SUBALGEBRAS AND FREE PRODUCT STRUCTURES OF A
GRAPH $W^*$-PROBABILITY SPACE

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Abstract. Let $G$ be a countable directed graph. Then we can construct the
graph $W^*$-algebra $W^*(G)$ and its diagonal subalgebra $D_G$. By defining
the conditional expectation $E : W^*(G) \to D_G$, we have the graph $W^*$-probability
space over $D_G$, $(W^*(G), E)$, as an amalgamated $W^*$-probability space over
$D_G$. The amalgamated freeness on $(W^*(G), E)$ is defined in the sense of Speicher.
In this paper, we will define the $D_G$-semicircular system and $D_G$-
valued $R$-diagonal system. If the graph $G$ contains $N$-mutually diagram-
distinct loops, then we can construct the $D_G$-semicircular system $L_N$ in
$(W^*(G), E)$ and the $D_G$-semicircular algebra $W^*(L_N, D_G)$ generated by $L_N$
and $D_G$, characterized by $(W^*(L_N, D_N), E_N) \otimes (D_G, 1)$, where $D_N \leq D_G$
is a subalgebra determined by $L_N$. If we have $N$-mutually distinct finite paths
$w_1, \ldots, w_N \notin \text{loop}(G)$, then we can construct the $D_G$-valued $R$-diagonal system
$R = \{L_{w_1}, L_{w_2}, \ldots, L_{w_N}, L_{w_N}^\ast\}$ satisfying that $\{L_{w_1}, L_{w_1}^\ast\}, \ldots, \{L_{w_N}, L_{w_N}^\ast\}$
are $D_G$-free from each other. The subalgebra $W^*(R, D_G)$ is observed. Pre-
cisely, we show that the graph $W^*$-algebra $W^*(G)$ is the $D_G$-free product of
$D_G$-free building blocks $D_G$ and $W^*(\{L_e\}, D_G)$, for all $e \in E(G)$. So, we can
observe the $D_G$-free product structure of $(W^*(G), E)$ by these building blocks.
Also, we can see that

$\left(W^*(G), E = (D_G, 1) \ast_{D_G} (\ast_{e \in E(G)} (W^*(\{L_e\}, D_G), E))\right)$

In [16], we constructed the graph $W^*$-probability spaces. The graph $W^*$-probability
theory is one of the good example of Speicher’s combinatorial free probability theory
with amalgamation. In [16], we observed how to compute the certain operator-
valued moments and cumulants of an arbitrary operator-valued random variables
in the graph $W^*$-probability space and observed the amalgamated freeness on the
graph $W^*$-probability space, with respect to the given conditional expectation.
Also, in [17], we consider certain operator-valued random variables of the graph
$W^*$-probability space, for example, semicircular elements, even elements and $R$-
diagonal elements. This shows that the graph $W^*$-probability spaces contain the
rich free probabilistic objects.

Throughout this paper, let $G$ be a countable directed graph and let $F^+(G)$ be
the free semigroupoid of $G$, in the sense of Kribs and Power. i.e., it is a collection
of all vertices of the graph $G$ as units and all admissible finite paths, under the
admissibility. The admissible product between two elements in the set $F^+(G)$ is

Key words and phrases. Graph $W^*$-Probability Spaces over the Diagonal Subalgebras, $D_G$-
valued Moments and Cumulants, $D_G$-Freeness, $D_G$-Semicircular Systems, $D_G$-Semicircular Sub-
algebras, $D_G$-valued $R$-diagonal Systems, $D_G$-valued $R$-diagonal Subalgebras.
the binary operation on \( \mathbb{F}^+(G) \). As a set, the free semigroupoid \( \mathbb{F}^+(G) \) can be decomposed by

\[
\mathbb{F}^+(G) = V(G) \cup FP(G),
\]

where \( V(G) \) is the vertex set of the graph \( G \) and \( FP(G) \) is the set of all admissible finite paths. Trivially the edge set \( E(G) \) of the graph \( G \) is properly contained in \( FP(G) \), since all edges of the graph can be regarded as finite paths with their length 1. Kribs and Power defined the graph Hilbert space \( H_G = l^2(\mathbb{F}^+(G)) \) with its Hilbert basis \( \{ \xi_w : w \in \mathbb{F}^+(G) \} \). In [16] and [17], we defined the creation operator \( L_w \) by the multiplication operator with its symbol \( \xi_w \) and its adjoint \( L_w^* \), the annihilation operator on \( H_G \). They have the following relation; if \( w = v_1 w v_2 \) in \( \mathbb{F}^+(G) \) with \( v_1, v_2 \in V(G) \), then

(i) \( L_w = L_{v_1} L_w L_{v_2} \)
(ii) \( L_w L_w^* = L_{v_1} \) and \( L_w^* L_w = L_{v_2} \)
(iii) \( L_w^* L_w = L_w^* \) if \( w = w w w \in V(G) \)
(iv) \( L_w L_w^* L_w = L_w^* \) if \( w \in FP(G) \).

We define a graph \( W^* \)-algebra of \( G \) by

\[
W^*(G) \overset{\text{def}}{=} \mathbb{C}[\{ L_w, L_w^* : w \in \mathbb{F}^+(G) \}].
\]

Notice that the creation operators induced by vertices are projections and the creation operators induced by finite paths are partial isometries. We can define the \( W^* \)-subalgebra \( D_G \) of \( W^*(G) \), which is called the diagonal subalgebra by

\[
D_G \overset{\text{def}}{=} \mathbb{C}[\{ L_v : v \in V(G) \}].
\]

Then each element \( a \) in the graph \( W^* \)-algebra \( W^*(G) \) is expressed by

\[
a = \sum_{v \in V(G)} p_v L_v + \sum_{w \in FP(G)} (p_w L_w + p_w^* L_w^*),
\]

for \( p_v, p_w, p_w^* \in \mathbb{C} \). Here, \( p_w^* \) is just a complex number. Remark that \( p_w^* \) is not a conjugate \( \overline{p_w} \) of \( p_w \) in \( \mathbb{C} \). The above expression of \( a \) is said to be the Fourier expansion of \( a \). Define the support \( \mathbb{F}^+(G : a) \) of \( a \) by

\[
\mathbb{F}^+(G : a) = V(G : a) \cup FP(G : a)
\]

where

\[
V(G : a) = \{ v \in V(G) : p_v \neq 0 \text{ in (0.1)} \}
\]

and

\[
FP(G : a) = \{ w \in FP(G) : p_w \neq 0 \text{ in (0.1)} \} \cup \{ w' \in FP(G) : p_w' \neq 0 \text{ in (0.1)} \}.
\]
Notice that if \( V(G : a) \neq \emptyset \), then \( \sum_{v \in V(G : a)} p_v L_v \) is contained in the diagonal subalgebra \( D_G \). Thus we have the canonical conditional expectation \( E : W^*(G) \to D_G \), defined by

\[
E(a) = \sum_{v \in V(G : a)} p_v L_v,
\]

for all \( a \) in \( W^*(G) \) with its Fourier expansion (0.1). Then the algebraic pair \( (W^*(G), E) \) is a \( W^* \)-probability space with amalgamation over \( D_G \) (See [16]). This structure is called the graph \( W^* \)-probability space over its diagonal subalgebra \( D_G \), and all elements in \( (W^*(G), E) \) are said to be \( D_G \)-valued random variables. It is easy to check that the conditional expectation \( E \) is faithful in the sense that if \( E(a^*a) = 0 \) \( D_G \), for \( a \in W^*(G) \), then \( a = 0 \) \( D_G \).

In [16] and [17], we computed the \( D_G \)-valued moments and cumulants of an arbitrary \( D_G \)-valued random variable \( a \) having its Fourier expansion (0.1). In particular, by using the \( D_G \)-cumulant formula, we got the \( D_G \)-valued mixed cumulants of \( D_G \)-valued random variables \( a_1 \) and \( a_2 \) and then we could find the \( D_G \)-freeness characterization which is so abstract to use. However, by using this characterization, we can characterize the \( D_G \)-freeness of generators \( L_{w_1} \) and \( L_{w_2} \) and then we could find the \( D_G \)-freeness condition:

\[
L_{w_1} \text{ and } L_{w_2} \text{ are free over } D_G \text{ in } (W^*(G), E) \iff w_1 \text{ and } w_2 \text{ in } F^+(G) \text{ are diagram-distinct},
\]

in the sense that \( w_1 \) and \( w_2 \) have different diagram on the graph \( G \), graphically. Based on this \( D_G \)-freeness condition, in this paper, we will observe the \( D_G \)-free structure of the given graph \( W^*- \) algebra \( W^*(G) \).

Define the subset \( \text{loop}(G) \) of \( FP(G) \), by the collection of all loop finite paths in \( FP(G) \). We will consider the family in \( \text{loop}(G) \)

\[
\mathcal{F} = \{ l_j \in \text{loop}(G) : \text{mutually diagram-distinct} \}_{j=1}^{N}
\]

and construct the corresponding \( D_G \)-semicircular system

\[
\mathcal{L}_N = \left\{ L_{l_j} + L_{l_j^*} : l_j \in F \right\}.
\]

Indeed, the elements in \( \mathcal{L}_N \) are free from each other over \( D_G \), by the diagram-distinctness of \( \mathcal{F} \) and they are all \( D_G \)-semicircular, by [17]. So, the family \( \mathcal{L}_N \) is the \( D_G \)-semicircular system in \( (W^*(G), E) \). We show that the free product structure of the \( W^* \)-subalgebra \( W^*(\mathcal{L}_N, D_G) \) generated by the \( D_G \)-semicircular system \( \mathcal{L}_N \) and the diagonal subalgebra \( D_G \);
\[(W^*(L_N, D_G), E) \simeq (W^*(L_N, D_N) \otimes D_G, E_N \otimes 1)\]
\[\simeq (W^*(L_N, D_N), E_N) \otimes (D_G, 1)\]
\[= \bigoplus_{j=1}^N \left( (W^*\{L_{l_j}\}, D_N), E_N \otimes (D_G, 1) \right)\]

where \(D_N = \mathbb{C}[L_{v_j} : l_j = v_jl_jv_j, l_j \in F]\), \(E_N = E_{D_N} \circ E\), and \(1\) is the identity map on \(D_G\).

Also we define a new family

\[\mathcal{R} = \{L_{w_j}, L_{w_j}^* : w_j \in \text{loop}^e(G)\}_{j=1}^N,\]

where \(\text{loop}^e(G) = FP(G) \setminus \text{loop}(G)\). Then since the distinctness of non-loop finite paths is the diagram-distinctness on them, \(\{L_{w_j}, L_{w_j}^*\}\)'s, for \(j = 1, \ldots, N\), in \(\mathcal{R}\) are free from each other over \(D_G\). Moreover, by [17], \(L_{w_j}\) and \(L_{w_j}^*\) are \(D_G\)-valued \(\mathcal{R}\)-diagonal. We call this family \(\mathcal{R}\), the \(D_G\)-valued \(\mathcal{R}\)-diagonal system. Similar to the \(D_G\)-semicircular system case, we will define a \(W^*\)-subalgebra \(W^*\{\mathcal{R}, D_G\}\) of the graph \(W^*\)-algebra \(W^*(G)\), called the \(\mathcal{R}\)-diagonal subalgebra of \((W^*(G), E)\).

We can see that

\[(W^*(\mathcal{R}, D_G), E_{\mathcal{R}}) = \left( \bigoplus_{j=1}^N \left( W^*\{L_{w_j}\}, D_{\mathcal{R}}\right), E_{D_{\mathcal{R}}} \right) \otimes (D_G, 1),\]

where \(D_{\mathcal{R}} = \mathbb{C}[L_{v_1}, L_{w_2} : w_j = v_1w_jv_2]\) and \(E_{\mathcal{R}} = E|_{W^*(\mathcal{R}, D_G)}\).

We will also define the \(D_G\)-free building blocks of the graph \(W^*\)-algebra \(W^*(G)\) and we prove that

\[(W^*(G), E) = (D_G, E)\]

\[= \bigoplus_{l \in \text{Loop}(G)} \left( W^*\{L_{l}\}, D_G\right)\]
\[= \bigoplus_{w \in \text{loop}^e(G)} \left( W^*\{L_{w}\}, D_G\right),\]

where \(\text{Loop}(G)\) is the set of all basic loops in \(\text{loop}(G)\). A loop \(l\) is basic if there is no loop \(w\) and a natural number \(k \in \mathbb{N} \setminus \{1\}\) such that \(l = w^k\). This free product structure of \((W^*(G), E)\) is nice for studying the subalgebras of \(W^*(G)\).

Finally, by considering the subalgebra inclusion, we can get the following free product structure of \((W^*(G), E)\),

\[(W^*(G), E) = \bigoplus_{l \in \text{Loop}(G)} \left( W^*\{L_{l}\}, D_G\right)\]
\[= \bigoplus_{w \in \text{loop}^e(G)} \left( W^*\{L_{w}\}, D_G\right).\]
(W∗(G), E) = (D_G; E) ∗_{D_G} \left( *_{D_G} (W^∗(\{L_e\}, D_G), E) \right).

1. Semicircular System

1.1. The D_G-Semicircular System. In this chapter, we will consider the amalgamated semicircular system observed by Shlyaktenko (See [10]), in our graph structure. Throughout this chapter, let G be a countable directed graph and (W∗(G), E), the graph W∗-probability space over the diagonal subalgebra D_G.

Definition 1.1. Let B be a von Neumann algebra and A, a von Neumann algebra over B and let F : A → B be a conditional expectation. Then (A, F) is the amalgamated W∗-probability space over B. The B-valued random variable x ∈ (A, F) is said to be a B-semicircular element if x is self-adjoint and if the only nonvanishing B-cumulant of x is the second B-cumulant of x. Let x_1, ..., x_s be self-adjoint B-valued random variables in (A, F), where s ∈ N. We say that the set S = \{x_1, ..., x_s\} is a **B-semicircular family** if all x_j’s are B-semicircular, for j = 1, ..., s. The B-semicircular family S is said to be a **B-semicircular system** if x_1, ..., x_s are free from each other over B, in (A, F). The algebra generated by a B-semicircular system and B is called the **B-semicircular (sub)algebra** of A.

Assume that we have a one-vertex directed graph H. Then the diagonal subalgebra D_H = C. So, in this case, the D_H-semicircularity is same as the Voiculescu’s semicircularity. We will define the lattice path model LP^∗_n.

LP^∗_n = \{L : \text{lattice path having the } ∗\text{-axis-property}\}

(See [16] and [17]). Take L ∈ LP^∗_n. Then we have a (non-unique) corresponding lattice path u_{\nu_1}^{\nu_n} of the D_G-valued random variable L_{\nu_1}^{\nu_n}, where u_j ∈ \{1, ∗\}, in some graph W∗-probability space (W^∗(G), E) over D_G.

Theorem 1.1. (See [17]) The D_G-valued random variables a_l = L_l + L_l^∗ are D_G-semicircular, for all l = vlv ∈ loop(G), with v ∈ V(G). In particular, we have that

\[ k_n (a_l, ..., a_l) = \begin{cases} 2L_v & \text{if } n = 2 \\ 0_{D_G} & \text{otherwise,} \end{cases} \]

and
\[ E(a_i^n) = \begin{cases} 
\frac{c_2}{(2L_2)^n} & \text{if } n \text{ is even} \\
0_{DG} & \text{if } n \text{ is odd,} 
\end{cases} \]

for all \( n \in \mathbb{N} \), where \( c_k = \frac{1}{k+1} \left( \begin{array}{c} 2k \\ k \end{array} \right) \) is the \( k \)-th Catalan number. \( \square \)

We will consider the \( DG \)-semicircular family \( \{a_1, \ldots, a_N\} \), for \( N \in \mathbb{N} \). Let \( l_j = v_jl_jv_j \in \text{loop}(G) \), for \( j = 1, \ldots, N \), with \( v_j \in V(G) \). Assume that loops \( l_1, \ldots, l_N \) are mutually diagram-distinct. Define \( DG \)-valued random variables \( a_1, \ldots, a_N \), \( a_j \overset{d}{=} L_{l_j} + L_{l_j}^* \), for all \( j = 1, \ldots, N \).

Again, remark that we assumed that \( l_j \)'s are mutually diagram-distinct. So, \( a_1, \ldots, a_N \) are free from each other over \( DG \) in \((W^*(G), E)\) and hence the \( DG \)-semicircular family \( \{a_1, \ldots, a_N\} \) is a \( DG \)-semicircular system in \((W^*(G), E)\). So, we have the \( DG \)-semicircular system, in \( W^*(G) \), induced by the mutually diagram-distinct loops in \( FP(G) \), i.e., the set

\[ \mathcal{L}_N = \{a_j : l_j \text{'s are diagram-distinct in } \text{loop}(G)\} \]

is the \( DG \)-semicircular system in \((W^*(G), E)\).

1.2. \( DG \)-Semicircular Subalgebra of \((W^*(G), E)\).

Now, we will construct the \( DG \)-semicircular algebra \( W^*(\mathcal{L}_N, DG) \), as a \( W^* \)-subalgebra of the graph \( W^*-\)algebra, generated by \( \mathcal{L}_N \) and \( DG \). Let

\[ \mathcal{F} = \{ l_j \in \text{loop}(G) : j = 1, \ldots, N \} \]

be a collection of mutually diagram-distinct loops in \( FP(G) \) and let

\[ \mathcal{L}_N = \{a_j = L_{l_j} + L_{l_j}^* : l_j \in \mathcal{F}\}. \]

Then the family \( \mathcal{L}_N \) is a \( DG \)-semicircular system in \((W^*(G), E)\) and the \( W^* \)-subalgebra \( W^*(\mathcal{L}_N, DG) \) is the \( DG \)-semicircular subalgebra of \( W^*(G) \). The \( DG \)-semicircular subalgebra \( W^*(\mathcal{L}_N, DG) \) have the following free product structure which is very natural by the very definition.

**Lemma 1.2.** Let \((W^*(G), E)\) be a graph \( W^*-\)probability space over the diagonal subalgebra \( DG \) and let

\[ \mathcal{L}_N = \{a_j = L_{l_j} + L_{l_j}^* : l_j \text{'s are diagram-distinct in } \text{loop}(G)\} \]
Then the $W^*$-subalgebra $W^*(\mathcal{L}_N, D_G)$ of $W^*(G)$ is a $D_G$-semicircular algebra satisfies that

$$W^*(\mathcal{L}_N, D_G) \simeq \bigotimes_{j=1}^N W^*(a_j, D_G).$$

\[\square\]

Let $\mathcal{L}_N$ be given as above. Assume that $l_j = v_j l_j v_j$, for $v_j \in V(G)$. (It is possible that $v_i = v_k$, for some $i, k$ in $\{1, ..., N\}$.) Define the subalgebra $D_N$ of the diagonal subalgebra $D_G$ by

$$D_N = \mathbb{C}\langle\{L_{v_j} : j = 1, ..., N\}\rangle^\omega.$$ 

Trivially, $D_N \subseteq D_G$, as von Neumann algebras.

**Proposition 1.3.** (*Also See [18]*) Let $\mathcal{L}_N$ be the given $D_G$-semicircular system and let $D_N$ be defined as above. As amalgamated $W^*$-probability spaces,

$$(W^*(\mathcal{L}_N, D_G), E) \simeq (W^*(\mathcal{L}_N, D_N) \otimes D_G, E_N \otimes 1) \simeq (W^*(\mathcal{L}_N), E_N) \otimes (D_G, 1),$$

where $E_N : W^*(\mathcal{L}_N) \to D_N$ is the conditional expectation defined by $E_N = E_{D_N}$.

**Proof.** As $W^*$-algebras, $W^*(\mathcal{L}_N, D_G) \simeq W^*(\mathcal{L}_N, D_N) \otimes D_G$. Indeed, without loss of generality, take $a \in W^*(\mathcal{L}_N, D_G)$ by

$$a = d_1 a^{k_1}_{l_{i_1}} d_2 a^{k_2}_{l_{i_2}} \cdots d_n a^{k_n}_{l_{i_n}} \text{ and } a_{l_j} = L_{l_j} + L^*_{l_j},$$

where $d_1, ..., d_n \in D_G, k_1, ..., k_n \in \mathbb{N}$ and $(i_1, ..., i_n) \in \{1, ..., N\}^n, n \in \mathbb{N}$. Observe that, for any $j \in \{1, ..., N\}$, we have that

$$a^k_{l_j} = \left(L_{l_j} + L^*_{l_j}\right)^k = L^k_{l_j} + L^*_{l_j} + Q\left(L_{l_j}, L^*_{l_j}\right),$$

where $Q \in \mathbb{C}[z_1, z_2]$. Also, observe that $L^{k_1}_{l_{i_1}} L^{k_2}_{l_{i_2}}$, for any $k_1, k_2 \in \mathbb{N}$, satisfies that

$$L^{k_1}_{l_j} L^{k_2}_{l_j} = L^{k_1}_{l_j} L^{k_2}_{l_j} = \begin{cases} L^{k_1-k_2}_{l_j} = L_{v_j} L^{k_1-k_2}_{l_j} \text{ if } k_1 > k_2 \\ L^{k_2-k_1}_{l_j} = L_{v_j} L^{k_2-k_1}_{l_j} \text{ if } k_1 < k_2 \\ L_{v_j} \text{ if } k_1 = k_2, \end{cases}$$

and similarly,
\[ L_{i_j}^* \prod_{j=1}^k L_{i_j}^* = L_{i_j}^* \prod_{j=1}^k \prod_{j'=1}^l \left\{ \begin{array}{ll}
L_{i_j}^{k_1-k_2} = L_{i_j}^* L_{i_j}^{k_1-k_2} & \text{if } k_1 > k_2 \\
L_{i_j}^{k_2-k_1} = L_{i_j}^* L_{i_j}^{k_2-k_1} & \text{if } k_1 < k_2 \\
L_{i_j} & \text{if } k_1 = k_2.
\end{array} \right. \]

So,

\[ Q(L_{i_j}, L_{i_j}^*) = L_{i_j} \left( Q(L_{i_j}, L_{i_j}^*) \right) L_{i_j}, \]

for all \( j = 1, \ldots, N \). Thus

\[(1.1) \quad a_{i_j}^k = L_{i_j} a_{i_j}^k = L_{i_j} a_{i_j}^k L_{i_j}, \text{ for all } j = 1, \ldots, N.\]

Now, consider that

\[ d_j = d_j^N + d_j', \forall j = 1, \ldots, N. \]

where \( d_j^N = \sum_{j=1}^N L_{i_j} d_j L_{i_j} \) and \( d_j' = d_j - d_j^N \) in \( D_G \). So, we can rewrite that

\[ a = (d_1^N + d_1') a_{i_1}^{k_1} (d_2^N + d_2') a_{i_2}^{k_2} \cdots (d_n^N + d_n') a_{i_n}^{k_n} \]

\[ = d_1^N a_{i_1}^{k_1} d_2^N a_{i_2}^{k_2} \cdots d_n^N a_{i_n}^{k_n} + d_1' a_{i_1}^{k_1} d_2' a_{i_2}^{k_2} \cdots d_n' a_{i_n}^{k_n} \]

by (1.1). This shows that \( a = a \otimes 1 \in W^*(L_N, D_N) \otimes 1 \) and

\[ E(a) = E_{D_N}^{D_G} \circ E(a) = E_N(a) = E_N \otimes 1(a \otimes 1). \]

Trivially, if \( a \in D_G \subset W^*(L_N, D_G) \), then \( a = 1 \otimes a \in 1 \otimes D_G \). Furthermore, if \( a \in D_G \) in \( W^*(L_N, D_G) \), then

\[ E(a) = a = 1 \otimes a = E_N \otimes 1(1 \otimes a). \]

Now, consider \( W^*(L_N, D_N) \). By the previous lemma, similarly, we have that

\[(1.2) \quad W^*(L_N, D_N) = W^*(\{a_1\}, D_N) \ast_{D_N} \cdots \ast_{D_N} W^*(\{a_N\}, D_N). \]

Indeed, the \( D_G \)-semicircular elements \( a_i \) and \( a_j \) in \( L_N \) are free over \( D_N \) in \( W^*(L_N, D_N) \). Clearly, since \( D_N \subset D_G \) and since \( a_i \) and \( a_j \) are free over \( D_G \), they are free over \( D_N \). Therefore, the formula (3.1.2) holds true with respect to the (compressed) conditional expectation

\[ E_N = E |_{W^*(L_N, D_N)} = E_{D_N}^{D_G} \circ E, \]

on \( W^*(L_N, D_N) \).
Corollary 1.4. Let $\mathcal{F} = \{l_j : l_j = v_0 l_j v_0 \}_{j=1}^N$ be a family of mutually diagram-distinct loops in $\text{loop}(G)$, where $v_0 \in V(G)$. If the collection

$$\mathcal{L}_N = \{ \frac{1}{\sqrt{2}} a_{l_j} : l_j \in \mathcal{F} \}$$

is a $D_G$-semicircular system induced by the family $\mathcal{F}$, then

$$(W^*(\mathcal{L}_N, D_G), E) \simeq (L(F_N), \text{tr}) \otimes (D_G, 1),$$

in the sense of Voiculescu, where $\text{tr}$ is the canonical $II_1$-trace of the free group factor $L(F_K)$, $\forall K \in \mathbb{N}$, and where $1(d) = d$, $\forall d \in D_G$. □

By the previous proposition and corollary, we can have the following fact:

Theorem 1.5. Let $\mathcal{F} = \{l_{k1}, \ldots, l_{kN} : l_{kj} = v_k l_{kj} v_k, j = 1, \ldots, n_k \}_{k=1}^N$ be the collection of $\sum_{k=1}^N n_k$-mutually diagram-distinct loops in $\text{FP}(G)$. Assume that $v_{k1} \neq v_{k2}$, for all pair $(k1, k2) \in \mathbb{N}^2$. If

$$\mathcal{L} = \{ \frac{1}{\sqrt{2}} a_{l_{kj}} : j = 1, \ldots, n_k \}_{k=1}^N$$

and

$$D_{\mathcal{L}} = \mathbb{C}[\{ L_{v_k} : k = 1, \ldots, N \}]^w$$

are the corresponding $D_G$-semicircular system and the subalgebra of $D_G$ by $L$, respectively, and

$$E_{\mathcal{L}} = E_{D_{\mathcal{L}}} \circ E,$$

then

$$(W^*(\mathcal{L}, D_G), E) \simeq \left( \bigotimes_{k=1}^N (L(F_{nk}), \text{tr}) \otimes (D_{\mathcal{L}}, 1) \right) \otimes (D_G, 1).$$

□

Corollary 1.6. Let $\mathcal{F}_1 = \{l_1^1, \ldots, l_1^N : l_1^j = v_1 l_1^j v_1 \}$ and $\mathcal{F}_2 = \{l_2^1, \ldots, l_2^N : l_2^j = v_2 l_2^j v_2 \}$ be the $N$-mutually diagram-distinct families of loops in $\text{FP}(G)$, where $v_1 \neq v_2 \in V(G)$ are fixed. Let

$$\mathcal{L}_1 = \{ a_k = \frac{1}{\sqrt{2}} (L_{l_1^k} + L_{l_1^k}^*) : k = 1, \ldots, N \}$$

and

$$\mathcal{L}_2 = \{ b_k = \frac{1}{\sqrt{2}} (L_{l_2^k} + L_{l_2^k}^*) : k = 1, \ldots, N \}$$

be the corresponding $D_G$-semicircular systems, respectively. Then two $D_G$-semicircular subalgebras $(W^*(\mathcal{L}_1, D_G), E)$ and $(W^*(\mathcal{L}_2, D_G), E)$ are free over $D_G$ and they are isomorphic, as amalgamated $W^*$-probability spaces. □
The above corollary shows us how to construct the isomorphic semicircular subalgebras in the graph $W^*$-probability space from two vertices having the same number of loops. (Assume that the vertex $v_1$ has its loops $l_{1i}^1, \ldots, l_{1i}^{n_1}$ and $v_2$ has its loops $l_{2i}^1, \ldots, l_{2i}^{n_2}$ and suppose that $n_1 < n_2$. Then we can choose $n_1$-loops of $v_2$, $l_{2i}^1, \ldots, l_{2i}^{n_1}$. And then we can apply the above corollary for them.)

One of the most interesting example of $D_G$-semicircular subalgebra is as follows;

**Example 1.1.** Let $G$ be a directed graph with

$$V(G) = \{v\} \text{ and } E(G) = \{l_1, \ldots, l_N : l_j = vl_jv\}.$$  

Note that $D_G = \Delta_1 = \mathbb{C}$. Also, note that the projection $L_v = 1_{DG} = 1_C = 1 \in \mathbb{C}$. We have the graph $W^*$-probability space $(W^*(G), E)$ over $\mathbb{C}$. Consider the $W^*$-subalgebra $W^*(L_N, D_G) = W^*(L_N)$, where

$$L_N = \left\{ \frac{1}{\sqrt{2}}a_j : a_j = L_{l_{ij}^n} + L_{l_{ij}^n}^* \right\}_{j=1}^N,$$

where $n_j \in \mathbb{N}, j = 1, \ldots, N$. Then $L_N$ is a $D_G$-semicircular system, too. Definitely it is a $D_G$-semicircular family. Since $F = \{l_j^n : j = 1, \ldots, N\}$ is consists of mutually diagram-distinct loops in $FP(G)$, $a_j$'s are free from each other and hence $L_N$ is a $D_G$-semicircular system. We have that

$$k_2 \left( \frac{1}{\sqrt{2}}a_j, \frac{1}{\sqrt{2}}a_j \right) = \frac{1}{2} k_2 (a_j, a_j)$$

$$= \frac{1}{2} k_2 \left( L_{l_{ij}^n} + L_{l_{ij}^n}^*, L_{l_{ij}^n} + L_{l_{ij}^n}^* \right)$$

$$= \frac{1}{2} \sum_{(r_1, r_2)\in\{1, \ast\}^2} k_2 \left( L_{l_{ij}^n}^{r_1}, L_{l_{ij}^n}^{r_2} \right)$$

$$= \frac{1}{2} \left( \mu_{l_{ij}^n, l_{ij}^n}^{1, \ast} \cdot E \left( L_{l_{ij}^n}^{r_1}, L_{l_{ij}^n}^{r_2} \right) + \mu_{l_{ij}^n, l_{ij}^n}^{\ast, 1} \cdot E \left( L_{l_{ij}^n}^{r_1}, L_{l_{ij}^n}^{r_2} \right) \right)$$

$$= \frac{1}{2} (L_v + L_v) = \frac{1}{2} \cdot 2 = 1,$$

for all $j = 1, \ldots, N$. So, we can get that

$$(W^*(L_N), E) = (W^*(L_N, D_G), E) = (W^*(L_N, D_N), E_N) \otimes (D_G, 1) = (W^*(L_N), E_N) \otimes (\mathbb{C}, 1) = (L(F_N), tr),$$
where \( tr : L(F_N) \rightarrow \mathbb{C} \) is the canonical II\(_1\)-trace on the free group factor \( L(F_K) \), \( \forall K \in \mathbb{N} \). So, the graph \( W^*\)-probability space \((W^*(G), E)\) contains the free group factor \( L(F_N) \) which is isomorphic to the \( D_G \)-semicircular subalgebra \( W^*(L_N) \), generated by \( L_N \).

2. R-diagonal Systems

In this chapter, similar to Chapter 2, we will consider the special \( W^*\)-subalgebra of the graph \( W^*\)-probability space \((W^*(G), E)\) over the diagonal subalgebra \( D_G \). As we defined the \( D_G \)-semicircular systems in \( W^*(G) \), we will define the \( (D_G\text{-valued}) \) R-diagonal systems in \( W^*(G) \). Recall that if \( w \in \text{loop}^*(G) \) is a (non-loop) finite path in \( F^+(G) \), then the \( D_G\text{-valued} \) random variables \( L_w \) and \( L_w^* \) are R-diagonal over \( D_G \) (See [17]). Take a finite family

\[
\mathcal{F} = \{ w_j : w_j \in \text{loop}^*(G) \}_{j=1}^N, \, N \in \mathbb{N}.
\]

Define a \( (D_G\text{-valued}) \) R-diagonal family induced by \( \mathcal{F} \) by

\[
R = \{ L_w, L_w^* : w \in \mathcal{F} \}.
\]

Notice that, by the (diagram-)distinctness of \( w_j \)'s in \( \mathcal{F} \), the subfamilies \( \{ L_{w_1}, L_{w_1}^* \} \), \( \ldots, \{ L_{w_N}, L_{w_N}^* \} \) are free from each other over \( D_G \) in \( (W^*(G), E) \). We will observe that the R-diagonal subalgebra \( W^*(R, D_G) \) satisfies that

\[
(W^*(R, D_G), E) = \bigotimes_{j=1}^N ((W^*(\{ L_{w_j} \}, D_R), E_R) \otimes (D_G, 1)),
\]

where \( D_R \) is the subalgebra generated by the projections

\[
\{ L_{v_1}, L_{v_2} : w = v_1wv_2, \forall w \in \mathcal{F} \}.
\]

and \( E_R = E |_{W^*(\{ L_{w_j} \}_{j=1}^N, D_R)} \) is the restricted conditional expectation onto \( D_R \).

2.1. \( D_G\text{-valued} \) R-diagonal Systems.
Let $G$ be a countable directed graph and $(W^*(G), E)$, the graph $W^*$-probability space over the diagonal subalgebra $D_G$. Throughout this section, we will fix the following finite family,

$$\mathcal{F} = \{w_j : w_j \in \text{loop}^c(G)\}_{j=1}^N \subset FP(G),$$

where $N \in \mathbb{N}$. Notice that all elements in the family $\mathcal{F}$ are non-loop finite paths and hence the corresponding $D_G$-valued random variables are $D_G$-valued R-diagonal elements (See [17]).

**Definition 2.1.** Let $a \in (W^*(G), E)$ be a $D_G$-valued random variable. The $D_G$-valued random variable is said to be an $(D_G$-valued) R-diagonal element if it has the only nonvanishing $D_G$-valued cumulants having their forms of

$$k_n(a, a^*, ..., a) \quad \text{and} \quad k_n(a^*, a, ..., a),$$

for all $n \in 2\mathbb{N}$ (See [19] and [17]). Clearly, if $a$ is a R-diagonal, then automatically $a^*$ is R-diagonal. (In other words, the pair $(a, a^*)$ is a $D_G$-valued R-diagonal pair. Also, see [19].) Suppose we have a collection

$$R = \{a_1, ..., a_N : a_j \text{ is R-diagonal over } D_G\}.$$

Then the family $R$ is called the $(D_G$-valued) R-diagonal system. The subalgebra $W^*(R, D_G)$ is called the $(D_G$-valued) R-diagonal subalgebra, induced by $R$ in $(W^*(G), E)$.

In [17], we showed the following theorem:

**Theorem 2.1.** Let $w \in FP(G)$. Then the $D_G$-valued random variables $L_w$ and $L_w^*$ are R-diagonal over $D_G$ in $(W^*(G), E)$. □

Notice that all $D_G$-semicircular elements are R-diagonal, by definition. But we will restrict our interests to the R-diagonal systems consisting of the non-loop finite paths. Recall that if $w_1 \neq w_2 \in \text{loop}^c(G)$, then the $D_G$-valued R-diagonal elements $L_{w_1}$ and $L_{w_2}$ are free over $D_G$, in $(W^*(G), E)$, since the distinctness of non-loop finite paths is equivalent to the diagram-distinctness of them.

2.2. $D_G$-valued R-diagonal Subalgebras.

In this section, we will consider the $D_G$-valued R-diagonal subalgebras of $W^*(G)$, generated by the fixed R-diagonal systems and $D_G$. As before, let $\mathcal{F}$ be a finite family consisting of $N$-mutually diagram-distinct non-loop finite paths and let $R = \{w_j : w_j \in \text{loop}^c(G)\}_{j=1}^N \subset FP(G)$.
\{L_w : w \in \mathcal{F}\}. As we saw in the previous section, the family \( R \) is the \( R \)-diagonal system in \((W^*(G), E)\). Therefore, we can get the following result:

**Proposition 2.2.** Let \( \mathcal{F} \) and \( R \) be given as before and let \( W^*(R, D_G) \) be the \( R \)-diagonal subalgebra of \( W^*(G) \). Then

\[
(W^*(R, D_G), E_R) = \bigotimes_{j=1}^N \left( W^*(\{L_w\}, D_G), E_j \right),
\]

where \( E_R = E \mid_{W^*(R, D_G)} \) and \( E_j = E \mid_{W^*(\{L_w\}, D_G)} \), for all \( j = 1, ..., N \).

Similar to the previous chapter, we will define

\[
D_R = \mathbb{C}[[L_v_1, L_v_2 : w = v_1 w v_2, w \in \mathcal{F}]].
\]

Notice that, for the given \( R \)-diagonal system \( R \), the von Neumann algebra \( D_R \) should not be \( \mathbb{C} \), because \( \mathcal{F} \) consists of all non-loop finite paths which are mutually distinct. For example, if the family \( \mathcal{F} = \{w_0 = v_1 w_0 v_2\} \), where \( v_1 \neq v_2 \) in \( V(G) \), and \( R = \{L_{w_0} : w_0 \in \mathcal{F}\} \). Then

\[
D_R = \mathbb{C}[[L_v_1, L_v_2]] \cong \Delta_2,
\]

where \( \Delta_2 \) is the subalgebra of the matricial algebra \( M_2(\mathbb{C}) \) generated by all diagonal matrices. Notice that, for the inclusion \( D_R \subset D_G \), there exists the well-determined canonical conditional expectation \( E_{D_R} \mid_{D_G} : D_G \to D_R \). Also, notice that \( D_R \subset W^*(R) \).

**Proposition 2.3.** Let \( \mathcal{F} \) and \( R \) be given as before and let \( W^*(R, D_G) \) is the \( R \)-diagonal subalgebra of \( W^*(G) \). Then

\[
(W^*(R, D_G), E_R) = \left( W^*(R, D_R), E_{D_R} \right) \otimes (D_G, 1),
\]

where \( E_R = E \mid_{W^*(R, D_G)} \) and \( 1 \) is the identity map on \( D_G \).

**Proof.** Let \( a \in (W^*(R, D_G), E_R) \) be the nonzero \( D_G \)-valued random variable such that

\[
a = d_1 L_{w_1}^{r_1} d_2 L_{w_2}^{r_2} ... d_n L_{w_n}^{r_n} \quad \text{or} \quad a \in D_G,
\]

where \( d_1, ..., d_n \in D_G, r_1, ..., r_n \in \{1, \ast\} \) and \( (i_1, ..., i_n) \in \{1, ..., N\}^n, n \in \mathbb{N} \).

Let \( d_j = \sum_{v_j \in V(G: d_j)} q_{v_j} L_{v_j} \), for all \( j = 1, ..., n \). Then
Proof. Notice that if two $D_G$-valued random variables $x$ and $y$ are free over $D_G$ in $(W^*(G), E)$, then they are free over $D_R$, since $D_R \subset D_G$, i.e., since all mixed
$D_G$-valued cumulants of $p(x, x^*)$ and $q(y, y^*)$ vanish, for all $p, q \in \mathbb{C}[z_1, z_2]$, all their mixed $D_R$-valued cumulants vanish, again. Thus, we have that

$$
\left( W^*(R, D_R), E_{D_R}^D \circ E \right) = \prod_{j=1}^N \left( W^*(\{L_{w_j}\}, D_G), E_{D_R}^D \circ E \right).
$$

Therefore, by the previous proposition, we can get that

$$
(W^*(R, D_G), E_R) = \left( W^*(R, D_R), E_{D_R}^D \circ E \right) \otimes (D_G, 1)
$$

$$
= \prod_{j=1}^N \left( W^*(\{L_{w_j}\}, D_G), E_{D_R}^D \circ E \right) \otimes (D_G, 1).
$$

Now, we will provide the following fundamental examples:

**Example 2.1.** Let $G$ be a directed graph with $V(G) = \{v_1, v_2\}$ and $E(G) = \{e = v_1 e v_2\}$. Let

$$
F = \{e\} \quad \text{and} \quad R = \{L_e, L_e^*\}.
$$

We can construct the $R$-diagonal subalgebra $W^*(R, D_G)$. It is easy to check that

$$
W^*(R, D_G) = W^*(\{L_e, L_e^*\}, D_G)
$$

$$
= W^*(\{L_e, L_e^*\}, D_R) \otimes D_G
$$

$$
= W^*(\{L_e, L_e^*, L_{v_1}, L_{v_2}\}).
$$

By the result from the final chapter, later, we can conclude that $W^*(R, D_G) = W^*(G)$. So, the graph $W^*$-algebra $W^*(G)$ is same as the $R$-diagonal subalgebra $W^*(R, D_G)$ of it, where $R$ is consists of all generators of $W^*(G)$ induced by the edge.

**Example 2.2.** Let $G$ be a directed graph with $V(G) = \{v_1, v_2\}$ and $E(G) = \{e_j = v_1 e_j v_2\}_{j=1}^N$. Let $n \leq N$. Then we have a $R$-diagonal system,

$$
R = \{L_{e_1}, L_{e_1}^*, \ldots, L_{e_n}, L_{e_n}^*\}, \forall n \leq N.
$$

The $R$-diagonal subalgebra $W^*(R, D_G)$ is trivially a $W^*$-subalgebra of $W^*(G)$. Also,

$$
(W^*(R, D_G), E_R) = \prod_{j=1}^n \left( W^*(\{L_{e_j}\}, D_G), E_j \right)
$$
\[
= \left( \bigotimes_{j=1}^{n} (W^*([L_{e_j}], D_{R_j}), E_{e_j}) \right) \otimes (D_G, 1).
\]

**Example 2.3.** Let \( G \) be a directed graph with \( V(G) = \{v_1, v_2, v_3\} \) and \( E(G) = \{e_j = v_1e_jv_2, \quad e'_k = v_2e'_kv_3 \quad : \quad j = 1, \ldots, n \not\equiv k = 1, \ldots, m\} \).

Take \( \mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \),

where \( \mathcal{F}_1 = \{e_1, \ldots, e_n\} \) and \( \mathcal{F}_2 = \{e'_1, \ldots, e'_m\} \).

If we construct the \( R \)-diagonal systems \( R_1 \) and \( R_2 \), induced by \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), respectively, then \( R_1 \) and \( R_2 \) are free over \( D_G \) in \( (W^*(G), E) \), because they are totally disjoint (See [16]). Therefore, the \( R \)-diagonal subalgebra generated by \( R_1 \cup R_2 \) is

\[
W^*(R_1 \cup R_2, D_G) = W^*(R_1, D_G) \ast_{D_G} W^*(R_2, D_G)
\]

\[
= \left( \bigotimes_{j=1}^{n} (W^*([L_{e_j}], D_{R_1}), E_{e_j}) \right) \otimes (D_G, 1)
\]

\[
\ast_{D_G} \left( \bigotimes_{j=1}^{m} (W^*([L_{e'_j}], D_{R_2}), E_{e'_j}) \right) \otimes (D_G, 1).
\]

### 3. Free Product Structures of \( (W^*(G), E) \)

Throughout this chapter, let \( G \) be a countable directed graph and \( (W^*(G), E) \), the graph \( W^* \)-probability space over the diagonal subalgebra \( D_G \). In this chapter, we will consider the building blocks of \( W^*(G) \). Notice that if \( w_1 \not\equiv w_2 \in FP(G) \) and if \( w_1, w_2 \in \mathbb{F}^+(G) \), then we can construct the \( D_G \)-valued random variable \( L_{w_1, w_2} \) which is same as \( L_{w_1}L_{w_2} \). Also notice that \( L_{w_1} \) and \( L_{w_2} \) are not free over \( D_G \), in general. But under the diagram-distinctness of \( w_1 \) and \( w_2 \), \( L_{w_1} \) and \( L_{w_2} \) are free over \( D_G \).

#### 3.1. The \( D_G \)-Free Product Structures of \( (W^*(G), E) \) I.
In this section, we will consider the $D_G$-semicircular algebras and $D_G$-valued R-diagonal algebras in $W^*(G)$, more in detail. Recall that the loop $l$ is basic if there is no other loop $w$ such that $l = w^k$, for some $k \in \mathbb{N} \setminus \{1\}$. Define

$$\text{Loop}(G) \overset{\text{def}}{=} \{ l \in \text{loop}(G) : l \text{ is basic} \}.$$ 

The following lemma is easily proved, by the very definition of basic loops:

**Lemma 3.1.** Let $w \in \text{loop}(G)$ with $w = l^k$, for some $l \in \text{Loop}(G)$ and $k \in \mathbb{N} \setminus \{1\}$. Then

$$W^*(\{L_w\}, D_G) \leq W^*(\{L_l\}, D_G).$$

□

Remark that since $w$ and $l$ are not diagram-distinct, $L_w$ and $L_l$ are not free over $D_G$ in $(W^*(G), E)$. In fact, $W^*(\{L_l\}, D_G)$ contains all $W^*(\{L_w\}, D_G)$, if $l \in \text{Loop}(G)$ and $w = l^k$, $\forall \ k \in \mathbb{N}$.

**Proposition 3.2.** $(W^*(\{L_l : l \in \text{loop}(G)\}, D_G), E) = \underset{l \in \text{Loop}(G)}{\ast_{D_G}} (W^*(\{L_l\}, D_G), E).$

**Proof.** Let

$$\mathcal{L} = \{ L_l : l \in \text{loop}(G) \},$$

$$\mathcal{L}_l = \{ L_{lk} : l \in \text{Loop}(G), k \in \mathbb{N} \}$$

and

$$\mathcal{L}_0 = \{ L_l : l \in \text{Loop}(G) \}.$$

Then

$$\mathcal{L} = \bigcup_{l \in \text{Loop}(G)} \mathcal{L}_l = \bigcup_{L_l \in \mathcal{L}_0} \left( \bigcup_{k=1}^{\infty} \{ L_l^k \} \right).$$

Thus

$$(W^*(\{L_l : l \in \text{loop}(G)\}, D_G), E) = (W^*(\mathcal{L}, D_G), E)$$

$$= \left( W^* \left( \bigcup_{l \in \text{Loop}(G)} \mathcal{L}_l, D_G \right), E \right)$$

$$= \left( W^* \left( \bigcup_{L_l \in \mathcal{L}_0} \left( \bigcup_{k=1}^{\infty} \{ L_l^k \} \right), D_G \right), E \right)$$

$$= \underset{L_l \in \mathcal{L}_0}{\ast_{D_G}} (W^* \left( \left( \bigcup_{k=1}^{\infty} \{ L_l^k \} \right), D_G \right), E)$$

by the fact that if $L_{l_1} \neq L_{l_2}$ in $\mathcal{L}_0$, then they are free over $D_G$, by the diagram-distinctness of $l_1 \neq l_2 \in \text{Loop}(G)$. 
\[
= \ast_{L_l \in \mathcal{L}_0} (W^* (\{L_l\}, D_G), E),
\]
since \(W^* (\bigcup_{k=1}^{\infty} \{L^k_l\}, D_G) = W^* (\{L_l\}, D_G)\). Therefore,
\[
(W^* (\{L_l : l \in \text{loop}(G)\}, D_G), E) = \ast_{l \in \text{Loop}(G)} (W^* (\{L_l\}, D_G), E).
\]

Finally, we can have the \(D_G\)-free product structure of the graph \(W^*\)-probability space \((W^* (G), E)\) over its diagonal subalgebra \(D_G\). By considering the \(D_G\)-freeness of generators of \(W^* (G)\), we can characterize the free product structure of \((W^* (G), E)\).

**Theorem 3.3.** Let \(G\) be a countable directed graph and \((W^* (G), E)\), the graph \(W^*\)-probability space over its diagonal subalgebra \(D_G\). Then
\[
(W^* (G), E) = (D_G, E)
\]
\[
\ast_{D_G} \left( \ast_{D_G} (W^* (\{L_l\}, D_G), E) \right)
\]
\[
\ast_{D_G} \left( \ast_{D_G} (W^* (\{L_w\}, D_G), E) \right).
\]

**Proof.** Recall that \(D_G\)-valued random variables \(L_{w_1}\) and \(L_{w_2}\) are free over \(D_G\) in \((W^* (G), E)\) if and only if \(w_1\) and \(w_2\) are diagram-distinct. So, for any loop \(l\) and non-loop finite path \(w\), \(L_l\) and \(L_w\) are free over \(D_G\). So,
\[
W^* (\{L_l : l \in \text{loop}(G)\}, D_G)
\]
and
\[
W^* (\{L_w : w \in \text{loop}^e (G)\}, D_G)
\]
are free over \(D_G\) in \((W^* (G), E)\). Denote the above subalgebras by \(L\) and \(R\), respectively. Therefore, we have that the free product space
\[
D_G \ast_{D_G} W^* (\{L_l : l \in \text{loop}(G)\}, D_G)
\]
\[
\ast_{D_G} W^* (\{L_w : w \in \text{loop}^e (G)\}, D_G)
\]
is contained in \(W^* (G)\). Since the generators of \(W^* (G)\) and those of \(D_G \ast_{D_G} \mathcal{L} \ast_{D_G} \mathcal{R}\) are same, we can conclude that
\[
(W^* (G), E) = (D_G \ast_{D_G} L \ast_{D_G} R, E).
\]
But, by the previous proposition, we obtained that
SUBALGEBRAS OF \((W^*(G), E)\)

\[
(L, E) = \bigast_{L \in \text{Loop}(G)} (W^*(\{L\}, D_G), E).
\]

Now, we will observe that

\[
(R, E) = \bigast_{w \in \text{loop}(G)} (W^*(\{L_w\}, D_G), E).
\]

Assume that \(L_{w_1}, L_{w_2} \in (R, E)\) are the generators. Then

\[
W^*(\{L_{w_1}, L_{w_2}\}, D_G) = W^*(\{L_{w_1}\}, D_G) \ast_{D_G} W^*(\{L_{w_2}\}, D_G).
\]

Therefore, we can get

\[
(R, E) \leq \bigast_{w \in \text{loop}(G)} (W^*(\{L_w\}, D_G), E).
\]

The subalgebra inclusion \(\geq\) is clear. So,

\[
(W^*(G), E) = (D_G \ast_{D_G} L \ast_{D_G} R, E)
\]

\[
= (D_G, E) \ast_{D_G} (L, E) \ast_{D_G} (R, E)
\]

\[
= (D_G, E) \ast_{D_G} \left( \bigast_{L \in \text{Loop}(G)} (W^*(\{L\}, D_G), E) \right)
\]

\[
\ast_{D_G} \left( \bigast_{w \in \text{loop}(G)} (W^*(\{L_w\}, D_G), E) \right).
\]

3.2. \textbf{D}_G-\textbf{Free Building Blocks of} \((W^*(G), E)\).

In this section, we will construct the \(D_G\)-free building blocks of the graph \(W^*\)-probability space \((W^*(G), E)\) over its diagonal subalgebra \(D_G\). Recall that

\[
(W^*(G), E) = (D_G, E)
\]

\[
\ast_{D_G} \left( \bigast_{L \in \text{Loop}(G)} (W^*(\{L\}, D_G), E) \right)
\]

\[
\ast_{D_G} \left( \bigast_{w \in \text{loop}(G)} (W^*(\{L_w\}, D_G), E) \right).
\]
where Loop(G) is the collection of all basic loops contained in loop(G). Notice that, even though we have a finite directed graph, Loop(G) and loop\(^{-}\)(G) may contain countably many elements. So, (W\(^{\ast}\)(G), E) is, in general, the \(D_{G}\)-free product of infinitely many \(D_{G}\)-free W\(^{\ast}\)-subalgebras. But, in the final chapter, we will show that this infinite free product of algebras can be contained in the finite free product of algebras.

**Definition 3.1.** Let G be a countable directed graph and let W\(^{\ast}\)(G) be the graph W\(^{\ast}\)-algebra. The diagonal subalgebra \(D_{G}\) and W\(^{\ast}\)-subalgebras, W\(^{\ast}\)(\{L\(_{i}\}\}, D\(_{G}\)) and W\(^{\ast}\)(\{L\(_{w}\}\}, D\(_{G}\)) for all \(l \in \text{Loop}(G)\) and \(w \in \text{loop}^{-}(G)\) are \(D_{G}\)-free building blocks of W\(^{\ast}\)(G).

As we observed in Chapter 1 and Chapter 2, we have that if \(l\) is a loop, then

\[
(3.1) \quad (W^{\ast}(\{L_{i}\}, D_{G}), E) = (W^{\ast}(\{L_{i}\}), tr) \otimes (D_{G}, 1),
\]

where \(tr = E \mid_{W^{\ast}(\{L_{i}\})}\) is a tracial linear functional on \(W^{\ast}(\{L_{i}\})\). We also have that if \(w\) is a non-loop finite path, then

\[
(3.2) \quad (W^{\ast}(\{L_{w}\}, D_{G}), E) = (W^{\ast}(\{L_{w}\}, D_{w}), E_{2}) \otimes (D_{G}, 1),
\]

where \(D_{w} = \mathbb{C}[L_{v_{1}, L_{v_{2}}}: w = v_{1}wv_{2}]\) is the W\(^{\ast}\)-subalgebra of \(D_{G}\) and \(E_{2} = E_{D_{G}} \circ E\). By (3.2), we can get the following proposition which shows us the vector space property of the non-loop \(D_{G}\)-free building blocks of W\(^{\ast}\)(G).

**Proposition 3.4.** Let \(w \in \text{loop}^{-}(G)\) be a non-loop finite path and let W\(^{\ast}\)(\{L\(_{w}\}\}, D\(_{G}\)) be the corresponding free building block. Then, as a topological vector space,

\[
W^{\ast}(\{L_{w}\}, D_{G}) = \mathbb{C}\{d, p(L_{w}, L_{w}^{*}) : d \in D_{G}, p \in \mathbb{C}_{1}[z_{1}, z_{2}]\}^{\psi},
\]

where

\[
\mathbb{C}_{1}[z_{1}, z_{2}] \overset{\text{def}}{=} \{p \in \mathbb{C}[z_{1}, z_{2}] : p(z_{1}, z_{2}) = \alpha_{0} + \alpha_{1}z_{1} + \alpha_{2}z_{2}\},
\]

for \(\alpha_{0}, \alpha_{1}, \alpha_{2} \in \mathbb{C}\). □

**Proof.** In general, if \(w \in FP(G)\) is a finite path, then the free building block W\(^{\ast}\)(\{L\(_{w}\}\}, D\(_{G}\)) is a W\(^{\ast}\)-subalgebra of the graph W\(^{\ast}\)-algebra W\(^{\ast}\)(G) such that

\[
W^{\ast}(\{L_{w}\}, D_{G}) = \text{span}\{d, p(L_{w}, L_{w}^{*}) : d \in D_{G}, p \in \mathbb{C}[z_{1}, z_{2}]\}^{\psi},
\]

as a topological vector space. Let \(w \in \text{loop}^{-}(G)\) be a non-loop finite path. Then \(w^{k} \notin \text{FP}^{-}(G)\), for all \(k \in \mathbb{N} \setminus \{1\}\). In other words, if \(k \neq 1\), then \(w^{k}\) is not a admissible finite path of the graph G. Thus \(L_{w}^{k} = L_{w}^{*} - 0_{D_{G}}\), for all \(k \in \mathbb{N} \setminus \{1\}\). Therefore, if \(q \in \mathbb{C}_{1}[z_{1}, z_{2}]\), then
$q(L_w, L_w^*) = \begin{cases} 
q_1(L_w, L_w^*) & \text{or} \\
\alpha \in \mathbb{C}, 
\end{cases}$

in general, where $q_1 \in \mathbb{C}[z_1, z_2]$, i.e.,

(i) if $q(z_1, z_2) = \alpha_0 + \sum_{k=2}^{\infty} (\alpha_k z_1^k + \alpha_k^2 z_2^k)$, then

$$q(L_w, L_w^*) = \alpha_0.$$ 

(ii) if $q(z_1, z_2) = \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k z_1^k + \alpha_k^2 z_2^k)$, then

$$q(L_w, L_w^*) = \alpha_0 + (\alpha_1^2 L_w + \alpha_2^2 L_w^*).$$

So, if we define $q_1 \in \mathbb{C}[z_1, z_2]$ by

$$q_1(z_1, z_2) = \alpha_0 + (\alpha_1^2 z_1 + \alpha_2^2 z_2),$$

then

$$q(L_w, L_w^*) = q_1(L_w, L_w^*).$$ 

3.3. The $D_G$-Free Product Structure of $(W^*(G), E)$ II.

By Section 3.1 and by (3.1) and (3.2), we can get the following theorem:

**Theorem 3.5.** Let $G$ be a countable directed graph and $(W^*(G), E)$, the graph $W^*$-probability space over its diagonal subalgebra $D_G$. Then

$$(W^*(G), E) = D_G *_{D_G} \left( *_{D_G} \left( (W^*(\{L_l\}), tr) \otimes (D_G, 1) \right) \right)$$

$$*_{D_G} \left( *_{D_G} \left( (W^*(\{L_w\}, D_w), E_w) \otimes (D_G, 1) \right) \right),$$

where $tr = E^{D_G}_{D_G} \circ E$ is a trace on $W^*(\{L_l\})$ and $E_w = E^{D_G}_{D_G} \circ E$ is a conditional expectation from $W^*(G)$ onto $D_w$ ($E^A_B$ means the conditional expectation from $A$ onto $B$) and

$$D_l = \mathbb{C}[[L_v : l = vlv]] = \mathbb{C}$$

and

$$D_w = \mathbb{C}[[L_{v_1}, L_{v_2} : w = v_1^w v_2^w]] = \Delta_2.$$
4. More About the Free Product Structure of \((W^*(G), E)\)

In this chapter, we will complete to observe the free product structure of the graph \(W^*\)-probability spaces. Let \(G\) be a countable directed graph and let \((W^*(G), E)\) be the graph \(W^*\)-probability space over its diagonal subalgebra \(D_G\). In Chapter 3, we showed that

\[
(W^*(G), E) = (D_G, 1)
\]

\[
*_{D_G} \left( *_{D_G} \left( (W^*(\{L_l\}, tr) \otimes (D_G, 1)) \right) \right)
\]

\[
*_{D_G, D_G} \left( *_{D_G, D_G} \left( (W^*(\{L_w\}, D_w, E_w) \otimes (D_G, 1)) \right) \right)
\]

In the previous chapter, we emphasize the roles of free building blocks and tried to consider each free building block. By using the characterization of the free building blocks, we could get the above free product structure of the graph \(W^*\)-probability space \((W^*(G), E)\). Without considering the structure of each free building blocks of \((W^*(G), E)\), by Section 3.1, we can rewrite the above formula as

\[
(W^*(G), E) = (D_G, E)
\]

\[
*_{D_G} \left( *_{D_G} \left( (W^*(\{L_l\}, D_G), E) \right) \right)
\]

\[
*_{D_G} \left( *_{D_G} \left( (W^*(\{L_w\}, D_G), E) \right) \right)
\]

In this chapter, we will show that

\[
(4.2)
\]
Subalgebras of $(W^*(G), E)$ 23

$$(W^*(G), E) = (D_G, E)$$

$$*_{D_G} \left( *_{D_G} \left( W^*(\{L_l\}, D_G), E \right) \right)$$

$$*_{D_G} \left( *_{D_G} \left( \right) \right)$$.

where

$$E_{\text{Loop}}(G) \stackrel{\text{def}}{=} E(G) \cap \text{Loop}(G)$$

and

$$E_{\text{loop}}^c(G) \stackrel{\text{def}}{=} E(G) \cap \text{loop}^c(G).$$

Equivalently, we will show that

$$\tag{4.3} (W^*(G), E) = (D_G, 1) *_{D_G} \left( *_{D_G} \left( \right) \right).$$

First, we will concentrate on proving the formula (4.1) is equivalent to the formula (4.2). For the convenience, define

$$\mathcal{L}_{\text{edge}}(G) \stackrel{\text{def}}{=} *_{D_G} \left( \right)$$

$$\mathcal{L}_{\text{edge}}^c(G) \stackrel{\text{def}}{=} *_{D_G} \left( \right)$$

$$\mathcal{L}(G) \stackrel{\text{def}}{=} *_{D_G} \left( \right)$$

and

$$\mathcal{L}^c(G) \stackrel{\text{def}}{=} *_{D_G} \left( \right).$$

**Theorem 4.1.** Let $G$ be a countable directed graph and let $(W^*(G), E)$ be the graph $W^*$-probability space over its diagonal subalgebra $D_G$. Then

$$(W^*(G), E) = (D_G, E) *_{D_G} (\mathcal{L}_{\text{edge}}(G), E) *_{D_G} (\mathcal{L}_{\text{edge}}^c(G), E).$$

**Proof.** Let $E_{\text{Loop}}(G) = E(G) \cap \text{Loop}(G)$ and $E_{\text{loop}}^c(G) = E(G) \cap \text{loop}^c(G)$. By (4.1),
(\(W^*(G, E) = (D_G, E) \ast_{D_G} (\mathcal{L}(G), E) \ast_{D_G} (\mathcal{L}^c(G), E)\)).

Since \(E_{\text{loop}}(G) \subset \text{loop}(G)\) and \(E_{\text{loop}}^c(G) \subset \text{loop}^c(G)\), we have that 
\[\mathcal{L}_{\text{edge}}(G) \leq \mathcal{L}(G)\quad \text{and}\quad \mathcal{L}_{\text{edge}}^c(G) \leq \mathcal{L}^c(G).\]

Therefore, we have the following subalgebra inclusion “\(\geq\)”; 
\[(4.4) (W^*(G, E) \geq (D_G, E) \ast_{D_G} (\mathcal{L}_{\text{edge}}(G), E) \ast_{D_G} (\mathcal{L}_{\text{edge}}^c(G), E)).\]

So, it suffices to show that we have the reverse subalgebra inclusion “\(\leq\”).

(Case I) Assume that \(l \in \text{Loop}(G)\). If \(l \in E_{\text{Loop}}(G)\), then
\[(4.5) W^*(\{L_l\}, D_G) < \mathcal{L}_{\text{edge}}(G).\]

If \(l = e_1, ..., e_n\) with \(e_1, ..., e_n \in E(G)\), such that \(e_j \in E_{\text{loop}}^c(G)\), for all \(j = 1, ..., n, n > 1\), then we have that
\[W^*(\{L_l\}, D_G) \leq \ast_{D_G} W^*(\{L_{e_j}\}, D_G).\]

Therefore, by the assumption that \(e_j \in E_{\text{loop}}^c(G)\), we have that
\[(4.6) W^*(\{L_l\}, D_G) < \mathcal{L}_{\text{edge}}^c(G).\]

By (4.5) and (4.6), we can conclude that if \(l \in \text{Loop}(G)\), then the subalgebra \(W^*(\{L_l\}, D_G)\) of \(W^*(G)\) is the subalgebra of \(\mathcal{L}_{\text{edge}}(G) \ast_{D_G} \mathcal{L}_{\text{edge}}^c(G)\). i.e.,
\[(4.7) (W^*(\{L_l\}, D_G)) \leq (\mathcal{L}_{\text{edge}}(G), E) \ast_{D_G} (\mathcal{L}_{\text{edge}}^c(G), E),\]

for all \(l \in \text{Loop}(G)\). Therefore,
\[(4.8) (\mathcal{L}(G), E) \leq (\mathcal{L}_{\text{edge}}(G), E) \ast_{D_G} (\mathcal{L}_{\text{edge}}^c(G), D_G).\]

(Case II) Now, assume that \(w \in \text{loop}^c(G)\). Suppose that \(w \in E_{\text{loop}}^c(G)\). Then, clearly,
\[(4.9) W^*(\{L_w\}, D_G) < \mathcal{L}_{\text{edge}}^c(G).\]

Now, assume that \(w = e_1...e_k \in \text{loop}^c(G)\) with \(e_1, ..., e_k \in E(G)\) are edges satisfying that the initial vertex of \(e_1\) and the final vertex of \(e_k\) are different. Then
\[(4.10) W^*(\{L_w\}, D_G) \leq \ast_{D_G} W^*(\{L_{e_j}\}, D_G).\]
This also shows that

\[(4.11) \quad W^* (\{L_w\}, D_G) < L_{\text{edge}}^c (G).\]

By (4.10) and (4.11), we can conclude that if \( w \in \text{loop}^c (G) \), then the subalgebra \( W^* (\{L_w\}, D_G) \) of \( W^* (G) \) is the subalgebra of \( L_{\text{edge}}^c (G) \). i.e.,

\[(4.12) \quad (W^* (\{L_w\}, D_G), E) \leq \left( L_{\text{edge}}^c (G), E \right),\]

for all \( w \in \text{loop}^c (G) \). Therefore, we have that

\[(4.13) \quad (L^c (G), E) \leq \left( L_{\text{edge}}^c (G), E \right).\]

As we considered in the previous two cases, we can get that

\[(4.14) \quad L(G) *_{D_G} L^c (G) \leq L_{\text{edge}}^c (G) *_{D_G} L_{\text{edge}}^c (G).\]

Therefore, by the relation (4.14), we can conclude that

\[(W^*(G), E) \leq (D_G, E) *_{D_G} (L_{\text{edge}}(G), E) *_{D_G} \left( L_{\text{edge}}^c (G), E \right).\]

The above theorem provides us that

\[(W^*(G), E) = (D_G, E) *_{D_G} \left( W^*(\{L_l\}, D_G), E \right) *_{D_G} \left( W^*(\{L_w\}, D_G), E \right) *_{D_G} \left( W^*(\{L_e\}, D_G), E \right).\]

Therefore, we can get the following simple free product structure of a graph \( W^* \)-probability space \((W^*(G), E)\);

**Corollary 4.2.** Let \( G \) be a countable directed graph and let \((W^*(G), E)\) be the graph \( W^* \)-probability space over its diagonal subalgebra \( D_G \). Then \((W^*(G), E)\) has the following free product structure:

\[(W^*(G), E) = (D_G, E) *_{D_G} \left( W^*(\{L_e\}, D_G), E \right).\]
Proof. Notice that the edge set $E(G)$ of the graph $G$ satisfies that

$$E(G) = (E(G) \cap \text{Loop}(G)) \cup (E\text{loop}^c(G)).$$

Assume that if $e \in E(G)$ is a loop-edge in $\text{loop}(G)$, then $e$ is a basic loop in $\text{Loop}(G)$. i.e.e,

$$E(G) \cap \text{Loop}(G) = E\text{loop}(G).$$

Thus we have that

$$E(G) = (E(G) \cap \text{Loop}(G)) \cup (E(G) \cap \text{loop}^c(G))$$

$$= E(G) \cap (\text{loop}(G) \cup \text{loop}^c(G))$$

$$= E(G) \cap \mathcal{F}P(G) = E(G).$$

The above theorem and corollary shows us that the free product structure of a graph $W^*$-probability space $(W^*(G), E)$ is totally depending on the admissibility on edges of the graph $G$.

In the rest of this section, we will consider the several examples ;

**Example 4.1.** Let $G$ be a directed one-vertex graph with $V(G) = \{v\}$ and $E(G) = \{l_1, \ldots, l_N\}$, where $l_j$’s are all loop edges, $j = 1, \ldots, N$. Then, by the previous theorem, we can have that

$$(W^*(G), E) = (D_G, E) \ast_{D_G} \left( N \sum_{j=1}^{N} (W^*(\{L_{c_j}\}, D_G), E) \right).$$

Notice that $D_G = \mathbb{C}$ and $E = tr$, where $tr$ is the canonical trace, induced by the given conditional expectation. Therefore,

$$(W^*(G), tr) = \ast_{D_G} \left( N \sum_{j=1}^{N} (W^*(\{L_{c_j}\}), tr) \right).$$

Trivially, it contains the free group factor $(L(F_N), \tau)$, where $\tau = tr |_A$, where

$$A = W^* \left( \{L_{c_j} + L_{c_j}^* : j = 1, \ldots, N\} \right).$$

**Example 4.2.** Suppose that we have a directed graph $G$ with
\[ V(G) = \{v_1, v_2\} \text{ and } E(G) = \{l_1^1, l_1^2, l_2^1, l_2^2, l_3^1, c\}, \]

where
\[ l_j^1 = v_1 l_j^1 v_1, \text{ for } j = 1, 2, \quad c = v_1 e v_2 \]

and
\[ l_j^2 = v_2 l_j^2 v_2, \text{ for } j = 1, 2, 3. \]

Then \( D_G = \Delta_2 \subset M_2(\mathbb{C}) \) and
\[
(W^*(G), E) = D_G *_{D_G} \left( \bigotimes_{i=1}^{2} (W^*({\{L_i\}})) \otimes D_G, tr \otimes 1 \right)
\]
\[
+ D_G \left( \bigotimes_{j=1}^{3} (W^*({\{L_j\}})) \otimes D_G, tr \otimes 1 \right)
\]
\[
+ (W^*({\{L_e\}}, \Delta_2), E_2) \otimes (D_G, 1) .
\]

This shows that our graph \( W^*-\)probability space \( (W^*(G), E) \) contains free group factors
\[
L(F_2) \leq 2 \bigotimes_{i=1} (W^*({\{L_i\}}), tr)
\]
and
\[
L(F_3) \leq 3 \bigotimes_{j=1} (W^*({\{L_j\}}), tr)
\]

(See Section 4.1). Notice that these free group factors \( L(F_2) \) and \( L(F_3) \) are free over \( D_G \) in \( (W^*(G), E) \). Therefore, \( (W^*(G), E) \) contains \( L(F_2) *_{D_G} L(F_3) \).

Remark that
\[
L(F_2) *_{D_G} L(F_3) \neq L(F_2) * L(F_3) = L(F_3).
\]

**Example 4.3.** Let \( C_N \) be the circular graph with \( V(C_N) = \{v_1, ..., v_N\} \) and

\[
E(C_N) = \{e_j : e_j = v_j e_j v_{j+1}, j = 1, ..., N - 1, e_N = v_N e_N v_1\}.
\]

Then \( D_G = \Delta_N \). Thus we have that
\[
\text{Loop}(G) = \{ l := e_1 ... e_N \}
\]
equivalently,
\[
\text{loop}^c(G) = \{ w \in FP(G) : w \neq l^k, k \in \mathbb{N} \}.
\]

So, for the canonical conditional expectation \( E \), we have that
\[
(W^*(C_N), E) = (\Delta_N, 1)
\]
\[
*_{\Delta_N} (W^*({\{L_i\}}) \otimes \Delta_N, E \otimes 1)\]
\[^{\ast}\Delta_N \left( \left. ^{\ast}D_G \right|_{w \in \text{loop}(G)} (W^\ast\{L_w\}, \Delta_2) \otimes \Delta_N, E_2 \otimes 1 \right) \]

\[= (\Delta_N, E) \ast^{\ast} \Delta_N \left( \bigotimes_{j=1}^N (W^\ast\{L_{E_j}\}, \Delta_N), E \right) \].

Reference

[1] A. Nica, R-transform in Free Probability, IHP course note, available at www.math.uwaterloo.ca/~anica.
[2] A. Nica and R. Speicher, R-diagonal Pair-A Common Approach to Haar Unitaries and Circular Elements, (1995), www.mast.queensu.ca/~speicher.
[3] B. Solel, You can see the arrows in a Quiver Operator Algebras, (2000), preprint.
[4] A. Nica, D. Shlyakhtenko and R. Speicher, R-cyclic Families of Matrices in Free Probability, J. of Funct Anal, 188 (2002), 227-271.
[5] D. Shlyakhtenko, Some Applications of Freeness with Amalgamation, J. Reine Angew. Math, 500 (1998), 191-212.
[6] D. Voiculescu, K. Dykemma and A. Nica, Free Random Variables, CRM Monograph Series Vol 1 (1992).
[7] D. Voiculescu, Operations on Certain Non-commuting Operator-Valued Random Variables, Astérisque, 232 (1995), 243-275.
[10] D. Shlyakhtenko, A-Valued Semicircular Systems, J. of Funct Anal, 166 (1999), 1-47.
[10] D.W. Kribs and M.T. Jury, Ideal Structure in Free Semigroupoid Algebras from Directed Graphs, preprint.
[10] D.W. Kribs and S.C. Power, Free Semigroupoid Algebras, preprint.
[11] I. Cho, Amalgamated Boxed Convolution and Amalgamated R-transform Theory, (2002), preprint.
[12] I. Cho, The Tower of Amalgamated Noncommutative Probability Spaces, (2002), Preprint.
[13] I. Cho, Free Perturbed R-transform Theory, (2003), Preprint.
[14] I. Cho, Compatibility of a Noncommutative Probability Space and a Noncommutative Probability Space with Amalgamation, (2003), Preprint.
[15] I. Cho, An Example of Scalar-Valued Moments, Under Compatibility, (2003), Preprint.
[16] I. Cho, Graph $W^\ast$-Probability Theory, (2004), Preprint.
[17] I. Cho, Random Variables in Graph $W^\ast$-Probability Spaces, (2004), Preprint.
[18] I. Cho, Amalgamated Semicircular Systems in Graph $W^\ast$-Probability Spaces, (2004), Preprint.
[19] I. Cho, Amalgamated R-diagonal Pairs, (2004), Preprint.
[20] I. Cho, Compressed Random Variables in Graph $W^*$-Probability Spaces, (2004), Preprint.

[21] P. Śniady and R. Speicher, Continuous Family of Invariant Subspaces for R-diagonal Operators, Invent Math, 146, (2001) 329-363.

[22] R. Speicher, Combinatorial Theory of the Free Product with Amalgamation and Operator-Valued Free Probability Theory, AMS Mem, Vol 132, Num 627, (1998).

[23] R. Speicher, Combinatorics of Free Probability Theory IHP course note, available at www.mast.queensu.ca/~speicher.