Simultaneous Elements Of Prescribed Multiplicative Orders

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Abstract: Let $u \neq \pm 1$, and $v \neq \pm 1$ be a pair of fixed relatively prime squarefree integers, and let $d \geq 1$, and $e \geq 1$ be a pair of fixed integers. It is shown that there are infinitely many primes $p \geq 2$ such that $u$ and $v$ have simultaneous prescribed multiplicative orders $\text{ord}_p u = (p-1)/d$ and $\text{ord}_p v = (p-1)/e$ respectively, unconditionally. In particular, a squarefree odd integer $u > 2$ and $v = 2$ are simultaneous primitive roots and quadratic residues (or quadratic nonresidues) modulo $p$ for infinitely many primes $p$, unconditionally.

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1 Introduction

The earliest study of admissible integers $k$-tuples $u_1, u_2, \ldots, u_k \in \mathbb{Z}$ of simultaneous primitive roots modulo a prime $p$ seems to be the conditional result in [21]. Much more general

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results for admissible rationals \( k \)-tuples \( u_1, u_2, \ldots, u_k \in \mathbb{Q} \) of simultaneous elements of independent (or pseudo independent) multiplicative orders modulo a prime \( p \geq 2 \) are considered in [26], and [14]. An important part of this problem is the characterization of admissible rationals \( k \)-tuples. There are various partial characterizations of admissible rationals \( k \)-tuples. For example, an important criterion states that a rationals \( k \)-tuple is admissible if and only if
\[
 u_1^{e_1} u_2^{e_2} \cdots u_k^{e_k} \neq -1,
\]
where \( e_1, e_2, \ldots, e_k \in \mathbb{Z} \) are integers. A more general characterization and the proof appears in [26, Proposition 14], [14, Lemma 5.1]. Other ad hoc techniques are explained in [15]. The characterization of admissible triple of rational numbers \( a, b, c \in \mathbb{Q} - \{-1, 0, 1\} \) with simultaneous equal multiplicative orders
\[
 \text{ord}_p a = \text{ord}_p b = \text{ord}_p c,
\]
known as the Schinzel-Wojcik problem, is an open problem, see [26, p. 2], and [8]. The specific relationship between the orders \( \text{ord}_p u | d \) and \( \text{ord}_p v | d \) of a pair of integers \( u, v \neq \pm 1 \) and some divisor \( d | p - 1 \), and the generalization to number fields, and abelian varieties are studied in several papers as [4], [1], et cetera, and counterexamples are produced in [4] and [20].

**Definition 1.1.** A \( k \)-tuple \( u_1, u_2, \ldots, u_k \neq \pm 1 \) of rational numbers is called an admissible \( k \)-tuple if the product is multiplicatively independent over the rational numbers:
\[
 u_1^{e_1} u_2^{e_2} \cdots u_k^{e_k} \neq 1,
\]
for any list of integer exponents \( e_1, e_2, \ldots, e_k \in \mathbb{Z}^\times \).

Relatively prime \( k \)-tuples, and relatively prime and squarefree \( k \)-tuples are automatically multiplicatively independent over the rational numbers. However, squarefree \( k \)-tuples are not necessarily multiplicatively independent over the rational numbers, for example, 3, 5, 15. Any \( k \)-tuple of integers that satisfies the Lang-Waldschmid conjecture is an admissible \( k \)-tuple. Specifically,
\[
 |u_1^{e_1} u_2^{e_2} \cdots u_k^{e_k} - 1| \geq \frac{C(\varepsilon)^k E}{|u_1 u_2 \cdots u_k | e_1 e_2 \cdots a e_k |^{1+\varepsilon}},
\]
where \( C(\varepsilon) > 0 \) is a constant, \( E = \max \{ |e_i| \} \), and \( \varepsilon > 0 \) is a small number, confer [31, Conjecture 2.5] for details.

**Definition 1.2.** Fix an admissible \( k \)-tuple \( u_1, u_2, \ldots, u_k \neq \pm 1 \) of rational numbers. The elements \( u_1, u_2, \ldots, u_k \in \mathbb{F}_p \) are said to be a simultaneous \( k \)-tuple of equal multiplicative orders modulo a prime \( p \geq 2 \), if \( \text{ord}_p u_1 | p - 1 \), and
\[
 \text{ord}_p u_1 = \text{ord}_p u_2 = \cdots = \text{ord}_p u_k,
\]
infinity often as \( p \to \infty \).

**Definition 1.3.** Fix an admissible \( k \)-tuple \( u_1, u_2, \ldots, u_k \neq \pm 1 \) of rational numbers. The elements \( u_1, u_2, \ldots, u_k \in \mathbb{F}_p \) are said to be a simultaneous \( k \)-tuple of decreasing multiplicative orders modulo a prime \( p \), if \( \text{ord}_p u_i | p - 1 \), and
\[
 \text{ord}_p u_1 > \text{ord}_p u_2 > \cdots > \text{ord}_p u_k,
\]
infinity often as \( p \to \infty \).
**Definition 1.4.** Fix an admissible $k$-tuple $u_1, u_2, \ldots, u_k \neq \pm 1$ of rational numbers, and fix an index integers $k$-tuple $d_1, d_2, \ldots, d_k \geq 1$. The elements $u_1, u_2, \ldots, u_k \in \mathbb{F}_p$ are called a simultaneous $k$-tuple of prescribed multiplicative orders modulo a prime $p$, if

$$\text{ord}_p u_1 = (p-1)/d_1, \quad \text{ord}_p u_2 = (p-1)/d_2, \quad \cdot \cdot \cdot \quad \text{ord}_p u_k = (p-1)/d_k,$$

(7)

infinitely often as $p \to \infty$.

Conditional on the GRH and or the $k$-tuple primes conjecture, several results for the existence and the densities of simultaneous rationals $k$-tuples have been proved in the literature, see [26], [8], and [14]. This note studies the unconditional asymptotic formulas for the number of primes with simultaneous elements of prescribed multiplicative orders.

**Theorem 1.1.** Fix a pair of relatively prime squarefree $u \neq \pm 1$, and $v \neq \pm 1$ rational numbers, and fix a pair of integers $d \geq 1$, and $e \geq 1$. If $x \geq 1$ is a sufficiently large number, and the indices $d, e \ll (\log x)^B$, with $B \geq 0$, then, the number of primes $p \in [x, 2x]$ with simultaneous elements $u$ and $v$ of prescribed multiplicative orders $\text{ord}_p u = (p-1)/d$ and $\text{ord}_p v = (p-1)/e$ modulo $p \geq 2$, has the asymptotic lower bound

$$R_2(x, u, v) \gg \frac{x}{(\log x)^{4B+1}(\log \log x)^2}$$

(8)

as $x \to \infty$, unconditionally.

In general, the number of primes $p \in [x, 2x]$ such that a fixed admissible $k$-tuples of rational numbers $u_1, u_2, \ldots, u_k \neq \pm 1$ has simultaneous prescribed multiplicative orders

$$\text{ord}_p u_1 = (p-1)/d_1, \quad \text{ord}_p u_2 = (p-1)/d_2, \quad \cdot \cdot \cdot \quad \text{ord}_p u_k = (p-1)/d_k,$$

(9)

where $d_i \ll (\log x)^B$ are fixed indices, and $B \geq 0$, has the lower bound

$$R_k(x, u, v) \gg \frac{x}{(\log x)^{2Bk+1}(\log \log x)^k}$$

(10)

as $x \to \infty$, unconditionally.

The maximal number of simultaneous primitive roots is bounded by $k = O(\log p)$, see [3, Section 14]. The same upper bound is expected to hold for any combination of simultaneous $k$-tuple prescribed of multiplicative orders, but it has not been verified yet. The average multiplicative order of a fixed rational number $u \neq \pm 1$ has the asymptotic lower bound

$$T_u(x) = \frac{1}{x} \sum_{n \leq x, \gcd(u, n) = 1} \text{ord}_n u \gg \frac{x}{\log x} e^{c(\log \log \log x)^{3/2}},$$

(11)

where $c > 0$ is a constant, the fine details are given in [17]. Other information on the average multiplicative orders in finite cyclic groups are given in [18], and the literature.

A special case illustrates the existence of simultaneously prescribed primitive roots and quadratic residues (or quadratic nonresidues) in the prime finite field $\mathbb{F}_p$ for infinitely many primes $p$.

**Corollary 1.1.** Let $u \geq 3$ be fixed a squarefree odd integer and let $v = 2$. If $x \geq 1$ is a large number, then elements $u, 2 \in \mathbb{F}_p$ have multiplicative orders $\text{ord}_p u = p - 1$
and \( \text{ord}_p 2 = (p - 1)/2 \), simultaneously. Moreover, the number of such primes has the asymptotic lower bound

\[
R_2(x, u, 2) \gg \frac{x}{(\log x)(\log \log x)^2}
\]

as \( x \to \infty \), unconditionally.

The unconditional number of primes with simultaneous admissible triple of rational numbers \( a, b, c \in \mathbb{Q} - \{-1, 0, 1\} \) of equal multiplicative orders

\[
\text{ord}_p a = \text{ord}_p b = \text{ord}_p c,
\]

such that \( \text{ord}_p a = (p - 1)/d \), with \( d \ll (\log x)^B \), and \( B \geq 0 \), has almost identical analysis as the proof of Theorem 11. In fact, any permutation of equality or inequality between the multiplicative orders can be produced by selecting any small fixed indices \( d, e, f \geq 1 \) to prescribe the multiplicative orders \( \text{ord}_p a = (p - 1)/d \), \( \text{ord}_p b = (p - 1)/e \), and \( \text{ord}_p c = (p - 1)/f \) respectively.

Some of the foundation works on the calculations of the implied constants in (3) appear in [26], [8], [14], et alii. The proof of Theorem 11 appears in Section 11; this result is completely unconditional. The other sections cover foundational and auxiliary materials.

## 2 Divisors Free Characteristic Function

**Definition 2.1.** The multiplicative order of an element in the cyclic group \( \mathbb{F}_p^\times \) is defined by \( \text{ord}_p(v) = \min\{k : v^k \equiv 1 \mod p\} \). Primitive elements in this cyclic group have order \( p - 1 = \#G \).

**Definition 2.2.** Let \( p \geq 2 \) be a prime, and let \( \mathbb{F}_p \) be the prime field of characteristic \( p \).

If \( d | p - 1 \) is a small divisor, the multiplicative subgroup of \( d \)-powers, and the subgroup index are defined by \( \mathbb{F}_p^d = \{\alpha^d : \alpha \in \mathbb{F}_p^\times\} \), and \( [\mathbb{F}_p^d : \mathbb{F}_p] = d \) respectively.

**Definition 2.3.** Fix an integer \( d \geq 1 \), and a rational number \( u \in \mathbb{Q}^\times \). An element \( u \in \mathbb{F}_p \) has index \( d = \text{ind}_d u \) modulo a prime \( p \) if and only if \( \text{ind}_p u = (p - 1)/\text{ord}_p u \). In particular, if \( d | p - 1 \), then the prime finite field \( \mathbb{F}_p \) contains \( \varphi(d) \) elements of index \( d \geq 1 \).

Each element \( u \in \mathbb{F}_p \) of index \( d \) is a \( d \)-power, but not conversely. A new divisor-free representation of the characteristic function of primitive element is introduced here. This representation can surpasses some of the limitations of the standard divisor-dependent representation of the characteristic function

\[
\Psi(u) = \frac{\varphi(p - 1)}{p - 1} \sum_{d|p-1} \frac{\mu(d)}{\varphi(d)} \sum_{\text{ord}(\chi) = d} \chi(u)
\]

or prime roots. The works in [15], and [12] attribute this formula to Vinogradov. The proof and other details on this representation of the characteristic function of primitive roots are given in [17] p. 863, [19] p. 258, [23] p. 18.

Equation (14) detects the multiplicative order \( \text{ord}_p(u) \) of the element \( u \in \mathbb{F}_p \) by means of the divisors of the totient \( p - 1 \). In contrast, the divisor-free representation of the characteristic function in (15) detects the multiplicative order \( \text{ord}_p(u) \geq 1 \) of the element \( u \in \mathbb{F}_p \) by means of the solutions of the equation \( \tau^n - u = 0 \) in \( \mathbb{F}_p \), where \( u, \tau \) are constants, and \( 1 \leq n < p - 1, \gcd(n, p - 1) = 1 \), is a variable.
Lemma 2.1. Let $p \geq 2$ be a prime, and let $\tau$ be a primitive root mod $p$. Let $\psi(z) = e^{i2\pi z/p} \neq 1$ be a nonprincipal additive character of order $\text{ord}\psi = p$. If $u \in \mathbb{F}_p$ is a nonzero element, then,

$$
\Psi(u) = \sum_{\gcd(n,p-1)=1} \frac{1}{p} \sum_{0 \leq k \leq p-1} \psi((\tau^n - u)k)
$$

(15)\hfill(15)

$$
= \begin{cases} 
 1 & \text{if } \text{ord}_p(u) = p-1, \\
 0 & \text{if } \text{ord}_p(u) \neq p-1.
\end{cases}
$$

Proof. As the index $n \geq 1$ ranges over the integers relatively prime to $p-1$, the element $\tau^n \in \mathbb{F}_p$ ranges over the primitive roots mod $p$. Ergo, the equation

$$
\tau^n - u = 0 \quad \text{(16)}
$$

has a solution if and only if the fixed element $u \in \mathbb{F}_p$ is a primitive root. Next, replace $\psi(z) = e^{i2\pi z/p}$ to obtain

$$
\Psi(u) = \sum_{\gcd(n,p-1)=1} \frac{1}{p} \sum_{0 \leq k \leq p-1} e^{i2\pi(\tau^n - u)k/p}
$$

(17)\hfill(17)

$$
= \begin{cases} 
 1 & \text{if } \text{ord}_p(u) = (p-1)/d, \\
 0 & \text{if } \text{ord}_p(u) \neq (p-1)/d.
\end{cases}
$$

This follows from the geometric series identity $\sum_{0 \leq k \leq N-1} w^k = (w^N - 1)/(w - 1)$ with $w \neq 1$, applied to the inner sum. \hfill\blacksquare

Let $d | p-1$. A new representation of the indicator function for $d$-power $v \in \mathbb{F}_p$ or elements of order $\text{ord}_p(v) = (p-1)/d$ is consider below.

Lemma 2.2. Let $p \geq 2$ be a prime, and let $\tau$ be a primitive root mod $p$. Let $\psi(z) = e^{i2\pi z/p} \neq 1$ be a nonprincipal additive character of order $\text{ord}\psi = p$. If $u \in \mathbb{F}_p$ is a $d$-power, then,

$$
\Psi(u,d) = \sum_{\gcd(n,(p-1)/d)=1} \frac{1}{p} \sum_{0 \leq k \leq (p-1)/d} \psi((\tau^{dn} - u)k)
$$

(18)\hfill(18)

$$
= \begin{cases} 
 1 & \text{if } \text{ord}_p(u) = (p-1)/d, \\
 0 & \text{if } \text{ord}_p(u) \neq (p-1)/d.
\end{cases}
$$

Proof. Similar to the proof of Lemma 2.1 mutatis mutandis. \hfill\blacksquare

3 Finite Summation Kernels

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a function, and let $q \in \mathbb{N}$ be a large integer. The finite Fourier transform

$$
\hat{f}(t) = \frac{1}{q} \sum_{0 \leq s \leq q-1} e^{i\pi st/q}
$$

(19)\hfill(19)

and its inverse are used here to derive a summation kernel function, which is almost identical to the Dirichlet kernel.

Definition 3.1. Let $p$ and $q$ be primes, and let $\omega = e^{i2\pi/q}$, and $\zeta = e^{i2\pi/p}$ be roots of unity. The finite summation kernel is defined by the finite Fourier transform identity

$$
K(f(n)) = \frac{1}{q} \sum_{0 \leq t \leq q-1} \sum_{0 \leq s \leq p-1} \omega^{t(n-s)} f(s) = f(n).
$$

(20)\hfill(20)
This simple identity is very effective in computing upper bounds of some exponential sums
\[ \sum_{n \leq x} f(n) = \sum_{n \leq x} K(f(n)), \]
(21)
where \( x \leq p < q \). This technique generalizes the sum of resolvents method used in [22]. Here, it is reformulated as a finite Fourier transform method, which is applicable to a wide range of functions.

**Lemma 3.1.** Let \( p \geq 2 \) and \( q = p + o(p) \) be large primes. Let \( \omega = e^{i\pi/q} \) be a \( q \)th root of unity, and let \( t \in [1, p - 1] \). Then,
\[
\begin{align*}
(i) & \quad \sum_{n \leq p-1} \omega^{tn} = \frac{\omega^t - \omega^{tp}}{1 - \omega^t}, \\
(ii) & \quad \left| \sum_{n \leq p-1} \omega^{tn} \right| \leq \frac{2q}{\pi t}.
\end{align*}
\]

**Proof.** (i) Use the geometric series to compute this simple exponential sum as
\[ \sum_{n \leq p-1} \omega^{tn} = \frac{\omega^t - \omega^{tp}}{1 - \omega^t}. \]
(ii) Observe that the parameters \( q = p + o(p) \) is prime, \( \omega = e^{i\pi/q} \), the integers \( t \in [1, p - 1] \), and \( d \leq p - 1 < q - 1 \). This data implies that \( \pi t/q \neq k\pi \) with \( k \in \mathbb{Z} \), so the sine function \( \sin(\pi t/q) \neq 0 \) is well defined. Using standard manipulations, and \( z/2 \leq \sin(z) < z \) for \( 0 < |z| < \pi/2 \), the last expression becomes
\[ \frac{2q}{\pi t} \leq \frac{2}{\sin(\pi t/q)} \leq \frac{2q}{\pi t}. \]  
(22)

**Lemma 3.2.** Let \( p \geq 2 \) and \( q = p + o(p) \) be large primes, and let \( \omega = e^{i\pi/q} \) be a \( q \)th root of unity. Then,
\[
\begin{align*}
(i) & \quad \sum_{\gcd(n, (p-1)/d) = 1} \omega^{tn} = \sum_{\mu(d) \omega^{rt} - \omega^{drt((p-1)/r(d+1))}} \mu(d) \omega^{drt((p-1)/r(d+1))} \frac{1}{1 - \omega^{drt}}, \\
(ii) & \quad \left| \sum_{\gcd(n, (p-1)/d) = 1} \omega^{tn} \right| \leq \frac{4q \log \log p}{\pi t},
\end{align*}
\]
where \( \mu(k) \) is the Mobius function, for any fixed pair \( d \mid p - 1 \) and \( t \in [1, p - 1] \).

**Proof.** (i) Use the inclusion exclusion principle to rewrite the exponential sum as
\[
\begin{align*}
\sum_{\gcd(n, (p-1)/d) = 1} \omega^{tn} = \sum_{n \leq p-1} \omega^{tn} \sum_{\mu(d) \omega^{drt((p-1)/r(d+1))}} \mu(d) \\
= \sum_{d \mid p-1} \mu(d) \sum_{n \leq p-1} \omega^{tn} \\
= \sum_{d \mid p-1} \mu(d) \sum_{n \leq p-1} \omega^{dtn} \\
= \sum_{d \mid p-1} \mu(d) \omega^{dtm} \frac{1 - \omega^{dtm}}{1 - \omega^{dt}}.
\end{align*}
\]
(23)
(ii) Observe that the parameters \( q = p + o(p) \) is prime, \( \omega = e^{i2\pi/q} \), the integers \( t \in [1, p-1] \), and \( d \leq p-1 < q-1 \). This data implies that \( \pi dt/q \neq k\pi \) with \( k \in \mathbb{Z} \), so the sine function \( \sin(\pi dt/q) \neq 0 \) is well defined. Using standard manipulations, and \( z/2 \leq \sin(z) < z \) for \( 0 < |z| \leq \pi/2 \), the last expression becomes

\[
\left| \frac{\omega^{dt} - \omega^{d(p-1)+1}}{1 - \omega^{dt}} \right| \leq \frac{2}{\sin(\pi dt/q)} \leq \frac{2q}{\pi dt}
\]

for \( 1 \leq d \leq p-1 \). Finally, the upper bound is

\[
\left| \sum_{d \mid p-1} \mu(d) \frac{\omega^{dt} - \omega^{d(p-1)/d} + 1}{1 - \omega^{dt}} \right| \leq \frac{2q}{\pi t} \sum_{d \mid p-1} \frac{1}{d} \leq 4q \log \log p \pi t.
\]

The last inequality uses the elementary estimate \( \sum_{d \mid n} d^{-1} \leq 2 \log \log n \). \( \blacksquare \)

## 4 Gaussian Sums, And Weil Sums

**Theorem 4.1.** (Gauss sums) Let \( p \geq 2 \) and \( q \geq 2 \) be large primes. Let \( \tau \) be a primitive root modulo \( p \). If \( \chi(t) = e^{i2\pi t/q} \) and \( \psi(t) = e^{i2\pi \tau^t/p} \) are a pair of characters, then, the Gaussian sum has the upper bound

\[
\left| \sum_{1 \leq t \leq q-1} \chi(t) \psi(t) \right| \leq 2q^{1/2} \log q.
\]

**Theorem 4.2.** (Weil sums) Let \( p \geq 2 \) and \( q \geq 2 \) be large primes, and let \( f(t) \) be a powerfree polynomial of degree \( \deg f = d \geq 1 \). If \( \chi(t) = e^{i2\pi t/q} \) and \( \psi(t) = e^{i2\pi f(t)/p} \) are a pair of characters. Then, the Weil sum has the upper bound

\[
\left| \sum_{1 \leq t \leq q-1} \chi(t) \psi(f(t)) \right| \leq 2dq^{1/2} \log q.
\]

**Theorem 4.3.** Let \( p \geq 2 \) and \( q \geq 2 \) be large primes. Let \( \tau \) be a primitive root modulo \( p \), and let \( \kappa = \tau^d \) be an element of large multiplicative order. If \( \chi(t) = e^{i2\pi t/q} \) and \( \psi(t) = e^{i2\pi t/p} \) are a pair of characters. Then, the exponential sum has the upper bound

\[
\left| \sum_{1 \leq t \leq q-1} \chi(t) \psi(\tau^dt) \right| \leq 2dq^{1/2} \log q.
\]

*Proof.* Use the change of variable \( z = \tau^t \) to rewrite the exponential sum as a Weil sum with a polynomial \( f(t) = z^d \) of degree \( d \). \( \blacksquare \)

## 5 Incomplete And Complete Exponential Sums

Two applications of the generalizes the sum of resolvents method used in \[22\], and \[28\], to estimate exponential sums are illustrated here. The first application is a nonlinear counterpart of the Polya-Vinogradov inequality

\[
\sum_{n \leq x} \chi(n) \leq 2p^{1/2} \log p
\]

for nonprincipal character \( \chi \neq 1 \) modulo \( p \).
Theorem 5.1. Let $p \geq 2$ be a large prime, and let $\kappa \in \mathbb{F}_p$ be an element of large multiplicative order $\text{ord}_p(\kappa) \mid p - 1$. Then, for any fixed integer $a \in [1, p - 1]$, and $x \leq p - 1$,

$$\sum_{n \leq x} e^{i2\pi \kappa n/p} \ll p^{1/2} \log^3 p. \quad (30)$$

Proof. Let $q = p + o(p)$ be a large prime, and write $f(n) = e^{i2\pi \tau^d n/p}$, where $\tau$ is a primitive root modulo $p$, and $\kappa = \tau^d$ has large multiplicative order modulo $p$ modulo $p$. Applying the finite summation kernel in Definition 3.1, yields

$$R(d, x) = \sum_{n \leq x} e^{i2\pi \kappa \tau^d n/p} \quad (31)$$

$$= \sum_{n \leq x} \frac{1}{q} \sum_{0 \leq t \leq q - 1, 1 \leq s \leq p - 1} \omega^{t(n-s)} e^{i2\pi \tau^d s/p}. \quad (32)$$

Use the Weil sum upper bound, see Theorem 4.3, to show that the value $t = 0$ contributes

$$\frac{1}{q} \sum_{n \leq x, 1 \leq s \leq p - 1} e^{i2\pi \tau^d s/p} = \frac{x}{q} \sum_{1 \leq s \leq p - 1} e^{i2\pi \tau^d s/p} \quad (33)$$

where $1/q \leq 2/p$. Replacing (32) into (31), and rearranging it, yield

$$R(d, x) = \sum_{n \leq x} e^{i2\pi \kappa \tau^d n/p}$$

$$= \frac{1}{q} \sum_{n \leq x, 1 \leq t \leq q - 1, 1 \leq s \leq p - 1} \omega^{t(n-s)} e^{i2\pi \kappa \tau^d s/p} + O \left( \frac{4x \log p}{p^{1/2}} \right)$$

$$= \frac{1}{q} \sum_{1 \leq t \leq q - 1} \left( \sum_{1 \leq s \leq p - 1} \omega^{-ts} e^{i2\pi \kappa \tau^d s/p} \right) \left( \sum_{n \leq x} \omega^t \right) + O \left( \frac{4x \log p}{p^{1/2}} \right).$$

Taking absolute value, and applying Lemma 3.1 to the inner sum, and Theorem 4.3 to the middle sum, yield

$$|R(d, x)| = \sum_{n \leq x} e^{i2\pi \kappa \tau^d n/p} \quad (34)$$

$$\ll \frac{1}{q} \sum_{1 \leq t \leq q - 1} \left( \sum_{1 \leq s \leq p - 1} \omega^{-ts} e^{i2\pi \kappa \tau^d s/p} \right) \cdot \left| \sum_{n \leq x} \omega^t \right| + O \left( \frac{4x \log p}{p^{1/2}} \right)$$

$$\ll \frac{1}{q} \sum_{1 \leq t \leq q - 1} \left( 2p^{1/2} \log^2 p \right) \cdot \left( \frac{2q}{\pi t} \right) + 4x \log p \frac{p^{1/2}}{p^{1/2}}$$

$$\ll p^{1/2} \log^3 p + 4x \log p \frac{p^{1/2}}{p^{1/2}}.$$
The last summation in (34) uses the estimate
\[ \sum_{1 \leq t \leq q-1} \frac{1}{t} \ll \log q \ll \log p \]
(35)
since \( q = p + o(p) \).

This result is nontrivial for \( x \geq p^{1/2+\delta} \), and elements of large multiplicative orders \( \text{ord}_p \kappa \geq p^{1/2+\delta} \), where \( \delta > 0 \). A similar upper bound for composite moduli \( p = m \) is also proved in [22, Equation (2.29)].

The second application is a complete exponential sum version of the previous one, but restricted to relatively prime arguments.

**Theorem 5.2.** Let \( p \geq 2 \) be a large prime, let \( \tau \) be a primitive root modulo \( p \), and let \( \kappa = \tau^d \) be an element of large multiplicative order modulo \( p \). Then,
\[ \sum_{\gcd(n,(p-1)/d)=1} e^{i2\pi a \tau^d n / p} \ll p^{1-\varepsilon} \]
(36)
for any fixed integer \( a \in [1, p-1] \), and any arbitrary small number \( \varepsilon \in (0, 1/2) \).

**Proof.** Let \( q = p + o(p) \) be a large prime, and write \( f(n) = e^{i2\pi a \tau^d n / p} \), where \( \tau \) is a primitive root modulo \( p \). Start with the representation
\[ \sum_{\gcd(n,(p-1)/d)=1} e^{i2\pi a \tau^d n / p} = \sum_{n \leq (p-1)/d} \frac{1}{q} \sum_{0 \leq t \leq q-1, 1 \leq s \leq p-1} \omega^{f(n-s)} e^{i2\pi a \tau^d s / p} \sum_{r | (p-1)/d} \mu(r), \]
(37)
see Definition [21]. Use the inclusion exclusion principle to rewrite the exponential sum as
\[ \sum_{\gcd(n,(p-1)/d)=1} e^{i2\pi a \tau^d n / p} = \sum_{n \leq (p-1)/d} \frac{1}{q} \sum_{0 \leq t \leq q-1, 1 \leq s \leq p-1} \omega^{f(n-s)} e^{i2\pi a \tau^d s / p} \sum_{r | (p-1)/d \ n} \mu(r). \]
(38)
The value \( t = 0 \) contributes
\[ T_0(d, p) = \sum_{n \leq (p-1)/d} \frac{1}{q} \sum_{1 \leq s \leq p-1} e^{i2\pi a \tau^d s / p} \sum_{r | (p-1)/d \ n} \mu(r) \]
\[ \leq \frac{1}{q} \sum_{r | (p-1)/d} \left| \sum_{1 \leq s \leq p-1} e^{i2\pi a \tau^d s / p} \right| \sum_{m \leq (p-1)/rd} 1 \]
(39)
\[ \leq \frac{1}{q} \frac{p-1}{d} \sum_{1 \leq s \leq p-1} e^{i2\pi a \tau^d s / p} \sum_{r | (p-1)/d} \frac{1}{r} \]
\[ \leq \frac{2p^{1/2} \log^2 p}{d}, \]
where the middle sum is a Weil sum, see Theorem [43] and \( (p-1)/q \leq 1 \). Replacing (39)
into (38), and rearranging it, yield
\[
\rho(d, p) = \sum_{\gcd(n, (p-1)/d)=1} e^{i2\pi a r^n / p} \sum_{\omega \equiv 1 \mod{p-1}} \omega^{t(n-s)} e^{i2\pi a r^s / p} \sum_{\mu(d) \sum_{n \equiv 1 \mod{p-1}/d}} \mu(d) + O\left(\frac{2p^{1/2\log^2 p}}{d}\right)
\]
\[
= \sum_{1 \leq t \leq q-1} \left( \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi a r^s / p} \right) \left( \sum_{r \equiv 1 \mod{p-1}/d} \mu(d) \sum_{n \equiv 1 \mod{p-1}/d}} \omega^{tn} \right) + O\left(\frac{2p^{1/2\log^2 p}}{d}\right).
\]
Taking absolute value, and applying Lemma 3.2 to the inner sum, and Theorem 4.3 to the middle sum, yield
\[
|\rho(d, p)| = \left| \sum_{\gcd(n, (p-1)/d)=1} e^{i2\pi a r^n / p} \right| \leq \frac{1}{q} \sum_{1 \leq t \leq q-1} \left| \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi a r^s / p} \right| \cdot \left| \sum_{r \equiv 1 \mod{p-1}/d} \mu(d) \sum_{n \equiv 1 \mod{p-1}/d} \omega^{tn} \right| + O\left(\frac{2p^{1/2\log^2 p}}{d}\right)
\]
\[
\ll \frac{1}{q} \sum_{1 \leq t \leq q-1} \left( 2p^{1/2\log^2 p} \right) \cdot \left( \frac{4q \log \log p}{\pi t} \right) + O\left(\frac{2p^{1/2\log^2 p}}{d}\right)
\ll \frac{p^{1/2\log^3 p}}{p^{1/2\log^3 p}}.
\]
The last summation in (41) uses the estimate
\[
\sum_{1 \leq t \leq q-1} \frac{1}{t} \ll \log q \ll \log p
\]
since \( q = p + o(p) \). This is restated in the simpler notation \( p^{1/2\log^3 p} \leq p^{1-\varepsilon} \) for any arbitrary small number \( \varepsilon \in (0, 1/2) \), as \( x \to \infty \).

The upper bound given in Theorem 5.2 for maximal \( \varepsilon < 1/2 \) seems to be optimum. A different proof, which has a weaker upper bound is included here as a reference for a second independent proof.

**Theorem 5.3.** ([10, Theorem 6]) Let \( p \geq 2 \) be a large prime, and let \( \tau \) be a primitive root modulo \( p \). Then,
\[
\sum_{\gcd(n, p-1)=1} e^{i2\pi a r^n / p} \ll p^{1-\varepsilon}
\]
for any integer \( a \in [1, p-1] \), and any arbitrary small number \( \varepsilon > 0 \) is a small number.
Other related results are given in [2], [9], [11], and [12, Theorem 1].

6 Explicit Exponential Sums

An explicit version of Theorem 5.2 and Theorem 5.3 is computed below.

**Theorem 6.1.** Let \( m \geq 1 \) be an integer, and let \( Q \geq 1 \) be the period of the element \( w \in \mathbb{Z}/m\mathbb{Z} \). If the number \( P < Q \), then

\[
\sum_{1 \leq n \leq P} e^{i2\pi an^m} \leq c_0 P^{1-\varepsilon}.
\]

(44)

where \( a \neq 0 \), \( \varepsilon = c_1 \log P/\log m < 1 \) is a small number, and \( c_0, c_1 > 0 \) are constants.

**Proof.** A discussion of this exponential sum and a proof appears in [16, p. 8]. ■

The complete exponential sum

\[
\sum_{1 \leq n \leq Q} e^{i2\pi an^m/m}
\]

(45)

is known to be a very small number or to vanish. An upper bound for a related and different exponential sum will be used in the analysis of the orders of elements in finite rings.

**Theorem 6.2.** Let \( m \geq 1 \) be an integer, and let \( Q \geq 1 \) be the period of the element \( w \in \mathbb{Z}/m\mathbb{Z} \). If the number \( P < Q \), then

\[
\sum_{1 \leq n \leq P} e^{i2\pi an^m/m} \leq c_0 m^\varepsilon P^{1-2\varepsilon}.
\]

(46)

where \( a \neq 0 \), \( 2\varepsilon = c_1 \log P/\log m < 1 \) is a small number, and \( c_0, c_1 > 0 \) are constants.

**Proof.** Let \( P = Q - 1 \), and rewrite the exponential sum in the form

\[
\sum_{1 \leq n \leq P} e^{i2\pi an^m/m} = \sum_{1 \leq n \leq P} \sum_{d \mid n} \mu(d) e^{i2\pi a d^n/m} = \sum_{d \mid \varphi(m)} \mu(d) \sum_{1 \leq n \leq P} e^{i2\pi a d^n/m}.
\]

(47)

Taking absolute value, and applying Theorem 6.1 to the inner exponential sum return

\[
\left| \sum_{1 \leq n \leq P} e^{i2\pi an^m/m} \right| \leq \sum_{d \mid \varphi(m)} \left| \sum_{1 \leq n \leq P} e^{i2\pi a d^n/m} \right| \\
\leq \sum_{d \mid \varphi(m)} 1 \cdot c_1 \left( \frac{P}{d} \right)^{1-2\varepsilon} \\
\leq c_2 P^{1-2\varepsilon} \sum_{d \mid \varphi(m)} \frac{1}{d^{1-2\varepsilon}} \\
\leq c_3 P^{1-2\varepsilon} \sum_{d \mid \varphi(m)} 1 \\
\leq c_4 P^{1-2\varepsilon} \varphi(m)^\varepsilon,
\]

(48)
where \( c_0 = c_4, c_1, c_2, c_3 > 0 \) are constants. Plugging the trivial upper bounds \( P \leq m \), and \( \varphi(m) \leq m \) complete the verification of the inequality

\[
\sum_{1 \leq n \leq P \atop \gcd(n, \varphi(m))} e^{i2\pi au_n/m} \leq c_4 P^{1-2\varepsilon} \varphi(m)^\varepsilon \leq c_4 m^{1-\varepsilon}, \tag{49}
\]

for sufficiently large \( m \).

7 Equivalent Exponential Sums

This section demonstrate that the exponential sums

\[
\sum_{\gcd(n, (p-1)/d) = 1} e^{i2\pi axn/p} \quad \text{and} \quad \sum_{\gcd(n, (p-1)/d) = 1} e^{i2\pi xn/p}, \tag{50}
\]

where \( d \mid p - 1 \), and \( a \neq 0 \), are asymptotically equivalent. This result expresses this exponential sum as a sum of simpler exponential sum and an error term. The proof is entirely based on established results and elementary techniques.

**Theorem 7.1.** Let \( p \geq 2 \) be a large primes, and let \( d \mid p - 1 \) be a small divisor. If \( \tau \) be a primitive root modulo \( p \), then,

\[
\sum_{\gcd(n, (p-1)/d) = 1} e^{i2\pi axn/p} = \sum_{\gcd(n, (p-1)/d) = 1} e^{i2\pi xn/p} + O(p^{1/2} \log^4 p), \tag{51}
\]

for any integer \( a \in [1, p - 1] \).

**Proof.** For any integer \( a \geq 1 \), the exponential sum has the representation

\[
\rho(a, d, p) = \sum_{\gcd(n, (p-1)/d) = 1} e^{i2\pi axn/p} = \frac{1}{q} \sum_{1 \leq t \leq q-1} \left( \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi axn/p} \right) \left( \sum_{r \mid (p-1)/d} \mu(r) \sum_{n \mid (p-1)/d, r \mid n} \omega^{tn} \right) + O \left( \frac{2p^{1/2} \log^2 p}{d} \right),
\]

confer equations (37) to (40) for details. And, for \( a = 1 \),

\[
\rho(1, d, p) = \sum_{\gcd(n, (p-1)/d) = 1} e^{i2\pi xn/p} = \frac{1}{q} \sum_{1 \leq t \leq q-1} \left( \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi xn/p} \right) \left( \sum_{r \mid (p-1)/d} \mu(r) \sum_{n \mid (p-1)/d, r \mid n} \omega^{tn} \right) + O \left( \frac{2p^{1/2} \log^2 p}{d} \right),
\]
respectively, see equations (37) to (40). Differencing (52) and (53) produces

\[ D(d, p) = \sum_{\gcd(n, (p-1)/d)=1} e^{2\pi \alpha a r n / p} - \sum_{\gcd(n, (p-1)/d)=1} e^{2\pi \alpha a r n / p} \] (54)

\[ = \frac{1}{q} \sum_{0 \leq t \leq q-1} \left( \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{2\pi \alpha a r n / p} - \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{2\pi \alpha a r n / p} \right) \times \left( \sum_{r \mid (p-1)/d} \sum_{n \leq (p-1)/d, \gcd(n, (p-1)/d)=1} \omega^{rn} \right). \]

By Lemma 3.2, the relatively prime summation kernel is bounded by

\[ \left| \sum_{r \mid (p-1)/d} \sum_{n \leq (p-1)/d, \gcd(n, (p-1)/d)=1} \omega^{rn} \right| \leq 4q \log \log p, \] (55)

and by Theorem 4.3, the difference of two Weil sums (or Gauss sums) is bounded by

\[ \delta(d, p) = \left| \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{2\pi \alpha a r n / p} - \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{2\pi \alpha a r n / p} \right| \]

\[ \leq 4p^{1/2} \log^2 p, \] (56)

where \( \chi(s) = e^{i\pi st / p} \) and \( \psi_a(s) = e^{2\pi \alpha a r n / p} \). Taking absolute value in (54) and replacing (55), and (56), return

\[ |D(d, p)| = \left| \sum_{\gcd(n, (p-1)/d)=1} e^{2\pi \alpha a r n / p} - \sum_{\gcd(n, (p-1)/d)=1} e^{2\pi \alpha a r n / p} \right| \] (57)

\[ \leq \frac{1}{q} \sum_{0 \leq t \leq q-1} \left( 4p^{1/2} \log^2 p \right) \cdot \left( \frac{4q \log \log p}{t} \right) \]

\[ \leq 16p^{1/2} (\log^2 p) (\log q) (\log \log p) \]

\[ \leq 16p^{1/2} \log^4 p, \]

where \( q = p + o(p) \).

8 Double Exponential Sums

**Lemma 8.1.** Given a small number \( \varepsilon > 0 \). Let \( p \) be a large prime number, and let \( d \mid p-1 \) be a small divisor. If \( \tau \) is a primitive root modulo \( p \), then,

\[ \sum_{0 < a < p \atop \gcd(n, \varphi(p)/d)=1} e^{2\pi \alpha a r n / p} \ll p^{1-\varepsilon}. \] (58)
Taking absolute value, and applying Theorem 5.2 (or Theorem 6.1), yield

\begin{equation}
\sum_{0 < a < p, \gcd(n, \varphi(p)/d) = 1} e^{2 \pi i x a u/p} = \sum_{0 < a < p} e^{-i2\pi au/p} \sum_{\gcd(n, \varphi(p)/d) = 1} e^{i2\pi x a p/n/p}. \tag{59}
\end{equation}

Applying Theorem 7.1 to remove the \( a \) dependence of the inner finite sum, yields

\begin{equation}
T(p) = \sum_{0 < a < p} e^{-i2\pi au/p} \sum_{\gcd(n, \varphi(p)/d) = 1} e^{i2\pi x a p/n/p} \tag{60}
\end{equation}

Taking absolute value, and applying Theorem 5.2 (or Theorem 6.1), yield

\begin{align}
|T(p)| & \leq \left| \sum_{0 < a < p} e^{-i2\pi au/p} \right| \left| \sum_{\gcd(n, \varphi(p)/d) = 1} e^{i2\pi x a p/n/p} \right| + O \left( p^{1/2} \log^4 p \right) \\
& \leq \sum_{0 < a < p} e^{-i2\pi au/p} \left( \sum_{\gcd(n, \varphi(p)/d) = 1} e^{i2\pi x a p/n/p} \right) + O \left( p^{1/2} \log^4 p \right) \\
& \ll |-1| \cdot p^{1-\varepsilon} + p^{1/2} \log^4 p \\
& \ll p^{1-\varepsilon}, \tag{61}
\end{align}

where \( \sum_{0 < a < p} e^{i2\pi au/p} = -1 \) for any \( u \neq 0 \), and \( \varepsilon < 1/2 \) is a small number.

\section{The Main Term}

The main term of two or more simultaneous elements requires the average order of a product of totient functions. A lower bound for the product two totients will be computed here.

\begin{lemma}
If \( x \geq 1 \) is a large number, and \( d, e \ll (\log x)^B \), with \( B \geq 0 \), then

\[ \sum_{x \leq p \leq 2x} \frac{\varphi((p-1)/d)}{p} \cdot \frac{\varphi((p-1)/e)}{p} \gg \frac{x}{(\log x)^{4B+1}(\log \log x)^2}. \tag{62} \]

\end{lemma}

\begin{proof}
The totient function has the lower bound \( \varphi(n)/n \gg 1/\log \log n \), see [27, Theorem 15]. Replacing this estimate yields

\begin{align}
M(x, u, v) &= \sum_{x \leq p \leq 2x} \frac{\varphi((p-1)/d)}{p} \cdot \frac{\varphi((p-1)/e)}{p} \tag{63} \\
&= \sum_{x \leq p \leq 2x} \frac{\varphi((p-1)/d)}{p-1} \cdot \frac{\varphi((p-1)/e)}{p-1} \left( 1 - \frac{1}{p} \right)^2 \\
&\gg \sum_{x \leq p \leq 2x} \frac{1}{d \log \log p} \cdot \frac{1}{e \log \log p} \\
&\gg \frac{1}{(\log x)^{2B}} \frac{1}{(\log \log x)^2} \sum_{x \leq p \leq 2x} \frac{1}{p \equiv 1 \mod de}.
\end{align}

\end{proof}
since \(d, e \ll (\log x)^B\), with \(B \geq 0\). Applying the prime number theorem on arithmetic progression over the short interval \([x, 2x]\), yields

\[
M(x, u, v) = \frac{1}{(\log x)^{2B}} \frac{1}{(\log \log x)^2} \sum_{\substack{x \leq p \leq 2x \text{ \(p \equiv 1 \mod de\)}}} 1 \\
\geq \frac{x}{(\log x)^{2B}(\log \log x)^2} \left( \frac{x}{\varphi(de) \log x} + O \left( xe^{-c\sqrt{\log x}} \right) \right) \\
\geq \frac{1}{(\log x)^{4B+1}(\log \log x)^2},
\]

where \(\varphi(de) \leq de \leq (\log x)^{2B}\), and \(c > 0\) is an absolute constant.

This analysis is effective and unconditional for the prescribed indices \(d, e \ll \log^B x\), where \(B \geq 0\) is a constant, as limited by the current version of the prime number theorem on arithmetic progressions, confer [6, Theorem 3.10].

The exact asymptotic for the average order of a product \(k\) totient functions over the primes is proved in [30], and related discussions are given in [25, p. 16]. The generalization to number fields appears in [13]. However, the exact asymptotic for the average order of a product of \(k\) totient functions over the primes in arithmetic progressions seems to be unknown. For example, for equal prescribed multiplicative orders \(((p-1)/d)_i\), it should have the form

\[
\sum_{x \leq p \leq 2x} \prod_{p \equiv a \mod q} \left( \frac{\varphi((p-1)/d_i)}{p-1} \right)^{g_i} \equiv A_k \frac{x}{\varphi(q) \log x} + O \left( xe^{-c\sqrt{\log x}} \right),
\]

where \(q = d_1d_2\cdots d_k \leq (\log x)^{Bk}\), and \(A_k = A_k(a, q) > 0\), which is slightly more complex.

### 10 The Error Terms

**Lemma 10.1.** Assume \(\text{ord}_p u \neq (p-1)/d\). If \(x\) is a large number, then

\[
E_1(x) = \sum_{x \leq p \leq 2x} \frac{1}{p} \sum_{0 \leq a < n \text{ \(\gcd(n,\varphi(p)/d) = 1\)}} \frac{1}{p} \sum_{0 \leq a \leq n} \sum_{\gcd(n,\varphi(p)/d) = 1} e^{i2\pi a^2 u p^{-1}} = 0.
\]

**Proof.** By hypothesis, \(\text{ord}_p u \neq (p-1)/d\), so the first finite sum

\[
\sum_{0 \leq a < n \text{ \(\gcd(n,\varphi(p)/d) = 1\)}} e^{i2\pi a^2 u p^{-1}} = 0
\]

vanishes, see Lemma 2.2.
Lemma 10.2. Assume \( \text{ord}_p v \neq (p - 1)/e \). If \( x \) is a large number, then

\[
E_2(x) = \sum_{x \leq p \leq 2x} \frac{1}{p} \sum_{0 < b < n, \gcd(m, \varphi(p)/e) = 1} e^{i2\pi b(x^{m-n})/p} \cdot \sum_{0 < b < \varphi(p)/e, \gcd(m, \varphi(p)/e) = 1} e^{i2\pi b(x^{m-n})/p} = 0.
\]

Proof. By hypothesis, \( \text{ord}_p v \neq (p - 1)/e \), so the second finite sum

\[
\sum_{0 < b < n, \gcd(m, \varphi(p)/e) = 1} e^{i2\pi b(x^{m-n})/p} = 0
\]

vanishes, see Lemma 2.2.

Lemma 10.3. If \( x \) is a large number, then

\[
E_3(x) = \sum_{x \leq p \leq 2x} \frac{1}{p} \sum_{0 < a < p, \gcd(n, \varphi(p)/d) = 1} e^{i2\pi a(x^{d-n})/p} \cdot \sum_{0 < b < \varphi(p)/e, \gcd(m, \varphi(p)/e) = 1} e^{i2\pi b(x^{m-n})/p} \ll x^{1-2\varepsilon}.
\]

Proof. To compute an upper bound, define the exponential sum

\[
T(d, p) = \sum_{0 < a < p, \gcd(n, \varphi(p)/d) = 1} e^{i2\pi a(x^{d-n})/p}.
\]

Now, apply Lemma 8.1 to each factor \( T(d, p) \) and \( T(e, p) \) to obtain the followings.

\[
E_3(x) = \sum_{x \leq p \leq 2x} \left( \frac{1}{p} \cdot T(d, p) \right) \cdot \left( \frac{1}{p} \cdot T(e, p) \right) \ll \sum_{x \leq p \leq 2x} \left( \frac{1}{p} \cdot p^{1-\varepsilon} \right) \cdot \left( \frac{1}{p} \cdot p^{1-\varepsilon} \right).
\]

Take an upper bound, and apply the prime number theorem:

\[
E_3(x) \ll \frac{1}{x^{2\varepsilon}} \sum_{x \leq p \leq 2x} 1 \ll x^{1-2\varepsilon},
\]

as \( x \to \infty \).

11 Simultaneous Prescribed Multiplicative Orders

The multiplicative order of an element \( u \in \mathbb{F}_p \) in finite field is the smallest integer \( n \geq 1 \) for which \( u^n = 1 \) in \( \mathbb{F}_p \), see Section 2 for additional details. The simpler case of simultaneous prescribed multiplicative orders of two admissible rational numbers is investigated in this section. The analysis of simultaneous and prescribed multiplicative orders for \( k \)-tuple of admissible rational numbers are similar to this analysis, but have bulky and cumbersome notation.
**Definition 11.1.** Fix a pair of rational numbers \( u, v \neq \pm 1 \) such that \( u^a v^b \neq \pm 1 \) for any \( a, b \in \mathbb{Z} \). The elements \( u, v \in \mathbb{F}_p \) are said to be simultaneous of equal orders \( \operatorname{ord}_p u \mid p - 1 \) and \( \operatorname{ord}_p v \mid p - 1 \) respectively, if \( \operatorname{ord}_p u = \operatorname{ord}_p v \) infinitely often as \( p \to \infty \). Otherwise, the elements are said to be simultaneous of unequal orders, if \( \operatorname{ord}_p u \neq \operatorname{ord}_p v \), infinitely often as \( p \to \infty \).

Given a pair of small integer indices \( d \geq 1 \) and \( e \geq 1 \), the number of primes \( x \leq p \leq 2x \), which have simultaneous elements \( u > 1 \) and \( v > 1 \) of prescribed orders \( \operatorname{ord}_p u \mid (p - 1)/d \) and \( \operatorname{ord}_p v \mid (p - 1)/e \) modulo \( p \geq 2 \), respectively, is defined by

\[
R(x, u, v) = \sum_{x \leq p \leq 2x} \Psi_p(u, d) \cdot \Psi_p(v, e) \tag{72}
\]

The small integers indices \( d \geq 1 \) and \( e \geq 1 \) prescribed the multiplicative orders of the fixed pair \( u > 1 \) and \( v > 1 \)

**Proof.** (Theorem 1.1) Substitute the indicator function \( \Psi_p(u, d) \) for elements \( u \) of order \( \operatorname{ord}_p u = (p - 1)/d \) modulo \( p \), and the indicator function \( \Psi_p(u, e) \) for elements \( v \) of order \( \operatorname{ord}_p v = (p - 1)/e \) modulo \( p \), see Lemma 2.2 to construct the associated counting function for the number of such primes \( p \in [x, 2x] \):

\[
R(x, u, v) = \sum_{x \leq p \leq 2x} \Psi_p(u, d) \cdot \Psi_p(v, e) = \sum_{x \leq p \leq 2x} \frac{1}{p} \sum_{0 \leq a < p, \gcd(a, \varphi(p)/d) = 1} e^{2\pi i a (x^{dn} - u)/p} \cdot \frac{1}{p} \sum_{0 \leq b < p, \gcd(b, \varphi(p)/e) = 1} e^{2\pi i b (x^{en} - v)/p} = M(x, u, v) + E_1(x) + E_2(x) + E_3(x). \tag{73}
\]

The main term \( M(x, u, v) \) is determined by \( (a, b) = (0, 0) \), the error terms \( E_1(x) \), \( E_2(x) \), and \( E_3(x) \) are determined by \( (a, b) = (0, b \neq 0) \), \( (a, b) = (a \neq 0, 0) \), and \( (a, b) \neq (0, 0) \), respectively.

Summing the main term \( M(x, u, v) \), estimated in Lemma 9.1 and the error terms \( E_1(x) \), \( E_2(x) \), and \( E_3(x) \) estimated in Lemma 10.1, Lemma 10.2 and Lemma 10.3 respectively, yield

\[
R(x, u, v) = M(x, u, v) + E_1(x) + E_2(x) + E_3(x) \tag{74}
\]

\[
\gg \frac{x}{(\log x)^{4B+1}(\log \log x)^2} + 0 + 0 + x^{1-2\epsilon}
\]

where \( \epsilon > 0 \) is a small number, as \( x \to \infty \).

The current analysis is unconditional for any indices product \( de \ll (\log x)^{2B} \), where \( B \geq 0 \) is a constant. Assuming the RH, it appears that this analysis can handled any indices product \( de \) as large as \( de \leq p^{1/2-\delta} \), where \( \delta > 0 \), but no effort was made to verify this observation, confer the proof for the lower bound of the main term in Section 9 for some information.
12 Probabilistic Results For Simultaneous Orders

**Theorem 12.1.** For any pair of random relatively prime integers \(a > 1\) and \(b > 1\), and any sufficiently large prime \(p\), the ratio

\[
\frac{\text{ord}_p(a)}{\text{ord}_p(b)} \neq 1
\]

is true with probability \(1 + o(1) > 1/2\).

**Proof.** Fix a large prime \(p\). Two random relatively prime integers \(a\) and \(b\) have the same multiplicative order if and only if \(\text{ord}_p(a) = \text{ord}_p(b) = (p-1)/d_0\) for at least one divisor \(d_0 | p-1\). Let \(\alpha_2 > 0\) denotes the probability that

\[
\frac{\text{ord}_p(a)}{\text{ord}_p(b)} = 1
\]

is true. Otherwise, \(\alpha_2 = 0\), and the claim in (75) is trivially true. Assuming statistical pseudo independence, this event occurs with probability

\[
c_0 \frac{1}{p-1} \cdot \frac{1}{p-1} < \alpha_2 = c_0 \frac{\varphi((p-1)/d_0)}{p-1} \cdot \frac{\varphi((p-1)/d_0)}{p-1} < \sum_{dp-1} c_d \left( \frac{\varphi((p-1)/d)}{p-1} \right)^2 \leq \sum_{dp-1} \frac{\varphi((p-1)/d)}{p-1} = \frac{\varphi(p-1)}{p-1} \leq \frac{1}{2},
\]

where \(c_i = c_i(a,b) \leq 1\) is a statistical independence correction factor. The lower bound \(1/(p-1)^2\) in (77) follows from the hypothesis \(a > 1\) and \(b > 1\). Hence, random relatively prime integers \(a\) and \(b\) of multiplicative order \(\text{ord}_p(a) \neq \text{ord}_p(b)\), satisfy the ratio

\[
\frac{\text{ord}_p(a)}{\text{ord}_p(b)} \neq 1
\]

with probability

\[
1 - \alpha_2 > \frac{1}{2}.
\]

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