Dirac Quantum Field on Curved Spacetime: 
Wick Rotation

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Abstract

For a linear Dirac field on a globally hyperbolic static space-time
the analytic continuation of its Wightman functions (Green functions)
to Schwinger functions and back at zero and finite temperature is
shown.

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1 Introduction

The Euclidean formulation of relativistic quantum field theory allows to study concrete models by an approach based on functional integration, which has lead to a wide variety of both rigorous and perturbative constructions. The connection between the Euclidean Schwinger functions and the Wightman distributions (Green functions) on Minkowski space has been established long ago by Osterwalder and Schrader [OS2], making use, among other properties, of the Poincaré symmetry present in the relativistic theory. Considering quantum fields on a Lorentzian space-time manifold, this symmetry is lacking, which prevents a direct adaption of the relations valid on Minkowski space. Restricting to space-time manifolds, the metric of which is invariant under time translations, however, the analytic continuation in the time variable can be performed.

The purpose of this paper is to show for a linear Dirac field on a general static space-time the analytic continuation of its Wightman distributions (Green functions) by a “Wick rotation” to Schwinger functions and back. Furthermore, proceeding analogously, the thermal equilibrium functions are considered, first introducing the real-time distributions by definition and identifying them by the KMS condition shown to hold. Aiming only at a Euclidean formulation of a (Lorentzian) quantum field theory, no attempt is made to construct a Euclidean Dirac field operator acting in a related Euclidean Fock space and yielding the Schwinger functions in the form of vacuum expectation values, as has been achieved on Minkowski space,[OS1]. Recently Jaffe and Ritter [JR], considering a “static” Riemannian manifold, have demonstrated that the inverse of the massive Dirac operator is reflection positive, thus providing the framework for a quantisation of the Dirac field. Moreover, within the algebraic approach to quantum field theory, various properties of the field algebra of a linear Dirac field on non-static Lorentzian manifolds have been worked out,[Di],[Ho],[Kr].

The paper is organized as follows: In Section 2 we collect elements of the classical Dirac equation on a Lorentzian manifold, mainly following [Di], which are used in the sequel. Aiming at a quantum theory, we restrict in Section 3 to a general static manifold. Bringing the Dirac equation into the form of a Schrödinger equation, we obtain a potential Hamilton operator, shown to be a symmetric operator in a Hilbert space related to the metric. We assume that it has a unique self-adjoint extension. Cook’s method of second quantization [Co] is used in Section 4 to convert the deficient quantum
theory into a consistent quantum field theory of the Dirac field, which acts in a particle-antiparticle Fock space, by attributing the Hamilton operator obtained before, which is unbounded from below, to a particle-antiparticle pair. In Section 5 we start from the 2-point Wightman functions, which determine the linear theory considered, and analytically continue their spectral representation to complex values of the time variables. At imaginary time a positivity property is exhibited. Moreover, proceeding similarly as Fulling and Ruijsemaars [FR] in the case of a scalar field, we deduce a single holomorphic function of complex time, which provides the initial Wightman functions as boundary values and at purely imaginary time the related Schwinger functions. In Section 6 thermal 2-point Wightman functions are defined and analytically continued to complex time with analogous steps as in the preceding section. The KMS condition is shown to hold. These analytically continued functions again form a single holomorphic function, which in this case is antiperiodic in imaginary time and provides at purely imaginary time the Schwinger functions.

2 The Dirac equation on curved spacetime

Although in the quantum domain we aim at a static Lorentzian manifold, in these preparatory classical steps we do not yet restrict to this particular class.

i) Let \( M \) be a differentiable manifold, connected and oriented, \( \dim M = 4 \), with chart \( (U, \varphi) \) and local coordinates \( x \),

\[
\varphi : U \subset M \longrightarrow \mathbb{R}^4 \ni x \equiv \{x^\mu\} = (x^0, x^1, x^2, x^3). \tag{1}
\]

providing in the space of vector fields \( T^1_0(M) \) and covector fields (1-forms) \( T^0_1(M) \) the natural dual basis systems

\[
\{ \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \}_{\mu=0,...,3} \subset T^1_0, \quad \{dx^\mu\}_{\mu=0,...,3} \subset T^0_1 : \quad \langle dx^\mu, \partial_\nu \rangle = \delta^\mu_\nu. \tag{2}
\]

ii) There is a Lorentzian metric

\[
g = g_{\mu\nu}(x) \, dx^\mu \otimes dx^\nu \subset T^0_2(M) \tag{3}
\]

with a symmetric nondegenerate 2-cotensor \( g_{\mu\nu}(x) \). The summation convention on a pair of equal upper and lower indices is used, as always in the sequel.
Due to the properties of the matrix $g_{\mu\nu}(x)$ there exists $(c_\alpha^\mu(x)) \in GL(4, \mathbb{R})$ such that

$$c_\alpha^\mu(x) g_{\mu\nu}(x) c_\beta^\nu(x) = \eta_{\alpha\beta}, \quad \tag{4}$$

where $(\eta_{\alpha\beta}) = \text{diag}(+1, -1, -1, -1)$ is the Minkowski metric. For the frame indices $a,b,\cdots \in \{0, \cdots, 3\}$ chosen from the beginning of the Roman alphabet we also use the corresponding summation convention. Defining

$$g^{\lambda\mu}(x) g_{\lambda\nu}(x) := \delta^\mu_\nu, \quad \eta^{ab} \eta_{bc} := \delta^a_c, \quad \vartheta^a_\mu(x) c^\nu_a(x) := \delta^\nu_\mu, \quad \vartheta^a_\mu(x) c^\mu_b(x) = \delta^a_b, \quad \tag{5}$$

we can invert (4)

$$g_{\mu\nu}(x) = \eta_{ab} \vartheta^a_\mu(x) \vartheta^b_\nu(x), \quad \rightarrow \quad g^{\mu\nu}(x) = \eta^{ab} c^\mu_a(x) c^\nu_b(x), \quad \tag{6}$$

and introduce in place of (2) new basis systems of vector fields and 1-forms, respectively, $a = 0,\ldots, 3$,

$$e_a = c^\mu_a(x) \partial_\mu, \quad \Theta^a = \vartheta^a_\mu(x) dx^\mu, \quad \tag{7}$$

implying

$$\Theta^a(e_b) \equiv \langle \Theta^a, e_b \rangle = \delta^a_b, \quad g = \eta_{ab} \Theta^a \otimes \Theta^b, \quad \tag{8}$$

hence providing an orthonormal basis in $T^1_0$.

From the Lorentzian metric (3) follow the associated Levi-Civita connection 1-forms

$$w^\lambda_\mu(\cdot) = \Gamma^\lambda_\mu_\rho(x) dx^\rho \in T^1_0, \quad \lambda, \mu = 0,\ldots, 3 \quad \tag{9}$$

with the Christoffel symbols

$$\Gamma^\lambda_\mu_\nu(x) = \frac{1}{2} g^{\lambda\sigma}(x) \{ \partial_\nu g_{\sigma\mu}(x) + \partial_\mu g_{\sigma\nu}(x) - \partial_\sigma g_{\mu\nu}(x) \} = \Gamma^\lambda_\nu_\mu(x). \quad \tag{10}$$

The covariant derivative $\nabla_V$ in direction of a vector field $V = v^\mu(x) \partial_\mu$ is determined when defined on a basis or, equivalently, on a cobasis,

$$\nabla_V \partial_\mu = w^\lambda_\mu(V) \partial_\lambda, \quad \nabla_V dx^\lambda = - w^\lambda_\mu(V) dx^\mu. \quad \tag{11}$$

Herefrom follows the covariant derivative of the orthonormal basis (7) as

$$\nabla_V e_b = \omega^a_b(V) e_a, \quad \nabla_V \Theta^a = - \omega^a_b(V) \Theta^b, \quad \tag{12}$$

implying

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with transformed connection forms

\[ \omega^a_{b}(V) = \partial^a_{\lambda} \omega^{\lambda}_{\mu}(V) c^\mu_b + \partial^a_{\lambda} (V c^\lambda_b). \]  

(13)

Using successively (13), (9), (7) provides the relation

\[ \omega^a_{b}(e_a) = c^\lambda_b = \text{div} e_b, \]  

(14)

the semicolon denoting the covariant derivative.

A lucid presentation of the classical Dirac equation on a manifold is contained in [Di]. Here we collect some elements necessary for a quantum version, thereby setting the notation used.

The generators of the Dirac algebra on Minkowski space (Dirac matrices) \( \gamma^a \in \text{Mat}_4(\mathbb{C}), a = 0, 1, 2, 3, \) satisfy

\[ \gamma^a \gamma^b + \gamma^b \gamma^a = 2 \eta^{ab} \text{id}. \]  

(15)

In addition, we require the property on Hermitean conjugation

\[ (\gamma^a)^* = \gamma^0 \gamma^a \gamma^0, \quad a = 0, \cdots, 3, \]  

(16)

but impose no further constraint, unless stated explicitly.

There is a two-to-one homomorphism of the Lie group \( \text{Spin}(1,3) \ni S \) onto the full Lorentz group \( \text{O}(1,3) \ni \Lambda, \)

\[ \det S = 1, \quad S^{-1} \gamma^a S = \Lambda^a_b \gamma^b. \]  

(17)

In the sequel we only consider the restriction of (17) to the respective subgroups connected to the identity, i.e. \( S \in \text{Spin}_0(1,3) \) and \( \Lambda \in \text{SO}(1,3)^\dagger. \) \( \text{Spin}_0(1,3) \) is isomorphic to \( \text{SL}(2,\mathbb{C}). \)

Given any differentiable function \( \Lambda : \mathcal{M} \rightarrow \text{SO}(1,3)^\dagger, \) a local 'gauge' transformation of the respective basis systems (7),(8) in \( T^*_0 T^1_0(\mathcal{M}) \) and \( T^*_1 T^0_0(\mathcal{M}), \)

\[ (\Theta')^a := \Lambda^a_b(x) \Theta^b, \quad e'_b := e_c(\Lambda^{-1}(x))^c_b \]  

(18)

yields equivalent basis systems.

A Dirac spinor field on a manifold is a function \( \psi : \mathcal{M} \rightarrow \mathbb{C}^4, \) which is required to transform as a Dirac spinor with respect to a local gauge transformation of the frames, (18). Thus, there is assumed an associated function \( S : \mathcal{M} \rightarrow \text{Spin}_0(1,3) \) such that the homomorphism (17), \( S \rightarrow \Lambda, \) holds locally. Then, introducing a moving frame \( E_A(x), A = 1, \cdots, 4, \) in the space of
spinor fields, $\psi(x) = \psi^A(x)E_A(x)$ (summation convention), the companion transformation to (18) is given by

$$
(\psi')^A = S(x)^A_B \psi^B, \quad E'_A = E_C(S(x)^{-1})_C^A.
$$

(19)

Depending on the global topological structure of $\mathcal{M}$, however, there may arise an obstruction to “lifting” the function $\Lambda$ from the non-simply connected group $SO(1, 3)^\dagger$ to its simply connected covering group $Spin_0(1, 3)$. These problems, dealt with in [Is], are not pursued here.

The components of the Dirac-adjoint spinor are defined as

$$
\psi^+_A := (\gamma^0)_B^A \psi^B = \sum_B \overline{\psi^B} (\gamma^0)_B^A.
$$

(20)

Under a frame transformation (19) they transform as $(\psi')^+_A = \psi^+_C (S^{-1})_C^A$, i.e. as the components of a cospinor, because of $\gamma^0 S^* \gamma^0 = S^{-1}$ implied by (17), (16). In the space of cospinors a dual moving frame $E^A, A = 1, \ldots, 4$, can be introduced, $E^A(E_B) = \delta^A_B$, then $\psi^+ = \psi^+_A E^A$.

From the connection forms (12) of the orthonormal frames via the map (17) follows the spin connection form, with $\omega_{ac}(V) := \eta_{ab} \omega_c^b(V)$,

$$
W(V) = \frac{1}{2} \omega_{ac}(V) \sigma^{ac}, \quad \sigma^{ab} = \frac{1}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a),
$$

(21)

determining the covariant derivatives of the spinor and cospinor frames, respectively, in direction of the vector field $V$,

$$
\nabla_V E_A = E_B W(V)_A^B, \quad \nabla_V E^A = -W(V)^A_B E^B.
$$

(22)

Denoting by $\nabla_a$ the covariant derivative in the direction $V = e_a$, (7), the Dirac equation for a spinor field $\psi = \psi^A E_A$ reads

$$
(-i \gamma^a \nabla_a + m) \psi = 0.
$$

(23)

Therefrom follow via the Leibniz rule and (22) the equations to be fulfilled by the spinor components,

$$
(-i \gamma^a (e_a + W(e_a)) + m)_B^A \psi^B = 0.
$$

(24)

\footnote{On a globally hyperbolic Lorentzian manifold this lifting can be shown to exist, [Ge].}
Equivalent with these equations the components of the Dirac-adjoint spinor satisfy

$$0 = \psi^+_A(i(\bar{e}_a - W(e_a)) \gamma^a + m)^A_B .$$

(25)

Given two solutions \((\psi^A), (\chi^A)\) of the Dirac equation \((24)\), or correspondingly of \((25)\), the bilinear current

$$J = j^\mu \partial_\mu, \quad j^\mu := c^\mu_a \psi_A^+ (\gamma^a)^A_B \chi^B$$

(26)

is covariantly conserved, (observe \((14)\)),

$$\text{div} J = j^\mu_{\mu} = 0 .$$

(27)

Furthermore, the Dirac operator \(\gamma^a \nabla_a\) satisfies the Lichnerowicz identity \([Li]\),

$$(-i\gamma^a \nabla_a + m)(i\gamma^b \nabla_b + m) \psi = (\eta^{ab} \nabla_a \nabla_b - \frac{1}{4} R + m^2) \psi,$$

(28)

where \(R\) denotes the scalar curvature of the manifold. The principal part of the spinor wave operator \(\eta^{ab} \nabla_a \nabla_b\) appearing on the r.h.s. is diagonal and coincides with the principal part of the scalar Klein-Gordon operator.

# 3 Quantum Theory

In the sequel we restrict to a static Lorentzian manifold \((\mathcal{M}, g)\), hence, topologically \(\mathcal{M} = R \times \Sigma\). With local coordinates \(x = (t, \vec{x})\) the metric \((3)\) has components of the form

$$g_{00} = (q(\vec{x}))^{-2}, \quad \mu, \nu \in \{1, 2, 3\} : \quad g_{0\nu} = 0, \quad g_{\mu\nu} = -h_{\mu\nu}(\vec{x}),$$

(29)

where \(q(\vec{x}) > 0\) and \((h_{\mu\nu}) \in Mat_3(R)\) is positive definite. This metric shows the formal reflection symmetry \(t \to -t\) and \(\partial_t\) is a Killing vector field. Upon requiring in addition

A1) \(0 < c_1 < q(x) < c_2 < \infty\),

A2) \((\Sigma, h)\) is a complete Riemannian (d=3) manifold, the space-time manifold \((\mathcal{M}, g)\), \((29)\), is globally hyperbolic, [Ka]. For simplicity \((\mathcal{M}, g)\) is taken to be \(C^\infty\). The orthonormal basis vector fields \((7)\) emerging from \((29)\) are

$$c^\mu_0 = q(\vec{x}) \delta^\mu_0, \quad r, s \in \{1, 2, 3\} : \quad c^0_r = 0, \quad c^\mu_r h_{\mu\nu}(\vec{x}) c^\nu_s = \delta_{rs},$$

(30)
implying the related dual basis according to (5).
We now refine our notation and denote frame indices which are confined to the (spatial) values 1, 2, 3 by letters \( r, s, \ldots \) from the end of the Roman alphabet and also use a summation convention covering these values only. From (29) follow the Levi-Civita connection forms (13),

\[
\omega^0_s(e_0) = - (\partial_\mu \ln q) \sigma^0_s, \quad \omega^r_s(e_0) = 0, \quad \omega^0_s(e_v) = 0, \quad \omega^r_s(e_v) \neq 0. \tag{31}
\]

Moreover, observing the Riemannian volume density \(|g|^{1/2} = q^{-1/2} |h|^{1/2}\) on \(\mathcal{M}\), we obtain from (14) together with \(\omega^0_s(e_0)\) from (31),

\[
\omega^s_r(e_s) = \text{div}_3 c_r := |h|^{-1/2} \partial_\lambda (|h|^{1/2} c^\lambda_r). \tag{32}
\]

The spin connection forms (21) follow as

\[
W(e_0) = -(\partial_\mu \ln q) c^\mu_s \sigma^{0s}, \quad W(e_v) = \frac{1}{2} \omega_{rs}(e_v) \sigma^{rs}. \tag{33}
\]

We write the Dirac equation (24) as an evolution equation to be read as a matrix equation, the spinor components \((\psi^B)\) forming a column matrix, and introduce the time-honoured Hermitian matrices \(\alpha^r = \gamma^0 \gamma^r, \quad \beta = \gamma^0\),

\[
\begin{align*}
\bar{q}(\vec{x}) \partial_t \psi &= (K - iW(e_0)) \psi, \\
K &= -i \alpha^r e_r - i \alpha^r W(e_r) + m \beta.
\end{align*} \tag{34}
\]

A Hilbert space \(\mathcal{H}\) formed of Dirac spinors \(\psi, \chi, \ldots\) emerges from the inner product (at fixed \(t\)),

\[
(\psi, \chi) := \int_\Sigma d^3 x |h|^{1/2} \sum_{A=1}^4 \overline{\psi}^A \chi^A, \tag{35}
\]

suggested by the conserved current (26), upon completion. We first notice, with \(\psi, \chi \in \mathcal{C}_0^\infty(\Sigma)^4\), i.e. smooth functions of compact support,

\[
(\psi, W(e_0) \chi) = (W(e_0) \psi, \chi), \quad (\psi, K \chi) = (K \psi, \chi), \tag{36}
\]

using (32) in the latter case. The multiplication operator \(q\) defined on \(\mathcal{H}\) is bounded, positive, self-adjoint and invertible. Furthermore, on the domain \(\mathcal{C}_0^\infty(\Sigma)^4 \subset \mathcal{H}\) holds the commutation relation

\[
\left[ K, \frac{1}{q} \right] = -2i \frac{1}{q} W(e_0). \tag{37}
\]
Hence, the operator
\[ H = \frac{1}{q} \left( K - iW(e_0) \right) = \frac{1}{2} \left( \frac{1}{q} K + K \frac{1}{q} \right) \]
(38)
is symmetric on this domain, too,
\[ (\psi, H \chi) = (H \psi, \chi) . \]
(39)
We assume, that it has a unique self adjoint extension, called the Hamilton operator in \( \mathcal{H} \) and denoted again by \( H \) in the sequel. For any initial value \( \psi_0 \in \mathcal{H} \) at \( t = 0 \) then
\[ \psi(t, \vec{x}) = e^{-itH} \psi_0 \]
(40)
is a solution of (34) in \( \mathcal{H} \), and the inner product (35) of two such solutions does not depend on time. The operator \( H \), however, is not bounded from below, hence does not allow to consider \( \mathcal{H} \) as a one-particle space. But \( \mathcal{H} \) has an inherent conjugation symmetry. To exhibit this symmetry, one observes that the generators of the Dirac algebra (15) in the standard representation \( \mathbb{R} \) (or in the Weyl representation) have in addition to (16) the property
\[ a = 0, 1, 3 : \quad \tilde{\gamma}^a = \gamma^a, \quad \tilde{\gamma}^2 = -\gamma^2 . \]
(42)
Henceforth, we require this property, too, shown by any representation resulting from the stated ones by a real unitary similarity transformation. Given such a representation, the real matrix \( C := i\gamma^2 \), hence \( C = C^* = C^{-1} \), implies
\[ C \tilde{\gamma}^a C^* = -\gamma^a, \quad a = 0, ..., 3 . \]
(43)
Defining then in \( \mathcal{H} \) the antiunitary involution \( A \),
\[ \psi \in \mathcal{H} : \quad A \psi = C \overline{\psi} , \]
(44)
the Hamilton operator (38) satisfies
\[ A H = -H A . \]
(45)
\[ \gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} , \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} , \quad k = 1, 2, 3 , \]
(41)
with \( \sigma_0 \) the \( 2 \times 2 \) unit matrix and \( \sigma_k \) the standard Pauli matrices.
The spectral representation of $H$ provides in $\mathcal{H}$ the orthogonal projection operators $P_+ > 0$ and $P_- < 0$ corresponding to the positive and negative part of the spectrum, respectively, $P_+ + P_- = id$, a separating gap is tacitly assumed. Defining $\mathcal{H}_+ = P_+ \mathcal{H}, \mathcal{H}_- = P_- \mathcal{H}$, the Hilbert space decomposes as a sum of two orthogonal subspaces,

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-.$$  \hspace{1cm} (46)

Furthermore, the conjugation (43),

$$A : \mathcal{H}_+ \to \mathcal{H}_-$$  \hspace{1cm} (47)

is a bijective map.

4 Quantum Field Theory

The deficient quantum theory of the previous section is elevated to a consistent quantum field theory using Cook’s method [Co] of second quantization. In addition to the Hilbert space $\mathcal{H}$, (46), a “physical” Hilbert space $\mathcal{H}'$ is introduced,

$$\mathcal{H}' = \mathcal{H}_p \oplus \mathcal{H}_{\bar{p}}.$$  \hspace{1cm} (48)

related to $\mathcal{H}$ by the map

$$\nu : \mathcal{H} \to \mathcal{H}', \quad \nu = I_p P_+ + I_{\bar{p}} A P_-,$$  \hspace{1cm} (49)

with identification maps $I_p$ on $\mathcal{H}_p$ and $I_{\bar{p}}$ on $\mathcal{H}_{\bar{p}}$. The Fock space is built on $\mathcal{H}'$ in the standard way,

$$\mathcal{F} = C\Omega \oplus \bigoplus_{n=1}^{\infty} (\mathcal{H}')^{\otimes n}_{\text{as}}$$  \hspace{1cm} (50)

with the one-dimensional subspace spanned by the vacuum vector $\Omega$ and a direct sum of totally antisymmetrized tensor products of $\mathcal{H}'$. Let $\langle \cdot, \cdot \rangle$ denote the scalar product of $\mathcal{F}$, (50), and the vacuum vector $\Omega$ being normalized, $\langle \Omega, \Omega \rangle = 1$. In $\mathcal{F}$ annihilation operators $c(f')$ with $f' = \nu f, f \in \mathcal{H}$ are defined, their adjoint $c^*(f')$ acting as creation operators. We split in notation, cf.(49), and denote by $a(P_+ f)$ and $b(AP_- f)$ the annihilation operator of a

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3early applications of this method are e.g. [Bo], [DM].
particle and an antiparticle, respectively. These operators and their adjoints satisfy the anticommutation relations, \( f, g \in \mathcal{H} \),
\[
\{ a(P_f), a^*(P_g) \} = (P_f, g) \tag{51}
\]
\[
\{ b(AP_f), b^*(AP_g) \} = (AP_f, AP_g) = (P_f, g),
\]
all other anticommutators vanishing. Given these operators acting in \( \mathcal{F} \), the (smeared) field operator at time \( t = 0 \) is defined as
\[
\Psi(f) := a(P_f) + b^*(AP_f), \tag{52}
\]
satisfying
\[
\{ \Psi(f), \Psi^*(g) \} = (f, g). \tag{53}
\]
One should notice that in the definition (52) of the smeared field operator \( f \in \mathcal{H} \) acts antilinearly on the operator-valued distribution \( (\Psi^A(\vec{x})) \),
\[
\Psi(f) = \int d^3x |h|^{1/2} \sum_A \overline{F^A(\vec{x})} \Psi^A(\vec{x}). \tag{54}
\]
Finally the time dependence of the field operator follows from (52) as
\[
\Psi(t, f) := \Psi(e^{itH} f), \tag{55}
\]
implying the linear field equation
\[
\Psi((i\partial_t + H) e^{itH} f) = 0. \tag{56}
\]
The anticommutation relations of the time dependent fields result from (53) as,
\[
\{ \Psi(t, f), \Psi^*(t', f') \} = (f, e^{-i(t-t')H} f'), \tag{57}
\]
the other ones vanishing. Moreover, if we restrict \( f, f' \in \mathcal{H} \) to functions \( f, f' \in C_0^\infty(\Sigma)^4 \) such that the spacetime domains \( (t, \text{supp } f) \) and \( (t', \text{supp } f') \) are causally disjoint (c.d.), i.e. there is no spacetime point in one of these domains, that can be connected with any point of the other by a forward or a backward time-like or light-like curve, then
\[
\{ \Psi(t, f), \Psi^*(t', f') \} = 0, \text{ if (c.d.) } (t, \text{supp } f) \text{ and } (t', \text{supp } f'). \tag{58}
\]
This property eventually is a consequence of the finite propagation speed of a hyperbolic differential equation. Although the Dirac equation itself is not
hyperbolic, the r.h.s. of the Lichnerowicz identity \((28)\) is, herefrom the claim can be deduced, see [Di].

In \(\mathcal{F}\) the operator \(U_c\) of particle-antiparticle conjugation, defined as

\[
U_c \Omega = \Omega, \quad U_c \Psi(f) U^*_c = \epsilon \Psi^*(Af)
\]

with a (physically irrelevant) phase factor \(\epsilon, |\epsilon| = 1\), and the antiunitary operator \(A\) in \(\mathcal{H}\) from \((44)\) is a unitary operator. From \((59)\) then follows the transformation of the Heisenberg field operator

\[
U_c \Psi(t, f) U^*_c = \epsilon \Psi^*(t, Af)
\]

5 Green functions and Schwinger functions

The theory considered being linear, all \(n\)-point Wightman functions are sums (with signs) of products of 2-point functions. Furthermore, a non-vanishing \(n\)-point function requires \(n = 2l, l \in \mathbb{N}\), and has to be composed of \(l\) factors \(\Psi\) and of \(l\) factors \(\Psi^*\). The non-vanishing 2-point functions follow as

\[
\langle \Omega, \Psi(t', f') \Psi^*(t, f) \Omega \rangle = \langle \Omega, \Psi(A'f') \Psi^*(A f) \Omega \rangle.
\]

The spectral representation \((61)\) of these 2-point functions can be analytically continued to complex values of the time coordinates, \(t \to t + i\tau, t' \to t' + i\tau'\), \(\tau, \tau' \in \mathbb{R}\). Defining \(z = t - t' + i(\tau - \tau')\) with \(s := \text{Im} z = (\tau - \tau')\) we obtain from \((61)\) by analytic continuation \(t - t' \to z\) the corresponding functions \(F^{(+)}(z; f, f')\), holomorphic in \(\text{Im} z < 0\), and \(F^{(-)}(z; f, f')\), holomorphic in \(\text{Im} z > 0\). The original 2-point functions, which are continuous functions of \(t - t'\), appear as boundary values on the real axis,

\[
\langle \Omega, \Psi(t, f) \Psi^*(t', f') \Omega \rangle = \lim_{\epsilon \to 0^+} F^{(+)}(t - t' - i\epsilon; f, f'),
\]

\[
\langle \Omega, \Psi^*(t', f') \Psi(t, f) \Omega \rangle = \lim_{\epsilon \to 0^-} F^{(-)}(t - t' + i\epsilon; f, f').
\]
Considering these holomorphic functions $F^{(\pm)}(z; f, f')$ in their respective domains at \textit{“ imaginary time”} $z = is$, we notice on the functional diagonal $f = f'$ the following positivity properties,

\begin{align}
F^{(+)}(is; f, f) &\geq 0, \quad s < 0, \\
F^{(-)}(is; f, f) &\geq 0, \quad s > 0. 
\end{align} \tag{64}

Since $s = \tau - \tau'$ we observe, that these positivity properties hold, if the product of fields entering the 2-point function is ordered (from left to right) according to \textit{increasing} values of \textit{imaginary time}. In particular, this ordering is given by requiring the factor on the right(left) to have positive(neg)ative \textit{imaginary time}. The resulting positivity then implies \textit{reflection positivity}, which forms the basis in an Euclidean approach on a \textit{“ static”} Riemannian manifold \cite{JR}.

Moreover, restricting now $f, f' \in \mathcal{H}$ to functions $f, f' \in C_0^\infty(\Sigma)^4$ which have disjoint supports, $\text{supp } f \cap \text{supp } f' = \emptyset$, then according to (58) there exists an intervall $|t - t'| < d \equiv d(\text{supp } f, \text{supp } f')$, such that for corresponding values of $t - t'$ holds

\begin{align}
\lim_{\epsilon \searrow 0} F^{(+)}(t - t' - i\epsilon; f, f') + \lim_{\epsilon \searrow 0} F^{(-)}(t - t' + i\epsilon; f, f') = 0. \tag{65}
\end{align}

Hence, due to a theorem of Painlevé, \cite[Theor.2.13]{SW} or \cite[Theor. 7.7.1]{Hi}, there is a single function $F(z; f, f')$, which is holomorphic in the \textit{cut plane} $z \in \mathbb{C} \setminus (\mathbb{R} \setminus I)$ and continuous at the boundary, where $I$ denotes the real intervall $I = (\text{Im } z = 0, |\text{Re } z| < d)$, such that

\begin{align}
F(z; f, f') = \begin{cases} 
+ F^{(+)}(z; f, f'), & \text{Im } z \leq 0, \\
- F^{(-)}(z; f, f'), & \text{Im } z \geq 0.
\end{cases} \tag{66}
\end{align}

Because of the cut on the real axis for $(z \in \mathbb{R}, |z| > d)$ there generically arise different boundary values from above and below.

In addition, we can define a \textit{“ Wick-rotated”} version of this function,

\begin{align}
F_E(\zeta; f, f') := F(i\zeta; f, f'), \quad \zeta \in \mathbb{C}, \tag{67}
\end{align}

with a domain of holomorphy resulting in an obvious way from that of the function $F$. For real values of $\zeta$, corresponding to \textit{“ imaginary time”}, then emerges the Schwinger function,

\begin{align}
s \in \mathbb{R}: \quad F_E(s; f, f') = \theta(-s)(f, P_+ e^{sH} f') - \theta(s)(f, P_- e^{sH} f'). \tag{68}
\end{align}
The Schwinger function (68) satisfies
\[ \partial_s F_E(s; f, f') - F_E(s; Hf, f') = -\delta(s) \langle f, f' \rangle, \]
where the r.h.s., however, vanishes because of the assumed support properties of the functions \( f, f' \). We notice, that the function (68) considered autonomously, is also well defined for functions \( f, f' \in C^\infty_0(\Sigma)^4 \) without the restriction on their supports.

From the Schwinger function (68) one recovers the Wightman functions (63) by analytic continuation, with \( \epsilon > 0 \),
\[ \lim_{\epsilon \searrow 0} F_E(-\epsilon - i(t - t'); f, f') = \lim_{\epsilon \searrow 0} F^{(+)}(t - t' - i\epsilon; f, f') \]
\[ \lim_{\epsilon \searrow 0} F_E(\epsilon - i(t - t'); f, f') = -\lim_{\epsilon \searrow 0} F^{(-)}(t - t' + i\epsilon; f, f'). \]

Wightman functions involving in place of the adjoint field operator \( \Psi^*(t, f) \) the Dirac-adjoint field field operator \( \Psi^+(t, f) \) are easily obtained from the former ones. By definition, cp.(20), the Dirac-adjoint of the field operator (55) is given by
\[ \Psi^+(t, f) = \Psi^*(t, \gamma^0 f). \]

Hence, substituting in (61) the function \( f' \) by \( \gamma^0 f' \) converts the adjoint field operator into the Dirac-adjoint field operator according to (71).

Finally, as a consequence of (70) the time-ordered product \(^4\) follows,
\[ \langle \Omega, T\Psi(t, f)\Psi^*(t', f') \Omega \rangle = \lim_{\epsilon \searrow 0} F_E(-i(t - t')(1 - i\epsilon); f, f'). \]

The approach includes Minkowski space, of course, as a particular instance. Then we have \( \Sigma = \mathbb{R}^3 \) and the metric (29) has the particular form
\[ q(\vec{x}) \equiv 1, \quad h_{\mu\nu}(\vec{x}) = \delta_{\mu\nu}, \quad \mu, \nu \in \{1, 2, 3\}. \]

The Wightman functions (63), transcribed to involve the Dirac-adjoint field, have the particular form
\[ \lim_{\epsilon \searrow 0} F^{(\pm)}(t - t' \mp i\epsilon; f, \gamma^0 f') = \]
\[ \frac{1}{i} \int d^3x \int d^3x' \sum_{A,B} \overline{f^A}(\vec{x}) S^{(\pm)}(t - t', \vec{x} - \vec{x}') A_B f'^B(\vec{x}') \]

\(^4\)with standard time ordering of fermion operators.
where \( S^{(\pm)}(x-x') \) are the familiar 2-point functions of the free Dirac field,

\[
\frac{1}{i} S^{(\pm)}(x-x') = (2\pi)^{-3} \int \frac{d^3 p}{2p^0} (\gamma p \pm m)e^{\mp ip(x-x')},
\]

with \( p^0 = (\vec{p}^2 + m^2)^{1/2} \), see eg. [IZ].

\section{Thermal Equilibrium}

With the notation introduced before, a real parameter \( \beta \), where \( 0 < \beta < \infty \), and \( f, f' \in C_0^\infty(\Sigma)^4 \), we define the functions

\[
F^{(+)}_{\beta}(z; f, f') := \left( f, P_+ \frac{e^{-izH}}{1 + e^{-\beta H} f'} \right) + \left( f, P_- \frac{e^{-izH+\beta H}}{1 + e^{\beta H} f'} \right),
\]

\[
\text{for } -\beta \leq \text{Im } z \leq 0;
\]

\[
F^{(-)}_{\beta}(z; f, f') := \left( f, P_+ \frac{e^{-izH-\beta H}}{1 + e^{-\beta H} f'} \right) + \left( f, P_- \frac{e^{-izH}}{1 + e^{\beta H} f'} \right),
\]

\[
\text{for } 0 \leq \text{Im } z \leq \beta.
\]

\( F^{(+)}_{\beta}(z; f, f') \) is a holomorphic function in the open strip \(-\beta < \text{Im } z < 0\), whereas \( F^{(-)}_{\beta}(z; f, f') \) is holomorphic in the open strip \( 0 < \text{Im } z < \beta \), and both functions are continuous on the respective boundary.

For \( \beta \rightarrow \infty \) these functions have as limits the functions \( F^{(\pm)} \) defined before after \[62\],

\[
\lim_{\beta \rightarrow \infty} F^{(+)}_{\beta}(z; f, f') = \left( f, P_+ e^{-i z H} f' \right) = F^{(+)}(z; f, f'), \quad \text{Im } z \leq 0,
\]

\[
\lim_{\beta \rightarrow \infty} F^{(-)}_{\beta}(z; f, f') = \left( f, P_- e^{-i z H} f' \right) = F^{(-)}(z; f, f'), \quad \text{Im } z \geq 0.
\]

Furthermore, considering the functions \[76\] at \( \text{“imriy maime time” } z = is \), we notice on the functional diagonal \( f = f' \), similarly as in \[64\], the positivity properties,

\[
F^{(+)}_{\beta}(is; f, f) \geq 0, \quad -\beta \leq s \leq 0,
\]

\[
F^{(-)}_{\beta}(is; f, f) \geq 0, \quad 0 \leq s \leq \beta.
\]

\[5\] These definitions are suggested by heuristically manipulating the Gibbs formula.
Reading again \( s = \tau - \tau' \) as difference in imaginary time, positivity obviously requires an ordering condition to be satisfied: \( 0 < \tau' - \tau < \beta \) in the first inequality, and \( 0 < \tau - \tau' < \beta \) in the second one. Moreover, with \( \beta \) fixed, due to their very definition \((76)\) the functions \( F_{\beta}^{(\pm)} \) are related in the respective closure of the domain of holomorphy,

\[
0 \leq \text{Im} \, z \leq \beta : \quad F_{\beta}^{(\pm)}(z - i\beta ; f, f') = F_{\beta}^{(\mp)}(z ; f, f') . \tag{79}
\]

This is the Kubo-Martin-Schwinger (KMS) condition [Ku],[MS] characterising thermal equilibrium at temperature \( T = 1/\beta \), [HHW]. Furthermore, in case of real values of \( z \) follows from the definitions \((76)\) with \( t - t' \in \mathbb{R} \) :

\[
\lim_{\varepsilon \searrow 0} F_{\beta}^{(+))(t-t'-i\varepsilon ; f, f')} + \lim_{\varepsilon \searrow 0} F_{\beta}^{(-)(t-t'+i\varepsilon ; f, f')} = (f, e^{-i(t-t')H} f') . \tag{80}
\]

Requiring now the functions \( f, f' \in C_{0}^{\infty}(\Sigma)^{4} \) to have disjoint supports, \( \text{supp} \, f \cap \text{supp} \, f' = \emptyset \), then as in the previous case \((65)\) the r.h.s. of \((80)\) vanishes for all \( t - t' \) satisfying \(|t - t'| < d \equiv d(\text{supp} \, f, \text{supp} \, f')\), because of finite propagation speed. (See the remarks after \((57)\), \((58)\).) Hence, similarly as in \((66)\), the Painlevé theorem again yields a single function \( F_{\beta}(z ; f, f') \), holomorphic in the cut open strip \((-\beta < \text{Im} \, z < \beta) \setminus (\mathbb{R} \setminus I)\) with a gate on the real axis given by the intervall \( I = (\text{Im} \, z = 0, |\text{Re} \, z| < d)\),

\[
F_{\beta}(z ; f, f') = \begin{cases} 
+ F_{\beta}^{(+)}(z ; f, f'), & -\beta \leq \text{Im} \, z \leq 0, \\
- F_{\beta}^{(-)}(z ; f, f'), & 0 \leq \text{Im} \, z \leq \beta.
\end{cases} \tag{81}
\]

The boundary values on the real axis are continuous, but the cut has to be observed. From the KMS condition \((79)\) then follows that this analytic function \( F_{\beta}(z ; f, f') \) satisfies

\[
0 \leq \text{Im} \, z \leq \beta : \quad F_{\beta}(z - i\beta ; f, f') = - F_{\beta}(z ; f, f') , \tag{82}
\]

i.e. is anti-periodic in imaginary time. The relation \((82)\) can also be used as a definition to extend the function \( F_{\beta}(z ; f, f') \) by analytic continuation from the strip \( 0 \leq \text{Im} \, z \leq \beta \) to the adjacent strip \( \beta \leq \text{Im} \, z \leq 2\beta \) and so on, as well as proceeding similarly from the strip \(-\beta \leq \text{Im} \, z \leq 0 \) downward. Thus results a function \( F_{\beta}(z ; f, f') \) which is holomorphic in the cut plane \( z \in \mathbb{C} \setminus L \) with the set of cuts \( L = \{ t + in\beta : t \in \mathbb{R}, |t| > d, n \in \mathbb{Z} \} \), and continuous on the boundary,
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