Fault Detection Scheme for Grid-Forming Inverters in Islanded Droop-Controlled AC Microgrids

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Abstract—An observer-based fault detection scheme for grid-forming inverters operating in islanded droop-controlled AC microgrids is proposed. The detection scheme uses linear matrix inequalities (LMIs) as constraints with $\mathcal{H}_\infty$ optimization to achieve sensitiveness to faults and robustness against disturbances or parametric uncertainties. We explore a nonlinear inverter model formulation based on the less-restrictive one-sided Lipschitz and quadratic inner-boundedness conditions instead of a formulation based on the Lipschitz condition. In this sense, we aim to overcome the sensitivity of observer-based schemes to the Lipschitz constant. The relation between these two formulations for fault detection is analyzed theoretically. We find the deterministic matrix expressions of different faults, such as three-phase, actuator, inverter bridge, and sensor faults. The performance of the proposed detection scheme is assessed in an islanded AC microgrid with four grid-forming inverters.

Index Terms—AC microgrids, grid-forming inverter, fault detection, one-sided Lipschitz, quadratic inner-boundedness.

I. INTRODUCTION

In the past few decades, the development of inverter-based resources (IBRs) has expedited the successful integration of distributed generation into microgrids [1] [2]. Microgrids are small-scale power systems that can operate in either grid-connected or islanded mode. A microgrid sets off in islanding mode when it is far away from the main grid, or a considerable disturbance occurs in the power system [3]. The stability of islanded inverter-based ac microgrids is at risk when unwanted events occur, such as disturbances or faults. These events can cause a sizeable imbalance between the energy supply and demand and produce a blackout under certain conditions. Moreover, the disconnection from the bulk power system reduces the fault current strength in the islanded microgrid that helps halt the decline of voltage magnitude and frequency following a severe event [4]. The situation is further aggravated because power inverters reduce the available electrical inertia in the system [5]. In this sense, a fault detection and location (FDL) scheme that is robust against disturbances and sensitive to faults becomes vital to promote a stable microgrid performance and avoid the energy supply interruption.

A. Literature Review

Several fault detection techniques have been developed in the last decade to guarantee an accurate and reliable operation of microgrids [6]. Despite data-driven techniques demanding vast training data, they have gained popularity for fault detection due to the advances in microgrids software and hardware assets [7]–[9]. Methods based on signal processing have proven to be an effective policy for fault detection in microgrids; however, their solutions are difficult to generalize because they depend on a case-by-case basis [10]–[12]. Other strategies rely on signal patterns and local measurements close to the generation units to build a reliable fault detection module [13]–[17]. Besides these techniques, model-based methods are considered an option for FD when a system model is available. Recently, these methods have become a popular technique for fault detection because they are built upon the physical relationships that govern the system dynamics [18]. Among the model-based methods, the observer-based methods have received considerable interest due to their fast detection capabilities, low-cost implementation, and the availability of powerful tools for observer design [19]–[23].

A model-based nonlinear observer method for induction motor drives that detect and isolate faults is studied in [19]. In [20], the authors propose a nonlinear observer for a grid-connected PV circuit that supervises the values of unmodeled fault signatures for measuring output deviations that might suggest the existence of a fault. However, the effect of disturbances and parametric uncertainties was not considered. Fault detection and identification method enhanced with $\mathcal{H}_\infty$ optimization that is able to respond to different component faults is proposed in [21]. Nevertheless, the study is carried out for DC microgrids, and the state-space model is linear. A time-varying observer for predicting the states of synchronous machines is developed in [22]. The study relies on a linearized model computed at each time step, which increases the computational complexity of the proposed approach. In addition, model uncertainties and disturbances have not been considered. The authors in [23] consider a Lipschitz equivalent nonlinear model to pose an optimization problem with matrix inequalities constraints and $\mathcal{L}_\infty$ performance indexes for detecting faults in a microgrid composed by synchronous machines. However, the study does not consider parametric uncertainties and fault influence in the nonlinear function that...
arises in the states transition model. Moreover, the study does not address how the restrictiveness of the Lipschitz condition may affect the feasibility of the constraints expressed as matrix inequalities.

B. Motivation

Grid-forming inverters (GFMs) technology are essential for grid performance, significantly when renewable energy sources increase their share in the energy mix. Inverter-based resources are fundamentally changing the physics of the grid, which may lead to power systems stability concerns. In 2021, the Department of Energy funded with $25 million the Universal Interoperability for Grid-Forming Inverters (UNIFI) consortium. The consortium aims to expand its understanding of the operation and integration of inverter-based resources in electric power grids. In this sense, we explore a fault detection strategy for grid-forming inverters to promote their reliable and secure operation in electric grids, namely in islanded droop-controlled ac microgrids.

C. Contribution

A main drawback of observer-based fault detection schemes for nonlinear systems is their sensitivity to the constant of the Lipschitz condition, leading to conservative observer designs [24], [25]. In [24], the authors introduce observer design with two less conservative conditions, the one-sided Lipschitz and quadratic inner-boundedness conditions. A full-order and reduced-order design problem for one-sided Lipschitz nonlinear systems using a Ricatti equation approach is addressed in [25]. This work proposes a novel fault detection scheme for GFMs based on the above observers. The contributions of our work are listed as follows:

1) We show that LMI-based techniques combined with $\mathcal{H}_\infty$ optimization can be successfully applied for fault detection in GFMs connected to islanded droop-controlled AC microgrids.
2) Our proposed observer-based fault detection scheme is robust against disturbances while sensitive to faults.
3) We consider the one-sided Lipschitz and quadratic inner-boundedness conditions to derive the GFM nonlinear model instead of the Lipschitz equivalent formulation. Furthermore, we study the relation between these two formulations when used in our proposed fault detection scheme.
4) We derive the deterministic matrix representation for different types of faults acting on GFMs.
5) We consider the effect of disturbances and parametric uncertainties on the system model, including the nonlinear function in the states transition equation.

II. GRID-FORMING INVERTERS MODEL

The dynamics of GFMs are formulated using the $d-q$ reference frame according to [26]. The following equations are indexed with the subscript $i$ indicating the variables corresponding to the $i$-th GFM. The reference frame of each GFM rotates at the frequency $\omega_i$. The block diagram of the inverters considered in this work is shown in Fig. 1.

Fig. 1: Grid-forming inverter block diagram.

- **LC Filter and Output Connector**: The differential equations governing the LC filter and output connector are given as follows:

\[
\begin{align*}
\dot{i}_{odi} &= -\frac{1}{L_f} K_{PC} K_{PV} n_{Q_i} Q_i + \frac{1}{L_f} K_{PC} K_{IV} \phi_{di} + \frac{1}{L_f} K_{IC} \gamma_{di} - \frac{1}{L_f} K_{PV} v_{ni} i_{odi} - \omega_b i_{odi} - m_p P_i i_{odi} + \frac{1}{L_f} K_{PC} v_{ni} v_{odi} - \frac{1}{L_f} K_{PV} C_{fi} v_{odi} - \omega_m i_{odi} - \left( \frac{1}{L_f} + \frac{1}{L_f} + \frac{1}{L_f} \right) v_{odi} \\
i_{odi} &= \frac{1}{L_f} K_{PC} K_{IV} \phi_{qi} + \frac{1}{L_f} K_{IC} \gamma_{qi} + \omega_m i_{odi} - \frac{1}{L_f} \bar{P}_i v_{odi} + \frac{1}{L_f} \bar{P}_i v_{odi} + m_p P_i i_{odi} - \frac{1}{L_f} K_{PC} C_{fi} v_{odi} \\
\dot{v}_{odi} &= \omega_m v_{odi} - m_p P_i v_{odi} + \frac{1}{L_f} v_{odi} + \frac{1}{L_f} i_{odi} - \frac{1}{L_f} v_{odi} \\
i_{odi} &= -\frac{1}{L_f} i_{odi} - \omega_n i_{odi} - m_p P_i i_{odi} + \frac{1}{L_f} v_{odi} - \frac{1}{L_f} v_{odi} \\
i_{phi} &= -\frac{1}{L_f} i_{phi} - \omega_n i_{phi} - m_p P_i i_{phi} + \frac{1}{L_f} v_{phi} - \frac{1}{L_f} v_{phi}
\end{align*}
\]

- **Current Controller**: The differential-algebraic equations that govern the current controller are shown as follows:

\[
\begin{align*}
\dot{\gamma}_{di} &= F_i i_{odi} - \omega_m C_{fi} v_{odi} + K_{PV} v_{qi} - K_{PV} v_{odi} + K_{IV} \phi_{di} - i_{odi} \\
\dot{\gamma}_{qi} &= F_i i_{phi} + \omega_m C_{fi} v_{odi} - K_{PV} v_{phi} + K_{IV} \phi_{qi} + \omega_m \gamma_{di} - i_{phi} \\
\dot{v}_{odi}^* &= -\omega_b L_f i_{odi} + K_{PC} F_i i_{odi} - K_{PC} C_{fi} v_{odi} + K_{PC} v_{odi} + K_{PV} C_{fi} v_{odi} \\
\dot{v}_{phi}^* &= -K_{PC} v_{odi} + K_{PC} C_{fi} v_{odi} - K_{PC} C_{fi} v_{odi} - K_{PC} \phi_{di} - K_{IC} \gamma_{di} \\
\dot{v}_{phi}^* &= \omega_b L_f i_{phi} + K_{PC} F_i i_{phi} + K_{PC} C_{fi} v_{phi} - K_{PC} C_{fi} v_{phi} - K_{PC} \phi_{qi} - K_{IC} \gamma_{qi}
\end{align*}
\]

where $\gamma_{di}$ and $\gamma_{qi}$ are the auxiliary state variables defined for the current PI controllers, $v_{odi}^*$ and $v_{phi}^*$ are the output
The dynamical model of each GFM can be expressed in the power controllers and propagate to the rest of blocks. Notice that the nonlinearities in (7) arise from the droop and frequency control inputs, respectively, which can be expressed as [27]:

\[
\begin{align*}
\dot{P}_i &= -\omega_c i + \omega_c (v_{odi} i_{odi} + v_{ogi} i_{ogi}) \\
\dot{Q}_i &= -\omega_c i + \omega_c (v_{odi} i_{odi} - v_{odi} i_{odi}) \\
v_{odi}^* &= V_{ni} - n_i Q_i \\
v_{ogi}^* &= 0 \\
\end{align*}
\]

where \(\omega_c\) is the cut-off frequency of two low-pass filters that extract the fundamental component of the active and reactive power expressed as \(P_i\) and \(Q_i\) respectively. \(v_{odi}, v_{ogi}, i_{odi},\) and \(i_{odi}\) are the \(d-q\) components of the GFM’s output voltage and current respectively.

### Frequency
The reference frame of one of the GFMs is selected as the common reference frame rotating at the frequency \(\omega_{com}\). The power angle between the \(i\)-th reference frame and the common reference frame is governed by:

\[
\delta_i = \omega_i - \omega_{com}
\]

where \(\omega_i\) is the angular frequency of the \(i\)-th GFM.

The dynamical model of each GFM can be expressed in the following state-space form:

\[
\begin{align*}
\dot{x} &= Ax + \phi(x, u) + Bu \\
y &= Cx + Du
\end{align*}
\]

where,

\[
x = [\alpha_i P_i Q_i \phi_{odi} \phi_{ogi} \gamma_{odi} \gamma_{odi} i_{odi} i_{odi} i_{odi}]^T \\
u = [\omega_{com} V_{ni} v_{odi} v_{odi} v_{odi}]^T \\
y = [\alpha_i \omega_i v_{odi} i_{odi} i_{odi} i_{odi} i_{odi} v_{odi} v_{odi}]^T
\]

The parametric matrices \(A, B, C, D\), and the nonlinear function \(\phi(x, u)\) are formed according to equations (??)-(??). Notice that the nonlinearities in (7) arise from the droop and power controllers and propagate to the rest of blocks.

## III. Preliminaries

### A. Observers for Nonlinear Systems

Let us consider the nonlinear model shown in (7). In practice, dynamical physical systems are subject to disturbances and faults. In this sense, the system representation in (7) is modified to incorporate both disturbances and faults as follows:

\[
\begin{align*}
\dot{x} &= Ax + Bu + \eta_1(x, u) + \phi(x, u) + E_w w + E_f f \\
y &= Cx + Du + \eta_2(x, u) + F_w w + F_f f
\end{align*}
\]

where, \(w\) is an unknown input vector, \(f\) is the fault vector, \(E_w\) and \(F_w\) are constant disturbance matrices, \(E_f\) and \(F_f\) are deterministic fault matrices. The unknown functions \(\eta_1\) and \(\eta_2\) represent the model uncertainties which are assumed to be bounded, i.e.:

\[
\begin{align*}
||\eta_1(x, u)|| &\leq \eta_1 < \infty, \quad \forall x \in \mathcal{D}, \forall u \in \mathcal{U} \\
||\eta_2(x, u)|| &\leq \eta_2 < \infty, \quad \forall x \in \mathcal{D}, \forall u \in \mathcal{U}
\end{align*}
\]

Such bounds are unnecessary to be known for the observer design. We define two convex sets \(\mathcal{D}\) and \(\mathcal{U}\) that consist of the intersection of all upper and lower bounds of all states and control inputs, respectively, which can be expressed as [27]:

\[
\begin{align*}
\mathcal{D} &= [x_1^{\min}, x_1^{\max}] \times \cdots \times [x_n^{\min}, x_n^{\max}] \\
\mathcal{U} &= [u_1^{\min}, u_1^{\max}] \times \cdots \times [u_p^{\min}, u_p^{\max}]
\end{align*}
\]

The sets \(\mathcal{D}\) and \(\mathcal{U}\) make up the feasible operating region of the system. This is a sound assumption for an array of dynamical physical systems. A Luenberger observer is expressed as follows:

\[
\begin{align*}
\dot{\hat{x}} &= A\hat{x} + Bu + \phi(\hat{x}, u) + L(y - \hat{y}) \\
\hat{y} &= C\hat{x} + Du
\end{align*}
\]

where \(\hat{x}\) is the vector of estimated states, \(\hat{y}\) is the vector of estimated measurements, and \(L\) is the observer gain matrix to be designed. We make the following assumptions about the system in (8):

- The pair \((A, C)\) is observable
- The signal vectors \(w\) and \(f\) are square-integrable functions \(L_2\) satisfying:

\[
\begin{align*}
||w||_{2,[0,t]} &< \infty \\
||f||_{2,[0,t]} &< \infty
\end{align*}
\]

where, \(\cdot \quad ||.||_{2,[0,t]} = \left(\int_0^t ||.||_2^2 \, dt\right)^{\frac{1}{2}}\), and \(\cdot \quad ||.||_2\) is the euclidean norm.

### B. Residual Generation

Defining the observation error as \(e = x - \hat{x}\), and the measurements residual as \(r = W(y - \hat{y})\), the error dynamics can be represented as:

\[
\begin{align*}
\dot{e} &= (A - LC)e + \phi(x, u) - \phi(\hat{x}, u) \\
&\quad + (E_w - LF_w)w + (E_f - LF_f)f \\
r &= W(y - C\hat{x} - Du)
\end{align*}
\]
where $W$ is a weight matrix according to [28]. We can rewrite the error dynamics in (12) as follows:

$$
\begin{align*}
\dot{\epsilon} &= \bar{A}\epsilon + \bar{F}_w \bar{w} + \bar{E}_f \bar{f} \\
\epsilon &= \bar{C}\epsilon + \bar{F}_w \bar{w} + \bar{F}_f \bar{f}
\end{align*}
$$

(13)

with $\Phi = \phi(x, u) - \phi(\bar{x}, \bar{u})$ and:

$$
\begin{align*}
\bar{A} &= A - LC \\
\bar{E}_w &= E_w - LF_w \\
\bar{E}_f &= E_f - LF_f \\
\bar{C} &= WC \\
\bar{F}_w &= WF_w \\
\bar{F}_f &= WF_f
\end{align*}
$$

(14)

The residual $r$ in (13) is used for fault detection.

C. Mixed $\mathcal{H}_- / \mathcal{H}_\infty$ Optimization for Observer Design

The fault detection problem becomes challenging when there is no clear distinction between disturbances and faults. Such a situation may mislead the fault detection filter triggering false alarms. To deal with this issue, we considered the mixed $\mathcal{H}_- / \mathcal{H}_\infty$ optimization framework for observer design [23]. This framework aims to simultaneously make the fault detector filter robust against disturbances and sensitive to faults by satisfying the following criteria:

1) Robustness against disturbances $\bar{w}$:

$$
\|\bar{r}_w\|_{2, [0, t]} \leq \alpha \|\bar{w}\|_{2, [0, t]}
$$

(15)

2) Sensitivity to faults $\bar{f}$:

$$
\|\bar{r}_f\|_{2, [0, t]} \geq \beta \|\bar{f}\|_{2, [0, t]}
$$

(16)

where,

$$
\begin{align*}
\bar{r}_w &= \bar{C}\epsilon + \bar{F}_w \bar{w} \\
\bar{r}_f &= \bar{C}\epsilon + \bar{F}_f \bar{f}
\end{align*}
$$

(17)

(18)

In the next section, we pose the problem of designing a fault detection filter as a convex optimization problem where sufficient conditions in the form of linear matrix inequalities are derived to demonstrate the existence of such a filter. In this sense, we transform conditions (15) and (16) into linear matrix inequalities such that we can effectively incorporate them in the fault detection filter design.

D. Residual Evaluation Function and Threshold Computation

We follow the procedure presented in [29] to decide the occurrence of faults. The residual evaluation function $J$ is defined as $J = \|\bar{r}\|_2$. Ideally, we might be able to detect a fault when $J$ is non-zero. However, disturbances make $J$ non-zero even in the absence of faults. To cope with this issue, we compute a threshold for $J$ considering the system is in a fault-free condition and subjected to disturbances:

$$
J_{th} = \sup_{\bar{w} \in \mathbb{L}_2} \|\bar{r}_f\|_2
$$

(19)

where,

$$
\bar{r}_f = \bar{C}\epsilon + \bar{F}_w \bar{w}
$$

(20)

Hence, the logic for fault detection is defined as:

$$
\begin{align*}
J &\leq J_{th} \Rightarrow \text{No faults} \\
J &> J_{th} \Rightarrow \text{Fault detected}
\end{align*}
$$

(21)

The threshold is considered an essential component in any fault detection scheme. A trustworthy threshold reduced the chances of false alarms and missed detection, enhancing the fault detection capability.

IV. Observer Design for Nonlinear Systems

A. Lipschitz Nonlinear Systems

Lipschitz systems cover a wide range of nonlinear systems and provide flexibility for observer design purposes. Let us assume the nonlinear function $\phi(\cdot)$ satisfies the Lipschitz condition given by the following definition:

**Definition 1.** The function $\phi(\cdot)$ is said to be Lipschitz continuous if there exist a Lipschitz constant $\gamma \in \mathbb{R}^+$ such that

$$
\|\phi(x, u) - \phi(\bar{x}, \bar{u})\| \leq \gamma\|x - \bar{x}\|
$$

(22)

The following proposition provides sufficient conditions for the existence of a filter gain matrix $L$ based on the mixed $\mathcal{H}_- / \mathcal{H}_\infty$ optimization framework for Lipschitz nonlinear systems:

**Proposition 1.** Consider a Lipschitz constant $\gamma$, the system in (8) satisfying the condition in (11), the observer defined in (12), and the residual generator in (13). If there exist positive definite matrices $P$ and $Q$, filter and post filter gain matrices $L$ and $W$ respectively, strictly positive scalars $\alpha$, $\beta$, $\varepsilon_1$, $\varepsilon_2$ such that:

$$
\begin{align*}
[\Omega_1] &= P\bar{E}_w + C^T \bar{F}_w P & P \\
&\leq -\alpha^2 I + \bar{F}_w^T \bar{F}_w & 0 \\
&\leq -\varepsilon_1 I \\
[\Omega_2] &= Q\bar{E}_f + \bar{C}^T \bar{F}_f Q & Q \\
&\leq -\beta^2 I + \bar{F}_f^T \bar{F}_f & 0 \\
&\leq -\varepsilon_2 I
\end{align*}
$$

(23)

(24)

where $\Omega_1 = \bar{A}^T P + P\bar{A} + \bar{C}^T \bar{C} + \varepsilon_1 \gamma^2 I$, $\Omega_2 = \bar{A}^T Q + Q\bar{A} - \bar{C}^T \bar{C} + \varepsilon_2 \gamma^2 I$. Then,

1) The residual generator is stable and,
2) The observer solves the following mixed constraints simultaneously:

$$
\begin{align*}
\|\bar{r}_w\|_{2, [0, t]} &\leq \alpha \|\bar{w}\|_{2, [0, t]} \\
\|\bar{r}_f\|_{2, [0, t]} &\geq \beta \|\bar{f}\|_{2, [0, t]}
\end{align*}
$$

(25)

The proof of Proposition 1 can be found [23] where the authors proposed an LMI solution for the observer design problem considering a Lipschitz nonlinear system with the $\mathcal{H}_- / \mathcal{H}_\infty$ framework.

B. Observer Design for one-sided Lipschitz Nonlinear Systems

As discussed before, the Lipschitz constant is very sensitive to the system’s operating region $\mathcal{D} \times \mathcal{U}$ and the parameters defining the nonlinear function $\phi(\cdot)$. In this situation, Proposition 1 may likely fail to stabilize the dynamics of the residual generator and find the observer matrix gain $L$. An alternative is to consider the less restrictive one-sided Lipschitz
and quadratic inner-boundedness conditions, which generalize the Lipschitz condition \[24] [25].

**Definition 2.** The function \( \phi(\cdot) \) is said to be one-sided Lipschitz continuous if there exist \( \rho \in \mathbb{R} \) such that \( \forall x, \hat{x} \in \mathcal{D}, \forall u \in \mathcal{U} \):

\[
\langle \phi(x, u) - \phi(\hat{x}, u), x - \hat{x} \rangle \leq \rho \|x - \hat{x}\|^2
\]

**Definition 3.** The function \( \phi(\cdot) \) is said to be quadratic inner-bounded if there exist \( \delta, \varphi \in \mathbb{R} \) such that \( \forall x, \hat{x} \in \mathcal{D}, \forall u \in \mathcal{U} \):

\[
\|\phi(x, u) - \phi(\hat{x}, u)\|^2 \leq \delta \|x - \hat{x}\|^2 + \varphi(x - \hat{x}, \phi(x, u) - \phi(\hat{x}, u))
\]

**Remark.** Notice that the constants \( \rho, \delta, \varphi \) are not strictly positive compared to the Lipschitz constant \( \gamma \), making it beneficial for a less conservative design of observers.

In the following theorem, we propose sufficient conditions that incorporate the \( H_\infty \) framework, and inequalities \[25\] and \[26\] to demonstrate the existence of the filter matrix gain \( L \).

**Theorem 1.** Consider the constants \( \rho, \delta, \varphi \), the system in \[8\] satisfying the conditions in \[25\] and \[26\], the observer defined in \[11\] and the residual generator in \[12\]. If there exist positive definite matrices \( P, Q \) and \( \Omega \) and filter post filter gain matrices \( L \) and \( W \) respectively, strictly positive scalars \( \alpha, \beta, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \) such that:

\[
\begin{bmatrix}
\Omega_1 & P\bar{E}_w + \bar{C}^\top\bar{F}_w & P + \frac{1}{2}(\epsilon_2\varphi - \epsilon_1)I \\
* & -\alpha^2 I + \bar{F}_w^\top\bar{F}_w & 0 - \epsilon_2 I \\
* & * & -\epsilon_2 I
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
\Omega_2 & Q\bar{E}_f - \bar{C}^\top\bar{F}_f & Q + \frac{1}{2}(\epsilon_4\varphi - \epsilon_3)I \\
* & -\beta^2 I + \bar{F}_f^\top\bar{F}_f & 0 - \epsilon_4 I \\
* & * & -\epsilon_4 I
\end{bmatrix} < 0
\]

where \( \Omega_1 = \bar{A}^\top P + P\bar{A} + \bar{C}^\top\bar{C} + (\epsilon_1\rho + \epsilon_2\delta)I \), \( \Omega_2 = \bar{A}^\top Q + QA - \bar{C}^\top\bar{C} + (\epsilon_3\rho + \epsilon_4\delta)I \). Then,

1) The residual generator is stable and,
2) The observer solves the following mixed constraints simultaneously:

\[
\|r_w\|_{L_2, [0, t]} \leq \alpha\|w\|_{L_2, [0, t]}
\]

\[
\|r_f\|_{L_2, [0, t]} \geq \beta\|f\|_{L_2, [0, t]}
\]

The proof of **Theorem 1** is presented in appendix \[A\]. The matrix inequalities in \[27\] and \[28\] are not linear in terms of \( P, Q \) and \( L \). Both \( P \) and \( Q \) matrices are positive definite, hence we can define \( P = Q \). Also, if we set \( \bar{Y} = PL \), then the inequalities \[27\] and \[28\] become linear matrix inequalities. Once the problem is solved, we can recover the filter matrix gain using \( L = P^{-1}Y \).

**C. Relation between Lipschitz and one-sided Lipschitz Observer Design for Nonlinear Systems**

The relation between **Proposition 1** and **Theorem 1** is established in **Theorem 2** which indicates that the latter is less conservative than the former.

**Theorem 2.** Assume the nonlinear function \( \phi(x, u) \) is Lipschitz continuous with Lipschitz constant \( \gamma \), and there exist the gain matrices \( L, P, Q \), non-negative scalars \( \epsilon_1, \epsilon_2 \) such that the inequalities \[23\] and \[24\] hold. Then, there exist scalars \( \rho \in \mathbb{R}, \delta \in \mathbb{R}, \varphi \in \mathbb{R} \), non-negative scalars \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \) together with the matrices \( L, P, Q \) such that the inequalities in \[27\] and \[28\] are satisfied.

**Proof.** The Lipschitz condition implies the one-sided Lipschitz, and the quadratic inner-boundedness conditions \[24\]. Using the Cauchy-Schwarz inequality and the assumption that \( \phi(x, u) \) is Lipschitz continuous in \( \mathcal{D} \times \mathcal{U} \), we have:

\[
\|\phi(x, u) - \phi(\hat{x}, u), x - \hat{x}\| \leq \gamma\|x - \hat{x}\|^2
\]

which is equivalent to:

\[
-\gamma\|x - \hat{x}\|^2 \leq \langle \phi(x, u) - \phi(\hat{x}, u), x - \hat{x}\rangle \leq \gamma\|x - \hat{x}\|^2
\]

According to **Definition 3** the one-sided Lipschitz constant \( \rho \) can be set equal to \( \pm \gamma \). Similarly, using **Definition 4** we have that:

\[
\|\phi(x, u) - \phi(\hat{x}, u)\|^2 \leq \delta\|x - \hat{x}\|^2 + \varphi(x - \hat{x}, \phi(x, u) - \phi(\hat{x}, u))
\]

\[
\leq \delta\|x - \hat{x}\|^2 + \varphi\|x - \hat{x}\|^2
\]

\[
\leq (\delta + \varphi\rho)\|x - \hat{x}\|^2
\]

Based on **Definition 1** we can set:

\[
\gamma^2 = (\delta + \varphi\rho)
\]

then, we can obtain the value of \( \delta \). Notice that the previous inequality is valid if \( \varphi > 0 \).

By comparing the entries of matrices in inequalities \[27\] and \[28\], we can establish the following relations:

\[
\epsilon_2\varphi - \epsilon_1 = 0
\]

\[
\epsilon_2 = \epsilon_1
\]

\[
\epsilon_1\rho + \epsilon_2\delta = \epsilon_1\gamma^2
\]

Using simple mathematical manipulations, we obtain:

\[
\epsilon_1 = \epsilon_1\varphi
\]

\[
\epsilon_2 = \epsilon_1
\]

\[
\gamma^2 = \delta + \varphi\rho
\]

Given that \( \varphi > 0 \), then \( \epsilon_1 > 0 \). A similar analysis can be done using the entries of inequalities \[23\] and \[24\].

**Remark.** One-sided Lipschitz continuity does not imply Lipschitz continuity, because the Lipschitz condition is a two-sided inequality \[24\]. Hence, the converse of **Theorem 2** does not hold, which implies that **Theorem 1** is less conservative than **Proposition 1**. \(\square\)
V. FAULTS MODEL

This section introduces five faults that are likely to affect the operating point of a GFM. Also, we express the faults and disturbances as a deterministic matrix representation using matrices $E_f$, $F_f$, $E_w$, and $F_w$.

A. Three-phase Faults

A three-phase fault affects the voltage phasor at the point of common coupling (PCC) with the microgrid; such voltage is represented by the inputs $v_{bdi}$ and $v_{bqi}$. The three-phase fault is modeled as follows:

$$
v'_{bdi} = v_{bdi} + \Delta v_{bdi} \tag{29}
$$

$$
v'_{bqi} = v_{bqi} + \Delta v_{bqi} \tag{30}
$$

$$
f = [\Delta v_{bdi} \Delta v_{bqi}]^\top
$$

Hence, by replacing (29) & (30) in (7), we find the fault matrices $E_f$ and $F_f$ as:

$$
E_f = \begin{bmatrix} 0_{1 \times 11} & -\frac{1}{E_{c,i}} & 0 \\ 0_{1 \times 11} & 0 & -\frac{1}{E_{c,i}} \end{bmatrix}
$$

$$
F_f = \begin{bmatrix} 0_{7 \times 2} \end{bmatrix}
$$

B. Actuator Faults

The actuator signals consist of the inputs $\omega_{ni}$ and $V_{ni}$, which set the GFM’s desired frequency and voltage magnitude, respectively. The actuator faults are modeled as follows:

$$
\omega'_{ni} = \omega_{ni} + \Delta \omega_{ni} \tag{31}
$$

$$
V'_{ni} = V_{ni} + \Delta V_{ni} \tag{32}
$$

The actuator fault on $\omega_{ni}$ affects the linear term $Bu$ and the nonlinear function $\phi(x,u)$ of (7). Hence, by replacing (31) in (7), and after some mathematical manipulations, we have:

$$
f_{\omega_{ni}} = \begin{bmatrix} \Delta \omega_{ni} \Delta \omega_{ni} \omega_{odi} \Delta \omega_{ni}v_{odi} \Delta \omega_{ni}v_{odi} \end{bmatrix}
$$

$$
E_{f_{\omega_{ni}}} = \begin{bmatrix} 1 & 0_{1 \times 6} \\ 0_{6 \times 1} & 0_{6 \times 6} \\ 0_{6 \times 1} & A_{\omega_{ni}} \end{bmatrix}
$$

$$
A_{\omega_{ni}} = diag([1 \ -1 \ 1 \ -1 \ 1 \ -1])
$$

$$
F_{f_{\omega_{ni}}} = \begin{bmatrix} 0_{6 \times 1} \ 0_{1 \times 5} \ 0_{6 \times 5} \end{bmatrix}
$$

The actuator fault on $V_{ni}$ affects the linear term $Bu$ of (7) exclusively. Similarly, by replacing (32) in (7), and after mathematical manipulations, we find the following fault vector and matrices:

$$
f_{V_{ni}} = \begin{bmatrix} \Delta V_{ni} \end{bmatrix}
$$

$$
E_{f_{V_{ni}}} = \begin{bmatrix} 0_{1 \times 3} & 1 & 0 & K_{PV_i} & 0 & \frac{1}{L_f}K_{PC_i}K_{PV_i} & 0_{1 \times 5} \end{bmatrix}
$$

$$
F_{f_{V_{ni}}} = \begin{bmatrix} 0_{1 \times 3} & 1 & 0 & K_{PV_i} & 0 & K_{PC_i}K_{PV_i} \end{bmatrix}
$$

C. Inverter Bridge Faults

Ideally, the $d-q$ output voltages of the current controller $v_{odi}^*$ and $v_{odi}^*$ are equal to the inverter bridge’s output voltage $v_{odi}$ and $v_{odi}$:

$$
v_{odi} = \eta_{odi}v_{odi}^* \quad \eta_{odi} = 1
$$

$$
v_{odi} = \eta_{odi}v_{odi}^* \quad \eta_{odi} = 1
$$

In this sense, we model inverter bridge’s faults as a parametric fault representing a reduction in the bridge’s efficiency ($\Delta \eta_{odi}, \Delta \eta_{odi}$):

$$
v_{odi} = (1 - \Delta \eta_{odi})v_{odi}^* \quad \Delta \eta_{odi} \in (0, 1]
$$

$$
v_{odi} = (1 - \Delta \eta_{odi})v_{odi}^* \quad \Delta \eta_{odi} \in (0, 1]
$$

We find the fault vector $f$, fault matrices $E_f$, and $F_f$ by modifying the corresponding dynamical equations (1) and (2) that depend on $v_{odi}$ and $v_{odi}^*$. Consequently, we split the analysis into two parts.

1) Fault vector and fault matrices for $v_{odi}$:

$$
f_{v_{odi}} = \Delta \eta_{odi} [Q_i \phi_{odi} \gamma_{odi} \omega_{odi} \eta_{odi} \omega_{odi}]
$$

$$
E_{f_{v_{odi}}} = \begin{bmatrix} 0_{9 \times 7} \xi_{v_{odi}} \ 0_{9 \times 5} \end{bmatrix}
$$

$$
F_{f_{v_{odi}}} = \begin{bmatrix} 0_{9 \times 5} \tau_{v_{odi}} \ 0_{9 \times 1} \end{bmatrix}
$$

2) Fault vector and fault matrices for $v_{odi}$:

$$
f_{v_{odi}} = \Delta \eta_{odi} [Q_i \phi_{odi} \gamma_{odi} \omega_{odi} \eta_{odi} \omega_{odi}]
$$

$$
E_{f_{v_{odi}}} = \begin{bmatrix} 0_{7 \times 8} \xi \ 0_{7 \times 4} \end{bmatrix}
$$

$$
F_{f_{v_{odi}}} = \begin{bmatrix} 0_{7 \times 6} \tau_{v_{odi}} \end{bmatrix}
$$

The inverter bridge fault affects both efficiencies simultaneously, so we merge previous fault vectors and matrices as follows:

$$
f = [f_{v_{odi}} \ f_{v_{odi}}]\top
$$

$$
E_f = \begin{bmatrix} E_{f_{v_{odi}}} \ E_{f_{v_{odi}}} \end{bmatrix}
$$

$$
F_f = \begin{bmatrix} F_{f_{v_{odi}}} \ F_{f_{v_{odi}}} \end{bmatrix}
$$

D. Sensor Faults

Sensor faults are considered atypical perturbations in the system measurements due to sensor malfunctioning. Sensors
provide important information about the internal behavior of the system. Hence, a sensor fault may cause a severe impact on the system’s control schemes [30] [31]. A fault in a sensor affects the measurements equation in (7) only. We model sensor faults as step functions for each measurement, so the fault matrices are given as follows:

\[ E_f = [0_{13 \times 7}] \]
\[ F_f = [I_{7 \times 7}] \]

E. Disturbances

The disturbances are modeled as white Gaussian noise on both the process and measurements equations [23]. The fault vector and matrices are presented as follows:

\[ E_f = B \]
\[ F_f = D \]
\[ f = [f_1 \ f_2 \ f_3 \ f_4 \ f_5]^T \]

where, \( f_i \sim \mathcal{N}[-\lambda_i, \lambda_i], \forall i = 1, \ldots, 5. \)

VI. RESULTS AND EXPERIMENTS

We use the islanded ac microgrid shown in Fig. 8 to test our proposed fault detection framework. The microgrid consists of four GFMs with inductive RL lines connecting the buses; the parameters and data of the microgrid can be found in [26]. Droop curves perform the power-sharing coordination among the GFMs. We run the experiments using a MacBook Pro 2019, 2.8 GHz Intel Core i7 processor, 16 GB 2133 MHz LPDDR3 RAM, and 1 TB hard disk drive. The experiments were performed using MATLAB, Simulink, and the YALMIP toolbox with the semidefinite programming solvers LMILAB, MOSEK, Sedum, and SDPT3 to solve optimization problems described by Proposition 1 and Theorem 1. The Lipschitz, one-sided Lipschitz, and quadratic inner-boundedness constants are presented in Table I. We follow the procedure specified in [27] to compute the constants. For a single type of fault, the fault simulations happen in this order: at time \( t = 4 \) for GFM #1, at time \( t = 5 \) for GFM #2, at time \( t = 6 \) for GFM #3, and at time \( t = 7 \) for GFM #4. All the faults are cleared after 0.2 seconds, except for the sensor faults.

Fig. 2 shows the response of the residual norm using a one-sided Lipschitz observer when a three-phase fault occurs at the PCC of all GFMs. From this figure, we observe that the four residuals are sensitive to the three-phase faults with residuals less than two. A mentionable observation is that the residual for a single GFM is robust against disturbances and parametric uncertainties and the faults occurring at the PCC of other GFMs. Fig. 3 presents the response of the residual norms against a three-phase fault using a Lipschitz observer. Although the thresholds are less than one, such a detection scheme remains idle when the faults occur. Such a situation may be explained because the squared Lipschitz constant in the last term of \( \Omega_1 \) and \( \Omega_2 \) in Proposition 1 restricts the search space in the feasible set for the observer matrix gain \( L \) to make \( \Omega_1 \) and \( \Omega_2 \) negative definite. Moreover, even though the solvers output a solution for solving the feasibility problem in

| Fault type         | One-sided Lipschitz | Lipschitz |
|--------------------|---------------------|-----------|
|                    | \( \mu \) (seconds) | \( \sigma \) (seconds) | \( \mu \) (seconds) | \( \sigma \) (seconds) |
| Three-phase        | 1444                | 7.77      | 3130                | 3.30                |
| \( \omega_{ni} \)  | 1736                | 3.85      | 4076                | 14.89               |
| \( V_{ni} \)       | 1472                | 3.11      | 3784                | 12.76               |
| Inverter bridge    | 1686                | 6.91      | 1916                | 6.02                |
| Sensor             | 1262                | 5.50      | 1498                | 5.63                |

TABLE I: Lipschitz, one-sided Lipschitz and quadratic inner-boundedness constants.

| Fault type         | One-sided Lipschitz | Lipschitz |
|--------------------|---------------------|-----------|
|                    | \( \mu \) (seconds) | \( \sigma \) (seconds) | \( \mu \) (seconds) | \( \sigma \) (seconds) |
| Three-phase        | 1144                | 7.77      | 3130                | 3.30                |
| \( \omega_{ni} \)  | 1736                | 3.85      | 4076                | 14.89               |
| \( V_{ni} \)       | 1472                | 3.11      | 3784                | 12.76               |
| Inverter bridge    | 1686                | 6.91      | 1916                | 6.02                |
| Sensor             | 1262                | 5.50      | 1498                | 5.63                |

TABLE II: 5-fold average \( \mu \), and standard deviation \( \sigma \) of the required computational time of one-sided Lipschitz and Lipschitz observers.

Proposition 1 the solvers report that the problem is infeasible. On the contrary, the constants in Theorem 1 are not squared, the constants are allowed to be non-positive, and there are two scalar variables per LMI, leading to less-restricted search space for the matrix \( L \).

Actuator faults considering the input signal \( \omega_{ni} \) for GFMs #2 and #4, and the input signal \( V_{ni} \) for GFMs #1 and #3 are shown in Figs. 4 and 5 respectively. We notice that the behavior of the residual norms is very similar for both the one-sided Lipschitz and Lipschitz observers. The residual thresholds are a little less for the one-sided Lipschitz observer. Notice that the residual norm of each GFM remains unaffected by the occurrence of the faults in the other GFMs. There is a significant difference in the computational time between the two types of observers, as shown in Table II.

A similar situation occurs with the inverter bridge fault, as shown in Figs. 6. The residual norms for both observers exhibit a comparable response when the fault occurs. The threshold obtained for the one-sided Lipschitz observer is almost a tenth greater than the threshold computed using the Lipschitz observer regarding the GFM #1. Considering the GFM #4, the threshold for the Lipschitz observer is slightly greater than its counterpart. Table II shows that the computational time required by the one-sided Lipschitz observer is almost 12% less relative to the Lipschitz observer.

Fig. 7 shows the performance of the residual norms using both types of observers under a sensor fault. The residual threshold is slightly greater for the one-sided Lipschitz observer than the threshold for the Lipschitz observer considering GFM #2, whereas the opposite situation occurs when considering GFM #3. The disturbance amplitude is considerably less after the fault than before the fault; however, both observers allow to distinguish faults from disturbances. Similar to the experiments of the other faults, the one-sided Lipschitz observer requires less computational effort than the Lipschitz observer, as shown in Table II.
VII. CONCLUSIONS

In this work, we introduce an observer-based fault detection scheme for grid-forming inverters operating in islanded ac microgrids. The nonlinear model of the GFM is expressed as a Lipschitz and one-sided Lipschitz formulation. The two formulations are combined with LMI constraints and $\mathcal{H}_\infty$ optimization so the observer achieves sensitivity to faults and robustness against disturbances or parametric uncertainties. The relation between the Lipschitz and one-sided Lipschitz observer is studied theoretically and experimentally showing that the latter allows for less-restrictive observer design. We derive the matrix expressions of five different faults affecting GFM and consider them in the observer design process. Our proposed approach is evaluated using an islanded droop-controlled microgrid with four GFMs. The numerical results corroborate our proposed approach’s effectiveness that yields a reliable fault detection strategy for GFMs. Despite these results, we acknowledge that future work is needed to make our approach more robust. For instance, this work can be extended by studying the effect of an adaptive threshold when the microgrid incorporates a distributed cooperative secondary control layer. Also, a more detailed analysis of the eigenvalues of the error dynamics transition matrix to identify
Rewriting (36) as in a matrix inequality:

\[
\dot{V} \leq \begin{bmatrix} e & w & \Phi \end{bmatrix}^T \begin{bmatrix}
E_e e^T + \frac{1}{2} e^T \Phi e
& P E_w + \frac{1}{2} (e^T w - e^T \Phi e)
& (e^T P + e^T \Phi e) e

- e^T \Phi e + \frac{1}{2} (e^T w - e^T \Phi e) - e^T w

P + \frac{1}{2} (e^T w - e^T \Phi e) I
0
\end{bmatrix} \begin{bmatrix} e & w & \Phi \end{bmatrix}
\]

(37)

Hence, \( \dot{V} < 0 \) if \( M < 0 \). Considering that \( V > 0 \) and \( r_w \) as in (17), we can satisfy constraint (15) rewriting it as:

\[
\int_0^t ||r_w||_2^2 dt - \int_0^t \alpha^2 ||w||_2^2 dt + V \leq 0
\]

\[
\int_0^t ||r_w||_2^2 dt - \int_0^t \alpha^2 ||w||_2^2 dt + \int_0^t \dot{V} dt \leq 0
\]

\[
\int_0^t \left( r_w^T e - w^T \phi + e^T \Phi e + \alpha^2 w^T e + \dot{V} \right) dt \leq 0
\]

\[
- \alpha^2 w^T w + e^T C^T \dot{C} + e^T \dot{F} \dot{w} + w^T \dot{F} \dot{w} + \dot{V} \leq 0
\]

Assuming zero initial conditions, combining (38) with (37) we have:

\[
\dot{V} \leq \begin{bmatrix} e & w & \Phi \end{bmatrix}^T \begin{bmatrix}
E_e e^T + \frac{1}{2} e^T \Phi e
& P E_w + \frac{1}{2} (e^T w - e^T \Phi e)
& (e^T P + e^T \Phi e) e

- e^T \Phi e + \frac{1}{2} (e^T w - e^T \Phi e) - e^T w

P + \frac{1}{2} (e^T w - e^T \Phi e) I
0
\end{bmatrix} \begin{bmatrix} e & w & \Phi \end{bmatrix}
\]

(39)

If \( M^* < 0 \), then \( \dot{V} < 0 \) which means that the residual generator is asymptotically stable in the Lyapunov sense. Therefore, the inequality \( M^* < 0 \) is equivalent to (27).
Similarly, defining the Lyapunov function $Y = e^t Q e$ with $Q > 0$ to demonstrate the internal stability of [12] in the disturbance-free case:

$$
\dot{Y} = e^t (A^T P + P A) e + e^t P E f + f^T E_f^T P e + P \Phi e + e^T \Phi P e$$

Adding (34) and (35) with strictly positive scalars $\epsilon_3$ and $\epsilon_4$ to the right-hand side of (40):

$$
\dot{Y} \leq e^t (A^T P + P A) e + e^t P E f + f^T E_f^T P e + P \Phi e + e^T \Phi P e + \epsilon_3 (\text{Pe} - e^T \Phi e) + e_4 \delta e^T e$$

And proceeding in a similar way as above for constraint (10) with Schur complements and mathematical manipulations yield the matrix inequality in (28).

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