Harmonic oscillator in a background magnetic field in noncommutative quantum phase-space

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Abstract – We solve explicitly the two-dimensional harmonic oscillator and the harmonic oscillator in a background magnetic field in noncommutative phase-space without making use of any type of representation. A key observation that we make is that for a specific choice of the noncommutative parameters, the time-reversal symmetry of the systems get restored since the energy spectrum becomes degenerate. This is in contrast to the noncommutative configuration space where the time-reversal symmetry of the harmonic oscillator is always broken.

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In the last few years, theories in noncommutative space have been extensively studied [1–15]. The motivation for this kind of investigation is that the effects of noncommutativity of space may appear in the very tiny string scale or at very high energies [2]. In particular, two-dimensional noncommutative harmonic oscillators have attracted a great deal of attention in the literature [5–7,12–15]. An interesting observation is that the introduction of noncommutative spatial coordinates breaks the time-reversal symmetry since the angular-momentum states with eigenvalue $-m$ and $+m$ do not have the same energy.

A description of two-dimensional noncommutative phase-space has been given in the literature [13–15] to study the noncommutative harmonic-oscillator and noncommutative Lorentz transformations. The generalized Bopp shift in this new formulation connecting the noncommutative variables to commutative variables has also been written down. Using this, the noncommutative harmonic-oscillator Hamiltonian has been mapped to an equivalent commutative Hamiltonian and solved explicitly. A disadvantage of this approach is that one lacks a clear physical interpretation of the commutative variables that are introduced in the Bopp shift and subsequently the interpretation of the quantum theory is obscured. Furthermore, more complicated potentials may lead to highly nonlocal Hamiltonians once the Bopp shift is performed. In [12], an algebraic approach was used to solve the noncommutative harmonic oscillator and within this framework an unambiguous physical interpretation was also given. With this background it is natural to ask if it is possible to solve the spectrum of the harmonic oscillator on noncommutative phase-space without invoking the Bopp-shift.

In this letter, we solve the two-dimensional harmonic oscillator and the harmonic oscillator in a background magnetic field in noncommutative phase-space working with noncommutative coordinates and momenta without making use of any representation. Our results are in conformity with the ones existing in the literature. We also show that there exists some interesting choices of the noncommutative parameters for which the time-reversal symmetry can be restored.

Let us start by considering a two-dimensional noncommutative space defined by the following commutation relation:

$$[\hat{x}, \hat{y}] = i \theta,$$

where $\theta$ is the noncommutative parameter.

It is useful to introduce the complex coordinates

$$\hat{z} = \hat{x} + i \hat{y}, \quad \hat{\bar{z}} = \hat{x} - i \hat{y}$$

so that one can infer from (1)

$$[\hat{z}, \hat{\bar{z}}] = 2 \theta.$$

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Next, we can introduce the pair of boson annihilation and creation operators \( b = (1/\sqrt{2\theta}) \hat{z} \) and \( b^\dagger = (1/\sqrt{2\theta}) \hat{z}^\dagger \) satisfying the Heisenberg-Fock algebra \([b, b^\dagger] = 1\). Therefore, the noncommutative configuration space \( \mathcal{H}_c \) is itself a Hilbert space isomorphic to the boson Fock space \( \mathcal{H}_c = \text{span} \{ |n\rangle , n \in \mathbb{N} \} \), with \(|n\rangle = (1/\sqrt{n!}) (b^\dagger)^n |0\rangle \).

The Hilbert space at the quantum level, denoted by \( \mathcal{H}_q \), is defined to be the space of Hilbert-Schmidt operators on \( \mathcal{H}_c \) [16]
\[
\mathcal{H}_q = \{ \psi(\hat{z}, \hat{\bar{z}}) : \psi(\hat{z}, \hat{\bar{z}}) \in \mathcal{B}(\mathcal{H}_c), \text{tr}_c(\psi(\hat{z}, \hat{\bar{z}})^\dagger \psi(\hat{z}, \hat{\bar{z}})) < \infty \},
\]
where \( \text{tr}_c \) stands for the trace over \( \mathcal{H}_c \) and \( \mathcal{B}(\mathcal{H}_c) \) is the set of bounded operators on \( \mathcal{H}_c \).

With this formalism at our disposal, we now consider the Hamiltonian \( \hat{H} \) of the two-dimensional harmonic oscillator
\[
\hat{H} = \frac{1}{2m} \hat{p}_i^2 + \frac{1}{2} m \omega^2 X_i^2 \quad (i = 1, 2),
\]
\[
\hat{X}_1 = X, \quad \hat{X}_2 = \hat{Y}, \quad \hat{P}_1 = \hat{P}_X, \quad \hat{P}_2 = \hat{P}_Y
\]
on the noncommutative phase-space
\[
[X, \hat{Y}] = i \theta, \quad [X, \hat{P}_X] = i \hbar [Y, \hat{P}_Y], \quad [\hat{P}_X, \hat{P}_Y] = -i \theta.
\]
where capital letters refer to quantum operators over \( \mathcal{H}_q \) and \( \hbar = \hbar (1 + \frac{\theta}{2\pi}) \) is the deformed Planck’s constant [13,14]. Note that the noncommutative parameter \( \theta \) can be treated as a constant magnetic field \( B \) (in units of \( \hbar = 1 \)) at a fundamental level as both govern momentum-momentum noncommutativity.

Introducing the complex operators \( \hat{Z} = X + i \hat{Y}, \quad \hat{\bar{Z}} = X - i \hat{Y}, \quad \hat{P}_z = \hat{P}_X - i \hat{P}_Y \) and \( \hat{P}_{\bar{z}} = \hat{P}_X + i \hat{P}_Y \), the algebra (6) can be rewritten as
\[
[\hat{Z}, \hat{\bar{Z}}] = 2\theta, \quad [\hat{Z}, \hat{P}_z] = 2 i \hbar [\hat{Z}, \hat{P}_\bar{z}], \quad [\hat{P}_z, \hat{P}_{\bar{z}}] = 2 \hbar.
\]
The Hamiltonian \( \hat{H} \) of the harmonic oscillator (5) in terms of the complex coordinates reads
\[
\hat{H} = \frac{1}{4m} (\hat{P}_z \hat{P}_{\bar{z}} + \hat{P}_{\bar{z}} \hat{P}_z) + \frac{1}{4} m \omega^2 (\hat{Z} \hat{\bar{Z}} + \hat{\bar{Z}} \hat{Z}).
\]
We now rewrite \( \hat{H} \) in the matrix form
\[
\hat{H} = \frac{1}{4m} \hat{Z}^\dagger \hat{Z}, \quad \hat{Z} = (\hat{Z}', \hat{\bar{Z}}', \hat{P}_z, \hat{P}_{\bar{z}})^t,
\]
\[
\hat{Z}' = m \omega \hat{Z}, \quad \hat{\bar{Z}}' = m \omega \hat{\bar{Z}},
\]
where the symbol \( t \) means the transpose operation. The next objective is the diagonalization of this Hamiltonian such that
\[
\hat{H} = \frac{1}{4m} A^\dagger D A, \quad A = (a_+, a_+, a_-, a_-)^t, \quad A = S \hat{Z},
\]
where \( D \) is some diagonal positive matrix, \( S \) is some linear transformation such that \((a_+, a_-)\) satisfy decoupled bosonic commutation relations \([a_+, a_-^\dagger] = 1\).

To begin with the factorization of the noncommutative Hamiltonian (9), we introduce the vectors
\[
A^+ = (a_+^t, a_-, a_-, a_-^t)^t = \Lambda A,
\]
\[
\tilde{Z}^+ = (\hat{Z}', \hat{\bar{Z}}', \hat{P}_z, \hat{P}_{\bar{z}})^t = \Lambda S^+\tilde{Z}
\]
with the permutation matrix
\[
\Lambda = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]
Note that \( \Lambda^2 = I_4 \) and \((A^+)^t = A^t \) and the linear transformation \( S \) is such that
\[
A^+ = S^+ \tilde{Z}^+
\]
with \( AS = S^* \). Another important ingredient in the factorization is the matrix \( g \) with entries
\[
g_{jk} = [3_l, 3^*_k], \quad l, k = 1, \ldots, 4.
\]
A simple verification shows that \( g \) is Hermitian and therefore has the natural property to be diagonalizable by a matrix of different orthogonal eigenvectors that we shall denote by \( u_i, i = 1, \ldots, 4 \), associated with real eigenvalues \( \lambda_i \). Now since \((a_+, a_-^t)\) satisfy the bosonic commutation relations, we have the following algebraic constraints on the pair \((A, A^+)\)
\[
[A_k, A_{m^*}] = (J_4)_{km}, \quad J_4 = \text{diag}(\sigma_3, \sigma_3), \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
which leads to the key identity
\[
S g S^t = J_4.
\]
Another important property of \( g \) which simplifies thing further reads
\[
\Lambda g \Lambda = -g^*
\]
from which it can be shown that if \( u_i \) is an eigenvector of \( g \) associated with the eigenvalue \( \lambda_i \), then \( \Lambda u_i^t \) is also an eigenvector of \( g \) associated with the eigenvalue \(-\lambda_i\). From this property, the structure of the eigenbasis of \( g \) follows to be \( \{u_1, \Lambda u_1^t, u_2, \Lambda u_2^t\} \) and its associated spectrum is \( \{\lambda_1, -\lambda_1, \lambda_2, -\lambda_2\} \).

The spectral structure of \( g \) allows us to diagonalize the Hamiltonian easily. We start by choosing the matrix \( S^t = (u_1, \Lambda u_1^t, u_2, \Lambda u_2^t) \) to be the eigenvector matrix of \( g \). A rapid checking proves that indeed \( AS = S^* \). Furthermore, from (16) we have \((S^*)^{-1} = J_4 S g \) and \( S^{-1} = g S^t J_4^t \) and from (10) and (13) follow \( 3 = S^{-1} A \) and \( 3^+ = (S^*)^{-1} A \). Substituting these relations in (9), we obtain the new operator
\[
\hat{H} = \frac{1}{4m} (A^+)^t J_4 S g^2 S^t J_4 A.
\]
It should be noted that the eigenvectors have not yet been normalized. The normalization conditions can be fixed such that, for \( i = 1, 2 \), we have
\[
(u_i)^\dagger u_i = 1/|\lambda_i|
\]
which also guarantees the positivity of the diagonal matrix \( D \).

This finally leads to the diagonalized Hamiltonian
\[
\hat{H} = \frac{1}{4m} \left\{ \lambda_+(a_+^\dagger a_+ + a_+a_+^\dagger) + \lambda_-(a_-^\dagger a_- + a_-a_-^\dagger) \right\},
\]
where \( \lambda_\pm \) are the positive eigenvalues of the matrix \( g \), which reads
\[
g = \begin{pmatrix}
a_0 & 0 & 0 & i b_\tilde{h} \\
0 & -a_0 & i b_\tilde{h} & 0 \\
0 & -i b_\tilde{h} & 2\tilde{\theta} & 0 \\
-i b_\tilde{h} & 0 & 0 & -2\tilde{\theta}
\end{pmatrix},
\]
where \( a_0 = 2m^2\omega^2\tilde{\theta} \) and \( b_\tilde{h} = 2\tilde{h}m\omega \). The eigenvalues of \( g \) can be easily determined and read
\[
\lambda_+ = \frac{1}{2} \left( -2\tilde{\theta} + a_0 + \sqrt{\Delta_{\theta,\tilde{h},\tilde{\theta}}} \right),
\lambda_- = \frac{1}{2} \left( 2\tilde{\theta} - a_0 + \sqrt{\Delta_{\theta,\tilde{h},\tilde{\theta}}} \right),
\Delta_{\theta,\tilde{h},\tilde{\theta}} = (2\tilde{\theta} + a_0)^2 + 4b_\tilde{h}^2,
\]
thereby achieving the diagonalization of the Hamiltonian (9). This operator is nothing but the sum of a pair of one-dimensional harmonic-oscillator Hamiltonians for each coordinate direction with different frequencies both encoding noncommutative parameters and deformed Planck’s constant (\( \theta, \tilde{h}, \tilde{\theta} \)). Using the Fock space basis \( |n_+, n_-\rangle \), one gets the energy spectrum of (9),
\[
E_{n_+, n_-} = \frac{1}{2m} \left\{ \frac{1}{2} \left( -2\tilde{\theta} + a_0 + \sqrt{\Delta_{\theta,\tilde{h},\tilde{\theta}}} \right) n_+ + \frac{1}{2} \left( 2\tilde{\theta} - a_0 + \sqrt{\Delta_{\theta,\tilde{h},\tilde{\theta}}} \right) n_- + \sqrt{\Delta_{\theta,\tilde{h},\tilde{\theta}}} \right\}.
\]

Setting \( \tilde{\theta} = 0 \), we get the spectrum of [12] computed by the same diagonalization technique. This also agrees with the result of [7] obtained via Bopp shift representation without momentum-momentum noncommutativity.

We now make another interesting observation. The energy spectrum (23) shows that the time-reversal symmetry is broken as expected for any noncommutative theory. Nevertheless, we find that there exists a particular choice of \( \tilde{\theta} \) for which the time-reversal symmetry gets restored. Indeed, setting \( \mp 2\tilde{\theta} + a_0 = 0 \), equivalently \( \tilde{\theta} = \mp 2m^2\omega^2\tilde{\theta} \), we get a degenerate energy spectrum. This result is in fact impossible to obtain in a quantum phase-space where we have only space-space noncommutativity. The choice \( \tilde{\theta} = -m^2\omega^2\tilde{\theta} \), however, leads to a simplified but non-degenerate energy spectrum:
\[
E_{n_+, n_-} = \hbar \omega \left( 1 - \left( \frac{m^2 \omega^2}{\hbar} \right)^2 \right) (n_+ + n_-) + m^2 \omega^2 (n_+ - n_-) + 8m(\hbar)^2 \left( 1 - \left( \frac{m^2 \omega^2}{\hbar} \right)^2 \right).
\]

Finally, we consider the problem of a charged particle on a noncommutative plane subjected to a magnetic field \( \vec{B} = B\hat{k} \) (\( \hat{k} \) is the unit vector along the \( z \)-direction) orthogonal to the plane along with a harmonic-oscillator potential of frequency \( \omega \). In the symmetric gauge \( \vec{A}(\vec{X}) = (1/2)\vec{X} \times \vec{B}, \vec{X} = (\vec{X}, \vec{Y}) \), the Hamiltonian reads
\[
\hat{H} = \frac{1}{2m} \left( \vec{P} + \frac{eB}{2} \epsilon_{ij} \vec{X}_j \right)^2 + \frac{1}{2} m^2 \omega^2 \vec{X}_i^2.
\]

We now show that the above method can be applied to diagonalize an even more general noncommutative Hamiltonian of this form. Using the algebra (6), we obtain the following algebra between \( \hat{\Pi}_i \) and the canonically conjugate momenta \( \hat{\Pi}_j \),
\[
[\hat{X}_i, \hat{X}_j] = i\theta \epsilon_{ij}, \quad [\hat{X}_i, \hat{\Pi}_j] = i\tilde{\theta} \delta_{ij}, \quad [\hat{\Pi}_i, \hat{\Pi}_j] = -i\tilde{\theta} \epsilon_{ij},
\]
where \( \tilde{\theta} = \tilde{\theta} + eB\theta/2 \) and \( \bar{\theta} = \bar{\theta} - eB\tilde{h} - e^2B^2\theta/4 \).

The complex operators \( \bar{Z}, \bar{\bar{Z}} \) (as defined earlier) and \( \bar{\Pi}_Z = \bar{\Pi}_X - i\bar{\Pi}_Y, \bar{\Pi}_\bar{Z} = \bar{\Pi}_X + i\bar{\Pi}_Y \) can then be used to recast the Hamiltonian (25) in the form (9) with \( \bar{3} = (\bar{Z}, \bar{\bar{Z}}, \bar{\Pi}_Z, \bar{\Pi}_\bar{Z}) \) and they satisfy the algebra
\[
[\bar{Z}, \bar{\bar{Z}}] = 2\bar{\theta}, \quad [\bar{Z}, \bar{\Pi}_Z] = 2i\tilde{\bar{h}} = [\bar{\bar{Z}}, \bar{\Pi}_\bar{Z}], \quad [\bar{\Pi}_Z, \bar{\Pi}_\bar{Z}] = 2\tilde{\theta}.
\]

One can now easily obtain the matrix \( g \) (21) by replacing \( \tilde{\theta} \rightarrow \bar{\theta} \) in \( b_\tilde{h} \) and \( \Delta_{\theta,\tilde{h},\tilde{\theta}} \) which finally leads to the energy spectrum
\[
E_{n_+, n_-} = \frac{1}{2m} \left\{ \frac{1}{2} \left( -2\bar{\theta} + a_0 + \sqrt{\Delta_{\theta,\bar{\theta},\tilde{\theta}}} \right) n_+ + \frac{1}{2} \left( 2\bar{\theta} - a_0 + \sqrt{\Delta_{\theta,\bar{\theta},\tilde{\theta}}} \right) n_- + \sqrt{\Delta_{\theta,\bar{\theta},\tilde{\theta}}} \right\}.
\]

Once again, we observe that there exists a choice of \( \bar{\theta} \) given by
\[
\bar{\theta} = \left( m^2 \omega^2 + \frac{e^2B^2}{4} \right) \theta + eB\tilde{h}
\]
for which the time-reversal symmetry is restored. The choice \( \bar{\theta} = -\left( m^2 \omega^2 - \frac{e^2B^2}{4} \right) \theta + eB\tilde{h} \), leads to a simplified
but nondegenerate energy spectrum as before:

\[
E_{n_+,n_-} = \omega \left( \frac{\hbar + eB\theta}{2} \right) (n_+ + n_-) \\
+ m\omega^2 \theta (n_+ - n_-) + 8m\omega^2 \left( \frac{\hbar + eB\theta}{2} \right)^2.
\]

(30)

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