OPERATOR SPLITTING FOR NON-AUTONOMOUS EVOLUTION EQUATIONS

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To Ulf Schlotterbeck, our inspirator, on his 70th birthday.

Abstract. We establish general product formulas for the solutions of non-autonomous abstract Cauchy problems. The main technical tools are evolution semigroups allowing the direct application of existing results on autonomous problems. The results obtained are illustrated by the example of an autonomous diffusion equation perturbed with time dependent potential. We also prove convergence rates for the sequential splitting applied to this problem.

1. Introduction

Operator splitting procedures are used to solve ordinary and partial differential equations numerically. They can be considered as certain finite difference methods which simplify or even make the numerical treatment of differential equations possible. The idea behind these procedures is the following. In many situations, a certain physical phenomenon can be considered as the combined effect of several processes. Hence the behavior of a physical quantity is described by a partial differential equation in which the time derivative depends on the sum of operators corresponding to the different processes. These operators usually are of different nature and for each sub-problem corresponding to each operator there might be an effective numerical method providing fast and accurate solutions. For the sum of these operators, however, it is not always possible to find an adequate and effective method. Hence, the idea of operator splitting procedures means that instead of the sum we treat the operators separately and the solution of the original problem is then to be recovered from the numerical solutions of these sub-problems. We refer to the recent monographs by Faragó and Havasi [11] or Holden et al. [15] for a detailed introduction to the theory and applications of operator splitting methods.

There was enormous progress in recent years in the theoretical investigation of operator splitting procedures. Especially, ordinary differential equations and autonomous linear evolution equations have been treated thoroughly, see also Bátkai, Csomós and Nickel [2] and the subsection below for a (certainly not complete) list of references.

The aim of the present paper is to investigate the above described splitting method for non-autonomous evolution equations of the form

\[
\begin{align*}
\frac{d}{dt}u(t) &= (A(t) + B(t))u(t), & t \geq s \in \mathbb{R}, \\
u(s) &= x \in X,
\end{align*}
\]

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on some Banach space \( X \). Our particular goal is to emphasize that non-autonomous evolution equations can often be rewritten as an autonomous abstract Cauchy problem by means of an appropriate choice for the state-space. Thus, by making use of so-called evolution semigroups, it is possible to apply existing results for autonomous problems.

First we summarize the necessary background on operator splitting for abstract Cauchy problems, i.e., operator splitting in the framework of strongly continuous operator semigroups. The key ingredient here is Chernoff’s Theorem 1.1. Then non-autonomous evolution equations and evolution semigroups are surveyed, providing the main technical tools for the succeeding sections.

A product representation is presented in Section 2, while operator splitting — strictly in the sense above — is considered in Section 3. To keep our presentation short, we mainly restrict ourselves to the case of the so-called sequential splitting, but in Section 4 we show how higher order splitting methods can be treated with essentially no difference. In that section, we also prove the convergence of the splitting methods when combined with spatial “discretization,” and make a quick outlook on the positivity of evolution families. Finally, as an illustration of the developed tools, we apply them to a diffusion equation with time dependent potential.

A word on notation: For a family of operators \( U_0, U_1, \ldots, U_{n-1} \in \mathcal{L}(X) \), we denote the (“time-ordered”) product of these operators by

\[
\prod_{p=0}^{n-1} U_p := U_{n-1} U_{n-2} \cdots U_1 U_0 \quad \text{and} \quad \prod_{p=n-1}^{0} U_p := U_0 U_1 \cdots U_{n-2} U_{n-1}.
\]

**Operator splitting for autonomous problems.** In this section, we recollect the main notions and results of operator splitting for autonomous equations. Consider the following abstract Cauchy problem on a given Banach space \( X \):

\[
\begin{aligned}
\frac{d}{dt} u(t) & = (A + B) u(t), \quad t \geq 0, \\
\quad u(0) & = x \in X,
\end{aligned}
\]

(ACP)

where the operators \( A, B \), and the closure \( C := A + B \) are supposed to be generators of strongly continuous semigroups \( T, S \), and \( U \), respectively. Our general reference on strongly continuous operator semigroups is the monograph Engel and Nagel [8].

As mentioned in the introduction, operator splitting means that we try to recover the solution semigroup \( U \) using the semigroups \( T \) and \( S \). As for splitting procedures we mention the most frequently used ones (for more details, see Bátkai, Csomós and Nickel [2, Section 2.2]):

- The **sequential splitting**, classically the Lie-Trotter product formula, is given by

  \[
  u_{n}^{\text{seq}}(t) := [S(t/n)T(t/n)]^n x,
  \]

- the **Strang** splitting is given by

  \[
  u_{n}^{\text{Str}}(t) := [T(t/2n)S(t/n)T(t/2n)]^n x,
  \]

- and — for a fixed parameter \( \Theta \in (0, 1) \) — the **weighted splitting** is

  \[
  u_{n}^{\Theta}(t) := [\Theta S(t/n)T(t/n) + (1 - \Theta)T(t/n)S(t/n)]^n x
  \]

with \( n \in \mathbb{N} \). In case \( \Theta = \frac{1}{2} \), it is also called symmetrically weighted splitting. The convergence of these procedures is usually ensured by the following classical result.
Theorem 1.1 (Chernoff [6], or see Engel and Nagel [8, Sec. III.]). Let $C$ be a linear operator in the Banach space $X$ and assume that $F : \mathbb{R}_+ \to \mathcal{L}(X)$ is a (strongly) continuous function with $F(0) = I$ and

$$
\|(F(t))^k\| \leq Me^{k\omega t} \text{ for all } t \geq 0 \text{ and } k \in \mathbb{N} \quad \text{(stability)}.
$$

Suppose that there is a dense subspace $D$, with $(\lambda - C)D$ being also dense for some (large) $\lambda > 0$. If for every $x \in D$ the limit

$$
\lim_{h \to 0} \frac{F(h)x - x}{h} = Cx \quad \text{(consistency)}
$$

exists, then $C$ is the generator of a $C_0$-semigroup $U$, the set $D$ is a core for the generator $C$, and we have

$$
\lim_{n \to \infty} (F(\frac{h}{n}))^n x = U(t)x \quad \text{(convergence)}.
$$

Note that if the closure of $C$ is already known to be a generator, as it is the case in problems motivated by numerical analysis, then the range condition is automatically satisfied.

The operator family $F$ is sometimes called a finite difference method. Clearly, the above mentioned splitting procedures have this form. For example, for the sequential splitting we take

$$
F^{\text{seq}}(h) = S(h)T(h).
$$

It is important to note that Chernoff’s Theorem does not yield anything a priori about the rate of convergence. The finite difference method $F$ is said to be of order $p > 0$, if for $x$ from a suitably large subset of $X$ there is $C > 0$ such that for all $t \in [0, t_0]$ we have

$$
\|F(\frac{1}{n})^n x - U(t)x\| \leq \frac{C}{n^p},
$$

or, as in many special cases, equivalently,

$$
\|F(h)x - U(h)x\| \leq C'h^{p+1}.
$$

The equivalence holds in special cases where it is possible to ensure the invariance of the above mentioned large subset $D$ of $X$ (for more details we refer to the Lax equivalence theorem which states that the above two definitions are equivalent for a finite different method if and only if the method is stable).

Different splitting procedures were introduced to increase the order of convergence. In the finite dimensional setting, it is well known that the sequential splitting is of first order, the Strang and the weighted splitting with $\Theta = \frac{1}{2}$ are of second order. Moreover, the weighted splitting allows also the use of parallel computing.

In the infinite dimensional case, however, no similar general statement can be made without additional assumptions. There has been intense research in this direction, and we mention the works by Bjørhus [4], Cachia and Zagrebnov [5], Faragó and Havasi [10], Hansen and Ostermann [13], Ichinose et al. [17], Jahnke and Lubich [18] or Neidhardt and Zagrebnov [27].

To obtain error estimates later for diffusion problems, we apply a result by Jahnke-Lubich, Hansen-Ostermann, which relies on commutator bounds. For simplicity, we mention here only the special case used later.

Theorem 1.2 (Jahnke and Lubich [18, Theorem 2.1], Hansen and Ostermann [13, Theorem 2.3]). Suppose that $A$ generates a strongly continuous contraction semigroup $e^{tA}$ in the Banach space $X$ and that $B \in \mathcal{L}(X)$ such that there exists an $\alpha > 0$ such that

$$
\|[A, B]v\| = \|(AB - BA)v\| \leq e\|(-A)^{\alpha}v\|
$$

(1)
for all $v \in D \subseteq D((-A)^{\alpha})$ (where $D$ is some dense subspace of $D((-A)^{\alpha})$ invariant under $e^{t(A+B)}$). Then one has first order convergence for the sequential and Strang splittings, i.e.,

$$\| e^{\frac{t}{n}B} e^{\frac{n-1}{n}A} v - e^{t(A+B)} v \| \leq \frac{C t^2}{n} \| (-A)^{\alpha} v \|,$$

$$\| e^{\frac{t}{n}A} e^{\frac{n-1}{n}B} v - e^{t(A+B)} v \| \leq \frac{C t^2}{n} \| (-A)^{\alpha} v \|.$$

Non-autonomous evolution equations and evolution semigroups. In this section we summarize the main results and definitions on non-autonomous evolution equations and evolution semigroups needed for our later exposition. For a detailed account and bibliographic references see, e.g., the survey by Schnaubelt in \[] Section VI.9]. Consider now the non-autonomous evolution equation

$$(\text{NCP}_{s,x}) \quad \left\{ \begin{array}{l}
\frac{d}{dt} u(t) = A(t) u(t), \quad t \geq s \in \mathbb{R}, \\
u(s) = x \in X,
\end{array} \right.$$ 

where $X$ is a Banach space, $(A(t), D(A(t)))$ is a family of (usually unbounded) linear operators on $X$.

Definition 1.3. A continuous function $u : [s, \infty) \to X$ is called a (classical) solution of $(\text{NCP}_{s,x})$ if $u \in C^1([s, \infty); X)$, $u(t) \in D(A(t))$ for all $t \geq s$, $u(s) = x$, and $\frac{d}{dt} u(t) = A(t) u(t)$ for $t \geq s$.

We use the following slight modification of Kellermann’s definition [20] Definition 1.1 for the well-posedness of the non-autonomous Cauchy problem (NCP).

Definition 1.4 (Well-posedness). For a family $(A(t), D(A(t)))_{t \in \mathbb{R}}$ of linear operators on the Banach space $X$ the non-autonomous Cauchy problem (NCP) is called well-posed (with regularity subspaces $(Y_s)_{s \in \mathbb{R}}$ and exponentially bounded solutions) if the following are true.

(i) (Existence) For all $s \in \mathbb{R}$ the subspace

$$Y_s := \left\{ y \in X : \text{ there exists a classical solution for } (\text{NCP})_{s,y} \right\} \subseteq D(A(s))$$

is dense in $X$.

(ii) (Uniqueness) For every $y \in Y_s$ the solution $u_s(\cdot, y)$ is unique.

(iii) (Continuous dependence) The solution depends continuously on $s$ and $y$, i.e., if $s_n \to s \in \mathbb{R}$, $y_n \to y \in Y_s$ with $y_n \in Y_{s_n}$, then we have

$$\| \hat{u}_{s_n}(t, y_n) - \hat{u}_s(t, y) \| \to 0$$

uniformly for $t$ in compact subsets of $\mathbb{R}$, where

$$\hat{u}_r(t, y) := \begin{cases}
  u_r(t, y) & \text{if } r \leq t, \\
y & \text{if } r > t.
\end{cases}$$

(iv) (Exponential boundedness) There exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\| u_s(t, y) \| \leq M e^{\omega (t-s)} \| y \|$$

for all $y \in Y_s$ and $t \geq s$.

As in the autonomous case, the operator family solving a non-autonomous Cauchy problem enjoys certain algebraic properties.

Definition 1.5 ( Evolution family). A family $U = (U(t, s))_{t \geq s}$ of linear, bounded operators on a Banach space $X$ is called an (exponentially bounded) evolution family if
Definition 1.6. An evolution family \( U = (U(t, s))_{t \geq s} \) is called evolution family solving \((\text{NCP})\) if for every \( s \in \mathbb{R} \) the regularity subspace
\[
Y_s := \left\{ y \in X : |s, \infty) \ni t \mapsto U(t, s)y \text{ solves } (\text{NCP})_{s,y} \right\}
\]
is dense in \( X \).

The well-posedness of \((\text{NCP})\) can now be characterized by the existence of a solving evolution family.

Proposition 1.7 (Nickel [29, Proposition 2.5]). Let \( X \) be a Banach space, and assume that \((A(t), D(A(t)))_{t \in \mathbb{R}}\) is a family of linear operators on \( X \) and consider the non-autonomous Cauchy problem \((\text{NCP})\). The following assertions are equivalent.

(i) The non-autonomous Cauchy problem \((\text{NCP})\) is well-posed.

(ii) There exists a unique evolution family \((U(t, s))_{t \geq s}\) solving \((\text{NCP})\).

To every evolution family we can associate \(C_0\)-semigroups on \( X\)-valued function spaces. These semigroups, which determine the behavior of the evolution family completely, are called evolution semigroups. Consider the Banach space
\[
\text{BUC}(\mathbb{R}; X) = \left\{ f : \mathbb{R} \to X : f \text{ is bounded and uniformly continuous} \right\}
\]
normed by
\[
\|f\| := \sup_{t \in \mathbb{R}} \|f(t)\|, \quad f \in \text{BUC}(\mathbb{R}; X);
\]
or any closed subspace of it that is invariant under the right translation semigroup \( \mathcal{R} \) defined by
\[
(\mathcal{R}(t)f)(s) := f(s-t) \quad \text{for } f \in \text{BUC}(\mathbb{R}; X) \text{ and } s \in \mathbb{R}, \ t \geq 0.
\]
In the following \( \mathcal{X} \) will denote such a closed subspace; we shall typically take \( \mathcal{X} = C_0(\mathbb{R}; X) \), the space of continuous functions vanishing at infinity.

It is easy to check that the following definition yields a strongly continuous semigroup.

Definition 1.8. For an evolution family \( U = (U(t, s))_{t \geq s} \) we define the corresponding evolution semigroup \( \mathcal{T} \) on the space \( \mathcal{X} \) by
\[
(\mathcal{T}(t)f)(s) := U(s, s-t)f(s-t)
\]
for \( f \in \mathcal{X}, \ s \in \mathbb{R} \) and \( t \geq 0 \). We denote its infinitesimal generator by \((\mathcal{G}, D(\mathcal{G}))\).

With the above notation, the evolution semigroup operators can be written as
\[
\mathcal{T}(t)f = U(\cdot, -t)\mathcal{R}(t)f.
\]
We can recover the evolution family from the evolution semigroup by choosing a function \( f \in \mathcal{X} \) with \( f(s) = x \). Then we obtain
\[
U(t, s)x = (\mathcal{R}(s-t)\mathcal{T}(t-s)f)(s)
\]
for every \( s \in \mathbb{R} \) and \( t \geq s \).
The generator of the right translation semigroup is essentially the differentiation \(-\frac{d}{dt}\) with domain
\[ D(-\frac{d}{dt}) := \mathcal{A}_1 := \{ f \in C^1(\mathbb{R}; X) : f, f' \in \mathcal{X} \}. \]

For a family \((A(t), D(A(t)))_{t \in \mathbb{R}}\) of unbounded operators on \(X\) we consider the corresponding multiplication operator \((A(\cdot), D(A(\cdot)))\) on the space \(\mathcal{X}\) with domain
\[ D(A(\cdot)) := \{ f \in \mathcal{X} : f(s) \in D(A(s)) \forall s \in \mathbb{R}, \text{ and } [s \mapsto A(s)f(s)] \in \mathcal{X} \}, \]
and defined by
\[ (A(\cdot)f)(s) := A(s)f(s) \text{ for all } s \in \mathbb{R}. \]

Now we characterize well-posedness for non-autonomous Cauchy problems.

**Theorem 1.9** (Nickel [[29], Theorem 2.9]). Given a Banach space \(X\), and a family of linear operators \((A(t), D(A(t)))_{t \in \mathbb{R}}\) on \(X\). The following assertions are equivalent.

(i) The non-autonomous Cauchy problem (NCP) for the family \((A(t))_{t \in \mathbb{R}}\) is well-posed (with exponentially bounded solutions).

(ii) There exists a unique evolution semigroup \(\mathcal{T}\) with generator \((\mathcal{G}, D(\mathcal{G}))\) and an invariant core \(D \subseteq \mathcal{A}_1 \cap D(\mathcal{G})\) such that
\[ \mathcal{G}f + f' = A(\cdot)f \]
for all \(f \in D\).

Conditions implying well-posedness are generally divided into assumptions of “parabolic” and of “hyperbolic” type. Roughly speaking, the main difference between these two types is that in the parabolic case we assume all \(A(t)\) being generators of analytic semigroups, while in the hyperbolic case we assume the stability for certain products instead. In both cases one has to add some continuity assumption on the mapping \(t \mapsto A(t)\). We mention only a typical and quite simple version for each type.

**Assumption 1.10 (Parabolic case).**

(P1) The domain \(D := D(A(t))\) is dense in \(X\) and is independent of \(t \in \mathbb{R}\).

(P2) For each \(t \in \mathbb{R}\) the operator \(A(t)\) is the generator of an analytic semigroup \(e^{A(t)}\). For all \(t \in \mathbb{R}\), the resolvent \(R(\lambda, A(t))\) exists for all \(\lambda \in \mathbb{C}\) with \(\Re \lambda \geq 0\) and there is a constant \(M \geq 1\) such that
\[ \|R(\lambda, A(t))\| \leq \frac{M}{|\lambda| + 1} \]
for \(\Re \lambda \geq 0, t \in \mathbb{R}\). The semigroups \(e^{A(t)}\) satisfy \(\|e^{\lambda A(t)}\| \leq M e^{\omega s}\) for absolute constants \(\omega < 0\) and \(M \geq 1\).

(P3) There exist constants \(L \geq 0\) and \(0 < \alpha \leq 1\) such that
\[ \|(A(t) - A(s))A(0)^{-1}\| \leq L|t - s|^{\alpha} \text{ for all } t, s \in \mathbb{R}. \]

**Assumption 1.11 (Hyperbolic case).**

(H1) The family \((A(t))_{t \in \mathbb{R}}\) is stable, i.e., all operators \(A(t)\) are generators of \(C_0\)-semigroups and there exist constants \(M \geq 1\) and \(\omega \in \mathbb{R}\) such that
\[ (\omega, \infty) \subset \rho(A(t)) \text{ for all } t \in \mathbb{R} \]
and
\[ \left\| \prod_{j=1}^{k} R(\lambda, A(t_j)) \right\| \leq M(\lambda - \omega)^{-k} \text{ for all } \lambda > \omega \]
and every finite sequence \(-\infty < t_1 \leq t_2 \leq \cdots \leq t_k < \infty, k \in \mathbb{N}\. \]
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dependent multiplication operators. Formally, this means that we assume that the
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we denote by using the exponential notation as e

take the time-step τ
Remark 1.12. Since the classical papers of Evans [9], Howland [16], and Neidhardt [24, 25, 26], evolution semigroups have been intensively used to study non-autonomous evolution equations. Here, various results on well-posedness as well as qualitative behavior of these equations were obtained. For a quite comprehensive overview and a long list of different variants we refer, e.g., to Nagel and Nickel [22], Neidhardt and Zagrebnov [28], Nickel [29, 30], Nickel and Schnaubelt [31] and Schnaubelt [34]. The recent article Neidhardt and Zagrebnov [28] focuses on (quite

autonomous Cauchy problem (NCP). In the case

which we solve again on

H2) There exists a densely embedded subspace Y ↪ X, which is a core for every
A(t) such that the family of the parts (A|Y (t))t∈R in Y is a stable family on
the space Y.

(H3) The mapping R ⊃ t ↦ A(t) ∈ L(Y, X) is uniformly continuous.

\textbf{Remark 1.12.} Since the classical papers of Evans [9], Howland [16], and Neidhardt [24, 25, 26], evolution semigroups have been intensively used to study non-autonomous evolution equations. Here, various results on well-posedness as well as qualitative behavior of these equations were obtained. For a quite comprehensive overview and a long list of different variants we refer, e.g., to Nagel and Nickel [22], Neidhardt and Zagrebnov [28], Nickel [29, 30], Nickel and Schnaubelt [31] and Schnaubelt [34]. The recent article Neidhardt and Zagrebnov [28] focuses on (quite general) assumptions of the “hyperbolic” type and obtains well-posedness results – in a general sense – for non-autonomous evolution equations by properly defining and analyzing the “sum” \( \frac{d}{dt} + A(\cdot) \) yielding the generator of the associated evolution semigroup. In contrast to that approach, in our paper we simply assume well-posedness of our evolution equation under any appropriate (parabolic or hyperbolic) condition. Therefore, the solving evolution family, the corresponding evolution semigroup, and its generator are well defined by assumption. Our main interest is then, how these solutions can be approximated (numerically) by splitting procedures.

2. A PRODUCT FORMULA

In this section we present a product formula for the solutions of the non-autonomous Cauchy problem (NCP). In the case \( B(t) \equiv 0 \), this formula essentially goes back to Kato [19]. This splitting-type formula is especially useful if for every time \( r \in \mathbb{R} \) we are able to solve effectively the autonomous Cauchy problems

\[
\frac{d}{dt} u(t) = A(r)u(t) \\
\frac{d}{dt} v(t) = B(r)v(t)
\]

with appropriate initial conditions. This is usually the case if the operators \( A(\cdot) \) and \( B(\cdot) \) are partial differential operators with time dependent coefficients or time dependent multiplication operators. Formally, this means that we assume that the operators \( A(r) \) and \( B(r) \) generate strongly continuous operator semigroups, which we denote by using the exponential notation as \( e^{A(r)} \) and \( e^{B(r)} \), respectively. We devote this section to the simplest product formula arising from the sequential splitting.

Suppose we want to determine the solution of (NCP) at time \( t + s > 0 \) and hence take the time-step \( \tau = t/n \). We start with the known initial value \( u^{\text{sq}}(s) = x \), then solve the first (Eq. 1) equation on the time interval \( [s, s + \tau] \) taking \( r = s \). Then we take the result \( u_1^{(i)}(s + \tau) \) as the initial value for the second equation (Eq. 2) which we solve again on \( [s, s + \tau] \). With this new result \( u^{\text{sq}}(s + \tau) := u_2^{(i)}(s + \tau) \) as initial value for (Eq. 1) we restart the procedure and iterate it \( n \) times. Formally:

\[
\begin{align*}
\frac{d}{dt} u_1^{(k)}(t) &= A(s + (k - 1)\tau)u_1^{(k)}(t), & t \in (s + (k - 1)\tau, s + k\tau], \\
u_1^{(k)}(s + (k - 1)\tau) &= u^{\text{sq}}(s + (k - 1)\tau), \\
\frac{d}{dt} u_2^{(k)}(t) &= B(s + (k - 1)\tau)u_2^{(k)}(t), & t \in (s + (k - 1)\tau, s + k\tau], \\
u_2^{(k)}(s + (k - 1)\tau) &= u_1^{(k)}(s + k\tau), \\
u^{\text{sq}}(s + k\tau) := u_2^{(k)}(s + k\tau),
\end{align*}
\]
with \( k = 1, 2, \ldots, n \). Using that for \( r \in [0, \tau] \),
\[
 u^{(k)}_1(s + (k - 1)\tau + r) = e^{rA(s+(k-1)\tau)}u^{(k)}_1(s + (k - 1)\tau),
\]
and that
\[
u^{(k)}_2(s + (k - 1)\tau + r) = e^{rB(s+(k-1)\tau)}u^{(k)}_1(s + k\tau)
\]
\[
e^{rB(s+(k-1)\tau)}e^{rA(s+(k-1)\tau)}u^{(k)}_2(s + (k - 1)\tau),
\]
we see by a simple induction argument that the split solution \( u^{(k)}(s + k\tau) \), obtained by applying the sequential splitting procedure, can be written as
\[
(3) \quad u^{(k)}(s + k\tau) = \prod_{p=0}^{k-1} e^{rB(s+p\tau)}e^{rA(s+p\tau)}x \quad \text{for } k \in \mathbb{N}, k\tau \leq t, \text{ and } x \in X.
\]
In what follows, we study the convergence of this expression.

**Assumption 2.1.** Suppose that
a) the non-autonomous Cauchy problem corresponding to the operators \( (A(\cdot) + B(\cdot)) \) is well-posed,
b) (Stability) the operators \( A(r) \) and \( B(r) \) are generators of \( C_{01} \)-semigroups \( e^{A(r)} \), \( e^{B(r)} \) of type \( (M, \omega) \) \( (M \geq 1 \text{ and } \omega \in \mathbb{R}) \) on the Banach space \( X \) and, therefore,
\[
(\omega, \infty) \subset \rho(A(r)) \cap \rho(B(r)) \quad \text{for all } r \in \mathbb{R}.
\]
Moreover, let
\[
\sup_{s \in \mathbb{R}} \left\| \prod_{p=n}^{1} (e^{\frac{1}{p}B(s-\frac{p-1}{p}r)}e^{\frac{1}{p}A(s-\frac{p-1}{p}r)}) \right\| \leq Me^{\omega t}, \text{ and}
\]
c) (Continuity) the maps
\[
t \mapsto R(\lambda, A(t))x, \quad t \mapsto R(\lambda, B(t))x
\]
are continuous for all \( \lambda > \omega \) and \( x \in X \).

We denote the evolution family solving \( \text{NCP} \) by \( W \) and the corresponding evolution semigroup, generated by the closure \( \hat{C} \) of \( C := \mathbb{R}^+ + A(\cdot) + B(\cdot) \), by \( \mathcal{W} \).

As we shall see in a moment, Assumption 2.1 yields that the multiplication operators \( A(\cdot), B(\cdot) \) with appropriate domain generate strongly continuous multiplication semigroups on \( C_0(\mathbb{R}; X) \) (for more on this matter we refer to Engel and Nagel [8, Sec. III.4.13] and Graser [12]).

**Theorem 2.2.** Under Assumption 2.1 one has the convergence
\[
(4) \quad W(t, s)x = \lim_{n \to \infty} \prod_{p=0}^{n-1} (e^{\frac{r+1}{n}B(s+\frac{r+1}{n}t)}e^{\frac{r+1}{n}A(s+\frac{r+1}{n}t)})x
\]
for all \( x \in X, \) locally uniformly in \( s, t \) with \( s \leq t \).

**Proof.** The main idea of the proof is analogous to the one in Nickel [8] Proposition 3.2. Consider the semigroups \( e^{tA(\cdot)} \) and \( e^{tB(\cdot)} \) for given \( r \in \mathbb{R} \). By the uniform growth assumption in 2.1(b) on the semigroups, for fixed \( t \geq 0 \) the function \( r \mapsto e^{tA(\cdot)}f(r) \) vanishes at infinity whenever \( f \) has this property. We also have that the function \( r \mapsto e^{tA(\cdot)} \) is strongly continuous. Indeed, by the Trotter-Kato Theorem (see Engel and Nagel [8 Thm. III.4.8]) we even obtain that \( \mathbb{R}^+ \times \mathbb{R} \ni (t, r) \mapsto e^{tA(\cdot)} \) is strongly continuous. All these reasoning are, of course, true if \( A(\cdot) \) is replaced by \( B(\cdot) \). Let now \( f \in \text{BUC}(\mathbb{R}; X) \). Then \( r \mapsto e^{tA(\cdot)}f(r) \) is continuous, too. We have therefore shown that the multiplication semigroups \( e^{tA(\cdot)} \) and \( e^{tB(\cdot)} \), generated
by the multiplication operators $A(\cdot)$ and $B(\cdot)$, both act on the space $\mathcal{X} = C_0(\mathbb{R}; X)$, see also Graser [12]. It can be seen by induction that
\[
(R_{123}^n)_{e^{x_1B(\cdot)}e^{x_2A(\cdot)}}f(\cdot) = \frac{1}{p=n} \left( e^{x_1B(-\frac{h}{p})}e^{x_2A(-\frac{h}{p})} \right) R(t)f(\cdot).
\]

The stability assumption [22]h) immediately implies the stability for the finite difference method $F(h) := R(h)e^{hB(\cdot)}e^{hA(\cdot)}$. Consistency is standard to check: take $f \in \mathcal{X}_i \cap D(A(\cdot)) \cap D(B(\cdot))$. Then we can write
\[
\lim_{h \downarrow 0} \frac{F(h)f - f}{h} = \lim_{h \downarrow 0} \left[ R(h)e^{hB(\cdot)}e^{hA(\cdot)}f - f + R(h)\frac{e^{hB(\cdot)}f - f + R(h)f - f}{h} \right]
= A(\cdot)f + B(\cdot)f - f'.
\]

By our well-posedness assumptions, the closure of the operator $\mathcal{C} = -\frac{d}{dt} + B(\cdot) + A(\cdot)$ generates a strongly continuous semigroup on $\mathcal{X}$, hence the set $(\lambda - \mathcal{C})D(\mathcal{C})$ is dense in $\mathcal{X}$. By the stability assumption we can apply Chernoff’s Theorem [1, 1] with the three operators $-\frac{d}{dt}$, $A(\cdot)$, $B(\cdot)$, and obtain that the evolution semigroup generated by $\mathcal{C}$ is given by
\[
W(t)f = \lim_{n \to \infty} \prod_{p=n}^{1} \left( e^{x_1B(\cdot)}e^{x_2A(\cdot)} \right) f(\cdot - t).
\]
The above limit is to be understood in the topology of $\mathcal{X}$, that is, in the uniform topology. By using this, and by applying the formula [2] from the previous section, we can recover the evolution family from the evolution semigroup and arrive at the formula
\[
W(t, s)x = \lim_{n \to \infty} \prod_{p=n}^{1} \left( e^{x_1B(t - \frac{pt + s}{n})}e^{x_2A(t - \frac{pt + s}{n})} \right)x,
\]
from which the assertion follows.

\[\square\]

**Remark 2.3.** In the proof of Theorem 2.2 we have used that the semigroups $e^{A(r)}$ and $e^{B(r)}$ map $C_0(\mathbb{R}; X)$ into itself. If $e^{A(r)}$ and $e^{B(r)}$ are uniformly strongly continuous in $r \in \mathbb{R}$, then one could also work on the space $\mathcal{X} = \text{BUC}(\mathbb{R}; X)$.

**Remark 2.4.** The stability condition b) is automatically satisfied, if $A(t)$ and $B(t)$ are generators of quasi-contractive semigroups with uniform exponential bound $\omega$ for all $t$.

**Remark 2.5.** In Vuillermot et al. [36, 37], the authors prove the representation formula [4] where $A(t)$ and $B(t)$ are generators of contraction semigroups, the family $A(\cdot)$ satisfies a version of the so-called parabolic condition and the family $B(\cdot)$ is a small perturbation. Theorem 2.2 can be seen as a generalization of this result and can be applied not only in a larger class of parabolic problems but also in the hyperbolic case. In [35] Vuillermot proves a Chernoff-type approximation theorem for time-dependent operator families. Under appropriate consistency and stability assumptions it is possible to derive formula [1] from this result (as done in [35]) instead of proving it by the application of the classical Chernoff’s Theorem to evolution semigroups. It is however amongst our aims to emphasize that semigroup techniques may be used to prove approximation results also for non-autonomous problems.

**Remark 2.6.** In case $B(t) \equiv 0$, we recover the well-known representation formula
\[
U(t, s)x = \lim_{n \to \infty} \prod_{p=0}^{n-1} e^{x_1A(s + \frac{(t + s)}{n})}x,
\]
see Nickel [30 Proposition 3.2] and Schnaubelt [33 Theorem 2.1]. Again, the stability condition reduces essentially to the classical stability condition of Kato [19].

**Remark 2.7.** It is straightforward to check that if one of the equations is autonomous, e.g., \( A(t) \equiv A \), then we arrive at the same product formula but we can split the original operator \( C \) into two (and not three) operators, namely into \(-\frac{d}{ds} + A\) and \( B(\cdot)\).

### 3. Operator Splitting

In this section we assume that we can solve the non-autonomous equations

\[
\text{(Eq. A)} \quad \frac{d}{dt} u(t) = A(t)u(t),
\]
\[
\text{(Eq. B)} \quad \frac{d}{dt} v(t) = B(t)v(t)
\]

and want to construct the solution of \((\text{NCP})\) applying an operator splitting procedure. For the sake of simplicity we only present the case of sequential splitting: We start with the initial value \( u^{\text{sol}}(s) = x \), then solve the first equation on the time interval \([s, s + \tau]\). Then we take this \( u^{(1)}_1(s + \tau) \) as the initial value for the second equation which we solve on \([s, s + \tau]\). With this result \( u^{\text{sol}}(s + \tau) := u^{(1)}_2(s + \tau) \) as initial value for \((\text{Eq. A})\) we restart the procedure and iterate it \( n \) times. Formally:

\[
\begin{align*}
\frac{d}{dt} u^{(1)}_1(t) &= A(t)u^{(1)}_1(t), & \text{for } t \in (s + (k - 1)\tau, s + k\tau], \\
u^{(1)}_1(s + (k - 1)\tau) &= u^{\text{sol}}(s + (k - 1)\tau), \\
\frac{d}{dt} u^{(1)}_2(t) &= B(t)u^{(1)}_2(t), & \text{for } t \in (s + (k - 1)\tau, s + k\tau], \\
u^{(1)}_2(s + (k - 1)\tau) &= u^{(1)}_1(s + k\tau).
\end{align*}
\]

for \( k = 1, 2, \ldots, n \). If \( U \) and \( V \) denote the evolution families solving the above equations (Eq. A)-(Eq. B), then we have

\[
u^{(1)}_1(r) = U(r, s, s + (k - 1)\tau)u^{\text{sol}}(s + (k - 1)\tau),
\]

and

\[
u^{(1)}_2(r) = V(r, s, s + (k - 1)\tau)u^{(1)}_1(s + k\tau) = V(r, s, s + (k - 1)\tau)U(s + k\tau, s + (k - 1)\tau)u^{\text{sol}}(s + (k - 1)\tau).
\]

By this the splitting solution \( u^{\text{sol}} \) can be written as

\[
u^{\text{sol}}(s + k\tau) = \prod_{p=0}^{k-1} \left( V(s + (p + 1)\tau, s + pr)U(s + (p + 1)\tau, s + pr)\right)x.
\]

In the following we analyze the convergence of this procedure.

**Assumption 3.1.** Suppose that

a) the non-autonomous Cauchy problems corresponding to the operators \( A(\cdot) + B(\cdot) \) are well-posed, and that

b) \((\text{Stability})\) there exist \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that

\[
\sup_{s \in [0,\omega]} \left\| \prod_{p=n-1}^{0} V(s - \frac{pt}{n}, s - \frac{(p+1)t}{n})U(s - \frac{pt}{n}, s - \frac{(p+1)t}{n}) \right\| \leq Me^{\omega t}.
\]
Here, again, the evolution family solving the Cauchy problem corresponding to \( A(\cdot) \) and \( B(\cdot) \), will be denoted by \( U \) and \( V \), respectively. Further, we denote the evolution family solving \((\text{NCP})\) by \( W \) and the corresponding evolution semigroup, generated by the closure of \( C = -\frac{d}{dt} + A(\cdot) + B(\cdot) \), by \( \mathcal{W} \).

**Theorem 3.2.** Under Assumptions \( \text{(A)} \) one has the convergence

\[
W(t, s)x = \lim_{n \to \infty} \lim_{p \to 0} \prod_{p=0}^{n-1} V(s + \frac{(p+1)(t-s)}{n}, s + \frac{p(t-s)}{n})U(s + \frac{(p+1)(t-s)}{n}, s + \frac{p(t-s)}{n})x
\]

for all \( x \in X \).

**Proof.** In the space \( X \), we define

\[
\mathcal{F}(t) := V(\cdot, -t) \quad \text{and} \quad \mathcal{G}(t) := U(\cdot, -t)\mathcal{R}(t).
\]

Inductively, one can see that

\[
(\mathcal{F}(\frac{t}{n})\mathcal{G}(\frac{t}{n}))^n f = \left(V(\cdot, -\frac{t}{n})U(\cdot, -\frac{t}{n})\mathcal{R}(\frac{t}{n})\right)^n f = \prod_{p=n-1}^{0} V(\cdot - \frac{pt}{n}, \cdot - \frac{(p+1)t}{n})U(\cdot - \frac{pt}{n}, \cdot - \frac{(p+1)t}{n})f(\cdot - t).
\]

By our assumptions, the closure \( \overline{\mathcal{C}} \) of the operator \( \mathcal{C} = -\frac{d}{dt} + A(\cdot) + B(\cdot) \) generates a strongly continuous semigroup on \( X \), and hence the set \( (\lambda - \mathcal{C})D(\mathcal{C}) \) is dense. Straightforward calculation analogous to the one in the proof of Theorem 2.2 yields that \( (\mathcal{F}(\cdot)\mathcal{G}(\cdot))(0)f = \mathcal{C}f \) for \( f \in D(\mathcal{C}) \). Hence, by the stability assumption, we can apply Chernoff’s Theorem to this function and obtain that the evolution semigroup generated by \( \overline{\mathcal{C}} \) is given by

\[
W(t)f = \lim_{n \to \infty} \prod_{p=n-1}^{0} V(\cdot - \frac{pt}{n}, \cdot - \frac{(p+1)t}{n})U(\cdot - \frac{pt}{n}, \cdot - \frac{(p+1)t}{n})f(\cdot - t).
\]

From this, by picking some \( f \in X \) with \( f(s) = x \), we obtain for the evolution family

\[
W(t, s)x = \lim_{n \to \infty} \prod_{p=n-1}^{0} V(t - \frac{p(t-s)}{n}, t - \frac{(p+1)(t-s)}{n})U(t - \frac{p(t-s)}{n}, t - \frac{(p+1)(t-s)}{n})x
\]

which was to be proved. \( \square \)

**Remark 3.3.** Note that the stability condition is trivially satisfied if the evolution families \( U \) and \( V \) are quasi-contractive, i.e., if \( M \leq 1 \) can be taken in Definition \( \text{(iii)} \). In general, as usual with stability assumptions, it is rather hard to verify.

Using similar arguments but a different decomposition, we arrive at a different splitting formula using evolution families corresponding to different (time-rescaled) evolution equations.

**Proposition 3.4.** Suppose that the operator families \( A(\cdot/2), B(\cdot/2) \) and \( A(\cdot) + B(\cdot) \) generate the evolution families \( \tilde{U}, \tilde{V} \) and \( W \), respectively. Assume furthermore that there is \( M > 1 \) and \( \omega \in \mathbb{R} \) such that

\[
\sup_{s \in \mathbb{R}} \left| \prod_{p=n-1}^{0} \tilde{V}(2s - \frac{2pt}{n}, 2s - \frac{(2p+1)t}{n})\tilde{U}(2s - \frac{(2p+1)t}{n}, 2s - \frac{(2p+2)t}{n}) \right| \leq Me^{\omega t}.
\]
Then we have
\[ W(t, s)x = \lim_{n \to \infty} \prod_{p=0}^{n-1} \tilde{V}(2s + \frac{2p(t-s)}{n}, 2s + \frac{(2p+1)(t-s)}{n}) \tilde{U}(2s + \frac{(2p+1)(t-s)}{n}, 2s + \frac{2(t-s)}{n})x. \]

Proof. In the space \( X \), we write formally
\[ -\frac{d}{ds} + A(\cdot) + B(\cdot) = \left(-\frac{d}{2ds} + A(\cdot) \right) + \left(-\frac{d}{2ds} + B(\cdot) \right) = A_1 + B_1. \]

Since the division by 2 in the formula means a rescaling of the corresponding evolution semigroups \( S \) and \( T \), we obtain the representation formulas
\[ S(t) = \tilde{V}(2, 2 \cdot -t)R(t/2) \]
\[ T(t) = \tilde{U}(2, 2 \cdot -t)R(t/2). \]

By induction one can see that
\[ (S(\frac{t}{n})T(\frac{t}{n}))^n f = (\tilde{V}(2, 2 \cdot -\frac{t}{n})R(t/2n)\tilde{U}(2, 2 \cdot -\frac{t}{n})R(t/2n))^n f \]
\[ = \prod_{p=0}^{n} \tilde{V}(2 \cdot -\frac{2pt}{n}, 2 \cdot -\frac{(2p+1)t}{n}) \tilde{U}(2 \cdot -\frac{(2p+1)t}{n}, 2 \cdot -\frac{2(t+2)}{n})f(\cdot - t). \]

Again, the closure \( \mathcal{C} \) of the operator \( \mathcal{C} = -\frac{d}{dt} + A(\cdot) + B(\cdot) \) generates a strongly continuous semigroup on \( X \), hence \( (\lambda - \mathcal{C})D(\mathcal{C}) \) is dense. By this and by the stability assumption Chernoff’s Theorem is applicable. We obtain that the evolution semigroup generated by \( \mathcal{C} \) is given by
\[ \mathcal{W}(t)f = \lim_{n \to \infty} \prod_{p=0}^{n} \tilde{V}(2 \cdot -\frac{2pt}{n}, 2 \cdot -\frac{(2p+1)t}{n}) \tilde{U}(2 \cdot -\frac{(2p+1)t}{n}, 2 \cdot -\frac{2(t+2)}{n})f(\cdot - t). \]

By passing to the evolution family we get the assertion:
\[ W(t, s)x = \lim_{n \to \infty} \prod_{p=0}^{n-1} \tilde{V}(2t \cdot -\frac{2pt(t-s)}{n}, 2t \cdot -\frac{(2p+1)(t-s)}{n}) \tilde{U}(2t \cdot -\frac{(2p+1)(t-s)}{n}, 2t \cdot -\frac{2(t+2)(t-s)}{n})x \]
\[ = \lim_{n \to \infty} \prod_{p=0}^{n-1} \tilde{V}(2s \cdot +\frac{2p(t-s)}{n}, 2s \cdot +\frac{(2p+1)(t-s)}{n}) \tilde{U}(2s \cdot +\frac{(2p+1)(t-s)}{n}, 2s \cdot +\frac{2(t-s)}{n})x. \]

\[ \square \]

Remark 3.5. Note that, in contrast to the autonomous case, there is no general connection between the evolution families \( U \) and \( \tilde{U} \), see Nickel [29].

4. Generalizations and Remarks

Higher order splitting methods. We now show how the previous results generalize to higher order splitting methods. The results are, using the stage set up previously, direct applications of the corresponding autonomous results applied to the evolution semigroups. We restrict ourselves to the Strang and symmetrically weighted splitting, but other splitting methods can be handled analogously. In any case only the stability condition has to be adapted. This, however, is always satisfied (and typically verifiable) if the operators involved are contractions.

Theorem 4.1. Suppose that Assumptions 3.1 a) and c) are satisfied, and that the stability condition holds in the following form:
in the case of the symmetrically weighted splitting. Then we have
\[ \sup_{s \in \mathbb{R}} \left\| \prod_{p=0}^{n-1} e^{\frac{1}{t} A(s - \frac{pt}{n})} e^{\frac{1}{n} B(s - \frac{pt}{n})} e^{\frac{1}{s} A(s - \frac{pt}{n})} \right\| \leq Me^{\omega t} \]
in the case of the Strang splitting, or:
\[ \sup_{s \in \mathbb{R}} \left\| \prod_{p=0}^{n-1} \left( e^{\frac{1}{t} A(s - \frac{pt}{n})} e^{\frac{1}{n} B(s - \frac{pt}{n})} + e^{\frac{1}{s} B(s - \frac{pt}{n})} e^{\frac{1}{t} A(s - \frac{pt}{n})} \right) \right\| \leq Me^{\omega t} \]
in the case of the symmetrically weighted splitting. Then we have
\[ W(t,s) = \lim_{n \to \infty} \frac{1}{2^n} \prod_{p=0}^{n-1} \left( e^{\frac{1}{t} A(s + \frac{pt}{n})} e^{\frac{1}{n} B(s + \frac{pt}{n})} + e^{\frac{1}{s} B(s + \frac{pt}{n})} e^{\frac{1}{t} A(s + \frac{pt}{n})} \right) x \]
for all \( x \in X \) in case of the Strang splitting; and we have
\[ W(t,s) = \lim_{n \to \infty} \frac{1}{2^n} \prod_{p=0}^{n-1} \left( e^{\frac{1}{t} A(s + \frac{pt}{n})} e^{\frac{1}{n} B(s + \frac{pt}{n})} + e^{\frac{1}{s} B(s + \frac{pt}{n})} e^{\frac{1}{t} A(s + \frac{pt}{n})} \right) x \]
for all \( x \in X \) in case of the symmetrically weighted splitting.

Proof. The statements follow immediately by the same reasonings as in the proof of Theorem 2.2 but now considering the expressions
\[ \left( R \left( \frac{1}{n} \right) e^{\frac{1}{n} A(\cdot)} e^{\frac{1}{n} B(\cdot)} e^{\frac{1}{n} A(\cdot)} \right)^n \]
for the Strang-splitting, and
\[ \frac{1}{2^n} \left( R \left( \frac{1}{n} \right) \left( e^{\frac{1}{n} A(\cdot)} e^{\frac{1}{n} B(\cdot)} + e^{\frac{1}{n} B(\cdot)} e^{\frac{1}{n} A(\cdot)} \right) \right)^n, \]
for the weighted splitting, respectively. \( \square \)

Remark 4.2. It can be shown by exactly the same arguments as in Csomós and Nickel [7] Lemma 2.3 that the stability condition \( (b') \) is equivalent to the stability condition in Assumption 2.1 b) for the sequential splitting.

Spatial approximations. Continuing earlier investigations started in Bátki, Csomós and Nickel [2], we show that operator splitting combined with spatial approximations is also convergent. We only concentrate on the formula (31) for the sequential splitting. Other methods can be considered analogously.

Assumption 4.3. Let \( X_m, m \in \mathbb{N} \) be Banach spaces and take operators
\[ P_m : X \to X_m \quad \text{and} \quad J_m : X_m \to X \]
fulfilling the following properties:

(i) \( P_m J_m = I_m \) for all \( m \in \mathbb{N} \), where \( I_m \) is the identity operator in \( X_m \),
(ii) \( \lim_{m \to \infty} J_m P_m x = x \) for all \( x \in X \),
(iii) \( \|J_m\| \leq K \) and \( \|P_m\| \leq K \) for all \( m \in \mathbb{N} \) and a suitable absolute constant \( K \geq 1 \).

The operators \( P_m \) together with the spaces \( X_m \) usually refer to a kind of spatial discretization method (triangulation, Galerkin approximation, Fourier coefficients, etc.), the spaces \( X_m \) are in most applications finite dimensional spaces, and the operators \( J_m \) refer to the interpolation method describing how we associate specific elements of the function space to the elements of the approximating spaces (linear/polynomial/spline interpolation, etc.).
Assumption 4.4. For each \( m \in \mathbb{N} \) and \( r \in \mathbb{R} \) let the operators \( A_m(r) \) and \( B_m(r) \) be generators of strongly continuous semigroups \( e^{A_m(r)} \) and \( e^{B_m(r)} \), respectively. Assume furthermore that

a) (Stability) there exist constants \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that
\[
\| e^{hA(r)} \|, \| e^{hA_m(r)} \|, \| e^{hB(r)} \|, \| e^{hB_m(r)} \| \leq Me^{\omega h}, \text{ for all } h > 0 \text{ and } r \in \mathbb{R},
\]

\[
\sup_{s \in \mathbb{R}} \left\| \prod_{p=0}^{n-1} \left( e^{\frac{h}{n} B_{m} (s - \frac{pt}{n})} e^{\frac{h}{n} A_{m} (s - \frac{pt}{n})} \right) \right\| \leq Me^{\omega t}, \text{ and that}
\]

b) (Consistency) the identities \( \lim_{m \to \infty} J_m A_m(\cdot) P_m f = A(\cdot) f \) for all \( f \in D(A(\cdot)) \), and \( \lim_{m \to \infty} J_m B_m(\cdot) P_m f = B(\cdot) f \) for all \( f \in D(B(\cdot)) \) hold.

As in Bátkai, Csomó and Nickel [2], stability and consistency implies convergence.

**Theorem 4.5.** Suppose that Assumption 4.4 is satisfied. Then one has the convergence
\[
W(t, s)x = \lim_{m \to \infty} \lim_{n \to \infty} J_m \prod_{p=0}^{n-1} \left( e^{\frac{1}{n} B_{m} (s + \frac{pt}{n})} e^{\frac{1}{n} A_{m} (s + \frac{pt}{n})} \right) P_m x
\]
for all \( x \in X \).

**Proof.** We will apply Bátkai, Csomó and Nickel [2, Theorem 3.6], the modified Chernoff’s Theorem directly. To this end, define the spaces
\[
\mathcal{X}_m = C_0(\mathbb{R}; X_m), \quad \mathcal{X} := C_0(\mathbb{R}; X)
\]
and the projection operators
\[
P_m = I \otimes P_m : \mathcal{X} \to \mathcal{X}_m, \quad (P_m f)(t) := P_m f(t),
\]
and interpolation operators
\[
J_m = I \otimes J_m : \mathcal{X}_m \to \mathcal{X}, \quad (J_m f_m)(t) := J_m f_m(t).
\]
We have to check that these operators satisfy the conditions in Assumption 4.3:

Conditions (i) and (iii) are immediate from the definitions. The \( (J_m P_m f)(s) \rightarrow f(s) \) is true pointwise. We have to show that the convergence holds in fact uniformly in \( s \in \mathbb{R} \). Take \( \varepsilon > 0 \). Let \( f \in \mathcal{X} \) and \( [a, b] \subset \mathbb{R} \) such that \( \| f(s) \| \leq \frac{\varepsilon}{M} \) for all \( s \in \mathbb{R} \setminus [a, b] \). Then
\[
\| J_m P_m f(s) - f(s) \| \leq \varepsilon
\]
for \( s \in \mathbb{R} \setminus [a, b] \). Since \( f \) is uniformly continuous, there is \( \delta > 0 \) such that for all \( s, t \in [a, b] \), \( |s - t| < \delta \), we have \( \| f(s) - f(t) \| \leq \frac{\varepsilon}{M} \). Take a partition \( a = s_0 < s_1 < \ldots < s_n = b \) such that \( |s_{i+1} - s_i| < \delta \). Then by definition, there is \( M > 0 \) such that for all \( m \geq M \)
\[
\| J_m P_m f(s_i) - f(s_i) \| \leq \varepsilon.
\]
Since for \( s \in [a, b] \) there is \( j \) such that \( s \in [s_j, s_{j+1}] \), we get for \( m \geq M \),
\[
\| J_m P_m f(s) - f(s) \|
\leq \| J_m P_m f(s) - f(s_j) \| + \| J_m P_m f(s_j) - f(s_j) \| + \| f(s_j) - f(s) \| \leq \varepsilon.
\]
Hence, \( \| J_m P_m f - f \|_\infty \leq \varepsilon \) holds for all \( m \geq M \).

The validity of Assumption 4.4 implies that the corresponding multiplication semigroups satisfy the necessary stability and consistency conditions. \( \square \)
Positivity preservation. As it was pointed out by W. Arendt (Ulm), the product and splitting formulas can be used to show positivity properties of evolution families. On the terminology and properties of positive operator semigroups see Arendt et al. [1] or Engel and Nagel [8, Section VI.1].

**Theorem 4.6.** Assume that $X$ is a Banach lattice.

1. Let the conditions of Assumptions 2.2 are satisfied and that all the operators $A(r)$ and $B(r)$ generate positive semigroups. Then the evolution family $W$ given by (4) in Theorem 2.2 is positive.

2. Let the conditions of Assumptions 3.1 are satisfied and that all the evolution families $U$ and $V$ are positive. Then the evolution family $W$ given by Theorem 3.2 is positive.

The proof is an immediate consequence of the fact that the corresponding multiplication, shift, and evolution semigroups are positive. It would be an important and interesting question whether similar results hold for shape preserving semigroups in the sense of Kovács [21, Definition 20].

5. A non-autonomous parabolic equation

In order to demonstrate the range of our results, we will consider an important and much studied parabolic equation

$$\partial_t u(x, t) = \Delta u(x, t) + V(x, t)u(x, t)$$

in $\mathbb{R}^d$ with appropriate initial conditions, where $V$ is a smooth and bounded function. Rewritten abstractly this takes the form

$$\frac{d}{dt} u(t) = \Delta u(t) + V(t)u(t)$$

with $u : \mathbb{R}_+ \to L^2(\mathbb{R}^d) := X$ a vector valued function. Hence a straightforward choice for the splitting for the evolution semigroups is

$$A := -\frac{d}{dt} + \Delta, \quad B := \text{the pointwise multiplication by } V(t).$$

These operators (with appropriate domain) generate the following semigroups on the Banach space $\mathcal{X} := \text{BUC}(\mathbb{R}; L^2(\mathbb{R}^d))$

$$[T(t)f](s) := e^{t\Delta}f(s-t) \quad \text{and} \quad [S(t)f](s) := e^{tV(s)}f(s).$$

We shall assume that $V \in \text{BUC}(\mathbb{R}; L^\infty(\mathbb{R}^d))$, so $B$ is bounded. The domain of the generator of $\mathcal{S}$ can be given explicitly, see Nagel, Nickel and Romanelli [23, Prop. 4.3]):

$$D(A) = \{ f \in \text{BUC}(\mathbb{R}; X) \cap \text{BUC}^1(\mathbb{R}; X_{-1}) : -f' + \Delta_{-1}f \in \text{BUC}(\mathbb{R}; X) \},$$

here $\Delta_{-1}$ with domain $L^2(\mathbb{R}^d)$ is the generator of the extrapolated semigroup, see Engel and Nagel [8, Section II.5.a] for the corresponding definitions.

As a corollary of Theorem 2.2 we obtain the convergence of the sequential (and also the Strang) splitting procedures.

**Proposition 5.1.** Suppose that the potential $V \in \text{BUC}(\mathbb{R}; L^\infty(\mathbb{R}^d))$. Let $W$ denote the semigroup generated by $A + B$ on $\text{BUC}(\mathbb{R}; L^2(\mathbb{R}^d))$. For every function $f \in \text{BUC}(\mathbb{R}; L^2(\mathbb{R}^d))$ we have the product formula

$$\lim_{n \to \infty} \left(\mathcal{S}(\frac{t}{n}) T(\frac{t}{n})\right)^n f = W(t) f,$$

where the convergence is uniform on compact time-intervals. Let $(W(t,s))_{t \geq s}$ denote the evolution system solving (11) on $L^2(\mathbb{R}^d)$. Then for every $u_0 \in L^2(\mathbb{R}^d)$ we have

$$\lim_{n \to \infty} \left\| W(t,s) u_0 - \prod_{p=0}^{n-1} e^{tV\left(\frac{s+p}{n}\right)} e^{t\Delta} u_0 \right\| = 0,$$
Theorem 5.2. Evolution semigroups.

The second assertion is a direct consequence of Theorem 2.2. □

\[ f \in \text{thereon}. \] We are interested in the Favard spaces \( X \).

Consider now the Banach space \( \| \cdot \| \) which becomes a Banach space if endowed with the norm \( \| \cdot \| \) of Theorem 5.1 and from the fact that the constant function \( f(s) = 0 \) is in the domain of \( A \), the following abstract identification of the domains of fractional powers of evolution semigroup generators.

\[ \text{Suppose that } V \in \text{BUC}(\mathbb{R}; W^{2,\infty}(\mathbb{R}^d)) \cap \text{BUC}^1(\mathbb{R}; L^\infty(\mathbb{R}^d)). \] If \( f \in \text{BUC}^1(\mathbb{R}; H^2(\mathbb{R}^d)) \), we obtain

\[
\left\| (S(\frac{t}{n})T(\frac{s}{n}))^n - W(t)f \right\| \leq \frac{Ct^2}{n} \| f \|_{\text{BUC}^1(\mathbb{R}; H^2(\mathbb{R}^d))}.
\]

Before we prove the theorem, let us first reformulate this product formula for the solutions of the non-autonomous problem.

Corollary 5.3. Consider the non-autonomous parabolic equation

\[
\begin{align*}
\partial_t u(x, t) &= \Delta u(x, t) + V(x, t)u(x, t), \quad t \geq s, \ x \in \mathbb{R}^d, \\
u(x, s) &= u_0(x), \quad x \in \mathbb{R}^d.
\end{align*}
\]

Suppose that \( V \in \text{BUC}(\mathbb{R}; W^{2,\infty}(\mathbb{R}^d)) \cap \text{BUC}^1(\mathbb{R}; L^\infty(\mathbb{R}^d)) \). If \( u_0 \in H^2(\mathbb{R}^d) \) then for the evolution family \( (W(t, s))_{t \geq s} \) solving the above problem we have

\[
\left\| W(t, s)u_0 - \prod_{p=0}^{n-1} e^{\frac{t-s}{n}V(s + \frac{p}{n})} e^{\frac{t-s}{n}\Delta} u_0 \right\| \leq \frac{C(t-s)^2}{n} \| u_0 \|_{H^2}.
\]

Proof. The assertion follows from Theorem 5.2, from the calculations in the proof of Theorem 5.1 and from the fact that the constant function \( f(s) := u_0 \in H^2(\mathbb{R}^d) \) is in the domain of \( A \). □

In order to prove Theorem 5.2 we have to verify the commutator condition in Theorem 1.2 for the generators of the evolution semigroups. To do this, we need the following abstract identification of the domains of fractional powers of evolution semigroup generators.

In what follows, let \( B(\mathbb{R}; Y) \), \( \text{BUC}^\alpha(\mathbb{R}; Y) \) etc. denote the space of bounded \( Y \)-valued functions, the space of \( \alpha \)-Hölder continuous \( Y \)-valued functions etc., where \( Y \) is some Banach space. Let \( X \) be a fixed Banach space, and let \( e^{tA} \) be a (contractive) analytic semigroup with generator \( (A, D(A)) \) thereon. The fractional powers of \( -A \) are denoted by \( ((-A)^\alpha, D((-A)^\alpha)) \). Denote by \( \mathcal{F}_\alpha \) the abstract Favard spaces for \( X \) and \( (e^{tA})_{t \geq 0} \), i.e.,

\[
\mathcal{F}_\alpha := \left\{ x \in X : \| x \|_\alpha := \| x \| + \sup_{t > 0} \left\| \frac{e^{tA} - e^{(t-b)A}}{t-b} \right\| < +\infty \right\},
\]

which becomes a Banach space if endowed with the norm \( \| \cdot \|_\alpha \). For every \( \alpha, \beta \in (0, 1) \) with \( \alpha > \beta \) we have continuous embeddings [see Engel and Nagel 8 Sec. II.5.]:

\[
\mathcal{F}_\alpha \hookrightarrow D((-A)^\beta) \hookrightarrow \mathcal{F}_\beta.
\]

Consider now the Banach space \( \mathcal{X} := \text{BUC}(\mathbb{R}; X) \) and the semigroup

\[
(T(t)f)(s) := e^{tA}f(s - t)
\]

thereon. We are interested in the Favard spaces \( \mathcal{X}_\alpha \) of this semigroup.
Proposition 5.4. In the above setting we have the following continuous inclusions:

\[ \text{BUC}(\mathbb{R}; D((-A)^{\alpha})) \cap \mathcal{X}_\alpha \hookrightarrow \text{BUC}(\mathbb{R}; D((-A)^{\beta})) \cap \text{BUC}^\beta(\mathbb{R}; X), \]

for all \( 0 < \beta \leq \alpha < 1 \), and

\[ \text{BUC}^\alpha(\mathbb{R}; X) \cap \text{BUC}(\mathbb{R}; D((-A)^{\alpha})) \hookrightarrow \text{BUC}(\mathbb{R}; D((-A)^{\beta})) \cap \mathcal{X}_\beta, \]

for all \( 0 < \beta \leq \alpha < 1 \).

**Proof.** We show the statement for \( \beta = \alpha \), the rest then immediately follows. We start with the second inclusion. For \( f \in \text{BUC}(\mathbb{R}; X) \) we can write

\[
\sup_{t > 0}\left\| \frac{T(t)f - f}{t^\alpha} \right\| = \sup_{t > 0}\sup_{s \in \mathbb{R}}\left\| e^{tA}f(s - t) - f(s) \right\| = \sup_{t > 0}\sup_{s \in \mathbb{R}}\left\| e^{tA}f(s) - f(s) + e^{tA}[f(s - t) - f(s)] \right\| \\
\leq \sup_{s \in \mathbb{R}}\|f(s)\|_{F_\alpha} + \|f\|_{\text{BUC}^\alpha}.
\]

This shows that if \( f \in B(\mathbb{R}; F_\alpha) \cap \text{BUC}^\alpha(\mathbb{R}; X) \), then \( f \in \mathcal{X}_\alpha \), and the inclusion is continuous, i.e.

\[ \|f\|_{\mathcal{X}_\alpha} \leq C\left(\|f\|_{B(\mathbb{R}; F_\alpha)} + \|f\|_{\text{BUC}^\alpha(\mathbb{R}; X)}\right). \]

To see the first inclusion we use now that \( A \) generates an analytic semigroup. If \( f \in \text{BUC}(\mathbb{R}; D((-A)^{\alpha})) \), then

\[
\sup_{t > 0}\left\| e^{tA}f(s - t) - f(s - t) \right\| = \sup_{t > 0}\left\| (e^{tA} - I)(-A)^{-\alpha}(-A)^{\alpha}f(s - t) \right\| \\
\leq C\sup_{t \in \mathbb{R}}\|(-A)^{\alpha}f(s - t)\| \leq C\|f\|_{\text{BUC}(\mathbb{R}; D((-A)^{\alpha}))}.
\]

This implies then

\[
\sup_{t > 0}\left\| \frac{f(s - t) - f(s)}{t^\alpha} \right\| \leq \sup_{s \in \mathbb{R}}\sup_{t > 0}\left\| \frac{T(t)f - f}{t^\alpha} \right\| + C\|f\|_{\text{BUC}(\mathbb{R}; D((-A)^{\alpha}))}.
\]

The proof is complete. \( \square \)

Now we are in the position to check the required commutator condition and thus to prove Theorem 5.2.

**Proof of Theorem 5.2.** Consider now the evolution semigroup corresponding to the non-autonomous equation (5). The corresponding generator is given formally as \(-\frac{d}{dt} + \Delta + V(t)\).

Take now \( f \in \text{BUC}^1(\mathbb{R}; H^2(\mathbb{R}^d)) \), and notice that then \( f \) belongs to the domain \( D(A) \). We calculate the commutator of \( A \) and \( B \). We have

\[ [A, B]f = -V'(t)f(t) + (\Delta V(t))f + 2\nabla V(t) \cdot \nabla f(t). \]

Now, if we assume that \( V \in \text{BUC}^1(\mathbb{R}; L^\infty(\mathbb{R}^d)) \) and \( V \in \text{BUC}(\mathbb{R}; W^{2,\infty}(\mathbb{R}^d)) \), then the first two terms can be estimated by \( c\|f\| \), so we have only to deal with the term \( 2\nabla V \cdot \nabla f \), for which it suffices to estimate \( \partial_i f(t) \) for \( i = 1, \ldots, d \). We have

\[ \|\partial_i f(t)\|_2 \leq c\|\Delta^{1/2} f(t)\|_2 \quad (\partial_i \text{ is } \Delta^{1/2}\text{-bounded on } L^2). \]

By Proposition 5.4 this completes the proof of the commutator condition (1) in the form

\[ \|[A, B]f\| \leq \|(-A)^{\alpha}f\| \quad \text{for all } f \in D(A) \text{ with some given } \alpha \geq 1/2. \]

Hence Theorem 1.2 yields the assertion. \( \square \)
6. Numerical examples illustrating the convergence

In Section 5 we already introduced the non-autonomous parabolic equation (sometimes also called imaginary time Schrödinger equation)

\[ \partial_t u(x,t) = \Delta u(x,t) + V(x,t)u(x,t) \]

in \( \mathbb{R}^d \) with appropriate initial conditions with \( V \) being a smooth and bounded function. In the following we will apply the sequential splitting introduced in Section 3 to the sub-operators

\[ A(t) := \Delta \quad \text{and} \quad B(t) := \text{multiplication by } V(x,t). \]

In Theorem 2.2 we showed that the product formula describing the sequential splitting is convergent also in the case if we are able to solve the corresponding autonomous Cauchy problems (Eq. 1)-(Eq. 2) with operators \( A(r) \) and \( B(r) \) for every time level \( r \in \mathbb{R} \). We will use this result when constructing our numerical scheme.

In order to illustrate numerically the convergence of the sequential splitting and give an estimate on its order, let us consider the following non-autonomous equation with boundary and initial conditions:

\[
\begin{aligned}
\partial_t u(x,t) &= \partial_t^2 u(x,t) + V(x,t)u(x,t), \quad t \geq 0, \quad x \in [0,1], \\
\tau u(0,t) &= u(1,t) = 0, \quad t \geq 0, \\
u(x,0) &= u_0(x), \quad x \in [0,1]
\end{aligned}
\]

(7)

with functions \( V(x,t) \) and \( u_0(x) \) given later on in the example.

6.1. Error analysis. Let \( (u^{\text{spl}})_n^i \) denote the approximation of the exact solution \( u(i\delta,n\tau) \) of problem (7) at time \( n\tau \) and at the grid point \( i\delta \) (with \( n = 0, \ldots, N - 1 \) and \( i = 0, \ldots, I - 1 \)) using sequential splitting. At this point the time-step \( \tau = \frac{1}{N - 1} \) and the grid size \( \delta = \frac{1}{I - 1} \) have certain given values. We call \( (u^{\text{spl}})^n = ((u^{\text{spl}})_0^n, (u^{\text{spl}})_1^n, \ldots, (u^{\text{spl}})_I^n) \), \( n = 0,1, \ldots, N - 1 \), the split solution of problem (7).

As already seen, the order of the splitting procedure can be estimated with the help of the splitting error defined by

\[ E^{n}_{\text{spl}} := \| u^n - u^{\text{spl}}_n \| \]

where \( u^n = (u_0^n, u_1^n, \ldots, u_{I-1}^n) \) with \( u_i^n = u(i\delta,n\tau), i = 0,1, \ldots, I - 1 \). With this notation the splitting procedure (or an arbitrary finite difference method) is of order \( p > 0 \) if for sufficiently smooth initial values there is a constant \( C > 0 \) such that for all \( t \in [0,t_0] \) we have

\[ E^{n}_{\text{spl}} \leq C \tau^{p+1}, \]

or, if the method is stable, equivalently,

\[ E^{1}_{\text{spl}} \leq C \tau^{p+1}. \]

In general, the exact solution of problem (7) is unknown, therefore, the local splitting error \( E^{1}_{\text{spl}} \) is to be estimated as well. To this end we compute a so-called reference solution \( u^{n}_{\text{ref}} \) on a finer space grid using no splitting procedure. Then the order \( p \) of the splitting procedure can be determined as follows. From the definition of \( p \) we have \( E^{1}_{\text{spl}} \leq C \tau^{p+1} \). Approximating \( u^n \) with \( u^{n}_{\text{ref}} \), we obtain \( E^{1}_{\text{spl}} \approx \tilde{E}^{1}_{\text{spl}} := \| u^{n}_{\text{ref}} - u^{1}_{\text{spl}} \| \leq C \tau^{p+1} \). Thus,

\[ \log \tilde{E}^{1}_{\text{spl}} \leq (p + 1) \log \tau + \log C. \]

Then we can estimate \( p \) by computing the approximate local splitting error \( \tilde{E}^{1}_{\text{spl}} \) for many different values of the time-step \( \tau \), plotting the logarithm of the results, and
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fitting a line of form \( y(w) = aw + b \) to them. Hence, \( a \approx p + 1 \) and \( b \approx \log C \).

Note, however, that the split solution contains not only the splitting error but also a certain amount of error originating from the spatial and temporal discretization. In what follows we show how to determine the numerical solutions \( u_{\text{ref}}^1 \) and \( u_{\text{spl}}^1 \).

We also note that it is reasonable to compute a relative local error defined as

\[
E_{\text{loc}} = \frac{\tilde{E}_1}{\| u_{\text{ref}}^1 \|}
\]

because this yields the ratio how the split solution differs from the reference solution.

6.2. Numerical scheme. In order to solve numerically the problem (7) we should discretize it in both space and time. For the temporal discretization we used the Crank-Nicholson method, and we chose the finite difference method for the spatial discretization.

6.2.1. Reference solution. As mentioned above, we need a reference solution \( u_{\text{ref}}^n \) computed without using splitting procedures. After discretizing the equation, we obtain the following numerical scheme for determining \( u_{\text{ref}}^n \):

\[
(u_{\text{ref}}^n)_{i+1} = (1 - (H_{\text{ref}})^n_{i+1})^{-1} (1 + (H_{\text{ref}})^n_{i}) (u_{\text{ref}}^n)_{i}
\]

with

\[
(H_{\text{ref}})^n_{i} = \frac{\tau}{2} \left( u_{i+1}^{n+1} - 2u_{i}^{n+1} + u_{i-1}^{n+1} + V_{i}^n \right),
\]

where \( V_{i}^n := V(i\delta, n\tau) \).

6.2.2. Split solution. Application of sequential splitting means that instead of the whole problem (7) two sub-problems are solved. In our examples the first sub-problem corresponds to the diffusion equation \( \partial_t u_A(x, t) = \partial_x^2 u_A(x, t) \). Its numerical solution \( u_A^n \) can also be computed using Crank-Nicholson temporal and finite difference spatial discretization methods. Then we obtain the following numerical scheme similar to (8):

\[
(u_A^{n+1})_{i} = (1 - (H_A)^{n+1}_i)^{-1} (1 + (H_A)^n_i) (u_A^n)_{i}
\]

with

\[
(H_A)^n_i = \frac{\tau}{2} \frac{u_{i+1}^{n+1} - 2u_{i}^{n+1} + u_{i-1}^{n+1}}{\delta^2}.
\]

The second sub-problem has the multiplication operator by \( V(x, t) \) on its right-hand side, i.e. \( \partial_t u_B(x, t) = V(x, t)u_B(x, t) \). We refer again to Theorem 2.2 and take the function \( V \) only at time levels \( t = n\tau, n = 0, 1, ..., N - 1 \). In this (autonomous) case the exact solution \( u_B(x, t) = e^{V(x, n\tau)}u_0(x) \) is known. At the \( n^{\text{th}} \) time level and on the space grid it has the form

\[
(u_B^n)_{i} = u_B(i\delta, n\tau) = e^{V(i\delta, n\tau)}u_0(i\delta).
\]
Due to the product formula (9), the split solution $u_{\text{spl}}^n$ is given by the following algorithm:

\[
\text{for } i = 0, \ldots, I - 1 \\
\text{initial function: } (u_A)_i^0 := u_0(i\delta) \\
\text{end} \\
\text{for } n = 0, 1, \ldots, N - 1 \\
\text{for } i = 0, 1, \ldots, I - 1 \\
\text{solve the first sub-problem using } (9) \Rightarrow (u_A)_i^n \\
\text{end} \\
\text{for } i = 0, 1, \ldots, I - 1 \\
\text{solve the second sub-problem using } (10) \Rightarrow (u_B)_i^n \\
\text{end} \\
\text{end} \\
\text{split solution: } u_{\text{spl}}^{N-1} := u_B^{N-1}
\]

6.3. Numerical results. Now we present some numerical results on the following example.

Choose

\[V(x, t) = t - 500x^2 \quad \text{and} \quad u_0(x) = e^{-50(x-0.4)^2}.
\]

Since the exact solution is unknown in this case, we should estimate the local splitting error using the reference solution instead of the exact one. Then the relative local splitting error $\mathcal{E}_{\text{loc}}$ and its order $p$ can be measured.

The dots correspond to $\log(\mathcal{E}_{\text{loc}})$ for the various step sizes. The line fitted to these points has the form $y(\log(\tau)) = a \log(\tau) + b$ with $a = 1.9470$ and $b = 3.25925$.

As mentioned above, the order of the splitting procedure $p$ can be estimated by $a - 1 \approx 1$, that is, the sequential splitting is of first order.

\[\text{Figure 1. Numerical solution of equation (7) at time levels } t = 0, t = 10^{-3}, t = 5 \cdot 10^{-3}, \text{ and } t = 10^{-2}, \text{ respectively.}\]
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Figure 2. Results obtained by applying the sequential splitting with various time steps (dots), and the line \( y(w) = aw + b \) fitted to them with parameters \( a = 1.9470 \approx p + 1 \) and \( b = 3.25925 \).

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