of $[r, +\infty)$ (resp. $(r, +\infty)$), by $[x_{c,r}, x_{c,r'}]$ (resp. $(x_{c,r}, x_{c,r'})$, $[x_{c,r}, x_{c,r'})$, $(x_{c,r}, x_{c,r'})$) the image of $[r, r')$ (resp. $(r, r')$, $[r, r')$, $(r, r')$).
Notation 2.7. Let $X$ be an affinoid domain of $\mathbb{A}^1_{k^{an}}$ and $f \in \mathcal{O}_X(X)$. We can see $f$ as an analytic morphism $X \to \mathbb{A}^1_{k^{an}}$ that we still denote by $f$. In particular, for a polynomial $\sum_i a_i T^i \in k[T]$ we denote by $\sum_i a_i x^i$, the image of $x$ by this polynomial.

2.1.3. Universal points and fiber of a point under extension of scalars.

Definition 2.8. A point $x \in \mathbb{A}^1_{k^{an}}$ is said to be universal if, for any $\Omega \in E(k)$, the tensor norm on the algebra $\mathcal{H}(x) \hat{\otimes}_{k^{alg}} \Omega$ is multiplicative. In this case, it defines a point of $\pi^{-1}_{\Omega/k}(x)$ in $\mathbb{A}^1_{\Omega^{alg}}$ that we denote by $\sigma_{\Omega/k}(x)$.

Proposition 2.9. If $k$ is algebraically closed, any point $x \in \mathbb{A}^1_{k^{an}}$ is universal.

Proof. See [Poi13, Corollary 3.14].

Theorem 2.10. Suppose that $k$ is algebraically closed. Let $\Omega \in E(k)$ algebraically closed.

- If $x$ is of type (i), where $i \in \{1, 2\}$, then so is $\sigma_{\Omega/k}(x)$. If $x$ is of type (j), where $j \in \{3, 4\}$, then $\sigma_{\Omega/k}(x)$ is of type (j) or (2).
- The fiber $\pi^{-1}_{\Omega/k}(x)$ is connected and the connected components of $\pi^{-1}_{\Omega/k}(x) \setminus \{\sigma_{\Omega/k}(x)\}$ are open disks with boundary $\{\sigma_{\Omega/k}(x)\}$. Moreover they are open in $\mathbb{A}^1_{\Omega^{alg}}$.

Proof. See [PP15, Theorem 2.2.9].

Corollary 2.11. Let $x \in \mathbb{A}^1_{k^{an}}$ be a point of type (i), where $i \in \{2, 3, 4\}$. Let $\Omega \in E(k)$ algebraically closed such that there is no isometric k-embedding $\mathcal{H}(x) \hookrightarrow \Omega$. Then

\begin{equation}
\pi^{-1}_{\Omega/k}(x) = \{\sigma_{\Omega/k}(x)\}.
\end{equation}

Proof. Recall that $\pi^{-1}_{\Omega/k}(x) \setminus \{\sigma_{\Omega/k}(x)\}$ is a disjoint union of open disks (cf. Theorem 2.10). Therefore, if it is not empty, it contains points of type (1) which gives rise to isometric k-embeddings $\mathcal{H}(x) \hookrightarrow \Omega$, which contradicts the hypothesis.

Lemma 2.12. Let $\Omega \in E(k)$ such that $k^{alg} \subset \Omega$. Let $x \in \mathbb{A}^1_{k^{an}}$. Then for any $y \in \pi^{-1}_{k^{alg}/k}(x)$ we have:

\begin{equation}
r_{\Omega}(\sigma_{\Omega/w^{alg}}(y)) = r_k(x).
\end{equation}

Proof. If $x$ is of type (1), then for any $y \in \pi^{-1}_{k^{alg}/k}(x)$, the point $\sigma_{\Omega/w^{alg}}(y)$ is of type (1). Hence, we obtain $r_{\Omega}(\sigma_{\Omega/w^{alg}}(y)) = r_k(x) = 0$.

If $x$ is of type (2) or (3), then any $y \in \pi^{-1}_{k^{alg}/k}(x)$ is of the form $x_{c,r_k(x)}$, where $c \in k^{alg}$. Since the morphism $\mathcal{O}(D^+_{k^{alg}}(c, r_k(x))) \to \mathcal{H}(y)$ is isometric, then so is

\begin{equation}
\mathcal{O}(D^+_{k^{alg}}(c, r_k(x))) \to \mathcal{H}(y) \hat{\otimes}_{k^{alg}} \Omega.
\end{equation}

Therefore, we have $\sigma_{\Omega/w^{alg}}(y) = x_{c,r_k(x)}$ in $\mathbb{A}^1_{\Omega^{alg}}$. Hence $r_{\Omega}(\sigma_{\Omega/w^{alg}}(y)) = r_k(x)$.

Now suppose that $x$ is a point of type (4), then for any $y \in \pi^{-1}_{k^{alg}/k}(x)$ there exists a family of nested disks $\mathcal{E}$ indexed by $(I, \leq)$ such that $\bigcap_{i \in I} D^+_{k^{alg}}(c_i, r_i) = \{y\}$. Note that we have $r_k(x) = r_k(y) = \inf_{i \in I} r_i$. Then we have:

\begin{equation}
\pi^{-1}_{\Omega/k^{alg}}(y) = \bigcap_{i \in I} D^+_{\Omega}(c_i, r_i).
\end{equation}

We distinguish two cases: the first is $\bigcap_{i \in I} D^+_{\Omega}(c_i, r_i) = \{\sigma_{k^{alg}/\Omega}(y)\}$. Then, we have:

\begin{equation}
r(\sigma_{\Omega/w^{alg}}(y)) = \inf_{i \in I} r_i = r_{k^{alg}}(y) = r_k(x).
\end{equation}
The second is \( \bigcap_{i \in I} D_{\Omega}^+(c, r_i) = D_{\Omega}^+(c, r_{k/\tilde{k}}(y)) \), where \( c \in \Omega \setminus \tilde{k} \). Here, \( \sigma_{\Omega/k} \) coincides with the Shilov boundary of \( D_{\Omega}^+(c, r_{k/\tilde{k}}(y)) \). Therefore we have
\[
(2.13) \quad r(\sigma_{\Omega/k}(y)) = r_{k/\tilde{k}}(y) = r_k(x). 
\]

2.14. Sheaf of differential forms and étale morphisms. Here we do not give the general definition of sheaf of differential forms given in [Ber93, §1.4], but only how it looks like in the case of an analytic domain of \( \mathbb{A}^1_{k/\tilde{k}} \). Let \( X \) be an analytic domain of \( \mathbb{A}^1_{k/\tilde{k}} \). Let \( T \) be the global coordinate function on \( \mathbb{A}^1_{k/\tilde{k}} \) fixed above. It induces a global coordinate function \( T \) on \( X \). The sheaf of differential forms \( \Omega_{X/k} \) of \( X \) is free with \( dT \) as a basis.

Let \( \frac{d}{dT} : \mathcal{O}_X \to \mathcal{O}_X \) be the formal derivation with respect to \( T \). In this setting the canonical derivation \( d_{X/k} \) satisfies:
\[
(2.14) \quad d_{X/k} : \mathcal{O}_X(U) \to \Omega_{X/k}(U), \quad f \mapsto \frac{dT}{\sigma_T}(f) \cdot dT, 
\]
where \( U \) is an open subset of \( X \).

**Lemma 2.13.** Let \( X \) and \( Y \) be two connected analytic domain of \( \mathbb{A}^1_{k/\tilde{k}} \) and let \( T \) (resp. \( S \)) be a coordinate function defined on \( X \) (resp. \( Y \)). Let \( \varphi : Y \to X \) be a finite morphism of \( k \)-analytic spaces and let \( \varphi^\# : \varphi^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y \) be the induced sheaves morphism. If \( \varphi \) is étale then for each analytic sub domain \( Y' \) of \( Y \)
\[
(2.15) \quad \varphi^*(\Omega_{X/k}\)(Y')) = (\varphi^{-1}(\Omega_{X/k}\)(Y')) \otimes_{\varphi^{-1}(\mathcal{O}_X\)(Y'))} \mathcal{O}_Y(Y') \to \Omega_{Y/k}(Y), \quad h.dT \otimes g \mapsto \frac{\partial}{\partial S}(\varphi^\#(T)) \varphi^\#(h) dS
\]
is an isomorphism of \( \mathcal{O}_Y(Y') \)-modules (resp. \( \mathcal{O}_Y(Y') \)-Banach module if \( Y' \) is an affinoid domain). If moreover \( X \) and \( Y \) are smooth, if the morphisms \( (2.15) \) are isomorphisms then \( \varphi \) is étale.

**Proof.** See [Ber93, Proposition 3.5.3].

**Remark 2.14.** Note that, any morphism \( \varphi : Y \to X \) between two connected open analytic domains of \( \mathbb{A}^1_{k/\tilde{k}} \) is obtained by a convenient choice of an element \( f \) of \( \mathcal{O}_Y(Y) \), which is the image of \( T \) by \( \varphi^\# \). In this setting, assume that \( \varphi \) is finite, then \( \varphi \) is étale if and only if \( \frac{\partial}{\partial S}(f) \) is invertible in \( \mathcal{O}_U(Y) \).

**Corollary 2.15.** Let \( \varphi : Y \to X \) be a finite morphism between connected open analytic domains of \( \mathbb{A}^1_{k/\tilde{k}} \). If \( \text{char}(k) = 0 \), then for each \( x \in X \) of type (2), (3) or (4) there exists an affinoid neighbourhood \( U \) of \( x \) in \( X \) such that \( \varphi^{-1}(U) : \varphi^{-1}(U) \to U \) is an étale morphism.

**Proof.** Let \( f \) be \( \varphi^\#(T) \). Since \( \text{char}(\tilde{k}) = 0 \), for each \( x \in X \) not of type (1) we have: \( \frac{\partial}{\partial S}(f)(x) = 0 \) if and only if \( f \in k \). Since \( \varphi \) is finite, \( f \notin k \). Hence, there exists an affinoid neighbourhood \( U \) of \( x \) such that \( \frac{\partial}{\partial S}(f) \) is invertible in \( \mathcal{O}_U \). The result follows by Remark 2.14.

**Lemma 2.16.** Let \( \varphi : Y \to X \) be a finite morphism between affinoid domains of \( \mathbb{A}^1_{k/\tilde{k}} \) and let \( T \) (resp. \( S \)) be a coordinate function defined on \( X \) (resp. \( Y \)). Let \( y \) be a point of type (2), (3) or (4). Then the induced extension \( \mathcal{H}(\varphi(y)) \hookrightarrow \mathcal{H}(y) \) is finite and we have
\[
(2.16) \quad \mathcal{H}(y) = \mathcal{H}(\varphi(y))(S(y)) \simeq \bigoplus_{i=0}^{n-1} \mathcal{H}(\varphi(y)) \cdot S(y)^i. 
\]

**Proof.** Since \( \varphi : Y \to X \) is finite, we have \( [\mathcal{H}(y) : \mathcal{H}(\varphi(y))] = n \) for some \( n \in \mathbb{N} \) (cf. [PP15, Lemma 2.24]). Therefore, \( S(y) \) is algebraic over \( \mathcal{H}(\varphi(y)) \). Hence, \( \mathcal{H}(\varphi(y))(S(y)) \) is a complete intermediate finite extension that contains \( k(S(y)) \). Because \( k(S(y)) \) is dense in \( \mathcal{H}(y) \), we obtain
\[
(2.17) \quad \mathcal{H}(y) = \mathcal{H}(\varphi(y))(S(y)) \simeq \bigoplus_{i=0}^{n-1} \mathcal{H}(\varphi(y)) \cdot S(y)^i. 
\]
Proposition 2.17. Assume that $k$ is algebraically closed. Let $\varphi : Y \to X$ be a finite étale cover between two analytic domains of $\mathbb{A}^1_{k,an}$. Let $y \in Y$ and $x = \varphi(y)$. Let $U$ be a neighbourhood of $y$ such that $U$ is a connected component of $\varphi^{-1}(\varphi(U))$. Then we have

$$[\mathcal{H}(y) : \mathcal{H}(x)] = \#U \cap \varphi^{-1}(b)$$

for each $b \in \varphi(U) \cap k$.

Proof. See [BR10, Corollary 9.17] and [Duc, (3.5.4.3)].

2.1.5. Differential equations over an affinoid domain. Let $X$ be an analytic domain of $\mathbb{A}^1_{k,an}$. A differential equation over $X$ is a locally free $\mathcal{O}_X$-module $\mathcal{F}$ of finite rank together with a connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \Omega_{X/k}$. Consider an isomorphism between $\mathcal{O}_X$ and $\Omega_{X/k}$ i.e. a morphism of $\mathcal{O}_X$-module of the form:

$$\Omega_{X/k}(U) \longrightarrow \mathcal{O}_X(U)$$

$$f \cdot dT \longmapsto g \cdot f,$$

where $U$ is an open subset of $X$ and $g$ is an invertible element of $\mathcal{O}_X(X)$. By composing with $d_{X/k}$ we obtain a $k$-linear derivation $d$ on $\mathcal{O}_X$, and we have $d = g \cdot dT$. For each $x \in X \setminus k$, $d$ extends to a $k$-linear bounded derivation on $\mathcal{H}(x)$. Moreover, by composing with the isomorphism (2.19) $(\mathcal{F}_x \otimes_{\mathcal{O}_X} \mathcal{H}(x), \nabla)$ can be seen as a differential module over $(\mathcal{H}(x), d)$.

2.1.6. Spectrum of a differential module. Let us recall the definition of the spectrum in the sense of Berkovich.

Definition 2.18. Let $E$ be a $k$-Banach algebra and $f \in E$. The spectrum of $f$ is the set $\Sigma_{f,k}(E)$ of points $x \in \mathbb{A}^1_{k,an}$ such that the element $f \otimes 1 - 1 \otimes T(x)$ is not invertible in the $k$-Banach algebra $E \hat{\otimes}_k \mathcal{H}(x)$.

Remark 2.19. If there is no confusion we denote the spectrum of $f$, as an element of $E$, just by $\Sigma_f$.

Remark 2.20. The set $\Sigma_f \cap k$ coincides with the classical spectrum, i.e.

$$\Sigma_f \cap k = \{ c \in k; f - c \text{ is not invertible in } E \}.$$ 

Let $\Omega \in E(k)$. As in our paper [Azz20], for each differential module $(M, \nabla)$ in $d - \text{Mod}(\Omega)$, we assign a spectrum in the sense of Berkovich in the following way: we can endow naturally $M$ by a structure of $\Omega$-Banach space, hence by a structure of $k$-Banach space, for which $\nabla : M \to M$ is a bounded, i.e. $\nabla$ is an element of $\mathcal{L}_k(M)$, $\mathcal{L}_k(M)$ is the $k$-Banach algebra of $k$-linear bounded operators with respect to operator norm. Then the spectrum of $(M, \nabla)$ is the spectrum of $\nabla$ as an element of $\mathcal{L}_k(M)$.\footnote{Since the $\Omega$-Banach structures on $M$ are equivalent, the associated spectrum is well defined.} This spectrum is a compact non-empty set, moreover the smallest closed disk centered at zero and containing $\Sigma_{\nabla,k}(\mathcal{L}_k(M))$ has radius equal to $\|\nabla\|_{\text{sp}} := \lim_{m \to +\infty} \|\nabla\|_{\text{op}}^m$ (c.f. [Ber90, Theorem 7.1.2]). Stressing also that this spectrum is invariant by bi-bounded isomorphisms of differential modules. However, it depends on the choice of the derivation. Indeed, for example the derivations $\frac{d}{dT}$ and $T \frac{d}{dT}$ defined over $\mathcal{H}(x_{0,r})$, both are associated to the trivial differential equation defined over some open neighborhood $X$ of $x_{0,r}$ (i.e $\mathcal{O}_X, d_{X/k}$), but we proved, in char($\tilde{k}$) = 0, that they have a different spectra (see [Azz20] and [Azz22]). In the next section we will see that, in some cases, two different derivations can be linked by an étale morphism, and so is for their spectra. Since the derivation $(T - c)\frac{d}{dT}$ with $(c \in k)$ has a good behavior under ramification of $(T - c)$, we will privilege the choice of these kind of derivations.

Let us recall some technical results from [Azz20], and another one that are very useful for the computation of the spectrum.
Lemma 2.21 ([Azz20, Lemma 2.20]). Let $E$ be a $k$-Banach algebra and $f \in E$. If $a \in \mathbb{A}^1_{k, \text{an}} \setminus \Sigma_f$, then the largest open disk centered at $a$ contained in $\mathbb{A}^1_{k, \text{an}} \setminus \Sigma_f$ has radius equal to $\|(f - a)^{-1}\|_{S^p}^{-1}$.

Lemma 2.22 ([Azz20, Lemma 2.29, Remark 2.30]). Let $(M, \varphi)$, $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$ be three $k$-Banach space endowed with a $k$-linear bounded operators, such that we have an exact sequence

$$0 \rightarrow M_1 \xrightarrow{i} M \xrightarrow{p} M_2 \rightarrow 0,$$

with $i \circ \varphi_1 = \varphi \circ i$ and $p \circ \varphi = \varphi_2 \circ p$. Then we have

$$\left( \Sigma_{\varphi_1,k}(L_k(M_1)) \cup \Sigma_{\varphi_2,k}(L_k(M_2)) \right) \setminus \left( \Sigma_{\varphi_1,k}(L_k(M_1)) \cap \Sigma_{\varphi_2,k}(L_k(M_2)) \right) \subset \Sigma_{\varphi,k}(L_k(M)).$$

Moreover, if $x \notin \Sigma_{\varphi,k}(L_k(M))$, then $\varphi_1 \otimes 1 - 1 \otimes T(x)$ is left invertible and $\varphi_2 \otimes 1 - 1 \otimes T(x)$ is right invertible.

Corollary 2.23. We keep the assumptions of Lemma 2.22. If in addition we have another exact sequence of the form:

$$0 \rightarrow M_2 \xrightarrow{i'} M \xrightarrow{p'} M_1 \rightarrow 0,$$

with $i' \circ \varphi_2 = \varphi \circ i'$ and $p' \circ \varphi = \varphi_1 \circ p$. Then we have $\Sigma_{\varphi,k}(L_k(M)) = \Sigma_{\varphi_1,k}(L_k(M_1)) \cup \Sigma_{\varphi_2,k}(L_k(M_2))$.

Remark 2.24. In particular, if $(M, \nabla) = (M_1, \nabla_1) \oplus (M_2, \nabla_2)$ as differential modules, then we have $\Sigma_\nabla = \Sigma_{\nabla_1} \cup \Sigma_{\nabla_2}$.

Lemma 2.25 ([Bou07, p.2]). Let $P(T) \in k[T]$. Let $E$ be a $k$-Banach algebra and let $f \in E$. Then we have:

$$\Sigma_{P(f),k}(E) = P(\Sigma_{f,k}(k)).$$

2.2. Push-forward and pull-back of a differential module and their spectra. Let $(\Omega, d)$ be a differential field and $(\Omega', d')$ be a finite differential extension of $(\Omega, d)$. Then we have two natural functors:

$$d - \text{Mod}(\Omega) \rightarrow d' - \text{Mod}(\Omega')$$

where $\nabla_{M \otimes \Omega'} := \nabla \otimes 1 + 1 \otimes d'$. Note that if $\dim \Omega M = n$, then so is $(M \otimes \Omega, \nabla_{M \otimes \Omega'})$.

$$d' - \text{Mod}(\Omega') \rightarrow d - \text{Mod}(\Omega)$$

where $M_\Omega$ is the restriction of scalars of $M$ via $\Omega \hookrightarrow \Omega'$, and $\nabla_\Omega = \nabla$ as $k$-linear maps. If $[\Omega' : \Omega] = n'$ and $\dim \Omega' M = n$, then $\dim \Omega M_\Omega = n.n'$.

Notation 2.26. From now on we will fix $S$ to be a coordinate function of the analytic domain (of the affine line) where the linear differential equation is defined, and $T$ to be a coordinate function on $\mathbb{A}^1_{k, \text{an}}$ for the computation of the spectrum.

Let $Y$ and $X$ be two connected affinoid domains of $\mathbb{A}^1_{k, \text{an}}$ and $Z$ (resp. $S$) a coordinate function on $X$ (resp. $Y$). Let $\varphi : Y \rightarrow X$ be a finite étale morphism and $\varphi^\#: O_X \rightarrow \varphi_* O_Y$ be the induced sheaves morphism. Let $f := \varphi^#(Z)$ and $f' := \frac{df}{dS}$. Since $\varphi$ is étale, $f'$ is invertible in $O_Y(Y)$ (cf. Lemma 2.13). To any bounded derivation $d = \frac{d}{dt}$ on $X$ we assign the bounded derivation on $Y$

$$\varphi^* d := \frac{\varphi^#(g)}{f'} \frac{d}{dS}$$

called the pull-back of $d$ by $\varphi$. 

Let $y \in Y$ and $x = \varphi(y)$. Since the derivation $\frac{d}{dy}$ (resp. $\frac{d}{dx}$) extends to a bounded derivation on $Y$ (resp. $X$), the derivation $d$ (resp. $\varphi^*d$) extends to a bounded derivation on $Y$ (resp. $X$). We have the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H}(x) & \longrightarrow & \mathcal{H}(y) \\
\downarrow d & & \downarrow \varphi^*d \\
\mathcal{H}(x) & \longrightarrow & \mathcal{H}(y)
\end{array}
\]

Since $\varphi^*$ induces a finite extension $\mathcal{H}(x) \hookrightarrow \mathcal{H}(y)$, we have the push-forward functor by $\varphi$ defined as in (2.27):

\[
\varphi_* : \varphi^*d - \text{Mod}(\mathcal{H}(y)) \longrightarrow d - \text{Mod}(\mathcal{H}(x))
\]

and the pull-back functor by $\varphi$ defined as in (2.26):

\[
\varphi^* : d - \text{Mod}(\mathcal{H}(x)) \longrightarrow \varphi^*d - \text{Mod}(\mathcal{H}(y))
\]

**Remark 2.27.** In Sections 3.1 and 5.2, we will describe the above functors more explicitly.

**Proposition 2.28.** We have the set-theoretic equality:

\[
\Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \Sigma_{\nabla, \varphi, k}(\mathcal{L}_k(\varphi_* M)).
\]

**Proof.** Since $M$ and $\varphi_* M$ are the same as $\mathcal{H}(x)$-Banach spaces, then they are isomorphic as $k$-Banach spaces. As $\nabla$ and $\varphi_* \nabla$ coincide as $k$-linear maps, the equality of spectra holds. \(\square\)

### 3. Spectrum of a differential module with regular singularities

**Convention 3.1.** We assume that $k$ is algebraically closed.

**Notation 3.2.** We set

\[
\omega := \lim_{n \to +\infty} |n|!^{\frac{1}{p}} = \begin{cases} |p|^{\frac{n-1}{p-1}} & \text{char}(\bar{k}) = p > 0 \\ 1 & \text{char}(\bar{k}) = 0 \end{cases}
\]

In this part we compute the spectrum of a differential module $(M, \nabla)$ over $(\mathcal{H}(x), S_{\frac{d}{\text{diff}}})$ with regular singularities, defined as below, for $x \in \mathbb{A}^{\text{lan}}_k \setminus k$. We will observe that if we fix a differential equation $(\mathcal{F}, \nabla)$ over $\mathbb{A}^{\text{lan}}_k \setminus \{0\}$ with regular singularities (we will explain the meaning later), we observe that the spectrum of $(\mathcal{F} \otimes \mathcal{H}(x), \nabla)$ (as differential module over $(\mathcal{H}(x), S_{\frac{d}{\text{diff}}})$) has an interesting behavior when we vary $x$.

**Definition 3.3.** A differential module $(M, \nabla)$ over $(\mathcal{H}(x), S_{\frac{d}{\text{diff}}})$ is said to be regular singular, if there exists a basis for which the associated matrix $G$, i.e.

\[
\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} S_{\frac{d}{\text{diff}}}f_1 \\ \vdots \\ S_{\frac{d}{\text{diff}}}f_n \end{pmatrix} + G \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},
\]

has constant entries (i.e $G \in \mathcal{M}_n(k)$).

**Remark 3.4.** As explained in [Azz20, Proposition 3.15], given a differential module over $(\Omega, d)$, with $\Omega \in E(k)$, such that:

\[
\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} + G \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},
\]
with $G \in \mathcal{M}_n(k)$ for some basis of $M$, then the computation of the spectrum of $\nabla$ is reduced to the computation of the spectrum of $d$. Indeed, the spectrum of $\nabla$ is $\Sigma_\nabla = \bigcup_{i=1}^{N} (a_i + \Sigma_d)$, where $\{a_1, \cdots, a_N\}$ are the eigenvalues of $G$.

### 3.1. Spectrum of the derivation $S \frac{d}{dS}$.

**Lemma 3.5.** Let $x = x_{0,r}$, with $r > 0$. The norm and spectral semi-norm of $S \frac{d}{dS}$ as an element of $\mathcal{L}_k(\mathcal{H}(x))$ satisfy:

$$
\|S \frac{d}{dS}\| = 1, \quad \|S \frac{d}{dS}\|_{Sp} = 1.
$$

**Proof.** Since $\|S\| = |S| = r$ and $\|\frac{d}{dS}\| = \frac{1}{r}$ (cf. [Pul15, Lemma 4.4.1]), we have $\|S \frac{d}{dS}\| \leq 1$. Hence also, $\|S \frac{d}{dS}\|_{Sp} \leq 1$. The map

$$
\mathcal{L}_k(\mathcal{H}(x)) \rightarrow \mathcal{L}_k(\mathcal{H}(x))
$$

is bi-bounded and induces a change of basis. Therefore, as $S^{-1} \circ (S \frac{d}{dS}) \circ S = S \frac{d}{dS} + 1$, we have $\|S \frac{d}{dS}\|_{Sp} = \|S \frac{d}{dS} + 1\|_{Sp}$. Since $1$ commutes with $S \frac{d}{dS}$, we have:

$$
1 = \|S \frac{d}{dS}\| + 1 - S \frac{d}{dS} \|_{Sp} \leq \max(\|S \frac{d}{dS}\| + 1, \|S \frac{d}{dS}\|_{Sp}) = 1.
$$

Consequently, we obtain

$$
\|S \frac{d}{dS}\| = \|S \frac{d}{dS}\|_{Sp} = 1.
$$

3.1.1. **The case of positive characteristic.** We assume here that $\text{char}(\bar{k}) = p > 0$.

**The case where** $x \in (0, \infty)$. We start with the case where $x = x_{0,r}$. In order to determine the spectrum of $S \frac{d}{dS}$, we will use the push-forward by the Frobenius map. We refer the reader to Section 6.2.2 for the definition of Frobenius map $\text{Frob}_p : \mathbb{A}_k^{1,\text{an}} \to \mathbb{A}_k^{1,\text{an}}$ and its properties. Since it induces an étale map $(\text{Frob}_p)^n : \mathbb{A}_k^{1,\text{an}} \to \mathbb{A}_k^{1,\text{an}}$ and with $r > 0$, the push-forward functor (cf. (2.30)) is well defined for any $x_{0,r} \in \mathbb{A}_k^{1,\text{an}}$ with $r > 0$. Recall that $(\text{Frob}_p)^n(x_{0,r}) = x_{0,r^n}$ and $[\mathcal{H}(x_{0,r^n}) : \mathcal{H}(x_{0,r})] = p^n$ (cf. Properties 6.13).

Let $x = x_{0,r}$ and $y := (\text{Frob}_p)^n(x)$. According to formula (2.28) the pull-back of the derivation $p^nS \frac{d}{dS} : \mathcal{H}(y) \to \mathcal{H}(y)$ is the derivation $S \frac{d}{dS} : \mathcal{H}(x) \to \mathcal{H}(x)$. To avoid confusion in the following, we set $p^nS(y) \frac{d}{dS(y)} := p^nS \frac{d}{dS}$ and $S(x) \frac{d}{dS(x)} := S \frac{d}{dS}$.

Now let $(M_{pn}, \nabla_{pn})$ be the push-forward of the differential module $(\mathcal{H}(x), S \frac{d}{dS})$ by $(\text{Frob}_p)^n$. Since $M_{pn} \simeq \mathcal{H}(x)_{p^n}$ as an $\mathcal{H}(y)$-Banach space, and according to Lemma 2.16, we can take $\{1, S(x), \cdots, S(x)^{p^n-1}\}$ as a basis of $(M_{pn}, \nabla_{pn})$. Since

$$
\nabla_{pn}(S(x)^i) = S(x) \frac{d}{dS(x)}(S(x)^i) = iS(x)^i,
$$

in this basis we have:

$$
\nabla_{pn} = \begin{pmatrix} f_1 \\ \vdots \\ f_{p^n} \end{pmatrix} = \begin{pmatrix} p^nS(y) \frac{d}{dS(y)}f_1 \\ \vdots \\ p^nS(y) \frac{d}{dS(y)}f_{p^n} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p^n - 1 \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_{p^n} \end{pmatrix}.
$$
On the other hand, we have an isomorphism of differential modules
\[
(M_{p^n}, \nabla_{p^n}) \simeq \bigoplus_{i=0}^{p^n-1} (\mathcal{H}(y), p^n S(y) \frac{d}{dS(y)} + i).
\]

**Notation 3.6.** For simplicity, here we still denote \(S(x) \frac{d}{dS(x)}\) by \(S \frac{d}{ds}\).

**Proposition 3.7.** Let \(r > 0\). Let \(x \in x_{0,r}\). The spectrum of \(S \frac{d}{ds}\) as an element of \(L_k(\mathcal{H}(x))\) is
\[
\Sigma_{S \frac{d}{ds}, k}(L_k(\mathcal{H}(x))) = \mathbb{Z}_p.
\]

**Proof.** Since for all \(l \in \mathbb{N}\) we have \(S \frac{d}{ds}(S(x)^l) - l(S(x)^l) = 0\), we obtain
\[
\mathbb{N} \subset \Sigma_{S \frac{d}{ds}, k}(L_k(\mathcal{H}(x))).
\]
By compactness of the spectrum, we obtain
\[
\mathbb{Z}_p \subset \Sigma_{S \frac{d}{ds}, k}(L_k(\mathcal{H}(x))).
\]

Let \(n \in \mathbb{N} \setminus \{0\}\). We set \(y := (\text{Frob}_p)^n(x) = x_{0.p^n}\). Let \((M_{p^n}, \nabla_{p^n})\) be the push-forward of \((\mathcal{H}(x), S \frac{d}{ds})\) by \((\text{Frob}_p)^n\). On the one hand, according to Proposition 2.28 we have
\[
\Sigma_{S \frac{d}{ds}, k}(L_k(\mathcal{H}(x))) = \Sigma_{\nabla_{p^n}, k}(L_k(M_{p^n})).
\]

On the other hand, since we have the isomorphism (3.10), we have
\[
\Sigma_{\nabla_{p^n}, k}(L_k(M_{p^n})) = \bigcup_{i=0}^{p^n-1} \Sigma_{p^n S(y) \frac{d}{dS(y)} + i, k}(L_k(\mathcal{H}(y))) = \bigcup_{i=0}^{p^n-1} p^n \Sigma_{S \frac{d}{ds}, k}(L_k(\mathcal{H}(y))) + i
\]
(cf. Remark 2.24 and Lemma 2.25). By Lemma 3.5, we know that \(\|S \frac{d}{ds}\|_{S_p} = 1\) in \(L_k(\mathcal{H}(y))\). Therefore, we have \(\Sigma_{S \frac{d}{ds}, k}(L_k(\mathcal{H}(y))) \subset D^+(0,1)\) (cf. [Ber90, Theorem 7.1.2]). Consequently, \(\Sigma_{\nabla_{p^n}, k}(L_k(M_{p^n})) \subset \bigcup_{i=0}^{p^n-1} D^+(i, |p|^n)\). Applying this process for all \(n\), we obtain
\[
\Sigma_{S \frac{d}{ds}, k}(L_k(\mathcal{H}(x))) \subset \bigcap_{n \in \mathbb{N} \setminus \{0\}} \bigcup_{i=0}^{p^n-1} D^+(i, |p|^n) = \mathbb{Z}_p,
\]
which ends the proof. \(\square\)

**The case where** \(x \notin (0, \infty)\). We now assume that \(x \in \mathbb{A}_k^{1,\text{an}}\) is a point of type (2), (3) or (4) not of the form \(x_{0,r}\). Then there exists \(c \in k \setminus \{0\}\) such that \(x \in D^-(c, |c|)\). The logarithm map (cf. Section 6.2.1)
\[
\text{Log}_c : D^-(c, |c|) \to \mathbb{A}_k^{1,\text{an}}
\]
is well defined and induces an infinite étale cover. Let \(y = \text{Log}_c(x)\). Let \(r_k : \mathbb{A}_k^{1,\text{an}} \to \mathbb{R}_+\) be the radius map (cf. Definition 2.3) and \(\omega = |p|^{-1/r_k}\). We have for \(n \in \mathbb{N} \setminus \{0\}\)
\bullet if \(0 < r_k(x) < |c|\omega\), then \(0 < r_k(y) < \omega\) and \([\mathcal{H}(x) : \mathcal{H}(y)] = 1,\)
\bullet if \(|c|\omega^{p^{-n-1}} \leq r_k(x) < |c|\omega^{-1}\), then \(|c|\omega^{p^{-n-1}} \leq r_k(y) < \omega^{-1}\) and \([\mathcal{H}(x) : \mathcal{H}(y)] = p^n\)
(cf. Properties 6.11). Note that, since \(|T(x)| = |c|\) the inequalities above do not depend on the choice of \(c\). According to Formula (2.28) the pull-back of the derivation \(S \frac{d}{ds} : \mathcal{H}(y) \to \mathcal{H}(y)\) is the derivation \(S \frac{d}{ds} : \mathcal{H}(x) \to \mathcal{H}(x)\).

Let \((M, \nabla)\) be the push-forward of the differential module \((\mathcal{H}(x), S \frac{d}{ds})\) by \(\text{Log}_c\). Assume that \(|c|\omega^{p^{-1}} \leq r_k(x) < |c|\omega^{-1}\). By Lemma 2.16, we can take \(\{1, S(x), \cdots, S(x)^{p^n-1}\}\) as a basis of \((M, \nabla)\). Since
\[
\nabla(S(x)^i) = S \frac{d}{ds}(S(x)^i) = iS(x)^i,
\]
in this basis we have:
\( \nabla \begin{pmatrix} f_1 \\ \vdots \\ f_{p^n} \end{pmatrix} = \begin{pmatrix} \frac{d}{ds} f_1 \\ \vdots \\ \frac{d}{ds} f_{p^n} \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p^n - 1 \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_{p^n} \end{pmatrix}. \)

Hence, \((M, \nabla)\) is a differential module with constant coefficient in the sense of [Azz20]. Let us recall the result concerning differential modules with constant coefficients:

**Theorem 3.8** ([Azz20, Theorem 4.14]). Let \( x \in \mathbb{R}^{1,\text{an}} \) be a point of type (2), (3) or (4). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), \frac{d}{ds})\) such that there exists a basis for which the associated matrix \( G \) has constant entries (i.e. \( G \in \mathcal{M}_p(k) \)), and let \( \{a_1, \cdots, a_N\} \) be the set of eigenvalues of \( G \). Then we have:

- if \( x \) is a point of type (2) or (3), then

\[
\Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \bigcup_{i=1}^{N} D^+(a_i, \frac{\omega}{r(x)}).
\]

- if \( x \) is a point of type (4), then

\[
\Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \begin{cases} \bigcup_{i=1}^{N} D^+(a_i, \frac{\omega}{r(x)}) & \text{if \( \text{char}(\bar{k}) > 0 \)} \\
\bigcup_{i=1}^{N} D^-(a_i, \frac{1}{r(x)}) & \text{if \( \text{char}(\bar{k}) = 0 \)} \end{cases}
\]

**Proposition 3.9.** Let \( x \in \mathbb{R}^{1,\text{an}} \) be a point of type (2), (3) or (4) not of the form \( x_{0,r} \). Let \( c \in k \setminus \{0\} \) such that \( x \in D^-(c, |c|) \). Let \( y = \log_c(x) \). If \( r_k(x) \leq |c|\omega \), then the spectrum of \( S_{\frac{d}{ds}, k}(\mathcal{H}(x)) \), as an element of \( \mathcal{L}_k(\mathcal{H}(x)) \), is

\[
\Sigma_{S_{\frac{d}{ds}, k}}(\mathcal{L}_k(\mathcal{H}(x))) = D^+(0, \frac{\omega}{r_k(y)}).
\]

If \( |c|\omega^{\frac{1}{p^n-1}} < r_k(x) \leq |c|\omega^{\frac{1}{p^n}} \) with \( n \in \mathbb{N} \setminus \{0\} \), then the spectrum is a disjoint union of \( p^n \) closed disks

\[
\Sigma_{S_{\frac{d}{ds}, k}}(\mathcal{L}_k(\mathcal{H}(x))) = \bigcup_{i=0}^{p^n-1} D^+(i, \frac{\omega}{r_k(y)}).
\]

**Proof.** Assume that \( r_k(x) < |c|\omega \). Since \( [\mathcal{H}(x) : \mathcal{H}(y)] = 1 \), the push-forward of \((\mathcal{H}(x), S_{\frac{d}{ds}})\) by \( \log_c \) is isomorphic to \((\mathcal{H}(y), \frac{d}{ds})\). Therefore, by Theorem 3.8 and Proposition 2.28 we obtain

\[
\Sigma_{S_{\frac{d}{ds}, k}}(\mathcal{L}_k(\mathcal{H}(x))) = \Sigma_{\frac{d}{ds}, k}(\mathcal{L}_k(\mathcal{H}(y))) = D^+(0, \frac{\omega}{r_k(y)}).
\]

We now assume that \( |c|\omega^{\frac{1}{p^n-1}} \leq r_k(x) < |c|\omega^{\frac{1}{p^n}} \) with \( n \in \mathbb{N} \setminus \{0\} \). Let \((M, \nabla)\) be the push-forward of \((\mathcal{H}(x), S_{\frac{d}{ds}})\) by \( \log_c \). Since we have the formula (3.19) and according to Propositions 2.28 and Theorem 3.8, we have

\[
\Sigma_{S_{\frac{d}{ds}, k}}(\mathcal{L}_k(\mathcal{H}(x))) = \Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \bigcup_{i=0}^{p^n-1} D^+(i, \frac{\omega}{r_k(y)}).
\]

If moreover \( |c|\omega^{\frac{1}{p^n-1}} < r_k(x) \), then \( |p|^n < \frac{\omega}{r_k(y)} < |p|^{n-1} \). Consequently, the spectrum \( \Sigma_{S_{\frac{d}{ds}, k}}(\mathcal{L}_k(\mathcal{H}(x))) \) is a disjoint union of \( p^n \) disks. For the case where \( r_k(x) = |c|\omega^{\frac{1}{p^n}} \) with \( n \in \mathbb{N} \), we have \( r_k(y) = \frac{\omega}{|p|^n} \) (cf.
Properties 6.11). For all $0 \leq i \leq p^n - 1$ and $1 \leq l \leq p - 1$, we have $D^+(i, |p|^n) = D^+(i + lp^i, |p|^n)$. Hence, we obtain

$$\sum_{\mathbb{A}_{\mathfrak{d}}^{\frac{1}{\mathfrak{d}}} k} (L_k(\mathcal{H}(x))) = \bigcup_{i=0}^{p^n-1} D^+(i, |p|^n) = \bigcup_{i=0}^{p^n-1} D^+(i, |p|^n),$$

which is obviously a disjoint union.

**Corollary 3.10.** Let $x \in \mathbb{A}_{\mathfrak{d}}^{\frac{1}{\mathfrak{d}}} k$ be a point of type (2), (3) or (4) not of the form $x_0$, $c$. Let $e \in k \setminus \{0\}$ such that $x \in D^-(c, |e|)$. Let $y$ be Log$_\mathfrak{c}(x)$. The spectrum of $S^{d}_{\mathfrak{d},d}^{e}$, as an element of $L_k(\mathcal{H}(x))$, is

$$\sum_{\mathbb{A}_{\mathfrak{d}}^{\frac{1}{\mathfrak{d}}} k} (L_k(\mathcal{H}(x))) = \bigcup_{i \in \mathbb{N}} D^+(i, \frac{\omega}{r_k(y)}).$$

**Proof.** By Proposition 3.9, we have $\sum_{\mathbb{A}_{\mathfrak{d}}^{\frac{1}{\mathfrak{d}}} k} (L_k(\mathcal{H}(x))) = \bigcup_{i=0}^{p^n-1} D^+(i, \frac{\omega}{r_k(y)})$ for some $n \in \mathbb{N}$. Since for all $l \in \mathbb{N}$ we have $S_{\mathfrak{d},d}^{e}(S(x)^l) - l(S(x)^l) = 0$, we obtain

$$\mathbb{N} \subset \sum_{\mathbb{A}_{\mathfrak{d}}^{\frac{1}{\mathfrak{d}}} k} (L_k(\mathcal{H}(x))).$$

Therefore, for each $l \in \mathbb{N}$ there exists $0 \leq i_l \leq p^n - 1$ such that $D^+(l, \frac{\omega}{r_k(y)}) = D^+(i_l, \frac{\omega}{r_k(y)})$. Consequently, we obtain $\bigcup_{i \in \mathbb{N}} D^+(i, \frac{\omega}{r_k(y)}) \subset \sum_{\mathbb{A}_{\mathfrak{d}}^{\frac{1}{\mathfrak{d}}} k} (L_k(\mathcal{H}(x)))$ which ends the proof. $\square$

**3.1.2. The case of residue characteristic zero.** We assume here that char($\mathfrak{k}$) = 0.

**The case where $x \in (0, \infty)$.**

**Proposition 3.11.** Let $x \in \mathbb{A}_{\mathfrak{d}}^{\frac{1}{\mathfrak{d}}} k$ be a point of type (2) of the form $x_0$, $c$. The spectrum of $S_{\mathfrak{d},d}^{e}$, as an element of $L_k(\mathcal{H}(x))$, is

$$\sum_{\mathbb{A}_{\mathfrak{d}}^{\frac{1}{\mathfrak{d}}} k} (L_k(\mathcal{H}(x))) = D^+(0, 1).$$

**Proof.** We set $d := S_{\mathfrak{d},d}^{e}$ and $\Sigma_d = \sum_{\mathbb{A}_{\mathfrak{d}}^{\frac{1}{\mathfrak{d}}} k} (L_k(\mathcal{H}(x)))$. Since $d(\mathfrak{d}) = 1$, we have $\Sigma_d \subset D^+(0, 1)$. Let $y \in D^+(0, 1)$. We set $A_{\mathcal{H}(y)} = \mathcal{H}(x) \otimes k \mathcal{H}(y)$ and $d_{\mathcal{H}(y)} = S_{\mathfrak{d},d}^{e} : A_{\mathcal{H}(y)} \rightarrow A_{\mathcal{H}(y)}$. From [Azz20, Lemma 2.5] we have a bounded morphism:

$$L_k(\mathcal{H}(x)) \otimes k \mathcal{H}(y) \rightarrow L_{\mathcal{H}(y)}(A_{\mathcal{H}(y)}).$$

The image of $d \otimes 1$ by this morphism is the derivation $d_{\mathcal{H}(y)}$. We now show that the image of $d \otimes 1 - 1 \otimes T(y)$ is not invertible in $L_{\mathcal{H}(y)}(A_{\mathcal{H}(y)})$. Let $\alpha$ be an element of $k$ that corresponds to the class $\alpha$ in $\mathfrak{k}$. We have the following decomposition

$$\mathcal{H}(x) = \bigoplus_{\mathfrak{d} \in \mathfrak{k}} \bigoplus_{i \in \mathbb{N}} \frac{a_{\mathfrak{d}i}}{(S + \gamma \alpha)^i} \left| a_{\mathfrak{d}i} \in k, \lim_{i \rightarrow +\infty} |a_{\mathfrak{d}i}|r^{-i} = 0 \right) \oplus O(D^+(0, r))$$

with $\gamma \in k$ and $|\gamma| = r$ (cf. [Chr83, Theorem 2.1.6], [Azz20, Proposition 2.10]). Therefore, we obtain the isometric isomorphism

$$\mathcal{H}(x) \otimes k \mathcal{H}(y) \simeq \bigoplus_{\mathfrak{d} \in \mathfrak{k}} \bigoplus_{i \in \mathbb{N}} \frac{a_{\mathfrak{d}i}}{(S + \gamma \alpha)^i} \left| a_{\mathfrak{d}i} \in \mathcal{H}(y), \lim_{i \rightarrow +\infty} |a_{\mathfrak{d}i}|r^{-i} = 0 \right) \oplus O(D^+(y)(0, r)).$$

Each Banach space of the completed direct sum is stable under $d_{\mathcal{H}(y)} - T(y)$. The operator $d_{\mathcal{H}(y)} - T(y)$ is not surjective. Indeed, let $c := \gamma \alpha$ with $\alpha \in k \setminus \{0\}$ and let $g = \frac{1}{S - c}$. If there exists $f \in \mathcal{H}(x) \otimes k \mathcal{H}(y)$ such that $(d_{\mathcal{H}(y)} - T(y))(f) = g$, then we can choose $f$ of the form $f = \sum_{i \in \mathbb{N} \setminus \{0\}} \frac{a_{\mathfrak{d}i}}{(S - \gamma)^i}$, such that for each $i \in \mathbb{N} \setminus \{0\}$ we have

$$a_{i} = \frac{(-c)^{i}(i - 1)!}{\prod_{j=1}^{i} (T(y) + j)}.$$
We observe that $|a_i|r^i \geq 1$ for each $i \in \mathbb{N} \setminus \{0\}$. This means that such $f$ does not exist in $\mathcal{H}(x) \otimes_k \mathcal{H}(y)$. Hence, $d \otimes 1 - 1 \otimes T(y)$ is not invertible in $\mathcal{L}_k(\mathcal{H}(x))$ and we conclude that $D^+(0,1) \subset \Sigma_d$. □

The proof of the following proposition is almost similar to [Azz22, Proposition 3.7], but it is in a more general context.

**Proposition 3.12.** Let $x \in \mathbb{A}_k^{1,an}$ be a point of type (3) of the form $x_{0,r}$. The spectrum of $S_d^{\frac{d}{ds}}$ as an element of $\mathcal{L}_k(\mathcal{H}(x))$ is

$$\Sigma_{S_d^{\frac{d}{ds}}}(\mathcal{L}_k(\mathcal{H}(x))) = \mathbb{Z} \cup \{x_{0,1}\}. \tag{3.34}$$

**Proof.** We set $d := S_d^{\frac{d}{ds}}$ and $\Sigma_{d-n} := \Sigma_{d-n,k}(\mathcal{L}_k(\mathcal{H}(x)))$. As $\|d\|_{Sp} = 1$ (cf. Lemma 3.5), we have $\Sigma_d \subset D^+(0,1)$. Recall that

$$\mathcal{H}(x) = \mathcal{O}(C^+(0,r,r)) = \{\sum_i a_i S_i^i; \lim_{|i| \to \infty} |a_i|^r = 0\}. \tag{3.35}$$

Let $a \in k \cap D^+(0,1)$. If $a \in \mathbb{Z}$, then we have $(d-a)(S^a) = 0$. Hence, $(d-a)$ is not injective and $\mathbb{Z} \subset \Sigma_d$. As the spectrum is compact, we have $\mathbb{Z} \cup \{x_{0,1}\} \subset \Sigma_d$. If $a \not\in \mathbb{Z}$, then $(d-a)$ is invertible in $\mathcal{L}_k(\mathcal{H}(x))$. Indeed, let $g(S) = \sum_{i \in \mathbb{Z}} b_i S^i \in \mathcal{H}(x)$, if there exists $f = \sum_{i \in \mathbb{Z}} a_i S^i \in \mathcal{H}(x)$ such that $(d-a)f = g$, then for each $i \in \mathbb{Z}$ we have

$$a_i = \frac{b_i}{(i-a)}. \tag{3.36}$$

If there exists $i_0 \in \mathbb{Z}$ such that $a \in D^-(i_0,1)$, then for each $i \not= i_0$ we have $|a_i| = |b_i|$ and $|a_{i_0}| = \frac{|b_{i_0}|}{|i_0-a|}$. Otherwise, for each $i \in \mathbb{Z}$ we have $|a_i| = |b_i|$. This means that $f$ is unique and converges in $\mathcal{H}(x)$. We obtain also $|f| \leq \frac{|g|}{|i_0-a|}$ or $|f| = |g|$. Consequently, the set theoretical inverse $(d-a)^{-1}$ is bounded. We claim that if $a \in D^-(i_0,1)$ then $\|(d-a)^{-1}\|_{Sp} = \frac{1}{|i_0-a|}$, otherwise $\|(d-a)^{-1}\|_{Sp} = 1$. Indeed, in the first case, similar computations show that $\|(d-a)^{-n}\| \leq \frac{1}{(|i_0-a|)^n}$. Since $(d-a)^{-n}(S^0) = \frac{S^0}{(|i_0-a|)^n}$, the equality holds and we obtain $\|(d-a)^{-1}\|_{Sp} = \frac{1}{|i_0-a|}$. In the second case, by the above computations $(d-a)^{-1}$ is an isometry. Therefore, we have $\|(d-a)^{-1}\|_{Sp} = 1$. Setting $R_a := \inf_{i \in \mathbb{Z}} |i-a|$, we have $\|(d-a)^{-1}\|_{Sp} = \frac{1}{R_a}$. According to Lemma 2.21, we have $D^-(a, R_a) \subset \mathbb{A}_k^{1,an} \setminus \Sigma_d$.

In order to end the proof, since $D^+(0,1) = \bigcup_{a \in k \setminus \mathbb{Z}} D^-(a, R_a) \cup \bigcup_{n \in \mathbb{Z}, [n, x_{0,1}]}$, it is enough to show that $(n, x_{0,1}) \subset \mathbb{A}_k^{1,an} \setminus \Sigma_d$ for all $n \in \mathbb{Z}$. Let $n \in \mathbb{Z}$. Then we have

$$\mathcal{H}(x) = k.S^n \oplus \bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i. \tag{3.37}$$

The operator $(d-n)$ stabilizes both $k.S^n$ and $\bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i$. We set $\nabla_1 := (d-n)|_{k.S^n}$ and $\nabla_2 := (d-n)|_{\bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i}$. We set $\Sigma_{\nabla_1} := \Sigma_{\nabla_1,k}(\mathcal{L}_k(k.S^n))$ and $\Sigma_{\nabla_2} := \Sigma_{\nabla_2,k}(\mathcal{L}_k(\bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i))$. We have $\nabla_1 = 0$. By Lemma 2.22, we have:

$$\Sigma_{d-n} = \Sigma_{\nabla_1} \cup \Sigma_{\nabla_2} = \{0\} \cup \Sigma_{\nabla_2}. \tag{3.38}$$

Now we prove that

$$D^-(0,1) \cap \Sigma_{\nabla_2} = \emptyset. \tag{3.39}$$

The operator $\nabla_2$ is invertible in $\mathcal{L}_k(\bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i)$. Indeed, let $g(S) = \sum_{i \in \mathbb{Z} \setminus \{n\}} b_i S^i \in \bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i$. If there exists $f = \sum_{i \in \mathbb{Z} \setminus \{n\}} a_i S^i \in \bigoplus_{i \in \mathbb{Z} \setminus \{n\}} k.S^i$ such that $\nabla_2(f) = g$, then for each $i \in \mathbb{Z} \setminus \{n\}$ we have

$$a_i = \frac{b_i}{(i-n)}. \tag{3.40}$$
Since \(|a_i| = |b_i|\), the element \(f\) exists and it is unique, moreover \(|f| = |g|\). Hence, \(\nabla_2\) is invertible in \(\mathcal{L}_k(\bigoplus_{i \in \mathbb{Z}\setminus\{n\}} kS^i)\) and as a \(k\)-linear map it is isometric. Therefore, we have \(\|\nabla_2^{-1}\|_{sp} = 1\). Hence, \(D^-(0, 1) \subset \mathbb{A}_k^{1, \text{an}} \setminus \Sigma_{\nabla_2}\) by Lemma 2.21. Consequently, \(D^-(0, 1) \cap \Sigma_{d-n} = \{0\}\). As \(\Sigma_d = \Sigma_{d-n} + n\), we have \(D^-(n, 1) \cap \Sigma_d = \{n\}\). Therefore, for all \(n \in \mathbb{Z}\) we have \((n, x_{0,1}) \subset \mathbb{A}_k^{1, \text{an}} \setminus \Sigma_d\) and the claim follows. 

\[\Box\]

The case where \(x \not\in (0, \infty)\).

**Proposition 3.13.** Let \(x \in \mathbb{A}_k^{1, \text{an}}\) be a point of type (2), (3) or (4) not of the form \(x_{0,r}\). Let \(c \in k \setminus \{0\}\) such that \(x \in D^-(c, |c|)\). The spectrum of \(S_{\frac{d}{dS}}A\) as an element of \(\mathcal{L}_k(\mathcal{H}(x))\) is

\[
\Sigma_{S_{\frac{d}{dS}}A}(\mathcal{L}_k(\mathcal{H}(x))) = \begin{cases} 
D^-(0, \frac{|c|}{r_k(x)}) & \text{if } x \text{ is of type (4)} \\
D^+(0, \frac{|c|}{r_k(x)}) & \text{otherwise}
\end{cases}
\]

**Proof.** Let \(\text{Log}_c : D^-(c, |c|) \to D^-(0, 1)\) be the logarithm, we set \(y := \text{Log}_c(x)\). Since \(\text{char}(\tilde{k}) = 0\), \(\text{Log}_c\) is an analytic isomorphism and \([\mathcal{H}(x) : \mathcal{H}(y)] = 1\). Therefore, the push-forward of \(\mathcal{H}(x), S_{\frac{d}{dS}}(y)\) by \(\text{Log}_c\) is isomorphic to \(\mathcal{H}(y), S_{\frac{d}{dS}}(y)\). Therefore, by Propositions 3.8 and 2.28 we obtain

\[
\Sigma_{S_{\frac{d}{dS}}A}(\mathcal{L}_k(\mathcal{H}(x))) = \Sigma_{A}(\mathcal{L}_k(\mathcal{H}(y))) = \begin{cases} 
D^-(0, \frac{1}{r_k(y)}) & \text{if } x \text{ is of type (4)} \\
D^+(0, \frac{1}{r_k(y)}) & \text{otherwise}
\end{cases}
\]

Since \(r_k(y) = \frac{r_k(x)}{|c|}\), the result follows. 

\[\Box\]

### 3.2. Spectrum of a regular singular differential module.

As mentioned at the beginning of the section, the computation of the spectrum of a regular differential module follows directly from the computation of the spectrum done above and Remark 3.4. In this section, we will summarize all the different cases discussed in the previous section. We will also discuss the variation of the spectrum.

**Notation 3.14.** We denote by \(\overline{\mathbb{Z}}\) the topological closure of \(\mathbb{Z}\) in \(\mathbb{A}_k^{1, \text{an}}\).

**Theorem 3.15.** Assume that \(\text{char}(\tilde{k}) = p > 0\). Let \(x \in \mathbb{A}_k^{1, \text{an}}\) be a point of type (2), (3) or (4). Let \((M, \nabla)\) be a regular singular differential module over \((\mathcal{H}(x), S_{\frac{d}{dS}})\). Let \(G\) be the matrix associated to \(\nabla\) with constant entries (i.e. \(G \in \mathcal{M}_d(k)\)), and let \(\{a_1, \cdots, a_N\}\) be the set of eigenvalues of \(G\).

If \(x\) is a point of the form \(x_{0,r}\), then we have

\[
\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \bigcup_{i=1}^{N} a_i + \overline{\mathbb{Z}}.
\]

Otherwise, let \(c \in k \setminus \{0\}\) such that \(x \in D^-(c, |c|)\) and \(y := \text{Log}_c(x)\). Then we have

\[
\Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \begin{cases} 
\bigcup_{j=1}^{N} D^+(a_j, \frac{\omega}{r_k(y)}) & \text{if } r_k(x) \in (0, |c| \omega] \\
\bigcup_{j=1}^{N} D^+(a_j + i, \frac{\omega}{r_k(y)}) & \text{if } r_k(x) \in (|c| \omega^{-1}, |c| \omega^{-1}] 
\end{cases}
\]

for \(n \in \mathbb{N}\) \(\setminus \{0\}\).

**Theorem 3.16.** Assume that \(\text{char}(\tilde{k}) = 0\). Let \(x \in \mathbb{A}_k^{1, \text{an}}\) be a point of type (2), (3) or (4). Let \((M, \nabla)\) be a regular singular differential module over \((\mathcal{H}(x), S_{\frac{d}{dS}})\). Let \(G\) be the matrix associated to \(\nabla\) with constant
entries (i.e. \( G \in \mathcal{M}_n(k) \)), and let \( \{ a_1, \cdots, a_N \} \) be the set of eigenvalues of \( G \).

If \( x \) is a point of type (2) of the form \( x_{a,r} \), then we have

\[
(3.51) \quad \Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \bigcup_{j=1}^{N} D^+(a_j, 1).
\]

If \( x \) is a point of type (3) of the form \( x_{a,r} \), then we have

\[
(3.46) \quad \Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \bigcup_{j=1}^{N} a_j + \mathbb{Z}.
\]

Otherwise, let \( c \in k \setminus \{ 0 \} \) such that \( x \in D^-(c, |c|) \). Then we have

\[
(3.47) \quad \Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \begin{cases} 
\bigcup_{j=1}^{N} D^-(a_j, |c|/r_k(x)) & \text{if } x \text{ is of type (4)}, \\
\bigcup_{j=1}^{N} D^+(a_j, |c|/r_k(x)) & \text{otherwise}.
\end{cases}
\]

Remark 3.17. Note that from the computation of the spectrum of \( S_{d_{\mathcal{D}S}} \), we observe that for a point \( x_{c,r} \in \mathbb{A}_k^{1,\text{an}} \setminus k \) with \( c \neq 0 \) and \(|c| > r\), it is better to choose \((S - c)\frac{d_{\mathcal{D}S}}{d_{\mathcal{D}S}}\) than \( S_{d_{\mathcal{D}S}} \).

3.2.1. Variation of the spectrum. Let \((\mathcal{F}, \nabla)\) be a differential equation over \( \mathbb{A}_k^{1,\text{an}} \setminus \{ 0 \} \). We fix the derivation \( S_{d_{\mathcal{D}S}} \) over \( \mathbb{A}_k^{1,\text{an}} \). For each \( x \in \mathbb{A}_k^{1,\text{an}} \setminus k \) we set \((M_x, \nabla_x) := (\mathcal{F}_x \otimes \mathcal{H}(x), \nabla)\) the differential module over \((\mathcal{H}(x), S_{d_{\mathcal{D}S}})\). We say that \((\mathcal{F}, \nabla)\) is a differential equation with regular singularities if there exists a matrix \( G \in \mathcal{M}_n(k) \) such that \( G \) is an associated matrix of \((M_x, \nabla_x)\) for each \( x \). Note that it is for its own interest to study the variation of the spectrum of \( \nabla_x \).

Notation 3.18. Let \( \mathcal{K}(\mathbb{A}_k^{1,\text{an}}) \) be the set of nonempty compact subsets of \( \mathbb{A}_k^{1,\text{an}} \). We endow \( \mathcal{K}(\mathbb{A}_k^{1,\text{an}}) \) with the exponential topology, the topology generated by the following family of sets:

\[
(3.48) \quad (U, \{ U_i \}_{i \in I}) = \{ \Sigma \in \mathcal{K}(\mathbb{A}_k^{1,\text{an}}) : \Sigma \subset U, \Sigma \cap U_i \neq \emptyset \forall i \},
\]

where \( U \) is an open of \( \mathbb{A}_k^{1,\text{an}} \) and \( \{ U_i \}_{i \in I} \) is a finite open cover of \( U \). In this case, since \( \mathbb{A}_k^{1,\text{an}} \) is a Hausdorff space, then so is for \( \mathcal{K}(\mathbb{A}_k^{1,\text{an}}) \).

Lemma 3.19. The following function is continuous

\[
(3.49) \quad \Upsilon : \mathcal{K}(\mathcal{T}) \times \mathcal{K}(\mathcal{T}) \rightarrow \mathcal{K}(\mathcal{T}) \quad (\Sigma, \Sigma') \mapsto \Sigma \cup \Sigma'
\]

Proof. Let \( \Sigma \) and \( \Sigma' \) be two non-empty compact subsets of \( \mathcal{T} \). Let \((U, \{ U_i \}_{i \in I})\) be an open neigbourhood of \( \Sigma \cup \Sigma' \). We set

\[
(3.50) \quad J := \{ i \in I : \Sigma \cap U_i \neq \emptyset \} \text{ and } J' := \{ i \in I : \Sigma' \cap U_i \neq \emptyset \}.
\]

Then \((U, \{ U_i \}_{i \in J})\) (resp. \((U, \{ U_i \}_{i \in J'})\)) is an open neigbourhood of \( \Sigma \) (resp. \( \Sigma' \)) and we have

\[
(3.51) \quad (U, \{ U_i \}_{i \in J}) \times (U, \{ U_i \}_{i \in J'}) \subset \Upsilon^{-1}((U, \{ U_i \}_{i \in I})).
\]

Hence we obtain the result. \( \square \)
The case of positive residue characteristic. Assume that \( \text{char}(\overline{k}) = p > 0 \). We observe from Theorem 3.15 that, although the spectrum is roughly different from the constant case studied in [Azz20], it satisfies analogous continuity properties.

**Theorem 3.20.** Let \((\mathcal{F}, \nabla)\) be a differential equation over \( \mathcal{A}_k^{1,\text{an}} \setminus k \) with regular singularities. Let \( x \in \mathcal{A}_k^{1,\text{an}} \setminus k \). We set:

\[
(3.52) \quad \Psi : [x, \infty) \rightarrow \mathcal{K}(\mathcal{A}_k^{1,\text{an}})
\]

\[
y \mapsto \Sigma_{\nabla, k}(\mathcal{L}_k(M_y))
\]

Let \( y \in [x, \infty) \), then we have:

- the restriction of \( \Psi \) to \([x, y]\) is continuous at \( y \),
- the map \( \Psi \) is continuous at \( y \) if and only if \( y \) is of type (3) or of the form \( x_0.R \).

**Proof.** We identify \([x, \infty)\) with the interval \([r(x), \infty)\) by the map \( y \mapsto r(y) \) (cf. Definition 2.3). Let \( y \in [x, \infty) \). Assume that there exists \( x' \in [x, \infty) \) such that \([x, x'] \cap (0, \infty) = \emptyset \) and \([x, y] \subset [x, x']\). Let \( y' \in [x, x'] \). By Theorem 3.15 and Corollary 3.10, we have \( \Psi(y) = \bigcup_{i=1}^{N} D^{+}(a_i, \phi(y)) \) and \( \Psi(y') = \bigcup_{i=1}^{N} D^{+}(a_i, \phi(y')) \), where \( \phi : [x, y] \rightarrow \mathbb{R}_+ \) is a decreasing continuous function and \( \phi(y) \notin [k] \) if \( y \) is of type (3). Therefore, the claims:

- \( \Psi \) is continuous at \( y \) if and only if \( y \) is of type (3),
- the restriction of \( \Psi \) to \([x, y]\) is continuous at \( y \),

holds by the continuity results of [Azz20, Theorem 5.3].

Now assume that \( y \in [x_0.R, \infty) \). In the case where \( y \neq x_0.R \), the restriction of \( \Psi \) to \([x_0.R, \infty) \) is constant (cf. Theorem 3.15). Hence the restriction of \( \Psi \) to \([x_0.R, \infty) \) is continuous. Otherwise, i.e. \( y = x_0.R \), on the one hand the restriction of \( \Psi \) to \([y, \infty) \) is continuous at \( y \). On the other hand, since for all \( y' \in [x, y] \) we have \( \Psi(y') = \bigcup_{i=1}^{N} \alpha_i + \Sigma_{\nabla, k}(\mathcal{L}_k(\mathcal{H}(y'))) \), it is enough to show the result for the differential module \((\mathcal{H}(y), S^{d_{\mathcal{H}}})\) (cf. Lemma 3.19). Hence, we reduce to the case where \( \Psi(y) = \mathbb{Z}_p \) and \( \Psi(y') = \bigcup_{\ell \in \mathbb{N}} D^{+}(\ell, \phi(y')) \), with \( \phi : [x, y] \rightarrow \mathbb{R}_+ \) a decreasing continuous function and \( \lim_{y' \rightarrow y} \phi(y') = 0 \) (cf. Theorem 3.15 and Corollary 3.10).

Let \((U, \{U_i\}_{i \in I})\) be an open neighbourhood of \( \Psi(y) \). Since \( \Psi(y) \) is a set of points of type (1), we can assume that \( U_i \) is an open disk for all \( i \in I \). Since \( \mathbb{Z}_p \) is the topological closure of \( \mathbb{N} \) in \( \mathcal{A}_k^{1,\text{an}} \), for all \( i \in I \) we have \( \mathbb{N} \cap U_i \neq \emptyset \). Therefore, for all \( i \in I \) we have \( \Psi(y') \cap U_i \neq \emptyset \). We now prove that there exists \( x' \in [x, y) \) such that for all \( y' \in (x', y) \) we have \( \Psi(y') \subset U \). Let \( L \) be the smallest radius of the disks \( U_i \). Since \( \phi \) is a decreasing continuous function, there exists \( y_L \) such that for all \( y' \in (y_L, y) \) we have \( \phi(y') < L \). Therefore, since \( \mathbb{N} \subset U = \bigcup_{i \in I} U_i \), for all \( j \in \mathbb{N} \), there exists \( i \in I \) such that \( D^{+}(j, \phi(y')) \subset U_i \). Consequently, we have \( \Psi(y') \subset U \) and \( \Psi(y') \subset (U, \{U_i\}_{i \in I}) \).

The case of residue characteristic zero. Assume that \( \text{char}(\overline{k}) = 0 \). We observe from Theorem 3.16 that the spectrum behaves differently from the case where \( \text{char}(\overline{k}) = p > 0 \). In the special case where \( k \) is not trivially valued and \( |k| \neq \mathbb{R}_+ \), the map

\[
(3.53) \quad \Psi : (0, \infty) \rightarrow \mathcal{K}(\mathcal{A}_k^{1,\text{an}})
\]

\[
y \mapsto \Sigma_{\nabla, k}(\mathcal{L}_k(M_y))
\]

is not continuous at all. Indeed, let \( y \in (0, \infty) \) be a point of type (2). Assume that \((\mathcal{F}, \nabla) = (\mathcal{O}_{\mathcal{A}_k^{1,\text{an}} \setminus \{0\}}, d_{\mathcal{A}_k^{1,\text{an}} \setminus \{0\}})\).

Then we have \( \Psi(y) = D^{+}(0, 1) \). Let \( U \) be an open neighbourhood of \( \Psi(y) \) in \( \mathcal{A}_k^{1,\text{an}} \). Let \( a \in (D^{+}(0, 1) \cap k) \setminus \mathbb{Z} \) and let \( 0 < r < 1 \) such that \( D^{-}(a, r) \cap \mathbb{Z} = \emptyset \). For any \( y' \in (0, \infty) \) of type (3) we have \( \Psi(y') = \mathbb{Z} \cup \{x_0, 1\} \), hence \( \Psi(y') \cap D^{-}(a, r) = \emptyset \). Therefore, \( \Psi(y') \notin (U, \{U, D^{+}(a, r)\}) \).

If \( k \) is trivially valued, the only point where there is no continuity is \( x_0.1 \). For the other points of \((0, \infty) \), since \( \Psi \) is constant, it is continuous on \((0, \infty) \setminus \{x_0, 1\}\).

For branches \((c, x_{0,[c]} \]) \) with \( c \in k \setminus \{0\} \), the map

\[
(3.54) \quad \Psi : (c, x_{0,[c]} \]) \rightarrow \mathcal{K}(\mathcal{A}_k^{1,\text{an}})
\]

\[
y \mapsto \Sigma_{\nabla, k}(\mathcal{L}_k(M_y))
\]
satisfies the same continuity properties as those of [Azz20, Theorem 5.3]. Indeed, for any \( y \in (c, x_{c, [c]}) \) we have \( \Psi(y) = \bigcup_{i=1}^{N} D^+(a_i, \varphi(y)) \) with \( \varphi : (c, x_{0, [c]}) \to \mathbb{R}_+ \) a decreasing continuous function and \( \varphi(y) \notin |k| \) if \( y \) is of type (3).

We have the following results:

**Theorem 3.21.** Assume that \( |k| = \mathbb{R}_+ \). Let \( (\mathcal{F}, \nabla) \) be a differential equation over \( \mathbb{A}_k^{1, \text{an}} \) with regular singularities. Let \( x \in \mathbb{A}_k^{1, \text{an}} \). We set:

\[
\Psi : [x, \infty) \longrightarrow \mathcal{K}(\mathbb{A}_k^{1, \text{an}}) \\
y \mapsto \Sigma y_k(\mathcal{L}_k(M_y))
\]

Let \( y \in [x, \infty) \), then we have:

- the restriction of \( \Psi \) to \([x, y]\) is continuous at \( y \),
- the map \( \Psi \) is continuous at \( y \) if and only if \( y \) is of type (4) or of the form \( x_0, R \).

**Proof.** The proof is analogous to the proof of Theorem 3.20. \( \square \)

### 4. Spectral version of Young’s theorem

In this part we give a spectral version of Young’s theorem [You92], [Ked10, Theorem 6.5.3], [CM02, Theorem 6.2], which states the following.

**Convention 4.1.** We still assume that \( k \) is algebraically closed.

**Theorem 4.2 (Young).** Let \( x \in \mathbb{A}_k^{1, \text{an}} \) be a point of type (2), (3) or (4). Let \( \mathcal{L} = \sum_{i=0}^{n} g_{n-i} \frac{d^i}{dx^i} \) with \( g_0 = 1 \) and \( g_i \in \mathcal{H}(x) \), and let \((M, \nabla)\) be the associated differential module over \((\mathcal{H}(x), \frac{d}{dx})\). We set \( |\mathcal{L}|_{\text{Sp}} = \max_{0 \leq i \leq n} |g_i|^{\frac{1}{i}} \). If \( |\mathcal{L}|_{\text{Sp}} > \frac{1}{\|
abla\|_{\text{Sp}}} \) then \( \|
abla\|_{\text{Sp}} = |\mathcal{L}|_{\text{Sp}} \).

In order to state and prove the main statement of the section, we will need the following additional results.

**Definition 4.3.** Let \( E \) be a \( k \)-Banach algebra and \( B \) a commutative \( k \)-subalgebra of \( E \). We say that \( B \) is a maximal commutative subalgebra of \( E \), if for any commutative \( k \)-subalgebra \( B' \) of \( E \) we have the following property:

\[
(B \subset B' \subset E) \Leftrightarrow (B' = B).
\]

**Remark 4.4.** A maximal subalgebra \( B \) is necessarily closed in \( E \), hence a \( k \)-Banach algebra.

**Proposition 4.5 ([Ber90, Proposition 7.2.4]).** Let \( E \) be a \( k \)-Banach algebra. For any maximal commutative subalgebra \( B \) of \( E \), we have:

\[
\forall f \in B, \quad \Sigma_f(B) = \Sigma_f(E).
\]

**Definition 4.6.** Let \( E \) be a \( k \)-algebra and let \( B \) be a \( k \)-subalgebra of \( E \). If any element of \( B \) invertible in \( E \) is also invertible in \( B \), we say that \( B \) is a saturated subalgebra of \( E \).

**Proposition 4.7 ([Ber90, Proposition 7.2.4]).** Assume that \( k \) is not trivially valued. Let \( E \) be a \( k \)-Banach algebra and let \( B \) be a saturated \( k \)-Banach subalgebra of \( E \). Then we have:

\[
\forall f \in B, \quad \Sigma_f(B) = \Sigma_f(E).
\]

**Lemma 4.8.** Let \( \Omega \in E(k) \), let \( E \) be an \( \Omega \)-Banach algebra and \( f \in E \). Then we have

\[
\Sigma_{f, \Omega}(E) = \pi_{\Omega/k}(\Sigma_f(\Omega)),
\]

where \( \pi_{\Omega/k} : \mathbb{A}_\Omega^{1, \text{an}} \to \mathbb{A}_k^{1, \text{an}} \) is the canonical projection.
Proof. Let $B$ be a maximal commutative $\Omega$-subalgebra of $E$ containing $f$. Let $B'$ be a commutative $k$-subalgebra of $E$ such that $B \subset B'$. Then $B'$ is also an $\Omega$-subalgebra of $E$. Therefore $B$ is also maximal as a commutative $k$-subalgebra of $E$. Let $B'$ be an isometric isomorphism $B \otimes_k \mathcal{H}(x) \simeq B \otimes_\Omega (\Omega \otimes_k \mathcal{H}(x))$ (cf. [BGR84, Section 2.1, Proposition 7]). For each $y \in \pi_{\Omega/k}^{-1}(x)$, we have a contracting map $\Omega \otimes_k \mathcal{H}(x) \to \mathcal{H}(y)$. Therefore, the induced map $B \otimes_k \mathcal{H}(x) \to B \otimes_\Omega \mathcal{H}(y)$ is contracting too. Hence, if $f \otimes 1 - 1 \otimes T(y)$ is not invertible in $B \otimes_\Omega \mathcal{H}(y)$, then $f \otimes 1 - 1 \otimes T(x)$ is not invertible in $B \otimes_k \mathcal{H}(x)$. Therefore, $\pi_{\Omega/k}(\Sigma_{f,k}(B)) \subset \Sigma_{f,k}(B)$. Now let $x \in \Sigma_{f,k}(B)$. Since $f \otimes 1 - 1 \otimes T(x)$ is not invertible in $B \otimes_k \mathcal{H}(x) \simeq B \otimes_\Omega (\Omega \otimes_k \mathcal{H}(x))$, according to Lemma 6.6, there exists $y \in \mathcal{M}(\Omega \otimes_k \mathcal{H}(x)) = \pi_{\Omega/k}^{-1}(x)$ such that $f \otimes 1 - 1 \otimes T(y)$ is not invertible in $B \otimes_\Omega \mathcal{H}(y)$). Therefore, $\Sigma_{f,k}(B) \subset \pi_{\Omega/k}^{-1}(\Sigma_{f,k}(B))$. Hence, by Proposition 4.5 we obtain

\[
\Sigma_{f,k}(E) = \pi_{\Omega/k}(\Sigma_{f,k}(E)).
\]

\[\square\]

**Definition 4.9.** Let $\Omega \in E(k)$ and $f \in \mathcal{M}_n(\Omega)$. Let $\{a_1, \cdots, a_N\}$ be the set of eigenvalues of $f$ in $\widehat{\Omega^{alg}}$. Let us call $\pi_{\Omega^{alg}/\Omega}(\{a_1, \cdots, a_N\})$ the set of eigenvalues of $f$ in $\mathbb{A}^{1,n}_\Omega$.

**Corollary 4.10.** Let $\Omega \in E(k)$ and $f \in \mathcal{M}_n(\Omega)$. Let $\{a_1, \cdots, a_N\}$ be the set of eigenvalues of $f$ in $\mathbb{A}^{1,n}_\Omega$. Then we have

\[
\Sigma_{f,k}(\mathcal{M}_n(\Omega)) = \pi_{\Omega/k}(\{a_1, \cdots, a_N\}).
\]

**Corollary 4.11.** Suppose that $k$ is not trivially valued. Let $\Omega \in E(k)$ and $f \in \mathcal{M}_n(\Omega)$. Let $\{a_1, \cdots, a_N\}$ be the set of rigid points of $\mathbb{A}^{1,n}_\Omega$ that correspond to the eigenvalues of $f$ in some finite extensions of $\Omega$. Then we have

\[
\Sigma_{f,k}(\mathcal{L}_k(\Omega^n)) = \pi_{\Omega/k}(\{a_1, \cdots, a_N\}).
\]

**Proof.** Since $\mathcal{M}_n(\Omega)$ is a saturated subalgebra of $\mathcal{L}_k(\Omega^n)$, according to Proposition 4.7, we have $\Sigma_{f,k}(\mathcal{M}_n(\Omega)) = \Sigma_{f,k}(\mathcal{L}_k(\Omega^n))$. The result follows by Corollary 4.10. \[\square\]

**Remark 4.12.** In the case where $k$ is trivially valued, we have at least the inclusion $\Sigma_{f,k}(\mathcal{L}_k(\Omega^n)) \subset \pi_{\Omega/k}(\{a_1, \cdots, a_N\})$.

**Lemma 4.13.** Let $(A, |.|)$ be a commutative $k$-Banach algebra. Let $P(T) = \sum_{i=0}^n a_i T^i \in A[T]$ with $a_n = 1$. Let $G \in \mathcal{M}_n(A)$ such that

\[
G = \begin{pmatrix}
0 & -a_0 & & \\
1 & 0 & -a_0 & \\
0 & 1 & \cdots & \\
0 & 0 & \cdots & -a_{n-1}
\end{pmatrix}
\]

If $a_0$ is invertible in $A$, then $G$ is invertible and we have $\|G^{-1}\| = \max_{0 \leq i \leq n} |a_i a_0^{-1}|$.

**Proof.** Since $\det(G) = (-1)^n a_0$ then $G$ is invertible if and only if $a_0$ is invertible in $A$. Assume now that $a_0$ is invertible. Then we have

\[
G^{-1} = \frac{1}{a_0} \begin{pmatrix}
-a_1 & a_0 & 0 & \cdots & \\
0 & 0 & 0 & \cdots & \\
0 & \cdots & 0 & 0 & \cdots \\
0 & \cdots & \cdots & 0 & 0 \\
-1 & \cdots & \cdots & \cdots & 0
\end{pmatrix}
\]
Hence, \( \|G^{-1}\| = \max_{0 \leq i \leq n} |a_i a_0^{-1}|. \)

\[ \square \]

**Lemma 4.14** ([BGR84, Proposition 2.1.8]). Let \( \Omega, \Omega' \in E(k) \). Then
\[
(\mathcal{M}_n(\Omega \otimes_k \Omega'), \|\cdot\|) \simeq (\mathcal{M}_n(\Omega) \otimes_k \Omega', \|\cdot\|'),
\]
where \( \|\cdot\| \) is the maximum norm and \( \|\cdot\|' \) is the tensor product norm.

**Theorem 4.15** (Spectral version of Young’s theorem (weak version)). Let \( \Omega \in E(k) \) and \( d : \Omega \to \Omega \) be a \( k \)-linear bounded derivation. Let \( (M, \nabla) \) be a differential module over \( (\Omega, d) \) with \( (M, \nabla) \simeq (\mathscr{D}_\Omega/\mathscr{D}_\Omega \cdot P(d), d) \) and \( P(d) = \sum_{i=0}^n a_i d^i \) with \( a_n = 1 \). Let \( \{z_1, \cdots, z_n\} \subset \Omega^{alg} \) be the multiset of roots of \( P(T) \) (the commutative polynomial associated to \( P(d) \)). If \( \min_i r_k(\pi_{\hat{\Omega}^{alg}/k}(z_i)) > \|d\| \), then
\[
(4.10) \quad \Sigma_{\nabla,k}(\mathcal{L}_k(M)) \subset \pi_{\hat{\Omega}^{alg}/k}(\{z_1, \cdots, z_n\}).
\]
In particular if \( \pi_{\hat{\Omega}^{alg}/k}(\{z_1, \cdots, z_n\}) = \{z\} \), we have \( \Sigma_{\nabla,k}(\mathcal{L}_k(M)) = \{z\} \).

**Proof.** Since \( (M, \nabla) \simeq (\mathscr{D}_\Omega/\mathscr{D}_\Omega \cdot P(d), d) \), there exists a cyclic basis \( \{m, \nabla(m), \cdots, \nabla^{n-1}(m)\} \) such that:
\[
(4.11) \quad \nabla = \begin{pmatrix} f_0 \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} df_0 \\ df_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -a_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ 0 \\ f_{n-1} \end{pmatrix}.
\]

As first step we will assume that \( k \) is not trivially valued. Then there exists \( \alpha \in k \) such that \( |\alpha| \cdot \min_i r_k(\pi_{\hat{\Omega}^{alg}/k}(z_i)) \geq 1 \) and \( \|\alpha d\| < 1 \). In order to prove the statement, it is enough to show that \( \Sigma_{\alpha \nabla,k}(\mathcal{L}_k(M)) \subset \pi_{\hat{\Omega}^{alg}/k}(\{\alpha z_1, \cdots, \alpha z_n\}) \). In the basis \( \{m, \alpha \nabla(m), \cdots, \alpha^{n-1} \nabla^{n-1}(m)\} \) we have:
\[
(4.12) \quad \alpha \nabla = \begin{pmatrix} f_0 \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha df_0 \\ \alpha df_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -a_0\alpha^n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ 0 \\ f_{n-1} \end{pmatrix}.
\]

We set
\[
(4.13) \quad G := \begin{pmatrix} 0 & 0 & -b_0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

with \( b_i = a_i \alpha^{n-i} \). We set \( w_i := \alpha z_i \) and \( \Delta := \alpha \nabla - G \). Then on the one hand, it is easy to see that \( \{w_1, \cdots, w_n\} \) are the eigenvalues of \( G \) in \( A_\Omega^{1,an} \). On the other hand, we have \( \|\Delta\| = \|\alpha d\| < 1 \) and \( \min_i r_k(\pi_{\hat{\Omega}^{alg}/k}(w_i)) \geq 1 \).

By Corollary 4.11, \( G \otimes 1 - 1 \otimes T(y) \) is invertible for all \( y \in A_\Omega^{1,an} \setminus \pi_{\hat{\Omega}^{alg}/k}(\{w_1, \cdots, w_n\}) \).
Let \( y \in A_{k,\text{an}}^1 \setminus \pi_{\text{alg}}/k(\{w_1, \ldots, w_n\}) \), now we show that \( \alpha \nabla \otimes 1 - 1 \otimes T(y) \) is invertible. Since we have \( M_n(\Omega \hat{\otimes}_k \mathcal{H}(y)) \simeq M_n(\Omega) \hat{\otimes}_k \mathcal{H}(y) \) (cf. Lemme 4.14), then we can write

\[
G \otimes 1 - 1 \otimes T(y) = \begin{pmatrix}
-1 \otimes T(y) & 1 \otimes T(y) & \cdots & 1 \otimes T(y) \\
1 & -1 \otimes T(y) & \cdots & 0 \\
0 & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & -b_{n-1} \otimes 1
\end{pmatrix}.
\]

Let \( Q_y(T) := \sum_i b_i^y T^i + T^n \in \Omega \hat{\otimes}_k \mathcal{H}(y)[T] \) such that \( Q_y(T) = P(T + 1 \otimes T(y)) \). For

\[
U = \begin{pmatrix}
1 & 1 \otimes (-T(y)) & 1 \otimes T(y)^2 & \cdots & 1 \otimes T(y)^{n-1} \\
0 & 1 & 2 \otimes (-T(y)) & \cdots & \binom{n-1}{1} \otimes (-T(y))^{n-2} \\
0 & \ldots & \ldots & \ldots & \binom{n-1}{n-2} \otimes (-T(y)) \\
0 & \ldots & \ldots & \ldots & 1
\end{pmatrix},
\]

then, it is easy to see that \( U \) is invertible in \( M_n(\Omega \hat{\otimes}_k \mathcal{H}(y)) \) and we have:

\[
G_y := U^{-1}(G \otimes 1 - 1 \otimes T(y))U = \begin{pmatrix}
0 & -b_0^y & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 1
\end{pmatrix}.
\]

Since \( U \in M_n(k \hat{\otimes}_k \mathcal{H}(y)) \), we have \( U^{-1} \Delta U = U^{-1}U \Delta = \Delta \). Then \( U^{-1}(\alpha \nabla \otimes 1 - 1 \otimes T(y))U = \Delta + G_y \).

Consequently, in order to proof that \( \alpha \nabla \otimes 1 - 1 \otimes T(y) \) is invertible, since \( \|\Delta\| < 1 \), it is enough to show that

\[
1 \leq \|G_y^{-1}\|^{-1}.
\]

Since \( k \) is algebraically closed, \( \Omega \hat{\otimes}_k \mathcal{H}(y) \) is a multiplicative \( k \)-algebra (cf. Proposition 2.9). By Lemma 4.13, we have \( \|G_y^{-1}\| = |b_0^y|^{-1}\|G_y\| \). Now we show that \( \|G_y^{-1}\| = 1 \), i.e. \( \|G_y\| = |b_0| \). Since \( \Omega \rightarrow \hat{\Omega}_{\text{alg}} \) is an isometry, so is \( \hat{\Omega}_{\text{alg}}(y) \rightarrow \hat{\Omega}_{\text{alg}} \hat{\otimes}_k \mathcal{H}(y) \) (cf. [Poi13, Lemme 3.1]). We have \( Q_y(T) = P(T + 1 \otimes T(y)) = \prod_{i=0}^n(T - (w_i \otimes 1 - 1 \otimes T(y))) \), hence

\[
b_{n-i}^y = \sum_j \prod_{i=1}^j (w_{ji} \otimes 1 - 1 \otimes T(y)).
\]

Therefore, if we show that \( |w_i \otimes 1 - 1 \otimes T(y)| \geq 1 \) for all \( i \), then we have automatically

\[
\|G_y\| = \max_{1 \leq i \leq n} |b_i^y| = |b_0^y|.
\]

Now let us show that \( |w_i \otimes 1 - 1 \otimes T(y)| \geq 1 \) for all \( i \). To avoid any confusion, we fix another coordinate function \( S \) on \( A_{k,\text{an}}^1 \). Note that we have an isometric embedding \( \mathcal{H}(\pi_{\text{alg}}/k(w_i)) \rightarrow \hat{\Omega}_{\text{alg}}, \)
that assigns \( w_i \) to \( S(\pi_{\Omega_{\text{alg}}/k}(w_i)) \). By above argument, we have an isometric embedding of \( k \)-algebra 
\[ \mathcal{H}(\pi_{\Omega_{\text{alg}}/k}(w_i)) \otimes_k \mathcal{H}(y) \to \Omega_{\text{alg}} \otimes_k \mathcal{H}(y), \]
and we have \( |w_i \otimes 1 - 1 \otimes T(y)| = |S(\pi_{\Omega_{\text{alg}}/k}(w_i)) \otimes 1 - 1 \otimes T(y)|. \)

The natural map \( \mathcal{H}(\pi_{\Omega_{\text{alg}}/k}(w_i)) \otimes_k \mathcal{H}(y) \to \mathcal{H}(\sigma_{\mathcal{H}(y)/k}(\pi_{\Omega_{\text{alg}}/k}(w_i))) \) is an isometry (see Definition 2.8), mapping \( S(\pi_{\Omega_{\text{alg}}/k}(w_i)) \otimes 1 - 1 \otimes T(y) \) to \( S(\sigma_{\mathcal{H}(y)/k}(\pi_{\Omega_{\text{alg}}/k}(w_i))) - T(y) \). By Lemma 2.11, we have \( r(\mathcal{H}(y)(\pi_{\mathcal{H}(y)/k}(\pi_{\Omega_{\text{alg}}/k}(w_i)))) = r_k(\pi_{\Omega_{\text{alg}}/k}(w_i)) \). Consequently, we obtain

\[ 1 \leq r_k(\pi_{\Omega_{\text{alg}}/k}(w_i)) \leq |S(\sigma_{\mathcal{H}(y)/k}(\pi_{\Omega_{\text{alg}}/k}(w_i))) - T(y)| = |w_i \otimes 1 - 1 \otimes T(y)|. \]

Hence, we obtain \( ||G^{-1}_y|| = 1 \) and conclude that \( \alpha \nabla \otimes 1 - 1 \otimes T(y) \) is invertible for all \( y \in \mathbb{A}^1_{\text{an}} \setminus \{w_1, \ldots, w_n\} \).

Now we assume that \( k \) is trivially valued. Let \( k' \in E(k) \) be algebraically closed, such that there exists \( \alpha \in k' \), with \( |\alpha| \min_i r_k(z_i) \geq 1 \) and \( |\alpha| |d| < 1 \). We know that \( \Sigma_{\nabla \otimes 1,k'}(L_k(M) \otimes k') = \pi_{k'/k}(\Sigma_{\nabla,k}(L_k(M))) \) (cf. [Ber90, Proposition 7.1.6]). Therefore, in order to show that \( \Sigma_{\nabla,k}(L_k(M)) \subset \pi_{\Omega_{\text{alg}}/k}(\{z_1, \ldots, z_n\}) \), it is enough to show that \( \Sigma_{\nabla \otimes 1,k'}(L_k(M) \otimes k') \subset \pi_{k'/k}(\pi_{\Omega_{\text{alg}}/k}(\{z_1, \ldots, z_n\})) \). For that we will consider \( \nabla \otimes \alpha \) and show, as for the non trivial case, that \( \Sigma_{\nabla \otimes \alpha,k'}(L_k(M) \otimes k') \subset \alpha \pi_{k'/k}(\pi_{\Omega_{\text{alg}}/k}(\{z_1, \ldots, z_n\})) \).

If in addition we assume that \( \pi_{\Omega_{\text{alg}}/k}(\{z_1, \ldots, z_n\}) = \{z\} \), then since the spectrum is not empty [Ber90, Theorem 7.1.2] we must have \( \Sigma_{\nabla,k}(L_k(M)) = \{z\} \). \( \square \)

Remark 4.16. Set the notations as in Theorem 4.15. If \( P(T) \) is irreducible as a commutative polynomial, then it is easy to see that \( |z_1 - a| = \cdots = |z_n - a| \) for all \( a \in k \). Hence \( \Sigma_{\nabla,k}(L_k(M)) = \pi_{\Omega_{\text{alg}}/k}(\{z_1, \ldots, z_n\}) = \{z\} \).

We need the following results to prove that the same result (as in Remark 4.16) hold for a monic irreducible differential polynomial \( P(d) \).

Lemma 4.17 ([Ked10, Theorem 6.4.4]). Let \( \Omega \in E(k) \) and \( d : \Omega \to \Omega \) be a k-linear bounded derivation. Let \( P(d) \in \mathcal{D}_\Omega \) be a monic differential polynomial. Let \( \{z_1, \ldots, z_n\} \) be the multiset\(^3\) of roots of \( P(T) \) (the commutative polynomial associated to \( P(d) \)), with \( |z_1| \leq \cdots \leq |z_n| \). Let \( r > R^+ \). If \( |d| < r \) and for some \( i_0 \) we have \( |z_i| < r < |z_{i_0} + 1| \), then there exists a unique factorization \( P(d) = Q(d)R(d) \) such that \( \{\omega_1, \ldots, \omega_{i_0}\} \) (resp. \( \{\omega_{i_0} + 1, \ldots, \omega_n\} \)) is the multiset of roots of \( R(T) \) (resp. \( Q(T) \)) with \( |\omega_i| < r \) (resp. \( |\omega_i| = |z_i| \)) for \( i < i_0 \) (resp. \( i > i_0 \)), where \( R(T) \) (resp. \( Q(T) \)) is the commutative polynomial associated to \( R(d) \) (resp. \( Q(d) \)). If moreover we have \( |z_i| > |d| \) for each \( i \) then \( |\omega_i| = |z_i| \) for each \( i \).

Lemma 4.18. Let \( \Omega \in E(k) \) and \( d : \Omega \to \Omega \) be a k-linear bounded derivation. Let \( P(d) \in \mathcal{D}_\Omega \) be a monic irreducible differential polynomial. Let \( \{z_1, \ldots, z_n\} \) be the multiset of roots of \( P(T) \) (the commutative polynomial associated to \( P(d) \)). If \( \min_i r_k(\pi_{\Omega_{\text{alg}}/k}(z_i)) > |d| \), then \( \pi_{\Omega_{\text{alg}}/k}(z_1, \ldots, z_n) = \{z\} \).

Proof. Since \( P(d) \) is irreducible, then so is for \( P_a(d) = P(d + a) \) for all \( a \in k \). Let \( a \in k \), we have \( P_a(T) = P(T + a) \), hence the roots of \( P_a \) as a commutative polynomial are exactly \( \{z_1 - a, \ldots, z_n - a\} \). Since

\[ \forall i, \ |z_i - a| \geq \min_i r_k(\pi_{\Omega_{\text{alg}}/k}(z_i)) > |d|, \]

then by Lemma 4.17 we have

\[ |z_1 - a| = \cdots = |z_n - a|. \]

Consequently, since \( k \) is algebraically closed, we have

\[ \pi_{\Omega_{\text{alg}}/k}(z_1) = \cdots = \pi_{\Omega_{\text{alg}}/k}(z_n) = \{z\} \]

for some \( z \in \mathbb{A}^1_{\text{an}} \setminus k \). \( \square \)

\(^3\)Counted with multiplicity.
Corollary 4.19. Let $\Omega \in E(k)$ and $d : \Omega \to \Omega$ be a $k$-linear bounded derivation. Let $(M, \nabla)$ be a differential module over $(\Omega, d)$ with $(M, \nabla) \simeq (\mathcal{O}_\Omega, \mathcal{O}_\Omega \cdot P(d), d)$ and $P(d) = \sum a_i d^i + d^a$. We assume that $P(d)$ is an irreducible differential polynomial. Let $\{z_1, \ldots, z_n\} \subset \Omega^{\text{alg}}$ be the multiset of roots of $P(T)$ (the commutative polynomial associated to $P(T)$). If $\min r_k(\pi_{\Omega^{\text{alg}}/k}(z_i)) > \|d\|$, then

\begin{equation}
\Sigma_{\nabla, k}(L_k(M)) = \pi_{\Omega^{\text{alg}}/k}(\{z_1, \ldots, z_n\}) = \{z\}.
\end{equation}

The first spectral version of Young's theorem can be refined as follows. Before we need the following results.

Lemma 4.20. Let $\{\omega_1, \ldots, \omega_n\} \subset A_k^{1, \text{an}}$ and assume that $r_k(\omega_1) \leq \cdots \leq r_k(\omega_n)$. Then for each $\omega_i$ there exists an $a_i \in k$ such that for all $j > i$ we have $|T(\omega_j) - a_i| > |T(\omega_j) - a_i|$. 

Proof. If $\omega_i$ is not a point of type (4), then there exists an $a_i$ such that $r_k(\omega_i) = |T(\omega_i) - a_i|$, and if $j > i$, then we have $r_k(\omega_j) = r_k(\omega_i) = |T(\omega_j) - a_i|$. If $r_k(\omega_i) < r_k(\omega_j)$ then it is clear that $|T(\omega_j) - a_i| > |T(\omega_j) - a_i|$. If $r_k(\omega_j) = r_k(\omega_i)$, then we must have $r_k(\omega_j) < |T(\omega_i) - a_i|$, otherwise we get $\omega_j = \omega_i$ which contradicts the hypothesis. Hence we have $|T(\omega_j) - a_i| > |T(\omega_i) - a_i|$. Now if $\omega_i$ is a point of type (4), then we choose $a_i$ such that $\omega_i \in D^+(a_i, r_k(\omega_i) + \varepsilon)$ with $\varepsilon > 0$ and for all $j \neq i$ we have $|T(\omega_j) - a_i| > |T(\omega_j) - a_i|$, which proves the statement.

Theorem 4.21. Let $\Omega \in E(k)$ and $d : \Omega \to \Omega$ be a $k$-linear bounded derivation. Let $P(d) \in \mathcal{O}_\Omega$ be a monic differential polynomial. Let $\{z_1, \ldots, z_n\}$ be the multiset of the roots of $P(T)$ (the commutative polynomial associated to $P(d)$). Assume that $\min r_k(\pi_{\Omega^{\text{alg}}/k}(z_i)) > \|d\|$. Let $\{\omega_1, \ldots, \omega_n\} := \pi_{\Omega^{\text{alg}}/k}(\{z_1, \ldots, z_n\})$ with $\omega_i \neq \omega_j$ if $i \neq j$ and $r_k(\omega_1) \leq \cdots \leq r_k(\omega_n)$. Then for each $\omega_i$ there exists a commutative monic polynomial $P_{\omega_i}(T)$ such that the projection of its roots in $A_k^{1, \text{an}}$ are equal to $\omega_i$, and we have $P(d) = P_{\omega_1}(d) \cdots P_{\omega_i}(d)$, where $P_{\omega_i}(d)$ are the differential polynomials associated to $P_{\omega_i}(T)$.

Proof. For each differential polynomial $Q(d)$, we denote by $\Sigma_Q$ the spectrum of the differential module $(\mathcal{O}_\Omega, \mathcal{O}_\Omega Q(d), d)$. Let $S$ be the multiset of roots of $P(T)$, in particular we have $\text{Card}(S) = \text{deg}(P)$, and let $S(\omega_i) := \{z_1, \ldots, z_{\text{ord} \omega_i}\}$ be the multiset of all the elements of $S$ such that $\pi_{\Omega^{\text{alg}}/k}(S(\omega_i)) = \{\omega_i\}$.

We will prove by induction that there exist $Q_i \in \Omega[T]$ where $S_i$ is the multiset of its roots, and $P_{\omega_i} \in \Omega[T]$ where $S_{\omega_i}$ is the multiset of its roots for each $j \in \{1, \ldots, i\}$, such that $P(d) = Q_1(d) \cdots Q_i(d) \cdots P_{\omega_i}(d)$, and $\text{Card}(S_i) \subset \text{Card}(S_{\omega_i})$ and $\text{deg}(P_{\omega_i}) = \text{Card}(S(\omega_i)) = \text{Card}(S(\omega_j))$ for each $j \in \{1, \ldots, i\}$. First of all, for each $\omega_i$ we will associate an $a_i \in k$ as in Lemma 4.20, and we set $r_1 = |T(\omega_i) - a_i|$. Now we prove the induction hypothesis for $i = 1$. For each $z_1 \in S$ with $\pi_{\Omega^{\text{alg}}/k}(z_1) \neq \omega_1$, we have $|z_1 - a_1| > r_1$. Since $r_1 > \|d\|$, then by Lemma 4.17 there exists a unique $Q, R \in \Omega[T]$ such that $P(d + a_1) = Q(d).R(d)$ and $\{a_1, \ldots, a_n\}$ (resp. $\{a_{n+1}, \ldots, a_n\}$) the multiset of roots of $R(T)$ (resp. $Q(T)$) with $|a_i| = r_1$ for all $l \leq n_1$ and $|a_i| > r_1$ for all $l > n_1$. We set $Q_1(T) = Q(T - a_1)$ and $P_{\omega_1}(T) = R(T - a_1)$ and $S_1$ (resp. $S_{\omega_i}$) the multiset of roots of $Q_1(T)$ (resp. $P_{\omega_1}(T)$). We clearly have $\text{Card}(S_1) = \text{Card}(S(\omega_1))$, it remains to prove that $\pi_{\Omega^{\text{alg}}/k}(S_{\omega_1}) = \{\omega_1\}$ and $\Sigma_{Q_1} \subset \{\omega_2, \ldots, \omega_n\}$. It is clear that $\pi_{\Omega^{\text{alg}}/k}(S_{\omega_1}) \subset \pi_{\Omega^{\text{alg}}/k}(S_{\omega_1}) = \emptyset$. Moreover if $z$ is a root of $Q_1(T)$ or $P_{\omega_1}(T)$ then we must have $r_k(z) > \|d\|$. On one hand we have the short exact sequence

\begin{equation}
0 \to (\mathcal{O}_\Omega/\mathcal{O}_\Omega Q_1(d) \to (\mathcal{O}_\Omega/\mathcal{O}_\Omega P(d), d) \to (\mathcal{O}_\Omega/\mathcal{O}_\Omega P_{\omega_1}(d), d) \to 0.
\end{equation}

On the other hand, by Theorem 4.15 we have $\Sigma_{Q_1} \subset \pi_{\Omega^{\text{alg}}/k}(S_{\omega_1})$, $\Sigma_{P_{\omega_1}} \subset \pi_{\Omega^{\text{alg}}/k}(S_{\omega_1})$ and $\Sigma P \subset \pi_{\Omega^{\text{alg}}/k}(S)$.

Since $\pi_{\Omega^{\text{alg}}/k}(S_{\omega_1}) \subset \pi_{\Omega^{\text{alg}}/k}(S_{\omega_1}) = \emptyset$, we must have $\Sigma_{Q_1} \subset \Sigma_{P_{\omega_1}} = \emptyset$. By Lemma 2.22, we have $\Sigma_{Q_1} \cup \Sigma_{P_{\omega_1}} = \Sigma P$. Hence $\Sigma_{Q_1}, \Sigma_{P_{\omega_1}} \subset \{\omega_1, \ldots, \omega_n\}$. Since $\omega_1$ is the only point such that $|T(\omega_1) - a_1| = r_1$, then we must have $\Sigma_{P_{\omega_1}} = \{\omega_1\}$ and $\Sigma_{Q_1} \subset \{\omega_2, \ldots, \omega_n\}$. If we suppose that there exists $\omega \in \pi_{\Omega^{\text{alg}}/k}(S_{\omega_1}) \setminus \{\omega_1\}$, then there exists $a \in k$ such that $|T(\omega) - a| < |T(\omega) - a_1| = r_1$. In particular we have $|a - a_1| = r_1$. This means
that there exists a root $z$ of $P(T)$ such that $\frac{\pi_{(\Omega_i^\mathcal{O}_d)/k}}{\pi_{(\Omega_i^\mathcal{O}_d)/k}}(z) \neq \omega_1$ and $|z - a| < r_1$, on the other hand we have

$$|z - a| = |z - a_1 + a_1 - a| = |z - a_1| > r_1,$$

which is impossible. Consequently, we have $\pi_{(\Omega_i^\mathcal{O}_d)/k}(\omega_1) = \{\omega_1\}$.

Assume now that we have $P(d) = Q_i(d) \cdot P_{\omega_1}(d) \cdots P_{\omega_1}(d)$ that satisfies the induction hypothesis for $\nu < i$. Let $S_{r+1}$ be the multiset of all the roots $z$ of $P(T)$ such that $|z - a_1| = r_1$. There exist $\text{Card}(S_{r+1}) - \text{Card}(S(\omega_1))$ roots $z \in S_{r+1}$ such that $\pi_{(\Omega_i^\mathcal{O}_d)/k}(z) = \omega_j$ with $j \leq i$, i.e., $|z - a_j| = r_j$. This means that if $R(d) = P_{\omega_1}(d) \cdots P_{\omega_1}(d)$, then $R(d)$ admits exactly $\text{Card}(S_{r+1}) - \text{Card}(S(\omega_1))$ roots $z$ counted with multiplicity such that $|z - a_1| = r_1$. Therefore $Q_i$ admits exactly $\text{Card}(S(\omega_1))$ roots $z$ counted with multiplicity such that $|z - a_1| = r_1$. On the other hand if $z \in S \setminus \bigcup_{j \leq i} S(\omega_j)$, then we have $|z - a_1| > r_1$. Hence, if $z \in S_i$ then $|z - a_1| \geq r_1$. By Lemma 4.17, there exists $Q_{i+1}$ (resp. $P_{\omega_1}(d)$) with $S_{i+1}$ (resp. $S_{i+1}$) the multiset of its roots, such that $Q_i(d) = Q_{i+1}(d) \cdot P_{\omega_1}(d)$, $\forall a \in S_{i+1}$ we have $|a - a_1| = r_1$, $\forall \alpha \in S_{i+1}$ we have $|a - a_1| > r_1$ and $\text{Card}(S_{i+1}) = \text{Card}(S(\omega_1))$. It remains to prove that $\pi_{(\Omega_i^\mathcal{O}_d)/k}(S_{i+1}) = \{\omega_1\}$ and $\Sigma_i \cap \{\omega_1, \cdots, \omega_n\}$. It is clear that $\pi_{(\Omega_i^\mathcal{O}_d)/k}(S_{i+1}) \cap \pi_{(\Omega_i^\mathcal{O}_d)/k}(S_{i+1}) = \emptyset$. Moreover if $\alpha$ is a root of $Q_{i+1}(T)$ or $P_{\omega_1}(T)$ then we must have $r_k(z) > ||d||$. On the other hand we have the short exact sequence

$$0 \rightarrow (\mathcal{O}_\Omega/\mathcal{O}_\Omega Q_{i+1}(d), d) \rightarrow (\mathcal{O}_\Omega/\mathcal{O}_\Omega Q_{i+1}(d), d) \rightarrow (\mathcal{O}_\Omega/\mathcal{O}_\Omega P_{\omega_1}(d), d) \rightarrow 0.$$  

On the other hand, by Theorem 4.15 we have $\Sigma_i \subset \pi_{(\Omega_i^\mathcal{O}_d)/k}(S_{i+1})$, $\Sigma_{P_{\omega_1}} \subset \pi_{(\Omega_i^\mathcal{O}_d)/k}(S_{i+1})$ and $\Sigma_P \subset \pi_{(\Omega_i^\mathcal{O}_d)/k}(S)$. Since $\pi_{(\Omega_i^\mathcal{O}_d)/k}(S) \cap \pi_{(\Omega_i^\mathcal{O}_d)/k}(S_{i+1}) = \emptyset$, we must have $\Sigma_i \subset \Sigma_{P_{\omega_1}} = \emptyset$. By Lemma 2.22, we have $\Sigma_i \cup \Sigma_{P_{\omega_1}} = \Sigma_i$. Hence $\Sigma_i \subset \{\omega_1, \cdots, \omega_n\}$. Since $\omega_1$ is the only point such that $|T(\omega_1) - a_1| = r_1$, then we must have $\Sigma_{P_{\omega_1}} = \{\omega_1\}$ and $\Sigma_{P_{\omega_1}} \subset \{\omega_2, \cdots, \omega_n\}$. If we suppose that there exists $\omega \in \pi_{(\Omega_i^\mathcal{O}_d)/k}(S_{i+1}) \setminus \{\omega_1\}$, then there exists $a \in k$ such that $|T(\omega) - a| < |T(\omega) - a_1| = r_1$. By Lemma 4.17, there exists $P_1$ and $P_2$ such that $P_{\omega_1}(d) = P_1(d)P_2(d)$ with the root $z$ of $P_1(d)$ (resp. $P_2(d)$) satisfying $|z - a| \geq r_1$ (resp. $|z - a| < r_1$). By the same argument as above we have $\Sigma_{P_1} \cup \Sigma_{P_2} = \emptyset$ and $\Sigma_{P_1} \cup \Sigma_{P_2} = \{\omega_1\}$, which is impossible. Therefore we have $\pi_{(\Omega_i^\mathcal{O}_d)/k}(S_{i+1}) = \{\omega_1\}$, which proves the induction for $i + 1$.

Let $Q_{i-1} \in \Omega[T]$ with $S_{i-1}$ the multiset of its roots and $P_{\omega_j} \in \Omega[T]$ with $S_{\omega_j}$ the multiset of its roots, for each $j \in \{1, \cdots, \nu - 1\}$, such that $P(d) = Q_{i-1}(d)P_{\omega_1}(d) \cdots P_{\omega_1}(d)$, $\Sigma_{Q_{i-1}} \subset \{\omega_j\}$, $\pi_{(\Omega_i^\mathcal{O}_d)/k}(S_{i-1}) = \{\omega_j\}$ and $\deg(P_{\omega_j}) = \text{Card}(S_{\omega_j}) = \text{Card}(S(\omega_j))$ for each $j \in \{1, \cdots, \nu - 1\}$. Then we must have $\deg(Q_{i-1}) = \text{Card}(S(\omega_i))$. Since $\Sigma_{Q_{i-1}} \neq \emptyset$, we have $\Sigma_{Q_{i-1}} = \{\omega_i\}$. We can prove as above that we must have $\pi_{(\Omega_i^\mathcal{O}_d)/k}(S_{i-1}) = \{\omega_i\}$. We set $P_{\omega_i} := Q_{i+1}$ and we get our decomposition $P(d) = P_{\omega_i}(d) \cdots P_{\omega_1}(d)$. \qed

Remark 4.22. This result can be seen as a strong version of the decomposition of a differential polynomial by its slopes.

Theorem 4.23 (Spectral version of Young’s theorem (strong version)). Let $\Omega \in E(k)$ and $d : \Omega \rightarrow \Omega$ be a $k$-linear bounded derivation. Let $(M, \nabla)$ be a differential module over $(\mathcal{O}_\Omega, \nabla) \cong (\mathcal{O}_\Omega/\mathcal{O}_\Omega.P(d), d)$ and $P(d) = \sum_{i=0}^n a_i d^i + \alpha$. Let $\{z_1, \cdots, z_n\} \subset \Omega^\mathcal{O}_d$ be the multiset of the roots of $P(T)$ (the commutative polynomial associated to $P(d)$). If $\min_i r_k(\pi_{(\Omega_i^\mathcal{O}_d)/k}(z_i)) > ||d||$, then

$$\pi_{(\Omega_i^\mathcal{O}_d)/k}(\Sigma_{\nabla,k}(M)) = \pi_{(\Omega_i^\mathcal{O}_d)/k}(\{z_1, \cdots, z_n\}).$$

Proof. We set $\pi_{(\Omega_i^\mathcal{O}_d)/k}(\{z_1, \cdots, z_n\}) = \{\omega_1, \cdots, \omega_n\}$ such that $r_k(\omega_1) \leq \cdots \leq r_k(\omega_n)$. By Theorem 4.21 we have $P(d) = P_{\omega_n}(d) \cdots P_{\omega_1}(d)$. We set $P_{\omega_1, \cdots, \omega_{i-1}}(d) := P_{\omega_i}(d) \cdots P_{\omega_1}(d)$ and $(M_{\omega_1, \cdots, \omega_{i-1}}, \nabla_{\omega_1, \cdots, \omega_{i-1}}) = (\mathcal{O}_\Omega/\mathcal{O}_\Omega.P_{\omega_1, \cdots, \omega_{i-1}}, d)$. By Theorem 4.15, we have $\Sigma_{\nabla,k}(M_{\omega_1, \cdots, \omega_{i-1}}) = \{\omega_i\}$. By induction and using Lemma 2.22 we obtain the result. \qed

Remark 4.24. Assume that $k$ is trivially valued. Let $r < 1$. If $(M, \nabla)$ is a differential module over $(\mathcal{H}(x_0, \mathcal{O}_d), \mathcal{O}_\mathcal{H})$ with irregular singularities, the use of Turrittin’s theorem in [Azz22] was unavoidable to determine the spectrum of $\nabla$. However, by using only Theorem 4.23 we can obtain the result directly.
5. Spectrum of differential module

The aim of this section is to determine the spectrum of a differential module \((M, \nabla)\) over \((\mathcal{H}(x), (S-c)\frac{d}{dx})\), where \(x \in (c, \infty)\) and \(c \in k\). Note that we can reduce the computation only for \(c = 0\).

The section is divided into four parts, the first one is to recall the definitions of subsidiary spectral radii of convergence. In the special case where they are small, we establish the link between the spectral radii and the spectrum. In the second, we determine the spectrum of the pull-back by the Frobenius of a differential module having small radii, and we establish the link between the spectrum and the spectral radii. In third part we announce and prove the main result of the paper. In the last part, we explain how we can deduce from the main result the shape of the spectrum of a differential module \((M, \nabla)\) over \((\mathcal{H}(x), d)\), where \(x\) is a point of a quasi-smooth curve of type (2) or (3), and \(d\) is well chosen \(k\)-linear bounded derivation.

**Convention 5.1.** We still assume that \(k\) is algebraically closed.

5.1. Link between the radius of convergence and the spectrum when the radii are small.

**Subsidiary spectral radii of convergence.** Let \(x \in k_{\text{an}} \setminus k\). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), \frac{d}{dx})\) with rank equal to \(n\). We set

\[
R^{(M,\nabla),\text{Sp}}_1(x) := \frac{\omega}{\|\nabla\|_{\text{Sp}}}
\]

Consider the following Jordan-Hölder sequence of \((M, \nabla)\)

\[
0 = M_0 \subset M_1 \subset \cdots \subset M_\nu = M.
\]

This means that for all \(i\), \(N_i := M_i/M_{i-1}\) has non trivial strict differential sub-modules. Let \(n_i\) be the rank of \(N_i\), and let \(R_i := R^{(N_i,\nabla_i),\text{Sp}}_1(x)\). Perform a permutation of the indexes to get \(R_1 \leq \cdots \leq R_\nu\). Let

\[
R^{(M,\nabla),\text{Sp}}(x) := \{R_1, \ldots, R_\nu\}
\]

be the sequence obtained from \(R_1 \leq \cdots \leq R_\nu\) by counting the value \(R_i\) with multiplicity \(n_i\), i.e.

\[
R^{(M,\nabla),\text{Sp}}(x) := \underbrace{R_1 = \cdots = R_1}_{n_1 \text{ times}} \leq \underbrace{R_2 = \cdots = R_2}_{n_2 \text{ times}} \leq \cdots \leq \underbrace{R_\nu = \cdots = R_\nu}_{n_\nu \text{ times}}.
\]

The values \(R^{(M,\nabla),\text{Sp}}_i(x)\) are called the subsidiary spectral radii of convergence of \((M, \nabla)\). We will just denote by \(R^{(M,\nabla),\text{Sp}}_i(x)\) when no confusion is possible.

**Definition 5.2.** For a differential module \((M, \nabla)\) over \((\mathcal{H}(x), g\frac{d}{dx})\) with \(g \in \mathcal{H}(x) \setminus \{0\}\), the subsidiary spectral radii of convergence of \((M, \nabla)\), that we still denote by \(R^{(M,\nabla),\text{Sp}}_i(x)\), are the subsidiary radii of \((M, g^{-1}\nabla)\) as a differential module over \((\mathcal{H}(x), \frac{d}{dx})\). If \(R^{(M,\nabla),\text{Sp}}_i(x) = \cdots = R^{(M,\nabla),\text{Sp}}_n(x)\), we say that \((M, \nabla)\) is pure.

**Lemma 5.3 ([Ked10, Lemma 6.2.3]).** Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), \frac{d}{dx})\). We have

\[
R^{(M,\nabla),\text{Sp}}_i(x) \leq r(x).
\]

**Definition 5.4.** Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), \frac{d}{dx})\). If \(R^{(M,\nabla),\text{Sp}}_i(x) = r(x)\) we say that \(R^{(M,\nabla),\text{Sp}}_i(x)\) is solvable.

5.1.1. Differential polynomial and radii of convergence. Let \(\Omega \in E(k)\) and let \(d : \Omega \rightarrow \Omega\) be a \(k\)-linear bounded derivation. Let \(\mathcal{L} := \sum_{i=0}^n f_{n-i} \cdot d^i\) be a differential operator with \(f_0 = 1\). Let \(\{\lambda_1, \ldots, \lambda_n\}\) be the multiset of roots of the commutative polynomial \(\tilde{\mathcal{L}} = \sum_{i=0}^n f_{n-i} T^i \in \Omega[T]\), and assume that \(|\lambda_n| \leq \cdots \leq |\lambda_1|\).

We set:

\[
R^{\mathcal{L},d}_i := \frac{\omega}{\max(\|d\|, |\lambda_i|)}.
\]
Lemma 5.5. Let \((M, \nabla)\) be a differential module over \((\Omega, d)\). Let \(L\) and \(L'\) be two attached differential operators of \((M, \nabla)\). Then for each \(i\) we have

\[
R_{i}^{L, d} = R_{i}^{L', d}
\]

Proof. See [Ked10, Corollary 6.5.4].

Theorem 5.6 ([You92], [Ked10, Theorem 6.5.3], [CM02, Theorem 6.2]). Let \(x \in \mathbb{A}^{1, \text{an}}_k \setminus k\), and \(L := \sum_{i=0}^{n-1} f_{n-i} \frac{d^i}{d\bar{s}^i} + \frac{d^n}{d\bar{s}^n}\) be a differential operator with coefficients in \(\mathcal{H}(x)\). Let \((M, \nabla)\) be the differential module over \((\mathcal{H}(x), \frac{d}{d\bar{s}})\) attached to \(L\). Then \(R_{i}^{L, \frac{d}{d\bar{s}}} < \omega \cdot r(x)\) if and only if \(R_{i}^{M, \text{Sp}}(x) < \omega \cdot r(x)\), and in this case we have

\[
R_{i}^{M, \text{Sp}}(x) = R_{i}^{L, \frac{d}{d\bar{s}}}.
\]

1.2. Link between the subsidiary radii and the spectrum, the case of small radii. Let \(x \in \mathbb{A}^{1, \text{an}}_k \setminus k\), and let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), \frac{d}{d\bar{s}})\). The starting point of our motivation to study the spectrum is the interesting relation between \(R_{i}^{M, \text{Sp}}(x)\) and the spectrum of \(\nabla\). Indeed, the smallest closed disk centered at zero and containing this component has radius equal to \(\frac{\omega}{R_{i}^{M, \text{Sp}}(x)}\). In our work [Azz20] we prove, in the case of constant coefficients, that if \(\Sigma_{\nabla} = \bigcup_i \Sigma_i\) where \(\Sigma_i\) are the connected component of the spectrum, then for each \(R_{j}^{M, \text{Sp}}(x)\) there exists \(\Sigma_j\) such that the smallest closed disk centered at zero and containing this component has radius equal to \(\frac{\omega}{R_{i}^{M, \text{Sp}}(x)}\), and conversely. In this paper we prove that this can be generalized to the case where the radii are small. Indeed from Theorem 4.23 and 5.6 we have the following result.

Proposition 5.7. Let \(x \in \mathbb{A}^{1, \text{an}}_k\) be a point of type (2), (3) or (4), and let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), \frac{d}{d\bar{s}})\). For each \(a \in k\) we set \((M_a, \nabla_a) := (M, \nabla - a)\). Suppose that \(R_{i}^{M_a, \text{Sp}} < \omega \cdot r(x)\) for each \(a\) and all \(i\). Then we have for each \(R_{i}^{M, \text{Sp}}\) there exists \(x_i \in \Sigma_{\nabla}\) such that

\[
|T(x_i)| = \frac{\omega}{R_{i}^{M, \text{Sp}}(x)},
\]

and conversely, the same holds for each \(x_i \in \Sigma_{\nabla}\).

If \(\text{char}(\bar{k}) = 0\), except if the radii are solvable, the radii are small and we can compute the spectrum of \(\nabla\). However, if \(\text{char}(\bar{k}) = p > 0\), there are many cases where the radii are neither small nor solvable. In such situation the use of the push-forward by \(\text{Frob}_p\) (cf. Section 6.2.2) is primordial to compute the radii, but it is not that easy to use the push-forward by \(\text{Frob}_p\) to determine the spectrum. Indeed, in order to use Proposition 2.28, we need to find \(g \frac{d}{d\bar{s}}\) such that \(\text{Frob}_p(g \frac{d}{d\bar{s}}) = \frac{d}{d\bar{s}}\), which is impossible. That is why it is more convenient to use the derivation \(S \frac{d}{d\bar{s}}\), in fact we have \(\text{Frob}_p(S \frac{d}{d\bar{s}}) = S \frac{d}{d\bar{s}}\).

In the following, we will show how we can recover the data of radii in the spectrum of a differential module \((M, \nabla)\) over \((\mathcal{H}(x), p^{l} \frac{d}{d\bar{s}})\).

Notation 5.8. We set \(s(i, j)\) and \(S(i, j)\) to be the numbers satisfying the following identities in \(\mathbb{N}[T]\):

\[
T(T - 1) \cdots (T - i + 1) = \sum_{j=0}^{i} s(i, j) T^j,
\]

\[
T^i = \sum_{j=0}^{i} S(i, j) T(T - 1) \cdots (T - j + 1).
\]

Lemma 5.9. Let \(x = x_0 + r \in \mathbb{A}^{1, \text{an}}_k \setminus k\). Let \((M, \nabla_M)\) (resp. \((N, \nabla_N)\)) be a differential module over \((\mathcal{H}(x), \frac{d}{d\bar{s}})\) (resp. \((\mathcal{H}(x), p^{l} \frac{d}{d\bar{s}})\)) with \(l \in \mathbb{N}\) such that \(L_{\nabla_M} := \sum_{i=0}^{n} f_{n-i} \frac{d^i}{d\bar{s}^i}\) with \(f_0 = 1\) (resp. \(L_{\nabla_N} := \sum_{i=0}^{n} g_{n-i} \frac{d^i}{d\bar{s}^i}\) with \(g_0 = 1\) (resp. \(L_{\nabla_M} := \sum_{i=0}^{n} h_{n-i} \frac{d^i}{d\bar{s}^i}\) with \(h_0 = 1\) (resp.)
\[ L_N := \sum_{i=0}^{n} g_{n-i}(p^i S \frac{d}{dS})^i \quad \text{with} \quad g_0 = 1 \] is an attached differential operator of \((M, \nabla_M)\) (resp. \((N, \nabla_N)\)). Then:

\[
L_p S \nabla_M := \sum_{i=0}^{n} p^{(n-i)} \left( \sum_{j=1}^{i} f_{n-j} S^{n-j} s(j, i) \right) \left( p^i S \frac{d}{dS} \right)^i
\]
is an attached differential operator of \((M, p^i S \nabla_M)\) (differential module over \((\mathcal{H}(x), p^i S \frac{d}{dS})\)), and

\[
L_{p^{-1} S^{-1}} \nabla_N := \sum_{i=0}^{n} S^{(i-n)} \left( \sum_{j=1}^{i} g_{n-j} p^{(j-n)} S(j, i) \right) \left( \frac{d}{dS} \right)^i
\]
is an attached differential operator of \((N, p^{-1} S^{-1} \nabla_N)\) (differential module over \((\mathcal{H}(x), \frac{d}{dS})\)).

**Proof.** We prove by induction that:

\[
(\nabla_M)^i = (S)^{-i} \sum_{j=1}^{i} p^{(i-j)} s(i, j) (p^j S \cdot \nabla_M)^j,
\]

\[
(\nabla_N)^i = p^i \sum_{j=1}^{i} S(i, j) S^j ((p^j S)^{-1} \cdot \nabla_N)^j.
\]

Let \(c\) be a cyclic vector of \((M, \nabla_M)\). Then:

\[
\sum_{i=0}^{n} f_{n-i} \nabla_M(c) = \sum_{i=0}^{n} f_{n-i} (S)^{-i} \sum_{j=1}^{i} p^{(i-j)} s(i, j) (p^j S \cdot \nabla_M)^j(c)
\]

\[
= \sum_{i=0}^{n} p^{(i-j)} \left( \sum_{j=1}^{i} f_{n-j} S^{n-j} s(j, i) \right) (p^i S \cdot \nabla_M)^i(c)
\]

\[=0.\]

Hence, multiplying the equality by \(p^n S^n\) we obtain

\[
\sum_{i=0}^{n} p^{(n-i)} \left( \sum_{j=1}^{i} f_{n-j} S^{n-j} s(j, i) \right) (p^i S \cdot \nabla_M)^i(c) = 0,
\]

and deduce that \(L_p S \nabla_M\) is an attached differential operator of \((M, p^i S \nabla_M)\).

Let \(e\) be a cyclic vector of \((N, \nabla_N)\). Then:

\[
\sum_{i=0}^{n} g_{n-i} \nabla_N(e) = \sum_{i=0}^{n} g_{n-i} p^i \sum_{j=1}^{i} S(i, j) S^j \left( (p^j S)^{-1} \cdot \nabla_N \right)^j(e)
\]

\[
= \sum_{i=0}^{n} S \sum_{j=1}^{i} g_{n-j} p^j S(j, i) \left( (p^j S)^{-1} \cdot \nabla_N \right)^i(e)
\]

\[=0.\]

Hence, by multiplying by \(p^{-1} S^{-1}\) we obtain

\[
\sum_{i=0}^{n} g_{n-i} \sum_{j=1}^{i} g_{n-j} p^{(j-n)} S(j, i) \left( (p^j S)^{-1} \cdot \nabla_N \right)^i(e) = 0,
\]

and deduce that \(L_{p^{-1} S^{-1}} \nabla_N\) is an attached differential module of \((N, p^{-1} S^{-1} \nabla_N)\).

**Remark 5.10.** We have a bijection

\[
\Lambda : \mathcal{D}_{\mathcal{H}(x), \frac{d}{dS}} P(\frac{d}{dS}) \quad \rightarrow \quad \mathcal{D}_{\mathcal{H}(x), p^i S \frac{d}{dS}},
\]

such that \(\Lambda(P)\) \((p^i S \frac{d}{dS})\) is the differential polynomial obtained from \(P(\frac{d}{dS})\) using the formula \((5.10)\), and \(\Lambda^{-1}(Q)(\frac{d}{dS})\) is the differential polynomial obtained from \(Q(p^i S \frac{d}{dS})\) using the formula \((5.11)\).
Definition 5.11. Let $(\Omega, |.|)$ be an ultrametric complete field. Let $P(T) = \sum_{i=0}^{n} a_i T^i \in \Omega[T]$. For $r > 0$, let us denote by $W_r(P)$ the width of $P$ under the $r$-Gauss norm $|.|_r$ (i.e., $|P|_r := \max_i |a_i| r^i$), as the difference between the maximum and minimum values of $i$ for which $\max_i |a_i| r^i$ is achieved.

Lemma 5.12. Let $(\Omega, |.|)$ be an ultrametric complete field. Let $P(T) = \sum_{i=0}^{n} a_i T^i \in \Omega[T]$ and let $r > 0$. If $W_r(P) = l$, then $P$ admits exactly $l$ roots, counted with multiplicity, whose absolute value are equal to $r$.

Proof. See [Ked10, Section 2.1]. \hfill \Box

Lemma 5.13. Let $x = x_{0,r} \in \mathbb{A}_k^{1,an} \setminus k$. Let $(M, \nabla_M)$ be a differential module over $(\mathcal{H}(x), \frac{d}{d\mathcal{H}})$ such that $\mathcal{L}_\nabla := \sum_{i=0}^{n} f_{n-i}(\frac{d}{d\mathcal{H}})^i$, with $f_0 = 1$, is an attached differential operator of $(M, \nabla)$. Let $\mathcal{L}_p^{\mathcal{S}}$ be an attached differential operator of $(M, p^l \mathcal{S})$. Suppose that $\mathcal{L}_\nabla = P_2(\frac{d}{d\mathcal{S}}) \cdot P_1(\frac{d}{d\mathcal{S}})$, Then there exists a decomposition of $\mathcal{R}_i^{Q_d} \mathcal{P}^d = \mathcal{R}_i^{Q_d, p^l \mathcal{S}^d} \mathcal{P}^d$.

Proof. Let $(M_1, \nabla_1) = (\mathcal{D}\mathcal{H}(x), \frac{d}{d\mathcal{H}}) / (\mathcal{D}\mathcal{H}(x), \frac{d}{d\mathcal{H}}) \cdot P_1(\frac{d}{d\mathcal{H}})$ and $(M_1, \nabla_2) = (\mathcal{D}\mathcal{H}(x), \frac{d}{d\mathcal{H}}) / (\mathcal{D}\mathcal{H}(x), \frac{d}{d\mathcal{H}}) \cdot P_2(\frac{d}{d\mathcal{H}})$. Then

(22) \quad 0 \rightarrow (M_2, \nabla_2) \rightarrow (M, \nabla) \rightarrow (M_1, \nabla_1) \rightarrow 0,

therefore

(23) \quad 0 \rightarrow (M_2, p^l \mathcal{S} \nabla_2) \rightarrow (M, p^l \mathcal{S} \nabla) \rightarrow (M_1, p^l \mathcal{S} \nabla_1) \rightarrow 0.

Consequently, on one hand there exist $Q_1(p^l \mathcal{S}^d)$ and $Q_2(p^l \mathcal{S}^d)$ such that $\mathcal{L}_p^{\mathcal{S}} = Q_2(p^l \mathcal{S}^d) \cdot Q_1(p^l \mathcal{S}^d)$,

(24) \quad (M_1, p^l \mathcal{S} \nabla_1) = (\mathcal{D}\mathcal{H}(x), p^l \mathcal{S}^d) / (\mathcal{D}\mathcal{H}(x), p^l \mathcal{S}^d) \cdot Q_1(p^l \mathcal{S}^d) \mathcal{P}^d.

On the other hand, by Lemma 5.9 we have $(M_1, p^l \mathcal{S} \nabla_1) = (\mathcal{D}\mathcal{H}(x), p^l \mathcal{S}^d) / (\mathcal{D}\mathcal{H}(x), p^l \mathcal{S}^d) \cdot \Lambda(\mathcal{P}_1, p^l \mathcal{S}^d)$.

Proposition 5.14. Let $x = x_{0,r} \in \mathbb{A}_k^{1,an} \setminus k$. Let $(M, \nabla_M)$ be a differential module over $(\mathcal{H}(x), \frac{d}{d\mathcal{H}})$ such that $\mathcal{L}_\nabla := \sum_{i=0}^{n} f_{n-i}(\frac{d}{d\mathcal{H}})^i$, with $f_0 = 1$, is an attached differential operator of $(M, \nabla)$. Let $\mathcal{L}_p^{\mathcal{S}}$ be an attached differential operator of $(M, p^l \mathcal{S} \nabla)$. Then $\mathcal{R}_i^{\mathcal{L} \cdot \mathcal{S}^d} < \omega \cdot r(x)$ if and only if $\mathcal{R}_i^{\mathcal{L}^{\mathcal{S}^d} \cdot p^l \mathcal{S}^d} < \omega \cdot |p|^{-l}$ and we have

(25) \quad \mathcal{R}_i^{\mathcal{L} \cdot \mathcal{S}^d} = \mathcal{R}_i^{\mathcal{L}^{\mathcal{S}^d} \cdot p^l \mathcal{S}^d}.

Proof. By Lemma 4.17 there exists a unique decomposition such that $\mathcal{L}_\nabla = \mathcal{P}_1 \cdot \mathcal{P}_0 \cdots \mathcal{P}_0$ with $\mathcal{R}_i^{\mathcal{P}_0 \cdot \mathcal{S}^d} = \mathcal{R}_i^{\mathcal{P}_0 \cdot \mathcal{S}^d} = \mathcal{P}_1 \cdot \mathcal{P}_0 \cdots \mathcal{P}_0$. For each $i \neq 0$, there exists $r_i < \omega r(x)$ such that $\mathcal{R}_i^{\mathcal{P}_1 \cdot \mathcal{S}^d} = r_i$. By induction and using Lemma 5.13 there exists a decomposition of $\mathcal{L}_p^{\mathcal{S} \nabla M} = \mathcal{P}_1 \cdot \mathcal{P}_0 \cdots \mathcal{P}_0$ such that $\mathcal{R}_i^{\mathcal{P}_1 \cdot \mathcal{S}^d} = \mathcal{R}_i^{\mathcal{P}_1 \cdot \mathcal{S}^d}$. Then in order to prove the statement it is enough to show that $\mathcal{R}_i^{\mathcal{P}_1 \cdot \mathcal{S}^d} = \mathcal{P}_1^{\mathcal{L}^{\mathcal{S}^d} \cdot p^l \mathcal{S}^d}$. Therefore, we can reduce to the case where $\mathcal{R}_i^{\mathcal{L} \cdot \mathcal{S}^d} = \rho$ for each $i$. We choose $\mathcal{L}_p^{\mathcal{S} \nabla \nabla M} = \Lambda(\mathcal{L} \cdot \mathcal{S}^d)$, we set

(26) \quad g_{n-i} = p^{(n-i)l} \sum_{j=i}^{n} f_{n-j} S^{n-j}(s, j, i),

then we get $\mathcal{L}_p^{\mathcal{S} \nabla \nabla M} = \sum_{i=0}^{n} g_{n-i} (p^l \mathcal{S}^d)^i$. We set $P(T) := \sum_{i=0}^{n} g_{n-i} T^i$. Suppose that $\rho = \omega r(x)$ in order to prove that $\mathcal{R}_i^{\mathcal{L}^{\mathcal{S}^d} \cdot p^l \mathcal{S}^d} = \omega |p|^{-l}$ for all $i$, we need to prove that the commutative polynomial $P$ does not admit a root $\lambda$ with $|\lambda| > |p|^l$. Since $\rho = \omega r(x)$, then

(27) \quad |f_{n-i} S^{n-i}| \leq 1.
Therefore we have

\[ |g_{n-i}| \leq |p|^{(n-i)t} \]

Let \( \alpha > |p|^t \) then we have

\[ \forall i, \quad |g_{n-i}| \alpha^i < |g_0| \alpha^n. \]

Hence, \( W_\alpha(P) = 0 \) and we conclude that \( P \) does not admit a root with absolute value greater than \( |p|^t \) (cf. Lemma 5.12). Consequently we obtain \( \mathcal{R}_i^{L_{p'\mathbb{S}^p\mathbb{S}^d_{\mathbb{Q}}}} = \frac{\omega}{|p'|^{S_{\mathbb{Q}}}} = \omega|p|^{-l} \). Now suppose that \( \rho < \omega.r(x) \).

Then we have

\[ |f_{n-i}S^{n-i}| \leq \left( \frac{\omega.r(x)}{\rho} \right)^{n-i}. \]

Consequently,

\[ |g_{n-i}| \leq |p|^{(n-i)t} \left( \frac{\omega.r(x)}{\rho} \right)^{n-i}. \]

In order to prove that \( \mathcal{R}_i^{L_{p'\mathbb{S}^p\mathbb{S}^d_{\mathbb{Q}}}} = \frac{\rho}{|p|^t} \), it is enough to prove that \( W_{|p|^t}\left( \frac{\omega.r(x)}{\rho} \right)(P) = n \). We have

\[ |g_{n-i}| |p|^l \left( \frac{\omega.r(x)}{\rho} \right)^i \leq |p|^{(n-i)t} \left( \frac{\omega.r(x)}{\rho} \right)^n \]

with \( |g_n| = |p|^n f_n S^n = |p|^{(n-l)t} \left( \frac{\omega.r(x)}{\rho} \right)^n \) and \( |g_0| |p|^l \left( \frac{\omega.r(x)}{\rho} \right)^n = |p|^{(n-l)t} \left( \frac{\omega.r(x)}{\rho} \right)^n \). Consequently,

\[ W_{|p|^t}\left( \frac{\omega.r(x)}{\rho} \right)(P) = n, \]

which completes the proof. \( \square \)

Now we can establish the link between the spectrum of a differential module over \( (\mathcal{H}(x), p' S_{\mathbb{Q}}^{d_{\mathbb{Q}}}) \) and the radii of convergence.

**Proposition 5.15.** Let \( x = x_{0.r} \in \mathbb{A}_k^{1,an} \setminus k \), and let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), p' S_{\mathbb{Q}}^{d_{\mathbb{Q}}})\). For each \( a \in k \) we set \((M_a, \nabla_a) := (M, \nabla - a)\). Suppose that \( \mathcal{R}_i^{M_a, Sp} < \omega.r(x) \) for each \( a \) and \( i \). Then for each \( \mathcal{R}_i^{M, Sp} \), there exists \( x_i \in \Sigma_\nabla \) such that

\[ \mathcal{R}_i^{M, Sp}(x) = \frac{\omega \cdot |p|^l r(x)}{|T(x_i)|}, \]

and conversely, the same holds for each \( x_i \in \Sigma_\nabla \).

**Proof.** The result holds directly from Theorem 4.23, 5.6 and Proposition 5.14. \( \square \)

**Corollary 5.16.** Let \( x = x_{0.r} \in \mathbb{A}_k^{1,an} \setminus k \), and let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), p' S_{\mathbb{Q}}^{d_{\mathbb{Q}}})\). For each \( a \in k \) we set \((M_a, \nabla_a) := (M, \nabla - a)\). If \((M_a, \nabla_a)\) is pure and with small radii for each \( a \in k \), then there exists \( z \in \mathbb{A}_k^{1,an} \setminus k \) such that \( \Sigma_\nabla = \{z\} \).

5.2. **Frobenius and spectrum.** As we explained before if \( \text{char}(\bar{k}) = 0 \), except the case where the radii are solvable, the condition of Theorem 4.23 are satisfied, which allows to determine the spectrum. However this is not the case when \( \text{char}(\bar{k}) = p > 0 \). That is why in this part we will focus on this case and assume from now on that \( \text{char}(\bar{k}) = p > 0 \).
Let \( x = x_{0,r} \) and \( y := \text{Frob}_p^i(x) \). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(y), p! S_{d_{\mathcal{H}}}/S_{d_{\mathcal{H}}}^n)\). According to formula (2.28), we have \((\text{Frob}_p^{i})^*(p! S_{d_{\mathcal{H}}}^n) = S_{d_{\mathcal{H}}}^n\). To avoid confusion, we set \(p! S_{d_{\mathcal{H}}}/S_{d_{\mathcal{H}}}^n : \mathcal{H}(y) \to \mathcal{H}(y)\) and \(S(x) d_{\mathcal{H}}/d_{\mathcal{H}}(x) : \mathcal{H}(x) \to \mathcal{H}(x)\). Recall from Lemma 2.16 that, we have

\[
\mathcal{H}(x) = \bigoplus_{i=0}^{p!-1} \mathcal{H}(y) \cdot S(x)^i.
\]

Let \( \{e_1, \ldots, e_n\} \) be a basis of \((M, \nabla)\) and let \( G \) be the associated matrix in this basis. Then \( \{e_1 \otimes 1, \ldots, e_n \otimes 1\} \) is a basis of \((\text{Frob}_p^i)^*(M, \nabla)\) for which the associated matrix is \( G \). Moreover, the following inequality holds [Ked10, Lemma 10.3.2]

\[
\mathcal{R}_1^{(\text{Frob}_p^i)^* M, Sp}(x) \geq \min(\mathcal{R}_1^{M, Sp}(y)_{p!}, p\mathcal{R}_1^{M, Sp}(y)).
\]

We set \((M_{p!}, \nabla_{p!}) := (\text{Frob}_p^i)(\text{Frob}_p^i)^*(M, \nabla)\). The associated matrix of \((M_{p!}, \nabla_{p!})\) in the basis \( \{e_1 \otimes 1, \ldots, e_n \otimes 1, e_1 \otimes S(x), \ldots, e_n \otimes S(x)\} \) is:

\[
\begin{pmatrix}
G & 0 & \cdots & 0 \\
0 & G + I_n & \cdots & 0 \\
0 & \cdots & 0 & G + (p! - 1) \cdot I_n
\end{pmatrix}
\]

Therefore we have the following isomorphism:

\[
(M_{p!}, \nabla_{p!}) \simeq \bigoplus_{i=0}^{p!-1} (M, \nabla + i).
\]

By Proposition 2.28 and Remark 2.24 we have

\[
\Sigma_{(\text{Frob}_p^i)^* \nabla, k}(\mathcal{L}_k((\text{Frob}_p^i)^* M)) = \Sigma_{\nabla, k}(\mathcal{L}_k(M_{p!})) = \bigcup_{i=0}^{p!-1} (\Sigma_{\nabla, k}(\mathcal{L}_k(M)) + i).
\]

**Theorem 5.17.** Let \( x = x_{0,r} \in \mathbb{A}^1_{\text{an}}/\kappa \), \( y := \text{Frob}_p^i(x) \) and consider the embedding \( \mathcal{H}(y) \hookrightarrow \mathcal{H}(x) \) induced by \( \text{Frob}_p^i \). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), S_{d_{\mathcal{H}}}^n)\). Assume that \((M, \nabla) \simeq (\mathcal{D}_{\mathcal{H}}(x)/\mathcal{D}_{\mathcal{H}}(x) \cdot P(S_{d_{\mathcal{H}}}^n), S_{d_{\mathcal{H}}}^n)\) with \( P(S_{d_{\mathcal{H}}}^n) = \sum_i a_i S_{d_{\mathcal{H}}}^n + S_{d_{\mathcal{H}}}^n \) and \( a_i \in \mathcal{H}(y) \). Let \( \{z_1, \ldots, z_n\} \subset \mathcal{H}(x)_{\text{alg}} \) be the multiset of the roots of \( P(T) \). If \( \min_i r_k(\pi_{\mathcal{H}(x)_{\text{alg}}/\kappa}(z_i)) > |p|! \), then

\[
\Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \bigcup_{i=0}^{p!-1} (\pi_{\mathcal{H}(x)_{\text{alg}}/\kappa}(\{z_1, \ldots, z_n\}) + i).
\]

**Proof.** Let \((N, \nabla_N) \simeq (\mathcal{D}_{\mathcal{H}}(y)/\mathcal{D}_{\mathcal{H}}(y) \cdot P(p! S_{d_{\mathcal{H}}}^n), p! S_{d_{\mathcal{H}}}^n)\). Then we have \((M, \nabla) \simeq (\text{Frob}_p^i)^*(N, \nabla_N)\). From (5.38) we have:

\[
\Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \bigcup_{i=0}^{p!-1} (\Sigma_{\nabla, k}(\mathcal{L}_k(N)) + i).
\]

It remains to prove that \( \Sigma_{\nabla, k}(\mathcal{L}_k(N)) = \pi_{\mathcal{H}(x)_{\text{alg}}/\kappa}(\{z_1, \ldots, z_n\}) \). Since \( \mathcal{H}(x) \) is an algebraic finite extension of \( \mathcal{H}(y) \) then we have \( \mathcal{H}(x)_{\text{alg}} \simeq \mathcal{H}(y)_{\text{alg}} \). Hence, we should have \( \pi_{\mathcal{H}(x)_{\text{alg}}/\kappa}(\{z_1, \ldots, z_n\}) = \pi_{\mathcal{H}(y)_{\text{alg}}/\kappa}(\{z_1, \ldots, z_n\}) \). Recall that we have \( |p! S_{d_{\mathcal{H}}}^n| = |p|! \) (cf. Lemma 3.5). Then by Theorem 4.23 we have \( \Sigma_{\nabla, k}(\mathcal{L}_k(N)) = \pi_{\mathcal{H}(x)_{\text{alg}}/\kappa}(\{z_1, \ldots, z_n\}) \), which ends the proof.
Now we need to establish, on the one hand, the link between the spectrum and the spectral radii of convergence. On the other hand, we are looking for conditions such that the hypothesis of Theorem 5.17 hold. For that we need the following result.

**Theorem 5.18** (after Christol and Dwork [Ked10, Theorem 10.4.2]). Let \( x = x_{0,r} \in \mathbb{A}^1_{k \text{an}} \setminus k \), \( y := \text{Frob}_p(x) \).

Let \((M, \nabla)\) be a finite differential module over \((\mathcal{H}(x), S_{\mathfrak{d}^n_{\mathcal{D}}})\) such that \( R^M_{i,\text{Sp}}(x) > \omega \cdot r(x) \). Then there exists a unique (up to isomorphism) \((N, \nabla_N)\) such that \((M, \nabla) \simeq (\text{Frob}_p)^*(N, \nabla_N)\) and \( R^N_{i,\text{Sp}}(y) > \omega^p \cdot r(y) \). Moreover,

\[
R^N_{i,\text{Sp}}(y) = R^M_{i,\text{Sp}}(x)^p.
\]

**Remark 5.19.** The differential module \((N, \nabla_N)\) in Theorem 5.18 is called the Frobenius antecedent of \((M, \nabla)\).

**Remark 5.20.** By induction we can prove that, if moreover \( R^M_{i,\text{Sp}}(x) > \omega^{p-1} \cdot r(x) \), then there exists a unique (up to isomorphism) \((N, \nabla_N)\) such that \((M, \nabla) \simeq (\text{Frob}_p)^*(N, \nabla_N)\) and \( R^N_{i,\text{Sp}}(y) > \omega^p \cdot r(y) \). Moreover, we have \( R^N_{i,\text{Sp}}(y) = R^M_{i,\text{Sp}}(x)^p \). We will call it the \( p \)-Frobenius antecedent of \((M, \nabla)\).

**Remark 5.21.** In particular if \((M, \nabla) = (\text{Frob}_p)^*(N, \nabla)\) and \( R^N_{i,\text{Sp}}(y) > \omega^p \cdot r(y) \), then we must have \( R^N_{i,\text{Sp}}(y) = R^M_{i,\text{Sp}}(x)^p \).

Therefore, the following proposition shows the link between the spectrum and the radius of convergence.

**Proposition 5.22.** Let \( x = x_{0,r} \in \mathbb{A}^1_{k \text{an}} \setminus k \), \( y := \text{Frob}_p(x) \) and we consider the embedding \( \mathcal{H}(y) \hookrightarrow \mathcal{H}(x) \) induced by \( \text{Frob}_p \).

Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), S_{\mathfrak{d}^n_{\mathcal{D}}})\). Assume that \((M, \nabla) \simeq (\mathcal{D}_{\mathcal{H}(x)}/\mathcal{D}_{\mathcal{H}(y)}) \cdot P(S_{\mathfrak{d}^n_{\mathcal{D}}}/S_{\mathfrak{d}^n_{\mathcal{D}}})\) with \( P(S_{\mathfrak{d}^n_{\mathcal{D}}}) = \sum_i a_i S_{\mathfrak{d}^n_{\mathcal{D}}} + S_{\mathfrak{d}^n_{\mathcal{D}}} \) and \( a_i \in \mathcal{H}(y) \). Let \( \{z_1, \ldots, z_n\} \subset \mathcal{H}(x)_{\text{alg}}^n \) be the multiset of the roots of \( P(T) \). If \( \min_i r_k(\pi_{\mathcal{H}(x)\mathcal{H}(y)/k}(z_i)) > |p|^{i} \) and \( \max_i |T(z_i)| < |p|^{r-1} \), then for each \( R^M_{i,\text{Sp}}(x) \) there exists \( z_i \) such that:

\[
R^M_{i,\text{Sp}}(x) = \frac{\frac{1}{\omega^p} \cdot |p|^{\frac{i}{p^r}} \cdot r(x)}{|T(z_i)|^{|p|}}.
\]

and conversely, the same holds for each \( z_i \).

**Proof.** Let \((N, \nabla_N) \simeq (\mathcal{D}_{\mathcal{H}(y)}/\mathcal{D}_{\mathcal{H}(y)}) \cdot P(S_{\mathfrak{d}^n_{\mathcal{D}}}/S_{\mathfrak{d}^n_{\mathcal{D}}})\). Then we have \((M, \nabla) \simeq (\text{Frob}_p)^*(N, \nabla_N)\). By Theorem 4.23 and Proposition 5.15 for each \( R^M_{i,\text{Sp}}(y) \) there exists \( z_i \) such that:

\[
R^N_{i,\text{Sp}}(y) = \frac{\omega \cdot |p|^{i} \cdot r(y)}{|T(z_i)|} = \frac{\omega \cdot |p|^{i} \cdot r(x)^p}{|T(z_i)|} > \omega^p \cdot r(x)^p = \omega \cdot r(y),
\]

By Remark 5.21 we obtain the result. \(\square\)

**Remark 5.23.** The condition \( \max_i |T(z_i)| < |p|^{r-1} \) of Proposition 5.22 means that if \((M, \nabla) \simeq (\text{Frob}_p)^*(N, \nabla_N)\), then \((N, \nabla_N)\) does not admit a Frobenius antecedent.

Note that in practice we can compute the spectrum in a more general case. However, it is not that easy to recover the link between the spectrum and the spectral radii of convergence as in formula (5.33) and (5.42). Indeed, let \( x = x_{0,r} \) with \( r > 0 \), given a differential module \((M, \nabla)\) over \((\mathcal{H}(x), S_{\mathfrak{d}^n_{\mathcal{D}}})\), the following results show how to compute the radii of \( \text{Frob}_p(M, \nabla)\).

**Proposition 5.24** ([Ked10, Theorem 10.5.1]). Let \( x = x_{0,r} \in \mathbb{A}^1_{k \text{an}} \setminus k \). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), S_{\mathfrak{d}^n_{\mathcal{D}}})\) of rank \( n \) with subsidiary spectral radii \( R^M_{1,\text{Sp}}(x) \leq \cdots \leq R^n_{\text{Sp}}(x) \). Then the multiset of spectral subsidiary radii of \( \text{Frob}_p(M, \nabla)\) is

\[
\bigcup_{i=1}^{n} \left\{ \left( R^M_{i,\text{Sp}}(x)^p, \omega^p r(x)^p (p-1 \text{ times}) \right) \cup \left\{ |p|^r(x)^p \left( R^M_{i,\text{Sp}}(x)^p \right) \right\} \right\}
\]

\[
R^M_{i,\text{Sp}}(x) \geq \omega r(x), \quad R^M_{i,\text{Sp}}(x) \leq \omega r(x).
\]
In practice, by induction, we can prove the following result.

**Corollary 5.25.** Let \( x = x_{0,r} \in A_k^{1,an} \setminus k \). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), S_{\frac{d}{dR}})\) of rank \(n\) with subsidiary radii \(R^M_{1,Sp}(x) \leq \cdots \leq R^M_{n,Sp}(x)\). Suppose that \((M, \nabla)\) is pure and \(\omega^{\rho_{M,S}}r(x) \leq R^M_{1,Sp}(x) \leq \omega^{\rho_{M,S}}r(x)\), with \(l \in \mathbb{N} \setminus \{0\}\). Then the multiset of subsidiary spectral radii of \((Frob_p)_*^l(M, \nabla)\) is

\[
\mathcal{R}^{l}_{\text{Sp}}(M, \nabla) = \bigcup_{i=2}^{l} \{ \rho \mid \omega r(x)^{\rho}(n(p-1)p^{i-1} \text{ times}) \} \cup \{ \rho \mid \omega r(x)^{\rho}(n(p-1) \text{ times}) \} \cup \{ \rho \mid \omega r(x)^{\rho}(n \text{ times}) \}.
\]

**Remark 5.26.** We keep the assumption of Corollary 5.25. We can observe from equation (5.45) that the greatest spectral subsidiary spectral radius of \((Frob_p)_*^l(M, \nabla)\) is equal to \(R^M_{1,Sp}(x)^{\rho_{M,S}}\). This criterion allows to recover easily the radii of convergence of \((M, \nabla)\).

**5.3. The main result.** We assume here that \(\text{char}(\tilde{k}) = p > 0\). Now we will determine the spectrum of any differential equation and establish the link with the subsidiary radii.

**Theorem 5.27** (Robba, [PP13a, Corollary 3.6.9], [Ked10, Theorem 10.6.2]). Let \( x = x_{0,r} \in A_k^{1,an} \setminus k \). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), S_{\frac{d}{dR}})\). Then there exists a unique decomposition

\[
(M, \nabla) = \bigoplus_{\rho \leq r(x)} (M_\rho, \nabla_\rho)
\]

of differential modules, such that every sub-quotient \((N, \nabla_N)\) of \((M_\rho, \nabla_\rho)\) satisfies \(R^M_{1,Sp}(x) = \rho\). This decomposition is called the spectral decomposition.

**Corollary 5.28.** The spectral decomposition (5.46) can be refined as follows:

\[
(M, \nabla) = \bigoplus_{i=1}^{\nu} (M_i, \nabla_i)
\]

such that for each \(i\) and for all \(a \in k\), every sub-quotient of \((M_i, \nabla_i - a)\) is pure with radius equal to \(R^{(M_i, \nabla_i - a), Sp}\).

**Proof.** We will proceed by contradiction and suppose that a decomposition

\[
(M, \nabla) = \bigoplus_{i=1}^{\nu} (M_i, \nabla_i)
\]

such that for each \(i\) and for all \(a \in k\) the differential module \((M_i, \nabla_i - a)\) is pure does not exist. Then we prove by induction that there exists a family of finite sets \((I_l)_{l \in \mathbb{N}}\) such that \(\text{Card}(I_l) < \text{Card}(I_{l+1})\) such that

\[
(M, \nabla) = \bigoplus_{i \in I_l} (M_i, \nabla_i),
\]

with \((M_i, \nabla_i) \neq 0\). We set \(I_0\) such that

\[
(M, \nabla) = \bigoplus_{i \in I_0} (M_i, \nabla_i),
\]

is the decomposition induced by Theorem 5.27. We suppose now that the induction hypothesis is true until \(l\), i.e. there exist \((I_j)_{j \leq l}\) with \(\text{Card}(I_j) < \text{Card}(I_{j+1})\) such that

\[
(M, \nabla) = \bigoplus_{i \in I_j} (M_i, \nabla_i),
\]

with \((M_i, \nabla_i) \neq 0\). By the contradiction hypothesis there exists \(i_0 \in I_l\) and \(a_l \in k\) such that \((M_{i_0}, \nabla_{i_0} - a)\) is not pure, then there exists a finite set \(J_l\) with \(\text{Card}(J_l) \geq 2\), such that

\[
(M_{i_0}, \nabla_{i_0} - a) = \bigoplus_{j \in J_l} (M_j, \nabla_j - a_l),
\]
with \((M_j, \nabla_j - a_l) \neq 0\). We set \(I_{i+1} = (I_i \setminus \{i_0\}) \coprod I_i\), then we get
\[
(M, \nabla) = \bigoplus_{i \in I_{i+1}} (M_i, \nabla_i),
\]
with \((M_i, \nabla_i) \neq 0\). Since for all \(l\) and all \(i \in I_l\) we have \((M_i, \nabla_i) \neq 0\), we should have for all \(l\) Card\((I_l) \leq \dim(M) = n\). Hence we obtain a strictly increasing sequence of bounded integers, which is absurd. We consider now a decomposition like in (5.48), then by Theorem 5.27 every sub-quotient of \((M_i, \nabla_i - a)\) is pure with radius equal to \(R^{(M_i, \nabla_i - a)\cdot Sp}\).

**Theorem 5.29** \(((\text{Christol-Dwork}) [\text{Ked10}, \text{Theorem 6.5.3}])\). Let \(\Omega \in E(k)\) and \(d : \Omega \to \Omega\) be a bounded derivation. Let \(\mathcal{L} \in \mathcal{D}_{\Omega} \setminus k\), we set \((M, \nabla) := (\mathcal{D}_{\Omega} \setminus \mathcal{D}_{\Omega} \cdot \mathcal{L}, d)\). Then we have
\[
\max(\|d\|, \|\nabla\|_{Sp}) = \frac{\omega}{R_1^{\mathcal{L}, d}}.
\]

**Remark** 3.30. From Theorem 5.29 we can deduce the following. Let \(x \in A^1_{k, \text{an}} \setminus k\). For a differential module \((M, \nabla)\) over \((\mathcal{H}(x), g_{\frac{d}{ds}})\), with \(g \in \mathcal{H}(x)\), if \(R^{1, Sp}_x(x) \geq \omega \cdot r(x)\), then we have
\[
\|\nabla\|_{Sp} \leq \|g_{\frac{d}{ds}}\|.
\]

**Lemma 5.31** \(((\text{Ked10}, \text{Corollary 10.6.3})\). Let \(x = x_{0, r} \in A^1_{k, \text{an}} \setminus k\). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), S_{\frac{d}{ds}})\). Suppose that all the spectral radii of \((M, \nabla)\) are not solvable, then \(\nabla\) is a bijective operator.

**Lemma 5.32.** Let \(x = x_{0, r} \in A^1_{k, \text{an}} \setminus k\) and we set \(x^p := \text{Frob}_p^l(x)\). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), S_{\frac{d}{ds}})\). For each \(a \in k\) we set \((M_a, \nabla_a) := (M, \nabla - a)\). Assume that \(R^{1, Sp}_x(M, \nabla)_a(x) < r\) for all \(i\) and \(a \in k\). There exists \(l \in \mathbb{N}\) such that all the radii of \((\text{Frob}_p^l)_* M, (\text{Frob}_p^l)_*(\nabla) - a\) are strictly less than \(\omega \cdot r(x^p)\) for each \(a \in k\). \(\square\)

**Proof.** First of all, since \(R^{1, Sp}_x(M, \nabla - a)\) is bijective. By Open Mapping Theorem (cf. [BGR84, Section 2.8.1]), \(\nabla - a\) is invertible in \(\mathcal{L}_k(M)\) for all \(a \in k\). Hence, \(\Sigma_{\nabla} \subset A^1_{k, \text{an}} \setminus k\). By Corollary 5.28 we can reduce to the case where \((M, \nabla - a)\) is pure for all \(a \in k\). If \((M, \nabla)\) satisfies the hypothesis of Theorem 4.23, it is clear that for each \(l \in \mathbb{N}\), all the radii of \((\text{Frob}_p^l)_*(M, \nabla - a)\) are strictly less than \(\omega \cdot r(x^p)\) for each \(a \in k\).

Now suppose that there exists \(a \in k\) such that \(R^{1, Sp}_x(M, \nabla - a)\) is bijective. We prove the statement by contradiction. We assume that for each \(l \in \mathbb{N}\) there exists \(a_l \in k\) such that \(R^{1, Sp}_x(M, \nabla - a) \geq \omega \cdot r(x^{p^l})\). By Remark 5.20, there exists a unique (up to an isomorphism) differential module \((N_{p^l}, \nabla_{p^l})\) over \((\mathcal{H}(x^{p^l}), p^l S_{\frac{d}{ds}})\) such that \((M_{a_l}, \nabla_{a_l}) = (\text{Frob}_p^l)_*(N_{p^l}, \nabla_{p^l})\). Hence, we have \(R^{1, Sp}_{p^l}(N_{p^l}, \nabla_{p^l}) \geq \omega r(x^{p^l}) > \omega r(x^{p^l})\). By Remark 5.30, we have
\[
\|\nabla_{p^l}\|_{Sp} \leq \|p^{l} S_{\frac{d}{ds}}\| = |p|^l.
\]

Therefore we obtain
\[
\Sigma_{\nabla_{p^l}} \subset D^+(0, |p|^l).
\]
Hence, by formula (5.38), for all \(l \in \mathbb{N}\) we have
\[
\Sigma_{\nabla} \subset \bigcup_{i=0}^{p^l-1} D^+(a_l + i, |p|^l).
\]
Consequently, we have
\[
\Sigma_{\nabla} \subset \bigcap_{l \in \mathbb{N}} \bigcup_{i=0}^{p^l-1} D^+(a_l + i, |p|^l).
\]
This means in particular that $\Sigma_{\nabla} \subset k$, which contradict the fact that $\Sigma_{\nabla} \subset \Lambda_k^{1,an} \setminus k$. \qed

**Remark 5.33.** Lemma 5.32 shows in particular that, if $(M_a, \nabla_a)$ is not solvable for all $a \in k$, then there exists $l \in \mathbb{N}$ such that for all $a \in k$ and all $i$ we have $R^a(M_a,Sp)(x) < \omega^p r(x)$.

**Definition 5.34.** Let $\Omega \in E(k)$. We define the following map

\begin{equation}
\delta_{\Omega} : \Omega \to \mathbb{R}_+, \quad z \mapsto \inf_{n \in \mathbb{Z}} |z - n|.
\end{equation}

We will drop the subscription $\Omega$ when no confusion is possible.

**Lemma 5.35.** Let $x = x_{0,r}$. Let $a \in k$, and $R_a(x)$ be the spectral radius of convergence of $(H(x), S_{dS}^{ \frac{d}{ds}} - a)$. We have

\begin{equation}
R_a(x) := \begin{cases} 
\frac{\omega(r)}{\delta(a)} & \text{if } a \in \mathbb{Z}_p, \\
\left(\frac{|p|^r}{\delta(a)}\right)^{\frac{1}{\omega_r}} r(x) & \text{if } |p| < \delta(a) \leq |p|^{-1}, \ l \in \mathbb{N} \setminus \{0\} \\
\omega_r(x) & \text{if } \delta(a) > 1.
\end{cases}
\end{equation}

**Proof.** Let $a \in k$. If $\delta(a) > 1$ then $\delta(a) = |a|$. By Theorem 5.6 and Proposition 5.14 we have $R_a(x) = \omega r(x)$. Let $(M, \nabla) = (\text{Frob}_p)^i_a(H(x), S_{dS}^{ \frac{d}{ds}} - a)$ with $l \in \mathbb{N}$. Then we have $(M, \nabla) = \bigoplus_{i=0}^l (H(x)^{p^l}, p^l S_{dS}^{ \frac{d}{ds}} - a + i)$. If we assume that $a \in \mathbb{Z}_p$, then we have $\delta(a) = 0$. This means that for each $l \in \mathbb{N}$, there exists $i_0 < p^l$ such that $|a - i_0| \leq |p|^l$. Since $|a - i_0| \leq |p|^l$, then $R_1(\omega(r)^{p^l} S_{dS}^{ \frac{d}{ds}} - a + i_0) \geq \omega r(x^{p^l})$. Therefore, $\max_i R_i^{M,Sp}(x^{p^l}) \geq \omega r(x^{p^l})$. Hence, for all $l \in \mathbb{N}$ we have $R_a(x) \geq \omega \frac{1}{\omega_r} r(x)$ (cf. Corollary 5.25), and we deduce that $R_a(x) = r$. Now if we suppose that $|p|^l < \delta(a) \leq |p|^{-1}$ for some $l \in \mathbb{N} \setminus \{0\}$. Then there exists $i_0 < p^l$ such that $|a - i_0| = |a - i_0|$. On the other hand we have $\max_i R_i^{M,Sp}(x^{p^l}) = R_1^{(\omega(r)^{p^l} S_{dS}^{ \frac{d}{ds}} - a + i_0)}(x^{p^l}) = \omega r(x^{p^l})$ (cf. Theorem 5.6 and Proposition 5.14). Hence, by Corollary 5.25 we obtain $R_a(x) = \left(\frac{|p|^r \omega r(x)}{\delta(a)}\right)^{\frac{1}{\omega_r}}$. \qed

**Proposition 5.36.** Let $x = x_{0,r} \in \mathbb{A}_k^{1,an} \setminus k$. Let $(M, \nabla)$ be a differential module over $(H(x), S_{dS}^{ \frac{d}{ds}})$. For each $a \in k$ we set $(M_a, \nabla_a) := (M, \nabla - a)$. Assume that $(M_a, \nabla_a)$ is pure and non-solvable for all $a \in k$. Then there exists $z := x_{c,\rho} \in \mathbb{A}_k^{1,an} \setminus k$ such that:

1. $\Sigma_{\nabla} = \{z\} + \mathbb{Z}_p$;
2. if $\rho > 1$, then $\Sigma_{\nabla} = \{z\}$, and we have $R_1^{M,Sp}(x) = \min(\frac{\omega}{\rho} r(x), R_{c-a}(x))$;
3. if $|p|^l < \rho < |p|^{-1}$, then $\Sigma_{\nabla} = \{x_{c,\rho}, \ldots, x_{c + p^l - 1, \rho}\}$, and we have $R_1^{M,Sp}(x) = \min \left(\omega^r r(x), R_{c-a}(x)\right)$;
4. if $\rho = |p|^l$, then $\Sigma_{\nabla} = \{x_{c,\rho}, \ldots, x_{c + p^l - 1, \rho}\}$, and we have $R_1^{M,Sp}(x) = \min(\omega^r r(x), R_{c-a}(x))$;
5. if $|p|^l < \rho \leq |p|^{-1}$ (resp. $\rho > 1$), let $P(d)$ be a differential polynomial associated to $\text{Frob}_p^i(M, \nabla)$ (resp. $(M, \nabla)$), with $d = p^l S_{dS}^{ \frac{d}{ds}}$. Let $P(T)$ be the commutative polynomial associated to $P(d)$, then the image by $\pi_{\text{Alg}(\nabla)^{alg}/k}$ (resp. $\pi_{\text{Alg}(\nabla)^{alg}/k}$) of all roots of $P(T)$ is the set $\Sigma_{\nabla}$.

**Remark 5.37.** The values of the radius of convergence in (2), (3) and (4) of Proposition 5.36 can be resumed by one formula as follows:

\begin{equation}
R_1^{M,Sp}(x) = \left(\frac{|p|^l \omega}{\delta(||T(z) - a||)}\right)^{\frac{1}{\omega_r}} r(x),
\end{equation}

where $l_a := \max(0, \left\lfloor \frac{\log(\delta(T(z) - a))}{\log |p|} \right\rfloor + 1)$.

**Proof of Proposition 5.36.** We set $x^{p^l} := \text{Frob}_p^i(x)$. By Remark 5.33 there exists $l \in \mathbb{N}$ such that for all $a \in k$ we have $R_1^{M,Sp}(x) \leq \omega^r r(x)$, we assume that $l$ is the smallest one verifying this inequality. Let $a \in k$
such that $\omega_{\frac{|a|}{p}} r(x) < R_{1}^{M_{a} Sp}(x) \leq \omega_{\frac{|a|}{p}} r(x)$, without loss of generality we can suppose that $a = 0$. This means that there exists a unique (up to isomorphism) $(N_{p'}, \nabla_{p'})$ such that $(M, \nabla) = (\text{Frob}_{p'})^{*} (N_{p'}, \nabla_{p'})$ and $R_{1}^{M_{a} Sp}(x)^{p} = R_{1}^{N_{p'} Sp}(x^{p})$. Since $(M, \nabla)$ is pure then so is for $(N_{p'}, \nabla_{p'})$.

The idea is to first prove that $\Sigma_{\nabla_{p'}} = \{z\}$, where $z \in \mathbb{A}_{k, \text{an}}^{1} \setminus k$. Firstly, we need to prove that for each $a \in k$ the differential module $(N_{p'}, \nabla_{p'} - a)$ is pure. Let $a \in k$, we proceed case by case.

1. **Case where $\delta(a) \geq |p|^{-1}$**
   In this case, the radius of $(\mathcal{H}(x^{p}), p^{l} S_{\frac{d}{dS}} - a)$ is equal to $\frac{|p^{l} \omega r(x)^{p}}{\delta(p)}$ (cf. Lemma 5.35). In particular, we have $\frac{\omega r(x^{p})}{\delta(p)} \leq |p| \omega r(x)^{p}$. Consequently, all the radii of $(N_{p'}, \nabla_{p'} - a)$ should be equal to $\frac{\omega r(x^{p})}{\delta(p)}$. Hence, $(N_{p'}, \nabla_{p'} - a)$ is pure with small radii.

2. **Case where $\delta(a) < |p|^{-1}$**
   Since here the radius of $(\mathcal{H}(x), S_{\frac{d}{dS}} - a)$ is strictly greater than $\omega_{\frac{|a|}{p}} r(x)$ (cf. Lemma 5.35), then $\omega_{\frac{|a|}{p}} r(x) < R_{1}^{M_{a} Sp}(x) \leq \omega_{\frac{|a|}{p}} r(x)$.
   (a) **If $\delta(a) = 0$, then there exists $i_{0} \in \mathbb{N}$ such that $i_{0} < p^{l}$ such that $a - i_{0}$ is divisible by $p^{l}$. Hence, $\delta(\frac{a - i_{0}}{p^{l}}) = 0$, then $(N_{p'}, \nabla_{p'} - a + i_{0})$ has the same radii as $(N_{p'}, \nabla_{p'})$, i.e. strictly greater than $|p| \omega r(x)^{p}$. Therefore, $(N_{p'}, \nabla_{p'} - a + i_{0})$ is the $l$-Frobenius antecedent of $(M_{a}, \nabla)$. Since $\delta(\frac{i_{0}}{p^{l}}) = \frac{|i_{0}|}{|p^{l}|} \geq |p|^{-1}$, the radii of $(\mathcal{H}(x^{p}), p^{l} S_{\frac{d}{dS}} + i_{0})$ are less or equal to $|p| \omega r(x)^{p}$. Therefore, $(N_{p'}, \nabla_{p'} - a)$ is pure with radii equal to $R_{i_{0}}(x^{p})$.
   (b) **If $\delta(a) = |a - i_{0}|$, with $i_{0} \in \mathbb{N}$ and $|i_{0}| \leq |p|^{l}$, then $\delta(\frac{a}{p^{l}}) = \frac{|a|}{|p^{l}|}$. Therefore the radius of $(\mathcal{H}(x^{p}), p^{l} S_{\frac{d}{dS}} - a)$ is strictly greater than $|p| \omega r(x)^{p}$. Hence, the smallest radius of $(N_{p'}, \nabla_{p'} - a)$ is strictly greater than $|p| \omega r(x)^{p}$. Consequently, $(N_{p'}, \nabla_{p'} - a)$ is the $l$-Frobenius antecedent of $(M_{a}, \nabla)$, which implies that it is pure.
   (c) **If $\delta(a) = |a - i_{0}|$, with $i_{0} \in \mathbb{N}$ and $|i_{0}| > |p|^{l}$, then the $l$-Frobenius antecedent of $(M_{a}, \nabla)$ is $N_{p'}(\nabla_{p'} - a + i_{0})$. This means that $(N_{p'}, \nabla_{p'} - a + i_{0})$ is pure. Since $(N_{p'}, \nabla_{p'} - a + i_{0}) = (N_{p'}, (\nabla_{p'} - a + i_{0}) - i_{0})$, by the same argument as above all the radii of $(N_{p'}, \nabla_{p'} - a)$ are small and should be equal to $\frac{\omega r(x^{p})}{\delta(p^{l})}$.

Then we conclude that for each $a \in k$, the differential module $(N_{p'}, \nabla_{p'} - a)$ is pure, moreover except the case (2), (b), the differential module has small radii.

If we assume that for all $a \in k$, we have $R_{1}^{M_{a} Sp}(x) < \omega_{\frac{|a|}{p^{l}}} r(x)$, then for each $a \in k$, the differential module $(N_{p'}, \nabla_{p'} - a)$ is pure with small radii. By Proposition 5.15 and Corollary 5.16 we have $\Sigma_{\nabla_{p'}} = \{z\}$ with

$$\frac{R_{1}^{(N_{p'}, \nabla_{p'} - a), Sp}(x^{p})}{|T(z) - a|} = \frac{|p|^{l} \omega r(x)^{p}}{|T(z) - a|}. \tag{5.58}$$

In this case we have $r(z) > |p|^{l}$.

Now if we assume that there exists $a_{0} \in k$ such that $R_{1}^{M_{a_{0}} Sp}(x) = \omega_{\frac{1}{p^{l}}} r(x)$. We set $(N, \nabla_{N}) = (\text{Frob}_{p})^{*} (N_{p'}, \nabla_{p'})$. Since $(N_{p'}, \nabla_{p'} - a)$ is pure with radii less than or equal to $\omega r(x^{p})$ for each $a \in k$, $(N, \nabla_{N} - a)$ is pure with small radii for each $a \in k$ (cf. Proposition 5.24). Hence, by Proposition 5.15, Corollary 5.16 and Proposition 2.28, we have $\Sigma_{\nabla_{p'}} = \Sigma_{\nabla} = \{z\}$ with

$$\frac{R_{1}^{(N, \nabla_{N} - a), Sp}(x^{p})}{|T(z) - a|} = \frac{|p|^{l+1} \omega r(x)^{p^{l+1}}}{|T(z) - a|}. \tag{5.59}$$
and by Proposition 5.24 we get

\[
\mathcal{R}^{(N_p, \nabla, p^t - a), \text{Sp}}_1(x^p) = \left| p^t \omega r(x)^p \right| \left| T(z) - a \right|.
\]

Since \( \mathcal{R}^{(N_p, \nabla, p^t - a), \text{Sp}}_1(x^p) \leq \omega r(x)^p \) for all \( a \in k \), and \( \mathcal{R}^{M_{a_0}, \text{Sp}}_1(x) = \omega^p r(x) \) then we have \( r(z) = \left| p^t \right| \). By formula (5.38) we obtain

\[
\Sigma = \{ z + 1, \ldots, z + p^t - 1 \} = \{ z \} + \mathbb{Z}_p.
\]

In the first cases i.e \( r(z) > \left| p^t \right| \), if \( P(p^t S \frac{d}{dz}) \) is a differential polynomial of \( (\text{Frob}_p)^* (M, \nabla) \) and \( P(T) \) its associated commutative polynomial, then \( \Sigma \) is the image of all the roots of \( P(T) \) by \( \pi \frac{H(x^p)}{\alpha_{\lambda} k} \). In the case where \( r(z) = \left| p^t \right| \), if \( P(p^t S \frac{d}{dz}) \) is a differential polynomial of \( (\text{Frob}_p)^* (M, \nabla) \) and \( P(T) \) its associated commutative polynomial, then \( \Sigma \) is the image of all the roots of \( P(T) \) by \( \pi \frac{H(x^p)}{\alpha_{\lambda} k} \).

Since \( r(z) \geq \left| p^t \right| \), then \( \delta(T(z) - a) = \min \{ |T(z) - a|, |T(z) - a + 1|, \ldots, |T(z) - a + p^t - 1| \} \). We have \( (\text{Frob}_p)^* (M, \nabla - a) = \bigoplus_{i=0}^{p^t-1} (N_{p^t}, \nabla_{p^t} - a + i) \) (cf. (5.37)) with maximal radius of convergence equal to

\[
\mathcal{R} = \frac{\left| p^t \omega r(x)^p \right|}{\delta(T(z) - a)}
\]

If \( \left| p^{t+1} \omega r(x)^p \right| < \mathcal{R} \leq \left| p^t \omega r(x)^p \right| \) with \( i \in \mathbb{N} \), on the one hand, we have \( |p|^t - i \leq \delta(T(z) - a) < |p|^t - i - 1 \). We set \( l_a = \left[ \frac{\log \delta(T(z) - a)}{\log |p|} \right] + 1 = l - i \). On the other hand, by induction, Proposition 5.24 and Remark 5.31, we have

\[
\mathcal{R}^{M_{a_0}, \text{Sp}}_1(x)^{p^t} = \frac{\mathcal{R}}{|p|^t r(x)^{p^t - p^t_a}} = \frac{|p|^t \omega r(x)^{p^t_a}}{\delta(T(z) - a)},
\]

giving the desired result.

**Corollary 5.38.** Let \( x = x_{0, r} \in \mathbb{A}_{k, \text{an}}^1 \setminus k \). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), S \frac{d}{dz})\). For each \( a \in k \) we set \((M_a, \nabla_a) := (M, \nabla - a)\). Assume that \((M_a, \nabla_a)\) is non-solvable for each \( a \in k \). If there exists \( z \in \mathbb{A}_{k, \text{an}}^1 \setminus k \) such that \( \Sigma = \{ z \} + \mathbb{Z}_p \), then \((M_a, \nabla_a)\) is pure for each \( a \in k \) and we have \( \mathcal{R}^{M_{a_0}, \text{Sp}}_1(x) = \left( \frac{|p|^t \omega}{\delta(T(z) - a)} \right)^{p^t} r(x) \) with \( l_a := \max(0, \left[ \frac{\log \delta(T(z) - a)}{\log |p|} \right] + 1) \).

**Remark 5.39.** Note that the condition \( \mathcal{R}^{M_{a_0}, \text{Sp}}_1(x) < r \) for all \( i \) and \( a \in k \), excludes the case of a differential module with regular singularities. However this does not mean that if \((M, \nabla)\) is not with regular singularities then \( \mathcal{R}^{M_{a_0}, \text{Sp}}_1(x) < r \). Indeed, for example for \( x = x_{0, \omega} \), the differential module \((\mathcal{H}(x), \nabla)\) with \( \nabla := S \frac{d}{dz} - S \) is solvable.

Now we treat the remaining case, the case where a differential module (or its translate) admits a solvable radii.

**Proposition 5.40.** Let \( x = x_{0, r} \in \mathbb{A}_{k, \text{an}}^1 \setminus k \). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), S \frac{d}{dz})\) such that \( \mathcal{R}^{M_{a_0}, \text{Sp}}_1(x) = r(x) \). Then we have

\[
\Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \mathbb{Z}_p.
\]

**Proof.** We set \( x^{p^t} = \text{Frob}_p^i(x) \). The proof is slightly like the proof of Proposition 3.7. Since \( \mathcal{R}^{M_{a_0}, \text{Sp}}_1(x) = r(x) \) by Remark 5.20, for each \( i \in \mathbb{N} \), there exists \((M_{p^t}, \nabla_{p^t})\) a differential module over \((\mathcal{H}(x^{p^t}), p^t S \frac{d}{dz})\) such that \((M, \nabla) = (\text{Frob}_p)^* (M_{p^t}, \nabla_{p^t})\), moreover \( \mathcal{R}^{M_{a_0}, \text{Sp}}_1(x^{p^t}) = r(x)^{p^t} = r(x^{p^t}) \). By Formula (5.38), we have

\[
\Sigma_{\nabla, k}(\mathcal{L}_k(M)) = \bigcup_{i=0}^{p^t-1} (\Sigma_{\nabla, k}^{p^t}(\mathcal{L}_k(M_{p^t}))) + i).
\]
Since \((M^p, \nabla_p)\) is a solvable differential module over \((\mathcal{H}(x^{p^r}), p^l S_{\frac{d}{d5}})\) and \(\|S_{\frac{d}{d5}}\| = |p|^l\), by Remark 5.30, we have
\[
\|\nabla\|_{S^p} \leq |p|^l.
\]
Consequently, we deduce that \(\Sigma_{\nabla_p}k(L_k(M^p)) \subset D^+(0, |p|^l)\). Then we obtain
\[
\Sigma_{\nabla,k}(L_k(M)) \cap \bigcap_{i \in \mathbb{N}} D^+(i, |p|^l) = Z_{p}.
\]
As \(\Sigma_{\nabla,k}(L_k(M)) \neq \emptyset\), there exists \(a \in Z_{p} \cap \Sigma_{\nabla,k}(L_k(M))\). Then we have \(a + Z_{p} = \Sigma_{\nabla,k}(L_k(M))\). Since the spectrum is compact, we have \(a + Z_{p} = \Sigma_{\nabla,k}(L_k(M))\). Then the result follows. \(\square\)

Corollary 5.41. Let \(x = x_{0,r} \in \mathbb{A}_k^1,_{\text{an}}\setminus k\). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), S_{\frac{d}{d5}})\). If \(\Sigma_{\nabla} = a + Z_{p}\) with \(a \in k\) then \((M, \nabla)\) is pure with radius equal to \(R_{a}(x)\).

Corollary 5.42. Let \(x = x_{0,r} \in \mathbb{A}_k^1,_{\text{an}}\setminus k\). Let \((M, \nabla)\) be a differential module over \((\mathcal{H}(x), S_{\frac{d}{d5}})\). If for all \(a \in k\), \((M, \nabla - a)\) is pure, then any sub-quotient of \((M, \nabla)\) has the same spectrum as \((M, \nabla)\).

Notation 5.43. Let \(S \subset \mathbb{A}_k^1,_{\text{an}}\), we denote by \(S/Z_{p}\) the quotient set of \(S\) by the equivalence relation:
\[
x \sim x' \iff \exists n \in Z_{p} ; \ x = x' + n.
\]
Remark 5.44. Note that if \(z, z' \in \mathbb{A}_k^1,_{\text{an}}\) and \(z \sim z'\), then we have \(\delta(T(z) - a) = \delta(T(z') - a)\) for all \(a \in k\).

Now we can announce the main theorem of the paper.

Theorem 5.45. Assume that \(\text{char}(\mathbb{k}) = p > 0\) and \(x := x_{0,r} \in \mathbb{A}_k^1,_{\text{an}}\setminus k\). Let \((M, \nabla)\) be a differential module \((\mathcal{H}(x), S_{\frac{d}{d5}})\). We denote \(\text{Frob}_{p}(x)\) by \(x^{p^r}\).

- There exist \(z_1, \cdots, z_v \in \mathbb{A}_k^1,_{\text{an}}\) and \(a_1, \cdots, a_{\mu} \in k\), such that
\[
\Sigma_{\nabla,k}(L_k(M)) = \{z_1, \cdots, z_v, a_1, \cdots, a_{\mu}\} + Z_{p},
\]
where \(z_i\) has the same type as \(x\), and \((\nu, \mu)\) is not equal to \((0, 0)\).
- We can choose \(z_i\) and \(a_j\) such that the set \(\{z_1, \cdots, z_v, a_1, \cdots, a_{\mu}\}\) has minimal cardinality. Indeed it is enough to keep only \(z_i\) and \(a_j\), for which we have \(\{z_i\} + Z_{p} \cap \{z_v\} + Z_{p} = \emptyset\) and \(\{a_j\} + Z_{p} \cap \{a_{j'}\} + Z_{p} = \emptyset\) for \(i \neq i'\) and \(j \neq j'\).
- We choose \(\{z_1, \cdots, z_v, a_1, \cdots, a_{\mu}\}\) to be minimal. Then we have a unique (up to an isomorphism) decomposition
\[
(M, \nabla) = \bigoplus_{i=1}^{\nu} (M_{z_i}, \nabla_{z_i}) \oplus \bigoplus_{j=1}^{\mu} (M_{a_j}, \nabla_{a_j}),
\]
such that, \(\Sigma_{\nabla_{z_i},k}(L_k(M_{z_i})) = \{z_i\} + Z_{p}\) and \(\Sigma_{\nabla_{a_j},k}(L_k(M_{a_j})) = \{a_j\} + Z_{p}\).
- Let \(c_i \in k\) and \(r_i > 0\) such that \(z_i = x_{c_i r_i}\). \(E_{p^l}^{p^l} \leq r_i < |p|^{l-1},\) with \(l \in \mathbb{N}\setminus\{0\}\), then \(\text{Card}(\{z_i\} + Z_{p}) = p^l\) and \(\{z_i\} + Z_{p} = \{x_{c_i r_i}, x_{c_i+1 r_i}, \cdots, x_{c_i p^{l-1} r_i}\}\). If \(r_i \geq 1\) then we have \(\text{Card}(\{z_i\} + Z_{p}) = 1\) and \(\{z_i\} + Z_{p} = \{x_{c_i r_i}\}\).
- If \(r_i > 1\), let \(P_{z_i}(S_{\frac{d}{d5}})\) be a differential polynomial associated to \((M_{z_i}, \nabla_{z_i})\). Then the image by \(\pi_{\mathcal{T}(x^{p^l})}^{\text{alg}_k}\) of all roots of \(P_{z_i}(S_{\frac{d}{d5}})\) \((\text{the commutative polynomial associated to } P_{z_i}(S_{\frac{d}{d5}})\) is equal to \(z_i\).
- If \(|p|^l < r_i \leq |p|^{l-1}\), let \(P_{z_i}(p S_{\frac{d}{d5}})\) be a differential polynomial associated to \((\text{Frob}_{p})_{*}(M, \nabla)\) \((\text{a differential module over } (\mathcal{H}(x^{p^r}), p^l S_{\frac{d}{d5}}))\). Then the image by \(\pi_{\mathcal{T}(x^{p^l})}^{\text{alg}_k}\) of all roots of \(P_{z_i}(T)\) \((\text{the commutative polynomial associated to } P_{z_i}(p S_{\frac{d}{d5}})\) is equal to \(\{x_{c_i r_i}, x_{c_i+1 r_i}, \cdots, x_{c_i p^{l-1} r_i}\}\). In the special case where \(r_i = |p|^{l-1}\) we have \(\{x_{c_i r_i}, x_{c_i+1 r_i}, \cdots, x_{c_i p^{l-1} r_i}\}\).
If \( r_i \geq 1 \). For all \( a \in k \), the differential module \((M_{zi}, \nabla_{zi} - a)\) is pure. For \( a \in D^+(c_i, r_i) \cap k \) we have
\[
\mathcal{R}_1^{(M_{zi}, \nabla_{zi} - a), \text{Sp}}(x) = \frac{\omega}{r_i},
\]
and for all \( a \in k \setminus D^+(c_i, r_i) \)
\[
\mathcal{R}_1^{(M_{zi}, \nabla_{zi} - a), \text{Sp}}(x) = \frac{\omega}{|a - c_i|}.
\]

If \(|p|^l \leq r_i < |p|^l - 1\). For all \( a \in k \), the differential module \((M_{zi}, \nabla_{zi} - a)\) is pure. We have, for all \( a \in \bigcup_{j=0}^{l-1} D^+(c_i + j, r_i) \cap k \),
\[
\mathcal{R}_1^{(M_{zi}, \nabla_{zi} - a), \text{Sp}}(x) = \left( \frac{|p|^l \omega}{r_i} \right)^{\frac{1}{r_i}},
\]
and for all \( a \in k \setminus \bigcup_{j=0}^{l-1} D^+(c_i + j, r_i) \)
\[
\mathcal{R}_1^{(M_{zi}, \nabla_{zi} - a), \text{Sp}}(x) = \mathcal{R}_a - c_i(x).
\]

For all \( a \in k \), the differential module \((M_{a_{i}}, \nabla_{a_i} - a)\) is pure. More precisely, for all \( a \in \{a_i\} + \mathbb{Z}_p \), \((M_{a_i}, \nabla_{a_i} - a)\) is solvable, and for all \( a \in k \setminus \{a_i\} + \mathbb{Z}_p \), we have \( \mathcal{R}_1^{(M_{a_i}, \nabla_{a_i} - a), \text{Sp}}(x) = \mathcal{R}_a - a_i(x) \).

**Proof.** By Propositions 5.36, 5.40, Corollaries 5.28, 5.38 and 5.41, there exists a decomposition
\[
(M, \nabla) = \bigoplus_{i=1}^{\mu} (M_{w_i}, \nabla'_{w_i}),
\]
with \( w_1, \ldots, w_\mu \in \mathbb{A}_k^{1, \text{an}} \), such that \((M_{w_i}, \nabla'_{w_i} - a)\) is pure for each \( a \in k \) and \( \Sigma\nabla_{w_i}(\mathcal{L}_k(M_{w_i})) = \{w_i\} + \mathbb{Z}_p \). Also, for each \( \omega_i \), \( \mathcal{R}_1^{(M_{w_i}, \nabla'_{w_i} - a), \text{Sp}}(x) \) satisfies equations (5.70), (5.71), (5.72) and (5.73). We set \((M_{w_i}, \nabla_{w_i}) := \bigoplus_{i=1}^{\mu} (M_{w_i}, \nabla'_{w_i}) \) (cf. (5.68)). Let \( \{z_1, \ldots, z_\mu\} \subset \{w_1, \ldots, w_\mu\} \) be the maximal subset such that \( \{\{z_1\} + \mathbb{Z}_p\} \cap \{\{z_j\} + \mathbb{Z}_p\} = \emptyset \). Then we have \( \Sigma\nabla_{k}(\mathcal{L}_k(M)) = \{z_1, \ldots, z_\mu\} + \mathbb{Z}_p \) and the decomposition
\[
(M, \nabla) = \bigoplus_{i=1}^{\mu} (M_{z_i}, \nabla_{z_i}),
\]
satisfies the properties of the theorem. It remains to prove the uniqueness up to isomorphisms of such decomposition. Let us proof that \((M_{z_i}, \nabla_{z_i})\) are uniquely determined up to isomorphisms. Suppose that there exists another decomposition satisfying the same properties of the theorem
\[
(M, \nabla) = \bigoplus_{i=1}^{\mu} (M'_{z_i}, \nabla'_{z_i}),
\]
Let \( \iota_{z_i} : M_{z_i} \rightarrow M \) (resp. \( \iota'_{z_i} : M'_{z_i} \rightarrow M \)) be the canonical injection and \( \pi_{z_i} : M \rightarrow M_{z_i} \) (resp. \( \pi'_{z_i} : M \rightarrow M'_{z_i} \)) be the canonical projection. Then for all \( i \neq j \), we have \( \iota_{z_i} \circ \pi'_{z_j} = 0 \) and \( \iota'_{z_j} \circ \pi_{z_j} = 0 \), because otherwise we will get a sub-quotient of \( M_{z_i} \) (resp. \( M'_{z_i} \)) with spectrum equal to \( \{z_j\} + \mathbb{Z}_p \) (resp. \( \{z_j\} + \mathbb{Z}_p \)) which is absurd (cf. Corollary 5.42). This means that \( \iota_{z_i} \circ \pi_{z_i} \) and \( \iota'_{z_i} \circ \pi_{z_i} \) are injective, hence \((M_{z_i}, \nabla_{z_i}) \cong (M'_{z_i}, \nabla'_{z_i}). \)

**Corollary 5.46.** Let \( x = x_0, r \in \mathbb{A}_k^{1, \text{an}} \setminus k \). Let \((M, \nabla_M) \) and \((N, \nabla_N) \) be a differential module over \((\mathcal{H}(x), \mathcal{S}_d^{\text{an}})\). Let \((M, \nabla_M) = \bigoplus_{i=1}^{\mu} (M_{z_i}, \nabla_{z_i}) \) (resp. \( (N, \nabla_N) = \bigoplus_{i=1}^{\mu} (N_{w_i}, \nabla_{w_i}) \) ) be a decomposition as in (5.71). Then for \( (\{z_1\} + \mathbb{Z}_p) \cap (\{w_1\} + \mathbb{Z}_p) = \emptyset \) we have \( \text{Hom}_{\mathcal{H}(x)}(M_{z_i}, N_{w_j}) = 0 \).

**Corollary 5.47.** Let \( x = x_0, r \in \mathbb{A}_k^{1, \text{an}} \setminus k \). Let \((M, \nabla) \) be a differential module over \((\mathcal{H}(x), \mathcal{S}_d^{\text{an}})\). If \((M_1, \nabla_1) \) and \((M_2, \nabla_2) \) are two differential modules over \((\mathcal{H}(x), \mathcal{S}_d^{\text{an}})\) such that
\[
0 \rightarrow (M_1, \nabla_1) \rightarrow (M, \nabla) \rightarrow (M_2, \nabla_2) \rightarrow 0,
\]
then we have \( \Sigma\nabla(\mathcal{L}_k(M)) = \Sigma\nabla_1(\mathcal{L}_k(M_1)) \cup \Sigma\nabla_2(\mathcal{L}_k(M_2)) \).
Proof. By Corollary 5.46, we have $\Sigma_{\mathcal{V}_i}(\mathcal{L}_k(M_1)) \subset \Sigma_{\mathcal{V}}(\mathcal{L}_k(M))$. The result follows by Lemma 2.22. □

Remark 5.48. As a direct consequence of Corollary 5.47, if $(N_i, \nabla_i)$ are the quotient differential module of a Jordan-Hölder sequence of $(M, \nabla)$, then we have

$$\Sigma_{\mathcal{V}}(\mathcal{L}_k(M)) = \bigcup_i \Sigma_{\mathcal{V}_i}(\mathcal{L}_k(N_i)).$$

5.4. Spectrum of a differential equation at a point of a quasi-smooth curve. Now let us say some words concerning the spectrum of a differential module defined over a point of a quasi-smooth curve. Consider a quasi-smooth curve $C$ and a differential equation $(\mathcal{F}, \nabla)$ (i.e $\mathcal{F}$ is a locally free $\mathcal{O}_C$-module of finite rank together with a connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_C} \Omega^1_C$). Let $x$ be a point of type 2 or 3 of $C$. In order to compute the spectrum of $(\mathcal{F}_x \otimes_{\mathcal{O}_C} \mathcal{H}(x), \nabla)$ we need to fix a bounded derivation $d$ defined in a neighbourhood of $x$. Recall that there exists an affine neighbourhood $Y$ of $x$ and an affine domain $X \subset \mathbb{A}^1_{\text{an}}$ such that there exists an étale map $\varphi : Y \to X$, where we can assume that $\varphi(x) = x_{0,r}$ and $0 \not\in X$ (cf. [PP15, Theorem 3.12]). Since $\varphi$ is étale we have the isomorphism of sheaves $\varphi^*\Omega^1_X \simeq \Omega^1_Y$ (cf. [Ber90, Proposition 3.5.3]). We fix a coordinate function $S$ on $X$. Then one of the most suitable derivation $d : \mathcal{O}_Y \to \mathcal{O}_Y$ is the one corresponding to the $\mathcal{O}_Y$-morphism $\Omega^1_Y \to \mathcal{O}_Y$, $S^{-1}dS \otimes 1 \mapsto 1$, which by construction extends $S\frac{\partial}{\partial S} : \mathcal{O}_X \to \mathcal{O}_X$. Therefore, $(\mathcal{H}(x), d)$ is an extension of $(\mathcal{H}(x_{0,r}), S\frac{\partial}{\partial S})$ as a differential field. Then we consider $(\mathcal{F}_x \otimes_{\mathcal{O}_C} \mathcal{H}(x), \nabla)$ as a differential module over $(\mathcal{H}(x), d)$. By a restriction of scalars, we can see $(\mathcal{F}_x \otimes_{\mathcal{O}_C} \mathcal{H}(x), \nabla)$ as a differential module over $(\mathcal{H}(x_{0,r}), S\frac{\partial}{\partial S})$, which by Proposition 2.28, does not affect the spectrum of $\nabla$. Hence, we can use our previous results to compute the spectrum of $\nabla$.

Theorem 5.49. Assume that $\text{char}(\bar{k}) = p > 0$. Let $C$ be a quasi-smooth curve and $x \in C$ of type (2) or (3). Let $(M, \nabla)$ be a differential module over $(\mathcal{H}(x), d)$, where $d = \psi^*(S\frac{\partial}{\partial S})$, $\psi$ is a finite étale morphism $\psi$ from a neighbourhood of $x$ to $\mathbb{A}^1_{\text{an}}$, with $\psi(x) = x_{0,r}$. Then there exist $z_1, \ldots, z_\mu \in \mathbb{A}^1_{\text{an}}$, with $(\{z_i\} + \mathbb{Z}_p) \cap (\{z_j\} + \mathbb{Z}_p) = \emptyset$ for $i \neq j$, such that:

$$\Sigma_{\mathcal{V}_k}(\mathcal{L}_k(M)) = \{z_1, \ldots, z_\mu\} + \mathbb{Z}_p.$$
Let \((A, \|\cdot\|)\) and \((B, \|\cdot\|')\) be two commutative Banach rings, let \(\varphi : (A, \|\cdot\|) \rightarrow (B, \|\cdot\|')\) be a bounded morphism of rings. Then \(\varphi\) induces a continuous map defined as follows:

\[
\varphi^* : M(B) \longrightarrow M(A) \\
x \mapsto f \mapsto |\varphi(f)(x)|.
\]

**Definition 6.3.** Let \((A, \|\cdot\|)\) be a normed ring. The spectral semi-norm associated to \(\|\cdot\|\) is the map:

\[
\|\cdot\|_{sp,A} : A \rightarrow \mathbb{R}_+ \\
f \mapsto \lim_{n \to +\infty} \|f^n\|^\frac{1}{n}.
\]

In the case where for all \(f \in A\) we have \(\|f^n\| = \|f\|^n\), (i.e. \(\|\cdot\| = \|\cdot\|_{sp}\)), we say that \((A, \|\cdot\|)\) is *uniform*.

The spectral semi-norm defined on a commutative Banach ring \((A, \|\cdot\|)\) (c.f (6.5)) satisfies the following property:

**Properties 6.4.** For all element \(f\) in \(A\) we have:

\[
\|f\|_{sp,A} = \max_{x \in M(A)} |f(x)|.
\]

**Corollary 6.5.** Let \(A\) be a commutative Banach ring. Then the spectral semi-norm satisfies:

- \(\forall f, g \in A; \|fg\|_{sp} \leq \|f\|_{sp} \cdot \|g\|_{sp}\).
- \(\forall f, g \in A; \|f + g\|_{sp} \leq \|f\|_{sp} + \|g\|_{sp}\).

**Lemma 6.6.** Let \((A, \|\cdot\|)\) be a Banach ring, let \((B, \|\cdot\|')\) and \((C, \|\cdot\|'')\) be two Banach \(A\)-algebras. Let \(f \in B \otimes_A C\). Then \(f\) is not invertible in \(B \otimes_A C\) if and only if there exists \(x \in M(C)\) such that the image of \(f\) by the natural map \(B \otimes_A C \rightarrow B \otimes_A \mathcal{H}(x)\) is not invertible.

**Proof.** It is obvious that if the image of \(f\) is not invertible in \(B \otimes_A \mathcal{H}(x)\) for some \(x \in M(C)\), then so is for \(f\) in \(B \otimes_A C\). We suppose now that \(f\) is not invertible in \(B \otimes_A C\). By Corollary 6.2 there exists \(z \in M(B \otimes_A C)\) such that \(f(z) = 0\) in \(\mathcal{H}(z)\). We have the following commutative diagram:

\[
\begin{array}{ccc}
B \otimes_A C & \longrightarrow & \mathcal{H}(z) \\
\downarrow \quad \ & \quad \ & \quad \ \downarrow \\
C & \longrightarrow & \mathcal{H}(x)
\end{array}
\]

By Remark 6.1 there exists \(x \in M(C)\) such that we have the following diagram:

\[
\begin{array}{ccc}
C & \longrightarrow & B \otimes_A C \\
\downarrow \quad \ & \quad \ & \quad \ \downarrow \\
\mathcal{H}(x) & \longrightarrow & \mathcal{H}(z)
\end{array}
\]

Therefore, we obtain the commutative diagram

\[
\begin{array}{ccc}
B \otimes_A C & \longrightarrow & \mathcal{H}(x) \\
\downarrow \quad \ & \quad \ & \quad \ \downarrow \\
C & \longrightarrow & \mathcal{H}(x)
\end{array}
\]

Then, \(f(z) = 0\) implies that the image of \(f\) in \(B \otimes_A \mathcal{H}(x)\) is not invertible. \(\square\)
Corollary 6.7. Let $(A, ||||)$ be a Banach ring, let $(B, ||||')$ and $(C, ||||''')$ be two Banach $A$-algebras. Let $f \in B \otimes_A C$. If $f$ is not invertible in $B \otimes_A C$ then there exists $x \in M(C)$ and $y \in M(B)$ such that the image of $f$ by the natural map $B \otimes_A C \rightarrow \mathcal{M}(y) \otimes_A \mathcal{M}(x)$ is not invertible.

6.2. Description of some étale morphisms. This part is devoted to summarize the properties of the following analytic morphisms: logarithm and power map.

6.2.1. Logarithm. Let $a \in k \setminus \{0\}$. We define the logarithm function $\Log_a : D^-(a, |a|) \rightarrow k^{1,\text{an}}$ to be the analytic map associated to the ring morphism:

\[
\begin{align*}
T & \mapsto O(D^-(a, |a|)) \\
\log(a) & \mapsto \sum_{n \in \mathbb{N} \setminus \{0\}} \frac{(-1)^{n-1}}{a^n} - (T - a)^n.
\end{align*}
\]

We define the exponential function $\exp_a : D^-(0, \omega) \rightarrow D^-(a, |a|)$ to be the analytic map associated to the ring morphism:

\[
\begin{align*}
O(D^-(a, |a|)) & \rightarrow O(D^-(0, \omega)) \\
\exp(a) & \mapsto \sum_{n \in \mathbb{N}} \frac{T^n}{n!},
\end{align*}
\]

where

\[
\omega = \begin{cases} 
|p|^{1/p} & \text{if } \text{char}(\bar{k}) = p, \\
1 & \text{if } \text{char}(\bar{k}) = 0.
\end{cases}
\]

Lemma 6.8. Let $b \in D^-(a, |a|) \cap k$. Then we have $\Log_b = \Log_a - \Log_a(b)$.

Proof. We have $\Log_a(T) = \Log_1(T_a)$, in particular $\Log_1(T_a)$ is well defined. Therefore,

\[
\Log_b(T) = \Log_1 \left( \frac{T_b}{a} \right) = \Log_1 \left( \frac{T_b}{a} \right) = \Log_1 \left( \frac{T_a}{a} \right) - \Log_1 \left( \frac{b}{a} \right) = \Log_a(T) - \Log_a(b).
\]

Lemma 6.9. The logarithm function $\Log_a$ induces an analytic isomorphism $D^-(a, |a|) \rightarrow D^-(0, \omega)$, whose reverse isomorphism is $\exp_a$.

Proof. Since $\exp_a : D^-(0, \omega) \rightarrow D^-(a, |a|)$ is surjective, we obtain the isomorphism.

Lemma 6.10. Assume that $k$ is algebraically closed and $\text{char}(\bar{k}) = p > 0$. Let $\zeta_{p^n}$ be a $p^n$th root of unity. Then we have

\[
\begin{align*}
\bullet \quad |\zeta_{p^n} - 1| = \omega^{\frac{1}{p^{n-1}}}; \\
\bullet \quad \text{if } x \in D^-(a, |a|) \cap k, \text{ then } \Log_a(x) = 0 \text{ if and only if } x = a\zeta_{p^n}.
\end{align*}
\]

Proof. It is easy to see that $\Log_a(a\zeta_{p^n}) = 0$. Indeed,

\[
\Log_a(a\zeta_{p^n}) = \Log_1(a\zeta_{p^n}) = \frac{1}{p^n} \Log_1(1) = 0.
\]

Since $\Log_a(T) = \Log_1(T_a)$, it is enough to show that $\Log_1(x) = 0$ if and only if $x = \zeta_{p^n}$. By Lemma 6.9, we have $\Log_1(x) = 0$ if and only if $x = 1$, if $x \in D^-(1, \omega) \cap k$. Now let $x \in D^-(1, 1) \cap k$, we have

\[
x^{p} - 1 = (x - 1)^{p} + p(x - 1) \sum_{i=1}^{p-1} \binom{p}{i} (x - 1)^{i-1}.
\]

Therefore,

\[
|x^{p} - 1| \leq \max(|p||x - 1|, |x - 1|^{p}).
\]

Hence, there exists $n \in \mathbb{N}$ such that $|x^{p^n} - 1| < \omega$. Since $\Log_1(x^{p^n}) = p^n \Log_1(x)$, if $\Log_1(x) = 0$ then $x^{p^n} = 1$. The result follows.

Properties 6.11. Assume that $k$ is algebraically closed and $\text{char}(\bar{k}) = p > 0$. Let $D^+(b, r)$ be a closed sub-disk of $D^-(a, |a|)$. Then:
• The logarithm function induces an étale cover \( D^{-}(a, |a|) \to \mathbb{A}^{1,\text{an}}_{k} \).

• \( \text{Log}_{a}(D^{+}(b, r)) = D^{+}(\text{Log}_{a}(b), \varphi(r)) \) where

\[
\varphi : [0, |a|] \to \mathbb{R}_{+} \\
r \mapsto |\text{Log}_{b}(x_{b, r})|.
\]

(6.17)

• The function \( \varphi \) depends only on the choice of the radius of the disk \( D^{+}(b, r) \). In particular, it is an increasing continuous function and piecewise logarithmically affine on \([0, |a|]\) and

• \( \varphi(|a|^{\frac{1}{p^n}}) = \frac{\omega}{|p^n|}, \) where \( n \in \mathbb{N} \).

• If \( |a|^{\frac{1}{p^n}} \leq r < |a|^{\frac{1}{p^{n+1}}}, \) then \( \text{Log}_{a}^{-1}(\text{Log}_{a}(b)) \cap D^{+}(b, r) = \{ b \zeta_{p^n}^{i} | 0 \leq i \leq p^n - 1 \} \) where \( \zeta_{p^n} \) is a \( p^n \)-th root of unity.

Proof.

• Since \( \frac{d}{dT} \text{Log}_{a}(T) = \frac{1}{T} \) is invertible in \( \mathcal{O}(D^{-}(a, |a|)) \), by Remark 2.14 \( \text{Log}_{a} \) is locally étale. Hence, it is an étale cover.

• We know that the image of the disk \( D^{+}(b, r) \) by the analytic map \( \text{Log}_{a} \) is the disk \( D^{+}(\text{Log}_{a}(b), \varphi(r)) \), with radius equal to \( \varphi(r) = |\text{Log}_{a}(x_{b, r}) - \text{Log}_{a}(b)| \). By Lemma 6.8 we obtain \( \varphi(r) = |\text{Log}_{b}(x_{b, r})| \).

• Since \( |b| = |a| \) and \( \varphi(r) = |\text{Log}_{b}(x_{b, r})| \), by construction it depends only on the value \( r \). Since \( \text{Log}_{b} \) is an analytic map well defined on \((b, x_{a, |a|})\), the map \( \varphi \) is an increasing continuous function piecewise logarithmically affine on \([0, |a|]\).

• We have \( \varphi(|a|^{\frac{1}{p^n}}) = \max_{i \in \mathbb{N} \setminus \{0\}} |i|^{-\frac{1}{p^n}} = \frac{\omega^{n}}{|p^n|} \).

• Since \( \text{Log}_{b} = \text{Log}_{a} - \text{Log}_{a}(b) \), we conclude by Lemma 6.10. \( \square \)

Proposition 6.12. Assume that \( \text{char}(\bar{k}) = p > 0 \). Let \( y \in D^{-}(a, |a|) \) and \( x = \text{Log}_{a}(y) \), then we have:

• If \( 0 < r_{k}(y) < |a|\omega \), then \( [\mathcal{H}(y) : \mathcal{H}(x)] = 1 \)

• If \( |a|^{\frac{1}{p^{n+1}}} \leq r_{k}(y) < |a|^{\frac{1}{p^{n}}} \) with \( n \in \mathbb{N} \setminus \{0\} \), then \( [\mathcal{H}(y) : \mathcal{H}(x)] = p^{n} \).

Prologue. It is a consequence of Propositions 6.11 and 2.17. \( \square \)

6.2.2. Power map. For the details of this part we refer the reader for example to [Pul15, Section 5] and [Ked10, Chapter 10]. We define the \( n \)-th power map \( \Delta_{n} : \mathbb{A}^{1,\text{an}}_{k} \to \mathbb{A}^{1,\text{an}}_{k} \) to be the analytic map associated to the ring morphism:

\[
k[T] \to k[T] \\
T \mapsto T^{n}
\]

(6.18)

Frobenius map. We assume here that \( \text{char}(\bar{k}) = p > 0 \), with \( p > 0 \).

We define the Frobenius map to be the \( p \)-th power map. We will denote it by \( \text{Frob}_{p} \).

Properties 6.13. Let \( a \in k \) and \( r \in \mathbb{R}^{+} \). The Frobenius map satisfies the following properties:

• It induces an finite étale morphism \( \mathbb{A}^{1,\text{an}}_{k} \setminus \{0\} \to \mathbb{A}^{1,\text{an}}_{k} \setminus \{0\} \).

• \( \text{Frob}_{p}(D^{+}(a, r)) = D^{+}(a^{p}, \varphi(a, r)) \) where \( \varphi(a, r) = \max(|p||a|^{p^{-1}r}, r^{p}) \).

• \( \text{Frob}_{p}(x_{a, r}) = x_{a^{p}, \varphi(a, r)} \).

Proposition 6.14. Let \( y := x_{a, r} \) with \( r > 0 \). We set \( x = \text{Frob}_{p}(y) \). Then we have:

• If \( r < |a| \omega \), then \( [\mathcal{H}(y) : \mathcal{H}(x)] = 1 \).

• If \( r \geq |a| \omega \), then \( [\mathcal{H}(y) : \mathcal{H}(x)] = p \).

Corollary 6.15. Let \( y := x_{0, r} \), with \( r > 0 \). Let \( n \in \mathbb{N} \setminus \{0\} \), we set \( x = (\text{Frob}_{p})^{n}(y) \). Then we have \( [\mathcal{H}(y) : \mathcal{H}(x)] = p^{n} \).
Tame case. Let $n \in \mathbb{N} \setminus \{0\}$. We assume that $n$ is coprime to $\text{char}(\overline{k})$.

Properties 6.16. Let $a \in k$ and $r \in \mathbb{R}_+^*$. The $n$th power map satisfies the following properties:

- It induces a finite étale morphism $\mathbb{A}_k^{1,n} \setminus \{0\} \rightarrow \mathbb{A}_k^{1,n} \setminus \{0\}$.
- $\Delta_n(D^+(a,r)) = D^+(a^n, \varphi(a,r))$ where $\varphi(a,r) = \max(|a|^{n-1}r, r^n)$.
- $\Delta_n(x_{a,r}) = x_{a^n, \varphi(a,r)}$.

Proposition 6.17. Let $y := x_{a,r}$ with $r > 0$. We set $x = \Delta_n(y)$, then we have:

- If $r < |a|$, then $[\mathcal{H}(y) : \mathcal{H}(x)] = 1$.
- If $r \geq |a|$, then $[\mathcal{H}(y) : \mathcal{H}(x)] = n$.

References

[Azz20] T. A. Azzouz. “Spectrum of a linear differential equation with constant coefficients”. In: Math. Z. 296.3-4 (2020), pp. 1613–1644. ISSN: 0025-5874. DOI: 10.1007/s00209-020-02482-z. URL: https://doi.org/10.1007/s00209-020-02482-z.

[Azz22] T. A. Azzouz. “Spectrum of a linear differential equation over a field of formal power series”. In: J. Number Theory 231 (2022), pp. 139–157. ISSN: 0022-314X. DOI: 10.1016/j.jnt.2020.11.021. URL: https://doi.org/10.1016/j.jnt.2020.11.021.

[Azz23] T. A. Azzouz. Spectrum of $p$-adic linear differential equations II: Variation of the spectrum. 2023. arXiv: 2303.06014 [math.NT].

[Ber90] V. G. Berkovich. Spectral Theory and Analytic Geometry Over non-Archimedean Fields. AMS Mathematical Surveys and Monographs 33. AMS, 1990.

[Ber93] V. G. Berkovich. “Étale cohomology for non-Archimedean analytic spaces.” English. In: Publ. Math., Inst. Hautes Étud. Sci. 78 (1993), pp. 5–161. ISSN: 0073-8301; 1618-1913/e. DOI: 10.1007/BF02712916.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert. Non-archimedean analysis : a systematic approach to rigid analytic geometry. Grundlehren der mathematischen Wissenschaften, 261. Springer-Verlag, 1984.

[Bou07] N. Bourbaki. Théories spectrales: Chapitres 1 et 2. Bourbaki, Nicolas. Springer Berlin Heidelberg, 2007. ISBN: 9783540353317.

[BR10] M. Baker and R. Rumely. Potential theory and dynamics on the Berkovich projective line. English. Providence, RI: American Mathematical Society (AMS), 2010, pp. xxxiii + 428. ISBN: 978-0-8218-4924-8/hbk.

[CD94] G. Christol and B. Dwork. “Modules différentiels sur les couronnes (Differential modules over annuli).” French. In: Ann. Inst. Fourier 44.3 (1994), pp. 663–701. ISSN: 0373-0956; 1777-5310/e. DOI: 10.5802/aif.1414.

[Chr83] G. Christol. “Modules différentiels et équations différentielles $p$-adiques”. In: Queen’s Papers in Pure and Applied Math (1983).

[CM02] G. Christol and Z. Mebkhout. “Équations différentielles $p$-adiques et coefficients $p$-adiques sur les courbes. ($p$-adic differential equations and $p$-adic coefficients over curves).” French. In: Cohomologies $p$-adiques et applications arithmétiques (II). Paris: Société Mathématique de France, 2002, pp. 125–183. ISBN: 2-85629-117-1/pbk.

[Duc] A. Ducros. La structure des courbes analytiques. URL: https://webusers.imj-prg.fr/~antoine.ducros/trirss.pdf

[Dwo73] B. Dwork. “On $p$-adic differential equations. II: The $p$-adic asymptotic behavior of solutions of ordinary linear differential equations with rational function coefficients”. English. In: Ann. Math. (2) 98 (1973), pp. 366–376. ISSN: 0003-486X. DOI: 10.2307/1970786.

[Ked10] K. S. Kedlaya. $p$-adic differential equations. English. Cambridge: Cambridge University Press, 2010, pp. xvii + 380. ISBN: 978-0-521-76879-5/hbk.

[Poi13] J. Poineau. “Les espaces de Berkovich sont angéliques”. In: Bull. soc. Math. France (2013).

[PP13a] J. Poineau and A. Pulita. “The convergence Newton polygon of a $p$-adic differential equation III : global decomposition and controlling graphs”. In: ArXiv e-prints (Aug. 2013). arXiv: 1308.0859 [math.NT].

[PP13b] J. Poineau and A. Pulita. “The convergence Newton polygon of a $p$-adic differential equation IV : local and global index theorems”. In: ArXiv e-prints (Sept. 2013). arXiv: 1309.3940 [math.NT].
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