Scaling at the OTOC Wavefront: Integrable versus chaotic models

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Out of time ordered correlators (OTOCs) are useful tools for investigating foundational questions such as thermalization in closed quantum systems because they can potentially distinguish between integrable and non-integrable dynamics. Here we discuss the properties of wavefronts of OTOCs by focusing on the region around the main wavefront at $x = v_B t$, where $v_B$ is the butterfly velocity. Using a Heisenberg spin model as an example, we find that a propagating Gaussian with the argument $-m(x) (x - v_B t)^2 + b(x)$ gives an excellent fit for both the integrable case and the chaotic case. However, the scaling in these two regimes is very different: in the integrable case the coefficients $m(x)$ and $b(x)$ have an inverse power law dependence on $x$ whereas in the chaotic case they decay exponentially. In fact, the wavefront in the integrable case is a rainbow caustic and catastrophe theory can be invoked to assert that power law scaling holds rigorously in that case. Thus, we conjecture that exponential scaling of the OTOC wavefront is a robust signature of a nonintegrable dynamics.

Introduction: The hallmark of chaos in classical dynamics is an exponential sensitivity to small changes in initial conditions (butterfly effect). This is at odds with quantum mechanics where unitary time evolution means that the overlap between two states is constant in time. Although quantum systems do not display chaos, there are qualitative differences in behavior depending upon whether their classical limit is integrable or nonintegrable (chaotic) [11]. In the latter case we have ‘quantum chaos’ which is well studied in single-particle quantum mechanics, including in experiments [2, 12]. On the theoretical side, the main approach has traditionally been through spectral statistics [13, 14]. These have universal properties that depend only on the symmetries of the Hamiltonian and show close agreement with the predictions of random matrix theory (RMT) [15, 18]. More recently, attention has shifted to many body quantum chaos and particularly its role in foundational issues such as thermalization in closed quantum systems. One limitation of RMT is that it does not describe thermodynamic quantities like temperature and energy that are needed for such analyses [19]. This is remedied by the eigenstate thermalization hypothesis (ETH) [20, 24] which has been numerically verified in a range of generic models [25, 28] but is violated in integrable and localized systems [29, 38], as expected. The ETH generalizes RMT and gives identical predictions if one focuses on a small enough region of the spectrum. Any diagnostic of quantum chaos should therefore clearly differentiate between the integrable and ETH cases. While the ETH does give rise to the notion of chaotic eigenstates, it is a time independent statement and does not resemble classical chaos. In fact, aside from the weak ETH (eigenstate typicality) [39, 41], it has no classical counterpart.

The out-of-time-ordered correlator (OTOC) has risen to prominence as a dynamical diagnostic for quantum many body chaos [42–51]. It takes the form

$$C(x, t) = \langle [\hat{A}(t), \hat{B}] [\hat{A}(t), \hat{B}] \rangle,$$

(1)

where $\hat{A}$ and $\hat{B}$ are operators that at $t = 0$ only have local support (act on different individual lattice sites) and hence commute. The average is usually taken over an ensemble diagonal in the energy basis, but some studies have considered pure states as well [52, 54]. As $\hat{A}$ evolves in time it picks up weight throughout the lattice, becoming non-local and causing $C(x, t)$ to become non-zero. This, in effect, tracks the tendency of dynamics to smear information across the system, and it becomes impossible to determine the initial conditions from local data alone. In this respect the OTOC resembles classical chaos where incomplete information leads to exponential inaccuracy. Indeed, the late time value of the OTOC in local spin models does appear to be an indicator of chaos [43, 52–64]. In the classical limit commutators become Poisson brackets which are a diagnostic for classical chaos, and the general expectation is therefore that OTOCs in nonintegrable models experience exponential growth [51].

$$C(0, t) \sim e^{\lambda_L t},$$

(2)

where $\lambda_L$ is the quantum Lyapunov exponent and obeys [51],

$$\lambda_L \leq 2\pi k_B T / \hbar.$$

(3)

Models that approach the bound are known as fast scramblers. However, doubt has been cast upon whether exponential growth of the OTOC really is unique to chaotic systems because integrable systems near unstable points behave similarly [65, 70].

An OTOC should also display spatial dependence as information propagates across the system. A new conjecture gives the early time growth of the OTOC wavefront as [71, 73]

$$C(x, t) \sim \exp \left[ -\lambda_L \frac{(x/v_B - t)^{1+p}}{t^{p}} \right].$$

(4)

This has been verified in several cases and used to study the many body localization transition [71, 82]. For interacting models Eq. (4) is usually fitted in regimes where $C(x, t) \ll 1 [72, 73]$, corresponding to early times significantly before the arrival of the main front. When the broadening coefficient takes the value $p = 0$, it reduces to the simple “Lyapunov-like” exponential growth of Eq. (2), but for quantum spin models expected to obey ETH it is believed that in general $p > 0$ [73]. However, broadening is not necessarily a general
indicator of how close one is to a chaotic model in the sense of ETH \[83\][84], and puzzles remain concerning the value of \( p \) in this early growth regime. For example, in two dimensions the values of \( p \) coincide in chaotic and integrable models, so the broadening coefficient is inadequate for distinguishing them \[73\], while some studies \[71\][72][85][88] differ on whether the distinction between values of \( p \) even exists in either regime.

The aim of this Letter is to show that the main wavefront (region around \( x = v_B t \) which is the edge of the OTOC “light cone”) carries information on integrability. While there can be signatures of chaos in OTOCs at late times, including long-time oscillations \[73\][84][89][91], it is preferable to examine the main front rather than the signal either at early times (exponentially small) or late times (more contamination from the environment or numerical errors). Recent numerical work in free models has shown that the OTOC in this region is well-fitted by a propagating Gaussian of the form \[53\][92].

\[
C_G(x, t) \sim e^{-m(x-v_B t)^2+b(x)t},
\]

where \( m(x) \) and \( b(x) \) are well behaved functions of \( x \). A Gaussian also occurs in random circuit models \[83\] and wavefront results suggest it would also be found in the critical Ising model \[59\]. Here we will employ rigorous arguments from catastrophe theory to show that in many models that can be mapped to free fermions the wavefront takes on a universal Airy function form that can be locally described by Eq. (5). This allows us to extract the scaling of \( m(x), b(x) \) analytically, verifying the findings of \[53\][92]. For the chaotic case, we numerically verify the Gaussian wave form of Eq. (5) and show that the scaling of \( m(x), b(x) \) is very different from the free model. In locally interacting models the Gaussian wave form Eq. (5) therefore carries signatures of whether the model is free or ETH-obeying.

**Model:** We consider a Heisenberg spin Hamiltonian with nearest and next nearest interactions:

\[
\hat{H} = \sum_{j=1}^{L-1} J_1 \left( \hat{S}_j^+ \hat{S}_{j+1}^- + h.c. \right) + \Delta \hat{S}_j^z \hat{S}_{j+1}^z + \\
\sum_{j=1}^{L-2} J_2 \left( \hat{S}_j^+ \hat{S}_{j+2}^- + h.c. \right) + \gamma \hat{S}_j^x \hat{S}_{j+2}^x,
\]

and open boundary conditions. This model has both free and non-integrable regimes. In particular, we consider two choices of the coefficient vector \( \vec{c} = (J_1, \Delta, J_2, \gamma) \). The first one, \( \vec{c}_f = (-1, 0, 0, 0) \) is the XX chain and is free while the second, \( \vec{c}_{\text{ETH}} = (-0.5, 1, -0.2, 0.5) \) has all parameters non-zero which has been verified to obey the ETH with periodic boundary conditions \[25\]. In the supplementary materials (SM) \[93\] we demonstrate that an alternative choice of parameters for \( \vec{c}_{\text{ETH}} \) leads to the same basic results. Suitable operators for \( \hat{A}(t) \) and \( \hat{B} \) must be chosen for the OTOC in Eq. (1). In the ETH regime we use spin operators \( \hat{A}(t) = \sigma_1^z \), and \( \hat{B} = \sigma_m^z \), where \( x \) is the distance between sites 1 and \( m \), and the average \( \langle \ldots \rangle \) is taken over the thermal ensemble restricted to eigenstates with zero magnetization, \( m_z = \sum_{j=1}^L \langle \hat{S}_j^z \rangle = 0 \).

In the integrable case we perform a Jordan-Wigner transformation from spins to fermions and for simplicity the OTOC we use in this case is

\[
C(x, t) = |a_{m,n}(t)|^2
\]

where \( a_{m,n}(t) = \{ \hat{f}_m(t), \hat{f}_{-n} \} \). Here, \( \hat{f}_m \) is the annihilation operator for a fermion on site \( m \). Note that if instead of Eq. (7) we use Eq. (1) with operators \( \sigma_{m}^z \), then in the case of a pure Gaussian state or a thermal ensemble the dominant dynamical term is in fact \( |a_{m,n}(t)|^2 \), see Refs. \[53\][92] for further details.

**Airy light cones in free systems:** In 1972 Lieb and Robinson \[93\] showed that quantum correlations in spin systems propagate at finite speeds and spread out in a light cone-like fashion. Pioneering experiments with ultracold atoms and trapped ions \[95\][101], where a sudden quench leads to a nonequilibrium state, have confirmed this behavior. In particular, the wavefront for interacting bosonic atoms in an optical lattice was measured to have an Airy function profile \[25\] in qualitative agreement with theoretical calculations which can be done analytically in certain limits \[101\]. The associated problem of domain wall propagation \[102\][109] also yields Airy functions or related kernels for the wavefront. The Airy function shape implies a dynamical scaling behavior, such as a \( t^{1/3} \) broadening of the magnetization domain wall in an XX chain \[102\]. This body of results has led to the notion of an **Airy universality class** for free systems \[110\][112].

A more general understanding of light cones can be gained by realizing that they are caustics \[113\]. These are singularities of the ray description of a wave, where in the present case the rays are trajectories of quasiparticles excited by the quench. Caustics are regions where rays coalesce, leading to a diverging probability density in the classical limit. Significantly, only certain morphologies of caustic are structurally stable and hence occur generically in nature; these form a hierarchy described by catastrophe theory where each catastrophe forms an equivalence class with its own scaling properties similar to universality classes for phase transitions \[114\][115]. The simplest catastrophe is the fold which occurs where rays coalesce in pairs and an everyday example of this is the rainbow, and another is a ship’s wake \[116\][117].

In the wave theory each caustic is dressed by a characteristic wavefunction, and in the case of the fold it is the Airy function \[118\]. To see how this works, consider the case where the quench excites a Bogoliubov fermion on the site at \( x = 0 \), say. The resulting wavefunction is

\[
\Psi(x, t) = \langle x | e^{-i\hat{H}_t \hat{t}} | 0 \rangle = \langle x | \sum_k e^{-i\epsilon(k)t} | k \rangle 
\]

\[
\approx \sqrt{\frac{a}{2\pi}} \int_{-\pi}^{\pi} dk e^{i[kx-\epsilon(k)t]} \tag{8}
\]

where \( a \) is the lattice constant. The operators \( \hat{b}_k \) are the linear combinations of \( \hat{f}_m \) and \( \hat{f}_{-m}^\dagger \) that diagonalize the Hamiltonian via a Bogoliubov transformation and \( \epsilon(k) \) is the Bogoliubov dispersion relation [for the XX chain \( \epsilon(k) = J_1 \cos ka \)].
Defining $\Phi(k, x, t) = k x - \epsilon(k) t$, a caustic occurs at quasi-momentum $k_c$, where two conditions are satisfied \([113]\):

\[
(\partial\Phi/\partial k)_{k_c} = 0 \quad \text{and} \quad (\partial^2\Phi/\partial k^2)_{k_c} = 0.
\]

The first condition is Fermat’s principle that gives classical rays as saddles of the action $k x - \epsilon(k)$ and the second defines the caustic as the place where saddles coalesce. These conditions correspond exactly to the Lieb-Robinson (LR) bound for a light cone as being determined by the maximum value of the group velocity $d\epsilon/dk$ of the fermions \([113]\).\([119]\)\([120]\). $v_{LR} = \max_k |d\epsilon/dk|$.

The fact that light cones are caustics allows a number of powerful results from catastrophe theory to be applied: i) The only structurally stable catastrophes in two dimensions (the space-time formed by $x$ and $t$) are fold lines that meet at cusp points, as anyone who has ironed a shirt knows. For a light cone the only place a cusp could occur is at the origin where the two edges meet. However, in the present case of the XX model the dispersion relation is so simple that only two rays can coalesce and no cusp occurs, just two pure fold lines that meet at $x = t = 0$. This result is special and if a symmetry breaking term is added the two folds will instead generically meet at a cusp (coalescence of three rays) and the back-to-back rays are locally replaced by a Pearcey function \([113]\); ii) The defining feature of a fold catastrophe is that the phase $\Phi(k, x, t)$ is cubic in $k$. This is why the Airy function is the universal wavefunction at a fold because $A_i(z) = (1/2\pi i) \int_{-\infty}^{\infty} ds \exp[i(z s + s^3/3)]$; iii) There exists a diffeomorphism from the physical variables $(k, x, t)$ to the canonical Airy cubic form $(s, z)$. Therefore, a Taylor expansion truncated at precisely third order about the caustic gives the exact semiclassical description in the neighborhood of that point. Performing the transformation of variables, $s^3 = 2(k - k_c)^3/[t d\epsilon(k_c)/dt]$ gives \([71]\)\([73]\)\([93]\).

\[
\Psi_{Ai}(x, t, z) = \sqrt{a} \left( \frac{-2}{(\partial^2\epsilon(k_c)/dt^2)} \right)^{1/3} e^{i\Phi(k_c, x, t)} Ai(z) , \quad (9)
\]

where $z = (x - v_B t)/[t d\epsilon(k_c)/dt]^{-1/3}$. \(10\)

In Fig. 1(a), we plot $|\Psi_{Ai}(x, t)|^2$ alongside the numerical result at the point $x = 10$, with the caustic ($z = 0$) marked by the vertical dotted line. The Airy wavefunction gradually goes out of phase at longer times because the Taylor expansion was made at a single point, but the range could be extended via a uniform mapping \([121]\). From the asymptotics of the Airy function as $z \to -\infty$ it follows that the amplitude of the OTOC wavefront decays as $|z|^{3/2}/(x - v_B t)^2 \sim 1/t$ (in agreement with Refs. \([59]\)\([60]\)). The fact that $x/t$ is constant can be seen in the numerical results. Keeping just the first term of the $z \to \infty$ asymptotic series for the Airy function \([93]\) gives the universal $p = 1/2$ form of the OTOC in Eq. \(4\) \([71]\)\([73]\).

While an Airy function has been derived for OTOCs before \([71]\)\([73]\)\([112]\), the point we emphasize here is that catastrophe theory guarantees that this result is rigorously true and robust to perturbations. Hence, deviations from it imply some qualitative change to the dynamics. One possibility is the presence of a symmetry breaking term which gives one of the higher catastrophes \([113]\) (such as a cusp in the XY model which has a double cone). Another possibility is nonintegrable dynamics, and it is to that case we now turn.

**Profile of the wavefront in the ETH case:** In Fig. 1(b) we plot the exact results for the OTOC for $\delta \epsilon_{\text{ETH}}$. Fringes are partially visible at smaller $x$ but the Airy nodes have disappeared. Although structural stability implies that catastrophes are stable against weak chaos, $\delta \epsilon_{\text{ETH}}$ corresponds to strong chaos which disrupts the rays and their interference significantly. At $x = 3$ the wavefront has quite a sharp slope, indicating that the process of scrambling (the increase in non-locality of the observable) is still in full swing. By $x = 8$, the slope of the OTOC at the wavefront has significantly decreased. The Gaussian waveform of Eq. \(5\) provides an excellent local fit to the wavefront in both the integrable \([53]\)\([92]\) and chaotic regimes, as seen from the dashed curves in Fig. 1(a) and 1(b), respectively. The fit is performed over the range $t = x/v_B \pm \Delta$.
where $\Delta \approx 0.5$ gives a reasonably large window to describe the shape of $C(x, t)$. Within the fitting window, for a fixed $x$, the parameters $m(x)$ and $b(x)$ in Eq. (5) can be determined with very high precision with errors on each term of the order of $10^{-7}$ to $10^{-9}$. A crucial ingredient to identify the parameters in the ETH case is to first determine the butterfly velocity $v_B$, which can be done using velocity-dependent Lyapunov exponents \cite{73, 122}, as demonstrated in the SM \cite{93}. We find that the velocity for the ETH model characterized by $\vec{c}_{ETH}$ is roughly $v_B \approx 1.28$ (in contrast to $v_B = 1$ for $\vec{c}_f$). Although the integrable and ETH wavefronts both display flattening, the scaling properties of $m(x)$ and $b(x)$ are fundamentally different in the two regimes as we now show.

**Scaling in Free Models:** By expanding the Airy wavefunction given in Eq. (9) about the caustic at $z = 0$ we obtain

$$m(x) = \frac{c_m}{x^{\gamma}}, \quad b(x) = \frac{c_b}{x^{\gamma}},$$

where $c_m$ and $c_b$ are constants that depend explicitly on the dispersion relation (see the SM \cite{93} for details). Due to the universality of the Airy wavefunction, this scaling is expected to hold for many models which can be written in terms of freely propagating quasiparticles. Furthermore, corrections beyond quadratic order in $x - v_B t$ can be obtained. However, the cubic term in the exponent falls off rapidly (at least as $x^{-1}$), and so it is reasonable, even at moderate distances, to keep only the Gaussian approximation. We have numerically verified Eq. (11) and the results are shown in Fig. 2(a). Fitting the scaling of each parameter for distances $0 < x \leq 650$ we find,

$$m(x) \propto \frac{1}{x^{\alpha_m}}, \quad b(x) \propto \frac{1}{x^{\alpha_b}},$$

with $\alpha_m = 0.68857 \pm 0.00008$, and $\alpha_b = 0.33043 \pm 0.00002$, indicating good agreement with the expected values. We also note that because $m(x) \propto b(x)^2$, $m(x)$ falls off significantly quicker than $b(x)$. This may point to an intermediate regime in $x$ where the OTOC is well described by $C(x, t) \sim e^{b(x) t}$.

**Scaling in ETH regime:** In Fig. 2(b) we show a plot of the data for the $\vec{c}_{ETH}$ case. A linear trend emerges, implying that the spatial dependence on $m(x)$ and $b(x)$ in the ETH regime exhibits exponential rather than power-law decay,

$$b(x) \sim e^{-c x}, \quad m(x) \sim e^{-w x},$$

where $c, w > 0$ are constants. We find that $c = 0.38 \pm 0.02$ and $w = 0.66 \pm 0.05$. Like the free case, $m(x) \propto b(x)^2$, however, as shown in the SM \cite{93}, this is not generally the case.

The exponentially decaying behavior of $m(x)$ and $b(x)$ is clearly distinct from the free fermion case. This indicates that the Gaussian waveform can distinguish ETH-obeying from free dynamics. In both Figs. 2(a) and (b) $m(x), b(x)$ decay by upwards of two orders of magnitude as a function of position, however the exponential decay in the ETH regime ensures that this occurs over a short distance of $x \approx 10$ while in the free model it takes a distance of $x \approx 600$. Thus, the general flattening of the OTOC at the wavefront (see e.g. Fig. 1) occurs much faster in thermalizing models.

**Conclusions:** Close to the wavefront, integrable and ETH models can be distinguished by the difference in scaling of the parameters $m(x), b(x)$ in Eq. (5). The ability of modern experiments to measure quantum light cone profiles \cite{95, 100} and OTOCs \cite{123, 128} holds out the possibility that this prediction can be tested in the laboratory. A remaining open question concerns the transition from integrable to ETH dynamics \cite{129, 131} and the degree to which structural stability protects the Airy wavefront. A resolution of this question would constitute a quantum version of the celebrated KAM theorem \cite{132}.

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[1] M. V. Berry, Quantum chaosology (The Bakerian Lecture), Proc. R. Soc. A 413, 183 (1987).
[2] R. A. Jalabert, H. U. Baranger, and A. D. Stone, Conduction fluctuations in the ballistic regime: A probe of quantum chaos?, Phys. Rev. Lett. 65, 2442 (1990).
[3] C. M. Marcus, A. J. Rimberg, R. M. Westervelt, P. F. Hopkins, and A. C. Gossard, Conduction fluctuations and chaotic scattering in ballistic microstructures, Phys. Rev. Lett. 69, 506 (1992).
[4] V. Milner, J. L. Hanssen, W. C. Campbell, and M. G. Raizen, Optical billiards for atoms, Phys. Rev. Lett. 86, 1514 (2001).
[5] N. Friedman, A. Kaplan, D. Carasso, and N. Davidson, Observation of chaotic and regular dynamics in atom-optics billiards, Phys. Rev. Lett. 86, 1518 (2001).
[6] H. J. Stockmann and J. Stein, “Quantum” chaos in billiards studied by microwave absorption, Phys. Rev. Lett. 64, 2215 (1990).
[7] S. Sridhar, Experimental observation of scarrd eigenfunctions of chaotic microwave cavities, Phys. Rev. Lett. 67, 785 (1991).
[8] F. L. Moore, J. C. Robinson, C. Bharucha, P. E. Williams, and M. G. Raizen, Observation of dynamical localization in atomic momentum transfer: A new testing ground for quantum chaos, Phys. Rev. Lett. 73, 2974 (1994).
[9] D. A. Steck, W. H. Oskay, and M. G. Raizen, Observation of chaotic-assisted tunneling between islands of stability, Science 293, 274 (2001).
[10] W. K. Hensinger, H. Häffner, A. Browaeys, N. R. Heckenberg, K. Helmerson, C. McKenzie, G. J. Milburn, W. D. Phillips, S. L. Rolston, H. Rubinsztein-Dunlop, and B. Uproft, Dynamical tunnelling of ultracold atoms, Nature 412, 52 (2001).
[11] S. Chaudhury, A. Smith, B. E. Anderson, S. Ghose, and P. S. Jessen, Quantum signatures of chaos in a kicked top, Nature 461, 768 (2009).
[12] Y. S. Weinstein, S. Lloyd, J. Emerson, and D. G. Cory, Experimental implementation of the quantum baker’s map, Phys. Rev. Lett. 89, 157902 (2002).
[13] M. C. Gutzwiller, Periodic orbits and classical quantization conditions, J. Math. Phys. 12, 343 (1971).
[14] M. V. Berry and M. Tabor, Closed orbits and the regular bound spectrum, Proc. R. Soc. A 349, 101 (1976).
[15] E. P. Wigner, On the statistical distribution of the widths and spacings of nuclear resonance levels, Proc. Cambridge Philos. Soc. 47, 790 (1951).
[16] C. E. Porter, Statistical theories of spectra: fluctuations (Academic Press, New York, 1965).
[17] M. V. Berry and M. Tabor, Level clustering in the regular spectrum, Proc. R. Soc. A 356, 375 (1977).
[18] O. Bohigas, M. J. Giannoni, and C. Schmit, Characterization of chaotic quantum spectra and universality of level fluctuation laws, Phys. Rev. Lett. 52, 1 (1984).
[19] L. D’Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, Advances in Physics 65, 239 (2016). https://doi.org/10.1080/00018732.2016.1198134
[20] J. M. Deutsch, Quantum statistical mechanics in a closed system, Phys. Rev. A 43, 2046 (1991).
[21] M. Srednicki, Chaos and quantum thermalization, Phys. Rev. E 50, 888 (1994).
[22] M. Srednicki, The approach to thermal equilibrium in quantized chaotic systems, Journal of Physics A: Mathematical and General 32, 1163 (1999).
[23] M. Rigol, V. Dunjko, and M. Olshanii, Thermalization and its mechanism for generic isolated quantum systems, Nature 452, 854 (2008).
[24] G. De Palma, A. Serafini, V. Giovannetti, and M. Cramer, Necessity of eigenstate thermalization, Phys. Rev. Lett. 115, 220401 (2015).
[25] T. LeBlond, K. Mallayya, L. Vidmar, and M. Rigol, Entanglement and matrix elements of observables in interacting integrable systems, Phys. Rev. E 100, 062134 (2019).
[26] H. Kim, M. N. Ikeda, and D. A. Huse, Testing whether all eigenstates obey the eigenstate thermalization hypothesis, Phys. Rev. E 90, 052105 (2014).
[27] R. Mondaini, K. R. Fratus, M. Srednicki, and M. Rigol, Eigenstate thermalization in the two-dimensional transverse field Ising model, Phys. Rev. E 93, 032104 (2016).
[28] K. Kaneko, E. Iyoda, and T. Sagawa, Work extraction from a single energy eigenstate, Phys. Rev. E 99, 032128 (2019).
[29] G. Biroli, C. Kollath, and A. M. Läuchli, Effect of rare fluctuations on the thermalization of isolated quantum systems, Phys. Rev. Lett. 105, 250401 (2010).
[30] H.-H. Lai and K. Yang, Entanglement entropy scaling laws and eigenstate typicality in free fermion systems, Phys. Rev. B 91, 081101(R) (2015).
[31] J. Riddell and M. P. Müller, Generalized eigenstate typicality in translation-invariant quasi-free fermion models, Phys. Rev. B 97, 035129 (2018).
[32] P. Ribeiro, M. Haque, and A. Lazarides, Strongly interacting bosons in multichromatic potentials supporting mobility edges: Localization, quasi-condensation, and expansion dynamics, Phys. Rev. A 87, 043635 (2013).
[33] L. Vidmar and M. Rigol, Generalized gibbs ensemble in integrable lattice models, Journal of Statistical Mechanics: Theory and Experiment 2016, 064007 (2016).
[34] R. Nandkishore and D. A. Huse, Many-body localization and thermalization in quantum statistical mechanics, Annual Review of Condensed Matter Physics 6, 15 (2015).
[35] S. Iyer, V. Oganesyan, G. Refael, and D. A. Huse, Many-body localization in a quasi-periodic system, Phys. Rev. B 87, 134202 (2013).
[36] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Observation of many-body localization of interacting fermions in a quasirandom optical lattice, Science 349, 842 (2015).
[37] H. P. Lüschen, P. Bordia, S. S. Hodgman, M. Schreiber, S. Sarkar, A. J. Daley, M. H. Fischer, E. Altman, I. Bloch, and U. Schneider, Signatures of Many-Body Localization in a Controlled Open Quantum System, Phys. Rev. X 7, 011034 (2017).
[38] H. P. Lüschen, P. Bordia, S. Scherg, F. Alet, E. Altman, U. Schneider, and I. Bloch, Observation of Slow Dynamics near the Many-Body Localization Transition in One-Dimensional Quasi-periodic Systems, Physical Review Letters 119, 260401 (2017).
[39] T. Mori, Weak eigenstate thermalization with large deviation bound (2016), arXiv:1609.09776.
[40] F. G. S. L. Brandão, E. Crosson, M. B. Şahinoğlu, and J. Bowen, Quantum error correcting codes in eigenstates of translation-invariant spin chains, Phys. Rev. Lett. 123, 110502 (2019).
[41] A. M. Alhambra, J. Riddell, and L. P. García-Pintos, Time evolution of correlation functions in quantum many-body systems, Phys. Rev. Lett. 124, 110605 (2020).

[42] B. Yoshida, Firewalls vs. scrambling, Journal of High Energy Physics 10, 132 (2019).

[43] B. Swingle and D. Chowdhury, Slow scrambling in disordered quantum systems, Phys. Rev. B 95, 060201(R) (2017).

[44] J. R. González Alonso, N. Yunger Halpern, and J. Dressel, Out-of-time-ordered-correlator quasi-probabilities robustly witness scrambling, Phys. Rev. Lett. 122, 040404 (2019).

[45] B. Yan, L. Cincio, and W. H. Zurek, Information scrambling and Loschmidt echo, Phys. Rev. Lett. 124, 160603 (2020).

[46] J. Tuziemski, Out-of-time-ordered correlation functions in open systems: A Feynman-Vernon influence functional approach, Phys. Rev. A 100, 062106 (2019).

[47] D. Mao, D. Chowdhury, and T. Senthil, Slow scrambling and hidden integrability in a random rotor model, Phys. Rev. B 102, 094306 (2020).

[48] R. J. Lewis-Swan, A. Safavi-Naini, J. J. Bollinger, and A. M. Rey, Unifying scrambling, thermalization and entanglement through measurement of fidelity out-of-time-order correlators in the dicke model, Nature Communications 10, 1581 (2019).

[49] S. Nakamura, E. Iyoda, T. Deguchi, and T. Sagawa, Universal scrambling in gapless quantum spin chains (2019), arXiv:1904.09778.

[50] R. Belyansky, P. Bienias, Y. A. Kharkov, A. V. Gorshkov, and B. Swingle, Minimal model for fast scrambling, Phys. Rev. Lett. 125, 130601 (2020).

[51] J. Maldacena, S. H. Shenker, and D. Stanford, A bound on chaos, Journal of High Energy Physics 2016, 106 (2016).

[52] J. Lee, D. Kim, and D.-H. Kim, Typical growth behavior of the out-of-time-ordered commutator in many-body localized systems, Phys. Rev. B 99, 184202 (2019).

[53] J. Riddell and E. S. Sørensen, Out-of-time ordered correlators and entanglement growth in the random-field XX spin chain, Phys. Rev. B 99, 054205 (2019).

[54] X. Chen, T. Zhou, D. A. Huse, and E. Fradkin, Out-of-time-order correlations in many-body localized and thermal phases, Annalen der Physik 529, 1600332 (2017).

[55] Y. Huang, Y.-L. Zhang, and X. Chen, Out-of-time-ordered correlators in many-body localized systems, Annalen der Physik 529, 1600318 (2017).

[56] R. Fan, P. Zhang, H. Shen, and H. Zhai, Out-of-time-order correlation for many-body localization, Science Bulletin 62, 707 (2017).

[57] Y. Chen, Universal logarithmic scrambling in many body localization (2016), arXiv:1608.02765.

[58] R.-Q. He and Z.-Y. Lu, Characterizing many-body localization by out-of-time-ordered correlation, Phys. Rev. B 95, 054201 (2017).

[59] C.-J. Lin and O. I. Motrunich, Out-of-time-ordered correlators in a quantum ising chain, Phys. Rev. B 97, 144304 (2018).

[60] J. Bao and C. Zhang, Out-of-time-order correlators in one-dimensional XY model (2019), arXiv:1901.09327.

[61] D. A. Roberts and B. Yoshida, Chaos and complexity by design, Journal of High Energy Physics 2017, 121 (2017).

[62] Y. Huang, F. G. S. L. Brandão, and Y.-L. Zhang, Finite-size scaling of out-of-time-ordered correlators at late times, Phys. Rev. Lett. 123, 010601 (2019).

[63] Y. Chen, Universal logarithmic scrambling in many-body localization (2016), arXiv:1608.02765.

[64] J. K. Max McGinley, Andreas Nunnerkamp, Slow growth of entanglement and out-of-time-order correlators in integrable disordered systems (2018), arXiv:1807.06039.

[65] S. Pappalardi, A. Russo-manno, B. B. Žunkovic, F. Iemini, A. Silva, and R. Fazio, Scrambling and entanglement spreading in long-range spin chains, Phys. Rev. B 98, 134303 (2018).

[66] Q. Hummel, B. Geiger, J. D. Urbina, and K. Richter, Reversible quantum information spreading in many-body systems near criticality, Phys. Rev. Lett. 123, 160401 (2019).

[67] K. Hashimoto, K.-B. Huh, K.-Y. Kim, and R. Watanabe, Exponential growth of out-of-time-order correlator without chaos: inverted harmonic oscillator, Journal of High Energy Physics 2020, 68 (2020).

[68] S. Pilatowsky-Canedo, J. Chávez-Carlos, M. A. Bastarrachea-Magnani, P. Stránský, S. Lerma-Hernández, L. F. Santos, and J. G. Hirsch, Positive quantum Lyapunov exponents in experimental systems with a regular classiﬁcal limit, Phys. Rev. E 101, 010202(R) (2020).

[69] T. Xu, T. Scaffidi, and X. Cao, Does scrambling equal chaos?, Phys. Rev. Lett. 124, 140602 (2020).

[70] W. Kirkby, D. H. J. O’Dell, and J. Mumford, False signals of chaos from quantum probes, Phys. Rev. A 104, 043308 (2021).

[71] S. Xu and B. Swingle, Accessing scrambling using matrix product operators, Nature Physics 16, 199–204 (2019).

[72] S. Xu and B. Swingle, Locality, quantum fluctuations, and scrambling, Physical Review X 9, 031048 (2019).

[73] V. Khemani, D. A. Huse, and A. Nahum, Velocity-dependent Lyapunov exponents in many-body quantum, semiclassical, and classical chaos, Phys. Rev. B 98, 144304 (2018).

[74] A. Nahum, S. Vijay, and J. Haah, Operator spreading in random unitary circuits, Phys. Rev. X 8, 021014 (2018).

[75] C. W. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. L. Sondhi, Operator hydrodynamics, OTOCs, and entanglement growth in systems without conservation laws, Phys. Rev. X 8, 021013 (2018).

[76] S.-K. Jian and H. Yao, Universal properties of many-body quantum chaos at gross-neveu criticality (2018), arXiv:1805.12299.

[77] Y. Gu, X.-L. Qi, and D. Stanford, Local criticality, diffusion and chaos in generalized sachdev-ye-kitaev models, Journal of High Energy Physics 2017, 125 (2017).

[78] S. Sahu, S. Xu, and B. Swingle, Scrambling dynamics across a thermalization-localization quantum phase transition, Phys. Rev. Lett. 123, 165902 (2019).

[79] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, Diffusive hydrodynamics of out-of-time-ordered correlators with charge conservation, Phys. Rev. X 8, 031058 (2018).

[80] S. H. Shenker and D. Stanford, Black holes and the butterfly effect, Journal of High Energy Physics 2014, 67 (2014).

[81] A. A. Patel, D. Chowdhury, S. Sachdev, and B. Swingle, Quantum butterfly effect in weakly interacting diffusive metals, Phys. Rev. X 7, 031047 (2017).

[82] D. Chowdhury and B. Swingle, Onset of many-body chaos in the O(N) model, Phys. Rev. D 96, 065005 (2017).

[83] V. Khemani, A. Vishwanath, and D. A. Huse, Operator spreading and the Emergence of Dissipative Hydrodynamics under Unitary Evolution with Conservation Laws, Physical Review X 8, 031057 (2018).

[84] N. Anand, G. Styliaris, M. Kumari, and P. Zanardi, Quantum coherence as a signature of chaos (2020), arXiv:2009.02760.

[85] S. Gopalakrishnan, D. A. Huse, V. Khemani, and R. Vasseur, Hydrodynamics of operator spreading and quasiparticle diffusion in interacting integrable systems, Phys. Rev. B 99, 220303(R) (2018).

[86] P. L. Doussal, S. N. Majumdar, and G. Schehr, Large deviations for the height in 1D Kardar-Parisi-Zhang growth at late times, EPL (Europhysics Letters) 113, 60004 (2016).
[87] C. Monthus and T. Garel, Probing the tails of the ground-state energy distribution for the directed polymer in a random medium of dimension $d = 1, 2, 3$ via a Monte Carlo procedure in the disorder, Phys. Rev. E 74, 051109 (2006).

[88] I. V. Kolokolov and S. E. Korshunov, Universal and nonuniversal tails of distribution functions in the directed polymer and Kardar-Parisi-Zhang problems, Phys. Rev. B 78, 024206 (2008).

[89] E. M. Fortes, I. García-Mata, R. A. Jalabert, and D. A. Wisniacki, Gauging classical and quantum integrability through out-of-time-ordered correlators, Phys. Rev. E 100, 042201 (2019).

[90] E. M. Fortes, I. García-Mata, R. A. Jalabert, and D. A. Wisniacki, Signatures of quantum chaos transition in short spin chains, EPL (Europhysics Letters) 130, 60001 (2020).

[91] J. Wang, G. Benenti, G. Casati, and W. ge Wang, Quantum chaos and the correspondence principle (2020), arXiv:2010.10360.

[92] J. Riddell and E. S. Sørensen, Out-of-time-order correlations in the quasiperiodic Aubry-André model, Phys. Rev. B 101, 024202 (2020).

[93] J. Riddell, W. Kirkby, D. H. J. O’Dell, and E. S. Sørensen, Supplementary material (2021).

[94] E. H. Lieb and D. W. Robinson, The finite group velocity of quantum many-body system, Nature 481, 484–487 (2012).

[95] T. Fukuhara, P. Schauß, M. Endres, S. Hild, M. Cheneau, I. Bloch, and C. Gross, Microscopic observation of magnon bound states and their dynamics, Nature 502, 76–79 (2013).

[96] M. Grättner, J. G. Bohnet, A. Safavi-Naini, M. L. Wall, J. J. Martinis, J. I. Cirac, J. M. Yepez, C. Carusotto, and D. Schuster, Quantum signatures of chaos on a Floquet synthetic lattice, Phys. Rev. X 9, 031011 (2019).

[97] D. J. Auckley, L.测, and D. Z. Hase, Quantum signatures of chaos on a Floquet synthetic lattice, Phys. Rev. X 9, 031011 (2019).

[98] J. Viti, J.-M. Stéphan, J. Dubail, and M. Haque, Inhomogeneous quenches in a free fermionic chain: Exact results, EPL (Europhysics Letters) 115, 40011 (2016).

[99] P. Barmettler, D. Poletti, M. Cheneau, and C. Kollath, Propagation front of correlations in an interacting Bose gas, Phys. Rev. E 96, 021238 (2017).

[100] J.-M. Stéphan, Free fermions at the edge of interacting systems, SciPost Phys. 6, 10.21468/scipostphys.6.5.057 (2019).

[101] M. Grättner, Higher-order generalized hydrodynamics in one dimension: The noninteracting test, Physical Review B 96, 220302(R) (2017).

[102] M. Kormos, Inhomogeneous quenches in the transverse field Ising chain: scaling and front dynamics, SciPost Physics 3, 10.21468/scipostphys.3.3.037 (2020).

[103] V. B. Bulchandani and C. Karrasch, Subdiffusive front scaling in interacting integrable models, Physical Review B 99, 121410(R) (2019).

[104] J. Viti, J.-M. Stéphan, J. Dubail, and M. Haque, Inhomogeneous quenches in a free fermionic chain: Exact results, EPL (Europhysics Letters) 115, 40011 (2016).

[105] G. Perfetto and A. Gambassi, Ballistic front dynamics after joining two semi-infinite quantum Ising chains, Physical Review E 96, 021238 (2017).

[106] M. Kormos, Inhomogeneous quenches in the transverse field Ising chain: scaling and front dynamics, SciPost Physics 3, 10.21468/scipostphys.3.3.037 (2020).

[107] V. B. Bulchandani and C. Karrasch, Subdiffusive front scal-...
M. Schreitl, I. Mazets, D. Adu Smith, E. Demler, and J. Schmiedmayer, Relaxation and pre-thermalization in an isolated quantum system, Science 337, 1318 (2012).

[130] F. H. L. Essler, S. Kehrein, S. R. Manmana, and N. J. Robinson, Quench dynamics in a model with tuneable integrability breaking.

[131] B. Bertini and M. Fagotti, Pre-relaxation in weakly interacting models, J. Stat. Mech., P07012 (2015).

[132] G. Brandino, J.-S. Caux, and R. Konik, Glimmers of a quantum KAM theorem: Insights from quantum quenches in one-dimensional Bose gases, Phys. Rev. X 5, 041043 (2015).