Alpha-Particle Condensation in Nuclear Systems

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**Abstract.** The onset of quartetting, i.e. $\alpha$-particle condensation, in symmetric and asymmetric nuclear matter is studied with the help of an in-medium modified four nucleon equation. It is found that at very low density quartetting wins over pairing, because of the strong binding of the $\alpha$-particles. The critical temperature can reach values up to around 8 MeV. The disappearance of $\alpha$-particles with increasing density, i.e. the Mott transition, is investigated. In finite nuclei the Hoyle state, that is the $0^+\text{ of }^{12}\text{C}$ is identified as an ‘$\alpha$-particle condensate’ state. It is conjectured that such states also exist in heavier $\alpha$-nuclei, like $^{16}\text{O}$, $^{20}\text{Ne}$, etc. The sixth $0^+$ state in $^{16}\text{O}$ is proposed as an analogue to the Hoyle state. The Gross-Pitaevski equation is employed to make an estimate of the maximum number of $\alpha$ particles a condensate state can contain. Possible quartet condensation in other systems is discussed briefly.

Keywords: nuclear matter, $\alpha$-matter, superfluidity, Bose-Einstein condensation, strongly coupled systems

1. Introduction

One of the most amazing phenomena in quantum many-particle systems is the formation of quantum condensates. At present, the formation of condensates is of particular interest in strongly coupled fermion systems in which the crossover from Bardeen-Cooper-Schrieffer (BCS) pairing to Bose-Einstein condensation (BEC) may be investigated. Among very different quantum systems such as the electron-hole-exciton system in excited semiconductors, atoms in traps at extremely low temperatures, etc., nuclear matter is especially well suited for the study of correlation effects in a quantum liquid.

Neutron matter, nuclear matter, but also finite nuclei are superfluid. However, at low density, nuclear matter will not cluster into pairs, i.e. deuterons but rather into $\alpha$ -particles which are much more stable. Also heavier clusters, starting with Carbon, may be of importance but are presently not considered for condensation phenomena. Therefore, one may ask the question...
whether there exists quartetting, i.e. α-particle condensation, in nuclei, analogous to nuclear pairing. The only nucleus which in its ground state has a pronounced α-cluster structure is $^8$Be. In section 4 we will show a figure of $^8$Be in the laboratory frame and in the intrinsic deformed frame. We will see that $^8$Be is formed out of two almost free α-particles roughly 4 fm apart and barely touching one another. Actually $^8$Be is slightly unstable and the two α’s only hold together via the Coulomb barrier. Because of the large distance of the two α-particles, the 0$^+$ ground state of $^8$Be has, in the laboratory frame, a spherical density distribution whose average is very low: about 1/3 of ordinary saturation density $\rho_0$. $^8$Be is, therefore, a very large object with an rms radius of about 3.7 fm to be compared with the nuclear systematics of $R = r_0A^{1/3} = 2.44$ fm. Definitely $^8$Be is a rather unusual and, in its kind, unique nucleus. One may ask the question what happens when one brings a third α-particle alongside of $^8$Be. We know the answer: the 3-α system collapses to the ground state of $^{12}$C which is much denser than $^8$Be and can not accommodate with its small radius of 2.4 fm three more or less free α-particles barely touching one another. One nevertheless may ask the question whether the dilute three α configuration $^8$Be- α, or $\alpha - \alpha - \alpha$, may not form an isomeric or excited state of $^{12}$C. That such a state indeed exists will be one of the main subjects of our considerations.

Once one accepts the idea of the existence of an α-gas state in $^{12}$C, there is no reason why equivalent states at low density should not also exist in heavier no-nuclei, like $^{16}$O, $^{20}$Ne, etc. In a mean field picture, i.e. all α’s being ideal bosons (in this context remember that the first excited state of an α-particle is at ~ 20 MeV, by factors higher than in all other nuclei), all α’s will occupy the lowest 0S-state, i.e. they will condense into this state. This forms, of course, not a macroscopic condensate but it can be understood in the same sense as we know that nuclei are superfluid because of the presence of a finite number of Cooper pairs. On the other hand, for example during the cooling process of compact stars [1], where one predicts the presence of α-particles [2], a real macroscopic phase of condensed α’s may be formed. In the present contribution we will mainly concentrate on nuclear systems but we also can think about the possibility of quartetting in other Fermi-systems. One should, however, keep in mind that a pre-requisite for its existence is, as in nuclear physics, that there exists a bound quartet in free space. This is facilitated with the existence of four different types of fermions, all attracting one another. For example to form quartets with cold atoms one could try to trap fermions in four different magnetic substates, a task which eventually seems possible [3].

In the next section we will investigate how the binding energy of various nuclear clusters changes with density. In section 3 we study the critical temperature of α-particle condensation in infinite matter via an in-medium four-nucleon equation (Thouless criterion) and in section 4 we treat α-particle condensation in finite nuclei. Finally in section 5 we conclude with outlook and further discussions.

2. Nuclear clusters in the medium

With increasing density of nuclear matter, medium modifications of single-particle states as well as of few-nucleon states become of importance. The self-energy of an $A$-particle cluster can in principal be deduced from contributions describing the single-particle self-energies as well as medium modifications of the interaction and the vertices. A guiding principle in incorporating medium effects is the construction of consistent (“conserving”) approximations, which treat medium corrections in the self-energy and in the interaction vertex at the same level of accuracy. This can be achieved in a systematic way using the Green functions formalism [4] [5]. At the mean-field level, we have only the Hartree-Fock self-energy $\Gamma_{HF} = \sum_{\nu} V(12,1\nu) f(2)$ together with the Pauli blocking factors, which modify the interaction from $V(12,1\nu)f(2)$ to $V(12,1\nu’)(1 - f(1) - f(2))$ with the $f(i)$ the Fermi-Dirac distribution. In the case of the two-nucleon system ($A = 2$), the resulting effective wave equation which includes those corrections reads...
This effective wave equation describes bound states as well as scattering states. The onset of pair condensation is achieved when the binding energy $E_{d,P=0}$ coincides with $2\mu$.

Similar equations have been derived from the Green function approach for the case $A=3$ and $A=4$, describing triton/helion ($^{3}\text{He}$) nuclei as well as $\alpha$-particles in nuclear matter. The effective wave equation contains in mean field approximation the Hartree-Fock self-energy shift of the single-particle energies as well as the Pauli blocking of the interaction. We give the effective wave equation for $A=4$,

$$\left[E_{\text{HF}}^{1} + E_{\text{HF}}^{2} - E_{2,n,P}^{2}\right] \psi_{2,n,P}(12) + \sum_{1'2'} \left[1 - f(1) - f(2) \right] V(12,1'2') \psi_{2,n,P}(1'2') = 0.$$  \hspace{1cm} (1)

$$\left[E_{\text{HF}}^{1} + E_{\text{HF}}^{2} + E_{\text{HF}}^{3} + E_{4,n,P}^{4} - E_{4,n,P}^{4}\right] \psi_{4,n,P}(1234) + \sum_{1'2'} \left[1 - f(1) - f(2) \right] V(12,1'2') \psi_{4,n,P}(1'2'34) + \text{permutations} = 0.$$  \hspace{1cm} (2)

A similar equation is obtained for $A=3$.

The effective wave equation has been solved using separable potentials for $A=2$ by integration. For $A=3,4$ we can use a Faddeev approach [6]. The shifts of binding energy can also be calculated approximately via perturbation theory. In Fig. 1 we show the shift of the binding energy of the light clusters ($d,t/h$ and $\alpha$) in symmetric nuclear matter as a function of density for temperature $T=10$ MeV.

3. Four-particle condensates and quartetting in nuclear matter

In general, it is necessary to take account of all bosonic clusters to gain a complete picture of the onset of superfluidity. As is well known, the deuteron is weakly bound compared to other nuclei. Higher $A$-clusters can arise that are more stable. In this section, we will consider the formation

![Figure 1. Shift of binding energies of the light clusters (d - dash dotted, t/h - dotted, and $\alpha$ - dashed: perturbation theory, full line : non-perturbative Faddeev-Yakubovskii equation) in symmetric nuclear matter as a function of density for given temperature $T = 10$ MeV.](image-url)
of \( \alpha \)-particles, which are of special importance because of their large binding energy per nucleon (7 MeV). We will not include tritons or helions, which are fermions and not so tightly bound. Moreover, we will not consider nuclei in the iron region, which have even larger binding energy per nucleon than the \( \alpha \)-particle and thus constitute, in principle, the dominant component at low temperatures and densities. However, the latter are complex structures of many particles and are strongly affected by the medium as the density increases, since their excitation spectrum starts at much lower energy than the one of the \( \alpha \) particle, for instance. So they are assumed not to be of relevance in the density region considered here.

The in-medium wave equation for the four-nucleon problem has been solved using the Faddeev-Yakubovski technique, with the inclusion of Pauli blocking. The binding energy of an \( \alpha \)-like cluster with zero c.o.m. momentum vanishes at around \( \rho_0/10 \), where \( \rho_0 \approx 0.16 \) nucleons/fm\(^3\) denotes the saturation density of isospin-symmetric nuclear matter, see Fig. 1. Thus, the four-body bound states make no significant contribution to the composition of the system above this density. Given the medium-modified bound-state energy \( E_{4,p} \), the bound-state contribution to the EOS is

\[
\rho_4(\beta, \mu) = \sum_p \left[ \epsilon^{\beta(E_{4,p}-2\mu_p-2\mu_n)} - 1 \right]^{-1}.
\]  

We will not include the contribution of the excited states or that of scattering states. Because of the large specific binding energy of the \( \alpha \) particle, low-density nuclear matter is predominantly composed of \( \alpha \) particles. This observation underlies the concept of \( \alpha \) matter and its relevance to diverse nuclear phenomena.

As exemplified by Eq. (2), the effect of the medium on the properties of an \( \alpha \) particle in mean-field approximation (i.e., for an uncorrelated medium) is produced by the Hartree-Fock self-energy shift and Pauli blocking. The shift of the \( \alpha \)-like bound state has been calculated using perturbation theory [7] as well as by solution of the Faddeev-Yakubovski equation [6]. It is found that this bound state merges with the continuum of scattering states at a Mott density \( \rho_{\text{Mott}} \approx \rho_0/10 \), see Fig. 1. The bound states of clusters \( d, t, \) and \( h \) with \( A < 4 \) are already dissolved at the density \( \rho_{\text{Mott}} \). Consequently, if we neglect the contribution of the four-particle scattering phase shifts in the different channels, we can now construct an equation of state \( \rho(T, \mu) = \rho^{\text{free}}(T, \mu) + \rho^{\text{bound},d}(T, \mu) + \rho^{\text{bound},\alpha}(T, \mu) \) such that \( \alpha \)-particles determine the behavior of symmetric nuclear matter at densities below \( \rho_{\text{Mott}} \) and temperatures below the binding energy per nucleon of the \( \alpha \)-particle. The formation of deuteron clusters alone gives an incorrect description because the deuteron binding energy is small, and the abundance of \( d \)-clusters is small compared with that of \( \alpha \)-clusters. In the low density region of the phase diagram, \( \alpha \)-matter emerges as an adequate model for describing the nuclear-matter equation of state.

With increasing density, the medium modifications – especially Pauli blocking – will lead to a deviation of the critical temperature \( T_c(\rho) \) from that of an ideal Bose gas of \( \alpha \)-particles (the analogous situation holds for deuteron clusters, i.e., in the isospin-singlet channel). Symmetric nuclear matter is characterized by the equality of the proton and neutron chemical potentials, i.e., \( \mu_p = \mu_n = \mu \). Then an extended Thouless condition based on the relation for the four-body T-matrix, equivalent to Eq. (2) at eigenvalue \( 4\mu \) serves to determine the onset of Bose condensation of \( \alpha \)-like clusters, noting that existence of a solution of this relation signals a divergence of the four-particle correlation function.

An approximate solution has been obtained by a variational approach, in which the wave function is taken of the following form [8] (see also [9] for another variational form):

\[
\psi(1234) = \delta(k_1 + k_2 + k_3 + k_4)\varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)\chi(1234)
\]  

(4)
where the $\varphi$'s are single particle wave functions in momentum space to be determined variationally out of the in medium four body wave equation and $\chi(1234)$ is the scalar spin-isospin function. The delta function is a projector on total momentum zero.

![Figure 2.](image)

Figure 2. (Left) Critical temperatures for $\alpha$ particle and deuteron condensation in symmetric nuclear matter as a function of $\mu$ (a) and density $n^{(0)}$ (b). (Right) Same as in Fig.2 but for asymmetric nuclear matter for various asymmetry parameters $\delta = (n_n - n_p)/(n_n + n_p)$. $\delta = 0$ (full line); $\delta = 0.5$ (broken line); $\delta = 0.9$ (dotted line).

The results for symmetric and asymmetric nuclear matter with a separable force which reproduces binding energy and radius of a free $\alpha$ particle are presented in Figs. 3 and 3. The crosses on the figure indicate an exact solution of the in medium four body equation with a realistic nucleon-nucleon force [8] [10]. The excellent agreement with the approximate solution justifies the choice of our variational wave function. An important feature is that at the lowest temperatures, Bose-Einstein condensation occurs for $\alpha$ particles rather than for deuterons. As the density increases within the low-temperature regime, the chemical potential $\mu$ first reaches $-7 \text{ MeV}$, where the $\alpha$'s Bose-condense. By contrast, Bose condensation of deuterons would not occur until $\mu$ rises to $-1.1 \text{ MeV}$. In the asymmetric case, it is interesting to note that at strong asymmetry $\alpha$ particle condensation wins over the deuteron condensation because of the strong binding of the $\alpha$'s.

The “quartetting” transition temperature sharply drops as the rising density approaches the critical Mott value ($\mu_0$) at which the four-body bound states disappear. At that point, pair formation in the isospin-singlet deuteron-like channel comes into play, and a deuteron condensate will exist below the critical temperature for BCS pairing up to densities above the nuclear-matter
saturation density $\rho_0$, as described in the previous Section. The critical density at which the $\alpha$ condensate disappears is estimated to be $\rho_0/5$. Therefore, $\alpha$-particle condensation primarily only exists in the Bose-Einstein-Condensed (BEC) phase and there does not seem to exist a phase where the quartets acquire a large extension as Cooper pairs do in the weak coupling regime. However, our variational approach on which this conclusion is based represents only a first attempt at the description of the transition from quartetting to pairing. The detailed nature of this fascinating transition remains to be clarified. Further discussions on this phenomenon can be found in [11].

A very intriguing question in relation with non existence of an $\alpha$ condensate at higher densities can be asked: is it possible that in heavy nuclei an $\alpha$ particle condensate exists in the nuclear surface, at least in its fluctuating form? The preformation assumption of $\alpha$ particles in the surface to explain $\alpha$ decay may give a hint to this.

4. Alpha-Particle Condensate States in Self-Conjugate 4n Nuclei

![Figure 3](image)

**Figure 3.** Contours of constant density (taken from Ref. [12]), plotted in cylindrical coordinates, for $^8\text{Be}(0^+)$). The left side (a) is in the "laboratory" frame while the right side (b) is in the intrinsic frame.

Let us discuss the possibility of quartetting in nuclei. The only nucleus having a pronounced $\alpha$-cluster structure in its ground state is $^8\text{Be}$. In Fig. 3(a), we show the result of an exact calculation of the density distribution of $^8\text{Be}$ in the laboratory frame. In Fig. 3(b) we show, for comparison, the result of the same calculation in the intrinsic, deformed frame. We see a pronounced two $\alpha$-cluster structure where the two $\alpha$'s are $\sim 4$ fm apart, giving rise to a very low average density $\rho \sim \rho_0/3$ as seen in Fig. 3(a). As already discussed in the introduction, $^8\text{Be}$ is a rather unusual and unique nucleus. One may be intrigued by the question, already raised earlier, whether loosely bound $\alpha$-particle configurations may not also exist in heavier no-nuclei, at least in excited states, naturally close to the no disintegration threshold. Since $\alpha$-particles are rather inert bosons (first excited state at $\sim 20$ MeV), these $\alpha$-particles then would all condense in the lowest $S$-wavefunction, very much in the same way as do bosonic atoms in magneto-optical traps [13]. This question and exploring related issues of quartetting in finite nuclei will consume most of the rest of the present outline on $\alpha$ particle condensation. In fact, we will be able to offer strong arguments that the $0^+_2$ state of $^{12}\text{C}$ at 7.654 MeV is a state of $\alpha$-condensate nature.

First, it should be understood that the $0^+_2$ state in $^{12}\text{C}$ is in fact hadronically unstable (as $^8\text{Be}$), being situated about 300 keV above the three $\alpha$-break up threshold. This state is stabilized only by the Coulomb barrier. It has a width of 8.7 eV and a corresponding lifetime of $7.6 \times 10^{-17}$ s. As well known, this state is of paramount astrophysical (and biological!) importance due to its role in the creation of $^{12}\text{C}$ in stellar nucleosynthesis. Its existence was predicted in 1953 by the astrophysicist Fred Hoyle [14]; his prediction was confirmed experimentally a few years later by Willy Fowler and coworkers at Caltech [15]. It is also well known that this Hoyle state, as it is called, is a notoriously difficult state for any nuclear theory to explain. For example, the most
modern no-core shell-model calculations predict the $0^+_2$ state in $^{12}$C to lie at around 17 MeV above the ground state – more than twice the actual value [16]. This fact alone tells us that the Hoyle state must have a very unusual structure. It is easy to understand that, should it indeed have the proposed loosely bound three $\alpha$-particle structure, a shell-model type of calculation would have great difficulties in reproducing its properties.

An important development bearing on this issue took place some thirty years ago. Two Japanese physicists, M. Kamimura [17] and K. Uegaki [18], along with their collaborators, almost simultaneously reproduced the Hoyle state from a microscopic theory. They employed a twelve-nucleon wave function together with a Hamiltonian containing an effective nucleon-nucleon interaction. At that time, their work did not attract the attention it deserved; the true importance of their achievement has been appreciated only recently. The two groups started from practically the same ansatz for the $^{12}$C wave function, which has the following three $\alpha$-cluster structure:

$$\langle r_1...r_{12}|^{12}\text{C} \rangle = \mathcal{A} \left| \chi(R, s)\phi_1\phi_2\phi_3 \right| .$$

(5)

In this expression, the operator $\mathcal{A}$ imposes antisymmetry in the nucleonic degrees of freedom and $\phi_i$, with $i = 1, 2, 3$ for the three $\alpha$’s, is an intrinsic $\alpha$-particle wave function of prescribed Gaussian form,

$$\phi(r_1, r_2, r_3, r_4) = \exp \left\{ - \frac{1}{2b^2} \sum (r_{ij} - r_{ik})^2 + r_{ij} + r_{ik} \right\} ,$$

where the size parameter $b$ is adjusted to fit the rms of the free $\alpha$-particle, and $\chi(R, s)$ is a yet-to-be determined three-body wave function for the c.o.m. motion of the three $\alpha$’s, their corresponding Jacobi coordinates being denoted by $R$ and $s$. The unknown function $\chi$ was determined via calculations based on the Generator Coordinate Method [18] (GCM) and the Resonating Group Method [17] (RGM) calculations using the Volkov I and Volkov II nucleon-nucleon forces, which fit $\alpha-\alpha$ phase shifts. The precise solution of this complicated three body problem, carried out three decades ago, was truly a pioneering achievement, with results fulfilling expectations. The position of the Hoyle state, as well as other properties including the inelastic form factor and transition probability, successfully reproduced the experimental data. Other states of $^{12}$C below and around the energy of the Hoyle state were also successfully described. Moreover, it was already recognized that the three $\alpha$’s in the Hoyle state form sort of a gas-like state. In fact, this feature had previously been noted by H. Horiuchi [19] prior to the appearance of Refs. [17, 18], based on results from the orthogonality condition model (OCM) [20]. All three Japanese research groups concluded from their studies that the linear-chain state of three $\alpha$-particles, postulated by Morinaga many years earlier [21], had to be rejected.

Although the evidence for interpreting the Hoyle state in terms of an $\alpha$ gas was stressed in the cited papers from the late 1970’s, two important aspects of the situation were missed at that time. First, because the three $\alpha$’s move in identical $S$-wave orbits, one is dealing with an $\alpha$-condensate state what may, in fact, be a general feature in $n\alpha$ nuclei. The second and most important point is that the complicated three-body wave function $\chi(R, s)$ for the c.o.m. motion of the three $\alpha$’s can be replaced by a structurally and conceptually very simple microscopic three-$\alpha$ wave function of the condensate type, which has practically 100 percent overlap with the previously constructed ones [22] [23] (see also Ref. [24]). We now describe this condensate wave function.

We start by examining the BCS wave function of ordinary fermion pairing, obtained by projecting the familiar BCS ground-state ansatz onto an $N$-particle subspace of Fock space. In the position representation, this wave function is

$$\langle r_1...r_N|\text{BCS} \rangle = \mathcal{A} \left[ \phi(r_1, r_2)\phi(r_3, r_4)...\phi(r_{N-1}, r_N) \right] ,$$

(7)
where $\phi(r_1, r_2)$ is the Cooper-pair wave function (including spin and isospin), which is to be determined variationally through the familiar BCS equations. The condensate character of the BCS ansatz is born out by the fact that within the antisymmetrizer $A$, one has a product of $N/2$ times the same pair wave function $\phi$, with one such function for each distinct pair in the reference partition of $\{1, 2, \ldots, N-1, N\}$. Formally, it is now a simple matter to generalize (7) to quartet or $\alpha$-particle condensation. We write

$$\langle r_1, \ldots, r_N|\Phi_{\alpha\alpha}\rangle = A[\phi_\alpha(r_1, r_2, r_3, r_4)\phi_\alpha(r_5, \ldots, r_8) \cdots \phi_\alpha(r_{N-3}, \ldots, r_N)],$$

where $\phi_\alpha$ is the wave function common to all condensed $\alpha$-particles. Of course, finding the variational solution for this function is, in general, extraordinarily more complicated than finding the Cooper pair-wave function $\phi$ of Eq. (7). Even so, in the present case that the $\alpha$-particle is the four-body cluster involved, and for applications to relatively light nuclei, the complexity of the problem can be reduced dramatically. This possibility stems from the fact that an excellent variational ansatz for the intrinsic wave function of the $\alpha$-particle is provided [as in Eq. (6)], by a Gaussian form with only the size parameter $b$ to be determined. The new aspect in [22] is that in addition even the center-of-mass motion of the system of $\alpha$-particles can be described very well by a Gaussian wave function with, this time, a size parameter $B \gg b$ to account for the motion over the nuclear space. This is a strong technical simplification and, at the same time, underlines the boson condensate character of the wave function. We therefore write

$$\phi_\alpha(r_1, r_2, r_3, r_4) = e^{-2R^2/B^2}\phi(r_1 - r_2, r_1 - r_3, \ldots),$$

where $R = (r_1 + r_2 + r_3 + r_4)/4$ is the c.o.m. coordinate of one $\alpha$-particle and $\phi(r_1 - r_2, \ldots)$ is the same intrinsic $\alpha$-particle wave function of Gaussian form as already used in Refs. [17, 18] and given explicitly in Eq. (6). Naturally, in Eq. (8) the center of mass $X_{\text{cm}}$ of the three $\alpha$’s, i.e., of the whole nucleus, should be eliminated; this is easily achieved by replacing $R$ by $R - X_{\text{cm}}$ in each of the $\alpha$ wave functions in Eq. (8). The $\alpha$-particle condensate wave function specified by Eqs. (8) and (9), proposed in Ref. [22] and henceforth called the THSR wave function, now depends on only two parameters, $B$ and $b$. The expectation value of an assumed microscopic Hamiltonian $H$,

$$\mathcal{H}(B, b) = \frac{\langle \Phi_{\alpha\alpha}(B, b)|H|\Phi_{\alpha\alpha}(B, b)\rangle}{\langle \Phi_{\alpha\alpha}|\Phi_{\alpha\alpha}\rangle},$$

can be evaluated, and the corresponding two-dimensional energy surface can be quantized using the two parameters $B$ and $b$ as Hill-Wheeler coordinates.

Before presenting the results, let us discuss the THSR wave function in somewhat more detail. This innocuous-looking variational ansatz, namely Eq. (8) together with Eq. (9), is actually more subtle than it might at first appear. One should realize that two limits are incorporated exactly. One is obtained by choosing $B = b$, for which Eq. (8) reduces to a standard Slater determinant with harmonic-oscillator single-nucleon wave functions, leaving the oscillator length $b$ as the single adjustable parameter. This holds because the right-hand-side of expression (9), with $B = b$, becomes a product of four identical Gaussians, and the antisymmetrization creates all the necessary $P$, $D$, etc. harmonic oscillator wave functions automatically [22]. On the other hand, when $B \gg b$, the density of $\alpha$-particles is very low, and in the limit $B \rightarrow \infty$, the average distance between $\alpha$’s is so large that the antisymmetrisation between them can be neglected, i.e., the operator $A$ in front of Eq. (8) becomes irrelevant and can be removed. In this limiting case, our wave function then describes an ideal gas of independent, condensed $\alpha$-particles – it is a pure product state of $\alpha$’s! An elucidating study on this aspect is given in Ref. [25].

Evidently, however, in realistic cases the antisymmetrizer $A$ cannot be neglected, and evaluation of the expectation value (10) becomes a nontrivial analytical task.
The absolute magnitude of the form factor; Fig. 5 illustrates this behavior with a plot showing the positions, rms values, and transition probabilities are given in Table 1 and compared to the data. Inspecting the rms radii, we see that the Hoyle state has a volume 3 to 4 larger than that of the ground state of $^{12}\text{C}$. This is the primary aspect of the dilute-gas state we highlighted above. Constructing a pure-state $\alpha$-particle density matrix $\rho(R, R')$ from our wave function, integrating out of the total density matrix all intrinsic $\alpha$-particle coordinates, and diagonalizing this reduced density matrix, we find that the corresponding $0S$ $\alpha$-particle orbit is occupied to 70 percent by the three $\alpha$-particles [25, 28] whereas the occupation of all other states is down by at least a factor of ten. This is a huge percentage, giving vivid support to the view that the Hoyle state is the correct one. Like the authors of Ref. [17], we reproduce very accurately the $\alpha$-particle occupation is about equally shared between the $0S$, $0D$, and $0G$ orbits, clearly invalidating a condensate picture of the ground state (it is important to note that the ground-state energy of $^{12}\text{C}$ is also reasonably reproduced by our theory).

Let us now discuss what to our mind is the most convincing evidence that our description of the Hoyle state is the correct one. Like the authors of Ref. [17], we reproduce very accurately the inelastic form factor $0^+_1 \rightarrow 0^+_2$ of $^{12}\text{C}$, as shown in Fig. 4. As such, the agreement with experiment is already quite impressive in view of the fact that we did not use a single adjustable parameter. Additionally, however, the following study was made, results from which are presented in Fig. 5. We artificially varied the extension of the Hoyle state and examined the influence on the form factor. It was found that the overall shape of the form factor shows little variation, for example in the position of the minimum. On the other hand, we found a strong dependence of the absolute magnitude of the form factor; Fig. 5 illustrates this behavior with a plot showing the

| $E$(MeV) | $0^+_1$ | $0^+_2$ | $0^+_1$ | $0^+_2$ | $0^+_1$ | $0^+_2$ |
|----------|----------|----------|----------|----------|----------|----------|
|          | $-89.52$ | $-81.79$ | $-89.4$  | $-84.6$  | $-89.4$  | $-84.6$  |
| $R_{r.m.s.}$(fm) | $0^+_1$ | $2.40$ | $2.40$ | $2.44$ | $0^+_2$ | $3.83$ | $3.47$ |
|          | $M(0^+_2 \rightarrow 0^+_1)$(fm$^2$) | $6.45$ | $6.7$ | $5.4$ |

Table 1. Comparison of the binding energies, rms radii ($R_{r.m.s.}$), and monopole matrix elements ($M(0^+_2 \rightarrow 0^+_1)$) for $^{12}\text{C}$ given by solving Hill-Wheeler equation based on Eq. (8) and by Ref. [17]. The effective two-nucleon force Volkov No. 2 was adopted in the two cases for which the $3\alpha$ threshold energy is calculated to be $-82.04$ MeV.
variation of the height of the first maximum of the inelastic form factor as a function of the percentage change of the rms radius of the Hoyle state [23]. It can be seen that a 20 percent increase of the rms radius produces a remarkable decrease of the maximum – by a factor of two! This strong sensitivity of the magnitude of the form factor to the size of the Hoyle state enhances our firm belief that the agreement with the actual measurement is tantamount to a proof that the calculated wide extension of the Hoyle state corresponds to reality.

We thus advocate and support the view that the Hoyle state can be regarded as the ground state of an $\alpha$-particle condensate [29]. We should, however, be aware of the fact that it is not an ideal condensate and that the $\alpha$’s occupy the lowest S-state only with 70 percent, as already discussed. The major part of the correlations comes from th Pauli principle and intermediate 8Be formation. It is by the way not clear whether a gas of $\alpha$ particles condenses as such or as ‘molecules’ of 8Be. The latter are also bosons, of course. We furthermore performed a deformed calculation to investigate the structure of the $2^+_2$ state in $^{12}$C. The state came at the right energy. Our analysis showed that this state essentially corresponds to exciting one $\alpha$-particle out of the condensate and putting it into the $0D$ orbit. Without going into details, we also affirm that the width of this state is correctly reproduced [30]. It should also be mentioned that this $2^+_2$ state has in our calculations an enormous extension with an rms radius of 4.3 fm what corresponds approximately to eight times the ground state volume of $^{12}$C or also to the size of $^{40}$Ca. As a matter of fact the properties of this state have been subject of a vivid debate among the experimentalists in the recent past. The situation seems clarified now [31][32][33]. It is for

Figure 4. Experimental values of inelastic form factor in $^{12}$C to the Hoyle state are compared with our values and those given by Kamimura et al. in Ref. [17] (RGM). In our result, the Hoyle-state wave function is obtained by solving the Hill-Wheeler equation based on Eq. (8).

Figure 5. The ratio of the value of the maximum height, theory versus experiment, of the inelastic form factor, i.e. $\frac{\text{max} |F(q)|^2}{\text{max} |F(q)|_{\text{exp}}^2}$, is plotted as a function of $\delta = (R_{r.m.s.} - R_0) / R_0$. Here $R_{r.m.s.}$ and $R_0$ are the rms radii corresponding, respectively, to the wave function of Eq. (8) and that obtained by solving the Hill-Wheeler equation based on Eq. (8).
instance, the very nice experiment by Moshe Gai [33] which seems to confirm the properties of the \(2^+_2\) state beyond any doubt. In that reference it also is given an estimate of the radius which agrees with our value. One can talk about an \(\alpha\) halo state.

It is tempting to imagine that the \(0^+_1\) state which – experimentally – is almost degenerate with the \(2^+_2\) state, is obtained by lifting one \(\alpha\)-particle into the \(1S\) orbit. Initial theoretical studies [34] indicate that this scenario might indeed apply. However, the width of the \(0^+_3\) state (\(~3\) MeV) is very broad, rendering a theoretical treatment rather delicate. Further investigations are necessary to validate or reject this picture. At any rate, it would be quite satisfying if the triplet of states \((0^+_1, 2^+_2, 0^+_3)\) could all be explained from the \(\alpha\)-particle perspective, since those three states are precisely the ones which cannot be reproduced within a (no core) shell-model approach [16].

Summarizing our inquiry into the possible role of \(\alpha\) clustering in \(^{12}\text{C}\), we have accumulated enough facts to be convinced that the Hoyle state is, indeed, what one may call to first approximation an \(\alpha\)-particle condensate state. At the same time, we acknowledge that referring to only three particles as a “condensate” constitutes a certain abuse of the word. However, in this regard it should be remembered that also in the case of nuclear Cooper pairing, only a few pairs are sufficient to obtain clear signatures of superfluidity in nuclei!

What about \(\alpha\)-particle condensation in heavier nuclei? Once one accepts the idea that the Hoyle state is essentially a state of three free \(\alpha\)-particles held together only by the Coulomb barrier, it is hard to see why analogous states would not also exist in heavier \(n\alpha\) nuclei like \(^{16}\text{O}, \, ^{20}\text{Ne}, \, ^{24}\text{Mg}, \, \text{etc.}\) In fact, our calculations on such nuclei systematically yield a \(0^+\)-state close to the \(\alpha\)-particle disintegration threshold. For example in \(^{16}\text{O}\) we obtain three \(0^+\)-states [22]: the ground state at \(E_0 = -124.8\) MeV (experimental value: \(127.62\) MeV), a second state at excitation energy \(E_{0^+_2} = 8.8\) MeV, and a third one at \(E_{0^+_3} = 14.1\) MeV. The threshold in \(^{16}\text{O}\) is at \(14.4\) MeV. Unfortunately, the relevant experimental information in \(^{18}\text{O}\) is not nearly so complete as in \(^{12}\text{C}\). In particular, no measurements are available for transition probabilities of \(0^+\)-states near the threshold or for inelastic form factors.

In contrast to the situation for \(^{12}\text{C}\), the THSR wave function is certainly not able to describe the structure of all \(0^+\)-states in \(^{16}\text{O}\) lying below the disintegration threshold. In \(^{12}\text{C}\) knocking loose one \(\alpha\) particle, the other two are also loosely bound, since what remains is \(^{8}\text{Be}\). However, in \(^{16}\text{O}\) this is not the case. Before reaching a four \(\alpha\) particle gas state, there will appear configurations where one \(\alpha\) particle orbits around a \(^{12}\text{C}\) core in its ground state or in excited states of particle- hole type. A case in point is the first excited state in \(^{16}\text{O}\), i.e., the \(0^+_2\)-state at \(6.06\) MeV, which is believed to have a structure corresponding to an \(\alpha\)-particle orbiting in an \(S\) wave around a \(^{12}\text{C}\) core in its ground state. Such a configuration is clearly missing from our wave function (8). As a matter of fact a calculation for the first six \(0^+\) states has been performed in the mean while employing a somewhat different, not completely microscopic approach [35]. This method is the so-called ‘orthogonality condition model (OCM)’ which generally works also quite well for the description of cluster states. The novelty with respect to former calculations of \(^{16}\text{O}\) states with this method was that the configuration space was strongly enlarged. In Fig. 6, we show the comparison of the calculated with the experimental sprectrum. In view of the fact that the states are complicated cluster states, the agreement between experiment and theory is very good. The interpretation goes as follows: the first four excited \(0^+\) states have a \(^{12}\text{C} + \alpha\) structure where the \(\alpha\) orbits in \(0S, \, 0D, \, 1S\) waves around the ground state core of \(^{12}\text{C}\) and also in a \(0P\) wave orbiting around the first \(1^-\) of \(^{12}\text{C}\). It is only the first state above the four \(\alpha\) particle threshold at \(15.1\) MeV which is interpreted as an \(\alpha\) gas state or a condensate. This state has, indeed, some analogies with the Hoyle state: it is just some hundreds of keV above threshold, it is strongly excited by inelastic electron scattering what means that monopole transition is quite large. Unfortunately the inelastic form factor has not been measured so far.
One interesting question that can be asked at this point is: How many $\alpha$'s can maximally exist in a self-bound $\alpha$-gas state? Seeking an answer, we performed a schematic investigation using an effective $\alpha$-$\alpha$ interaction of the Ali-Bodmer form \cite{36} within an $\alpha$-gas mean-field calculation of the Gross-Pitaevskii type \cite{37}. The parameters of the force were slightly adjusted to reproduce our microscopic results for $^{12}$C. The corresponding $\alpha$ mean-field potential is shown in Fig. 4. One sees the $0S$-state lying slightly above threshold but below the Coulomb barrier. As more $\alpha$-particles are added, the Coulomb repulsion drives the loosely bound system of $\alpha$-particles farther and farther apart, so that the Coulomb barrier fades away. According to our estimate \cite{38}, a maximum of eight to ten $\alpha$-particles can be held together in a condensate. However, there may be ways to lend additional stability to such systems. We know that in the case of $^8$Be, adding one or two neutrons produces extra binding without seriously disturbing the pronounced $\alpha$-cluster structure. Therefore, one has reason to speculate that adding a few of neutrons to a many-$\alpha$ state may stabilize the condensate. But again, state-of-the-art microscopic investigations are necessary before anything definite can be said about how extra neutrons will...
influence an $\alpha$-particle condensate.

Another interesting idea concerning $\alpha$-particle condensates was put forward by von Oertzen and collaborators [39, 40]. Adding more and more $\alpha$-particles to the, e.g., $^{40}$Ca core, one will arrive sooner or later at the point of $\alpha$-particle drip. Therefore minimal further excitation may be sufficient to shake loose some $\alpha$-particles, so that an $n\alpha$-condensate could be created on top of an inert $^{40}$Ca core. Similar ideas also have been advanced by Ogloblin [41], who envisions a three-$\alpha$-particle condensate on top of $^{106}$Sn, and earlier by Brenner and Gridnev, who have presented evidence of experimental detection of gaseous $\alpha$-particles in $^{28}$Si and $^{32}$S on top of an inert $^{16}$O core [42].

5. Conclusions, Discussion, Outlook

We have investigated the role that pairing and multiparticle correlations may play in nuclear matter existing in dense astrophysical objects and in finite nuclei. A complete and quantitative description of nuclear matter must allow for the presence of clusters of nucleons, bound or metastable, possibly forming a quantum condensate. In particular, quartetting correlations, responsible for the emergence of $\alpha$-like clusters, are identified as uniquely important in determining the behavior of nuclear matter in the limiting regime of low density and low temperature. We have calculated the transition temperature for the onset of quantum condensates made up of $\alpha$-like and deuteron-like bosonic clusters, and considered in considerable detail the intriguing example of Bose-Einstein condensation of $\alpha$ particles. It turns out that contrary to pairing, quartet condensation primarily exists in the BEC phase at low density. In which way quartet condensation is lost by increasing the density is still an open question. It may be similar to a liquid-gas phase transition. Anyway, it is clear that there can not exist a condensate of quartets with a long coherence length for arbitrarily small attraction as this is the case for pairing in the BCS phase. It is inevitable that under increasing density or pressure, the bound $\alpha$, $d$, or other nuclidic clusters present at low density experience significant modification due to the background medium (and eventually merge with it). We have shown how self-energy corrections and Pauli blocking alter the properties of cluster states, and we have formulated a cluster mean-field approximation to provide an initial description of this process. One result of special interest is the suppression of the $\alpha$-like condensate, which is dominant at lower densities, as the density reaches and exceeds the Mott value, allowing the pairing transition to occur. Even at lower densities $\alpha$-particle condensation may be influenced by neutron excess, i.e. in the case of asymmetric nuclear matter, see Fig. 3. A genuine theory for the quartet order parameter in homogeneous infinite matter, similar to BCS theory, is demanded. First results in this direction have been published [11].

A truly remarkable manifestation of $\alpha$-particle condensation seems to be present in finite nuclei. Indeed, the so-called Hoyle state ($0_{2}^{+}$) in $^{12}$C at 7.654 MeV is very likely a dilute gas of three $\alpha$-particles, held together only by the Coulomb barrier. This view is encouraged by the fact that we can explain all the experimental data in terms of a conceptually simple wave function of the quartet-condensate type. Within the same model, we also systematically predict such states in heavier $n\alpha$ nuclei, and the search is on for their experimental identification. With the more phenomenological OCM method, we found that in $^{16}$O the sixth $0^{+}$ state at 15.1 MeV should be the candidate for an $\alpha$ condensed state. It is quite natural that such states should exist up to some maximum number of $\alpha$ particles. We estimate that the phenomenon will terminate at about eight to ten $\alpha$’s as the confining Coulomb barrier fades away. However, there is the possibility that larger condensates could be stabilized by addition of a few neutrons. Indeed, consider $^{9}$Be, which, contrary to $^{8}$Be, is bound by $\sim 1.5$ MeV, still showing a pronounced two $\alpha$-structure similar to the one of Fig. 3 (b). One could imagine ten $\alpha$’s or more, stabilised by two or four extra neutrons in a low density phase. However, even without being stabilised, if a compressed hot nuclear blob as e.g. produced in a central Heavy Ion collision expands and
cools, it may turn on its way out, at a certain low density, into an expanding α condensed state where all α’s are in relative S-waves. One may also, in one way or the other (photons?) excite, e.g., $^{40}$Ca to the 10 α threshold at about 60 MeV where a slow Coulomb explosion of an α gas would then take place. This would be an analogous situation to an expanding Bose condensate of atoms, once the trapping potential has been switched off. Future dedicated experiments with high resolution multiparticle detectors will tell whether such scenarios can be realised. Intriguing news in this respect come from GANIL where one may have achieved the disintegration of $^{56}$Ni into 14 α’s (seven α’s have been detected with high yield, the other seven may not have been seen because of the detectors were not sensitive to very low energy α particles [43]). Other possibilities of loose α-gas states may exist on top of particularly stable cores, like $^{16}$O or $^{40}$Ca. Indeed in adding α’s to e.g. $^{40}$Ca, one will reach the α-particle drip line. Compound states of heavy N = Z nuclei of this kind may be produced in heavy ion reactions and an enhanced α-decay rate may reveal the existence of an α-particle condensate. Ideas of this type are presently promoted by von Oertzen [40], and also M. Brenner [42], and A. Ogloblin [41]. However, coincidence measurements of multiple α’s of decaying lighter nuclei like $^{16}$O may also be very useful [44][45] [46] to detect at least one additional α- condensate state beyond the only one that has been identified so far, namely the $^{0}_{2}^{+}$-state in $^{12}$C.

Another issue which may be raised in the context of α-particle condensation is the question, also discussed in condensed matter physics [47], whether α’s condense as singles or as doubles, i.e. as $^{8}$Be. In microscopic studies of $^{12}$C one, indeed, can see that in the $^{0}_{2}^{+}$-state two of the three α’s are slightly more closer to one another than to the third one [48]. The question is definitely very interesting and deserves future investigation. However, quantitatively, it constitutes probably only a slight modification over the present formulation of α-condensation.

What about ‘ab initio’ calculations of the Hoyle state? Presently several groups are on the track [49][50]. The Los Alamos group has achieved to calculate the density of the Hoyle state, though it is not yet converged in the far tail [50]. It agrees very well with the density from the THSR wave function besides in the far tail.

In conclusion, we see that the idea of α-particle condensation in nuclei and nuclear systems has already triggered many new ideas, calculations, and experiments, in spite of the fact that, so far, a compelling case for such a state has only been made in $^{12}$C. Even so, the possible existence of a completely new nuclear phase in which α-particles play the role of quasi-elementary constituents is surely fascinating. Hopefully, many more α-particle states of nuclei will be detected in the near future, bringing deeper insights into the role of clustering and quantum condensates in systems of strongly interacting fermions.

Let us mention in the end that a more elaborate report on α particle condensation can be found in [51]

6. Acknowledgements
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The fact that this condensate state is actually an excited state of $^{12}$C does not invalidate this picture, since, on nuclear scales, this state is long lived. Actually, also atomic Bose condensates in traps find themselves in meta-stable states, see e.g. Ref. [13].