Yet another additivity conjecture

Keiji Matsumoto\textsuperscript{1,2}

keiji@nii.ac.jp

April 1, 2022

Abstract
In quantum information theory, there are several important open problems which center around whether certain quantities are additive or not. Especially, additivity conjecture about Holevo capacity and additivity/strong superadditivity conjecture about entanglement of formation have been attracting many researchers. It recently turned out that these are equivalent to the additivity of the minimum output entropy of quantum channels, which is mathematically simpler. This paper suggests yet another additivity conjecture which is equivalent to those, and is mathematically simple. This conjecture might be easier than other conjectures to solve, for this can be proven for almost all the examples where one of these conjectures are proven.

1 Introduction

In quantum information theory, several open problems center around whether certain quantities are additive or not. The additivity of Holevo capacity is the oldest of those. As mathematical lemma to study this conjecture, many researchers have been studying the additivity of the minimum output entropy of quantum channels.

Also, the entanglement of formation (EoF) is conjectured to be additive by many. For this quantity, the property called strong superadditivity is conjectured\textsuperscript{5}, too. Roughly speaking, this conjecture insists that the sum of entanglement of the subsystems should not be larger than entanglement of the whole system.

As is proved in\textsuperscript{4}, these additivity conjectures are equivalent. Based on this result, now, many authors are working on additivity of the minimum output entropy of quantum channels, for its mathematical simplicity.

In this talk, we suggest yet another entanglement quantity, whose strong superadditivity and additivity are equivalent to additivity of the quantities mentioned above. The motivations are as follows. First, in

\textsuperscript{1}ERATO Quantum Computation and Information Project, JST.
Hongo White Building, 5-28-3 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan.

\textsuperscript{2}Quantum Information Science Group, National Institute of Informatics.
2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan.
existing proofs of additivity conjectures for specific examples, they are essentially proving additivity of this quantity. Second, this quantity seems at least as simple as the output minimum entropy. Third, the author prefer entanglement measures rather than quantities about channels.

2 Additivity conjectures

Let \( \rho \) be a bipartite state on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). The entanglement of formation (EoF) of \( \rho \) is defined as

\[
E_f(\rho) := \min_{\{p_i, \pi_i\}} \sum_i p_i E(\pi_i)
\]

(1)

where \( \{p_i, \pi_i\} \) runs all over the ensembles of pure bipartite states with \( \sum_i p_i \pi_i = \rho \), and the (entropy of) entanglement for a pure bipartite state \( \pi \) is defined as

\[
E(\pi) := S(tr_{\mathcal{H}_2}\pi) = S(tr_{\mathcal{H}_1}\pi).
\]

Let \( \rho \) be a state on \( \mathcal{H} \otimes \mathcal{H}' \), where \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) and \( \mathcal{H}' = \mathcal{H}_1' \otimes \mathcal{H}_2' \). Then the strong superadditivity \[5\] means that

\[
E_f(\rho) \geq E_f(tr_{\mathcal{H}'_2}\rho) + E_f(tr_{\mathcal{H}_2}\rho),
\]

(2)

where all entanglement of formation are understood with respect to the 1–2–partition of the respective system. The “weaker version” of additivity conjecture of EoF states,

\[
E_f(\rho \otimes \rho') = E_f(\rho) + E_f(\rho').
\]

(3)

Let \( \Lambda \) be a CPTP map from \( \mathcal{B}(\mathcal{K}) \) to \( \mathcal{B}(\mathcal{H}_1) \). The minimum output entropy is defined as,

\[
S_{\min}(\Lambda) := \min_{\rho \in \mathcal{S}(\mathcal{K})} S(\Lambda(\rho)).
\]

The additivity conjecture about this quantity means,

\[
S_{\min}(\Lambda \otimes \Lambda') = S_{\min}(\Lambda) + S_{\min}(\Lambda'),
\]

(4)

Shor\[4\] had proven that \(2, 3, 4\) and additivity of the Holevo capacity are equivalent with each other. In the proof, the correspondence between quantum channels and entangled states are made via Stinespring dilation, as is first proposed Matsumot et. al. \[2\]. Due to the Stinespring dilation, a CPTP map \( \Lambda \) is expressed as the composition of the isometric embedding \( U \) followed by the partial trace,

\[
\mathcal{B}(\mathcal{K}) \xrightarrow{U} \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \xrightarrow{tr_{\mathcal{H}_1}} \mathcal{B}(\mathcal{H}_2).
\]

(5)

Below, we denote \( UK \) simply by \( K \), so far no confusion arises. In this correspondence,

\[
S(\Lambda(\pi)) = E(\pi).
\]

(6)
3 Yet another additivity conjecture

Now, we propose a new entanglement quantity,

\[ E_m(\rho) := \min_{\{p_i, \pi_i\}} \min E(\pi_i), \tag{7} \]

where \(\{p_i, \pi_i\}\) runs all over the ensembles of pure bipartite states with \(\sum_i p_i \pi_i = \rho\). The additivity and the strong superadditivity of this quantity means,

\[ E_m(\rho \otimes \rho') = E_m(\rho) + E_m(\rho'), \tag{8} \]

and

\[ E_m(\rho) \geq E_m(\text{tr}_{\mathcal{H}'} \rho) + E_m(\text{tr}_\mathcal{H} \rho), \tag{9} \]

respectively. Note that,

\[ E_m(\rho) = \min_\pi E(\pi), \tag{10} \]

where \(\pi\) runs all over the pure states living in the support of \(\rho\). This expression strongly suggest the close tie between \(E_m\) and the output minimum entropy.

**Theorem 1** (main theorem) The followings are equivalent.

(i) \[ \text{(9)} \] for all the pure states.

(ii) \[ \text{(9)} \] for all the states.

(iii) \[ \text{(8)} \] for all the states.

(iv) \[ \text{(8)} \] for all the quantum channels.

(v) \[ \text{(8)} \] for all the states.

Combining this theorem with the main theorem of [4], we can conclude the additivity of the new entanglement quantity is equivalent to all the other additivity conjectures.

**Proof** For (iv) ⇔ (v) due to [4], it suffices to show (v) ⇔ (i) ⇔ (ii) ⇔ (iii) ⇔ (iv).

In the following, let \(\rho \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}')\).

(i) ⇔ (i): Let \(\rho\) be a pure state. Then,

\[ E_m(\rho) = E(\rho) = E_f(\rho) \geq E_f(\text{tr}_{\mathcal{H}'} \rho) + E_f(\text{tr}_\mathcal{H} \rho) \]
\[ \geq E_m(\text{tr}_{\mathcal{H}'} \rho) + E_m(\text{tr}_\mathcal{H} \rho). \]

(i) ⇔ (ii): Let \(\pi_*\) be a pure state living in the support of \(\rho\) with \(E_m(\rho) = E(\pi_*)\). Then,

\[ E_m(\rho) = E(\pi_*) \geq E_m(\text{tr}_{\mathcal{H}'} \pi_*) + E_m(\text{tr}_\mathcal{H} \pi_*) \]
\[ \geq E_m(\text{tr}_{\mathcal{H}'} \rho) + E_m(\text{tr}_\mathcal{H} \rho), \]

in which the second inequality comes from the assumption, and the third inequality due to the fact that the support of \(\text{tr}_{\mathcal{H}'} \pi_*\) is a subset of the support of \(\text{tr}_{\mathcal{H}'} \rho\).

(ii) ⇐ (i), (ii) ⇒ (iii): trivial.
Let $\Lambda'$ be a CPTP map from $B(K')$ to $B(H'_2)$, and consider an isometric embedding like (iii). Let $\rho$ and $\rho'$ be the state whose support is $K$ and $K'$, respectively. Then, (i) and (iv) imply,

$$E_m(\rho) = \min_{\phi \in K} S\left(\text{tr}_{H_2}|\phi\rangle\langle\phi|\right)$$

$$= \min_{\phi \in K} S\left(\Lambda(|\phi\rangle\langle\phi|)\right)$$

$$= S_{\min}(\Lambda'),$$

and

$$E_m(\rho') = S_{\min}(\Lambda').$$

In addition, for the support of $\rho \otimes \rho'$ is $K \otimes K'$, we have,

$$E_m(\rho \otimes \rho') = S_{\min}(\Lambda \otimes \Lambda').$$

Combining these equations, we have the assertion.

4 Properties of $E_m$

For a quantity to be a proper entanglement measure, that quantity should be

(i) equal to $E$ for the pure states.

(ii) monotone by the application of LOCC.

(iii) asymptotic continuity.

Our new quantity $E_m$ trivially satisfy (i). Also, (ii) is satisfied, for, letting $\Omega$ be a LOCC operation, and $\pi_*$ be the pure state with $E_m(\pi_*) = E_m(\rho)$, we have,

$$E_m(\Omega(\rho)) \leq E_m(\Omega(\pi_*)) \leq E_f(\Omega(\pi_*)) \leq E_f(\pi_*) = E(\pi_*) = E_m(\rho).$$

However, it is obvious that (iii) cannot be satisfied.

On the other hand, this quantity satisfies convexity,

$$E_m(p\rho_1 + (1 - p)\rho_2) \leq \min\{E_m(\rho_1), E_m(\rho_2)\} \leq pE_m(\rho_1) + (1 - p)E_m(\rho_2).$$

5 Discussions

Among all the additivity conjectures which are equivalent with each other, many people are focusing on additivity of the minimum output entropy. However, in the existing proofs of this additivity conjecture for the special cases (e.g., [3, 4, 5, 6, 7]), they first show the strong superadditivity of $E_m$ for all the pure states, living in $K \otimes K'$,

$$E_m(\rho) \geq E_m(\text{tr}_{H'}\rho) + E_m(\text{tr}_H\rho),$$

which naturally leads to the additivity of the minimum output entropy.

Also, in many states for which the additivity or the strong superadditivity of $E_m$ is shown, $E_m$ is equal to $E_{\text{EoF}}$ (e.g., [9, 10]).

Hence, the additivity or the strong superadditivity of $E_m$ can be another good equivalent statement of the additivity conjecture. However, its operational meaning is hard to find out.
References

[1] D. Fattal, K. Matsumoto, I. Chuang, Y Yamamoto, “Strong superadditivity of stabilizer states”, unpublished.

[2] K. Matsumoto, T. Shimono, and A. Winter, Comm, Math, Phys. 2003.

[3] P. W. Shor, “Additivity of the Classical Capacity of Entanglement-Breaking Quantum Channels”, J. Math. Phys. Vol. 43, 4334-4340 (2002), e–print quant-ph/0201149.

[4] P. W. Shor, “Equivalence of Additivity Questions in Quantum Information Theory,” Comm. Math. Phys., 2003.

[5] K. G. H. Vollbrecht, R. F. Werner, “Entanglement Measures under Symmetry”, Phys. Rev. A, vol. 64, 062307, 2001.

[6] C. King, “Additivity for unital qubit channels”, J. Math. Phys. vol. 43, no. 10, 4641 – 4653 (2002)

[7] C. King, “The capacity of the quantum depolarizing channel”, IEEE Transaction on Information Theory, vol. 49, no. 1, 221 – 229, 2003.

[8] K. Matsumoto and F. Yura “Entanglement Cost of Antisymmetric States and Additivity of Capacity of Some Quantum Channels”, J. Phys. A, 37, L167, 2004, e–print quant-ph/0306009

[9] G. Vidal, W. Dür, J. I. Cirac, “Entanglement cost of mixed states”, Phys. Rev. Let., vol. 89, no. 2, 027901, 2002.