Gauge Symmetries, Topology and Quantisation

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Abstract

The following two loosely connected sets of topics are reviewed in these lecture notes: 1) Gauge invariance, its treatment in field theories and its implications for internal symmetries and edge states such as those in the quantum Hall effect. 2) Quantisation on multiply connected spaces and a topological proof the spin-statistics theorem which avoids quantum field theory and relativity. Under 1), after explaining the meaning of gauge invariance and the theory of constraints, we discuss boundary conditions on gauge transformations and the definition of internal symmetries in gauge field theories. We then show how the edge states in the quantum Hall effect can be derived from the Chern-Simons action using the preceding ideas. Under 2), after explaining the significance of fibre bundles for quantum physics, we review quantisation on multiply connected spaces in detail, explaining also mathematical ideas such as those of the universal covering space and the fundamental group. These ideas are then used to prove the aforementioned topological spin-statistics theorem.

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I. INTRODUCTION

In recent years, there have been several important developments in low dimensional quantum physics such as those associated with conformal and Chern-Simons field theories, the quantum Hall effect and anyon physics. These lecture notes will address certain aspects of these developments, in particular those concerning gauge invariance and multiple connectivity and their consequences for low dimensional physics.

The material in these notes is organised as follows. In Chapter 2, we discuss the meaning of gauge symmetries and their distinction from conventional symmetries in general terms. The reason why gauge invariance leads to constrained Hamiltonian dynamics is also pointed out using qualitative arguments. An important tool for the quantisation of theories with constraints is the Dirac-Bergmann theory of constraints and that is briefly reviewed in Chapter 3.

Chapters 4 and 5 deal with important technical aspects regarding the treatment of constraints in gauge field theories and some of their physical consequences. The intimate and beautiful relationship between symmetries and gauge invariance is clarified and the general theory illustrated by examples from electrodynamics and the quantum Hall effect. The relation of the edge states and source excitations in the Hall system to gauge invariance is in particular explained in Chapter 5.

The remaining Chapters deal with quantisation of classical theories in multiply connected configuration spaces. As indicated previously, this topic has assumed importance in low dimensional physics. It has an especially crucial role in Hall effect and anyon physics where fractional statistics has a basic significance, statistics being a manifestation of configuration space connectivity. The notes conclude with a proof of the spin-statistics theorem in Chapter 7 using topological methods. This proof avoids the use of relativistic quantum fields and seems well adapted to condensed matter systems where such fields are not generally of any relevance.

II. MEANING OF GAUGE INVARIANCE

The subject matter of our first few Chapters is gauge invariance and its physical implications. We will introduce the topic of gauge transformations
in this Chapter, discussing it in general conceptual terms [following ref. 1] and emphasizing its distinction from ordinary (global) symmetry transformations.

A. The Action

The action $S$ is a functional of fields with values in a suitable range space. The domain of the fields is a suitable parameter space.

Thus for a nonrelativistic particle, the range space may be $\mathbb{R}^3$, a point of which denotes the position of the particle. The parameter space is $\mathbb{R}^1$, a point of which denotes an instant $t$ of time. The fields $q$ are functions from $\mathbb{R}^1$ to $\mathbb{R}^3$. Thus, if $F(\mathbb{R}^1, \mathbb{R}^3)$ is the collection of these fields,

$$F(\mathbb{R}^1, \mathbb{R}^3) = \{q\}, \quad q = (q_1, q_2, q_3), q(t) \in \mathbb{R}^3. \quad (1)$$

In other words, each field $q$ assigns a point $q(t)$ in $\mathbb{R}^3$ to each instant of time $t$.

For a real scalar field theory in Minkowski space $M^4$, the parameter space is $M^4$, the range space is $\mathbb{R}^1$ and the set of fields $F(\mathbb{R}^4, \mathbb{R}^1)$ is the set of functions from $\mathbb{R}^4$ to $\mathbb{R}^1$.

Let us denote the parameter space by $D$, the range space by $R$ and the set of fields by $F(D, R)$. Then the action $S$ is a function on $F(D, R)$ with values in $\mathbb{R}^1$. It assigns a real number $S(f)$ to each $f \in F(D, R)$. For instance, in the nonrelativistic example cited above,

$$S(q) = \frac{m}{2} \int dt dq_i(t) \frac{dq_i(t)}{dt}. \quad (2)$$

[The action also depends on the limits of time integration. Since these limits are not important for us, they have been ignored here. If necessary, they can be introduced by restricting $D$ suitably. In this case, for example, instead of $\mathbb{R}^1$, we can choose the interval $t_1 \leq t \leq t_2$ for $D$.]

The concept of a global symmetry group may be defined as follows: Suppose $G = \{g\}$ is a group with a specified action $r \rightarrow gr$ on $R \equiv \{r\}$. Then, $G$ has a natural action $f \rightarrow gf$ on $F(D, R)$, where $(gf)(t) = gf(t)$. This group of transformations on $F(D, R)$ is the global group associated with $G$. We denote it by the same symbol $G$. We say further that $G$ is a global symmetry group if
\[ S(f) = S(gf) \quad (3) \]

up to surface terms. For simplicity, we will assume hereafter that \( G \) is a connected Lie group.

As an example, consider the nonrelativistic free particle with \( D = \{ t \mid -\infty < t < \infty \} \), \( R = \mathbb{R}^3 \) and \( G = SO(3) \). The rotation group has a standard action on \( \mathbb{R}^3 \). It can be “lifted” to the action \( q \to gq \) on \( F(\mathbb{R}^1, \mathbb{R}^3) \), where

\[ [gq](t) = gq(t) \quad [\equiv (g_{ij}q_j(t))] . \quad (4) \]

Thus in the usual language, \( g \) is a global rotation. Further, \( SO(3) \) is a global symmetry group since for \( (2.4) \),

\[ S(q) = S(gq) . \quad (5) \]

In contrast, the gauge group \( \hat{G} \) associated with a global group \( G \) is defined to be the set of all functions \( F(D,G) = \{ h \} \) from \( D \) to \( G \) [with a group composition law to be defined below]. An element \( h \) of \( F(D,G) \) thus assigns an element \( h(d) \) of \( G \) for each point \( d \) in \( D \):

\[ D \ni d \mapsto h(d) \in G. \quad (6) \]

[The hat for \( \hat{G} \) is put there to distinguish it from \( G \) which will occur later.]

The group multiplication in \( \hat{G} \) is defined by \((hh')(d) = h(d)h'(d)\). This group as well has a natural action \( f \to hf \) on \( F(D,R) \) defined by \((hf)(d) = h(d)f(d)\). If \( S \) is invariant under \( \hat{G} \) (up to surface terms), that is, if \( S(hf) = S(f) \) + possible surface terms, then the gauge group is a gauge symmetry group.

It is possible that the sort of boundary conditions we impose on the set of functions in the gauge group can have serious consequences for the theory as we shall see in Chapter 4. See also ref. 2.

Let \( \hat{G} \) be a gauge symmetry group and let \( \Gamma \) be a global symmetry group where \( \hat{G} \) is not necessarily associated with \( \Gamma \). Recall that the parameter space contains a coordinate which we identify as time \( t \). The profound difference between \( \hat{G} \) and \( \Gamma \) is due to the fact that \( \hat{G} \) contains time dependent transformations unlike \( \Gamma \). It affects the deterministic aspects of the theory and also
has its impact on Noether’s derivation of conservation laws. These twin aspects are manifested as constraints in the Hamiltonian framework. We can illustrate these remarks as follows:

1. **Determinism**

A trajectory, by which we mean a solution to the equations of motion, is a function $\bar{f} \in F(D, R)$ at which the action is an extremum. [The extremum is defined relative to a certain class of variations around $\bar{f}$. We will not discuss the details of these variations here.]

Suppose that $\bar{f}$ is a possible trajectory for a specified set of initial conditions $\frac{d^k \bar{f}}{dt^k} \bigg|_{t=0}, k = 0, 1, \ldots, n$. Since $\hat{G}$ is a gauge symmetry group, $h\bar{f}$ is also a trajectory. Further, since the time dependence of $h$ is at our disposal, we can choose $h$ such that

$$\left. \frac{d^k (h\bar{f})}{dt^k} \right|_{t=0} = \left. \frac{d^k \bar{f}}{dt^k} \right|_{t=0}, \quad k = 0, 1, \ldots, n. \quad (7)$$

This does not constrain $h$ to be trivial for all time [so that we can have $h\bar{f} \neq \bar{f}$]. The conclusion is that there are several possible trajectories for specified initial conditions. [We assume of course that $\hat{G}$ acts nontrivially on fields.] In this sense, the theory does not determine the future from the present if the state of the system is given by the values of $\bar{f}$ and its derivatives at a given time.

In the customary formulation, determinism is restored by considering only those functions which are invariant under $\hat{G}$. These gauge invariant functions and their derivatives at a given time are then defined to constitute the observables of the theory. (Such a definition of observables seems to have little direct bearing on whether they are accessible to experimental observation. It is a definition which is internal to the theory and dictated by requirements of determinism.)

In a Hamiltonian formulation with no constraints, the specification of Cauchy data (a point of phase space) allows us to uniquely specify the future state of the system (at least for sufficiently small times). The existence of a gauge symmetry group for the action $S$ thus suggests that $S$ should lead to a constrained Hamiltonian dynamics. This is in fact generally the case. An
orderly way to treat constrained dynamics is due to Dirac and Bergmann. We will explain it briefly in the next Chapter.

2. Conservation Laws

The infinitesimal variation of $S$ under a gauge transformation is characterized by arbitrary functions $\epsilon_\alpha$. If $\hat{G}$ is a gauge symmetry, Noether’s argument shows that there is a charge

$$Q = \int_\mathcal{D} \epsilon_\alpha Q_\alpha$$

which is a constant of motion:

$$\frac{dQ}{dt} = 0 .$$

Here $\mathcal{D}$ is a fixed time slice of $D$. Since the $\epsilon_\alpha$’s are arbitrary functions, we can conclude that

$$Q_\alpha = 0 .$$

Thus the generators of the gauge symmetry group vanish.

In electromagnetism, the analogues of (2.10) are Gauss’ law

$$\nabla \cdot \vec{E} + J_0 = 0$$

and the vanishing of the canonical momentum $\pi^0$ conjugate to $A_0$. The nonabelian generalizations of these equations are well known.

In the Hamiltonian framework, the equations $Q_\alpha = 0$ become first class constraints [cf. Chapter 3]. Quantization of the system often becomes highly nontrivial in their presence.

B. The Lagrangian

We will assume as previously that the theories we consider admit a choice of time. The configuration space in such a theory is usually identified with $F(\mathcal{D}, \mathbb{R}^4)$, where $\mathcal{D}$ is a fixed time slice of $D$. It is clear however that for precision, we should write $\mathcal{D}_t$ for the slice of $D$ at time $t$. The customary hypothesis is that $\mathcal{D}_t$ for different $t$ are diffeomorphic and that there is a natural
identification of points of \( \mathcal{D}_t \) for different times. Under these circumstances (which we assume), we are justified in writing \( \mathcal{D} \).

As an example, consider a field theory on a four dimensional manifold with the topology of Minkowski space \( M^4 \). Slices at different times \( t \) give different three dimensional subspaces \( \mathbb{R}^3_t \). Without further considerations, there is no natural identification of points of these spaces, that is, there is as yet no obvious meaning to the identity of spatial points for observations at different times. What is done in practice is as follows: On \( M^4 \), there is an action of the time translation group \( \{ U_\tau \mid -\infty < \tau < \infty \} \). The latter maps \( \mathbb{R}^3_t \) to \( \mathbb{R}^3_{t+\tau} \) in a smooth, invertible way. We then identify all points in \( \mathbb{R}^3_t \) and \( \mathbb{R}^3_{t+\tau} \) which are carried into each other by time translations \( U_{\pm\tau} \). In the conventional coordinates \( (\vec{x}, t) \),

\[
U_\tau(\vec{x}, t) = (\vec{x}, t + \tau)
\]

(12)

and we think of \( \vec{x} \) as referring to the same three dimensional point for all times.

A field \( f \in F(D, R) \) restricted to a given time \( t \) is a function on \( \mathcal{D}_t \). Since we have an identification of points of \( \mathcal{D}_t \) for different \( t \), the field \( f \) can be regarded as a one dimensional family of functions \( f_t \in F(\mathcal{D}, R) \) parametrized by time. We have thus established a correspondence

\[
F(D, R) \rightarrow F(\mathbb{R}^1, F(\mathcal{D}, R))
\]

(13)

between functions appropriate to the action principle and curves in the configuration space \( F(\mathcal{D}, R) \).

The Lagrangian is a function of “coordinates and velocities.” That is, it is a function of a point \( \alpha \in F(\mathcal{D}, R) \) on the configuration space and of the tangent \( \dot{\alpha} \) to this space at this point. This new space (a point of which is a point and a tangent at that point of the configuration space) is called the tangent bundle \( T F(\mathcal{D}, R) \) on the configuration space.

When the action is reconstructed from the Lagrangian by the formula

\[
S = \int dt \ L(\alpha(t), \dot{\alpha}(t)),
\]

(14)

we are integrating \( L \) along curves in the tangent bundle. This curve is not arbitrary since we require that \( \dot{\alpha}(t) = d\alpha(t)/dt \). Such a curve in the tangent
bundle is the “lift of a curve” from the configuration space. (It is defined by a “second order” vector field in the tangent bundle). With this restriction on curves, a curve in the tangent bundle is uniquely determined by a curve in $F(D, R)$. Since such a curve in turn defines a function in $F(D, R)$, we recover the original interpretation of the action as a function on $F(D, R)$.

We need to investigate the action of the gauge group on the tangent bundle. It turns out that in its action on the tangent bundle, the gauge group, in its simplest version, is associated to the global group

$$G \circledast G = \{(\ell, h) \mid \ell \in G, \quad h \in G\}$$

where $G$ is the global group appropriate for $\hat{G}$, $\mathfrak{g}$ is its Lie algebra and the group multiplication is

$$(\ell', h')(\ell, h) = (\ell' + \text{Ad } h' \ell, h'h)$$

[The sense in which the gauge group appropriate for the Lagrangian formalism can be thought of as associated with (2.15) will be explained below.] Here $\text{Ad } h'$ is the adjoint action of $h'$ on $G$. In the notation common in physics literature,

$$\text{Ad } h' \ell = h' \ell h'^{-1}.$$ 

Thus $G \circledast G$ is the semi-direct product of $G$ with $G$. This result has been discussed before by Sudarshan and Mukunda.

We denote the gauge group associated to $\hat{G}$ at a given time by $G$. It consists of functions $F(D, G) = \{h\}$ with group multiplication defined by

$$(hh')(\vec{d}) = h(\vec{d})h'(\vec{d}), \quad \vec{d} \in D.$$ 

The Lie algebra $\mathfrak{g}$ is a group under addition and its associated gauge group $F(D, \mathfrak{g})$ at a given time will be denoted by $\mathfrak{g}$. Finally the gauge group associated to $G \circledast G$ at a given time will be denoted by $\hat{G} \circledast G$.

In contrast to elements of $G$, elements of the group $\hat{G}$ introduced earlier had arbitrary time dependence. These two groups are to be carefully distinguished although both have been called gauge groups.

The group law (2.16) can be established by examining the way the action of the gauge group $\hat{G}$ “projects down” to an action on coordinates
and velocities. A function \( f \in F(D, R) \) is transformed to \( hf \). Thus the curve \( \{ \alpha(t) \in F(\mathbb{D}, \mathbb{R}^1) \} \) (\( t \) being time) is transformed into \( \{(ha)(t)\} \) where \( h(t) \in G \) is time dependent. Thus a point of the tangent bundle is transformed according to

\[
(\alpha(t), \frac{d\alpha(t)}{dt} = \dot{\alpha}(t)) \rightarrow (h(t)\alpha(t), h(t)\frac{d\alpha(t)}{dt} + \ell(t)h(t)\alpha(t)) \quad (19)
\]

where \( \ell(t) \equiv \frac{dh(t)}{dt}h(t)^{-1} \in \mathcal{G} \). In (2.19), all time dependences can henceforth be ignored since we are examining the action of the gauge group restricted to \( TF(\mathbb{D}, R) \) at a given time. In writing (2.19), we have also assumed that the action of the gauge group is local in time, that is that

\[
(ha)(t) = h(t)\alpha(t) . \quad (20)
\]

If \( (ha)(t) \) depends on \( h(t) \) as well as (say) its derivatives \( d^kh(t)/dt^k \), (2.19) will have to be modified. For Yang-Mills theories, this actually happens. (See below). We prefer to illustrate the idea without this complication. With this assumption, we can write

\[
(\ell, h) \in \mathcal{G} \otimes \mathcal{G}, \quad (\ell, h)(\alpha, \dot{\alpha}) = (h\alpha, h\dot{\alpha} + \ell(h\alpha)) . \quad (21)
\]

The group multiplication (2.16) follows from

\[
(\ell', h')(h\alpha, h\dot{\alpha} + \ell(h\alpha)) = (h'ha, h'h\dot{\alpha} + (h'\ell h'^{-1})(h'ha) + \ell'(h'ha)) = (h'h\alpha, h'h\dot{\alpha} + (\ell' + Adh'\ell)(h'ha)) = (\ell' + Adh'\ell, h'h)\alpha, \dot{\alpha}) \quad (22)
\]

The preceding considerations are easily illustrated by Yang-Mills theory where the vector potential \( A_\mu \) has values in the Lie algebra \( \mathcal{G} \) of the global group \( G \) and transforms as follows:

\[
A_\mu \rightarrow hA_\mu h^{-1} + h\partial_\mu h^{-1} . \quad (23)
\]

Thus at a fixed time,

\[
(\ell, h)A_i = hA_ih^{-1} , \quad (24)
\]
\( (\ell, h) A_0 = h A_0 h^{-1} - \ell \) \hspace{1cm} (25)

where

\[ \ell = \dot{h} h^{-1} . \] \hspace{1cm} (26)

The group multiplication law (2.21) follows by considering the application of \((\ell', h')\) to the left hand sides of (2.24) and (2.25).

The transformation (2.25) on the configuration space variable \( A_0 \) is not local in time since (2.26) involves \( dh/dt \). Nonetheless, the group multiplication (2.21) is unaffected.

The space on which the group is supposed to act however is not the space of \( A_\mu \), but of \((A_\mu, \dot{A}_\mu)\). If we consider the subspace \((A_i, \dot{A}_i)\), since (2.24) does not involve \( \dot{h} \), we find the group \( \mathcal{G} \otimes \mathcal{G} \). However, the argument has to be modified if \( \dot{A}_0 \) is considered since its transformation involves \( \dot{\ell} \). An element of the gauge group is now a triple \((\ell, \dot{\ell}, h)\) with the action

\[ (\ell, \dot{\ell}, h)(A_0, \dot{A}_0) = (h A_0 h^{-1} - \ell, h \dot{A}_0 h^{-1} + [\ell, h A_0 h^{-1}] - \dot{\ell}) \] \hspace{1cm} (27)

and the multiplication law

\[ (\ell_1, \dot{\ell}_1, h_1)(\ell_2, \dot{\ell}_2, h_2) = (\ell_1 + h_1 \ell_2 h_1^{-1}, \dot{\ell}_1 + [\ell_1, h_1 \ell_2 h_1^{-1}] + h_1 \dot{\ell}_2 h_1^{-1}, h_1 h_2) \] \hspace{1cm} (28)

The action of \((\ell, \dot{\ell}, h)\) on \((A_i, \dot{A}_i)\) is obtained from taking the derivative of (2.24). In this action, \( \dot{\ell} \) is passive.

The general gauge group \( \mathcal{G}_L \) at the Lagrangian level can thus in general involve \( \ell, \dot{\ell}, \dot{h}, \ldots \).

The group of constant functions from \( \overline{D} \) to \( G \) is what is often called the global symmetry group. Since it is isomorphic to \( G \), we can denote it by the same symbol \( G \). It is a subgroup of \( \mathcal{G} \) if all constant functions are allowed in \( \mathcal{G} \). Thus, if the boundary conditions do not eliminate any such constant function, we can conclude the following: Since observables are gauge invariant or invariant under \( \mathcal{G} \), they are invariant under the global group \( G \). That is, all observables are globally neutral. But note however that there are as a rule conditions on the elements of \( \mathcal{G} \) so that this conclusion is not always warranted.
C. The Hamiltonian

The Hamiltonian framework provides an algebraic formulation of the classical theory in terms of Poisson brackets (PB's). It is the essential step in the quantization of the classical theory.

In Chapter 3, we outline Dirac’s procedure for setting up the canonical formalism in the presence of constraints. Certain subtle, but important aspects of this procedure involving the aforementioned boundary conditions will be explained in Chapter 4 and illustrated in Chapter 5.

III. THE DIRAC-BERGMANN THEORY OF CONSTRAINTS

A. Introduction

Constraints appear in the Hamiltonian formulation of all gauge theories we know of. We shall be applying the Dirac-Bergmann constraint theory for the treatment of these constraints. For readers unfamiliar with the subject, we give a very brief summary of this theory of constraints in the discussion which follows. [See refs. 3 and 2 for reviews and applications. They also contain further references on this subject.]

Let $M$ be the space of “coordinates” appropriate to a Lagrangian $L$. It is the space $Q$ on which equations of motion give trajectories if the Lagrangian is of the sort treated in elementary classical mechanics. More generally, it can be different from $Q$ especially for gauge invariant systems. We denote the points of $M$ by $m = (m_1, m_2, ...)$.

Now given any manifold $M$, it is possible to associate two spaces $TM$ and $T^*M$ to $M$. The space $TM$ is called the tangent bundle over $M$. The coordinate of a point $(m, \dot{m})[\dot{m} = (\dot{m}_1, \dot{m}_2, ...)]$ of $TM$ can be interpreted as a position and a velocity. The Lagrangian is a function on $TM$. The space $T^*M$ is called the cotangent bundle over $M$. The coordinate of a point $(m, p)[p = (p_1, p_2, ...)]$ of $T^*M$ can be interpreted as a coordinate (or a “position”) and a momentum so that in physicists’ language, $T^*M$ is the phase space. At each $m, p$ belongs to the vector space dual to the vector space of velocities.
Poisson brackets (PB’s) can be defined for any cotangent bundle $T^*M$. In the notation familiar to physicists, they read

$$\{m_i, m_j\} = \{p_i, p_j\} = 0, \quad \{m_i, p_j\} = \delta_{ij}. \quad (29)$$

Now given a Lagrangian $L$, there exists a map from $TM$ to $T^*M$ defined by

$$(m, \dot{m}) \to (m, \frac{\partial L}{\partial \dot{m}}(m, \dot{m})). \quad (30)$$

If this map is globally one to one and onto, the image of $TM$ is $T^*M$ and we can express velocity as a function of position and momentum. This is the case in elementary mechanics and leads to the familiar rules for the passage from Lagrangian to Hamiltonian mechanics.

### B. Constraint Analysis

It may happen, however, that the image of $TM$ under the map (3.2) is not all of $T^*M$. Suppose for instance, that it is a submanifold of $T^*M$ defined by the equations

$$P_j(m, p) = 0, \quad j = 1, 2, \ldots. \quad (31)$$

Then we are dealing with a theory with constraints. The constraints $P_j$ are said to be primary.

The functions $P_j$ do not identically vanish on $T^*M$. Rather their zeros define a submanifold of $T^*M$. A reflection of the fact that $P_j$ are not zero functions on $T^*M$ is that there exist functions $g$ on $T^*M$ such that $\{g, P_j\}$ do not vanish on the surface $P_j = 0$. These functions $g$ generate canonical transformations which take a point of the surface $P_j = 0$ out of this surface. It follows that it is incorrect to take PB’s of arbitrary functions with both sides of the equations $P_j = 0$ and equate them. This fact is emphasized by rewriting (3.3), replacing the “strong” equality signs $=$ of these equations by “weak” equality signs $\approx$:

$$P_j(m, p) \approx 0. \quad (32)$$
When $P_j(m, p)$ are weakly zero, we can in general set $P_j(m, p)$ equal to zero only after evaluating all PB’s.

In the presence of constraints, the Hamiltonian can be shown to be

$$H = \dot{m}_j \frac{\partial L}{\partial \dot{m}_j}(m, \dot{m}) - L(m, \dot{m}) + v_j P_j(m, p)$$

$$\equiv H_0(m, p) + v_j P_j(m, p).$$

(33)

In obtaining $H_0$ from the first two terms of the first line, one can freely use the primary constraints. The functions $v_j$ are as yet undetermined Lagrange multipliers. Some of them may get determined later in the analysis while the remaining ones will continue to be unknown functions with even their time dependence arbitrary.

Consistency of dynamics requires that the primary constraints are preserved in time. Thus we require that

$$\{P_m, H\} \approx 0.$$  

(34)

These equations may determine some of the $v_j$ or they may hold identically when the constraints $P_j \approx 0$ are imposed. Yet another possibility is that they lead to the “secondary constraints”

$$P'_m(q, p) \approx 0.$$  

(35)

The requirement $\{P'_m, H\} \approx 0$ may determine more of the Lagrange multipliers, lead to tertiary constraints or be identically satisfied when (3.6) and (3.7) are imposed. We proceed in this fashion until no more new constraints are generated.

Let us denote all the constraints one obtains in this way by

$$C_k \approx 0.$$  

(36)

Dirac divides these constraints into first class and second class constraints. First class constraints $F_\alpha \approx 0$ are those for which

$$\{F_\alpha, C_k\} \approx 0, \quad \forall k.$$  

(37)
In other words, the Poisson brackets of $F_\alpha$ with $C_k$ vanish on the surface defined by (3.8). The remaining constraints $S_a$ are defined to be second class.

It can be shown that

$$\{F_\alpha, F_\beta\} = C_{\alpha\beta}^\gamma F_\gamma,$$

(38)

where $C_{\alpha\beta}^\gamma = -C_{\beta\alpha}^\gamma$ are functions on $T^*M$. The proof is as follows: Eq.(3.9) implies that $\{F_\alpha, F_\beta\} = C_{\alpha\beta}^\gamma F_\gamma + D_{\alpha\beta}^a S_a$. But on using the Jacobi identity

$$\{F_\alpha, \{F_\beta, S_a\}\} + \{F_\beta, \{S_a, F_\alpha\}\} + \{S_a, \{F_\alpha, F_\beta\}\} = 0,$$

we find,

$$0 \approx \{S_a, \{F_\alpha, F_\beta\}\} \approx D_{\alpha\beta}^b \{S_a, S_b\}.$$

In obtaining this result, we have used (3.9) which implies that $\{F_\alpha, S_a\}$ is of the form $\sum_k \xi_k a C_k$. Now as regards $S_a$, we have the basic property

$$\det(\{S_a, S_b\}) \neq 0$$

(39)

on the surface $C_k \approx 0$. Thus the matrix ($\{S_a, S_b\}$) is nonsingular on the surface $C_k \approx 0$. It then follows that $D_{\alpha\beta}^b$ weakly vanishes, proving (3.10).

Let $\mathcal{C}$ be the submanifold of $T^*M$ defined by the constraints:

$$\mathcal{C} = \{(m, p) \mid C_k(m, p) = 0\}.$$ 

(40)

Then since the canonical transformations generated by $F_\alpha$ preserve the constraints, a point of $\mathcal{C}$ is mapped onto another point of $\mathcal{C}$ under the canonical transformations generated by $F_\alpha$. Since the canonical transformations generated by $S_a$ do not preserve the constraints, such is not the case for $S_a$.

Second class constraints can be eliminated by introducing the so-called Dirac brackets. They have the basic property that the Dirac bracket of $S_a$ with any function on $T^*M$ is weakly zero. We will not go into their details having no use for them in these lectures. Instead, we shall later follow the alternative route of finding all functions $\mathcal{F}$ with zero PB’s with $S_a$. So long as we work with only such functions, we can use the constraints $S_a \approx 0$ as strong constraints $S_a = 0$ and eliminate variables using them even before taking PB’s. Assuming that there are no first class constraints, the number
$N$ of functionally independent functions $\mathcal{F}$ is dimension of $T^*M$ – number of $S_a$, $N = \dim(T^*M) - s$, $s$ being the range of $a$. Thus $s$ second class constraints eliminate $s$ variables. Since $(\{S_a, S_b\})$ is nonsingular and antisymmetric, $s$ is even. Since $\dim(T^*M)$ is even as well, $N$ is even.

C. Quantization Procedure

Let us now imagine that there are only first class constraints and that $\mathcal{C}$ is defined exclusively by the zeros of $F_\alpha$. (If there are second class constraints $S_a$ as well, they can first be eliminated in the manner indicated above.) Dirac’s prescription for the implementation of first class constraints in quantum theory is that they be imposed as conditions on the physically allowed states $| \cdot >$

$$\hat{F}_\alpha | \cdot > = 0.$$  (41)

Here $\hat{F}_\alpha$ is the quantum operator corresponding to the classical function $F_\alpha$.

The following may be observed in connection with (3.12). In writing it, there is the assumption that functions on $T^*M$ have been realised (in some suitable sense) as operators on a vector space.

Since the PB’s between $F$’s involve only $F$’s, this prescription is consistent (modulo factor ordering problems). That is, both sides of the equation

$$[\hat{F}_\alpha, \hat{F}_\beta] = iC_{\alpha\beta}^\gamma \hat{F}_\gamma$$  (42)

annihilate the physical states. Here the commutator brackets $[,]$ are obtained from the PB’s using the standard prescription of Dirac. [A similar argument shows that we cannot impose the conditions $\hat{S}_a | \cdot > = 0$ on physical states where $\hat{S}_a$ is the operator corresponding to the function $S_a$.]

An observable $\hat{O}$ of the theory must preserve the condition (3.13) on the physical states. Requiring that $\hat{O} | \cdot >$ is physical if $| \cdot >$ is, we find, for the set of quantum observables $\hat{A}$, the condition

$$[\hat{O}, \hat{F}_\alpha] = id^\gamma_\alpha(\hat{O})\hat{F}_\gamma, \quad \hat{O} \in \hat{A}.$$  (43)

For classical observables $\mathcal{O}$, this becomes

$$\{\mathcal{O}, F_\alpha\} = d^\gamma_\alpha(\mathcal{O})F_\gamma.$$  (44)
Since the right hand side is zero on $\mathcal{C}$, we can regard $\mathcal{O}$ as a function on $\mathcal{C}$ which is constant on the orbits generated by $F_\alpha$. If we regard these orbits as generating an equivalence relation $\sim$ between points of $\mathcal{C}$, then the classical observables are functions on the quotient of $\mathcal{C}$ by $\sim$. This quotient $\mathcal{C}/\sim$ may be regarded as the physical phase space. Note that if there are $f$ first class constraints, then the dimension $\dim[\mathcal{C}/\sim]$ of the physical phase space is $\dim(T^*M) - 2f$, $\mathcal{C}$ having dimension $\dim(T^*M) - f$ and each orbit in $\mathcal{C}$ having dimension $f$. [Here we assume that there is no nontrivial subgroup of the group of canonical transformations generated by $F_\alpha$ which leaves a point of this orbit invariant.]

An alternative method to deal with $F_\alpha$ consists in directly finding all the classical observables $\mathcal{O}$ and the corresponding classical PB algebra $\mathcal{A}$ of observables. This is the algebra of functions on $\mathcal{C}/\sim$. We then quantize it by replacing $\{\ldots\}$ by $-\imath\{\ldots\}$ and thus find $\hat{\mathcal{A}}$, and then look for a suitable representation of $\hat{\mathcal{A}}$ on a Hilbert space. In this approach, unlike in Dirac’s approach, we do not first find a vector space $V$ of vectors $|\cdot\rangle$ with the property $\hat{F}_\alpha |\cdot\rangle = 0$. Rather, we directly look for a representation of $\hat{\mathcal{A}}$.

In many examples, $\mathcal{C}^\alpha_{\alpha\beta}$ are constants so that $F_\alpha$ generate a Lie algebra over reals and are associated with a group in a familiar manner. This group is in fact the Hamiltonian version of the group of gauge transformations for the action. Hence one says that first class constraints generate gauge transformations. An important fact one can prove is that the only undetermined Lagrange multipliers in $\mathcal{H}$ at the end of the constraint analysis multiply first class constraints. Since $\{\mathcal{O}, F_\alpha\} \approx 0$ for an observable, it follows that the time evolution of $\mathcal{O}$ does not depend on these arbitrary functions. Thus a well defined Cauchy problem can be posed on $\mathcal{A}$ and the time evolution of $\mathcal{O}$ can be determined uniquely from suitable initial data. The theory is therefore deterministic if we consider only $\mathcal{A}$. This ceases to be the case when nonobservables are also considered since their time evolution is influenced by the unknown Lagrange multipliers $v_j$. See Chapter 1 also in connection with these remarks.

Finally, we notice that there is an important symmetry structure associated with the first class constrained surfaces in phase space, the so-called BRST symmetry. [See ref. 2 for literature on this subject.] It is frequently used in the quantization of gauge theories, which are typically theories with first class constraints. We will not touch upon these considerations since we
shall have no compelling reason for using the BRST approach to quantiza-

tion.

IV. GAUGE CONSTRAINTS IN FIELD THEORIES

A. Gauss Law Generates Asymptotically Trivial Gauge Transfor-

mations

In previous Chapters, we have outlined the physical reasons which lead to

important distinctions between gauge invariance and invariance under time

independent symmetry transformations. We have also sketched the classical

theory of constraints and its extension to the quantum domain.

In this Chapter, we look more closely at gauge constraints in field the-

ories. In field theories, even classical field theories, not all functions of fields

and their conjugate momenta are admissible in the Hamiltonian formalism

[3]. This is because not all functions generate well defined canonical transfor-

mations classically. Such functions, one presumes, are ill defined in quantum

theory as well and are thus to be excluded. The restriction of allowed phase

space functions using considerations along these lines has profound conse-

quences for gauge field theories. It is this restriction which leads to the

possibility of QCD $\theta$-states and fractionally charged dyons, and to the edge

states of Chern-Simons dynamics. The purpose of this Chapter is to explain

this restriction and its physical implications.

It may be remarked that there are similar constraints on functions on the

phase space $\mathcal{P}$ in classical mechanics as well. Thus in classical mechanics, we

almost always deal with infinitely differentiable functions on $\mathcal{P}$ in order that

all PB’s and the finite canonical transformations obtained therefrom are well

defined. The field theoretic conditions to be found below are conditions of

this kind, and are therefore to be expected.

The sort of constraints we have in mind are best illustrated by a spe-

cific example. Let us consider the free electromagnetic field in 3+1 dimen-
sional Minkowski space. Let the vector potential $A_\mu$ describe this field. The

Lagrangian for this system contains no time derivative of $A_0$ so that the

momentum field $\pi_0$ conjugate to $A_0$ vanishes weakly:

$$\pi_0 \approx 0.$$ (45)
The momentum field $\pi_i$ conjugate to $A_i$ is the electric field and it has the equal time PB

$$\{A_i(x), E_j(y)\} = \delta_{ij} \delta^3(x - y), \quad x^0 = y^0$$

(46)

with $A_i$. [All fields in this Section hereafter are at equal times and $x$ for example is the same as $\vec{x}$.] The fields $E_i$ are not all independent, but are also subject to the Gauss law constraint

$$\partial_i E_i \approx 0.$$  

(47)

The equations (4.1) and (4.3) constitute all the constraints in this system. They are first class, as their mutual PB is zero.

The constraint (4.1) is easy to deal with. Its PB with $A_o$ is non-zero:

$$\{A_0(x), \pi_0(y)\} = \delta^3(x - y), x^0 = y^0.$$  

(48)

It follows that $A_0$ is not an observable and that we can ignore it and $\pi_0$ as well hereafter and consider only functions of $A_i$ and $E_i$. The latter have zero PB’s with $\pi_0$ and are thus candidates for observables.

The constraint which merits delicacy of treatment is (4.3). Let us first rewrite it by smearing it with a ‘test function’ $\Lambda^\infty$:

$$g^\infty(\Lambda^\infty) = \int d^3 x \, \Lambda^\infty \partial_i E_i \approx 0.$$  

(49)

$g^\infty(\Lambda^\infty)$ is a generator of gauge transformations on $A_i$ and $E_i$ as shown by the PB’s

$$\{A_i, g^\infty(\Lambda^\infty)\} = -\partial_i \Lambda^\infty,$$

$$\{E_i, g^\infty(\Lambda^\infty)\} = 0.$$  

(50)

The underline on $g^\infty$ has been put to indicate that it is associated with the Lie algebra of the gauge group rather than with the gauge group. The superscripts $\infty$ are to indicate certain boundary conditions at infinity which will emerge below.

The PB’s of $g^\infty$ with all quantities of interest are not well defined unless $\Lambda^\infty$ is suitably restricted at spatial infinity. Such a restriction does not show
up in (4.6) as it involves only the local fields $A_i$ and $E_i$. Thus, consider for example the canonical expressions

$$J_i = \int d^3x E_j[(\vec{x} \times \vec{\nabla})_i \delta_{jk} + \theta(i)_{jk}]A_k,$$

$$\theta(i)_{jk} = \epsilon_{ijk},$$

for generators of rotations (components of angular momentum). The PB of $J_i$ with $g^\infty(\Lambda^\infty)$ can be computed by first evaluating it with $\partial_i E_i$ and then multiplying by $\Lambda^\infty$ and integrating over $x_i$. Since

$$\{J_i, \partial \cdot E(x)\} = -\epsilon_{ijk} x_j \partial_k \partial \cdot E(x)$$

(52)

where $\partial \cdot E \equiv \partial_i E_i$, this method gives

$$\{J_i, g^\infty(\Lambda^\infty)\} = \int d^3x \Lambda^\infty(x) \{J_i, \partial \cdot E\}(x)$$

$$= -\int d^3x \Lambda^\infty(x)(\vec{x} \times \vec{\nabla})_i \partial \cdot E(x)$$

$$= -\int |\vec{x}| \rightarrow \infty d\Omega |\vec{x}|^2 \Lambda^\infty(x)(\vec{x} \times \vec{\nabla})_i \partial \cdot E(x)$$

(53)

$$+ \int d^3x [(\vec{x} \times \vec{\nabla})_i \Lambda^\infty(x)] \partial \cdot E(x)$$

$$= g^\infty((\vec{x} \times \vec{\nabla})_i \Lambda^\infty)$$

where $d\Omega$ is the usual volume form on a two-sphere and $(\vec{x} \times \vec{\nabla})_i \Lambda^\infty$ is the function with value $(\vec{x} \times \vec{\nabla})_i \Lambda^\infty(x)$ at $x$.

We can also compute the PB $\{J_i, g^\infty(\Lambda^\infty)\}$ by first evaluating the PB of $g^\infty(\Lambda^\infty)$ with the integrand of $J_i$:

$$\{J_i, g^\infty(\Lambda^\infty)\} = \{\int d^3x \{E_j[(\vec{x} \times \vec{\nabla})_i \delta_{jk} + \theta(i)_{jk}]A_k, g^\infty(\Lambda^\infty)\}$$

$$= -\int d^3x \quad E_j[(\vec{x} \times \vec{\nabla})_i \delta_{jk} + \theta(i)_{jk}]\partial_k \Lambda^\infty$$

(54)

$$= -\int |\vec{x}| \rightarrow \infty d\Omega |\vec{x}|^2 \frac{\vec{x} \cdot E}{|\vec{x}|}(\vec{x} \times \vec{\nabla})_i \Lambda^\infty + g^\infty((\vec{x} \times \vec{\nabla})_i \Lambda^\infty).$$
Thus the interchange of orders of integration in the evaluation of this PB changes its value unless conditions are imposed on $\Lambda^\infty$. [See Chapter 5 (cf. Eq. (5.27)) or ref. 4 for another such example.] The simplest such condition is

$$\Lambda^\infty (x) \to 0 \text{ as } |\vec{x}| \to \infty$$

(55)

at some suitable rate. [We will not have to be more specific about this rate for the purposes of these notes.]

The condition (4.11) seems reasonable for our purposes. Besides $J_i$, there are also other functions such as momenta $P_i$ and Lorentz boosts $K_i$ which we must require to have well define PB’s with $g^\infty(\Lambda^\infty)$, and they too can lead to boundary terms containing $\Lambda^\infty$ like the one in (4.10). The condition (4.11) can serve to eliminate all these terms and to lead to well behaved PB’s.

There is another way to look upon the boundary condition (4.11). Consider the variation of $g^\infty(\Lambda^\infty)$ under a variation $\delta E_i$ of $E_i$:

$$\delta g^\infty(\Lambda^\infty) = \int_{|\vec{x}| \to \infty} d\Omega |\vec{x}|^2 \Lambda^\infty \frac{\vec{x} \cdot \delta \vec{E}}{|\vec{x}|} - \int d^3 x \partial_i \Lambda^\infty \delta E_i.$$  \hspace{1cm} (56)

Now a function (or “functional”) $F$ of a collection of fields $\varphi^{(a)}$ is said to be differentiable in $\varphi^{(a)}$ if and only if we are able to write the variation $\delta F$ of $F$ under a variation $\delta \varphi^{(a)}$ of $\varphi^{(a)}$ in the form

$$\delta F = \int d^3 x F_\alpha \delta \varphi^\alpha.$$ \hspace{1cm} (57)

If (4.13) is possible, we then define the functional derivative $\delta F/\delta \varphi^\alpha$ as $F_\alpha$:

$$\frac{\delta F}{\delta \varphi^{(a)}(x)} = F_\alpha[\varphi(x)], \varphi(x) = \varphi^1(x), \varphi^2(x), ... .$$ \hspace{1cm} (58)

Differentiability of phase space functions in field theory is analogous to differentiability of phase space functions in classical mechanics and is among the simplest conditions we can impose to obtain well defined PB’s. Comparison of (4.12) and (4.13) leads to the condition (4.11) when $g^\infty(\Lambda^\infty)$ is required to be differentiable.

Analogous considerations involving multiple PB’s suggest that phase space functions may have to be infinitely differentiable while the requirement that
they generate well defined canonical transformations can lead to more sophis-
ticated conditions.

It is important to remark that if for some reason we exclude functions like \( J_i \) from consideration, then there is no reason to impose (4.11). Thus we really must examine the collection of all functionals of possible interest and their PB’s before deciding on appropriate boundary conditions (BC’s).

We will not study such difficult matters here, and will content ourselves with the BC (4.11). Let \( T^\infty \) denote the class of test functions \( \Lambda^\infty \) which fulfill the BC (4.11). Then the weak equality (4.5) is thus valid only if \( \Lambda^\infty \in T^\infty \). Using the same symbols for quantum and classical objects, it follows also that the quantum states \( | \cdot > \) are annihilated only by such \( g^\infty(\Lambda^\infty) \):

\[
g^\infty(\Lambda^\infty) | \cdot > = 0 \iff \Lambda^\infty \in T^\infty. \tag{59}\]

Furthermore, as we saw in Chapter 2, the observables commute with \( g^\infty(\Lambda^\infty) \).

The charge operator in electrodynamics is closely related to the Gauss law operator \( g^\infty(\Lambda^\infty) \). It is best discussed after first coupling the electromagnetic field to a charged field \( \psi \) with charge density \( J_0 \). The Gauss law and the physical state constraints (4.5) and (4.15) are then changed to

\[
g^\infty(\Lambda^\infty) = \int d^3x \Lambda^\infty[\partial_i E_i + J_0] \approx 0, \quad \Lambda^\infty \in T^\infty, \tag{60}\]

while the observables now commute with this \( g^\infty(\Lambda^\infty) \).

**B. Internal Symmetries in Gauge Theories**

It is convenient at this point to introduce some definitions. A general element \( e^{i\Lambda} \) of the group \( G \) of gauge transformations (at a fixed time) in electrodynamics is a function (at a fixed time) on \( \mathbb{R}^3 \) with values in \( U(1) \):

\[
e^{i\Lambda} : \mathbb{R}^3 \to U(1),
\]

\[
x \to e^{i\Lambda(x)}. \tag{61}\]

It acts on \( A_i \) and \( \psi \) according to
\[ A_i \rightarrow A_i + \partial_i \Lambda, \]
\[ \psi \rightarrow e^{ie\Lambda} \psi. \]

(62)

We now wish to give names to several of its subgroups of particular interest, assuming as above that the spatial slice of spacetime is \( \mathbb{R}^3 \).

The group \( \mathcal{G}^c \): The elements of \( \mathcal{G}^c \) approach constant values as \( |\vec{x}| \to \infty \).

If \( e^{i\Lambda} \in \mathcal{G}^c \), we thus have

\[ \Lambda^c(x) \rightarrow \text{constant as } |\vec{x}| \to \infty. \]

(63)

The group \( \mathcal{G}^\infty \): The elements of \( \mathcal{G}^\infty \) approach 1 as \( |\vec{x}| \to \infty \). Because of this boundary condition, we can identify \( \mathcal{G}^\infty \) with the group of maps of the three-sphere \( S^3 \) to \( U(1) \). This sphere is the one obtained by identifying all “points at \( \infty \)” of \( \mathbb{R}^3 \), that is by compactifying \( \mathbb{R}^3 \) to \( S^3 \) by adding a “point at \( \infty \)”.

The group \( \mathcal{G}_0^\infty \): This is the subgroup of \( \mathcal{G}^\infty \) which is continuously connected to the identity. The generators of its Lie algebra are the Gauss law constraints \( g^\infty(\Lambda^\infty) \) for all choices of \( \Lambda^\infty \in T^\infty \).

The group \( G \) which is gauged in electrodynamics is \( U(1) \), while it is \( SU(3) \), in chromodynamics. All the preceding groups can be defined (in an obvious way) for the latter as well, and indeed for any choice of a Lie group \( G \). In every case, it is easy to verify the important result that \( \mathcal{G}^\infty \) is a normal subgroup of \( \mathcal{G} \) (and hence of \( \mathcal{G}^c \)):

\[ \mathcal{G}^{(\infty)} \triangleleft \mathcal{G}. \]

(64)

And of course we have the standard result

\[ \mathcal{G}_0^\infty \triangleleft \mathcal{G}^\infty. \]

(65)

But whereas \( \mathcal{G}^\infty \) is the same as \( \mathcal{G}_0^\infty \) when \( G \) is abelian, that is not the case when \( G \) is simple. When \( G \) is simple, we have instead the important result

\[ \mathcal{G}^\infty / \mathcal{G}_0^\infty = \mathbb{Z}, \]

(66)
\( \mathbb{Z} \) being the group of integers under addition. [We assume throughout this Chapter that the spatial manifold has the topology of \( \mathbb{R}^3 \).] For such a \( G \), a typical element of \( \mathcal{G}^\infty \) which is distinct from \( \mathcal{G}^\infty_0 \) is a winding number one transformation. Let us display such a transformation explicitly for \( G = SU(2) \). If \( \tau_\alpha \) are Pauli matrices, then a winding number one element of \( \mathcal{G}^\infty \) is \( \hat{g}^\infty \) where

\[
\hat{g}^\infty(x) = e^{i\psi(r)\tau_\alpha \hat{x}_\alpha},
\]

\[
r = |\vec{x}|, \quad \hat{x}_\alpha = \frac{x_\alpha}{r},
\]

and

\[
\psi(0) = 0, \quad \psi(\infty) = 2\pi.
\]

The group generated by \( \hat{g}^\infty \mathcal{G}^\infty_0 \) is the group \( \mathbb{Z} \).

Note that \( \hat{g}^\infty \) is well defined at \( r = 0 \) because of the condition on \( \psi(0) \) and becomes 1 at \( r = \infty \), as it should being an element of \( \mathcal{G}^\infty \).

The expression (4.23) is identical to Skyrme’s ansatz in Skyrmion physics [2].

The generalization of (4.23) to simple Lie groups such as \( G = SU(3) \) can be constructed by looking for example at its \( SU(2) \) subgroups. Thus, if \( \tau_\alpha \) in (4.23) is replaced by \( \lambda_\alpha \quad (1 \leq \alpha \leq 3) \) where

\[
\lambda_\alpha = \begin{bmatrix} \tau_\alpha & 0 \\ 0 & 0 \end{bmatrix},
\]

then, for all \( x \), \( \hat{g}^\infty(x) \) is contained in a fixed \( SU(2) \) subgroup of \( SU(3) \) (realised as \( 3 \times 3 \) unitary matrices of determinant 1) and \( \hat{g}^\infty \mathcal{G}^\infty_0 \) generates \( \mathbb{Z} \).

Now any connected Lie group is the quotient of the direct product of simple and abelian Lie groups by discrete abelian groups (which could be trivial). Using this fact, the preceding results can be generalized to arbitrary Lie groups.

We turn next to the examination of these groups in the canonical formalism and in quantum theory, limiting ourselves to \( G = U(1) \) at this stage.
Closely associated to the Gauss law generator \( g^\infty(\Lambda^\infty) \) is another function obtained therefrom by partial integration and subsequent substitution of a new test function \( \xi \) for \( \Lambda^\infty \). We thus consider
\[
g(\xi) = \int d^3x [ - \partial_i \xi E_i + \xi J_0 ] .
\] (70)

It is clear that \( g(\xi) \) generates gauge transformations just as \( g^\infty(\Lambda^\infty) \) does:
\[
\{ A_i, g(\xi) \} = - \partial_i \xi ,
\]
\[
\{ E_i, g(\xi) \} = 0 ,
\]
\[
\{ \psi, g(\xi) \} = \xi \psi .
\] (71)

It furthermore appears to have no problems of differentiability in \( E_i \) regardless of boundary conditions on \( \xi \), in contrast to what we found with \( g^\infty \).

Thus we seem at first sight to have discovered the generators of \( \mathcal{G} \).

But this conclusion is not quite correct. In electrodynamics, we encounter electric fields \( E_i \) which fall like \( 1/r^2 \) as \( r \equiv |\vec{x}| \to \infty \). If there is a charge distribution of compact support with total charge \( Q \), its Coulomb field for example behaves as follows:
\[
E_i = \frac{Q}{r^2} \hat{x}_i + 0 \left( \frac{1}{r^3} \right) \text{ as } |\vec{x}| \to \infty , \hat{x}_i = \frac{x_i}{r} .
\] (72)

This field in a moving frame has the behavior
\[
E_i = \frac{Q}{r^2} v_i^{(-)}(\hat{x}) + 0 \left( \frac{1}{r^3} \right) \text{ as } r \to \infty
\] (73)

where \( v_i^{(-)} \) is an odd function of its argument. The existence of these fields implies that \( g(\xi) \) will diverge unless we constrain \( \xi \) suitably. The simplest constraint for this purpose is
\[
\xi = \Lambda^c .
\] (74)

It is also what is universally assumed. It may be that there are more general permissible conditions on \( \xi \) compatible with the existence of \( g(\xi) \) and with Poincaré invariance. We will not however pursue this issue further here, but content ourselves with (4.29).
\( G^c \) is thus a canonically implementable group and presumably can be realised in quantum theory as well. As it acts on fields as a group of gauge transformations, it is also an invariance group of the Hamiltonian. In contrast, the full group \( G \) cannot be canonically implemented.

But \( G_0^\infty \) acts trivially on states and observables because of the Gauss law constraint. It is hence only the group

\[
G^c / G_0^\infty
\]

which has a nontrivial action in the theory. As it is an invariance group of the Hamiltonian as well, we thus conclude that \( G^c / G_0^\infty \) is the symmetry group of electrodynamics associated to \( G = U(1) \). We will call it the internal symmetry group. [The full symmetry group is larger, containing for instance the Poincaré group.]

It is important to appreciate that the internal symmetry group in a gauge theory is a group like \( G^c / G_0^\infty \). It is not necessarily \( G \) and may not even contain \( G \). The examples below will illustrate these points.

But for \( G = U(1) \) and when the spatial slice of spacetime is \( \mathbb{R}^3 \), it is not difficult to show that

\[
G^c / G_0^\infty = U(1) \quad .
\]

The proof is as follows. As mentioned previously, \( G^\infty \) and \( G_0^\infty \) are identical for this case. Now if two elements \( e^{i\Lambda_j^c} \) (\( j = 1, 2 \)) of \( G^c \) are both characterized by the same boundary values of \( \Lambda_j^c \) at spatial infinity, then

\[
\left( e^{i\Lambda_1^c} \right) \left( e^{i\Lambda_2^c} \right)^{-1} = e^{i(\Lambda_1^c - \Lambda_2^c)}
\]

approaches the value 1 at infinity and is hence an element of \( G_0^\infty \). Thus a coset \( e^{i\Lambda^c} G_0^\infty \) is entirely fixed by the value \( e^{i\Lambda^c} \bigg|_\infty \) of any of its elements \( e^{i\Lambda^c} \) at infinity, this value being independent of the choice of this element. Furthermore, the group multiplication law

\[
\left( e^{i\Lambda_1^c} G_0^\infty \right) \left( e^{i\Lambda_2^c} G_0^\infty \right) = e^{i(\Lambda_1^c + \Lambda_2^c)} G_0^\infty
\]

in \( G^c / G_0^\infty \) shows that the group multiplication law for the coset labels \( e^{i\Lambda^c} \bigg|_\infty \) is the standard multiplication of phases. We have thus the result (4.32).
Let \( g^c(\Lambda^c) \) denote the generators of \( G^c \). Our discussion shows that in quantum theory, when acting on states subject to the Gauss law constraint, all that matters is the asymptotic value \( \Lambda^c |_{\infty} = \lambda \) of \( \Lambda^c \). So we may as well take the function with the constant value \( \lambda \) for \( \Lambda^c \) as the test function in \( g^c(\Lambda^c) \) and call it the generator \( Q(\lambda) \) of \( G^c/G^\infty_0 \). From (4.26), we see that

\[
Q(\lambda) = \lambda \int d^3x J_0 .
\]

Hence \( Q(1) \) is what is called the charge \( Q \) in electrodynamics.

C. Nonabelian Examples

We will here limit ourselves to brief sketches about the structure of the symmetry group when we stray away from electrodynamics.

In chromodynamics, with \( \mathbf{R}^3 \) as the spatial slice, the discussion of test functions like \( \Lambda^\infty \) and \( \Lambda^c \) is similar to their discussion in electrodynamics. For example, the Gauss law generator in chromodynamics which generalizes \( g^\infty(\Lambda^\infty) \) is

\[
g^\infty(\Lambda^\infty) = \int d^3x \, Tr \Lambda^\infty D_i E_i , \; \Lambda^\infty \to 0 \text{ as } r \to \infty .
\]

Here we have not changed the notation for the generator or \( \Lambda^\infty, D_i E_i = \partial_i E_i + [A_i, E_i] \) and \( \Lambda^\infty, A_i \) and \( E_i \) are valued in the Lie algebra of \( SU(3) \). For example, \( \Lambda^\infty = \Lambda^\alpha \lambda_\alpha \) where \( \lambda_\alpha \) are the Gell-Mann matrices.

The symmetry group as before is \( G^c/G^\infty_0 \). But in this case, \( G^\infty/G^\infty_0 \) is \( \mathbb{Z} \) instead of being trivial. Now one can easily show, as for electrodynamics, that \( G^c/G^\infty \) is \( G \). It is also easy to show that \( G^c/G^\infty \) is \( (G^c/G^\infty_0)/(G^\infty/G^\infty_0) \). In other words, the symmetry group \( G^c/G^\infty_0 \) is an extension of \( G = SU(3) \) by \( \mathbb{Z} = G^\infty/G^\infty_0 \). As elements of this \( \mathbb{Z} \) are readily seen to commute with elements of \( G^c/G^\infty_0 \), the extension is central. The symmetry group is thus a central extension of \( SU(3) \) by \( \mathbb{Z} \).

This extension is actually trivial:

\[
G^c/G^\infty_0 = SU(3) \times \mathbb{Z} .
\]

The generators of \( SU(3) \) in (4.36) can be obtained from the nonabelian analogue of (4.26) with the help of constant test functions [valued in the Lie algebra of \( SU(3) \)].
The states in quantum theory can be associated with the unitary representations of the symmetry group (4.37).

The group $\mathbb{Z}$ has unitary irreducible representations $\rho_\theta$ which are in one-to-one correspondence with the points of the circle $S^1$. The image of $n \in \mathbb{Z}$ in the UIR $\rho_\theta$ is $e^{in\theta}$:

$$\rho_\theta : n \rightarrow e^{in\theta}. \quad (82)$$

The angle $\theta$ here is the famous QCD $\theta$ parameter.

The UIR’s of $SU(3)$ in (4.37) account for colour in QCD.

A more complicated and interesting example is the 't Hooft-Polyakov model for monopoles [5]. It is a model of an $SO(3)$ or $SU(2)$ gauge theory which in its simplest version contains a real Higgs field $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ transforming like an $SU(2)$ triplet. The vacuum value $< \varphi >$ of $\varphi$ is a constant nonzero vector which we may take to be $(0, 0, v)$, $v \neq 0$. The $U(1)$ or $SO(2)$ group of rotations in the 1-2 plane leaves this $< \varphi >$ invariant so that $SO(3)$ is said to be spontaneously broken to $U(1)$ in this model.

It was shown by 't Hooft and Polyakov that the model admits finite energy configurations of $\varphi_i$ and the gauge potential $A_i$ with the asymptotic conditions

$$\varphi_i(x) \rightarrow \varphi_i^\infty(x) = v \hat{x}_i, \quad (83)$$

$$A_i \equiv A_\alpha^i \tau_\alpha \rightarrow \frac{1}{2} \vec{\tau} \cdot \hat{x} \partial_i \vec{\tau} \cdot \hat{x}$$

for large $r$. It was also established that these configurations provide a field theoretic version of Dirac’s magnetic monopole.

A general element of $\mathcal{G}$ for the 't Hooft-Polyakov model is a map

$$g : R^3 \rightarrow SU(2), \quad (84)$$

$$x \rightarrow g(x),$$

while, for the boundary conditions (4.39), the analogue of $\mathcal{G}^c$ is a certain group of gauge transformations which leave $\varphi^\infty$ invariant at $\infty$. Let $\mathcal{G}^c$ still denote this group. It is defined as follows. Let $g^c \in \mathcal{G}^c$. Then
\[ g^c : \mathbb{R}^3 \to SU(2), \]
\[ x \to g^c(x), \]
\[ g^c(x) \to e^{i\alpha_\infty \vec{r} \cdot \hat{x}}, \]
(85)
\[ \alpha_\infty \text{ being a constant independent of } \hat{x}. \text{ Such a } g_c \text{ clearly leaves } \varphi_\infty \text{ invariant.} \]

Next suppose that \( g_j^c \ (j = 1, 2) \) have both the same asymptotic limit as \( r \to \infty \). Then
\[ g_1^c(x)^{-1} g_2^c(x) \to 1 \]
(87)
so that
\[ g_1^{c-1} g_2^c \in G^\infty \]
(88)
where the elements of \( G^\infty \) as before go to identity at infinity. A generic \( g^c \) with the asymptotic behaviour (4.42) is therefore given by
\[ g^c = g_0^c g^\infty, g^\infty \in G^\infty, \]
(89)
g_0^c being a particular solution of the condition (4.42).

One such particular solution is
\[ g_0^c(x) = e^{i\alpha(r) \vec{r} \cdot \hat{x}}, \]
(90)
where for \( \alpha(r) \) we insert any one function with the properties
\[ \alpha(0) = 0, \]
\[ 0 \leq \alpha(\infty) \equiv \alpha_\infty < 2\pi, \]
(91)
the last condition here eliminating the ambiguity in the determination of \( \alpha_\infty \) from the asymptotic limit of \( g_0^c \).

The symmetry group is \( G^c/G_0^\infty \). An element of this group is \( g_0^c g^\infty G_0^\infty \).

Now if \( h^\infty \) and \( k^\infty \) are two elements of \( G^\infty \) with the same winding number, then \( h^\infty = k^\infty g_0^\infty \) for some \( g_0^\infty \in G_0^\infty \). Hence \( h^\infty G_0^\infty = k^\infty G_0^\infty \), so that we can choose any one typical winding number \( n \) map for \( g^\infty \). One such choice is specified by
\[ g^\infty(x) = (\hat{g}^\infty(x))^n = e^{i\psi(r)\vec{r} \cdot \hat{x}}. \] (92)

We may thus choose \( g_0^c \) \( g^\infty \) according to
\[ g_0^c(x)g^\infty(x) = e^{i\beta(r)\vec{r} \cdot \hat{x}}, \] (93)
\[ \beta(0) = 0 \]
for insertion into the expression \( g_0^c \) \( g^\infty \) \( G_0^\infty \). In contrast to (4.47), we here do not restrict \( \beta(\infty) \). Further, as two \( \beta(r) \) with the same \( \beta(\infty) \) give the same element of the symmetry group, it suffices to consider one \( \beta(r) \) for each \( \beta(\infty) \).

For each \( \beta(\infty) \), we have thus an element \( \gamma G_0^\infty \) of the symmetry group with \( \gamma(x) = e^{i\beta(r)\vec{r} \cdot \hat{x}} \), this correspondence being onto the group. It is also one-to-one. For suppose that the images \( \gamma_j G_0^\infty (j = 1, 2) \) of \( \beta_j(\infty) \) are equal, \( \gamma_j \) being defined by
\[ \gamma_j(x) = e^{i\beta_j(r)\vec{r} \cdot \hat{x}} \] (94)
[\( \beta_j(0) \) being of course zero.] Then
\[ \gamma_1\gamma_2^{-1} \in G_0^\infty. \] (95)
Since \( \gamma_1\gamma_2^{-1}(x) = e^{i[\beta_1(r)-\beta_2(r)]\vec{r} \cdot \hat{x}} \), it follows that
\[ \beta_1(\infty) = \beta_2(\infty). \] (96)

Thus the elements of \( G^c/G_0^\infty \) are all uniquely labelled by a real number \( \beta(\infty) \). A formula similar to (4.34) also shows that the group composition in \( G^c/G_0^\infty \) induces addition as the group composition on \( \beta(\infty) \).

We have thus proved the remarkable result due to Witten [5] that the symmetry group \( G^c/G_0^\infty \) is the additive group \( \mathbb{R}^1 \) of real numbers:
\[ G^c/G_0^\infty = \mathbb{R}^1. \] (97)
This result is to be contrasted with (4.32), (4.37). In analogy to those expressions, we might have anticipated the symmetry group here to be \( U(1) \times \mathbb{Z} \). But it is not, it is rather the nontrivial central extension \( \mathbb{R}^1 \) of \( U(1) \) by \( \mathbb{Z} \).
The critical fact which leads to this result is that the map $\gamma$ defined above becomes a winding number one transformation when $\beta(\infty) = 2\pi$. Had it instead been an element of $G_0^\infty$, it would have acted trivially on states. In such a case, there would be periodicity of the elements of the symmetry group in $\beta(\infty)$ and this group would contain $U(1)$.

There are striking physical consequences of (4.53). The charges associated with $U(1)$ are quantized whereas those associated with $R_1$ are not. Therefore, there is the possibility of fractionally charged excitations (dyons) of the 't Hooft-Polyakov monopole as first established by Witten [5].

It is to be noted that the symmetry group $R_1$ of the monopole sector does not contain $U(1)$ as a subgroup even though $\varphi$ was supposed to spontaneously break $SU(2)$ to $U(1)$.

The result (4.53) is valid in the monopole sector. In the vacuum sector, the symmetry group is $U(1)$ as one can readily show.

In Chapter 5, we will illustrate the application of some of the ideas developed here to Chern-Simons theories.

V. THE QUANTUM HALL EFFECT AND THE EDGE STATES OF CHERN-SIMONS THEORY

A. Introduction

In this Chapter, we will review certain results due to Friedman, Sokoloff, Widom and Srivastava, Fröhlich and Kerler, and Fröhlich and Zee [cf. ref. 6 and citations therein] who show that the quantum Hall (QH) system is related to the pure Chern Simons (CS) gauge theory. We then consider CS theory on a disk, and using the methods of Chapter 4, show that there are chiral currents of a conformal field theory at the edge of the disk. This result is originally due to Witten. For the QH system, the existence of these currents has been demonstrated from microscopic considerations by Halperin.

As we set the speed of light $c$ equal to 1, the magnetic flux can be measured in units of $\hbar/e$. 

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B. Chern-Simons Field Theory and the Quantum Hall System

Let us begin our discussion by examining a QH system characterized by zero longitudinal resistance. The conductivity tensor $\sigma$ can then be written as

$$\sigma = \begin{pmatrix} 0 & \sigma_H \\ -\sigma_H & 0 \end{pmatrix}. \quad (98)$$

In QH systems, $\sigma_H$ is quantized and is a rational multiple of $e^2/h$. The idea which emerges from the works mentioned above is that this fact may have a universal explanation emerging from rational conformal field theories.

As the longitudinal conductivity $\sigma_L$ is zero for a two-dimensional system with $\sigma$ given by Eq. (5.1), the current density $j$ induced by an electric field $E$ is given by

$$j^a(\vec{x},t) = \sigma_H \epsilon^{ab} E_b(\vec{x},t); \quad a, b = 1, 2; \quad \epsilon^{ab} = -\epsilon^{ba}, \epsilon^{12} = 1. \quad (99)$$

Here $E_a = -F_{ba}, F_{\mu\nu}$ being the electromagnetic field strength tensor.

Now if $j^0$ is the charge density, then we have the continuity equation

$$\frac{\partial j^0}{\partial x^0} + \vec{\nabla} \cdot \vec{j} = 0, \quad x^0 = t. \quad (100)$$

Also $B$ and $E$ are related by the Maxwell’s equation

$$\frac{\partial B}{\partial x^0} = -\epsilon^{ab} \partial_a E_b, \quad (101)$$

where $B = F_{12}$. Equations (5.2), (5.3) and (5.4) give

$$\sigma_H \frac{\partial B}{\partial x^0} = \frac{\partial}{\partial x^0} j^0. \quad (102)$$

We thus obtain

$$j^0 = \sigma_H (B + B_c). \quad (103)$$

Here $B_c$ is an integration constant representing a time independent background magnetic field.
Let us assume that the three-dimensional manifold \( M \) has the topology of \( R^1 \times D \) with \( D \) characterizing the two-dimensional space of the sample, and \( R^1 \) describing time. Furthermore, let \( \eta = (\eta_{\mu\nu}) \) be any metric of Euclidean or Lorentzian signature on \( M \). Then Eqs. (5.2) and (5.6) can be extended to a generally covariant form valid for arbitrary metrics as well as follows.

Let

\[
J_{\alpha\beta}(x) = \left| \text{Det} \eta(x) \right|^{-1/2} \epsilon_{\alpha\beta\gamma} j^\gamma(x), \ x = \vec{x}, t \tag{104}
\]

and

\[
j^\alpha(x) = \frac{1}{2} \left| \text{Det} \eta(x) \right|^{1/2} \sigma_H \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}(x). \tag{105}
\]

Here, \( \alpha, \beta, \gamma = 0, 1, 2 \), \( \epsilon_{\alpha\beta\gamma} \) is the totally antisymmetric symbol with \( \epsilon_{012} = 1 \) and \( t = x^0 \) is time. By (5.7) and (5.8),

\[
J_{\alpha\beta}(x) = \sigma_H F_{\alpha\beta}(x). \tag{106}
\]

(5.9) reduces to (5.2) and (5.6) for a flat metric.

Using the language of differential forms, we can write Eqs. (5.9) and (5.7) as

\[
J = \sigma_H F, \tag{107}
\]

\[
J = \ast j
\]

where \( J = \frac{1}{2} J_{\alpha\beta} \ dx^\alpha \wedge dx^\beta \) and \( \ast \) is the Hodge dual. The one form \( j(x) \) is defined as

\[
j(x) = \sum_\alpha \left( \sum_\beta \eta_{\alpha\beta}(x) j^\beta(x) \right) dx^\alpha. \tag{108}
\]

The continuity equation (5.3) can be written as

\[
dJ = 0 \tag{109}
\]

where \( d \) is the exterior derivative.

We shall assume that \( \sigma_H \) is a constant. Equation (5.10) then gives the Maxwell equations
Here, we can write \( F = dA', A' = A + A_c \) where \( A_c \) is the vector potential corresponding to a constant magnetic field \( B_c \) (see Eq. 5.6), \( A \) represents the vector potential of a fluctuation field due to localized sources and \( A' \) the total vector potential.

Now, Eq. (5.12) implies that

\[
J = da
\]

(111)

where \( a \) is a one form. Equation (5.10) can then be written in terms of the one forms \( a \) and \( A' \) as

\[
da = \sigma_H dA'.
\]

(112)

We now note that this last equation can be obtained from an action principle with the action \( S_{CS} \) given by

\[
S_{CS} = \frac{1}{2\sigma_H} \int_M \epsilon^{\alpha\beta\gamma}(a - \sigma_H A'_\alpha) \partial_\beta(a - \sigma_H A'_\gamma) \ d^3x.
\]

(113)

or in terms of components,

\[
S_{CS} = \frac{1}{2\sigma_H} \int_M \epsilon^{\alpha\beta\gamma}(a - \sigma_H A'_\alpha) \partial_\beta(a - \sigma_H A'_\gamma) \ d^3x.
\]

(114)

The overall normalization of \( S_{CS} \) is here fixed by the requirement that the coupling of \( A'_\mu \) to \( j^\mu \) is by the term \(-j^\mu A'_\mu \) in the Lagrangian density.

The action \( S_{CS} \) is the Chern-Simons action for the gauge field \( a - \sigma_H A' \).

It is important to note at this step that the derivation of Eq. (5.16) from the QH effect is valid only in the scaling limit when both length and 1/frequency scales are large. This is because although the continuity equation (5.12) is exact, Eq. (5.2) is experimentally observed to be valid only at large distance and time scales.

The action \( S_{CS} \) can be naturally generalized to the case where there are several independently conserved electric current densities \( j^{(i)} \), \( i = 1, ... m \). For example, for \( m \) filled Landau levels, if one neglects mixing of levels (which is a good approximation due to the large gaps between Landau levels), each level can be treated as dynamically independent with electric currents in each level
being separately conserved. We will not however pursue such generalizations here.

We will continue in the next Section with the exploration of the relationship between the Quantum Hall system and the Chern-Simons theory and we will demonstrate how the edge currents in a Quantum Hall system arise naturally from the Chern-Simons theory. This result is first due to Witten. We follow the approach of Balachandran et al. [4] [see also ref. 6] who derive further results in Chern-Simons theory using this approach.

C. Conformal Edge Currents

The Lagrangians considered in this Section follow from (5.16) by setting

$$\bar{a} = (a - \sigma_H A') [2\pi | k \sigma_H |]^{1/2}$$

and calling $\bar{a}$ again as $a$, $k$ being $| k | (| \sigma_H |)$. We do so in order to be consistent with the form of the Chern-Simons Lagrangian most frequently encountered in the literature.

In this Section, we will use natural units where $\hbar = c = 1$.

1. The Canonical Formalism

Let us start with a $U(1)$ Chern-Simons (CS) theory on the solid cylinder $D \times R^1$ ($D$ being a disk) with action given by

$$S = \frac{k}{4\pi} \int_{D \times R^1} ada, \ a = a_\mu dx^\mu, \ ada \equiv a \wedge da$$

where $a_\mu$ is a real field.

The action $S$ is invariant under diffeos of the solid cylinder and does not permit a natural choice of a time function. As time is all the same indispensable in the canonical approach, we arbitrarily choose a time function denoted henceforth by $x^0$. Any constant $x^0$ slice of the solid cylinder is then the disc $D$ with coordinates $x^1, x^2$.

It is well known that the phase space of the action $S$ is described by the equal time Poisson brackets (PB's)
\{a_i(x), a_j(y)\} = \epsilon_{ij} \frac{2\pi}{k} \delta^2(x - y) \text{ for } i, j = 1, 2, \epsilon_{12} = -\epsilon_{21} = 1 \tag{117}

and the constraint

\partial_i a_j(x) - \partial_j a_i(x) \equiv f_{ij}(x) \approx 0 \tag{118}

where \(\approx\) denotes weak equality in the sense of Dirac. [Cf. Chapter 3.] All fields are evaluated at the same time \(x^0\) in these equations, and this will continue to be the case when dealing with the canonical formalism or quantum operators in the remainder of the paper. The connection \(a_0\) does not occur as a coordinate of this phase space. This is because, just as in electrodynamics, its conjugate momentum is weakly zero and first class and hence eliminates \(a_0\) as an observable.

The constraint (5.21) is somewhat loosely stated. As emphasized in Chapter 4, it is important to formulate it more accurately by first smearing it with a suitable class of “test” functions \(\Lambda^{(0)}\). Thus we write, instead of (5.21),

\[ g(\Lambda^{(0)}) := \frac{k}{2\pi} \int_D \Lambda^{(0)}(x) da(x) \approx 0. \tag{119} \]

It remains to state the space \(T^{(0)}\) of test functions \(\Lambda^{(0)}\). For this purpose, we recall from Chapter 4 that a functional on phase space can be relied on to generate well defined canonical transformations only if it is differentiable. The meaning and implications of this remark can be illustrated here by varying \(g(\Lambda^{(0)})\) with respect to \(a^\mu\),

\[ \delta g(\Lambda^{(0)}) = \frac{k}{2\pi} \left( \int_{\partial D} \Lambda^{(0)}(x) \delta a(x) - \int_D d\Lambda^{(0)}(x) \delta a(x) \right), \tag{120} \]

\(\partial D\) being the boundary of \(D\). By definition, \(g(\Lambda^{(0)})\) is differentiable in \(a\) only if the boundary term - the first term - in (5.23) is zero. We do not wish to constrain the phase space by legislating \(\delta a\) itself to be zero on \(\partial D\) to achieve this goal. This is because we have a vital interest in regarding fluctuations of \(a\) on \(\partial D\) as dynamical and hence allowing canonical transformations which change boundary values of \(a\). We are thus led to the following condition on functions \(\Lambda^{(0)}\) in \(T^{(0)}\):
\[ \Lambda^{(0)} |_{\partial D} = 0. \] (121)

It is useful to illustrate the sort of troubles we will encounter if (5.24) is dropped. Consider

\[ q(\Lambda) = \frac{k}{2\pi} \int_D d\Lambda a. \] (122)

It is perfectly differentiable in \( a \) even if the function \( \Lambda \) is nonzero on \( \partial D \). It creates fluctuations

\[ \delta a |_{\partial D} = d\Lambda |_{\partial D} \] (123)

of \( a \) on \( \partial D \) by canonical transformations. It is a function we wish to admit in our canonical approach. Now consider its PB with \( g(\Lambda^{(0)}) \):

\[ \{ g(\Lambda^{(0)}), q(\Lambda) \} = \frac{k}{2\pi} \int d^2 x d^2 y \Lambda^{(0)}(x) \epsilon^{ij} \left[ \frac{\partial}{\partial x^i} \delta^2(x-y) \right] \] (124)

where \( \epsilon^{ij} = \epsilon_{ij} \). This expression is quite ill defined if

\[ \Lambda^{(0)} |_{\partial D} \neq 0. \] (125)

Thus integration on \( y \) gives zero for (5.27). But if we integrate on \( x \) first, treating derivatives of distributions by usual rules, one finds instead,

\[ -\int d\Lambda^0 d\Lambda = -\int_{\partial D} \Lambda^0 d\Lambda. \] (126)

Thus consistency requires the condition (5.24).

The constraints \( g(\Lambda^{(0)}) \) are first class since

\[ \{ g(\Lambda_1^{(0)}), g(\Lambda_2^{(0)}) \} = \frac{k}{2\pi} \int_D d\Lambda_1^{(0)} d\Lambda_2^{(0)} \] (127)

\[ = \frac{k}{2\pi} \int_{\partial D} \Lambda_1^{(0)} d\Lambda_2^{(0)} \]

\[ = 0 \text{ for } \Lambda_1^{(0)}, \Lambda_2^{(0)} \in T^{(0)}. \]

\( g(\Lambda^{(0)}) \) generates the gauge transformation \( a \to a + d\Lambda^{(0)} \) of \( a \).
Next consider $q(\Lambda)$ where $\Lambda \mid_{\partial D}$ is not necessarily zero. Since

$$\{ q(\Lambda), g(\Lambda^{(0)}) \} = -\frac{k}{2\pi} \int_D d\Lambda d\Lambda^{(0)} = \frac{k}{2\pi} \int_{\partial D} \Lambda^{(0)} d\Lambda = 0 \text{ for } \Lambda^{(0)} \in \mathcal{T}^{(0)},$$

they are first class or the observables of the theory. More precisely, observables are obtained after identifying $q(\Lambda_1)$ with $q(\Lambda_2)$ if $(\Lambda_1 - \Lambda_2) \in \mathcal{T}^{(0)}$. For then,

$$q(\Lambda_1) - q(\Lambda_2) = -g(\Lambda_1 - \Lambda_2) \approx 0. \quad (129)$$

The functions $q(\Lambda)$ generate gauge transformations $a \to a + d\Lambda$ involving $\Lambda$’s which do not necessarily vanish on $\partial D$.

It may be remarked that the expression for $q(\Lambda)$ is obtained from $g(\Lambda^{(0)})$ after a partial integration and a subsequent substitution of $\Lambda$ for $\Lambda^{(0)}$. It too generates gauge transformations like $g(\Lambda^{(0)})$, but the test function spaces for the two are different. The pair $q(\Lambda), g(\Lambda^{(0)})$ thus resemble the pair $g^c(\Lambda^c), g^\infty(\Lambda^\infty)$ of electrodynamics discussed in Chapter 4. The resemblance suggests that we think of $q(\Lambda)$ as akin to the generator of a global symmetry transformation. It is natural to do so for another reason as well: the Hamiltonian is a constraint for a first order Lagrangian such as the one we have here, and for this Hamiltonian, $q(\Lambda)$ is a constant of motion.

In quantum gravity, for asymptotically flat spatial slices, it is often the practice to include a surface term in the Hamiltonian which would otherwise have been a constraint and led to trivial evolution. However, we know of no natural choice of such a surface term, except zero, for the CS theory.

The PB’s of $q(\Lambda)$’s are easy to compute:

$$\{ q(\Lambda_1), q(\Lambda_2) \} = \frac{k}{2\pi} \int_D d\Lambda_1 d\Lambda_2 = \frac{k}{2\pi} \int_{\partial D} \Lambda_1 d\Lambda_2. \quad (130)$$

Remembering that the observables are characterized by boundary values of test functions, (5.33) shows that the observables generate a $U(1)$ Kac-Moody algebra localized on $\partial D$. [Literature must be consulted for information on Kac-Moody algebras. Knowledge of these algebras is not important for understanding this Chapter.] Note that it is a Kac-Moody algebra for “zero momentum” or “charge”. For if $\Lambda \mid_{\partial D}$ is a constant, it can be extended as a constant function to all of $D$ and then $q(\Lambda) = 0$. The central charges (given
by the right hand side of (5.33)) and hence the representation of (5.33) are different for \( k > 0 \) and \( k < 0 \), a fact which reflects parity violation by the action \( S \).

Let \( \theta \mod 2\pi \) be the coordinate on \( \partial D \) and \( \phi \) a free massless scalar field moving with speed \( v \) on \( \partial D \) and obeying the equal time PB’s

\[
\{ \phi(\theta), \dot{\phi}(\theta') \} = \delta(\theta - \theta') .
\] (131)

If \( \mu_i \) are test functions on \( \partial D \) and \( \partial_{\pm} = \partial_{x^0} \pm v \partial_{\theta} \), then

\[
\left\{ \frac{1}{v} \int \mu_1(\theta) \partial_{\pm} \phi(\theta), \frac{1}{v} \int \mu_2(\theta) \partial_{\pm} \phi(\theta) \right\} = \pm 2 \int \mu_1(\theta) d\mu_2(\theta),
\] (132)

the remaining PB’s involving \( \partial_{\pm} \phi \) being zero. Also \( \partial_{\pm} \partial_{\pm} \phi = 0 \). Thus the algebra of observables is isomorphic to that generated by the left moving \( \partial_+ \phi \) or the right moving \( \partial_- \phi \).

2. Quantization

Our strategy for quantization relies on the observation that if

\[
\Lambda |_{\partial D}(\theta) = e^{iN\theta},
\] (133)

then the PB’s (5.33) become those of creation and annihilation operators. These latter can be identified with the similar operators of the chiral fields \( \partial_{\pm} \phi \).

Thus let \( \Lambda_N \) be any function on \( D \) with boundary value \( e^{iN\theta} \):

\[
\Lambda_N |_{\partial D}(\theta) = e^{iN\theta}, N \in \mathbb{Z} - \{0\}.
\] (134)

\([N = 0 \text{ is excluded here in view of a remark above, } \Lambda_0 |_{\partial D} \text{ being a constant.} \]

These \( \Lambda_N \)'s exist. All \( q(\Lambda_N) \) with the same \( \Lambda_N |_{\partial D} \) are weakly equal and define the same observable. Let \( \langle q(\Lambda_N) \rangle \) be this equivalence class of weakly equal \( q(\Lambda_N) \) and \( q \) any member thereof. \([q_N \text{ can also be regarded as the equivalence class itself.} \]

Their PB’s follow from (5.33):

\[
\{ q_N, q_M \} = -iNk \delta_{N+M,0} .
\] (135)
The \( q_N \)'s are the CS constructions of the Fourier modes of a massless chiral scalar field on the circle \( S^1 \).

We can now proceed to quantum field theory. Let \( G(\Lambda^{(0)}) \), \( Q(\Lambda_N) \) and \( Q_N \) denote the quantum operators for \( g(\Lambda^{(0)}) \), \( q(\Lambda_N) \) and \( q_N \). We then impose the constraints

\[
G(\Lambda^{(0)}) \, | \cdot \rangle = 0 \tag{136}
\]

on all quantum states. It is an expression of their gauge invariance. Because of this equation, \( Q(\Lambda_N) \) and \( Q(\Lambda'_N) \) have the same action on the states if \( \Lambda_N \) and \( \Lambda'_N \) have the same boundary values. We can hence write

\[
Q_N \, | \cdot \rangle = Q(\Lambda_N) \, | \cdot \rangle. \tag{137}
\]

Here, in view of (5.38), the commutator brackets of \( Q_N \) are

\[
[Q_N, Q_M] = Nk\delta_{N+M,0}. \tag{138}
\]

Thus if \( k > 0 \) \( (k < 0) \), \( Q_N \) for \( N > 0 \)\( (N < 0) \) are annihilation operators (up to a normalization) and \( Q_{-N} \) creation operators. The “vacuum” \( | 0 \rangle \) can therefore be defined by

\[
Q_N \, | 0 \rangle = 0 \text{ if } Nk > 0. \tag{139}
\]

The excitations are obtained by applying \( Q_{-N} \) to the vacuum.

When the spatial slice is a disc, the observables are all given by \( Q_N \) and our quantization is complete. When it is not simply connected, however, there are further observables associated with the holonomies of the connection \( a \) and they affect quantization. We will not examine quantization for non-simply connected spatial slices here.

The CS interaction does not fix the speed \( v \) of the scalar field in (5.34) and so its Hamiltonian, a point previously emphasized by Fröhlich and Kerler, and Fröhlich and Zee. This is but reasonable. For if we could fix \( v \), the Hamiltonian \( H \) for \( \phi \) could naturally be taken to be the one for a free massless chiral scalar field moving with speed \( v \). It could then be used to evolve the CS observables using the correspondence between this field and the latter. But we have seen that no natural nonzero Hamiltonian exists for the CS system. It is thus satisfying that we cannot fix \( v \) and hence a nonzero \( H \).
D. The Chern-Simons Source as a Conformal Family

From the physical point of view, it is of great interest to study Chern-Simons dynamics in the presence of point sources. It is known that the statistics and spin of particles are changed by interaction with the CS field and that they acquire fractional statistics and spin for suitable choices of the coupling strength. [Cf. ref 4 and references therein.]. For this reason, Chern-Simons dynamics with sources can provide a useful means to describe anyons. Also we saw in Section 5.2 that the abelian CS field theory furnishes a description of the QH effect. The sources of the CS field can therefore be thought of as quasiparticle excitations in the QH system, giving us another reason to study these sources. One can also argue that there are sound mathematical reasons for studying these sources, since their spacetime history are connected to Wilson lines and Wilson lines are important for knot theory.

As mentioned above, when a point source is immersed in the CS field, its statistics is affected thereby. As interaction renormalizes statistics, it must renormalize spin as well if, as some of us may conservatively desire, CS dynamics incorporates the canonical spin-statistics connection. One purpose of this Section is to discuss this spin renormalization using a generalization of the canonical approach to source free quantum CS dynamics developed in the last Section. We will see that the specific mechanism for spin renormalization is a novel one: the configuration space of particle mechanics is enlarged by a circle $S^1$. A point of $S^1$ can be regarded as parametrizing a tangent direction or an orthonormal frame (although not canonically). A spinless source thus ends up acquiring a configuration space which is that of a two-dimensional rotor with translations added on. What occurs in CS theory is a massless chiral (conformal) quantum field on this $S^1$ (and time) with ability to change its location in space and with precisely the right spin to maintain the spin-statistics connection. The necessity for framing the particle has been emphasized in the literature before. The qualitative reason for the emergence of this frame is regularization, which surrounds the particle with a tiny hole $H$ which is eventually shrunk to a point. The CS action is then no longer for a disc $D$, but for $D\setminus H$, which is a disc with a hole. In contrast to $D$, the latter has an additional boundary $\partial H$, which is the circle $S^1$ mentioned above. Just as $\partial D$, this boundary as well can be associated with a massless chiral scalar field. The internal states of a CS anyon for a fixed location on $D$ thus form an infinite dimensional family of quantum states and are not
described by just a single ray. This remark was first stated by Witten and applies with equal force to the Quantum Hall quasiparticle if described in the Chern-Simons framework. It is also noteworthy that the CS source is not a first quantized framed particle, but is better regarded as a “particle” with a first quantized position and a second quantized frame. One intention of this Section is to explain these striking results with hopefully transparent arguments.

Suppose that a spinless point source with coordinate $z$ is coupled to $A_\mu$ with coupling $eA_\mu(z(x^0))\dot{z}^\mu$, $z^0 = x^0$. The field equation $\partial_1 A_2 - \partial_2 A_1 = 0$ is thereby changed to

$$\partial_1 A_2 - \partial_2 A_1 = -\frac{2\pi e}{k} \delta^2(x - z).$$  \hspace{1cm} (140)$$

If $C$ is a contour enclosing $z$ with positive orientation, then by (5.43),

$$\oint_C A = -\frac{2\pi e}{k}.$$  \hspace{1cm} (141)

On letting $C$ shrink to a point, it now follows that $A(x) = A_j(x)dx^j$ has no definite limit when $x$ approaches $z$. This singularity of $A$ demands regularization. A good way to regularize is to punch a hole $H$ containing $z$, and eventually to shrink the hole to a point.

Once this hole is made, the action is no longer for a disc $D$, but for $D\setminus H$, a disc with a hole. $D\setminus H$ has a new boundary $\partial H$ and it must be treated exactly like $\partial D$. The Gauss law must accordingly be changed to

$$g(\Lambda^{(1)}) \approx 0$$  \hspace{1cm} (142)$$

where the new test function space $T^{(1)}$ for $\Lambda^{(1)}$ is defined by

$$\Lambda^{(1)}|_{\partial D} = \Lambda^{(1)}|_{\partial H} = 0.$$  \hspace{1cm} (143)$$

The quantum operator $G(\Lambda^{(1)})$ for $g(\Lambda^{(1)})$ annihilates all the physical states.

There are now two KM algebras of the type (5.33), one each for $\partial D$ and $\partial H$. The former is defined by observables $q(\xi^{(0)})$ with test functions $\xi^{(0)}$ which vanish on $\partial H$, the latter by observables $q(\xi^{(1)})$ with test functions $\xi^{(1)}$ which vanish on $\partial D$. Let us now define the KM generators for the outer and inner boundaries as
\[ q_N^{(0)} \equiv q(\xi_N^{(0)}), \quad \xi_N^{(0)}(\theta) \mid_{\partial D} = e^{iN\theta}, \quad \xi_N^{(0)} \mid_{\partial H} = 0; \]
\[ \xi_N^{(1)}(\theta) \mid_{\partial H} = e^{-iN\theta}, \quad \xi_N^{(1)} \mid_{\partial D} = 0, \quad \theta \pmod{2\pi} \]

\( \theta \) being an angular coordinate on \( \partial H \). [The coordinates \( \theta \) on both \( \partial D \) and \( \partial H \) increase, say, in the anticlockwise sense.] The corresponding quantum operators will be denoted by \( Q_N^{(0)} \) and \( Q_N^{(1)} \). Note that the boundary conditions exclude the choice \( \xi^\alpha = \) a constant nonzero function on \( D \setminus H \). Hence we may not exclude \( N = 0 \) now.

An interpretation of the observables localized on \( \partial H \) is as follows. Let \( \theta \pmod{2\pi} \) be an angular coordinate on \( D \) which reduces to the \( \theta \) coordinates we have fixed on \( \partial D \) and \( \partial H \). A typical \( A \) compatible with (5.44) has a blip \(-2\pi \delta(\theta - \theta_0) d\theta\) localized on \( \partial H \) at \( \theta_0 \). The behaviour of a general \( A \) on \( \partial H \) can be duplicated by an appropriate superposition of these blips. The observable \( q(\xi^{(1)}) \) has zero PB with the left side of (5.44) and hence preserves the flux enclosed by \( C \). In fact, the finite canonical transformation generated by \( q(\xi^{(1)}) \) changes \( A \) to \( A + d\xi^{(1)} \) where the fluctuation \( d\xi^{(1)} \) creates zero net flux through \( C \). All \( A \) compatible with (5.44) can be generated from any one \( A \), such as an \( A \) with a blip, by these transformations. Thus the KM algebra of observables \( Q_N^{(1)} \) on \( \partial H \) generates all connections on \( \partial H \) with a fixed flux from any one of these connections.

We have now reproduced Witten’s observation that the CS anyon or the CS version of the quantum Hall quasiparticle is a conformal family.

A point of \( \partial H \) can be regarded as a frame (alluded to previously) attached to the particle. The restriction (pull back) of a connection \( A \) to \( \partial H \) can be regarded as a field on these frames. It follows from this remark that the observables localised at \( \partial H \) can be regarded as describing spin excitations.

We refer the reader to the original papers [4] for further developments of the approach outlined here.

VI. QUANTIZATION AND MULTIPLY CONNECTED CONFIGURATION SPACES

It has been mentioned in Chapter 5 that the sources coupled to the CS field have their statistical properties changed in a way compatible with their
spin renormalisation and the spin-statistics theorem (although for reasons of length of the article, and time available for the lectures, we have not gone into the details of this statistics renormalisation). It is thus natural at this point to examine the theoretical foundations of statistics [7] and the spin-statistics theorem [8,9]. The remaining Chapters of this review will be devoted to this task.

It has been known for some time that the statistics of identical particles in two or more dimensions can be understood in terms of the topology of their configuration space $Q$, their connectivity playing a particularly significant role. In this Chapter, after having first explained why topology, and in particular connectivity, is important for quantisation, we will systematically develop a method of quantisation on multiply connected spaces, providing the necessary mathematical background along the way. The Chapter concludes with several examples of physical systems for which multiple connectivity is significant. Chapter 7 will be our final Chapter. There we outline a purely topological proof of the spin-statistics theorem which completely avoids relativistic quantum field theory (RQFT) and is entirely based on the topology of the configuration space. There are several interesting physical systems governed by nonrelativistic dynamics such as those of holes in a Fermi sea or excitations above that sea. The topological proof discussed here is applicable to many of these systems whereas a RQFT proof looks at best contrived.

Further discussion of the material of this and subsequent Chapter and pertinent references can be found in refs. 2 and 7 to 10.

A. Configuration Space and Quantum Theory

The dynamics of a system in classical mechanics can be described by equations of motion on a configuration space $Q$. These equations are generally of second order in time. Thus if the position $q(t_0)$ of the system in $Q$ and its velocity $\dot{q}(t_0)$ are known at some time $t_0$, then the equations of motion uniquely determine the trajectory $q(t)$ for all time $t$.

When the classical system is quantized, the state of a system at time $t_0$ is not specified by a position in $Q$ and a velocity. Rather, it is described by a wave function $\psi$ which in elementary quantum mechanics is a (normalized) function on $Q$. The correspondence between the quantum states and wave functions however is not one to one since two wave functions which differ by
a phase describe the same state. The quantum state of a system is thus an equivalence class \( \{ e^{i\alpha} \psi \mid \alpha \text{ real} \} \) of normalized wave functions. The physical reason for this circumstance is that experimental observables correspond to functions like \( \psi^* \psi \) which are insensitive to this phase.

In discussing the transformation properties of wave functions, it is often convenient to enlarge the domain of definition of wave functions in elementary quantum mechanics in such a way as to naturally describe all the wave functions of an equivalence class. Thus instead of considering wave functions as functions on \( Q \), we can regard them as functions on a larger space \( \hat{Q} = Q \times S^1 \equiv \{(q, e^{i\alpha})\} \). The space \( \hat{Q} \) is obtained by associating circles \( S^1 \) to each point of \( Q \) and is said to be a \( U(1) \) bundle on \( Q \). Wave functions on \( \hat{Q} \) are not completely general functions on \( \hat{Q} \), rather they are functions with the property \( \psi(q, e^{i(\alpha+\theta)}) = \psi(q, e^{i\alpha})e^{i\theta} \). [Here we can also replace \( e^{i\theta} \) by \( e^{in\theta} \) where \( n \) is a fixed integer]. Because of this property, experimental observables like \( \psi^* \psi \) are independent of the extra phase and are functions on \( Q \) as they should be. The standard elementary treatment which deals with functions on \( Q \) is recovered by restricting the wave functions to a surface \( \{(q, e^{i\alpha_0}) \mid q \in Q\} \) in \( Q \) where \( \alpha_0 \) has a fixed value. Such a choice \( \alpha_0 \) of \( \alpha \) corresponds to a phase convention in the elementary approach.

When the topology of \( Q \) is nontrivial, it is often possible to associate circles \( S^1 \) to each point of \( Q \) so that the resultant space \( \hat{Q} = \{\hat{q}\} \) is not \( Q \times S^1 \), although there is still an action of \( U(1) \) on \( \hat{Q} \). We shall indicate this action by \( \hat{q} \to \hat{q}e^{i\theta} \). It is the analogue of the transformation \( (q, e^{i\alpha}) \to (q, e^{i\alpha}e^{i\theta}) \) we encountered earlier. We shall require this action to be free, which means that \( \hat{q}e^{i\theta} = \hat{q} \) if and only if \( e^{i\theta} \) is the identity of \( U(1) \). When \( \hat{Q} \neq Q \times S^1 \), the \( U(1) \) bundle \( \hat{Q} \) over \( Q \) is said to be twisted. It is possible to contemplate wave functions which are functions on \( \hat{Q} \) even when this bundle is twisted provided they satisfy the constraint \( \psi(\hat{q}e^{i\theta}) = \psi(\hat{q})e^{in\theta} \) for some fixed integer \( n \). If this constraint is satisfied, experimental observables being invariant under the \( U(1) \) action are functions on \( Q \) as we require. However, when the bundle is twisted, it does not admit globally valid coordinates of the form \( (q, e^{i\alpha}) \) so that it is not possible (modulo certain technical qualifications) to make a global phase choice, as we did earlier. In other words, it is not possible to regard wave functions as functions on \( Q \) when \( \hat{Q} \) is twisted.

The classical Lagrangian \( L \) often contains complete information on the nature of the bundle \( \hat{Q} \). We can regard the classical Lagrangian as a function
on the tangent bundle $T\hat{Q}$ of $\hat{Q}$. The space $T\hat{Q}$ is the space of positions in $\hat{Q}$ and the associated velocities. When $\hat{Q}$ is trivial, it is possible to reduce any such Lagrangian to a Lagrangian on the space $TQ$ of positions and velocities associated with $Q$ thereby obtaining the familiar description. On the other hand, when $\hat{Q}$ is twisted, such a reduction is in general impossible. Since the equations of motion deal with trajectories on $Q$ and not on $\hat{Q}$, it is necessary that there is some principle which renders the additional $U(1)$ degree of freedom in such a Lagrangian nondynamical. This principle is the principle of gauge invariance for the gauged group $U(1)$. Thus under the gauge transformation $\hat{q}(t) \rightarrow \hat{q}(t)e^{i\theta(t)}$, these Lagrangians change by constant times $d\theta/dt$, where $t$ is time. Since the equations of motion therefore involve only gauge invariant quantities which can be regarded as functions of positions and velocities associated with $Q$, these equations describe dynamics on $Q$. The Lagrangians we often deal with split into two terms $L_0$ and $L_{WZ}$, where $L_0$ is gauge invariant while $L_{WZ}$ changes as indicated above. This term $L_{WZ}$ has a geometrical interpretation. It is the one which is associated with the nature of the bundle $\hat{Q}$.

In particle physics, such a topological term was first discovered by Wess and Zumino in their investigation of nonabelian anomalies in gauge theories. The importance and remarkable properties of such “Wess-Zumino terms” have been forcefully brought to the attention of particle physicists in recent years because of the realization that they play a critical role in creating fermionic states in a theory with bosonic fields and in determining the anomaly structure of effective field theories.

In point particle mechanics, the existence and significance of Wess-Zumino terms have long been understood. For example, such terms play an essential role in the program of geometric quantization and related investigations which study the Hamiltonian or Lagrangian description of particles of fixed spin. A similar term occurs in the description of the charge-monopole system and has also been discussed in the literature. Such terms have been found in dual string models as well.

The Wess-Zumino term affects the equations of motion and has significant dynamical consequences already at the classical level. Its impact however is most dramatic in quantum theory where, as was indicated above, it affects the structure of the state space. For example, in the $SU(3)$ chiral model, it is this term which is responsible for the fermionic nature of the Skyrmion.
The preceding remarks on the nature of wave functions in quantum theory can be generalized by replacing the group $U(1)$ by more general abelian or nonabelian groups. A particularly important class of physical systems where such groups are discrete are those with multiply connected configuration spaces.

Multiply connected configuration spaces play an important role in many branches of physics. Examples are molecular physics, condensed matter and quantum field theories, and quantum gravity. Exotic statistics, which has recently assumed an important physical role in condensed matter theory for example, can be understood in terms of the multiple connectivity of the configuration space. In this Chapter, we will also give a few examples of such physical systems.

As a prelude to the discussion of multiply connected configuration spaces, we shall first generalize the preceding remarks on the nature of wave functions.

The arguments above which led to the consideration of $U(1)$ bundles on $Q$ were based on the observation that since only observables like $\psi^* \psi$ are required to be functions on $Q$, it is permissible to consider wave functions $\psi$ which are functions on a $U(1)$ bundle $\hat{Q}$ over $Q$ provided all wave functions fulfill the property $\psi(\hat{q} e^{i\theta}) = \psi(\hat{q}) e^{in\theta}$. We shall now show that we can meet this requirements on observables even with vector valued wave functions $\psi = (\psi_1, ..., \psi_K)$ which are functions on an $H$ bundle $\bar{Q}$ over $Q$, the group $H$ not being necessarily $U(1)$.

The general definition of an $H$ bundle $\bar{Q}$ over $Q$ is as follows. In an $H$ bundle $\bar{Q} = \{\bar{q}\}$ over $Q$, there is an action $\bar{q} \to \bar{q}h$ of the group $H = \{h\}$ on $\bar{Q}$ with the property

$$\bar{q} = \bar{q}h \text{ if and only if } h = \text{ identity e.} \quad (145)$$

As indicated earlier, such an action of a group $H$ is said to be free. Furthermore, in an $H$ bundle, when all points of $\bar{Q}$ connected by this $H$ action are identified, we get back the space $Q$. The space $Q$ is thus the quotient of $\bar{Q}$ by the $H$ action:

$$Q = \bar{Q}/H. \quad (146)$$

A point of $Q$ can be thought of the set of all points $\{\bar{q}h \mid h \in H\} \equiv \bar{q}H$.
connected to $\bar{q}$ by the $H$ action.

If the action of $H$ on $\bar{Q}$ is written in the form $\bar{q} \rightarrow \rho(h^{-1})\bar{q} \equiv \bar{q}h$, then $\rho(h_1)\rho(h_2) = \rho(h_1h_2)$. Hence the map $\rho : h \rightarrow \rho(h)$ from $H$ to these transformations on $Q$ is a homomorphism. It is in fact an isomorphism in view of (6.1). Note that the image of $h$ under $\rho$ acts on $\bar{q}$ according to $\bar{q} \rightarrow \bar{q}h^{-1}$ and not according to $\bar{q} \rightarrow \bar{q}h$. Nevertheless, following the convention in the mathematical literature, we shall often regard the action of $h$ on $\bar{Q}$ as being given by $\bar{q} \rightarrow \bar{q}h$.

An example of $\bar{Q}$ is the trivial $H$ bundle $\bar{Q} = Q \times H = \{(q,s) \mid s \in H\}$. It carries the free $H$ action $(q,s) \rightarrow (q,s)h \equiv (q,sh)$. The quotient of $Q$ by this action is $Q$. A point of $Q$ is $q$ which can be identified with $(q,s)H$ (for any $s$).

In the mathematical literature, the space $\bar{Q}$ is known as the bundle space and $H$ is known as the structure group. The map

$$\pi : \bar{Q} \rightarrow Q,$$

$$\bar{q} \rightarrow \bar{q}H$$

is known as the projection map. The set of points in $\bar{Q}$ which project to the same point $q$ of $Q$ under $\pi$ is known as the fibre over $q$. The entire structure $(\bar{Q}, \pi, Q, H)$ is known as a principal fibre bundle. We shall however call $\bar{Q}$ itself as a principal fibre bundle (or as an $H$ bundle).

It follows from the relation (6.2) between $\bar{Q}$ and $Q$ that any function $\sigma$ on $\bar{Q}$ which is invariant under the $H$ action $[\sigma(\bar{q}h) = \sigma(\bar{q})$ for all $h \in H]$ can be regarded as a function on $Q$. Let $h \rightarrow D(h)$ define a representation $\Gamma$ of $H = \{h\}$ by $K \times K$ unitary matrices. Let us demand of our wave functions that they transform by $\Gamma$ under the action of $H$:

$$\psi_i(\bar{q}h) = \psi_j(\bar{q})D_{ij}(h).$$

Then for any two wave functions $\psi$ and $\psi'$, the expression

$$<\psi, \psi'>(\bar{q}) \equiv \psi_i^*(\bar{q})\psi_i'(\bar{q})$$

is invariant under $H$ and $<\psi, \psi'>$ may be thought of as a function on $Q$. If we define the scalar product $(\psi, \psi')$ on wave functions by appropriately inte-
grating $<\psi,\psi'>$ over $Q$, then it is clear that there is no obvious conceptual problem in working with wave functions of this sort.

We shall see that such vector valued wave functions with $N \geq 2$ will occur in the general theory of multiply connected configuration spaces if $H$ is nonabelian. When that happens, as Sorkin has proved, the space of wave functions we have described above is too large when the dimension of $\Gamma$ exceeds 1, even when $\Gamma$ is irreducible. The reduction of this space to its proper size will also be described following Sorkin and will be seen to lead to interesting consequences.

A result of particular importance we shall see later and which merits emphasis is that the quantum theory of systems with multiply connected configuration spaces is ambiguous, there being as many inequivalent ways of quantizing the system as there are distinct unitary irreducible representations (UIR’s) of $\pi_1(Q)$. The angle $\theta$ which labels the vacua in QCD, for example, can be thought as the label of the distinct UIR’s of $Z$, $Z$ being $\pi_1(Q)$ for such a theory. As is well-known, the quantum theories associated with different $e^{i\theta}$ are inequivalent.

**B. The Universal Covering Space and the Fundamental Group**

Given any manifold such as a configuration space $Q$, it is possible to associate another manifold $\bar{Q}$ to $Q$ which is simply connected. The space $\bar{Q}$ is known as the universal covering space of $Q$. The group $\pi_1(Q) = H$ acts freely on $\bar{Q}$ and the quotient of $\bar{Q}$ by this action is $Q$. Thus $\bar{Q}$ is a principal fibre bundle over $Q$ with structure group $H$. The space $\bar{Q}$ plays an important role in the construction of possible quantum theories associated with $Q$. In this Section, we shall describe the construction of $\bar{Q}$. We shall also explain the concept of the fundamental group $\pi_1(Q)$ of $Q$ and its action on $\bar{Q}$.

We shall assume in what follows that $Q$ is path-connected, that is that if $q_0, q_1$ are any two points of $Q$, we can find a continuous curve $q(t) \in Q$ with $q(0) = q_0, q(1) = q_1$.

The first step in the construction of $\bar{Q}$ is the construction of the path space $\mathcal{P}Q$ associated with $Q$. Let $q_0$ be any point of $Q$ which once chosen is held fixed in all subsequent considerations. Then $\mathcal{P}Q$ is the collection of all paths which start at $q_0$ and end at any point $q$ of $Q$. We shall denote
the paths ending at \( q \) by \( \Gamma_q, \tilde{\Gamma}_q, \Gamma'_q \) etc. It is to be noted that these paths \( \Gamma_q \) are oriented and unparametrized. The former means that they are to be regarded as starting at the base point \( q_0 \) and ending at \( q \). Each of these paths has thus an arrow attached pointing from \( q_0 \) to \( q \). The implication of the statement that \( \Gamma_q \) is “unparametrized” is that (besides its orientation) only its geographical location in \( Q \) matters. If we introduce a parameter \( s \) to label points of \( \Gamma_q \) and write the associated parametrized path as

\[
\gamma_q = \{ \gamma_q(s) \mid \gamma_q(0) = q_0, \ \gamma_q(1) = q \}, \tag{150}
\]

then \( \Gamma_q \) is the equivalence class of all such parametrized paths (with parameters compatible with the orientation of \( \Gamma_q \)) with the same location in \( Q \).

We next introduce an equivalence relation \( \sim \) on the paths known as homotopy equivalence. We say that two paths \( \Gamma_q \) and \( \tilde{\Gamma}_q \) with the same end point \( q \) are homotopic and write

\[
\Gamma_q \sim \tilde{\Gamma}_q \tag{151}
\]

if \( \Gamma_q \) can be continuously deformed to \( \tilde{\Gamma}_q \) while holding \( q \) (and of course \( q_0 \)) fixed.

A more formal definition of homotopy equivalence is the following: If there exists a continuous family of paths \( \Gamma_q(t) \ [0 \leq t \leq 1] \) in \( Q \) (all from \( q_0 \) to \( q \)) such that

\[
\Gamma_q(0) = \Gamma_q, \ \Gamma_q(1) = \tilde{\Gamma}_q, \tag{152}
\]

then \( \Gamma_q \sim \tilde{\Gamma}_q \).

Let \( [\Gamma_q] \) denote the equivalence class of all paths ending at \( q \) which are homotopic to \( \Gamma_q \). The universal covering space \( \tilde{Q} \) of \( Q \) is just the collection of all these equivalence classes:

\[
\tilde{Q} = \{ [\Gamma_q] \}. \tag{153}
\]

It can be shown that \( \tilde{Q} \) is simply connected.

Of particular interest to us are the equivalence classes \( [\Gamma_{q_0}] \) of all loops \( \Gamma_{q_0} \) starting and ending at \( q_0 \). These equivalence classes have a natural group structure. The group product is defined by
\[ [\Gamma_{q_0}][\tilde{\Gamma}_{q_0}] = [\Gamma_{q_0} \cup \tilde{\Gamma}_{q_0}], \]  
(154)  
where in the loop \( \Gamma_{q_0} \cup \tilde{\Gamma}_{q_0} \), we first traverse \( \Gamma_{q_0} \) and then traverse \( \tilde{\Gamma}_{q_0} \). The inverse is defined by  
\[ [\Gamma_{q_0}]^{-1} = [\Gamma^{-1}_{q_0}], \]  
(155)  
where the loop \( \Gamma^{-1}_{q_0} \) has the same geographical location in \( Q \) as \( \Gamma_{q_0} \), but has the opposite orientation. The identity \( e \) is the equivalence class of the loop consisting of the single point \( q_0 \). It is clear that  
\[ [\Gamma_{q_0}][\Gamma^{-1}_{q_0}] = [\Gamma^{-1}_{q_0}][\Gamma_{q_0}] = e. \]  
(156)  
The group \( \pi_1(Q) \) with elements \([\Gamma_{q_0}]\) and the group structure defined above is known as the fundamental group of \( Q \). If \( \pi_1(Q) \) is nontrivial \([\pi_1(Q) \neq \{e\}]\), the space \( Q \) is said to be multiply connected. We shall see examples of multiply connected spaces in Section 6.3. They will show in particular that \( \pi_1(Q) \) can be abelian or nonabelian. In any case, it is always discrete.

The group \( \pi_1(Q) \) has a free action on \( \tilde{Q} \). It is defined by  
\[ [\Gamma_{q_0}] : [\Gamma_q] \rightarrow [\Gamma_{q_0}][\Gamma_q] \equiv [\Gamma_{q_0} \cup \Gamma_q], \]  
(157)  
where in \( \Gamma_{q_0} \cup \Gamma_q \), we first traverse \( \Gamma_{q_0} \) and then traverse \( \Gamma_q \). It is a simple exercise to show that this action is free.

We now claim that the quotient of \( \tilde{Q} \) by this action is \( Q \), the associated projection map \( \pi : Q \rightarrow Q \) being defined by  
\[ \pi : [\Gamma_q] \rightarrow \pi([\Gamma_q]) = q. \]  
(158)  
This means the following: a) All the points \([\Gamma_q],[\tilde{\Gamma}_q],... \) with the same image \( q \) under \( \pi \) are related by \( \pi_1(Q) \) action, and b) these are the only points related by \( \pi_1(Q) \) action. To show a), let \( \tilde{\Gamma}_q \cup \Gamma_q^{-1} \) be the loop based at \( q_0 \) where we first go along \( \tilde{\Gamma}_q \) from \( q_0 \) to \( q \) and then return to \( q_0 \) along \( \Gamma_q \) (in a sense opposite to the orientation of \( \Gamma_q \)). It is clear that  
\[ [\tilde{\Gamma}_q] = [\tilde{\Gamma}_q \cup \Gamma_q^{-1}][\Gamma_q], \quad [\Gamma_q \cup \Gamma_q^{-1}] \in \pi_1(Q). \]  
(159)
This proves a). As regards b), elements of $\pi_1(Q)$ act by attaching loops at the starting point $q_0$ of $\Gamma_q$ and hence map $[\Gamma_q]$ to some $[\tilde{\Gamma}_q]$. Both $[\Gamma_q]$ and $[\tilde{\Gamma}_q]$ project under $\pi$ to the same point $q$ of $Q$. This proves b).

We have now proved that $\bar{Q}$ is a principal fibre bundle over $Q$ with structure group $\pi_1(Q)$.

C. Examples of Multiply Connected Configuration Spaces

It is appropriate at this point to give some examples of multiply connected spaces. We will avoid examples from gauge and gravity theories for reasons of simplicity. There are several such relevant examples and we shall pick three.

1. Let $x_1, x_2, ..., x_N$ be $N$ distinct points in the plane $\mathbb{R}^2$ and let $Q$ be the complement of the set $\{x_1, x_2, ..., x_N\}$ in $\mathbb{R}^2$:

$$Q = \mathbb{R}^2 \backslash \{x_1, x_2, ..., x_N\}.$$  \hspace{1cm} (160)

Thus $Q$ is the plane with $N$ holes $x_1, x_2, ..., x_N$. The fundamental group $\pi_1(Q)$ of this $Q$ is of infinite order. It is nonabelian for $N \geq 2$. The generators of this group are constructed as follows: Let $q_0$ be any fixed point of $Q$ and let $C_M$ be any closed curve from $q_0$ to $q_0$ which encloses $x_M$ and none of the remaining holes. It is understood that $C_M$ winds around $x_M$ exactly once with a particular orientation. Let $C_M^{-1}$ be the curve with orientation opposite to $C_M$, but otherwise the same as $C_M$. Let $[C_M]$ and $[C_M^{-1}] = [C_M]^{-1}$ be the homotopy classes of $C_M$ and $C_M^{-1}$. Then $\pi_1(Q)$ consists of all possible products like $[C_M][C_M'][C_M'']^{-1}...$ and is the free group with generators $[C_M]$. The products of homotopy classes are defined here as in the last Section. For example, $[C_M][C_M'] = [C_M \cup C_M']$ where $C_M \cup C_M'$ is the curve where we first trace $C_M$ and then trace $C_M'$. For $N = 1$, the group $\pi_1(Q)$ has one generator and is $\mathbb{Z}$. The relevance of this $Q$ for the treatment of the Aharonov-Bohm effect should be evident.

2. In the collective model of nuclei, one considers nuclei with asymmetric shapes with three distinct moments of inertia $I_i$ along the three principal axes. There are also polyatomic molecules such as the ethylene molecule $C_2H_4$ which can be described as such asymmetric rotors. The configuration space $Q$ in these cases is the space of orientations of the nucleus or the
molecule. These orientations can be described by a real symmetric $3 \times 3$ matrix $T$ (the moment of inertia tensor) with three distinct but fixed eigenvalues $I_i$. We now show that this $Q$ has a nonabelian fundamental group.

Any $T \in Q$ can be written in the form

$$T \equiv R T_0 R^{-1},$$

$$T_0 = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \\ I_3 & 0 \end{bmatrix},$$

(161)

where $R$ being in $SO(3)$ is regarded as a real orthogonal matrix of determinant 1. Hence $Q$ is the orbit of $T_0$ under the action of $SO(3)$ given by (6.17). If $R_i(\pi)$ is the rotation by $\pi$ around the $i^{th}$ axis,

$$R_1(\pi) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & -1 \end{bmatrix}, \quad R_2(\pi) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$R_3(\pi) = \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix},$$

(162)

then $T$ is invariant under the substitution $R \rightarrow R R_i(\pi)$. So $Q$ is the space of cosets of $SO(3)$ with respect to the four element subgroup $\{1, R_1(\pi), R_2(\pi), R_3(\pi)\}$.

It is convenient to view this coset space as the coset space $SU(2)/H$ of $SU(2)$ with regard to an appropriate subgroup $H$. For this purpose let us introduce the standard homomorphism $R : SU(2) \rightarrow SO(3)$. The definition of $R$ is

$$s \tau_i s^{-1} = \tau_j R_{ji}(s), \ s \in SU(2),$$

(163)

$\tau_i$ being Pauli matrices. [Here we think of $SU(2)$ concretely as the group of $2 \times 2$ unitary matrices of determinant 1.] Then we can write any $T$ in the form

$$T = R(s) T_0 R(s^{-1})$$

(164)
and hence view $Q$ as the orbit of $T_0$ under $SU(2)$. Since by (6.19),

$$R(-s) = R(s),$$

$$R(\pm si\tau_i) = R(\pm se^{i\pi \tau_i/2}) = R(s)R_i(\pi),$$

the stability group $H$ of $T_0$ is the quaternion (or binary dihedral) group $D^*_8$:

$$H = D^*_8 = \{\pm 1, \pm i\tau_1, \pm i\tau_2, \pm i\tau_3\}.$$  

(166)

Thus

$$Q = SU(2)/D^*_8. \quad \text{(167)}$$

It is well known that $SU(2)$ is simply connected $[\pi_1(SU(2)) = \{e\}]$. A consequence of this fact [which will not be proved here] is that

$$\pi_1(Q) = D^*_8. \quad \text{(168)}$$

The loops in $Q$ associated with the elements of $D^*_8$ can be constructed as follows. Consider a curve $\{s(t)\}$ in $SU(2)$ from identity to $h \in D^*_8$:

$$s(t) \in SU(2), \; s(0) = 1, \; s(1) = h. \quad \text{(169)}$$

The image of this curve in $Q$ is $\{T(t)\}$ where

$$T(t) = R[s(t)]T_0R[s(t)^{-1}]. \quad \text{(170)}$$

Since $T(0) = T(1) = T_0$, this is a loop in $Q$ based at $T_0$. Two loops $T(t)$ and $T'(t)$ with different $s(1) \in D^*_8$ are not homotopic, whereas all loops $T(t)$ and $T'(t)$ with the same $s(1) \in D^*_8$ are homotopic and form a homotopy class. Such homotopy classes can be thought of as the elements $h$ of $\pi_1(Q)$.

The relation (6.23) shows that $Q$ is the quotient of $SU(2)$ by the free action

$$s \to sh, \; s \in SU(2), \; h \in D^*_8 \quad \text{(171)}$$

of $D^*_8$. Furthermore $\pi_1(Q) = D^*_8$. Therefore in this example, $SU(2)$ as a manifold is the universal covering space of $Q$. 

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There are molecules with configuration spaces $Q$ such that $\pi_1(Q)$ is any one of the finite discrete subgroups of $SU(2)$, the binary dihedral group being just one of these possibilities. Reference [10] can be consulted for further discussion of this fact and for citations to the literature.

3. The last example we shall give is relevant for discussing possible statistics of particles in $k$ spatial dimensions. Consider $N$ identical spinless particles in $\mathbb{R}^k$ [for $N \geq 2$] and assume first that $k \geq 3$. A configuration of these particles is given by the unordered set $\{x_1, x_2, ..., x_N\}$ where $x_j \in \mathbb{R}^k$. The set must be regarded as unordered (so that for example $\{x_1, x_2, ..., x_N\} = \{x_2, x_1, ..., x_N\}$) because of the assumed indistinguishability of the particles. Let us also assume that no two particles can occupy the same position so that $x_i \neq x_j$ if $i \neq j$. The resultant space of these sets can be regarded as the configuration space $Q$ of this system. It can be shown that $\pi_1(Q)$ is identical to the permutation group $S_N$. The closed curves in $Q$ associated with the transpositions $s_{ij} \in S_N$ of two particles can be constructed as follows. Choose the base point $q_0$ to be $\{x_1, x_2, ..., x_N\}$. Let $\{\gamma_{ij}(t); 0 \leq t \leq 1\}$ be the loop in $Q$ defined by

$$\gamma_{ij}(t) = [x_1^0, x_2^0, ..., x_{i-1}^0, x_i(t), x_{i+1}^0, ..., x_{j-1}^0, x_j(t), x_{j+1}^0, ..., x_N^0] ,$$

$$x_i(0) = x_i^0 , \quad x_i(1) = x_j^0 ,$$

$$x_j(0) = x_j^0 , \quad x_j(1) = x_i^0 .$$

$\{\gamma_{ij}(t)\}$ is a loop since the set $\{x_1^0, x_2^0, ..., x_N^0\}$ is unordered. The homotopy class of this loop can be identified with $s_{ij}$.

The distinct quantum theories of this system are labelled by the UIR's of $S_N$ and are associated with parastatistics. Special cases of these theories describe bosons and fermions.

We can describe the configuration space of $N$ identical particles for $k = 2$ as well in a similar way. The fundamental group $\pi_1(Q)$ for $k = 2$ however is not $S_N$, but a very different (infinite) group known as the braid group $B_N$. It is because $\pi_1(Q) = B_N$ for $k = 2$ that remarkable possibilities for statistics (such as fractional statistics) arise in two spatial dimensions.

For $N = 2$, it is simple to illustrate the difference between $B_2$ and $S_2$. The discussion also shows why fractional statistics is possible in two dimensions.
Thus consider the square $s_{12}^2$ of the transposition for two particles. It is easy

to see that it is the homotopy class of the curve where $x_1^0$ is held fixed, say

at the origin, and $x_2$ goes around it from $x_2^0$ to $x_2^0$. For $k = 2$, that is in a
plane, this curve is a loop with $x_1^0$ at its middle. It can not be shrunk to

a point since $x_i \neq x_j$ for points of $Q$. Thus $s_{12}^2 \neq$ identity $e$ for $k = 2$. A
similar argument shows that no power of $s_{12}$ is $e$. The group $B_2$ is abelian
and is generated by $s_{12}$. Its UIR's are given by $s_{12} \rightarrow e^{i\theta}$ where $\theta$ is real. All
real $\theta$ are allowed since we have argued above that no power of $s_{12}$ is $e$. We
therefore have the possibility of fractional statistics which describe neither
bosons (for which $s_{12} \rightarrow 1$) nor fermions (for which $s_{12} \rightarrow -1$) for $k = 2$. [The next two Sections describe how to realise quantum theories for distinct
UIR's of $\pi_1(Q).$]

Now for $k > 2$, $s_{12}^2$ is still the homotopy class of a loop like the one
described above. But this loop can be shrunk to a point for $k > 2$. For
example, it can be taken to be in a plane not enclosing $x_1^0$, if necessary after
first deforming it. It can then be shrunk to a point on this plane. Thus $s_{12}^2 =
$ identity $e$ and the corresponding $\pi_1(Q)$ is $S_2 = \mathbb{Z}_2$. There are only two UIR's
of $S_2$ and they are given by $s_{12} \rightarrow 1$ and $s_{12} \rightarrow -1$. They describe bosons
and fermions respectively.

D. Quantization on Multiply Connected Configuration Spaces

We shall now describe the general approach to quantization when the
configuration space $Q$ is multiply connected.

As indicated previously, this quantization can be carried out by intro-
ducing a Hilbert space $\mathcal{H}$ of complex functions on $\bar{Q}$ with a suitable scalar
product and realizing the classical observables as quantum operators on this
space. Since the classical configuration space is $Q$ and not $\bar{Q}$, classical ob-
servables are functions of $q \in Q$ and of their conjugate momenta. Let us
concentrate on functions of $q$. Let $\alpha(q)$ define a function of $q$ and let $\hat{\alpha}$ be
the corresponding quantum operator. The definition of $\hat{\alpha}$ consists in speci-
fying the transformed function $\hat{\alpha}f$ for a generic function $f \in \mathcal{H}$. Thus given
the function $f$, we have to specify the value of $\hat{\alpha}f$ at every $\bar{q}$. This is done
by the rule
\((\hat{\alpha} f)(\bar{q}) = \alpha[\pi(\bar{q})f(\bar{q})]. \quad (173)\)

The group \(\pi_1(Q)\) acts on \(\mathcal{H}\). Let \(t\) denote a generic element of \(\pi_1(Q)\). If \(\hat{t}\) is the operator which represents \(t\) on \(\mathcal{H}\), and \(\hat{t} f\) is the transform of a function \(f \in \mathcal{H}\) by \(\hat{t}\), \(\hat{t}\) is defined by specifying the function \(\hat{t} f\) as follows:

\[(tf)(\bar{q}) \equiv f(\bar{qt}). \quad (174)\]

Now \(\hat{\alpha}\) commutes with \(\hat{t}\):

\[
(\hat{\alpha} \hat{t} f)(\bar{q}) = \alpha[\pi(\bar{q})](\hat{t} f)(\bar{q}) \\
= \alpha[\pi(\bar{q})]f(\bar{qt}) , \\
(\hat{t} \hat{\alpha} f)(\bar{q}) = (\hat{\alpha} f)(\bar{qt}) \\
= \alpha[\pi(\bar{qt})]f(\bar{qt}) \\
= \alpha[\pi(\bar{q})]f(\bar{qt}) \\
= (\hat{\alpha} \hat{t} f)(\bar{q}). \quad (175)
\]

Here we have used the fact that \(\pi(\bar{qt}) = \pi(\bar{q}). \) [See (6.14) and the remarks which follow.]

Since the operators \(\hat{t}\) are not all multiples of the identity operator, Schur’s lemma tells us that this representation of the observables \(\hat{\alpha}\) on \(\mathcal{H}\) is not irreducible. We can proceed in the following way to reduce it to its irreducible components. Let \(\Gamma_1, \Gamma_2, \ldots\) denote the distinct irreducible representations of \(\pi_1(Q)\). Let \(\mathcal{H}_\beta^\ell (\beta = 1, 2, \ldots)\) be the subspaces of \(\mathcal{H}\) which transform by \(\Gamma_\ell\), \(\beta\) being an index to account for multiple occurrences of \(\Gamma_\ell\) in the reduction. Let us also define

\[
\mathcal{H}^{(\ell)} = \bigoplus_\beta \mathcal{H}_\beta^{(\ell)}. \quad (176)
\]

Then

\[
\mathcal{H} = \bigoplus_\ell \mathcal{H}^{(\ell)}. \quad (177)
\]

Since \(\hat{\alpha}\) commutes with \(\hat{t}\), it can not map a vector transforming \(\Gamma_\ell\) to one transforming by \(\Gamma_m (m \neq \ell)\) since \(\Gamma_\ell\) and \(\Gamma_m\) are inequivalent. Thus

\[
\hat{\alpha} \mathcal{H}^{(\ell)} \subset \mathcal{H}^{(\ell)}. \quad (178)
\]
In other words, we can realize our observables on any one subspace $\mathcal{H}^{(\ell)}$ and ignore the remaining subspaces. Quantization on the subspaces $\mathcal{H}^{(\ell)}$ and $\mathcal{H}^{(m)}$ are known to be inequivalent when $\ell \neq m$. Thus there are at least as many distinct ways to quantize the system as the number of inequivalent irreducible representations of $\pi_1(Q)$. It may also be shown that the representation of the algebra of observables on any one $\mathcal{H}^{(\ell)}$ is irreducible if $\pi_1(Q)$ is abelian, while some additional reduction is possible if it is nonabelian as shown by Sorkin and as we shall see below.

Here we have not discussed how the momentum variables conjugate to the coordinates are realized on $\mathcal{H}^{(\ell)}$. It can be shown that for the problems at hand, these momentum variables can also be consistently realized.

**E. Nonabelian Fundamental Groups**

Let us now consider nonabelian $\pi_1(Q)$ in more detail. Let $\gamma_\ell (\ell = 1, 2, \ldots)$ denote its distinct one dimensional UIR’s and let $\tilde{\gamma}_\alpha (\alpha = 1, \ldots)$ denote its distinct UIR’s of dimension greater than 1. [For simplicity, we assume here that the indexing sets for both abelian and nonabelian UIR’s are countable.]

The subspaces of $\mathcal{H}$ which carry $\gamma_\ell$ will be called $h^{(\ell)}_k$ and the subspaces which carry $\tilde{\gamma}_\alpha$ will be called $\tilde{h}^{(\alpha)}_{\sigma}$, $k$ and $\sigma$ being indices to account for multiple occurrences of a given UIR in the reduction of $\mathcal{H}$. If we set

$$h^{(\ell)} = \bigoplus_k h^{(\ell)}_k,$$

then as in the abelian case the algebra of observables is represented irreducibly on $h^{(\ell)}$, and the representations on different $h^{(\ell)}$ are inequivalent. The novelty is associated with the representations on

$$\tilde{h}^{(\alpha)} = \bigoplus_{\sigma} \tilde{h}^{(\alpha)}_{\sigma}.$$  

They are inequivalent for different $\alpha$, but they are not irreducible. We now show this fact.

Let $e_\sigma(j)(j = 1, 2, \ldots, n > 1)$ be a basis for $\tilde{h}^{(\alpha)}_{\sigma}$ chosen so that they transform in the same way under $\pi_1(Q)$ for different $\sigma$:

$$\hat{t}e_\sigma(j) = e_\sigma(k) D(t)_{kj}.$$ (181)
Here $t \to D(t)$ defines the representation $\gamma_\alpha$. [Since $\alpha$ can be held fixed in the ensuing discussion, an index $\alpha$ has not been put on the vectors $e_\sigma(j)$ or on the matrices $D(t)$.]

Now if $\hat{L}$ is any linear operator such that $\hat{L}e_\sigma(j)$ transforms in the same way as $e_\sigma(j)$,

\[ i\hat{L}e_\sigma(j) = [\hat{L}e_\sigma(k)]D_{kj}(t), \quad (182) \]

that is if $[\hat{L}, \hat{t}] = 0$, then by Schur’s lemma $\hat{L}$ acts only on the index $\sigma$:

\[ \hat{L}e_\sigma(j) = e_\lambda(j)D_{\lambda\sigma}(\hat{L}). \quad (183) \]

Furthermore, again by Schur’s lemma, $D(\hat{L})$ is independent of $j$. Since $\hat{\alpha}$ in (6.29) shares the preceding property of $\hat{L}$, it follows that

\[ \hat{\alpha}e_\sigma(j) = e_\lambda(j)D_{\lambda\sigma}(\hat{\alpha}). \quad (184) \]

It can be shown that there is a similar formula for momentum observables as well.

Thus the subspace spanned by the vectors $e_\sigma(j)$ [$\sigma = 1, 2, \ldots$] for any fixed $j$ is invariant under the action of observables. Also, since $D(\hat{\alpha})$ is independent of $j$, the representation of the algebra of observables on the subspaces associated with different $j$ are equivalent. It is thus sufficient to retain just one such subspace, the remaining ones may be discarded. When we do so, we also obtain an irreducible representation of the algebra of observables.

Further insight into the nature of this representation is gained by working with a “basis” for $\mathcal{H}$ consisting of states localized at points of $Q$. These are analogous to the states $|\vec{x}>$ which are localized at positions $\vec{x}$ in the standard nonrelativistic quantum mechanics of spinless particles. But while there is only one such linearly independent state for a given $\vec{x}$, we have $\dim \pi_1(Q)$ [\equiv dimension of $\pi_1(Q)$] worth of such linearly independent states $\{|\vec{q}t>\}$ localized at $q$, because under $\pi$, $\vec{q}t$ projects to $q$ independently of $t$. [Here $\vec{q}$ is any conveniently chosen point of $\vec{Q}$ with $\pi(\vec{q}) = q$.] The group $\pi_1(Q)$ acts on these states according to

\[ \hat{s} |\vec{q}> = |\vec{q}s^{-1}>, \quad s \in \pi_1(Q). \quad (185) \]

Clearly this representation of $\pi_1(Q)$ on the subspace spanned by the vectors $\{|\vec{q}t>\} = \{|\vec{q}s^{-1}>\}$ (for fixed $\vec{q}$) is isomorphic to the regular representation
of $\pi_1(Q)$. As is well known, when this representation is fully reduced, each UIR occurs as often as its dimension. Thus each $\gamma_\ell$ occurs once and is carried by a one dimensional vector space with basis $F(\ell)$ say, while each $\bar{\gamma}_\alpha$ occurs $\dim \bar{\gamma}_\alpha$ times and is carried by a vector space with basis $E^{(\alpha)}(j)$ say, for a fixed $\ell$ and a fixed $\alpha$.

The transformation law of $E^{(\alpha)}(j)$ under $\pi_1(Q)$ is

$$iE^{(\alpha)}(j) = E^{(\alpha)}(k)D_{kj}(t).$$

(186)

According to our previous argument, the reduction of the representation of the algebra of observables is achieved by retaining only the subspace $V_j(q)$ spanned by the vectors $E^{(\alpha)}(j)$ for a fixed $j$ [and a fixed $\alpha$].

Now every nonzero vector in $V_j(q)$ is localized at $q$. Thus even after this reduction, there are $\dim \bar{\gamma}_\alpha$ linearly independent vectors localized at $q$. In nonrelativistic quantum mechanics, if the system has internal symmetry (or quantum numbers like intrinsic spin), the linearly independent states localized at $\vec{x}$ are of the form $|\vec{x},m\rangle$ ($m = 1,2,...,\dim \bar{\gamma}_\alpha$) where the index $m$ carries the representation of internal symmetry. In this case, there are $k$ linearly independent vectors localized at $\vec{x}$. The situation we are finding when $\pi_1(q)$ is nonabelian has points of resemblance to this familiar quantum mechanical situation in the sense that here as well there are many states localized at $q$.

It is of interest to know the physical observables $\hat{O}$ which mix the indices $\sigma$ of the basis $E^{(\alpha)}(j)$. That is, it is of interest to find the observables $\hat{O}$ with the property

$$\hat{O}E^{(\alpha)}(j) = E^{(\alpha)}(\lambda)D_{\lambda\sigma}(\hat{O})$$

(187)

such that their representation on $V_j(q)$ is irreducible. There is an elegant, but local, geometrical construction for a family of such operators which we now describe. Consider loops from $q$ to $q$, they can be divided into homotopy classes $[C_t(q)]$ [$t \in \pi_1(Q)$] labelled by elements of $\pi_1(Q)$. The class $[C_t(q)]$ consists of closed loops which are homotopic to each other. The labels can be so chosen that $[C_s(q)][C_t(q)] = [C_{st}(q)]$ where the multiplication of homotopy classes has been described in Section 6.2. [Note however that the loops $C_t(q)$ are based at $q$ and not at the base point $q_0$ of Section 6.2.] Pick one closed curve $C_t(q)$ from $[C_t(q)]$ and consider the operator which parallel transports wave functions around $C_t(q)$. It can be shown that the change of a wave
function as a result of parallel transporting it around a loop in \( C_t(q) \in [C_t(q)] \) is independent of the choice of the loop in the class \([C_t(q)]\). Thus the parallel transport operator depends only on the homotopy class \([C_t(q)]\) and not on the choice of the closed curve in \([C_t(q)]\). It can hence be denoted by \( \hat{O}_t \). These operators \( \hat{O}_t \) can serve as the observables we are seeking.

The above description of the operators \( \hat{O}_t \) is rather loose however since \( \hat{O}_t \) is defined only if the transform \( \hat{O}_t \psi \) of a wave function \( \psi \) is defined and this involves specifying \((\hat{O}_t \psi)(\bar{q})\) for all \( \bar{q} \). Hence we must associate a homotopy class \([C_t(q)]\) to each \( t \in \pi_1(Q) \) and all \( q \). This association must be smooth in \( q \) and fulfill the property \([C_s(q)] [C_t(q)] = [C_{st}(q)]\). Consider what happens if we smoothly change \([C_t(q)]\) as \( q \) is taken around a closed loop in the homotopy class \([C_s(q)]\), \( s \in \pi_1(Q) \). It is then easy to convince oneself that \([C_t(q)]\) evolves into the homotopy class \([C_{sts^{-1}}(q)]\). When \( \pi_1(Q) \) is nonabelian, \([C_t(q)]\) will not be equal to \([C_{sts^{-1}}(q)]\) for all \( t \) and \( s \). A consequence is that the operators \( \hat{O}_t \) are not all well defined when the UIR of \( \pi_1(Q) \) defining the quantum theory is nonabelian. [Nonetheless, the representation of the algebra of observables we have described can be shown to be irreducible.] The obstruction in defining all the operators \( \hat{O}_t \) here is similar to the obstruction in defining the colour group in the presence of nonabelian monopoles or the helicity group for massless particles in higher dimensions.[See ref. 2 for references on these topics.]

It is remarkable that when \( \pi_1(Q) \) is nonabelian, quantization can lead to a multiplicity of states all localized at the same point. The consequences of this multiplicity have not yet been sufficiently explored in the literature.

**F. The Case of the Asymmetric Rotor**

We shall now briefly illustrate these ideas by the example of the asymmetric rotor described in Section 6.3. The treatment given here is equivalent for example to the standard treatment molecules with \( D_8^* \) as the symmetry group [that is, \( \pi_1(Q) \)] or of nuclei with three distinct moments of inertia in the collective model approach to nuclei. See ref. 10 in particular in this connection.

Let \( \bar{Q} \) be the manifold of the group \( SU(2) \) and let \( s \) denote a point of \( \bar{Q} \). We regard \( s \) as a \( 2 \times 2 \) unitary matrix of determinant 1. Let \( D_8^* \) be the quaternion subgroup of \( SU(2) \):

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It has the free action

\[ s \to sh, \ h \in D_8^* \equiv H \]  \hspace{1cm} (189)

on \( \bar{Q} \). If we identify all the eight points which are taken into each other by this action, we get a space \( Q \) which as we saw in Section 6.3 is the configuration space of the asymptotic rotor.

The group \( D_8^* \) has five inequivalent UIR’s. Four of these are abelian and may be described as follows. In one, the trivial one, all elements of \( D_8^* \) are represented by the unit operator. In one of the remaining three, \( \pm 1 \) and \( \pm i\tau_1 \) are represented by +1 while \( \pm i\tau_2 \) and \( \pm i\tau_3 \) are represented by −1. The two remaining one dimensional UIR’s are constructed similarly, \( \pm 1 \) and \( \pm i\tau_2 \) being represented by +1 in one and \( \pm 1 \) and \( \pm i\tau_3 \) being represented by +1 in the other. As regards the two dimensional UIR, it is the defining representation (6.44) involving Pauli matrices.

There are thus five ways of quantizing this system. We now concentrate on the quantization method involving the two dimensional nonabelian UIR of \( D_8^* \).

A basis for all functions on \( SU(2) \) are the matrix elements \( D_{\rho\sigma}^j(s) [s \in SU(2)] \) of the rotation matrices. The group \( D_8^* = \{ h \} \) acts by operators \( \hat{h} \) on these functions according to the rule

\[ (\hat{h}D_{\rho\sigma}^j)(s) = D_{\rho\sigma}^j(sh). \]  \hspace{1cm} (190)

Since

\[ D_{\rho\sigma}^j(sh) = D_{\rho\lambda}^j(s)D_{\lambda\sigma}^j(h) \]  \hspace{1cm} (191)

and since for integer \( j \), \( h \to D^j(h) \) for \( h \in D_8^* \) defines an abelian representation of \( D_8^* \), we can and shall restrict \( j \) to half odd integer values.

The next step is to reduce the representation \( h \to D^j(h) \) into its irreducible components. It then splits into a direct sum of the two dimensional UIR’s (6.44). [Only the two dimensional UIR’s occur in this reduction. This is because the image of \((i\tau_i)^2 \) being a \( 2\pi \) rotation is represented by \(-1\), \( j \) being half an odd integer.] The basis vectors for the vector spaces which
carry such UIR’s are of the form \( e^j_{\rho,m,a} \), \( m = 1, 2, \ldots, N \); \( a = 1, 2 \) where \( 2N \) equals \( 2j + 1 \). Under the transformations \( s \rightarrow sh \), their behavior is given by

\[
e^j_{\rho,m,a}(sh) = e^j_{\rho,m,b}(s)h_{ba}.
\]

The vector space which carries the algebra of observables irreducibly is spanned by \( e^j_{\rho,m,a_0} \) with one fixed value \( a_0 \) and with \( j, \rho, \) and \( m \) taking on all allowed values. The vectors \( e^j_{\rho,m,a'} \) with the remaining values \( a' \) for \( a \) are to be discarded.

When the asymmetric rotor model is used to describe nuclei, \( m \) can be interpreted in terms of the third component of angular momentum in the body fixed frame.

We have not discussed a scalar product for this vector space. A suitable scalar product may be

\[
(\alpha, \beta) = \int_{SU(2)} d\mu(s) \alpha^*(s)\beta(s).
\]

Here we have regarded the elements of our vector space as functions on \( SU(2) \) and \( d\mu(s) \) is the invariant measure on \( SU(2) \).

In the preceding discussion, we have not referred to a Lagrangian or a Hamiltonian. They are of course important from a dynamical point of view. They do now however play a critical role in the construction of the vector space for wave functions that we have outlined because this construction is valid for a large class of Lagrangians and Hamiltonians.

**VII. TOPOLOGICAL SPIN-STATISTICS THEOREMS**

In nonrelativistic quantum mechanics or relativistic quantum field theory (RQFT) in three or more (spatial) dimensions, one encounters two sorts of particles or localized solitonic excitations. One of these is characterized by tensorial states, which are invariant under \( 2\pi \) rotation, and the other by spinorial states, which change sign under this rotation. If we limit ourselves to Bose and Fermi systems, the spin-statistics correlation in three or more dimensions amounts to the assertion that the former are bosons and the latter are fermions. Thus according to this assertion, the change in the phase of a state under the exchange of two identical systems of spin \( S \) is \( \exp[i2\pi S] \).
In two dimensions, there are more general possibilities for spin and statistics such as fractional spin and fractional statistics. But here as well, the above correlation asserts that the exchange operation is associated with the phase $\exp[i2\pi S]$ for a spin $S$ “anyon” subject to fractional statistics. [It may be emphasized here however that the notions of spin and statistics are more fragile in two dimensions. There the assignment of a well-defined statistics ceases to make sense when generic, velocity-dependent forces (“magnetic fields”) are present; and spin is subject to a similar loss of meaning. In such situations, the spin-statistics correlation is vacuous and our discussion will not apply.]

There are different sorts of proofs of this correlation currently available in the literature. One class of proofs typically uses RQFT in one of its formulations such as the one initiated by Wightman, or the algebraic formulation of quantum field theory. In the Wightman framework, for example, it is shown that tensorial fields commute and spinorial ones anticommute for space like separations, and this result is interpreted as a proof of the spin-statistics connection. A second approach to the spin-statistics theorem due to Finkelstein and Rubinstein applies to solitons or “kinks”. It is a “topological” proof which does not use the heavy machinery of RQFT. It examines the fundamental group $\pi_1(Q)$ of the configuration space $Q$ appropriate for solitons and shows that $2\pi$ rotation of a soliton and exchange of two identical solitons are the same element of $\pi_1(Q)$. This proof in particular does not use relativistic invariance, but does use the facts that solitons are continuous structures in field theories and that each soliton has its antisoliton.

The spin-statistics theorem is pertinent in disciplines such as atomic physics where relativity or field theory does not play a significant role. It is therefore desirable to prove it for point particles in a topological manner that would dispense with these assumptions. We may also hope that such a proof would make sense for topological geons in quantum gravity, where again the assumptions of flat space quantum field theory are too restrictive. Indeed there are reasons to hope that, once we see how such a derivation would go, we will have an important clue to the dynamical rules governing the change of spacetime topology. A derivation of this sort will be outlined in this Section.

References 8 and 9 can be consulted for citations to the literature on topological spin-statistics theorems, including those discussed here.

The existence of an antiparticle is an indispensable ingredient in the topo-
logical proofs for solitons, and will be so here as well. The concept of antiparticle in this context can be associated with any state which on suitable pairing with a particle state acquires the quantum numbers of the ground state. The proof below is thus applicable to condensed matter systems with particle-hole excitations. There are however many situations in low energy physics where even such antiparticles are not available. Electron pair production energies being several orders of magnitude larger than typical energies in atomic physics for example, the spin-statistics connection hence seems to provide us an example where a high energy result has a profound influence on low energy physics.

For purposes of simplicity, we shall assume here that the particle and antiparticle are distinct when they have spin, although this assumption can be dispensed with. We will not use such an assumption here when the particles are spinless.

We may at this juncture point out an important implication of the topological spin-statistics theorems for particles moving in $\mathbb{R}^d$ ($d \geq 2$): They exclude “nonabelian” statistics. Thus according to these theorems, paraparticles of order 2 and more for $d \geq 3$, and particles associated with non-abelian braid group representations for $d = 2$, could not exist in nature.

Let us first outline the proof for spinless particles with distinct antiparticles. As discussed in Section 6.3, in one conventional approach to statistics in particle mechanics, the configuration space $Q_M$ for $M$ identical spinless particles in $\mathbb{R}^d$ ($d \geq 2$) is

$$Q_M = \left\{ \begin{bmatrix} x^{(1)}, x^{(2)}, \ldots, x^{(M)} \end{bmatrix} \mid x^{(i)} \in \mathbb{R}^d; x^{(i)} \neq x^{(j)} \text{ if } i \neq j; \right\}.$$  \hspace{1cm} (194)

The configuration space $\bar{Q}_N$ for $N$ spinless antiparticles is obtained from (7.1) by replacing $M$ by $N$ and $x^{(i)}$’s by $\bar{x}^{(i)}$’s. Next consider the Cartesian product

$$Q_{M,N} = Q_M \times \bar{Q}_N, \quad M, N \geq 1.$$  \hspace{1cm} (195)
Define also

\[ Q_{M,0} = Q_M, \quad M \geq 1, \]  

\[ Q_{0,N} = Q_N, \quad N \geq 1, \]  

and introduce the vacuum ("VAC") by setting

\[ Q_{0,0} = \{VAC\}. \]  

The final configuration space \( C_K \) is obtained by imposing an equivalence relation \( \sim \) on the disjoint union

\[ \bigoplus_{K, \text{fixed}} Q_{N+K,N} \]  

which makes creation and annihilation processes possible. According to this relation, a particle and an antiparticle at the same location "annihilate" to VAC, and conversely they emerge from VAC by separating from an identical location. This is illustrated in Fig. 1 and can also be expressed in equations as follows:

\[ ([x]; [\bar{x}]) \sim \text{VAC} \text{ if } x = \bar{x}, \]

\[ ([x^{(1)}, ..., x^{(i)}, ..., x^{(N+K)}]; [\bar{x}^{(1)}, ..., \bar{x}^{(j)}, ..., \bar{x}^{(N)}]) \]

\[ \sim ([x^{(1)}, ..., x^{(i)}, ..., x^{(N+K)}]; [\bar{x}^{(1)}, ..., \bar{x}^{(j)}, ..., \bar{x}^{(N)}]) \]

if \( x^{(i)} = \bar{x}^{(j)}. \)

Here the underlined entries are to be deleted and equations such as \( ([x^{(1)}, x^{(2)}, x^{(3)}]; [\bar{x}]) = [x^{(1)}, x^{(2)}] \) are to be understood. \( C_K \) is the quotient of (7.5) by this equivalence relation. Elements of \( C_K \) which are equivalence classes containing points such as \( ([x^{(1)}, ..., x^{(N+K)}]; [\bar{x}^{(1)}, ..., \bar{x}^{(N)}]) \) will be denoted by \( [x^{(1)}, ..., x^{(N+K)}]; [\bar{x}^{(1)}, ..., \bar{x}^{(N)}] \). They fulfill identities which follow from (7.6). The significance of \( K \) is that the particle number (= number of particles-number of antiparticles) for a point of \( C_K \) is \( K \). Note that \( C_K \) is infinite dimensional and not a manifold.
The spin-statistics connection for spinless particles reduces to the statement that the particles are bosons. To establish this we will show that the exchange operation is associated with the trivial element of \( \pi_1(C_K) \). That this topological triviality of exchange does in fact entail Bose statistics in the ordinary sense is not something we will prove here, plausible though it is. Chapter 6 can be consulted regarding this point.

The result that particle interchange is trivial in \( \pi_1(C_K) \) will be shown in \( C_2 \) adopting the following conventions, the proof for any \( C_K \) being similar. The homotopy parameter \( t \) will increase upwards in the figures, their horizontal sections being \( \mathbb{R}^d \). Following Feynman, a “particle travelling backward in \( t \)” will be used to represent an antiparticle. We will sometimes refer to \( t \) as time. The base point for homotopy will correspond to two particles located say on the 1-axis.

The curve for exchange is Fig. 2(a) whereas the trivial curve describing static particles is Fig. 2(g). Figures (a-g) show how to deform the first to the last of these figures thereby demonstrating the theorem. Exchanges being the identity of \( \pi_1(C_K) \), nonabelian statistics are also excluded. Note that \( p_1 \) and \( p_2 \) \([q_1 \text{ and } q_2]\) are VAC, and superposing them as in the passage from Fig. 2(b) to 2(c) \([2(e) \text{ to } 2(f)]\) is a legitimate activity.

If the particle and its antiparticle are not distinct, the configuration space is

\[
D_K = \bigcup_{M=K \mod 2} \left\{ \begin{bmatrix} x^{(1)}, x^{(2)}, \ldots, x^{(M)} \\ x^{(1)}, x^{(i)}, \ldots, x^{(j)}, \ldots, x^{(M)} \end{bmatrix} \right. \\
= \left. \begin{bmatrix} x^{(1)}, \ldots, x^{(i)}, \ldots, x^{(j)}, \ldots, x^{(M)} \end{bmatrix} \right. ; \\
= \left. \begin{bmatrix} x^{(1)}, \ldots, x^{(i)}, \ldots, x^{(j)}, \ldots, x^{(M)} \end{bmatrix} \right. \\
\text{if } x^{(i)} = x^{(j)}. 
\]

Here \( K \) is either 0 or 1, underlined entries are as usual to be deleted and we employed the convention that \([x^{(1)}, x^{(2)}, \ldots, x^{(M)}] := \text{VAC} \) when \( M = 0 \). The spin-statistics connection and the exclusion of nonabelian statistics can be

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proved for $D_K$ exactly as before.

We now turn to particles with spin. We will in a well-known way account for spin by attaching a frame to each particle. [The physical origin of these frames is documented further in the second paper of ref. 8.] Let $\mathcal{F}^d$ be the set of all frames, orthonormal with respect to the Euclidean metric on $\mathbb{R}^d$ and with a fixed orientation. The generalization $Q_{M}^{\text{SPIN}}$ of $Q_{M}$ to spinning particles is then

$$Q_{M}^{\text{SPIN}} = \{[(x^{(1)}, F^{(1)}), \ldots, (x^{(M)}, F^{(M)})]\}$$

(201)

where $x^{(i)} \in \mathbb{R}^d$, $F^{(i)} \in \mathcal{F}^d$, the elements of $Q_{M}^{\text{SPIN}}$ are invariant under permutations of the $(x^{(i)}, F^{(i)})$ and we require $x^{(i)} \neq x^{(j)}$ if $i \neq j$. The antiparticle space $\bar{Q}_{N}^{\text{SPIN}}$ which generalizes $\bar{Q}_{N}$ is similarly obtained. Its elements are denoted by $[[(\bar{x}^{(1)}, \bar{F}^{(1)}), \ldots, (\bar{x}^{(N)}, \bar{F}^{(N)})]]$, $\bar{F}^{(i)} \in \mathcal{F}^d$ where now $\mathcal{F}^d$ is the set of orthonormal frames oppositely oriented to elements of $\mathcal{F}^d$. Such an orientation reversal is suggested by the fact that the CP or CPT transform of a left handed particle is a right handed antiparticle. It is also suggested by the Finkelstein-Rubinstein work. The particle and antiparticle are distinct since $\mathcal{F}^d \neq \bar{F}^d$.

Our final spinning particle configuration space $C_{K}^{\text{SPIN}}$ for particle number $K$ is obtained from the disjoint union

$$\bigoplus_{K \text{ Fixed}} \bigoplus_{N \geq 0} Q_{N+K,N}^{\text{SPIN}}$$

(202)

by specifying a condition which makes annihilation and creation possible. [The definition of $Q_{M,N}^{\text{SPIN}}$ is essentially analogous to the definition of $Q_{M,N}$. See the second or third paper of ref. 8 for a more precise treatment.] For this purpose, consider for simplicity a particle $i$ and an antiparticle $j$ moving towards each other along a straight line $L$ and colliding at $\xi$. Let $\mathcal{P}$ be the plane through $\xi$ normal to $L$. Our central assumption is that $i$ and $j$ annihilate at $\xi$ if and only if the antiparticle frame $\bar{F}^{(j)}$ approaches the reflection of the particle frame $F^{(i)}$ in $\mathcal{P}$. There is a similar rule for pair production. These assumptions are shown in Fig. 3 for $d = 2$. [The axes of the particle (antiparticle) frames are drawn in figures with single (double) lines]. They imply equations such as
\[
\lim_{x(i),\bar{x}(j) \to \xi} [(x^{(i)}, F^{(i)}); (\bar{x}^{(j)}, \bar{F}^{(j)})] = VAC;
\]
\[
\lim_{x(i),\bar{x}(j) \to \xi} [(x^{(1)}, F^{(1)}), ..., (x^{(i)}, F^{(i)}), ..., (\bar{x}^{(1)}, \bar{F}^{(1)}), ..., (\bar{x}^{(j)}, \bar{F}^{(j)}), ...]
= [(x^{(1)}, F^{(1)}), ..., (x^{(i)}, F^{(i)}), ..., (\bar{x}^{(1)}, \bar{F}^{(1)}), ..., (\bar{x}^{(j)}, \bar{F}^{(j)}), ...]
\]
(203)

where the limit is taken with \(x^{(i)}\) and \(\bar{x}^{(j)}\) approaching \(\xi\) along \(L\) and the antiparticle frame approaching the appropriate reflection of the particle frame (explained above) in the limit. The rest of the new notation follows the earlier one.

The exchange diagram Fig. 2(a) is as before homotopic to Fig. 2(e) where now an appropriate frame is supposed to be attached to each point of these figures. We now show that the left hand side Fig. 4(a) of Fig. 2(e) is homotopic to Fig. 4(b,c) where the frame of the outgoing particle undergoes \(2\pi\) rotation as \(t\) evolves, thereby showing the theorem.

The homotopy of Fig. 4(b) to Fig. 4(c) is obtained by coalescing \(C\) and \(D\). We must thus prove the homotopy of Fig. 4(a) and Fig. 4(b). For this purpose, it is convenient to assume that the particles and antiparticle in these pictures are moving along the 1-axis except within the dashed circle when the particle created by pair production takes a little excursion in the 1-2 plane and then returns to the 1-axis.

The process in Fig. 4(a) is redrawn in Fig. 5, which shows only the first two axes of the frames. At times \(t < t_1\), a particle, call it 1, is moving to the right on 1-axis. A pair is produced at \(t = t_1\), with the particle 2 of the pair to the left of antiparticle \(\bar{2}\). As \(t\) evolves, 1 and \(\bar{2}\) annihilate at \(t = t_2\) while 2 moves to the left on 1-axis, makes a detour in the 1-2 plane and then returns to the 1-axis. Fig. 4(b) is likewise redrawn in Fig. 6. Note that the alignment of the frames in Figs. 5 and 6 is consistent with (7.10).

A comparison of these figures shows that the left-right order of the \(2 - \bar{2}\) pair at the moment of production is reversed in going from Fig. 5 to Fig. 6. The homotopy of Fig. 5 to Fig. 6 thus involves gradually changing the production angle \(\theta\) of 2 from \(\pi\) as in Fig. 5 to zero as in Fig. 6. [We assume that 2 is produced in the 1-2 plane in the successive stages of the homotopy.] Fig. 7 shows the frame of 2 as \(\theta\) is so changed, the \(\bar{2}\) frame being held fixed. Clearly, because of the mirror rule involved in (7.10), the frame of 2 rotates
by $2(\theta_1 - \pi)$ when $\theta$ decreases from $\pi$ to $\theta_1$. This means that when Fig. 5 is deformed so that $\theta$ becomes $\theta_1$, the frame of 2 will rotate by $2(\pi - \theta_1)$ before 2 reaches its final destination. This is shown in Fig. 8. This rotation being $2\pi$ for $\theta = 0$, the homotopy of Figs. 4(a,b) is thus established.

Nonabelian statistics can be shown to be excluded by a simple extension of the preceding arguments. Thus consider $M$ particles in $C^{SPIN}_K$ say. By the above, the exchange $\sigma_{ij}$ of particles $i$ and $j$ is equal to $2\pi$ rotation $R^{(i)}_{2\pi}$ of the frame $i$. Repeating the argument, we have further $R^{(i)}_{2\pi} = \sigma_{1i} = R^{(1)}_{2\pi}$ whence all exchanges and all rotations are homotopic to each other. This shows that all exchanges commute thereby establishing the result.

In a more complete treatment, we must define suitable topologies for $C_K$ and $C^{SPIN}_K$ and derive equations like (7.10) as consequences of these topologies. This task is carried out in the second paper of ref. 8.

There are several physical systems of interest other than point particles in $\mathbb{R}^d$ to which the techniques outlined here can be extended. It has been shown elsewhere for example [2] that there exist exotic possibilities for the statistics of strings in $\mathbb{R}^3$ if antistrings are ignored. [These strings can be vortex rings in $He^4$ or strings produced in GUT’s during phase transitions.] A spin-statistics theorem for these strings as well has been proved in ref. 9 by including antistrings and creation-annihilation processes, and it will rule out these exotic statistics.

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