Partition of enstrophy between zonal and turbulent components

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Abstract

The partition of enstrophy between zonal (ordered) and wavy (turbulent) components of vorticity has been studied for the beta-plane model of two-dimensional barotropic flow. An analytic estimate of the minimum value for the zonal component has been derived. The energy, angular momentum, circulation, as well as the total enstrophy are invoked as constraints for the minimization of the zonal enstrophy. The corresponding variational principle has an interesting mathematical structure in that the target functional, the zonal enstrophy is not coercive with respect to the norm of enstrophy, by which the constraints work differently than usual variational principles. A discrete set of zonal enstrophy levels is generated by the energy constraint; each level is specified by an eigenvalue that represents the lamination period of zonal flow. However, the value itself of the zonal enstrophy level is a function of only angular momentum and circulation, being independent of the energy (and total enstrophy). Instead, the energy works in selecting the “level” (eigenvalue) of the relaxed state. Comparison with numerical simulations shows that the theory gives a proper estimate of the zonal enstrophy in the relaxed state.

1 Introduction

The creation of zonal flow in the planetary atmosphere is a spectacular example of the self-organization in physical systems [1]. The inverse-cascade model illustrates such a process. Because of the approximate two-dimensional geometry (due to the scale separation between the shallow vertical direction and wide horizontal directions), the vortex dynamics is free from the stretching effect. Then, the energy of flow velocity tends to accumulate into large-scale vortices —Fourier-transforming to the wave-number space, we observe that the energy cascades inversely toward small wave numbers [2]. On rotating planets, the
Coriolis force introduces latitude/longitude anisotropy. The energy of large-scale flow is then converted to a Rossby wave, resulting in zonal flow [3, 4, 5].

There is a strong analogy between the Rossby wave turbulence and the electrostatic turbulence of magnetized plasma in the plane perpendicular to an ambient magnetic field; this similarity manifests as the mathematical equivalence of the Charney equation of Rossby waves [6] and the Hasegawa-Mima equation of plasma drift waves [7]. Because the generation of zonal flow (coherent structure) affects the turbulent transport in magnetized plasmas, it is gaining new interest in the context of plasma confinement [8].

In parallel with simulation studies demonstrating the self-organization, there have been theoretical attempts to nail down the “target” of the spontaneous process. This can be done by formulating a variational principle with a target functional to be minimized (or maximized) under appropriate constraints. A well-known example is the entropy maximization in the microcanonical ensemble (the constraints are total particle number and total energy), which gives the Gibbs distribution. However, because of the essential non-equilibrium property of turbulence, the entropy is not an effective tool to dictate the process. The notion of selective dissipation guides us to find an appropriate target functional [1]. We begin by making a list of conservation laws that apply in the ideal (i.e. dissipation-less) limit. Adding finite dissipation breaks most of the conservation laws. However, there may be differences in fragility among the constants of motion. We choose the most fragile one as the target functional, and the others as (approximate) constraints. The Taylor state of magneto-fluid [9, 10] is the prototype of such a model of self-organization, which minimizes the magnetic field energy \( E = \frac{1}{2} \int |\mathbf{B}|^2 \, d^3 x = \frac{1}{2} \int |\nabla \times \mathbf{A}|^2 \, d^3 x \), where \( \mathbf{B} = \nabla \times \mathbf{A} \) is the magnetic field) under the constraint on the magnetic helicity \( H = \frac{1}{2} \int \mathbf{A} \cdot \mathbf{B} \, d^3 x \). The reason why \( E \) is more fragile than \( H \) is because \( E \) includes another differential operator curl in the integrand. When small-scale electromagnetic vortices (i.e. magnetic fields) are dissipated, \( E \) responds more sensitively. This model explains the relaxed states of magnetized plasmas in various systems, ranging from laboratory experiments to astronomical objects. A variety of similar models have been proposed for other self-organizing systems (see e.g. [1] for a review; see also Remark [1]).

However, the target of this study is different. Whereas we formulate a variational principle using the list of ideal conservation laws, the target functional is not such an ideal constant of motion. The aim is to estimate the minimum of the enstrophy given to the zonal component (which we call the zonal enstrophy). Our target functional is a part of the total enstrophy; the total is an ideal constant of motion, but the zonal part is not. If there is no constraint on partitioning, the zonal enstrophy may be minimized to zero. But some constraints prevent this to occur. We will identify the “key constraints” that determine the reasonable estimate of the zonal enstrophy.

An equivalent variational principle, the maximization of the complementary wavy enstrophy (= total enstrophy – zonal enstrophy) was first studied by Shepherd [11] with a different motivation, i.e. to estimate upper bounds on instabilities in nonlinear regime. The conservation of the pseudo-momentum
was invoked as the essential constraint. Improved estimates have been pro-
posed by taking into account more general set of invariants which are known
as Casimirs [12] (see Remark 1). However, we invoke a different constant of mo-
tion, the energy, as the principal constraint (in addition to other ones such as
angular momentum, which have been already studied). The reason is clear be-
cause the self-organization is a spontaneous process in which the redistribution
of the enstrophy between the zonal and wavy components can occur only if the
energetics admits. Moreover, the energy constraint imparts a mathematically
peculiar property to the variational principle, which is the other incentive of
this study.

In the next section, we will start by reviewing basic concepts of variational
principle. As mentioned above, the combination of the zonal enstrophy as the
target and the energy as the constraint constitutes a mathematically interesting
variational principle. Using some prototypical examples, we will explain the
non-triviality of our problem. Section 3 is devoted for basic formulation and
preliminaries. In Sec. 3.1 we will review the model equation. In Sec. 3.2 we
will give the list of the conservation laws pertinent to the vortex dynamics. In
Sec. 3.3 we will prepare mathematical tools for separating the zonal and wavy
components of vorticity. Section 4 describes the main results; we will derive
theoretical estimates of the minimum and maximum zonal enstrophy. We will
propose the notion of discrete zonal enstrophy levels (in analogy of energy levels
of quantum states); the relaxation into lower levels corresponds to the inverse
cascade, which is driven by the nonlinearity of vortex dynamics, and continues
until the linear Rossby wave term overcomes the nonlinear term. In Sec. 5 we
will compare the analytical results with numerical simulation. The conventional
Rhines scale will be revisited to give an accurate estimate of the relaxed zonal
enstrophy level. Section 7 concludes this paper.

Remark 1 (Casimir invariant) The minimization of the magnetic energy \( E \)
for a given magnetic helicity \( H \), was first studied by Wortjer [13] in a context
different from selective dissipation. When we chose energy as the target func-
tional, its stationary point is an equilibrium state. If we impose a constraint,
the actual range of dynamics is limited to a level-set (leaf) of the constraint
functional (constant of motion), on which the stationary point of the energy
gives a somewhat nontrivial equilibrium. Hamiltonian mechanics describes this
story in a rigorous formalism. The ideal magnetohydrodynamics (MHD) is cer-
tainly a Hamiltonian system, where the magnetic helicity is a Casimir invariant
(center of the Poisson algebra) [14, 15]. On the other hand, the energy (Hamil-
tonian) consists of the magnetic field energy, kinetic energy, and the thermal
energy. Minimizing the Hamiltonian as the target functional under the helicity
constraint yields the minimum \( E \) for a given \( H \), because the constraint \( H \)
only includes the magnetic field, so the kinetic energy and the thermal energy are
minimized to zero. Consequently, the minimizer of \( E \) for a given \( H \) is an equi-
librium state. Although introduced by a different argument, the Taylor relaxed
state is an equilibrium.
2 Preparation for variational principle

2.1 Target functional and constraint

To see the essence of the mathematical interest pertinent to our variational principle, let us start by recalling the basic relation between the target functional and constraint in a textbook example. The isoperimetric problem is to (1) maximize the surface area $S$ with a constraint on the periphery length $L$, or (2) minimize the peripheral length $L$ with a constraint on the surface area $S$. Both problems have the same solution, i.e. a circular disk or its periphery. Notice that reversing the target and constraint in each setting results in an ill-posed problem; one can make $L$ infinitely long without changing $S$, or one can make $S$ infinitely small without changing $L$. Let us concentrate on minimization problems. For a variational principle to be well-posed, the target ($L$) must be more “fragile” than the constraint ($S$). Here the fragility speaks of the sensitivity to small-scale perturbations. Suppose that we make pleats on a periphery; then $L$ is increased, but $S$ is not necessarily changed. In analytical formalism, a fragile functional includes a larger number of differentiations—derivatives are sensitive to small-scale perturbations. In the foregoing example, we may formally write

$$S = \int_{\mathbb{R}^2} \mathbb{I}_M d^2x, \quad L = \int_{\mathbb{R}^2} |\nabla \mathbb{I}_M| d^2x,$$

where $M$ is a simply-connected domain $\subset \mathbb{R}^2$ that should be optimized to minimize $L$ for some given value of $S$. We see that $L$ is more fragile including $\nabla$ in the integrand.

2.2 Coerciveness and continuity

To make the argument more precise, we introduce the notion of coercive functionals (cf. [16, 17]). Let $u$ be a real-valued function (state vector) belonging to a function space (phase space) $V$, which is a Banach space with a norm $\|u\|$. A real-valued functional $G(u)$ is said coercive, if

$$\|u\|^2 \leq c G(u),$$

where $c$ is some positive constant. On the other hand, a real-valued functional $H(u)$ is continuous, if

$$|H(u + \delta) - H(u)| \to 0 \quad (\|\delta\| \to 0).$$

We can formulate a well-posed minimization problem with a coercive target functional $G(u)$ and a continuous constraining functional $H(u)$ (we may also consider multiple constraints with continuous functionals).

To see how the coerciveness and continuity influence variational principles, let us consider an example with two functionals

$$G(u) = \int_M |\nabla u(x)|^2 d^n x, \quad H(u) = \int_M |u(x)|^2 d^n x,$$
where $u$ is a scalar function defined in a smoothly bounded open set $M \subset \mathbb{R}^n$. We assume that $u = 0$ on the boundary $\partial M$. Notice that $H(u)^{1/2}$ is the $L^2$ norm $\|u\|$. Therefore, $H(u)$ is a continuous functional on the function space $V = L^2(M)$. By the Poincaré inequality, we have

$$\|u\|^2 \leq c\|\nabla u\|^2 = cG(u)$$

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with a positive constant $c$. Therefore, $G(u)$ is a coercive functional.

First, we seek for a minimizer of $G(u)$ with the constraint $H(u) = 1$. This is a well-posed problem. The minimizer is found by the variational principle

$$\delta [G(u) - \lambda H(u)] = 0, \quad (3)$$

where $\lambda$ is a Lagrange multiplier. The Euler-Lagrange equation

$$-\Delta u = \lambda u,$$

together with the above-mentioned boundary condition, constitute an eigenvalue problem. We can easily show that every eigenvalue $\lambda$ is positive. Let $\lambda_j$ be an eigenvalue and $\varphi_j$ be the corresponding normalized eigenfunction ($\|\varphi_j\|^2 = 1$). With setting $u = a\varphi_j$, and demanding $H(u) = 1$, we obtain $a = 1$ and $G(u) = \lambda_j$. The smallest $\lambda_j$, then, yields the minimum $G(u)$.

The reversed problem of finding a minimizer of $H(u)$ with the restriction $G(u) = 1$ is ill-posed, because the constraint is posed by a functional $G(u)$ that is not continuous in the topology of $L^2(M)$. Let us elucidate the pathology. The variational principle $\delta [H(u) - \mu G(u)] = 0$ ($\mu$ is a Lagrange multiplier) yields the Euler-Lagrange equation $-\Delta u = \mu^{-1} u$. Let $\mu^{-1} = \lambda_j$ (an eigenvalue of $-\Delta$), and $u = a\varphi_j$. The condition $G(u) = 1$ yields $a = \lambda_j^{-1/2}$, and $H(u) = 1/\lambda_j$. Hence, the minimum of $H(u)$ is achieved by the largest eigenvalue that is unbounded, viz., $\inf H(u) = 0$ and the minimizer $\lim_{\lambda_j \to \infty} \lambda_j^{-1/2} \varphi_j = 0$ is nothing but the minimizer of $H(u)$ without any restriction. The constraint $G(u) = 1$ plays no role in this minimization problem.

### 2.3 Non-coercive target functional

Let us modify the target functional of (3) to a non-coercive functional. Let $V_k = \text{span} \{ \varphi_1, \cdots, \varphi_k \}$, which is a closed (finite-dimension) subspace of $V = L^2(M)$. We denote the orthogonal complement by $V'$, i.e. we decompose $V = V_k \oplus V'$. Let $P$ be the orthogonal projector $V \rightarrow V'$. Consider

$$G'(u) = \|\nabla(Pu)\|^2 = \int_M (-\Delta Pu)(Pu) \, d^n x = \sum_{j > k} \lambda_j(u, \varphi_j)^2,$$

where $(f, g) = \int_M f(\mathbf{x}) g(\mathbf{x}) \, d^n x$ is the inner product of $L^2(M)$. Evidently, $G'(u)$ is not coercive. The modified variational principle

$$\delta [G'(u) - \lambda H(u)] = 0 \quad (4)$$
yields the Euler-Lagrange equation that reads, after expanding with eigenfunctions,

\[ \lambda_j'(u, \varphi_j) = \lambda(u, \varphi_j) \quad (j = 1, 2, \ldots), \]

where the “modified eigenvalues” are

\[ \lambda_j' = \begin{cases} 
0 & (j = 1, \ldots, k), \\
\lambda_j & (j > k). 
\end{cases} \]

The minimizer of \( G'(u) \) is a solution of (5) such that \( \lambda = 0 \) and

\[ u = \sum_{j=1}^{k} a_j \varphi_j, \]

where constants \( a_1, \ldots, a_k \) can be arbitrarily chosen provided that \( \sum_{j=1}^{k} |a_j|^2 = 1 \) in order to satisfy the constraint \( H(u) = 1 \). We obtain \( \min G' = 0 \), but the minimizer is not a unique function.

This prototypical example elucidates the essence of the pathology in a variational principle with non-coercive target functional. We will encounter a similar non-uniqueness (degeneracy) problem in Sec. 5.1, where we seek the minimum of zonal enstrophy. Interestingly, however, we will find in Sec. 5.3 that the energy constraint brings about a dramatical change in the mathematical structure to remove the degeneracy.

3 Basic formulation and preliminaries

3.1 Vortex dynamics on a beta plane

We consider a barotropic fluid on a beta-plane

\[ M = \{ \mathbf{z} = (x, y)^T; x \in [0, 1), y \in (0, 1) \}. \]

Here, \( x \) is the azimuthal coordinate (longitude) and \( y \) is the meridional coordinate (latitude). Identifying the points \( (0, y)^T = (1, y)^T, \) \( M \) is a torus; all functions on \( M \) is periodic in \( x \). The boundary is \( \partial M = \Gamma_0 \cup \Gamma_1 \) with

\[ \Gamma_0 = \{ \mathbf{z} = (x, 0)^T; x \in [0, 1) \}, \quad \Gamma_1 = \{ \mathbf{z} = (x, 1)^T; x \in [0, 1) \}. \]

We will denote the standard \( L^2 \) inner product by \( \langle f, g \rangle \):

\[ \langle f, g \rangle = \int_M f(\mathbf{z}) g(\mathbf{z}) \, d^2z, \]

and the \( L^2 \) norm by \( \| f \| = (\langle f, f \rangle)^{1/2} \). We will also use the standard notation of the Sobolev spaces such as \( H^k \) and \( H^k_0 \).

The state vector is the fluid vorticity \( \omega \in L^2(M) \). We define the Gauss potential \( \phi \) by

\[ -\Delta \phi = \omega, \]

\[ \text{Sec. } 5.1, \text{ where we seek the minimum of zonal enstrophy. Interestingly, however, we will find in Sec. 5.3 that the energy constraint brings about a dramatical change in the mathematical structure to remove the degeneracy.} \]

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\[ -\Delta \phi = \omega, \]
where $\Delta = \partial_x^2 + \partial_y^2$. The flow velocity is given by
\[
v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \nabla \perp \phi = \begin{pmatrix} \partial_y \phi \\ -\partial_x \phi \end{pmatrix}.
\]
Adding a normal coordinate $z$, we embed $x$-$y$ plain in $\mathbb{R}^3$, and consider a 3-vector $\tilde{v} = (v_x, v_y, 0)^T$ such that $\partial_z \tilde{v} = 0$. Then, we may calculate $\nabla \times \tilde{v} = (0, 0, -\Delta \phi)^T = (0, 0, \omega)^T$, justifying that we call $\omega$ the vorticity.

To determine $\phi$ by (6), we impose a homogeneous Dirichlet boundary condition
\[
\phi |_{\Gamma_0} = \phi |_{\Gamma_1} = 0.
\]
Since $M$ is periodic in $x$, we have
\[
D\phi |_{x=0} = D\phi |_{x=1},
\]
where $D$ is an arbitrary linear operator. We note that (8) implies that the flow is confined in the domain (i.e. $n \cdot v |_{\partial M} = v_y |_{\partial M} = 0$; $n$ is the unit normal vector on $\partial M$), and has zero meridian flux:
\[
\int_0^1 v_x \, dy = \int_0^1 \partial_y \phi \, dy = \left[ \begin{array}{c} \phi \\ \gamma = 1 \end{array} \right]_{y=0}^{y=1} = 0.
\]
We note that a weaker boundary condition such that $\phi |_{\Gamma_0} = a$, $\phi |_{\Gamma_1} = b$ ($a$ and $b$ are some real constants) maintains $v_y |_{\Gamma_0} = v_y |_{\Gamma_1} = 0$, but allows a finite meridian flux (cf. Remark 3).

We define a Laplacian as an operator such that
\[
-\Delta : H_0^1(M) \cap H^2(M) \to L^2(M).
\]
Its unique inverse $K = (-\Delta)^{-1}$ is a compact self-adjoint operator, by which we can solve (6) for $\phi$. As $\phi = K \omega \in H_0^1(M)$, $\phi$ satisfies the boundary condition (8).

Taking into account the Coriolis force, the governing equation of $\omega$ is
\[
\partial_t \omega + \{ \omega + \beta y, \phi \} = 0,
\]
where $\{f, g\} = (\partial_x f)(\partial_y g) - (\partial_x g)(\partial_y f)$, and $\beta$ is a real constant number measuring the meridional variation of the Coriolis force (see Remark 2). When $\beta = 0$, (12) reduces into the standard vorticity equation. A finite $\beta$ introduces anisotropy to the system, resulting in creation of zonal flow.

Inverting (6) by $K = (-\Delta)^{-1}$, we may rewrite (12) as
\[
\partial_t \omega + \{ \omega + \beta y, K \omega \} = 0.
\]
We call
\[
\omega_t := \omega + \beta y
\]
the total vorticity, which is the sum of the fluid part $\omega$ and the ambient part $\beta y$ (the latter is due to the rotation of the system; see Remark 2).
The following identity will be useful in the later calculations:

$$\langle f, \{g, h\} \rangle = \langle g, \{h, f\} \rangle,$$

(15)

where \( f, g \) and \( h \) are \( C^1 \)-class functions in \( M \), and either \( f \) or \( g \) satisfy (8).

**Remark 2 (Euler’s equation with Coriolis force)** The vortex equation (12) is derived from Euler’s equation of incompressible \((\nabla \cdot \mathbf{v} = 0)\) inviscid flow with a Coriolis force:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + 2\mathbf{v} \times \boldsymbol{\Omega},$$

(16)

where \( \boldsymbol{\Omega} \) is the angular velocity of rotating frame, and \( p \) is the pressure of the fluid. Putting \( 2\boldsymbol{\Omega} = \beta y e_z \) (\( e_z \) is the unit vector normal to the \( x-y \) plane, which we will call the \( z \)-direction), assuming a two-dimensional flow (7), and operating curl on the both sides of (16), we obtain (12) from the \( z \)-component of the equation. Notice that the Coriolis force is directed perpendicular to \( \mathbf{v} \), so it does not change the energy of the flow; hence, Coriolis force resembles the Lorentz force \( \mathbf{v} \times \mathbf{B} \).

**3.2 Conservation laws and symmetries**

**Proposition 1 (constants of motion)** The following functionals are constants of motion of the evolution equation (13):

1. **Energy:**

$$E(\omega) := \frac{1}{2} \langle \omega, K \omega \rangle.$$  

(17)

By rewriting

$$E = \frac{1}{2} \langle (-\Delta \phi), \phi \rangle = \frac{1}{2} \int_M |\nabla \phi|^2 d^2 z = \frac{1}{2} \int_M |\nabla_{\perp} \phi|^2 d^2 z = \frac{1}{2} \int_M |\mathbf{v}|^2 d^2 z,$$

we find that \( E \) evaluates the kinetic energy of the flow \( \mathbf{v} \).

2. **Longitudinal momentum:**

$$P(\omega) := \int_M \partial_y (K \omega) d^2 z,$$

(18)

We may rewrite

$$P = \int_M \partial_y \phi d^2 z = \int_M v_x d^2 z$$

to see that \( P \) is the integral of the longitudinal momentum. By (10), \( P \) must be constantly zero.

3. **Circulation:**

$$F(\omega) := \langle 1, \omega \rangle.$$  

(19)

Integrating by parts, we may write

$$F = \int_0^1 \left[ v_x \right]_{y=0}^{y=1} dx,$$

which evaluates the circulation of the flow \( \mathbf{v} \) along the boundary \( \partial M \).
4. Angular momentum:

\[ L(\omega) := \langle y, \omega \rangle. \quad (20) \]

Integrating by parts and using the boundary conditions (8) and (9), we may rewrite

\[ L = \int_M y(\partial_x v_y - \partial_y v_x) \, d^2 z = \int_M v_x \, d^2 z - \int_0^1 \left[ yv_x \right]_{y=0}^{y=1} \, dx. \]

The first term on the right-hand side is \( P \), which vanishes by (10). Hence, \( L \) corresponds to the angular momentum \( z \times \mathbf{v} \) averaged over the boundary.

5. Generalized enstrophy:

\[ G_\beta(\omega) := \int_M f(\omega + \beta y) \, d^2 z, \quad (21) \]

where \( f \) is an arbitrary \( C^1 \)-class function, and the argument \( \omega + \beta y \) is the total vorticity including the ambient term \( \beta y \); see Remark 3. For \( f(u) = u^2/2 \), \( G_\beta(\omega) \) is the conventional enstrophy of the total vorticity.

6. Fluid enstrophy:

\[ Q(\omega) := \frac{1}{2} \| \omega \|^2. \quad (22) \]

(proof) While these conservation laws are well known, we give the proof to see how they originate. Suppose that \( \omega \) is a \( C^1 \)-class solution of (13). Rewriting (13) in terms of the total vorticity \( \omega_t = \omega + \beta y \), we have

\[ \partial_t \omega_t + \{ \omega_t, \phi \} = 0 \]

(1) Using the self-adjointness of \( \mathcal{K} \), we may calculate

\[ \frac{d}{dt} E = \langle \mathcal{K} \omega, \partial_t \omega \rangle = \langle \phi, \{ \phi, \omega_t \} \rangle = \langle \omega_t, \{ \phi, \phi \} \rangle = 0. \]

(2) To evaluate \( \frac{d}{dt} P = \int_M (\partial_t v_x) \, d^2 z \), we invoke the \( x \)-component of Euler's equation (16):

\[ \partial_t v_x = -v_x \partial_x v_x - v_y \partial_y v_x + \beta yv_y - \partial_x p. \]

Integrating by parts with the boundary conditions (8) and (9), we observe

\[ \frac{d}{dt} P = \int_M ( -v_x \partial_x v_x - v_y \partial_y v_x + \beta yv_y - \partial_x p ) \, d^2 z \]
\[ = \int_M [ -\partial_x (v_x^2 + p) + \beta yv_y ] \, d^2 z \]
\[ = -\beta \int_0^1 [ y\phi ]_{x=0}^{x=1} \, dy = 0. \]

To derive the second line, we have used \( \nabla \cdot \mathbf{v} = 0 \) to put \( \partial_y v_y = -\partial_x v_x \).
(3) Using (15), we obtain
\[ \frac{d}{dt} \langle 1, \omega_t \rangle = \langle 1, \{ \varphi, \omega_t \} \rangle = \langle \varphi, \{ \omega_t, 1 \} \rangle = 0. \]

(4) Similarly we obtain
\[
\frac{d}{dt} L = \langle y, \{ \varphi, \omega + \beta y \} \rangle = \langle y, \{ \varphi, \omega \} \rangle = \langle \varphi, \{ \omega, y \} \rangle = \int_M \varphi \partial_x \omega \, d^2 z = \int_M v_y \omega \, d^2 z = \frac{1}{2} \int_M \partial_x (v_y^2 - v_x^2) \, d^2 z = 0.
\]

(5) Using (15), we obtain
\[
\frac{d}{dt} G_\beta = \langle f'(\omega_t), \partial_t \omega_t \rangle = \langle f'(\omega_t), \{ \varphi, \omega_t \} \rangle = \langle \varphi, \{ \omega_t, f'(\omega_t) \} \rangle = 0.
\]

(6) The generalized enstrophy for \( f(\xi) = \xi^2/2 \) may be written as
\[
G_\beta(\omega) = \frac{1}{2} \| \omega + \beta y \|^2 = \frac{1}{2} \| \omega \|^2 + \beta \langle y, \omega \rangle + \frac{\beta^2}{6}
= Q(\omega) + \beta L(\omega) + \frac{\beta^2}{6}
\]
Since \( G_\beta(\omega) \) and \( L(\omega) \) are constants, \( Q(\omega) \) is also a constant.

\[ \square \]

Evidently, we have

**Lemma 1 (translational symmetry)** The constants of motion \( E, P, F, L, G_\beta, \) and \( W \) are invariant against the transformation
\[ T(\tau) : \omega(x, y) \mapsto \omega(x + \tau, y), \quad (\tau \in \mathbb{R}). \] (23)

**Remark 3 (Galilean symmetry)** Notice that \( P \equiv 0 \) is an immediate consequence of (10) that comes form the homogeneous Dirichlet boundary condition (8). However, in the proof of the constancy of \( P \) (Proposition 1 (3)), we used only \( v_y|_{r_0} = v_y|_{r_1} = 0 \), which may be guaranteed by a weaker boundary condition \( \phi|_{r_0} = a, \phi|_{r_1} = b \) \((a \text{ and } b \text{ are some real constants)}\). Hence, in a more general setting of boundary condition (or the definition of \( K \)), \( P \) may assume a general (non-zero) constant value. Then, a question arises: Does the homogeneous Dirichlet condition (8) violate the generality of vortex dynamics? The answer is no: The Galilean symmetry of the system subsumes the freedom of the foregoing \( a \) and \( b \). First, the transformation \( \phi \mapsto \phi - a \) does not change \( \nu = \nabla_\perp \phi \), so we may set a generalized boundary condition to be
$\phi|_{\Gamma_0} = 0$, $\phi|_{\Gamma_1} = c$. With $\phi_c := cy$, we decompose $\phi = \phi_0 + \phi_c$ so that $\phi_0$ satisfies the homogenized boundary condition (8). We have $\nabla x \phi_c = c\nabla x$, a constant velocity in the longitudinal direction, and $\omega = -\Delta \phi = -\Delta \phi_0$. Inserting this into (12), we obtain

$$\partial_t \omega + \{\omega + \beta y, \phi_0\} + c\{\omega, y\} = 0,$$

The distraction $c\{\omega, y\} = c\partial_x \omega$ can be cleared by Galilean boost $x \mapsto x - ct$. In the inertial frame, we may put $\phi_0 = K\omega$ to reproduce (13).

4 Zonal and wavy components

The phase space of the vorticity $\omega$ is

$$V = L^2(M).$$

We say that $\omega$ is zonal when $\partial_x \omega \equiv 0$ in $M$. The totality of zonal flows defines a closed subspace $V^z \subset V$. The zonal average

$$P^z \omega := \int_0^1 \omega(x, y) \, dx$$

may be regarded as a projection from $V$ onto $V^z$. By the orthogonal decomposition $V = V^z \oplus V^w$, we define the orthogonal complement $V^w$, i.e., $\omega_w \in V^w$, iff $\langle \omega_w, \omega \rangle = 0$ for all $\omega \in V^z$. We call $\omega_w \in V^w$ a wavy component, which has zero zonal average: $P^z \omega_w = 0$. We will denote

$$P^w = I - P^z,$$

which is the projector onto $V^w$. Now we may write

$$V = V^z \oplus V^w = (P^z V) \oplus (P^w V).$$

Being projectors, $P^z$ and $P^w$ satisfy $P^z P^z = P^z$, $P^w P^w = P^w$, and $P^z P^w = P^w P^z = 0$. We also have the following useful identity:

**Lemma 2 (commutativity)** Let $M$ be a beta-plane (which is periodic in $x$). For $\phi \in H^2(M)$, we have

$$P^z \Delta \phi = \Delta P^z \phi.$$  \hspace{1cm} (26)

For $\omega \in L^2(M)$, we have

$$P^z K \omega = K P^z \omega,$$  \hspace{1cm} (27)

$$P^w K \omega = K P^w \omega.$$  \hspace{1cm} (28)

(proof) By the periodicity in $x$, we may calculate as

$$P^z \Delta \phi = \int_0^1 (\partial_x^2 \phi + \partial_y^2 \phi) \, dx = \left[ \partial_x^2 \phi \right]_{x=0}^{x=1} + \partial_y^2 \int_0^1 \phi \, dx = \Delta P^z \phi.$$

Putting $\phi = K\omega$, (26) reads $-P^z \omega = \Delta P^z K \omega$. Operating $K$ on both sides yields (27). Using this, we obtain $P^w K \omega = (1 - P^z) K \omega = K (\omega - P^z) \omega = K P^w \omega$. 

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The following properties are useful:

**Lemma 3 (partition laws)** Let us decompose $\omega = \omega_z + \omega_w$ ($\omega_z = P_z \omega \in V_z$, $\omega_w = P_w \omega \in V_w$).

1. The circulation is occupied by the zonal component $\omega_z$, i.e.,
$$F(\omega) = F(\omega_z).$$
   (29)

2. The angular momentum is occupied by the zonal component $\omega_z$, i.e.,
$$L(\omega) = L(\omega_z).$$
   (30)

3. The fluid enstrophy is simply separated as
$$Q(\omega) = Q(\omega_z) + Q(\omega_w).$$
   (31)

4. The energy is simply separated as
$$E(\omega) = E(\omega_z) + E(\omega_w).$$
   (32)

(proof) The first three relations are clear. The energy partition (32) is due to
$$\langle \omega_z, K \omega_z \rangle = \langle \omega_w, K \omega_z \rangle = 0,$$
which follows from (27).

**Remark 4 (stationary state)** Evidently, $\partial_z (K \omega_z) = 0$ for $\omega_z \in V_z$. Hence, $\{\omega_z + \beta y, K \omega_z\} = 0$, implying that every member $\omega_z \in V_z$ is a stationary solution of (13).

5 Estimate of zonal enstrophy

5.1 Zonal enstrophy vs. wavy enstrophy

The aim of this work is to find the minimum of the zonal enstrophy defined by
$$Z(\omega) := \frac{1}{2} \| P_z \omega \|^2.$$  
(33)

The complementary wavy enstrophy is $W(\omega) = \frac{1}{2} \| P_w \omega \|^2$. By (31), the total enstrophy is
$$Q(\omega) = Q(P_z \omega) + Q(P_w \omega) = Z(\omega) + W(\omega).$$

When the total enstrophy $Q(\omega)$ is conserved (see Proposition 1(5)), the minimum of $Z(\omega)$ gives the maximum of $W(\omega)$.
The simplest version of the minimization problem is to find the minimum \( Z(\omega) \) under the constraint of \( Q(\omega) = C_Q \neq 0 \). Introducing a Lagrange multiplier \( \nu \), we minimize

\[ Z(\omega) - \nu Q(\omega). \]  

Using the self-adjointness of \( P_z \), we obtain the the Euler-Lagrange equation

\[ P_z \omega - \nu \omega = 0. \]  

Operating \( P_z \) on (35) yields

\[ (1 - \nu) P_z \omega = 0. \]

On the other hand, operating \( P_w \) yields

\[ \nu P_w \omega = 0. \]

There are two possibilities of solving these simultaneous equations.

1. \( \nu = 0 \): Then, \( P_z \omega = \omega_z = 0 \) and \( P_w \omega = \omega_w \) is an arbitrary function satisfying \( Q(\omega_w) = C_Q \); hence, \( \min Z(\omega) = 0 \).

2. \( \nu = 1 \): Then, \( P_w \omega = \omega_w = 0 \) and \( P_z \omega = \omega_z \) is an arbitrary function satisfying \( Q(\omega_z) = Z(\omega) = C_Q \); hence, this solution gives the “maximum” of \( Z(\omega) \).

This simple exercise reveals an unusual aspect of the present variational principle, which is caused by the non-coerciveness of the functional \( Z(\omega) \) to be minimized. Notice that the minimizer is not unique, because \( P_z \) has nontrivial kernel, i.e. \( \text{Ker}(P_z) = V_w \); every \( \omega_w \in V_w \) satisfies (35).

To obtain a nontrivial estimate of the minimum \( Z(\omega) \), we have to take into account “constraints” posed on the dynamics of redistributing enstrophy. Guided by Proposition, we start with some simple ones.

### 5.2 Constraints by circulation and angular momentum

Let us consider the circulation and angular momentum as constraints.

**Theorem 1** The minimizer of the zonal enstrophy \( Z(\omega) \) under the constraints on the circulation \( F(\omega) = C_F \), the angular momentum \( L(\omega) = C_L \), as well as the total enstrophy \( Q(\omega) = C_Q \) is a vorticity \( \omega \) such that

\[ P_z \omega = a + by, \quad (a = 4C_F - 6C_L, \ b = 12C_L - 6C_F), \]  

which gives

\[ Z_0 := \min Z(\omega) = 2C_F^2 - 6C_FC_L + 6C_L^2. \]
Let us minimize
\[ Z(\omega) - \nu Q(\omega) - \mu_0 F(\omega) - \mu_1 L(\omega). \] (38)

The Euler-Lagrange equation is
\[ \mathcal{P}_z \omega - \nu \omega = \mu_0 + \mu_1 y. \] (39)

Operating \( \mathcal{P}_z \) on both sides of (39) yields
\[ (1 - \nu) \mathcal{P}_z \omega = \mu_0 + \mu_1 y. \] (40)

On the other hand, operating \( \mathcal{P}_w \) yields
\[ \nu \mathcal{P}_w \omega = 0. \] (41)

First, assume that \( 1 - \nu \neq 0 \). Inserting \( \mathcal{P}_z \omega \) of (40) into the definition of \( F(\omega) = F(\mathcal{P}_z \omega) \) and \( L(\omega) = L(\mathcal{P}_z \omega) \) (see Lemma 3 (1) and (2)), we determine \( \mu_0 \) and \( \mu_1 \) of to match the constraint \( \langle 1, \omega \rangle = CF \) and \( \langle y, \omega \rangle = CL \); we obtain
\[ a \equiv \mu_0/(1 - \nu) = 4CF - 6CL \text{ and } b \equiv \mu_1/(1 - \nu) = 12CL - 6CF. \]

Inserting this \( \omega_z = a + by \) into \( Z(\omega) \), we obtain the minimum (37). On the other hand, (41) is satisfied by \( \nu = 0 \) (consistent with the forgoing assumption \( 1 - \nu \neq 0 \)) and an arbitrary \( \omega_w = \mathcal{P}_w \omega \) such that
\[ \frac{1}{2} \| \omega_w \|^2 = CQ - (2C_F^2 - 6CFCL + 6CL^2). \] (42)

The right-hand side is non-negative, if the constraints \( F(\omega) = CF, L(\omega) = CL \) and \( Q(\omega) = CQ \) are consistent. It is only when the constants \( CF, CL \) and \( CQ \) are given so that the right-hand side of (42) is zero, that the other assumption \( 1 - \nu = 0 \) applies; then, the unique solution \( \mathcal{P}_w \omega = 0 \) (hence, \( \omega = \mathcal{P}_z \omega \)) is obtained.

Notice that the minimizer is still non-unique (excepting the special case mentioned in the proof); every \( a + by + \omega_w \) \( \forall \omega_w \in V_w \) such that (42) holds satisfies (36). However, the minimum value (37) is uniquely determined.

### 5.3 Constraint by energy

The situation changes dramatically, when we include the energy constraint \( E(\omega) = CE \); laminated vorticity distribution, epitomizing the structure of zonal flow, is created by the energy constraint. The number of lamination (jet number) is identified by the “eigenvalue” of the Euler-Lagrange equation, which specifies the “level” of the zonal enstrophy (in analogy of the quantum number of discrete energy in quantum mechanics). To highlight the role of the energy constraint, we first omit the constraints on the circulation and angular momentum.
Taking into account the energy and total enstrophy constraint, we seek the critical points of
\[ Z(\omega) - \nu Q(\omega) - \mu_2 E(\omega). \]
The Euler-Lagrange equation is
\[ \mathcal{P}_z \omega - \nu \omega - \mu_2 \mathcal{K} \omega = 0. \quad (43) \]
Operating \( \mathcal{P}_z \) yields (denoting \( \omega_z = \mathcal{P}_z \omega \))
\[ \omega_z - \nu \omega_z - \mu_2 \mathcal{K} \omega_z = 0. \quad (44) \]
On the other hand, \( \omega_w = \mathcal{P}_w \omega \) must satisfy
\[ \nu \omega_w + \mu_2 \mathcal{K} \omega_w = 0. \quad (45) \]
Putting \( \omega_z = -\partial_y^2 \phi_z(y) \) in (44), we obtain
\[ \partial_y^2 \phi_z + \lambda^2 \phi_z = 0, \quad \lambda^2 = \frac{\mu_2}{1 - \nu}. \quad (46) \]
The solution satisfying the boundary conditions \( \phi_z(0) = \phi_z(1) = 0 \) is
\[ \phi_z = A \sin \lambda y \quad (47) \]
with eigenvalues
\[ \lambda = n_1 \pi \quad (n_1 \in \mathbb{Z}). \]
The corresponding zonal vorticity is
\[ \omega_z = A \lambda^2 \sin \lambda y. \quad (48) \]
On the other hand, putting \( \omega_w = -\Delta \phi_w \), \( \Delta \phi_w + k^2 \phi_w = 0, \quad k^2 = -\frac{\mu_2}{\nu}. \quad (49) \]
The solution satisfying the boundary conditions \( \phi_w(x, 0) = \phi_w(x, 1) = 0 \), as well as the periodicity in \( x \), is given by (as the equivalent class of the translational symmetry in \( x \); see Lemma\[1\])
\[ \phi_w = B \sin k_x x \sin k_y y, \quad k^2 = k_x^2 + k_y^2 \quad (50) \]
with eigenvalues
\[ k_x = 2n_2 \pi, \quad k_y = n_3 \pi \quad (n_2, \ n_3 \in \mathbb{Z}). \]
The corresponding wavy vorticity is
\[ \omega_w = B k^2 \sin k_x x \sin k_y y. \quad (51) \]
Summing the zonal and wavy components, we obtain
\[ \phi = A \sin \lambda y + B \sin k_x x \sin k_y y, \quad (52) \]
\[ \omega = A \lambda^2 \sin \lambda y + B k^2 \sin k_x x \sin k_y y. \] (53)

The two amplitudes \( A \) and \( B \) are determined by the constraints \( E(\omega) = C_E \) and \( Q(\omega) = C_Q \); inserting (52) and (53) into the definitions of \( E(\omega) \) and \( Q(\omega) \), we obtain

\[ C_E = \frac{A^2 \lambda^2}{4} + \frac{B^2 k^2}{8}, \] (54)

\[ C_Q = \frac{A^2 \lambda^4}{4} + \frac{B^2 k^4}{8}. \] (55)

Solving (54) and (55) for \( A \) and \( B \), and inserting the solution into the zonal enstrophy \( Z(\omega) \) and wavy enstrophy \( W(\omega) \), we obtain the critical values

\[ Z_{\lambda, \epsilon} = \frac{\lambda^2 C_E - \epsilon C_Q}{1 - \epsilon}, \] (56)

\[ W_{\lambda, \epsilon} = \frac{C_Q - \lambda^2 C_E}{1 - \epsilon}, \] (57)

where \( \epsilon = \frac{\lambda^2}{k^2} \), scaling the ratio of the wave length of the zonal components to that of the wavy components. For \( Z_{\lambda, \epsilon} \geq 0 \) and \( W_{\lambda, \epsilon} \geq 0 \), there are two possibilities: \( \epsilon \leq (\lambda^2 C_E)/C_Q \leq 1 \) or \( \epsilon \geq (\lambda^2 C_E)/C_Q \geq 1 \). Here, the former regime of \( \epsilon \) is relevant, because we assume that the wavy components have smaller scales in comparison with the zonal component (i.e. \( \epsilon < 1 \)). Then, \( Z_{\lambda, \epsilon} \) of (56) increases monotonically as \( \epsilon \) decreases (or \( k^2 \) increases; see Fig. 1), and we have

\[ \lim_{\epsilon \to 0} Z_{\lambda, \epsilon} = \lambda^2 C_E. \] (58)

Notice that this limit gives the upper bound for \( Z(\omega) \) of the corresponding eigenvalue \( \lambda \), which is achieved when the wavy component has the smallest scale \( \epsilon \to 0 \). For actual wavy components, \( Z(\omega) \) takes a smaller value than \( \lambda^2 C_E \), i.e.

\[ Z(\omega) \leq \lambda^2 C_E. \] (59)
Remembering the observation in Sec. 2, the target functional including higher order derivatives (therefore, more fragile) with respect to a constrained functional must find its “minimum” rather than the maximum. Here, what really minimizes is the wavy enstrophy $W(\omega)$; hence (59) gives the maximum of $C_Q - W(\omega) = Z(\omega)$. The wavy enstrophy is indeed the effective measure of “disorder” that is minimized in the self-organized state. Notice that the critical values of $Z(\omega)$ (or $W(\omega)$) are “quantized” by the eigenvalue $\lambda$ that measures the lamination period of the zonal flow. This enstrophy level is introduced by the energy constraint.

5.4 Constraints by energy, circulation, angular momentum and total enstrophy

Now we study the minimum of the zonal enstrophy $Z(\omega)$ under the all constraints of energy, circulation, angular momentum, and total enstrophy. In contrast to the observation of Sec. 5.3 (where the minimum of $Z(\omega)$ is not determined by the energy $C_E$), we will find that the minimum of $Z(\omega)$ is determined by the circulation $C_F$ and angular momentum $C_L$. In comparison with the result of Sec. 5.2 however, we have a discrete set of enstrophy levels (each of them corresponds to different lamination number of zonal flow). Whereas they are due to the energy constraint, $Z(\omega)$ itself does not depend on the values of the energy $C_E$.

Introducing Lagrange multipliers, we seek the minimizer of

$$Z(\omega) - \nu Q(\omega) - \mu_0 F(\omega) - \mu_1 L(\omega) - \mu_2 E(\omega).$$

The Euler-Lagrange equation is

$$P_z \omega - \nu \omega - \mu_0 - \mu_1 y - \mu_2 k \omega = 0. \quad (60)$$

The solution satisfying the boundary conditions $\phi(x, 0) = \phi(x, 1) = 0$, as well as the periodicity in $x$, is $\phi = \phi_z + \phi_w$ with

$$\begin{align*}
\phi_z &= A_1 \cos \lambda y + A_2 \sin \lambda y - \frac{(1 - \nu)(\mu_0 + \mu_1 y)}{\mu_2}, \\
\phi_w &= B \sin k_x x \sin k_y y,
\end{align*} \quad (61)$$

where

$$\lambda = \sqrt{\frac{\mu_2}{1 - \nu}}, \quad k^2 = k_x^2 + k_y^2 = -\frac{\mu_2}{\nu},$$

and

$$k_x = 2n_2 \pi, \quad k_y = n_3 \pi \quad (n_2, n_3 \in \mathbb{Z}).$$

The corresponding vorticities are

$$\begin{align*}
\omega_z &= A_1 \lambda^2 \cos \lambda y + A_2 \lambda^2 \sin \lambda y, \\
\omega_w &= B k^2 \sin k_x x \sin k_y y.
\end{align*} \quad (64)$$
The zonal enstrophy \( Z(\omega) \) of the minimizer is

\[
Z(\omega) = \frac{A_1^2\lambda^3}{8}(2\lambda + \sin 2\lambda) + \frac{A_2^2\lambda^3}{8}(2\lambda - \sin 2\lambda)
+ \frac{A_1 A_2 \lambda^3}{4}(1 - \cos 2\lambda).
\]  

(65)

We have yet to determine the eigenvalue \( \lambda \) and the coefficients \( A_1, A_2 \) and \( B \). Inserting \( \phi = \phi_z + \phi_w \) and \( \omega = \omega_z + \omega_w \) into the constraints \( F(\omega) = C_F, L(\omega) = C_L, E(\omega) = C_E, \) and \( Q(\omega) = C_Q \), we obtain

\[
\begin{align*}
C_F &= A_1 \lambda \sin \lambda + A_2 \lambda (1 - \cos \lambda), \\
C_L &= A_1 (\lambda \sin \lambda + \cos \lambda - 1) + A_2 (\sin \lambda - \lambda \cos \lambda), \\
C_E &= \frac{A_1^2 \lambda^3}{8}(2\lambda + \sin 2\lambda) + \frac{A_2^2 \lambda^3}{8}(2\lambda - \sin 2\lambda)
+ \frac{A_1 A_2 \lambda^3}{4}(1 - \cos 2\lambda) - \frac{A_1 C_F}{2}
+ \frac{B^2 k^2}{8}, \\
C_Q &= \frac{A_1^2 \lambda^3}{8}(2\lambda + \sin 2\lambda) + \frac{A_2^2 \lambda^3}{8}(2\lambda - \sin 2\lambda)
+ \frac{A_1 A_2 \lambda^3}{4}(1 - \cos 2\lambda) + \frac{B^2 k^4}{8}.
\end{align*}
\]  

(66, 67, 68, 69)

We may write (66) and (67) as

\[
\begin{pmatrix} C_F \\ C_L \end{pmatrix} = D(\lambda) \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},
\]

(70)

with

\[
D(\lambda) := \begin{pmatrix} \lambda \sin \lambda & \lambda(1 - \cos \lambda) \\ \lambda \sin \lambda + \cos \lambda - 1 \sin \lambda - \lambda \cos \lambda \end{pmatrix}.
\]  

(71)

For given \( C_F \) and \( C_L \), we solve (70) to determine the amplitudes of zonal vorticity:

\[
\begin{align*}
A_1 &= \frac{C_F(\sin \lambda - \lambda \cos \lambda) + C_L(-\lambda + \lambda \cos \lambda)}{\det D(\lambda)}, \\
A_2 &= \frac{C_F(-\lambda \sin \lambda - \cos \lambda + 1) + C_L \lambda \sin \lambda}{\det D(\lambda)},
\end{align*}
\]  

(72, 73)

where \( \det D(\lambda) = \lambda(2 - \lambda \sin \lambda - 2 \cos \lambda) \). Inserting (72) and (73) into (65), we obtain the zonal enstrophy evaluated as a function of \( \lambda \), which we denote by \( Z_\lambda \). The critical points (local minimums) of \( Z(\omega) \), given by

\[
\frac{d}{d\lambda} Z_\lambda = 0,
\]

(74)
determine the eigenvalues $\lambda$ characterizing the enstrophy levels.

Instead of displaying the lengthy expression of $Z_\lambda$, we will show its graphs for typical choices of the parameters $C_F$ and $C_L$. Notice that $Z_\lambda$ depends only on $C_F$ (circulation) and $C_L$ (angular momentum); it does not contain $C_E$ (energy) and $C_Q$ (enstrophy) as parameters. First, we pay attention to the singularities given by $\det D(\lambda) = 0$, where $A_1 \to \infty$ and $A_0 \to \infty$, hence $Z_\lambda \to \infty$ (there is an exception, as discussed later). We show the graph of $\det D(\lambda)$ in Fig. 2.

![Graph of det D](image)

Figure 2: The graph of $\det D$.

There are two types of solutions:

$$\lambda = \begin{cases} 
\Lambda_{2n} = 2n\pi, \\
\Lambda_{2n+1} = (2n + 1)\pi - \delta_n, 
\end{cases} \quad (n = 0, 1, \ldots),$$

where each $\delta_n$ is a small positive number such that $\delta_n \to 0$ as $n \to \infty$. The minimums of $Z_\lambda$ appear in every interval $(\Lambda_{2n}, \Lambda_{2n+1})$. However, if $C_F = 2C_L$, $Z_\lambda$ remains finite at $\lambda = \Lambda_{2n+1}$. In this special case, the minimums of $Z_\lambda$ appear in intervals $(\Lambda_{2n}, \Lambda_{2n+2})$.

In Fig. 3, we show examples of $Z_\lambda$ calculated for (left) $C_F = 0.21$ and $C_L = 0.0525$, (right) $C_F = 0.21$ and $C_L = 0.105$ ($C_F = 2C_L$).
At \( \lambda = 0 \), \( Z_\lambda \) reproduces the result of Theorem 1, i.e.

\[
\lim_{\lambda \to 0} Z_\lambda = Z_0 = 2C_F^2 - 6C_FC_L + 6C_L^2,
\]

which is the absolute minimum of the zonal enstrophy under the constraints on the circulation \( F(\omega) = C_F \), the angular momentum \( L(\omega) = C_L \), and the total enstrophy \( Q(\omega) = C_Q \).

The role of the energy constraint \( E(\omega) = C_E \) is to create eigenvalues of \( \lambda \) at which \( Z_\lambda \) takes local minimum values. However, the value of \( C_E \) does not influence the value of \( Z_\lambda \) directly. As we have seen in (59), it poses a constraint on the maximum:

\[
Z(\omega) \leq \lambda^2 C_E,
\]

in addition to the other implicit constraint \( Z(\omega) \leq C_Q \). Instead of the zonal component \( \omega_z \) of (63), \( C_E \) and \( C_Q \) work for determining the complementary wavy component \( \omega_w \) of (64). By (68) and (69), we obtain

\[
k^2 &= \frac{C_Q - Z_\lambda}{C_E - E_{z,\lambda}}, \tag{77}
\]
\[
B^2 &= \frac{8(C_E - E_{z,\lambda})^2}{C_Q - Z_\lambda}, \tag{78}
\]

where \( E_{z,\lambda} \) is the energy of the zonal component \( \omega_z \) evaluated at the eigenvalue \( \lambda \).

**Remark 5 (trivial constraints)** When \( C_F = 0 \) and \( C_L = 0 \), a laminated zonal flow \( (A_1 \neq 0 \text{ and/or } A_2 \neq 0) \) can occur only if

\[
\det D(\lambda) = \lambda(2 - \lambda \sin \lambda - 2 \cos \lambda) = 0.
\]

Then, the eigenvalues are \( \lambda = \Lambda_{2n} \) and \( \Lambda_{2n+1} \) \( (n = 0, 1, 2, \ldots) \), the previous singular points; see Fig. 2. For \( \lambda = \Lambda_{2n} \) \((\lambda = 0 \text{ gives the trivial solution } \omega_z = 0)\),
D(λ) = \( \begin{pmatrix} 0 & 0 \\ 0 & -λ \end{pmatrix} \),

hence, \( A_2 = 0 \). On the other hand, for \( λ = Λ_{2n+1} \),

\[
D(λ) = \frac{1}{4} λ \sin λ \begin{pmatrix} 4 & 2λ \\ 2 & λ \end{pmatrix},
\]

and then \( A_2 = -2A_1/Λ_{2n} \). In both cases, \( A_1 \) is arbitrary, so we cannot determine the amplitude of the zonal vorticity \( ω_z \). Therefore, the trivial conditions \( C_F = 0 \) and \( C_L = 0 \) reproduce the situation of “no-constraint” discussed in Sec. 5.3. We only have the estimate of the maximum (59).

The foregoing results are summarized as:

**Theorem 2** For a given set of constants \( F(ω) = C_F, L(ω) = C_L, E(ω) = C_E, \) and \( Q(ω) = C_Q \), the zonal enstrophy \( Z(ω) \) has a discrete set of critical (local minimum) values quantized by the eigenvalue \( λ \) measuring the lamination period of the zonal vorticity.

1. When \( C_F \neq 0 \) or \( C_L \neq 0 \), the eigenvalue \( λ \) is given by (74) as a function of \( C_F \) and \( C_L \). The corresponding eigenfunction \( ω_z \), and the critical value of \( Z(ω) \) are determined by \( C_F \) and \( C_L \); see (63), (65), (72) and (73). The other constants \( C_E \) and \( C_Q \) determine upper bounds \( C_Eλ^2 \geq Z(ω) \) and \( C_Q \geq Z(ω) \).

2. For the special values \( C_F = C_L = 0 \), additional eigenvalues \( λ = 2nπ \) and \( λ = Λ_n \ (n = 1, 2, \cdots) \) occur. However, the eigenfunctions \( ω_z \) and the critical values of \( Z(ω) \) are no longer determined by such \( C_F \) and \( C_L \); we only have estimates of upper bounds \( C_Eλ^2 \geq Z(ω) \) and \( C_Q \geq Z(ω) \).

**5.5 Determination of the zonal enstrophy level**

To apply Theorem 2 to the estimation of attainable zonal enstrophy, we have to determine the eigenvalue \( λ \) that identifies the zonal enstrophy level. Here, we suggest the following method (which we will examine and improve in Sec. 6).

The self-organization of zonal flow can be seen as a relaxation process of the zonal enstrophy level, which parallels the inverse cascade in the meridional wave number space. Just as the transition of the quantum energy level is caused by photon emission, the relaxation of the zonal enstrophy level is due to the emission of wavy vorticity, which is driven by the nonlinear coupling of the zonal and wavy components. Therefore, the relaxation can proceed as far as the nonlinear term \( \{ω, φ\} \) dominates the evolution equation (12). Relative to the concomitant linear term \( \{βy, φ\} \), the nonlinear term becomes weaker as the length scale increases (i.e., the inverse cascade proceeds). On the Rhines scale [3]

\[
L_R = \sqrt{\frac{2U}{β}}, \quad (79)
\]
the linear and nonlinear terms have comparable magnitudes, where \( U \) is the representative magnitude of the zonal flow velocity.

Since the energy is conserved, we may estimate \( U = \sqrt{2C_E} \). Hence, we have an *a priori* estimate

\[
\lambda \sim \frac{\pi}{L_R} = \pi \sqrt{\frac{\beta}{2\sqrt{2C_E}}}. 
\]

(80)

Notice the influence of the energy \( C_E \) on the eigenvalue \( \lambda \). Although each value of the zonal enstrophy level is independent to \( C_E \), the selection of the level is made by \( C_E \).

In the next section, we will examine the theoretical estimates by comparing numerical simulation results.

6 Comparison with numerical simulations

6.1 Simulation model

In this section, we compare the forgoing theoretical estimates with numerical simulation results. With a system size \( L \) and a rotation period \( T \), we normalize the variables as

\[
\tilde{x} = \frac{x}{L}, \quad \tilde{y} = \frac{y}{L}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{\omega} = \omega T, \quad \tilde{\phi} = \frac{\phi T}{L^2},
\]

(81)

by which the vorticity equation reads

\[
\partial_t \tilde{\omega} + \{\tilde{\omega} + \beta \tilde{y}, \tilde{\phi}\} = \nu \nabla \tilde{\omega},
\]

(82)

where \( \nu \) represents the viscosity (reciprocal Reynolds number). For simplicity, we will omit the normalization symbol \( \tilde{\cdot} \) in the following description. Whereas our theoretical analysis is based on the dissipation-free model (12), we add a finite viscosity \( \nu \) for numerical stability (typically, we put \( \nu = 1.0 \times 10^{-6} \)). A finite viscosity is also indispensable for the self-organization process, because the ideal (zero viscosity) dynamics is constrained by infinite number of Casimirs (local circulations), preventing changes in streamline topology. The theoretical model, however, ignores the dissipation by assuming the robustness of the invariants that are used as constraints (see Proposition\[\text{I}]. The influence of dissipation will be examined carefully when we compare the theory and numerical simulation.

In the following simulation, we assume parameters comparable to Jovian atmosphere; \( L = 4.4 \times 10^8 \text{m}, T = 8.6 \times 10^5 \text{sec} \). The parameter \( \beta \) is determined as

\[
\beta = \frac{2\Omega}{R} (\cos \theta) LT,
\]

where \( \Omega \) is the angular vorticity of rotating frame and \( R \) is the radius and \( \theta \) is latitude. For \( L \sim 2\pi R \) and \( \theta \sim 0 \), we obtain \( \beta \sim 10^2 \).
6.2 Self-organized zonal flow

As we have seen, the theoretical estimate of the minimum $Z(\omega)$ changes dramatically depending on whether $C_F$ and $C_L$ are finite or not (Sec. 5.4). First, we study the general case where both $C_F$ and $C_L$ are finite (the special case of $C_F = 0$ and $C_L = 0$ will be examined in Sec. 6.4). We assume an initial condition such that

$$\omega|_{t=0} = 5.0 \sin 15\pi y + \sum_{m,n} \alpha_{mn} e^{imx} \sin n\pi y,$$

with random $\alpha_{mn}(|\alpha_{mn}| \in [0, 50)$ for $5 \leq m, n \leq 10)$, which yields $C_E = 3.6 \times 10^{-2}$, $C_F = 0.21$ and $C_L = 0.11$.

In Fig. 4 we show the evolution of the “ideal” constants. Because of a finite viscosity ($\nu = 1.0 \times 10^{-6}$), the total enstrophy $C_Q$ changes significantly. But it is not essential for the present purpose of comparison, because the theoretical estimate of minimum $Z(\omega)$ is independent of the $C_Q$ (see Remark 6). The total energy $C_E$ is well conserved. The changes in $C_F$ and $C_L$ are also tolerable.

**Remark 6 (Total enstrophy)** As shown in Sec. 5.4, the total enstrophy $Q(\omega) = C_Q$ only implies a trivial upper bound $Z(\omega) \leq C_Q$. The difference $C_Q - Z(\omega)$ is the enstrophy given to the wavy component, if $C_Q$ is conserved. We may evaluate the amplitude of wavy component as (78) using $C_Q$, but all calculations pertinent to the zonal component are independent of $C_Q$. As the simulation shows, the “dissipation” of the total enstrophy is even the signature of the relaxation, when we consider a finite viscosity. We may interpret the dissipation as the scale separation between the visible scale and micro scale; the latter is separated from the vortex dynamics model by suppressing the amplitudes of micro-scale vortices. This scenario is consistent with the local interaction model; the nonlinear dynamics is dominated by interactions among similarly sized vortices (i.e., local in the Fourier space) within the inertial range, so it is not influenced by vortices of far smaller scales.
Figure 4: The evolution of the “ideal” constants in the simulation. Each value is normalized by the corresponding initial value.

Figure 5 shows the self-organized state \((t = 20)\), where an appreciable zonal component manifests. In Fig. 7, we compare the Fourier spectrum of the zonal component \(\omega_z = P_z \omega\) in the initial and self-organized states. We find the redistribution of the spectrum into lower \(\lambda\) modes (i.e., inverse cascade). A comparison with the Rhines scale will be described later.
Figure 5: Self-organization of zonal flow (gray level represents to the local value of $\omega$). (left) Initial condition with finite circulation $C_F = 0.21$ and angular momentum $C_L = 0.11$. (right) Creation of zonal flow observed at $t = 20$. 
Figure 6: The Fourier spectrum of the zonal vorticity $\omega_Z = P_z \omega$ in the self-organized state ($t = 20$). The eigenvalue $\lambda \sim 5\pi$ is dominant.

To make comparison with the theoretical estimate of zonal enstrophy, we plot $Z_\lambda$ (the theoretical minimum of zonal enstrophy) and $C_E \lambda^2$ (the theoretical maximum of zonal enstrophy), evaluated for the parameters determined by the given initial condition, in Fig. 6. As $\lambda = 5\pi$ is the dominant mode (Fig. 6), we obtain $Z_\lambda = 0.69$ and $C_E \lambda^2 = 8.8$. In Fig. 8 we compare the simulation result and the theoretical estimates, demonstrating that the actual zonal enstrophy $Z(\omega)$ stays between the theoretical minimum and maximum; the estimate of the lower bound is reasonably accurate.

Figure 7: The graphs of $Z_\lambda$ and $C_E \lambda^2$, evaluated for the parameters corresponding to the simulation of Fig. 5.
Figure 8: Evolution of the zonal enstrophy $Z(\omega)$, and its comparison with the theoretical minimum $Z_\lambda$ and the maximum $C_E \lambda^2$ evaluated for the self-organized state $\lambda \sim 5\pi$.

### 6.3 Improved Rhines scale

The foregoing discussion depends on the a posteriori estimate of the eigenvalue $\lambda$. As discussed in Sec. 5.5, however, we need an a priori estimate of $\lambda$ to make the theory useful. While the Rhines scale $L_R$ has been proposed to estimate $\lambda \sim \pi/L_R$, it turns out to be too crude. Here, we propose an improved Rhines scale to make more accurate estimate. Figure 9 compares the dominant scale in the final state obtained by simulation and the Rhines scale for different values of $\beta$. It is shown that the dominant scale is approximately 3 times of the Rhines scale.

The Rhines scale (79) is the length scale $L_R$ at which the magnitudes of the nonlinear term $\{\omega, \phi\}$ and the linear term $\beta \{y, \phi\}$ become comparable. However, it seems that the function of the nonlinear term, that derives the relaxation of the enstrophy level $\lambda$, does not end immediately at $L_R$; the numerical experiment shows that the relaxation continues up to $\sim 2\sqrt{2} \times L_R$, where the magnitude of the nonlinear term becomes about one eighth of the linear term. Therefore, we propose to use $L_R^* = 2\sqrt{2} L_R$ for the a priori estimate $\lambda = \pi/L_R^*$; modifying (80), we estimate

$$\lambda \sim \frac{\pi}{4} \sqrt[4]{\frac{\beta}{2C_E}}.$$  \hspace{1cm} (83)
Remark 7 (Rhines scale) Here we make a short survey of various debates about the Rhines scale. Two different categories must be distinguished; one is the unforced, free decaying turbulence, and the other is the forced, quasi-stationary turbulence. Our estimate of the improved Rhines scale is about the former case (because the aim of present study is to nail down the principal conservation laws that determine the zonal enstrophy, we study the robustness/fragility of the ideal constants; cf. Remark 6). For the latter case, one has to include some dissipation mechanism for large scale flows in order to remove the energy accumulating in the large scale regime by the inverse cascade. The usual viscosity only works for short scale flows, so something like “friction” is added to the model (however, which mechanism works in a realistic planetary system is still controversial). For the free decaying case, early simulation results [5, 18, 19] demonstrated the self-organization of zonal flow, and found that the scale of zonal flow has similar scaling with Rhines’ estimate. However, the quantitative comparison between the Rhines scale and the zonal flow scale was left unclear. Comparing the parameters of these earlier studies with those of present simulations (where we find more clear correlation with the Rhines scale; see Fig. 9), we find that the total kinetic energy in earlier studies was not large enough to obtain the clear scaling. On the other hand, in the forced turbulence case, more complex relation has been found, because of the influence of the dissipation mechanism for large scale flows; see [20, 21].

6.4 Degenerate case: $C_F = 0$ and $C_L = 0$

Finally, we examine the degenerate case of $C_F = 0$ and $C_L = 0$, where we cannot provide nontrivial estimate of the minimum zonal enstrophy. However,
we still observe self-organization of zonal flow, and the corresponding enstrophy satisfies the maximum condition.

Figure 10 shows the creation of zonal flow from an initial condition

\[
\omega|_{t=0} = \sum_{m,n} \alpha_{mn} e^{imx} \sin n\pi y,
\]

with random \(\alpha_{mn} (|\alpha_{mn}| \in [0, 50) \text{ for } 5 \leq m, n \leq 10)\) which is free from zonal component (\(\omega_z = 0 \text{ at } t = 0\)). The symmetry also yields \(C_F = 0 \text{ and } C_L = 0\), so that the special condition of Remark 5 applies. We only have a nontrivial estimate of the upper bound of \(Z(\omega)\).

In Fig. 11, we plot the evolution of the zonal enstrophy \(Z(\omega)\), and compare it with the theoretical maximum (59). Here we used \(\lambda = 5\pi \sim 1/L^*_R\) by the improved Rhines estimate; in Fig. 12 we show the Fourier spectrum of \(\omega_z\), which supports the choice. We observe that the time-asymptotic value of \(Z(\omega)\) stays below the upper bound \(C_F \lambda^2\).
Figure 10: Self-organization of zonal flow (gray level corresponds to the local value of $\omega$). (left) Initial condition with zero zonal component $\omega_z = 0$. (right) Creation of zonal flow observed at $t = 50$. 
7 Conclusion

We have found a discrete set of zonal enstrophy levels that are quantized by the eigenvalue $\lambda$ measuring the lamination period. Just as the quantum energy level of an orbital electron is lowered by photon emission, the relaxation of the zonal enstrophy level is caused by the emission of short-scale wavy vorticity (known as the forward cascade of enstrophy, resulting in the inverse cascade of the energy spectrum; cf. Remark 6), which continues down to the reciprocal improved Rhines scale $\lambda \sim \pi/L_R^*$. 
Comparing Theorems 1 and 2, we find that the energy constraint $E(\omega) = C_E$ plays an essential role in creating the discrete zonal enstrophy levels $Z_\lambda$. Interestingly, the value $C_E$ does not influence the value of each zonal enstrophy $Z_\lambda$, which is determined only by the other constants $C_F$ (circulation) and $C_L$ (angular momentum). Instead, $C_E$ works in selecting the eigenvalue $\lambda$ of the relaxed state (as well as posing the upper bound $Z(\omega) \leq C_E \lambda^2$). In absence of the energy constraint, we only have the “ground state” $\lambda = 0$ as given in Theorem 1. This unusual phenomenon in variational principle is cause by the co-coerciveness of the target functional $Z(\omega)$ with respect to the norm $\|\omega\|$. By comparing with simulation results, we verified that the theoretical value $Z_\lambda$ gives a reasonable estimate of the zonal enstrophy.

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