Interference of a thermal Tonks gas on a ring

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A nonzero temperature generalization of the Fermi-Bose mapping theorem is used to study the exact quantum statistical dynamics of a one-dimensional gas of impenetrable bosons on a ring. We investigate the interference produced when an initially trapped gas localized on one side of the ring is released, split via an optical-dipole grating, and recombined on the other side of the ring. Nonzero temperature is shown not to be a limitation to obtaining high visibility fringes.

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A fundamental assumption at the heart of current proposals to realize integrated atom sensors is that the guided atom wavepackets will display interference phenomena when they are split, propagated, and recombined. On the other hand it is well known that the coherence properties of atomic gases are affected by dimensionality, there being no true off-diagonal-long-range order in less than three dimensions, and this raises the issue of interference in restricted geometries. Motivated by recent theoretical arguments demonstrating that several stimulated processes for matter waves such as four-wave mixing, superradiance, and matter-wave amplification can be achieved in degenerate fermion gases as well as in Bose-condensed gases, we have recently shown that a one-dimensional gas of hard-core bosons, or Tonks gas, at zero temperature can exhibit high visibility interference fringes. These results suggest that the same mechanisms might be capable of overcoming the weakening of interference due to thermal excitation. The quantum Tonks gas is realized in a regime essentially opposite from that required for BEC, namely, the regime of low temperatures and densities and large positive scattering lengths where the transverse mode becomes frozen and the many-body Schrödinger dynamics becomes exactly solvable via a Fermi-Bose mapping theorem. We shall extend that theorem so as to obtain exact results for a model atom interferometer using a thermal Tonks gas. In particular, we apply our results to the interference produced when an initially trapped gas localized on one side of a ring is released, split via an optical-dipole grating, and recombined on the other side of the ring. Such a study is currently of relevance due to experimental efforts to fabricate atomic waveguides for matter wave interferometers.

Model: The model consists of a 1D gas of N₀ hard core bosonic atoms on a ring. This situation can be realized physically using a toroidal trap of high aspect ratio R = 2L/ℓ₀ where 2L is the torus circumference and ℓ₀ the transverse oscillator length ℓ₀ = \( \sqrt{\hbar/m\omega₀} \) with \( \omega₀ \) the frequency of transverse oscillations, assumed to be harmonic. The longitudinal (circumferential) motion can be described by a 1D coordinate \( x \) along the ring with periodic boundary conditions. The pure-state quantum dynamics of the system is described by the time-dependent many-body Schrödinger equation (TDMBSE)

\[
i\hbar\frac{\partial}{\partial t} \hat{\Psi}_B = \hat{H}\hat{\Psi}_B\]  

with Hamiltonian

\[
\hat{H} = -\frac{\hbar^2}{2m} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + V(x_1, \ldots, x_N; t).
\]

Here \( x_j \) is the 1D position of the \( j^{th} \) particle, \( \psi_B(x_1, \ldots, x_N; t) \) is the N-particle wave function with periodic boundary conditions,

\[
\Psi_B(x_1 \ldots x_j \pm 2L \ldots x_N; t) = \Psi_B(x_1 \ldots x_j \ldots x_N; t),
\]

which is also symmetric under exchange of any two particle coordinates, as is the many-body potential \( V \). We consider the case of impenetrable two-particle interactions, the so-called Tonks-gas regime, and this is conveniently treated as a constraint on allowed wave functions:

\[
\Psi_B = 0 \quad \text{if} \quad x_j = x_k, \quad 1 \leq j < k \leq N,
\]

rather than as an infinite contribution to \( V \), which then consists of all other (finite) interactions and external potentials.

We assume preparation of the system such that its initial state at time \( t = 0 \) is expressed by a statistical density operator \( \hat{\rho}_0 = \sum_\alpha w_\alpha |\Psi_{B\alpha}(0)\rangle \langle \Psi_{B\alpha}(0) | \) where the \( |\Psi_{B\alpha}(0)\rangle \) are a complete set of orthonormal many-boson states with label \( \alpha \) that stands for a set of quantum numbers, and the \( w_\alpha \) are nonnegative statistical weights summing to unity. Then the statistical average of any observable \( \hat{O} \) at any later time \( t > 0 \) is

\[
\langle \hat{O}(t) \rangle = \sum_\alpha w_\alpha \langle \Psi_{B\alpha}(t) | \hat{O} | \Psi_{B\alpha}(t) \rangle
\]

where the states \( |\Psi_{B\alpha}(t)\rangle \) evolve according to the TDMBSE with Hamiltonian.

Statistical Fermi-Bose mapping theorem: The Fermi-Bose mapping theorem for pure quantum states is a mapping \( |\Psi_B\rangle = A|\Psi_F\rangle \) from the Hilbert space of one-dimensional many-fermion states \( |\Psi_F\rangle \) to

\[
|\Psi_B\rangle = A|\Psi_F\rangle
\]
that of one-dimensional many-boson states $|\Psi_B\rangle$, holding if the interparticle interaction contains a hard core and the Hamiltonian is of the form $H$. In Schrödinger representation the mapping operator $A$ on $N$-particle states consists of multiplication by the “unit antisymmetric function”

$$A(x_1, \cdots, x_N) = \prod_{1 \leq k < l \leq N} \text{sgn}(x_k - x_l),$$

where $\text{sgn}(x)$ is the algebraic sign of $x = x_k - x_l$, i.e., it is $+1(-1)$ if $x > 0(x < 0)$. Since $A$ is constant except at nodes of the wavefunctions, the mapping converts antisymmetric fermionic solutions $\Psi_F(x_1, \cdots, x_N; t)$ of the TDMBSE into symmetric bosonic solutions $\Psi_B(x_1, \cdots, x_N; t)$ satisfying the same TDMBSE with the same Hamiltonian $\hat{H}$, and satisfying the same boundary conditions and constraint (3).

In the Olshanii limit [5] (low density, tightly confining atom gas) to that of the impenetrable point Bose gas, which is the Olshanii limit [5] (low density, tightly confining atom gas) to that of the same observable in an ensemble, with $\sum_{\nu} \rho_{\nu,v}(x,t) |\phi_{\nu,v}(x,t)|^2$ being the sum of the energies $\epsilon_{\nu}$ of the single particle states $\phi_{\nu,v}(x,t)$. Some manipulation of the sums leads to the familiar form

$$\langle \rho(x,t) \rangle = \sum_{n=0}^{\infty} f_n |\phi_n(x,t)|^2,$$

where $\sum_{n=0}^{\infty} f_n = N_0$ with $N_0$ atoms at temperature $T$, and for $t \leq t_0$ they are confined to a narrow segment of the ring by a trapping potential $V(x,t) < 0^\circ$ assumed harmonic, with natural frequency $\omega$, over the spatial extent of the initial trapped gas. The grand canonical ensemble in Eqs. (8) and (9) describing the state of the ring at $t = 0$ is therefore characterized by the single particle energies $\epsilon_n = (n + 1/2) \hbar \omega$. The basic configuration is shown schematically in Fig. 3(b).

In order to discuss the time-evolution of the system, we design the normal modes of a class of 1D harmonic oscillators by $u_n(x-x, \omega)$ where $x$ are their mean positions and the parameter $\omega$ relates their widths $x_0^2 = 1 + w^2$ to the width of the initial trap $x_0 = \sqrt{\hbar/m\omega}$. Thus the modes of the initial trap potential are $u_n(x-x, \omega = 0).$ We choose the circumferential coordinates to have the trap center at $x = L = -L$. On unwrapping the ring about $x = 0$ in Fig. 3(b), the initial Hermite-Gaussian orbitals are split into two parts at the ends of the fundamental periodicity cell $[-L, L]$, and can be written as

$$\phi_n(x,t = 0) = u_n(x + L, 0) + u_n(x - L, 0).$$

The asymptotic dependence of the harmonic oscilla-
tor modes \( u_n(x) \approx \cos(\sqrt{2n}x/x_0) \) suggests the definition of a critical wavevector for a thermal distribution, 
\[ k = \sqrt{2n}/x_0, \]
which would give a measure of the characteristic rate of expansion of atoms released from the initial longitudinal trap. We take it to correspond to \( f_n = f_0/2 \), so that for temperatures large compared to the Fermi temperature \((T_F)\) the critical wavevector coincides with the thermal wavevector \( \vec{k} \approx \sqrt{2kB/T}/\hbar \omega/x_0 \), and for \( T \rightarrow 0 \) we get the wavevector corresponding to the highest occupied mode \( \vec{k} \rightarrow \sqrt{2n_0}/x_0 \). If the initially trapped gas is allowed to expand freely along the ring, the critical wave-vector gives a measure of the time to wrap around the ring \( t_{\text{wrap}} = L/(\hbar k/m) \).

Some necessary conditions need to be satisfied in order that the initial atom cloud be accurately represented by a Tonks gas \([1, 2]\). First, the longitudinal energy must be small compared with the transverse excitation energy: at zero temperature this requires \( N\hbar \omega \ll \hbar \omega_0 \) and at finite temperatures \( k_BT \ll \hbar \omega_0 \). Second, for the initial gas to be accurately described as an impenetrable gas of bosons we require \( \vec{k}|a_{1D}| \ll 1 \), i.e. \( k_BT \ll \hbar \omega_0^2a^2/2 \) \([3]\).

**Optical-dipole grating:** In order to produce interference from the initial trapped gas we turn off the harmonic trap at \( t = 0 \) and apply a temporally short but intense spatially periodic potential of wavevector \( k \). This spatially periodic grating may be produced over the spatial extent of the trapped gas, for example, using intersecting and off-resonant pulsed laser beams to produce an optical-dipole grating whose wavevector may be tuned by varying the intersection angle. The applied periodic potential then produces counter-propagating scattered atomic waves, or daughter waves, from the initial gas, or mother, with momenta \( \pm \hbar k \) and these recombine on the opposite side of the ring at a time \( t_r = L/(\hbar k/m) = (\vec{k}/k)t_{\text{wrap}} \). For times \( t < 2t_{\text{wrap}} \) the mother packet has not yet encircled the ring and the gas expands freely as if on an infinite line.

Following Rojo et al. \([4]\) we use a delta-function approximation for the short pulse excitation of the periodic grating at \( t = 0 \), and implement the optical-dipole grating using a phase-imprinting scheme according to which each orbital just after the pulse at \( t = 0^+ \) is changed to
\[ \phi_n(x, 0^+) = e^{iη\cos[k(x-L)]} \phi_n(x, 0) \]
\[ \approx \left[ 1 + \frac{iη}{2} \left( e^{ik(x-L)} + e^{-ik(x-L)} \right) \right] \phi_n(x, 0), \tag{11} \]
where \( η \) is the grating amplitude. The second line assumes \( η < 1 \) in which case the optical-dipole grating predominantly produces two scattered copies of the initial mode travelling to the left and right with wavevectors \( \pm k \) in addition to the initial parent mode \( \phi_n(x, 0) \).

**Interference on a ring:** We wish to examine the presence or absence of interference at the opposite side of the ring at time \( t = t_r \) and how this depends on the temperature \( T \) and the grating vector \( k \). After the grating is applied the trapping potential is turned off, \( V(x, t > 0) = 0 \), so the subsequent quantum dynamics of the system is traced by freely propagating each orbital \( \phi_n(x, 0^+) \). Using the time evolved orbitals \( \phi_n(x, t_r) \) in Eq. (11) we obtain the atom density at time \( t_r \) as a sum of three terms
\[ \rho(x, t_r) = \rho_m(x, t_r) + \rho_d(x, t_r) + \rho_{md}(x, t_r), \]
with
\[ \rho_m(x, t_r) = \sum_{n=0}^{\infty} f_n \left[ u_n^2(x+L, \omega t_r) + u_n^2(x-L, \omega t_r) \right] \]
from an initial harmonic trap of frequency \( \nu_0 \) Hz. This corresponds to an oscillator length of
\[
x_0 = \frac{2\pi^{3/2}}{\xi_0(1+\omega^2)}
\]
\( \mu \) atoms on a ring of circumference \( 2L \). We find that for temperatures and applied wavevectors
\( T \), \( \bar{k} \)
producing the expression for it above. In conclusion, we have shown that even at nonzero temperature a strongly interacting 1D gas of impenetrable bosons can show high visibility interference fringes on a ring. This remarkable result is of importance for current schemes to realize integrated atom interferometers in that it shows that neither many-body interactions nor nonzero temperature are fundamental limitations, in 1D at least. Our results followed from a generalization of the zero temperature Fermi-Bose mapping which maps the strongly interacting 1D boson problem to a 1D gas of free fermions, and it is an open problem whether our conclusions can be extended to 2D and 3D interferometers.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fringe_visibility.png}
\caption{Estimate of the visibility in the region of detection, as a function of temperature for three values of the optical-dipole grating wavevector (a) \( x_0 \bar{k} = 20 \), (b) 30 and (c) 45.}
\end{figure}

The first term \( \rho_m \) is due to the expanding mother packet overlapping with itself, \( \rho_d \) is due to the overlap of the daughters, and \( \rho_{md} \) arises from the overlap of the mother and the daughter packets. In general, for \( t_{\text{wrap}} > t_r \), this is desirable so that the mother packet does not wrap at the same time as the daughters recombine, \( \rho_{md} \) is small compared to the other terms: This term is present in our simulation below but we have refrained from reproducing the expression for it above.

As a concrete example we consider \( N_0 = 100 \) sodium atoms on a ring of circumference \( 2L = 100x_0 \) released from an initial harmonic trap of frequency \( \omega = 2\pi \times 4 \) Hz. This corresponds to an oscillator length of \( x_0 = 10.4 \) \( \mu \)m and Fermi temperature \( T_F = 19.1 \) nK. In general, we find that for temperatures and applied wavevectors such that \( k \ll \bar{k} \) minimal interference fringes result when the daughter packets recombine at time \( t_r \). Recalling that the wrap time for the mother packet is given by \( t_{\text{wrap}} = (k/\bar{k})t_r \), then for \( k \ll \bar{k} \) the mother packet wraps around significantly by the time the daughters recombine, so \( \rho_m \) provides a pedestal for interference fringes due to the overlap of the daughters \( \rho_d \), leading to reduced fringe visibility \[ \text{[8]} \]. In contrast for \( k \geq \bar{k} \) pronounced interference fringes can appear when the daughter packets are recombined. This is illustrated in Fig.\[ \text{[8]} \]a) where we fix the wavevector \( k \) of the optical grating and vary the temperature or alternatively \( k \). Unit visibility fringes are seen in Fig.\[ \text{[8]} \]b,c) where \( T = 20, 45 \) nK.

By noting that around \( x = 0 \) the fringe pattern maximum of \( \rho(x, t_r) \) is \( \rho_m(0, t_r) + \rho_d(0, t_r) \), and the minimum \( \rho_m(0, t_r) \), we obtain a measure of the fringe visibility as
\[
\Gamma = \frac{\rho_d(0, t_r)}{2\rho_m(0, t_r) + \rho_d(0, t_r)}.
\]
This fringe visibility is plotted in Fig.\[ \text{[8]} \] as a function of temperature for three values of the grating vector \( k \), with the clear and physically obvious trend that higher temperatures require a higher grating wavevector to obtain high visibility interference fringes. Physically there are of course limitations: For sodium atoms, the condition for a Tonks-gas is \( T \ll (2 \times 10^{-5} \times \nu_0)^2 \) nK, for a transverse trapping frequency \( \omega_0 = 2\pi \nu_0 \) Hz, and current experimental limits are about \( \nu_0 \sim 2 \times 10^4 \) Hz \[ \text{[23]} \].
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