Extensional proofs
in a propositional logic modulo isomorphisms*

Alejandro Díaz-Caro\textsuperscript{a,b}, Gilles Dowek\textsuperscript{c}

\textsuperscript{a}ICC, CONICET–Universidad de Buenos Aires, Argentina
\textsuperscript{b}Departamento de Ciencia y Tecnología, Universidad Nacional de Quilmes, Argentina
\textsuperscript{c}Inria, LMF, ENS Paris-Saclay, Gif-sur-Yvette, France

Abstract
System I is a proof language for a fragment of propositional logic where isomorph-
phic propositions, such as $A \land B$ and $B \land A$, or $A \Rightarrow (B \land C)$ and $(A \Rightarrow B) \land (A \Rightarrow C)$ are made equal. System I enjoys the strong normalization property. This is
sufficient to prove the existence of empty types, but not to prove the introduc-
tion property (every closed term in normal form is an introduction). Moreover,
a severe restriction had to be made on the types of the variables in order to ob-
tain the existence of empty types. We show here that adding $\eta$-expansion rules
to System I permits to drop this restriction, and yields a strongly normalizing
calculus with enjoying the introduction property.

Keywords: Simply typed lambda calculus, Isomorphisms, Logic,
Cut-elimination, Proof-reduction, Eta-expansion, Strong normalization

2020 MSC: 03F05, 03B40, 03B38

1. Introduction
1.1. Making connectives algebraic

Operations in mathematics are often associative, commutative, equipped
with a neutral element, etc. In contrast, the logical connectives have no algebraic
properties. Of course, if the proposition $A \land B$ has a proof, then so does the
proposition $B \land A$, but if $r$ is a proof of $A \land B$, then it is not a proof of $B \land A$.
Thus, if we consider two propositions equal when they have the same proofs,
the propositions $A \land B$ and $B \land A$ are different. This lack of algebraic properties
of the connectives distinguishes logic, among the mathematical theories.

Our project is to bring logic closer to algebra, by making, for example, the
conjunction commutative, that is the propositions $A \land B$ and $B \land A$ equal.
This extends the project of Martin-Löf’s type theory \cite{24}, the Calculus of Con-
structions \cite{8}, Deduction modulo theory \cite{19, 20}, etc. that makes definitionally
equivalent propositions equal.

*Partially funded by PIP 11220200100368CO, PICT 2019-1272, STIC-AmSud 21STIC10 Qapla’, ECOS-Sud A17C03 QuCa, and the French-Argentinian IRP SINFIN.
The propositions $A \land B$ and $B \land A$ already are equal in some models: in Boolean algebras, or in Heyting algebras, conjunction and disjunction are associative and commutative, they are distributive one over the other, etc. Thus, they are genuine algebraic operations. In categorical models, the Cartesian product is associative and commutative, but only modulo isomorphisms. And, on the syntactic side, conjunction and disjunction are neither associative nor commutative.

Our long term objective is to understand how identifying some propositions impacts proof-theory: how must the notion of proof-reduction, that of reducibility candidate, etc. be modified. To explore such questions, we start with a simple case: constructive propositional logic with implication and conjunction.

1.2. Logical isomorphism

The first step in such a project is to understand which propositions can be identified. An answer to this question is given by the notion of logical isomorphism. Two propositions $C$ and $D$ are said to be isomorphic when there exist proofs of $C \Rightarrow D$ and $D \Rightarrow C$, whose composition, in both ways, is semantically equivalent to the identity. For instance, the propositions $A \land B$ and $B \land A$ are isomorphic.

This notion of isomorphism has been studied by M. Rittti [30], who has shown that identifying isomorphic propositions simplified the search for a lemma in a database of mathematical results. Then, such isomorphisms, for different constructive systems, have been characterized by K. Bruce, G. Longo, and R. Di Cosmo [6, 11, 12]. O. Laurent has then extended this characterization to classical logic [23].

In the case of constructive propositional logic with implication and conjunction four isomorphisms can be considered.

\[
\begin{align*}
A \land B &\equiv B \land A \\
A \land (B \land C) &\equiv (A \land B) \land C \\
A \Rightarrow (B \land C) &\equiv (A \Rightarrow B) \land (A \Rightarrow C) \\
(A \land B) \Rightarrow C &\equiv A \Rightarrow B \Rightarrow C
\end{align*}
\]

1.3. Non-deterministic proof-reduction

Another question that arises in such a project is that of the determinism of proof-reduction. The first models of computations: Turing machines, $\lambda$-calculus, etc. were often deterministic. But, quickly, some non-deterministic variants were introduced. This non-determinism then became essential with the rise of quantum computing and asynchronous parallel computing [5, 7, 9, 10, 25].

In proof-languages, in contrast, the reduction is still deterministic. Yet, there are several situations where non-determinism is natural. For example, if we diagonalize the conjunction, introducing a unary connective $\hat{\land}$ such that $\hat{\land}A = A \land A$, then the introduction rule of the conjunction becomes

\[
\frac{A}{\hat{\land}A} \land_i
\]
and its first elimination rule

\[ \frac{\hat{\lambda} A}{A} \ \hat{\lambda}-e1 \]

Then, the proof

\[ \frac{\pi_1 \ \pi_2}{A} \ \frac{\hat{\lambda} A}{A} \ \frac{A}{A} \ \hat{\lambda}-i \]

reduces to \( \pi_1 \). But, thanks to the diagonalization, it can also be reduced to \( \pi_2 \). As we shall see, making conjunction commutative introduces non-determinism in a similar way.

1.4. System I

System I [15] is a first attempt to identify isomorphic propositions in constructive propositional logic with implication and conjunction. The usual proof-language of this logic is simply typed lambda-calculus with Cartesian product. In this calculus, the term \( \lambda x : A. r \times \lambda x : A. s \), where we write \( u \times v \) for the pair of two terms \( u \) and \( v \), has type \( (A \Rightarrow B) \land (A \Rightarrow C) \). In System I, as \( (A \Rightarrow B) \land (A \Rightarrow C) \equiv A \Rightarrow (B \land C) \), this term also has type \( A \Rightarrow (B \land C) \) and it can be applied to \( t \) of type \( A \), yielding the term \( (\lambda x : A. r \times \lambda x : A. s) t \) of type \( B \land C \). With the usual reduction rules of lambda-calculus with pairs, such a mixed cut (an introduction followed by the elimination of another connective) would be in normal form, but we also extended the reduction relation, with an equation \( (\lambda x : A. r \times \lambda x : A. s) \leftrightarrow \lambda x : A. (r \times s) \), so that this term can be \( \beta \)-reduced, taking inspiration from rules well-known in the area of program transformation, for instance in G. Révész [28, 29], K. Støvring [32], and others.

One of the difficulties in the design of System I was the definition of the elimination rule for the conjunction. We cannot use a rule like “if \( r : A \land B \) then \( \pi_1(r) : A \)”. Indeed, if \( A \) and \( B \) are two arbitrary types, \( s \) a term of type \( A \) and \( t \) a term of type \( B \), then \( s \times t \) has both type \( A \land B \) and type \( B \land A \), thus \( \pi_1(s \times t) \) would have both type \( A \) and type \( B \). The solution is to consider explicitly typed (Church style) terms, and parameterize the projection by the type: if \( r : A \land B \) then \( \pi_A(r) : A \) and the reduction rule is then that \( \pi_A(s \times t) \) reduces to \( s \) if \( s \) has type \( A \). Thus, \( \pi \)-reduction is type driven, and \( \beta \)-reduction as well.

This rule makes reduction non-deterministic. Indeed, in the particular case where \( A \) is equal to \( B \), then both \( s \) and \( t \) have type \( A \) and \( \pi_A(s \times t) \) reduces both to \( s \) and to \( t \). Unlike in the lambda-calculus we cannot specify which reduct we get, but in any case, we eventually get a term in normal form of type \( A \), that is a cut-free proof of \( A \). Therefore, System I is a non-deterministic calculus and our pair-construction operator \( \times \) is also the parallel composition operator of a non-deterministic calculus. More precisely, the non-determinism does not come from one operator, but from the interaction of two operators, \( \times \) and \( \pi \). In this respect, System I is close to quantum and algebraic \( \lambda \)-calculi [1–4, 14, 17, 18, 33] where
the non-determinism comes from the interaction of superposition and projective measurement.

In [16], we have implemented an early version of System I, extended with general recursion. We showed with a couple of examples, how this language can be helpful as a realistic programming language. On the one hand, the language has a sort of partial application, which can start the computation as soon as it receives a parameter, in any order. On the other hand, the language enables to reuse code by projecting functions and discarding unused code prior to its usage. For example, a function calculating the quotient and the rest of two natural numbers can be projected out into a function calculating only the quotient, discarding the code calculating the rest. Even in the case of general recursion and mutual recursion, the language will unfold the recursion as needed to discard the unused code.

1.5. The drawbacks of System I

In [15] we succeeded in proving the strong normalization and the consistency of System I, that is, the existence of a proposition that has no closed proof. However, System I still has some drawbacks.

- As the propositions $A \Rightarrow B \Rightarrow A$ and $B \Rightarrow A \Rightarrow A$ are isomorphic, the term $(\lambda x^A.\lambda y^B.x)r$ where $r$ has type $B$ is well-typed, but it cannot be $\beta$-reduced. In System I, this term is in normal form, so System I does not verify the introduction property (a closed term in normal form is either an abstraction of a pair). Only when such a term is applied to a term $s$ of type $A$, to make a closed term of atomic type, it can be reduced: $(\lambda x^A.\lambda y^B.x)sr$, being equivalent to $(\lambda x^A.\lambda y^B.x)sr$, can be reduced to $(\lambda y^B.s)r$, and then to $s$. A solution has been explored in [16]: “delayed $\beta$-reduction” that reduces $(\lambda x^A.\lambda y^B.x)r$ to $\lambda x^A.((\lambda x^A.\lambda y^B.x)r)$ and then to $\lambda x^A.x$. A similar equivalence has been proposed before in the context of proof-nets [27].

- As the types $(A \land B) \Rightarrow (A \land B)$ and $A \Rightarrow B \Rightarrow (A \land B)$ are isomorphic, the term $(\lambda x^{A\land B}.x)r$ where $r$ has type $A$ is well-typed (of type $B \Rightarrow (A \land B)$), but it cannot be $\beta$-reduced as the term $r$ of type $A$ cannot be substituted for the variable $x$ of type $A \land B$. In System I variables have so called “prime types”, that is, types that do not contain a conjunction at head position. Thus, the above term can only be written as $(\lambda y^B.\lambda z^B.x \times z)r$, and it reduces to $\lambda z^B.x \times z$. Another possibility has been explored in [16]: “partial $\beta$-reduction” that reduces directly $(\lambda x^{A\land B}.x)r$ to $\lambda z^B.x \times z$.

1.6. System I$^\eta$

In this paper, we show these two drawbacks are symptoms of the lack of extensionality in System I. This leads us to introduce the System I$^\eta$ that extends System I with an $\eta$-expansion rule, and a surjective pairing $\delta$-expansion rule.

In System I$^\eta$, the term $(\lambda x^A.\lambda y^B.x)r$ $\eta$-expands to $\lambda x^A.((\lambda x^A.\lambda y^B.x)r)x$, that is equivalent to $\lambda x^A.((\lambda x^A.\lambda y^B.x)r)x$, and reduces to $\lambda x^A.x$. In the same
way, the term \((\lambda x^{A \land B} . x)r \eta\)-expands to \(\lambda y^B . (\lambda x^{A \land B} . x)ry\), that is equivalent to \(\lambda y^B . (\lambda x^{A \land B} . x) (r \times y)\), and reduces to \(\lambda y^B . r \times y\). This way, we do not need to constrain variables to have prime types. Dropping this restriction, makes the mixed cut \(\pi_\tau (\tau \land \tau \Rightarrow \tau (\lambda x^{\tau \land \tau} . x))\) well-typed, since \((A \land B) \Rightarrow C\) is isomorphic to \(A \Rightarrow B \Rightarrow C\) and variables can have any type. However, using the \(\delta\)-rule this term expands to \(\pi_\tau (\tau \land \tau \Rightarrow \tau ((\lambda x^{\tau \land \tau} . \pi_\tau (x)) \times (\lambda y^{\tau \land \tau} . \pi_\tau (x))))\) that is equivalent to \(\pi_\tau (\tau \land \tau \Rightarrow \tau ((\lambda x^{\tau \land \tau} . \pi_\tau (x)) \times (\lambda y^{\tau \land \tau} . \pi_\tau (x))))\), and reduces to \(\lambda x^{\tau \land \tau} . \pi_\tau (x)\) that is an introduction.

Designing System I\(^\eta\) yet led us to make a few choices. For instance, if the terms \(r\) and \(s\) are not introductions, then \((r \times s)t\), where \(t\) has type \(A\), \(\eta\)-expands to \((\lambda x^A . (r x) \times \lambda x^A . (s x))t\), that is equivalent to \(\lambda x^A . ((r x) \times (s x))t\) and \(\beta\)-reduces to \((rt) \times (st)\). But, if one of them is an abstraction on a type different from \(A\), then the term cannot be reduced. For instance \(((\lambda x^{\tau \Rightarrow \tau} . \lambda y^{\tau} . x) \times (\lambda y^{\tau} . y))t\), where \(t\) is a term of type \(\tau\), cannot be reduced. So we could either introduce a symmetric rule to commute the two abstractions or introduce a distributivity rule transforming the elimination \(((\lambda x^{\tau \Rightarrow \tau} . \lambda y^{\tau} . x) \times (\lambda y^{\tau} . y))t\) into the introduction \(((\lambda x^{\tau \Rightarrow \tau} . \lambda y^{\tau} . x)y \times (\lambda y^{\tau} . y)t\). We have chosen the second option, as we favoured reduction over equivalence. But both choices make sense.

Our main results are the normalization proof of System I\(^\eta\), developing ideas from [15, 22] and the introduction property, showing that System I\(^\eta\) solves the problems of System I.

2. Type isomorphisms

We first define the types and their equivalence, and state properties on this relation. Some of these properties have been proved in [15], and others are new.

2.1. Types and isomorphisms

Types are defined by the following grammar, where \(\tau\) is the only atomic type, \(\Rightarrow\) is the constructor of the type of functions, and \(\land\) is the constructor of the type for pairs.

\[
A = \tau \mid A \Rightarrow A \mid A \land A
\]

**Definition 2.1 (Size of a type).** The size of a type is defined as usual by

\[
s(\tau) = 1
s(A \Rightarrow B) = s(A) + s(B) + 1
s(A \land B) = s(A) + s(B) + 1
\]

**Definition 2.2 (Type equivalence).** The equivalence between types is the smallest congruence such that:

\[
A \land B \equiv B \land A
A \land (B \land C) \equiv (A \land B) \land C
A \Rightarrow (B \land C) \equiv (A \Rightarrow B) \land (A \Rightarrow C)
(A \land B) \Rightarrow C \equiv A \Rightarrow B \Rightarrow C
\]

5
Lemma 2.6. For all \( A \equiv B \), then \( \text{PF}(A) \sim \text{PF}(B) \).

Proof. First we check that \( \text{PF}(A \land B) \sim \text{PF}(B \land A) \) and similar for the other three isomorphisms. Then we prove by structural induction that if \( A \) and \( B \) are equivalent in one step, then \( \text{PF}(A) \sim \text{PF}(B) \). We conclude by an induction on the length of the derivation of the equivalence \( A \equiv B \).

Remark 2.3. This equivalence relation is decidable [11, Theorem 6.4.5] as these equivalences can be oriented as rewrite rules yielding a normal form modulo associativity and commutativity. We also have defined a notion of canonical form in [16] to implement an earlier version of System I. However, as different orientations can be chosen for distributivity and curryfication we prefer, in this theoretical presentation, to give the typing rules for the equivalence relation and not for a specific choice of a canonical form.

2.2. Prime factors

We recall a lemma proved in [15] stating that any type is equivalent to a conjunction of prime types [15].

This transformation of a type into a conjunction of prime types can be compared to the transformation of a proposition as a conjunction of clauses, except that we use the equivalence \( \equiv \) and not logical equivalence.

Definition 2.4 (Prime types). A prime type is a type of the form \( C_1 \Rightarrow \cdots \Rightarrow C_n \Rightarrow \tau \), with \( n \geq 0 \).

A prime type is equivalent to \( (C_1 \land \cdots \land C_n) \Rightarrow \tau \), which is either equivalent to \( \tau \) or to \( C \Rightarrow \tau \), for some \( C \). For uniformity, we may write \( \emptyset \Rightarrow \tau \) for \( \tau \).

We prove that each type can be decomposed into a conjunction of prime types.

Definition 2.5 (Prime factors). The multiset of prime factors of a type \( A \) is inductively defined as follows, with the convention that \( A \land \emptyset = A \).

\[
\text{PF}(\tau) = [\tau]
\]
\[
\text{PF}(A \Rightarrow B) = [(A \land B_i) \Rightarrow \tau]_{i=1}^n \quad \text{where } [B_i \Rightarrow \tau]_{i=1}^n = \text{PF}(B)
\]
\[
\text{PF}(A \land B) = \text{PF}(A) \uplus \text{PF}(B)
\]

Lemma 2.6. For all \( A \), \( A \equiv \text{conj}(\text{PF}(A)) \).

Proof. By induction on \( s(A) \).

- If \( A = \tau \), then \( \text{PF}(\tau) = [\tau] \), and so \( \text{conj}(\text{PF}(\tau)) = \tau \).

- If \( A = B \Rightarrow C \), then \( \text{PF}(A) = [(B \land C_i) \Rightarrow \tau]_i \), where \( [C_i \Rightarrow \tau]_i = \text{PF}(C) \).

By the i.h., \( C \equiv \bigwedge_i (C_i \Rightarrow \tau) \), hence, \( A = B \Rightarrow C \equiv B \Rightarrow \bigwedge_i (C_i \Rightarrow \tau) \equiv \bigwedge_i (B \Rightarrow C_i \Rightarrow \tau) \).

- If \( A = B \land C \), then \( \text{PF}(A) = \text{PF}(B) \uplus \text{PF}(C) \).

By the i.h., \( B \equiv \text{conj}(\text{PF}(B)) \), and \( C \equiv \text{conj}(\text{PF}(C)) \). Therefore, \( A = B \land C \equiv \text{conj}(\text{PF}(B) \land \text{conj}(\text{PF}(C)) \equiv \text{conj}(\text{PF}(B \land C)) \equiv \text{conj}(\text{PF}(B) \uplus \text{PF}(C)) = \text{conj}(\text{PF}(A)) \).
2.3. Measure of types

The size of a type is not preserved by equivalence. For instance, \((\tau \Rightarrow \tau) \equiv (\tau \Rightarrow \tau) \land (\tau \Rightarrow \tau)\), but \(s(\tau \Rightarrow (\tau \land \tau)) = 5\) and \(s((\tau \Rightarrow (\tau \land \tau)) \land (\tau \Rightarrow \tau)) = 7\). Thus, we define another notion of measure of a type, conforming the usual relation.

**Definition 2.8 (Measure of a type).** The measure of a type is defined as follows

\[
m(A) = \sum_i (m(C_i) + 1) \quad \text{where } [C_i \Rightarrow \tau] = PF(A)
\]

with the convention that \(m(\emptyset) = 0\).

The following lemma states that the given measure conforms the usual relation.

**Lemma 2.9.**

1. \(m(A \land B) > m(A)\)
2. \(m(A \Rightarrow B) > m(A)\)
3. \(m(A \Rightarrow B) > m(B)\)
4. if \(A \equiv B\), \(m(A) = m(B)\)

**Proof.**

1. \(PF(A)\) is a strict submultiset of \(PF(A \land B)\).
2. Let \(PF(B) = [C_i \Rightarrow \tau]_{i=1}^n\). Then, \(PF(A \Rightarrow B) = [(A \land C_i) \Rightarrow \tau]_{i=1}^n\). Hence,
   \[
m(A \Rightarrow B) \geq m(A \land C_1) + 1 > m(A \land C_1) \geq m(A).
\]
3. \(m(A \Rightarrow B) = \sum_i m(A \land C_i) + 1 > \sum_i m(C_i) + 1 = m(B)\).
4. By induction on \(s(A)\). Let \(PF(A) = [C_i \Rightarrow \tau]_i\) and \(PF(B) = [D_j \Rightarrow \tau]_j\).
   By Lemma 2.7, \([C_i \Rightarrow \tau]_i \sim [D_i \Rightarrow \tau]_i\). Without loss of generality, take \(C_i \equiv D_i\). By the induction hypothesis, \(m(C_i) = m(D_i)\). Then,
   \[
m(A) = \sum_i (m(C_i) + 1) = \sum_i (m(D_i) + 1) = m(B).
\]

2.4. Decomposition properties on types

In simply typed lambda calculus, the implication and the conjunction are constructors, that is \(A \Rightarrow B\) is never equal to \(C \land D\), if \(A \Rightarrow B = A' \Rightarrow B'\), then \(A = A'\) and \(B = B'\), and the same holds for the conjunction. This is not the case in System \(\Pi^0\), where \(\tau \Rightarrow (\tau \land \tau) \equiv (\tau \Rightarrow \tau) \land (\tau \Rightarrow \tau)\), but the connectors still have some coherence properties, which are proved in this section.

**Lemma 2.10.** If \(A \Rightarrow B \equiv C_1 \land C_2\), then \(C_1 \equiv A \Rightarrow B_1\) and \(C_2 \equiv A \Rightarrow B_2\) where \(B \equiv B_1 \land B_2\).

**Proof.** By Lemma 2.7, \(PF(A \Rightarrow B) \sim PF(C_1 \land C_2) = PF(C_1) \uplus PF(C_2)\). Let \(PF(B) = [D_i \Rightarrow \tau]_{i=1}^n\), so \(PF(A \Rightarrow B) = [(A \land D_i) \Rightarrow \tau]_{i=1}^n\). Without lost of generality, take \(PF(C_1) \sim [(A \land D_i) \Rightarrow \tau]_{i=1}^n\) and \(PF(C_2) \sim [(A \land D_i) \Rightarrow \tau]_{i=k+1}^n\). Therefore, by Lemma 2.6, we have \(A \Rightarrow B \equiv \bigwedge_{i=1}^k ((A \land D_i) \Rightarrow \tau) \land \bigwedge_{i=k+1}^n ((A \land D_i) \Rightarrow \tau) \equiv (A \Rightarrow \bigwedge_{i=1}^k (D_i \Rightarrow \tau)) \land (A \Rightarrow \bigwedge_{i=k+1}^n (D_i \Rightarrow \tau))\). Take \(B_1 = \bigwedge_{i=1}^k D_i \Rightarrow \tau\) and \(B_2 = \bigwedge_{i=k+1}^n D_i \Rightarrow \tau\). Remark that \(C_1 \equiv A \Rightarrow B_1\), \(C_2 \equiv A \Rightarrow B_2\) and \(B \equiv B_1 \land B_2\).
Lemma 2.11. If \( A \land B \equiv C \land D \) then one of the following cases happens

1. \( A \equiv C_1 \land D_1 \) and \( B \equiv C_2 \land D_2 \), with \( C \equiv C_1 \land C_2 \) and \( D \equiv D_1 \land D_2 \).
2. \( B \equiv C \land D_2 \), with \( D \equiv A \land D_2 \).
3. \( B \equiv C_2 \land D \), with \( C \equiv A \land C_2 \).
4. \( A \equiv C \land D_1 \), with \( D \equiv D_1 \land D \).
5. \( A \equiv C_1 \land D \), with \( C \equiv C_1 \land B \).
6. \( A \equiv C \) and \( B \equiv D \).
7. \( A \equiv D \) and \( B \equiv C \).

Proof. Let \( \text{PF}(A) = R \), \( \text{PF}(B) = S \), \( \text{PF}(C) = T \), and \( \text{PF}(D) = U \). By Lemma 2.7, we have \( R \uplus S \sim T \uplus U \). We prove first that there exist four multisets \( V \), \( W \), \( X \), and \( Y \) such that \( R = V \uplus X \), \( S = W \uplus Y \), \( T = V \uplus W \), and \( U = X \uplus Y \). Notice that \( V \) and \( X \) cannot be both empty, \( W \) and \( Y \) cannot be both empty, \( V \) and \( W \) cannot be both empty, and \( X \) and \( Y \) cannot be both empty.

We have \( T \uplus (S \uplus U) = (T \uplus S) \cap (T \uplus U) \sim (T \uplus S) \cap (R \uplus S) = (T \cap R) \uplus S \). Thus, \( T \setminus (T \cap R) \sim S \setminus (S \cap U) \). In the same way, \( R \setminus (R \cap T) \sim U \setminus (S \cap U) \).

We take \( V = R \setminus T \), \( Y = S \setminus U \), \( W = T \setminus V \setminus U \), \( X = R \setminus V \setminus U \setminus Y \).

Now, if \( V, W, X, Y \) are all non-empty, we let \( C_1 = \text{conj}(V) \), \( C_2 = \text{conj}(W) \), \( D_1 = \text{conj}(X) \), and \( D_2 = \text{conj}(Y) \), and we are in the first case.

If \( V \) is empty and the others are not, then we have \( T = W \), \( R = X \), so \( A = \text{conj}(X) \) and \( C = \text{conj}(W) \). We let \( D_2 = \text{conj}(Y) \), hence we are in the second case.

The cases where \( W, X, \) or \( Y \) are empty, but the others are not, are symmetric.

Finally, if \( X \) and \( W \) are both empty, then \( A \equiv C \) and \( B \equiv D \), and we are in the case 6. If \( V \) and \( Y \) are both empty, then \( A \equiv D \) and \( B \equiv C \), and we are in case 7.

Lemma 2.12. If \( A \Rightarrow B \equiv C \Rightarrow \tau \), then either \( (A \equiv C \text{ and } B \equiv \tau) \), or \( (C \equiv A \land B' \text{ and } B \equiv B' \Rightarrow \tau) \).

Proof. By Lemma 2.7, \( \text{PF}(A \Rightarrow B) \sim \text{PF}(C \Rightarrow \tau) = [C \Rightarrow \tau]_i \). Let \( \text{PF}(B) = [B \Rightarrow \tau]_i \). Then \( \text{PF}(A \Rightarrow B) = [(A \land B_i) \Rightarrow \tau]_i \). Therefore, \( n = 1 \) and \( A \land B_1 \equiv C \). If \( B_1 = \emptyset \), then \( A \equiv C \) and \( B \equiv \tau \). If \( B_1 \neq \emptyset \), then \( A \land B_1 \equiv C \) and \( B \equiv B_1 \Rightarrow \tau \).

Lemma 2.13. If \( A \land B \equiv A \land C \), then \( B \equiv C \).

Proof. By Lemma 2.7, \( \text{PF}(A \land B) = \text{PF}(A) \uplus \text{PF}(B) \sim \text{PF}(A) \uplus \text{PF}(C) = \text{PF}(A \land C) \). Then \( \text{PF}(B) \sim \text{PF}(C) \), and so, by Lemma 2.6, \( B \equiv C \).

Lemma 2.14. If \( A \Rightarrow B \equiv A \Rightarrow C \), then \( B \equiv C \).

Proof. Let \( \text{PF}(A \Rightarrow B) = [(A \land B_i) \Rightarrow \tau]_i \), with \( [B_i \Rightarrow \tau]_i \) \( = \text{PF}(B) \), and \( \text{PF}(A \Rightarrow C) = [(A \land C_i) \Rightarrow \tau']_i \), with \( [C_i \Rightarrow \tau']_i \) \( = \text{PF}(C) \). By Lemma 2.7, \( n = m \) and, without lost of generality, we can consider that \( (A \land B_i) \Rightarrow \tau \equiv \tau \).
Table 1: The type system.

\[
\begin{array}{c}
| x \in V_A | x : A \quad (ax) & | A \equiv B | r : A \quad (\equiv) & | r : B |
\end{array}
\]

\[
\begin{array}{c}
r : A \Rightarrow B \quad s : A \quad (\Rightarrow) & r \times s : A \wedge B \quad (\wedge) & r : A \wedge B \quad \pi_A(r) : A \quad (\wedge)
\end{array}
\]

(\(A \wedge C_i\) \(\Rightarrow\) \(\tau\)). Then, by Lemma 2.12, \(A \wedge B_i \equiv A \wedge C_i\), so, by Lemma 2.13, \(B_i \equiv C_i\). Therefore, by Lemma 2.6, \(B \equiv (B_1 \Rightarrow \tau) \wedge \cdots \wedge (B_n \Rightarrow \tau) \equiv (C_1 \Rightarrow \tau) \wedge \cdots \wedge (C_n \Rightarrow \tau) \equiv C\).

3. The System \(\eta\)

3.1. Syntax

We associate to each type \(A\) (up to equivalence) an infinite set of variables \(V_A\) such that if \(A \equiv B\) then \(V_A = V_B\) and if \(A \not\equiv B\) then \(V_A \cap V_B = \emptyset\). The set of preterms is defined by

\[r = x \mid \lambda x.A.r \mid rr \mid r \times s \mid \pi_A(r)\]

These terms are called respectively, variables, abstractions, applications, products and projections. An introduction is either an abstraction or a product. An elimination is either an application or a projection. We recall the type on binding occurrences of variables and write \(\lambda x.A.t\) for \(\lambda x.t\) when \(x \in V_A\). The set of free variables of \(r\) is written \(FV(r)\). \(\alpha\)-equivalence and substitution are defined as usual. The type system is given in Table 1. We use a presentation of typing rules without explicit context following [21, 26], hence the typing judgments have the form \(r : A\). The well-typed preterms are called terms.

3.2. Operational semantics

The operational semantics of the calculus is defined by two relations: an equivalence relation, and a reduction relation.

**Definition 3.1.** The symmetric relation \(\simeq\) is the smallest contextually closed relation defined by the rules given in Table 2.

Because of the associativity property of \(\times\), the term \(r \times (s \times t)\) is equivalent to the term \((r \times s) \times t\), so we can just write it \(r \times s \times t\).

The size of a term \(S(r)\), defined, as usual, by \(S(x) = 1\), \(S(\lambda x.A.r) = S(\pi_A(r)) = 1 + S(r)\), \(S(rs) = S(r \times s) = 1 + S(r) + S(s)\), is not invariant through the equivalence \(\simeq\). Hence, we introduce a measure \(M(\cdot)\) (given in Table 3) which relies on a measure \(P(\cdot)\) counting the number of pairs in a term.

**Lemma 3.2.** If \(r \simeq s\) then \(M(r) = M(s)\).
and finally the relations \( \rightsquigarrow \) and \( \rightrightarrows \) cannot neither be terms that are the left part of an application or the body of a projection.

Definition 3.5. This relation \( \Rightarrow \) be read in three steps: first we define the relation \( \Rightarrow \), then the relation \( \Rightarrow \), and finally the relations \( \Rightarrow \) and \( \Rightarrow \) in a mutually dependent way. Like in [22] this relation \( \Rightarrow \) forbids \( \eta \)-expansions and \( \delta \)-expansion and is used to reduce terms that are the left part of an application or the body of a projection.

Since, in System \( \Gamma \), an abstraction can be equivalent to a product, a subterm can neither be \( \eta \)-expanded nor \( \delta \)-expanded, if it is either an abstraction or a product, or if it occurs at left of an application or in the body of a projection [13].

Definition 3.5. We write \( \leadsto \) for the relation \( \Rightarrow \) modulo \( \Rightarrow \) (i.e. \( r \leadsto s \) iff \( r \Rightarrow s \) and \( \delta \)-expanded, if it is either an abstraction or a product, or if it occurs at left of an application or in the body of a projection.

Lemma 3.3. For any term \( r \), the set \( \{ s \mid s \Rightarrow r \} \) is finite (modulo \( \alpha \)-equivalence).

Proof. Let \( F = \text{FS}(r) \) and \( n = M(r) \). We have \( \{ s \mid s \Rightarrow r \} \subseteq \{ s \mid \text{FS}(s) = F \} \). Hence, it is finite.

Table 2: Symmetric relation.

|       |       |
|-------|-------|
| \( P(x) \) | \( M(x) \) |
| \( P(\lambda x^A.r) \) | \( P(r) \) |
| \( P(rs) \) | \( 0 \) |
| \( P(r \times s) \) | \( 1 + P(r) + P(s) \) |
| \( P(\pi_A(r)) \) | \( 0 \) |

Table 3: Measure on terms.

\[
\begin{array}{c|c}
\text{Term} & \text{Measure} \\
\hline
P(x) & 0 \\
P(\lambda x^A.r) & P(r) \\
P(rs) & 0 \\
P(r \times s) & 1 + P(r) + P(s) \\
P(\pi_A(r)) & 0 \\
M(x) & 1 \\
M(\lambda x^A.r) & 1 + M(r) + P(r) \\
M(rs) & 1 + M(r) + M(s) \\
M(r \times s) & 1 + M(r) + M(s) \\
M(\pi_A(r)) & 1 + M(r) \\
\end{array}
\]
Remark 3.6. By Lemma 3.3, a term has a finite number of one-step reducts and these reducts can be computed.

Finally, notice that unlike in System I, the $\zeta$-rule transforming an elimination into an introduction is a reduction rule and not an equivalence rule. Hence, variables, applications, and projections are preserved by $\iff$. In contrast, an abstraction can be equivalent to a product, but, introductions are preserved.

4. Subject Reduction

The set of types assigned to a term is preserved under $\iff$ and $\rightarrow$. Before proving this property, we prove the unicity of types (Lemma 4.1), the generation lemma (Lemma 4.2), and the substitution lemma (Lemma 4.3).

Lemma 4.1 (Unicity). If $r : A$ and $r : B$, then $A \equiv B$.

Proof.

- If the last rule of the derivation of $r : A$ is ($\equiv$), then we have a shorter derivation of $r : C$ with $C \equiv A$, and, by the i.h., $C \equiv B$, hence $A \equiv B$.

- If the last rule of the derivation of $r : B$ is ($\equiv$) we proceed in the same way.

- All the remaining cases are syntax directed. \hfill $\Box$

Lemma 4.2 (Generation).

1. If $x \in V_A$ and $x : B$, then $A \equiv B$.
2. If $\lambda x^A \cdot r : B$, then $B \equiv A \Rightarrow C$ and $r : C$.
3. If $rs : B$, then $r : A \Rightarrow B$ and $s : A$. 

Table 4: Reduction relation.
Proof. Each statement is proved by induction on the typing derivation. For the statement 1, we have \( x \in \mathcal{V}_A \) and \( x : B \). The only way to type this term is either by the rule \((ax)\) or \((\equiv)\).

- In the first case, \( A = B \), hence \( A \equiv B \).
- In the second case, there exists \( B' \) such that \( x : B' \) has a shorter derivation, and \( B \equiv B' \). By the i.h., \( A \equiv B' \equiv B \).

For the statement 2, we have \( \lambda x^A . r : B \). The only way to type this term is either by rule \((\Rightarrow_i)\), \((\equiv)\).

- In the first case, we have \( B = A \Rightarrow C \) for some, \( C \) and \( r : C \).
- In the second, there exists \( B' \) such that \( \lambda x^A . r : B' \) has a shorter derivation, and \( B \equiv B' \). By the i.h., \( B' \equiv A \Rightarrow C \) and \( r : C \). Thus, \( B \equiv B' \equiv A \Rightarrow C \).

The three other statements are similar. \( \square \)

**Lemma 4.3** (Substitution). If \( r : A \), \( s : B \), and \( x \in \mathcal{V}_B \), then \( r[s/x] : A \).

**Proof.** By structural induction on \( r \).

- Let \( r = x \). By Lemma 4.2, \( A \equiv B \), thus \( s : A \). We have \( x[s/x] = s \), so \( x[s/x] : A \).
- Let \( r = y \), with \( y \neq x \). We have \( y[s/x] = y \), so \( y[s/x] : A \).
- Let \( r = \lambda y^C . r' \). By Lemma 4.2, \( A \equiv C \Rightarrow D \), with \( r' : D \). By the i.h., \( r'[s/x] : D \), and so, by rule \((\Rightarrow_i)\), \( \lambda y^C . r'[s/x] : C \Rightarrow D \). Since \( \lambda y^C . r'[s/x] = (\lambda y^C . r')[s/x] \), using rule \((\equiv)\), \( (\lambda y^C . r')[s/x] : A \).
- Let \( r = r_1 \times r_2 \). By Lemma 4.2, \( r_1 : C \Rightarrow A \) and \( r_2 : C \). By the i.h., \( r_1[s/x] : C \Rightarrow A \) and \( r_2[s/x] : C \), and so, by rule \((\Rightarrow_e)\), \( (r_1[s/x])(r_2[s/x]) : A \). Since \( (r_1[s/x])(r_2[s/x]) = (r_1 \times r_2)[s/x] \), we have \( (r_1 \times r_2)[s/x] : A \).
- Let \( r = \pi_A(r') \). By Lemma 4.2, \( r' : A \＆ C \). Hence, by the i.h., \( r'[s/x] : A \＆ C \). Hence, by rule \((\land_e)\), \( \pi_A(r'[s/x]) : A \). Since \( \pi_A(r'[s/x]) = \pi_A(r')[s/x] \), we have \( \pi_A(r')[s/x] : A \).

**Theorem 4.4** (Subject reduction). If \( r : A \) and \( r \leftrightarrow s \) or \( r \equiv s \) then \( s : A \).

**Proof.** By induction on the rewrite relation.
• (comm): If \( r \times s : A \), then by Lemma 4.2, \( A \equiv A_1 \land A_2 \equiv A_2 \land A_1 \), with \( r : A_1 \) and \( s : A_2 \). Then, \( s \times r : A_2 \land A_1 \equiv A \).

• (asso):

(\( \rightarrow \)) If \( (r \times s) \times t : A \), then by Lemma 4.2, \( A \equiv (A_1 \land A_2) \land A_3 \equiv A_1 \land (A_2 \land A_3) \), with \( r : A_1 \), \( s : A_2 \) and \( t : A_3 \). Then, \( r \times (s \times t) : A_1 \land (A_2 \land A_3) \equiv A \).

\( \leftarrow \) Analogous to \( \rightarrow \).

• (dist):

(\( \rightarrow \)) If \( \lambda x^B\. (r \times s) : A \), then by Lemma 4.2, we have \( A \equiv (B \Rightarrow (C_1 \land C_2)) \equiv ((B \Rightarrow C_1) \land (B \Rightarrow C_2)) \), with \( r : C_1 \) and \( s : C_2 \). Then, \( \lambda x^B\. r \times \lambda x^B\. s : (B \Rightarrow C_1) \land (B \Rightarrow C_2) \equiv A \).

\( \leftarrow \) If \( \lambda x^B\. r \times \lambda x^B\. s : A \), then by Lemma 4.2, \( A \equiv ((B \Rightarrow C_1) \land (B \Rightarrow C_2)) \equiv (B \Rightarrow (C_1 \land C_2)) \), with \( r : C_1 \) and \( s : C_2 \). Then, \( \lambda x^B\. (r \times s) : B \Rightarrow (C_1 \land C_2) \equiv A \).

• (curry):

(\( \rightarrow \)) If \( rst : A \), then by Lemma 4.2, \( r : B \Rightarrow C \Rightarrow A \equiv (B \land C) \Rightarrow A \), \( s : B \) and \( t : C \). Then, \( r(s \times t) : A \).

\( \leftarrow \) If \( r(s \times t) : A \), then by Lemma 4.2, \( r : (B \land C) \Rightarrow A \equiv (B \Rightarrow C \Rightarrow A) \), \( s : B \) and \( t : C \). Then \( rst : A \).

• (λ): If \( (\lambda x^B\. r)s : A \), then by Lemma 4.2, \( \lambda x^B\. r : B \Rightarrow A \), and by Lemma 4.2 again, \( r : A \). Then by Lemma 4.3, \( r[s/x^B] : A \).

• (π): If \( \pi_B(r \times s) : A \), then by Lemma 4.2, \( A \equiv B \), and so, by rule (\( \equiv \)), \( r : A \).

• (c): If \( (r \times s)t : A \), then by Lemma 4.2, \( r \times s : B \Rightarrow A \), and \( t : B \). Hence, by Lemma 4.2 again, \( B \Rightarrow A \equiv C_1 \land C_2 \), and so by Lemma 2.10, \( A \equiv A_1 \land A_2 \), with \( r : B \Rightarrow A_1 \) and \( s : B \Rightarrow A_2 \). Then, \( rt \times st : A_1 \land A_2 \equiv A \).

• (a): If \( r : A \Rightarrow B \), then, by rules (\( \Rightarrow_c \)) and (\( \Rightarrow_1 \)), \( \lambda x^A\. (rx) : A \Rightarrow B \).

• (s): If \( r : A \land B \), then by rules (\( \land_c \)) and (\( \land_i \)), \( \pi_A(r) \times \pi_B(r) : A \land B \).

• Contextual closure: Let \( t \rightarrow r \), where \( \rightarrow \) is either \( \Rightarrow \) or \( \Leftarrow \).
  
  - Let \( \lambda x^B\. t \rightarrow \lambda x^B\. r \): If \( \lambda x^B\. t : A \), then by Lemma 4.2, \( A \equiv (B \Rightarrow C) \) and \( t : C \), hence by the i.h., \( r : C \) and so \( \lambda x^B\. r : B \Rightarrow C \equiv A \).
  
  - Let \( ts \rightarrow rs \): If \( ts \) \( A \) then by Lemma 4.2, \( t : B \Rightarrow A \) and \( s : B \), hence by the i.h., \( r : B \Rightarrow A \) and so \( rs : A \).
  
  - Let \( st \rightarrow st \): If \( st \) \( A \) then by Lemma 4.2, \( s : B \Rightarrow A \) and \( t : B \), hence by the i.h. \( r : B \) and so \( sr : A \).
Let $t \times s \rightarrow r \times s$: If $t \times s : A$ then by Lemma 4.2, $A \equiv A_1 \land A_2$, $t : A_1$, and $s : A_2$, hence by the i.h., $r : A_1$ and so $r \times s : A_1 \land A_2 \equiv A$.

Let $s \times t \rightarrow s \times r$: Analogous to previous case.

Let $\pi_B(t) \rightarrow \pi_B(r)$: If $\pi_B(t) : A$ then by Lemma 4.2, $A \equiv \tau$ and $t : \tau \Rightarrow (\tau \land \tau)$, hence by the i.h. $r : \tau \land \tau$. Therefore, $\pi_B(r) : \tau \Rightarrow (\tau \land \tau)$.

5. Strong Normalization

We now prove the strong normalization of reduction $\Rightarrow$.

Road-map of the proof. We associate, as usual, a set $[A]$ of strongly normalizing terms to each type $A$. We then prove an adequacy lemma stating that every term of type $A$ is in $[A]$. Compared with the proof for simply typed lambda-calculus with pairs our proof presents several novelties.

- In simply typed lambda-calculus, proving that if $r_1$ and $r_2$ are strongly normalizing, then so is $r_1 \times r_2$ is easy. However, like in System I, in System I$\eta$ this property is harder to prove, as it requires a characterization of the terms equivalent to the product $r_1 \times r_2$ and of all its reducts. This will be the first part of our proof (Lemmas 5.1, 5.2 and Corollary 5.3).

- As usual we associate to each type $A$ a set $[A]$ of reducible terms, but this definition has to take into account the equivalence between types. For instance, $r \in [\tau \Rightarrow (\tau \land \tau)]$, if and only if, $r : \tau \Rightarrow (\tau \land \tau)$, for all $s \in [\tau]$, $rs \in [\tau \land \tau]$, and, moreover, $\pi_B[r] \in [\tau \Rightarrow \tau]$ as $\tau \Rightarrow (\tau \land \tau) \equiv (\tau \Rightarrow \tau) \land (\tau \Rightarrow \tau)$ (Definition 5.6).

- In the strong normalization proof of simply typed lambda-calculus the so-called properties CR1, CR2, and CR3, the adequacy of product, and the adequacy of abstraction are five independent lemmas. Like in [22], we have to prove these properties in a huge single induction (Lemma 5.8).

- In simply typed lambda-calculus, neutral terms are those which are neither abstractions nor pairs. The reason is that such terms can be put in any context without creating a redex. In our case, the applications are not always neutral. For example, if $r : A$, $(\lambda x^{A \land B}.x)r$ is not neutral. Indeed, if $s : B$, $(\lambda x^{A \land B}.x)rs \rightleftharpoons (\lambda x^{A \land B}.x)(r \times s) \leftrightarrow r \times s$. This leads us to generalize the induction hypothesis in the proof of the adequacy of product and of abstraction.

The set of strongly normalizing terms is written $SN$. The size of the longest reduction issued from $t \in SN$ is written $|t|$. Recall that each term has a finite number of one-step reducts (Remark 3.6).

Lemma 5.1. If $r \times s \rightleftharpoons t$ then either

1. $t = u \times v$ where either
Lemma 5.2. If \( r_1 \times r_2 \models s \leftrightarrow t \), there exists \( u_1, u_2 \) such that \( t \models u_1 \times u_2 \) and either \( r_1 \models u_1 \) and \( r_2 \models u_2 \), or \( r_1 \models u_1 \) and \( r_2 \models u_2 \).

Proof. By a double induction, first on \( M(t) \) and then on the length of the derivation of \( r \times s \models t \). Consider an equivalence proof \( r \times s \models t' \models t \) with a shorter proof \( r \times s \models t' \). By the second i.h. (induction hypothesis), the term \( t' \) has the form prescribed by the lemma. We consider the three cases and in each case, the possible rules transforming \( t' \) in \( t \).

1. Let \( r \times s \models u \times v \models t \). The possible equivalences from \( u \times v \) are
   - \( t = u' \times v \) or \( u \times v' \) with \( u \models u' \) and \( v \models v' \), and so the term \( t \) is in case 1.
   - Rules (comm) and (asso) preserve the conditions of case 1.
   - \( t = \lambda x^A.a \) and \( a \models a_1 \times a_2 \) with \( r \models \lambda x^A.a_1 \) and \( s \models \lambda x^A.a_2 \). Hence, possible equivalences from \( \lambda x^A.a \) to \( t \) are
     - \( t = \lambda x^A.a' \) with \( a \models a' \), hence \( a' \models a_1 \times a_2 \), and so the term \( t \) is in case 2.
     - \( t = \lambda x^A.u \times \lambda x^A.v \), with \( a_1 \times a_2 \models a = u \times v \). Hence, by the first i.h. (since \( M(a) < M(t) \)), either
       - \( a_1 \models u \) and \( a_2 \models v \), and so \( r \models \lambda x^A.u \) and \( s \models \lambda x^A.v \), or
       - \( v \models t_1 \times t_2 \) with \( a_1 \models u \times t_1 \) and \( a_2 \models v \times t_2 \), and so \( \lambda x^A.v \models \lambda x^A.t_1 \times \lambda x^A.t_2 \), \( r \) \models \lambda x^A.u \times \lambda x^A.t_1 \) and \( s \models \lambda x^A.t_2 \), or
       - \( u \models t_1 \times t_2 \) with \( a_1 \models u_1 \times t_1 \) and \( a_2 \models u_2 \times t_2 \), and so \( \lambda x^A.u \models \lambda x^A.t_1 \times \lambda x^A.t_2 \), \( r \) \models \lambda x^A.t_1 \times \lambda x^A.t_2 \), and \( s \models \lambda x^A.t_1 \times \lambda x^A.t_2 \) (the symmetric cases are analogous), and so the term \( t \) is in case 1.

\( \square \)
Proof. By induction on \( M(r_1 \times r_2) \). By Lemma 5.1, \( s \) is either a product \( s_1 \times s_2 \) or an abstraction \( \lambda x^A.a \) with the conditions given in the lemma. The different terms \( s \) reducible by \( \leftrightarrow \) are \( s_1 \times s_2 \) or \( \lambda x^A.a \), with a reduction in the subterm \( s_1, s_2, \) or \( a \).

Notice that no rule can be applied in head position. Indeed, rule nor (\( \beta \)) nor (\( \zeta \)) can apply, since \( s \) is not an application, rule (\( \varepsilon \)) cannot apply since \( s \) is not a projection, and rules (\( \eta \)) and (\( \delta \)) cannot apply since \( s \) is an introduction.

We consider each case:

- \( s = s_1 \times s_2, t = t_1 \times t_2 \) or \( t = s_1 \times t_2 \), with \( s_1 \leftrightarrow t_1 \) and \( s_2 \leftrightarrow t_2 \). We only consider the first case since the other is analogous. One of the following cases happen
  
  \( a \) \( r_1 \rightarrow^* w_11 \times w_{21}, r_2 \rightarrow^* w_{12} \times w_{22}, s_1 = w_11 \times w_{12} \) and \( s_2 = w_21 \times w_{22} \). Hence, by the i.h., either \( t_1 = w_11 \times w_{12} \) or \( t_1 = w_{11} \times w_{12} \), with \( w_11 \leftrightarrow w_{11} \) and \( w_{12} \leftrightarrow w_{12} \). We take, in the first case \( u_1 = w_11 \times w_{21} \) and \( u_2 = w_{12} \times w_{22} \), in the second case \( u_1 = w_{11} \times w_{21} \rightarrow^* r_1 \) and \( u_2 = w_{12} \times w_{22} \).

  \( b \) We consider two cases, since the other two are symmetric.

  - \( r_1 \rightarrow^* s_1 \times w \) and \( s_2 \rightarrow^* w \times r_2 \), in which case we take \( u_1 = t_1 \times w \) and \( u_2 = r_2 \).
  - \( r_2 \rightarrow^* w \times s_2 \) and \( s_1 = r_1 \times w \). Hence, by the i.h., either \( t_1 = r_1 \times w \), or \( t_1 = r_1 \times w' \), with \( r_1 \leftrightarrow r_1' \) and \( w \leftrightarrow w' \). We take, in the first case \( u_1 = r_1' \) and \( u_2 = w \times s_2 \), and in the second case \( u_1 = r_1 \) and \( u_2 = w' \times s_2 \).

  \( c \) \( r_1 \rightarrow^* s_1 \) and \( r_2 \rightarrow^* s_2 \), in which case we take \( u_1 = t_1 \) and \( u_2 = s_2 \).

- \( s = \lambda x^A.s', t = \lambda x^A.t', \) and \( s' \leftrightarrow t' \), with \( s' \rightarrow^* s'_1 \times s'_2 \) and \( s = \lambda x^A.s'_1 \times \lambda x^A.s'_2 \). Therefore, by the i.h., there exists \( u'_1, u'_2 \) such that either \( (s_1 \rightarrow^* u'_1) \) and \( (s'_2 \rightarrow^* u'_2) \) or \( (s'_1 \rightarrow^* u'_1') \) and \( (s_2 \rightarrow^* u'_2') \). Therefore, we take \( u_1 = \lambda x^A.u'_1 \) and \( u_2 = \lambda x^A.u'_2 \).

Corollary 5.3. If \( r_1 \in SN \) and \( r_2 \in SN \), then \( r_1 \times r_2 \in SN \).

Proof. By Lemma 5.2, from a reduction sequence starting from \( r_1 \times r_2 \), we can extract one starting from \( r_1, r_2 \), or both. Hence, this reduction sequence is finite.

Lemma 5.4. If \( r \in SN \), then \( \lambda x^A.r \in SN \).

Proof. By induction on the length of the derivation we prove that if \( \lambda x^A.r \rightarrow^* s \), then \( s = (\lambda x^A.s_1) \times \cdots \times (\lambda x^A.s_n) \), where \( r \rightarrow^* s_1 \times \cdots \times s_n \). Thus, if \( \lambda x^A.r \rightarrow^* s \leftrightarrow t \), the reduction is in some \( s_i \), thus \( t \rightarrow^* \lambda x^A.r' \) where \( r \leftrightarrow r' \). Therefore, \( \lambda x^A.r \in SN \).

Lemma 5.5. Let \( r \) and \( t \) be introductions, then if \( rs \rightarrow^* tu \), then \( r \rightarrow^* t \) and \( s \rightarrow^* u \).
Proof. We proceed by induction on the length of the derivation \( rs \rightsquigarrow v \rightsquigarrow^* tu \). So, the possibilities for \( v \) are:

1. If \( v = v' s \) or \( v = rs' \), with \( r \rightsquigarrow r' \) and \( s \rightsquigarrow s' \), the i.h. applies.
2. If \( v \) is obtained by \((\text{Curry})\), then either \( r = r_1 r_2 \), which is impossible since no elimination is equivalent to an introduction, or \( s = s_1 \times s_2 \), and \( v = rs_1 s_2 \), then by the i.h., we have \( rs_1 \rightsquigarrow^* t \), which is impossible since no elimination is equivalent to an introduction. \( \square \)

**Definition 5.6 (Reducibility).** The set \( \llbracket A \rrbracket \) of reducible terms of type \( A \) is defined by induction on \( m(A) \) as follows: \( t \in \llbracket A \rrbracket \) if and only if \( t : A \) and

- if \( A = \tau \), then \( t \in \text{SN} \),
- for all \( B, C \), if \( A = B \Rightarrow C \), then for all \( r \in \llbracket B \rrbracket \), \( tr \in \llbracket C \rrbracket \),
- for all \( B, C \), if \( A = B \land C \), then \( \pi_B(t) \in \llbracket B \rrbracket \).

Note that, by construction, if \( A = B \), then \( \llbracket A \rrbracket = \llbracket B \rrbracket \).

**Definition 5.7 (Neutral term).** A term \( t \) is neutral if no term of the form \( tr \) or \( \pi_A(t) \), can be \( \rightsquigarrow_{\Delta} \)-reduced at head position.

The variables and the projections are always neutral, but, as we have discussed in the road-map of the proof, applications are not necessarily neutral. For example if \( r : A \), then \( (\lambda x^{A \land B} . x) r \) is not.

**Lemma 5.8.** For all types \( T \), we have

- (CR1) \( \llbracket T \rrbracket \subseteq \text{SN} \).
- (CR2) If \( t \in \llbracket T \rrbracket \) and \( t \rightsquigarrow t' \), then \( t' \in \llbracket T \rrbracket \).
- (CR3) If \( t : T \) is neutral, and for all \( t' \) such that \( t \rightsquigarrow_{\Delta} t' \), \( t' \in \llbracket T \rrbracket \), we have \( t \in \llbracket T \rrbracket \).
- (Adequacy of product) If \( T = A \land B \), then for all \( r \in \llbracket A \rrbracket \) and \( s \in \llbracket B \rrbracket \), \( r \times s \in \llbracket T \rrbracket \).
- (Adequacy of abstraction) If \( T = A \Rightarrow B \), then for all \( t \in \llbracket B \rrbracket \), if for all \( r \in \llbracket A \rrbracket \), \( t[r/x] \in \llbracket B \rrbracket \), then \( \lambda x^A . t \in \llbracket T \rrbracket \).

**Proof.** By induction on \( m(T) \).

**Proof of (CR1).** Let \( t \in \llbracket T \rrbracket \). We want to prove that \( t \in \text{SN} \).

- If \( T = \tau \), then \( t \in \llbracket T \rrbracket = \text{SN} \).
- If \( T = A \Rightarrow B \), then, by the i.h. (CR3), we have \( x^A \in \llbracket A \rrbracket \). Hence, \( tx \in \llbracket B \rrbracket \), then, by the i.h., \( tx \in \text{SN} \). We prove by a second induction on \( |tx| \) that all the one-step \( \rightsquigarrow \)-reducts of \( t \) are in \( \text{SN} \).
  - If \( t \rightsquigarrow_{\Delta} t' \), then \( tx \rightsquigarrow_{\Delta} t'x \), so by the second i.h., \( t' \in \text{SN} \).
Let \( (A \text{dequacy of product}) \)\( (\equiv) \llbracket \text{all in} \) Proof of (CR3').

Cases:

- If \( t \sim_\eta \lambda y^C.(ty) \), where \( T \equiv C \Rightarrow D \). Since \( t \in \llbracket T \rrbracket \), and, by the i.h. (CR3'), \( y \in \llbracket C \rrbracket \), so \( ty \in \llbracket D \rrbracket \), which, by the i.h. is a subset of \( SN \). Therefore, by Lemma 5.4, \( \lambda y^C.(ty) \in SN \).
- If \( t \sim_\delta \pi_C(t) \times \pi_D(t) \), where \( T \equiv C \land D \). Since \( t \in \llbracket T \rrbracket \), we have \( \pi_C(t) \in \llbracket C \rrbracket \), and by the i.h., \( \pi_C(t) \in SN \). In the same way, \( \pi_D(t) \in SN \), so by Corollary 5.3, \( \pi_C(t) \times \pi_D(t) \in SN \).

- If \( T = A \land B \), then \( \pi_A(t) \in [A] \) and \( \pi_B(t) \in [B] \), by the i.h., \( [A] \subseteq SN \), and so we proceed by a second induction on \([\pi_A(t)]\) to prove that all the one-step \( \sim \)-reducts of \( t \) are in \( SN \).

- If \( t \sim_\Delta t' \), \( \pi_A(t) \sim_\Delta \pi_A(t') \), so by the second i.h., \( t' \in SN \).
- If \( t \sim_\eta \lambda y^C.(ty) \), where \( T \equiv C \Rightarrow D \). Since \( t \in \llbracket T \rrbracket \), and, by the i.h. (CR3'), \( y \in \llbracket C \rrbracket \), so \( ty \in \llbracket D \rrbracket \), which, by the i.h. is a subset of \( SN \). Therefore, by Lemma 5.4, \( \lambda y^C.(ty) \in SN \).
- If \( t \sim_\delta \pi_C(t) \times \pi_D(t) \), where \( T \equiv C \land D \). Since \( t \in \llbracket T \rrbracket \), we have \( \pi_C(t) \in \llbracket C \rrbracket \), and by the i.h., \( \pi_C(t) \in SN \). In the same way, \( \pi_D(t) \in SN \), so by Corollary 5.3, \( \pi_C(t) \times \pi_D(t) \in SN \).

**Proof of (CR2).** Let \( t \in \llbracket T \rrbracket \) and \( t \sim t' \). We want to prove that \( t' \in \llbracket T \rrbracket \).

Cases:

- \( t \sim_\Delta t' \). We want to prove that \( t' \in \llbracket T \rrbracket \). That is, if \( T \equiv \tau \), then \( t' \in SN \), if \( T \equiv A \Rightarrow B \), then for all \( r \in [A] \), \( t'r \in [B] \), and if \( T \equiv A \land B \), then \( \pi_A(t') \in [A] \).
  - If \( T \equiv \tau \), then since \( t \in SN \), we have \( t' \in SN \).
  - If \( T \equiv A \Rightarrow B \), then let \( r \in [A] \), we need to prove \( t'r \in [B] \). Since \( t \in \llbracket T \rrbracket = \llbracket A \Rightarrow B \rrbracket \), we have \( tr \in [B] \). Then, by the i.h. in \([B]\), and the fact that \( tr \sim_\Delta t'r \), we have \( t'r \in [B] \).
  - If \( T \equiv A \land B \), then we need to prove \( \pi_A(t') \in [A] \). Since \( t \in \llbracket T \rrbracket = \llbracket A \land B \rrbracket \), we have \( \pi_A(t) \in [A] \). Then, by the i.h. in \([A]\), and the fact that \( \pi_A(t) \sim_\Delta \pi_A(t') \), we have \( \pi_A(t') \in [A] \).

- \( t \sim_\eta \lambda x^A.tx \). Then, \( T \equiv A \Rightarrow B \). Since \( t \in \llbracket T \rrbracket = \llbracket A \Rightarrow B \rrbracket \), for any \( s \in [A] \), \( ts \in [B] \), and, since \( x \notin \text{FV}(t) \), we have \( ts = (tx)[s/x] \). Then, by i.h. (Adequacy of abstraction), \( \lambda x^A.tx \in [A \Rightarrow B] = \llbracket T \rrbracket \).

- \( t \sim_\delta \pi_A(t) \times \pi_B(t) \). Then, \( T \equiv A \land B \). Since \( t \in \llbracket T \rrbracket = \llbracket A \land B \rrbracket \), we have \( \pi_A(t) \in [A] \) and \( \pi_B(t) \in [B] \). Then, by the i.h. (Adequacy of product), \( \pi_A(t) \times \pi_B(t) \in [A \land B] = \llbracket T \rrbracket \).

**Proof of (CR3').** Let \( t : T \) be a neutral term whose \( \sim_\Delta \)-one-step reducts \( t' \) are all in \( \llbracket T \rrbracket \). We want to prove that \( t \in \llbracket T \rrbracket \). That is, if \( T \equiv \tau \), then \( t \in SN \), if \( T \equiv A \Rightarrow B \), then for all \( r \in [A] \), \( tr \in [B] \), and if \( T \equiv A \land B \), then \( \pi_A(t) \in [A] \).

- If \( T \equiv \tau \), we need to prove that all the one-step reducts of \( t \) are in \( SN \).
  Since \( T \equiv \tau \), these reducts are neither \((\eta)\) reducts nor \((\delta)\) reducts, but \( \sim_\Delta \)-reducts, which are in \( SN \).
• If $T \equiv A \Rightarrow B$, we know that for all $r \in [A]$, we have $t'r \in [B]$. By the i.h. (CR1) in $[A]$, we know $r \in \text{SN}$. So we proceed by induction on $|r|$ to prove that $tr \in [B]$. by the i.h., it suffices to check that every term $s$ such that $tr \rightsquigarrow \triangle s$ is in $[B]$. Since the reduction is $\rightsquigarrow \triangle$, and the term $t$ is neutral, there is no possible head reduction. So, the possible cases are

\begin{itemize}
  \item $s = tr'$ with $r \rightsquigarrow r'$, then the i.h. applies.
  \item $s = t'r$, with $t \rightsquigarrow t'$. As $t$ cannot reduce to $t'$ by (\delta) or (\eta), we have $t \rightsquigarrow \triangle t'$, and $t'r \in [B]$ by hypothesis.
\end{itemize}

• If $T \equiv A \wedge B$, then we know that $\pi_A(t') \in [A]$. by the i.h., it suffices to check that every term $s$ such that $\pi_A(t) \rightsquigarrow \triangle s$ is in $[A]$. Since the reduction is $\rightsquigarrow \triangle$, and the term $t$ is neutral, there is no possible head reduction. So, the only possible case is $s = \pi_A(t')$ with $t \rightsquigarrow t'$. As $t$ cannot reduce to $t'$ by (\delta) or (\eta), we have $t \rightsquigarrow \triangle t'$, and $\pi_A(t') \in [B]$ by hypothesis.

\textbf{Proof of (Adequacy of product).} If $T = A \wedge B$, we want to prove that for all $r \in [A]$ and $s \in [B]$, we have $r \times s \in [T]$. We prove, more generally, by a simultaneous second induction on $m(D)$ that for all types $D$

1. if $T = A \wedge B \equiv D$, then $v = r \times s \in [D]$, and
2. if $T = A \wedge B \equiv C \Rightarrow D$, then for all $t \in [C]$ we have $v = (r \times s)t \in [D]$.

To prove that $v \in [D]$, we need to prove that if $D \equiv \tau$, then $v \in \text{SN}$, if $D \equiv E \Rightarrow F$, then for all $u \in [E]$, $vu \in [F]$, and if $D \equiv E \wedge F$, then $\pi_E(v) \in [E]$.

• $D \not\equiv \tau$, since, in case 1, it is equivalent to a conjunction, and also in case 2, by Lemma 2.10.

• If $D \equiv E \Rightarrow F$, in both cases we must prove that for all $u \in [E]$, $vu \in [F]$.

1. In case 1, we want to prove that $(r \times s)u \in [F]$. Since $m(F) < m(D)$, the second i.h. applies.
2. In case 2, we want to prove that $(r \times s)tu \in [F]$. As $m(C \wedge E) < m((C \wedge E) \Rightarrow F) = m(T)$, by the i.h., $t \times u \in [C \wedge E]$, and so, since $m(F) < m(D)$, by the second i.h., we have $(r \times s)(t \times u) \in [F]$. Then, by the i.h. (CR2), $(r \times s)tu \in [F]$.

• If $D \equiv E \wedge F$, in both cases we must prove that $\pi_E(v) \in [E]$.

  – In case 1, we want to prove that $\pi_E(r \times s) \in [E]$. by the i.h. (CR3') it suffices to prove that every one-step $\rightsquigarrow \triangle$ reduct of $\pi_E(r \times s)$ is in $[E]$. by the i.h. (CR1), $r, s \in \text{SN}$, so we proceed with a third induction on $|r| + |s|$.

A $\rightsquigarrow \triangle$-reduction issued from $\pi_E(r \times s)$ cannot be a $\beta$-reduction or $\zeta$-reduction at head position, since a projection is not equivalent to an application (by rule inspection). Therefore, the possible $\rightsquigarrow \triangle$-reductions issued from $\pi_E(r \times s)$ are:
* A reduction in \( r \times s \), then, by Lemma 5.2, the reduction takes place either in \( r \) or in \( s \), and the third i.h. applies.

* \( \pi_E(r \times s) \xrightarrow{\ast} \pi_E(w_1 \times w_2) \leadsto w_1 \). Then, \( r \times s \xrightarrow{\ast} w_1 \times w_2 \). We need to prove that \( w_1 \in \langle E \rangle \). By Lemma 5.1, we have either:
  
  - \( w_1 \xrightarrow{\ast} r_1 \times s_1 \), with \( r \xrightarrow{\ast} r_1 \times r_2 \) and \( s \xrightarrow{\ast} s_1 \times s_2 \). In such a case, by Lemma 4.2, \( A \equiv A_1 \land A_2 \) and \( B \equiv B_1 \land B_2 \), with \( E \equiv A_1 \land B_1 \), and \( F \equiv A_2 \land B_2 \). Since \( r \in \llbracket A \rrbracket = \llbracket A_1 \land A_2 \rrbracket \), we have \( \pi_A(r) \in \llbracket A_1 \rrbracket \). Then, by the i.h. \((\text{CR2})\) in \( \llbracket A_1 \rrbracket \), we have \( r_1 \in \llbracket A_1 \rrbracket \). Similarly \( s_1 \in \llbracket B_1 \rrbracket \). Then, by the i.h., the i.h. \((\text{CR2})\), \( r_1 \times s_1 \xrightarrow{\ast} w_1 \in \llbracket A_1 \land B_1 \rrbracket = \langle E \rangle \).
  
  - \( w_1 \xrightarrow{\ast} r \times s_1 \), with \( s \xrightarrow{\ast} s_1 \times s_2 \). Then, by Lemma 4.2, \( B \equiv B_1 \land B_2 \), with \( E \equiv D_1 \). Since \( s \in \llbracket B \rrbracket = \llbracket B_1 \land B_2 \rrbracket \), we have \( \pi_B(s) \in \llbracket B_1 \rrbracket \). Then, by the i.h. \((\text{CR2})\) in \( \llbracket B_1 \rrbracket \), we have \( s_1 \in \llbracket B_1 \rrbracket \). Since, \( r \in \llbracket A \rrbracket \), by the i.h. and the i.h. \((\text{CR2})\), \( r \times s_1 \xrightarrow{\ast} w_1 \in \llbracket D_1 \rrbracket = \langle E \rangle \).
  
  - \( w_1 \xrightarrow{\ast} r_1 \times s \), with \( r \xrightarrow{\ast} r_1 \times r_2 \). This case is analogous to the previous one.
  
  - \( r \xrightarrow{\ast} w_1 \times r_2 \), in which case, by Lemma 4.2, \( A \equiv E \land A_2 \), since \( r \in \llbracket A \rrbracket \), we have \( \pi_E(r) \in \llbracket E \rrbracket \), so by the i.h. \((\text{CR2})\) in \( \llbracket E \rrbracket \), \( w_1 \in \llbracket E \rrbracket \).
  
  - \( s \xrightarrow{\ast} w_1 \times s_2 \). This case is analogous to the previous case.
  
  - \( w_1 \xrightarrow{\ast} r \in \llbracket A \rrbracket = \langle E \rangle \).
  
  - \( w_1 \xrightarrow{\ast} s \in \llbracket B \rrbracket = \langle E \rangle \).

- In case 2, we want to prove that \( \pi_E((r \times s)t) \in \langle E \rangle \). Since \( T = A \land B \equiv C \Rightarrow D \), by Lemma 2.10, \( D \equiv D_1 \land D_2 \), with \( A \equiv C \Rightarrow D_1 \) and \( B \equiv C \Rightarrow D_2 \). Since a projection is always neutral, and \( m(E) < m(E \land F) = m(D) < m(C \Rightarrow D) = m(T) \), by i.h. \((\text{CR3}')\), it suffices to prove that every one-step \( \rightsquigarrow_{\Delta} \) reduction issued from \( \pi_E((r \times s)t) \) is in \( \langle E \rangle \). By the i.h. \((\text{CR1})\), \( r, s, t \in \text{SN} \). Therefore, we can proceed by a third induction on \(|r| + |s| + |t| \). The reduction cannot happen at head position since a projection is not equivalent to an application, to apply \( \beta \) or \( \zeta \), and an application is not equivalent to a product to apply \( \pi \). Hence, the reduction must happen in \((r \times s)t\). Therefore, we must prove that the one-step \( \rightsquigarrow_{\Delta} \)-reductions of \((r \times s)t\) are in \( \llbracket D \rrbracket = \llbracket E \land F \rrbracket \), from which we conclude that \( \pi_E((r \times s)t) \in \langle E \rangle \).

A \( \rightsquigarrow_{\Delta} \)-reduction in \((r \times s)t\) cannot be a \( \pi \)-reduction in head position, since an application is not equivalent to a projection. Then, the possible \( \rightsquigarrow_{\Delta} \) reductions issued from \((r \times s)t\) are:

* A reduction in \( r \times s \), in which case, by Lemma 5.2 it takes place either in \( r \) or in \( s \), and then the third i.h. applies.

* A reduction in \( t \), then the third i.h. also applies.

* If the reduction is a \( \beta \)-reduction at head position, then we have \((r \times s)t \xrightarrow{\ast} (\lambda x^C . w_1)w_2 \). Hence, by Lemma 5.5, \( r \times s \xrightarrow{\ast} \lambda x^A . w_1 \) and \( t \xrightarrow{\ast} w_2 \). By Lemma 5.1, \( r \xrightarrow{\ast} \lambda x^C . r' \), \( s \xrightarrow{\ast} \lambda x^C . s' \).
and \( w_1 \rightharpoonup^* r' \times s' \). Therefore, \((r \times s)t \rightharpoonup^* (\lambda x^C . r' \times s')t \rightarrow r'[t/x] \times s'[t/x]\). Since \((\lambda x^C . r')t \times (\lambda x^C . s')t \rightharpoonup^* r'[t/x] \times s'[t/x]\), by the i.h. \((\text{CR}2)\) in \([D]\), it is enough to prove that \((\lambda x^C . r')t \times (\lambda x^C . s')t \in [D]\). By the i.h. \((\text{CR}2)\), since \( r \in [A] \) and \( s \in [B] \), we have \( r \rightharpoonup^* \lambda x^C . r' \in [A] = [C \Rightarrow D_1], \) and \( s \rightharpoonup^* \lambda x^C . s' \in [B] = [C \Rightarrow D_2] \). Therefore, by definition, \((\lambda x^C . r')t \in [D_1] \) and \((\lambda x^C . s')t \in [D_2] \). Since \( m(D) < m(T) \), by the i.h., we have \((\lambda x^C . r')t \times (\lambda x^C . s')t \in [D]\).

* If the reduction is a \( \zeta \)-reduction at head position, then \((r \times s)t \rightharpoonup^* (u_1 \times u_2)w \). By Lemma 5.5, \( r \times s \rightharpoonup^* u_1 \times u_2 \) and \( t \rightharpoonup^* w \). By Lemma 5.1, the possibilities are:

- \( r \rightharpoonup^* r_1 \times r_2 \), \( s \rightharpoonup^* s_1 \times s_2 \), \( u_1 \rightharpoonup^* r_1 \times s_1 \) and \( u_2 \rightharpoonup^* r_2 \times s_2 \). Then, \((u_1 \times u_2)w \Rightarrow^* u_1w \times u_2w \rightharpoonup^* (r_1 \times s_1)w \times (r_2 \times s_2)w \). By Lemmas 4.2 and 2.10, we have \( D_1 \equiv D_{11} \land D_{12} \) and \( D_2 \equiv D_{21} \land D_{22} \). So, since \( r \in [A] = [C \Rightarrow D_1] = [(C \Rightarrow D_{11}) \land (C \Rightarrow D_{12})], \) we have \( \pi_{C \Rightarrow D_{11}}(r) \in [C \Rightarrow D_{11}] \), so, by the i.h. \((\text{CR}2)\), \( r_1 \in [C \Rightarrow D_{11}] \). Similarly, \( r_2 \in [C \Rightarrow D_{21}], s_1 \in [C \Rightarrow D_{21}] \) and \( s_2 \in [C \Rightarrow D_{22}] \). Therefore, by the i.h., \( r_1 \times s_1 \in [(C \Rightarrow D_{11}) \land (C \Rightarrow D_{21})] = [C \Rightarrow (D_{11} \land D_{21})]\) hence, by the i.h. \((\text{CR}2)\), we have \( u_1 \in [C \Rightarrow (D_{11} \land D_{21})]. \) Therefore, \( u_1w \in [D_{11} \land D_{21}] \). Similarly, \( u_2w \in [D_{12} \land D_{22}] \). So, by the i.h. again, \( u_1w \times u_2w \in [D_{11} \land D_{21} \land D_{12} \land D_{22}] = [D]. \) The other three cases are symmetric.

- \( s \rightharpoonup^* s_1 \times u_2 \), \( u_1 \rightharpoonup^* r \times s_1 \). Then, \((u_1 \times u_2)w \Rightarrow^* u_1w \times u_2w \rightharpoonup^* (r \times s_1)w \times u_2w \). By Lemmas 4.2 and 2.10, we have \( D_2 \equiv D_{21} \land D_{22} \). So, since \( s \in [B] = [C \Rightarrow D_2] = [(C \Rightarrow D_{21}) \land (C \Rightarrow D_{22})], \) we have \( \pi_{C \Rightarrow D_{21}}(s) \in [C \Rightarrow D_{21}] \), so, by the i.h. \((\text{CR}2)\), \( s_1 \in [C \Rightarrow D_{21}] \). Similarly, \( u_2 \in [C \Rightarrow D_{22}] \). Therefore, by the i.h., we have that \( r \times s_1 \in [(C \Rightarrow D_{1}) \land (C \Rightarrow D_{21})] = [C \Rightarrow (D_1 \land D_{21})]\) hence, by the i.h. \((\text{CR}2)\), \( u_1 \in [C \Rightarrow (D_1 \land D_{21})]. \) Therefore, \( u_1w \in [D_1 \land D_{21}] \). Similarly, \( u_2w \in [D_{22}] \). So, by the i.h. again, \( u_1w \times u_2w \in [D_1 \land D_{21} \land D_{22}] = [D]. \) The other three cases are symmetric.

Proof of (Adequacy of abstraction). If \( T = A \Rightarrow B \), we want to prove that for all \( t \in [B] \), if for all \( r \in [A], t[r/x] \in [B] \), we have \( \lambda x^A . t \in [T] \). We prove, more generally, by a simultaneous second induction on \( m(D) \) that for all type \( D \)

1. if \( T = A \Rightarrow B \equiv D \), then \( v = \lambda x^A . t \in [D] \), and
2. if \( T = A \Rightarrow B \equiv C \Rightarrow D \), then for all \( u \in [C] \) we have \( v = (\lambda x^A . t)u \in [D]. \)
To prove that $v \in [D]$, we need to prove that if $D \equiv \tau$, then $v \in \text{SN}$, if $D \equiv E \Rightarrow F$, then for all $s \in [E]$, $vs \in [F]$, and if $D \equiv E \land F$, then $\pi_E(v) \in [E]$.

- **If $D \equiv \tau$, in both cases we must prove that $v \in \text{SN}$.**
  1. Case 1 is impossible, by Lemma 4.2.
  2. In case 2, we have to prove that $v = (\lambda x^A.t)u \in \text{SN}$, so it suffices to prove that every one-step $\Rightarrow_\Delta$ reduction issued from $(\lambda x^A.t)u$ is in $\text{SN}$, by the i.h. (CR1), $t, u \in \text{SN}$. Therefore, we can proceed by third induction on $|t| + |u|$. The possible $\Rightarrow_\Delta$ reductions issued from $(\lambda x^A.t)u$ are:
    - Reducing $t$, or $u$, then the third i.h. applies.
    - $(\lambda x^A.t)u \Rightarrow t[u/x]$, then, by Lemma 4.2, we have $A \equiv C$, and by Lemma 2.12, $B \equiv D$. Then, since by hypothesis $t[u/x] \in [B]$, we have $t[u/x] \in [D] = \text{SN}$.
    - $(\lambda x^A.t)u \Rightarrow t(u_1/x)u_2$, with $u \Rightarrow_* u_1 \times u_2$. Then, by Lemmas 4.2 and 2.12, $C \equiv A \land C''$, and $C' \Rightarrow D \equiv B$ so, by definition of reducibility, $\pi_A(u) \in [A]$ and $\pi_{C'}(u) \in [C']$. Therefore, by the i.h. (CR2), $u_1 \in [A]$ and $u_2 \in [C']$. So, since $t[u_1/x] \in [B] = [C' \Rightarrow D]$, we have $t[u_1/x]u_2 \in [D] = \text{SN}$.
    - Notice that the reduction cannot be a $\zeta$-reduction in head position since, by $D \equiv \tau$ and so, by Lemma 4.2, $t \not\Rightarrow_* t \times t_2$.

- **If $D \equiv E \Rightarrow F$, in both cases we must prove that for all $s \in [E]$, we have $vs \in [F]$.**
  1. In case 1, we have to prove that $(\lambda x^A.t)s \in [F]$, which is a consequence of the second i.h., since $m(F) < m(D)$.
  2. In case 2, we have to prove that $(\lambda x^A.t)s \in [F]$. Since $m(C \land E) < m((C \land E) \Rightarrow F) = m(T)$, by the i.h. (Adequacy of product), $u \times s \in [C \land E]$, then by the second i.h., since $m(F) < m(D)$, we have $(\lambda x^A.t)(u \times s) \in [F]$, so, by the i.h. (CR2), $(\lambda x^A.t)s \in [F]$.

- **If $D \equiv E \land F$, in both cases we must prove that $\pi_E(v) \in [E]$.**
  1. In case 1, we have to prove that $\pi_E(\lambda x^A.t) \in [E]$, by the i.h. (CR3’) it suffices to prove that every one-step $\Rightarrow_\Delta$ reduction issued from $\pi_E(\lambda x^A.t)$ is in $[E]$. by the i.h. (CR1), $t \in \text{SN}$. Therefore, we can proceed by third induction on $|t|$. The possible $\Rightarrow_\Delta$ reductions issued from $\pi_E(\lambda x^A.t)$ are:
    - A reduction in $t$, in which case, the third i.h. applies.
    - $\pi_E(\lambda x^A.t) \Rightarrow_\Delta \pi_E(\lambda x^A.t_1 \times \lambda x^A.t_2) \Rightarrow \lambda x^A.t_1$. By Lemmas 4.2 and 2.10, $E \equiv A \Rightarrow E'$ and $F \equiv A \Rightarrow F'$, with $t_1 : E'$ and $t_2 : F'$. In addition, since $A \Rightarrow B \equiv T \equiv D \equiv E \land F \equiv A \Rightarrow (E' \land F')$, by Lemma 2.14, we have $B \equiv E' \land F'$. Therefore, since $t[r/x] \in [B]$,
\[ \pi_E(t[r/x]) \in [E'], \text{ by the i.h. (CR2)}, \ t_1[r/x] \in [E']. \] We have
\[ m(A \Rightarrow E') = m(E) < m(D) = m(A \Rightarrow B), \text{ hence by the i.h., } \lambda x^A.t_1 \in [E]. \]

2. In case 2, we have to prove that \( \pi_E((\lambda x^A.t)u) \in [E] \). by the i.h. (CR3') it suffices to prove that every one-step \( \Rightarrow \Delta \) reduction issued from \( \pi_E((\lambda x^A.t)u) \) is in \([E]\). by the i.h. (CR1), \( t, u \in \mathsf{SN} \). Therefore, we can proceed by third induction on \(|t| + |u| \). The possible \( \Rightarrow \Delta \) reductions issued from \( \pi_E((\lambda x^A.t)u) \) are:

- A reduction in \( t \) or in \( u \), in which case, the third i.h. applies.

- \( \pi_E((\lambda x^A.t)u) \Rightarrow \pi_E(t[u/x]), \) hence by Lemmas 4.2 and 4.1, \( A \equiv C \), and so, by Lemma 2.14, \( B \equiv D \equiv E \land F \). Since \( t[u/x] \in [B] \), we have \( \pi_E(t[u/x]) \in [E] \).

- \( \pi_E((\lambda x^A.t)u) \Rightarrow \pi_E(t[u_1/x]u_2), \) with \( u \Rightarrow^* u_1 \times u_2 \), hence by Lemmas 4.2 and 4.1, \( C \equiv A \land C', \) with \( u_1 : A \) and \( u_2 : C' \). Therefore, by Lemma 2.14, \( B \equiv C' \Rightarrow (E \land F) \). Since \( u \in [C] \), we have \( \pi_A(u) \in [A] \) and \( \pi_{C'}(u) \in [C'] \). Then, by the i.h. (CR2), \( u_1 \in [A] \) and \( u_2 \in [C'] \). Then, \( t[u_1/x] \in [B] = [C' \Rightarrow (E \land F)] \), so \( \pi_E(t[u_1/x]u_2) \in [E] \).

- \( \pi_E((\lambda x^A.t)u) \Rightarrow \pi_E((\lambda x^A.t_1)u \times (\lambda x^A.t_2)u), \) with \( t \Rightarrow^* t_1 \times t_2 \). Hence, by Lemmas 4.2 and 4.1, \( B \equiv B_1 \land B_2 \), with \( t_1 : B_1, t_2 : B_2 \). Since \( t \in [B] = [B_1 \land B_2] \), then \( \pi_B(t) \in [B_1] \), and so, by the i.h. (CR2), \( t \in [B_1] \). In the same way, since \( t[r/x] \in [B] \), \( t_1[r/x] \in [B_1] \). Since \( (A \Rightarrow B_1) \land (A \Rightarrow B_2) \equiv C \Rightarrow D, \) we have, by Lemma 2.10, \( D \equiv D_1 \land D_2 \), and \( A \Rightarrow B_1 \equiv C \Rightarrow D_1 \). Then, by the i.h., \( (\lambda x^A.t_1)u \in [D_1] \) and \( (\lambda x^A.t_2)u \in [D_2] \). Therefore, since \( m(D_1 \times D_2) = m(D) \leq m(C \Rightarrow D) = m(T), \) by the i.h. (Adequacy of product), \( (\lambda x^A.t_1)u \times (\lambda x^A.t_2)u \in [D_1 \land D_2] = [D] = [E \land F] \), so, by definition, \( \pi_E((\lambda x^A.t_1)u \times (\lambda x^A.t_2)u) \in [E] \).

We finally prove the adequacy lemma and the strong normalization theorem.

**Definition 5.9 (Adequate substitution).** A substitution \( \sigma \) is adequate if for all \( x \in \mathcal{V}_A, \ \sigma(x) \in [A] \).

**Lemma 5.10 (Adequacy).** If \( r : A \), then for all \( \sigma \) adequate, \( \sigma r \in [A] \).

**Proof.** By induction on \( r \).

- If \( r \) is a variable \( x \in \mathcal{V}_A \), then, since \( \sigma \) is adequate, we have \( \sigma r \in [A] \).

- If \( r \) is a product \( s \times t \), then by Lemma 4.2, \( s : B, t : C \), and \( A \equiv B \land C \), then by the i.h., \( \sigma s \in [B] \) and \( \sigma t \in [C] \). By Lemma 5.8 (adequacy of product), \( (\sigma s \times \sigma t) \in [B \land C] \), hence, \( \sigma r \in [A] \).

- If \( r \) is a projection \( \pi_A(s) \), then by Lemma 4.2, \( s : A \land B \), and by the i.h., \( \sigma s \in [A \land B] \). Therefore, \( \sigma(\pi_A(s)) = \pi_A(\sigma s) \in [A] \).
• If $r$ is an abstraction $\lambda x^B.s$, with $s : C$, then by Lemma 4.2, $A \equiv B \Rightarrow C$, hence by the i.h., for all $\sigma$, and for all $t \in \llbracket B \rrbracket$, $(\sigma s)[t/x] \in \llbracket C \rrbracket$. Hence, by Lemma 5.8 (adequacy of abstraction), $\lambda x^B.\sigma s \in \llbracket B \Rightarrow C \rrbracket$, hence, $\sigma r \in \llbracket A \rrbracket$.

• If $r$ is an application $st$, then by Lemma 4.2, $s : B \Rightarrow A$ and $t : B$, then by the i.h., $\sigma s \in \llbracket B \Rightarrow A \rrbracket$ and $\sigma t \in \llbracket B \rrbracket$. Then $\sigma(st) = \sigma s \sigma t \in \llbracket A \rrbracket$. □

**Theorem 5.11** (Strong normalization). If $r : A$, then $r \in SN$.

**Proof.** By Lemma 5.8 (CR3'), for all type $B$, $x^B \in \llbracket B \rrbracket$, so the identity substitution is adequate. Thus, by Lemma 5.10 and Lemma 5.8 (CR1), $r \in \llbracket A \rrbracket \subseteq SN$. □

6. Consistency

We say that a term is $\sim_\Delta$-normal whenever it cannot continue reducing by relation $\sim_\Delta$, that is, a term that cannot be $\beta$, $\pi$, or $\zeta$-reduced, but may be expanded by rules $\eta$ or $\delta$.

**Lemma 6.1.** If $r : A \land B$ is closed $\sim_\Delta$-normal, then $r \equiv^* r_1 \times r_2$, with $r_1 : A$ and $r_2 : B$.

**Proof.** We proceed by induction on $M(r)$.

- $r$ cannot be a variable, since it is closed.

- If $r = u \times v$, then by Lemma 4.2, $u : C, v : D$, and $C \land D \equiv A \land B$. Then, by Lemma 2.11, one of the following cases happens

  - $A \equiv C_1 \land D_1$ and $B \equiv C_2 \land D_2$, with $C \equiv C_1 \land C_2$ and $D \equiv D_1 \land D_2$. Then, by the i.h., $u \equiv^* u_1 \times u_2$ with $u_1 : C_1$ and $u_2 : C_2$, and $v \equiv^* v_1 \times v_2$ with $v_1 : D_1$ and $v_2 : D_2$. So, take $r_1 = u_1 \times v_1$ and $r_2 = u_2 \times v_2$.

  - $B \equiv C \land D_2$, with $D \equiv A \land D_2$. Then, by the i.h., $v \equiv^* v_1 \times v_2$. Take $r_1 = v_1$ and $r_2 = u \times v_2$. Three other cases are symmetric.

  - $A \equiv C$ and $B \equiv D$, take $r_1 = u$ and $r_2 = v$. The last case is symmetric.

- If $r = \lambda x^C.r'$, then, by Lemma 4.2, $A \land B \equiv C \Rightarrow D$, and so, by Lemma 2.10, $D \equiv D_1 \land D_2$, with $A \equiv C \Rightarrow D_1$ and $B \equiv C \Rightarrow D_2$. Hence, by the i.h., $r' \equiv^* r'_1 \times r'_2$ with $r'_1 : D_1$ and $r'_2 : D_2$. Therefore, $r \equiv^* (\lambda x^C.r'_1) \times (\lambda x^C.r'_2)$, with $\lambda x^C.r'_1 : C \Rightarrow D_1 \equiv A$ and $\lambda x^C.r'_2 : C \Rightarrow D_2 \equiv B$.

- If $r = r_1 r_2$, then by Lemma 4.2, $r_1 : C \Rightarrow A \land B \equiv (C \Rightarrow A) \land (C \Rightarrow B)$, so, by the i.h., $r_1 \equiv^* s \times t$, and so $(s \times t)r_2 \leftrightarrow sr_2 \times tr_2$, so $r$ is not $\sim_\Delta$-normal.
• If \( r = \pi_A \land B(r') \), then, by Lemma 4.2, \( r' : A \land B \land C \), so, by the i.h.,
\( r' \leadsto^* s_1 \times s_2 \), with \( s_1 : A \land B \), and so \( r \) is not \( \leadsto_\Delta \)-normal.

**Theorem 6.2** (Consistency). There is no closed term in normal form of type \( \tau \).

**Proof.** Consider a closed term in normal form \( r \) of type \( \tau \).

- If \( r \) is a variable, it is not closed.
- If \( r \) is an abstraction or a product, then by Lemma 4.2, it does not have type \( \tau \).
- If \( r \) is a projection \( r = \pi_\tau(r') \), then, by Lemma 4.2, \( r' : \tau \land A \). Hence, since \( r \) is in normal form, \( r' \) is \( \leadsto_\Delta \)-normal, so, by Lemma 6.1, \( r' \leftrightarrow^* r_1 \times r_2 \) with \( r_1 : \tau \), hence \( r \) is not in normal form.
- If \( r \) is an application, \( r = st_1 \ldots t_n \), with \( n \geq 1 \), and \( s \not\leftrightarrow^* s_1s_2 \), then let \( t = t_1 \times \cdots \times t_n \), so we have \( r \leftrightarrow^* st \), and consider the cases for \( s \).
  - \( s \) cannot be a variable, since the term is closed.
  - \( s \) cannot be an abstraction \( \lambda x : C . s' \), since, by Lemmas 4.2 and 2.12, \( t : C \), or \( t : C \land D \). In the first case, the term \( r \) is a \( \beta \)-redex, hence it is not in normal form, in the second case, we have that since \( r \) and \( t \) are in normal form, so it is also \( \leadsto_\Delta \)-normal, and by Lemma 6.1, \( t \leftrightarrow^* u \times v \), with \( u : C \), so \( r \leftrightarrow^* (\lambda x : C . s')uv \), which contains a \( \beta \)-redex.
  - \( s \) cannot be an application, by hypothesis.
  - \( s \) cannot be a product, since \( st \) would be a \( \zeta \)-redex.
  - \( s \) cannot be a projection \( \pi_A(s') \), since in such a case, by Lemma 4.2, \( s' : A \land B \), and it would be \( \leadsto_\Delta \)-normal, so, by Lemma 6.1, \( s' \leftrightarrow^* s_1 \times s_2 \) with \( s_1 : A \), and so, \( r \) would contain a \( \pi \)-redex.

Note that, in the proof of Theorem 6.2, we need Lemma 6.1 to handle the case of the projection, but no analogous lemma for implication is needed, as both \( (\lambda x : A . s)t \) and \( (s_1 \times s_2)t \) can be reduced.

**Theorem 6.3** (Introduction property).

- If \( r \) is a closed term in normal form of type \( A \Rightarrow B \), then \( r \) is an introduction.
- If \( r \) is a closed term in normal form of type \( A \land B \), then \( r \) is an introduction.

**Proof.** If \( r \) has type \( A \Rightarrow B \) and it is not an introduction then it can be \( \eta \)-expanded and it is not normal. If \( r \) has type \( A \land B \) and it is not an introduction it can be \( \delta \)-expanded and it is not normal.

**Corollary 6.4.** If \( r : A \) is a closed term in normal form, then \( r \) is an introduction.

**Proof.** Since \( r \) is a closed term in normal form, by Theorem 6.2, \( A \neq \tau \). Hence \( A = B \Rightarrow C \) or \( A = B \land C \). We conclude with Theorem 6.3.
7. Conclusion

We have proposed a calculus, System $\eta$, where conjunction is associative and commutative, where implication distributes over conjunction and where currified and uncurrified proofs are equated. In this calculus, reduction is non-deterministic, but it enjoys termination and subject reduction: a proof of a proposition $A$ always reduces to a proof of this same proposition.

Compared with System I, a first version of this calculus without the extensionality rules, System $\eta$, also enjoys the introduction property, and abstractions are not restricted to prime types. This means that, unlike in simply typed lambda-calculus, where the $\eta$-rule can be considered or not, when isomorphic types are equated, this rule seems to be mandatory to unblock terms like $(\lambda x^A.\lambda y^B.x)r$, where $r : B$, $(\lambda x^{A\land B}.x)r$, where $r : A$, or $\pi_{A\Rightarrow B}(\lambda x^A.r)$, where $r : B \land C$.

In this preliminary work, we consider implication and conjunction only. This system needs to be extended to other connectives and quantifiers of predicate logic and, possibly, to more complex systems, such as dependent type theory. A first step in this direction is the Polymorphic System I [31], that adds the universal quantifiers at the level of types to System I.

Yet, with these two connectives the proofs are more complex than for simply typed lambda-calculus, but the work on Polymorphic System I shows that they scale, at least for the case of the universal quantifiers at the level of types.

Finally, we have addressed in this paper the syntactic properties of System $\eta$ only. The construction of a model for this system is left for future work.

Acknowledgements

The authors would like to thank Jean-Baptiste Joinet for useful comments and discussions.

References

[1] P. Arrighi and A. Díaz-Caro. A System F accounting for scalars. *Logical Methods in Computer Science*, 8(1:11), 2012.

[2] P. Arrighi and G. Dowek. Linear-algebraic lambda-calculus: higher-order, encodings, and confluence. In A. Voronkov, editor, *Proceedings of RTA 2008*, volume 5117 of *LNCS*, pages 17–31, 2008.

[3] P. Arrighi and G. Dowek. Lineal: A linear-algebraic lambda-calculus. *Logical Methods in Computer Science*, 13(1:8), 2017.

[4] P. Arrighi, A. Díaz-Caro, and B. Valiron. The vectorial lambda-calculus. *Information and Computation*, 254(1):105–139, 2017.

[5] G. Boudol. Lambda-calculi for (strict) parallel functions. *Information and Computation*, 108(1):51–127, 1994.
[6] K. B. Bruce, R. Di Cosmo, and G. Longo. Provable isomorphisms of types. *Mathematical Structures in Computer Science*, 2(2):231–247, 1992.

[7] A. Bucciarelli, T. Ehrhard, and G. Manzonetto. A relational semantics for parallelism and non-determinism in a functional setting. *Annals of Pure and Applied Logic*, 163(7):918–934, 2012.

[8] T. Coquand and G. Huet. The calculus of constructions. *Information and Computation*, 76(2–3):95–120, 1988.

[9] U. de'Liguoro and A. Piperno. Non deterministic extensions of untyped $\lambda$-calculus. *Information and Computation*, 122(2):149–177, 1995.

[10] M. Dezani-Ciancaglini, U. de'Liguoro, and A. Piperno. A filter model for concurrent $\lambda$-calculus. *SIAM Journal on Computing*, 27(5):1376–1419, 1998.

[11] R. Di Cosmo. *Isomorphisms of types: from $\lambda$-calculus to information retrieval and language design*. Progress in Theoretical Computer Science. Birkhauser, 1995.

[12] R. Di Cosmo. A short survey of isomorphisms of types. *Mathematical Structures in Computer Science*, 15(5):825–838, 2005.

[13] R. Di Cosmo and D. Kesner. Simulating expansions without expansions. *Mathematical Structures in Computer Science*, 4(3):315–362, 1994.

[14] A. Díaz-Caro and G. Dowek. Typing quantum superpositions and measurement. In C. Martín-Vide, R. Neruda, and M. A. Vega-Rodríguez, editors, *Proceedings of TPNC 2017*, volume 10687 of *LNCS*, pages 281–293, 2017.

[15] A. Díaz-Caro and G. Dowek. Proof normalisation in a logic identifying isomorphic propositions. In H. Geuvers, editor, *4th International Conference on Formal Structures for Computation and Deduction (FSCD 2019)*, volume 131 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 14:1–14:23. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2019.

[16] A. Díaz-Caro and P. E. Martínez López. Isomorphisms considered as equalities: Projecting functions and enhancing partial application through an implementation of $\lambda^\dagger$. In *Proceedings of the 27th Symposium on the Implementation and Application of Functional Programming Languages*, IFL ’15, pages 9:1–9:11. ACM, 2015.

[17] A. Díaz-Caro and B. Petit. Linearity in the non-deterministic call-by-value setting. In L. Ong and R. de Queiroz, editors, *Proceedings of WoLLIC 2012*, volume 7456 of *LNCS*, pages 216–231, 2012.

[18] A. Díaz-Caro, M. Guillermo, A. Miquel, and B. Valiron. Realizability in the unitary sphere. In *Proceedings of the 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2019)*, pages 1–13, 2019.
[19] G. Dowek and B. Werner. Proof normalization modulo. *The Journal of Symbolic Logic*, 68(4):1289–1316, 2003.

[20] G. Dowek, T. Hardin, and C. Kirchner. Theorem proving modulo. *Journal of Automated Reasoning*, 31(1):33–72, 2003.

[21] H. Geuvers, R. Krebbers, J. McKinna, and F. Wiedijk. Pure type systems without explicit contexts. In K. Crary and M. Miculan, editors, *Proceedings of LFMTP 2010*, volume 34 of *EPTCS*, pages 53–67, 2010.

[22] C. B. Jay and N. Ghani. The virtues of eta-expansion. *Journal of Functional Programming*, 5(2):135–154, 1995.

[23] O. Laurent. Classical isomorphisms of types. *Mathematical Structures in Computer Science*, 15(5):969–1004, 2005.

[24] P. Martin-Löf. *Intuitionistic type theory*. Studies in proof theory. Bibliopolis, 1984.

[25] M. Pagani and S. Ronchi Della Rocca. Linearity, non-determinism and solvability. *Fundamental Informaticae*, 103(1–4):173–202, 2010.

[26] J. Park, J. Seo, S. Park, and G. Lee. Mechanizing metatheory without typing contexts. *Journal of Automated Reasoning*, 52(2):215–239, 2014.

[27] L. Regnier. Une équivalence sur les lambda-termes. *Theoretical Computer Science*, 126(2):281–292, 1994.

[28] G. E. Révész. A list-oriented extension of the lambda-calculus satisfying the Church-Rosser theorem. *Theoretical Computer Science*, 93(1):75–89, 1992.

[29] G. E. Révész. Categorical combinations with explicit products. *Fundamenta Informaticae*, 22(1/2):153–166, 1995.

[30] M. Rittri. Retrieving library identifiers via equational matching of types. In *Proceedings of CADE 1990*, volume 449 of *LNCS*, pages 603–617, 1990.

[31] C. F. Sottile, A. Díaz-Caro, and P. E. Martínez López. Polymorphic System I. In *IFL 2020: Proceedings of the 32nd Symposium on Implementation and Application of Functional Languages*, IFL 2020, pages 127–137. ACM, 2020.

[32] K. Støvring. Extending the extensional lambda calculus with surjective pairing is conservative. In *Logical Methods in Computer Science. Superseeds*, pages 05–35, 2006.

[33] L. Vaux. The algebraic lambda calculus. *Mathematical Structures in Computer Science*, 19(5):1029–1059, 2009.