Exact inversion of the conical Radon transform with a fixed opening angle

Rim Gouia-Zarrad\textsuperscript{1} and Gaik Ambartsoumian\textsuperscript{2}

\textsuperscript{1} Department of Mathematics and Statistics, American University of Sharjah, PO Box 26666, Sharjah, UAE
\textsuperscript{2} Department of Mathematics, University of Texas at Arlington, Arlington, TX 76017-0408, USA
E-mail: rgouia@aus.edu and gambarts@uta.edu

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Abstract
We study a new class of Radon transforms defined on circular cones called the conical Radon transform. In $\mathbb{R}^3$ it maps a function to its surface integrals over circular cones, and in $\mathbb{R}^2$ it maps a function to its integrals along two rays with a common vertex. Such transforms appear in various mathematical models arising in medical imaging, nuclear industry and homeland security. This paper contains new results about inversion of conical Radon transform with fixed opening angle and vertical central axis in $\mathbb{R}^2$ and $\mathbb{R}^3$. New simple explicit inversion formulae are presented in these cases. Numerical simulations were performed to demonstrate the efficiency of the suggested algorithm in 2D.

Keywords: conical Radon transform, V-line transform, broken-ray transform, integral geometry, image reconstruction

(Some figures may appear in colour only in the online journal)

1. Introduction
The conical Radon transform (CRT) integrates a function over circular cones in $\mathbb{R}^3$, and along coupled rays with a common vertex in $\mathbb{R}^2$. The interest towards such transforms during the last decade was triggered by the connection between the CRT and some mathematical models arising in medical imaging [4–8, 10–12, 20, 22, 23, 26], homeland security [2], nuclear industry and astrophysics [25].

The first work concerning CRT was done by Cree and Bones who considered the three-dimensional problem in relation to Compton cameras [8]. They showed that in the case when the vertices of integration cones are restricted to a plane, while their symmetry axis is normal to the plane and the opening angles vary, the CRT can be uniquely inverted.

Basko and collaborators analyzed the related problem for the planar CRT with the vertices of coupled rays restricted to a line and the image function $f$ supported on one side of that line.
They showed that the problem of inverting the CRT in this setup can be reduced to the inversion of the Radon transform of another function related to $f$ by a certain symmetry. They extended their idea to the 3D case in [7] with the aid of a spherical harmonics expansion, in order to convert the cone-surface integrals into the classical Radon projections.

Some further results for 2D and 3D CRT in these setups were presented by Nguyen, Truong and collaborators. As in the previous works, they considered CRT with vertices restricted to a plane in 3D and a line in 2D, and varying opening angle. We refer the readers to [20, 22, 23, 26] for further details.

In a series of works [10–12] Florescu, Markel, and Schotland investigated image reconstruction problems from CRT appearing in 2D optical tomography. Here the axes of symmetry and the opening angle between the coupled rays of integration were fixed, while the locations of their vertices were not restricted. The authors proved that in this setup CRT can be inverted and provided an exact reconstruction formula.

Ambartsoumian studied the related problem in a circular setup, where the opening angle of CRT was fixed, one of the coupled rays was normal to the unit circle, while the locations of vertices varied inside the disc [4]. For functions supported sufficiently deep inside the disc he showed that the inversion problem of CRT can be reduced to the inversion of the classical Radon transform. In a subsequent work [5] Ambartsoumian and Moon showed that CRT in this setup has a unique inversion without any restriction on the support of $f$ and provided an exact inversion formula.

It must be noted that different authors use different terminology to denote CRT in 2D. Nguyen, Truong et al., as well as Ambartsoumian and Moon call it a V-line transform (since the integration is done along V-shaped trajectories). Florescu et al call this transform a broken-ray transform. The latter term is used by other authors to denote a transform integrating along trajectories made of multiple connected linear intervals, e.g. see [16, 17] and the references there. To avoid this type of confusion in this paper we will follow the first choice and call the 2D CRT a V-line transform.

In this paper we concentrate on the problem of recovering a function from its conical projections in the case of a fixed opening angle and a vertical central axis in both 2D and 3D. The geometry and the setup of CRT in $\mathbb{R}^3$ is different from the ones considered in previous works and we provide new inversion results for this case. In $\mathbb{R}^2$ the CRT that we consider is similar to the one studied by Florescu et al [10–12], but we provide a much simpler inversion formula, which holds for a wider range of opening angles, and demonstrate its efficiency with numerical simulations. For the details about the physical aspects of the proposed setup (fixed central axis and opening angle) and its applicability in imaging we refer the readers to [10–12], whose authors are experts (among other things) in physics and biomedical engineering.

The rest of this paper is organized as follows. Section 2 is devoted to the introduction of the concept of CRT in $\mathbb{R}^2$ and $\mathbb{R}^3$. In section 3, we present new mathematical results on the inversion of the two-dimensional CRT and derive an explicit inversion formula. The proof is based on the use of the Fourier transform and the theory of integral equations. In section 4, we study the three-dimensional case. New explicit inversion formula is derived for the CRT on a special class of cones. In section 5, numerical simulations are presented to demonstrate the efficiency of the suggested 2D algorithm. Additional remarks are given in the last section with acknowledgments and bibliography.

2. Formulation of the problem

Consider an $n$-dimensional $(x, z)$ space, where $n = 2$ or $3$, and $x = (x_1, \ldots, x_{n-1})$. We consider a family of circular cones $C(V, \beta, n)$ parameterized by a vertex $V$ of
coordinates \((x_v, z_v)\), central axis vector \(n\), and a half opening angle \(\beta \in (0, \frac{\pi}{2})\) (see figure 1).

If \(n = 3\) we define the CRT of a given function \(f(x, z)\) to be the surface integral of the function along a cone \(C(V, \beta, n)\)

\[
g(V, \beta, n) = \int_{C(V, \beta, n)} f(x, z) \, ds,
\]

where \(ds\) is the surface element on the cone \(C(V, \beta, n)\).

If \(n = 2\) the above integration is done along a pair of rays with the same vertex \(V\) symmetrically located on two sides on the normal vector \(n\). Here \(ds\) is an arc length element along the rays, and we call the CRT a V-line Radon transform of \(f\).

This paper is dedicated to the problem of recovering the unknown function \(f(x, z)\) from the values of an \(n\)-dimensional family of its conical projections \(g(V, \beta, n)\). As with many other generalized Radon transforms with translation or rotation invariance (e.g. see [3, 5, 10, 13, 18, 19, 21, 24]), our proofs are based on Fourier techniques both for \(n = 2\) and for \(n = 3\).

3. Exact inversion formula in 2D

In two-dimensional setting, a circular cone simply consists of two half-lines \(L_1\) and \(L_2\) with common vertex of coordinates \((x_v, y_v)\) (see figure 2). We will use the existing terminology and call our transform the V-line Radon transform [4, 5, 20, 26]. If \(g(x_v, y_v, \beta, n)\) is known for all possible values of its five arguments then the reconstruction of the function \(f(x, y)\) of two variables is an overdetermined problem. To match the dimensions of the data and the function of interest, one can apply additional restrictions on \(g\). There are many different ways
of reducing the number of parameters, e.g. by considering the vertex \( V \) on the \( x \)-axis, by fixing the half-angle \( \beta \) or by fixing the central axis etc. All of these approaches lead to interesting mathematical problems about the invertibility of the V-line Radon transform.

In this paper, we shall consider the family of cones \( C(x_v, y_v, \beta, n) \) with fixed half opening angle \( \beta \in (0, \frac{\pi}{2}) \), vertical central axis (i.e. parallel to \( y \)-axis) and no restriction on the vertex. This problem is similar to the one considered in [10], but we use a different approach to solve it, our inversion formula is much simpler than the one obtained there, and it holds for a wider range of opening angles, since the inversion in [10] holds only for \( \beta \in (\pi/4, \pi/2) \).

**Theorem 1.** Consider a function \( f \in C^\infty(\mathbb{R}^2) \) supported in \( D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq x_{\text{max}}, 0 \leq y \leq y_{\text{max}}\} \). For \( (x_v, y_v) \in \mathbb{R}^2 \) we define the V-line Radon transform as follows

\[
g(x_v, y_v) = \int_{C(x_v, y_v)} f(x, y) \, ds. \tag{2}\]

An exact solution of the inversion problem for the V-line Radon transform is given by the formula

\[
f(x, y) = -\frac{\cos \beta}{2} \left( \frac{\partial}{\partial y} g(x, y) + \tan^2(\beta) \int_y^{y_{\text{max}}} \frac{\partial^2}{\partial x^2} g(x, t) \, dt \right). \tag{3}\]

**Proof.** The V-line Radon transform is

\[
g(x_v, y_v) = \int_{L_1} f(x, y) \, dl + \int_{L_2} f(x, y) \, dl. \tag{4}\]

We can represent the equation of the half-lines \( L_1 \) and \( L_2 \) in the form

\[x = x_v \pm r \sin(\beta), \quad y = y_v \pm r \cos(\beta)\]

where \( r \geq 0 \) is the distance along \( L_1 \) or \( L_2 \) measured from the vertex \( (x_v, y_v) \). Since \( \tan(\beta) \) is constant, it is convenient to denote \( t = \tan(\beta) \) and \( x = x_v \pm (y - y_v)t \). Let us rewrite the equation (4) as

\[
g(x_v, y_v) = \int_{y_v}^{y_{\text{max}}} \frac{f(x_v + (y - y_v)t, y)}{\cos(\beta)} \, dy + \int_{y_v}^{y_{\text{max}}} \frac{f(x_v - (y - y_v)t, y)}{\cos(\beta)} \, dy.\]
The first step consists of applying the Fourier transform with respect to the variable \( x_v \) where the Fourier transform \( \hat{g}(y_v) \) is denoted by

\[
\hat{g}(y_v) = \int_{-\infty}^{\infty} g(x_v, y_v) e^{-i\lambda x_v} \, dx_v.
\]

This yields

\[
\hat{g}_x(y_v) = \int_{-\infty}^{\infty} \int_{y_v}^{\infty} f(x_v + (y - y_v)t, y) \, e^{-i\lambda x_v} \, dy \, dx_v
\]

\[
= \int_{y_v}^{\infty} \int_{-\infty}^{\infty} f(x_v - (y - y_v)t, y) \, e^{-i\lambda x_v} \, dy \, dx_v
\]

\[
= \int_{y_v}^{\infty} \int_{-\infty}^{\infty} f(x_v + (y - y_v)t, y) \, e^{-i\lambda x_v} \, dx_v \, dy
\]

\[
+ \int_{y_v}^{\infty} \int_{-\infty}^{\infty} f(x_v - (y - y_v)t, y) \, e^{-i\lambda x_v} \, dx_v \, dy.
\]

The equation may be given another form with the following change of variables \( X = x_v + (y - y_v)t \) for the first part and \( X = x_v - (y - y_v)t \) for the second part

\[
\hat{g}_x(y_v) \cos(\beta) = \int_{y_v}^{\infty} \int_{-\infty}^{\infty} f(X, y) e^{-i\lambda X} \, e^{i\lambda (y - y_v)t} \, dX \, dy
\]

\[
+ \int_{y_v}^{\infty} \int_{-\infty}^{\infty} f(X, y) e^{-i\lambda X} \, e^{-i\lambda (y - y_v)t} \, dX \, dy.
\]

This result is further simplified using the Fourier transform

\[
\hat{f}_x(y) = \int_{-\infty}^{\infty} f(X, y) e^{-i\lambda X} \, dX
\]

to obtain

\[
\hat{g}_x(y_v) \cos(\beta) = \int_{y_v}^{\infty} \hat{f}_x(y) \, e^{i\lambda (y - y_v)t} \, dy + \int_{y_v}^{\infty} \hat{f}_x(y) \, e^{-i\lambda (y - y_v)t} \, dy.
\]

To simplify the notation we introduce the function \( G_x(y_v) \) defined by the following formula:

\[
G_x(y_v) = \frac{\hat{g}_x(y_v) \cos(\beta)}{2}.
\]

This yields

\[
G_x(y_v) = \int_{y_v}^{\infty} \hat{f}_x(y) \cos(\lambda (y - y_v)t) \, dy. \tag{5}
\]

In order to obtain an explicit formula for \( \hat{f}_x(y) \), we first differentiate the equation with respect to \( y_v \)

\[
G'_x(y_v) = \int_{y_v}^{\infty} \hat{f}_x(y) \lambda t \, \sin(\lambda (y - y_v)t) \, dy - \hat{f}_x(y_v), \tag{6}
\]

and combine it with

\[
\int_{y_v}^{\infty} G_x(y_v) \, dy_v = \frac{1}{\lambda t} \int_{y_v}^{\infty} \hat{f}_x(y) \, \sin(\lambda (y - z)t) \, dy. \tag{7}
\]

Finally, we arrive at the following formula

\[
\hat{f}_x(z) = -G'_x(z) + (\lambda t)^2 \int_{y_v}^{\infty} G_x(y_v) \, dy_v. \tag{8}
\]

By inverse Fourier transform, the solution is

\[
f(x, z) = -\frac{\cos \beta}{2} \left( \frac{\partial}{\partial z} g(x, z) + \tan(\beta) \int_{y_v}^{\infty} \frac{\partial^2}{\partial x^2} g(x, y) \, dy \right). \tag{9}
\]
4. Exact inversion formula in 3D

In the 3D case of this paper, we consider the family of cones \( C(x_v, y_v, z_v, \beta, z) \) with fixed half opening angle \( \beta \in (0, \frac{\pi}{2}) \), vertical central axis \( z \) and no restriction on the vertex. The problem of integral geometry considered in this section is to reconstruct the unknown function \( f(x, y, z) \) in terms of the conical projections \( g(x_v, y_v, z_v) \) (see figure 3). To the best of our knowledge, no exact inversion formula is known for this reconstruction problem.

**Theorem 2.** Consider a function \( f \in C^\infty(\mathbb{R}^3) \) supported in \( D = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq x_{\text{max}}, 0 \leq y \leq y_{\text{max}}, 0 \leq z \leq z_{\text{max}}\} \). For \( (x_v, y_v, z_v) \in \mathbb{R}^3 \) we define the 3D conical Radon transform by

\[
g(x_v, y_v, z_v) = \int_{C(x_v, y_v, z_v)} f(x, y, z) \, ds. \tag{10}
\]

An exact solution of the inversion problem for CRT is given by

\[
\hat{f}_{\lambda, \mu}(z) = \frac{\cos \beta}{2\pi \tan \beta} \int_{z_{\text{max}}}^{z} J_0(u(z - x)) \left[ \frac{d^2}{dx^2} + u^2 \right]^2 \int_{z_{\text{max}}}^{x} \hat{g}_{\lambda, \mu}(z_v) \, dz_v \, dx,
\]

where \( \hat{g}_{\lambda, \mu}(z_v) \) and \( \hat{f}_{\lambda, \mu}(z) \) are the 2D Fourier transforms of the functions \( g(x_v, y_v, z_v) \) and \( f(x, y, z) \) with respect to the first two variables.

**Proof.** Due to the invariance of the family \( C(x_v, y_v, z_v) \) with respect to translations along the hyperplane \( z = 0 \), the CRT can be simplified as follows

\[
g(x_v, y_v, z_v) = \int_{C(0,0,z_v)} f(x + x_v, y + y_v, z) \, ds.
\]

Then we apply the 2D Fourier transform with respect to the variables \( x_v \) and \( y_v \) to get

\[
\hat{g}_{\lambda, \mu}(z_v) = \int_{C(0,0,z_v)} \hat{f}_{\lambda, \mu}(z) e^{i\lambda x} e^{i\mu y} \, ds \tag{11}
\]

where \( \hat{g}_{\lambda, \mu}(z_v) \) and \( \hat{f}_{\lambda, \mu}(z) \) are the Fourier transforms of the functions \( g(x_v, y_v, z_v) \) and \( f(x, y, z) \).
Let us denote by $S(k, z_e)$ the circles of the intersection of $C(0, 0, z_e)$ and the hyperplane $z = k$. We can split the surface element $ds$ into an integration with respect to $dz$ and $dl$, which is the arc length along the circle $S(z, z_e)$,

$$ds = \frac{dl}{cos \beta}.$$

Equation (11) becomes

$$\hat{g}_{\lambda, \mu}(z_e) = \int_{z_e}^{z_{\text{max}}} f_{\lambda, \mu}(z) K_{\lambda, \mu}(z, z_e) dz.$$  \hspace{1cm} (12)

The kernel is given by the formula

$$K_{\lambda, \mu}(z, z_e) = \int_{S(z_e)} e^{i(\lambda \beta + \mu \gamma)} \frac{dl}{cos \beta}.$$  \hspace{1cm} (13)

The latter is easily calculated using the equations of the circles $S(k, z_e)$ (we set $x_e = y_e = 0$) and introducing the polar coordinates

$$x = r \cos \varphi \quad y = r \sin \varphi \quad r = (z - z_e) \tan \beta.$$

We can calculate $dl$ as follows

$$dl = r d\varphi = (z - z_e) \tan \beta \, d\varphi.$$

Equation (13) becomes

$$K_{\lambda, \mu}(z, z_e) = \int_{0}^{2\pi} e^{i(\lambda \varphi + \mu \cos \varphi)} (z - z_e) \tan \beta \, d\varphi$$

or

$$K_{\lambda, \mu}(z, z_e) = \frac{(z - z_e) \tan \beta}{cos \beta} \int_{0}^{2\pi} e^{i((z - z_e) \tan \beta) (\lambda \cos \varphi + \mu \sin \varphi)} d\varphi.$$

In the case when $(\lambda, \mu) \neq (0, 0)$ let us introduce a new variable $\omega$ as follows

$$\lambda = \sqrt{\lambda^2 + \mu^2} \cos \omega, \quad \mu = \sqrt{\lambda^2 + \mu^2} \sin \omega.$$

Then the expression of the kernel becomes

$$K_{\lambda, \mu}(z, z_e) = \frac{(z - z_e) \tan \beta}{cos \beta} \int_{0}^{2\pi} e^{i((z - z_e) \tan \beta) \sqrt{\lambda^2 + \mu^2} \cos \omega \cos \varphi + \sin \omega \sin \varphi)} d\varphi,$$

which yields

$$K_{\lambda, \mu}(z, z_e) = \frac{(z - z_e) \tan \beta}{cos \beta} \int_{0}^{2\pi} e^{i((z - z_e) \tan \beta) \sqrt{\lambda^2 + \mu^2} \cos (\omega - \varphi)} d\varphi.$$

Using the integral representation of a Bessel function of first kind

$$J_0(z) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{i \cos \theta} d\theta$$

(see [1], p 360), we can express this integral in terms of $J_0$

$$K_{\lambda, \mu}(z, z_e) = \frac{2\pi \tan \beta}{cos \beta} (z - z_e) J_0((z - z_e) \tan \beta \sqrt{\lambda^2 + \mu^2}).$$

As a result our problem breaks down to the following set of one-dimensional integral equations

$$\hat{g}_{\lambda, \mu}(z_e) = \frac{2\pi \tan \beta}{cos \beta} \int_{z_e}^{z_{\text{max}}} f_{\lambda, \mu}(z) (z - z_e) J_0((z - z_e) \tan \beta \sqrt{\lambda^2 + \mu^2}) dz.$$  \hspace{1cm} (14)

By setting

$$G_{\lambda, \mu}(z_e) = \frac{cos \beta}{2\pi \tan \beta} \hat{g}_{\lambda, \mu}(z_e)$$
and
\[ u = \tan \beta \sqrt{\lambda^2 + \mu^2}, \]
one can transform equation (14) to the following expression
\[ H_{\lambda, \mu}(z) = \int_{z_1}^{z_{\text{max}}} \hat{J}_{\lambda, \mu}(z) \hat{J}_0(u(z - z_v)) \, dz. \]  
(15)

We now show how to find \( \hat{f}_{\lambda, \mu}(z) \). For this purpose we introduce the operator \( \mathcal{H} \) defined by
\[ \mathcal{H}(F) \equiv \left( \frac{d^2}{dx^2} + u^2 \right) F(x) \]
and apply it to \( \int_x^{z_{\text{max}}} G_{\lambda, \mu}(z_v) \, dz_v \). First, we differentiate our integral using Leibnitz’s rule as follows
\[ \frac{d^2}{dx^2} \left( \int_x^{z_{\text{max}}} G_{\lambda, \mu}(z_v) \, dz_v \right) = \int_x^{z_{\text{max}}} \hat{f}_{\lambda, \mu}(z) \hat{J}_0(u(z - x)) \, dz - u \int_x^{z_{\text{max}}} \hat{f}_{\lambda, \mu}(z) \hat{J}_1(u(x - z)) \, dz. \]
(16)

Here we used the fact that \( \hat{J}_0(z) = -\hat{J}_1(z) \) (see [1] p 361). We also write
\[ \int_x^{z_{\text{max}}} G_{\lambda, \mu}(z_v) \, dz_v = \int_x^{z_{\text{max}}} \int_{z_1}^{z_{\text{max}}} \hat{f}_{\lambda, \mu}(z) (z - z_v) \hat{J}_0(u(z - z_v)) \, dz \, dz_v, \]
which can be simplified by interchanging the integrals
\[ \int_x^{z_{\text{max}}} G_{\lambda, \mu}(z_v) \, dz_v = \int_x^{z_{\text{max}}} \hat{f}_{\lambda, \mu}(z) \int_x^{z_{\text{max}}} \hat{J}_0(u(z - z_v)) \, dz_v. \]  
(17)

Using the integral identity of the Bessel function \( \int_0^x v \hat{J}_0(v) \, dv = x \hat{J}_1(x) \), we can write equation (17) as follows
\[ \int_x^{z_{\text{max}}} G_{\lambda, \mu}(z_v) \, dz_v = \frac{1}{u} \int_x^{z_{\text{max}}} \hat{f}_{\lambda, \mu}(z) (z - x) \hat{J}_1(u(z - x)) \, dz. \]  
(18)

Adding equations (18) and (16) we get
\[ \mathcal{H} \left( \int_x^{z_{\text{max}}} G_{\lambda, \mu}(z_v) \, dz_v \right) = \int_x^{z_{\text{max}}} \hat{f}_{\lambda, \mu}(z) \hat{J}_0(u(z - x)) \, dz. \]  
(19)

We again apply the operator \( \mathcal{H} \) to equation (19) to obtain
\[ \mathcal{H}^2 \left( \int_x^{z_{\text{max}}} G_{\lambda, \mu}(z_v) \, dz_v \right) = -\hat{f}_{\lambda, \mu}(x) + u^2 \int_x^{z_{\text{max}}} \hat{f}_{\lambda, \mu}(z) \left[ \hat{J}_0(u(z - x)) + \hat{J}_1'(u(z - x)) \right] \, dz. \]

Then we integrate our equation as follows
\[ \int_t^{z_{\text{max}}} J_0(u(t - x)) \mathcal{H}^2 \left( \int_x^{z_{\text{max}}} G_{\lambda, \mu}(z_v) \, dz_v \right) \, dx = -\int_t^{z_{\text{max}}} J_0(u(t - x)) \hat{f}_{\lambda, \mu}(x) \, dx 
+ u^2 \int_t^{z_{\text{max}}} J_0(u(t - x)) \int_x^{z_{\text{max}}} \hat{f}_{\lambda, \mu}(z) \left[ \hat{J}_0(u(z - x)) + \hat{J}_1'(u(z - x)) \right] \, dz \, dx. \]  
(20)

Let us simplify the RHS of equation (20). First, we simplify the first term of the RHS by integration by parts
\[ \int_t^{z_{\text{max}}} J_0(u(t - x)) \hat{f}_{\lambda, \mu}(x) \, dx = \hat{f}_{\lambda, \mu}(t) - u \int_t^{z_{\text{max}}} J_1(u(t - x)) \hat{f}_{\lambda, \mu}(x) \, dx. \]
We used the conditions imposed upon the function \( \hat{f}_{\lambda,\mu}(z_{\text{max}}) = 0 \). Then we simplify the second term of the RHS by interchanging the order of integration
\[
\int_{t}^{\tilde{t}} J_{0}(u(t - x)) \int_{x}^{\tilde{t}} \hat{f}_{\lambda,\mu}(z) [J_{0}(u(z - x)) + J_{0}''(u(z - x))] \, dz \, dx = \int_{\tilde{t}}^{t} \hat{f}_{\lambda,\mu}(z) \, dz \int_{x}^{\tilde{t}} J_{0}(u(t - x)) [J_{0}(u(z - x)) + J_{0}''(u(z - x))] \, dx.
\]
Then we use the recurrence formula of Bessel function \( J_{0}(x) + J_{0}(x) = -J_{1}(x)/x \) and the formula \( \int_{0}^{\tilde{t}} J_{1}(x) J_{0}(z - x) \, dx = J_{1}(z) \) (for more details about Bessel functions see [14] p 671). After simplification, we obtain that for \( (\lambda, \mu) \neq (0, 0) \)
\[
\hat{f}_{\lambda,\mu}(t) = \int_{0}^{\tilde{t}} J_{0}(u(t - x)) \mathcal{H}^{2} \left( \int_{x}^{\tilde{t}} G_{\lambda,\mu}(z_{\epsilon}) \, dz_{\epsilon} \right) \, dx.
\]
Since \( f \) is compactly supported, its Fourier transform at \( (\lambda, \mu) = (0, 0) \) is uniquely defined by continuity.

5. Numerical implementation

To validate our inversion formulas in 2D we run a few simple numerical simulations. Here we present the results of these experiments for smooth phantoms supported within a square \( \Omega = [-1, 1] \times [-1, 1] \) and defined by functions of the form
\[
f(x, y) = \begin{cases} 
\exp \left( \frac{-r^{2}}{r^{2} - [(x - x_{c})^{2} + (y - y_{c})^{2}]} \right) & \text{if } (x - x_{c})^{2} + (y - y_{c})^{2} < r^{2}, \\
0 & \text{otherwise}.
\end{cases}
\]

In all experiments we used a fixed opening angle \( \beta = \pi/8 \). The V-line Radon transform was computed numerically from an \( N \times N \) discretized version of the function \( f \). The differentiation operator in the inversion formula (3) was approximated by the standard first order forward difference formula
\[
\frac{\partial}{\partial z} g(x, z) \approx \frac{g(x, z + \Delta z) - g(x, z)}{\Delta z},
\]
and the integration was done using the trapezoidal rule.

Figure 4 shows the results for a phantom with \( r = 0.25 \) and the center \( (x_{c}, y_{c}) = (0.2, 0.1) \) using two different values of discretization \( N = 60 \) and \( N = 120 \) pixels.

We also used a phantom consisting of two circles of varying centers and intensities as shown in table 1. The images were constructed with \( 220 \times 220 \) pixels (figure 5).

The artifacts that appear in our reconstructions are typical for numerical inversions of this type of transforms (e.g. see [10]). They appear at and around the locations of the vertices of V-lines that tangentially touch the support of \( f \). Hence the numerical errors of differentiation in the inversion formula are more emphasized at those locations. Notice that although our phantom is theoretically infinitely smooth, the numerical discretization results in jump discontinuity across the edge of the support.

| Circle | Coordinates of the center \((x_c, y_c)\) | intensity |
|--------|-------------------------------------|-----------|
| 1      | \((0.5, 0.3)\)                     | 3         |
| 2      | \((-0.2, -0.2)\)                   | 4         |

Table 1. Parameters for the phantom image.
The authors of [9] have developed a very nice technique for reducing the artifacts that occur during the inversion of an operator $R$ that appears in slant hole SPECT. The connection here is that in their case, the normal operator $R^*R$ integrates along cones in $R^3$, and the method developed there may be applicable in our case as well. We plan to investigate this in future work.

6. Additional remarks

(i) The smoothness and decay conditions for $f$ in both inversion formulae (for 2D and 3D) are not optimized. The formulas may hold with weaker requirements, e.g. for $f$ in Schwartz space, or compactly supported functions that have only a few orders of derivatives.

(ii) The main results of this paper have potential to be generalized to dimensions higher than $n = 3$. The authors plan to address this issue in future work.

(iii) The numerical implementation of the 3D inversion formula presented here is a challenging task in itself and the authors plan to address this in future work.

(iv) In this paper, we consider the case when the half opening angle $\beta \in (0, \frac{\pi}{2})$. The case of $\beta = \frac{\pi}{4}$ corresponds to the classical Radon transform along lines with the same fixed angular coefficient, and the inversion of the transform is impossible. The case when $\beta = 0$ corresponds to integrals of $f$ along parallel rays with vertices at all possible positions $\mathbf{V}$.
in the support of $f$. The latter transform can be inverted trivially by differentiating the data with respect to the vertical component of $V$.

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