ON PRIMITIVITY OF SETS OF MATRICES
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Abstract. A nonnegative matrix $A$ is called primitive if $A^k$ is positive for some integer $k > 0$. A generalization of this concept to sets of matrices is as follows: a set of matrices $\mathcal{M} = \{A_1, A_2, \ldots, A_m\}$ is primitive if $A_{i_1}A_{i_2} \cdots A_{i_k}$ is positive for some indices $i_1, i_2, \ldots, i_k$. The concept of primitive sets of matrices is of importance in several applications, including the problem of computing the Lyapunov exponents of stochastic switching systems. In this paper, we analyze the computational complexity of deciding if a given set of matrices is primitive and we derive bounds on the length of the shortest positive product.

We show that while primitivity is algorithmically decidable, unless $P = NP$ it is not possible to decide primitivity of a matrix set in polynomial time. Moreover, we show that the length of the shortest positive sequence can be superpolynomial in the dimension of the matrices. On the other hand, defining $\mathcal{P}$ to be the set of matrices with no zero rows or columns, we give a simple combinatorial proof that primitivity can be tested in polynomial time when all the matrices are in $\mathcal{P}$. This latter observation is related to the well-known 1964 conjecture of Černý on synchronizing automata; in fact, any bound on the minimal length of a synchronizing word for synchronizable automata immediately translates into a bound on the length of the shortest positive product of a primitive set of matrices in $\mathcal{P}$. In particular, any primitive set of $n \times n$ matrices in $\mathcal{P}$ has a positive product of length $O(n^3)$.

1. Introduction. A $n \times n$ nonnegative matrix $A \geq 0$ is said to be primitive if $A^k$ is positive (i.e. every entry in $A^k$ is positive, which we denote $A^k > 0$) for some positive integer $k$. It is well-known (see [11], Corollary 8.5.9) that this is the case if and only if $A^{n^2-2n+2} > 0$ and so the primitivity of a matrix is easy to verify algorithmically. A straightforward generalization of the concept of primitive matrix to sets of matrices is the following [21]: a finite set of nonnegative matrices $\mathcal{M} = \{A_1, A_2, \ldots, A_m\}$ is primitive if $A_{i_1}A_{i_2} \cdots A_{i_k} > 0$ for some indices $i_1, i_2, \ldots, i_k \in \{1, \ldots, m\}$.

The property of primitivity of a set of matrices is important in several applications. In particular, its presence enables one to use efficient algorithms for the computation of the Lyapunov exponent of a stochastic switching system (See [12],[15],[24] for introductions on switching systems). Given a finite set of matrices $\mathcal{M} \subset \mathbb{R}^{n \times n}$, one can define a stochastic switched system as:

$$x_{k+1} = A_{i_k}x_k, \quad A_{i_k} \in \mathcal{M}.$$  (1.1)

where for simplicity we assume each $A_{i_k}$ is uniformly distributed in $\mathcal{M}$. Such models are commonly used throughout stochastic control; for example, they are a common choice for modeling manufacturing systems with random component failures (see [5], Chapter 1). The Lyapunov exponent of this system is defined by the following limit:

$$\lambda = \lim_{k \to \infty} \frac{1}{k} \mathbb{E} \log \| A_{d_k} \cdots A_{d_1} \|.$$  (1.2)

The Lyapunov exponent characterizes the rate of growth of the switching system. More precisely, we have the following theorem:

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Theorem 1.1. \cite{11} Suppose that the sequence $A_t$ of matrices appearing in Eq. (1.1) is i.i.d. Then, $x_t$ converges to zero with probability one if and only if
\[
\lambda = \lim_{k \to \infty} \frac{1}{t} E \log \| A_{d_k} \cdots A_{d_1} \| < 1, \tag{1.3}
\]
and moreover, with probability one, we have
\[
\lim \log |x_k|/k = \lambda.
\]

While the Lyapunov exponent $\lambda$ is hard to compute in general \cite{27}, it turns out that in the particular case of primitive sets of matrices, efficient algorithms can be obtained \cite{17,18,20,22}.

In this paper, we study the problem of recognizing primitivity and related problems. Given a set of $n \times n$ nonnegative matrices $\mathcal{M} = \{A_1, \ldots, A_m\}$ one would like to determine, efficiently if possible, whether or not $\mathcal{M}$ is primitive. This is closely related to the problem of bounding the length of the shortest positive product of matrices from $\mathcal{M}$, which we denote by $l(\mathcal{M})$. Indeed, an upper bound on $l(\mathcal{M})$ immediately translates into algorithms for checking primitivity by simply checking every possible product of length smaller or equal to this bound.

1.1. Our results. This paper is consequently concerned with upper bounds on the length of $l(\mathcal{M})$ as well as algorithms and complexity of verifying existence of a positive product of matrices taken in a given set $\mathcal{M}$. Our main results are:

1. We show in Section II that recognizing primitivity is decidable but NP-hard as soon as there are three matrices in the set. Primitivity can be decided in polynomial time for one matrix and so we leave the computational complexity of the case of two matrices unresolved.

2. We also show in Section II that the shortest positive product may have a length that is superpolynomial in the dimension of the matrices, even with a bounded number of matrices in the set.

3. We consider in Section III the special case of matrices in the set $\mathcal{P}$ of matrices with no zero rows or columns. We provide a combinatorial proof that, for such matrices, primitivity can then be decided in polynomial time. This resolves an open question of Protasov and Voynov \cite{21}.

4. We also prove in Section III that for primitive sets of matrices in $\mathcal{P}$, the shortest positive product has length $O(n^3)$. Moreover, we show that in this case the length of the shortest positive product is related to the well-known (and unresolved) conjecture of Černý on synchronizing automata. In particular, we show that resolution of the Černý conjecture would improve the above bound to $O(n^2)$. Moreover, any upper bound on the length of shortest synchronizing word for a synchronizable automaton immediately translates into a bound on the length of the shortest positive product of a set of primitive matrices in $\mathcal{P}$.

1.2. Related work. The concept of primitive matrix families was pioneered in the recent paper \cite{21}, which extended the classical Perron-Frobenius theory and provided a structure theorem for the primitive matrix sets in $\mathcal{P}$. A consequence of this theorem was that for matrices in $\mathcal{P}$ primitivity can be tested in polynomial time. The proofs were based on a somewhat involved spectral analysis, and the question of finding a combinatorial proof was left open.

A follow-up paper \cite{19} considered the related question of characterizing nonnegative families which have a positive product of length equal to the number of matrices
Two recent papers, appearing simultaneously with the conference version of this paper [3], have also resolved parts of items 3 and 4 above. The paper [28] proved an $O(n^3)$ bound on the length of the shortest positive product of a set of primitive matrices in $P$, and the paper [1] provided a combinatorial proof that primitivity can be tested in polynomial time for this class. However, we remark that the combinatorial proof in this paper appears to be considerably shorter than the one from [1], and that our upper bound is tighter than the one from [28].

2. The general case. In this section, we study the problem of recognizing primitivity: given a set of $n \times n$ nonnegative matrices $M = \{A_1, \ldots, A_m\}$ does there exist an efficient algorithm that determines whether $M$ is primitive? This is closely related to the problem of bounding the length of the shortest positive product of matrices from $M$, which we denote by $l(M)$. Indeed, upper bounds on $l(M)$ immediately translate into algorithms for checking primitivity by simply checking every possible product of length smaller than this bound.

Unfortunately, without any further assumptions on the matrices $A_1, \ldots, A_m$ our main results in this section are rather pessimistic. Theorem 2.6 shows that whenever the number of matrices $m$ is at least 3, testing primitivity is NP-hard; thus there exists no algorithm for recognizing the primivity of a set of $m$ matrices in $\mathbb{R}^{n \times n}$ with running time polynomial in $m$ and $n$ unless $P = NP$. Furthermore, Theorem 2.10 shows that the length of the shortest positive product can be exponentially large in the dimension $n$.

While these results demonstrate that the problem of checking primitivity is intractable in general, we note that it may become tractable under additional assumptions on the matrices $A_1, \ldots, A_m$; indeed, Section 3 is dedicated to the study of a class of matrices for which recognizing primitivity has polynomial complexity and the length of the shortest possible product is polynomial in $m$ and $n$.

We now begin with a sequence of definitions and lemmas which will ultimately result in a proof of the aforementioned results, namely Theorem 2.6 on NP-hardness and Theorem 2.10 on the length of the shortest positive product. We will find it more convenient to make our arguments in terms of graphs rather than matrices; our starting point is the following definition which gives a natural way to associate matrices with directed graphs.

**Definition 2.1.** Given a (directed) graph $G = (V, E)$, the adjacency matrix of $G$, denoted by $A(G)$, is defined as

$$[A(G)]_{ij} = \begin{cases} 1, & \text{if } j \in N_i(G), \\ 0, & \text{otherwise.} \end{cases}$$

where $N_i(G)$ is the out-neighborhood of node $i$ in $G$ (i.e., $N_i(G) = \{j \mid (i, j) \in E\}$). Conversely, given a nonnegative matrix $M \in \mathbb{R}^{n \times n}$, we will use $G(M)$ to denote the (directed) graph with vertex set $\{1, \ldots, n\}$ and edge set $\{(i, j) \mid M_{ij} > 0\}$.

It is standard observation that entries of the product $A[G_1]A[G_2]\cdots A[G_l]$ are related to the number of paths in the graph sequence $G_1, G_2, G_3, \ldots, G_l$. After formally defining the notion of a path in a graph sequence next, we state the relationship in a lemma.
Definition 2.2. Let $G_1, G_2, \ldots, G_l$ be a sequence of graphs all with the same vertex set $V$, and let us adopt the notation $E_k$ for the edge set of $G_k$. For vertices $a, b \in V$ we will say that there exists a path from $a$ to $b$ in $G_1, \ldots, G_l$ if there exists a sequence of vertices $i_1, \ldots, i_{l+1}$ such that

- $i_1 = a$ and $i_{l+1} = b$.
- For each $k = 1, \ldots, l$ we have that $(i_k, i_{k+1}) \in E_k$.

We will say that a node $b$ is reachable from $a$ in the sequence $G_1, \ldots, G_l$ if there exists a path from $a$ to $b$ in that sequence. Given the graph sequence $G_1, \ldots, G_l$ and node $a \in V$, we will use the notation $R_a(k)$ to denote the set of reachable vertices in the sequence $G_1, \ldots, G_k$, where $k \leq l$. We will adopt the convention that $R_a(0) = \{a\}$ for all $a \in V$. Finally, we will say that $b$ is reachable from $a$ in $l$ steps of $G_1, \ldots, G_p$ if there exists a sequence of length $l$ consisting of the graphs from $\{G_1, \ldots, G_p\}$ in which $b$ is reachable from $a$.

The following lemma (which we state without proof and which follows straightforwardly from the definition of matrix multiplication) states the usual correspondence between entries of $A[G_1] \cdots A[G_l]$ and paths in the sequence $G_1, \ldots, G_l$.

Lemma 2.3. The number of paths from $i$ to $j$ in the sequence $G_1, G_2, \ldots, G_l$ is the $i, j$'th entry of the product $A[G_1] A[G_2] \cdots A[G_l]$. Moreover, there exists a path from $i$ to $j$ in the sequence $G_1, \ldots, G_l$ if and only if the $i, j$'th entry of the product $P_1 \cdots P_l$ is positive, where for $i = 1, \ldots, l$, $P_i$ is any nonnegative matrix satisfying $G(P_i) = G_i$.

Thus the primitivity problem for the matrix set $\{A[G_1], \ldots, A[G_k]\}$ is equivalent to the problem of finding a sequence of the graphs $G_1, \ldots, G_k$ such that there is at least one path from every node to every other node. We will make use of this interpretation shortly.

We now define the 3-SAT satisfiability problem, which is well known to be NP-hard. We will prove below that the primitivity problem is NP-hard by reducing the 3-SAT problem to it.

Definition 2.4. Let $x_1, \ldots, x_n$ be Boolean variables. Both $x_i$ and its negation $\overline{x_i}$ are called literals. A clause is a disjunction (logical OR) of three literals, for example $x_1 \lor x_2 \lor x_3$ or $\overline{x_1} \lor x_2 \lor x_3$. A 3-CNF formula is the conjunction (logical AND) of clauses. For example,

$$f = (x_1 \lor x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3})$$

is a 3-CNF formula, being a conjunction of two clauses. The number of clauses in a 3-CNF formula is usually denoted by $K$. Given a 3-CNF formula $f$, the 3-SAT problem asks for an assignment of the values $\{0, 1\}$ to the variables $x_1, \ldots, x_n$ so that the formula evaluates to 1. If such an assignment exists, the formula is called satisfiable and the corresponding assignment is called a satisfying assignment.

Our first step is to associate several graphs with a given 3-CNF formula, as explained in the following definition.
Definition 2.5. Given a 3-CNF formula \( f \) on \( n \) variables with \( K \) clauses, we define three graphs \( G_1(f) \), \( G_2(f) \), \( G_3(f) \). Figures 2.1 and 2.2 show the graphs for the formula \( (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \). We recommend the reader to refer to the figures while going through our description below.

All three graphs will have the same vertex set. We will have a “source node” \( u \). For each \( i = 1, \ldots, K \), we will have the \( n \) nodes \( u^i_1, \ldots, u^i_n \) and the \( n - 1 \) nodes \( l^i_2, \ldots, l^i_n \). We will also have the “failure node” \( f^i \) and the “success node” \( s^i \); these nodes will also be referred to as \( u^i_{n+1} \) and \( l^i_{n+1} \), respectively.

For each \( i = 1, \ldots, K \) and \( j = 1, \ldots, n \), if clause \( i \) is satisfied by setting \( x_j = 1 \), we put an edge going from \( u^i_j \) to \( l^i_{j+1} \) in \( G_1(f) \). Else, we put an edge going from \( u^i_j \) to \( u^i_{j+1} \) in \( G_1(f) \).

Similarly, for each \( i = 1, \ldots, K \) and \( j = 1, \ldots, n \), if clause \( i \) is satisfied by setting \( x_j = 0 \), we put an edge going from \( u^i_j \) to \( l^i_{j+1} \) in \( G_2(f) \). Else, we put an edge going from \( u^i_j \) to \( u^i_{j+1} \) in \( G_2(f) \).

We then add the following edges to both \( G_1(f) \) and \( G_2(f) \): edges from \( l^i_j \) to \( l^i_{j+1} \) for all \( j = 2, \ldots, n \) and \( i = 1, \ldots, K \); self-loops at all nodes \( f^i \); edges leading from each \( s^i \) to each \( f^i \); and edges leading from \( u \) to all \( u^i_1 \).

Finally, \( G_3(f) \) has edges leading from each \( f^i \) and each \( s^i \) to \( u \), as well as edges leading from each \( s^i \) to every node which bears the superscript \( i \). Note that \( G_3(f) \) does not depend on \( f \) in the sense that it is the same for all formulas with the same number of variables and same number of clauses.

We remark that this construction is a variation of one of the constructions from the earlier work [27]. It appears somewhat unwieldy at first glance, but the subsequent lemmas will provide some insights into it. First, however, we state our first main result.

\[\text{Note that clause } i \text{ may not contain } x_j \text{ or its negation } \neg x_j; \text{ in that case, neither setting the variable } x_j \text{ to zero nor to one will satisfy the clause, and consequently we will not have a link from } u^i_j \text{ to } l^i_{j+1} \text{ in either of } G_1(f) \text{ and } G_2(f).\]
of this section, Theorem 2.6, which provides a reduction from 3-SAT to checking primitivity of a set of three matrices.

**Theorem 2.6.** The 3-SAT formula \( f \) has a satisfying assignment if and only if the matrix set \( \{ A[G_1(f)], A[G_2(f)], A[G_3(f)] \} \) is primitive. Consequently, there is no algorithm for deciding matrix primitivity which scales polynomially in \( n \) unless \( P = NP \).

We now begin the proof of this theorem. We will assume henceforth that \( f \) is a fixed formula, and correspondingly we will simply write \( G_1, G_2, G_3 \) for the three graphs. We begin with the key lemma which encapsulates the most important property of these graphs. We remark that this is a variation of a lemma from [427] used to establish the complexity of closely related problems.

**Lemma 2.7.** Consider a sequence of length \( n \) of graphs from \( \{ G_1, G_2 \} \) and set \( x_i = 1 \) if the \( i \)’th graph is \( G_1 \), and \( x_i = 0 \) if the \( i \)’th graph is \( G_2 \). We have that \( s' \) is reachable
from $u_i$ in this sequence if and only if the $i$’th clause of $f$ is satisfied by the assignment $x_1, \ldots, x_n$.

Proof. By construction, an edge goes from $u_i$ to $l_i+1$ in $G_1$ whenever setting $x_j = 1$ satisfies the $i$’th clause of $f$, and the same edge is present in $G_2$ whenever setting $x_j = 0$ satisfies the clause of $f$. Thus clause $i$ is satisfied by the assignment of $x_1, \ldots, x_n$ defined in this lemma if and only if the corresponding sequence of $G_1, G_2$ includes an edge from some $u_i$ to $l_i+1$. But then the presence of edges from each $l_i$ to $l_{i+1}$ implies this happens if and only if $s^i = l_{n+1}$ is reachable after $n$ steps from $u_1$.

This simple lemma is an important ingredient of our proof of Theorem 2.6. Indeed, to prove this theorem we need to relate the satisfiability of $f$ to the primitivity of the matrix set $\{A[G_1], A[G_2], A[G_3]\}$. The latter, as a consequence of Lemma 2.3, can be recast as a question about the existence of a sequence with a path from every node to every other node; we thus need to somehow relate path-existence questions to satisfiability questions. This is precisely what is done by this previous lemma.

We sharpen the conclusions of this lemma with the following corollary, which follows from the fact that $\{u_1, \ldots, u^K\}$ is the set of out-neighbours of $u$:

Corollary 2.8. Consider a sequence of length $n + 1$ of graphs from $\{G_1, G_2\}$, and define $x_i = 1$ if the $i + 1$’st graph is $G_1$, and $x_i = 0$ if the $i + 1$st graph is $G_2$. We have that all $s^i$ are reachable from $u$ in this sequence if and only if this $x_1, \ldots, x_n$ is a satisfying assignment for $f$.

We are now essentially ready to provide a proof of Theorem 2.6. However, before embarking on the details of the proof, we collect a number of straightforward observations about the graphs $G_1, G_2, G_3$ in a remark.

Remark 2.9.

- The set of reachable nodes from $u$ in strictly more than $n + 1$ steps of $G_1, G_2$ is a nonempty subset of the failure nodes $f^i$. Indeed, this follows from the previous item and the observation that the only outgoing link from $s^i$ in these graphs leads to $f^i$, and the only outgoing link from $f^i$ leads to itself.

- The nodes reachable from a node $v \neq u$ in $n + 1$ steps of $G_1, G_2$ or more are a nonempty subset of the set of failure nodes. The argument for this is identical to the argument for the previous item.

- Consider the sequence $G_1, G_1, \ldots$ repeated $n + 1$ times. Regardless of the starting vertex, the only reachable vertices are success nodes and failure nodes. This follows by the previous item and the fact that $G_3$ has a single outgoing edge from each failure node to $u$. 

Proof of Theorem 2.6. Suppose first that the 3-SAT problem has a satisfying assumption. Consider the graph sequence of length $2n+4$ defined as follows:

1. First, we repeat the graph $G_1$ $n+1$ times.
2. The $n+2$nd graph equals $G_3$ and the $n+3$rd graph is $G_1$
3. For $k = 1, \ldots, n$ the $n+3+k$th graph is $G_1$ if $x_k = 1$ in the satisfying assignment and $G_2$ if $x_k = 0$ in the satisfying assignment.
4. Finally, $G_{2n+4} = G_3$.

For example, for the formula $(x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_2 \lor \neg x_3)$, we have the satisfying assumption $x_1 = 0, x_2 = 0, x_3 = 1$; the corresponding sequence is

$$G_1, G_1, G_1, G_1, G_2, G_2, G_1, G_3.$$

We claim that there are paths from every node to every other node in this sequence, which by Lemma 2.3 implies that the corresponding matrix product is positive. Indeed, as noted in Remark 2.9 after repeating $G_1$ $n+1$ times, we have that for any node $a \in V$, $R_a(n+1)$ is a nonempty subset of the set of failure nodes and success nodes. After graph $G_3$ on the $n+2$nd step, we have $u \in R_a(n+2)$; and after $G_1$ on the $n+3$rd step, we have $R_a(n+3) = \{u_1', u_2', \ldots, u_k\}$.

Now appealing to Lemma 2.7 we see that $R_a(2n+3)$ contains the set of all the success node $\{s^1, \ldots, s^n\}$. When we apply the graph $G_3$ on the $2n+4$th step, it follows that every node becomes reachable.

Conversely, suppose that there exists a positive product of $A[G_1], A[G_2], A[G_3]$. By Lemma 2.3 there exists a sequence of $G_1, G_2, G_3$ such that, in particular, there is a path from $u$ to every other node. Consider one such sequence of minimal length; say it has length $p$.

Note that since neither $G_1$ nor $G_2$ have an edge incoming to $u$, it follows that the $p$th graph in this sequence must be $G_3$. Consider the last time $G_3$ appeared before time $p$. Let us say that this happened at time $k$, i.e., the $k$th graph was $G_3$ and $G_3$ did not appear in the sequence at any time $l$ which satisfies $k < l < p$. In the event that $G_3$ never appeared before time $p$, we will set $k = 0$.

Now starting from $u$ the set of reachable nodes $R_u(k)$ is certainly non-empty: this is trivially true if $k = 0$, and otherwise true because $R_u(p) = V$ by assumption. Consequently, due to the structure of $G_3$, we have $u \in R_u(k)$. Moreover, since $R_u(k) \neq V$ (else $p$ would not be the shortest length of a positive product), the structure of $G_3$ implies that some $s^i$ does not belong to $R_u(k)$. Without loss of generality, suppose $s^i \notin R_u(k)$. Yet once again appealing to the structure of $G_3$, this implies all nodes with superscript 1 do not belong to $R_u(k)$.

We next argue that $p−1−k = n+1$, i.e., there are exactly $n+1$ graphs between $k$ and $p−1$. Indeed, we know that $R_u(p) = V$; but in order for $s^i \notin R_u(p)$, by the structure of $G_3$ we must have $s^i \in R_u(p−1)$. Since in any sequence of $G_1, G_2$ (i) the only node without a superscript of 1 that has a path to $s^i$ of any length is $u$ (ii) the only path from $u$ to $s^i$ has length $n+1$, we must have that $(p−1)−k = n+1$, as claimed.

Furthermore - once again by the structure of $G_3$ - we must have that every $s^i$ belongs to $R_u(p−1)$ in order to have $R_u(p) = V$. As pointed out earlier in Remark 2.9 starting from any node other than $u$ there are no paths of length $n+1$ to any $s^i$. It follows that in the minimal length sequence we are considering there must be a path of length $n+1$ from $u$ to all $s^i$.

Now we appeal to Corollary 2.8 to get a satisfying assignment for $f$.  \(\blacksquare\)
We now turn to the question of bounding \( l(\mathcal{M}) \), the length of the shortest positive product of matrices from \( \mathcal{M} \); we will adopt the convention that \( l(\mathcal{M}) = +\infty \) when no product of matrices from \( \mathcal{M} \) is positive. As we remarked earlier, any upper bound on \( l(\mathcal{M}) \) can be translated into an algorithm for checking primitivity simply by checking all products of length \( l(\mathcal{M}) \) from \( \mathcal{M} \). Unfortunately, our results are once again quite pessimistic: while upper bounds exist that show matrix primitivity is decidable, we construct four nonnegative matrices for which the shortest positive product has length at least exponential in the dimension.

We define \( l(m, n) \) to be the largest \( l(\mathcal{M}) \) over all sets \( \mathcal{M} \) with \( m \) matrices of size \( n \times n \) with \( l(\mathcal{M}) < \infty \). Our second main result of this section is the following theorem.

**Theorem 2.10.** We have that for all \( m, n \),

\[
l(m, n) \leq 2^{n^2}.
\]

Moreover, if \( m \geq 4 \), then for all \( \epsilon > 0 \) there exists a sequence of positive integers \( n_1, n_2, \ldots \), tending to infinity such that

\[
((1 - \epsilon)e)^{\sqrt{n_n/2}} \leq l(m, n_k).
\]

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((1 - \epsilon)e)^{\sqrt{n_n/2}} \leq l(m, n_k).
\]

We see that \( l(m, n) \) does not in general strongly depend on the number of matrices \( m \), in the sense that in the case of \( m \geq 4 \) it can be bounded above and below by exponentials independent of \( m \). An obvious consequence of this theorem is that matrix primitivity is decidable, but the natural algorithm which tests all products of length \( l(m, n) \) can take doubly exponential time in the dimension to halt.

We conclude this section with a proof of this theorem.

**Proof of Theorem 2.10.** We first do the easy direction, namely we prove inequality

\[
l(m, n) \leq 2^{n^2}.
\]

Let \( P_1P_2\ldots P_k \) be a shortest-length positive product of matrices from \( \mathcal{M} \). For \( S \subset \{1, \ldots, k\} \), we will use the notation \( P_S \) to denote the product

\[
P_S = \prod_{i \in S} P_i,
\]

where the indices are taken in increasing order. We argue that we cannot have

\[
R_1(i_1) = R_1(i_2), \quad R_2(i_1) = R_2(i_2), \quad \ldots, \quad R_n(i_1) = R_n(i_2) \quad (2.1)
\]

for some \( 1 \leq i_1 < i_2 \leq k \). Indeed, let us proceed by contradiction; assume Eq. (2.1) holds for such \( i_1, i_2 \); we then argue that the product \( P_{\{1,\ldots,k\}\setminus\{i_1+1,i_1+2,\ldots,i_2\}} \) is also positive. Indeed, we know by Lemma 2.3 that positivity of the product \( A_1, A_2, \ldots, A_l \) is equivalent to the existence of a path from every \( i \) to every \( j \) in the sequence \( G(A_1), G(A_2), \ldots, G(A_l) \). But since Eq. (2.1) implies that the set of reachable points by times \( i_1 \) and \( i_2 \) is the same, this means there is a path from every \( i \) to every \( j \) in the subsequence of \( G(P_1), \ldots, G(P_k) \) which omits the graphs \( G(P_{i_1+1}), \ldots, G(P_{i_2}) \). Consequently, \( P_{\{1,\ldots,k\}\setminus\{i_1+1,i_1+2,\ldots,i_2\}} \) is positive. However, \( P_1 \cdots P_k \) was assumed to
be a minimum length positive product; we therefore conclude that indeed Eq. (2.1) cannot hold.

This means that the tuple \((R_1(i), R_2(i), \ldots, R_n(i))\) takes on distinct values for each \(i = 1, \ldots, k\). But since each set \(R_i\) can assume at most \(2^n\) possible values, the tuple has at most \(2^{n^2}\) possible values and thus \(k\) cannot be larger than \(2^{n^2}\). This proves the upper bound.

We now turn our attention to the lower bound of the theorem. Let \(n_k = 1 + \sum_{i=2}^k p(i)\) where \(p(i)\) is the \(i\)th prime and \(k \geq 3\). We will establish the lower bound as follows: we will describe four graphs on \(n_k\) nodes whose adjacency matrices have a positive product, but the shortest length positive product has length at least \((1 - \epsilon)e^{\sqrt{n_k/2}}\) as long as \(n_k\) is large enough as a function of \(\epsilon\).

We next describe the graphs. We refer the reader to Figures 2.3 and 2.4 and we recommend that the reader refer to figures in while following our description. We will have a source node \(u\) and \(k - 1\) cycles, the \(i\)th on \(p(i)\) nodes. \(G_1\) will have edges going around each cycle counterclockwise. \(G_2\) will have edges from the last node in each cycle to all the nodes in its cycle, as well as an edge from the last node in each cycle to \(u\). \(G_3\) will have edges from every node to \(u\), and \(G_4\) will have edges from \(u\) to the first node in each cycle. We let \(M\) be \(\{A[G_1], A[G_2], A[G_3], A[G_4]\}\), i.e., the adjacency matrices of these four graphs.

We first show that \(l(m, n_k)\) is finite by exhibiting a positive product. We pick the following sequence of graphs. The first is \(G_3\), which ensures that \(R_i(1) = u\) for any \(i\); the second is \(G_4\) which ensures that \(R_i(2)\) is exactly the set of the first nodes in all cycles for any \(i\). We then pick \(G_1\) and repeat it \(P(k) = 2 \cdot 3 \cdot 5 \cdots p(k) - 1\) times. Observe that (i) for each \(i = 2, \ldots, k\), \(P(k) - (p(i) - 1)\) is divisible by \(p(i)\); and (ii) starting from the first node in each cycle, \(p(i) - 1\) steps bring us to the last node in that cycle, and then a multiple of \(p(i)\) steps returns us to the same node in that cycle. Therefore, the set of nodes reachable (from any node) after the last repetition of \(G_1\) is exactly the set of the last nodes in each cycle. Applying \(G_2\) then means that the set of reachable nodes equals the set of all nodes. We have just exhibited a sequence with the property that any node is reachable from any node; by Lemma 2.30 the corresponding matrix product of \(A[G_1], A[G_2], A[G_3], A[G_4]\) is positive.

We now proceed to prove a lower bound on the length of the shortest positive product of \(A[G_1], A[G_2], A[G_3], A[G_4]\). Consider such a product; it corresponds to a sequence of graphs from \(\{G_1, G_2, G_3, G_4\}\) of shortest length in which there is a path from any node to any other node. As we have argued previously, the fact that this sequence has minimal length implies the tuple of reachable sets never repeats.

We can immediately assert that the first graph is \(G_3\); if any other graph appears as first then some \(R_i(1)\) equals the empty set, and so equals the empty set thereafter. Similarly, the second graph then cannot be \(G_1\) or \(G_2\) because that would give \(R_i(2) = \emptyset\) for all \(i\); it therefore must either be \(G_3\) or \(G_4\). It cannot be \(G_3\) because that would contradict the minimality of the sequence; so it is \(G_4\).

We thus have that \(R_i(2)\) is the set of the first nodes of every cycle for every node \(i\). We now argue that \(G_3\) never occurs again, since that would make the reachable set from any node \(i\) equal to \(R_i(1)\) and once again contradict the minimality of the graph sequence we are considering. Thus the remainder of the sequence is composed of just \(G_1, G_2, G_4\).
We now argue that in the remainder of the sequence $G_2$ appears only once, and on the last step. Indeed, it is clearly true that $G_2$ must appear on the last step since each of the other graphs has at least one node without any incoming edges. On the other hand, suppose $G_2$ appears before the last step, at time $k$. Now either we have that for all $i$, $R_i(k-1)$ includes all of the last nodes in all the cycles, or some $R_i(k-1)$ does not include some last node in some cycle. In the former case, $R_i(k)$ is the set of all nodes for every $i$, which contradicts the minimality of the sequence. In the latter case, there is at least one cycle such that all $R_i(k)$ do not include any node in that cycle; without loss of generality, suppose it is the cycle of length 2. This means the graph $G_4$ appears at some future time as it is the only graph with an edge incoming to a node on the length 2 cycle. But after applying $G_4$ the set of reachable nodes is either empty or consists of all the first nodes in all the cycles, both of which cannot be: the first contradicts the eventual existence of paths from every node to every node, and the second contradicts the minimality of the sequence since this tuple of reachable sets has already occurred.

We recap: we have shown that the sequence is of the form $G_3, G_4, \ast, \ldots, \ast, G_2$ where $\ast$'s are either $G_1$ or $G_4$. But we can now immediately conclude that every $\ast$ above is in fact $G_1$: if some $\ast$ were $G_4$, the set of reachable nodes is empty.

We have thus concluded that the sequence must be of the form

$$G_3, G_4, G_1, \ldots, G_1, G_2.$$  

To be able to bound its length, we have to consider just how many repetitions of $G_1$ this sequence has. Note that $R_i(2)$ consists of the first nodes in each cycle, while after the last $G_1$, we have that $R_i$ has to include the last node in each cycle. This means that the number of times $G_1$ is repeated has to be of the form $p(i) - 1 + j(i)p(i)$ for each $i = 2, \ldots, k$, where $j(i)$ is a nonnegative integer: $p(i) - 1$ times to reach the last node in the cycle and $j(i) \geq 0$ additional round trips of length $p(i)$ around the cycle.

Note that since

$$p(2) - 1 + j(2)p(2) = p(3) - 1 + j(3)p(3)$$

we have

$$(1 + j(2))p(2) = (1 + j(3))p(3)$$

and consequently $p(3)$ divides $1 + j(2)$. Repeating this argument, we see that

$$1 + j(2) \geq \prod_{i=3}^{k} p(i)$$

so that the number of times $G_1$ is repeated is at least

$$p(2) - 1 + \left(-1 + \sum_{i=3}^{k} p(i)\right) p(2) = -1 + \sum_{i=2}^{k} p(i).$$

Thus the total length of the sequence is at least $2 + \prod_{i=2}^{k} p(i)$ while the number of nodes is $1 + \sum_{i=2}^{k} p(i)$. Now we can conclude using the prime product limit [23],

$$\lim_{k \to \infty} \left(\prod_{i=2}^{k} p(i)\right)^{1/p(k)} = e$$
and the fact that for any $\epsilon > 0$, we have that for large enough $i$,

$$i \ln i \leq p(i) \leq (1 + \epsilon) i \ln i$$

which implies that for any $\epsilon > 0$ for large enough $k$ [2],

$$\prod_{i=2}^{k} p(i) \geq ((1 - \epsilon)e)^{k \log k}$$

and also that (again for large enough $k$),

$$1 + \sum_{i=2}^{k} p(i) \leq 2k^2 \log k.$$

Thus with $n_k = 1 + \sum_{i=2}^{k} p(i)$ we have shown that the minimal length of a positive product is finite and at least $((1 - \epsilon)e)^{\sqrt{n_k}/2}$ as long as $n_k$ is large enough as a function of $\epsilon$; this concludes the proof of this theorem.

---

**Fig. 2.3.** Proof of Theorem 2.10. On the left, graph $G_1$ is shown with edges in blue; on the right, graph $G_2$ is shown with edges in red.

**Fig. 2.4.** Proof of Theorem 2.10: on the left is $G_3$ and on the right is $G_4$. 

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This concludes our section on general nonnegative matrices. We have proved that not only it is NP-hard to decide whether a set is primitive, but even more, the size of such a minimal product can be exponential in the size of the matrices. In the next section we show that under an additional assumption, the situation changes, and primitivity becomes much more amenable on an algorithmic point of view.

3. Sets with no zero rows nor zero columns. In this section we focus on sets of matrices that satisfy the following assumption:

**Assumption 1.** No matrix $A \in \mathcal{M}$ has a row or a column identically equal to zero:

\[
\forall A \in \mathcal{M}, \forall 1 \leq i \leq n, \exists j: A_{i,j} > 0,
\]

\[
\forall A \in \mathcal{M}, \forall 1 \leq j \leq n, \exists i: A_{i,j} > 0.
\]

We start with an easy lemma. In the following, we write $\mathcal{M}^t$ for the set of matrices which are products of length $t$ of matrices taken in $\mathcal{M}$, and $\mathcal{M}^*$ for the set of products of arbitrary length of matrices in $\mathcal{M}$.

**Lemma 3.1.** If a set of matrices $\mathcal{M}$ satisfies assumption 1 then every matrix in $\mathcal{M}^*$ satisfies it.

3.1. Polynomial time recognizability. The following theorem provides a structural characterization of primitivity for sets of matrices satisfying Assumption 1. It was first proved in [21] (after a conjecture of [18]), where the authors show that it leads to an efficient algorithm for recognizing such sets. The proof in that paper is long, and involves linear algebraic and geometric considerations. The authors also ask whether a simple combinatorial proof is possible for this result. We provide here such a combinatorial and self-contained proof.

**Theorem 3.2.** [21] A set of nonnegative matrices $\mathcal{M} \subset \mathbb{R}^{n \times n}_+$ satisfying Assumption 1 fails to be primitive if and only if one of the following conditions holds:

- There exists a permutation matrix $P$ such that all the matrices $A \in \mathcal{M}$ can be put in the same block triangular structure. Equivalently, there exists a partition $\{N_1, N_2\}$ of $\{1, \ldots, n\}$ such that

\[
\forall A \in \mathcal{M}, i \in N_1, j \in N_2 \Rightarrow A_{i,j} = 0.
\]

- There exists a permutation matrix $P$ such that all the matrices $A \in \mathcal{M}$ can be put in the same block permutation structure. Equivalently, there exists a partition $\{N_1, N_2, \ldots, N_k\}$ such that for all $A \in \mathcal{M}$,

\[
\forall i \in N_i, j \in N_j, \quad k \notin N_i, l \notin N_j,
A_{i,j} \neq 0 \Rightarrow A_{i,l} = 0, A_{k,j} = 0.
\]

**Proof.** The two conditions in the theorem are obvious sufficient conditions for imprimitiveness. Indeed, it is easy to see that the product of two matrices satisfying any of these conditions still satisfies it, and thus cannot be primitive.

---

2A partition of a set $S$ is a set $\mathcal{P} = \{N_1, N_2, \ldots, N_k\}$, $k > 1$, such that $N_i \subset S$, $N_i \cap N_j = \emptyset \forall i \neq j$, and $N_1 \cup N_2 \cdots \cup N_k = S$.  

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Suppose now that the set is imprimitive, and the first condition is violated (that is, the set is irreducible). We prove that the second condition holds. First, it is relatively easy to show (see [21, Lemma 4]) that under Assumption [1] if the set is irreducible and imprimitive, there is actually no product with a positive row, nor a positive column.

Let then $n' < n$ be the maximum number of nonzero entries in a row or a column of any matrix $A \in \mathcal{M}'$.

We claim that, up to relabelling of the entries, there is a product $A$ such that $\forall 1 \leq i, j \leq n', A_{i,j} > 0$. Let us take a product $B \in \mathcal{M}'$ and an index $k \in \{1, \ldots, n\}$ with a row (say, the $k$th rom) having $n'$ positive entries. We denote by $S$ the set of indices of these entries:

$$\forall i \in S, \ B_{k,i} > 0. \quad |S| = n'.$$

(The proof goes the same if there is a column with $n'$ nonzero entries.) First, we can suppose that $B(k,k) > 0$ (if needed, postmultiply $B$ with a matrix $C$ in $\mathcal{M}'$ such that $C_{i,k} > 0$ for some $i \in S$. This is always possible since $\mathcal{M}$ is irreducible.)

Now, it turns out that

$$\forall i \in S, \ \exists j \in S : B_{i,j} > 0.$$

Indeed, in the opposite case, there would be a $j \notin S$ such that $B_{i,j} > 0$, and thus $B^2$ would have $n' + 1$ nonzero entries in the $k$th row.

We take $k = 1$ without loss of generality. Now, for any $i \in S$, we can build another matrix in $\mathcal{M}'$ such that not only the first, but also the $i$th row are positive: Take the corresponding $j$ in the equation above, and a matrix $D \in \mathcal{M}'$ such that $D_{j,1} > 0$. We have that $BDB_{r,s} > 0$ for all couples $(r,s)$ such that $r \in \{1,i\}, 1 \leq s \leq n'$. By repeating this argument for all $i$ in $S$, we arrive at a matrix $A$ satisfying the claim.

Now, since $n'$ is the maximum number of nonzero entries in any row and column, the matrix $A$ is actually block-diagonal (with blocks $\{1, \ldots, n'\}$ and $\{n'+1, \ldots, n\}$), which implies that there is a nonzero entry in every row and column of the lower right diagonal block (because every matrix in the semigroup has nonzero rows and nonzero columns).

We claim that every matrix in the set $\mathcal{M}$ is a permutation on the sets

$$\{\{1, \ldots, n'\}, \{n'+1, \ldots, n\}\}.$$

Suppose on the contrary (without loss of generality) that there exists a matrix $B \in \mathcal{M}$ together with $1 \leq i, j, k \leq n', l > n'$, such that $B_{i,j}, B_{k,l} > 0$. Then one can check that the product $ABA$ has $n' + 1$ nonzero entries in its $n'$ first rows, which violates the assumption on $n'$.

\[\square\]

### 3.2. Bounds on the length of the product.

We now turn to the problem of obtaining tight bounds on the length of a shortest strictly positive product, as a function of the dimension of the matrices. For this purpose, we make connections with a well known concept in TCS, namely, Synchronizing Automata. Our result also suggests that an exact answer to that problem is probably very hard to obtain.

A (deterministic, finite state, complete) automaton is a set of $m$ row-stochastic matrices $\mathcal{M} \subset \{0,1\}^{n \times n}$ (where $m, n$ are positive integers). That is, the matrices in $\mathcal{M}$ have binary entries, and they satisfy $A e = e$, where $e$ is the all-ones (column) vector. For convenience of product representation, to each matrix $A_e \in \mathcal{M}$ is associated a letter $c$, such that the product $A_{c_1} \ldots A_{c_t} \in \mathcal{M}'$ can be written $A_{c_{1, \ldots, c_t}}$. 


Definition 3.3. An automaton $M \subset \{0, 1\}^{n \times n}$ is synchronizing if there is a finite product $A = A_{c_1} \ldots A_{c_T} : A_{c_i} \in M$ which satisfies

$$A = ee_i^T,$$

where $e$ is the all-ones vector and $e_i$ is the $i$th standard basis vector.

In this case, the sequence of letters $c_1 \ldots c_T$ is said to be a synchronizing word.

We recall the following conjecture which has raised a large interest in the TCS community [6, 9, 13, 14, 25]. It has been proved to hold in many particular cases, but the general case remains open.

Conjecture 1. Černý's conjecture, 1964 [7] Let $M \subset \{0, 1\}^{n \times n}$ be a synchronizing automaton. Then, there is a synchronizing word of length at most $(n - 1)^2$.

In fact, it is even not known whether there is a valid bound with a quadratic growth in $n$, and we study in the rest of this paper the weaker following conjecture.

Conjecture 2. Let $M \subset \{0, 1\}^{n \times n}$ be a synchronizing automaton. Then, there is a synchronizing word of length at most $Kn^2$ for some fixed $K > 0$.

Fig. 3.1 (a) represents a synchronizing automaton whose shorter synchronizing word is of length $(n - 1)^2$, as proved in [7]. Thus, if Conjecture 1 is true, the bound in the conjecture is tight. We first present a technical result which makes the bridge between the combinatorial problem studied in the present paper and the notion of synchronizing automaton.

Theorem 3.4. For any primitive set of nonnegative matrices $M = \{A_1, \ldots, A_m\} \subset \{0, 1\}^{n \times n}$ satisfying Assumption [1] there exists a synchronizing automaton $M' = \{A'_1, \ldots, A'_t\}$ such that

$$\forall 1 \leq s \leq t, \exists l \in \{1, \ldots, m\} : A'_s \leq A_l \text{ (entrywise)}.$$ 

We attract the attention of the reader to the fact that the number of matrices in the automaton is not necessarily the same as the number of matrices in the initial set $M$.

Proof. Let us consider the positive product $A_{i_1}A_{i_2} \ldots A_{i_t} \in M^*$. We will keep the different matrices $A_{i_l} : l = 1 \ldots t$ for the construction of our automaton. Since the product is positive, there are actually paths from nodes 1, \ldots, $n$ to node (say) 1, in the sequence of graphs $G_{i_1}, G_{i_2} \ldots G_{i_t}$. In order to obtain our automaton, we have to remove edges (i.e., put some entries to zero in our constructed matrices $A'_i$) so that one and only one entry in every row is equal to one. If we manage to do that while keeping the $n$ paths in the sequence of our corresponding graphs $G'_{i_1}, G'_{i_2} \ldots G'_{i_t}$, then we will have a synchronizing automaton.

In order to do that, we simply keep in each matrix $A_{i_l}$ all the edges that are part of the above mentioned paths. If there is a node $v$ in the graph $G_{i_l}$ such that $v$ is on none of these paths (at level $l$), we can just pick any edge leaving $v$ in order to define a valid automaton. Such an edge exists because all matrices in $M$ have nonzero rows and columns.

Now, there might be some graphs $G_{i_l}$ in which two edges leaving the same node $v$ have
been kept. However, this could occur only if there are two separate paths leaving $v$ (at the level $l$) and reaching node 1 at level $t$. Thus, one can safely iteratively remove these edges in excess, and making sure at the same time that if a node was connected to the node 1 at level $t$, it remains connected by at least one path throughout this process.

**Theorem 3.5.** For any primitive set of matrices $M$ of dimension $n$ satisfying Assumption 1 there is a product of length smaller than $2f(n)+n-1$ with positive entries, where $f(n)$ is any upper bound on the minimal length of a synchronizing word for $n$-dimensional automata.

*Proof.* From Theorem 3.4 above, let us consider the automaton $A'$ whose matrices are smaller (entrywise) than matrices from $M$. There is a product $B_1$ of length $f(n)$ with a positive column (say, the $i$th one). Now, reasoning on the set $M^T$, there is a product $B_2$ of length $f(n)$ with a positive row (say, the $j$th one). Now one can take a product $C$ (of length smaller than $n$) such that $C_{i,j} > 0$, and one obtains

$$B_1CB_2 > 0.$$  

**Corollary 3.6.** For any primitive set of matrices of dimension $n$ satisfying Assumption 1 there is a product of length smaller than

$$(n(7n^2 + 6n + 8) - 24)/24 = O(n^3)$$

with positive entries.

*Proof.* It is known [26] that any synchronizing automaton has a synchronizing word of length smaller or equal to

$$f(n) = n(7n^2 + 6n - 16)/48.$$  

By combining this bound with Theorem 3.5 above, we obtain the result.

We did not try to optimize the bound in the above theorem. Most probably simple arguments could allow to lower it with the same general ideas. Hence, Conjecture 2 being true would imply the following weaker conjecture:

**Conjecture 3.** There is a constant $K$ such that for any set of primitive matrices of dimension $n$ satisfying Assumption 1 there is a product of length smaller than $Kn^2$ with positive entries.

We finish by providing a lower bound for the shortest length of a positive product.

**Example 1.** Fig. 3.1 (b) represents a set of matrices which is primitive, but the length of any positive product is at least $n^2/2$. To see this, let us consider only the first row of a product of length $t$, which we denote $v_t$, and start with the empty product (i.e. the Identity matrix). Let us denote $A_n, A_b$ the two matrices corresponding to the graph in Fig. 3.1 (b). Observe that right-multiplying our product with $A_b$ shifts all the entries to the right, that is,

$$v_tA_b(i) = v_t(j) : j = i - 1 \mod n.$$
Fig. 3.1. Construction of our extremal example. (a) Černý’s well known automaton, whose shortest synchronizing words are of length \((n-1)^2\). This set of matrices is not primitive. (b) Our extremal example, which is primitive. It has no positive product of length shorter than \((n^2/2)\).

Also, multiplying our product with \(A_0\) leaves \(v_t\) unchanged, except if \(v_t(1) = 0, v_t(n) > 0\), in which case \(v_{t+1}(1) = v_t(n)\). Thus, it is straightforward to prove inductively that the only way to increase the number of nonzero entries in a vector \(v_t = (1, 0, \ldots, 0, 1, \ldots, 1)\) is to apply \(A_0\) \(n-1\) times (or \(k(n-1)\) for some natural number \(k\)), followed by \(A_n\) in which case the vector has the same general shape \(v_t = (1, 0, \ldots, 0, 1, \ldots, 1)\), with one more 1. Thus, any positive product having all its entries in the first row positive, this process has to be repeated \(n-1\) times, which brings a lower bound of \(n(n-1)\).

**Corollary 3.7.** The upper bound in Conjecture 3 cannot be \(o(n^2)\).

**Proof.** The proof is a direct consequence of Example 1.

4. Conclusion. We have shown that the concept of primitivity of a matrix semigroup is hard to handle algorithmically: Sets enjoying this property are hard to recognize, and can have positive products only for very long lengths, which is another sign of the algorithmic difficulty of the concept.

We have then studied a class of matrices which has better properties with respect to primitivity (see Assumption 1). We have provided a simpler combinatorial proof of a recent structure theorem allowing to recognize primitivity in polynomial time for this special class of matrices. We have connected the primitivity problem with a well known and studied problem in TCS, namely synchronizing automata. This allowed us to derive bounds on the length of the shortest positive product in this case, which are much shorter than in the general case. We hope that this connection will bring some insight on synchronizing automata, which we defer to further work.
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