The Gödelizing Quantum-Mechanical Automata

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Abstract

Using Albert results we argue that we don’t need new physics to understand Gödelization. Albert quantum automaton can ”understand” both a formal system and a Gödel proposition which can’t be obtained within this system. There are two significant conclusions. The first speaks ”against Penrose” whereas the second speaks for him.

1 Introduction

In his two books [1] Penrose has argued that we need new physics in order to understand the ”mind”. The crucial point of his approach is the Gödel theorem and the natural ability of our mind to do the Gödel procedure which will be referred as Gödelization. What Gödel showed [2] was how to transcend any system of formalized rules. And because our mind can do it, than it cannot be the ”formal system” itself. But our physics is nothing but some (very complex and intricate) ”formal system” so it should be incomplete and we need new physics which must include our mind in.

This is Penrose point, but I don’t think so (with the proviso in last Section). In this report I argue that quantum-mechanical automaton do something which can be called as Gödelization. The argument below is far from being a rigorous proof but I believe that it makes it rather plausible. The crucial point will be Albert approach to quantum-mechanical automata [3]. What Albert showed was that ”there are some combination of facts that can in principle be predicted by an (quantum-mechanical - A.Y.) automaton only about itself” [3]. I’ll show that we can interpret it as an quantum automaton’s ability to Gödelize, so we don’t need new physics and ”new mind” to understand the mystery of Gödelization.

2 Gödel theorem and quantum automaton

Let $p_n(w)$ is the propositional function applied to number $w$, where $n$ will be called further the Gödel number. The strings of propositions which constitute a proof of some theorem (for example, $p_n(w)$, if this statement is true in our system) be $U(x)$, where $x$ is the Gödel number of the proof. We denote our (classical) formal system as $W_c$.

The fundamental Gödel proposition has the form

$$p_w(w) = \sim \exists x [U(x) \text{ proves } p_w(w)], \quad (1)$$
where $\sim$, $\exists$ are usual logical ”quantifiers”: $\exists$ mean ”there exists...such that”, $\sim$ mean ”it is not so that...”.

What Gödel showed was that (1) can be strongly written in the special formal system $W_c$. It’s clear that $p_w(u)$ must be true as an arithmetical proposition (we suppose that $W_c$ is good, reconcilable system and there are not proofs of false statements) and therefore, by the implication of the statement (1), it can’t be proves within the system $W_c$.

It is seems that Gödel proposition (1) can be constructed in any formal system therefore any formal system will be limited by the Gödel theorem but it is not the case for us because we can Gödelize and, consequently, transcend any system of formalized rules.

Now let consider the quantum machine which must proves theorems in formal system $W_c$. Let $S$ is quantum system with Hilbert space $H_S$. We introduce observable $P_k(w)$ such that for any $|w;k\rangle_S \in H_S$ it will be true that

$$P_k(w)|w;k\rangle_S = k|w;k\rangle_S. \tag{2}$$

So eigenvalues of $P_k(w)$ will be Gödel numbers of the propositions $p_k(w)$. In the same Hilbert space we define logical ”gaits” $U_j$. The entire list of all such ”gaits” will be referred as quantum formal system $W_q$. We’ll say that quantum formal system $W_q$ can proves the proposition $p_k(w)$ (about number $w$) if such chain of ”gaits” exist

$$U = U(w,k) \equiv \prod_j U_j,$$

that

$$|\psi\rangle_S \rightarrow |w;k\rangle_S = U|\psi\rangle_S, \tag{3}$$

for some initial state $|\psi\rangle_S \in H_S$. In this case we can start out from $|\psi\rangle_S$ to find $|w;k\rangle_S$; then to measure the average value of $P_k(w)$, to obtain the Gödel number $k$

$$k = s\langle\psi|U^+P_k(w)U|\psi\rangle_S,$$

which allow one to reconstruct the proposition $p_k(w)$. It may not be simple task but we can do it as a matter of principle.

When this procedure is realizable one? To understand this we suppose that the quantum formal system $W_q$ is universal one and it is suitable for testing $|w;k\rangle_S$ to be solution of the (2). Substituting $|w;k\rangle_S$ in place of $|\psi\rangle_S$ into the (3) we must obtain the same vector

$$U|w;k\rangle_S = u(w;k)|w;k\rangle_S, \tag{4}$$

so it must be

$$[P_k(w),U] = 0. \tag{5}$$

The equation (5) is at one with uncertainty relation. Indeed, let the quantum formal system $W_q$ is instructed to measure (or calculate: it is the same in our case) the value of $p_k(w)$. The observable of $W_q$ is $U$ so both $P_k(w)$ and $U$ must be measurable simultaneously. This is another way to obtain (5).

What about Gödel proposition (1)? It is propositional function applied to number $w$ with Gödel number $w$ which can’t be obtained (proved) in formal systems $W_c$ and $W_q$. Write $P_w \equiv P_w(w)$. It would be

$$[P_w,U] \neq 0. \tag{6}$$

Otherwise we can use $U$ to find $|w\rangle_S \equiv |w;w\rangle_S$ and, after all, to calculate Gödel proposition $p_w(w)$. But it is impossible because of the Gödel theorem.

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1May be one get another vector $|w;m\rangle_S$ with $m \neq k$. In this case we can redefine $U$ to exclude this outcome.
3 Albert quantum automaton and auto-description

Let consider the automaton $A$ which is instructed to measure and to record the value of $P_w$ and $U$. It being known that

$$U|\psi\rangle_S = u|\psi\rangle_S, \quad |\psi\rangle_S = \sum_w c_w |w\rangle_S, \quad P_w|w\rangle_S = w|w\rangle_S,$$

(6)

where $P_w$, are observables which are comparing to G"odel propositions $p_w$. Let us take the state of the system $S$ to be $|w\rangle_S$. When measurement of the $P_w$ is finished, the state of the composite system $S + A$ will be [3]

$$|w^{(1)}\rangle = |w\rangle_S \otimes |w\rangle_{P_w},$$

(7)

where $|w\rangle_{P_w} \in H_A$, $H_A$ is the Hilbert space of the automaton $A$ and $|w\rangle_{P_w}$ is the eigenvector of the ”Albert observable” $G(P_w)$ (in [3] Albert has used another notation),

$$G(P_w)|w\rangle_{P_w} = \tilde{w}|w\rangle_{P_w}.$$  

(8)

This observable measures the value predicted by the automaton for the $P_w$. The prediction will accurate if

$$E(P_w)|w^{(1)}\rangle = (\tilde{w} - w)|w^{(1)}\rangle = 0,$$

(9)

with

$$E(P_w) \equiv G(P_w) - P_w.$$  

(10)

Below we’ll restricting the accurate predictions ($\tilde{w} = w$).

Albert observables do commute with the rest observables $^2$,

$$[G(P_w), P_w] = [G(P_w), U] = [G(P_w), G(U)] = [G(U), P_w] = [G(U), U] = 0,$$

so

$$[E(P_w), E(U)] = [P_w, U] \neq 0,$$

(11)

what is at one with the uncertainty relations and, at the same time, with the G"odel theorem. Indeed, a G"odel proposition $p_w$ has no proof within formal system $W_q$ so the predictions about $U$ and $P_w$ can’t both be accurate. Thus we have the same situation as above. If that’s the case, why did we introduce the second quantum automaton $A$? This because Albert has demonstrated that quantum automaton $A$ can transcend the restriction (11) in a certain sense. To do it the automaton must measure something about itself.

Let us take the initial state of the system $S$ to be the eigenvector of $U$: $|\psi\rangle_S$. It mean that $W_q$ is ”in action” and $P_w$ is beyond the system. In spite of this the $A$ is instructed to measure $P_w$. When this measurement is finished the state of composite system $S + A$ will be

$$|\psi^{(1)}\rangle = \sum_w c_w |w^{(1)}\rangle,$$

(12)

because of linearity of quantum-mechanical equations. In (12) $|w^{(1)}\rangle$ are defined by the (7). Now one can introduce new observable $U^{(1)}$ such that

$$U^{(1)}|\psi^{(1)}\rangle = u^{(1)}|\psi^{(1)}\rangle.$$  

(13)

$^2$We stress that $G(P_w)$ is not the function of the observable $P_w$ as well as $G(U)$ is not function of $U$.  


What is the \( U^{(1)} \)? We can call it the ”gait” of new formal system \( W_q^{(1)} \) which is, partially, recorded in the memory of the automaton \( A \). It’s clear that

\[
\left[ P_w, U^{(1)} \right] \neq 0, \tag{14}
\]

so it's seems that the new formal system \( W_q^{(1)} \) is not more powerful system that initial one \( (W_q) \). But now, if the automaton will measure the observable \( U^{(1)} \) then we get

\[
|\psi^{(1)}\rangle \rightarrow |\psi^{(2)}\rangle = |\psi^{(1)}\rangle_{U^{(1)}} \bigotimes |\psi^{(1)}\rangle, \tag{15}
\]

(see (7)), where

\[
G(U^{(1)})|\psi^{(1)}\rangle_{U^{(1)}} = u^{(1)}|\psi^{(1)}\rangle_{U^{(1)}}, \tag{16}
\]

therefore

\[
E(P_w)|\psi^{(2)}\rangle = E(U^{(1)})|\psi^{(2)}\rangle = 0, \tag{17}
\]

thought \( P_w \) and \( U^{(1)} \) do not commute (see (11))!

Thus the automaton in the state \( |\psi^{(2)}\rangle \) can predict accurate both \( P_w \) and \( U^{(1)} \). I believe we can call it ”Gödelization” in a sense. It is not mean that the automaton can proves \( p_w \) using within \( W_q^{(1)} \). What is implied by this that \( P_w \) (and therefore the proposition \( p_w \)) and \( U^{(1)} \) are jointly satisfiable for the automaton \( A \). In other words, the automaton \( A \) can ”understand” both a Gödel proposition \( p_w \) and a formal system \( W_q^{(1)} \). The classical automaton can’t do it. To force classical automaton ”understand” \( p_w \) and \( W_q^{(1)} \) we need to load it with more powerful ”formal system” \( W_c^{(1)} \). It is not the case when we deal with quantum automaton.

An last but not least. The Gödelization above is the personal file of the automaton \( A \). The observable \( U^{(1)} \) and observable \( P_w \) are external ones for the another automaton \( \tilde{A} \), so it can’t to ‘understand’ both \( U^{(1)} \) and \( P_w \). To Gödelize it must find its personal state \( |\tilde{\psi}^{(2)}\rangle \).

### 4 Conclusion

There are two significant conclusions. The first speaks ”against Penrose” whereas the second speaks for him.

**Against.** We don’t need new physics to understand Gödelization. Albert quantum automaton can ”understand” both a formal system and a Gödel proposition which can’t be obtained within this system.

**For.** If Gödelization is unalgorithmic procedure then, at least, we can admit that quantum mechanics really containing ”something unalgorithmic”.

### References

1. R. Penrose *New Emperor’s Mind*, Oxford University Press, (1989); R. Penrose *Shadows of the Mind*, Oxford University Press, (1994).

2. K. Gödel, *Über formal unentscheidbare Sätze per Principia Mathematica und verwandter System I*. Monatshefte für Mathematik und Physic, 38, 173-98 (1931).

3. D. Albert, *On Quantum-Mechanical Automata*. Phys. Lett. A 98, 249-52 (1983).