Variational method for solving the boundary value problem of hydrodynamics

D V Sysoev¹, A A Sysoeva¹, S A Sazonova², A V Zvyagintseva³,⁴ and N V Mozgovoj²

¹Department of Applied Mathematics and Mechanics, Voronezh State Technical University, 84 October 20th Anniversary Street, Voronezh, 394006, Russia
²Department of Technosphere and Fire Safety, Voronezh State Technical University, 84 October 20th Anniversary Street, Voronezh, 394006, Russia
³Department of Chemistry and Chemical Technology, Voronezh State Technical University, 84 October 20th Anniversary Street, Voronezh, 394006, Russia
⁴E-mail: zvygincevaav@mail.ru

Abstract. The analysis of the application of the variational method for solving the boundary value problem of hydrodynamics is carried out. From the point of view of numerical research of mathematical physics problems, these variation formulations are considered as the basis of projection methods (the Ritz method). The article presents the main techniques that allow reducing the cost of machine time and speed up the convergence of the computational process when calculating the hydrodynamic characteristics of cavities of various configurations. Using the Trefts method allows you to reduce the calculation time of the boundary value problem. The green transformation allows us to reduce the three-dimensional integral to a one-dimensional. This creates a universal method for determining hydrodynamic coefficients for rotation cavities with an arbitrary contour of the Meridian section. However, for most configurations of cavities, the convergence rate is satisfactory and provides numerical values with a high degree of accuracy.

1. Introduction

In the boundary-value problem of hydrodynamics, the most difficult is to satisfy the boundary conditions on the surface of the cavity of arbitrary configuration. Methods for solving the problem [1-3] using modern computer technology lead to satisfactory numerical results for a number of specific configurations of the cavities, but they are quite time-consuming, however. The problem of efficiently calculating hydrodynamic coefficients has arisen for an arbitrary body containing a liquid, in connection with the growing needs of practice. The method of universalization of computing programs was considered in [4]. The problems of increasing accuracy and speeding up the computational process are discussed below.

Considered in the task is reduced to research of wave vibrations of the free surface of an ideal incompressible liquid located inside an axisymmetric cavity and exposed to a uniform field of mass forces.

Given that the Bessel functions give fast convergence, but are not complete, and the Legendre polynomials have completeness, the construction of a variation series based on a "mixed" system of
functions was implemented. This technique allowed us to significantly speed up and improve the convergence process for adverse cases when using each coordinate function system separately did not lead to success.

2. Convergence of the Ritz method

A solid hollow body partially filled with an ideal incompressible fluid, in a homogeneous. This system is conservative. The Ostrogradsky variation principle can be applied to it. Knowing that functions with the maximum value of the Hamilton action fit the homogeneous boundary value problem with the parameter described below [5]:

\[ \Delta \varphi = 0, \quad \frac{\partial \varphi}{\partial \vartheta} \bigg|_x = 0, \quad \frac{\partial \varphi}{\partial \varSigma} = \Xi \varphi. \tag{1} \]

inhomogeneous boundary value problem

\[ \Delta \psi_i = 0, \quad \frac{\partial \psi_i}{\partial \vartheta} \bigg|_x + \varSigma = (R \times \vartheta)_i, \quad (i = 1, 2, 3). \tag{2} \]

Here \( s \) is the part of the surface of the cavity moistened with liquid; \( \Sigma \) is the free surface of the liquid; \( R \) is the radius vector; \( \vartheta \) is the unit vector of the outer normal to the surface of the liquid; \( \Xi \) is the frequency parameter representing the eigenvalues of a boundary value problem (1).

The boundary-value problem (1), due to the self-adjointness and positivity of operators, has a discrete set of positive eigenvalues \( \Xi_n \), each of which corresponds to an eigenfunction \( \varphi_n \). In the future, we will consider only the lowest eigenvalue \( (n = 1) \) and the corresponding eigenfunction, since higher harmonics in a rigid cavity are practically not important. As a result, we remove the index \( n \). The boundary value problem (2) is satisfied by functions called Zhukovsky potentials [3, 6]. To solve variational problems, the Ritz – Trefts method is used. For example, the function \( \varphi \) is represented as a linear combination of coordinate functions \( \gamma_j \).

\[ \varphi = \sum_{j=1}^{k} a_j \gamma_j. \tag{3} \]

Here \( a_j \) are the indefinite constants forming \( k \) - the dimensional column vector of \( a \).

The elements of square \( k \) - dimensional matrices \( A \) and \( B \) are calculated.

\[ \varphi_{ij} = \int_{\tau} \nabla \gamma_j \nabla \gamma_i d\tau; \quad \beta_{ij} = \int_{Z} \gamma_j \gamma_i dS. \tag{4} \]

Then from solving a linear homogeneous system of algebraic equations

\[ Aa + \Xi Ba = 0 \]

the eigenvalue \( \Xi \) and the eigenvector \( a \) are determined. Similarly, the Zhukovsky potentials \( \psi_i \), The convergence rate of the Ritz method depending on the type of the chosen sequence of the function \( \gamma_j \) are determined. To ensure convergence of the computational process, on average, only the completeness condition of the applied system of coordinate functions is required and it is not necessary that the functions \( \gamma_j \) satisfy the boundary conditions of problem (1), since the latter are natural. However, in the general case, convergence may turn out to be slow, and the computational process becomes unstable due to the inevitably accumulating rounding errors [5].
3. Trefts method

Consider the basic techniques to reduce the cost of computer time and accelerate the convergence of the computational process when calculating the hydrodynamic characteristics of cavities of various configurations. First of all, in formulas (4) for \( a_{ij} \) three-dimensional volume integrals are constructed, the calculation of which even on modern computers requires a lot of time. If we restrict ourselves to the class of axisymmetric cavities, we can reduce the circular coordinate \( \eta \) by integrating the cylindrical coordinate system \( x, r, \eta \) by the volume coordinate \( \tau \) to two-dimensional integrals by the area of the meridional section \( G \), and reduce the two-dimensional integrals by the free surface \( \Sigma \) to the one-dimensional integral by her outline \( G_0 \). The same transformation is performed in the case of the compartment of the rotation cavity formed by two continuous radial partitions [3].

In addition, the time spent on computation can be reduced by applying the method Tripsta. The difference between this method and the Ritz method is that the coordinate functions must fit the differential equation. Then the system of coordinate functions for (1), (2) is chosen from the set of harmonic functions. If the cavity is a two-dimensional integral (4), then it can be reduced to a one-dimensional one using the green transform and dividing the circle coordinate \( \eta \) [5]:

\[
a_{ij} = \oint_{G} \frac{\partial \gamma_j}{\partial \theta} \frac{\partial \gamma_i}{\partial \theta} \, dS. \tag{5}
\]

Here \( G \) is the contour of the meridian section of the fluid volume. More than often, two systems of coordinate functions are used in the Treftz method: cylindrical functions

\[
\gamma_j = \begin{cases} \sin \left( \frac{\pi x}{2} \right) \left[ J_n \left( \frac{\pi r}{2} \right) \right] & \sin m \eta \\ \cos \left( \frac{\pi x}{2} \right) \left[ N_n \left( \frac{\pi r}{2} \right) \right] & \cos m \eta \end{cases}, \tag{6}
\]

and spherical functions

\[
\gamma_j = \begin{cases} P_n^m \left( \frac{\pi y}{2} \right) & \sin m \eta \\ Q_n^m \left( \frac{\pi y}{2} \right) & \cos m \eta \end{cases}; \quad R = \sqrt{x^2 + r^2} \quad \gamma = \frac{x}{\sqrt{x^2 + r^2}}. \tag{7}
\]

Here, the index \( j \) is determined by the two-parameter set of integers natural numbers \( n \) and \( m \).

Convergence increases when the coordinate functions fit the maximum number of boundary value problem conditions. To determine the configuration of the cavity, you can choose a more optimal sequence of coordinate functions in each case specifically. Faster convergence can be provided by functions that do not even have the completeness property. But they are suitable for the largest number of conditions of the boundary value problem. For example, if we consider the problem of conic cavities, the method of variations converges at a high rate. Provided that the system of "conic" coordinates is used. It must also be suitable for the boundary conditions of the wetted surface and the differential equation [5]. Good convergence in the solution homogeneous problem (1) was obtained for the cylindrical cavities with flat bottoms when using the system cylinder functions (6) satisfying both equation and the boundary condition on the free surface [3]. However, the system of functions (6) does not have the completeness property on an arbitrary surface and, as a result; the results of solving the inhomogeneous boundary value problem (2) were incorrect [6]. This system of functions becomes complete if more functions are included in it.

\[
\left\{ r^m, x^m, I_m(\xi_n r), K_m(\xi_n r) \right\} \left[ \begin{array}{c} \sin m \eta \\ \cos m \eta \end{array} \right].
\]

Here \( I_m, K_n \) are the modified Bessel and Hankel functions.
To solve boundary value problems (1), (2) by the Trefftz method, the system of spherical functions (7), which has the completeness property, which was used by some authors to determine the hydrodynamic characteristics of a number of specific cavities [3], has proven itself well. To construct a practically convergent sequence of spherical functions, the “test” method was proposed [4], in which, when building up the variational series, that function was chosen from four types of successive functions, which gives the minimum value of the eigenfrequency. This and other techniques have allowed us to create a universal method for determining the hydrodynamic coefficients for cavities of revolution with an arbitrary contour of the meridional section. Moreover, for most cavity configurations, the convergence rate was satisfactory and provided numerical values with a high degree of accuracy, however, sometimes k members of the series (3) became more than $20-25$, and satisfactory convergence of the computational process was not achieved. Poor conditionality of the matrices $A$ and $B$, at such values of $k$, a loss of count stability occurs. The low convergence can be explained by the fact that the initial functions of a number of variations practically did not satisfy the boundary conditions.

When solving a homogeneous boundary value problem (1), the boundary condition breaks when moving from the free surface of the liquid to the wetted one. Let’s assume that there is a function whose derivative has the same gap that is perpendicular to the surface of the volume of the entire liquid. In this case, the series will converge faster. This is due to the fact that the use of cylindrical functions indicates a rapid convergence of the method of variations [3, 6]. Because their normal derivative has a gap on the contour of the free surface. This results in good results of the approximate method [7]. In this method, the solution of the boundary value problem is written as a cylindrical function of the first kind [5], and the value of the parameter is found after the boundary conditions are met on the contour of the free surface. Then the boundary condition and the solution on the boundary have the same gap. With a more accurate approximate solution [8], it is proposed to satisfy the boundary condition on the contour of the free surface of the liquid not only in the angle of inclination of the tangent, but also in the curvature of the walls of the cavity. In the latter case, the difference between the true solution at the boundary and the approximate solution has smoothness not only in the first, but also in the second derivative. However, these solutions give satisfactory accuracy only for the case of a deep liquid and a sufficiently smooth surface of the cavity walls. For example, these approximate methods are not suitable for a cylindrical cavity with concave bottoms [5].

4. The "mixed" system functions

From the analysis of the available results, it can be seen that the convergence rate of the computational process depends on how well the first functions are chosen. Considering that the Bessel functions (6) give fast convergence, but are not complete, and the Legendre polynomials (7) are complete, the construction of the variational series (3) using the “mixed” system of functions, including both cylindrical and spherical functions, was implemented. This technique made it possible to significantly accelerate and improve the convergence process for adverse cases when using each system of coordinate functions separately did not lead to success. However, it was necessary to use the Bessel functions (6) for the first few roots $\xi_n$.

As noted in [3], if an approximate solution is constructed from the already known value of the eigenfrequency of free oscillations of the liquid, it will also be expressed in terms of the Bessel function (6), but with the parameter $\xi$ that does not coincide with the parameter $\xi_n$. In the same way, but to a lesser extent, the parameter $\xi$ obtained [7, 8] does not coincide, which is found from the condition that the normal derivative of the cavity vanishes on the normal surface contour.

$$\left(\ctg(\theta) + \frac{1}{\xi}\right) J_1(\xi) - J_0(\xi) = 0,$$

(8)

where $\theta$ is the angle of inclination of the tangent to the surface of the cavity on the contour $\Sigma$.

Based on the analysis of the dependence of the solution of the boundary value problem (1) on the value of the parameter $\zeta$, the first function can be represented as
Here \( \eta_0 \) is the radius of the free surface of the liquid; \( h \) is the maximum liquid depth; \( x^\Sigma \) is the coordinate of the free surface; \( \zeta \) should be found by varying from the condition of minimum the eigenvalue \( \mathcal{W} \).

This approach made it possible to obtain fairly accurate results almost from the one-term approximation, i.e. Rayleigh method.

Even more accurate results were obtained using the first spherical function (7) of the second type, which has the following form:

\[
\gamma_I = r \left[ (x-L)^2 + r^2 \right]^{-3/2} \sin(\eta). \tag{10}
\]

Here \( L \) is a parameter characterizing the shift of the origin, which should be varied to ensure a minimum of the free value of \( \mathcal{W} \).

The boundaries of the values of the parameter \( L \) should be chosen so that the feature of function \( (10) \) is outside the region occupied by the liquid.

The parameter \( \zeta \) in the function \( (9) \) and the parameter \( L \) in the function \( (10) \) allow us to bring these functions as close as possible to the boundary conditions of the boundary value problem \( (1) \). Regardless of the type of resonator \([5]\), thus determine the frequency value in the first approximation with an accuracy of \( 1 \div 3\% \). Lines without breaks in figure 1 determine the exact values of the parameter \( \mathcal{W} \) for both configurations of cylindrical cavities. The first one with a conical bottom, which has an opening angle equal to \( \pi/4 \). The second one has two concave spherical bottoms. The ratio of their radius to the radius of the cylinder is 1, 4. The total elongation of the cavities is 3.0. In this case, the known approximate methods \([9-15]\) do not take into account the geometry of the bottoms in the tanks and give too rough results.

**Figure 1.** The exact values of the frequency parameter \( \mathcal{W} \) for two configurations of cylindrical cavities: 1) a cylindrical cavity with a conical bottom whose opening angle is \( \pi/4 \) and 2) a cylindrical cavity with two concave spherical bottoms, the ratio of the radius of which to the radius of the cylinder is 1, 4.
The results obtained using the optimally constructed function (9) are shown by a dotted line with a dash, and using the optimally constructed function (10) they practically coincided with the exact results, and therefore are not visible. For less complex cavity configurations with convex bottoms such as a sphere, cone, or cylinder, the optimal functions (9) or (10) provide even higher accuracy. The cost of computer time when choosing the optimal value of the parameter $\zeta$ or $L$ is insignificant, since the dependence $r(\zeta)$ or $r(L)$ has a pronounced extremum representing the magnitude of the parabola. The cost of computer time is minimal when constructing the optimal function (10), which has a very simple expression.

5. Conclusion
As a result, both functions (9) or (10) fit the Laplace equation and satisfy the boundary conditions of the boundary value problem (1). This speeds up the convergence of the Trefts method. Thus, they themselves provide almost exact values of the integral characteristics [5].

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