Convergence of Batch Greenkhorn for Regularized Multimarginal Optimal Transport

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Abstract

In this work we propose a batch version of the Greenkhorn algorithm for multimarginal regularized optimal transport problems. Our framework is general enough to cover, as particular cases, some existing algorithms like Sinkhorn and Greenkhorn algorithm for the bi-marginal setting, and (greedy) MultiSinkhorn for multimarginal optimal transport. We provide a complete convergence analysis, which is based on the properties of the iterative Bregman projections (IBP) method with greedy control. Global linear rate of convergence and explicit bound on the iteration complexity are obtained. When specialized to above mentioned algorithms, our results give new insights and/or improve existing ones.

1 Introduction

Over the recent years the field of optimal transport (OT) [24] has received significant attention in machine learning and data science, as it provides natural and powerful tools to compare probability distributions. In this paper we study a general class of OT problems known as multimarginal optimal transport (MOT), whereby several probability distributions are coupled together in order to compute a measure of their association, see e.g. [20, 5]. MOT is receiving increasing interest due to its numerous applications, ranging from density functional theory in quantum chemistry, to fluid dynamics, to economics, to image processing, just to name a few, see [21] and references therein. Particularly, in machine learning, MOT is important for generative adversarial networks (GANs) [8], domain adaptation [15], Wasserstein barycenters [1], clustering [19], and Bayesian inference of joint distributions [14], among others.

We focus on discrete MOT, in which, given $m$ finitely supported probability distributions, we wish to compute an optimal joint distribution which solves a linear program, whose objective function involves a cost tensor and the constraint set requires the joint distribution to have the given individual ones as its marginals. It is well known that addressing directly the MOT problem is computationally prohibitive. Furthermore, unlike the bi-marginal case, MOT is NP-Hard for certain costs, even approximately [2]. To overcome this issue, regularization techniques have been widely considered.

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we propose a batch version of the popular Greenkhorn algorithm for RMOT that greedily selects at each iteration a marginal and a batch of its components of prescribed sizes. In Sec. 3 we establish its linear convergence, providing also explicit rates depending on problem data in two important cases: 1) batch Greenkhorn for bi-marginal optimal transport and 2) (greedy) MultiSinkhorn of [17]. Moreover, in these two cases we also provide iteration complexity bounds that improve state-of-the-art results. We stress that the explicit rate we obtain for the greedy multimarginal Sinkhorn is strictly better than the one recently derived in [4], for cyclic multimarginal Sinkhorn. This shows the effectiveness of the greedy option for speeding up the convergence of this type of algorithms.

Contribution. After the introduction of RMOT as a Bregman projection problem in Sec. 2, in Sec. 3 we propose a batch version of the popular Greenkhorn algorithm for RMOT that greedily selects at each iteration a marginal and a batch of its components of prescribed sizes. In Sec. 4, we establish its linear convergence, providing also explicit rates depending on problem data in two important cases: 1) batch Greenkhorn for bi-marginal optimal transport and 2) (greedy) MultiSinkhorn of [17]. Moreover, in these two cases we also provide iteration complexity bounds that improve state-of-the-art results. We stress that the explicit rate we obtain for the greedy multimarginal Sinkhorn is strictly better than the one recently derived in [4], for cyclic multimarginal Sinkhorn. This shows the effectiveness of the greedy option for speeding up the convergence of this type of algorithms.

Notation. Given $n \in \mathbb{N}$, let $[n] := \{1,2,\ldots,n\}$, and let $\Delta_n = \{x \in \mathbb{R}_+^n : \|x\|_1 = 1\}$ be the unit simplex. Next, let $m \geq 2$ and $n_1,\ldots,n_m \in \mathbb{N}$. The space of tensors (of order $m$) is denoted by $\mathcal{X} := \mathbb{R}^{n_1 \times \cdots \times n_m}$. For the sake of brevity we set $\mathcal{J} = [n_1] \times \cdots \times [n_m]$ and we denote by $j = (j_1,\ldots,j_m)$ a general multi-index in $\mathcal{J}$. We set $\mathcal{X}_+ = \{\pi \in \mathcal{X} : \pi_j \geq 0, \text{ for every } j \in \mathcal{J}\}$ and $\mathcal{X}_{++} = \{\pi \in \mathcal{X} : \pi_j > 0, \text{ for every } j \in \mathcal{J}\}$. The space $\mathcal{X}$ is an Euclidean space when endowed with the standard scalar product and norm, defined as

$$\langle \pi, \pi' \rangle = \sum_{j \in \mathcal{J}} \pi_j \pi'_j \quad \|\pi\|^2 = \sum_{j \in \mathcal{J}} \pi_j^2 \quad \forall \pi, \pi' \in \mathcal{X}.$$  \hspace{1cm} (1.1)

For every $\pi, \pi'$ we denote by $\pi \odot \pi' \in \mathcal{X}$, the Hadamard product of $\pi$ and $\pi'$, that is, $(\pi \odot \pi')_j = \pi_j \pi'_j$. Moreover, if $v_1 = (v_{1,j_1})_{j_1 \in [n_1]} \in \mathbb{R}^{n_1}, \ldots, v_m = (v_{m,j_m})_{j_m \in [n_m]} \in \mathbb{R}^{n_m}$, we set $\otimes_{k=1}^m v_k \in \mathcal{X}$ such that $(\otimes_{k=1}^m v_k)_j = \sum_{k=1}^m v_{k,j_k}$, and $\otimes_{k=1}^m v_k \in \mathcal{X}$ such that $(\otimes_{k=1}^m v_k)_j = \prod_{k=1}^m v_{k,j_k}$ For a function
while its KL distance to 
where $k \forall \xi \in \mathbb{R}_{++}$ is the 
version using the (negative) Boltzmann-Shannon entropy $H$ and considering that $\text{dom } H = \mathbb{X}_+$. For a cost tensor $C \in \mathbb{X}_+$, the regularized multimarginal optimal transport (RMOT) problem consists in computing

$$
\pi^* = \text{arg min}_{\pi \in \Pi(a_1, \ldots, a_m)} \langle C, \pi \rangle + \eta H(\pi),
$$

where $\eta > 0$ is a regularization parameter. Now, recalling that the Kulback-Leibler (KL) divergence $\text{KL} : \mathbb{X} \times \mathbb{X} \rightarrow [0, +\infty]$ is the Bregman distance associated to the (negative) entropy $H$, i.e.,

$$
\text{KL}(\pi, \pi') = \begin{cases} 
    H(\pi) - H(\pi') - \langle \pi - \pi', \nabla H(\pi') \rangle & \text{if } \pi' \in \text{int(dom H)} \\
    +\infty & \text{otherwise}
\end{cases}
$$

and considering that $\text{dom } H = \mathbb{X}_+$, it easy to check that (2.4) can be re-written as

$$
\pi^* = \text{arg min}_{R_k(\pi) = a_k, k \in [m]} \text{KL}(\pi, \xi),
$$

where $\xi = \nabla H^*(-C/\eta) = \exp(-C/\eta) \in \mathbb{X}_+$ is the Gibbs kernel tensor. Hence, the solution of (2.6) is nothing but the KL projection of $\xi$ onto the affine set $\{ \pi \in \mathbb{X} | (\forall k \in [m]) R_k(\pi) = a_k \}$. Indeed, in general, for an arbitrary closed convex set $C \subset \mathbb{X}$ such that $C \cap \mathbb{X}_+ \neq \emptyset$ and a point $\pi \in \mathbb{X}_+$, the KL projection of $\pi$ onto $C$ is defined as

$$
P_C(\pi) := \text{arg min}_{\gamma \in C} \text{KL}(\gamma, \pi),
$$

while its KL distance to $C$ is defined as $\text{KL}_C(\pi) := \text{KL}(P_C(\pi), \pi)$. 

2 Entropic RMOT as the Bregman projection problem

In this section we formally introduce the discrete multimarginal optimal transport problem and its regularized versions, emphasizing its connection with the Bregman projection problem.

Let $a_k \in \Delta_{n_k}$, $k \in [m]$ be prescribed histograms. MOT consists in solving a linear program

$$
\min_{\pi \in \Pi(a_1, \ldots, a_m)} \langle C, \pi \rangle,
$$

where $C \in \mathbb{X}$ is a given cost tensor and $\Pi(a_1, \ldots, a_m)$, called transport polytope, is the convex set of nonnegative tensors in $\mathbb{X}$ whose marginals are $a_1, \ldots, a_m$. More specifically

$$
\Pi(a_1, \ldots, a_m) = \{ \pi \in \mathbb{X}_+ | R_k(\pi) = a_k, \text{ for every } k \in [m] \},
$$

where for all $k \in [m]$, we denote by $R_k : \mathbb{X} \rightarrow \mathbb{R}^{n_k}$ the $k$-th push-forward projection operator, defined as

$$
(\forall j_k \in [n_k]) \quad R_k(\pi)_{j_k} = \sum_{j \in J_k} \pi_{(j-k \cdot j_k)},
$$

so that $R_k(\pi)$ is the $k$-th marginal of $\pi$.

As noted in the introduction, problem (2.1) may be hard to solve. We thus consider a regularized version using the (negative) Boltzmann-Shannon entropy $H(\pi) := \sum_j \pi_j (\log \pi_j - 1)$, $\pi \in \text{dom } H = \mathbb{X}_+$. For a cost tensor $C \in \mathbb{X}_+$, the regularized multimarginal optimal transport (RMOT) problem consists in computing

$$
\pi^* = \text{arg min}_{\pi \in \Pi(a_1, \ldots, a_m)} \langle C, \pi \rangle + \eta H(\pi),
$$

where $\eta > 0$ is a regularization parameter. Now, recalling that the Kulback-Leibler (KL) divergence $\text{KL} : \mathbb{X} \times \mathbb{X} \rightarrow [0, +\infty]$ is the Bregman distance associated to the (negative) entropy $H$, i.e.,

$$
\text{KL}(\pi, \pi') = \begin{cases} 
    H(\pi) - H(\pi') - \langle \pi - \pi', \nabla H(\pi') \rangle & \text{if } \pi' \in \text{int(dom H)} \\
    +\infty & \text{otherwise}
\end{cases}
$$

and considering that $\text{dom } H = \mathbb{X}_+$, it easy to check that (2.4) can be re-written as

$$
\pi^* = \text{arg min}_{R_k(\pi) = a_k, k \in [m]} \text{KL}(\pi, \xi),
$$

where $\xi = \nabla H^*(-C/\eta) = \exp(-C/\eta) \in \mathbb{X}_+$ is the Gibbs kernel tensor. Hence, the solution of (2.6) is nothing but the KL projection of $\xi$ onto the affine set $\{ \pi \in \mathbb{X} | (\forall k \in [m]) R_k(\pi) = a_k \}$. Indeed, in general, for an arbitrary closed convex set $C \subset \mathbb{X}$ such that $C \cap \mathbb{X}_+ \neq \emptyset$ and a point $\pi \in \mathbb{X}_+$, the KL projection of $\pi$ onto $C$ is defined as

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P_C(\pi) := \text{arg min}_{\gamma \in C} \text{KL}(\gamma, \pi),
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while its KL distance to $C$ is defined as $\text{KL}_C(\pi) := \text{KL}(P_C(\pi), \pi)$.
3 Greedy KL projections for entropic RMOT

Now we focus on the entropic-regularized MOT problem (2.4), in the equivalent form (2.6). Recalling definition (2.3), we introduce the linear operator

$$R: X \to \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}, \quad R(\pi) = (R_1(\pi), \ldots, R_m(\pi)).$$

(3.1)

Then, the constraints in (2.6) define the affine set

$$\Pi = \{ \pi \in X \mid R(\pi) = (a_1, \ldots, a_m) \}.$$  

(3.2)

We recall that the positiveness constrain embodied in problem (2.4) can be absorbed in the entropy function $H$, since $\text{dom} \ H = X_+$. So that problem (2.4) is equivalent to the computation of the KL projection of the Gibbs kernel $\xi$ onto the affine set $\Pi$. However, it is possible to prove (see equation (A.11) in Appendix A) that $\mathcal{P}_\Pi(\xi) = \mathcal{P}_\Pi(\xi \circ \otimes_{k=1}^n a_k)$. Therefore, we will equivalently target the computation of the KL projection of the normalized Gibbs kernel $\xi \circ \otimes_{k=1}^n a_k$.

In order to compute such projection, in the following, we will represent the set $\Pi$ as an intersection of simpler affine sets and then we will use the alternating KL projection algorithm. More precisely, we will rewrite the set (3.2) as an intersection of affine sets, obtained via specific sketching, on which KL projections have closed forms.

We consider sketches that cover several popular algorithms (multimarginal Sinkhorn of [4], greedy MultiSinkhorn of [17], bi-marginal Greenkhorn of [3]). To that purpose, for each $k \in [m]$ (which refers to the $k$-th marginal) and each batch $L \subset [n_k]$ we consider the canonical injection

$$S_{(k,L)}: \mathbb{R}^L \to \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$$

of $\mathbb{R}^L$ into $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$ meaning that for each $u = (u_{j_k})_{j_k \in L} \in \mathbb{R}^L$, $S_{(k,L)} u$ is the vector of $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$ obtained from the completion of $u$ with zero entries. Then, since the adjoint operator $S_{(k,L)}^*: \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} \to \mathbb{R}^L$ is the standard projection, we can define

$$R_{(k,L)} := S_{(k,L)}^* R: X \to \mathbb{R}^L,$$

and the set

$$\Pi_{(k,L)} := \{ \pi \in X \mid S_{(k,L)}^* R(\pi) = S_{(k,L)}^* (a_1, \ldots, a_m) \} = \{ \pi \in X \mid R_{(k,L)}(\pi) = a_k_{|L} \} = \{ \pi \in X \mid (R_k(\pi))_{|L} = a_k_{|L} \}. $$

(3.4)

Note that in the definition of $\Pi_{(k,L)}$ we ask for the $k$-th marginal of $\pi$ to be equal to $a_k$ only on the components in $L$. Now, given $\tau = (\tau_k)_{1 \leq k \leq m}$, a vector of batch sizes, we set

$$\mathcal{I}(\tau) = \{(k,L) \mid k \in [m], L \subset [n_k] \mid |L| \leq \tau_k \}$$

and obtain $\Pi = \bigcap_{(k,L) \in \mathcal{I}(\tau)} \Pi_{(k,L)}$.

Hence, the iterative Bregman projections (IBP) algorithm, for problem (2.6), leads to the following procedure. Let $\pi^0 = \xi \otimes \otimes_{k=1}^n a_k = e^{-C/\eta} \otimes \otimes_{k=1}^n a_k \in \text{int}(\text{dom} \ H) = X_+$ and define the sequence $\pi^t$ recursively as follows

$$\begin{align*}
\text{for } t = 0, 1, \ldots & \\
\text{choose } (k_t, L_t) \in \mathcal{I}(\tau), & \\
\pi^{t+1} = \mathcal{P}_{\Pi_{(k_t, L_t)}}(\pi^t). &
\end{align*}$$

(3.5)
Since the generalized Pythagoras’s theorem for Bregman projections (see equation (A.10) in Appendix A) yields that $KL_{\Pi}(\pi^t) = KL_{\Pi(k,L)}(\pi^t) + KL_{\Pi}(\mathcal{P}_{\Pi(k,L)}(\pi^t))$, in (3.5) one may choose the sets in a greedy manner as

$$(k_t, L_t) = \arg \max_{(k,L) \in \mathcal{I}(\tau)} KL_{\Pi(k,L)}(\pi^t),$$

so that

$$(k_t, L_t) = \arg \min_{(k,L) \in \mathcal{I}(\tau)} KL_{\Pi}(\mathcal{P}_{\Pi(k,L)}(\pi^t)) \text{ and } KL_{\Pi}(\pi^{t+1}) = \min_{(k,L) \in \mathcal{I}(\tau)} KL_{\Pi}(\mathcal{P}_{\Pi(k,L)}(\pi^t)).$$

This means that the next iterate is chosen, among the possible projections, as the one which is the closest to the target set $\Pi$. Notable examples of existing algorithms that fit in this framework are greedy multimarginal Sinkhorn of [17] ($\tau_k = n_k$) and bi-marginal Greenkhorn of [3] ($m = 2$, $\tau_1 = \tau_2 = 1$). We emphasize that this greedy strategy typically leads to the best performance, provided that it can be implemented efficiently. In the following proposition and subsequent remark we show that the projection onto the sets $\Pi(k,L)$ can be computed in a closed form and that the greedy choice of the sets $\Pi(k,L)$’s can indeed be implemented efficiently. The proof is postponed in Appendix B.

**Proposition 3.1.** For every $\pi \in \mathcal{X}_+$, $k \in [m]$ and $L \subset [n_k]$,

$$\mathcal{P}_{\Pi(k,L)}(\pi) = \nabla H^*(\nabla H(\pi) + R_{k,L}^*(\bar{u})) = \pi \circ \exp \left(R_{k,L}^*(\bar{u})\right),$$

where

$$\bar{u} = \arg \min_{u \in \mathbb{R}^L} H^*(\nabla H(\pi) + R_{k,L}^*(\bar{u})) - H^*(\nabla H(\pi)) - \langle a_{k|L}, u \rangle = \log \frac{a_{k|L}}{R_k(\pi)|L},$$

and, consequently, for every $j \in \mathcal{J}$,

$$(\mathcal{P}_{\Pi(k,L)}(\pi))_j = \pi_j \times \begin{cases} \frac{a_{k,j_k}}{R_k(\pi)_{j_k}} & \text{if } j_k \in L, \\ 1 & \text{otherwise}. \end{cases}$$

Moreover,

$$KL_{\Pi(k,L)}(\pi) = KL(\mathcal{P}_{\Pi(k,L)}(\pi), \pi) = KL(a_{k|L}, R_k(\pi)|L).$$

**Remark 3.2.** It follows from (3.11) that the greedy choice described above can be implemented by computing $m$ vectors of sizes $n_k$ and then choosing $k_t$ among $m$ as the index of the vector that has the maximal sum of the largest $\tau_k$ components. More formally one let $d_k = (KL(a_{k,1}, (R_k(\pi^t)))_1, \ldots, KL(a_{k,n_k}, (R_k(\pi^t)))_{n_k}) \in \mathbb{R}^{n_k}$ and consider the vector $d_{k,t} \in \mathbb{R}^{n_k}$ which has the components of $d_k$ arranged in a decreasing order. Then $k_t = \arg \max_{k \in [m]} \left(\sum_{j_k=1}^{\tau_k} (d_{k,t})_{j_k}\right)$ and $L_t$ corresponds to the indexes of the largest $\tau_k$ components of $d_{k,t}$.

To ensure better numerical stability, especially for small regularization parameters $\eta$, it is more convenient to work with the dual variables $\nabla H(\pi) = \log(\pi)$. More precisely, according to (3.8) we have that each iteration of the IBP algorithm (3.5) can be parameterized as

$$\pi^t = \exp \left(-C/\eta + \bigoplus_{k=1}^{m} v_k^t\right) \odot \bigotimes_{k=1}^{m} a_k, \ t \in \mathbb{N},$$

and one can implement the algorithm by updating only the dual variables $v_k^t = (v_k^t)_{1 \leq j \leq n_k} \in \mathbb{R}^{n_k}$, $k \in [m]$, also known as potentials (see Proposition B.1). Thus, in the end, the IBP algorithm (3.5)-(3.6) can be written as the following Algorithm 1.
Algorithm 1: BatchGreenkhorn($a_1, \ldots, a_m, C, \eta, \tau$)

Input: $(a_1, \ldots, a_m)$, $C \in X_+ = \eta > 0$, $\tau = (\tau_1, \ldots, \tau_m)$, $1 \leq \tau_k \leq n_k$

1 Initialisation: $t = 0$, $v_k^t = 0$, $k \in [m]$, $r_k^t = R_k(\exp(-C/\eta + \bigoplus_{k=1}^m v_k^t) \odot \bigotimes_{k=1}^m a_k)$

2 for $t = 0, 1, \ldots$

3 Compute $(k_t, L_t) = \arg \max_{(k,L) \in I(\tau)} KL(a_k|L, r_k^t)$

4 Set $v_{k_t}^{t+1} = v_{k_t}^t$, $k \in [m]$ and update $\nu_{k_t}^{t+1} = v_{k_t}^{t+1} + \log(a_{k_t}|L_{k_t}) - \log(r_{k_t}|L_{k_t})$

5 For $k \in [m]$ compute $r_k^{t+1} = R_k(\exp(-C/\eta + \bigoplus_{k=1}^m v_{k_t}^{t+1}) \odot \bigotimes_{k=1}^m a_k)$

6 end

Output: $\pi^t = \exp(-C/\eta + \bigoplus_{k=1}^m v_k^t) \odot \bigotimes_{k=1}^m a_k$

4 Convergence theory for Batch Greenkhorn algorithm

Results on the convergence of general IBP are typically without any rates [6, 9, 10], with the notable exception of [16] where the explicit local linear rate for the greedy IBP was derived. In the following we prove global linear convergence and derive the explicit dependence of the rate on the given data in two important cases. Moreover, we also provide an analysis of the iteration complexity of Algorithm 1.

Based on the properties of operators $R$ and $R_{(k,L)}$ we can derive the main results using the properties of KL as Bregman divergence. The proofs are given in Appendix C.

Theorem 4.1. Algorithm 1 converges linearly. More precisely, if $(v_k^t)_{k \in [m]}$ are generated by Algorithm 1, then the primal iterates given by (3.12) converge linearly in KL divergence to $\pi^*$ given by (2.6), i.e.,

$$\forall t \in \mathbb{N}, \quad KL(\pi^*, \nu^t) \leq \left(1 - \frac{e^{-2\|C\|_{\infty}/\eta + 3M_1}}{b_r - 1}\right)^t KL(\pi^*, \pi^0),$$

(4.1)

where $b_r = \sum_{k \in [m]} [n_k/\tau_k]$, and $0 < M_1 < +\infty$ is a constant independent of the batch sizes that satisfies $\|\bigoplus_{k=1}^m v_k^t\|_{\infty}, \|\bigoplus_{k=1}^m v_k^{t+1}\|_{\infty} \leq M_1$ for $t \in \mathbb{N}$.

Theorem 4.2. Let $\varepsilon > 0$ and suppose that $\eta > \varepsilon$. For Algorithm 1, the number of iterations required to reach the stopping criterion $d_t := \max_{k \in [m]} \|v_k^t - R_k(\pi^t)\|_1 \leq \varepsilon$ satisfies

$$t \leq 2 + \max_{k \in [m]} \frac{n_k}{\tau_k} \frac{5M_2}{\varepsilon} (2 + M_2 \eta),$$

(4.2)

where $0 < M_2 < +\infty$ is a constant independent of the batch sizes such that $\sum_{k \in [m]} \|v_k^* - v_k^t\| \leq M_2$, for all $t \in \mathbb{N}$.

Remark 4.3. We stress that the constants $M_1$ and $M_2$ considered in the above theorems always exist (see the proofs in the Appendix C), but we could not have obtained general explicit expression for them depending on the problem data, which are valid for any $m$ and $(\tau_k)_{k \in [m]}$. On the other hand, in the following, we show the important cases $m = 2$ and $(\tau_k)_{k \in [m]}$ arbitrary and $m > 2$ and $(\tau_k)_{k \in [m]} = (n_k)_{k \in [m]}$ for which we do provide explicit dependence on the problem data.

A natural issue for the BatchGreenkhorn algorithm is that of optimizing the batch size. As extreme cases we have full batch $(\tau_k = n_k)$, which yields the (greedy) MultiSinkhorn algorithm proposed in [17], and $\tau_k = 1$, which, in the bi-marginal case, is known as the Greenkhorn algorithm.
In this respect we observe that, since step 3 in Algorithm 1 can be efficiently implemented, as discussed in Remark 3.2, the largest computational cost lies in the computation of the marginals \( r_{k+1} \) in step 5. For simplicity, let us assume that \( n_k = n \) and \( \tau_k = \tau \), for every \( k \in [m] \). Then, computing it naively yields \( O(mn^m) \) operations, but indeed it can be done more efficiently in \( O(\tau n^{m-1}) \) as we show in Appendix B. This way \( n/\tau \) iterations with batch size \( \tau \) have the same computational cost of one iteration with a full batch \( n \) and consequently \( b_\tau = mn/\tau \) iterations of \( \text{BatchGreenkhorn} \) with batch sizes \( \tau \) corresponds to one cycle of cyclic multimarginal Sinkhorn. Hence, the number of normalized cycles \( T = t/b_\tau \) required to satisfy the stopping criterion given in Theorem 4.2 is

\[
T \leq 1 + \frac{5M_2}{m\varepsilon}(2 + M_2),
\]

which we stress is independent on the batch-sizes and the dimension \( n \). Whereas, the rate (4.1) w.r.t the normalized cycles becomes

\[
\text{KL}(\pi^*, \pi_{b_\tau T}) \leq \left(1 - e^{-2|C|_\infty/\eta + 3M_1}/b_\tau - 1\right) b_\tau^T \text{KL}(\pi^*, \pi^0).
\]

We now introduce the analysis of the special cases discussed in Remark 4.3. The proofs are based on bounding the potentials and are detailed in Appendix C.

**Theorem 4.4.** If \( m = 2 \), then algorithm \( \text{BatchGreenkhorn}(a_1, \ldots, a_m, C, \eta, \tau) \) converges linearly with the global rate

\[
(\forall t \in \mathbb{N}) \quad \text{KL}(\pi^*, \pi^t) \leq \left(1 - e^{-20|C|_\infty/\eta}/b_\tau - 1\right) t \text{KL}(\pi^*, \pi^0).
\]

Moreover, when \( \eta > \varepsilon \), the number of iterations required to reach the stopping criterion \( d_t \leq \varepsilon \) satisfies

\[
t \leq 1 + 8\left(4m - 3\right)|C|_\infty \frac{\eta}{\varepsilon}.
\]

The following theorem provides new insights into a known algorithm.

**Theorem 4.5.** If for all \( k \in [m] \) \( \tau_k = n_k \), then algorithm \( \text{BatchGreenkhorn}(a_1, \ldots, a_m, C, \eta, \tau) \), i.e. MultiSinkhorn algorithm of [17], converges linearly with the global rate

\[
(\forall t \in \mathbb{N}) \quad \text{KL}(\pi^*, \pi^t) \leq \left(1 - e^{-12m-7}|C|_\infty/\eta}/m - 1\right) t \text{KL}(\pi^*, \pi^0).
\]

Moreover, the number of iterations required to reach stopping criterion \( d_t \leq \varepsilon \) satisfies

\[
t \leq 1 + \frac{8(4m - 3)|C|_\infty}{\eta\varepsilon}.
\]

**Remark 4.6.** Concerning the linear convergence rate, we notice that when the batch is full (\( \tau_k = n_k \), \( k \in [m] \)), cyclic Sinkhorn of [4, 7] and (greedy) MultiSinkhorn algorithm of [17] generally differ (unless \( m = 2 \)). Moreover, our results shows (for the first time) that the rate of convergence of
(greedy) MultiSinkhorn algorithm is strictly better than that of the cyclic Sinkhorn algorithm obtained in [7]. Indeed in [7], the following rate was provided

$$\forall T \in \mathbb{N} \quad KL(\pi^*, \pi^{mT}) \leq \left(1 - \frac{e^{-8(2m-1)||C||_{\infty}/\eta}}{m}\right)^T KL(\pi^*, \pi^0),$$

(4.9)

where $T$ counts the number of cycles, each one consisting of $m$ KL projections on the given marginals. Whereas, it follows from our rate (4.7) that for the (greedy) MultiSinkhorn algorithm we have

$$\forall T \in \mathbb{N} \quad KL(\pi^*, \pi^{mT}) \leq \left(1 - \frac{e^{-8(2m-7)||C||_{\infty}/\eta}}{m-1}\right)^m T KL(\pi^*, \pi^0),$$

(4.10)

which mainly gains an $m$-root in the rate. Note that in the bi-marginal case, the rate of the classical Sinkhorn algorithm according to [7] is $1 - \frac{1}{2}e^{-24||C||_{\infty}/\eta}$, while we obtain $1 - e^{-17||C||_{\infty}/\eta}^2$.

**Remark 4.7.** Concerning iteration complexity, we note that in literature the stopping criteria concerns $\sum_{k \in [m]} a_k - R_k(\pi^T)\|_1$ and the assumptions usually demand $n_k = n$ and $\tau_k = \tau$, for every $k \in [m]$. In this setting, Theorems 4.4 and 4.5 provide the following bounds (in terms of normalized cycles)

$$T \leq 1 + \frac{15||C||_{\infty}(2 + 3||C||_{\infty})}{\eta \varepsilon} \quad \text{and} \quad T \leq 1 + \frac{8(4m - 3)||C||_{\infty}}{\eta \varepsilon}.$$

(4.11)

respectively. Those bounds improve existing results from [17], related to (greedy) MultiSinkhorn, and [18], for bi-marginal Sinkhorn and Greenkhorn. Indeed, defining $a_{\min} = \min_{k \in [m], j \in [n]} a_{k,j}$, in [13, 17], the bi-marginal Sinkhorn and Greenkhorn algorithm are shown to feature the following iteration complexity

$$T \leq 1 + \frac{2(||C||_{\infty}/\eta + \log(a_{\min}^{-1}))}{\varepsilon} \quad \text{and} \quad T \leq 1 + \frac{56(||C||_{\infty}/\eta + \log(n) + 2\log(a_{\min}^{-1}))}{\varepsilon}$$

whereas, in [18], for the MultiSinkhorn algorithm, the following bound is provided

$$T \leq 1 + \frac{2m(||C||_{\infty}/\eta + \log(1/a_{\min}))}{\varepsilon}.$$

(4.12)

All those bounds contain the term $\log(a_{\min}^{-1})$ which at the best is $\log(n)$. So they all depend on the dimension of the problem. Our results remove this dependency.

**Remark 4.8.** Here we look at the computational complexity in terms of arithmetic operations. To this purpose we let $O_\xi$ be the number of arithmetic operations needed to compute a full marginal using the Gibbs kernel $\xi$. When we factor this number out of the total computational complexity, what remains is the number of iterations normalized with respect to the full batch, meaning, $\tilde{t} = \tau t/n$. Then according to equation (4.3) the total computational complexity is given by

$$\left[m + \frac{5M_2}{\varepsilon}(2 + M_2)\right] O_\xi.$$

(4.13)

We note that in the worst case $O_\xi$ is of the order of $n^m$, but this cost can be significantly reduced for structured costs [21].
5 Conclusions, limitations and future work

We present a new algorithm for solving multimarginal entropic regularized optimal transport problems, called batch Greenkhorn, which is an extension of the popular Greenkhorn algorithm, that can handle multiple marginals and at each iteration select in a greedy fashion a batch of components of a marginal. We study the convergence of the algorithm in the framework of the iterative Bregman projections method, providing novel linear rate of convergence as well as iteration complexity bounds. We made a comprehensive comparison with existing results showing the improvements over the state-of-the-art. A problem which remains open is that of deriving bounds on the dual variables with an explicit dependence on the given problem data for \( m \geq 3 \) when the batch is not full. According to our general Theorems 4.1 and 4.2, this will allow to have explicit linear rate and iteration complexity in all possible cases. Additional research directions are the extension of our analysis to infinite dimensions and general convex regularizers, implementing batch Greenkhorn with structured costs, and analyze the impact of parallel computations.

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Appendices

This supplementary material is organized as follows:

- In Appendix A we provide some basic facts on Bregman divergences.
- Appendix B contains the proofs of the results stated in Section 3, notably Proposition 3.1, and gives more information on the implementation of Algorithm 1.
- Finally, in Appendix C we provide the proof of the main results of Sec. 4, concerning the linear convergence and iteration complexity of Algorithm 1.

For the reader’s convenience all results presented in the main body of the paper are restated in this supplementary material.

A Bregman divergences and Bregman projections

In this section we recall few facts on Bregman distance and Bregman projections onto affine sets. In the following $X$ is an Euclidean space and $\phi: X \to [-\infty, +\infty]$ is an extended-real valued function. The set of minimizers of the function $\phi$ is denoted by $\{x \in X \mid \phi(x) < +\infty\}$ and $\phi$ is proper when $\text{dom} \phi \neq \emptyset$. The function $\phi$ is convex if $\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$ for all $x, y \in \text{dom} \phi$ and $t \in [0,1]$. If the above inequality is strict when $0 < t < 1$ and $x \neq y$, the function is strictly convex. The function $\phi$ is closed if the sublevel sets $\{x \in X \mid \phi(x) \leq t\}$ are closed in $X$ for any $t \in \mathbb{R}$. For a convex function $\phi: X \to [-\infty, +\infty]$, we denote by $\phi^*$ its Fenchel conjugate, that is, $\phi^*: X \to [-\infty, +\infty]$, $\phi^*(y) := \sup_{x \in X} \{\langle x, y \rangle - \phi(x)\}$. The conjugate of a convex function is always closed and convex, and if $\phi$ is proper closed and convex, then $(\phi^*)^* = \phi$.

A proper closed and convex function $\phi$ is essentially smooth if it is differentiable on $\text{int}(\text{dom} \phi) \neq \emptyset$, and $\|\nabla \phi(x_n)\| \to +\infty$ whenever $x_n \in \text{int}(\text{dom} \phi)$ and $x_n \to x \in \text{bdry}(\text{dom} \phi)$. The function $\phi$ is essentially strictly convex if $\text{int}(\text{dom} \phi^*) \neq \emptyset$ and is strictly convex on every convex subset of $\text{dom} \phi^*$. A Legendre function is a proper closed and convex function which is also essentially smooth and essentially strictly convex. A function is Legendre if and only if its conjugate is so. Moreover, if $\phi$ is a Legendre function, then $\nabla \phi: \text{int}(\text{dom} \phi) \to \text{int}(\text{dom} \phi^*)$ and $\nabla \phi^* : \text{int}(\text{dom} \phi^*) \to \text{int}(\text{dom} \phi)$ are bijective, inverses of each other, and continuous. Given a Legendre function $\phi$, the Bregman distance associated to $\phi$ is the function $D_\phi: X \times X \to [0, +\infty]$ such that

$$D_\phi(x, y) = \begin{cases} \phi(x) - \phi(y) - \langle x - y, \nabla \phi(y) \rangle & \text{if } y \in \text{int}(\text{dom} \phi) \\ +\infty & \text{otherwise}. \end{cases} \tag{A.1}$$

**Fact A.1.** Let $\phi$ be a Legendre function on $X$. Then the following hold

(i) $(\forall \pi, \gamma \in \text{dom} \phi) \quad D_\phi(\pi, \gamma) + D_\phi(\gamma, \pi) = \langle \pi - \gamma, \nabla \phi(\pi) - \nabla \phi(\gamma) \rangle$.

(ii) $(\forall \pi, \gamma \in \text{dom} \phi) \quad D_\phi(\pi, \gamma) = D_{\phi^*}(\nabla \phi(\gamma), \nabla \phi(\pi))$.

(iii) If $\phi$ is twice differentiable, then

$$D_\phi(\pi, \gamma) = \frac{1}{2} \langle \nabla^2 \phi(\xi)(\pi - \gamma), \pi - \gamma \rangle,$$

where $[\pi, \gamma] = \{(1-\alpha)\pi + \alpha\gamma \mid \alpha \in [0,1]\}$ is the segment with end points $\pi$ and $\gamma$. 

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(iv) Suppose that $\phi$ is twice differentiable on $\text{int}(\text{dom } \phi)$. Then
\[(\forall \pi \in \text{int}(\text{dom } \phi), \nabla^2 \phi(\pi) \text{ is invertible}) \Leftrightarrow (\phi^* \text{ is twice differentiable}). \tag{A.2}\]

**Fact A.2.** Let $\phi$ be a Legendre function on $X$ and $\phi^*$ be its Fenchel conjugate. If $\text{dom } \phi^*$ is open, then the following hold

(i) For every $\pi \in \text{int}(\text{dom } \phi)$, the sublevel sets of $D_\phi(\pi, \cdot)$ are compact.

(ii) For every $\pi \in \text{int}(\text{dom } \phi)$, and every sequence $(\gamma_k)_{k \in \mathbb{N}}$ in $\text{int}(\text{dom } \phi)$
\[D_\phi(\pi, \gamma_k) \to 0 \Rightarrow \gamma_k \to \pi. \tag{A.3}\]

Let $C \subset X$ be an affine set, represented as follows
\[A: \ X \to Y, \quad b \in \text{Im}(A), \quad C := \{\pi \in X | A\pi = b\}, \tag{A.4}\]
for some linear operator $A$ between $X$ and another Euclidean space $Y$. Given a Legendre function $\phi: \ X \to ]-\infty, +\infty]$ and $\pi \in \text{int}(\text{dom } \phi)$, the Bregman projection of $\pi$ onto $C$ is defined as the unique solution, denoted by $P_C^\phi(\pi)$, of the optimization problem
\[\min_{\gamma \in C} D_\phi(\gamma, \pi) = \min_{\gamma \in C} \phi(\gamma) - \phi(\pi) - \langle \gamma - \pi, \nabla \phi(\pi) \rangle \tag{A.5}\]
and the optimal value defines the Bregman distance from $\pi$ to $C$ and is denoted by $D_C^\phi(\pi)$. The dual of the above problem is
\[\min_{\lambda \in Y} \phi^*(\nabla \phi(\pi) + A^* \lambda) - \phi^*(\nabla \phi(\pi)) - \langle b, \lambda \rangle \tag{A.6}\]
and strong duality holds, meaning that
\[\min_{\gamma \in C} D_\phi(\gamma, \pi) = -\min_{\lambda \in Y} [\phi^*(\nabla \phi(\pi) + A^* \lambda) - \phi^*(\nabla \phi(\pi)) - \langle b, \lambda \rangle]. \tag{A.7}\]

Moreover, the following KKT conditions hold for a couple $(\pi^*, \lambda^*)$ solving the primal and dual problem above
\[\pi^* \in \text{int}(\text{dom } \phi), \quad A\pi^* = b \quad \text{and} \quad \nabla \phi(\pi) + A^* \lambda^* = \nabla \phi(\pi^*). \tag{A.8}\]

Note that the KKT conditions characterizes the projection, so that
\[\pi^* = P_C^\phi(\pi) \Leftrightarrow (\pi^* \in \text{int}(\text{dom } \phi), \ A\pi^* = b, \ \text{and} \ \nabla \phi(\pi) - \nabla \phi(\pi^*) \in \text{Im}(A^*)). \tag{A.9}\]

Finally we mention the generalized Pythagora's theorem. If $C_1$ is an affine set such that $C \subset C_1$, then, for every $\pi \in \text{int}(\text{dom } \phi)$ it holds
\[D_C^\phi(\pi) = D_{C_1}^\phi(\pi) + D_C^\phi(P_{C_1}^\phi(\pi)). \tag{A.10}\]

Moreover, in this case $P_C^\phi(\pi) = P_C^\phi(P_{C_1}^\phi(\pi)))$, and
\[(\forall \gamma \in \text{int}(\text{dom } \phi)) \ \nabla \phi(\gamma) - \nabla \phi(\pi) \in \text{Im}(A^*) \iff P_C(\gamma) = P_C(\pi). \tag{A.11}\]
In the following we let \( \phi: \mathbb{R} \to ]-\infty, +\infty[ \) be the (negative) Boltzmann-Shannon entropy, that is, 
\[
\phi(t) = \begin{cases} 
  t \log t - t & \text{if } t > 0 \\
  0 & \text{if } t = 0 \\
  +\infty & \text{if } t < 0.
\end{cases}
\]

It is clear that \( \phi^*(s) = \exp(s) \). We define the Bregman distance associated to \( \phi \)
\[
D_\phi(s, t) = \begin{cases} 
  \phi(s) - \phi(t) - \phi'(t)(s - t) & \text{if } t > 0 \\
  +\infty & \text{otherwise},
\end{cases}
\]
which is nothing but the Kullback-Leibler divergence on \( \mathbb{R} \).

**Proposition A.3.** Let \( M > 0 \). The following hold.

(i) The function \( \phi \) is strongly convex on the interval \( [0, M] \) with modulus of strong convexity equal to \( 1/M \). Moreover, for every \( a > 0 \) and \( s, t \in \mathbb{R}_++ \), \( D_\phi(s, t) = aD_\phi(s/a, t/a) \).

(ii) The function \( \phi^* \) is strongly convex on the interval \( [-M, +\infty[ \) with modulus of strong convexity equal to \( \exp(-M) \).

**Proof.** (i): It follows from the fact that the second derivative of \( \phi \) is \( \phi''(t) = 1/t \), which is bounded from below away from zero on the interval \( [0, M] \) by the constant \( 1/M \). The second part follows directly from the definition (A.12).

(ii): It follows from the fact that the second derivative of \( \exp \) is bounded from below away from zero on the interval \( [-M, +\infty[ \) by \( \exp(-M) \). \( \square \)

The negative entropy and the Kullback-Leibler divergence on \( \mathbb{X} \) are
\[
H(\gamma) = \sum_j \phi(\gamma_j) \quad \text{and} \quad \text{KL}(\gamma, \pi) = \sum_j D_\phi(\gamma_j, \pi_j). \tag{A.13}
\]

**Lemma A.4.** Let \( \pi, \gamma, \alpha \in \mathbb{X}_++ \) and suppose that
\[
0 < M_1 \leq \min_j \frac{\min\{\pi_j, \gamma_j\}}{\alpha_j} \leq \max_j \frac{\max\{\pi_j, \gamma_j\}}{\alpha_j} \leq M_2.
\]
Then, setting \( A = \alpha \odot (\cdot): \mathbb{X} \to \mathbb{X} \) (which is a positive diagonal operator), we have
\[
\text{KL}(\pi, \gamma) \geq \max \left\{ \frac{M_1}{2} \|\log \pi - \log \gamma\|_A^2, \frac{1}{2M_2} \|\pi - \gamma\|_{A^{-1}}^2 \right\}. \tag{A.14}
\]

**Proof.** It follows from Proposition A.3(i) that, for \( a > 0 \) and \( s, t > 0 \) such that \( s/a, t/a \leq M \), we have \( D_\phi(s, t) = aD_\phi(s/a, t/a) \geq a(2M)^{-1}|s/a-t/a|^2 = (2M)^{-1}a^{-1}|s-t|^2 \). Thus, since \( \gamma_j/\alpha_j, \pi_j/\alpha_j \leq M_2 \), we have
\[
\text{KL}(\pi, \gamma) = \sum_j D_\phi(\pi_j, \gamma_j) \geq \frac{1}{2M_2} \sum_j \frac{1}{\alpha_j} |\pi_j - \gamma_j|^2 = \frac{1}{2M_2} \|\pi - \gamma\|_{A^{-1}}^2.
\]
Now, it follows from Proposition \textbf{A.3(ii)} that for every $a, s, t > 0$ such that $s/a, t/a \geq e^{-M}$ we have $\log(s/a), \log(t/a) \geq -M$ and hence $D_\phi(s, t) = aD_\phi(s/a, t/a) = aD_\phi(\log(s/a), \log(t/a)) \geq a(e^{-M}/2)\log s - \log t|^{2}$. Therefore, since $\pi_j/\alpha_j, \gamma_j/\alpha_j \geq M_1$, we have

\[ KL(\pi, \gamma) = \sum_j D_\phi(\pi_j, \gamma_j) \geq \frac{M_1}{2} \sum_j \alpha_j |\log \pi_j - \log \gamma_j|^2 = \frac{M_1}{2} \|\log \pi - \log \gamma\|_A^2. \]

\[ \Box \]

\section{BatchGreenkhorn algorithm and its implementation}

Here we provide proofs of the results in Sec. 3.

\textbf{Proposition 3.1.} For every $\pi \in \mathcal{X}_+$, $k \in [m]$ and $L \subset [n_k]$, \n
\[ \mathcal{P}_{\Pi_{(k, L)}}(\pi) = \nabla H^* (\nabla H(\pi) + R^*_{(k, L)}(\bar{u})) = \pi \circ \exp \left( R^*_{(k, L)}(\bar{u}) \right), \] \n
where

\[ \bar{u} = \arg \min_{u \in \mathbb{R}^L} H^* (\nabla H(\pi) + R^*_{(k, L)}(u)) - H^* (\nabla H(\pi)) - \langle a_{k|L}, u \rangle = \log \frac{a_{k|L}}{R_{k}(\pi)|_{L}}, \]

and, consequently, for every $j \in \mathcal{J}$,

\[ \left( \mathcal{P}_{\Pi_{(k, L)}}(\pi) \right)_j = \pi_j \times \begin{cases} \frac{a_{k|L}}{R_{k}(\pi)|_{L}} & \text{if } j_k \in L, \\ 1 & \text{otherwise}. \end{cases} \]

Moreover,

\[ KL_{\Pi_{(k, L)}}(\pi) = KL(\mathcal{P}_{\Pi_{(k, L)}}(\pi), \pi) = KL(a_{k|L}, R_{k}(\pi)|_{L}). \]

\textbf{Proof.} It follows from (3.4) that

\[ \Pi_{(k, L)} = \left\{ \pi \in \mathcal{X} \mid R_{(k, L)}(\pi) = a_{k|L} \right\}. \]

Then the first equality in (3.8) follows directly from the KKT conditions (A.8), which in this case yields

\[ R_{(k, L)}(\bar{\pi}) = a_{k|L} \quad \text{and} \quad \bar{\pi} = \nabla H^*(\nabla H(\pi) + R^*_{(k, L)}(\bar{u})), \]

where, according to (A.6), the dual parameter $\bar{u} \in \mathbb{R}^L$ solves the minimization problem in (3.9). Now, since for every $j \in \mathcal{J}$, $(\nabla H(\pi))_j = \log(\pi_j)$ and $(\nabla H^*(\gamma))_j = \exp(\gamma_j)$, then (B.2) gives

\[ \bar{\pi}_j = \exp(\log(\pi_j) + R^*_{(k, L)}(\bar{u}))_j = \pi_j \exp \left( \langle R^*_{(k, L)}(\bar{u}) \rangle_j \right) \]

and the second equality in (3.8) follows.

Now, let $J_{L}^{(k)} : \mathbb{R}^L \rightarrow \mathbb{R}^{n_k}$ be the canonical injection of $\mathbb{R}^L$ into $\mathbb{R}^{n_k}$. Then, recalling the definition of $R_{(k, L)}$ in (3.3), we have $R_{(k, L)} = J_{L}^{(k)*} R_{k}$ and hence $R^*_{(k, L)} = R^*_{k} J_{L}^{(k)}$, where $R^*_{k} : \mathbb{R}^{n_k} \rightarrow \mathcal{X}$ acts as $(R^*_{k} v)_j = v_{j_k}$. Therefore, for every $j \in \mathcal{J}$,

\[ (R^*_{(k, L)} \bar{u})_j = (R^*_{k} J_{L}^{(k)} \bar{u})_j = (J_{L}^{(k)} \bar{u})_{j_k} = \begin{cases} \bar{u}_{j_k} & \text{if } j_k \in L, \\ 0 & \text{otherwise}. \end{cases} \]
Hence,
\[ \bar{\pi}_j = \pi_j \times \begin{cases} e^{\bar{a}_{jk}} & \text{if } j_k \in \mathcal{L} \\ 1 & \text{otherwise.} \end{cases} \] (B.3)

On the other hand, since \( a_{k|\mathcal{L}} = J_{\mathcal{L}}^{(k)} \ast R_k \bar{\pi} \), by (B.3), we derive that, for every \( j_k \in \mathcal{L} \),
\[ a_{k,j_k} = (R_k \bar{\pi})_{j_k} = \sum_{j_{-k} \in J_{-k}} \bar{\pi}_{(j_{-k}, j_k)} = e^{\bar{a}_{jk}} (R_k \pi)_j, \] (B.4)
so that \( e^{\bar{a}_{jk}} = a_{k,j_k} / (R_k \pi)_{j_k} \). Hence now (3.10) follows from (B.3). Concerning the formula for the distance, by (3.10), we have that,
\[ D_{\Pi(k,L)}(\pi) = D^\phi(\bar{\pi}, \pi) \]
\[ = \sum_{j_k \in \mathcal{J}} \bar{\pi}_{j_k} \log \left( \frac{\bar{\pi}_{j_k}}{\pi_{j_k}} \right) - \bar{\pi}_{j_k} + \pi_{j_k} \]
\[ = \sum_{j_k \in \mathcal{L}_{j_k} \in \mathcal{J}_{-k}} \bar{\pi}_{(j_{-k}, j_k)} \log \left( \frac{a_{k,j_k}}{(R_k \pi)_{j_k}} \right) - \bar{\pi}_{(j_{-k}, j_k)} + \pi_{(j_{-k}, j_k)} \]
\[ = \sum_{j_k \in \mathcal{L}} a_{k,j_k} \log \left( \frac{a_{k,j_k}}{(R_k \pi)_{j_k}} \right) - a_{k,j_k} + (R_k \pi)_{j_k} \]
\[ = KL(a_{k|\mathcal{L}}, R_k(\pi)_{|\mathcal{L}}), \]
which completes the proof. \( \square \)

The following proposition justifies equation (3.12) and Algorithm 1.

**Proposition B.1.** Let \((\pi^t)_{t \in \mathbb{N}}\) be defined according to algorithm (3.5). Then, we have
\[ (\forall t \in \mathbb{N}) \quad \pi^t = \exp \left( - \frac{C}{\eta} + \sum_{k=1}^{m} v_{k}^t \right) \odot \bigotimes_{k=1}^{m} a_k, \] (B.5)
where \( v_k^t = (v_{k,j_k})_{1 \leq j_k \leq n_k} \in \mathbb{R}^{n_k} \) and
\[ v_{k}^{t+1} = \delta_{k,k_t} J_{L_t}^{(k_t)} u^t + v_{k}, \quad u^t = \log a_{k|\mathcal{L}_t} - \log(R_{k_t} \pi^t)_{|\mathcal{L}_t} \] (B.6)
\( J_{L_t}^{(k_t)} : \mathbb{R}^{L_t} \rightarrow \mathbb{R}^{n_{k_t}} \) is the canonical injection, \( \delta_{k,k_t} \) is the Kronecker symbol, and \( v_k^0, k \in \{m\} \) are arbitrary.

**Proof.** By definition of \( \pi^{t+1} \) we have
\[ \pi^{t+1} = \Pi_{\Pi(k_t,L_t)}(\pi^t), \quad \Pi_{(k_t,L_t)} = \{ \pi \in \mathbb{X} \mid J_{L_t}^{(k_t)} R_{k_t} \pi = a_{k_t|\mathcal{L}_t} \}. \] (B.7)
Then it follows from Proposition 3.1 that
\[ \nabla H(\pi^{t+1}) = \nabla H(\pi^t) + R_{k_t}^* J_{L_t}^{(k_t)} u^t \] (B.8)
with $u' \in \mathbb{R}^{J_t}$. Therefore, applying the above equation recursively, we get

\[
\nabla H(\pi^t) = \nabla H(\pi^0) + \sum_{s=0}^{t-1} R^s_k J^{(k_s)} u^s
\]

\[
= \nabla H(\pi^0) + \sum_{s=0}^{t-1} \sum_{k=1}^{m} R^s_k \delta_{k,k_s} J^{(k_s)} u^s
\]

\[
= \nabla H(\pi^0) + \sum_{k=1}^{m} \sum_{s=0}^{t-1} R^s_k \delta_{k,k_s} J^{(k_s)} u^s
\]

\[
= \nabla H(\pi^0) + \sum_{k=1}^{m} R^s_k \left( \sum_{s=0}^{t-1} \delta_{k,k_s} J^{(k_s)} u^s \right).
\]

Then if we set $v_k^t = \sum_{s=0}^{t-1} \delta_{k,k_s} J^{(k_s)} u^s \in \mathbb{R}^{m_k}$, we have (recalling that $\pi_0 = e^{-C/\eta} \otimes \prod_{k=1}^{m} a_k$)

\[
\pi_j^t = \exp \left( \log(\pi_j^0) + \sum_{k=1}^{m} (R^s_k v_k^t) \right) = \exp \left( - C_j / \eta + \sum_{k=1}^{m} v_k^t \right) \prod_{k=1}^{m} a_{k,j_k}.
\]

Moreover, it is clear that

\[
v_k^{t+1} = \sum_{s=0}^{t} \delta_{k,k_s} J^{(k_s)} u^s = \delta_{k,k_t} J^{(k_t)} u^t + v_k^t
\]

Finally, it follows from (3.9) that, for every $j_{k_t} \in L_t$, $e^{u_{j_{k_t}}} = a_{k_t,j_{k_t}}/(R_k \pi^t)_{j_{k_t}}$, so that

\[(\forall j_{k_t} \in L_t) \quad u^t_{j_{k_t}} = \log a_{k_t,j_{k_t}} - \log ((R_k \pi^t)_{j_{k_t}}).
\]

The statement follows.

Next we give more detailed implementation of the batch Greenkhorn given in Algorithm 1.

**Remark B.2** (Implementation details on Algorithm 1). **The most delicate part is to avoid recomputing the marginals $R_k(\pi^t) = r_k^t = (r_{k,j_{k_t}})_{j_{k_t} \in [m_k]}$, $k \in [m]$, (step 5) that are necessary for making the greedy choice in step 3. Now, due to equation (3.10) in Proposition 3.1, we have that**

\[
\pi_j^{t+1} = \pi_j^t \times \begin{cases} 
\frac{a_{k_t,j_{k_t}}}{R_k(\pi^t)_{j_{k_t}}} & \text{if } j_{k_t} \in L_t, \\
1 & \text{otherwise}
\end{cases}
\]

and hence

\[
r_{k_t,j_{k_t}}^{t+1} = \begin{cases} 
\frac{a_{k_t,j_{k_t}}}{r_{k_t,j_{k_t}}} & \text{if } j_{k_t} \in L_t, \\
r_{k_t,j_{k_t}}^t & \text{otherwise.}
\end{cases}
\]

**To derive update formula for the other marginals, observe that for all $k \neq k_t$ it follows from (B.5) that**

\[
r_{k,j_h}^{t+1} = \sum_{j_{j_h} \in J_{-k}} \exp \left( \log \pi_{(j_{-k},j_h)}^0 + \sum_{h \neq k} v_{h,j_h}^t + v_{k,j_h}^t \right)
\]

\[
= \sum_{j_{j_h} \in J_{-k}} \exp \left( \log \pi_{(j_{-k},j_h)}^0 + \sum_{h \notin \{k, k_t\}} v_{h,j_h}^t + v_{k,j_h}^t + v_{k_t,j_{k_t}}^{t+1} + v_{k_t,j_{k_t}}^t \right).
\]
So, since according to \((B.6)\), 

\[ v_{kj}^{t+1} = v_{kj}^t \quad \text{if} \quad j_k \notin L_t \quad \text{and} \quad v_{kj}^{t+1} = v_{kj}^t + \log(a_{kj} / r_{kj}^t) \quad \text{if} \quad j_k \in L_t, \]

we have

\[
x_{k,j}^{t+1} = \sum_{j \in \mathcal{J} - j_k \notin L_t} \exp \left( \log \pi^0_{(j-k,j)} + \sum_{h \neq k} v_{h,j}^t + v_{k,j}^t \right)
+ \sum_{j \in \mathcal{J} - j_k \in L_t} \exp \left( \log \pi^0_{(j-k,j)} + \sum_{h \neq k} v_{h,j}^t + v_{k,j}^t \right) \frac{a_{kj}}{r_{kj}^t}
= \sum_{j \in \mathcal{J} - j_k \in L_t} \exp \left( \log \pi^0_{(j-k,j)} + \sum_{h \neq k} v_{h,j}^t + v_{k,j}^t \right) \left( \frac{a_{kj}}{r_{kj}^t} - 1 \right) + r_{k,j}^t.
\]

Therefore, at each iteration \(t \geq 0\) we will construct an auxiliary tensor \(\tilde{\pi}^t \in \mathbb{R}^{n_1 \times \ldots \times n_k_1 - 1 \times \ldots \times n_k_t \times \ldots \times n_m}\) by

\[
\tilde{\pi}_{j_1 \ldots j_{k-1},j_{k+1},...,j_m}^t = \sum_{j_k \in L_t} \exp \left( \log \pi^0_{j_1 \ldots j_{k-1},j_{k+1},...,j_m} + \sum_{k \in [m]} v_{k,j_k}^t \right)
+ \log |a_{k,j_k} - r_{k,j_k}^t| - \log(r_{k,j_k}^t) \sgn(a_{k,j_k} - r_{k,j_k}^t),
\]

in order to obtain that for every \(k \neq k_t\), 

\[ r_{k}^{t+1} = r_{k}^t + R_k(\tilde{\pi}^t). \]

Hence, we can use \(\tilde{\pi}^t\) to efficiently update non-active marginals without recomputing them from scratch. Moreover, note that using \((B.11)-(B.12)\) one avoids excessive numerical errors when \(a_{k,j} \approx r_{k,j}^t\). These observations lead us to the following implementation of \textsc{BatchGreenkhorn}.

**Algorithm 2: BatchGreenkhorn(C, \eta, \rho, \tau, \varepsilon)**

1. **Initialization:** \(t = 0\), \(v_k^0 = 0\), \(k \in [m]\), \(r_k^0 = (r_{k,j}^0)_{j \in [n_k]} = R_k(\exp(-C/\eta) \odot \odot_{k=1}^m \mathbf{a}_k)\)
2. while \(\sum_{k \in [m]} |a_k - r_k^t| > \varepsilon \) do
   3. for \(k \in [m]\) do
      4. Compute vectors \(p_k\) as \(p_{k,j} := \text{KL}(a_{k,j}, r_{k,j}^t)\), for \(j \in [n_k]\)
      5. Take \(L'_k\) to be \(\tau_k\) largest elements of \(p_k\)
   end
5. Choose the marginal with the best batch: \(k_t \leftarrow \arg \max_{k \in [m]} \|p_k|_{L'_k}\|_1\) and \(L_t = L_{k_t}'\)
6. Set \(v_{k_t}^{t+1} = v_{k_t}^t\) and update \(v_{k_t}^{t+1} |_{L_t} \leftarrow v_{k_t}^{t+1} |_{L_t} + \log(a_{k_t}|_{L_t}) - \log(r_{k_t}^t |_{L_t})\)
7. Set \(r_{k_t}^{t+1} = r_{k_t}^t\) and update \(r_{k_t}^{t+1} |_{L_t} = a_{k_t}|_{L_t}\)
8. For \(k \in [m] \setminus \{k_t\}\) update \(r_{k}^{t+1} \leftarrow r_{k}^t + R_k(\tilde{\pi}^t)\), where \(\tilde{\pi}^t\) is given by \((B.11)-(B.12)\)
9. Set \(t \leftarrow t + 1\)
end
Result: \(\{v_k^t\}_{k \in [m]}\)

**Remark B.3.** Let us assume that \(\tau_k = \tau\) and \(n_k = n\) for all \(k \in [m]\) and that \(m << n\). Then, we can conclude that the cost of one iteration of \textsc{BatchGreenkhorn} is essentially determined by step 10 of
Algorithm 2 which is performed in $\mathcal{O}(\tau n^{m-1})$ operations. Hence, one iteration of the MultiSinkhorn (i.e., BatchGreenkhorn with a full batch $\tau = n$) has the same order of computational cost as $n/\tau$ iterations of BatchGreenkhorn with a batch size $\tau$. So, we can introduce the normalized iteration counter as $t = t_\tau n/\tau$, where $t_\tau$ is the iteration counter for the BatchGreenkhorn with a batch size $\tau$.

C Convergence of Batch Greenkhorn algorithm

Here we provide proofs of the main results given in Sec. 4. We first set notation for the rest of the section. Given $k \in [m]$ and $L \subset [n_k]$, we denote by

$$J_k : \mathbb{R}^{n_k} \to \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \times \cdots \times \mathbb{R}^m, \quad v_k \mapsto (0, \ldots, 0, v_k, 0, \ldots, 0),$$

the canonical injection of $\mathbb{R}^{n_k}$ into $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k} \times \cdots \times \mathbb{R}^m$ and by

$$J_L^{(k)} : \mathbb{R}^L \to \mathbb{R}^{n_k}$$

the canonical injection of $\mathbb{R}^L$ into $\mathbb{R}^{n_k}$.

We note that, referring to the operators $R$ and $R_k$ defined in (3.1) and (2.3), respectively, we have

$$R^* : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} \to \mathbb{X}, \quad R^*(v_1, \ldots, v_m) = \sum_{k=1}^m R_k^*(v_k) = \bigotimes_{k=1}^m v_k,$$

where $(R_k^*(v_k))_{j_1, \ldots, j_k \ldots, j_m} = v_{k,j_k}$. Indeed the second equality in (C.3) follows from the fact that for every $j \in J$, $(R^*(v_1, \ldots, v_m))_j = \sum_{k=1}^m (R_k^*(v_k))_j = \sum_{k=1}^m v_{k,j_k} = (\bigoplus_{k=1}^m v_k)_j$. We note also that $R_k = J_L^{(k)} \circ R$, since $J_L^{(k)}$ is the $k$-th canonical projection.

Then we provide a result concerning the properties of optimal potentials.

Lemma C.1. Let $\pi^*$ be the solution of RMOT given by (2.4). Then $\pi^* = \mathcal{P}_{\Pi}(\xi \odot \bigotimes_{k=1}^m a_k)$ and, for every $k \in [m]$, there exist $v_k^* = (v_k^*_{j,k})_{1 \leq j \leq n_k} \in \mathbb{R}^{n_k}$, such that

$$\pi^* = \exp \left( - \frac{C}{\eta} + \bigoplus_{k=1}^m v_k^* \right) \odot \bigotimes_{k=1}^m a_k,$$

and the $v_k^*$'s, can be chosen so that

$$\sum_{k \in [m]} \|v_k^*\|_\infty \leq (4m - 3) \frac{\|C\|_\infty}{\eta}.$$

Moreover, if $m = 2$, then $v_1^*$ and $v_2^*$ can be chosen such that

$$\max_{k \in [m]} \|v_k^*\|_\infty \leq \frac{3}{2} \frac{\|C\|_\infty}{\eta}.$$

Proof. Since, by definition $\pi^* = \mathcal{P}_{\Pi}(\xi)$, it easy to see, from the characterization of the projection given in (A.9), that

$$\pi^* = \mathcal{P}_{\Pi}(\xi \odot \bigotimes_{k=1}^m a_k) \Leftrightarrow \nabla \mathcal{H}(\xi \odot \bigotimes_{k=1}^m a_k) - \nabla \mathcal{H}(\xi) \in \text{Im}(R^*)$$
Thus, since $\nabla H(\xi \otimes \otimes_{k=1}^{m} a_k) - \nabla H(\xi) = \log \otimes_{k=1}^{m} a_k = \oplus_{k=1}^{m} \log a_k = R^*(\log a_1, \ldots, \log a_m) \in \text{Im}(R^*)$, we have that $P_{\Pi}(\xi \otimes \otimes_{k=1}^{m} a_k) = P_{\Pi}(\xi) = \pi^*$. Now, it follows from the KKT conditions (A.8) for the projection of $\xi \otimes \otimes_{k=1}^{m} a_k$ onto affine set $\Pi$, that

$$\pi^* = \nabla H^*(\nabla H(\xi \otimes \otimes_{k=1}^{m} a_k) + R^*(v_1^*, \ldots, v_m^*))$$

for some $(v_1^*, \ldots, v_m^*) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$. Since $\nabla H^* = \exp$ and $\nabla H = \log$, (C.4) follows. Next, observe that for every $k \in [m]$, since $R_k(\pi^*) = a_k$, using (C.4), we obtain that for every $j_k \in [n_k]$,

$$\exp(v_{k,j_k}^*) \sum_{j_{-k} \in J_{-k}} \exp(-C_{j_{-k},j_k}/\eta + \sum_{h \neq k}^m v_{h,j_h}^*) a_{h,j_h} = 1.$$

(C.7)

Hence, the vectors $v_1^*, \ldots, v_m^*$ solve a (discrete) Schrödinger system, and we can apply the results from [7, Lemma 3.1] and [12, Theorem 2.8] to obtain (C.5) and (C.6), respectively.

**Lemma C.2.** Let $A : \mathbb{X} \rightarrow \mathbb{X}$ and $A_k : \mathbb{R}^{n_k} \rightarrow \mathbb{R}^{n_k}$ be the diagonal and positive operators defined as $A(\pi) = \pi \otimes \otimes_{k=1}^{m} a_k$ and $A_k v_k = v_k \otimes a_k$, respectively. Let $(v_1, \ldots, v_m) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$. Then

$$\langle v_k, a_k \rangle = \langle v_k, a_k \rangle = 1$$

Moreover, if $\langle v_k, a_k \rangle = 0$ for every $k = 1, \ldots, m - 1$, then

$$\|R^*_k v_k\|_A^2 = \sum_{k=1}^m \|R^*_k v_k\|_A^2 = \sum_{k=1}^m \|v_k\|_A^2.$$

(C.8)

**Proof.** Let $k \in [m]$. Then, recalling that $(R^*_k(v_k))_j = v_{k,j}$, we have

$$\|R^*_k v_k\|_A^2 = \sum_{j \in J} v_{k,j}^2 \prod_{\ell=1}^m a_{\ell,j}\ell = \sum_{j_{-k} \in J_{-k}} v_{k,j_k}^2 a_{k,j_k} \prod_{\ell \neq k} a_{\ell,j}\ell = \prod_{\ell \neq k} \left( \sum_{\ell=1}^{n_\ell} a_{\ell,j}\ell \right) \sum_{j_{-k} \in J_{-k}} v_{k,j_k}^2 a_{k,j_k}.$$

Since $\sum_{\ell=1}^{n_\ell} a_{\ell,j}\ell = 1$, we get $\|R^*_k v_k\|_A^2 = \|v_k\|_A^2$ and the first part of the statement follows. Concerning the second part, equation (C.9) will follow if we prove that, for every $k, h \in [m]$ with $k \neq h$, we have that $R^*_k(v_k)$ and $R^*_h(v_h)$ are orthogonal in the metric $\langle \cdot, \cdot \rangle_A$. Thus, let $k, h \in [m]$ and suppose (w.l.o.g.) that $k < h$. Then

$$\langle R^*_k(v_k), R^*_h(v_h) \rangle_A = \sum_{j \in J} v_{k,j} v_{h,j} \prod_{\ell=1}^m a_{\ell,j}\ell = \sum_{j_{-k} \in J_{-k}} \sum_{j_{-h} \in J_{-h}} \sum_{j_{k,h} \in J_{k,h}} v_{k,j_k} v_{h,j_h} a_{k,j_k} a_{h,j_h} \prod_{\ell \neq k, \ell \neq h} a_{\ell,j}\ell = \left( \sum_{j_{k,h} \in J_{k,h}} v_{k,j_k} \right) \left( \sum_{j_{h} \in J_{h}} v_{h,j_h} \right) \prod_{\ell \neq k, \ell \neq h} \left( \sum_{\ell=1}^{n_\ell} a_{\ell,j}\ell \right).$$

Since $\sum_{\ell=1}^{n_\ell} a_{\ell,j}\ell = 1$ and for $k < m$, $\sum_{j_{k} \in J_{k}} v_{k,j} a_{k,j_k} = \langle v_k, a_k \rangle = 0$, we have $\langle R^*_k(v_k), R^*_h(v_h) \rangle_A = 0$ and hence the statement follows. \hfill \Box
Theorem 4.1. Algorithm 1 converges linearly. More precisely, if \((y_k^t)_{k \in [m]}\) are generated by Algorithm 1, then the primal iterates given by (3.12) converge linearly in KL divergence to \(\pi^*\) given by (2.6), i.e.,

\[
(\forall t \in \mathbb{N}) \quad KL(\pi^*, \pi^t) \leq \left(1 - \frac{e^{-2\|C\|_\infty/\eta + 3M_1}}{b_T - 1}\right)^t KL(\pi^*, \pi^0),
\]

where \(b_T = \sum_{k \in [m]} \left|\frac{n_k}{\tau_k}\right|\), and \(0 < M_1 < +\infty\) is a constant independent of the batch sizes that satisfies \(\|\bigoplus_{k=1}^m v_k^t\|_\infty \leq M_1\) for \(t \in \mathbb{N}\).

Proof. We start by recalling the two formulas

\[
\pi^t = \exp\left(-\frac{C}{\eta} + V^t\right) \odot \alpha \quad \text{and} \quad \pi^* = \exp\left(-\frac{C}{\eta} + V^*\right) \odot \alpha,
\]

where \(\alpha := \bigotimes_{k=1}^m a_k\), \(V^t := \bigoplus_{k=1}^m v_k^t\), and \(V^* := \bigoplus_{k=1}^m v_k^*\). Moreover, since for every \((\lambda_k)_{k \in \mathbb{N}} \in \mathbb{R}^m\) such that \(\sum_{k=1}^m \lambda_k = 0\), we have \(\bigoplus_{k=1}^m (v_k^t + \lambda_k) = \bigoplus_{k=1}^m v_k^t\) and \(\bigoplus_{k=1}^m (v_k^* + \lambda_k) = \bigoplus_{k=1}^m v_k^*\), we can choose the dual variables \((v_k^t)_{k \in [n_k]}\) and \((v_k^*)_{k \in [n_k]}\) so that

\[
(\forall k = 1, \ldots, m - 1) \quad \langle v_k^t, a_k \rangle = 0 \quad \text{and} \quad \langle v_k^*, a_k \rangle = 0.
\]

First, observe that Pythagoras’ theorem yields that \(KL(\pi^{t+1}) = KL(\pi^t) - D_{KL}(\log \pi^t, \log \pi^* \leq KL(\pi^0) \leq +\infty\). However, since \(H^*\) is a Legendre function, the sublevel sets of \(D_{KL}(\cdot, \log \pi^*)\) are bounded, and hence the sequence \((\log \pi^t)_{t \in \mathbb{N}}\) is bounded in \(X\). Now, since the first of (C.10) yields that \(\log \pi^t = -\frac{C}{\eta} + V^t + \log \alpha\), we have that also the sequence \((V^t)_{t \in \mathbb{N}}\) is bounded in \(X\). Thus let \(M_1 > 0\) be such that

\[
\|V^*\|_\infty, \|V^t\|_\infty \leq M_1 \quad (\forall t \in \mathbb{N}).
\]

Then, recalling (C.10),

\[
\frac{\pi^t}{\alpha} = \exp\left(-\frac{C}{\eta} + V^t\right) \geq \exp\left(-\frac{\|C\|_\infty}{\eta} - \|V^t\|_\infty\right) \geq \exp\left(-\frac{\|C\|_\infty}{\eta} - M_1\right)
\]

and

\[
\frac{\pi^*}{\alpha} = \exp\left(-\frac{C}{\eta} + V^*\right) \geq \exp\left(-\frac{\|C\|_\infty}{\eta} - \|V^*\|_\infty\right) \geq \exp\left(-\frac{\|C\|_\infty}{\eta} - M_1\right)
\]

and hence

\[
\exp\left(-\frac{\|C\|_\infty}{\eta} - M_1\right) \leq \min\left\{\frac{\pi^t}{\alpha}, \frac{\pi^*}{\alpha}\right\} \quad (\forall t \in \mathbb{N}).
\]

Let \(t \in \mathbb{N}, k \in [m]\) and \(L \subset [n_k]\). It follows from (3.10) that

\[
(\forall j \in J) \quad (P_{\Pi([k,n])}(\pi^t)_j) = \pi^t_j \times \left\{\begin{array}{ll}
\frac{a_{k,jk}}{R_k(\pi)_j} & \text{if } j_k \in L, \\
1 & \text{otherwise}
\end{array}\right.
\]

and hence

\[
(\forall j \in J) \quad \frac{(P_{\Pi([k,n])}(\pi^t)_j)}{\alpha_j} \leq \frac{\pi^t_j}{\alpha_j} \max\left\{1, \frac{a_{k,jk}}{R_k(\pi)_j}\right\}.
\]
Now, since $C \geq 0$, we have
\[
\frac{\pi^t}{\alpha} = \exp \left( - \frac{C}{\eta} + V^t \right) \leq \exp(V^t) \leq \exp(\|V\|_{\infty}) \leq \exp(M_1)
\] (C.15)
and
\[
\frac{R_k(\pi)_{jk}}{a_{k,jk}} = \sum_{j-k \in \mathcal{J}_{-k}} \frac{\pi^t_{(j-k,jk)}}{a_{k,jk}} = \sum_{j-k \in \mathcal{J}_{-k}} \exp \left( - \frac{C_{(j-k,jk)}}{\eta} + V^t_{(j-k,jk)} \right) \prod_{h \neq k}^m a_{h,jh}
\]
\[
\geq \exp \left( - \|C\|_{\infty}/\eta - M_1 \right) \sum_{j-k \in \mathcal{J}_{-k}} \prod_{h \neq k}^m a_{h,jh}
\]
\[
= \exp \left( - \|C\|_{\infty}/\eta - M_1 \right) \prod_{h \neq k}^m \left( \sum_{j_h=1}^{n_h} a_{h,jh} \right)
\]
\[
= \exp \left( - \|C\|_{\infty}/\eta - M_1 \right),
\] (C.16)
since $\sum_{j_h=1}^{n_h} a_{h,jh} = 1$. Therefore, by (C.14), (C.15), and (C.16),
\[
\frac{P_{\Pi I(h\cdot l)}(\pi^t)}{\alpha} \leq \exp(M_1) \exp(\|C\|_{\infty}/\eta + M_1) = \exp(\|C\|_{\infty}/\eta + 2M_1)
\]
and hence, recalling (C.15),
\[
\max \left\{ \frac{\pi^t}{\alpha}, \frac{P_{\Pi I(h\cdot l)}(\pi^t)}{\alpha} \right\} \leq \exp \left( \|C\|_{\infty}/\eta + 2M_1 \right).
\] (C.17)

We now prove that
\[
\exp \left( - 2\|C\|_{\infty} - 3M_1 \right) b^{-1}_r KL_{\Pi}(\pi^t) = KL_{\Pi_{(h\cdot l)}}(\pi^t) = \max_{(k,l)\in I(\tau)} KL(P_{\Pi_{(k,l)}}(\pi^t), \pi^t).
\] (C.18)

From this inequality it will follow, using the Pythagora's theorem $KL_{\Pi}(\pi^{t+1}) + KL_{\Pi_{(h\cdot l)}}(\pi^t) = KL_{\Pi}(\pi^t)$, that
\[
\exp(-2\|C\|_{\infty} - 3M_1) b^{-1}_r KL_{\Pi}(\pi^t) = KL_{\Pi}(\pi^t) - KL_{\Pi}(\pi^{t+1})
\] (C.19)
and hence
\[
KL_{\Pi}(\pi^{t+1}) \leq (1 - \exp(-2\|C\|_{\infty} - 3M_1) b^{-1}_r) KL_{\Pi}(\pi^t),
\] (C.20)
which gives the statement. Thus, it remains to prove (C.18). Let $t \in \mathbb{N}$ and, for the sake of brevity set
\[
\pi := \pi^t, \quad \pi^h := P_{\Pi_{(k,h)}}(\pi^t) \quad v_k := v_k^t - v_k^h, \quad \text{and} \quad v_k^h := J_k(v_k^t).
\] (C.21)
Let, for every $k \in [m]$, $(L^h_k)_{1 \leq h \leq s_k}$ be a partition of $[n_k]$ made of non empty sets of cardinality exactly $\tau_k$ possibly except for the last one, such that $L^h_{k-1} = L_{t-1}$, where $s_k = \lfloor n_k/\tau_k \rfloor$. Then, it follows from (C.10) that

$$\frac{\pi^*}{\pi} = \frac{\exp(-C/\eta) \exp(V^*)}{\exp(-C/\eta) \exp(V^t)} = \exp(V^* - V^t).$$

(C.22)

Hence, recalling that $R^*(v_1, \ldots, v_m) = \sum_{k=1}^m R^*_k v_k = \bigoplus_{k=1}^m v_k = \bigoplus_{k=1}^m (v^*_k - v^t_k)$, we have

$$K\!L_\Pi(\pi) + K\!L(\pi, \pi^*) = K\!L(\pi^*, \pi) + K\!L(\pi, \pi^*)$$

$$= \langle \pi^* - \pi, \log(\pi^*/\pi) \rangle$$

$$= \langle \pi^* - \pi, V^* - V^t \rangle$$

$$= \langle \pi^* - \pi, R^*(v_1, \ldots, v_m) \rangle$$

$$= \sum_{k=1}^m \langle \pi^* - \pi, R^*_k(v_k) \rangle.$$  \hspace{1cm} (C.23)

Moreover, recalling that $R^*_{(k,L^h_k)} = J^{(k)*}_{L^h_k} \circ R^*_k$ and $v^*_k = J^{(k)*}_{L^h_k}(v_k)$, we have

$$R^*_k(v_k) = R^*_k \left( \sum_{h=1}^{s_k} J^{(k)*}_{L^h_k} \circ J^{(k)*}_{L^h_k}(v_k) \right) = \sum_{h=1}^{s_k} R^*_{(k,L^h_k)}(v^*_k)$$

(C.24)

and hence

$$K\!L_\Pi(\pi) + K\!L(\pi, \pi^*) = \sum_{k=1}^m \sum_{h=1}^{s_k} \langle \pi^* - \pi, R^*_{(k,L^h_k)}(v^*_k) \rangle.$$  \hspace{1cm} (C.25)

Now, recalling the general definition of $\Pi_{(k,L)}$ in (3.4), since $\pi^h_k$ and $\pi^*$ both belong to $\Pi_{(k,L^h_k)}$, we have that $\pi^* - \pi^h_k \in \text{Ker}(R^*_{(k,L^h_k)}) = \text{Im}(R^*_{(k,L^h_k)})$ and hence

$$\langle \pi^* - \pi, R^*_{(k,L^h_k)}(v^*_k) \rangle = \langle \pi^* - \pi^h_k, R^*_{(k,L^h_k)}(v^*_k) \rangle + \langle \pi^h_k - \pi, R^*_{(k,L^h_k)}(v^*_k) \rangle = \langle \pi^h_k - \pi, R^*_{(k,L^h_k)}(v^*_k) \rangle$$

and hence

$$K\!L_\Pi(\pi) + K\!L(\pi, \pi^*) = \sum_{k=1}^m \sum_{h=1}^{s_k} \langle \pi^h_k - \pi, R^*_{(k,L^h_k)}(v^*_k) \rangle$$

$$= \sum_{k=1}^m \sum_{h=1}^{s_k} \langle A^{-1}(\pi^h_k - \pi), R^*_{(k,L^h_k)}(v^*_k) \rangle A,$$  \hspace{1cm} (C.26)

where $A$ is the positive diagonal operator defined in Lemma C.2. Now, it follows from (C.12), Lemma A.4, and Lemma C.2 that

$$K\!L(\pi^t, \pi^*) \geq (1/2) \exp(-\|C\|_\infty/\eta - M_1) \|\log \pi^* - \log \pi^t\|_A^2$$

$$= (1/2) \exp(-\|C\|_\infty/\eta - M_1) \|V^* - V^t\|_A^2$$

$$= (1/2) \exp(-\|C\|_\infty/\eta - M_1) \|R^*(v_1, \ldots, v_m)\|_A^2$$

$$= (1/2) \exp(-\|C\|_\infty/\eta - M_1) \sum_{k=1}^m \|v_k\|_{A_k}^2.$$
Moreover, recalling the definition of $v_k^h$ in (C.21), since $v_k = \sum_{h=1}^{s_k} J_{(k,L_k^h)} v_k^h$ and $(J_{(k,L_k^h)} v_k^h)_{h \in [s_k]}$ is a finite orthogonal sequence in $\mathbb{R}^{n_k}$ w.r.t. the metric $(\cdot, \cdot)_{A_k}$, we have

$$\|v_k\|_{A_k}^2 = \sum_{h=1}^{s_k} \|J_{(k,L_k^h)} v_k^h\|_{A_k}^2 = \sum_{h=1}^{s_k} \|R_{(k,L_k^h)} v_k^h\|_{A_k}^2 = \sum_{h=1}^{s_k} \|R_{(k,L_k^h)} v_k^h\|_{A_k}^2,$$

where we used equation (C.8) from Lemma C.2 applied to $J_{(k,L_k^h)} v_k^h$ and the fact that, by definition, $R_{(j,L_j^h)} = J_{(j,L_j^h)} R_{L_j^h}$. Overall we get that

$$KL(\pi^t, \pi^*) \geq (1/2) \exp\left(-\|C\|_{\infty}/\eta - M_1\right) \sum_{k=1}^{m} \sum_{h=1}^{s_k} \|R_{(k,L_k^h)} v_k^h\|_{A_k}^2$$

and hence (C.26) yields

$$KL_{\Pi}(\pi) \leq \sum_{k=1}^{m} \sum_{h=1}^{s_k} \langle A^{-1}(\pi_k^h - \pi), R_{(k,L_k^h)}(v_k^h) \rangle_{A} - KL(\pi, \pi^*)$$

$$\leq \sum_{k=1}^{m} \sum_{h=1}^{s_k} \langle A^{-1}(\pi_k^h - \pi), R_{(k,L_k^h)}(v_k^h) \rangle_{A} - \frac{1}{2} \exp\left(-\|C\|_{\infty}/\eta - M_1\right) \|R_{(k,L_k^h)} v_k^h\|_{A_k}^2$$

$$\leq \frac{\exp\left(\|C\|_{\infty}/\eta + M_1\right)}{2} \sum_{k=1}^{m} \sum_{h=1}^{s_k} \|A^{-1}(\pi_k^h - \pi)\|_{A_k}^2$$

$$= \frac{\exp\left(\|C\|_{\infty}/\eta + M_1\right)}{2} \sum_{k=1}^{m} \sum_{h=1}^{s_k} \|\pi_k^h - \pi\|_{A_k}^2,$$

where in the last inequality we used the Young-Fenchel inequality $\langle a, b \rangle_A \leq \frac{\delta}{2} \|a\|_A^2 + \frac{1}{2\delta} \|b\|_A^2$. Now, recalling that we set $\pi = \pi^t$ and $\pi_k^h = \mathcal{P}_{\Pi_{(k,L_k^h)}}(\pi^t)$, it follows from (C.17) and Lemma A.4 that

$$\frac{1}{2} \|\pi_k^h - \pi\|_{A_k}^2 \leq \exp\left(\|C\|_{\infty}/\eta + 2M_1\right) KL(\pi_k^h, \pi),$$

and consequently

$$KL_{\Pi}(\pi^t) \leq \exp\left(2\|C\|_{\infty}/\eta + 3M_1\right) \sum_{k \in [m]} \sum_{h \in [s_k]} KL(\pi_k^h, \pi)$$

$$\leq \exp\left(2\|C\|_{\infty}/\eta + 3M_1\right) \left(\sum_{k \in [m]} s_k - 1\right) \max_{k \in [m]} \max_{h \in [s_k]} KL(\pi_k^h, \pi)$$

$$= \exp\left(2\|C\|_{\infty}/\eta + 3M_1\right) \left(\sum_{k \in [m]} \left[n_k/\tau_k\right] - 1\right) \max_{k \in [m]} \max_{h \in [s_k]} KL(\mathcal{P}_{\Pi_{(k,L_k^h)}}(\pi^t), \pi^t)$$

$$\leq (b_r - 1) \exp\left(2\|C\|_{\infty}/\eta + 3M_1\right) \max_{(k,L) \in I(\tau)} \ KL(\mathcal{P}_{\Pi_{(k,L)}}(\pi^t), \pi^t),$$

where in the second inequality we used that, for $k = k_{l-1}$ and $h = 1$, $\pi_k^h = \mathcal{P}_{\Pi_{(k_{l-1}, L_{l-1})}}(\pi^t) = \pi^t$ (since by definition $\pi^t \in \Pi_{(k_{l-1}, L_{l-1})}$), so that $KL(\pi_k^h, \pi) = 0$. This proves (C.18) and the proof is complete.

We now provide a result concerning the convergence of numerical sequences which is critical to analyze the iteration complexity of the algorithm. This result has been first showed implicitly in [13]. We provide here a more explicit version together with a complete proof for the reader’s convenience.
Lemma C.3. Let $M, C > 0$ and let $(\delta_t)_{t \in \mathbb{N}}$ and $(d_t)_{t \in \mathbb{N}}$ be two sequences of positive numbers such that, for every $t \in \mathbb{N}$,

(i) $\delta_t - \delta_{t+1} \geq \left(\frac{d_t}{C}\right)^2$, 

(ii) $\delta_t \leq Md_t$.

Let $\varepsilon > 0$ and set $\bar{t} = \min \{t \in \mathbb{N} \mid d_t \leq \varepsilon\}$. Then $\bar{t} \leq 1 + 2MC^2/\varepsilon$.

Proof. Items (i) and (ii) imply that

$$\delta_t - \delta_{t+1} \geq \left(\frac{\delta_t}{MC}\right)^2.$$ 

Therefore, since $\delta_t \geq \delta_{t+1}$, we have

$$\delta_t - \delta_{t+1} \geq \frac{\delta_t^2}{M^2C^2} \geq \frac{\delta_t \delta_{t+1}}{M^2C^2}$$

and hence, dividing by $\delta_t \delta_{t+1}$,

$$\frac{1}{\delta_{t+1}} - \frac{1}{\delta_t} \geq \frac{1}{M^2C^2}.$$

Thus,

$$\frac{1}{\delta_t} - \frac{1}{\delta_0} = \sum_{i=0}^{t-1} \left(\frac{1}{\delta_i} - \frac{1}{\delta_{i+1}}\right) \geq \frac{t}{M^2C^2}$$

and hence we get the following rate of convergence for the sequence $(\delta_t)_{t \in \mathbb{N}}$

$$\delta_t \leq \left(\frac{1}{\delta_0} + \frac{t}{M^2C^2}\right)^{-1}. \quad \text{(C.28)}$$

Now, we let $\delta \in [0, \delta_0]$. We wish to determine the number of iterations such that $\delta_t \leq \delta$. It follows from (C.28) that

$$\left(\frac{1}{\delta_0} + \frac{t}{M^2C^2}\right)^{-1} \leq \delta \Leftrightarrow \frac{1}{\delta_0} + \frac{t}{M^2C^2} \geq \frac{1}{\delta} \Leftrightarrow t \geq M^2C^2 \left(\frac{1}{\delta} - \frac{1}{\delta_0}\right). \quad \text{(C.29)}$$

This means that if we take $t \geq M^2C^2(1/\delta - 1/\delta_0)$, we have $\delta_t \leq \delta$ as desired. So we set $t = [M^2C^2(1/\delta - 1/\delta_0)] + 1$. Then, we have $\delta_t \leq \delta$. Now we have to cases. Suppose that $t < \bar{t}$ and let $s \in \mathbb{N}$ be such that $t + s = \bar{t} - 1$. Then, for every $i = 0, \ldots, s$, since $t + i < \bar{t}$, we have $d_{t+i} > \varepsilon$ and hence, using (i),

$$\delta \geq \delta_t - \delta_{t+1} = \sum_{i=0}^{s} \left(\frac{1}{\delta_{t+i}} - \frac{1}{\delta_{t+i+1}}\right) \geq \sum_{i=0}^{s} \frac{d_{t+i}^2}{M^2C^2} \geq (s+1)\frac{\varepsilon^2}{M^2C^2}, \quad \text{(C.30)}$$

which implies that $s + 1 \leq C^2\delta/\varepsilon^2$. Overall we have

$$\bar{t} = t + s + 1 \leq \left[M^2C^2 \left(\frac{1}{\delta} - \frac{1}{\delta_0}\right)\right] + 1 + C^2\frac{\delta}{\varepsilon^2} \leq 1 + \frac{M^2C^2}{\delta} - \frac{M^2C^2}{\delta_0} + \frac{C^2\delta}{\varepsilon^2}. \quad \text{(C.31)}$$
Note that this inequality is true for any $\delta \in [0, \delta_0]$. Now, suppose that $M\varepsilon \leq \delta_0$. Then we have
\[
\bar{t} \leq 1 + \min_{\delta \in [0, \delta_0]} \left( \frac{M^2C^2}{\delta} + \frac{C^2\delta}{\varepsilon^2} \right) = 1 + 2 \frac{MC^2}{\varepsilon},
\] (C.32)
where the minimum is attained at $\delta = M\varepsilon \in [0, \delta_0]$. On the other hand, if $\delta_0 < M\varepsilon$, then the minimum on the right hand side of (C.31) is attained at $\delta = \delta_0$ and hence
\[
\bar{t} \leq 1 + \frac{C^2\delta_0}{\varepsilon^2} \leq 1 + \frac{C^2M\varepsilon}{\varepsilon^2} = 1 + \frac{MC^2}{\varepsilon}.\] (C.33)
In any case, the statement follows.

**Remark C.4.** The statement of Lemma C.3 is equivalent to the fact that the sequence $(\min_{0 \leq s \leq t} d_s)_{t \in \mathbb{N}}$ converge to zero with rate $O(1/t)$, i.e., that for every integer $t > 1$,
\[
\min_{0 \leq s \leq t} d_s \leq \frac{2MC^2}{t-1}.
\]

Next, we prove the main result on the iteration complexity.

**Theorem 4.2.** Let $\varepsilon > 0$ and suppose that $\eta > \varepsilon$. For Algorithm 1, the number of iterations required to reach the stopping criterion $t = \max_{k \in [m]} ||a_k - R_k(\pi^t)||_1 \leq \varepsilon$ satisfies
\[
t \leq 2 + \max_{k \in [m]} \left( \frac{n_k}{\tau_k} \right) \frac{5M_2}{\varepsilon} (2 + M_2\eta),
\] (4.2)
where $0 < M_2 < +\infty$ is a constant independent of the batch sizes such that $\sum_{k \in [m]} ||v^*_k - v^t_k|| \leq M_2$, for all $t \in \mathbb{N}$.

**Proof.** For the sake of brevity let $\bar{b} = \max_{k \in [m]} [n_k/\tau_k]$ and set, for every $t \in \mathbb{N}$, $\delta_t := \text{KL}^t(\pi^t)$. Let $t \in \mathbb{N}$ be arbitrary. Recalling (C.23), we have that
\[
\delta_t = \text{KL}^t(\pi^t) \leq \sum_{k \in [m]} (\pi^* - \pi^t, R^t_k(v^*_k - v^t_k)) = \sum_{k \in [m]} (a_k - R_k(\pi^t), v^*_k - v^t_k),
\]
which, using Holder’s inequality, yields
\[
\delta_t \leq \sum_{k \in [m]} ||a_k - R_k(\pi^t)||_1 ||v^*_k - v^t_k||_{\infty} \leq M_2d_t.
\] (C.34)

Now, we prove
\[
\delta_t - \delta_{t+1} \geq \min \left\{ \frac{d_t^2}{5b}, \frac{\delta_t^2}{4M^2b} \right\} \geq \frac{\delta_t^2}{5M^2b}.
\] (C.35)

Let for every $k \in [m]$, $(L^t_k)_{1 \leq h \leq [s_k]}$, $s_k := [n_k/\tau_k]$, be a partition of $[n_k]$ made of nonempty sets of cardinality exactly $\tau_k$, except maybe for the last one, such that $L^t_k = L_t$ (not that necessarily the cardinality of $L_t$ is $\tau_{k_t}$). Then, according to the greedy choice of $(k_t, L_t)$ we have that
\[
\bar{b} \text{KL}^t_{(k_t, L_t)}(\pi^t) \geq \max_{k \in [m]} \max_{h \in [s_k]} \text{KL}^t_{(k, h, k_t)}(\pi^t) \geq \max_{k \in [m]} \sum_{h \in [s_k]} \text{KL}^t_{(k, h, k_t)}(\pi^t).
\]
Thus, equation (3.11) of Proposition 3.1 yields
\[ \bar{b}\text{KL}_{\Pi_{(h_t, L_t)}}(\pi^t) = \max_{k \in [m]} \sum_{h \in [s_k]} \text{KL}(a_k|L_k^h, R_k(\pi^t)|L_k^h) = \max_{k \in [m]} \text{KL}(a_k, R_k(\pi^t)). \] (C.36)

Now Pinsker’s inequality guaranties that, for every \( k \in [m] \)
\[ \text{KL}(a_k, R_k(\pi^t)) \geq \frac{\|a_k - R_k(\pi^t)\|_2^2}{2\|a_k\|_1 + \frac{4}{3}\|R_k(\pi^t)\|_1} = \frac{\|a_k - R_k(\pi^t)\|_2^2}{2 \left( \frac{2}{3} + \frac{4}{3}\|\pi^t\|_1 \right)} \geq \frac{\|a_k - R_k(\pi^t)\|_2^2}{2 + \frac{4}{3}\|a_k - R_k(\pi^t)\|_1}. \] (C.37)

where in the second inequality we used that \( \|a_k - R_k(\pi^t)\|_1 \geq \|R_k(\pi^t)\|_1 - \|a_k\|_1 = \|\pi^t\|_1 - 1 \). Thus, solving the quadratic inequality in \( \|a_k - R_k(\pi^t)\|_1 \geq 0 \) we can conclude that
\[ \|a_k - R_k(\pi^t)\|_1 \leq \frac{2}{3}\text{KL}(a_k, R_k(\pi^t)) + \sqrt{\left(\frac{2}{3}\text{KL}(a_k, R_k(\pi^t))\right)^2 + 2\text{KL}(a_k, R_k(\pi^t))}. \]

Therefore, if \( \max_{k \in [m]} \text{KL}(a_k, R_k(\pi^t)) \leq 1 \), then \( 2 + 4d_4/3 \leq 5 \), and consequently, \( \max_{k \in [m]} \text{KL}(a_k, R_k(\pi^t)) \geq d_2/5 \), which, using Pythagoras theorem and (C.36), yields
\[ \delta_t - \delta_{t+1} = \text{KL}_{\Pi}(\pi^t) - \text{KL}_{\Pi}(\pi^{t+1}) = \text{KL}_{\Pi(h_t, L_t)}(\pi^t) \geq \frac{d_2^2}{5b}. \] (C.38)

On the other hand, if \( \max_{k \in [m]} \text{KL}(a_k, R_k(\pi^t)) > 1 \), it follows again from Pythagoras theorem and (C.36), that \( \delta_t - \delta_{t+1} \geq 1/\bar{b} \). Moreover, since \( \delta_t \leq \delta_0 \leq M_2 \bar{d}_0 \leq M_2(1 + \|\pi^0\|_1) \leq 2M_2 \), we have that \( 1 \geq \delta_t^2/(4M_2^2) \), and (C.35) follows.

Now, similarly to what was done in the proof of Lemma C.3 we can derive from (C.35) that
\[ \delta_t \leq \left( \frac{1}{\delta_0} + \frac{t}{5M_2^2 \bar{b}} \right)^{-1}. \] (C.39)

Thus, if we take \( r = \lfloor 5M_2^2 \bar{b} \rfloor + 1 \) we have \( \delta_r \leq 1 \). Then, by (3.11) with \( L = [n_k] \) and Pythagoras theorem we have that, for every \( t \in \mathbb{N} \),
\[ \text{KL}(a_k, R_k(\pi^{r+t})) = \text{KL}(\mathcal{P}_{\Pi_{(h_k, L_k)}}(\pi^r \circ \pi^{t}), \pi^{r+t}) \leq \text{KL}(\pi^*, \pi^{r+t}) = \delta_{r+t} \leq \delta_r \leq 1. \]

Thus, \( \max_{k \in [m]} \text{KL}(a_k, R_k(\pi^{r+t})) \leq 1 \) and for what we already saw,
\[ \delta_{r+t} - \delta_{r+t+1} \geq \frac{d_2^2}{5b}. \] (C.40)

In the end the sequence \( (\delta_{r+t})_{t \in \mathbb{N}} \) satisfies the two assumptions of Lemma C.3 with \( C = \sqrt{5b} \) and \( M = M_2 \). Thus, we can conclude that the smallest \( t \) so that \( d_{r+t} \leq \epsilon \) satisfies \( t \leq 1 + 10M_2 \bar{d}/\epsilon \). Hence
\[ r + t \leq 2 + \frac{10M_2 \bar{d}}{\epsilon} + 5M_2^2 \bar{b} \leq 2 + \frac{10M_2 \bar{d}}{\epsilon} + \frac{5M_2^2 \bar{b} \eta}{\epsilon} = 2 + \frac{5M_2 \bar{d}}{\epsilon}(2 + M_2 \eta). \]

The next two results are based on novel bounds on potentials that imply explicit dependence of constant \( M > 0 \) in the global rate (4.1) on the given data: \( a_1, \ldots, a_m, C \) and \( \eta \).
Theorem 4.4. If \( m = 2 \), then algorithm BatchGreenkhorn\( (a_1, \ldots, a_m, C, \eta, \tau) \) converges linearly with the global rate

\[
\forall t \in \mathbb{N} \quad \text{KL}(\pi^*, \pi^t) \leq \left( 1 - \frac{e^{-20\|C\|_\infty / \eta}}{b_r - 1} \right)^t \text{KL}(\pi^*, \pi^0).
\]

Moreover, when \( \eta > \varepsilon \), the number of iterations required to reach the stopping criterion \( d_t \leq \varepsilon \) satisfies

\[
t \leq 2 + \max_{k \in [m]} \left[ \frac{n_k}{\tau_k} \right] \frac{15\|C\|_\infty (2 + 3\|C\|_\infty)}{\eta \varepsilon}.
\]

Proof. Let \( v_k^t, k \in [m], t \geq 0 \) be given by Algorithm 1. Then from Proposition B.1 we have that for every \( t \geq 0 \)

\[
\pi^{t+1} = \exp \left( -\frac{C}{\eta} + \bigoplus_{k=1}^m v_k^{t+1} \right) \bigotimes_{k=1}^m a_k,
\]

with \( v_k^0 = 0 \) and for \( t \geq 0, k \in [m] \) and \( j_k \in [n_k]\),

\[
v_{k,j_k}^{t+1} = \begin{cases} v_{k,j_k}^t + \log(a_{k,j_k}) - \log(R_{k}(\pi^t)) & k = k_t, j_k \in L_t, \\ v_{k,j_k}^t & \text{otherwise}. \end{cases}
\]

So, to bound \( \log \pi^t \), we will bound \( v_k^t, k \in [m] \). Since \( R_{k_t}(\pi^{t+1})_{j_{k_t}} = a_{k_t,j_{k_t}} \) for all \( j_{k_t} \in L_t \), (C.41) implies that

\[
1 = \exp(\sum_{j_{k_t} \in J_{-k_t}} \exp \left( -\frac{C_{(j_{-k_t}, j_{k_t})}}{\eta} + \sum_{k \neq k_t} v_{k,j_k}^t \right) \prod_{k \neq k_t} a_{k,j_k},
\]

and, hence,

\[
\exp(-v_{k_t,j_{k_t}}^{t+1}) = \sum_{j_{k_t} \in J_{-k_t}} \exp \left( -\frac{C_{(j_{-k_t}, j_{k_t})}}{\eta} + \sum_{k \neq k_t} v_{k,j_k}^t \right) \prod_{k \neq k_t} a_{k,j_k}.
\]

So, using (C.7) we obtain that for every \( j_{k_t} \in L_t \)

\[
\exp(v_{k_t,j_{k_t}}^{t+1} - v_{k_t,j_{k_t}}^*) = \frac{\sum_{j_{k_t} \in J_{-k_t}} \exp \left( -\frac{C_{(j_{-k_t}, j_{k_t})}}{\eta} + \sum_{k \neq k_t} v_{k,j_k}^t \right)}{\sum_{j_{k_t} \in J_{-k_t}} \exp \left( -\frac{C_{(j_{-k_t}, j_{k_t})}}{\eta} + \sum_{k \neq k_t} v_{k,j_k}^t \right)},
\]

while

\[
\exp(v_{k_t,j_{k_t}}^* - v_{k_t,j_{k_t}}^{t+1}) = \frac{\sum_{j_{k_t} \in J_{-k_t}} \exp \left( -\frac{C_{(j_{-k_t}, j_{k_t})}}{\eta} + \sum_{k \neq k_t} v_{k,j_k}^t \right)}{\sum_{j_{k_t} \in J_{-k_t}} \exp \left( -\frac{C_{(j_{-k_t}, j_{k_t})}}{\eta} + \sum_{k \neq k_t} v_{k,j_k}^* \right)}.
\]
However, since in general, $\alpha_i, \beta_i > 0$ implies that $(\sum_i \alpha_i)/(\sum_i \beta_i) \leq \max_i \alpha_i/\beta_i$, the last two equalities give
\[
\exp(v_{k_i,k_j}^{t+1} - v_{k_i,k_j}^*) \leq \max_{j \neq k_i} \exp\left( \sum_{k \neq k_i} (v_{k,j}^* - v_{k,j}^t) \right)
\]
and
\[
\exp(v_{k_i,k_j}^* - v_{k_i,k_j}^{t+1}) \leq \max_{j \neq k_i} \exp\left( \sum_{k \neq k_i} (v_{k,j}^t - v_{k,j}^*) \right).
\]
Hence, taking the logarithm we obtain that for every $j_{k_i} \in L_k$,
\[
|v_{k_i,k_j}^{t+1} - v_{k_i,k_j}^*| \leq \max_{j \neq k_i} \left| \sum_{k \neq k_i} (v_{k,j}^* - v_{k,j}^t) \right| \leq \max_{j \neq k_i} \left| \sum_{k \neq k_i} v_{k,j}^* - v_{k,j}^t \right| = \sum_{k \neq k_i} \|v_k^* - v_k^t\|_\infty.
\]
Therefore, since $v_{k_i,k_j}^{t+1} = v_{k_i,k_j}^*$ if $k \neq k_i$ or $j_{k_i} \notin L_k$,
\[
\max \left\{ \|v_{k_i}^{t+1} - v_{k_i}^*\|_\infty, \sum_{k \neq k_i} \|v_{k_i}^{t+1} - v_{k_i}^*\|_\infty \right\} \leq \max \left\{ \|v_{k_i}^t - v_{k_i}^*\|_\infty, \sum_{k \neq k_i} \|v_k^t - v_k^*\|_\infty \right\},
\]
and, since $m = 2$,
\[
\max_{k \in [m]} \|v_k^{t+1} - v_k^*\|_\infty \leq \max_{k \in [m]} \|v_k^t - v_k^*\|_\infty,
\]
which implies, recalling that $v^0 = 0$, that, for all $t \geq 0$, $\max_{k \in [m]} \|v_k^t - v_k^*\|_\infty \leq \max_{k \in [m]} \|v_k^*\|_\infty$. Now, in view of (C.6) in Lemma C.1, we have
\[
\max_{k \in [m]} \|v_k^*\|_\infty \leq \frac{3}{2} \frac{\|C\|_\infty}{\eta} \tag{C.43}
\]
and hence, since $\|v_k^t\|_\infty \leq \|v_k^t - v_k^*\|_\infty + \|v_k^*\|_\infty$, $\max_{k \in [m]} \|v_k^t\|_\infty \leq 2 \max_{k \in [m]} \|v_k^*\|_\infty \leq 3 \|C\|_\infty/\eta$. In the end, since for every $(v_{k_{k_i}})_{k \in [m]} \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$,
\[
\left\| \bigoplus_{k=1}^m v_k \right\|_\infty = \max_{j \in J} \left| \sum_{k=1}^m v_{k,j} \right| \leq \max_{j \in J} \left| \sum_{k=1}^m v_{k,j} \right| = \sum_{k=1}^m \|v_k\|_\infty \leq m \max_{k \in [m]} \|v_k\|_\infty, \tag{C.44}
\]
we can satisfy the boundedness assumptions on the dual variables of Theorem 4.1 with $M = 6\|C\|_\infty/\eta$ and (4.5) follows from (4.1). Concerning the iteration complexity, again by (C.43), since $\max_{k \in [m]} \|v_k^* - v_k^t\|_\infty \leq \max_{k \in [m]} \|v_k^*\|_\infty$, we have $\sum_{k \in [m]} \|v_k^* - v_k^t\|_\infty \leq 3\|C\|_\infty/\eta$. So, using $\eta \geq \varepsilon$, (4.8) follows directly from (4.2) with $M_2 = 3\|C\|_\infty/\eta$. \hfill $\square$

**Theorem 4.5.** If for all $k \in [m]$ $\tau_k = n_k$, then algorithm BatchGreenhorn($a_1, \ldots, a_m, C, \eta, \tau$), i.e. MultiSinkhorn algorithm of [17], converges linearly with the global rate
\[
(\forall t \in \mathbb{N}) \quad \text{KL}(\pi^*, \pi^t) \leq \left( 1 - \frac{e^{-(12m-7)\|C\|_\infty/\eta}}{m-1} \right)^t \text{KL}(\pi^*, \pi^0). \tag{4.7}
\]
Moreover, the number of iterations required to reach stopping criterion $d_t \leq \varepsilon$ satisfies
\[
t \leq 1 + \frac{8(4m - 3)\|C\|_\infty}{\eta \varepsilon}. \tag{4.8}
\]
Proof. Using the same notation as in the previous proof, we first show that

\[
(\forall t \in \mathbb{N})(\forall k \in [m])(\forall j_k, \ell_k \in [n_k]) \quad v_{k,j_k}^t - v_{k,\ell_k}^t \leq 2\|C\|_{\infty} / \eta. \tag{C.45}
\]

Indeed, since for \( t = 0 \), \( v_{k,j_k}^0 - v_{k,\ell_k}^0 = 0 \), we proceed by induction assuming that (C.45) holds for \( t \) and proving it for \( t + 1 \). Noting that for every \( k \in [m] \), every \( j_k \in \mathcal{J}_k \) and every \( j, \ell \in [n_k] \)

\[
C_{(j-k,j_k)} - C_{(j-k,\ell_k)} \leq 2\|C\|_{\infty},
\]

and that \( L_t = [n_k] \), from (C.42) we have that \( v_{k,j_k}^{t+1} - v_{k,\ell_k}^{t+1} \leq 2\|C\|_{\infty} / \eta \) holds for every \( j_k, \ell_k \in [n_k] \). On the other hand for every \( k \neq k_t \), \( v_{k,j_k}^{t+1} = v_{k,j_k}^t \), which using the inductive hypothesis (C.45) yields \( v_{k,j_k}^{t+1} - v_{k,\ell_k}^{t+1} \leq 2\|C\|_{\infty} / \eta \). In any case (C.45) holds for \( t + 1 \).

Next, let \( t \in \mathbb{N} \) and define the normalizing constants \( \lambda_1^{t+1}, \ldots, \lambda_m^{t+1} \in \mathbb{R} \) as \( \lambda_k^{t+1} := -\langle a_k, v_{k,j_k}^{t+1} \rangle \) for \( k \neq k_t \), and \( \lambda_{k_t}^{t+1} := -\sum_{k \neq k_t} \lambda_k^{t+1} \). Then denoting \( u_{k,j_k}^{t+1} := v_{k,j_k}^{t+1} + \lambda_k^{t+1} \), \( k \in [m] \), since \( \sum_{k \in [m]} \lambda_k^{t+1} = 0 \), we have \( \sum_{k=1}^m u_{k,j_k}^{t+1} = 1 \) and hence, recalling (B.5),

\[
\pi^{t+1} = \exp \left( -\frac{C}{\eta} + \sum_{k=1}^m u_{k,j_k}^{t+1} \right) \odot \bigotimes_{k=1}^m a_k, \tag{C.46}
\]

Moreover, from (C.45) we have that for every \( k \neq k_t \) and every \( j, \ell \in [n_k] \)

\[
u_{k,j_k}^{t+1} - u_{k,j_k}^{t+1} = v_{k,j_k}^{t+1} - v_{k,\ell_k}^{t+1} \leq 2\|C\|_{\infty} / \eta,
\]

which, using \( \sum_{j \in [n_k]} a_{k,j} = 1 \) and the fact that the \( \lambda_k^{t+1} \)'s are chosen so that \( \langle a_k, u_{k,j_k}^{t+1} \rangle = 0 \) for all \( k \neq k_t \), implies

\[
u_{k,j_k}^{t+1} - u_{k,j_k}^{t+1} = \sum_{j_k \in [n_k]} a_{k,j_k} (u_{k,j_k}^{t+1} - u_{k,\ell_k}^{t+1}) \leq 2\|C\|_{\infty} / \eta, \quad \ell_k \in [n_k], \tag{C.47}
\]

and

\[
u_{k,j_k}^{t+1} = \sum_{\ell_k \in [n_k]} a_{k,\ell_k} (u_{k,j_k}^{t+1} - u_{k,\ell_k}^{t+1}) \leq 2\|C\|_{\infty} / \eta, \quad j_k \in [n_k]. \tag{C.48}
\]

Therefore, we have obtained that \( \|u_{k,j_k}^{t+1}\|_{\infty} \leq 2\|C\|_{\infty} / \eta \) for \( k \neq k_t \). On the other hand, similar to what was done in the proof of Theorem 4.4 we can derive that, for every \( j, k_t \in L_t = [n_k] \),

\[
\exp(-v_{k,j_k}^{t+1}) = \sum_{j_k \in \mathcal{J}_j} \exp \left( -\frac{C_{(j-k,j_k)}}{\eta} + \sum_{k \neq k_t} u_{k,j_k}^{t+1} \right) \prod_{k \neq k_t} a_{k,j_k}.
\]

Since, recalling (C.47) and (C.48),

\[
\exp(-(2m - 1)\|C\|_{\infty} / \eta) \leq \exp \left( -\frac{C_{(j-k,j_k)}}{\eta} + \sum_{k \neq k_t} u_{k,j_k}^{t+1} \right) \leq \exp((2m - 1)\|C\|_{\infty} / \eta),
\]

and \( \sum_{j_k \in \mathcal{J}_j} \prod_{k \neq k_t} a_{k,j_k} = 1 \), we have

\[
\exp(-(2m - 1)\|C\|_{\infty} / \eta) \leq \exp(-u_{k,j_k}^{t+1}) \leq \exp((2m - 1)\|C\|_{\infty} / \eta).
\]
Therefore,
\[
\exp \left( |u_{k_i,j_{k_i}}^{t+1}| \right) = \max \left\{ \exp \left( u_{k_i,j_{k_i}}^{t+1} \right), \exp \left( -u_{k_i,j_{k_i}}^{t+1} \right) \right\} \leq \exp((2m - 1)\|\mathbb{C}\|_{\infty}/\eta)
\] (C.49)
and hence
\[
\|u_{k_i}^{t+1}\|_{\infty} \leq (2m - 1)\|\mathbb{C}\|_{\infty}/\eta.
\]
Therefore, we have \(\sum_{k \in [m]} \|u_k^t\|_{\infty} \leq (4m - 3)\|\mathbb{C}\|_{\infty}/\eta =: M\) and due to (C.46) and the computation (C.44), we can use \(M_1 = M\) in Theorem 4.1 and get (4.7). Concerning iteration complexity, recalling (C.5) we have that \(\sum_{k \in [m]} \|u_k - v_k^\star\|_{\infty} \leq 2(4m - 3)\|\mathbb{C}\|_{\infty}/\eta\) and hence as done in (C.34) we have
\[
\delta_t \leq \frac{2(4m - 3)\|\mathbb{C}\|_{\infty}}{\eta} d_t.
\]
Moreover, since \(\|\pi^t\|_1 = 1\) and \(\bar{b} = 1\), it follows from (C.36), (C.37) and (C.38) that
\[
\delta_t - \delta_{t+1} \geq \frac{d_t^2}{2}.
\] (C.50)
Thus, the statement follows from Lemma C.3.