A GRADED PULLBACK STRUCTURE OF LEAVITT PATH ALGEBRAS OF TRIMMABLE GRAPHS

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Abstract. Motivated by recent results in graph C*-algebras concerning an equivariant pushout structure of the Vaksman-Soibelman quantum odd spheres, we introduce a class of graphs called trimmable. Then we show that the Leavitt path algebra of a trimmable graph is graded-isomorphic to a pullback algebra of simpler Leavitt path algebras and their tensor products.

Introduction

The goal of this paper is to introduce and apply the concept of a trimmable graph. We begin by recalling the fundamental concepts of path algebras [5] and Leavitt path algebras [7, 2, 1, 3]. Then we define a trimmable graph, and prove our main result: There is a \( \mathbb{Z} \)-graded algebra isomorphism from the Leavitt path algebra of a trimmable graph to an appropriate pullback algebra. The graph C*-algebraic version of this result is proven in [6], where it was used to analyze the generators of K-groups of quantum complex projective spaces.

1. LEAVITT PATH ALGEBRAS

Definition 1.1. A graph \( Q \) is a quadruple \( (Q_0, Q_1, s, t) \) consisting of the set of vertices \( Q_0 \), the set of edges \( Q_1 \), and the source and target maps \( s, t : Q_1 \to Q_0 \) assigning to each edge its source and target vertex respectively.

We say that a graph \( Q' = (Q'_0, Q'_1, s', t') \) is a sub-graph of a graph \( Q = (Q_0, Q_1, s, t) \) iff \( Q'_0 \subseteq Q_0, Q'_1 \subseteq Q_1 \), and the source and target maps \( s' \) and \( t' \) are respective restrictions-corestrictions of the source and target maps \( s \) and \( t \). Furthermore, we say that two edges are composable if the end of one of them is the beginning of the other. Now we can define a path in a graph as a sequence of composable edges. The length of a path is the number of edges it consists of, infinity included. We treat vertices as zero-length paths that begin and end in themselves.

Definition 1.2. Let \( k \) be a field and \( Q \) a graph. The path algebra \( kQ \) is the \( k \)-algebra whose underlying vector space has as its basis the set of all finite-length paths \( \text{Path}(Q) \). The product is given by the composition of paths when the end of one path matches the beginning of the other path. The product is defined to be zero otherwise.
One can check that the path algebra $kQ$ is unital if and only if the set of vertices $Q_0$ is finite. Then the unit is the sum of all vertices. It is also straightforward to verify that $kQ$ is $\mathbb{N}$-graded by the path length.

To define a Leavitt path algebra, we need ghost edges. For any graph $Q = (Q_0, Q_1, s, t)$, we create a new set $Q_1^* := \{ x^* \mid x \in Q_1 \}$ and call its elements ghost edges. Now, the source and the target maps for the extended graph $\hat{Q} := (Q_0, Q_1 \bigsqcup Q_1^*, \hat{s}, \hat{t})$ are defined as follows:

$$\hat{s}(x) := s(x), \quad \hat{t}(x) := t(x), \quad \hat{s}(x^*) := \hat{t}(x^*) := \hat{t}(x). \quad (1.1)$$

**Definition 1.3.** Let $k$ be a field and $Q$ a graph. The Leavitt path algebra $L_k(Q)$ of a graph $Q$ is the path algebra of the extended graph $\hat{Q}$ divided by the ideal generated by the relations:

1. For all edges $x_i, x_j \in Q_1$, we have $x_i^* x_j = \delta_{ij} t(x_i)$.
2. For every vertex $v \in Q_0$ whose preimage $s^{-1}(v)$ is not empty and finite, we have

$$\sum_{x \in s^{-1}(v)} xx^* = v.$$

In other words, the Leavitt path algebra $L_k(Q)$ of a graph $Q$ is the universal $k$-algebra generated by the elements $v \in Q_0$, $x \in Q_1$, $x^* \in Q_1^*$, subject to relations:

1. $v_i v_j = \delta_{ij} v_i$ for all $v_i, v_j \in Q_0$,
2. $s(x) x = x t(x) = x$ for all $x \in Q_1$,
3. $t(x) x^* = x^* s(x) = x^*$ for all $x^* \in Q_1^*$,
4. $x_i^* x_j = \delta_{ij} t(x_i)$ for all $x_i, x_j \in Q_1$, and
5. $\sum_{x \in s^{-1}(v)} xx^* = v$ for all $v \in Q_0$ such that $s^{-1}(v)$ is finite and nonempty.

Furthermore, note that the $\mathbb{N}$-grading of the path algebra $k\hat{Q}$ induces a $\mathbb{Z}$-grading of the Leavitt path algebra $L_k(Q)$ by counting the length of any ghost edge as $-1$ (see [2, Lemma 1.7]). Let us recall now the Graded Uniqueness Theorem [9, Theorem 4.8] that shows the importance of this grading. We will need it in the next section.

**Theorem 1.4 ([9]).** Let $Q$ be an arbitrary graph and $k$ be any field. If $A$ is a $\mathbb{Z}$-graded ring, and $f : L_k(E) \to A$ is a graded ring homomorphism with $f(v) \neq 0$ for every vertex $v \in Q_0$, then $f$ is injective.

Recall that a vertex $v \in Q_0$ is called a sink if $s^{-1}(v) = \emptyset$. Next, let $x = x_1 x_2 \ldots x_n$ be a path in $Q$. If the length of $x$ is at least 1, and if $s(x) = v = t(x) \in Q_0$, we say that $x$ is a closed path based at $v$. If in addition $s(x_i) \neq s(x_j)$ for every $i \neq j$, then $x$ is called a cycle based at $v$.

Next, if $v \in Q_0$ is either a sink or a base of a cycle of length 1 (a loop), then a singleton set $\{v\} \subseteq Q_0$ is a basic example of a hereditary subset of $Q_0$ [3, Definition 2.0.5 (i)], and it follows from Corollary 2.4.13 (i) in [3] (cf. [9, Theorem 5.7 (2)]) that

$$L_k(Q)/I(v) \cong L_k(Q \setminus \{v\}). \quad (1.2)$$
Here the graph $Q \setminus \{v\}$ is obtained by removing from $Q$ the vertex $v$ and every edge that ends in $v$. In other words,

$$\left( Q \setminus \{v\} \right)_0 = Q_0 \setminus \{v\} \quad \text{and} \quad \left( Q \setminus \{v\} \right)_1 = \{x \in Q_1 : t(x) \neq v\}. \quad (1.3)$$

By Lemma 2.4.1 in [3] (cf. [9, Lemma 5.6]), we also know that

$$I(v) = \left\{ \sum_{i=1}^{n} k_i x_i y_i^* \mid n \geq 1, x_i, y_i \in \text{Path}(Q), \hat{i}(x_i) = \hat{s}(y_i^*) = v \right\}. \quad (1.4)$$

2. Trimmmable graphs

We are now ready for the main definition of the paper. Merely to focus attention, we assume henceforth that graphs are finite, i.e. that the set of vertices and the set of edges are both finite.

**Definition 2.1.** Let $Q$ be a finite graph consisting of a sub-graph $Q'$ emitting at least one edge to an external vertex $v_0$ whose only outgoing edge $x_0$ is a loop. We call such a graph $(Q', v_0)$-trimmable iff all edges from $Q'$ to $v_0$ begin in a vertex emitting an edge that ends inside $Q'$.

In symbols, a trimmable graph is described as follows:

$$Q_0 = Q'_0 \cup \{v_0\}, \quad v_0 \notin Q'_0, \quad Q_1 = Q'_1 \cup t^{-1}(v_0), \quad (2.1)$$

$$s^{-1}(v_0) = \{x_0\}, \quad t(x_0) = v_0, \quad t^{-1}(v_0) \setminus \{x_0\} \neq \emptyset, \quad (2.2)$$

$$\forall v \in s(t^{-1}(v_0) \setminus \{x_0\}) : s^{-1}(v) \setminus t^{-1}(v_0) \neq \emptyset. \quad (2.3)$$

The condition for a trimmable graph guarantees the fact that when we remove the distinguished vertex $v_0$, the resulting graph does not have new sinks. One can imagine a $(Q', v_0)$-trimmable graph like this:

![Diagram](image)

The following graph is a simple example of a trimmable graph:

![Diagram](image)
Note that the direction of the edge joining vertices $v_1$ and $v_2$ is important as the following graph is no longer trimmable:

\[
\begin{array}{ccc}
  v_2 & \downarrow & v_0 \\
  v_1 & \longrightarrow & v_0
\end{array}
\]

We need the trimmability condition to guarantee the existence of maps given in the lemma below. This lemma is an algebraic incarnation of a graph-C*-algebraic lemma proved in [6]. The only difference between their proofs is that instead of using the Gauge Uniqueness Theorem [9, Theorem 4.8] we use the Graded Uniqueness Theorem [4, Theorem 2.3] (Theorem 1.4).

**Lemma 2.2.** Let $Q$ be a $(Q'_0, v_0)$-trimmable graph. Denote by $Q''$ the sub-graph of $Q$ obtained by removing the edge $x_0$. The following formulas define homomorphisms of algebras:

1. $\pi_1: L_k(Q) \to L_k(Q')$,
   \[
   \pi_1(\alpha) = \begin{cases} 
   \alpha & \text{if } \alpha \in Q'_0 \cup Q'_1 \cup Q'^*_1, \\
   0 & \text{otherwise.}
   \end{cases}
   \]

2. $\pi_2: L_k(Q'') \to L_k(Q')$,
   \[
   \pi_2(\alpha) = \begin{cases} 
   \alpha & \text{if } \alpha \in Q'_0 \cup Q'_1 \cup Q'^*_1, \\
   0 & \text{otherwise.}
   \end{cases}
   \]

3. $f: L_k(Q) \to L_k(Q'') \otimes k[u, u^{-1}]$,
   \[
   f(\alpha) = \begin{cases} 
   \alpha \otimes 1 & \text{if } \alpha \in Q_0, \\
   v_0 \otimes u & \text{if } \alpha = x_0, \\
   v_0 \otimes u^{-1} & \text{if } \alpha = x_0^*, \\
   \alpha \otimes u & \text{if } \alpha \in Q_1 \setminus \{x_0\}, \\
   \alpha \otimes u^{-1} & \text{if } \alpha \in Q^*_1 \setminus \{x_0^*\}.
   \end{cases}
   \]

4. $\delta: L_k(Q') \to L_k(Q') \otimes k[u, u^{-1}]$,
   \[
   \delta(\alpha) = \begin{cases} 
   \alpha \otimes 1 & \text{if } \alpha \in Q'_0, \\
   \alpha \otimes u & \text{if } \alpha \in Q'_1, \\
   \alpha \otimes u^{-1} & \text{if } \alpha \in (Q'_1)^*.
   \end{cases}
   \]

These morphisms are $\mathbb{Z}$-graded for the standard grading on $L_k(Q)$, $L_k(Q')$, $L_k(Q'')$, and the gradings on $L_k(Q') \otimes k[u, u^{-1}]$ and $L_k(Q'') \otimes k[u, u^{-1}]$ given by the standard grading of the rightmost tensorand. Furthermore, $\pi_1$ and $\pi_2$ are surjective, and $f$ and $\delta$ are injective.
3. A graded pullback structure

To prove the theorem of the paper, we need a general lemma along the lines of [8, Proposition 3.1]. We omit its routine proof.

**Lemma 3.1.** Let $A_1$, $A_2$, $B$ and $P$ be abelian groups. A commutative diagram of group homomorphisms

\[
\begin{array}{ccc}
P & & \\
| & p_1 & p_2 \downarrow \\
A_1 & & A_2 \\
& q_1 & q_2 \downarrow \\
B & & \\
\end{array}
\]

is a pullback diagram if and only if the following conditions hold:

\[
\ker(p_1) \cap \ker(p_2) = \{0\}, \quad (3.1)
\]
\[
q_1^{-1}(q_2(A_2)) = p_1(P), \quad (3.2)
\]
\[
p_2(\ker(p_1)) = \ker(q_2). \quad (3.3)
\]

Recall that to prove that an algebra $P$ is a pullback algebra, one can proceed as follows. The first step is to establish the existence of a commutative diagram of algebra homomorphisms as above. This implies that $p_1$ and $p_2$ define an algebra homomorphism $p$ into the pullback algebra of $A_1$ and $A_2$ over $B$. Then one only needs to prove that the three conditions of Lemma 3.1 are satisfied to conclude that $p$ is an isomorphism. Note that (3.1) is equivalent to the injectivity of $p$, whereas the conjunction of (3.2) and (3.3) is equivalent to the surjectivity of $p$.

Much as Lemma 2.2, the theorem of the paper is an algebraic incarnation of a graph-C*-algebraic theorem proved in [6]. This time, the only difference between their proofs is that instead of using [4, Lemma 3.1] we use [3, Lemma 2.4.1] (see (1.4), cf. [9, Lemma 5.6]).

**Theorem 3.2.** Let $\pi_1$, $\pi_2$, $f$ and $\delta$ be as in Lemma 2.2. Then the commutative diagram

\[
\begin{array}{ccc}
L_k(Q) & & L_k(Q'' \otimes k[u, u^{-1}]) \\
| & f & \\
L_k(Q') & & \\
\downarrow & & \downarrow \pi_2 \otimes \text{id} \\
L_k(Q'') \otimes k[u, u^{-1}] & & \\
\end{array}
\]

of graded algebra homomorphisms is a pullback diagram.

Representing pictorially the Leavitt path algebras by their respective graphs, the above diagram becomes:
Note that the only non-standard map in this diagram is $f$. It can be described verbally by the assignment

\begin{align*}
\text{vertex} & \mapsto \text{vertex} \otimes 1, \\
\nu_0\text{-emitted edge} & \mapsto \nu_0 \otimes u, \\
\text{other edge} & \mapsto \text{other edge} \otimes u.
\end{align*}

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