Nonuniformly expanding 1D maps with logarithmic singularities

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Abstract

For a certain parametrized family of maps on a circle with critical points and logarithmic singularities where derivatives blow up to infinity, we construct a positive measure set of parameters corresponding to maps which exhibit nonuniformly expanding behaviour. This implies the existence of “chaotic” dynamics in dissipative homoclinic tangles in periodically perturbed differential equations.

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1. Introduction

Let \( f_a : \mathbb{R} \to \mathbb{R} \) be such that

\[
    f_a : x \mapsto x + a + L \cdot \ln |\Phi(x)|, \quad L > 0,
\]

where \( a \in [0, 1] \) and \( \Phi : \mathbb{R} \to \mathbb{R} \) is \( C^2 \) satisfying: (i) \( \Phi(x + 1) = \Phi(x) \); (ii) \( \Phi'(x) \neq 0 \) if \( \Phi(x) = 0 \); (iii) \( \Phi''(x) \neq 0 \) if \( \Phi'(x) = 0 \). The family \( (f_a) \) induces a parametrized family of maps from \( S^1 = \mathbb{R}/\mathbb{Z} \) to itself. We study the abundance of nonuniform hyperbolicity in this family of circle maps.

Our study of \( (f_a) \) is motivated by the recent studies of [17–20] on homoclinic tangles and strange attractors in periodically perturbed differential equations \( (S^1 \) reflects the time periodicity of the force). When a homoclinic solution of a dissipative saddle is periodically perturbed, the perturbation either pulls the stable and the unstable manifold of the saddle fix point completely apart, or it creates chaos through homoclinic intersections. In both cases, first-return maps of the flow induced by the solutions of the perturbed equation in the extended phase space (introducing time as a new variable) to appropriate cross-sections are a family of partially defined two-dimensional maps. Taking a singular limit, one obtains a family...
of partially defined one-dimensional maps in the form of (1), with $\ln \Phi(x)$ in the place of $\ln |\Phi(x)|$. Let $\mu$ be a small parameter representing the magnitude of the perturbation and $\omega$ be the forcing frequency. We have $a \sim \omega \ln \mu^{-1} \pmod{1}$, $L \sim \omega$; and $\Phi$ is the classical Melnikov function [18–20].

When we start with two unperturbed homoclinic loops and assume symmetry, then the separatrix maps are a family of annulus maps, the singular limit of which is precisely $f_\mu$ in (1) (see [17]). If the stable and unstable manifolds of the perturbed saddle are pulled completely apart by the forcing function, then $\Phi(x) \neq 0$ for all $x$. In this case we obtain strange attractors, to which the theory of rank one maps developed in [23] apply. If the stable and unstable manifolds intersect each other, then $\Phi(x) = 0$ is allowed and the strange attractors are associated with homoclinic intersections. For the modern theory of chaos and dynamical systems, this is a case of historical and practical importance [3, 11, 12]. In this case, which is the focus of this paper, the theory of rank one maps in [23] does not apply because of the existence of the singularities of $f_\mu$. Our ultimate goal is to develop a theory that can be applied to the separatrix maps allowing $\Phi(x) = 0$. This paper is the first step, in which we develop a 1D theory.

For $f = f_\mu$, let $C(f) = \{f'(x) = 0\}$ be the set of critical points and $S(f) = \{\Phi(x) = 0\}$ be the set of singular points. In this paper we are interested in the case $L \gg 1$. As $L$ gets larger, the contracting region gets smaller and the dynamics is more and more expanding in most of the phase space. Nevertheless, the recurrence of the critical points is inevitable, and thus infinitesimal changes of dynamics occur when $a$ is varied. In addition, the logarithmic nature of the singular set $S$ assists in the search of a property (nonexistence of nontrivial coboundary [13]) which is unknown to occur for smooth one-dimensional maps with critical points.

Our main result states that nonuniform expansion prevails for ‘most’ parameters, provided $L \gg 1$. Let $\lambda = 10^{-3}$ and let $|\cdot|$ denote the one-dimensional Lebesgue measure.

**Theorem.** For all large $L$ there exists a set $\Delta = \Delta(L) \subset [0, 1)$ of $\mu$-values with $|\Delta| > 0$ such that if $\mu \in \Delta$ then for $f = f_\mu$ and each $c \in C$, $|(f^n)'(f^c)| \geq L^{a_n}$ holds for every $n \geq 0$. In addition, $|\Delta| \to 1$ holds as $L \to \infty$.

For the maps corresponding to the parameters in $\Delta$, our argument shows a nonuniform expansion, i.e. for Lebesgue a.e. $x \in S^1$, $\lim_{n \to \infty} \frac{1}{n} \ln |(f^n)'x| \geq \frac{1}{3} \ln L$. In addition, combining our argument with an argument in [21, section 3] one can construct invariant probability measures absolutely continuous with respect to Lebesgue measure (acips). The main difference from the smooth case is to bound distortions, which can be handled with lemma 2.2 in this paper. A careful construction exploiting the largeness of $L$ shows the uniqueness of acips and advanced statistical limit theorems [13].

Since the pioneering work of Jakobson [5], there have been quite a few number of papers over the last 30 years dedicated to proving the abundance of chaotic dynamics in increasingly general families of smooth one-dimensional maps [1, 2, 4, 6, 10, 14–16, 21]. Families of maps with critical and singular sets were studied in [7–9]. In all these papers, inductive constructions of parameter sets start with relatively small intervals around special parameters for which all critical points are nonrecurrent, or recurrent slowly. In order to get the asymptotic estimate $|\Delta| \to 1$ as $L \to \infty$, we start with a large parameter set, denoted by $\Delta_N$, which is a union of a finite number of intervals. This necessitates additional works on establishing the uniform hyperbolicity outside of a neighbourhood of $C$ (see lemma 2.5), which is rather straightforward in some of the classical setups.
The rest of this paper consists of two sections. In section 2 we perform a phase space analysis. In section 3 we construct the parameter set \( \Delta \) by induction. To estimate the measure of the set of parameters excluded at each step, instead of the approach of Benedicks and Carleson [1, 2] we elect to follow that of Tsujii [15, 16], primarily because partitions depend on \( a \), and the extension of this approach is more transparent in our dealing with the issues related to the singularities. Unlike [1, 2], the current strategy relies on a geometric structure of the set of parameters excluded at each step. In addition, there is no longer the need for a large deviation argument, introduced originally in [2] as an independent step of parameter exclusions.

2. Phase space analysis

In this section we perform a phase space analysis. Elementary facts on \( f_a \) are introduced in section 2.1. In section 2.2 we prove a statement on distortion. In section 2.3 we discuss an initial setup. In section 2.4 we introduce three conditions, which will be taken as assumptions of induction for the construction of \( \Delta_1 \), and then develop a binding argument. In section 2.5 we study global dynamical properties of maps satisfying these conditions.

2.1. Elementary facts

For \( \varepsilon > 0 \), we use \( C_\varepsilon \) and \( S_\varepsilon \) to denote the \( \varepsilon \)-neighbourhoods of \( C \) and \( S \), respectively. The distances from \( x \in S_1 \) to \( C \) and \( S \) are denoted as \( d_C(x) \) and \( d_S(x) \), respectively. We take \( L \) as a base of \( \log(\cdot) \).

Lemma 2.1. There exist \( K_0 > 1 \) and \( \varepsilon_0 > 0 \) such that the following holds for all sufficiently large \( L \) and \( f = f_a \):

(a) for all \( x \in S_1 \),

\[
K_0^{-1} L \frac{d_C(x)}{d_S(x)} \leq |f'(x)| \leq K_0 L \frac{d_C(x)}{d_S(x)}, \quad |f''(x)| \leq K_0 L \frac{d_S(x)^2}{d_C(x)^2}.
\]

(b) for all \( \varepsilon > 0 \) and \( x \notin C_\varepsilon \), \( |f'(x)| \geq K_0^{-1} L \varepsilon \);

(c) for all \( x \in C_{\varepsilon_0} \), \( K_0^{-1} L < |f''(x)| < K_0 L \).

Proof. This lemma follows immediately from

\[
f' = 1 + L \cdot \frac{\Phi'}{\Phi}, \quad f'' = L \cdot \frac{\Phi'' - (\Phi')^2}{\Phi^2},
\]

and our assumptions on \( \Phi \) in the beginning of the introduction. \( \square \)

2.2. Bounded distortion

We frequently use the following notation: for \( c \in C \) and \( n \geq 1 \), \( c_0 = f c \) and \( c_n = f^n c_0 \): for \( x \in S^1 \) and \( n \geq 1 \), \( J(x) = |f'(x)| \) and \( J^n(x) = J(f(x))J(f(x)) \cdots J(f^{n-1}(x)) \).

Let \( c \in C \), \( c_0 = f c \), and \( n \geq 1 \). Let

\[
D_n(c_0) = \frac{1}{\sqrt{L}} \left[ \sum_{i=0}^{n-1} d_i^{-1}(c_0) \right]^{-1} \quad \text{where} \quad d_i(c_0) = \frac{d_C(c_i) \cdot d_S(c_i)}{J(c_0)}.
\] (2)

Lemma 2.2. For all \( x, y \in [c_0 - D_n(c_0), c_0 + D_n(c_0)] \) we have \( J^n(x) \leq 2 J^n(y) \), provided that \( c_i \notin C \cup S \) for every \( 0 \leq i < n \).
Proof. Write \( D_n \) for \( D_n(c_0) \), and let \( I = [c_0 - D_n, c_0 + D_n] \). Then

\[
\log \frac{J^n(x)}{J^n(y)} = \sum_{j=0}^{n-1} \frac{J(f^j(x))}{J(f^j(y))} \leq \sum_{j=0}^{n-1} \left| f^j I \right| \sup_{\phi \in f^j I} \frac{|f'' \phi|}{|f' \phi|}.
\]

Lemma 2.2 would hold if for all \( j \leq n - 1 \) we have \( f^j I \cap (S \cup C) = \emptyset \) and

\[
\left| f^j I \right| \sup_{\phi \in f^j I} \frac{|f'' \phi|}{|f' \phi|} \leq \log 2 \cdot d^{-1}(c_0) \left( \sum_{i=0}^{n-1} d^{-1}_i(c_0) \right)^{-1}.
\]

We prove (3) by induction on \( j \). Assume (3) holds for all \( j < k \). Summing (3) over \( j = 0, 1, \ldots, k - 1 \) implies \( \frac{1}{2} \leq \frac{J^k(\eta)}{J^k(c_0)} \leq 2 \) for all \( \eta \in I \). We have

\[
\left| f^k I \right| \leq 2J^k(c_0)D_n = 2d^{-1}_k \cdot d(C(c_k)d_S(c_k))D_n \leq \frac{2K^2_0}{\sqrt{L}}d(C(c_k)d_S(c_k)).
\]

We have \( f^k I \cap (C \cup S) = \emptyset \) from (4), and for \( \phi \in f^k I \),

\[
\left| f^k I \right| \frac{|f'' \phi|}{|f' \phi|} \leq 2d^{-1}_k d(C(c_k)d_S(c_k)) \cdot \frac{K^2_0}{d(C(\phi)d_S(\phi)}
\]

\[
= 2K^2_0 d^{-1}_k D_n \cdot \frac{d(C(c_k)d_S(c_k))}{d(C(\phi)d_S(\phi))} \leq \frac{2K^2_0}{\sqrt{L}} d^{-1}_k \left( \sum_{i=0}^{n-1} d^{-1}_i \right)^{-1},
\]

where we used lemma 2.1(a) for \( \frac{|f'' \phi|}{|f' \phi|} \) for the first inequality. For the last inequality we observe that the second factor on the left-hand side is \( < 2 \) by (4).

\[ \Box \]

2.3. Initial setup

In one-dimensional dynamics, a general strategy for constructing positive measure sets of ‘good’ parameters is to start an inductive construction in small parameter intervals, in which the orbits of critical points are kept out of bad sets for certain number of iterates. One way to find these intervals is to first look for Misiuriewicz parameters, for which all critical orbits stay out of the bad sets under any positive iterate. We would then restrict ourselves to small parameter intervals containing the Misiuriewicz parameters, and would eventually prove that the Misiuriewicz parameters are Lebesgue density points of the set of good parameters. However, this approach for initial setups has some drawbacks. First, for a one-parameter family of maps with multiple critical points, the Misiuriewicz parameters are relatively hard to find because of the need of controlling multiple critical orbits with one parameter. Although the argument in [22] is readily extended to cover our family, we are nevertheless up to a hard start. Second, with the rest of the study confined in a small parameter interval containing a Misiuriewicz parameter, it is not clear how we could prove the asymptotic measure estimate \((|\Delta| \to 1 \text{ as } L \to \infty)\) of the theorem.

An alternative route that is made possible by the approach of this paper is to start with a rather straightforward and relatively weak assumption. Set \( \sigma = L^{-\frac{1}{4}} \), and let \( N \) be a large integer independent of \( L \). For \( 0 \leq n \leq N \), define

\[ \Delta_n = \{ a \in [0, 1) : f_a^{i+1}(C) \cap (C_{\sigma} \cup S_{\sigma}) = \emptyset \text{ for every } 0 \leq i \leq n \}. \]

Observe that \( \Delta_n \) is a union of intervals unless it is empty. We start with \( \Delta_N \). We shall show that \( |\Delta_N| \geq 1 - L^{-\frac{1}{4}} \) (see lemma 3.4). This approach for initial setups is easier, and leads to the desired asymptotic measure estimate on \( \Delta \) as \( L \to \infty \).
Set $\alpha = 10^{-6}$ and $\delta = L^{-\alpha N}$. In what follows we suppose $N$ to be a large integer such that $\delta < \alpha$. The value of $N$ will be replaced if necessary, but only a finite number of times. The letter $K$ will be used to denote generic constants which are independent of $N$ and $L$.

**Hypotheses for the rest of this section:** $L \gg N \gg 1$, $a \in \Delta_N$, $f = f_a$.

### 2.4. Recovering expansion

For $f = f_a$, $c \in C$ and $n \geq N$ we introduce three conditions:

1. $(G1)_{n,c} \ J^{i-j}(c_i) \geq L \min[\sigma, L^{-ai}]$ for all $0 \leq i < j \leq n + 1$;
2. $(G2)_{n,c} \ J^i(c_0) \geq L^{2i}$ for every $0 < i \leq n + 1$;
3. $(G3)_{n,c} \ d\delta(c_i) \geq L^{-4ai}$ for every $N \leq i \leq n$.

We say $f$ satisfies $(G1)_n$ if $(G1)_{n,c}$ holds for every $c \in C$. The definitions of $(G2)_n$, $(G3)_n$ are analogous. These conditions are taken as inductive assumptions in the construction of the parameter set $\Delta$.

We establish a recovery estimate of expansion. Let $c \in C$, $c_0 = fc$ and assume that $(G1)_{n,c} - (G3)_{n,c}$. For $p \in [2, n]$, let

$$I_p(c) = \begin{cases} f^{-1}[c_0 + D_{p-1}(c_0), c_0 + D_p(c_0)) & \text{if } c \text{ is a local minima of } f \\ f^{-1}(c_0 - D_p(c_0), c_0 - D_{p-1}(c_0)] & \text{if } c \text{ is a local maxima of } f \end{cases}$$

By the nondegeneracy of $c$, $I_p(c)$ is the union of two intervals lying on both sides of $c$. By lemma 2.2, if $x \in I_p(c)$ then the derivatives along the orbit of $fx$ shadow that of the orbit of $c_0$ for $p - 1$ iterates. We regard the orbit of $x$ as been bound to the orbit of $c$ up to time $p$, and call $p$ the bound period of $x$ to $c$.

**Lemma 2.3.** If $(G1)_{n,c} - (G3)_{n,c}$ holds, then for $p \in [2, n]$ and $x \in I_p(c)$ we have:

1. $(a) \ p \leq \log |c - x|^{1/2};$
2. $(b) \ |x| \leq \frac{|c - x|^{1 + \frac{2p}{L}}}{L^{p}}$.

**Proof.** By definition we have

$$|c - x|^2 \leq D_{p-1}(c_0) \leq L^{-\frac{1}{2}}d_{p-2}(c_0) < L^{-\frac{1}{2}}J^{p-2}(c_0)^{-1}. $$

Then by (G2),

$$|c - x|^2 \leq L^{-\frac{1}{2} - \lambda(p-2)} \leq L^{-\lambda p}. $$

From which (a) follows. The second inequality of (b) follows from (5).

**Sublemma 2.4.** For $0 \leq i \leq n$ we have:

1. $(a) \ d_i(c_i) \geq K^{-1}aL^{-ai};$
2. $(b) \ J^i(c_0)D_{i+1}(c_0) \geq L^{-1-\gamma_{ai}}.$

We finish the proof of lemma 2.3 assuming the conclusion of this sublemma. We have

$$J^p(x) = J^{p-1}(fx)J(x) \geq K^{-1}J^{p-1}(c_0) \cdot |c - x| \geq K^{-1}J^{p-1}(c_0) \cdot |c - x|^{-1}D_p(c_0),$$

where for the first inequality we use lemma 2.2 and lemma 2.1(c), and for the last inequality we use $x \in I_{p-1}(c) \cup I_p(c)$. Using sublemma 2.4(b) for $i = p - 1$ we obtain

$$J^p(x) \geq K^{-1}L^{-1-\gamma_{p-1}}|c - x|^{-1}. $$

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Substituting into this the upper estimate of $p$ in lemma 2.3(a) we obtain
\[ J^p(x) \geq K^{-1} L^{-1} |c - x|^{-1+\frac{15}{\lambda}} \geq |c - x|^{-1+\frac{16}{\lambda}}. \]
We have used $|c - x| \leq \delta = L^{-\alpha N}$ for the last inequality.

It is left to prove the sublemma. (G1) implies $|f’c_i| \geq L \min\{\sigma, L^{-\alpha i}\}$. Then (a) follows from lemma 2.1(a). As for (b), let $j \in [0, i]$. By definition,
\[ J^j(c_0)d_j(c_0) = \frac{J^j(c_0)}{J(c_0)} d_c(c_j) d_{\bar{c}}(c_j). \]
We have $\frac{J^j(c_0)}{J(c_0)} \geq L^{-\alpha j} \sigma$ from (G1); $d_c(c_j) \geq K^{-1} \sigma L^{-\alpha j}$ from (a); $d_{\bar{c}}(c_j) \geq \sigma L^{-4\alpha j}$ from (G3). Hence, $J^j(c_0)d_j(c_0) \geq K^{-1} \sigma^3 L^{-6\alpha j}$, and thus
\[ \sum_{j=0}^i J^j(c_0)^{-1} d_{\bar{c}}^{-1}(c_0) \leq \sigma^{-3} L^{3\alpha}. \]
Taking reciprocals implies (b). \[ \square \]

2.5. Global dynamical properties

At step $n$ of induction, we wish to exclude all parameters for which one of (G1)$_n$–(G3)$_n$ fails for some $c \in C$, and to estimate the measure of the parameters deleted. Conditions (G1)$_n$, (G2)$_n$, however, cannot be used directly as rules for exclusion, since they do not care about cumulative effects of ‘shallow returns’. Hence we introduce a stronger condition called (R)$_n$, based on the notion of deep returns, and will use it as a rule for exclusion in section 3.

Hypothesis in section 2.5: $f = f_a$ is such that $a \in \Delta_N$. $n \geq N$ and (G1)$_{n-1}$–(G3)$_{n-1}$ hold for all $c \in C$. For all $c \in C$ we have:

(i) $f^{i+1}c \notin C \cup S$ for all $0 \leq i \leq n$;
(ii) for the orbit of $c_0 = fc$, the bound period initiated at all returns to $C_\delta$ before $n$ is $\ll n$.

(Bound/free structure) We divide the orbit of $c_0$ into alternative bound/free segments as follows. Let $n_1$ be the smallest $j \geq 0$ such that $c_j \in C_\delta$. For $k > 1$, we define free return times $n_k$ inductively as follows. Let $p_k$ be the bound period of $c_{n_k-1}$, and let $n_k$ be the smallest $j \geq n_{k-1} + p_k$ such that $c_j \in C_\delta$. We decompose the orbit of $c_0$ into bound segments corresponding to time intervals $(n_k, n_k+p_k)$ and free segments corresponding to time intervals $[n_k, n_{k+1}]$. The times $n_k$ are the free return times. We have
\[ n_1 < n_1 + p_1 < n_2 < n_2 + p_2 < \cdots. \]

We have already obtained derivative estimates for bound segments (lemma 2.3(b)). The next lemma ensures an exponential growth of derivatives for free segments which are outside of $C_\delta$.

Lemma 2.5. If $L$ is sufficiently large and $f = f_a$ is such that $a \in \Delta_N(L)$, then the following holds:

(a) if $n \geq 1$ and $x, f x, \ldots, f^{n-1}x \notin C_\delta$, then $J^n(x) \geq \delta L^{2kn}$;
(b) if moreover $f^n x \in C_\delta$, then $J^n(x) \geq L^{2kn}$. 
Proof. Let \( \delta_0 = L^{-\frac{m}{n}} \). For the orbit of \( x \) lying out of \( C_\delta \) we first introduce a bound/free structure similarly to the above, using \( C_{\delta_0} \) in the place of \( C_\delta \). We then prove a result that is similar to lemma 2.3, from which lemma 2.5 would follow directly. For this argument to work, we first need to show that the bound periods on \( C_{\delta_0} \setminus C_\delta \), as defined in section 2.4, are \( \leq N \).

Sublemma 2.6. If \( f = f_\mu \) is such that \( a \in \Delta_\nu \), then for each \( c \in C \) we have \([c - \delta_0, c - \delta] \cup [c + \delta, c + \delta_0] \subset \bigcup_{1 \leq \rho \leq N} I_\rho(c)\).

Proof. Let \( c_0 = f c \). It suffices to show that
\[
L^{-1} D_N(c_0) < \delta^2 < \delta_0^2 < L^{-1} D_1(c_0).
\]
Since \( a \in \Delta_\nu \) we have \( J^{N-1}(c_0) \geq (KL\sigma)^{N-1} \) from lemma 2.1. It then follows that \( D_N(c_0) \leq d_{N-1}(c_0) \leq (KL\sigma)^{-N+1} < L\delta^2 \).

On the other hand we have
\[
D_1(c_0) = \frac{1}{\sqrt{L}} \cdot d_{c_0} d_\delta(c_0) \geq L^{-1}\sigma^2,
\]
from which the last inequality of (6) follows directly. \( \square \)

With the help of sublemma 2.6 we know that the bound/free structure for the orbit of \( x \) out of \( C_\delta \) is well defined.

Sublemma 2.7. Let \( p \geq 2 \) be the bound period for \( y \in C_{\delta_0} \setminus C_\delta \). Then:
(a) for \( i \in [1, p] \), we have \( d_c(f^i y) > \delta_0 \) and \( J^{-1}(c_0) D_1(c_0) \geq L^{-\frac{m}{n}} \sigma^2 \);
(b) \( J^p(y) \geq K^{-1} L^{-\frac{m}{n}} \sigma^2 |c - y|^{-1} \geq K^{-1} L^{-\frac{m}{n}} \sigma^2 (K_0^{-1} L\sigma)^{p-2} \);
(c) \( J^p(y) \geq L^{\frac{m}{n}} \).

Proof. (a) is a version of sublemma 2.4. The estimates are better here because we have \( J^{-1}(c_\rho) \geq (KL\sigma)^{-1} \), and \( d_c(c_\rho), d_\delta(c_\rho) > \sigma \). As for (b) we have
\[
J^p(y) \geq K^{-1} L|c - y| J^{p-1}(c_0) |c - y|^{-1} J^{p-1}(c_0) D_\rho(c_0),
\]
where for the last inequality we use \( D_\rho(c_0) \leq KL|c - y|^2 \) by the definition of \( p \). The first inequality of (b) then follows by using (a). The second inequality of (b) follows from
\[
|c - y| \leq D_{p-1}(c_0) \leq \frac{d_{p-2}(c_0)}{\sqrt{L}} \leq \frac{(K_0^{-1} L\sigma)^{-p+2}}{\sqrt{L}}.
\]
Here, the last inequality is because \( a \in \Delta_\nu \).

If \( p > 10 \), then (c) is much weaker than the second inequality of (b). If \( p < 10 \), then (c) follows from the first inequality of (b) and the fact that \( |c - y|^{-1} \geq \delta_0^{-1} = L^{\frac{m}{n}} \). \( \square \)

We are in a position to complete the proof of lemma 2.5. If \( f^p x \) is free (this includes the case \( f^n x \in C_\delta \)), then we use sublemma 2.7(c) for bound segments and \( |f^p| > K^{-1} L\gamma^n \) for iterates in free segments, which are out of \( C_{\delta_0} \). This proves lemma 2.5(b). If \( f^p x \) is bound, then there is a drop of a factor \( > \delta \) at the last free return that cannot be recovered. In this case we need the factor \( \delta \) in lemma 2.5(a). \( \square \)

(Deep returns and condition (R)\( _n \)) We now introduce a condition which is used as a rule of exclusion at step \( n \). Let \( c \in C \) and assume that \( c_0 = f c \) makes a free return to \( C_\delta \) at time \( v \leq n \). We say \( v \) is a deep return of \( c_0 \) if for every free return \( i \in [0, v - 1] \);
\[
\sum_{j \in [i+1, v]} \max_{\text{free return}} 2 \log d_c(c_j) \leq \log d_c(c_i).
\]

(7)
We say \((R)_{n,c}\) holds if
\[\prod_{i \in [0,j] : \text{deep return}} d_C(c_i) \geq L^{-\frac{1}{20} \lambda \alpha j} \quad \text{for every } N_0 \leq j \leq n.\]
We say \((R)_{n}\) holds if \((R)_{n,c}\) holds for every \(c \in C\).

**Lemma 2.8.** If \((R)_{n,c}\) holds, then \((G1)_{n,c}\), \((G2)_{n,c}\) hold.

**Proof.** The first step is to show
\[\prod_{N_0 < i \leq n \text{ free return}} d_C(c_i) \geq L^{-\alpha n}. \tag{8}\]
We call a free return shallow if it is not a deep free return. Let \(\mu \in (0, n)\) be a shallow free return time, and \(i(\mu)\) be the largest deep free return time < \(\mu\). We claim that
\[\sum_{i(\mu) + 1 \leq j \leq \mu \text{ shallow return}} \log d_C(c_j) \geq \sum_{0 \leq i \leq k \text{ deep return}} \log d_C(c_i) \tag{9}\]
where the first inequality is from (9). We then have
\[\sum_{0 \leq j \leq n \text{ free return}} \log d_C(c_j) \geq 2 \sum_{0 \leq j \leq n \text{ free return}} \log d_C(c_j) \geq \alpha n, \tag{10}\]
where the last inequality is from \((R)_{n,c}\).

To prove (9), we let \(\beta_1\) be the smallest free return time \(\leq \mu - 1\) such that
\[\sum_{\beta_1 + 1 \leq j \leq \mu \text{ free return}} 2 \log d_C(c_j) > \log d_C(c_{\beta_1}). \tag{11}\]
We claim that no deep free return occurs during the period \([\beta_1 + 1, \mu]\). This is because if \(i' \in [\beta_1 + 1, \mu]\) is a deep return, then we must have
\[\sum_{\beta_1 + 1 \leq j \leq i' \text{ free return}} 2 \log d_C(c_j) \leq \sum_{\beta_1 + 1 \leq j \leq i' \text{ free return}} 2 \log d_C(c_j) \leq \log d_C(c_{\beta_1}), \]
contradicting (11). If \(\beta_1\) is a deep return, we are done. Otherwise we find a \(\beta_2 < \beta_1\) so that
\[\sum_{\beta_2 + 1 \leq j \leq \beta_1 \text{ free return}} 2 \log d_C(c_j) > \log d_C(c_{\beta_2}), \tag{12}\]
and so on. This process will end at a deep free return time, which we denote as \(\beta_q := i(\mu)\). (9) follows from adding (11), (12) and so on up to the time for \(\beta_q = i(\mu)\).

We first prove \((G1)_{n,c}\). Let \(0 \leq i < j \leq n + 1\). Observe that the bound periods for all returns to \(C_d\) for the orbit of \(c_0\) up to time \(n\) is \(\leq \frac{2N}{K} n \ll n\). This follows from (8) and
lemma 2.3(a). Hence, it is possible to introduce the bound–free structure starting from \( c_i \) to \( c_j \). We consider the following two cases separately.

**Case I:** \( j \) is free. For free segments we use lemma 2.5, and for bound segments we use lemma 2.3(c). We obtain exponential growth of derivatives from time \( i \) to \( j \), which is much better than what is asserted by (G1)_{n,c}.

**Case II:** \( j \) is bound. Let \( \hat{j} \) denote the free return with a bound period \( p \) such that \( j \in [\hat{j}+1, \hat{j}+p] \). We have \( j \leq n \), for otherwise \( j > n + 1 \). Consequently,

\[
J_{j-i}(c_i) = \frac{J_{i}(c_0)}{J_{j}(c_0)} \cdot \frac{J_{j+1}(c_0)}{J_{j+1}(c_0)} > L^{i(j-i)} \cdot K^{-1} L_d(c_{\hat{j}}) \cdot K^{-1} L^{j-(j-1)}
\]

where for the last inequality, we use lemma 2.3(c) combined with lemma 2.5 for the first factor. For the third factor we use bound distortion and (G2)_{n-1} for the binding critical orbit. It then follows that \( J^{-1}(c_0) \geq L^{(n-1)-n} \geq L^{-n} \). Hence (G1)_{n,c} holds.

As for (G2)_{n,c}, we introduce the bound–free structure starting from \( c_0 \) to \( c_{n+1} \). Observe that the sum of the lengths of all bound periods for the orbit of \( c_0 \) up to time \( n \) is \( \leq \frac{2\alpha}{\lambda} n \ll n \). This follows from (8) and lemma 2.3(a). Using lemma 2.3(b) for each free segment and lemma 2.5 for each free segment in between two consecutive free returns, we have

\[
J^{n+1}(c_0) \geq \delta L^{2\alpha(1-\frac{\lambda}{\alpha})} \geq L^{\lambda n}.
\]

This completes the proof of lemma 2.8. \( \square \)

The next expansion estimate at deep return times will be used in a crucial way in the construction of the parameter set \( \Delta \).

**Lemma 2.9.** If \( c \in C \) and \( v \leq n + 1 \) is a deep return time of \( c_0 = f^c \), then

\[
J^v(c_0) \cdot D_v(c_0) \geq \sqrt{d}(c_0).
\]

**Proof.** Let \( 0 < n_1 \leq \cdots \leq n_j < n \) denote all free returns in the first \( v \) iterates of \( c_0 \), with \( p_1, \ldots, p_l \) the corresponding bound periods. Let

\[
\Theta_n = \sum_{i=n_k}^{n_k+p_k-1} d_i^{-1}(c_0) \quad \text{and} \quad \Theta_0 = \sum_{i=0}^{v-1} d_i^{-1}(c_0) - \sum_{k=1}^{l} \Theta_{n_k}.
\]

**Step 1 (Estimate for bound segments).** Observe that

\[
\Theta_{n_k} = \frac{1}{J^{n_k+p_k}(c_0)} = \frac{1}{J^{n_k}(c_{n_k}) d_c(c_{n_k}) d_c(c_{n_k})} + \sum_{i=n_{k+1}}^{n_k+p_k-1} \frac{1}{J^{n_k+p_k-i}(c_{i}) d_c(c_{i}) d_s(c_{i})}.
\]

To estimate the first term we use lemma 2.3(b) to obtain

\[
\frac{1}{J^{n_k}(c_{n_k}) d_c(c_{n_k}) d_s(c_{n_k})} \leq \sigma^{-1} d_c(c_{n_k}) \left| \frac{\lambda}{\alpha} \right|.
\]

To estimate the second term we let \( \bar{c} \) be the critical point to which \( c_{n_k} \) is bound. Using lemma 2.2 and (4) in the proof of lemma 2.2, which implies \( d_c(c_{i}) \geq \frac{1}{2} d_c(c_{i-1}) \) and \( d_s(c_{i}) \geq \frac{1}{2} d_s(c_{i-1}) \) for \( i \in [n_k+1, n_k + p_k - 1] \), we have

\[
J^{n_k+p_k-i}(c_{i}) d_c(c_{i}) d_s(c_{i}) \geq K^{-1} J^{n_k+p_k-i}(\bar{c}_{i-n_k-1}) d_c(\bar{c}_{i-n_k-1}) d_s(\bar{c}_{i-n_k-1}) \geq \sigma^2 L^{-S(n_k)}.
\]
where the last inequality is obtained using (G1), lemma 2.4(a) and (G3) for \( \tilde{c} \). Summing this estimate over all \( i \) and combining the result with (13),

\[
\left| \left( f^{n_i + p_i} \right) c_0 \right|^{-1} \Theta_{n_i} \leq \sigma^{-1} |dC(c_{n_i})|^{-\frac{18}{20}} + \sigma^{-2} L^{6n_i p_i} \leq |dC(c_{n_i})|^{-\frac{18}{20}}.
\]  

(14)

Here for the last inequality we use \( \sigma \gg \delta \) and \( L^{6n_i p_i} \leq |dC(c_{n_i})|^{-\frac{12}{5}} \) from lemma 2.3(a).

Step 2 (Estimate for free segments). By definition,

\[
\Theta_0 = \frac{1}{J^v(c_0)} = \sum_{i \in [0,v-1] \setminus \{i, n_i, n_i + p_i-1\}} \frac{1}{J^{v-i}(c_i) dC(c_i) dS(c_i)}.
\]

(15)

Here we cannot simply use (G3) for \( \delta \) in proving (16). We observe, instead, that either

\[ J^{v-i}(c_i) = J^{v-i+1}(c_{i+1}) J(c_i) \geq K^{-1} L^\frac{1}{\lambda(v-i+1)} |dS(c_i)|^{-1}. \]

It then follows, by using \( dC(c_i) > \delta \), that

\[
\Theta_0 \leq \sum_{i \in [0,v-1] \setminus \{i, n_i, n_i + p_i-1\}} KL^{-\frac{1}{\lambda(v-i)}} (\sigma \delta)^{-1} > \frac{1}{\sigma \delta}.
\]

(16)

Step 3 (Proof of the lemma). From the assumption that \( v \) is a deep free return, we have

\[
|dC(c_{n_i})|^{-1} \leq |dC(c_v)|^{-2} \prod_{j : n_j \in (n_i, v)} |dC(c_{n_j})|^{-2}.
\]

(17)

Substituting this into (14) gives

\[
J^{n_i + p_i}(c_0) \left| \Theta_{n_i} \leq |dC(c_v)|^{-\frac{2n_i}{36}} \prod_{j : n_j \in (n_i, v)} |dC(c_{n_j})|^{-\frac{2n_i}{36}}.\right.
\]

(18)

Meanwhile, splitting the orbit from time \( n_k + p_k + 1 \) to \( v \) into bound and free segments we have

\[
J^{v-n_k-p_k}(c_{n_k+p_k})^{-1} \leq \left( \prod_{j : n_j \in (n_k, v)} J^v(c_{n_j}) \right)^{-1}.
\]

(19)

Multiplying (17) with (18) gives

\[
J^v(c_0)^{-1} \Theta_{n_k} \leq |dC(c_v)|^{-\frac{2n_k}{36}} \prod_{j : n_j \in (n_k, v)} \left( J^v(c_{n_j}) \cdot |dC(c_{n_j})|^{\frac{2n_k}{36}} \right)^{-1} \leq |dC(c_v)|^{-\frac{2n_k}{36}} \prod_{j : n_j \in (n_k, v)} |dC(c_{n_j})|^{\frac{1}{2}} \leq \delta^{(v-k)/2} |dC(c_v)|^{-\frac{2n_k}{36}},
\]

where for the second inequality we use lemma 2.3(b) for \( J^v(c_{n_j}) \), and for the last we use \( dC(c_{n_j}) < \delta \). Thus

\[
\sum_{n_k \in [0,v-1]} J^v(c_0)^{-1} \Theta_{n_k} \leq |dC(c_v)|^{-\frac{2n_k}{36}} \sum_{k=1}^{v} \delta^{(v-k)/2} \leq 2|dC(c_v)|^{-\frac{2n_k}{36}}.
\]

(20)
Combining this with (16) we obtain
\[ J^\nu(c_0)^{-1}D_{\nu}^{-1} = \sqrt{L} \left( \sum_{1 \leq k \leq t} J^\nu(c_0)^{-1}\Theta_{nk} + J^\nu(c_0)^{-1}\Theta_0 \right) \leq \frac{1}{\sqrt{dc(c_0)}}. \]
This completes the proof of lemma 2.9.

\[ \square \]

3. Measure of the set of excluded parameters

In this last section we complete the proof of the theorem. We first estimate the measure of \( \Delta_N \). Then, for \( n > N \), define
\[ \Delta_n = \{ a \in \Delta_N : (R1)_n \text{ and } (G3)_n \text{ hold} \}, \]
and set \( \Delta = \bigcap_{n \geq N} \Delta_n \). We show that \( |\Delta| \to 1 \) as \( L \to \infty \). By lemma 2.8, if \( a \in \Delta \) then \( |(f^n)'(fc)| \geq L^n c \) holds for every \( n \geq 0 \), as in the statement of the theorem.

This section is organized as follows. In section 3.1 we establish two fundamental estimates: an equivalence between phase-space derivatives and a bounded distortion in parameter space. In section 3.2 we estimate the measure of \( \Delta_N \). In section 3.3 we deal with the exclusion on account of condition (R). In section 3.4 we deal with the exclusion on account of (G3).

**Remark 3.1.** In the transition from step \( n - 1 \) to step \( n \) we deal with the parameters in \( \Delta_n - 1 \).

If \( n > N \), then \((R)_n - 1\) holds by definition, and thus by lemma 2.8, \((G1)_n - 1\), \((G2)_n - 1\) hold.

### 3.1. Equivalence of derivatives and distortion in parameter space

For \( c \in C \) and \( k \geq 0 \), we define \( \gamma_k^{(c)} : \Delta_N \to S^1 \) by \( \gamma_k^{(c)}(a) = f_k^{a}c \). Let \( \tau_k^{(c)}(a) = \frac{d\gamma_k^{(c)}(a)}{da} \).

**Lemma 3.2.** Let \( a \in \Delta_n - 1 \). Then, for all \( c \in C \) and \( k \leq n \) we have
\[ \frac{1}{2} \leq \frac{|\tau_k^{(c)}(a)|}{|\gamma_k^{(c)}(a)|} \leq 2. \]

**Proof.** Let \( \gamma_k^{(c)}(a) = \gamma_k(a) \) and \( \tau_k^{(c)}(a) = \tau_k(a) \). We have
\[ \tau_k(a) = 1 + f_k^{a} \gamma_{k-1}(a) \tau_{k-1}(a). \] (19)

Using this inductively and then dividing the result by \( (f_k^{a})'\gamma_0(a) \) we obtain
\[ \frac{\tau_k(a)}{(f_k^{a})'\gamma_0(a)} = 1 + \sum_{i=1}^{k} \frac{1}{(f_i^{a})'\gamma_0(a)}. \]
To deduce the inequalities we use \( |(f_k^{a})'\gamma_0(a)| \geq L^k \), which follows from \((G2)_{n-1}\) for \( n > N \) and from lemma 2.5 for \( n \leq N \). \( \square \)

For \( a_* \in [0, 1) \), \( c \in C \) and \( c_0 = fc \), define
\[ I_n(a_*, c) = [a_* - D_n(c_0), a_* + D_n(c_0)] \]
where \( D_n(c_0) \) is the same as the one in (2) with \( f = f_{a_*} \).

**Lemma 3.3.** Let \( a_* \in \Delta_{n-1} \). For all \( c \in C \), \( a \in I_n(a_*, c) \) and \( k \leq n \) we have
\[ \frac{1}{2} \leq \frac{|\tau_k^{(c)}(a)|}{|\tau_k^{(c)}(a_*)|} \leq 2. \]
Proof. For \( j = 0, 1, \ldots, k - 1 \) we assume
\[
\frac{|\tau_{j+1}(a)|}{|\tau_j(a)|} \leq 2, \quad \text{for all } a \in I_n(a_*, c).
\] (20)

We then prove the same estimate for \( j = k \). For all \( a \in I_n := I_n(a_*, c) \) we have
\[
\left| \log \frac{|\tau_{j+1}(a)|}{|\tau_j(a)|} \right| \leq \frac{|\tau_{j+1}(a) - \tau_j(a)|}{|\tau_j(a)|} \leq (I)_a + (II),
\]

where
\[
(I)_a = \left| \log \frac{|\tau_{j+1}(a)|}{|\tau_j(a)|} - \frac{f_{j+1}'(a)}{f_j'(a)} \right|, \quad (II) = \left| \log \frac{(f_{j+1}'(a))^2}{(f_j'(a))^2} \right|.
\]

We claim that
\[
|\tau_j(a)| \leq 2L^{-\frac{1}{3}} \cdot d^{-\frac{1}{3}} \left[ \sum_{i=0}^{n-1} d_i^{-1} \right]^{-1},
\] (21)

where \( d_i \) is the same as the one in (2) with \( f = f_{a_*} \).

To prove (21), first we use (20) and lemma 3.2 to obtain
\[
|\gamma_j(I_n)| \leq 2|\tau_j(a_*)| |I_n| \leq 4 |(f_j')'(a_*)| |\gamma_j(a_*)| |I_n| \leq 2L^{-\frac{1}{3}} d_C(\gamma_j(a_*)) d_S(\gamma_j(a_*)).
\] (22)

This implies \( d_S(\gamma_j(a_*)) \geq \frac{1}{2} d_S(\gamma_j(a_*)) \) for all \( a \in I_n \). Thus from lemma 2.1(b),
\[
|f''| \leq \frac{KL}{d_S(\gamma_j(a_*))^2} \quad \text{on } \gamma_j(I_n).
\]

It then follows that
\[
|f_{j+1}'(a) - f_j'(a)| = \frac{|f_{j+1}'(a) - f_j'(a)|}{|f_j'(a)|} \leq \frac{KL}{d_S(\gamma_j(a_*))^2} \cdot |\gamma_j(I_n)|
\]
\[
\leq \frac{KL}{d_S(\gamma_j(a_*))^2} |\tau_j(a_*)| \cdot |I_n|,
\]

where (20) is again used for the last inequality. Then
\[
|f_{j+1}'(a) - f_j'(a)| \leq \frac{KL}{d_S(\gamma_j(a_*))^2} \left[ \sum_{i=0}^{n-1} d_i^{-1} \right]^{-1}
\]
\[
\leq L^{-\frac{1}{3}} |f_{j+1}'(a)| d^{-\frac{1}{3}} \left[ \sum_{i=0}^{n-1} d_i^{-1} \right]^{-1},
\]

where for the last inequality we used lemma 2.1(a). (21) follows directly from the last estimate.

As for \( (I)_a \), (19) gives
\[
(I)_a \leq \frac{1}{|(f_{a_*})'(\gamma_j(a_*))| \cdot |\tau_j(a_*)|}.
\]

For the first factor of the denominator we have
\[
|f_{a_*}'(\gamma_j(a_*))| \geq \frac{1}{2} |f_{a_*}'(\gamma_j(a_*))| \geq \frac{K_{a_*}^{-1} L}{2} \cdot \min[\sigma, L^{-\alpha}] .
\]
where the second inequality follows from (G1) for $n > N$ and $\gamma_j(a_n) \notin C_n$ for $n \leq N$.

For the second factor, using the inductive assumption (20) and lemma 3.2 we have $|\tau_j(a)| \geq \frac{1}{2}|\tau_j(a_n)| \geq \frac{1}{2}L^{|j|}$. Plugging these two estimates into the denominator we obtain

$$ (I)_{a} \leq L^{-\frac{1}{2}}. \tag{23} $$

Now the desired estimate $\frac{|\gamma_j(a)|}{|\Delta_1(a_n)|} \leq 2$ follows from combining (21) and (23) for $j = 0, 1, \ldots, k = 1$.

\[\square\]

3.2. Measure estimate of $\Delta_N$

Recall that, for $0 \leq n \leq N$,

$$ \Delta_n = \{a \in [0, 1); J^i_n(C) \cap (C_n \cup S_n) = \emptyset \text{ for every } 0 \leq i \leq n\}. $$

In this subsection we estimate the measure of $\Delta_N$.

**Lemma 3.4.** For any large integer $N$ there exists $L_0 = L_0(N) \gg 1$ such that if $L \geq L_0$, then $|\Delta_N| \geq 1 - L^{-\frac{1}{4}}$.

**Proof.** Let $1 \leq n \leq N$. For $c \in C, s \in C \cup S$, let $E_n(c, s)$ denote the set of all $a \in \Delta_{n-1} \setminus \Delta_n$ such that $d(\gamma_i(c)(a), s) \leq \sigma$. Obviously, $\Delta_{n-1} \setminus \Delta_n = \bigcup_{c \in C, s \in C \cup S} E_n(c, s)$ holds. To estimate the measure of $\Delta_{n-1} \setminus \Delta_n$, we first estimate the measure of $E_n(c, s)$, and then sum it up over all combinations of $(c, s)$.

**Notation.** For a compact interval $I$ centred at $a$ and $r > 0$, let $r \cdot I$ denote the interval of length $r|I|$ centred at $a$.

For $a_n \in \Delta_{n-1}$, define a parameter interval $I_n(a_n)$ as follows. If $|\gamma_i(c)(I_n(a_n), c))| < \frac{1}{4}$, then let $I_n(a_n) = I_n(a_n, c)$. Otherwise, let $I_n(a_n) = \frac{1}{|\gamma_i(c)(I_n(a_n), c))|} \cdot I_n(a_n, c)$. We show that:

(A1) if $a_n \in E_n(c, s)$, then $I_n(a_n) \setminus L^{-1/4} \cdot I_n(a_n)$ does not intersect $E_n(c, s)$;

(A2) if $a_1, a_2 \in E_n(c, s)$ and $a_2 \notin I_n(a_1)$, then $I_n(a_1) \cap I_n(a_2) = \emptyset$.

(A2) implies that $E_n(c, s)$ is covered by a countable number of pairwise disjoint intervals of the form $I_n(a_n), a_n \in E_n(c, s)$. Then (A1) implies $|E_n(c, s)| \leq L^{-1/4}$. Therefore

$$ |\Delta_{n-1} \setminus \Delta_n| \leq \bigcup_{c \in C, s \in C \cup S} E_n(c, s) \leq \#C(\#C + \#S)L^{-1/4}. $$

Using this inductively for $1 \leq n \leq N$ we obtain

$$ |\Delta_N| \geq 1 - \sum_{n=1}^{N} |\Delta_{n-1} \setminus \Delta_n| \geq 1 - L^{-\frac{1}{4}}. $$

To prove (A1), (A2) we need the next better expansion estimate in parameter space.

**Sublemma 3.5.** If $1 \leq n \leq N, a_n \in \Delta_{n-1}, c \in C$, then $|\gamma_i(c)(I_n(a_n))| |I_n(a_n)| \geq L^{1/4}$.\[\square\]

**Proof.** Let $f = f_{a_n}, c = f^{i+1}c$ and $J^i(c_0) = |(f^i)'(c_0)|$. Lemma 2.1 gives

$$ J^{i-1}(c_0) \geq (K^{-1}L\sigma)^{i-1} \quad 0 \leq i < \forall j \leq n. $$

Hence, for $0 \leq i < n$ we have

$$ J^{n}(c_0)d_i(c_0) = J^{n-1}(c_0)d_c(c_0)d_s(c_0) \geq (K_0^{-1}L\sigma)^{n-i}\sigma^2, $$
where \( d_i(c_0) \) is the same as the one in (2) with \( f = f_a \). Taking reciprocals and then summing the result over all \( 0 \leq i < n \) we have
\[
\sum_{i=0}^{n-1} J^n(c_0)^{-1} d_i(c_0)^{-1} \leq \frac{K_0}{L^3}.
\]
Lemma 3.2 gives \( |\gamma_n^{(c)}(a_s)| \geq \frac{1}{2} L^n(c_0) \). Hence
\[
|\gamma_n^{(c)}(a_s)| |I_n(a_s, c)|^{-1} \leq \frac{1}{2} L^n(c_0) - D_n^{-1}(c_0) = 2\sqrt{L} \cdot \sum_{i=0}^{n-1} J^n(c_0)^{-1} d_i(c_0)^{-1} \leq \frac{1}{L^2+\sigma},
\]
where the last inequality follows from \( \sigma = L^{-\frac{1}{2}} \). Hence the desired inequality holds if \( I_n(a_s) = I_n(a_s, c) \). Otherwise, \( |\gamma_n^{(c)}(a_s)| |I_n(a_s, c)| = \frac{|\gamma_n^{(c)}(a_s)|}{|I_n(a_s, c)|} |I_n(a_s, c)| \geq \frac{1}{2L} - L^\frac{1}{2} \sigma \). □

**Proof of (A1).** Let \( a \in I_n(a_s) \setminus L^{-\frac{1}{2}} \cdot I_n(a_s) \). Then \( |a_s - a| \geq L^{-\frac{1}{2}} |I_n(a_s)| \) holds. Using the mean value theorem, lemma 3.3 and sublemma 3.5 we have
\[
|\gamma_n^{(c)}(a_s) - \gamma_n^{(c)}(a)| \geq \frac{1}{2} |\gamma_n^{(c)}(a_s)||a_s - a| \geq \frac{1}{2} L^{-\frac{1}{2}} |\gamma_n^{(c)}(a_s)| |I_n(a_s)| \geq \frac{1}{2} L^{\frac{1}{2}} \sigma.
\]
Hence
\[
|\gamma_n^{(c)}(a) - s| \geq |\gamma_n^{(c)}(a_s) - \gamma_n^{(c)}(a)| - |\gamma_n^{(c)}(a_s) - s| \geq \frac{1}{2} L^\frac{1}{2} \sigma - \sigma > \sigma,
\]
and therefore \( a \notin E_n(c, s) \).

**Proof of (A2).** We argue by contradiction assuming \( I_n(a_1) \cap I_n(a_2) \neq \emptyset \). Lemmas 3.2 and 3.3 together imply \( \frac{1}{8} \leq |\gamma_n^{(c)}(a_1)| \leq 8 \). Since \( a_2 \notin I_n(a_1) \) it follows that \( L^{-\frac{1}{2}} \cdot I_n(a_1) \cap L^{-\frac{1}{2}} \cdot I_n(a_2) = \emptyset \).

While the proof of (A1) implies that \( (\gamma_n^{(c)}(I_n(a_1)))^{-1}(s) \) is a point and is contained in \( L^{-\frac{1}{2}} \cdot I_n(a_1) \), and the same for \( a_2 \). Hence \( (\gamma_n^{(c)}(I_n(a_1) \cup I_n(a_2)))^{-1}(s) \) consists of two distinct points. A contradiction arises because \( \gamma_n^{(c)} \) is injective on \( I_n(a_1) \cup I_n(a_2) \). □

### 3.3. Exclusion on account of \((R)\)

In this subsection and the next we estimate the measure of \( \Delta_{n-1} \setminus \Delta_n \) for every \( n > N \). This involves similar ideas used in the proof of lemma 3.4.

Define
\[
E_n = \{ a \in \Delta_{n-1} \setminus \Delta_n : (R)_n \text{ fails for } f_a \} \quad \text{and} \quad E'_n = \{ a \in \Delta_{n-1} \setminus \Delta_n : (G3)_n \text{ fails for } f_a \}.
\]

Obviously, \( \Delta_{n-1} \setminus \Delta_n \subset E_n \cup E'_n \) holds. In this subsection we estimate the measure of \( E_n \). We first express \( E_n \) as the union of conveniently chosen subsets. We then estimate the measure of each subset separately, and sum it up over all the subsets.

The subsets are defined as follows. Given \( q \geq 1 \), a sequence \( 0 < t_1 < t_2 < \cdots < t_q \leq n \) of integers, a sequence \( c^{(1)}, c^{(2)}, \ldots, c^{(q)} \) of critical points and a sequence \( r_1, r_2, \ldots, r_q \) of positive integers, define \( E_n(c, *, *) \) to be the set of all \( a \in E_n \) such that:

- the orbit of \( \gamma_{t_1}^{(c)}(a) \) makes deep returns up to time \( n \) exactly at times \( t_1 < t_2 < \cdots < t_q \);
- for each \( 1 \leq i \leq q \), \( \gamma_{t_i}^{(c)}(a) \) is bound to \( c^{(i)} \), and \( r_i \) is the unique integer such that \( |\gamma_{t_i}^{(c)}(a) - c^{(i)}| \in (L^{-n-i}, L^{-(n+i)}] \).

Observe that \( E_n = \bigcup E_n(c, *, *) \), where the union runs over all \( c \in C \) and all feasible sequences \( t_1, \ldots, t_q; r_1, \ldots, r_q; c^{(1)}, \ldots, c^{(q)} \).
Lemma 3.6. \(|E_n(c, \star)| \leq L^{-\frac{1}{2}R}\), where \(R = r_1 + r_2 + \cdots + r_q\).

To count the number of these subsets we need the following.

Lemma 3.7. Assume that \(f_\alpha\) is such that \(a \in \Delta_N\). Then the lengths of any given bound period due to a return to \(C_\delta\) is \(\geq \frac{1}{2}a\alpha N\).

Proof. From \(a \in \Delta_N\) and lemma 2.1(a) we have \(J_\alpha(c_\star)(c_0) \leq (\sigma^{-1}L)^{\frac{1}{2}a\alpha N}\) where \(\sigma = L^{-\frac{1}{2}}\).

It then follows that
\[
D_\alpha(c_\star)(c_0) \geq \frac{L^{-\frac{1}{2}\alpha^2}}{J_\alpha(c_\star)(c_0)} \geq L^{-\frac{1}{2}\alpha^2}(L^{-\frac{1}{2}})^{\frac{1}{2}a\alpha N} \ni \delta.
\]

This implies the lemma.

\((\Box)\)

Lemma 3.7 implies \(q \leq \frac{2n}{\alpha N}\). The number of all feasible sequences \(t_1, \ldots, t_q\) is \(\leq \left(\begin{array}{c}n \\alpha N \\end{array}\right)\), and the number of all feasible sequences \(r_1, \ldots, r_q\) satisfying \(r_1 + \cdots + r_q = R\) is \(\leq \left(\begin{array}{c}R + q \\end{array}\right)\).

Since \((R\alpha)\) fails, we have \(R > \frac{\alpha N}{20}\). By Stirling’s formula for factorials, \(\left(\begin{array}{c}n \\alpha N \\end{array}\right) \leq e^\theta(N)^N\) and \(\left(\begin{array}{c}R + q \\end{array}\right) \leq e^{\theta(N)R\alpha}\), where \(\beta(N) \to 0\) as \(N \to \infty\). Using these and lemma 3.6 we conclude that
\[
|E_n| \leq \sum_{c \in C} |E_n(c, \star)| \leq \#C \sum_{1 \leq q \leq \frac{n}{20}} \sum_{\alpha N} (\#C)\left(\begin{array}{c}n \\alpha N \\end{array}\right) \left(\begin{array}{c}R + q \\end{array}\right) L^{-\frac{q}{2}} \leq \frac{1}{20},
\]

where the last inequality holds for sufficiently large \(N\).

Proof of lemma 3.6. Similarly to the proof of lemma 3.4, for \(a_n \in \Delta_{n=N}\) and \(1 \leq k \leq q\) define a parameter interval \(I_k(a_n)\) as follows: if \(|y_k^{(c)}(I_k(a_n, c))| < \frac{1}{2}\), then let \(I_k(a_n) = I_k(a_n, c)\).

Otherwise, let \(I_k(a_n) = \frac{1}{10|y_k^{(c)}(I_k(a_n, c))|} : I_k(a_n, c)\). We show that

(B1) if \(a_n \in E_n(c, \star)\), then \(I_k(a_n) \cap L^{\frac{1}{2}c} \cdot I_k(a_n)\) does not intersect \(E_n(c, \star)\);

(B2) if \(a_1, a_2 \in E_n(c, \star)\) and \(a_2 \notin I_k(a_1)\), then \(I_k(a_1) \cap I_k(a_2) = \emptyset\);

(B3) if \(a_1, a_2 \in E_n(c, \star)\) and \(a_2 \in L^{-\frac{1}{3}} \cdot I_k(a_1)\), then \(I_k(a_1) \subset 2L^{-\frac{1}{3}} \cdot I_k(a_1)\).

Proof of (B1). Let \(a \in I_k(a_n) \cap L^{\frac{1}{2}c} \cdot I_k(a_n)\). Then \(|a_n - a| > L^{-\frac{1}{2}c} |I_k(a_n)|\) holds. If \(I_k(a_n) = I_k(a_n, c)\), then using the mean value theorem, lemmas 3.3, 3.2 and 2.9 we have
\[
|y_k^{(c)}(a_n) - y_k^{(c)}(a)| > \frac{1}{2}|c_1^{(c)}(a_n)||a_n - a| > \frac{1}{2}L^{-1/2} |f_k'_{\alpha N}(f_{\alpha N}(c_{\star}))| |I_k(a_n)| \geq L^{-\frac{3}{2}c_n},
\]

and therefore \(a \notin E_n(c, \star)\). The same conclusion remains to hold in the case \(I_k(a_n) \neq I_k(a_n, c)\), because \(|c_1^{(c)}(a_n)||I_k(a_n)| = \frac{1}{10|y_k^{(c)}(I_k(a_n, c))|} |I_k(a_n, c)| \geq \frac{1}{20}\).

Proof of (B2). We argue by contradiction assuming \(I_k(a_1) \cap I_k(a_2) \neq \emptyset\). Lemmas 3.2 and 3.3 give \(\frac{1}{8} < \frac{1}{10|y_k^{(c)}(I_k(a_1))|} \leq 8\). This and \(a_2 \notin I_k(a_1)\) together imply \(L^{-\frac{1}{3}} I_k(a_1) \cap L^{-\frac{1}{3}} I_k(a_2) = \emptyset\).

While the proof of (B1) implies that \((y_k^{(c)} |I_k(a_1))^{-1}(c_{\star})\) is a point and is contained in \(L^{-\frac{1}{3}} \cdot I_k(a_1)\), and the same for \(a_2\). Hence \((y_k^{(c)} |I_k(a_1) \cup I_k(a_2))^{-1}(c_{\star})\) consists of two distinct points. A contradiction arises because \(y_k^{(c)}\) is injective on \(I_k(a_1) \cup I_k(a_2)\).

Proof of (B3). It follows from lemmas 3.2 and 3.3 that if \(a \in I_k(a_1)\) then \(|y_k^{(c)}(a) - c_{\star}| \geq L^{-2c_n}\). Hence \((y_k^{(c)} |I_k(a_1))^{-1}(c_{\star}) \notin I_k(a_2)\) holds. This and the assumption together imply
that one of the connected components of $I_{n+1}(a_2) \setminus \{a_2\}$ is contained in $L^{-\alpha/3} \cdot I_n(a_1)$. This implies the inclusion.

For each $k \in [1, q]$ we choose a countable subset $[a_1^{(k)}, a_2^{(k)}, \ldots]$ of $E_n(c, \ast)$ with the following properties:

(i) the corresponding intervals $I_{n_k}(a_1^{(k)}), I_{n_k}(a_2^{(k)}), \ldots$ are pairwise disjoint and $E_n(c, \ast) \subset \bigcup \limits \{L^{-\alpha/3} \cdot I_{n_k}(a_k^{(k)})\}$,

(ii) for each $k \in [2, q]$ and $a_k^{(k)} (i = 1, 2, \ldots)$ there exists $a_j^{(k-1)}$ such that $I_{n_k}(a_k^{(k)}) \subset 2L^{-n_{k-1}/3} \cdot I_{n_{k-1}}(a_{j(k-1)}^{(k-1)})$.

Then the desired estimate in lemma 3.6 readily follows from these.

Start with $k = 1$. Pick an arbitrary parameter in $E_n(c, \ast)$ and denote it by $a_1^{(1)}$. If $E_n(c, \ast) \subset I_{1_k}(a_1^{(1)})$, then we are done. Otherwise, pick a parameter in $E_n(c, \ast) \setminus I_{1_k}(a_1^{(1)})$ and denote it by $a_2^{(1)}$. If $E_n(c, \ast) \subset I_{1_k}(a_1^{(1)}) \cup I_{1_k}(a_2^{(1)})$, then we are done. Otherwise, pick a parameter in $E_n(c, \ast) \setminus I_{1_k}(a_1^{(1)}) \cup I_{1_k}(a_2^{(1)})$ and denote it by $a_{1_2}^{(1)}$, and so on. By (B2) we end up with a countable number of pairwise disjoint intervals which altogether cover $E_n(c, \ast)$. In addition, if $a \in I_{1_k}(a_{1_1}) \setminus L^{-\alpha/3} \cdot I_{1_k}(a_{1_1}) (i = 1, 2, \ldots)$, then the inclusion in (i) follows from (B1).

Given $a_1^{(k-1)}, a_2^{(k-1)}, \ldots$, we choose $a_1^{(k)}, a_2^{(k)}, \ldots$ as follows. Similarly to the previous paragraph, for each $a_j^{(k-1)} (j = 1, 2, \ldots)$ it is possible to choose parameters $a_{1j}, a_{2j}, \ldots$ in $E_n(c, \ast) \cap L^{-n_{k-1}/3} \cdot I_{n_{k-1}}(a_{j(k-1)})$ such that the corresponding intervals $I_{n_k}(a_{1j}), I_{n_k}(a_{2j}), \ldots$ are pairwise disjoint and altogether cover $E_n(c, \ast) \cap L^{-n_{k-1}/3} \cdot I_{n_{k-1}}(a_{j(k-1)})$. Let $[a_1^{(k)}, a_2^{(k)}, \ldots] = \bigcup \limits \{a_{1j}, a_{2j}, \ldots\}$. (ii) follows from (B3).

### 3.4. Exclusion on account of (G3)

The estimate of $|E_n|$ goes much in parallel to that of $|\Delta_n|$, $1 \leq n \leq N$ done in the proof of lemma 3.4. For $c \in C$ and $s \in S$, let $E_n(c, s)$ denote the set of all $a \in E_n$ such that $d(y_n^{(s)}(a), s) < L^{-4\alpha n}$. We have $E_n = \bigcup \limits \{E_n(c, s) \mid c \in C, s \in S\}$.

**Lemma 3.8.** If $n > N$, $a_* \in \Delta_{n-1}$, $c \in C$, then $|\gamma_n(c)(a_*)| |I_n(a_*)| \geq L^{-3\alpha n}$.

**Proof.** Write $f = f_{a_*}$, $c = f^{r_{a_*}}$. Since $a_* \in \Delta_{n-1}$, (G1)$_{n-1}$ and (G2)$_{n-1}$ hold for $c_0$ by lemma 2.8. Using lemma 3.2 we have

$$|\gamma_n(c)(a_*)| |I_n(a_*, c)| \geq 2J^n(c_0) |I_n(a_*, c)| \geq \frac{1}{\sqrt{L}} \left( \sum_{i=0}^{n-1} \frac{1}{J^n(c_0) d_i(c_0)} \right)^{-1}.$$

If $c_1 \notin S_{\sigma}$, then

$$J^n(c_0) d_i(c_0) = J^{n_i(c)}(c) d_i(c_0) \geq K L^{-2\alpha \sigma}.$$

If $c_1 \in S_{\sigma}$, then $d_5(c_1) \geq |f^{r_{c_1}}|$ from lemma 2.1(a), and we have

$$J^n(c_0) d_i(c_0) = J^{n_i(c)}(c) d_i(c_0) \geq J^{n_i(c)+1}(c_1) d_i(c_0) \geq L^{-\alpha i+1}.$$

where (G1)$_{n-1}$ is used for the last inequality. Hence we obtain

$$|\gamma_n(c)(a_*)| |I_n(a_*, c)| \geq \frac{1}{\sqrt{L}} \left( \sum_{i=0}^{n-1} K L^{2\alpha \sigma} \right)^{-1} \geq L^{-3\alpha n}.$$
Hence the desired inequality holds if $I_n(\alpha_c) = I_n(\alpha_c, c)$. Otherwise, $|r_n^{\alpha}(\alpha_c)||I_n(\alpha_c)| = \frac{|I_n^{\alpha}(\alpha_c)||I_n(\alpha_c, c)|}{|I_n^{\alpha}(\alpha_c)|} \geq \frac{1}{M} > L^{-3\alpha n}$. □

Using lemma 3.8 and arguing similarly to the proofs of (A1), (A2) it is possible to show:

(C1) if $a^* \in E_n'(c, s)$, then $I_n(\alpha_c) \setminus L^{-\frac{1}{4}n} \cdot I_n(\alpha_c)$ does not intersect $E_n'(c, s)$;
(C2) if $a_1, a_2 \in E_n'(c, s)$ and $a_2 \notin I_n(\alpha_1)$, then $I_n(\alpha_1) \cap I_n(\alpha_2) = \emptyset$.

It then follows that $|E_n'(c, s)| \leq L^{-\frac{1}{4}n}$, and therefore

$$|E_n'| < \#C\#S \cdot L^{-\frac{1}{4}n} \leq L^{-\frac{1}{4}n}.$$ (25)

Lemma 3.4, (24) and (25) yield

$$|\Delta| = |\Delta_N| - \sum_{n=N+1}^{\infty} |\Delta_{n-1} \setminus \Delta_n| \geq 1 - L^{-\frac{1}{4}} \sum_{n=N+1}^{\infty} (L^{-\frac{1}{4}n} + L^{-\frac{1}{4}n}),$$

which goes to 1 as $L \to \infty$.

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