Edge number report 1: state of the art estimates for \( n \leq 43 \).

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Abstract

This first extracted report contains all lower and upper bounds for e-numbers \( e(3, k; n) \), for \( n \leq 43 \), that I know. All but 24 of them are known (exactly). Very little of the proofs is given. A few consequences for upper classical Ramsey number bounds are mentioned.

1 Introduction.

Throughout the years, I have investigated e-numbers, and updated my tables of these and of properties for graphs with edge numbers close to the respective e-number. The results have been collected in the various updated versions of [1]. However, that work is not easily accessible; not only since I have not made it public, but since it is large, and based on a somewhat complex terminology, both for graph objects and for methods for dealing with them.

At present, I’m integrating the consequences of Goedgebeur’s and Radziszowski’s investigations in [4] into my tables. This is slow work; I have now more or less finished it up to vertex number 43. This has yielded a few improvements, compared both to [4] and to older versions of [1].

I have received some criticism for not making my results more accessible. In this report, I indeed try to present the more recent ones, as regards e-number bounds; but not the further Ramsey graph properties. I believe that this makes it easier to access the conclusions; but it makes it harder to reproduce or improve the proofs. I outline a few proof examples; they may at least illustrate the ‘Ramsey calculus’ methods.

Moreover I also discuss upper bounds for e-numbers. This is an area not equally well covered by the literature, I think, and I’m not sure of how good the upper bounds I give here are, compared to the state-of-the-art.

Finally, the terminology is a bit experimental. I try to make it more conformant to other recent state-of-the-art articles, and (against my instincts) leave a good bit undefined. I’ll be very thankful for comments, both on this, and on the factual content of this report.
2 Definitions.

Throughout this work, all graphs $G = (V, E)$ are finite, simple, and undirected; and they are triangle-free; i.e., the clique number $\omega(G) \leq 2$.

The second degree of a vertex $v$ in a graph $G$ is

$$\deg^2(v) = \deg^2_G(v) := \sum_{w \in N(v)} \deg(w),$$

where $N(v)$ is the set of vertices adjacent to $v$. (The second degree is denoted $Z(v)$ in e.g. [4].) The induced $G$ subgraph on $V \setminus (N(v) \cup \{v\})$ is denoted $G_v$.

$G$ is an $(i, j; n, e)$-graph and an $(i, j; n)$-graph, if $\omega(G) < i$, its independence number $\alpha(G) < j$, $n(G) := |V| = n$, and $e(G) := |E| = e$.

For any positive integers $i$, $j$, and $n$, the e-number $e(i, j; n)$ is the minimal number $e$, such that there are $(i, j; n, e)$-graphs, or $\infty$, if no $(i, j; n)$-graphs exist. They are of great interest for finding improved bounds of Ramsey numbers

$$R(i, j) := \min(n : e(i, j; n) = \infty),$$

but are also of interest in themselves.

In this report, we only discuss the e-numbers $e(3, j; n)$. For the estimates, we shall use a few linear or ‘piecewise linear’ functions on two integer variables, namely,

$$f_1(n, k) = \max(0, n - k, 3n - 5k, 5n - 10k, 6n - 13k);$$

$$f_2(n, k) = 8n - 19.5k;$$

$$f_3(n, k) = 9n - 23k; \text{ and}$$

$$f_4(n, k) = 6.8n - 15.6k.$$

Note, that $f_1(n, k) = 6n - 13k$, if $n \geq 3k$.

Occasionally, we mention the “linear graph invariant”

$$t(G) := e(G) - 6n(G) + 13\alpha(G).$$

$W_{13,1,5}$ denotes the cyclic graph with 13 vertices (conventionally named $u_1, \ldots, u_{13}$), and with two vertices forming an edge if the absolute value of their indices counted modulo 13 is either 1 or 5. (This graph very often is denoted $H_{13}$.)

For other concepts, background, et cetera, see the bibliography. In particular, we shall discuss some graphs given by means of extension patterns, which provide recipes for constructing them step-by-step; but neither the patterns and nor the corresponding graphs are formally described here.
3 Known general values.

For \( n \leq 3.25k + 1.5 \), all e-numbers are known. (This indeed includes all \( e(3, k + 1; n) \) with \( n \leq 43 \) and \( k \geq 13 \).) To begin with, we have

**Proposition 1.** For all positive integers \( n \) and \( k \),

\[
e(3, k + 1; n) \geq f_1(n, k).
\]

The values are exact if and only if \( n < R(3, k + 1) \), and moreover either \( n \leq 3.25k - 1 \), or \( n = 3.25n \).

For a proof, see e.g. [10]. Note, that part of the result is the fact that \( t(G) \geq 0 \) for all (triangle-free) \( G \).

**Lemma 3.1.** Let \( k \) and \( n \) be positive integers, such that \( 3k \leq n < R(3, k + 1) \), but \( e(3, k + 1; n) > f_1(n, k) \). Then \( e(3, k + 1; n) = f_1(n, k) + 1 \iff -1 < n - 3.25k < 0 \), \( e(3, k + 1; n) = f_1(n, k) + 2 \iff 0 < n - 3.25k \leq 0.5 \), and \( e(3, k + 1; n) \geq f_1(n, k) + 3 \iff 0.5 < n - 3.25k \).

The proof depends on deriving properties for graphs with \( t(G) \leq 2 \). In [1], indeed, all \( G \) with \( t(G) \leq 1 \) are characterised, and sufficient restrictions are found for those with \( t(G) = 2 \). (Actually, the complete characterising of the graphs with \( t(G) = 0 \) also is the main object of the stand-alone manuscript [2]. The \( t(G) = 2 \) result partly employs [4].)

Employing some constructions, we find that the lower bound in the last part of lemma 3.1 is exact in a few cases:

**Lemma 3.2.** If \( 3k \leq n < R(3, k + 1) \) and \( 0.5 < n - 3.25k \leq 1.5 \), then \( e(3, k + 1; n) = f_1(n, k) + 3 \).

If \( n > 3.25k + 1.5 \), and moreover \( k \leq 12 \), then \( e(3, k + 1; n) > f_1(n, k) + 3 \); and I find it likely that this should hold also for all higher \( k \). Moreover, I guess that

\[
e(3, k + 1; n) \geq \max(f_2(n, k), f_3(n, k)), \tag{1}
\]

too; but I am far from being able to prove this. The best general result I have for \( n - 3.25k \gg 0 \) is

**Lemma 3.3.** For any \( n \) and \( k \),

\[
e(3, k + 1; n) \geq f_4(n, k).
\]

(This is contained in [1, proposition 13.5], which is proved by means of a somewhat complicated induction argument).
4 The other values for $n \leq 34$.

For $n \leq 34$, all $e(3, k + 1; n)$ are known. Actually, only 15 of them are ‘sporadic’, i.e., not given by the known Ramsey numbers, or in section 3; and they all have $n \geq 22$ and $6 \leq k \leq 9$. Thus, they are included in the following $e(3, l; n)$ table (where $l = k + 1$):

| $n \backslash l$ | 7   | 8   | 9   | 10  |
|-----------------|-----|-----|-----|-----|
| 22              | 60  | 42  | 30  | 21  |
| 23              | $\infty$ | 49  | 35  | 25  |
| 24              | $\infty$ | 56  | 40  | 30  |
| 25              | $\infty$ | 65  | 46  | 35  |
| 26              | $\infty$ | 73  | 52  | 40  |
| 27              | $\infty$ | 85  | 61  | 45  |
| 28              | $\infty$ | $\infty$ | 68  | 51  |
| 29              | $\infty$ | $\infty$ | 77  | 58  |
| 30              | $\infty$ | $\infty$ | 86  | 66  |
| 31              | $\infty$ | $\infty$ | 95  | 73  |
| 32              | $\infty$ | $\infty$ | 104 | 81  |
| 33              | $\infty$ | $\infty$ | 118 | 90  |
| 34              | $\infty$ | $\infty$ | 129 | 99  |

Note, that all items under an $\infty$ in a column also are $\infty$. In the sequel, in each column, just the top $\infty$ (if any) is printed.

5 The other values and estimates for $35 \leq n \leq 43$.

In the table, a single value indicates that this is the exact e-value. Two values separated by a dash (–) are the best known lower and upper bounds of the respective e-value. Again, $l = k + 1$.

| $n \backslash l$ | 9   | 10   | 11   | 12   | 13   |
|-----------------|-----|------|------|------|------|
| 35              | 140 | 107–108 | 84–85 | 68  | 55  |
| 36              | $\infty$ | 117–119 | 92–94 | 75  | 60  |
| 37              | 128–(132) | 100–103 | 82  | 66  |
| 38              | 139–(143) | 109–112 | 89–90 | 72  |
| 39              | 151–161 | 119–121 | 96–98 | 78  |
| 40              | 161–$\infty$ | 128–130 | 103–107 | 87  |
| 41              | 172–$\infty$ | 139–(150) | 111–116 | 94  |
| 42              | $\infty$ | 149–(160) | 120–125 | 101–102 |
| 43              | $\infty$ | 159–(171) | 129–134 | 108–111 |

The upper bounds within parentheses are rather preliminary; they are achieved by crude constructions, made more or less on the fly, since I am too ignorant to know where to look for the best actually achieved upper bounds. I expect there to have been constructions
or computer enumerations around for a while, giving better upper bounds for all five or most of them.

6 Consequences for Ramsey numbers.

By hand calculations or by means of e.g. the matlab programme FRANK ([6])\(^1\), it is fairly easy to check for consequences for upper bounds on Ramsey numbers for any improvement of lower bounds of e-numbers. As compared to the combined values from [4] and older versions of [1], the sharper bounds presented here yield just two improved upper Ramsey number bounds.

It turned out that the improvement of the lower bound for \(e(3, 12; 43)\) from 128 to 129 was crucial for deducing that

\[ R(3, 19) \leq 132, \]

as reported in the latest dynamic survey on small Ramsey numbers ([8]).

The improvement of lower \(e(3, 11; 39)\) bound from 117 ([4]) to 119 suffices to prove that

\[ R(3, 16) \leq 97. \]

This bound is not (yet) included in the dynamic survey.

7 A few proof hints.

7.1 Lower bounds.

Most of the ‘sporadic’ lower bounds are found in [4]; and/or are direct consequences of lower bounds for smaller independence numbers. The exceptions are the lower bounds for \(e(3, 11; 35)\), \(e(3, 12; 38)\), \(e(3, 12; 39)\), \(e(3, 13; 41)\), \(e(3, 13; 42)\), \(e(3, 12; 43)\), \(e(3, 11; 39)\), and \(e(3, 11; 41)\).

The first six of these bounds, as well as the ‘general’ bounds, depend partly on theoretical classification of some ‘lower’ graphs, i.e., graphs with lower independence and vertex numbers; likewise, the two last ones depend on computational classification of some lower graphs. In all cases, there is some use of properties deduced for some lower graphs; and the general proof technique is to assume the existence of a graph \(G\) with ‘offendingly’ low \(e(G)\), and then to deduce more and more precise conditions for \(G\), until finally a contradiction is achieved. I’ll provide a few examples.

First, assume that \(G\) is a \((3,11;35)\)-graph with \(e(G) \leq 83\); whence actually equality must hold. We then successively may prove:

\(^1\)The version of FRANK that I employ includes a test for raising the lower e-number bound in a few cases, where the only formally possible degree distributions all would have to contain either a triangle of low-degree vertices, or a low-degree vertex with too few low-degree neighbours (and thus a too high second degree). In practice, this only may happen, when the unraised e-number bound would be close to, but slightly less than, the e-value for some regular graph. This tweak yielded e.g. \(e(3, 13; 51) \geq 179\).
\[(a) \quad \delta(G) > 2; \]
\[(b) \quad \delta(G) > 3; \]
\[(c) \quad \text{any vertex of degree 4 has at most one neighbour of degree } \geq 5; \]
\[(d) \quad G_v \text{ has no } W_{13,1,5} \text{ component for any vertex of degree 5; and} \]
\[(e) \quad \text{if } \deg(v) = 5, \text{ then } \deg^2(v) \leq 24.} \]

Property \((a)\) is immediate from the \(e(3, 10; n)\) values.

\((b)\) follows from \((a)\), and from the fact that any \((3, 10; 31)\)-graph \(H\) with \(e(H) \leq 74\) has \(\delta(H) \geq 2\), strictly if \(e(H) = 73\); and that there are at most two vertices of degree 2 in \(H\), which (if indeed there are two of them) moreover must be adjacent.

\((c)\) is immediate from \((b)\), and the fact that \(\deg^2(v) \leq 17\) for any vertex of degree 4.

\((e)\) is an immediate consequence of \((d)\), and of the fact that any \((3, 10; 29, 58)\)-graph does contain a \(W_{13,1,5}\) component. On the other hand, \((e)\) directly yields a contradiction, since it means that we could calculate as if \(e(3, 10; 29)\) were at least 59.

This just leaves the deduction of \((d)\) from \((b)\) and \((c)\), which is somewhat less immediate. Assume for a contradiction that \(\deg(v) = 5\), and that \(G_v\) has a \(W_{13,1,5}\) component. Let \(N(v) = \{w_1, \ldots, w_5\}\), and let \(U\) be the set of vertices in \(W_{13,1,5}\), which are not adjacent to any \(w_i\); in other words, \(U = \{u \in V(W_{13,1,5}) : \deg_G(u) = 4\}\).

Now, \(|U| \leq 8\), since \(U\) cannot contain an independent 4-set; if it did, any edge between \(U\) and \(N(v)\) would be redundant (in the sense that removing it from \(G\) would leave a graph which also did not contain an independent 11-set), but \(G\) can contain neither a redundant edge, nor a \(W_{13,1,5}\) component. Thus, and by inspection of \(W_{13,1,5}\), if \(U\) were non-empty, then there were a \(u_j \in U\) with at most two neighbours in \(U\), and therefore at least two neighbours of degrees \(\geq 5\), contradicting \((c)\).

Thus, instead, \(U = \emptyset\); i.e., each vertex in \(W_{13,1,5}\) is adjacent to at least one \(w_i\). This makes it possible to apply a “discharging” argument. ‘Charge’ each \(u_j\) with a unit charge, 1; and then ‘discharge’ each \(u_j\) by distributing its charge in equal proportions to its \(w_i\) neighbours. The total charge after discharging must stay 13. However, no \(w_i\) can receive a charge larger than 2.5; which means that \(N(v)\) in total cannot carry a higher charge than 12.5. This is a contradiction; which indeed proves \((d)\).

For a second example, assume that \(G\) is a \((3, 11; 41)\)-graph with \(e(G) = 138\). There are few theoretic ways for such a graph to be ‘realised numerically’; in other words, if we let the degree distribution (degree sequence) of the graph be \((n_0, n_1, \ldots, n_{10})\), then there are just a handful possible such sequences, for which the resulting Graver-Yackel defect \(\gamma(G)\) would be non-negative (cf. [5] and [4]). In fact, also employing that a single vertex \(v\) of degree 8 would have \(\deg^2(v) \leq 8 \cdot 7 = 56\), and thus a positive defect, and repressing all leading and trailing zeroes in the distributions, we would have one of

\[ (11, 30), (12, 28, 1), (1, 9, 31), (2, 7, 32), \text{ and } (3, 5, 33) \]

as degree distribution, with the total defect \(\gamma(G) = 3, 1, 2, 1, \) and 0, respectively.

Put \(F := \{v \in V : \deg(v) = 7 \text{ and } \deg^2(v) = 48\}\). In other words, \(F\) is the set of non-defect vertices of degree 7. Counting directly yields that \(|F| \geq 27\), in each one of the cases.
For any \( f \in F \), \( G_f \) is a \((3,10;33,90)\)-graph. Now, Goedgebeur and Radziszowski classified all these graphs, and made a list of all 57099 of them available on the House of Graphs ([4]). Running the NAUTY ([7]) command `countg --Jd` on this list reveals that any such graph \( H \) contains an induced \( K_{2,4} \), and has \( \delta(H) \geq 4 \). Moreover, a theoretical analysis shows that for any vertex \( v \) with \( 5 \leq \deg(v) \leq 7 \), either \( \delta(G_v) \geq 3 \), or \( \delta(G_v) = 2 \) and \( \gamma(v) = 3 \), or \( \gamma(v) > 3 \).

Now, choose such an \( f \); if there is a vertex \( x \) of degree 8, actually choose \( f \in F \cap N(x) \); choose a \( K_{2,4} \subset V_f \subset V \), with \( V(K_{2,4}) = \{a_1, a_2; b_1, \ldots, b_4\} \) and \( \deg(a_1) \leq \deg(a_2) \leq 7 \), say. We now note, that

\[
\delta(G_{a_i}) \leq \deg(a)_{3-i} - 4, \text{ for } i = 1, 2;
\]

and employ this in estimating the defects of the \( a_i \).

If \( \deg(a_1) = 5 \), then \( \gamma(a_2) \geq 4 > 3 \geq \gamma(G) \), a contradiction. Likewise, if \( \deg(a_1) = 6 \), then \( \gamma(a_2) = 3 \), whence then \( \gamma(a_1) = 0 \); whence anyhow

\[
6 \leq \deg(a_1) \leq 7 = \deg(a_2).
\]

If \( \deg(a_1) = 7 \), then both \( a_1 \) and \( a_2 \) are defective, and the further defects in \( G \) sum up to at most 1, whence in particular then \( \Delta(G) = 7 \). Moreover, if \( \deg(a_1) = 7 \), then not both \( a_1 \) and \( a_2 \) may have defects \( \geq 2 \), whence instead then at least one of them has second valency 47, and thus at least five neighbours of degree 7, of which at least four belong to \( F \). Thus, in this case, we may assume that \( f' := b_4 \in F \); while if \( \deg(a_1) = 6 \), then let \( f' \) be arbitrarily chosen in \( F \cap \{\gamma a_2\} \). In either case, there is some \( K_{2,4} \) in \( V_{f'} \), and this would also carry a defect at least 2, which would yield a total defect at least 4 in \( G \), a contradiction.

### 7.2 Upper bounds.

For \( n \leq 4k = 4l - 4n \) (but excepting \( (n,l) \in \{(17,6), (22,7), (27,8)\}\)), there are constructions, whose connected components either are described by their extension patterns, or are one or the other of two exceptional graphs: The cyclic graph \( W_{13,1.5} \) (the unique \((3,5;13,26)\)-graph), and the twisted tesseract \( (a \,(3,6;16,32)\)-graph). (The twisted tesseract also is denoted \( 2W_{8,1,4} \) in [1]; i.e., it consists of two disjoint copies of \( W_{8,1,4} \), with the \( i \)’th vertex in the first copy connected to the \( 5i \)’th one in the second copy by an edge; where indices are taken modulo 8.)

The extension pattern of a graph \( G \) of the kind we consider here includes a triangle free graph \( T \), such that

\[
e(T) \leq 2n(T),
\]

\[
\alpha(G) = n(T),
\]

\[
n(G) = 2n(T) + e(T), \text{ and }
\]

\[
e(G) = n(T) + 2e(T) + \frac{1}{2} \sum_{x \in V(T)} \deg(x)^2.
\]
This yields that the graphs with only patterned and/or exceptional graphs as components indeed fulfil (1). In fact, for ‘most’ \( k \) and \( n \) with \( 3.25k \leq n \leq 4k \), we have such graphs realising equality in (1). However, there are some irregularities, for two reasons. First, each \( \mathcal{W}_{13,1.5} \) component contributes 4 to the independence number of the graph; and there may not be an integer number of such components that realises equality in (1). Second, in general, for a connected patterned graph \( G \) with \( 3.25 \alpha(G) \leq n(G) \leq 4 \alpha(G) \), equality only can be achieved by having only vertices of degrees 3 and 4 in the pattern graph \( T \) (since other degree distributions yield higher \( \sum_{V(T)} \deg(x)^2 \)); which for \((3, 10; 36)\)-graphs would force the pattern graph to be 4-regular, on 9 vertices. By inspection, there is no such triangle-free graph; the closest possible degree distribution is \((2,5,2)\) vertices of degrees \((3,4,5)\), respectively.

The upper bound 161 for \( e(3, 10; 39) \) is reported by Goedgebeur and Radziszowski in [4], where it is noted that both they and Exoo have found huge amounts of \((3, 10; 39, 161)\)-graphs \( G \), but no \((3, 10; 39)\)-graph with a lower number of edges.

For the five upper bounds within parentheses, let \( L \) be the regular \((3^8)\)-type lace with constant offsets \((1,3)\), a \((3,9;32,104)\)-graph. (Laces are defined and investigated in [1]; they form a special class of patterned graphs.) Its family \( (v_1, \ldots, v_8) \) of apices consists of non-adjacent vertices of degree 6, where moreover \( \text{dist}(v_i, v_j) \geq 3 \), if \( i \) and \( j \) have the same parity. The upper \( e(3,10;37) \) (\( e(3,10;38) \)) bounds are achieved by a 4-extension (5-extension) of \( H \), employing 3 (all 4) of the odd-indexed \( v_i \), respectively; and the upper \( e(3,11;41—43) \) bounds by making a further extension of one of these, employing the \( v_i \) with even indices.

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