Coded Caching for Combination Networks with Multiaccess

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Abstract

In a traditional \((H, r)\) combination network, each user is connected to a unique set of \(r\) relays. However, few research efforts to consider \((H, r, u)\) multiaccess combination network problem where each \(u\) users are connected to a unique set of \(r\) relays. A naive strategy to obtain a coded caching scheme for \((H, r, u)\) multiaccess combination network is by \(u\) times repeated application of a coded caching scheme for a traditional \((H, r)\) combination network. Obviously, the transmission load for each relay of this trivial scheme is exactly \(u\) times that of the original scheme, which implies that as the number of users multiplies, the transmission load for each relay will also multiply. Therefore, it is very meaningful to design a coded caching scheme for \((H, r, u)\) multiaccess combination network with lower transmission load for each relay. In this paper, by directly applying the well known coding method (proposed by Zewail and Yener) for \((H, r)\) combination network, a coded caching scheme (ZY scheme) for \((H, r, u)\) multiaccess combination network is obtained. However, the subpacketization of this scheme has exponential order with the number of users, which leads to a high implementation complexity. In order to reduce the subpacketization, a direct construction of a coded caching scheme for \((H, r, u)\) multiaccess combination network is proposed by means of Combinational Design Theory, where the parameter \(u\) must be a combinatorial number. For arbitrary parameter \(u\), a hybrid construction of a coded caching scheme for \((H, r, u)\) multiaccess combination network is proposed based on our direct construction. Theoretical and numerical analysis show that our last two schemes have smaller transmission load for each relay compared with the trivial scheme, and have much lower subpacketization compared with ZY scheme.

Index Terms

Coded caching, Placement delivery array, Combination networks, Multiaccess.

I. INTRODUCTION

Coded caching proposed by Maddah-Ali and Niesen (MN) in [1] not only utilizes the users’ cache memories in shifting some of the network traffic to off-peak hours, but also creates multicast opportunities that further reduce network congestion during peak traffic hours. The first coded caching system studied is the following shared-link broadcast network with end-user-caches: there exists a single sever with \(N\) files with the same length connecting to \(K\) users through a shared error-free broadcast link, where each user has a cache of size \(M\) files. An \(F\)-division \((K, M, N)\) coded caching scheme consists of two phases: placement phase during the off-peak traffic times and delivery phase during peak traffic times. In the placement phase, the server divides each file into \(F\) packets (\(F\) is referred to as the subpacketization) with the same length, and then places some packets of size \(M\) files in each user’s cache without knowledge of later users’ demands. If the server directly places some packets in each user’s cache, without coding, the placement is said to be uncoded. Otherwise, the placement is called coded. In the delivery phase, each user’s demand arrives at the server. According to the users’ demands, the server transmits coded packets (XOR of the required packets) in order to satisfy all users’ demands with the help of their caches. The maximum normalized transmission amount over all possible demands is defined as the transmission load \(R\), and the coded caching gain is defined as \(\frac{K\left(1-\frac{M}{N}\right)}{R}\), where \(K\left(1-\frac{M}{N}\right)\) is the transmission load of the conventional uncoded caching scheme. The goal is to design a coded caching scheme with transmission load \(R\) and the subpacketization \(F\) as small as possible due to the requirements of efficiency of transmission and low implementation complexity. The above coded caching model is called a \((K, M, N)\) caching or MN caching system, for which Maddah-Ali and Niesen [1] proposed the first coded caching scheme (called MN scheme), which is optimal under the constraint of uncoded placement and \(K \leq N\) [2]. Yan et al. [3] proposed a combinatorial structure called placement delivery array (PDA) to design a coded caching scheme for MN caching system. It is worth noting that MN scheme can be represented by a special PDA which is called the MN PDA.

A. Traditional combination network caching system

Compared with the MN caching system, a more practical system, where users may communicate with the server through intermediate relays, has gained attention. Since the analysis of relay networks with arbitrary topologies is challenging, many
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worth noting that Scheme B not only has the same coded caching gain among the users who are connected to different relay subsets containing at least one common relay as Scheme A, but also fully considers the coded caching gain among the users who are connected to the same relay by increasing some subpacketization.

The rest of this paper is organized as follows. In Section II, the traditional combination network caching system is extended to the multiaccess combination network caching system, and the concepts of PDA and CPDA are briefly reviewed. In Section III, three schemes for the multiaccess combination network are proposed. In Section IV, performance analysis is provided. Finally, conclusion is drawn in Section V while some proofs are provided in the Appendices.

II. PRELIMINARIES

In this section, the multiaccess combination network caching system and the concepts of PDA and CPDA are introduced. First, the following notations are useful in this paper.

- \([a : b] := \{a, a+1, \ldots, b\}\), \([a] := \{1, 2, \ldots, a\}\).
- \(\text{mod}(a, q)\) denotes the least non-negative residue of \(a\) modulo \(q\). \(< a > := \text{mod}(a, q)\) if \(\text{mod}(a, q) \neq 0\), otherwise \(< a > := q\).
- For any set \(\mathcal{H}\) and for any positive integer \(r\) with \(r < |\mathcal{H}|\), \(\binom{\mathcal{H}}{r} := \{A | A \subseteq \mathcal{H}, |A| = r\}\), i.e., \(\binom{\mathcal{H}}{r}\) is the collection of \(r\)-sized subsets of \(\mathcal{H}\), where \(|\cdot|\) denotes the cardinality of a set.
- Given an array \(P = (p_{j,k})_{j \in [F], k \in [K]}\) with alphabet \([S] \cup \{\ast\}\), we define \(P + a = (p_{j,k} + a)_{j \in [F], k \in [K]}\) and \(P \times a = (p_{j,k} \times a)_{j \in [F], k \in [K]}\) for any integer \(a\), where \(a + \ast = \ast, a \times \ast = \ast\).
- For any two vectors \(x, y\) with the same length, \(d(x, y)\) is the number of coordinates in which \(x\) and \(y\) differ, \(wt(x)\) is the weight of \(x\), i.e., the number of nonzero coordinates of \(x\). For example, if \(x = (0, 1, 2, 1)\) and \(y = (0, 1, 3, 4)\), then \(d(x, y) = 2\) and \(wt(x) = 3\).
- For any vector \(f\) with length \(H\) and for any nonempty subset \(T \subseteq [H]\), \(f|T\) is a vector with length \(|T|\) obtained by taking only the coordinates with subscript \(j \in T\). For example, if \(f = (0, 2, 3, 1)\) and \(T = \{1, 3\}\), then \(f|T = (0, 3)\).

A. Placement Delivery Array

Yan et al. [3] proposed a combinatorial structure, called placement delivery array (PDA), to characterize the placement phase and delivery phase of a scheme for the MN caching system simultaneously.

**Definition 1:** (PDA, [3]) For positive integers \(K, F, Z, S\) and \(S\), an \(F \times K\) array \(P = (p_{j,k})\), \(j \in [F], k \in [K]\), composed of a specific symbol "\(\ast\)" called star and \(S\) integers in \([S]\), is called a \((K, F, Z, S)\) placement delivery array (PDA) if it satisfies the following conditions:

C1. Each column has exactly \(Z\) stars.

C2. Each integer in \([S]\) occurs at lease once.

C3. For any two distinct entries \(p_{j1,k1}\) and \(p_{j2,k2}\), \(p_{j1,k1} = p_{j2,k2} = s\) is an integer only if
   a. \(j_1 \neq j_2, k_1 \neq k_2\), i.e., they lie in distinct rows and distinct columns; and
   b. \(p_{j1,k2} = p_{j2,k1} = \ast\), i.e., the corresponding \(2 \times 2\) subarray formed by rows \(j_1, j_2\) and columns \(k_1, k_2\) must be of the following form

   \[
   \begin{pmatrix}
   s & \ast \\
   \ast & s
   \end{pmatrix}
   \]

   or

   \[
   \begin{pmatrix}
   \ast & s \\
   * & \ast
   \end{pmatrix}
   \]

**Example 1:** It is easy to verify that the following array is a \((6, 4, 2, 4)\) PDA,

\[
P = \begin{pmatrix}
\ast & \ast & 1 & 2 & 3 \\
\ast & 1 & 2 & \ast & 4 \\
1 & \ast & 3 & 4 & \ast \\
2 & 3 & 4 & \ast & \ast
\end{pmatrix}
\]

The first PDA was proposed by Maddah-Ali and Niesen in [1], [3].

**Lemma 1:** (MN PDA, MN scheme [1], [3]) For any positive integers \(K, M\) and \(N\) with \(M < N\), if \(KM/N\) is an integer, there exists a \((K, (K_{M/N}), (K_{M/N-1}), (K_{M/N+1}))\) PDA, which leads to a \((K, M, N)\) coded caching scheme (MN scheme) with subpacketization \(F = (K_{M/N})\) and transmission load \(R = \frac{K(1-M/N)}{KM/N+1}\) for the MN caching system. \(\square\)
B. Multiaccess combination network caching system

We consider an $(H, r, u, M, N)$ multiaccess combination network caching system (see Fig. 1) in which a server $S_{er}$ containing $N$ files $\mathcal{W} = \{W_1, W_2, \ldots, W_N\}$, each of which is uniformly distributed in $\{0,1\}^H$ for some positive integer $B$, connects $H$ relays through $H$ error-free and interference-free links. Each set of $r$ relays connects $u$ users, so the total number of users is $K = u(H)$. Each user has a storage capacity of size $M$ files where $0 < M < N$ and all relays have no cache memory. Each relay $h$ could broadcast the intermediate signals from the server to the users who is connected to it. The users is denoted by $\mathcal{K} = \{k = (A, i) \mid A \in ([H]), i \in [u]\}$ and user $(A, i)$ is connected to relay $h$ if and only if $h \in A$. The set of relays which connect user $k \in \mathcal{K}$ is denoted by $\mathcal{A}_k$.

An $F$-division $(H, r, u, M, N)$ multiaccess combination network caching scheme contains the following two phases.

- **Placement phase**: Each file is divided into $F$ packets with equal size. Then each user $k \in \mathcal{K}$ directly accesses to the file library $\mathcal{W}$ and stores some packets or linear combinations of some packets of the $N$ files in its cache. The set of cached packets by user $k$ is denoted by $Z_k$ whose size is at most $M$ files. That is, for each user $k \in \mathcal{K}$, there exists a function $\phi_k : \{0,1\}^{NB} \rightarrow \{0,1\}^{MB}$ to generate the cache contents $Z_k = \phi_k((W_n)_{n \in [N]})$. Let $\mathcal{Z} = \{Z_k \mid k \in \mathcal{K}\}$.

- **Delivery phase**: Assume that each user $k = (A, i) \in \mathcal{K}$ randomly requests one file from the file library $\mathcal{W}$. The demand vector is represented by $d = (d_k)_{k \in \mathcal{K}}$, which implies that user $k$ requests the $d_k$ file $W_{d_k}$ where $d_k \in [N]$. Given $(\mathcal{Z}, d)$, the server sends a message $X_{S_{er} \rightarrow h}$ of size $L_{S_{er} \rightarrow h}$ bits to relay $h \in [H]$. Then relay $h \in [H]$ forwards $X_{S_{er} \rightarrow h}$ to its connecting users. User $k$ can recover its desired file $W_{d_k}$ by $\{X_{S_{er} \rightarrow h} \mid h \in A\}$ with help of $Z_k$. This phase can be represented by the following encoding functions and decoding functions.

  - The $H$ encoding functions: For each relay $h \in [H]$, 
    \[
    \psi_{S_{er} \rightarrow h} : \{0,1\}^{NB} \times \{0,1\}^{KMB} \times [N]^K \rightarrow \{0,1\}^{L_{S_{er} \rightarrow h}}
    \]
    generates the transmitted message $X_{S_{er} \rightarrow h} \triangleq \psi_{S_{er} \rightarrow h}(\mathcal{W}, \mathcal{Z}, d)$ from the server to the relay $h$. It is a function of the library $\mathcal{W}$, the cached contents of all users $\mathcal{Z}$ and the demand vector $d$.

  - The $K$ decoding functions: For each user $k = (A, i) \in \mathcal{K}$, 
    \[
    \mu_k : \{0,1\}^{\sum_{h \in A} L_{S_{er} \rightarrow h}} \times \{0,1\}^{MB} \times [N]^K \rightarrow \{0,1\}^{B}
    \]
    decodes the requested file of user $k$ from all messages received by user $k$ and its own cache, i.e., 
    \[
    W_{d_k} = \mu_k (\{X_{S_{er} \rightarrow h} \mid h \in A\}, Z_k, d).
    \]

The transmission load for each relay is defined as

\[
R = \max_{d \in [N]^K, h \in [H]} \left\{ \frac{L_{S_{er} \rightarrow h}}{B} \right\}.
\]
C. Combinatorial placement delivery array

Definition 2: (CPDA, [12]) For any positive integers $H, r$ and $u$ with $r < H$, a $(K = u(H), F, Z, S)$ PDA $P$ is called a $(K, F, Z, S)$ combinatorial placement delivery array (CPDA) if it satisfies the following condition.

C4. All the columns can be labeled by $K = \{ (A, i) \mid A \in (H), i \in [u] \}$ such that for any $s \in [S]$, the intersection of the first coordinate $A$ of all column labels $(A, i)$ satisfying that $s$ appears in column $(A, i)$, denoted by $I_s$, is not empty.

The definition of CPDA is proposed based on the definition of PDA. So a CPDA must be a PDA. However, the converse does not hold necessarily, since CPDA has another requirement, i.e., the condition C4. The detailed discussion of the relationship between PDA and CPDA could be found in [13]. Based on a CPDA, a scheme for the multiaccess combination network can be obtained by using Algorithm 1.

Algorithm 1 Caching scheme based on $(K, F_1, Z, S)$ CPDA in [13]

1: procedure Placement$(P, W)$
2: 
3: for $k \in K$ do
4: 
5: end for
6: end procedure
7: procedure Delivery$(P, W, d)$
8: 
9: for $s \in [S]$ do
10: 
11: 
12: for $l \in [\mu_s]$ do
13: 
14: 
15: end for
16: end for
17: end procedure

From Algorithm 1 a $(K, F_1, Z, S)$ CPDA $P$ can be explained intuitively as follows.

- Each row $j \in [F_1]$ represents the index of the $j$th packet of all files and each column $k \in K$ represents user $k$. If $p_{j,k} = *$, then user $k$ has cached the $j$th packet of all files. So the condition C1 of Definition 1 implies that each user caches $M = \frac{F_1}{r}N$ files.

- If $p_{j,k} = s$ is an integer, it means that the $j$th packet of all files is not stored by user $k$. Then the XOR of the requested packets indicated by $s$ is generated by the server at time slot $s$, denoted by $X_s$. If the set $I_s$ defined in the condition C4 of Definition 2 is $\{h_{s,1}, h_{s,2}, \ldots, h_{s,\mu_s}\}$ where $1 \leq \mu_s \leq r$, then $X_s$ is divided into $\mu_s$ sub-packets, i.e., $X_s = \{X_{s,1}, X_{s,2}, \ldots, X_{s,\mu_s}\}$. Finally, for any $l \in [\mu_s]$, the server sends $X_{s,l}$ to relay $h_{s,l}$ and relay $h_{s,l}$ forwards $X_{s,l}$ to its connecting users. So the subpacketization is at most $rF_1$. The condition C4 of Definition 2 guarantees that user $k$ can receive the whole message $X_s$, since user $k$ is connected to each relay in $I_s$. The condition C3 of Definition 1 guarantees that user $k$ can recover its requested packet indicated by $s$ from $X_s$, since it has cached all the other packets in the message $X_s$ except its requested one. The occurrence number of integer $s$ in $P$, denoted by $g_s$, is the coded caching gain at time slot $s$, since the message $X_s$ is useful for $g_s$ users.

- The condition C2 of Definition 1 implies that the number of messages $X_s$ sent by the server is exactly $S$. If the size of $I_s$ is a constant for each $s \in [S]$, assume that $|I_s| = \mu$, and if the number of $I_s$ containing $h$ is a constant for each relay $h \in [H]$, assume that $|\{I_s \mid h \in I_s, s \in [S]\}| = \nu$, then we have $\nu H = \mu S$. So the transmission load for each relay is $R = \nu \frac{1}{\mu F_1} = \frac{S}{F_1}$. Lemma 2: ([13]) Given a $(K = u(H), F_1, Z, S)$ CPDA for any positive integers $H, r$ and $u$ with $r < H$, we have an $(H, r, u, M, N)$ multiaccess combination network caching scheme with memory ratio $M = \frac{F}{r}$, subpacketization $F \leq rF_1$. Moreover, if the size of $I_s$, which is defined in Definition 2, is a constant for each $s \in [S]$ and if the number of $I_s$ containing $h$ is a constant for each relay $h \in [H]$, the transmission load for each relay is $R = \frac{S}{F_1}$. □

In a PDA, a star not contained in any subarray shown as $\square$ in C3-b of Definition 1 is called useless. The authors in [17] pointed out that useless stars not only make no contribution to reducing the transmission load, but also result in a high
subpacketization. If each column of a $(K, F, Z)$ PDA has $Z'$ useless stars, the authors in [17] improved the scheme in [14] by deleting all the useless stars and using an $[F, F - Z']_q$ maximum distance separable (MDS) code [18] for some prime power $q$, and came up with a new coded caching scheme with smaller transmission load and subpacketization than the original scheme in [14] for the same number of users and memory ratio. In fact, this idea also works for the multiaccess combination network caching system.

**Lemma 3:** Given a $(K = u(H), F_1, Z, S)$ CPDA $P$ for any positive integers $H$, $r$ and $u$ with $r < H$, assume that there exist $Z'$ useless stars in each column of $P$. Then we have an $(H, r, u, M, N)$ multiaccess combination network caching scheme with memory ratio $\frac{M}{N} = \frac{Z - Z'}{F_1 - Z'}$, subpacketization $F \leq r(F_1 - Z')$. If the size of $I_s$, which is defined in Definition 2, is a constant for each integer $s \in [S]$ and if the number of $I_s$ containing $h$ is a constant for each relay $h \in [H]$, the transmission load for each relay is $R = \frac{K(Z - Z')}{H(F_1 - Z')}$. □

**Proof.** Assume that $P$ is a $(K = u(H), F_1, Z, S)$ CPDA where each column has $Z'$ useless stars. Delete the $Z'$ useless stars in each column, we obtain a new array $P' = \langle p'_{j, k} \rangle j \in [F_1], k \in K \rangle$. Clearly each column of $P'$ has $Z'$ blanks, $Z - Z'$ stars and $F_1 - Z$ integers. Based on $P'$, we modify the placement strategy in Algorithm 1 as follows: the server divides each file into $F_1 - Z'$ equal-sized packets and then encodes them using an $[F_1, F_1 - Z']_q$ MDS code for some prime power $q$. The resulting encoded packets are denoted by $W_{n, 1}, W_{n, 2}, \ldots, W_{n, F_1}$ for each file $W_n$ where $n \in [N]$. Using the caching strategy in Lines 3-5 in Algorithm 1, each user $k$ caches $Z_k = \langle W_{n, j}p'_{j, k} = *, j \in [F_1], n \in [N] \rangle$. Clearly, the memory ratio of each user is $\frac{M}{N} = \frac{Z - Z'}{F_1 - Z'}$. Now let us consider its subpacketization and transmission load for each relay. For any request vector $d$ in the delivery phase, we use the same delivery strategy as in Algorithm 1. Then each user can get exactly $F_1 - Z$ required coded packets. From the property of an $[F_1, F_1 - Z']_q$ MDS code, each user can recover its requested file. Since the size of $I_s$ is a constant for each integer $s \in [S]$ and the number of $I_s$ containing $h$ is a constant for each relay $h \in [H]$, the transmission load for each relay is $R = \frac{K(Z - Z')}{H(F_1 - Z')}$. □

**Remark 1:** Given a $(K, F_1, Z, S)$ CPDA $P$, if each column of $P$ has $Z'$ useless stars, then the scheme in Lemma 3 has smaller memory ratio and subpacketization than the scheme in Lemma 2 since $\frac{Z}{Z'} > \frac{Z - Z'}{F_1 - Z'}$ and $F_1 > F_1 - Z'$ always hold for any positive integer $Z' < Z$. It is worth noting that for the scheme in Lemma 3, the operation field must be $O(rF_1)$, hence the size of each packet must be proximately of length $\log_2(rF_1)$ bits. This implies that the size of each file in the server must be more than $\log_2(rF_1)(rF_1 - Z')$. □

**III. THE SCHEMES FOR THE MULTIACCESS COMBINATION NETWORK**

In this section, we will propose three coded caching schemes for the multiaccess combination network caching system. The first one is obtained directly from the idea in [5]. The second one is from a direct construction of CPDA via combinatorial design theory and the third one is from a hybrid construction of CPDA based on the CPDA directly constructed before and the MN PDA.

**A. The first scheme from the idea in [5]**

Since the scheme in [5] can be used to deal with any relay networks with arbitrary topologies, it also works for the multiaccess combination network caching setting. Specifically, from the idea in [5], we can get an $(H, r, u, M, N)$ multiaccess combination network caching scheme as follows: Firstly, the server divides each file into $r$ packets and then encodes them into $H$ coded packets by an $[H, r]_q$ MDS code for some prime power $q$. Denote the $h^{th}$ coded packet of all files by $W^{(h)} = \{W^{(h)}_n | n \in [N]\}$. It is worth noting that the size of each coded packet is $\frac{H}{r}$ bits. Secondly, for each relay $h$, taking the $h^{th}$ coded packet of all files $W^{(h)}$ as the file library and using the $(K = u(H - 1), M, N)$ MN scheme, the server sends all the required coded signals to relay $h$, then relay $h$ forwards them to its connecting users. Let $t = \frac{KM}{N}$, then each user caches exactly $r^{(K - 1)} r^{(h)} r^{(K)} T_{MB}$ bits, since each user is just connected to $r$ relays. Thus the transmission load for each relay is $R = \frac{K(1 - M/N)}{r(KM/N + 1)}$. That is the following result.

**Lemma 4:** (ZY scheme) For any positive integers $H$, $r$, $u$, $M$ and $N$ with $r < H$ and $M < N$, let $K = u(H - 1)$, if $t = \frac{KM}{N}$ is an integer, we have an $(H, r, u, M, N)$ multiaccess combination network caching scheme with subpacketization $F = r^{(K - 1)}$ and transmission load for each relay $R = \frac{K(1 - M/N)}{r(KM/N + 1)}$. □

It is easy to verify that the transmission load for each relay of ZY scheme is approximately $\frac{1}{r}$ of the transmission load of MN scheme. However, the subpacketization of ZY scheme is exponential order with $K = u(H - 1)$, which may lead to high implementation complexity and infeasibility in reality. Therefore, reducing the subpacketization while slightly increasing the transmission load for each relay as the tradeoff is very meaningful.
B. The second scheme via direct construction of CPDA

In this section, we propose a direct construction of CPDA by means of Combinatorial Design Theory, which leads to a multiaccess combination network caching scheme with much lower subpacketization than ZY scheme in Lemma 4.

Construction 1: For any positive integers $H$, $r$, $a$, $\omega$ and $\lambda$ with $\max\{\omega, \lambda\} \leq r < H$ and $a < H$, let the row index set $\mathcal{F}$ and the column index set $\mathcal{K}$ be
\[ \mathcal{F} = \{ f = (f_1, f_2, \ldots, f_H) \in \{0,1\}^H \mid wt(f) = a \}, \]
\[ \mathcal{K} = \{ k = (\mathcal{T}, \mathbf{b}) \mid \mathcal{T} \in \binom{[H]}{r}, \mathbf{b} \in \{0,1\}^r, wt(\mathbf{b}) = r - \omega \}, \]
respectively. An $(H^r_a \times \omega^r)$ array $\mathbf{P} = (p_{f,k})_{f \in \mathcal{F}, k \in \mathcal{K}}$ with $\mathcal{T} = \{ \delta_1, \delta_2, \ldots, \delta_r \} \subset [H]$, $\delta_1 < \delta_2 < \cdots < \delta_r$ and $\mathbf{b} = (b_1, b_2, \ldots, b_r)$, is defined as
\[ p_{f,k} = \begin{cases} (e, C) & \text{if } d(f|_{\mathcal{T}}, \mathbf{b}) = r - \lambda, \\ * & \text{otherwise,} \end{cases} \]
where $e = (e_1, e_2, \ldots, e_H) \in \{0,1\}^H$ is defined as
\[ e_i = \begin{cases} b_v & \text{if } i = \delta_v, v \in [r], \\ f_i & \text{otherwise} \end{cases} \]
and
\[ C = \{ \delta_v \mid f_{\delta_v} = b_v, v \in [r] \} \subseteq \{ \delta_1, \delta_2, \ldots, \delta_r \}. \]

Clearly, we have $|C| = \lambda$ from (5).

Example 2: When $H = 4$, $a = 2$, $r = 2$, $\omega = 1$ and $\lambda = 1$, the array generated by Construction 1 is shown in Table I.

| Table I: The $(12, 6, 4, 8)$ CPDA with $H = 4$, $a = r = 2$, $\omega = 1$ and $\lambda = 1$. |
|---|
| $f$ | $\mathcal{T}$, $\mathbf{b}$ | $(1,2)$ | $(1,3)$ | $(1,4)$ | $(2,3)$ | $(2,4)$ | $(3,4)$ |
| $(0,1,1)$ | 0111,1 1011,2 | * * * * | * * * * | 0000,1 0100,2 | * * | * * | * * |
| $(0,1,0)$ | * * 0111,1 1101,3 | * * * * | 0100,3 0100,2 | * * * * | * * | * * |
| $(1,0,1)$ | * * * 0111,1 1101,3 | 0100,3 0100,2 | 0010,4 1000,1 | * * | * * | * * |
| $(1,0,0)$ | * 0010,3 1000,1 | * * * * | 1011,2 1110,4 | * * | * * | * * |
| $(1,1,0)$ | 0100,2 1000,1 | * * * * | 1101,3 1110,4 | * * | * * | * * |

For instance, when $f = (0, 0, 1, 1)$ and $(\mathcal{T}, \mathbf{b}) = (\{1,2\}, \{1,0\})$, since $d(f|_{\mathcal{T}}, (1,0)) = d((0,0), (1,0)) = 1 = r - \lambda$, we have $C = \{2\}$ from (5), then from (4) and (3) we have $p_{f,k} = ((1,0,1,1), \{2\})$, abbreviated as 1011, 2. It is easy to verify that the array in Table I is a $(12, 6, 4, 8)$ CPDA, which leads to an $(H, r, u, M, N)$ multiaccess combination network caching scheme with $u = (\omega) = 2$, memory ratio $M \overline{N} = 2$, subpacketization $F = 6$ and transmission load for each relay $R = 8 \overline{r} = \frac{8}{4} = \frac{2}{3}$. In this case, the subpacketization and transmission load for each relay of ZY scheme in Lemma 4 are $F_{ZY} = 30$ and $R_{ZY} = \frac{1}{3}$, respectively.

In general, the array generated by Construction 1 is a CPDA, where each column has the same number of useless stars. Hence, from Lemma 3 we have the following result, whose proof could be found in Appendix A.

Theorem 1: (Scheme A) For any positive integers $H$, $r$, $a$, $\omega$ and $\lambda$ with $\max\{\omega, \lambda\} \leq r < H$ and $a < H$, there exists a $(\binom{H}{a}, \binom{H}{r}, Z, S)$ CPDA with $Z = \binom{H}{a} - \sum_{\lambda_1 \in [x:y]} \binom{\omega}{\lambda_1} \binom{H-r}{\lambda_1 - \lambda_1}$ and $S = \sum_{\lambda_1 \in [x:y]} \binom{H}{a+r-2\omega+2\lambda_1-\lambda_1}$.
$$P = \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{array} \right) \quad \mathbf{A} = \left( \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right)$$

$$(a + r - 2\omega + 2\lambda_1 - \lambda),$$ where $x = \max\{0, \lambda - r + \omega\}, y = \min\{\omega, \lambda\}$. It can lead to an $(H, r, u = (\frac{H}{r}), M, N)$ multiaccess combination network caching scheme with memory ratio, subpacketization and transmission load for each relay being

$$\frac{M}{N} = 1 - \frac{\sum_{\lambda_1 \in [x:y]} (\frac{H - r}{a}) \left( \frac{r - \omega}{\lambda - \lambda_1} \right) - \sum_{\lambda' \in [0, \lambda - 1]} \lambda_1 \in [x:y]}{\sum_{\lambda_1 \in [x:y]} (\frac{H - r}{a}) \left( \frac{r - \omega}{\lambda - \lambda_1} \right) \times \left( \frac{\lambda_1}{\lambda_1} \right) \times \left( \frac{H - r}{a - \omega + 2\lambda_1 \lambda - \lambda_1} \right)}$$

$$F = \sum_{\lambda' \in [0, \lambda - 1]} \lambda_1 \in [x:y]} \left( \frac{H - (a + r - 2\omega + 2\lambda_1 - \lambda)}{\lambda_1} \right) \left( \frac{H - r}{a - \omega + 2\lambda_1 - \lambda'} \right)$$

$$R = \sum_{\lambda_1 \in [x:y]} \left( \frac{H - (a + r - 2\omega + 2\lambda_1 - \lambda)}{\lambda_1} \right) \left( \frac{H - r}{a + r - 2\omega + 2\lambda_1 - \lambda} \right)$$

respectively, where $x' = \max\{0, \lambda' - r + \omega\}$ and $y' = \min\{\omega, \lambda'\}$. □

**Remark 2:**
- When $\omega = 0$, from Theorem 1 we can obtain an $(\frac{H}{r}, \frac{H}{r}, \frac{H}{r} - (\frac{H}{r}), \frac{H}{r} - (\frac{H}{r}) - (\frac{H}{r}) - (\frac{H}{r}))$ CPDA which is exactly the strongly coloring PDA in [4] by the fact $(\frac{H}{r} - (\frac{H}{r}) - (\frac{H}{r})) - (\frac{H}{r}) - (\frac{H}{r})$. 
- The row index set $\mathcal{F}$ in (2) is determined by the value of $a$. In fact, $\mathcal{F}$ can be chosen more flexibly. For example, let $\mathcal{F} = \{(f_1, f_2, \ldots, f_H) \mid f_1, f_2, \ldots, f_{H-1} \in [0 : q - 1], f_H = \text{mod}(\sum_{i=1}^{H-1} f_i, q)\}$ and $\mathcal{K} = (\frac{H}{r}) \times [0 : q - 1]^r$ for any positive integer $q \geq 2$, when $H = 4, r = 2$ and $\lambda = 1$, according to (3), (4) and (5), a $(24, 8, 4, 32)$ CPDA can be obtained, which is shown in Table II

![Table II: The $(24, 8, 4, 32)$ CPDA.](image)

- From the **footnote** [1] in Appendix A the scheme from the CPDA generated by Construction 1 omits the coded caching gain among the users who are connected to the same $r$ relays. □

**C. The third scheme via hybrid construction of CPDA**

From Theorem 1 an $(H, r, u = (\frac{H}{r}), M, N)$ multiaccess combination network caching scheme can be obtained. However, it has a limitation of the parameter $u$, i.e., $u = (\frac{H}{r})$. In this subsection, we will propose a hybrid construction of CPDA for any parameter $u$ based on a PDA and a CPDA. Let us use an example to illustrate the main idea of the construction.

**Example 3:** Given the $(3, 3, 1, 3)$ CPDA $\mathbf{P}$ with $H = 3, r = 2, u = 1$, and the $(2, 2, 1, 1)$ PDA $\mathbf{A}$ in [9], by replacing each entry in $\mathbf{P}$, i.e., $p_{j_1, k_1}$ where $j_1 \in [3], k_1 \in (\frac{3}{9})$, with an array $\mathbf{A} + p_{j_1, k_1} = (a_{j_2, k_2} + p_{j_1, k_1})_{j_2 \in [2], k_2 \in [2]}$, where $x + * = *$ for any integer $x$, we can obtain the following $(6, 6, 3, 3)$ CPDA $\mathbf{L}$ with $H = 3, r = 2$ and $u = 2$.

$$\mathbf{L} = \left( \begin{array}{cccccc} 1 & 1 & 1 & 2 & 3 & 2 \\ 2 & 2 & 3 & 2 & 2 & 2 \\ 3 & 3 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{array} \right)$$

\[(1, 1) \quad (1, 2) \quad (2, 3, 1) \quad (2, 3, 2) \quad (2, 3, 2) \]
It is easy to verify that the scheme from $L$ creates broadcast opportunities among the users who are connected to the same $r$ relays, since $l_{(2,2),((1,2),1)} = l_{(2,1),((1,2),2)} = 1$.

Mathematically, the construction is given as follows.

**Construction 2:** Suppose that there exists a $(K_1 = u((H), F_1, Z_1, S_1)$ CPDA $P = (p_{j_1,k_1}, j_1 \in [F_1], k_1 \in (\binom{H}{r}) \times [u])$, and a $(K_2, F_2, Z_2, S_2)$ PDA $A = (a_{j_2,k_2}, j_2 \in [F_2], k_2 \in [K_2])$, the array $L = (l_{(j_1,j_2),((k_1,k_2)})$, $(j_1,j_2) \in [F_1] \times [F_2]$, $(k_1, k_2) \in (\binom{H}{r}) \times [u]) \times [K_2]$, is defined as

$$l_{(j_1,j_2),((k_1,k_2)} = \begin{cases} a_{j_2,k_2} + (p_{j_1,k_1} - 1)S_2 & \text{if } p_{j_1,k_1} \neq *, a_{j_2,k_2} \neq *, \\ 0 & \text{otherwise}. \end{cases}$$

Let us return to Example 3, when $j_1 = 1$, $k_1 = \{1, 2\}$, $j_2 = 1, k_2 = 1$, since $p_{j_1,k_1} = p_{1,1,2} = 1$, we have $l_{(j_1,j_2),((k_1,k_2)} = 1$ from (10); when $j_1 = 2, k_1 = \{1, 2\}$, $j_2 = 2, k_2 = 1$, since $p_{j_1,k_1} = p_{2,2,1}$ and $a_{j_2,k_2} = a_{2,1,2} = 1$, we have $l_{(j_1,j_2),((k_1,k_2)} = a_{j_2,k_2} + (p_{j_1,k_1} - 1)S_2 = 1$ from (10).

By Construction 2, we have the following result, whose proof could be found in Appendix B.

**Theorem 2:** Given a $(K_1 = u((H), F_1, Z_1, S_1)$ CPDA $P = (p_{j_1,k_1}, j_1 \in [F_1], k_1 \in (\binom{H}{r}) \times [u])$ and a $(K_2, F_2, Z_2, S_2)$ PDA $A$, there always exists a $(uK_2((H), r))$ CPDA $P = (p_{j_1,k_1}, j_1 \in [F_1], k_1 \in (\binom{H}{r}) \times [u])$ and a $(K_2, F_2, Z_2, S_2)$ PDA $A$, such that the subpacketization and transmission load for each relay being

$$R = \frac{S_1S_2}{P(D, F_2)}.$$

From Theorem 2, the following result can be obtained based on the CPDA in Theorem 1 and the $(K_2, (K_2, (K_2, t_2), (K_2, t_2 + 1))$ MN PDA for any $K_2, t_2 \in \mathbb{Z}^+$ with $t_2 < K_2$.

**Theorem 3:** (Scheme B) For any positive integers $H, r, a, \omega, \lambda, t_2$ and $K_2$ with $\max\{\omega, \lambda\} \leq r < H$, $a < H$ and $t_2 < K_2$, there exists an $(H, r, u = K_2((H), M, N)$ multaccess combination network caching scheme with the memory ratio $\frac{M}{N} = 1 - \frac{K_2 - t_2}{K_2} - \sum_{j \in [x]} \frac{H}{a} - \sum_{j \in [x]} \frac{H}{a} - \sum_{j \in [x]} \frac{H}{a} - \sum_{j \in [x]} \frac{H}{a}$ and transmission load $R = \frac{K_2 - t_2}{t_2 + 1} + \sum_{j \in [x]} \frac{H}{a} - \sum_{j \in [x]} \frac{H}{a} - \sum_{j \in [x]} \frac{H}{a} - \sum_{j \in [x]} \frac{H}{a}$, respectively, where $x = \max\{0, \omega - r + \omega\}$, $y = \min\{\omega, \lambda\}$, $x' = \max\{0, \lambda' - r + \omega\}$ and $y' = \min\{\omega, \lambda'\}$.

**IV. PERFORMANCE ANALYSIS**

In this section, we compare Scheme A in Theorem 1 and Scheme B in Theorem 3 with ZY scheme in Lemma 4 and the naive scheme of repeatedly using the scheme in [17] (referred to as Scheme C).
A. Analytic Comparison of Scheme A, B with ZY scheme

Since it is difficult to give an analytic comparison of Scheme A and Scheme B with ZY scheme for general parameters $\lambda$ and $\omega$, we focus on the case of $\lambda = 1, \omega = 1$. In this case, the memory ratio, subpacketization and transmission load for each relay of Scheme A are

$$
M/N = 1 - \frac{(H-r) + (r-1)(H-r)}{(H) - (H-a-1)}, \quad F_{Th1} = \left(\frac{H}{a}\right) - \left(\frac{H-r}{a-1}\right),
$$

$$
R_{Th1} = \frac{1 - \frac{M}{N}}{H} \cdot \left(\frac{H}{a-r-1}\right) (H-a-r+1) + \left(\frac{H}{a+r-3}\right) (a+r-3)
$$

respectively from (6), (7) and (8); from (11), (12) and (13), Scheme B has memory ratio, subpacketization and transmission load for each relay as follows,

$$
M/N = 1 - \frac{K_2 - t_2}{K_2} \cdot \frac{(H-r) + (r-1)(H-r)}{(H) - (H-a-1)}, \quad F_{Th3} = \left(\frac{K_2}{t_2}\right) \left(\frac{H}{a}\right) - \left(\frac{H-r}{a-1}\right),
$$

$$
R_{Th3} = \frac{K_2(1 - \frac{M}{N})}{H(t_2 + 1)} \cdot \left(\frac{H}{a+r-1}\right) (H-a-r+1) + \left(\frac{H}{a+r-3}\right) (a+r-3)
$$

$$
= \left(\frac{rM}{HN} + \frac{1}{K}\right) H(H-1)\cdots(H-r+1) \frac{(H-a-r+1)(H-a-r+2)(H-a-r+1)}{(a+r-1)!} + \frac{1}{(a+r-4)!}
$$

$$
< \left(\frac{rM}{HN} + \frac{1}{K}\right) H(H-1)\cdots(H-r+1)
$$

and

$$
\frac{R_{Th3}}{R_{ZY}} < \frac{K_2}{t_2 + 1} \left(\frac{rM}{HN} + \frac{1}{K}\right) H(H-1)\cdots(H-r+1)
$$

$$
\left(\frac{a+r-1}{(a+r-1)!}\right) \cdots \left(\frac{a+r-1}{(a+r-1)!}\right)
$$

when $H$ is large enough. And we have

$$
\frac{F_{Th1}}{F_{ZY}} = \frac{(H-1) - (H-r)}{r\left(\frac{H-r}{a-1}\right) M} < \frac{(H-1) - (H-r)}{r\left(\frac{H-r}{a-1}\right) M}
$$

and

$$
\frac{F_{Th3}}{F_{ZY}} < \frac{K_2}{t_2 + 1} \left(\frac{H}{a}\right) - \left(\frac{H-r}{a-1}\right)
$$

Clearly, for Scheme A and Scheme B, the transmission load for each relay is just multiplied while the subpacketization is reduced exponentially compared to ZY scheme. Moreover, it is easy to verify that $\left(\frac{rM}{HN} + \frac{1}{K}\right) H(H-1)\cdots(H-r+1) \ll \frac{(H-r)M}{N}$ and $\frac{K_2}{t_2 + 1} \left(\frac{rM}{HN} + \frac{1}{K}\right) H(H-1)\cdots(H-r+1) \ll \frac{K_2r(H-r)M}{N}$, i.e., in Scheme A and Scheme B the growth multiple of transmission loads for each relay are much less than the reduction exponent of the subpacketization.

B. Numerical comparison

The numerical comparison of Scheme A and Scheme B with ZY scheme is given in Table III. It is easy to see from Table III that $\frac{R_{Th1}}{R_{ZY}} \ll \ln \frac{F_{Th1}}{F_{ZY}}$ and $\frac{R_{Th3}}{R_{ZY}} \ll \ln \frac{F_{Th3}}{F_{ZY}}$, which coincides with the analytic comparison.
TABLE III: Numerical comparison of Scheme A and Scheme B when $\omega = 1$, $\lambda = 1$ with ZY scheme

| $(H, r, a)$ | $K$ | $\frac{R_{ZY}}{F_{ZY}}$ | $\frac{R_{Th}}{F_{Th}}$ | $\ln \frac{F_{ZY}}{F_{Th}}$ | $(H, r, a, K, t_2)$ | $\frac{R_{ZY}}{F_{ZY}}$ | $\frac{R_{Th}}{F_{Th}}$ | $\ln \frac{F_{ZY}}{F_{Th}}$ |
|-------------|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 15,2,4      | 210 | 0.27            | 1.59            | 11.75           | 20,3,3,8,1      | 27360           | 0.38            | 53.68           |
| 16,3,6      | 1680| 0.43            | 8.19            | 297.35          | 20,3,3,6,1      | 20520           | 0.41            | 43.39           |
| 18,3,6      | 2448| 0.50            | 5.98            | 391.00          | 20,3,3,5,1      | 17100           | 0.43            | 38.24           |
| 20,3,12     | 3420| 0.56            | 4.03            | 487.29          | 20,3,3,3,1      | 10260           | 0.53            | 27.95           |

Finally, when $H = 14$, $r = 4$, $u = 6$ and $K = u(\frac{H}{r}) = 6006$, the memory-load and memory-subpacketization tradeoffs are given in Fig. 2. From Fig. 2(a), we can see that for the same number of users and memory ratio, the schemes with transmission load for each relay from small to large are ZY scheme, Scheme B, Scheme A and Scheme C in turn. That is because ZY scheme fully utilizes the multicast opportunities among the users who are connected to the same relay, then its coded caching gain is the largest, so it has the minimum transmission load for each relay; Scheme A and Scheme C only utilizes the multicast opportunities among the users who are connected to different sets of $r$ relays, while omits the multicast opportunities among the users who are connected to the same $r$ relays, and Scheme A utilizes the multicast opportunities among the users who are connected to different sets of $r$ relays more efficiently than Scheme C, so the transmission load for each relay of Scheme A is lower than that of Scheme C; Scheme B not only utilizes the multicast opportunities among the users who are connected to different sets of $r$ relays as Scheme A, but also utilizes the multicast opportunities among the users who are connected to the same $r$ relays, so the transmission load for each relay of Scheme B is lower than that of Scheme A. From Fig. 2(b), we can see that for the same number of users and memory ratio, the schemes with subpacketization from small to large are Scheme A, Scheme C, Scheme B and ZY scheme in turn. So Scheme A and Scheme B have significant advantages in subpacketization compared with ZY scheme and have significant advantages in transmission load for each relay compared with Scheme C. Moreover, the subpacketization of Scheme A is lower than that of Scheme C.

![Fig. 2](https://via.placeholder.com/150)

(a) Transmission load for each relay.

![Fig. 2](https://via.placeholder.com/150)

(b) Subpacketization.

**Fig. 2:** Comparison of Scheme A and Scheme B with ZY scheme and Scheme C when $H = 14$, $r = 4$, $u = 6$.

V. CONCLUSION

In this paper, we extended a traditional $(H, r)$ combination network to an $(H, r, u)$ multiaccess combination network, where each $u$ users are connected to a unique set of $r$ relays, and proposed three schemes for such a network, i.e., ZY scheme, Scheme A and Scheme B. The transmission load for each relay of ZY scheme obtained from the idea in [5] is approximately $\frac{1}{r}$ of the transmission load of MN scheme, but its subpacketization increases exponentially with the number of users. Scheme A obtained by a direct construction of CPDA has significant advantages in subpacketization compared with ZY scheme, where the parameter $u$ must be a combinational number. Scheme B obtained by a hybrid construction of CPDA works for arbitrary parameter $u$ and also has significant advantages in subpacketization compared with ZY scheme. In addition, Scheme A and Scheme B have significant advantages in transmission load for each relay compared with Scheme C of repeatedly using the scheme in [17]. Moreover, the subpacketization of Scheme A is lower than that of Scheme C. It is worth noting that the hybrid construction is based on a given CPDA and PDA, and the memory ratio of the resulting CPDA is greater than that of the original CPDA. It is great significance to look for a direct construction of CPDA which works for arbitrary parameter $u$. 
Proof. Let the array generated by Construction 1 be \( P = (p_{f,k}) \) where \( f \in F \) and \( k \in K \). For any non-star entry \((e,C)\) occurring in \( P \), i.e., there exists some \( f \in F \) and \( k = (T, b) \in K \) such that \( p_{f,(T,b)} = (e,C) \). Let \( C = C_1 \cup C_2 \) where \( C_1 \) is the subset of \( C \) such that \( f_i = 0 \) for each \( i \in C_1 \) and \( C_2 \) is the subset of \( C \) such that \( f_i = 1 \) for each \( i \in C_2 \). Let \( |C_1| = \lambda_1 \), then \( |C_2| = \lambda - \lambda_1 \) since \( C_1 \cap C_2 = \emptyset \). In addition, \( 0 \leq \lambda_1 \leq \omega \) and \( 0 \leq \lambda - \lambda_1 \leq r - \omega \). This implies that 
\[ \max\{0, \lambda - r + \omega\} \leq \lambda_1 \leq \min\{\lambda, \omega\} \]. Let \( x = \max\{0, \lambda - r + \omega\} \) and \( y = \min\{\lambda, \omega\} \), then \( \lambda_1 \in [x : y] \).

Next we will prove that the array \( P \) satisfies the conditions of Definition 2. Let us consider C1 of Definition 1 first. For any column label \( k = (T, b) \), without loss of generality, assume that \( b = (b_1, b_2, \ldots, b_r) \) satisfying \( b_1 = b_2 = \ldots = b_\omega = 0 \) and \( b_{\omega+1} = \ldots = b_r = 1 \). Let \( T = \{ \delta_1, \delta_2, \ldots, \delta_t \} \) with \( \delta_1 < \delta_2 < \ldots < \delta_t \). From (3) there are 

\begin{equation}
Z = \sum_{\lambda_1 \in [x:y]} \left( \frac{\omega}{X_1} \right) \times \left( \frac{r - \omega}{X_2} \right) \times \left( \frac{H - r}{X_3} \right) \times \left( \frac{\lambda_1 - \lambda}{X_4} \right) \times \left( \frac{\lambda - \lambda_1}{X_5} \right)
\end{equation}

vectors \( f = (f_1, f_2, \ldots, f_H) \in F \) satisfying \( d(f|_T, b) = r - \lambda \), where \( X_1 \) is the number of subvectors \( f|_{\{\delta_1, \delta_2, \ldots, \delta_t\}} \) satisfying \( d(f|_{\{\delta_1, \delta_2, \ldots, \delta_t\}}, b|_{[\omega: \omega]}) = \omega - \lambda_1 \); \( X_2 \) is the number of subvectors \( f|_{\{\delta_1, \delta_2, \ldots, \delta_t\}} \) satisfying \( d(f|_{\{\delta_1, \delta_2, \ldots, \delta_t\}}, b|_{[\omega+1:r]}) = r - \omega - (\lambda - \lambda_1) \); and \( X_3 \) is the number of subvectors \( f|_{[H]|T} \) satisfying \( wt(f|_{[H]|T}) = a - (\lambda - \lambda_1) = a - \omega + 2\lambda_1 - \lambda \). Thus, there are \( Z = (|H| - Z) \) stars in each column. So the condition C1 holds.

For any two distinct entries, say \( (f, c) \) and \( (f', c') \), assume that \( p_{f,(T,b)} = p_{f',(T',b')} = (e,C) \). Denote 
\[ f = (f_1, f_2, \ldots, f_H), \quad T = \{ \delta_1, \delta_2, \ldots, \delta_t \}, \quad b = (b_1, b_2, \ldots, b_r), \quad f' = (f'_1, f'_2, \ldots, f'_H), \quad T' = \{ \delta'_1, \delta'_2, \ldots, \delta'_t \}, \quad b' = (b'_1, b'_2, \ldots, b'_r) \]
If \( T = T' \), we have \( e|_T = b = b' \) from (3). Then we have \( f = f' \) since 
\[ e|_{[H]|T} = f|_{[H]|T} \neq f'|_{[H]|T}, \quad e|_{T|C} = c = c'|_{T|C} \neq e|_{T'|C} \].
This contradicts our hypothesis[1]. So we have \( T \neq T' \), then there must exist two distinct integers, say \( i, i' \in [H] \), satisfying \( i \in T \cap C \) and \( i' \in T' \cap C \). Without loss of generality, assume that \( i = \delta \) and \( i' = \delta' \). From Construction 1 we have \( f_{\delta_i} \neq f_1 = e_{\delta_i} = f_{\delta_i}' \) and \( f'_{\delta'_i} \neq f_1' = e_{\delta'_i} = f_{\delta'_i}' \). From (3), (4) and (5) we have \( C \subseteq T \cap T' \) and \( e|_C = b|_C = e|_C = f'|_C = b'|_C \). Then \( d(f|_{[H]|T}, b) < r - \lambda \) and \( d(f'|_{[H]|T}, b) < r - \lambda \) hold since \( |C| = \lambda \). So we have \( p_{f,(T,b)} = p_{f',(T',b')} = * \) from [3]. Clearly \( f \neq f' \) holds. So the condition C3 of Definition 1 holds.

Furthermore, it implies that each useful star \( p_{f,(T,b)} = * \) satisfies \( d(f|_T, b) < r - \lambda \). In other words, each star \( p_{f,(T,b)} = * \) satisfying \( d(f|_T, b) > r - \lambda \) is useless. Similar to (14), there are exactly \( Z' = \sum_{\lambda_1 \in [0 : \lambda_1 - 1]} \sum_{\lambda_1 \in [x : y']} \prod_{i \in [x : y']} (\lambda_1 - \lambda_i) \) vectors \( f \in F \) satisfying \( d(f|_T, b) > r - \lambda \), where \( x' = \max\{0, \lambda_1 - r + \omega\} \) and \( y' = \min\{\lambda', \omega\} \). Deleting the useless stars, there are exactly \( Z - Z' \) stars in each column.

For any non-star entry \((e,C)\) occurring in \( P \), assume that \( p_{f,(T,b)} = (e,C) \), from Construction 1 we have \( C \subseteq T \), \( |C| = \lambda \), \( e_1 = f_1 \) for each \( i \in C \) \( (|H| \setminus T) \) and \( e_1 \neq f_1 \) for each \( i \in T \setminus C \). Since \( C = C_1 \cup C_2 \) where \( C_2 \) is the subset of \( C \) such that \( f_i = 0 \) for each \( i \in C_1 \), \( |C_1| = \lambda_1 \in [x : y] \) and \( C_2 \) is the subset of \( C \) such that \( f_i = 1 \) for each \( i \in C_2 \), we have \( wt(f|_C) = \lambda - \lambda_1 \). Then \( wt(f|_{[H]|T}) = \omega - \lambda_1 \) since \( wt(b) = r - \omega \) and \( d(f|_T, b) = r - \lambda \). So \( wt(f|_T) = \lambda = \lambda_1 + \omega - \lambda_1 \) and \( wt(f|_{[H]|T}) = a - (\lambda + \omega - 2\lambda_1) \) since \( wt(f) = a \). Since \( wt(e|_T) = wt(b) = r - \omega \) and \( wt(e|_{[H]|T}) = wt(f|_{[H]|T}) = a - (\lambda + \omega - 2\lambda_1) \), we have \( wt(e) = r - \omega + a - (\lambda + \omega - 2\lambda_1) = a + r - 2\omega + 2\lambda_1 - \lambda \). For any \( \lambda_1 \in [x : y] \) and for each vector \( e \) with \( wt(e) = a + r - 2\omega + 2\lambda_1 - \lambda \), let us consider any possible coordinate sets \( C_1 \) and \( C_2 \) such that 1) each coordinate of \( e|_{C_1} \) is zero; 2) each coordinate of \( e|_{C_2} \) is one; 3) \( |C_1| = \lambda_1 \) and \( |C_2| = \lambda - \lambda_1 \). Let \( C = C_1 \cup C_2 \). Next we will prove that (e, C) occurs in P. Without loss of generality, assume that \( C_1 = [1 : \lambda_1] \), \( C_2 = [\lambda_1 + 1 : \lambda] \) and \( e_1 = e_2 = \ldots = e_{\lambda_1} = 0 \), \( e_{\lambda_1 + 1} = e_{\lambda_1 + 2} = \ldots = e_\lambda = \ldots = e_{a + r - 2\omega + 3\lambda_1 - \lambda} = 1 \) and \( e_{a + r - 2\omega + 3\lambda_1 - \lambda + 1} = e_{a + r - 2\omega + 3\lambda_1 - \lambda + 2} = \ldots = e_H = 0 \). That is, 
\[ e = (0, \ldots, 0, 1, \ldots, 1, \ldots, 0) \].

Define 
\[ T_C = \left\{ C \cup T_1 \cup T_2 \mid T_1 \in \left( [a + r - 2\omega + 3\lambda_1 - \lambda + 1 : H] - \lambda_1 \right), T_2 \in \left( \lambda + 1 : a + r - 2\omega + 3\lambda_1 - \lambda \right) \right\} \].
It is easy to check that for any $T = C \cup T_1 \cup T_2 \in \mathcal{C}$, we have $|T| = r$, $wt(e_T) = r - \omega$ and there is an $H$-dimensional vector $f$ defined as

\[
f_i = \begin{cases} 
1 & \text{if } i \in T_1, \\
0 & \text{if } i \in T_2, \\
e_i & \text{otherwise}
\end{cases}
\]

satisfying $wt(f) = a$, then $f \in F$, $(T, e_T) \in K$ and $d(f, (T, e_T)) = r - \omega$. Consequently, we have $p_T(T, e_T) = (e, c)$ by Construction $[\ref{construction}]$ and $(e, c)$ satisfies $H^{-1}(a - r + 2\omega + 2\lambda_1 - \lambda_1)$ times in $P$. Since the intersection of any element of $H^{-1}(a - r + 2\omega + 2\lambda_1 - \lambda_1)$ and any element of $H^{-1}(\lambda + 1 + a - 2\omega + 3\lambda_1 - \lambda_1)$ is empty, we have $\bigcap_{T \in \mathcal{C}} T = C$. This implies that the set $\mathfrak{I}_c(T, e_T) = C$. Recall that $\mathfrak{I}_c(T, e_T) = C$. This implies that the set $\mathfrak{I}_c(T, e_T)$ is the intersection of the first coordinate $T$ of all column labels $(T, b)$ satisfying that $(e, c)$ appears in column $(T, b)$, i.e., $\mathfrak{I}_c(T, e_T) = \bigcap_{p_T(T, e_T) = (e, c), c \in \mathfrak{F}} A(T, b)$, where $A(T, b)$ is the set of relays which connect user $(T, b)$. The condition C4 of Definition $[\ref{definition}]$ holds.

From the above analysis, there are $S = \sum_{\lambda_1 \in [x:y]} \binom{H}{a + r - 2\omega + 2\lambda_1 - \lambda_1} (H^{-1}(a + r - 2\omega + 2\lambda_1 - \lambda_1))$ different non-star entries in $P$. The condition C2 of Definition $[\ref{definition}]$ holds.

Hence, $P$ is a CPDA with the parameters in Theorem $[\ref{theorem}]$ For each non-star entry $(e, c)$ in $P$, since $\mathfrak{I}_c(T, e_T) = C$, we have $|\mathfrak{I}_c(T, e_T)| = \lambda$, which is a constant. Next we will prove that for each relay $h \in [H]$, the number of $\mathfrak{I}_c(T, e_T)$ containing $h$ is also a constant. In fact, the number of $\mathfrak{I}_c(T, e_T)$ containing relay $h$ is exactly

\[
\sum_{\lambda_1 \in [x:y]} \binom{H - 1}{a + r - 2\omega + 2\lambda_1 - \lambda_1} (H^{-1}(a + r - 2\omega + 2\lambda_1 - \lambda_1)) \\
+ \sum_{\lambda_1 \in [x:y]} \binom{H - 1}{a + r - 2\omega + 2\lambda_1 - \lambda_1} (H^{-1}(a + r - 2\omega + 2\lambda_1 - \lambda_1))
\]

which is also a constant. From Lemma $[\ref{lemma}]$, Theorem $[\ref{theorem}]$ is proved.

\[\square\]

**APPENDIX B**

**PROOF OF THEOREM 2**

Proof. Firstly, we will prove that the array $L$ generated by Construction $[\ref{construction}]$ is a $(uK_2(\frac{H}{1}), F_1 F_2, Z_1 F_2 + (F_1 - Z_1)Z_2, S_1 S_2)$ CPDA. Clearly, $L$ is a $uK_2(\frac{H}{1}) \times F_1 F_2$ array. Furthermore, each column of $L$ has $Z_1 F_2 + (F_1 - Z_1)Z_2$ stars since each column of $P$ has $Z_1$ stars and each column of $A$ has $Z_2$ stars. There are $S = S_1 S_2$ integers in $L$, since there are $S_1$ integers in $P$ and there are $S_2$ integers in $L$. The Conditions C1 and C2 of Definition $[\ref{definition}]$ hold.

For any two distinct entries $l_{(j_1, k_1)}(j_2, k_2)$ and $l_{(j_1', k_1')}(j_2', k_2')$. If $l_{(j_1, j_2)}(k_1, k_2) = l_{(j_1', j_2')}(k_1', k_2') = s \in [S]$, then all entries $a_{j_2, k_2}, a_{j_2', k_2}, p_{j_1, k_1}, p_{j_1', k_1'}$ are integers and $s = a_{j_2, k_2} + p_{j_1, k_1} - 1S_2 = a_{j_2, k_2} + p_{j_1, k_1} - 1S_2$ from $[10]$. Since $a_{j_2, k_2} \leq S_2$ and $a_{j_2', k_2} \leq S_2$, we have $a_{j_2, k_2} = a_{j_2', k_2} \in [S_2]$ and $p_{j_1, k_1} = p_{j_1', k_1'} \in [S_1]$. Then from the definition of PDA and CPDA, we have that (1) either $j_2 = j_2', k_2 = k_2'$ or $a_{j_2, k_2} = a_{j_2', k_2} = *$ holds; (2) either $j_1 = j_1', k_1 = k_1'$ or $p_{j_1, k_1} = p_{j_1', k_1'} = *$ holds. Since $l_{(j_1, j_2)}(k_1, k_2)$ and $l_{(j_1', j_2')}(k_1', k_2')$ are two distinct entries, $j_2 = j_2', k_2 = k_2'$ and $j_1 = j_1', k_1 = k_1'$ can not hold simultaneously, which implies that either $a_{j_2, k_2} = a_{j_2', k_2} = *$ or $p_{j_1, k_1} = p_{j_1', k_1'} = *$ holds. Consequently, we have $l_{(j_1, j_2)}(k_1, k_2) = l_{(j_1', j_2')}(k_1', k_2') = *$ from $[10]$. The condition C3 of Definition $[\ref{definition}]$ holds.

For any $s \in [S]$, assume that $s$ appears $g_s$ times in $L$, say $l_{(j_1^{(s)}(1), j_2^{(s)}(1))}(k_1^{(s)}, k_2^{(s)(1)}) = l_{(j_1^{(s)}(2), j_2^{(s)(2)})(k_1^{(s)(2)}, k_2^{(s)(2)})} = \ldots = l_{(j_1^{(s)}(s), j_2^{(s)(s)})(k_1^{(s)(s)}, k_2^{(s)(s)})} = s$, then $p_{j_1^{(s)(1)}, k_1^{(s)}} = p_{j_2^{(s)(1)}, k_2^{(s)}} = \ldots = p_{j_1^{(s)(s)}, k_1^{(s)(s)}} = \frac{s - \sum_{\lambda_1 \in [x:y]} \binom{H - 1}{a + r - 2\omega + 2\lambda_1 - \lambda_1}}{S_2} + 1 \equiv s_1$ from $[10]$. Assume that $\mu = (T_0, \mu)$ for each $\mu \in [g_s]$, we have $\mathfrak{I}_c(L) = \mathfrak{I}_c(P) = \bigcap_{\mu \in [g_s]} T_\mu$. Since $P$ is a CPDA, from condition C4 of Definition $[\ref{definition}]$ the set $\mathfrak{I}_c(P)$ is not empty. Then $\mathfrak{I}_c(L)$ is not empty and the condition C4 of Definition $[\ref{definition}]$ holds. Hence, the array $L$ is a $(K, F, Z, S) = (uK_2(\frac{H}{1}), F_1 F_2, Z_1 F_2 + (F_1 - Z_1)Z_2, S_1 S_2)$ CPDA.

Secondly, if $|\mathfrak{I}_c(P)| = \mu$ is a constant for each $s_1 \in [S_1]$ and the number of $\mathfrak{I}_c(P)$ containing $h$ is a constant for each $h \in [H]$, say $|\mathfrak{I}_c(P)\{h \in \mathfrak{I}_c(P), s_1 \in [S_1]\}| = \nu$ for each $h \in [H]$, then we have $\mathfrak{I}_c(L) = |\mathfrak{I}_c(P)| = \mu$ for each $s_1 \in [S_1 S_2]$ where $s_1 = \sum_{\lambda_1 \in [x:y]} \binom{H - 1}{a + r - 2\omega + 2\lambda_1 - \lambda_1} + 1$ and $|\mathfrak{I}_c(L)\{h \in \mathfrak{I}_c(L), s_1 \in [S_1 S_2]\}| = \nu S_2$ for each $h \in [H]$. If there are $Z_1'$ useless stars in each column of $P$, there are $Z_1' F_2$ useless stars in each column of $L$. From Lemma $[\ref{lemma}]$, $L$ leads to an $(H, r, uK_2, M, N)$ multiaccess combination network caching scheme with memory ratio $\frac{M}{N} = \frac{Z_1 F_2 + (F_1 - Z_1)Z_2}{Z_1 F_2} = \frac{Z_1 - Z_1'}{F_1 - Z_1'}$, subpacketization $F = \mu(F_1 - Z_1') F_2$ and transmission load for each relay $R = H(F_1 - Z_1') F_2$.

\[\square\]
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