Lame problem for a multilayer viscoelastic hollow ball with regard to inhomogeneous aging

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Abstract. Determination of characteristics of the stress strain state of compound viscoelastic bodies is of both theoretical and practical interest. In the present paper, the Lamé problem is investigated for an uneven-aged multilayer viscoelastic hollow ball in the framework of N. Kh. Arutyunyan’s theory of creep of nonhomogeneously aging bodies [1, 2]. Solving this problem reduces to solving an inhomogeneous finite-difference equation of second order that contains operators with coordinates of time and space. The obtained formulas allow one to determine the required contact stresses and other mechanical characteristics of the problem related to uneven age of contacting balls.

1. Problem formulation and derivation of basic equations

In a spherical coordinate system \((r, \varphi, \theta)\), consider a composite shaped as a package of an arbitrary \(n\) number of viscoelastic hollow balls

\[
(r_{k-1} \leq r \leq r_k, \ 0 < \varphi \leq 2\pi, \ 0 < \theta \leq 2\pi) \quad (k = 1, n)
\]

with instant deformation moduli \(E_k(t)\) soldered together through their surfaces. We assume that the contacting balls possess the property of creep and are of different ages. Assume further that, at time \(t = \tau_0\), uniformly distributed forces of intensities \(P_0(t)\) and \(P_n(t)\) are respectively applied to the composite surfaces \(r = r_0\) and \(r = r_n\), i.e.,

\[
\sigma_r(r, t)\big|_{r=r_0} = -P_0(t), \quad \sigma_r(r, t)\big|_{r=r_n} = -P_n(t),
\]

where \(\sigma_r(r, t)\) is the radial stress component, and the tangential stresses are equal to zero due to symmetry.

To derive the governing equations of the problem, we consider the \(k\)th layer of the package. Then to determine the elastic instant stresses and displacements of the layers, according to Lamé formulas, we have the expressions [3]

\[
\begin{align*}
\sigma_r^{(k)}(r, t) &= \frac{\lambda_k(t)}{\nu_k} A_k(t) + \frac{\lambda_k(t)(3\nu_k - 2)}{\nu_k} \frac{B_k(t)}{r^3}, \\
\sigma_\varphi^{(k)}(r, t) &= \sigma_\theta^{(k)}(r, t) = \frac{\lambda_k(t)}{\nu_k} \left[ A_k(t) + (1 - 3\nu_k) \frac{B_k(t)}{r^3} \right], \\
U_r^{(k)}(r, t) &= A_k(t) \frac{B_k(t)}{r^2}, \quad \lambda_k(t) = \frac{E_k(t)\nu_k}{(1 + \nu_k)(1 - 2\nu_k)} \quad (k = 1, n), \\
\sigma_r^{(k)}(r, t)\big|_{r=r_{k-1}} &= -P_{k-1}(t), \quad \sigma_r^{(k)}(r, t)\big|_{r=r_k} = -P_k(t).
\end{align*}
\]
Here $A_k(t)$ and $B_k(t)$ are unknown coefficients to be determined, $U_r^{(k)}(r,t)$ is the elastic instant displacement of the layer, $P_{k-1}(t)$ and $P_k(t)$ are unknown radial contact stresses on the surfaces $r = r_{k-1}$ and $r = r_k$ of the layer, respectively, $\nu_k$ is Poisson ratio of the material of the layer, and $E_k(t)$ is the elastic instant modulus of deformation of the layer.

Satisfying the boundary-value conditions in equation (3), for the radial displacement from equation (2) we obtain

$$U_r^{(k)}(r,t) = \frac{\lambda_k(t)}{r_k} \left[ \int_0^t \rho_i(t+\rho_i) Y(u) du \right] (r, t),$$

$$U_r^{(k+1)}(r,t) = \frac{\lambda_k(t)}{r_{k+1}} \left[ \int_0^t \rho_i(t+\rho_i) Y(u) du \right] (r, t).$$

where

$$c_k(t) = \frac{\lambda_k(t)}{r_k}, \quad d_k(t) = \frac{\lambda_k(t)(3\nu_k - 2)}{r_k}, \quad a_k = r_k^3 - r_{k-1}^3.$$

Now the condition of displacement continuity on the surface of contact $r = r_k$ of the two successive balls, with regard to the creep, can be written as

$$(I - L_k)U_r^{(k)}(r,t) = (I - L_{k+1})U_r^{(k+1)}(r,t) \quad (k = 1, n - 1).$$

Here

$$L_i[Y(t)] = \int_0^t E_i(u) K_i(t + \rho_i, u + \rho_i) Y(u) du, \quad \rho_i = \tau_i - \tau_0,$$

$$C_i(t, u) = \left( \frac{1}{E_i(u)} + C_i(t, u) \right) (i = 1, n),$$

where $\tau_i$ are the ages and $C_i(t, u)$ are the amounts of creep ($i = 1, n$) of the layers.

We use equation (4) to rewrite contact condition (5) as

$$\frac{1}{L_k} (I - L_k) \left[ \frac{r_{k-1}^3 P_k(t) - r_{k-1}^3 P_{k-1}(t)}{d_k(t)} - \frac{r_k^3 P_k(t) - r_k^3 P_{k-1}(t)}{c_k(t)} \right] = \frac{1}{L_{k+1}} (I - L_{k+1}) \left[ \frac{r_{k+1}^3 P_{k+1}(t) - r_{k+1}^3 P_k(t)}{d_{k+1}(t)} - \frac{r_{k+1}^3 P_{k+1}(t) - r_{k+1}^3 P_k(t)}{c_{k+1}(t)} \right] \quad (k = 1, n - 1).$$

The latter can be reduced to the following second-order finite-difference equations [4]:

$$(I - L_k) [a_k(t) q_{k-1}(t) + b_k(t) q_k(t)] = f_k(t),$$

$$(I - L_{k+1}) [a_{k+1}(t) q_k(t) + a_k(t) q_{k+1}(t)] = g_{k+1}(t).$$

Under the conditions

$$f_k(t) + g_{k+1}(t) = 0 \quad (k = 1, n - 1),$$

equations (6) and (7) can be rewritten as

$$a_k(t) q_{k-1}(t) + b_k(t) q_k(t) = (I + R_k)f_k(t),$$

$$\delta_{k+1}(t) q_{k+1}(t) + a_{k+1}(t) q_{k+1}(t) = (I + R_{k+1})g_{k+1}(t).$$

Here $R_{s}(t, u)$ is the resolvent of the kernel $K_{s}(t, u) \quad (s = 1, n)$, and the following notation is introduced:

$$a_k(t) = \frac{1}{c_k(t)} - \frac{1}{d_k(t)} \frac{1}{l_k}, \quad q_k(t) = \frac{r_k^3 P_k(t)}{r_k}, \quad \varphi_k = \frac{r_{k-1}^3}{r_k^3},$$

$$b_k(t) = \frac{\varphi_k}{d_k(t)} - \frac{1}{c_k(t)} \frac{1}{l_k}, \quad \delta_k(t) = \frac{1}{\varphi_k d_k(t)} - \frac{1}{c_k(t)} \frac{1}{l_k}. $$
2. Solution of the finite-difference equations

The solution of the first-order finite-difference equations (9) and (10) is constructed according to the well-known method [4, 5]. It can be represented in the form

\[ q_k(t) = [1 - F(k, t)]q_{k-1}(t) + \frac{(I + R_k)f_k(t)}{b_k(t)}. \] (11)

Here

\[ F(k, t) = \frac{[a_k(t) + b_k(t)]}{b_k(t)} \quad (k = \overline{1, n-1}). \]

By analogy with [4], we find the general solution of equation (11)

\[ q_k(t) = \prod_{j=1}^{k} [1 - F(j, t)]\left\{ \sum_{i=1}^{k} (I + R_i)f_i(t) \left[ b_i(t) \prod_{r=1}^{i} (1 - F(j, t)) \right]^{-1} + P_0(t) \right\} (k = \overline{1, n-1}). \] (12)

Quite similarly, the solution of equation (10) can be represented as

\[ q_k(t) = \prod_{j=k}^{n-1} [1 - \tilde{F}(j, t)]^{-1}\left\{ \sum_{i=k}^{n-1} (I + R_{i+1})g_{i+1}(t) \left[ a_{i+1}(t) \prod_{r=i+1}^{n-1} (1 - \tilde{F}(r, t))^{-1} \right]^{-1} + P_n(t) \right\}. \] (13)

Here

\[ \tilde{F}(k, t) = \frac{[a_{k+1}(t) + \delta_{k+1}(t)]}{a_{k+1}(t)} \quad (k = \overline{1, n-1}). \]

Now equating expressions (12) and (13) and taking (8) into account, we obtain the following system of Volterra integral equations of the second kind with respect to \( f_i(t) \) \((i = \overline{1, n-1})\):

\[ \sum_{i=1}^{k} \frac{1}{b_i(t)} \prod_{j=1}^{k} \frac{a_j(t)}{b_j(t)} \prod_{r=1}^{i} \frac{b_r(t)}{a_r(t)} (I + R_i)f_i(t) + \sum_{i=k}^{n-1} \frac{1}{a_{i+1}(t)} \prod_{j=k}^{n-1} \frac{a_{j+1}(t)}{d_{j+1}(t)} \prod_{r=i+1}^{n-1} \frac{\delta_{r+1}(t)}{a_{r+1}(t)} (I + R_{i+1})f_i(t) \]

\[ = P_0(t) \prod_{j=1}^{k} \frac{a_j(t)}{b_j(t)} - P_n(t) \prod_{j=k}^{n-1} \frac{a_{j+1}(t)}{b_{j+1}(t)} \quad (k = \overline{1, n-1}) \] (14)

or

\[ Q_k(t)(I + R_k)f_k(t) + \sum_{i=1}^{k-1} Q_i(t) \frac{M_i(t)}{M_k(t)} (I + R_i)f_i(t) \]

\[ + \sum_{i=k}^{n-1} H_i(t) \frac{N_{i+1}(t)}{N_k(t)} (I + R_{i+1})f_i(t) = \frac{P_0(t)}{M_k(t)} - \frac{P_n(t)}{N_k(t)} \quad (k = \overline{1, n-1}). \] (15)

Here the following notation is used:

\[ Q_i(t) = \frac{1}{b_i(t)}, \quad H_i(t) = \frac{1}{a_{i+1}(t)}, \]

\[ M_i(t) = \prod_{r=1}^{i} \frac{b_r(t)}{a_r(t)} \quad (i = \overline{1, k}), \quad N_i(t) = \prod_{r=1}^{n-1} \frac{\delta_{r+1}(t)}{a_{r+1}(t)} \quad (i = \overline{k, n-1}). \]

Once the governing system of integral equations (15) is solved with respect to \( f_i(t) \), the radial contact stresses \( q_k(t) \) \((k = \overline{1, n-1})\) are determined from equations (9) and (10).
3. Particular case

Let us consider a particular case of the problem where the composite consists of two heterogeneous balls, i.e., \( n = 2 \). Note that a similar problem is considered in [6]. In our case of two layers, we determine \( f_1(t) \) from equation (11). Next, we find the radial contact stress \( q_1(t) \) from equations (12) and/or (13) through \( P_0(t) \) and \( P_2(t) \). The expression for \( q_1(t) \) can be obtained as well from equations (6) and (7) with \( n = 2 \). In this case, to determine the unknown pressure \( q_1(t) \), according to equation (5), we obtain the following Volterra integral equations of the second kind

\[
\alpha(t)q_1(t) - \int_{t_0}^{t} \left[ \alpha_1(\tau)E_1(\tau)K_1(t, \tau) + \alpha_2(\tau)E_2(\tau)K_2(t, \tau) \right] q_1(\tau) \, d\tau = F(t).
\]

Here the following notation is used:

\[
F(t) = \beta_1(t)P_0(t) + \beta_1(t)P_2(t) - \int_{t_0}^{t} \left[ \beta_1(\tau)E_1(\tau)K_1(t, \tau)P_0(t) - \beta_2(\tau)E_2(\tau)K_2(t, \tau)P_2(\tau) \right] d\tau,
\]

\[
\alpha_1(t) = \frac{\left[ r_0^3 - (3\nu_1 - 2)r_0^2(1 + \nu_1)(1 - 2\nu_1) \right]}{r_1^2(r_1^4 - r_0^4)(3\nu_1 - 2)E_1(t)}, \quad \alpha_2(t) = \frac{\left[ r_0^3 - (3\nu_2 - 2)r_0^2(1 + \nu_2)(1 - 2\nu_2) \right]}{r_1^2(r_1^4 - r_0^4)(3\nu_2 - 2)E_2(t)},
\]

\[
\beta_1(t) = \frac{3r_1r_0^2(1 - \nu_1^2)(1 - 2\nu_1)}{(r_1^4 - r_0^4)(3\nu_1 - 2)E_1(t)}, \quad \beta_2(t) = \frac{3r_1r_0^2(1 - \nu_2^2)(1 - 2\nu_2)}{(r_1^4 - r_0^4)(3\nu_2 - 2)E_2(t)},
\]

\[
\alpha(t) = \alpha_1(t) + \alpha_2(t).
\]

To solve integral equation (16), we assume that the kernels of the creep of materials have the form [1]

\[
K_i(t, \tau) = \frac{\partial}{\partial \tau} \left[ \frac{1}{E_i(\tau)} + C_i(t, \tau) \right], \quad C_i(t, \tau) = \varphi_i(\tau)(1 - e^{-\gamma(t-\tau)}) \quad (i = 1, 2),
\]

where \( \varphi_1(\tau) \) and \( \varphi_2(\tau) \) are the functions of aging for the materials of the outer and inner layers, respectively. The meaning of the other parameters well-known in the theory of creep is given in [1]. Later, we also assume that \( E_1(t) = E_1 = \text{const} \) and \( E_2(t) = E_2 = \text{const} \).

Since the kernels determined by formula (17) are singular, the Volterra integral equations of the second kind (16) can be reduced to a differential equation with variable coefficients.

For simplicity, let us consider a particular case of external load. Assume that, at time \( t = t_0 \), a concentrated radial force of intensity \( T_0 \) is applied to the outer sphere of the compound hollow ball. Afterwards, the applied force remains constant in time. Then \( P_0(t) = P_2(t) = T_0H(t - t_0) \). Here \( H(t) \) is the unit Heaviside function. Also, assume that \( \rho_1 = 0 \). Under the applied load, the solution of integral equation (16) becomes

\[
q_1(t) = T_0 \left\{ (\beta_1 + \beta_2)\lambda_0 + \lambda_0^2(\beta_1\alpha_2 - \beta_2\alpha_1)(E_1\varphi_1(\tau_0) - E_2\varphi_2(\tau_2)) \right\} \int_{t_0}^{t} e^{-\eta(z)} \, dz,
\]

where

\[
\eta(t) = \gamma \int_{t_0}^{t} \left[ 1 + \lambda_0\alpha_1 E_1\varphi_1(\tau) + \lambda_0\alpha_2 E_1\varphi_2(\tau + \rho_2) \right] d\tau, \quad \lambda_0 = \frac{1}{\alpha_1 + \alpha_2}.
\]

It is easy to see that when the creep deformation of two contacting bodies is caused by constant stresses applied to the layers from inside and outside simultaneously (at the age of \( t_0 \) and \( t_2 \), respectively) and proportionally to their elastic deformation, i.e., \( E_1\varphi_1(\tau_0) = E_2\varphi_2(\tau_2) \), then the solution of the above-stated problem coincides with the solution of the corresponding elastic problem.
If we assume that \( P_2(t) = 0 \), then from the general solution of equation (18) we obtain

\[
q_1(t) = T_0 \beta_1 \lambda_0 \left\{ 1 + \lambda_0 \alpha_2 [E_1 \varphi_1(\tau_0) - E_2 \varphi_2(\tau_2)] \int_{\tau_0}^{t} e^{-\eta(z)} \, dz \right\}.
\]

(19)

Meanwhile, equation (19) implies that, as a function of time, the radial pressure \( q_1(t) \) increases for \( E_1 \varphi_1(\tau_0) > E_2 \varphi_2(\tau_2) \) and, vice versa, it decreases for \( E_1 \varphi_1(\tau_0) < E_2 \varphi_2(\tau_2) \).

Conclusions

In the paper an inhomogeneously aging multilayer viscoelastic hollow ball was investigated when stressed uniaxially. In particular, the problem was considered for two-layered composite with the creep property. Simple formulas were obtained for unknown contact stresses. It was shown that if the creep deformations and the elastic deformations of the layers are proportional, then the solution of the formulated problem coincides with that of the corresponding elastic problem.

References

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