Small quartic planar graphs
that are not circle representable

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Abstract

A quartic planar graph \( G \) is said to be circle representable if there exists a collection of circles
drawn on the plane such that the kissing and crossing points correspond to the vertices of \( G \),
and the circular arcs between those points give the edges of \( G \). Lovász (1970) conjectured that
every quartic planar graph has a circle representation, but an infinite family of counterexamples
were given by Bekos and Raftopoulou (2015). We reduce the order of the smallest known
counterexamples among simple graphs from 822 to 68 based on a multigraph counterexample
of order 12.

1 Introduction

The graphs in this paper are simple, finite and undirected. When we wish to allow loops and
parallel edges, we will use the term multigraph. For a graph (multigraph) \( G \), let \( V(G) \) denote the
vertex set of \( G \) and \( E(G) \) denote the (multi)set of edges of \( G \).

We are interested in certain geometric representations of quartic planar graphs by circles. In
particular:

**Definition 1.1.** A *circle representation* of a graph \( G \) is a collection of circles embedded in \( \mathbb{R}^2 \)
such that at most two circles intersect at any point, the set of points belonging to two circles is
in bijective correspondence with \( V(G) \), and the circular arcs between those points correspond to
the edges of \( G \). A point in the intersection of two circles is called a *kissing* point if it is the only
point at which those circles intersect, and a *crossing* point if it is one of two points at which they
intersect.

Circle representations are closely related to the classical *coin representations* of graphs, which
are a collection of interior-disjoint circles in \( \mathbb{R}^2 \) such that the circles are in bijective correspondence
with \( V(G) \), and two vertices are adjacent in \( G \) if and only if their corresponding circles are tangent
to each other. An example of both types of representations is shown in Figure 1, which also

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illustrates the key observation made by Bekos and Raftopoulou that, roughly speaking, for a quartic planar graph $G$, a coin representation of its inverse medial graph (denoted $M^{-1}G$) gives a circle representation of $G$.

![Diagram](image1)

Figure 1: A coin representation of $M^{-1}G$ is a circle representation of $G$.

The remarkable fact that all simple planar graphs admit a coin representation is known as the Circle Packing Theorem, and was originally proved by Koebe [3]. In 1970, Lovász conjectured the analogous result for circle representations.

**Conjecture 1.2 (Lovász [2]).** Every quartic planar graph admits a circle representation.

Bekos and Raftopoulou [1] showed that the conjecture holds for 3-connected quartic planar graphs, and in the general case presented two infinite families of counterexamples, one of which consists of 2-connected graphs. The constructions for the smallest members of each of these families is depicted in Figure 2, by replacing each dotted line in (c) with a copy of either gadget (a) or (b), giving a total of 822 vertices.

![Diagram](image2)

Figure 2: Gadgets and base of the counterexample.

They also ask for the smallest counterexample to Lovász’ conjecture. As a step toward answering this question, we construct two counterexamples on 68 vertices, one of which is 2-connected. The structure of our counterexamples is very similar to those of Bekos and Raftopoulou, and is based
on the configuration along one edge of their base octahedron. In Section 2, we give a multigraph of
order 12 that is not circle representable. By attaching gadget subgraphs to this base multigraph,
we obtain simple quartic planar graphs with the same property, presented in Section 3.

2 A base multigraph

Although Lovász’ conjecture is usually taken as referring to simple graphs, the definition of a
circle representation applies without change to multigraphs. Working with multigraphs is actually
somewhat easier because there are a very limited number of ways in which a loop or digon can be
represented. We use this to our advantage in the following multigraph.

Theorem 2.1. The multigraph $M$ shown in Figure 3(a) is not circle representable.

![Figure 3: A non-simple counterexample](image)

The idea of the proof is to first show that any circle representation of this multigraph must have
a particular structure forced by the neighbouring digons. This corresponds to a particular config-
uration of circles, indicated in Figure 3(b). The problem is then reduced to showing geometrically
that this configuration is not realisable. We will handle the geometric aspect first.

Lemma 2.2. Suppose we have four circles $\{C_i(r_i, t_i)\}_{i=1,2,3,4}$ in the plane with radii $r_i > 0$ and
which are tangent to the $x$-axis at points $(t_i, 0)$ respectively, and assume the circles are numbered
so that $t_1 < t_2 < t_3 < t_4$. In addition, suppose that $C_1$ is tangent to $C_2$ and $C_4$, and $C_3$ is tangent
to $C_2$ and $C_4$. Let $N = t_4 - t_1$, $M = t_3 - t_2$, $L = t_2 - t_1$ and $R = t_4 - t_3$. Then:

(i) $MN = LR$.

(ii) $\{L, R\} = \{f^+(N, M), f^-(N, M)\}$, where $f^+(N, M) = \frac{N-M+\sqrt{(M-N)^2-4MN}}{2}$ and $f^-(N, M) = \frac{N-M-\sqrt{(M-N)^2-4MN}}{2}$ with $N > 0$ and $0 < M \leq (3 - 2\sqrt{2})N$.

(iii) $f^-(N, M)$ is increasing in $M$, and decreasing in $N$.

Proof. From Figure 4, we observe that

$$(t_4 - t_1)^2(t_3 - t_2)^2 = (4r_1r_4)(4r_2r_3) = (4r_1r_2)(4r_3r_4) = (t_2 - t_1)^2(t_4 - t_3)^2,$$

which implies (i).
Figure 4: Configuration of circles in Lemma 2.2.

For (ii) we can substitute \( L = N - M - R \), or symmetrically \( R = N - M - L \), into (i) to obtain an equation of the form \((N - M - X)X = MN\) where \( X \) can be \( L \) or \( R \). This has solutions \( X = f^{-}(N, M) \) and \( X = f^{+}(N, M) \), one of which is \( L \) whilst the other is \( R \). Note that we restrict the domain of the functions \( f^{-} \) and \( f^{+} \) so that \( N, M, L \) and \( R \) are all positive real values.

From the explicit expression for \( f^{-}(N, M) \), we compute

\[
\frac{\partial}{\partial M} f^{-} = -\frac{1}{2} \left( \frac{2(M - N) - 4N}{2\sqrt{(M - N)^2 - 4MN}} + 1 \right).
\]

The inequality \( \frac{\partial}{\partial M} f^{-} > 0 \) has as a solution \( M < (3 - 2\sqrt{2})N \) and \( N > 0 \). This implies the first statement of (iii). Similarly, one can verify that \( \frac{\partial}{\partial N} f^{-} < 0 \) on the specified interval to establish the second statement.

These basic results allow us to show the impossibility of the configuration depicted in Figure 5a, in which each numbered simple closed curve should be regarded as a circle. The configuration consists of two sets of circles, \( \{C_1, C_4, C_5, C_8\} \) and \( \{C_2, C_3, C_6, C_7\} \). As before, let circle \( C_i \) have radius \( r_i > 0 \) and be tangent to the \( x \)-axis at \((t_i, 0)\). The numbering of the circles shown in figure is chosen so that \( t_1 < t_2 < \cdots < t_8 \), and the eight red points represent all of the kissing points between pairs of circles. In particular, each set of circles satisfies the conditions of Lemma 2.2, and circles of different sets do not intersect.

**Lemma 2.3.** The configuration described above (Figure 5a) cannot be realised by circles.

**Proof.** By scaling and translating horizontally, we may assume that \( t_2 = 0 \) and \( t_7 = 1 \). Let \( M = t_6 - t_3, M' = t_5 - t_4 \) and \( N' = t_8 - t_1 \). Since \( t_3 < t_4 < t_5 < t_6 \) we know that \( M' < M \). Also, as \( t_1 < 0 \) and \( t_8 = t_1 + N' > 1 \) we must have \( N' > 1 \). Lemma 2.2(ii) tells us that the lengths \( t_4 - t_1 \) and \( t_8 - t_5 \) are given by \( \frac{1}{2}(N' - M' \pm \sqrt{(M' - N')^2 - 4M'N'}) \), so using our earlier notation the length of the shorter interval is \( f^{-}(N', M') \). Applying the second then first statement of Lemma 2.2(iii),
we find that
\[ f^{-}(N', M') < f^{-}(1, M') < f^{-}(1, M). \]
On the other hand, we know that \( f^{-}(1, M) = \min(t_3 - t_2, t_7 - t_6) \), so \( f^{-}(N', M') < t_3 - t_2 \) and \( f^{-}(N', M') < t_7 - t_6 \). Now if \( f^{-}(N', M') = t_4 - t_1 \), then we would have
\[ t_4 = t_1 + (t_4 - t_1) < 0 + f^{-}(N', M') < t_3 - t_2 = t_3, \]
which is a contradiction. Similarly, if \( f^{-}(N', M') = t_8 - t_5 \) then
\[ t_5 = t_8 - (t_8 - t_5) > 1 - f^{-}(N', M') > 1 - (t_7 - t_6) = t_6, \]
so we reach a contradiction in both cases.

**Corollary 2.4.** The configuration shown in Figure 5b cannot be realised by circles.

**Proof.** It is enough to show that it is possible to adjust the circles to create new points of intersection while preserving their order along the axis, so that we obtain the configuration in the preceding lemma (Figure 5a). Below the axis, fix \( C_2 \) and \( C_6 \) and replace \( C_3 \) and \( C_7 \) with two new circles \( C'_3 \) and \( C'_7 \) that are both tangent to \( C_2 \), \( C_6 \) and the line, as shown in Figure 6. Then \( r'_3 > r_3 \), which implies that \( t_6 - t_3 = 2\sqrt{r_3r_6} < 2\sqrt{r'_3r_6} < t_6 - t'_3 \), and hence \( t'_3 < t_3 < t_4 \). Also, we know that \( t_2 < t'_3 \) by choice of \( C'_3 \) being tangent to \( C_2 \) and \( C_6 \), so this replacement preserves the order of the circles. Similarly, we have \( r'_7 < r_7 \) from which we deduce that \( t_6 < t'_7 < t_7 < t_8 \). Since the inequalities are strict at each step, no circle above the axis is tangent to any circle below the axis. Doing the same thing above the axis, by fixing \( C_1 \) and \( C_8 \) and replacing \( C_1 \) and \( C_5 \), produces the desired configuration. Then we are done by Lemma 2.3. \( \square \)
Figure 6: Replacing $C_3$ and $C_7$ by the circles $C'_3$ and $C'_7$ with dashed edges.

We can now prove the main result of this section.

Proof of Theorem 2.1. We begin by making two observations. Firstly, any pair of digons sharing exactly one vertex must be realised by two circles that are tangent at that shared vertex. This is because if one of the digons is produced by crossing circles, then the edges of the neighbouring digon are realised by arcs of the same two circles, which is only possible if the two digons share both of their vertices. We also observe that if a pair of vertices that do not form a 2-cut are joined by exactly two parallel edges, then in any plane embedding, the parallel edges must appear consecutively in the cyclic ordering at both of those vertices.

Let \{v_i\}_{i=1,...,8} be the set of vertices in $M$ incident to only one digon, numbered so that $v_1v_2...v_8v_1$ is the cycle consisting of simple edges, and let $e_i$ be the edge directly after $v_i$ in this cycle. The first observation implies that any circle representation of $M$ must have one circle representing each digon. One of the circles on which $v_1$ lies therefore corresponds to a digon, so the other edges $e_1$ and $e_8$ must lie on the same circle. Furthermore, by the second observation it follows that $v_1$ is a kissing point. The same argument applies at each $v_i$ for $i = 1, 2, \ldots, 8$ so we find that $e_1, e_2, \ldots, e_8$ all lie on the same circle, and hence the cycle $e_1, e_2, \ldots, e_8$ must be represented by a single circle. In addition, all vertices have now been shown to be kissing points, so no circles cross. Now if a circle representation of $M$ exists, it must have one circle $C_0$ corresponding to the 8-cycle, and then 8 more circles $C_1, \ldots, C_8$ labelled so that $v_i$ is the tangent point of $C_i$ with $C_0$, and these points occur in the cyclic order around $C_0$. In addition, $(C_1, C_4)$, $(C_2, C_3)$, $(C_5, C_8)$ and $(C_6, C_7)$ are pairs of kissing circles, and no other circles kiss.

These tangency relationships are the same as those in the configuration in Figure 5b if we take $C_0$ to be the line via a Möbius transformation. Explicitly, if we view the circle representation on the sphere, rotate so that a point on $C_0$ strictly between $v_1$ and $v_8$ is at the North Pole and then apply a stereographic projection, then the resulting planar drawing satisfies the conditions of Corollary 2.4. As that configuration was not realisable by circles, no circle representation of $M$ can exist.


3 Small simple counterexamples

From our multigraph counterexample, we now proceed to construct simple counterexamples by subdividing to obtain a simple graph, and then planting certain gadgets at the degree-2 vertices to recover 4-regularity. The smallest counterexamples are obtained by attaching the small gadgets shown on the right of Figures 7(a) and (b), which are based on the graph of the octahedron. More generally though, for a quartic planar multigraph $G$ we define a gadget associated to either a cutvertex or minimal 2-cut as follows:

(a) For a cutvertex $a$ and component $C$ of $G - a$, the induced submultigraph $G[V(C) \cup \{a\}] \subset G$ is a gadget. There are therefore two possible gadgets for each cutvertex.

(b) For a minimal 2-cut $\{a, b\}$ such that $ab \in E(G)$, let $C$ be the component of $G - \{a, b\}$ with four edges incident with the cut. Then the induced submultigraph $G[V(C) \cup \{a, b\}] \subset G$ is a gadget. Note that for degree parity reasons, it is not possible for any side of a 2-cut of adjacent vertices in a quartic multigraph to have three edges incident with the cut, so each such cut gives rise to one gadget.

In Figure 7, the octahedral gadgets (right) are shown alongside their corresponding general versions (left). Here, the solid lines and vertices are gadget vertices and gadget edges, whilst the broken lines and uncoloured vertices will be described as non-gadget. The shaded region represents the component $C$, which is the rest of the gadget not explicitly drawn.

![Figure 7: General gadget subgraphs and octahedral gadgets](image_url)

We will show that gadgets can be detached in such a way as to preserve circle representability. In a circle representation, every edge is an arc of exactly one circle, so the edges along a circle represents a simple cycle in the multigraph. Let’s call such a cycle circular, and let $C_{xy}$ denote the unique circle that contains the edge $xy$. Geometrically, one should think of $C_{xy}$ as an honest circle, but we will use the same notation to denote the edge set of the circular cycle. If a gadget is associated to a cutvertex $a$, then a circular cycle that contains both gadget and non-gadget edges would have to contain $a$ twice (in order to ‘enter’ and ‘exit’ the gadget, if we orient the cycle), which is impossible. By similar reasoning, if a gadget is associated to a 2-cut of adjacent vertices, then any circular cycle containing both gadget and non-gadget edges must enter the gadget at one
of the vertices in the cut and leave at the other in order to avoid subcycles. That is, both vertices in the cut lie on the circle. We have thus established the following lemma.

**Lemma 3.1.** Suppose we have a circle representation of a quartic planar multigraph containing a gadget subgraph $H$.

(i) If $H$ is associated to a cutvertex, then every circle in the representation contains either gadget-edges only, or non-gadget edges only.

(ii) If $H$ is associated to a 2-cut of adjacent vertices, then every circle containing both gadget and non-gadget edges must contain both vertices in the cut.

**Theorem 3.2.** Let $G$ be a quartic planar multigraph with a specified gadget. Using the labelling in Figure 7, let $G'$ be the multigraph obtained from $G$ by removing the gadget vertices together with all incident edges, and adding a possibly parallel edge $vw$. If $G$ has a circle representation, then so does $G'$.

**Proof.** Suppose we have a circle representation of $G$ and a specified gadget $H_a$ associated to a cutvertex $a$. Then Lemma 3.1(i) tells us that every circle contains either gadget edges only, or non-gadget edges only. By deleting the circles containing only gadget edges, we are left with a circle representation of $G - H_a$ but with an additional edge between $v$ and $w$. This is precisely $G'$.

We now consider the case where the gadget is associated to a 2-cut of adjacent vertices, say $a$ and $b$. Observe that $\{va, bw\}$ is a 2-edge-cut, so $va$ is a bridge in $G - bw$ and therefore cannot lie on any cycle in $G - bw$. We know, however, that $va$ does lie on the circular cycle $C_{va}$ in $G$, so it follows that $bw$ must also lie on $C_{va}$. Hence, $C_{va} = C_{bw}$, and from now on we shall simply refer to this circle as $C$.

Applying Lemma 3.1(ii), any circle containing both gadget and non-gadget edges must be either $C$ or $C_{ab}$. If $C = C_{ab}$, then no circles contain both gadget and non-gadget edges. As in the previous case, deleting all arcs in the circle representation of $G$ that represent gadget edges gives us a circle representation of $G'$. If $C \neq C_{ab}$, then this same deletion nearly gives the required circle representation of $G'$, except that the Jordan curve formed by the arc representing the non-gadget edges of $C$ and the arc representing $ab$ is not a circle. This is easily fixed by transforming the latter arc into that needed to complete the circle $C$.

**Corollary 3.3.** The two multigraphs shown in Figure 8 do not have circle representations.

**Proof.** If either multigraph has a circle representation, then eight applications of Theorem 3.2, specifying each of the octahedral gadgets in turn, would give a circle representation of the base multigraph $M$ from Figure 3(a). This contradicts Theorem 2.1.
One can generate more counterexamples on 68 vertices by using combinations of these two octahedral gadgets, and infinitely many larger counterexamples by attaching larger gadgets to the same base multigraph. We conjecture that all simple quartic planar graphs of order less than 68 are circle representable, and that all quartic planar multigraphs of order less than 12 are circle representable.

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