Compatible Dubrovin–Novikov Hamiltonian operators, Lie derivative and integrable systems of hydrodynamic type

O. I. Mokhov

1 Introduction

In the present paper, we prove that a local Hamiltonian operator of hydrodynamic type $K^{ij}_1$ (Dubrovin–Novikov Hamiltonian operator) is compatible with a nondegenerate local Hamiltonian operator of hydrodynamic type $K^{ij}_2$ if and only if the operator $K^{ij}_1$ is locally the Lie derivative of the operator $K^{ij}_2$ along a vector field in the corresponding domain of local coordinates. This result gives, first of all, a convenient general invariant criterion of the compatibility for the Dubrovin–Novikov Hamiltonian operators and, in addition, this gives a natural invariant definition of the class of special flat manifolds corresponding to all the class of compatible Dubrovin–Novikov Hamiltonian operators (the Frobenius–Dubrovin manifolds naturally belong to this class of flat manifolds). There is an integrable bi-Hamiltonian hierarchy corresponding to every flat manifold of this class. The integrable systems are also studied in the present paper. This class of integrable systems is explicitly given by solutions of the nonlinear system of equations, which is integrated by the method of inverse scattering problem. The corresponding results on compatible nonlocal Hamiltonian operators of hydrodynamic type and other Hamiltonian and symplectic differential-geometric type operators related by the Lie derivative and the results on the corresponding to them integrable bi-Hamiltonian systems will be published in other our works.

Recall that an operator $K^{ij}[u(x)]$ is called Hamiltonian if it defines a Poisson bracket (skew-symmetric and satisfying the Jacobi identity)

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} K^{ij}[u(x)] \frac{\delta J}{\delta u^j(x)} dx$$

for arbitrary functionals $I[u(x)]$ and $J[u(x)]$ on the space of functions (fields) $u(x) = \{u^i(x), 1 \leq i \leq N\}$, where $u^1, ..., u^N$ are local coordinates on a certain given smooth $N$-dimensional manifold $M$. It is obvious that any Hamiltonian operator always behaves as

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a contravariant two-valent operator tensor field with respect to arbitrary local changes of coordinates \( u^i = u^i(v^1, ..., v^N) \), \( 1 \leq i \leq N \), on the manifold \( M \):

\[
\tilde{K}^{sr}[v(x)] = \frac{\partial v^s}{\partial u^i} K^{ij}[u(v(x))] \circ \frac{\partial v^r}{\partial u^j},
\]

where the symbol \( \circ \) means the operator multiplication (it is important to note that in contrast to the usual tensor fields the components of operators in (1.2) multiply only in the indicated order). Accordingly, the classical tensor constructions, in particular, the Lie derivative along a vector field, are applied to Hamiltonian operators (see, for example, the monograph [2] and also section 3 below).

Hamiltonian operators are called compatible if any their linear combination is also a Hamiltonian operator (Magri, [3]; see also [2]). In the theory of Hamiltonian operators, there is the following general fact (see, for example, [2]), which is important for applications: if the second cohomology group of the corresponding complex of formal variational calculus is trivial, then from the compatibility of two Hamiltonian operators \( K_{ij}^1 \) and \( K_{ij}^2 \), where \( K_{ij}^2 \) is an invertible operator, it follows that there exists a formal vector field \( X[u(x)] \) depending, generally speaking, in an arbitrary and nonlocal way, on the fields \( u(x) \) and their derivatives and such that \( K_1 = L_X K_2 \), where \( L_X K \) is the Lie derivative of a Hamiltonian operator \( K \) along a formal vector field \( X \). Moreover, if an operator \( K \) is Hamiltonian and, in addition, the operator \( L_X K \), where \( X \) is a certain arbitrary formal vector field, is also Hamiltonian, then the Hamiltonian operators \( K \) and \( L_X K \) are always compatible. If an operator \( K \) is Hamiltonian and, besides, \( L_X^2 K = 0 \), where \( X \) is a certain vector field, then the operator \( L_X K \) is always Hamiltonian, and consequently the Hamiltonian operators \( K \) and \( L_X K \) are compatible in this case. This beautiful construction plays the very important role in applications and the corresponding class of compatible Hamiltonian operators (the pairs of operators of the form \( K_{ij} \), \( (L_X K)_{ij} \), such that \( (L_X^2 K)_{ij} = 0 \)) deserves a separate study.

An important special partial case of this construction arose afterwards in the Dubrovin theory of Frobenius manifolds (the quasihomogeneous compatible nondegenerate Dubrovin–Novikov Hamiltonian operators) [4]–[6]. The study of general compatible Dubrovin–Novikov Hamiltonian operators of this class, that is, of the form \( B_{ij} \) and \( (L_X B)_{ij} \) and such that \( L_X^2 B = 0 \), was started by the present author and Fordy in [7], where the first classification results were obtained.

2 Compatible Dubrovin–Novikov Hamiltonian operators

Recall that a Hamiltonian operator given by an arbitrary matrix homogeneous first-order ordinary differential operator, that is, a Hamiltonian operator of the form

\[
P^{ij}[u(x)] = g^{ij}(u(x)) \frac{d}{dx} + b_k^{ij}(u(x)) u^k,
\]

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is called a local Hamiltonian operator of hydrodynamic type or Dubrovin–Novikov Hamiltonian operator [1]. This definition does not depend on the choice of local coordinates \( u^1, ..., u^N \) on the manifold \( M \), since it follows from (2.2) that the form of operator (2.1) is invariant with respect to local changes of coordinates on \( M \). Operator (2.1) is called nondegenerate if \( \det(g^{ij}(u)) \neq 0 \). If \( \det(g^{ij}(u)) \neq 0 \), then operator (2.1) is Hamiltonian if and only if 1) \( g^{ij}(u) \) is an arbitrary contravariant flat pseudo-Riemannian metric (a metric of zero Riemannian curvature), 2) \( b^{ij}_k(u) = -g^{ik}(u)\Gamma^j_{sk}(u) \), where \( \Gamma^j_{sk}(u) \) is the Levi-Civita connection generated by the metric \( g^{ij}(u) \) (the Dubrovin–Novikov theorem [1]). In particular, it follows from the Dubrovin–Novikov theorem that for any nondegenerate local Hamiltonian operator of hydrodynamic type there always exist local coordinates \( v^1, ..., v^N \) (flat coordinates of the metric \( g^{ij}(u) \)) in which all the coefficients of the operator are constant:

\[
\tilde{g}^{ij}(v) = \eta^{ij} = \text{const}, \quad \tilde{\Gamma}^i_{jk}(v) = 0, \quad \tilde{b}^{ij}_k(v) = 0, \quad (2.2)
\]

that is the corresponding Poisson bracket has the form

\[
\{I, J\} = \int \frac{\delta I}{\delta v^i(x)} \eta^{ij} \frac{d}{dx} \frac{\delta J}{\delta v^j(x)} dx, \quad (2.3)
\]

where \( (\eta^{ij}) \) is a nondegenerate symmetric constant matrix:

\[
\eta^{ij} = \eta^{ji}, \quad \eta^{ij} = \text{const}, \quad \det(\eta^{ij}) \neq 0. \quad (2.4)
\]

Moreover, it immediately follows from the Dubrovin–Novikov theorem that any two nondegenerate Dubrovin–Novikov Hamiltonian operators \( P^{ij}_1[u(x)] \) and \( P^{ij}_2[u(x)] \) generated by flat contravariant metrics \( g^{ij}_1(u) \) and \( g^{ij}_2(u) \) respectively are compatible if and only if 1) any linear combination of these flat metrics

\[
g^{ij}(u) = \lambda_1 g^{ij}_1(u) + \lambda_2 g^{ij}_2(u), \quad (2.5)
\]

where \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants for which \( \det(g^{ij}(u)) \neq 0 \), is also a flat metric, 2) the coefficients of the corresponding Levi-Civita connections are related by the same linear formula:

\[
\Gamma^{ij}_k(u) = \lambda_1 \Gamma^{ij}_{1,k}(u) + \lambda_2 \Gamma^{ij}_{2,k}(u). \quad (2.6)
\]

The derived purely differential-geometric conditions (2.5) and (2.6) on flat metrics \( g^{ij}_1(u) \) and \( g^{ij}_2(u) \) define a flat pencil of metrics \( \mathcal{G} \). In this case, we shall also say that the flat metrics \( g^{ij}_1(u) \) and \( g^{ij}_2(u) \) are compatible (see [8]).

So the problem of description for compatible nondegenerate local Hamiltonian operators of hydrodynamic type is the purely differential-geometric problem of description of general flat pencils of metrics (see [8]). In [4, 5] Dubrovin considered all the tensor relations for the general flat pencils of metrics.

First of all, let us introduce the necessary notation. Let \( \nabla_1 \) and \( \nabla_2 \) be the operators of covariant differentiation given by the Levi-Civita connections \( \Gamma^{ij}_{1,k}(u) \) and \( \Gamma^{ij}_{2,k}(u) \) generated...
by the metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ respectively. The indices of the covariant differentials are raised and lowered by the corresponding metrics: $\nabla_1 = g_1^{is}(u)\nabla_{1,s}$, $\nabla_2 = g_2^{is}(u)\nabla_{2,s}$. Consider the tensor

$$\Delta^{ijk}(u) = g_1^{is}(u)g_2^{jp}(u)\left(\Gamma_{2,ps}^{k}(u) - \Gamma_{1,ps}^{k}(u)\right),$$

introduced by Dubrovin in [4], [5].

**Theorem 2.1 (Dubrovin [4], [5])** If metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ form a flat pencil, then there exists a vector field $f^i(u)$ such that the tensor $\Delta^{ijk}(u)$ and the metric $g_1^{ij}(u)$ have the form

$$\Delta^{ijk}(u) = \nabla_1^{i}f^k(u),$$

$$g_1^{ij}(u) = \nabla_2^{i}f^j(u) + \nabla_2^{j}f^i(u) + cg_2^{ij}(u),$$

where $c$ is a certain constant, and the vector field $f^i(u)$ satisfies the equations

$$\Delta^{ij}(u)\Delta^{ik}(u) = \Delta^{jk}(u)\Delta^{ij}(u),$$

where

$$\Delta^{ij}(u) = g_{2,ks}(u)\Delta^{ks}(u) = \nabla_{2,k}\nabla_{2}^{i}f^j(u),$$

and

$$(g_1^{is}(u)g_2^{jp}(u) - g_2^{is}(u)g_1^{jp}(u))\nabla_{2,s}\nabla_{2,p}f^k(u) = 0.$$ (2.12)

Conversely, for the flat metric $g_2^{ij}(u)$ and the vector field $f^i(u)$ that is a solution of the system of equations (2.11) and (2.12), the metrics $g_2^{ij}(u)$ and $g_1^{ij}(u)$ form a flat pencil.

In [6] Dubrovin proved that the theory of Frobenius manifolds constructed him in [4] (the Frobenius manifolds correspond to two-dimensional topological field theories) is equivalent to the theory of quasihomogeneous compatible nondegenerate Dubrovin–Novikov Hamiltonian operators or, in other words, quasihomogeneous flat pencils of metrics.

A flat pencil of metrics generated by flat metrics $g_1^{ij}(u)$ and $g_2^{ij}(u)$ is called quasihomogeneous of degree $d$ if there exists a function $\tau(u)$ such that for the vector fields

$$E = \nabla_1\tau, \quad E^i = g_1^{is}(u)\frac{\partial\tau}{\partial u^s}, \quad e = \nabla_2\tau, \quad e^i = g_2^{is}(u)\frac{\partial\tau}{\partial u^s},$$

the following conditions are satisfied:

1) $[e, E] = e$,
2) $L_\xi g_1 = (d - 1)g_1$,
3) $L_\xi g_1 = g_2$,
4) $L_\xi g_2 = 0$,

where $L_\xi g$ is the Lie derivative of a metric $g$ along a vector field $\xi$ (Dubrovin, [6]). Note that in the quasihomogeneous case we always have $L_\xi Q_1 = Q_2$, $L_\xi Q_2 = 0$, $L_\xi^2 Q_1 = 0$, for
the corresponding quasihomogeneous compatible nondegenerate Dubrovin–Novikov Hamiltonian operators \( Q_{ij}^1[u(x)] \) and \( Q_{ij}^2[u(x)] \) (more in detail about this important class of quasihomogeneous compatible nondegenerate Dubrovin–Novikov Hamiltonian operators see \([7]\)).

In the present author’s work \([9]\), a necessary for further, explicit and simple criterion of compatibility for two local Poisson brackets of hydrodynamic type was stated in a suitable for us way, that is, it is shown what an explicit form is sufficient and necessary for two local Hamiltonian operators of hydrodynamic type to be compatible (see also theorem \([2]\)).

**Lemma 2.1 (\([9]\)) (explicit criterion of compatibility for local Poisson brackets of hydrodynamic type)** Any local Poisson bracket of hydrodynamic type \( \{I, J\}_2 \) is compatible with the constant nondegenerate Poisson bracket \((2.3)\) if and only if it has the form

\[
\{I, J\}_2 = \int \frac{\delta I}{\delta v^i(x)} \left( \eta^i s \frac{\partial h^j}{\partial v^s} + \eta^i s \frac{\partial h^j}{\partial v^s} \frac{d}{dx} + \eta^i s \frac{\partial^2 h^j}{\partial v^s \partial v^k} u^k \right) \frac{\delta J}{\delta v^j(x)} \ dx, \tag{2.14}
\]

where \( h^i(v) \), \( 1 \leq i \leq N \), are smooth functions defined in a certain domain of local coordinates.

We do not require in lemma \([2]\) that the Poisson bracket of hydrodynamic type \( \{I, J\}_2 \) is nondegenerate. Besides, it is important to note that this statement is local.

### 3 Lie derivative and Dubrovin–Novikov Hamiltonian operators

Let \( K^{ij}[u(x)] \) be an arbitrary Hamiltonian operator, \( \xi(u) = \{\xi^i(u), 1 \leq i \leq N\} \) is an arbitrary smooth vector field on the manifold \( M \). The Lie derivative of the Hamiltonian operator \( K^{ij}[u(x)] \) (just as any contravariant two-valent operator tensor field of type \((1,2)\)) along the vector field \( \xi(u) \) is the operator

\[
(L_\xi K)^{ij}[u(x)] = \left( \frac{d}{dt} \left[ \left( \delta^i_t - t \frac{\partial \xi^i}{\partial u^s} \right) K^{st}[u(x)] + t \xi(u(x)) \right] \right) \bigg|_{t=0}, \tag{3.1}
\]

which also always behaves as contravariant two-valent operator tensor field of type \((1,2)\) with respect to arbitrary local changes of coordinates on the manifold \( M \).

For the Lie derivative of a local Hamiltonian operator of hydrodynamic type \( P^{ij}[u(x)] \) along the vector field \( \xi(u) \) we get

\[
(L_\xi P)^{ij}[u(x)] = \left( \xi^s \frac{\partial g^{ij}}{\partial u^s} - g^{sj} \frac{\partial \xi^i}{\partial u^s} - g^{is} \frac{\partial \xi^j}{\partial u^s} \right) \frac{d}{dx} \\
+ \left( \xi^s \frac{\partial b^{ij}}{\partial u^s} - b^{ij} \frac{\partial \xi^i}{\partial u^s} - b^{is} \frac{\partial \xi^j}{\partial u^s} + b^{ij} \frac{\partial^2 \xi^l}{\partial u^s \partial u^k} u^k \right). \tag{3.2}
\]
Theorem 3.1 Any Dubrovin–Novikov Hamiltonian operator $P^{ij}_1$ is compatible with a non-degenerate Dubrovin–Novikov Hamiltonian operator $P^{ij}_2$ if and only if there locally exists a vector field $\xi(u)$ such that

$$P^{ij}_1 = (L_\xi P^2)^{ij}_i.$$  \hfill (3.3)

In a different form, which is not connected with the Lie derivative, the general compatibility conditions for the Dubrovin–Novikov Hamiltonian operators were shown in [4], [8]–[12] (see also section 2 above).

Question. If there always globally exists such a smooth vector field on every flat manifold $M$, on the loop space of which are globally defined compatible Dubrovin–Novikov Hamiltonian operators $P^{ij}_1$ and $P^{ij}_2$? If not always, then it is necessary to investigate the corresponding differential-geometric and topology obstructions and also to single out and study all the flat manifolds on which there globally exists a vector field such that the Lie derivative of the Dubrovin–Novikov Hamiltonian operator generated by the flat metric of the manifold along the vector field is also a Hamiltonian operator. In particular, it is interesting to study all such vector fields in $\mathbb{R}^N$ or $\mathbb{R}^{N, k, N-k}$, and also in domains of these spaces. Locally, any such vector field is a solution of the nonlinear system of equations integrable by the method of inverse scattering problem (see [10]–[12]) and generates integrable bi-Hamiltonian systems of hydrodynamic type (we shall consider them in the next section).

Here let us prove theorem 3.1. Let Dubrovin–Novikov Hamiltonian operators $P^{ij}_1[u(x)]$ and $P^{ij}_2[u(x)]$ be compatible and the operator $P^{ij}_2[u(x)]$ be nondegenerate. Consider the local coordinates $v = (v^1, ..., v^N)$ in which the nondegenerate Dubrovin–Novikov Hamiltonian operator $P^{ij}_2[v(x)]$ is reduced to the constant form (2.3):

$$P^{ij}_2[v(x)] = \eta^{ij} \frac{d}{dx},$$  \hfill (3.4)

where $(\eta^{ij})$ is an arbitrary nondegenerate constant symmetric matrix: $\det(\eta^{ij}) \neq 0$, $\eta^{ij} = \text{const}$, $\eta^{ij} = \eta^{ji}$ (there always exist such local coordinates by the Dubrovin–Novikov theorem). In these coordinates, according to formula (3.2), for the Lie derivative of the operator $P^{ij}_2[v(x)]$ along an arbitrary vector field $\xi(v) = (\xi^1(v), ..., \xi^N(v))$ we get

$$(L_\xi P^2)^{ij}_i[v(x)] = \left(-\eta^{ij} \frac{\partial \xi^i}{\partial v^s} - \eta^{is} \frac{\partial \xi^j}{\partial v^s}\right) \frac{d}{dx} - \eta^{is} \frac{\partial^2 \xi^j}{\partial v^s \partial v^k} v^k.$$  \hfill (3.5)

According to lemma 2.1 any Dubrovin–Novikov Hamiltonian operator $P^{ij}_1[v(x)]$ compatible with the Hamiltonian operator (3.4) must have namely such the form (3.3) in the local coordinates $v = (v^1, ..., v^N)$ (in formula (2.14) $h^i(v) = -\xi^i(v)$, $1 \leq i \leq N$). Thus there exists a vector field $\xi^i(v)$, $1 \leq i \leq N$, such that

$$P^{ij}_1[v(x)] = (L_\xi P^2)^{ij}_i[v(x)].$$  \hfill (3.6)

Then by virtue of tensor invariance of the Lie derivative formula (3.3) is valid in arbitrary local coordinates.
Conversely, let two Dubrovin–Novikov Hamiltonian operators $P^{1 ij}[u(x)]$ and $P^{2 ij}[u(x)]$ are related by formula (3.3) and the operator $P^{2 ij}[u(x)]$ is nondegenerate. Then, in the local coordinates $v = (v^1, ..., v^N)$ in which the nondegenerate Dubrovin–Novikov Hamiltonian operator $P^{2 ij}[u(x)]$ is reduced to the constant form (3.4), the Hamiltonian operator $P^{1 ij}[v(x)]$ has the form (3.5). According to lemma 2.1 a pair of Hamiltonian operators $P^{1 ij}[v(x)]$ and $P^{2 ij}[v(x)]$ of such the form are necessarily compatible. Thus theorem 3.1 is proved.

Now the special class of flat manifolds which corresponds to the class of all compatible Dubrovin–Novikov Hamiltonian operators and generalizes the class of Frobenius–Dubrovin manifolds is naturally singled out. Theorem 3.1 gives the following invariant definition of these manifolds. We consider the manifolds $(M, g, \xi)$, where $M$ is a flat manifold with a flat metric $g$ equipped also with a vector field $\xi$ such that, for the nondegenerate Dubrovin–Novikov Hamiltonian operator $P^{ij}[u(x)]$ generated by the flat metric $g$, the operator $(\mathcal{L}_\xi P)^{ij}[u(x)]$ is also Hamiltonian (that is, the operator defines a Poisson bracket on the corresponding loop space of the manifold $M$). Generally speaking, for the description of all compatible Dubrovin–Novikov Hamiltonian operators it is necessary to consider the following more weak condition: an existence of such vector field locally, in a neighbourhood of every point of the manifold. Locally, such vector fields are described by the nonlinear system integrable by the method of inverse scattering problem (see [10]–[12]).

4 Class of integrable bi-Hamiltonian systems of hydrodynamic type

An arbitrary pair of compatible Dubrovin–Novikov Hamiltonian operators $P^{ij}_1$ and $P^{ij}_2$, one of which (let us assume $P^{ij}_1$) is nondegenerate, can be reduced to the following special form by a local change of coordinates:

$$P^{ij}_2[v(x)] = \eta^{ij} \frac{d}{dx}, \quad (4.1)$$

$$P^{ij}_1[v(x)] = \left(\eta^{is} \frac{\partial h^s}{\partial v^i} + \eta^{is} \frac{\partial h^i}{\partial v^s}\right) \frac{d}{dx} + \eta^{is} \frac{\partial^2 h^j}{\partial v^i \partial v^k} v^k, \quad (4.2)$$

where $(\eta^{ij})$ is an arbitrary nondegenerate constant symmetric matrix: $\det(\eta^{ij}) \neq 0$, $\eta^{ij} = \text{const}$, $\eta^{ij} = \eta^{ji}$; $h^i(v)$, $1 \leq i \leq N$, are smooth functions given in a certain domain of local coordinates such that operator (4.2) is Hamiltonian (lemma 2.1, see also theorem 2.1 above). An operator of form (4.2) is Hamiltonian if and only if

$$\eta^{sr} \frac{\partial^2 h^j}{\partial v^s \partial v^r} \frac{\partial^2 h^k}{\partial v^i \partial v^r} = \eta^{sr} \frac{\partial^2 h^k}{\partial v^s \partial v^r} \frac{\partial^2 h^j}{\partial v^i \partial v^r}, \quad (4.3)$$

$$\left(\eta^{ip} \frac{\partial h^s}{\partial v^p} + \eta^{ip} \frac{\partial h^i}{\partial v^p}\right) \frac{\partial^2 h^r}{\partial v^i \partial v^s} = \left(\eta^{ip} \frac{\partial h^s}{\partial v^p} + \eta^{ip} \frac{\partial h^i}{\partial v^p}\right) \frac{\partial^2 h^r}{\partial v^i \partial v^s}. \quad (4.4)$$
The system of nonlinear equations (4.3), (4.4), as was conjectured in the present author’s work [13] (there was stated the corresponding conjecture in [13]), is integrable by the method of inverse scattering problem. The procedure of integrating this system was presented in the author’s work [10], [11]. In the work [12] the Lax pair for this system was demonstrated. Note that the associativity equations of two-dimensional topological field theory (see [4]) are a natural reduction of equations (4.3) for “potential” vector fields $h_i(v)$, $1 \leq i \leq N$, of the special form

$$h_i(v) = \eta_{ij} \frac{\partial \Phi}{\partial v_j},$$  

(4.5)

where $\Phi(v)$ is a certain smooth function (“the potential”) (see also [9], [13]–[15]).

Consider the recursion operator generated by the “canonical” compatible Dubrovin–Novikov Hamiltonian operators (4.1), (4.2):

$$R^i_l = \left[ P_1[v(x)] \right. \left( P_2[v(x)] \right)^{-1} \right]^{i}_l = \left( \eta^{is} \frac{\partial h^j}{\partial v^s} + \eta^{js} \frac{\partial h^i}{\partial v^s} \right) \left( \frac{d}{dx} + \eta^{is} \frac{\partial^2 h^i}{\partial v^s \partial v^k} v^k_x \right) \eta_{jl} \left( \frac{d}{dx} \right)^{-1},$$  

(4.6)

where $(\eta_{ij})$ is the matrix which is inverse to the matrix $(\eta^{ij})$: $\eta^{is} \eta_{sj} = \delta^i_j$ (see [2], [16]–[20] about recursion operators generated by pairs of compatible Hamiltonian operators).

Let us apply the derived recursion operator (4.6) to the system of translations in $x$, that is, the system of hydrodynamic type

$$v^i_t = v^i_x,$$  

(4.7)

which is, obviously, Hamiltonian with the Hamiltonian operator (4.1):

$$v^i_t = \delta_H \frac{\delta H}{\delta v^j(x)}, \quad H = \frac{1}{2} \int \eta_{jl} v^j(x) v^l(x) dx.$$  

(4.8)

Any system from the hierarchy

$$v^i_{tn} = (R^n)^j_l v^j_x, \quad n \in \mathbb{Z},$$  

(4.9)

is a multi-Hamiltonian integrable system.

In particular, any system of the form

$$v^i_{tn} = \left( R^n \right)^j_l v^j_x,$$  

(4.10)

that is, the system of hydrodynamic type

$$v^i_t = \left( \eta^{is} \frac{\partial h^j}{\partial v^s} + \eta^{js} \frac{\partial h^i}{\partial v^s} \right) \frac{d}{dx} + \eta^{is} \frac{\partial^2 h^i}{\partial v^s \partial v^k} v^k_x \eta_{jl} v^j$$

$$\equiv \left( \eta^{is} \frac{\partial h^j}{\partial v^s} \eta_{jk} + \eta^{is} \frac{\partial h^i}{\partial v^s} \frac{\partial^2 h^j}{\partial v^s \partial v^k} v^k_x \right) v^j_x \equiv \left( h^i(v) + \eta^{is} \frac{\partial h^j}{\partial v^s} \right) x,$$  

(4.11)
where $h^i(v)$, $1 \leq i \leq N$, is an arbitrary solution of the integrable system (4.3), (4.4), is integrable.

This system of hydrodynamic type is bi-Hamiltonian with the pair of “canonical” Dubrovin–Novikov Hamiltonian operators (4.1), (4.2):

$$v^i_t = \left( \left( \begin{array}{c} \frac{\partial h^j}{\partial v^s} + \frac{\partial h^j}{\partial v^i} \\
\frac{\partial^2 h^j}{\partial v^s \partial v^k} v^k_x \end{array} \right) \frac{d}{dx} + \frac{\partial^2 h^k}{\partial v^i \partial v^j} v^j_x \right) \frac{\delta H_1}{\delta v^j(x)}, \quad H_1 = \frac{1}{2} \int \eta_{jl} v^j(x) v^l(x) dx, \quad (4.12)$$

$$v^i_t = \eta_{i}^j \frac{d}{dx} \frac{\delta H_2}{\delta v^j(x)}, \quad H_2 = \int \eta_{jk} h^k(v(x)) v^j(x) dx. \quad (4.13)$$

The next system in the hierarchy (4.9) is the integrable system of hydrodynamic type

$$v^i_{t_2} = \left( \left( \begin{array}{c} \frac{\partial h^j}{\partial v^s} + \frac{\partial h^j}{\partial v^i} \\
\frac{\partial^2 h^j}{\partial v^s \partial v^k} v^k_x \end{array} \right) \frac{d}{dx} + \frac{\partial^2 h^k}{\partial v^i \partial v^j} v^j_x \right) \eta_{jl} \left( h^l(v) + \eta_{lp} \frac{\partial h^r}{\partial v^p} \eta_{rq} v^q \right)$$

$$\equiv \left( \left( \begin{array}{c} \frac{\partial h^j}{\partial v^s} + \frac{\partial h^j}{\partial v^i} \\
\frac{\partial^2 h^k}{\partial v^i \partial v^j} \end{array} \right) \eta_{jl} \left( h^l(v) + \eta_{lp} \frac{\partial h^r}{\partial v^p} \eta_{rq} v^q \right) + \eta_{is} \frac{\partial^2 h^j}{\partial v^s \partial v^k} \left( \eta_{jl} h^l(v) + \eta_{lp} \frac{\partial h^r}{\partial v^p} \eta_{rq} v^q \right) \right) v^k_x. \quad (4.14)$$

The hierarchy of integrable systems (4.9) is “canonical” for all bi-Hamiltonian systems of hydrodynamic type possessing pairs of compatible local Hamiltonian operators of hydrodynamic type.

We have also realized a completely similar explicit construction of the corresponding class of integrable bi-Hamiltonian systems of hydrodynamic type in the case of compatible nonlocal Hamiltonian operators of hydrodynamic type (see [21]–[26] about the nonlocal Hamiltonian operators of hydrodynamic type). These results will be published somewhere else.

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Centre for Nonlinear Studies,
L.D. Landau Institute for Theoretical Physics,
Russian Academy of Sciences
e-mail: mokhov@mi.ras.ru; mokhov@landau.ac.ru