SMALL CIRCULANT COMPLEX HADAMARD MATRICES OF BUTSON TYPE

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Abstract. We study the circulant complex Hadamard matrices of order \( n \) whose entries are \( l \)-th roots of unity. For \( n = l \) prime we prove that the only such matrix, up to equivalence, is the Fourier matrix, while for \( n = p + q, l = pq \) with \( p, q \) distinct primes there is no such matrix. We then provide a list of equivalence classes of such matrices, for small values of \( n, l \).

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1. Introduction and results

A complex Hadamard matrix of order \( n \) is a matrix \( H \) having as entries complex numbers of modulus 1, such that \( H/\sqrt{n} \) is unitary. Among complex Hadamard matrices, those with all entries roots of unity are said to be of Butson type. The basic example here is the Fourier matrix, \( F_n = (w^{ij})_{ij} \) with \( w = e^{2\pi i/n} \):

\[
F_n = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & w & w^2 & \cdots & w^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2}
\end{pmatrix}.
\]

We denote by \( C_n(l) \) the set of complex Hadamard matrices of order \( n \), with all entries being \( l \)-th roots of unity. As a first example here, observe that we have \( F_n \in C_n(n) \).

In general, the complex Hadamard matrices are known to have applications to a wide array of questions, ranging from electrical engineering to von Neumann algebras and quantum physics, and the Butson ones are known to be at the “core” of the theory. For further details here, we

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recommend the excellent survey article by Tadej and Życzkowski [TZ06], and the subsequent website, made and maintained in collaboration with Bruzda[1].

We are more specifically interested in understanding the complex Hadamard matrices of Butson type which are circulant, that is, of the form \((H_{ij})_{i,j=1,...,n}\) with \(H_{ij}\) depending only on \(i-j\). We denote by \(C_n^{\text{circ}}(l)\) the set of circulant matrices in \(C_n(l)\), and by \(C_n^{\text{circ},1}(l)\) the set of matrices from \(C_n^{\text{circ}}(l)\) with 1 on the diagonal.

Regarding the motivations for the study of such matrices, let us mention: (1) their key importance for the construction of complex Hadamard matrices, see [TZ06], (2) their relation with cyclic \(n\)-roots and their applications, see [Bj¨ o90], and (3) their relation with the Circulant Hadamard Conjecture, a beautiful mathematical problem, to be explained now.

In the real case – that is, for \(l=2\) – only one example is known, at \(n=4\), namely:

\[
K_4 = \begin{pmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1
\end{pmatrix}.
\]

For larger values of \(n\) it is conjectured that there is no example:

**Conjecture 1.1** (Circulant Hadamard Conjecture). \(C_n^{\text{circ}}(2) = \emptyset\) for all \(n > 4\).

We explain below how a certain duality between circulant complex Hadamard matrices makes it possible to reformulate this conjecture in terms of Hermitian circulant Hadamard matrices, therefore opening new ways of trying to solve it. We believe that this duality is actually a good reason to study the circulant complex Hadamard matrices.

For larger values of \(l\), however, things are different and, to a large extent, mysterious. Our main goal here is to make some progress on the understanding of the values of \(n, l\) which allow the existence of circulant Hadamard matrices of Butson type.

It is known since [Bac89], [Fau01] that \(F_n\) is equivalent to a circulant matrix, and it follows that \(C_n^{\text{circ}}(n) \neq \emptyset\). We will prove below that when \(n\) is prime, the Fourier matrix (in circulant form) is the only example, that is, any matrix in \(C_n^{\text{circ}}(n)\) is equivalent to the Fourier matrix.

We describe in Section 3 a duality which is not new (see McKay and Wang [MWS1]) but can serve as a motivation for studying the complex circulant Hadamard matrices, and can suggest new approaches for important problems like the Circulant Hadamard Conjecture. This duality is a map \(C_n^{\text{circ}}(\infty) \to C_n^{\text{circ}}(\infty)\) which has a number of interesting properties, in particular it sends \(C_n^{\text{circ}}(2)\), the set of real circulant Hadamard matrices, to \(C_n^{\text{circ},sa}(\infty)\), the set of complex Hermitian circulant Hadamard matrices. So Conjecture 1.1 can be reformulated as stating the non-existence of Hermitian complex circulant Hadamard matrices of order \(n > 4\). Section 4 describes some other properties of this duality.

The first classification result that we present is for circulant \(p \times p\) Butson matrices based on a \(p\)-th root of unity, where \(p\) is a prime number:

**Theorem 1.2.** Any matrix in \(C_p^{\text{circ}}(p)\) is equivalent to the Fourier matrix \(F_p\).

Here, as in the rest of the paper, the equivalence relation which is considered among circulant matrices is the cyclic permutation of rows and columns, and multiplication of all entries by a constant. However, since the Fourier matrix is not circulant, we need a wider notion of

[1]http://chaos.if.uj.edu.pl/~karol/hadamard
equivalence in the statement of this theorem. More precisely, we use the standard notion of equivalence for (non-circulant) complex Hadamard matrices, that is, permutation of the rows and columns, and multiplication of each row and each column by a constant.

Our second result is a non-existence statement for circulant Butson matrices based on two different prime numbers:

**Theorem 1.3.** Let \( p, q \geq 5 \) be two distinct primes. Then \( C_{p+q}^{\text{circ}}(pq) = \emptyset \).

A large part of the proof is actually valid for non-circulant matrices. To put this result in perspective, recall the following result [BBS09, Theorem 7.9]. Assume \( C_n(l) \neq \emptyset \).

1. If \( n = p + 2 \) with \( p \geq 3 \) prime, then \( l \neq 2p^b \).
2. If \( n = 2q \) with \( p > q \geq 3 \) primes, then \( l \neq 2^a p^b \).

It follows from the first result that for \( p \geq 3 \) prime, \( C_{p+2}(2p) = \emptyset \), so \( C_{p+2}^{\text{circ}}(2p) = \emptyset \). However it remains unclear whether \( C_{p+3}(3p) = \emptyset \) for \( p \geq 5 \) prime.

**Question 1.4.** Let \( p, q \) be two distinct primes. Is it true that \( C_{p+q}(pq) = \emptyset \)?

The last main contribution of the present paper is an “experimental” study of the set of equivalence classes of matrices in \( C_n^{\text{circ}}(l) \), for small values of \( n, l \). See Section 7. This study was done using a computer program, designed to find all possible equivalence classes of circulant matrices which can contain a matrix of Butson type. At the counting level, the main results are presented in Table 1.

The classification results obtained by this experimental approach lead to a number of quite natural questions, for instance the following ones:

- What are the obstructions to the existence of a matrix in \( C_n^{\text{circ}}(l) \) for various values of \( n, l \)? In the non-circulant case, there are a few obstructions known, which explain quite well, for small \( n, l \), when \( C_n(l) = \emptyset \). In the circulant case, however, few obstructions are known beyond those known for general (non-circulant) Butson matrices.
- For \( p \) prime and \( k \in \mathbb{N} \), does \( C_p^{\text{circ}}(kp) \) contain any matrix other than those obtained from the Fourier matrix \( F_p \)?
- What is the number of elements of \( C_4^{\text{circ}}(l), C_8^{\text{circ}}(l), C_9^{\text{circ}}(l) \), with \( l \in \mathbb{N} \) arbitrary? For instance, what are the next terms in the sequence defined as the number of equivalence classes in \( C_9^{\text{circ}}(3k) \), with first terms 6, 24, 62, 108, 172?

Finding answers to these questions might ultimately lead to progress on Conjecture 1.1.

**Acknowledgements.** Theorem 5.1 was obtained thanks to the website MathOverflow. Shortly after we posted a question\(^2\) which is a more elementary but equivalent statement of Theorem 5.1, Noam Elkies posted an answer to this question which is basically a proof of Theorem 5.1. Peter Müller then posted another answer pointing to the papers by Gluck, by Rónyai and Szőnyi, and by Hiramine, containing the “original” proofs of this result. We are grateful to Noam Elkies and Peter Müller for their contributions.

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\(^2\)http://mathoverflow.net/questions/135949
2. Circulant Butson matrices

We consider in this paper various $n \times n$ matrices over the real or complex numbers. The matrix indices will range in $\{0, 1, \ldots, n - 1\}$, and will be taken modulo $n$.

Definition 2.1. A complex Hadamard matrix is a square matrix $H \in M_n(C)$, all whose entries are on the unit circle, $|H_{ij}| = 1$, and whose rows are pairwise orthogonal.

It follows from basic linear algebra that the columns are pairwise orthogonal as well. In fact, the $n \times n$ complex Hadamard matrices form a real algebraic manifold, given by:

$$C_n(\infty) = M_n(T) \cap \sqrt{n}U(n)$$

Here, and in what follows, we denote by $T$ the unit circle in the complex plane.

The basic example is the Fourier matrix, $F_n = (w^{ij})_{ij}$ with $w = e^{2\pi i/n}$. There are many other interesting examples, see [TZ06]. These examples are often constructed by using roots of unity, and we have the following definition, going back to Butson’s work [But62]:

Definition 2.2. The Butson class $C_n(l)$ with $l \in \{2, 3, \ldots, \infty\}$ consists of the $n \times n$ complex Hadamard matrices having as entries the $l$-th roots of unity. In particular:

1. $C_n(2)$ is the set of usual (real) $n \times n$ Hadamard matrices.
2. $C_n(4)$ is the set of $n \times n$ Turyn matrices, over $\{\pm 1, \pm i\}$.
3. $C_n(\infty)$ is the set of all $n \times n$ complex Hadamard matrices.

As explained in the introduction, we will be mostly interested in the circulant case, with the Circulant Hadamard Conjecture (CHC) in mind. Observe that the CHC states that, for $n > 4$:

$$C_n^{\text{circ}}(2) = \emptyset.$$

Our idea here will be to study the set $C_n^{\text{circ}}(l)$, for general exponents $l \in \{2, 3, 4, \ldots\}$, and then to connect this study to the CHC itself, via a Fourier-type duality.

In order to discuss a few examples, we use Björck’s formalism [Bjö90]:

Proposition 2.3. Assume that $H \in M_n(T)$ is circulant, $H_{ij} = \xi_{j-i}$. Then $H$ is complex Hadamard if and only if the vector $(z_0, z_1, \ldots, z_{n-1})$ given by $z_i = \xi_i/\xi_{i-1}$ satisfies:

$$z_0 + z_1 + \cdots + z_{n-1} = 0$$
$$z_0z_1 + z_1z_2 + \cdots + z_{n-1}z_0 = 0$$
$$\cdots$$
$$z_0z_1 \cdots z_{n-2} + \cdots + z_{n-1}z_0 \cdots z_{n-3} = 0$$
$$z_0z_1 \cdots z_{n-1} = 1.$$

If so is the case, we say that $z = (z_0, \ldots, z_{n-1})$ is a cyclic $n$-root.

We refer to [Bjö90] for the proof of this result. Observe that the set of cyclic $n$-roots is given by $X_n = C_n^{\text{circ}}(\infty)/T$, the quotient by $T$ coming from the choice of $\xi_0$. Given a cyclic $n$-root $z = (z_0, \ldots, z_{n-1})$ it is customary to set $\xi_0 = 1$, so that:

$$\xi = (1, z_1, z_1z_2, z_1z_2z_3, \ldots, z_1z_2 \cdots z_{n-1}).$$
On the other hand, it is also customary to dephase the complex Hadamard matrices. If we dephase the matrix associated to a cyclic \( n \)-root, we obtain:

\[
H_{ij} = \frac{z_{n-i+1}z_{n-i+2} \cdots z_n}{z_{j-i+1}z_{j-i+2} \cdots z_j}.
\]

As a basic application, we have the following result:

**Proposition 2.4.** The Fourier matrix \( F_n \) can be put in circulant form, the corresponding cyclic \( n \)-root being \( z_k = e^{2\pi i/n} \) for \( n \) odd, and \( z_k = e^{(2k+1)\pi i/n} \) for \( n \) even.

**Proof.** Let \( w = e^{2\pi i/n} \) and set \( z_k = w^k \). Then the above = 0 equations for a cyclic \( n \)-root are all satisfied, and we have \( z_0z_1 \cdots z_{n-1} = w^{\frac{n(n-1)}{2}} = e^{\pi i(n-1)} = (-1)^{n-1} \).

Thus when \( n \) is odd we obtain a cyclic \( n \)-root, which is the one in the statement. In the case where \( n \) is even the above product is \(-1\), so by replacing \( z_k \rightarrow e^{\pi i/n}z_k \) the product becomes 1, and we obtain a cyclic \( n \)-root, which is the one in the statement.

The associated circulant matrix is then \( H_{ij} = (w^{n-j})^i = w^{-ij} \), and this matrix is equivalent to \( F_n \), by interchanging the columns \((i, n-i)\), for any \( i = 1, 2, \ldots, [n/2] \).

The following generalization of Proposition 2.4 comes from Backelin [Bac89]:

**Proposition 2.5.** For \( m|n \) the \( nm \times nm \) matrix \( H_{ia,jb} = w^{-(mij+ib+ja)} \), with \( w = e^{2\pi i/n} \), can be put in circulant form, the corresponding cyclic \( mn \)-root being

\[
\rho = \left( \frac{1, \ldots, 1, w, \ldots, w^2, \ldots, w^{n-1}, \ldots, w^{n-1}}{m, \ldots, m, m, \ldots, m} \right)
\]

where \( \rho = e^{\pi i/mn} \), and where \( c = m(n-1) \mod 2 \).

**Proof.** First, the = 0 equations for a cyclic root are satisfied, see [Bac89] or [Fan01]. The = 1 equation is satisfied as well, because the total product is given by:

\[
P = \rho^{mn}(1 \cdot w \cdot w^2 \cdots w^{n-1})^m = (-1)^c w^{\frac{mn(n-1)}{2}} = (-1)^{c+m(n-1)} = 1.
\]

Regarding now the corresponding matrix, observe first that, in double index notation, with \( ia, jb \in \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\} \), the general formula is:

\[
H_{ia,jb} = \frac{z_{(ia)-0}z_{(ia)+1}z_{(ia)+2} \cdots z_{n0}}{z_{(jb)-0}z_{(jb)+1}z_{(jb)+2} \cdots z_{j0}}.
\]

More precisely, we have \( H_{ia,jb} = \frac{L_{ia,jb}}{L_{ia,jb}} \), where \( L_{ia,jb} \) is the denominator of the above fraction. This denominator is best written backwards, as follows:

\[
L_{ia,jb} = \frac{z_{jb}z_{j,b-1} \cdots z_{j,0}}{z_{j-1,m-1}z_{j-1,m-2} \cdots z_{j-1,0}}
\]

\[
\cdots z_{j-i+1,m-1}z_{j-i+1,m-2} \cdots z_{j-i+1,0}
\]

\[
\cdots z_{j-i,m-1}z_{j-i,m-2} \cdots z_{j-i,b-a+1}.
\]

Indeed, the indices in the above product are decreasing, starting from \((jb)\), and their number is \((b + 1) + m(i - 1) + (m - b + a - 1) = mi + a\), which is the correct one.
Now by getting back to the cyclic root in the statement, in double index notation this is given by $z_{ia} = \rho^c w^i$. Since the $\rho^c$ factors will cancel in the expression of $H_{ia,jb}$, we can assume for the purposes of our computation that we have $z_{ia} = w^i$, and we obtain:

\[
L_{ia,jb} = (w^j)^{b+1}(w^{j-1})^m \cdots (w^{j-i+1})^m (w^{j-i})^{m-b+a-1} = w^{i(b+1)}(w^{(j-1)+(j-2)+\cdots+(j-i+1)})^m w^{(j-i)(m-b+a-1)} = w^{i(b+1)+mj(i-1)-\frac{i(i-1)}{2}+m(j-i)(m-b+a-1)}.
\]

In particular at $(jb) = (n0)$ we obtain, by using $w^n = 1$:

\[
L_{ia,n0} = w^{-\frac{i(i-1)}{2}+mj(n+a-1)}.
\]

Now by using $H_{ia,jb} = \frac{L_{ia,n0}}{L_{ia,jb}}$, we obtain the following formula for our matrix:

\[
H_{ia,jb} = w^{-i(m+a-1)-j(b+1)-mj(i-1)-(j-i)(m-b+a-1)} = w^{-(mj+ib+ja)}.
\]

Thus we have obtained the formula in the statement, and we are done.

The construction in [Bac89] is in fact a bit more general, with some free parameters belonging to $\mathbb{T}$ added, somehow in the spirit of the Dia deformation method [Dit04], and with the main consequence being the fact that for $m|n$ and $m \geq 2$, we have $\#C_{mn}^{circ}(\infty) = \infty$. See [Bac89], [Fau01].

Finally, we have the following result, due to Haagerup [Haa08]:

**Theorem 2.6.** When $n = p$ is prime we have $\#C_p^{circ,1}(\infty) \leq \left(\frac{2^{p-2}}{p-1}\right)$.

The proof of this result uses algebraic geometry, and a theorem of Chebotarev, the idea being that for $p$ prime, the number of circulant $p \times p$ complex Hadamard matrices, counted with certain algebraic geometry multiplicities, is exactly $\left(\frac{2^{p-2}}{p-1}\right)$. See [Haa08].

The above results suggest the following conjecture:

**Conjecture 2.7.** $C_n^{circ}(\infty)$ is finite if and only if $n = p_1 \cdots p_k$, with $p_i$ distinct primes. In addition, in this case, there should be an explicit bound, of type $\#C_n^{circ,1}(\infty) \leq 4^n$.

This conjecture is compatible with above-mentioned results in [Bac89], [Haa08], and is compatible as well with various numeric findings, reported in [Fau01], [Haa08].

Regarding now the Butson matrix case, a key problem here concerns the obstructions coming from the work of Lam-Leung [LL00], de Launey [Lau94] and Turyn [Tur65]:

**Problem 2.8.** What are the precise, final statements of the Lam-Leung, de Launey and Turyn obstructions, for the circulant Butson matrices?

We refer to our previous paper [BNS12] for technical details regarding these questions, and for various advances on them. See also [ALM02], [LS12], [Sch99].

### 3. Eigenvalues, duality

We fix $n \in \mathbb{N}$ and we denote by $F = F_n/\sqrt{n}$ the normalized Fourier matrix.

Given a column vector $q = (q_0, \ldots, q_{n-1})^t$, we agree to denote by $Q \in M_n(\mathbb{C})$ the diagonal matrix $Q = \text{diag}(q_0, \ldots, q_{n-1})$. With these notations, we have the following well-known result:

**Proposition 3.1.** The circulant matrices are Fourier-diagonal, as follows:
(1) $M_n(\mathbb{C})^\text{circ} = \{FQF^*| q \in \mathbb{C}^n\}$.
(2) $U(n)^\text{circ} = \{FQF^*| q \in \mathbb{T^n}\}$.
(3) $O(n)^\text{circ} = \{FQF^*| q \in \mathbb{T^n}, q_i = q_{-i}, \forall i\}$.

In addition, the first row vector of $FQF^*$ is given by $\xi = Fq/\sqrt{n}$.

**Proof.** Here (1) and the last assertion are well-known, (2) is clear, and (3) comes from
the formula $Fq = Fq'$, with $q'_i = q_{-i}$. See e.g. [BNS12].

The following result basically goes back to the paper of McKay and Wang [MW81]:

**Theorem 3.2.** We have a duality

$$C_n^\text{circ}(\infty) \simeq C_n^\text{circ}(\infty)$$

$$\cup \quad \cup$$

$$C_n^\text{circ}(2) \simeq C_n^\text{circ,sa}(\infty)$$

given at the level of first row vectors by \(x \to Fx\).

**Proof.** Consider the map \(\Phi : M_n(\mathbb{C})^\text{circ} \to M_n(\mathbb{C})^\text{circ}\) given by \(\Phi((x_{j-i})_{ij}) = ((Fx)_{j-i})_{ij}\), as in
the statement. By using the formulæ in Proposition 3.1, we have:

\[H \in \sqrt{n}U(n) \iff \Phi(H) \in M_n(\mathbb{T}),\]

\[H \in \sqrt{n}O(n) \iff \Phi(H) \in M_n(\mathbb{T})^{\text{sa}}.\]

Now since the inverse of \(\Phi\) is implemented by \(y \to F^*y\), which is of the same nature as
\(x \to Fx\), these equivalences hold as well in the reverse sense, and this gives the result. \qed

One question raised by the above result is that of finding an intermediate duality $X \simeq Y$,
with the sets $X, Y$ at the same time “small”, and so prone to investigation, but not empty.

One answer here comes from the following computation, with $y = Fx$:

$$\sum_i y_i^p = n^{-p/2} \sum_{k_1, \ldots, k_p} w^{(k_1+\cdots+k_p)}x_{k_1} \cdots x_{k_p} = n^{1-p/2} \sum_{k_1+\cdots+k_p=0} x_{k_1} \cdots x_{k_p}.\]$$

Indeed, by using this formula at $p = 1, 2$ we obtain:

**Proposition 3.3.** The following sets are stable by the duality:

(1) $X_1 = \{(x_{j-i}) \in C_n^\text{circ}(\infty)| \sum_i x_i \in \mathbb{R}, x_0 \in \mathbb{R}\}$.
(2) $X_2 = \{(x_{j-i}) \in C_n^\text{circ}(\infty)| \sum_{ij} x_i x_j \in \mathbb{R}, \sum_{i+j=0} x_i x_j \in \mathbb{R}, \sum_i x_i^2 \in \mathbb{R}, x_0^2 \in \mathbb{R}\}$.

**Proof.** We have $\sum_i y_i = \sqrt{n}x_0$, and by symmetry we have as well $\sum_i x_i = \sqrt{n}y_0$, and this gives (1). Also, we have $\sum_{ij} y_i y_j = (\sum_i y_i)^2 = nx_0^2$ and $\sum_i y_i^2 = \sum_{i+j} x_i x_j$, and by symmetry we have as well $\sum_{ij} x_i x_j = ny_0^2$ and $\sum_i x_i^2 = \sum_{i+j} y_i y_j$, and this gives (2). \qed

In the general case now, where $p \in \mathbb{N}$ is arbitrary, given a partition $\pi \in P(p)$ and a multi-
index $i = (i_1, \ldots, i_p)$ we write $i \in \pi$ if $a \sim_{\pi} b \implies i_a = i_b$, and we write $i \vdash \pi$ whenever
$\sum_{r \in \beta} i_r = 0$, for any block $\beta$ of $\pi$. With these notations, we have:
Proposition 3.4. For any \( p \in \mathbb{N} \) the duality leaves stable the set
\[
X_p = \{(x_{j-i}) \in C_n^{\text{circ}}(\infty) \mid f_\pi(x), g_\pi(x) \in \mathbb{R}, \forall \pi \in P(p)\}
\]
where \( f_\pi(x) = \sum_{i \in \pi} x_i \cdots x_{i_p} \), and where \( g_\pi = \sum_{i \not\in \pi} x_i \cdots x_{i_p} \).

Proof. Let \( \beta \) denote the partition with one block, \( \beta = \{1, \ldots, p\} \). Since \( f_\beta(x) = \sum x_i^p \) and \( g_\beta(x) = \sum x_i x_{i+i} \cdots x_{i_p} \), the above general formula \( \sum_i y_i^p = n^{1-p/2} \sum k_1 + \cdots + k_p = 0 x_{k_1} \cdots x_{k_p} \) reads \( f_\beta(y) = n^{1-p/2} t \beta(x) \). Now by reasoning over blocks, this gives \( f_\pi(y) = n^{1-p/2} g_\pi(x) \), where \(|.|\) is the number of blocks. By symmetry we have as well \( g_\pi(y) = n^{1-p/2} f_\pi(x) \), and this gives the result. \( \square \)

As a main consequence, for any \( I \subset \mathbb{N} \), the set \( X_I = \bigcup_{p \in I} X_p \) is stable by duality:
\[
C_n^{\text{circ}}(\infty) \simeq C_n^{\text{circ}}(\infty) \quad \cup \quad X_I \simeq X_I
\]

Observe that \( C_n^{\text{circ}}(2) \subset X_I \), for any \( I \). We believe that a systematic study of the sets \( X_I \) can be of help in connection with the CHC, but we have no further results here.

4. Hermitian circulant matrices

We have seen in the previous section that we have a duality \( C_n^{\text{circ}}(2) \simeq C_n^{\text{circ,sa}}(\infty) \). This duality suggests the following two-step approach to the CHC:

Problem 4.1. Consider the Butson class \( C_n(l) \), with \( l \in \{2, 3, \ldots, \infty\} \).

(1) Why \( H \in C_n^{\text{circ,sa}}(\infty) \) should imply \( H \in C_n(l) \), for some \( l < \infty \)?

(2) For finite exponents \( l < \infty \), why should \( C_n^{\text{circ,sa}}(l) \) be empty, for \( n > 4 \)?

Observe that solving both the above questions (1) and (2) would be the same as solving the CHC. Regarding the first question, here we don’t have any idea for the moment.

Regarding the second question, the straightforward approach would be by first understanding the class \( C_n^{\text{circ}}(l) \), and then restricting attention to the self-adjoint case. And, from this point of view, the various questions raised in section 1 above are of course of great interest.

Let us recall now the answer to question (2) for the exponents \( l = 2, 4 \), where the situation is quite special, and doesn’t require a systematic study of \( C_n^{\text{circ}}(l) \):

Theorem 4.2. The following hold, for any \( n > 4 \):

(1) Brualdi and Newman [Bru65]: \( C_n^{\text{circ,sa}}(2) = \emptyset \).

(2) Craigen and Kharaghani [CK93]: \( C_n^{\text{circ,sa}}(4) = \emptyset \).

Proof. (1) There are several proofs of this fact, see [Bru65], [MW87]. In what follows we will view this statement as a particular case of the \( l = 4 \) result in [CK93], explained below.

(2) Here is the proof in [CK93]. Let \( H \in C_n^{\text{circ,sa}}(\infty) \), and write \( H = A + iB \) with \( A, B \in M_n(\mathbb{R}) \). The claim is that \( K = A + B \) belongs to \( \sqrt{n}O(n) \), \( K^2 \) is symmetric, and \( K^4 = n^2 I \).

Indeed, observe first that \( H = H^* \) implies \( A = A^t \) and \( B = -B^t \). Also, \( A, B \) inherit the circulant property from \( H \), so in particular they commute, and we have:
\[
H^2 = (A + iB)^2 = A^2 - B^2 + 2iAB.
\]
Thus from $H^2 = nI$ we obtain $A^2 - B^2 = nI$ and $AB = 0$, and it follows that:

$$KK^t = (A + B)(A - B) = A^2 - B^2 = nI.$$  

This shows that we have $K \in \sqrt{n}O(n)$, as claimed. Regarding now the second assertion, this follows from $K^2 = A^2 + B^2$, because both $A^2, B^2$ are symmetric. Finally, the last assertion follows from the first two ones, because $K^4 = K^2(K^2)^t = (KK^t)^2$.

With this claim in hand, let $H \in C_n^{circ,sa}(4)$, and write $H = A + iB$, and set $K = A + B$, as above. From the $l = 4$ Butson matrix condition we obtain $A, B \in M_n(0, 1)$, with disjoint supports, and it follows that $K$ is a $\pm 1$ matrix. Thus $K \in C_n^{circ}(2)$, and $K^4 = n^2I$.

Consider now the 0-1 matrix $A = (K + J)/2$. Since $A$ is circulant it is stochastic, i.e. we have $KJ = JK = rJ$ for some integer $r$, where $J$ is the all-one matrix. Together with $K^4 = n^2I$ this shows that $A$ satisfies an equation of the following type, with $x, y \in \mathbb{Z}$:

$$A^4 = xI + yJ.$$  

Now by [Ma84], $A$ must be either $0, P, J$ or $P - J$, for some permutation matrix $P$. Thus $K$ itself must be either $\pm J$ or $\pm (J - 2P)$, and it follows that $r$ must be $\pm n$ or $\pm (n - 2)$. But, since the circulant Hadamard matrices are $\pm \sqrt{n}$-stochastic, we have $\sqrt{n} = n - 4$, and so $n = 4$. □

The above result and its proof raise the following question:

**Problem 4.3.** Is there a Ma-type formulation and proof of $C_n^{circ,sa}(l) = \emptyset$ at $n > 4$, for other finite exponents, $l \neq 2, 4$?

The exponents $l = 3, 8$ are probably those to be tried first, but we have no results.

### 5. Circulant matrices of prime order

We fix a prime number $p$, and we let $\omega = e^{2\pi i/p}$. The proof of Theorem 1.2 is based on:

**Theorem 5.1.** Let $M = \{(m_{ij})_{i,j=1}^n\} \in M_p(\mathbb{Z}/p\mathbb{Z})$ be a circulant matrix. Then $U = (\omega^{m_{ij}})_{i,j=0}^{n-1}$ is a circulant complex Hadamard matrix if and only if its first row is given by a polynomial of degree 2, that is, if and only if there are $a, b, c \in \mathbb{Z}/p\mathbb{Z}$ with $a \neq 0$ such that, for all $j \in \mathbb{Z}/p\mathbb{Z},$

$$m_{0j} = aj^2 + bj + c.$$  

We first reformulate this theorem in a more combinatorial but clearly equivalent form:

**Theorem 5.2.** Let $p \geq 3$ be a prime number, and let $u : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be a map such that, for all $l \in \mathbb{Z}/p\mathbb{Z}, l \neq 0$, the map $k \mapsto u(k + l) - u(k)$ is a permutation. Then $u$ is a polynomial of degree 2.

Under this form, the statement is well-known to specialists of finite geometry, who call the above functions “planar”. The fact that any planar function on a field of prime order is a quadratic polynomial was proved independently by Gluck [Glu90], by Rónyai and Szőnyi [RS89] and by Hiramine [Hir89]. See the introduction/acknowledgements.

**Theorem 5.3.** Let $M = \{(m_{ij})_{i,j=0}^{n-1}\} \in M_p(\mathbb{Z}/p\mathbb{Z})$ be a circulant matrix. If the first row of $M$ is given by a polynomial of degree 2 then $U = (\omega^{m_{ij}})_{i,j=0}^{n-1}$ is equivalent to the Fourier matrix.
Suppose first that the first row of $M$ is given by a polynomial of degree 2. This means that there are elements $a, b, c \in \mathbb{Z}/p\mathbb{Z}$ with $a \neq 0$ such that
\[
\forall j \in \mathbb{Z}/p\mathbb{Z}, \quad m_{0j} = aj^2 + bj + c.
\]
Since $M$ is circulant, this means that
\[
\forall i, j \in \mathbb{Z}/p\mathbb{Z}, \quad m_{ij} = a(j-i)^2 + b(j-i) + c.
\]

First, we remove from each entry of $M$ the first entry of the same column. We get a new matrix $M'$ equivalent to $M$, with entries given by
\[
m'_{ij} = m_{ij} - m_{0j} = a((j-i)^2 - j^2) - bi = a^2 - 2aij - bi.
\]

Second, we remove from each entry of this matrix $M'$ the first entry of its row. We obtain a new matrix $M''$ which is equivalent to $M'$ and therefore to $M$, with entries
\[
m''_{ij} = m'_{ij} - m'_{i0} = -2aij.
\]

Third, we permute the rows by sending row $i$ to row $-2ai$. We finally obtain a matrix $M'''$, still equivalent to $M$, with entries
\[
m'''_{ij} = ij.
\]

Clearly this is the Fourier matrix, so $M$ is equivalent to $F_p$. \hfill \Box

6. Matrices based on two prime numbers

We now turn to the proof of Theorem 1.3. Note that most of the proof deals with non-circulant complex Hadamard matrices of Butson type.

In what follows, $p$ and $q$ denote two distinct primes. We consider a matrix $M \in C_{p+q}(pq)$, and use additive notations, so that $M \in M_{p+q}(\mathbb{Z}/pq\mathbb{Z})$. We call $L_i, i = 0, \ldots, p+q-1$ the rows of $M$, and for $i, j \in \mathbb{Z}/(p+q)\mathbb{Z}$ we set $L_{ij} = L_j - L_i$.

The fact that $M$ is Hadamard translates as the following basic property.

**Lemma 6.1.** For all distinct $i, j \in \mathbb{Z}/(p+q)\mathbb{Z}$, there exists a partition $\mathbb{Z}/(p+q)\mathbb{Z} = P_{ij} \sqcup Q_{ij} \sqcup R_{ij}$ and an element $r \in \mathbb{Z}/pq\mathbb{Z}$ such that:
- $\#R_{ij} = 2$, and $L_{ij}(k) = r$ for all $k \in R_{ij},$
- $\#P_{ij} = p-1$, and $\{L_{ij}(k), k \in P_{ij}\} = \{r + q(\mathbb{Z}/pq\mathbb{Z})\} \setminus \{r\},$
- $\#Q_{ij} = q-1$, and $\{L_{ij}(k), k \in Q_{ij}\} = \{r + p(\mathbb{Z}/pq\mathbb{Z})\} \setminus \{r\}$.

**Proof.** This follows from the Chinese Remainder Theorem. \hfill \Box

Thus, the values of $L_{ij}$ on $P_{ij}$ go through all elements of a $p$-cycle in $\mathbb{Z}/pq\mathbb{Z}$ with the exception of $r$, the values of $L_{ij}$ on the elements of $Q_{ij}$ go through all elements of a $q$-cycle in $\mathbb{Z}/pq\mathbb{Z}$ with the exception of $r$, and the value $r$ is taken twice, exactly at the two elements of $R_{ij}$.

To simplify a little the notations, we set $P_{ij}^+ = P_{ij} \cup R_{ij}$ and $Q_{ij}^+ = Q_{ij} \cup R_{ij}$.

**Lemma 6.2.** Suppose that $i, j \in \mathbb{Z}/(p+q)\mathbb{Z}$ are distinct and that $a, b \in \mathbb{Z}/(p+q)\mathbb{Z}$ are distinct and such that $L_{ij}(b) - L_{ij}(a) \in q(\mathbb{Z}/pq\mathbb{Z})$. Then $a, b \in P_{ij}^+$.
Proof. If \( a, b \in Q_{ij} \), then \( L_{ij}(b) - L_{ij}(a) = p(\mathbb{Z}/pq\mathbb{Z}) \). But \( p(\mathbb{Z}/pq\mathbb{Z}) \cap q(\mathbb{Z}/pq\mathbb{Z}) = \{0\} \) since \( p, q \) are prime. So \( L_{ij}(b) - L_{ij}(a) = 0 \), which is not possible if \( a, b \in Q_{ij} \) are distinct. So either \( a \) or \( b \) is in \( P_{ij}^+ \). But then it follows from the definition of \( P_{ij}^+ \), and from the fact that \( L_{ij}(b) - L_{ij}(a) \in q(\mathbb{Z}/pq\mathbb{Z}) \), that the other is in \( P_{ij}^+ \), too.

**Lemma 6.3.** Let \( i, j, k \in \mathbb{Z}/(p + q)\mathbb{Z} \) be distinct. Then \#(\( P_{ij} \cap Q_{jk}^+ \)) \( \leq 2 \).

**Proof.** Suppose by contradiction that \( P_{ij} \cap Q_{jk}^+ \) contains three distinct elements \( a, b, c \). But then either two of them are in \( P_{ij} \cap Q_{jk}^+ \cap P_{ki}^+ \) or two of them are in \( P_{ij} \cap Q_{jk}^+ \cap Q_{ki}^+ \). We suppose for instance that \( a, b \in P_{ij} \cap Q_{jk}^+ \cap P_{ki}^+ \), the same argument applies in the other cases.

Since \( a, b \in P_{ij} \), \( L_{ij}(b) - L_{ij}(a) \in q(\mathbb{Z}/pq\mathbb{Z}) \). Similarly, since \( a, b \in P_{ki}^+ \), \( L_{ki}(b) - L_{ki}(a) \in q(\mathbb{Z}/pq\mathbb{Z}) \), so that \( L_{jk}(b) - L_{jk}(a) = -(L_{ij}(b) - L_{ij}(a)) - (L_{ki}(b) - L_{ki}(a)) \in q(\mathbb{Z}/pq\mathbb{Z}) \). However we also know that \( a, b \in Q_{jk}^+ \), so that \( L_{jk}(b) - L_{jk}(a) \in p(\mathbb{Z}/pq\mathbb{Z}) \), and therefore \( L_{jk}(b) - L_{jk}(a) = 0 \), which contradicts the fact that \( a, b \in P_{ij} \) and \( a \neq b \).

**Lemma 6.4.** Suppose that \( p \geq 5 \). Then for \( i, j, k \in \mathbb{Z}/(p + q)\mathbb{Z} \) distinct, \#(\( P_{ij} \cap P_{jk} \)) \( \geq 2 \).

**Proof.** By definition,

\[
P_{ij} = P_{ij} \cap (P_{jk} \cup Q_{jk}^+) = (P_{ij} \cap P_{jk}) \cup (P_{ij} \cap Q_{jk}^+) \,.
\]

However \#\( P_{ij} = p - 1 \), while \#(\( P_{ij} \cap Q_{jk}^+ \)) \( \leq 2 \) by Lemma 6.3. The result follows.

**Corollary 6.5.** If \( p \geq 5 \) then for \( i, j, k \in \mathbb{Z}/(p + q)\mathbb{Z} \) distinct, \( P_{ij} \cap P_{jk}^+ \subset P_{ik}^+ \).

**Proof.** According to Lemma 6.4 \#(\( P_{ij}^+ \cap P_{jk}^+ \)) \( \geq 2 \). Moreover if \( a, b \in P_{ij}^+ \cap P_{jk}^+ \) are distinct, then \( L_{ij}(b) - L_{ij}(a) \in q(\mathbb{Z}/pq\mathbb{Z}) \) and \( L_{jk}(b) - L_{jk}(a) \in q(\mathbb{Z}/pq\mathbb{Z}) \). It follows that \( L_{ik}(b) - L_{ik}(a) \in q(\mathbb{Z}/pq\mathbb{Z}) \) and therefore, from Lemma 6.2, that \( a \) and \( b \) are in \( P_{ik}^+ \).

**Lemma 6.6.** Suppose that \( p, q \geq 5 \), and let \( i, j, k \in \mathbb{Z}/(p + q)\mathbb{Z} \) be distinct. Then \( P_{ij} \cap Q_{jk} = \emptyset \).

**Proof.** Let \( a \in P_{ij} \cap P_{jk} \) and let \( b \in Q_{ij} \cap Q_{jk} \) — such elements exist by Lemma 6.4. We know that \( a \in P_{ik}^+ \), \( b \in Q_{ik}^+ \) by Corollary 6.5.

Suppose now by contradiction that \( c \in P_{ij} \cap Q_{jk} \). Then \( a, c \in P_{ij} \) so that

\[
L_{ij}(c) - L_{ij}(a) \in q(\mathbb{Z}/pq\mathbb{Z}) \,.
\]

and \( a \in P_{jk} \), \( c \in Q_{jk} \), so that

\[
L_{jk}(c) - L_{jk}(a) \notin q(\mathbb{Z}/pq\mathbb{Z}) \,.
\]

So

\[
L_{ik}(c) - L_{ik}(a) = (L_{jk}(c) - L_{jk}(a)) + (L_{ij}(c) - L_{ij}(a)) \notin q(\mathbb{Z}/pq\mathbb{Z}) \,.
\]

Since \( a \in P_{ik}^+ \), it follows that \( c \notin P_{ik}^+ \).

The same argument with \( b \) instead of \( a \) shows that \( c \notin Q_{ik}^+ \), which is a contradiction.

**Corollary 6.7.** Suppose that \( p, q \geq 5 \). Then for all \( i, j, k \in \mathbb{Z}/pq\mathbb{Z} \) distinct, \( P_{ij} \subset P_{jk}^+ \) and \( Q_{ij} \subset Q_{jk}^+ \).

**Proof.** We know that \( P_{ij} \subset P_{jk}^+ \) by Lemma 6.6. It follows that \( P_{ij} \subset P_{ij}^+ \cap P_{jk}^+ \), and therefore that \( P_{ij} \subset P_{jk}^+ \) by \( P_{ik}^+ \) by Corollary 6.5. The same argument works for \( Q_{ij} \).
Corollary 6.8. Suppose that \( p, q \geq 5 \). Then for all \( i, j, k, l \in \mathbb{Z}/pq\mathbb{Z} \), not necessarily distinct, \( \#(P_{ij} \cap P_{kl}) \geq p - 3 \) and \( \#(Q_{ij} \cap Q_{kl}) \geq q - 3 \).

Proof. We have \( P_{ij} \subset P_{jk}^+ \) and \( P_{kl} \subset P_{jk}^+ \) by Corollary 6.7. The first result follows because \( \#P_{ij} = \#P_{kl} = p - 1 \) while \( \#P_{jk}^+ = p + 1 \). The same argument used with \( Q_{ij} \) gives the second result. \(\)

We now prove a result about sets having large intersections with all their circular rotations.

Proposition 6.9. Let \( A \subset \mathbb{Z}/n\mathbb{Z} \) be a set with \( \#A = a \) such that, for all \( y \in \mathbb{Z}/n\mathbb{Z} \), \( \#(A \cap (y + A)) \geq b \), for some fixed integer \( b \). Then, we have that \( a \geq b\sqrt{n} \).

Proof. Let \( \pi : \mathbb{Z}/n\mathbb{Z} \to \{0, 1\} \) be the indicator function of \( A \), \( \pi(x) = 1 \) if \( x \in A \), \( \pi(x) = 0 \) otherwise. Let \( \pi' : \mathbb{Z}/n\mathbb{Z} \to \{0, 1\} \) be the indicator function of \( -A \), \( \pi'(x) = \pi(-x) \).

In terms of indicator functions, the hypothesis about the cardinality of intersections reads
\[
\forall y \in \mathbb{Z}/n\mathbb{Z}, \sum_{x \in \mathbb{Z}/n\mathbb{Z}} \pi(x)\pi(x + y) \geq b,
\]
or in other terms \( (\pi * \pi')(-y) \geq b \). Taking the value at 0 of the Fourier transform and using the fact that \( \hat{\pi}' = \hat{\pi} \), we find that
\[
|\hat{\pi}(0)|^2 \geq b\sqrt{n}.
\]
However, we also know that
\[
||\hat{\pi}||^2 = ||\pi||^2 = \sum_{x \in \mathbb{Z}/n\mathbb{Z}} \pi(x)^2 = a.
\]
The conclusion \( b\sqrt{n} \leq a \) follows from the previous two relations. \(\)

We now have all the elements needed to prove Theorem 1.3.

Proof of Theorem 1.3. Consider a matrix \( M \in C_{pq}^{\text{circ}}(pq) \), and put, with the notations of Lemma 6.1, \( A = P_{01} \subset \mathbb{Z}/(p + q)\mathbb{Z} \). Note that, for all \( i \in \mathbb{Z}/(p + q)\mathbb{Z} \), the sets \( P_{i,i+1} \) are exactly the circular rotations of \( A \): \( P_{i,i+1} = i + A \), and, by Corollary 6.8, we have \( \#(P_{01} \cap P_{i,i+1}) \geq p - 3 \).

From Proposition 6.9 with \( a = \#A = p - 1 \), \( b = p - 3 \) and \( n = p + q \), it follows that
\[
p - 1 \geq \sqrt{p + q(p - 3)},
\]
and, since \( q \geq 5 \), that
\[
(p - 1)^2 \geq (p - 3)^2(p + 5).
\]
However a direct examination shows that
\[
(x - 3)^2(x + 5) - (x - 1)^2
\]
is strictly positive for \( x \geq 5 \), and a contradiction follows. \(\)
Table 1. Existence and number of circulant Butson matrices. Here $\times$ is the Lam-Leung obstruction, $\times_l, \times_h, \times_s$ are the de Launey, Haagerup and Sylvester obstructions, and $\times_{pq}$ denote the Sylvester’ and Sylvester” obstructions.

7. COUNTING AND CLASSIFICATION RESULTS FOR SMALL MATRICES

There are a number of known obstructions to the existence of a matrix in $C_n(l)$:

**Theorem 7.1.** We have the following results:

1. **Lam-Leung:** If $C_n(l) \neq \emptyset$ and $l = p_1^{a_1} \cdots p_k^{a_k}$ then $n \in p_1 N + \ldots + p_k N$.
2. **de Launey:** If $C_n(l) \neq \emptyset$ then there is $d \in \mathbb{Z}[e^{2\pi i/l}]$ such that $|d|^2 = n^n$.
3. **Sylvester:** If $C_n(2) \neq \emptyset$ then $n = 2$ or $4n$.
4. **Sylvester’**: If $C_n(l) \neq \emptyset$ and $n = p + 2$ with $p \geq 3$ prime, then $l \neq 2 p^b$.
5. **Sylvester”**: If $C_n(l) \neq \emptyset$ and $n = 2q$ with $p > q \geq 3$ primes, then $l \neq 2^a p^b$.
6. **Haagerup:** If $C_5(l) \neq \emptyset$ then $5|l$.

**Proof.** See respectively [LL00], [Lau94], [Syl87], [BBS09], [BBS09], [Haa97]. □

Table 1 above describes for each $n, l$ either an obstruction to the existence of matrices in $C_n^{circ}(l)$, or “0” if no obstruction is known but there is no such matrix, or the number of equivalence classes of those matrices. The blank cells indicate that we have no results.

The symbol “$F_p$” is used when $n = l$ is prime, so that Theorem 1.2 indicates that the only circulant Hadamard matrix is the Fourier matrix or one of its multiples (up to equivalence). The symbol “$(F_p)$” was used to indicate that the same uniqueness result applies but when $n$ is prime and $l$ is a multiple of $n$, so that Theorem 1.2 does not apply directly. In those cases the uniqueness which is claimed results from computations.
A list, for small values of \( n, l \) for which no obstruction is known, of the first rows of matrices in \( C_n^{\text{circ}} (l) \), is available online\(^3\). The classification results were obtained by using a computer program.

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\(^3\)http://math.uni.lu/schlenker/programs/circbut/circbut.html
SMALL CIRCULANT COMPLEX HADAMARD MATRICES OF BUTSON TYPE

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