The Light Lexicographic path Ordering

E.A. Cichon and J-Y. Marion *

February 1, 2008

Abstract

We introduce syntactic restrictions of the lexicographic path ordering to obtain the Light Lexicographic Path Ordering. We show that the light lexicographic path ordering leads to a characterisation of the functions computable in space bounded by a polynomial in the size of the inputs.

1 Introduction

Termination orderings have been particularly successful inventions for proving the termination of rewrite systems. Their success is mainly due to their ease of implementation and they are the principal tool used in modern completion-based theorem provers. The two best known termination orderings are the multiset path ordering, MPO1, and the lexicographic path ordering, LPO2, (and their variations or derivatives).

Termination orderings give rise to interesting theoretical questions concerning the classes of rewrite algorithms for which they provide termination proofs. It has been shown that MPO gives rise to a characterisation of primitive recursion (see [4]) and that LPO characterises the multiply recursive functions (see [16]). While both of these classes contain functions which are highly unfeasible, the fact remains that many feasible algorithms can be successfully treated using one or both of MPO or LPO.

If we compare different syntactic characterisations of function classes we observe that termination orderings can have a remarkable advantage over other

---

1 Suppose that $<_{\Sigma}$ is an ordering on the signature $\Sigma$. The multiset path ordering, $<_{\text{mpo}}$, on $T(\Sigma)$ is defined recursively as follows: (i) $f(t_1, \ldots, t_n) <_{\text{mpo}} t \rightarrow (\forall i \in 1..n) t_i <_{\text{mpo}} t$, (ii) $g <_{\Sigma} f$ and $(\forall i \in 1..m) s_i <_{\text{mpo}} f(t_1, \ldots, t_n) \rightarrow g(s_1, \ldots, s_m) <_{\text{mpo}} f(t_1, \ldots, t_n)$, (iii) $\{s_1, \ldots, s_m\} <_{\text{mpo}} \{t_1, \ldots, t_n\} \rightarrow f(s_1, \ldots, s_m) <_{\text{mpo}} f(t_1, \ldots, t_n)$, where $<_{\text{mpo}}$ is the multiset ordering induced by $<_{\text{mpo}}$.

2 Suppose that $<_{\Sigma}$ is an ordering on the signature $\Sigma$. The lexicographic path ordering, $<_{\text{ipo}}$, on $T(\Sigma)$ is defined recursively as follows: (i) $f(t_1, \ldots, t_n) <_{\text{ipo}} t \rightarrow (\forall i \in 1..n) t_i <_{\text{ipo}} t$, (ii) $g <_{\Sigma} f$ and $(\forall i \in 1..m) s_i <_{\text{ipo}} f(t_1, \ldots, t_n) \rightarrow g(s_1, \ldots, s_m) <_{\text{ipo}} f(t_1, \ldots, t_n)$, (iii) for some $i \in 1..n$, $s_1 = t_1, \ldots, s_{i-1} = t_{i-1}, s_i <_{\text{ipo}} t_i$, and $s_{i+1} <_{\text{ipo}} t, \ldots, s_m <_{\text{ipo}} t \rightarrow f(s_1, \ldots, s_m) <_{\text{ipo}} f(t_1, \ldots, t_n)$. 
characterisations. Let us compare MPO with Strict Primitive Recursion, SPR$^3$. MPO is more general than SPR. MPO is applicable to arbitrary equational specifications. There is no a priori dependence on the data-type semantics. MPO always proves termination of an SPR programme over the tally numbers$^4$, as was observed by Plaisted$^5$, but it can also prove termination of other algorithms where the intended semantics are not those of functions over the natural numbers. So MPO can be thought of as a generalisation of SPR which allows a broader class of algorithms$^5$.

Now, the results of Hofbauer and Weiermann, $^4, ^6$, indicate that LPO is considerably more powerful than MPO from the point of view of computational complexity, and one might consider such extra power to be redundant - one can show that LPO easily proves termination of the Ackermann function$^6$ whereas no MPO termination proof for this function is possible, but one criticism of this is to say “so what, since the Ackermann function is not feasibly computable?”. Yet, in some cases, the builders of theorem provers have preferred LPO. This is because of LPO’s applicability to a wide range of naturally arising algorithms. For example, we can paraphrase the Ackermann result stated above: for syntactic reasons, MPO cannot prove termination of a simple feasible function when its algorithm is based on a straightforward tail-recursion whereas LPO will, in general, succeed.

In recent years, the lower complexity classes have been studied from a syntactical point of view. Bellantoni and Cook$^1$ restricted the schemes for defining the primitive recursive functions to obtain a syntactic characterisation of the functions computable in time bounded by a polynomial in the size of the inputs. In parallel, Leivant$^7$ devised a method of data-tiering - a process which, roughly speaking, assigns types to inputs and outputs in such a way as to restrict the class of allowable algorithms within a given syntactic framework (strict primitive recursion, for example).

In the present paper we introduce syntactic restrictions of the lexicographic path ordering to obtain what we call the Light Lexicographic Path Ordering. This is a follow-up to the work of Marion in $^7$ where he imposes syntactic restrictions on MPO to obtain a characterisation of the polynomial time computable functions. We show that the light lexicographic path ordering leads to a characterisation of the functions computable in space bounded by a polynomial in the size of the inputs. The proof depends essentially on the characterisation

$^3$The schemes for defining the primitive recursive functions over the natural numbers are as follows. The Initial Functions are \{ $Z(x) = 0$, $S(x) = x'$, $U^n(x_0, \ldots, x_{n-1}) = x_i$ where $0 \leq i \leq n - 1$ \}. The primitive recursive function are closed under (i) Composition: $f(\vec{x}) = g(h_1(\vec{x}), \ldots, h_m(\vec{x}))$, and (ii) Primitive Recursion: $f(\vec{x}, 0) = g(\vec{x})$, $f(\vec{x}, y + 1) = h(\vec{x}, y, f(\vec{x}, y))$.

$^4$ The tally numbers are obtained using a single constant and a single, unary, successor function symbol.

$^5$MPO is intimately linked with the SPR functions in the following way: the computation time of an MPO algorithm is primitive recursive in the size of the input.

$^6$A(0, y) = y + 1, A(x + 1, 0) = A(x, 1), A(x + 1, y + 1) = A(x, A(x + 1, y))$

$^7$ We presuppose no familiarity with this work
of the polynomial space computable functions given in [10]. This characterization relies on the ability to capture recursion with parameter substitution, as in [11, 12].

2 First order functional programming

The set of terms built up from a signature \( S \) and from a set of variables \( V \) is \( T(S, V) \). A program, is defined by a quadruplet \( \langle \rightarrow, C, D, f \rangle \) thus. The set of data is the term algebra \( T(C) \) where symbols in \( C \) are called constructors. We shall always assume that \( C \) contains, at least, a 0-ary constructor, i.e. a constant, and all other constructors are unary, i.e. successors.\(^8\). For example, binary words will be represented by terms built up from \( \{ \varepsilon, s_0, s_1 \} \) where \( \varepsilon \) is a constant denoting the empty words, and \( s_0, s_1 \) are two successors. \( D \) is the set of function symbols of fixed arity \( > 0 \). So, the full signature is \( S = C \cup D \).

Rewrite rules are given by the binary relation \( \rightarrow \). Each rewrite rule is of the form \( g(p_1, \ldots, p_n) \rightarrow s \) where \( g \in D \), the \( p_i \)'s are patterns, that is terms of \( T(C, V) \), and \( s \) are terms of \( T(C \cup D, V) \). Moreover, \( \text{Var}(g(t_1, \ldots, t_n)) \subseteq \text{Var}(s) \) where \( \text{Var}(t) \) is the set of variables in a term \( t \). Lastly, the function symbol \( f \in D \) is the main function symbol.

We define \( u \rightarrow v \) to say that the term \( v \) is obtained from \( u \) by applying a rewrite rule. The relation \( \xrightarrow{\rightarrow} \) (\( \xrightarrow{\rightarrow} \)) denotes the transitive (reflexive-transitive) closure of \( \rightarrow \). We write \( s \xrightarrow{\rightarrow} t \) to mean that \( s \xrightarrow{\rightarrow} t \) and \( t \) is in normal form. A (ground) substitution \( \sigma \) is a function from \( V \) into \( T(C \cup D, V) \) (resp. \( T(C \cup D) \)).

Say that a program is confluent if the relation \( \rightarrow \) is confluent. To give a semantic to programs, we just consider, as meaningful, the normal forms which are in the data set \( T(C) \).

**Definition 1.** A confluent program \( \langle \rightarrow, C, D, f \rangle \) computes the function \( \{ f \} : T(C)^n \rightarrow T(C) \) which is defined as follows. For all \( u_1, \ldots, u_n \in T(C), \)

\[
\{ f \}(u_1, \ldots, u_n) = v \text{ if } f(u_1, \ldots, u_n) \xrightarrow{\rightarrow} v, \text{ otherwise } \{ f \}(u_1, \ldots, u_n) \text{ is undefined.}
\]

A program is terminating if there is no infinite derivation, that is there is no infinite sequence of terms such that \( t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \cdots \). One might consult [2] for a survey about rewriting termination, and [3] about general references on rewriting.

3 Light Lexicographic Path Ordering

We now describe our restriction of LPO which we call the **Light Lexicographic Path Ordering** (LLPO),

---

\(^8\)The result presented can be extended easily to lists. It might be possible to establish the same result on trees data structure by representing trees as directed acyclic graphs.
Definition 2. The *valency* of a function symbol $f$, of arity $n$, is a mapping $\nu(f) : \{1, \ldots, n\} \rightarrow \{0, 1\}$. We write $\nu(f,i)$ to denote the valency of $f$ at its $i^{th}$ argument.

Valencies will allow us to combine two kinds of orderings to prove the termination of a program. Valencies are to some extent related to the Kamin-Levy notion of functionals on orders. Indeed, a functional on orderings can be defined as a status function on $D$ which indicates how to compare terms, either in a lexicographic or in a multiset way. Similarly, the valency of a function will also indicate how to compare terms.

Above all, the notion of valency resembles the notion of data tiering which was introduced by Leivant in [7, 8]. The data tiering discipline ensures that the types of terms are also tiered. Actually, function valencies are much more like normal and safe position arguments as defined by Bellantoni and Cook in [1]. Function valencies generalise this concept to functions defined by means of recursive equations. For the sake of readability, we use a notational convention similar to that of [1], and write $f(x_1, \ldots, x_n; y_1, \ldots, y_m)$, with a semi-colon separating two lists of arguments, to indicate that $\nu(f,i) = 1$ for $i \leq n$, and $\nu(f, n + j) = 0$ for $j \leq m$. We shall write $f(\ldots, t_i, \ldots)$ to mean that the term $t_i$ occurs at position $i$ in $f$, which is a position of valency $\nu(f,i)$.

Throughout, we shall always assume that there is a precedence, $\preceq_{D}$ on $D$ which is a total pre-order. As usual the strict precedence $\prec_{D}$ is defined by $g \prec_{D} f$ if $g \preceq_{D} f$ and $f \not\preceq_{D} g$. Also, the equivalence relation $\approx$ is defined by $g \approx f$ if $g \preceq_{D} f$ and $f \preceq_{D} g$. We also assume that $\approx$ respects the function symbol arities and the valencies. That is, if $f \approx g$ then the arity of $f$ and $g$ is $n$ and $\nu(f,i) = \nu(g,i)$ for all $i \leq n$. On the other hand, the constructors of $C$ will be incomparable, and that the only equivalence relation on $C$ will be that of syntactic equality.

Definition 3. The partial ordering $\prec_{1}$ on $T(C \cup D, V)$ is defined recursively as follows:

1. If $s \preceq_{1} t$ and if $c \in C$, then $s \prec_{1} c(t)$.
2. If $c \in C$ and $s \prec_{1} f(t_1, \ldots, t_m)$, then $c(s) \prec_{1} f(t_1, \ldots, t_m)$.
3. If $s \preceq_{1} t_i$ and if $\nu(f,i) = 1$ then $s \prec_{1} f(\ldots, t_i, \ldots)$.
4. If $(g \prec_{D} f)$ and if for all $i \leq n$, $s_i \prec_{1} f(t_1, \ldots, t_m)$, then $g(s_1, \ldots, s_n) \prec_{1} f(t_1, \ldots, t_m)$.

Here $\preceq_{1} = \prec_{1} \cup =$, where $=$ is the syntactic identity, and $f \in D$.

Examples 4.

1. Two distinct terms of $T(C)$ are incomparable by $\prec_{1}$, except if there are subterms.
2. $g(x, x;) \prec_{1} f(x;)$ if $g \prec_{D} f$. 

4
3. \( h(x;x) \triangleleft_1 f(x) \) if \( h \triangleleft D f \).

4. \( f(x) \not\triangleleft_1 k(y;x) \) because \( x \) is of valency 0 in \( k \).

5. Two terms with the same root symbol are incomparable with respect to \( \triangleleft_1 \). So \( \triangleleft_1 \) is not monotonic.

**Definition 5.** For each \( f \in D \), we define

\[
D[f = \{ g : g \triangleleft_D f \}]
\]

**Definition 6.** The partial ordering \( \triangleleft_0 \) on \( (C \cup D, V) \) is defined recursively as follows:

1. If \( s \leq_0 t \) and if \( f \in C \cup D \), then \( s \triangleleft_0 f(t_1, \ldots, t_m) \).

2. If \( c \in C \) and \( s \triangleleft_0 f(t_1, \ldots, t_m) \), then \( c(s) \triangleleft_0 f(t_1, \ldots, t_m) \).

3. If \( g \triangleleft_D f \) and for all \( i \leq n \), \( s_i \triangleleft_{\nu(g,0)} f(t_1, \ldots, t_m) \), then \( g(s_1, \ldots, s_n) \triangleleft_0 f(t_1, \ldots, t_m) \).

4. If \( g \approx f \) and if \( s_1 = t_1, \ldots, s_{p-1} = t_{p-1} \) and \( s_p \triangleleft_1 t_p \), where \( \nu(f, p) = 1 \), and for all \( 1 \leq j \leq n - p \), either \( s_{p+j} \leq_D t_{p+j} \) and \( \nu(f, p + j) = 1 \), or \( s_{p+j} \in (C \cup D)[f, V] \) and \( s_{p+j} \triangleleft_0 f(t_1, \ldots, t_n) \) with \( \nu(f, p + j) = 0 \), then \( g(s_1, \ldots, s_n) \triangleleft_0 f(t_1, \ldots, t_n) \).

where \( \leq_0 = \triangleleft_0 \cup = \), where \( = \) is the syntactic identity, and \( f \in D \).

The ordering \( \triangleleft_0 \) possesses the subterm property, i.e. \( t \triangleleft_0 f(t_1, \ldots, t_m) \) for all terms \( t \in (C \cup D, V) \) and \( f \in C \cup D \). The ordering \( \triangleleft_0 \) is monotonic with respect to arguments of valency 1, that is, if \( s_1 \triangleleft_1 t_1 \), where \( \nu(f, i) = 1 \), then \( f(s_1, \ldots, s_n) \triangleleft_0 f(t_1, \ldots, t_n) \).

**Proposition 7.** The ordering \( \triangleleft_0 \) is an extension of \( \triangleleft_1 \), that is, if \( s \triangleleft_1 t \) then \( s \triangleleft_0 t \).

**Proposition 8.** The lexicographic path ordering \( \triangleleft_{\text{lpo}} \) is an extension of \( \triangleleft_0 \), that is if \( s \triangleleft_0 t \) then \( s \triangleleft_{\text{lpo}} t \).

LLPO is the ordering formed by \((\leq_1, \leq_0)\). Termination follows from the fact that LLPO is a restriction of LPO.

**Definition 9.** An LLPO-program is defined by a tuple \( (\rightarrow, C, D, f, \nu, \triangleleft_D) \), where \( (\rightarrow, C, D, f) \) is a confluent program, \( \nu \) is a valency function on \( D \), \( \leq_D \) is a precedence on \( D \), and such that for each rule \( l \rightarrow r, r \triangleleft_D l \).

**Theorem 10.** Each LLPO-program \( (\rightarrow, D, C, f, \nu, \triangleleft_D) \) is terminating.
To use LLPO for a proof of termination, we must find an appropriate precedence $\leq_D$ such that the program is terminating by LPO. Then, it remains to determine a correct valency function over $D$.

**Examples 11.**

1. Suppose that $\mathcal{C} = \{\epsilon, 0, 1\}$.

The set of constructor terms of $T(\mathcal{C})$ represents binary words. The program below computes a function which reverses a binary word.

\[
\begin{align*}
\text{reverse}(\epsilon; y) & \rightarrow y \\
\text{reverse}(i(x); y) & \rightarrow \text{reverse}(x; i(y))
\end{align*}
\]

It is easy to see that each rule is ordered by $\prec_0$. Notice that reverse is defined by a tail recursion whose termination cannot be proved using MPO.

2. The following rules which define a recurrence with substitution of parameters. This schema is important because it might be seen as a template for simulating the computation of an alternating Turing machine. Again, MPO does not prove the termination of this schema.

\[
\begin{align*}
f(b; y) & \rightarrow y \\
f(c(x); y) & \rightarrow h(x; y, f(x; \delta_0(y)), f(x; \delta_1(y)))
\end{align*}
\]

By setting, $h \prec_D f$ and $\delta_i \prec_D f$, the schema is ordered by LLPO. Indeed, $f(x; \delta_i(y)) \prec_0 f(c(x); y)$ because (i) $\nu(f, 2) = 0$, (ii) $\delta_i(y) \in T(\mathcal{C} \cup D[f, \mathcal{V}])$ and (iii) $\delta_i(y) \prec_0 f(c(x); y)$. The ordering proof of the last equation is displayed below in sequent style.

\[
\begin{array}{c}
x = x \\
\hline
x \prec_1 c(x) \quad y = y
\end{array}
\quad
\begin{array}{c}
x = x \\
\hline
y \prec_0 f(x; y)
\end{array}
\quad
\begin{array}{c}
x = x \\
\hline
\delta_i(y) \prec_0 f(x; y)
\end{array}
\quad
\begin{array}{c}
\delta_i(y) \prec_0 f(c(x); y)
\end{array}
\]

3. In this last example, we consider a program that computes $2^x + y$.

\[
\begin{align*}
f(b, y) & \rightarrow c(y) \\
f(c(x), y) & \rightarrow f(x, f(x, y))
\end{align*}
\]

The termination of the above program cannot be shown by $\prec_0$. Indeed, if we use LPO, we have to (i) compare lexicographically $(x, f(x, y))$ with $(c(x), y)$ and then (ii) to show that $f(x, y)$ is less than $f(c(x), y)$. But $f(x, y) \not\in T(\mathcal{C} \cup D[f, \mathcal{V}])$ and so, this condition, which is imposed by $\prec_0$, does not hold.
4 Characterisation of Pspace

Computational resources, that is, time and space, are measured relative to the size of the input arguments. The size, $|t|$, of a term $t$ is the number of symbols in $t$.

$$|t| = \begin{cases} 1 & \text{if } t \text{ is a constant} \\ \sum_{i=1}^{n} |t_i| + 1 & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

A LLPO-program $(\rightarrow, \mathcal{C}, D, f, \nu, \prec_D)$ is computable in polynomial space, if there is a Turing Machine $M$ such that for each input $a_1, \ldots, a_n$ of $T(C)$, $M$ computes $\{f\}(a_1, \ldots, a_n)$ in space bounded by $P(\sum_{i \leq n} |a_i|)$. We now state our main result.

**Theorem 12.** Each LLPO-program is computable in polynomial space and, conversely, each polynomial space function is computed by an LLPO-program.

The remainder of the paper is devoted to the proof of Theorem above.

4.1 Poly-space functions are LLPO-computable

**Theorem 13.** Each function $\phi$ which is computable in polynomial space is represented by a LLPO-program.

**Proof.** The proof is based on the characterisation of polynomial space computable functions by means of ramified recurrence, reported in [14]. We shall represent ramified functions by LLPO-programs. Firstly, let us recall briefly how ramified functions are specified. Let $\mathcal{C}$ be the set of constructors, and suppose that each constructor is of arity 0 or 1. Let $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_k, \ldots$ be copies of the set $\mathcal{C}$. We shall say that $\mathcal{C}_i$ is (a copy) of tier $i$.

A ramified function $F$ of arity $n$ must satisfy the following condition: The domain of a ramified function $F$ of arity $n$ is $T(\mathcal{C}_{i_1}) \times \cdots \times T(\mathcal{C}_{i_n})$, and the range is $T(\mathcal{C}_k)$ where the output tier is $k$ and $k = \min_{j=1,n} (i_j)$.

The class of ramified functions is generated from constructors in $\mathcal{C}_i$, for all tiers $i$, and is closed under composition, recursion with parameter substitution and flat recursion.

A ramified function $F$ is defined by flat recursion if

$$F(b, \bar{x}) = G(b, \bar{x}) \quad \text{where } b \text{ is a 0-ary constructor in } \mathcal{C}_p$$
$$F(c(t), \bar{x}) = H(t, \bar{x}) \quad \text{where } c \text{ is a unary constructor in } \mathcal{C}_p$$

where $\bar{x} = x_1, \ldots, x_n$ and $G$ and $H$ are defined previously. We see that the flat recursion template is ordered by $\prec_0$ because we can set $G, H \prec_D F$.

A ramified function $F$ of output tier $k$ is defined by recursion with substitution of parameters if for some $p > k$

$$F(b, \bar{x}) = G(\bar{x}) \quad \text{where } b \text{ is a 0-ary constructor in } \mathcal{C}_p$$
$$F(c(t), \bar{x}) = H(t, \bar{x}, A_1, \cdots, A_m) \quad \text{where } c \text{ is a unary constructor in } \mathcal{C}_p$$
where \(A_j = F(t, \sigma^i_1(\bar{x}), \ldots, \sigma^i_n(\bar{x}))\). The functions \(G, H\) and \((\sigma^i_j)_{i \leq n; j \leq m}\) are ramified functions. The crucial requirement imposed on the ramified recursion schema is that the output tier \(k\) of \(F\) be strictly smaller than the tier \(p\) of the recursion parameter. It follows that \(p > 0\).

Now the above template is ordered by \(\prec_0\) by putting \(F\) valencies thus.

\[
\nu(F, p) = \begin{cases} 
0 & \text{if the tier of the } p\text{th argument is } k \\
1 & \text{otherwise}
\end{cases}
\]

The termination proof is similar to the proof of example 11(2).

We conclude that each ramified function is computable by an LLPO-program, and so, following [10], each polynomial-space computable function is represented by an LLPO-program.

### 4.2 LLPO-programs are Poly-space computable

The height \(ht(t)\) of a term \(t\) is defined as the length of the longest branch in the tree \(t\).

\[
ht(t) = \begin{cases} 
1 & \text{if } t \text{ is a constant or a variable} \\
\max_{i=1}^n ht(t_i) + 1 & \text{if } t = f(t_1, \ldots, t_n)
\end{cases}
\]

**Theorem 14.** Let \(<\rightarrow, C, D, f, \nu, \prec_D>\) be a LLPO-program. There is a polynomial \(P\) such that for all inputs \(a_1, \ldots, a_n \in T(C)\), we have

1. If \(f(x_1, \ldots, x_n) \sigma^* \rightarrow t\) then \(ht(t) \leq P(\sum_{i \leq n} |a_i|)\).

2. If \(f(x_1, \ldots, x_n) \sigma^* \rightarrow u\) and \(u \in T(C)\) is a subterm of \(t\), then \(|u| \leq P(\sum_{i \leq n} |a_i|)\).

**Proof.** (1) is a consequence of Theorem 17 of Lemma 18 and of Lemma 19. (2) is a consequence of (1) by observing that for each \(u \in T(C)\), we have \(|u| = ht(u)|.\)

**Theorem 15.** Let \(<\rightarrow, C, D, f, \nu, \prec_D>\) be an LLPO-program. For all inputs \(a_1, \ldots, a_n \in T(C)\), the computation of \(f(a_1, \ldots, a_n) \rightarrow v\), where \(v \in T(C)\), is performed in space bounded by \(P(\sum_{i \leq n} |a_i|)\), where \(P\) is some polynomial which depends on the program.

**Proof.** Given a program \(<\rightarrow, C, D, f, \nu, \prec_D>\), the operational semantics of call by value are provided by a relation \(\vdash^\sigma \subseteq T(C \cup D, V) \times T(C)\) where \(\sigma\) is a substitution over \(T(C)\). The relation \(\vdash^\sigma\) is defined as the union of the family \(\vdash^\sigma_h\) defined below:

- \(x \vdash^\sigma_h \sigma(x)\), if \(x \in V\) and \(h = |\sigma(x)|\).
- \(b \vdash^\sigma_h b\), if \(b\) is a 0-ary constant of \(C \cup D\).
\begin{itemize}
  \item $\mathbf{c}(t) \vdash_{h+1}^\sigma \mathbf{c}(u)$ if $t \vdash_h^\sigma u$
  \item $f(t_1, \ldots, t_n) \vdash_{h}^\sigma u$, if $t_1 \vdash_{h1}^\sigma u_1, \ldots, t_n \vdash_{h_n}^\sigma u_n$, and $s \vdash_h^\sigma u$
    where $f(u_1, \ldots, u_n) = t\theta t \rightarrow s$ is a rulw, and $h = \max_{i=1,n}(h_i + 1, h_0)$.
\end{itemize}

and $\vdash_{h}^\sigma = \bigcup_{h \geq 0} \vdash_{h}^\sigma$. It is routine to verify that $t\sigma \vdash_{h}^\sigma v$ if $t \vdash_{h}^\sigma v$, where $v \in T(\mathcal{C})$.

The rules of the operational semantics described above form a recursive algorithm which is an interpreter of LLPO-programs. Put $\sigma_0(x_i) = a_i$. The computation of $f(a_1, \ldots, a_n)$ consists in determining $u$ such that $f(x_1, \ldots, x_n) \vdash_{h_0}^\sigma u$ for some $h$. Actually, $h$ is the height of the computation tree.

It remains to show that this evaluation procedure runs in space bounded by a polynomial in the sum of sizes of the inputs.

For this, put $DH(t) = \max\{\text{ht}(u) : t\sigma \vdash_{h}^\sigma u\}$. By induction on $h$, we can establish that $t \vdash_{h}^\sigma u$ implies $h \leq DH(\sigma)$. As an immediate consequence of Theorem 14(1), we have $h \leq P(\sum_{i \leq n} |\sigma_0(x_i)|)$.

Now, at any stage of the evaluation, the number of variables assigned by a substitution is less or equal to (the maximal arity of a function symbol \times $h$). Next, the size of the value of each variable is bounded $P(\sum_{i \leq n} |\sigma_0(x_i)|)$ because of Theorem 14(2). Consequently, the space required to store a substitution is always less than $O(P(\sum_{i \leq n} |\sigma_0(x_i)|)^2)$. We conclude that the whole runspace is bounded by $O(P(\sum_{i \leq n} |\sigma_0(x_i)|)^3)$.

\section{The bounding theorem}

The purpose of this section is to prove Theorem 14. We introduce a polynomial quasi-interpretation of LLPO-programs parameterised by $d \geq 2$. Given a LLPO-program $(\rightarrow, \mathcal{C}, \mathcal{D}, f, \nu, <_D)$. we take $d$ such that for each rule $l \rightarrow r$, we have $d > |r|$.

The quasi-interpretation is given by a sequence of polynomials.

\begin{align*}
F_0(X) &= X^d \\
F_{k+1}(X) &= F_k(X)
\end{align*}

where $F_k(X)$ means $a$ iterations of $F_k$, i.e. $F_k(\cdots(F_k(X))\cdots)$.

Define $\mathcal{D}_1, \ldots, \mathcal{D}_k$ as the partition of $\mathcal{D}$ determined by $\leq_D$ such that $g \in \mathcal{D}_q$ and $f \in \mathcal{D}_{q+1}$ iff $g <_D f$, and $f \approx g$ iff $f$ and $g$ are in $\mathcal{D}_q$. We say that if $f$ is in $\mathcal{D}_q$ then $f$ is of rank $q$. Intuitively, we consider constructors as symbols of rank 0. Now, the interpretation \([\cdot]\) on terms of $T(\mathcal{C} \cup \mathcal{D})$ is defined as follows.

\begin{itemize}
  \item $[b] = d$ for every constant $b$ of $\mathcal{C}$.
  \item $[c(t)] = [t] + d$ for every unary constructor $c$ of $\mathcal{C}$.
  \item $[f(t_1, \ldots, t_n)] = F_k(\max(d, \sum_{\nu(f,i)=1} [t_i]) + \max_{\nu(f,i)=0}([t_i])$ for every $f \in \mathcal{D}$ of rank $k$.
\end{itemize}

Lemma 16.
1. If \([s] \leq [t]\) then \([f(\cdots, s, \cdots)] \leq [f(\cdots, t, \cdots)]\).

2. \([t] \leq [f(\cdots t, \cdots)]\).

**Theorem 17.** Let \(\langle \to, C, D, f, \nu, \prec_D \rangle\) be an LLPO-program. For every substitution \(\sigma\), if \(v \sigma^* \to u \sigma\) then \([u \sigma] \leq [v \sigma]\).

**Proof.** The demonstration of the theorem above is tedious. The theorem is a consequence of Theorem 24 whose proof is detailed in the three next subsections.

For each ground substitution \(\sigma\) and rule \(l \to r\), we have \([l \sigma] > [r \sigma]\) by Theorem 24 below. The result follows from the monotonicity of the interpretation as stated in Lemma 16(1).

**Lemma 18.** Let \(\langle \to, C, D, f, \nu, \prec_D \rangle\) be a LLPO-program. For all inputs \(a_1, \ldots, a_n \in T(C)\),

\[
[f(a_1, \ldots, a_n)] \leq P(\sum_{i \leq n} |a_i|) \quad (1)
\]

**Proof.** Suppose that \(f\) is a function symbol of rank \(k\). Put \(P(X) = F_k(d \cdot X) + d \cdot X\). By definition, we have \([f(x_1, \cdots, x_n) \sigma] = F_k(\sum_{i=1}^{n} |a_i|) + \max_{\nu(f(i))=0}([a_i])\). Thus, \([f(x_1, \cdots, x_n) \sigma] \leq P(\sum_{i \leq n} |a_i|)\), since \([a_i] = |a_i|\). \(\square\)

**Lemma 19.** \(\text{ht}(t) \leq [t]\)

**Proof.** Straightforward by induction on \(\text{ht}(t)\). \(\square\)

### 5.1 Properties of \(F_k\)

**Proposition 20.** For all \(X, k\) and \(d \geq 2\),

1. \(F_k(X) = X^{d^k}\)

2. \(F_{k+1}^\alpha(X) = F_k^{\alpha-d}(X)\)

3. For all \(j \leq k\) and \(X \leq Y\), we have \(F_j(X) \leq F_k(Y)\)

4. For all \(X \geq d\) and \(\alpha, \beta > 0\), \(F_k^\alpha(X) + F_k^\beta(X) \leq F_k^{\alpha+\beta}(X)\)

5. For all \(X\) and \(\alpha < d\), \(F_k^\alpha(X+1) + F_{k+1}(X) \leq F_{k+1}(X+1)\)
5.2 Bounds on arguments of valency 1

Lemma 21. Let \( s \) and \( t = f(t_1, \cdots, t_n) \) be two terms of \( T(C \cup D, V) \) such that \( \text{Var}(s) \subseteq \text{Var}(t) \). Suppose that \( s \preceq_1 t \). Let \( \sigma \) be a ground substitution.

Assume that for all terms \( u, u \preceq_1 t_i \) implies \( [u] \leq [t_i] \), for each \( i \leq n \).
Then, \( [s] \leq F^{|s|}_k(A) \) where \( f \) is of rank \( k + 1 \) and \( A = \max(d, \sum_{\nu(f,i) = 1} [t_i]) \).

Proof. The proof is by induction on \(|s|\).

Assume \(|s| = 1\).

Suppose that \( s \) is a constant of \( C \). We have \( [s] = d \leq F^{|s|}_k(A) \), by Proposition 20(3).

Suppose that \( s \) is a variable of \( V \). We have \( s \preceq_1 t_i \) for some \( i \) satisfying \( \nu(f,i) = 1 \) and \( s \in \text{Var}(t_i) \). By the hypotheses of the lemma, \( [s] \leq [t_i] \). So, \( s \leq F^{|s|}_k(A) \) by Proposition 20(3).

Assume \(|s| > 1\).

Suppose that \( s \preceq_1 t_i \) and that \( \nu(f,i) = 1 \). Again, by the hypotheses of the lemma, \( [s] \leq [t_i] \) where \( \nu(f,i) = 1 \). So, \( [s] \leq F^{|s|}_k(A) \) by Proposition 20(3).

Suppose \( s = g(s_1, \cdots, s_m) \) where \( g \) is a function symbol of \( D \) of rank \( j \leq k \) and that for all \( i \leq m, s_i \preceq_1 t \). We have

\[
[s] = F^{|s|}_j(\sum_{i \leq m} [s_i]) \quad \text{by definition}
\]

\[
\leq F^{|s|}_j(\sum_{i \leq m} F^{|s_i|}_k(A)) \quad \text{by induction hypothesis and by Prop. 20(3)}
\]

\[
\leq F^{|s|}_j(F^{|s|}_{k}(\sum_{i \leq m} [s_i]) = F^{|s|}_j(\sum_{i \leq m} F^{|s_i|}_{k}(A)) \quad \text{by Prop. 20(3)}
\]

\[
\leq F^{|s|}_{k}(\sum_{i \leq m} [s_i]+1(A) \quad \text{by Prop. 20(3)}
\]

Lastly, the case when \( s = c(s') \) where \( c \) is a constructor of \( C \), is similar to the previous one, and so we skip it.

\( \Box \)

5.3 An upper bound on the interpretation

Lemma 22. Let \( s \) and \( t = f(t_1, \cdots, t_n) \) be two terms of \( T(C \cup D, V) \) such that \( \text{Var}(s) \subseteq \text{Var}(t) \) and \( s \in T(C \cup D | f, V) \). Suppose that \( s \preceq_0 t \). Let \( \sigma \) be a ground substitution.

Assume also, that for all \( i \leq n \), and all terms \( u, u \preceq_0 t_i \) implies \([u] \leq [t_i] \).
Then,

\[
|s| \leq F^{|s|}_k(A) + \max_{\nu(f,i) = 0}([t_i]) \quad \text{(2)}
\]

where \( A = \max(d, \sum_{\nu(f,i) = 1} [t_i]) \) and \( f \) is a function symbol of rank \( k + 1 \).
Proof. By induction on $|s|$.

Assume $|s| = 1$.

Suppose that $s$ is the constant $b \in C$. We obtain $[s\sigma] = [b] = d \leq F_k(A)$, by $20(3)$ and so (2) holds.

Suppose that $s$ is a variable. We have $s \prec_0 t_i$ for some $i$. By the hypotheses of the lemma we have $[s\sigma] \leq [t_i\sigma]$. Therefore $[s\sigma] \leq \max_{\nu(f,i)=0}([t_i\sigma])$.

Assume $|s| > 1$. Suppose that $s \preceq_0 t_i$. The hypotheses of the lemma give $[s\sigma] \leq [t_i\sigma]$. So (2) holds.

Suppose that $s = f_j(s_1, \ldots, s_m)$ where $f_j$ is a function symbol of $D$ of rank $j \leq k$. Then, for all $i \leq m$, if $\nu(f,i) = 1$, we have $s_i \preceq t_i$. Lemma 21 yields $[s_i\sigma] \leq F_k^{[s_i]}(A)$, so we have

$$F_j\left(\sum_{\nu(f_j,i)=1} [s_i\sigma]\right) \leq F_j\left(\sum_{\nu(f_j,i)=1} F_k^{[s_i]}(A)\right) \text{ repl. } [s_i\sigma] \text{ by } F_k^{[s_i]}(A)$$

$$\leq F_j\left(F_k^{\sum_{\nu(f_j,i)=1} [s_i]}(A)\right) \text{ by } 20(3)$$

$$\leq F_k^{1+\sum_{\nu(f_j,i)=1} [s_i]}(A) \text{ by } 20(3)$$

Otherwise, $\nu(f,i) = 0$ and we have $s_i \preceq t_i$. It follows by the induction hypothesis and by monotonicity of $F_k$ that

$$\max_{\nu(f_j,i)=0} (\max(d, [s_i\sigma])) \leq F_k^{\sum_{\nu(f_j,i)=0} [s_i]}(A) + \max_{\nu(f,i)=0} ([t_i\sigma])$$

By definition, $[s\sigma] = F_j(\sum_{\nu(f_j,i)=1} [s_i\sigma])$. From the above inequalities, we get the following bound on $[s\sigma]$

$$[s\sigma] \leq F_k^{1+\sum_{i\leq m} [s_i]}(A) + \max_{\nu(f,i)=0} ([t_i\sigma])$$

The case when $s = c(s')$ is similar to the case above.

\[\square\]

Lemma 23. Let $s$ and $t = f(t_1, \ldots, t_n)$ be two terms of $T(C \cup D, V)$ such that $\text{Var}(s) \subseteq \text{Var}(t)$. Suppose that $s \preceq_0 t$. Let $\sigma$ be a substitution which assigns to each variable of $\text{Var}(t)$ a ground term of $T(C \cup D)$. Assume also, that for all $i \leq n$, and all terms $u$, $u \preceq_0 t_i$ implies $[u\sigma] \leq [t_i\sigma]$. Then,

$$[s\sigma] \leq F_k^{[s]}(A) + F_{k+1}(A - 1) + \max_{\nu(f,i)=0} ([t_i\sigma])$$

where $A = \max(d, \sum_{\nu(f,i)=1} [t_i\sigma])$ and $f$ is a function symbol of rank $k + 1$. 

12
Proof. The proof goes by induction on $|s|$. However all the cases are the same as in lemma 24, except the following one.

Suppose that $s = g(s_1, \cdots, s_m)$ where $g$ is a function symbol of $D$ of rank $k + 1$.

We have $s_i \leq t_i$ for each position $i$ such that $\nu(f, i) = 1$. Now, $s_i \leq t_i$ implies $s_i \leq t_i$, by Proposition 3. So we can apply the lemma hypothesis, and we obtain

$$\sum_{\nu(g, i) = 1} [s_i] < \sum_{\nu(f, i) = 1} [t_i] = A.$$  

The inequality is strict because $s_p \not\leq t_p$ for at least one $p$ such that $\nu(f, p) = 1$.

On the other hand, if $\nu(f, j) = 0$ we know that $s_j \in T(C \cup D[f, \nu])$. By Lemma 22,

$$|s_j\sigma| \leq F_k[|s_j|](\sum_{\nu(f, i) = 1}(|t_i|)) + \max_{\nu(f, i) = 0}(|t_i|).$$

Consequently,

$$\max_{\nu(f, j) = 0}(|s_j\sigma|) \leq F_k[|s_j|](A) + \max_{\nu(f, j) = 0}(|t_j|).$$

From both former inequalities, we conclude that

$$[s\sigma] \leq F_{k+1}(A - 1) + F_k[|s_j|](A) + \max_{\nu(f, j) = 0}(|t_j|).$$

\[\square\]

Theorem 24. Let $s$ and $t$ be two terms of $T(C \cup D, \nu)$ such that $|s| < d$. For every ground substitution $\sigma$, if $s \not\leq t\sigma$ then $|s\sigma| < |t\sigma|$.

Proof. The proof is by induction on $|t|$. We have $|t| > 1$. Assume that $t = c(t')$, $c \in C$. We have $s \leq c\sigma'$, and so $[s\sigma] \leq [t'\sigma]$ by induction hypothesis.

Assume that $t = f(t_1, \cdots, t_n)$. Lemma 22 yields that $[s\sigma] \leq F_k[|s|](A) + F_{k+1}(A - 1) + \max_{\nu(f, i) = 0}(|t_i|)$. By Proposition 20, $F_k[|s|](A) + F_{k+1}(A - 1) < F_{k+1}(A)$. Therefore, $[s\sigma] < F_{k+1}(A) + \max_{\nu(f, i) = 0}(|t_i|) = [f(t_1, \cdots, t_n)\sigma]$.  

\[\square\]

6 conclusion

The result gives a purely syntactic characterisation of functions computed in polynomial space by mean of programs interpreted over infinite domains. There are few such characterisations like Leivant-Marion [9, 11], Leivant [9], and Jones [13]. The other characterisations of functions computed in polynomial space deal with finite model theory or with bounded recursions.

Actually, we think that the main interest of this result lies on the fact that we have illustrated that the notion of valency and of termination ordering is a tool to analyse programs. Putting things quickly, the role of valencies is to predict argument behaviour, and the role of termination ordering is to capture algorithmic patterns.

Finally, it is worth noticing that a consequence of this study and of the recent work [14] is the following : A program which terminates by LPO and admits a quasi interpretation bounded by a polynomial runs in polynomial space, and conversely.

13
References

[1] S. Bellantoni and S. Cook. A new recursion-theoretic characterization of the poly-time functions. *Computational Complexity*, 2:97–110, 1992.

[2] N. Dershowitz. Termination of rewriting. *Journal of symbolic computation*, pages 69–115, 1987.

[3] N. Dershowitz and J-P Jouannaud. *Handbook of Theoretical Computer Science vol.B*, chapter Rewrite systems, pages 243–320. Elsevier Science Publishers B. V. (NorthHolland), 1990.

[4] D. Hofbauer. Termination proofs with multiset path orderings imply primitive recursive derivation lengths. *Theoretical Computer Science*, 105(1):129–140, 1992.

[5] N. Jones. The expressive power of higher order types. To appear in *Journal of Functional Programming*, 2000.

[6] S. Kamin and J-J Lévy. Attempts for generalising the recursive path orderings. Technical report, University of Illinois, Urbana, 1980. Unpublished note.

[7] D. Leivant. Functions over free algebras definable in the simply typed lambda calculus. *Theoretical Computer Science*, 121:309–321, 1993.

[8] D. Leivant. Predicative recurrence and computational complexity I: Word recurrence and poly-time. In Peter Clote and Jeffery Remmel, editors, *Feasible Mathematics II*, pages 320–343. Birkhäuser, 1994.

[9] D. Leivant. Applicative control and computational complexity. In *Computer Science Logic*, LNCS, 1999.

[10] D. Leivant and J.-Y. Marion. Ramified recurrence and computational complexity ii: substitution and poly-space. In L. Pacholski and J. Tiuryn, editors, *Computer Science Logic (CSL’94)*, pages pp 486–500. LNCS 933, Springer-Verlag, April 1995.

[11] D. Leivant and J.-Y. Marion. Predicative functional recurrence and poly-space. In M. Bidoit and M. Dauchet, editors, *TAPSOFT’97, Theory and Practice of Software Development, CAAP*, LNCS 1214, pages 369–380. Springer-Verlag, April 1997.

[12] D. Leivant and J-Y Marion. A characterization of alogtime by pure ramified recurrence. *Theoretical Computer Science*, 236(1-2):192–208, April 2000.

[13] J-Y Marion. Analysing the implicit complexity of programs. Technical report, Loria, 2000. Presented at the workshop Implicit Computational Complexity (ICC’99), FLOC, Trento, Italy. The paper is submitted and accessible from *http://www.loria.fr/~marionjy*
[14] J-Y Moyen and J-Y Marion. Efficient first order functional program interpreter with time bound certifications. In *LPAR*, 2000.

[15] D. Plaisted. A recursively defined ordering for proving termination of term rewriting systems. Technical Report R-78-943, Department of Computer Science, University of Illinois, 1978.

[16] A. Weiermann. Termination proofs by lexicographic path orderings yield multiply recursive derivation lengths. *Theoretical Computer Science*, 139:335–362, 1995.