In this paper we study the question of effective field assignment to measured or nonperturbatively calculated spectral functions. The straightforward procedure is to approximate it by a sum of independent Breit-Wigner resonances, and assign an independent field to each of these resonances. The problem with this idea is that it introduces new conserved quantities in the free model (the new particle numbers), therefore it changes the symmetry of the system. We avoid this inconsistency by representing each quantum channel with a single effective field, no matter how complicated the spectral function is. Thermodynamical characterization of the system will be computed with this representation method, and its relation to the independent resonance approximation will be discussed.

I. INTRODUCTION

A fundamental field theory is defined through its microscopic field content, representing the fundamental degrees of freedom (dof), and the interaction between them. But microscopic dof are not always observable: in low energy QCD we find hadrons instead of quarks and gluons. Also in an ordinary gas we might encounter rather molecules than separate electrons and nuclei (even nucleons). Although this statement reflects a trivial fact, it is hard to consistently implement the rather smooth shift/overlap in the nature of the relevant dof in an interacting theory. If we want to work directly with the observable dof, we need to build effective theories.

A primary principle of construction of the effective theories is that the symmetries of the fundamental system should be present unchanged in the effective model. This ensures that they belong to the same universality class, i.e. they describe the same physics. In fact, the by-now textbook examples of the QCD effective models like the nonlinear or linear sigma models, chiral sigma models, Polyakov loop models, Walecka model, etc, use this strict guiding principle to represent low-energy forces generated by QCD [1, 2]. In these models one defines the field content to represent the low-lying bound states of QCD, and construct a Lagrangian which respects the symmetries of QCD. The parameters of the Lagrangian are eventually tuned to reproduce most observed features of the particle spectra.

In the examples listed above effective fields represent not just the (lowest lying) observable hadrons with the corresponding quantum numbers but actually the complete quantum channel. We get into trouble, however, when we want to represent individually also the higher energy states in a given channel; in case of QCD the plethora of excited hadrons. The representation of the observed particles would require to assign a field to each hadron. The manifestation of this idea, treated in the ideal gas approximation is the widely used hadron resonance gas (HRG) model, which indeed provides surprisingly accurate description to certain quantities in QCD in the low temperature regime [3-6]. For a more realistic description, in particular to describe transport properties, one has to consider also the widths of the peaks (off-shell transport) [7, 8]. The problem of conservation of energy-momentum tensor and other conserved currents can be consistently solved in the framework of Kadanoff-Baym equations [9]. In this extended version one still assigns a field (Breit-Wigner resonance) to each peak of the measured spectral function separately [10].

In these approaches, however, one gives up the principle of keeping the symmetries of the fundamental theory, by generating a large number of new conserved quantities at tree level, namely the particle numbers of the excited hadrons. With other words: each quantum channel is represented multiple times. If we do not want this unphysical extra symmetry to appear on the level of observables, the tree level contributions must be compensated by the effect of interactions. This may require, however, strong radiative corrections of O(1), eventually resulting in a nonperturbative system. Then tree level results cannot be considered reliable, whatsoever nice is the arising description. This situation would question the effective nature of the chosen model.

There are cases when the idea that we assign a field (a new dof) to each peak of the spectral functions indeed leads to false conclusions. A well-known example is the hydrogen atom: if we took all bound states into account as independent dof then we would obtain false thermodynamics [11, 12]. In the standard solution [11, 12] we quit at some scale the framework of quantum mechanics, and explicitly assume that bound states with certain size cannot exist in the matter. Although physically motivated, nonetheless this requirement is not self-consistent, and it is hard
to imagine that it can be phrased formally on the level of the Lagrangian. In a similar spirit, one can try to improve HRG by cutting off the number of hadronic resonances above an appropriately chosen energy scale \([13]\).

We can study this question within the scattering theory, too. In the quantum mechanical approach, bound states appear as a jump in the scattering phase. Their contribution to free energy is described by Beth-Uhlenbeck formula \([12]\) provided the scattering phases of resonances are simply additive \([14]\). In the generic situation, however, this is not the case. Finite width peaks should be included in the S-matrix as Breit-Wigner resonances with complex amplitudes \([15, 16]\). The unitarity of the S-matrix then yields a consistency constraint, which should be solved. The resulting amplitudes are thus results of interference effects which might decrease the “weight” of the individual finite lifetime dof. If the finite width peaks of the spectral function are well separated, then a careful analysis can show \([17]\) that the additivity of their scattering phases is indeed true.

In this paper we approach this question from the field theory point of view. The potential problem coming from the introduction of new conserved quantities can be avoided by simply representing each quantum channel by a single field – no matter, how many peaks are in the corresponding spectral function. As we have discussed above, associating a single field with the singled out quantum channel works well in a lot of effective models of QCD. The representation of a complicated spectral function by a single field, technically is similar to the perturbative 2PI technique \([18]\), where the propagator (e.g. the spectral function) of a channel is a single dynamical object. However, in this case we go beyond the reorganization of the perturbative series, we shall construct a non-perturbative scheme where the spectral function is the input. We expect (and actually will prove) that our scheme shares with the 2PI technique a number of good features like energy-momentum conservation, causality, unitarity and renormalizability.

In this paper we try to use this idea to build up a consistent one-dof bosonic effective model. We start with a very simple analysis, demonstrating that the correct representation of the spectral function is relevant (Section III). In Section III we present the one-dof model and prove its consistency. In Section IV we calculate the thermodynamical observables (energy density, pressure etc.) from this model, and compare the results with the ones assuming independent excitations. The comparison is facilitated by a prescription how to measure the “number of dof”. We will see that overlapping dof are indeed not independent, and they always behave as if they were less than one degree of freedom. In Section VI we give the conclusions.

II. WHY THE REPRESENTATION OF THE SPECTRAL FUNCTION IS IMPORTANT?

The goal of this Section is to demonstrate that the spectral function alone does not determine the physics completely: the symmetry properties influence certain observables, in particular the thermodynamics crucially.

First let us define the spectral function: we choose an operator \(A\) with fixed quantum numbers, and compute the \(\varrho_{AA^\dagger}(t) = \langle [A(t), A^\dagger(0)] \rangle\) expectation value, and take its Fourier transform \(\varrho_{AA^\dagger}(p_0)\). If the system is spatially translation invariant then the spatial momentum is part of the quantum numbers, too. As it is well known \([19]\), this expectation value at zero temperature is proportional to the density of states, at finite dimensional quantum mechanical systems it consists of discrete Dirac-deltas at the position of the energy levels. Choosing different operators to measure the spectrum results in different weights of the Dirac-deltas, but their position will be the same.

Now consider a one-dof free scalar Klein-Gordon model. Its Lagrangian reads

\[
\mathcal{L} = \frac{1}{2} \varphi K \varphi, \quad K = -\partial_t^2 - \omega_k^2, \quad \omega_k^2 = k^2 + m^2.
\]  

We can use the operator \(\varphi(t, k)\) to measure the spectral function in the momentum \(k\), spin 0 quantum channel. It reads \(\varrho(p_0) = 2\pi \text{sgn}(p_0) \delta(p_0^2 - \omega_k^2)\). We remark that because of the canonical commutation relations this spectral function satisfies a sum rule \([10]\).

As a second example we take two scalar degrees of freedom \(\varphi_1\) and \(\varphi_2\) with the same mass, and construct their free model:

\[
\mathcal{L} = \frac{1}{2} (\varphi_1, \varphi_2) \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.
\]

To measure the spectral function in the momentum \(k\), spin 0 quantum channel, we can choose for example the operator \(A(t, k) = \varphi_1(t, k) \cos \theta + \varphi_2(t, k) \sin \theta\) for arbitrary angles. The measured spectral function will be, independently on the angle, \(\varrho_{AA^\dagger}(p_0) = 2\pi \text{sgn}(p_0) \delta(p_0^2 - \omega_k^2)\), the same result as before.

In this model there is an extra internal symmetry, the \(O(2) \equiv U(1)\) rotation of the two fields. If this symmetry is physical, then there may exist a measurable quantity which makes a distinction between the two fields; sometimes it is the electric charge attached to the \(U(1)\) rotation. But if we introduce the two fields exclusively to represent the spectrum, the extra symmetry is non-physical, and the physical operators do not see the difference between the two degrees of freedom. In this case physically there exists only one quantum channel with \(k\) momentum and spin 0.
Now we have two models which give the same spectral function. The thermodynamics, described by these models, is still different. For example the Stefan-Boltzmann (SB) limit for the energy density would be $\varepsilon = \pi^2 T^4/30$ in the first case and twice as much in the second case. This can be physical, of course, if the U(1) symmetry has a physical background. But if the two degrees of freedom were introduced exclusively to represent the spectrum, then the extra symmetry is nonphysical, and we have false results for thermodynamics, despite the fact that we correctly described the spectrum.

For practical reasons we may still want to choose a two-particle representation even if the physics do not support the new symmetry. For example if we have two Dirac-delta peaks in the spectral function then, according to [17], we may use a two-particle representation. In the limiting case when the two peaks merge, we arrive to the second model. Since the nonphysical symmetries appear in the observables, in particular in thermodynamical quantities, to a correct two-field description we have to choose an interaction which breaks the artificial symmetry. Now we consider two examples for this symmetry breaking interaction within the framework of a quadratic model, where the interaction appears as off-diagonal elements of the kernel matrix.

Consider for example the following model:

\[
L = \frac{1}{2}(\varphi_1, \varphi_2) \begin{pmatrix} KK \\ K \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.
\]

This Lagrangian has no more U(1) symmetry. A special property of this model is that the kernel has no inverse. This means that one degree of freedom (now the $\varphi_1 - \varphi_2$ mode) has no dynamics, it completely decouples from the system. The remaining mode $\varphi_1 + \varphi_2$ has the same dynamics as in (1), so the physics is the same too. In particular the spectral function (up to a normalization factor) and the thermodynamics is the same.

Another example is the model

\[
L = \frac{1}{2}(\varphi_1, \varphi_2) \begin{pmatrix} K - \lambda & -\lambda \\ -\lambda & K - \lambda \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.
\]

The two eigenvalues are $K$ and $K - 2\lambda$, so the spectral function exhibits two peaks. But in case $\lambda \to \infty$, the mass of one eigenstate remains $m$, while the other has a mass-squared $m^2 + 2\lambda \to \infty$ which decouples from the system. So the measurable part of the spectral function as well as the thermodynamics at any finite temperature behave as if we had one dof.

So apparently we can work with more dof than the physical ones. The price we have to pay for it is that we have strong interactions. This must be so, since in the non-interacting (diagonal) case the SB limit is twice as much as in the interacting case: so the interactions give $O(1)$ corrections relative to the free case. The lesson is that wrong assignment of the dof’s alone may lead to an apparently strongly interacting theory, even if with correctly chosen fields the theory is free.

### III. REPRESENTATION OF A GENERIC SPECTRAL FUNCTION WITH A SINGLE FIELD

Let us try to generalize the observations of the previous Section for the case when we observe, after fixing all possible quantum numbers, a generic spectral function. Let us assume that we can identify finite width peaks. Heuristically, the width of the peaks can be treated as a “resolution power”. If, according to this resolution, the separation of the peaks is large, then earlier studies [13, 17] tell us that these behave as distinct objects: the independent particle approximation works well. If the separation of the peaks starts to be comparable with their width, then by their own resolution they are more and more like one object, and we arrive at the case discussed above. Then the lesson of the examples of Section [11] tells us that if we further insist to a multi-dof description, the theory which correctly describes physics becomes strongly interacting, nonperturbative.

This complication can be avoided if we do not introduce new dof into the system, and assign a single field to each quantum channel: then the symmetries of the original system is not altered. As a price, however, we have to work with a more complicated quadratic theory which can represent a generic spectral function. In this Section we construct a 1-dof system which can represent a generic spectral function, then we demonstrate that it describes a consistent theory, if the input spectral function comes from a physical system.

---

1. We can determine the spectral function measured by the the above defined $A$, it results in $\varrho_{AA}(p_0) = \pi \sin^2(\theta + \frac{\pi}{4}) \text{sgn}(p_0) \delta(p_0^2 - \omega_k^2)$. This depends on $\theta$ since the action is not U(1) symmetric.
A. The Lagrangian

Let us assume that we have measured a scalar spectral function, and there are no other hidden quantum numbers in the system. This means that the system must be represented with a single scalar field $\phi$. We are looking for a quadratic Lagrangian which could represent the given spectral function (interactions can be necessary to correctly describe spectral functions of other quantum channels, which is not the topic of the present paper). We want moreover a system where energy and momentum are conserved, so we want space and time translation symmetry. Then the most generic scalar field theory can be written as:

$$S[\phi] = \frac{1}{2} \int d^4 x \phi(x) K(i\partial) \phi(x).$$

(5)

In Fourier space we can write

$$S[\phi] = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \phi^*(p) K(p) \phi(p).$$

(6)

If $K$ is not a polynomial, then this action is not local. Since $\phi(x)$ is real, $\phi(-p) = \phi^*(p)$; since $S$ is also real, the kernel must satisfy $K^*(p) = K(-p) = K(p)$.

The spectral function is defined as

$$\rho(x-y) = \langle [\phi(x), \phi(y)] \rangle .$$

(7)

Since the theory is quadratic, so the commutator $[\phi(x), \phi(y)]$ is a c-number, thus we can also omit the expectation value. Using the standard procedure [19], we compute the retarded retarded Green’s function (Landau prescription) as

$$G_{R}(p) = \mathcal{K}^{-1}(p_0 + i\varepsilon, p).$$

(8)

The spectral function is the discontinuity of the retarded Green’s function

$$\rho(p) = \text{Disc} G_{R}(p) = \lim_{\varepsilon \to 0} \left[ \mathcal{K}^{-1}(p_0 + i\varepsilon, p) - \mathcal{K}^{-1}(p_0 - i\varepsilon, p) \right].$$

(9)

That means that a kernel determines the spectral function in a unique way.

In our approach the spectral function is the input, and we want to construct the corresponding kernel. The retarded Green’s function can be obtained via the Kramers-Kronig relation:

$$G_{R}(p) = \int \frac{d\omega}{2\pi} \frac{\varrho(\omega, p)}{p_0 - \omega + i\varepsilon}. $$

(10)

Using (5), the kernel can be determined as

$$K(p) = \left( P \int \frac{d\omega}{2\pi} \frac{\varrho(\omega, p)}{p_0 - \omega} \right)^{-1},$$

(11)

where $P$ means principal value. That means that the kernel is also completely determined by the spectral function.

B. Consistency

When one works with a nonlocal theory like (5), one has to worry about the consistency of the model which, in case of a field theory, means that the requirements of energy- and momentum conservation, causality, unitarity, and Lorentz-invariance (in case of zero temperature) must be satisfied.

First we remark that within perturbation theory it is also possible to change the tree level propagator and work with a self-consistently determined generic propagator instead: this is the 2PI resummation [18]. This method is known to be a fully consistent way of treating the system perturbatively. Since one can formulate 2PI resummation as a model with nonlocal kernel [20], the present method can be thought to be a nonperturbative realization of the consistent 2PI resummation method.

But, independently on the perturbative treatment we can prove the consistency criteria one-by-one.
a. Energy and momentum conservation  This requirement is satisfied if the action is time and space translation invariant. From the definition \(\text{(4)}\) this fact is trivial.

b. Causality:  Causality means in relativistic invariant systems that measurements at space-like separated points must not influence each other (in non-relativistic systems causality is not an issue). Mathematically this requires that local operators at space-like separated positions must commute. We show that if the input spectral function is causal then the system is causal, too.

A local operator built up from a single scalar field has the form \(\hat{O}(x) = \prod_{i=1}^{n} A_i\), where \(A_i = a_i(x, i\partial)\varphi(x)\) where \(a_i(x, i\partial)\) is a local differential operator (i.e. at most polynomial in the derivatives). We define \(\hat{Q}(y)\) in a similar way as \(\prod_{i=1}^{m} B_i\). Then we can use the following identity

\[
[\hat{O}(x), \hat{Q}(y)] = \sum_{i=1}^{n} \sum_{j=1}^{m} B_1 \ldots B_{j-1} A_1 \ldots A_{i-1} [A_i, B_j] A_{i+1} \ldots A_n B_{j+1} \ldots B_m. \tag{12}
\]

The commutator can be expressed as \([A_i, B_j] = a_i(x, i\partial_x)b_j(y, i\partial_y)[\varphi(x), \varphi(y)]\). Since \(a_i\) and \(b_j\) are local differential operators, an infinitesimally small opened neighborhood of the point \(x\) is enough to compute them from the field \(\varphi\).

But if \(x\) and \(y\) are spatially separated points, i.e. \((x - y)^2 < 0\), then a sufficiently small opened neighborhood is also spatially separated. Since \(\varphi(x - y) = [\varphi(x), \varphi(y)]\) is the input spectral function, it is guaranteed that it is zero for spatial separation. This means that in a small neighborhood it is identically zero, therefore all derivatives are zero. This means that \([A_i, B_j]\) is zero \(\forall \ i, j\), and finally \([\hat{O}(x), \hat{Q}(y)] = 0\), too.

c. Unitarity  Physically unitarity means that there is no loss of probability in the system. If we use, for example, finite lifetime particles, then the time evolution will not be unitary, since the decaying state will be missing from the final states. To circumvent this problem we have to keep all possible energy states, then a decaying state turns into another state and the complete probability will be the same. As a dynamical quantity, therefore, we have to work with the complete spectral function.

Mathematically one can examine the unitarity (and the corresponding reflection positivity for euclidean systems) of models where the kernel is an arbitrary polynomial \([21]\). The result is, formulating in the language of the spectral function, that for unitarity the spectral function must be positive for all positive frequencies \(\varrho(\omega > 0, p) \geq 0\). Since we use the spectral function as an input stemming from a real quantum field theory, this requirement will be satisfied. Therefore unitarity is also granted.

d. Lorentz invariance  From the definition \(\text{(5)}\) it is clear that if the kernel is Lorentz invariant, then the action is Lorentz invariant, too. For the reverse argumentation, in Lorentz invariant systems the input spectral function leads to a Lorentz invariant retarded Green’s function, which leads to a Lorentz invariant kernel. So, for a physically sensible zero temperature spectral function the Lorentz invariance of our model is also granted.

The input spectral function, if it comes from a finite temperature system, need not be Lorentz invariant. Still, in most part of this paper we will use a Lorentz invariant input.

We see, that all requirement of a consistent field theory could be satisfied with the Ansatz \(\text{(5)}\), provided we use an input spectral function which is physically sensible. Now we are in the position that we can use it for calculating physical observables.

IV. THERMODYNAMICS

Given the Lagrangian of the system, in principle we can calculate the expectation value of any physical observable, and, in particular, approach thermodynamics. But, because of time-nonlocality, some standard techniques do not work in this case. First of all we do not have canonical formalism (more precisely we would need infinitely many fields to represent the canonical formalism). Therefore there is not a direct connection between the imaginary time formalism and the thermodynamics: in fact, the standard derivation is based on the fact that the Hamiltonian density depends on the canonical momentum as \(\Pi^2/2\) \([19]\). Therefore thermodynamics should be started from something which is meaningful also microscopically: this can be the energy density. Known for all temperatures, this is then enough to calculate all the other thermodynamical quantities like pressure or entropy density.

A. Energy-momentum tensor

The energy density is the 00 component of the energy-momentum tensor, which represents the Noether currents belonging to the time and space translation invariance, respectively. We can use the standard procedure to determine the energy-momentum tensor: we consider position dependent translation \(a^\mu(x)\) with which \(x^\mu = x^\mu + a^\mu(x)\) and the
shifted field satisfies $\varphi'(x') = \varphi(x)$. Then we can calculate the action with the shifted field $S[\varphi']$, for which we expect the behavior $S[\varphi'] - S[\varphi] = \int d^4x T^{\mu\nu}(x) \partial_\mu \partial_\nu$, where $T^{\mu\nu}$ is the energy momentum tensor.

**Statement:** The energy momentum tensor belonging to the action (5) is

$$T^{\mu\nu}(x) = \frac{1}{2} \varphi(x) D_{\mu\nu} K(i\partial) \varphi(x)$$

where

$$D_{\mu\nu} K(i\partial) = \left[ \frac{\partial K(p)}{\partial p^\mu} \bigg|_{p\rightarrow i\partial} \right]_{\text{sym}} i\partial_\nu - g_{\mu\nu} K(i\partial),$$

and the symmetrized derivative is defined as

$$f(x)[(i\partial)^n]_{\text{sym}} g(x) = \frac{1}{n+1} \sum_{a=0}^{n} [(-i\partial)^a f(x)][(i\partial)^{n-a} g(x)].$$

**Proof:** We represent the kernel in Fourier space as a power series:

$$K(p) = \sum c^{\mu_1 \ldots \mu_n}_{(n)} p_{\mu_1} \ldots p_{\mu_n} \Rightarrow K(i\partial) = \sum c^{\mu_1 \ldots \mu_n}_{(n)} (i\partial_{\mu_1}) \ldots (i\partial_{\mu_n}),$$

where $c$ is a completely symmetric tensor. Now we consider $S[\varphi']$ with integration variable $x'$, then change to $x' \rightarrow x$ where $x' = x + a$ with position dependent $a$. The corresponding Jacobian is $\partial x'_\mu / \partial x^\mu = \delta^\mu_\mu + \partial_\mu a^\nu + O(a^2)$, and the transformation of the derivative reads $\partial'_\mu = \partial_\mu - (\partial_\mu a^\nu) \partial_\nu + O(a^2)$. We obtain

$$S[\varphi'] = \sum c^{\mu_1 \ldots \mu_n}_{(n)} \int d^4x' \varphi'(x')(i\partial'_{\mu_1}) \ldots (i\partial'_{\mu_n}) \varphi'(x') = \sum c^{\mu_1 \ldots \mu_n}_{(n)} \int d^4x (1 + \partial a) \varphi(x) (i\partial_{\mu_1} - \partial_\mu a^\nu i\partial_\nu) \ldots (i\partial_{\mu_n} - \partial_\mu a^\nu i\partial_\nu) \varphi(x).$$

We need the difference linear in $\partial a$. We find

$$\delta S = \int d^4x \partial a L - \sum c^{\mu_1 \ldots \mu_n}_{(n)} \int d^4x \varphi(x) \sum_{i=1}^{n} (i\partial_{\mu_1}) \ldots (i\partial_{\mu_{i-1}}) \partial_\mu a^\nu i\partial_\nu (i\partial_{\mu_{i+1}}) \ldots (i\partial_{\mu_n}) \varphi(x).$$

Now we perform partial integration in the second term, and find

$$- \sum c^{\mu_1 \ldots \mu_n}_{(n)} \int d^4x \sum_{i=1}^{n} \partial_\mu a^\nu \left[ (-i\partial_{\mu_1}) \ldots (-i\partial_{\mu_{i-1}}) \varphi(x) \right] \left[ (i\partial_{\mu_{i+1}}) \ldots (i\partial_{\mu_n}) \varphi(x) \right].$$

Since $c^{\mu_1 \ldots \mu_n}_{(n)}$ is completely symmetric in the indexes, we can put $i \rightarrow n$ and find

$$- \sum c^{\mu_1 \ldots \mu_{n-1} \mu}_{(n)} \int d^4x \partial_\mu a^\nu \sum_{i=1}^{n-1} \left[ (-i\partial_{\mu_1}) \ldots (-i\partial_{\mu_{i-1}}) \varphi(x) \right] \left[ (i\partial_{\mu_i}) \ldots (i\partial_{\mu_{n-1}}) (i\partial_\nu \varphi(x)) \right].$$

Introducing the symmetrized derivative notion we find

$$- \sum c^{\mu_1 \ldots \mu_{n-1} \mu}_{(n)} \int d^4x \partial_\mu a^\nu \varphi(x) \left[ (i\partial_{\mu_1}) \ldots (i\partial_{\mu_{n-1}}) \right]_{\text{sym}} (i\partial_\nu \varphi(x)) = \int d^4x \partial_\mu a^\nu \varphi(x) \left[ \frac{1}{2} \frac{\partial K(p)}{\partial p^\mu} \bigg|_{p\rightarrow i\partial} \right]_{\text{sym}} (i\partial_\nu \varphi(x)).$$

Finally

$$\delta S = \frac{1}{2} \int d^4x \partial a^\nu \left\{ g_{\mu\nu} \varphi(x) K(i\partial) \varphi(x) - \varphi(x) \left[ \frac{\partial K(p)}{\partial p^\mu} \bigg|_{p\rightarrow i\partial} \right]_{\text{sym}} (i\partial_\nu \varphi(x)) \right\},$$

so we find

$$T^{\mu\nu}(x) = \frac{1}{2} \varphi(x) \left\{ \left[ \frac{\partial K(p)}{\partial p^\mu} \bigg|_{p\rightarrow i\partial} \right]_{\text{sym}} i\partial_\nu - g_{\mu\nu} K(i\partial) \right\} \varphi(x).$$

This is the equation we had to prove.

**QED**
B. Finite temperature expectation value

Now we will compute the expectation value of the energy-momentum tensor in equilibrium. For that we change into Fourier space:

$$\langle T_{\mu\nu}(k) \rangle = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta(k - p - q) \varphi(q) D_{\mu\nu} K(q, p) \varphi(p),$$  \hfill (24)

where the symmetrization of the derivatives is inherited to the momentum dependent kernel as

$$[p^n]_{\text{sym}} \rightarrow \frac{1}{n+1} \sum_{i=0}^{n} (-q)^i p^{n-i}. \hfill (25)$$

We can take its expectation value:

$$\langle T_{\mu\nu}(k) \rangle = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta(k - p - q) D_{\mu\nu} K(q, p) \langle \varphi(q) \varphi(p) \rangle = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} D_{\mu\nu} K(q = -p, p) iG_<(p)(2\pi)^4 \delta(k). \hfill (26)$$

In the expression of $K(q = -p, p)$, according to (25), the symmetrization of the derivative can be omitted, so we simply denote

$$D_{\mu\nu} K(p) = p_\mu \frac{\partial K}{\partial p^\nu} - g_{\mu\nu} K. \hfill (27)$$

Since (26) is proportional to $\delta(k)$, in real space it is position-independent: $\langle T_{\mu\nu}(x) \rangle = \langle T_{\mu\nu} \rangle$. We remark that if the kernel is relativistic invariant (i.e. depends only on $p^2$) then $\langle T_{\mu\nu} \rangle$ is a symmetric tensor.

To determine the propagator at finite temperature we use KMS relation [19]. We note here that the formal correspondence between the time translation $e^{-\beta H}$ and statistical operator $e^{-\beta H}$ is still valid, and so the KMS relation remains true. With the KMS relation we find:

$$\langle T_{\mu\nu} \rangle = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} D_{\mu\nu} K(p) n(p_0) \varphi(p), \hfill (28)$$

where $n(p_0) = (e^{\beta p_0} - 1)^{-1}$ is the Bose-Einstein distribution. Using the fact that $K$ is a symmetric real function, $D_\mu K$ is also symmetric real function, while $\varphi(p_0)$ is a real antisymmetric function. Therefore we can average the positive and negative energy parts as

$$n(p_0) \varphi(p_0, p) \rightarrow \left( \frac{1}{2} + n(p_0) \right) \varphi(p_0, p), \hfill (29)$$

which is a completely symmetric function. We can then restrict ourselves to the positive frequency part as

$$\langle T_{\mu\nu} \rangle = \int_{p^+} D_{\mu\nu} K(p) \left( \frac{1}{2} + n(p_0) \right) \varphi(p), \text{ where } \int_{p^+} = \int_{p^0}^{+\infty} \frac{dp_0}{2\pi} \int \frac{d^3p}{(2\pi)^3}. \hfill (30)$$

The leading $1/2$ represents the vacuum energy. To obtain a finite expression we have to renormalize the expression which means “normal ordering”, i.e. subtract the expectation value at zero temperature:

$$\langle T_{\mu\nu} \rangle_{\text{ren}} = \int_{p^+} D_{\mu\nu} K(p) \left[ n(p_0) \varphi(p) + \frac{1}{2} \delta \varphi(p) \right], \text{ where } \delta \varphi(p) = \varphi(p) - \varphi_0(p). \hfill (31)$$

In a rotationally invariant system only the diagonal elements survive. The energy density is the 00 component of the energy-momentum tensor:

$$\varepsilon = \int_{p^+} D K(p) \left[ n(p_0) \varphi(p) + \frac{1}{2} \delta \varphi(p) \right], \text{ } D K(p) = p_0 \frac{\partial K}{\partial p^0} - K, \hfill (32)$$
This is a completely finite expression for the energy density. This formula is the main result of this paper.

In case of relativistic invariance the kernel is a function of $p^2$ alone, then we can write

$$DK(p) = 2p_0^2 \frac{\partial K}{\partial p^2} - K. \quad (33)$$

Note, that (32) is non-linear functional of the input spectral function, since apart from the explicit $\varrho$, the kernel $K$ also depends on it. This nonlinearity has the interesting consequence that the result is independent of the overall normalization of the spectral function. Indeed, if we change the normalization of $\varrho$, i.e. we multiply the original spectral function by $Z$, then $G_R \to ZG_R$ and $K \to K/Z$, and so $Z$ drops out from the result. This fact can be interpreted that thermodynamics is not sensitive to the normalization of the fields, only on the density of states.

Once we know the $\varepsilon(T)$ relation, we can use thermodynamics to calculate other quantities. For the explicit formulae we will not consider thermal variation of the spectrum, and so assume $\delta\varrho = 0$. In that case the pressure is

$$p = -T \int \frac{1}{p_0} DK(p) \ln \left(1 - e^{-\beta p_0}\right) \varrho(p), \quad (34)$$

and the entropy density reads

$$s = \int \frac{1}{p_0} DK(p) \left[\beta p_0 n(p_0) - \ln \left(1 - e^{-\beta p_0}\right)\right] \varrho(p). \quad (35)$$

These expressions are also well defined finite quantities.

C. Number of degrees of freedom

The real observables are the energy density and the corresponding thermodynamical quantities. In order to help understanding the physics of the effective system, other, more loosely defined quantities can be useful like “number of dof”.

For stable free particles this notion is well defined: it is the number of energy levels at fixed quantum numbers, in particular at fixed momentum. The spectral function $\varrho(\omega, k)$ consists of Dirac-delta peaks in this case, and the number of dof is the number of the Dirac-delta peaks. The heights do not matter, they come from the normalization of the operator with which we measure the spectrum. Note however, that the normalization independence means that the number of dof must be a nonlinear functional of the spectral function!

A general input spectral function can have arbitrary structure, usually broadened peaks and other “continuum” contributions. The question of the number of dof is thus not well-defined there. We may hope, however, that plausible definitions yield approximately the same physical picture.

A plausible choice is, for example, to compute the energy density $\varepsilon$ of the system. For a free theory with $N_{dof}$ degrees of freedom the energy density is $N_{dof}\varepsilon_0(m, T)$, where $\varepsilon_0(m, T)$ is the free energy density of a gas with mass $m$ at temperature $T$. This quantity, therefore, is proportional to the number of dof, so we can define a “thermodynamical dof” at any temperature. The definition depends on the mass $m$, unless we go to asymptotic high temperatures, where $N_{dof}(T \to \infty) = 30\varepsilon(T \to \infty)/(\pi^2 T^4)$.

Another proposal is to use the formula of the energy density and use the ideas of Williams and Weizsacker equivalent photon number [22]. The corresponding definition is the “zero temperature equivalent dof” with the Ansatz:

$$N_{dof} = \frac{\int_0^\infty d\omega d\mathbf{k}}{2\pi} \frac{1}{p_0} DK(p) \varrho(p), \quad (36)$$

This is a dimensionless, temperature independent quantity which, as we will see later, in case of the discrete spectrum yields indeed the number of energy levels.

V. INDEPENDENCE OF THE DEGREES OF FREEDOM

We have all the formulae to answer physical questions. The main issue will be to compare the results coming from the independent resonance approximation and the one-field representation.
A. Dirac-delta peaks

The first problem that we should check is whether a system that contains \( N \) stable excitations with different dispersion relations can be represented with \( N \) free fields. According to [17] we expect that it must be true.

Consider now a spectral function consisting of \( N \) Dirac-delta peaks with dispersion relations \( \omega_i(p) \) (\( \omega_i \neq \omega_j \) for \( i \neq j \)), allowing also different normalization:

\[
\rho(p) = \sum_i 2\pi Z_i \delta(p_0 - \omega_i(p)).
\] (37)

In the independent resonance approximation we assign the following Lagrangian to this spectral function:

\[
\mathcal{L}_{\text{indep}} = \sum_{i=1}^{N} \frac{1}{2} \phi_i \left( i \partial_0 - \omega_i(i \partial) \right) \phi_i.
\] (38)

The normalization is irrelevant, since we can rescale the fields to achieve the standard form. Using this Lagrangian and measuring the spectral function of the operator \( \varphi = \sum_i \sqrt{Z_i} \phi_i \) we indeed recover (37).

This Lagrangian represents an \( N \)-dof system. If we calculate the thermodynamical observables, the energy density and the pressure we obtain sum of the partial energy densities and pressure:

\[
\varepsilon = \sum_{i=1}^{N} \varepsilon_i, \quad P = \sum_{i=1}^{N} P_i,
\] (39)

where

\[
\varepsilon_i = \int \frac{d^3p}{(2\pi)^3} \omega_i n(\omega_i), \quad P_i = \int \frac{d^3p}{(2\pi)^3} \frac{p \omega_i}{3} d\omega_i n(\omega_i).
\] (40)

Let us check whether we obtain the same result in the single-field representation, where – correctly – we do not introduce new conserved charges into the system. Now we have a single field; from the Kramers-Kronig relation (10) we find for the retarded Green’s function

\[
G_R = \sum_i \frac{Z_i}{p_0 - \omega_i + i\varepsilon}.
\] (41)

The kernel is its inverse: it is now a complicated, nonlocal object which have zeroes at \( p_0 = \omega_i(p) \) dispersion relations. Therefore we immediately see that \( K\varphi = 0 \). The nonzero contribution to the energy density (32), pressure (34) or number of dof (36) is the derivative of the kernel multiplied by the spectral function. In case of the energy density we need

\[
\frac{p_0}{G_R} \frac{dK}{dp_0} = -\frac{p_0}{G_R} \frac{dG_R}{dp_0} = \frac{p_0}{G_R} \sum_i \frac{Z_i}{(p_0 - \omega_i)^2}.
\] (42)

The spectral function evaluates this formula at each \( p_0 = \omega_i \). We find

\[
\lim_{p_0 \to \omega_i} \frac{p_0}{G_R} \frac{dK}{dp_0} = \lim_{p_0 \to \omega_i} \frac{\omega_i}{Z_i/(p_0 - \omega_i)^2 + \text{finite}} \left[ \frac{Z_i}{(p_0 - \omega_i)^2 + \text{finite}} \right] = \frac{\omega_i}{Z_i}.
\] (43)

That means

\[
DK(p) \rho(p) = \sum_{i=1}^{N} \lim_{p_0 \to \omega_i} \left( \frac{p_0}{G_R} \frac{dK}{dp_0} \right) Z_i(2\pi) \delta(p_0 - \omega_i) = \sum_{i=1}^{N} \omega_i(2\pi) \delta(p_0 - \omega_i).
\] (44)

The wave function renormalization factors disappear, and the energy becomes \( \omega_i \). Therefore the number of degrees of freedom from (36) is reduced to the integral

\[
N_{\text{dof}} = \sum_{i=1}^{N} \int_{0}^{\infty} \frac{dp_0}{2\pi} \frac{1}{p_0} \omega_i(2\pi) \delta(p_0 - \omega_i) = N,
\] (45)
supporting the $N$ independent degrees of freedom approximation. The formula of the energy density \[32\] yields

$$
\varepsilon = \sum_{i=1}^{N} \int_{p} n(p_0) \omega_i (2\pi) \delta(p_0 - \omega_i) = \sum_{i=1}^{N} \int \frac{d^3p}{(2\pi)^3} \omega_i n(\omega_i),
$$

(46)

which agrees with \[39\] with \[40\]. Similarly for the pressure, using \[34\], we recover the pressure formula in \[39\] with \[40\].

We can conclude that for a spectral function which has $N$ Delta-peaks, the independent resonance approximation exactly reproduces the results of the one-field representation. This also agrees with the earlier results \[17\].

**B. Single particle with finite width**

Now let us start to study the problem of overlapping peaks. The basic building block is the case of a single particle with finite width. There the spectral function has the (relativistic) Breit-Wigner form

$$
\varrho(p) = \frac{4p_0\Gamma}{(p_0^2 - \Gamma^2 - \omega_p^2)^2 + 4p_0^2\Gamma^2}.
$$

(47)

From this spectral function we obtain the retarded Greens function and the kernel as

$$
G_R(p) = \frac{1}{(p_0 + i\Gamma)^2 - \omega_p^2}, \quad K(p) = p_0^2 - \Gamma^2 - \omega_p^2, \quad DK = p_0^2 + \Gamma^2 + \omega_p^2.
$$

(48)

The formula for the number of degrees of freedom \[36\] can be exactly evaluated:

$$
N_{dof} = \frac{\int dp_0}{2\pi} \frac{p_0^2 + \Gamma^2 + \omega_p^2}{(p_0^2 - \Gamma^2 - \omega_p^2)^2 + 4p_0^2\Gamma^2} = 1,
$$

(49)

it is independent on the width of the Lorentzian.

![Fig. 1](image.png)

**FIG. 1.** The $T^4$ normalized (a.) energy density and (b.) pressure of a system with spectral function of a single Lorentzian, compared to the same mass, zero width case. SB means Stefan-Boltzmann limit.

For the energy density and pressure we have to do numerical calculations. The result can be seen on Fig. 1. For comparison we have depicted the zero width (ie. Dirac-delta) case. We can see that despite the relatively large width ($\Gamma/m = 0.2$), the energy density and pressure are very close to the free case, especially in the low energy region. This makes understandable that the hadron resonance gas approximation works well in the low energy regime even for particles with relatively large width like the rho-meson.
C. Two Lorentzians

Now let us take two Lorentzians in the same quantum channel. The spectral function which we use consists of two independent Breit-Wigner functions:

\[ \rho(p) = \frac{4Z_{1p0}\Gamma_1}{(p_0^2 - \Gamma_1^2 - \omega_1^2)^2 + 4p_0^2\Gamma_1^2} + \frac{4Z_{2p0}\Gamma_2}{(p_0^2 - \Gamma_2^2 - \omega_2^2)^2 + 4p_0^2\Gamma_2^2}. \]  

(50)

The spectral function is shown on Fig. 2. Now the spectral function can be exactly represented by two finite width particles. The independent resonance approximation then would yield the sum of the single Lorentzian results: the number of degrees of freedom is 2, and the energy-density and pressure is the sum of the one-Lorentzian results:

\[ \varepsilon = \varepsilon_{1-Lorentzian}(m_1) + \varepsilon_{1-Lorentzian}(m_2) \]  

and \( p = p_{1-Lorentzian}(m_1) + p_{1-Lorentzian}(m_2) \).

Now we work out the result coming from the one-field representation which does not introduce new conserved quantities. The corresponding retarded Greens function is:

\[ G_R(p) = \frac{Z_1}{(p_0 + i\Gamma_1)^2 - \omega_1^2} + \frac{Z_2}{(p_0 + i\Gamma_2)^2 - \omega_2^2}, \]  

(51)

its inverse is the kernel of the Lagrangian, according to (11). From there we can calculate the number of dof (36), the energy density (32) and pressure (34). These latter need numerical calculations, the number of dof, however, can be worked out analytically, and it reflects well, what is going on when the two peaks start to merge.

By representing the spectral function with its pole structure, the formula for the number of dof (36) reads in this case \( N_{\text{dof}}^{(1)} = N_{\text{dof}}^{(1)} + N_{\text{dof}}^{(2)} \) where

\[ N_{\text{dof}}^{(1)} = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{DK(p)}{p_0} \frac{Z_1}{4im_1 p_0} \left[ \frac{1}{p_0 - i\Gamma_1 - m_1} - \frac{1}{p_0 + i\Gamma_1 + m_1} - \frac{1}{p_0 - i\Gamma_1 - m_1} + \frac{1}{p_0 + i\Gamma_1 + m_1} \right]. \]  

(52)

and \( N_{\text{dof}}^{(2)} \) is a similar expression with \( 1 \leftrightarrow 2 \) index change. We can close the contour of the integration from above (or below, the result is the same) to find

\[ N_{\text{dof}}^{(1)} = \frac{Z_1}{4m_1} \left[ \frac{DK(p)}{p_0} \bigg|_{p_0 = m_1 + i\Gamma_1} - \frac{DK(p)}{p_0} \bigg|_{p_0 = -m_1 + i\Gamma_1} \right]. \]  

(53)

Since \( DK(-p_0) = DK(p_0) \), the above expression is real. The actual expression is rather complicated, but with help of Mathematica one can work out the analytical formulae. The generic formula is rather lengthy, here we analyze the \( Z_1 = Z_2 \) and \( \Gamma_1 = \Gamma_2 \) case. Then the result reads

\[ N_{\text{dof}}^{(1)} = \frac{(2048\Gamma^8 + (m_1^2 - m_2^2)^4 + 16\Gamma^2(m_1^2 - m_2^2)^2(m_1^2 + m_2^2) + 512\Gamma^6(5m_1^2 + 3m_2^2) + 32\Gamma^4(23m_1^4 + 34m_1^2m_2^2 + 7m_2^4)}{(64\Gamma^4 + (m_1^2 - m_2^2)^2 + 16\Gamma^2(3m_1^4 + m_2^4))^2}. \]  

(54)
and for \(N_{\text{dof}}^{(2)}\) we obtain the same formula with \(1 \leftrightarrow 2\) change. When we choose \(m_1 = 1\), \(m_2 = 2\) then we can plot the degrees of freedom, it is shown in Fig. 3. To understand better these formulæ it is worth to work out the approximate formula valid in the \(\Gamma, |m_1 - m_2| \ll m_{1,2}\). We obtain

\[
N_{\text{dof}} = 1 + \frac{1 - y^2}{(1 + y^2)^2}, \quad \text{where} \quad y = \frac{4\Gamma}{|m_2 - m_1|}.
\]

Although it is just an approximate formula, it approximates well the correct expression, as Fig. 3 shows.

The robust feature of the number of dof is that for small width it starts at 2: this is the case of two Dirac deltas analyzed in Subsection VII A. On the other hand as \(\Gamma \rightarrow \infty\) the number of dof goes to one. The two peaks merge, and there is no way to say whether they were two separate peaks or not. Then the situation is similar to the one we considered in Section II. We can therefore trace how a two-degrees-of-freedom system becomes a one-degree of freedom system dynamically, without changing the number of representing fields manually.

At non-extremal cases, ie. when \(0 < \Gamma < \infty\), the number of dof is not a well-defined physical observable. As we can see, with the present definition (36) the number of dof can be smaller than one. Still, in order to determine the width when the the originally two-dof system turns into a one-dof system, this definition gives probably a sensible answer. Therefore we can claim that already at \(\Gamma/\Delta m \approx 0.2\) we have an approximately one-dof system, although in the spectral function the two peaks are clearly identifiable yet (cf. Fig. 2).

We remark here that this phenomenon is closely related to the resolution of the Gibbs paradox. The paradox is that if in a two dof system we continuously make the “difference” of the dof to disappear, then for arbitrarily small difference we observe a 2-dof system which suddenly, non-analytically changes to a 1-dof system at zero difference. The resolution, according to the above analysis is that this change is in fact analytical if we consider finite lifetime particles. A similar conclusion is drawn in [23]. Physically it means that we cannot perform a sufficiently long measurement which is accurate enough to identify the two separate energy levels in case of finite lifetime.

Although the number of dof describes well, what is going on, but we still have to see, how it is realized in the physically observable quantities. To this end we determined the energy density and pressure for the case of small and large width. The result can be seen on Fig. 4. The exact result for \(\Gamma = 0.2\) is the curve labeled by iii.). If we assumed that the two Lorentzians are independently contribute to the thermodynamics, then we would obtain curve ii.). For comparison we also show the results of the small width case (ie. two Dirac-deltas) as curve i.) and the result of one Lorentzian case with mass \(m = 1.2\) and width \(\Gamma = 0.2\) as curve iv.). It can be clearly seen that in the overlapping Lorentzian case the independent resonance approximation fails. While the results of the independent resonance approximation is close to the two Dirac-delta peaks case, the exact result is closer to the one Lorentzian case with appropriate parameters: ie. a one-dof approximation. Therefore the results of the thermodynamical calculation which provides physical observables supports the conclusion coming from the analysis of the number of dof (36): for \(\Gamma/\Delta m = 0.2\) the system with two Lorentzian peaks practically behave as a one-dof system.

D. Thresholds

We have seen in the previous subsection that two Breit-Wigner resonances in general are combined in a non-linear way. This phenomenon is in accordance with the scattering theory predictions [15], where the S-matrix contributions
FIG. 4. The $T^4$ normalized (a.) energy density and (b.) pressure of a system with spectral function consisting of two Lorentzian peaks with $m_1 = 1$, $m_2 = 2$ and common $\Gamma$ width, cf. eq. (50). The result for $\Gamma = 0.2$ is in both figures denoted by (iii.), the other curves are here for comparison. (i.) is the result of the two Dirac-delta case. (ii.) is the result of the independent resonance approximation. (iv.) is the result of one Lorentzian case with mass $m = 1.2$ and width $\Gamma = 0.2$. SB means Stefan-Boltzmann limit, 2SB is twice the Stefan-Boltzmann limit.

FIG. 5. The spectral function of a single threshold accuracy, with a number of massively overlapping Lorentzian peaks. Since the overall normalization drops out from the formulae, we do not even have to care about the correct normalization. The threshold is, however, very far from a particle-like spectral function: there is no peak, it does not vanish at infinity. We actually do not even expect that it describes a physical system on its own: rather it serves as a mathematical example for the strongly correlated peaks.

When we evaluate the number of dof using (36) we obtain for $m = 1$ the value $N_{dof} = 0.98$. However, when we use the formula for the energy density and pressure we obtain the curves of Fig. 6. These curves are very far from the multi-dof expectations. The most striking difference is in the amplitude: instead of having several times the Stefan-Boltzmann limit, we are below the $0.01 \times$ SB limit. This further supports the hadron resonance gas approximation where one neglects the contribution of the branch cuts to the thermodynamics. We see that in this pathological case the number of dof coming from (36) was also not informative, this seems to work only for particle-like systems.
is almost completely washed out. To understand how is it possible we recall that the single resonance contributions are complex (cf. [15]), but their sum finally must give a real number. So we have a large number of uncorrelated complex phases, and study their sum: it is then plausible that finally we obtain a near-zero net contribution.

We emphasize that this last example is non-physical. It serves for providing a warning signal that it can be very misleading if one tries to represent a spectral function with a large number of heavily overlapping peaks.

VI. CONCLUSIONS

In this paper we discussed the question of what kind of effective model can be built on an experimentally determined spectral function. The spectral function in general can consist of multiple finite width peaks and non-particle-like continuum contributions. One procedure can be to represent the spectral function with the sum of several Breit-Wigner spectra, and assign a new dof for each of these Breit-Wigner peaks which interact with each other in a way that the correct widths come out (independent resonance approximation). However, in this case each of the new degrees of freedom at tree level introduces a new conserved quantity (the corresponding particle number), which may alter the dynamical behavior of the system. While for large separation this procedure is known to work (cf. [17]), with massively overlapping peaks the independence of these resonances is questionable.

A more conservative way is to find a representation which does not use extra, non-physical dof. As a price we have to describe multiple peaks with a single field. This can be done consistently, defining carefully a non-local effective theory, and checking all the consistency requirements. In this case the number of dof is not the number of the representing fields (which is one), we can define it from the spectral function.

Having a physical representation at hand, we can study how well the independent resonance approximation works. We have found that for infinite lifetime excitations (i.e. a spectral function which consists of discrete Dirac-delta peaks), the two methods provide identical results. So, as we have expected, infinitely well separated peaks indeed describe independent dof. But if the peaks are finite width Breit-Wigner resonances and the peak separation becomes comparable with the width, this nice agreement starts to fail. As we have shown, the relevant parameter is $\Gamma/\Delta m$: a two-peak spectral function with $\Gamma/\Delta m = 0.2$ already behaves from the point of view of thermodynamics as a one-dof system. An even more striking example for the non-independence of the overlapping Breit-Wigner peaks is the case of a mathematical construction of a threshold-like spectral function. Although it could be represented with a sum of multiple, overlapping Breit-Wigner resonances, the energy density of this model is still about 100-times smaller than that of a one-particle system.

We may conclude finally that a multi-dof representation of the spectral functions can yield very misleading results. According to the numerical evidences of this paper, the peaks of the spectral function represent independent dof only in the case when the width of the peaks is smaller than about $0.1 \times$ the separation of the peaks. In other cases one has to use a one-dof representation in order not to change the original symmetries of the system, and obtain physically correct results.
ACKNOWLEDGMENTS

The author is grateful to V. Koch for long and instructive discussions, and for kind hospitality at BNL. The author thanks D. Anchishkin, T.S. Biró, S.D. Katz, A. Patkós, P. Petreczky and Zs. Szép for useful discussions. This work is supported by the Hungarian Research Fund (OTKA) under contract No. K68108.

[1] J. I. Kapusta and C. Gale, “Finite-temperature field theory: Principles and applications”, Cambridge, UK: Univ. Pr. (2006) 428 p
[2] B. W. Lee, “Chiral dynamics”, Gordon and Breach Science Publishers (1972) 117 p
[3] A. Andronic, P. Braun-Munzinger, K. Redlich and J. Stachel, Phys. Lett. B 571, 36(2003) nucl-th/0303036
[4] F. Karsch, K. Redlich and A. Tawfik, Eur. Phys. J. C 29, 549(2003) hep-ph/0303108
[5] P. Huovinen and P. Petreczky, Nucl. Phys. A 837, 26 (2010) [arXiv:0912.2541 [hep-ph]]
[6] S. Borsányi, G. Endrodi, Z. Fodor, A. Jakovac, S. D. Katz, S. Krieg, C. Ratti and K. K. Szabo, JHEP 1011, 077 (2010) [arXiv:1007.2580 [hep-lat]].
[7] Y. B. Ivanov, J. Knoll and D. N. Voskresensky, Nucl. Phys. A 657, 413 (1999) [hep-ph/9807351].
[8] Y. B. Ivanov, J. Knoll and D. N. Voskresensky, Nucl. Phys. A 672, 313 (2000) [nucl-th/9905028].
[9] J. Knoll, Y. B. Ivanov and D. N. Voskresensky, Annals Phys. 293, 126 (2001) [nucl-th/0102044].
[10] J. Knoll, F. Riek, Y. B. Ivanov and D. N. Voskresensky, J. Phys. Conf. Ser. 35, 357 (2006) [nucl-th/0512040].
[11] R.E. Peierls, Surprises in Theoretical Physics (Princeton University Press, 1979)
[12] L.D. Landau, E.M. Lifshitz, Course of Theoretical Physics, Vol. 3: Quantum mechanics (Butterworth-Heinemann, 1977)
[13] J. Cleymans and D. Worku, Mod. Phys. Lett. A 26, 1197 (2011) [arXiv:1103.1463 [hep-ph]].
[14] N. Hu, Phys. Rev. 74, 131 (1948).
[15] M. Svec, Phys. Rev. D 64, 096003 (2001) [hep-ph/0009275].
[16] H. Feshbach, Ann. Phys. 43, 110 (1967), L. Rosenfeld, Acta Phys. Polonica A 38, 603 (1970).
[17] R.F. Dashen and R. Rajaraman, Phys.Rev. D10 (1974) 694; Phys.Rev. D10 (1974) 708.
[18] J. M. Luttinger and J. C. Ward, Phys. Rev. 118 (1966) 1417. G. Baym, Phys. Rev. 127 1391 (1962); J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D10 2428 (1974).
[19] M. Le Bellac, Thermal Field Theory, (Cambridge Univ. Press, 1996.)
[20] A. Jakovac, Phys. Rev. D 76, 125004 (2007) [hep-ph/0612268].
[21] J. Polonyi and A. Siwek, Phys. Rev. D 81, 085040 (2010) [arXiv:1004.0789 [hep-th]].
[22] C.F. von Weizsacker, Z. Phys. 88, 612-625 (1934), E.J. Williams, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. 13, No. 4 (1935).
[23] A.E. Allahverdyan, Th.M. Nieuwenhuizen, Phys. Rev. E 73, 066119 (2006)