A Faster Parameterized Algorithm for Temporal Matching

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Abstract
A temporal graph is a sequence of graphs (called layers) over the same vertex set—describing a graph topology which is subject to discrete changes over time. A $\Delta$-temporal matching $M$ is a set of time edges $(e, t)$ (an edge $e$ paired up with a point in time $t$) such that for all distinct time edges $(e, t), (e', t') \in M$ we have that $e$ and $e'$ do not share an endpoint, or the time-labels $t$ and $t'$ are at least $\Delta$ time units apart. Mertzios et al. [STACS ’20] provided a $2^{O(\Delta \nu)} \cdot |G|^{O(1)}$-time algorithm to compute the maximum size of $\Delta$-temporal matching in a temporal graph $G$, where $|G|$ denotes the size of $G$, and $\nu$ is the $\Delta$-vertex cover number of $G$. The $\Delta$-vertex cover number is the minimum number of vertices which are needed to hit (or cover) all edges in any $\Delta$ consecutive layers of the temporal graph. We show an improved algorithm to compute a $\Delta$-temporal matching of maximum size with a running time of $\Delta^{O(\nu)} \cdot |G|$ and hence provide an exponential speedup in terms of $\Delta$.

1 Introduction

Matchings are one of the most fundamental and best studied notions in graph theory, see Lovász and Plummer [10] and Schrijver [13] for an overview. Recently, Baste et al. [3] and Mertzios et al. [11] studied matchings in temporal graphs. A temporal graph $G = (V,(E_t)_{t=1}^\tau)$ consists of a set $V$ of vertices and an ordered list of $\tau$ edge sets $E_1, E_2, \ldots, E_\tau$. A tuple $(e, t)$ is a time edge of $G$ if $e \in E_t$, $t \in \{1, 2, \ldots, \tau\}$. Two time edges $(e, t)$ and $(e', t')$ are $\Delta$-independent whenever the edges $e, e'$ do not share an endpoint or their time labels $t, t'$ are at least $\Delta$ time units apart from each other (that is, $|t - t'| \geq \Delta$). A $\Delta$-temporal matching $M$ of a temporal graph $G$ is a set of time edges of $G$ which are pairwise $\Delta$-independent. This leads naturally to the following decision problem, introduced by Mertzios et al. [11].

Throughout the paper, $\Delta$ always refers to this number, and never to the maximum degree of a static graph (which is another common use of $\Delta$).
Without loss of generality, we assume that $\Delta \leq \tau$. While TEMPORAL MATCHING is polynomial-time solvable if the temporal graph has $\tau \leq 2$ layers, it becomes NP-hard, even if $\tau = 3$ and $\Delta = 2$ [11]. Driven by this NP-hardness, Mertzios et al. [11] showed an FPT-algorithm for TEMPORAL MATCHING, when parameterized by $\Delta$ and the maximum matching size of the underlying graph $G^\downarrow(G) := (V, \bigcup_{t=1}^\tau E_t)$ of the input temporal graph $G = (V, (E_t)_{t=1}^\tau)$. On a historical note, one has to mention that Baste et al. [3] introduced temporal matchings in a slightly different way. The main difference to the model of Mertzios et al. [11] which we also adopt here is that the model of Baste et al. [3] requires edges to exist in at least $\Delta$ consecutive time steps in order for them to be eligible for a matching. However, with little preprocessing an instance of the model of Baste et al. [3] can be reduced to our model and the algorithmic ideas presented by Baste et al. [3] apply as well. Notably, there is also the related problem MULTISTAGE MATCHINGS: this is a radically different way to lift the notion of matchings into the temporal setting. Here, we are given a temporal graph $G = (V, (E_t)_{t=1}^\tau)$ and we want to find a perfect (or maximum) matching for each layer $(V, E_t)$ such that the symmetric differences of matchings for consecutive layers are small [2, 5, 8, 9].

In this paper, we consider the vertex cover number \footnote{That is, the minimum number of vertices needed to cover all edges of a graph.} to measure the width of local sections (that is, $\Delta$ consecutive layers) in temporal graphs. We call this the $\Delta$-vertex cover number of a temporal graph $G = (V, (E_t)_{t=1}^\tau)$. Intuitively, this is the minimum number of vertices which we need to hit (or cover) all edges in any $\Delta$ consecutive layers of the temporal graph. Formally, the $\Delta$-vertex cover number of $G$ is the minimum number $\nu$ such that for all $i \in [\tau - \Delta + 1]$ there is a $\nu$-size vertex set $S$ such that all edges in $\bigcup_{t=i}^{i+\Delta-1} E_t$ are incident to at least one vertex in $S$. Observe that the $\Delta$-vertex cover number can be smaller but not larger than the smallest sliding $\Delta$-window vertex cover, see Akrida et al. [1] for details. Note that also VERTEX COVER has been studied in the multistage setting [6].

It is easy to check that the running time of the algorithm for TEMPORAL MATCHING of Mertzios et al. [11] can be upper-bounded by $2^{O(\nu \Delta)} \cdot |G|^{O(1)}$, where $\nu$ is the $\Delta$-vertex cover number of $G$. This paper contributes an improved algorithm for TEMPORAL MATCHING with a running time of $\Delta^{O(\nu)} \cdot |G|$. Hence, this is an exponential speedup in term of $\Delta$ compared to the algorithm of Mertzios et al. [11]. Before we describe the details of the algorithm in Section 3, we introduce further basic notations in the next section.
2 Preliminaries

We denote by $\log(x)$ the ceiling of the binary logarithm of $x (\lfloor \log_2(x) \rfloor)$. A $p$-family is a family of sets where each set has size $p$. We refer to a set of consecutive natural numbers $[i, j] := \{k \in \mathbb{N} | i \leq k \leq j\}$ for some $i, j \in \mathbb{N}$ as an interval. If $i = 1$, then we denote $[i, j]$ simply by $[j]$. The neighborhood of a vertex $v$ and a vertex set $X$ in a graph $G = (V, E)$ is denoted by $N_G(v) := \{u \in V | \{v, u\} \in E\}$ and $N_G(X) := (\bigcup_{v \in X} N_G(v)) \setminus X$, respectively.

The lifetime of a temporal graph $G = (V, (E_t)_{t=1}^\tau)$ is $\tau$. The size of a temporal graph $G = (V, (E_t)_{t=1}^\tau)$ is $|G| := |V| + \sum_{t=1}^\tau |E_t|$. Furthermore, in accordance with the literature [14, 15], we assume that the lists of labels are given in ascending order. The set of time edges $E(G)$ of a temporal graph $G = (V, (E_t)_{t=1}^\tau)$ is defined as $\{(e, t) | e \in E_t\}$. A pair $(v, t)$ is a vertex appearance in a temporal graph $G = (V, (E_t)_{t=1}^\tau)$ of $v$ at time $t$ if $v \in V$ and $t \in [\tau]$. A time edge $(e, t)$ $\Delta$-blocks a vertex appearance $(v, t)$ (or $(v, t')$) is $\Delta$-blocked by $(e, t)$ if $v \in e$ and $|t - t'| \leq \Delta - 1$. For a time edge set $E$ and integers $a$ and $b$, we denote by $E[a, b] := \{(e, t) \in E | a \leq t \leq b\}$ the subset of $E$ between the time steps $a$ and $b$. Analogously, for a temporal graph $G = (V, (E_t)_{t=1}^\tau)$ we denote by $G[a, b]$ the temporal graph on the vertex set $V$ with the time edge set $E(G)[a, b]$.

A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where $\Sigma$ is a finite alphabet. The second component is called the parameter of the problem. A parameterized problem $L$ is fixed-parameter tractable if we can decide in $f(k) \cdot |x|^{O(1)}$ time whether a given instance $(x, k)$ is in $L$, where $f$ is an arbitrary function depending only on $k$. An algorithm is an FPT-algorithm for parameter $k$ if its running time is upper-bounded by $f(k) \cdot n^{O(1)}$, where $n$ is the input size and $f$ is a computable function depending only on $k$.

3 The Algorithm

Mertzios et al. [11] provided a $2^{O(\Delta \nu)}|\mathcal{G}|^{O(1)}$-time algorithm for Temporal Matching. We now develop an improved algorithm which runs in $\Delta^{O(\nu)}|\mathcal{G}|$. Formally, we show the following.

**Theorem 1.** Temporal Matching can be solved in $\Delta^{O(\nu)} \cdot |\mathcal{G}|$ time, where $\nu$ is the $\Delta$-vertex cover number of $\mathcal{G}$.

The proof of Theorem 1 is deferred to the end of the section. Formally, we solve the decision variant of Temporal Matching as it is defined in Section I. However, the algorithm actually computes the maximum size of a $\Delta$-temporal matching in a temporal graph and with a straight-forward adjustment the algorithm can also output a $\Delta$-temporal matching of maximum size.

Similarly to the algorithm of Mertzios et al. [11], the algorithm behind Theorem 1 works in three major steps:

1. Divide the temporal graph into disjoint $\Delta$-windows.
2. For each of these $\Delta$-windows compute a small family of $\Delta$-temporal matchings.
3. Based on the families of the Step 2, by dynamic programming compute the maximum size of a $\Delta$-temporal matching for the whole temporal graph. While Step 1 is trivial and Step 3 is similar to the algorithm of Mertzios et al.\cite{11}, Step 2 is where we provide new ideas leading to an improved overall running time. In the next two subsections, we describe Step 2 and Step 3 in detail. Afterwards, we put everything together and prove Theorem 1.

3.1 Step 2: Families of $d$-complete $\Delta$-temporal matchings.

In a nutshell, the core of Step 2 consists of an iterative computation of a small (bounded by $\Delta O(\nu)$) family of $\Delta$-temporal matchings for an arbitrary $\Delta$-window such that at least one of them is “extendable” to a maximum $\Delta$-temporal matching for the whole temporal graph. Let $G = (V, \{(E_t)_{t=1}^\tau\})$ be a temporal graph of lifetime $\tau$, and $d$ and let $\Delta$ be two natural numbers such that $d\Delta \leq \tau$. A family $M$ of $\Delta$-temporal matchings is $d$-complete for $G$ if for any $\Delta$-temporal matching $M$ of $G$ there is an $M' \in M$ such that $(M \setminus M[\Delta(d-1)+1, \Delta d]) \cup M'$ is a $\Delta$-temporal matching of $G$ of size at least $|M|$. The central technical contribution of this paper is a procedure to compute in $\Delta O(\nu) \cdot |E(G[\Delta(d-1)+1, \Delta d])|$ time such a $d$-complete family $M$ of size at most $\Delta O(\nu)$, where $\nu$ is the $\Delta$-vertex cover number of $G$. Formally, we aim for the following theorem.

**Theorem 2.** Given two natural numbers $d, \Delta$ and a temporal graph $G$ of lifetime at least $d\Delta$ and $\Delta$-vertex cover number $\nu$, one can compute in $\Delta O(\nu) \cdot |E(G[\Delta(d-1)+1, \Delta d])|$ time a family of $\Delta$-temporal matchings which is $d$-complete for $G$ and of size at most $\Delta O(\nu)$.

To this end, we define a binary tree where the leaves have a fixed ordering. An order of the leaves of a rooted tree is in **post order** if a depth-first search traversal started at the root can visit the leaves in this order. A **postfix order** of the leaves of a rooted tree is an arbitrarily chosen fixed ordering which is in post order.

**Definition 1.** Let $\Delta \in \mathbb{N}$. A $\Delta$-postfix tree $T$ is a rooted full binary tree of depth at most $\log(\Delta)$ with $\Delta$ many leaves $v_i, i \in [\Delta]$, such that $v_1, v_2, \ldots, v_\Delta$ is the postfix order of the leaves.

Later, the algorithm will construct a $\Delta$-postfix tree $T_v$ for each vertex $v$ in the temporal graph. Here, each leaf in $T_v$ represents a vertex appearance $v$. For example, the leaf $v_t$ represents the vertex appearance $(v, t)$. To encode that vertex appearances of $v$ are $\Delta$-blocked until (or since) some point in the time, we will use specific “separators”.

**Definition 2.** Let $T$ be $\Delta$-postfix tree rooted at $v$ with leaves $v_1, v_2, \ldots, v_\Delta$ in postfix order. Then, a $[a, b]$-separator of $T$ is given by $S := N_T(\bigcup_{i \in [\Delta] \setminus [a, b]} V(P_i))$, where $P_i$ is the $v_i$-$v$ path in $T$.

We now consider an example, depicted in Figure 1, to develop some intuition on $\Delta$-postfix trees and $[a, b]$-separators. In Figure 1, we see an auxiliary graph,
which is constructed for some $\Delta$-window in a temporal graph with only two vertices $v$ and $u$, where $\Delta = 8$. For both vertices we constructed $\Delta$-postfix trees visualized by the dashed edges. Moreover, there is an edge between $v$ and $u$ in the first, fifth, and sixth layer of this $\Delta$-window. This is depicted by the straight edges $\{u_i, v_i\}, i \in \{1, 5, 6\}$. In our algorithm a path from $u$ to $v$ (the roots of our trees) represents a time edge between $u$ and $v$ in the $\Delta$-window. The $[a, b]$-separators will represent that a vertex is blocked by some time edge outside of the $\Delta$-window. In Figure 1 we look at the situation where the vertex $u$ (vertex $v$) is blocked in the first three and last two (first four and last three) layers of the $\Delta$-window. To encode these blocks, we use a $[1, 3]$-separator (blue/stars) and a $[7, 8]$-separator (orange/triangle) in the $\Delta$-postfix tree of $u$ and a $[1, 4]$-separator (green/diamond) and a $[6, 8]$-separator in the $\Delta$-postfix tree of $v$. Note that paths from $u$ to $v$ not intersecting one of the separator vertices correspond to time edges which can be taken into a matching even if $u$ and $v$ are blocked by some other time edges outside of the $\Delta$-window, as depicted by the gray areas in Figure 1.

It is crucial for our algorithm that the $[a, b]$-separators are small in terms of $\Delta$.

Lemma 1. Let $T$ be a $\Delta$-postfix tree $T$ rooted at $v$ with leaves $v_1, v_2, \ldots, v_\Delta$ in postfix order, and let $S$ be the $[a, b]$-separator of $T$, where $[a, b] \subset [\Delta]$ with $a = 1$ or $b = \Delta$. Then,

1. $|S| \leq \log(\Delta) + 1$, and

2. the $v_1$-$v$ path in $T$ contains a vertex from $S$ if and only if $i \in [a, b]$. 

Figure 1: Illustration how the $\Delta$-postfix trees are used for a $\Delta$-window of a temporal graph with only two vertices $u$ and $v$. The edge $\{u, v\}$ is in the first, fifth, and sixth edge set of the $\Delta$-window.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Illustration of $\Delta$-postfix trees for a $\Delta$-window in a temporal graph with two vertices $u$ and $v$. The edge $\{u, v\}$ is in the first, fifth, and sixth edge set of the $\Delta$-window.}
\end{figure}
such that the size of the q algorithms for the parameter $N$ is not empty. Hence, $S$ cannot contain two vertices with the same distance to the root $v$, otherwise $|\Delta| \setminus [a, b]$ is not an interval or we have that $a \neq 1$ and $b \neq \Delta$. Thus, $|S| \leq \log(\Delta) + 1$. The rest of the lemma follows simply by the fact that the root $v$ is in the set $\bigcup_{i \in [\Delta]|\setminus [a, b]} V(P_i)$ and that $\left( \bigcup_{i \in [\Delta]|\setminus [a, b]} V(P_i) \right) \cap N(\bigcup_{i \in [\Delta]|\setminus [a, b]} V(P_i)) = \emptyset$, where $P_i$ is the $v_i$-v path in $T$.

Our intermediate goal now is to compute a family of paths between roots of $\Delta$-postfix trees in the auxiliary graph such that if we are given set of $[a, b]$-separators $S$, then the family shall contain a path between two roots which avoids the vertices in $S$ (if there is one). We will use representative families for this.

Before we jump into the formal definition of representative families, we build some intuition for representative families by considering an illustrative game played by Alice and Bob. Let $U$ and a $p$-family $S \subseteq 2^U$. He shows $U$ and $S$ once to Alice. Afterwards, Bob puts $q$ elements from $U$ on the table $Y$ and asks Alice whether there is a set in $S$ which does not contain any element on the table $Y$. Alice wins if and only if she can answer the question correctly. The goal of Alice is to win this game while remembering as little as possible from $S$. This is represented by a set $\hat{S} \subseteq S$. Intuitively, a representative family $\hat{S} \subseteq S$ guarantees to Alice that there is at least one set in $\hat{S}$ which does not contain an element on the table $Y$ if there is a set in $S$ which does not contain an element on the table $Y$. Formally, we define representative families as follows.

**Definition 3.** Let $S$ be a $p$-family and $\omega : S \to \mathbb{N}$. A subfamily $\hat{S} \subseteq S$ is a max $q$-representative with respect to $\omega$ if for each set $Y$ of size at most $q$ it holds true that if there is a set $X \in \hat{S}$ with $X \cap Y = \emptyset$, then there is an $\hat{X} \in \hat{S}$ such that $\hat{X} \cap Y = \emptyset$ and $\omega(\hat{X}) \geq \omega(X)$.

Representative families are algorithmically useful because there are FPT-algorithms for the parameter $p + q$ to compute $q$-representatives of $p$-families such that the size of the $q$-representative only depends on $p + q$.

An algorithm of Fomin et al. [7] can be iteratively applied to show the following.

**Proposition 1.** [7, Proposition 4.8] Let $\alpha$, $\beta$, and $\gamma$ be non-negative integers such that $r = (\alpha + \beta)\gamma \geq 1$, and let $\omega : U \to \mathbb{N}$ be a weight function. Furthermore, let $U$ be a set, $\mathcal{H} \subseteq 2^U$ be a $\gamma$-family of size $t$, and let $\mathcal{S} = \{ S = \bigcup_{i \in [\alpha]} H_j | H_j \in \mathcal{H} \forall j \in [\alpha] \}$. Then, we can compute a max $\beta\gamma$-representative $\hat{S}$ of $\mathcal{S}$ with respect to $\omega'$ in $2^{O(r)} \cdot t$ time such that $|\hat{S}| \leq \left( \frac{r}{\alpha} \right)^{\frac{1}{\beta}}$, where $\omega'(X) := \sum_{x \in X} \omega(x)$.

Note that van Bevern et al. [4] actually showed a more general version of Proposition 1. However, for the following algorithm, we only need Proposition 1 (that is, Proposition 4.8 of van Bevern et al. [4] in the special case of a uniform matroid represented over a large enough prime field).

Now, we can describe the algorithm behind Theorem 2 in detail.
Algorithm 1 (Algorithm behind Theorem 2). Let \( G = (V, (E_i)_{i=1}^{\tau}) \) be a temporal graph of lifetime \( \tau \), and let \( d \) and \( \Delta \) be two natural numbers such that \( d\Delta \leq \tau \). Furthermore let \( G' := G[\Delta(d-1)+1, \Delta d] \), and let \( \nu \) be the \( \Delta \)-vertex cover number of \( G \).

(i) For each vertex \( u \in V \), we construct the \( \Delta \)-postfix tree \( T_u \) with \( \Delta \) many leaves. These trees have pair-wise disjoint vertex sets. The root of \( T_u \) is \( u_1 \), and the leaves in postfix order are \( u_{1}, u_{2}, \ldots, u_{\Delta} \).

(ii) Construct a \((2 \log \Delta + 3)\)-family \( \mathcal{H} := \mathcal{H}_E \cup \mathcal{H}_D \) such that \( \mathcal{H}_D \) contains \( \nu \) pairwise disjoint sets of fresh vertices, and \( \mathcal{H}_E := \{ E_{(\{u,w\}, t)} \mid (\{u, w\}, t) \) is time edge of \( G' \} \), where \( E_{(\{u,w\}, t)} := \bigcup_{y \in (u, w)} \{ x \in V(T_y) \mid x \) is on the \( y-y_{t-\Delta(d-1)} \)-path in \( T_y \} \cup \{ (\{u, w\}, t) \} \), for all time edges \((\{u, w\}, t)\) of \( G' \).

(iii) Let \( \omega : 2^U \to \mathbb{N} \) with \( \omega(X) := |X \cap \mathcal{E}(G')| \), for all \( X \in 2^U \) be a weight function, where \( U := \bigcup_{A \in \mathcal{H}} A \).

(iv) Compute the max \( 4\nu(\log \Delta + 1) \)-representative \( \tilde{S} \) of

\[
S := \left\{ \bigcup_{i=1}^{\nu} H_i \mid \emptyset \neq H_i \in \mathcal{H}, i \in [\nu] \right\}
\]

with respect to \( \omega \) (using Proposition 1).

(v) Output \( M := \{ S \cap \mathcal{E}(G') \mid S \in \tilde{S} \} \).

Towards the correctness of Algorithm 1 we observe the following.

Lemma 2. Let \( S, G', \) and \( \nu \) be defined as in Algorithm 1 for some temporal graph \( G \) and \( d, \Delta \in \mathbb{N} \). Then, \( M \) is a \( \Delta \)-temporal matching in \( G' \) if and only if there is an \( S \in \tilde{S} \) such that \( M = \mathcal{E}(G') \cap S \) and \( \omega(S) = |M| \).

Proof. (\( \Leftarrow \)): Let \( S \in \tilde{S} \) and set \( M = \mathcal{E}(G') \cap S \). Clearly, \( \omega(S) = |M| \) and for two distinct time edges \((e, t), (e', t') \) we have that \( e \cap e' = \emptyset \), because otherwise \( E_{(e, t)} \cap E_{(e', t')} \neq \emptyset \) and hence \( S \not\subseteq \mathcal{S} \).

(\( \Rightarrow \)): Let \( M \) be a \( \Delta \)-temporal matching in \( G' \). Since all time edges of \( G' \) are in \( \Delta \) consecutive time steps, we know that for all \((e, t), (e', t') \in M \) we have that \( e \cap e' = \emptyset \). Hence, \(|M| \leq \nu \) and \( E_{(e, t)} \cap E_{(e', t')} = \emptyset \). Thus, \( S := \bigcup_{(e, t) \in M} E_{(e, t)} \cup \bigcup_{i=1}^{\nu(|M|)} D_i \in \mathcal{S} \) and \( \omega(S) = |M| \), where \( D_1, \ldots, D_{\nu-|M|} \in \mathcal{H}_D \) are pair-wise disjoint.

We now show the correctness of Algorithm 1.

Lemma 3. Let \( M, G', \) and \( \nu \) be defined as in Algorithm 1 for some temporal graph \( G \), and \( d, \Delta \in \mathbb{N} \). Then, \( M \) is a \( d \)-complete family of \( \Delta \)-temporal matchings for \( G' \).
Proof. By Lemma 2 together with Step (iv) and (v) of Algorithm 1, the family $M$ only contains $\Delta$-temporal matchings in $G'$. To show that $M$ is $d$-complete, let $M$ be a $\Delta$-temporal matching for the whole temporal graph $G$. Then, $M' := M[\Delta(d-1)+1, \Delta d]$ is the $\Delta$-temporal matching in $G'$ which is included in $M$. If $d > 1$, then let $M^- := M[\Delta(d - 2) + 1, \Delta(d - 1)]$ or otherwise $M^- := \emptyset$. If $d < \gamma/\Delta$, then let $M^+ := M[\Delta d + 1, \Delta(d + 1)]$ or otherwise $M^+ := \emptyset$. Observe that $|M^-|, |M'|, |M^+| \leq \nu$ and that vertex appearances in $G'$ which are incident to an arbitrary time edge can only be $\Delta$-blocked by time edges in $M^- \cup M^+$. Hence, there at most $4\nu$ vertices for which some vertex appearances in $G'$ are $\Delta$-blocked by time edges in $M^- \cup M^+$. Let

$$B := \{(v, [1, t - \Delta d]) \mid (e, t) \in M^-, v \in e\} \cup \{(v, [t + 1 - \Delta d, \Delta]) \mid (e, t) \in M^+, v \in e\}.$$ 

Thus, a vertex appearance $v_t$ from $G'$ is $\Delta$-blocked by some time edge in $M^- \cup M^+$ if and only if there is a $(v, [a, b]) \in B$ with $t - \Delta(d - 1) \in [a, b]$. Now, let $Y := \bigcup_{(v, [a, b]) \in B} S_{(v, [a, b])}$, where $S_{(v, [a, b])}$ is an $(a, b)$-separator in the $\Delta$-postfix tree $T_v$. Furthermore, by Lemma 2, there is an $S \in S$ such that $M' = \mathcal{E}(G') \cap S$, and $\omega(S) = |M'|$. We now show that $S \cap Y = \emptyset$. Assume towards a contradiction that $S \cap Y \neq \emptyset$. Hence, there is an $(e, t) \in M'$ such that there is a $u \in E(e, t) \cap Y$. Since $u \in Y$, there is a $v \in e$ such that $u \in V(T_v)$, and a $(v, [a, b]) \in B$ such that $u \in S(v, [a, b])$. From $u \in E(e, t)$ we know that $u$ is on the $v-v_{t - \Delta(d - 1)}$ path in $T_v$. Hence, by Lemma 1, $t \in [a, b]$ and thus there is a time edge $(e', t') \in M^- \cup M^+$ which is not $\Delta$-independent with $(e, t)$. This contradicts $M$ being a $\Delta$-temporal matching. Thus, $S \cap Y = \emptyset$. Since $S \cap Y = \emptyset$, $S \in S$, $|Y| \leq 4\nu(\log \Delta + 1)$, and $\mathcal{S}$ is a max $4\nu(\log \Delta + 1)$-representative of $S$ with respect to $\omega$, we know that there is an $\hat{S} \in \mathcal{S}$ such that $\hat{S} \cap Y = \emptyset$ and $\omega(\hat{S}) \geq \omega(S)$. By the construction of $\mathcal{M}$ in Algorithm 1 and by Lemma 2 we know that there is an $\hat{M} \in \mathcal{M}$ such that $\hat{S} \cap E(G') = \hat{M}$ and $|\hat{M}| = \omega(\hat{S}) \geq \omega(S) = |M'|$. Hence, $|(M \setminus M') \cup \hat{M}| \geq |M|$. We now show that $(M \setminus M') \cup \hat{M}$ is a $\Delta$-temporal matching. Suppose not. Then there are time edges $(e, t) \in M^-$ or $(\hat{e}, \hat{t}) \in \hat{M}$ with $v \in e \cap \hat{e}$ such that the vertex appearance $v_t$ is $\Delta$-blocked by $(e, t)$. Hence, there is a $(v, [a, b]) \in B$ with $\hat{t} - \Delta(d - 1) \in [a, b]$. By Lemma 1 this contradicts $\hat{S} \cap Y = \emptyset$. Hence, $(M \setminus M') \cup \hat{M}$ is a $\Delta$-temporal matching and thus $\mathcal{M}$ is $d$-complete.

The running time of the dynamic program defined in 1 will be discussed directly in the following proof of Theorem 2.

Proof of Theorem 2 By Lemma 3 we can use Algorithm 1 to compute a $d$-complete family $M$ of $\Delta$-temporal matchings in $G[\Delta(d - 1) + 1, \Delta d]$. It is easy to verify that we can compute $\mathcal{H}$ in $O(\nu + |\mathcal{E}(G')[\Delta(d - 1) + 1, \Delta d]|) \log \Delta$ time (by ignoring isolated vertices). Finally, we compute $\mathcal{S}$ with Proposition 1 in $2^{O(\nu \log \Delta)} \cdot |\mathcal{E}(G')[\Delta(d - 1) + 1, \Delta d]|$ time, by setting $\alpha$ to $2 \log \Delta + 3$, $\beta$ to $2\nu$, 

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and γ to ν. By Proposition 1 the size of $\hat{S}$ is at most $2^{O(\nu \cdot \log \Delta)} = \Delta^{O(\nu)}$. Hence, we end up with an overall running time of $\Delta^{O(\nu)} \cdot |E(G[\Delta(d-1)+1, \Delta d])|$. □

3.2 Step 3: The dynamic program

In this section we describe Step 3 of the algorithm behind Theorem 1, see Section 3.

Let $G = (V, (E_t)_{t=1}^\tau)$ be a temporal graph such that $\tau$ is a multiple of $\Delta \in \mathbb{N}$. Assume that we already computed for all $d \in [\tau/\Delta]$ a family $\mathcal{M}_d$ of $\Delta$-temporal matchings which is $d$-complete for $G$.

For all $i \in [\tau/\Delta] \setminus \{0\}$ and $M \in \mathcal{M}_i$, for $i > 1$ let

$$T_i[M] := \max(A(M) \cup \{0\}), \text{ where } A(M) := \{|M| + T_{i-1}[M'] \mid M' \in \mathcal{M}_{i-1}, M \cup M' \text{ is a } \Delta\text{-temporal matching}\}.$$ 

(1)

Towards the correctness of the dynamic program specified in (1), we observe the following.

**Lemma 4.** There is a $\Delta$-temporal matching of size at least $k$ in $G$ if and only if $\max_{M \in \mathcal{M}_k} T_\Delta[M] \geq k$.

**Proof.** $(\Rightarrow)$: We show by induction over $i$ that if there is a $\Delta$-temporal matching $M$ in $G$, then there is an $M' \in \mathcal{M}_i$ such that $T_i[M'] \geq |M[1, \Delta i]|$ and $M[1, \Delta(i-1)] \cup M' \cup M[\Delta i + 1, \tau]$ is a $\Delta$-temporal matching of size at least $|M|$. By (1), this is clearly the case for $i = 1$, because $\mathcal{M}_1$ is 1-complete for $G$.

For the induction step, let $i > 1$ and assume that if there is a $\Delta$-temporal matching $M$ in $G$, then there is an $M' \in \mathcal{M}_{i-1}$ such that

(i) $T_{i-1}[M'] \geq |M[1, \Delta(i-1)]|$, and

(ii) $M[1, \Delta(i-2)] \cup M' \cup M[\Delta(i-1) + 1, \tau]$ is a $\Delta$-temporal matching of size at least $|M|$.

Let $M''$ be a $\Delta$-temporal matching for $G$. By the induction hypothesis, there is an $M' \in \mathcal{M}_{i-1}$ such that $T_{i-1}[M'] \geq |M''[1, \Delta(i-1)]|$ and $\hat{M} := M''[1, \Delta(i-2)] \cup M' \cup M''[\Delta(i-1) + 1, \tau]$ is a $\Delta$-temporal matching of size at least $|M''|$. Since $\mathcal{M}_i$ is $i$-complete for $G$, there is an $M'' \in \mathcal{M}_i$ such that $\hat{M}[1, \Delta(i-1)] \cup M'' \cup \hat{M}[\Delta i + 1, \tau]$ is a $\Delta$-temporal matching of size at least $|\hat{M}| \geq |M''|$. By (1), we have that $T_i[M''] \geq |\hat{M}[1, \Delta i]| \geq |M''[1, \Delta i]|$. Hence, if there is a $\Delta$-temporal matching of size $k$ in $G$, then there is an $M' \in \mathcal{M}_k$ such that $T_\Delta[M'] \geq k$.

$(\Leftarrow)$: We show by induction over $i$ that if $T_i[\hat{M}] > 0$, then there is a $\Delta$-temporal matching $M$ in $G[1, \Delta(i-1)]$ such that $|M| = T_i[M']$ and $M[\Delta(i-1) + 1, \Delta i] = M'$, where $M' \in \mathcal{M}_i$. By (1), this is clearly the case for $i = 1$, because $\mathcal{M}_1$ is 1-complete for $G$.

For the induction step, let $i > 1$ and assume that if $T_{i-1}[M'] > 0$, then there is a $\Delta$-temporal matching $M$ in $G[1, \Delta(i-1)]$ such that $|M| = T_{i-1}[M']$ and
$M[\Delta(i-2)+1, \Delta(i-1)] = M'$, where $M' \in \mathcal{M}_{i-1}$. Let $T_i[M'] > 0$, for some $M'' \in \mathcal{M}_i$. By Corollary 1, there is an $M' \in \mathcal{M}_{i-1}$ such that $M'' \cup M'$ is a $\Delta$-temporal matching and $T_i[M''] = T_{i-1}[M'] + |M''|$. By the induction hypothesis, there is a $\Delta$-temporal matching $M$ in $\mathcal{G}[1, \Delta(i-2)]$ such that $|M| = T_{i-1}[M']$ and $M[\Delta(i-2)+1, \Delta(i-1)] = M'$. Since $M[\Delta(i-2)+1, \Delta(i-1)] = M'$ and $M' \cup M''$ is a $\Delta$-temporal matching, $M \cup M''$ is a $\Delta$-temporal matching of size $T_{i-1}[M'] + |M''| = |T_i[M'']|$. Hence, if $\max_{M \in \mathcal{M}} T_{\frac{\Delta}{2}}[M] \geq k$, then there is a $\Delta$-temporal matching of size at least $k$ in $\mathcal{G}$.

We now are ready to show Theorem 1.

**Proof of Theorem 1** Let $(\mathcal{G}, k, \Delta)$ be an instance of Temporal Matching. We assume without loss of generality that there is no $\Delta$-window in $\mathcal{G}$ which does not contain any time edge, otherwise we can split $\mathcal{G}$ into two parts, compute the maximum size of a $\Delta$-temporal matching in each part separately, and check whether the sum is at least $k$. Moreover, we assume without loss of generality that the lifetime $\tau$ of $\mathcal{G}$ is a multiple of $\Delta$, otherwise we can add some empty layers at the end of $\mathcal{G}$.

We start by splitting $\mathcal{G}$ into $\lceil \tau/\Delta \rceil$ many $\Delta$-windows: for all $i \in [\lceil \tau/\Delta \rceil]$ let $\mathcal{G}_i := \mathcal{G}[\Delta(i-1)+1, \Delta i]$. This can be done in $O(|\mathcal{G}|)$ time. To compute the $\Delta$-vertex cover number, we first compute the vertex cover number of the underlying graph of $\mathcal{G}_i$, for all $i \in [\lceil \tau/\Delta \rceil]$. Let $\nu'$ be the maximum vertex cover number over all underlying graphs $\mathcal{G}_i$, where $i \in [\lceil \tau/\Delta \rceil]$. Note that the $\Delta$-vertex cover number $\nu$ of $\mathcal{G}$ is at least $\nu'$ and at most $2\nu'$. Hence, in $2^{O(\nu)} \cdot |\mathcal{G}|$ time, we can compute $\nu'$ and then guess $\nu$. Next, we compute with Theorem 2 for each $i \in [\lceil \tau/\Delta \rceil]$ a family $\mathcal{M}_i$ of $\Delta$-temporal matching of size $\Delta^O(\nu)$ which is $i$-complete for $\mathcal{G}$ in $\Delta^O(\nu) \cdot |\mathcal{E}(\mathcal{G}_i)|$. Hence, it takes $\Delta^O(\nu) \cdot |\mathcal{G}|$ time to compute the families $\mathcal{M}_1, \ldots, \mathcal{M}_{\lceil \tau/\Delta \rceil}$.

Now we have met the preconditions to compute the dynamic program specified in Corollary 1. By Lemma 4, there is a $\Delta$-temporal matching of size at least $k$ in $\mathcal{G}$ if and only if $\max_{M \in \mathcal{M}} T_{\frac{\Delta}{2}}[M] \geq k$. Note that $\max_{M \in \mathcal{M}} T_{\frac{\Delta}{2}}[M]$ can be computed in $\Delta^O(\nu) \cdot \sum_{i=1}^{\lceil \tau/\Delta \rceil} |\mathcal{E}(\mathcal{G}[\Delta(i-1), \Delta i])|$ time. Since each $\Delta$-window contains at least one time edge, we arrive at an overall running time of $\Delta^O(\nu) \cdot |\mathcal{G}|$. This completes the proof.

It is easy to check that if we additionally store for the table entry $T_i[M'], i \in [\lceil \tau/\Delta \rceil], M' \in \mathcal{M}_i$ a $\Delta$-temporal matching $M$ in $\mathcal{G}[1, \Delta i]$ of size $T_i[M']$ such that $M[\Delta(i-1)+1, \Delta i] = M'$, then the dynamic program also computes a $\Delta$-temporal matching of maximum size and not just the size. Thus, we can solve the optimization variant of Temporal Matching.

**Corollary 1.** Given a temporal graph $\mathcal{G}$ and an integer $\Delta$, we can compute in $\Delta^O(\nu) \cdot |\mathcal{G}|$ time a maximum-cardinality $\Delta$-temporal matching in $\mathcal{G}$, where $\nu$ is the $\Delta$-vertex cover number.
4 Conclusion

While we could improve the running time to solve Temporal Matching exponentially in terms of $\Delta$ compared to the algorithm of Mertzios et al. [11], we left open whether in Theorem 1 we can get rid off the running time dependence on $\Delta$.

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