Bidimensional intermittent search processes: an alternative to Lévy flights strategies

O. Bénichou,1 C. Loverdo,1 M. Moreau,1 and R. Voituriez1

1Laboratoire de Physique Théorique de la Matière Condensée, UMR CNRS 7600, Université Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris, France

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Lévy flights are known to be optimal search strategies in the particular case of revisitable targets. In the relevant situation of non revisitable targets, we propose an alternative model of bidimensional search processes, which explicitly relies on the widely observed intermittent behavior of foraging animals. We show analytically that intermittent strategies can minimize the search time, and therefore do constitute real optimal strategies. We study two representative modes of target detection, and determine which features of the search time are robust and do not depend on the specific characteristics of detection mechanisms. In particular, both modes lead to a global minimum of the search time as a function of the typical times spent in each state, for the same optimal duration of the ballistic phase. This last quantity could be a universal feature of bidimensional intermittent search strategies.

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Search processes, involving a searcher and a target of unknown position, play an important role in many physical, chemical or biological problems. This is for instance the case of reactants diffusing in a solvent until they get close enough to react, or of a protein searching for its specific target site on DNA. One can also mention animals searching for food, or coast-guards trying to locate wreck victims. In all these examples, it is of great importance to minimize the search time. Since the pioneering works of Viswanathan et al., the question of determining optimal search strategies has appealed a growing attention.

In this context, Lévy flights strategies have been proved to play a crucial role in such optimization problems. However, two limitations of these strategies have to be mentioned. First, Lévy flights trajectories have been shown to optimize the search efficiency, but only in the particular case where the targets are regenerated at the same location after a finite time, which can not be taken as a general rule. Indeed, in the case of destructive search where each target can be found only once, or in the case of a single target, the optimal strategy proposed in is not anymore of Lévy type, but reduces to a linear ballistic motion. Second, as for the applications to behavioral ecology, the destructive search is relevant to many situations. However, the purely ballistic strategy predicted in that case can not account for the generally observed reoriented animal trajectories.

Alternatively to these Lévy strategies, it has been observed that intermittent search strategies are widely used by foraging animals. Many searchers combine phases of fast displacement, non reactive to the targets, and slow reactive search phases. Everyday-life examples also confirm that we instinctively adopt such intermittent behavior when looking for a lost object: we search carefully around one location, then move quickly to another unvisited area and then search again.

Up to now, only 1D models of such intermittent search have been developed, providing a satisfactory agreement with experimental data from behavioral ecology. Here we develop a model of 2D intermittent search strategies, which encompasses a much broader field of applications, in particular for animal or human searchers. We show that bidimensional intermittent search strategies do optimize the search time for non revisitable targets. We explicitly determine how to share the time between the phases of non reactive displacement and of reactive search to find a target in the quickest way.

Following, we consider a two state searcher (see Fig.1) of position that performs slow reactive phases (denoted 1), randomly interrupted by fast relocating ballistic flights of constant velocity and random direction (phases 2). We assume the duration of each phase to be exponentially distributed with mean . As fast motion usually strongly degrades perception abilities, we consider that the searcher is able to find a target only during reactive phases 1. The detection phase involves complex biological processes that we do not aim at modeling accurately here. However, essentially two modes of detection can be put forward, and lead to distinguish between two types of reactive phases 1. The first one, referred to in the following as the "dynamic mode", corresponds to a diffusive modeling (with diffusion coefficient of the search phase as recently proposed in
in agreement with observations for vision \[10\], tactile sense or olfaction \[5\]. The detection is assumed to be infinitely efficient. The mean first passage time (MFPT) at a target is obtained by combining both modes and considering a diffusion coefficient \(D\) and finite reaction rate \(k\).

We now present the basic equations combining the two search modes introduced above in the case of a point-like target placed with a finite rate \(\tau\) at position \(r\) with velocity \(v\). \(I_a(r) = 1\) if \(|r| \leq a\) and \(I_a(r) = 0\) if \(|r| > a\). In the present form, these integro-differential equations do not seem to allow for an exact solution with standard methods. We propose here an approximate resolution based on the introduction of the following auxiliary functions:

\[
s(r) = \frac{1}{2\pi} \int_0^{2\pi} t_2 d\theta_\nu, \quad d(r) = \frac{1}{2\pi} \int_0^{2\pi} t_2 v d\theta_\nu. \tag{3}
\]

Averaging Eq. (2) and Eq. (2) times \(v\) over \(\theta_\nu\), one successively gets

\[
\nabla \cdot d - \frac{1}{\tau_2} (s(r) - I_a(r)) = -\frac{1}{\tau_2} \int_0^{2\pi} (v \cdot \nabla t_2) v d\theta_\nu, \tag{4}
\]

which gives in turn

\[
\nabla \cdot d = \frac{\tau_2}{2\pi} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (v_i v_j t_2) \theta_\nu, \tag{5}
\]

where \(\langle \rangle_{\theta_\nu}\) stands for the average over \(\theta_\nu\). We now make the following decoupling assumption

\[
\langle v_i v_j t_2 \rangle_{\theta_\nu} \approx \langle v_i v_j \rangle_{\theta_\nu} \langle t_2 \rangle_{\theta_\nu} = \frac{v^2}{2} \delta_{ij} s(r) \tag{6}
\]

which leads, together with Eq. (4), to the diffusion-like equation:

\[
\tilde{D} \nabla^2 s(r) - \frac{1}{\tau_2} (s(r) - I_a(r)) = -1 \tag{7}
\]

where \(\tilde{D} = v^2 \tau_2 / 2\). Rewriting Eq. (1) as

\[
D \nabla^2 t_1 + \frac{1}{\tau_1} (s(r) - I_a(r)) t_1 = -1 \tag{8}
\]

Eqs. (7) and (8) together with vanishing normal derivatives at \(|r| = b\) provide a closed system for the variables \(s\) and \(t_1\), whose resolution is lengthy but standard. The validity domain of assumption (3) is much broader than the "Brownian" limit \(v \to \infty\) and \(\tau_2 \to 0\) with \(\tilde{D}\) fixed, in which \(t_2\) is independent of the direction of \(v\). Indeed, it is also valid in the limit \(v \tau_2 \gg b\), in which a ballistic phase includes many reorientations due to successive reflections on the boundary \(r = b\). In addition, it can be shown that in one dimension this assumption is exact.

We first present the solution of Eqs (7) in the "dynamic mode" \((k \to \infty)\). The search time \(\langle t \rangle\), defined as \(t_1\) uniformly averaged over the initial position of the searcher (note that this last averaging reflects the complete ignorance of the target position), reads in this case:
(9)

\[ \langle t \rangle = \frac{\tau_1 + \tau_2}{2k\tau_1 y^2} \left\{ \frac{1}{x} (1 + k\tau_1)(y^2 - x^2)^2 \frac{I_0(x)}{I_1(x)} + \frac{1}{4} \left[ 8y^2 + (1 + k\tau_1) \left( 4y^4 \ln(y/x) + (y^2 - x^2)(x^2 - 3y^2 + 8y) \right) \right] \right\} \]

with \( L_\pm = I_0(a/\sqrt{D\tau_2}) (I_1(ba)K_1(aa) - I_1(aa)K_1(ba)) \pm a\sqrt{D\tau_2} I_1(a/\sqrt{D\tau_2}) (I_1(ba)K_0(aa) + I_0(aa)K_1(ba)) \)

and \( M = I_0(a/\sqrt{D\tau_2}) (I_1(ba)K_0(aa) + I_0(aa)K_1(ba)) - 4\frac{a^2\sqrt{D\tau_2}}{a(b^2 - a^2)^2} I_1(a/\sqrt{D\tau_2}) (I_1(ba)K_1(aa) - I_1(aa)K_1(ba)) \)

where \( \alpha = (1/(D\tau_1) + 1/(\tilde{D}\tau_2))^{1/2} \), and \( I_i \) and \( K_i \) are modified Bessel functions. This expression (9) has proved to be in very good agreement with numerical simulations for a wide range of the parameters (see Fig. 2). The optimization of the explicit expression (9) leads to simple forms in the following situations, depending on the relative magnitude of the three characteristic lengths of

\[ \tilde{\tau}_1 \text{min} \sim \frac{D}{2\nu^2} \frac{\ln(b/a)}{\ln(b/a) - 1}, \quad \tilde{\tau}_2 \text{min} \sim \frac{a}{\nu} (\ln(b/a) - 1/2)^{1/2}, \]

leads to a qualitative change of the search time which can be rendered arbitrarily smaller than the non intermittent search time when \( \nu \to \infty \). This optimal strategy corresponds to a scaling law

\[ \frac{\tilde{\tau}_1 \text{min}}{\tilde{\tau}_2 \text{min}} \sim \frac{D}{a^2 (2 - 1/\ln(b/a))} \]

which here does not depend on \( \nu \).

We now turn to the "static mode" \( (D \to 0) \), which leads to the following expression for the search time

\[ \langle t \rangle = \tau_1 + \tau_2 \left\{ \frac{1}{x} (1 + k\tau_1)(y^2 - x^2)^2 \frac{I_0(x)}{I_1(x)} + \frac{1}{4} \left[ 8y^2 + (1 + k\tau_1) \left( 4y^4 \ln(y/x) + (y^2 - x^2)(x^2 - 3y^2 + 8y) \right) \right] \right\} \]

where \( x = \sqrt{\frac{2k\tau_1}{1 + k\tau_1} \frac{a}{v\nu \tau_2}} \) and \( y = \sqrt{\frac{2k\tau_1}{1 + k\tau_1} \frac{b}{v\nu \tau_2}} \)

Here again, this expression (12) is in very good agreement with numerical simulations for a wide range of the parameters (see Fig. 3). In that case, intermittence is trivially necessary to find the target, and the optimiza-

\[ \tau_{1 \text{min}} = \left( \frac{a}{\nu k} \right)^{1/2} \left( \frac{2 \ln(b/a) - 1}{8} \right)^{1/4}, \]

\[ \tau_{2 \text{min}} = \frac{a}{\nu} (\ln(b/a) - 1/2)^{1/2}, \]
which corresponds to the scaling law $\tau_{2,\text{min}} = 2k\tau_{1,\text{min}}$, which still does not depend on $v$.

The main results (11) and (14), (15) obtained in the two modes of search lead to the following remarkable characteristics of intermittent search processes: (i) In both cases the search time $\langle t \rangle$ presents a global minimum for finite values of the $\tau_i$, which means that intermittence is an optimal strategy. (ii) A very striking and non intuitive feature is that both modes of search lead to the same optimal value of $\tau_{2,\text{min}}$. As this optimal time does not depend on the specific characteristics $D$ and $k$ of the search mode, it seems to constitute a general property of intermittent search strategies. (iii) The optimal $\tau_{1,\text{min}}$ are different and depend explicitly on $D$ and $k$, leading to different scaling laws which are susceptible to discriminate between the two search modes.

Finally we remark that this model provides as a by-product an approximation for the MFPT for a Pearson type random walk in the spherical geometry previously defined: the searcher performs ballistic flights reoriented at exponentially distributed times, and, as opposed to standard Pearson walks, the target can be found only when the distance between the target and a reorientation point is less than $a$. This quantity, obtained here straightforwardly by taking $k \to \infty$ and $\tau_1 \to 0$ in Eq. (12), writes:

$$\langle t \rangle = \frac{v\tau_2^2}{4b^2} \left( \sqrt{\frac{2}{a}} (b^2 - a^2) \frac{I_0(a\sqrt{2}/v\tau_2)}{I_1(a\sqrt{2}/v\tau_2)} + \frac{1}{\sqrt{3\tau_2^2}} (b^4 \ln(b/a) + (b^2 - a^2)(a^2 - 3b^2 + 4v^2\tau_2^2)) \right)$$ (16)

To our knowledge, a similar result for standard Pearson walks is still missing. Note that in the limit $v \to \infty$, $\tau_2 \to 0$ with $D = v^2\tau_2/2$ fixed, the approximate expression (16) gives back the well known exact expression for the MFPT of a Brownian particle between concentric spheres. Moreover, for $b \gg a$, the search time (16) is minimized again for the same value (11) and (15) of $\tau_2$, in agreement with the limit $k \to \infty$ of Eqs. (14), (15).

To conclude, we have proposed a two state model of search processes for non revisitable targets, which closely relies on the experimentally observed intermittent strategies adopted by foraging animals. Using a decoupling approximation numerically validated, we have shown analytically that in the physically most relevant bidimensional geometry, intermittent strategies minimize the search time, and therefore constitute optimal strategies, as opposed to Lévy flights which are optimal only for revisitable targets. We studied two representative modes of search, and determined which features of the corresponding optimal strategies are robust and do not depend on the specific characteristics of the search mode. In particular both modes lead to a global minimum of the search time as a function of the typical times spent in each state, and the optimal duration of the ballistic relocation phase is the same for these both modes. As this last time does not depend on the nature of the search mode, it could be a universal feature of bidimensional intermittent search strategies.

[1] S.A.Rice, Diffusion-Limited Reactions, in: Compr. Chem. Kinetics 25, eds.: C.H.Bamford, C.F.H.Tipper, and R.G.Compton (Elsevier, New York, 1985)
[2] O.G. Berg, R.B. Winter and P.H. von Hippel 20, 6929-6948 (1981)
[3] S.E. Halford and J.F. Marko, Nucleic Acids Res. 32, 3040 (2004)
[4] M. Coppey, O. Bénichou, R. Voituriez, and M. Moreau, Biophys. J. 87, 1640 (2004).
[5] J.W. Bell, Searching Behaviour, the behavioural ecology of finding resources, (Chapman and Hall Animal Behaviour Series 1991)
[6] W.J. O'Brien, H.I. Brownman and B.I. Evans. American Scientist 78, 152 (1990)
[7] G.M. Viswanathan et al, Nature 381, 413 (1996); Nature (London) 401, 911 (1999).
[8] F. Bartumeus et al Phys. Rev. Lett. 88, 097901 (2002)
[9] E.P. Raposo et al, Phys. Rev. Lett. 91, 240601 (2003); M.C. Santos, G. M. Viswanathan, E. P. Raposo, and M. G. E. da Luz Phys. Rev. E 72, 046143 (2005)
[10] O. Bénichou et al, Phys. Rev. Lett. 94, 198101 (2005).
[11] J. R. Frost and L. D. Stone. Review of search theory: Advances and applications to search and rescue decision Support. URL http://www.rdc.usgs.gov/reports/2001/cq11501dpdexsum.pdf.
[12] M. A. Lomholt, T. Ambjørnsson, and R. Metzler Phys. Rev. Lett. 95, 260603 (2005)
[13] H.-X. Zhou and A. Szabo Phys. Rev. Lett. 93, 178101 (2004)
[14] IM Sokolov ,R Metzler ,K Pant ,MC. Williams Biophys J. 89(2) (2005)
[15] M. Slutsky and L.A. Mirny, Biophys. J. 87, 1640 (2004)
[16] M. Schlesinger,Y. Klafter , in Lévy walks vs Lévy flights, pp. 279-283, Eds. H.E. Stanley and N. Ostrowski, On growth and Form (Martinus Nijhoff Publishers, Amsterdam, 1986)
[17] Y. Klafter,M. Schlesinger,G. Zumofen, Physics Today 49,
[18] P. Levitz, J. Phys. Condens. Mat. 17, 4059 (2005)
[19] L.D. Kramer and R.L. McLaughlin, Amer. Zool. 41, 137 (2001)
[20] K. Pearson, Nature 72, 294 (1905)
[21] S. Blanco and R. Fournier, Europhys. Lett. 61, 168 (2003)
[22] A. Mazzolo, Europhys. Lett. 68, 350 (2004)
[23] O. Bénichou et al, Europhys. Lett. 70, 42 (2005)
[24] S. N. Majumdar, A. Comtet and R. N. Ziff, cond-mat/0509613
[25] R.B. Huey The psychology and pedagogy of reading (MIT Press 1968).
[26] S. Redner, A guide to first passage time processes (Cambridge University Press 2001).