ON THE HILBERT SCHEME OF A PRYM VARIETY

H. Lange  E. Sernesi

Dedicated to the memory of Fabio Bardelli

Introduction

We work over the field of complex numbers. In this paper we consider the Prym map \( P : \mathcal{R}_g \to \mathcal{A}_{g-1} \) from the moduli space of unramified double covers of projective irreducible and nonsingular curves of genus \( g \geq 6 \) to the moduli space of principally polarized abelian varieties of dimension \( g-1 \). If \( \pi : \tilde{C} \to C \) is such a double cover with \( C \) non hyperelliptic we consider the natural embedding \( \tilde{C} \subset P \) (defined up to translation) of \( \tilde{C} \) into the Prym variety \( P \) of \( \pi \) and we study the local structure of the Hilbert scheme \( \text{Hilb}^P \) of \( P \) at the point \([\tilde{C}]\) (here and through the paper we adopt the notation \([-]\) for the point of a moduli space or of a Hilbert scheme which parametrizes the object \(-\)). We show that this structure is related with the local geometry of the Prym map, or more precisely with the validity of the infinitesimal version of Torelli’s theorem for Pryms at \([\pi]\) (see §3 for the definitions).

The results we prove are the following.

**Proposition** If the infinitesimal Torelli theorem for Pryms holds at \([\pi]\) then \( \text{Hilb}^P \) is nonsingular of dimension \( g-1 \) at \([\tilde{C}]\) (i.e. \( \tilde{C} \) is unobstructed) and the only deformations of \( \tilde{C} \) in \( P \) are translations.

It is known that if the Clifford index of \( C \) is at least 3 then the condition of the Proposition is satisfied. Therefore we have in particular:

**Corollary** If \( \text{Cliff}(C) \geq 3 \) then \( \text{Hilb}^P \) is nonsingular of dimension \( g-1 \) at \([\tilde{C}]\) (i.e. \( \tilde{C} \) is unobstructed) and the only deformations of \( \tilde{C} \) in \( P \) are translations.

On the other side we have the following result:

**Theorem** Assume that the following conditions are satisfied:
(a) the infinitesimal Torelli theorem for Pryms fails at \([\pi]\);
(b) \([\pi]\) is an isolated point of the fibre \( P^{-1}(P([\pi])) \).

Then \( \tilde{C} \) is obstructed. Moreover the only local deformations of \( \tilde{C} \) in \( P \) are translations and the only irreducible component of \( \text{Hilb}^P \) containing \([\tilde{C}]\) is everywhere non reduced.
Conversely, if $\tilde{C}$ is obstructed then the infinitesimal Torelli theorem for Pryms fails at $[\pi]$.

Using these results we give some examples in which $\tilde{C} \subset P$ is obstructed and some in which we have unobstructedness but the infinitesimal Torelli theorem for Pryms fails. The examples we construct are obtained from double covers belonging to $R_6$ and to $R_7$. For their construction we make use of a result proved in §4 which is a slight extension of a theorem of Recillas (see [R]).

The paper is divided into 5 sections. In §1 we discuss the Hilbert scheme of curves Abel-Jacobi embedded in their jacobian. We prove that such curves are obstructed precisely when they are hyperelliptic of genus $g \geq 3$. This case is not relevant for what follows but it is worth keeping in mind the analogies between the two cases. In §2 we consider our problem and we study the conditions for the unobstructedness of $[\tilde{C}]$. We use the well-known cohomological description of certain tangent spaces and of maps between them. In §3 we relate these results with the infinitesimal Torelli theorem and we prove our main result. In §4 we give a proof of the extension of Recillas' theorem. The final §5 contains the examples.

1. The case of curves in their jacobians

Consider a projective nonsingular irreducible curve $C$ of genus $g \geq 2$, let $JC := \text{Pic}^0(C)$ be the jacobian variety of $C$, and let $j : C \to JC$ be an Abel-Jacobi map. We want to study the local structure of the Hilbert scheme $HilbJC$ of $JC$ at $[j(C)]$ (the point parametrizing $j(C)$). Since $j$ is an embedding we will identify $C$ with $j(C)$. We have an exact sequence of locally free sheaves on $C$:

$$0 \to T_C \to T_{JC|C} \to N_C \to 0$$

We have a canonical isomorphism $T_{JC|C} \cong H^1(O_C) \otimes O_C$ and therefore the cohomology sequence of (1) is:

$$0 \to H^1(O_C) \to H^0(N_C) \xrightarrow{\delta} H^1(T_C) \xrightarrow{\sigma} H^1(O_C) \otimes H^1(O_C) \to H^1(N_C)$$

The family of translations of $C$ in $JC$ is parametrized by $JC$ itself, and the map $H^1(O_C) \to H^0(N_C)$ in (2) is precisely the characteristic map of this family at the point 0. Therefore we have the following Lemma, whose proof is obvious:

1.1 Lemma The following conditions are equivalent:
(a) $HilbJC$ is nonsingular of dimension $g$ at $[C]$.
(b) $\delta = 0$
(c) $\sigma$ is injective
If these conditions are satisfied the only local deformations of $C$ in $JC$ are translations.
Using this Lemma we can prove the following

1.2 Theorem Suppose that $C$ has genus $g \geq 3$.

(a) If $C$ is non hyperelliptic then $\text{Hilb}^{JC}$ is nonsingular of dimension $g$ at $[C]$.

(b) If $C$ is hyperelliptic then the connected component of $\text{Hilb}^{JC}$ containing $[C]$ is irreducible of dimension $g$ and everywhere non reduced with Zariski tangent space of dimension $2g - 2$.

In both cases the only deformations of $C$ in $JC$ are translations.

Proof. The transpose of $\sigma$ is the multiplication map:

$$\sigma^\vee : H^0(\omega_C) \otimes H^0(\omega_C) \to H^0(\omega_C^{\otimes 2})$$

(see [G], Lemma 3). This map, by Noether’s theorem, is surjective if $C$ is non hyperelliptic and has corank $g - 2$ if $C$ is hyperelliptic (see [ACGH]). Therefore, in view of Lemma 1.1, part (a) follows.

Now assume that $C$ is hyperelliptic and that $\overline{C} \subset JC$ is a closed subscheme such that $[\overline{C}]$ belongs to the connected component of $\text{Hilb}^{JC}$ containing $[C]$. By the criterion of Matsusaka-Ran (see [LB]) $\overline{C} = C_1 \cup \cdots \cup C_r$ is a reduced curve of compact type, and $JC$ and $JC_1 \times \cdots \times JC_r$ are isomorphic as ppav’s. Then it follows that $r = 1$ and $\overline{C}$ is irreducible and nonsingular because $C$ is. Now we apply Torelli’s theorem to conclude that $\overline{C}$ is a translate of $C$. It follows that the connected component of $\text{Hilb}^{JC}$ containing $[C]$ is irreducible of dimension $g$ and parametrizes the translates of $C$. On the other hand by (2) we have $h^0(N_C) = 2g - 2 > g$. The conclusion follows. qed

Theorem 1.2 can be interpreted in terms of the Torelli morphism

$$\tau : M_g \to A_g$$

from the moduli stack of projective nonsingular curves of genus $g$ to the moduli stack of principally polarized abelian varieties of dimension $g$. The surjectivity of $\sigma^\vee$ is equivalent to that of the multiplication map

$$S^2 H^0(\omega_C) \to H^0(\omega_C^{\otimes 2})$$

which is the codifferential of $\tau$ at $[C]$. Hence the surjectivity of this map is equivalent to the infinitesimal Torelli theorem for $C$ (see [OS]). Therefore Theorem 1.2 implies the following:

1.3 Corollary $C$ is unobstructed in $JC$ if and only if the infinitesimal Torelli theorem holds for $C$.

Remarks. (i) The proof of Theorem 1.2(a) appeared already in [G], but the argument does not appear to be complete. A proof is also given in [Bl] using the semiregularity map, but it is more complicated; moreover the semiregularity map does not seem to be able to detect what happens in case (b).
(ii) In the case $g = 2$ we have that $C$ is unobstructed in $JC$ because the semiregularity map $H^1(N_C) \to H^2(O_{JC})$ is injective since $H^1(O_{JC}(C)) = 0$ by the ampleness of $C$ in $JC$.

2. The Hilbert scheme of the Prym variety at $[\tilde{C}]$

Let now $\pi : \tilde{C} \to C$ be an unramified double cover of a projective nonsingular irreducible curve $C$ of genus $g \geq 3$, so that $\tilde{C}$ has genus $\tilde{g} = 2g - 1$. Let $\eta \in \text{Pic}^0(C)$ be the 2-division point corresponding to $\pi$. We have a canonical isogeny $J\tilde{C} \to JC \times P$, where $P$ is the Prym variety of $\pi$. Throughout this section we assume $C$ to be non hyperelliptic. Under this hypothesis we have an embedding $\alpha : \tilde{C} \to P$ which is obtained as the composition $\tilde{C} \to J\tilde{C} \to JC \times P \to P$ (see [LB]). We will identify $\tilde{C}$ with $\alpha(\tilde{C})$. We want to study the Hilbert scheme $\text{Hilb}^P$ locally at the point $[\tilde{C}]$.

In analogy with the situation studied in §1, we consider the exact sequence of locally free sheaves on $\tilde{C}$:

$$0 \to T_{\tilde{C}} \to T_{P|\tilde{C}} \to N_{\tilde{C}} \to 0$$

We have a canonical isomorphism $T_{P|\tilde{C}} \cong H^1(C, \eta) \otimes O_{\tilde{C}}$ so that the cohomology sequence of (3) becomes:

$$0 \to H^1(C, \eta) \to H^0(\tilde{C}, N_{\tilde{C}}) \overset{\delta}{\to} H^1(\tilde{C}, T_{\tilde{C}}) \overset{\sigma}{\to} H^1(C, \eta) \otimes H^1(\tilde{C}, O_{\tilde{C}}) \to H^1(\tilde{C}, N_{\tilde{C}})$$

Along the same lines of §1 we can state the following

2.1 Lemma The following conditions are equivalent:

(a) $\text{Hilb}^P$ is nonsingular of dimension $g - 1$ at $[\tilde{C}]$.
(b) $\delta = 0$
(c) $\sigma$ is injective

If these conditions are satisfied the only local deformations of $\tilde{C}$ in $P$ are translations.

In order to understand the conditions of Lemma 2.1 we must study the map $\sigma$, or equivalently its transpose $\sigma^\vee$. We have canonical isomorphisms:

$$H^1(\tilde{C}, O_{\tilde{C}})^\vee \cong H^0(\tilde{C}, \omega_{\tilde{C}}) \cong H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes \eta)$$

and

$$H^1(\tilde{C}, T_{\tilde{C}})^\vee \cong H^0(\tilde{C}, \omega^{\otimes 2}_{\tilde{C}}) \cong H^0(C, \omega^{\otimes 2}_C \otimes \eta) \oplus H^0(C, \omega^{\otimes 2}_C)$$

corresponding to the decompositions into $+1$ and $-1$ eigenvalues under the action induced by the involution on $\tilde{C}$. Hence

$$\sigma^\vee : H^0(C, \omega_C \otimes \eta) \bigotimes [H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes \eta)] \to H^0(C, \omega^{\otimes 2}_C \otimes \eta) \oplus H^0(C, \omega^{\otimes 2}_C)$$
and it is induced by multiplication of sections ([B], page 38 2). Therefore, after decomposing the domain of $\sigma^\vee$ as

$$H^0(C, \omega_C \otimes \eta) \otimes [H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes \eta)] =$$

$$= [H^0(C, \omega_C \otimes \eta) \otimes H^0(C, \omega_C)] \oplus [H^0(C, \omega_C \otimes \eta) \otimes H^0(C, \omega_C \otimes \eta)]$$

we see that $\sigma^\vee = \mu \oplus \nu$ where:

$$\mu : H^0(C, \omega_C \otimes \eta) \otimes H^0(C, \omega_C) \to H^0(C, \omega_C \otimes 2 \otimes \eta)$$

and

$$\nu : H^0(C, \omega_C \otimes \eta) \otimes H^0(C, \omega_C \otimes \eta) \to H^0(C, \omega_C \otimes 2)$$

The following Lemma is well known (see [B], Prop. 7.7):

**2.2 Lemma** The sheaf $\omega_C \otimes \eta$ is not very ample if and only if there exist points $x, y, z, t \in C$ such that $\eta \simeq \mathcal{O}_C(x + y - z - t)$. If these conditions are satisfied then the map $\nu$ is not surjective.

**2.3 Proposition** (i) In each of the following cases the map $\mu$ is surjective:

(a) $C$ is not bielliptic

(b) $\nu$ is surjective.

(ii) If $C$ is not bielliptic then $\text{cork}(\sigma^\vee) = \text{cork}(\nu)$. In particular if $C$ is not bielliptic the surjectivity of $\sigma^\vee$ is equivalent to the surjectivity of $\nu$.

**Proof.** (i) Note first that in both cases (a) and (b) the linear series $|\omega_C \otimes \eta|$ is base point free and is not composed with an involution: in fact, in case (a) since $C$ is not hyperelliptic $|\omega_C \otimes \eta|$ is base point free; moreover if it were composed with an involution then, since $\deg(\omega_C \otimes \eta) = 2g - 2$, the morphism

$$\phi_\eta : C \to \mathbb{P}^{g-2}$$

would be of degree 2 onto a curve of degree $g - 1$, which has genus $\leq 1$, a contradiction. In case (b) the assertion is true by Lemma 2.2.

Let $\underline{b} := P_1 + \cdots + P_{g-3} \in C^{(g-3)}$ be general. Consider the exact sequence on $C$:

$$0 \to \omega_C \otimes \eta(-\underline{b}) \to \omega_C \otimes \eta \to T \to 0$$

where $T$ is a torsion sheaf supported on $\underline{b}$. Multiplying firstly by $H^0(\omega_C)$ and taking cohomology, and secondly by $\omega_C$ and taking cohomology, we obtain the following commutative diagram with exact rows, where the vertical maps are given by multiplication:

$$
\begin{array}{cccccccc}
0 & \to & H^0(\omega_C \otimes \eta(-\underline{b})) \otimes H^0(\omega_C) & \to & H^0(\omega_C \otimes \eta) \otimes H^0(\omega_C) & \to & H^0(T) \otimes H^0(\omega_C) & \to & 0 \\
\downarrow & & \downarrow \mu & & \downarrow & & \downarrow \mu & & \\
0 & \to & H^0(\omega_C^{\otimes 2} \otimes \eta(-\underline{b})) & \to & H^0(\omega_C^{\otimes 2} \otimes \eta) & \to & H^0(T \otimes \omega_C) & \to & 0
\end{array}
$$
Since $|\omega_C \otimes \eta|$ is not composed with an involution and $b$ is generic, $\omega_C \otimes \eta(-b)$ is base point free, and by the base point free pencil trick we find:

$$\ker(\mu_b) = H^0(b \otimes \eta) = 0$$

hence:

$$\text{rk}(\mu_b) = 2 = h^0(\omega_C^2 \otimes \eta(-b))$$

i.e. $\mu_b$ is surjective. On the other hand $\bar{\mu}$ is surjective because $\omega_C$ is globally generated.

The conclusion follows from the above diagram.

(ii) follows immediately from part (i) and from the relation between the maps $\sigma, \mu, \nu$. $\boxed{}$

Collecting all we have said so far we can state the following:

2.4 Corollary If $\nu$ is surjective then $\text{Hilb}^P$ is nonsingular of dimension $g - 1$ at $[\tilde{C}]$ (i.e. $\tilde{C}$ is unobstructed) and the only local deformations of $\tilde{C}$ in $P$ are translations.

As an application we can prove the following:

2.5 Corollary If $\text{Cliff}(C) \geq 3$ then $\text{Hilb}^P$ is nonsingular of dimension $g - 1$ at $[\tilde{C}]$ (i.e. $\tilde{C}$ is unobstructed) and the only local deformations of $\tilde{C}$ in $P$ are translations.

Proof. It follows easily from a result of [GL] (see e.g. [LS]) that if $\text{Cliff}(C) \geq 3$ then the map $\nu$ is surjective. Therefore the Corollary follows from Corollary 2.4. $\boxed{}$

3. Hilb$^P$ and the infinitesimal Torelli theorem for Pryms

We keep the notations of §2. Consider the Prym morphism:

$$\mathcal{P} : \mathcal{R}_g \to \mathcal{A}_{g-1}$$

which goes from the coarse moduli scheme of étale double covers of curves of genus $g$ to the coarse moduli scheme of ppav of dimension $g - 1$, $g \geq 6$. These schemes have singularities due to the presence of automorphisms of the objects they classify. Therefore if we want to study the infinitesimal properties of $\mathcal{P}$ it is more natural to consider the corresponding moduli stacks $\mathcal{R}_g, \mathcal{A}_{g-1}$. The Prym construction defines a morphism of stacks

$$\mathcal{Pr} : \mathcal{R}_g \to \mathcal{A}_{g-1}$$

Then the map $\nu$ considered in §2 coincides with the codifferential of $\mathcal{Pr}$ at $[\pi]$ (see [B], Prop. 7.5, which implies this statement modulo obvious modifications). Therefore the surjectivity of $\nu$ is equivalent to $\mathcal{Pr}$ being an immersion at $[\pi]$ (see [B], 7.6). In this case we say that the infinitesimal Torelli theorem for Pryms holds at $[\pi]$, according to the
terminology most commonly used nowadays. In view of Corollary 2.4 we can therefore state the following:

3.1 Proposition If the infinitesimal Torelli theorem for Pryms holds at \([\pi]\) then \(\text{Hilb}^P\) is nonsingular of dimension \(g-1\) at \([\tilde{C}]\) (i.e. \(\tilde{C}\) is unobstructed) and the only local deformations of \(\tilde{C}\) in \(P\) are translations.

In the case \(\text{Cliff}(C) \leq 2\) the infinitesimal Torelli theorem for Pryms in general fails, i.e. in general \(\nu\) is not surjective. Our next goal is to relate the obstructedness of \(\tilde{C}\) in \(P\) to the failure of the infinitesimal Torelli theorem for Pryms. The main result of this section is the following:

3.2 Theorem Assume that the following conditions are satisfied:
(a) the infinitesimal Torelli theorem for Pryms fails at \([\pi]\);
(b) \([\pi]\) is an isolated point of the fibre \(P^{-1}(P([\pi]))\).

Then \(\tilde{C}\) is obstructed. Moreover the only local deformations of \(\tilde{C}\) in \(P\) are translations; in particular the only irreducible component of \(\text{Hilb}^P\) containing \([\tilde{C}]\) is everywhere non reduced of dimension \(g-1\).

Conversely, if \(\tilde{C}\) is obstructed then the infinitesimal Torelli theorem fails at \([\pi]\).

Proof. By (a) the map \(\delta\) in the exact sequence (4) is non zero. Assume by contradiction that \([\tilde{C}]\) is unobstructed. Then we can find a nonsingular curve \(S \subset \text{Hilb}^P\) passing through \([\tilde{C}]\) such that \(\delta(T_{S,[\tilde{C}]}) \neq 0\). This condition implies that the functorial morphism \(S \to \mathcal{M}_{\tilde{C}}\) defined by the family of curves \(C \to S\) (which can be assumed to be smooth) is not constant and therefore this family does not consist of curves all isomorphic to \(\tilde{C}\). But this is impossible: in fact for each curve \(\tilde{C}'\) in the family we have

\[
\tilde{C}' \equiv_{\text{num}} \tilde{C} \equiv_{\text{num}} \frac{2}{(g-2)!} \Xi^{g-2}
\]

so that by a theorem of Welters (see [W]) there is an étale double cover \(\pi' : \tilde{C}' \to C'\), \((P, \Xi)\) is the Prym variety of \(\pi'\) and \(\tilde{C}'\) is Prym embedded. But this contradicts condition (b) if \(\tilde{C}'\) is not isomorphic to \(\tilde{C}\) because \([\pi'] \in P^{-1}(P([\pi]))\).

This analysis also shows that locally the only deformations of \(\tilde{C}\) in \(P\) are translations; and since \(\delta \neq 0\) the Zariski tangent space of \(\text{Hilb}^P\) at \([\tilde{C}]\) has dimension larger than \(g-1\). This proves also the last assertion.

The converse is a special case of Prop. 3.1. \(\text{qed}\)

The Theorem does not say anything in the case when \([\pi]\) is a non isolated point of the fibre \(P^{-1}(P([\pi]))\). We will see in \S 5 that in this case there are examples where \([\tilde{C}]\) is unobstructed.

4. Further considerations.
A Theorem of Recillas [R] says that if \( \pi : \tilde{C} \to C \) is a double cover with \( C \) trigonal (but not hyperelliptic) of genus \( g \) then \( \mathcal{P}([\pi]) = [JX] \) with \( X \) a 4-gonal curve, and the pair \((X, g_1^1)\) is uniquely determined. A consequence of this result and of a theorem of Mumford [M] which gives a list of the Prym varieties which are Jacobians, is that

\[
\mathcal{P}^{-1}([JX]) = W_4^1(X)
\]

set-theoretically if \( g - 1 \geq 6 \). This says in particular that for \( g \geq 11 \) the Prym map is 1-1 on \( \mathcal{R}_{g,T} \) (=the locus of étale double covers of trigonal curves): this follows from the fact that \( W_4^1(X) \) consists of at most one point if \( g(X) \geq 10 \). If \( g - 1 = 5 \) then (5) is not true but we have a strict inclusion \( \supset \) (see §4). The following Proposition gives some further information which will suffice for some applications.

**4.1 Proposition** Assume that \( X \) is a nonsingular irreducible curve of genus \( g - 1 \geq 5 \), non hyperelliptic nor trigonal, and such that every \( g_1^1 \) on \( X \) has no divisors of the form \( 2P + 2Q \) or \( 4P \) and let \( \pi : \tilde{C} \to C \) be an unramified double cover, with \( C \) trigonal of genus \( g \), such that \( \mathcal{P}([\pi]) = [JX] \). Then there is a canonical isomorphism between the kernel of the differential of \( \text{Pr} \) at \([\pi]\) and the Zariski tangent space of \( W_4^1(X) \) at the line bundle \( L \) corresponding to \( \pi \).

**Proof.** Since \( X \) is not trigonal we may view \( W_4^1(X) \) as parametrizing 4-1 morphisms of \( X \) into \( \mathbb{P}^1 \). Let \( \varphi : X \to \mathbb{P}^1 \) be the 4-1 cover defined by \( L \).

Let \( B = \text{Spec}(\mathbb{C}[\varepsilon]) \) and consider a family of deformations of \( \varphi \) parametrized by \( B \):

\[
X \times B \rightarrow \mathbb{P}^1 \times B
\]

\[
\searrow \quad \swarrow
\]

\[
B
\]

To this family we can associate a family of deformations of \( \pi \) just extending Recillas’ construction, as follows. Consider the second relative symmetric product over \( \mathbb{P}^1 \times B \):

\[
\tilde{\mathcal{C}} := S_{\mathbb{P}^1 \times B}^{(2)}(X \times B)
\]

which comes endowed with an induced family of morphisms of degree 6:

\[
f^{(2)} : \tilde{\mathcal{C}} \rightarrow \mathbb{P}^1 \times B
\]

\[
\searrow \quad \swarrow
\]

\[
B
\]

On \( \tilde{\mathcal{C}} \) there is a natural involution \( \iota \) commuting with the projection to \( B \). Letting \( \mathcal{C} = \tilde{\mathcal{C}} / \iota \), we obtain a family parametrized by \( B \):

\[
\tilde{\mathcal{C}} \xrightarrow{\Pi} \mathcal{C}
\]

\[
\searrow \quad \swarrow
\]

\[
B
\]

(6)
such that \( f^{(2)} \) factors through \( \Pi \). Therefore (6) is a first order deformation of \( \pi \); moreover (6) is contained in \( \mathcal{P}^{-1}([JX]) \) by construction and therefore it is an element of \( \ker(d\text{Pr}_{[\pi]}) \).

Conversely, assume a family (6) given, and assume that (6) is contained in \( \ker(d\text{Pr}_{[\pi]}) \). Then we also have a family of triple covers of \( \mathbb{P}^1 \):

\[
\begin{array}{c}
\mathcal{C} \\
\downarrow
\end{array} \quad \rightarrow \quad \begin{array}{c}
\mathbb{P}^1 \times B \\
\downarrow
\end{array}
\begin{array}{c}
B \\
\leftarrow
\end{array}
\]

Correspondingly we have an inclusion \( \mathbb{P}^1 \times B \subset S^{(3)}_B(\mathcal{C}) \), and an étale morphism of degree 8

\[
\Pi^{(3)} : S^{(3)}_B(\tilde{\mathcal{C}}) \to S^{(3)}_B(\mathcal{C})
\]

induced by \( \Pi \). Let \( \mathcal{D} := \Pi^{(3)}(\mathbb{P}^1 \times B) \). The involution \( \iota \) on \( \tilde{\mathcal{C}} \) induces an involution on \( S^{(3)}_B(\tilde{\mathcal{C}}) \) which commutes with \( \Pi^{(3)} \) and induces an involution on \( \mathcal{D} \). We obtain a commutative diagram:

\[
\begin{array}{c}
\mathcal{D} \\
\downarrow
\end{array} \quad \rightarrow \quad \begin{array}{c}
\mathbb{P}^1 \times B \\
\downarrow
\end{array}
\begin{array}{c}
\mathcal{D}/\iota \\
\leftarrow
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
B \\
\leftarrow
\end{array}
\]

where the diagonal morphism defines a family of deformations of \( X \) with an assigned \( g^1_4 \) on the family. The assumption that (6) is contained in \( \ker(d\text{Pr}_{[\pi]}) \) means that the family of \( \text{ppav} \) obtained as Pryms of the family (6) is the family of jacobians of \( \mathcal{D}/\iota \to B \) and that it is trivial. By the infinitesimal Torelli theorem for jacobians \( \mathcal{D}/\iota \to B \) is the trivial family as well. Therefore diagram (7) gives us a family of deformations of \( \varphi \). \( \text{qed} \)

5. Examples

5.1 Let \( \bar{X} \subset \mathbb{P}^2 \) be an irreducible sextic having 4 distinct nodes \( N_1, \ldots, N_4 \), and let \( X \) be the normalization of \( \bar{X} \), which has genus 6. If no three among \( N_1, \ldots, N_4 \) are on a line then \( W^1_4(X) \) consists of five nonsingular points, the \( g^1_4 \)'s cut by the four pencils of lines through each of the nodes and by the pencil of conics containing \( N_1, \ldots, N_4 \). From Proposition 4.1 it then follows that the infinitesimal Torelli theorem holds at the five double covers \( \pi : \tilde{C} \to C \) of trigonal curves of genus 7 such that \( \mathcal{P}([\pi]) = [JX] \).

Assume now that \( \bar{X} \) has three of the nodes, say \( N_1, N_2, N_3 \), on a line. Then the \( g^1_4 \) defined by \( N_4 \) and that defined by the pencil of conics are identified to a unique element \( L \) of \( W^1_4(X) \) with a 1-dimensional Zariski tangent space. Applying Proposition 4.1 to this case we see that the infinitesimal Torelli theorem fails at the double cover \( \pi \) corresponding to \( (X, L) \) in the fibre \( \mathcal{P}^{-1}([JX]) \). Moreover, since equality (5) implies that \( \mathcal{P}^{-1}([JX]) \) is finite, from Theorem 3.2 we deduce that \( \tilde{C} \) is obstructed in \( JX \) and that the only component of \( \text{Hilb}^P \) containing \([\tilde{C}]\) is everywhere non reduced and consists of translates of \([\tilde{C}]\).
A count of parameters shows that in this way we get a 14-dimensional locus where the infinitesimal Torelli theorem fails inside the 18-dimensional space $\mathcal{R}_7$.

5.2 Let’s consider the Prym map $\mathcal{P} : \mathcal{R}_6 \to A_5$. This case has been extensively studied in [DS] and offers a wide variety of examples, but it is not yet completely understood from the point of view of the infinitesimal Torelli theorem. Recall that both domain and codomain are irreducible of dimension 15. Some loci where the infinitesimal Torelli theorem fails are the following.

5.2.1 - Consider a nonsingular curve $C \subset \mathbb{P}^4$ obtained as a general hyperplane section of a Reye congruence in $\mathbb{P}^5$, i.e. of an Enriques surface $S$ of degree 10 contained in a nonsingular quadric. Then $\tilde{C}$ is a curve of genus 6, embedded with a Prym canonical linear series $|\omega \otimes \eta|$; since $C$ is contained in a quadric it follows that the map $\nu$ is not surjective, and therefore the infinitesimal Torelli theorem fails at the double cover $\pi : \tilde{C} \to C$ associated to $\eta$.

Naranjo-Verra proved that the fibre $\mathcal{P}^{-1}(\pi)$ is discrete [NV]. Therefore from Theorem 3.2 it follows that $\text{Hilb}^P$ is obstructed at $\tilde{C}$.

Note that a count of parameters shows that the locus of double covers $\pi$ constructed in this way has dimension $14 = 9 + 5$ (9 for the moduli of Enriques surfaces and 5 for the hyperplane sections), i.e. it is a divisor in $\mathcal{R}_6$. In particular it follows that a general such curve $\tilde{C}$ is not trigonal since trigonal curves depend on 13 parameters.

5.2.2 - Consider an irreducible sextic $\tilde{C} \subset \mathbb{P}_2$ with four nodes such that two of its bitangents meet in one of the nodes, say $N$. Then the normalization $C$ has genus 6 and the $g_1^1$ defined by the pencil of lines through $N$ has two divisors of the form $2P + 2Q$. It follows that there is a 2-division point $\eta \in \text{Pic}(C)$ such that $\omega \otimes \eta$ is not very ample and the map $\nu$ is not surjective (use Lemma 2.2). Therefore the infinitesimal Torelli theorem fails at the double cover $\pi : \tilde{C} \to C$ associated to $\eta$. The locus in $\mathcal{R}_6$ defined by this family of examples is disjoint from the previous one because there the line bundles $\omega \otimes \eta$ were very ample. It is not clear to us what the dimensions of the fibres $\mathcal{P}^{-1}(\mathcal{P}(\pi))$ are and therefore whether $[\pi]$ is obstructed in this case.

5.2.3 - Another locus where the infinitesimal Torelli fails is $\mathcal{R}_{6,T}$, the locus of double covers of trigonal curves. It has dimension 13, and the restriction of $\mathcal{P}$ to $\mathcal{R}_{6,T}$ has general fibre of dimension 1, as it follows from Recillas’ Theorem recalling that $W_4^1(X)$ for a curve $X$ of genus 5 has dimension 1.

What is interesting here is that $W_4^1(X) = \Theta_{sing}$, the singular locus of the theta divisor of $JX$: for a general $X$ this is a nonsingular curve of genus 11 which has an involution $\iota$ with quotient a nonsingular plane quintic $C$. The double cover $\pi : W_4^1(X) \to C$ is associated to a 2-division point $\eta$ such that $\mathcal{O}(1) \otimes \eta$ is an even theta-characteristic (see [DS] for details). Moreover $\mathcal{P}(\pi) = [JX]$ again, by [M]. Therefore we see that for a general $X$ of genus 5 we have:

$$\mathcal{P}^{-1}([JX]) = W_4^1(X) \cup \{[\pi]\}$$

In particular the fibre of $\mathcal{P}$ is not equidimensional. Moreover $\nu$ is surjective at $[\pi]$ (see [DS], part II, §5) and therefore the curve $W_4^1(X) = \Theta_{sing}$ is unobstructed in $JX$. Note
that this gives an example of a double cover \( \pi \) of a curve of Clifford index 1 (namely a nonsingular plane quintic) at which the infinitesimal Torelli theorem holds.

Note also that, since \( \text{cork}(\nu) \) is 1-dimensional if \( \pi : \tilde{C} \to C \) is a double cover of a general trigonal curve of genus 6, we have that \( \text{cork}(\sigma) \) is 1-dimensional as well, by Prop. 2.3 (clearly a general trigonal \( C \) is not bielliptic). Therefore \( \delta \) has rank 1 and

\[
h^0(\tilde{C}, N\tilde{C}) = g
\]

With some extra effort one can easily show that in this case \( \tilde{C} \) is unobstructed in \( JX \). In fact consider a small (1-dimensional) neighborhood \( A \) of \( [\pi] \) in the fibre \( P^{-1}([JX]) \) and let

\[
\begin{align*}
\tilde{C} & \subset JX \times A \\
\downarrow & \\
A
\end{align*}
\]

be the corresponding 1-parameter family of deformations of \( \tilde{C} \) in \( JX \). Since this family has varying moduli, in the exact sequence (4) we have \( 0 \neq \delta(v) \in H^1(\tilde{C}, T\tilde{C}) \) if \( v \neq 0 \) is a tangent vector to \( A \) at the point \( a_0 \) parametrizing \( \tilde{C} \), and \( \delta(v) \) generates \( \text{Im}(\delta) \). Now consider a small neighborhood \( B \) of 0 in \( JX \) and build a new family:

\[
\begin{align*}
\tilde{C}' & \subset JX \times A \times B \\
\downarrow & \\
A \times B
\end{align*}
\]

whose fibre over \( (a,b) \) is the curve \( t_b^*(\tilde{C}_a) \), i.e. the translate by \( b \) of the fibre \( \tilde{C}_a \) of the family (8). It is clear that the characteristic map

\[
T_{A\times B,(a_0,0)} \to H^0(\tilde{C}, N\tilde{C})
\]

is an isomorphism, proving that \( \tilde{C} \) is unobstructed. Note that \( h^0(\tilde{C}, N\tilde{C}) > g - 1 \) in this case, and \( \tilde{C} \) has non trivial moduli.

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Mathematisches Institut
Bismarckstr. 1½, D-91054 Erlangen (Germany)
lange@mi.uni-erlangen.de

Dipartimento di Matematica, Università Roma Tre
L.go S.L. Murialdo 1, 00146 Roma (Italy)
sernesi@mat.uniroma3.it