1-DIMENSIONAL HARNACK ESTIMATES

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Dedicated to the memory of our friend Alfredo Lorenzi

Abstract. Let \( u \) be a non-negative super-solution to a 1-dimensional singular parabolic equation of \( p \)-Laplacian type \( (1 < p < 2) \). If \( u \) is bounded below on a time-segment \( \{ y \} \times (0, T] \) by a positive number \( M \), then it has a power-like decay of order \( \frac{1}{p^2} \) with respect to the space variable \( x \) in \( \mathbb{R} \times \left[ T/2, T \right] \). This fact, stated quantitatively in Proposition 1.2, is a “sidewise spreading of positivity” of solutions to such singular equations, and can be considered as a form of Harnack inequality. The proof of such an effect is based on geometrical ideas.

1. Introduction. Let \( E = (\alpha, \beta) \) and define \( E_{-\tau_0, T} = E \times (-\tau_0, T] \), for \( \tau_0, T > 0 \). Consider the non-linear diffusion equation

\[
 u_t - (|u_x|^{p-2}u_x)_x = 0, \quad 1 < p < 2. \tag{1.1}
\]

A function \( u \in C_{\text{loc}}(-\tau_0, T; L^2_{\text{loc}}(E)) \cap L^p_{\text{loc}}(-\tau_0, T; W^{1,p}_{\text{loc}}(E)) \) is a local, weak super-solution to 1.1, if for every compact set \( K \subset E \) and every sub-interval \( [t_1, t_2] \subset (-\tau_0, T] \)

\[
 \int_K u \varphi dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \left[ -u \varphi_t + |u_x|^{p-2}u_x \varphi_x \right] dx dt \geq 0 \tag{1.3}
\]

for all non-negative test functions \( \varphi \in W^{1,2}_{\text{loc}}(-\tau_0, T; L^2(K)) \cap L^p_{\text{loc}}(-\tau_0, T; W^{1,p}_{\text{loc}}(K)) \).

This guarantees that all the integrals in 1.3 are convergent. These equations are termed singular since, for \( 1 < p < 2 \), the modulus of ellipticity \( |u_x|^{p-2} \to \infty \) as \( |u_x| \to 0 \).

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Remark 1.1. Since we are working with local solutions, we consider the domain $E_{-\tau_o,T} = E \times (-\tau_o,T]$, instead of dealing with the more natural $E_T = E \times (0,T]$, in order to avoid problems with the initial conditions. The only role played by $\tau_o > 0$ is precisely to get rid of any difficulty at $t = 0$, and its precise value plays no role in the argument to follow.

Proposition 1.2. Let $u$ be a non-negative, local, weak super-solution to 1.1 in $E_{-\tau_o,T}$, in the sense of 1.2–1.3, satisfying $u(y,t) > M \quad \forall \ t \in (0,T/2]$ \hspace{1cm} (1.4) for some $y \in E$, and for some $M > 0$. Let $\bar{\rho} \overset{\text{def}}{=} \left( \frac{2^{\frac{2}{p-2}} M}{2^{\frac{2}{p-2}}-p} \right)^{\frac{1}{p}}$, take $\rho \geq 4\bar{\rho}$, and assume that $B_\rho(\bar{x}) \subset B_{4\rho}(y) \subset E$, where $\text{dist}(\bar{x},y) = 2\rho$.

There exists $\bar{\sigma} \in (0,1)$, that can be determined a priori, quantitatively only in terms of the data, and independent of $M$ and $T$, such that $u(x,t) \geq \bar{\sigma} M \left( \frac{\bar{\rho}}{\rho} \right)^{\frac{1}{p}}$ for all $(x,t) \in B_\frac{\rho}{4}(\bar{x}) \times \left[ \frac{T}{4}, \frac{T}{2} \right]$ \hspace{1cm} (1.5)

Remark 1.3. Strictly speaking, it might not be possible to satisfy the assumption $\rho \geq 4\bar{\rho}$ and $B_{4\rho}(y) \subset E$, if $E$ were too small; nevertheless, we can always assume it without loss of generality. Indeed, if it were not satisfied, we would decompose the interval $(0,T/2]$ in smaller subintervals, each of width $\tau$, such that the previous requirement is satisfied working with $\bar{\rho}$ replaced by $\hat{\rho} = \left( \frac{2^{\frac{2}{p-2}} \rho}{M^{2-p}} \right)^{\frac{1}{p}}$.

1.1. Novelty and significance. The measure theoretical information on the “positivity set” in $\{y\} \times (0,T/2]$ implies that such a positivity set actually “expands” sidewise in $\mathbb{R} \times \left[ \frac{T}{4}, \frac{T}{2} \right]$, with a power-like decay of order $\frac{p}{p-2}$ with respect to the space variable $x$. Although considered a sort of natural fact, to our knowledge this result has never been proven before; it is the analogue of the power-like decay of order $\frac{1}{p-2}$ with respect to the time variable $t$, known in the degenerate setting $p > 2$ (see [2], [3, Chapter 4, Section 4], [7]). As the $t^{-\frac{1}{p-2}}$-decay is at the heart of the Harnack estimate for $p > 2$, so Proposition 1.2 could be used to give a more streamlined proof of the Harnack inequality in the singular, super-critical range $2 < p < 2$. This will be the object of future work, where we plan to address the general $N$-dimensional case.

The proof is based on geometrical ideas, originally introduced in two different contexts: the energy estimates of § 2 and the decay of § 3 rely on a method introduced in [8] in order to prove the Hölder continuity of solutions to an anisotropic elliptic equation, and further developed in [5, 6]; the change of variable used in the actual proof of Proposition 1.2 was used in [4].
1.2. Further generalization. Consider partial differential equations of the form
\[ u_t - (A(x, t, u, u_x))_x = 0 \quad \text{weakly in } E_{-\tau_2, T}, \]  
where the function \( A : E_{-\tau_2, T} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is only assumed to be measurable and subject to the structure condition
\[
\begin{align*}
\{ & A(x, t, u, u_x) u_x \geq C_\alpha |u_x|^p, \\
& |A(x, t, u, u_x)| \leq C_1 |u_x|^{p-1} \end{align*}
\quad \text{a.e. in } E_{-\tau_2, T},
\]
where \( 1 < p < 2, \) \( C_\alpha \) and \( C_1 \) are given positive constants. It is not hard to show that Proposition 1.2 holds also for weak super-solutions to 1.6–1.7, since our proof is entirely based on the structural properties of 1.1, and the explicit dependence on \( u_x \) plays no role. However, to keep the exposition simple, we have limited ourselves to the prototype case.

2. Energy estimates. Let \( u \) be a non-negative, local, weak super-solution in \( E_{-\tau_2, T}, \) set
\[ 0 \leq \mu_- = \inf_{E_{-\tau_2, T}} u, \]
and let \( 0 < \omega < +\infty. \) Without loss of generality we may assume that \( 0 \in (\alpha, \beta). \) For \( \rho \) sufficiently small, so that \((-\rho, \rho) \subset (\alpha, \beta),\) let
\[
\begin{align*}
B_\rho &= (-\rho, \rho), \\
Q &= B_\rho \times (0, T], \\
B_\rho(y) &= (y - \rho, y + \rho), \\
Q(y) &= B_\rho(y) \times (0, T], \\
\alpha &\in (0, 1), \\
\tau_1, \tau_2 &\in (0, 1] \quad \text{parameters that will be fixed in the following,} \\
A &= \{(x, t) \in Q(y) : u(x, t) < \mu_- + (1 - a)H\omega\}, \\
A(\tau) &= \{x \in B_\rho(y) : u(x, \tau) < \mu_- + (1 - a)H\omega\}, \\
0 &\leq \tau \leq T.
\end{align*}
\]

**Proposition 2.1.** Let \( u \) be a non-negative, local, weak super-solution to 1.1 in \( E_{-\tau_2, T}, \) in the sense of 1.2–1.3. There exists a positive constant \( \gamma = \gamma(p), \) such that for every cylinder \( Q(y) = B_\rho(y) \times (0, T] \subset E_{-\tau_2, T}, \) and every piecewise smooth, cutoff function \( \zeta \) vanishing on \( \partial B_\rho(y), \) such that \( 0 \leq \zeta \leq 1, \) and \( \zeta_\ell \leq 0, \)
\[
\begin{align*}
\int_{B_\rho(y) \cap \{(x, 0) < \mu_- + (1 - a)H\omega\}} &\frac{(u(x, 0) - \mu_- + a\omega H)^{2-p}}{2 - p} \\
&\cdot \bigg[ \frac{u(x, 0) - \mu_- + a\omega H}{(\omega H)^{p-1}} \bigg] \zeta_p(x, 0) dx + \int_{A} |u_x|^p \zeta^p dx + \int_{A} (u - \mu_- + a\omega H)^p \zeta^p = 0
\end{align*}
\]  
\[
\leq \gamma \int_{A} |\zeta_x|^p dx + \gamma \int_{A} (u - \mu_- + a\omega H)^{2-p} |\zeta_\ell| dx.
\]

**Proof.** Without loss of generality, we may assume \( y = 0. \) In the weak formulation of 1.1 take \( \varphi = G(u)\zeta^p \) as test function, with
\[
G(u) = \frac{1}{(u - \mu_- + a\omega H)^{p-1}} - \frac{1}{(\omega H)^{p-1}},
\]
and \( \zeta \) a piecewise smooth, cutoff function vanishing on \( \partial B_\rho, \) and on \( B_\rho \times \{T\}, \) such that \( 0 \leq \zeta \leq 1, \) and \( \zeta_\ell \leq 0. \) It is easy to see that we have
\[
G'(u) = -\frac{p - 1}{(u - \mu_- + a\omega H)^p} \chi_A.
\]
Modulo a Steklov averaging process, we have

\[
\int_Q u_t G(u) \zeta^p \, dx \, dt + \int_Q \zeta^p G'(u) |u_x|^p \, dx \, dt + p \int_Q G(u) |u_x|^{p-2} \zeta^{p-1} u_x \cdot \zeta \, dx \, dt \geq 0,
\]

\[
(p - 1) \int_A \frac{|u_x|^p}{(u - \mu_+ + a\omega H)^p} \zeta^p \, dx \, dt \leq p \int_A \zeta^{p-1} \frac{|u_x|^{p-1}}{(u - \mu_+ + a\omega H)^{p-1}} |\zeta_x| \, dx \, dt + \int_A \frac{|u_t|}{(u - \mu_+ + a\omega H)^{p-1}} \zeta^p \, dx \, dt - \int_A \frac{|u_t|}{(\omega H)^{p-1}} \zeta^p \, dx \, dt,
\]

\[
(p - 1) \int_A \frac{|u_x|^p}{(u - \mu_+ + a\omega H)^p} \zeta^p \, dx \, dt \leq p \int_A \zeta^{p-1} \frac{|u_x|^{p-1}}{(u - \mu_+ + a\omega H)^{p-1}} |\zeta_x| \, dx \, dt + \int_A \partial_t \left[ \frac{(u - \mu_+ + a\omega H)^{2-p}}{2-p} - \frac{u - \mu_+}{(\omega H)^{p-1}} \right] \zeta^p \, dx \, dt,
\]

\[
(p - 1) \int_A \frac{|u_x|^p}{(u - \mu_+ + a\omega H)^p} \zeta^p \, dx \, dt \leq p \int_A \zeta^{p-1} \frac{|u_x|^{p-1}}{(u - \mu_+ + a\omega H)^{p-1}} |\zeta_x| \, dx \, dt + \int_{A(T)} \frac{(u(x, T) - \mu_+ + a\omega H)^{2-p}}{2-p} - \frac{u(x, T) - \mu_+}{(\omega H)^{p-1}} \zeta^p(x, T) \, dx \]

\[
- \int_{A(0)} \left[ \frac{(u(x, 0) - \mu_+ + a\omega H)^{2-p}}{2-p} - \frac{u(x, 0) - \mu_+}{(\omega H)^{p-1}} \right] \zeta^p(x, 0) \, dx
\]

\[-p \int_A \left[ \frac{(u - \mu_+ + a\omega H)^{2-p}}{2-p} - \frac{u - \mu_+}{(\omega H)^{p-1}} \right] \zeta^{p-1} \zeta_t \, dx \, dt.
\]

The second term on the right-hand side vanishes, as \( \zeta(x, T) = 0 \). An application of Young’s inequality yields

\[
(p - 1) \int_A \frac{|u_x|^p}{(u - \mu_+ + a\omega H)^p} \zeta^p \, dx \, dt + \int_{B} \frac{(u(x, 0) - \mu_+ + a\omega H)^{2-p}}{2-p} \frac{|u_x|^p}{(\omega H)^{p-1}} \zeta^p(x, 0) \, dx \leq \frac{p - 1}{2} \int_A \frac{|u_x|^p}{(u - \mu_+ + a\omega H)^p} \zeta^p \, dx \, dt
\]

\[
+ \gamma \int_A |\zeta_x|^p \, dx \, dt + p \int_A \frac{(u - \mu_+ + a\omega H)^{2-p}}{2-p} \zeta^{p-1} |\zeta_t| \, dx \, dt,
\]
where we have taken into account that $\zeta_t \leq 0$. Therefore, we conclude

$$
\int_{B_{\rho} \cap \{u(x,0) < \mu_-(1-a)H\omega\}} \left[ \frac{(u(x,0) - \mu_- + a\omega H)^{2-p}}{2-p} - \frac{u(x,0) - \mu_-}{(\omega H)^{p-1}} \right] \zeta^p(x,t) \, dx + \frac{p-1}{2} \int_A |u_x|^p \zeta^p \, dx\,dt
\leq \gamma \int_A |\zeta|^p \, dx\,dt + \gamma \int_A \left( u - \mu_- + a\omega H \right)^{2-p} \zeta^{p-1} |\zeta_t| \, dx\,dt.
$$

Notice that the first term on the left-hand side is non–negative. Indeed, since $1 < p < 2$, first of all we have

$$
\frac{(u(x,0) - \mu_- + a\omega H)^{2-p}}{2-p} - \frac{u(x,0) - \mu_-}{(\omega H)^{p-1}} \geq (u(x,0) - \mu_- + a\omega H)^{2-p} - \frac{u(x,0) - \mu_-}{(\omega H)^{p-1}}.
$$

Now, if we let $v = u(x,0) - \mu_-$, we have

$$
(u(x,0) - \mu_- + a\omega H)^{2-p} - \frac{u(x,0) - \mu_-}{(\omega H)^{p-1}} = \frac{v}{(\omega H)^{p-1}} \left[ \left( \frac{v}{\omega H} + a \right)^{2-p} - 1 \right].
$$

To conclude, it suffices to remark that for $0 < s < 1 - a < 1$ the function $f(s) = \frac{(s+a)^{2-p}}{s}$ is monotone decreasing, and $f(1-a) = \frac{1}{1-a} > 1$. \hfill \Box

**Remark 2.2.** The constant $\gamma$ deteriorates, as $p \to 1$.

**Remark 2.3.** Even though in the next Section $H$ basically plays no role, we chose to state the previous Proposition with an explicit dependence also on $H$ for future applications. The same applies to $\omega$: in the next Section it will play the role of the lower bound $M$ for $u$ on a proper set, and we could have directly used such a notation, as indicated below. However, we have in mind future applications, where $\omega$ will have a more general meaning.

3. **A decay lemma.** Without loss of generality, we may assume $\mu_- = 0$. Let $M = \omega$, $L \leq \frac{M}{2}$, and suppose that

$$
u(0, t) > M \quad \forall t \in (0, \frac{T}{2}] \quad (3.1)
$$

Now, let $s_o$ be an integer to be chosen, define

$$
F_{s_o} = \{ t \in (0, \frac{T}{2}] : \exists x \in B_{\frac{T}{2}}, u(x,t) < \frac{L}{2^{s_o}} \}
$$

$$
F(t) = \{ x \in B_{\frac{T}{2}} : u(x,t) < L(1 - \frac{1}{2^{s_o}}) \}, \quad t \in (0, \frac{T}{2}],
$$

and notice that with the previous choices,

$$
A = \{ (x,t) \in B_{\rho} \times (0,T) : u(x,t) < L(1 - \frac{1}{2^{s_o}}) \}.
$$
Lemma 3.1. Let $u$ be a non-negative, local, weak super-solution to 1.1 in $E_{-\tau_0,T}$, in the sense of 1.2–1.3. Let 3.1 hold and take

$$L \leq \min\left\{ \frac{M}{2}, \left(\frac{T}{\rho^p}\right)^{\frac{1}{1-p}} \right\}.$$ 

Then, for any $\nu \in (0,1)$, there exists a positive integer $s_o$ such that

$$|\{t \in (0,T] : \exists x \in B_{\frac{\rho}{2}}, u(x,t) \leq \frac{L}{2s_o} \}| \leq \nu |(0,T]|,$$

where $|G|$ denotes the $N$-dimensional Lebesgue measure of $G \subset \mathbb{R}^N$, with $N = 1$ or $N = 2$.

Proof. Take $t \in F_{s_o}$: by definition, there exists $\bar{x} \in B_{\frac{\rho}{2}}$ such that $u(\bar{x},t) < \frac{L}{2s_o}$.

On the other hand, by assumption $u(0,t) > 2L$, and therefore, $u(0,t) + (L/2s_o) > L$. Hence

$$\ln u(0,t) + \frac{L}{2s_o} > (s_o - 1) \ln 2,$$

and we obtain

$$(s_o - 1) \ln 2 \leq \ln \left( \frac{L}{u(\bar{x},t) + \frac{L}{2s_o}} \right) - \ln \left( \frac{L}{u(0,t) + \frac{L}{2s_o}} \right)$$

$$= \int_{0}^{\bar{x}} \frac{\partial}{\partial x} \left( \ln \left( \frac{L}{u(\xi,t) + \frac{L}{2s_o}} \right) \right) d\xi$$

$$\leq \int_{-\frac{\rho}{2}}^{\frac{\rho}{2}} \left| \frac{\partial}{\partial x} \left( \ln \left( \frac{L}{u(x,t) + \frac{L}{2s_o}} \right) \right) \right| dx$$

$$= \int_{B_{\frac{\rho}{2}} \cap F(t)} \left| \frac{\partial}{\partial x} \left( \ln \left( \frac{L}{u(x,t) + \frac{L}{2s_o}} \right) \right) \right| dx.$$

If we integrate with respect to time over the set $F_{s_o}$, we have

$$(s_o - 1)|F_{s_o}| \ln 2 \leq \int_{0}^{T} \int_{B_{\frac{\rho}{2}} \cap F(t)} \left| \frac{\partial}{\partial x} \left( \ln \left( \frac{L}{u(x,t) + \frac{L}{2s_o}} \right) \right) \right| dx dt$$

$$\leq \left[ \int_{0}^{T} \int_{B_{\frac{\rho}{2}} \cap F(t)} \left| \frac{\partial}{\partial x} \left( \ln \left( \frac{L}{u(x,t) + \frac{L}{2s_o}} \right) \right) \right|^p dx dt \right]^{\frac{1}{p}} \left| Q \right|^{\frac{p-1}{p}}$$

$$\leq \left[ \int_{Q \cap A} \frac{|u_x|^p}{(u + \frac{L}{2s_o})^p} \zeta^p dx dt \right]^{\frac{1}{p}} \left| Q \right|^{\frac{p-1}{p}},$$

where $\zeta$ is as in Proposition 2.1, and is chosen such that $\zeta = \zeta_1(x)\zeta_2(t)$, where $\zeta_1$ vanishes outside $B_{\rho}$ and satisfies

$$0 \leq \zeta_1 \leq 1, \quad \zeta_1 = 1 \text{ in } B_{\frac{\rho}{2}}, \quad |\partial_x \zeta_1| \leq \frac{\gamma_1}{\rho},$$

for an absolute constant $\gamma_1$ independent of $\rho$, and $\zeta_2$ is monotone decreasing, and satisfies

$$0 \leq \zeta_2 \leq 1, \quad \zeta_2 = 1 \text{ in } (0,\frac{T}{2}], \quad \zeta_2 = 0 \text{ for } t \geq T, \quad |\partial_t \zeta_2| \leq \frac{\gamma_2}{T},$$

for an absolute constant $\gamma_2$ independent of $T$. 
Apply estimates 2.1 with $a = \frac{1}{2}$, $H\omega = HM = L$. The requirement $H \leq 1$ is satisfied, since $L \leq \frac{M}{2}$. They yield

$$
(s_o - 1)|F_{s_o}| \leq \gamma |Q| \frac{\nu \omega}{p} \left[ \int_A |\zeta_x|^p dx \right] \frac{1}{p} + \gamma |Q| \frac{\nu \omega}{p} \left[ \int_A (u + \frac{L}{2s_o})^2 |\zeta_x| dx \right] \frac{1}{p}.
$$

By the choice of $\zeta$ we have

$$
(s_o - 1)|F_{s_o}| \leq \gamma |Q| \frac{\nu \omega}{p} + \gamma |Q| \frac{\nu \omega}{p} \left( \frac{L^2 - p}{T} \right) \frac{1}{p} |Q| \frac{1}{p}.
$$

If we require $L \leq \left( \frac{T}{\nu \omega} \right)^{\frac{1}{p-1}}$, and we substitute it back in the previous estimate, we have

$$
(s_o - 1)|F_{s_o}| \leq \gamma |Q| \frac{\nu \omega}{p} + \gamma |Q| \frac{\nu \omega}{p} \left( \frac{L^2 - p}{T} \right) \frac{1}{p} |Q| \frac{1}{p}.
$$

Therefore, if we want that $|F_{s_o}| \leq \nu |(0, T/2]|$, it is enough to require that $s_o = \frac{\nu \omega}{\nu \omega} + 1$.

The previous result can also be rewritten as

**Lemma 3.2.** Let $u$ be a non-negative, local, weak super-solution to 1.1 in $E-\tau_o, T$, in the sense of 1.2–1.3. Let 3.1 hold. For any $\nu \in (0, 1)$, there exists a positive integer $s_o$ such that

$$
|\{ t \in (0, T/2] : \exists x \in B_2, u(x, t) \leq \left( \frac{T}{\rho \omega} \right)^{\frac{1}{p}} \} | \leq \nu |(0, T/2]|,
$$

provided $\rho > 0$ is so large that $\left( \frac{T}{\rho \omega} \right)^{\frac{1}{p}} \leq \frac{M}{2}$.

Now let $\bar{\rho}$ be such that

$$
\left( \frac{T}{\bar{\rho} \omega} \right)^{\frac{1}{p}} = \frac{M}{2} \quad \Rightarrow \quad \bar{\rho} = \left( \frac{2^{2-p} T}{M^2} \right)^{\frac{1}{p}},
$$

and assume that $B_{\bar{\rho}} \subset (\alpha, \beta)$. Then Lemmas 3.1–3.2 can be rephrased as

**Lemma 3.3.** Let $u$ be a non-negative, local, weak super-solution to 1.1 in $E-\tau_o, T$, in the sense of 1.2–1.3. Let 3.1 hold. For any $\nu \in (0, 1)$, there exists a positive integer $s_o$ such that for any $\rho > \bar{\rho}$

$$
|\{ t \in (0, T/2] : \exists x \in B_2, u(x, t) \leq \left( \frac{\rho \omega}{\bar{\rho} \omega} \right)^{\frac{1}{p}} \} | \leq \nu |(0, T/2]|,
$$

provided that $B_{\bar{\rho}} \subset (\alpha, \beta)$.

**Remark 3.4.** The previous corollary gives us the power-like decay, required in Proposition 1.2.
Let us now set $F_{s_o} \overset{\text{def}}{=} (0, \frac{T}{2}] \backslash F_{s_o}$. Then, if 3.1 holds, we conclude that for any $t \in F_{s_o}$ and for any $x \in B_{\hat{\rho}}$ with $\rho > \hat{\rho}$

$$u(x, t) \geq \frac{M}{2^{s_o+1}} \left( \frac{\hat{\rho}}{\rho} \right)^{\frac{s_o}{p}}.$$ (3.3)

Let $c \geq 4$ denote a positive parameter, choose $\bar{x} \in (\alpha, \beta)$ such that $|\bar{x}| = 2c\rho$, and consider $B_{c\hat{\rho}}(\bar{x})$. Then, by 3.3

$$\forall x \in B_{c\hat{\rho}}(\bar{x}), \forall t \in F_{s_o}^c \; u(x, t) \geq \frac{M}{2^{s_o+1}} \left( \frac{2}{5c} \right)^{\frac{s_o}{p}}.$$ (3.4)

provided 3.1 holds, and $B_{c\rho}(\bar{x}) \subset (\alpha, \beta)$.

4. A DeGiorgi-Type lemma. Assume that some information is available on the “initial data” relative to the cylinder $B_{2\rho}(y) \times (s, s + \theta \rho^p]$, say for example

$$u(x, s) \geq M \quad \text{for a.e.} \; x \in B_{2\rho}(y) \quad (4.1)$$

for some $M > 0$. Then, the following Proposition is proved in [3, Chapter 3, Lemma 4.1].

**Lemma 4.1.** Let $u$ be a non-negative, local, weak super-solution to 1.1, and $M$ be a positive number such that 4.1 holds. Then

$$u \geq \frac{1}{2} M \quad \text{a.e. in} \; B_{\rho}(y) \times (s, s + \theta (4\rho)^p],$$

where

$$\theta = \delta M^{2-p}.$$ (4.2)

for a constant $\delta \in (0, 1)$ depending only upon $p$, and independent of $M$ and $\rho$.

**Remark 4.2.** Lemma 4.1 is based on the energy estimates and Proposition 3.1 of [1], Chapter I, which continue to hold in a stable manner for $p \to 1$. These results are therefore valid for all $p \geq 1$, including a seamless transition from the singular range $p < 2$ to the degenerate range $p > 2$.

5. Proof of Proposition 1.2. Fix $y \in E$, define $\hat{\rho}$ as in 3.2, and choose a positive parameter $C \geq 4$, such that the cylindrical domain

$$B_{\frac{2}{\sqrt{p}C\hat{\rho}}} (y) \times (0, \frac{T}{2}] \subset E_{-\tau_o, T}. \quad (5.1)$$

This is an assumption both on the size of the reference ball $B_{\frac{2}{\sqrt{p}C\hat{\rho}}} (y)$ and on $T$; we can always assume it without loss of generality. Indeed, as we have already pointed out in Remark 1.3, if 5.1 were not satisfied, we would decompose the interval $(0, \frac{T}{2}]$ in smaller subintervals, each of width $\tau$, such that 5.1 is satisfied working with $\hat{\rho}$ replaced by

$$\hat{\rho} = \left( \frac{2^{2-p} \tau}{M^{2-p}} \right)^{\frac{1}{p}}.$$ 

The only role of $C$ is in determining a sufficiently large reference domain

$$B_{\frac{2}{\sqrt{p}C\hat{\rho}}} (y) \subset E,$$

which contains the smaller ball we will actually work with, and will play no other role; in particular the structural constants will not depend on $C$. 

Now, introduce the change of variables and the new unknown function
\[ z = \frac{x - y}{\rho}, \quad -e^{-\tau} = \frac{t - T}{T}, \quad v(z, \tau) = \frac{1}{M} u(x, t) e^{\frac{x - y}{\rho}}. \] (5.2)

This maps the cylinder in 5.1 into \( B_C \times (0, \infty) \) and transforms 1.1 into
\[ v_{\tau} - \frac{1}{2} |v_z|^p v_z = \frac{1}{2 - p} v \quad \text{weakly in} \quad B_C \times (0, \infty). \] (5.3)

The only effect of the factor \( \frac{1}{2} \) in front of \( |v_z|^p v_z \) is to modify the constant \( \gamma \) in Proposition 2.1, and consequently \( s_o \) in Lemmas 3.1–3.3. By the previous change of variable, assumption 1.4 of Proposition 1.2 becomes
\[ v(0, \tau) \geq e^{\frac{x - y}{\rho}} \quad \text{for all} \quad \tau \in (0, +\infty). \] (5.4)

Let \( \tau_o > 0 \) to be chosen and set
\[ k = e^{\frac{x - y}{\rho}}. \]

With this symbolism, 5.4 implies
\[ v(0, \tau) \geq k \quad \text{for all} \quad \tau \in (\tau_o, +\infty). \] (5.5)

Now consider the segment
\[ I \overset{\text{def}}{=} \{ 0 \} \times (\tau_o, \tau_o + \frac{1}{2} k^{2-p}). \]

Let \( \nu = \frac{1}{2} \) and \( s_o \) be the corresponding quantity introduced in Lemma 3.1. We can then apply Lemmas 3.1–3.3 with \( T = k^{2-p} \), \( M \) substituted by \( k \),
\[ F_{s_o} = \{ \tau \in (\tau_o, \tau_o + \frac{1}{2} k^{2-p}) : \exists z \in B_{\frac{1}{2}k}, \ v(z, \tau) < \frac{k}{2^{s_o + 1}} \} \quad \text{for} \quad \rho > \rho_s, \]
with \( \rho_s \overset{\text{def}}{=} 2^{\frac{2-p}{p}}. \) Therefore, if \( c \geq 4 \) denotes a positive parameter, we choose \( \bar{z} \in B_C \) such that \( |\bar{z}| = 2c\rho_s \), and consider \( B_{cp_s}(\bar{z}) \), by 3.3
\[ \forall z \in B_{c, \frac{1}{2}k}(\bar{z}), \ \forall \tau \in F_{s_o}^c \ v(z, \tau) \geq \frac{k}{2^{s_o + 1}} \left( \frac{2}{5c} \right)^{2-p}, \] (5.6)
provided \( B_{cp_s}(\bar{z}) \subset B_C \). Summarising, there exists at least a time level \( \tau_1 \) in the range
\[ \tau_o < \tau_1 < \tau_o + \frac{1}{2} k^{2-p} \] (5.7)
such that
\[ \forall z \in B_{c, \frac{1}{2}k}(\bar{z}), \ v(z, \tau_1) \geq \sigma_o e^{\frac{x - y}{\rho}} \quad \text{where} \quad \sigma_o = \frac{1}{2^{s_o + 1}} \left( \frac{2}{5c} \right)^{\frac{p}{2-p}}. \]

**Remark 5.1.** Notice that \( \sigma_o \) is determined only in terms of the data and is independent of the parameter \( \tau_o \), which is still to be chosen.
5.1. Returning to the original coordinates. In terms of the original coordinates and the original function $u(x, t)$, this implies

$$u(\cdot, t_1) \geq \sigma_o M e^{\frac{-\tau_1 - \bar{\tau}_o}{2p}} = M_o \quad \text{in } B_{\frac{\rho}{\delta}}(\bar{x})$$

where the time $t_1$ corresponding to $\tau_1$ is computed from 5.2 and 5.7, and $\text{dist}(\bar{x}, y) = 2c\bar{\rho}$. Now, apply Lemma 4.1 with $M$ replaced by $M_o$ over the cylinder $B_{\frac{\rho}{\delta}}(\bar{x}) \times (t_1, t_1 + \theta(c\bar{\rho})^p)$. By choosing

$$\theta = \delta M_o^{2-p} \quad \text{where } \delta = \delta(\text{data}),$$

the assumption 4.2 is satisfied, and Lemma 4.1 yields

$$u(\cdot, t) \geq \frac{1}{2} M_o = \frac{1}{2} \sigma_o M e^{\frac{-\bar{\tau}_o}{4p}} \geq \frac{1}{2^{s_o+2}} \left( \frac{2}{5} \right)^p e^{-2p} \varepsilon^{\bar{\tau}_o} M \quad \text{in } B_{\frac{\rho}{\delta}}(\bar{x}) \quad (5.8)$$

for all times

$$t_1 \leq t \leq t_1 + \delta \left( \frac{2}{5} \right)^p e^{-\left(\tau_1 - \tau_o\right)} \frac{T}{2}. \quad (5.9)$$

Notice that 5.8 can be rewritten as

$$u(\cdot, t) \geq \bar{\sigma} \left( \frac{\bar{\rho}}{\rho} \right)^{\frac{p}{2p}} M \quad \text{in } B_{\frac{\rho}{\delta}}(\bar{x}), \quad (5.10)$$

with

$$\bar{\sigma} \defeq \frac{1}{2^{s_o+2}} \left( \frac{2}{5} \right)^p e^{-2p} \varepsilon^{\bar{\tau}_o} \quad (5.11)$$

If the right hand side of 5.9 equals $\frac{T}{2}$, then 5.8 holds for all times in

$$\left( \frac{T}{2} - \varepsilon M^{2-p}(c\bar{\rho})^p, \frac{T}{2} \right) \quad \text{where } \varepsilon = \delta \sigma_o^{2-p} e^{-e^{\tau_o}}; \quad (5.12)$$

taking into account the expression for $\bar{\rho}$ and $\sigma_o$, we conclude that 5.8 holds for all times in the interval

$$\left( \frac{T}{2} - e^{-e^{\tau_o}} \frac{\delta}{2^{s_o(2-p)}} \left( \frac{2}{5} \right)^p \frac{T}{2}, \frac{T}{2} \right]. \quad (5.13)$$

Thus, the conclusion of Proposition 1.2 holds, provided the upper time level in 5.9 equals $\frac{T}{2}$. The transformed $\tau_o$ level is still undetermined, and it will be so chosen as to verify such a requirement. Precisely, taking into account 5.2

$$\frac{T}{2} e^{-\tau_1} = -\left( T_1 - T \right) = \delta \left( \frac{2}{5} \right)^p e^{-\left(\tau_1 - \tau_o\right)} \frac{T}{2} \implies e^{\tau_o} = \left( \frac{5}{2} \right)^p \frac{2^{s_o(2-p)}}{\delta}. \quad (5.14)$$

This determines quantitatively $\tau_o = \tau_o(\text{data})$, and inserting such a $\tau_o$ on the right-hand side of 5.11 and 5.13, yields a bound below that depends only on the data; 5.11 and 5.13 have been obtained relying on the bound below for $u$ along the segment $\{y\} \times (0, \frac{T}{2})$. However, the same argument on the bound along the shorter segment $\{y\} \times (0, s)$ for any $\frac{T}{4} \leq s < \frac{T}{2}$ yields the same result with $\frac{T}{2}$ substituted by $s$: the proof of Proposition 1.2 is then completed.

Remark 5.2. In the proof of Proposition 1.2, the parameter $c$ basically measures the relative size of $\rho$ with respect to $\bar{\rho}$. 

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684 FATMA GAMZE DÜZGÜN, UGO GIANAZZA AND VINCENZO VESPRI

Miscellaneous notes about the PDF layout and formatting.
5.2. A remark about the limit as $p \to 2$. The change of variables 5.2 and the subsequent arguments, yield constants that deteriorate as $p \to 2$. This is no surprise, as the decay of solutions to linear parabolic equations is not power-like, but rather exponential-like, as in the fundamental solution of the heat equation.

Nevertheless, our estimates can be stabilised, in order to recover the correct exponential decay in the $p = 2$ limit. However, this would require a careful tracing of all the functional dependencies in our estimates, and we postpone it to a future work.

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