The moduli space of non-abelian vortices in Yang–Mills–Chern–Simons–Higgs theory

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Abstract
We determine the dimension of the moduli space of non-abelian vortices in Yang–Mills–Chern–Simons–Higgs theory in 2 + 1 dimensions for gauge groups \( G = (U(1) \times G')/\mathbb{Z}_{n_0} \) with \( G' \) being an arbitrary semi-simple group and \( n_0 \) the greatest common divisor of the abelian charges of the \( G' \) invariants. The calculation is carried out using a Callias-type index theorem, the moduli matrix approach and a D-brane setup in type IIB string theory. We prove that the index theorem gives the number of zeromodes or moduli of the non-abelian vortices, extend the moduli matrix approach to the Yang–Mills–Chern–Simons–Higgs theory and finally derive the effective Lagrangian of Collie and Tong using string theory.

Keywords: moduli space of non-abelian vortices, index theorem, D-brane construction

1. Introduction

Chern–Simons (CS) field theory is an interesting topological field theory that is nondynamical. Once coupled with Yang–Mills (Maxwell) theory, it becomes a nontrivial theory giving mass to the gauge field, without resort to the Higgs mechanism [1]. Once matter fields and the Higgs mechanism is included as well, the gauge fields possess a complicated pole structure, with two mass poles in their propagator. Abelian CS theory has proved its importance in the subject of the fractional quantum Hall effect [2], whereas the non-abelian CS term is crucial for...
in the Witten–Sakai–Sugimoto model [3, 4] as it reproduces the chiral anomaly of quantum chromodynamics and the Wess–Zumino–Witten term of the Skyrme model at low energies.

In this paper, we shall focus on two-dimensional solitons in gauge theory, namely vortices. The prime example of such solitons are Abrikosov vortices in type-II superconductors and their relativistic generalization called Nielsen–Olesen or abelian-Higgs vortices, for reviews see e.g. references [5, 6]. The CS vortices were first studied in the abelian Chern–Simons–Higgs theory [7] and Yang–Mills–Chern–Simons–Higgs (YMCSH) theory, but with the scalar fields in the adjoint representation of SU(N), hence giving rise to Z_N vortices [8–10]. An interesting discovery was that Abrikosov–Nielsen–Olesen vortices, that are magnetic vortices and electrically neutral, become electrically charged when the CS term is introduced in the theory. This is a mere consequence of Gauss’ law. Abelian selfdual Chern–Simons–Higgs vortices were found later on with a special sixth-order potential and a vortex Bogomol’nyi–Prasad–Sommerfield (BPS) equation that has a fourth-order scalar-field dependence [11, 12]. Abelian selfdual Maxwell–Chern–Simons–Higgs vortices were then discovered [13–15], where the new ingredient is not a sixth-order potential as in the pure CS case, but a normal fourth-order potential and the addition of an extra neutral scalar field as well as an interaction potential between the Higgs and the neutral scalar field. Although non-abelian theories with YMCSH vortices were studied already in references [8–10], they are not non-abelian vortices with orientational moduli. The latter appear in U(N) gauge theories (as opposed to SU(N) gauge theories), where the crucial extra U(1) factor allows for the fundamental group π_1(G) ≃ ℤ, namely any number of vortices, whereas SU(N) vortices has a fundamental group π_1(SU(N)/ZN) ≃ Z_N. The genuine type of non-abelian vortex with orientational moduli was first found in references [16–18] in the context of Yang–Mills–Higgs theory. Such genuine non-abelian Chern–Simons–Higgs vortices were found later on in references [19–21] and with both Yang–Mills and CS terms in references [22, 23]. An important distinction between Yang–Mills–Higgs vortices and Chern–Simons–Higgs vortices is that the former are topological vortices, whereas the latter can be topological or nontopological. This feature is in general expected to persist in YMCSH theory, because of the existence of an unbroken phase, although we are not aware of any explicit solutions of nontopological vortices in this theory. Nevertheless, in this paper we shall focus on the topological vortices in YMCSH theory only and this means that we study the theory in the gapped, asymmetric or broken phase. For nice reviews on CS vortices, see e.g. references [24–27].

The genuine non-abelian vortices discussed so far are all in U(N) gauge theories, which in some sense is the most natural choice for this type of non-abelian vortices with orientational (CP^{N-1}_N) moduli. The generalization of Yang–Mills–Higgs vortices from U(N) to (U(1) × G)/ZN, theories was made in references [28–33] because of the group theoretical nontriviality of the corresponding non-abelian monopoles that the vortex strings can end on, for example in a cascading symmetry breaking theory considered in references [28, 34]. A special feature of non-abelian vortices in theories with G being a smaller group than SU(N), like for instance SO(N) or USp(2M) (with N = 2M), is that they generically contain semilocal or size moduli [30] and that the moduli space contains fractional vortices [30, 35]. The parallel development in non-abelian Chern–Simons–Higgs theory of generalizing the gauge group from U(N) to (U(1) × G)/ZN, was made in references [36–38].

In this paper, we work out the general equations for the BPS vortices in YMCSH theory and mainly focus on determining the dimension of the moduli space, that is, the number of zeromodes contained in solutions to these BPS equations. In doing so, we address the problem

3Here n_0 is the greatest common divisor of the abelian charges of the G' invariants.
at hand using a Callias-type index theorem, the moduli matrix approach as well as a brane construction in type IIB string theory. The index theorem calculation is a non-abelianization of the calculation carried out in reference [14], however, instead of splitting the equations up into real and imaginary parts, we retain the complex structure, which is crucial for the next step. The index only calculates the difference between the number of zeromodes of the operator that corresponds to the fluctuation equations of the BPS equations and the number of zeromodes of its adjoint. This only gives a lower bound on the number of zeromodes, unless we can prove that the adjoint operator does not possess normalizable zeromodes (which was the next step referred to above).

A main result of this paper is the proof that the adjoint operator does indeed not possess any normalizable zeromodes, which is established by theorems 1 and 2 and their corresponding proofs. The proof partially uses the vacua-in-gauge-theory-approach utilized in references [16, 30], but a crucial point was to find a useful gauge fixing condition for the fluctuation spectrum. The rest of the proof uses a Hermitian constraint on the fluctuations of the gauge fields, which on top of the BPS vortex background solutions is strong enough to eliminate the possibilities of fluctuations of the adjoint operator. After establishing the number of zeromodes, which is also the dimension of the moduli space (of vortex solutions), we rewrite the BPS equations into a master form using the moduli matrix approach. The moduli matrix approach is essentially the non-abelian solution to the selfdual equation that is shared among most BPS solitons and fully determines the non-abelian gauge field in terms of a moduli matrix and a residual field that does not contain moduli. This approach was used for non-abelian domain walls [39], composite solitons [40], and non-abelian vortices [41] in the U(1) Yang–Mills–Higgs theory, see reference [42] for a review. It was also used in references [29, 30, 35] where it was applied to the non-abelian vortices in Yang–Mills–Higgs theory and in references [36–38] in non-abelian Chern–Simons–Higgs theory, both with gauge groups (U(1) × G′)/Zn. All these well-studied vortices are included in our BPS theory as several limits. For example, the non-abelian vortices in Yang–Mills–Higgs theory [41] are obtained in the weak gauge coupling limit (and setting the adjoint scalar field to zero). The non-abelian Chern–Simons–Higgs vortex [36], on the other hand, is obtained by taking the strong gauge coupling limit. In this sense, the theory at hand is the most generic theory and we derive in this paper the most generic form of the moduli matrix method for the non-abelian vortices.

The final approach to the moduli space of non-abelian vortices in YMCSH theory is a generalization of the brane construction by Hanany and Tong [16] from having only Yang–Mills theory to having both Yang–Mills and CS terms in the theory on the brane (the vortex). The D-brane construction of supersymmetric three-dimensional CS theory in type IIB string theory has been studied in references [43–45], which involves studying so-called (p, q)-branes, where the CS level κ is related to the ratio p/q. Abelian CS vortices were studied in this context in references [44, 46]. T-duality can be used to make the brane construction of non-abelian vortices in Type IIA string theory [47] instead of in type IIB string theory [16] and the same holds true also for the abelian vortices in CS theory [48]. Furthermore, the D-brane construction also allows us to write down the effective theory on the world-line, exactly giving rise to the result obtained in reference [22].

The organization of the paper is as follows. In section 2, we introduce YMCSH theory, its various limiting theories, its mass spectrum and the BPS equations for vortices. In section 3, we calculate the index of the operator corresponding to the zeromodes of the vortices and prove that this index exactly counts this number. In section 4, we extend the moduli matrix approach to the YMCSH theory by including an extra adjoint field in the now coupled master equations. We also give some examples of vortex equations at the center of orientational patches on standard Taubes-like equation form. In section 5, we construct the D-brane setup for the vortices in type
IIB string theory, read off the number of zeromodes and its low-energy effective Lagrangian from the vortex brane.

2. The model

We consider $\mathcal{N} = 2$ supersymmetric Maxwell–YMCSH theory in $d = 2 + 1$ dimensions with the gauge group $G = (U(1) \times G')/\mathbb{Z}_{n_0}$ and often take $G'$ to be $SU(N)$, but will keep it general in most equations. $n_0$ is the greatest common divisor of the abelian charges of the invariants of $G'$. In particular, for $G' = SU(N)$ it is $n_0 = N$, for $G' = SO(2M), USp(2M)$ it is $n_0 = 2$ and for $G' = SO(2M + 1)$ it is $n_0 = 1$. $M \in \mathbb{Z}_{>0}$ is taken to be a non-negative integer, whereas $N$ is always the rank of $G$.

The bosonic part of the Lagrangian density reads

\[
\mathcal{L} = -\frac{1}{4g^2}(F_{\mu\nu}^\prime)^2 - \frac{1}{4e^2}(F_{\mu\nu}^0)^2 - \frac{\mu}{8\pi} e^{i\nu_\rho} A_\rho^\prime \partial_\mu A_\nu^\prime - \frac{1}{3} f^{abc} A_\mu^a A_\nu^b A_\rho^c + \frac{\kappa}{8\pi} e^{0} A_\rho^0 \partial_\mu A_\mu^0 + \frac{1}{2g^2} (D_\mu \phi^0)^2 + \frac{1}{2e^2}(\partial_\mu \phi^0)^2 + \text{tr} [iD^\prime H)(DH)^\dagger] \\
- \text{tr} [(\phi H - Hm)(\phi H - Hm)^\dagger] - \frac{g^2}{2} \left( \text{tr}[HH^\dagger \phi^0] - \frac{\mu}{4\pi} \phi^0 \right)^2 \\
- \frac{\kappa^2}{2} \left( \text{tr}[HH^\dagger \phi^0] - \frac{\kappa}{4\pi} \phi^0 - \frac{\xi}{\sqrt{2N}} \right)^2,
\]

(2.1)

where $a = 1, 2, \ldots, \dim(G')$ is the gauge index for $G'$ and the index 0 denotes the U(1) gauge group. The gauge couplings $e, g$ are for the U(1) and the $G'$ part of the Yang–Mills term, respectively. Similarly, the two CS couplings $\kappa, \mu$ are for the U(1) and the $G'$ part of the CS term, respectively, and take values in real and integer numbers: $\kappa \in \mathbb{R}$ and $\mu \in \mathbb{Z}$. The integral condition on the coupling renders the non-abelian CS term invariant under large gauge transformations [1]. The $N_f = N$ complex scalar fields $H$ are in the fundamental representation of $G'$ and are therefore $N \times N_f$ complex matrices and the real scalar field $\phi^0$ is in the adjoint representation of $G'$ ($\phi^0$ is a singlet). The mass matrix $m$ is a complex $N_f \times N_f$ matrix which gives rise to the mass-deformed version of the theory at hand (see e.g. reference [33] for a mass-deformed version of Yang–Mills–Higgs theory with arbitrary gauge groups), but in this paper we will consider only the case of $m = 0$. We fix our conventions as

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu],
\]

(2.2)

\[
D_\mu H = (\partial_\mu + iA_\mu)H,
\]

(2.3)

\[
D_\mu \phi = \partial_\mu \phi + i[A_\mu, \phi],
\]

(2.4)

with $\phi = \phi^a t^a$ and $A_\mu = A_\mu^a t^a$. Here we choose a standard notation

\[
\text{tr}(t^\alpha t^\beta) = \frac{1}{2} \delta^{\alpha\beta}, \quad \phi^0 = \frac{1}{\sqrt{2N}},
\]

(2.5)

for $\alpha, \beta = 0, 1, 2, \ldots, \dim(G')$, which are gauge group indices running over both U(1) and $G'$. The so-called Fayet–Illiopoulos (FI) parameter $\xi > 0$ is positive, which ensures stable supersymmetric vacua. To simplify the notation, we define the following parameter

\[
v \equiv \sqrt{\frac{\xi}{N}} > 0.
\]

(2.6)
Let us briefly discuss the spectrum of the theory in various limits in which the above theory reduces to other well-known theories.

- Maxwell–Yang–Mills–Higgs ($\kappa \to 0$ and $\mu \to 0$, $m = 0$):
  
  There exists a unique Higgs vacuum $H = v1_N$ and $\phi = 0$. The mass spectrum is
  
  $$m_e = ev, \quad m_h = gv,$$
  
  which are the masses of the abelian and non-abelian perturbative excitations of the fields, respectively. Due to supersymmetry, the vector multiplets and the chiral multiplets have the same masses. Namely, the abelian gauge field $A_0^\mu$, the real scalar field $\phi^0$ and the real part of the trace part of $H$ all have the same mass $m_e$. The vortex in this model is a non-abelian $G$-vortex which is a natural extension of the Nielsen–Olesen vortex of the abelian-Higgs model. The $G$-vortex has internal orientational modes associated with the spontaneously broken global symmetry in addition to the zeromode corresponding to the center of mass position. In the $G' = SU(N)$ case, these zeromodes of a single non-abelian vortex live on $CP^{N-1}$, whereas for $G' = SO(2M)$ and $G' = USp(2M)$, $(N = 2M)$ they live on $SO(2M)/U(M)$ and $USp(2M)/U(M)$, respectively.

- Maxwell–Yang–Mills–Chern–Simons ($H = 0$, $m = 0$):
  
  If we switch off the Higgs fields $H$ from the full Lagrangian (2.1), the model reduces to the Maxwell–Yang–Mills–Chern–Simons model. The FI term in equation (2.1) can be absorbed by a shift of $\phi^0$, which is equivalent to setting $\xi = 0$. The vacuum is in the symmetric phase with $\phi = 0$. No symmetries are broken. Although the Higgs mechanism is not at work, the vector multiplet acquires a topological mass due to the CS term. The masses are given by
  
  $$m_e = \frac{\kappa e^2}{4\pi}, \quad m_\mu = \frac{\mu g^2}{4\pi}.$$  
  (2.8)

$m_e$ is the mass of the abelian gauge field $A_0^\mu$ and also the real scalar field $\phi^0$. $m_\mu$ is the mass of $G'$ gauge field $A_\mu$, as well as the real adjoint field $\phi^a$.

- Chern–Simons–Higgs ($e \to \infty$ and $g \to \infty$, $m = 0$):
  
  The Maxwell and Yang–Mills kinetic terms (2.1) disappear in this limit. Hence, the Lagrangian (2.1) reduces to the Chern–Simons–Higgs model (see e.g. reference [36]). Furthermore, the adjoint scalar fields $\phi^a$ become nondynamical fields and can be integrated out as follows
  
  $$\phi^0 = \frac{4\pi}{\kappa} \left( \text{tr}(HH^T) - \sqrt{\frac{N}{2}}v^2 \right), \quad \phi^a = \frac{4\pi}{\mu} \text{tr}(HH^T)^a.$$  
  (2.9)

Plugging these back into equation (2.1), one obtains the sixth-order scalar potential

$$V_{e,g \to \infty} = \text{tr}[\phi^2HH^T].$$  
  (2.10)

There exist three types of vacua in this model. One is in the completely symmetric phase where $H = 0$ and no symmetries are broken. The vector multiplets are decoupled, so the Higgs fields are the only dynamical degrees of freedom. Their masses are

$$m_H = |\phi_0| = \frac{4\pi v^2}{\kappa} \sqrt{\frac{N}{2}}.$$  
  (2.11)
In between there is a variety of partially broken phases, which we will not discuss in detail here. Finally, there is the vacuum in the Higgs phase, where the gauge symmetry is completely broken. The longitudinal part of the gauge field acquires a mass through the Higgs mechanism. Moreover, the mass of the Higgs field is the same as that of the gauge field due to supersymmetry
\[ m_{\kappa}\infty = \frac{m_\kappa^2}{m_\kappa}, \quad m_{\mu}\infty = \frac{m_\mu^2}{m_\mu}. \]  
(2.12)

In the symmetric phase of the \( N = 1 \) (abelian) case, it is known that nontopological vortices exist. On the other hand, there exist topological solitons in the broken phase in the abelian \([11, 12]\) as well as the non-abelian theory \([19, 20]\) (see also references \([8–10, 21]\) for a non-abelian variant of the theory with the matter fields in the adjoint representation).

The topological and non-topological vortices in the Maxwell–Chern–Simons–Higgs model have also been paid great attention in the literature \([13–15]\) (\( N = 1, \mu = g = 0 \) in Lagrangian (2.1)). Its non-abelian generalization has also been studied \([22, 23]\):

- \( \kappa = \mu, e = g \) and \( m = 0 \): the Maxwell–Yang–Mills–Chern–Simons vortex.
- \( \kappa = \mu, e = g \) and \( m \neq 0 \): the dyonic non-abelian CS vortex.

In the following we will set \( m = 0 \),

\[ (2.13) \]

and set the number of flavors \( N_f := N \) so that \( H \) is a complex square matrix.

Now we are ready to identify the mass spectrum in the full Lagrangian (2.1) with the coupling constants kept at generic finite (and nonvanishing) values. There are two ground states respecting the full supersymmetry of the theory. The first one is an unbroken phase (i.e. the symmetric phase) and the second one is the Higgs phase (i.e. the broken phase). The unbroken phase is given by

\[ \begin{align*}
\text{Unbroken phase:} \quad & \phi^0 = -\frac{4\pi e^2}{\kappa}\sqrt{\frac{N}{2}}, \quad \phi^a = 0, \quad H = 0. \\
\end{align*} \]  
(2.14)

The gauge and flavor symmetries are not broken in this phase. Although the gauge symmetry is not broken, the gauge fields acquire topologically induced masses. The abelian gauge fields and the real scalar field \( \phi^0 \) have the same mass \( m_\kappa \), while the \( G' \) gauge fields and the adjoint scalar \( \phi^a \) have the same mass \( m_\mu \). This mass degeneracy is of course due to \( N = 2 \) supersymmetry.

On the other hand, the gauge symmetry is broken in the Higgs phase (i.e. the asymmetric or broken phase) where the fields develop the following vacuum expectation values (VEVs):

\[ \begin{align*}
\text{Broken phase:} \quad & \phi^0 = 0, \quad \phi^a = 0, \quad H = v1_N. \\
\end{align*} \]  
(2.15)

The symmetry breaking pattern of the continuous part of the symmetry is:

\[ \text{U}(1)_c \times G' \times SU(N)_H \rightarrow G'_{c+f}, \]  
(2.16)

hence the broken phase is in the color-flavor locking phase and this was the main reason for setting \( N_f = N \). Explicitly, for \( G' = SU(N) \) we have

\[ \text{U}(1)_c \times SU(N)_c \times SU(N)_H \rightarrow SU(N)_{c+f}. \]

Let us now identify the mass spectrum in the broken phase. Note that \( 1 + \dim G' \) real degrees of freedom in \( H \) are eaten by the gauge fields via the Higgs mechanism, which is \( N^2 \) real degrees
of freedom for \( G' = \text{SU}(N) \) and exactly half of the real degrees of freedom of \( H \). For other classical continuous groups, like \( G' = \text{SO}(N) \) and \( G' = \text{USp}(2M) \), it is smaller: \( N(N - 1)/2 \) and \( M(1 + 2M) \), respectively. The gauge fields acquire their masses through both the CS topological mechanism and the Higgs mechanism. Their mass poles of propagators of the abelian and the \( G' \) gauge fields split into two poles for each propagator respectively as (see references [49, 50] for the abelian case)

\[
m_{\text{vac}} = \sqrt{m_e^2 + m_\mu^2} \pm \frac{m_e}{2}, \quad m_{\text{vac}} = \sqrt{m_g^2 + m_\mu^2} \pm \frac{m_\mu}{2}, \tag{2.17}
\]

where \( m_e, m_\mu \) are given in equation (2.7) and \( m_e, m_\mu \) are given in equation (2.8). Note that we can reproduce the topological masses of equation (2.12) as the limit where the couplings \( e \) and \( g \) are sent to infinity

\[
\lim_{e \to \infty} m_{\text{vac}} = m_{\text{vac}}, \quad \lim_{e \to \infty} m_{\text{vac}} = \infty, \quad \lim_{g \to \infty} m_{\text{vac}} = m_{\text{vac}}, \quad \lim_{g \to \infty} m_{\text{vac}} = \infty, \tag{2.18}
\]

where in each mass, one pole is sent off to infinity and the other pole converges to the topological mass.

In order to identify the masses of the scalar fields, let us consider the following small fluctuations

\[
H = v 1_N + (\delta H^0 + i \delta H^e) e^\ell, \quad \phi = -e \delta \phi^0 - g \delta \phi^t. \tag{2.19}
\]

Let us furthermore define \( \delta \Phi^\nu \equiv (\delta H^0, \delta \phi^0) \). Then the quadratic terms in the fluctuations of the Lagrangian in the scalar sector are found to be

\[
\mathcal{L}^{(2)} = \frac{1}{2} \partial_\mu \delta H^\nu + \partial^\nu \delta H^\mu + \frac{1}{2} \partial_\mu \partial^\nu \delta \Phi^\rho \delta \Phi^{\rho \nu} - \frac{1}{2} \partial_\mu \partial^\nu \delta \Phi^\rho \delta \Phi^{\rho t} - \frac{1}{2} \delta \Phi^\rho M^2_{\Phi} (\delta \Phi^\rho)^T, \tag{2.20}
\]

where we have introduced the following Hermitian mass-squared matrices

\[
M^2_e \equiv \begin{pmatrix} m_e^2 & m_e m_\mu \\ m_e m_\mu & m_\mu^2 \end{pmatrix}, \quad M^2_\mu \equiv \begin{pmatrix} m_\mu^2 & m_\mu m_e \\ m_\mu m_e & m_e^2 \end{pmatrix}. \tag{2.21}
\]

It is then possible to read off the mass spectrum by diagonalization of these mass-squared matrices. For later convenience, let us rewrite the mass matrices in the following way

\[
M^2_e = U_e \begin{pmatrix} m_{\text{vac}}^2 & m_{\text{vac}} \\ m_{\text{vac}} & m_{\text{vac}}^2 \end{pmatrix} U^{-1}_e, \quad M^2_\mu = U_\mu \begin{pmatrix} m_{\text{vac}}^2 & m_{\text{vac}}^2 \\ m_{\text{vac}}^2 & m_{\text{vac}}^2 \end{pmatrix} U^{-1}_\mu, \tag{2.22}
\]

where the SO(2) matrices \( U_e, U_\mu \) are defined as follows

\[
U_e \equiv \begin{pmatrix} \cos u_e & \sin u_e \\ -\sin u_e & \cos u_e \end{pmatrix}, \quad \tan 2u_e = \frac{2m_e}{m_{\text{vac}}}, \quad u_e \in \left[ 0, \frac{\pi}{4} \right]. \tag{2.23}
\]

\[\text{Note that if } G' \text{ is a subgroup of } \text{SU}(N), \text{ then all the complimentary elements to } G' \text{ of the matrix } H \text{ remain massless, whereas for } G' = \text{SU}(N) \text{ all elements of } H \text{ acquire a mass.}\]
Again the degeneracy between the masses of the vector multiplets and the chiral multiplets occurs because the Lagrangian respects $\mathcal{N} = 2$ supersymmetry. Hence we have four different characteristic mass scales in our system. Let us introduce the following definition

$$\gamma_{i/j} = m_i^2/m_j^2, \quad i, j = e, g, \mu, \kappa, e\kappa\pm, g\mu\pm,$$

which are dimensionless parameters characterizing the solutions. However, only three of these parameters $\gamma_{i/j}$ are independent.

We are interested in vortex-type solitons in the Higgs vacuum. Since $(U(1)_c \times G')/\mathbb{Z}_{n_0}$ is spontaneously broken, we expect topologically stable vortices, which are supported by

$$Z = \pi_1 \left( U(1)_c \right) \subset \pi_1 \left( (U(1)_c \times G')/\mathbb{Z}_{n_0} \right).$$

Thanks to $\mathcal{N} = 2$ supersymmetry, the vortices in this system are of the BPS type of soliton, which preserves half of supersymmetry. Static interactions between multiple vortices are completely canceled out, hence we can construct static multi-vortices. Instead of solving half of the supersymmetry-preserving conditions on the fermions, we can derive the BPS equations by performing a standard Bogomol’nyi completion of the energy functional

$$T = \int_C \left( \frac{1}{2g^2} \left[ F_{12}^0 - g^2 \left( \text{tr}(HH') - \frac{\mu}{4\pi} \phi^0 \right) \right]^2 + \frac{1}{2g^2} \left[ F_{ij}^0 + D_i \phi^0 \right]^2 \right. \\
+ \frac{1}{2e^2} \left[ F_{12}^0 - e^2 \left( \text{tr}(HH') - \frac{\kappa}{4\pi} \phi^0 - \frac{\xi}{\sqrt{2N}} \right) \right]^2 + \frac{1}{2e^2} (D_i \phi^0)^2 \right) \left. + 4 \text{tr}|D_i H|^2 + \text{tr}|D_i H - i\phi H|^2 + \frac{1}{2g^2} (D_i \phi^0)^2 + \frac{1}{2e^2} (D_i \phi^0)^2 \right) \left. - \frac{\xi}{\sqrt{2N}} D_{ij}^0 - \frac{1}{e^2} \partial_i (F_{ij}^0 \phi^0) - \frac{1}{e^2} \partial_i (F_{ij}^0 \phi^0) - i \epsilon^{ij} \text{tr} \partial_i \left( (D_j H) H' \right) \right] \right) \, d^2x,$$

where $i, j = 1, 2$, we have used Gauss’ law (2.36) and we have defined

$$D_i = D_i + iD_i.$$

The tension is bounded from below by the Bogomol’nyi bound

$$T \geq T_{\text{BPS}} = -\frac{\xi}{\sqrt{2N}} \int_C F_{12}^0 \, d^2x = 2\pi \nu = \frac{2\pi \xi k}{n_0}, \quad k \in \mathbb{Z}_{\geq 0},$$

with

$$\nu = -\frac{1}{2\pi \sqrt{2N}} \int_C \epsilon_{12} \, d^2x = \frac{k}{n_0}.$$

being the U(1) winding number and $n_0$ is the greatest common divisor of the abelian charges of the $G'$ invariants. This result was established in reference [29] and the analysis looks at the winding of all $G'$ invariants at spatial infinity. The smallest ‘closed loop’ is made by having a U(1) winding of $1/n_0$ and returning inside the group $G'$. Any smaller winding numbers would thus make some invariants non-single-valued at infinity. Due to the normalization of the U(1) gauge fields (2.5), the factor of $1/\sqrt{2N}$ appears explicitly in the definition of the U(1) winding number. For $G' = SU(N)$, $n_0 = N$ and the BPS tension is simply $T_{\text{BPS}} = 2\pi \xi k/N = 2\pi \nu^2 k$. 

\[\text{(2.24)}\]

\[\text{(2.25)}\]

\[\text{(2.26)}\]

\[\text{(2.27)}\]

\[\text{(2.28)}\]

\[\text{(2.29)}\]
The inequality is saturated when the BPS equations are satisfied:

\[ D^\ast_{\bar{z}} H = 0, \quad (2.30) \]

\[ \hat{F}_{12} - g^2 \left( (HH)_{\alpha} - \frac{\mu}{4\pi} \right) = 0, \quad (2.31) \]

\[ F_{0\bar{z}}^0 - e^2 \left( \frac{1}{\sqrt{2N}} \text{tr}(HH) - \frac{\kappa}{4\pi} \phi^0 - \frac{\xi}{\sqrt{2N}} \right) = 0, \quad (2.32) \]

\[ F_{0\bar{z}}^i + D_i \phi = 0, \quad (D_0 - i\phi) H = 0, \quad D_0 \phi = 0, \quad (2.33) \]

where we have introduced the notation

\[ \hat{\phi} = \phi^{\alpha\rho}, \quad (2.34) \]

for the \( G' \) part of fields (and similarly for \( F_{12} \)), as well as the projection operator

\[ \langle X \rangle_{G'} = \text{tr} \left[ X^{\alpha} t^{\alpha} \right], \quad (2.35) \]

for an arbitrary matrix \( X \). Note that the above BPS equations are to be accompanied by Gauss' law

\[ \frac{1}{g^2} D_0 \hat{F}_{0\bar{z}} = \frac{\mu}{4\pi} \hat{F}_{12} + \frac{i}{g^2} \left[ \hat{\phi}, D_0 \hat{\phi} \right] = j_0, \quad \frac{1}{e^2} \partial_i F_{0\bar{z}}^i - \frac{\kappa}{4\pi} F_{0\bar{z}}^0 = j_0^0, \quad (2.36) \]

where we have defined the Noether current

\[ j_0 = j_0^{\alpha\rho} \equiv i \langle (D_0 H) H^\dagger - H (D_0 H)^\dagger \rangle_{G'}, \quad (2.37) \]

and the \( G \) projection operator

\[ \langle X \rangle_G \equiv \text{tr} \left[ X^{\alpha} \right]^{\alpha} = \langle X \rangle_{G'} + \frac{1}{\sqrt{2N}} \text{tr}(X) I_N, \quad (2.38) \]

and the covariant derivative acts on the field strength as

\[ \partial_i \hat{F}_{0\bar{z}} = \partial_i \hat{F}_{0\bar{z}} + i \left[ A_i, \hat{F}_{0\bar{z}} \right]. \quad (2.39) \]

Anti-BPS vortices can be obtained in similar way by a different choice of signs in writing the squares when doing the Bogomol'nyi completion. An important feature of the vortices in CS theory is that they carry electric charges. This fact can be seen directly from Gauss' law. The electric charge is fractionally quantized since it is related to the magnetic flux as

\[ Q \equiv \frac{1}{\sqrt{2N}} \int_C j_0^0 \, d^2x = - \frac{\kappa}{4\pi \sqrt{2N}} \int_C F_{12}^0 \, d^2x = \frac{\kappa k}{2\pi n_0}. \quad (2.40) \]

In order to simplify the BPS equations, we can consistently choose \( \phi = A_0 \) in temporal gauge. Moreover, we will only consider static solutions in this paper. Hence, the BPS equations combined with Gauss' law yield the following system of equations

\[ D_0 H = 0, \quad (2.41) \]

\[ \hat{F}_{12} = g^2 \left( (HH)_{\alpha} - \frac{\mu}{4\pi} \phi \right), \quad F_{12}^{0\bar{z}} = \frac{e^2}{\sqrt{2N}} \left( \text{tr}(HH) - \frac{\kappa \sqrt{2N}}{4\pi} \phi^0 - \xi \right), \quad (2.42) \]
\[
\frac{1}{g^2} D_\phi^2 + \frac{\mu}{4\pi} F_{12} = \left\langle \{ HH^\dagger, \phi \} \right\rangle_{\phi}, \quad \frac{1}{e^2} \partial_\phi^2 \phi^0 + \frac{\kappa}{4\pi} F_{12} = \sqrt{2} \frac{N}{N} tr( \{ HH^\dagger, \phi \}),
\]

(2.43)

For later convenience we write equation (2.43) as follows

\[
4D_z D_{\bar{z}} \hat{\phi} = m^2 \mu \hat{\phi} - g^2 m^2 \phi + \frac{e^2}{\sqrt{2N}} \left[ (2\phi - m_\mu 1_N) HH^\dagger \right] + \sqrt{N} \frac{m^2}{2} m_\mu 1_N,
\]

(2.44)

and the former simplifies for \( G' = SU(N) \)

\[
4D_z D_{\bar{z}} \hat{\phi} = m^2 \mu \hat{\phi} + g^2 \left( \phi - m_\mu 1_N \right) HH^\dagger + \frac{1}{2} m^2 m_\mu 1_N.
\]

(2.45)

where we have used the identity \( D_\phi^2 \hat{\phi} = 4D_z D_{\bar{z}} \hat{\phi} + [ \hat{F}_{12}, \hat{\phi} ] \), and equation (2.42).

The equations simplify for the \( G = U(N) \) case, by which we mean that \( G' = SU(N) \) and the abelian couplings equal their non-abelian counterparts: \( e = g \) and \( \kappa = \mu \). In that case we can write

\[
D_z H = 0,
\]

(3.1)

\[
F_{12} = \frac{g^2}{2} HH^\dagger - m_\mu \phi - \frac{1}{2} m_s^2 1_N,
\]

(3.2)

\[
4D_z D_{\bar{z}} \phi = m^2 \phi + g^2 \left( \phi - \frac{1}{2} m_\mu 1_N \right) HH^\dagger + \frac{1}{2} m^2 m_\mu 1_N.
\]

(3.3)

3. The index theorem

We will now turn to the fluctuation spectrum around an arbitrary BPS solution in the YMCSH theory. In order to avoid a cluttered notation, we will carry out the calculation for the \( G = U(N) \) case, but the \( G' \) dependence in the end result will remain transparent, as we will see shortly.

In order to get a real equation for \( \phi \) and its corresponding fluctuations, we will add the Hermitian conjugate of the equation to itself and divide by four, obtaining:

\[
D_{[\phi} D_{\phi]} \phi = D_z D_\phi \phi + D_{\phi} D_z \phi = \frac{m^2}{2} \phi + \frac{g^2}{4} \{ \phi, HH^\dagger \} - \frac{m_\mu g^2}{4} HH^\dagger
\]

\[+ \frac{m_\mu m^2}{4} 1_N,
\]

(3.1)

where the curly brackets on the indices mean symmetrization.

It will prove convenient and simplify the notation, if we do the following rescaling

\[
z \rightarrow \frac{z}{m_s}, \quad \partial_z \rightarrow m_s \partial_z, \quad A_z \rightarrow m_\mu A_z, \quad H \rightarrow vH, \quad \phi \rightarrow \frac{m^2}{m_\mu} \phi,
\]

(3.2)

for which the BPS system of equations reads

\[
D_z H = 0,
\]

(3.3)
\[ F_{12} = \frac{1}{2}HH^\dagger - \phi - \frac{1}{2}1_N, \] (3.4)

\[ D_{\{D\}}\phi = \frac{\tau}{2}\phi + \frac{1}{4}\{\phi, HH^\dagger\} - \frac{\tau}{4}HH^\dagger + \frac{\tau}{4}1_N, \] (3.5)

where we have defined

\[ \tau \equiv \frac{m_+^2}{m_g}. \] (3.6)

Varying the left-hand-side of equation (3.1) yields

\[ \delta[D_{\{D\}}\phi] = D_{\{D\}}\delta\phi + i[D_{\{D\}}\delta A_\zeta, \phi] + i2[\delta A_\zeta, D_{\{D\}}\phi] + i2[\delta A_\zeta, D_{\{D\}}\phi], \] (3.7)

where \( \delta\phi, \delta A_\zeta \) and \( \delta A_\zeta \) denote the perturbations (fluctuations) of the fields \( \phi, A_\zeta \) and \( A_\zeta \), respectively, and the fluctuations still have to obey the BPS equations on the background of a given solution where they describe moduli (zeromodes) of the solution. The covariant derivative acts on the gauge field (which is adjoint valued), and hence on the fluctuations of the gauge field, as

\[ D_{\{D\}}\delta A_\zeta = \partial_{\zeta}\delta A_\zeta + i[A_\zeta, \delta A_\zeta]. \] (3.8)

To perform the calculation, we need to fix the gauge of the fluctuations and we choose

\[ D_{\{D\}}\delta A_\zeta + \delta HX + Y \delta H^\dagger = 0, \] (3.9)

where \( X \) and \( Y \) are unspecified functions of the background fields (i.e. solutions to the BPS equations).

The linear fluctuation equations thus read

\[ iD_{\{D\}}\delta H - \delta A_\zeta H = 0, \] (3.10)

\[ iD_{\{D\}}\delta A_\zeta - iD_{\{D\}}\delta A_\zeta - \frac{1}{4}\delta HH^\dagger - \frac{1}{4}H\delta H^\dagger + \frac{1}{2}\delta\phi = 0, \] (3.11)

\[ D_{\{D\}}\delta H^\dagger - \frac{\tau}{2}\delta\phi - \frac{1}{4}\{\delta HH^\dagger, \phi\} - \frac{1}{4}\{H\delta H^\dagger, \phi\} - \frac{1}{4}\{HH^\dagger, \delta\phi\} = 0, \] (3.12)

where we have used the gauge condition for the fluctuations (3.9), and in turn the operator equation is

\[ \Box \eta = 0, \] (3.13)

with the operator defined by

\[ \Box = iD + K, \] (3.14)
\[ D \equiv \begin{pmatrix} \partial_z & 0 & 0 & 0 & 0 \\ 0 & \partial_z & 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & -i2D_z \partial_z - i2D_\bar{z} \partial_\bar{z} + i2\partial_\bar{z} \partial_z \end{pmatrix}, \quad (3.15) \]

\[ K \equiv \begin{pmatrix} -A_z & 0 & -\circ H & 0 & 0 \\ 0 & -\circ A_\bar{z} & 0 & \circ H^\dagger & 0 \\ -1/4\sqrt{2} \circ H^\dagger & -1/4\sqrt{2} \circ H & \frac{1}{\sqrt{2}} [A_\circ, \circ] & -\frac{1}{\sqrt{2}} [A_\circ, \circ] & 1/2\sqrt{2} \\ -\circ X & \circ Y & -\frac{1}{\sqrt{2}} [A_\circ, \circ] & -\frac{1}{\sqrt{2}} [A_\circ, \circ] & 0 \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{pmatrix}, \quad (3.16) \]

\[ K_{51} \equiv \frac{\tau}{4} \circ H^\dagger - \frac{1}{4} \{ \circ H^\dagger, \phi \} - i[\circ X, \phi], \quad (3.17) \]

\[ K_{52} \equiv \frac{\tau}{4} \circ H - \frac{1}{4} \{ H, \circ \} - i[Y, \phi], \quad (3.18) \]

\[ K_{53} \equiv i[\circ, D_\circ \phi], \quad (3.19) \]

\[ K_{54} \equiv i[\circ, D_\circ \phi], \quad (3.20) \]

\[ K_{55} \equiv -\frac{\tau}{2} - \frac{1}{4} \{ HH^\dagger, \circ \} - [A_\circ, [A_\circ, \circ]] - [A_\circ, [A_\circ, \circ]], \quad (3.21) \]

acting on

\[ \eta \equiv \begin{pmatrix} \delta H \\ \delta H^\dagger \\ \delta A_\circ \\ \delta A_\circ \\ \delta \phi \end{pmatrix}, \quad (3.22) \]

where the rows of the operator correspond to the fluctuations of the self-dual equation (3.10), its Hermitian conjugate, the fluctuations of the magnetic flux equation (3.11), the gauge fixing condition (3.9) and the fluctuations of the adjoint scalar field equation (3.12), and \( \circ \) is a placeholder for the matrix that the operator acts on in \( \eta \), which makes left and right actions more convenient in matrix notation. We have used the Coulomb gauge condition for the background gauge field in equation (3.21), i.e. \( D_\circ [A_\circ] = 0 \). The index of this operator will result in the number of real zeromodes, because we have included the complex self-dual BPS equation twice.

The definition of the split of the operator \( \mathcal{D} \) into \( D \) and \( K \) is that all derivative operators are placed in \( D \) and the rest in \( K \).

The adjoint operator is found to be

\[ \mathcal{D}^\dagger = iD^\dagger + K^\dagger, \quad (3.23) \]
First it is demonstrated that \( D^\dagger D = DD^\dagger \) is the following diagonal operator

\[
D^\dagger \equiv \begin{pmatrix}
\partial_z & 0 & 0 & 0 & 0 \\
0 & \partial_z & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} \partial_{z_2} & \frac{1}{\sqrt{2}} \partial_{z_2} & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} \partial_{z_2} & \frac{1}{\sqrt{2}} \partial_{z_2} & 0 \\
0 & 0 & 0 & 0 & -i2\partial_z \partial_{z_2} - i2\partial_z \partial_{z_2} + i2\partial_z \partial_{z_2}
\end{pmatrix}, \tag{3.24}
\]

\[
K^\dagger \equiv \begin{pmatrix}
-A_z & 0 & -\frac{1}{4\sqrt{2}} \phi H & \phi X^\dagger & K^\dagger_{15} \\
0 & \phi A_z & -\frac{1}{4\sqrt{2}} H^\dagger & y^\dagger & K^\dagger_{25} \\
-\phi H^\dagger & 0 & \frac{1}{\sqrt{2}} [A_z, \phi] & -\frac{1}{\sqrt{2}} [A_z, \phi] & K^\dagger_{35} \\
0 & H & -\frac{1}{\sqrt{2}} [A_z, \phi] & -\frac{1}{\sqrt{2}} [A_z, \phi] & K^\dagger_{35} \\
0 & 0 & \frac{1}{2\sqrt{2}} & 0 & K^\dagger_{55}
\end{pmatrix}, \tag{3.25}
\]

\[
K^\dagger_{15} \equiv \frac{\tau}{4} \phi H - \frac{1}{4} \{\phi, \phi\} H + i\{\phi, \phi\} X^\dagger, \tag{3.26}
\]

\[
K^\dagger_{25} \equiv \frac{\tau}{4} H^\dagger - \frac{1}{4} H^\dagger \{\phi, \phi\} + iY^\dagger \{\phi, \phi\}, \tag{3.27}
\]

\[
K^\dagger_{35} \equiv -i2\{\phi, D_z \phi\}, \tag{3.28}
\]

\[
K^\dagger_{45} \equiv -i2\{\phi, D_z \phi\}. \tag{3.29}
\]

The index that we will calculate here is an extension of the index that was calculated in reference [38], which in turn is an extension of the Atiyah–Singer index to \( \mathbb{R}^2 \), i.e. of Callias type, specialized to non-abelian systems whose fluctuations are described by \( D \). The index is formally given by

\[
I = \dim \ker D - \dim \ker D^\dagger. \tag{3.32}
\]

If we can show that the dimension of the kernel of the adjoint operator, \( D^\dagger \), is zero, then the index is equal to the number of zeromodes of the operator \( D \). If not, the index only gives a lower bound on the number of zeromodes.

It will prove convenient to write the index as

\[
I = \dim \ker D^\dagger D - \dim \ker DD^\dagger, \tag{3.33}
\]
which can be seen to hold based on the considerations
\[ \langle \eta, D\eta \rangle = \langle D\eta, \eta \rangle = \|\eta\|^2 \geq 0, \]  
(3.34)
and hence the kernel of \( D\eta \) must coincide with that of \( \eta \). Similarly, the kernel of \( DD\eta \) must
coincide with that of \( D\eta \). We will now write the index in the form
\[ I = \lim_{M \to \infty} I(M^2) = \lim_{M \to \infty} \text{Tr} \left( \frac{M^2}{D\eta} + M^2 \right) - \text{Tr} \left( \frac{M^2}{DD\eta} + M^2 \right) \]
(3.35)
where \( \text{Tr} \) denotes both a trace over states as well as the trace over matrices, whereas \( \text{tr} \) is the
usual matrix trace and we have defined
\[ M^2 = \text{block-diag} \left[ M^2, M^2, M^2, M^2, M^2, M^2 \right]. \]
(3.36)
Notice that the last block contains \( M^4 \), matching the higher power of derivatives in \( \Delta \) (see
equation (3.30)).

Subtleties arise in the case that the continuum part of the spectrum extends down to zero
and more care has to be taken. This will not be a problem when the system possesses a mass
gap, which is the case in the fully Higgsed phase.

Using the decompositions (3.14) and (3.23), we have
\[ D\eta = -D^\dagger D + iD^\dagger K + iK^\dagger D + K^\dagger K \equiv \Delta - L_1, \]
(3.37)
\[ DD\eta = -DD^\dagger + iD^\dagger K + iK^\dagger D + KK^\dagger \equiv \Delta - L_2, \]
(3.38)
and \( L_{1,2} \) are at most linear in the operators \( D \) and \( D^\dagger \). We will now follow reference [38] and
expand equation (3.35) as follows
\[ \frac{M^2}{D\eta} + M^2 = M^2 \left( P^{-1} + P^{-1}L_1P^{-1} + P^{-1}L_1P^{-1}L_1P^{-1} + \cdots \right), \]
(3.39)
\[ \frac{M^2}{DD\eta} + M^2 = M^2 \left( P^{-1} + P^{-1}L_2P^{-1} + P^{-1}L_2P^{-1}L_2P^{-1} + \cdots \right), \]
(3.40)
where we have defined \( P \equiv \Delta + M^2 \). All terms beyond the second order in \( L_{1,2} \) will not con-
tribute to the index (3.35) in the \( M \to \infty \) limit. The symbolic manipulation of the terms given
in equations (3.39) and (3.40) follow analogously to that in references [14, 38], because we
have that
\[ [P^{-1}, D] = [P^{-1}, D^\dagger] = 0, \quad DP^{-1}D^\dagger = D^\dagger P^{-1}D = -1 + M^2P^{-1} = -1 + P^{-1}M^2. \]
(3.41)
The result of the algebraic manipulations is [14, 38]
\[ I = \lim_{M \to \infty} \int_C \text{tr} M^2 \left( x \left| \frac{1}{\Delta + M^2} \left( DK^\dagger - K^\dagger D - D^\dagger K + KD^\dagger \right) \frac{1}{\Delta + M^2} \right| x \right) d^2 x. \]
(3.42)
Calculating explicitly the matrix $DK^\dagger - K^\dagger D - D^\dagger K + KD^\dagger$, and using the fact that $^5K_{55}^\dagger = K_{34}, K_{43}^\dagger = K_{43}$ are self-adjoint operators, we find that the only contributions to the trace are

$$\text{block-diag} \left[ \partial \bar{z}K^\dagger_{11} - \partial zK_{11}, \partial \bar{z}K^\dagger_{22} - \partial zK_{22}, \frac{\partial zK_{33} - \partial \bar{z}K_{33}}{\sqrt{2}}, \frac{\partial \bar{z}K_{44}^\dagger - \partial zK_{44}}{\sqrt{2}}, 0 \right]. \quad (3.43)$$

Notice that the last block is identically zero, which has the consequence that the $\square$ operator does not contribute to the trace at all. This is due to the form of the operators $D, D^\dagger$ and the self-adjointness of $K_{55}$. Furthermore, the matrix trace of the third and fourth block vanishes

$$\text{tr} \left( \partial \bar{z}K_{33} - \partial zK_{33} \right) = 0, \quad \text{tr} \left( \partial \bar{z}K_{44}^\dagger - \partial zK_{44} \right) = 0, \quad (3.44)$$

simply because the trace of a commutator vanishes. Thus the index simplifies to

$$I = \lim_{M \to \infty} -2iM^2 \int_C \text{tr} \left( \partial \bar{A}_z - \partial zA_\bar{z} \right) \langle x | (-\partial _z \partial_{\bar{z}} + M^2)^{-2} | x \rangle \; d^2x$$

$$= \lim_{M \to \infty} -M^2 \frac{N}{\sqrt{2N}} \sum_{j=1}^{N_f} \int_{C_{12}} F^0_{12} \; d^2x \int \frac{d^2k}{(2\pi)^2} \frac{1}{\left( \frac{1}{4} k^2 + M^2 \right)^2}$$

$$= -N_f N \frac{1}{\pi \sqrt{2N}} \int_{C_{12}} F^0_{12} \; d^2x$$

$$= 2N_f N \nu = \frac{2N_f N \nu}{n_0}. \quad (3.45)$$

Now, recall that the operator $D$ contained both the self-dual equation as well as its Hermitian conjugate equation, $\delta(D, H)^\dagger$, and therefore the index is counting the number of real zeromodes.

Notice that leaving the gauge group $G'$ unspecified, the calculation remains unaltered and depends only on $F^0_{12}$, although the operator $D$ now contains projections $\langle \cdot \rangle_{G'}$. Carrying out the index calculation for generic $G'$ thus gives the result (3.45) and the $G'$ dependence is simply encoded in the greatest common divisor of the abelian charges of the $G'$ invariants, which is exactly $n_0$. For $G' = SU(N)$, the index is $2N_f k$ and thus independent of $N$. However, we assumed a fully Higgsed theory, which requires $N_\ell \geq N$. For the square-matrix case, $N_\ell = N$, the index is simply $2Nk$.

In order to establish that $I$ is in fact the number of real zeromodes, we first need the following theorem.

**Theorem 1.** The kernel of $D^\dagger$, in the case of $G = U(N)$ and $N_\ell \geq N$, has dimension zero.

^5 Here we adopt the notation that the indices $ij$ denote the matrix block of the operator.
Proof. First we act on a state with the adjoint operator $D^\dagger$:

$$
D^\dagger \begin{pmatrix} \alpha \\ \alpha^\dagger \\ \beta \\ \beta^\dagger \end{pmatrix} = \begin{pmatrix}
    iD_\alpha - \frac{1}{4\sqrt{2}}(\beta H + \beta^\dagger X^\dagger + \frac{\tau}{4}\omega H - \frac{1}{4}\{\omega, \phi\}H + i[\omega, \phi]X^\dagger) \\
    iD_\beta + \frac{1}{4\sqrt{2}}H^\dagger \beta + Y^\dagger \beta^\dagger + \frac{2}{4}H^\dagger \omega - \frac{1}{4}(H^\dagger \{\omega, \phi\} + iY^\dagger [\omega, \phi]) \\
    -\alpha H^\dagger - \frac{i}{\sqrt{2}}D_\beta + \frac{i}{\sqrt{2}}D_\beta^\dagger - i2[\omega, D_\phi] \\
    H_{\alpha}^\dagger + \frac{i}{\sqrt{2}}D_\beta + \frac{i}{\sqrt{2}}D_\beta^\dagger - i2[\omega, D_\phi] \\
    D_\alpha D_\omega + D_\beta D_\omega + \frac{1}{2\sqrt{2}}(\beta - \frac{\tau}{2}\omega - \frac{1}{4}\{HH^\dagger, \omega\})
\end{pmatrix} = 0,
$$

(3.46)

where $\omega = \omega^\dagger$. Writing combinations of the above equations yields

$$
D_\alpha \alpha = 0, \quad (3.47)
$$

$$
\left(\frac{1}{\sqrt{2}}(\beta + \beta^\dagger) - \tau \omega + \{\omega, \phi\} - \frac{i}{\sqrt{2}}[\omega, \phi]\right)H = 0, \quad (3.48)
$$

$$
D_\beta \beta^\dagger + i\sqrt{2}\alpha H^\dagger = 0, \quad (3.49)
$$

$$
D_\alpha \beta + 2\sqrt{2}[\omega, D_\phi] = 0, \quad (3.50)
$$

$$
D_\alpha D_\omega + D_\beta D_\omega + \frac{1}{2\sqrt{2}}(\beta - \frac{\tau}{2}\omega - \frac{1}{4}\{HH^\dagger, \omega\}) = 0, \quad (3.51)
$$

where we have chosen the gauge fixing functions as

$$
X^\dagger = Y = -\frac{1}{4\sqrt{2}}H. \quad (3.52)
$$

Writing the complex norm-squared of equation (3.49), we have

$$
0 = \int_C \left(\frac{1}{\sqrt{2}}(\beta + \beta^\dagger) - \tau \omega + \{\omega, \phi\} - \frac{i}{\sqrt{2}}[\omega, \phi]\right) \left(D_\beta \beta^\dagger + i\sqrt{2}\alpha H^\dagger\right) d^2x
$$

$$
= \|D_\beta \beta^\dagger\|^2 + 2\|\alpha H^\dagger\|^2 - i\sqrt{2}\int_C \partial_x(H_{\alpha}^\dagger \beta^\dagger) d^2x + i\sqrt{2}\int_C \partial_x(\beta \alpha H^\dagger) d^2x
$$

$$
= \|D_\beta \beta^\dagger\|^2 + 2\|\alpha H^\dagger\|^2, \quad (3.53)
$$

where in the second line, we have used the self-dual equation $D_\alpha H = 0$, equation (3.47) and their Hermitian conjugates, and we have defined

$$
\|X\|^2 \equiv \int_C X^\dagger X d^2x. \quad (3.54)
$$

In the third line, we have used that the boundary conditions for the fluctuations are

$$
\lim_{|x|\to\infty} \alpha = \lim_{|x|\to\infty} \beta = \lim_{|x|\to\infty} \omega = 0. \quad (3.55)
$$

From the third line in equation (3.53), we can conclude that

$$
\alpha = 0, \quad (3.56)
$$
since $H$ has full rank almost everywhere (except at vortex positions) and that $D_z \beta^\dagger = 0$. Taking the Hermitian conjugate of equation (3.51), we can conclude that

$$\beta = \beta^\dagger,$$  \hspace{1cm} (3.57)

is a real-valued adjoint scalar field and therefore we have

$$D_z \beta^\dagger = D_z \beta = D_z \beta^\dagger = D_z \beta = 0.$$  \hspace{1cm} (3.58)

From this result, we have from equation (3.50) that

$$[\omega, D_z \phi] = 0,$$  \hspace{1cm} (3.59)

which is not quite strong enough a condition to conclude that $\omega$ vanishes. Considering first $D_z \beta = 0$ in the vortex background, using the solution to the gauge field (4.1), we have

$$\partial_z \beta + [S(z, \bar{z})^{-1} \partial_z S(z, \bar{z}), \beta] = 0,$$  \hspace{1cm} (3.60)

which yields the solution

$$\beta = S(z, \bar{z})^{-1} \beta_0(z) S(z, \bar{z}).$$  \hspace{1cm} (3.61)

Considering instead $D_z \beta = 0$ in the vortex background, using the Hermitian conjugate of the solution to the gauge field (4.1), we obtain

$$\partial_z \beta - [\partial_z S(z, \bar{z})^\dagger S(z, \bar{z})^{-1}, \beta] = 0,$$  \hspace{1cm} (3.62)

yielding the different solution

$$\beta = S(z, \bar{z})^\dagger \bar{\beta}_0(\bar{z}) S(z, \bar{z})^{-1}.$$  \hspace{1cm} (3.63)

Equating the two solutions, we have

$$\bar{\beta}_0(\bar{z}) = \Omega^{-1} \beta_0(z) \Omega,$$  \hspace{1cm} (3.64)

where $\Omega$ is the Hermitian invertible matrix defined in equation (4.4). There is no Hermitian matrix that can transform a holomorphic function in $z$ into an anti-holomorphic function. We can thus conclude that $\beta_0(z)$ and $\bar{\beta}_0(\bar{z})$ must be independent of $z$ and $\bar{z}$ and hence be constant matrices. That is not sufficient, however, for meeting the constraint that $\beta = \beta^\dagger$ and hence, we must have that $S(z, \bar{z})^{-1} = S(z, \bar{z})^\dagger$ and $\beta_0 = \bar{\beta}_0$ (and $\beta_0 = \bar{\beta}_0$), which is generically not the case for a background solution. If $S(z, \bar{z})^{-1} \neq S(z, \bar{z})^\dagger$, we must take

$$\beta_0 = \bar{\beta}_0 = c \mathbf{1}_N,$$  \hspace{1cm} (3.65)

which means that

$$\beta = c \mathbf{1}_N,$$  \hspace{1cm} (3.66)

or else we can have

$$\beta = S(z, \bar{z}) \beta_0 S(z, \bar{z})^{-1} = S(z, \bar{z}) \beta_0 S(z, \bar{z})^\dagger,$$  \hspace{1cm} (3.67)

with $\beta_0 = \beta_0^\dagger$ a constant matrix. Using the boundary conditions (3.55), we have $c = 0$ or $\beta_0 = 0$ and hence $\beta = 0$ in any case. This can be seen by the fact that $S(z, \bar{z})$ in the latter case acts as
a unitary transformation of a Hermitian constant matrix that must have real eigenvalues: the only way the matrix can vanish at spatial infinity is if all the eigenvalues vanish.

Since \( \beta = 0 \) and \( H \) has full rank, almost everywhere (except at vortex positions), we conclude from equation (3.48) that \( \omega = 0 \). This can be seen by choosing generic values of \( \tau \) for which cancellations are certainly impossible. Thus all fluctuations \( \alpha = \beta = \omega = 0 \) vanish which completes the proof.

The extension to unequal abelian and non-abelian couplings \( e, \kappa \) and \( g, \mu \) is straightforward and theorem 1 still holds. The extension to generic semi-simple gauge groups \( G' \) alters the operators \( D \) and \( D^\dagger \) in a straightforward fashion and various places the projection operator \( \langle \phi \rangle_G \) appears, but we can establish

**Theorem 2.** The kernel of \( D^\dagger \), in the case of generic gauge groups \( (U(1) \times G')/\mathbb{Z}_{3N_0} \) with \( G' \) a semi-simple group and \( N_0 \geq N \), has dimension zero.

**Proof.** The proof is analogous to that of theorem 1, with the following changes. For simplicity, we will provide the proof for the case with the abelian couplings equal to the non-abelian ones. First of all, the \( g \)-valued equations (where \( g \) is the algebra corresponding to the gauge group \( G \)) now contain a projection of the Higgs fields to the group \( G \) by means of \( \langle \phi \rangle_G \). This amounts to the \( K \) matrix of the form

\[
K \equiv \begin{pmatrix}
-A_z & 0 & -\langle \phi \rangle & 0 & 0 \\
0 & -\langle \phi \rangle & H & 0 & 0 \\
\frac{1}{4\sqrt{2}} \langle \phi \rangle^\dagger & -\frac{1}{4\sqrt{2}} H & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] \\
2 \langle \phi \rangle^\dagger & 2 \langle \phi \rangle & -\frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & 0 \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55}
\end{pmatrix}
\]

(3.68)

\[
K_{51} \equiv \frac{\tau}{2} \langle \phi \rangle^\dagger_G - \frac{1}{2} \langle \{\phi^\dagger, \phi\} \rangle_G - i2\langle \phi^\dagger_X \rangle_G, \\
K_{52} \equiv \frac{\tau}{2} \langle \phi \rangle_G - \frac{1}{2} \langle \{\phi, \phi\} \rangle_G - i2\langle \phi^\dagger_Y \rangle_G, \\
K_{55} \equiv \frac{\tau}{2} - \frac{1}{2} \langle \{\phi^\dagger, \phi\} \rangle_G - [A_z, [A_z, \langle \phi \rangle]] - [A_z, [A_z, \langle \phi \rangle]],
\]

(3.69), (3.70), (3.71)

with \( D, K_{33} \) and \( K_{44} \) unchanged and given in equations (3.15), (3.19) and (3.20), respectively. The adjoint operator is found straightforwardly and we have

\[
K^\dagger \equiv \begin{pmatrix}
-A_z & 0 & -\frac{1}{4\sqrt{2}} \langle \phi \rangle & \frac{1}{4\sqrt{2}} H & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & K_{15} \\
0 & -\langle \phi \rangle^\dagger & -\frac{1}{4\sqrt{2}} H & \frac{1}{4\sqrt{2}} X & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & K_{25} \\
-2 \langle \phi \rangle^\dagger_G & 0 & -\frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & K_{35} \\
0 & 2 \langle \phi \rangle_G & -\frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & \frac{1}{\sqrt{2}} [A_z, \langle \phi \rangle] & K_{45} \\
0 & 0 & 0 & 0 & 0 & 0 & K_{55}
\end{pmatrix}
\]

(3.72)
with $D^i$, $K_{15}^i$, $K_{25}^i$, $K_{35}^i$, $K_{45}^i$ and $K_{55}^i$ unchanged and given in equations (3.24), (3.26)–(3.29), and (3.21), respectively.

Acting on $\eta$ with $D^i$ corresponding to the general gauge group case at hand, we obtain that the equations (3.47), (3.48) and (3.50) remain unaltered, whereas equation (3.49) changes into

$$D_\beta^\dagger + i 2\sqrt{2} \langle \alpha H^\dagger \rangle_G = 0,$$  \hspace{1cm} (3.73)

and equation (3.51) remains Hermitian apart from $\beta$. Equation (3.53), in the general gauge group case, can be written as

$$0 = \operatorname{tr} \int_C \left( D_\beta^\dagger + i 2\sqrt{2} \langle \alpha H^\dagger \rangle_G \right)^\dagger \left( D_\beta^\dagger + i 2\sqrt{2} \langle \alpha H^\dagger \rangle_G \right) d^2x$$

$$= \operatorname{tr} ||D_\beta^\dagger||^2 + 8 \operatorname{tr} ||\langle \alpha H^\dagger \rangle_G||^2 - i\sqrt{2} \operatorname{tr} \int_C \partial_\alpha \langle H^\dagger \beta^\dagger \rangle d^2x$$

$$+ i\sqrt{2} \operatorname{tr} \int_C \partial_\beta \langle \alpha H^\dagger \rangle d^2x$$

$$= \operatorname{tr} ||D_\beta^\dagger||^2 + 8 \operatorname{tr} ||\langle \alpha H^\dagger \rangle_G||^2. \hspace{1cm} (3.74)$$

We thus get $D_\beta^\dagger = 0$ as before, but now we have $\langle \alpha H^\dagger \rangle_G = 0$, which is exactly the condition found in reference [30]. In the latter reference, it was shown that $\alpha$ must vanish if there are no independent $G'$ invariants with positive $U(1)$ windings. This is the case for $G' = SU(N)$, $G' = SO(N)$ and $G' = USp(2M)$.

Repeating now the logic of the proof of theorem 1, we have $\alpha = 0$ and $D_\beta^\dagger = 0$ as before. Taking the Hermitian conjugate of equation (3.51), which now contains some projection operators to $G$, reveals that $\beta = \beta^\dagger$ and hence we have $D_\beta \beta = D_\beta \beta = 0$ as before. Using the remaining arguments of the proof of theorem 1, we thus again reach the conclusion that $\alpha = \beta = \omega = 0$. \hspace{1cm} \square

**Corollary 3.** The operator $D^i$ does not contain zeromodes as shown by theorems 1 and 2, and therefore the index (3.45) counts the number of real zeromodes of $D$ and therefore of the BPS equations (2.41), (2.42), and (2.44). The number of zeromodes is the dimension of the moduli space, which is

$$\dim_{\mathbb{C}} \mathcal{M}_{k,N,N_f} = \frac{kNN_f}{n_0}. \hspace{1cm} (3.75)$$

### 4. The moduli matrix method

The BPS equations obtained in section 2 have many solutions with degenerate energy. The space of solutions forms a manifold which is the so-called the moduli space\(^6\). We will now solve the BPS equations and write down the master equations for the YMCSH vortices of vorticity $k$.

The moduli matrix approach solves the first BPS equation (2.41):

$$H = S^{-1} H_0(z), \quad A_\ell = -i S^{-1} \partial_\ell S, \quad S \in G^c. \hspace{1cm} (4.1)$$

\(^6\)By moduli space, we here mean the moduli space of solutions and not the moduli space of vacua which is the manifold of the vacua in the theory.
decompose the abelian and non-abelian parts as
where \( U_c \in G^c \) and \( U_t \in SU(N_t) \). Furthermore, there is a residual gauge symmetry, which is denoted \( V \)-equivalence and it acts as

\[
S \sim V(z)S, \quad H_0 \sim V(z)H_0, \quad V(z) \in G^c,
\]
with \( V(z) \) being holomorphic with respect to the coordinate \( z \). We are now left with the remaining four BPS equations (2.42) and (2.44).

In order to rewrite these equations in a gauge invariant fashion, let us introduce the following gauge invariants:

\[
\Omega = SS^\dagger, \quad \Omega_0 = \nu^{-2}H_0 H_0^\dagger, \quad \Upsilon = S\phi S^{-1}.
\]

In terms of these invariants, the field strength reads

\[
F_{\Omega} = -2 [D_\Omega, D_\Omega] = -2 S^{-1} \partial_t (\partial_t \Omega^{-1}) S,
\]

\[
F_{\Omega} = -i D_\Omega [D_\Omega, D_\Omega] = F_{\Omega} + i F_{\Omega2} = -D_\phi = -S^{-1} \partial_t \Upsilon S,
\]

where we have used \( D_\phi H = i \phi H \) and that we are considering only the static case \( \partial_t = 0 \). The complex covariant Laplacian on \( \phi \) is

\[
D_\Omega D_\Omega \phi = i \partial_t [D_\Omega, D_\Omega] = S^{-1} (\partial_t \partial_t \Upsilon + [\partial_t \Upsilon, \partial_t \Omega^{-1}]) S.
\]

We can now rewrite equations (2.42) and (2.44) using the gauge invariants. To this end, let us decompose the abelian and non-abelian parts as \( S = sS \) with \( s \in U(1)^c \) and \( S \in G^c \):

\[
\Omega = \omega \hat{\Omega}, \quad \omega = |s|^2, \quad \hat{\Omega} = \hat{S} \hat{S}^\dagger, \quad \Upsilon = \Upsilon^0 + \hat{\Upsilon}, \quad \Upsilon^0 = \phi^0, \quad \hat{\Upsilon} = \hat{S} \hat{S} \phi^0.
\]

This leads to the system of master equations

\[
\partial_t (\partial_t \Omega^{-1}) = -\frac{m_\Omega}{2} (\Omega_0 \Omega^{-1})_G + \frac{m_\phi}{2} \hat{\Upsilon},
\]

\[
\partial_t \partial_t \log \omega = \frac{m_\Omega^2}{4N} \text{tr}[\Omega - \Omega_0 \Omega^{-1}] + \frac{m_\phi}{2\sqrt{2N}} \Upsilon^0,
\]

\[
\partial_t \partial_t \hat{\Upsilon} + [\partial_t \hat{\Upsilon}, \partial_t \Omega^{-1}] = \frac{m_\Omega^2}{4} \hat{\Upsilon} - \frac{m_\Omega^2 m_\phi}{4} (\Omega_0 \Omega^{-1})_G + \frac{m_\phi^2}{4} \{\Omega_0 \Omega^{-1}, \Upsilon\}_G + \frac{m_\phi^2}{4} (\Omega_0 \Omega^{-1})_G.
\]

\[
\partial_t \partial_t \Upsilon^0 = \frac{1}{4\sqrt{2N}} \text{tr}[2(m_\Omega^2 + m_\phi^2 \Omega_0 \Omega^{-1}) \Upsilon + m_\phi^2 m_\phi (\Omega - \Omega_0 \Omega^{-1})]_G.
\]
where, for convenience, we recall the definitions of the mass parameters of the theory

\[ m_g \equiv g v, \quad m_e \equiv e v, \quad m_\mu \equiv \frac{\mu g^2}{4\pi}, \quad m_\kappa \equiv \frac{\kappa e^2}{4\pi}, \]  

(4.13)

and the boundary conditions for the PDEs are given by

\[
\lim_{|z| \to \infty} \Omega = \Omega_0, \quad \lim_{|z| \to \infty} \Upsilon = 0.
\]  

(4.14)

A first simplification of the equations happens when considering the equal coupling case, \( e = g \) and \( \kappa = \mu \), for which we can write

\[
\partial_\bar{z}(\partial_z \Omega \Omega^{-1}) = \frac{m_g^2}{2}(1_N - \Omega_0 \Omega^{-1})G + \frac{m_\mu}{2} \Upsilon,
\]

(4.15)

\[
\partial_\bar{z} \partial_\bar{z} \Upsilon + [\partial_z \Upsilon, \partial_\bar{z} \Omega \Omega^{-1}] = \frac{m_g^2}{4} \Upsilon + \frac{m_\mu^2 m_\kappa}{4}(1_N - \Omega_0 \Omega^{-1})G
\]

\[- \frac{m_\mu^2}{4} \left(\{\Omega_0 \Omega^{-1}, \Upsilon\}\right) + \frac{m_\mu^2}{4} \left(\{\Omega_0 \Omega^{-1}, \Upsilon\}\right) G^\dagger.
\]

(4.16)

A further simplification happens when \( G' = SU(N) \), so the \( G' \) projections simply give (a half times) the traceless part of the matrix expressions. This corresponds to the case of \( G = U(N) \) (due to the equal couplings):

\[
\partial_\bar{z}(\partial_z \Omega \Omega^{-1}) = \frac{m_g^2}{2}(1_N - \Omega_0 \Omega^{-1}) + \frac{m_\mu}{2} \Upsilon,
\]

(4.17)

\[
\partial_\bar{z} \partial_\bar{z} \Upsilon + [\partial_z \Upsilon, \partial_\bar{z} \Omega \Omega^{-1}] = \frac{m_g^2}{4} \Upsilon + \frac{m_\mu^2 m_\kappa}{8}(1_N - \Omega_0 \Omega^{-1}) + \frac{m_\mu^2}{4} \Upsilon \Omega_0 \Omega^{-1}.
\]

(4.18)

The zeromodes or moduli are all contained in the moduli matrix \( H_0(z) \), which appears in the combination \( \Omega_0 = v^{-2} H_0 H_0^\dagger \) in the master equations above. Although we have not proved from the PDE point-of-view that the field \( \Omega \) and \( \Upsilon \) are uniquely determined by the above PDEs for given \( H_0(z) \), we know that the number of moduli parameters available in the moduli matrix are (see references [29, 30, 36, 38, 41, 42])

\[
\dim_{\mathbb{C}} \mathcal{M}_{k,N,N} = \frac{kNN_0}{n_0},
\]

(4.19)

which is in perfect agreement with the index theorem of section 3. For explicit realizations of the moduli in terms of the moduli matrices \( H_0(z) \) for \( G' = SU(N) \), see references [41, 42], whereas for \( G' = SO(N) \) and \( G' = USp(2M) \), see references [30, 38].

4.1. Examples

The master equations are in general quite complicated due to their non-abelian nature. A nice simplification happens when considering the centers of a patch of the moduli space, for which
all matrices may be taken to be diagonal matrices. For simplicity, we will take the $G = U(N)$ case, for which we have

\[
\nabla^2 u_i = \frac{m_i^2}{2} \left(e^{2u_i} - 1\right) - m_i \Upsilon_i + \frac{k_i}{2} \sum_{r=1}^{k_i} \delta^{(2)}(z - Z_i), \quad i = 1, 2, \ldots, N
\]

(4.20)

\[
\nabla^2 \Upsilon_i = m_i^2 \Upsilon_i - \frac{m_i^2 m_\mu}{2} \left(e^{2u_i} - 1\right) + m_\mu^2 \Upsilon_i e^{2u_i},
\]

(4.21)

where $i$ is nowhere summed over, we have defined the field

\[
u_i \equiv \frac{1}{2} \log (\Omega_i (\Omega^{-1})),
\]

(4.22)

the subscript (index) $i$ denotes here the $i$th diagonal element of each diagonal matrix, and the vortex positions are encoded in the position moduli $Z_i$. Notice that only the equation for $u_i$ contains delta function sources for the vortices. Notice also that there are no orientational moduli in this example, because considering the center of a patch of the moduli space simply means setting the orientational moduli to zero in some local coordinates. For analysis or numerical computations, it is convenient to perform the rescaling (3.2), yielding

\[
\nabla^2 u_i = \frac{1}{2} \left(e^{2u_i} - 1\right) - \Upsilon_i + \frac{k_i}{2} \sum_{r=1}^{k_i} \delta^{(2)}(z - Z_i), \quad i = 1, 2, \ldots, N
\]

(4.23)

\[
\nabla^2 \Upsilon_i = \tau \Upsilon_i - \frac{\tau^2}{2} \left(e^{2u_i} - 1\right) + \Upsilon_i e^{2u_i},
\]

(4.24)

where $\tau \geq 0$ is defined in equation (3.6).

An example of vortex equations in the case of unequal couplings in the $G' = SU(2)$ theory (for simplicity) is

\[
\nabla^2 u_1 = \frac{m_1^2}{2} \left(e^{2u_1} + e^{2u_2} - 2\right) + \frac{m_2^2}{2} \left(e^{2u_1} - e^{2u_2}\right) - \frac{m_\mu}{2} \Upsilon^0 + m_\mu \Upsilon^3
\]

\[
+ 2\pi \sum_{i=1}^{k_1} \delta^{(2)}(z - Z_i),
\]

\[
\nabla^2 u_2 = \frac{m_1^2}{2} \left(e^{2u_1} + e^{2u_2} - 2\right) - \frac{m_2^2}{2} \left(e^{2u_1} - e^{2u_2}\right) + \frac{m_\mu}{2} \Upsilon^0 + m_\mu \Upsilon^3
\]

\[
+ 2\pi \sum_{i=1}^{k_1} \delta^{(2)}(z - Z_i),
\]

\[
\nabla^2 \Upsilon^0 = m_\mu ^2 \Upsilon^0 + \frac{m_\mu ^2}{2} \left(e^{2u_1} + e^{2u_2}\right) (\Upsilon^0 - m_\mu) + \frac{m_\mu ^2}{2} \left(e^{2u_1} - e^{2u_2}\right) \Upsilon^3
\]

\[
+ m_\mu^2 m_\nu,
\]

\[
\nabla^2 \Upsilon^3 = m_\mu ^2 \Upsilon^3 + \frac{m_\mu ^2}{2} \left(e^{2u_1} - e^{2u_2}\right) (\Upsilon^0 - m_\mu) + \frac{m_\mu ^2}{2} \left(e^{2u_1} + e^{2u_2}\right) \Upsilon^3,
\]

(4.25)
where \( u_{1,2} \) are given by equation (4.22) and \( \Upsilon = \frac{1}{2} \Upsilon^0 1_2 + \frac{1}{2} \Upsilon^3 \sigma^3 \) with \( \sigma^3 \) being the third Pauli spin matrix. Notice that this example correctly reduces to equations (4.20) and (4.21) in the equal coupling case \( (m_\kappa = m_e \text{ and } m_\mu = m_e) \). Performing a rescaling

\[
\partial_\kappa \rightarrow m_\kappa^2 \partial_\kappa, \quad \Upsilon^0 \rightarrow \frac{m_\kappa^2}{m_e} \Upsilon^0, \quad \Upsilon^3 \rightarrow \frac{m_\mu^2}{m_e} \Upsilon^3,
\]

(4.26)

convenient for further analysis, we get

\[
\nabla^2 u_1 = \frac{\gamma_{\epsilon/g}}{4} (e^{2u_1} + e^{2u_2} - 2) + \frac{1}{4} (e^{2u_1} - e^{2u_2}) - \frac{1}{2} \Upsilon^0 - \frac{1}{2} \Upsilon^3 + 2\pi \sum_{r=1}^{k_1} \delta(z - Z_r^1),
\]

\[
\nabla^2 u_2 = \frac{\gamma_{\epsilon/g}}{4} (e^{2u_1} + e^{2u_2} - 2) - \frac{1}{4} (e^{2u_1} - e^{2u_2}) - \frac{1}{2} \Upsilon^0 + \frac{1}{2} \Upsilon^3 + 2\pi \sum_{r=1}^{k_2} \delta(z - Z_r^2),
\]

\[
\nabla^2 \Upsilon^0 = \gamma_{\kappa/g} \Upsilon^0 + \frac{\gamma_{\epsilon/g}}{2} (e^{2u_1} + e^{2u_2}) (\Upsilon^0 - \gamma_{\kappa/g}) + \frac{\gamma_{\epsilon/g}}{2} (e^{2u_1} - e^{2u_2}) \Upsilon^3 + \gamma_{\epsilon/g} \gamma_{\kappa/g},
\]

\[
\nabla^2 \Upsilon^3 = \gamma_{\mu/g} \Upsilon^3 + \frac{1}{2} (e^{2u_1} - e^{2u_2}) (\Upsilon^0 - \gamma_{\mu/g}) + \frac{1}{2} (e^{2u_1} + e^{2u_2}) \Upsilon^3,
\]

(4.27)

where \( \gamma_{i/j} \) are defined in equation (2.24). Notice that there are exactly three dimensionless couplings: \( \gamma_{\epsilon/g} \), \( \gamma_{\kappa/g} \) and \( \gamma_{\mu/g} \), which parametrize the most general coupling space of the theory.

5. The D-brane picture

We will now study the YMCSH vortices from a D-brane perspective with which we can geometrically understand some properties of solitons in a low energy effective theory of the D-brane configuration. We will show that the results in the previous sections can be reproduced by making use of the language in type IIB string theory which is simple, intuitive and geometrical. Our construction here is in some sense a non-abelianization of the construction made in reference [44], where abelian Chern–Simons–Higgs vortices were constructed in type IIB string theory.

The type IIB D-brane configuration of interest is summarized in the table below.

| Type IIB | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|----------|---|---|---|---|---|---|---|---|---|
| NS5      | ♥ | ♥ | ♥ | ♥ | ♥ | — | — | — | — |
| (κ,1)5   | ♥ | ♥ | ♥ | cos θ | — | — | — | — | sin θ |
| N D3     | ♥ | ♥ | — | — | — | | | | |
| N1 D5    | ♥ | ♥ | — | — | — | | | | |
The \((\kappa, 1)S\)-brane which is the bound state of an NS5-brane (spanned in the directions 012389) and \(\kappa D\)-brane (spanned in the directions 012789) is orientated at an angle \(\theta\) in \((x^5, x^\theta)\)-space. The D3-brane is suspended by the ‘left’ NS5-brane at \(x^\theta = 0\) and the ‘right’ \((\kappa, 1)D\)-brane at \(x^\theta = L_\theta\).

A low energy effective theory on \(N\) D3-branes suspended by the five branes is a \(d = 2 + 1\) \(\mathcal{N} = 2\) supersymmetric \(U(N)\) YMCS theory in a limit where massive string modes of order \(l_s^{-1}\) (\(l_s\) is the string scale) are decoupled. The \(U(N)\) gauge coupling is given by

\[
\frac{1}{g^2} = \frac{L_\theta}{2\pi g_s},
\]

(5.1)

with \(g_s\) being the string coupling constant of type IIB string theory. Note that \(g_s\) is a dimensionless coupling constant, so \(g^2\) is dimensionful in \(2 + 1\) dimension, as it should be.

The CS interaction appears in the low-energy effective theory because of the D3-branes suspended between an NS5-brane and a \((\kappa, 1)S\)-brane \([43]\). Let us briefly review how the CS term appears \([43]\). To this end, we take the simple example where a D3-brane is suspended by the 5-branes. From the supergravity (SUGRA) solution of the \((p, q)S\)-brane, there is a non-trivial VEV of the axion field \([51]\)

\[
\chi(x^d, x^5, x^6, x^\theta) = \frac{\sin \theta \cos \theta(1 - H)}{g_s \sin^2 \theta + H \cos^2 \theta},
\]

(5.2)

with a harmonic function \(H = 1 + l_s^2/r^2\) on the \((x^d, x^5, x^6, x^\theta)\)-space \((r\) is the distance from the \((p, q)S\)-brane). Clearly, the axion VEV vanishes at \(r \gg l_s\) and it develops a nonzero VEV only near the \((p, q)S\)-brane as \(\chi|_{r \ll l_s} = -\frac{1}{g_s} \tan \theta\). The low energy effective action contains

\[
S_{D3} = \int_0^{L_6} \int d^3 x d^6 \chi \left( -\frac{1}{4g_s^2} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2g_4^2} F_{\mu \nu} F^{\mu \nu} + \frac{1}{4\pi} \epsilon^{\mu \nu \rho \chi} A_\mu \partial_\nu \partial_\rho \chi + \cdots \right)
\]

(5.3)

where the integration range on \(x^d\) is from \(x^d = 0\) to \(x^d = L_6\). Here \(g_s^2 = 2\pi g_s\) is the \(d = 3 + 1\) gauge coupling constant on the D3-branes. Ignoring all massive Kaluza–Klein modes of order \(1/L_6\) or smaller reduces the theory to the three dimensional Lagrangian with gauge coupling constant \(g^2 = g_s^2/L_\theta\) as in equation (5.1).

The gauge transformation for \(A_\mu\) reads

\[
\delta S_{D3} = \int_0^{L_6} \left[ \delta A_\mu \left( -\frac{1}{8g_s^2} \partial_\rho F^{\rho \mu} + \frac{1}{4\pi} \epsilon^{\mu \nu \rho \chi} \partial_\nu \partial_\rho \chi \right) \right] d^3 x d^6 \chi
\]

(5.4)

We must require this to vanish which gives the following boundary conditions

\[
F_{\rho \theta}|_{\theta = 0} = 0, \quad F_{\rho \theta}|_{\theta = L_\theta} = \frac{g_s^2}{4\pi} \chi(L_\theta) \epsilon_{\rho \mu \nu} F^{\mu \nu} = -\frac{g_s^2}{4\pi g_s} \tan \theta \epsilon_{\rho \mu \nu} F^{\mu \nu}.
\]

(5.5)

The former is nothing but the boundary condition for the D3-brane ending on the NS5-brane at \(x^\theta = 0\), while the latter can be understood as an \(SL(2, \mathbb{Z})\) transformation of the boundary
conditions by the NS5-brane and D5-branes, namely the \((p, q)\)-brane. From equation (5.3), we find that the CS coupling constant induced on the D3-brane is

\[
\kappa' = -\frac{1}{g_s} \tan \theta = -\frac{p}{q}
\]  

Thus, for the choice \((p, q) = (\kappa, 1)\) the CS coupling constant is \(\kappa\). The extension to multiple D3-branes, namely the non-abelian theory, is straightforward.

The hypermultiplets including squark fields, \((H, \tilde{H}^\dagger)\), which are \(N \times N_f\) matrix-valued fields, correspond to excitations of open strings between \(N\) D3-branes and \(N_f\) D5-branes. The FI parameter is related to the distance \(L_7\) in the \(x^7\) direction (we place the left NS5-brane at \(x_7^1 = 0\) and the right \((\kappa \to 0, 1)\)-brane at \(x_7^1 = L_7\) between the left NS5-brane and right \((\kappa, 1)\)-brane

\[
v^2 = \frac{L_7}{4\pi^2 g_s l_s^2}
\]  

The schematic picture is shown in figure 1.

The field theory limit can be taken by \(g_s \to 0\) and \(l_s \to 0\) with the field theory parameters

\[
g^2 \sim \frac{g_s}{L_6}, \quad \kappa \sim \frac{g_s}{\theta}, \quad v^2 \sim \frac{L_7}{g_s l_s^2},
\]  

being fixed. The scalar potential in the low-energy effective theory is given by

\[
V = \text{tr} \left[ \frac{g^2}{2} \left( HH^\dagger - \bar{H}^\dagger \bar{H} - \frac{\kappa}{4\pi} X_3 - v^2 1_N \right)^2 + (HH^\dagger + \bar{H}^\dagger \bar{H}) X_3^2 \right],
\]  

where \(X_3\) corresponds to the positions of D3-branes in the \(x^3\) direction and is in the adjoint representation of \(U(N)\). If we identify \(X_3\) with \(\phi\) and set \(\bar{H} = 0\), this corresponds to the theory with the equal gauge coupling constants, \(e = g\), also considered in previous sections.

From figure 1, one can easily find three phases of the classical supersymmetric vacua \(V = 0\):

(a) asymmetric phase where \(U(N)\) is completely broken.
Figure 2. The Hanany–Witten shift from figure 1(a). The \((\kappa, 1)\)-string (bold dashed line) corresponds to the topological non-abelian local (for \(N_f = N_f\)) and semilocal (for \(N_f > N_f\)) vortices.

(b) partially symmetric phase where \(U(N)\) is broken to \(U(n)\) with \(n \in [1, N_f - 1]\).

(c) symmetric phase where the gauge group is fully unbroken.

One can also read off quantum effects by taking into account the s-rule.

Since we are interested in the topological solitons which arise due to the spontaneously broken symmetry, we will consider only the asymmetric phase in the rest of this section. The topological vortex in the abelian case was identified with a \((\kappa, 1)\)-string suspended by the \(N_f = 1\) D3-brane split on the \(N_f = 1\) D5-brane [44]. The axion background field again affects a D-string and it acquires the electric charge proportional to \(\chi\) by an analogue of Witten’s effect. Indeed, the tension formula for the \((p, q)\)-string in the constant axion field background is given by [51]

\[
T_{(p,q)} = \frac{1}{2\pi l_s^2} \sqrt{\left( \frac{q}{g_s} \right)^2 + (p + q\chi)^2}.
\]

Thus the minimal tension is not that of a D-string but of a \((\kappa, 1)\)-string [44]. This is consistent with the field theory result that the mass of a \((\kappa, 1)\)-string between the D3-branes is

\[
L_7 T_{(\kappa,1)} = \frac{L_7^2}{2\pi g_s l_s^4} = 2\pi \nu^2.
\]

This is exactly the tension of a single topological vortex.

Now we will extend this result from the abelian case to the non-abelian case. There are now \(N\) D3-branes and \(N_f\) D5-branes. The topological vortices are again identified with a \((\kappa, 1)\)-string, and their electric charge and tension can be read off of the string theory as done above.

The aim of the remainder of this section is to find the moduli space of YMCSH vortices. From the D-brane perspective, the dimension of the moduli space is easily counted as follows. Let us first move the D5-brane beyond the right \((\kappa, 1)\)5-brane in the \(x^6\) direction. Taking the Hanany–Witten effect into account, the brane configuration results as shown in figure 2.

The topological vortex is identified with the \((\kappa, 1)\)-string extending in \((x^1)\)-space and suspended by the D3-branes as well as the \((\kappa, 1)\)5-brane. Note that if we send \(\theta \to 0 (\kappa \to 0)\), figure 2 reduces to the D-brane configuration of reference [16], which is that describing non-abelian
vortices in $\mathcal{N} = 2$ supersymmetric Yang–Mills–Higgs theory without the CS term. As in reference [16], the zeromodes of vortices are identified with the zeromodes on the $(\kappa, 1)$-string. Open strings between $k$ $(\kappa, 1)$-strings yield a $k \times k$ matrix field $Z$ and those between $k$ $(\kappa, 1)$-strings and $N$ D3-branes $(N_f - N$ D3-branes) yield a $k \times N$ matrix field $\psi ((N_f - N) \times k$ matrix field $\tilde{\psi})$. The field $Z$ is in the adjoint representation of the $U(k)$ gauge symmetry while $\psi, \tilde{\psi}$ are in the fundamental representation. These fields obey $k^2$ real D-term conditions. Thus the complex dimension of the moduli space is

$$\text{dim}_C \mathcal{M}_{k,N,N_f} = k^2 + kN + (N_f - N)k - k^2 = kN_f. \quad (5.12)$$

This moduli space is for the gauge group $G = U(N)$, for which $n_0 = N$ and hence $kNN_f/n_0 = kN_f$. The string theory result thus precisely coincides with the previous results on the field theory side.

As we have seen, the D-brane picture serves as an easy way of counting the dimensions of moduli space of topological solitons. Nevertheless, it is also possible to construct a low-energy effective theory on world volume of the soliton. For example, an effective theory for the non-abelian vortices was made using a type IIB D-brane configuration in reference [16].

Collie and Tong [22] have derived a low energy effective action on the world-volume of the non-abelian CS vortex shown in figure 2. Let us first see that a CS term again appears in the low-energy effective theory on a single $(\kappa, 1)$-string. To this end, we begin with the low-energy effective action on the $(\kappa, 1)$-string

$$S_{(\kappa, 1)|1} = \int_{L_7}^t \int \left( \frac{1}{2g_2^2} f_{07}^2 + \frac{1}{4\pi} a_0 \partial_7 \chi + \cdots \right) \text{d}t \text{d}x^7, \quad (5.13)$$

where the integration range of $x^7$ is from 0 to $L_7$; the $(\kappa, 1)$-string ends on the NS5-brane at $x^7 = 0$ and on the $(\kappa, 1)5$-brane at $x^7 = L_7$. Here $a_0$ ($\alpha = 0, 7$) are the abelian gauge fields on the $(\kappa, 1)$-string and $g_2$ stands for the gauge coupling constant $g_2^2 = g_s l_s^2$. Varying the action, we get

$$\delta S_{(\kappa, 1)|1} = \int \delta a_0 \left[ \frac{1}{g_2^2} f_{07} + \frac{1}{4\pi} \chi \right]_{x^7 = 0} \text{d}t. \quad (5.14)$$

As before, the axion field acquires a nonzero VEV only in the region close to the $(\kappa, 1)5$-brane at $x^7 = L_7$. Then we can read off the boundary condition at $x^7 = L_7$ as

$$f_{07} \big|_{x^7 = L_7} = \frac{g_2^2}{4\pi} \chi(L_7) = - \frac{g_2^2}{4\pi} \kappa. \quad (5.15)$$

This electric flux, proportional to $\kappa$, is responsible for the electrically charged D-string, namely the $(\kappa, 1)$-string. From equation (5.13), the CS term in $d = 1$ spacetime is found as

$$S_{d=1}^{|CS} = - \frac{\kappa}{4\pi} \int a = - \frac{\kappa}{4\pi} \int a_0 \text{d}t. \quad (5.16)$$
We are now ready to construct the effective action for a single non-abelian CS vortex. Note that, for the $(\kappa, 1)$-string, the distance $L_7$ is related to the $U(1)$ gauge coupling constant while $L_6$ is the FI term. The $U(1)$ gauge coupling constant $g_1$ in one dimension is given by

$$g_1^2 = \frac{g_s^2}{L_7} = \frac{g_s}{k^2 L_7}.$$  

(5.17)

This diverges in the decoupling limit $l_s \to 0$, since we are keeping $v_1^2 \sim g_s l_s^2/L_7$ fixed. The FI term in one dimension $v_1^2 = L_6/g_s$ is related to the three-dimensional gauge coupling constant via equation (5.1) as $v_1^2 = 2\pi/g^2$ [16]. Thus, the $U(1)$ vector multiplet is infinitely heavy, which forces the theory to remain on the Higgs branch, as in reference [16]. Thus the $N$ chiral multiplets $\psi_i$ (zeromodes of F-strings between the D3 branes and the $(\kappa, 1)$-string) must obey the D-term constraint

$$\sum_{i=1}^{N} |\psi_i|^2 = \frac{2\pi}{g^2}.$$  

(5.18)

This, together with the $U(1)$ gauge symmetry $\psi_i \sim e^{i\alpha} \psi_i$, means that $\psi_i$ are the fields taking value in $\mathbb{C}P^{N-1}$. In this way we obtain the $d = 1$ effective Lagrangian

$$L = \sum_{i=1}^{N} |D_0 \psi_i|^2 - \frac{\kappa}{4\pi} a_0,$$  

(5.19)

with $D_0 \psi_i = \partial_0 \psi_i + i\alpha \psi_i$. Note that the gauge kinetic term vanishes because of the strong coupling limit $g_1 \to \infty$. This is the main result of this section and is in full agreement with the result of reference [22].

Note that we have shown, using string theory, that the CS couplings in one dimension and in three dimensions are related and induced by the $(\kappa, 1)$-brane. This is also in agreement with the results of reference [22], where field theory was used.

6. Conclusion and discussion

In this paper, we have studied the zeromodes or moduli of the YMCSH vortices. After setting up the model, we discussed many limits of the theory, like the Yang–Mills–Higgs and non-abelian Chern–Simons–Higgs theories, found the mass spectrum of the theory and wrote down the BPS equations in convenient form, valid for any gauge group $G = (U(1) \times G')/\mathbb{Z}_n$, with $G'$ a semi-simple gauge group. We then studied the zeromodes or moduli using a Callias-type index theorem, the moduli matrix approach and a D-brane construction in type IIB string theory. The main result of the paper is given by theorems 1 and 2, which prove that the index counts the number of real zeromodes of an operator that is constructed from the linear fluctuation equations, derived from the BPS equations. The index really counts the difference between the dimension of the kernel of the operator that corresponds to the fluctuations and that of its adjoint. Theorems 1 and 2 prove that the adjoint operator does not have zeromodes and hence the index is the total number of zeromodes in the vortex solutions. The number of zeromodes or the dimension of the moduli space, coincides with that of Yang–Mills–Higgs theory and Chern–Simons–Higgs theory. The vanishing theorem in the case of Yang–Mills–Higgs theory is rigorously proved (see refs. [16, 30]), whereas in the Chern–Simons–Higgs case, the trick of writing the adjoint operator squared as a positive definite sum of terms plus boundary terms that vanish upon integration over the plane, does not work and hence no rigorous proof has
been made (see reference [38]). In YMCSH theory studied in this paper, the trick also does not work, but a suitable gauge fixing choice made it possible to simplify the fluctuation equation for the Higgs field enough so that the proof could be established. We then extended the moduli matrix formalism (see e.g. reference [41]) to include the extra adjoint field that is possessed by YMCSH theory and the result is consistent with the result of the index theorem calculation, that the moduli are all contained in the moduli matrix, that comes from the self-dual BPS equation. That is, the adjoint field is fully determined by the other fields. Finally, we extend the construction of reference [44] of the abelian Chern–Simons–Higgs vortices in type IIB string theory to the non-abelian case, where we confirm the dimension of the moduli space (the number of zeromodes). We then construct the low-energy effective Lagrangian on the vortices in agreement with the result of Collie and Tong, who found the same result using field theory [22].

The obvious missing piece in the construction of non-abelian vortices, is the generalization from $G = \text{U}(N)$ to $G = (\text{U}(1) \times G')/\mathbb{Z}_n$ (with $G'$ a semi-simple gauge group) in type IIB string theory using a D-brane setup. This has not been possible yet, with or without the CS term and it is not known what should be done in string theory to make such a generalization possible.

In this paper, we have restricted to the case of $N_f = N$ flavors, which simplifies the vacuum equations that for $N_f > N$ possess vacuum moduli. The $N_f > N$ case also always contains semilocal zeromodes. The index theorem and the moduli matrix method studied in this paper, are already valid for the $N_f > N$ case, but we have not considered the vacuum moduli and the repercussions of that carefully here. The brane construction was made for any $N_f \geq N$, but we have not derived the low-energy effective theory on the vortex brane for the $N_f > N$ case. We leave such extensions to future work.

Finally, we have written down the PDEs as master equations and given examples in specific patches of certain moduli spaces. The constructed system of equations account for exactly the right number of moduli, found by the index calculation, by means of the moduli matrix, barring that the master equation fields $\Omega$ and $\Upsilon$ do not possess moduli. However, we have not proved this. That is, we have not proved uniqueness of the master equations for given fixed moduli matrix, fixing the $kNN_f/\mathbb{n}_0$ known moduli. We leave this problem as a direction of future work.

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Data availability statement

No new data were created or analysed in this study.
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