Exotic $\mathbb{R}^4$'s and positive isotropic curvature

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Abstract

We show that no exotic $\mathbb{R}^4$ admits a complete Riemannian metric with uniformly positive isotropic curvature and with bounded geometry. This is essentially a corollary of the main result in [Hu1], and was stated in [Hu2] without proof. In the process of the proof we also show that the diffeomorphism type of an infinite connected sum of some connected smooth $n$-manifolds ($n \geq 2$) according to a locally finite graph does not depend on the gluing maps used.

Key words: exotic $\mathbb{R}^4$'s, positive isotropic curvature, infinite connected sum

1 Introduction

In [Hu1] we proved the following result which extends [CZ] to the noncompact case.

Theorem 1.1. Let $X$ be a complete, connected, non-compact 4-manifold with uniformly positive isotropic curvature, with bounded geometry and with no essential incompressible space form. Then $X$ is diffeomorphic to an infinite connected sum of $\mathbb{S}^4$, $\mathbb{R}P^4$, $\mathbb{S}^3 \times \mathbb{S}^1$, and/or $\mathbb{S}^3 \tilde\times \mathbb{S}^1$.

Note that here we use the standard smooth structures of $\mathbb{S}^4$, $\mathbb{R}P^4$, $\mathbb{S}^3 \times \mathbb{S}^1$ and $\mathbb{S}^3 \tilde\times \mathbb{S}^1$.

Now we explain the notion of infinite connected sum used here (compare [BBM] and [Hu2]). Let $G$ be a countably infinite graph which is connected and locally finite (here we allow an edge to connect a vertex to itself, and allow more than one edge to connect two vertices (or connect one vertex to itself)), and let $\mathcal{X}$ be a class of connected, smooth $n$-manifolds ($n \geq 2$). We associate an element $X_v \in \mathcal{X}$ to each vertex $v$ of $G$. For each edge of $G$, suppose it connects the vertices $v_1$ and $v_2$ (it may be that $v_1 = v_2$), we do a connected sum of $X_{v_1}$ and $X_{v_2}$ (as in pp. 102-106 in [BJ]). The result is a connected, smooth $n$-manifold, which is called an infinite connected sum of members of $\mathcal{X}$ according to the graph $G$.

The following result is stated in [Hu2] without proof. It is essentially a corollary of Theorem 1.1.

Theorem 1.2. No exotic $\mathbb{R}^4$ admits a complete Riemannian metric with uniformly positive isotropic curvature and with bounded geometry.
To prove Theorem 1.2 we also need the fact that the diffeomorphism type of an infinite connected sum of some connected smooth $n$-manifolds ($n \geq 2$) according to a locally finite graph does not depend on the gluing maps used. This fact is proved in Section 2. Theorem 1.2 itself is proved in Section 3.

2 Infinite connected sum

We give more details of the definition of infinite connected sum. Let $G$ be as in the Introduction. Let $\{v_1, v_2, \cdots\}$ be the set of the vertices in $G$. If the connected, smooth $n$-manifold ($n \geq 2$) $X_{v_i}$ associated to the vertex $v_i$ is orientable, we choose an orientation of it. For each pair $(i, j)$ with $i \leq j$, let $\{e_{ij}^k | k = 1, 2, \cdots, m_{ij}\}$ be the set of edges connecting $v_i$ to $v_j$ (of course, if there is no edge connecting $v_i$ to $v_j$, $m_{ij} = 0$, and in this case this set is empty). To each edge $e_{ij}^k$ we associate a pair of smooth embeddings $f_{e_{ij}^k} : \mathbb{R}^n \to X_{v_i}$ and $g_{e_{ij}^k} : \mathbb{R}^n \to X_{v_j}$ from the standard $\mathbb{R}^n$. We fix an orientation of the standard $\mathbb{R}^n$. If $X_{v_i}$ is oriented, we let $f_{e_{ij}^k}$ ($j \geq i$, $k = 1, 2, \cdots, m_{ij}$) be orientation-preserving, and let $g_{e_{ij}^l}$ ($p \leq i$, $l = 1, 2, \cdots, m_{pi}$) be orientation-reversing. We assume that the images of all these embeddings are disjoint from each other.

Let $D^n$ be the closed unit ball with center the origin in the standard $\mathbb{R}^n$. For each $i$, let

$$Y_i := X_{v_i} \setminus (\cup_{j \geq i} \cup_{k=1}^{m_{ij}} f_{e_{ij}^k}(\frac{1}{3}D^n) \cup \cup_{p \leq i} \cup_{l=1}^{m_{pi}} g_{e_{ij}^l}(\frac{1}{3}D^n)),$$

and let $Y$ be the infinite disjoint union $\bigsqcup Y_i$.

We define an equivalence relation $\sim$ in $Y$ by setting

$$f_{e_{ij}^k}(tu) \sim g_{e_{ij}^l}((1-t)u)$$

for all $(i, j)$ with $i \leq j$, $k = 1, 2, \cdots, m_{ij}$, $t \in (\frac{1}{3}, \frac{2}{3})$ and $u \in S^{n-1}$. Let $X$ be the quotient space $Y/\sim$. We call $X$ the connected sum of $X_{v_i}$ according to the graph $G$ via $\{f_{e_{ij}^k}, g_{e_{ij}^k}\}$, and denote it by $\#GX_{v_i}(f_{e_{ij}^k}, g_{e_{ij}^k})$.

Let $M^n$ be a connected $n$-manifold, and $\varphi = \bigsqcup_{i=1}^k \varphi_i : \bigsqcup_{i=1}^k D^n \to M^n$ and $\tilde{\varphi} = \bigsqcup_{i=1}^k \tilde{\varphi}_i : \bigsqcup_{i=1}^k D^n \to M^n$ be two embeddings from the disjoint union of $k$ copies of the standard $n$-disk. As in [BJ, Definition (10.1)] we say $\varphi$ and $\tilde{\varphi}$ are compatibly oriented if either $M^n$ is not orientable, or, for each $i$ ($1 \leq i \leq k$), $\varphi_i$ and $\tilde{\varphi}_i$ are both orientation preserving or both orientation reversing (relative to fixed orientations of $D^n$ and $M^n$).

The following result is well-known, see for example Theorem 3.2 in Chapter 8 of [H].

**Proposition 2.1.** Let $\varphi = \bigsqcup_{i=1}^k \varphi_i : \bigsqcup_{i=1}^k D^n \to M^n$ and $\tilde{\varphi} = \bigsqcup_{i=1}^k \tilde{\varphi}_i : \bigsqcup_{i=1}^k D^n \to M^n$ be two (smooth) embeddings from the disjoint union of $k$ ($< \infty$) copies of the standard $n$-disk to a connected, smooth $n$-manifold $M^n$ ($n \geq 2$). Suppose that $\varphi$ and $\tilde{\varphi}$ are compatibly oriented. Then there is a diffeotopy $H$ of $M^n$, which is fixed outside of a compact subset of $M^n$ such that $H(\cdot, 1) \circ \varphi = \tilde{\varphi}$. 


Chapters 9 and 10 in [BJ], Chapter III in [K] and Theorem 3.1 in Chapter 8 of [H]). Suppose the result is true for \(k = p\). (For definition of diffeotopy (or ambient isotopy), see [BJ, Definition (9.3)] and p.178 of [H].)

**Proof** We follow closely the proof of Theorem 3.2 in Chapter 8 of [H]. We do induction on \(k\). The \(k = 1\) case is due to Cerf and Palais (for expositions see Chapters 9 and 10 in [BJ], Chapter III in [K] and Theorem 3.1 in Chapter 8 of [H]). Suppose the result is true for \(k = j\). Now we consider the case \(k = j + 1\). By assumption there exists a diffeotopy \(\tilde{H}\) of \(M^n\), which is fixed outside of a compact subset of \(M^n\) such that \(\tilde{H}(\cdot, 1) \circ \varphi_i|\cup_{i=1}^j D^n = \tilde{\varphi}|\cup_{i=1}^j D^n.\) (In particular, it follows that \(\tilde{H}(\cdot, 1)(\varphi_{j+1}(D^n)) \subset M^n \setminus \cup_{i=1}^j \tilde{\varphi}_i(D^n).\) Since \(n \geq 2\), \(M^n \setminus \cup_{i=1}^j \tilde{\varphi}_i(D^n)\) is connected. We apply the \(k = 1\) case to the two embeddings

\[
\tilde{H}(\cdot, 1) \circ \varphi_{j+1}, \tilde{\varphi}_{j+1}: D^n \to M^n \setminus \cup_{i=1}^j \tilde{\varphi}_i(D^n),
\]

and get a diffeotopy \(\bar{H}\) of \(M^n \setminus \cup_{i=1}^j \tilde{\varphi}_i(D^n)\) which is fixed outside of a compact subset of \(M^n \setminus \cup_{i=1}^j \tilde{\varphi}_i(D^n)\) such that \(\bar{H}(\cdot, 1) \circ \tilde{H}(\cdot, 1) \circ \varphi_{j+1} = \tilde{\varphi}_{j+1}\). Clearly \(\bar{H}\) extends to a diffeotopy of \(M^n\) which leaves \(\cup_{i=1}^j \tilde{\varphi}_i(D^n)\) fixed. Then \(H_t := \bar{H}_t \circ \tilde{H}_t\) is the desired diffeotopy.

**Theorem 2.2.** The infinite connected sum \(\mathbb{G}_{G}X_{v_i}(f_{\epsilon_{ij}}^k, g_{\epsilon_{ij}}^k)\) is a connected, smooth manifold, and oriented if all \(X_{v_i}\) are oriented. Its diffeomorphism type (oriented if relevant) does not depend on the choice of embeddings \(f_{\epsilon_{ij}}^k\) and \(g_{\epsilon_{ij}}^k\).

**Proof** The first claim can be shown as in pp. 103-104 in [BJ] and pp. 90-91 in [K]. Now we show the second claim. Suppose that to each edge \(e_{ij}^k\) we associate another pair of smooth embeddings \(\bar{f}_{\epsilon_{ij}}\) : \(\mathbb{R}^n \to X_{v_i}\) and \(\bar{g}_{\epsilon_{ij}}\) : \(\mathbb{R}^n \to X_{v_j}\) from the standard \(\mathbb{R}^n\). If \(X_{v_i}\) is oriented, we let \(f_{\epsilon_{ij}}\) \((j \geq i, k = 1, 2, \cdots, m_{ij})\) be orientation-preserving, and let \(\bar{g}_{\epsilon_{ij}}\) \((p \leq i, l = 1, 2, \cdots, m_{pi})\) be orientation-reversing. We assume that the images of all these embeddings \(\bar{f}_{\epsilon_{ij}}\) and \(\bar{g}_{\epsilon_{ij}}\) are disjoint from each other.

We define \(\bar{\bar{Y}}_i\) and \(\bar{\bar{Y}}\) as before using \(\bar{f}_{\epsilon_{ij}}\) and \(\bar{g}_{\epsilon_{ij}}\). We also introduce an equivalence relation in \(\bar{\bar{Y}}\) as before using \(\bar{f}_{\epsilon_{ij}}\) and \(\bar{g}_{\epsilon_{ij}}\), and still denote it by \(\sim\). Finally we let \(\mathbb{G}_{G}X_{v_i}(\bar{f}_{\epsilon_{ij}}^k, \bar{g}_{\epsilon_{ij}}^k) := \bar{\bar{Y}} / \sim\). For each \(i\) let

\[
\varphi_i := \bigcup_{j \geq i} \bigcup_{k=1}^{m_{ij}} f_{\epsilon_{ij}}^k \cup \bigcup_{p \leq i} \bigcup_{l=1}^{m_{pi}} g_{\epsilon_{ij}}^l : \bigcup \mathbb{R}^n \to X_{v_i}
\]

and

\[
\tilde{\varphi}_i := \bigcup_{j \geq i} \bigcup_{k=1}^{m_{ij}} \bar{f}_{\epsilon_{ij}}^k \cup \bigcup_{p \leq i} \bigcup_{l=1}^{m_{pi}} \bar{g}_{\epsilon_{ij}}^l : \bigcup \mathbb{R}^n \to X_{v_i}.
\]

Since \(G\) is locally finite, for each \(i\), the above \(\bigcup \mathbb{R}^n\) is a finite disjoint union; we consider the finite disjoint union \(\bigcup D^n\) contained in it. For each \(i\), we can apply Proposition 2.1 to \(\varphi_i|\cup D^n\) and \(\tilde{\varphi}_i|\cup D^n\), and get a diffeotopy \(H_i(\cdot, t)\) of \(X_{v_i}\), which is fixed outside a compact subset of \(X_{v_i}\), such that

\[
\tilde{\varphi}_i|\cup D^n = H_i(\cdot, 1) \circ \varphi_i|\cup D^n. \tag{2.1}
\]
By equation (2.1) we can define a map $F : Y = \sqcup Y_i \to \tilde{Y} = \sqcup \tilde{Y}_i$ via

$$F(x) = H_i(x, 1) \quad \text{when} \quad x \in Y_i \quad \text{for some} \quad i.$$ 

Clearly $F$ is a diffeomorphism. Note that by equation (2.1) again $F$ is compatible with the equivalence relations in $Y$ and in $\tilde{Y}$. So $F$ induces a diffeomorphism

$$\tilde{F} : \sharp_G X_{v_i} (f_{e_{ij}}^k, g_{e_{ij}}^k) \to \sharp_G X_{v_i} (\tilde{f}_{e_{ij}}^k, \tilde{g}_{e_{ij}}^k).$$

\[\square\]

**Remark** In general, if each $X_{v_i}$ is orientable, the diffeomorphism type of the infinite connected sum $\sharp_G X_{v_i}$ may depend on the choice of the orientations of $X_{v_i}$. But it is easy to see that if each $X_{v_i}$ is orientable and admits an orientation-reversing diffeomorphism, then the (unoriented) diffeomorphism type of $\sharp_G X_{v_i}$ does not depend on the choice of the orientations of $X_{v_i}$.

## 3 Proof of Theorem 1.2

Let $X$ be a smooth 4-manifold which is homeomorphic to the standard $\mathbb{R}^4$. Assume that $X$ admits a complete Riemannian metric with uniformly positive isotropic curvature and with bounded geometry. Clearly $X$ contains no essential incompressible space form. By Theorem 1.1, $X$ is diffeomorphic to an infinite connected sum of $S^4$, $\mathbb{R}P^4$, $S^3 \times S^1$, and/or $S^3 \times S^1$ according to a locally finite graph $G$. Since the fundamental group of $X$ is trivial, the graph $G$ must be a tree, and the smooth manifold $X_v$ associated to any vertex $v$ in $G$ must be diffeomorphic to the standard $S^4$.

We know that any topological manifold homeomorphic to $\mathbb{R}^4$ has exactly one (topological) end (for definition see for example, [DK]). It follows that the tree $G$ has exactly one topological end also. But for a locally finite graph, there is a natural bijection between its topological ends and its graph-theoretical ends, cf. [DK] and the references therein. So the tree $G$ has only one graph-theoretical end. (There should be a more direct argument for this fact.) Now we choose a ray $\gamma$ in $G$, which is essentially unique. Let $w_0, w_1, w_2, \cdots$ be the set of vertices along the ray $\gamma$. For each $i$, there are only finite vertices which can be connected to $w_i$ via a sequence of edges not contained in the ray $\gamma$. For each $i$, we do connected sum of all $S^4$'s associated to these finite vertices (including $w_i$), the result is diffeomorphic to the $S^4$ associated to the vertex $w_i$ via a diffeomorphism not affecting the part of this $S^4$ where its connected sum with the two $S^4$'s associated to $w_{i-1}$ and $w_{i+1}$ occurs. Then we see that $X$ is diffeomorphic to an infinite connected sum of $S^4$'s according to the ray $[0, +\infty)$ with a vertex $w_i$ at $i$ ($i = 0, 1, 2, \cdots$) using some gluing maps.

We know that the infinite connected sum of $S^4$'s (all with the standard orientation) according to the ray $[0, +\infty)$ using certain special gluing maps actually produces the standard $\mathbb{R}^4$. (Represent the standard $\mathbb{R}^4$ as the union of the unit
4-ball and the closed subspaces $A_i$ bounded by the two 3-spheres with radius $i$ and $i+1$ (and each with center the origin), $i = 1, 2, \cdots$. Note that each $A_i$ may be seen as $S^4$ with two open 4-balls removed.) We also know that $S^4$ admits an orientation-reversing diffeomorphism. So by Theorem 2.2 and the Remark following it, $X$ is diffeomorphic to the standard $\mathbb{R}^4$.

**Remark** It is interesting to see whether or not the condition ‘with bounded geometry’ in Theorem 1.2 can be removed.

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