COALGEBRAIC METHODS FOR RAMSEY DEGREES OF UNARY ALGEBRAS

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Abstract. In this paper we prove the existence of small and big Ramsey degrees of classes of finite unary algebras in an arbitrary (not necessarily finite) algebraic language Ω. Our results generalize some Ramsey-type results of M. Sokić concerning finite unary algebras over finite languages. To do so we develop a completely new strategy that relies on the fact that right adjoints preserve the Ramsey property. We then treat unary algebras as Eilenberg-Moore coalgebras for a functor with comultiplication and using pre-adjunctions transport the Ramsey properties we are interested in from the category of finite or countably infinite chains of order type ω. Moreover, we show that finite objects have finite big Ramsey degrees in the corresponding cofree structures over countably many generators.

1. Introduction

Almost any reasonable class of finite relational structures has some kind of Ramsey property. In the realm of finite algebras the picture is considerably different – in the past 50 years the Ramsey property has been established in only a handful of cases: finite Boolean algebras [5]; finite vector spaces over a fixed finite field [6]; finite Boolean lattices [18]; finite unary algebras over a finite language [21]; finite G-sets for a finite group G [21]; and finite semilattices [20]. It is notable that, despite a long history [4, 17, 22], the following question is still open:

Open problem. Is it true that the class of finite groups has small Ramsey degrees?

In this paper we generalize some of the results of Sokić from [21] and show that for an arbitrary monoid M (finite or infinite) the class of all finite M-sets has finite small Ramsey degrees. This immediately implies that the class of all finite G-sets has finite small Ramsey degrees for an arbitrary group G (finite or infinite), and that the class of all finite unary algebras over an arbitrary (finite or infinite) algebraic language Ω has finite small Ramsey degrees. Moreover, we show that in all of the three contexts finite objects have finite big Ramsey degrees in the corresponding cofree structure over countably many generators.

All our results are spelled out using the categorical reinterpretation of the Ramsey property as proposed in [13]. Actually, it was Leeb who pointed out already in 1970 that the use of category theory can be quite helpful both in the formulation and in the proofs of results pertaining to structural Ramsey theory [8].

In order to prove the existence of small and big Ramsey degrees in classes of M-sets (and, thus, immediately get the corresponding results about G-sets for an arbitrary group G, and Ω-algebras for an arbitrary unary algebraic language Ω) we use a completely new strategy developed in [13] that relies on transporting the Ramsey property from a category to the associated category of Eilenberg-Moore (co)algebras for a (co)monad. In [13] we show how dual Ramsey phenomena

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transport from a category to the category of Eilenberg-Moore algebras for a monad. In this paper we are interested in analogous results for comonads. Although comonads are categorical duals of monads, the combinatorics involved is of a different kind, and modeling algebraic phenomena by co-algebras for a comonad is far more challenging. For a monoid \( M \) we consider \( M \)-sets as Eilenberg-Moore co-algebras for monoid action comonad (to be described below). The results then follow from the Finite Ramsey Theorem and the Infinite Ramsey Theorem by building an appropriate endofunctor and a comultiplication on the category of finite and countably infinite chains of order type \( \omega \).

After fixing standard notions and notation in Section 2, we present our proof strategy in Section 3. Our starting point is the observation from [14] that right adjoints preserve the Ramsey property. We then observe that if \( E \) is a comonad on a category with the Ramsey property, both the Kleisli category and the Eilenberg-Moore category for the comonad have the Ramsey property. Unfortunately, these two simple results are not very useful: for the categorical treatment of the Ramsey property it is essential to restrict the attention to categories where the morphisms are mono, and counits for comonads we are interested in cannot be expected to consist of monos for cardinality reasons (see Example 2). This makes it impossible to model phenomena we are interested in by comonads over categories whose morphisms are mono. Therefore, we relax the context by proving that the Ramsey property carries over from the category of chains to the category of weak Eilenberg-Moore coalgebras defined for functors with comultiplication and no counit, which are straightforward weakenings of comonads. Our main results are presented in Section 4.

2. Preliminaries

2.1. Chains. A chain is a linearly ordered set \((A, <)\). Finite or countably infinite chains will sometimes be denoted as \(\{a_1 < \cdots < a_n < \ldots\}\). For example, \(\omega = \{0 < 1 < 2 < \ldots\}\). Every strict linear order \(<\) induces the reflexive version \(\leq\) in the obvious way. A map \(f : A \to B\) from a chain \((A, <)\) to a chain \((B, <)\) is monotone if \(a_1 < a_2 \Rightarrow f(a_1) < f(a_2)\) for all \(a_1, a_2 \in A\).

Let \((A, <)\) be a chain. Sequences of elements of \(A\) can be lexicographically ordered as follows. Let \(\alpha\) and \(\beta\) be ordinals. For a pair of distinct sequences \((a_\xi)_{\xi < \alpha} \in A^\alpha\) and \((b_\xi)_{\xi < \beta} \in A^\beta\) (not necessarily of the same length) we write \((a_\xi)_{\xi < \alpha} <_{\text{lex}} (b_\xi)_{\xi < \beta}\) if

- \(\alpha < \beta\) and \(a_\xi = b_\xi\) for all \(\xi < \alpha\); or
- there is a \(\xi\) such that \(a_\xi \neq b_\xi\) in which case we let \(\eta = \min\{\xi : a_\xi \neq b_\xi\}\) and declare \((a_\xi)_{\xi < \alpha} <_{\text{lex}} (b_\xi)_{\xi < \alpha}\) if and only if \(a_\eta < b_\eta\).

In particular, for every chain \((A, <)\) and every well-ordered chain \((S, <)\) the set \(A^S\) of all functions \(f : S \to A\) can be ordered lexicographically and the corresponding lexicographic ordering on \(A^S\) will be denoted by \(<_{\text{lex}}^A\). For every finite ordinal \(n\) and every well-ordered chain \((A, <)\) the chain \((A^n, <_{\text{lex}}^A)\) is also well-ordered.

2.2. Unary algebras, G-sets and M-sets. Let \(\Omega\) be an algebraic language, that is, the set of constant and functional symbols. A \(\Omega\)-algebra is a structure \((A, \Omega^A)\) where \(\Omega^A = \{f^A : f \in \Omega\}\) is a set of operations on \(A\) such that the arity of each operation \(f^A\) coincides with the arity of the corresponding functional symbol \(f \in \Omega\). We say that \(A\) is a carrier of an \(\Omega\)-algebra \((A, \Omega^A)\).

A language \(\Omega = \{f_i : i \in I\}\) is unary if every symbol in \(\Omega\) is a unary functional symbol; and it is monounary if \(\Omega\) consists of a single unary functional symbol. Correspondingly, a (mono)unary algebra is an \(\Omega\)-algebra where \(\Omega\) is a (mono)unary algebraic language.

Let \((M, \cdot, 1)\) be a monoid. An \(M\)-set is a pair \((A, \alpha)\) where \(\alpha : M \times A \to A\) is the structure map and has the following properties: \(\alpha(1, a) = a\) and \(\alpha(m_1, \alpha(m_2, a)) = \alpha(m_2 \cdot m_1, a)\) for all \(a \in A\) and \(m_1, m_2 \in M\). A mapping \(f : A \to B\) is a morphism between \(M\)-sets \((A, \alpha)\) and
(\mathcal{B}, \beta)$ if $f(\alpha(m, a)) = \beta(m, f(a))$ for all $a \in A$ and all $m \in M$. An injective morphism of $M$-sets will be referred to as an embedding. A cofree $M$-set on the set of generators $X$ is the $M$-set $\mathcal{E}(X) = (X^M, \gamma)$ where $\gamma(m, h) \in X^M$ is given by $\gamma(m, h)(m') = h(m \cdot m')$. See Lemma [2.1] and Example [3] below. In particular, a $G$-set is an $M$-set where $M = G$ is a group.

Let $\Omega = \{f_i : i \in I\}$ be an arbitrary unary algebraic language. Let $M = \Omega^*$ be the free monoid of finite words over $\Omega$ and let $1 \in \Omega^*$ denote the empty word. Every $\Omega$-algebra $\mathcal{A} = (A, \Omega^A)$ can be represented by an $M$-set $(A, \alpha)$ where the structure map $\alpha : M \times A \to A$ evaluates words from $M = \Omega^*$ in $\mathcal{A}$ in the obvious way: $\alpha(1, a) = a^1 = a$ and $\alpha(f_{i_1} \ldots f_{i_k}, a) = a^{f_{i_1} \cdots f_{i_k}} = (f_{i_k}^{a_1} \circ \cdots \circ f_{i_1}^{a_k})(a)$. Note that a map is a homomorphism between two unary algebras if and only if it is a morphism of corresponding $M$-sets.

2.3. Categories and functors. For basic category-theoretic notions and notation (category, functor, homset, natural transformation etc) we refer the reader to [1]. Let us here quickly fix some notation. Let $\mathcal{C}$ be a category. By $\text{Ob}(\mathcal{C})$ we denote the class of all the objects in $\mathcal{C}$. Homsets in $\mathcal{C}$ will be denoted by $\text{hom}_{\mathcal{C}}(A, B)$, or simply $\text{hom}(A, B)$ when $\mathcal{C}$ is clear from the context. The identity morphism will be denoted by $\text{id}_A$ and the composition of morphisms by $\cdot$ (dot). If $\text{hom}_{\mathcal{C}}(A, B) \neq \emptyset$ we write $A \to B$.

A morphism $f \in \text{hom}_{\mathcal{C}}(A, B)$ is mono if it is left cancellable in the following sense: for all $C \in \text{Ob}(\mathcal{C})$ and all $g, h \in \text{hom}_{\mathcal{C}}(C, A)$, if $f \cdot g = f \cdot h$ then $g = h$; and it is iso or invertible if there exists an $f' \in \text{hom}_{\mathcal{C}}(B, A)$ such that $f \cdot f' = \text{id}_B$ and $f' \cdot f = \text{id}_A$. Let $\text{iso}_{\mathcal{C}}(A, B)$ denote the set of all the invertible morphisms $A \to B$, and let $\text{Aut}_{\mathcal{C}}(A) = \text{iso}(A, A)$ denote the set of all the invertible morphisms $A \to A$. An object $A \in \text{Ob}(\mathcal{C})$ is rigid if $\text{Aut}_{\mathcal{C}}(A) = \{\text{id}_A\}$. As usual, $\mathcal{C}^{\text{op}}$ denotes the opposite category. If $\mathcal{C}$ is a category of structures, where by a structure we mean a set together with some additional information, by $\mathcal{C}^{\text{fin}}$ we denote the full subcategory of $\mathcal{C}$ spanned by its finite members.

A category $\mathcal{D}$ is a subcategory of a category $\mathcal{C}$ if $\text{Ob}(\mathcal{D}) \subseteq \text{Ob}(\mathcal{C})$ and $\text{hom}_{\mathcal{D}}(A, B) \subseteq \text{hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \text{Ob}(\mathcal{D})$; and it is a full subcategory if it is a subcategory of $\mathcal{C}$ such that $\text{hom}_{\mathcal{D}}(A, B) = \text{hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \text{Ob}(\mathcal{D})$. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. An $S \in \text{Ob}(\mathcal{C})$ is universal for $\mathcal{D}$ if for every $D \in \text{Ob}(\mathcal{D})$ the set $\text{hom}_{\mathcal{C}}(D, S)$ is nonempty and consists of monos only. Note that if there exists an $S \in \text{Ob}(\mathcal{C})$ universal for $\mathcal{D}$ then all the morphisms in $\mathcal{D}$ are mono. If $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ the we say that $\mathcal{C}$ is an ambient category for $\mathcal{D}$. An ambient category $\mathcal{C}$ is usually a category in which we can perform certain operations that are not possible in $\mathcal{D}$, or which contains an object universal for $\mathcal{D}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ maps $\text{Ob}(\mathcal{C})$ to $\text{Ob}(\mathcal{D})$ and maps morphisms of $\mathcal{C}$ to morphisms of $\mathcal{D}$ so that $F(f) \in \text{hom}_{\mathcal{D}}(F(A), F(B))$ whenever $f \in \text{hom}_{\mathcal{C}}(A, B)$, $F(f \cdot g) = F(f) \cdot F(g)$ whenever $f \cdot g$ is defined, and $F(\text{id}_A) = \text{id}_{F(A)}$. A functor $F : \mathcal{C} \to \mathcal{D}$ is faithful if it is injective on homsets in the sense that for all $A, B \in \text{Ob}(\mathcal{C})$ and all $f, g \in \text{hom}_{\mathcal{C}}(A, B)$, if $F(f) = F(g)$ then $f = g$. A functor $U : \mathcal{C} \to \mathcal{D}$ is forgetful if it is faithful and surjective on objects (that is, for every $D \in \text{Ob}(\mathcal{D})$ there is a $C \in \text{Ob}(\mathcal{C})$ with $U(C) = D$).

If $U : \mathcal{C} \to \mathcal{D}$ is a forgetful functor we may actually assume that $\text{hom}_{\mathcal{C}}(A, B) \subseteq \text{hom}_{\mathcal{D}}(U(A), U(B))$ for all $A, B \in \text{Ob}(\mathcal{C})$. The intuition behind this point of view is that $\mathcal{C}$ is a category of objects with a lot of structure, say topological groups, $\mathcal{D}$ is a category of objects with less structure, say groups, and $U$ takes a “rich” object $A$ and “forgets” the extra structure (topology, say) to produce its “poor” version $A$. Then for every morphism $f : A \to B$ in $\mathcal{C}$ the same map is a morphism $f : A \to B$ in $\mathcal{D}$. Therefore, if $U$ is a forgetful functor we shall always take that $U(f) = f$. In particular, $U(\text{id}_A) = \text{id}_{U(A)}$ and we, therefore, identify $\text{id}_A$ with $\text{id}_{U(A)}$. Also, if $U : \mathcal{C} \to \mathcal{D}$ is a forgetful functor and all the morphisms in $\mathcal{D}$ are mono, then all the morphisms in $\mathcal{C}$ are mono.

Example 1. (1) $\text{Set}$ is the category of sets and all set functions. $\text{Set}_{\text{fin}}$ is the category of sets and injective functions; this is a subcategory of $\text{Set}$, although not a full one.
(2) \( Ch_{emb} \) is the category whose objects are chains and whose morphisms are embeddings.

(3) For a monoid \( M \) let \( \text{Set}_{emb}(M) \) denote the category of \( M \)-sets and embeddings of \( M \)-sets, and for a group \( G \) let \( \text{Set}_{emb}(G) \) denote the category of \( G \)-sets and embeddings.

(4) For an algebraic language \( \Omega \) let \( \text{Alg}_{emb}(\Omega) \) denote the category of \( \Omega \)-algebras and embeddings, and let \( \text{Oalg}_{emb}(\Omega) \) denote the category of ordered \( \Omega \)-algebras and embeddings.

2.4. Adjunctions, monads and comonads. An adjunction between categories \( B \) and \( C \) consists of a pair of functors \( F : B \rightarrow C : H \) together with a family of isomorphisms \( \Phi_{X,Y} : \text{hom}_C(F(X), Y) \cong \text{hom}_B(X, H(Y)) \) indexed by pairs \((X,Y) \in \text{Ob}(B) \times \text{Ob}(C)\) and natural in both \( X \) and \( Y \). The functor \( F \) is then left adjoint (to \( H \)) and \( H \) is right adjoint (to \( F \)).

Let \( C \) be a category and \( T : C \rightarrow C \) a functor. By dualizing the notions of multiplication for an endofunctor and the monad we arrive at the notions of the comultiplication and comonad as follows. A comultiplication for an endofunctor \( E : C \rightarrow C \) is a natural transformation \( \delta : E \rightarrow EE \) such that for each \( A \in \text{Ob}(C) \) the diagram on the left commutes:

\[
\begin{array}{ccc}
E(A) & \xrightarrow{\delta_A} & EE(A) \\
\downarrow{\delta_{E(A)}} & & \downarrow{\delta_{E(A)}} \\
EE(A) & \xrightarrow{E(\delta_A)} & EEE(A)
\end{array}
\]

A natural transformation \( \varepsilon : E \rightarrow ID \) is a counit for \( \delta \) if the diagram on the right above commutes for each \( A \in \text{Ob}(C) \). A comonad on a category \( C \) is a triple \((E, \delta, \varepsilon)\) where \( E : C \rightarrow C \) is a functor, \( \delta \) is a comultiplication for \( E \) and \( \varepsilon \) is a counit for \( \delta \).

Let \( F : C \rightarrow C \) be a functor. An \( F \)-coalgebra is a pair \((A, \alpha)\) where \( \alpha \in \text{hom}_C(F(A), A) \), while an \( F \)-coalgba is a pair \((A, \alpha)\) where \( \alpha \in \text{hom}_C(A, F(A)) \). An algebraic homomorphism between \( F \)-algebras \((A, \alpha)\) and \((B, \beta)\) is a morphism \( f \in \text{hom}_C(A, B) \) such that the diagram on the left commutes:

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
\downarrow{\alpha} & & \downarrow{\beta} \\
A & \xrightarrow{f} & B
\end{array}
\]

A coalgebraic homomorphism between \( F \)-coalgebras \((A, \alpha)\) and \((B, \beta)\) is a morphism \( f \in \text{hom}_C(A, B) \) such that the diagram on the right commutes.

To each comonad \((E, \delta, \varepsilon)\) we can straightforwardly assign the Kleisli category \( K = K(E, \delta, \varepsilon) \) and the Eilenberg-Moore category \( \text{EM} = \text{EM}(E, \delta, \varepsilon) \) by dualizing the corresponding standard constructions for a monad. Explicitly, the objects of the Kleisli category \( K(E, \delta, \varepsilon) \) are the same as the objects of \( C \), morphisms are defined by \( \text{hom}_K(A, B) = \text{hom}_C(E(A), B) \) and the composition in \( K \) for \( f \in \text{hom}_K(A, B) \) and \( g \in \text{hom}_K(B, C) \) is given by \( g \cdot_K f = g \cdot E(f) \cdot \delta_A \). The objects of the Eilenberg-Moore category \( \text{EM}(E, \delta, \varepsilon) \) are Eilenberg-Moore \( E \)-coalgebras (special \( E \)-coalgebras to be defined immediately), morphisms are coalgebraic homomorphisms and the composition is as in \( C \). An Eilenberg-Moore \( E \)-coalgebra is an \( E \)-coalgebra for which the following two diagrams commute:

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & E(A) \\
\downarrow{\alpha} & & \downarrow{\delta_A} \\
E(A) & \xrightarrow{E(\alpha)} & EE(A)
\end{array}
\]

\[
\begin{array}{ccc}
A & \xleftarrow{\varepsilon_A} & E(A) \\
\uparrow{\alpha} & & \uparrow{\varepsilon_A} \\
A & \xrightarrow{\alpha} & E(A)
\end{array}
\]
An weak Eilenberg-Moore E-coalgebra is an E-coalgebra for which only the diagram on the left commutes. Let \(EM^w(E, \delta)\) denote the category of weak Eilenberg-Moore E-coalgebras and coalgebraic homomorphisms.

**Example 2.** The list comonad is a comonad that provides the necessary infrastructure for dealing with finite lists. For any set \(A\) let \(E(A) = A^+\) be the set of all the nonempty finite sequences of elements of \(A\), and for a set-function \(f : A \to B\) let \(E(f)\) act coordinatewise: \(E(f)(a_1, \ldots , a_n) = (f(a_1), \ldots , f(a_n))\). Define \(\delta_A : E(A) \to EE(A)\) by \(\delta_A(a_1, \ldots , a_n) = ((a_1, a_2, \ldots , a_n), (a_2, \ldots , a_n), \ldots , (a_n))\) and \(\varepsilon_A : E(A) \to A\) by \(\varepsilon_A(a_1, \ldots , a_n) = a_1\). Then it is easy to check that \((E, \delta, \varepsilon)\) is a Set-coMonad. The Eilenberg-Moore coalgebras for this comonad correspond to rooted forests (see Section[3]). Note that \(\varepsilon_A\) cannot be mono for cardinality reasons. So, in order to model algebraic phenomena we have to work with functors with comultiplication and no counit.

**Example 3.** Fix a monoid \(M\). Each \(M\)-set can be represented as an algebra for a monad as well as a coalgebra for a comonad. Define \(T : \text{Set} \to \text{Set}\) by \(T(A) = M \times A\) on objects, while for a mapping \(f : A \to B\) let \(T(f) : M \times A \to M \times B\) by \((m, a) \mapsto (m, f(a))\). Define \(\mu_A : TT(A) \to T(A)\) by \(\mu_A(m_1, (m_2, a)) = (m_2 \cdot m_1, a)\) and \(\eta_A : A \to T(A)\) by \(\eta(a) = (1, a)\). Then \((T, \mu, \eta)\) is a monad whose Eilenberg-Moore algebras correspond precisely to \(M\)-sets. Using the natural isomorphism \(\text{hom}(M \times A, B) \cong \text{hom}(A, B^M)\), \(M\)-sets can be represented by Eilenberg-Moore coalgebras for the following comonad. Define \(E : \text{Set} \to \text{Set}\) by \(E(A) = A^M\) on objects, while for a mapping \(f : A \to B\) let \(E(f) : A^M \to B^M\) by \(h \mapsto h \circ f\). Next, define \(\delta_A : E(A) \to EE(A)\) by \(\delta_A(h)(m_1)(m_2) = h(m_2 \cdot m_1)\) and \(\varepsilon_A : E(A) \to A\) by \(\varepsilon_A(h) = h(1)\). Then \((E, \delta, \varepsilon)\) is a comonad whose Eilenberg-Moore coalgebras correspond precisely to \(M\)-sets. Finally, recall that for a unary algebraic language \(\Omega\) every \(\Omega\)-algebra \(A = (A, \Omega^A)\) can be represented by an \(M\)-set \(\alpha : M \times A \to A\) where \(M = \Omega^*\) is the free monoid of finite words over \(\Omega\), and hence by an \(E\)-coalgebra for the comonad we have just described. Then it easily follows that \(\text{Alg}_{\text{emb}}(\Omega) \cong EM(E, \delta, \varepsilon)\).

A cofree Eilenberg-Moore E-coalgebra over a set of generators \(X\) is the Eilenberg-Moore E-coalgebra \(E(X) = (E(X), \delta_X)\). The following lemma motivates the choice of the terminology. We will not use this lemma later in the paper – the reason we include it is just to motivate the name.

**Lemma 2.1.** Let \(A = (A, \alpha)\) be an Eilenberg-Moore E-coalgebra and \(X\) a set. For every mapping \(f : A \to X\) there is a unique coalgebra homomorphism \(f^\# : A \to E(X)\) such that \(\varepsilon_X \cdot f^\# = f\). □

**Example 4.** Let us describe the cofree coalgebra on \(\omega\) generators for the monoid action comonad and, in particular, the cofree unary coalgebra on \(\omega\) generators for a unary language \(\Omega\). Given a monoid \(M\), the carrier of the cofree coalgebra on \(\omega\) generators is the set \(C = \omega^M\) and \(M\) acts on \(C\) so that \((m \cdot f)(x) = f(xm)\) for all \(f \in C\) and \(x, m \in M\). Now, given a unary algebraic language \(\Omega\) the carrier of the cofree \(\Omega\)-coalgebra on \(\omega\) generators is \(D = \omega^\Omega^*\) and for an \(s \in \Omega\) the corresponding unary operation is given by \(s(f)(w) = f(ws)\) for all \(f \in D\) and \(w \in \Omega^*\).

3. RAMSEY PROPERTIES IN A CATEGORY

In this section we collect and prove several results about the Ramsey property, Ramsey degrees, dual Ramsey property and dual small Ramsey degrees in a category. We then use the results of this section as the main tool to obtain new Ramsey results about \(G\)-sets in Section[4].

For \(k \in \mathbb{N}\), a \(k\)-coloring of a set \(S\) is any mapping \(\chi : S \to k\), where, as usual, we identify \(k\) with \(\{0, 1, \ldots , k-1\}\). Let \(C\) be a locally small category. For integers \(k \geq 2\) and \(t \geq 1\), and objects \(A, B, C \in \text{Ob}(C)\) we write \(C \rightarrow (B)^A_k\), to denote that for every \(k\)-coloring \(\chi : \text{hom}(A, C) \to k\) there is a morphism \(w \in \text{hom}(B, C)\) such that \(|\chi(w \cdot \text{hom}(A, B))| \leq t\). For a set of morphisms \(F\) we let \(w \cdot F = \{w \cdot f : f \in F\}\). In case \(t = 1\) we write \(C \rightarrow (B)^A_k\). We write \(C \leftarrow (B)^A_{k,t}\), resp. \(C \leftrightarrow (B)^A_{k,t}\), to denote that \(C \rightarrow (B)^A_{k,t}\), resp. \(C \rightarrow (B)^A_k\), in \(C^{op}\).
Lemma 3.1. [14] Lemma 2.4] Let $C$ be a locally small category such that all the morphisms in $C$ are mono and let $A, B, C, D \in \text{Ob}(C)$. If $C \rightarrow (B)^A_k$ for some $k, t \geq 2$ and if $C \rightarrow D$, then $D \rightarrow (B)^A_{k,t}$.

The above lemma tells us that with the arrow notation we can always go “up to a superstructure” of $C$. In some cases we can also “go down to a substructure” of $C$. Let $A$ be a subcategory of $C$ and let $C \in \text{Ob}(C)$. An object $B \in \text{Ob}(A)$ together with a morphism $c : B \rightarrow C$ is a coreflection of $C$ in $A$ if for every $A \in \text{Ob}(A)$ and every morphism $f \in \text{hom}_A(A, C)$ there is a unique morphism $g \in \text{hom}_A(A, B)$ such that $c \cdot g = f$.

Lemma 3.2. [13] Let $C$ be a locally small category such that all the morphisms in $C$ are mono. Let $A$ be a full subcategory of $C$, let $A, B, D \in \text{Ob}(A)$ and $C \in \text{Ob}(C)$. If $C \rightarrow (B)^A_k$ for some $k, t \geq 2$ and if $c : D \rightarrow C$ is a coreflection of $C$ in $A$ then $D \rightarrow (B)^A_{k,t}$.

A category $C$ has the (finite) Ramsey property if for every integer $k \geq 2$ and all $A, B \in \text{Ob}(C)$ there is a $C \in \text{Ob}(C)$ such that $C \rightarrow (B)^A_k$.

For $A \in \text{Ob}(C)$ let $t_C(A)$ denote the least positive integer $n$ such that for all $k \geq 2$ and all $B \in \text{Ob}(C)$ there exists a $C \in \text{Ob}(C)$ such that $C \rightarrow (B)^A_k$, if such an integer exists. Otherwise put $t_C(A) = \infty$. The number $t_C(A)$ is referred to as the small Ramsey degree of $A$ in $C$. A category $C$ has the finite small Ramsey degrees if $t_C(A) < \infty$ for all $A \in \text{Ob}(C)$. Clearly, a category $C$ has the Ramsey property if and only if $t_C(A) = 1$ for all $A \in \text{Ob}(C)$. In this parlance the Finite Ramsey Theorem takes the following form.

Theorem 3.1 (The Finite Ramsey Theorem [11]). The category $\text{Ch}^{fin}_{emb}$ has the Ramsey property.

The Ramsey property for ordered structures implies the existence of finite small Ramsey degrees for the corresponding unordered structures. This was first observed for categories of structures in [2], and generalized to arbitrary categories in [11].

Let us outline the main tool we employ to obtain results of this form. Following [7, 2, 11] we say that an expansion of a category $C$ is a category $C^*$ together with a forgetful functor $U : C^* \rightarrow C$. We shall generally follow the convention that $A, B, C, \ldots$ denote objects from $C$ while $A, B, C, \ldots$ denote objects from $C^*$. Since $U$ is injective on hom-sets we may safely assume that $\text{hom}_{C^*}(A, B) \subseteq \text{hom}_C(A, B)$ where $A = U(A), B = U(B)$. In particular, $\text{id}_{A} = \text{id}_{U(A)}$ for $A = U(A)$. Moreover, it is safe to drop subscripts $C$ and $C^*$ in $\text{hom}_C(A, B)$ and $\text{hom}_{C^*}(A, B)$, so we shall simply write $\text{hom}(A, B)$ and $\text{hom}(A, B)$, respectively. Let $U^{-1}(A) = \{ A \in \text{Ob}(C^*) : U(A) = A \}$. Note that this is not necessarily a set.

An expansion $U : C^* \rightarrow C$ is reasonable (cf. [7, 11]) if for every $e \in \text{hom}(A, B)$ and every $A \in U^{-1}(A)$ there is a $B \in U^{-1}(B)$ such that $e \in \text{hom}(A, B)$. An expansion $U : C^* \rightarrow C$ has unique restrictions [11] if for every $B \in \text{Ob}(C^*)$ and every $e \in \text{hom}(A, U(B))$ there is a unique $A \in U^{-1}(A)$ such that $e \in \text{hom}(A, B)$. We denote this unique $A$ by $B|_e$ and refer to it as the restriction of $B$ along $e$.

The following result was first proved for categories of structures in [2], and for general categories in [11].

Theorem 3.2. [2, 11] Let $C$ and $C^*$ be locally small categories such that all the morphisms in $C$ and $C^*$ are mono. Let $U : C^* \rightarrow C$ be a reasonable expansion with unique restrictions. Then $t_C(A) \leq \sum_{A \in U^{-1}(A)} t_{C^*}(A)$ for all $A \in \text{Ob}(C)$. Consequently, if $U^{-1}(A)$ is finite and $t_{C^*}(A) < \infty$ for all $A \in U^{-1}(A)$ then $t_C(A) < \infty$.

In particular, if $U : C^* \rightarrow C$ is a reasonable expansion with unique restrictions such that $C^*$ has the Ramsey property and $U^{-1}(A)$ is finite for all $A \in \text{Ob}(C)$ then $C$ has finite small Ramsey degrees.
Let $C$ be a locally small category. For $A, S \in \text{Ob}(C)$ let $T_C(A, S)$ denote the least positive integer $n$ such that $S \to (S)^k$ for all $k \geq 2$, if such an integer exists. Otherwise put $T_C(A, S) = \infty$. The number $T_C(A, S)$ is referred to as the big Ramsey degree of $A$ in $S$. In this parlance the Infinite Ramsey Theorem takes the following form.

**Theorem 3.3** (The Infinite Ramsey Theorem \[19\]). In the category $\text{Ch}_{\text{emb}}$ we have that $T(A, \omega) = 1$ for every finite chain $A$. □

The following result was first proved for categories of structures in \[2\], and for general categories in \[12\].

**Theorem 3.4.** (cf. \[2, 12\]) Let $C$ and $C^*$ be locally small categories such that all the morphisms in $C$ and $C^*$ are mono. Let $U : C^* \to C$ be an expansion with unique restrictions. For $A \in \text{Ob}(C)$, $S^* \in \text{Ob}(C^*)$ and $S = U(S^*)$, if $U^{-1}(A)$ is finite then $T_C(A, S) \leq \sum_{A^* \in U^{-1}(A)} T_{C^*}(A^*, S^*)$. □

Our major tool for transporting the Ramsey property from one context to another is to establish an adjunction-like relationship between the corresponding categories.

**Theorem 3.5.** \[14\] Right adjoints preserve the Ramsey property while left adjoints preserve the dual Ramsey property. More precisely, let $B$ and $C$ be locally small categories and let $F : B \rightleftharpoons C : H$ be an adjunction.

(a) If $C$ has the Ramsey property then so does $B$

(b) If $B$ has the dual Ramsey property then so does $C$. □

**Theorem 3.6.** \[13\] Let $C$ be a locally small category, $(E, \delta, \varepsilon)$ a comonad on $C$, and let $K = K(E, \delta, \varepsilon)$ and $EM = EM(E, \delta, \varepsilon)$ be the Kleisli category and the Eilenberg-Moore category, respectively, for the comonad. If $C$ has the Ramsey property then so do both $K$ and $EM$.

However, more is true in case of the Eilenberg-Moore construction. We are now going to show that the Ramsey property carries over from $C$ to a more general context of coalgebras for functors with comultiplication, which are straightforward weakenings of comonads. The proof relies on the following weakening of the notion of adjunction.

**Definition 1.** \[10\] Let $B$ and $C$ be locally small categories. A pair of maps $F : \text{Ob}(B) \rightleftharpoons \text{Ob}(C) : H$ is a pre-adjunction between $B$ and $C$ provided there is a family of maps $\Phi_{X,Y} : \text{hom}_C(F(X), Y) \to \text{hom}_B(X, H(Y))$ indexed by the pairs $(X, Y) \in \text{Ob}(B) \times \text{Ob}(C)$ and satisfying the following:

(PA) for every $C \in \text{Ob}(C)$, every $A, B \in \text{Ob}(B)$, every $u \in \text{hom}_C(F(B), C)$ and every $f \in \text{hom}_B(A, B)$ there is a $v \in \text{hom}_C(F(A), F(B))$ satisfying $\Phi_{B,C}(u) \cdot f = \Phi_{A,C}(u \cdot v)$.

(Note that in a pre-adjunction $F$ and $H$ are not required to be functors, just maps from the class of objects of one of the two categories into the class of objects of the other category; also $\Phi$ is not required to be a natural isomorphism, just a family of maps between hom-sets satisfying the requirement above.)

The proof of the following result is a straightforward generalization of \[10\] Theorem 3.2, but we shall include it for completeness.

**Theorem 3.7.** Let $B$ and $C$ be locally small categories and let $F : \text{Ob}(B) \rightleftharpoons \text{Ob}(C) : H$ be a pre-adjunction.

(a) Let $t, k \geq 2$ be integers, let $A, B \in \text{Ob}(B)$ and $C \in \text{Ob}(C)$. If $C \to (F(B))^F(A)$ in $C$ then $H(C) \to (B)^F(A)$ in $B$.

(b) $t_B(A) \leq t_C(F(A))$ for all $A \in \text{Ob}(B)$.

(c) If $C$ has the Ramsey property then so does $B$. □
Proof. (a) Assume that \( A, B \in \text{Ob}(D) \) and \( k, t \geq 2 \) satisfy \( C \to (F(B))^F_{F(A)} \) and let us show that \( H(C) \to (B)^t_k \). Take any \( \chi : \text{hom}_B(A, H(C)) \to k \) and define \( \chi' : \text{hom}_C(F(A), C) \to k \) by 
\[
|\chi'(u \cdot \text{hom}_C(F(A), F(B)))| \leq t.
\]
Let us show that 
\[
\chi(\Phi_B,F_C(u) \cdot \text{hom}_B(A, B)) \subseteq \chi'(u \cdot \text{hom}_C(F(A), F(B))).
\]
Take any \( f \in \text{hom}_B(A, B) \). Since \( F : \text{Ob}(B) \rightleftarrows \text{Ob}(C) : H \) is a pre-adjunction, there is a \( v \in \text{hom}_C(F(A), F(B)) \) such that \( \Phi_{B,C}(u) \cdot f = \Phi_{A,C}(u \cdot v) \). Then \( \chi(\Phi_B,F_C(u) \cdot f) = \chi(\Phi_{A,C}(u \cdot v)) = \chi'(u \cdot v) \), which proves (3.2). Finally, (3.2) together with (3.1) implies 
\[
|\chi(\Phi_B,F_C(u) \cdot \text{hom}_B(A, B))| \leq t.
\]

(b) and (c) are now immediate. \( \square \)

The proof of the dual version of the theorem below is given in [13]. Just as an illustration we provide the proof of the “direct” version here.

**Theorem 3.8.** (cf. [13] Theorem 3.15) Let \( C \) be a locally small category, \( E : C \to C \) a functor and \( \delta : E \to E E \) a comultiplication for \( E \). If \( C \) has the Ramsey property then so does every full subcategory of \( \mathbf{EM}^\delta(E, \delta) \) which contains all the \( E \)-coalgebras of the form \((E(C), \delta_C)\), \( C \in \text{Ob}(C) \).

Proof. Let \( B \) be a full subcategory of \( \mathbf{EM}^\delta(E, \delta) \) such that all the \( E \)-coalgebras of the form \((E(C), \delta_C)\), \( C \in \text{Ob}(C) \), are in \( B \). By Theorem 3.7 in order to show that \( B \) has the Ramsey property it suffices to construct a pre-adjunction \( F : \text{Ob}(B) \rightleftarrows \text{Ob}(C) : H \). For \( (B, \beta) \in \text{Ob}(B) \) put \( F(B, \beta) = B \), for \( C \in \text{Ob}(C) \) put \( H(C) = (E(C), \delta_C) \) and for \( u \in \text{hom}_C(B, C) \) put \( \Phi_{(B, \beta),C}(u) = E(u) \cdot \beta \).

Let us first show that the definition of \( \Phi \) is correct by showing that for each \( u \in \text{hom}_C(B, C) \) we have that \( \Phi_{(B, \beta),C}(u) \) is a coalgebraic homomorphism from \((B, \beta)\) to \( H(C) \), that is:

\[
\begin{array}{ccc}
B & \xrightarrow{E(u) \cdot \beta} & E(C) \\
\beta \downarrow & & \downarrow \delta_C \\
E(B) & \xrightarrow{E(E(u) \cdot \beta)} & EE(C)
\end{array}
\]

The following two diagrams commute because \((B, \beta)\) is a weak \( E \)-coalgebra and because \( \delta : E \to E E \) is a natural transformation, respectively:

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & E(B) \\
\beta \downarrow & & \downarrow E(\beta) \\
E(B) & \xrightarrow{\delta_B} & EE(B)
\end{array} \quad \begin{array}{ccc}
E(B) & \xrightarrow{E(u)} & E(C) \\
\delta_B \downarrow & & \downarrow \delta_C \\
EE(B) & \xrightarrow{EE(u)} & EE(C)
\end{array}
\]

It now easily follows that \( EE(u) \cdot \beta \cdot \delta_B = EE(u) \cdot \delta_B \cdot \beta = \delta \cdot E(u) \cdot \beta \).

To complete the proof we still have to show that the condition (PA) of Definition[1] is satisfied. Take any \( C \in \text{Ob}(C) \) and \((A, \alpha), (B, \beta) \in \text{Ob}(B) \), take arbitrary \( u \in \text{hom}_C(B, C) \) and an arbitrary coalgebraic homomorphism \( f \in \text{hom}_B((A, \alpha), (B, \beta)) \). Then, \( f \in \text{hom}_C(A, B) \) and \( \Phi_{(B, \beta),C}(u) \cdot f = E(u) \cdot \beta \cdot f = E(u) \cdot E(f) \cdot \alpha = E(u \cdot E(f)) \cdot \alpha = E(f(u \cdot E(f))) \cdot \alpha = \Phi_{(A, \alpha),C}(u \cdot f) \alpha \), having in mind that \( \beta \cdot f = E(f) \cdot \alpha \) because \( f \) is a coalgebraic homomorphism. This completes the proof. \( \square \)
As a warm-up for the discussion that follows we conclude the section by reproving a simple and well-known Ramsey result about finite trees \(3\) (see also \(9\) p. 28)). Let \((A, f)\) be a monounary algebra. We shall say that \(r \in A\) is a root if \(f(r) = r\). A rooted forest is a monounary algebra \((A, f)\) such that for every \(a \in A\) there is an \(n \geq 0\) and a root \(r\) such that \(f^n(a) = r\). A rooted tree is a rooted forest with a single root. Let \(Rf_{emb}\) denote the category of rooted forests with embeddings, and let \(Rf^\text{fin}_{emb}\) denote the full subcategory of \(Rf_{emb}\) spanned by its finite members.

An ordered rooted forest (resp. tree) is a rooted forest (resp. tree) together with a linear ordering of its vertices. An embedding between two ordered rooted forests is every monotone embedding of the underlying rooted forests. Let \(Orf_{emb}\) denote the category of ordered rooted forests with embeddings, and let \(Orf^\text{fin}_{emb}\) denote the full subcategory of \(Orf_{emb}\) spanned by its finite members. We are now going to prove the Ramsey property for the class of all finite ordered rooted forests by representing them as \(E\)-coalgebras for a modification of the comonad presented in Example 2.

**Example 5.** Fig. 1 depicts an ordered rooted forest. Note that every ordered rooted forest can be represented by an \(E\)-coalgebra straightforwardly: if \((A, f)\) is an ordered rooted forest define \(\alpha : A \to E(A)\) by \(\alpha(a) = \) the unique path from \(a\) to the root of its connected component.

**Theorem 3.9.** (\(3\); cf. \(9\) p. 28) The category \(Orf^\text{fin}_{emb}\) of finite ordered rooted forests and embeddings has the Ramsey property.

**Proof.** For a set \(A\) let \(A^\dagger\) denote the set of all finite sequences of pairwise distinct elements of \(A\). (Note that if \(A\) is finite then so is \(A^\dagger\).) Let \(E : Ch^\text{fin}_{emb} \to Ch^\text{fin}_{emb}\) be the functor such that \(E(A, <^A) = (A^\dagger, <_{lex}^A)\) and \(E(f)(a_1, \ldots, a_k) = (f(a_1), \ldots, f(a_k))\). If \(f : (A, <^A) \to (B, <^B)\) is an embedding then \(E(f) : (A^\dagger, <_{lex}^A) \to (B^\dagger, <_{lex}^B)\) is also an embedding, so the definition of \(E\) is correct. Finally, let us define comultiplication \(\delta : E \to EE\) as in Example 2

\[
\delta_A(a_1, \ldots, a_n) = ((a_1, a_2, \ldots, a_n), (a_2, \ldots, a_n), \ldots, (a_n))
\]

It is obvious that, given a chain \((A, <^A)\), the map \(\delta_A : E(A, <^A) \to EE(A, <^A)\) is an embedding of chains, so the definition of \(\delta\) is also correct.

It follows immediately from the construction that each \(\alpha\) is an embedding between the corres-ponding chains, and hence, a morphism in \(Ch^\text{fin}_{emb}\). Moreover, every coalgebra constructed in this manner is a weak Eilenberg-Moore coalgebra for \(E\) and \(\delta\). It is also easy to see that every embedding between two ordered rooted forests is clearly a coalgebraic homomorphism and vice versa. Moreover, every cofree Eilenberg-Moore \(E\)-coalgebra is a \(E\)-coalgebra representation of an ordered rooted forest. Therefore, \(Orf^\text{fin}_{emb}\) is a full subcategory of \(E\), the category whose objects are weak Eilenberg-Moore \(E\)-coalgebras and morphisms are coalgebraic homomorphisms, and contains all the cofree Eilenberg-Moore \(E\)-coalgebras. The claim now follows by Theorem 3.8 \(\square\)
Corollary 3.9.1. The category \( \text{Rf}^{\text{fin}}_{\text{emb}} \) of finite rooted forests and embeddings has finite Ramsey degrees.

Proof. Let \( U : \text{Orf}^{\text{fin}}_{\text{emb}} \to \text{Rf}^{\text{fin}}_{\text{emb}} \) be the forgetful functor that forgets the order. It is now easy to see that \( U \) is a reasonable expansion with unique restrictions. Since \( \text{Orf}^{\text{fin}}_{\text{emb}} \) has the Ramsey property and \( U^{-1}(A) \) is finite for every finite rooted forest \( A \) (because there are only finitely many linear orders on a finite set) Theorem 3.2 implies that \( \text{Rf}^{\text{fin}}_{\text{emb}} \) has finite Ramsey degrees. \( \square \)

4. RAMSEY PROPERTIES OF \( M \)-SETS

In this section we apply the proof strategies outlined in Section 3 by proving several Ramsey results for categories of \( M \)-sets. In 2016 Sokić proved that for a finite monoid \( M \) the class of all ordered finite \( M \)-sets has the Ramsey property [21]. Using a completely different strategy, in this section we prove that for any monoid \( M \) (finite or infinite) the category of all finite ordered \( M \)-sets with embeddings has the Ramsey property, and from that conclude that (unordered) finite \( M \)-sets have finite small Ramsey degrees. Moreover, we prove that for any monoid \( M \) (finite or infinite) finite ordered \( M \)-sets have finite big Ramsey degrees in the ordered cofree \( M \)-set \( \mathcal{E}(\omega) \) on \( \omega \)-generators and again infer the corresponding result for the unordered case.

Our proof mimics the proof of the fact that the category of weak Eilenberg-Moore coalgebras for a monad has the Ramsey property (Theorem 3.2). Unfortunately, we are unable to apply Theorem 3.2 directly because the construction in this section relies on the comonad \( E(A) = A^M \) of Example 3, which has the unpleasant property that \( E(A) \) is infinite in case \( M \) is an infinite monoid; and treating the case of infinite monoids is the key motivation for the paper.

We shall bypass this issue using the following compactness argument which was first proved for categories of structures in [15] (see also [16]), and for general categories in [11]. An \( \infty \)-ordered \( M \)-set has the Ramsey property [21]. Using a completely different strategy, in this section we apply the proof strategies outlined in Section 3 by proving several Ramsey properties of \( M \)-sets.

Let \( \mathbf{C} \) be a locally small category whose morphisms are mono, let \( \mathbf{B} \) be a full subcategory of \( \mathbf{C} \) such that \( \text{hom}(A, B) \) is finite for all \( A, B \in \text{Ob}(\mathbf{B}) \), and let \( F \in \text{Ob}(\mathbf{C}) \) be universal and weakly locally finite for \( \mathbf{B} \). Then for all \( A \in \text{Ob}(\mathbf{B}) \) and \( t \geq 2 \), \( \text{t}_{\mathbf{B}}(A) \leq t \) if and only if \( \mathbf{C} \to (\mathbf{B})_{t, t} \) for all \( k \geq 2 \) and all \( B \in \text{Ob}(\mathbf{B}) \) such that \( A \to B \).

Let \( \mathbf{M} \) be a monoid (finite or infinite). As we have already seen in Example 3 every \( \mathbf{M} \)-set can be represented by an Eilenberg-Moore coalgebra for the comonad \( E : \text{Set} \to \text{Set} \) defined by \( E(A) = A^M \) on objects and by \( E(f) : A^M \to B^M \) for \( f \in \text{mor}(A, B) \). In the proofs below we shall move freely between the two representations of \( \mathbf{M} \)-sets.

Let us now upgrade \( E \) and \( \delta \) to \( \text{Ch}_{\text{emb}} \). Take an arbitrary but fixed well-ordering of \( M \) such that \( \min M = 1 \). Recall that for every chain \( (X, <) \) the set \( X^M \) can be ordered lexicographically as follows: for \( f, g \in X^M \), \( f \neq g \) if \( f \prec_{\text{lex}} g \) if \( f(v) < g(v) \), where \( v = \min\{w \in M : f(w) \neq g(w)\} \). For a chain \( (X, <) \) let \( \hat{E}(X, <) = (X^M, <_{\text{lex}}) \). This is how \( \hat{E} \) acts on objects. For an embedding \( h : (X, <) \to (Y, <) \) define \( \hat{E}(h) : \hat{E}(X, <) \to \hat{E}(Y, <) \) as \( \hat{E}(h)(f) = E(h)(f) = h \cdot f \). To see that \( \hat{E} \) is well-defined take \( f, g \in X^M \) such that \( f \prec_{\text{lex}} g \). Let \( v = \min\{w \in M : f(w) \neq g(w)\} \). Then \( f(w) < g(w) \) for \( w < v \) and \( f(v) = g(v) \). Since \( h \) is an embedding we immediately get that \( h(f(w)) < h(g(w)) \) for \( w < v \) and \( h(f(v)) = h(g(v)) \). Whence \( \hat{E}(h)(f) <_{\text{lex}} \hat{E}(h)(g) \). In other words, \( \hat{E}(h) \) is an embedding \((X^M, <_{\text{lex}}) \to (Y^M, <_{\text{lex}})\).

Following Example 3 let us define comultiplication \( \hat{\delta}_{(X, <)} : \hat{E}(X, <) \to \hat{E}\hat{E}(X, <) \) by \( \hat{\delta}_{(X, <)}(h)(v)(w) = h(wv) \). Let us show that the definition is correct, that is, that \( \hat{\delta}_{(X, <)} \) is an embedding. Note that
\[\delta(X,<)(h)(1) = h.\] Take any \(f, g \in \hat{E}(X, <)\) such that \(f \leq_{lex} g\). Then \(\delta(X,<)(f)(1) < \delta(X,<)(g)(1)\), whence \(\hat{E}(X,<)(f) \leq_{lex} \hat{E}(X,<)(g)\) because the ordering of \(\hat{E}\hat{E}(X, <) = (\hat{E}(X, <))^M\) is lexicographic and \(M\) is well-ordered so that \(1 = \min M\).

**Lemma 4.2.** Every ordered \(M\)-set \(A = (A, a', <)\) where \(<\) is a linear order on \(A\), can be represented by a weak Eilenberg-Moore \(\hat{E}\)-coalgebra \(((A, <), \alpha)\), where the structure map \(\alpha : (A, <) \to \hat{E}(A, <)\) is defined by \(\alpha(a)(g) = a'(g, a)\).

**Proof.** Clearly, we only have to check that \(\alpha\) is an embedding. Take \(a_1, a_2 \in A\) such that \(a_1 < a_2\). Then \(\alpha(a_1)(1) = a_1\) and \(\alpha(a_2)(1) = a_2\), whence \(\alpha(a_1) \leq_{lex} \alpha(a_2)\) because the ordering of \(\hat{E}(A, <) = (A, <)^M\) is lexicographic and \(1 = \min M\).

Let \(\text{Oset}_{\text{emb}}(M)\) denote the category of ordered \(M\)-sets and embeddings understood as a full subcategory of \(\text{EM}^{\text{fin}}_{\text{emb}}(\hat{E}, \hat{\delta})\) using the correspondence given in the above lemma.

**Lemma 4.3.** Taking \(\text{EM}^{\text{fin}}_{\text{emb}}(\hat{E}, \hat{\delta})\) as the ambient category, for all \(\mathcal{U}, \mathcal{V} \in \text{Oset}_{\text{emb}}^{\text{fin}}(M)\) and all \(k \geq 2\) we have that \((\hat{E}(\omega), \hat{\delta}_\omega) \to (\mathcal{V})^M_k\).

**Proof.** Let \(B = \text{Oset}_{\text{emb}}(M)\) and let \(F : \text{Ob}(B) \to \text{Ob}(\text{Ch}_{\text{emb}}) : H\) be a pre-adjunction constructed as in the proof of Theorem 3.7 for \(B = ((B, <), \beta) \in \text{Ob}(B)\) put \(F(B) = (B, <)\), for \((C, <) \in \text{Ob}(\text{Ch}_{\text{emb}})\) put \(H(C, <) = (\hat{E}(C, <), \delta(C, <))\) and for \(u \in \text{hom}_{\text{Ch}_{\text{emb}}}((B, <), (C, <))\) put \(\Phi_{u, (C, <)}(u) = \hat{E}(u)\).

Take any \(\mathcal{U}, \mathcal{V} \in \text{Oset}_{\text{emb}}^{\text{fin}}(M)\) and any \(k \geq 2\). Since \(\text{Ch}_{\text{emb}}^{\text{fin}}\) has the Ramsey property there is a finite chain \((W, <)\) such that \((W, <) \to (F(V))^M_k\). By Theorem 3.7 (a) it then follows that \(H(W, <) = (\hat{E}(W, <), \delta(W, <)) \to (\mathcal{V})^M_k\). Now, take any embedding \(f : (W, <) \to \omega\). The fact that \(\hat{\delta}\) is natural yields that \(\hat{E}(f) : \hat{E}(W, <) \to \hat{E}(\omega)\) is a morphism in \(B\) from \((\hat{E}(W, <), \delta(W, <))\) to \((\hat{E}(\omega), \delta_\omega)\). Lemma 3.1 now ensures that \((\hat{E}(\omega), \delta_\omega) \to (\mathcal{V})^M_k\).

**Lemma 4.4.** Taking \(\text{EM}^{\text{fin}}_{\text{emb}}(\hat{E}, \hat{\delta})\) as the ambient category, \((\hat{E}(\omega), \hat{\delta}_\omega)\) is universal and weakly locally finite for \(\text{Oset}_{\text{emb}}^{\text{fin}}(M)\).

**Proof.** Let \(B = \text{Oset}_{\text{emb}}^{\text{fin}}(M)\). To see that \((\hat{E}(\omega), \hat{\delta}_\omega)\) is universal for \(B\) take any \(\mathcal{B} = ((B, <), \beta) \in \text{Ob}(B)\) and any embedding \(f : (B, <) \to (\omega, <)\). Then the square on the right commutes because \(\hat{\delta}\) is natural, while the square on the left commutes since \(\mathcal{B}\) is a weak Eilenberg-Moore \(\hat{E}\)-coalgebra:

\[
\begin{array}{ccc}
(B, <) & \xrightarrow{\beta} & \hat{E}(B, <) & \xrightarrow{\hat{E}(f)} & \hat{E}(\omega) \\
\downarrow & & \downarrow & & \\
\hat{E}(B, <) & \xrightarrow{\hat{E}(\beta)} & \hat{E}\hat{E}(B, <) & \xrightarrow{\hat{E}(E(f))} & \hat{E}\hat{E}(\omega)
\end{array}
\]

Therefore, \(\hat{E}(f) \cdot \beta \in \text{hom}(B, (\hat{E}(\omega), \hat{\delta}_\omega))\).

To see that \((\hat{E}(\omega), \hat{\delta}_\omega)\) is weakly locally finite for \(B\) take any \(\mathcal{A} = ((A, <), \alpha)\) and \(\mathcal{B} = ((B, <), \beta)\) in \(\text{Ob}(B)\) and arbitrary morphisms \(f : A \to (\hat{E}(\omega), \hat{\delta}_\omega)\) and \(g : B \to (\hat{E}(\omega), \hat{\delta}_\omega)\). Then \(f : (A, \alpha) \to (E(\omega), \delta_\omega)\) and \(g : (B, \beta) \to (E(\omega), \delta_\omega)\) are embeddings in \(\text{Set}_{\text{emb}}(M)\) of (unordered) \(M\)-sets and embeddings. So, \(f(A)\) and \(g(B)\) are carriers of two finite subcoalgebras of \((E(\omega), \delta_\omega)\). It is easy to see that \(C = f(A) \cup g(B)\) is then also a carrier of a finite subcoalgebra of \((E(\omega), \delta_\omega)\), so let \(\gamma : C \to E(C)\) be the structure map that turns \(C\) into a subcoalgebra \((C, \gamma)\) of \((E(\omega), \delta_\omega)\). Therefore, the following diagram commutes in \(\text{Set}_{\text{emb}}(M)\), where \(f_C : (A, \alpha) \to (C, \gamma)\) and \(g_C : (B, \beta) \to (C, \gamma)\) are codomain restrictions of \(f\) and \(g\), respectively:
Finally, let us order $C$ by restricting the linear ordering of $\hat{E}(\omega)$ to $C$. Then $((C, <), \gamma) \in \text{Ob}(B)$ and all the morphisms in the diagram above are embeddings. This concludes the proof that $(\hat{E}(\omega), \delta_\omega)$ is locally finite for $B$.

\[ \begin{array}{ccc}
\hat{C} & \xrightarrow{\delta} & \hat{C} \\
\uparrow & & \uparrow \gamma \\
(A, \alpha) & \xrightarrow{f_c} & (B, \beta) \\
\end{array} \]

**Theorem 4.1.** (a) Let $M$ be an arbitrary monoid (finite or infinite). Then the category $\text{Oset}_{\text{emb}}^\text{fin}(M)$ of ordered finite $M$-sets and embeddings has the Ramsey property.

(b) For every group $G$ the category $\text{Oset}_{\text{emb}}^\text{fin}(G)$ of ordered finite $G$-sets and embeddings has the Ramsey property (for finite groups this was proved in [21]).

(c) For every unary algebraic language $\Omega$ the category $\text{Oalg}_{\text{emb}}^\text{fin}(\Omega)$ of ordered finite $\Omega$-algebras and embeddings has the Ramsey property (for finite unary languages $\Omega$ this was proved in [21]).

**Proof.** (a) Fix a well-ordering of $M$ such that $1 = \min M$. Using this well-ordering construct the functor $\hat{E} : \text{Ch}_{\text{emb}} \to \text{Ch}_{\text{emb}}$ and the comultiplication $\hat{\delta} : \hat{E} \to \hat{E}\hat{E}$ as above. For notational convenience let $C = \text{Oset}_{\text{emb}}^\text{fin}(M)$. Recall that $C$ is a category of finite weak Eilenberg-Moore $\hat{E}$-coalgebras (see Lemma 4.2 and the remark that follows). Take $\text{EM} = \text{EM}_{\text{emb}}^w(\hat{E}, \hat{\delta})$ as the ambient category. We have seen in Lemma 4.1 that $(\hat{E}(\omega), \hat{\delta}_\omega)$ is universal and locally finite for $C$. Lemma 4.3 shows that for all $A, B \in \text{Ob}(C)$ and all $k \geq 2$ we have that $(\hat{E}(\omega), \hat{\delta}_\omega) \to (B)^A_{\hat{E}}$. Therefore, by Lemma 4.4 we have that $t_C(A) = 1$ for all $A \in \text{Ob}(C)$. This is just another way of saying that $C$ has the Ramsey property.

(b) is a special case of (a) where $M = G$ is a group.

(c) is a special case of (a) where $M = \Omega^*$.

**Corollary 4.1.1.** For every monoid $M$ the category $\text{Set}_{\text{emb}}^\text{fin}(M)$ of finite $M$-sets and embeddings has finite small Ramsey degrees. In particular,

- for every group $G$ the category $\text{Set}_{\text{emb}}^\text{fin}(G)$ of finite $G$-sets and embeddings has small Ramsey degrees; and
- for every unary algebraic language $\Omega$ the category $\text{Alg}_{\text{emb}}^\text{fin}(\Omega)$ of finite $\Omega$-algebras and embeddings has small Ramsey degrees.

**Proof.** Let $U : \text{Oset}_{\text{emb}}^\text{fin}(M) \to \text{Set}_{\text{emb}}^\text{fin}(M)$ be the forgetful functor that forgets the order. It is now easy to see that $U$ is a reasonable expansion with unique restrictions. Since $\text{Oset}_{\text{emb}}^\text{fin}(M)$ has the Ramsey property and $U^{-1}(A)$ is finite for every finite $M$-set $A$ (because there are only finitely many linear orders on a finite set) Theorem 4.2 implies that $\text{Set}_{\text{emb}}^\text{fin}(M)$ has finite small Ramsey degrees.

**Theorem 4.2.** Let $M$ be an arbitrary monoid (finite or infinite). There exists an ordering $\hat{E}(\omega)$ of the cofree $M$-set $E(\omega)$ on $\omega$ generators such that every finite ordered $M$-set has finite big Ramsey degree in $\hat{E}(\omega)$. More precisely, for every finite ordered $M$-set $A = (A, \alpha, <)$ we have $T(A, \hat{E}(\omega)) \leq 2^{|A|} - 1$ in $\text{Oset}_{\text{emb}}(M)$.

**Proof.** Fix a well-ordering of $M$ such that $1 = \min M$. Using this well-ordering construct the functor $\hat{E} : \text{Ch}_{\text{emb}} \to \text{Ch}_{\text{emb}}$ and the comultiplication $\hat{\delta} : \hat{E} \to \hat{E}\hat{E}$ as above. For notational convenience let $C = \text{Oset}_{\text{emb}}^\text{fin}(M)$. Recall that $C$ is a category of finite weak Eilenberg-Moore $\hat{E}$-coalgebras (see Lemma 4.2 and the remark that follows). Let $\hat{E}(\omega) = (\hat{E}(\omega), \hat{\delta}_\omega)$.
Take any $A = ((A, <), \alpha) \in C^{\text{fin}}$ where $(A, <) = \{a_1 < a_2 < \cdots < a_s\}, s = |A|$, and let us show that $T_C(A, E(\omega)) \leq 2^{s-1}$. Let $\chi : \text{hom}(A, E(\omega)) \rightarrow k, k \geq 2$, be an arbitrary coloring.

Let $n = 2^{s-1}$ and let $A_1, \ldots, A_n$ be all the subchains of $A$ that contain $a_1$. As a notational convenience, let $R = \text{hom}(A, E(\omega))$ and let $S_i = \text{hom}_{\text{ch-mon}}(A_i, \omega), 1 \leq i \leq n$. Take any $f \in R$ and let $f(a_i) = h_i \in \omega^M$, $1 \leq i \leq s$. Since $f$ is an embedding and $\omega^M$ is ordered lexicographically, we have that $h_1(1) \leq h_2(1) \leq \cdots \leq h_s(1)$. Define an equivalence relation $\rho$ on $A$ by $(a_i, a_j) \in \rho$ iff $h_i(1) = h_j(1)$ and let $A/\rho = \{B_1, \ldots, B_m\}$ where $\min B_1 < \cdots < \min B_m$. Let $\min B_i = a_{j_i}, 1 \leq i \leq m$ and note that $j_1 = 1$. Therefore, $\{a_{j_1}, \ldots, a_{j_m}\}$ is a subset of $A$ that contains $a_1$, say, $A_\ell = \{a_{j_1}, \ldots, a_{j_m}\}$. Finally, $f^* : A_\ell \rightarrow \omega : a_{j_1} \mapsto h_j(1)$ is clearly an embedding, see Fig. 2. This defines a mapping $\pi : R \rightarrow \bigcup_{\ell=1}^n S_\ell : f \mapsto f^*$.

Claim 1. $\pi$ is injective.

Proof. Assume that $\pi(g_1) = \pi(g_2) = f^*$ where $f^* : A_\ell \rightarrow \omega$ for some $A_\ell = \{a_{j_1} < \cdots < a_{j_m}\} \subseteq A$ with $j_1 = 1$. Then $g_1$ and $g_2$ are coalgebraic homomorphisms between the unordered coalgebras $(A, \alpha)$ and $E(\omega)$ constructed for the Set-monad $(E, \delta, \varepsilon)$, Example 3. Define a mapping $f : A \rightarrow \omega$ so that

$$f^*(a_{j_1}) = f(a_{j_1}) = f(a_{j_1+1}) = \cdots = f(a_{j_2-1}),$$

$$f^*(a_{j_2}) = f(a_{j_2}) = f(a_{j_2+1}) = \cdots = f(a_{j_3-1}),$$

$$\vdots$$

$$f^*(a_{j_m}) = f(a_{j_m}) = f(a_{j_m+1}) = \cdots = f(a_s).$$

Then the definition of $\pi$ implies that $\varepsilon_\omega \cdot g_1 = f$ and $\varepsilon_\omega \cdot g_2 = f$. Since $E(X)$ is a cofree $E$-coalgebra, $g_1 = g_2$ by Lemma 2.1. This concludes the proof of Claim 1.

Define $\gamma : \pi(R) \rightarrow k$ by $\gamma(\pi(f)) = \chi(f)$ so that $\chi(R) = \gamma(\pi(R))$ and then define $\gamma_i : S_i \rightarrow k, 1 \leq i \leq n$, by

$$\gamma_i(h) = \begin{cases} 
\gamma(h), & h \in \pi(R) \cap S_i, \\
0, & \text{otherwise}.
\end{cases}$$

Let us construct $\gamma_i : S_i \rightarrow k$ and $w_i \in \text{hom}_{\text{ch-mon}}(\omega, \omega)$, $i \in \{1, \ldots, n\}$, inductively as follows. First, put $\gamma'_i = \gamma_n$. Given a coloring $\gamma'_i : S_i \rightarrow k$, construct $w_i$ by the Infinite Ramsey Theorem (Theorem 3.3): since $\omega \rightarrow (\omega)^k_A$, there is a $w_i \in \text{hom}_{\text{ch-mon}}(\omega, \omega)$ such that $|\gamma'_i(w_i \cdot S_i)| \leq 1$. Finally, given $w_i \in \text{hom}_{\text{ch-mon}}(\omega, \omega)$ define $\gamma'_{i-1} : S_{i-1} \rightarrow k$ by $\gamma'_{i-1}(f) = \gamma_{i-1}(w_n \cdot \cdots \cdot w_i \cdot f)$. Now, put $u = w_n \cdot \cdots \cdot w_1 \in \text{hom}_{\text{ch-mon}}(\omega, \omega)$ and let us show that $|\chi(\tilde{E}(u) \cdot R)| \leq n$.

Claim 2. $\pi(\tilde{E}(u) \cdot R) = u \cdot \pi(R) \subseteq \pi(R)$.

Proof. Since $\tilde{E}(u) \cdot R \subseteq R$ it immediately follows that $\pi(\tilde{E}(u) \cdot R) \subseteq \pi(R)$. To see that $\pi(\tilde{E}(u) \cdot R) = u \cdot \pi(R)$ take any $f \in R$, let $g = \tilde{E}(u) \cdot f$ and let us show that $g^* = u \cdot f^*$. Following
The definition of $f^*$ let $f(a_i) = h_i$, $1 \leq i \leq s$. Then $g(a_i) = (\hat{E}(u) \cdot f)(a_i) = \hat{E}(u)(f(a_i)) = \hat{E}(u)(h_i) = u \cdot h_i$. As above, we easily conclude that $h_1(1) \leq h_2(1) \leq \cdots \leq h_s(1)$ because $f$ is an embedding and $\omega^M$ is ordered lexicographically. Since $u : \omega \to \omega$ is an embedding, $u \cdot h_i(1) \leq u \cdot h_2(1) \leq \cdots \leq u \cdot h_s(1)$. Moreover, $h_i(1) = h_j(1)$ iff $u \cdot h_i(1) = u \cdot h_j(1)$ for all $1 \leq i, j \leq s$. The definition of $f^*$ then immediately gives us that $g^* = u \cdot f^*$. This concludes the proof of the claim.

Therefore, $\chi(\hat{E}(u) \cdot R) = \gamma(\pi(\hat{E}(u) \cdot R)) = \gamma(u \cdot \pi(R))$. Since $\pi(R) \subseteq \bigcup_{i=1}^n S_i$,

$$|\gamma(u \cdot \pi(R))| \leq |\gamma(u \cdot \bigcup_{i=1}^n S_i)| = |\bigcup_{i=1}^n \gamma(u \cdot S_i)| \leq \sum_{i=1}^n |\gamma(u \cdot S_i)|.$$  

Fix an $i \in \{1, \ldots, n\}$. Clearly, $u \cdot S_i \subseteq S_i$ and $w_1 \cdot \ldots \cdot w_i \cdot S_i \subseteq w_1 \cdot S_i$, whence

$$|\gamma(u \cdot S_i)| = |\gamma_i(w_1 \cdot \ldots \cdot w_i \cdot S_i)| = |\gamma_i'(w_1 \cdot \ldots \cdot w_i \cdot S_i)| \leq |\gamma_i'(w_1 \cdot S_i)| \leq 1.$$

Putting it all together, we finally get $|\chi(\hat{E}(u) \cdot R)| = |\gamma(u \cdot \pi(R))| \leq \sum_{i=1}^n |\gamma(u \cdot S_i)| \leq n = 2^{s-1}$.

This completes the proof. \hfill $\Box$

**Corollary 4.2.1.** Let $M$ be an arbitrary monoid (finite or infinite). Every finite $M$-set has a finite big Ramsey degree in $\mathcal{E}(\omega)$, the cofree $M$-set on $\omega$ generators. More precisely, for every finite $M$-set $A$ with $n$ elements we have $T(A, \mathcal{E}(\omega)) \leq n! \cdot 2^{n-1}$ in $\text{Set}_{\text{emb}}(M)$.

In particular, for every finite $G$-set has a finite big Ramsey degree in the cofree $G$-set on $\omega$ generators; and for every unary algebraic language $\Omega$ every finite $\Omega$-algebra has a finite big Ramsey degree in the cofree $\Omega$-algebra on $\omega$ generators.

**Proof.** As a matter of notational convenience put $\mathbf{C} = \text{Oset}_{\text{emb}}(M)$ and $\mathbf{B} = \text{Set}_{\text{emb}}(M)$. Let $U : \mathbf{C}^{\text{op}} \to \mathbf{B}^{\text{op}}$ be the forgetful functor that forgets the order. To see that $U$ has unique restrictions we can repeat the argument from the proof of Corollary 1.1.1

Let $A = (A, \Omega^A) \in \text{Ob}(\mathbf{B}^{\text{fin}})$ be an arbitrary finite $M$-set. As we have seen in Theorem 4.2 there exists a linear ordering $\hat{E}(\omega)$ of $\mathcal{E}(\omega)$ such that every finite ordered $M$-set with $n$ elements has a big dual Ramsey degree in $\hat{E}(\omega)$ which does not exceed $2^{n-1}$. Therefore, for every linear ordering $\prec$ of $A$ we have that $T_{\mathcal{C}}((A, \alpha, \prec), \hat{E}(\omega)) \leq 2^{n-1}$. Since $U^{-1}(A)$ is finite (because there are only finitely many linear orders on a finite set) Theorem 3.4 tells us that $T_{\mathbf{B}}(A, \mathcal{E}(\omega)) \leq \sum_{A^* \in U^{-1}(A)} T_{\mathcal{C}}(A^*, \hat{E}(\omega)) \leq n! \cdot 2^{n-1}$, where $n = |A|$. This completes the proof. \hfill $\Box$

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