1. Introduction: Interpretation versus Reconstruction

Almost from the inception of quantum mechanics, it has been clear that it does not merely represent a theory of new phenomena, but rather, an entirely novel way of theory-building. There is now wide agreement that certain assumptions and conceptions, implicit in the Newtonian, classical framework, can no longer be upheld—albeit, and perhaps shockingly, there is as yet no consensus on what, precisely, those are.

In coming to terms with the novelty of quantum mechanics, the dominant strategy has been that of interpretation: roughly, the attempt of matching the formalism to an underlying reality (whatever that, exactly, may mean). However, the plethora of interpretations on the market—the Wikipedia article [1] currently lists 14 ‘mainstream’ interpretations—indicates that this project is still far from completion.

Sometimes, the inverse of a hard problem is more easily solved. Instead of trying to infer the underlying ontology to match the quantum formalism, one might thus take a constructive road and explore which phenomena arise naturally in certain ‘model’ or ‘toy’ settings, with the aim of eventually zeroing in on QM. This is the project of reconstructing quantum mechanics: finding one or more foundational principles such that the quantum predictions naturally follow.

In contrast to the project of interpretation, this search has, it seems, produced a significant convergence of ideas. As pointed out by Grinbaum [2], two principles are common to several recent attempts (see references in [3]):

1. Finiteness: There is a finite maximum of information that can be obtained about any given system.
2. Extensibility: It is always possible to acquire new information about any system.

At first glance, these seem contradictory: how can we obtain additional information, if we already possess the maximum possible information about a system? The answer, as we will see, is closely related to one of the central puzzles of quantum mechanics: there must be a mechanism such that ‘old’ information becomes obsolete—which, in QM, is just the hotly-debated ‘collapse’ of the wave function.

Compare this to the situation of an observer on the spherical Earth: moving closer to their horizon, they lose sight of what they’ve left behind. Although we typically expect that which has slipped beyond the horizon to remain largely unchanged, and thus, our information about it to remain accurate—but of course, this may not be the case.
2. Horizons of our Understanding

I do not propose to present a detailed reconstruction of the formalism of quantum mechanics here. However, I want to at least present an intuition as to how such a reconstruction, starting from the principles 1 and 2, might proceed.

To this end, consider as a toy model a (classical) point particle of mass \( m \) moving in one dimension. Its state can be completely described by giving its position \( x \) and velocity \( v \)—or, as is more common, its momentum \( p = mv \). The space spanned by the particle’s possible positions and momenta is called its phase space. Each point in phase space gives a tuple \((x_0, p_0)\) uniquely determining the particle’s state (see Fig. 1 (a)).

From this starting point, we impose principles 1 and 2. Upon requiring that there be a maximum amount of information that can be obtained about a system, we can no longer localize its state within phase space with perfect precision—the space effectively becomes discretized (see Fig. 1 (b)). Imposing then that we can always obtain additional information entails that we can increase our information about, say, its position—but to compensate, must lose information about its momentum (see Fig. 1 (c)).

Thus, we do not simply obtain a discretized phase space, but rather, there is a minimum area of localization, whose shape is determined by the information obtained about each coordinate. Since position has units of length [m], while momentum has units of mass \( \cdot \) velocity [kg \( \cdot \) m/s], this area of maximum localizability has units of [kg \( \cdot \) m\(^2\) s\(^{-1}\)]—which is the dimension of Planck’s famous constant, \( \hbar \). Hence, maximum localizability in phase space is bounded by \( \hbar \), which entails for the uncertainties \( \Delta x \) and \( \Delta p \)

\[
\Delta x \Delta p \gtrsim \hbar,
\]

which is of course nothing but Heisenberg’s famous uncertainty relation. In this way, assumptions 1 and 2 carry us the first step of the way towards quantization.

This is, of course, an entirely heuristic picture. However, it will help, in the following, to have an intuition about the sort of project being outlined here.

2.1. Superposition

Having now had a glimpse of how quantum phenomena emerge due to the restriction of information available about a system, it is time to consider some characteristic aspects of quantum mechanics in detail. The first step along this road will be to discuss how the impossibility of associating a definite value to every possible property of a system emerges from an argument trading on inconsistent self-reference, in much the same way as Gödelian incompleteness [6] and Turing undecidability [7].

Suppose, for simplicity, that a given system \( \mathcal{S} \) can be in countably\(^*\) many different states \( \{ s_i \}_{i \in \mathbb{N}} \)—that is, there exists an enumeration \( \{ s_1, s_2, \ldots \} \) of states of \( \mathcal{S} \).

\[^*\text{Note, however, that the argument can be generalized beyond countable sets [3].}\]
Furthermore, suppose there exists likewise an enumeration of possible measurements \( \{m_j\}_{j \in \mathbb{N}} \). We will suppose that these are dichotomic: that is, each yields either 1 or \(-1\) as outcome. This is not a restriction: we can always decompose a many-valued measurement into an appropriate set of dichotomic ones. Measurements are then functions that take states as input and return values, \( m_n(s_k) \in \{1, -1\} \).

Think, as an example, of a coin: after we flip it, we make a measurement (that is, we look to see which side is up), and denote ‘heads’ as 1, ‘tails’ as \(-1\). For this system, there exist only two states—\( s_1 \) for heads and \( s_2 \) for tails—and one measurement \( m_1 \), and we have
\[
\begin{align*}
m_1(s_1) &= 1 \\
m_1(s_2) &= -1
\end{align*}
\]

We now introduce the following assumption:

**Assumption 1** (Classicality). *For every state \( s_k \) and measurement \( m_n \), there exists a function \( f \) such that \( f(n, k) = m_n(s_k) \).*

We can think of this \( f \) as a universal prediction machine for \( S \): given the index of a state and a measurement, it spits out the result the measurement will produce. For our coin example, this function is given by Table 1:

| \( f(n, k) \) | \( s_1 \) | \( s_2 \) |
|---------------|-------|-------|
| \( m_1 \)     | 1     | \(-1\) |

Consequently, \( f(1, 1) = 1 \) (in state \( s_1 \), the coin shows heads), and \( f(1, 2) = -1 \) (in state \( s_2 \), the coin shows tails).

For the general case, with \( i, j \in \mathbb{N} \), we obtain Table 2.

| \( f(n, k) \) | \( s_1 \) | \( s_2 \) | \( s_3 \) | \( s_4 \) | \( s_5 \) | \ldots |
|---------------|-------|-------|-------|-------|-------|-------|
| \( m_1 \)     | (1)   | \(-1\) | 1     | 1     | \ldots | 1     | \ldots |
| \( m_2 \)     | 1     | (-1)  | 1     | \(-1\) | \(-1\) | \(-1\) |
| \( m_3 \)     | \(-1\) | 1     | (-1)  | \(-1\) | \ldots | 1     |
| \( m_4 \)     | 1     | \(-1\) | \(-1\) | (1)   | 1     | \ldots |
| \( m_5 \)     | \(-1\) | \(-1\) | \(-1\) | 1     | (1)   | \ldots |
| \ldots   | \ldots | \ldots | \ldots | \ldots | \ldots |
| \( m_g \)     | \(-1\) | 1     | \(-1\) | \(-1\) | \ldots | (f)   | \ldots |
| \ldots   | \ldots | \ldots | \ldots | \ldots | \ldots |

We can now lead Assumption 1 to a contradiction. To do so, we must first observe that we can construct new measurements by means of logical operations. For this, it is convenient to think of the values 1 and \(-1\) as representing ‘true’ and ‘false’, respectively. Then, we can consider \( m_n(s_k) = 1 \) to mean that the proposition ‘\( S \) has property \( n \) in state \( k \)’ is true, and \( m_n(s_k) = -1 \) consequently that it is false. Each measurement thus tests whether a system in a given state has or fails to have a certain property. Since properties and measurements are thus in one-to-one correspondence, we will on occasion abuse notation and speak of the ‘property \( m_n \)’.

We can equivalently look at this in terms of subsets (or -regions) of the state space introduced in Fig. 1. Each measurement essentially tests whether the system is in some region of that space. For instance, the region with \( p \) smaller than \( \sqrt{2mE_0} \) corresponds to the set of states with energy \( E \) less than \( E_0 \); a measurement that yields 1 for all states in that region (and \(-1\) otherwise) then indicates the truth of the proposition ‘\( S \) has energy less than \( E_0 \)’.
This enables us to construct a logical calculus for the properties of the system. From two measurements \( m_1 \) and \( m_2 \), we can, for instance, construct \( m_{12} = m_1 \oplus m_2 \), where the operator \( \oplus \) is taken to signify the logical xor: that is, \( m_{12} = 1 \) if \( m_1 \neq m_2 \), and \( m_{12} = -1 \) if \( m_1 = m_2 \). For ease of notation, we indicate the property values by superscripts; see Fig. 2.

Moreover, we can give an explicit measurement procedure for each property: simply measure momentum and position up to the precision necessary to localize the state within the respective subset.

But then, this means that we can construct the following measurement \( m_g \): for each \( m_i \), \( m_g(s_i) \) is just the opposite of \( m_i(s_i) \). That is, if \( m_1(s_1) \) yields 1, \( m_g(s_1) \) yields \(-1\); if \( m_2(s_2) \) yields \(-1\), then \( m_g(s_2) \) yields 1. The construction of this measurement is then shown in Table 2.

If we now hold fast to our assumption that \( f(n,k) \) enumerates all possible measurement outcomes, then \( m_g \) itself must correspond to some row of Table 2. However, it cannot correspond to the first row, as it differs from \( m_1 \) in the value associated to \( s_1 \); it cannot correspond to the second row, as it differs in the value associated to \( s_2 \); and so on, for any particular row of that table.

We might now suppose that, having infinitely many rows, we can just add the missing measurement. But, as is of course familiar, this move will not get us out of trouble: we can always just repeat the construction, finding a further measurement not on the list already.

But this means that there exists some state \( s_g \) and measurement \( m_g \) such that the value of \( m_g(s_g) \) cannot be predicted by \( f \). Thus, our ‘universal prediction machine’ cannot, in fact, exist; there are measurements such that their outcome for certain states cannot be predicted. They are, in other words, undecidable.

The above has the form of a *diagonal argument*. Diagonalization was first introduced by Cantor in his famous proof of the existence of uncountable sets, and lies at the heart of Gödel’s (first) incompleteness theorem, the undecidability of the halting problem, and many others. The precise structure of such arguments in a category-theoretic setting was brought to the fore by Lawvere by means of a fixed-point theorem [4]. This can be directly adapted to the present setting, yielding a somewhat more general argument than the above; for details, see [3] and Appendix A.

An intuitive way to understand this result is the following. Consider that we can define every measurement by listing the states that lie within the corresponding subregion of state space. Then, note that we can, correspondingly, define each state via measurements—say, listing all the measurements that yield a +1-outcome (all the properties the system possesses in that state). Thus, we can define a measurement in terms of a state defined in terms of that very measurement—yielding the paradoxical circularity characteristic of self-reference.

This has intriguing consequences. First of all, we cannot consistently assign to \( s_g \) either a value of 1 or \(-1\) for \( m_g \), as supposing it ought to be 1 yields the conclusion that it must be \(-1\), and vice versa. Thus, when faced with the question whether the system has property \( m_g \), we find that we can neither affirm nor deny. This is, of course, just the situation Schrödinger’s infamous and much-abused cat finds itself in: we can neither claim it is alive, nor that it is not. Thus, we may consider the system to be in a *superposition* with respect to \( m_g \).

Suppose now we perform a measurement of \( m_g \). Any possible outcome will be inconsistent with the system being in state \( s_g \)—since, as we had surmised, no outcome can consistently be associated with that state. Hence, after the measurement has yielded a result, it follows that the system can no longer be in
the state \( s_g \)—that is, post measurement state change (‘wave-function collapse’) is a direct consequence of the preceding considerations.

It is important to note that quantum mechanics, itself, does not again fall prey to the same issues. There are two salient factors accounting for this: first, the proof depends on the possibility of ‘duplicating’ the index \( g \) to construct \( m_g(s_g) \)—which, physically, represents a cloning operation that is famously impossible in quantum mechanics [8]. Second, we must be able to invert the value of a measurement—take the value 1 to \(-1\), and vice versa. That is, every possible value assigned to a property must be negated.

But this likewise is impossible in quantum mechanics [9]. Let us replace the classical outcomes with orthogonal quantum states \(|1\rangle\) and \(|-1\rangle\). Then, the operator

\[
U_{\text{NOT}} = |1\rangle \langle -1| + |-1\rangle \langle 1|
\]

takes \(|1\rangle\) to \(|-1\rangle\), and vice versa. However, applied to the state \( \frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle) \), we get

\[
U_{\text{NOT}} \frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle).
\]

Hence, the superposition yields a fixed point for \( U_{\text{NOT}} \)—thus evading the inconsistent assignment of Table 2.

This is, of course, only the first whiff of quantum phenomena. The picture can be developed further. Complementarity, the impossibility to simultaneously assign definite values to certain properties, can be obtained by considering a form of the above argument in the context of sequences of measurements. Furthermore, the uncertainty principle emerges as a finite bound on the information available about a system—the number of simultaneously definite properties—by appealing to Chaitin’s version of the incompleteness theorem [10]. For details, see [3]. In the following, we will consider another paradigmatically quantum feature that has, so far, not been considered: entanglement.

For this, it will be useful to consider a simple ‘toy’ system. Thus, take the extreme case of a system \( \mathcal{S} \) such that only one of its properties is decidable. We are then in the situation of Fig. 2: one bit of information decides one of three possible mutually exclusive measurements on the system. Appreciating the parallel to the orthogonal measurements for a single qubit, we will name these three properties \( x_S \), \( y_S \), and \( z_S \).

### 2.2. Entanglement

So far, we have only considered single, individual systems. One might therefore ask what this framework entails once one investigates composite systems instead. Thus, take two systems, \( \mathcal{A} \) and \( \mathcal{B} \). To keep matters simple, we will conjugate our discussion here to ‘toy systems’ of the kind introduced above—that is, systems described by a single definite property.

Consequently, \( \mathcal{A} \) is described by either of \( x_A \), \( y_A \), or \( z_A \) having a definite value, while \( \mathcal{B} \)’s state is given by one out of \( x_B \), \( y_B \), and \( z_B \). A possible state of the compound system \( \mathcal{A} \otimes \mathcal{B} \) would then be \( (x_A^+, x_B^+) \), where we used the superscript notation to indicate property values.

As we had surmised, however, we can use elementary Boolean logic to construct new properties. Thus, let us consider the property \( x_{AB} = x_A \oplus x_B \). This indicates a correlation between the two \( x \)-values: it signifies that one must be the opposite of the other. We can then e. g. give a complete description of the system as \( (x_A^+, x_{AB}^+) \), signifying the state where \( x_A = 1 \), and \( x_B \) must be the opposite, hence \(-1\).

This is, as yet, a completely classical situation. Picture the case of two colored cards, one red, and one green, in two envelopes: once you open one, you immediately know the color of the card within the other, even if the latter is located on Pluto. There is in particular nothing nonlocal about you having this knowledge.

However, consider now the state \( (x_{AB}^+, z_{AB}^+) \). Here, the two bits of information we have available to describe the state are entirely taken up by the correlations: we know that the two \( x \)-values, as well as the two \( z \)-values, are opposed to one another; but we know nothing whatever about any individual \( x \)- or \( z \)-value!

This is precisely the situation of an entangled two-particle system (cf. [11]). Our \( f(n, k) \), which, for this system, can only determine two properties, only provides values for \( x_{AB} \) and \( z_{AB} \), but leaves, e.
g., $z_B$ undecidable. However, once we have performed the requisite measurement, the considerations of the previous sections tell us that something remarkable must happen: whatever outcome is produced, one bit of information must now be taken up by the value of $z_B$; but, due to the (anti-)correlation between $z$-values, this then immediately tells us the value of $z_A$, as well! Furthermore, as all information available is now taken up by (e.g.) $(z_B^+, z_B^-)$, it follows that nothing about the $x$-values can be known: the correlation there is destroyed.

### 3. Does this Ring a Bell?

We now have the tools in hand to investigate one of the most famous expressions of quantum ‘weirdness’: Bell’s theorem [12], or the failure of ‘local realism’. We will start with a slightly different view on Bell inequalities [13].

Consider, to this end, again that there exists a function $f(n, k)$ providing values to all possible measurements. In particular, consider the above bipartite system and the properties $x_A, z_A, x_B$ and $z_B$. With respect to these properties, every state can be written as a four-tuple $(x_A, z_A, x_B, z_B)$, corresponding to a column in Table 2. That is, there are 16 possible states, from $f(n, 1) = (x_A^+, z_A^+, x_B^+, z_B^+)$ to $f(n, 16) = (x_A, z_A, x_B, z_B)$, which we label $\lambda_i$.

In any given experiment, each of these states may be present with a certain probability $P(\lambda_i) = p_i$. See Table 3 for an enumeration.

| State | $x_A$ | $z_A$ | $x_B$ | $z_B$ | $P(\lambda_i)$ |
|-------|-------|-------|-------|-------|----------------|
| $\lambda_1$ | 1     | 1     | 1     | 1     | $p_1$          |
| $\lambda_2$ | 1     | 1     | 1     | -1    | $p_2$          |
| $\lambda_3$ | 1     | 1     | -1    | 1     | $p_3$          |
| $\lambda_4$ | 1     | 1     | -1    | -1    | $p_4$          |
| $\lambda_5$ | 1     | -1    | 1     | 1     | $p_5$          |
| $\lambda_6$ | 1     | -1    | 1     | -1    | $p_6$          |
| $\lambda_7$ | 1     | -1    | -1    | 1     | $p_7$          |
| $\lambda_8$ | 1     | -1    | -1    | -1    | $p_8$          |
| $\lambda_9$ | -1    | 1     | 1     | 1     | $p_9$          |
| $\lambda_{10}$ | -1    | 1     | 1     | -1    | $p_{10}$       |
| $\lambda_{11}$ | -1    | 1     | -1    | 1     | $p_{11}$       |
| $\lambda_{12}$ | -1    | 1     | -1    | -1    | $p_{12}$       |
| $\lambda_{13}$ | -1    | -1    | 1     | 1     | $p_{13}$       |
| $\lambda_{14}$ | -1    | -1    | 1     | -1    | $p_{14}$       |
| $\lambda_{15}$ | -1    | -1    | -1    | 1     | $p_{15}$       |
| $\lambda_{16}$ | -1    | -1    | -1    | -1    | $p_{16}$       |

With this, we can compute probabilities for individual outcomes by marginalization—that is, summing over all probabilities for states that contain the desired outcome. Therefore, the probability to find $x_A = 1$ is equal to $P(x_A^+ = 1) = \sum_{i=1}^{16} p_i = p_1 + p_2 + \ldots + p_{16}$, as states $\lambda_1$ through $\lambda_8$ have $x_A = 1$. We can likewise compute probabilities for joint events: $P(x_A^+, x_B^+) = p_3 + p_4 + p_7 + p_8$.

Finally, we can compute expectation values for such joint events:

$$\langle x_A x_B \rangle = \sum_{r,s \in \{1,-1\}} rs P(x_A^r, x_B^s)$$

$$= P(x_A^+, x_B^+) + P(x_A^-, x_B^-) - P(x_A^+, x_B^-) - P(x_A^-, x_B^+)$$

$$= p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8 - p_9 - p_{10} + p_{11} + p_{12} - p_{13} - p_{14} + p_{15} + p_{16}$$

These expectation values carry information about the correlation between the two properties: if $\langle x_A x_B \rangle = 1$, only the $p_i$ with a positive sign are nonzero, and thus, $x_A = x_B$ for all states in the ensemble; for $\langle x_A x_B \rangle = -1$, we obtain $x_A \neq x_B$. If $\langle x_A x_B \rangle = 0$, the value of $x_A$ tells us nothing about $x_B$, and vice versa.
With this, it is easy to compute the quantity
\[
\langle C_{CHSH} \rangle = \langle x_A x_B \rangle + \langle x_A z_B \rangle + \langle z_A x_B \rangle - \langle z_A z_B \rangle = 2 - 4(p_3 + p_4 + p_6 + p_8 + p_9 + p_{11} + p_{13} + p_{14})
\]
\[= 4(p_1 + p_2 + p_5 + p_7 + p_{10} + p_{12} + p_{15} + p_{16}) - 2.
\]
Since \(\sum_i p_i \leq 1\), this immediately yields
\[-2 \leq \langle C_{CHSH} \rangle \leq 2.
\]

This is, of course, nothing but the famous CHSH-Bell inequality [14].

This should strike us as somewhat remarkable: only the assumption that there exists a \(f(n, k)\) assigning values to all observables turns out to be enough to derive a bound on the above expression. Thus, Bell inequalities precisely delineate the set of theories for which there exists \(f(n, k)\) such that it yields the values for all possible measurements. Contrariwise, Bell inequality violations certify that no such \(f(n, k)\) for all values can exist—or at least, be probed by experiment.

The undecidability of these values then allows for the violation of this expression—as is, indeed, observed in quantum mechanics. Mathematically, the bounds on \(\langle C_{CHSH} \rangle\) correspond to necessary conditions for the existence of a joint probability distribution (Table 3); their violation means that no consistent assignment of probabilities to the \(\lambda_i\) is possible.

This should not surprise us: we have already seen that, for instance, the event \(\langle x_A^+, z_B^+ \rangle\) cannot occur—\(f(n, k)\) does not assign simultaneous values to both elements. But if the above probability distribution were to exist, we could easily obtain
\[
P(x_A^+, z_B^+) = p_1 + p_2 + p_3 + p_4.
\]

But what could it mean to assign a probability to an impossible event?

It is more usual to attribute violations of Bell inequalities to the failure of either locality or realism. What does the above probability distribution have to do with either?

‘Realism’ is ultimately simply the possibility of assigning values to all observables. If such an assignment is possible, each of the rows in Table 3 designates a valid state, and can be assigned a probability, leading to the above considerations.

But how is the failure of locality supposed to avoid this trouble? The resolution here is that we have implicitly assumed that we can fairly sample from the above probability distribution. However, if outcome probabilities on \(\mathcal{B}\) were to change due to measurements on \(\mathcal{A}\), then we could no longer carry the argument through. Hence, one usually makes an assumption that a measurement on one part of the system does not influence measurements carried out on the other; to make this assumption sensible, one ensures that both parts of the system are far away from one another, such that no influence, propagating at the speed of light, could travel between them. Should there then be any instantaneous influence despite these precautions, we speak of a failure of locality.

4. EPistemic HoRizons: Incomplete Quantum Mechanics?

It is sometimes proposed that Bell’s theorem only hinges on the assumption of locality, and hence, its violation suffices to conclude that nature is nonlocal (e. g. [15]). The reasoning here is typically that ‘realism’ is not a separate requirement that could fail on its own, but rather, is already established by the famous argument due to Einstein, Podolski, and Rosen (EPR) [16].

Let us take a lightning-quick review of the argument adapted to the present formalism. EPR take a system in the state \(\langle x_A^+, z_B^+ \rangle\), and consider measurements on one of its parts (say \(\mathcal{A}\)). Upon measuring \(x_A\), we obtain the \(x\)-value for \(\mathcal{A}\), and due to the correlation given by \(x_A B\), can immediately infer \(x_B\); likewise for \(z\). However, the quantum formalism does not permit us to speak of simultaneous values for \(x_B\) and \(z_B\). But how, then, is \(\mathcal{B}\) supposed to know to ‘produce’ the right value in each case?

The EPR-argument hinges on a bit of counterfactual reasoning: had we measured \(z_A\) (instead of \(x_A\)), we would have been able to predict a definite value for \(z_B\) (instead of \(x_B\)). Due to the absence of any disturbance on \(\mathcal{B}\) due to our actions on \(\mathcal{A}\) (locality), we then conclude that \(\mathcal{B}\) cannot just spontaneously
‘decide’ which value to produce, and hence, both $x_B$ and $z_B$ must have had a definite value—in EPR’s parlance, an ‘element of reality’—associated to them all along.

To illustrate this puzzle, Schrödinger introduced the analogy of the fatigued student [17]: quizzed in an oral examination, they will get the first answer right with certainty, after which, however, any further answer will be random. Even though we only get one correct answer out in any case, we still must conclude that the student knew the answer to every question, in order to produce this performance: had we asked a different first question, then nevertheless the student would have produced the right answer.

Applied to quantum mechanics, this would entail that the description of the correlated system $A \otimes B$ must be incomplete: $B$ must, to give the right answer in each of these cases, ‘know’ the correct values for $x_B$ and $z_B$ in advance, these answers simply being hidden to the quantum formalism.

If this is correct, then nonlocality is our only out in the case of Bell’s theorem: there are simultaneous values for all observables—$f(n, k)$ does not tell the whole story—and measuring one part of a system must influence the value distribution of the distant part.

One way to attempt to defuse the force of EPR’s argument is to deny that the sort of counterfactual inference that allows us to reason about what would have happened had our measurement choice been different is valid, at least in a quantum context. However, without further substantiation regarding why that should be the case, simply denying the validity of a certain form of argument to avoid an unwelcome conclusion hardly seems fair.

While I do not presume to settle this controversy once and for all, I believe the present framework offers a fresh perspective on the matter. For consider what happens in each of the two cases. The initial state $(x_{AB}^+, z_{AB}^+)$ becomes, say, $(z_{A}^+, z_{AB}^-)$, respectively $(x_{A}^+, x_{AB}^+)$. The only change is thus in the properties of the local system $A$, about which we have gained new information.

Such a state is one in which we have the following two items of knowledge: ‘the $x/z$-value of $A$ is 1’ and ‘the $x/z$-value of $B$ is opposite that of $A$’. This differs from a state like $(x_{A}^+, x_{B}^-)$ in a subtle, but crucial, way. In that state, our knowledge is given by ‘the $x$-value of $A$ is 1’ and ‘the $x$-value of $B$ is -1’. The difference emerges if we imagine varying the first of each set of propositions—that is, engage in counterfactual reasoning. In case of a state like $(x_{A}^+, x_{B}^-)$, we can say that had we obtained a value of 1 for the $z$-value instead, we could still validly speak of the $x$-value of $B$ being -1.

That is not the case for the state $(x_{A}^+, x_{AB}^+)$: varying the first proposition, but leaving the second constant, would lead us to a state in which we have no information about the $x$-value of $B$. Consequently, the two states differ in the counterfactuals they support: the state $(x_{A}^+, x_{B}^-)$ allows us to say that, had the first value been different, the second would have been the same (absent any disturbance), leading to e. g. $(z_{A}^+, x_{B}^-)$. However, in the state $(x_{A}^+, z_{AB}^-)$, as soon as we imagine exchanging $x_A$, we lose any ability to make determinations of $x_B$, as this value is specified only contingently on that of $x_A$. In a state like $(z_{A}^+, x_{AB}^-)$, $x_B$ would simply not have any determinate value at all.

Only given that one has actually measured $x_A$ is reasoning about the value of $x_B$ possible. In this sense, the present framework gives a natural meaning to Bohr’s somewhat opaque ‘influence on the precise conditions which define the possible types of prediction which regard the subsequent behaviour of the system’ [18]. We naturally imagine it to be possible to change one thing, while keeping something else equal; but in this case, the ‘one thing’ (the definite value of $x_A$) is part of the antecedent conditions for making determinations about that ‘something else’ (the value of $x_B$).

The EPR argument, then, essentially trades on a conflation of $(x_{A}^+, x_{AB}^+)$ with $(x_{A}^+, x_{B}^-)$. Only the latter state supports the reasoning that leads us to conclude that the distant particle must have known the value of both $x_B$ and $z_B$ all along.

5. Conclusion

We have considered the application of self-referential arguments to physical systems, and found that many paradigmatically quantum phenomena seem to gain a natural explanation from this perspective. This idea is not entirely new: John Wheeler himself proposed the undecidable propositions of mathematical logic as a candidate for a ‘quantum principle’, from which to derive the phenomenology of quantum mechanics [19]—a proposal which, as legend has it, got him thrown out of Gödel’s office [20]. For a brief review of these efforts, see [3] and references therein.
What this program, if successful, shows is that there is a common thread behind mathematical undecidability and physical unknowability—that, in other words, the epistemic horizons the pure mathematician and the experimental physicist find delimiting their perspectives are not separated, but instead, derive from a common thread.

In an intriguing sense, the incompleteness of mathematics may then come to the rescue of physics, allowing it in turn to yield a complete picture: the incompleteness the EPR-argument seeks to establish is averted by the horizon that bars counterfactual reasoning about unperformed experiments—which, hence, famously ‘have no results’ [21]. It is as if Schrödinger’s student does not know the answer to any questions, as such, but knows each answer only relative to that question being asked.

This motivates a proposal of relative realism: assign ‘elements of reality’ only where $f(n,k)$ yields a definite value. In this way, we get as close to the classical ideal of local realism as is possible in a quantum world.
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Technical Endnotes

A. The Lawvere Fixed-Point Argument

We will explicitly construct a measurement \( m_g(s_k) \), that is, a function \( m_g : \Sigma_S \to \{1, -1\} \), where \( \Sigma_S \) denotes the state space of \( S \), such that it differs from \( f(n,k) \) for at least one \( s_k \).

Suppose that there exists a function \( f(n,k) : \mathbb{N} \times \mathbb{N} \to \{1, -1\} \) such that it is equal to the outcome of the \( n \)th measurement for the \( k \)th state. Furthermore, we introduce the arbitrary map \( \alpha : \{1, -1\} \to \{1, -1\} \), and the map \( \Delta : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \) that takes \( n \in \mathbb{N} \) to the tuple \( (n,n) \in \mathbb{N} \times \mathbb{N} \). With these, we construct \( g \) as the map that makes the following diagram commute:

\[
\begin{array}{ccc}
\mathbb{N} \times \mathbb{N} & \xrightarrow{f} & \{1, -1\} \\
\downarrow{\Delta} & & \downarrow{\alpha} \\
\mathbb{N} & \xrightarrow{g} & \{1, -1\}
\end{array}
\]

The map \( g \) constructed in this way then yields sequentially values for a certain measurement, \( m_g \), if performed on states of \( S \), i.e. \( g(k) = m_g(s_k) \). If \( f \) yields the value of every measurement applied to every state, then there must be some \( n \) such that \( g(k) = f(n,k) \) for all states \( s_k \). Choose now \( k = n \) and evaluate \( g(n) \):

\[ f(n,n) = g(n) = \alpha(f(n,n)) \]

The first equality is simply our stipulation that \( g \) should encode some measurement, and that \( f(n,n) \) yields the outcome of the \( n \)th measurement on the \( n \)th state. The above then shows that the map \( \alpha \) must have a fixed point at \( f(n,n) \) for the construction to be consistent.

However, we are free in our choice of \( \alpha \), and consequently, may choose the negation \( \neg(1) = -1, \neg(-1) = 1 \). But this clearly has no fixed point, and we obtain the contradiction

\[ f(n,n) = \neg(f(n,n)) \]

But then, this means that no \( f \) reproducing every measurement outcome can exist.