The emergent Weyl fermions in condensed matter generally breaks the Lorentz invariance resulting a tilted (type-I) or over-tilted (type-II) energy dispersion. It has remarkable effects on quantum transport properties. Here, we study the Andreev bound state (ABS) and Josephson current in a Weyl superconductor-Weyl (semi)metal-Weyl superconductor junction of a time reversal (TR) broken type-I Weyl semimetal. Two types of pairings are considered in Weyl superconductor, zero momentum BCS-like pairing and finite momentum Fulde-Ferrell-Larkin-Oshinnikov (FFLO)-like pairing. Using Bogoliubov-de Gennes equation, we analytically derive the ABSs and study the current phase relations (CPRs) for the two pairings. We examine both inversion symmetric and inversion breaking tilting effects in CPRs. We demonstrate that the Josephson current $0 - \pi$ transition and Josephson $\phi$ junction would be feasible for BCS-like pairing with inversion symmetric tilt by tuning the length or doping of the normal region. However, these anomalous Josephson effects are absent for inversion breaking tilt and FFLO-like pairing in Weyl superconductor. Our study provides a theoretical framework for understanding unconventional superconducting pairing mechanisms of TR-broken Weyl semimetals. Moreover, the tilt in Weyl nodes naturally lead anomalous current phase relations in this model without magnetic manipulation.

I. INTRODUCTION

A Weyl semimetal (WSM) hosts three dimensional gapless topological states emanated from $k$-space singularities. These singularities persist due to merging of valance and conduction band at some specific $k$-points in the Brillouin Zone, known as Weyl nodes. The nodes always appear in pairs, carry opposite topological charges, and are protected due to either time-reversal (TR) or inversion (IR) symmetry. The energy has linear dispersion with momentum and form Weyl cones around the nodes. The realization of these gapless topologically nontrivial states draw much attention. Initially, WSMs discovered theoretically\cite{1,2} and meanwhile observed experimentally in wide range of materials\cite{3–5}. However, most of the experimential findings Weyl fermions have anisotropic and tilted (type-I) or over tilted (type-II) energy dispersion\cite{6,7}. In type-I, the Weyl cone is weakly tilted so that the Fermi surface encloses either electron-like or hole-like energy separately whereas, in type-II, the tilting is so strong that the electron and hole energy coexist. The Fermi surface in type-II phase is unbounded and has a large density of states near the Weyl points. The tilt in the Weyl cone does not affect the topology of the band structure rather it shows astonishing behaviour in quantum transport including Klein tunneling\cite{8,9}, spin transport\cite{10}, Andreev reflection\cite{11} magnetotransport\cite{12}.

A semimetalic phase is realized when valance band is filled and conduction band is empty. The Fermi label is situated at the touching point i.e., at the Weyl nodes. This ideal situation in WSMs is unstable to impurities which causes the Fermi label enter into the conduction or valance band. Thus the density of states at the Weyl nodes become finite initiating the phase Weyl metals i.e., the lightly doped WSMs. The finite density of states at the nodes are very generic in tilted WSMs, even in absence of impurities. These naturally motivates to query about the superconducting states in Weyl metals. Moreover, the non-trivial topology and nondegenerete bands may trigger unconventional superconducting states in Weyl metals. Many works have been devoted to understand the superconducting pairing mechanisms of Weyl metals\cite{14,15}. Two distinct types of cooper pairings were predicted\cite{16–22} a even/odd parity BCS(Bardeen-Cooper-Schrieffer)-like pairing with zero cooper pair momentum and a FFLO-like pairing with finite-momentum pairs. In BCS state, the electron at momentum $k$ near one Weyl node pairs with another Weyl node electron at momenta $-k$ (intrane) with same energy (let assume TR-symmetry is broken but IR-symmetry is preserved) whereas in FFLO state, pairs are formed from the same Weyl node (intrane). The mean field calculation predicts that local phonon-mediated attractive interaction favours finite momentum FFLO-like pairing over the even-parity BCS state\cite{19}. In contrast, Ref\cite{21} predicts that non-local interaction enforce odd parity BCS state over FFLO state. In this reference, it was shown that the BCS state vanishes identically for local interactions. The odd parity BCS ground state is also predicted in inversion symmetric WSM\cite{22}. The FFLO state is only pairing term when both IR and TR symmetry are broken. In this situation two opposite chiral Weyl nodes are shifted to different energy value. However, further experiments are required to understand the proper superconducting mechanisms in Weyl metals.

The Andreev reflection and Josephson effect are basic tools to investigate the unconventional superconducting pairings. Recently, there are several works have been reported in this context\cite{23,24}. Ref\cite{25} show that the Josephson effect for FFLO-like pairing of Weyl SNS junction is closely resemble to the theory of Josephson effect of graphene or topological insulators. On the other hand, the critical current is independent of chemical potential for BCS-like pairing and thus the effect is different from
entirely different signatures for the two pairing mecha-
nisms. This prompt us to study the tilting effect in
the Josephson current. Most importantly, we explore
whether the tilt induced Josephson effect can be used as
a tool to distinguish the distinct types of pairing mecha-
nisms in these scenarios.

In a superconductor-normal-superconductor junction,
the Andreev bound states exist in the middle region due
to the interference effect of Andreev reflected electrons
and holes within subgap energies of the two supercon-
ductors. The ABS leads to a dc Josephson current car-
rried by the cooper pairs which depends on the macro-
scopic phase difference between two superconductors (φ).

The current phase relation (CPR) generally given by
J = Jc sin φ with Jc is critical current of the junction.
The ground state of such junction is φ = 0. In certain
situations the critical current become negative with
ground state at φ = π, known as Josephson π junction
with CPR: J = |Jc| sin(φ + π). If the ground state appear
at φ = φ0 then the corresponding CPR is given J =
Jc sin(φ − φ0), known as Josephson φ junction. The
Josephson current in a superconductor-ferromagnetic-
superconductor junction or in unconventional super-
conductor Josephson junction stimulates these
anomalous CPR. These junction can exhibits Josephson
current 0 − π transition. The supercurrent reversal
also predicted in irradiated WSM or strong spin-orbit
coupled 2d material Josephson junctions. Recently,
it was shown that a chirality imbalanced potential also
lead 0 − π transition for BCS-like pairing of a TR broken
WSM. The anomalous CPRs are studied recently in
tilted WSM Josephson junction with proximity induced
superconductors. In those references, the supercur-
rent reversal and Josephson φ junction were realized by
tuning the magnetic field or other relevant parameters
in the Weyl Hamiltonian.

In the present work, we study the Josephson effect of
a TR-broken type-I WSM. We consider both FFLO and
BCS-like pairings in Weyl superconductor. We demon-
strate that if the cones are tilting oppositely (i.e., IR sym-
metry breaking tilt). We discuss the critical
current dependencies on the length of normal Weyl metal
region and anticipate the qualitative differences between
two pairings. The findings presented here will be useful
to distinguish the unconventional pairing mechanism of
TR-broken WSMs.

This paper is organized as follows. In Sec-II and Sec-
III, the model, theory and basic formulas for Andreev
bound states and Josephson current are constructed. In
Sec-IV, we discuss the results and finally in Sec-V the

conclusion of this work is given.

II. THEORY

We consider Josephson junction made of type-I WSM
with a slab of normal type-I Weyl (semi)metal for
0 < z < L is sandwiched between two type-I heavily
doped Weyl superconducting regions. The left and right
superconducting regions extend semi-infinetly along z-
direction. Here, we have taken TR-broken WSMs and
assume that the two opposite chiral Weyl nodes are situated
at ±K0 on qz−qz plane with K0 = K00(εx cos α + εy sin α)
with α is angle between the crystal coordinate and junc-
tion coordinate. The normal-state two band Hamiltonian
with momenta k = ±K0 + q around the Weyl nodes at
±K0 reads,

\[ H_0 = \sum_{\chi} \sum_q \Psi_{\chi}^\dagger(q) h_{\chi}^W \Psi_{\chi}(q) \]  

and

\[ h_{\chi}^W = h(a_1 q_1 + a_3 q_3) \sigma_0 + h v(q_1 \sigma_1 + q_2 \sigma_2 + q_3 \sigma_3) - \mu = h_t + h_x \]  

Here, χ = ± defines the chirality of the Weyl nodes. The
first term in Eq. (2), \( h_t = h(a_1 q_1 + a_3 q_3) \sigma_0 \) is responsible
for tilting. For simplicity, we consider tilting is along
q1 and q3 direction with strength a1 and a3 respectively. \( \sigma_0 \) and \( \sigma_y \)'s are unit and Pauli matrices acting on the
spin space, respectively. \( \Psi_{\chi}^\dagger(q) = (c_{\chi,\uparrow}(q), c_{\chi,\downarrow}(q)) \) is the spinor basis with \( c_{\chi,\sigma}(q) \) the creation operator for an
electron.

\[ q_1 = q_x \cos \alpha - q_z \sin \alpha \]  

\[ q_2 = a_y \]  

\[ q_3 = q_x \cos \alpha + q_z \sin \alpha \]  

and, similarly, \( \sigma_1 = \sigma_x \cos \alpha - \sigma_z \sin \alpha \), \( \sigma_2 = \sigma_y \), \( \sigma_3 = \sigma_z \cos \alpha + \sigma_x \sin \alpha \). The pairing term for BCS and FFLO
pairings are given

\[ \mathcal{H}_{pair}^B = \sum_{\chi,q} \Delta(z) c_{\chi,\uparrow}(q) c_{\chi,\downarrow}(-q) + h.c. \]  

\[ \mathcal{H}_{pair}^F = \sum_{\chi,q} \Delta(z) c_{\chi,\uparrow}(q) c_{\chi,\downarrow}(-q) + h.c. \]  

where the subscript B and F correspond to BCS and
FFLO-like pairing, respectively. \( \Delta(z) \) is the pairing potential. The BdG Hamiltonian in the
basis \( (c_{\chi,\uparrow}(q), c_{\chi,\downarrow}(q), c_{\chi,\downarrow}(-q), -c_{\chi,\downarrow}(-q)) \) and
\( (c_{\chi,\downarrow}(q), -c_{\chi,\downarrow}(-q), c_{\chi,\uparrow}(-q), -c_{\chi,\uparrow}(-q)) \) are given, both
BCS and FFLO-like pairings,

\[ H_B^\pm = \begin{pmatrix} h_{B}^W (i \nabla \mp K_0) & \Delta(z) \sigma \pm 2i K_0 \tau \end{pmatrix}, \quad -h_{B}^W \Delta(z), \quad \mp \end{pmatrix} \]  

\[ H_F^\pm = \begin{pmatrix} h_{F}^W (i \nabla \mp K_0) & \Delta(z) e^{\mp 2i K_0 \tau} \end{pmatrix}, \quad h_{F}^W \Delta(z), \quad \mp \end{pmatrix} \]  

where \( \tau \) is the Pauli matrix in the spin space.
A gauge transformation removes the large momentum $K_0$ from the BdG Hamiltonian in Eqs. (5,6). The transformation for BCS and FFLO pairings are,

$$H_B^\uparrow \rightarrow \tilde{H}_B^\uparrow = e^{i\pm K_0 \cdot \tau} H_B^\uparrow e^{i\mp K_0 \cdot \tau}$$
$$H_B^\downarrow \rightarrow \tilde{H}_B^\downarrow = e^{i\pm \sigma_z K_0 \cdot \tau} H_B^\downarrow e^{i\mp \sigma_z K_0 \cdot \tau}$$

respectively, which gives the transformed Hamiltonian,

$$\tilde{H}_B^\uparrow = \begin{pmatrix} h_{\uparrow}^\uparrow (z) & \Delta(z) \\ \Delta(z)^* & -h_{\uparrow}^\downarrow (z) \end{pmatrix}$$
$$\tilde{H}_B^\downarrow = \begin{pmatrix} h_{\downarrow}^\downarrow (z) & \Delta(z) \\ \Delta(z)^* & -h_{\downarrow}^\uparrow (z) \end{pmatrix}$$

The Hamiltonian $h_{\uparrow}$ is independent of $\alpha$. For BCS pairing we perform an extra unitary transformation

$$\tilde{H}_B^\uparrow \rightarrow \tilde{U}_\alpha^+ \tilde{H}_B^\uparrow \tilde{U}_\alpha^-$$

with

$$\tilde{U}_\alpha^\pm = \frac{1}{2} \left[ (\tau_0 \pm \tau_z) \sigma_x e^{i\alpha \sigma_y} + (\tau_0 \mp \tau_z) \right]$$

The unit matrix $\tau_0$ and Pauli matrix $\tau_i$ are acting on particle-hole space. The resulting BdG Hamiltonian become,

$$\tilde{H}_B^\uparrow = \begin{pmatrix} h_1 + h_2 (z) & \Delta(z) \\ \Delta(z)^* & -h_1 - h_2 (z) \end{pmatrix}$$

and similarly, we can write down the Hamiltonian for $\tilde{H}_B^\downarrow$. Here, $\tilde{\Delta}(z) = \Delta(z) \sigma_x \cos \alpha - \Delta(z) \sigma_z \sin \alpha \approx -\Delta(z) \sigma_z \sin \alpha$ and $\tilde{h}_\uparrow = \hbar v (q_x \sigma_x - q_y \sigma_y + q_z \sigma_z)$. The tilting Hamiltonian part is: $h_1 = \hbar (a_1 \cos \alpha + a_3 \sin \alpha) q_x + (-a_1 \sin \alpha + a_3 \cos \alpha) q_z$. In the rest of the paper we take $\alpha = \pi/2$ and consider tilting is only along the transport direction (z-axis) i.e., $h_1 = \hbar C_q q_z$. A WSM is in type-I phase if $C_q < v$ and in type-II phase if $C_q > v$. With this simplification, the normal part Hamiltonian (see Eq. (2)) in junction coordinate is inversion symmetric (i.e., $\sigma_z h_{\uparrow}^\uparrow (q) \sigma_z = h_{\downarrow}^\downarrow (-q)$) if $C_q = -C_-$ (Case-I). In this case, the opposite chiral Weyl nodes tilting oppositely. The inversion symmetry is broken if $C_+ = C_-$(Case-II). In this case, the opposite chiral Weyl nodes tilting parallelly. We discuss both tilting cases in the Josephson effect. In the Josephson junction, we assume a step-like model for $\tilde{\Delta}(z)$ and $\mu$. We denote pairing potential: $\tilde{\Delta}(z) = \Delta_0 [\Theta(-z) \epsilon^{i\phi/2} + \Theta(z-L) \epsilon^{-i\phi/2}]$ with $\Delta_0$ is the superconducting gap and $\phi$ is the phase difference of superconducting order parameter. The chemical potential is given: $\mu = \mu_N (L - |z|) + \mu_S \Theta(|z| - L)$. We have taken $h = v = 1$ and put them back when necessary.

III. ANDREEV BOUND STATE AND JOSEPHSON CURRENT

The Josephson current in the junction is obtained by calculating the Andreev-bound state in the normal region. This is done by matching the wave functions at the interface between three different regions. Explicitly, the wave functions in three different regions are given,

$$\Psi_S^\pm = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}$$

where $\Psi_N = a_1 \Psi_+^e + a_2 \Psi_-^e + a_3 \Psi_+^h + a_4 \Psi_-^h$.

Here, $\Psi_S^{L(R)}$ is the wave function in the left (right) superconducting region and $\Psi_N$ is the wave function in the normal region. $t_i$’s and $a_i$’s are the scattering coefficients of quasiparticles (electron or hole) in different regions. The subscript $\pm$ on the wave function in the normal region indicates the direction of quasiparticles motion (group velocity). We now look for an energy eigenvalues $\epsilon$ which gives a non zero solution for the boundary conditions: $\Psi_S^L = \Psi_N$ at $z = 0$ and $\Psi_N = \Psi_S^R$ at $z = L$. These boundary conditions leads to $8 \times 8$ matrix $\mathcal{M}$.

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 \\ \mathcal{M}_3 & \mathcal{M}_4 \end{pmatrix}$$

where every elements $\mathcal{M}_i$ are the $4 \times 4$ matrix. $\text{Det} [\mathcal{M}]_{tb} = 0$ gives the non-trivial relation between $\epsilon_b$ and superconducting phase difference $\phi$. It is known that, the Josephson current at low temperature ($T \ll \Delta_0/k_B$, with $k_B$ is the Boltzmann constant) is determined solely by the bound states ($\epsilon_b$) and is given by,

$$I(\phi) = -\frac{2e}{h} \sum_b \frac{\partial \epsilon_b}{\partial \phi} f(\epsilon_b)$$

where $f(\epsilon_b)$ is the Fermi-Dirac distribution function. The Josephson current can be obtained as,

$$J(\phi) = \frac{W^2}{(2\pi)^2} \int I(\phi) dk_x dk_y$$

with $W$ is the dimension in both $x$ and $y$ direction. We define critical supercurrent as $J_c = \text{max} \{|J(\phi)|\}$. We take the limit $\mu_N, \mu_S \gg \Delta$. We also consider the short-junction limit i.e., $L \ll \xi = \hbar v/\Delta_0$, which allow us to neglect the Josephson current contribution from states $\epsilon_b > \Delta_0$. In the following, using this method we calculate Josephson current both in FFLO and BCS-like pairings. Here we focus on zero temperature.

A. FFLO-like Pairing

We first consider the BdG Hamiltonian for FFLO-like pairing given in Eq. (15) and write down the wave functions in three different regions. The electron and hole wave functions in the normal region are read,

$$\Psi_S^{e(+)\downarrow}_{\text{in(out)}} = e^{iK_{\downarrow}(z)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\Psi_S^{h(+)\uparrow}_{\text{in(out)}} = e^{iK_{\uparrow}(z)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(18)
Here, ± sign in superscript corresponds to the Hamiltonian $H_B^\pm$ in Eq.(10) and $i\phi$ (out) denotes the inward and outward particles motion. The elements $\mathcal{P}_i^{(+)}$ are given by

$$\mathcal{P}_1^{(+)} = \frac{k_p e^{i\theta}}{k^{(+)} + k_1}; \mathcal{P}_2^{(+)} = \frac{k_p e^{i\theta}}{k^{(+)} + k_2}$$

$$\mathcal{P}_3^{(+)} = \frac{k_p e^{i\theta}}{k^{(+)} + k_3}; \mathcal{P}_4^{(+)} = \frac{k_p e^{i\theta}}{k^{(+)} + k_4};$$

where $k_p = \sqrt{k_x^2 + k_y^2}$ is the conserved transversed momenta and $\theta = \tan^{-1}(k_y/k_x)$. The wavevectors $k_i^{(+)}$'s are obtained by solving eigenvalues equations of electron and hole hamiltonian separately, which are given by

$$k^{(+)} = \frac{C_{+}(\mu + \mu_N) \mp \sqrt{(E + \mu_N)^2 + (C^{(2)}_{+} - 1)k_p^2}}{(C^{(2)}_{+} - 1)}$$

(19)

with

$$k^{(+)}_{1(2)} = \frac{C_{+}(\mu - E) \pm \sqrt{(E - \mu_N)^2 + (C^{(2)}_{+} - 1)k_p^2}}{(C^{(2)}_{+} - 1)}$$

(20)

The quasiparticle energy spectrum of BdG hamiltonian is obtained by diagonalizing the hamiltonian of Eq.(10). The energy eigenvalues are given by

$$\mathcal{E}^{(+)} = \pm \sqrt{(\mu_s - C_{+}(\pm k_z \pm k))^2 + \Delta^2}$$

(22)

where $k = \sqrt{k_x^2 + k_y^2}$. In the following, we consider heavily doped superconducting regions ($\mu_s \gg \mu_N \gg \Delta$). The basis spinor for $z < 0$ takes a simple form as follows,

$$\Psi_1^{(+)} = e^{i k^{(+)} x} \begin{pmatrix} e^{-i \phi/2} & 0 & e^{-i \gamma_p^{(+)} \gamma_p^{(-)}} \\ 0 & e^{i \phi/2} & 0 \end{pmatrix}$$

$$\Psi_2^{(+)} = e^{i k^{(+)} y} \begin{pmatrix} 0 & e^{-i \phi/2} & 0 \end{pmatrix}$$

(23)

and similarly for $z > 0$ the spinors are read as,

$$\Psi_3^{(+)} = e^{i k^{(+)} z} \begin{pmatrix} e^{i \phi/2} & 0 & e^{i \gamma_p^{(-)} \gamma_p^{(+)}} \\ 0 & 0 & e^{-i \phi/2} \end{pmatrix}$$

$$\Psi_4^{(+)} = e^{i k^{(+)} x} \begin{pmatrix} 0 & e^{i \phi/2} & 0 \\ e^{-i \phi/2} & 0 & 0 \end{pmatrix}$$

(24)

where,

$$\gamma_p^{(+)} = -\cos^{-1} \frac{\mathcal{E}^{(+)}}{\Delta}$$

(25)

and the wavevectors $k_i^{(+)}$'s can obtained from Eq.(22). We now calculate ABSs using the method discussed in Sec-III. We work in the short junction limit. In this limit, one can easily show that in both Case-I and Case-II, $k_3^{(+)} = k_2^{(+)}$ and $\mathcal{P}_3^{(+)} = \mathcal{P}_2^{(+)}$. The ABS energy is given by,

$$E^{(+)} = \Delta \sqrt{1 - \Gamma^{(+)} \sin^2 \frac{\phi}{2}}$$

(26)

where the transmission probability ($\Gamma$) is given by,

$$\Gamma^{(+)} = \frac{(\mathcal{P}_1^{(+)} - \mathcal{P}_2^{(+)})^2}{(\mathcal{P}_1^{(+)} - \mathcal{P}_2^{(-)} + 4\mathcal{P}_1^{(+)} \mathcal{P}_2^{(-)} \sin^2(\Delta k L))}$$

(27)

where

$$\Delta k = \frac{(k_2^{(+)} - k_3^{(+)})}{2} = \frac{\sqrt{\mu_s^2 + (C^2 - 1)k_p^2}}{(C^2 - 1)}$$

(28)

Here, $C$ is the absolute value of $C_{+}$. One can check that $\Gamma^{+} = \Gamma^{-}$ and hence $E^{ABS} = E^{ABS}$ for both Case-I and Case-II. So, the contribution of Josephson current (See Eq.(16)) from two chirality sectors are same. Using Eqs.(16,17) we calculate the Josephson current. The total Josephson current is now given,

$$J = (J_+ + J_-) \sin \phi$$

$$= 2J_0 \sin \phi$$

(29)

Here, $\phi$ is the phase difference between two superconductors.

B. BCS-like Pairing

We execute similar calculations for the BCS-like pairing superconductor also. We calculate ABSs and Josephson current for $H_B^+$ and $H_B^-$ separately. Using Eq.(13), the electron and hole wave functions in the normal region are read,

$$\Psi_{in(out)}^{(+)} = e^{ik_{in(out)}^+} \begin{pmatrix} 1 & Q_{1(2)} & 0 \end{pmatrix}$$

$$\Psi_{in(out)}^{(-)} = e^{ik_{in(out)}^+} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} Q_{3(4)}$$

(30)

Here, + sign in superscript corresponds to the Hamiltonian $H_B^+$. The elements $Q_i^\pm$ are follows,

$$Q_1^+ = \frac{q_p e^{i\theta}}{q_+ + q_1}; Q_2^+ = \frac{q_p e^{i\theta}}{q_- + q_2}$$

$$Q_3^+ = \frac{q_p e^{-i\theta}}{q_+ + q_3}; Q_4^+ = \frac{q_p e^{-i\theta}}{q_- + q_4}$$

(31)

where $q_p = \sqrt{q_x^2 + q_y^2}$ is the conserved transversed momenta and $\theta = \tan^{-1}(q_y/q_x)$. The wavevectors $q_i^\pm$'s are
obtained by solving eigenvalues equations of electron and hole Hamiltonian separately, which are given,

\[ q_{1(2)}^+ = \frac{C_+ (E + \mu_N) \mp \sqrt{(E + \mu_N)^2 + (C_+^2 - 1)q_p^2}}{(C_+^2 - 1)} \]
\[ q_{3(4)}^+ = \frac{C_- (\mu_N - E) \pm \sqrt{(E - \mu_N)^2 + (C_-^2 - 1)q_p^2}}{(C_-^2 - 1)} \]

with

\[ q_{r(+)}^+ = \frac{C_+^2 q_p^2 + q_{r(2)}^2}{\sqrt{C_+^2 q_p^2 + q_{r(2)}^2}} \]
\[ q_{r(-)}^+ = \frac{C_-^2 q_p^2 + q_{r(2)}^2}{\sqrt{C_-^2 q_p^2 + q_{r(2)}^2}} \]

The quasiparticles energy spectrum is obtained by diagonalizing Eq.(13) and given (we consider \( \mu_s \) is large),

\[ \mathcal{E}^+ = \frac{1}{2} [(C_+ - C_-) q_z \pm \sqrt{4 \Delta^2 + (q_z (C_+ + C_- \pm 2 - 2 \mu_s)^2} \]

Eq.(34) takes the following form for Case-II (i.e., inversion breaking tilt),

\[ \mathcal{E}^+ = \sqrt{\Delta^2 + (\mu_s - C q_z \pm 1)^2} \]

which is equal for FFLO pairing in Eq.(22) for large \( \mu_s \) and for Case-I (i.e, inversion symmetric tilt),

\[ \mathcal{E}^+ = C q_z \pm \sqrt{\Delta^2 + (\mu_s \pm q_z)^2} \]

The basis spinors for \( z < 0 \) are

\[ \Psi_1^+ = e^{i\phi_{1z}} \begin{pmatrix} e^{-i\phi/2} & 0 & e^{-i\gamma_{1b}^+} & 0 \end{pmatrix} \]
\[ \Psi_2^+ = e^{i\phi_{2z}} \begin{pmatrix} 0 & e^{-i\phi/2} & 0 & -e^{i\gamma_{1b}^+} \end{pmatrix} \]

and similarly, for \( z > 0 \) are

\[ \Psi_3^+ = e^{i\phi_{3z}} \begin{pmatrix} e^{i\phi/2} & 0 & e^{i\gamma_{2b}^+} & 0 \end{pmatrix} \]
\[ \Psi_4^+ = e^{i\phi_{4z}} \begin{pmatrix} 0 & e^{i\phi/2} & 0 & -e^{-i\gamma_{2b}^+} \end{pmatrix} \]

where \( \gamma_{1b}^+ = -\cos^{-1}(\mathcal{E}^+ / \Delta) \) and the wave vectors \( q_+^+ \) is obtained from Eq.(34). Now, in short junction limit the wavevectors in Eq.(32) are satisfied \( q_{3(4)}^+ = q_{2(1)}^+ \) in Case-II and \( q_{3(4)}^+ = -q_{1(2)}^+ \) in Case-I, respectively. The wavevectors follow same relation as for FFLO pairing in Case-II and thus the ABS spectrum will have similar nature. In Case-I, two pair electrons of cooper pair at Fermi surface have finite wavevector shift i.e., \( |q_{1(2)}^+ - q_{3(4)}^+| \) is finite while it is zero for Case-II or in FFLO pairing. This additional phase should alter the phase relation of ABS and the Josephson current. In Case-I, the analytical form of ABS is given by,

\[ \mathcal{E}_{ABS}^+ = \Delta \sqrt{\frac{C}{A} - \frac{B}{A} \sin^2 \phi_B} \]

where the expression of \( A \) and \( C \) are given by,

\[ A = (Q_1 Q_3 + Q_2 Q_4) \cos(\Delta q L) - (Q_2 Q_3 + Q_1 Q_4) \]
\[ B = (Q_1 - Q_2)(Q_3 - Q_4) \]
\[ C = (Q_1 Q_3 + Q_2 Q_4) \cos^2(\frac{\Delta q L}{2}) \]
\[ + (Q_1 Q_2 + Q_3 Q_4) \sin^2(\frac{\Delta q L}{2}) \]
\[ - (Q_1 Q_4 + Q_2 Q_3) \]

with \( \Delta q = (q_1^+ - q_2^+) / 2 \). The expression of phase \( \phi_B^+ \) is given,

\[ \phi_B^+ = \frac{(q_1^+ + q_2^+) L}{2} - \frac{\phi}{2} \]

with \( \phi = \mu N C L / (C^2 - 1) \). The ABS for \( H_B^- \) is obtained by replacing \( Q_1 \leftrightarrow Q_3 \) and \( Q_2 \leftrightarrow Q_4 \) in Eqs.(39),(40) and \( \phi_{1z} \rightarrow -\phi_{1z} \) in Eq.(41). The total Josephson current is given by,

\[ J = J_+ + J_- \]
\[ = J_0^+ \sin(\phi + \phi_{1z}) + J_0^- \sin(\phi - \phi_{1z}) \]

and the difference between two currents is,

\[ J_{diff} = J_+ - J_- \]
\[ = \frac{J_0^+}{2} \sin(\phi + \phi_{1z}) - \frac{J_0^-}{2} \sin(\phi - \phi_{1z}) \]

The current \( J_{diff} \) is zero for FFLO pairing since \( J_+ \) and \( J_- \) are equal.

IV. RESULTS AND DISCUSSIONS

The numerical results of Josephson current with \( \phi \) for FFLO pairing are shown in Fig.(11). In the left panel, we have shown the results for different lengths with a fixed
value of $C$. In the right panel, we have shown these for different values of $C$ with a fixed $L/\xi$. The total current follows the relation $J = J_0 \sin \phi$ in which $J_0$ is only dependent on the length and tilted parameter. Therefore, the junction always behaves as a 0-junction.

To understand the non-trivial features for BCS state in Case-I, we will first discuss the ABSs. We have shown the numerical results of phase dependency of ABSs as a parameter of $L/\xi$ with a fixed $C = 0.3$ in Fig. (2). The ABSs for two sectors $H_B^+$ and $H_B^-$ are degenerate with the two conditions i.e., if $\phi_1 = n\pi$ or $\phi_1 = (2n + 1)\pi/2$. With the first condition, the ABS spectrum has negative (positive) slope and with the second condition the spectrum has positive (negative) slope for $\phi \in [0, \pi] (\phi \in [\pi, 2\pi])$ as shown in upper and below left panel of Fig. (2) respectively. The corresponding Josephson current remains positive and negative for $\phi \in [0, \pi]$ which represents the 0 and $\pi$-junction, respectively. On the other hand, if $L/\xi$ is not satisfied the above conditions, then it is difficult to say whether the $0-\pi$ transition happens or not. We have shown these in upper and below right panel of Fig. (2) for $L/\xi = 0.01\pi$ and $0.02\pi$ respectively. The linear crossing points are shifted oppositely in $\phi$-plane for these value of $L/\xi$. It is found that the some ABSs spectra have mixture of positive and negative slope for $\phi \in [0, \pi]$ or $\phi \in [\pi, 2\pi]$. Also, the spectra shows two nondegenerate minima for $\phi \in [0, 2\pi]$.

From the above discussion, we found that ABS slope can be positive, negative or mixture of these two. Now these cause Josephson 0, $\pi$ and $\phi$ junction, respectively. In Fig. (3) we have shown the numerical results of the Josephson currents. It is found that the supercurrent reversal occurs with tuning the length for a fixed $C$ (left panel of Fig. (3)) or likewise, changing $C$ with a fixed length (right panel of Fig. (3)). Interestingly, the difference of the Josephson current from the two sectors $H_B^+$ and $H_B^-$ lead Josephon $\phi$ junction when either the conditions $\phi_1 = n\pi$ or $\phi_1 = (2n + 1)\pi/2$ are not satisfied. We have shown this in Fig. (4) for two different values of $L/\xi$ with a fixed value $C$. In this case the junction allows a finite supercurrent even if the superconducting phase difference is zero. In order to access the experimental signature of dc Josephson current we have shown the characteristics of junction length dependent of critical current $J_c$ in Fig. (5). The peacks in the critical current plot for BCS state gives a clear indication of Josephson current $0-\pi$ transition. For example, the right panel of Fig. (5) shows a peak around $L/\xi = 0.015\pi$ where the supercurrent reversed (see also Fig. (4)). The period of oscillation of $J_c$ is compatible with the relation $J_c = 2J_0\cos \phi_1$ for a finite value of $C$. The characteristics of critical current for FFLO-like pairing are shown in left panel of Fig. (5). The critical current $J_c$ oscillates and the amplitude decreases rapidly with $L/\xi$. The critical current in absence of tilting, shown by blue solid line in Fig. (3), have same pattern for the two pairing mechanisms. The experimental studies of $J_c$ can be used to distinguish the BCS and FFLO-like pairing in WSM.

The proximity effect of an s-wave superconductor on a magnetic WSM has been studied by Bovenzi et al. They have shown that if the vector connecting the Weyl nodes has a component parallel to the interface, the Josephson current from the bulk states supressed by the phenomenon ‘chirality blockade’. In contrast to their work, where the superconductivity is extrinsic, we rather...
focused here on the intrinsic superconductivity of the doped WSMs. In this manifestation the superconducting pairing potential of odd parity BCS state has pseudoscalar nature as classified in Ref.\(^\text{41}\). Also the superconducting gap \(\Delta_B = |\Delta| \sin \alpha\) in BCS state has maximum at \(\alpha = \pi/2\) i.e., vector connecting opposite chiral nodes is perpendicular to the interface. In both the cases the chirality blockade will be absent as discussed in Ref.\(^\text{40,42}\). This also explains the absence of chirality blockade in our model and in earlier studies.\(^\text{23,24}\)

In the present work we restrict our study for type-I WSMs. The results can be generalized for type-II WSMs. The wave vectors of electrons and holes given in Eq.\(^\text{32}\) also holds for type-II WSMs which cause a finite wave vector shift. So, our results of supercurrent reversal and Josephson \(\phi_0\) junction will be valid for type-II WSMs as well. However, due to the open Fermi surface and large density of states of type-II Weyl nodes, it may bring interesting physics over the type-I case. Also, in type-II WSMs there is a critical tilt orientation at which the two nodes in the superconducting gap function disappear by merging in the Brillouin zone\(^\text{43}\). We will report the Josephson effects of a type-II WSM, keeping all these issues, elsewhere.

V. CONCLUSIONS

We investigate the Josephson effect of a TR-broken type-I WSM in a WSC-WSM-WSC junction. We consider two types of hitherto known possible pairings of a TR-broken WSM: FFLO-like and BCS-like pairing. For BCS-like pairing, a finite inversion symmetric tilt results a wave-vector shift between the electron and the hole in the Andreev Bound state. This tilt induced extra phase in wave vector modifies the phase relation of ABS and therefore the Josephson current. We found three kind of slopes in \(\phi\)-dependent ABS spectrum: positive, negative and mixture of these two for \(\phi \in [0, \pi]\). The three kind of slopes are corresponds to Josephson 0, \(\pi\) and \(\phi\) junction. However, this tilt induced phase vanishes in inversion asymmetric tilt. In this case, the Josephson effect of BCS state is quite akin to FFLO-like pairing. We have also shown that, the junction always have a 0-junction for FFLO-like pairing. Since the tilt in WSM is ubiquitous, our results can easily verify in experiments. Finally, the tilt induced supercurrent reversal and Josephson \(\phi\) junction are novel findings here which can provides an efficient way to understand the unconventional superconductor pairing mechanism of Weyl semimetals. Furthermore, here the anomalous CPRs are very natural in tilted WSMs and can realized in absence of magnetic terms in the Hamiltonian.

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