Communication Structures, its graph representation and decomposition possibilities

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Abstract. Communication networks are modelled using graph structures. The efficiency of a network can be determined using the parameters of the graph model. The article reviews the modelling of various communication networks, their graph parameters, classical topologies and their available properties.

If the communication is in one direction, the model is known as simplex model. Its graphical representation requires directed graphs, a kind of generalized graphs. The large networks become complicated for analysis because of having large parametric values. To overcome this difficulty more generalized graphs called hypergraphs are required.

1. Introduction
Broadcasting and Gossiping are two important problems in interconnection networks for disseminating information between nodes. A network can be modelled as a graph \( G = (V, E) \), where \( V \) is the set of nodes and \( E \) is the set of communication lines. In the information dissemination four main problems have widely studied in literature.

(i) Broadcasting: Let \( G = (V, E) \) be the network with \( V \) as the set of nodes and \( E \) is the set of lines. Let \( v \in V \) knows a piece of information \( I(v) \). Broadcasting is the problem of determining a protocol in which all nodes in \( G \) learn the information \( I(v) \) in minimum possible time.

(ii) Accumulation: Let \( G = (V, E) \) represent the communication network and let \( v \in V \) be a node in \( G \). Let each node \( u \in V \) have a piece of information \( I(u) \) and for every \( x, y \in V \), \( I(x) \) and \( I(y) \) are independent. The set \( I(G) = \{I(w)/w \in V\} \) is called the cumulative message of \( G \). The accumulation problem is to determine a communication strategy in which the node \( v \) gets the cumulative message of \( G \).

(iii) Gossiping: The gossip problem is to identify a protocol such that all nodes in \( V \) learn the cumulative message in minimum possible time.

(iv) Multi-casting: The multi-casting problem is to determine the communication strategy such that some nodes in \( G, u \in S' \subseteq V \), learn the message \( I(v) \) in minimum possible time.
Communication networks are classified based on the ability of the nodes to communicate simultaneously with their neighbors. One is called Whispering or processor bond or 1-port which can communicate with only one neighbor at a time. The other is called Shouting or Link bond or n-port that can communicate with all its neighbors simultaneously.

The communication networks can be characterized based on the time for transferring the message between nodes. In constant model, the time required to transmit the message is constant. In linear model, the time is modeled as $T = \beta + L\tau$ where $\beta$ is the cost of preparing the message and $\tau$ is the propagation time of a data of unit length.

Another classification the model is in to simplex or full-duplex. In simplex model a communication link sends messages in a particular direction. The simplex network is modeled by directed graphs. In the full duplex model communication flows in both directions and hence graph model for the network is undirected.

In full duplex model the broadcast time of a vertex is the minimum time required by $u$ to broadcast its information in the graph $G$. Broadcast time is denoted by $b(u)$, and $b(G) = \max\{b(u)/u \in V(G)\}$, represents the broadcast time of the network $G$. The broadcast time of the graph $G$, $b(G)$ will be the maximum broadcast time among all vertices of $G$. A graph $G = (V,E)$ with the broadcast time $b(G) = b(K_n)$ is a broadcast graph. As the less number of communication links reduce the cost of broadcast scheme, broadcast graph with minimum edges is of great interest. Since $b(K_n) = \lceil\log_2 n\rceil$, a broadcast graph with $n$ nodes and minimum edges which has the broadcast time $b(G) = \lceil\log_2 n\rceil$ is the minimum broadcast graph. The number of edges of minimum broadcast graph is denoted by $B(n)$, where $n$ represents the number of vertices.

In gossiping every node knows a unique information and requires to communicate it to every other nodes. The gossip time of $G$ is the minimum time required to gossip in $G$. It is denoted by $g(G)$ in unit cost model and $g_{\beta,\tau}(G)$ in linear cost model. A graph $G$ is a gossip graph if $g(G) = g(K_n)$. A linear gossip graph is graph $G$ such that $g_{\beta,\tau}(G) = g_{\beta,\tau}(K_n)$. As $g(K_n) = \lceil\log_2 n\rceil$ for even $n$ and $g(K_n) = \lceil\log_2 n\rceil + 1$, for odd $n$, a minimum gossip graph $G$ has the gossip time $g(G) = \lceil\log_2 n\rceil$, for even $n$ and $g(G) = \lceil\log_2 n\rceil + 1$, for odd $n$. For even $n$, $g_{\beta,\tau}(K_n) \in \{2^{k-1}, 2^k - 1\}$. In a constant time, 1-port full duplex model Fertin et al., [21] studied about the structure of the networks that gossiping in minimum time using minimum number of links $G(n)$. Using the structural properties Fertin [21] succeeded to provide bounds for $G(n)$ when $1 \leq n \leq 32$.

The competence of a communication network is determined by many graph parameters such as Diameter, Broadcast time, vertex and edge connectivity, degree centrality etc[2].

The major graph topologies used in communication networks comprise hypercube, cube connected cycles, De-Bruijn Graph, shuffle exchange graph, d-grid graph, d-torus graph, Star graph, Recursive circulant graph, Cayley Graph, Butterfly graph, Knodel Graph, r-modulo bipartite graph, Fibonacci Graph, etc. More efficient communication systems are developed with the help of these graphs and their parameters.

Graph decomposition is the partitioning of the edge set so that each subset induces a subgraph whose union is the given graph. If the subgraphs are isomorphic to each other the decomposition is said to be an isomorphic decomposition, otherwise it is a non-isomorphic decomposition. Reji Kumar and Jasmine[6] have studied the isomorphic path and star decomposition of Fibonacci graphs. The decomposition of complete bipartite graphs and complete graphs in to some classes of stars have been studied by the same authors. The path and star decomposition of Knodel and Fibonacci digraphs have been discussed in [14]. Knodel and Fibonacci hypergraphs are
defined by Reji Kumar and Jasmine. The star decomposition of Fibonacci, Knodel and r-partite complete hypergraphs are discussed in [7].

There are numerous models for representing communication structures. We discuss few among them in the following section.

2. Important Communication Models

The $k$-broadcasting in trees had been introduced by Proskurowski[26]. In $k$-broadcasting an informed vertex can communicate simultaneously up to $k$ neighbors in a single round. A $k$-broadcast graph is an $n$ vertex communication network that supports a broadcast from one vertex to all in optimal time $\lceil \log_{k+1} n \rceil$. A minimum $k$-broadcast graph is a $k$-broadcast graph having minimum edges. The number of edges in a minimum $k$-broadcast graph is denoted by $B_k(n)$. In [24], Harutyunyan and Liestman gave bounds for $B_k(n)$. In [25] Harutyunyan et al. defined $k$-broadcast center. A $k$-broadcast center of a graph is the set of vertices that can initiate a minimum time $k$-broadcast within the graph. They provide a linear time algorithm for finding the $k$-broadcast time of any vertex of the tree and thus the $k$-broadcast time of the tree.

Farely [3] introduced a model in which each time unit an informed vertex $u$ may send the message to an uninformed vertex $v$ via a path of arbitrary length. The model is called open path model. It is a generalization of a classical model. The path used in each time unit must be vertex disjoint. Hence open path model is also known as vertex disjoint path mode broadcasting. The open line model is similar to open path model. In this model paths used for sending message in each time unit must be edge disjoint. It is also known as edge disjoint path mode broadcasting. This model was introduced by Farley [3] in 1983.

In classical models it is assumed that each node knows the communication topology, the originator of the message and the time at which the message was sent. But in messy scheme the nodes are unaware of the communication topology and transmit the message to random neighbors at each round of broadcasting. It first appeared in [1] in 1994. The behavior of the node depends only on its neighbors. Messy broadcasting uses three models depending on the quality of information about its neighbors available for each node.

- Model 1: Each node knows the informed and uninformed neighbors. In this model each node sends the message to the uninformed neighbors only.
- Model 2: In this model each node has a list of vertices from which the message received or sent. So it sends the message to a neighbor that is absent in the list in each time unit.
- Model 3: In model 3 each node keeps a list of all neighbors to which it sent the message. It sends the message to a neighbor in each time unit, that is not present in the list.

If a node sends message to an already informed node it is a price to the broadcast protocol. Models 1 and 2 may have to pay some price. The messy broadcast time in the worst cases of complete graphs, cycles, paths and complete $d$-ary trees are given by Harutyunyan and Liestman [24]. The worst case messy broadcast time of hypercubes, multi dimensional tori and complete bipartite graphs are studied by Comellas, Hart, Hell, Harutyunyan and Liestman.

Mareo and Pele discuss a similar model with knowledge of the $r$-neighborhood. In this model a node knows the structure of the network within $r$ radius from it.

If the informed vertex disseminate an information to its neighbor in a predefined order but not depend on the originator, is known as broadcasting with universal lists. This model can be classified under model 3 of messy broadcasting. A function assigns an ordered list of neighbors to each vertex which is the broadcast scheme and the list is known as the universal list. Each informed vertex informs its neighbors in the order given by the universal list regardless of the originator. There are two versions for the model of broadcasting with universal lists; adaptive and non-adaptive. In the adaptive version, each informed vertex tracks the vertices from which
it receives the message and skips them from its universal list. The non-adaptive version sends message to all of its neighbors.

In the fault tolerant broadcasting the nodes or links are not reliable in a network, then a faulty link may fail to transmit messages or a faulty node may stop to send or receive messages. Then the communication between nodes will fail. A network which continues to communicate even $k$ links or nodes fails to transmit the message, the network is said to be $k$-tolerant. The fault tolerant broadcasting was introduced by Liestman [28] in 1985. The $k$-tolerant broadcast function $B_k(n)$ is the minimum number of links in a network having $n$ nodes, which is $k$-tolerant. Pelc has presented a detailed survey on the fault tolerant communication model in view of broadcasting and gossiping. Fault tolerance in Hypercubes was studied by Stephen Dobrev and Imrich Vrto [29].

In the radio broadcasting model an informed node can send the message to all its neighbors simultaneously. A node cannot send the message to a strict subset of its neighbors. A node receives information from only one node in a time unit. Because the message from more than one neighbor in the same time unit gets corrupted. In radio broadcasting nodes does not know the topology or size of the network or even immediate neighborhood. The initial knowledge of each node is its own label only. Such networks are called ad-hoc networks. Chlebus et al. [30] studied the time of deterministic broadcasting under these assumptions. Dessmark and Pelc [32] developed algorithms to check the execution time and their number of transmissions. In [31] they presented the impact of knowledge radius $s$ on the time of deterministic broadcasting in a Geometric Radio Networks with $n$ nodes and eccentricity $D$ of the source. David Peleg reviewed the literature on time-efficient broadcasting algorithms for radio networks under various models and assumptions.

Graphs can be used to represent the communication networks. Many graph parameters are used to study nature and effectiveness of communication networks. We mention some of them in the following section.

2.1. Important Communication Parameters

Broadcast time, Diameter, degree centrality, vertex and edge connectivity are few among the graph parameters that are used to analyze the efficiency of a communication networks.

The distance between two nodes in a graph $G$ is the length of the shortest path connecting them. The distance between the most distanced nodes is the diameter. In other words, it is the maximum distance among the distances between pairs of vertices in a given graph and is denoted by $D(G)$. $D(G) = \max\{d(x,y)/d(x,y) = \text{length of shortest path between } x,y\}$. The diameter of the network $G$ represents the minimum number of rounds required to reach all nodes. The number of links attached to a node is called the degree of the node. A node become more important in a network as the degree increases, because of its connectivity with other nodes. Nodes of higher degree are known as hubs and terminal vertices are of low degree. Vertex connectivity is the measurement of flexibility of the network. It shows the ability of the network to continue the communication process even though some nodes or links are faulty. Vertex connectivity is the minimum number of nodes whose faultiness leads to the failure of disseminating the information. $k(G)$ denotes vertex connectivity. The edge connectivity is the minimum number of edges whose failure will interrupt transmission of messages in the network. It is denoted by $\lambda(G)$. If $\delta(G)$ is the minimum degree in $G$, then $k(G) \leq \lambda(G) \leq \delta(G)$ for any graph $G$. Mean inter-node distance (MID) is the average distance between two distinct nodes. MID indicates the average delay in transmission under uniform message distribution.

If an originator sends a message to another node in a network, the message must cross some nodes and links before reaching the destination. Consider all source destination pairs that exchange messages. Count the number of visits to each nodes and links. Calculate the probability of node visit or link visit by an average message. The number of visits to a node
(edge) by an average message is called the visit ratio of the node (edge). The maximum of the visit ratios over all nodes (edges) is called the node visit ratio (edge visit ratio) in the network. These ratios help to identify crucial nodes and edges. The maximum number of paths of a given routing $R$ passing through any vertex of $G$ is regarded as vertex forwarding index of the network $G$. It is denoted by $\zeta(G)$. In [33] Chung et al. defined vertex forwarding index. It refers to the maximum amount of forwarding done by a node in transmitting messages. Vertex forwarding indexes of rings (Cycles), cubes, generalized de Bruijn Graph in some restricted cases had given by Chung et al. The maximum number of paths of the routing $R$ passing through every edge of $G$ is the edge forwarding index of $G$. It represents the load of the network. Heydemann et al. [34] analyze the graphs with minimum edge forwarding index. The edge bisection width Bisw($G$) of a graph $G$ is of order $N$ is the minimum number of edges whose removal splits the graph into two components such that each component having roughly half vertices. In other words it is the number of edges connecting the partition $V_1$ and $V_2$ of $V(G)$, such that $|V_1| = |V_2| \pm 1$.

2.2. Classical Topologies that are used in communication networks

The Hypercube of dimension $n$ have $2^n$ vertices which are represented by $2^n$ bit strings of length $n$, $n \geq 1$. Two vertices are adjacent if their bit strings differ by exactly one bit position. $H_n$ has $n$ vertices, $n2^{n-1}$ edges, diameter $n$, and $n$-regular. It is a minimum broadcast graph and hence $b(H_n) = \lceil \frac{n}{2} \rceil = n$. It is a vertex transitive graph with edge connectivity $n$.

When each vertex of a hypercube $H_n$ is replaced with a cycle of $n$ vertices, the resultant graph is called cube connected cycles, denoted by $CCC_n$. The $i$-th dimensional edge incident to a vertex of the hypercube is then connected to the $i$-th vertex of the corresponding cycle of $CCC_n$. Thus $CCC_n$ has $2^n n$ vertices, $3n2^{n-1}$ edges and maximum degree $3$. The diameter $D(CCC_n) = 2n + \lceil \frac{n}{2} \rceil - 2$ and $b(CCC_n) = \lceil \frac{5n}{2} \rceil - 1$.

The vertices of a shuffle exchange graph is represented by binary strings of length $n$. A vertex $\beta a$ is connected to the vertex $\beta c$, where $\beta$ is a binary string of length $n - 1$, $a \in \{0, 1\}$, $c$ is the binary complement of $a$. It is denoted by $SE_n$. $SE_n$ has $2^n$ vertices, $3.2^{d-1}$ and maximum degree is $3$. Diameter $D(SE_n) = 2n - 1$ and $b(SE_n) \leq 2n - 1$.

The nodes of De Bruijn graph $DB_n$ is represented by binary strings of length $n$. An edge connects a node $\beta a$ to another node $\beta b$; where $\beta$ is a binary string of length $n - 1$ and $a \in \{0, 1\}$. $DB_n$ has $2^n$ vertices, $2^{d+1}$ with maximum degree $4$ and diameter $n$. Piene at el [27] provides lower bound for broadcasting time of $DB_n$. 1.3171$n \leq b(DB_n)$. Bermond and Peyrat [2] have given the upper bound $b(DB_n) \leq 1.5n + 1.5$.

d-Grid is denoted by $G[a_1 \times a_2 \times \cdots \times a_d]$, $a_i$’s are positive integers. The nodes of $d$-dimensional grid are $d$-tuples of positive integers $(z_1, z_2, \cdots, z_d)$ where $0 \leq z_i \leq a_i$, for all $i(1 \leq i \leq d)$. The edges connect two nodes if the $n$-tuples of both differ exactly by co-ordinate one. It has $a_1 \times a_2 \times \cdots \times a_d$ vertices, $(a_1 - 1)(a_2 - 1) \cdots (a_d - 1)$ edges and has degree $2d$. The diameter of $d$-grid graph is $(a_1 - 1) + (a_2 - 1) + \cdots + (a_d - 1)$. Hedetniemi at el shown that the broadcast time $b(a_1 \times a_2) = a_1 + a_2 - 2$. The $d$-grid graph is the cartesian product of paths, $G[a_1 \times a_2 \times \cdots \times a_d] \cong P_{a_1} \times P_{a_2} \times \cdots \times P_{a_d}$.

If both ends of the rows and columns of a $d$-grid graph are connected by edges, the resultant graph is called $d$-torus graph. $d$-torus graph is denoted by $T(a_1, a_2, \cdots, a_d)$. It is isomorphic to the cartesian product of cycles, i.e, $T(a_1, a_2, \cdots, a_d) \cong C_{a_1} \times C_{a_2} \times \cdots \times C_{a_d}$. Gu at el derived the diameter of $T(k, k)$ as $D(T(k, k)) = \lceil \frac{k}{2} \rceil$. Farley and Hedetniemi shown that the broadcast time of 2-torus graph is $\lceil \frac{a_1}{2} \rceil + \lceil \frac{a_2}{2} \rceil$; where $a_1$ or $a_2$ is even and $\lceil \frac{a_1}{2} \rceil + \lceil \frac{a_2}{2} \rceil - 1$, when both $a_1$ and $a_2$ are odd.

Recursive Circulant graph is a circulant graph $G(n, d)$; where $n$ is the number of vertices and $d$ is the degree of each vertex as it is $d$-regular. Let the vertices be labeled by $0, 1, \cdots, n - 1$; two vertices $x$ and $y$ are connected by an edge if $x + d^i \equiv y \pmod{n}$ for some $0 \leq i \leq \lceil \log_d n \rceil - 1$. Park and Chwa[41] established the diameter of recursive circulant graph in the following special
cases. If $d$ is odd $D(G(cd^m, d)) = \left\lceil \frac{d}{2} \right\rceil m + \left\lfloor \frac{d}{2} \right\rfloor$. When $d$ is even and $c$ is odd it is $\left\lceil \frac{d+1}{2} \right\rceil m + \left\lfloor \frac{d}{2} \right\rfloor$. If both $d$ and $c$ are even, then the diameter is $\left\lceil \frac{d-1}{2} \right\rceil m + \left\lfloor \frac{d}{2} \right\rfloor$. The degree of $G(2^m, 4)$ is $m$, and has $2^d$ vertices, $d2^d-1$ edges and maximum connectivity. $D(G(2^m, 4)) = \left\lceil \frac{3m-1}{2} \right\rceil$ and the broadcast time is $m$.

Star Graph is a Cayley graph on the group $S_d$ consisting of all permutations of $d$ symbols. It was proposed by Akers, Harel and Krishnamurthy [35]. Sur and Srimani [36] defined star broadcast time for $W$. The Cayley graph $\Gamma(g,h)$ is edge transitive if $1 \leq \alpha < n$, $\alpha \equiv 0 \mod n$. The edge forwarding index of $\Gamma(g,h)$ is $\frac{\alpha n}{2} + 1 \mod n$. The broadcast time is $D(\Gamma(g,h)) = \frac{n+\alpha}{2}$. For each $\alpha \in \{1,2,\ldots,n, n-1\}$, $\alpha \equiv 0 \mod n$, $\alpha$ and $\alpha + 1 \mod n$ are connected by a straight edge with vertex $\alpha$.

### Remark

In this article, star is the bipartite graph $K_{1,n}$ denoted by $S_{n+1}$ with $n+1$ vertices and $n$ edges.

According to Klasing et al.[27] Butterfly graph is defined by $V(BF_n) = \{(i,\alpha)/i = 0, 1, 2, \ldots,n-1, \alpha \in \{0, 1\}^n\}$ and $E = \{((i, \alpha_1, \alpha_2, \ldots, \alpha_d), (i+1(mod d), \alpha_1, \alpha_2, \ldots, \alpha_d))/j = 1, 2, \ldots, n\}$; where $\{0, 1\}^n$ denotes the set of length $n$ binary strings. For each vertex $V = \langle i, \alpha \rangle \in V_n$, $i \in \{0, 1, 2, \ldots,n-1\}$, $\alpha \in \{0, 1\}^n$, we call $i$ the level and $\alpha$ the position within the level string of $V$. For each $i \in \{0, 1, 2, \ldots,n-1\}$ and each $\alpha = \alpha_1\alpha_2\cdots\alpha_n \in \{0, 1\}^n$, the vertex $i, \alpha > on level i of BF(n) is connected by a straight edge with vertex $i \equiv 1 + (mod n)$, $\alpha > and by a cross edge with vertex $i \equiv 1 + (mod n), \alpha(i) > on level i + 1 (mod n)$. BF(n) has $n.2^n$ nodes, diameter $\left\lceil \frac{3n-1}{2} \right\rceil$ and maximum node degree 4.

The Cayley graph [38] $\Gamma = \Gamma(G, \Omega)$ is the simple graph whose vertex-set and edge-set are defined as follows: $V(\Gamma) = G$; $E(\Gamma) = \{(g,h)/g^{-1}h \in \Omega\}$. If $G$ is a symmetric group $S_3$, and $\Omega = \{(12), (23), (13)\}$, then the Cayley graph $\Gamma(G, \Omega)$ is isomorphic to $K_{3,3}$.

Knodel graphs are the underlying topology from Knodel’s construction [5] of time optimal algorithm for gossiping among $n$ nodes, with even $n$. But it is formally defined by Fraigniaud and Peters [39] in 2001. The Knodel graph on $n \geq 2$, vertices $(n$ even) and of maximum degree $d$, where $1 \leq d \leq \log_2 n$ is denoted by $W_{d,n}$. The vertices of $W_{d,n}$ are pairs $(i, j)$ with $i = 0, 1$ and $0 \leq j \leq \frac{n}{2} - 1$. And the set of edges $E = \{(0, j), (1, j')/j' \equiv j + 2^k - 1 (mod \frac{n}{2}); 0 \leq i, j \leq \frac{n}{2} - 1; 0 \leq k \leq d - 1\}$. Knodel graphs are bipartite $d$-regular graphs.

Among Knodel graphs $W_{k,2^k}$ are of great interest due to their communication properties. $W_{k,2^k}$, $W_{k-1,2^k-2}$, $W_{k-1,2^k-4}$ are minimum gossip graph and minimum linear gossip graph. $W_{k,2^k}$ and $W_{k-1,2^k-2}$ are minimum broadcast graphs. $W_{k-1,2^k-6}$ is also minimum linear gossip graph. Knodel graph is a Cayley graph. As every Cayley graph is vertex transitive $W_{d,n}$ is also vertex transitive. But $W_{d,n}$ is edge transitive only when $n = 2^k - 2$, $W_{k-1,n}$. $W_{d,n}$ has edge connectivity $\Lambda(W_{d,n}) = d$ and vertex connectivity $\frac{2d^2}{3} \leq \kappa(W_{d,n}) \leq d$. Fertin et al. [4] derived the diameter of $W_{k,2^k}$ as $\left\lceil \frac{k+2}{2} \right\rceil$, which is the lowest among the similar topologies. Oad [40] deduced the diameter of $W_{3,n}$ as $D(W_{3,n}) = \left\lceil \frac{n}{n^2 + 1} \right\rceil$, for $n > 8$. By using simulation techniques Oad produced diameter of some specific Knodel graphs such as $D(W_{d-1,2^d-2}) = D(W_{d-1,2^d}) = D(W_{d,2^d+4}) = D(W_{d,2^{d+2}+2^d-1}) = \left\lceil \frac{d+2}{2} \right\rceil$ and $D(W_{d,2^d+2}) = \left\lceil \frac{d+2}{2} \right\rceil$, for $d \leq 24$. Oad also developed the broadcast time for $W_{3,n}$. $b(W_{3,n}) = D(W_{3,n}) + 1 = \frac{2n}{2} + 2$ for $n \equiv 0, 2 (mod 6)$ and $b(W_{n,n}) = D(W_{3,n}) = \frac{n}{2} + 1$ for $n > 16$, $n \equiv 4 (mod 6)$. Cohen et al. [39] developed algorithm to recognize Knodel graph $W_{log_2(n),n}$, with complexity: $O(n log_2^3(n))$, if $n = 2^k$ and $O(n log_2^3(n))$, otherwise. The edge forwarding index of $W_{k,2^k}$ is $II(W_{k,2^k}) = 2^k$, for any $k$.
vertex forwarding index of $W_{k,2^k}$ is $\zeta(W_{k,2^k}) = (\frac{1}{2^k})\Sigma_{(u,v) \in V \times V} d(u,v) - (2^k - 1)$, for any $k$. The edge bisection width of $W_{k,2^k}$ is $Bisu(G) = 2^{k-1}$, for any $k$.

The Fibonacci graph [39] on $n \geq 2$, vertices ($n$ even) and of maximum degree $d$, where $1 \leq d \leq k$; where $k = F^{-1}(n)$ is denoted by $F_{d,n}$. The vertices of $F_{d,n}$ are pairs $(i,j)$ with $i = 0, 1$ and $0 \leq j \leq 2^n - 1$. And the set of edges $E = \{(0,j), (1,j')\} / j' = j + F(r) - 1(\text{mod} 2^k); 0 \leq j \leq 2^n - 1; 1 \leq r \leq d$. Fibonacci graphs are also $d$-regular bipartite graphs. The literature of Fibonacci graph is in its infancy. Similar parameters are open. A recognition algorithm for both Fibonacci and Knodel graphs was given by Cohen et al. The isomorphic $P_{d+1}$-decomposition, star decomposition are given by Rejikumar and Jasmine in [6]. Hamilton decomposition of Fibonacci graphs under certain constraints are also proved.

$r$-modulo bipartite graphs have introduced by Rejikumar and Jasmine. The vertices of $r$-modulo bipartite graph on $n \geq 2$, $M_{r,2n}$ can be labeled by pairs $(i,j)$ with $i = 0, 1$ and $0 \leq j \leq n - 1$. And the set of edges $E = \{(0,j), (1,j')\} / j' = j + k(\text{mod} n); 0 \leq k \leq r - 1$, $r \leq n$. The isomorphic $P_{r}$-decomposition, star decomposition and Hamilton decomposition of $r$-modulo bipartite graphs are proved by Jasmine and Rejikumar. It is $r$-regular bipartite graph. Communication parameters of $r$-modulo bipartite graphs are open.

Until now full duplex model in which the graph model is undirected was discussed. In simplex model a communication link sends messages in a particular direction and the network is modelled by directed graphs.

3. Simplex Models-Digraphs
A symmetric digraph produced by replacing each edge of an undirected Knodel graph [39] by a symmetric pair of directed edges is a Knodel digraph. A symmetric digraph obtained by replacing each edge of an undirected Fibonacci graph by a symmetric pair of directed edges is a Fibonacci digraph.

A directed graph or digraph [23], is a graph $D = (V,A)$, if $A$ is a set of the ordered pairs of elements from $V$. The ordered pair $a = (v_j, v_i)$ is an arc. $v_j$ is tail and $v_i$ is head of $a$. The tail and head are the end vertices of an arc. Two arcs with same head and same tail are known as parallel arcs. The graph $G$ corresponding to a digraph $D$ is the underlying graph of $D$. Conversely, for a graph $G$ if the digraph $D$ obtained by specifying an order for each arc, such a digraph is an orientation of $G$. The order of $D$ is the number of vertices and the size of $D$ is the number of arcs. Order is represented by $n$ and size is denoted by $m$. The number of arcs having head $v$ is the in-degree $d^+_G(v)$ of a vertex $v$ in $D$. The number of arcs having tail $v$ is the out-degree $d^-_G(v)$ of a vertex $v$. $\Delta^-(D)$ and $\Delta^-(D)$ denote the maximum and minimum in-degrees. Likewise the maximum and minimum out-degrees are denoted by $\Delta^+(D)$ and $\Delta^+/(D)$. If a digraph has no loops or no two arcs with the same ends have the same direction, it is strict.

A tournament is an orientation of a complete graph. A directed path which passes through every vertex of $D$ is a directed Hamilton path. A directed cycle which passes through every vertex of $D$ is a directed Hamilton cycle of $D$. A digraph produced by replacing each edge $e$ of $G$ by two arcs with opposite directions is the associated digraph (symmetric digraph) $D(G)$ of a graph $G$. There exists a one-one relation between directed paths in $D(G)$ and paths in $G$.

Knodel graphs are optimal samples of minimum broadcast graph and minimum gossip graph. A symmetric digraph produced by replacing each edge of an undirected Knodel graph by a symmetric pair of arcs is a Knodel digraph. Let $|V_0^*| = |V_1^*| = \frac{n}{2}$, $n$ is even; $V^* = V_0^* \cup V_1^*$ be a bipartition of $V^*$. Each vertex is an ordered pair $(i,j)$; $i = 0$ if $(i,j) \in V_0^*$ and $i = 1$ if $(i,j) \in V_1^*$, $j = 0,1,2,\ldots, \frac{n}{2} - 1$. The edge set consisting of all ordered pairs of vertices $\{(0,j), (1,j')\} / j' = j + 2^k - 1(\text{mod} \frac{n}{2}) \cup \{(1,j'), (0,j'')\} / j'' = j' - 2^k + 1(\text{mod} \frac{n}{2})$, $0 \leq k \leq d$, $1 \leq d \leq \lfloor \log_2 n \rfloor$. Knodel Digraphs are symbolized by $W_{d,n}^*$. It is a bipartite symmetric digraph having same indegree $(v)$ and outdegree $(v)$ equal to $d$; where $0 \leq d \leq \lfloor \log_2 n \rfloor$, for each vertex $v = (i,j)$; consequently $2d$-regular. The dimension[14] of an arc $((0,j), (1,j'))$ is positive.
is positive $k$ for each vertex $j$. $k$ is the dimension of an arc $((i, j'))$, $(0, j'')$) is negative $k^-$ if $j'' \equiv j' - 2^k + 1 \pmod{\frac{n}{2}}$. $G_{k^+}$ denotes the sub-graph induced by the set of all edges of dimension $k^+$ and $G_{k^-}$ represents the sub-graph induced by the set of all edges of dimension $k^-$. The decomposition possibilities of Knodel digraph $W_{d,n}^*$ have been discussed by Rejikumar and Jasmine in [14]. Major results are given below without proof.

**Theorem 3.1.** $W_{d,n}^*$ is $P_{d+1}^*$ decomposable, for $0 \leq d \leq n - 1$.

The classification of edges into positive and negative dimensions helps us to decompose Knodel digraphs into isomorphic directed paths. Selection of edges of different dimensions consecutively produces the path decomposition. The succeeding theorem decomposes $W_{d,n}^*$ to isomorphic stars of size $d$.

**Theorem 3.2.** $W_{d,n}^*$ is decomposable in to $S_{d+1}^*$, for $0 \leq d \leq n - 1$.

By picking each vertex along with its adjacent edges we produce star decomposition of Knodel digraphs. The Bipartite complement of digraphs is defined as follows. Let $G = (V_0, V_1, E)$ be a bipartite digraph. $G^{bc} = (V_0^*, V_1^*, E^{bc})$ is the bipartite complement of $G^*$; where $E^{bc} = \{(x, y) \in E((K^*)/x \in V_0^*, y \in V_1^*, and (x, y) \notin E^*, p = |V_0^*|, q = |V_1^*|\}$

The upcoming theorem decomposes complete bipartite digraph $K_{n,n}^*$ in to two isomorphic families of stars.

**Theorem 3.3.** $K_{n,n}^*$ is decomposable in to $S_{d+1}^*$ and $S_{n-d+1}^*$, where $0 \leq d \leq n - 1$.

The star decomposition of Knodel digraphs leads to the decomposition of $K_{n,n}^*$ in to $S_{d+1}^*$ and $S_{n-d+1}^*$.

**Fibonacci Digraphs** [14] are similar topologies in which $2^k$ is replaced by $F(k)$ the $k^{th}$ Fibonacci number. It can be formally defined as follows: Label the nodes of Fibonacci digraphs by ordered pairs $(i, j)$; $i = 0$ if $(i, j) \in V_0^*$ and $i = 1$ if $(i, j) \in V_1^*$, where $V = V_0^* \cup V_1^*$ and the arc set comprising of all ordered pairs of vertices $A = \{((0, j'), (1, j''))/j' \equiv j + F(k) - 1(\pmod{\frac{n}{2}})\}$

We represent Fibonacci Digraphs with the notation $F_{d,n}^*$. Fibonacci Digraph is also a bipartite symmetric digraph having in-degree $(v)$ and out-degree $(v)$ equal to $d$; where $0 \leq d \leq F^{-1}(n)$, for each vertex $v = (i, j)$. Hence it is a regular digraph. The dimension of an arc $((0, j'), (1, j''))$ is positive $k^+$ if $j' \equiv j + F(k) - 1(\pmod{\frac{n}{2}})$ and the dimension of an arc $((1, j'), (0, j''))$ is negative $k^-$ if $j'' \equiv j' = F(k) + 1(\pmod{\frac{n}{2}})$.

The succeeding theorem decomposes Fibonacci digraphs in to Isomorphic paths.

**Theorem 3.4.** $F_{d,n}^*$ is decomposable in to $P_{d+1}^*$.

Elect a vertex and consecutively choosing edges of different dimensions produces path decomposition of Fibonacci digraphs. Next theorem decomposes Fibonacci digraphs in to stars.

**Theorem 3.5.** $F_{d,n}^*$ is $S_{d+1}^*$-decomposable.

By choosing each vertex along with its adjacent edges Rejikumar and Jasmine produced star decomposition of Fibonacci digraphs.

The succeeding theorem decomposes Complete bipartite digraphs $K_{n,n}^*$ in to two classes of isomorphic stars; $d$ is an integer $1 \leq d \leq F^{-1}(n)$.

**Theorem 3.6.** $K_{n,n}^*$ is decomposable in to two families of stars $S_{d+1}^*$ and $S_{n-d+1}^*$.

The real life networks are complicated with large parameters.
4. Hypergraphs

Hypergraphs are a kind of generalization of the ordinary graphs. An edge in a graph is an
un-ordered pair of vertices whereas a hyperedge is a subset of vertices with any cardinality
$0 \leq |e_i| \leq n$; where $n$ is the order of the hypergraph. The hypergraphs have been studied from
late 1950, but was formally defined by Claude Berge. In computational geometry Hypergraphs
are known as range spaces and hyperedges as ranges. In social choice theory Simple co-
operative games can be modelled as hypergraphs. Hyper-edges are also termed as hyperlinks
and connectors. For definitions and terminology the reader may refer to the book Graphs and
Hypergraphs by C.Berge[8].

The structure of large number of problems can be described with hypergraphs rather than
ordinary graphs. The Indian railway system can be modelled as a hypergraph with stations are
vertices and trains are hyperedges. Ravindran and Nageswar[9] observed that the diameter of
247 in a base graph model was changed to 5 in a hypergraph model. 75 latent communities had
reduced to 5. In Social Networks, for clustering users besides posts, group memberships can be
used for learning. These groups are multi-way relationships.

A r-uniform hypergraph is a Steiner system, in which every pair of nodes is contained in
exactly single edge. In machine learning, Hypergraphs have been extensively used as the data
model and classifier regularization. They have applications in image retrieval [11], recommender
systems [10], and bioinformatics [12]. For extensive hypergraphs, a disseminated framework[13]
built using Apache Spark is available. Graph decomposition is the partitioning of the edge
set so that each subset induces a subgraph whose union is the given graph. If the subgraphs
are isomorphic to each other the decomposition is said to be an isomorphic decomposition,
otherwise it is a non-isomorphic decomposition. The isomorphic star decomposition of Knodel
and Fibonacci hypergraphs [7] had developed by Rejikumar and Jasmine.

Crescenzo and Galdi[15] investigated the construction of systematic secret sharing schemes
using hypergraph decomposition. The implementation of this method allows to acquire secret
sharing schemes for various classes of access structures such as hypercycles, hyperpaths,
hyperstars and acyclic hypergraphs with better efficiency. They secured some fundamental
characterization of the perfect access structures between the hyperstars.

Chee et al.,[16] studied the least number of acyclic hypergraphs that decompose a given
hypergraph, called the arboricity of hypergraphs. When $k = n − 3$, the arboricity of the complete
$k$-uniform hypergraph is set on. The complete $k$-uniform hypergraph’s arboricity is determined
for order $n$ essentially when $k = n = O(\log^{1-\delta} n)$, $\delta$ positive.

A systematic Decomposition for hypergraphs was given by Peter Jeavons, Marc Gyssens and
David Cohen [17]. Decomposition of regular hypergraphs was studied by Choi and West[18].
The necessary and sufficient conditions for a decomposition of complete 3-uniform hypergraph
of order $n$ in to 4-cycles was given by Jordon and Newkirk[19]. Brandt[20] had studied the
intersecting hypergraphs and decompositions of complete uniform hypergraphs in her thesis.
Gottlob, G., Leone, N., and Scarcello, F.[22] surveyed on Hypertree decompositions.

The the notion of Knodel and Fibonacci hypergraphs was introduced by Rejikumar and
Jasmine in [7]. The r-partite hypergraphs, Knodel hypergraphs and Fibonacci hyprgraphs are
decomposed in to isomorphic stars.

A hyper graph on a set $X = \{x_1, x_2, \cdots, x_n\}$, is a family $H = \{E_1, E_2, \cdots, E_m\}$ of subsets
of $X$ such that

(i) $E_i \neq \phi$, for $i = 1, 2, \cdots, m$
(ii) $\bigcup_{i=1}^{m} E_i = X$

A hypergraph is a simple hypergraph if

(iii) $E_i \subseteq E_j \Rightarrow i = j$
The elements \(x_1, x_2, \ldots, x_n\) of \(X\) are called vertices and the sets \(E_1, E_2, \ldots, E_m\) are the edges of the hypergraph. If \(|E_i| \leq 2\), for every \(i = 1, 2, \ldots, m\), then the simple hypergraph become simple graph.

A hypergraph \(H\) may be drawn as a set of points representing the vertices. The edge \(E_j\) is represented by a continuous curve joining the two elements if \(|E_j| = 2\), by a loop if \(|E_j| = 1\) and by a simple closed curve enclosing the elements if \(|E_j| \geq 3\). The incidence matrix \(A = (a_{ij}^j)\) with columns representing edges \(E_1, E_2, \ldots, E_m\) and rows representing the vertices \(x_1, x_2, \ldots, x_n\); where \(a_{ij}^j = 1\) if \(x_i \in E_j\), otherwise zero. The order of \(H\) is denoted by \(n = n(H)\), is the number of vertices. \(m = m(H)\) is the number of edges. Further the rank is \(r(H) = \max_j |E_j|\) and anti-rank is \(s(H) = \min_j |E_i|\). If \(r(H) = s(H)\), then \(H\) is a uniform hypergraph. If \(|E_j| = r, \forall j = 1, 2, \ldots, m\), then \(H\) is uniform.

Partial hypergraph is defined as follows; Let \(J \subseteq \{1, 2, \ldots, m\}\). The family \(H' = \{E_j/j \in J\}\) is a partial graph generated by \(J\). For a set \(A \subset X\), the family \(H_A = \{E_j/E_j \cap A \neq \phi, 1 \leq j \leq m\}\), the subgraph induced by the set \(A\).

The definition of a star in a hypergraph by Jenson is upcoming. For \(x \in X\), define the star \(H(x)\) with center \(x\) to be the partial hypergraph formed by the edges containing \(x\). The degree \(d_H(x)\) of \(x\) is the number of edges of \(H(x)\), so \(d_H(x) = m(H(x))\). The maximum degree of a hypergraph \(H\) is denoted by \(\Delta(H) = \max_{x \in H} d_H(x)\). If the degree of every vertex of a hypergraph is same, then it is a regular hypergraph.

Let \(r, n\) be integers with \(1 \leq r \leq n\). The \(r\)-uniform complete hypergraph of order \(n\) (or \(r\)-complete hypergraph) consisting of all \(r\)-subsets of a set \(X\) of cardinality \(n\). A hypergraph \(H\) is separable if for every vertex \(x\), the intersection of the edges containing \(x\) is the singleton \(\{x\}\). i.e, if \(\bigcap_{E \in H(x)} E = \{x\}\).

**Remark 2.** A Steiner system \(S(2, r, n)\) is an \(r\)-uniform hyper graph on \(X\) with \(|X| = n\), in which every pair of vertices is contained in exactly one edge.

An intersecting family is a set set of edges of a hypergraph having non-empty pairwise intersection. For example, for every vertex \(x\) of \(H\), the star \(H(x) = \{E/E \in H, x \in E\}\) is an intersecting family. \(r\)-partite complete hypergraph The \(r\)-partite complete hypergraph denoted by \(K^r_{n_1,n_2,\ldots,n_r}\) is defined by \(H = (V^h, E^h)\). The vertex set \(V^h = \{X^i\}\), with \(|X| = n_i\), for \(i = 1, 2, \ldots, r\) and the edge set \(E^h = \{x^i, x^j, \ldots, x^r\}\) with each \(x^i \in X^i\), for \(i = 1, 2, \ldots, r\).

**Traversals Hypergraphs** Let \(H = (E_1, E_2, \ldots, E_r)\) be a hypergraph on \(X\). A set \(T \subset X\) is a traversal of \(H\) if it meets all edges. i.e, \(T \bigcap_{E_j \neq \phi} E_j = \{1, 2, \ldots, m\}\). The family of minimal traversals of \(H\) constitutes a simple hypergraph on \(X\), called the traversal hypergraph of \(H\) and is denoted by \(Tr(H)\).

\[ Tr(K^r_n) = K_n^{n-r+1} \text{ and } Tr(K^r_{n_1,n_2,\ldots,n_r}) = \{X^1, X^2, \ldots, X^r\} \]

The fan of rank \(r\) is the hypergraph \(K_r^{r} \) having \(r\) edges of cardinality \(2\) and one edge of cardinality \(r\). A matching in a hypergraph \(H\) is a family of pairwise disjoint edges and maximum cardinality of a matching is denoted by \(\nu(H)\). A matching is a partial hypergraph with \(\Delta(H_0) = 1\).

Succeeding theorem decomposes \(r\)-partite complete hypergraph into stars.

**Theorem 4.1.** \(r\)-partite complete hyper graph is decomposable into stars.

An \(r\)-partite hypergraph \(H^r_{n_1,n_2,\ldots,n_r}\) is said to be \(r\)-equivpartite hypergraph if \(n_1 = n_2 = \ldots = n_r = n\), i.e, each part has the same cardinality.

Let us label the vertices of the first part by \(1^1, 1^2, \ldots, 1^n\), the vertices of the second part by \(2^1, 2^2, \ldots, 2^n\), and so on. In general the vertices of \(r\)-th part are denoted by \(r^1, r^2, \ldots, r^n\). The definition of Knodel Hypergraph given by Rejikumar and Jasmine is upcoming.

An \(r\)-equivpartite hypergraph is said to be Knodel hypergraph if each edge \(E = (1^{n_1}, 2^{n_2}, \ldots, r^{n_r})\), with \(n_m \in \{1, 2, \ldots, n\}\) is a collection of \(r\) vertices satisfying the following
conditions. \( \exists \) a pair \((n_i, n_j)\) such that \(n_i \equiv n_j + 2^k - 1 (\text{mod} n)\), \(1 \leq k \leq \lfloor \log_2(n) \rfloor\), \(\forall\) other \(n_l\), either \(n_l = n_i\) or \(n_l = n_j\).

**Theorem 4.2.** Knodel hypergraph \(W_{d,rn}^h\) is star decomposable.

Rejikumar and Jasmine [7] defined Fibonacci hypergraphs as follows: An \(r\)-equipartite hypergraph is said to be Fibonacci hypergraph if each edge \(E = (1^n_1, 2^n_2, \cdots, r^n_r)\), each \(n_m \in \{1, 2, \cdots, n\}\) is a collection of \(r\) vertices satisfying the following conditions. \( \exists \) a pair \((n_i, n_j)\) such that \(n_i \equiv n_j + F(k) - 1 (\text{mod} n)\), \(1 \leq k \leq F^{-1}(n)\), \(\forall\) other \(n_l\), either \(n_l = n_i\) or \(n_l = n_j\).

The next theorem presents the star decomposition of Fibonacci hypergraphs.

**Theorem 4.3.** Fibonacci hypergraph \(F_{d,rn}^h\) is star decomposable.

5. Conclusion

Graph Theory is a branch of Mathematics with applications in social networks, data mining, communication systems, Economics, Chemistry, Physics, Social Science, Optimization, job allocation problems and so forth. We review the graph representation of communication systems in the introduction. Important graph models that are used in communication systems, dominant graph topologies and their communication parameters in terms of graphs are discussed in the second section. Third section deals more general representation including direction of communication called digraphs. Fourth section is devoted to another generalization known as hypergraphs that enables to reduce huge graph parameters in large networks. We have discussed many parameters of undirected graphs that improves the efficiency of communication structures in second section. The literature of parametric data of digraphs and hypergraphs is in its infancy. Researchers can find wide variety of problems such as diameter of Knodel digraphs are open.

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