On deterministic nature of the intermittent geodesic acoustic mode observed in L- mode discharge near tokamak edge

Part I  Zonal flow driven by ion temperature gradient turbulence

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Abstract

Through a carefully designed numerical experiment, we demonstrate that a transition between two distinct phases of energy concentration in a zonal flow-drift wave system (caviton and instanton) may play a key role in the intermittent excitation of geodesic acoustic mode (GAM) that are observed in tokamaks. The two energy structures - the caviton, a slowly breathing spatial local structure of ‘negative’ energy, and the instanton, a fine radial structure of short lifetime in rapid propagation, were recently identified in [Y. Z. Zhang, Z. Y. Liu, T. Xie, S. M. Mahajan, and J. Liu, Physics of Plasmas 24, 122304 (2017)]; the preceding work is based on the micro-turbulence associated with ion temperature gradient (ITG) mode, and slab-based phenomenological model of zonal flow. When toroidal effect are introduced into the system, two branches of zonal flow emerge: the torus-modified low frequency zonal flow (TLFZF), and GAM, necessitating a unified exploration of GAM and TLFZF. Indeed, we observe that the transition (decay) from the caviton to instanton is triggered by a rapid zero-crossing of radial group velocity of drift wave and is found to be strongly correlated with the GAM onset; it happens if and only if the intensity of TLFZF is above a certain level. Many features peculiar to intermittent GAMs, observed in real machines, are thus identified in the numerical experiment; the results will be displayed in figures and in a movie. Since one cannot make a foolproof case that the experimentally observed zonal flows must originate in ITG driven micro-turbulence (Appendix A), we have also explored the possibility that this phenomenon could be caused by electron drift waves, instead; the latter will make part II of this work.

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I Introduction

In this paper we construct a possible theoretical-computational pathway for the intermittent excitation of geodesic acoustic mode (GAM), observed, routinely, on several tokamaks such as ASDEX\textsuperscript{[1]}, T-10\textsuperscript{[2]}, JFT-2M \textsuperscript{[3]}, HL-2A\textsuperscript{[4]}, DIII-D\textsuperscript{[5]}, JET\textsuperscript{[6]}, EAST\textsuperscript{[7]}. Experimentally, the GAM is an intermittent (periodic stopping), random, discrete temporal structure. More specifically, the GAM in the frequency range 10-20 KHz does not last long; typically, it lasts a few milliseconds (e.g., 0.5-5ms) before disappearance, and then reappears in a shorter period of time. The entire cycle - the intermittency period - is less than 1ms near the plasma edge \textsuperscript{[4]} and longer than 1ms away from the plasma edge \textsuperscript{[1, 2, 7]}. During its occurrence, the GAM amplitude varies with no recognizable (so far) pattern. In this paper we emphasize the words ‘intermittent, random’, since irregularity away from GAM frequency is not the topic of this paper.

We begin with presenting a theoretical framework for extending the framework of Ref.\textsuperscript{8} – the zonal flow-drift wave system \textsuperscript{[8]} – to include GAM. ‘Extending’ simply means that much of the content developed in Ref.\textsuperscript{8} will be used to build the system suitable for the study of GAM. In particular, we will: (I) use all microscopic information including linear ITG eigenmode equations and ballooning solutions for both eigenvalue and 2D mode structure. This type of knowledge is obtained from Ref.\textsuperscript{26}, (II) calculate the Reynolds stress generated by the ITG mode structure obtained in Ref.\textsuperscript{25-26}, (III) and use the derivative expansion method for the two-scale system of ITG and for the calculation of ITG group velocity. This latter part will essentially follow Ref.\textsuperscript{8}. It is important to mention that the exact Eq.(9) in Ref.\textsuperscript{8} will be the starting point of this work; what will be different from Ref.\textsuperscript{8} is the zonal flow equation (14), the mode in mesoscale.

The toroidal effects are introduced into the two basic moment equations (charge and number conservation) for the axisymmetric mode (a synonym of zonal flow). To study the GAM behavior pertaining to a given rational surface, it is appropriate to incorporate toroidal effects on the zonal flow. The original system \textsuperscript{[8]} consisted of two equations: the (slab) zonal flow (\(\mathbf{u}\)) equation, and the equation of drift wave energy (\(\phi\)) modulated by zonal flow; these correspond, respectively, to Eqs.(14), (9) in Ref.\textsuperscript{8}. For the toroidal system within the same framework, however, the zonal flow equation, due to geodesic curvature, is coupled to the first harmonic sinusoidal component of sound wave (\(\chi^{(1)}\)) resulting in two new branches, namely TLFZF and GAM, for two coupled fields, the zonal flow and sound wave. Thus,
the single LFZF equation, Eq. (14) in Ref.8, becomes two equations Eqs. (18), (19) in section II. The equation of the drift wave energy modulated by zonal flow, Eq. (9) in Ref.8, remains intact in form. It is not directly modulated by \( \chi_z^{(i)} \). If the background turbulence generating the zonal flow is not ITG, then the corresponding Reynolds stress and group velocity must be calculated as we do in the second part of this paper for the electron drift wave turbulence. Such an extension, consisting of three coupled equations Eqs.(18-20), is fully described in section II. The set of three equations, derived in section III, represents a well-posed initial value problem under specific boundary conditions. The numerical methods, though essentially the same as in Ref.8, are described in detail so that this paper is self-contained. In section IV, use is made of figures and one movie to discuss leading characteristics of GAM generated by ITG, in particular, the intermittent excitation observed in tokamaks. Summary and discussions are presented in section V. In Appendix A, we critically examine what may be the most likely source of micro-turbulence generating zonal flows-GAM. Data on GAM experiments from nine discharges on 7 machines were collected and analyzed. Because of high collisionality, the well-known collisionless trapped electron mode (TEM or simply CTEM), and the dissipative trapped electron mode (DTEM) are excluded, from consideration. We also provide a few experimental data showing that GAM occurs in ITG unstable region. Appendix B is devoted to a derivation of the charge and particle number conservation equations for axisymmetric electrostatic mode; these equations constitute the basics for understanding the close relationship between TLFZF and GAM. A somewhat technical calculation for poloidal moments of Reynolds stress is given in Appendix C. In Appendix D it is shown that the low frequency limit of zonal flow equation set reduces to one equation of TLFZF. It is consistent with Eq.(14) of Ref.8, where the free parameter is determined \( a_{neo} = 1 + 2q^2 \).

II  The zonal flow equation set in tokamak

In the existing literature, the LFZF equation and equation for GAM are derived separately. However, we will soon show (based on a hybrid Braginskii-kinetic model, i.e., the Braginskii two-fluid equations\(^9\) with kinetic modifications) that these equations are simply two branches of a unified zonal flow system. We will assume a geometry of concentric circular magnetic surfaces in a toroidal coordinate system. Such a simplified framework should be sufficient for a paper meant to explore the qualitative features of GAM intermittency. The starting point of the investigation is the set of two coupled conservation equations for the axisymmetric electrostatic mode\(^10\), derived in Appendix B.

In a toroidal coordinate system \((r, \Theta, \zeta)\), where \(r, \Theta, \zeta\) are respectively the radial, poloidal and toroidal coordinates, the charge conservation and particle conservation equations are (see Eqs. (B.16)-(B.17)),

\[ \]
\[
\left( \frac{\partial}{\partial \theta} - \frac{r \sin \theta}{R} \right) \frac{\sigma_\delta}{q^2} \frac{\partial^2}{\partial \theta^2} \left( \bar{n} - \bar{\omega} \right) - 2 \rho_s (1 + \tau_s) \sin \theta \frac{\partial \bar{n}}{\partial r} - \frac{R \rho_s^2}{c_s} \frac{\partial}{\partial t} \frac{\partial^2 \bar{n}}{\partial r^2} (\bar{\phi} + \tau_s \bar{n}) = \frac{R \rho_s^2}{c_s^2} \nabla \cdot \bar{\mathbf{u}}_{\text{NP}}.
\]
(1)

The particle number conservation equation is

\[
\frac{R^2}{c_s^2} \frac{\partial^2 \bar{n}}{\partial t^2} - \frac{1 + \tau_s}{q^2} \left( \frac{\partial}{\partial \theta} - \frac{r \sin \theta}{R} \right) \frac{\partial \bar{n}}{\partial \theta} - \frac{2 R}{\omega_s} \sin \theta \frac{\partial}{\partial t} \frac{\partial (\bar{\phi} + \tau_s \bar{n})}{\partial r} = \frac{R^2}{\omega_s^2} \frac{\partial^2 \bar{\phi}}{\partial t^2} + \frac{R^2 \rho_s^2}{c_s^2} \frac{\partial}{\partial t} \nabla \cdot \bar{\mathbf{u}}_{\text{NP}}.
\]
(2)

In Eq.(1) \( \sigma_\delta \equiv 2 m c_s / m_e v_e R \) is the normalized parallel conductivity, \( \bar{n} (\bar{\phi}) \) is the normalized density (potential) fluctuation associated with the axisymmetric mode in mesoscale, \( \bar{\mathbf{u}}_{\text{NP}} \equiv \left\{ \rho_s^3 c_s^2 \left( \mathbf{b} \times \nabla \bar{\phi} \cdot \nabla \right) \right\}_{\text{em}} \) is the ensemble average of nonlinear polarization drift, \( \bar{\phi} \) is electrostatic potential of micro-turbulence such as drift waves. For other symbols, the reader is referred to Appendix B.

Due to the large parallel conductivity (e.g., \( \sigma_\delta > 10^2 \)) in tokamak plasmas, the leading term in Eq. (1) is

\[
\left( \frac{\partial}{\partial \theta} - \frac{r \sin \theta}{R} \right) \frac{\sigma_\delta}{q^2} \frac{\partial}{\partial \theta} \left( \bar{n} - \bar{\omega} \right) = 0.
\]
(3)

For this study, it is natural to split \( \bar{\omega} \) and \( \bar{n} \) into two parts- the zonal flow part \( (\bar{n}, \bar{\omega}) \), no poloidal dependence) and the part that depends on the poloidal angle \(^{[10]}\):

\[
\bar{n}(r,\theta,t) = \bar{n}(r,t) + \eta(r,\theta,t), \quad \bar{\omega}(r,\theta,t) = \bar{\omega}(r,t) + \bar{\phi}(r,\theta,t).
\]
(4)

Eq.(3), containing only \( \theta \) derivatives, will require \( \eta = \bar{\omega} \); the leading order response is, thus, \textit{adiabatic}.

On averaging over the magnetic surface \(^{[10]}\) (\( \varepsilon \equiv r/R \ll 1 \), the inverse aspect ratio)

\[
\langle \ldots \rangle_\theta \equiv \oint d\theta \ldots (1 + \varepsilon \cos \theta)
\]
(5)

on Eq.(1) yields

\[
2 \rho_s (1 + \tau_s) \frac{\partial}{\partial r} \left( \sin \theta \bar{\omega}_\theta \right) + \frac{R \rho_s^2}{c_s} \frac{\partial}{\partial t} \frac{\partial^2 \bar{n}}{\partial r^2} \left[ \bar{\omega} + \tau_s \bar{n} + (1 + \tau_s) \langle \bar{\phi} \rangle_{\theta} \right] + R \rho_s^3 \langle \Pi (r, \theta) \rangle_{\theta} = 0.
\]
(6)

In Eq.(6) \( \Pi (r, \theta) \equiv \nabla \cdot \left[ \left( \mathbf{b} \times \nabla \bar{\phi} \cdot \nabla \right) \nabla \bar{\phi} \right] \). Noticeably, this step annihilates the leading term of Eq.(1); the leftover Eq.(6) is the charge conservation equation for an axisymmetric mode that contains a geodesic curvature induced coupling to the sinusoidal wave. It will be further simplified after discussing some relations arising from particle number conservation, Eq.(2).
The magnetic surface average of Eq.(2) yields
\[
\frac{R^2}{C_z^2} \frac{\partial^2}{\partial t^2} \left( \bar{n} + \langle \varphi \rangle_\vartheta \right) - \frac{2R}{\omega_z} \frac{\partial}{\partial r} \left( 1 + \tau_z \right) \left( \sin \vartheta \varphi \right)_\vartheta = \frac{R^2}{C_z^2} \frac{\partial^2}{\partial t^2} \left( \bar{n} + \langle \varphi \rangle_\vartheta \right) + \frac{R^2}{C_z^2} \frac{\partial}{\partial t} \left( P(r, \vartheta) \right)_\vartheta. 
\]
(7)

By acting \( \left( R / C_z \right) (\partial / \partial t) \) on Eq.(6) and adding the result to Eq.(7), a straightforward algebra leads to a rather simple equation
\[
\left( \partial^2 / \partial r^2 \right) \left[ 1 + \tau_z \rho_z^2 \left( \partial^2 / \partial r^2 \right) \right] \left( \bar{n} + \langle \varphi \rangle_\vartheta \right) = 0. \]

We choose the trivial solution
\[
\bar{n} = -\langle \varphi \rangle_\vartheta = -\epsilon \int d\vartheta \cos \vartheta \varphi. 
\]
(8)

Eq.(8) implies that the zonal density arises from the first cosinoidal component of \( \varphi \), however, at the order of \( O(\epsilon) \).

Eq.(8) is now used to eliminate \( \bar{n} \) from Eq.(2) to yield
\[
\frac{R^2}{C_z^2} \frac{\partial^2}{\partial t^2} \left( \varphi - \epsilon \int d\vartheta \cos \vartheta \varphi \right) - \frac{1 + \tau_z}{q^2} \left( \frac{\partial}{\partial \vartheta} - \epsilon \sin \vartheta \right) \frac{\partial \varphi}{\partial \vartheta} = \frac{2 \sin \vartheta}{\omega_z} \frac{\partial}{\partial t} \frac{\partial}{\partial r} \left( \bar{n} + \langle \varphi \rangle_\vartheta \right) + \rho_z \frac{\partial}{\partial t} \left( P(r, \vartheta) \right)_\vartheta. 
\]
(9)

For mesoscale (\( \rho_z (\partial / \partial r) \ll 1 \)) solution all terms containing \( \rho_z \) can be neglected in Eq. (9). It becomes
\[
\frac{R^2}{C_z^2} \frac{\partial^2}{\partial t^2} \left( \varphi - \epsilon \int d\vartheta \cos \vartheta \varphi \right) - \frac{1 + \tau_z}{q^2} \left( \frac{\partial}{\partial \vartheta} - \epsilon \sin \vartheta \right) \frac{\partial \varphi}{\partial \vartheta} = 0. 
\]
(10)

Since the coefficients of Eq.(10) depends on \( \omega \), \( \varphi \) can be separated as
\[
\varphi(r, \omega, t) = \tilde{\alpha}_v^{(a)}(r, t) F_v^{(a)}(\omega). 
\]
(11)

Interestingly, if we further split the \( \omega \) dependence as \( F_v^{(a)}(\omega) = \exp(-\epsilon \cos \omega / 2) f_v^{(a)}(\omega) \), the homogeneous part of Eq.(10) can be cast into the canonical Mathieu equation
\[
\frac{d^2 f_v^{(a)}}{dz^2} + \left( 4 \tilde{\lambda}_v^{(a)} - 2 \epsilon \cos 2z \right) f_v^{(a)} = 0. 
\]
(12)

In Eq.(12) \( \omega = 2z + \pi \), \( \tilde{\lambda}_v^{(a)} = q^2 R^2 \omega^2 / \left( 1 + \tau_z \right) c_z^2 \), \( \alpha \) and \( \nu \) are characteristic exponents. The inhomogeneous part of Eq.(10) only contributes to the cosinoidal component of \( \varphi \). Characteristic functions and characteristic values of Eq.(10) are listed in Table I.
Table I: Characteristic functions and values of Eq.(10)

| Characteristic functions | Characteristic values |
|--------------------------|-----------------------|
| 1st sine \( f_1^{(s)} = \sin \vartheta + \frac{\varepsilon}{12} \sin 2\vartheta \) | \( \lambda_1^{(s)} = 1 + O(\varepsilon^2) \) |
| 1st cosine \( F_1^{(c)} = \cos \vartheta - \frac{\varepsilon}{6} \cos 2\vartheta \) | \( \lambda_1^{(c)} = 1 + O(\varepsilon^2) \) |
| 2nd sine \( f_2^{(s)} = \sin 2\vartheta + \varepsilon \left( \frac{1}{20} \sin 3\vartheta - \frac{1}{12} \sin \vartheta \right) \) | \( \lambda_2^{(s)} = 4 + O(\varepsilon^2) \) |

Substituting these characteristic functions into Eq.(9) yields

\[
\frac{R^2 \partial^2 \chi_1^{(a)}}{c_s^2 \partial t^2} \left( F_v^{(a)} - \varepsilon \int d\vartheta \cos \vartheta F_v^{(a)} \right) + \frac{(1 + \tau_i)}{q^2} \lambda_1^{(a)} \chi_1^{(a)} \left[ F_v^{(a)} - \varepsilon \int d\vartheta \cos \vartheta F_v^{(a)} \right]

-2 \sin \vartheta \frac{R \rho_s}{c_s} \frac{\partial}{\partial r} \left( \bar{\varphi} + \left[ (1 + \tau_i) F_v^{(a)} - \varepsilon \int d\vartheta \cos \vartheta F_v^{(a)} \right] \chi_1^{(a)} \right)

= \frac{R \rho_s^2}{c_s^2} \left( \frac{\partial^2}{\partial t^2} \frac{\partial \bar{\varphi}}{\partial r^2} + \frac{\partial^2}{\partial t^2} \frac{\partial^2 \chi_1^{(a)}}{\partial r^2} \right) F_v^{(a)} + \rho_s c_s \frac{\partial}{\partial r} \Pi(r, \vartheta). \tag{13}

Because the Mathieu characteristic functions constitute an orthogonal complete set, multiplying Eq.(13) by \( F_v^{(a)} \equiv \exp(\varepsilon \cos \vartheta / 2) f_1^{(a)} \) and integrating over \( \vartheta \) yields the radial equation of each harmonic. For the first sinusoidal component, \( \nu = 1 \) and \( \alpha := s \), the radial equation becomes

\[
\left( \frac{R^2}{c_s^2} \frac{\partial^2}{\partial t^2} + \frac{1 + \tau_i}{q^2} \right) \chi_1^{(s)} - \frac{R \rho_s^2}{c_s^2} \frac{\partial^2}{\partial t^2} \frac{\partial^2 \chi_1^{(s)}}{\partial r^2} - \frac{2R \rho_s}{c_s} \frac{\partial}{\partial r} \bar{\varphi} + \rho_s c_s \frac{\partial}{\partial r} \Pi(r, \vartheta)

= \frac{2R \rho_s^3}{c_s} \frac{\partial}{\partial t} \int d\vartheta \exp \left( \frac{\varepsilon}{2} \cos \vartheta \right) \sin \vartheta + \frac{\varepsilon}{12} \sin 2\vartheta \Pi(r, \vartheta). \tag{14}

The same substitution \( \varphi \rightarrow \chi_1^{(s)}(r, t) F_1^{(s)}(\vartheta) \) - can be similarly applied to the charge conservation Eq.(6), which on \( \vartheta \) integration, reduces to (in lowest order)

\[
\rho_s (1 + \tau_i) \frac{\partial \chi_1^{(s)}}{\partial r} + \frac{R \rho_s^2}{c_s} \frac{\partial}{\partial t} \frac{\partial^2 \bar{\varphi}}{\partial r^2} + R \rho_s \left[ \Pi(r, \vartheta) \right]_\vartheta = 0. \tag{15}

In Eq.(15) the last term

\[
\int d\vartheta \Pi(r, \vartheta) - \frac{\partial^2}{\partial r^2} \left( \int \frac{\partial \bar{\varphi}}{\partial \vartheta} \frac{\partial \bar{\varphi}}{\partial \vartheta} \right)

\tag{16}

is the average micro-turbulence drive that generates the zonal flow and GAM; the quantity behind the second order radial derivative in Eq.(15) is the well-known Reynolds stress.
The poloidal moment of the Reynolds stress (introduced by the toroidal coupling) is obtained upon performing surface average on Eq.(14)

\[
\mathcal{R} \left[ K \right] \equiv -\rho_c^2 c_s^2 \int d\mathcal{O} \left( \frac{\partial \hat{\Phi}}{r \partial \mathcal{O}} \frac{\partial \hat{\Phi}}{\partial r} \right) K (\mathcal{O}).
\] (17)

In Eq.(17) \( K = 1, \sin \mathcal{O}, \cos \mathcal{O}, \sin 2 \mathcal{O} \). The analytical expression for \( \mathcal{R} \left[ K \right] \) can be obtained straightforwardly as shown in Appendix C.

As long as the group velocities of the drift wave and zonal flows are real in spatiotemporal representation, the modulation of drift wave energy by zonal flow comes through modulation with \( \cos^2 \Theta \) (equivalently \( \cos (2 \Theta) / 2 \)) for a single rational surface.

Eqs.(14),(15) constitute the zonal flow-sound wave system. However, there are two caveats we should address before justifying the theoretical framework. The first is the missing viscosity term in zonal flow equation Eq.(15). This can be readily seen by taking the slab limit (removing the coupling to sound wave) as compared with Eq.(14) of Ref.8. This term is important, at least, to suppress numerical problems at short scales. Another issue will be explained after Eq.(20).

After adding the viscosity term, the equivalent zonal flow equation pertaining to the single rational surface, Eq.(15) becomes

\[
\frac{\partial \hat{\nabla}}{\partial t} - \mu \frac{\partial^2 \hat{\nabla}}{\partial r^2} = -\left( 1 + \tau_i \right) \chi_1^{(i)} - \frac{1}{2} \frac{\partial}{\partial r} \left( \mathcal{R} [1] + e \mathcal{R} [\cos \mathcal{O}] \cos 2 \Theta \right).
\] (18)

In Eq.(18) \( \psi \equiv 1 - \left( r - r_j \right) / L_{T_e} \) describes the (linear) electron temperature profile, \( L_{T_e} \) is the electron temperature gradient length. In the slab limit (by dropping \( \chi_1^{(i)} \) and \( \mathcal{R} [\cos \mathcal{O}] \)) Eq.(18) reduces to Eq.(19) of Ref.8 with \( a_{\text{neq}} \equiv 1 \).

Substituting Eq.(18) into Eq.(14) yields the equation for the first harmonic sinusoidal component of sound wave

\[
\left[ \frac{\partial^2}{\partial t^2} \left( 1 + D(\tau_i) \rho_i^2 \frac{\partial^2}{\partial r^2} \right) + 2 \left( 1 + \tau_i \right) \frac{\psi c_s^2}{R^2} \left( 1 + \frac{1}{2q^2} \right) \right] \chi_1^{(i)} =
\]

\[
\frac{2\mu}{R} \frac{\partial^2 \hat{\nabla}}{\partial r^2} - \frac{1}{R} \frac{\partial}{\partial r} \left( \mathcal{R} [1] + e \mathcal{R} [\cos \mathcal{O}] \cos 2 \Theta \right)
+ \frac{\rho_i}{c_i} \frac{\partial}{\partial t} \frac{\partial^2}{\partial r^2} \left( \mathcal{R} [\sin \mathcal{O}] + \frac{e}{3} \mathcal{R} [\sin 2 \mathcal{O}] \right) \cos 2 \Theta.
\] (19)
In Eq.(19) \( \Theta \) is the eikonal of the drift wave energy \(^{[8]}\)

\[
\Theta(r,t) = k_0 \int_0^1 \left( r - \int_{s_0}^s \nu_{gs} \left( \mathcal{G}(s) \right) \right) d\mathcal{G}(s)/ds = \nu_{gs} \left( \mathcal{G} \right) / r_j .
\]  

with \( \mathcal{G}(s = 0) = 0 \), \( \nu_{gs} \) is the radial (poloidal) group velocity. \( \bar{\mathcal{G}} \equiv \rho_j c_s \partial \mathcal{G} / \partial r \) is the zonal flow. \( \mu \) is the perpendicular viscosity; for classical fluid model, \( \mu \rightarrow \mu_B \equiv 3

\nu_i \rho_i^2 / 10 \), where \( \nu_i \) is the ion-ion collision frequency and \( \rho_i \equiv \sqrt{T_i / m_i \omega_{ci}^2} \) is the ion Larmor radius. For a real plasma, the viscosity is likely to be anomalous, \( \mu = \mu_B \). \( a_\mu \) being the measure of anomaly. In the preceding equations, \( k_\beta \equiv m / r_j \) is the poloidal wave vector, and \( n \) is for toroidal mode number. At the mode rational surface, the poloidal mode number \( m = nq(r_j) \), \( q(r_j) \) being the safety factor.

A crucial comment, regarding the weak dispersive (finite Larmor radius) term in Eq.(19), is in order. The first harmonic sinusoidal component in the sound wave equation (in this section) is derived from a fluid model that implies \( D(\tau_\epsilon) \rho_i^2 \rightarrow -\rho_i^2 \), a relationship valid only for cold ions. For warm (and possibly kinetic) ions, there have been many attempts to derive an analytic form of \( D(\tau_\epsilon) \) (see Ref.13 and cited references for attempts based on the gyro-kinetic theory).

The coefficient \( D(\tau_\epsilon) \) is a function of \( \tau_\epsilon = T_e / T_i \), introduced in Ref.13 to signify the change of sign to the term. There are some forms of \( D(\tau_\epsilon) \) which are always positive regardless the value of \( \tau_\epsilon \), apparently not suitable in cold ion limit. Here we simply change the electron Larmor radius \( -\rho_i^2 \) to \( D(\tau_\epsilon) \rho_i^2 \) as the same form in Eq. (19), leaving \( D(\tau_\epsilon) \) to be variable.

Eqs.(18), (19) are two components of the zonal flow equation set in tokamak. This is in contrast to Ref.8 where only one component exists. The second component comes into the system because of the toroidal coupling to sinusoidal component of sound wave owing to the geodesic curvature. The two component set contains two branches in different frequency regime GAM and TLFZF. Interested readers may wonder if the low frequency branch of this set is consistent with the discussion in Ref.8. The answer to this question is ‘yes’ as briefly discussed in Appendix D.
It would be worth emphasizing that the zonal flow equations Eqs. (18), (19) are not, strictly, a rigorous consequence of the ‘first principles’ Braginskii model:

1. The weak dispersive term $-\rho_s^2$, characteristic of the Braginskii model, is replaced by the heuristic $D(\tau_s)\rho_s^2$. This substitution is motivated by to include/capture the essence of kinetic effects (not included in Braginskii model); the relative negative sign is a major qualitative change, and it plays a significant role in GAM propagation.

2. The second modification is the addition of the dissipation $\mu$-term at the last step, so that Eq. (18) reduces to Eq. (19) of Ref. 8 when the toroidal coupling is removed. This term is important at small scale, however, the qualitative features of numerical results are not found sensitive to the value of $a_\mu$, say in range of 3-10.

Before carrying out numerical calculations, it may be appropriate to briefly describe the physics involving decay of a pair of caviton into instantons, and the role played by radial group velocity etc. (for details, please see Sec. V of Ref. 8).

As shown in Eq. (20), the zonal flow modulates drift wave energy in phase ($\phi \sim \cos \Theta$, where $\phi$ is the amplitude of the drift wave). The radial group velocity $\nu_{gr}$ appears as the argument of $\vec{\nu}$ under integral, describing the movement of drift wave energy along the radial characteristic line $r_g(t) = \int_r^t ds \nu_{gr} \left( \Theta(s) \right)$. According to the calculation of drift wave group velocity (ITG fluid model in Ref. 8 and in this paper, electron drift wave model in Part II), $\nu_{gr}$ always consists of two consecutive distinct phases: a long slowly varying part (at high level) and a short sudden spike crossing zero. Notice that the zonal flow $\vec{\nu}$ (in this paragraph refers to LFZF only) is localized around the region where the Reynolds stress is not small (reaction region). Before crossing zero, $\nu_{gr}$ is large, $r_g$ would run out of the reaction region depending on the reference position $r$, where $\vec{\nu}$ is too small to contribute to the integral in Eq. (20). This process corresponds to formation of a pair of caviton as $\Theta$ is slowly varying. Upon $\nu_{gr}$ zero-crossing, the sign is changed, making $r_g$ smaller, and pulling the local integrand $\vec{\nu}$ back to the reaction region, making $\vec{\nu}$ contribute to $\Theta$ again. Such a process occurs on different instants at different reference positions, just like wave propagation. It annihilates a pair of caviton into instantons and propagates along with $r_g$, taking $\Theta$ far away from reaction region. The details can be seen from Fig. 6-10 in Ref. 8, in particular Fig. 7.

III Boundary conditions and numerical methods for the dimensionless zonal flow equation set
We first list the various normalization introduced in Ref.8. The zonal flow speed is normalized to $ar{V}_z = \rho \omega x k_s \sqrt{I_m(\tau_0)}$ ($\bar{V} = V \bar{V}_z$), time is normalized to $\tau = \omega_z k_s (t \omega_z = \tau)$, the zonal flow Reynolds number is defined as $R_z \equiv \omega_z / \mu k_s^2 \delta^2$ ($\mu = 3a_r V_a \rho_i^2 / 10$), and the dimensionless Reynolds stress is $R[K] \equiv \hat{\gamma}[K] / \rho_i c_s^2 k_s^2 \delta^2 I_m(x_0)$. Then we define the dimensionless first harmonic sinusoidal component of sound wave to be $\chi \equiv k_s R \chi_1^{(1)}$. By introducing two frequencies

$$\omega_0^2 \equiv 2\left(1 + \tau_0\right) c_s^2 R^2 \left(1 + \frac{1}{2q^2}\right), \quad \sigma_0^2 \equiv \frac{1 + \tau_1}{q^2} c_s^2 R^2,$$

and $\hat{\omega}_0 \equiv \omega_0 / \omega_z$, $\hat{\sigma}_0 \equiv \sigma_0 / \omega_z$, Eqs.(18-20) can be cast into the dimensionless form

$$\frac{\partial V}{\partial \tau} - \frac{1}{R_c \delta^2 \tan^2} \frac{1}{\delta^2 \tan^2} = -\psi q^2 \hat{\omega}_0^2 \chi - \frac{1}{2} \frac{\partial}{\partial x} \left\{\left[R[1] + \varepsilon R[\cos \theta] \cos 2\Theta\right]\right\}, \quad \Theta(x, \tau) = \int_0^1 d\tau V \left(x - \delta, \frac{d}{d\tau} \hat{\omega}_0 \left(\theta(\xi)\right), \tau\right), \quad \frac{d\theta(\xi)}{d\xi} = \hat{\omega}_0 \left(\theta(\xi)\right) / \left(k_s \tau_1\right),$$

$$\left[\frac{\partial^2}{\partial \tau^2} \left(1 + D(\tau)\right) \delta^2 \frac{\partial^2}{\partial x^2}\right] + \gamma_{LD} \frac{\partial}{\partial \tau} + \psi \omega_0^2 \chi = \frac{2}{R_c} \frac{\partial^2 V}{\partial x^2} - \frac{\partial}{\partial x} \left\{\left[R[1] + \varepsilon R[\cos \theta] \cos 2\Theta\right]\right\},$$

where $\hat{\omega}_0 \equiv \omega_0 / \bar{V}_z$, $\hat{\sigma}_0 \equiv \sigma_0 / \bar{V}_z$, $\delta \equiv \rho \omega z \omega y / \omega_k$, $c_R \equiv R k_s \delta z / \omega_y$, and a new term associated with $\gamma_{LD}$ is introduced to simulate the Landau damping on GAM in low q region [14,15]. Before solving the preceding equations, we make use of the 2D mode structure of weakly asymmetric ballooning theory (WABT [25]); In Appendix C, the Reynolds stress and the drift wave group velocities (Eqs.(10-11) in Ref.8) are calculated.

The solution of the initial value problem is worked out for the initial conditions: $V(x, 0) = 0$, $\Theta(x, 0) = 0$ and $\chi(x, 0) = 0$. Since the phase modulation of drift wave energy is significant only inside the so-called reaction region, where the Reynolds stress is not small, the boundary condition for drift wave energy is not important; the latter is relevant only in the instanton phase that has a vanishing coupling to zonal flow outside reaction region. However, the signal of phase function $\Theta$ could propagate to a place far away from the reaction region denoted by $x_{\text{ref}}$, where the cut-off has been introduced ($\Theta(x_{\text{ref}}, \tau) = 0$) [8].
The boundary condition for the zonal flow equation set has to be set up differently with respect to left and right side in contrast to Ref.8, since the data of GAM are measured not too far away from plasma edge, and edge effects on GAM could be important.

The term $\gamma_{LD}(r) \partial / \partial \tau$ in Eq.(24) is a model for Landau damping on the sound wave. The original form $\exp(-q^2)$ is simulated by Eq. (25), i.e., as approaching plasma edge, $q$ gets so big that the Landau damping vanishes; it is much bigger away from the edge.

$$\gamma_{LD}(r) = \begin{cases} 0 & r \in [r_{j-10}, a] \\ \hat{\partial}_r \left[ \frac{(r - r_{j-10})}{(r_{j-12} - r_{j-10})} \right]^2 & r \in [r_{j-12}, r_{j-10}] \end{cases}.$$  \tag{25}

In Eq.(25) $r_{j,\ell} \equiv r_j + (x_0 + \ell \sqrt{n \sigma}) / |k_p \delta|$, $\ell = 0, \pm 1, \ldots, \pm 12$, $r_{j,0}$ stands for the peak position of the central driving torque $(\partial R / \partial r)$, see Eq.(17); the region from $r_{j,-2}$ to $r_{j,2}$ is more or less the reaction region and $\sqrt{n \sigma} / |k_p \delta|$ stands for the half width of the Gaussian peak. As a result, the boundary condition can be set up as $V(x_-, \tau) = 0$, $\chi(x_-, \tau) = 0$.

On the right side, the zero Dirichlet (reflecting) boundary condition is chosen at the plasma edge for $V(x_{edge}, \tau) = 0$, $\chi(x_{edge}, \tau) = 0$, where $x_{edge} \equiv a$ denotes the position of plasma edge.

The dimensionless set of Eqs.(22-24), combined with the assumed boundary conditions, constitutes a well-posed initial value problem, which is solved by making use of the finite difference methods, where the spatiotemporal grids are discretized as $(r_k, t_m)$, where $r_k \equiv r_{j,-12} + k \cdot \Delta r$, $t_m \equiv m \cdot \Delta t$, $k=0,1,\ldots,K$ and $m=0,1,\ldots,M$ are integers, in this paper we choose $K = 512$, $M = 12000$, $\Delta r = (a - r_{j,-12}) / K$, $\Delta t = 5\mu s$. The dimensionless spatiotemporal step sizes are $\Delta x \equiv |k_p \delta| \cdot \Delta r$ and $\Delta \tau \equiv \omega_z \cdot \Delta t$ respectively.

The zonal flow equation set Eqs.(22),(24) is solved by making use of the 2nd order Crank-Nicolson method \[16\] with accuracy up to $6 \times 10^{-3}$ for $\omega_o \approx 15kHz$. The wave energy equation Eq.(23) is directly integrated as shown in Appendix C of Ref.8.
For illustrative purposes the numerical experiment is performed for the parameters corresponding to the shot #141958 on DIII-D [5] shown in Table II.

Table II: Basic equilibrium parameters

| \( \hat{s} \) | q | \( \bar{\eta}_i \) | \( \varepsilon_n \) | R [m] | a [m] | B [T] | \( \tau_e \) | n | \( r_j \) [cm] | \( T_e \) [eV] | \( N_e \) \( [10^{19} \text{m}^{-3}] \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 4 | 4.3 | 0.12 | 1.7 | 0.6 | 1.8 | 1 | -60 | 54 | 140 | 1.2 |

Some ITG related parameters required for Reynolds stress are listed in Table III.

Table III: Partial mode related parameters

| \( \eta^2 \) | \( k_x(x_0) \) | \( \beta_1 \) | \( \beta_2 \) | \( \sigma \) | \( x_0 \) |
|---|---|---|---|---|---|
| -4+2i | -0.002-0.002i | 0.21-0.002i | 0.43-0.08i | 0.65 | -12.9 |
| \( \omega \) [kHz] | \( I_m(x_0) \) | \( a \eta \) | \( \omega_c \) [kHz] | \( L_r \) [cm] | \( D(\tau_e) \) |
| -270 | \( 5 \times 10^6 \) | 3 | 15.6 | 6 | 1 |

Table IV: The five radial points near the peak of static Reynolds stress \( r_{j,0} \) and the radial position of \( x_{\pm \infty} \).

| \( x_{-\infty} \equiv r_{j,-12} \) [cm] | \( r_{j,-2} \) [cm] | \( r_{j,-1} \) [cm] | \( r_{j,0} \) [cm] | \( r_{j,1} \) [cm] | \( r_{j,2} \) [cm] | \( x_{\pm \infty} \equiv a \) [cm] |
|---|---|---|---|---|---|---|
| 47.1 | 54.1 | 54.8 | 55.5 | 56.2 | 56.9 | 60 |

**IV The intermittent excitation of GAM in ITG background turbulence**

In this section ITG is assumed to be the micro-turbulence generating zonal flow as suggested by Appendix A. The intermittent excitation of GAM can be clearly seen in Fig.1, and Fig.2 at five distinct radial positions (defined in Table IV, in which the radial positions of \( x_{\pm \infty} \) are also given). One can also see that the intermittent excitation of GAM is highly correlated with the downward as well as upward zero-crossing of radial group velocity, and the typical period ~ 4ms, is within the experimentally observed range, e.g., 2-5ms in ASDEX-U [1], 2-4ms in T-10 [2], 1-3ms in HL-2A [4]. The structure of temporal evolution is spatially-dependent, this is also consistent with experimental observations in the sense that no temporal pattern is recognized so far. One can also observe that the GAM amplitudes increase with higher level of TLFZF from Fig.1. The downward (upward) zero-crossing results in stronger (weaker) and longer (shorter)-lasting
GAM. The similar patterns shown in Fig. 2 are also reported in JFT-2M (Fig. 2 (d) of second paper of Ref. 3), JET (Fig. 7 of second paper of Ref. 6), and EAST (Fig. 8 of Ref. 7).

Fig. 1 (a) Temporal evolution of radial group velocity and (b-f) zonal flow with intermittent excitation of GAM at five radial positions, (b) $r_{j,-2}$, (c) $r_{j,-1}$, (d) $r_{j,0}$, (e) $r_{j,1}$, (f) $r_{j,2}$
The frequency-time spectrogram, like that shown in Fig.3 (a), has been reported in ASDEX-U (Fig.13 (a) of Ref.1) and in T-10 (Fig.20 of Ref.2) with similar intermittency characteristics. It is also reported in HL-2A (Fig.19 of first paper and Fig.3 (a) of second paper of Ref.4), however, with quite different intermittency periods. Noticeably, the measurement in HL-2A is done very close to the plasma edge.

The temporal evolution of spatial structure for the zonal flow $\vec{D}$ and its high frequency component $\vec{u}_{GAM}$ is presented in Fig.3 (b), (c) respectively. Similar spatiotemporal patterns, however, have not yet been reported in experiments - perhaps because of its fine radial structure, and inadequate spatial resolution of diagnostics. Right after 54.7ms, the so-called pre-GAM (around GAM frequency, but irregular spatial structure) is generated moving inwardly as driven by the ingoing instantons (see corresponding wave energy pattern in the movie in the caption of Fig.5 (Multimedia view), and also in snapshots in Fig.5 (a), (b) and (c)). When the wave front hits the turning points, only a portion is reflected back outwardly. The coherent pattern is thus formed and evolves into a semi-eigenmode till ~ 56.83ms when the caviton decays into outgoing instantons. A shorter life-time pre-GAM is generated, but it gradually dissipates away. The
reflection boundary at plasma edge has little effect on the subsequent behavior because the incoming wave near plasma edge is not strong enough to support a reflected wave that could reach the inner WKB turning point to yield what could be classed as a full eigenmode \cite{17}.

Fig. 3 (a) Frequency-time spectrogram at $r_{j,0}$, (b) and (c) for spatiotemporal evolution for the zonal flow $\vec{v}$ and its high frequency component $\vec{v}_{GAM}$ respectively.

As mentioned in the beginning of introduction, “Experimentally, the GAM is an intermittent (periodic stopping), random, discrete temporal structure”. Notice that randomness and not perfect periodicity is the defining character.

The quantitative study of randomness has been documented in Ref. 5 by making use of auto-correlation function. The auto-correlation function defined in Eq.(B.2) in Appendix B of Ref. 5 is a very useful concept in experiment. It could reveal the nature of the observed intermittency, i.e. whether stochastic or deterministic. The measurement is displayed in Fig. 15 (a) and (b) of Ref. 5 for two different radial positions, also reported in Fig. 9 of Ref. 7. Similar ‘numerical measurements’ are carried out and presented in Fig. 4 for two spatial positions. While the patterns are somewhat different
from those in experiments they share one very important feature, a long tail after non-exponential quick fall-off. The non-exponential quick fall-off is consistent with non-Gaussian process; the long tail can be interpreted to be residue oscillations supported by TLFZF shown in Fig.2; both belong to deterministic system Eqs.(18-20).

![Auto-correlation function for different radial position](image.png)

Fig.4 Auto-correlation function for different radial position at (a) $r_{j,-1}$ and (b) $r_{j,0}$

In addition to Fig.3 (c) the spatiotemporal evolution of GAM can, equivalently, be represented by the movie - ‘GAM spatiotemporal evolution’ in the caption of Fig.5 (Multimedia view). Not only it captures the clear radial structure of GAM as a snapshot at any time, but also captures the temporal correlation to the motion of drift wave energy. Since the
movie can only be viewed online, nine distinct snapshots are selected and displayed in Fig.5. Two physical quantities, the high frequency zonal flow $\vec{U}_{\text{GAM}}$ and the normalized wave energy represented by $\cos^2 \Theta$, are displayed jointly on the same time base during a cycle; the temporal (spatial) range is 54-60ms (48-60cm). This choice of spatial dimension is reasonable because: (1) the plasma edge is at 60cm, set to be the right boundary, i.e., the reflection boundary for outgoing zonal flow; (2) the so-called WKB turning point lies within 52-55cm as seen from Fig.3 (c); the latter serves to be the reflection layer for the ingoing pre-GAM. Let us see how the entire dynamics unfolds.

Initially, between 54-54.7ms, a pair of slowly breathing cavitons emerge, and begin to grow (Fig.5 (a)). During a very short period of time 0.1ms (from 54.7 to 54.8ms) three events occur, almost simultaneously: (1) The amplitude of normalized wave energy grows – crosses half and eventually becomes unity, (2) the pre-GAM starts to form rapidly in the reaction region (where static Reynolds stress is not small), and (3) the caviton pair decays into instantons (Fig.5 (b)). Right after 54.85ms instantons quickly propagate inwards and bring up the ingoing pre-GAM (Fig.5 (c)). The ingoing instantons disappear after 55ms and a new caviton pair starts to form and breathe slowly in the reaction region (Fig.5 (d)). At this moment, the pre-GAM front reaches the WKB turning point; the penetrated part is then absorbed somewhere further inward, while the reflected part moves outwards. After that, the interference between the ingoing and outgoing zonal flow around GAM frequency results in the transit phase of forming an “eigenmode” between the WKB turning point and the right edge of reaction region. At 56.7ms (Fig.5 (e)), the outgoing GAM reaches the right zero boundary (GAM cannot propagate outside the last closed magnetic surface), and then is reflected back to form semi-eigenmode between plasma edge and the inner turning point (till 56.82ms (Fig.5 (f)). At this moment a caviton pair starts first grows and then decays into outgoing instantons. Right after 56.9ms, the instantons quickly propagate outwards (Fig.5 (g)) inducing right moving pre-GAM overlapped with the existing outgoing GAM. Such an interference results in a rather complicated pattern till 57.03ms (Fig.5 (h)), at that moment the outgoing instantons run outside the reaction region. Afterwards, the GAM gradually dissipated away. The process mentioned above occurs almost repetitively at 59ms when another caviton pair grows and decays into ingoing instantons (Fig.5 (i)). A new instance of pre-GAM is generated and moves inwardly just at that moment.
Fig. 5 Snapshots of 9 time points in the movie, (a) 54.70 ms, (b) 54.80 ms, (c) 54.85 ms, (d) 55.00 ms, (e) 56.70 ms, (f) 56.82 ms, (g) 56.90 ms, (h) 57.03 ms, (i) 59.00 ms. The time evolution for 54-60ms can be seen via the link [GAM spatiotemporal evolution] (Multimedia view)

V Summary and discussions

We conclude this paper by emphasizing three new high-level qualitative results in intermittent excitation of GAM:

1. The numerical experiment based on the deterministic GAM system (Eqs.(18-20)) seems to reproduce many of the characteristics associated with the intermittent geodesic acoustic mode observed in experiments in a variety of tokamaks (Ref. 1-7).

2. GAM amplitude increases with higher level of TLFZF, the torus-modified low frequency zonal flow. Since GAM is observable only if TLFZF grows beyond a certain level, toroidal effects are fundamental to the understanding of the experiments. The generated GAM is an intrinsic nonlinear spatiotemporal structure (near GAM frequency) accompanying TLFZF.

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1 http://home.ustc.edu.cn/~lzy0928/GAM%20spatiotemporal%20evolution.mp4
(3) The GAM generation is triggered when the radial group velocity of the drift wave crosses zero; it is the precise moment for the phase transition from a caviton pair into instantons - there appears to be a causal relationship between the phase transition and GAM generation.

Since essential features of the experimental observation are accessible to a deterministic physics model, it is quite legitimate to claim that the intermittency of GAM observed in tokamaks is deterministic.

Finally, we must apprise the reader of the somewhat limited scope of this paper. Many interesting GAM phenomena observed in experiments - such as ‘continuum’ versus ‘eigenmode’, coexistence of multiple modes, radial extension of ‘eigenmode’ and the scaling law etc. as well as the effects arising from nonlinear coupling between rational surfaces - are not discussed here. We have concentrated here (through the numerical solutions of the three equations) only on a subset of GAM related issues. A more complete and systematic exploration (including for example working out the parameter dependences), is deferred to a future paper. Furthermore, an invariant turbulence level was assumed to calculate the time evolution of the system. Since the generated shear flow tends to suppress turbulence, this assumption is not fully valid - we expect quantitative modifications to our results through shear suppression. However, the theory developed in Ref.8 as well as in this paper, limited to the like-mode coupling either in calculation of Reynolds stress and concepts regarding drift wave energy, will give a qualitative knowledge of the dynamics. Unlike-mode coupling is surely possible and it may lead to new features - a resonant mechanism [18], for instance.

Appendix A  Conditions for detrapping due to collision and occurrence of ITG in L-mode discharge near tokamak edge

There is circulating a belief that GAM is likely generated by TEM not by ITG mode, because at the tokamak edge \( \eta_i \) is normally less than 2, the threshold for instability.

In fact, in L-mode discharges near tokamak edge, the collision frequency is high enough that particle’s trapping is minimal. Mathematically, this condition translates into \( \omega_b \tau_{\text{detrap}} < 1 \), where the bounce frequency \( \omega_b = \left( \frac{v_{Te}}{qR} \right) \left( \frac{r}{R} \right)^{1/2} \) and the de-trapping time \( \tau_{\text{detrap}} \approx \left( \frac{r}{R} \right) \tau_e \) [ \( v_{Te} \equiv \sqrt{T_e/m_e} \) is the electron thermal speed and \( \tau_e \sim \tau_e^{3/2} / N_e Z_{\text{eff}}^2 \) is the Coulomb collision time]. For typical tokamak geometry and plasma densities, the
critical temperature, below which particle trapping is a minor effect, is a few hundred eV \[19\]. Experimental Data on a variety of tokamaks studying zonal flows is shown in Table V (with \( Z_{\text{eff}} = 1 \)). Since \( \omega_b \tau_{\text{detrap}} \approx 1 \) is true for all cases, the trapped density will most likely be too small to support a TEM.

Table V  
Equilibrium parameters of L-mode (Ohmic) discharge in the edge (\( r / a \approx 0.9 \)) of 7 tokamak machines

| Tokamaks      | Shots                  | \( q \) | \( N_e [\times 10^{19} \text{m}^{-3}] \) | \( T_e [\text{eV}] \) | \( \nu_e [\text{kHz}] \) | \( \omega_b [\text{kHz}] \) | \( \omega_b \tau_{\text{detrap}} \) |
|---------------|------------------------|--------|-----------------------------------|------------------|-----------------|----------------------|---------------------|
| ASDEX\[20\]  | #20787 (Ohmic)         | 3.0    | 1.3                               | 180              | 227             | 595                  | 0.72                |
| T-10\[2\]    | #36819 (L-mode)        | 3.5    | 1.0                               | 200              | 151             | 480                  | 0.57                |
| JFT-2M\[3\]  | (L-mode)               | 2.8    | 1.0                               | 170              | 190             | 687                  | 0.75                |
| HL-2A\[4\]   | (L-mode)               | 3.0    | 0.65                              | 96               | 285             | 390                  | 0.30                |
| DIII-D\[5\]  | #141958 (L-mode)       | 4.0    | 1.2                               | 140              | 300             | 412                  | 0.44                |
|              | #142121 (L-mode)       | 4.0    | 2.6                               | 220              | 332             | 517                  | 0.49                |
| JET\[21,22\] | #86470 (Ohmic)         | 3.2    | 2.0                               | 300              | 165             | 424                  | 0.78                |
|              | #90492 (L-mode)        | 3.0    | 2.7                               | 300              | 220             | 452                  | 0.62                |
| EAST\[7\]    | #74036 (L-mode)        | 4.9    | 1.1                               | 100              | 447             | 212                  | 0.10                |

The case for the ITG turbulence is a little stronger; there are published papers measuring GAMs in which the edge (\( \rho \equiv r / a \approx 0.9 \)) conditions are such \( \eta_i = L_n / L_{\tau_i} > 2 \) that unstable ITG can readily exist. Table VI lists edge-conditions for four machines; the data sources are: Fig.5 of Ref.20 for ASDEX, Fig.3 of Ref.5 for DIII-D, Fig.15 of second paper of Ref.6 for JET, and Fig.3 of Ref.7 for EAST. All shots, except for #18813 that provides ion temperature profile, are analyzed by assuming \( L_{\tau_i} = L_{\tau_i} \).

Table VI  \( \eta_i \) in L-mode (Ohmic) discharge near tokamak edge (\( \rho \equiv r / a \approx 0.9 \)) on 4 machines

| Tokamaks      | Shots                  | \( L_n [\text{cm}] \) | \( L_{\tau} [\text{cm}] \) | \( \eta_i \) |
|---------------|------------------------|-----------------------|--------------------------|-------------|
| ASDEX\[20\]  | #20787 (Ohmic)         | 9.7                   | 4.3                      | 2.2         |
|              | #18813 (Ohmic)         | 18.7                  | 7.4                      | 2.5         |
| DIII-D\[5\]  | #141958 (L-mode)       | 20.1                  | 5.9                      | 3.4         |
|              | #142121 (L-mode)       | 37.8                  | 10.2                     | 3.7         |
| JET\[6\]     | #87802 (Ohmic)         | 22.3                  | 7.3                      | 3.1         |
| EAST\[7\]    | #74036 (L-mode)        | 8.2                   | 3.8                      | 2.2         |
Appendix B  Derivation of charge and particle number conservation equation for axisymmetric electrostatic mode

In this Appendix the derivation of charge and particle number conservation for low frequency electrostatic axisymmetric mode, Eqs.(1),(2) respectively, is presented on basis of Braginskii two-fluid model [9] essentially same as Ref.10, but extended from cold ion to warm ion.

It begins with symbol definition. The quantities having both overbar and tilde stand for fluctuations in meso-scale, while those without overbar and tilde stand for equilibrium with subscript \(i\) \((e)\) representing ion (electron). \(u\), \(J\), \(N\), \(P\) and \(\varphi\) are fluid velocity, current, density, pressure and potential respectively, \(\Pi\) is viscosity tensor, \(R\) is friction force \((R = -R_e)\), \(B\) is magnetic field, \(b\) is the unit vector along the equilibrium magnetic field, \(T_{i,e}\) is ion (electron) temperature, \(c\) is the speed of light, and \(e\) is the unit electric charge.

The ion momentum equation is \([9]\)

\[
\frac{D}{Dt} + \nabla \cdot \mathbf{u}_i = eN\left(\mathbf{E} - \frac{1}{c} \mathbf{u}_i \times \mathbf{B} \right) - \nabla \mathbf{P} + \nabla \cdot \mathbf{\Pi}_i + \mathbf{R}_e. \tag{B.1}
\]

To the leading order for low frequency waves \(\omega \ll \omega_{ci}\), where \(\omega_{ci} = eB / cm_i\) is the ion cyclotron frequency, the perpendicular component of Eq.(B.1) is chosen to be the electrostatic and diamagnetic drift velocity

\[
\mathbf{u}_{i,\perp}^{(0)} = \mathbf{u}_K + \mathbf{u}_{i,D}, \quad \mathbf{u}_K \equiv \frac{c}{B} \mathbf{b} \times \nabla \varphi \quad \mathbf{u}_{i,D} \equiv \frac{c}{eNB} \mathbf{b} \times \nabla \mathbf{P}_i. \tag{B.2}
\]

Neglecting friction force \(\mathbf{R}_e\), and substituting \(\mathbf{u}_{i,\perp}^{(0)}\) into the inertia term and viscosity tensor \(\mathbf{\Pi}_i\) yield

\[
\mathbf{u}_{i,\perp} = \mathbf{u}_K + \mathbf{u}_{i,D} + \frac{1}{\omega_{ci}} \mathbf{b} \times \left(\frac{D}{Dt} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla\right) \mathbf{u}_K + \frac{c}{eNB} \mathbf{b} \times \nabla \cdot \left(\mathbf{\Pi}_{i\parallel} + \mathbf{\Pi}_{i\perp}\right). \tag{B.3}
\]

In Eq.(B.3) \(\mathbf{\Pi}_{i\parallel}\) and \(\mathbf{\Pi}_{i\perp}\) are parallel and perpendicular viscosity tensor respectively. Eq.(B.3) is achieved by making use of the so-called Hinton-Horton cancellation \([23]\)

\[
m_i N \left(\frac{D}{Dt} + \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla\right) \mathbf{u}_{i,D} + \nabla \cdot \mathbf{\Pi}_{i,g} = 0. \tag{B.4}
\]

In Eq.(B.4) \(\mathbf{\Pi}_{i,g}\) is the gyro-viscosity. For axisymmetric mode \(\mathbf{b} \times \mathbf{u}_{i,\perp}^{(0)} \cdot \nabla \mathbf{u}_K\) can be neglected. Then, suppressing the other two viscosity tensors temporarily yield the expression for perpendicular velocity in terms of electrostatic potential and density fluctuation
\[
\vec{u}_{\perp} = \frac{c}{B} \left( \vec{b} \times \nabla \vec{\phi} + \frac{T_e}{eN} \vec{b} \times \nabla \vec{N} \right) - \frac{1}{\omega_{ci}} \left( \frac{c}{B} \frac{\partial}{\partial t} \nabla \cdot \vec{\phi} + \vec{u}_{NP} \right),
\]

where \( \vec{u}_{NP} \equiv \left\langle \left( c^2 / B^2 \right) \left( \vec{b} \times \nabla \vec{\phi} \cdot \nabla \vec{\phi} \right) \right\rangle_{\text{en}} \) is called the nonlinear polarization drift velocity. This is the term not covered by previous derivation in mesoscale, and represents the process that two microscale electrostatic potentials \( \vec{\phi} \) are annihilated into one velocity field in mesoscale, known as ‘three wave interaction’ in literature. \( \left\langle \ldots \right\rangle_{\text{en}} \) stands for ensemble average over microscopic scale, \( i.e. \vec{u}_{NP} \) is in mesoscale. Note that the microscale \( \vec{\phi} \) can only enter the mesoscale equation through the nonlinear terms under ensemble average. At this stage we would like to mention that such viscosity suppression does not sound plausible physically, since it could lead to numerical divergence in small scale. This will be discussed further in section II regarding Eqs.\((18),(19)\).

The electron momentum equation is

\[
m_e \vec{N} \left( \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right) \vec{u} = -eN \left( \vec{E} + \frac{1}{c} \vec{u} \times \vec{B} \right) - \left( \nabla \vec{P} + \nabla \cdot \vec{\Pi} + \vec{R} \right).
\]

The parallel Ohm law can be derived from the parallel component of Eq.\((B.6)\) straightforwardly by invoking the concept of conductivity \( \vec{R}_{||} = -eN \eta_{||} \vec{J}_{||} \), where \( \vec{J}_{||} \) is the parallel current fluctuation, \( \eta_{||} = m_e \nu_e / 2Ne^2 \), \( \nu_e \) is the electron-ion collision frequency. Simply, neglecting the electron inertia and viscosity tensor yields

\[
\eta_{||} \vec{J}_{||} = \left( \vec{b} \cdot \nabla \right) \left( \frac{T_e}{eN} \vec{N} - \vec{\phi} \right).
\]

In Eq.\((B.7)\) use is made of the assumed state equation \( \vec{P} := T_c \vec{N} \).

The one-fluid equation of motion can be derived by adding Eq.\((B.1)\) to Eq.\((B.6)\) with dropping electron inertia term and viscosity tensor

\[
m_N \left( \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla \vec{P} + \frac{1}{c} \vec{J} \times \vec{B} \cdot \vec{P} := \left( T_c + T_i \right) \vec{N}.
\]

In Eq.\((B.8)\) \( \vec{J} = eN \left( \vec{u}_i - \vec{u}_c \right) \), the subscript \( i \) of \( \vec{u}_i \) has been omitted.

In the above manipulations the assumed state equation introduced cut-off to energy conservation. This makes sense provided that the thermal physics merely plays minor role for physics of zonal flow.
The two conservation equations, namely the particle number conservation and charge density conservation, are originally

$$\frac{\partial \tilde{N}}{\partial t} + \tilde{u} \cdot \nabla \tilde{N} + N \nabla \cdot \tilde{u} = 0,$$  \hspace{1cm}  \text{(B.9)}$$

and

$$\nabla \cdot \tilde{J} = 0.$$  \hspace{1cm}  \text{(B.10)}$$

By taking the vector product of Eq.(B.8) with \( \mathbf{B} \), we obtain

$$\mathbf{J}_{\perp} = \frac{c}{B^2} \left[ \mathbf{B} \times \nabla \tilde{P} - cm_t N \frac{\partial}{\partial t} \left( \nabla_{\perp} \tilde{\phi} + \frac{N_t}{eN} \nabla_{\perp} \tilde{N} \right) \right] - nm_t \frac{c}{B} \tilde{u}_{NP}.$$  \hspace{1cm}  \text{(B.11)}$$

In Eq.(B.11) the polarization drift in \( \tilde{u}_{\perp} \) proportional to \( (\omega / \omega_t)^2 \) has been neglected.

Since the term

$$\frac{c}{B^2} \nabla \cdot (\mathbf{B} \times \nabla \tilde{P}) = \frac{4\pi}{B^2} \int \frac{c}{B} \frac{dP}{dr} \left( 1 - \frac{2r}{R} \cos \vartheta \right) + \left( E_{\parallel} / \eta_{\parallel} \right) \frac{r}{qR} \frac{dP}{d\vartheta}$$  \hspace{1cm}  \text{(B.12)}$$

can be neglected for low-\( \beta \) plasmas \[24] (\( \beta \) is the ratio of thermal pressure to magnetic pressure), the main contribution to divergence of diamagnetic current comes from the curvature of magnetic field in tokamak. For the divergence of linear and nonlinear polarization current, curvature effect can be neglected because \( |\kappa| \ll |\nabla| \), where \( |\kappa| \) and \( |\nabla| \) correspond to equilibrium scale and mesoscale respectively for \( \kappa := \nabla B / B \) is the curvature of magnetic field. Substituting the divergence of Eq.(B.7) and (B.11) into (B.10) yields charge conservation equation

$$\nabla \cdot \left( \frac{\tilde{b} (\mathbf{b} \cdot \nabla)}{\eta_{\parallel}} \left( \frac{T_e}{eN} \nabla_{\perp} \tilde{N} - \tilde{\phi} \right) \right) - \frac{2c}{B^2} \tilde{\kappa} \cdot (\mathbf{b} \times \nabla \tilde{P}) - m_t N \frac{c^2}{B^2} \frac{\partial}{\partial t} \left( \nabla_{\perp}^2 \tilde{\phi} + \frac{N_t}{eN} \nabla_{\perp}^2 \tilde{N} \right) = nm_t \frac{c}{B} \nabla \cdot \tilde{u}_{NP}.$$  \hspace{1cm}  \text{(B.13)}$$

Similar procedure is adopted for obtaining the divergence of \( \tilde{u}_{\perp} \). Substituting it into Eq.(B.9) yields particle number conservation equation

$$\frac{1}{N} \frac{\partial \tilde{N}}{\partial t} + \nabla \cdot (\tilde{u} \mathbf{b}) - \frac{2c}{B^2} \tilde{\kappa} \cdot \left( \mathbf{b} \times \nabla \tilde{\phi} + \frac{N_t}{eN} \mathbf{b} \times \nabla \tilde{N} \right) = \frac{1}{\omega_{\parallel}} \left( \frac{c}{B} \frac{\partial}{\partial t} \nabla_{\parallel}^2 \tilde{\phi} + \nabla \cdot \tilde{u}_{NP} \right).$$  \hspace{1cm}  \text{(B.14)}$$

In Eq.(B.14) the term \( \tilde{u} \cdot \nabla N \) is neglected for zonal flow which is in poloidal direction.

The parallel component of Eq.(B.8) is

$$m_t N \frac{\partial \tilde{u}_{\parallel}}{\partial t} = -(T_e + T_i) \nabla_{\parallel} \tilde{N}.$$  \hspace{1cm}  \text{(B.15)}$$
In terms of the dimensionless quantities, $\bar{N}/N \rightarrow \bar{N}$, $\bar{\phi}/T_e \rightarrow \bar{\phi}$, $\bar{\phi}/T_e \rightarrow \bar{\phi}$ and defining $\tau_i = T_i/T_e$, $\rho_i \equiv c_i/\omega_i$, $c_i \equiv \sqrt{T_e/m_i}$, the two conservation equations in toroidal coordinate system $(r, \theta, z)$ can be readily obtained for axisymmetric zonal flow on basis of Eq.(B.13-B.15) by invoking $\partial/\partial z \rightarrow 0$, and $\left| \partial/\partial r \right| \gg (1/r) \partial/\partial \theta$, i.e. radial variation is much faster than poloidal variation.

Charge conservation equation is

$$\left( \frac{\partial}{\partial \theta} - \frac{r \sin \theta}{R} \right) \sigma_s \frac{\partial}{\partial \theta} (\bar{n} - \bar{\phi}) - 2 \rho_s (1 + \tau_s) \sin \theta \frac{\partial}{\partial r} (\frac{\partial n}{\partial r} - \frac{R \rho_s^2}{c_s} \frac{\partial}{\partial t} \frac{\partial^2}{\partial r^2} (\bar{\phi} + \tau_r \bar{n})) = \frac{R \rho}{c_s^2} \nabla \cdot \bar{u}_{\text{NP}}. \quad (B.16)$$

Particle number conservation equation is

$$\frac{R^2}{c_s^2} \frac{\partial^2 \bar{n}}{\partial t^2} - \frac{(1 + \tau_s)}{q_s^2} \left( \frac{\partial}{\partial \theta} - \frac{r \sin \theta}{R} \right) \frac{\partial \bar{n}}{\partial \theta} - \frac{2R}{\omega_s} \sin \theta \frac{\partial}{\partial t} \frac{\partial}{\partial r} (\bar{\phi} + \tau_r \bar{n}) = \frac{R^2 \rho_s^2}{c_s^2} \frac{\partial^2 \bar{\phi}}{\partial r^2} + \frac{R^2 \rho_s}{c_s^2} \frac{\partial}{\partial t} \nabla \cdot \bar{u}_{\text{NP}}. \quad (B.17)$$

In Eq.(B.17) $\sigma_s = 2m_s c_s / m_e v_e R$, $R$ is the major radius, $\sin \theta$ in the second term of Eq.(B.16) results from the geodesic curvature. In nonlinear polarization current $\bar{u}_{\text{NP}} \equiv \left\{ \rho_s^2 c_s^2 \left( \bar{b} \times \nabla \bar{\phi} \cdot \nabla \bar{\phi} \right) \right\}_{\text{en}}$, $\bar{\phi}$ is 2D mode structure of micro-turbulence, which provides mesoscale driving force through three wave couplings, as depicted in Appendix C with the example of fluid ITG.

**Appendix C  Poloidal moments of Reynolds stress**

In this appendix use is made of the methodology of Ref.25 to calculate analytic formulae of poloidal moments of Reynolds stress induced by ITG micro-turbulence $\bar{\phi}$

$$\mathfrak{M}[K] \equiv -\rho_s^2 c_s^2 \bar{\phi} d \theta \frac{\partial \bar{\phi}}{r \partial \theta} \frac{\partial \bar{\phi}}{\partial r} K(\theta), \quad (C.1)$$

the same as Eq.(17) in section II. For reader’s convenience, the symbols are the same as those in Ref.25 throughout this appendix, where the 2D mode structure is

$$\bar{\phi}(r, \theta) = \sum \bar{\phi}_m(r) \cos [S(r) - (m + 1) \theta]. \quad (C.2)$$

The methodology used in Ref.25 has been numerically verified by comparing the results of the analytic formula with the direct numerical integral of Eq.(C.1) for $K(\theta) := 1$ satisfactorily.
Substituting Eq. (C.2) into Eq. (C.1) yields
\[
\mathcal{R}[K] = -\rho^2 c_s^2 \sum_{l,l'} \int d\theta \left( \frac{m+l}{r} \right) |\vec{\phi}(r)| \sin \left[ S(r) - (m+l)\theta \right] K(\theta) \times 
\left\{ \frac{d|\vec{\phi}_l(r)|}{dr} \cos \left[ S(r) - (m+l')\theta \right] - |\vec{\phi}_l(r)| \frac{dS(r)}{dr} \sin \left[ S(r) - (m+l')\theta \right] \right\}.
\]  
(C.3)

This form will be simplified by defining two new functionals
\[ P_{l,l'}[K] = \oint d\theta \sin \left[ S(r) - (m+l)\theta \right] \cos \left[ S(r) - (m+l')\theta \right] K(\theta). \]  
(C.4)
\[ Q_{l,l'}[K] = \oint d\theta \sin \left[ S(r) - (m+l)\theta \right] \sin \left[ S(r) - (m+l')\theta \right] K(\theta). \]  
(C.5)

For micro-turbulence \( m \gg 1 \), integral of high order harmonic vanishes, Eq. (C.4) and (C.5) become
\[ P_{l,l'}[K] = \frac{1}{2} \oint d\theta \sin \left[ (l'-l)\theta \right] K(\theta). \]  
(C.6)
\[ Q_{l,l'}[K] = \frac{1}{2} \oint d\theta \cos \left[ (l'-l)\theta \right] K(\theta). \]  
(C.7)

For \( k \)’th sinusoidal harmonic
\[ P_{l,l'}[\sin k\theta] = \frac{1}{4} \left( \delta_{l,l'-k} - \delta_{l,l'+k} \right), \quad Q_{l,l'}[\sin k\theta] = 0. \]  
(C.8)

Substituting Eq. (C.4) and (C.5) into Eq. (C.3) yields
\[
\mathcal{R}[K] = -\rho^2 c_s^2 \sum_{l,l'} \left( \frac{m+l}{r} \right) \left\{ \frac{d|\vec{\phi}_l(r)|}{dr} \right\} P_{l,l'}[K] - |\vec{\phi}_l(r)| \frac{dS(r)}{dr} Q_{l,l'}[K].
\]  
(C.9)

Since \( l/m \approx 1/\sqrt{n} \ll 1 \) \cite{25,26}, for \( k \)’th cosinoidal and sinusoidal component, Eq. (C.9) can be written respectively as,

Cosinoidal moment of Reynolds stress:
\[
\mathcal{R}[\cos k\theta] = \frac{1}{4} \rho^2 c_s^2 k_s^2 \delta^2 \sum_{p=\pm 1} r \int dx \left| \frac{dS(r)}{dx} \right| |\vec{\phi}_p(x)| |\vec{\phi}_{s,pk}(x)|. \]  
(C.10)

Sinusoidal moment of Reynolds stress:
\[
\mathcal{R}[\sin k\theta] = \frac{1}{4} \rho^2 c_s^2 k_s^2 \delta^2 \sum_{p=\pm 1} r \int dx \left| \frac{d\vec{\phi}_p(x)}{dx} \right| |\vec{\phi}_{s,pk}(x)|. \]  
(C.11)

In Eq. (C.11) \( x = k_s \delta \left( r - r_j \right) \), the dimensionless micro-radius in the ballooning theory \cite{25,26}.

Some factors in Eqs. (C.10), (C.11) can be found in Ref. 25. They are
\[
S(r) = \text{Re} \left[ k_s (\lambda_s)(x-l) - \frac{1}{2\eta^2} (x-l)^3 \right]. \]  
(C.12)
\[ |\tilde{\phi}(r)| = \exp \left\{ -\text{Im} \left[ k_r(\lambda_r)(x - l) - \frac{1}{2\eta^2} (x - l)^2 \right] \right\} \exp \left\{ \text{Re} \left[ -\frac{n}{2\beta_z} \left( \beta_i + \frac{l}{n} \right)^2 \right] \right\}. \tag{C.13} \]

\[ \frac{dS(r)}{dx} = \frac{\text{Im} \left[ \eta^2 k_r(\lambda_r) \right]}{\text{Im} \eta^2} + \frac{\text{Re} \eta^2}{2\text{Im} \eta^2} \frac{d\ln |\tilde{\phi}(x)|^2}{dx}. \tag{C.14} \]

\[ I_n(x) \equiv \int \! d\vartheta |\tilde{\phi}(r, \vartheta)|^2 = \frac{1}{2} \sum_i |\tilde{\phi}_i(x)|^2 - \Gamma(x). \tag{C.15} \]

In Eq. (C.15) ‘—’ denotes that a constant factor before \( \Gamma(x) \) is not important and neglected,

\[ \Gamma(x) \equiv \sum_i \exp \left\{ -\bar{\alpha} (x - l - k_0) \right\} \exp \left\{ -\bar{\beta} (l - x_\beta) \right\}. \tag{C.16} \]

In Eq. (C.16) \( \bar{\alpha} \equiv \text{Im} \eta^2 / |\eta|^2 \), \( \bar{\beta} \equiv \text{Re} \beta / \left(n |\beta_z|^2 \right) \), \( k_0 \equiv -|\eta|^2 \text{Im} \bar{k} / \text{Im} \eta^2 \), \( \bar{k} \equiv k_r(\bar{\lambda}_r) \), \( \bar{\lambda}_r \equiv -i\beta_i / \beta_z \), \( n = |\eta| \), \( x_\beta = -n (\text{Re} \beta_i + \text{Im} \beta_i) / \beta_z \).

Now, we are ready to calculate Eqs. (C.10), (C.11). Use is made of Eq. (C.13) to obtain

\[ \left| \frac{\tilde{\phi}_{r,p,k}(x)}{\tilde{\phi}_i(x)} \right| = \exp \left\{ pk \left[ \bar{\alpha} (x - l) + \bar{\beta} (x_\beta - l) + \text{Im} \bar{k}_r(\lambda_r) \right] - \frac{(\bar{\alpha} + \bar{\beta})k^2}{2} \right\}. \tag{C.17} \]

From Eq. (C.15) and (C.16), we obtain

\[ |\tilde{\phi}_i(x)|^2 - 2\exp \left[ -\bar{\alpha} (x - l - k_0) \right] \exp \left[ -\bar{\beta} (l - x_\beta) \right]. \tag{C.18} \]

Multiplying Eq. (C.17) by (C.18) yields

\[ \sum_i \tilde{\phi}_i(x) |\tilde{\phi}_{r,p,k}(x)| - 2\Gamma_{p,k}(x) \exp \left\{ pk \left[ \bar{\alpha} k_0 + \text{Im} \bar{k}_r(\lambda_r) \right] - \frac{1}{4} (\bar{\alpha} + \bar{\beta})k^2 \right\}. \tag{C.19} \]

with

\[ \Gamma_{p,k}(x) \equiv \sum_i \exp \left\{ -\bar{\alpha} (x - l - k_p) \right\} \exp \left[ -\bar{\beta} (x_\beta - l) \right] \]. \tag{C.20} \]

In Eq. (C.20) \( k_p \equiv k_0 + pk / 2 \), \( x_{\beta,p} \equiv x_\beta - pk / 2 \). Eq. (C.20) can also be written as

\[ \Gamma_{p,k}(x) = \exp \left[ -\frac{\bar{\alpha} \bar{\beta}}{(\bar{\alpha} + \bar{\beta})} \left( x - x_{\beta,p} - k_p \right)^2 \right] \Pi_{p,k}(x). \tag{C.21} \]

In Eq. (C.21)

\[ \Pi_{p,k}(x) \equiv \sum_i \exp \left\{ -\left( \bar{\alpha} + \bar{\beta} \right) \left[ \frac{\bar{\alpha}}{\alpha + \bar{\beta}} (x - l) - \frac{\bar{\alpha} k_p - \bar{\beta} x_{\beta,p}}{(\bar{\alpha} + \bar{\beta})} \right]^2 \right\}. \tag{C.22} \]
The function $\Pi_{p,k}(x)$ has been discussed in the Appendix B in Ref.25 in detail, and is shown close to a constant for $(\alpha + \beta) < 2$. This is still true for other integer $p$, because the translational invariance of this function is independent of $p$. Neglecting the small variation of $\Pi_{p,k}(x)$, Eq.(C.19) becomes

$$
\sum_l |\phi_l(x)||\phi_{l+pk}(x)| \approx 2I_m(x_0)\exp\left[-\frac{(x-x_0)^2}{n\sigma}\right]\exp\left\{pk\left[\alpha k_0 + \text{Im} k_+(\lambda)\right] - \frac{1}{4}(\alpha + \beta)k^2\right\}. \tag{C.23}
$$

In Eq.(C.23) $x_0 \equiv x_\beta + k_0$, $\sigma \equiv (\alpha + \beta)/(n\alpha\beta) = |\beta_2|^2/\text{Re} \beta_2 + |\eta|^2/(n\text{Im} \eta^2)$. Substituting Eq.(C.18) into (C.14) yields

$$
\frac{dS(x)}{dx} = \frac{\text{Im}\eta^2k_+(\lambda)}{\text{Im} \eta^2} - \frac{\text{Re} \eta^2}{\text{Im} \eta^2}\alpha(x-l-k_0), \tag{C.24}
$$

in which $\alpha(x-l-k_0)$ can be equivalently replaced by the operator $(-d/dx + pk\alpha)/2$,

$$
\sum_i \frac{dS(x)}{dx} |\phi_i(x)||\phi_{i+pk}(x)| = \left\{\frac{\text{Im}\eta^2k_+(\lambda)}{\text{Im} \eta^2} + \frac{\text{Re} \eta^2}{2\text{Im} \eta^2}\left(\frac{d}{dx} - pk\alpha\right)\right\} \sum_k |\phi_k(x)||\phi_{k+pk}(x)|. \tag{C.25}
$$

In Eq.(C.25) $k_+(\lambda) = \sum_i k_+(\lambda)|\phi_i(x)||\phi_{i+pk}(x)|/\sum_k |\phi_k(x)||\phi_{k+pk}(x)|$. As discussed in Ref.25, $k_+(\lambda)$ is a slowly varying function of radius, it can be further approximated by a simple constant $k_+(x_i)$ with $x_i = -i(\beta_1 + x_\beta/n)/\beta_2$.

Substituting Eq.(B.23) into (B.25), then into Eq.(C.10), the analytic expression of cosinoidal moment of Reynolds stress is found to be

$$
\Re[k\theta] = \frac{1}{2} \rho_s c_f^2 k_0^2 2\delta I_m(x_0) \sum_{p=-1}^{p=1} \left\{\frac{\text{Im}\eta^2k_+(x_i)}{\text{Im} \eta^2} - \frac{\text{Re} \eta^2}{\text{Im} \eta^2}\frac{(x-x_0)}{n\sigma} - \frac{\text{Re} \eta^2}{2|\eta|^2}\right\} \times \exp\left[-\frac{(x-x_0)^2}{n\sigma}\right]\exp\left\{pk\left[\alpha k_0 + \text{Im} k_+(\lambda)\right] - \frac{1}{4}(\alpha + \beta)k^2\right\}. \tag{C.26}
$$

For $k = 0$, Eq.(B.26) becomes

$$
\Re[1] = \rho_s c_f^2 k_0^2 2\delta I_m(x_0) \left\{\frac{\text{Im}\eta^2k_+(x_i)}{\text{Im} \eta^2} - \frac{\text{Re} \eta^2}{\text{Im} \eta^2}\frac{(x-x_0)}{n\sigma}\right\} \exp\left[-\frac{(x-x_0)^2}{n\sigma}\right]. \tag{C.27}
$$

which is the same result as Eq.(42) in Ref.25.

The analytic expression of sinusoidal moment of Reynolds stress can be obtained similarly as
Both the cosinoidal and sinusoidal moment are composed of a monopole and a dipole. However, the dipole is much larger than the monopole for cosinoidal moment, since \( k_s(x_s) \) is small. But for sinusoidal moment, the monopole structure looks more apparent.

### Appendix D  Low frequency limit of the zonal flow equation set

As stated in the introductory section, the framework of the present paper in studying GAM is the extension of equations of Ref.\(^8\) to include toroidal coupling of the first harmonic sinusoidal component of sound wave owing to geodesic curvature. A question may arise from the concerns how far the toroidal coupling would modify the model used in Ref.\(^8\). Whether or not so many feathers derived in Ref.\(^8\) regarding LFZF might survive etc. The quick answer is that in the low frequency limit the toroidal effect on the TLFZF branch is simply a quantitative change of the inertia of zonal flow as shown below. Since a free parameter \( a_{neo} \) has been introduced in Ref.\(^8\) to get the theory suitable for various models of zonal flow inertia, the results obtained in Ref.\(^8\) are still valid.

Eqs.\((18)\) , \((19)\) can be merged into a single equation by eliminating \( \chi^{(i)}_t \)

\[
\left[ \frac{\partial^2}{\partial t^2} \left( 1 + D(\tau_s) \right) \rho^2 \frac{\partial^2}{\partial r^2} + 2(1 + \tau_s) \frac{\psi c^2}{R^2} \left( 1 + \frac{1}{2q^2} \right) \right] \times \\
\left[ \frac{\partial \bar{\nu}}{\partial t} - \mu \frac{\partial^2 \bar{\nu}}{\partial r^2} + \frac{1}{2} \frac{\partial}{\partial r} \left\{ \left( \bar{\Re}[1] + \epsilon \bar{\Re}[^{\cos \theta}] \right) \cos 2\Theta \right\} \right] = \\
-2 \frac{\psi c^2}{R^2} (1 + \tau_s) \left[ \mu \frac{\partial^2 \bar{\nu}}{\partial r^2} - \frac{1}{2} \frac{\partial}{\partial r} \left\{ \left( \bar{\Re}[1] + \epsilon \bar{\Re}[\cos \theta] \right) \cos 2\Theta \right\} \right] \\
- \frac{\psi \rho^2 c^2}{R} (1 + \tau_s) \frac{\partial}{\partial t} \frac{\partial^2}{\partial r^2} \left( \bar{\Im}[\sin \theta] - \frac{\epsilon}{3} \bar{\Im}[\sin 2\theta] \cos 2\Theta \right). \tag{D.1}
\]
\[
(1 + 2q^2) \frac{\partial \vec{v}}{\partial t} - \mu \frac{\partial^2 \vec{v}}{\partial r^2} + \frac{1}{2} \frac{\partial}{\partial r} \left( \left\{ \hat{K}[1] + \varepsilon \hat{K}[\cos \vartheta] \right\} \cos 2\Theta \right) = \]
\[
-q^2 R \frac{\partial}{\partial t} \frac{\partial}{\partial r^2} \left( \left\{ \hat{K}[\sin \vartheta] + \frac{\varepsilon}{3} \hat{K}[\sin 2\vartheta] \right\} \cos 2\Theta \right). \tag{D.2}
\]

Since the r.h.s. of Eq.(D.2) is associated with a low frequency, and the toroidal correction is associated with a small parameter \( \varepsilon \), we readily obtain the final equation for the low frequency branch by neglecting these small quantitative corrections

\[
(1 + 2q^2) \frac{\partial \vec{v}}{\partial t} - \mu \frac{\partial^2 \vec{v}}{\partial r^2} + \frac{1}{2} \frac{\partial}{\partial r} \left\{ \hat{K}[1] \cos 2\Theta \right\} = 0. \tag{D.3}
\]

It is precisely the leading order equation of TLFZF under fluid model in simple tokamak configuration. Compared to Eq.(19) of Ref.8, \( a_{\text{neo}} := 1 + 2q^2 \).

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**Data availability**

All the data that support the findings of this study are available from the corresponding author.

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