On critical double phase Kirchhoff problems with singular nonlinearity

Rakesh Arora¹ · Alessio Fiscella² · Tuhina Mukherjee³ · Patrick Winkert⁴

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Abstract
The paper deals with the following double phase problem

\[- m \left( \int_{\Omega} \left( \frac{|\nabla u|^p}{p} + a(x) \frac{|\nabla u|^q}{q} \right) \, dx \right) \text{div} \left( |\nabla u|^{p-2} \nabla u + a(x) |\nabla u|^{q-2} \nabla u \right) = \lambda u^{-r} + u^{p^*-1} \]

\[= \lambda u^{-\gamma} + u^{p^* - 1} \quad \text{in} \quad \Omega, \]

\[u > 0 \quad \text{in} \quad \Omega, \]

\[u = 0 \quad \text{on} \quad \partial \Omega, \]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with Lipschitz boundary \( \partial \Omega \), \( N \geq 2 \), \( m \) represents a Kirchhoff coefficient, \( 1 < p < q < p^* \) with \( p^* = Np/(N - p) \) being the critical Sobolev exponent to \( p \), a bounded weight \( a(\cdot) \geq 0 \), \( \lambda > 0 \) and \( \gamma \in (0, 1) \). By the Nehari manifold approach, we establish the existence of at least one weak solution.

Keywords Critical growth · Double phase operator · Fibering method · Nehari manifold · Nonlocal Kirchhoff term · Singular problem

Mathematics Subject Classification 35A15 · 35J15 · 35J60 · 35J62 · 35J75
1 Introduction

In this paper, we combine the effects of a nonlocal Kirchhoff coefficient and a double phase operator with a singular term and a critical Sobolev nonlinearity. Precisely, we study the problem

\[-m \left[ \int_{\Omega} \left( \frac{|\nabla u|^p}{p} + a(x) \frac{|\nabla u|^q}{q} \right) \, dx \right] \mathcal{L}_{p,q}^\alpha(u) = \lambda u^{-\gamma} + u^{p^* - 1} \quad \text{in } \Omega,\]

\[u > 0 \quad \text{in } \Omega,\]

\[u = 0 \quad \text{on } \partial \Omega,\]

where along the paper, and without further mentioning, \(\Omega \subset \mathbb{R}^N\) is a bounded domain with Lipschitz boundary \(\partial \Omega\), dimension \(N \geq 2\), \(\lambda > 0\) is a real parameter and exponent \(\gamma \in (0, 1)\). The main operator \(\mathcal{L}_{p,q}^u\) is the so-called double phase operator given by

\[\mathcal{L}_{p,q}^u(u) := \text{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u), \quad u \in W^{1,N}_0(\Omega),\] (1.1)

with \(W^{1,N}_0(\Omega)\) being the homogeneous Musielak-Orlicz Sobolev space where we assume that

\((h_1)\) \(1 < p < N, \ p < q < p^*\) and \(0 \leq a(\cdot) \in L^\infty(\Omega)\) with \(p^*\) being the critical Sobolev exponent to \(p\) given by

\[p^* = \frac{Np}{N - p}.\] (1.2)

While the nonlocal term \(m\) in \((P_\lambda)\) denotes a Kirchhoff coefficient satisfying

\((h_2)\) \(m : [0, \infty) \to [0, \infty)\) is a continuous function defined by

\[m(t) = a_0 + b_0 t^{\theta - 1} \quad \text{for all } t \geq 0,\]

where \(a_0 \geq 0, b_0 > 0\) with \(\theta \in [1, p^*/q)\).

Problem \((P_\lambda)\) is said to be of double phase type because of the presence of two different elliptic growths \(p\) and \(q\). The study of double phase problems and related functionals originates from the seminal paper by Zhikov [25], where he introduced for the first time in literature the related energy functional to (1.1) defined by

\[\omega \mapsto \int_{\Omega} (|\nabla \omega|^p + a(x)|\nabla \omega|^q) \, dx.\] (1.3)

This kind of functional has been used to describe models for strongly anisotropic materials in the context of homogenization and elasticity. Indeed, the modulating coefficient \(a(\cdot)\) dictates the geometry of composites made of two different materials with distinct power hardening exponents \(p\) and \(q\). From the mathematical point of view, the behavior of (1.3) is related to the sets on which the weight function \(a(\cdot)\) vanishes or not. In this direction, Zhikov found other mathematical applications for (1.3) in the study of duality theory and of the Lavrentiev gap phenomenon, as shown in [26, 27]. Also, (1.3) belongs to the class of the integral functionals with nonstandard growth condition, according to Marcellini’s terminology [22, 23]. Following this line of research, Mingione et al. provide famous results.
in the regularity theory of local minimizers of (1.3), see, for example, the works of Baroni-Colombo-Mingione [4, 5] and Colombo-Mingione [9, 10].

Starting from [25], several authors studied existence and multiplicity results for nonlinear problems driven by (1.1) with the help of different variational techniques. In particular, Fiscella-Pinamonti [18] introduced two different double phase problems of Kirchhoff type, with the same variational structure set in $W_0^{1,\gamma}(\Omega)$. By the mountain pass and fountain theorems, existence and multiplicity results are provided in [18]. Following this direction, in [17] Fiscella-Marino-Pinamonti-Verzellesi consider some classes of Kirchhoff type problems on a double phase setting but with nonlinear boundary conditions. Combining variational methods, truncation arguments and topological tools, different multiplicity results are established. Recently, the authors [2] were able to study a Kirchhoff problem like $(P_{\lambda})$, but involving a subcritical term. By a suitable Nehari manifold decomposition, the existence of two different solutions is obtained in [12]. We also mention the works of Cammaroto-Vilasi [7], Isernia-Repovš [20] and Ambrosio-Isernia [1] for Kirchhoff type problems driven by the $p(\cdot)$-Laplacian or the $(p, q)$-Laplacian.

The main novelty, as well as the main difficulty, of problem $(P_{\lambda})$ is the presence of a critical Sobolev nonlinearity. Indeed, in order to overcome the lack of compactness at critical levels arising from the presence of the critical term in $(P_{\lambda})$, the same fibering analysis used in [2] cannot work. For this, we exploit other variational tools inspired by more recent situations as in [14]. For this, Farkas-Fiscella-Winkert [14] used a suitable convergence analysis of gradients in order to handle the critical Sobolev nonlinearity of problem

$$-\text{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = \lambda |u|^{q-2}u + |u|^{r-2}u \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega.$$  

Following this direction, we mention [15, 16] concerning existence results for critical double phase problems involving a singular term and defined on Minkowski spaces in terms of Finsler manifolds, that is driven by the Finsler double phase operator

$$L^F_{p,q}(u) := \text{div}(F^{p-1}(\nabla u)\nabla F(\nabla u) + a(x)F^{q-1}(\nabla u)\nabla F(\nabla u)),$$

where $(\mathbb{R}^N, F)$ stands for a Minkowski space. While, Crespo-Blanco-Papageorgiou-Winkert [12] consider a nonhomogeneous singular Neumann double phase problem with critical growth on the boundary, given by

$$-\text{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) + a(x)u^{p-1} = \zeta(x)u^{-\gamma} + \lambda u^{q-1} \quad \text{in } \Omega,$$

$$|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u \cdot v = -\beta(x)u^{p-1} \quad \text{on } \partial \Omega. \quad (1.4)$$

By the fibering approach introduced by Drábek-Pohozaev [13] along with a Nehari manifold decomposition, the existence of at least two solutions of (1.4) is obtained in [12].

Inspired by the above papers, we solve problem $(P_{\lambda})$ by a variational approach. Indeed, a function $u \in W_0^{1,\gamma}(\Omega)$ is said to be a weak solution of problem $(P_{\lambda})$ if $u^{-\gamma}\varphi \in L^1(\Omega)$, $u > 0$ a.e. in $\Omega$ and

$$m(\phi_{\gamma}(\nabla u))\left\langle L^a_{p,q}(u), \varphi \right\rangle = \lambda \int_{\Omega} u^{-\gamma}\varphi \, dx + \int_{\Omega} u^{p-1}\varphi \, dx$$

is satisfied for all $\varphi \in W_0^{1,\gamma}(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{1,\gamma}(\Omega)$ and its dual space $W_0^{1,\gamma}(\Omega)^\ast$. In particular, the weak solutions of $(P_{\lambda})$ are the critical points of the energy functional $J_{\lambda} : W_0^{1,\gamma}(\Omega) \to \mathbb{R}$ given by
\[ J_{\lambda}(u) = \left[ a_{0}\phi_{H}(\nabla u) + \frac{b_{0}}{\theta}\phi_{H}(\nabla u) \right] - \frac{\lambda}{1-\gamma} \int_{\Omega} |u|^{1-\gamma} \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx, \]

for any \( u \in W^{1,\mathcal{H}}_{0}(\Omega) \), where
\[
\phi_{H}(u) = \int_{\Omega} \left( \frac{|u|^p}{p} + a(x) \frac{|u|^q}{q} \right) \, dx.
\]

Hence, the main result reads as follows.

**Theorem 1.1** Let hypotheses \((h_1)-(h_2)\) be satisfied. Then there exists \( \lambda^* > 0 \) such that for all \( \lambda \in (0, \lambda^*] \) problem \((P)\) has at least one weak solution \( u^* \) such that \( J_{\lambda}(u^*) < 0 \).

The proof of Theorem 1.1 is based on a suitable minimization argument on the Nehari manifold. For this, we extract a minimizing sequence whose energy values converge to a negative number. However, in order to verify that the sequence actually converges to a solution of \((P)\) we need a truncation argument combined with a delicate gradient analysis, inspired by [14].

The paper is organized as follows. In Sect. 2, we recall the main properties of Musielak-Orlicz Sobolev spaces \( W^{1,\mathcal{H}}_{0}(\Omega) \) and state the main embeddings concerning these spaces. Section 3 gives a detailed analysis of the fibering map, presents the main properties of suitable subsets of the Nehari manifold and finally shows the existence of a weak solution of problem \((P)\).

## 2 Preliminaries

In this section, we will present the main properties and embedding results for Musielak-Orlicz Sobolev spaces. First, we denote by \( L^{r}(\Omega) = L^{r}(\Omega;\mathbb{R}) \) and \( L^{r}(\Omega;\mathbb{R}^N) \) the usual Lebesgue spaces with the norm ||·||_r, and the corresponding Sobolev space \( W^{1,r}_{0}(\Omega) \) is equipped with the norm ||∇·||_r, for \( 1 \leq r \leq \infty \).

Suppose hypothesis \((h_1)\) and consider the nonlinear function \( \mathcal{H} : \Omega \times [0, \infty) \to [0, \infty) \) defined by
\[
\mathcal{H}(x,t) = t^p + a(x)t^q.
\]

The Musielak-Orlicz Lebesgue space \( L^{\mathcal{H}}(\Omega) \) is given by
\[
L^{\mathcal{H}}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \phi_{\mathcal{H}}(u) < \infty \right\}
\]

equipped with the Luxemburg norm
\[
\|u\|_{\mathcal{H}} = \inf \left\{ \tau > 0 \mid \phi_{\mathcal{H}} \left( \frac{u}{\tau} \right) \leq 1 \right\},
\]

where the modular function is given by
\[
\phi_{\mathcal{H}}(u) := \int_{\Omega} \mathcal{H}(x,|u|) \, dx = \int_{\Omega} \left( |u|^p + a(x)|u|^q \right) \, dx.
\]
Next, we recall the relation between the norm $\| \cdot \|_{\mathcal{H}}$ and the modular function $\varphi_{\mathcal{H}}$, see Liu-Dai [21, Proposition 2.1] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [11, Proposition 2.13].

**Proposition 2.1** Let $(h_1)$ be satisfied, $u \in L^{\mathcal{H}}(\Omega)$ and $c > 0$. Then the following hold:

(i) If $u \neq 0$, then $\| u \|_{\mathcal{H}} = c$ if and only if $\varphi_{\mathcal{H}}(\frac{u}{c}) = 1$;
(ii) $\| u \|_{\mathcal{H}} < 1$ (resp. $> 1$, $= 1$) if and only if $\varphi_{\mathcal{H}}(u) < 1$ (resp. $> 1$, $= 1$);
(iii) If $\| u \|_{\mathcal{H}} < 1$, then $\| u \|_{\mathcal{H}}^q \leq \varphi_{\mathcal{H}}(u) \leq \| u \|_{\mathcal{H}}^p$;
(iv) If $\| u \|_{\mathcal{H}} > 1$, then $\| u \|_{\mathcal{H}} \leq \varphi_{\mathcal{H}}(u) \leq \| u \|_{\mathcal{H}}$;
(v) $\| u \|_{\mathcal{H}} \to 0$ if and only if $\varphi_{\mathcal{H}}(u) \to 0$;
(vi) $\| u \|_{\mathcal{H}} \to \infty$ if and only if $\varphi_{\mathcal{H}}(u) \to \infty$.

Moreover, we define the weighted space

$$L^q_a(\Omega) = \left\{ u : \Omega \to \mathbb{R} \mid u \text{ is measurable and } \int_{\Omega} a(x)|u|^q \, dx < \infty \right\}$$

equipped with the seminorm

$$\| u \|_{q,a} = \left( \int_{\Omega} a(x)|u|^q \, dx \right)^{\frac{1}{q}}.$$

The corresponding Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{ u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\}$$
equipped with the norm

$$\| u \|_{1,\mathcal{H}} = \| \nabla u \|_{\mathcal{H}} + \| u \|_{\mathcal{H}},$$
where $\| \nabla u \|_{\mathcal{H}} = \| |\nabla u| \|_{\mathcal{H}}$. In addition, we denote by $W^{1,\mathcal{H}}_0(\Omega)$ the completion of $C^\infty_0(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$. Thanks to hypothesis $(h_1)$, we know that

$$\| u \| = \| \nabla u \|_{\mathcal{H}},$$
is an equivalent norm in $W^{1,\mathcal{H}}_0(\Omega)$, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [11, Proposition 2.16(ii)]. Furthermore, it is known that $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$ are uniformly convex and so reflexive Banach spaces, see Colasuonno-Squassina [8, Proposition 2.14] or Harjulehto-Hästö [19, Theorem 6.1.4].

Finally, we recall some useful embedding results for the spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$, see Colasuonno-Squassina [8, Proposition 2.15] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [11, Propositions 2.17 and 2.19].

**Proposition 2.2** Let $(h_1)$ be satisfied and let $p^*$ be the critical exponent to $p$ given in (1.2). Then the following embeddings hold:

(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega) \hookrightarrow W^{1,r}_0(\Omega)$ are continuous for all $r \in [1, p^*)$.
(ii) $W^{1,n} (\Omega) \hookrightarrow L^r (\Omega)$ is continuous for all $r \in [1, p^*)$ and compact for all $r \in [1, p^*)$;
(iii) $L^q (\Omega) \hookrightarrow L^q_0 (\Omega)$ is continuous;
(iv) $L^q (\Omega) \hookrightarrow L^q_0 (\Omega)$ is continuous.

3 Proof the main result

In order to solve problem $(P_\alpha)$, we apply a minimization argument for $J_\alpha$ on a suitable subset of $W^{1,n} _0 (\Omega)$. For this, we define the fibering function $\psi_u : [0, \infty) \to \mathbb{R}$ defined by

$$\psi_u (t) = J_\alpha (tu) \quad \text{for all } t \geq 0,$$

which gives

$$\psi_u (t) = \left[ a_0 \phi_{\nu t} (t \nabla u) + \frac{b_0}{\theta} \phi_{\nu t} (t \nabla u) \right] - \lambda \frac{1-\gamma}{1-\gamma} \int_\Omega |u|^{1-\gamma} \, dx - \frac{\rho^*}{\rho^*} \int_\Omega |u|^{p^*} \, dx.$$

It is easy to see that $\psi_u \in C^\infty ((0, \infty))$. In particular, we have for $t > 0$

$$\psi'_u (t) = \left[ a_0 \phi_{\nu t} (t \nabla u) \right] \left( (p-1)t^{p-2} \| \nabla u \|^p_p + (q-1)t^{q-2} \| \nabla u \|^q_{q,a} \right)$$

and

$$\psi''_u (t) = \left[ a_0 \phi_{\nu t} (t \nabla u) \right] \left( (p-1)t^{p-2} \| \nabla u \|^p_p + (q-1)t^{q-2} \| \nabla u \|^q_{q,a} \right)$$

Thus, we can introduce the Nehari manifold related to our problem which is defined by

$$\mathcal{N}_\alpha = \left\{ u \in W^{1,n} _0 (\Omega) \setminus \{0\} : \psi'_u (1) = 0 \right\}.$$ 

In particular, we have $u \in \mathcal{N}_\alpha$ if and only if

$$\left[ a_0 \phi_{\nu t} (\nabla u) \right] \left( \| \nabla u \|^p_p + \| \nabla u \|^q_{q,a} \right) = \lambda \int_\Omega |u|^{1-\gamma} \, dx + \int_\Omega |u|^{p^*} \, dx.$$

Also $tu \in \mathcal{N}_\alpha$ if and only if $\psi'_u (1) = 0$. Observe that $\mathcal{N}_\alpha$ contains all weak solutions of $(P_\alpha)$. Moreover, we define the following subsets of $\mathcal{N}_\alpha$

$$\mathcal{N}_\alpha^+ = \left\{ u \in \mathcal{N}_\alpha : \psi''_u (1) > 0 \right\} \quad \text{and} \quad \mathcal{N}_\alpha^- = \left\{ u \in \mathcal{N}_\alpha : \psi''_u (1) < 0 \right\}.$$

In contrast to [2] we are not going to study the set $\mathcal{N}_\alpha^- = \left\{ u \in \mathcal{N}_\alpha : \psi''_u (1) < 0 \right\}$. The next Lemma can be shown as in [2, Lemmas 3.1 and 3.2] replacing $r$ by $p^*$.

Lemma 3.1 Let hypotheses (h1)-(h2) be satisfied.
(i) The functional $J_{\lambda}^{\alpha} \mid_{N_\lambda}$ is coercive and bounded from below for any $\lambda > 0$.

(ii) There exists $\Lambda_1 > 0$ such that $N_\lambda^\alpha = \emptyset$ for all $\lambda \in (0, \Lambda_1)$.

Let $S$ be the best Sobolev constant in $W_0^{1,p}(\Omega)$ defined as

$$S = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|\nabla u\|^p_p}{\|u\|^p_p}. \quad (3.1)$$

Note that we can write $\psi_u'(t)$ in the form

$$\psi_u'(t) = t^{-\gamma} \left( \sigma_u(t) - \lambda \int_\Omega |u|^{1-\gamma} \, dx \right), \quad t > 0,$$

where

$$\sigma_u(t) = [a_0 + b_0 \phi_{H_1}^{\beta-1}(t\nabla u)] \left( p^{\beta-1+\gamma} \|\nabla u\|^p_p + r^{\beta-1+\gamma} \|\nabla u\|^q_{q,a} \right) - p^{\beta-1+\gamma} \int_\Omega |u|^p \, dx.$$ 

From this definition we see that $t u \in N_\lambda^\alpha$ if and only if

$$\sigma_u(t) = \lambda \int_\Omega |u|^{1-\gamma} \, dx. \quad (3.3)$$

The next Lemma shows that $N_\lambda^\alpha$ is nonempty whenever $\lambda$ is sufficiently small.

**Lemma 3.2** Let hypotheses (h1)-(h2) be satisfied and let $u \in W_0^{1,J}(\Omega) \setminus \{0\}$. Then there exist $\Lambda_2 > 0$ and unique $t_\lambda^a < t_{\max}^a < t_\lambda^a$ such that

$$0 < \sigma_u(t_\lambda^a) = (t_\lambda^a)^{\gamma} \psi_u''(t_\lambda^a), \quad 0 > \sigma_u(t_{\max}^a) = (t_{\max}^a)\gamma \psi_u''(t_{\max}^a) \quad \text{and} \quad \sigma_u(t_{\max}^a) = \max_{t > 0} \sigma_u(t)$$

whenever $\lambda \in (0, \Lambda_2)$. In particular, $t_\lambda^a u \in N_\lambda^\alpha$ for $\lambda \in (0, \Lambda_2)$.

**Proof** For $u \in W_0^{1,J}(\Omega) \setminus \{0\}$ the equation

$$0 = \sigma_u'(t) = [a_0 + b_0 \phi_{H_1}^{\beta-1}(t\nabla u)] \left( (p - 1 + \gamma) t^{\beta-2+\gamma} \|\nabla u\|^p_p + (q - 1 + \gamma) t^{\beta-2+\gamma} \|\nabla u\|^q_{q,a} \right)$$

$$+ b_0(\theta - 1) \phi_{H_1}^{\theta-2}(t\nabla u) \left( t^{\beta-1+\gamma} \|\nabla u\|^p_p + t^{\beta-1+\gamma} \|\nabla u\|^q_{q,a} \right)$$

$$- (t^{\theta-1+\gamma} \|\nabla u\|^p_p + t^{\theta-1+\gamma} \|\nabla u\|^q_{q,a})$$

can be equivalently written as

$$[a_0 + b_0 \phi_{H_1}^{\beta-1}(t\nabla u)] \left( (p - 1 + \gamma) t^{\beta-2+\gamma} \|\nabla u\|^p_p + (q - 1 + \gamma) t^{\beta-2+\gamma} \|\nabla u\|^q_{q,a} \right)$$

$$+ b_0(\theta - 1) \phi_{H_1}^{\theta-2}(t\nabla u) \left( t^{\beta-1+\gamma} \|\nabla u\|^p_p + t^{\beta-1+\gamma} \|\nabla u\|^q_{q,a} \right)$$

$$= (t^{\theta-1+\gamma} \|\nabla u\|^p_p + t^{\theta-1+\gamma} \|\nabla u\|^q_{q,a}).$$ 

$(3.4)$
From $p^* > q\theta$ and $\theta \geq 1$ we see that
\[
    p(\theta - 1) + p - p^* < \min \{p(\theta - 1) + q - p^*, q(\theta - 1) + p - p^*\}
\leq \max \{p(\theta - 1) + q - p^*, q(\theta - 1) + p - p^*\}
\leq q(\theta - 1) + q - p^* = q\theta - p^* < 0.
\tag{3.5}
\]

We denote the left-hand side of (3.4) by
\[
T_u(t) = [a_0 + b_0 \phi_{\gamma t}^\beta(t\nabla u)] \left[ (p - 1 + \gamma)p^{r - p^*} \|\nabla u\|_p^p + (q - 1 + \gamma)q^{r^* - p^*} \|\nabla u\|_q^q \right] + b_0(\theta - 1) \phi_{\gamma t}^{\beta - 2}(t\nabla u) \left( p^{r - p^* + 1} \|\nabla u\|_p^p + q^{r^* - p^* + 1} \|\nabla u\|_q^q \right)
\left( p^{r - 1} \|\nabla u\|_p^p + q^{r^* - 1} \|\nabla u\|_q^q \right).
\]

Then, from (3.5) and $0 < \gamma < 1 < p < q < p^*$, we know that

(i) $\lim_{t \to 0^+} T_u(t) = \infty$,
(ii) $\lim_{t \to \infty} T_u(t) = 0$,
(iii) $T_u(t) < 0$ for all $t > 0$.

From the intermediate value theorem along with (i) and (ii) we can find $r_{\max}^u > 0$ such that (3.4) holds. In addition, (iii) implies that $r_{\max}^u$ is unique due to the injectivity of $T_u$. Moreover, if we consider $\sigma_u(t) > 0$, then in place of (3.4) we get
\[
T_u(t) > (p^* - 1 + \gamma) \int_\Omega |u|^{p^*} \ dx.
\]

Since $T_u$ is strictly decreasing, this holds for all $t < r_{\max}^u$. The same can be said for $\sigma_u'(t) > 0$ and $t > r_{\max}^u$. Hence, $\sigma_u$ is injective in $(0, r_{\max}^u)$ and in $(r_{\max}^u, \infty)$. Furthermore,
\[
\sigma_u(r_{\max}^u) = \max_{t > 0} \sigma_u(t)
\]
with the global maximum $r_{\max}^u > 0$ of $\sigma_u$. Moreover, we have
\[
\lim_{t \to 0^+} \sigma_u(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} \sigma_u(t) = -\infty.
\]

Applying the estimate $p \phi_{\gamma t}(\nabla u) \geq \|\nabla u\|_p^p$ we obtain
\[
\sigma_u'(t) \geq \frac{b_0}{p^{\beta - 1}} (p\theta - 1 + \gamma)p^{\beta - 2 + \gamma} \|\nabla u\|_p^p - (p^* - 1 + \gamma)p^{\beta - 2 + \gamma} \int_\Omega |u|^{p^*} \ dx, \tag{3.6}
\]
which by using Hölder’s inequality and (3.1) results in
\[
r_{\max}^u \geq \frac{1}{\|\nabla u\|_p^p} \left( \frac{b_0(p\theta - 1 + \gamma)S^\beta}{p^{\beta - 1}(p^* - 1 + \gamma)} \right)^{\frac{1}{p^{\beta - \beta^*}}} := r_{\max}^u_0. \tag{3.7}
\]

Note that $\sigma_u$ is increasing on $(0, r_{\max}^u)$. Hence from $p \phi_{\gamma t}(\nabla u) \geq \|\nabla u\|_p^p$, $p < q$, Hölder’s inequality, (3.1) and the representation of $r_{\max}^u_0$ in (3.7) we have
Due to (3.3) we have
whenever
From the considerations above we conclude that
Finally, since
Let hypotheses
Lemma 3.3
\[ D_1 = D_1(\lambda) > 0 \]
such that
\[ \sigma_u(t^*_{\text{max}}) \geq \sigma_u(t^*_{0}) \geq \frac{b_0}{p^{\theta-1}}(t^*_{0})^{p\theta-1+\gamma} \left\| \nabla u \right\|_{p}^{p\theta} - (t^*_{0})^{p\gamma-1+\gamma} \int_{\Omega} |u|^{p}\ dx \]
\[ \geq (t^*_{0})^{p\theta-1+\gamma} \left\| \nabla u \right\|_{p}^{p\theta} \left( \frac{b_0}{p^{\theta-1}} - \frac{(t^*_{0})^{p\gamma-p\gamma} S^\frac{p\gamma}{p}}{\left\| \nabla u \right\|_{p}^{p\gamma-p\gamma}} \right) \]
\[ \geq \left( \frac{p^*-p\theta}{p^*-1+\gamma} \right) \frac{b_0}{p^{\theta-1}}(t^*_{0})^{p\theta-1+\gamma} \left\| \nabla u \right\|_{p}^{p\theta} \]
\[ > \left( \frac{p^*-q\theta}{p^*-1+\gamma} \right) \frac{b_0}{p^{\theta-1}}(t^*_{0})^{p\theta-1+\gamma} \left\| \nabla u \right\|_{p}^{p\theta} \]
\[ = \left( \frac{p^*-q\theta}{p^*-1+\gamma} \right) \left\| \nabla u \right\|_{p}^{1-\gamma} \frac{b_0}{p^{\theta-1}} \left( \frac{b_0(p\theta-1+\gamma)S^\frac{p\gamma}{p}}{p^{\theta-1}(p^*-1+\gamma)} \right) \]
\[ \geq \Lambda_2 \int_{\Omega} |u|^{1-\gamma} \ dx, \]
where \( \Lambda_2 \) is given by
\[ \Lambda_2 = \frac{b_0}{p^{\theta-1}} \left( \frac{p^*-q\theta}{p^*-1+\gamma} \right) \left( \frac{b_0(p\theta-1+\gamma)S^\frac{p\gamma}{p}}{p^{\theta-1}(p^*-1+\gamma)} \right) \frac{S^{-\frac{1-\gamma}{p-1+\gamma}}}{|\Omega|^{\frac{1-\gamma}{p-1+\gamma}}}. \]
From the considerations above we conclude that
\[ \sigma_u(t^*_{\text{max}}) > \lambda \int_{\Omega} |u|^{1-\gamma} \ dx \]
whenever \( \lambda \in (0, \Lambda_2) \). Since \( \sigma_u \) is injective in \((0, t^*_{\text{max}})\) and in \((t^*_{\text{max}}, \infty)\), we can find unique \( t^*_1, t^*_2 > 0 \) such that
\[ \sigma_u(t^*_i) = \lambda \int_{\Omega} |u|^{1-\gamma} \ dx = \sigma_u(t^*_2) \quad \text{with} \quad \sigma'_u(t^*_i) < 0 < \sigma'_u(t^*_2). \]
Due to (3.3) we have \( t^*_1 u \in \mathcal{N}_\lambda^+ \). Then, from the representation in (3.2), we observe that
\[ \sigma_u'(t^*_i) = t^* \psi''_u(t^*_i) + \gamma t^* \gamma^{-1} \psi'_u(t). \]
Finally, since \( \psi'_u(t^*_1) = \psi'_u(t^*_2) = 0 \) and \( \sigma'_u(t^*_2) < 0 < \sigma'_u(t^*_1) \) we derive that
\[ 0 < \sigma'_u(t^*_1) = (t^*_1)^\gamma \psi''_u(t^*_1) \quad \text{and} \quad 0 > \sigma'_u(t^*_2) = (t^*_2)^\gamma \psi''_u(t^*_2). \]
This shows, in particular, that \( t^*_1 u \in \mathcal{N}_\lambda^+ \) for \( \lambda \in (0, \Lambda_2) \).

Next we show that the modular \( \phi_H(\nabla \cdot) \) is upper bounded with respect to the elements of \( \mathcal{N}_\lambda^+ \). The proof is similar to that in [2, Proposition 3.4] and so we omitted it.

**Lemma 3.3** Let hypotheses (h_1)-(h_2) be satisfied. Then there exist \( \Lambda_3 > 0 \) and constant \( D_1 = D_1(\lambda) > 0 \) such that
\[ \phi_H(\nabla u) = \left\| \nabla u \right\|_{p}^{p} + \left\| \nabla u \right\|_{q,\alpha}^{q} < D_1 \]
for every \( u \in \mathcal{N}_\lambda^+ \) and for every \( \lambda \in (0, \Lambda_3) \).
By Lemma 3.1(ii), we observe that $\mathcal{N}_\lambda^+$ is closed in $W_0^{1,\mathcal{H}}(\Omega)$ for $\lambda > 0$ small enough. We define

$$\Theta_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u).$$

The next proposition shows that $\Theta_\lambda^+ < 0$. We refer to [2, Proposition 4.1] for its proof.

**Proposition 3.4** Let hypotheses (h_1)-(h_2) be satisfied and let $\lambda \in (0, \min\{\Lambda_1, \Lambda_2\})$, with $\Lambda_1$, $\Lambda_2$ given in Lemmas 3.1(ii) and 3.2. Then $\Theta_\lambda^+ < 0$.

Based on the implicit function theorem in its version stated in Berger [6, p. 115] we can prove the following Lemma which proof is similar to the one in [2, Lemma 4.2].

**Lemma 3.5** Let hypotheses (h_1)-(h_2) be satisfied and let $\lambda > 0$. Let us consider $u \in \mathcal{N}_\lambda^+$. Then there exist $\epsilon > 0$ and a continuous function $\zeta : B_\epsilon(0) \to (0, \infty)$ such that

$$\zeta(0) = 1 \quad \text{and} \quad \zeta(v)(u + v) \in \mathcal{N}_\lambda^+ \quad \text{for all} \quad v \in B_\epsilon(0),$$

where $B_\epsilon(0) := \{v \in W_0^{1,\mathcal{H}}(\Omega) : ||v|| < \epsilon\}$.

Now, we set $\Lambda^* := \min\{\Lambda_1, \Lambda_2, \Lambda_3\}$ with $\Lambda_1$, $\Lambda_2$ and $\Lambda_3 > 0$ given in Lemmas 3.1(ii), 3.2 and 3.3. Let $\lambda \in (0, \Lambda^*)$. Applying Ekeland’s variational principle, we obtain a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{N}_\lambda^+$ satisfying

$$\Theta_\lambda^+ < J_\lambda(u_n) < \Theta_\lambda^+ + \frac{1}{n}, \quad (3.8)$$

$$J_\lambda(u) \geq J_\lambda(u_n) + \frac{||u - u_n||}{n} \quad (3.9)$$

for any $u \in \mathcal{N}_\lambda^+$. By Lemma 3.1(i), we know that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Hence, by Proposition 2.2(ii) along with the reflexivity of $W_0^{1,\mathcal{H}}(\Omega)$, there exist a subsequence, still denoted by $\{u_n\}_{n \in \mathbb{N}}$, and an element $u_\lambda \in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$u_n \rightharpoonup u_\lambda \quad \text{in} \quad W_0^{1,\mathcal{H}}(\Omega), \quad u_n \to u_\lambda \quad \text{in} \quad L^s(\Omega) \quad \text{and} \quad u_n \to u_\lambda \quad \text{a.e. in} \quad \Omega \quad (3.10)$$

for any $s \in [1, p^*)$. By the coercivity given in Lemma 3.1(i), we can assume that there exist $E_1, E_2 \geq 0$ such that

$$\lim_{n \to \infty} ||u_n||_p^p = E_1 \quad \text{and} \quad \lim_{n \to \infty} ||u_n||_q^q = E_2. \quad (3.11)$$

We get the following technical results.

**Lemma 3.6** Let hypotheses (h_1)-(h_2) be satisfied, let $\lambda \in (0, \Lambda^*)$ and let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{N}_\lambda^+$ be a sequence satisfying (3.8)-(3.9). Then $u_\lambda \neq 0$.

**Proof** Let us assume by contradiction that $u_\lambda = 0$. Then $\psi_{u_\lambda}'(1) = 0$ implies

$$[a_0 + b_0 \phi_H^{\theta - 1}(\nabla u_n)] (||u_n||_p^p + ||u_n||_q^q) - \lambda \int_\Omega |u_n|^{1-\gamma} \,dx - \int_\Omega |u_n|^p \,dx = 0.$$
Using (3.10), (3.11) and letting \( n \to \infty \), we get
\[
\left[a_0 + b_0 \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{-1} \right] (E_1 + E_2) - d^{p^*} = 0,
\]
where we set
\[
\lim_{n \to \infty} \int_{\Omega} |u_n|^{p^*} \, dx =: d^{p^*} \geq 0.
\]
Moreover by (3.8) we have
\[
\lim_{n \to \infty} J_\lambda(u_n) = \Theta^+ < 0,
\]
which implies that
\[
\left[a_0 \left( \frac{E_1}{p} + \frac{E_2}{q} \right) + b_0 \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta} \right] - \frac{d^{p^*}}{p^*} < 0.
\]
Recall that \( E_1, E_2 \geq 0 \). Then, taking the value of \( d^{p^*} \) from (3.12) into (3.13), we derive that
\[
\left[a_0 \left( \frac{E_1}{p} + \frac{E_2}{q} \right) + b_0 \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta} \right] - \left[a_0 + b_0 \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta-1} \right] \frac{E_1 + E_2}{p^*} < 0.
\]
This implies
\[
a_0 \left[ \frac{E_1}{p} + \frac{E_2}{q} - \frac{E_1 + E_2}{p^*} \right] + b_0 \left[ \frac{1}{\theta} \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta} - \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta-1} \frac{E_1 + E_2}{p^*} \right] < 0
\]
and so
\[
a_0 \left[ \frac{1}{p} \left( \frac{1}{\theta} \left( \frac{1}{p} - \frac{1}{p^*} \right) + \frac{E_2}{q} \left( \frac{1}{q} - \frac{1}{p^*} \right) \right) \right] + b_0 \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta-1}
\]
\[
\left[ \frac{1}{p} \left( \frac{1}{\theta} \left( \frac{1}{p} - \frac{1}{p^*} \right) + \frac{E_2}{q} \left( \frac{1}{q} - \frac{1}{p^*} \right) \right) \right] < 0,
\]
which is a contradiction because of \( p < q \leq q \theta < p^* \).

\[\square\]

**Lemma 3.7** Let hypotheses (h1)–(h2) be satisfied, let \( \lambda \in (0, \Lambda^+) \) and let \( \{u_n\}_{n \in \mathbb{N}} \subset \mathcal{N}_\lambda^+ \) be a sequence satisfying (3.8)–(3.9). Then \( \liminf_{n \to \infty} \psi''_{u_n}(1) > 0 \), that is,
\[
\liminf_{n \to \infty} \left\{ a_0 + b_0 \phi^{-1}_H(\nabla u_n) \left[ (p - 1 + \gamma) \|\nabla u_n\|_p^p + (q - 1 + \gamma) \|\nabla u_n\|_q^q \right] \right. \\
+ b_0 (\theta - 1) \phi^{-2}_H(\nabla u_n) \left[ (\|\nabla u_n\|_p^p + \|\nabla u_n\|_q^q)^2 - (p^* - 1 + \gamma) \right] \int_{\Omega} |u_n|^{p^*} \, dx \left. \right\} > 0.
\]

**Proof** Since \( \{u_n\}_{n \in \mathbb{N}} \subset \mathcal{N}_\lambda^+ \), we have \( \psi'_{u_n}(1) = 0 \) and \( \psi''_{u_n}(1) > 0 \), that is,
\[ [a_0 + b_0\phi^{\theta-1}_\theta(\nabla u_n)] \left[ (p - 1 + \gamma)\|\nabla u_n\|_p^p + (q - 1 + \gamma)\|\nabla u_n\|_{q,a}^q \right] + b_0(\theta - 1)\phi^{\theta-2}_\theta(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 - (p^* - 1 + \gamma)\int_\Omega |u_n|^{p^*} \, dx > 0 \]

and

\[ [a_0 + b_0\phi^{\theta-1}_\theta(\nabla u_n)] \left[ (p - p^*)\|\nabla u_n\|_p^p + (q - p^*)\|\nabla u_n\|_{q,a}^q \right] + b_0(\theta - 1)\phi^{\theta-2}_\theta(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 + \lambda(p^* - 1 + \gamma)\int_\Omega |u_n|^{1-\gamma} \, dx > 0. \]

(3.14)

Thus, in order to prove the lemma, it is enough to show that

\[
\liminf_{n \to \infty} \left\{ \left[ a_0 + b_0\phi^{\theta-1}_\theta(\nabla u_n) \right] \left[ (p - p^*)\|\nabla u_n\|_p^p + (q - p^*)\|\nabla u_n\|_{q,a}^q \right] + b_0(\theta - 1)\phi^{\theta-2}_\theta(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 + \lambda(p^* - 1 + \gamma)\int_\Omega |u_n|^{1-\gamma} \, dx \right\} > 0.
\]

By contradicting (3.14), let us assume that

\[
\liminf_{n \to \infty} \left\{ \left[ a_0 + b_0\phi^{\theta-1}_\theta(\nabla u_n) \right] \left[ (p - p^*)\|\nabla u_n\|_p^p + (q - p^*)\|\nabla u_n\|_{q,a}^q \right] + b_0(\theta - 1)\phi^{\theta-2}_\theta(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 + \lambda(p^* - 1 + \gamma)\int_\Omega |u_n|^{1-\gamma} \, dx \right\} = 0.
\]

(3.15)

By Lebesgue dominated convergence theorem, we obtain

\[
\lim_{n \to \infty} \int_\Omega |u_n|^{1-\gamma} \, dx = \int_\Omega |u_1|^{1-\gamma} \, dx.
\]

(3.16)

Using (3.16) in (3.15), we get

\[
\liminf_{n \to \infty} \left\{ \left[ a_0 + b_0\phi^{\theta-1}_\theta(\nabla u_n) \right] \left[ (p - p^*)\|\nabla u_n\|_p^p + (q - p^*)\|\nabla u_n\|_{q,a}^q \right] + b_0(\theta - 1)\phi^{\theta-2}_\theta(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 \right\} = -\lambda(p^* - 1 + \gamma)\int_\Omega |u_1|^{1-\gamma} \, dx,
\]

which yields, by applying (3.11),

\[
\begin{align*}
-\lambda \int_\Omega |u_1|^{1-\gamma} \, dx &= \left[ a_0 + b_0\left( \frac{E_1}{p} + \frac{E_2}{q} \right) \theta^{-1} \right] \left[ (p - p^*)E_1 + (q - p^*)E_2 \right] (p^* - 1 + \gamma) \\
&\quad + b_0(\theta - 1) \left( \frac{E_1}{p} + \frac{E_2}{q} \right) \theta^{-2} (E_1 + E_2)^2.
\end{align*}
\]

(3.17)
From this, due to $p < q < p^*$, we have
\[
-\lambda \int_{\Omega} |u_{\lambda}|^{1-\gamma} \, dx \leq b_0 \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta-1} \left[ \frac{(q-p^*)(E_1 + E_2)}{(p^*-1 + \gamma)} + \frac{b_0(\theta - 1)q(E_1 + E_2)}{(p^* + \gamma - 1)} \right]
= \frac{b_0(q - p^*)(E_1 + E_2)}{(p^* + \gamma - 1)} \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta-1}.
\]

(3.18)

Considering $\psi'(u_w)(1) = 0$ and (3.16), we have
\[
\lim_{n \to \infty} \int_{\Omega} |u_n|^{p^*} \, dx = \left[ a_0 + b_0 \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta-1} \right] [E_1 + E_2] - \lambda \int_{\Omega} |u_{\lambda}|^{1-\gamma} \, dx.
\]

From this and (3.17), we obtain
\[
\lim_{n \to \infty} \int_{\Omega} |u_n|^{p^*} \, dx \geq \left[ a_0 + b_0 \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta-1} \right] [E_1 + E_2] + \frac{b_0(p\theta - p)}{p^* - 1 + \gamma} \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta-1} (E_1 + E_2)
= \frac{b_0(p\theta + \gamma - 1)}{p^* + \gamma - 1} \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta-1} (E_1 + E_2).
\]

(3.19)

For any fixed $w \in W_0^{1,q}(\Omega) \setminus \{0\}$, we know that there exists a unique $t_{\max} > 0$ such that $\sigma'_w(t_{\max}) = 0$. From this and (3.6), we conclude that
\[
t_{\max} \geq \left( \frac{b_0(p\theta + \gamma - 1)\|\nabla w\|_{p^*}^{\theta}}{p^* - 1 + \gamma} \right)^{1/p^*} = t_{00}.
\]

(3.20)

It is easy to verify that $t_{\max} \geq t_{00} \geq t_w$ as defined in (3.7) and from the proof of Lemma 3.2, we know that $\Lambda_2 > 0$ is chosen in such a way that
\[
\frac{b_0(p^* - q\theta)}{p^* - 1 + \gamma} \left( t_{00} \right)^{q^* + \gamma - 1} \|\nabla w\|_{p^*}^{q^*} \geq \Lambda_2 \int_{\Omega} |w|^{1-\gamma} \, dx.
\]

We define
\[
S(w) := \frac{b_0(p^* - q\theta)}{p^* - 1 + \gamma} \left( t_{00} \right)^{q^* + \gamma - 1} \|\nabla w\|_{p^*}^{q^*} - \Lambda_2 \int_{\Omega} |w|^{1-\gamma} \, dx \geq 0
\]
for all $w \in W_0^{1,q}(\Omega)$,
with \( t_{00} \) given in (3.20). Taking \( w = u_n \) in (3.21) and then passing to the limit as \( n \to \infty \) we get

\[
\lim_{n \to \infty} S(u_n) \geq 0.
\]

On the other hand, by Lemma 3.6 and (3.11), we have that at least one of \( E_1 \) and \( E_2 \) is not zero. Let us assume, without any loss of generality, that \( E_1 > 0, E_2 \geq 0 \). Then by (3.18), (3.19), (3.20) along with \( q\theta < p^* \) and \( \lambda \in (0, \Lambda_2) \), we obtain

\[
\lim_{n \to \infty} S(u_n) \leq \frac{\lambda_2 b_0(q\theta - p\gamma)(E_1 + E_2) \left( \frac{E_1}{p} + \frac{E_2}{q} \right)^{\theta - 1}}{p^{\theta - 1}(p^* + \gamma - 1)} + \frac{b_0(p - q\theta)}{p^{\theta - 1}(p^* + \gamma - 1)} E_1^{\theta} + \frac{b_0(q\theta - p\gamma)E^0_1}{p^{\theta - 1}(p^* + \gamma - 1)} = 0.
\]

This proves the assertion of the lemma.

Let \( h \in W^{1,\gamma}(\Omega) \) be nonnegative. From Lemma 3.5 there exists a sequence of maps \( \{\xi_n\}_{n \in \mathbb{N}} \) such that \( \xi_n(0) = 1 \) and \( \xi_n(\theta)(u_n + th) \in \mathcal{N}_\lambda^+ \) for sufficiently small \( \theta > 0 \) and for each \( n \in \mathbb{N} \). From this and \( u_n, n \in \mathcal{N}_\lambda \), we have the equations

\[
[a_0 + b_0 \phi_{\gamma}^{\theta - 1}(\nabla u_n)] \left( \|\nabla u_n\|_p + \|\nabla u_n\|_q \right) - \lambda \int_\Omega |u_n|^{1 - \gamma} \, dx - \int_\Omega |u_n|^p \, dx = 0
\]

(3.22)

and

\[
[a_0 + b_0 \phi_{\gamma}^{\theta - 1}(\xi_n(\theta)\nabla w_n)] \left( \xi_n(\theta)\|\nabla w_n\|_p + \xi_n(\theta)\|\nabla w_n\|_q \right) - \lambda \xi_n^{-\gamma}(\theta) \int_\Omega |w_n|^{1 - \gamma} \, dx - \xi_n^{p'}(\theta) \int_\Omega |w_n|^p \, dx = 0
\]

(3.23)

where \( w_n = u_n + th \).

**Lemma 3.8** Let hypotheses (h1)-(h2) be satisfied, let \( \lambda \in (0, \Lambda^*) \) and let \( \{u_n\}_{n \in \mathbb{N}} \subset \mathcal{N}_\lambda^+ \) be a sequence satisfying (3.8)–(3.9). For any nonnegative function \( h \in W^{1,\gamma}(\Omega) \), the sequence \( \{\langle \xi_n(0), h \rangle\}_{n \in \mathbb{N}} \) is uniformly bounded.

**Proof** Subtracting (3.22) from (3.23), we get
\( (a_0 + b_0 \phi_{\frac{1}{\alpha}}(\nabla u_n)) \left( \| \nabla w_n \|^p_p - \| \nabla u_n \|^p_p \right) + (\| \nabla w_n \|^q_{q,a} - \| \nabla u_n \|^q_{q,a} ) \)

\[ + (\zeta_0^n(th) - 1) \| \nabla w_n \|^p_p + (\zeta_0^n(th) - 1) \| \nabla w_n \|^q_{q,a} \]

\[ + b_0 \left[ \phi_{\frac{1}{\alpha}}(\zeta_n(th) \nabla w_n) - \phi_{\frac{1}{\alpha}}(\nabla u_n) \right] \left( \zeta_0^n(th) \| \nabla w_n \|^p_p + \zeta_0^n(th) \| \nabla w_n \|^q_{q,a} \right) \]

\[ - \lambda \left( \zeta_1^n(\theta - 1) \int_{\Omega} |w_n|^{1-\gamma} \, dx - \lambda \int_{\Omega} (\|w_n\|^{1-\gamma} - |u_n|^{1-\gamma}) \, dx \right) \]

\[ - (\zeta_2^n(th) - 1) \int_{\Omega} |w_n|^\tau \, dx - \int_{\Omega} (\|w_n\|^\tau - |u_n|^\tau) \, dx = 0. \]

For notational convenience, we set

\[ \langle u_n, h \rangle_p = \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \, dx \quad \text{and} \quad \langle u_n, h \rangle_{q,a} = \int_{\Omega} a(x)|\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \, dx. \]

We have the following limits

\[ \lim_{t \to 0} \frac{\phi_{\frac{1}{\alpha}}(\zeta_n(th) \nabla w_n) - \phi_{\frac{1}{\alpha}}(\nabla u_n)}{t} = (\zeta_0^n(0), h)(\theta - 1) \phi_{\frac{1}{\alpha}}(\nabla u_n)(\| \nabla u_n \|^p_p + \| \nabla u_n \|^q_{q,a}) \]

\[ + (\theta - 1) \phi_{\frac{1}{\alpha}}(\nabla u_n)(\langle u_n, h \rangle_p + \langle u_n, h \rangle_{q,a}). \]

\[ \lim_{t \to 0} \frac{\| \nabla w_n \|^p_p - \| \nabla u_n \|^p_p}{t} = p \langle u_n, h \rangle_p, \]

\[ \lim_{t \to 0} \frac{\| \nabla w_n \|^q_{q,a} - \| \nabla u_n \|^q_{q,a}}{t} = q \langle u_n, h \rangle_{q,a}, \]

\[ \lim_{t \to 0} \left( |w_n|^\tau - |u_n|^\tau \right) \, dx = p^* \int_{\Omega} |u_n|^{p-2} u_n h \, dx, \]

\[ \lim_{t \to 0} \left( \frac{\zeta_1^n(th)}{t} - 1 \right) = s(\zeta_0^n(0), h) \quad \text{for any } s > 1. \]

Taking into account

\[ \int_{\Omega} (\|w_n\|^{1-\gamma} - |u_n|^{1-\gamma}) \, dx \geq 0 \]

since \( h \) is nonnegative, dividing both sides of (3.24) by \( t > 0 \) and then passing the limit as \( t \to 0^+ \), we obtain

\[ 0 \leq (a_0 + b_0 \phi_{\frac{1}{\alpha}}(\nabla u_n)) \left( p \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla h \, dx + q \int_{\Omega} a(x)|\nabla u_n|^{q-2} \nabla u_n \nabla h \, dx \right) \]

\[ + p (\zeta_0^n(0), h) \| \nabla u_n \|^p_p + q(\zeta_0^n(0), h) \| \nabla u_n \|^q_{q,a} ) \]

\[ + b_0(\theta - 1) \phi_{\frac{1}{\alpha}}(\nabla u_n)(\zeta_1^n(0), h)(\| \nabla u_n \|^p_p + \| \nabla u_n \|^q_{q,a})^2 \]

\[ - \lambda (1 - \gamma)(\zeta_2^n(0), h) \int_{\Omega} |u_n|^{1-\gamma} \, dx - p^*(\zeta_2^n(0), h) \int_{\Omega} |u_n|^\tau \, dx - p^* \int_{\Omega} |u_n|^{p-2} u_n h \, dx. \]

This implies
\[ 0 \leq \langle \zeta_n'(0), h \rangle \left[ (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \left[ p \| \nabla u_n \|^p + q \| \nabla u_n \|^q \right] + b_0(\theta - 1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n) \left( \| \nabla u_n \|^p + \| \nabla u_n \|^q \right)^2 \right. \]
\[ \left. - \lambda(1 - \gamma) \int_{\Omega} |u_n|^{1-\gamma} \, dx - p^* \int_{\Omega} |u_n|^{p^*} \, dx \right] + (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \]
\[ \left( p \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \, dx + q \int_{\Omega} a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \, dx \right) \]
\[ - p^* \int_{\Omega} |u_n|^{p^* - 2} u_n h \, dx. \]

Therefore, using the fact that \( u_n \in \mathcal{N}_\lambda \), we have

\[ 0 \leq \langle \zeta_n'(0), h \rangle \left\{ (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \left( (p + \gamma - 1)\| \nabla u_n \|^p + (q + \gamma - 1)\| \nabla u_n \|^q \right) \right. \]
\[ \left. + b_0(\theta - 1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n) \left( \| \nabla u_n \|^p + \| \nabla u_n \|^q \right)^2 - (p^* + \gamma - 1) \int_{\Omega} |u_n|^{p^*} \, dx \right\} \]
\[ + \left[ (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \left( p \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \, dx + q \int_{\Omega} a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \, dx \right) \right. \]
\[ - p^* \int_{\Omega} |u_n|^{p^* - 2} u_n h \, dx \right]. \]

Now using Lemma 3.7 and taking into account the boundedness of \( \{ u_n \}_{n \in \mathbb{N}} \) in \( W_0^{1,\mathcal{H}}(\Omega) \), we infer that \( \{ \langle \zeta_n'(0), h \rangle \}_{n \in \mathbb{N}} \) is bounded below for any nonnegative \( h \in W_0^{1,\mathcal{H}}(\Omega) \).

It remains to show that \( \{ \langle \zeta_n'(0), h \rangle \}_{n \in \mathbb{N}} \) is bounded above for any nonnegative \( h \in W_0^{1,\mathcal{H}}(\Omega) \). Assume by contradiction that \( \limsup_{n \to \infty} \langle \zeta_n'(0), h \rangle = \infty \). Thus, without loss of generality, we can consider \( \zeta_n(th) > \zeta_n(0) = 1 \) for \( n \in \mathbb{N} \) large enough. It is easy to see that

\[ |\zeta_n(th) - 1| \| u_n \| + \zeta_n(th) \| h \| \geq \| (\zeta_n(th) - 1) u_n + th\zeta_n(th) \| = \| \zeta_n(th) w_n - u_n \|. \]

Applying this in (3.9) with \( u = \zeta_n(th) w_n \), we get

\[ |\zeta_n(th) - 1| \| u_n \| + \zeta_n(th) \| h \| \]
\[ \geq J_\lambda(u_n) - J_\lambda(\zeta_n(th) w_n) \]
\[ = a_0 \left[ \phi_{\mathcal{H}}(\nabla u_n) - \phi_{\mathcal{H}}(\zeta_n(th) \nabla w_n) \right] + \frac{b_0}{\theta} \left[ \phi_{\mathcal{H}}^\theta(\nabla u_n) - \phi_{\mathcal{H}}^\theta(\zeta_n(th) \nabla w_n) \right] \]
\[ - \frac{\lambda}{1 - \gamma} \int_{\Omega} |u_n|^{1-\gamma} - |\zeta_n(th) w_n|^{1-\gamma} \, dx - \frac{1}{p^*} \int_{\Omega} |u_n|^{p^*} - |\zeta_n(th) w_n|^{p^*} \, dx. \]

Using (3.22) and (3.23) in the inequality above, we obtain
|ζ_n(th) - 1| \frac{||u_n||}{n} + |ζ_n(th)\frac{th}{n} |

= a_0 \left[ \phi_{\gamma}(\nabla u_n) - \phi_{\gamma}(\zeta_n(th)\nabla w_n) - \frac{1}{1-\gamma} \right.

\left( \||\nabla u_n||_p^p + ||\nabla u_n||_q^q - \zeta_n^p(th)||\nabla w_n||_p^p - \zeta_n^q(th)||\nabla w_n||_q^q \right)

+ b_0 \left[ \phi_{\gamma}(\nabla u_n) - \phi_{\gamma}(\zeta_n(th)\nabla w_n) \right] - \frac{\phi_{\gamma}(\nabla u_n)}{1-\gamma} \left( \||\nabla u_n||_p^p + ||\nabla u_n||_q^q \right)

+ \frac{\phi_{\gamma}(\zeta_n(th)\nabla w_n)}{1-\gamma} \left( \zeta_n^p(th)||\nabla w_n||_p^p + \zeta_n^q(th)||\nabla w_n||_q^q \right)

- \left( \frac{1}{1-\gamma} - \frac{1}{p^*} \right) \int_\Omega \left( |ζ_n(th)w_n|^{p^*} - |u_n|^{p^*} \right) dx.

Now dividing the above inequality by \( t > 0 \), passing to the limit as \( t \to 0^+ \) and using (3.25), we have

\[ \frac{||h||}{n} \geq a_0 \left[ \langle u_n, h \rangle_p + \langle u_n, h \rangle_{q,a} - \langle \zeta_n^p(0), h \rangle \left( \||\nabla u_n||_p^p + ||\nabla u_n||_q^q \right) \right. 

\left. + \frac{1}{1-\gamma} \left\{ \langle \zeta_n^p(0), h \rangle \left( p||\nabla u_n||_p^p + q||\nabla u_n||_q^q \right) + p\langle u_n, h \rangle_p + q\langle u_n, h \rangle_{q,a} \right\} \right] 

+ b_0 \left[ \phi_{\gamma}(\nabla u_n)(\zeta_n^p(0), h) \left( p||\nabla u_n||_p^p + q||\nabla u_n||_q^q \right) \right. 

\left. + \frac{1}{1-\gamma} \left\{ \langle \zeta_n^p(0), h \rangle (\theta - 1)\phi_{\gamma}(\nabla u_n)(||\nabla u_n||_p^p + ||\nabla u_n||_q^q)^2 \right. 

\left. + \phi_{\gamma}(\nabla u_n)(\zeta_n^p(0), h) \left( p||\nabla u_n||_p^p + q||\nabla u_n||_q^q \right) + \phi_{\gamma}(\nabla u_n) \right] 

\left. (p\langle u_n, h \rangle_p + q\langle u_n, h \rangle_{q,a} \right) \right\] 

- \left( \frac{p^* - 1 + \gamma}{1-\gamma} \right) \left[ \langle \zeta_n^p(0), h \rangle \int_\Omega |u_n|^{p^*} dx + \int_\Omega |u_n|^{p^*-2}u_nh dx \right] 

= \frac{\langle \zeta_n^p(0), h \rangle}{1-\gamma} \left[ \langle a_0 + \phi_{\gamma}^{-1}(\nabla u_n) \rangle \left( (p - 1 + \gamma)||\nabla u_n||_p^p + (q - 1 + \gamma)||\nabla u_n||_q^q \right) \right. 

\left. + b_0(\theta - 1)\phi_{\gamma}^{-2}(\nabla u_n)(||\nabla u_n||_p^p + ||\nabla u_n||_q^q)^2 \right. 

\left. - (p^* - 1 + \gamma) \int_\Omega |u_n|^{p^*} dx - \frac{(1-\gamma)||u_n||}{n} \right] 

+ \frac{a_0}{1-\gamma} \left[ (p - \gamma + 1)\langle u_n, h \rangle_p + (q - \gamma + 1)\langle u_n, h \rangle_{q,a} \right] 

+ \frac{b_0\phi_{\gamma}^{-1}(\nabla u_n)}{1-\gamma} \left[ p\langle u_n, h \rangle_p + q\langle u_n, h \rangle_{q,a} \right] 

- \left( \frac{p^* - 1 + \gamma}{1-\gamma} \right) \int_\Omega |u_n|^{p^*-2}u_nh dx. \]
which gives a contradiction if we take the limits \( n \to \infty \) on both sides, considering 
\[
\limsup_{n \to \infty} (\zeta_n(0), h) = \infty,
\]
since by Lemma 3.7 and the boundedness of \( \{u_n\}_{n \in \mathbb{N}} \), there exists some \( M_1 > 0 \) such that
\[
\begin{align*}
\left[(a_0 + \phi_{\mathcal{H}}^0(\nabla u_n)) & \right] \left( (p - 1 + \gamma) \| \nabla u_n \|_p^p + (q - 1 + \gamma) \| \nabla u_n \|_{q,a}^q \right) \\
+ b_0 (\theta - 1) \phi_{\mathcal{H}}^0(\nabla u_n)(\| \nabla u_n \|_p^p + \| \nabla u_n \|_{q,a}^q)^2 \\
- (p^* - 1 + \gamma) \int_\Omega |u_n|^{p^*} \, dx - \frac{(1 - \gamma) \| u_n \|}{n} \right] > M_1
\end{align*}
\]
for \( n \in \mathbb{N} \) large enough. Thus \( \{ (\zeta_n(0), h) \}_{n \in \mathbb{N}} \) must be bounded above. \( \square \)

Since \( J_\lambda(u_n) = J_\lambda(|u_n|) \), without loss of generality, we may assume that \( u_n \geq 0 \) a.e. in \( \Omega \) and so, \( u_n \geq 0 \) a.e. in \( \Omega \). With this assumption, we state our next result.

**Lemma 3.9** Let hypotheses (h1)-(h2) be satisfied, let \( \lambda \in (0, \Lambda^\ast) \) and let \( \{u_n\}_{n \in \mathbb{N}} \subset \mathcal{N}_\lambda^+ \) be a sequence satisfying (3.8)–(3.9). For any \( h \in W_0^{1,\mathcal{H}}(\Omega) \) and \( n \in \mathbb{N} \), \( u_n^{-\gamma} h \in L^1(\Omega) \) and as \( n \to \infty \)
\[
\begin{align*}
(a_0 + b_0 \phi_{\mathcal{H}}^0(\nabla u_n)) \left[ \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \, dx + \int_\Omega a(x)|\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \, dx \right] \\
- \lambda \int_\Omega u_n^{-\gamma} h \, dx - \int_\Omega u_n^{-\beta - 1} h \, dx = o_n(1).
\end{align*}
\]

**Proof** Let \( h \in W_0^{1,\mathcal{H}}(\Omega) \) be nonnegative and recall the following estimate from the proof of Lemma 3.8
\[
|\zeta_n(th) - 1| \frac{\| u_n \|}{n} + \zeta_n(th) \frac{\| th \|}{n}
\]
\[
\begin{align*}
& \geq a_0 \left[ \phi_{\mathcal{H}}(\nabla u_n) - \phi_{\mathcal{H}}(\zeta_n(th) \nabla w_n) \right] + \frac{b_0}{\theta} \left[ \phi_{\mathcal{H}}(\nabla u_n) - \phi_{\mathcal{H}}(\zeta_n(th) \nabla w_n) \right] \\
& - \frac{\lambda}{1 - \gamma} \int_\Omega \left[ |u_n|^{1-\gamma} - |\zeta_n(th)w_n|^{1-\gamma} \right] \, dx - \frac{1}{p^*} \int_\Omega \left[ |u_n|^{p^*} - |\zeta_n(th)w_n|^{p^*} \right] \, dx \\
& = a_0 \left[ \left( \phi_{\mathcal{H}}(\nabla u_n) - \phi_{\mathcal{H}}(\nabla w_n) \right) + \left( \phi_{\mathcal{H}}(\nabla w_n) - \phi_{\mathcal{H}}(\zeta_n(th) \nabla w_n) \right) \right] \\
& + \frac{b_0}{\theta} \left[ \left( \phi_{\mathcal{H}}(\nabla u_n) - \phi_{\mathcal{H}}(\nabla w_n) \right) + \left( \phi_{\mathcal{H}}(\nabla w_n) - \phi_{\mathcal{H}}(\zeta_n(th) \nabla w_n) \right) \right] \\
& - \frac{\lambda}{1 - \gamma} \int_\Omega \left[ |u_n|^{1-\gamma} - |w_n|^{1-\gamma} \right] \, dx - \frac{\lambda}{1 - \gamma} \int_\Omega \left[ |w_n|^{1-\gamma} - |\zeta_n(th)w_n|^{1-\gamma} \right] \, dx \\
& - \frac{1}{p^*} \int_\Omega \left[ |u_n|^{p^*} - |w_n|^{p^*} \right] \, dx - \frac{1}{p^*} \int_\Omega \left[ |w_n|^{p^*} - |\zeta_n(th)w_n|^{p^*} \right] \, dx.
\end{align*}
\]
Dividing the above equation with \( t > 0 \) and then passing to limit as \( t \to 0^+ \), we get
\[
|\langle \zeta'_n(0), h \rangle| \frac{\|u_n\|}{n} + \frac{\|h\|}{n} \\
\geq -(a_0 + b_0 \Phi_{\chi}^{\theta_1}(\nabla u_n)) \left[ \langle u_n, h \rangle_p + \langle u_n, h \rangle_{q,a} + \langle \zeta'_n(0), h \rangle (\|u_n\|_p^p + \|u_n\|_{q,a}^q) \right] \\
- \frac{\lambda}{1 - \gamma} \liminf_{t \to 0^+} \int_\Omega \frac{[u_n^{1-\gamma} - |w_n|^{1-\gamma}]}{t} \, dx + \lambda \langle \zeta'_n(0), h \rangle \int_\Omega |u_n|^{1-\gamma} \, dx \\
+ \langle \zeta'_n(0), h \rangle \int_\Omega |u_n|^q \, dx + \int_\Omega u_n^{p-1} h \, dx \\
= -\langle \zeta'_n(0), h \rangle \left[ (a_0 + b_0 \Phi_{\chi}^{\theta_1}(\nabla u_n)) \left[ (\|u_n\|_p^p + \|u_n\|_{q,a}^q) \right] \\
- \frac{\lambda}{1 - \gamma} \liminf_{t \to 0^+} \int_\Omega \frac{[u_n^{1-\gamma} - |w_n|^{1-\gamma}]}{t} \, dx + \int_\Omega u_n^{p-1} h \, dx \\
- (a_0 + b_0 \Phi_{\chi}^{\theta_1}(\nabla u_n)) \left[ (\|u_n\|_p^p + \|u_n\|_{q,a}^q) \right] \\
- \frac{\lambda}{1 - \gamma} \liminf_{t \to 0^+} \int_\Omega \frac{[u_n^{1-\gamma} - |w_n|^{1-\gamma}]}{t} \, dx + \int_\Omega u_n^{p-1} h \, dx, \\
\]

where we used \( u_n \in N_{\lambda} \) that is \( \psi'_{u_n}(1) = 0 \). This implies
\[
\frac{\lambda}{1 - \gamma} \liminf_{t \to 0^+} \int_\Omega \frac{[u_n + th]^{1-\gamma} - |u_n|^{1-\gamma}}{t} \, dx \\
\leq (a_0 + b_0 \Phi_{\chi}^{\theta_1}(\nabla u_n)) \left[ (\|u_n\|_p^p + \|u_n\|_{q,a}^q) \right] \\
- \int_\Omega u_n^{p-1} h \, dx + |\langle \zeta'_n(0), h \rangle| \left[ \frac{\|u_n\|}{n} + \frac{\|h\|}{n} \right]. \\
(3.27)
\]

Observe that \( |u_n + th|^{1-\gamma} - |u_n|^{1-\gamma} \geq 0 \), so we can use Fatou’s lemma in (3.27) to obtain
\[
\frac{\lambda}{1 - \gamma} \int_\Omega u_n^{p-1} h \, dx \leq (a_0 + b_0 \Phi_{\chi}^{\theta_1}(\nabla u_n)) \left[ (\|u_n\|_p^p + \|u_n\|_{q,a}^q) \right] \\
- \int_\Omega u_n^{p-1} h \, dx + |\langle \zeta'_n(0), h \rangle| \left[ \frac{\|u_n\|}{n} + \frac{\|h\|}{n} \right].
\]

Recall that \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( W^{1,\chi}_0(\Omega) \). Then, passing to the limit as \( n \to \infty \) in the above estimate, we obtain
\[
(a_0 + b_0 \Phi_{\chi}^{\theta_1}(\nabla u_n)) \left[ \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \, dx + \int_\Omega a(x)|\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \, dx \right] \\
- \lambda \int_\Omega u_n^{p-1} h \, dx - \int_\Omega u_n^{p-1} h \, dx \geq o_n(1), \\
(3.28)
\]

for each nonnegative \( h \in W^{1,\chi}_0(\Omega) \), where we used the uniform boundedness from Lemma 3.8.
We aim to establish that (3.28) holds true for any arbitrary \( h \in W_{0}^{1,\gamma}(\Omega) \). For this, we replace \( h \) in (3.28) by \((u_{n} + \varepsilon h)^{+}\) with \( \varepsilon > 0 \) and \( h \in W_{0}^{1,\gamma}(\Omega) \). Renaming as \( \hat{h}_{\varepsilon} = u_{n} + \varepsilon h \) and using (3.28), we get
\[
o_{n}(1) \leq (a_{0} + b_{0} \Phi_{\hat{h}_{\varepsilon}}^{\theta_{n}^{-1}}(\nabla u_{n})) \]
\[
\left[ \int_{\Omega} |\nabla u_{n}|^{p-2} \nabla u_{n} \cdot \nabla h_{\varepsilon} \, dx + \int_{\Omega} a(x) |\nabla u_{n}|^{q-2} \nabla u_{n} \cdot \nabla h_{\varepsilon} \, dx \right] \]
\[ - \lambda \int_{\Omega} u_{n}^{-\gamma} h_{\varepsilon} \, dx - \int_{\Omega} u_{n}^{p-1} h_{\varepsilon} \, dx \]
\[ = (a_{0} + b_{0} \Phi_{\hat{h}_{\varepsilon}}^{\theta_{n}^{-1}}(\nabla u_{n})) \]
\[
\left[ \int_{\Omega} |\nabla u_{n}|^{p-2} \nabla u_{n} \cdot \nabla h_{\varepsilon} \, dx + \int_{\Omega} a(x) |\nabla u_{n}|^{q-2} \nabla u_{n} \cdot \nabla h_{\varepsilon} \, dx \right] \]
\[ + (a_{0} + b_{0} \Phi_{\hat{h}_{\varepsilon}}^{\theta_{n}^{-1}}(\nabla u_{n})) \]
\[
\int_{\Omega} |\nabla u_{n}|^{p-2} \nabla u_{n} \cdot \nabla h_{\varepsilon} \, dx + \int_{\Omega} \left[ (\|u_{n}\|_{p}^{\theta} + \|u_{n}\|_{q,n}^{\theta}) \right] - \lambda \int_{\Omega} |u_{n}|^{1-\gamma} \, dx - \int_{\Omega} |u_{n}|^{p} \, dx \]
\[ + \varepsilon \left\{ (a_{0} + b_{0} \Phi_{\hat{h}_{\varepsilon}}^{\theta_{n}^{-1}}(\nabla u_{n})) \int_{\Omega} |\nabla u_{n}|^{p-2} \nabla u_{n} \cdot \nabla h \, dx \right. \]
\[ - \lambda \int_{\Omega} u_{n}^{-\gamma} h \, dx - \int_{\Omega} u_{n}^{p-1} h \, dx \right\} - \lambda \int_{\Omega} u_{n}^{-\gamma} h_{\varepsilon} \, dx - \int_{\Omega} u_{n}^{p-1} h_{\varepsilon} \, dx \]
\[ + (a_{0} + b_{0} \Phi_{\hat{h}_{\varepsilon}}^{\theta_{n}^{-1}}(\nabla u_{n})) \]
\[
\left. \left[ \int_{\Omega} |\nabla u_{n}|^{p-2} \nabla u_{n} \cdot \nabla h_{\varepsilon} \, dx + \int_{\Omega} a(x) |\nabla u_{n}|^{q-2} \nabla u_{n} \cdot \nabla h_{\varepsilon} \, dx \right] \right). \]

We define \( \Omega_{\varepsilon} = \{ x \in \Omega : u_{n} + \varepsilon h \leq 0 \} \). Using \( u_{n} \in \mathcal{N}_{\lambda} \) and \( \int_{\Omega} u_{n}^{-\gamma} h_{\varepsilon} \, dx \geq 0 \) in the above estimate, we get
\[
o_{n}(1) \leq \varepsilon \left\{ (a_{0} + b_{0} \Phi_{\hat{h}_{\varepsilon}}^{\theta_{n}^{-1}}(\nabla u_{n})) \int_{\Omega_{\varepsilon}} |\nabla u_{n}|^{p-2} \nabla u_{n} \cdot \nabla h \, dx \right. \]
\[ - \lambda \int_{\Omega} u_{n}^{-\gamma} h \, dx - \int_{\Omega} u_{n}^{p-1} h \, dx \right\} + \int_{\Omega_{\varepsilon}} w_{n}^{p-1} h_{\varepsilon} \, dx - (a_{0} + b_{0} \Phi_{\hat{h}_{\varepsilon}}^{\theta_{n}^{-1}}(\nabla u_{n})) \]
\[
\left. \left[ \int_{\Omega_{\varepsilon}} |\nabla u_{n}|^{p-2} \nabla u_{n} \cdot \nabla h_{\varepsilon} \, dx + \int_{\Omega_{\varepsilon}} a(x) |\nabla u_{n}|^{q-2} \nabla u_{n} \cdot \nabla h_{\varepsilon} \, dx \right) \right]. \]

Note that

\[ (3.29) \]
Putting all these in (3.29), we infer that

\[- \int_{\Omega} |\nabla u_n|^\rho - 2 \nabla u_n \cdot \nabla h \, dx = - \int_{\Omega} |\nabla u_n|^\rho - 2 \nabla u_n \cdot \nabla (u_n + \varepsilon h) \, dx\]

\[- \int_{\Omega} |\nabla u_n|^\rho \, dx - \varepsilon \int_{\Omega} |\nabla u_n|^\rho - 2 \nabla u_n \cdot \nabla h \, dx\]

\[\leq - \varepsilon \int_{\Omega} |\nabla u_n|^\rho - 2 \nabla u_n \cdot \nabla h \, dx\]

and similarly,

\[- \int_{\Omega} a(x)|\nabla u_n|^q - 2 \nabla u_n \cdot \nabla h \, dx \leq - \varepsilon \int_{\Omega} a(x)|\nabla u_n|^q - 2 \nabla u_n \cdot \nabla h \, dx.\]

Moreover, applying Hölder’s inequality and \(u_n \leq - \varepsilon h\) in \(\Omega^\varepsilon\), we have

\[\left| \int_{\Omega^\varepsilon} u_n^{\rho - 1} h \, dx \right| \leq \left| \int_{\Omega^\varepsilon} u_n^\rho \, dx \right| + \varepsilon \left| \int_{\Omega^\varepsilon} u_n^{\rho - 1} |h| \, dx \right|\]

\[\leq \varepsilon \rho \int_{\Omega^\varepsilon} |h|^\rho \, dx + \varepsilon \left( \int_{\Omega^\varepsilon} u_n^\rho \, dx \right)^{\frac{\rho - 1}{\rho}} \left( \int_{\Omega^\varepsilon} |h|^\rho \, dx \right)^{\frac{1}{\rho}}.\]

Putting all these in (3.29), we infer that

\[o_n(1) \leq \varepsilon \left\{ (a_0 + b_0 \phi_{\mathcal{H}}^{\rho - 1} (\nabla u_n)) \left[ \int_{\Omega} |\nabla u_n|^\rho - 2 \nabla u_n \cdot \nabla h \, dx + \int_{\Omega} a(x)|\nabla u_n|^q - 2 \nabla u_n \cdot \nabla h \, dx \right] \right\} \]

\[- \lambda \int_{\Omega} u_n^{\rho - 1} h \, dx \left( \int_{\Omega} |h|^\rho \, dx \right)^{\frac{\rho - 1}{\rho}} - \varepsilon \left( \int_{\Omega^\varepsilon} u_n^{\rho - 1} \, dx \right)^{\frac{\rho - 1}{\rho}} \]

\[+ \varepsilon \left( \int_{\Omega^\varepsilon} u_n^\rho \, dx \right)^{\frac{\rho - 1}{\rho}} \left( \int_{\Omega^\varepsilon} |h|^\rho \, dx \right)^{\frac{1}{\rho}} - \varepsilon (a_0 + b_0 \phi_{\mathcal{H}}^{\rho - 1} (\nabla u_n)) \]

\[\left[ \int_{\Omega^\varepsilon} |\nabla u_n|^\rho - 2 \nabla u_n \cdot \nabla h \, dx + \int_{\Omega^\varepsilon} a(x)|\nabla u_n|^q - 2 \nabla u_n \cdot \nabla h \, dx \right].\]

(3.30)

Since \(|\Omega^\varepsilon| \to 0\) as \(\varepsilon \to 0^+\) and by the boundedness of \(\{u_n\}_{n \in \mathbb{N}}\) in \(W^{1,2}_0(\Omega)\), if we divide (3.30) by \(\varepsilon > 0\) and then pass to the limit as \(\varepsilon \to 0^+\), we obtain

\[(a_0 + b_0 \phi_{\mathcal{H}}^{\rho - 1} (\nabla u_n)) \left[ \int_{\Omega} |\nabla u_n|^\rho - 2 \nabla u_n \cdot \nabla h \, dx + \int_{\Omega} a(x)|\nabla u_n|^q - 2 \nabla u_n \cdot \nabla h \, dx \right] \]

\[- \lambda \int_{\Omega} u_n^{\rho - 1} h \, dx \geq a_n(1),\]

as \(n \to \infty\). By the arbitrariness of \(h \in W^{1,2}_0(\Omega)\), (3.31) actually implies (3.26) which completes the proof. 

Now, we prove the compactness property of the energy functional \(J_\lambda\) in a suitable range of \(\lambda\). For this purpose, we set for any \(\lambda > 0\)

\[c_\lambda := a_2 - a_1 \lambda \frac{\rho^*}{\rho^* - 1 + \gamma}\]
where
\[
a_0 := \left(\frac{1}{q\theta} - \frac{1}{p^*}\right), \quad a_1 := \left(\frac{p^* - 1 + \gamma}{q^*} - 1\right) \left(\frac{\theta - 1 + \gamma}{\theta(1 - \gamma)}\right)^{\frac{\rho^*}{\rho^* - 1} - 1}\frac{1 - \gamma}{\rho^* - 1 + \gamma}\]
\( (3.32) \)
and
\[
a_2 := a_0 \left(\frac{Sb_0}{p^{\theta - 1}}\right)^{\frac{p^*}{p^* - \theta}} \left(S^{\rho^* - \theta} \left(\frac{b_0}{q^{\theta - 1}}\right)^{p^* - \theta}\right)\left(\frac{\gamma - 1}{p^* - \theta}\right).
\( (3.33) \)

Also, for any \( k \in \mathbb{N} \), let \( T_k \) be the truncation defined by
\[
T_k(t) := \begin{cases} t & \text{if } |t| \leq k, \\ k \frac{t}{|t|} & \text{if } |t| > k. \end{cases}
\]

**Proposition 3.10** Let hypotheses \((h_1)-(h_2)\) be satisfied, let \( \lambda \in (0, \Lambda^*) \) and let \( \{u_n\}_{n \in \mathbb{N}} \subset \mathcal{N}_\lambda \) be a sequence satisfying \((3.8)-(3.9)\) and
\[
J_\lambda(u_n) \to c < c_\lambda \quad \text{as } n \to \infty.
\( (3.34) \)

Then \( \{u_n\}_{n \in \mathbb{N}} \) possesses a strongly convergent subsequence in \( W^{1,\mathcal{H}}(\Omega) \).

**Proof** Fixing \( k \in \mathbb{N} \) and taking \( h = T_k(u_n - u_\lambda) \in W^{1,\mathcal{H}}(\Omega) \) as a test function in \((3.26)\), we get
\[
o_n(1) = (a_0 + b_0 \phi^{\theta - 1}_\lambda(\nabla u_n)) \left[ \int_\Omega |\nabla u_n|^{\rho - 2} \nabla u_n \nabla T_k(u_n - u_\lambda) \, dx \right.
\]
\[
+ \int_\Omega a(x) |\nabla u_n|^{\theta - 2} \nabla u_n \nabla T_k(u_n - u_\lambda) \, dx \right]
\]
\[
- \lambda \int u_n^{\rho - 1} T_k(u_n - u_\lambda) \, dx - \int u_n^{\rho - 1} T_k(u_n - u_\lambda) \, dx := I - J - K \quad \text{as } n \to \infty.
\( (3.35) \)

Using Young’s inequality, Propositions 2.1(iii)–(iv), 2.2(ii) and boundedness of the sequences \( \{u_n\}_{n \in \mathbb{N}}, \{T_k(u_n - u_\lambda)\}_{n \in \mathbb{N}} \) in \( W^{1,\mathcal{H}}(\Omega) \), we obtain
\[
|J| \leq |I| + |K| + o_n(1)
\]
\[
\leq (a_0 + b_0 \phi^{\theta - 1}_\lambda(\nabla u_n)) \int_\Omega |\nabla u_n|^{\rho - 1} |\nabla T_k(u_n - u_\lambda)| \, dx
\]
\[
+ \int_\Omega a(x) |\nabla u_n|^{\theta - 1} |\nabla T_k(u_n - u_\lambda)| \, dx
\]
\[
+ \int_\Omega |u_n|^{\rho - 1} T_k(u_n - u_\lambda) \, dx + o_n(1)
\]
\[
\leq (a_0 + b_0 \phi^{\theta - 1}_\lambda(\nabla u_n)) (\rho_\mathcal{H}(\nabla u_n) + \rho_\mathcal{H}(\nabla T_k(u_n - u_\lambda)))
\]
\[
+ k \int u_n^{\rho - 1} \, dx + o_n(1) \leq C(1 + k)
\]
\( (3.36) \)
with a constant \( C \) independent of \( n \) and \( k \), that is, the sequence \( \{ u_n^{-r} T_k(u_n - u_\lambda) \}_{n \in \mathbb{N}} \) is uniformly integrable. Then, using (3.10) and Vitali’s convergence theorem, we get
\[
\int_{\Omega} u_n^{-r} T_k(u_n - u_\lambda) \, dx \to 0.
\]

By Hölder’s inequality, we observe that
\[
[L^p(\Omega)]^N \ni g \mapsto \int_{\Omega} (|\nabla u_\lambda|^{p-2} + a(x)|\nabla u_\lambda|^{q-2}) \nabla u_\lambda \cdot g \, dx
\]
is a bounded linear functional. From (3.10), we see that \( \nabla T_k(u_n - u_\lambda) \to 0 \) in \([L^p(\Omega)]^N\), so we can get
\[
\lim_{n \to \infty} \int_{\Omega} (|\nabla u_\lambda|^{p-2} + a(x)|\nabla u_\lambda|^{q-2}) \nabla u_\lambda \cdot \nabla T_k(u_n - u_\lambda) \, dx = 0. \tag{3.37}
\]

Let \( \phi_{H}(\nabla u_\lambda) \to \beta := \frac{E_1}{p} + \frac{E_2}{q} \) as \( n \to \infty \), where \( E_1 \) and \( E_2 \) are defined in (3.11). Thus, by using (3.36)–(3.37) in (3.35) and the fact that \( a_0 \geq 0, b_0 > 0, \beta > 0 \), we get
\[
(a_0 + b_0 \beta^{\theta-1}) \limsup_{n \to \infty} \left[ \int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_\lambda|^{p-2} \nabla u_\lambda) \cdot \nabla T_k(u_n - u_\lambda) \, dx 
\right. \\
\left. + \int_{\Omega} a(x)(|\nabla u_n|^{q-2} \nabla u_n - |\nabla u_\lambda|^{q-2} \nabla u_\lambda) \cdot \nabla T_k(u_n - u_\lambda) \, dx \right] \\
= \limsup_{n \to \infty} \int_{\Omega} u_n^{p-1} T_k(u_n - u_\lambda) \, dx \leq Ck.
\]

By Simon’s inequalities, see [24, formula (2.2)], we rewrite the above estimate as
\[
\limsup_{n \to \infty} \left[ \int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_\lambda|^{p-2} \nabla u_\lambda) \cdot \nabla T_k(u_n - u_\lambda) \, dx \right] \leq \frac{Ck}{(a_0 + b_0 \beta^{\theta-1})}. \tag{3.38}
\]

Set
\[
s_n(x) = (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_\lambda|^{p-2} \nabla u_\lambda) \cdot \nabla (u_n - u_\lambda).
\]

Note that \( s_n(x) \geq 0 \) a. e. in \( \Omega \). We divide the set \( \Omega \) by
\[
E_n^k = \{ x \in \Omega : |u_n(x) - u_\lambda(x)| \leq k \} \quad \text{and} \quad F_n^k = \{ x \in \Omega : |u_n(x) - u_\lambda(x)| > k \},
\]
where \( k, n \in \mathbb{N} \) are fixed. Let \( \eta \in (0, 1) \). Then, from the definition of \( T_k \), Hölder’s inequality, (3.38) and the fact that \( \lim_{n \to \infty} |F_n^k| = 0 \), we get
\[
\limsup_{n \to \infty} \int_{\Omega} s_n^\eta \, dx \leq \limsup_{n \to \infty} \left( \int_{E_n^k} s_n \, dx \right)^\eta |E_n^k|^{1-\eta} + \limsup_{n \to \infty} \left( \int_{F_n^k} s_n \, dx \right)^\eta |F_n^k|^{1-\eta} \, dx.
\]
\[
\leq \left( \frac{Ck}{(a_0 + b_0 \beta^{\theta-1})} \right)^\eta |\Omega|^{1-\eta}.
\]
Letting \( k \to 0^+ \), we obtain that \( s_n^\gamma \to 0 \) in \( L^1(\Omega) \). Thus, we may assume that \( s_n \to 0 \) a.e. in \( \Omega \) (up to a subsequence) which along with Simon’s inequalities [24, formula (2.2)] gives that

\[
\nabla u_n \to \nabla u_\lambda \quad \text{a.e. in } \Omega.
\]

(3.39)

Let \( M \) be the nodal set of the weight function \( a(\cdot) \) given by

\[
M := \{ x \in \Omega : a(x) = 0 \}.
\]

Since, the sequences \( \{ |\nabla u_n|^{p-2} \nabla u_n \}_{n \in \mathbb{N}} \) and \( \{ |\nabla u_n|^{q-2} \nabla u_n \}_{n \in \mathbb{N}} \) are bounded in \( L^{p^*}(\Omega) \) and \( L^{q^*}(\Omega \setminus M, a(x) \, dx) \), respectively, then by using (3.39) and [3, Proposition A.8], we conclude that

\[
\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u_\lambda = \| \nabla u_\lambda \|_p^p
\]

and

\[
\int_{\Omega} a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla u_\lambda = \int_{\Omega \setminus M} a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla u_\lambda = \| \nabla u_\lambda \|_{q,a}^q.
\]

Furthermore, using (3.10), (3.39) and the Brezis-Lieb Lemma, we obtain

\[
\rho_H(\nabla u_n) - \rho_H(\nabla u_n - \nabla u_\lambda) = \rho_H(\nabla u_\lambda) + o_n(1),
\]

\[
\| u_n \|_{p^*}^p - \| u_n - u_\lambda \|_{p^*}^p = \| u_\lambda \|_{p^*}^p + o_n(1)
\]

as \( n \to \infty \). Let \( \| u_n - u_\lambda \|_{p^*} \to \varepsilon^* \) for some \( \varepsilon^* \geq 0 \). Now, by taking \( u_n - u_\lambda \) as a test function in (3.26), we get

\[
o_n(1) = (a_0 + b_0 \theta^{-1}(\nabla u_n)) \left[ \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u_\lambda) \, dx + \int_{\Omega} a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla (u_n - u_\lambda) \, dx \right]
\]

\[
- \lambda \int_{\Omega} u_n^{-\gamma} (u_n - u_\lambda) \, dx - \int_{\Omega} u_n^{p-1} (u_n - u_\lambda) \, dx
\]

\[
= (a_0 + b_0 \theta^{-1}) \left[ \rho_H(\nabla u_n) - \rho_H(\nabla u_\lambda) + o_n(1) \right] - \| u_n \|_{p^*}^p + \| u_\lambda \|_{p^*}^p + o_n(1)
\]

as \( n \to \infty \). Hence, by (3.10) and (3.40) it follows that

\[
(a_0 + b_0 \theta^{-1}) \left[ \rho_H(\nabla u_n) - \rho_H(\nabla u_\lambda) \right] = \varepsilon^{p^*} + o_n(1) \quad \text{as } n \to \infty,
\]

(3.41)

which further gives

\[
(a_0 + b_0 \theta^{-1}) \lim_{n \to \infty} \left( \| \nabla u_n - \nabla u_\lambda \|_p^p + \| \nabla u_n - \nabla u_\lambda \|_{q,a}^q \right) \leq \varepsilon^{p^*}.
\]

(3.42)

Now, we claim that \( \varepsilon^* = 0 \). Assume by contradiction that \( \varepsilon^* > 0 \). By (3.1) and (3.42), we have

\[
Sa_0 \varepsilon^p \leq S(a_0 + b_0 \theta^{-1}) \varepsilon^p \leq (a_0 + b_0 \theta^{-1}) \lim_{n \to \infty} \| \nabla u_n - \nabla u_\lambda \|_p^p \leq \varepsilon^{p^*}.
\]

(3.43)

Note that (3.42) implies that
On critical double phase Kirchhoff problems with singular...

Using (3.43) in (3.44), we get

\[
\left(\epsilon_1 \right)^{\theta-1} \geq \left( a_0 + b_0 \right) \left( E_1 + E_2 - \|\nabla u\|_p - 2 \|\nabla u\|_{q,a} \right) \geq \epsilon_1^{\theta-1}.
\]

(3.44)

Using (3.43) in (3.44), we get

\[
\left( \epsilon_1 \right)^{\theta-1} \geq \left( a_0 + b_0 \right) \left( E_1 + E_2 - \|\nabla u\|_p - 2 \|\nabla u\|_{q,a} \right) \geq \epsilon_1^{\theta-1}.
\]

(3.43)

Using (3.43) in (3.44), we get

\[
\left( \epsilon_1 \right)^{\theta-1} \geq \left( a_0 + b_0 \right) \left( E_1 + E_2 - \|\nabla u\|_p - 2 \|\nabla u\|_{q,a} \right) \geq \epsilon_1^{\theta-1}.
\]

(3.45)

From (3.45) and (3.1), we obtain

\[
E_1 \geq \left( E_1 - \|\nabla u\|_p \right) \geq \epsilon_1^{\theta-1}.
\]

This gives

\[
E_1 \geq S \left( a_0 + b_0 \right) \geq \epsilon_1^{\theta-1}.
\]

(3.46)

and so we have

\[
E_1 \geq \left[ S \left( \frac{b_0}{p^{\theta-1}} \right) \right]^{1/\theta-1}.
\]

(3.47)

Combining (3.45) and (3.46), we obtain

\[
\epsilon_1^{\theta-1} \geq S \left( a_0 + b_0 \right) \geq \epsilon_1^{\theta-1}.
\]

(3.48)

For any \( n \in \mathbb{N} \), we have
\[ J_\lambda(u_n) - \frac{1}{q\theta} \langle J'_\lambda(u_n), u_n \rangle = a_0 \phi_{\frac{b_0}{\theta}}(\nabla u_n) + b_0 \phi_{\frac{b_0}{\theta}}(\nabla u_n) - \frac{1}{q\theta} m(\phi_{\frac{b_0}{\theta}}(\nabla u_n)) \left( L_{p,q}^a(u_n), u_n \right) \\
\quad - \lambda \left( \frac{1}{1-\gamma} - \frac{1}{q\theta} \right) \int_{\Omega} u_n^{1-\gamma} \, dx + \left( \frac{1}{q\theta} - \frac{1}{p^*} \right) \int_{\Omega} u_n^{p^*} \, dx \\
\quad \geq \left( \frac{1}{q\theta} - \frac{1}{p^*} \right) \| u_n \|_{p^*}^{p^*} - \lambda \left( \frac{1}{1-\gamma} - \frac{1}{q\theta} \right) \int_{\Omega} u_n^{1-\gamma} \, dx. \]

From this, as \( n \to \infty \), by (3.47), (3.40), Hölder’s and Young’s inequality, we derive
\[
c = \lim_{n \to \infty} \left( J_\lambda(u_n) - \frac{1}{q\theta} \langle J'_\lambda(u_n), u_n \rangle \right) \\
\quad \geq a_0 \left( \varepsilon^{p^*} + \| u_\lambda \|_{p^*}^{p^*} \right) - \lambda \left( \frac{1}{1-\gamma} - \frac{1}{q\theta} \right) \| \Omega \|^{\frac{\gamma-1\gamma}{p^*}} \| u_\lambda \|_{p^*}^{1-\gamma} \\
\quad \geq a_0 \varepsilon^{p^*} - a_1 \lambda^{\frac{p^*}{1-\gamma}} \\
\quad \geq a_0 \left( \frac{Sb_0}{p^{\theta-1}} \right)^{\frac{p^*}{p-\theta}} \left[ \frac{b_0}{p^{\theta-1}} \right]^{p} \left( \frac{m(\phi_{b_0/\theta}(\nabla u_\lambda))}{p^{\theta-1}\phi_{b_0/\theta}^{p^*}} \right) - a_1 \lambda^{\frac{p^*}{1-\gamma}} = c_\lambda,
\]

where \( a_0, a_1 \) are defined in (3.32). The above estimates gives a contradiction to (3.34). Hence \( \varepsilon = 0 \) and using (3.41) and Proposition 2.1(v), we conclude the proof. \( \square \)

**Remark 3.11** By taking \( \lambda \in (0, \Lambda_\lambda) \) with \( \Lambda_\lambda := (\alpha_2 \alpha_1^{-1})^{\frac{1}{p^*}} \) and \( a_1, \alpha_2 \) are defined in (3.32) and (3.33) respectively, we have \( c_\lambda > 0 \).

**Proof** (Proof of Theorem 1.1) Fix \( \lambda < \lambda^* := \min \{ \Lambda^*, \Lambda_\lambda \} \). From Lemma 3.1(ii) and Ekeland’s variational principle there exists a minimizing sequence \( \{ u_n \}_{n \in \mathbb{N}} \in \mathcal{N}_\lambda^0 \setminus \{ 0 \} \) verifying (3.8), (3.9), (3.10) and (3.34) with \( c = \Theta^*_\lambda \). Hence, by combining Propositions 3.4 and 3.10, we obtain \( u_n \rightharpoonup u_\lambda \) strongly in \( W_0^{1,\mathcal{H}}(\Omega) \) (up to a subsequence). This further implies that \( u_\lambda \in \mathcal{N}_\lambda^0 \) and by Lemma 3.7, we get \( u_\lambda \in \mathcal{N}_\lambda^{p^*} \) with \( u_\lambda \) achieving \( \Theta^*_\lambda \) since \( J_\lambda \) is continuous on \( W_0^{1,\mathcal{H}}(\Omega) \). Since \( 0 \notin \mathcal{N}_\lambda^0 \) and \( u_n \geq 0 \) we have \( u_\lambda \neq 0 \) and \( u_\lambda \geq 0 \). Letting \( n \to \infty \) in (3.26), we obtain that \( u_\lambda \) satisfies \( u_\lambda^{-\gamma} \varphi \in L^1(\Omega) \) and
\[
m(\phi_{\mathcal{H}}(\nabla u_\lambda)) \left( L_{p,q}^a(u_\lambda), \varphi \right) = \lambda \int_{\Omega} u_\lambda^{-\gamma} \varphi \, dx + \int_{\Omega} u_\lambda^{1-\gamma} \varphi \, dx
\]
for all \( \varphi \in W_0^{1,\mathcal{H}}(\Omega) \). Finally, by using Proposition 3.4, Lemma 3.5 and by repeating the proof of [2, Proposition 4.3 and Proposition 4.4, Step 1], we obtain \( u_\lambda > 0 \) a.e. in \( \Omega \). \( \square \)

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