THE SECOND YANG-BAXTER HOMOLOGY FOR THE HOMFLYPT POLYNOMIAL

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Abstract. In this article, we adjust the Yang-Baxter operators constructed by Jones for the HOMFLYPT polynomial. Then we compute the second homology for this family of Yang-Baxter operators. It has the potential to yield 2-cocycle invariant for links.

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1. Introduction

The Yang-Baxter equation was first introduced independently in a study of many body quantum system by Yang [15] and statistical mechanics by Baxter [1]1. Since the discovery of the Jones polynomial [7] in 1984, solutions to Yang-Baxter equation have become important for knot theory. In particular, Jones [7] and Turaev [13] built a machinery to construct link invariants using Yang-Baxter operators and the family of Yang-Baxter operators from the representation of $A_1^1$ series lead to $sl(m)$ polynomial invariants whose “limit” is the Homflypt polynomial [6, 11]. Racks and quandles give special examples of Yang-Baxter operators. Homology theory of rack and quandles were introduced in [5, 6]. Carter, Elhamdadi and Saito [2] defined a (co)homology theory for set-theoretic Yang-Baxter operators generalizing this homology, and they constructed cocycle link invariants. The homology theory of general Yang-Baxter operators were developed by Lebed [9] and Przytycki [10] independently and this homology theory is equivalent to the one defined in [2] when restricted to the set-theoretic Yang-Baxter operators [12]. In the first part of this paper, we give a detailed proof that after modifying the Yang-Baxter matrix obtained from $A_1$ series to be column unital, they are still Yang-Baxter operators. Furthermore, this new family of operators also lead to the $sl(m)$ polynomial invariants [14], which is implicit in [7]. The homology can be defined for any column unital Yang-Baxter operators, see [10]. In the second part of the paper, we compute the second homology of the column unital Yang-Baxter operators corresponding to $sl_m$ link invariants denoted by $R_{(m)}$(see Theorem [3,3]).

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1The name Yang-Baxter equation was coined by Ludvig Faddeev
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2. Column unital Yang-Baxter operators

Inspired by statistical mechanics, Jones constructed the Yang-Baxter operators leading to the Jones and HOMFLYPT polynomials, see [8, 13] for more information on the use of Yang-Baxter operators in knot theory.

Definition 2.1. Let \( k \) be a commutative ring and \( V \) be a \( k \)-module. If a \( k \)-linear map, \( R : V \otimes V \rightarrow V \otimes V \), satisfies the following equation called Yang-Baxter equation

\[
(R \otimes \text{Id}_V) \circ (\text{Id}_V \otimes R) \circ (R \otimes \text{Id}_V) = (\text{Id}_V \otimes R) \circ (R \otimes \text{Id}_V) \circ (\text{Id}_V \otimes R),
\]

then we say \( R \) is a pre-Yang-Baxter operator. If, in addition, \( R \) is invertible, then we say \( R \) is a Yang-Baxter operator.

Jones’ Yang-Baxter operator on level \( m \) is given by the following formula,

\[
R_{ab}^{cd} = \begin{cases} 
-q, & \text{if } a=b=c=d; \\
1, & \text{if } d=a \neq b=c; \\
q^{-1} - q, & \text{if } c=a<b=d; \\
0, & \text{otherwise.}
\end{cases}
\]

where \( 1 \leq a, b, c, d \leq m \)

In this section, we give a detailed proof that the family of column unital operators defined in Theorem 2.2 are Yang-Baxter operators. These operators are obtained from the Jones’ Yang-Baxter operators by dividing each column by the sum of elements in the column and substitution \( y^2 = \frac{1}{1+q^{-1}-q} \).

Theorem 2.2. Let \( k = \mathbb{Z}[y, y^{-1}] \), \( m \) be a positive integer, and \( V_m \) be the free \( k \) module generated by the set \( X_m = \{v_1, \ldots, v_m\} \) with ordering \( v_a \leq v_b \) if and only if \( a \leq b \). Then the \( k \) linear operator \( R_{(m)} : V_m \otimes V_m \rightarrow V_m \otimes V_m \) given by the coefficients

\[
R_{ab}^{cd} = \begin{cases} 
1, & \text{if } d=a \geq b=c; \\
y^2, & \text{if } d=a < b=c; \\
1 - y^2, & \text{if } c=a < b=d; \\
0, & \text{otherwise.}
\end{cases}
\]

is a Yang-Baxter operator for each \( m \geq 1 \).

One can check directly that the inverse of the these operators is

\[
(R^{-1})_{ab}^{cd} = \begin{cases} 
1, & \text{if } d=a \leq b=c; \\
y^{-2}, & \text{if } d=a > b=c; \\
1 - y^{-2}, & \text{if } c=a > b=d; \\
0, & \text{otherwise.}
\end{cases}
\]

Before the proof of Theorem 2.2 we set up some notations and give Proposition 2.3. Throughout the paper, we will write \( R, V, X \) for \( R_{(m)}, V_m, X_m \) defined in Theorem 2.2 respectively. In any statement, whenever we use \( R, V, X \), it implies the statement is true for \( R_{(m)}, V_m, X_m \), \( \forall m = 2, 3, \ldots \). We will use integers \( 1 \leq a, b, c \leq m \) to represent the basis elements \( v_a, v_b, v_c \) and \((a, b, c)\) for the tensor product \( v_a \otimes v_b \otimes v_c \).

Proposition 2.3. \( R(a, a) = (a, a) \) agrees with the formulas \( R(a, b) = (1 - y^2)(a, b) + y^2(b, a) \) when \( a < b \), \( R(a, b) = (b, a) \) when \( a > b \) by substituting \( b = a \).
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\[
\begin{align*}
\frac{1}{3} & \quad \text{(a,b,c)} \\
\frac{1}{y^2} & \quad \text{(b,c,a)} \\
\text{(b,a,c)} & \quad \frac{1}{y^2} \\
\text{(c,b,a)} & \quad \text{(a,b,c)} \\
\text{(a,b,c)} & \quad \text{(a,c,b)} \\
\text{(c,a,b)} & \quad \text{(b,c,a)}
\end{align*}
\]

**Figure 1.** Computational tree for the left-hand-side of the Yang-Baxter equation of \((a,b,c)\)

**Proof.** \(R(a,a) = (a,a) = (1-y^2)(a,a) + y^2(a,a)\).

Now, we prove Theorem 2.2

**Proof.** For \(m = 1\), the Yang-Baxter equation hold trivially.

We consider the cases of \(m \geq 2\).

Let \(a \leq b \leq c\) for \(a, b, c \in X_{(m)}\), then by Proposition 2.3 we need to check in total six cases for the Yang-Baxter equation, which are \((a, b, c); (b, a, c); (a, c, b); (b, c, a); (c, a, b); (c, b, a) \in X_{(m)}^3\).

We start from the case of \((a, b, c)\). From the left-hand-side of the Yang-Baxter equation, computing terms by terms, we get the following (see Figure 1),

\[
\begin{align*}
(R \otimes Id_V) & \circ (Id_V \otimes R) \circ (R \otimes Id_V)(a, b, c) = (R \otimes Id_V) \circ (Id_V \otimes R)(y^2(b, a, c) + (1-y^2)(a, b, c))) \\
(R \otimes Id_V) & \circ (Id_V \otimes R)(y^2(b, a, c)) = (R \otimes Id_V)(y^2y^2(b, c, a) + (1-y^2)y^2(b, a, c)) \\
((R \otimes Id_V)(y^2y^2(b, c, a))) & = y^2y^2y^2(c, b, a) + (1-y^2)y^2y^2(b, c, a) \\
((R \otimes Id_V)(((1-y^2)y^2(b, a, c)))) & = (1-y^2)y^2(a, b, c),
\end{align*}
\]

and

\[
\begin{align*}
(R \otimes Id_V) & \circ (Id_V \otimes R)((1-y^2)(a, b, c)) = (R \otimes Id_V)(y^2(1-y^2)(a, c, b) + (1-y^2)(1-y^2)(a, b, c)) \\
(R \otimes Id_V)(y^2(1-y^2)(a, c, b)) & = y^2y^2(1-y^2)(c, a, b) + (1-y^2)y^2(1-y^2)(a, c, b) \\
(R \otimes Id_V)((1-y^2)(1-y^2)(a, b, c)) & = y^2(1-y^2)(1-y^2)(b, a, c) + (1-y^2)(1-y^2)(1-y^2)(a, b, c)
\end{align*}
\]
Similarly, we deal with the right-hand-side of the Yang-Baxter equation by terms, we get the following (see Figure 2),
\[(Id_V \otimes R) \circ (R \otimes Id_V) (a, b, c) = (Id_V \otimes R) \circ (R \otimes Id_V) (y^2(a, c, b) + (1 - y^2(a, b, c)))\]
\[(Id_V \otimes R) \circ (R \otimes Id_V) (y^2(a, c, b)) = Id_V \otimes R (y^2y^2(c, a, b) + (1 - y^2)y^2(a, c, b))\]
\[(Id_V \otimes R)(y^2y^2(c, a, b)) = y^2y^2y^2(c, b, a) + (1 - y^2)(1 - y^2)y^2(c, a, b)\]
and
\[(Id_V \otimes R)((1 - y^2)y^2(a, c, b)) = (1 - y^2)y^2(a, b, c),\]

\[(Id_V \otimes R)(1 - y^2(a, b, c)) = Id_V \otimes R (y^2(1 - y^2(b, a, c)) + (1 - y^2)(1 - y^2(a, b, c)))\]
\[Id_V \otimes R)(y^2(1 - y^2(b, a, c)) = y^2y^2(1 - y^2(b, c, a)) + (1 - y^2)y^2(1 - y^2(b, a, c))\]
\[Id_V \otimes R)((1 - y^2)(1 - y^2(a, b, c)) = y^2(1 - y^2)(1 - y^2(a, c, b)) + (1 - y^2)(1 - y^2)(1 - y^2(a, b, c))\]
Both expressions are equal, thus prove Yang-Baxter equation holds for \((a, b, c)\). The other cases can be checked directly in a similar way.

\[\square\]

**Remark 2.4.** From our proof and Figure 1 and Figure 2, we can conclude more.

\[(R \otimes Id_V) \circ (Id_V \otimes R) \circ (R \otimes Id_V)(a, b, c) =\]
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\[ ([y^2y^2y^2(c, b) + (1 - y^2)y^2y^2(b, c, a)] + [(1 - y^2)y^2(a, b, c)]) + \\
([y^2y^2(1 - y^2)(c, a, b) + (1 - y^2)y^2(1 - y^2)(a, c, b)] + [y^2(1 - y^2)(1 - y^2)(b, a, c) + (1 - y^2)(1 - y^2)(1 - y^2)(a, b, c)]) \\
\]

\[ (Id_V \otimes R) \circ (R \otimes Id_V) \circ (Id_V \otimes R)(a, b, c) = \\
([y^2y^2y^2(c, b, a) + (1 - y^2)(1 - y^2)y^2(c, a, b)] + [(1 - y^2)y^2(a, b, c)]) + \\
([y^2y^2(1 - y^2)(b, c, a) + (1 - y^2)y^2(1 - y^2)(b, a, c)] + [y^2(1 - y^2)(1 - y^2)(a, c, b) + (1 - y^2)(1 - y^2)(1 - y^2)(a, b, c)]) \]

Terms in the sum correspond to the leaves of the computational tree. Square brackets group terms according to the structure of the tree (see Figure 7 and Figure 8).

Important observation is that if we transform the result of the left-hand-side of \((a, b, c)\) by first switching the position of \(a\) and \(c\) and then reversing the order of the triple, we obtain exactly the result of right-hand-side of \((a, b, c)\) square bracket-wisely. This observation can actually reduce the number of cases to check, which is important for us to compute the higher level homology in the future.

As mentioned before, the family of Yang-Baxter operators, \(R_{(m)}\), have the property that summation of elements in each column of the matrix presentation equals to 1. They are obtained from the Yang-Baxter operators leading to the Jones and HOMFLYPT polynomials \(\mathcal{J}\) \(\mathcal{H}\) by normalizing each column. However, normalizing columns of Yang-Baxter operators does not always produce Yang-Baxter operators in general.

**Counterexample 2.5.** The following Yang-Baxter operator leading to the Kauffman two-variable polynomial (see \(\mathcal{H}\) for detail) with substitution \(m = 4, \nu = -1\) is a counterexample.

\[
\begin{pmatrix}
q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q - q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q - q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q - 2q^{-1} + (q^{-3}) & 0 & 0 & (q^{-2} - 1) & 0 & 0 & (q^{-2} - 1) & 0 & 0 & q^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (q^{-2} - 1) & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (q^{-2} - 1) & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q
\end{pmatrix}
\]

This matrix as a \(k\)-linear operator from \(V \otimes V\) to \(V \otimes V\), with \(k = \mathbb{Z}[q, q^{-1}]\) and \(V = k\{v_1, v_2, v_3, v_4\}\) the free \(k\)-module generated by four elements, is a Yang-Baxter operator with the standard basis in tensor product of \(V \otimes V\). However, if we divide the elements of each column by the summation of the elements in the corresponding column, it is no longer a Yang-Baxter operator. We have checked this by using Mathematica directly.

3. Computation of homology for Yang-Baxter operators leading to HOMFLYPT polynomial

In this section, we are interested in the second homology of \(R_{(m)}\). First we recall the definition of Yang-Baxter homology for column unital operators. Let \(k\) be a commutative ring, \(V = kX\)
be the free $k$–module generated by basis in $X$, and let the chain modules be $C_n(R) = V^\otimes n$. The boundary homomorphism $\partial_n : C_n(R) \to C_{n-1}(R)$ is given as follows,

$$\partial_n = \sum_{i=1}^{n} (-1)^i (d_{i,n} - d'_{i,n}).$$

The face maps $d_{i,n}$ and $d'_{i,n}$ are illustrated in Figure 3, where going from top to bottom, and whenever we meet a crossing we apply the Yang-Baxter operator $R$ and we delete the first tensor factor or the last tensor factor at the bottom for $d_{i,n}$ and $d'_{i,n}$, respectively. See [10] [12] for detail.

Consider pre-Yang-Baxter operators $R : V \otimes V \to V \otimes V$ given on the basis $X^2$ by

$$R(a, b) = \sum_{c,d} R_{c,d}^{a,b} \cdot (c, d)$$

with column unital condition, that is $\sum_{c,d} R_{c,d}^{a,b} = 1$ for every $(a, b) \in X^2$. Now $C_1(R) = V$ and $C_2(R) = V^\otimes 2$ and

$$\partial_2(a, b) = \sum_{i=1}^{2} (-1)^i (d_{i}(a, b) - d'_{i}(a, b)) =$$

$$(b) - \sum_{c,d} R_{c,d}^{a,b}(d) - \left( \sum_{c,d} R_{c,d}^{a,b}(c) - (a) \right) =$$

$$(a) + (b) - \sum_{c,d} R_{c,d}^{a,b}((c) + (d))$$

and

$$\partial_3(a, b, c) = \sum_{i=1}^{3} (-1)^i (d_{i}(a, b, c) - d'_{i}(a, b, c)).$$

Now we go back to analysis of the chain complex for the column unital matrices $R_{(m)}$ in Theorem 2.2. Recall that

$$R_{cd}^{ab} = \begin{cases} 
1, & \text{if } d=a \geq b=c; \\
y^2, & \text{if } d=a < b=c; \\
1-y^2, & \text{if } c=a < b=d; \\
0, & \text{otherwise.}
\end{cases}$$
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In particular, for \( m = 2 \) we have the matrix

\[
R_{(2)} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - y^2 & 1 & 0 \\
0 & y^2 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad R_{(2)}^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & y^{-2} & 0 \\
0 & 1 & 1 & - y^{-2} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

In Lemma 3.1 we prove that the second boundary map is trivial.

**Lemma 3.1.** For the family of column unital Yang-Baxter operators in Theorem 2.2, \( \partial_2 = 0 \) and

\[
H_1(R) = C_1(R_{(m)}) = V
\]

and

\[
\ker \partial_2 = C_2(R_{(m)}) = V^{\otimes 2}
\]

**Proof.** We check now that \( \partial_2(a, b) = 0 \) for any pair \( a, b \in X^2 \). The main reason for \( \partial_2 = 0 \) is that if \( \{a, b\} \neq \{c, d\} \) then \( R_{c,d}^{a,b} = 0 \) and the column unital property. That is for \( a, b \in X \):

\[
\sum_{c,d} R_{c,d}^{a,b}(c, d) = R_{a,b}^{a,b}(a, b) + R_{b,a}^{a,b}(b, a) \quad \text{with} \quad R_{a,b}^{a,b} + R_{b,a}^{a,b} = 1.
\]

so \( \partial_2(a, b) = (a) + (b) - (R_{a,b}^{a,b} + R_{b,a}^{a,b})(a + (b)) = 0 \).

Thus

\[
H_1(R_{(m)}) = C_1(R_{(m)}) = V
\]

and

\[
\ker \partial_2 = C_2(R_{(m)}) = V^2.
\]

To compute \( H_2(R_{(m)}) \), we need to understand \( \im \partial_3 \). The following lemma will be used later in computation,

**Lemma 3.2.** For the column unital Yang-Baxter operators in Theorem 2.2 we have

1. \( \partial_3(v_m, a_1, a_2) = 0 \) and \( \partial_3(a_1, a_2, v_1) = 0 \), for all \( a_1, a_2 \in X \), where as before \( v_m \) is the largest element and \( v_1 \) is the smallest element in \( X \);

2. \( \partial_3(a_1, a_2, a_3) = 0 \) if either \( a_1 \geq a_i \) for all \( i = 1, 2, 3 \) or \( a_3 \leq a_j \) for all \( j = 1, 2, 3 \), for all \( a_1, a_2, a_3 \in X \).

**Proof.**

Part (1) follows from Lemma 3.1 \( \partial_3(v_m, a_1, a_2) = [d^1_1 - d^1_0](v_m, a_1, a_2) - v_m \otimes \partial_2(a_1, a_2) \), by Lemma 3.1 \( \partial_3(v_m, a_1, a_2) = [d^1_1 - d^1_0](v_m, a_1, a_2) \). Note that \( R(a_1, a_2) = (a_2, a_1) \) whenever \( a_1 \geq a_2 \), \( \partial_3(v_m, a_1, a_2) = [d^1_1 - d^1_0](v_m, a_1, a_2) = (a_1, a_2) - (a_1, a_2) = 0 \). Similarly, \( \partial_3(a_1, a_2, v_1) = 0 \).

Part (2) follows from part (1) by considering the subchain complex given by the subspace \( \{v_1, v_2, ..., a_1\} \) of \( V_m \) or \( \{a_3, ..., a_{m-1}, a_m\} \) of \( V_m \) respectively.

The main result of this section is as follows. Notice that the ring \( k \) can be either \( \mathbb{Z}[y^\pm] \) or \( \mathbb{Z}[y] \).
Theorem 3.3. Let $R$ be a unital Yang-Baxter operator giving Homflypt polynomial on level $m$ in Theorem 2.2, then

$$H_2(R) = k^{1+\binom{m}{2}} \oplus \left( k/(1 - y^2) \right)^{\binom{m}{2}} \oplus \left( k/(1 - y^4) \right)^{m-1}.$$  

Proof. First, we compute $\partial_3$. Let $a < b < c$, we need to consider 13 cases, which are:

- $(a, b, c)$;
- $(b, c, a)$;
- $(c, a, b)$;
- $(a, c, b)$;
- $(b, a, c)$;
- $(a, b, a)$;
- $(b, a, a)$;
- $(a, b, b)$;
- $(b, a, b)$;
- $(a, a, a)$;
- $(a, b, a)$;
- $(b, a, b)$;
- $(a, a, a)$

By Lemma 3.2, we have $\partial_3(b, c, a) = \partial_3(c, a, b) = \partial_3(b, b, a) = \partial_3(b, a, a) = 0$.

$\partial_3$ provides non-trivial relations in the homology for the 5 remaining cases (however, they are not all linearly independent). Let us demonstrate the calculation of $\partial_3(a, b, c)$.

We make calculation easy by considering graphical interpretation of face maps $d_i^e$, starting from the defining formula:

$$\partial_3 = d_1^e + d_2^e + d_3^e - (d_3^r + d_2^r + d_1^r).$$

From this diagrams we compute (keeping the terms in the same order as in figures):

$$\partial_3(a, b, c) = (b, c) + \left( y^2(a, c) + (1 - y^2)(ab) \right) +$$

$$\left( y^4(a, b) + y^2(1 - y^2)(c, b) + (1 - y^2)y^2(a, c) + (1 - y^2)^2(b, c) \right) -$$

$$(a, b) - \left( y^2(a, c) + (1 - y^2)(b, c) \right) -$$

$$\left( y^4(b, c) + y^2(1 - y^2)(b, a) + (1 - y^2)y^2(a, c) + (1 - y^2)^2(a, b) \right) =$$

$$(1 - y^2) \left( (b, c) - (a, b) + y^2((c, b) - (b, a)) \right).$$

The longest calculation is that of $d_3^e$ and $d_1^r$. In the next picture we illustrate how to compute quickly $d_3^e$: 
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\[ d_3^{\ell} = y^4(a, b) + y^2(1 - y^2)(c, b) + (1 - y^2)y^2(a, c) + (1 - y^2)^2(b, c) \]

The computation for \( d_1^r \) is similar. We can also use the symmetry that is \( a \) and \( c \) switch roles and \( (x, y) \) goes to \( (y, x) \). Thus we get

\[ d_1^r = y^4(b, c) + y^2(1 - y^2)(b, a) + (1 - y^2)y^2(a, c) + (1 - y^2)^2(a, b) \]

With some efforts, we get the following non-trivial differentials of elements with three distinct letters:

\[ \partial_3(a, b, c) = (1 - y^2) \left( (b, c) - (a, b) + y^2((c, b) - (b, a)) \right) \]

\[ \partial_3(a, c, b) = (1 - y^2) \left( (b, c) - (a, c) + y^2((c, b) - (c, a)) \right) \]

\[ \partial_3(b, a, c) = (1 - y^2) \left( (a, c) - (a, b) + y^2((c, a) - (b, a)) \right) \]

They are not independent as:

\[ \partial_3(a, b, c) - \partial_3(a, c, b) - \partial_3(b, a, c) = 0. \]
Also, by Proposition 2.4, we have:
\[
\partial_3(a,a,b) = (1 - y^2) \left( (a,b) - (a,a) + y^2((b,a) - (a,a)) \right).
\]
\[
\partial_3(a,b,b) = (1 - y^2) \left( (b,b) - (a,b) + y^2((b,b) - (b,a)) \right).
\]
From the following two equations, we see that the relations given by \( \partial_3 \) are generated by the images of \((a,a,b)\) and \((a,b,b)\) as follows.
\[
\partial_3(a,b,c) = (1 - y^2) \left( (b,c) - (a,b) + y^2((c,b) - (b,a)) \right) = \partial_3(b,b,c) + \partial_3(a,b,b),
\]
and
\[
\partial_3(b,a,c) = (1 - y^2) \left( (a,c) - (a,b) + y^2((c,a) - (b,a)) \right) = \partial_3(a,a,c) - \partial_3(a,a,b).
\]
Let us summarize the structure of the image \( \partial_3(C_3) \). It is generated by
\[
\partial_3(v_i, v_j) = (1 - y^2)((v_i, v_j) - (v_i, v_i) + y^2((v_j, v_i) - (v_i, v_i)))\]
for \( i < j \), and
\[
\partial_3((v_i, v_j) + (v_i, v_j, v_j)) = (1 - y^4)((v_j, v_j) - (v_i, v_i))\]
for \( i < j \).
We notice quickly that \( \partial_3((v_i, v_j) + (v_i, v_j, v_j)) \) is generated by \( m - 1 \) elements \((v_j, v_j) - (v_i, v_1)\) with \( m \geq j > 1 \).
Consider the following new basis of \( kX^2 \) consisting of three groups of basis elements:
\[
X_0 = \{(v_1, v_1), (v_j, v_i) \text{ for } i < j \} \text{ that is } \binom{m}{2} + 1 \text{ elements.}
\]
\[
X_1 = \{(v_i, v_j) - (v_i, v_i) + y^2((v_j, v_i) - (v_i, v_i)) \text{ for } i < j \} \text{ that is } \binom{m}{2} \text{ elements.}
\]
\[
X_2 = \{(v_j, v_j) - (v_i, v_1) \text{ with } m \geq j > 1 \} \text{ that is } m - 1 \text{ elements.}
\]
Clearly \( X_0 \sqcup X_1 \sqcup X_2 \) form a basis of \( kX^2 \).
We look now at relations: in our basis, the matrix of relations is diagonal with 0 for elements in \( X_0, 1 - y^2 \) for elements in \( X_1 \), and \((1 - y^4)\) for elements in \( X_2 \). Thus not only we proved that
\[
H_2(R) = k^{1+\binom{m}{2}} \oplus \left(k/(1 - y^2)^{\binom{m}{2}} \oplus k/(1 - y^4)^{m-1}\right).
\]
but we also found a basis of \( C_2 = kX^2 \) realizing the decomposition into cyclic submodules. \( \square \)

From Theorem 3.3, we can easily see the rank of \( ker\partial_3(R(m)) \).

**Corollary 3.4.** \( \text{Rank}(ker\partial_3(R(m))) = \frac{(m+1)(2m^2-3m+2)}{2} \).

**Proof.** Rank of the kernel \( \partial_3 \) is the rank of \( C_3 \) minus the number of nonzero elements in the diagonal relation matrix of \( \partial_3 \), which is exactly the numbers of \((1 - y^2)\) and \((1 - y^4)\). Thus
\[
\text{Rank}(ker\partial_3(R(m))) = m^3 - \binom{m}{2} - (m-1) = \frac{(m+1)(2m^2-3m+2)}{2}
\]
\( \square \)
The second Yang-Baxter homology for the Homflypt polynomial

4. FURTHER COMPUTATIONS AND FUTURE WORK

Here we summarise all data obtained with the help of computer. Because of the limitation of
the computation program, the computation were done over the ring $\mathbb{Q}[y]$. In [12], we formulated
a conjecture about the homology of $R_m$ when $m = 2$ as follows,

**Conjecture 4.1.** [12] When $m = 2$, $H_n = k^2 \bigoplus (k/(1 - y^2))^{s_n} \bigoplus (k/(1 - y^4))^{s_{n-2}}$, where $s_n = \sum_{i=1}^{n+1} f_i$ is the partial sum of Fibonacci sequence, where $f_1 = 1 = f_2$ and $a_n$ is given by $2^n = 2 + a_{n-1} + s_{n-3} + a_n + s_{n-2}$ with $a_1 = 0$.

This conjecture is verified upto $n \leq 11$ by computer. In the paper [4], there is a discussion
of various aspects of this conjecture. More computation is shown in Table 4, where $(x, y, z)$
represents decomposition into $x$ copies of $k$, $y$ copies of $k/(1 - y^2)$ torsion and $z$ copies of $k/(1 - y^4)$ torsion.

| $H_n$ | $m=3$ | $m=4$ | $m=5$ | $m=6$ | $m=7$ |
|-------|-------|-------|-------|-------|-------|
| $H_2$ | (4,3,2) | (7,6,3) | (11,10,4) | (16,15,5) | (22,21,6) |
| $H_3$ | (4,12,6) | (8,35,12) | (15,76,20) | (26,140,30) | (42,232,42) |

Table 4

From the first row of the table, we can see that the results match with that of the formula in Theorem 3.3. From the second row of the table, we conjecture the formula for $H_3$ as follows,

**Conjecture 4.2.** $H_3(R_m) = \bigoplus (k/(1 - y^2))^{(m^2 - 1)(5m - 6)/6} \bigoplus (k/(1 - y^4))^{m(m-1)}$

**Remark 4.3.**

1. The ranks in Conjecture 4.2 sum up to the rank of $\ker \partial_3$.
2. The rank of $H_3(R_2)$ in Conjecture 4.1 agrees with the rank of $H_3(R_2)$ in Conjecture 4.2.

Computations and patterns observed so far suggest that there are only two types of torsion
elements $k/(1 - y^2)$ and $k/(1 - y^4)$. However, this is only checked upto the strength of computer
program. By analyzing the boundary maps in general, we hope to gain more information about $H_n(R_m)$. The first step towards this goal is the following observation.

**Remark 4.4.** The factor $(1 - y^2)$ divides every element in $\text{Im} (\partial_n)$. This follows from the fact that when setting $1 - y^2 = 0$, $d_i^k = d_i^r$. Thus, we have $\partial_n(a_1, ..., a_n) \subset (1 - y^2)V^n$, where $a_i \in X_m$, $i = 1, 2, ..., m$. One possible approach to compute $H_n(R_m)$ is to decompose the boundary map along the factors $(1 - y^2)^i$. In the first step, we ignore the branches with factor $(1 - y^2)$ in the computational tree, see Figure 8. Generally, in the $i$-th step, we ignore the paths in the computational tree which are going $i$ or more times to the right.

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