Fully Mechanized Proofs of Dilworth’s Theorem
and Mirsky’s Theorem

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Abstract. We present two fully mechanized proofs of Dilworth’s and Mirsky’s theorems in the Coq proof assistant. Dilworth’s Theorem states that in any finite partially ordered set (poset), the size of a smallest chain cover and a largest antichain are the same. Mirsky’s Theorem is a dual of Dilworth’s Theorem. We formalize the proofs by Perles [2] (for Dilworth’s Theorem) and by Mirsky [5] (for the dual theorem). We also come up with a library of definitions and facts that can be used as a framework for formalizing other theorems on finite posets.

1 Introduction

Formalization of any mathematical theory requires a lot of time and effort. The length of formal proofs blow up significantly. In combinatorics, the task becomes even more difficult due to the lack of structure in the theory. There is no established order to formalize the theory. Some statements often admit more than one proof using completely different ideas. Thus, exploring the dependencies among the results may help in identifying an effective order to formalize them. Dilworth’s Theorem is a well-known result in combinatorics. It relates the size of a chain cover and an antichain in a poset. The original version, which talks about the chain cover, was first proved by Dilworth [1]. Since then, the theorem has attracted significant attention and several new proofs [2,3,4] were found. Besides being popular, Dilworth’s Theorem is an important result as it reveals the structure of a general poset. It has been successfully used to give intuitive and concise proofs of some other important results such as Hall’s Theorem [6,7], Erdős-Szekeres Theorem [9] and Sperner’s Lemma [8]. In this sense, it is a central theorem and a good candidate for formalization. For Dilworth’s Theorem we have formalized the proof by Perles [2]. We also formalize a dual of the Dilworth’s Theorem (Mirsky’s Theorem [5]) which relates the size of an antichain cover and a chain in a poset.

Formalization of the known mathematical results can be traced back to the systems Automath and Mizar [18]. Mizar hosts the largest repository of the formalized mathematics. It uses a declarative proof style close to the language that mathematicians understand. Mizar supports some built in automation to

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save time during proof development. However, this results in a large kernel (core) and reduces our faith in the system. The Coq proof assistant deals with this problem in a nice way. It separates the process of proof development from proof checking. Some small scale proof automation is also possible in Coq. However, every proof process finally yields a proof-term which is verified using a small kernel. Thus, the part (kernel) of the code we need to trust remains small.

In addition to a small kernel, the Coq proof assistant also has some other distinctive features that we found useful:

**Records (Dependent):** It helps us to pack mathematical objects and their properties in one definition. For example, in the Coq standard library different components of a partial order and their properties are expressed using a single definition of dependent record (PO) [12].

**Coercions:** It helps us in defining hierarchy among mathematical structure. For example, in our formalization finite partial orders are defined by extending the definition of partial orders with finiteness condition. This avoids redefining similar things at different places. It is similar to the concept of inheritance in programming languages.

**Ltac:** A language support [14] for creating new tactics while being completely inside Coq. It helps in small scale proof automation and proof search, hence reduces the time of proof development significantly.

**Library:** The Coq system hosts a standard library [19] that contains a bunch of useful definitions and results. We use this facility and avoid new definitions, unless absolutely essential. For example, we use the modules for Sets and Basic Peano arithmetic extensively in this work.

In this paper, we give the details of our mechanized proofs of Dilworth’s and Mirsky’s Theorems. In section §2 we present the original theorems and their proofs by Perles [2] and Mirsky [5]. In section §3 we describe every term that appears in the formal statements of these theorems. We provide a detailed account of some general results on finite partial orders in section §4. Finally, we review the related works in section §5 and conclude in section §6.

## 2 Dilworth’s and Mirsky’s Theorem

### 2.1 Partially ordered sets (poset)

A poset \((P, \leq)\) consists of a set \(P\) together with a binary relation \(\leq\) satisfying reflexivity, antisymmetry and transitivity properties. Elements \(a, b \in P\) are said to be *comparable* if \(a \leq b\) or \(b \leq a\). Otherwise, they are *incomparable*. A *chain* is a subset of \(P\) any two of whose elements are comparable. A subset of \(P\) in which no two distinct elements are comparable is called an *antichain*. Note that

- A chain and an antichain can have at most one element in common.

A *chain cover* is a collection of chains such that their union is the entire poset. Similarly, an *antichain cover* is a collection of antichains such that their union is the entire poset.
The width of a poset $P$, $\text{width}(P)$, is the size of a largest antichain in $P$.

The height of a poset, $\text{height}(P)$, is the size of a largest chain in $P$.

An element $b \in P$ is called a maximal element if there is no $a \in P$ (different from $b$) such that $b \leq a$. Similarly, an element $a \in P$ is called a minimal element if there is no $b \in P$ (different from $a$) such that $b \leq a$.

### 2.2 Dilworth's Theorem

**Theorem** (Dilworth): In any poset, the maximum size of an antichain is equal to the minimum number of chains in any chain cover. In other words, if $c(P)$ represents the size of a smallest chain cover of $P$, then $\text{width}(P) = c(P)$.

**Proof** (Perles): The equality will follow if one can prove:

1. Size of an antichain $\leq$ Size of a chain cover, and
2. There is a chain cover of size equal to $\text{width}(P)$.

It is easy to see why (1) is true. Assume otherwise, i.e, there is an antichain $A$ of size bigger than the size of a smallest chain cover $C_V$. Then $A$ will have more elements than the number of chains in $C_V$. Hence, there must exist a chain $C$ in $C_V$ which covers two element of $A$. However, this cannot be true since a chain and an antichain (in this case $C$ and $A$) can have at most one element in common.

Proof of (2) is more involved. We will prove (2) using strong induction on the size of $P$. Let $m$ be the size of the largest antichain in $P$, i.e, $m = \text{width}(P)$.

- Induction hypothesis: For all posets $P'$ of size at most $n$, there exists a chain cover of size equal to $\text{width}(P')$.

**Induction step:** Fix a poset $P$ of size at most $n + 1$. Let maximal($P$) and minimal($P$) represents respectively the set of all maximal and the set of all minimal elements of $P$. Now, one of the following two cases might occur,

1. There exists an antichain $A$ of size $m$ which is neither maximal($P$) nor minimal($P$).
2. No antichain other than maximal($P$) or minimal($P$) has size $m$.

**Case-1:** For the first case we define the sets $P^+$ and $P^-$ as follows:

\[
P^+ = \{x \in P : x \geq y \text{ for some } y \in A\}
\]

\[
P^- = \{x \in P : x \leq y \text{ for some } y \in A\}
\]

Here $P^+$ captures the notion of being above $A$ and $P^-$ captures the notion of being below $A$. Note that the elements of $A$ are both above and below $A$, i.e, $A \subseteq P^+ \cap P^-$. For any arbitrary element $x \in P$

- If $x \in A$ then $x \in P^+ \cap P^-$ and hence $x \in P^+ \cup P^-$. 


– If \( x \notin A \) then \( x \) must be comparable to some element in \( A \); otherwise \( \{x\} \cup A \) will be an antichain of size \( m + 1 \). Hence, if \( x \notin A \) then \( x \in P^+ \cup P^- \).

Therefore, \( P^+ \cup P^- = P \). Since there is at least one minimal element not in \( A \), \( P^+ \neq P \). Similarly \( P^- \neq P \). Thus \( |P^+| < |P| \) and \( |P^-| < |P| \), hence we will be able to apply induction hypothesis to them. Observe that \( A \) is also a largest antichain in the poset restricted to \( P^+ \); because if there was a larger one, it would have been larger in \( P \) also. Therefore by induction,

– There exists a chain cover of size \( m \) for \( P^+ \), say \( \bigcup_{i=1}^{m} C_i \).
– Similarly, there is a chain cover of size \( m \) for \( P^- \), say \( \bigcup_{i=1}^{m} D_i \).

Elements of \( A \) are the minimal elements of the chains \( C_i \) and the maximal elements of the chains \( D_i \). Therefore we can join the chains \( C_i \) and \( D_i \) together in pairs to form \( m \) chains which form a chain cover for the original poset \( P \).

Case-2: In this case we can’t have an antichain of size \( m \) which is different from both maximal(\( P \)) and minimal(\( P \)). Consider a minimal element \( x \). Choose a maximal element \( y \) such that \( x \leq y \). Such a \( y \) always exists. Remove the chain \( \{x, y\} \) from \( P \) to get the poset \( P' \). Then \( P' \) contains an antichain of size \( m - 1 \). Also note that \( P' \) can’t have an antichain of size \( m \).

– Because if there was an antichain of size \( m \) in \( P' \), then that would also be an antichain in \( P \) which is different from both maximal(\( P \)) and minimal(\( P \)), and hence we would have been in the first case (i.e, Case-1).

Hence by induction hypothesis we get a chain decomposition of \( P' \) of size \( m - 1 \). These chains, together with \( \{x, y\} \), give a decomposition of \( P \) into \( m \) chains. □

### 2.3 Mirsky’s Theorem

**Theorem** (Dual-Dilworth): In any poset, the maximum size of a chain is equal to the minimum number of antichains in any antichain cover. In other words, if \( c(P) \) represents the size of a smallest antichain cover of \( P \), then \( \text{height}(P) = c(P) \).

**Proof** (Mirsky): The equality will follow if one can prove:

1. Size of a chain \( \leq \) Size of an antichain cover, and
2. There is an antichain cover of size equal to \( \text{height}(P) \).

Again, it is easy to see why (1) is true. Any chain shares at most one element with each antichain from an antichain cover. Moreover, every element of the chain must be covered by some antichain from the antichain cover. Hence, the size of any chain is smaller than or equal to the size of any antichain cover.

We will prove (2) using strong induction on the size of the largest chain of \( P \). Let \( m \) be the size of the largest chain in \( P \), i.e, \( m = \text{height}(P) \).

– Induction hypothesis: For all posets \( P' \) of height at most \( m - 1 \), there exists an antichain cover of size equal to \( \text{height}(P') \).
Induction Step: Let $M$ denotes the set of all maximal elements of $P$, i.e, $M = \text{maximal}(P)$. Observe that $M$ is a non empty antichain and shares an element with every largest chain of $P$. Consider now the partially ordered set $(P - M, \leq)$. The length of the largest chain in $P - M$ is at most $m - 1$. On the other hand, if the length of the largest chain in $P - M$ is less than $m - 1$, $M$ must contain two or more elements that are members of the same chain, which is a contradiction. Hence, we conclude that the length of largest chain in $P - M$ is $m - 1$. Using induction hypothesis there we get an antichain cover $\mathcal{A}_C$ of size $m - 1$ for $P - M$. Thus, we get an antichain cover $\mathcal{A}_C \cup \{M\}$ of size $m$ for $P$. \square

3 Definitions and Formalization

3.1 Definitions

Once a statement is proved in Coq, the proof is certified without having to go through the proof-script. However, one needs to verify whether the statement being proved correctly represents the original theorem. In this section we try to explain the definition of each term that appears in the formal statement of the Dilworth’s theorem.

Definitions from Standard Library We have used the *Sets* module from the Coq Standard Library [19], where a declaration $\text{S: Ensemble U}$ is used to represent a set $S$.

– Sets are treated as predicates, i.e, $x \in S$ iff $S x$ is provable. Set belongingness is written as $\text{In S x}$ instead of just writing $S x$.

Empty set is defined as a predicate $\text{Empty_set}$ which is not provable anywhere. $\text{Singleton x}$ and $\text{Couple x y}$ represents the sets $\{x\}$ and $\{x, y\}$ respectively. Partial Orders are defined as a dependent record type in the standard library. It has four fields,

Record $\text{PO (U : Type)} : \text{Type} := \text{Definition_of_PO}$

\{ $\text{Carrier_of : Ensemble U; \}
\text{Rel_of : Relation U; \}
\text{PO_cond1 : Inhabited U Carrier_of; \}
\text{PO_cond2 : Order U Rel_of }$.\}

First two fields represents the carrier set $S$ and the binary relation $\leq$ of a partially ordered set $(S, \leq)$. Third field $\text{PO_cond1}$ is a proof that $S$ is a non-empty set. Similarly, $\text{PO_cond2}$ is a proof that $\leq$ is an order (i.e, reflexive, transitive and antisymmetric). Using the coercion feature of Coq we extend this definition to define finite partial orders as

Record $\text{FPO (U : Type)} : \text{Type} := \text{Definition_of_FPO}$

\{ $\text{PO_of :> PO U ; \}
\text{FPO_cond : Finite _ (Carrier_of _ PO_of ) }$.\}

□
It has two components; a partial order and a proof that the carrier set of the partial order is finite. Note that FPO is defined as a dependent record type which inherits all the fields of PO. Hence,

– an object of type FPO can appear in any context where an object of type PO is expected.

Some more definitions For a finite partial order \( P \): FPO \( U \) on some type \( U \) let \( C := \text{Carrier} \_ \text{of} \ U \ P \) and \( R := \text{Rel} \_ \text{of} \ U \ P \). Then, we have the following definitions:

Definition \( \text{Is} \_ \text{a} \_ \text{chain} \_ \text{in} \ (e: \text{Ensemble} \ U) \): Prop:=
\[
\begin{align*}
\text{In} \_ \text{cluded} \ U \ e \ C & \land \ \text{In} \_ \text{habited} \ U \ e \\
(\forall x, y:U, (\text{In} \_ \text{cluded} \ U (\text{Cou} \_ \text{ple} \ U x y) \ e) & \rightarrow R x y \lor R y x).
\end{align*}
\]

– Note that \( \text{Is} \_ \text{a} \_ \text{chain} \_ \text{in} \) is defined as a predicate. It becomes true for any set \( e: \text{Ensemble} \ U \) iff \( e \) is a non-empty set included in \( C \) which satisfies the chain condition.

– Also note that the first parameter in the above definition is a finite partial order. We can avoid writing it using the section mechanism of Coq.

Similarly we have,

Definition \( \text{Is} \_ \text{an} \_ \text{anti} \_ \text{chain} \_ \text{in} \ (e: \text{Ensemble} \ U) \): Prop :=
\[
\begin{align*}
\text{In} \_ \text{cluded} \ U \ e \ C & \land \ \text{In} \_ \text{habited} \ U \ e \\
(\forall x, y:U, (\text{In} \_ \text{cluded} \ U (\text{Cou} \_ \text{ple} \ U x y) \ e) \rightarrow (R x y \lor R y x))
\rightarrow x=y).
\end{align*}
\]

Inductive \( \text{Is} \_ \text{l} \text{argest} \_ \text{chain} \_ \text{in} \ (e: \text{Ensemble} \ U) \): Prop:=
\[
\begin{align*}
\text{largest} \_ \text{chain} \_ \text{cond}:
\text{Is} \_ \text{a} \_ \text{chain} \_ \text{in} \ e \\
(\forall e1: \text{Ensemble} \ U) (n n1:nat), \text{Is} \_ \text{a} \_ \text{chain} \_ \text{in} \ e1 \rightarrow \text{cardinal } e n \rightarrow \text{cardinal } e1 n1 \rightarrow n1\leq n)
\rightarrow \text{Is} \_ \text{l} \text{argest} \_ \text{chain} \_ \text{in} \ e.
\end{align*}
\]

– In the above definition cardinal is a predicate which becomes true for a set \( S \) and a natural number \( n \) iff \( n \) is the size of the set \( S \).

Inductive \( \text{Is} \_ \text{l} \text{argest} \_ \text{anti} \_ \text{chain} \_ \text{in} \ (e: \text{Ensemble} \ U) \): Prop:=
\[
\begin{align*}
\text{largest} \_ \text{anti} \_ \text{chain} \_ \text{cond}:
\text{Is} \_ \text{an} \_ \text{anti} \_ \text{chain} \_ \text{in} \ e \\
(\forall e1: \text{Ensemble} \ U) (n n1: nat), \text{Is} \_ \text{an} \_ \text{anti} \_ \text{chain} \_ \text{in} \ e1 \rightarrow \\
\text{cardinal } e n \rightarrow \text{cardinal } e1 n1 \rightarrow n1\leq n)
\rightarrow \text{Is} \_ \text{l} \text{argest} \_ \text{anti} \_ \text{chain} \_ \text{in} \ e.
\end{align*}
\]

Inductive \( \text{Is} \_ \text{a} \_ \text{chain} \_ \text{cover} \ (\text{cover}: \text{Ensemble} (\text{Ensemble} \ U)) \): Prop:=
\[
\begin{align*}
\text{cover} \_ \text{cond}:
(\forall e: \text{Ensemble} \ U), \text{In} \_ \text{cover} e \rightarrow \text{Is} \_ \text{a} \_ \text{chain} \_ \text{in} \ e) \\
(\forall x:U, \text{In} \_ \text{C} x \rightarrow (\exists e: \text{Ensemble} \ U, \text{In} \_ \text{cover} e \land \text{In} \_ \text{e} x))
\rightarrow \text{Is} \_ \text{a} \_ \text{chain} \_ \text{cover} \ e.
\end{align*}
\]
In the above definition \textit{cover} is a collection of sets.

Here \textit{Is\_a\_chain\_cover} is a predicate which becomes true for \textit{cover} iff it satisfies the \textit{cover\_cond}. To satisfy the \textit{cover\_cond} every member of \textit{cover} should be a chain and every element of the carrier set \textit{C} must be covered by some chain in the cover. Similarly we have,

\begin{itemize}
  \item Inductive \textit{Is\_an\_antichain\_cover} (cover: Ensemble (Ensemble U)): Prop:=
    \textit{AC\_cover\_cond}:
    $(\forall (e: Ensemble U), \text{In\_cover} e -> \text{Is\_an\_antichain\_in} e) -> (\forall x:U, \text{In\_C} x -> (\exists e: Ensemble U, \text{In\_cover} e /\text{In\_e} x))$
    -> \text{Is\_an\_antichain\_cover} cover.
  \item Inductive \textit{Is\_a\_smallest\_chain\_cover} (scover: Ensemble (Ensemble U)): Prop:=
    Prop:=
    \textit{smallest\_cover\_cond}:
    \text{Is\_a\_chain\_cover} P scover ->
    $(\forall (cover: Ensemble (Ensemble U))(sn n: nat), \text{Is\_a\_chain\_cover} P cover /\text{cardinal\_scover} sn /	ext{cardinal\_cover} n) -> (sn <=n) )$
    -> \text{Is\_a\_smallest\_chain\_cover} P scover.
  \item Inductive \textit{Is\_a\_smallest\_antichain\_cover} (scover: Ensemble (Ensemble U)): Prop:=
    Prop:=
    \textit{smallest\_cover\_cond\_AC}:
    \text{Is\_an\_antichain\_cover} P scover ->
    $(\forall (cover: Ensemble (Ensemble U))(sn n: nat), \text{Is\_an\_antichain\_cover} P cover /\text{cardinal\_scover} sn /	ext{cardinal\_cover} n) -> (sn <=n) )$
    -> \text{Is\_a\_smallest\_antichain\_cover} P scover.
  \item Inductive \textit{Is\_height} (n: nat) : Prop:=
    \textit{H\_cond}:
    $(\exists lc: Ensemble U, \text{Is\_largest\_chain\_in} P lc /\text{cardinal\_lc} n) -> (\text{Is\_height} P n)$.
  \item Inductive \textit{Is\_width} (n: nat) :Prop :=
    \textit{W\_cond}:
    $(\exists la: Ensemble U, \text{Is\_largest\_antichain\_in} P la /\text{cardinal\_la} n) -> (\text{Is\_width} P n)$.
\end{itemize}

\section{Theorem statements}

The definitions, we have seen so far, are sufficient to express the formal statement of Dilworth’s Theorem in Coq.

\textbf{Theorem Dilworth:} $\forall (P: \text{FPO \_U}), \text{Dilworth\_statement} P$.

where \textit{Dilworth\_statement} is defined as,
Definition Dilworth_statement:=
  fun (P: FPO U)=>
    ∀ (m n:nat), (Is_width P m) ->
    (∃ cover: Ensemble (Ensemble U), (Is_a_smallest_chain_cover P cover)\/
       (cardinal _ cover n))
    -> m=n.

At this point one can easily verify that the combined meaning of all the terms
in the above statement corresponds to the actual Dilworth's Theorem. Similarly
for dual-Dilworth we proved,

Theorem Dual_Dilworth: ∀ (P: FPO U), Dual_Dilworth_statement P.

where, Dual_Dilworth_statement is defined as,

Definition Dual_Dilworth_statement:=
  fun (P : FPO U)=>
    ∀ (m n:nat), (Is_height P m) ->
    (∃ cover: Ensemble (Ensemble U), (Is_a_smallest_antichain_cover
       P cover)\/
       (cardinal _ cover n))
    -> m=n.

The original proofs of these theorems are classical in nature, hence, we need the
principal of excluded middle at many places. At certain points, we also need
to extract functions from relations. Therefore, we imported the Classical
and ClassicalChoice modules of the standard library, which assumes the following
three axioms:

Axiom classic : ∀ P:Prop, P \/ ~ P.
Axiom dependent_unique_choice :
  ∀ (A : Type) (B : A -> Type) (R : ∀ x:A, B x -> Prop),
  (∀ x : A, ∃! y : B x, R x y) ->
  (∃ f : (∀ x:A, B x), ∀ x:A, R x (f x)).
Axiom relational_choice :
  ∀ (A B : Type) (R : A->B->Prop),
  (∀ x : A, ∃ y : B, R x y) ->
  ∃ R' : A->B->Prop, subrelation R' R \/ ∀ x : A, ∃! y : B, R' x y.

4 Some results on finite partial orders

In this section we explain some results of general nature on finite partial orders.
These results are used at more than one places in the proof of Dilworth’s The-
orem. They are proved as Lemmas and compiled in a separate file. For most of
the Lemmas their statements can be inferred from their names. Here, we only
provide an English language description of some of them.
Existence proofs

A large number of lemmas are concerned with the existence of a defined object. For example, in our proof when we say “Let A be an antichain of the poset P...” we assume that there exists an antichain for the poset P. However, in a formal system like Coq, we need a proof of existence of such object before we can instantiate it. Following is a partial list of such results:

**Lemma-1** *Chain_exists*: There exists a chain in every finite partial order (FPO).
*Proof.* Trivial.

**Lemma-2** *Chain_cover_exists*: There exists a chain cover for every FPO.
*Proof.* Trivial.

**Lemma-3** *Minimal_element_exists*: The set minimal(P) is non-empty for every P: FPO.
*Proof.* Using induction on the size of P.

**Lemma-4** *Maximal_element_exists*: The set maximal(P) is non-empty for every P: FPO.
*Proof.* Using induction on the size of P.

**Lemma-5** *Largest_element_exists*: If a finite partial order is also totally ordered then there exists a largest element in it. *Proof.* The maximal element becomes the largest element and we know that there exists a maximal element.

**Lemma-6** *Minimal_for_every_y*: For every element y of a finite partial order P there exists an element x in P such that x ≤ y and x ∈ minimal(P).
*Proof.* Let X = \{x : P | x ≤ y\}. Then the poset (X, ≤) will have a minimal element, say x₀. It is also a minimal element of P.

**Lemma-7** *Maximal_for_every_x*: For every element x of a finite partial order P there exists an element y in P such that x ≤ y and y ∈ maximal(P).
*Proof.* Let Y = \{y : P | x ≤ y\}. Then the poset (Y, ≤) will have a maximal element, say y₀. It is also a maximal element of P.

**Lemma-8** *Largest_set_exists*: There exists a largest set (by cardinality) in a finite and non-empty collection of finite sets. *Proof.* Consider the collection of sets together with the strict set-inclusion relation. This forms a finite partial order. Any maximal element of this finite partial order will be a largest set. Moreover, such a maximal element exists due to Lemma-4.

**Lemma-9** *exists_largest_antichain*: In every finite partial order there exists a largest antichain. *Proof.* Note that this statement is not true for partial orders. The proof is similar to Lemma-7.

**Lemma-10** *exists_largest_chain*: In every finite partial order there exists a largest antichain. *Proof.* Again, its true only for finite partial orders. Proof is similar to Lemma-7.

Some other proofs

When dealing with sets the set-inclusion relations occurs more naturally than the comparison based on the set sizes. Therefore, we defined a binary relations *Inside* (≺) on the collection of all the finite partial orders.

- We say P₁ ≺ P₂ iff carrier set of P₁ is strictly included in the carrier set of P₂ and both the posets are defined on the same binary relation.
Then to use well-founded induction we proved that the relation $\prec$ is well founded.

**Lemma-11** *Inside_is_WF*: The binary relation Inside (i.e., $\prec$) is well founded on the set of all finite partial orders. **Proof.** Using strong induction on the size of finite partial orders.

**Lemma-12** *Largest_antichain_remains*: If $A$ is a largest antichain of $P_2$ and $P_1 \prec P_2$ then $A$ is also a largest antichain in $P_1$ provided $A \subset P_1$.

**Proof.** Assume otherwise, then there will be a larger antichain say $A'$ in $P_1$. This will also be larger in $P_2$, which contradicts.

**Lemma-13** *NoTwoCommon*: A chain and an antichain can have at most one element in common. **Proof.** Trivial.

**Lemma-14** *Minimal_is_antichain*: Minimal($P$) is an antichain in $P$.

**Proof.** Trivial.

**Lemma-15** *Maximal_is_antichain*: Maximal($P$) is an antichain in $P$.

**Proof.** Trivial.

**Lemma-16** *exists_disjoint_cover*: If $C_P$ is a smallest chain cover of size $m$ for $P$, then there also exists a disjoint chain cover $C_P'$ of size $m$ for $P$.

**Proof.** Using induction on $m$.

**Wrapping Up**

This work is done in the Coq Proof General (Version 4.4pre). We have used the Company-Coq extension [20] for the Proof General. The proof is split into the following nine files:

1. *PigeonHole.v*: It contains some variants of the Pigeonhole Principal.
2. *BasicFacts.v*: Contains some useful properties on numbers and sets. It also contains strong induction and some variants of Choice theorem.
3. *BasicFacts2.v*: Contains some more facts about power-sets and binomial coefficients.
4. *FPQ_Facts.v*: Most of the definitions and some results on finite partial orders are proved in this file.
5. *FPQ_Facts2.v*: Contains most of the lemmas that we discussed in this section.
6. *FiniteDilworth_AB.v*: Contains the proofs of forward and backward directions of Dilworth’s Theorem.
7. *FiniteDilworth.v*: Contains the proof of the main statement of Dilworth’s Theorem.
8. *Dual_Dilworth.v*: Contains the proof of the Dual-Dilworth Theorem.
9. *Combi_1.v*: Some new tactics are defined to automate the proofs of some trivial facts on numbers, logic, sets and finite partial orders.

The Coq code for this work is available at [21]. The files can be safely compiled in the given order.
5 Related Work

Rudnicki [15] presents a formalization of Dilworth’s Theorem in Mizar. In the same paper they also provide a proof of Erdős-Szekeres Theorem [9] using Dilworth’s Theorem. A separate proof of the Hall’s marriage Theorem in Mizar appeared in [16]. Jiang and Nipkov [17] also presented two different proofs of Hall’s Theorem in HOL/Isabella. All these formalizations stand isolated because they are built on different platforms using different libraries of facts and definitions. Our work aims to create a homogeneous set of definitions that can be used efficiently in the formalization of all these and many other important results. Moreover, we selected a different theorem prover, the Coq Proof Assistant, to reduce the time and effort of formalization.

6 Conclusions

Formalization of any mathematical theory is a difficult task. It requires a lot of time and effort. In combinatorics, the task becomes even more difficult due to the lack of structure in the theory. Some statements often admit more than one proof using completely different ideas. There is no predefined order to mechanize the theory. Any such effort would require to explore the dependencies among the results and identify an effective order to formalize them. Our work aims in this direction and the main contributions of this paper are:

1. Fully formalized proofs of Dilworth’s and Mirsky’s decomposition theorems in Coq, together with a detailed account of all the definitions and the theorem statement.
2. A clear compilation of more general results and definitions which could be useful in other similar formalizations.

The Coq code for this work is accessible online at [21]. In the immediate future, one can use this work to formalize other related results like Hall’s Theorem [6,7], Erdős-Szekeres Theorem [9] and Sperner’s Lemma [10,8]. One can also attempt to formalize the infinite version of Dilworth’s Theorem.

In the long run we would like to see most of the well known results from this field formalized and organized in the form a library. This would significantly reduce the time of formalization of any new result from this area.

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