Variations on a theme of Gel’fand and Na˘ımark

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Abstract

$C^\ast$-algebras are widely used in mathematical physics to represent the observables of physical systems, and are sometimes taken as the starting point for rigorous formulations of quantum mechanics and classical statistical mechanics. Nevertheless, in many cases the naïve choice of an algebra of observables does not admit a $C^\ast$-algebra structure, and some massaging is necessary. In this paper we investigate what properties of $C^\ast$-algebras carry over to more general algebras and what modifications of the Gel’fand theory of normed algebras are necessary. We use category theory as a guide and, by replacing the ordinary definition of the Gel’fand spectrum with a manifestly functorial definition, we succeed in generalizing the Gel’fand–Na˘ımark theorem to locally convex $\ast$-algebras. We also recall a little-known but potentially very useful generalization of the Stone–Weierstrass theorem to completely regular, Hausdorff spaces.

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1 Introduction

In rigorous formulations of quantum theory, non-commutative $C^\ast$-algebras are used extensively to represent the observables of physical systems [Haa96, Emc72, JCBZ92, Wal94]. This approach was pioneered by Segal [Seg47], who also advocated formalizing probability theory in terms of commutative algebras of bounded random variables [Seg54], this leads naturally to a $C^\ast$-algebraic formulation of classical statistical mechanics [Emc72]. Once the $C^\ast$-algebra of observables is specified, the formal development of either theory is well-understood. In the case of non-commutative $C^\ast$-algebras, the Gel’fand–Na˘ımark–Segal construction produces representations of the $C^\ast$-algebra as bounded operators on a Hilbert-space, and the problem is to identify and analyze the physically relevant ones. In the commutative case, the Gel’fand transform provides a geometric interpretation of the algebra of observables as continuous bounded functions on
a compact space, and the Gel'fand–Naimark–Segal construction gives this algebra of random variables a Hilbert-space structure with the covariance as inner product.

In physics, the hardest problem is often finding a suitable $C^*$-algebra of observables in the first place, when the only input is a geometrical or operational description of a physical system. As mentioned before, the Gel'fand theory of normed algebras provides a natural interpretation of Abelian $C^*$-algebras as algebras of continuous functions on a compact Hausdorff space, the Gel'fand spectrum, which in applications to classical mechanics would be the phase space of the system under consideration. It should be obvious, however, that compact phase spaces are very rare as, even when the configuration space is bounded, momentum is usually unbounded. Even worse, in the case of field theories or the mechanics of continuous media, the phase space is an infinite-dimensional manifold and is not even locally compact! Also, in many instances, such as when using Poisson brackets, one is interested in algebras of smooth functions which, while tending to be metrizable, do not admit a norm and so cannot be $C^*$-algebras.

Given an algebra of classical observables which is not a $C^*$-algebra, two related approaches can be taken. The first is to study the manipulation required to turn the given algebra into a $C^*$-algebra. The process may involve loss of information (as when taking equivalence classes), new information (as when extending or completing the algebra), or arbitrary choices (such as a choice of complex structure on a real space), and one should pay attention to the physical interpretation of these manipulations. The second approach, which we develop in this paper, is trying to extend as much as possible of the theory of $C^*$-algebras to more general algebras, possibly changing the key definitions in the theory so they apply more generally. For the mathematical side of this exploration we use category theory, which provides notation and concepts tailored to asking and answering questions about naturality of mathematical operations. As for the physical interpretation, whether a manipulation is unphysical can only be answered in each particular instance, but hopefully mathematically sensible manipulations will turn out to be physically sensible.

In the Gel'fand theory of normed algebras [Ric60], the Gel'fand spectrum of maximal ideals of an algebra plays a central rôle. As we have indicated, it is given a natural compact, Hausdorff topology, and the elements of the algebra can be naturally interpreted as continuous complex functions on it. One of the key results in the theory of $C^*$-algebras is the Gel'fand–Naimark theorem, which states that every Abelian $C^*$-algebra is isometrically $*$-isomorphic to the $*$-algebra of bounded continuous functions on its spectrum, with the normed topology of uniform convergence. The content of the Gel'fand–Naimark theorem really is that the quotient of an Abelian $C^*$-algebra by a maximal ideal is a continuous $*$-homomorphism into the complex numbers. This is a striking connection between algebra and analysis, but the functorial properties of the Gel'fand transform depend not on the fact that the spectrum consists of maximal ideals, but on the fact that it is a hom-set in the category of $C^*$-algebras. Accordingly, in more general settings than $C^*$-algebras it is more productive to
simply restrict one’s attention to continuous \( * \)-homomorphisms into the complex numbers than to study maximal ideals. Large parts of the Gel’fand–Naĭmark theory, including the Gel’fand–Naĭmark–Segal construction, then carry over.

Our main results, theorems 6 and 8, imply that, under rather general hypothesis, given a \( * \)-algebra \( A \) one can find a \( * \)-homomorphism injecting it as a dense \( * \)-subalgebra of continuous complex functions on a Tychonoff (completely regular, Hausdorff) space, with the compact-open topology. This \( * \)-homomorphism always exists, but it may have a nontrivial kernel.

The contents of this paper are summarized in the following diagram of functors, each cell of which roughly corresponds to one section:

\[
\begin{array}{c}
\text{AbAlg} \xrightarrow{F} \text{Ab}^* \text{Alb} \\
\omega(-,-^*) \downarrow \\
\text{LCAbAlg} \xrightarrow{F} \text{LCAb}^* \text{Alg} \\
\Delta \downarrow \\
\text{Tych} \xrightarrow{\Delta} \text{AbLC}^* \text{Alg}
\end{array}
\]

In section 2 we define the categories \( \text{AbAlg} \) and \( \text{Ab}^* \text{Alb} \) of unital Abelian algebras and \( * \)-algebras, and study the adjoint pair of functors, “underlying” and “free”, going between them.

Section 3 deals with the square cell at the top of the above diagram of functors. We use the weak topology to make every algebra and \( * \)-algebra locally convex, hence the names \( \text{LCAbAlg} \) and \( \text{LCAb}^* \text{Alg} \), in such a way that the underlying and free functors commute with the operation of adding the weak topology.

The Gel’fand spectrum is defined as a hom-set in section 4, associated to the triangular cell to the left of the diagram of functors. This definition entails that the spectrum of a \( * \)-algebra is, in general, strictly contained in the spectrum of its underlying algebra. It is then shown that the Gel’fand spectrum is a Tychonoff space, the next best thing after compact Hausdorff spaces, and that it is a weak*-closed subset of the topological dual of the algebra. Because of the functorial definition of the spectrum, the Gel’fand transform automatically becomes a \( * \)-algebra \( * \)-homomorphism when applied to a \( * \)-algebra, in a setting when the lack of a norm makes the usual techniques applied to \( C^* \)-algebras break down. We also discuss the interpretation of the statement “every \( * \)-homomorphism is a homomorphism” as a natural transformation.

The triangular cell on the bottom-right of the diagram is discussed in section 5. We study the difference between the usual definitions of the Gel’fand transform and ours, and use a generalization of the Stone–Weierstrass theorem from the case of a compact, Hausdorff space (the spectrum of a \( C^* \)-algebra) to the case of a Tychonoff space (the spectrum of any algebra), to show that the image of the Gel’fand transform is dense in the continuous functions on the
spectrum. The generalization of the Stone–Weierstrass theorem involves replacing uniform convergence by the compact-open topology (uniform convergence on compact sets), which is not a surprise since this is a topology that is extensively used in complex analysis. For lack of a better name, we call the space of complex continuous functions on a Tychonoff spaces an “Abelian LC*-algebra” (LC for locally convex, or for “locally C*”).

Finally, in section 6 we study the notion of a state and apply the Gelfand–Naïmark–Segal construction to the *-algebra of complex functions on a Tychonoff space. The states are realized as compactly-supported Borel probability measures, which is related to the fact that the restriction of the algebra of continuous functions to a compact set is a C*-algebra. This illustrates the sense in which we are dealing with “locally C*” algebras.

While lacking an intrinsic, algebraic characterization of LC*-algebras (such as is available for C*-algebras), our discussion shows that there is life outside the world of C*-algebras in the sense that the basic operations that mathematical physicists need to perform on algebras of observables can be carried out for LC*-algebras. Also, it illustrates how category theory can be a powerful guide to find the right definitions making it possible to extend impressive results like the Gelfand–Naïmark theory to situations where few, if any, of the specific techniques used in the original proofs are available.

2 Abelian algebras and *-algebras

For the purposes of this paper, an algebra will be a complex vector space A with an associative, bilinear multiplication and a unit 1_A. An algebra is Abelian if ab = ba for all a, b ∈ A. A linear map φ: A → B between algebras is an algebra homomorphism if, and only if, φ(aa’) = φ(a)φ(a’) and φ(1_A) = 1_B.

Note that we are assuming that all algebras are unital, and that all algebra homomorphisms map units to units. This is partly because the category of unital algebras with unit-preserving homomorphisms is relatively nice among the possible categories of algebras. We denote this category by Alg, and the category of Abelian algebras with algebra homomorphisms by AbAlg.

An algebra A is a *-algebra if it has an involutive anti-linear anti-homomorphism *: A → A. What this means is that a** = a, that (z1_A)** = ¯z1_A, and that (ab)* = b*a*. Although in general the term “involution” refers just to * being its own inverse, in this context a *-algebra is usually called “an algebra with an involution”, and the operation *: a → a* is called “the involution”. The complex numbers is naturally a *-algebra, with involution given by complex conjugation, z*: = ¯z.

An algebra homomorphism f: A → B between *-algebras is an algebra *-homomorphism if, and only if, f(a*) = f(a)* for all a ∈ A. Note that our definition of the involution includes the requirement that the “unit map” e_A: C → A, such that e_A(z) = z1_A, be a *-homomorphism. There is a category of *-algebras and *-homomorphisms, which we denote Alg*, and a category of Abelian *-algebras with *-homomorphisms denoted Ab*Alb.

It is clear that every (Abelian) *-algebra is an (Abelian) algebra, and that
every $\ast$-homomorphism is a homomorphism. Hence, the process of considering a $\ast$-algebra as an algebra is a forgetful functor $U: \text{Alg}^* \to \text{Alg}$ restricting to a functor between the categories of Abelian algebras $U: \text{Ab}^* \text{Alb} \to \text{AbAlg}$. If $A$ is an (Abelian) $\ast$-algebra, $U(A)$ is called its underlying (Abelian) algebra. As it happens generally in algebra, these forgetful functors have left-adjoint functors. In the Abelian case $F: \text{AbAlg} \to \text{Ab}^* \text{Alb}$ is such that there is a natural isomorphism

$$\text{AbAlg}(A, U(B)) \simeq \text{Ab}^* \text{Alb}(F(A), B)$$

for all Abelian algebras $A$ and Abelian $\ast$-algebras $B$ (following [Lan98], we denote by $\text{Xmpl}(A, B)$ the set of morphisms $f: A \to B$ in the category $\text{Xmpl}$). The functor $F$ is said to be the left adjoint of $U$, and its interpretation is that $F(A)$ is the free Abelian $\ast$-algebra generated by $A$.

The existence of the functor $F$ is equivalent to the following universal property: for every Abelian algebra $A$ there exists an Abelian $\ast$-algebra $F(A)$ and such that, for every Abelian $\ast$-algebra $B$, and for every algebra homomorphism $f: A \to U(B)$, there exists a unique $\ast$-homomorphism $f': F(A) \to B$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\iota} & U(F(A)) \\
\downarrow{f} & & \downarrow{U(f')} \\
U(B) & & \\
\end{array}
$$

where $\iota$ is “the unit of the adjunction”. Uniqueness of $F(A)$ up to isomorphism follows from abstract nonsense [Lan98].

Existence of $F(A)$ is proved by construction. In the Abelian case, it suffices to add to $A$ a new element $a^*$ for each $a \in A$ and all the products of the form $a^*b$, and to consider the collection of all linear combinations of those three kinds of elements, subject only to the relations necessary to enforce that $a \mapsto a^*$ is an involution on the new algebra. Note that, although the “underlying algebra” functor is the same for Abelian and non-Abelian algebras, the “free algebra” functor is very different from the “free Abelian algebra” functor. In particular, the free $\ast$-algebra generated by an Abelian algebra is non-Abelian, and much larger than the free Abelian $\ast$-algebra generated on it, because in the free Abelian $\ast$-algebra, $a^*b = ba^*$ but not so in the free $\ast$-algebra. Finally, note that neither $UF$ nor $FU$ are the identity. We summarize the situation thus:

$$\text{AbAlg} \xrightarrow{F} \text{Ab}^* \text{Alb} \xleftarrow{U} \text{AbAlg}$$

### 2.1 Examples

The simplest example of this involves algebras of polynomials with complex coefficients.
C[x] Let us start by considering the $*$-algebra of complex polynomials on one self-adjoint variable $x$ (satisfying $x^* = x$), so the involution maps a polynomial $a_0 + a_1x + \cdots + a_nx^n$ to $\overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n$. We denote this $*$-algebra by $A = C[x]$.

In application to classical mechanics, this is the polynomials on one real configuration variable.

C[z] The underlying algebra $B = U(A)$ is the same algebra of polynomials, except that it is “forgotten” that one can apply the involution to them. So, the two polynomials $a_0 + a_1z + \cdots + a_nz^n$ and $\overline{a_0} + \overline{a_1}z + \cdots + \overline{a_n}z^n$ are both elements of $B$, but now they are not related by any operation on $B$ unless all the $a_i$ are real and the polynomials are actually the same. We denote this algebra by $B = C[z]$, which is actually the “free Abelian algebra on one generator”.

In classical mechanics, this is the algebra of polynomials on one complex phase variable $z = q + ip$.

C[z, z*] We now consider $C = F(B)$, the free $*$-algebra on $B$ or the “free $*$-algebra on one generator”. To $z$ we must add a distinct adjoint $z^*$, and then build the free Abelian algebra generated by the two. A typical polynomial in this algebra is on $a_{00} + a_{10}z + a_{01}z^* + a_{20}z^2 + a_{11}zz^* + a_{02}(z^*)^2 + \cdots$, and the effect of the involution on it is now $\overline{a_{00}} + \overline{a_{10}}z^* + \overline{a_{01}}z + \overline{a_{20}}(z^*)^2 + \overline{a_{11}}zz^* + \overline{a_{02}}z^2 + \cdots$. We denote this $*$-algebra by $C = C[z, z^*]$.

In classical mechanics this is the algebra of polynomials on phase space, since we can interpret $z = q + ip$ and $z^* = q - ip$ with $p = p^*$ and $q = q^*$, and indeed in that case

$$C[z, z^*] \simeq C[q, p].$$

where $q$ and $p$ are self-adjoint generators. Accordingly, in physical applications this algebra could also be associated to a two-dimensional real configuration space.

C[z, w] The underlying algebra of $C[z, z^*]$ is $D = C[z, w]$, where we still have two generators but we forget that they are related by the involution (or, equivalently, we forget that the two generators are self-adjoint). This is algebra of complex polynomials on two variables.

In classical mechanics, this would be the algebra of polynomials on a phase space of two degrees of freedom, with $z = q_1 + ip_1$ and $w = q_2 + ip_2$.

The situation is summarized by the following diagram:
It is apparent that, in this example, $F$ doubles the complex dimension of the algebra as a vector space, while $U$ leaves it unchanged.

Another interesting series of examples, this time related to quantum field theory, is that of algebras where the generators form a Hilbert space. These are associated to the Fock space representation of systems with variable numbers of particles, such as are used in particle physics, quantum optics or solid-state physics. As algebras, they are the algebras of creation operators, which are Abelian subalgebras of the full-blown algebras of observables on Fock space.

Let $H$ be a complex Hilbert space. If $H$ has an anti-unitary involution $\ast$, $H$ decomposes as $H \simeq H^\sharp \oplus_{R} i H^\sharp$, where $H^\sharp$ is the real eigenspace of vectors such that $a^\ast = a$. We denote the polynomials on $H$ by $C[H^\sharp]$ if $H$ has an involution, and $C[H]$ otherwise. The analogue of the preceding diagram is

\[
\begin{array}{ccc}
C[H^\sharp] & \xrightarrow{U} & C[H] \\
\downarrow^F & & \downarrow^U \\
C[H^\sharp, H^\ast] & \simeq & C[H_1^\sharp \oplus H_2^\sharp] \\
\end{array}
\]

and the physical interpretation of each of the algebras is as follows.

$C[H^\sharp]$ The space $H^\sharp$ is a real Hilbert space of states of a truly neutral particle (which is its own antiparticle), and its complexification $H$ is the complex vector space of all quantum states. The algebra $C[H^\sharp]$ is a dense subspace of the corresponding Fock space or, alternatively, the algebra of creation operators on it.

$C[H]$ The complex Hilbert space $H$ is the Hilbert space of single-particle states for a charged particle, and the algebra $C[H]$ is dense in the subspace of Fock space not including any antiparticles. As an algebra, $C[H]$ is the algebra of creation operators of particles, but it contains no creation operators of antiparticles.

$C[H \oplus H^\ast]$ The dual space $H^\ast$ is the Hilbert space of the single-antiparticle states associated to $H$, and $C[H \oplus H^\ast]$ is dense in the full Fock space of particles and antiparticles. The isomorphism $C[H \oplus H^\ast] \simeq C[H_1^\sharp \oplus H_2^\sharp]$ is associated with the possibility of representing a complex charged fields by a pair of real neutral fields, and conversely. Again, as an algebra, this is the algebra of creation operators of one species of charged particles and antiparticles, or two species of truly neutral particles.

$C[H_1 \oplus H_2]$ This is dense in the Fock space of two charged particles which are not antiparticles of each other, and it does not include the antiparticle states. As an algebra, it is the algebra of creation operators of two charged particles, with no creation operators for antiparticles.
3 The weak topology

In order to use tools from analysis it is necessary that all algebras under consideration have a topology making all the algebra operations continuous, and that the involution on a $*$-algebra be continuous, too. This is not much of a restriction, since any vector space $V$ can be given the (locally convex) weak topology $\omega(V, V^*)$ induced by its algebraic dual $V^*$, which then coincides with the topological dual. Not only that, but every linear map $f: V \to W$ is continuous with respect to the weak topologies on $V$ and $W$. Indeed, $f: V \to W$ is continuous with respect to the weak topology on $W$ if, and only if, $g \circ f: V \to \mathbb{C}$ is continuous for all $g: W \to \mathbb{C}$, but the weak topology on $V$ makes every linear functional on it, and in particular $g \circ f$, continuous by definition. Since the multiplication and unit maps are linear, the weak topology provides a functor from algebras with homomorphisms to locally convex algebras with continuous algebra homomorphisms. The same applies to Abelian algebras and $*$-algebras, and the weak topology defines functors from Abelian algebras or $*$-algebras to locally convex Abelian algebras or $*$-algebras. Because the weak topology can be put on every algebra in a way that makes all homomorphisms continuous, we have the following commutative diagram of functors:

$$
\begin{array}{ccc}
\text{AbAlg} & \xrightarrow{F} & \text{AbAlg}^* \\
\downarrow{\omega((-,-^*}} & & \downarrow{\omega((-,-^*)} \\
\text{LCAbAlg} & \xrightarrow{U} & \text{LCAb^*Alg}
\end{array}
$$

On the bottom row of this diagram, the “underlying” functor $U: \text{LCAb^*Alg} \to \text{LCAbAlg}$ takes any locally convex $*$-algebra $B$ to its underlying algebra $U(B)$ with the weak topology defined by the collection of all $U(f)$, where $f: B \to \mathbb{C}$ is a $*$-homomorphism. Given a locally convex algebra $A$, the “free” locally convex $*$-algebra $F(A)$ is the free $*$-algebra with the weak topology defined by all the $*$-homomorphisms $F(f)$ where $f: A \to \mathbb{C}$ is a homomorphism.

3.1 Examples

Algebras of polynomials on finitely many variables, such as $\mathbb{C}[z]$, are isomorphic as vector spaces to the space of complex sequences with finitely many nonzero entries, usually denoted $c_{00}$. A linear functional on this space assigns to any sequence a linear combination of its (finitely many) nonzero entries, and the algebraic dual $c_{00}^*$ is isomorphic to the space of all sequences $l_0$, with unrestricted complex coefficients. The space $c_{00}$ is weakly complete, since for any $a \in c_{00}$ there is an $\alpha \in l_0$ obtained by replacing each nonzero element of $a$ with its inverse, so that $\alpha(a)$ is the number of nonzero elements of $a$. Since $a \in c_{00}$ and $\alpha \in c_{00}^*$, $\alpha(a)$ must be finite, and so $a \in c_{00}$ already.

In the case of $\mathbb{C}[H]$, the set of generators of the polynomial algebra is not just any infinite set, but it forms a Hilbert space. We have a homogeneous decomposition $\mathbb{C}[H] \simeq \bigoplus_{n \geq 0} H^\otimes n$, where $H^\otimes n$ (the symmetric tensor power of $H$)
has a natural inner product derived from that of $H$, and any element of $C[H]$ is finite linear combination of monomials. We define $C[H]^* := \prod_{n \geq 0} (H^\otimes n)^*$, that is, each element of $C[H]^*$ consists of one element of the topological dual $(H^\otimes n)^*$ for each $n$. The resulting weak topology makes $C[H]$ complete, as in the finitely-generated case.

4 The Gel’fand spectrum

If $A$ is an Abelian algebra or $\ast$-algebra $A$ with the weak topology, we define its Gel’fand spectrum, denoted $\Delta_A$, as the collection of morphisms from it into the complex numbers in the appropriate category. Specifically, if $A$ is a commutative algebra with a topology, we define the Gel’fand spectrum of $A$ to be the collection of all continuous algebra homomorphisms into the complex numbers. In symbols,

$$\Delta_A = \text{LCAbAlg}(A, \mathbb{C}).$$

Similarly, if $A$ is a commutative $\ast$-algebra with a topology, we define its Gel’fand spectrum $\Delta_A$ to be the collection of all continuous $\ast$-algebra $\ast$-homomorphisms into the complex numbers, or

$$\Delta_A = \text{LCAbAlg}^\ast(A, \mathbb{C}).$$

These are both instances of hom-sets so, by abstract nonsense, they are contravariant functors to $\text{Set}$, meaning that algebra homomorphisms induce natural set maps going between the spectra in the opposite direction. Precisely, if $f: A \to B$ is an continuous homomorphism (or $\ast$-homomorphism) between Abelian algebras (or $\ast$-algebras), then there is a function $\Delta_f: \Delta_B \to \Delta_A$ given by $\Delta_f(p) = p \circ f: A \to \mathbb{C} \in \Delta_A$ for any continuous homomorphism (or $\ast$-homomorphism) $p \in \Delta_B$. In the literature, the Gel’fand spectrum is normally defined as the collection of maximal ideals. The functorial definition given here is much more restrictive, and it coincides with the usual definitions only for $C^\ast$-algebras or normed algebras. We discuss this in greater detail in the next section.

Consider now the evaluation map

$$e: \Delta_A \times A \to \mathbb{C} \quad (p, a) \mapsto p(a).$$

Equivalent to this is the Gel’fand transform, which associates to each element $a \in A$ the function $e(\cdot, a): \Delta_A \to \mathbb{C}$. The Gel’fand transform

$$\hat{\cdot}: A \to C^{\Delta_A} \quad a \mapsto e(\cdot, a)$$

is an algebra homomorphism (or $\ast$-homomorphism) into the $\ast$-algebra of all complex functions on $\Delta_A$ (with pointwise complex conjugation as involution).

The evaluation map induces a natural topology on $\Delta_A$, namely the weakest topology making every $\hat{a} \in \hat{A}$ continuous. Note that $\Delta_A$ is a subset of the
dual $A^*$, and that the spectral topology just defined is the same as the one induced on $\Delta_A$ as a subset of $A^*$ with the weak$^*$ topology. Since the weak$^*$ topology separates points—because given two different homomorphisms $p, q \in \Delta_A$, there must be an $a \in A$ such that $\hat{a}(p) = p(a) \neq q(a) = \hat{a}(q)$—, $\Delta_A$ is Hausdorff.

**Proposition 1 (Completeness)** The spectrum $\Delta_A$ is a weak$^*$ closed subset of $A^*$.

**Proof** This argument is essentially the first half of the proof of the Banach–Alaoglu theorem [Rud91 §3.15].

Let $\{p_\lambda\}_{\lambda \in \Lambda}$ be a net in $\Delta_A$ converging in the weak$^*$ topology; i.e., for every $a \in A$, $p_\lambda(a) \to p(a)$ for some $p(a) \in C$. Then,

$$p(a + b) - p(a) - p(b) = [p(a + b) - p_\lambda(a + b)] + [p_\lambda(a) - p(a)] + [p_\lambda(b) - p(b)]$$

implies that $p$ is linear, so $p \in A^*$. Also,

$$p(ab) - p(a)p(b) = [p(ab) - p_\lambda(ab)] + [p_\lambda(a) - p(a)] [p_\lambda(b) - p(b)] + [p_\lambda(a) - p(a)] [p_\lambda(b) - p(b)]$$

together with the trivial observation that $p_\lambda(1) = 1$ for all $\lambda \in \Lambda$ so $p(1) = 1$, implies that $p$ is an algebra homomorphism. This completes the proof in the category of algebras and continuous algebra homomorphisms. If $A$ is a $*$-algebra,

$$p(a^*) - \overline{p(a)} = [p(a^*) - p_\lambda(a^*)] + [\overline{p_\lambda(a)} - \overline{p(a)}]$$

implies that $p$ is a $*$-algebra $*$-homomorphism. \(\square\)

The second part of the proof of the Banach–Alaoglu theorem [Rud91 §3.15] provides a characterization of compact subsets of the spectrum.

**Proposition 2 (Compactness)** With respect to the weak$^*$ topology on $\Delta_A$, a closed subset $F \subseteq \Delta_A$ is compact if, and only if, every $\hat{a} \in \hat{A}$ is bounded on it.

**Proof**

$\Rightarrow$) the continuous image of a compact set is compact, and compact sets of $C$ are bounded; and

$\Leftarrow$) we can use $\hat{A}$ to embed $F$ homeomorphically as a closed subset of a cube which is compact by Tychonoff’s theorem, and Hausdorff. \(\square\)

It follows that a subset of the spectrum has compact closure if, and only if, every $\hat{a} \in \hat{A}$ is bounded on it; and a point of the spectrum has a basis of compact neighbourhoods if, and only if, it has an neighbourhood on which every $\hat{a} \in \hat{A}$ is bounded. Also, this result implies that the restriction of $\hat{A}$ to a compact subset of $\Delta_A$ is a normed algebra.

We now turn to the question whether the set map $\Delta_f : \Delta_B \to \Delta_A$ defined above is a continuous map with respect to the weak$^*$ topologies on $\Delta_A$ and $\Delta_B$. This is all that is required to show that $\Delta$ is a functor not only into $\textbf{Set}$, but into $\textbf{Top}$.
Proposition 3 (Functoriality) If \( f : A \to B \) is a continuous homomorphism (or \(*\)-homomorphism) of Abelian algebras (or \(*\)-algebras), then the set map
\[
\Delta_f : \Delta_B \to \Delta_A \\
p \mapsto fp
\]
is continuous with respect to the weak\(^*\) topologies on \( \Delta_A \) and \( \Delta_B \).

**Proof** The weak\(^*\) topology on \( \Delta_A \) admits a sub-base consisting of sets of the form \( U = \hat{a}^{-1}(G) \), where \( a \in A \) and \( G \) is open in \( C \). We need to show that \( V = (\Delta_f)^{-1}(U) \) is open with respect to the weak\(^*\) topology on \( \Delta_B \). In fact, a stronger statement is true, namely, \( (\Delta_f)^{-1}(U) = \hat{b}^{-1}(G) \) where \( b = f(a) \).

Indeed, \( p : B \to C \) is in \( V \) if, and only if, \( \Delta_f(p) = fp \in U \), that is, \( p(f(a)) \in G \) or, equivalently, \( \hat{b}(p) \in G \). \( \square \)

Proposition 4 (Separation and regularity) With respect to the weak\(^*\) topology, \( \Delta_A \) is a Tychonoff (completely regular, Hausdorff) space.

**Proof** The topology on \( \Delta_A \) is the weak topology defined by the complex functions \( \hat{a} \in \hat{A} \). However, the same topology is obtained if the image is considered to be not the complex plane, but the complex sphere, which is compact metric and so Tychonoff. By means of the family of all \( \hat{a} \in A \), \( \Delta_A \) can be homeomorphically embedded as a subset of a product of Tychonoff spaces, and so is a Tychonoff space \([\text{Wil70}, \S14]\). \( \square \)

The situation is this:

\[
\begin{array}{ccc}
\text{LCAbAlg} & \overset{F}{\longrightarrow} & \text{LCAb}^*\text{Alg} \\
\Delta \downarrow & & \Delta \\
\text{Tych} & \overset{\Delta}{\longrightarrow} & \text{Tych}
\end{array}
\]

The diagram commutes in one direction only, namely, for any locally convex Abelian algebra \( A \), it is true that \( \Delta_{F(A)} = \Delta_A \) because each continuous algebra homomorphism \( f : A \to B \) extends to a unique continuous \(*\)-algebra \(*\)-homomorphism \( F(f) : F(A) \to F(B) \) whose restriction to \( A \) is precisely \( f \). On the other hand, if \( A \) is a general locally convex Abelian \(*\)-algebra, \( \Delta_A \not\cong \Delta_{U(A)} \). However, the next best thing is true: there is a natural transformation \( j : \Delta \Rightarrow \Delta_{U(A)} \) associated to the fact that every continuous \(*\)-algebra \(*\)-homomorphism is an ordinary continuous homomorphism of the underlying algebra. In other words,

Proposition 5 (Naturality) If \( A \) is any locally convex \(*\)-algebra and \( U(A) \) is the underlying locally convex algebra, there is a continuous inclusion map \( j_A : \Delta_A \to \Delta_{U(A)} \) such that, for every continuous \(*\)-algebra \(*\)-homomorphism \( f : A \to B \) the
following diagram commutes

\[
\begin{array}{ccc}
A & \xrightarrow{\Delta_A} & \Delta_U(A) \\
\downarrow f & & \downarrow \Delta_U(f) \\
B & \xrightarrow{\Delta_B} & \Delta_U(B)
\end{array}
\]

**Proof** Recall that \(U(f):U(A) \to U(B)\) is the continuous algebra homomorphism between the underlying locally convex algebras associated to \(f:A \to B\). Recall also that \(\Delta_f:\Delta_B \to \Delta_A\) is the continuous map obtained by composing with \(f\); that is, if \(p:B \to C\) is in \(\Delta_B\), then \(\Delta_f(p) = fp:A \to C\) is in \(\Delta_A\). Similarly, \(\Delta_{U(f)}:\Delta_{U(B)} \to \Delta_{U(A)}\) is the continuous function mapping \(p:U(B) \to C\) to \(U(f)p:U(A) \to C\).

To show that the diagram commutes, let \(p:B \to C\) be a continuous \(*\)-algebra \(*\)-homomorphism. Then, \(\Delta_f(p) = fp:A \to C\), and \(j_A(fp):U(A) \to C\) is the associated continuous algebra homomorphism. On the other hand, \(j_B(p):U(B) \to C\) is the continuous algebra homomorphism associated to \(p\), and \(\Delta_{U(f)}(j_B(p)) = U(f)j_B(p):U(A) \to C\). It remains only to show that \(j_A(fp) = U(f)j_B(p):U(A) \to C\), but this is because, as set maps, \(j_A(fp) = fp\), \(j_B(p) = p\), and \(U(f) = f\).

We can sum up the content of this section in the following theorem.

**Theorem 6 (Gel’fand spectrum)** Let \(A\) be a locally convex algebra or \(*\)-algebra, and let its Gel’fand spectrum \(\Delta_A\) be the hom-set \(\text{hom}(A,C)\) in the appropriate category. Then, \(\Delta_A\) is a weak\(^*\)-closed subset of the topological dual \(A^*\) and inherits a Tychonoff space topology.

**4.1 Examples**

Consider the \(*\)-algebra \(A = \mathbb{C}[x]\) where \(x^* = x\). A \(*\)-algebra \(*\)-homomorphism \(p:A \to C\) is uniquely determined by \(p(x)\), which must be a real number since \(p(x) = p(x^*) = p(x)\). Other than that, \(p(x) \in \mathbb{R}\) is unrestricted, and so \(\Delta_A \simeq \mathbb{R}\).

Similarly, it can be shown that the spectrum of \(B = \mathbb{C}[z]\) is \(\Delta_B \simeq \mathbb{C}\). Since \(\mathbb{R}\) is strictly contained in \(\mathbb{C}\) and \(B = U(A)\), we have an example of how \(\Delta_{U(A)} \neq \Delta_A\).

Next we consider the spectrum of \(C = \mathbb{C}[z,z^*]\). An algebra \(*\)-homomorphism \(p:C \to \mathbb{C}\) is determined by \(p(z) \in \mathbb{C}\), and the condition that \(p(z^*) = p(z)\) does not restrict the possible value of \(p(z)\). Hence, \(\Delta_C \simeq \mathbb{C}\). This is expected, as \(C = F(B)\) and we know that \(\Delta_{F(B)} = \Delta_B\).

We can use these three examples to illustrate a principle: real analysis is all about \(*\)-algebras, and complex analysis is all about algebras. Also, analysis on the complex plane is done by going back and forth between the algebra \(\mathbb{C}[z]\) and the \(*\)-algebra \(\mathbb{C}[z,z^*]\), whose spectra are both isomorphic to \(\mathbb{C}\). The difference is that \(\mathbb{C}[z,z^*]\) is used to study the structure of \(\mathbb{C}\) as a two-dimensional real manifold, while \(\mathbb{C}[z]\) is used to study the structure of \(\mathbb{C}\) as a one-dimensional...
complex manifold. In complex analysis, nominally one is studying holomorphic functions, which are limits of polynomials in $\mathbb{C}[z]$. However, often one needs to use the real and imaginary parts, which live in $\mathbb{C}[z, z^*] \simeq \mathbb{C}[x, y]$. A case in point is the Cauchy–Riemann equations $\partial f / \partial \bar{z} = 0$, which characterizes the image of $\mathbb{C}[z]$ inside $F(\mathbb{C}[z]) = \mathbb{C}[z, \bar{z}]$. In other words, the following sequence is exact:

$$
\begin{array}{c}
\mathbb{C}[z] 
\xrightarrow{F} 
\mathbb{C}[z, z^*] 
\xrightarrow{\partial / \partial z^*} 
\mathbb{C}[z, z^*]
\end{array}
$$

The case of $\mathbb{C}[H]$, where $H$ is a Hilbert space, is interesting because its spectrum is not locally compact. Indeed, just as in the case of polynomials on finitely many generators, an algebra homomorphism $p: \mathbb{C}[H] \rightarrow \mathbb{C}$ is uniquely determined by $p|_H \in H^*$, and so $\Delta_{\mathbb{C}[H]} \simeq H^*$, with the weak* topology. We know that locally compact, Hausdorff topological vector spaces must be finite-dimensional, so in this case the spectrum is not locally compact. Incidentally, since every unital Banach algebra has compact spectrum, this shows that the algebra of creation operators on Fock space cannot be a Banach algebra.

## 5 The Gel’fand transform

We now study in detail the Gel’fand transform, which is the algebra homomorphism (or $\ast$-algebra $\ast$-homomorphism) given by

$$
\begin{array}{ccc}
\hat{\cdot}: & A & \rightarrow & C(\Delta_A) \\
\quad & a & \mapsto & e(\cdot, a)
\end{array}
\quad \text{such that} \quad
\begin{array}{ccc}
\hat{a}: & \Delta_A & \rightarrow & \mathbb{C} \\
\quad & p & \mapsto & p(a)
\end{array}
$$

where $C(\Delta_A) \subseteq C^{\Delta_A}$ denotes the $\ast$-algebra of continuous complex functions on $\Delta_A$ or its underlying algebra. To fully understand this homomorphism we need to characterize its kernel and its image.

### 5.1 Ideals and homomorphisms

The kernel of a homomorphism of Abelian algebras is an ideal, that is, closed under addition and preserved by multiplication by elements of the algebra. Conversely, the quotient of an Abelian algebra by an ideal is an Abelian algebra homomorphism. The kernel of a $\ast$-homomorphism is closed under the involution and, if an ideal is closed under the involution the quotient is a $\ast$-homomorphism.

All nilpotent elements of $A$ must be in the kernel of the Gel’fand transform, as the equation $a^n = 0$ translates into the complex equation $p(a)^n = 0$, for all $p \in \Delta_A$, which is equivalent to $p(a) = 0$ for all $p \in \Delta_A$, or $\hat{a} = 0$. Although the nilpotent elements form an ideal, it is possible that the kernel of the Gel’fand transform contains other elements. If the Gel’fand transform is one-to-one, we say the algebra $A$ is semisimple.

We have defined the Gel’fand spectrum as the collection of continuous algebra homomorphisms (or $\ast$-algebra $\ast$-homomorphisms) into $\mathbb{C}$. We call the kernels of these homomorphisms Gel’fand ideals, and they are characterized by being closed, codimension-1 ideals and, in the case of $\ast$-algebras, closed under
the involution. The kernel of the Gel’fand transform, called the Gel’fand radical of $A$, consist of precisely those $a \in A$ on which every $p \in \Delta_A$ vanishes. Being the intersection of all the Gel’fand ideals, it is a closed ideal and, if $A$ is a $*$-algebra, it is closed under the involution. In sum,

**Gel’fand ideal** An ideal $I$ in $A$ of codimension 1, closed if $A$ has a topology and closed under the involution if $A$ is a $*$-algebra. The quotient $A/I$ is $C$, and the quotient map is a $*$-homomorphism if $A$ is a $*$-algebra.

**Gel’fand radical** The Gel’fand radical $R$ is the kernel of the Gel’fand transform, and it is the intersection of all Gel’fand ideals. If $A$ has a topology, the Gel’fand radical is closed; if it has an involution, the Gel’fand radical is closed under it.

Let us now analyze in more detail the difference between maximal ideals and Gel’fand ideals or, equivalently, the difference between the Gel’fand radical and the Jacobson radical. Recall the following concepts from commutative algebra [AM69]:

**Maximal ideal** A proper ideal $I$ in $A$ is maximal iff it is maximal among proper ideals with respect to inclusion. The algebra $A/I$ is a field. Every Gel’fand ideal is maximal, but maximal ideals may fail to be closed or have codimension 1.

**Prime ideal** A proper ideal $I$ in $A$ is prime iff $a, b \notin I$ implies $ab \notin I$. In the algebra $A/I$, the product of nonzero elements is nonzero. Every maximal ideal is prime.

**Radical ideal** If $I$ is an ideal in $A$, the radical of $I$ is the ideal

$$\text{rad}(I) = \{a \in A \mid \exists n > 0, a^n \in I\}.$$ 

It is the intersection of the prime ideals containing $I$.

**Jacobson radical** The Jacobson radical of $A$ is the ideal $J$ obtained by taking the intersection of all maximal ideals of $A$, and it is contained in the Gel’fand radical.

**Nilradical** The nilradical of $A$ is the ideal $N$ consisting of all nilpotent elements of $A$ (i.e., the radical ideal of the zero ideal). It is the intersection of all prime ideals. The algebra $A/N$ has no nilpotent elements. It is contained in the Jacobson radical.

We have already indicated that the Gel’fand spectrum is usually defined as the collection of all maximal ideals, the implication being that the Gel’fand radical coincides with the Jacobson radical. This is because maximal ideals of a normed algebra are closed, and because $C$ is the only normed field extension of $C$ (the Gel’fand–Mazur theorem), every maximal ideal of a normed algebra is a Gel’fand ideal. In the presence of an involution, the construction only works
for $C^*$-algebras because only then it can be proved that every homomorphism is a $*$-homomorphism.

Since every $*$-homomorphism is a homomorphism, for $*$-algebras what we have called the Gelfand spectrum is in general smaller than usually defined, and the Gelfand radical larger than usual. Sometimes the spectrum of an algebra is defined as the collection of all algebra homomorphisms into $\mathbb{C}$ (also called characters), continuous or not, irrespective of whether the algebra under consideration has an involution. Because of the inclusion of ordinary homomorphisms in the spectrum of a $*$-algebra, it can happen that the Gelfand transform is not a $*$-homomorphism. This is fixed by removing, as we do, from the spectrum of a $*$-algebra all the homomorphisms which are not $*$-homomorphisms.

In ring theory it is remarked that the spectrum of prime ideals is functorial because the inverse image of a prime ideal by a homomorphism is a prime ideal, but not so for maximal ideals so the maximal spectrum is not functorial. The point of our redefinition of the Gelfand transform is to show that, by insisting on a functorial definition that extends beyond the realm of $C^*$-algebras, some of the important conclusions of the Gelfand–Naimark theorem can also be extended.

### 5.2 Topologies on $C(\Delta)$

The space $C(\Delta_A)$ has a weak topology making all the evaluation maps continuous, which can be easily seen to be associated to pointwise convergence on the spectrum; the image of the Gelfand transform $\hat{A}$ inherits this topology. On the other hand, since the Gelfand radical $G$ is a closed ideal of the locally convex algebra $A$, the image of the Gelfand transform $\hat{A} \approx A/G$ has a locally convex quotient topology. These two topologies on $\hat{A}$ coincide. Note, however, that the space of continuous functions is rarely closed under the topology of pointwise convergence. A stronger topology is needed to make the algebra $C(\Delta_A)$ closed, but then it is no longer obvious that the Gelfand transform is continuous.

The natural stronger topology on $C(\Delta_A)$ is the compact-open topology (i.e., uniform convergence on compact sets), which is the locally convex topology defined by the seminorms

$$|f|_K = \sup_{p \in K} |f(p)|,$$

where $K$ is any compact subset of $\Delta_A$, and the algebra operations are continuous with respect to this topology. Since $\hat{A}$ is a subalgebra of $C(\Delta_A)$, it inherits the compact-open topology. The original (weak) topology on $A$ is strictly weaker than the compact-open topology on $\hat{A}$ unless the only compact subsets of the spectrum $\Delta_A$ are the finite subsets.

The compact-open topology is natural in another, more interesting sense, and that is the existence of a Stone–Weierstrass theorem for Tychonoff spaces (see [Wei70 §44B] for a sketch of the proof). Since the spectrum $\Delta_A$ of any algebra $A$ is a Tychonoff space, it follows that $\hat{A}$ is dense in $C(\Delta_A)$ with the compact-open topology.

**Proposition 7 (Stone–Weierstrass for Tychonoff spaces)** If $\Delta$ is a T-
chonoff space and \( A \) is a \(*\)-subalgebra of \( C(\Delta) \) which separates points of \( \Delta \), and contains the constant functions, then \( A \) is dense in \( C(\Delta) \) with the compact-open topology.

The conclusion is that the following diagram of functors commutes in both directions.

\[
\begin{array}{ccc}
\text{LCAb}^*\text{Alg} & \xrightarrow{\Delta} & \text{Tych} \\
\downarrow & & \downarrow \Delta \\
\text{AbLC}^*\text{Alg} & \xrightarrow{A \mapsto \hat{A}} & \text{C}(\cdot)
\end{array}
\]

That is, because of the Stone–Weierstrass theorem for Tychonoff spaces, the closure of \( \hat{A} \) is the space of continuous functions on the Gel’fand spectrum; and then there is the rather trivial observation that the spectrum of \( C(\Delta) \) is precisely \( \Delta \). This last observation implies that the Gel’fand functor from Abelian \( C^*\)-algebras to compact Hausdorff spaces, and its inverse, extend to functors between Abelian \( LC^*\)-algebras and Tychonoff spaces:

\[
\begin{array}{ccc}
\text{AbC}^*\text{Alg} & \xrightarrow{\Delta} & \text{CompT}_2 \\
\downarrow & & \downarrow \Delta \\
\text{AbLC}^*\text{Alg} & \xrightarrow{\text{C}(\cdot)} & \text{Tych}
\end{array}
\]

We can summarize the content of this section in the following theorem. For lack of a better name, we call the algebra of continuous complex functions on a Tychonoff space an “\( LC^*\)-algebra”, for “locally convex” and “locally \( C^*\)”.

**Theorem 8 (Generalized Gel’fand–Naǐmark theorem)** If \( \Delta_A \) is the Gel’fand spectrum of a semisimple, locally convex \(*\)-algebra \( A \), the Gel’fand transform is \(*\)-homomorphism of \( A \) into a dense \(*\)-subalgebra of \( C(\Delta_A) \), the \(*\)-algebra of continuous complex functions with the compact-open topology.

### 5.3 Examples

In the case of the \(*\)-algebra \( A = C[x] \), the results of this section translate into the fact that complex polynomials on \( \mathbb{R} \) (the Gel’fand spectrum of \( A \)) are dense in the space of all continuous functions from \( \mathbb{R} \) to \( \mathbb{C} \) with the compact-open topology. Similarly, the space of polynomials \( C[z, z^*] \) is dense in the continuous functions on \( \mathbb{C} \) with the compact-open topology.

In the infinite-dimensional case, we get the more interesting result that \( C[H, H^*] \) is dense (with the compact-open topology) in the space of all continuous complex functions on the Hilbert space \( H \). This goes a long way towards reducing nonlinear analysis on Hilbert spaces to algebra.
6 The Gel’fand–Na˘ımark–Segal construction

The Gel’fand–Na˘ımark–Segal theorem is based on the concept of a state on a $C^*$-algebra, which in the commutative setting has the interpretation of a classical expectation value on a family of bounded random variables. Since the definition of state does not require the algebra to be a $C^*$-algebra, it applies without modification to our setting. An intuitively appealing characterization of states which uses the Riesz representation theorem is that any state on a $C^*$-algebra is realized as a Borel probability measure on the Gel’fand spectrum. As we have seen, all that is lost when dropping the $C^*$ hypothesis is the compactness of the spectrum, but the next best result is true: the Gel’fand spectrum, if correctly defined, is always a Tychonoff space.

A state $E$ on a $*$-algebra $A$ is a positive, normalized, compact-open continuous linear functional on $\hat{A}$. That is:

- $E \in A^*$,
- $E(1) = 1$, and
- $E(\hat{a}^*\hat{a}) \geq 0$ for all $a \in A$.
- there are compact subsets $K_1, \ldots, K_n \subseteq \Delta_A$ and positive numbers $C_1, \ldots, C_n$ such that $\max\{C_i|\hat{a}|_{K_i}\} < 1$ implies $|E(\hat{a})| < 1$;

The compact-open continuity condition is equivalent to

- there is a compact subsets $K \subseteq \Delta_A$ and a positive number $C$ such that $C|\hat{a}|_K < 1$ implies $|E(\hat{a})| < 1$

(just let $C = \max\{C_i\}$ and $K = \cup\{K_i\}$). By the Riesz representation theorem, any such linear functional is of the form

$$E(a) = \int_K \hat{a}(p)\mu_E(dp)$$

where $\mu_E$ is a Borel probability measure on $K$. In other words, a state on $A$ is a state on the $C^*$-algebra $A|_K$. In this way, once the Gel’fand spectrum of an algebra or $*$-algebra is known, an ample supply of states becomes available.

Given a state $E$ on the $*$-algebra $\hat{A}$, one can define an inner product on $A$ by the formula $(a, b) := E(a^*b)$ for all $a, b \in A$. To obtain a Hilbert space one must complete $A$ with respect to the inner product, and take the quotient by the ideal of zero-norm elements of $A$. This is the Gel’fand–Na˘ımark–Segal construction. As in the case of $C^*$-algebras [Rud91 §12.41], given that for every $\hat{a} \in \hat{A}$ it is always possible to find a state that does not vanish on it, one can find a (possibly non-separable) Hilbert space on which $\hat{A}$ and $C(\Delta_A)$ are faithfully represented as algebras of unbounded operators.

If $\Delta_A$ is not compact, Borel probability measures on $\Delta_A$ are associated with densely defined states, meaning positive, normalized, linear functionals on $C(\Delta_A)$ which are densely defined and not necessarily continuous in the
compact-open topology. The GNS construction can be carried out normally in that case, with due attention being paid to subtleties about domains of unbounded operators.

6.1 Examples

It might be surprising that states on the algebra of polynomials must be compactly-supported measures on the spectrum. Where did the ubiquitous Gaussian measure go? The answer is that the Gaussian probability density function has an inverse which can be approximated by polynomials uniformly on compact sets, and that means that the Gaussian measure cannot be a continuous linear functional with respect to the compact-open topology. However, the integral of the Gaussian density times any polynomial is finite, so the Gaussian measure is a densely-defined state. A similar argument holds for the two-dimensional Gaussian and the algebra $\mathbb{C}[z, z^*]$. With due care, the Gaussian measure and other measures with non-compact support are no harder to deal with than compactly-supported measures.

7 Conclusions

In this paper we have shown that the Gel’fand–Naĭmark theorem generalizes from $C^*$-algebras to any semisimple $*$-algebra, and that the Gel’fand spectrum and Gel’fand transform are well-behaved for virtually any algebra or $*$-algebra. The key to obtaining these results is to define the Gel’fand spectrum in a manifestly functorial way which, if nothing else, shows the power of elementary category theory as an aid to generalization and to the formulation of the right definitions.

The ability to generalize the Gel’fand–Naĭmark theorem to essentially arbitrary algebras is important for applications in physics and probability theory, where often the requirement that observables be bounded seems rather unnatural and can be justified only on the grounds of mathematical convenience. It would be desirable to more fully illustrate the usefulness of our results with problems where the $C^*$-algebraic formulation of probability theory is awkward because of the essential presence of unbounded random variables. On the mathematical side, the question remains open whether there is an intrinsic characterization of semisimple, locally convex $*$-algebras.

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