Littelmann’s path crystal and combinatorics of certain integrable $\widehat{sl}_{\ell+1}$ modules of level zero

by

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1. Introduction
Throughout the introduction, the base field is assumed to be $\mathbb{C}$.

1.1. The aim of the present paper is to construct a subcrystal of Littelmann’s path crystal, whose formal character coincides with that of a certain simple integrable module of level zero over the untwisted affine algebra associated with $\mathfrak{sl}_{t+1}$, and to study the decomposition of the tensor product of that crystal with a highest weight crystal.

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra with a Cartan subalgebra $\mathfrak{h}$ and let $\pi \subset \mathfrak{h}^\ast$ be a set of simple roots of $\mathfrak{g}$. If $\alpha \in \pi$, denote the corresponding simple coroot by $\alpha^\vee$ and let $x_{\pm \alpha} \in \mathfrak{g}_{\pm \alpha} \setminus \{0\}$ be the corresponding Chevalley generators of $\mathfrak{g}$. Fix a weight lattice $P(\pi)$ of $\mathfrak{g}$ and let $P^+(\pi) = \{ \lambda \in P(\pi) : \alpha^\vee(\lambda) \geq 0 \}$ be the set of dominant weights. A $\mathfrak{g}$ module $M$ is called integrable if $M$ is a direct sum of its weight spaces $M_\nu$, $\nu \in P(\pi)$ and the $x_{\pm \alpha}$ act locally nilpotently on $M$ for all $\alpha \in \pi$. One can also define, in a similar way, a notion of an integrable module over the quantized enveloping algebra $U_q(\mathfrak{g})$ associated with $\mathfrak{g}$.

If $\dim M_\nu < \infty$ for all $\nu \in P(\pi)$, we call $M$ admissible and define its formal character by

$$\text{ch } M = \sum_{\nu \in P(\pi)} (\dim M_\nu) e^\nu.$$

Given $\lambda \in P^+(\pi)$, denote by $V(\lambda)$ (respectively, $V(-\lambda)$) the unique, up to an isomorphism, highest (respectively, lowest) weight simple integrable module over $\mathfrak{g}$ or over $U_q(\mathfrak{g})$. Its character is given by the famous Weyl-Kac formula (cf. [13, Chap. 10]) and, moreover, determines $V(\lambda)$ up to an isomorphism. Another important property of $V(\lambda)$ is that it admits a crystal basis and a canonical basis (cf. [9, 21, 15]).

1.2. Littelmann’s path model provides a combinatorial realisation of the crystal basis of $V(\lambda)$, which reflects the above properties of that module. Namely, let $\mathbb{P}$ be the set of all piecewise-linear continuous paths $b : [0, 1] \rightarrow \mathbb{R}P(\pi)$ such that $b(0) = 0$ and $b(1) \in P(\pi)$, where one identifies $b$ and $b'$ if $b = b'$ up to a reparametrization. After Littelmann [18, 19], one can endow $\mathbb{P}$ with a structure of a normal crystal, which will be henceforth referred to as Littelmann’s path crystal, by defining crystal operators $e_\alpha, f_\alpha$ for all $\alpha \in \pi$. Given a subcrystal $B$ of $\mathbb{P}$, define its formal character by

$$\text{ch } B = \sum_{b \in B} e^{b(1)}.$$

Let $\mathcal{A}$ be the associative monoid generated by the $e_\alpha, f_\alpha : \alpha \in \pi$. If $\lambda \in P^+(\pi)$ and $b_\lambda \in \mathbb{P}$ is a linear path connecting the origin with $\lambda$, then the formal character of the subcrystal $B(\lambda) = \mathcal{A}b_\lambda$ of $\mathbb{P}$ coincides with that of $V(\lambda)$ ([13]). Moreover, $B(\lambda)$ provides a combinatorial model for the crystal basis of $V(\lambda)$ and allows one to construct a standard monomial basis of $V(\lambda)$ ([20]).

One of the fundamental properties of $B(\lambda)$ is its independence of the choice of $b_\lambda$. Namely, let $b$ be a path in $\mathbb{P}$, whose image lies in the dominant Weyl chamber, that
is $\alpha^\vee(b(\tau)) \geq 0$ for all $\tau \in [0, 1]$. If $b' \in \mathbb{P}$ is another such a path then, by the Isomorphism Theorem of Littelmann ([19, Theorem 7.1]), $Ab$ is isomorphic to $Ab'$ if and only if $b(1) = b'(1)$. In particular, if the image of $b$ lies in the dominant Weyl chamber and $b(1) = \lambda$, then $Ab$ is isomorphic to $B(\lambda)$. Thus, similarly to $V(\lambda)$, $B(\lambda)$ is uniquely determined, up to an isomorphism, by its formal character.

1.3. If $\mathfrak{g}$ is not finite dimensional, there might exist simple admissible integrable modules which are neither highest nor lowest weight. The interest in this class of modules is due to the observation that they occur as submodules in $\mathfrak{g}$ modules $\text{Hom}(V(\lambda), V(\mu))$, $\lambda, \mu \in P^+(\pi)$ (see for example [11, 5.12] or [12, 3.1]). Namely, if $V$ is a simple admissible integrable $\mathfrak{g}$ module, denote by $V^\# = \bigoplus_{\nu \in P(\pi)} V^\nu_\nu \subset V^*$ its graded dual. Then $V^\#$ is also simple, admissible and integrable and $\text{Hom}_{\mathfrak{g}}(V^\#, \text{Hom}(V(\lambda), V(\mu)))$ is isomorphic to the subspace $V^\mu_{\lambda-\mu} = \{v \in V_{\lambda-\mu} : x^\alpha v = 0\}$.

In particular, if $\lambda = \mu$, then $V$ must have a non trivial weight subspace of weight zero, which cannot occur in the highest or lowest weight case. The embeddings of $V^\#$ into $\text{End} V(\lambda)$ play a crucial role in the construction of KPRV determinants in the affine case (cf. [11, 22]). Thus one would like to be able to describe the subspaces $V^\mu_0$ or, more generally, $V^\mu_{\lambda-\mu}$. That problem is rather difficult for modules, but is likely to simplify significantly if one is able to pass to crystals.

1.4. Suppose now that $\mathfrak{g}$ is a Kac-Moody algebra of an untwisted affine type and denote by $\hat{\mathfrak{g}}$ its underlying finite dimensional simple Lie algebra. By [13, Theorem 7.4], $\mathfrak{g}$ can be constructed from $\hat{\mathfrak{g}}$ in the following way. Given a vector space $V$, set $L(V) := V \otimes \mathbb{C}[z, z^{-1}]$. Then $\mathfrak{g}$ is the universal central extension of the semi-direct sum of $L(\hat{\mathfrak{g}})$ and the one-dimensional space spanned by the Euler operator $D = z \frac{d}{dz}$. Let $K$ be a central element of $\mathfrak{g}$. Then $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = L(\hat{\mathfrak{g}}) \oplus \mathbb{C} K$. A $\mathfrak{g}$ module $M$ is said to be of level zero if $K$ acts trivially on $M$. One can easily see that a highest or lowest weight module of level zero is necessarily one-dimensional.

In the affine case, simple admissible integrable modules were classified by V. Chari and A. Pressley ([24, 25, 26]). Moreover, the modules of that type which are neither highest nor lowest weight can be constructed as follows (cf. [24]). For any $a := (a_1, \ldots, a_m)$, $a_i \in \mathbb{C}$, define a homomorphism of Lie algebras $\text{ev}_a : \mathfrak{g}' \rightarrow \hat{\mathfrak{g}}^{\otimes m}$ by $\text{ev}_a(K) = 0$ and $\text{ev}_a(x \otimes z^k) = (a_1^k x, \ldots, a_m^k x)$, for all $x \in \hat{\mathfrak{g}}$, $k \in \mathbb{Z}$. Let $V = (V_1, \ldots, V_m)$ be a collection of finite-dimensional simple $\hat{\mathfrak{g}}$ modules. Then $V_1 \otimes \cdots \otimes V_m$ is a simple $\hat{\mathfrak{g}}^{\otimes m}$ module and we can endow it with a structure of a $\mathfrak{g}'$ module by taking the pull-back by the homomorphism $\text{ev}_a$. The resulting module is simple provided that all the $a_j$ are distinct.

Furthermore, the loop space $L(V_1 \otimes \cdots \otimes V_m)$ becomes a $\mathfrak{g}$ module, which we denote by $L(V, a)$, if we set

$$(x \otimes z^k)(v \otimes z^n) = (\text{ev}_a(x \otimes z^k))(v) \otimes z^{k+n}, \quad D(v \otimes z^n) = nv \otimes z^n,$$
for all $x \in \mathfrak{g}$, $v \in V_1 \otimes \cdots \otimes V_m$, $k, n \in \mathbb{Z}$. If all the $a_j$ are distinct, $L(V, a)$ is said to be generic and is completely reducible. Simple submodules of modules of that type exhausts all simple admissible integrable $\mathfrak{g}$ modules which are neither highest nor lowest weight modules.

Following [11, 7.2], we call these modules bounded since their weights satisfy the following condition. By [13], there exists a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ which is positive semidefinite on the root lattice and may be assumed to be rational-valued on $P(\pi)$. A module $M = \bigoplus_{\nu \in P(\pi)} M_\nu$ is called bounded if $(\nu, \nu) \leq n$ for some $n \in \mathbb{N}$ fixed and for all $\nu \in P(\pi)$ such that $M_\nu$ is non-trivial. If $M$ is simple then the bound is actually attained ([11, 7.2]) that is, there exists a weight $\lambda \in P(\pi)$ of $M$ called maximal such that $(\nu, \nu) \leq (\lambda, \lambda)$ for all weights $\nu$ of $M$. For example, $V(\lambda)$ is always bounded and $\lambda$ is its maximal weight by [13, Proposition 11.4]. One can show that a simple integrable module is admissible if and only if it is bounded (cf. [10]).

Formal characters of simple generic bounded modules were computed in [3, 4]. It turns out that, unlike the modules $V(\lambda)$, these modules are not in general determined by their formal characters up to an isomorphism. Besides, their construction arises from the realisation of $\mathfrak{g}$ as a central extension of a loop algebra, which is peculiar to Kac-Moody algebras of affine type. Thus, one should not expect that a combinatorial model similar to that of Littelmann for $V(\lambda)$ exists for an arbitrary simple admissible integrable module of level zero.

1.5. Suppose now that $a_i = \zeta^i$ where $\zeta$ is an $m$th primitive root of unity and that $V_1 \cong \cdots \cong V_m \cong V$. Then $L(V, a)$ becomes a direct sum of simple components $L(V, m)^r$, $r = 0, \ldots, m - 1$, where $L(V, m)^r$ is a cyclic submodule generated by $v \otimes m \otimes z^r$ and $v$ is a highest weight vector of $V$. The interest of this particular case is due to the fact that the $L(V, m)^r$ are determined by their formal characters up to an isomorphism.

In the present paper we consider the case of $\mathfrak{g} \cong \mathfrak{sl}_{\ell+1}$ and $V$ isomorphic to the natural representation $\mathbb{C}^{\ell+1}$ of $\mathfrak{g}$. Henceforth we will denote the corresponding modules $L(V, m)^n$ by $L(\ell, m; n)$. We show that $L(\ell, m; n)$ does admit a combinatorial model in the framework of Littelmann’s path crystal. Namely, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\omega$ be the highest weight of $V$ with respect to $\mathfrak{g}$. Extend $\omega$ to the Cartan subalgebra $\mathfrak{h} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$ of $\mathfrak{g}$ by $\omega(K) = \omega(D) = 0$. Furthermore, let $\delta \in \mathfrak{h}^*$ be the unique element defined by the conditions $\delta(D) = 1$, $\delta|_{\mathfrak{h} \oplus \mathbb{C}K} = 0$. Then $m\omega + n\delta$ is a maximal weight of $L(\ell, m; n)$. Needless to say, $m\omega + n\delta$ does not lie in the dominant Weyl chamber. Our main result is the following

**Theorem.** Let $p_{\ell,m,n}$ be the linear path in $\mathbb{P}$ connecting the origin to $m\omega + n\delta$. Then the formal character of the subcrystal $\hat{B}(m)^n = \mathcal{A}p_{\ell,m,n}$ of $\mathbb{P}$ equals the formal character of $L(\ell, m; n)$.

As a byproduct, we obtain (Lemma [A.2]) a nice combinatorial interpretation of the dimensions of weight spaces of $L(\ell, m; n)$. A similar result holds for $L(V^*, m)^n$, which is isomorphic to $L(\ell, m; m - n)^\#$. 
A natural question is how the module $L(\ell, m; n)$ is related to the crystal $\widehat{B}(m)^n$, apart from the equality of their formal characters. It is shown in [3] that $L(\ell, m; n)$ has a quantum analogue which, in turn, admits a pseudo-crystal basis. The crystal $\widehat{B}(m)^n$ provides a combinatorial model for that basis (cf. [3, 4.8–4.10]).

1.6. As a first application of the above result, we consider the decomposition of $B(\lambda) \otimes \widehat{B}(m)$ where $\widehat{B}(m) = \bigsqcup_{m=0}^{n-1} \widehat{B}(m)^n$ and the tensor product is understood as concatenation of paths. We obtain the following

**Theorem.** Let $\lambda \in P(\pi)$ be a dominant weight which is not a multiple of $\delta$ and let $\widehat{B}(m)\,^\lambda$ be the set of paths $b \in \widehat{B}(m)$ satisfying $\alpha^\vee(\lambda + b(\tau)) \geq 0$ for all $\alpha \in \pi$ and for all $\tau \in [0, 1]$. Then the decomposition of the tensor product of $B(\lambda)$ and $\widehat{B}(m)$ is given by

$$B(\lambda) \otimes \widehat{B}(m) \xrightarrow{\sim} \prod_{b \in \widehat{B}(m)\,^\lambda} B(\lambda + b(1)).$$

We also obtain (Proposition 5.7) an explicit description of $\widehat{B}(m)\,^\lambda$ for the case when $\lambda$ is a fundamental weight.

The above decomposition should be compared with the Decomposition rule (cf. [18, 19]), which generalizes the Littlewood-Richardson rule, and with [8, Theorem 3.1]. The main difference with the latter is that the crystal involved in our situation is not finite. Besides, we consider an entirely different framework, namely that of Littelmann’s path crystal, and our proof is not based on the theory of perfect crystals. On the other hand, unlike that of the Decomposition rule of [18, 19], the meaning of our decomposition for modules is not yet understood. We expect, however, that it will allow one to extract some information about embeddings of $L(\ell, m; n)$ or its graded dual into $\text{Hom}(V(\lambda), V(\mu))$, $\lambda, \mu \in P^+(\pi)$ discussed in 1.3.

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2. Preliminaries

In this section we recall the definition and some basic properties of crystals and fix the notations which will be used throughout the rest of the paper. Henceforth, $\mathbb{N}$ stands for the set of non-negative integers and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. The cardinality of a finite set $S$ will be denoted by $\#S$.

2.1. Let $I$ be a finite index set and let $A = (a_{ij})_{i,j \in I}$ be a generalised Cartan matrix, that is, $a_{ii} = 2$, $a_{ij} \in -\mathbb{N}$ if $i \neq j$ and $a_{ij} = 0$ if and only if $a_{ji} = 0$. We will assume that $A$ is symmetrisable, that is, there exist $d_i : i \in I$ such that the matrix $(d_i a_{ij})_{i,j \in I}$ is symmetric.

Consider a triple $(\mathfrak{h}, \pi, \pi^\vee)$, where $\mathfrak{h}$ is a $\mathbb{Q}$-vector space, $\pi = \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ and $\pi^\vee = \{\alpha^\vee_i\}_{i \in I}$ is a linearly independent subset of $\mathfrak{h}$. We would like to emphasize that $\pi$ is not assumed to be linearly independent. We call such a triple a realisation of $A$ if $a_{ij} = \alpha^\vee_i(\alpha_j)$, for all $i, j \in I$. The realisation becomes unique, up to an isomorphism, if we require both sets $\pi$ and $\pi^\vee$ to be linearly independent and $\dim \mathfrak{h} = 2\#I - \text{rk} A$ (cf. [13, Chap. 1]).

Given $A$ and its realisation $(\mathfrak{h}, \pi, \pi^\vee)$, fix $\Lambda_i \in \mathfrak{h}^*$, $i \in I$ such that $\alpha^\vee_i(\Lambda_j) = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker’s symbol. Set $P_0(\pi) = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i$. Complete the set $\{\Lambda_i : i \in I\}$ to a basis of $\mathfrak{h}^*$, and let $P(\pi)$ be the free abelian group generated by that basis.

Endow $\mathbb{Z} \cup \{-\infty\}$ with a structure of an ordered semi-group such that $-\infty$ is the smallest element, $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$ and $\mathbb{Z}$ is given its natural order.

**Definition** (cf. [13, Definition 1.2.1] or [9, 5.2.1]). A crystal $B$ is a set endowed with the maps $e_i, f_i : B \to B \cup \{0\}$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$, $\text{wt} : B \to P(\pi)$, for all $i \in I$ which satisfy the following rules

(C1) For any $b \in B$, $\varphi_i(b) = \varepsilon_i(b) + \alpha^\vee_i(\text{wt} b)$, for all $i \in I$.

(C2) If $b \in B$ and $e_ib \in B$ (respectively, $f_ib \in B$), then $\text{wt} e_ib = \text{wt} b + \alpha_i$, $\varepsilon_i(e_ib) = \varepsilon_i(b) - 1$ and $\varphi_i(e_ib) = \varphi_i(b) + 1$ (respectively, $\text{wt} f_ib = \text{wt} b - \alpha_i$, $\varepsilon_i(f_ib) = \varepsilon_i(b) + 1$ and $\varphi_i(f_ib) = \varphi_i(b) - 1$).

(C3) For $b, b' \in B$ and $i \in I$, $b' = e_ib$ if and only if $b = f_ib'$.

(C4) If $\varphi_i(b) = -\infty$, then $e_ib = f_ib = 0$.

Given $b \in B$, the value of $\text{wt} b$ is called the weight of $b$.

A crystal is said to be upper (respectively, lower) normal if $\varepsilon_i(b) = \max\{n : e^n_ib \neq 0\}$ (respectively, $\varphi_i(b) = \max\{n : f^n_ib \neq 0\}$). A crystal is normal if it is both upper and lower normal.

2.2. Let $B$ a crystal. For any $\lambda \in P(\pi)$, set $B_\lambda = \{b \in B : \text{wt} b = \lambda\}$. If $\#B_\lambda < \infty$ for all $\lambda \in P(\pi)$, one can define a formal character of $B$ as

$$
\text{ch} B = \sum_{b \in B} e^{\text{wt} b} = \sum_{\lambda \in P(\pi)} \#B_\lambda e^\lambda.
$$
We say that $\lambda \in P(\pi)$ is a weight of $B$ if $B_\lambda$ is non-empty. Denote by $\Omega(B)$ the set of all weights of $B$.

2.3. Let $B_1, \ldots, B_n$ be crystals. The set $B_1 \times \cdots \times B_n$ can be endowed with a structure of a crystal which will be denoted by $B_1 \otimes \cdots \otimes B_n$ and called the tensor product of crystals $B_1, \ldots, B_n$. The crystal maps are defined as follows (cf. \[15, Proposition 1.3\]).

Given $b = b_1 \otimes \cdots \otimes b_n \in B_1 \otimes \cdots \otimes B_n$, define the Kashiwara functions $b \mapsto r^i_k(b) : i \in I$, $k \in \{1, \ldots, n\}$ by

$$r^i_k(b) = \varepsilon_i(b_k) - \sum_{1 \leq j < k} \alpha^\vee_i(\text{wt } b_j).$$

Then

(T1) $\varepsilon_i(b) = \max \{r^i_k(b) : 1 \leq k \leq n\}$.

(T2) $\text{wt } b = \sum_k \text{wt } b_k$.

(T3) $e_i b = b_1 \otimes \cdots \otimes b_{r-1} \otimes e_i b_r \otimes b_{r+1} \otimes \cdots \otimes b_n$, where $r = \min \{k : r^i_k(b) = \varepsilon_i(b)\}$, that is, $e_i$ acts in the leftmost place where the maximal value of $r^i_k(b)$ is attained.

(T4) $f_i b = b_1 \otimes \cdots \otimes b_{r-1} \otimes f_i b_r \otimes b_{r+1} \otimes \cdots \otimes b_n$, where $r = \max \{k : r^i_k(b) = \varepsilon_i(b)\}$, that is, $f_i$ acts in the rightmost place where the maximal value of $r^i_k(b)$ is attained.

In the above we identify $b_1 \otimes \cdots \otimes b_{r-1} \otimes 0 \otimes b_{r+1} \otimes \cdots \otimes b_n$ with 0. Since $\varphi_i(b) = \varepsilon_i(b) + \alpha^\vee_i(\text{wt } b)$, these rules take a particularly nice form for the product $B_1 \otimes B_2$, namely

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes e_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases} \quad (2.1)$$

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes f_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases} \quad (2.2)$$

whilst $\varepsilon_i(b_1 \otimes b_2) = \max \{\varepsilon_i(b_1), \varepsilon_i(b_2) - \alpha^\vee_i(\text{wt } b_1)\}$. The tensor product of crystals is associative (cf. \[15, Proposition 1.3.1\]) and a tensor product of normal crystals is also normal (cf. for example \[3, Lemma 5.2.6\]).

2.4. Let $A$ be the associative monoid generated by the $e_i, f_i : i \in I$. We say that a crystal $B$ is generated by $b \in B$ over a submonoid $A'$ of $A$ if $B = A'b := \{fb : f \in A'\} \setminus \{0\}$. If $B$ is generated by $b$ over $A$ we will say that $B$ is generated by $b$.

Let $B$ be a crystal. An element $b \in B$ is said to be of a highest (respectively, lowest) weight $\lambda \in P(\pi)$ if $\text{wt } b = \lambda$ and $e_i b = 0$ (respectively, $f_i b = 0$) for all $i \in I$. Let $E$ (respectively, $F$) be the submonoid of $A$ generated by the $e_i$ (respectively, by the $f_i$), $i \in I$. We call $B$ a highest (respectively, lowest) weight crystal of highest (respectively, lowest) weight $\lambda$ if there exists an element $b_\lambda$ of highest (respectively, lowest) weight $\lambda$ such that $B = Fb_\lambda$ (respectively, $B = Eb_\lambda$).
Lemma. Let $B$ be a normal crystal and assume that there exists $b_0 \in B$ such that $b_0^{\otimes m} \otimes b \in \mathcal{F}b_0^{\otimes m+1}$, for all $b \in B$ and for all $m \geq 0$. Then $B^{\otimes m}$ is generated by $b_0^{\otimes m}$ over $\mathcal{F}$ for all $m > 0$. Similarly, if there exists $b_0 \in B$ such that $b \otimes b_0^{\otimes m} \in \mathcal{E}b_0^{\otimes m}$ for all $b \in B$ and for all $m \geq 0$ then $B^{\otimes m}$ is generated by $b_0^{\otimes m}$ over $\mathcal{E}$.

Proof. The proof is by induction on $m$. The induction base is given by the assumption. Suppose that $m > 1$ and assume that some $b' \in B^{\otimes m-1}$ satisfies $b' \otimes b'' \in \mathcal{F}b_0^{\otimes m}$ for all $b'' \in B$. We claim that $f_i b' \otimes b'' \in \mathcal{F}b_0^{\otimes m}$ for all $b'' \in B$ and for all $i \in I$ such that $f_i b' \neq 0$. Indeed, $\varphi_i(b') > 0$ since $f_i b' \in B^{\otimes m-1}$ and $B$ is normal. If $b'' \in B$ satisfies $\varepsilon_i(b'') < \varphi_i(b')$, then $f_i b' \otimes b'' = f_i (b' \otimes b'') \in \mathcal{F}b_0^{\otimes m}$ by (2.2). Otherwise set $k = \varepsilon_i(b'') - \varphi_i(b') + 1$. Then $0 < k \leq \varepsilon_i(b'')$, whence $b'' := e_i^{k} b'' \in B$ by normality of $B$. It follows that $b' \otimes b'' \in \mathcal{F}b_0^{\otimes m}$.

On the other hand $\varepsilon_i(b'') = \varepsilon_i(b'') - k = \varphi_i(b') - 1 < \varphi_i(b')$, whence $f_i^{k+1}(b' \otimes b'') = f_i^{k} (f_i b' \otimes b'') = f_i b' \otimes f_i b'' = f_i b' \otimes b''$ by (2.2). Therefore, $f_i b' \otimes b'' \in \mathcal{F}b_0^{\otimes m}$.

Furthermore, $b_0^{\otimes m-1} \otimes b \in \mathcal{F}b_0^{\otimes m}$ for all $b \in B$ by assumption. Then it follows from the claim by induction on $k$ that $f_i \cdots f_k b_0^{\otimes m-1} \otimes b \in \mathcal{F}b_0^{\otimes m}$ for all $b \in B$ provided that $f_i \cdots f_k b_0^{\otimes m-1} \neq 0$. The assertion follows since $B^{\otimes m-1} = \mathcal{F}b_0^{\otimes m-1}$ by the induction hypothesis.

Similarly, for the second part it is enough to prove that, for $b' \in B^{\otimes m-1}$ fixed, $b' \otimes b \in \mathcal{E}b_0^{\otimes m}$ for all $b'' \in B$ implies that $b'' \otimes e_i b' \in \mathcal{E}b_0^{\otimes m}$ for all $b'' \in B$ and for all $i \in I$ such that $e_i b' \neq 0$ (or, equivalently, $\varepsilon_i(b') > 0$). For, observe that, for $b'' \in B$ such that $\varphi_i(b'') < \varepsilon_i(b')$, $b'' \otimes e_i b' = e_i (b'' \otimes b') \in \mathcal{E}b_0^{\otimes m}$ by (2.1). Furthermore, assume that $\varphi_i(b'') \geq \varepsilon_i(b')$ and set $n = \varphi_i(b'') - \varepsilon_i(b') + 1$. Then $0 < n \leq \varphi_i(b'')$, whence $b'' = f_i^n b'' \in B$ and $\varphi_i(b'') = \varphi_i(b'') - n = \varepsilon_i(b') - 1 < \varepsilon_i(b')$. Then, by (2.3), $e_i^{n+1} (b'' \otimes b') = e_i^n (b'' \otimes e_i b') = e_i^n b'' \otimes e_i b' = b' \otimes e_i b'$, and so $b'' \otimes e_i b' \in \mathcal{E}b_0^{\otimes m}$.

2.5. A morphism of crystals $\psi$ (cf. [13, 1.2.1]) is a map $\psi : B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ such that $\psi(0) = 0$ and, for all $i \in I$,

(M1) If $b \in B_1$ and $\psi(b) \in B_2$ then $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, wt $\psi(b) = \mathrm{wt} b$.
(M2) For all $b \in B_1$, $\psi(e_i b) = e_i \psi(b)$ provided that $\psi(e_i b), \psi(b) \in B_2$.
(M3) For all $b \in B_1$, $\psi(f_i b) = f_i \psi(b)$, provided that $\psi(f_i b), \psi(b) \in B_2$.

A morphism is said to be strict if it commutes with the $e_i, f_i : i \in I$. Evidently, any morphism of normal crystals is strict ([13, Lemma 1.2.3]). Throughout the rest of the paper, all morphisms of crystals will be assumed to be strict.

Let $B'$ and $B$ be crystals. We say that $B'$ is a subcrystal of $B$ if there is an injective morphism of crystals from $B'$ to $B$. In particular, a subset $B' \subset B$ will be called a subcrystal of $B$ if $B'$ is a crystal with respect to the operations $e_i, f_i, \varepsilon_i, \varphi : i \in I$ and wt of $B$ restricted to $B'$. A crystal is said to be indecomposable if it does not admit a non-empty subcrystal different from itself.
Observe that a crystal $B$ is indecomposable if and only if it is generated by an element $b \in B$. Indeed, if $B$ is indecomposable then for any $b \in B$, $B' = \mathcal{A}b \supset b$ is a non-empty subcrystal of $B$, hence coincides with $B$. On the other hand, suppose that $B = \mathcal{A}b$ for some $b \in B$ and that $B' \not\subset B$ is a non-empty subcrystal of $B$. Then $b \notin B'$ for otherwise $B = \mathcal{A}b \subset B'$. On the other hand, for any $b' \in B'$, there exists a monomial $f \in \mathcal{A}$ such that $b = fb'$. It follows from $[2.2]$ (C1) that there exists a monomial $f' \in \mathcal{A}$ such that $b = f'b'$. Since $B'$ is a crystal, we conclude that $b \in B'$, which is a contradiction. In particular, it follows that if $B$ admits a decomposition as a disjoint union of finitely many indecomposable crystals, then such a decomposition is unique up to a permutation of the components.

3. The crystal $\hat{B}_\ell(m)$ and its combinatorics

3.1. Set $I = \{0, \ldots, \ell\}$. Henceforth we identify $I$ with $\mathbb{Z}/(\ell + 1)\mathbb{Z}$ in the sense that $i + k, i \in I$, $k \in \mathbb{Z}$ is understood as $i + k \pmod{\ell + 1}$. Let $A = (a_{ij})_{i,j \in I}$ be the Cartan matrix of the affine Lie algebra $\mathfrak{g} = \mathfrak{sl}_{\ell+1}$. Explicitly, $a_{ij} = 2\delta_{i,j} - \delta_{i,j-1} - \delta_{i,j+1}$, $i,j \in I$ (for example, $A = (\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix})$ for $\ell = 1$). We will use two different realisations of $A$. The first one is the realisation in the sense of $[13]$, Chap. 1, that is, we consider a triple $(\mathfrak{h}, \pi, \pi^\vee)$ where $\dim_{\mathbb{Q}} \mathfrak{h} = \ell + 2$. Observe that $\delta = \alpha_0 + \cdots + \alpha_\ell \in \mathfrak{h}^*$ satisfies $\alpha_i^\vee(\delta) = 0$ for all $i \in I$. Fix $\Lambda_i \in \mathfrak{h}^*$ as in $[2.1]$. Then $\Lambda_0, \ldots, \Lambda_\ell, \delta$ form a basis of $\mathfrak{h}$. Throughout the rest of the paper we take $P(\pi) = P_0(\pi) \oplus \mathbb{Z}\delta$. The corresponding crystals will be called affine.

The other realisation is obtained by replacing $\mathfrak{h}$ by $\mathfrak{h}' = \mathbb{Q}\pi^\vee$. Then $\mathfrak{h}^* = \mathbb{Q}P_0(\pi) \cong \mathfrak{h}^*/\mathbb{Q}\delta$. We will use the same notations for the elements of $\pi$ and $\pi^\vee$ in both realisations. The corresponding crystals will be referred to as finite. The image of the weight map for finite crystals is contained in $P_0(\pi) \cong P(\pi)/\mathbb{Z}\delta$.

3.2. The finite crystal $B_\ell$ is a set indexed by $I$. The elements of $B_\ell$ will be denoted by $b_i : i \in I$. The crystal operators $e_i, f_i, \varepsilon_i, \text{wt}$ on $B_\ell$ are defined by the following formulae:

$$
e_i b_j = \delta_{i,j}b_{j-1}, \quad f_i b_j = \delta_{i-1,j}b_i,$$

$$\varepsilon_i (b_j) = \delta_{i,j}, \quad \text{wt } b_j = \Lambda_1 - \Lambda_0 - \sum_{1 \leq k < j} \alpha_k = -\Lambda_j + \Lambda_{j+1}.$$

One can easily check that $B_\ell$ is a normal crystal. Moreover, if we consider $B_\ell$ as a crystal with respect to the operations $e_i, f_i, \varepsilon_i : i \in I \setminus \{0\}$ and the realisation $(\mathfrak{h}_0, \pi_0, \pi_0^\vee)$ of $A_0 = (a_{ij})_{i,j \in I \setminus \{0\}}$, where $\pi_0 = \pi \setminus \{\alpha_0\}$, $\pi_0^\vee = \pi \setminus \{\alpha_0^\vee\}$ and $\mathfrak{h}_0 = \mathbb{Q}\pi_0^\vee$, then $B_\ell$ can be realised as a crystal basis of the natural representation $\mathbb{C}^{\ell+1}$ of the finite dimensional simple Lie algebra $\mathfrak{sl}_{\ell+1}$ (cf. $[7, 22]$).

For any $m \in \mathbb{N}^+$, consider the crystal $B_\ell(m) := B_\ell \otimes m$, the operations being defined as in $[2.3]$. Then $B_\ell(m)$ is normal as a tensor product of normal crystals.

**Lemma.** The crystal $B_\ell(m) : m > 0$ is neither a highest weight nor a lowest weight crystal.
Proof. Suppose that there exists $b \in B_\ell(m)$ such that $\varepsilon_i(b) = 0$ for all $i \in I$. Then, by normality, $\varepsilon_i(b) = 0$ for all $i \in I$. Indeed, suppose that $f_i b = 0$ for all $i \in I$. Then, by normality, $\varepsilon_i(b) = 0$ for all $i \in I$. We claim that $\sum_{i \in I} \alpha_i^\vee(\text{wt } b) = 0$ for all $b \in B_\ell(m)$. Indeed, for all $j \in I$ one has $\sum_{i \in I} \alpha_i^\vee(\text{wt } b_j) = \sum_{i \in I} (\delta_{i,j+1} - \delta_{i,j}) = \sum_{i \in I} (\delta_{i-1,j} - \delta_{i,j}) = 0$. The claim now follows by $\sum_{i \in I} \varepsilon_i(b) = 0$ by 2.3. Then $\sum_{i \in I} \varepsilon_i(b) = 0$ by 2.4 (T). Yet $\varepsilon_i(b) \geq 0$ by normality of $B_\ell(m)$ and so $\varepsilon_i(b) = 0$ for all $i \in I$, which is a contradiction by the first part.

3.3. Even though $B_\ell(m)$ is not a highest weight crystal, it turns out to be generated by its element over $\mathcal{F}$ or $\mathcal{C}$.

Proposition. The crystal $B_\ell(m)$ is generated by $b_0^{\otimes m}$ over $\mathcal{F}$. Furthermore, $B_\ell(m)$ is also generated by $b_0^{\otimes m}$ over $\mathcal{C}$.

Proof. By Lemma 2.4 it suffices to prove that $b_0^{\otimes m-1} \otimes b_i \in \mathcal{F} b_0^{\otimes m}$, for all $i \in I$, $m > 0$. The cases $m = 1$ and $m > 1$, $i = 0$ are trivial. Suppose further that $m > 1$ and $i \neq 0$. Since $\varphi_i(b_0^{\otimes m-1}) = 0 = \varepsilon_i(b_{i-1})$, $i \in I \setminus \{0,1\}$, $f_i(b_0^{\otimes m-1} \otimes b_{i-1}) = b_0^{\otimes m-1} \otimes b_i$ by (2.2), whence $b_0^{\otimes m-1} \otimes b_i = f_i \cdots f_m(b_0^{\otimes m-1} \otimes b_1), i \in I \setminus \{0,1\}$. Thus, it suffices to prove that $b_0^{\otimes m-1} \otimes b_i \in \mathcal{F} b_0^{\otimes m}$. Indeed, observe that $\varphi_i(b_i^{\otimes k}) = k$, whence $f_i b_i^{\otimes k} = b_i^{\otimes k}$ for all $i \in I$, $k > 0$. In particular, $f_i b_i^{\otimes m} = b_i^{\otimes m}$. Furthermore, for all $i \in I \setminus \{1\}$, $\varphi_i(b_i^{\otimes m-1}) = m-1 > \varepsilon_i(b_1) = 0$, whence by (2.2) $f_i^{m-1}(b_i^{\otimes m-1} \otimes b_1) = f_i^{m-1} b_i^{\otimes m-1} \otimes b_1 = b_i^{\otimes m-1} \otimes b_1$. It follows that $b_0^{\otimes m-1} \otimes b_1 = f_0^{m-1} f_2^{m-1} \cdots f_i^{m-1} f_1 b_0^{\otimes m} \in \mathcal{F} b_0^{\otimes m}$ as required.

For the second part, it is sufficient to prove, by Lemma 2.4, that $b_i \otimes b_0^{\otimes m-1} \in \mathcal{C} b_0^{\otimes m-1}$ for all $i \in I$ and $m \geq 1$. Suppose that $m > 1$ and $i \neq 0$, the other cases being obvious. Since $\varepsilon_i(b_i \otimes b_0^{\otimes m-1}) = b_{i-1} \otimes b_0^{\otimes m-1}$, it is sufficient to prove that $b_i \otimes b_0^{\otimes m-1} \in \mathcal{C} b_0^{\otimes m}$ for all $m > 1$. Indeed, observe that, for all $i \in I$ and $k > 0$, $\varepsilon_i(b_i^{\otimes k}) = k$, whence $e_i^{k} b_i^{\otimes k} = b_i^{\otimes k}$. In particular, $b_0^{\otimes m} = b_\ell^{\otimes m}$. Furthermore, for all $i \in I \setminus \{0\}$, $\varphi_i(b_i^{\otimes m-1}) = m-1 > \varepsilon_i(b_1) = 0$, whence by (2.1) $e_i^{m-1}(b_i \otimes b_i^{\otimes m-1}) = b_i \otimes e_i^{m-1} b_i^{\otimes m-1} = b_i \otimes b_i^{\otimes m-1}$. It follows that $b_\ell \otimes b_0^{\otimes m-1} = e_1^{m-1} \cdots e_\ell^{m-1} b_0^{\otimes m} \in \mathcal{C} b_0^{\otimes m}$ as required.

Remark. One can show that $B_\ell$ is a perfect crystal (cf. [14, 4.6]). Then $B_\ell^{\otimes m}$ is indecomposable for all $m > 0$ by [14, Corollary 4.6.3]. However, we need a stronger version of this result, namely that $B_\ell^{\otimes m}$ is generated by some element over $\mathcal{F}$ and not just over $\mathcal{A}$. Besides, our proof does not use the fact that $B_\ell$ is perfect.

3.4. The affine crystal $B_\ell(m)$, which we are about to define, provides the affinisation of $B_\ell(m)$ in the sense of [14, 3.3]. Set $B_\ell(m) = B_\ell(m) \times \mathbb{Z}$ and define the crystal operations as
follows. Denote an element \((b, n) : b \in B_\ell(m), n \in \mathbb{Z}\) by \(b \otimes z^n\). Then \(\text{wt}(b \otimes z^n) = \text{wt}\ b + n\delta \in P(\pi)\), which is compatible with the decomposition \(P(\pi) = B_0(\pi) \oplus \mathbb{Z}\delta\). Furthermore, set

\[
e_i(b \otimes z^n) = \begin{cases} e_ib \otimes z^{n+\delta}, & \text{if } e_i b \in B_\ell(m) \\ 0, & \text{if } e_i b = 0 \end{cases}
\]

\[
f_i(b \otimes z^n) = \begin{cases} f_ib \otimes z^{n-\delta}, & \text{if } f_i b \in B_\ell(m) \\ 0, & \text{if } f_i b = 0 \end{cases}
\]

and \(\varepsilon_i(b \otimes z^n) = \varepsilon_i(b), i \in I\). Evidently, \(\widehat{B_\ell(m)}\) is a normal crystal.

**Proposition.** The crystal \(\widehat{B_\ell(m)}\) is the disjoint union of indecomposable normal subcrystals \(\widehat{B_\ell(m)^n}\), \(n = 0, \ldots, m - 1\), where \(\widehat{B_\ell(m)^k} : k \in \mathbb{Z}\) is the subcrystal of \(\widehat{B_\ell(m)}\) generated by \(b_0^\otimes \otimes z^k\).

The sections 3.5–3.7 are devoted to the prove of the above Proposition.

**3.5.** The \(\widehat{B_\ell(m)^n}\) are indecomposable by 2.5 and normal as subcrystals of a normal crystal. So, it remains to prove that \(\widehat{B_\ell(m)^r} = \widehat{B_\ell(m)^s}\) if \(s = r \pmod m\), \(\widehat{B_\ell(m)^r} \cap \widehat{B_\ell(m)^s} = \emptyset\) otherwise, and that every element of \(\widehat{B_\ell(m)}\) lies in some \(\widehat{B_\ell(m)^k}\).

**Lemma.** The crystal \(\widehat{B_\ell(m)}\) is a union of \(\widehat{B_\ell(m)^n} : n = 0, \ldots, m - 1\).

**Proof.** Let \(b \in B_\ell(m)\). By Proposition 3.3, there exists \(f = f_i \cdots f_k \in \mathcal{F}\) such that \(b = fb_0^\otimes\). Define

\[
n_f(b) := \#\{t : i_t = 0\}.
\]

Then \(b \otimes z^r = f(b_0^\otimes \otimes z^{r + n_f(b)}) \in \widehat{B_\ell(m)^{r + n_f(b)}}\).

Furthermore, an elementary computation shows that

\[
(e_1^m \cdots e_\ell^m e_0^m)^r(b_0^\otimes \otimes z^k) = b_0^\otimes \otimes z^{k + rm},
\]

\[
(f_1^m f_\ell^m \cdots f_1^m)^r(b_0^\otimes \otimes z^k) = b_0^\otimes \otimes z^{k - rm}, \quad r > 0.
\]

It follows immediately that \(\widehat{B_\ell(m)^r} = \widehat{B_\ell(m)^s}\) if \(r = s \pmod m\). \(\Box\)

**3.6.** Notice that \(n_f(b)\) depends on \(f\). For example, one has \(f_0^m f_\ell^m \cdots f_1^m b_0^\otimes = b_0^\otimes\). However, it turns out that the residue class of \(n_f(b)\) modulo \(m\) does not depend on \(f\), which allows one to introduce a function \(N : \widehat{B_\ell(m)} \rightarrow \mathbb{Z}/m\mathbb{Z}\) such that \(N(b) = n\) if and only if \(b \in \widehat{B_\ell(m)^n}\). That function also plays a crucial role in the computation of characters of the indecomposable subcrystals of \(\widehat{B_\ell(m)}\) and in the construction of a subcrystal of Littelmann’s path crystal isomorphic to \(\widehat{B_\ell(m)}\).

Given a product \(u = b_{j_k} \otimes \cdots \otimes b_{j_1}\), define \(t(u) = j_k, h(u) = j_1\) and \(|u| = k\).
**Definition.** Let $b = b_{i_m} \otimes \cdots \otimes b_{i_1}$ be an element of $B_\ell(m)$. Define

$$\text{desc}(b) := \{ r : 1 \leq r < m, \ i_r > i_{r+1} \}. \quad (3.1)$$

Furthermore, set $k = \# \text{desc}(b) + 1$ and write $\text{desc}(b) = \{ n_1 < \cdots < n_{k-1} \}$ if $k > 1$. Set $n_0 = 0$, $n_k = |b| = m$ and define

$$\text{\tilde{desc}(b)} := \{ n_0, \ldots, n_k \}$$

$$N(b) := \sum_{r=1}^{k} r(n_r - n_{r-1}).$$

**Proposition.** Let $b \in B_\ell(m)$. Then

1. $N(e_i b) = N(b) - \delta_{i,0} \pmod{m}$ provided that $e_i b \in B_\ell(m)$.
2. $N(f_i b) = N(b) + \delta_{i,0} \pmod{m}$ provided that $f_i b \in B_\ell(m)$.

**Proof.** Observe that the second statement follows from the first. Indeed, if $b' = f_i b \in B$, then $b = e_i b'$ by $2.1$ (T), whence $N(f_i b) = N(b') = N(e_i b') + \delta_{i,0} \pmod{m} = N(b) + \delta_{i,0} \pmod{m}$.

Suppose that $b = b'' \otimes b_i \otimes b'$ for some $b'$, $b''$, possibly empty. We claim that if $e_i b = b'' \otimes b_{i-1} \otimes b'$ then $h(b'') \neq i - 1$ and $t(b') \neq i$. Indeed, suppose that $b'' = b'' \otimes b_{i-1}$ and $e_i b = b'' \otimes b_{i-1} \otimes b'$. Then by $2.3$ (T), we must have, in particular, $\varepsilon_i(b_{i-1}) - \alpha_i^{\gamma}(\text{wt } b'') < \varepsilon_i(b_{i}) - \alpha_i^{\gamma}(\text{wt } b'' \otimes b_{i-1})$. That inequality reduces to $0 < 1 - \alpha_i^{\gamma}(\text{wt } b_{i-1}) = 1 - \alpha_i^{\gamma}(\Lambda_i - \Lambda_{i-1}) = 0$, which is a contradiction. Similarly, if $t(b') = i$, that is $b' = b_i \otimes b''$, and $e_i b = b'' \otimes b_{i-1} \otimes b_i \otimes b''$ then we have, by $2.3$ (T), $\varepsilon_i(b_{i}) - \alpha_i^{\gamma}(\text{wt } b'') \geq \varepsilon_i(b_{i}) - \alpha_i^{\gamma}(\text{wt } b'') + 1$, which is absurd.

Suppose that $e_i b \neq 0$. Since $e_i b_j = 0$ if $j \neq i$, $b = b'' \otimes b_i \otimes b'$ for some $b'$, $b''$ such that $e_i b = b'' \otimes e_i b_i \otimes b' = b'' \otimes b_{i-1} \otimes b'$. First, consider the case $i \neq 0$. Since $h(b'') \neq i - 1$ and $t(b') \neq i$ by the above, we conclude that desc$(b) = \text{desc}(e_i b)$ whence $N(e_i b) = N(b)$.

Suppose now that $i = 0$ and retain the notations from the above definition.

1. Assume first that $b = b_0 \otimes b'$ and $e_0 b = b_0 \otimes b'$. By the above claim, $t(b') \neq 0$. Then $n_{k-1} = m - 1$ and desc$(e_0 b) = \{ n_0, \ldots, n_{k-2}, n_k \}$. Therefore,

$$N(e_0 b) = \sum_{r=1}^{k-2} r(n_r - n_{r-1}) + (k - 1)(n_k - n_{k-2}) = \sum_{r=1}^{k-1} r(n_r - n_{r-1}) + k - 1 = \sum_{r=1}^{k-1} r(n_r - n_{r-1}) + k(n_k - n_{k-1}) - 1 = N(b) - 1.$$

2. Suppose that $b = b'' \otimes b_0 \otimes b'$, where $|b'|, |b''| > 0$, and $e_0 b = b'' \otimes b_0 \otimes b'$. Since $t(b') \neq 0$ by the above claim, $|b'| \in \text{desc}(b)$. Suppose that $n_s = |b'|$ in our notations for the elements of desc$(b)$. On the other hand, $h(b'') < \ell$ by the claim we proved above. Since $\ell \geq t(b')$, it
follows that \( \text{desc}(e_0b) = \{n_0, \ldots, n_{s-1}, n_s + 1, n_{s+1}, \ldots, n_k\} \), whence

\[
N(e_0b) = \sum_{r=1}^{s-1} r(n_r - n_{r-1}) + s(n_s - n_{s-1} + 1) + (s + 1)(n_{s+1} - n_s - 1) + \sum_{r=s+1}^{k} r(n_r - n_{r-1})
\]

\[
= \sum_{r=1}^{k} r(n_r - n_{r-1}) - 1 = N(b) - 1.
\]

3. Finally, assume that \( b = b'' \otimes b_0 \) and \( e_0b = b'' \otimes b_\ell \). Evidently, \( 1 \notin \text{desc}(b) \). On the other hand, \( h(b'') < \ell \) by our claim, whence \( \text{desc}(e_0b) = \{n_0, 1, \ldots, n_k\} \). Therefore,

\[
N(e_0b) = 1 + 2(n_1 - 1) + \sum_{r=2}^{k} (r + 1)(n_r - n_{r-1}) = -1 + \sum_{r=1}^{k} (r + 1)(n_r - n_{r-1})
\]

\[
= -1 + N(b) + \sum_{r=1}^{k} (n_r - n_{r-1}) = N(b) + m - 1 = N(b) - 1 \pmod{m}.
\]

Corollary. Let \( b \) be an element of \( B_\ell(m) \) and let \( f \in F \) be a monomial such that \( b = fb_0^{\otimes m} \). Then \( n_f(b) = N(b) \pmod{m} \). In particular, the residue class of \( n_f(b) \mod{m} \) does not depend on \( f \).

Proof. It suffices to observe that \( N(b_0^{\otimes m}) = m = 0 \pmod{m} \).

3.7. Now we are able to complete the proof of Proposition 3.4.

Proof. By Lemma 3.5, \( \hat{B}_r(m) \) is a union of \( \hat{B}_r(m)^n : n = 0, \ldots, m - 1 \). It only remains to prove that \( \hat{B}_r(m)^r \cap \hat{B}_r(m)^s \) is empty if \( r \neq s \pmod{m} \). Given \( b = b' \otimes z^k, b' \in \hat{B}_r(m), k \in \mathbb{Z} \), set \( N(b) = N(b') + k \) and define

\[
C_n := \{b \in \hat{B}_r(m) : N(b) = n \pmod{m}\}.
\]

Evidently, \( C_r \cap C_s \) is empty if \( r \neq s \pmod{m} \). The idea is to prove that \( C_n = \hat{B}_r(m)^n \).

First, let us prove that \( C_n \) is a subcrystal of \( \hat{B}_r(m) \). Indeed, let \( b = b' \otimes z^k \in C_n \) and suppose that \( e_i b \neq 0 \). Then \( e_i b = e_i b' \otimes z^{k+\delta_{i,0}} \) and \( e_i b' \neq 0 \). It follows from Proposition 3.4 that \( N(e_i b') = N(b') - \delta_{i,0} \pmod{m} \). Therefore, \( N(e_i b) = N(e_i b') + k + \delta_{i,0} = N(b) \pmod{m} \), hence \( e_i b \in C_n \). Similarly, if \( f_j b \neq 0 \), then \( f_j b' \neq 0 \) and \( N(f_j b') = N(b') + \delta_{i,0} \pmod{m} \), whence \( N(f_j b) = N(f_j b') + k - \delta_{i,0} = N(b) \pmod{m} \).

Furthermore, \( N(b_0^{\otimes m}) = 0 \pmod{m} \) hence \( C_n \) contains \( b_0^{\otimes m} \otimes z^n \). By the proof of Lemma 3.3, the \( b_0^{\otimes m} \otimes z^{n+rm} \) lie in the subcrystal of \( \hat{B}_r(m) \) generated by \( b_0^{\otimes m} \otimes z^n \) for all \( r \in \mathbb{Z} \) which is contained in \( C_n \). Take \( b = b' \otimes z^k \in C_n \) and let \( f \in F \) be a monomial such that \( b' = fb_0^{\otimes m} \). Then \( b \otimes z^k = f(b_0^{\otimes m} \otimes z^{k+n_f(b')}) \). On the other hand, \( k + n_f(b') = k + N(b') \pmod{m} = N(b) \pmod{m} = n \) by Corollary 3.4. Thus, \( C_n \) is generated by \( b_0^{\otimes m} \otimes z^n \),
hence is indecomposable by 2.3, and contains $\hat{B}_\ell(m)^n$ as a subcrystal by the definition of the latter.

\[\text{Corollary.} \quad \text{The indecomposable subcrystals of } \hat{B}_\ell(m) \text{ are given explicitly as}\]

$$\hat{B}_\ell(m)^n = \{ b \otimes z^k : b \in B_\ell(m), k \in \mathbb{Z}, N(b) = n - k \pmod{m} \}, \quad n = 0, \ldots, m - 1.$$  

\[\text{Remark.} \quad \text{The decomposition of } \hat{B}_\ell(m) \text{ of Proposition 3.4 appears in [22, Corollary 6.25] in the case } \ell = 1. \text{ However, our proof for arbitrary } \ell \text{ does not use the theory of perfect crystals, yields an efficient explicit description of the indecomposable subcrystals and allows one to compute their formal characters.}\]

\[\text{3.8.} \quad \text{Our present aim is to compute the formal character of } \hat{B}_\ell(m)^n. \text{ For, we calculate first the cardinalities of the sets } B_\ell(m)_\nu^n := \{ b \in B_\ell(m) : \text{wt } b = \nu, N(b) = n \pmod{m} \}, \text{ where } \nu \in P_0(\pi) \text{ and } n = 0, \ldots, m - 1. \text{ Let } b = b_{i_m} \otimes \cdots \otimes b_{i_1} \text{ be an element of } B_\ell(m) \text{ and set } k_i = \# \{ r : i_r = i \}. \text{ Then } \text{wt} b = \sum_{i \in I} k_i \text{wt} b_i = \sum_{i \in I} k_i (\Lambda_{i+1} - \Lambda_i). \text{ On the other hand, the numbers } k_i : i \in I \text{ are uniquely determined by } \text{wt } b \text{ and } m. \text{ Indeed, write } \text{wt } b = \sum_{i \in I} k_i \text{wt } b_i. \text{ Then } \alpha^\nu_\ell(\text{wt } b) = k_{i-1} - k_i, i \in I \setminus \{ 0 \}, \text{ whence } k_i = k_\ell + \sum_{j > i} \alpha^\nu_\ell(\text{wt } b), i \in I \setminus \{ \ell \}. \text{ Yet } \sum_{i \in I} k_i = m, \text{ whence } (\ell+1)k_\ell = m - (\alpha^\nu_1(\text{wt } b) + 2\alpha^\nu_2(\text{wt } b) + \cdots + \ell\alpha^\nu_\ell(\text{wt } b)).\]

Thus, there is a bijection between the set of weights of $B_\ell(m)$ and the set $\{(k_0, \ldots, k_\ell) \in \mathbb{N}^{\ell+1} : \sum_{i \in I} k_i = m \}$. We will identify a weight $\nu$ of $B_\ell(m)$ with the tuple $(k_0, \ldots, k_\ell)$. It follows immediately that $\#B_\ell(m)_\nu = \binom{m}{k_0, \ldots, k_\ell}$. Indeed, the multinomial coefficient $\binom{m}{k_0, \ldots, k_\ell}$ gives the number of distinct permutations the word $0^{k_0} \cdots \ell^{k_\ell}$. By the above there is a bijection between this set and the set of all elements of $B_\ell(m)$ of weight $\nu = (k_0, \ldots, k_\ell)$.

\[\text{Proposition.} \quad \text{Let } \nu = (k_0, \ldots, k_\ell) \text{ be a weight of } B_\ell(m). \text{ Then}\]

$$\#B_\ell(m)_\nu^n := \frac{1}{m} \sum_{d | \gcd(k_0, \ldots, k_\ell)} \varphi_d(n) \binom{m}{k_0/d, \ldots, k_\ell/d}, \quad \varphi_d(n) = \frac{\mu(d/\gcd(d, r))}{\varphi(d/\gcd(d, r))},$$

where $\varphi$ is the Euler function and $\mu$ is the Möbius function, $\mu(k) = 0$ if $k$ is divisible by a square and $\mu(k) = (-1)^\nu$ if $k$ is a product of $r$ distinct primes.

The proof of this proposition is not based on the theory of crystals, and for that reason is given in the Appendix.

\[\text{3.9.} \quad \text{Retain the notations of 1.3.}\]

\[\text{Theorem.} \quad \text{The formal character of } \hat{B}_\ell(m)^n, n = 0, \ldots, m - 1 \text{ equals that of the simple integrable module } L(\ell, m; n) \text{ described in 1.3.}\]

\[\text{Proof.} \quad \text{Evidently, } \Omega(\hat{B}_\ell(m)^n) \subset \{ \nu + k\delta : \nu \in \Omega(B_\ell(m)), k \in \mathbb{Z} \}. \text{ Let } \nu = \sum_{i=0}^\ell k_i(\Lambda_{i+1} - \Lambda_i) \text{ be a weight of } B_\ell(m). \text{ By Corollary 3.7, } \hat{B}_\ell(m)^n_{\nu+k\delta} = \{ b \otimes z^k : b \in B_\ell(m)_\nu, N(b) = n - k \}.$$


by Proposition 3.8. On the other hand, \( \nu + k \delta \) is a weight of \( L(\ell, m; n) \) and the dimension of the corresponding weight space equals the right hand side of the above expression by [7, Theorem 4.4].

4. Littelmann’s path crystal and \( \hat{B}_\ell(m) \)

4.1. Let us briefly recall the definition of Littelmann’s path crystal ([18, 19]).

Fix a realisation \((\mathfrak{h}, \pi, \pi^\vee)\) of a symmetrizable Cartan matrix \( A \) and denote by \([a, b]\) the set \( \{ \tau \in \mathbb{Q} : a \leq \tau \leq b \} \). Let \( \mathbb{P} \) be the set of piecewise-linear continuous paths \( b : [0, 1] \rightarrow \mathfrak{h} \) such that \( b(0) = 0 \) and \( b(1) \in P(\pi) \). Two paths \( b_1, b_2 \) are considered to be identical if there exists a piecewise-linear, nondecreasing, surjective continuous map \( \varphi : [0, 1] \rightarrow [0, 1] \) such that \( b_1 = b_2 \circ \varphi \).

One can endow \( \mathbb{P} \) with a structure of a normal crystal in the following way. For all \( i \in I \), define the Littelmann function \( h_i^b : [0, 1] \rightarrow \mathbb{Q} \), \( h_i^b(\tau) = -\alpha_i^\vee(b(\tau)) \) and set \( \varepsilon_i(b) = \max\{h_i^b(\tau) \cap \mathbb{Z} : \tau \in [0, 1]\} \). Furthermore, define \( e_i^+(b) = \min\{\tau \in [0, 1] : h_i^b(\tau) = \varepsilon_i(b)\} \). If \( e_i^+(b) = 0 \), define \( e_i b = 0 \). Otherwise let \( e_i^-(b) = \max\{\tau \in [0, e_i^+(b)] : h_i^b(\tau) = \varepsilon_i(b) - 1\} \) and define

\[
(e_i b)(\tau) := \begin{cases} b(\tau), & \tau \in [0, e_i^+(b)] \\ s_i(b(\tau) - b(e_i^-(b))) + b(e_i^-(b)) & \tau \in [e_i^-(b), e_i^+(b)] \\ b(\tau) + \alpha_i, & \tau \in [e_i^+(b), 1], \end{cases}
\]

where \( s_i \) is the simple reflection corresponding to \( \alpha_i \), \( s_i \lambda = \lambda - \alpha_i^\vee(\lambda) \alpha_i \) for all \( \lambda \in \mathfrak{h}^* \) and \( s_i(b(\tau)) \) is taken point-wise. In particular, \( s_i(b(\tau)) = b(\tau) + h_i^b(\tau) \alpha_i \).

Similarly, define \( f_i^+(b) = \max\{\tau \in [0, 1] : h_i^b(\tau) = \varepsilon_i(b)\} \). If \( f_i^+(b) = 1 \), set \( f_i b = 0 \). Otherwise, let \( f_i^-(b) = \min\{\tau \in [f_i^+(b), 1] : h_i^b(\tau) = \varepsilon_i(b) - 1\} \) and define

\[
(f_i b)(\tau) := \begin{cases} b(\tau), & \tau \in [0, f_i^+(b)] \\ s_i(b(\tau) - b(f_i^+(b))) + b(f_i^+(b)) & \tau \in [f_i^+(b), f_i^-(b)] \\ b(\tau) - \alpha_i, & \tau \in [f_i^-(b), 1]. \end{cases}
\]

Finally, set \( \text{wt} b(\tau) = b(1) \).

Remark. We use the definition of crystal operations on \( \mathbb{P} \) given in [9, 6.4.4] which differs by the sign of \( h_i^b \) from the original definition of [18, 1.2]. That choice is more convenient for the proof of Proposition 4.3.
A path $b \in \mathbb{P}$ is said to have the integrality property (cf. [13, 2.6]) if the maximal value of $h_i^j(\tau)$ is an integer for all $i \in I$. If that condition holds for every $b \in B$ where $B$ is a subcrystal of $\mathbb{P}$, we say that $B$ has the integrality property.

4.2. For any $b_1, b_2 \in \mathbb{P}$, let $b_1 \ast b_2$ denote their concatenation, that is, a path defined by

$$(b_1 \ast b_2)(\tau) = \begin{cases} b_1(\tau/\sigma), & \tau \in [0, \sigma] \\ b_1(1) + b_2((\tau - \sigma)/(1 - \sigma)), & \tau \in [\sigma, 1], \end{cases}$$

where $\sigma \in (0, 1)$. One may check that the resulting path does not depend on $\sigma$, up to a reparametrisation. Moreover, the concatenation of paths is compatible with the tensor product rules listed in 2.3 (cf. [19, 2.6]). Henceforth we will use the notation $b_1 \otimes b_2$ for the concatenation of $b_1, b_2$.

4.3. Retain the notations of 3.1–3.7. Our present aim is to define an isomorphism between $B_\ell(\mathbb{P})$ and a certain subcrystal of Littelmann’s path crystal $\mathbb{P}$.

Let $\lambda = (\lambda_0, \ldots, \lambda_r)$ be a tuple of elements of $\mathfrak{h}^* = \mathbb{Q}P(\pi)$. We assume that $\lambda_0 = 0$ and $\lambda_r \in P(\pi)$. Given $a = (a_0, \ldots, a_r)$, $a_s \in \mathbb{Q}$ such that $0 = a_0 < a_1 < \cdots < a_r = 1$, define a path $p_{\lambda, a}(\tau) \in \mathbb{P}$ as

$$p_{\lambda, a}(\tau) = \lambda_{r-1} + \left( \frac{\tau - a_{r-1}}{a_r - a_{r-1}} \right) (\lambda_r - \lambda_{r-1}), \quad \tau \in [a_{r-1}, a_r].$$

Evidently $p_{\lambda, a} \in \mathbb{P}$. We shall omit $a$ if $a_s = s/r$, $s = 0, \ldots, r$.

Let $b = b_{i_m} \otimes \cdots \otimes b_{i_1}$ be an element of $B_\ell(\mathbb{P})$. We associate to $b \otimes z^n$ a path $p_{\lambda, \lambda} = (\lambda_0, \ldots, \lambda_m)$, where $\lambda_s = \sum_{t=m-s+1}^m \text{wt } b_i + \kappa_s(b \otimes z^n)\delta$, $s = 0, \ldots, m$ and the $\kappa_s(b \otimes z^n)$ are defined recursively in the following way. Set $\kappa_0(b \otimes z^n) = 0$. Furthermore, write $\text{desc}(b) = \{n_0, \ldots, n_k\}$ as in Definition 3.6 and let $\rho_s(b)$ be the unique $r$, $1 \leq r \leq k$ such that $n_{r-1} < m - s + 1 \leq n_r$. Then

$$\kappa_s(b \otimes z^n) = \kappa_{s-1}(b \otimes z^n) - (\rho_s(b) - 1) + (N(b) + n - m)/m, \quad s = 1, \ldots, m. \quad (4.1)$$

Proposition. The map $\psi : B_\ell(\mathbb{P}) \longrightarrow \mathbb{P}$ given by $b \otimes z^n \longmapsto p_{\lambda}$ with $\lambda$ defined as above is an injective morphism of normal crystals and the image of $\psi$ has the integrality property.

Proof. The injectivity of $\psi$ is obvious. Let $b = b_{i_m} \otimes \cdots \otimes b_{i_1} \in B_\ell(\mathbb{P})$. By 2.3 (11),

$$\varepsilon_i(b) = \max_{1 \leq s \leq m} \left\{ \varepsilon_i(b_s) - \sum_{t > s} \alpha^\vee_t(\text{wt } b_i) \right\} = \max_{1 \leq s \leq m} \left\{ \delta_{i,s} - \alpha^\vee_i(\lambda_{m-s} - \kappa_{m-s}(b \otimes z^n)\delta) \right\}$$

$$= \max_{1 \leq s \leq m} \left\{ \delta_{i,s} - \alpha^\vee_i(\lambda_{m-s}) \right\}.$$
On the other hand,

\[
    h^i_{p\lambda}(\tau) = -\alpha^\vee_i(\lambda_{s-1}) - (m\tau - s + 1)\alpha^\vee_i(wt b_{i_{m-s+1}}) \\
    = -\alpha^\vee_i(\lambda_{s-1}) + (m\tau - s + 1)(\delta_{i_{m-s+1}} - \delta_{i_{m-s+1}-1}), \quad \tau \in [(s-1)/m, s/m].
\]

It follows immediately that \(\max\{h^i_{p\lambda}(\tau) : \tau \in [(s-1)/m, s/m]\} = \delta_{i_{m-s+1}} - \alpha^\vee_i(\lambda_{s-1}) \in \mathbb{Z}\).

Since \(h^i_{p\lambda}\) is linear on the intervals \([(t-1)/m, t/m], t = 1, \ldots, m\) and all the local maxima are integral, we conclude that \(p\lambda\) has the integrality property and that the maximal (integer) value of \(h^i_{p\lambda}\) is attained at \(t/m\) for some \(t\). Therefore,

\[
    \varepsilon_i(p\lambda) = \max_{1 \leq s \leq m} \{\delta_{i_{m-s+1}} - \alpha^\vee_i(\lambda_{s-1})\} = \max_{1 \leq s < m} \{\delta_{i_s} - \alpha^\vee_i(\lambda_{s})\} = \varepsilon_i(b) = \varepsilon_i(b \otimes z^n). \quad (4.2)
\]

Thus, \(\varepsilon_i\) commutes with \(\psi\) for all \(i \in I\). Furthermore, by the definition of \(\kappa_s(b \otimes z^n)\)

\[
    \kappa_m(b \otimes z^n) = -\sum_{r=1}^k (r-1)(n_r - n_{r-1}) + N(b) - m + n = n,
\]

whence \(wt p\lambda = p\lambda(1) = \sum_{s=1}^m wt b_s + n\delta = wt b + n\delta = wt(b \otimes z^n)\). Thus, \(\psi\) commutes with \(wt\) and hence with the \(\varphi_i\) for all \(i \in I\).

Since both \(B_i(m)\) and \(\mathbb{P}\) are normal, it remains to prove that \(\psi\) commutes with the \(e_i, f_i : i \in I\). By \([2.3](C)\) it suffices to prove that \(\psi\) commutes with the \(e_i\). Suppose that \(e_i b \neq 0\). Then \(e_i b = b_m \otimes \cdots \otimes e_i b_{i_s} \otimes \cdots \otimes b_{i_1}\) and \(i_s = i\). By \([2.3](1)\) \(s\) is the largest element of the set \(\{1, \ldots, m\}\) such that \(\varepsilon_i(b_{i_s}) = \sum_{t \geq s} \alpha^\vee_i(wt b_{i_t}) = \varepsilon_i(b)\). It follows from \((4.2)\) and \([1]\) that \(e^+_i(p\lambda) = (m - s + 1)/m\) and \(e^-_i(p\lambda) = (m-s)/m\). Using the definition of \(e_i\) given in \([4.1]\) we obtain

\[
    e_i p\lambda = p\lambda',
\]

where

\[
    \lambda'_t = \begin{cases} 
        \lambda_t, & t = 0, \ldots, m-s, \\
        \lambda_t + \alpha_i, & t = m-s+1, \ldots, m. 
    \end{cases} \quad (4.3)
\]

Let us prove that \(\lambda' = \mu\) where \(p\mu = \psi(e_i b \otimes z^{n+\delta_i,0})\). Indeed, since \(e_i b = b_m \otimes \cdots \otimes b_{i+1} \otimes b_{i-1} \otimes b_{i-1} \otimes \cdots \otimes b_i\) and \(wt b_{i-1} - wt b_i = 2\lambda_i - \Lambda_i - \Lambda_{i+1}\) which equals \(\alpha_i - \delta_{i,0}\delta\) as an element of \(P(\pi)\), it follows from the definition of the \(\lambda_t\) and \((4.3)\) that

\[
    \mu_t = \begin{cases} 
        \lambda'_t + (k'_t - k_t)\delta, & t = 0, \ldots, m-s, \\
        \lambda'_t + (k'_t - k_t - \delta_{i,0})\delta, & t = m-s+1, \ldots, m, 
    \end{cases} \quad (4.4)
\]

where \(k'_t = \kappa_t(e_i(b \otimes z^n)) = \kappa_t(e_i(b \otimes z^{n+\delta_i,0})\) and \(k_t = \kappa_t(b \otimes z^n)\). Denote also \(r_t = \rho_t(b), r'_t = \rho_t(e_i b)\).

Suppose first that \(i \neq 0\). Then \(N(e_i b) = N(b)\) and, as we saw in the proof of Proposition \([3.6](C)\), \(\text{desc}(e_i b) = \text{desc}(b)\), whence \(r'_t = r_t\) for all \(t = 1, \ldots, m\). It follows that \(k'_t = k_t\) for all \(t = 0, \ldots, m\) and so \(\mu = \lambda'\).
Assume that \( i = 0 \). First, if \( s = 1 \), that is \( e_0 b = b_{i_m} \otimes \cdots \otimes e_0 b_{i_1} \), then by the proof of Proposition \([3.6]^{-}\), \( N(e_0 b) = N(b) + m - 1 \) and \( \text{desc}(e_0 b) = \text{desc}(b) \cup \{1\} \). It follows that \( r'_t = r_t + 1, t = 1, \ldots, m - 1 \). Then
\[
k'_t = k'_{t-1} - (r'_t - 1) + (N(e_0 b) - m + n + 1)/m = k'_{t-1} - r_t + (N(b) + n)/m
\]
whence \( k'_t = k_t \) for all \( t \). Furthermore, \( r'_m = 1 = r_m \), whence
\[
k'_m = k'_{m-1} + (N(b) + n)/m = k_{m-1} + (N(b) + n)/m = k_m + 1.
\]
Then \( \mu = \lambda' \) by \([4.4]\).

Finally, assume that \( s > 1 \). Then, by the proof of Proposition \([3.6]\), \( N(e_0 b) = N(b) - 1 \) and \( r'_t = r_t, t \neq m - s + 1 \) whilst \( r'_{m-s+1} = r_{m-s+1} - 1 \). One has
\[
k'_t = k'_{t-1} - (r'_t - 1) + (N(e_0 b) + n + 1 - m)/m
\]
for all \( t = 1, \ldots, m - s, m - s + 2, \ldots, m \). It follows that \( k'_t = k_t \) for all \( t, m - s + 1, \ldots, m \). Therefore, \( \mu = \lambda' \) by \([4.4]\).

\[\Box\]

4.4. Our main result (Theorem \([1.5]\)) follows immediately from Theorem \([3.9]\) and

**Theorem.** The indecomposable crystal \( \widehat{\mathcal{B}_c(m)^n} \) is isomorphic to a subcrystal of the Littelmann’s crystal \( \mathbb{P} \) generated by the path \( p_{\ell,m,n}(\tau) := (m(\Lambda_1 - \Lambda_0) + n\delta)\tau, \tau \in [0,1] \).

**Proof.** Let \( p_{\lambda} \) be the image of \( b_0^{\otimes m} \otimes \mathbb{Z}^n \) under the map constructed in \([4.3]\). Then \( \lambda = (\lambda_0, \ldots, \lambda_m) \), where \( \lambda_t = t(\Lambda_1 - \Lambda_0) + (nt/m)\delta, t = 0, \ldots, m \). Since \( (\lambda_t - \lambda_{t-1})/(a_t - a_{t-1}) = m(\Lambda_1 - \Lambda_0 + (n/m)\delta) = m(\Lambda_1 - \Lambda_0) + n\delta \), it follows that \( p_{\lambda} \) coincides with \( p_{\ell,m,n} \).

On the other hand, \( b_0^{\otimes m} \otimes \mathbb{Z}^n \) generates an indecomposable subcrystal \( \widehat{\mathcal{B}_c(m)^n} \) of \( \widehat{\mathcal{B}_c(m)} \), whence \( p_{\lambda} \) generates an indecomposable subcrystal of \( \mathbb{P} \), which is isomorphic to \( \widehat{\mathcal{B}_c(m)^n} \) by Proposition \([4.3]\). \[\Box\]

5. Decomposition of \( B(\Lambda) \otimes \widehat{\mathcal{B}_c(m)} \)

5.1. Let \( P^+(\pi) = \{ \lambda \in P(\pi) : \alpha_i^\vee(\lambda) \geq 0, \forall i \in I \} \) and \( \mathbb{P}^+ = \{ b \in \mathbb{P} : \alpha_i^\vee(b(\tau)) \geq 0, \forall i \in I, \forall \tau \in [0,1] \} \). Take \( \lambda \in P^+(\pi) \). A path \( b \in \mathbb{P} \) is said to be \( \lambda \)-dominant (cf. \([18]\))
if \( \alpha^\vee_i(\lambda + b(\tau)) \geq 0 \) for all \( \tau \in [0, 1] \) and for all \( i \in I \). The following Lemma is rather standard (cf. for example [19, 6.4.14])

**Lemma.** Let \( \lambda \in P^+(\pi) \) and \( b_\lambda \in \mathbb{P}^+ \) such that \( \text{wt } b_\lambda = \lambda \). Let \( b \in \mathbb{P} \) and suppose that \( b \) has the integrality property. Then the following are equivalent

(i) \( \varepsilon_i(b \otimes b) = 0 \) for all \( i \in I \).
(ii) \( \varepsilon_i(b) \leq \alpha^\vee_i(\lambda) \) for all \( i \in I \).
(iii) \( b \) is \( \lambda \)-dominant.
(iv) \( b_\lambda \otimes b \in \mathbb{P}^+ \).

**Proof.** Suppose that (i) holds. By 2.3 (Theorem 7.1), \( b_\lambda \) remains to prove that if \( \varepsilon_i(b) \leq \alpha^\vee_i(\lambda) \) for all \( i \in I \).

Furthermore, \( b_\lambda \otimes b = b_\lambda(\tau/\sigma) \), \( \tau \in [0, \sigma] \), whence \( \alpha^\vee_i((b_\lambda \otimes b(\tau)) \geq 0 \) for all \( \tau \in [0, \sigma] \). On the other hand, \( b_\lambda \otimes b = b_\lambda(1) + b((\tau - \sigma)/(1 - \sigma)), \tau \in [\sigma, 1] \), whence \( \alpha^\vee_i((b_\lambda \otimes b(\tau)) = \alpha^\vee_i(\lambda) + \alpha^\vee_i(b(\tau')) \geq 0 \), where \( \tau' = (\tau - \sigma)/(1 - \sigma) \in [0, 1] \), provided that \( b \) is \( \lambda \)-dominant. Thus, (iii) leads to (iv). Finally, if \( b_\lambda \otimes b \in \mathbb{P}^+ \), then \(-\alpha^\vee_i((b_\lambda \otimes b(\tau)) \leq 0 \) for all \( i \in I \) and for all \( \tau \in [0, 1] \), whence the maximum of \( h^i_{b_\lambda \otimes b}(0) = 0 \), we conclude that \( \varepsilon_i(b_\lambda \otimes b) = 0 \). Thus, (iv) implies (i). \( \blacksquare \)

5.2. Take \( \lambda \in P^+(\pi) \) and let \( b_\lambda \in \mathbb{P}^+ \) be any path satisfying \( \text{wt } b_\lambda = \lambda \). By the Isomorphism Theorem of Littelmann (cf. [19, Theorem 7.1]), the subcrystal of \( \mathbb{P} \) generated by \( b_\lambda \) is isomorphic to the subcrystal \( B(\lambda) \) of \( \mathbb{P} \) generated by the path \( \tau \mapsto \lambda \tau \). Moreover, by [19, 7, Corollary 1 b]), \( b_\lambda \) is the unique highest weight element of the subcrystal of \( \mathbb{P} \) it generates. Observe that if \( \lambda \in \mathbb{Z}\delta \), then the corresponding crystal \( B(\lambda) \) is trivial.

Let \( B \) be a subcrystal of \( \mathbb{P} \). Given \( \lambda \in P^+(\pi) \setminus \mathbb{Z}\delta \), let \( B^\lambda \) be the set of \( \lambda \)-dominant paths in \( B \).

**Lemma.** Let \( B \) be a subcrystal of \( \mathbb{P} \) and suppose that \( B \) has the integrality property. Then

\[
B(\lambda) \otimes B \longrightarrow \bigcap_{b \in B^\lambda} B(\lambda + \text{wt } b)
\]

and the crystals which appear in the right-hand side are the only highest weight subcrystals of \( B(\lambda) \otimes B \).

**Proof.** Let \( b_\lambda = \lambda \tau, \tau \in [0, 1] \). By Lemma 5.1(i), \( b_\lambda \otimes b \), where \( b \in B^\lambda \), is a highest weight element and, by the Isomorphism Theorem [19, Theorem 7.1] and Lemma 5.1(iv) it generates a highest weight subcrystal of \( \mathbb{P} \) isomorphic to \( B(\mu) \) where \( \mu = (b_\lambda \otimes b)(1) = \lambda + \text{wt } b \). It remains to prove that if \( b' \otimes b \in B(\lambda) \otimes B \) lies entirely in the dominant Weyl chamber then \( b' = b_\lambda \) and \( b \in B^\lambda \). For, assume that \( b' \otimes b \in \mathbb{P}^+ \). Then \( \varepsilon_i(b' \otimes b) = 0 \) by Lemma 5.1. On the other hand, \( \varepsilon_i(b' \otimes b) = \max\{\varepsilon_i(b'), \varepsilon_i(b) - \alpha^\vee_i(\text{wt } b')\} \geq \varepsilon_i(b') \) for all \( i \in I \) by 2.3 (Theorem 7.1).
Therefore, $\varepsilon_i(b') = 0$ for all $i \in I$, whence $b' = b_\lambda$ by [19, 7, Corollary 1 b)]. It remains to apply Lemma 5.1(iii).

---

5.3. Henceforth, let $B$ be the image of $\hat{B}_E(m)$ inside $\mathbb{P}$ under the morphism $\psi$ constructed in 4.3. By Proposition 4.3 $B$ has the integrality property, hence we immediately obtain a surjective morphism of $B(\lambda) \otimes B$ onto a disjoint union of highest weight crystals $B(\mu)$ where $\mu = \lambda + \text{wt } b$ for some $b \in B^\lambda$. Our goal now is to prove that this surjective morphism is actually an isomorphism. By Lemma 5.2, it suffices to prove that $B(\lambda) \otimes B$ is generated by its highest weight elements over $\mathcal{F}$ and that $B^\lambda$ is not empty for all $\lambda \in P^+(\pi) \setminus \mathbb{Z}\delta$.

**Lemma.** Let $b$ be an element of $B_E(m)$. Then

$$
\varepsilon_j(b \otimes b_i) = \begin{cases} 
\varepsilon_j(b) + 1, & \text{if } i = j \text{ and } \varphi_i(b) = 0 \\
\varepsilon_j(b), & \text{otherwise.}
\end{cases}
$$

In particular, if $b \in B_E(m)$ then there exists $i \in I$ such that $\varepsilon_j(b \otimes b_i) = \varepsilon_j(b)$ for all $j \in I$.

**Proof.** By 2.1 (C) and 2.3 (T), $\varepsilon_j(b \otimes b_i) = \max\{\varepsilon_j(b), \varepsilon_j(b_i) - \alpha_j^\vee(\text{wt } b)\} = \varepsilon_j(b) + \max\{0, \delta_{ij} - \varphi_j(b)\}$. The first statement follows immediately since $B_E(m)$ is a normal crystal. The second statement follows from the first and Lemma 3.2.

5.4. Set $B_E(m)^\lambda = \{b \in B_E(m) : \varepsilon_i(b) \leq \alpha_i^\vee(\lambda), \forall i \in I\}$. The next step is to prove that every element of $B_E(m)$ can be transformed into an element of $B_E(m)^\lambda$ provided that the latter is not empty by applying some special monomial $e \in \mathcal{E}$.

**Lemma.** Let $\lambda \in P(\pi)^+ \setminus \mathbb{Z}\delta$ and $b \in B_E(m)$. Then there exist $i_1, \ldots, i_k \in I$ such that $e_{i_k} \cdots e_{i_1} b \in B_E(m)^\lambda$ and $m_r = \varepsilon_{i_r}(u_{r-1}) - \alpha_{i_r}^\vee(\lambda) > 0$, where $u_0 = b$ and $u_r = e_{i_r}^m u_{r-1}$, $r = 1, \ldots, k$.

**Proof.** Write $b = b_{j_1} \otimes \cdots \otimes b_{j_m}$ and choose $1 \leq s \leq m$ maximal such that $b' = b_{j_1} \otimes \cdots \otimes b_{j_s}$ satisfies $\varepsilon_j(b') \leq \alpha_j^\vee(\lambda)$ for all $j \in I$. If $s = m$ then there is nothing to prove. Otherwise, write $b = b' \otimes b_i \otimes b''$ where $i = j_{s+1}$. By Lemma 5.3, $\varepsilon_j(b' \otimes b_i) = \varepsilon_j(b') \leq \alpha_j^\vee(\lambda)$ for all $j \neq i$.

On the other hand, $\varepsilon_j(b' \otimes b_i) > \alpha_j^\vee(\lambda)$ for some $j \in I$ by the choice of $b'$. It follows from Lemma 5.3 that $j = i, \varepsilon_i(b' \otimes b_i) = \varepsilon_i(b') + 1, \varepsilon_i(b') = \alpha_i^\vee(\lambda)$ and $\varphi_i(b') = 0$. In particular, $e_i(b' \otimes b_i) = b' \otimes b_{i-1}$ by (2.1). Furthermore,

$$
\varepsilon_i(b) = \max\{\varepsilon_i(b' \otimes b_i), \varepsilon_i(b'') - \alpha_i^\vee(\text{wt } b') - \alpha_i^\vee(\text{wt } b_i)\}
$$

$$
= \varepsilon_i(b') + 1 + \max\{0, \varepsilon_i(b'') - \varphi_i(b')\} = \alpha_i^\vee(\lambda) + 1 + \max\{0, \varepsilon_i(b'')\}.
$$

Set $i_1 = i$. Then $m_1 = \varepsilon_i(b) - \alpha_i^\vee(\lambda) = \max\{0, \varepsilon_i(b'')\} + 1 > 0$. Furthermore, by 2.3 (T),

$$
e_{i_1}^{m_1} b = e_{i_1}^m b_i = e_i^\vee(b' \otimes b_i) \otimes e_i^{\varepsilon_i(b')} b'' = b' \otimes b_{i-1} \otimes u''_r.
$$

If $\varepsilon_j(b' \otimes b_{i-1}) = \varepsilon_j(b') \leq \alpha_j^\vee(\lambda)$ for all $j \in I$ then we can use induction on $|b'|$. Otherwise, we repeat the above argument for $r = 1, \ldots, k$ where $k \in \mathbb{N}$ is minimal such that $i - k$
satisfies $\varepsilon_j(b' \otimes b_{i-k}) = \varepsilon_j(b') \leq \alpha_j^\gamma(\lambda)$ for all $j \in I$. The existence of such $i - k \in I$ is guaranteed by Lemma 5.3. As a result we obtain a monomial $e = e_{i_k}^{m_k} \cdots e_{i_1}^{m_1}$ where $m_{i_r} = \varepsilon_i, (u_{r-1}) - \alpha_i^\gamma(\lambda)$, $u_r = e_{i_r}^{m_r} \cdots e_{i_1}^{m_1} b$ and $u_0 = b$ such that $eb = b' \otimes u_k^\gamma$ where $\varepsilon_j(b') \leq \alpha_j^\gamma(\lambda)$ for all $j \in I$ and $|b'| > |b|$. The assertion follows by induction on $|b'|$. 

\[\Box\]

5.5. The following Lemma allows one to prove that $B^\lambda$ is not empty for each $\lambda \in P^+(\pi) \setminus \mathbb{Z}\delta$.

**Lemma.** For all $i \in I$, $B_\ell(m)^{A_i} = \{b(i, m)\}$ where $b(i, m) = b_i \otimes b_{i+1} \otimes \cdots \otimes b_{i+m-1}$.

**Proof.** One has, for all $t = 1, \ldots, m$,

$$r_j^\ell(b(i, m)) = \varepsilon_j(b_{i+t-1}) - \sum_{s=1}^{t-1} \alpha_j^\gamma(\text{wt } b_{i+s-1}) = \delta_{j,i+t-1} + \delta_{j,i} - \delta_{j,i+t-1} = \delta_{i,j},$$

whence $\varepsilon_j(b(i, m)) = \max_i \{r_j^\ell(b(i, m))\} = \delta_{i,j}$. Therefore, $b(i, m) \in B_\ell(m)^{A_i}$.

We prove that $b(i, m)$ is the only element of $B_\ell(m)^{A_i}$ by induction on $m$. The induction base is given by (3.2). Suppose that $b \in B_\ell(m) : m > 1$ satisfies $\varepsilon_j(b) \leq \delta_{i,j}$. Then $\varepsilon_j(b) = \delta_{i,j}$ for otherwise $b$ is a highest weight element of $B_\ell(m)$ by normality of the latter, which is a contradiction by Lemma 3.2. Write $b = b_i \otimes b'$ where $b' \in B_\ell(m-1)$. Since $\varepsilon_j(b) = \max\{\varepsilon_j(b_r), \varepsilon_j(b') + \delta_{j,r} - \delta_{j,r+1}\} = \max\{0, \varepsilon_j(b') - \delta_{j,r+1}\} + \delta_{j,r}$ it follows that $\varepsilon_r(b') > 0$, whence $r = i$. Then $\varepsilon_j(b) = \delta_{i,j} + \max\{0, \varepsilon_j(b') - \delta_{j,i+1}\}$, hence we must also have $\varepsilon_j(b') = 0$ if $j \neq i + 1$ and $\varepsilon_{i+1}(b') \leq 1$. Therefore, $b' \in B_\ell(m-1)^{A_{i+1}}$, whence $b' = b(i+1, m-1)$ by the induction hypothesis, and so $b = b_i \otimes b(i+1, m-1) = b(i, m)$. 

\[\Box\]

5.6. Now we are able to prove Theorem 1.6.

**Proof.** By Proposition 4.3 and Lemma 5.1, $B^\lambda = \{\psi(b \otimes z^k) : b \in B_\ell(m)^\lambda, k \in \mathbb{Z}\}$. Yet for all $\lambda \in P^+(\pi) \setminus \mathbb{Z}\delta$, there exists $i \in I$ such that $\alpha_i^\gamma(\lambda) > 0$. It follows that $B_\ell(m)^{A_i}$, which is non-empty by Lemma 5.3, is contained in $B_\ell(m)^\lambda$. Therefore, $B^\lambda$ is not empty for all $\lambda \in P^+(\pi) \setminus \mathbb{Z}\delta$.

By 5.3 and Lemma 5.2, it remains to prove that $B(\lambda) \otimes B$ is generated over $\mathcal{F}$ by its highest weight elements $b_\lambda \otimes p$ where $p \in B$ is $\lambda$-dominant. That is equivalent to proving that for all $b \in B(\lambda)$ and for all $p \in B$ there exists $e \in \mathcal{E}$ such that $e(b \otimes p) = b_\lambda \otimes p'$ where $p'$ is $\lambda$-dominant. By Lemma 5.1 and Proposition 4.3, $p'$ is $\lambda$-dominant if and only if $p' = \psi(b' \otimes z^k)$ where $b' \in B_\ell(m)^\lambda$ and $k \in \mathbb{Z}$.

Take arbitrary $b \in B(\lambda), p \in B$ and let us prove first that there exists a monomial $e \in \mathcal{E}$ such that $e(b \otimes p) = b_\lambda \otimes p'$ for some $p' \in B$. Indeed, by (2.7), $e_j^{k+1}(b \otimes p) = e_j b \otimes e_j^k p = e_j b \otimes p'$, where $k = \max\{0, \varepsilon_j(p) - \varphi_j(b)\} \leq \varepsilon_j(p)$. Since $B(\lambda)$ is generated by $b_\lambda$ over $\mathcal{F}$ by (19, 7, Corollary 1 c)], there exists a monomial $e \in \mathcal{E}$ such that $eb = b_\lambda$. By the above, there exists a monomial $e'$ such that $e'(b \otimes p) = eb \otimes p' = b_\lambda \otimes p'$ and $p' \neq 0$. 

\[\Box\]
It remains to prove that for all \( p \in B \), there exist \( e \in \mathcal{E} \) such that \( e(b_\lambda \otimes p) = b_\lambda \otimes p' \) where \( p' \in B^\lambda \). Set \( m_j(p) = \max\{0, \varepsilon_j(p) - \alpha_j^\vee(\lambda)\} \). Then

\[
e_j^{m_j(p)}(b_\lambda \otimes p) = b_\lambda \otimes e_j^{m_j(p)} p.
\]

Indeed, \( \varepsilon_j(b_\lambda \otimes p) = \max\{0, \varepsilon_j(p) - \alpha_j^\vee(\lambda)\} = m_j(p) \) and, since \( e_j b_\lambda = 0 \), we conclude that \( e_j^{m_j(p)}(b_\lambda \otimes p) = b_\lambda \otimes e_j^{m_j(p)} p \). Furthermore, write \( p = \psi(u \otimes z^n) \). By Lemma 5.4, there exists a monomial \( e = e_i^{m_k} \cdots e_i^{m_1} \) such that \( e(u \otimes z^n) = u' \otimes z^{n+s} \) where \( u' \in B_t(m)^\lambda \), \( s = \sum_t m_t \delta_{0,i_t} \) and \( m_t = m_{i_t}(u_t-1) > 0 \) where \( u_0 = u, u_t = e_{i_t}^{m_{i_t}} \cdots e_{i_1}^{m_{i_1}} u, t = 1, \ldots, k \). Then, using (5.1) repeatedly, we obtain

\[
e(b_\lambda \otimes p) = e_i^{m_k} \cdots e_i^{m_2} (b_\lambda \otimes e_i^{m_1} p) = \cdots = b_\lambda \otimes ep = b_\lambda \otimes p',
\]

where \( p' = \psi(u' \otimes z^{n+s}) \in B^\lambda \).

**5.7.** In the case \( \lambda = \Lambda_i \) we are able to describe the decomposition of Theorem 1.6 more explicitly.

**Proposition.** For all \( i \in I \),

\[
B(\Lambda_i) \otimes \overline{B_t(m)} \xrightarrow{\sim} \prod_{k \in \mathbb{Z}} B(\Lambda_{i+m} + k\delta).
\]

**Proof.** Observe that \( wt \, b(i, m) = \Lambda_{i+m} - \Lambda_i \). The assertion follows immediately from Theorem 1.6 and Lemma 5.3.

**Appendix. Proof of Proposition 3.8**

**A.1.** Given \( b \in B_t(m) \), define its major index \( \operatorname{Maj}(b) := \sum_{r=1}^{k-1} n_r \) where \( \widetilde{\operatorname{desc}}(b) = \{n_0 < \cdots < n_k\} \) (cf. Definition 3.6). In other words, \( \operatorname{Maj}(b) \) equals the sum of all elements in \( \operatorname{desc}(b) \) and zero if \( \operatorname{desc}(b) \) is empty. Our definition of \( \operatorname{Maj}(b) \) is just the standard definition of the major index of a word in a free monoid over a completely ordered alphabet.

**Lemma.** For all \( b \in B_t(m) \), \( \operatorname{Maj}(b) = -N(b) \pmod{m} \).

**Proof.** Indeed, write \( \widetilde{\operatorname{desc}}(b) = \{n_0, \ldots, n_k\} \) and recall that \( n_0 = 0, n_k = m \). Then

\[
N(b) = \sum_{r=1}^k r(n_r - n_{r-1}) = \sum_{r=1}^k r n_r - \sum_{r=1}^{k-1} (r + 1)n_r
\]

\[
= kn_k - \sum_{r=1}^{k-1} n_r = km - \operatorname{Maj}(b) = -\operatorname{Maj}(b) \pmod{m}.
\]
A.2. Set $[n] = (q^n - 1)/(q - 1) = 1 + q + \cdots + q^{n-1}$ and define

$$[n]! := [n][n-1]\cdots[1], \quad \left[ \begin{array}{c} n \\ n_1, \ldots, n_k \end{array} \right] = \frac{[n]!}{[n_1]!\cdots[n_k]!},$$

where $n = n_1 + \cdots + n_k$. It is convenient to assume that $\left[ \begin{array}{c} n \\ n_1, \ldots, n_k \end{array} \right] = 0$ if $n \neq n_1 + \cdots + n_k$.

We will also use the notation $[n]_{q^r} := (q^{nr} - 1)/(q^r - 1) = 1 + q^r + \cdots + q^{(n-1)r}$.

**Lemma.** The cardinality of the set $B_\ell(m)^\nu = \{b \in B_\ell(m)_\nu : N(b) = -n \pmod{m}\}$ equals the coefficient of $q^n$ in the polynomial

$$\left[ \begin{array}{c} m \\ k_0, \ldots, k_\ell \end{array} \right] \pmod{q^m - 1},$$

where $\nu = (k_0, \ldots, k_\ell)$.

**Proof.** Let $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$ be a completely ordered alphabet and let $R$ be a set of all distinct permutations of the word $\gamma_1^{k_1} \cdots \gamma_r^{k_r}$ in the free monoid over $\Gamma$. Then

$$\sum_{w \in R} q^{\text{Maj}(w)} = \left[ \begin{array}{c} m_1 + \cdots + m_r \\ m_1, \ldots, m_r \end{array} \right]$$

by a classical theorem of MacMahon (cf. for example [1, Theorem 3.7]). Apply this theorem to $\Gamma = \{b_\ell, \ldots, b_0\}$ and the word $b_\ell^{k_\ell} \otimes \cdots \otimes b_0^{k_0}$ whose distinct permutations form the set $B_\ell(m)_\nu$ where $\nu = \sum_{i \in I} k_i (\Lambda_{i+1} - \Lambda_i)$. The result then follows immediately from Lemma [A.1].

A.3. Let $r_0, \ldots, r_\ell$ be non-negative integers and denote their sum by $r$. Set, for all $n \in \mathbb{Z}$,

$$C(r_0, \ldots, r_\ell; n) := \frac{1}{r} \sum_{d|r} \varphi_n(d) \left( \frac{r_0}{d}, \ldots, \frac{r_\ell}{d} \right),$$

where $\varphi_n(d)$ is defined as in Proposition [3.8]. Furthermore, for all $d$ dividing $r$ set

$$\tilde{C}(r_0, \ldots, r_\ell; d) := \frac{d}{r} \sum_{dd'|r} \mu(d') \left( \frac{r_0}{dd'}, \ldots, \frac{r_\ell}{dd'} \right).$$

From now on we adopt the convention that a multinomial coefficient equals zero if any of the rational numbers involved is not an integer.

**Lemma.** Fix $m \in \mathbb{N}^+$ and let $k_0, \ldots, k_\ell$ be non-negative integers such that $k_0 + \cdots + k_\ell = m$. Then

$$\sum_{n=0}^{m-1} C(k_0, \ldots, k_\ell; n) q^n = \sum_{d|m} \tilde{C}(k_0, \ldots, k_\ell; d)[m/d]q^d. \quad (A.1)$$
Furthermore, set \( (cf. \ [21, \text{Lemma } 34.1.2]) \)

\[
\sum_{d|n} \tilde{C}(k_0, \ldots, k_\ell; d) = \sum_{d|n} \sum_{d'|m} \frac{d}{m} \mu(d') \left( \frac{m}{dd'} \right) \left( \frac{k_0}{dd'}, \ldots, \frac{k_\ell}{dd'} \right)
\]

\[
= \frac{1}{m} \sum_{d|m} \left( \frac{m}{d} \right) \left( d \sum_{d'|d} \frac{\mu(d')}{d'} \right).
\]

The inner sum equals \( \varphi_n(d) \) (cf. the proof of \([3\text{ Corollary } 4.2])\).

\[\square\]

**A.4.** The next step of our proof is the following

**Lemma.** Let \( d \) be a divisor of \( m \) and denote by \( \Phi_d(q) \) the \( d \)th cyclotomic polynomial. Let \( \psi_d \) be the canonical projection \( \mathbb{Q}[q] \to \mathbb{Q}[q]/(\Phi_d(q)) \cong \mathbb{Q}(\zeta_d) \), where \( \zeta_d \) is a \( d \)th primitive root of unity. Then

\[
\sum_{r|m} \tilde{C}(k_0, \ldots, k_\ell; r) \psi_d([m/r]q^r) = \left( \frac{m}{d} \right) \left( \frac{k_0}{d}, \ldots, \frac{k_\ell}{d} \right).
\]

**Proof.** Set \( P(q) := \sum_{r|m} \tilde{C}(k_0, \ldots, k_\ell; r)[m/r]q^r \). Then \( \psi_d(P(q)) \) identifies with \( P(\zeta_d) \). If \( d \) divides \( r \), then \( \psi_d([m/r]q^r) = m/r \). Otherwise, \( \zeta_d^m \neq 1 \) and so \( \psi_d([m/r]q^r) = (\zeta_d^m - 1)/(\zeta_d - 1) = 0 \) since \( d|m \). Therefore,

\[
P(\zeta_d) = \sum_{rd|m} \tilde{C}(k_0, \ldots, k_\ell; rd) \frac{m}{rd} = \sum_{r|m} \tilde{C}(k_0, \ldots, k_\ell; r) \frac{m}{rd}.
\]

Furthermore, set \( n = m/d, \ n_i = k_i/d, \ i = 0, \ldots, \ell \). Then

\[
P(\zeta_d) = \sum_{r|n} \sum_{d'r|n} \mu(d') \left( \frac{n}{d'r} \right) \left( \frac{n}{r} \right) \sum_{r'|r} \mu(r/r').
\]

It remains to apply the fundamental property of the Möbius function, namely, \( \sum_{d|n} \mu(d) = 0 \) if \( n > 1 \) and 1 otherwise.

\[\square\]

**A.5.** The following Lemma has been adapted from \([21\text{ Lemma } 34.1.2])\). We deem it necessary to present its proof here since we use the definition \([1, \text{3.3}]\) of the \( q \)-multinomial coefficients, which differs from that of \([21, \text{1.3}]\) by a power of \( q \).

**Lemma** (cf. \([21, \text{Lemma } 34.1.2])\). Let \( m, d \) be non-negative integers and let \( \psi_d \) be the map defined in \( \text{A.4} \).

1° Let \( k_1, \ldots, k_r \in \mathbb{N} \) and suppose that \( d \) does not divide \( \gcd(k_1, \ldots, k_r) \). Then

\[
\psi_d\left( \left[ \frac{md}{k_1, \ldots, k_r} \right] \right) = 0.
\]

2° For all \( k_1, \ldots, k_r \in \mathbb{N} \)

\[
\psi_d\left( \left[ \frac{md}{k_1d, \ldots, k_rd} \right] \right) = \left( \frac{m}{k_1, \ldots, k_r} \right).
\]
**Proof.** First, let us prove both assertions for \( r = 2 \). Set \( \begin{bmatrix} m \\ k, m - k \end{bmatrix} := \begin{bmatrix} m \\ k \end{bmatrix} \).

1° Obviously, \( \begin{bmatrix} md \\ k \end{bmatrix} = 0 \) if \( m = 0, 1 \) and \( d \) does not divide \( k \). Suppose that the assertion holds for all non-negative integers \( < m \) and for all \( k < (m - 1)d \) not divisible by \( d \). Then by \([1, Theorem 3.4]\) or \([21, 1.3.1]\),

\[
\psi_d\left( \begin{bmatrix} md \\ k \end{bmatrix} \right) = \sum_{t=0}^{k} \psi_d\left( \begin{bmatrix} (m - 1)d \\ t \end{bmatrix} \right) \psi_d\left( \begin{bmatrix} d \\ k - t \end{bmatrix} \right) = 0,
\]

since at least one of \( t, k - t \) is not divisible by \( d \).

2° Since \( \begin{bmatrix} md \\ kd \end{bmatrix} = 0 \) if \( k > m \), we immediately conclude that the second assertion holds for \( m = 0, 1 \). Furthermore,

\[
\psi_d\left( \begin{bmatrix} md \\ kd \end{bmatrix} \right) = \sum_{t=0}^{kd} \psi_d\left( \begin{bmatrix} (m - 1)d \\ t \end{bmatrix} \right) \psi_d\left( \begin{bmatrix} d \\ (k - t)d \end{bmatrix} \right) = \sum_{t=0}^{k} \psi_d\left( \begin{bmatrix} (m - 1)d \\ td \end{bmatrix} \right) \psi_d\left( \begin{bmatrix} d \\ (k - t)d \end{bmatrix} \right),
\]

where we applied the first part. Notice that \( \psi_d(q^d) = 1 \). Then, by the induction hypothesis,

\[
\psi_d\left( \begin{bmatrix} md \\ kd \end{bmatrix} \right) = \sum_{t=0}^{k} \left( \frac{m - 1}{t} \right) \left( \frac{1}{k - t} \right) = \left( \frac{m - 1}{k} \right) + \left( \frac{m - 1}{k - 1} \right) = \left( \frac{m}{k} \right).
\]

Suppose now that \( r > 2 \) and observe that

\[
\begin{bmatrix} m \\ k_1, \ldots, k_r \end{bmatrix} = \begin{bmatrix} m \\ k_1 \end{bmatrix} \begin{bmatrix} m - k_1 \\ k_2, \ldots, k_r \end{bmatrix}.
\]

The assertion follows immediately by induction on \( r \). \( \blacksquare \)

**A.6.** Now we are able to prove Proposition 3.8

**Proof.** By Lemma A.2, \( \# B_{\ell}(m)^{-n} \) equals the coefficient of \( q^n \) in the polynomial \( \begin{bmatrix} m \\ k_0, \ldots, k_{\ell} \end{bmatrix} \) (mod \( q^m - 1 \)). It follows from Lemmata A.3, A.4 and A.5 that

\[
\psi_d\left( \sum_{n=0}^{m-1} C(k_0, \ldots, k_{\ell}; n) q^n \right) = \psi_d\left( \begin{bmatrix} m \\ k_0, \ldots, k_{\ell} \end{bmatrix} \right),
\]

for all \( d \) dividing \( m \). On the other hand, \( q^m - 1 = \prod_{d|m} \Phi_d(q) \). Since cyclotomic polynomials are irreducible over \( \mathbb{Q} \) and \( \mathbb{Q}[q] \) is a unique factorisation domain, it follows that

\[
\sum_{n=0}^{m-1} C(k_0, \ldots, k_{\ell}; n) q^n = \begin{bmatrix} m \\ k_0, \ldots, k_{\ell} \end{bmatrix} \mod q^m - 1.
\]
Therefore, \( \#B_\ell(m)^{-n} = C(k_0, \ldots, k_\ell; n) \).

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