CATEGORICAL ASPECTS OF COINTEGRALS ON QUASI-HOPF
ALGEBRAS

TAIKI SHIBATA AND KENICHI SHIMIZU

Abstract. We discuss relations between some category-theoretical notions for
a finite tensor category and cointegrals on a quasi-Hopf algebra. Specifically,
for a finite-dimensional quasi-Hopf algebra $H$, we give an explicit description
of categorical cointegrals of the category $\mathcal{H}_m$ of left $H$-modules in terms of
coinTEGRALS on $H$. Provided that $H$ is unimodular, we also express the Frobe-
nius structure of the ‘adjoint algebra’ in the Yetter-Drinfeld category $\mathcal{H}_m\mathcal{YD}$
by using an integral in $H$ and a cointegral on $H$. Finally, we give a description
of the twisted modified trace for projective $H$-modules in terms of cointegrals
on $H$.

1. Introduction

Results on Hopf algebras have been reexamined from the viewpoint of the theory
of tensor categories since a result on Hopf algebras generalized to tensor categories
is expected to be useful in applications to, e.g., low-dimensional topology and con-
formal field theories. Integrals and cointegrals for Hopf algebras are introduced by
Sweedler [Swe69] and play an important role in the study of Hopf algebras. Let $H$
be a finite-dimensional Hopf algebra. Recently, we have observed that cointegrals
on $H$ appear in several results on finite tensor categories. For example, it is known
that, if $\mathcal{C}$ is a unimodular finite tensor category, then there is a Frobenius algebra
$A$ in the Drinfeld center $Z(\mathcal{C})$ of $\mathcal{C}$ arising from an adjunction between $Z(\mathcal{C})$ and $\mathcal{C}$
[IM14, Shi17b]. The Frobenius structure of $A$ is written in terms of integrals and
coinTEGRALS on $H$ if $\mathcal{C} = \mathcal{H}_m\mathcal{YD}$.

As another example, we mention that the space of modified traces on a pivotal finite
tensor category $\mathcal{C}$ can be identified with the space of ‘$\mu$-symmetrized’ cointegrals
on $H$ if $\mathcal{C} = \mathcal{H}_m\mathcal{YD}$ and $\mu$ is the modular function on $H$ [BBG17, FG18].

The above-mentioned results may be explained by the integral theory for finite
tensor categories established by the second-named author [Shi17b, Shi18]. He intro-
duced the notions of categorical integrals and categorical cointegrals for a finite
tensor category $\mathcal{C}$ and showed that they can be identified with ordinary (co)integrals
if $\mathcal{C} = \mathcal{H}_m\mathcal{YD}$. As demonstrated in [Shi18], several results in the Hopf algebra theory
are extended to finite tensor categories by using categorical (co)integrals. However,
a relation between modified traces and categorical cointegrals is still open. We also
remark that an explicit description of a categorical cointegral is not known except
the case where $\mathcal{C} = \mathcal{H}_m\mathcal{YD}$ or if $\mathcal{C}$ is a fusion category.

In this paper, we discuss these problems in the case where $\mathcal{C}$ is the category $\mathcal{H}_m$ of
left modules over a finite-dimensional quasi-Hopf algebra $H$ over a field $k$. In this
case, the Drinfeld center $Z(\mathcal{C})$ is identified with the category $\mathcal{H}_m\mathcal{YD}$ of Yetter-Drinfeld
modules over $H$. By using the fundamental theorem for quasi-Hopf bimodules, we
give a convenient expression of a right adjoint $R : \mathcal{H}_m \rightarrow \mathcal{H}_m\mathcal{YD}$ of the forgetful
functor from $^H_y\mathcal{YD}$ to $^H_M$ (Theorem 5.3). As an application, we express the space of categorical cointegrals in terms of ordinary cointegrals on $H$ (Theorem 5.11). Under the assumption that $H$ is unimodular, we give an explicit description of the Frobenius structure of the algebra $A = R(k)$ in terms of (co)integrals of $H$ (Theorem 5.12). Finally, we show that $\mu$-twisted modified traces on $^H_M$ are expressed by `$\mu$-symmetrized' cointegrals on $H$, where $\mu$ is the modular function on $H$ (Theorem 6.4).

Recently, several interesting examples of finite-dimensional quasi-Hopf algebras are introduced and studied from the viewpoint of logarithmic conformal field theories [FGR17, CGR17, Neg18]. We hope our abstract results are useful in future study of the representation theory of these algebras and their applications to conformal field theories.

This paper is organized as follows: In Section 2 we recall basic notions in the theory of monoidal categories and their module categories. In Section 3, we recall the definition of quasi-Hopf algebras from [Dri89, Kas95] and elementary results on the representation theory of quasi-Hopf algebras. We also collect useful identities in a quasi-Hopf algebra.

In Section 4, we review the integral theory for quasi-Hopf algebras. Let $H$ be a quasi-Hopf algebra and, for simplicity, assume that $H$ is finite-dimensional in this Introduction. The category $^H_M$ of $H$-bimodules is a monoidal category as it can be identified with the category of left modules over the quasi-Hopf algebra $H \otimes H^{op}$. The axioms of a quasi-Hopf algebra ensure that $H$ is a coalgebra in $^H_M$. The categories $^H_M$ and $^H_M$ of left and right quasi-Hopf bimodules are defined as the categories of left $H$-comodules and right $H$-comodules in $^H_M$, respectively. The fundamental theorem for quasi-Hopf bimodules [HN99] gives equivalences $^H_M \approx H^M \approx ^H_M$. Cointegrals on $H$ are defined by using these equivalences. We give some characterizations of cointegrals for later use.

In Section 5, we give applications of the integral theory to Yetter-Drinfeld modules over a quasi-Hopf algebra. The monoidal category $^H_M$ acts on $^H_M$ in such a way that the equivalence $^H_M \approx ^H_M$ is $^H_M$-equivariant. Schauenburg [Sch02] described this action explicitly and established an isomorphism between the category of left $H$-comodules in $^H_M$ and the category $^H_y\mathcal{YD}$ of Yetter-Drinfeld modules over $H$. Hence we obtain a commutative diagram

\[
\begin{array}{ccc}
^H_y\mathcal{YD} & \approx & \text{(the category of $H$-bicomodules in $^H_M$)} \\
\downarrow & & \downarrow \\
^H_M & \approx & \text{(the category of right $H$-comodules in $^H_M$)},
\end{array}
\]

where the vertical arrows are the forgetful functors. This commutative diagram shows that the free left $H$-comodule functor $R : ^H_M \rightarrow ^H_y\mathcal{YD}$ is right adjoint to the forgetful functor $F : ^H_y\mathcal{YD} \rightarrow ^H_M$ (Theorem 5.3). By using the above commutative diagram, we also show that the adjunction $F \dashv R$ is a co-Hopf adjunction (Theorem 5.5), the dual notion of a Hopf adjunction [BLV11]. We also express the monoidal structure of $R$ and the structure of the algebra $A = R(k) \in ^H_y\mathcal{YD}$ explicitly (Theorem 5.6). It turns out the algebra $A$ is identical to the algebra $H_0$ given in [BCP05, BCP06]. We deduce some results on the algebra $A$ from the general theory of monoidal categories (Corollaries 5.7, 5.8 and 5.9).
The above commutative diagram also shows that the functor $L : H \mathcal{M} \to H \mathcal{M}$ given by tensoring $H^\vee \in H \mathcal{M}_H$ is left adjoint to $F$ (Theorem 5.10). Let $\mu : H \to k$ be the modular function on $H$ and regard it as a one-dimensional left $H$-module. The fundamental theorem for quasi-Hopf bimodules gives a relation between the isomorphisms $\Phi_{X,Y,Z}$ and $\overline{r}_X : X \otimes 1 \to X$ satisfying the pentagon and the triangle axioms. According to $[\text{FG18}]$, such a trace is constructed from a ‘$\mu$-symmetrized’ cointegral on $H$ if $H$ is an ordinary Hopf algebra. The main result of this section is a generalization of this result to the case where $H$ is a quasi-Hopf algebra (Theorem 6.3). The proof goes along almost the same way as $[\text{FG18}]$ but also uses a certain technical result on cointegrals on a quasi-Hopf algebra discussed in Section 4.

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2. Preliminaries

2.1. Monoidal categories. We recall basic results on monoidal categories from $[\text{ML98, Kas95, EGN15}]$. A monoidal category is a data $\mathcal{C} = (\otimes, \mathbb{1}, \Phi, l, r)$ consisting of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $\mathbb{1} \in \mathcal{C}$ and natural isomorphisms $\Phi_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X : \mathbb{1} \otimes X \to X$ and $r_X : X \otimes \mathbb{1} \to X$ satisfying the pentagon and the triangle axioms. The natural isomorphisms $\Phi, l$ and $r$ are called the associator, the left unit isomorphism and the right unit isomorphism, respectively. We always assume that the left and the right unit isomorphisms are identities. Although the associator is often assumed to be the identity in the study of monoidal categories, we do not so in this paper as our main example is the category of modules over a quasi-Hopf algebra.

Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, F^{(0)}, F^{(2)})$ consisting of a functor $F : \mathcal{C} \to \mathcal{D}$, a morphism $F^{(0)} : \mathbb{1} \to F(\mathbb{1})$ in $\mathcal{D}$ and a natural transformation $F^{(2)}_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ ($X, Y \in \mathcal{C}$).
such that the equations
\[ F(\Phi_{X,Y,Z}) \circ F^{(2)}_{X,Y,Z} \circ (F^{(2)}_{X,Y} \otimes \text{id}_F(Z)) = F^{(2)}_{X,Y} \otimes (\text{id}_F(X) \otimes F^{(2)}_{Y,Z}) \circ \Phi_{F(X),F(Y),F(Z)}, \]
\[ F^{(2)}_{1,X} \circ (\eta^{(0)} \otimes \text{id}_F(X)) = \text{id}_F(X) = F^{(2)}_{1,X} \circ (\text{id}_F(X) \otimes F^{(0)}) \]
hold for all objects \( X, Y, Z \in \mathcal{C} \). A monoidal functor \( F : \mathcal{C} \rightarrow \mathcal{D} \) is said to be strong if \( F^{(0)} \) and \( F^{(2)} \) are invertible. It is said to be strict if \( F^{(0)} \) and \( F^{(2)} \) are identities.

Let \( F, G : \mathcal{C} \rightarrow \mathcal{D} \) be monoidal functors. A \textit{monoidal natural transformation} from \( F \) to \( G \) is a natural transformation \( \xi : F \rightarrow G \) such that the equations \( \xi_1 \circ F^{(0)} = G^{(0)} \) and \( \xi_X \otimes Y \circ F^{(2)}_{X,Y} = G^{(2)}_{X,Y} \circ (\xi_X \otimes \xi_Y) \) hold for all objects \( X, Y \in \mathcal{C} \). Suppose that a strong monoidal functor \( F : \mathcal{C} \rightarrow \mathcal{D} \) admits a right adjoint \( R : \mathcal{D} \rightarrow \mathcal{C} \). Let \( \eta : \text{id}_C \rightarrow RF \) and \( \varepsilon : FR \rightarrow \text{id}_D \) be the unit and the counit of the adjunction, respectively. The functor \( R \) has a monoidal structure given by
\[ R^{(0)} = R((F^{(0)})^{-1}) \circ \eta_1, \]
\[ R^{(2)}_{X,Y} = R(\varepsilon_X \otimes \varepsilon_Y) \circ R((F^{(2)}_{R(X),R(Y)})^{-1}) \circ \eta_{R(X) \otimes R(Y)} \]
for \( X, Y \in \mathcal{C} \). The adjunction \( (F, R, \varepsilon, \eta) \) is in fact a \textit{monoidal adjunction} in the sense that \( F \) and \( R \) are monoidal functors and \( \eta \) and \( \varepsilon \) are monoidal natural transformations.

2.2. Duality in a monoidal category. Let \( \mathcal{C} \) be a monoidal category, and let \( X \) be an object of \( \mathcal{C} \). A \textit{left dual object} of \( X \) is a triple \( (Y, e, c) \) consisting of an object \( Y \in \mathcal{C} \) and morphisms \( e : Y \otimes X \rightarrow \mathbb{1} \) and \( c : \mathbb{1} \rightarrow X \otimes Y \) in \( \mathcal{C} \) such that the equations
\[ (\text{id}_X \otimes e) \Phi_{X,Y,X}(c \otimes \text{id}_X) = \text{id}_X \quad \text{and} \quad (e \otimes \text{id}_Y) \Phi^{-1}_{Y,X,Y}(\text{id}_Y \otimes c) = \text{id}_Y \]
hold. The morphisms \( e \) and \( c \) are referred to as the \textit{evaluation} and the \textit{coevaluation}, respectively. A left dual object of \( X \) is, if it exists, unique up to unique isomorphism in the following sense: If \((Y, e, c)\) and \((Y', e', c')\) are both left dual objects of \( X \), then there exists a unique isomorphism \( f : Y \rightarrow Y' \) in \( \mathcal{C} \) such that \( e' \circ (f \otimes \text{id}_X) = e \) and \( e' = (\text{id}_X \otimes f) \circ c \).

Now we suppose that \( X \) has a left dual object \((X^\vee, ev_X, coev_X)\). Given an object \( A \in \mathcal{C} \), we denote by \( \mathcal{L}_A : \mathcal{C} \rightarrow \mathcal{C} \) the functor defined by \( \mathcal{L}_A(V) = A \otimes V \). The definition of a left dual object implies that the functor \( \mathcal{L}_{X^\vee} \) is left adjoint to \( \mathcal{L}_X \). More precisely, there is a natural isomorphism
\[ Q = Q_{V,W,X} : \text{Hom}_C(V, X \otimes W) \rightarrow \text{Hom}_C(X^\vee \otimes V, W), \]
\[ Q(f) = (ev_X \otimes \text{id}_W) \circ \Phi^{-1}_{X^\vee,X,W}(\text{id}_X \otimes f) \] (2.1)
for \( V, W \in \mathcal{C} \). The inverse of \( Q \) is given by
\[ Q^{-1}(g) = (\text{id}_X \otimes g) \circ \Phi_{X^\vee,X,W}(coev_X \otimes \text{id}_V). \]
If we use the graphical calculus (see, e.g., [Kas02]), then the natural isomorphism \( Q \) and its inverse are expressed by string diagrams as follows:

\[ Q \left( \begin{array}{c}
\begin{array}{c}
\mathbb{1}
\end{array}
\end{array} \right) = \begin{array}{c}
\begin{array}{c}
\mathbb{1}
\end{array}
\end{array} \quad Q^{-1} \left( \begin{array}{c}
\begin{array}{c}
\mathbb{1}
\end{array}
\end{array} \right) = \begin{array}{c}
\begin{array}{c}
\mathbb{1}
\end{array}
\end{array} \]
When we express a morphism by such a string diagram, we adopt the convention that a morphism goes from the top to the bottom of the diagram. The evaluation and the coevaluation are represented by a cup (∪) and a cap (∩), respectively. Although the graphical calculus is a very useful tool in the theory of monoidal categories, it hides the associator that should not be ignored in the study of quasi-Hopf algebras. For this reason, we use string diagrams only to give readers graphical intuition.

There is also a natural isomorphism

\[ \mathbb{P} = \mathbb{P}_{V,W,X} : \text{Hom}_C(V \otimes X, W) \to \text{Hom}_C(V, W \otimes X^\vee), \]  

(2.2)

\[ \mathbb{P}(f) = (f \otimes \text{id}_{X^\vee}) \circ \Phi_{V,X,X^\vee}^{-1} \circ (\text{id}_V \otimes \text{coev}_X) \]

for \( V, W \in C \). The inverse of \( \mathbb{P} \) is given by

\[ \mathbb{P}^{-1}(g) = (\text{id}_W \otimes \text{ev}_X) \circ \Phi_{W,X^\vee,X} \circ (g \otimes \text{id}_X). \]

The isomorphism \( \mathbb{P} \) and its inverse are expressed as follows:

\[ \mathbb{P} \left( \begin{array}{c} V \\ f \\ X \\ W \end{array} \right) = \left( \begin{array}{c} V \\ f \\ W \\ X^\vee \end{array} \right), \quad \mathbb{P}^{-1} \left( \begin{array}{c} V \\ g \\ X^\vee \\ W \end{array} \right) = \left( \begin{array}{c} V \\ g \\ W \\ X^\vee \end{array} \right). \]

A right dual object of \( X \) is a triple \((Y, e, c)\) consisting of an object \( Y \in C \) and morphisms \( e : X \otimes Y \to 1 \) and \( c : 1 \to Y \otimes X \) such that, in a word, the triple \((X, e, c)\) is a left dual object of \( Y \). Let \( \text{C}^{\text{op}} \) be the monoidal category obtained from \( C \) by reversing the order of the tensor product. A right dual object of \( X \) is nothing but a left dual object of \( X \in \text{C}^{\text{op}} \). Thus, if \( ^\vee X \) is a right dual object of \( X \), then there are natural isomorphisms

\[ \text{Hom}_C(X \otimes V, W) \cong \text{Hom}_C(V, ^\vee X \otimes W), \]

\[ \text{Hom}_C(V, W \otimes X) \cong \text{Hom}_C(V \otimes ^\vee X, W) \]

for \( V, W \in C \).

2.3. Modules over an algebra. Let \( C \) be a monoidal category. A left \( C \)-module category is a category \( \mathcal{M} \) endowed with a functor \( \otimes : C \times \mathcal{M} \to \mathcal{M} \) (called the action) and natural isomorphisms \( \Omega_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M) \) and \( l_M : 1 \otimes M \to M (X,Y \in C, M \in \mathcal{M}) \) obeying certain axioms similar to those for monoidal categories. A right \( C \)-module category and a \( C \)-bimodule category are defined analogously. We omit the definitions of \( C \)-module functors and their morphisms; see \([EGNO13]\).

An algebra in \( C \) is a triple \((A, m, u)\) consisting of an object \( A \in C \) and morphisms \( m : A \otimes A \to A \) and \( u : 1 \to A \) in \( C \) such that the following equations hold:

\[ m \circ (m \otimes \text{id}_A) = m \circ (\text{id}_A \otimes m) \circ \Phi_{A,A,A}, \]

\[ m \circ (u \otimes \text{id}_A) = \text{id}_A = m \circ (\text{id}_A \otimes u). \]

Dually, a coalgebra in \( C \) is a triple \((C, \Delta, \varepsilon)\) consisting of an object \( C \in C \) and morphisms \( \Delta : C \to C \otimes C \) and \( \varepsilon : C \to 1 \) in \( C \) such that, in a word, \((C, \Delta, \varepsilon)\) is an algebra in \( C^{\text{op}} \). Now let \( \mathcal{M} \) be a left \( C \)-module category. Given an object \( A \in C \), we define the functor \( \mathfrak{L}_A : \mathcal{M} \to \mathcal{M} \) by \( \mathfrak{L}_A(M) = A \otimes M \) for \( M \in \mathcal{M} \). If \( A \) is an algebra in \( C \), then the functor \( \mathfrak{L}_A \) is a monad \([ML98, VI.1]\) on \( \mathcal{M} \) in a natural way.
Similarly, if \(C\) is a coalgebra in \(C\), then the functor \(\mathcal{L}_C\) has a natural structure of a comonad on \(M\).

**Definition 2.1.** Given an algebra \(A\) in \(C\), we define the category \(\mathcal{A}\mathcal{M}\) of left \(A\)-modules in \(M\) to be the category of \(\mathcal{L}_A\)-modules (= the Eilenberg-Moore category of \(\mathcal{L}_A\)) \(^{[92]}\). Given a coalgebra \(C\) in \(C\), we define the category \(\mathcal{C}\mathcal{M}\) of left \(C\)-comodules in \(M\) to be the category of \(\mathcal{L}_C\)-comodules. The category of right (co)modules in a right \(C\)-module category is defined and denoted in an analogous way.

We note that \(C\) is a \(C\)-bimodule category by the tensor product. Thus, given an algebra \(A\) in \(C\), the notions of a left \(A\)-module in \(C\) and a right \(C\)-module in \(C\) are defined. Let \(X\) be a left \(A\)-module in \(C\) with action \(\rho : A \otimes X \to X\). If \(X\) has a left dual object \((X^\vee, ev_X, coev_X)\), then we define \(\rho^\#: X^\vee \otimes A \to X^\vee\) to be the morphism corresponding to \(\rho\) via

\[
\text{Hom}_C(A \otimes X, X) \xrightarrow{\cong} \text{Hom}_C(A, X \otimes X^\vee) \xrightarrow{\cong} \text{Hom}_C(X^\vee \otimes A, X^\vee).
\]

It is known that \(X^\vee\) is a right \(A\)-module in \(C\) by the action \(\rho^\#\). Graphically, the morphism \(\rho^\#\) can be expressed as follows:

\[
\begin{array}{ccc}
X^\vee & A & X^\vee \\
\rho^\# & & \\
X^\vee & \circlearrowright & \\
\end{array}
\]

There is a similar construction for comodules: Let \(C\) be a coalgebra in \(C\), and let \(X\) be a right \(C\)-comodule in \(C\) with coaction \(\delta : X \to X \otimes C\). If \(X\) has a left dual object \(X^\vee\), then \(X^\vee\) is a left \(C\)-comodule by the coaction defined to be the morphism corresponding to \(\delta\) via

\[
\text{Hom}_C(X, X \otimes C) \xrightarrow{\cong} \text{Hom}_C(X^\vee \otimes X, C) \xrightarrow{\cong} \text{Hom}_C(X^\vee, C \otimes X^\vee).
\]

3. Quasi-Hopf algebras

3.1. **Notation.** Throughout this paper, we work over a fixed field \(k\). Unless otherwise noted, the symbol \(\otimes\) means the tensor product over \(k\). For a vector space \(V\) over \(k\) and a positive integer \(n\), we denote by \(V^{\otimes n}\) the \(n\)-th tensor power of \(V\) over \(k\).

An element \(x \in V^{\otimes n}\) is written symbolically as

\[
x = x_1 \otimes \cdots \otimes x_n,
\]

although an element of \(V^{\otimes n}\) is a sum of the form \(\sum_i x_{i_1} \otimes \cdots \otimes x_{i_n}\) in general. Given a permutation \(\sigma\) on the set \(\{1, \ldots, n\}\), we write \(x_{\sigma(1)} \cdots x_{\sigma(n)} = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}\).

By a \(k\)-algebra, we always mean an associative unital algebra over the field \(k\). If \(A\) is a \(k\)-algebra and \(x\) is an invertible element of the \(k\)-algebra \(A^{\otimes n}\), then we write its inverse as

\[
x^{-1} = \bar{x}_1 \otimes \cdots \otimes \bar{x}_n.
\]

Given a vector space \(M\), we denote its dual space by \(M^* = \text{Hom}_k(M, k)\). For \(\xi \in M^*\) and \(m \in M\), we often write \(\xi(m)\) as \(\langle \xi, m \rangle\). If \(M\) is a left \(A\)-module, then \(M^*\) is a right \(A\)-module by the action \(\leftarrow\) given by \(\langle \xi \leftarrow a, m \rangle = \langle \xi, am \rangle\) for \(\xi \in M^*\),
a ∈ A and m ∈ M. Similarly, if M is a right A-module, then M∗ is a left A-module by (a → ξ, m) = (ξ, ma) for a ∈ A, ξ ∈ M∗ and m ∈ M.

3.2. Quasi-Hopf algebras. A quasi-Hopf algebra [Dr89, Kas95] is a data $H = (H, Δ, e, φ, S, α, β)$ consisting of a $k$-algebra $H$, algebra maps $Δ : H → H^⊗2$ and $e : H → k$, an invertible element $φ ∈ H^⊗3$, an anti-algebra map $S : H → H$, and elements $α, β ∈ H$ satisfying the equations

$$h_{(1)} ⊗ Δ(h_{(2)}) = φ · (Δ(h_{(1)}) ⊗ h_{(2)}) · φ^{-1},$$

$$e(h_{(1)})h_{(2)} = h = h_{(1)}e(h_{(2)}),$$

$$φ(h_{(1)}h_{(2)}) = φ(1 ⊗ 1) = (e ⊗ id ⊗ id)(φ) = (id ⊗ e ⊗ id)(φ) = 1 ⊗ 1,$$

$$S(h_{(1)})αh_{(2)} = e(h)α, \quad h_{(1)}βS(h_{(2)}) = e(h)β,$$

$$φ_1βS(φ_2αφ_3) = 1 = S(φ_1)αφ_2βS(φ_3),$$

for all $h ∈ H$, where $h_{(1)} ⊗ h_{(2)}$ is the symbolic notation for $Δ(h)$. The maps $Δ, e$ and $S$ are called the comultiplication, the counit and the antipode of $H$, respectively. Equations (3.2)–(3.4) implies

$$(id ⊗ id ⊗ e)(φ) = 1 ⊗ 1 = (e ⊗ id ⊗ id)(φ).$$

By applying $e$ to equation (3.6), we have $e(α)e(β) = 1$. Hence the equation

$$e(α) = e(β) = 1$$

holds if we replace the elements $α$ and $β$ with $e(β)α$ and $e(α)β$, respectively. For simplicity, all quasi-Hopf algebras in this paper are assumed to satisfy (3.8). A category-theoretical meaning of this assumption is explained in Remark 3.4.

Now we fix a quasi-Hopf algebra $H = (H, Δ, e, φ, S, α, β)$, and assume that the antipode $S$ is bijective with inverse $S$ (this assumption is automatically satisfied when $H$ is finite-dimensional; see [BCO03] and [Sch04]). We define the quasi-Hopf algebras $H^{op}$ and $H^{cop}$ by

$$H^{op} = (H^{op}, Δ, e, φ^{-1}, Sφ, S(β), S(α)),$$

$$H^{cop} = (H, Δ^{cop}, e, φ_{321}, Sφ, S(β)),$$

respectively, where $Δ^{cop}(h) = h_{(2)} ⊗ h_{(1)}$ ($h ∈ H$). Hence,

$$H^{op,cop} := (H^{op})^{cop} = (H^{op}, Δ^{cop}, e, φ_{321}, S, β, α) = (H^{cop})^{op}.$$

To express iterated comultiplications, we use the following variant of the Sweedler notation: For $h ∈ H$, we write $h_{(1)} ⊗ Δ(h_{(2)}) = h_{(1)} ⊗ h_{(2,1)} ⊗ h_{(2,2)}$ and $Δ(h_{(1)}) ⊗ h_{(2)} = h_{(1,1)} ⊗ h_{(1,2)} ⊗ h_{(2)}$. If $H$ is an ordinary Hopf algebra, then the equation $h_{(1,1)} ⊗ h_{(1,2)}S(h_{(2)}) = h ⊗ 1$ holds for all $h ∈ H$. To state quasi-Hopf analogues of such equations, we introduce:

$$p^R := φ_1 ⊗ φ_2βS(φ_3),$$

$$q^R := φ_1S(φ_2φ_3)φ_2,$$

$$p^L := φ_2S(φ_1β) ⊗ φ_3,$$

$$q^L := S(φ_1)αφ_2 ⊗ φ_3.$$
Then, for all \( h \in H \), we have
\[
\begin{align*}
\Delta F(h(1)) &= \Delta h(1) \otimes \Delta h(2) = \Delta^R(h \otimes 1), \\
\Delta F(h(2)) &= \Delta(h \otimes 1) \Delta^R, \\
\Delta F(h(1)) \Delta F(h(2)) &= (1 \otimes h) \Delta^L.
\end{align*}
\]
(3.15)
(3.16)
(3.17)
(3.18)

The following formulas are also useful:
\[
\begin{align*}
\Delta F(h(1)) \otimes \Delta F(h(2)) &= 1 \otimes 1, \\
\Delta F(h(1)) \otimes \Delta F(h(2)) &= 1 \otimes 1, \\
\Delta F(h(1)) \otimes \Delta F(h(2)) &= 1 \otimes 1, \\
\Delta F(h(1)) \otimes \Delta F(h(2)) &= 1 \otimes 1.
\end{align*}
\]
(3.19)
(3.20)
(3.21)
(3.22)

Equations (3.15)–(3.18) are equivalent in the following sense: Let \( \Delta^R, \Delta^L \) and \( \Delta^\text{cop} \) be the elements \( \Delta^R, \Delta^L \) and \( \Delta^\text{cop} \) for \( H^\text{op} \), respectively. We use the same notation for \( H^\text{cop} \). Then we have
\[
\begin{align*}
\Delta^R &= \Delta^L, \\
\Delta^L &= \Delta^R, \\
\Delta^\text{cop} &= \Delta^\text{cop},
\end{align*}
\]
(3.23)
(3.24)

by defining formulas (3.9)–(3.11). Thus (3.16), (3.17) and (3.18) are obtained by applying (3.9) to the quasi-Hopf algebras \( H^\text{op} \), \( H^\text{cop} \) and \( H^\text{op, cop} \), respectively. Equations (3.19)–(3.22) are also equivalent in the same sense.

The antipode of a Hopf algebra is known to be an anti-coalgebra map. To extend this result to quasi-Hopf algebras, it is convenient to introduce the twisting operation for quasi-Hopf algebras \cite{Dri89, Kas95}. A gauge transformation of \( H \) is an invertible element \( F \in H^\otimes 2 \) such that \( \epsilon(F_1)F_2 = 1 = F_1 \epsilon(F_2) \). Given a gauge transformation \( F \) of \( H \), we define
\[
\Delta^F(h) = F \Delta(h) F^{-1} \quad (h \in H), \quad \Delta^F = S(\Phi_1) \alpha \Phi_2, \quad \beta^F = F_1 \beta S(F_2),
\]
\[
\phi^F = (1 \otimes F)(\text{id} \otimes \Delta)(F \phi(\Delta \otimes \text{id})(F^{-1})(F^{-1} \otimes 1)).
\]
Then \( H^F := (H, \Delta^F, \epsilon, \phi^F, S, \alpha^F, \beta^F) \) is a quasi-Hopf algebra, called the twist of \( H \) by \( F \). Drinfeld \cite{Dri89} introduced a special gauge transformation \( \Phi \) given as follows: We define \( a, b \in H^\otimes 2 \) by
\[
\begin{align*}
a &= S(\Phi_1 \phi_2) \alpha \Phi_2 \phi_3 \phi_4(1) \otimes S(\phi_1) \alpha \Phi_3 \phi_4(2), \\
b &= \Phi_1(3) \beta S(\phi_3) \otimes \phi_1(1) \Phi_2 \beta S(\phi_2 \Phi_3), \\
e &= \Phi_1(1) \beta S(\Phi_2 \phi_3) \otimes \Phi_1(2) \Phi_2 \beta S(\phi_2 \Phi_3).
\end{align*}
\]
(3.25)
(3.26)

Then \( \Phi \) and its inverse are given by the following formulas:
\[
\phi = (S \otimes S) \Delta^\text{cop}(\Phi_1(1)) a \Delta(\Phi_2(2)), \quad \phi^{-1} = \Delta(\Phi_2(1)) b (S \otimes S) \Delta^\text{cop}(\Phi_1(1)).
\]
(3.27)
The antipode of $H$ is shown to be an isomorphism $S : H^{\text{op, cop}} \to H^f$ of quasi-Hopf algebras. Namely, we have

\[
\Delta^f(S(h)) = S(h_{(2)}) \otimes S(h_{(1)}), \quad \epsilon(S(h)) = \epsilon(h), \quad (3.28)
\]

\[
S(\phi_3) \otimes S(\phi_2) \otimes S(\phi_1) = \phi^f, \quad (3.29)
\]

\[
S(\beta) = S(\overline{\alpha}) \alpha, \quad S(\alpha) = f_1 \beta S(\overline{\beta}), \quad (3.30)
\]

for all $h \in H$. The following equations are also useful:

\[
\Delta(\alpha) = f^{-1} \alpha, \quad \Delta(\beta) = \beta f. \quad (3.31)
\]

For later use, we prove the following three lemmas:

**Lemma 3.1.** Let $f_\text{op}$ and $f_\text{cop}$ be the elements $f$ for the quasi-Hopf algebras $H^{\text{op}}$ and $H^{\text{cop}}$, respectively. These elements are given explicitly by

\[
f_\text{op} = \overline{S}(f_2) \otimes \overline{S}(f_1), \quad f_\text{cop} = S(f_1) \otimes S(f_2). \quad (3.32)
\]

**Proof.** By (3.23), (3.25), (3.27), and the definition of $H^{\text{op}}$, we have

\[
f_\text{op} = \Delta(a^R_2) \cdot (\overline{S}_3(\phi_2)S(\phi_1)) \otimes \overline{S}_3(\phi_3S(\phi_1)) \cdot (S \otimes S)\Delta^\text{cop}(a^R_1). \quad (3.32)
\]

Thus we compute:

\[
(S \otimes S)(f_{\text{op}}) = \Delta^\text{cop}(a^R_2) \cdot (\overline{S}(\phi_2)S(\phi_1)) \otimes \overline{S}_3(\phi_3S(\phi_1)) \cdot (S \otimes S)\Delta(a^R_2)
\]

\[
= \Delta^\text{cop}(a^R_2) \cdot b_{21} \cdot (S \otimes S)\Delta(a^R_2)
\]

\[
= \Delta^\text{cop}(a^R_2) \cdot b_{21} \cdot \Delta^\text{cop}(S(a^R_2)) \cdot f_{21}^{-1}
\]

\[
= \Delta^\text{cop}(a^R_2) \cdot \Delta^\text{cop}(b_{21}) \cdot \Delta^\text{cop}(S(a^R_2)) \cdot f_{21}^{-1}
\]

\[
= \Delta^\text{cop}(a^R_2) \cdot \Delta^\text{cop}(f_{21}^{-1}) \cdot f_{21} \quad (3.33)
\]

This shows the first equation of (3.32). To prove the second one, we note:

\[
f_{\text{cop}} = (S \otimes S)(f_{\text{cop}}) = (S \otimes S)\Delta(a^L_1) \cdot (S \otimes S)(\phi_2)\phi_2(\phi_1(1)) \otimes S(\phi_3)\overline{S}(\phi_1(1)) \cdot \Delta^\text{cop}(p^R_1).
\]

Thus we have

\[
(S \otimes S)(f_{\text{cop}}) = (S \otimes S)\Delta^\text{cop}(p^L_1) \cdot (S \otimes S)(\phi_2)\phi_2(\phi_1(1)) \otimes S(\phi_3)\overline{S}(\phi_1(1)) \cdot \Delta^\text{cop}(p^R_1)
\]

\[
= \Delta(S(p^L_1)) \cdot S(\overline{S}(\phi_1(1))) \cdot S(\phi_3) \phi_2(\phi_1(1)) \otimes S(\phi_3) \phi_2(\phi_1(1)) \cdot \Delta^\text{cop}(p^L_1)
\]

\[
= \Delta(S(p^L_1)) \cdot \Delta(\alpha) \cdot \Delta(p^L_1)
\]

\[
= \Delta(S(p^L_1)) \cdot \Delta(\alpha) \cdot \Delta(p^L_1)
\]

\[
= \Delta(S(p^L_1)) \cdot \alpha p^L_1 = f. \quad (3.34)
\]

**Lemma 3.2.** The following equations hold:

\[
q_{11}^R \overline{\phi}_1 \otimes q_{12}^R \overline{\phi}_2 \otimes q_{23}^R \overline{\phi}_3 = \phi_1 \otimes q_{11}^R \phi_2 \otimes S(\phi_3)q_{23}^R \phi_2, \quad (3.33)
\]

\[
\phi_1 p_{11}^R \otimes \phi_2 p_{12}^R \otimes \phi_3 p_{23}^R = \overline{\phi}_1 \otimes \phi_2 \overline{\phi}_2 \otimes \overline{\phi}_3 p_{23}^R S(\overline{\phi}_3). \quad (3.34)
\]
Proof. The first equation is verified as follows:

\[
\begin{align*}
q_1^R \Phi_1 & \otimes q_1^R \Phi_2 \otimes q_2^R \Phi_3
\end{align*}
\]

where \(\Phi\) is a copy of \(\phi\). The second one is obtained by applying (3.33) to \(H^{op}\).

\[\tag{3.35}\]

Lemma 3.3. The following equations hold:

\[
\begin{align*}
S(p_1^R q_1^R (p_2^R \otimes q_2^R)) & = S(p_1^R) \otimes S(p_2^R),
\end{align*}
\]

where \(\Phi\) and \(\Phi'\) are copies of \(\phi\). Equations (3.36), (3.37), and (3.38) are obtained by applying the equation (3.35) to \(H^{op}, H^{cop}\), and \(H^{op,cop}\), respectively.

3.3. Modules over a quasi-Hopf algebra. Let \(H\) be a quasi-Hopf algebra with bijective antipode. If \(V\) and \(W\) are left \(H\)-modules, then their tensor product \(V \otimes W\) is a left \(H\)-module by the action given by \(h \cdot (v \otimes w) = h_{(1)} v \otimes h_{(2)} w\) for \(h \in H, v \in V, w \in W\). The vector space \(V := k\) is a left \(H\)-module through the counit \(\epsilon : H \rightarrow k\).
The category \( _H \mathcal{M} \) of left \( H \)-modules is a monoidal category with the associator \( \Phi \), the left unit isomorphism \( l \) and the right unit isomorphism \( r \) given by

\[
\Phi_{X,Y,Z}((x \otimes y) \otimes z) = \phi_1 x \otimes (\phi_2 y \otimes \phi_3 z),
\]

\[
l_X(1 \otimes x) = x, \quad \text{and} \quad r_X(x \otimes 1) = x,
\]

respectively, for \( X, Y, Z \in _H \mathcal{M} \), \( x \in X \), \( y \in Y \) and \( z \in Z \).

For a finite-dimensional left \( H \)-module \( X \), we define the left \( H \)-module \( X^\ast \) to be the vector space \( X^\ast = X^\ast \) with the left \( H \)-module structure given by \( h \cdot x = \xi \leftarrow S(h) \) for \( h \in H \) and \( \xi \in X^\ast \). We fix a basis \( \{x_i\} \) of \( X \) and let \( \{x^i\} \) be the dual basis of \( \{x_i\} \). We define \( \text{ev}_X \) and \( \text{coev}_X \) by

\[
\text{ev}_X : X^\ast \otimes X \to 1, \quad \text{ev}_X(\xi \otimes x) = \langle \xi, \alpha x \rangle,
\]

\[
\text{coev}_X : 1 \to X \otimes X^\ast, \quad \text{coev}_X(1) = \beta x_i \otimes x^i,
\]

respectively, for \( \xi \in X^\ast \) and \( x \in X \), where we have used the Einstein convention to suppress the sum over \( i \). Equations (3.5) and (3.6) imply that \((X^\ast, \text{ev}_X, \text{coev}_X)\) is a left dual object of \( X \).

**Remark 3.4.** A representation-theoretical meaning of the condition (3.5) is explained as follows: The triple \((1, r_1, r_1^{-1})\) is a left dual object of \( 1 \). Thus, by the uniqueness of a left dual object (see Subsection 2.2), there is an isomorphism \( f : 1 \to 1^\ast \) in \( _H \mathcal{M} \) such that \((id_1 \otimes f)r_1^{-1} = \text{coev}_1 \) and \( r_1 = \text{ev}_1(f \otimes id_1) \). By the former equation, \( f \) is given by \((f(c), c') = \epsilon(\beta)c' \) for \( c, c' \in k \). Thus (3.8) is equivalent to that the canonical isomorphism \( f : 1 \to 1^\ast \) in \( _H \mathcal{M} \) coincides with the canonical isomorphism \( k \cong k^* \) of vector spaces.

Let \( X \) be a finite-dimensional left \( H \)-module with basis \( \{x_i\} \), and let \( \{x^i\} \) be the dual basis of \( X^\ast \). As we have recalled in Subsection 2.2, there is a natural isomorphism

\[
\mathcal{Q} = \mathcal{Q}_{V,W,X} : \text{Hom}_H(V,X \otimes W) \to \text{Hom}_H(X^\ast \otimes V,W)
\]

for \( V, W \in _H \mathcal{M} \). We express this isomorphism explicitly. Given a morphism \( f : A \to B \otimes C \in _H \mathcal{M} \), where \( A, B, C \in _H \mathcal{M} \), we write \( f(a) \) for \( a \in A \) symbolically as \( f(a)_B \otimes f(a)_C \).

**Lemma 3.5.** The isomorphism \( \mathcal{Q} \) and its inverse are given by

\[
\mathcal{Q}(f)(\xi \otimes v) = \langle \xi, \alpha_1^1 f(v)_X \rangle \alpha_1^2 f(v)_W,
\]

\[
\mathcal{Q}^{-1}(g)(v) = S(p_1^1)x_i \otimes g(x^i \otimes p_2^v v)
\]

for \( f \in \text{Hom}_H(V,X \otimes W) \), \( g \in \text{Hom}_H(X^\ast \otimes V,W) \), \( v \in V \) and \( \xi \in X^\ast \).

**Proof.** For \( f \in \text{Hom}_H(V,X \otimes W) \), \( \xi \in X^\ast \) and \( v \in V \), we compute:

\[
\mathcal{Q}(f)(\xi \otimes v) = ((\text{ev}_X \otimes id_W)\Phi_{X^\ast,X,W}^{-1}(\text{id}_{X^\ast} \otimes f))((\xi \otimes v)
\]

\[
= (\text{ev}_X \otimes id_W)(\Phi_1^1 \xi \otimes \Phi_2^2 f(v)_X \otimes \Phi_3^3 f(v)_W)
\]

\[
= \langle \xi, S(\Phi_1) \alpha_2 f(v)_X \rangle \Phi_3 f(v)_W
\]

\[
= \langle \xi, \alpha_1^1 f(v)_X \rangle \alpha_1^2 f(v)_W.
\]
To verify the expression for $Q^{-1}$, we note that the equation $x_i \otimes (x^i \otimes T) = T(x_i) \otimes x^i$ holds in $X \otimes X^*$ for all $T \in \text{End}_k(X)$. In particular, we have $x_i \otimes h x^i = S(h) x_i \otimes x^i$ for all $h \in H$. Thus we have

$$Q^{-1}(g)(v) = \left((id_X \otimes g)\Phi_X,\psi,\nu(\text{coev}_X \otimes id_1)\right)(v)$$

$$= (id_X \otimes g)(\phi_1 \beta x_i \otimes \phi_2 x^i \otimes \phi_3 v)$$

$$= \phi_1 \beta x_i \otimes g(\phi_2 x^i \otimes \phi_3 v)$$

$$= \phi_1 \beta S(\phi_2) x_i \otimes g(x^i \otimes \phi_3 v)$$

$$= S(p_1^T) x_i \otimes g(x^i \otimes p_2^Tv)$$

for $g \in \text{Hom}_H(X^\vee \otimes V, W)$ and $v \in V$. \hfill \Box

The following lemma is proved in a similar manner:

**Lemma 3.6.** The natural isomorphism

$$P = P_{V,W,X} : \text{Hom}_H(V \otimes X, W) \rightarrow \text{Hom}_H(V, W \otimes X^\vee)$$

and its inverse are given by

$$P(f)(v) = f(p_1^R v \otimes p_2^R x_i) \otimes x^i,$$

$$P^{-1}(g)(v \otimes x) = q_1^R g(v) \otimes (g(x)_{X^\vee} \otimes S(q_2^R)x)$$

for $f \in \text{Hom}_H(V \otimes X, W)$, $g \in \text{Hom}_H(V, W \otimes X^\vee)$, $v \in V$ and $x \in X$.

A left $H$-module (co)algebra is a synonym for a (co)algebra in the monoidal category $H, \mathcal{M}$. As an application of Lemmas 3.5 and 3.6, we give a description of the dual (co)module in $H, \mathcal{M}$. For this purpose, we introduce the following notation:

$$u_l^T = T_1 S(q_2^R) \otimes T_2 S(q_1^R),$$

$$v^R = S(p_2^R) T_1 \otimes S(p_1^T) T_2.$$  

**Lemma 3.7.** (i) Let $A$ be a left $H$-module algebra. If $X$ is a finite-dimensional left $A$-module in $H, \mathcal{M}$ with action $\triangleright$, then the right $A$-module structure of $X^\vee$ is given by

$$\langle \xi \triangleright a, x \rangle = \langle \xi, (u_l^T a) \triangleright (u_2^T x) \rangle \quad (a \in A, \xi \in X^\vee, x \in X).$$

(ii) Let $C$ be a left $H$-module coalgebra. If $X$ is a finite-dimensional right $C$-comodule in $H, \mathcal{M}$ with coaction $x \mapsto x(0) \otimes x(1)$, then the left $C$-comodule structure of $X^\vee$ is given by

$$\xi \mapsto \langle \xi, (u_l v^R)_{x(0)} \triangleright (u_2^R x(1)) \otimes x^i \rangle \quad (\xi \in X^\vee),$$

where $\{x_i\}$ is a basis of $X$ and $\{x^i\}$ is the dual basis of $\{x_i\}$.

**Proof.** We write $\rho(a \otimes x) = a \triangleright x$. We have $h \rho(a \otimes x) = \rho(h(1)a \otimes h(2) x)$ for all $h \in H$, $a \in A$ and $x \in X$ since $\rho : A \otimes X \rightarrow X$ is a morphism in $H, \mathcal{M}$. By Lemmas 3.5 and 3.6, we have

$$\langle \xi \triangleright a, x \rangle = \langle (Q_A,\psi,\nu_X) P_{A,X}(\rho)(\xi \otimes a), x \rangle$$

$$= \langle \xi, q_1^R \rho(p_1^R a \otimes p_2^R x_i) \rangle \langle q_2^R x_i, x \rangle$$

$$= \langle \xi, \rho(q_1^R(1) p_1^R a \otimes q_1^R(2) p_2^R S(q_2^R x_i)) \rangle \langle x^i, x \rangle$$

$$= \langle \xi, \rho(T_1 S(q_2^R) a \otimes S(T_2 S(q_1^R) x_i)) \rangle$$

$$= \langle \xi, (u_l^T a) \triangleright (u_2^T x) \rangle.$$
for $\xi \in X^\vee$, $a \in A$ and $x \in X$. Hence Part (i) is proved. To prove Part (ii), we let $\delta : X \to X \otimes C$ be the coaction of $C$ on $X$ and set $\delta^3 = Q_{X,C,X}(\delta)$. Then the coaction of $C$ on $X^\vee$ is computed as follows:

$$
\xi \mapsto \mathcal{P}_{X^\vee,C,X}(\delta^3)(\xi) = \delta^3(\mathcal{P}^R \xi \otimes \mathcal{P}^R_2 x_i) \otimes x^i
$$

$$
= \langle \mathcal{P}^R_1 \xi, q_1^L(q_2^R x_i(0)) q_2^L_2(\mathcal{P}^R_2 x_i(1)) \otimes x^i
$$

$$
= \langle \xi, \mathcal{S}(\mathcal{P}^R_1 q_1^L q_1^R_2 x_i(0)) q_2^L_2 \mathcal{P}^R_2 x_i(1) \otimes x^i
$$

$$
= \langle \xi, \mathcal{S}(\mathcal{P}^R_1 q_1^L q_1^R_2 x_i(0)) \mathcal{S}(\mathcal{P}^R_2) x_i(1) \otimes x^i
$$

for all $\xi \in X^\vee$. Thus the proof of Part (ii) is done. \(\square\)

For a finite-dimensional left $H$-module $X$, we also define the left $H$-module $X^\vee$ to be the vector space $X^* \otimes C$ with the left $H$-module structure given by $h \cdot \xi = \xi \otimes S(h)$ for $h \in H$ and $\xi \in X^\vee$. We fix a basis $\{x_i\}$ of $X$ and define $\text{ev}'_X$ and $\text{coev}'_X$ by

$$
\text{ev}'_X : X \otimes X^\vee \to 1, \quad \text{ev}'_X(x \otimes \xi) = \langle \xi, \mathcal{S}(\alpha) x \rangle,
$$

$$
\text{coev}'_X : 1 \to X^\vee \otimes X, \quad \text{coev}'_X(1) = x^i \otimes \mathcal{S}(\beta) x_i
$$

for $x \in X$ and $\xi \in X^\vee$, where $\{x^i\}$ is the dual basis of $\{x_i\}$. Then $(X^\vee, \text{ev}'_X, \text{coev}'_X)$ is a right dual object of $X$ in $H$-$\mathcal{M}$. We remark that a right dual object of $X$ is a left dual object of $X$ in $H$-$\mathcal{M}^{rev}$ and the monoidal category $(H$-$\mathcal{M}$)$_{rev}$ can be identified with $H$-$\mathcal{M}^{cop}$. We now define

$$
u^R = \mathcal{S}(q_1^R f_2) \otimes \mathcal{S}(q_1^L 1), \quad \nu^L = \mathcal{S}(f_2 q_2^R) \otimes \mathcal{S}(1 q_1^R).$$

If we denote by $\nu^L_{\text{cop}}$ and $\nu^R_{\text{cop}}$ the elements $\nu^L$ and $\nu^R$ for $H$-$\mathcal{M}^{cop}$, respectively, then we have

$$
\nu^L_{\text{cop}} = \nu^R_{21} \quad \text{and} \quad \nu^R_{\text{cop}} = \nu^L_{21}
$$

by equations (3.48) and (3.49). By applying Lemma 3.7 to $H$-$\mathcal{M}^{cop}$, we obtain the following lemma:

**Lemma 3.8.** (i) Let $A$ be a left $H$-module algebra. If $X$ is a finite-dimensional right $A$-module in $H$-$\mathcal{M}$ with action $\langle$, then the left $A$-module structure of $X^\vee$ is given by

$$
\langle a \triangleright \xi, x \rangle = \langle \xi, (q_1^R a) \otimes \mathcal{S}(q_1^L 1) \rangle \quad (a \in A, \xi \in X^\vee, x \in X).
$$

(ii) Let $C$ be a left $H$-module coalgebra. If $X$ is a finite-dimensional left $C$-comodule in $H$-$\mathcal{M}$ with coaction $\Delta x \mapsto x_{(-1)} \otimes x_{(0)}$, then the right $C$-comodule structure of $X^\vee$ is given by

$$
\xi \mapsto x^i \otimes \mathcal{S}(q_1^L 1) \langle \xi, \mathcal{S}(q_1^R) x_i(0) \rangle \quad (\xi \in X^\vee),
$$

where $\{x_i\}$ is a basis of $X$ and $\{x^i\}$ is the dual basis of $\{x_i\}$.

4. Integral theory for quasi-Hopf algebras

4.1. Integrals. The notions of integrals and cointegrals for Hopf algebras play an important role in the Hopf algebra theory and its applications. The integral theory for quasi-Hopf algebras is established by Hausser and Nill in [HN99]. The definition of integrals in a quasi-Hopf algebra is completely same as the case of Hopf algebras:
Definition 4.1. Let $H$ be a quasi-Hopf algebra. A left integral in $H$ is an element $\Lambda \in H$ such that $h \Lambda = \epsilon(h)\Lambda$ for all $h \in H$. Analogously, a right integral in $H$ is an element $\Lambda \in H$ such that $\Lambda h = \epsilon(h)\Lambda$ for all $h \in H$.

The definition of cointegrals on a quasi-Hopf algebra is more complicated than the Hopf case. The original definition of Hausser and Nill is based on the fundamental theorem for quasi-Hopf bimodules. There are several formulas defining cointegrals, however, the original definition is convenient from the theoretical point of view. In this section, following [HN99, BC03, BC12], we review basic results on quasi-Hopf bimodules and (co)integrals for quasi-Hopf algebras.

4.2. Quasi-Hopf bimodules. Let $H$ be a quasi-Hopf algebra with bijective antipode. The category $\mathcal{H}_H$ of $H$-bimodules is identified with the category of left modules over $H^r := H \otimes H^{op}$. Since $H^r$ is naturally a quasi-Hopf algebra, the category $\mathcal{H}_H$ is a monoidal category. To be precise, for $M, N \in \mathcal{H}_H$, their tensor product $M \otimes N \in \mathcal{H}_H$ is the vector space $M \otimes N$ endowed with the $H$-bimodule structure given by

$$h \cdot (m \otimes n) \cdot h' = h_1 n h'_1 \otimes h_2 n h'_2$$

for $h, h' \in H$, $m \in M$ and $n \in N$. The unit object $\mathbb{1}$ is the base field $k$ regarded as an $H$-bimodule by the counit of $H$. The associator of $(\mathcal{H}_H, \otimes, \mathbb{1})$ is given by

$$\Phi_{L,M,N} : (L \otimes M) \otimes N \to (L \otimes M) \otimes N,$$

$$(a \otimes b) \otimes c \mapsto \phi_1 a \phi_1 \otimes (\phi_2 b \phi_3 c \phi_4)$$

for $L, M, N \in \mathcal{H}_H$, $a \in L$, $b \in M$ and $c \in N$. The left and the right unit isomorphisms of $\mathcal{H}_H$ are same as those of the monoidal category of vector spaces over $k$.

Equations (3.1) and (3.2) imply that the $H$-bimodule $H$ is a coalgebra in the monoidal category $(\mathcal{H}_H, \otimes, \mathbb{1})$ with respect to $\Delta$ and $\epsilon$. The category $\mathcal{M}_H$ of right quasi-Hopf bimodules over $H$ is defined as the category of right $H$-comodules in $\mathcal{H}_H$. Given a left $H$-module $V$, we regard it as an $H$-bimodule by defining the right action of $H$ through the counit of $H$. Then the $H$-bimodule $V \otimes H$ is a right $H$-comodule in $\mathcal{H}_H$ by the coaction

$$V \otimes H \xrightarrow{V \otimes \Delta} V \otimes (H \otimes H) \xrightarrow{\Phi_{V,H,H}^{-1}} (V \otimes H) \otimes H. \quad (4.1)$$

Thus we have a functor

$$\mathcal{H}_H \to \mathcal{H}_H^H, \quad V \mapsto V \otimes H. \quad (4.2)$$

For $M \in \mathcal{H}_H^H$, we define the linear map $E_M : M \to M$ by

$$E_M(m) = q^R m_{(0)} \beta S(q^R m_{(1)}), \quad (m \in M),$$

where $m \mapsto m_{(0)} \otimes m_{(1)}$ is the coaction. The subspace $M^{coH} := \text{Im}(E_M)$ is called the space of coinvariants. Although $M^{coH}$ is not a submodule of $M$, it is a left $H$-module by the action $\triangleright$ given by $h \triangleright m = E_M(hm)$ for $h \in H$ and $m \in M^{coH}$. This construction extends to the functor

$$\mathcal{H}_H^H \to \mathcal{H}_H, \quad M \mapsto M^{coH}. \quad (4.3)$$

The fundamental theorem for quasi-Hopf bimodules [HN99, Section 3] states that the functors (4.2) and (4.3) are mutually quasi-inverse to each other. Furthermore,
they form an adjoint equivalence between $H\mathcal{H}$ and $H\mathcal{H}_H$ in the unit and the counit given respectively by

$$V \to (V \otimes H)^{\text{co}H}, \quad v \mapsto v \otimes 1 \quad (v \in V),$$

$$\co^H \otimes H \to M, \quad m \otimes h \mapsto mh \quad (m \in \co^H, h \in H).$$

The category $H\mathcal{H}_H$ of left quasi-Hopf bimodules over $H$ is defined to be the category of left $H$-comodules in $H\mathcal{H}_H$. This category can be identified with the category of right quasi-Hopf modules over $K = H^{\text{cop}}$. Thus, by applying the fundamental theorem to $K$, we see that the functor

$$H\mathcal{H} \to H\mathcal{H}_H, \quad V \mapsto H \hat{\otimes} V$$

is an equivalence of categories. A quasi-inverse of (4.4), which we denote by

$$\text{co}^H(-) : H\mathcal{H}_H \to H\mathcal{H},$$

is given as follows: For $M \in H\mathcal{H}_H$, we define the linear map $E_M^\text{co} : M \to M$ by

$$E_M^\text{co}(m) = q_{2}^{\beta m_{(0)}(\alpha)} \overline{\alpha}(q_{1}^{\beta} m_{(1)}) \quad (m \in M),$$

where $m \mapsto m_{(-1)} \otimes m_{(0)}$ is the coaction. Then $\text{co}^H M := \text{Im}(E_M^\text{co})$ as a vector space. The left $H$-module structure is given by $h \triangleright m = E_M^\text{co}(hm)$. The functors (4.6) and (4.7) actually form an adjoint equivalence with the unit and the counit given respectively by

$$V \to \text{co}^H(H \hat{\otimes} V), \quad v \mapsto 1 \otimes v \quad (v \in V),$$

$$H \hat{\otimes} \text{co}^H M \to M, \quad h \otimes m \mapsto mh \quad (m \in \text{co}^H, h \in H).$$

### 4.3. Cointegrals

Let $H$ be a quasi-Hopf algebra with bijective antipode. Since $H\mathcal{H}_H$ is isomorphic to the category of left $H^r$-modules as a monoidal category, every finite-dimensional object of $H\mathcal{H}_H$ admits a left dual object and a right dual object. Specifically, if $M$ is a finite-dimensional $H$-bimodule, then its left dual object $(M^\vee, ev_M, coev_M)$ is given as follows: As a vector space, $M^\vee = M^*$. We fix a basis $\{m_i\}$ of $M$ and let $\{m^i\}$ be the dual basis of $\{m_i\}$. Then the $H$-bimodule structure of $M^\vee$, the evaluation morphism and the coevaluation morphism are given by

$$h \cdot \xi \cdot h' = \overline{S}(h') \mapsto \xi \leftarrow S(h),$$

$$ev_M : M^\vee \otimes M \to k, \quad ev_M(\xi \otimes m) = (\xi, \alpha m \overline{S}(\beta)), \quad (4.10)$$

$$coev_M : k \to M \hat{\otimes} M^\vee, \quad coev_M(1) = \beta m_i \overline{S}(\alpha) \otimes m^i, \quad (4.11)$$

respectively, for $h, h' \in H$, $m \in M$ and $\xi \in M^\vee$, where the Einstein notation is used to suppress the sum over $i$. Analogously, the right dual object $\vee^* M$ of $M$ is defined by $\vee^* M = M^*$ as a vector space. The $H$-bimodule structure, the evaluation and the coevaluation are given by

$$h \cdot \xi \cdot h' = S(h') \mapsto \xi \leftarrow \overline{S}(h),$$

$$ev'_M : M \otimes \vee^* M \to k, \quad ev'_M(m \otimes \xi) = (\xi, \overline{S}(\alpha)m\beta), \quad (4.13)$$

$$coev'_M : k \to \vee^* M \hat{\otimes} M, \quad coev'_M(1) = m^i \otimes \overline{S}(\beta)m_i \alpha, \quad (4.14)$$

respectively, for $h, h' \in H$, $m \in M$ and $\xi \in \vee^* M$. 

Lemma 4.2. (i) If $M$ is a finite-dimensional right quasi-Hopf bimodule over $H$, then the $H$-bimodule $M^\vee$ is a left quasi-Hopf bimodule by the coaction

$$\xi \mapsto (\xi, \psi^R m_{i(0)} u^R_1) \psi^R m_{i(1)} u^R_i \otimes m^i \quad (\xi \in M^\vee),$$

where $\{m_i\}$ is a basis of $M$, $\{m^i\}$ is the dual basis of $M^*$.

(ii) If $M$ is a finite-dimensional left quasi-Hopf bimodule over $H$, then the $H$-bimodule $M^\vee$ is a right quasi-Hopf bimodule by the coaction

$$\xi \mapsto m^i \otimes \psi^L m_{i(-1)} u^L_1 \psi^L m_{i(0)} u^L_2 \quad (\xi \in M),$$

where $\{m_i\}$ is a basis of $M$, $\{m^i\}$ is the dual basis of $M^*$.

See (4.14), (4.15), (4.16) and (5.41) for the definitions of the elements $u^L$, $u^R$, $\psi^R$ and $\psi^L$ of $H \otimes H$, respectively.

Proof. The result follows from Lemmas 3.7 and 3.8 applied to $H^c$. One can also verify this lemma by a direct computation along the same way as [BC12, Proposition 3.2].

Now we suppose that $H$ is a finite-dimensional quasi-Hopf algebra. Then $H$ is a finite-dimensional left and right quasi-Hopf bimodule over $H$. Thus $H^\vee$ and $\vee H$ are a left and a right quasi-Hopf bimodule over $H$, respectively.

Definition 4.3. We set $f^L := (\vee H)^{\text{co} H}$ and $f^R := \text{co} H(H^\vee)$ and call them the spaces of left cointegrals and right cointegrals, respectively.

The fundamental theorem gives an isomorphism $\vee H \cong f^L \otimes H$ in $H_0^H$. By counting dimensions, we see that $f^L$ is one-dimensional. Thus there is an algebra map $\mu : H \to k$ such that $h \mapsto \mu(h)$ for all $h \in H$ and $\lambda \in f^L$.

Definition 4.4. We call $\mu$ the modular function on $H$. We say that $H$ is unimodular if the modular function on $H$ is identical to the counit of $H$.

By the fundamental theorem, there is also an isomorphism $H^\vee \cong H \hat{\otimes} f^R$ in $H_1^H$. Hence there is an algebra map $\mu' : H \to k$ such that $h \mapsto \mu'(h)\lambda$ for all $h \in H$ and $\lambda \in f^R$. The following lemma shows that $\mu'$ coincides with the modular function.

Lemma 4.5. $f^L \cong f^R$ as left $H$-modules.

Proof. We have isomorphisms

$$\text{co} f^R \otimes f^L \cong \text{co} f^R \otimes (f^L \otimes H) \cong \vee f^R \otimes H \cong \vee H \cong \vee H \cong H$$

of right quasi-Hopf bimodules. By the fundamental theorem, we have an isomorphism $\text{co} f^R \otimes f^L \cong 1$ of left $H$-modules. Thus $f^L \cong f^R$ as left $H$-modules. □

4.4. Properties of cointegrals. Let $H$ be a finite-dimensional quasi-Hopf algebra. We note that the antipode of $H$ is bijective if this is the case [BC03, Sch04]. By the fundamental theorem for quasi-Hopf bimodules, the map

$$\Xi : f^L \otimes H \to \vee H, \quad \lambda \otimes h \mapsto S(h) \mapsto \lambda \quad (\lambda \in f^L, h \in H)$$

is an isomorphism in the category $H_0^H$. We fix a non-zero left cointegral $\lambda$ on $H$. Since the map $\Xi$ is bijective, and since the antipode $S$ of $H$ is bijective, the map $H \to H^*$ given by $h \mapsto h \mapsto \lambda (h \in H)$ is also bijective. This means that the algebra $H$ is a Frobenius algebra with Frobenius form $\lambda$. 

\"\"\"
We briefly recall basic results on Frobenius algebras. Let $A$ be a Frobenius algebra with Frobenius form $\lambda : A \to k$. By definition, the map $\Theta : A \to A^*$ defined by $\Theta(a) = a \to \lambda (a \in A)$ is bijective. Given an algebra map $\chi : A \to k$, we define $I_\chi = \{ t \in A \mid at = \chi(a)t \text{ for all } a \in A \}$ and $J_\chi = \{ f \in A^* \mid a \to f = \chi(a)f \text{ for all } a \in A \}$. Since $\Theta$ is an isomorphism of left $A$-modules, it induces an isomorphism between $I_\chi$ and $J_\chi$. It is easy to see that $J_\chi$ is spanned by $\chi$. Hence $I_\chi$ is spanned by $t_\chi := \Theta^{-1}(\chi)$. If $t \in I_\chi$ is a non-zero element, then it is a non-zero scalar multiple of $t_\chi$. Since

$$\langle \lambda, t_\chi \rangle = (t_\chi \to \lambda)(1) = (\Theta \Theta^{-1}(\chi))(1) = \chi(1) = 1,$$

we have $\lambda(t) \neq 0$ for all non-zero elements $t \in I_\chi$. The **Nakayama automorphism of $A$ (with respect to the Frobenius form $\lambda$)** is the algebra automorphism $\nu : A \to A$ determined by

$$\lambda(ab) = \lambda(b \cdot \nu(a)) \quad (a, b \in A). \quad (4.19)$$

Let $\tau$ be the inverse of $\nu$. Then we have

$$\Theta(t_\chi a)(b) = \lambda(bt_\chi a) = \lambda(\tau(a)bt_\chi) = \chi(\tau(a))\chi(b)\lambda(t_\chi) = \chi(\tau(a))\chi(b)$$

for all $a, b \in A$. Thus we have $\tau a = \chi(\tau(a))t$ for all $t \in I_\chi$ and $a \in A$. Now we apply these results to the Frobenius algebra $H$.

**Theorem 4.6.** Let $\lambda$ be a non-zero left cointegral on $H$. Then we have:

1. $H$ is a Frobenius algebra with Frobenius form $\lambda$. The **Nakayama automorphism of $H$ with respect to $\lambda$** is given by

   $$\nu(h) = S(S(h) \leftarrow \mu) \quad (h \in H),$$

   where $\mu$ is the modular function on $H$ and

   $$h \leftarrow \xi = \langle \xi, h_{(1)} \rangle h_{(2)} \quad (\xi \in H^*, h \in H).$$

2. The space of left integrals in $H$ is one-dimensional. If $\Lambda$ is a non-zero left integral in $H$, then we have $\lambda(\Lambda) \neq 0$ and $\Lambda h = \mu(h)\Lambda$ for all $h \in H$.

3. The space of right integrals in $H$ is also one-dimensional. If $\Lambda$ is a non-zero right integral in $H$, then we have $\lambda(\Lambda) \neq 0$ and $h\Lambda = \overline{\mu}(h)\Lambda$ for all $h \in H$, where $\overline{\mu} = \mu \circ S$.

**Proof.** We have proved that $H$ is a Frobenius algebra with Frobenius form $\lambda$. We verify the given expression of the Nakayama automorphism. Since the map $\Xi$ given by \cite{4.18} is left $H$-linear, we have

$$\lambda(\overline{S}(h)S(h')) = (h \cdot \Xi(\lambda \otimes h'))(1) = \Xi(h \cdot (\lambda \otimes h'))(1)$$

$$= \Xi(h_{(1)} \triangleright \lambda \otimes h_{(2)}h')(1) = \mu(h_{(1)})\lambda(S(h_{(2)}h')) = \lambda(S(h')S(h \leftarrow \mu))$$

for all $h, h' \in H$. If we replace $h$ and $h'$ with $S(h)$ and $\overline{S}(h')$, respectively, then we obtain $\lambda(hh') = \lambda(h' S(S(h) \leftarrow \mu))$ as desired. Thus Part (1) is proved.

To prove Part (2), we note that the space of left integrals in $H$ is $I_\chi$ with $\chi = \epsilon$ in the above notation. By the above-mentioned results on Frobenius algebras, $I_\epsilon$ is one-dimensional. Furthermore, if $\Lambda$ is a non-zero left integral in $H$, then we have $\lambda(\Lambda) \neq 0$ and $\Lambda h = \epsilon(\tau(h))\Lambda = \mu(h)\Lambda$ for all $h \in H$. Thus the proof of Part (2) is done. Part (3) is proved by applying the same argument to $H^{op}$, which is also a Frobenius algebra with the same Frobenius form $\lambda$. \hfill $\Box$
Let $\lambda$ be a non-zero left cointegral on $H$. In summary, the bijectivity of the map $\Xi$ given by (4.18) implies that $H$ is a Frobenius algebra with Frobenius form $\lambda$. As we have seen in the proof of the above theorem, an expression of the Nakayama automorphism is obtained from the left $H$-linearity of $\Xi$. Other results in this subsection follow from general theory of Frobenius algebras.

The $H$-co-linearity of $\Xi$ has not been used yet. This property gives the following equation:

**Lemma 4.7.** Let $\lambda$ be a left cointegral on $H$. Then the equation

$$v_1^h h_{(1)} u_1^i (\lambda, v_2^h h_{(2)} u_2^l S(h')) = \mu(\overline{\Phi}_1)(\lambda, h S(\overline{\phi}_2 h_{(1)})) \overline{\phi}_3 h_{(2)}$$

(4.20)

holds for all $h, h' \in H$.

**Proof.** We fix a left cointegral $\lambda$ on $H$ and elements $h, h' \in H$. Let $\delta$ and $\delta'$ be the right coactions of $H^*$ and $H$ and $\forall H$, respectively. Since the map $\Xi$ preserves the coactions, we have $F = G$, where

$$F = ((\Xi \otimes \text{id}_H) \circ \delta)(\lambda \otimes h') \quad \text{and} \quad G = (\delta' \circ \Xi)(\lambda \otimes h').$$

We recall that $\delta$ is given by (4.11) with $V = f^k$, and the $H$-bimodule structure of $f^k$ is given by $x y = \mu(x) \epsilon(y) \lambda$ for $x, y \in H$. Thus we have

$$\delta(\lambda \otimes h') = (\overline{\Phi}_1 \lambda \Phi_1 \otimes \overline{\Phi}_3 h_{(3)} (\Phi_2) \otimes \overline{\Phi}_3 h_{(2)} \Phi_3 = (\mu(\overline{\Phi}_1) \lambda \otimes \overline{\phi}_3 h_{(1)}(\lambda) \otimes \overline{\phi}_3 h_{(2)}(\lambda)).$$

Now we define a linear map $T_h : H^* \otimes H \to H$ by $T_h(\xi \otimes a) = \xi(h) a$. Then,

$$T_h(F) = \Xi(\mu(\overline{\Phi}_1) \lambda \otimes \overline{\phi}_3 h_{(1)}(h)) \cdot \overline{\phi}_3 h_{(2)}(h) = \mu(\overline{\Phi}_1)(\lambda, h S(\overline{\phi}_2 h_{(1)})) \overline{\phi}_3 h_{(2)}.$$

We fix a basis $\{e_i\}$ of $H$ and let $\{e^i\}$ be the dual basis of $\{e_i\}$. Then we have

$$T_h(G) = T_h(\delta'(S(h') \to \lambda)) = (e^i, h) v_1^h e_{(1)} u_1^i (\lambda, v_2^h e_{(2)} u_2^l)$$

$$= v_1^h h_{(1)} u_1^i (\lambda, v_2^h h_{(2)} u_2^l S(h'))$$

by Lemma 4.2. Now equation (4.20) follows from $T_h(F) = T_h(G)$. \hfill $\square$

### 4.5. Characterizations of cointegrals

If $\lambda \in H^*$ is a left cointegral on $H$, then we have

$$v_1^h h_{(1)} u_1^i (\lambda, v_2^h h_{(2)} u_2^l) = \mu(\overline{\Phi}_1)(\lambda, h S(\overline{\phi}_2)) \overline{\phi}_3$$

(4.21)

for all $h \in H$ by (4.20). The first appearance of (4.21) seems to be the proof of [BC03 Proposition 3.4]. Later, equation (4.21) has been used as a definition of left cointegrals on $H$ in some literature including [BT04, BT08]. However, it is not trivial that equation (4.21) characterizes left cointegrals. One can prove such a characterization by putting some arguments of [HN99, BC03, BC12] together. For reader’s convenience, we here give a self-contained proof of the fact that (4.21) characterizes left cointegrals on $H$ and give some more characterizations of cointegrals.

We first begin with the following technical lemma:

**Lemma 4.8.** If $\lambda \in H^*$ satisfies (4.21), then the equation

$$q_1^R h_{(1)} p^1 (\lambda, q_2^R h_{(2)} p_2^l) = \mu(\overline{\Phi}_1)(\lambda, \overline{S}(q_1^1) h S(\overline{\phi}_2 p_1^1)) q_2^l \overline{\phi}_3 p_2^l$$

(4.22)

holds for all $h \in H$. 

Theorem 4.10. Let $\lambda$ be a non-zero left cointegral on $H$. Then the linear maps $\Theta_L, \Theta_R : H \to H^*$, $\Theta_L(h) = h \to \lambda$, $\Theta_R(h) = \lambda \to h$ are bijective. Let $\Lambda$ be a left integral in $H$ such that $\langle \lambda, \Lambda \rangle = \mu(\beta)^{-1}$, and set $\overline{\Lambda} = q_1^R\Lambda(1)_1 p_1^R \otimes q_2^R\Lambda(2)_2 p_2^R$. (4.25)
Then the inverses of $\Theta_L$ and $\Theta_R$ are given respectively by

$$
\Theta_L^{-1}(\xi) = S(\tilde{\lambda}_1 \leftarrow \mu)(\xi, \tilde{\lambda}_2) \quad \text{and} \quad \Theta_R^{-1}(\xi) = \overline{S}(\tilde{\lambda}_1)(\xi, \tilde{\lambda}_2) \quad (\xi \in H^*).
$$

**Proof.** Since $H$ is a Frobenius algebra with Frobenius form $\lambda$, the maps $\Theta_L$ and $\Theta_R$ are bijective. Let $\nu$ be the Nakayama automorphism of $H$ with respect to $\lambda$. The defining formula (4.10) of $\nu$ implies the equation $\Theta_R = \Theta_L \circ \nu$. Hence,

$$
\Theta_L^{-1} = \nu \circ \Theta_R^{-1}.
$$

To prove the formula for $\Theta_R^{-1}$, we note:

$$
h\tilde{\lambda}_1 \otimes \tilde{\lambda}_2 = \tilde{\lambda}_1 \otimes \overline{S}(h)\tilde{\lambda}_2 \quad (h \in H).
$$

Indeed, for all $h \in H$, we have

$$
h\tilde{\lambda}_1 \otimes \tilde{\lambda}_2 = hq_1^R\Lambda(1)\phi_1^R \otimes q_2^R\Lambda(2)\phi_2^R.
$$

By Lemmas 4.8 and 4.9, equation (4.24) holds. Hence, $\Theta_R(h) = \overline{S}(\tilde{\lambda}_1)(\lambda, h\tilde{\lambda}_2) = \overline{S}(\tilde{\lambda}_1)\tilde{\lambda}_2 = \overline{S}(\tilde{\lambda}_1)(\lambda, h) = h$. This implies $\Theta_R^{-1} = \Theta$. The expression for $\Theta_L^{-1}$ is obtained by (4.26) and the explicit description of the Nakayama automorphism $\nu$ given in Theorem 4.6. \qed

Now we give the following characterizations of cointegrals:

**Theorem 4.11.** For $\lambda \in H^*$, the following are equivalent:

1. $\lambda$ is a left cointegral on $H$.
2. For all $h \in H$, the following equation holds:

$$
v_1^Lh(1)u_1^L(\lambda, v_2^Lh(2)u_2^L) = \mu(\phi_1)(\lambda, hS(\phi_2))(\overline{\lambda}, h).
$$

3. For all $h \in H$, the following equation holds:

$$
q_1^R\phi_1(\lambda, q_2^R\phi_2)(h) = \mu(\phi_1)(\lambda, \overline{S}(q_1^R\phi_1))q_2^R(\overline{\phi}_2).
$$

4. For all $h \in H$, the following equation holds:

$$
v_1^Lh(1)w_1^L(\lambda, v_2^Lh(2)w_2^L) = \langle \lambda, h \rangle 1,
$$

where $w_1^L$ is the element of $H^{\otimes 2}$ defined by

$$
w_1^L = \mu(\phi_1)\overline{S}(q_2^R\overline{\phi}_1) \otimes \overline{S}(q_1^R) \leftarrow \mu(\overline{\phi}_2).
$$

5. For all left integrals $\Lambda$ in $H$, the following equation holds:

$$
q_1^R\Lambda(1)\phi_1(\lambda, q_2^R\Lambda(2)\phi_2) = \mu(\beta)\Lambda(1).
$$

**Proof.** We prove this theorem in the following way:

(1) $\implies$ (2) $\implies$ (3) $\implies$ (5) $\implies$ (1), \quad (2) $\iff$ (4),
We fix a non-zero cointegral to prove (5) ⇒ (1). Suppose that λ ∈ H∗ satisfies (4.24) for any left integral Λ in H. Thus we have Lemmas 4.7, 4.8 and 4.9 prove (1) and define ΘR : H → H∗ by ΘR(h) = λ0 − h for h ∈ H. Then, by Theorem 4.11, ΘR(λ) is a scalar multiple of the unit 1 ∈ H. Thus we have λ ∈ ΘR(k1) = kλ0 = fλ, that is, λ is a left cointegral on H.

(2) ⇔ (4). To prove this, we require the following relations:

\[ w^l = \mu(\phi_1)S(\phi_2)_1u^l_1\phi_3 \otimes S(\phi_2)_2u^l_2, \]

\[ u^l = \mu(\bar{\phi}_1)S(\bar{\phi}_2)_1w^l_1\phi_3 \otimes S(\bar{\phi}_2)_2w^l_2. \]

The first equation is proved as follows:

\[ \mu(\phi_1)S(\phi_2)_1u^l_1\phi_3 \otimes S(\phi_2)_2u^l_2 = \mu(\phi_1)S(\phi_2)_1u^l_1\phi_3 \otimes S(\phi_2)_2F_2S(q_1^R) \]

\[ = \mu(\phi_1)F_1S(\phi_2)_1S(q_1^R)\phi_3 \otimes F_2S(\phi_2)_2S(q_1^R) \]

\[ = \mu(\phi_1)F_1S(\phi_3)q_1^R\phi_2(2) \otimes F_2S(q_1^R\phi_2(1)) \]

\[ = \mu(q_1^R(\bar{\phi}_1))F_1S(q_1^R\phi_3) \otimes F_2S(q_1^R\phi_2(2)) = w^l. \]

The second one easily follows from the first one. Now we suppose that (2) holds. Then we have

\[ \langle \lambda, h \rangle 1 = \mu(\phi_1) : \mu(\bar{\phi}_1)(\lambda, hS(\phi_2)_1S(\phi_2)_2) : \phi_3 \]

\[ = \mu(\phi_1)\phi_2^l(\lambda, hS(\phi_2)_1u^l_1(\lambda, \phi_2^l(\lambda, hS(\phi_2)_2)u^l_2))\phi_3 \]

\[ = \mu(\phi_1)\phi_2^l(\lambda, \phi_2^l(\lambda, hS(\phi_2)_2u^l_2)) \]

\[ = \phi_1^l(\lambda, hS(\phi_2)_2u^l_2). \]

for all h ∈ H. Thus (4) holds. If, conversely, (4) holds, then we prove that (2) holds as follows:

\[ \mu(\bar{\phi}_1)(\lambda, hS(\phi_2)_2) \bar{\phi}_3 \]

\[ = \phi_1^l(\lambda, hS(\phi_2)_2u^l_2) \phi_3 \]

\[ = \phi_1^l(\lambda, \phi_2^l(\lambda, hS(\phi_2)_2u^l_2)) \phi_3 \]

\[ = \phi_1^l(\lambda, hS(\phi_2)_2u^l_2). \]

We note that a right cointegral on H is just a left cointegral on the quasi-Hopf algebra Hcop. By rephrasing the above theorem for Hcop by using (3.10), (3.24), (3.32) and (3.50), we obtain the following theorem:

**Theorem 4.12.** For λ ∈ H∗, the following are equivalent:

1. λ is a right cointegral on H.
2. For all h ∈ H, the following equation holds:

\[ \langle \lambda, \phi_1^R(\lambda, hS(\phi_2)_2) \rangle = \phi_1(\lambda, hS(\phi_2)_2) \mu(\phi_3). \]

3. For all h ∈ H, the following equation holds:

\[ \langle \lambda, \phi_1^R(\lambda, hS(\phi_2)_2) \rangle = q_1^R \phi_1^R(\lambda, \phi_2^R(\lambda, hS(\phi_2)_2)) \mu(\phi_3). \]

4. For all h ∈ H, the following equation holds:

\[ \phi_1^R(\lambda, hS(\phi_2)_2w^R_2) = (\lambda, h) 1, \]
where \( w^R \) is the element of \( H \otimes 2 \) defined by
\[
 w^R = \mu(\phi_1) \overline{S}(\mu \to q_1^L) \phi_2 \overline{S}(q_1^L \phi_1 \overline{T})
\]  
(4.35)

(5) For all left integrals \( \Lambda \) in \( H \), the following equation holds:
\[
(\lambda, q_1^L \Lambda(1)) q_2^L \Lambda(2) = \mu(\overline{S}(\beta)) (\lambda, \Lambda).
\]  
(4.36)

5. Yetter-Drinfel’d category

5.1. Yetter-Drinfel’d modules of the first kind. Let \( H \) be a quasi-Hopf algebra with bijective antipode. In this section, we give some applications of the integral theory for quasi-Hopf algebras to Yetter-Drinfel’d modules over \( H \). As in the case of ordinary Hopf algebras, a Yetter-Drinfel’d module over \( H \) is defined to be a left \( H \)-module \( V \) equipped with a linear map \( \delta : V \to H \otimes V \), denoted by \( \delta(v) = v_{(-1)} \otimes v_{(0)} \), such that the family
\[
\{ V \otimes M \to M \otimes V, v \otimes m \mapsto v_{(-1)} m \otimes v_{(0)} \}_{M \in H \cdot \mathcal{M}}
\]
of maps makes \( V \) into an object of the Drinfeld center of \( H \cdot \mathcal{M} \). An explicit definition in this spirit is found, e.g., in Majid [Maj98] and Bulacu-Caenepeel-Pannier [BCP05] [BCP06]. On the other hand, Schauenburg [Sch02] introduced another kind of Yetter-Drinfel’d module over \( H \) from the viewpoint of the fundamental theorem for quasi-Hopf bimodules. Although these two kinds of Yetter-Drinfel’d modules are equivalent notions, each of them has advantages and disadvantages. In this paper, we use both of them. We first recall the definition of the first kind:

Definition 5.1 [Maj98] [BCP05] [BCP06]. A Yetter-Drinfel’d module of the first kind is a left \( H \)-module \( V \) endowed with a linear map
\[
\delta^{1st}_V : V \to H \otimes V,
\]
such that the following equations hold for all \( h \in H \) and \( v \in V \).
\[
\phi_1 v_{(-1)} \otimes (\phi_2 v_{(0)})_{(-1)} \phi_3 \otimes (\phi_2 v_{(0)})_{(0)} = \phi_1' (\phi_1 v)_{(-1)(1)} \phi_2 \otimes \phi_2' (\phi_1 v)_{(-1)(2)} \phi_3 \otimes \phi_3' (\phi_1 v)_{(0)},
\]
(5.1)
\[
e(v_{(-1)}) v_{(0)} = v,
\]
(5.2)
\[
h_{(1)} v_{(-1)} \otimes h_{(2)} v_{(0)} = (h_{(1)} v)_{(-1)} h_{(2)} \otimes (h_{(1)} v)_{(0)},
\]
(5.3)
where \( \phi' \) is a copy of \( \phi \). The map \( \delta^{1st}_V \) is called the coaction of the first kind. We denote by \( \mathcal{H} \mathcal{D} \) the category of Yetter-Drinfel’d modules of the first kind and \( k \)-linear maps that preserves the action and the coaction of \( H \).

The trivial \( H \)-module \( 1 = k \) is a Yetter-Drinfel’d module of the first kind by the coaction determined by \( \delta^{1st}_1(1) = 1 \otimes 1 \). If \( V \) and \( W \) are Yetter-Drinfel’d modules of the first kind, then their tensor product \( H \)-module \( V \otimes W \) is a Yetter-Drinfel’d module of the first kind by the coaction given by
\[
\delta^{1st}_{V \otimes W}(v \otimes w) = \phi_1 (\overline{\phi}_1 (v)_{(-1)} \overline{\phi}_2 (v)_{(-1)} \phi_3 \otimes \phi_3' (w)_{(0)} \otimes \phi_3' (w)_{(0)}
\]
(5.4)
for \( v \in V \) and \( w \in W \) (see [Maj98] Proposition 2.2). The category \( \mathcal{H} \mathcal{D} \) is a monoidal category with this tensor product.

Given \( V \in \mathcal{H} \mathcal{D} \) and \( X \in \mathcal{H} \mathcal{M} \), we define \( \sigma_{V \otimes X} : V \otimes X \to X \otimes V \) by \( \sigma_{V \otimes X}(v \otimes x) = v_{(-1)} x \otimes v_{(0)} \) for \( v \in V \) and \( x \in X \). The family \( \sigma_V = \{ \sigma_{V \otimes X} \}_{X} \) is natural in \( X \in \mathcal{H} \mathcal{M} \) and the pair \((V, \sigma_V)\) is in fact an object of the Drinfeld center \( \mathcal{Z}(H \cdot \mathcal{M}) \) of
Moreover, the assignment \((V, \delta^1_V) \mapsto (V, \sigma_V)\) is an isomorphism of monoidal categories from \(\mathcal{H}_H\) to \(\mathcal{Z}(\mathcal{H})\).

### 5.2. Yetter-Drinfeld modules of the second kind.

The category \(\mathcal{H}_H\) is naturally a left \(\mathcal{H}_H\)-module category. Hence \(\mathcal{H}_H\) is also a left \(\mathcal{H}_H\)-module category in such a way that the category equivalence \(\Psi\) is \(\mathcal{H}_H\)-equivariant.

Schauenburg \([\text{Sch02}]\) gave an explicit description of such an action of \(\mathcal{H}_H\) on \(\mathcal{H}_H\). Given \(M \in \mathcal{H}_H\), we denote by \(\text{ad} M\) the vector space \(M\) equipped with the left \(H\)-module structure given by

\[ h \triangleright m = h(1) m S(h(2)) \quad (h \in H, m \in M). \]

Then the action \(\otimes : \mathcal{H}_H \times \mathcal{H}_H \to \mathcal{H}_H\) is defined by

\[ M \otimes V = \text{ad}(M \otimes V) \quad (M \in \mathcal{H}_H, V \in \mathcal{H}_H). \]

To describe the associator \(\Omega\) of \(\mathcal{H}_H\), we introduce the following element:

\[ \omega = (1 \otimes 1 \otimes 1 \otimes \hat{T}_1 \otimes \hat{T}_2) \cdot (\text{id} \otimes \Delta \otimes S \otimes S)(\chi) \cdot (\phi \otimes 1 \otimes 1) \in H^{\otimes 5}, \tag{5.5} \]

where \(\chi\) is the element of \(H^{\otimes 4}\) given by

\[ \chi = (\text{id} \otimes \Delta \otimes \text{id})(\phi^{-1}) \cdot (1 \otimes \phi^{-1}) \cdot (\text{id} \otimes \text{id} \otimes \Delta)(\phi) \]

\[ \omega_3 \] given by

\[ \omega = (\phi \otimes 1) \cdot (\Delta \otimes \text{id} \otimes \text{id})(\phi^{-1}). \tag{5.6} \]

The associator \(\Omega\) is then given by

\[ \Omega_{M,N,V}(m \otimes n \otimes v) = \omega_1 m \omega_5 \otimes \omega_2 n \omega_4 \otimes \omega_3 v \quad (m \in M, n \in N, v \in V). \]

**Definition 5.2** (Schauenburg \([\text{Sch02}]\)). We define the category \(\mathcal{H}_H\) to be the category of left \(H\)-comodules in the left \(\mathcal{H}_H\)-module category \(\mathcal{H}_H\). An object of \(\mathcal{H}_H\) is referred to as a *Yetter-Drinfeld module of the second kind*. Stated differently, a Yetter-Drinfeld module of the second kind is a left \(H\)-module \(V\) endowed with a linear map \(\delta^2_V : V \to H \otimes V\), expressed as \(v \mapsto v_{[-1]} \otimes v_{[0]}\), such that the equations

\[ v_{[-1]} \otimes v_{[0]} \otimes v_{[0]}[0] = \omega_1(v_{[-1]})_{(1)} \omega_5 \otimes \omega_2(v_{[-1]})_{(2)} \omega_4 \otimes \omega_3 v_{[0]}, \tag{5.7} \]

\[ \varepsilon(v_{[-1]})v_{[0]} = v, \tag{5.8} \]

\[ \delta^2_V(hv) = h(1,1)v_{[-1]}S(h(2)) \otimes h(1,2)v \tag{5.9} \]

hold for all \(h \in H\) and \(v \in V\).

Schauenburg \([\text{Sch02}]\) showed that there is an isomorphism \(\Psi : \mathcal{H}_H \to \mathcal{H}_H\) of categories. The isomorphism \(\Psi\) is the identity on morphisms and keeps the underlying \(H\)-module unchanged. Given \(V \in \mathcal{H}_H\), the isomorphism \(\Psi\) replaces the coaction of \(V\) of the first kind with

\[ \delta^2_V(v) = (p_1^R v)(-1)p_2^R \otimes (p_1^R v)(0) \quad (v \in V). \tag{5.10} \]

The inverse of the isomorphism \(\Psi\) replaces the second kind coaction of \(V \in \mathcal{H}_H\) with

\[ \delta^1_V(v) = q_1^R(v_{[-1]}S(q_1^R) \otimes q_1^R v_{[0]} \quad (v \in V). \tag{5.11} \]

From now on, we identify \(\mathcal{H}_H\) with \(\mathcal{H}_H\), and denote both of them by \(\mathcal{H}_H\).

An object of \(\mathcal{H}_H\) is thought of as a left \(H\)-module \(V\) equipped with two kinds of coaction \(\delta^1_V\) and \(\delta^2_V\) related to each other by equations \(\text{(5.10)}\) and \(\text{(5.11)}\).
5.3. Induction to the Yetter-Drinfeld category, I. Given \( V \in \mathcal{M}_H \), we define \( R(V) \in \mathcal{M}_H \) by \( R(V) = H \otimes V \). The left \( H \)-module \( R(V) \) is a Yetter-Drinfeld module of the second kind by the free left \( H \)-coaction given by

\[
R(V) = H \otimes V \xrightarrow{\Delta \otimes id_V} (H \hat{\otimes} H) \otimes V \xrightarrow{\Omega_{H,H,V}} H \otimes (H \otimes V) = H \otimes R(V).
\]

Specifically, the action of \( H \) and the second kind coaction of \( H \) on \( R(V) \) are given by

\[
h \triangleright (a \otimes v) = h_{(1,1)}aS(h_{(2)}) \otimes h_{(1,2)}v, \quad \delta^{2nd}_{R(V)}(a \otimes v) = \omega_1 a_{(1)} \omega_5 \otimes \omega_2 a_{(2)} \omega_4 \otimes \omega_3 v \tag{5.12}
\]

for \( a, h \in H \) and \( v \in V \). By (5.11), the first kind coaction is given by

\[
\delta^{1st}_{R(V)}(a \otimes v) = q^R_{1(1)} \omega_1 a_{(1)} \omega_5 S(q^R_{2(2)}) \otimes q^R_{1(2)} \triangleright (\omega_2 a_{(2)} \omega_4 \otimes \omega_3 v) \tag{5.13}
\]

Now let \( F : \mathcal{H}_H \mathcal{D} \rightarrow \mathcal{M}_H \) be the forgetful functor. We recall that \( \mathcal{H}_H \mathcal{D} = \mathcal{H}_H \mathcal{D}_2 \) is the category of left \( H \)-comodules in \( \mathcal{M}_H \). Hence the free \( H \)-comodule functor is right adjoint to \( F \). Namely, we have:

**Theorem 5.3.** The assignment \( V \mapsto R(V) \) extends to a functor from \( \mathcal{M}_H \) to \( \mathcal{H}_H \mathcal{D} \). This functor is right adjoint to \( F \) with the unit \( \eta \) and the counit \( \varepsilon \) given by

\[
\eta_M : M \rightarrow RF(M), \quad m \mapsto m_{[-1]} \otimes m_{[0]} \quad (m \in M \in \mathcal{H}_H \mathcal{D}), \tag{5.15}
\]

\[
\varepsilon_V : FR(V) 
\rightarrow \mathcal{M}_H, \quad a \otimes v \mapsto \varepsilon(a)v \quad (a \in H, v \in \mathcal{H}_H \mathcal{D}). \tag{5.16}
\]

As we have recalled in Subsection 2.1, the functor \( R \) is a monoidal functor as a right adjoint of the strict monoidal functor \( F \). The structure morphisms

\[
R^{(0)} : \mathbb{1} \rightarrow R(\mathbb{1}) \quad \text{and} \quad R^{(2)}_{X,Y} : R(X) \otimes R(Y) \rightarrow R(X \otimes Y) \quad (X, Y \in \mathcal{M}_H)
\]

are given as follows:

**Lemma 5.4.** The morphism \( R^{(0)} \) is determined by

\[
R^{(0)}(1) = \beta \otimes 1. \tag{5.17}
\]

The natural transformation \( R^{(2)} \) is given by

\[
R^{(2)}_{X,Y}((a \otimes x) \otimes (b \otimes y)) = \phi_1 \overline{\Phi}_{1(1)} q^R_{1(1)} \overline{\Phi}_{1(1,1)} aS(q^R_{2(2)}) \overline{\Phi}_{2} \phi'_1 bS(\overline{\Phi}_{3} \overline{\Phi}_{3(2)} \phi'_3) \otimes \phi_2 \overline{\Phi}_{1(2)} q^R_{1(2)} \overline{\Phi}_{1(1,2)} x \otimes \phi_3 \overline{\Phi}_{2} \overline{\Phi}_{3(1)} \phi'_2 y \tag{5.18}
\]

for \( X, Y \in \mathcal{M}_H \), \( a, b \in H \), \( x \in X \) and \( y \in Y \), where \( \phi' \) is a copy of \( \phi \).

**Proof.** We note that the unit \( \eta \) is given by the coaction of the second kind. Since the first kind coaction of the trivial Yetter-Drinfeld module \( \mathbb{1} = k \) is determined by \( \delta^{1st}_1(1) = 1 \otimes 1 \), we have

\[
R^{(0)}(1) = \delta^{2nd}_1(1) = \beta \otimes 1.
\]
Thus (5.17) is proved. We verify (5.18). For simplicity, we write \( m = m_H \otimes m_V \in H \otimes V \) for an element \( m \in R(V) \). The following equations hold:

\[
m_{(-1)} \otimes \varepsilon_V(h \triangleright m_{(0)}) = q_{(1)}^R m_H S(q_{(2)}^R) \otimes h q_{(2)}^R m_V, \\
q_{(1)}^R (h_{(1)} \triangleright m) H S(q_{(2)}^R) h_{(2)} \otimes q_{(2)}^R (h_{(1)} \triangleright m)V = h_{(1)} q_{(1)}^R m_H S(q_{(2)}^R) \otimes h_{(2)} q_{(2)}^R m_V, \\
q_{(1)}^R (p_1^R \triangleright m) H S(q_{(2)}^R) p_2^R \otimes q_{(2)}^R (p_1^R \triangleright m)V = m
\]

for \( h \in H \) and \( m \in R(V) \). Indeed, the first one is proved as follows:

\[
m_{(-1)} \otimes \varepsilon_V(h \triangleright m_{(0)}) = q_{(1)}^R \omega_1 m_H(1) \omega_5 S(q_{(2)}^R) \otimes \varepsilon_V(\omega_1 m_H(2) \omega_4 \otimes \omega_3 m_V)
\]

The second one is proved as follows:

\[
q_{(1)}^R (h_{(1)} \triangleright m) H S(q_{(2)}^R) h_{(2)} \otimes q_{(2)}^R (h_{(1)} \triangleright m)V = h_{(1)} m_{(-1)} \otimes \varepsilon_V(h_{(2)} \triangleright m_{(0)})
\]

The third one is proved as follows:

\[
q_{(1)}^R (p_1^R \triangleright m) H S(q_{(2)}^R) p_2^R \otimes q_{(2)}^R (p_1^R \triangleright m)V = m_H \otimes m_V = m
\]

For \( v \in R(X) \) and \( w \in R(Y) \), we have

\[
\delta_{R(X) \otimes R(Y)}^{\text{2nd}}(v \otimes w) = \left(p_1^R \cdot (v \otimes w) \right)(-1) p_2^R \otimes \left(p_1^R \cdot (v \otimes w) \right)(0)
\]

\[
= \phi_1(\overline{\phi_1^R}^R \cdot v)(-1) \overline{\phi_2^R}(\overline{\phi_2^R}^R \cdot w)(-1) \phi_3^R
\]

\[
= \phi_1(\overline{\phi_1^R}^R \cdot v)(-1) \overline{\phi_2^R}(\overline{\phi_2^R}^R \cdot w)(-1) \phi_3^R
\]

where \( t \in H^{\otimes 3} \) is given by

\[
t := \phi_1(\overline{\phi_1^R}^R) \otimes \phi_2^R \otimes \phi_3^R \overline{T_1 \otimes \overline{T_2} \otimes \overline{T_3}} R(X) R(Y) S(T_3).
\]

Set \( q = q^R \). We verify (5.18) as follows:

\[
R_{X,Y}^{(2)}(v \otimes w) = R(\varepsilon_X \otimes \varepsilon_Y) \delta_{R(X) \otimes R(Y)}^{\text{2nd}}(v \otimes w)
\]

\[
= \phi_1(\overline{\phi_1^R}^R \cdot v)(-1) \overline{\phi_2^R}(\overline{\phi_2^R}^R \cdot w)(-1) \phi_3^R
\]
say that the monoidal adjunction $\mathcal{U} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathcal{V}$ is co-Hopf if and only if the comonoidal adjunction $\mathcal{V} \rightleftarrows \mathcal{C} : \mathcal{W}$.

5.4. Hopf-type property of the adjunction. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, and let $U : \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor admitting a right adjoint $T : \mathcal{D} \to \mathcal{C}$. The left Hopf operator $\mathbb{H}^{(l)}$ and the right Hopf operator $\mathbb{H}^{(r)}$ for the monoidal adjunction $U \dashv T$ are the natural transformations defined by

$$
\mathbb{H}^{(l)}_{X,M} = T_{X,U(M)}^{(2)} \circ (\text{id}_{T(X)} \otimes i_M) \quad \text{and} \quad \mathbb{H}^{(r)}_{X,M} = T_{U(M),X}^{(2)} \circ (i_M \otimes \text{id}_{T(X)}),
$$

respectively, for $X \in \mathcal{D}$ and $M \in \mathcal{C}$, where $i : \text{id}_{\mathcal{C}} \to TU$ is the unit of $U \dashv T$. We say that the monoidal adjunction $U \dashv T$ is co-Hopf if $\mathbb{H}^{(l)}$ and $\mathbb{H}^{(r)}$ are invertible. We note that $U \dashv T$ is co-Hopf if and only if the comonoidal adjunction $U^{\text{op}} \dashv T^{\text{op}}$ is a Hopf adjunction in the sense of [BLV11]. Thus any results on a Hopf adjunction can be translated into a result on a co-Hopf adjunction.

A monoidal adjunction enjoys several favorable properties when it is co-Hopf. Thus it is important to know whether a given monoidal adjunction is co-Hopf. We shall consider the monoidal adjunction $F \dashv R : H\mathcal{M} \rightleftarrows H\mathcal{M}^H$ given in Theorem 5.3.

Theorem 5.5. The monoidal adjunction $F \dashv R$ is co-Hopf.

Proof. Let $H_H^{\mathcal{M}}$ be the category of $H$-bicomodules in $H\mathcal{M}^H$, and let $U : H_H^{\mathcal{M}} \to H\mathcal{M}_H^H$ be the forgetful functor. Since $H_H^{\mathcal{M}}$ is identified with the category of left $H$-comodules in the left $H\mathcal{M}$-module category $H\mathcal{M}_H^H$, a right adjoint of $U$ is given by the free left $H$-comodule functor

$$
T : H\mathcal{M}_H^H \to H_H^{\mathcal{M}}, \quad M \mapsto H \otimes M.
$$
If $M, N \in \mathcal{H}_{\mathcal{H}}^H$, then their tensor product $M \otimes_H N$ over $H$ is also an object of $\mathcal{H}_{\mathcal{H}}^H$ by the usual $H$-bimodule structure and the coaction given by

$$\delta_{M \otimes_H N} (m \otimes_H n) = (m_{(0)} \otimes_H n_{(0)}) \hat{\otimes} m_{(1)} n_{(1)}$$

for $m \in M$ and $n \in N$. The category $\mathcal{H}_{\mathcal{H}}^H$ is a monoidal category with respect to this tensor product and the equivalence (4.2) is in fact an equivalence of monoidal categories [HN99]. The category $\mathcal{H}_{\mathcal{H}}^H$ is also a monoidal category in such a way that the forgetful functor $U$ is strict monoidal. In summary, we have the following commutative diagram of monoidal functors:

Thus the co-Hopfness of $F \dashv R$ is equivalent to that of $U \dashv T$. Since our expression of the monoidal structure of $R$ is quite complicated, we shall show that $U \dashv T$ is co-Hopf. We first note that the unit $\nu$ and the counit $\varphi$ of $T \dashv U$ are given as follows:

$$\nu_M : M \rightarrow TU(M), \quad m \mapsto m_{(-1)} \hat{\otimes} m_{(0)} \quad (m \in M \in \mathcal{H}_{\mathcal{H}}^H),$$

$$\varphi_X : UT(X) \rightarrow X, \quad h \hat{\otimes} x \mapsto e(h)x \quad (x \in X \in \mathcal{H}_{\mathcal{H}}^H, h \in H).$$

The monoidal structure $T^{(2)}$ and Hopf operators are given as follows:

$$T^{(2)}_{X,Y} : T(X) \otimes_H T(Y) \rightarrow T(X \otimes_H Y),$$

$$h \otimes x \otimes_{H} (h' \otimes y) \mapsto hh' \otimes x \otimes_{H} y,$$

$$\mathbb{H}^{(l)}_{X,M} : (H \otimes X) \otimes_{H} M \rightarrow H \otimes (X \otimes_{H} U(M)),

(h \otimes x) \otimes_{H} m \mapsto hm_{(-1)} \hat{\otimes} (x \otimes_{H} m_{(0)}),$$

$$\mathbb{H}^{(r)}_{M,X} : M \otimes_{H} (H \otimes X) \rightarrow H \otimes (U(M) \otimes_{H} X),$$

$$m \otimes_{H} (h \otimes x) \mapsto m_{(-1)}h \otimes (m_{(0)} \otimes_{H} x)$$

for $h, h' \in H, X, Y \in \mathcal{H}_{\mathcal{H}}^H; M \in \mathcal{H}_{\mathcal{H}}^H, x \in X, y \in Y$ and $m \in M$. To give the inverse of the left Hopf operator, we first define the linear map

$$\mathbb{G}^{(l)}_{X,M} : H \otimes X \otimes M \rightarrow (H \otimes X) \otimes_{H} M,$$

$$h \otimes x \otimes m \mapsto (hS(m_{(-1)}p_{1}^{l})q_{1}^{l} \hat{\otimes} xq_{2}^{l}) \otimes_{H} m_{(0)}p_{2}^{l}.$$ 

For $h, h' \in H, x \in X$ and $m \in M$, we compute

$$\mathbb{G}^{(l)}_{X,M}(h \otimes x \otimes h'm) = (hS(h'_{(1)}m_{(-1)}p_{1}^{l})q_{1}^{l} \hat{\otimes} xq_{2}^{l}) \otimes_{H} h'_{(2)}m_{(0)}p_{2}^{l}$$

$$= (hS(m_{(-1)}p_{1}^{l})S(h'_{(1)})q_{1}^{l}h'_{(1)} \hat{\otimes} xq_{2}^{l}h'_{(2)}q_{2}^{l}) \otimes_{H} m_{(0)}p_{2}^{l}$$

$$= (hS(m_{(-1)}p_{1}^{l})q_{1}^{l} \otimes xh'q_{2}^{l}) \otimes_{H} m_{(0)}p_{2}^{l}$$

$$= \mathbb{G}^{(l)}_{X,M}(h \otimes xh' \otimes m).$$

Hence the following linear map is well-defined:

$$\mathbb{G}^{(l)}_{X,M} : H \otimes (X \otimes_{H} M) \rightarrow (H \otimes X) \otimes_{H} M,$$
are given respectively by

\[ A \text{ the adjunction given in Theorem 5.3. The Yetter-Drinfeld module is well-defined and is the inverse of the second kind, the multiplication} \ \\
\] holds for all \( m \in M \). By using this equation instead of (3.11), one can verify that the equations

\[ m_{(-1)} \otimes m_{(0,-1)} \otimes m_{(0,0)} = \Phi_1 m_{(-1,1)} \Phi_1 \otimes \Phi_2 m_{(-1,2)} \Phi_2 \otimes \Phi_3 m_{(0)} \Phi_3 \]

(5.23)

\[ S(m_{(-1)})q_1^1 m_{(0)} \otimes m_{(0,0)} q_2^2(2) = q_1^1 \otimes m q_2^2 \]

(5.24)

hold for all \( m \in M \) in a similar way as (5.17) and (3.18). Now we prove that \( \mathcal{G}^{(t)} \) is the inverse of \( \mathcal{H}^{(t)} \) as follows: For \( h \in H, x \in X \) and \( m \in M \),

\[ \mathcal{G}^{(t)} \mathcal{H}^{(t)}((h \hat{\otimes} x) \otimes_H m) \]

\[ = (hm_{(-1)}S(m_{(0,-1)})q_1^1 \otimes xq_2^2) \otimes_H m_{(0,0)}q_2^2 \]

(5.23)

\[ = (hS(m_{(-1)})q_1^1 m_{(0)} \otimes xq_2^2) \otimes_H m_{(0,0)}q_2^2 \]

(3.22)

\[ \mathcal{H}^{(t)} \mathcal{G}^{(t)}(h \hat{\otimes} (x \otimes_H m)) \]

\[ = hS(m_{(-1)})q_1^1 m_{(0)} \otimes xq_2^2 \otimes_H m_{(0,0)}q_2^2 \]

(5.24)

\[ = hS(m_{(-1)})q_1^1 m_{(0)} \otimes xq_2^2 \otimes_H m_{(0,0)}q_2^2 \]

(3.22)

In a similar manner, one can verify that the linear map

\[ \mathcal{G}^{(t)}_{M,X} : H \hat{\otimes} (U(M) \otimes_H X) \rightarrow M \otimes_H (H \hat{\otimes} X), \]

\[ h \hat{\otimes} (m \otimes_H x) \mapsto q_2^2 m_{(0)} \otimes_H (p_1^1 S(q_1^1 m_{(-1)})h \hat{\otimes} p_2^2 x) \]

is well-defined and is the inverse of \( \mathcal{H}^{(t)}_{M,X} \). The proof is done. \( \square \)

5.5. The adjoint algebra and class functions. Let \( F \dashv R : H \mathcal{M} \rightarrow H \mathcal{M}^{\mathcal{A}} \) be the adjunction given in Theorem 5.3. The Yetter-Drinfeld module \( \mathbf{A} := R(1) \) is an algebra in \( H \mathcal{M}^{\mathcal{A}} \) as the image of the trivial algebra \( 1 \) under a monoidal functor. We identify \( \mathbf{A} = H \) as a vector space. Then the structure of \( \mathbf{A} \) is given as follows:

**Theorem 5.6.** The action \( \triangleright \), the coaction \( \delta^{1st}_\mathbf{A} \) of the first kind, the coaction \( \delta^{2nd}_\mathbf{A} \) of the second kind, the multiplication \( * \), and the unit \( 1_\mathbf{A} \) of the algebra \( \mathbf{A} \in H \mathcal{M}^{\mathcal{A}} \) are given respectively by

\[ h \triangleright a = h_{(1)} a S(h_{(2)}), \]

(5.25)

\[ \delta^{1st}_\mathbf{A}(a) = \Phi_1 \Phi_1' a_{(1)} (1) \Phi_3 S(q_1^R \Phi_2(2)) \Phi_3' \otimes \Phi_2 \Phi_2(1) a_{(2)} (1) \Phi_3 S(q_1^R \Phi_2(1)), \]

(5.26)

\[ \delta^{2nd}_\mathbf{A}(a) = \Phi_1 \Phi_1' a_{(1)} (1) \Phi_3 S(\Phi_3) \otimes \Phi_2 \Phi_2(1) a_{(2)} (1) \Phi_3 S(\Phi_3), \]

(5.27)

\[ a \star b = \Phi_1 a S(\Phi_1 \Phi_2) a \Phi_2 b S(\Phi_3), \]

(5.28)

\[ 1_\mathbf{A} = \beta \]

(5.29)
for \( a, b \in \mathcal{A} \) and \( h \in H \), where \( \phi' = \phi \).

Thus our algebra \( \mathcal{A} \in \mathcal{H} \Phi \otimes \) is identical to the algebra \( H_0 \) of \[ \text{[BCP05]}. \]

**Proof.** Equation \[ (5.20) \] is obvious from the definition of the functor \( R \). It also follows from the definition of \( R \) that the second kind coaction of \( \mathcal{A} \) is given by

\[
\delta^{\text{2nd}}(a) = \omega_1 a_{(1)} \omega_5 \otimes \omega_2 a_{(2)} \omega_4 \epsilon(\omega_3)
\]

\[
\text{and } = \chi_1 a_{(1)} \bar{T}_1 S(\chi_4) \otimes \chi_2 a_{(2)} \bar{T}_2 S(\chi_3)
\]

\[
\text{and } = \phi_1 \bar{T}_1 a_{(1)} \bar{T}_3 S(\bar{T}_4) \otimes \phi_2 \bar{T}_2(2) \bar{T}_3 S(\phi_3 \bar{T}_2)
\]

for \( a \in \mathcal{A} \). Thus \[ (5.27) \] is proved. We verify \[ (5.20) \] as follows: For \( a \in \mathcal{A} \),

\[
\delta^{\text{1st}}(a) = \bigstar a_{(1)} a_{(2)} \bigstar
\]

\[
q_{1,1}^R \phi_1 \bar{T}_1 a_{(1)} \bar{T}_1 S(\phi_2) S(q_{1,2}^R) \otimes q_{1,2}^R(2) \bigstar (\phi_2 \bar{T}_2(2) a_{(2)} \bar{T}_2 S(\phi_3 \bar{T}_2))
\]

\[
q_{1,1}^R \phi_1 \bar{T}_1 a_{(1)} \bar{T}_1 S(q_{1,2}^R \phi_3) \otimes \phi_2 \bar{T}_2(2) a_{(2)} \bar{T}_2 S(\phi_3 q_{1,2}^R \phi_2)
\]

\[
\phi_1 \bar{T}_1 a_{(1)} \bar{T}_1 S(q_{1,2}^R \phi_3) \otimes \phi_2 q_{1,1}^R \phi_3 \bar{T}_2 S(\phi_3 q_{1,2}^R \phi_2)
\]

\[
\phi_1 \bigstar a_{(1)} \bigstar S(\phi_3 q_{1,2}^R \phi_2)
\]

\[
= \text{(the right-hand side of } (5.20)).
\]

Equation \[ (5.28) \] is proved as follows: For \( a, b \in \mathcal{A} \),

\[
a \ast b = R_{(2)}^0(a \otimes b)
\]

\[
= \phi_1 \bar{T}_1(1) q_{1,1}^R \bar{T}_1(1,a) S(q_{1,2}^R \bar{T}_2) \bar{T}_2 \phi_2 \bar{T}_2(2) S(\phi_3 \bar{T}_3 \phi_2)
\]

\[
\epsilon(\phi_2 \bar{T}_2(2) q_{1,2}^R \phi_3 \bar{T}_3 \phi_2)
\]

\[
= q_{1,1}^R \phi_1 a S(q_{1,2}^R \phi_2) \bar{T}_2 b S(\phi_3)
\]

\[
= \phi_1 \bar{T}_1(1) a S(\phi_2 \bar{T}_2) a \phi_3 \bar{T}_3 b S(\phi_3)
\]

\[
= \phi_1 \bar{T}_1(1) a S(\phi_2 \bar{T}_2) a \phi_3 \bar{T}_3 b S(\phi_3)
\]

\[
= \phi_1 \bar{T}_1(1) a S(\phi_2 \bar{T}_2) a \phi_3 \bar{T}_3 b S(\phi_3)
\]

\[
= \text{(the right-hand side of } (5.28)).
\]

The unit of \( \mathcal{A} \) is \( R^{(0)} : 1 \rightarrow R(1) \). Thus \[ (5.24) \] follows from \[ (5.17) \].

Theorem \[ 5.6 \] gives a category-theoretical origin of the algebra \( H_0 \) of \[ \text{[BCP05]}. \]

Since \[ (5.25) \] is usually called the adjoint action of \( H \), we call \( \mathcal{A} \) the *adjoint algebra*. We now demonstrate that some properties of the adjoint algebra are derived from the general theory of monoidal categories. Let \( \mathcal{C} \) be a monoidal category such that the forgetful functor \( Z(\mathcal{C}) \rightarrow \mathcal{C} \) admits a right adjoint, say \( I : \mathcal{C} \rightarrow Z(\mathcal{C}) \). It is known that the object \( I(\mathbb{1}) \) is a commutative algebra in \( Z(\mathcal{C}) \) (a proof for fusion categories is found in \[ \text{DMNO13, Lemma 3.5} \], but the same proof can be applied for the general case). Applying this result to the monoidal category \( \mathcal{C} =_{H \Phi} \mathcal{M} \), we obtain:

**Corollary 5.7.** The algebra \( \mathcal{A} \) is commutative.
The algebra $A$ acts on an object of the form $R(V)$, $V \in H\mathcal{M}$, from the right by $R_{V,1}^{(2)} : R(V) \otimes A \to R(V)$. We denote by $R(V)_A$ the right $A$-module obtained in this way. This construction gives rise to a functor

$$K : H\mathcal{M} \to H^H\mathcal{P}_A, \quad V \mapsto R(V)_A$$

from $H\mathcal{M}$ to the category $H^H\mathcal{P}_A$ of right $A$-modules in $H^H\mathcal{P}$.

**Corollary 5.8.** The functor $K$ is an equivalence.

**Proof.** We have proved that $F \dashv R$ is a co-Hopf adjunction. Hence $R^{op} \dashv F^{op}$ is Hopf. The claim is proved just by applying the fundamental theorem for Hopf modules \cite{BLV11} to the Hopf monad associated to this Hopf adjunction.

The vector space $\text{CF}(H) := \text{Hom}_H(A, 1)$ is called the *space of class functions* as it coincides with the space of class functions in the usual sense when $H$ is a group algebra \cite{Shi17a}. We introduce the binary operation $\star$ on $\text{CF}(H)$ by

$$(\xi \star \zeta, a) = \langle \xi, \phi_1 \overline{\Phi}(1) a(1) \overline{\Phi}(3) \rangle \langle \zeta, \phi_2 \overline{\Phi}(2) a(2) \overline{\Phi}(2) \rangle$$

for $\xi, \zeta \in \text{CF}(H)$ and $a \in A$.

**Corollary 5.9.** $\text{CF}(H)$ is an associative unital algebra with respect to $\star$, and the map

$$\text{CF}(H) \to H^H\mathcal{P}(A, A), \quad \xi \mapsto (\text{id}_H \otimes \xi) \circ \delta_{2nd}$$

is an isomorphism of algebras.

**Proof.** Let $\eta$ be the unit of the adjunction $F \dashv R$. By Theorems 5.5 and 5.6, we have

$$\xi \cdot \zeta = \xi \circ FR(\zeta) \circ F(\eta_A)$$

for $\xi, \zeta \in \text{CF}(H)$. Namely, the binary operation $\star$ is in fact the composition of morphisms in the co-Kleisli category of the adjunction $F \dashv R$. The adjunction isomorphism of $F \dashv R$ is given by

$$\text{Hom}_H(U(M), X) \cong H^H\mathcal{P}(M, R(X)), \quad \xi \mapsto R(\xi) \circ \eta_M$$

for $X \in H\mathcal{M}$ and $M \in H^H\mathcal{P}$, and the map (5.30) is just the case where $M = A$ and $X = 1$. Now the claim of this lemma follows from the well-known compatibility between the adjunction isomorphism and the composition in the co-Kleisli category.

### 5.6. Induction to the Yetter-Drinfeld category, II.

Suppose that the quasi-Hopf algebra $H$ is finite-dimensional. Given $V \in H\mathcal{M}$, we define $L(V) \in H\mathcal{M}$ by $L(V) = H^V \otimes V$. We recall that $H^V$ is a left quasi-Hopf bimodule over $H$. Thus $L(V)$ is a Yetter-Drinfeld module over $H$ by the second kind coaction

$$L(V) = H^V \otimes V \xrightarrow{\delta \otimes \text{id}_V} (H \otimes H^V) \otimes V \xrightarrow{\alpha_{H^V,V}} H \otimes (H^V \otimes V) = H \otimes L(V),$$

where $\delta : H^V \to H \otimes H^V$ is the left coaction of $H$ given by Lemma 4.2 (i) with $M = H$. Specifically, the action and the second kind coaction on $L(V)$ are given by

$$h \triangleright (\xi \otimes v) = (\overline{\mathcal{S}}(h(2)) \to \xi \leftarrow \mathcal{S}(h(1,1))) \otimes h(1,2)v$$

$$\delta_{2nd}^{2nd}(\xi \otimes v) = \langle \xi, v_1^2 h_{i(2)} v_2^1 \rangle \omega_1 v_1^1 h_{i(1)} u_1^1 \omega_5$$

$$\otimes (\overline{\mathcal{S}}(\omega_4) \to h^1 \leftarrow \mathcal{S}(\omega_2)) \otimes \omega_3 v$$
for \( h \in H, \xi \in H^\vee \) and \( v \in V \), where \( \{h_i\} \) is a basis of \( H \) and \( \{h^i\} \) is the dual basis of \( H^\vee \).

**Theorem 5.10.** (i) The assignment \( V \mapsto L(V) \) extends to a functor from \( H, \mathcal{M} \to H_H \mathcal{D} \). This functor is left adjoint to the forgetful functor \( F \) with the unit \( \eta' \) and the counit \( \varepsilon' \) given by

\[
\eta' : V \to FL(V), \quad v \mapsto \epsilon \otimes v \quad (v \in V \in H, \mathcal{M}),
\]

\[
\varepsilon' : LF(M) \to M, \quad \xi \otimes m \mapsto (\xi, t_1 m_{[-1]} t_3 t_2 m_{[0]} \quad (m \in M \in H_H \mathcal{D}),
\]

where

\[
t = S(\Phi_1)q_1^{L \Phi_2(1)} \otimes q_1^R q_2^{L \Phi_2(2)} \otimes S(q_2^R \Phi_3).
\]

(ii) For \( V \in H, \mathcal{M} \), there is a natural isomorphism

\[
R(f^R \otimes V) \to L(V), \quad a \otimes (\lambda \otimes v) \mapsto (S(\Phi_1)\mu(\Phi_2) \to \lambda) \otimes \Phi_3 v.
\]

**Proof.** (i) The \( H \)-bimodule \( H^\vee \) is an algebra in the monoidal category \((H, \mathcal{M}, \hat{\otimes}, \text{Id})\) as the left dual object of \( H \). A Yetter-Drinfeld module \( M \) is naturally a left \( H \)-module in \( H, \mathcal{M} \) by the action

\[
a_M := (H^\vee \otimes M \xrightarrow{id_{H^\vee} \otimes \Delta_{2nd}} H^\vee \otimes (H \otimes M) \xrightarrow{(\Omega_{H^\vee,H,\mathcal{M}})^{-1}} (H^\vee \hat{\otimes} H) \otimes M \xrightarrow{ev_H \otimes id_M} M),
\]

and the category \( H_H \mathcal{D} \) is identified with the category \( H(H, \mathcal{M}) \) of left \( H \)-modules in \( H, \mathcal{M} \). Under this identification, a left adjoint of the forgetful functor \( F \) is given by the free \( H \)-module functor, that is, the functor \( L \). The unit of \( L \) is given by the unit of \( H^\vee \), that is, the counit of \( H \). Thus \( \eta' \) is given as stated. The counit of \( L \) is given by \( \varepsilon' = a_M \) for \( M \in H_H \mathcal{D} \), where \( a_M \) is given in the above. To verify the expression for \( a_M \), we introduce the element \( \hat{\omega} \in H^{\otimes 5} \) defined by

\[
\hat{\omega} = \Phi_1 \chi_3 \otimes \Phi_2 \chi_2(1) \otimes \Phi_3 \chi_2(2) \otimes S(\chi_3)\Phi_2 \otimes S(\chi_4)\Phi_1,
\]

where \( \chi \in H^{\otimes 4} \) is given by (5.6). The element \( \hat{\omega} \) is actually the inverse of the element \( \omega \in H \hat{\otimes} H \hat{\otimes} H \hat{\otimes} H^{op} \hat{\otimes} H^{op} \). Hence the inverse of the natural isomorphism \( \Omega \) is given by

\[
\Omega^{-1}_{M,N,V} : M \otimes (N \otimes V) \to (M \hat{\otimes} N) \hat{\otimes} V,
\]

\[
m \otimes (n \otimes v) \mapsto (\hat{\omega}_1 \xi \hat{\omega}_5 \otimes \hat{\omega}_2 n \hat{\omega}_4) \otimes \hat{\omega}_3 v
\]

for \( M, N \in H, \mathcal{M}, V \in H, \mathcal{M}, m \in M, n \in N \) and \( v \in V \). Now let \( M \) be a Yetter-Drinfeld \( H \)-module. For \( \xi \in H^\vee \) and \( m \in M \), we have:

\[
a_M(\xi \otimes m) = ev_H(\hat{\omega}_1 \xi \hat{\omega}_5 \otimes \hat{\omega}_2 m_{[-1]} \hat{\omega}_4) \hat{\omega}_3 m_{[0]}
\]

\[
= \langle S(\hat{\omega}_5) \to \xi \leftarrow S(\hat{\omega}_1), \alpha \hat{\omega}_2 m_{[-1]} \hat{\omega}_3 S(\hat{\beta}) \rangle \hat{\omega}_3 m_{[0]}
\]

\[
= \langle \xi, t_1 m_{[-1]} t_3 t_2 m_{[0]} \rangle,
\]
where $t' = S(\bar{\omega}_1)\alpha_\omega \otimes \omega_3 \otimes \bar{\omega}_4 S(\beta) S(\bar{\omega}_5)$. We prove (3.32) by showing $t' = t$ as follows:

\[
\begin{align*}
t'(3.32) &= S(\bar{\phi}_1 \chi_1) \alpha_\phi \bar{\phi}_2 \chi_{2(1)} \otimes \bar{\phi}_3 \chi_{2(2)} \otimes S(\chi_3) \phi_2 S(f_1(\beta) \chi_4) \\
&= S(\bar{\phi}_1 \chi_1) \alpha_\phi \bar{\phi}_2 \chi_{2(1)} \otimes \bar{\phi}_3 \chi_{2(2)} \otimes S(\chi_3) \alpha \chi_4 \\
&= S(\bar{\phi}_1 \chi_1) \alpha_\phi \bar{\phi}_2 \chi_{2(1)} \otimes \bar{\phi}_3 \chi_{2(2)} \otimes S(\chi_3) \alpha_\phi \chi_4 \\
&= (\bar{\phi}_1(1,1,1 \chi_1) \alpha_\phi \bar{\phi}_2 \chi_{2(1)} \otimes \bar{\phi}_3 \chi_{2(2)} \otimes S(\chi_3) \alpha_\phi \chi_4) \\
&= (\bar{\phi}_1(1,1,1 \chi_1) \alpha_\phi \bar{\phi}_2 \chi_{2(1)} \otimes \bar{\phi}_3 \chi_{2(2)} \otimes S(\chi_3) \alpha_\phi \chi_4) \\
&= (\bar{\phi}_1(1,1,1 \chi_1) \alpha_\phi \bar{\phi}_2 \chi_{2(1)} \otimes \bar{\phi}_3 \chi_{2(2)} \otimes S(\chi_3) \alpha_\phi \chi_4) \\
&= S(\bar{\phi}_1) q_1 \chi_{2(1)} \otimes q_1 \phi_2 \chi_{2(2)} \otimes S(q_1 \chi_3) = t.
\end{align*}
\]

(ii) By the fundamental theorem for quasi-Hopf bimodules, the map

\[
\Xi_R : H \otimes f^R \to H^\vee, \quad h \otimes \lambda \mapsto S(h) \to \lambda
\]

is an isomorphism of left quasi-Hopf bimodules. Thus the map

\[
\xi_V := \left( R(f^R \otimes V) = H \otimes (f^R \otimes V) \xrightarrow{\Omega^{-1}} (H \otimes f^R) \otimes V \xrightarrow{\Xi_R} H^\vee \otimes V = L(V) \right)
\]

is an isomorphism of Yetter-Drinfeld $H$-modules that is natural in the variable $V \in \mathcal{M}$. The map $\xi_V$ actually coincides with the map (3.33). Indeed, for $a \in H$, $\lambda \in f^R$ and $v \in V$, we have

\[
\xi_V(a \otimes (\lambda \otimes v)) = \Xi_R(\bar{\omega}_1 a \bar{\omega}_5 \otimes \omega_2 \lambda \bar{\omega}_4) \bar{\omega}_3 v = \mu(\bar{\omega}_2) \epsilon(\bar{\omega}_4) (S(\bar{\omega}_1 a \bar{\omega}_5) \to \lambda) \bar{\omega}_3 v = (S(\bar{\phi}_1) \mu(\bar{\phi}_2) \to \lambda) \chi_3 v.
\]

Here, the last equality follows from the following computation:

\[
\begin{align*}
(id_H \otimes id_H \otimes id_H \otimes \epsilon \otimes id_H)(\bar{\omega}) \\
= \bar{\phi}_1 \chi_1 \otimes \bar{\phi}_2 \chi_{2(1)} \otimes \bar{\phi}_3 \chi_{2(2)} \otimes \epsilon(S(\chi_3) \phi_2) S(\chi_4) \phi_1 \\
= \bar{\phi}_1 \otimes \bar{\phi}_2 \otimes \bar{\phi}_3 \otimes 1.
\end{align*}
\]

5.7. Categorical cointegrals of $\mathcal{M}$. Let $H$ be a finite-dimensional quasi-Hopf algebra. Given an algebra map $\gamma : H \to k$, we regard it as a one-dimensional left $H$-module and set $A_\gamma := R(\gamma) \in H \mathcal{G}$. According to [Shi18, Shi17b], we call

\[
f_{\text{cat}} := H \mathcal{G}(A_\mu, \mathbb{1})
\]

the space of categorical cointegrals. If $H$ is an ordinary Hopf algebra, then $f_{\text{cat}}$ is identified with the space of left cointegrals on $H$ [Shi18]. We now establish analogous results for quasi-Hopf algebras as follows:

**Theorem 5.11.** Let $\lambda$ be a non-zero right cointegral on $H$, and define

\[
\lambda_{\text{cat}} : A_\mu \to \mathbb{1}, \quad \lambda_{\text{cat}}(a) = (\lambda, \alpha S(a)) \quad (a \in A_\mu).
\]

Then $f_{\text{cat}}$ is the one-dimensional vector space spanned by $\lambda_{\text{cat}}$.

**Proof.** Let $\lambda$ be a non-zero right cointegral on $H$. Since $f^R$ is the one-dimensional left $H$-module spanned by $\lambda$ and is isomorphic to $\mu$ as a left $H$-module, there is an
isomorphism $A_μ ≅ R(f^R)$ of Yetter-Drinfeld $H$-modules given by $a ↦ a ⊗ λ$. Thus we have isomorphisms

$$f^{\text{cat}} = H_H^M(A_μ, 1) ≅ H_H^M(R(f_μ), 1)$$ \[\overset{\text{Theorem 5.10}}{\longrightarrow} H_H^M(L(1), 1) ≅ \text{Hom}_H (1, 1) \cong k.\]

The element of $f^{\text{cat}}$ corresponding to $1 \in k$ is the composition

$$l := \left( A_μ \overset{\alpha ⊗ λ}{\longrightarrow} R(f^R) \overset{ε_1}{\longrightarrow} L(1) \overset{ε_1'}{\longrightarrow} 1 \right),$$

where $ε'$ is the counit of $L ⊗ F$ given by (5.32). We compute this composition: For $a ∈ A_μ$

$$l(a) = ε_1' (\mathcal{S}(\Phi_1) a) μ(\Phi_2) → λ) ε(\Phi_3)$$

Thus $f^{\text{cat}}$ is the one-dimensional vector space spanned by $λ^{\text{cat}}$. □

5.8. Frobenius structure of the adjoint algebra. Let $H$ be a finite-dimensional unimodular Hopf algebra. Ishii and Masuoka observed that the commutative algebra $A$ of Theorem 5.6 is in fact a Frobenius algebra in $H_H^M$ and use this fact to construct an invariant of handlebody-knots [IM14]. By the result of [Shi17b], the algebra $A$ is still Frobenius even in the case where $H$ is a finite-dimensional unimodular quasi-Hopf algebra. However, an explicit Frobenius structure is not yet known. We answer this problem as follows:

**Theorem 5.12.** Suppose that $H$ is a finite-dimensional unimodular quasi-Hopf algebra. We fix a non-zero right cointegral $λ$ on $H$, and let $λ$ the left integral in $H$ such that $(λ, λ) = 1$. Then the algebra $A ∈ H_H^M$ of Theorem 5.6 is a self-dual object in $H_H^M$ with the evaluation morphism $e : A ⊗ A → 1$ and the coevaluation morphism $d : 1 → A ⊗ A$ given by

$$e(a ⊗ b) = (λ^{\text{cat}}, a ⊗ b)$$

$$d(1) = β ⊗ S(q_2q_1^λ A(1)β) ⊗ q_2^β A(2)β$$

respectively, for $a, b ∈ A$, where $λ^{\text{cat}} = (λ ⊗ λ) ∪ \mathcal{S}$. In particular, the algebra $A$ is a commutative Frobenius algebra in $H_H^M$ with Frobenius form $λ^{\text{cat}} : A → 1$.

**Proof.** The monoidal category $H_H^M$ is rigid as it is isomorphic to the Drinfeld center of $H_M$. By the description of the isomorphism $H_H^M ≅ Z(H_M)$, the left dual object of $M ∈ H_H^M$ is, as a left $H$-module, identical to $M'$. By Theorem 5.6 $λ^{\text{cat}}$ belongs to $H_H^M(A, 1)$. Thus the map $e$ is a morphism in $H_H^M$. Since $A$ is finite-dimensional, it has a left dual object in $H_H^M$. We now consider the morphism

$$Θ := \left( A \overset{id_A ⊗ \text{coev}_A}{\longrightarrow} A ⊗ (A ⊗ A') \overset{φ_{A,A,A'}^{-1}}{\longrightarrow} (A ⊗ A) ⊗ A' \overset{ε ⊗ id_{A'}}{\longrightarrow} A' \right)$$

of Yetter-Drinfeld modules. Since $H$ is unimodular, we have

$$λ(hh') = λ(h'S^2(h))$$

(5.36)
for all \( h, h' \in H \) by Theorem 6.10. We choose a basis \( \{ a_i \} \) of \( A \) and let \( \{ a^i \} \) be the dual basis. Using the Einstein convention, we compute

\[
\langle \tilde{\Theta}(h), h' \rangle = (\lambda^{\text{cat}}, (\tilde{\Phi}_1 \triangleright h) * (\tilde{\Phi}_2 \triangleright a_i)) \langle \tilde{\Phi}_3 \triangleright a^i, h' \rangle
\]

\[
= (\lambda^{\text{cat}}, (\tilde{\Phi}_1 \triangleright h) * (\tilde{\Phi}_2 \triangleright a_i)) \langle a^i, S(\tilde{\Phi}_3) \triangleright h' \rangle
\]

\[
= (\lambda^{\text{cat}}, (\tilde{\Phi}_1 \triangleright h) * (\tilde{\Phi}_2 \triangleright S(\tilde{\Phi}_3)) \triangleright h')
\]

6.10 \( 6.30 \)

\[
= (\lambda S, ((p^R_1 \triangleright h) * (p^R_2 \triangleright h')) S(\alpha))
\]

6.20

\[
= (\lambda S, \Phi_1 (p^R_1 \triangleright h) S(\Phi_2) \alpha \Phi_2 (p^R_1 (p^R_2 \triangleright h')) S(\Phi_2 (p^R_1) S(\alpha))
\]

6.25

\[
= (\lambda S, \Phi_1 (p^R_1) h S(p^R_2) S(\Phi_2) \alpha \Phi_2 (p^R_1) P^R_2 h' S(p^R_2) S(\Phi_2) S(\alpha))
\]

6.30

\[
= (\lambda S, (a \Phi_3 (p^R_2) P^R_2) \Phi_1 (p^R_1) h S(P^R_2) S(\Phi_2) \alpha \Phi_2 (p^R_1) P^R_2 h')
\]

6.24

\[
= (\lambda S, (S(\Phi_1 P^R_1) a \Phi_3 (p^R_2) P^R_2) h S(\Phi_2) \alpha \Phi_2 (p^R_1) P^R_2 h')
\]

for \( h, h' \in H \). Hence, we finally obtain

\[
\tilde{\Theta}(h) = (S(S(e_2) h e_1) \rightarrow \lambda) \circ S = \Theta_L (S(S(e_2) h e_1)) \circ S
\]

for \( h \in H \), where \( \Theta_L(x) = x \rightarrow \lambda (x \in H) \). We note that \( \lambda \) is a left cointegral on \( H^{\text{cop}} \). Thus, by applying Theorem 6.10 to \( H^{\text{cop}} \), we see that the inverse of \( \Theta_L \) is given by

\[
\Theta_L^{-1}(\xi) = S(\tilde{\Lambda}_1)(\xi, \tilde{\Lambda}_2)
\]

for \( \xi \in H^* \), where \( \tilde{\Lambda} = q^2 \Lambda(2) p^2_2 \otimes q^1 \Lambda(1) p^1_1 \). By (4.27) applied to \( H^{\text{cop}} \), we have

\[
h \tilde{\Lambda}_1 \otimes \tilde{\Lambda}_2 = \tilde{\Lambda}_1 \otimes S(h) \tilde{\Lambda}_2
\]

(5.37)

for all \( h \in H \). Since \( \Theta_L \), \( S \) and \( \mathfrak{f} \) are invertible, \( \tilde{\Theta} \) is invertible. Explicitly, the inverse of \( \Theta \) is given by

\[
\tilde{\Theta}^{-1}(\xi) = S(\tilde{\mathfrak{f}}_2) \cdot S(\Theta_L^{-1}(\xi) \circ S) \cdot \tilde{\mathfrak{f}}_1
\]

\[
= S(\tilde{\mathfrak{f}}_2) \cdot S(S(\tilde{\Lambda}_1)(\xi \circ S, \tilde{\Lambda}_2)) \cdot \tilde{\mathfrak{f}}_1
\]

\[
= S(\tilde{\mathfrak{f}}_2) \tilde{\Lambda}_1 \tilde{\mathfrak{f}}_1(\xi, S(\tilde{\Lambda}_2))
\]

(6.28)

\[
= \tilde{\Lambda}_1 \tilde{\mathfrak{f}}_1(\xi, S(\tilde{\mathfrak{f}}_2 \tilde{\Lambda}_2))
\]

for \( \xi \in A^\vee \). Thus \( A \) is a self-dual object with evaluation \( e \) and the coevaluation

\[
(id_A \otimes \tilde{\Theta}^{-1})\text{coev}_A(1) = \beta \triangleright a_i \otimes \tilde{\Theta}^{-1}(a^i)
\]

\[
= \beta \triangleright a_i \otimes \tilde{\Lambda}_1 \tilde{\mathfrak{f}}_1(a^i, S(\tilde{\mathfrak{f}}_2 \tilde{\Lambda}_2))
\]

\[
= \beta \triangleright S(\tilde{\mathfrak{f}}_2 \tilde{\Lambda}_2) \otimes \tilde{\Lambda}_1 \tilde{\mathfrak{f}}_1.
\]

\[
6.1. \textbf{Modified trace on module categories.} \text{ Let } C \text{ be a rigid monoidal category. Then the assignment } X \mapsto X^{**} \text{ canonically extends to a monoidal autoequivalence on } C. \text{ A pivotal structure of } C \text{ is an isomorphism } X \mapsto X^{**} (X \in C) \text{ of monoidal functors. A pivotal monoidal category is a rigid monoidal category equipped with a pivotal structure. Now let } C \text{ be a pivotal monoidal category with pivotal structure.}
\]
Graphically, the partial pivotal trace is expressed as follows:

\[ p\text{tr}_{X,Y|V}^{C} : \text{Hom}_{C}(V \otimes X, W \otimes X) \to \text{Hom}_{C}(V, W), \]

\[ f \mapsto (\text{id}_{W} \otimes \text{ev}_{X}(j_{X} \otimes \text{id}_{X}))\Phi_{W,X,X^{\vee}}(f \otimes \text{id}_{X^{\vee}})\Phi_{V,X,X^{\vee}}^{-1}(\text{id}_{V} \otimes \text{coev}_{X}). \]

Graphically, the partial pivotal trace is expressed as follows:

\[ \begin{array}{cc}
V & X \\
W & X
\end{array} \]

The partial pivotal trace is widely used to construct invariants of knots and closed 3-manifolds so-called quantum invariants \cite{T}. Although such a construction of quantum invariants works in a quite general setting, most of interesting quantum invariants originate from a semisimple \(k\)-linear pivotal monoidal category. In fact, the partial pivotal trace often vanishes in the non-semisimple case. To construct a meaningful invariant from a non-semisimple category, some authors considered a modification of the partial pivotal trace \cite{BBG17, CGPT18, GKP18, FG18}.

We suppose that \(C\) is a \(k\)-linear pivotal monoidal category with finite-dimensional Hom-spaces. In this paper, we mention a recent result of Fontalvo Orozco and Gainutdinov \cite{FG18}. They introduced the notion of a module trace \cite{FG18}, which we call a module trace on a \(k\)-linear pivotal monoidal category. For simplicity, we consider the case where \(M\) is a tensor ideal of \(C\), that is, a full subcategory of \(C\) such that \(P \otimes V\) and \(V \otimes P\) belong to \(M\) whenever \(P \in M\) and \(V \in C\). We moreover restrict ourselves to the case where \(\Sigma = D \otimes (-)\) for some object \(D \in C\). Then the notion of a module trace \cite{FG18}, which we call a \(D\)-twisted modified trace, is defined as follows:

**Definition 6.1.** (a) Let \(M\) be a \(k\)-linear category, and let \(\Sigma : M \to M\) be a \(k\)-linear endofunctor on \(M\). A \(\Sigma\)-twisted trace \cite{BK16} on \(M\) is a family \(t = \{t_{P} : \text{Hom}_{M}(P, \Sigma(P)) \to k\}_{P \in M}\) of \(k\)-linear maps such that the equation

\[ t_{P}\left(P \xrightarrow{g} Q \xrightarrow{f} \Sigma(P)\right) = t_{Q}\left(Q \xrightarrow{f} \Sigma(P) \xrightarrow{\Sigma(g)} \Sigma(Q)\right) \quad (6.1) \]

holds for all morphisms \(g : P \to Q\) and \(f : Q \to \Sigma(P)\) in \(M\). We denote by \(\text{HH}_{D}(M, \Sigma)\) the class of \(\Sigma\)-twisted traces on \(M\). We say that \(t \in \text{HH}_{D}(M, \Sigma)\) is non-degenerate if the bilinear form

\[ \text{Hom}_{M}(Q, \Sigma(P)) \times \text{Hom}_{M}(P, Q) \to k; \quad (f, g) \mapsto t_{P}(fg) \]

is non-degenerate for all objects \(P, Q \in M\).

(b) Let \(I\) be a tensor ideal of \(C\), and let \(D\) be an object of \(C\). A \(D\)-twisted module trace on \(I\) is a \(\Sigma\)-twisted trace on \(I\), where \(\Sigma = D \otimes (-)\), such that the equation

\[ t_{P \otimes V}(f) = t_{P} \left(p\text{tr}_{D \otimes P, P|V}^{C}(\Phi_{D,P,V}^{-1} \circ f)\right) \quad (6.2) \]

holds for all objects \(P \in I, V \in C\) and morphisms \(f : P \otimes V \to D \otimes (P \otimes V)\).

One of the main results of \cite{FG18} classifies \(D\)-twisted module traces on \(I\) in the case where \(C = H_{H} \mathcal{M}_{\text{fd}}\) for some finite-dimensional pivotal Hopf algebra \(H, \mathcal{I}\)
is the class of projective objects of $\mathcal{C}$, and $D$ is the one-dimensional left $H$-module associated to the modular function $\mu$ on $H$. According to [FG18], the space of such a trace is identified with the space of “$\mu$-symmetrized” cointegrals on $H$. The aim of this section is to give the same description of such a trace in the case where $H$ is a finite-dimensional quasi-Hopf algebra.

6.2. Pivotal quasi-Hopf algebras. We recall that a pivotal Hopf algebra is a Hopf algebra $H$ equipped with a grouplike element $g \in H$ such that $ghg^{-1} = S^2(h)$ for all $h \in H$. If $H$ is a pivotal Hopf algebra, then $H\mathcal{M}_d$ is a pivotal monoidal category by the pivotal structure given by $g$. The definition of a pivotal quasi-Hopf algebra is a little more complicated than the ordinary case because of the fact that the canonical isomorphism $(V \otimes W)^{\vee \vee} \cong V^{\vee \vee} \otimes W^{\vee \vee}$ is non-trivial in the quasi-Hopf case.

Definition 6.2 (cf. [BCT09 Proposition 4.2]). Let $H$ be a quasi-Hopf algebra. A pivotal element of $H$ is an invertible element $g \in H$ such that the equations

$$ghg^{-1} = S^2(h) \quad \text{and} \quad \Delta(g) = T_1 S(f_2)g \otimes T_2 S(f_1)g$$

(6.3)

hold for all element $h \in H$. A pivotal quasi-Hopf algebra is a quasi-Hopf algebra equipped with a pivotal element.

The category $\mathbf{Vec} := k\mathcal{M}_d$ of finite-dimensional vector spaces over $k$ is a pivotal monoidal category with the canonical pivotal structure $j_V : V \to V^{**}$ given by $\langle j_V(v), \xi \rangle = \langle \xi, v \rangle$ for $v \in V$ and $\xi \in V^*$. Let $f : V \otimes X \to W \otimes X$ be a morphism in $\mathbf{Vec}$. We write $f(m) = f(m)_W \otimes f(m)_X$ for $m \in V \otimes X$. The partial pivotal trace over $X$ (with respect to the canonical pivotal structure $j$) is given by

$$\text{ptr}_{V,W\mid X}(f)(x) = f(v \otimes x_i)_W \langle x^i, f(v \otimes x_i)_X \rangle$$

(6.4)

for $v \in V$, where $\{x_i\}$ is a basis of $X$, $\{x^i\}$ is the dual basis of $\{x_i\}$ and the Einstein summation convention is used to suppress the sum.

Let $H$ be a quasi-Hopf algebra. Given a pivotal element $g$ of $H$, we define the natural isomorphism $g_V : V \to V^{\vee \vee}$ $(V \in H\mathcal{M}_d)$ by $g_V(v) = j_V(gv)$ for $v \in V$. Then $g = \{g_V\}$ is a pivotal structure on $H\mathcal{M}_d$ [BCT09 Proposition 4.2]. Moreover, if $H$ is finite-dimensional, then every pivotal structure of $H\mathcal{M}_d$ is obtained in this way from a pivotal element of $H$.

Now we fix a pivotal element $g$ of $H$ and compute the partial pivotal trace in $H\mathcal{M}_d$ with respect to the pivotal structure associated to $g$. In the following lemma, given left $H$-modules $X$ and $Y$, we regard $X \otimes Y$ as a left $H \otimes H$-module by $(h \otimes h')(x \otimes y) = hx \otimes h'y$ for $h, h' \in H$, $x \in X$ and $y \in Y$.

Lemma 6.3. The partial trace of $f : V \otimes X \to W \otimes X$ in $\mathcal{C} := H\mathcal{M}_d$ is given by

$$\text{ptr}^\mathbf{Vec}_{V,W\mid X}(f) = \text{ptr}^\mathbf{Vec}_{V,W\mid X}(\tilde{f}),$$

(6.5)

where $\tilde{f} : V \otimes X \to W \otimes X$ is the linear map defined by

$$\tilde{f}(m) = (1 \otimes g)q^R f(g^R m) \quad (m \in V \otimes X).$$

(6.6)

Proof. We recall that there is a canonical isomorphism

$$\text{Hom}_H(V \otimes X, W \otimes X) \cong \text{Hom}_H(V, (W \otimes X) \otimes X^\vee).$$

Let $f^2 : V \to (W \otimes X) \otimes X^\vee$ be the morphism in $H\mathcal{M}_d$ corresponding to $f : V \otimes X \to W \otimes X$ via this isomorphism. Although an explicit formula of $f^2$ has been given in
Lemma 3.6 we also express $f^t$ by $f^t(v) = (f^t(v)_W \otimes f^t(v)_X) \otimes f^t(v)_{X^v}$ for $v \in V$. We fix a basis $\{x_i\}$ of $X$ and let $\{x^i\}$ be the dual basis to $\{x_i\}$. By the definition of the partial pivotal trace, we compute as follows: For $v \in V$,

$$\text{ptr}^t_{W\mid X}(f)(v) = (\text{id}_W \otimes \text{ev}_{X^v})(g_X \otimes X^v))\Phi^{-1}_{W \cdot X, X^v} f^t(v)$$

$$= \phi_1 f^t(v)_W \langle g_X(\phi_2 f^t(v)_X), \alpha \phi_3 f^t(v)_{X^v} \rangle$$

$$= \phi_1 f^t(v)_W \langle f^t(v)_{X^v}, S(\alpha \phi_3) g g \phi_2 f^t(v)_X \rangle$$

$$= \phi_1 f^t(v)_W \langle f^t(v)_{X^v}, g g S(\alpha \phi_3) f^t(v)_X \rangle$$

$$= \phi_1 f^t(v)_W \langle f^t(v)_X, g g R f^t(v)_X \rangle$$

$$= \phi_1 f^t(v)_W \langle f^t(v)_X, g g R f^t(v)_X \rangle$$

$$= \phi_1 f^t(v)_W \langle f^t(v)_X, g g R f^t(v)_X \rangle$$

$$= \phi_1 f^t(v)_W \langle f^t(v)_X, g g R f^t(v)_X \rangle$$

$$= \phi_1 f^t(v)_W \langle f^t(v)_X, g g R f^t(v)_X \rangle$$

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$$= \phi_1 f^t(v)_W \langle f^t(v)_X, g g R f^t(v)_X \rangle$$

$$= \phi_1 f^t(v)_W \langle f^t(v)_X, g g R f^t(v)_X \rangle$$

$\square$

6.3. **Twisted modified trace on** $H \cdot M_{\text{id}}$. Let $A$ be a finite-dimensional algebra. We denote by $A \cdot \mathcal{P}_{\text{id}}$ the full subcategory of projective objects of $A \cdot \mathcal{M}_{\text{id}}$. Given an algebra automorphism $\chi : A \to A$ and a left $A$-module $V$, we define $\chi^*(V)$ to be the vector space $V$ equipped with the left action $\chi$ of $A$ given by $a \cdot v = \chi(a)v$ for $a \in H$ and $v \in V$. It is easy to see that the assignment $V \mapsto \chi^*(V)$ extends to a $k$-linear autoequivalence on $A \cdot \mathcal{M}_{\text{id}}$ preserving the full subcategory $A \cdot \mathcal{P}_{\text{id}}$. We say that a linear form $t : A \to k$ is $\chi$-symmetric if $t(ab) = t(\chi(b)a)$ for all $a, b \in A$.

According to [FG18], we denote by $HH_0(A, \chi)$ the space of $\chi$-symmetric linear forms on $A$. Given a $\chi$-symmetric linear form $t$ on $A$ and $P \in A \cdot \mathcal{P}_{\text{id}}$, we define the linear map $t_P : \text{Hom}_A(P, \chi^*(P)) \to k$ as follows: By the dual basis lemma, there are a natural number $n$ and $A$-linear maps $a_i : A \to P$ and $b_i : P \to A$ ($i = 1, \ldots, n$) such that $\sum_{i=1}^n a_i b_i = \text{id}_P$. Choose such a system $\{a_i, b_i\}_{i=1}^n$ and define

$$t_P(f) = \sum_{i=1}^n t(b_i f a_i(1))$$

(6.7)

for $f \in \text{Hom}_A(P, \chi^*(P))$. The family $t = \{t_P\}$ of $k$-linear maps is actually a $\chi^*$-twisted trace on $A \cdot \mathcal{P}_{\text{id}}$ and this construction gives an isomorphism

$$HH_0(A, \chi) \to HH_0(A \cdot \mathcal{P}_{\text{id}}, \chi^*)$$

of vector spaces. Moreover, a $\chi^*$-twisted trace is non-degenerate if and only if the corresponding element of $HH_0(A, \chi)$ is a non-degenerate linear form on $A$.

Now we consider the case where $A = H$ is a finite-dimensional pivotal quasi-Hopf algebra and the automorphism $\chi$ is given by $\chi(h) = h \cdot \mu$ for $h \in H$. By abuse of notation, we write

$$HH_0(H, \mu) := HH_0(H, \chi) \quad \text{and} \quad HH_0(H \cdot \mathcal{P}_{\text{id}}, \mu) := HH_0(H \cdot \mathcal{P}_{\text{id}}, \chi^*).$$

We note that $\mathcal{P} := H \cdot \mathcal{P}_{\text{id}}$ is a tensor ideal of $H \cdot \mathcal{M}_{\text{id}}$. Again by abuse of notation, we denote by $\mu$ the one-dimensional left $H$-module associated to $\mu$. Let $HH_0^{\text{mod}}(\mathcal{P}, \mu)$ be the class of $\mu$-twisted module traces on $\mathcal{P}$. Since the autoequivalence $\chi^*$ on $\mathcal{P}$ is identified with $\mu \otimes (-)$, the class $HH_0^{\text{mod}}(\mathcal{P}, \mu)$ is the subspace of $HH_0(\mathcal{P}, \mu)$ consisting of elements satisfying the module trace property (6.2). By the reduction lemma [FG18, Lemma 2.9], to check the module trace property, it is enough to
verify that the equation
\[ t_{H \otimes H}(f) = t_H \left( \text{ptr}^C_{(\mu \otimes H) \otimes H, H \otimes H}(\Phi^{-1}_{\mu, H, H} \circ f) \right) \]  
holds for all \( f \in \text{Hom}_{H}(H \otimes H, \mu \otimes (H \otimes H)) \). Thus we have
\[ \text{HH}^0_{\mu} (P, \mu) = \{ t \in \text{HH}_0(P, \mu) \mid t \text{ satisfies (6.8)} \}. \]

6.4. Twisted modified trace and cointegrals. Let \( H \) be a finite-dimensional pivotal quasi-Hopf algebra with pivotal element \( g \). Let \( \lambda \) be a left cointegral on \( H \), and set \( t = (\lambda \circ \mathcal{S}) \leftarrow g \). Then we have
\[ (t, hh') = (\lambda, \mathcal{S}(ghh')) = (\lambda, \mathcal{S}(h'h)\mathcal{S}(gh)) \]
(\text{Theorem 6.9})
\[ = (\lambda, \mathcal{S}(gh)\mathcal{S}(h' \leftarrow \mu)) \]
\[ = (\lambda, \mathcal{S}(S^2(h' \leftarrow \mu)gh)) \]
\[ = (\lambda, \mathcal{S}(gh'(h' \leftarrow \mu)h)) = (t, (h' \leftarrow \mu)h) \]
for all \( h, h' \in H \). Thus the vector space
\[ f^{\mu, \text{sym}} := \{ (\lambda \circ \mathcal{S}) \leftarrow g \mid \lambda \in f^1 \} \]
is a subspace of \( \text{HH}_0(H, \mu) \). We call \( f^{\mu, \text{sym}} \) the space of \( \mu \)-symmetrized cointegrals on \( H \). We recall that there is an isomorphism between \( \text{HH}_0(H, \mu) \) and \( \text{HH}_0(\mu \mathcal{P}_d, \mu) \).

Now the main result of this section is stated as follows:

\textbf{Theorem 6.4.} The isomorphism \( \text{HH}_0(H, \mu) \cong \text{HH}_0(\mu \mathcal{P}_d, \mu) \) restricts to an isomorphism
\[ f^{\mu, \text{sym}} \cong \text{HH}^0_{\mu}(\mu \mathcal{P}_d, \mu). \]

We fix a \( \mu \)-symmetric linear form \( t \in \text{HH}_0(H, \mu) \) and then define \( t \in \text{HH}_0(\mu \mathcal{P}_d, \mu) \) from \( t \) as in the previous subsection. To prove the above theorem, we first derive a necessary and sufficient condition for \( t \) satisfying (6.8) in terms of the linear form \( t \). The following lemma is useful:

\textbf{Lemma 6.5.} For a left \( H \)-module \( X \), we define the linear map \( \theta_X \) by
\[ \theta_X : H \otimes X_0 \to H \otimes X, \quad h \otimes x \mapsto h_{(1)}p^R_1 \otimes h_{(2)}p^R_2 x \quad (h \in H, x \in X), \]
where \( X_0 \) is the vector space \( X \) regarded as a left \( H \)-module by \( \varepsilon \). Then the family \( \theta = \{ \theta_X \} \) is a natural isomorphism. The inverse of \( \theta_X \) is given by
\[ \theta_X^{-1}(h \otimes x) = q^R_1 h_{(1)} \otimes \mathcal{S}(q^R_2 h_{(2)}) x \quad (h \in H, x \in X). \]

\textit{Proof.} A left-right switched version is found at \[ \text{Sch02}, \text{p.3356} \]. One can also verify this lemma directly by using equations \( (3.15), (3.16), (3.19) \) and \( (3.20) \). \( \square \)

Set \( M = \mu \otimes (H \otimes H_0) \) for simplicity. We may identify \( M \) with the vector space \( H \otimes H \) equipped with the left \( H \)-action given by \( h \cdot (x \otimes y) = (\mu, h_{(1)})h_{(2)}x \otimes y \) \( (h, x, y \in H) \). There are isomorphisms
\[ \text{Hom}_{H}(H \otimes H_0, M) \cong \text{Hom}_{k}(H, M) \cong H^* \otimes M \cong H^* \otimes H \otimes H \]
of vector spaces. Let \( \Gamma_{\xi, x, y} \in \text{Hom}_{H}(H \otimes H_0, M) \) be the element corresponding to \( \xi \otimes x \otimes y \in H^* \otimes H \otimes H \) via the above isomorphism. Explicitly, it is given by
\[ \Gamma_{\xi, x, y}(h \otimes h') = (\xi, h')\langle \mu, h_{(1)} \rangle h_{(2)}x \otimes y \quad (h, h' \in H). \]
By the above lemma, we have
\[ \text{Hom}_H(H \otimes H, M) = \{ (\mu \otimes \theta_H) \circ f \circ \theta_H^{-1} \mid f \in \text{Hom}_H(H \otimes H_0, \mu \otimes (H \otimes H_0)) \} \]
where \( \Gamma_{\xi,x,y} := (\mu \otimes \theta_H) \circ \Gamma_{\xi,x,y} \circ \theta_H^{-1} \). Thus \( t \) satisfies (6.8) if and only if the equation
\[ t_{H \otimes H}(\Gamma_{\xi,x,y}^\prime) = t_H \left( \text{ptr}_{(\mu \otimes H) \otimes H} \circ \Gamma_{\xi,x,y} \right) \]  
(6.11)
holds for all \( \xi \in H^* \) and \( x, y \in H \). We now compute the left and the right hand sides of (6.11).

**Claim 6.6.** The left-hand side of (6.11) is equal to \( t(x)\xi(y) \).

**Proof.** Let \( \{ h_i \}_{i=1}^n \) be a basis of \( H \), and let \( \{ h^i \} \) be the dual basis of \( \{ h_i \} \). For each \( i \), we define
\[ a_i : H \rightarrow H \otimes H_0, \quad a_i(h) = h \otimes h_i, \]
\[ b_i : H \otimes H_0 \rightarrow H, \quad b_i(h \otimes h') = (h^i, h') h \]
for \( h, h' \in H \). Then \( a_i \) and \( b_i \) are \( H \)-linear maps such that \( \sum_{i=1}^n a_i b_i = \text{id}_H \). Since the trace \( t \) satisfies (6.11) with \( \Sigma = \mu \otimes (-) \), we have
\[ t_{H \otimes H}(\Gamma_{\xi,x,y}^\prime) = t_H \circ \Gamma_{\xi,x,y}^\prime = t_{H \otimes H_0}(\Gamma_{\xi,x,y}) \]
(6.11)
\[ \sum_{i=1}^n t(b_i \Gamma_{\xi,x,y} a_i(1)) = \sum_{i=1}^n t(x(h^i, y)\xi(h_i)) = t(x)\xi(y). \]

**Claim 6.7.** The right-hand side of (6.11) is equal to
\[ \mu(\Phi_1)\xi, \emptyset \{ q_{\Phi_2}R x(2) p_{\Phi_2}R y \} / t, (q_{\Phi_1}R \leftarrow \mu) \Phi_2 x(1) p_{\Phi_1}R. \]  
(6.12)

**Proof.** For simplicity, we set \( f := \Phi^{-1}_{\mu,H,H} \circ \Gamma_{\xi,x,y}^\prime \) and \( r := \text{ptr}_{H,\mu \otimes H}(f) \). By the definition of the trace \( t \), the right-hand side of (6.11) is equal to \( t(r(1)) \). To compute \( r(1) \), we note:
\[ \theta_H^{-1}(p_{\Phi_1}^R \otimes p_{\Phi_2}^R h) = q_{\Phi_1}^R \Phi_1(1) \otimes S(q_{\Phi_2}^R \Phi_2(1)) p_{\Phi_2}^R h = 1 \otimes h \quad (h \in H). \]

Hence, for all \( h \in H \), we have
\[ f(p_{\Phi_1}^R \otimes p_{\Phi_2}^R h) = (\Phi^{-1}_{\mu,H,H}(\mu \otimes \theta_H) \Gamma_{\xi,x,y})(1 \otimes h) \]
\[ = \langle \xi, h \rangle \Phi^{-1}_{\mu,H,H}(\mu \otimes \theta_H)(1 \otimes (x \otimes y)) \]
\[ = \langle \xi, h \rangle \Phi^{-1}_{\mu,H,H}(1 \otimes (x(1) p_{\Phi_1}^R \otimes x(2) p_{\Phi_2}^R y)) \]
\[ = \langle \xi, h \rangle \Phi^{-1}_{\mu,H,H}(1 \otimes \Phi_2 x(1) p_{\Phi_1}^R \otimes \Phi_3 x(2) p_{\Phi_2}^R y). \]

Now we define \( \tilde{f} \) by (6.6) with \( V = H \) and \( W = \mu \otimes H \). Then,
\[ \tilde{f}(1 \otimes h) = (\mu, \Phi_1)\xi, h, (1 \otimes g) q_{\Phi_1}^R \cdot (1 \otimes \Phi_2 x(1) p_{\Phi_1}^R \otimes \Phi_3 x(2) p_{\Phi_2}^R y) \]
\[ = (\mu, q_{\Phi_1(1)}^R \Phi_1)\xi, h, ((1 \otimes q_{\Phi_2(1)}^R \Phi_2 x(1) p_{\Phi_1}^R) \otimes g q_{\Phi_2(2)}^R \Phi_3 x(2) p_{\Phi_2}^R). \]
We fix a basis \( \{ h_i \} \) of \( H \) and let \( \{ h^i \} \) be the dual basis of \( \{ h_i \} \). We identify \( \mu \otimes H \) with \( H \) as a vector space. By Lemma 6.3, we compute \( r(1) \) as follows:

\[
    r(1) = \text{pt}^\text{Ysc}_{H,H/H}(\tilde{f})(1) \quad (\text{by identifying } \mu \otimes H \text{ with } H)
\]

By the above argument, it is sufficient to show the proof of Theorem 6.4.

Hence the right-hand side of (6.11) coincides with (6.12).

The above two claims show that the trace \( t \) satisfies (6.11) if and only if the equation

\[
    \langle t, h \rangle = \mu(\Phi_1)(t, (q_i^R - \mu)\Phi_2 h(1)_1) \otimes \langle q_i^R, \Phi_3 h(2) \rangle^R
\]

holds for all \( h \in H \). Now we prove Theorem 6.4.

**Proof of Theorem 6.4.** By the above argument, it is sufficient to show

\[
    f^{\mu_{-sym}} = \{ t \in HH_0(H, \mu) \mid t \text{ satisfies (6.13)} \}
\]

to prove this theorem. Suppose that \( t \in f^{\mu_{-sym}} \). Then, by definition, \( t = (\lambda \text{Sc}) - \mathfrak{g} \) for some left cointegral \( \lambda \) on \( H \). Let \( h \) be an arbitrary element of \( H \), and set \( h' = \text{Sc}(gh) \). Then we have

\[
    \text{Sc}(f_1)h'_2 \otimes \text{Sc}(f_2)h'_1 = \text{Sc}(f_1)\text{Sc}(gh)_2 h'_2 \otimes \text{Sc}(f_2)\text{Sc}(gh)_1 h'_1
\]

\[
    = \text{Sc}(g_1)h'_1 \text{Sc}(f_1)h'_2 \otimes \text{Sc}(g_2)h'_2 \text{Sc}(f_2)h'_1
\]

\[
    = \text{Sc}(\text{Sc}(f_2)g_1 h'_1) \otimes \text{Sc}(\text{Sc}(f_1)g_2 h'_2)
\]

\[
    = \text{Sc}(g_1 h'_1) \otimes \text{Sc}(g_2 h'_2).
\]

We now show that \( t \) satisfies (6.13) as follows:

\[
    \mu(\Phi_1)(t, (q_i^R - \mu)\Phi_2 h(1)_1) \otimes \langle q_i^R, \Phi_3 h(2) \rangle^R
\]

\[
    = \mu(\Phi_1)(\lambda, \text{Sc}(gh)_1) \text{Sc}(\langle q_i^R - \mu \Phi_2 \rangle) \text{Sc}(\langle q_i^R - \mu \Phi_2 \rangle) \text{Sc}(\langle q_i^R - \mu \Phi_2 \rangle)
\]

\[
    = \mu(\Phi_1)(\lambda, \text{Sc}(\mathfrak{g}_1^R) \text{Sc}(\mathfrak{g}_2^R) \text{Sc}(\mathfrak{g}_3^R) \text{Sc}(\mathfrak{g}_4^R)) \text{Sc}(\mathfrak{g}_5^R) \text{Sc}(\mathfrak{g}_6^R) \text{Sc}(\mathfrak{g}_7^R) \text{Sc}(\mathfrak{g}_8^R)
\]

\[
    = \text{Sc}(\mathfrak{g}_1^R) \text{Sc}(\mathfrak{g}_2^R) \text{Sc}(\mathfrak{g}_3^R) \text{Sc}(\mathfrak{g}_4^R) \text{Sc}(\mathfrak{g}_5^R) \text{Sc}(\mathfrak{g}_6^R) \text{Sc}(\mathfrak{g}_7^R) \text{Sc}(\mathfrak{g}_8^R)
\]

Hence “\( \subset \)” of (6.14) is proved. We shall prove the converse inclusion. Let \( t \in HH_0(H, \mu) \) be an element satisfying (6.13), and set \( \lambda = (t - g^{-1}) \circ S \). Then, for
all $h \in H$, we have

$$(\lambda, h) 1 \cong (t, g^{-1} S(h)) S(1)$$

$$= \mu(f_1)(t, (q_1^R \leftarrow -\mu) f_2 g^{-1} S(h))_{(1)} p_1 R S(g_2^{-1} S(1) S(h))_{(2)} p_2 R$$

$$= \mu(f_1)(t, (q_1^R \leftarrow -\mu) f_2 g^{-1} S(h))_{(1)} p_1 R S(g_2^{-1} S(h))_{(2)} p_2 R$$

$$= \mu(f_1)(t, (q_1^R \leftarrow -\mu) f_2 g^{-1} S(h))_{(1)} p_1 R S(h_{(1)}) f_2 R p_2 R$$

$$= \mu(f_1)(t, g^{-1} S^2(q_1^R \leftarrow -\mu) f_2 g^{-1} S(h_{(1)}) f_2 R p_2 R)$$

$$= \mu(f_1)(t, g^{-1} S^2(q_1^R \leftarrow -\mu) S(h_{(1)}) f_2 R p_2 R)$$

$$= \mu(f_1)(t, g^{-1} S^2(q_1^R \leftarrow -\mu) S(h_{(1)}) f_2 R p_2 R)$$

Thus, by Theorem 3.11 the linear form $\lambda$ is a left cointegral on $H$. Since $t = (\lambda \circ S) \leftarrow g$, the linear form $t$ belongs to $f_{\mu,sym}$. The proof is done. \qed

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(T. Shibata) Department of Applied Mathematics, Okayama University of Science, 1-1 Ridai-cho, Kita-ku, Okayama-shi, Okayama 700-0005, Japan.

E-mail address: shibata@xmath.ous.ac.jp

(K. Shimizu) Department of Mathematical Sciences, Shibaura Institute of Technology, 307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570, Japan.

E-mail address: kshimizu@shibaura-it.ac.jp