Compatibility of Larmor’s formula with radiation reaction for a radiating charge

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It is shown that the well-known disparity in classical electrodynamics between Larmor’s formula for the power radiated in electromagnetic fields of a charge undergoing a non-uniform motion and the power-loss calculated from the radiation reaction on the charge, is successfully resolved when a proper distinction is made between quantities expressed in terms of a retarded time and those expressed in terms of a “real time”. Larmor’s formula, in general, gives radiation crossing a surface surrounding the charge, in terms of the time-retarded value of the acceleration of the charge, while the rate of work done by the system against its self-forces, considered to be the source of radiation reaction, is expressed in terms of the “present” values of the acceleration and its time derivative. We show here that Larmor’s formula and the rate of work done against the self-forces are consistent with each other, provided both are expressed in terms of either time-retarded or “present” values of quantities specifying the charge motion.

I. INTRODUCTION

One of the most curious and perhaps an equally annoying problem in classical electrodynamics is that the power emitted from an accelerated charge does not appear to exactly match with the radiation reaction on the charge. In the standard, Larmor’s radiation formula (generalized to Liénard’s formula in the case of a relativistic motion), the radiated power is directly proportional to the square of acceleration of the charged particle. From the energy conservation law, it is to be surely expected that the power emitted in radiation fields equals the power loss of the radiating charge. But from the radiation reaction equation, the power loss of a radiating charge is directly proportional to its rate of change of acceleration (see e.g., [1, 2]). Although the two formulæ do yield the same value of energy when integrated over a time interval chosen such that the scalar product of velocity and acceleration vectors of the charge is the same at the beginning as at the end of the interval (a periodic motion!), the two calculations do not match when the charge is still undergoing a non-uniform, non-periodic motion at either end of the time interval [3]. In any case the functional forms of the two formulæ appear totally different. This enigma has defied a satisfactory solution despite the continuous efforts for the last 100 years or so. It is generally thought that the root-cause of this problem may lie in the radiation-reaction equation, whose derivation is considered to be not as rigorous as that of the formula for power radiated. Some interesting proposals for the “removal” of the above discrepancy include the ad hoc assumption of an acceleration dependent term either in a modified form of the Lorentz-force formula [4], or in the radiation-reaction equation in the form of a “bound-energy” term for an accelerated charge [5, 6] (also called as an internal-energy term or simply a “Schott-term”, based on the first such suggestion by Schott [7]), or a somewhat related proposition [8] that even the proper-mass of a radiating charged particle (e.g., of an electron) varies, or even a combination of some of these propositions [9]. Alternately it has been suggested [10] that there may be some fundamental difference in the electrodynamics of a continuous charge distribution and of a “point charge” (with a somewhat different stress-energy tensor for the latter). It is interesting to note that an understanding of this anomaly has sometimes been sought beyond the border of the classical electrodynamics (e.g., in the vacuum-fluctuations of the electromagnetic fields in quantum theory [11]). Ideally one would expect the classical electrodynamics to be mathematically consistent within itself, even though it might not be adequate to explain all experimental phenomena observed for an elementary charged particle. Because of the vastness of literature on this subject, we refer the reader to a review article or text-books [12, 13] for further references on these and other interesting ideas that have been proposed to remove this seemingly inconsistency in classical electrodynamics.

We intend to show here that the difference perceived in the two power formulæ is merely a reflection of the fact that the two are calculated in terms of two different time systems. While the radiation reaction formulæ is expressed in terms of acceleration and its temporal derivative being evaluated at the “real time” of the charged particle, Larmor’s radiation formulæ is written actually in terms of acceleration dependent term either in a modified form of the Lorentz-force formula [14], or in the radiation-reaction equation in the form of a “bound-energy” term for an accelerated charge [15, 16] (also called as an internal-energy term or simply a “Schott-term”, based on the first such suggestion by Schott [17]), or a...
but the formulation can be generalized to a relativistic case, by using the condition of relativistic covariance (see e.g., [2]).

II. POWER RADIATED BY THE CHARGE

The electromagnetic field \((\mathbf{E}, \mathbf{B})\) of an arbitrarily moving charge \(e\) is given by \([1,13]\),

\[
\mathbf{E} = \frac{e}{r^2} \frac{(n - \beta)}{\gamma^2 (1 - \beta \cdot n)^3} + \frac{n \times \{(n - \beta) \times \dot{\beta}\}}{rc (1 - \beta \cdot n)^3} \big|_{ret},
\]

\[
\mathbf{B} = n \times \mathbf{E},
\]

where the quantities in square brackets on the right hand side are to be evaluated at the retarded time. More specifically, \(\beta = v/c, \dot{\beta} = \dot{v}/c, \text{ and } \gamma = 1/\sqrt{1 - \beta^2}\) represent respectively the velocity, acceleration and the Lorentz factor of the charge at the retarded time, and \(r = mn\) is the radial vector from the retarded position of the charge to the field point where electromagnetic fields are being evaluated.

Let \(\mathcal{K}\) be an inertial frame, which is the instantaneous rest-frame for the charge (i.e., \(\beta = 0\), though the charge have a finite acceleration \(\mathbf{v}_0\) at say, \(t = 0\)), then we can write the electromagnetic fields as \([15]\),

\[
\mathbf{E} = \frac{e}{r^2} \frac{n}{r c^2} \left\{ \mathbf{n} \times \left( \frac{\mathbf{n} \times \mathbf{v}_0}{r c^2} \right) \right\},
\]

\[
\mathbf{B} = -\frac{e}{r c^2} \frac{\mathbf{n} \times \mathbf{v}_0}{r c}. \tag{4}
\]

The simple relations \((3)\) and \((4)\) do not give the electromagnetic field for all field points in \(\mathcal{K}\) at all times. Instead these yield the electromagnetic field for events in space-time causally connected to the charge when it had an acceleration \(\mathbf{v}_0\). Thus these give for any time \(t = \tau\), the electromagnetic field on a spherical surface \(\Sigma\) of radius \(r = cr\) centred on the charge position at \(t = 0\).

Then from Poynting vector \(\mathbf{P}\) one gets the electromagnetic power passing through \(\Sigma\) at \(t = \tau\),

\[
P = \frac{2e^2}{3c^3} \mathbf{v}_0^2 \cdot \tag{5}
\]

The power passing through the spherical surface is independent of its radius \(r\), which could be made vanishingly small around the charge position at \(t = 0\), and then from the causality argument one intuitively concludes that this is the energy loss rate of the charge at \(t = 0\). This is Larmor’s famous result for an accelerated charge that the power loss at any time is proportional to the square of its acceleration at that instant.

III. THE CALCULATION OF SELF-FORCE

Poynting’s theorem allows us to relate the rate of electromagnetic energy outflow through the surface boundary of a charge distribution to the rate of change in the mechanical energy of the enclosed charges due to the electromagnetic fields within the volume. In our case here, the electromagnetic field within the volume is that of the charge itself, so for the charge to lose energy its self-field must cause some net force on it. For simplicity we shall consider here for the charge particle, a classical uniform spherical-shell model of radius \(r_0\), which may be made vanishingly small in the limit. We need to consider the force on each infinitesimal element of the charged sphere due to the fields from all its remainder parts, with the positions, velocities and accelerations of the latter calculated at the retarded times (à la expression \((1)\)). Then the net force on the charge is calculated by an integration over the whole sphere. For a simplification, the force calculations are usually done in the instantaneous rest-frame of the charge, say, at \(t = 0\).

Actually the mathematical details of the calculations of self-force, carried out first by Lorentz \([17]\) and done later more meticulously by Schott \([7]\), are available, as series in powers of \(r_0\), in various forms in more modern text-books \([13]\). In such calculations it is generally assumed that the motion of the charged particle varies slowly so that during the light-travel time across the particle, any change in its velocity, acceleration and other higher time derivatives is relatively small. This is equivalent to the conditions that \(|\dot{v}|\tau \ll c, |\ddot{v}|\tau \ll |v|\), etc., where \(\tau = r_0/c\).

Alternately one can calculate the electric field outside as well as inside of an instantaneously stationary charged spherical shell. For a permanently stationary charged sphere, it is a simple radial Coulomb electric field outside the shell,

\[
\mathbf{E} = \frac{e}{r_0^2}. \tag{6}
\]

while the field inside the shell is zero. The above Coulomb field gives rise to an outward repulsive force on each charge element of the shell. However, the net force is zero due to the spherical symmetry (we ignore here the question of the stability of the charged sphere against the force of self-repulsion).

However for an accelerated charge, there is an additional acceleration–dependent field component, which gives rise to a non-zero electric field inside the shell \([18]\),

\[
\mathbf{E} = e \left[ -\frac{2v}{3r_0c^2} + \frac{2\ddot{v}}{3c^3} + \cdots \right]. \tag{7}
\]

This electric field is continuous at the surface and is constant, both in direction and magnitude, at all points on the charged sphere. However, the magnetic field to this order is zero both inside and on the surface of the sphere at this instant, i.e., at \(t = 0\).

\[
\mathbf{B} = 0. \tag{8}
\]
Due to the self-electric field there is a net force on the charge,
\[ \mathbf{f} = -\frac{2e^2}{3r_0c^2} \mathbf{v} + \frac{2e^2}{3c^3} \mathbf{\ddot{v}}. \] (9)

Here we have dropped terms of order \( r_0 \) or higher, which will become zero for a vanishingly small \( r_0 \). This self-force can be written as,
\[ \mathbf{f} = \frac{4U_{el}}{3c^2} \mathbf{v} + \frac{2e^2}{3c^3} \mathbf{\ddot{v}} = -m_{el} \mathbf{v} + \frac{2e^2}{3c^3} \mathbf{\ddot{v}}, \] (10)

where \( U_{el} = e^2/2r_0 \) represents the electromagnetic self-energy in Coulomb fields of a stationary spherical-shell charge and \( m_{el} = 4U_{el}/3c^2 \) the inertial mass because of the electromagnetic self-energy of the charge \( 20 \).

**IV. THE RATE OF WORK DONE AGAINST THE SELF-FORCE**

To calculate the instantaneous rate of work being done against the self-force of a moving charge in an inertial frame, one has to take the scalar product of the self-force \( \mathbf{f} \) and the instantaneous velocity vector, \( \mathbf{v} \), both measured in that frame. For a non-relativistic motion, the expression for force can be used directly from the electromagnetic self-energy of the charge \( 20 \).

\[ \frac{dW}{dt} = \frac{2e^2}{3r_0c^2} \mathbf{\dot{v}} \cdot \mathbf{v} - \frac{2e^2}{3c^3} \mathbf{\ddot{v}} \cdot \mathbf{v}, \] (11)

or
\[ \frac{dW}{dt} = m_{el} \mathbf{\dot{v}} \cdot \mathbf{v} - \frac{2e^2}{3c^3} \mathbf{\ddot{v}} \cdot \mathbf{v}. \] (12)

The first term on the right hand side in Eq. (12) represents the change in the self-Coulomb field energy of the charge as its velocity changes. This term when combined with the additional work done during a changing Lorentz contraction (not included here in Eq. (12)) against the Coulomb self-repulsion force of the charged particle, on integration leads to the correct expression for energy in fields of a uniformly moving charged particle \( 16 \). Thus it is only the second term \( -(2e^2/3c^3) \mathbf{\dot{v}} \cdot \mathbf{v} \) on the right hand side of Eq. (12) that represents the “excess” power \( \text{see , e.g., } 19 \) going into the electromagnetic fields of a charge with a non-uniform motion. This conforms with the conclusions arrived at otherwise that a uniformly accelerated charge does not radiate \( 21 \ 22 \).

It has always seemed enigmatic that if Larmor’s formula indeed represents the instantaneous rate of power loss for an accelerated charge, why the above term contains \( -\mathbf{\dot{v}} \cdot \mathbf{v} \) instead of \( \mathbf{\ddot{v}}^2 \) (cf. Eq. 5). It should be noted that while Eq. 5 was derived for a “point” charge, Eq. (12) is for a charged sphere, though the term \( -\mathbf{\dot{v}} \cdot \mathbf{v} \) is independent of radius \( r_0 \) of the sphere. But that should not be the source of the discrepancy, as a comparison of the electromagnetic field of a charged sphere of a vanishing small radius with that of a “point” charge in their instantaneous rest-frames shows them to be identical, at least in regions external to the sphere \( 18 \). Therefore, we take it that the electromagnetic field external to a spherical charged-shell of vanishing small radius is, in general, given by Eqs. (1) and (2).

**V. LARMOR’S FORMULA FOR RADIATED POWER VS. POWER LOSS DUE TO RADIATION REACTION**

In frame \( K \), where the charge was at rest at \( t = 0 \), the rate of energy flow through surface \( \Sigma \) is given by Eq. 5. That is, the radiation passing through the spherical surface at \( t = r/c \) involves \( \mathbf{v}_0 \), the acceleration evaluated at retarded time \( t = 0 \). This electromagnetic power passing the surface \( \Sigma \) at time \( t = r/c \) was equated to the energy loss rate of the charge at \( t = 0 \) to arrive at Larmor’s formula. There is something amiss here otherwise how could the charge have undergone any power loss at \( t = 0 \) since it had no kinetic energy at that instant that could have been lost? Even if we assume an external agency responsible for the acceleration of the electromagnetic inertial mass arising because of the acceleration-dependent self-forces, it could not have provided the power necessary for radiation since work done by this external agency will also be zero as the charge has a zero velocity at that instant. Moreover, equating the Poynting flux through a surface at a time \( \tau \) to the loss rate of the mechanical energy of the charge at another instant (viz. \( t = 0 \)) is not a mathematically correct approach, because Poynting’s theorem relates the electromagnetic power passing a closed surface to the energy changes within the enclosed volume \emph{at the same instant} only. Poynting’s theorem for an electromagnetic system states that at any particular instant the rate of energy flow out of a surface plus the time rate of increase of electromagnetic energy within the enclosed volume is equal to negative of the mechanical work done by the field on the charges within the volume \( 1 \). Therefore the sum total of the rate of change of the volume integral of electromagnetic field energy (\( \mathcal{E}_{em} \)) and mechanical energy (\( \mathcal{E}_{me} \)) of the charge within the enclosed volume, \emph{evaluated only at } \( t = r/c \), should be equated to the negative of the surface integral of the Poynting flow in Eq. 5.

\[ \frac{d\mathcal{E}_{me}}{dt} + \frac{d\mathcal{E}_{em}}{dt} = \frac{2e^2}{3c^3} \mathbf{v}_0^2, \] (13)

We calculate the volume integral of the electromagnetic field energy \( \mathcal{E}_{em} \), from Eq. 2 within the charged sphere, and find it to be proportional to \( r_0^3 \) (or its higher powers) that can therefore be neglected for a vanishing small \( r_0 \).
implying $d\mathcal{E}_{\text{em}}/dt = 0$. Therefore applying Poynting’s theorem (Eq. (23)) to the sphere of radius $r_0$, we get for the mechanical work done on the charged sphere,

$$\frac{dc_{\text{me}}}{dt} = -\frac{2e^2}{3c^3}(\mathbf{v}_0 \cdot \dot{\mathbf{v}}_0).$$  

(14)

Now while the power on either side in Eq. (14) is for $t = \tau$, the right hand side is expressed in terms of $\mathbf{v}_0$, the value of the acceleration at time $t = 0$. However, we can write the acceleration in terms of its value at time $t = \tau$ as,

$$\dot{\mathbf{v}}_0 = \mathbf{v} - \mathbf{v} \tau + \cdots.$$  

(15)

Further, the charge at time $t = \tau$ is moving with a velocity $\mathbf{v}$ given by,

$$\mathbf{v} = \mathbf{v}_0 \tau + \cdots,$$  

(16)

where $\tau = r_0/c$ and we dropped terms of order $\tau^2$ or higher for a small $r_0$. Substituting the values of $\dot{\mathbf{v}}$ from Eqs. (15) and (16) into Eq. (14), we can write for the rate of change of the energy of the charge,

$$\frac{d\mathcal{E}_{\text{me}}}{dt} = -\frac{2e^2}{3c^3} \left( (\mathbf{v} - \mathbf{v} \tau) \cdot \mathbf{v} \right),$$  

(17)

or

$$\frac{d\mathcal{E}_{\text{me}}}{dt} = -\left[ \frac{2e^2}{3r_0c^2} \mathbf{v} \cdot \mathbf{v} - \frac{2e^2}{3c^2} \dot{\mathbf{v}} \cdot \mathbf{v} \right].$$  

(18)

Thus we see that the rate of decrease of the energy of the charge (Eq. (15)), inferred from the Larmor’s radiation formula (Eq. (5)), is the same as the rate of work (Eq. (11)) done against the self-force of the charge.

With the hindsight one can understand the radiation reaction equation now more clearly. Using Eq. (10), we can write the self-electric field in Eq. (7) as,

$$\mathbf{E} = \frac{-2e}{3r_0c^2} \dot{\mathbf{v}}_0,$$  

(19)

and thereby the force on the charge from Eq. (14) as,

$$\mathbf{f} = -\frac{2e^2}{3r_0c^2} \dot{\mathbf{v}}_0 = -m_{\text{el}} \dot{\mathbf{v}}_0,$$  

(20)

and accordingly a rate of work done,

$$\frac{dW_{\text{el}}}{dt} = \frac{2e^2}{3r_0c^2} \dot{\mathbf{v}}_0 \cdot \mathbf{v} = m_{\text{el}} \dot{\mathbf{v}}_0 \cdot \mathbf{v}.$$  

(21)

which of course is consistent with Larmor’s formula (Eq. (5)) when we make use of Eq. (10).

One can also examine the consistency of radiated power to that of nil rate of work being done against the self-forces of the charge at $t = 0$, because the charge has a zero instantaneous velocity. Actually at that instant the magnetic field is zero on the surface of the charge (Eq. (5),[18]), therefore the Poynting vector is zero with a nil electromagnetic radiation crossing the surface at that instant. This of course makes the two methods of calculating power consistent with each other even at $t = 0$.

VI. MAXWELL STRESS TENSOR AND THE SELF-FORCE

We examined the consistency of instantaneous rate of work done, against the self-force, with Larmor’s formula, calculated from Poynting flux through surface $\Sigma$. Likewise, we can also examine the consistency of the self-force itself (Eq. (9) or equivalently Eq. (23)), on the charge, with the Maxwell stress tensor, the momentum flux density. The latter lets us calculate the flow of momentum across the boundary surface $\Sigma$ into the enclosed volume which in turn represents the force acting on the combined system of particles and fields within the volume $\Sigma$. Thus the force on the charge can be calculated, in an alternative method, by making use of the conservation of the electro-mechanical momentum of the system. From a surface integral of Maxwell stress tensor $T$ on $\Sigma$ at $t = \tau$, we write the force on the charge enclosed within surface $\Sigma$ as $\Sigma$,

$$\mathbf{f} = \frac{d\mathcal{P}_{\text{me}}}{dt} = \int_{\Sigma} d\Sigma \mathbf{n} \cdot \mathbf{T} - \frac{d\mathcal{P}_{\text{em}}}{dt},$$  

(22)

Here $\mathcal{P}_{\text{em}} = (1/4\pi c) \int dV \mathbf{E} \times \mathbf{B}$ is the volume integral of the electromagnetic field momentum within the charged sphere, proportional to $r_0$ (or its higher powers), and can therefore be neglected for a vanishing small $r_0$, implying $d\mathcal{P}_{\text{em}}/dt = 0$.

Then

$$\mathbf{f} = \int_{\Sigma} \frac{d\mathcal{P}}{4\pi} \left[ (\mathbf{n} \cdot \mathbf{E}) \mathbf{E} + (\mathbf{n} \cdot \mathbf{B}) \mathbf{B} - \frac{1}{2} \mathbf{n} (E^2 + B^2) \right],$$  

(23)

where $\mathbf{E}$ and $\mathbf{B}$, electromagnetic fields on the surface $\Sigma$ of radius $r_0 = ct$ centred on the charge instantaneously stationay at $t = 0$, are given by Eqs. (9) and (14).

We can then write Eq. (22) as,

$$\mathbf{f} = \frac{e^2}{2} \int_0^\tau d\theta \sin \theta \left[ \frac{\mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{v}}_0)}{r_0c^2} + \frac{\mathbf{n}}{2r_0} - \frac{\mathbf{n} \cdot \dot{\mathbf{v}}_0 \sin^2 \theta}{c^4} \right],$$  

(24)

where $\theta$ is the angle that $\mathbf{n}$ makes with the direction of acceleration $\dot{\mathbf{v}}_0$. Now either of the second and third terms ($\propto \mathbf{n}$) on the right hand side, when integrated over two halves of the surface $\Sigma$ (i.e., for $\theta$ varying from 0 to $\pi/2$ and from $\pi/2$ to $\pi$), yields equal and opposite values. Thence their contributions become zero when integrated over the whole sphere. Then from Eq. (24) we get,

$$\mathbf{f} = \frac{e^2}{2} \int_0^\pi d\theta \sin \theta \left[ \frac{\mathbf{n} \dot{\mathbf{v}}_0 \cos \theta - \dot{\mathbf{v}}_0}{r_0c^2} \right],$$  

(25)

or

$$\mathbf{f} = \frac{e^2}{r_0c^2} \left[ \frac{1}{3} - 1 \right] \dot{\mathbf{v}}_0 = -\frac{2e^2}{3r_0c^2} \dot{\mathbf{v}}_0.$$  

(26)

This expression for force on the charge enclosed within the surface $\Sigma$ is the same as obtained from a detailed
derivation of the self-force of the accelerated charge (Eq. (20)).

What this implies is that an accelerated charged sphere of vanishingly small radius $r_0$, experiences a self-force proportional to the acceleration it had at a time $r_0/c$ earlier. Accordingly, the rate of work done on the charge is not proportional to the value of the acceleration at the “real time” $t$ (as normally expected for the dynamics of a particle in classical mechanics), but is instead proportional to the time-retarded acceleration $[\dot{v}]_\text{ret}$ (corresponding to an earlier time $t - r_0/c$).

$$\frac{dW}{dt} = \frac{2e^2}{3\epsilon_0 c^2} [\dot{v}]_\text{ret} \cdot \dot{v} = m_{\text{el}} [\dot{v}]_\text{ret} \cdot \dot{v}. \quad (27)$$

Now if we rewrite the Eq. (27) by expressing the retarded value of acceleration in terms of the “present” values of acceleration and its time derivative (Eq. (15)), we get the work done against the self-force of the charge at that particular instant (Eq. (12)). However, if we kept the time-retarded value of acceleration and instead express the velocity also in terms of its time-retarded value (to the required order in $r_0/c$),

$$v = [\dot{v}]_\text{ret} + \frac{r_0}{c} [\ddot{v}]_\text{ret}, \quad (28)$$

we get,

$$\frac{dW}{dt} = \left[m_{\text{el}} \ddot{v} \cdot \dot{v} + \frac{2e^2}{3c^2} \dot{v}^2\right]_\text{ret}, \quad (29)$$

which, of course, reduces to (Eq. (1)) when the time-retarded value of velocity is zero, i.e., when $[\dot{v}]_\text{ret} = 0$.

Thus if we examine the rate of work done on a charged particle at some given instant, then Eq. (29) yields the familiar Larmor’s radiation formula (second term on the right hand side), but at a cost that the rate of work done is expressed in terms of quantities specifying the motion of the charge at a retarded time, and not in terms of their real-time values.

This not only resolves the almost a century-old apparent “discrepancy” in the two power formulae, but also shows an intimate relation between the energy in the radiation fields and that in the Coulomb fields. In particular, the factor of $4/3$ in the electromagnetic mass ($m_{\text{el}} = 4U_{\text{el}}/3c^2$ [20]) of a spherical charge in classical electrodynamics is intimately connected with the factor of $2/3$ found both in Larmor’s formula and in the radiation-reaction formula. For long, this “mysterious” factor of $4/3$ has been considered to be undesirable and modifications in classical electromagnetic theory have been suggested to get rid of this factor (see e.g., references cited in [1, 12, 16]). If one does adopt such a modification, then one cannot see the relation between Larmor’s formula and the radiation-reaction formula. Moreover, as explicitly shown in [16], the factor of $4/3$ in the electromagnetic inertial mass naturally arises in the conventional electromagnetic theory when a full account is taken of all the energy-momentum contributions of the electromagnetic forces during the process of attaining the motion of the charged sphere.

To summarize, the instantaneous rate of work done in order to accelerate a charged spherical shell of vanishingly small radius $r_0$, at any time $t$ is proportional not to the real-time value of acceleration at time $t$, but to the time-retarded value of the acceleration at a time $t - r_0/c$. This rate of work done, when written in terms of the real-time value of the acceleration, comprises an additional term proportional to the time derivative of the acceleration. Alternatively, the rate of work done at time $t$ can as well be written in terms of Larmor’s expression for power radiated, however, at the cost that the rate of work done at time $t$ is to be expressed in terms of quantities specifying the motion of the charge at a retarded time $t - r_0/c$.

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