Cosmological solutions with time-delay

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We introduce a time-delay function in bulk viscosity cosmology. Even for bulk viscosity functions where closed-form solutions are known, because of the time-delay term the exact solutions are lost. Therefore in order to study the cosmological evolution of the resulting models we perform a detail analysis of the stability of the critical points, which describe de Sitter solutions, by using Lindstedt's method. We find that for the stability of the critical points it depends also on the time-delay parameter, where a critical time-delay value is found which play the role of a bifurcation point. For time-delay values near to the critical value, the cosmological evolution has a periodic evolution, this oscillating behaviour is because of the time-delay function. We find a new behaviour near the exponential expansion point, which can be seen also as an alternative way to exit the exponential inflation.

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1. INTRODUCTION

I will discuss time-delay bulk viscosity cosmology. Time-delay in cosmological dynamics for the construction of periodic solutions was one of the last issues examined by the late John D. Barrow. This paper is dedicated to his memory.

The need to have a theoretically description of the cosmological observations, has motivated cosmologists the last decades to propose different gravitational models, for a re-

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Models with a running vacuum and a particle production have drawn the attention of the scientific society. The main idea behind these models is that there is production of particles as the universe is expanded because of the interaction between the gravitational field of the expanding universe and the quantum vacuum. In the case of a Friedmann–Lemaître–Robertson–Walker universe the particle production term is described by a bulk viscosity term for the cosmological fluid in the gravitational field equations [7–10]. In the running vacuum cosmological models the cosmological $\Lambda(t)$ term is introduced in the field equations which is an arbitrary function of time and provide a particle-like production mechanism for the cosmological fluid [11–14].

In a isotropic and homogeneous universe, bulk viscosity term is induced by a divergence of the velocity field for the cosmological fluid which introduce a effective pressure term for the cosmic fluid. Some first analysis of the bulk viscosity cosmology based on Eckart’s theory [15] can be found in [16, 17]. Bulk viscosity cosmology covers applications for the early universe and the late-time universe as well. Moreover, other cosmological models such as Chaplygin gas-like models [18–21] are included in the bulk viscosity.

Eckart’s theory is recovered by the first approximation of the causal theory of bulk viscosity described by Israel and Stewart formalism [22]. Israel-Stewart theory introduce additional degrees of freedom in the field equations and consequently, it can provide a better explanation on the physical variables of the observable universe and solves different problems of Eckart’s description, see the discussion in [23, 24]. Recently a new approach on the cosmological bulk viscosity proposed in [25], the interesting cosmological applications of that model is that reduces to zero viscosity in the case of vacuum or a stiff fluid, while for an ideal gas future singularities can be produced, for more applications of the latter approach we refer the reader in [26].

In this work we focus on the most simple bulk viscosity scenario described by Eckart’s formulation, where we assume that the viscosity term introduce a that time-delay function in the field equations. The resulting field equations in terms of the Hubble function for a system of first-order delay differential equations. Time-delay functions play a significant role in control theory and they are related with the finite time delay response of a stimulated by the action of an inducer. Applications of time-delay systems cover all areas of applied mathematics with emphasis in biological systems, for a review see [27]. An interesting discussion on the application of delay differential equation equations in relativistic physics
is given in [28].

The field equations are reduced in one nonlinear first-order time-delay differential equation. Exact solutions for time-delay differential equations are known only for linear time-delay equations. Therefore, we study the evolution of the nonlinear system near the critical points where for Eckart’s theory the critical points of the field equations describe de Sitter universes. We find that the evolution of the non time-delay system is recovered, while when the time-delay is greater or lower from a critical delay value then we have a new approach to de Sitter solution which is described by oscillations around the exponential expansion. The critical delay value is the bifurcation point of the dynamical system, while from time-delay near to the critical value the oscillations reach an amplitude which can be seen as constant for large time. Moreover, the stability of the de Sitter solutions is different in the presence of the time-delay function from the classical case, and the oscillating behaviour around a unstable de Sitter point can be seen as a new way to exit the inflationary era in the early universe. Cyclic cosmological solutions has been widely studied before, however previous studies are related with cyclic universes around a static Einstein universe [29–34]. See also the application of the averaging approach for the determination of periodic behaviours [35–37]. In [29], it has been proposed a cyclic cosmological model where the scale factor has an exponential growth in every cyclic. It was found that such approach solves various problems of the early universe such as the horizon, isotropy and flatness problems. The plan of the paper is as follows.

In Section 2 we give the basic properties and definitions of the bulk viscosity cosmology. The time-delay bulk viscosity term is introduced in Section 3 where Lindstedt’s method is applied [38] in order to study the stability and the evolution of the dynamical system near the critical solutions. For the time-delay functions of our consideration there are two families of critical points which describe the empty space, and the de Sitter universe. We find that the time-delay function affects only the evolution and the stability of the de Sitter point. In Section 4 we repeat our analysis by using dimensionless variables for the dynamical system from where are able to consider a more general bulk viscosity function. Finally in Section 5 we discuss our results and we extend our discussion by comparing our results with the simplest formulation of Israel-Stewart viscosity approach.
2. BULK VISCOSITY COSMOLOGY

In large scales the universe is described by a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime with line element

\[ ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right), \]  

(1)

where for the cosmological fluid source we assume the energy momentum tensor

\[ T_{\mu\nu} = \rho u_{\mu} u_{\nu} + (p + \eta) h_{\mu\nu} \]  

(2)

in which \( u^\mu = \delta^\mu_t \) is the comoving observer and \( h_{\mu\nu} = g_{\mu\nu} + u_{\mu} u_{\nu} \) is the projective tensor. Functions \( \rho \) and \( p \) describe the energy density and pressure of the perfect fluid while \( \eta = \eta(\rho) \) is the bulk viscosity term [39–44].

The introduction of the bulk viscosity term modify the Einstein field equations as follows

\[ 3H^2 = \rho \]  

(3)

\[ 2\dot{H} + 3H^2 = -p_{eff} \]  

(4)

where now \( p_{eff} \) is the effective pressure term given by the expression \( p_{eff} = p - \eta(t) \).

The bulk viscosity term can take various functional forms such that to describe other cosmological models as the Chaplygin gas and its modifications [45–49]. Chaplygin gas model is recovered when the perfect fluid is an ideal gas, i.e. \( p = (\gamma - 1) \rho \) and \( \eta = \eta_0 \rho - \lambda \). The later model has been proposed a unified dark matter theory, while it is possible to be applied in the early universe [50], and specifically in the description of inflation [18–21].

The conservation law \( T^\mu_{\mu\nu} = 0 \) has the following nonzero component

\[ \dot{\rho} + 3H(\rho + p) = 3H\eta. \]  

(5)

where we can see that the rhs of the conservation law introduce a particle creation/destroy term. Particle production process might be considered as another approach to explain different phases of the universe’s evolution and it has various applications in the early and late universe also [51–59].

We assume that the perfect fluid is an ideal gas and \( \eta(t) = \eta(H) \), then from (3) and (4) we end with the following first-order ordinary differential equation

\[ 2\dot{H} + 3\gamma H^2 - \eta(H) = 0. \]  

(6)
Equation (6) can be solved explicitly by quadratures, indeed the generic solution is
\[ \int \frac{2dH}{\eta(H) - 3\gamma H^2} = t - t_0 \] (7)

For specific forms of the bulk viscosity the differential equation (6) takes the form of some known equations form where we can write the analytic solution in a closed-form expression. For instance when \( \eta(H) \) is linear, then (6) takes the form of the Riccati first-order ODE, while when \( \eta(H) \) is a polynomial function of order three, equation (6) is the Abel equation.

Let \( H_0 \in \mathbb{R} \) be a zero of the function \( f(H_0) = 0 \), where
\[ f(H) = \frac{\eta(H) - 3\gamma H^2}{2} \] (8)

From a physical point of view, for \( H_0 \neq 0 \), the critical point describe a de Sitter universe while for \( H_0 = 0 \) the resulting spacetime is the empty Minkowski space. The critical point of the differential equation (6) will be an attractor when \( \frac{df}{dH}|_{H=H_0} < 0 \), or a saddle point when \( \frac{df}{dH}|_{H=H_0} > 0 \). Since function \( \eta(H) \) is a real function then there is not any possibility around the critical point to have a periodic behaviour. The question that arise is if it is possible to get a periodic behaviour for the first-order differential equation without assuming any complex function \( \eta(H) \). At this point it is important to mention that we refer to a periodic behaviour for the Hubble function and not for the scalar factor, thus the periodic behaviour does not refer necessary to a bounce cosmology, but in general to a cyclic universe.

In other natural sciences, such as biology, epidemiology or either in signal analysis, a mathematical approach to achieve such periodic behaviour around is thought the introduction of a time-delay term. Time delays reflects the time taken to detect the error signal and the time taken to act on it [27]. Applications of time-delay can be found also in astronomy and astrophysics, for instance see in [63, 64] and references therein.

We demonstrate how the introduction of a time-delay can introduce a periodic behaviour by considering the first-order linear delay ordinary differential equation
\[ \dot{x}(t) + \alpha x(t - T) = 0. \] (9)

In the special case where \( a = 1 \), \( T = \frac{\pi}{2} \) then we can see that a solution of (9) is \( x(t) = x_0 \cos(t) \).

The generic solution of equation (9) is given by \( x(t) = x_0 e^{\lambda t} \) where \( \lambda \) is a solution of the characteristic equation
\[ \lambda + \alpha e^{-\lambda T} = 0. \] (10)
The characteristic equation can provide complex values for $\lambda$ when $\alpha T > 0$, which means that $x(t)$ can be a periodic function.

3. **BULK VISCOSITY $\eta(H)$ WITH TIME-DELAY**

We make use of the free-function which describes the viscosity in order to introduce the time-delay in the cosmological studies. The delay parameter measures the time-difference between the compression or expansion of the fluid and the necessary time in order the equilibrium to be restored.

3.1. **Linear bulk viscosity**

Consider the simplest time-delay bulk viscosity function $\eta(H) = 2\eta_0 H(t - T)$ which introduces a delay term on the proposed model in [66]. Equation (6) becomes

$$2\dot{H}(t) + 3\gamma (H(t))^2 - 2\eta_0 H(t - T) = 0.$$  \hspace{1cm} (11)

Equation (11) admits two critical points, they are $H_A = 0$ and $H_B = \frac{2\eta_0}{3\gamma}$.

Point $H_A$ describes an empty universe, while $H_B$ denotes a de Sitter phase, which correspond to the de Sitter inflation [67, 68].

At the critical point $H_A$, the linearized equation is written as

$$\dot{H}(t) - \eta_0 H(t - T) = 0.$$ \hspace{1cm} (12)

Hence, according to the discussion we did before there is a periodic behaviour around $H_A$ iff $\eta_0 T < 0$.

We perform the change of variable $H(t) = y(t) + \frac{2\eta_0}{3\gamma}$, such that $H_B$ corresponds to $y_B = 0$. In the new variable, equation (11) is written

$$\dot{y} + \frac{3}{2}\gamma y^2 + 2\eta_0 y - \eta_0 y(t - T) = 0,$$ \hspace{1cm} (13)

where near to the critical point $y_B$ the later equation is linearized as follows

$$\dot{y} + 2\eta_0 y - \eta_0 y(t - T) = 0.$$ \hspace{1cm} (14)

Equation (14) admits an oscillating solutions when the parameters $\eta_0$ and $T$ satisfy the condition [65]

$$-\eta_0 T e^{2\eta_0 T} > e^{-1}.$$ \hspace{1cm} (15)
We infer that the later expression can not be true for any $\eta, T$, that is, there are not oscillations around the critical point $H_B$.

We demonstrate it by replacing $y(t) = A \cos(\omega t)$ in (14) from where find the algebraic equations

$$2 - \cos(\omega T) = 0, \quad \omega + \eta_0 \sin(\omega T) = 0$$

with solution

$$\omega^2 = -\sqrt{3} \quad \text{and} \quad T^2 = -\frac{\arccos 2}{\sqrt{3}}$$

Because $\omega$ is a pure imaginary number we conclude that there is not any oscillating behaviour around the critical point $H_B$.

The stability of the critical point $H_B$ can be determined easily from the linearized equation (14). Equation admits the exact solution $y(t) = y_0 e^{\lambda t}$ where $\lambda$ is a solution of the algebraic equation

$$\left(2 - e^{-\lambda T}\right) \eta_0 + \lambda = 0.$$ (18)

From the last expression we find that the point $H_B$ is a source when $\lambda < 0$, that is $\eta_0 T > 0$, $T > 0$, or $\eta_0 T < 0$, $T < 0$. We can see that there are differences in the stability when the time-delay is positive or negative.

Recall that for that specific model with linear bulk viscosity for the nondelay equation point $H_A$ is an attractor when $\eta_0 < 0$, while point $H_B$ is an attractor when $\eta_0 > 0$.

In equation (11) we perform the scale transformation $t = -\frac{\tau}{\omega \eta_0}$ and the rescale $H = \varepsilon h$ such that equation (11) to be written as follows

$$\omega \frac{dh}{d\tau} - \frac{3\gamma}{2\eta_0} \varepsilon (h(\tau))^2 + h(\tau + \omega \eta_0 T) = 0$$ (19)

where $\varepsilon << 1$ in order $H$ to take values near the critical point $H_A$.

Assume now that $-\eta_0 T = \frac{\pi}{2} + \varepsilon^2 \mu$, and $\omega = 1 + \varepsilon \omega_1 + ...$ then by performing a Taylor expansion of $h = h_0 + \varepsilon h_1 + ...$ we get

$$h_0' + h_0 \left(\tau - \frac{\pi}{2}\right) = 0$$ (20)

$$h_1' + h_1 \left(\tau - \frac{\pi}{2}\right) + \frac{3\gamma}{2\eta_0} h_0^2 = 0$$ (21)

$$h_2' + h_2 \left(\tau - \frac{\pi}{2}\right) + \omega_2 h_0' - \left(\frac{\pi}{2} \omega_2 + \mu\right) h_0 \left(\tau - \frac{\pi}{2}\right) - \frac{3\gamma}{\eta_0} h_0 h_1 = 0$$ (22)

$$...$$ (23)
From the two first equations we get

\[ h_0 (\tau) = A \cos \tau , \]  
\[ h_1 (\tau) = \frac{3\gamma}{4\eta_0} A^2 (1 + \cos (2\tau)) \]  

and by replacing in the differential equation for \( h_2 (\tau) \) and require the coefficients of \( \cos \tau \) and \( \sin \tau \) to be zero we get

\[ \omega_2 = 0 , \quad A^2 = -\left( \frac{2\eta_0}{3\gamma} \right)^2 \mu \]  

From where we infer that the critical point is stable when \( \mu < 0 \). That means that for \( -\eta_0 T < \frac{\pi}{2} \) the critical point is a source. The amplitude in the original coordinates is given by the expression

\[ A_H^2 = \left( \frac{2\eta_0}{3\gamma} \right)^2 \frac{\pi}{2} - \eta_0 T \].

The stability of the critical point is demonstrated in Fig. 1 where the evolution of \( H (\tau) \) is given around the critical point. We observe that for \( -\eta_0 T < \frac{\pi}{2} \) the solution oscillates around the Minkowski solution, while in the second case where \( -\eta_0 T > \frac{\pi}{2} \) the critical point is unstable. In the phase-space diagram for the solution near the critical point \( H_A \) is presented.

With the use of (24) and (25) the scale factor around the critical point is given by the following expression

\[ \ln \left( \frac{a (\tau)}{a_0} \right) = \frac{\eta_0}{6\gamma} \left( 4\sqrt{\frac{\pi}{2} - \eta_0 T} \sin \tau + \varepsilon \left( \frac{\pi}{2} - \eta_0 T \right) (2\tau + \sin (2\tau)) \right) + O (\varepsilon^2) \]  

from where it is clear that for \( -\eta_0 T < \frac{\pi}{2} \) the scale factor oscillates around the constant solution.

3.2. Bulk viscosity \( \eta (H) = -3\eta_1 H^2 (t - T) + 2\eta_0 H (t - T) \)

Consider now the bulk viscosity to be \( \eta (H) = -3\eta_1 H^2 (t - T) + 2\eta_0 H (t - T) \), \( \gamma + \eta_1 \neq 0 \), where now equation (6) becomes

\[ 2\dot{H} (t) + 3\gamma (H (t))^2 + 3\eta_1 H^2 (t - T) - 2\eta_0 H (t - T) = 0. \]  

When there is not any delay, i.e. \( T = 0 \), the later dynamical system can been as a linear bulk viscosity function \( \eta (H) \) for an ideal gas with equation of state parameter \( \bar{\gamma} = \gamma + \eta_1 \).
FIG. 1: Numerical simulation of equation (11) around the critical point $H_A = 0$. The simulation is for $\gamma = 1$ and $\eta_0 = 1$. Upper figures the evolution of $H(\tau)$ is presented, while in the lower figures the scale factor $\frac{a(\tau)}{a_0} = e^{\int H(\tau) d\tau}$ is given. Left Figs. are for delay $-\eta_0 T = \frac{\pi}{2} + 0.01$, and the critical point is unstable while right Figs. are for $-\eta_0 T = \frac{\pi}{2} - 0.01$ where the critical point is an attractor. The value $-\eta_0 T = \frac{\pi}{2}$ is a critical delay and a Hopf bifurcation point.

Thus there are two critical points, the static solution $\bar{H}_A = 0$ and the de Sitter point with $\bar{H}_B = \frac{2\eta_0}{3(\gamma + \eta_1)}$.

For the static solution, i.e. $\bar{H}_A$ the stability analysis is that for the linear bulk viscosity function, hence we omit it.

We focus on the stability analysis for the de Sitter point $\bar{H}_B$. We perform the change of
FIG. 2: Phase-space diagram for the solution of equation (11) near the critical point $H_A = 0$. The simulation is for $\gamma = 1$ and $\eta_0 = 1$. Left Fig. are for delay $-\eta_0 T = \frac{\pi}{2} + 0.01$, and the critical point is unstable while right Fig. is for $-\eta_0 T = \frac{\pi}{2} - 0.01$ where the critical point is an attractor.

variables $H(t) = \bar{H}_B + y(t)$ and equation (29) becomes

$$2 y' + \frac{4 \eta_0 \gamma}{\eta_1 + \gamma} y + 2 \eta_0 \left( \frac{\eta_1 - \gamma}{\eta_1 + \gamma} \right) y(t - T) + 3 \gamma y^2 + 3 \eta_1 y^2(t - T) = 0.$$  \hspace{1cm} (30)

Near to the critical point $y_B = 0$, for linearized system we replace $y(t) = A \cos(\omega t)$ from where we find

$$\cos(\omega T_{cr}) = -\frac{2 \gamma}{\eta_1 - \gamma}, \quad \sin(\omega T_{cr}) = \frac{(1 + \eta_1)}{\eta_0 (\eta_1 - 1)}$$  \hspace{1cm} (31)

where we find

$$\omega^2 = (\eta_1 - 3 \gamma) (\eta_1 + \gamma) \left( \frac{\eta_0 (\eta_1 - 1)}{(\gamma - \eta_1)^2 (\eta_1 + 1)} \right)^2,$$

$$T_{cr} = \frac{(\gamma - \eta_1) (\eta_1 + 1)}{2 \gamma (\eta_1 - 1)}.$$  \hspace{1cm} (32)

Therefore, at this case it is possible to have a periodic behaviour the de Sitter solution assuming that the following relation holds

$$(\eta_1 - 3 \gamma) (\eta_1 + \gamma) > 0.$$  \hspace{1cm} (33)

In order to study the stability of the solution we work as before and we do the change of variables $\tau = \Omega t$ such that equation (30) becomes

$$2 \Omega y' + \frac{4 \eta_0 \gamma}{\eta_1 + \gamma} y + 2 \eta_0 \left( \frac{\eta_1 - \gamma}{\eta_1 + \gamma} \right) y(t - \Omega T) + 3 \gamma y^2 + 3 \eta_1 y^2(t - \Omega T) = 0.$$  \hspace{1cm} (34)
Similarly with above we replace \( \Omega = \omega + \varepsilon^2 k_2 + \ldots \), \( y = \varepsilon y_0 + \varepsilon^2 y_1 + \ldots \) and \( T = T_{cr} + \varepsilon^2 \mu + \ldots \) in the later equation where we find that the amplitude \( A \) of the oscillations is expressed as follows
\[
A^2 = \frac{8\eta_0^4 (\eta_1 - \gamma) \omega^2}{9P(A)} \mu
\] (35)
and
\[
P(A) = (3\gamma^3 - \gamma \eta_1 + 2\eta_1^3) (\gamma + 2\eta_0 T_{cr} + \eta_1) \gamma \eta_0^3 + \eta_1^2 T_{cr} (\gamma + \eta_1)^3 \omega^4 \\
+ \eta_0 \eta_1 (\gamma + \eta_1) (\gamma^3 (3\eta_0 T_{cr} - 1) - 3\gamma \eta_1^2 (\eta_0 T_{cr} - 1) + 2\eta_0 \eta_1^3 T_{cr} + 2\eta_1 \gamma^2 (3\eta_0 T_{cr} + 1)) \omega^2.
\] (36)
Recall that for \( A^2 > 0 \) the critical point is an attractor, while when \( A^2 < 0 \), the critical point is a source.

4. EVOLUTION IN DIMENSIONLESS VARIABLES

We continue our analysis by considering the dimensionless variables
\[
\Omega = \frac{\rho}{3H^2}, \quad \Delta = \frac{\eta}{3\rho}.
\] (37)

Such variables are useful because the existence of ideal gas solutions, power-law solutions, can be investigated. For the bulk viscosity function we assume the case where \( \eta \sim \rho (t - T)^\nu \), the field equations are reduced to the following first order time-delay ordinary differential equation
\[
\Delta' = 3\Delta [3 (\nu \Delta (N - T) - \Delta) - (\nu - 1) \gamma]
\] (38)
where \( \Delta' = \frac{d\Delta}{dN} \), \( N = \ln a \), while from the first Friedmann equation \( \Omega = 1 \), and
\[
\frac{2\dot{H}}{3H^2} = (3\Delta - \gamma)
\] (39)
Equation (38) admits two critical points, point \( P_1 \), with \( \Delta = 0 \) and \( P_2 \) with \( \Delta = \frac{\gamma}{3} \).

The solution at \( P_1 \) describes a universe without the bulk viscosity term, while point \( P_2 \) a de Sitter universe where the bulk viscosity term contributes in the evolution of the universe.

As far as the stability of the critical point \( P_1 \) is concerned, we replace \( \Delta = \varepsilon \delta \) in (38) and we linearize around \( \varepsilon = 0 \), we find
\[
\delta' = 3 (1 - \nu) \gamma \delta (N),
\] (40)
from where can infer that the point is stable when \((1 - \nu) \gamma < 0\).

For \(P_2\) we perform the change of variable \(\Delta = \frac{\gamma}{3} + \varepsilon \delta\) with \(\varepsilon^2 = 0\), we find

\[
\delta' = 3\gamma (\delta (N) - \nu \delta (N - T)).
\] (41)

An exact solution of the latter equation is \(\delta (t) = \delta_0 \cos (\omega t)\), where

\[
\left(\frac{\omega}{3\gamma}\right)^2 = \nu^2 - 1, \quad \text{and} \quad T = T_{cr} = -\frac{1}{\omega} \arccos \frac{1}{\nu}.
\] (42)

Consequently, there is a periodic behaviour around the de Sitter point if and only if \(|\nu| > 1\). For \(\nu < 1\) the time-delay is positive, while for \(\nu > 1\) the time-delay parameter is negative in order a periodic behaviour to exist.

In order to study the stability of the critical point \(P_2\) we follow the same steps as above from where we find that the amplitude of the oscillations is given by the expression

\[
(\delta_0)^2 = -\frac{4\gamma^2 \mu \nu (4\nu^4 + 13\nu^2 - 8)}{\nu (\nu (8\nu^2 + 4\nu + 47) + 10) - 64} - 32.
\] (43)

where for \((\delta_0)^2 > 0\) the critical point is stable and \(\mu = T - T_{cr}\).

Easily we observe that for \(\nu = \frac{1}{2}\) in which \(\eta \sim H (N - T)\) there is not a a periodic behaviour around the de Sitter solution, a result in agreement with the analysis did above for the critical point \(H_B\).

Note that the Minkowski spacetime is not recovered, since \(\Delta\) is finite. That can be recovered by doing the change of variables. Since such solution is not of our interest we omit the analysis.

In Fig. 3 we plot the numerical evolution of equation (38) for \(\Delta = \frac{\gamma}{3} + \delta (N)\), for four different values of the delay parameter. Moreover, by using expression (39) we can write the equation of state parameter to be \(w = -1 + \delta (N)\), from where we observe that there are oscillations around the de Sitter solution. Which is another way to reach or exit the exponential inflation point in bulk viscosity theory [67, 68].

Function \(w (N)\) near to the critical point is approximated as follows

\[
w (N) = -1 + \varepsilon \delta_0 \cos (\omega N) + O (\varepsilon^2).
\] (44)

Equation of state parameters described by periodic functions studied before in [71] which were compared with the cosmological observation.
FIG. 3: Numerical simulation of equation (38) around the de Sitter point $\Delta = \frac{\gamma}{3} + \delta(N)$. The simulation is for $\gamma = 1$ and $\nu = -3$ where $T_{cr} = 0.23$. Upper figures are for $T < T_{cr}$ where the critical point is an attractor while lower figures are for $T > T_{cr}$ and the critical point is unstable.

5. CONCLUSION

In this work we introduced a time-delay in Eckart’s formulation of bulk viscosity cosmology. Time-delay has many applications in the optimal control theory for the applied sciences, where the delay describe the finite time-response of actuators which have been used in the implementation of the control law.

In the presence of the time-delay, the theory remains of second-order in terms of the scale factor or of first-order in terms of the Hubble function. Without the time-delay function such cosmological models describe singular solutions and a de Sitter evolution which correspond to
an exponential inflation era. However, when a small parameter is introduced the evolution of the cosmological model changes dramatically near the de Sitter solution. Specifically, because of the time-delay function the Hubble function has an oscillating behaviour around a constant value which correspond to the de Sitter universe. Such a behaviour can not be provided by a real bulk viscosity term in Eckart’s theory.

There are other treatments for the bulk viscosity theory, such is the Israel-Steward formalism, where the bulk viscosity term $\eta(t)$ satisfies the first-order ordinary differential equation \[ \tau \dot{\eta} + \eta = 3\xi H \] where $\tau$ is the relaxation time and it is assumed to be $\tau = \xi \rho^{-1}$ where $\xi$ is the bulk viscosity coefficient. Equation (45) can equivalently written as follows

\[
\dot{H} - Y = 0
\] (46)
\[
2\dot{Y} + 6\gamma HY + 3H^2 \left( (2Y + 3\gamma H^2) \xi^{-1} - H \right) = 0.
\] (47)

For $\xi = 3\gamma \xi_0 \rho^n$, the latter system admits the real critical point which describe the de Sitter solution $Y = 0$, $H_P = \xi_0^{-1/\gamma}$. The matrix of the linearized system near the critical point admits imaginary eigenvalues when $\kappa < -1 - \frac{1}{3\gamma^2} - 3\gamma^2$. That means that the point which describe the de Sitter solution is a spiral. However from phenomenological point of view parameter $\kappa$ should be positive \[39\], i.e. $\kappa > 0$, where there is not any oscillating behaviour. In contrary with the time-delay model studied before where oscillating behaviour was found also for positive values of $\kappa$.

In the full formulation of the Israel-Steward theory the bulk viscosity satisfies the ordinary differential equation

\[
\tau \dot{\eta} + \eta + \frac{1}{2} \left( 3 + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T} \right) \tau \eta = \xi H.
\] (48)

Because of the arbitrariness of the functions $\tau$ and $\xi$, different cosmological evolutions can be recovered \[69\]. The effects of a time-delay term in the full Israel-Steward theory is the subject of study of a future work.

In \[70\] inspired by the bulk viscosity cosmology, a new class of exact solutions of the field equations determined for a fluid with a equation of state parameter $\rho(t) + p(t) = \alpha \rho(t)^\nu$. 
The field equation reduce to the following first-order differential equation $\dot{H} + \bar{\alpha} H^{2\nu} = 0$ where $\bar{\alpha} = 2^{-1} 3^{\nu} \alpha$. The latter equation has the following solution

$$H (t) = (\bar{\alpha} (2\nu - 1) (t - t_0))^{\frac{1}{1 - 2\nu}} , \nu \neq \frac{1}{2}, \quad (49)$$

$$H (t) = e^{-\bar{\alpha} t} , \nu = \frac{1}{2}. \quad (50)$$

However, by introducing a time-delay term in the equation of state parameter, i.e. $\rho (t) + p (t) = \alpha \rho (t - T)^\nu$, the field equations becomes

$$\dot{H} + \bar{\alpha} H (t - T)^{2\nu} = 0 \quad (51)$$

where now exact solution exists only when $\nu = \frac{1}{2}$, when reduce to the linear equation (9).

Consequently, $H (t) = H_0 \cos (\omega t + \theta)$, that is,

$$a (t) = a_0 \exp \left( \frac{H_0}{\omega} \sin (\omega t + \theta) \right) \quad (52)$$

from where it follows, that there is a periodic behaviour around the Minkowski spacetime with radius $e^{H_0}$. Constants $\omega$ and $\theta$ are determined by (10).

We conclude that the introduction of the time-delay parameter provides new behaviours in the evolution of the cosmological model, which are of special interest. In the bulk viscosity model of consideration the time-delay provides periodic behaviour when we reach or going far from a de Sitter universe, the latter can be seen as an alternative way to escape the exponential inflationary era.

Moreover, a cyclic universe around an exponential growth was proposed recently in [6]. Indeed, such consideration can solve various problems of modern cosmology, from the homogeneity, isotropy and others. This work can be seen as a mathematical description in order to achieve the a similar universe with that proposed in [6]. However, in our model, the periodic oscillation decays for specific values of the time-delay. Specifically, the periodic is a stationary solution, however a singular behaviour for the scale factor can be recovered. Indeed, for large values of the Hubble function, equation (13) becomes $\dot{y} + \frac{3}{2} \gamma y^2 \simeq 0$, thus the scaling solution $H (t) \simeq \frac{1}{t}$ is recovered which describes a universe dominated by an ideal gas.

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