Abstract

An extension of the theory of the Iterated Logarithmic Algebra gives the logarithmic analog of a Sheffer or Appell sequence of polynomials. This leads to several examples including Stirling's formula and a logarithmic version of the Euler-MacLaurin summation formula.

L'Algèbre des Logarithmes Itérés II: Les Suites de Sheffer Grâce à une généralisation de la théorie de l'algèbre des logarithmes itérés, on définit un analogue logarithmique des suites de polynômes de Sheffer et d'Appell. Quelques exemples d'applications permettent de déduire la formule de Stirling ainsi qu'un version logarithmique de la formule de sommation de Euler–MacLaurin.

Dedicated to
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1 Introduction

Just as many of the important polynomial sequences of mathematics—generalized Laguerre, Hermite, Bernoulli—are not quite sequences of binomial type, and forced us to invent the theory of Sheffer and Appell sequences; we now have in [1] two complete theory of graded sequences of logarithmic series of binomial type. Both the theories contained in [1]—continuous and discrete—allow us to do work in a field involving the iterated logarithms; however, the continuous theory also allows us to use $x$ to a real power. Nevertheless, neither theory can yet help us compute any Euler-MacLaurin-like summation formulas, since the logarithmic generalization of the Bernoulli polynomials are not of binomial type. Again, we are forced to augment our theory through the use of Sheffer and Appell sequences.

Along the way, we will determine a sort of generating function for them, and for all ordinary graded sequence of formal power series of logarithmic type. We conclude by presenting several examples and applications.

The paper [1] contains the necessary background for this material.

As in [1], we will study two theories in parallel. All results and sections regarding the discrete iterated logarithmic algebra will be denoted “Discrete” and the results regarding the more general continuous iterated logarithmic algebra will be denoted “Continuous.” Readers interested in only one of these theories may safely omit all material pertaining to the other. Sections and results numbered with a $D$ and paragraphs starting with Discrete are relevant only to the discrete theory whereas sections and results numbered with a $C$ and paragraphs starting with Continuous are relevant only to the continuous theory. Otherwise, the remainder of this article may be interpreted discretely by supposing that all variables $a, b, c, \ldots$ are integers, and that $\alpha, \beta$ are vectors of integers, or it may be interpreted continuously by supposing that all variables $a, b, c, \ldots$ are real numbers and that $\alpha, \beta$ are vectors of real numbers.

Digressions and all sections marked with the word “Appendix” or the letter “A” are independent of all later material, and are included for their own sake.
2 Sheffer Graded Sequences

We begin by giving one definition of a Sheffer graded sequence of formal power series of logarithmic type.

**Definition 2.1** (Sheffer Graded Sequences) Suppose $s_\alpha^\alpha(x)$ is a graded sequence such that if there is a Roman graded sequence $p_\alpha^\alpha(x)$ and an Artinian operator $g(D)$ of degree zero with $s_\alpha^\alpha(x) = g(D)p_\alpha^\alpha(x)$. Then we say that $s_\alpha^\alpha(x)$ is Sheffer for $g(D)$ with respect to $p_\alpha^\alpha(x)$.

If $p_\alpha^\alpha(x)$ is the $(n^{th})$ associated sequence for the delta operator $f(D)$, then we also say that $s_\alpha^\alpha(x)$ is Sheffer for $g(D)$ with respect to $f(D)$ (and $n$).

Graded Sheffer sequences admit a number of equivalent characterizations:

**Theorem 2.2** For a graded sequence $s_\alpha^\alpha(x)$, the following are equivalent:

1. $s_\alpha^\alpha(x)$ is a Sheffer graded sequence.

2. There are Artinian operators $f(D)$ and $g(D)$ of degrees 1 and 0 respectively and an integer $n$ such that
   \[ \langle g(D)f(D)^{k,n}s_\alpha^\alpha(x) \rangle_\alpha = [a]! \delta_{ab} \]
   for all $a$ and $b$.

3. There is a delta operator $f(D)$ such that
   \[ f(D)s_\alpha^\alpha(x) = [a] s_{\alpha-1}^\alpha(x) \]
   for all $a$ and $\alpha$.

4. There is a Roman sequence $p_\alpha^\alpha(x)$ such that
   \[ E^zs_\alpha^\alpha(x) = \sum_b \left[ \begin{array}{c} a \\ b \end{array} \right] \langle E^z p_b^{(0)}(x) \rangle_0 s_{\alpha-b}^\alpha(x) \]
   for all complex numbers $z$, and all $a$ and $\alpha$. 
Proof: (4 implies 1) Let $T$ be the continuous, linear operator in $\mathcal{I}$ mapping $p_{\alpha}^a(x)$ to $\lambda_{\alpha}^a(x)$. We have:

$$TE^z s_{\alpha}^a(x) = \sum_b \left[ \begin{array}{c} a \\ b \end{array} \right] \lambda_{\alpha}^a(x) T s_{\alpha-b}^a(x)$$

$$= \sum_b \left[ \begin{array}{c} a \\ b \end{array} \right] \lambda_{\alpha}^a(x) T p_{\alpha-b}^a(x)$$

$$= E^z p_{\alpha}^a(x)$$

$$= E^z T s_{\alpha}^a(x)$$

By the characterization of differential operators in [1], $T$ is a differential operator of degree 0. Thus, we have

$$s_{\alpha}^a(x) = T^{-1} p_{\alpha}^a(x). \quad (2)$$

(1 implies 3) We have the following series of identities:

$$f(D)s_{\alpha}^a(x) = f(D)T^{-1} p_{\alpha}^a(x)$$

$$= T^{-1} f(D)p_{\alpha}^a(x)$$

$$= [a] T^{-1} p_{\alpha-1}^a(x)$$

$$= [a] s_{\alpha-1}^a(x).$$

(3 implies 4) Since $E^z$ is expressible in term of $f(D)^{a;n}$, the result is immediate.

(2 is equivalent to 1) See [1].

3 Expansion Theorem

As with the harmonic logarithms and all Roman graded sequences, we have formulas which give the coefficients of an arbitrary Artinian operator or logarithmic series in terms of a Sheffer graded sequence and its operators.

**Theorem 3.1** (Expansion Theorem) Let the graded sequence $s_{\alpha}^a(x)$ be Sheffer for $g(D)$ with respect to $f(D)$ (and n). Then for all Artinian operators $h(D)$ and vectors $\alpha \neq (0)$ we have the following convergent sum.

$$h(D) = \sum_a \frac{\langle h(D)s_{\alpha}^a(x) \rangle}{[a]!} g(D)^{-1} f(D)^{a;n}. \quad (3)$$

When $\alpha = (0)$, equation (3) holds for all differential operators $h(D)$.

Proof: See the Expansion Theorem of [1].
**Theorem 3.2** (Taylor’s Theorem) Let the graded sequence $s_{\alpha}^{a}(x)$ be Sheffer for $g(D)$ with respect to $f(D)$ (and $n$). Then for every formal power series of logarithmic type $p(x) \in \mathcal{I}^{+}$ we have the following convergent sum.

$$p(x) = \sum_{\alpha \neq (0)} \sum_{a} \frac{(g(D)^{-1} f(D)^{a;n} p(x))_{a}}{[a]!} s_{\alpha}^{a}(x).$$

*Proof:* See the Taylor’s Theorem of [1].

### 4 Generating Functions

**Definition 4.1** (Roman Exponential Series) For all $\alpha$,

**Discrete** A vector of integers, define the Roman exponential series of $\alpha$ to be the formal series

$$[e]_{\alpha}^{y} = [exp]_{\alpha}(y) = (E_{\alpha(0)} + E_{\alpha(1)}) \sum_{n} \frac{y^{n}}{[n]!}.$$

**Continuous** A vector of real $\alpha$, define the $n^{th}$ Roman exponential series of $\alpha$ to be the formal series

$$[e]_{\alpha}^{yn} = [exp]_{\alpha}(y;n) = (E_{\alpha(0)} + E_{\alpha(1)}) \sum_{a} \frac{y^{a;n}}{[a]!}.$$

Thus, the $(0^{th})$ exponential series is a generating function of $\alpha$ at $x$ is a generating function for the harmonic logarithms of order $\alpha$.

Note that this is not a formal power series of logarithmic type in fact is not any sort of Noetherian or Artinian series; it is merely a formal series whose coefficients are homogeneous logarithmic series of order $\alpha$. In general, products of this form are not well defined. For example, $[e]_{\alpha}^{x} [e]_{\alpha}^{x}$ is not well defined.

In [5, Corollary 3 to Theorem 2], it is shown that ordinary (nonlogarithmic) sequences of binomial type for the operator $f(D)$ are given by the generating function

$$\sum_{n} \frac{p_{n}(x)}{n!} y^{n} = \exp(x f^{(-1)}(y)).$$

In our present context, we have:
**Theorem 4.2** (Generating Function for Roman Graded Sequences) Let be $p^\alpha_n(x)$ be the ($n$th) associated sequence of the delta operator $f(D)$. Then

\[
\begin{cases}
\text{Discrete} & \sum_k \frac{p^\alpha_k(x)}{[n]!} y^n = [\exp]_\alpha \left(x f^{(-1)}(y)\right) \\
\text{Continuous} & \sum_a \frac{p^\alpha_a(x)}{[a]!} y^a = [\exp]_\alpha^{x f^{(-1)}(y):n}.
\end{cases}
\]

**Proof:** By consideration of the expansion Theorem [1],

\[ p(x, y) = \sum_a \frac{\tilde{p}_a(x)}{a!} y^a = e^{x f^{(-1;n)}(y):n}. \]

The result follows now from regularity.

**Theorem 4.3** (Generating Function for Sheffer Graded Sequences) Let $s^\alpha_n(x)$ be Sheffer for $g(D)$ with respect to $f(D)$ (and $n$). Then

\[
\begin{cases}
\text{Discrete} & \sum_k \frac{p^\alpha_k(x)}{[n]!} y^n = g(f^{-1}(y))[\exp]_\alpha \left(x f^{(-1)}(y)\right) \\
\text{Continuous} & \sum_a \frac{p^\alpha_a(x)}{[a]!} y^a = g(f^{-1;n}(y))[\exp]_\alpha^{x f^{(-1)}(y):n}.
\end{cases}
\]

**Proof:** By the generalized expansion theorem of [1],

\[
D^b g(D) = \sum_a \frac{\langle D^b g(D) p^\alpha_a(x) \rangle_a}{[a]!} f(D)^a : n
\]

\[
f^{(-1;n)}(D)^b g(f^{(-1;n)}(D)) = \sum_a \frac{\langle D^b g(D) p^\alpha_a(x) \rangle_a}{[a]!} D^a
\]

\[= \sum_a \frac{\langle D^b s^\alpha_a(x) \rangle_a}{[a]!} D^a. \]

5 Appell Graded Sequences

The most interesting types of Sheffer graded sequences are the Roman graded sequences themselves and the Appell graded sequences. The name Appell is given to all Sheffer graded sequences with respect to $D$ and 0, or equivalently with respect to the harmonic logarithms $\lambda^\alpha_n(x)$. 


They are the logarithmic generalization of the notion of an Appell sequence of polynomials, that is, a sequence \((p_n(x))_{n \geq 0}\) of polynomials satisfying the identity
\[
p_n(x + a) = \sum_{k \geq 0} \binom{n}{k} a^{n-k} p_k(x).
\]

By [1], the action of the derivative operator \(D\) on the space \(\mathcal{I}\) of formal power series of logarithmic type is naturally decomposes as a direct sum of the minimal invariant subspaces \(\mathcal{I}^\alpha\). For each \(\alpha\), the subspace \(\mathcal{I}^\alpha\) is the minimal invariant subspace of \(\mathcal{I}^+\) under the action of the operators \(D^a\) which contains \(\ell^\alpha\). It is not yet clear, however, that the pseudobasis of the spaces \(\mathcal{I}^\alpha\) provided by the harmonic logarithms \(\lambda^\alpha(x)\) is determined by intrinsic algebraic properties. In order to derive the properties that single out the harmonic logarithms as the natural pseudobasis for \(\mathcal{I}^\alpha\), we are led to this generalization to formal power series of logarithmic type of the classical theory of Appell polynomials.

**Proposition 5.1** Let \(p_\alpha^a(x)\) be a graded sequence of formal power series of logarithmic type. Then the following statements are equivalent:

1. \(p_\alpha^a(x)\) is an Appell graded sequence.
2. There is an Artinian operator \(g(D)\) of degree 0 such that
\[
\langle g(D)D^b p_\alpha^a(x) \rangle_\alpha = \lfloor a \rfloor! \delta_{ab}
\]
for all \(a\) and \(b\), and for all \(\alpha \neq (0)\).
3. For all \(a\) and \(\alpha\),
\[
D p_\alpha^a(x) = \lfloor a \rfloor! p_{\alpha-1}^a(x).
\]
4. For all complex numbers \(z\), and all \(a\) and \(\alpha\),
\[
E^z p_\alpha^a(x) = \sum_{n \geq 0} \frac{z^n \lfloor a \rfloor!}{\lfloor a - n \rfloor!} p_{\alpha-b}^a(x).
\]

**Proof:** Theorem 2.2. \(\blacksquare\)

We now deduce an explicit expression for an Appell graded sequence as a linear combination of harmonic logarithms.

**Corollary 5.2** Let \(p_\alpha^a(x)\) be an Appell graded sequence, then for all \(a\) and \(\alpha\),
\[
p_\alpha^a(x) = \sum_b \binom{a}{b} \langle p_0^{(0)}(x) \rangle_{(0)} \lambda^\alpha_{a-b}(x). \] \(\blacksquare\)
6 Examples

6.1 Harmonic Graded Sequence

We begin by giving examples of Appell graded sequences. The only Appell graded sequence which is Roman is the graded sequence of harmonic logarithms. It is characterized among all Appell graded sequences by the fact that

\[ \langle \lambda_0^a(x) \rangle_n = [a]^1 \delta_{a,0}. \]

This is the characterization of the harmonic graded sequence we had previously announced.

6.2 Bernoulli Graded Sequence

Next, we consider the logarithmic extension of Bernoulli polynomials.

**Definition 6.1** (Logarithmic Bernoulli Graded Sequence) Define the Bernoulli operator \( J \) by

\[ Jp(x) = \int_x^{x+1} p(t)dt \]

for \( p(x) \in \mathcal{I} \), that is,

\[ J = e^D - I = \frac{e^D - I}{D}. \]

The Appell graded sequence \( B_n^\alpha(x) \) defined as

\[ B_n^\alpha(x) = J^{-1} \lambda_n^\alpha(x), \]

is be called the logarithmic Bernoulli graded sequence. In particular, we obtain the ordinary Bernoulli polynomials, \( B_n(x) = B_n^{(0)}(x) \) and the Bernoulli numbers \( B_n = B_n(0) \).

The logarithmic Bernoulli graded sequence can be computed by Proposition 5.2:

\[ B_n^\alpha(x) = \sum_{k \geq 0} \binom{n}{k} B_k \lambda_{n-k}^\alpha(x). \]
Since the Euler-MacLaurin formula can be written as

\[ p(x) = p(x + 1) - \Delta p(x) + \int_x^{x+1} p(t)dt + \frac{1}{2} B_2 \frac{p''(x)}{2!} + \frac{1}{6} B_3 \frac{p'''(x)}{3!} + \cdots \]

Applying it to a discrete formal power series of logarithmic type \( p(x) \) we obtain

\[ B_0 \left[ \log(x + n + 1) - \log(x) \right] + B_1 \left[ (x + n + 1)^{-1} - x^{-1} \right] + B_2 \frac{n(n+1)}{2!} + \frac{B_3}{3!} \left[ 2(x + n + 1)^{-3} - 2x^{-3} \right] + \cdots \]

We stress the fact that this formula is an identity in the logarithmic algebra, and not just an asymptotic formula. For example, for \( p(x) = 1/x \), we obtain:

\[ \frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+n} = \]

\[ B_0 \log(x + n + 1) - \log(x) + B_1 \left[ (x + n + 1)^{-1} - x^{-1} \right] + B_2 \frac{n(n+1)}{2!} + \frac{B_3}{3!} \left[ 2(x + n + 1)^{-3} - 2x^{-3} \right] + \cdots \]

(7)

For another example, let \( p(x) = \log x \). We then obtain a version of Stirling’s formula

\[ \log(x(x+1)\cdots(x+n)) = \]

\[ B_0((x + n + 1)\log(x + n + 1) - x\log x - n - 1) + B_1((x + n + 1)\log x + B_2 \frac{1}{x + n + 1} - \frac{1}{x}) + \cdots \]

(8)
Finally, let \( p(x) = \log \log x \).

\[
\log(\log x \log (x+1) \cdots \log (x+n)) =
B_0 \left[ \frac{y \log y}{y \log y} \right]_{y=x}^{y=n+1} + B_1 \left[ \log \log y \right]_{y=x}^{y=n+1} + B_2 \left[ \frac{1}{y \log y} \right]_{y=x}^{y=n+1} + \cdots
\]

We note that \( J = \Delta D^{-1} \), and thus for \( \alpha \neq (0) \),
\[
\Delta B^\alpha_a(x) = \Delta J^{-1} \lambda^\alpha_a(x)
= D \lambda^\alpha_a(x)
= \lfloor a \rfloor \lambda^\alpha_{a-1}(x).
\]

Summing, we obtain
\[
\lambda^\alpha_a(x) + \lambda^\alpha_a(x+1) + \cdots + \lambda^\alpha_a(x+k) = \lfloor a \rfloor^{-1} \left[ B^\alpha_{a+1}(x+k+1) - B^\alpha_{a+1}(x) \right].
\]

For example,
\[
\log(x(x+1) \cdots (x+k)) = B^{(1)}_1(x+k+1) - B^{(1)}_1(x)
\]
\[
\log(\log x \log (x+1) \cdots \log (x+n)) = B^{(0,1)}_1(x+k+1) - B^{(0,1)}_1(x)
\]
\[
(\log x)^2 + \cdots + (\log(x+n+1))^2 = B^{(2)}_1(x+k+1) - B^{(2)}_1(x)
\]

Clearly, any similar sum of \( \ell^{(a),\alpha} \) can be evaluated in terms of the logarithmic Bernoulli graded sequence.

### 6.3 Hermite Graded Sequence

Our next example of an Appell graded sequence is the logarithmic Hermite graded sequence.

**Definition 6.2** (Logarithmic Hermite Graded Sequence) Let the Weirstrass operator be \( W = e^{D^2/2} \). The logarithmic Hermite graded sequence \( H^\alpha_a(x) \) is defined as
\[
H^\alpha_a(x) = W^{-1} \lambda^\alpha_a(x).
\]
6.3 Hermite Graded Sequence

Table 2: Logarithmic Hermite Graded Sequence $H_n^\alpha(x)$

| $H_0^\alpha(x)$ | $\lambda_0^\alpha(x)$ | $\lambda_0^\alpha(x)$ |
|-----------------|----------------------|----------------------|
| $H_1^\alpha(x)$ | $\lambda_1^\alpha(x)$ | $\lambda_1^\alpha(x)$ |
| $H_2^\alpha(x)$ | $\lambda_2^\alpha(x)$ | $\lambda_2^\alpha(x)$ |

For $n$ a nonnegative integer, $H_n^{(0)}(x) = H_n(x)$ is the usual Hermite polynomial. From them we define the Hermite numbers $H_n(x)$. The logarithmic Hermite graded sequence can be computed via Proposition 5.2:

$$H_n^\alpha(x) = \sum_{k \geq 0} \left[ \frac{\alpha}{k} \right] H_k \lambda_n^\alpha(x).$$

(See Table 2.) Recall that

$$E_0 H_n^\alpha(x) = H_n(x),$$

in other words, the “positive” terms of equation (9) equal the ordinary Hermite polynomials, except that $\lambda_n^\alpha(x)$ replaces $x^n$. Thus, we have a logarithmic generalization of the Hermite polynomials. All classical properties of Hermite polynomials may be extended to the logarithmic Hermite graded sequence. For example, we have the explicit expression

$$H_n^\alpha(x) = e^{-D^2/2} \lambda_n^\alpha(x)$$

$$= \sum_{k \geq 0} \left( -\frac{1}{2} \right)^k \frac{|\alpha|!}{k! |\alpha - 2k|!} \lambda_n^{\alpha-2k}(x).$$

so that for $n$ a negative integer,

$$H_n^{(1)}(x) = \sum_{k \geq 0} \left( -\frac{1}{2} \right)^k \frac{(2k - n - 1)!}{k!(-n - 1)!} \lambda_n^{\alpha-2k}(x).$$

or

$$H_n^\alpha(x) = \sum_{k \geq 0} \left( -\frac{1}{2} \right)^k \frac{(2k - n - 1)!}{k!(-n - 1)!} \lambda_n^{\alpha-2k}(x)$$

The right side of equation (11) is the classical asymptotic expansion of the Hermite series $H_n^{(1)}(x)$; in the present context, it is a convergent series, and one term of the Hermite graded sequence. We trivially have

$$D H_n^\alpha(x) = [\alpha] H_n^{\alpha-1}(x),$$

and

$$E^z H_n^\alpha(x) = \sum_{k \geq 0} \left[ \frac{\alpha}{k} \right] z^k H_n^{\alpha-k}(x).$$
Finally, every logarithmic series \( p(x) \) has a unique convergent expansion in terms of the logarithmic Hermite graded sequence:

\[
p(x) = \sum_{a,\alpha} \frac{\langle WD^a p(x) \rangle_{\alpha}}{a!} H^a_{\alpha}(x).
\]

### 6.4 Laguerre Graded Sequence

We conclude by introducing a Sheffer sequence which is neither Appell nor Roman. The generalized Laguerre graded sequence is Sheffer for the ordinary Laguerre graded sequence [1].

**Definition 6.3 (Laguerre Graded Sequence of Grade \( b \))** Given a real number \( b \), define the Laguerre sequence of grade \( b \) \( L^a; b(x) \) to be the Sheffer graded sequence for \((1 - D)^{b + 1}; 0\) with respect to the Laguerre graded sequence.

By the transfer formula [1],

\[
L^a; b(x) = (1 - D)^{b + 1}; 0 K' \left( \frac{\text{D}}{\text{K}} \right)^{a + 1}; 0 \lambda_{a + 1}^a(x)
\]

\[
= (-1)^{a}; 0 (1 - D)^{a + b} \lambda_a^a(x)
\]

\[
= \sum_{k \geq 0} \binom{a + b}{k} \frac{|a|!}{|a - k|!} (-1)^{a - k}; 0 \lambda_{a - k}^a(x)
\]

The generating function for the Laguerre graded sequence of grade \( b \) is

\[
(1 - y)^{-b - 1}; 0 \exp_a(xy/(y - 1)) = \sum_a \frac{L^a; b_a}{|a|!} y^a.
\]

We hope the preceding examples display the utility of the theory of formal power series of logarithmic type.
References

[1] D. Loeb, *The Iterated Logarithmic Algebra*, To appear.

[2] J. B. Miller, *The Standard Summation Operator, the Euler-MacLaurin Sum Formula, and the Laplace Transformation*, Journal of the Australian Mathematical Society, 39 (1985), 376–390.

[3] T. R. Prabhakar, and Reva, *An Appell Cross-sequence Suggested by the Bernoulli and Euler Polynomials of General Order*, Indian Journal of Pure and Applied Mathematics, 10 (1979), 1216–1227.

[4] N. Ray, *Extensions of Umbral Calculus, Penumbral Coalgebras and Generalized Bernoulli Numbers*, Advances in Mathematics, 61 (1986), 49–100.

[5] G.-C. Rota, “Finite Operator Calculus,” Academic Press, 1975.

Please refer to [1] for a more complete list of references on this topic.