Abstract

Graviton-dilaton background field equations in three space-time dimensions, following from the string effective action are solved when the metric has only time dependence. By taking one of the two space dimensions as compact, our solution reproduces a recently discovered charged black hole solution in two space-time dimensions. Solutions in presence of nonvanishing three dimensional background antisymmetric tensor field are also discussed.
Black hole solutions in string theory have recently been found both as an exact conformal field theory solution\textsuperscript{1,2} and as a solution of the field equations in effective field theory\textsuperscript{3} of strings. Exact conformal field theory solution is obtained as an SL(2,R)/U(1) coset conformal field theory, formulated as a gauged Wess-Zumino-Witten (WZW) model for SL(2,R) group with gauging of its U(1) subgroup. In the effective field theory approach, these solutions are characterized by the nontrivial classical values for the metric and dilaton fields which satisfy the background field equations.

Recently, a generalization of these results was done by finding the two dimensional charged black hole solution\textsuperscript{4} as an [SL(2,R)×U(1)\textsubscript{i}]/U(1) coset conformal field theory, where the group U(1)\textsubscript{i} corresponds to an internal compact dimension. This solution is also obtained as a gauged WZW model. The gauged U(1) subgroup now is a combination of U(1)\textsubscript{i} and a U(1) subgroup of SL(2,R).

In a related development recently, Meissner and Veneziano\textsuperscript{5} wrote down the field equations for the D-dimensional superstring effective action in a general time dependent graviton, dilaton and antisymmetric tensor background and showed its invariance under an O(d,d) group of transformations, where d is the number of space dimensions (d=D-1). In ref. [6] these equations were written down and solved for arbitrary D, when dilaton and diagonal components of the metric are present as background. For D=2 these solutions are identical to the black hole solutions\textsuperscript{5}.

In this paper we solve the field equations of Meissner and Veneziano\textsuperscript{5} in three dimensions when metric has off diagonal components as well. After compactification of one of the space dimensions, our solution can be interpreted as two dimensional charged black hole\textsuperscript{4}. We also show that, by making O(2,2) transformations, one can obtain solutions with nonvanishing background antisymmetric tensor field.

Our starting point is the genus zero low energy effective action for closed superstrings in the limit when string tension $\alpha' \to 0$. Restricting to the graviton, dilaton and antisymmetric
tensor field, this action in D-dimensions is written as,

\[ S = \int d^Dx \sqrt{-detGe^{-\phi}} [V - R - G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}] \] (1)

where \( V \) contains cosmological constant as well as the dilaton potential, which for our solution turns out to be a constant. \( \phi \) is the dilaton field, \( G_{\mu\nu} \) is the D-dimensional metric and \( H_{\mu\nu\rho} \) is the field strength for the antisymmetric tensor field \( B_{\mu\nu} \):

\[ H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \text{cyclic}. \] (2)

We consider the case when \( D=3 \). In general, when \( D \) is less than the critical dimension of (super)strings, massless gauge and Higgs fields can also be present. However we consider the case when they have vanishing background values.

As in refs.[5,6], we now look for the solutions of the field equations when \( G \) and \( B \) are functions of time only. In this case, gauge symmetries of the action, eqn. (1), allow \( G \) and \( B \) to be always brought in the form:

\[ G = \begin{pmatrix} -1 & 0 \\ 0 & G(t) \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & B(t) \end{pmatrix}. \] (3)

Then, action (1) can be rewritten as

\[ S = \int dt \sqrt{det G} e^{-\phi} [V - 2\partial_0^2 (ln\sqrt{det G}) - (\partial_0 (ln\sqrt{det G}))^2 + \frac{1}{4} Tr(\partial_0 G)(\partial_0 G^{-1}) + (\partial_0 \phi)^2 \]

\[ + \frac{1}{4} Tr(G^{-1}(\partial_0 B) G^{-1}(\partial_0 B))] \]. (4)

We will first consider the case of vanishing background \( B \) field, and discuss later on the case when it has nonzero value. By redefining the dilaton field:

\[ \Phi = \phi - \ln\sqrt{det G}, \] (5)

one can then write eqn. (4) as:
\[ S = \int dte^{-\phi}[V + \dot{\phi}^2 - \frac{1}{4}Tr(G^{-1}\dot{G})^2]. \]  

Equations of motion for these fields\(^5\) are:

\[ (\dot{\phi})^2 - \frac{1}{4}Tr[(G^{-1}\dot{G})(G^{-1}\dot{G})] - V = 0 \]  

\[ (\dot{\phi})^2 - 2\ddot{\phi} + \frac{1}{4}Tr[(G^{-1}\dot{G})(G^{-1}\dot{G})] - V = 0 \]  

and

\[ -\dot{\phi}\dot{G} + \ddot{G} - \dot{G}G^{-1}\dot{G} = 0. \]  

where eqns. (8) and (9) follow from the variations with respect to the fields \( \Phi \) and \( G_{ij} \) of action (6). Equation (7), which is obtained directly from the variation of the action (1) with respect to \( G_{00} \), is also called the "zero energy condition"\(^5\). It has also been pointed out in ref. [5] that eqns. (7)-(9) are the complete set of equations of motion. In ref. [6] these equations were solved for arbitrary \( D \) when \( G \) is diagonal. For \( D=2 \) the solution is identical to the uncharged black hole solution when roles of space and time are interchanged.

We now obtain a solution of eqns.(7)-(9) when \( G \) is a \( 2 \times 2 \) symmetric matrix of the form:

\[ G(t) = \begin{pmatrix} g_1(t) & \frac{1}{2}A(t) \\ \frac{1}{2}A(t) & g_2(t) \end{pmatrix} \]  

In general, in terms of these components, eqns. (7)-(9) will be very complicated. To get these equations in a rather convenient form, we now redefine:

\[ g_1 = \tilde{g}_1 + \frac{1}{4}A^2. \]  

Above form of the metric has also been used for constant backgrounds in ref. [7]. The determinant of \( G \) is then simply,
\[ \det G = \tilde{g}_1 g_2, \quad (12) \]

the inverse metric is given as,

\[ G^{-1} = \frac{1}{\tilde{g}_1 g_2} \left( \begin{array}{cc} g_2 & -\frac{1}{2} A \\ -\frac{1}{2} A & g_1 \end{array} \right), \quad (13) \]

and \( \dot{\Phi} = \dot{\phi} - \frac{1}{2} \frac{\dot{\tilde{g}}_1}{\tilde{g}_1} - \frac{1}{2} \frac{\dot{g}_2}{g_2} \). In component form, eqns. (7)-(9) can therefore be written as,

\[ (\dot{\Phi})^2 - \frac{1}{4} \left[ \frac{\dot{\tilde{g}}_1^2}{g_1^2} + \frac{1}{2} \frac{\dot{A}^2}{g_1 g_2} - \frac{A \dot{A}}{g_1 g_2} + \frac{1}{2} A^2 \frac{\dot{g}_2^2}{g_2^2} + \frac{\dot{g}_2^2}{g_2^2} \right] - V = 0 \quad (14) \]

\[ (\dot{\Phi})^2 - 2 \dot{\Phi} + \frac{1}{4} \left[ \frac{\dot{g}_1^2}{g_1^2} + \frac{1}{2} \frac{\dot{A}^2}{\tilde{g}_1 g_2} - \frac{A \dot{A}}{\tilde{g}_1 g_2} + \frac{1}{2} A^2 \frac{\dot{g}_2^2}{g_2^2} + \frac{\dot{g}_2^2}{g_2^2} \right] - V = 0 \quad (15) \]

\[ -\dot{\Phi} \dot{g}_1 + \ddot{g}_1 - \frac{1}{\tilde{g}_1 g_2} (g_2 \dot{g}_1^2 - \frac{1}{2} A \dot{g}_1 g_1 + \frac{1}{4} g_1 \dot{A}^2) = 0 \quad (16) \]

\[ -\dot{\Phi} \dot{g}_2 + \ddot{g}_2 - \frac{1}{\tilde{g}_1 g_2} (g_1 \dot{g}_2^2 - \frac{1}{2} A \dot{g}_2 g_2 + \frac{1}{4} g_2 \dot{A}^2) = 0 \quad (17) \]

\[ -\dot{\Phi} \dot{A} + \ddot{A} - \frac{1}{\tilde{g}_1 g_2} (g_2 \dot{A} \dot{g}_1 - A \dot{g}_1 \dot{g}_2 + g_1 \dot{A} \dot{g}_2 - \frac{1}{4} A \dot{A}^2) = 0 \quad (18) \]

where eqns. (14) and (15) follow from eqns. (7) and (8) respectively, and eqns. (16)-(18) are the component form of eqn. (9). We now show that eqns. (14)-(18) have a solution of the following form:

\[ g_1 = a_0 + b_0 \text{tanh}^2 t \quad (19) \]

\[ A = b_1 \text{tanh}^2 t \quad (20) \]

\[ g_2 = b_2 \text{tanh}^2 t \quad (21) \]

and

\[ \phi = -\log(\cosh^2 t) + a \quad (22) \]

with

\[ 4b_0 b_2 = b_1^2. \quad (23) \]
Putting eqns. (19)-(23) into eqn. (11), we get \( \tilde{g}_1 = a_0 \), which is a constant. For the above choice of solution, eqns. (16)-(18) then imply:

\[
(4b_0 - \frac{b_1^2}{b_2}) \frac{1}{\cosh^4 t} - (\frac{b_0}{a_0 b_2})(4b_2b_0 - b_1^2) \frac{\tanh^2 t}{\cosh^4 t} = 0
\]

\[
- \frac{1}{a_0} (4b_2b_0 - b_1^2) \frac{\tanh^2 t}{\cosh^4 t} = 0
\]

and

\[
(4b_2b_0 - b_1^2) \frac{\tanh^2 t}{\cosh^4 t} = 0
\]

and are therefore satisfied due to the condition (23). Similarly, since

\[
\dot{\Phi}^2 = (\dot{\phi} - \frac{1}{2} \frac{\dot{g}_1}{g_1} - \frac{1}{2} \frac{\dot{g}_2}{g_2})^2 = \frac{1}{\sinh^2 t \cosh^2 t} + 4
\]

\[
\ddot{\Phi} = (\ddot{\phi} - \frac{1}{2} \frac{\ddot{g}_1}{g_1} - \frac{1}{2} \frac{\ddot{g}_2}{g_2} + \frac{1}{2} \frac{\ddot{g}_1^2}{g_1^3} + \frac{1}{2} \frac{\ddot{g}_2^2}{g_2^3}) = \frac{1}{\sinh^2 t \cosh^2 t}
\]

and

\[
\frac{1}{4} Tr(G^{-1} \dot{G})^2 = \frac{1}{4} \left[ \frac{\dot{g}_1^2}{g_1^2} + \frac{\dot{A}^2}{A^2} - \frac{A \ddot{g}_1}{g_1 g_2^2} + \frac{1}{2} \frac{A^2 g_2^2}{g_1^3} + \frac{\dot{g}_2^2}{g_2^3} \right] = \frac{1}{\sinh^2 t \cosh^2 t}
\]

therefore eqns. (14) and (15) are also satisfied for \( V = 4 \). This value of \( V \) is same as when \( g_2 \) and \( \phi \) are the only nonzero background fields. \( g_1 \) and \( A \) fields therefore do not contribute to \( V \).

Our solution, eqns. (19)-(23), appear to have four independent parameters. However, since equations of motion (14)-(18) have the following scaling symmetry: \( g_2 \to \alpha g_2, \ g_1 \to \alpha^{-1} g_1, \ \tilde{g}_1 \to \alpha^{-1} \tilde{g}_1, \ A \to A \) and \( \Phi \to \Phi \), hence one of the parameters can be scaled away. It will be later on seen that the number of parameters for our solution is same as that of the charged black hole of ref. [4].

Now, to discuss the charged black hole interpretation of our solution, eqns. (19)-(23), let us consider the world sheet string action for a three dimensional target space, written in presence of background metric, gauge and Higgs fields as,

\[
I = \int d^2 z \left[ \sum_{i,j=0,2} G_{ij} \partial X^i \tilde{\partial} X^j + \sum_{i=0,2} \frac{1}{2} A_i (\partial X^i \tilde{\partial} X^i + \partial X^1 \tilde{\partial} X^i) + \psi \partial X^1 \tilde{\partial} X^1 \right]
\]
where indices (0,2) denote the space-time index and $X^1$ is a compact direction. Comparing our solution, eqns. (19)-(23) with eqn. (30), we get a background space-time metric of the form,

$$ G_{ij} = \begin{pmatrix} -1 & 0 \\ 0 & b_2 \tanh^2 t \end{pmatrix}, $$ (31)

background gauge field:

$$ A_i = (0, b_1 \tanh^2 t), $$ (32)

and scalar field:

$$ \psi = a_0 + b_0 \tanh^2 t = (a_0 + b_0) - \frac{b_0}{\cosh^2 t}. $$ (33)

In addition, there is also a background dilaton:

$$ \phi = -\log(\cosh^2 t) + a. $$ (34)

Our solution can therefore be identified with the charged black hole solution of ref.[4], if roles of space and time are interchanged and following identifications are also made:

$$ b_0 = \frac{e^2}{4k^2} $$
$$ b_1 = -\frac{e}{k} $$
$$ b_2 = 1 $$

and

$$ a_0 = -\frac{1}{2k} - \frac{e^2}{4k^2} $$

It can also be verified that the equations of motion (14)-(18) have another solution when two dimensional metric ($\hat{G}_{ij}$), gauge field ($\hat{A}_i$), Higgs ($\hat{\psi}$) and dilaton ($\hat{\phi}$) have the following form:

$$ \hat{G}_{ij} = \begin{pmatrix} -1 & 0 \\ 0 & \hat{b}_2 \coth^2 t \end{pmatrix}, $$ (36)

$$ \hat{A}_i = (0, \hat{b}_1 \coth^2 t) $$ (37)
\[
\hat{\psi} = \hat{a}_0 + \hat{b}_0 \coth^2 t \tag{38}
\]
\[
\hat{\phi} = -\log(\sinh^2 t) + \hat{a}. \tag{39}
\]

and coefficients $\hat{b}_i$'s satisfy $4\hat{b}_0 \hat{b}_2 = \hat{b}_1^2$. We note that the above solution is the "dual" black hole solution of refs. \cite{1,2,3} with nontrivial gauge and scalar background.

So far we have discussed solutions which correspond to the case when $B$ field is set to zero in the three dimensional effective action (1). However, it has been pointed out before in refs. \cite{5,8,9}, that nonzero $B$ field can be generated by using an $O(d,d)$ symmetry of the effective action as well as the equations of motion. This $O(d,d)$ transformation acts as:

\[
M' = \Omega M \Omega^T \tag{40}
\]

where

\[
M \equiv \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \tag{41}
\]

and $\Omega$ is a 2d×2d matrix satisfying

\[
\Omega^T \eta \Omega = \eta \tag{42}
\]

with

\[
\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{43}
\]

We now apply the above procedure to generate nonzero "$B" field from our solution. $B$ being zero in our original solution, $M$ is of the form:

\[
M = \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix}. \tag{44}
\]

By making an $O(2,2)$ transformation with $\Omega$:

\[
\Omega = \begin{pmatrix} \Pi & 1 - \Pi \\ 1 - \Pi & \Pi \end{pmatrix} \tag{45}
\]

and

\[
\Pi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{46}
\]

8
one obtains,

\[
M' \equiv \begin{pmatrix}
G'^{-1} & -G'^{-1}B' \\
B'G'^{-1} & G' - B'G'^{-1}B'
\end{pmatrix},
\]

(47)

where

\[
G' = \begin{pmatrix}
g_1 & 0 \\
0 & \frac{1}{g_2}
\end{pmatrix},
\]

(48)

and

\[
B' = \begin{pmatrix}
0 & \frac{1}{2} A \\
-\frac{1}{2} g_2 & 0
\end{pmatrix}.
\]

(49)

Also, since under O(d,d) \( \Phi \to \Phi \), hence new dilaton field is given as

\[
\phi' = \phi - \ln g_2 + \text{const}.
\]

(50)

It is interesting to note that given our earlier solution, eqn. (10) and (19)-(23), \( G' \) and \( B' \) also describe a black hole solution. The gauge field coupling in eqn. (30) is now antisymmetric.

One can generate several solutions in the above manner.

Finally, one can also try to give an alternative interpretation of our solution when roles of the string co-ordinates \( X^1 \) and \( X^2 \), in eqn. (30) are interchanged, so that \( X^2 \) is now the compact direction. In this case the two dimensional space-time metric, now specified by the indices (0,1), takes the form:

\[
\tilde{G}_{ij} = \begin{pmatrix}
-1 & 0 \\
0 & a_0 + b_0 \tanh^2 t
\end{pmatrix}.
\]

(51)

We now note that, unlike the previous case, eqn.(31), the new metric (51) and dilaton (34) are not a solution of the two dimensional graviton-dilaton equation of motion for \( a_0, b_0 \neq 0 \). Only after background gauge field

\[
\tilde{A}_i = (0, b_1 \tanh^2 t)
\]

(52)

and Higgs field

\[
\tilde{\psi} = b_2 \tanh^2 t
\]

(53)
are also added, that the equations of motion are satisfied. It will be interesting to further analyze this solution and study the space-time geometry. A generalization of the results of this paper to arbitrary number of dimensions is being investigated also.

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