Rigidity of Stable Lyapunov Exponents and Integrability for Anosov Maps

Jinpeng An\textsuperscript{1}, Shaobo Gan\textsuperscript{1}, Ruihao Gu\textsuperscript{1}, Yi Shi\textsuperscript{2}

\textsuperscript{1} School of Mathematical Sciences, Peking University, Beijing 100871, China. E-mail: anjinpeng@gmail.com; gansb@pku.edu.cn; rhgu@pku.edu.cn
\textsuperscript{2} School of Mathematics, Sichuan University, Chengdu 610065, China. E-mail: shiyi@math.pku.edu.cn

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Abstract: Let $f$ be a non-invertible irreducible Anosov map on $d$-torus. We show that if the stable bundle of $f$ is one-dimensional, then $f$ has the integrable unstable bundle, if and only if, every periodic point of $f$ admits the same Lyapunov exponent on the stable bundle as its linearization. For higher-dimensional stable bundle case, we get the same result on the assumption that $f$ is a $C^1$-perturbation of a linear Anosov map with real simple Lyapunov spectrum on the stable bundle. In both cases, this implies if $f$ is topologically conjugate to its linearization, then the conjugacy is smooth on the stable bundle.

1. Introduction

Let $M$ be a $d$-dimensional smooth closed Riemannian manifold. A diffeomorphism $f : M \to M$ is Anosov if there exists a continuous $Df$-invariant splitting $TM = E^s \oplus E^u$ such that $Df$ is uniformly contracting in $E^s$ and $Df$ is uniformly expanding in $E^u$. The classical Stable Manifold Theorem (e.g. [35]) shows that both $E^s$ and $E^u$ are uniquely integrable. So there are $f$-invariant stable and unstable foliations tangent to $E^s$ and $E^u$ respectively.

The most well-known example of Anosov diffeomorphisms is a linear automorphism $A \in \text{GL}_d(\mathbb{Z})$ with all eigenvalues whose absolute values are not equal to 1. The induced diffeomorphism $A : \mathbb{T}^d \to \mathbb{T}^d$ is Anosov. All known Anosov diffeomorphisms are conjugate to affine automorphisms of infra-nilmanifolds. In particular, every Anosov diffeomorphism with $\text{dim } E^s = 1$ or $\text{dim } E^u = 1$ must be supported on $\mathbb{T}^d$ [12], and every Anosov diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ is topologically conjugate to its linearization $f_* : \pi_1(\mathbb{T}^d) \to \pi_1(\mathbb{T}^d)$ acting on $\mathbb{T}^d$ [11,29].

In 1974, Mañé and Pugh extended the concept of Anosov diffeomorphisms to non-invertible Anosov maps.
Definition 1.1 [28]. A $C^1$ local diffeomorphism $f : M \to M$ is called Anosov map, if there exists a $Df$-invariant continuous subbundle $E^s \subset TM$ such that it is uniformly $Df$-contracting and its quotient bundle $TM/E^s$ is uniformly $Df$-expanding.

The set of Anosov maps on $M$ is $C^1$-open in the space $C^r(M)$ which consists of all $C^r$-maps of $M$ with $r \geq 1$. All known Anosov maps are homotopic to affine endomorphisms of infra-nilmanifolds.

Differing from Anosov diffeomorphisms, there is a priori no $Df$-expanding subbundle $E^u \subset TM$ for a non-invertible Anosov map because the negative orbit for a point is not unique. For instance, Przytycki [36] constructed a class of Anosov maps on torus which has infinitely many expanding directions on certain points. In fact, the set of expanding directions on a certain point in Przytycki’s example contains a curve homeomorphic to interval in the $(\dim M - \dim E^s)$-Grassman space [36, Theorem 2.15]. In the same paper [36], Przytycki defined Anosov maps in the way of orbit space (see Definition 2.11) which allowed us to define the unstable bundle along every orbit. However, these unstable bundles are not integrable in general when projected on the manifold $M$.

We say an Anosov map $f$ has an integrable unstable bundle, if there exists a continuous $Df$-invariant splitting $TM = E^s \oplus E^u$, such that $Df$ is uniformly contracting on $E^s$ (stable bundle) and expanding on $E^u$ (unstable bundle). Here $E^u$ is uniquely integrable, see [36]. For example,

$$A_0 = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2,$$

is an Anosov map on torus with integrable unstable bundle.

There are plenty of Anosov maps without integrable unstable bundles. Actually, Przytycki [36, Theorem 2.18] showed that any non-invertible Anosov map $f$ on any manifold $M$ with non-trivial stable bundle can be $C^1$-approximated by Anosov maps without integrable unstable bundles. Moreover, for every transitive Anosov map without integrable unstable bundle, it must have a residual set in the manifold in which every point has infinitely many expanding directions ([32]).

In this paper, we give an equivalent characterization for a class of Anosov maps on $d$-torus $\mathbb{T}^d$ which has integrable unstable bundle.

Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be an Anosov map on torus, then $f$ is homotopic to a linear toral map $A = f_* : \pi_1(\mathbb{T}^d) \to \pi_1(\mathbb{T}^d)$. Here $A$ is also an Anosov map [1, Theorem 8.1.1] and is called the linearization of $f$. A toral Anosov map $f$ is called irreducible, if its linearization $A \in GL_d(\mathbb{R}) \cap M_d(\mathbb{Z})$ has irreducible characteristic polynomial over $\mathbb{Q}$.

Theorem 1.1. Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a $C^{1+\alpha}$ irreducible non-invertible Anosov map with one-dimensional stable bundle. Then $f$ has integrable unstable bundle, if and only if, every periodic point of $f$ admits the same Lyapunov exponent on the stable bundle.

Remark 1.2. We remark here that if every periodic point of $f$ admits the same Lyapunov exponent on the stable bundle, then the periodic stable Lyapunov exponent is equal to the stable Lyapunov exponent of its linearization $f_*$ (see Theorem 3.1).

Remark 1.3. The necessity of Theorem 1.1 only needs $C^1$ regularity of $f$, i.e., if a $C^1$ irreducible non-invertible Anosov map $f : \mathbb{T}^d \to \mathbb{T}^d$ with $\dim E^s = 1$ has integrable unstable bundle, then every periodic point of $f$ admits the same stable Lyapunov exponent to its linearization $f_*$ (also see Theorem 3.1).
Remark 1.4. Our proof of the sufficient part of Theorem 1.1 relies on the Livscheit Theorem for transitive Anosov maps (Proposition 4.1). Hence we need \( f \) be \( C^{1+\alpha} \) smooth to get the integrable unstable bundle from the periodic data on the stable bundle.

A corollary is the following, which is an interesting example of rigidity in smooth dynamics, in the sense of “weak equivalence” (topological conjugacy) implies “strong equivalence” (smooth conjugacy).

**Corollary 1.5.** Let \( f : T^d \to T^d \) be a \( C^{1+\alpha} \) irreducible non-invertible Anosov map with one-dimensional stable bundle. If \( f \) is topologically conjugate to its linearization \( f_* : T^d \to T^d \), then the conjugacy is \( C^{1+\alpha} \)-smooth along stable foliation.

In fact, an Anosov map on torus \( f \) is conjugate to its linearization if and only if it admits an integrable unstable bundle [33,40]. By Theorem 1.1 and Remark 1.2, one has that the periodic stable Lyapunov exponents of \( f \) are equal to ones of \( A \). As the classical work [10] in rigidity of Anosov diffeomorphisms, this implies that the conjugacy is smooth, as the same regularity as \( f \), along the stable foliation. We will prove Corollary 1.5 precisely in Sect. 5.

In particular, we have the following corollary on \( T^2 \).

**Corollary 1.6.** Let \( f : T^2 \to T^2 \) be a \( C^{1+\alpha} \) non-invertible Anosov map, then the following are equivalent:

- \( f \) has integrable unstable bundle;
- \( f \) is topologically conjugate to its linearization \( f_* : T^2 \to T^2 \).

Both of them imply the conjugacy between \( f \) and \( f_* \) is \( C^{1+\alpha} \)-smooth along the stable foliation.

Remark 1.7. Recently, Micena [31, Theorem 1.10] shows that for a \( C^\infty \) non-invertible Anosov map \( f : T^2 \to T^2 \) with integrable unstable bundle, if it admits periodic data on the stable and unstable bundle, then \( f \) is \( C^\infty \)-conjugate to \( f_* \) (also see [30, Theorem C]). Our result shows that we only need to assume \( f \) admits periodic data on the unstable bundle, then it is \( C^\infty \)-conjugate to \( f_* \).

For higher-dimensional stable bundle case, we prove a local rigidity result for linear Anosov maps on \( T^d \) with real simple spectrum. We say a hyperbolic matrix \( A \in M_d(\mathbb{Z}) \cap GL_d(\mathbb{R}) \) has real simple spectrum on stable bundle, if all eigenvalues on the stable bundle are real and have mutually distinct moduli. Then \( A \) admits a dominated splitting

\[
T_{T^d} = L^s_1 \oplus \cdots \oplus L^s_k \oplus L^u,
\]

with \( \dim L^s_i = 1 \) for \( i = 1, \ldots, k \).

If a map \( f : T^d \to T^d \) is \( C^1 \)-close to a hyperbolic \( A \in M_d(\mathbb{Z}) \cap GL_d(\mathbb{R}) \) with real simple spectrum on stable bundle, then \( f \) is Anosov and has \( k \)-Lyapunov exponents on the stable bundle:

\[
\lambda^s_1(p, f) < \lambda^s_2(p, f) < \cdots < \lambda^s_k(p, f) < 0, \quad \forall p \in \text{Per}(f).
\]

We say \( f \) has spectral rigidity on stable bundle if for every periodic point \( p \in \text{Per}(f) \), it satisfies

\[
\lambda^s_i(p, f) = \log |\mu_i|, \quad i = 1, \ldots, k.
\]

Here \( \mu_i \) is the eigenvalue of \( A \) in the eigenspace \( L^s_i \).
**Theorem 1.2.** Let $A \in M_d(\mathbb{Z}) \cap GL_d(\mathbb{R})$ be hyperbolic and irreducible with real simple spectrum on stable bundle. If $A : \mathbb{T}^d \to \mathbb{T}^d$ is non-invertible, then for every $f \in C^{1+\alpha}(\mathbb{T}^d)$ which is $C^1$-close to $A$, $f$ has integrable unstable bundle, if and only if, it has spectral rigidity on stable bundle.

Again, the necessity of Theorem 1.2 only needs that $f$ is $C^1$ smooth, see Theorem 3.1. As Corollary 1.5, we have the following “bootstrap” property, since $f$ having integrable unstable bundle is equivalent to $f$ being topologically conjugate to $A$.

**Corollary 1.8.** Let $A \in M_d(\mathbb{Z}) \cap GL_d(\mathbb{R})$ be hyperbolic and irreducible with real simple spectrum on stable bundle. Assume $A : \mathbb{T}^d \to \mathbb{T}^d$ is non-invertible and $f \in C^{1+\alpha}(\mathbb{T}^d)$ is $C^1$-close to $A$. If $f$ is topologically conjugate to $A$, then the conjugacy is $C^{1+\beta}$-smooth along stable foliation, for some $0 < \beta \leq \alpha$.

**Remark 1.9.** Here we lose the regularity of conjugacy because the weak stable bundle may only be $C^\beta$ continuous for some $0 < \beta < \alpha$. In particular, when the stable bundle of $A$ is one-dimensional, one has $\beta = \alpha$. We will prove Corollary 1.8 in Sect. 5.

Here we would like to discuss the irreducible condition. Firstly we give an example from which one can see some necessity on irreducibility. Let

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} : \mathbb{T}^3 \to \mathbb{T}^3,$$

be a non-invertible Anosov map with one-dimensional stable bundle and integrable unstable bundle, and it can be treated as a product system on $\mathbb{T}^2 \times S^1$. Note that one of its factor systems

$$A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2,$$

is an Anosov diffeomorphism. So, we can make a smooth perturbation $f : \mathbb{T}^2 \to \mathbb{T}^2$ of $A_2$ which is not smooth conjugate to $A_2$. Then the product map

$$F = \begin{bmatrix} f & 0 \\ 0 & 2 \end{bmatrix} : \mathbb{T}^3 \to \mathbb{T}^3,$$

is still an Anosov map with integrable unstable bundle, but it loses the rigidity of Lyapunov exponents on stable bundle.

**Remark 1.10.** Instead of the irreducibility of $A$, to get the spectrum rigidity on the stable bundle from the integrability of the unstable bundle, we need the linearization $A$ satisfy that its preimage set is becoming dense exponentially, i.e., there exist $C > 0$ and $0 < \alpha < 1$ such that for every $l \geq 1$ and every $x_0 \in \mathbb{T}^d$, the $l$-preimage set $\{x \in \mathbb{T}^d | A^l(x) = x_0\}$ is $C \cdot \alpha^l$-dense in $\mathbb{T}^d$, see Theorem 3.1. Note that when $A$ is irreducible, we do have this density property (see Proposition 2.10). For example, the following Anosov map

$$A_3 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} : \mathbb{T}^3 \to \mathbb{T}^3,$$

is reducible but its preimage set is becoming dense exponentially. Hence by Theorem 3.1, if an Anosov map $g : \mathbb{T}^3 \to \mathbb{T}^3$ is homotopic to $A_3$ and has the integrable unstable bundle, then $\lambda^g(p, g) = \lambda^3(A_3)$ for every $p \in \text{Per}(g)$. Note that the preimage set of $A_1$ is not becoming dense exponentially.
Remark 1.11. Our proof of getting the integrable unstable bundle from the spectrum rigidity on the stable bundle needs only the assumption that each one-dimensional stable subfoliation of $A$ is minimal, i.e., the line induced by $L_s^i$ is dense in $\mathbb{T}^d$. Note that the irreducibility ensure this minimal property [13]. Differing from the Anosov diffeomorphism case in which $\dim L_s^i = 1$ implies the stable foliation is minimal, the stable foliation of $A_3$ is one-dimensional but not minimal. Moreover, there exists an Anosov map $g : \mathbb{T}^3 \to \mathbb{T}^3$ such that $g_* = A_3$ and $\lambda^s(p, g) = \lambda^s(A_3)$ for every $p \in \text{Per}(g)$, but $g$ has no integrable unstable bundle (see Sect. 2.4).

We would like to give another view of Theorem 1.1 and Theorem 1.2. In [17], Gogolev and Guysinsky show that for an Anosov diffeomorphism $g : \mathbb{T}^3 \to \mathbb{T}^3$ with partially hyperbolic splitting $T\mathbb{T}^3 = E^{ss} \oplus E^{us} \oplus E^u$, if $g$ has spectral rigidity on the weak stable bundle $E^{us}$, then $E^{ss} \oplus E^u$ is integrable. See [15,18] for higher-dimensional Anosov diffeomorphisms on $\mathbb{T}^d$. Conversely, Gan and Shi [14] proved that if $E^{ss} \oplus E^u$ is integrable, then $g$ has spectral rigidity on $E^{us}$. See [20] for higher-dimensional Anosov diffeomorphisms on $\mathbb{T}^d$.

For a non-invertible Anosov map $f : \mathbb{T}^d \to \mathbb{T}^d$ and every $x \in \mathbb{T}^d$, we can see the preimage set of $x$:

$$\text{Preimage}(x) = \bigcup_{k \geq 0} P_k(x) \quad \text{where} \quad P_k(x) = \{z \in \mathbb{T}^d | f^k(z) = f^k(x)\}$$

as the strongest stable manifold of $x$. So the stable bundle of $f$ is corresponding to the weak stable bundle $E^{us}$ of the Anosov diffeomorphism $g$. It is clear that $f$ has integrable unstable bundle implies for every $x \in \mathbb{T}^d$, the unstable bundle of $x$ is independent of the choice of negative orbits of $x$. So we can see the unstable bundle is jointly integrable with the strongest stable bundle, which is corresponding to the case that $E^{ss} \oplus E^u$ is integrable for the Anosov diffeomorphism $g$. Thus we can expect the Anosov map has some rigidity on the stable bundle.

The regularity of conjugacy for Anosov diffeomorphisms under the assumption of rigidity for Lyapunov exponents of periodic points has been extensively studied by many researchers e.g. [10,15,16,18]. Recently, there are elegant works about the smooth conjugacy for conservative Anosov diffeomorphisms under the assumption of rigidity for Lyapunov exponents with respect to Lebesgue measures e.g. [19,39]. Our Corollary 1.5 and Corollary 1.8 show that we only need to assume spectral rigidity on the unstable bundle to get smooth conjugacy for non-invertible Anosov map with integrable unstable bundles, see Remark 1.7 and [31].

Finally, we would like to mention that Anosov maps on $\mathbb{T}^2$ are a special class of partially hyperbolic maps on surfaces. A series of impressive results on SRB measures and statistical properties for partially hyperbolic maps on surfaces have been obtained, see [8,9,41]. Meanwhile, the classification of partially hyperbolic endomorphisms on $\mathbb{T}^2$ up to leaf conjugacy has also been studied, see [21,22,24]. It will be interesting to classify all partially hyperbolic maps with integrable unstable bundle on surfaces.

Organization of this paper: In Sect. 2, we recall some general properties of Anosov maps and give some useful properties on the assumptions of Theorem 1.1 and Theorem 1.2. In Sect. 3, we prove the "necessary" parts of Theorem 1.1 and Theorem 1.2 with $C^1$ regularity, which state that the existence of integrable unstable bundle implies the spectral rigidity on stable bundle. In Sect. 4, on the assumption of spectral rigidity on stable bundle, we endow an affine metric on each leaf of the lifting stable foliations, which will be useful for the proof of sufficient parts of our theorems. In Sect. 5, we prove
the "sufficient" parts of Theorem 1.1 and Theorem 1.2, which state that the rigidity of periodic stable Lyapunov spectrums implies the existence of integrable unstable bundle and we also prove Corollary 1.5 and Corollary 1.8 in this section.

2. Preliminaries

For short, an Anosov map \( f : M \to M \) is called special, if it has the integrable unstable bundle.

2.1. Global properties. For studying an Anosov map, one can lift it to the universal cover. In fact, Mañé and Pugh proved the following proposition which allows us to observe the dynamics on the universal cover.

**Proposition 2.1** [28]. Let \( \tilde{M} \) be the universal cover of \( M \) and \( F : \tilde{M} \to \tilde{M} \) be a lift of \( f : M \to M \). Then \( f \) is an Anosov map if and only if \( F \) is an Anosov diffeomorphism.

As usual, we define the stable manifolds of the Anosov diffeomorphism \( F \), denoted by \( \tilde{F}^s(x) \),

\[
\tilde{F}^s(x) := \{ y \in \tilde{M} \mid d(F^k(y), F^k(x)) \to 0 \text{ as } k \to +\infty \},
\]

for all \( x \in \tilde{M} \), and the unstable manifolds \( \tilde{F}^u(x) \) by iterating backward. And we define the local (un)stable manifolds with size \( \delta \), denoted by \( \tilde{F}^{s/u}(x, \delta) \),

\[
\tilde{F}^{s/u}(x, \delta) := \{ y \in \tilde{F}^{s/u}(x) \mid d_{\tilde{F}^{s/u}}(x, y) \leq \delta \},
\]

for all \( x \in \tilde{M} \), where \( d_{\tilde{F}^{s/u}}(\cdot, \cdot) \) is induced by the metric on \( \tilde{M} \).

In the rest of this paper, we restrict the manifolds \( M \) to be a \( d \)-torus \( T^d \). Let \( f : T^d \to T^d \) be an Anosov map and \( A : T^d \to T^d \) be its linearization. The following proposition exhibits the equivalent condition of an Anosov map being conjugate with its linearization.

**Proposition 2.2** ([33, 40]). Let \( f \) be an Anosov map on torus, then \( f \) is conjugate to its linearization if and only if \( f \) is special.

We mention that [1] has proved this result on the extra assumption that the stable foliation of \( f \) is minimal. We will give the sketch of the proof of Proposition 2.2 after some preparations.

On the other hand, from the observation of Proposition 2.1, we can expect that the liftings of \( f \) and \( A \) are conjugate. Let \( F : \mathbb{R}^d \to \mathbb{R}^d \) be a lift of \( f \) and \( \bar{A} : \mathbb{R}^d \to \mathbb{R}^d \) be the lift of \( A : \mathbb{T}^d \to \mathbb{T}^d \) induced by the same projection \( \pi : \mathbb{R}^d \to \mathbb{T}^d \). It means that \( \pi \circ F = f \circ \pi \) and \( \pi \circ \bar{A} = A \circ \pi \). For short, we denote \( \bar{A} \) by \( A \) if there is no confusion. The following proposition [1, Proposition 8.2.1 and Proposition 8.4.2] says that we do have a conjugacy between \( F \) and \( A \).

**Proposition 2.3** [1]. Let \( f : \mathbb{T}^d \to \mathbb{T}^d \) be an Anosov map with lifting \( F : \mathbb{R}^d \to \mathbb{R}^d \) and \( A \) be its linearization. There is a unique bijection \( H : \mathbb{R}^d \to \mathbb{R}^d \) such that

1. \( A \circ H = H \circ F \).
2. \( H \) and \( H^{-1} \) are both uniformly continuous.
3. There exists $C > 0$ such that $\|H - Id\| < C$ and $\|H^{-1} - Id\| < C$.

Remark 2.4. Without losing generality, we can always assume that $F(0) = 0$ and $H(0) = 0$.

By proposition 2.2, it is clear that $f$ is special if and only if $H$ is commutative with $\mathbb{Z}^d$-action, namely,

$$H(x + n) = H(x) + n, \quad \forall x \in \mathbb{R}^d \quad \text{and} \quad \forall n \in \mathbb{Z}^d.$$  

Although in general $H$ cannot be commutative with $\mathbb{Z}^d$-action, we will see it can be commutative with $\mathbb{Z}^d$-action as a stable leaf conjugacy.

Notation. Denote the stable/unstable bundles and foliations of $A$ on $\mathbb{T}^d$ by $L^{s/u}$, $\mathcal{L}^{s/u}$ and on $\mathbb{R}^d$ by $\tilde{L}^{s/u}$, $\tilde{\mathcal{L}}^{s/u}$ respectively.

It is clear that $H$ is a stable/unstable leaf conjugacy between $F$ and $A$, namely,

$$H(\tilde{F}^{s/u}) = \tilde{\mathcal{L}}^{s/u} \quad \text{and} \quad H(\tilde{F}^{s/u}(Fx)) = A(\tilde{\mathcal{L}}^{s/u}(Hx)).$$  \hspace{1cm} (2.3)

Indeed, by the topological character of stable/unstable foliations (2.1) for $A$ and $F$, one can get (2.3), directly. Especially, we mention that $\tilde{F}^s$ and $\tilde{F}^u$ admit the Global Product Structure, namely, any two leaves $\tilde{F}^s(x)$ and $\tilde{F}^u(y)$ intersect transversely at a unique point in $\mathbb{R}^d$.

Proposition 2.5. Assume that $H$ is given by Proposition 2.3. Then for every $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$,

$$H(x + n) - n \in \tilde{\mathcal{L}}^s(H(x)) \quad \text{and} \quad H^{-1}(x + n) - n \in \tilde{\mathcal{F}}^s(H^{-1}(x)).$$

Proof. By Proposition 2.3, let $H : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the unique conjugacy with $\|H^{-1} - Id\| < C_0$. Now, we iterate these two points $H^{-1}(x + n) - n$ and $H^{-1}(x)$ forward by $F$. Note that since $f$ and $A$ are homotopic, we have $F^k(x + n) = F^k(x) + A^k n$, for all $x \in \mathbb{R}^d$, $n \in \mathbb{Z}^d$ and $k \in \mathbb{N}$. It follows that

$$d\left(F^k \circ H^{-1}(x), F^k(H^{-1}(x + n) - n)\right),$$

$$= d\left(H^{-1}(A^k x) + A^k n, H^{-1}(A^k x + A^k n)\right),$$

$$\leq d\left(H^{-1}(A^k x) + A^k n, A^k x + A^k n\right) + d\left(A^k x + A^k n, H^{-1}(A^k x + A^k n)\right),$$

$$= d\left(H^{-1}(A^k x), A^k x\right) + d\left(A^k x + A^k n, H^{-1}(A^k x + A^k n)\right) \leq 2C_0.$$  

Let $k$ tend to infinity, the fact that $d\left(F^k(H^{-1}(x) + n), F^k(H^{-1}(x + n))\right)$ is always bounded by a uniform constant is sufficient to prove $H^{-1}(x + n) - n \in \tilde{\mathcal{L}}^s(H^{-1}(x))$.

The proof for $H(x + n) - n \in \tilde{\mathcal{L}}^s(H(x))$ deduces from the fact that $H(\tilde{F}^s) = \tilde{\mathcal{L}}^s$.  \hspace{1cm} $\Box$

The following three propositions are all related to approaching by “special” $\mathbb{Z}^d$-sequences which will be useful in Sect. 5.
Proposition 2.6. Let \( f : \mathbb{T}^d \to \mathbb{T}^d \) be an Anosov map with linearization \( A \), and \( H \) be the conjugacy between its lifting \( F \) and \( A \). There exist \( C > 0 \) and \( \{ \varepsilon_m \} \) with \( \varepsilon_m \to 0 \) as \( m \to +\infty \), such that for every \( x \in \mathbb{R}^d \) and every \( n_m \in \mathbb{Z}^d \) satisfying \( A^{-i}n_m \in \mathbb{Z}^d, \forall 1 \leq i \leq m \), the following two inequalities hold
\[
|H(x + n_m) - H(x) - n_m| < C \cdot \|A\|_{L^1}^m,
\]
and
\[
|H^{-1}(x + n_m) - H^{-1}(x) - n_m| < \varepsilon_m.
\]

Proof. By \( A^{-i}n_m \in \mathbb{Z}^d \) for all \( 1 \leq i \leq m \), we have \( F^{-i}(x + n_m) = A^{-i}n_m \in \mathbb{Z}^d \), for all \( 1 \leq i \leq m \). By \( |H - id| < C_0 \), we have
\[
\|H(A^{-m}x + A^{-m}n_m) - H(F^{-m}x + A^{-m}n_m)\| \leq 2C_0.
\]
By Proposition 2.5, one has \( H(x) + n \in \mathcal{L}^s(H(x) + n) \), for every \( x \in \mathbb{R}^d \) and \( n \in \mathbb{Z}^d \).

Hence,
\[
d\left(A^m \left(H(F^{-m}x + A^{-m}n_m)\right), A^m \left(H(F^{-m}x + A^{-m}n_m)\right)\right) \leq 2C_0 \cdot \|A\|_{L^1}^m.
\]
That is
\[
|H(x + n_m) - H(x) - n_m| < 2C_0 \cdot \|A\|_{L^1}^m.
\]
On the other hand, by the uniform continuity of \( H^{-1} \), there exists \( \varepsilon_m \) satisfying \( \varepsilon_m \to 0 \) as \( m \to +\infty \) such that if \( d(x, y) < 2C_0 \cdot \|A\|_{L^1}^m \), then \( d(H^{-1}(x), H^{-1}(y)) < \varepsilon_m \). Thus,
\[
d\left(x + n_m, H^{-1}(H(x) + n_m)\right) < \varepsilon_m.
\]
Equivalently, for every \( x \in \mathbb{R}^d \), we have
\[
|H^{-1}(x + n_m) - H^{-1}(x) - n_m| < \varepsilon_m.
\]
\[
\square
\]

The following proposition is a corollary of Proposition 2.6. It says that although the \( F \)-invariant foliation may not be commutative with \( \mathbb{Z}^d \)-actions, it can "almost" be commutative with \( n_m \)-actions if \( H \) maps it to an \( A \)-invariant linear foliation.

Proposition 2.7. Under the assumption of Proposition 2.6 and assume that \( \tilde{\mathcal{L}} \) is an \( A \)-invariant linear foliation and the sequence \( \{n_m\}_{m \in \mathbb{N}} \subset \mathbb{Z}^d \) satisfies \( n_m \in A^m\mathbb{Z}^d \). Let \( \tilde{\mathcal{F}} := H^{-1}(\tilde{\mathcal{L}}) \) be an \( F \)-invariant foliation. Then for every \( x \in \mathbb{R}^d \) and every \( R > 0 \),
\[
\tilde{\mathcal{F}}_R(x + n_m) - n_m \to \tilde{\mathcal{F}}_R(x), \text{ in } C^0\text{-topology},
\]
as \( m \to +\infty \), where the local topological manifold \( \tilde{\mathcal{F}}_R(x) \) is the component of \( \{y \in \tilde{\mathcal{F}}(x) \mid d(x, y) \leq R\} \) containing \( x \).
**Proof.** By Proposition 2.6, when \( m \to +\infty \),
\[
d_H \left( H \left( \tilde{F}_R(x + n_m) - n_m \right), \ H \left( \tilde{F}_R(x + n_m) \right) - n_m \right) \to 0, \tag{2.4}
\]
where \( d_H(\cdot, \cdot) \) is Hausdorff distance. Note that \( \mathcal{F} := H \left( \tilde{F}_R(x + n_m) \right) \) is a local leaf on \( \tilde{L}(H(x + n_m)) \). Since \( \tilde{L} \) is commutative with \( \mathbb{Z}^d \)-actions, the set \( (\mathcal{F} - n_m) \) is a copy of \( \mathcal{F} \) on \( \tilde{L}(H(x + n_m) - n_m) \). Again, by Proposition 2.6, as \( m \to +\infty \),
\[
d_H \left( \mathcal{F} - n_m, \ \mathcal{F} - H(x + n_m) + H(x) \right) \to 0. \tag{2.5}
\]
Then, combining (2.4) and (2.5), one has
\[
d_H \left( H \left( \tilde{F}_R(x + n_m) - n_m \right), \ \mathcal{F} - H(x + n_m) + H(x) \right) \to 0.
\]
By the uniform continuity of \( H^{-1} \), it follows that
\[
d_H \left( \tilde{F}_R(x + n_m) - n_m, \ H^{-1} \left( \mathcal{F} - H(x + n_m) + H(x) \right) \right) \to 0.
\]
Since \( (\mathcal{F} - H(x + n_m) + H(x)) \) is a copy of \( \mathcal{F} \) on \( \tilde{L}(H(x)) \), the set \( H^{-1} \left( \mathcal{F} - H(x + n_m) + H(x) \right) \) is a local leaf on \( \tilde{F}(x) \).

Moreover, the size of \( H^{-1} \left( \mathcal{F} - H(x + n_m) + H(x) \right) \) tends to \( R \) as \( m \to +\infty \). Indeed, for any point \( y \in \tilde{F}(x) \), one has
\[
y \in H^{-1} \left( \mathcal{F} - H(x + n_m) + H(x) \right) \iff d(H^{-1}(H(y) + H(x + n_m) - H(x)) - n_m, x) < R,
\]
restricted on a connected component. Note that by Proposition 2.6 and the uniform continuity of \( H^\pm \),
\[
d(\ H^{-1}(H(y) + H(x + n_m) - H(x)) - n_m, y) \to 0, \ m \to +\infty.
\]
It follows that as \( m \to +\infty \)
\[
d(y, x) < R \iff d(\ H^{-1}(H(y) + H(x + n_m) - H(x)) - n_m, x) < R.
\]
This implies that \( H^{-1} \left( \mathcal{F} - H(x + n_m) + H(x) \right) \to \tilde{F}_R(x) \) as \( m \to +\infty \) and hence \( \tilde{F}_R(x + n_m) - n_m \to \tilde{F}_R(x) \) in \( C^0 \)-topology. □

**Remark 2.8.** The foliation \( \tilde{F} \) in Proposition 2.7 can be the unstable foliation or the center foliation of \( F \), where \( F : \mathbb{R}^d \to \mathbb{R}^d \) is a lifting of an Anosov map on torus. We mention in advance that in the case of center foliations and unstable foliations, one has \( \tilde{F}_R(x + n_m) - n_m \to \tilde{F}_R(x) \) in \( C^1 \)-topology, also see Remark 2.16.

Recall that a foliation \( \mathcal{F} \) on \( \mathbb{T}^d \) is called *minimal*, if its every leaf is dense. It is clear that if \( A \) is irreducible, then every \( A \)-invariant linear foliation on \( \mathbb{T}^d \) is minimal (a complete proof is available in [13]). The following proposition actually says that the projection of each leaf of \( F \)-invariant foliation onto \( \mathbb{T}^d \) is dense if \( H \) maps it to an \( A \)-invariant linear foliation.
Proposition 2.9. Under the assumption of Proposition 2.6 and assume that $A$ is irreducible and $\mathcal{L}$ is an $A$-invariant linear foliation. Let $\mathcal{F} = H^{-1}(\mathcal{L})$ be a foliation on $\mathbb{R}^d$. Then for any $x, y \in \mathbb{R}^d$,

1. There exist $x_m \in \mathcal{F}(x)$ and $n_m \in A^m \mathbb{Z}^d$, such that $(x_m + n_m) \to y$ as $m \to +\infty$.
2. There exist $z_m \in \mathcal{F}(x)$ and $n_m \in A^m \mathbb{Z}^d$, such that $(z_m + n_m) \to y$ as $m \to +\infty$.

Proof. Since $A$ is irreducible, the set $\{\mathcal{L}(0) + \mathbb{Z}^d\}$ is dense in $\mathbb{R}^d$. Fix $m \in \mathbb{N}$, one has that the set $A^m \{\mathcal{L}(0) + \mathbb{Z}^d\}$ is dense in $\mathbb{R}^d$. It follows that $\{\mathcal{L}(0) + A^m \mathbb{Z}^d\}$ is dense in $\mathbb{R}^d$, since $A^m \mathcal{L}(0) = \mathcal{L}(0)$. This completes the proof for the first item.

Now, applying the first item for points $H(x)$ and $H(y)$, we can take $n_m \in A^m \mathbb{Z}^d$ and $x'_m \in \mathcal{L}(H(x))$ with $x'_m + n_m \to H(y)$. Let $z_m = H^{-1}(x'_m) \in \mathcal{F}(x)$. By Proposition 2.6, one has $d(H(z_m + n_m), H(z_m) + n_m) \to 0$ as $m \to +\infty$. By the uniform continuity of $H^{-1}$, we get $(z_m + n_m) \to y$. $\square$

Now we can give the sketch of the proof of Proposition 2.2. We refer readers to [33] for more details. Note that when $f$ is special, the fact that $\bigcup_{n \in \mathbb{Z}^d} \mathcal{F}^u(n)$ is dense in $\mathbb{R}^d$ is also needed in [33] where the authors get it by different methods from Proposition 2.9.

Sketch of Proof of Proposition 2.2. It is clear that if $f$ is conjugate to its linearization, then $f$ is special, since the conjugacy, as a homeomorphism, preserves the unstable manifolds (see [1, Remark 6.8.5]).

Assume that $f$ is special, then we have that

$$\mathcal{F}^u(x + n) = \mathcal{F}^u(x) + n, \quad \forall x \in \mathbb{R}^d \text{ and } \forall n \in \mathbb{Z}^d.$$ (2.6)

Let $H : \mathbb{R}^d \to \mathbb{R}^d$ is given by Proposition 2.3. Denote $H_n(x) = H(x - n) + n$ for any $x \in \mathcal{F}^u(n)$ and any $n \in \mathbb{Z}^d$. By (2.6), it is easy to get

- $H_n(x) \in \mathcal{F}^u(n)$ for any $x \in \mathcal{F}^u(n)$ and any $n \in \mathbb{Z}^d$.
- $H_n(x) = H_m(x)$ for $x \in \mathcal{F}^u(n) \cap \mathcal{F}^u(m)$ and some $n, m \in \mathbb{Z}^d$.

This allows us to define the map

$$\overline{H} : \bigcup_{n \in \mathbb{Z}^d} \mathcal{F}^u(n) \longrightarrow \bigcup_{n \in \mathbb{Z}^d} \mathcal{L}^u(n),$$

by $\overline{H}(x) = H(x - n) + n$. It is not hard to check that

1. $A \circ \overline{H}(x) = \overline{H} \circ F(x)$, for any $x \in \bigcup_{n \in \mathbb{Z}^d} \mathcal{F}^u(n)$.
2. $\overline{H}(n) = n$, for all $n \in \mathbb{Z}^d$.
3. There exists $C > 0$ such that $\|\overline{H} - \text{Id}|_{\bigcup_{n \in \mathbb{Z}^d} \mathcal{F}^u(n)}\| < C$.

Moreover, $\overline{H}$ is uniformly continuous [33, Proposition 3.6]. Note that $\bigcup_{n \in \mathbb{Z}^d} \mathcal{F}^u(n)$ is dense in $\mathbb{R}^d$ by Proposition 2.9. Hence $\overline{H}$ can be extended to $\mathbb{R}^d$ continuously. By the uniqueness stated in Proposition 2.3, we get that $\overline{H} = H$ and $H(n) = n$ for all $n \in \mathbb{Z}^d$.

For any $x \in \bigcup_{m \in \mathbb{Z}^d} \mathcal{F}^u(m)$ and $n \in \mathbb{Z}^d$, let $x \in \mathcal{F}^u(n_x)$ where $n_x \in \mathbb{Z}^d$. We have that $H(x + n) \in \mathcal{L}^u(H(n_x + n)) = \mathcal{L}^u(n_x + n)$ and $H(x) + n \in \mathcal{L}^u(H(n_x)) + n = \mathcal{L}^u(n_x) + n = \mathcal{L}^u(n_x + n)$. It follows from Proposition 2.5 and the Global Product
Structure that \( H(x + n) = H(x) + n \). Again by the density of \( \bigcup_{m \in \mathbb{Z}^d} \mathcal{F}^u(m) \), one has \( H(x + n) = H(x) + n \) for any \( x \in \mathbb{R}^d \) and \( n \in \mathbb{Z}^d \). This implies that \( H \) can descend to \( \mathbb{T}^d \), namely \( h(x) := \pi \circ H \circ \pi^{-1} : \mathbb{T}^d \to \mathbb{T}^d \) is a conjugacy between \( f \) and its linearization \( A \).

To end this subsection, we prove that the preimage set of an irreducible and non-invertible toral endomorphism (may not be Anosov) is becoming dense exponentially.

**Proposition 2.10.** Let \( A \in M_d(\mathbb{Z}) \cap \text{GL}_d(\mathbb{R}) \) be irreducible over \( \mathbb{Q} \). It induces a torus endomorphism \( A : \mathbb{T}^d \to \mathbb{T}^d \). Then there exists \( C > 0 \) such that for every \( k \geq 1 \) and every \( x_0 \in \mathbb{T}^d \), the \( k \)-preimage set of \( x_0 \)

\[
\{ x \in \mathbb{T}^d | A^k(x) = x_0 \}
\]

is \( C |\det(A)|^{-k/d} \)-dense in \( \mathbb{T}^d \).

**Proof.** Consider the \( d \)-dimensional real vector space

\[
V = \text{span}_{\mathbb{R}} \{ I_d, A, ..., A^{d-1} \}
\]

and the lattice \( V_\mathbb{Z} := V \cap M_d(\mathbb{Z}) \) in \( V \). We claim that nonzero matrices in \( V_\mathbb{Z} \) are invertible in \( \text{GL}_d(\mathbb{R}) \). In fact, if \( M \in V_\mathbb{Z} - \{ 0 \} \), then there is a nonzero rational polynomial \( m \in \mathbb{Q}[x] \) with \( \deg(m) \leq d - 1 \) such that \( M = m(A) \). Note that \( m \) and the characteristic polynomial \( \chi \) of \( A \) are coprime over \( \mathbb{Q} \), since \( A \) is irreducible. So there exist \( g, h \in \mathbb{Q}[x] \) such that \( mg + \chi h = 1 \). It follows that \( m(A)g(A) = I_d \). So \( M = m(A) \) is invertible.

Let \( \mathcal{H} \subset V \) be a compact convex symmetric subset with \( \text{vol}(\mathcal{H}) \geq 2^d \text{vol}(V/V_\mathbb{Z}) \), and let

\[
C = \sqrt{d} \cdot \sup_{K \in \mathcal{H}, v \in \mathbb{R}^d, \|v\|=1} \|Kv\|.
\]

We prove that for every \( k \geq 1 \), the \( k \)-preimage set of every \( x_0 \in \mathbb{T}^d \) is \( C |\det(A)|^{-k/d} \)-dense in \( \mathbb{T}^d \). It suffices to prove that \( A^{-k} \mathbb{Z}^d \) is \( C |\det(A)|^{-k/d} \)-dense in \( \mathbb{R}^d \).

Let \( L : V \to V \) be the linear map defined by \( L(X) = AX \). The matrix of \( L \) relative to the basis \( \{ I_d, A, ..., A^{d-1} \} \) of \( V \) is the companion matrix of \( \chi \). So \( \det(L) = \det(A) \). It follows that the compact convex symmetric subset \( |\det(A)|^{-k/d} \cdot L^k(\mathcal{H}) \) has volume

\[
\text{vol} \left( |\det(A)|^{-k/d} \cdot L^k(\mathcal{H}) \right) = |\det(A)|^{-k} \cdot \text{vol} \left( L^k(\mathcal{H}) \right) = |\det(A)|^{-k} |\det(L)|^k \cdot \text{vol}(\mathcal{H}) = \text{vol}(\mathcal{H}) \geq 2^d \text{vol}(V/V_\mathbb{Z}).
\]

By Minkowski’s convex body theorem, it contains a nonzero matrix in \( V_\mathbb{Z} \). Namely, there exist \( K \in \mathcal{H} \) and \( M \in V_\mathbb{Z} - \{ 0 \} \) such that \( |\det(A)|^{-k/d} A^{-k} K = M \). Note that \( M \) is invertible, so \( K \) is also invertible. For \( r > 0 \), let \( B(r) \subset \mathbb{R}^d \) denote the open ball with radius \( r \) centered at the origin. Then

\[
\mathbb{R}^d = |\det(A)|^{-k/d} K \mathbb{R}^d = |\det(A)|^{-k/d} K (\mathbb{Z}^d + B(\sqrt{d})) = A^{-k} M \mathbb{Z}^d + |\det(A)|^{-k/d} K B(\sqrt{d}) \subseteq A^{-k} \mathbb{Z}^d + B(C |\det(A)|^{-k/d}).
\]

This means that \( A^{-k} \mathbb{Z}^d \) is \( C |\det(A)|^{-k/d} \)-dense in \( \mathbb{R}^d \). \( \square \)
2.2. Dominated splitting on the inverse limit space. Now we introduce the dynamics on the inverse limit space. Note that the inverse limit space has compactness which the universal cover lacks.

Firstly, we clear the definition of inverse limit space. Let \((M, d)\) be a compact metric space and \(M^\mathbb{Z} := \{(x_i) \mid x_i \in M, \forall i \in \mathbb{Z}\}\) be the product topological space. \(M^\mathbb{Z}\) is compact by Tychonoff theorem and it can be metrizable by the metric
\[
\tilde{d}((x_i), (y_i)) = \sum_{j=-\infty}^{+\infty} \frac{d(x_i, y_i)}{2^{|j|}}.
\]

Let \(\sigma: M^\mathbb{Z} \to M^\mathbb{Z}\) be the (left) shift homeomorphism by \((\sigma(x_i))_j = x_{j+1}\), for all \(j \in \mathbb{Z}\). For a continuous map \(f: M \to M\), the inverse limit space of \(f\) is
\[
M_f := \{ (x_i) \mid x_i \in M \text{ and } f(x_i) = x_{i+1}, \forall i \in \mathbb{Z} \}.
\]

With the metric \(\tilde{d}(\cdot, \cdot)\), the inverse limit space \((M_f, \tilde{d})\) is a closed subset of \((M^\mathbb{Z}, \tilde{d})\). So it is a compact metric space. It is clear that \((M_f, \tilde{d})\) is \(\sigma\)-invariant.

**Definition 2.11** ([36]). A \(C^1\) local diffeomorphism \(f: M \to M\) is called Anosov map, if there exist constants \(C > 0\) and \(0 < \mu < 1\) such that, for every \(\bar{x} = (x_i) \in M_f\), there exists a hyperbolic splitting
\[
T_{x_i}M = E^s(x_i, \bar{x}) \oplus E^u(x_i, \bar{x}), \quad \forall i \in \mathbb{Z},
\]
which is \(Df\)-invariant
\[
D_{x_i}f(E^s(x_i, \bar{x})) = E^s(x_{i+1}, \bar{x}) \quad \text{and} \quad D_{x_i}f(E^u(x_i, \bar{x})) = E^u(x_{i+1}, \bar{x}), \quad \forall i \in \mathbb{Z},
\]
and for all \(n > 0\) the following estimates hold:
\[
\|D_{x_i} f^n(v)\| \leq C \mu^n \|v\|, \quad \forall v \in E^s(x_i, \bar{x}), \quad \forall i \in \mathbb{Z},
\]
\[
\|D_{x_i} f^n(v)\| \geq C^{-1} \mu^{-n} \|v\|, \quad \forall v \in E^u(x_i, \bar{x}), \quad \forall i \in \mathbb{Z}.
\]

We extend the hyperbolic splitting on the inverse limit space to the dominated splitting case. We say a local diffeomorphism \(g: M \to M\) admits a dominated splitting, if there exist constants \(C > 0\) and \(\mu \in (0, 1)\) such that for any \(\bar{x} = (x_i) \in M_g\) there exists a \(Dg\)-invariant splitting
\[
TM = E_1(x_i, \bar{x}) \oplus E_2(x_i, \bar{x}) \oplus \cdots \oplus E_n(x_i, \bar{x}), \quad \forall i \in \mathbb{Z},
\]
and for all \(n \in \mathbb{N}\), any unit vectors \(u \in E_j(x_i, \bar{x})\) and \(v \in E_{j+1}(x_i, \bar{x})\) (\(1 \leq j \leq m - 1\)),
\[
\|D_{x_i} g^n u\| \leq C \mu^n \|D_{x_i} g^n v\|, \quad \forall n \in \mathbb{N}.
\]

Let \(A: \mathbb{T}^d \to \mathbb{T}^d\) be a linear Anosov map. It is clear that \(A\) admits a dominated splitting. We further assume that \(A\) admits the finest (on stable bundle) dominated splitting,
\[
T\mathbb{T}^d = L^s_1 \oplus L^s_2 \oplus \cdots \oplus L^s_k \oplus L^u, \tag{2.7}
\]
where \(\dim L^s_i = 1, 1 \leq i \leq k\), such that
\[
0 < |\mu^s_1(A)| < |\mu^s_2(A)| < \ldots < |\mu^s_k(A)| < 1 < |\mu^u_{\min}(A)|, \tag{2.8}
\]
where \(\mu^s_i(A)\) is the eigenvalue with respect to \(L^s_i\) and \(\mu^u_{\min}(A) = m(A|_{L^u})\) the co-norm of \(A\) restricted on \(L^u\). Denote by \(\mu^u_{\max}(A) = \|A\|\), the norm of \(A\). And denote by \(\lambda^s_i(A) = \log|\mu^s_i(A)|\) the stable Lyapunov exponent of the subbundle \(L^s_i\).
Notation. We use the following notations to denote the joint bundles of $A$ which are analogical for $L$, $L$, $\tilde{L}$ and the bundles and foliations of $f$ and $F$, if they are well defined.

1. $L^s_{(i,j)} := L^s_i \oplus L^s_{i+1} \oplus \cdots \oplus L^s_j$, $1 \leq i \leq j \leq k$.
2. $L^{s,u}_{(i,j)} := L^s_i \oplus L^s_{i+1} \oplus \cdots \oplus L^s_j \oplus L^u$, $1 \leq i \leq j \leq k$.
3. $L^{s,u}_{(i,j)} = L^s_i \oplus L^{s,u}_{i,j} = L^s_i \oplus L^u$, $1 \leq i \leq k$.

Proposition 2.12. Let $A \in M_d(\mathbb{Z}) \cap GL_d(\mathbb{R})$ induce a toral Anosov map with the finest (on stable bundle) special dominated splitting satisfying (2.7) and (2.8). Then there exists a $C^1$ neighborhood $\mathcal{U} \subset C^1(\mathbb{T}^d)$ of $A$ such that for all $f \in \mathcal{U}$ and all $\bar{x} = (x_n) \in \mathbb{T}_f^d$, there exists a hyperbolic dominated splitting

\[ T_{x_n} \mathbb{T}^d = E^s_1(x_n, \bar{x}) \oplus \cdots \oplus E^s_k(x_n, \bar{x}) \oplus E^u(x_n, \bar{x}), \quad \forall n \in \mathbb{Z}, \]

where $\text{dim} E^s_i = 1, 1 \leq i \leq k$ and $E^s_i(x_n, \bar{x}) \oplus \cdots \oplus E^s_k(x_n, \bar{x}) = E^s(x_n, \bar{x})$. And the bundles $E^s_i(x_n, \bar{x})$ and $E^u(x_n, \bar{x})$ are continuous with $\bar{x}$. Especially, there exist bundles $E^s_{(1,i)}(1 \leq i \leq k)$ defined on $\mathbb{T}^d$ such that

\[ E^s_1 \subset E^s_{(1,2)} \subset \cdots \subset E^s_{(1,k-1)} \subset E^s_{(1,k)} = E^s. \tag{2.9} \]

Moreover, for any $\alpha \in (0, \frac{\pi}{2})$ and $\delta > 0$, there exists $\mathcal{U} \subset C^1(\mathbb{T}^d)$ such that for every $f \in \mathcal{U}$ and $\bar{x} = (x_n) \in \mathbb{T}_f^d$ and $1 \leq i \leq k$,

\[ \angle(E^s_i(x_0, \bar{x}), L^s_i(x_0)) \leq \alpha \quad \text{and} \quad \| Df \|_{E^s_i(x_0, \bar{x})} \| \in [\mu^s_i(A) - \delta, \mu^s_i(A) + \delta]. \tag{2.10} \]

And,

\[ \angle(E^u(x_0, \bar{x}), L^u(x_0)) \leq \alpha \quad \text{and} \quad \| Df \|_{E^u(x_0, \bar{x})} \| \in [\mu^u_{\max}(A) - \delta, \mu^u_{\max}(A) + \delta]. \tag{2.11} \]

Proof. We use invariant cone-fields to complete this proof. One may find more details in [3, Appendix B.1] and [7, Section 2.2]. Let $S \oplus U$ be a dominated splitting on $T \mathbb{T}^d$ of $A$ and $C_\alpha$ be an $A$-invariant cone-fields contain $U$ with size $\alpha$. That is

\[ C_\alpha(x) = \{ v = v_S + v_U \in T_x \mathbb{T}^d : \|v_S\| \leq \alpha \|v_U\| \} \quad \text{and} \quad DA(C_\alpha(x)) \subset C_{\mu \cdot \alpha}(x), \]

for some $\mu < 1$. So, there exists a $C^1$-neighborhood $\mathcal{U}$ of $A$ such that for any $f \in \mathcal{U}$ and any $x \in \mathbb{T}^d$, one has $Df(C_\alpha(x)) \subset \text{int}(C_\alpha(f(x)))$. This implies the existence of dominated splitting (see [7, Section 2.2]). We mention that we can make a larger perturbation as long as $C_\alpha$ can be contracted into itself by finite iterations of $Df$. And let $C_\alpha^*$ be the closure of the complement of $C_\alpha$. It is clear that $C_\alpha^*$ is also a cone-field which is contracted by $Df^{-1}$, if the preimage is given. Fix an $f$-orbit $\bar{x} = (x_n) \in \mathbb{T}_f^d$ and let

\[ S_f(x_0, \bar{x}) := \bigcap_{n \leq 0} Df^{-n}(C_\alpha^*(x_n)), \quad U_f(x_0, \bar{x}) := \bigcap_{n \geq 0} Df^n(C_\alpha(x_{-n})). \]

Then $S_f \oplus U_f$ give a dominated splitting on the inverse limits space $\mathbb{T}_f^d$. Note that $S_f(x_0, \bar{x})$ is independent of the choice of orbits for $x_0$. 

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For finding out $E^s_i(x_0, \bar{x})$, we consider these two dominated splittings for $A$ 

$$L^s_{(1,i)} \oplus L^{s,u}_{(i+1,k)}$$

and 

$$L^s_{(1,i-1)} \oplus L^{s,u}_{(i,k)}.$$ 

So, we can get the following two dominated splittings under the $C^1$-perturbation, 

$$E^s_{(1,i)}(x_0, \bar{x}) \oplus E^{s,u}_{(i+1,k)}(x_0, \bar{x})$$

and 

$$E^s_{(1,i-1)}(x_0, \bar{x}) \oplus E^{s,u}_{(i,k)}(x_0, \bar{x}).$$

Hence, we get 

$$E^s_i(x_0, \bar{x}) = E^s_{(1,i)}(x_0, \bar{x}) \cap E^{s,u}_{(i,k)}(x_0, \bar{x}).$$

It is clear that this dominated splitting is continuous with respect to orbits. Meanwhile, 

$$E^s_{(1,i)}(x_0, \bar{x})$$

only depends on $x_0$. It follows that the bundle $E^s_{(1,i)}$ is well defined on $\mathbb{T}^d$.

The previous proof also allows the control (2.10) and (2.11) for bundles of $f$. Indeed, fix $1 \leq i \leq k$, let $f$ be $C^1$ close to $A$, denote by $\varepsilon := d_{C^1}(f, A)$. By the view of cone-field, one has 

$$\angle(E^s_i(x_0, \bar{x}), L^s_i(x_0)) \leq \beta = \beta(\varepsilon)$$

for any $\bar{x} \in \mathbb{T}^d_f$ and some $\beta(\varepsilon) > 0$, where $\beta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Taking a unit vector $v \in E^s_i(x_0, \bar{x})$, one has 

$$-\varepsilon + \|Av\| \leq \|Dx_0fv\| \leq \varepsilon + \|Av\|$$

and it follows that 

$$-\varepsilon + (\cos \beta) \cdot \mu^s_i(A) - (\sin \beta) \cdot \mu^u_{\text{max}}(A) \leq \|Dx_0fv\| \leq \varepsilon + (\cos \beta) \cdot \mu^s_i(A) + (\sin \beta) \cdot \mu^u_{\text{max}}(A).$$

Hence for given $\alpha > 0$ and $\delta > 0$ we can choose $\varepsilon$ small such that (2.10) holds when $d_{C^1}(f, A) < \varepsilon$. And it is similar to find the neighborhood $\mathcal{U}$ to guarantee (2.11). \qed

**Remark 2.13.** In general, the weak stable bundles rely on the orbits. We refer to [6, Theorem B] where the authors proved that the absence of center bundle is dense in the set of non-invertible partially hyperbolic endomorphisms on a certain manifold. Here we briefly give an example. Let $A : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a non-invertible linear Anosov maps with hyperbolic dominated splitting 

$$L^{ss} \oplus L^{us} \oplus L^u,$$

where each subbundle is one-dimensional and $L^s = L^{ss} \oplus L^{us}$. Let $w \neq 0 \in \mathbb{T}^3$ such that $A(w) = A(0) = 0$. Let $0 \notin U \subset \mathbb{T}^3$ be a small open ball of $w$ such that there exists an $A$-orbit $\bar{w} = (w_i)$ of $w_0 = w$ with $w_i \notin U$ for all $i < 0$. Let $R_0 : \mathbb{T}^3 \rightarrow \mathbb{T}^3$ be a $C^\infty$ map supported on $U$ such that it is a small rotation restricted on each slice of $\mathbb{T}^3 \cap U$. Let 

$$g = A \circ R_0 : \mathbb{T}^3 \rightarrow \mathbb{T}^3.$$ 

It is clear that $g$ is Anosov and the stable bundle $E^s(x)$ of $g$ at any point $x \in \mathbb{T}^3$ is equal to $L^s(x)$. Note that in the view of cone-field, for any orbit $\bar{x} = (x_i) \in \mathbb{T}^3_g$, one can find the weak stable bundle $E^{us}(x_0, \bar{x})$ via the cone-field restricted on $E^s$, namely, there exists a cone-field $C^u_{\alpha}(x) \subset E^s(x)$ such that 

$$Dg(C^u_{\alpha}(x_i)) \subset \text{int}(C^u_{\alpha}(x_{i+1})), \forall i \in \mathbb{Z} \quad \text{and} \quad E^{us}(x_0, \bar{x}) = \bigcap_{i \geq 0} Dg^i(C^u_{\alpha}(x_{-i})).$$

Since $w_i \notin U$ for all $i < 0$, we have 

$$E^{us}(w, \bar{w}) = \bigcap_{i \geq 0} DA^i(C^u_{\alpha}(w_{-i})) = L^{us}(w).$$

And by the rotation, 

$$E^{us}(0, \sigma^g(\bar{w})) = DgE^{us}(w, \bar{w}) \neq E^{us}(0, (0)) = L^{us}(0),$$

where $(0) \in \mathbb{T}^3_g$ is the fixed point of $\sigma^g$. Hence the weak stable bundles of $g$ depends on the orbits.
We say a local diffeomorphism \( g : M \to M \) admits a *special dominated splitting*

\[ TM = E_1 \oplus E_2 \oplus \cdots \oplus E_m, \]

if for every \( i = 1, \ldots, m \), each subbundle \( E_i \) is \( Dg \)-invariant and there exist \( C > 0 \), \( 0 < \mu < 1 \) such that for any \( x \in M \) and any unit vectors \( u \in E_i(x) \) and \( v \in E_{i+1}(x) \),

\[ \|D_x g^n u\| \leq C \mu^n \|D_x g^n v\|, \quad \forall n \in \mathbb{N}. \]

As the Anosov case, we still use "special" to note that this splitting is independent with the negative orbits, i.e., it is well defined on \( TM \).

### 2.3. Foliations on the universal cover.

In this subsection, we always assume that \( A \in M_d(\mathbb{Z}) \cap GL_d(\mathbb{R}) \) admits the finest (on stable bundle) dominated splitting (see (2.7) and (2.8)). It is clear that there exists a \( C^1 \)-neighborhood \( \mathcal{U} \) of \( A \) such that for every \( f \in \mathcal{U} \), its lifting \( F : \mathbb{R}^d \to \mathbb{R}^d \) admits dominated splitting

\[ T\mathbb{R}^d = \tilde{E}^s_1 \oplus \tilde{E}^s_2 \oplus \cdots \oplus \tilde{E}^s_k \oplus \tilde{E}^u. \]

Let \( H : \mathbb{R}^d \to \mathbb{R}^d \) be the unique conjugacy between \( F \) and \( A \) guaranteed by Proposition 2.3.

Let \( \pi : \mathbb{R}^d \to \mathbb{T}^d \) be the natural projection such that \( \pi \circ F = f \circ \pi \). Note that any \( f \)-orbit \( \bar{x} \in \mathbb{T}^d_f \) can be approached by \( F \)-orbits. Hence for every \( 1 \leq i \leq j \leq k \), one can get

\[ \bigcup_{\bar{x} \in \mathbb{T}^d_f, (\bar{x})_0 = x_0} E^s_{(i, j)}(x_0, \bar{x}) = \bigcup_{\pi(y) = x_0} D_y \pi \left( \tilde{E}^s_{(i, j)}(y) \right), \quad \forall x_0 \in \mathbb{T}^d. \quad (2.12) \]

It is similar to \( E^u(x_0, \bar{x}) \). We refer to [32, Proposition 2.5] for more details. This projection allows us to get properties of \( F \) from ones of \( \mathbb{T}^d_f \). Especially, we have the following remarks.

**Remark 2.14.** By Proposition 2.12, we can choose \( \mathcal{U} \subset C^1(\mathbb{T}^d) \) small enough such that

\[ \mu^s_i(A) - \delta \leq \|DF|_{\tilde{E}^s_i(x)}\| \leq \mu^s_i(A) + \delta \quad \text{and} \quad \angle \left( \tilde{E}^s_i(x), \tilde{E}^s_i(x) \right) < \alpha < \frac{\pi}{2}, \quad (2.13) \]

for every \( x \in \mathbb{R}^d \) and \( 1 \leq i \leq k \), where \( \delta = \frac{1}{3} \min \{1 - \mu^s_k(A), \mu^s_{i+1}(A) - \mu^s_i(A) : 1 \leq i \leq k - 1\} \).

**Remark 2.15.** The distributions \( \tilde{E}^s_{(i, j)}(1 \leq i \leq j \leq k) \) and \( \tilde{E}^u \) are all uniformly continuous. If we further assume that \( f \) is \( C^{1+\alpha} \), then these distributions are all Hölder continuous on \( T\mathbb{R}^d \). One can get this from [35, Theorem 2.3] with the fact that the angle \( \angle(\tilde{E}_1, \tilde{E}_2) \) is uniformly away from 0, where \( \tilde{E}_1, \tilde{E}_2 \) are two different bundles in \( \{ \tilde{E}^u, \tilde{E}^s_i : 1 \leq i \leq k \} \) (also see Remark 2.18).
Remark 2.16. Let the sequence \( \{n_m\}_{m \in \mathbb{N}} \subset \mathbb{Z}^d \) satisfy \( n_m \in A^m \mathbb{Z}^d \) and \( \tilde{\mathcal{F}} \) be an \( F \)-invariant foliation tangent to \( \tilde{E} \), where \( \tilde{E} \) is a joint bundle of some subbundles in \( \\{ \tilde{E}^u, \tilde{E}_i^s : 1 \leq i \leq k \} \). If for every \( x \in \mathbb{R}^d \) and every \( R > 0 \),
\[
\tilde{\mathcal{F}}_R(x + n_m) - n_m \to \tilde{\mathcal{F}}_R(x), \quad \text{in } C^0 - \text{topology},
\]
(2.14)
as \( m \to +\infty \), in which the local manifold is defined as Proposition 2.7 where (2.14) holds. Then one has
\[
\tilde{\mathcal{F}}_R(x + n_m) - n_m \to \tilde{\mathcal{F}}_R(x), \quad \text{in } C^1 - \text{topology}.
\]
(2.15)
Indeed, the projections of orbits \( \text{Orb}_F(x) \) and \( \text{Orb}_F(x + n_m) \) onto \( \mathbb{T}_f \) get closer as \( m \to +\infty \). Hence by Proposition 2.12, the distribution \( \tilde{E} \) of \( \tilde{\mathcal{F}} \) has that for all \( x \in \mathbb{R}^d \)
\[
DT_{n_m}^{-1} \tilde{E}(x + n_m) \to \tilde{E}(x), \quad m \to +\infty,
\]
(2.16)
and it converges uniformly with respect to \( x \), where \( T_n : \mathbb{R}^d \to \mathbb{R}^d \) \((n \in \mathbb{Z}^d)\) is the translation
\[
T_n(x) = x + n, \quad \forall x \in \mathbb{R}^d \text{ and } n \in \mathbb{Z}^d.
\]
Then (2.15) follows from (2.14) and (2.16).

We say \( \tilde{\mathcal{F}} \) is a \textit{quasi-isometric} foliation on \( \mathbb{R}^d \), if there exist contants \( a, b > 0 \) such that
\[
d_{\tilde{\mathcal{F}}}(x, y) \leq a \cdot d(x, y) + b, \quad \forall x \in \mathbb{R}^d, y \in \tilde{\mathcal{F}}(x).
\]
(2.17)
A foliation \( \tilde{\mathcal{F}} \) defined on \( \mathbb{R}^d \) is called \textit{\( \mathbb{Z}^d \)-periodic} (equivalently, \textit{commutative with \( \mathbb{Z}^d \)-actions}), if
\[
\tilde{\mathcal{F}}(x + n) = \tilde{\mathcal{F}}(x) + n, \quad \forall x \in \mathbb{R}^d, \quad \forall n \in \mathbb{Z}^d.
\]

Similarly, a \textit{\( \mathbb{Z}^d \)-periodic} bundle \( \tilde{E} \) on \( T\mathbb{R}^d \) means that
\[
\tilde{E}(x + n) = DT_n \left( \tilde{E}(x) \right), \quad \forall x \in \mathbb{R}^d, \quad \forall n \in \mathbb{Z}^d.
\]

**Proposition 2.17.** There exists a \( C^1 \) neighborhood \( \mathcal{U} \subset C^1(\mathbb{T}^d) \) of \( A \) such that for every \( f \in \mathcal{U} \) and its lifting \( F \), we have the followings,

1. The \( F \)-invariant bundles \( \tilde{E}^s_{(i,j)}(1 \leq i \leq j \leq k) \) and \( \tilde{E}^u \) are uniquely integrable. Denote the integral foliations by \( \tilde{\mathcal{F}}^s_{(i,j)}, \tilde{\mathcal{F}}^u \) and \( \tilde{\mathcal{F}}^s_{i} \) \((i \leq j)\). Moreover, the foliation \( \tilde{\mathcal{F}}^s_{(i_0,j_0)} \) is subfoliated by \( \tilde{\mathcal{F}}^s_{(i,j)} \), for any \( 1 \leq i_0 \leq i \leq j \leq j_0 \leq k \).
2. The strong stable bundle \( \tilde{E}^s_{(1,i)} \) and strong stable foliation \( \tilde{\mathcal{F}}^s_{(1,i)} \) \((1 \leq i \leq k)\) are both \( \mathbb{Z}^d \)-periodic.
3. The foliation \( \tilde{\mathcal{F}}^s_{(i,j)} \) \((1 \leq i \leq j \leq k)\) is quasi-isometric. Especially, for every Anosov map on torus with one-dimensional stable bundle, the lifting of stable foliation is quasi-isometric.
4. \( H \) preserves the weak stable foliations. It means that \( H(\tilde{\mathcal{F}}^s_{(i,k)}) = \tilde{\mathcal{L}}^s_{(i,k)} \), for all \( 1 \leq i \leq k \).
Proof. Let $\mathcal{U}$ be given by Remark 2.14. Consider the dominated splitting

$$\tilde{E}^s_{(i-1)} \oplus \tilde{E}^s_{(i,k)} \oplus \tilde{E}^u,$$

where $2 \leq i \leq k$. By the Stable Manifold Theorem e.g. [35, Theorem 4.1 and Theorem 4.8], we have that $\tilde{E}^s_{(i-1)}$ and $\tilde{E}^u$ are always integrable. Although [35] prove it for diffeomorphism, in our case Remark 2.14 and Remark 2.15 provides the uniform continuity and domination of bundles to replace the compactness. Moreover, by Proposition 2.12 and (2.12), the bundle $\tilde{E}^s_{(i,i)}$ and foliation $\tilde{F}^s_{(i,i)} (1 \leq i \leq k)$ are $\mathbb{Z}^d$-periodic.

For the integrability of $\tilde{E}^s_{(i,k)}$, in the case of diffeomorphism, the linear Anosov system $A : \mathbb{R}^d \to \mathbb{R}^d$ is robustly dynamically coherent (see [25, Theorem 7.6] also [37, Proposition 3.2]) and this also holds for our case by the same reason of integrability for strong stable bundles. Hence, we have that

$$\tilde{F}^s_{(i,j)} = \tilde{F}^s_{(1,j)} \cap \tilde{F}^s_{(i,k)}$$

is a foliation tangent to $\tilde{E}^s_{(i,j)}$. So $\tilde{E}^s_{(i,j)}$ is integrable for all $1 \leq i \leq j \leq k$. We refer to [15, Lemma 6.1] for uniquely integrable property which is proved on the universal cover and also holds for non-invertible Anosov maps. It is clear that $\tilde{F}^s_{(0,j_0)}$ is subfoliated by $\tilde{F}^s_{(i,j)}$ for any $1 \leq i_0 \leq i \leq j \leq j_0 \leq k$. Moreover, $\tilde{F}^s_{(0,i)}$ and $\tilde{F}^s_{(i+1,j_0)}$ admit the Global Product Structure on $\tilde{F}^s_{(0,j_0)}$.

For a small perturbation, in particular from (2.13), we have that the foliation $\tilde{F}^s_{(i,j)}$ is uniformly transverse to $\tilde{F}^s_{(i-1)} \oplus \tilde{F}^s_{(i+1,k)}$. By [4, Proposition 4], we have the quasi-isometric property for $\tilde{F}^s_{(i,j)}$. We mention that the proof of this actually need $\tilde{F}^s_{(i,j)}$ has a uniformly transverse plane and it is uniformly continuous (see Remark 2.18).

Then we prove the third item for the case of $\dim E^s = 1$, namely, every Anosov map on torus with one-dimensional stable bundle has the quasi-isometric stable foliation. Since $|H - Id|$ is bounded, the unstable foliation $\tilde{F}^u$ is uniformly bounded by $\tilde{F}^u$. Namely, there exists $C_0 > 0$ such that $\tilde{F}^u(x)$ and $\tilde{F}^u(x)$ are contained in the $C_0$-neighborhoods of each other, for all $x \in \mathbb{R}^d$. Fix $n_0 \in \mathbb{Z}^d$ such that $d(x, x + n_0) \geq 3C_0$, for all $x \in \mathbb{R}^d$. It follows that the Hausdorff distance between $\tilde{F}^u(x)$ and $\tilde{F}^u(x + n_0)$ is bigger than $C_0$. Since the stable foliation $\tilde{F}^s$ and the unstable foliation $\tilde{F}^u$ admit the Global Product Structure, there exists $L_0$ such that for any $x \in \mathbb{R}^d$ with $L(x) \leq L_0$, one has $\tilde{F}^s(x, L(x))$ intersects $\tilde{F}^u(x + n_0)$ exactly once and the distance between $x$ and the intersection is bigger than $C_0$. Thus the one-dimensional foliation $\tilde{F}^s$ is always quasi-isometric, whether $f$ is a small perturbation or not. We refer readers to [5] for more details.

Finally, we prove that $H$ preserves the weak stable foliations. Let $y \in \tilde{F}^s(x)$ and we always have $H(y) \in \tilde{F}^s(H(x))$. Note that $H(y) \in \tilde{F}^s_{(i,k)}(H(x))$ if and only if

$$d\left(A^{-n}(Hy), A^{-n}(Hx)\right) \leq \left(\mu_i^s(A)\right)^{-n} \cdot d\left(Hy, Hx\right), \quad \forall n \in \mathbb{N}.$$

By Proposition 2.3, let $|H - id| < C_0$. One has that $H(y) \in \tilde{F}^s_{(i,k)}(H(x))$ if and only if

$$d\left(F^{-n}(y), F^{-n}(x)\right) \leq \left(\mu_i^s(A)\right)^{-n} \cdot d\left(Hy, Hx\right) + 2C_0, \quad \forall n \in \mathbb{N}. \quad (2.18)$$
It implies that \( H(y) \in \mathcal{F}^s_{i,k}(H(x)) \) if and only if \( y \in \mathcal{F}^s_{i,k}(x) \). Indeed, if \( y \notin \mathcal{F}^s_{i,k}(x) \), then there exists the unique point \( z \in \mathcal{F}^s_{i,k}(x) \cap \mathcal{F}^s_{i,k}(y) \) with \( a := \mathcal{d}_{\mathcal{F}^s_{i,k}}(x, z) > 0 \). Let \( b = \mathcal{d}_{\mathcal{F}^s_{i,k}}(z, y) \), note that \( b \) may be zero. For \( n \in \mathbb{N} \) big enough, one has

\[
d(F^{-n}y, F^{-n}z) \leq \mathcal{d}_{\mathcal{F}^s_{i,k}}(F^{-n}y, F^{-n}z) \leq \left( \mu^s_i(A) - \frac{\varepsilon_0}{2} \right)^{-n} \cdot b,
\]

where \( \varepsilon_0 \) is given by (2.13) and

\[
d_{\mathcal{F}^s_{i,k}}(F^{-n}x, F^{-n}z) \geq \left( \mu^s_{i-1}(A) + \frac{\varepsilon_0}{2} \right)^{-n} \cdot a.
\]

Since \( \mathcal{F}^s_{i,k} \) is quasi-isometric, there exists \( 0 < C_1 < 1 \) such that

\[
d(F^{-n}x, F^{-n}z) \geq C_1 \left( \mu^s_{i-1}(A) + \frac{\varepsilon_0}{2} \right)^{-n} \cdot a.
\]

Hence by (2.19) and (2.20), one has

\[
d(F^{-n}x, F^{-n}y) \geq C_1 \left( \mu^s_{i-1}(A) + \frac{\varepsilon_0}{2} \right)^{-n} \cdot a - \left( \mu^s_i(A) - \frac{\varepsilon_0}{2} \right)^{-n} \cdot b,
\]

which contradicts with (2.18). ☐

**Remark 2.18.** We state the uniform continuity of foliation as follow. For given \( \mathcal{F}^s_{i,k} \), \( 1 \leq i \leq k \) and any constant \( C > 0 \), there exists \( \delta > 0 \) such that for every \( x \in \mathbb{R}^d \) and \( y \in \mathcal{F}^s_{i,k} \) with \( d_{\mathcal{F}^s_{i,k}}(x, y) > C \), we have \( d(x, y) > \delta \). Just note that, by the choice of the neighborhood \( \mathcal{U} \), the angle \( \angle(\bar{E}^s_i, \bar{L}^s_i) \) is uniformly bounded by \( \alpha \). In fact, for any Anosov map (may not be a small perturbation) with a dominated splitting along orbit, \( T_{x_n} \mathbb{T}^d = E^s_i(x_n, \bar{x}) \oplus \cdots \oplus E^u_i(x_n, \bar{x}) \oplus E^u_i(x_n, \bar{x}) \) the angle between any two distinct subbundles is uniformly away from 0, where \( \dim E^s_i \) may bigger than one.

The following proposition says that the same periodic Lyapunov exponent implies it coincides with one of the linearization under the assumption that \( H \) preserves the corresponding foliation.

**Proposition 2.19.** Let \( \mathcal{F} \) be the neighborhood \( \mathcal{U} \) given by Proposition 2.17. Fix \( 1 \leq i \leq k \) and suppose that \( H(\mathcal{F}^s_{i,k}) = \mathcal{L}^s_i \) and \( \lambda^s_i(p, f) = \lambda^s_i(q, f) \) for every \( p, q \in \text{Per}(f) \). Then \( \lambda^s_i(p, f) = \lambda^s_i(A) \), for all \( p \in \text{Per}(f) \). Especially, for every Anosov map \( f \) on torus with \( \dim E^s_i = 1 \), if \( \lambda^s_i(p, f) = \lambda^s_i(q, f) \) for every \( p, q \in \text{Per}(f) \), then \( \lambda^s_i(p, f) = \lambda^s_i(A) \), for all \( p \in \text{Per}(f) \).

**Proof of Proposition 2.19.** We first claim that there exists an adapted metric on \( \mathbb{R}^d \).

**Claim 2.20.** Let \( \mu_1 \) and \( \mu_2 \) be the infimum and the supremum of the set \( \{ \exp(\lambda^s_i(p, f)) : p \in \text{Per}(f) \} \), respectively. For any \( \delta > 0 \), there exists a smooth adapted Riemannian metric on \( T\mathbb{R}^d \) such that

\[
\mu_1 \cdot (1 + \delta)^{-1} < \| DF_{|E^s_i(x)} \| < \mu_2 \cdot (1 + \delta), \quad \forall x \in \mathbb{R}^d.
\]
Proof of Claim 2.20. Since we have the Shadowing Lemma for Anosov maps (see [1]), it can be proved as the existence of adapted metrics for Anosov diffeomorphisms. For the convenience of readers, we prove it as follows.

Fix in advance a Riemannian metric on $T\mathbb{R}^d$ which induces a norm $| \cdot |$. Let

$$
\mu_+ := \sup_{x \in \mathbb{R}^d} |DF|_E^i(x)| \quad \text{and} \quad \mu_- := \inf_{x \in \mathbb{R}^d} |DF|_E^i(x)|.
$$

By (2.12) and the compactness of $\mathbb{T}_T^d$, one has $\mu_+ < +\infty$ and $\mu_- > 0$. By Remark 2.15 the subbundle $E^i_j(x)$ is uniformly continuous with respect to $x \in \mathbb{R}^d$, Then for any $\delta > 0$ there exists $\alpha > 0$ such that

$$(1 + \delta/2)^{-1} \leq \frac{|DF|_{E^i_j(x)}}{|DF|_{E^i_j(y)}} \leq 1 + \delta/2, \quad (2.21)$$

for any $x, y \in \mathbb{R}^d$ with $d(x, y) < \alpha$.

By the Shadowing Lemma, for given $\alpha > 0$, there exists $\beta > 0$ such that each periodic $\beta$-pseudo-orbit in $\mathbb{T}_T^d$ can be $\alpha$-shadowing by a periodic orbit. Let $B_1, \ldots, B_{n(\beta)}$ be finite many open $\beta$-periodic $\alpha$-shadowing periodic $\beta$-pseudo-orbit and is $\alpha$-shadowing by a periodic orbit. Let $B_1, \ldots, B_{n(\beta)}$ be finite many open $\beta$-pseudo-orbits cover $\mathbb{T}_T^d$. Since $f$ is transitive (also see Sect. 4), there exists $N_1 \in \mathbb{N}$ such that for any $B_i$ and $B_j$, $B_i$ can intersect $B_j$ within $N_1$-times iteration by $f$.

Let $\pi : \mathbb{R}^d \rightarrow \mathbb{T}_T^d$ be the natural projection. For any $x \in \mathbb{R}^d$ and $N_0 \in \mathbb{N}$ with $x_0 := \pi(x) \in B_{i_0}$ and $f^{N_0}(x_0) \in B_{i_1}$, there exists $y_0 \in B_{i_1}$ and $N_2 \in [0, N_1]$ such that $f^{N_2}y_0 \in B_{i_0}$. It follows that

$$
\{x_0, f(x_0), \ldots, f^{N_0-1}(x_0), y_0, f(y_0), \ldots, f^{N_2-1}y_0\}
$$

is a periodic $\beta$-pseudo-orbit and is $\alpha$-shadowing by a periodic orbit $p_0 \in \mathbb{T}_T^d$ with period $N(p_0) := N_0 + N_2$. Hence, there exists $p \in \pi^{-1}(p_0)$ such that $d\left(\pi(F^j x), \pi(F^j p)\right) < \alpha$, for all $j \in [0, N(p_0)]$.

Now, let $N = N_0 + N_1$ and

$$
\|v\|_N := \prod_{n=0}^{N-1} |DF^n|_E^i(x)|^{\frac{1}{N}}, \quad \forall v \in E^i_N(x).
$$

We use this norm $\| \cdot \|_N$ instead of the classical adapted norm which sums the first $N$-steps norm $| \cdot |$, since $\| \cdot \|_N$ can adapt the Lyapunov exponents better. We refer to [2, (3.6)] for more details of this construction. Note that $|DF^{N_0}(p_0)|_{E^i_N(p)} \in [\mu_1^{N_0}(p_0), \mu_2^{N_0}(p_0)]$, one has

$$
|DF^{N_0}|_{E^i(p)}| \leq \left[\mu_1^{N_0} \cdot \left(\frac{\mu_1}{\mu_+}\right)^{N_2} \cdot \mu_2^{N_0} \cdot \left(\frac{\mu_2}{\mu_-}\right)^{N_2}\right]. \quad (2.22)
$$

Taking $v \in E^i_N(x) - \{0\}$, by (2.21) and (2.22), we calculate directly,

$$
\left|\frac{D_x F(v)}{\|v\|_N}\right| = \left(\frac{|DF^{N_0}(x_0)F^{N_0} \circ D_x F^{N_0}(v)|}{|v|^{1/N}}\right)^{1/N
$$
Hence, there exists $N_0$ big enough such that
\[
\frac{\|D_x F(v)\|_N}{\|v\|_N} \in \left( \mu_1 \cdot (1 + \delta)^{-1}, \mu_2 \cdot (1 + \delta) \right).
\]

There exists a norm $\| \cdot \|$ whose restriction on subbundle $\tilde{E}_i^s$ is $\| \cdot \|_N$ gives the smooth adapted Riemannian metric we want. \hfill \square

Since $\lambda_i^s(p, f) = \lambda_i^s(q, f)$ for all $p, q \in \text{Per}(f)$, $\mu := \exp(\lambda_i^s(f, p))$ is a constant. Assume that $\mu \neq |\mu_i^s(A)| = \exp(\lambda_i^s(A))$. Fix $\delta < \min \{ |\frac{\mu}{\mu_i^s(A)} - 1|, |\frac{\mu_i^s(A)}{\mu} - 1| \}$ and an adapted norm $\| \cdot \|$ from Claim 2.20, namely,
\[
\mu \cdot (1 + \delta)^{-1} < \|DF|_{\tilde{E}_i^s(x)}\| < \mu \cdot (1 + \delta), \quad \forall x \in \mathbb{R}^d.
\]

Let $H : \mathbb{R}^d \to \mathbb{R}^d$ be the conjugacy defined in Proposition 2.3 satisfying $|H - I d| \leq C_0$. We take two points $x, y \in \mathbb{R}^d$ such that $y \in \tilde{\mathcal{F}}_i^s(x)$. One has
\[
\mu^{-k}(1 + \delta)^{-k} \cdot d^s(x, y) \leq d^s(F^{-k}x, F^{-k}y) \leq \mu^{-k}(1 + \delta)^k \cdot d^s(x, y),
\]

further,
\[
a \cdot \mu^{-k}(1 + \delta)^{-k} \cdot d^s(x, y) \leq d(F^{-k}x, F^{-k}y) \leq \mu^{-k}(1 + \delta)^k \cdot d^s(x, y), \quad (2.23)
\]

where $a$ is given by (2.17), since $\tilde{\mathcal{F}}_i^s$ is quasi-isometric (by Proposition 2.17). Meanwhile, since $H(\tilde{\mathcal{F}}_i^s) = \tilde{\mathcal{F}}_i^s$ (when $\dim E^s = 1$, $H(\tilde{\mathcal{F}}^s) = \tilde{\mathcal{F}}^s$ always holds),
\[
d\left( H(F^{-k}x), H(F^{-k}y) \right) = d\left( A^{-k}(Hx), A^{-k}(Hy) \right) = (\mu_i^s(A))^{-k} \cdot d(Hx, Hy). \quad (2.24)
\]

The formulas (2.23) and (2.24) jointly contradict with the fact
\[
\left| d(F^{-k}x, F^{-k}y) - d\left( H(F^{-k}x), H(F^{-k}y) \right) \right| \leq 2C_0.
\]

For the convenience of readers, we state Journé Lemma [26] as the following proposition which will be useful in Sect. 4 and Sect. 5.

**Proposition 2.21** [26]. Let $M_i$ ($i = 1, 2$) be smooth manifolds and $\mathcal{F}_i^s$, $\mathcal{F}_i^u$ be continuous transverse foliations on $M_i$ with uniformly $C^{r+\alpha}$-smooth leaves ($r \geq 1$, $0 < \alpha < 1$). Assume that $h : M_1 \to M_2$ is a homeomorphism and maps $\mathcal{F}_1^s$ to $\mathcal{F}_2^s$ ($\sigma = s, u$). If $h$ restricted on leaves of both $\mathcal{F}_1^s$ and $\mathcal{F}_1^u$ is uniformly $C^{r+\alpha}$, then $h$ is $C^{r+\alpha}$-smooth.
2.4. A reducible counter-example. As claimed in Remark 1.11, we will give a non-invertible reducible Anosov map on $\mathbb{T}^3$ whose periodic stable Lyapunov exponents are equal to ones of its linearization, but it is not special. We refer readers to [10,15] for two reducible Anosov diffeomorphisms on $\mathbb{T}^4$ with same periodic data which are not smoothly conjugate. Let

$$A_3 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} : \mathbb{T}^3 \to \mathbb{T}^3.$$ 

Note that $A_3$ is a reducible Anosov map with one-dimensional stable bundle and its preimage set is becoming dense exponentially. Let

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} : \mathbb{T}^2 \to \mathbb{T}^2.$$ 

Denote the smallest eigenvalue of $B$ and the corresponding unit eigenvector by $\mu^s \in (0, 1)$ and $v^s \in \mathbb{R}^2$, respectively. Denote by $v^u$ the unit eigenvector corresponding to the biggest eigenvalue of $B$. Let $\varphi_\varepsilon : S^1 \to S^1, \varphi_\varepsilon(y) = \varepsilon \sin(2\pi y)$ where $\varepsilon > 0$ close to 0 arbitrarily. Let

$$g_\varepsilon : \mathbb{T}^3 \to \mathbb{T}^3, \quad g_\varepsilon(x, y) = (Bx + \varphi_\varepsilon(y)v^s, 2y).$$

where $(x, y) \in \mathbb{T}^2 \times S^1.$

**Proposition 2.22.** The Anosov map $g_\varepsilon$ is $C^\infty$ close to $A_3$ (as $\varepsilon \to 0$) with

$$\lambda^s(p, g) = \lambda^s(A_3), \quad \forall p \in \text{Per}(g),$$

but $g_\varepsilon$ is not special.

**Proof.** Note that

$$D_{(x,y)}g_\varepsilon = \begin{bmatrix} B & \varphi'(y)v^s \\ 0 & 2 \end{bmatrix}.$$ 

Hence $\lambda^s(p, g) = \lambda^s(B) = \lambda^s(A_3)$, for all $p \in \text{Per}(g)$. Moreover, for any $z_0 = (x_0, y_0) \in \mathbb{T}^3$ and $\bar{z} = (z_i) \in \mathbb{T}^3_{g_\varepsilon}$ with $z_i = (x_i, y_i)$, one has the vector

$$\begin{bmatrix} v^u \\ 0 \end{bmatrix} \in E^u_{g}(z_0, \bar{z}) \quad \text{and} \quad \begin{bmatrix} v^s \\ 0 \end{bmatrix} \in E^s_{g}(z_0).$$

And in the view of cone-field (see the proof of Proposition 2.12), one has

$$\lim_{i \to +\infty} (D_{(x_{i-1},y_{i-1})}g_\varepsilon) \cdots (D_{(x_{i+1},y_{i+1})}g_\varepsilon)(D_{(x_{i},y_{i})}g_\varepsilon) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in E^u_{g}(z_0, \bar{z}).$$

It follows that

$$\begin{bmatrix} \eta(\bar{z})v^s \\ 1 \end{bmatrix} \in E^u_{g}(z_0, \bar{z}), \quad \eta(\bar{z}) = \sum_{i=1}^{+\infty} \frac{\mu^s}{2}^{i-1} \frac{\varphi'(y_{i-1})}{2}.$$ 

It is clear that $\eta(\bar{z})$ is well defined for any $\bar{z} \in \mathbb{T}_{g_\varepsilon}^s$. 


Now we consider two $\varphi_x$-orbits of $0 = (0, 0) \in \mathbb{T}^2 \times S^1$. Let $\tilde{z}_1 = (z_1^1)$ with $z_i^1 = (0, 0) \in \mathbb{T}^3$ for all $i < 0$. And there exists $\tilde{z}_2 = (z_2^2)$ with $z_i^2 = (x_i, y_i) \in \mathbb{T}^3$ such that $y_i = 2^i$ for all $i < 0$. It is clear that $\eta(\tilde{z}_1) \neq \eta(\tilde{z}_2)$. Hence

$$\text{span}_{\mathbb{R}} \left\{ \begin{bmatrix} \eta(\tilde{z}_1)^s \vspace{1mm} \\ 1 \end{bmatrix}, \begin{bmatrix} \varepsilon^s \vspace{1mm} \\ 0 \end{bmatrix} \right\} = E_{g_x}^u(0, \tilde{z}_1) \neq E_{g_x}^u(0, \tilde{z}_2) = \text{span}_{\mathbb{R}} \left\{ \begin{bmatrix} \eta(\tilde{z}_2)^s \vspace{1mm} \\ 1 \end{bmatrix}, \begin{bmatrix} \varepsilon^s \vspace{1mm} \\ 0 \end{bmatrix} \right\},$$

and $g_x$ is not special. □

3. Spectral Rigidity on Stable Bundle

In this section, we prove the necessary parts of both Theorem 1.1 and Theorem 1.2. As mentioned, we can actually prove them under the assumptions that $f$ is $C^1$-smooth (see Remark 1.3) and the preimage set of linearization $A : \mathbb{T}^d \to \mathbb{T}^d$ is becoming dense exponentially (see Remark 1.10), i.e., there exist $C > 0$ and $0 < \alpha < 1$ such that for every $l \geq 1$ and every $x_0 \in \mathbb{T}^d$, the set $\{x \in \mathbb{T}^d | A^l(x) = x_0\}$ is $C \cdot \alpha^l$-dense in $\mathbb{T}^d$. For convenience, we restate these as follow.

**Theorem 3.1.** Let $A : \mathbb{T}^d \to \mathbb{T}^d$ be a linear non-invertible Anosov map. Assume that the preimage set of $A$ is becoming dense exponentially and $A$ admits the finest (on stable bundle) dominated splitting,

$$T\mathbb{T}^d = L_1^s \oplus L_2^s \oplus \cdots \oplus L_k^s \oplus L^u,$$

where $\dim L_i^s = 1$, $1 \leq i \leq k$.

Then there exists a $C^1$ neighborhood $\mathcal{U} \subset C^1(\mathbb{T}^d)$ of $A$ such that for every $f \in \mathcal{U}$, if $f$ is special, then $\lambda_i^s(p, f) = \lambda_i^s(A)$, for all $p \in \text{Per}(f)$ and all $1 \leq i \leq k$. Moreover, $f$ admits the finest (on stable bundle) special dominated splitting,

$$T\mathbb{T}^d = E_1^s \oplus E_2^s \oplus \cdots \oplus E_k^s \oplus E^u,$$

where $\dim E_i^s = 1$, for all $1 < i < k$.

Moreover, when $k = 1$, for every $C^1$-smooth non-invertible Anosov map $f$ with linearization $A$, if $f$ is special and the preimage set of $A$ is becoming dense exponentially, then $\lambda_i^s(p, f) = \lambda_i^s(A)$, for all $p \in \text{Per}(f)$.

**Remark 3.1.** By Proposition 2.10, when $A$ is irreducible, the preimage set of $A$ is becoming dense with exponent $|\det(A)|^{-d}$. Hence Theorem 3.1 implies the necessary part of Theorem 1.1 and Theorem 1.2.

Firstly, we give the scheme of our proof. To get the spectral rigidity on stable bundle, we prove that every periodic point $p \in \text{Per}(f)$ has the same stable Lyapunov spectrum $\{\lambda_i^s(p, f) : i = 1, 2, \ldots, k\}$, under the assumption that there exists integrable subbundle $E_i^s$ ($i = 1, 2, \ldots, k$).

**Proposition 3.2.** Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a non-invertible Anosov map. Assume that the preimage set of its linearization is becoming dense exponentially. Let $E_i^s \subset E^s$ be a $Df$-invariant one-dimensional subbundle. If $f$ is special and there exists an $f$-invariant foliation on $\mathbb{T}^d$ tangent to $E_i^s$, then $\lambda_i^s(p, f) = \lambda_i^s(q, f)$, for all $p, q \in \text{Per}(f)$, where $\lambda_i^s(p, f)$ is the Lyapunov exponent of $f$ for $p$ corresponding the bundle $E_i^s$. 
Remark 3.3. We emphasize here that in the proof of Proposition 3.2, \( f \) need not be a small perturbation of \( A \), but we need the subbundle \( E^s_i \) exists, i.e., the direction \( E^s_i(x_0, \bar{x}) \) is independent with the negative orbits. In particular, Proposition 3.2 always holds for toral special Anosov maps with one-dimensional stable bundles.

To get that every periodic point of \( f \) has the same stable Lyapunov spectrum through Proposition 3.2, we need that \( f \) admits the finest (on stable bundle) dominated splitting and each stable subbundle is integrable. Recall that generally the weak stable bundle does not exist, see Remark 2.13.

**Proposition 3.4.** Let \( A : \mathbb{T}^d \to \mathbb{T}^d \) be a linear non-invertible Anosov map and admit the finest (on stable bundle) dominated splitting. Then there exists a \( C^1 \) neighborhood \( \mathcal{U} \subset C^1(\mathbb{T}^d) \) of \( A \) such that for every \( f \in \mathcal{U} \), if it is special, then it admits the finest (on stable bundle) special dominated splitting

\[
T\mathbb{T}^d = E^s_1 \oplus E^s_2 \oplus \cdots \oplus E^s_k \oplus E^u,
\]

where \( E^s_i \) is one-dimensional and integrable, for all \( 1 \leq i \leq k \).

We leave the proofs of Proposition 3.2 and Proposition 3.4 in Sect. 3.1.

To obtain the relationship between the periodic stable Lyapunov spectrum and one of its linearization, we can use Proposition 2.19. For a special \( f \in \mathcal{U} \) given by Proposition 3.4, let \( h \) be the conjugacy between \( f \) and \( A \) given by Proposition 2.2. We need to prove that the conjugacy \( h \) is also a leaf conjugacy between \( \mathcal{F}^s_i \) and \( \mathcal{L}^s_i \).

**Proposition 3.5.** Let \( f \in \mathcal{U} \) given by Proposition 3.4 be special. Further assume that the preimage set of its linearization \( A \) is becoming dense exponentially. Then \( h(\mathcal{F}^s_i) = \mathcal{L}^s_i \), for every \( 1 \leq i \leq k \).

We leave the proof of Proposition 3.5 in Sect. 3.2. Now, we can prove Theorem 3.1.

**Proof of Theorem 3.1.** For one-dimensional stable bundle case, the special Anosov map \( f \) admits the dominated splitting \( T\mathbb{T}^d = E^s \oplus E^u \) and \( H(\mathcal{F}^s) = \mathcal{L}^s \) always holds whether \( f \) is a small perturbation of its linearization or not. Hence by Proposition 2.19 and Proposition 3.2, we get \( \lambda^s(p, f) = \lambda^s(A) \), for all \( p \in \text{Per}(f) \), immediately.

For higher-dimensional stable bundle case, by Proposition 3.4, there exists a \( C^1 \) neighborhood \( \mathcal{U} \) of \( A \) such that every special \( f \in \mathcal{U} \) admits the finest (on stable bundle) dominated splitting. Thus by Proposition 3.2, every periodic point \( p \in \text{Per}(f) \) has the same stable Lyapunov spectrum. Now, combining Proposition 2.19 and Proposition 3.5, we have that \( \lambda^s_i(p, f) = \lambda^s_i(A) \), for every \( p \in \text{Per}(f) \) and every \( 1 \leq i \leq k \). \( \square \)

3.1. Periodic stable Lyapunov spectrums coincide. In this subsection, we prove Proposition 3.2 and Proposition 3.4. Let \( f : \mathbb{T}^d \to \mathbb{T}^d \) be a special non-invertible Anosov map.

Fixing \( 1 \leq i \leq k \), to prove Proposition 3.2, we assume that the one-dimensional subbundle \( E^s_i \subset E^s \) is \( Df \)-invariant and integrable. Let \( \mathcal{F}^s_i \) be an \( f \)-invariant integral foliation for \( E^s_i \). For short, we denote \( \mu_i^s(p, f) := \exp(\lambda_i^s(p, f)) \) by \( \mu_i^s(p) \), for all \( p \in \text{Per}(f) \).

**Proof of Proposition 3.2.** By the assumption, there exist \( C_0 > 0 \) and \( 0 < \alpha < 1 \) such that for every \( l \geq 1 \) and every \( x_0 \in \mathbb{T}^d \), the set \( \{ x \in \mathbb{T}^d : A^l(x) = x_0 \} \) is \( C_0 \cdot \alpha^l \)-dense
in $\mathbb{T}^d$. Since $f$ is special, by Proposition 2.2, $f$ is conjugate to $A$. Let $h : \mathbb{T}^d \to \mathbb{T}^d$ be the conjugacy between $f$ and $A$ with $h \circ f = A \circ h$.

We assume that there exist $p, q \in \text{Per}(f)$ such that $\mu^+_i(p) < \mu^+_i(q)$, and get a contradiction.

For convenience, we can assume that $p, q$ are both fixed points. Otherwise we can go through the rest of this proof by using $f^{n_0}$ instead of $f$, where $n_0$ is the minimal common period of $p$ and $q$. By the assumption of the existence of different periodic stable Lyapunov exponents, the infimum $\mu_-$ and the supremum $\mu_+$ of the set $\{\mu^+_i(p) : p \in \text{Per}(f)\}$ satisfy $0 < \mu_- < \mu_+ < 1$. Given $\delta > 0$ arbitrarily small, we can choose two periodic points $p, q$ of $f$ such that

$$\mu^+_i(p) \leq \mu_- \cdot (1 + \delta) \quad \text{and} \quad \mu^+_i(q) \geq \mu_+ \cdot (1 + \delta)^{-1},$$

By Claim 2.20, there exists a smooth adapted Riemannian metric such that

$$\mu_- \cdot (1 + \delta)^{-1} < \|Df|_{E^s_i(x)}\| < \mu_+ \cdot (1 + \delta), \quad \forall x \in \mathbb{T}^d.$$

Here we fix $\delta > 0$ such that $(1 + \delta)^4 \cdot \left(\frac{\mu_+}{\mu_-}\right)^C_1 > 1$, where $C_1 = \frac{-\ln(\delta)}{2\inf\mu_{\max}(A)}$ and $\mu_{\max}^u(A)$ is the module of the maximal eigenvalue of $A|_{E^u_i}$.

For given $\delta$, fix $\epsilon_0 > 0$ small enough such that for any $x_1, x_2 \in \mathbb{T}^d$ with $d(x_1, x_2) \leq \epsilon_0$, we have

$$(1 + \delta)^{-1} \leq \frac{\|Df|_{E^s_i(x_1)}\|}{\|Df|_{E^s_i(x_2)}\|} \leq 1 + \delta.$$

For any $\epsilon > 0(\epsilon \ll \eta_0)$, there exists $x_\epsilon$ in the $\epsilon$-Ball $B_\epsilon(q)$ and $k_\epsilon = k(\epsilon, x_\epsilon) > 0$ such that $f^{k_\epsilon}(x_\epsilon) = p$. Indeed, the preimage set of $p$ for $f$ is becoming dense (maybe not exponentially), since $f$ is conjugate to $A$.

Let $\epsilon$ small enough such that one has the local product structure, namely, there exists $\eta > 0$ such that the local unstable leaf $\mathcal{F}^u_i(x, \eta) \subset B_\eta(q)$ intersects with the local stable leaf $\mathcal{F}^s_i(x, \eta) \subset B_\eta(q)$ at the unique point $y_\epsilon$, i.e., $y_\epsilon = \mathcal{F}^s_i(x_\epsilon, \eta) \cap \mathcal{F}^u_i(x_\epsilon, \eta)$. Note that one has $d_{\mathcal{F}^u_i}(y_\epsilon, q) \leq d(\epsilon)$, where $d(\epsilon) \to 0$ as $\epsilon \to 0$.

Since we have assumed that the foliation $\mathcal{F}^s_i$ exists, one can choose a point $z_\epsilon \in \mathcal{F}^s_i(x_\epsilon, \eta)$ such that $d_{\mathcal{F}^s_i}(z_\epsilon, q) \geq \eta/3$. We denote by $I_\epsilon$ the curve in $\mathcal{F}^s_i(x_\epsilon)$ from $x_\epsilon$ to $z_\epsilon$. Since $x_\epsilon$ is a $k_\epsilon$-preimage of the fixed point $p$, we can find a curve $J_\epsilon$ in $\mathcal{F}^s_i(p)$ such that

$$f^{k_\epsilon}(J_\epsilon) = f^{k_\epsilon}(I_\epsilon).$$

For short, denote $h(x_\epsilon), h(y_\epsilon), h(p), h(q), h(I_\epsilon)$ and $h(J_\epsilon)$ by $x'_\epsilon, y'_\epsilon, p', q', I'_\epsilon$ and $J'_\epsilon$, respectively. The homeomorphism $h$ maps $B_\eta(q') \subset B_\eta(q)$ to be two neighborhoods of $q'$ such that we can choose two su-foliation boxes of $A$, $B_{\eta_0}(q') \subset h(B_{\eta_0}(q))$ and $B_{\epsilon'}(q') \subset h(B_{\epsilon'}(q))$, see Fig. 1. Note that $\eta_0$ is fixed by $\eta_0$ and independent with $\epsilon'$, since $\epsilon'$ tends to 0 following $\epsilon$. Thus we can shorten $I_\epsilon$ slightly such that $I'_\epsilon = h(I_\epsilon) \subset B_{\eta_0}(q') \cap \mathcal{L}^s(x_\epsilon')$ and the distance between two endpoinits of $I'_\epsilon$ is equal to $\frac{\eta_0}{3}$. Moreover, by the continuity of $h$, we can assume that the distance between any two points on $I'_\epsilon$ is less than $\frac{\eta_0}{3}$. Note that $I'_\epsilon$ is not a priori a $C^1$ curve.
Claim 3.6. For given \( \delta \) and \( \eta_0 \), there exists \( C_2 > 1 \) such that
\[
\frac{|I_e|}{|J_e|} \in \left[ C_2^{-1}, C_2 \right], \quad \forall \varepsilon \ll \eta_0.
\]

Proof of Claim 3.6. Let \( h : \mathbb{T}^d \to \mathbb{T}^d \) be the conjugacy between \( f \) and \( A \) with \( h \circ f = A \circ h \). By the construction of \( J_e \), we have \( h \circ f^{K_e'}(I_e') = h \circ f^{K_e'}(I_e') \), equivalently, \( A^{K_e'}(I_e') = A^{K_e'}(I_e') \). This implies that \( I_e' \) is just a translation of \( I_e' \), thus the distance between endpoints of \( J_e \) is equal to \( \frac{\eta_0}{2} \). Since \( h^{-1} \) is uniformly continuous, by the uniform continuity for \( F_i \) (see Remark 2.18), we have that \( \frac{|I_e|}{|J_e|} \) is uniformly bounded away from 0.

Remark 3.7. From the proof of Claim 3.6, we can assume \( J_e \subseteq B_{\delta_0}(p) \). Otherwise, by the uniform continuity of \( \delta \) again, we can shorten the length of \( I_e \) such that \( J_e \subseteq B_{\eta_0}(p) \), meanwhile ensure that the length of \( I_e \) is independent with \( \varepsilon \).

Now for given \( \varepsilon > 0 \) (\( \varepsilon \ll \eta_0 \)), let \( K_e \) be the minimal positive integer such that \( \{ x \in \mathbb{T}^d | f^{K_e}(x) = p \} \cap B_{\delta}(q) \neq \emptyset \). We choose \( k_e = K_e \) and \( x_e \in \mathbb{T}^d \) such that \( f^{K_e}(x_e) = p \) and \( x_e \in B_{\delta}(q) \). And then we fix \( I_e \) and \( J_e \) such that they satisfy the all conditions before mentioned. Let \( N_e \) be the maximal positive integer such that \( d(f^j(w), f^j(q)) < \eta_0 \), for all \( w \in I_e \) and \( j \in [0, N_e] \).

Claim 3.8. There exists a constant \( C_3 > 0 \) such that for given \( \delta \) and \( \eta_0 \), there exists \( \varepsilon_0 \) such that
\[
\frac{N_e}{K_e} \geq C_3, \quad \forall \varepsilon \leq \varepsilon_0.
\]
In particular, we can choose \( C_3 = C_1 = \frac{-\ln \eta_0}{2 \ln \mu_{\text{max}}(A)} \).

We estimate the upper bound and the lower bound of \( K_e \) and \( N_e \), respectively. Note that we can get the lower bound of \( N_e \) by controlling the distance of \( f^{N_e}(y_e) \) and \( f^{N_e}(q) \) along unstable leaves directly. However, it is difficult to estimate the upper bound of \( K_e \) under the dynamics of \( f \), while it is convenient in linear systems. Thus, we calculate the "\( N_e \)" and "\( K_e \)" of \( A \). A direct way to get Claim 3.8 is using the Hölder continuity of \( h \), but here we prove it by only uniform continuity. See Fig. 1.

Proof of Claim 3.8. Let \( N'_e \) be the maximal positive integer such that \( d(A^j(w'), A^j(q')) < \eta'_0 \), for all \( w' \in I'_e \) and \( j \in [0, N'_e] \). Let \( K'_e \) be the minimal positive integer such that \( \{ x \in \mathbb{T}^d | A^{K'_e}x = p' \} \cap B_{\delta}(q') \neq \emptyset \), where \( p' := h(p) \). It is clear that \( N_e \geq N'_e \) and \( K_e \leq K'_e \). So, we get \( N_e / K_e \geq N'_e / K'_e \).

For every \( w' \in I'_e \) and \( j \in [0, N'_e] \), we have
\[
d \left( A^j(w'), A^j(q') \right) \leq d_{\mathcal{L}_s} \left( A^j(w'), A^j(y'_e) \right) + d_{\mathcal{L}_u} \left( A^j(y'_e), A^j(q') \right) \leq \frac{\eta'_0}{2} + d(x'_e, y'_e) + \varepsilon' \cdot \left( \mu_{\text{max}}(A) \right)^j.
\]

Note that as \( \varepsilon \) small enough, since the distance between any two points on \( I'_e \) is less than \( \frac{\eta'_0}{2} \), we have \( d(x'_e, y'_e) \leq \frac{\eta'_0}{6} \). So, the maximal positive integer \( N'_e \) such that
\[
\frac{\eta'_0}{2} + d(x'_e, y'_e) + \varepsilon' \cdot \left( \mu_{\text{max}}(A) \right)^j \leq \frac{2\eta'_0}{3} + \varepsilon' \cdot \left( \mu_{\text{max}}(A) \right)^j \leq \eta'_0,
\]
holds should satisfy

\[ N'_e \geq \frac{\ln \eta'_0 - \ln (3\varepsilon')}{\ln \mu_{\max} (A)}. \]

Note that for every \( \varepsilon' > 0 \),

\[ K'_e \leq \frac{\ln \varepsilon' - \ln C_0}{\ln \alpha}. \]

Thus, as \( \varepsilon' \) tending to 0, there exists \( C_3 > 0 \) such that,

\[ \frac{N_e}{K_e} \geq \frac{N'_e}{K'_e} \geq C_3, \]

where \( C_3 \) could be close to \( \frac{-\ln \alpha}{\ln \mu_{\max} (A)} \) arbitrarily.

Using the uniform lower bound of the time ratio \( N_e/K_e \) of \( I_\varepsilon \) being around \( q \) to reaching \( p \), we can get an exponential error between \( |f^K_\varepsilon (I_\varepsilon)| \) and \( |f^K_\varepsilon (J_\varepsilon)| \).

Since we have assumed that \( p, q \) are fixed points of \( f \), then \( f^j (J_\varepsilon) \subset B_{\eta_0} (p) \), for every \( j \geq 0 \). So,

\[ |f^K_\varepsilon (J_\varepsilon)| \leq (\mu_- \cdot (1 + \delta)^2)^K_e |J_\varepsilon|. \]

And,

\[ |f^K_\varepsilon (I_\varepsilon)| = |f^{K_e-N_e} \circ f^{N_e} (I_\varepsilon)| \geq \left( \frac{\mu_-}{1 + \delta} \right)^{K_e-N_e} \cdot \left( \frac{\mu_+}{(1 + \delta)^2} \right)^{N_e} |I_\varepsilon|. \]

Consequently,

\[ \frac{|f^K_\varepsilon (I_\varepsilon)|}{|f^K_\varepsilon (J_\varepsilon)|} \geq \frac{1}{(1 + \delta)^{3K_e+N_e}} \cdot \left( \frac{\mu_+}{\mu_-} \right)^{N_e} \cdot \frac{|I_\varepsilon|}{|J_\varepsilon|} \geq \frac{1}{(1 + \delta)^{4K_e}} \cdot \left( \frac{\mu_+}{\mu_-} \right)^{N_e} \cdot \frac{1}{C_2} \]
Proof of Proposition 3.4.

We can assume that \( K_\varepsilon \geq N_\varepsilon \), so that we have the second inequality. Otherwise, when \( K_\varepsilon < N_\varepsilon \), the whole estimation of \( \frac{|f^{K_\varepsilon}(I_\varepsilon)|}{|f^{K_\varepsilon}(J_\varepsilon)|} \) is trivial.

Note that \((1 + \delta)^{-4} \cdot \left( \frac{\mu_+}{\mu_-} \right)^{C_1} > 1\). Let \( \varepsilon \) tend to zero. We have \( K_\varepsilon \to +\infty \), hence \( \frac{|f^{K_\varepsilon}(I_\varepsilon)|}{|f^{K_\varepsilon}(J_\varepsilon)|} \to +\infty \). This contradicts the fact that \( f^{K_\varepsilon}(I_\varepsilon) = f^{K_\varepsilon}(J_\varepsilon) \). This finishes the proof of Proposition 3.2.

Now, we show that there exists a \( C^1 \) neighborhood \( \mathcal{U} \) of \( A \) in which every special \( f \) admits the finest (on stable bundle) dominated splitting. Combining with Proposition 3.2, if \( f \in \mathcal{U} \) is special, then \( \lambda_1^s(p, f) = \lambda_1^s(q, f) \), for all \( p, q \in \text{Per}(f) \) and \( 1 \leq i \leq k \).

**Proof of Proposition 3.4.** Since \( E_i^s = E_{(1, i)}^s \cap E_{(i, k)}^s \), it suffices to prove the following lemma.

**Lemma 3.9.** There exists a \( C^1 \) neighborhood \( \mathcal{U} \subset C^1(\mathbb{T}^d) \) of \( A \) such that, for every \( f \in \mathcal{U} \) and \( 1 \leq i \leq k - 1 \), if \( f \) is special, then it admits the following special dominated splitting

\[
E_{(1, i)}^s \oplus E_{(i+1, k)}^s \oplus E^u,
\]

and \( E_{(i+1, k)}^s \) is integrable.

**Proof of Lemma 3.9.** By Proposition 2.12, \( E_{(1, i)}^s \) is well defined on \( \mathbb{T}^d \). By the assumption that \( f \) is special, the unstable bundle \( E^u \) is also well defined on \( \mathbb{T}^d \).

Let \( F \) be a lifting of \( f \) and \( H \) be the conjugacy between \( F \) and \( A \). As the proof of Proposition 2.17, there exists a \( C^1 \) neighborhood \( \mathcal{U} \subset C^1(\mathbb{T}^d) \) of \( A \) such that for every \( f \in \mathcal{U} \), its lifting \( F \) admits a dominated splitting

\[
\tilde{E}_{(1, i)}^s \oplus \tilde{E}_{(i+1, k)}^s \oplus E^u,
\]

and \( E_{(i+1, k)}^s \) is integrable. Moreover, by the forth item of Proposition 2.17, \( H(y) \in \tilde{\mathcal{L}}_{(i+1, k)}^s(H(x)) \) if and only if \( y \in \tilde{\mathcal{F}}_{(i+1, k)}^s(x) \). Note that, since \( f \) is special, \( H(x + n) = H(x) + n \), for every \( x \in \mathbb{R}^d \) and \( n \in \mathbb{Z}^d \). Thus, we have that

\[
y + n \in \tilde{\mathcal{F}}_{(i+1, k)}^s(x + n) \iff H(y + n) \in \tilde{\mathcal{L}}_{(i+1, k)}^s(H(x) + n),
\]

\[
\iff H(y) + n \in \tilde{\mathcal{L}}_{(i+1, k)}^s(H(x) + n),
\]

\[
\iff H(y) \in \tilde{\mathcal{L}}_{(i+1, k)}^s(H(x)),
\]

\[
\iff y \in \tilde{\mathcal{F}}_{(i+1, k)}^s(x).
\]

It means that \( \tilde{E}_{(i+1, k)}^s \) is \( \mathbb{Z}^d \)-periodic, hence it can descend to \( \mathbb{T}^d \) through (2.12). \( \square \)

Lemma 3.9 and the fact \( E_i^s = E_{(1, i)}^s \cap E_{(i, k)}^s \) finish the proof of Proposition 3.4. \( \square \)
3.2. The conjugacy preserves strong stable foliations. Now we prove Proposition 3.5 that is \( h(\mathcal{F}_i^s) = \mathcal{L}_i^s \), for all \( 1 \leq i \leq k \), where \( h \) is the conjugacy between the special \( f \) and its linearization \( A \). Note that we assume that the preimage set of \( A \) is becoming dense exponentially in this subsection.

**Proof of Proposition 3.5.** By Proposition 2.17, we already have \( h\left(\mathcal{F}_{(i,k)}^s\right) = \mathcal{L}_{(i,k)}^s \), for every \( 1 \leq i \leq k \). Since \( \mathcal{F}_i^s = \mathcal{F}_{(1,i)}^s \cap \mathcal{F}_{(i,k)}^s \), it suffices to prove the following lemma.

**Lemma 3.10.** Let \( f \in \mathcal{U} \) given by Proposition 3.4 be special. Then for every \( 1 \leq i \leq k \), \( h(\mathcal{F}_{(1,i)}^s) = \mathcal{L}_{(1,i)}^s \).

**Proof of Lemma 3.10.** Fix \( 1 \leq i \leq k - 1 \). Firstly, we prove the joint integrability of the bundle \( E_{(1,i)}^s \oplus E^u \).

**Claim 3.11.** The bundle \( E_{(1,i)}^s \oplus E_{(1,i)}^u \) is jointly integrable.

**Proof of Claim 3.11.** For any \( x \in \mathbb{T}^d \), \( y \in \mathcal{F}_{(1,i)}^s(x,\delta) \setminus \{x\} \) and \( x' \in \mathcal{F}_{(1,i)}^u(x) \setminus \{x\} \), let \( y' = \text{Hol}_{x,x'}^u(y) \in \mathcal{F}_{(1,i)}^s(x') \), where \( \text{Hol}_{x,x'}^u : \mathcal{F}_{(1,i)}^s(x,\delta) \to \mathcal{F}_{(1,i)}^s(x',\delta) \) is the holonomy map along the unstable foliation \( \mathcal{F}_{(1,i)}^u \). It suffices to show that \( y' \in \mathcal{F}_{(1,i)}^s(x') \).

Let \( I \) be any curve homeomorphic to \([0,1] \) and laying on \( \mathcal{F}_{(1,i)}^s(x,\delta) \) with endpoints \( x, y \). Let \( J = \text{Hol}_{x,x'}^u(I) \). Since the conjugacy \( h \) maps \( \mathcal{F}_{(1,i)}^u \) to \( \mathcal{L}_{(1,i)}^u \), the curve \( h(J) \) is a translation of \( h(I) \). Now, by assumption, there exists \( z_n \to h(x') \) with \( A^n z_n = A^n h(x) \). Moreover, we can pick curves \( I_n \) with \( z_n \in I_n \) such that \( A^n I_n = A^n h(I) \). Hence, the other endpoint \( w_n \neq z_n \) of \( I_n \) also has \( A^n w_n = A^n h(y) \). Since \( I_n \) is also a translation of \( h(I) \), one has \( I_n \to h(J) \) and \( w_n \to h(y') \). See Fig. 2.

Let \( x_n = h^{-1}(z_n) \) and \( y_n = h^{-1}(w_n) \). We have that \( x_n \to x' \) and \( y_n \to y' \). Since \( A^n I_n = A^n h(I) \), one has \( f^n(h^{-1} I_n) = f^n(I) \). It follows that \( h^{-1}(I_n) \) is a curve laying on \( \mathcal{F}_{(1,i)}^s(x_n) \) with endpoints \( x_n \) and \( y_n \). Moreover, by the continuity of \( h \) and \( h^{-1} \), the curve \( h^{-1}(I_n) \) is located in a small tubular neighborhood of \( J \), when \( n \) is big enough. Hence, we have that \( h^{-1}(I_n) \to J \). By the continuity of the foliation \( \mathcal{F}_{(1,i)}^s \), we get \( J \subset \mathcal{F}_{(1,i)}^s(x') \) and \( y' \in \mathcal{F}_{(1,i)}^s(x') \).

Now, by the fact that \( \mathcal{F}_{(1,i)}^s \) is subfoliated by the unstable foliation \( \mathcal{F}_{(1,i)}^u \) which is minimal, we can get that \( h(\mathcal{F}_{(1,i)}^s) \) is a linear foliation.

**Claim 3.12.** \( h(\mathcal{F}_{(1,i)}^s) \) is a linear foliation.
Proof of Claim 3.12. For convenience, we prove it on the universal cover space $\mathbb{R}^d$. Let $\tilde{\mathcal{W}} = H(\tilde{\mathcal{F}}^{s}_{(1,i)})$. Note that

$$H(\tilde{\mathcal{F}}^s) = \tilde{\mathcal{L}}^s, \quad H(\tilde{\mathcal{F}}^u) = \tilde{\mathcal{L}}^u, \quad H(\tilde{\mathcal{F}}^{s}_{(1,i)}) = \mathcal{W} \cap \tilde{\mathcal{L}}^s.$$

We claim that $\tilde{\mathcal{W}}(0)$ is a closed connected subgroup of $\mathbb{R}^d$, that is $x + y \in \tilde{\mathcal{W}}(0)$ and $-x \in \tilde{\mathcal{W}}(0)$, for all $x, y \in \tilde{\mathcal{W}}(0)$. Indeed, since $\tilde{\mathcal{L}}^u$ is linear and $\tilde{\mathcal{F}}^u(x+n) = \tilde{\mathcal{F}}^u(x)+n$, for all $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$, we have

1. $\tilde{\mathcal{W}}(x+n) = \tilde{\mathcal{W}}(x) + n$, for all $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$.
2. $\tilde{\mathcal{W}}(x+v) = \tilde{\mathcal{W}}(x)$, for all $x \in \mathbb{R}^d$ and $v \in \tilde{\mathcal{L}}^u(0)$.

Let $x, y \in \tilde{\mathcal{W}}(0)$. Since the foliation $\mathcal{L}^u$ is minimal, there exist $n \in \mathbb{Z}^d$ and $v_n \in \tilde{\mathcal{L}}^u(0)$ such that $n + v_n \to x$. By $y \in \tilde{\mathcal{W}}(0)$, we have $v_n + y \in \tilde{\mathcal{W}}(0)$. Hence, $n + v_n + y \in \tilde{\mathcal{W}}(n) = \tilde{\mathcal{W}}(n + v_n).$ Let $n + v_n \to x$, it follows that $x + y \in \tilde{\mathcal{W}}(x) = \tilde{\mathcal{W}}(0)$.

The fact that $\tilde{\mathcal{W}}(0)$ is a closed connected subgroup of $\mathbb{R}^d$ implies that $\tilde{\mathcal{W}}(0)$ is linear and we refer to [23, Lemma 6.2] for a precise proof on this. It follows that the foliation $\tilde{\mathcal{W}}$ is linear, since $\pi(\tilde{\mathcal{W}}(0)) \subset \mathcal{L}^u(0)$ is dense in $\mathbb{T}^d$ and $\tilde{\mathcal{W}}$ is continuous. Hence $H(\tilde{\mathcal{F}}^{s}_{(1,i)}) = \tilde{\mathcal{L}}^s \cap \tilde{\mathcal{W}}$ is a linear foliation.

Now, we can finish our proof of Lemma 3.10. Note that the linear foliation $H(\tilde{\mathcal{F}}^{s}_{(1,i)}) \subset \tilde{\mathcal{L}}^s$ is $A$-invariant. So, it must be a union foliation of $i$ subfoliations of $\mathcal{L}^s$, that is $H(\tilde{\mathcal{F}}^{s}_{(1,i)}) = \mathcal{L}^s_{i_1} \oplus \ldots \oplus \mathcal{L}^s_{i_k}$. On the other hand, by Proposition 2.17, we have $H(\tilde{\mathcal{F}}^{s}_{(i+1,k)}) = \mathcal{L}^s_{(i+1,k)}$. Since $H$ is a homeomorphism, we get $H(\tilde{\mathcal{F}}^{s}_{(1,i)}) = \mathcal{L}^s_{(1,i)}$. □ □

4. Affine Metric

Let $f$ belong to $\mathcal{W}$ given by Proposition 2.17. Applying Livschitz Theorem for Anosov maps (Proposition 4.1), the spectral rigidity on the stable bundle implies that we can endow with a metric to each stable foliation $\tilde{\mathcal{F}}^{s}_i$ such that $F$ is affine restricted on it. Especially, for the case of $\dim E^s = 1$, the existence of the affine metric is just like the diffeomorphism case [14, Lemma 3.1] whether $f$ is a small perturbation or not. However, for the case of $\dim E^s > 1$, there are something quite different: we do not a priori have foliation $\mathcal{F}^{s}_i$ on $\mathbb{T}^d$ for non-invertible Anosov maps. So, we will give this affine metric on the lifting $\mathcal{F}^{s}_i$. Moreover, lack of the bundle $E^p_i(2 \leq i \leq k)$ on $\mathbb{T}^d$ prevents us from defining the contracting-rate function for $E^p_i$. We overcome this by using quotient dynamics.

Proposition 4.1 (Livschitz Theorem). Let $M$ be a closed Riemannian manifold, $f : M \to M$ be a $C^{1+\alpha}$ transitive Anosov map and $\phi : M \to \mathbb{R}$ be a $\alpha$-Hölder continuous function. Suppose that, for every $p \in \text{Per}(f)$ with period $\pi(p)$, $\sum_{i=0}^{\pi(p)-1} \phi(f^i(p)) = 0$. Then there exists a continuous function $\psi : M \to \mathbb{R}$ such that $\phi = \psi \circ f - \psi$. Moreover $\psi$ is unique up to an additive constant and $\alpha$-Hölder continuous.

The sketch of proof of Proposition 4.1. Note that the proof of this proposition is quite similar to the Livschitz Theorem for transitive Anosov diffeomorphisms whose complete proof can be found in [27, Corollary 6.4.17 and Theorem 19.2.1]. Here we just give the sketch of our proof.
1. Note that we still have the (un)stable manifolds (with respect to orbits) theorem and the local product structure for Anosov maps, see \[36, \text{Theorem 2.1 and Proposition 2.3}\]. For convenience, we state them as follow.

- (Unstable Manifold Theorem) For any \(\bar{x} = (x_i) \in M_f\) and \(R > 0\), the set
  \[
  \mathcal{F}_R^u(x_0, \bar{x}) = \{ y \in M \mid \exists (y_i) \in M_f \text{ with } y_0 = y \text{ such that } d(x_{-i}, y_{-i}) < R, \forall i \geq 0 \},
  \]
  is a submanifold, called unstable manifold with respect to \(\bar{x}\), and tangent to \(E^u(x_0, \bar{x})\). The unstable manifolds vary continuously in \(C^{1+\alpha}\)-topology, namely, for a sequence of orbits \(\bar{x}^k = (x_i^k)_{i \in \mathbb{Z}} \in M_f(k \in \mathbb{N})\) with \(\bar{x}^k \rightarrow \bar{y} = (y_i) \in M_f\) as \(k \rightarrow +\infty\) and for any \(R > 0\),
  \[
  \mathcal{F}_R^u(x_0^k, \bar{x}^k) \rightarrow \mathcal{F}_R^u(y_0, \bar{y}),
  \]
  in \(C^1\)-topology as \(k \rightarrow +\infty\).

- (Local Product Structure) There exist \(\delta > 0\) and \(0 < \varepsilon < \delta\) such that for any \(x_0 \in M\) and any \(\bar{y} = (y_i) \in M_f\), if \(d(x_0, y_0) < \varepsilon\), then \(\mathcal{F}_\delta^s(y_0, \bar{y})\) transversely intersects with the stable manifold \(\mathcal{F}_\delta^s(x_0) := \{ y \in M \mid d(f^i(x_0), f^i(y)) < \delta, \forall i \geq 0 \}\) at a unique point \(z(x_0, \bar{y})\). In particular, \(z(x_0, \bar{y})\) is continuous with respect to \((x_0, \bar{y}) \in M \times M_f\). Moreover, there exists a unique orbit \(\bar{z} = (z_i) \in M_f\) such that \(z_0 = z(x_0, \bar{y})\) and \(d(z_{-i}, y_{-i}) < \delta\) for all \(i \geq 0\).

2. We also have the following Anosov Closing Lemma. There exist constants \(C > 0\) and \(\varepsilon_0 > 0\) such that for every \(\varepsilon < \varepsilon_0\) and any periodic \(\varepsilon\)-pseudo orbit \((x_0, \ldots, x_m)\) on \(M\), i.e, \(d(f(x_i), x_{i+1}) < \varepsilon\) for \(0 \leq i \leq m - 1\) and \(d(f(x_m), x_0) < \varepsilon\), there exists a point \(y \in M\) such that \(f^m(y) = y\) and \(d(f^i(y), x_i) < C\varepsilon\) for every \(0 \leq i \leq m - 1\).

3. Now we can prove Livschitz Theorem using the classical method (see \[27, \text{Theorem 19.2.1}\]). Choosing a transitive point \(x_0\) of \(f\) and a value \(\psi(x_0) \in \mathbb{R}\), let \(\psi(f^n(x_0)) := \psi(x_0) + \sum_{i=0}^{n-1} \phi(f^i(x_0))\). By the Anosov exponential closing lemma and the periodic data, there exists \(C_0 > 0\) such that if \(d(f^n(x_0), f^m(x_0)) < \varepsilon_0\) where \(\varepsilon_0\) is given by the second item, one has
  \[
  \left| \psi(f^n(x_0)) - \psi(f^m(x_0)) \right| < C_0 \cdot d(f^n(x_0), f^m(x_0))^\alpha.
  \]
  Since the forward orbit \(\text{Orb}^+(x_0)\) is dense in \(M\), the \(\alpha\)-Hölder continuous function \(\psi|_{\text{Orb}^+(x_0)}\) can extend uniquely to a \(\alpha\)-Hölder continuous function. And this is a desired function.

Remark 4.2. We also mention that every Anosov map on torus is transitive, so that the Livschitz Theorem always holds for toral Anosov maps. If one need, we refer to \[1, \text{Theorem 6.8.1}\] which says that \(\sigma_f : \mathbb{T}^d_f \rightarrow \mathbb{T}^d_f\) the inverse limit system of \(f\) is topologically conjugate to \(\sigma_A : \mathbb{T}_A^d \rightarrow \mathbb{T}_A^d\) the inverse limit system of \(A\). Combining the fact that the toral Anosov map \(A\) is transitive, we get the transitivity of \(f\). We also refer to \[34, \text{Proposition 1.2}\] to get a proof for transitivity of Anosov maps on infra-nilmanifolds.
Now, using quotient dynamics, we can apply Livschitz Theorem to $F : \mathbb{R}^d \to \mathbb{R}^d$.

**Proposition 4.3.** Let $f \in \mathcal{U}$ be given by Proposition 2.17 be $C^{1+s_\text{C}}$-smooth. Fix $1 \leq i \leq k$, if $\lambda_i^s(p, f) = \lambda_i^s(A)$, for every $p \in \text{Per}(f)$, then there exists a Hölder continuous function $\Psi : \mathbb{R}^d \to \mathbb{R}$, such that

$$\log \|Df|_{E_i^s(x)}\| = \lambda_i^s(A) + \Psi(x) - \Psi(f(x)),$$

for every $x \in \mathbb{R}^d$. Moreover, $\Psi$ is bounded on $\mathbb{R}^d$.

**Proof.** For $i = 1$, since $E_1^s$ is well defined on $TT^d$ (see Proposition 2.12), we can use Livschitz Theorem (Proposition 4.1) for function $\|Df|_{E_1^s(x)}\|$. By the assumption that $\lambda_1^s(p, f) = \lambda_1^s(A)$ for all $p \in \text{Per}(f)$, there exists a Hölder continuous function $\psi : \mathbb{T}^d \to \mathbb{R}$ such that $\log \|Df|_{E_1^s(x)}\| = \lambda_1^s(A) + \psi(x) - \psi(f(x))$ for all $x \in \mathbb{T}^d$. The lifting of $\psi$, $\Psi : \mathbb{R}^d \to \mathbb{R}$ is the function we need.

Fix $2 \leq i \leq k$. Since the strong stable bundle $E_{i-1}^s$ is always well defined on $TT^d$, we can define $N \subset TT^d$ the normal bundle of $E_{i-1}^s \subset E_i^s$. Let $\pi^N : E_i^s \to N$ be the natural projection and $\overline{D}f : E_i^s \to N$ defined as $\overline{D}f(x, v) = \pi^N \circ D_x f(v)$, for all $x \in \mathbb{T}^d$ and $v \in E_{i-1}^s(x)$.

Let $\mu(x) := \|\overline{D}f|_{N(x)}\|$ and $E_i^s(p)$ be certained by the return map of the periodic point $p$ with period $\pi(p)$, that is the eigenspace of the eigenvalue $\exp(\lambda_{i}^p(f, p)) = \mu_{i}^p(A)$ for $D_p f^{\pi(p)}$. For any unit vector $v \in N(p)$, let $v = \sum_{j=1}^{i} v_j$, where $v_j \in E_j^s(p)$.

One has

$$D_p f^{\pi(p)}(v) = \sum_{j=1}^{i} (\mu_{j}^p(A))^{\pi(p)} v_j,$$

(4.1)

Note that

$$\overline{D}f(x, w) = \overline{D}f \circ \pi^N(x, w), \quad \forall w \in E_{i-1}^s(x).$$

(4.2)

Indeed, let $w \in E_{i-1}^s(x)$ and $w = w_{i-1}^s + w^N$ be the decomposition in $E_{i-1}^s \oplus N$, one has

$$\pi_{f x}^N \circ D_x f(w) = \pi_{f x}^N \circ D_x f \left( w_{i-1}^s + w^N \right) = \pi_{f x}^N \left( D_x f w_{i-1}^s + D_x f w^N \right),$$

$$= \pi_{f x}^N \circ D_x f w_{i-1}^s + \pi_{f x}^N \circ D_x f w^N = \pi_{f x}^N \circ D_x f (w_N).$$

Hence, by (4.1) and (4.2),

$$(\overline{D}f)^{\pi(p)}(x, v) = \pi_p^N \circ D_p f^{\pi(p)}(v) = (\mu_{i}^p(A))^{\pi(p)} \pi_p^N (v_i) = (\mu_{i}^p(A))^{\pi(p)} v.$$

It follows that $\sum_{i=0}^{n-1} \log \mu(f^i(p)) = \pi(p) \cdot \mu_{i}^p(A)$ for all $p \in \text{Per}(f)$. Now, using Livschitz Theorem for $\log(\mu(x))$, there exists a Hölder continuous function $\phi : \mathbb{T}^d \to \mathbb{R}$ such that

$$\log \mu(x) = \lambda_{i}^p(A) + \phi(x) - \phi(f(x)).$$
Let $\bar{D}F : \tilde{E}^s_{(1,i)} \to \tilde{N}$ be the lifting of $Df : E^s_{(1,i)} \to N$, where $\tilde{N} \subset T\mathbb{R}^d$ is the lifting of $N$. Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be the lifting of $\phi : \mathbb{T}^d \to \mathbb{R}$. Denote $\tilde{\mu}(x) := \|\bar{D}F|\tilde{N}(x)\|$. Thus, we have

$$\log \tilde{\mu}(x) = \lambda^s_i(A) + \Phi(x) - \Phi(F(x)).$$

(4.3)

Note that $\Phi$ is bounded and Hölder continuous on $\mathbb{R}^d$.

Let $\alpha(x) := \log \cos \angle(\tilde{N}(x), \tilde{E}^s_i(x))$, we claim that

$$\log \|D\bar{F}|E^s_i(x)\| = \log \tilde{\mu}(x) + \alpha(x) - \alpha(F(x)).$$

(4.4)

Indeed, it is just a linear algebraic calculation. Let $v \in \tilde{E}^s_i(x)$, we have

$$\|v^{\tilde{N}}\| = \cos \angle(\tilde{N}(x), \tilde{E}^s_i(x)) \cdot \|v\| = e^{\alpha(x)} \cdot \|v\|,$$

and by (4.2) (this equation can also lift on $T\mathbb{R}^d$),

$$\tilde{\mu}(x)\|v^{\tilde{N}}\| = \pi_{\tilde{N}}^x \circ D_x F(v) = \cos \angle(\tilde{N}(F(x)), \tilde{E}^s_i(F(x))) \cdot \|D_x Fv\| = e^{\alpha(F(x))} \cdot \|D_x Fv\|.$$ 

Thus, we get (4.4).

Now, by (4.3) and (4.4), $\Psi = \Phi + \alpha$ is a function satisfying

$$\log \|D\bar{F}|E^s_i(x)\| = \lambda^s_i(A) + \Psi(x) - \Psi(F(x)).$$

Moreover, since the angle $\angle(\tilde{N}(x), \tilde{E}^s_i(x))$ is uniformly away from $\pi/2$ (see Proposition 2.12 and (2.13)), $\alpha(x)$ is a bounded function defined on $\mathbb{R}^d$. And, so is $\Psi$. The Hölder continuity of both $\angle(\tilde{N}(x), \tilde{E}^s_i(x))$ (see Remark 2.15) and function $\Phi$ implies one of $\Psi$. \qed

Now, we can endow an affine metric to each leaf of $\tilde{F}^s_i$ and it would be invariant under some certain holonomy maps. Let foliation $\tilde{\mathcal{F}}$ be subfoliated by foliations $\tilde{F}^1_1$ and $\tilde{F}^2$ which admit the Global Product Structure on $\tilde{\mathcal{F}}$. We define the holonomy map of $\tilde{\mathcal{F}}_1$ along $\tilde{\mathcal{F}}^2$ restricted on $\tilde{\mathcal{F}}$ as

$$\text{Hol}_{x,x'} : \tilde{\mathcal{F}}_1(x) \to \tilde{\mathcal{F}}_1(x') \quad \text{with} \quad \text{Hol}_{x,x'}(y) = \tilde{\mathcal{F}}_1(x') \cap \tilde{\mathcal{F}}_2(y),$$

for every $x \in \mathbb{R}^d$, $x' \in \tilde{\mathcal{F}}_2(y)$ and $y \in \tilde{\mathcal{F}}_1(x)$. Let $d^s_i(\cdot, \cdot)$ be a metric defined on each leaf of $\tilde{F}^s_i$, we say $d^s_i(\cdot, \cdot)$ is continuous, if for any $\varepsilon > 0$, $x \in \mathbb{R}^d$ and $y \in \tilde{F}^s_i(x)$, there exists $\delta > 0$ such that

$$|d^s_i(x, y) - d^s_i(x', y')| < \varepsilon,$$

for all $x' \in B(x, \delta)$ and $y' \in B(y, \delta)$ with $y' \in \tilde{F}^s_i(x')$.

**Proposition 4.4.** Let $f \in \mathcal{U}$ given by Proposition 2.17 be $C^{1+\alpha}$-smooth. Fix $1 \leq i \leq k$, if $\lambda^s_i(p, f) = \lambda^s_i(A)$, for every $p \in \text{Per}(f)$, then there exists a continuous metric $d^s_i(\cdot, \cdot)$ defined on each leaf of $\tilde{F}^s_i$ satisfying,

1. There exists a constant $K > 1$, such that $1/K \cdot d_{\tilde{\mathcal{F}}_1}(x, y) < d^s_i(x, y) < K \cdot d_{\tilde{\mathcal{F}}_1}(x, y)$, for every $x \in \mathbb{R}^d$ and $y \in \tilde{F}^s_i(x)$. 

2. \( d^s_i(Fx, Fy) = \exp(\lambda^s_i(A)) \cdot d^s_i(x, y), \) for every \( x \in \mathbb{R}^d \) and \( y \in \tilde{\mathcal{F}}^s_i(x) \).

3. The holonomy maps of \( \tilde{\mathcal{F}}^s_i \) along \( \tilde{\mathcal{F}}^s_{(1, i-1)} \) \( (2 \leq i \leq k) \) restricted on \( \tilde{\mathcal{F}}^s_{(1, i)} \) are isometric under the metric \( d^s_i(\cdot, \cdot) \).

4. If \( \tilde{E}^s_i \oplus \tilde{E}^u \) is integrable, then the holonomy maps of \( \tilde{\mathcal{F}}^s_i \) along \( \tilde{\mathcal{F}}^u \) restricted on \( \tilde{\mathcal{F}}^s_i \oplus \tilde{\mathcal{F}}^u \) are isometric under the metric \( d^s_i(\cdot, \cdot) \).

Especially, when \( \dim E^s = 1 \), if \( \lambda^s(p, f) = \lambda^s(A) \), for every \( p \in \text{Per}(f) \), there exists a continuous metric \( d^s(\cdot, \cdot) \) defined on each leaf of \( \tilde{\mathcal{F}}^s \) satisfying the first two items. Moreover, the holonomy maps of \( \tilde{\mathcal{F}}^s \) along \( \tilde{\mathcal{F}}^u \) are isometric under the metric \( d^s(\cdot, \cdot) \).

**Proof of Proposition 4.4.** Denote the Lebesgue measure on each leaf of \( \tilde{\mathcal{F}}^s_i \) by \( \text{Leb}^s_i \). For every \( x \in \mathbb{R}^d \) and \( y \in \tilde{\mathcal{F}}^s_i(x) \), using the same notations in Proposition 4.3, the following formula

\[
d^s_i(x, y) := \int_x^y e^{\Psi(t)} d\text{Leb}^s_i(t),
\]

defines the metric we need. It is clear that the metric \( d^s_i(\cdot, \cdot) \) is continuous.

Since \( \Psi \) is a bounded function defined on \( \mathbb{R}^d \), say \( \|\Psi\|_{C_0} \leq \log K \), the metric \( d^s_i(\cdot, \cdot) \) is \( K \)-equivalent to \( d^{\tilde{\mathcal{F}}^s_i}(\cdot, \cdot) \). So, we get the first item. For the second one, we calculate directly. Let \( y \in \tilde{\mathcal{F}}^s_i(x) \),

\[
d^s_i(F(x), F(y)) = \int_{F(x)}^{F(y)} e^{\Psi(t)} d\text{Leb}^s_i(t)
= \int_x^y e^{\Psi \circ F(t)} \cdot \|DF|_{\tilde{E}^s_i(t)}\| d\text{Leb}^s_i(t)
= \int_x^y \exp(\lambda^s_i(A)) \cdot e^{\Psi(t)} \cdot \|DF|_{\tilde{E}^s_i(t)}\| d\text{Leb}^s_i(t)
= \exp(\lambda^s_i(A))d^s_i(x, y).
\]

Denote the holonomy maps of \( \tilde{\mathcal{F}}^s_i \) along \( \tilde{\mathcal{F}}^s_{(1, i-1)} \) restricted on \( \tilde{\mathcal{F}}^s_{(1, i)} \) and the holonomy maps of \( \tilde{\mathcal{F}}^u_i \) along \( \tilde{\mathcal{F}}^u \) restricted on \( \tilde{\mathcal{F}}^u_i \) by \( \text{Hol}^s_i \) and \( \text{Hol}^u_i \), respectively.

**Claim 4.5.** Assume that \( \tilde{E}^s_i \oplus \tilde{E}^u \) is integrable. For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for any two curves on \( \tilde{\mathcal{F}}^s_i \), \( \gamma_1 : [0, 1] \to \tilde{\mathcal{F}}^s_i \) and \( \gamma_2 : [0, 1] \to \tilde{\mathcal{F}}^s_i(x') \), one has that each one of the following conditions,

1. \( \gamma_2(\tau) = \text{Hol}^s_i_{x, x'} \circ \gamma_1(\tau), \) and \( d_{\tilde{\mathcal{F}}^s_{(1, i-1)}}(\gamma_1(\tau), \gamma_2(\tau)) < \delta, \forall \tau \in [0, 1] \).

2. \( \gamma_2(\tau) = \text{Hol}^u_i_{x, x'} \circ \gamma_1(\tau), \) and \( d_{\tilde{\mathcal{F}}^u}(\gamma_1(\tau), \gamma_2(\tau)) < \delta, \forall \tau \in [0, 1] \).

implies,

\[
\frac{d^s_i(\gamma_1(0), \gamma_1(1))}{d^s_i(\gamma_2(0), \gamma_2(1))} \in (1 - \varepsilon, 1 + \varepsilon).
\]
Proof of Claim 4.5. Firstly, by the uniform continuity of $\Psi$ (see Proposition 4.3), for any $\varepsilon_0 > 0$, there exists $\delta > 0$ such that $\|\Psi \circ \gamma_1 - \Psi \circ \gamma_2\|_{C_0} \leq \varepsilon_0$.

Then, we need control the deviation of holonomy maps. Applying [38, Theorem B] for dominated splitting

$$\tilde{E}^s_{(1,i-1)} \oplus \tilde{E}^s_i \oplus \tilde{E}^{s,u}_{(i+1,k)},$$

we get that $\text{Hol}^s_{x,x'}$ is $C^1$-smooth. Hence, we can assume there exists $\delta > 0$ such that $|D\text{Hol}^s_{x,x'}(\gamma_1(\tau))| \in (1 - \varepsilon_0, 1 + \varepsilon_0)$, $\forall \tau \in [0, 1]$.

Finally, we compute by definition,

$$\frac{d^s_i(\gamma_1(0), \gamma_1(1))}{d^s_i(\gamma_2(0), \gamma_2(1))} = \frac{\int_{\gamma_1(0)}^{\gamma_1(1)} e^{\Psi(t)} d\text{Leb}^s_i(t)}{\int_{\gamma_2(0)}^{\gamma_2(1)} e^{\Psi(t)} d\text{Leb}^s_i(t)}, \quad (4.5)$$

$$\in \left[ (1 + \varepsilon_0)^{-1} e^{-\varepsilon_0}, \ (1 - \varepsilon_0)^{-1} e^{\varepsilon_0} \right]. \quad (4.6)$$

For the second case, note that by Journé Lemma [26](see Proposition 2.21), if $\tilde{E}^s_i \oplus \tilde{E}^u$ is integrable, then each leaf of $\tilde{F}^s_i \oplus \tilde{F}^u$ is $C^{1+\alpha}$. Therefore, we can use the same way of [35, Theorem 7.1] to get the absolute continuity of $\text{Hol}^u_{x,x'}$. In fact, we have the Radon-Nikodym derivative of $\text{Hol}^u_{x,x'}$ as follow,

$$\text{Jac}(\text{Hol}^u_{x,x'})(x) = \prod_{n=0}^{\infty} \left\| DF_{\tilde{E}^s_i(F^{-n}z)} \right\|,$$

where $z = \text{Hol}^u_{x,x'}(x)$. Since the distribution $\tilde{E}^s_i$ is Hölder continuous (Remark 2.15), by the standard distortion control techniques, there exists $\delta > 0$ such that $|\text{Jac}(\text{Hol}^u_{x,x'})(\gamma_1(\tau))| \in (1 - \varepsilon_0, 1 + \varepsilon_0)$, $\forall \tau \in [0, 1]$. The rest of proof is similar to (4.6), since $\tilde{F}^s_i$ is one-dimensional. Note that the case of $\gamma_2(\tau) = \text{Hol}^s_{x,x'} \circ \gamma_1(\tau)$ can also be proved by using only the absolute continuity of the strong stable foliation $\tilde{F}^s_{(1,i-1)}$, namely the Radon-Nikodym derivative of $\text{Hol}^u_{x,x'}$.

Now, we prove that $\text{Hol}^u_{x,x'}$ is isometric under the metric $d^u_i(\cdot, \cdot)$ by iterating backward. An analogical way can prove one for $\text{Hol}^s_{x,x'}$ by iterating forward.

If there exist $y \in \tilde{F}^s_i(x)$ and $y' \in \tilde{F}^s_i(x')$ with $\text{Hol}^u_{x,x'}(x) = x'$ and $\text{Hol}^u_{x,x'}(y) = y'$ such that $d^u_i(x, y) \neq d^u_i(x', y')$. Iterating these points backward, we can assume that $\gamma_1(0) = F^{-n}(x)$, $\gamma_1(1) = F^{-n}(y)$ and $\gamma_2(0) = F^{-n}(x')$, $\gamma_2(1) = F^{-n}(y')$ satisfy the conditions of Claim 4.5, for a large $n \in \mathbb{N}$. Since $F$ is affine along $\tilde{F}^s_i$ under the metric $d^u_i(\cdot, \cdot)$, one has

$$\frac{d^s_i(F^{-n}x, F^{-n}y)}{d^s_i(F^{-n}x', F^{-n}y')} = \frac{d^s_i(x, y)}{d^s_i(x', y')}. \quad (4.7)$$

If we pick $\varepsilon$ small enough in Claim 4.5, then it contradicts with (4.7). This complete the proof.
5. Existence of Integrable Subbundles

In this section, we prove the sufficient parts of Theorem 1.1 and Theorem 1.2 under the assumption that every periodic point of \( f \) has the same Lyapunov spectrum on the stable bundle (\textit{a priori}, need not be equal to one of the linearization) and we restate as follow.

**Theorem 5.1.** Let \( A : \mathbb{T}^d \to \mathbb{T}^d \) be an irreducible linear Anosov map. Assume that \( A \) admits the finest (on stable bundle) dominated splitting,

\[
T\mathbb{T}^d = L^s_1 \oplus L^s_2 \oplus \cdots \oplus L^s_k \oplus L^u,
\]

where \( \dim L^s_i = 1, 1 \leq i \leq k \).

There exists a \( C^1 \) neighborhood \( \mathcal{U} \subset C^1(\mathbb{T}^d) \) of \( A \) such that for every \( C^{1+\alpha} \)-smooth \( f \in \mathcal{U} \), if \( \lambda^s_i(p, f) = \lambda^s_i(q, f) \), for all \( p, q \in \text{Per}(f) \) and all \( 1 \leq i \leq k \), then \( f \) admits the finest (on stable bundle) special dominated splitting,

\[
T\mathbb{T}^d = E^s_1 \oplus E^s_2 \oplus \cdots \oplus E^s_k \oplus E^u,
\]

where \( \dim E^s_i = 1, 1 \leq i \leq k \). Especially, \( f \) is special.

Moreover, when \( k = 1 \), for every \( C^{1+\alpha} \) Anosov map \( f \) with linearization \( A \), if \( \lambda^s(p, f) = \lambda^s(q, f) \), for every \( p, q \in \text{Per}(f) \), then \( f \) is special.

We mention that in Theorem 5.1, \( f \) can be invertible since an Anosov diffeomorphism is always a special Anosov map. Note that in general the weak stable bundle does not exist, see Remark 2.13.

It is convenient to give the scheme of our proof. In this section we always assume that \( A \) is irreducible and has the finest (on stable bundle) dominated splitting. Let \( f \in \mathcal{U} \) given by Proposition 2.17 and \( F : \mathbb{R}^d \to \mathbb{R}^d \) be a lifting of \( f \) and \( H : \mathbb{R}^d \to \mathbb{R}^d \) be the conjugacy between \( F \) and \( A \).

Firstly, we show that \( H \) maps every one-dimensional stable foliation \( \tilde{\mathcal{F}}_i^s(1 \leq i \leq k) \) to one of the linearization \( \mathcal{L}_i^s \). Moreover, it is an isometry along each leaf of \( \tilde{\mathcal{F}}_i^s \). As proved in Proposition 2.17, we already have that \( H \) preserves weak stable foliations. For reducing to each single leaf, we need the following two propositions. The idea to reduce the leaf conjugacy originated from [15] (also see [18]). A main tool in [15] is the minimal property of the foliation \( \mathcal{F}_i^s \). However, in our case, there is a priori no \( \mathcal{F}_i^s(i \geq 2) \) on \( \mathbb{T}^d \). So, we cannot use the minimal property, directly. This obstruction can be overcome by using a special \( \mathbb{Z}^d \)-sequence described in Proposition 2.6, Proposition 2.7 (also Remark 2.8) and Proposition 2.9.

Again, by Proposition 2.17, we already have \( H(\tilde{\mathcal{F}}_i^s) = \mathcal{L}_i^s \). We show that it is an isometry restricted on each leaf of \( \tilde{\mathcal{F}}_k^s \) under the metric \( d_k^s(\cdot, \cdot) \) given by Proposition 4.4. Generally, we have the following proposition.

**Proposition 5.1.** Let \( f \in \mathcal{U} \) given by Proposition 2.17. Fix \( 1 \leq i \leq k \), assume that \( H(\tilde{\mathcal{F}}_i^s) = \mathcal{L}_i^s \) and \( \lambda^s_i(p, f) = \lambda^s_i(A) \) for all \( p \in \text{Per}(f) \). Then \( H \) is isometric restricted on each leaf of \( \tilde{\mathcal{F}}_i^s \) under the metric \( d_i^s(\cdot, \cdot) \) given by Proposition 4.4. Especially, for an irreducible Anosov map \( f \) with \( \dim E^s = 1 \), if \( \lambda^s(p, f) = \lambda^s(A) \) for all \( p \in \text{Per}(f) \), then \( H \) is isometric restricted on each leaf of \( \tilde{\mathcal{F}}^s \) under the metric \( d^s(\cdot, \cdot) \).

The following proposition allow us to reduce the leaf conjugacy by induction.
Proposition 5.2. Let \( f \in \mathcal{U} \) given by Proposition 2.17. Fix \( 1 < i \leq k \), assume that \( H(\tilde{\mathcal{F}}^s_{(1,i)}) = \mathcal{L}^s_{(1,i)} \) and \( H(\tilde{\mathcal{F}}^s_i) = \mathcal{L}^s_i \). If \( H \) is isometric restricted on each leaf of \( \tilde{\mathcal{F}}^s_i \) under the metric \( d^s_i(\cdot, \cdot) \), then \( H(\tilde{\mathcal{F}}^s_{(1,i-1)}) = \mathcal{L}^s_{(1,i-1)} \).

We leave the proofs for Propositions 5.1 and 5.2 in Sect. 5.1. Combining these two propositions, we can prove that \( H \) preserves every one-dimensional stable foliation \( \tilde{\mathcal{F}}^s_i \) and in fact is an isometry restricted on each leaf of \( \tilde{\mathcal{F}}^s_i \).

Corollary 5.3. Let \( f \in \mathcal{U} \) given by Proposition 2.17 with \( \lambda^s_i(p,f) = \lambda^s_i(q,f) \) for all \( p, q \in \text{Per}(f) \) and all \( 1 \leq i \leq k \). Then, for every \( 1 \leq i \leq k \), \( H(\tilde{\mathcal{F}}^s_i) = \mathcal{L}^s_i \) and \( H \) is isometric along each leaf of \( \tilde{\mathcal{F}}^s_i \) under the metric \( d^s_i(\cdot,\cdot) \).

Proof of Corollary 5.3. We get the proof by induction. The beginning of the induction is \( H(\tilde{\mathcal{F}}^s_{(1,k)}) = \mathcal{L}^s_{(1,k)} \) (see Proposition 2.17), especially, \( H(\tilde{\mathcal{F}}^s_k) = \mathcal{L}^s_k \).

Now, by Proposition 5.1, \( H : \tilde{\mathcal{F}}^s_k \rightarrow \mathcal{L}^s_k \) is isometric. Thus, by Proposition 5.2, \( H(\tilde{\mathcal{F}}^s_{(1,k)}) = \mathcal{L}^s_{(1,k)} \) implies \( H(\tilde{\mathcal{F}}^s_{(1,k-1)}) = \mathcal{L}^s_{(1,k-1)} \). Moreover, since \( H \) preserves the weak stable foliation \( H(\tilde{\mathcal{F}}^s_{(k-1,k)}) = \mathcal{L}^s_{(k-1,k)}, \) we have

\[
H(\tilde{\mathcal{F}}^s_{k-1}) = H(\tilde{\mathcal{F}}^s_{(1,k-1)} \cap \tilde{\mathcal{F}}^s_{(k-1,k)}) = \mathcal{L}^s_{(1,k-1)} \cap \mathcal{L}^s_{(k-1,k)} = \mathcal{L}^s_{k-1}.
\]

Applying the preceding methods to \( \tilde{\mathcal{F}}^s_{k-1} \) and \( \tilde{\mathcal{F}}^s_{(k-1,k)} \), we have \( H(\tilde{\mathcal{F}}^s_{(1,k-2)}) = \mathcal{L}^s_{(1,k-2)} \). Moreover, by intersecting with \( \tilde{\mathcal{F}}^s_{(k-2,k)} \), we have \( H(\tilde{\mathcal{F}}^s_{k-2}) = \mathcal{L}^s_{k-2} \).

Consequently, we can finish our proof by induction. \( \square \)

Using the isometry \( H \) along each leaf of single stable foliations, we also can show that all stable foliations \( \tilde{\mathcal{F}}^s_i \) (\( 1 \leq i \leq k \)) are \( \mathbb{Z}^d \)-periodic. Moreover, there is no deviation between \( H^{-1}(x+n) \) and \( H^{-1}(x)+n \) along \( \tilde{\mathcal{F}}^s_1 \), for all \( x \in \mathbb{R}^d \) and \( n \in \mathbb{Z}^d \). More precisely, we have the following two propositions.

Proposition 5.4. Let \( f \in \mathcal{U} \) given by Proposition 2.17. Assume that for every \( 1 \leq j \leq k \), \( H(\tilde{\mathcal{F}}^s_j) = \mathcal{L}^s_j \) and \( H \) is isometric along each leaf of \( \tilde{\mathcal{F}}^s_j \) under the metric \( d^s_j(\cdot,\cdot) \). Fix \( 1 \leq i \leq k \), if \( \tilde{\mathcal{F}}^s_{(i,k)} \) is \( \mathbb{Z}^d \)-periodic, then so is \( \tilde{\mathcal{F}}^s_{(i+1,k)} \).

Remark 5.5. Using Proposition 5.4 and by induction beginning with the fact that \( \tilde{\mathcal{F}}^s_{(1,k)} = \tilde{\mathcal{F}}^s \) is \( \mathbb{Z}^d \)-periodic, we have that \( \tilde{\mathcal{F}}^s_{(i,k)} \) (\( 1 \leq i \leq k \)) is \( \mathbb{Z}^d \)-periodic. Note that \( \tilde{\mathcal{F}}^s_{(i,i)} \) is always \( \mathbb{Z}^d \)-periodic (see Proposition 2.17). Thus, \( \tilde{\mathcal{F}}^s_i = \tilde{\mathcal{F}}^s_{(1,i)} \cap \tilde{\mathcal{F}}^s_{(i,k)} \) is also \( \mathbb{Z}^d \)-periodic, for all \( 1 \leq i \leq k \). It follows that for every \( 1 \leq i \leq k \), \( E^s_i \) is \( \mathbb{Z}^d \)-periodic. Hence, by (2.12), we get the bundle \( E^s_i \) defined well on \( \mathbb{T}^d \).

By Corollary 5.3, \( E^s_{(1,i-1)} \oplus \tilde{E}^s_{(i+1,k)} \) is integrable and denote the \( F \)-invariant integral foliations by \( \tilde{\mathcal{F}}^s_{i,-} \). We also denote \( \mathcal{L}^s_{(1,i-1)} \oplus \mathcal{L}^s_{(i+1,k)} \) by \( \mathcal{L}^s_{i,-} \).
Proposition 5.6. Let $f \in \mathcal{U}$ given by Proposition 2.17. Assume that for every $1 \leq i \leq k$, $H(\tilde{F}^s_i) = \tilde{L}^s_i$ and $H$ is isometric along each leaf of $\tilde{F}^s_i$ under the metric $d^s_i(\cdot, \cdot)$. Then

$$H^{-1}(x + n) - n \in \tilde{F}^s_i \setminus (H^{-1}(x)),$$

for all $1 \leq i \leq k$, $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$. Especially, for every irreducible Anosov map $f$ on torus with one dimensional stable bundle, if $H$ is isometric along each leaf of $\tilde{F}^s$ under the metric $d^s(\cdot, \cdot)$ then

$$H^{-1}(x + n) - n = H^{-1}(x),$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$.

We leave the proofs for Proposition 5.4 and Proposition 5.6 in Sect. 5.2. Now, by the previous propositions, we can prove Theorem 5.1.

Proof of Theorem 5.1. In the case of dim$E^s = 1$, since $\lambda^s(p, f) = \lambda^s(q, f)$ for all $p, q \in \text{Per}(f)$, combining Proposition 5.1 and Proposition 5.6, we have $H^{-1}(x + n) = H^{-1}(x) + n$, for all $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$. It means that $H$ can descend to $\mathbb{T}^d$. By Proposition 2.2, $f$ is special.

In the case of higher-dimensional stable bundle, by Corollary 5.3 and Proposition 5.6, we have

$$H^{-1}(x + n) - n \in \bigcap_{i=1}^k \left( \tilde{F}^s_{i, -1}(H^{-1}(x)) \right) = \{ H^{-1}(x) \},$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$. Hence, $f$ is special. And by Remark 5.5, $f$ admits $T\mathbb{T}^d = E^s_1 \oplus \ldots \oplus E^s_k \oplus E^u$.

Combining Theorem 3.1 and Corollary 5.3, we can show that, if the conjugacy between the non-invertible Anosov maps $f$ and $A$ exists, then it must be smooth along the stable foliation (see Corollary 1.5 and Corollary 1.8). We mention that it can be proved without Proposition 5.4 and Proposition 5.6.

Proof of Corollary 1.5 and Corollary 1.8. Assume that $f$ is $C^{1+\alpha}$-smooth. Let $h$ be a conjugacy between $f$ and $A$. By Proposition 2.2, $f$ is special. Then, by Theorem 3.1 and Remark 3.1, we have the spectral rigidity on stable bundle for $f$ which is exactly the condition stated in Theorem 5.1. Moreover, $f$ admits the finest (on stable bundle) dominated splitting and the conjugacy $h$ maps each stable foliation $\tilde{F}^s_i$ to $\tilde{L}^s_i$ (see Proposition 3.5). Since the bundle $E_i^s(1 \leq i \leq k)$ is Hölder continuous (see Remark 2.15) with exponent $\beta$, for some $0 < \beta \leq \alpha$, by Proposition 4.1 and the construction of $d_i^s(\cdot, \cdot)$ (see Proposition 4.4), the metric $d_i^s(\cdot, \cdot)$ is $C^{1+\beta}$-smooth along each leaf of $\tilde{F}^s_i$. So, Corollary 5.3 actually shows that $h$ is $C^{1+\beta}$-smooth along $\tilde{F}^s_i$, $1 \leq i \leq k$. It follows that $h$ is $C^{1+\beta}$-smooth along the stable foliation $\tilde{F}^s$, by Journé Lemma [26]. In the case of dim$E^s = 1$, since each leaf of the stable foliation $\tilde{F}^s$ is $C^{1+\alpha}$-smooth, the metric $d^s(\cdot, \cdot)$ is $C^{1+\alpha}$-smooth along each leaf of the stable foliation $\tilde{F}^s$ and so is $h$. $\Box$
5.1. Induction of the leaf conjugacy. In this subsection, we prove Proposition 5.1 and Proposition 5.2. Let \( f \in \mathcal{U} \) given by Proposition 2.17. Fix \( 1 \leq i \leq k \), assume that 
\[
H(\mathcal{F}_i^s) = \mathcal{L}_i^s \text{ and } \lambda_i^s(p, f) = \lambda_i^s(A) \text{ for all } p \in \text{Per}(f).
\]
We show that \( H \) is isometric restricted on \( \mathcal{F}_i^s \).

**Proof of Proposition 5.1.** By the existence of affine metric \( d_i^s(\cdot, \cdot) \) given by Proposition 4.4, we have that \( H \) is bi-Lipschitz along \( \mathcal{F}_i^s \). Indeed, one can just iterate any two points \( x_0, y_0 \in \mathbb{R}^d \) such that they are away an almost fixed distance, say \( d_i^s(F^n(x_0), F^n(y_0)) \in [C_1, C_2] \). By the uniform continuity of \( H \) and \( \mathcal{F}_i^s \) (see Remark 2.18), there exist \( C_1', C_2' > 0 \) such that for every \( x, y \in \mathbb{R}^d \), if \( d_i^s(x, y) \in [C_1, C_2] \) then \( d(H(x), H(y)) \in [C_1', C_2'] \). Hence, by Proposition 4.4 we have

\[
\frac{d(H(x_0), H(y_0))}{d_i^s(x_0, y_0)} = \frac{d(H \circ F^n(x_0), H \circ F^n(y_0))}{d_i^s(F^n(x_0), F^n(y_0))} \in \left[ \frac{C_1'}{C_2, C_1} \right].
\]

Similarly, we have \( H^{-1} \) is Lipschitz along \( \mathcal{F}_i^s \).

It is convenient to prove that \( H^{-1} \) is isometric along \( \mathcal{F}_i^s \). Thus, so is \( H \) along \( \mathcal{F}_i^s \).

By Lipschitz continuity, there exists a point \( x \in \mathcal{L}_i^s(0) \) such that \( H^{-1}|_{\mathcal{L}_i^s} \) is differentiable at \( x \) and we assume \((H^{-1}|_{\mathcal{L}_i^s})'(x) = C_0 \). It means that, for any \( \varepsilon > 0 \) small enough, there exists \( \delta > 0 \) such that

\[
\left| \frac{d_i^s(H^{-1}(x), H^{-1}(w))}{d_i^s(x, w)} - C_0 \right| < \frac{\varepsilon}{2}, \tag{5.1}
\]

for every \( w \in \mathcal{L}_i^s(x, \delta) \). Fix \( y \in \mathbb{R}^d \), for every \( z \in \mathcal{L}_i^s(y) \) and \( n_m \in \mathbb{Z}^d \), we denote

\[
y_m = \mathcal{L}_i^s(y + n_m) \cap \mathcal{L}_i^u(x), \quad z_m = \mathcal{L}_i^s(y_m) \cap \mathcal{L}_i^s, u^s(z + n_m) \quad \text{and} \quad x_m = \mathcal{L}_i^s(x) \cap \mathcal{L}_i^u(z_m).
\]

**Claim 5.7.** There exists a sequence \( \{n_m\} \subset \mathbb{Z}^d \) with \( n_m \in A^m \mathbb{Z}^d \), such that when \( m \to +\infty \),

\[
d(x, x_m) \to d(y, z), \tag{5.2}
\]

and

\[
d_i^s(H^{-1}(x), H^{-1}(x_m)) \to d_i^s(H^{-1}(z), H^{-1}(y)). \tag{5.3}
\]

**Proof of Claim 5.7.** Fix \( m \in \mathbb{N} \), we consider the point \( A^{-m}y \). Note that the unstable foliation \( \mathcal{L}_i^u \) is minimal on \( \mathbb{T}^d \). Thus we can choose \( n_m' \in \mathbb{Z}^d \) such that \( d(A^{-m}y + n_m', \mathcal{L}_i^u(A^{-m}x)) \leq 1 \). Let \( y(m) = \mathcal{L}_i^s(A^{-m}y + n_m') \cap \mathcal{L}_i^u(A^{-m}x) \) and \( n_m = A^m n_m' \). Note that \( y_m = A^m y(m) \) and

\[
d(y_m, y + n_m) \leq \|A\|_L^m d(y(m), A^{-m}y + n_m') \to 0,
\]

as \( m \to +\infty \). The sequence \( \{n_m\} \) is what we need. Let \( m \to +\infty \), we have

\[
d(y_m, y + n_m) \to 0 \quad \text{and} \quad d(z + n_m, z_m) \to 0. \tag{5.4}
\]
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Moreover,

\[ |d(y + n_m, z + n_m) - d(y_m, z_m)| \to 0. \]

Note that \( d(x, x_m) = d(y_m, z_m) \), then we get (5.2). See Fig. 3.

Now, by the uniform continuity of \( H^{-1} \) and (5.4), one has

\[ d\left( H^{-1}(z_m), H^{-1}(z + n_m) \right) \to 0 \quad \text{and} \quad d\left( H^{-1}(y_m), H^{-1}(y + n_m) \right) \to 0, \quad (5.5) \]

as \( m \to +\infty \). By \( H^{-1}(\tilde{\mathcal{L}}^s_i) = \tilde{\mathcal{F}}^s_i \) and \( H^{-1}(\tilde{\mathcal{L}}^u) = \tilde{\mathcal{F}}^u \), we have \( \tilde{\mathcal{E}}^s_i \oplus \tilde{\mathcal{E}}^u \) is jointly integrable. Thus, by the forth item of Proposition 4.4, one has

\[ d_i^s \left( H^{-1}(x), H^{-1}(x_m) \right) = d_i^s \left( H^{-1}(z_m), H^{-1}(y_m) \right). \quad (5.6) \]

Combining (5.5) and (5.6), we get

\[ \left| d_i^s \left( H^{-1}(x), H^{-1}(x_m) \right) - d_i^s \left( H^{-1}(z + n_m), H^{-1}(y + n_m) \right) \right| \to 0. \quad (5.7) \]

On the other hand, by Proposition 2.6, we have

\[ \left( H^{-1}(z + n_m) - n_m \right) \to H^{-1}(z) \quad \text{and} \quad \left( H^{-1}(y + n_m) - n_m \right) \to H^{-1}(y), \quad (5.8) \]

as \( m \to +\infty \). Note that \( H^{-1}(z) + n_m \) may not belong to \( \tilde{\mathcal{F}}^s_i(H^{-1}(y) + n_m) \). But, by Proposition 2.7,

\[ d \left( H^{-1}(z + n_m), \tilde{\mathcal{F}}^s_i(H^{-1}(y) + n_m) \right) \to 0. \quad (5.9) \]

Hence, by (5.8) and (5.9), one has

\[ d_i^s \left( H^{-1}(z + n_m), H^{-1}(y + n_m) \right) \to d_i^s \left( H^{-1}(z), H^{-1}(y) \right). \quad (5.10) \]

Consequently, according to (5.7) and (5.10), when \( m \to +\infty \), we get (5.3). \( \square \)
Now, by Claim 5.7, for any $\varepsilon > 0$ and $z \in \mathcal{L}^s_i(y)$, there exists $N_0 \in \mathbb{N}$ such that when $m > N_0$, one has
\[
\left| \frac{d_i^s(H^{-1}(y), H^{-1}(z))}{d(y, z)} - \frac{d_i^s(H^{-1}(x), H^{-1}(x_m))}{d(x, x_m)} \right| < \frac{\varepsilon}{2}.
\] (5.11)

Moreover, let $z \in \mathcal{L}^s_i(y, \frac{\varepsilon}{2})$, we have $d(x, x_m) < \delta$, for all $m > N_0$. Combining (5.1) and (5.11), for every $y \in \mathbb{R}^d$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that
\[
\left| \frac{d_i^s(H^{-1}(y), H^{-1}(z))}{d(y, z)} - C_0 \right| < \varepsilon,
\]
for every $z \in \mathcal{L}^s_i(y, \frac{\varepsilon}{2})$. It follows that $H$ is differentiable along $\mathcal{L}^s_i$ and the derivative is the constant $C_0$. Note that we can change the metric $d_i^s(\cdot, \cdot)$ by scaling such that $C_0 = 1$. \(\square\)

Now, fix $1 < i \leq k$, suppose that $H(\hat{\mathcal{F}}^s(1,i)) = \mathcal{L}^s_i(1,i)$. Note that since $H$ always preserves weak stable foliations, $H(\hat{\mathcal{F}}^s(1,i)) = \mathcal{L}^s_i(1,i)$ implies $H(\hat{\mathcal{F}}^s_i) = H(\hat{\mathcal{F}}^s(1,i) \cap \hat{\mathcal{F}}^s(i,k)) = \mathcal{L}^s_i$. Assume that $H$ is isometric restricted on each leaf of $\hat{\mathcal{F}}^s_i$ under the metric $d_i^s(\cdot, \cdot)$. We show $H(\hat{\mathcal{F}}^s(1,i-1)) = \mathcal{L}^s_i(1,i-1)$.

**Proof of Proposition 5.2.** By the assumption $H(\hat{\mathcal{F}}^s(1,i)) = \mathcal{L}^s_i(1,i)$, the foliation $\hat{\mathcal{F}} = H(\hat{\mathcal{F}}^s(1,i))$ is a sub-foliation of $\mathcal{L}^s_i(1,i)$. It is clear that $\hat{\mathcal{F}}$ and $\mathcal{L}^s_i$ give the Global Product Structure on $\mathcal{L}^s_i(1,i)$. By Proposition 4.4, the holonomy maps of $\hat{\mathcal{F}}^s_i$ along $\mathcal{L}^s_i(1,i-1)$ restricted on $\mathcal{L}^s_i(1,i)$ are isometric under the metric $d_i^s(\cdot, \cdot)$. Combining with the assumption that $H^{-1} : \mathcal{L}^s_i \to \hat{\mathcal{F}}^s_i$ is isometric, we have that the holonomy maps of $\mathcal{L}^s_i$ along $\hat{\mathcal{F}}$ restricted on $\mathcal{L}^s_i(1,i)$ are isometric.

Assume that $H^{-1}(\mathcal{L}^s_i(1,i-1)) \neq \mathcal{L}^s_i(1,i-1)$. It follows that there exist points $x_0$ and $x_1$ such that $x_1 \in \mathcal{F}(x_0) \setminus \mathcal{L}^s_i(1,i-1)(x_0)$.

**Claim 5.8.** There exist $b_m \in \mathcal{L}^s_i(x_0)$, $k_m \in A^m \mathbb{Z}^d$ and $x_2 \in \mathcal{F}(x_0)$ such that when $m \to +\infty$,
\begin{enumerate}
  \item $(b_m + k_m) \to x_1.$
  \item $d \left( y_m + k_m, \hat{\mathcal{F}}(b_m + k_m) \right) \to 0$, where $y_m = \hat{\mathcal{F}}(b_m) \cap \mathcal{L}^s_i(x_1)$.
  \item $(y_m + k_m) \to x_2.$
\end{enumerate}

**Proof of Claim 5.8.** By the first item of Proposition 2.9, we can choose $k_m \in A^m \mathbb{Z}^d$ and $b_m \in \mathcal{L}^s_i(x_0)$ such that when $m \to +\infty$, $(b_m + k_m) \to x_1$.

Now, let $y_m = \hat{\mathcal{F}}(b_m) \cap \mathcal{L}^s_i(x_1)$. Since $\|H^{-1} - Id\|$ is bounded, $H^{-1}(\mathcal{L}^s_i(1,i))$ is located in a neighborhood of $H^{-1}(\mathcal{L}^s_i(x_0))$. Hence, there exists $R > 0$ such that $d_{\mathcal{L}^s_i(1,i)}(H^{-1}(b_m), H^{-1}(y_m)) < R$, for all $b_m \in \mathcal{L}^s_i(x_0)$. By Proposition 2.6, $y_m \in \hat{\mathcal{F}}(b_m)$ implies
\[
d \left( H^{-1}(y_m), \hat{\mathcal{F}}(1,i-1) \left( H^{-1}(b_m + k_m) - k_m, R \right) \right) \to 0.
\]
By the proof of Claim 5.8, we actually have

\[ d \left( H^{-1}(y_m) + k_m, \mathcal{F}_{(1,i-1)}(H^{-1}(b_m + k_m), R) \right) \to 0. \]

Again, by Proposition 2.6,

\[ d \left( H^{-1}(y_m + k_m), \mathcal{F}_{(1,i-1)}(H^{-1}(b_m + k_m), R) \right) \to 0. \]  (5.12)

It follows that

\[ d(y_m + k_m, \mathcal{F}(b_m + k_m)) \to 0. \]

So, we complete the second item of claim.

Note that since \((b_m + k_m) \to x_1, (5.12)\) also implies

\[ d \left( H^{-1}(y_m + k_m), \mathcal{F}_{(1,i-1)}(H^{-1}(x_1), R) \right) \to 0. \]

Hence, \(d(y_m + k_m, \mathcal{F}(x_0)) \to 0\) and we get the third item. \(\square\)

Let \(z_1 = \mathcal{F}_{i}(x_1) \cap \mathcal{F}_{(1,i-1)}(x_0)\) and \(z_2 = \mathcal{F}_{i}(x_2) \cap \mathcal{F}_{(1,i-1)}(x_0)\). Since \(x_1 \notin \mathcal{F}_{(1,i-1)}(x_0)\), we have \(d(z_1, x_1) > 0\). See Fig. 4.

**Claim 5.9.** \(d(z_2, x_2) = 2 \cdot d(z_1, x_1)\) and \(d(x_0, z_2) = 2 \cdot d(x_0, z_1)\).

**Proof of Claim 5.9.** Let \(y_m \in \mathbb{Z}^d\) and \(b_m \in \mathcal{F}_{i}(x_0)\) given by Claim 5.8. Denote

\[ c_m = \mathcal{F}_{i}(b_m) \cap \mathcal{F}_{i}(x_1), \quad z'_2 = \mathcal{F}_{i}(x_1) \cap \mathcal{F}_{i}(x_2) \quad \text{and} \]

\[ y'_m = \mathcal{F}(b_m + k_m) \cap \mathcal{F}(y_m + k_m). \]

By the proof of Claim 5.8, we actually have \(d(y'_m, y_m + k_m) \to 0, y'_m \to x_2\) and \(c_m + k_m \to z'_2\). Since the holonomy maps of \(\mathcal{F}_{i}\) along \(\mathcal{F}\) restricted on \(\mathcal{F}_{(1,i)}\) are isometric, we have \(d(c_m, y'_m) = d(z_1, x_1).\) It follows that \(d(c_m + k_m, y_m + k_m) = d(z_1, x_1).\) Hence,

\[ d(z'_2, x_2) = \lim_{m \to +\infty} d(c_m + k_m, y'_m) = \lim_{m \to +\infty} d(c_m + k_m, y_m + k_m) = d(z_1, x_1). \]
Note that \( z_2' \) is between \( z_2 \) and \( x_2 \) and it follows that
\[
d(z_2, x_2) = d(z_2, z_2') + d(z_2', x_2) = 2d(z_1, x_1).
\]
Indeed, let \( \prec \) denote an order on each leaf of \( \mathcal{L}_i^s \) since the foliation \( \mathcal{L}_i^s \) is one-dimensional. Assume that each leaf has the same orientation. By \( z_1 \neq x_1 \) we can assume that \( z_1 \prec x_1 \prec c_m \). It follows from the linear foliations \( \mathcal{L}_i^s \) and \( \mathcal{L}_{(i,1-i)}^s \) that \( z_2 \prec z_2' \). On the other hand, one has that \( (c_m + k_m) \prec (y_m + k_m) \) since \( c_m \prec y_m \). Hence \( z_2' \prec x_2 \).

As a result, we get \( z_2 \prec z_2' \prec x_2 \).

For the other equality,
\[
d(z_1, z_2) = d(x_1, z_2') = \lim_{m \to +\infty} d(b_m + k_m, c_m + k_m) = \lim_{m \to +\infty} d(b_m, c_m) = d(x_0, z_1).
\]

Note that the dimension of \( \mathcal{L}_{(1,i-1)}^s \) could be more than one, but the line \( \overline{x_0z_1} \) is parallel to the line \( b_m c_m \) and also the line \( (b_m + k_m)(c_m + k_m) \). Hence, \( \overline{x_0z_1} \) is parallel to \( \overline{x_1z_2} \) and also \( \overline{z_1z_2} \). Thus, we have \( d(x_0, z_2) = d(x_0, z_1) + d(z_1, z_2) = 2d(x_0, z_1) \).

Repeating the construction in Claim 5.8 and Claim 5.9, there exists a sequence \( \{x_l\} \subset \mathcal{F}(x_0), l \in \mathbb{N} \) with \( z_l = \mathcal{L}_i^s(x_l) \cap \mathcal{L}_{(1,i-1)}^s(x_0) \) such that \( d(z_l, x_l) = l \cdot d(z_1, x_1) \) and \( d(x_0, z_l) = l \cdot d(x_0, z_1) \). Fix \( \delta > 0 \), let \( N_l > 0 \) be the minimal number such that \( A^N_l z_l \subset \mathcal{L}_{(1,i-1)}^s(A^N_l x_0, \delta) \). Since \( \mu^i_s(A) > \mu^i_{i-1}(A) \), we have \( d(A^N_l z_l, A^N_l x_1) \to +\infty \). It contradicts with the fact that \( d(y, y') \) is bounded, for every \( y \in \mathcal{L}_{(1,i-1)}^s(x_0, \delta) \) and \( y' = \mathcal{L}_i^s(y) \cap \mathcal{F}(x_0) \).

5.2. \( \mathbb{Z}^d \)-periodic foliations. Let \( \varphi \) given by Proposition 2.17 with \( \lambda^i_s(p, f) = \lambda^s_i(q, f) \), for all \( p, q \in \text{Per}(f) \) and \( 1 \leq i \leq k \). Now, we already have gotten Corollary 5.3 that for each \( 1 \leq i \leq k \), \( H(\mathcal{F}_i^s) = \mathcal{L}_i^s \) and \( H \) is isometric along \( \mathcal{F}_i^s \) under the metric \( d_i^s(\cdot, \cdot) \). It follows that the holonomy maps of \( \mathcal{F}_i^s \) along \( \mathcal{F}_{j_1}^s \oplus \cdots \oplus \mathcal{F}_{j_l}^s \) restricted on \( (\mathcal{F}_{j_1}^s \oplus \cdots \oplus \mathcal{F}_{j_l}^s \oplus \mathcal{F}_{i}^s) \) are isometric under \( d_i^s(\cdot, \cdot) \), where \( 1 \leq j_1 < j_2 < \cdots < j_l \leq k \).

Fix \( 1 \leq i < k \), assume that \( \mathcal{F}_{(i,k)}^s \) is \( \mathbb{Z}^d \)-periodic. Note that the assumption \( \mathbb{Z}^d \)-periodic property for foliation \( \mathcal{F}_{(i,k)}^s \) implies periodicity for \( \mathcal{F}_i^s = \mathcal{F}_{(i,1)}^s \cap \mathcal{F}_{(i,k)}^s \).

**Proof of Proposition 5.4.** Assume that there exist \( x_0 \in \mathbb{R}^d, n \in \mathbb{Z}^d \) and \( x_1 \in \mathcal{F}_{(i+1,k)}^s(x_0 + n) - n \), but \( x_1 \notin \mathcal{F}_{(i+1,k)}^s(x_0) \). Note that, for every \( x \in \mathbb{R}^d \) and \( n \in \mathbb{Z}^d \), \( \mathcal{F}_{(i+1,k)}^s(x + n) - n \subset \mathcal{F}_{(i,k)}^s(x + n) - n = \mathcal{F}_{(i,k)}^s(x) \). Thus we have \( x_1 \in \mathcal{F}_{(i,k)}^s(x_0) \). Let \( z_1 = \mathcal{F}_{(i,k)}^s(x_1) \cap \mathcal{F}_{(i+1,k)}^s(x_0) \). By the assumption, \( a := d_i^s(x_1, z_1) > 0 \).

For every \( b_m \in \mathcal{F}_{i}^s(x_0) \), we denote
\[
c_m = \mathcal{F}_{(i+1,k)}^s(b_m) \cap \mathcal{F}_{i}^s(x_1) \quad \text{and} \quad y_m = \left( \mathcal{F}_{(i+1,k)}^s(n + b_m) - n \right) \cap \mathcal{F}_{i}^s(x_1).
\]
We claim \( d_i^s(y_m, c_m) = a \). Indeed, since \( d_i^s(\cdot, \cdot) \) is holonomy invariant, we have
\[
d_i^s(x_0, b_m) = d_i^s(z_1, c_m) \quad \text{and} \quad d_i^s(x_0 + n, b_m + n) = d_i^s(x_1 + n, y_m + n).
\]
Again, $\tilde{\mathcal{F}}^s_{(i,k)}$ is $\mathbb{Z}^d$-periodic implies that for $\tilde{\mathcal{F}}^s_i$. It follows that
\[ d_i^s(x_0, b_m) = d_i^s(x_0 + n, b_m + n) \quad \text{and} \quad d_i^s(x_1 + n, y_m + n) = d_i^s(x_1, y_m). \]
Hence, $d_i^s(y_m, c_m) = d_i^s(x_1, z_1) = a$. See Fig. 5.

Since we already have $H(\tilde{\mathcal{F}}^s_i) = \tilde{\mathcal{F}}^s_i$, by the second item of Proposition 2.9, we can choose $b_m \in \tilde{\mathcal{F}}^s_{(i,k)}(x_0)$ and $k_m \in A^m \mathbb{Z}^d$ such that $(b_m + k_m) \to x_1$. By Proposition 2.7, for a fixed size $R > 0$, one has
\[ d_H \left( \tilde{\mathcal{F}}^s_{(i+1,k)}(b_m + n, R) + k_m, \tilde{\mathcal{F}}^s_{(i+1,k)}(b_m + n + k_m, R) \right) \to 0, \]
as $m \to +\infty$, where $d_H(\cdot, \cdot)$ is Hausdorff distance. Since $(b_m + k_m) \to x_1$ and $(x_1 + n) \in \tilde{\mathcal{F}}^s_{(i+1,k)}(x_0 + n)$, we have that
\[ d_H \left( \tilde{\mathcal{F}}^s_{(i+1,k)}(b_m + n, R) + k_m - n, \tilde{\mathcal{F}}^s_{(i+1,k)}(x_1 + n, R) - n \right) \to 0. \]
It means that the sequence $(y_m + k_m) \in \tilde{\mathcal{F}}^s_{(i+1,k)}(b_m + n) + k_m - n$ converges to $x_2 \in \tilde{\mathcal{F}}^s_{(i+1,k)}(x_0 + n) - n$.

Similarly, we can get
\[ d_H \left( \tilde{\mathcal{F}}^s_{(i+1,k)}(b_m, R) + k_m, \tilde{\mathcal{F}}^s_{(i+1,k)}(x_1, R) \right) \to 0. \]
It follows that $(c_m + k_m) \in \tilde{\mathcal{F}}^s_{(i+1,k)}(b_m) + k_m$ converges to $z_2' \in \tilde{\mathcal{F}}^s_{(i+1,k)}(x_1)$. Moreover, since $\tilde{\mathcal{F}}^s_i$ is $\mathbb{Z}^d$-periodic, we have
\[ d_i^s(x_2, z_2') = \lim_{m \to +\infty} d_i^s(y_m + k_m, c_m + k_m) = \lim_{m \to +\infty} d_i^s(y_m, c_m) = a. \]
Let $z_2 = \tilde{\mathcal{F}}^s_{(i)}(x_2) \cap \tilde{\mathcal{F}}^s_{(i+1,k)}(x_0)$, by the holonomy-invariant metric, we have $d_i^s(x_2, z_2) = 2a$.

Repeating the preceding method, we can pick $x_l \in \tilde{\mathcal{F}}^s_{(i+1,k)}(x_0 + n) - n$ and $z_l = \tilde{\mathcal{F}}^s_{(i)}(x_l) \cap \tilde{\mathcal{F}}^s_{(i+1,k)}(x_0)$ such that
\[ d_i^s(x_l, z_l) = l\alpha \to +\infty, \]
as \( l \to +\infty \). Since \( \tilde{F}_i^{s} \) is quasi-isometric (Proposition 2.17), it follows that
\[
d(x_l, z_l) \to +\infty, \quad \text{as} \quad l \to +\infty. \tag{5.13}
\]

On the other hand, since there exists \( C_0 > 0 \) such that \(|H - Id| < C_0\), one has
\[
\tilde{F}_i^{s} (x_0 + n) \subset B_{C_0} \left( \tilde{\mathcal{L}}^s_i (x_0 + n) \right),
\]
\[
\subset B_{3C_0} \left( \tilde{\mathcal{L}}^s_i (x_0 + n) \right),
\]
\[
= B_{3C_0} \left( \tilde{\mathcal{L}}^s_i (x_0) \right) + n \subset B_{4C_0} \left( \tilde{F}_i^{s} (x_0) \right) + n.
\]
It follows that there exists \( C \geq 4C_0 \) such that \( d(x_l, z_l) \leq C \), for all \( l \in \mathbb{N} \). This contradicts with (5.13). \( \square \)

By Remark 5.5, we have that \( \tilde{F}_i^{s} (1 \leq i \leq k) \) is \( \mathbb{Z}^d \)-periodic. By Corollary 5.3, \( H : \tilde{F}_i^{s} \to \tilde{\mathcal{L}}_i^{s} \) is isometric along each leaf of \( \tilde{F}_i^{s} \). Now, we can show that there is no deviation between \( H^{-1} (x + n) \) and \( H^{-1} (x) + n \) along \( \tilde{F}_i^{s} \). We mention that the following proof can also apply for the case of \( \dim E^s i = 1 \) without small perturbation, since \( H : \tilde{F}^{s} \to \tilde{\mathcal{L}}^{s} \) is also isometric by Proposition 5.1.

**Proof of Proposition 5.6.** Recall that \( H^{-1} (x + n) - n \in \tilde{F}^{s} (H^{-1} (x)) \) (see Proposition 2.5). Hence, we just need focus on \( \tilde{F}_i^{s} \). For any \( x \in \mathbb{R}^d \) and \( y \in \tilde{F}_i^{s} (x) \), let
\[
d_i^s (x, y) := d_i^s (x, y'),
\]
where \( y' = \tilde{F}_i^{s, \perp} (y) \cap \tilde{F}_i^{s} (x) \). Note that the pseudo-metric \( d_i^s (\cdot, \cdot) \) is well defined, namely

- \( d_i^s (x, y) = d_i^s (y, x) \), \( \forall x \in \mathbb{R}^d \) and \( \forall y \in \tilde{F}_i^{s} (x) \).
- \( d_i^s (x, z) \leq d_i^s (x, y) + d_i^s (y, z) \), \( \forall x \in \mathbb{R}^d \) and \( \forall y, z \in \tilde{F}_i^{s} (x) \).
- \( \forall x \in \mathbb{R}^d \) and \( \forall y \in \tilde{F}_i^{s} (x) \), one has \( d_i^s (x, y) = 0 \) if and only if \( y \in \tilde{F}_i^{s, \perp} (x) \),

since the holonomy maps of \( \tilde{F}_i^{s} \) along \( \tilde{F}_i^{s, \perp} \) restricted on \( \tilde{F}_i^{s} \) are isometric under \( d_i^s (\cdot, \cdot) \).

For proving Proposition 5.6, it suffices to show that \( d_i^s \left( H^{-1} (x), H^{-1} (x + n) - n \right) = 0 \), for all \( x \in \mathbb{R}^d \) and \( n \in \mathbb{Z}^d \).

Assume that there exist \( x \in \mathbb{R}^d \) and \( n \in \mathbb{Z}^d \) such that
\[
d_i^s \left( H^{-1} (x), H^{-1} (x + n) - n \right) = \alpha \neq 0.
\]

**Claim 5.10.** For every \( y \in \mathbb{R}^d \), \( d_i^s \left( H^{-1} (y), H^{-1} (y + n) - n \right) = \alpha \).

**Proof of Claim 5.10.** By Proposition 2.9, there exist \( z_m \in \tilde{\mathcal{L}}_i^{s} (x) \) and \( k_m \in A^m \mathbb{Z}^d \) such that \( (z_m + k_m) \to y \). By Proposition 2.6 and the uniform continuity of \( H \), as \( m \to +\infty \),
\[
d \left( H^{-1} (z_m + k_m), H^{-1} (z_m) + k_m \right) \to 0 \quad \text{and} \quad d \left( H^{-1} (z_m + k_m), H^{-1} (y) \right) \to 0,
\]
\[
d \left( H^{-1} (z_m + n) + k_m, H^{-1} (z_m + n + k_m) \right) \to 0 \quad \text{and} \quad d \left( H^{-1} (z_m + k_m + n), H^{-1} (y + n) \right) \to 0.
\]
Hence,

\[ d \left( H^{-1}(z_m) + k_m, \ H^{-1}(z_m + n) - n + k_m \right) \to d \left( H^{-1}(y), \ H^{-1}(y + n) - n \right). \]  

(5.14)

On the other hand, since foliations \( \tilde{\mathcal{F}}^s_i \) and \( \tilde{\mathcal{F}}^{s, \perp}_i \) are both \( \mathbb{Z}^d \)-periodic,

\[ \tilde{d}_i^s \left( H^{-1}(z_m) + k_m, \ H^{-1}(z_m + n) - n + k_m \right) = \tilde{d}_i^s \left( H^{-1}(z_m), \ H^{-1}(z_m + n) - n \right). \]  

(5.15)

Since \( H^{-1} \) is isometric along \( \tilde{\mathcal{F}}^s_i \), we have

\[ \tilde{d}_i^s \left( H^{-1}(z_m), \ H^{-1}(x) \right) = \tilde{d}_i^s \left( H^{-1}(z_m + n), \ H^{-1}(x + n) \right), \]

\[ = \tilde{d}_i^s \left( H^{-1}(z_m + n) - n, \ H^{-1}(x + n) - n \right). \]

It follows that

\[ \tilde{d}_i^s \left( H^{-1}(z_m), \ H^{-1}(z_m + n) - n \right) = \tilde{d}_i^s \left( H^{-1}(x), \ H^{-1}(x + n) - n \right) = \alpha. \]  

(5.16)

Now, combining (5.14) with (5.15) and (5.16), we get

\[ \tilde{d}_i^s \left( H^{-1}(y), \ H^{-1}(y + n) - n \right) = \lim_{m \to +\infty} \tilde{d}_i^s \left( H^{-1}(z_m) + k_m, \ H^{-1}(z_m + n) - n + k_m \right) = \alpha. \]

\[ \square \]

For every \( l \in \mathbb{N} \), applying the previous claim for \( H^{-1}y = H^{-1}(x + (l-1)n) - (l-1)n \), one has

\[ \tilde{d}_i^s \left( H^{-1}(x + ln) - ln, \ H^{-1}(x) \right) = \tilde{d}_i^s \left( H^{-1}y, \ H^{-1}(x) \right) \]

\[ + \tilde{d}_i^s \left( H^{-1}y, \ H^{-1}(x + ln) - ln \right), \]

\[ = (l - 1) \cdot \alpha + \alpha = l \cdot \alpha \to +\infty, \]

as \( l \to +\infty \). Since \( \tilde{\mathcal{F}}^s_i \) is quasi-isometric (Proposition 2.17), it contradicts with the fact that for all \( x \in \mathbb{R}^d \) and all \( m \in \mathbb{Z}^d \), \( d \left( H^{-1}(x + m) - m, \ H^{-1}(x) \right) \) is bounded. \[ \square \]

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References

1. Aoki, N., Hiraide, K.: Topological Theory of Dynamical Systems, Recent Advances, North-Holland Mathematical Library, vol. 52. North-Holland, Amsterdam (1994)
2. Bochi, J., Katok, A., Rodríguez Hertz, F.: Flexibility of Lyapunov exponents. Ergod. Theory Dyn. Syst. 42(2), 554–591 (2022)
3. Bonatti, C., Díaz, L.J., Viana, M.: Dynamics Beyond Uniform Hyperbolicity, a Global Geometric and Probabilistic Perspective, Encyclopaedia of Mathematical Sciences, vol. 102. Mathematical Physics, III. Springer, Berlin (2005)
4. Brin, M.: On dynamical coherency. Ergod. Theory Dyn. Syst. 23(2), 395–401 (2003)
5. Brin, M., Burago, D., Ivanov, S.: Dynamical coherence of partially hyperbolic diffeomorphisms of the 3-torus. J. Model. Dyn. 3(1), 1–11 (2009)
6. Costa, J.S.C., Micena, F.: Some generic properties of partially hyperbolic endomorphisms. Nonlinearity 35(10), 5297–5310 (2022)
7. Crovisier, S., Potrie, R.: Introduction to Partially Hyperbolic Dynamics. Unpublished Course Notes. School on Dynamical Systems, ICTP, Trieste (2015)
8. De Simoi, J., Liverani, C.: Statistical properties of mostly contracting fast-slow partially hyperbolic systems. Invent. Math. 206(1), 147–227 (2016)
9. De Simoi, J., Liverani, C.: Limit theorems for fast-slow partially hyperbolic systems. Invent. Math. 213(3), 811–1016 (2018)
10. de la Llave, R.: Smooth conjugacy and S–R–B measures for uniformly and non-uniformly hyperbolic systems. Comm. Math. Phys. 150(2), 289–320 (1992)
11. Franks, J.: Anosov diffeomorphisms on tori. Trans. Am. Math. Soc. 145, 117–124 (1969)
12. Franks, J.: Anosov diffeomorphisms, 1970 Global Analysis (Proceedings of the Symposium Pure Mathematics, Vol. XIV, Berkeley, California, 1968), pp. 61–93. American Mathematical Society, Providence
13. Gan, S., Shi, Y.: Rigidity of center Lyapunov exponents and su-integrability. Comment. Math. Helv. 95(3), 569–592 (2020)
14. Gogolev, A.: Smooth conjugacy of Anosov diffeomorphisms on higher-dimensional tori. J. Mod. Dyn. 2(4), 645–700 (2008)
15. Hall, L., Hammerlindl, A.: Partially hyperbolic surface endomorphisms. Theory Dyn. Syst. 41(1), 272–282 (2021)
16. Hall, L., Hammerlindl, A.: Classification of partially hyperbolic surface endomorphisms. Geom. Dedic. 216(3), Paper No. 29 (2022)
17. Hirsch, M., Pugh, C., Shub, M.: Invariant Manifolds. Lecture Notes in Mathematics, vol. 583. Springer, Berlin (1977)
18. Journé, J.-L.: A regularity lemma for functions of several variables. Rev. Mat. Iberoamericana 4(2), 187–193 (1988)
19. Katok, A., Hasselblatt, B.: Introduction to the Modern Theory of Dynamical Systems, with a Supplementary Chapter by Katok and Leonardo Mendoza, Encyclopedia of Mathematics and Its Applications, vol. 102. Cambridge University Press, Cambridge (1995)
20. Mañé, R., Pugh, C.: Stability of Endomorphisms, Dynamical Systems—Warwick 1974 (Proceedings of the Symposium Applications Topology and Dynamical Systems, University of Warwick, Coventry, 1973/1974; presented to E. C. Zeeman on his Fiftieth Birthday), pp 175–184, Lecture Notes in Mathematics, vol. 468. Springer, Berlin (1975)
21. Manning, A.: There are no new Anosov diffeomorphisms on tori. Am. J. Math. 96, 422–429 (1974)
30. Marisa, C., Régis, V.: Anosov Endomorphisms on the 2-torus: regularity of foliations and rigidity, Preprint, arXiv:2104.01693 (2021)
31. Micena, F.: Rigidity for some cases of Anosov endomorphisms of torus, Preprint, arXiv:2006.00407 (2022)
32. Micena, F., Tahzibi, A.: On the unstable directions and Lyapunov exponents of Anosov endomorphisms. Fund. Math. 235(1), 37–48 (2016)
33. Moosavi, S., Tajbakhsh, K.: Classification of special Anosov endomorphisms of nil-manifolds. Acta Math. Sin. Engl. Ser. 35(12), 1871–1890 (2019)
34. Moosavi, S., Tajbakhsh, K.: Robust special Anosov endomorphisms. Commun. Korean Math. Soc. 34(3), 897–910 (2019)
35. Pesin, Y.: Lectures on Partial Hyperbolicity and Stable Ergodicity, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich
36. Przytycki, F.: Anosov endomorphisms. Stud. Math. 58(3), 249–285 (1976)
37. Pugh, C., Shub, M.: Stably ergodic dynamical systems and partial hyperbolicity. J. Complex. 13(1), 125–179 (1997)
38. Pugh, C., Shub, M., Wilkinson, A.: Hölder foliations. Duke Math. J. 86(3), 517–546 (1997)
39. Saghin, R., Yang, J.: Lyapunov exponents and rigidity of Anosov automorphisms and skew products. Adv. Math. 355, 106764 (2019)
40. Sumi, N.: Linearization of Expansive Maps of Tori, Dynamical Systems and Chaos, vol. 1 (Hachioji, pp. 243–248, 1995), World Science Publications, River Edge (1994)
41. Tsujii, M.: Physical measures for partially hyperbolic surface endomorphisms. Acta Math. 194(1), 37–132 (2005)