Singular McKean-Vlasov SDEs: Well-Posedness, Regularities and Wang’s Harnack Inequality*

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Abstract

The well-posedness and regularity estimates in initial distributions are derived for singular McKean-Vlasov SDEs, where the drift contains a locally standard integrable term and a superlinear term in the spatial variable, and is Lipschitz continuous in the distribution variable with respect to a weighted variation distance. When the superlinear term is strengthened to be Lipschitz continuous, Wang’s Harnack inequality is established. These results are new also for the classical Itô SDEs where the coefficients are distribution independent.

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1 Introduction and main results

In recent years, singular SDEs have been intensively investigated by using Zvonkin’s transform [24] and Krykov’s estimate [8] developed from [13] for bounded drift, [9, 22] for integrable drift, and [20, 21] for locally integrable drift.

In this paper, we aim to improve existing results on the well-posedness and regularity estimates derived for singular SDEs, and make extensions to McKean-Vlasov SDEs (also called mean field SDEs or distribution dependent SDEs), a hot research object due to its essential links to nonlinear Fokker-Planck equations and mean field particle systems, see for instance the monographs [12], [1] and the survey [3].

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Let $\mathcal{P}$ be the space of probability measures on $\mathbb{R}^d$ equipped with the weak topology. For $T \in (0, \infty)$, consider

$$\begin{equation}
\text{(1.1)}
\frac{dX_t}{dt} = b_t(X_t, \mathcal{L}_{X_t})dt + \sigma_t(X_t)dW_t, \quad t \in [0, T],
\end{equation}$$

where $W_t$ is an $m$-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $\mathcal{L}_\xi$ is the distribution (i.e. the law) of a random variable $\xi$, and

$$b : [0, \infty) \times \mathbb{R}^d \times \mathcal{P} \to \mathbb{R}^d, \quad \sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^m$$

are measurable.

To characterize the dependence on the distribution variable, we introduce some probability distances. For a measurable function $V \geq 1$, let

$$\mathcal{P}_V := \left\{ \mu \in \mathcal{P} : \mu(V) := \int_{\mathbb{R}^d} V d\mu < \infty \right\}$$

be equipped with the $V$-weighted variation metric

$$\|\mu - \nu\|_V := \sup_{|f| \leq V} |\mu(f) - \nu(f)|, \quad \mu, \nu \in \mathcal{P}_V.$$ 

When $V = 1$ it reduces to the total variation norm, i.e. $\| \cdot \|_1 = \| \cdot \|_{\text{var}}$. We often take $V$ to be a compact function, i.e. its level sets $\{V \leq r\}$ for $r > 0$ are compact.

Next, for any $k \in [1, \infty)$, let $\mathcal{P}_k = \mathcal{P}_V$ for $V := 1 + | \cdot |^k$, i.e.

$$\mathcal{P}_k := \left\{ \mu \in \mathcal{P} : \|\mu\|_k := \mu(| \cdot |^k)^{\frac{1}{k}} < \infty \right\}.$$ 

In this case, we denote $\| \cdot \|_V = \| \cdot \|_{k, \text{var}}$. Consider the $L^k$-Wasserstein distance

$$\mathbb{W}_k(\mu, \nu) := \inf_{\pi \in C(\mu, \nu)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^k \pi(dx, dy) \right)^{\frac{1}{k}}, \quad \mu, \nu \in \mathcal{P}_k,$$

where $C(\mu, \nu)$ is the set of all couplings for $\mu$ and $\nu$. According to [14, Theorem 6.15], there exists a constant $c > 0$ such that

$$\|\mu - \nu\|_{\text{var}} + \mathbb{W}_k(\mu, \nu)^k \leq c\|\mu - \nu\|_{k, \text{var}}, \quad \mu, \nu \in \mathcal{P}_k.$$

To measure the singularity of the drift, we recall some functional spaces introduced in [20]. For any $p \geq 1$, $L^p(\mathbb{R}^d)$ is the class of measurable functions $f$ on $\mathbb{R}^d$ such that

$$\|f\|_{L^p(\mathbb{R}^d)} := \left( \int_{\mathbb{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$ 

For any $\epsilon > 0$ and $p \geq 1$, let $H^{\epsilon,p}(\mathbb{R}^d) := (1 - \Delta)^{-\frac{\epsilon}{2}} L^p(\mathbb{R}^d)$ with

$$\|f\|_{H^{\epsilon,p}(\mathbb{R}^d)} := \|(1 - \Delta)^{-\frac{\epsilon}{2}} f\|_{L^p(\mathbb{R}^d)} < \infty, \quad f \in H^{\epsilon,p}(\mathbb{R}^d).$$
For any \( z \in \mathbb{R}^d \) and \( r > 0 \), let \( B(z, r) := \{ x \in \mathbb{R}^d : |x - z| < r \} \) be the open ball centered at \( z \) with radius \( r \). For any \( p, q \geq 1 \), let \( L^p_q \) denote the class of measurable functions \( f \) on \([0, T] \times \mathbb{R}^d \) such that

\[
\|f\|_{L^p_q} := \sup_{z \in \mathbb{R}^d} \left( \int_0^T \|1_{B(z, 1)} f_t\|_{L^p(\mathbb{R}^d)}^q dt \right)^{\frac{1}{q}} < \infty.
\]

For any \( \epsilon > 0 \), let \( \tilde{H}^{\epsilon,p}_q \) be the space of \( f \in \tilde{L}^p_q \) with

\[
\|f\|_{\tilde{H}^{\epsilon,p}_q} := \sup_{z \in \mathbb{R}^d} \left( \int_0^T \|g(z + \cdot) f_t\|_{H^{\epsilon,p}(\mathbb{R}^d)}^q dt \right)^{\frac{1}{q}} < \infty
\]

for some \( g \in C_0^\infty(\mathbb{R}^d) \) satisfying \( g|_{B(0, 1)} = 1 \), where \( C_0^\infty(\mathbb{R}^d) \) is the class of \( C^\infty \) functions on \( \mathbb{R}^d \) with compact support. We remark that the space \( \tilde{H}^{\epsilon,p}_q \) does not depend on the choice of \( g \).

We will take \( (p, q) \) from the class

\[
\mathcal{K} := \{ (p, q) : p, q \in (2, \infty), \frac{d}{p} + \frac{2}{q} < 1 \}.
\]

### 1.1 Well-posedness

Let us first recall the definition of well-posedness.

**Definition 1.1.** A continuous adapted process \((X_{s,t})_{T \geq t \geq s}\) is called a solution of (1.1) from time \( s \), if

\[
\int_s^t \mathbb{E} \left[ |b_r(X_{s,r}, \mathcal{L}_{X_{s,r}})| + \|\sigma_r(X_{s,r})\|^2 \right] dr < \infty, \quad T \geq t \geq s,
\]

and \( \mathbb{P} \)-a.s.

\[
X_{s,t} = X_{s,s} + \int_s^t b_r(X_{s,r}, \mathcal{L}_{X_{s,r}}) dr + \int_s^t \sigma_r(X_{s,r}) dW_r, \quad T \geq t \geq s.
\]

When \( s = 0 \) we simply denote \( X_t = X_{0,t} \).

A couple \((\tilde{X}_{s,t}, \tilde{W}_t)_{T \geq t \geq s}\) is called a weak solution of (1.1) from time \( s \), if \( \tilde{W}_t \) is the \( m \)-dimensional Brownian motion on a complete filtration probability space \((\tilde{\Omega}, \{ \tilde{\mathcal{F}}_t \}_{t \in [0,T]}, \tilde{\mathbb{P}}) \) such that \((\tilde{X}_{s,t})_{T \geq t \geq s}\) is a solution of (1.1) from time \( s \) for \((\tilde{W}_t, \tilde{\mathbb{P}})\) replacing \((W_t, \mathbb{P})\). (1.1) is called weakly unique for an initial distribution \( \nu \in \mathcal{P} \), if all weak solutions with distribution \( \nu \) at time \( s \) are equal in law.

Let \( \mathcal{P} \) be a subspace of \( \mathcal{P} \). (1.1) is called strongly (respectively, weakly) well-posed for distributions in \( \mathcal{P} \), if for any \( s \in [0, T) \) and \( \mathcal{F}_s \)-measurable \( X_{s,s} \) with \( \mathcal{L}_{X_{s,s}} \in \mathcal{P} \) (respectively, any initial distribution \( \nu \in \hat{\mathcal{P}} \) at time \( s \)), it has a unique strong (respectively, weak) solution.

(1.1) is called well-posed for distributions in \( \hat{\mathcal{P}} \), if it is both strongly and weakly well-posed for distributions in \( \hat{\mathcal{P}} \).

To prove the well-posedness, we make the following assumptions.
(H1) \( \sigma_t(x) \) and \( b_t(x, \mu) = b_t^{(0)}(x) + b_t^{(1)}(x, \mu) \) satisfy the following conditions for a compact function \( 1 \leq V \in C^2(\mathbb{R}^d ; [1, \infty)) \).

(1) \( a := \sigma \sigma^* \) is invertible with \( \|a\|_\infty + \|a^{-1}\|_\infty < \infty \), where \( \sigma^* \) is the transposition of \( \sigma \), and
\[
\lim_{\varepsilon \to 0} \sup_{|x-y| \leq \varepsilon, t \in [0, T]} \|a_t(x) - a_t(y)\| = 0.
\]

(2) \( |b^{(0)}| \in \hat{L}^{p_0}_{q_0} \) for some \( (p_0, q_0) \in \mathcal{K} \). Moreover, \( \sigma \) is weakly differentiable such that
\[
\sum_{i=1}^{l} f_i \leq c \sigma(t)
\]
holds for some \( l \in \mathbb{N} \) and \( 0 \leq f_i \in \hat{L}^{p_i}_{q_i} \) with \( (p_i, q_i) \in \mathcal{K}, \ 1 \leq i \leq l \).

(3) for any \( \mu \in C([0, T]; \mathcal{P}_V) \), \( b_t^{(1)}(x, \mu) \) is locally bounded in \( (t, x) \in [0, T] \times \mathbb{R}^d \). Moreover, there exist constants \( K, \varepsilon > 0 \) a compact function \( V \in C^2(\mathbb{R}^d ; [1, \infty)) \) such that
\[
\sup_{|y-x| \leq \varepsilon} \{ |\nabla V(y)| + |\nabla^2 V(y)| \} \leq K V(x),
\]
\[
\langle b_t^{(1)}(x, \mu), \nabla V(x) \rangle + \varepsilon |b_t^{(1)}(x, \mu)| \sup_{B(x, \varepsilon)} \{ |
abla V| + |\nabla^2 V| \}
\]
\[
\leq K \{ V(x) + \mu(V) \}, \quad x \in \mathbb{R}^d, \mu \in \mathcal{P}_V.
\]

(4) there exists a constant \( \kappa > 0 \) such that
\[
|b_t(x, \mu) - b_t(x, \nu)| \leq \kappa \|\mu - \nu\|_V, \quad \mu, \nu \in \mathcal{P}_V, x \in \mathbb{R}^d.
\]

**Theorem 1.1.** Assume (H1). Then (1.1) is well-posed for distributions in \( \mathcal{P}_V \). Moreover:

(1) for any \( n \geq 1 \) there exists a constant \( c(n) > 0 \) such that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} V(X_t)^n \middle| X_0 \right] \leq c(n) \{ (\mathbb{E}[V(X_0)])^n + V(X_0)^n \}
\]
holds for any solution \( X_t \) of (1.1) with \( \mathcal{L}_{X_0} \in \mathcal{P}_V \).

(2) for any sequence \( \{\mu_n\}_{n \geq 1} \subset \mathcal{P}_V \) with bounded \( \mu_n(V^p) \) for some \( p > 1 \) such that \( \mu_n \to \mu \) weakly,

\[
\lim_{n \to \infty} \|P_t^* \mu_n - P_t^* \mu\|_V = 0.
\]

(3) if there exists a constant \( K > 0 \) such that

\[
|b_t(x, \mu) - b_t(x, \nu)| \leq K \|\mu - \nu\|_{\text{var}}, \quad \mu, \nu \in \mathcal{P}_V,
\]
then (1.4) holds for some constant \( c > 0 \) independent of \( \mu \), and

\[
\lim_{\nu \to \mu \text{ weakly}} \|P_t^* \mu - P_t^* \nu\|_{\text{var}} = 0.
\]
Remark 1.1. (1) Theorem 1.1 extends existing well-posedness results derived for singular McKean-Vlasov SDEs, for instance:

(a) [5] and [7] with \( l = 1 \) in \((H_1)(2)\), and \((H_1)(4)\) with \( V := (1 + | \cdot |^2) \frac{\delta}{2} \), and the following stronger condition stronger than \((H_1)(3)\):

\[
\sup_{t \in [0,T], x \neq y} \left\{ |b_t^{(1)}(0, \delta_0)| + \frac{|b_t^{(1)}(x, \delta_0) - b_t^{(1)}(y, \delta_0)|}{|x - y|} \right\} < \infty,
\]

where \( \delta_0 \) is the Dirac measure at 0.

(b) [10] as well as [23] for Hölder continuous \( \sigma_t \) and \( \sup_\mu \| b(\cdot, \mu) \| \tilde{L}^p_0 < \infty \).

(2) The regularity property included in (1.5) and (1.7) is new in this general situation. Under (1.8) replacing \((H_1)(4)\), the log-Harnack inequality was established in [19] so that

\[
\| P^*_t \mu - P^*_t \nu \|_{\text{var}} \leq \frac{c}{\sqrt{t}} \mathbb{W}_2(\mu, \nu)
\]

holds for some constant \( c > 0 \), which is incomparable with (1.5) since \( \| \cdot \|_{\text{var}} \) is essentially smaller than \( \| \cdot \|_V \).

(3) Theorem 1.1 is new even in the setting of singular SDEs, see comments before Theorem 2.1.

1.2 Wang’s Harnack inequality

Since 1997 when the dimension-free Harnack inequality of type

\[
|P_t f(x)|^p \leq (P_t |f|^p(y)) e^{\frac{c}{t} \rho(x,y)^2}
\]

was found in Wang [15] for diffusion semigroups \( P_t \) on Riemannian manifolds, this type inequality has been intensively developed and applied to many different models, see [17] for a general theory on the study. In recent years, Wang’s inequality has been established for McKean-Vlasov SDEs in [18] under monotone conditions as well as in [4] for bounded \( b \) which is Dini continuous in the space variable and \( \mathbb{W}_2 \)-Lipschitz continuous in the distribution variable.

In this paper, we establish dimension-free Harnack inequality in a more general situation, which is new even for classical SDEs, see comments before Theorem 2.1 in the next section.

\((H_2)\) \((H_1)(1)-(2), (1.8)\) and the following conditions hold.

(1) There exists increasing \( \Phi \in C^2([0, \infty); [1, \infty)) \) with

\[
\limsup_{r \to \infty} \frac{\Phi'(r) + |\Phi''(r)|}{\Phi(r)} < \infty,
\]

such that for some constant \( \kappa > 0 \) and \( V := \Phi(| \cdot |^2) \),

\[
|b_t(x, \mu) - b_t(x, \nu)| \leq \kappa \| \mu - \nu \|_V, \quad (t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_V.
\]
(2) There exists increasing \( \varphi \in C([0, \infty); [0, \infty)) \) satisfying \( \varphi(0) = 0, \varphi(r) > 0 \) for \( r > 0 \), \( \psi(r) := \frac{r^2}{\varphi(r)^2} \) is increasing in \( r > 0 \) and \( \int_0^1 \frac{(\varphi(\psi^{-1}(s)))^2}{s} \, ds < \infty \), such that

\[
\|\sigma_t(x) - \sigma_t(y)\| \leq \varphi(|x - y|), \quad x, y \in \mathbb{R}^d, t \in [0, T].
\]

Typical examples of \( \varphi \) in (H2)(2) include \( \varphi(r) = r^\alpha \) for \( \alpha \in (0, 1) \) and \( \varphi(r) = \log^{-\theta}(e + r^{-1}) \) for \( \theta > 1 \), where in the first case \( \sigma_t \) is Hölder continuous and in the second case it is only Dini continuous.

By Theorem 1.1 (H2) implies the well-posedness of (1.1) for distributions in \( \mathcal{P}_V \). Consider

\[
P_t f(\mu) := \int_{\mathbb{R}^d} f(dP_t^\mu), \quad t \in [0, T], f \in \mathcal{B}_b(\mathbb{R}^d), \mu \in \mathcal{P}_V.
\]

**Theorem 1.2.** Assume (H2). Then the following assertions hold.

1. There exist constants \( c, p > 1 \) such that for any \( t \in (0, T], f \in \mathcal{B}_b(\mathbb{R}^d) \),

\[
|P_t f|^p(\mu) \leq \{P_t|f|^p(\nu)\} \inf_{\pi \in \mathcal{P}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{c + \frac{p}{2}|x - y|^2} \pi(dx, dy), \quad \mu, \nu \in \mathcal{P}_V.
\]

2. If \( \Phi \) is bounded then there exists a constant \( c > 0 \) such that for any \( t \in (0, T] \),

\[
\|P_t^\mu - P_t^\nu\|_{\mathcal{V}_2}^2 \leq c\left(t^{-1} - \log[1 \land \mathbb{W}_2(\mu, \nu)]\right)\mathbb{W}_2(\mu, \nu)^2, \quad \mu, \nu \in \mathcal{P}_V.
\]

**Remark 1.2.** (1) By the proof of [17, Theorem 1.4.2], if the right hand side in (1.11) is finite, then \( P_t^{\mu, \nu} := \frac{dP_t^\mu}{dP_t^\nu} \) exists and satisfies

\[
\{P_t(\rho_t^{\mu, \nu})^{\frac{1}{p-1}}(\mu)\}^{p-1} \leq \inf_{\pi \in \mathcal{P}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{c + \frac{p}{2}|x - y|^2} \pi(dx, dy).
\]

The Harnack inequality (1.11) is new even for the classical distribution dependent SDEs, see comments before Theorem 2.2.

(2) By (1.12), for any \( f \in \mathcal{B}_b(\mathbb{R}^d) \) and \( t \in (0, T] \), \( P_t f \) is nearly Lipschitz continuous in \( \mathbb{W}_2 \) in the sense that

\[
|P_t f(\mu) - P_t f(\nu)| \leq \|f\|_{\mathcal{V}_2} \sqrt{c\left(t^{-1} - \log[1 \land \mathbb{W}_2(\mu, \nu)]\right)\mathbb{W}_2(\mu, \nu)}.
\]

When (1.10) holds for \( \mathbb{W}_2(\mu, \nu) \) replacing \( \|\mu - \nu\|_{\mathcal{V}_2} \), Theorem 4.1 in [19] implies the exact Lipschitz continuity of \( P_t f \) in \( \mathbb{W}_2 \). See also [4] and [6] for the \( \mathbb{W}_2 \)-Lipschitz continuity of \( P_t f \) under stronger conditions on \( b \), where [6] allows \( \sigma \) to be distribution dependent.

In the following three sections, we first prove the above results for singular SDEs where \( b_t(x, \mu) = b_t(x) \) does not depend on \( \mu \), then extend to the distribution dependent setting to prove the above two theorems.
2 Singular SDEs

Consider the following SDE on \( \mathbb{R}^d \):

\[
\begin{align*}
\dot{X}_t &= b_t(X_t)dt + \sigma_t(X_t)dW_t, \quad t \in [0, T].
\end{align*}
\]

There are a plenty of papers studying the well-posedness of this SDE. In the following we mention two typical results under weak monotone condition and locally integrable condition respectively.

According to [2], when \( b \) and \( \sigma \) are continuous satisfying the following weak semi-Lipschitz continuous condition:

\[
\begin{align*}
2\langle x - y, b_t(x) - b_t(y) \rangle + \|\sigma_t(x) - \sigma_t(y)\|_{HS}^2 &\leq K|x - y|^2 \log(2 + |x - y|^{-1}), \\
2\langle b_t(x), x \rangle + \|\sigma_t(x)\|_{HS}^2 &\leq K(1 + |x|^2) \log(2 + |x|^2), \quad t \in [0, T], x, y \in \mathbb{R}^d,
\end{align*}
\]

then (2.1) is well-posed.

On the other hand, in recent years (2.1) has been intensively studied under locally integrable conditions. According to [21], see [20, 22] and references within for earlier results, the well-posedness of (2.1) holds under the following assumption. We remark that these papers (also related existing references) only consider the case \( l = 1 \) in condition (2) below, but the proof applies to \( l \geq 2 \) by replacing \( \|\nabla \sigma\| \) with \( \sum_{i=1}^{l} f_i \) and applying Khasminskii’s estimate to each \( f_i \) respectively.

(A1) Let \( a_t(x) \) := \((\sigma_t\sigma^*_t)(x) \) and \( b_t(x) = b_t^{(0)}(x) + b_t^{(1)}(x) \).

(1) \( a \) is invertible with \( \|a\|_{\infty} + \|a^{-1}\|_{\infty} < \infty \) and uniformly continuous in \( x \):

\[
\lim_{\epsilon \to 0} \sup_{|x - y| \leq \epsilon, t \in [0, T]} \|a_t(x) - a_t(y)\| = 0.
\]

(2) There exist \( l \in \mathbb{N}, \{(p_i, q_i)\}_{0 \leq i \leq l} \subset \mathcal{K} \) and \( 0 \leq f_i \in \tilde{L}^{p_i}_{q_i}, 1 \leq i \leq l \) such that

\[
|b^{(0)}| \in \tilde{L}^{p_0}_{q_0}, \quad \|\nabla \sigma\| \leq \sum_{i=1}^{l} f_i.
\]

(3) \( b_t^{(1)} \) is Lipschitz continuous with

\[
\sup_{t \in [0, T]} \left\{ |b_t^{(1)}(0)| + \|\nabla b_t^{(1)}\|_{\infty} \right\} < \infty,
\]

where \( \|\nabla b_t^{(1)}\|_{\infty} \) is the Lipschitz constant of \( b_t^{(1)} \).
Remark 2.1. (A_1) does not include \((2.2)\). Our first result ensures the well-posedness under the following condition, which extends both (A_1) and \((2.2)\). Indeed, when \(\sigma\) is bounded, \((3')\) holds for \(V(x) = 1 + |x|^2\) if there exist constants \(\varepsilon, C > 0\) such that
\[
\langle b_t^{(1)}(x), x \rangle + \varepsilon |b_t^{(1)}(x)| \leq C\phi(1 + |x|^2), \quad x \in \mathbb{R}^d, t \in [0, T],
\]
which extends \((2.2)\) by allowing a singular term for \(\phi(s) := \log(e + s)\) and covers (A_1) for \(\phi(s) := 1 + s\).

(A_2) Assume (A_1)(1)-(2) and \(b^{(1)}\) is locally bounded such that the following condition holds.
\((3')\) There exist constants \(K, \varepsilon > 0\), increasing \(\phi \in C^1([0, \infty); [1, \infty))\) with \(\int_0^\infty \frac{ds}{s + \phi(s)} = \infty\), and \(V \in C^2(\mathbb{R}^d; [1, \infty))\) having compact level sets, such that
\[
\sup_{|y-x| \leq \varepsilon} \left\{ |\nabla V(y)| + \|\nabla^2 V(y)\| \right\} \leq KV(x),
\]
\[
\langle b^{(1)}(x), \nabla V(x) \rangle + \varepsilon |b^{(1)}(x)| \sup_{B(x, \varepsilon)} \left\{ |\nabla V| + |\nabla^2 V| \right\} \leq K\phi(V(x)), \quad x \in \mathbb{R}^d.
\]

In the following, we prove the well-posedness with strong Feller property and moment estimates under assumption (A_2) and establish Wang’s Harnack inequality under (A_1).

2.1 Well-posedness

**Theorem 2.1.** Assume (A_2). Then (2.1) is well-posed. Moreover:

1. For any \(n \geq 1\) and \(B_n := \{x \in \mathbb{R}^d : |x| \leq n\},\)
\[
\lim_{\varepsilon \downarrow 0} \sup_{x \in B_n, |x| \leq \varepsilon} \mathbb{E} \left[ \sup_{t \in [0, T]} |X_t^x - X_t^y| \right] = 0,
\]
where \(X_t^x\) is the solution starting at \(x\).

2. Let \(P_t^x\delta_x = \mathcal{L}_{X_t^x}\) be the distribution of \(X_t^x\). Then
\[
\lim_{y \to x} \|P_t^x\delta_x - P_t^y\delta_y\|_{\text{var}} = 0, \quad t \in (0, T], x \in \mathbb{R}^d.
\]

Equivalently, the associated semigroup \((P_t)_{t \in (0, T]}\) is strong Feller, i.e. \(P_t \mathcal{B}_b(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)\).

3. If \(\phi(r) = r\), then for any \(k \geq 1\) there exists a constant \(c(k) > 0\) such that
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} V(X_t^x)^k \right] \leq c(k)V(x)^k, \quad x \in \mathbb{R}^d.
\]

**Proof.** (a) For any \(n \geq 1\), let \(B_n := \{x : |x| \leq n\}\) and
\[
b_t^n := 1_{B_n}b_t^{(1)} + b_t^{(0)}, \quad t \in [0, T].
\]
By Theorem 1.1 in [20], for any \( x \in \mathbb{R}^d \), the following SDE is well-posed:

\[
(2.3) \quad dX_t = b^n(X^n_t)dt + \sigma(X^n_t)dW_t,
\]

and for \( X_{t}^{x,n} \) being the solution starting at \( x \),

\[
(2.4) \quad \sup_{x \neq y} \mathbb{E} \left[ \sup_{t \in [0,T]} \frac{|X_{t}^{x,n} - X_{t}^{y,n}|^k}{|x - y|^k} \right] < \infty, \quad k \geq 1.
\]

As we already mentioned before that [20] only considers \( l = 1 \) in condition \((A_1)(2)\), but the proof works also for \( l \geq 2 \) by applying Khasminskii’s estimate to each \( f_i \) replacing \( \|\nabla \sigma\| \).

Let \( \tau^n \equiv \inf\{t \geq 0 : T \land |X_{t}^{x,n}| > n\} \). Then \( X_{t}^{x,n} \) solves \((2.1)\) up to time \( \tau^n \), and by the uniqueness we have

\[ X_{t}^{x,n} = X_{t}^{x,m}, \quad t \leq \tau^n \land \tau^n_{m}, n, m \geq 1. \]

So, it suffices to prove that \( \tau^n \to T \) as \( n \to \infty \).

Let

\[ L_t := \frac{1}{2} \text{tr}\{\sigma_t \sigma_t^* \nabla^2 \} + \nabla \theta_t(0). \]

By [20, Theorem 3.1] and \((A_1)(1)-(2)\), for any \( \lambda \geq 0 \) the PDE

\[
(2.5) \quad (\partial_t + L_t)u_t = \lambda u_t - b^{(0)}_t, \quad t \in [0,T], u_T = 0
\]

has a unique solution \( u \in \mathcal{H}^{\text{loc}}_{\|\cdot\|}(T) \), and there exist constants \( \lambda_0, c, \theta > 0 \) such that

\[
(2.6) \quad \lambda^\theta(\|u\|_\infty + \|\nabla u\|_\infty) + \|\partial_t u\|_{\mathcal{L}^{\text{loc}}_{\|\cdot\|}(T)} + \|\nabla^2 u\|_{\mathcal{L}^{\text{loc}}_{\|\cdot\|}(T)} \leq c, \quad \lambda \geq \lambda_0.
\]

So, we may take \( \lambda \geq \lambda_0 \) such that

\[
(2.7) \quad \|u\|_\infty + \|\nabla u\|_\infty \leq \varepsilon.
\]

Let \( \Theta_t(z) = z + u_t(z) \) for \( (t, z) \in [0,T] \times \mathbb{R}^d \). By Itô’s formula in [20, Theorem 4.1(ii)], \( Y^n_t := \Theta(X^n_t) \) satisfies

\[
(2.8) \quad dY^n_t = \{1_B, b^{(1)} + \lambda u_t + 1_{B^c} \nabla b^{(1)} u_t\}(X^n_t)dt + \{(\nabla \Theta_t) \sigma\}(X^n_t)dW_t.
\]

By \((2.7)\) and \((A_2)(3')\), there exist \( c_0, c_1, c_1 > 0 \) such that for some martingale \( M_t \),

\[
\begin{align*}
\sup_{B_{(X^n_t, \varepsilon)}} \left\langle b^{(1)}, \nabla V(X^n_t) \right\rangle + \varepsilon\|b^{(1)}(X^n_t)\| &\sup_{B_{(X^n_t, \varepsilon)}} \left( |\nabla V| + \|\nabla^2 V|\right) + c_0 K V(Y^n_t) \right\rangle dt \\
&\leq \left\langle K \phi(V(X^n_t)) + c_0 K V(Y^n_t) \right\rangle dt \leq K \left\langle \phi((1 + \varepsilon K) V(Y^n_t)) + c_0 V(Y^n_t) \right\rangle dt, \quad t \leq \tau^n.
\end{align*}
\]

Let \( H(r) := \int_0^r \frac{ds}{s + \phi((1 + \varepsilon K)s)} \). Then \( \int_0^\infty \frac{ds}{s + \phi(s)} = \infty \) implies

\[
(2.9) \quad H(\infty) := \lim_{r \to \infty} H(r) = \infty.
\]
Since \( \phi \in C^1([0, \infty); [1, \infty)) \) is increasing, by Itô’s formula we obtain
\[
dH(V(Y^n_t)) \leq c_3 dt + d\tilde{M}_t, \quad t \in [0, \tau^n]\]
for some constant \( c_3 > 0 \) and some martingale \( \tilde{M}_t \). Then
\[
\mathbb{E}[(H \circ V)(Y^n_{t \wedge \tau^n})] \leq V(x + u_0(x)) + c_3 t, \quad n \geq 1, t \in [0, T].
\]
Since (2.10) and \( |z| \geq n \) imply \( |\Theta_t(z)| \geq |z| - |u(z)| \geq n - \varepsilon \), we derive
\[
P(\tau^n < t) \leq \frac{V(x + u_0(x)) + c_3 t}{\inf_{|y| \geq n - \varepsilon} H(Y(y))} =: \varepsilon_{t,n}(x), \quad t \in [0, T].
\]
Since \( \lim_{|x| \to \infty} H(Y)(x) = \infty \), we have \( \lim_{n \to \infty} \varepsilon_{t,n}(x) = 0 \). Therefore, \( \tau^n \to T \) as \( n \to \infty \) as desired.

(b) Let \( X^n_t \) and \( X^n_y \) solve (2.1) with initial values \( x, y \) respectively. Then
\[
X^{x,n}_t = X^n_t, \quad X^{y,n}_t = X^n_t, \quad t \in [0, T \wedge \tau^n \wedge \tau^n].
\]
Combining this with (2.4) and (2.10) for some \( c(n) > 0 \), we obtain
\[
sup_{x,y \in B_k, |x-y| \leq \varepsilon} \frac{\mathbb{E} \left[ \sup_{t \in [0,T]} |X^{x}_t - X^{y}_t| \wedge 1 \right]}{\inf_{|y| \geq n - \varepsilon} H(Y(y))} \leq c(n) \varepsilon + \varepsilon_{T,n}(x) + \varepsilon_{T,n}(y), \quad n \geq 1.
\]
By letting first \( \varepsilon \downarrow 0 \) then \( n \to \infty \), we prove assertion (1).

(c) Let \( P^n_t \) be associated with \( X^n_t \). By the Bismut formula in Theorem 1.1 (iii) of [20], we find some constant \( c_n > 0 \) such that
\[
\|\nabla P^n_t f\|_\infty \leq \frac{c_n}{\sqrt{t}} \|f\|_\infty, \quad t \in (0, T], f \in \mathcal{B}_b(\mathbb{R}^d).
\]
Equivalently,
\[
\| (P^n_t)^* \delta_x - (P^n_t)^* \delta_y \|_{\text{var}} \leq \frac{c_n}{\sqrt{t}} |x - y|, \quad x, y \in \mathbb{R}^d, t \in (0, T].
\]
Next, by (2.10) and \( X_t = X^n_t \) for \( t \leq \tau^n \), we obtain
\[
|P_t f(x) - P^n_t f(x)| \leq 2 \|f\|_\infty P(\tau^n \leq t) \leq 2 \|f\|_\infty \varepsilon_{t,n}(x) \to 0 \text{ as } n \to \infty.
\]
Then
\[
\lim_{y \to x} \sup_{y \to x} \|P^n_t \delta_x - P^n_t \delta_y \|_{\text{var}} \leq \lim_{n \to \infty} \sup_{y \to x} \sup_{|f| \leq 1} \left\{ |P^n_t f(x) - P^n_t f(y)| + 2 \|f\|_\infty \varepsilon_{t,n}(x) + 2 \|f\|_\infty \varepsilon_{t,n}(y) \right\} = 0, \quad t \in (0, T].
\]
So, assertion (2) is proved.

(d) When $\phi(r) = r$, by $(A_2)(3')$, (2.8) and Itô’s formula, for any $k \geq 1$ we find a constant $c_1(k) > 0$ such that

\[ d\{V(Y^n_t)^{\frac{1}{k}}\} \leq c_1(k)V(Y^n_t)^{\frac{1}{k}}dt + dM^k_t \]

for some martingale $M^k_t$ with $d\langle M^k \rangle_t \leq \{c_1(k)V(Y^n_t)^{\frac{1}{k}}\}^2dt$. Combining this with BDG’s inequality, (2.7) and $(A_2)(3)$, we find constants $c_2(k), c_3(k) > 0$ such that

\[
\mathbb{E}\left[ \sup_{t \in [0,T]} V(X^n_t)^{\frac{1}{k}} \right] \leq (1+\varepsilon K)\mathbb{E}\left[ \sup_{t \in [0,T]} V(Y^n_t)^{\frac{1}{k}} \right] \\
\leq c_2(k)V(x + u_0^\lambda(x)) \leq c_3(k)V(x), \quad n \geq 1
\]

By Fatou’s lemma with $n \to \infty$, we prove assertion (3) for some constant $c(k) > 0$.

\[ \square \]

## 2.2 Wang’s Harnack Inequality

Under the monotone condition

\[
2\langle b_t(x) - b_t(y), x - y \rangle + \|\sigma_t(x) - \sigma_t(y)\|_{HS}^2 \leq K|x-y|^2, \\
\|\sigma_t(x) - \sigma_t(y)\|^2 \leq K|x-y|^2, \quad x, y \in \mathbb{R}^d, t \in [0,T],
\]

the following Wang’s Harnack inequality was established in [16] for large $p > 1$ and some constant $c > 0$:

\[
|P_t f(y)|^p \leq e^{c\frac{|x-y|^2}{t}}P_t|f|^p(x), \quad x, y \in \mathbb{R}^d, t \in (0,T], f \in \mathcal{B}_b(\mathbb{R}^d).
\]

Our next result extends this inequality to the singular setting, which generalizes the main result in [11] for Lipschitz continuous $\sigma_t$ as well as the corresponding result in [21] Theorem 4.3(1) for $\frac{1}{2}$-Hölder $\sigma_t$, since $(H2)(2)$ allows $\sigma_t$ to only have a Dini type continuity.

**Theorem 2.2.** Assume $(A_1)$ and $(H2)(2)$. Then there exist constants $\hat{p} > 1$ and $c > 0$ such that for all $p \geq \hat{p}$,

\[
|P_t f(y)|^p \leq e^{c\frac{|x-y|^2}{t}}P_t|f|^p(x), \quad x, y \in \mathbb{R}^d, t \in (0,T], f \in \mathcal{B}_b(\mathbb{R}^d).
\]

**Proof.** (a) We first observe that it suffices to prove for $b^{(0)} = 0$. Indeed, let $\hat{P}_t$ be the semigroup associated with the SDE

\[
dX^x_t = b^{(1)}(X^x_t)dt + \sigma_t(X^x_t)dW_t, \quad t \in [0,T].
\]

Let

\[
R^x := e^{\int_0^T \langle \sigma_t^{(1)}(\sigma_t^{(1)})^{-1}b^{(0)}(X_t)\rangle dW_t - \frac{1}{2}\int_0^T \langle \sigma_t^{(1)}(\sigma_t^{(1)})^{-1}b^{(0)}(X_t)\rangle^2 dt}.
\]

By $(A_1)$ and Khasminskii’s estimate Lemma 4.1 in [20], we have

\[
\sup_{x \in \mathbb{R}^d} \mathbb{E}|R^x|^q < \infty, \quad q > 1.
\]
By Girsanov’s theorem, for any \( p > 1 \) there exists \( c(p) > 0 \) such that
\[
|P_t f|^p(x) = |\mathbb{E}[R^x f(X^x_t)]|^p \leq (\mathbb{E}[\|R^x f(X^x_t)\|^{p-1}])^{p-1} \mathbb{E}[\|f\|^p(x)] \leq c(p) \hat{P}_t|f|^p(x), \quad p > 1.
\]
Similarly, the same inequality holds by exchanging positions of \( P_t \) and \( \hat{P}_t \). Thus, if the desired assertion holds for \( \hat{P}_t \), it also holds for \( P_t \).

(b) Now, we consider the regular case that \( b = b^{(1)} \). In this case, there exists a constant \( K > 0 \) such that for any \( x, y \in \mathbb{R}^d \),
\[
(2.12) \quad 2 \langle x - y, b_t(x) - b_t(y) \rangle + \|\sigma_t(x) - \sigma_t(y)\|_{HS}^2 \leq K(|x - y|^2 + \varphi(|x - y|^2)).
\]
For fixed \( t \in (0, T] \), let
\[
(2.13) \quad \gamma_s = \frac{1 - e^{K(s-t)}}{K}, \quad s \in [0, t],
\]
so that for some constant \( K_1 > 1 \)
\[
K \gamma_s - 2 - \gamma'_s = -1, \quad K_1(t - s) \geq \gamma_s \geq K_1^{-1}(t - s), \quad s \in [0, t].
\]
Since the coefficients of the following SDE are continuous and of linear growth in \( x \) locally uniformly in \( s \in [0, t] \), it has a weak solution (note that in general it is not well-posed)
\[
(2.15) \quad \begin{cases}
    dX_s = b_s(X_s)ds + \sigma_s(X_s)dW_s, & X_0 = x, \\
    dY_s = \left\{ b_s(Y_s) + \sigma_s(Y_s)\xi_s \right\} ds + \sigma_s(Y_s)dW_s, & Y_0 = y, s \in [0, t),
\end{cases}
\]
where
\[
(2.16) \quad \xi_s := \left\{ \frac{\sigma_s^*(\sigma_s\sigma_s^*)^{-1}}{\gamma_s}(X_s)(X_s - Y_s), \right. \quad s \in [0, t].
\]
The coupling \((2.15)\) is modified from \([16]\), we will show that it implies \( X_t = Y_t \) which is crucial to establish Wang’s Harnack inequality. Let
\[
(2.17) \quad \tau_n = \frac{nt}{n+1} \land \inf \left\{ s \geq 0 : |X_s| \lor |Y_s| \geq n \right\}, \\
R_r := e^{-\int_0^r (\xi_s.dW_s) - \frac{1}{2} \int_0^r |\xi_s|^2 ds}, \quad r \in [0, t].
\]
By Girsanov’s theorem,
\[
\hat{W}_s := W_s + \int_0^{s \land \tau_n} \xi_s dr, \quad s \in [0, t]
\]
is an \( m \)-dimensional Brownian motion under the probability \( \mathbb{Q}_n := R_{\tau_n} \mathbb{P} \). So, before time \( \tau_n \), \((2.15)\) is reformulated as
\[
(2.18) \quad \begin{cases}
    dX_s = \left\{ b_s(X_s) - \frac{X_s - Y_s}{\gamma_s} \right\} ds + \sigma_s(X_s)d\hat{W}_s, & X_0 = x, \\
    dY_s = b_s(Y_s)ds + \sigma_s(Y_s)d\hat{W}_s, & Y_0 = y, s \in [0, \tau_n].
\end{cases}
\]
By (2.12) and Itô’s formula, we obtain
\[ \mathrm{d}|X_s - Y_s|^2 \leq \left\{ \frac{K(|X_s - Y_s|^2 + \varphi(|X_s - Y_s|)^2)}{\gamma_s} - \frac{2|X_s - Y_s|^2}{\gamma_s} \right\} \mathrm{d}s + \mathrm{d}M_s \]
for \( s \in [0, \tau_n] \) and the \( \mathbb{Q}_n \)-martingale
\[ \mathrm{d}M_s := 2\langle X_s - Y_s, (\sigma_s(X_s) - \sigma_s(Y_s)) \rangle \mathrm{d}\tilde{W}_s \]
satisfying
\[ \text{TMT} \ (2.18) \quad \mathrm{d} \langle M \rangle_s \leq K^2 |X_s - Y_s|^2 \mathrm{d}s, \quad s \in [0, \tau_n]. \]
Combining this with Itô’s formula, we obtain
\[ \mathrm{d}\left\{ \frac{|X_s - Y_s|^2}{\gamma_s} \right\} \leq \left\{ \frac{(K\gamma_s - 2 - \gamma_s')}{\gamma_s^2} |X_s - Y_s|^2 + \frac{K\varphi(|X_s - Y_s|)^2}{\gamma_s} \right\} \mathrm{d}s + \frac{\mathrm{d}M_s}{\gamma_s}. \]

On the other hand, we observe that
\[ \frac{K\varphi(|X_s - Y_s|)^2}{\gamma_s} - \frac{|X_s - Y_s|^2}{2\gamma_s^2} \]
\[ \leq \sup_{r > 0} \left\{ \frac{K\varphi(r)^2}{\gamma_s} - \frac{r^2}{2\gamma_s^2} \right\} \leq \frac{K(\varphi \circ \psi^{-1})^2(2K\gamma_s)}{\gamma_s} := g_t(s). \]

Indeed, since \( \psi(r) := \frac{r^2}{\varphi(r)^2} \) is increasing in \( r \), for \( r \geq \psi^{-1}(2K\gamma_s) \) we have
\[ \frac{K\varphi(r)^2}{\gamma_s} - \frac{r^2}{2\gamma_s^2} = \frac{K\varphi(r)^2}{\gamma_s} \left( 1 - \frac{\psi(r)}{2K\gamma_s} \right) \leq \frac{K\varphi(r)^2}{\gamma_s} \left( 1 - \frac{\psi^{-1}(2K\gamma_s)}{2K\gamma_s} \right) = 0, \]
while for \( r < \psi^{-1}(2K\gamma_s) \)
\[ \frac{K\varphi(r)^2}{\gamma_s} - \frac{r^2}{2\gamma_s^2} \leq \frac{K\varphi(\psi^{-1}(2K\gamma_s))^2}{\gamma_s}, \]
so that (2.20) holds. Combining (2.19) and (2.20), and noting that \( \int_0^t \frac{\varphi(\psi^{-1}(s))^2}{s} \mathrm{d}s < \infty \) implies
\[ \int_0^t g_t(s) \mathrm{d}s < \infty \]
by (2.14), we have
\[ \int_0^{\tau_n} \frac{|X_s - Y_s|^2}{2\gamma_s^2} \mathrm{d}s \leq \frac{|x - y|^2}{2\gamma_0^2} + c_1 + \int_0^{\tau_n} \frac{\mathrm{d}M_s}{\gamma_s}, \quad s \in [0, \tau_n]. \]
By this and (2.18), for any \( \lambda > 0 \) we have
\[
e^{- \left( \lambda c_1 + \frac{\lambda |x - y|^2}{\gamma_0^2} \right)} E_{Q_n} \left[ e^{\lambda \int_0^{T_n} \frac{|X_s - Y_s|^2}{\gamma_s^2} ds} \right] \leq E_{Q_n} \left[ e^{\lambda \int_0^{T_n} \frac{dM_s}{\gamma_s}} \right].
\]
\[
\leq \left( E_{Q_n} \left[ e^{2(M \gamma_s)} \right] \right)^{\frac{1}{2}} \leq \left( E_{Q_n} \left[ e^{2K^2 \lambda^2 \int_0^{T_n} \frac{|X_s - Y_s|^2}{\gamma_s^2} ds} \right] \right)^{\frac{1}{2}}.
\]
Taking \( \lambda = (2K^2)^{-1} \) and noting that (2.14) implies \( \gamma_0 \geq K_1 t \), we find a constant \( c_2 > 0 \) such that
\[
WRE (2.21) \sup_{n \geq 1} E_{Q_n} \left[ e^{\lambda \int_0^{T_n} \frac{|X_s - Y_s|^2}{\gamma_s^2} ds} \right] \leq e^{c_2 + \frac{c_2 |x - y|^2}{t^2}}.
\]
Since (2.16) implies
\[
|\xi_s|^2 \leq \frac{c_3 |X_s - Y_s|^2}{\gamma_s^2}
\]
for some constant \( c_3 > 0 \), this implies that for some constants \( q, c_4 > 1 \),
\[
PAY (2.22) \sup_{n \geq 1} E \left[ |R_{\tau_n}|^q \right] \leq e^{c_4 + \frac{c_4 |x - y|^2}{t^2}}.
\]
By the martingale convergence theorem, this implies that \( (R_s)_{s \in [0,t]} \) is a martingale with
\[
RTT (2.23) \mathbb{E}[R_t^q] \leq e^{c_4 + \frac{c_4 |x - y|^2}{t^2}},
\]
such that Girsanov's theorem implies that \( (\tilde{W}_s)_{s \in [0,t]} \) is an \( m \)-dimensional Brownian motion under \( Q := R_t \mathbb{P} \), and
\[
dY_s = b_s(Y_s) ds + \sigma_s(Y_s) d\tilde{W}_s, \quad Y_0 = y, s \in [0,t]
\]
holds so that \( P_t f(y) = \mathbb{E}_Q[f(Y_t)] \), and furthermore (2.22) ensures
\[
\mathbb{E}_Q \left[ e^{\lambda \int_0^t \frac{|X_s - Y_s|^2}{\gamma_s^2} ds} \right] < \infty.
\]
Since \( \int_0^t \frac{ds}{\gamma_s^2} = \infty \) and \( |X_s - Y_s|^2 \) is continuous in \( s \), this implies \( Q(X_t = Y_t) = 1 \). Combining this with (2.23) and \( P_t f(y) = \mathbb{E}_Q[f(Y_t)] \), we find a constant \( c > 0 \) such that for any \( p \geq \frac{q}{q-1} \), Hölder's inequality yields
\[
|P_t f(y)|^p = |\mathbb{E}[R_t f(Y_t)]|^p = |\mathbb{E}[R_t f(X_t)]|^p \leq \left( \mathbb{E}[R_t^q] \mathbb{E}[|f|^p(X_t)] \right)^{\frac{p}{q}} \leq (P_t |f|^p)(x)e^{c + \frac{c_4 |x - y|^2}{t^2}}.
\]
\[\square\]
3 Proof of Theorem 1.1

Proof of Theorem 1.1(1). Let $X_0$ be $\mathcal{F}_0$-measurable with $\gamma := L_{X_0} \in \mathcal{P}_V$. Let

$$\mathcal{C}^\gamma := \{ \mu \in C([0, T]; \mathcal{P}_V) : \mu_0 = \gamma \}. $$

For any $\mu \in C([0, T]; \mathcal{P}_V)$, by Theorem 2.1 (A2) implies that the following SDE is well-posed

$$dX_t^\mu = b_t(X_t^\mu, \mu_t)dt + \sigma_t(X_t^\mu)dW_t, \quad X_0^\mu = X_0. $$

Denote $\Phi_t(\mu) := Z_{X_t^\mu}$. By Theorem 2.1 for the well-posedness of (1.1) and estimate (1.4), it suffices to prove that $\Phi$ has a unique fixed point in $\mathcal{C}^\gamma$. To this end, following the line of [7] and [19], we approximate $\mathcal{C}_T^\gamma$ by bounded subsets

$$\mathcal{C}_N^\gamma := \{ \mu \in \mathcal{C}^\gamma : \sup_{t \in [0, T]} \mu_t(V)e^{-Nt} \leq N(1 + \gamma(V)) \}, \quad N \geq 1. $$

(1a) We claimed that for some constant $N_0 \geq 1$, $\Phi \mathcal{C}_N^\gamma \subset \mathcal{C}_N^\gamma$ for $N \geq N_0$. To this end, let

$$L_t^\mu := \nabla b_t^{(0)}(\cdot, \mu_t) + \frac{1}{2} \text{tr}\{\sigma_t \sigma_t^\gamma \nabla^2 \}, $$

and consider the Zvonkin’s transform of $X_t^\mu$ and the Kolmogorov backward equation as follows,

$$Y_t^\mu = X_t^\mu + u_t^\mu(X_t^\mu), \quad u_t^\mu \in \hat{H}_0. $$

$$(\partial_t + L_t^\mu)u_t^\mu = \lambda u_t^\mu - b_t^{(0)}(\cdot, \mu_t), \quad t \in [0, T], \quad u_T^\mu = 0, $$

for $\lambda > 0$ such that $\|u_t^\mu\|_\infty + \|\nabla u_t^\mu\|_\infty \leq \frac{1}{2}$. By $(H_1)(3)$ and Itô’s formula, we find a constant $c_1 > 0$ such that

$$d\{V(Y_t^\mu)\}^2 \leq c_1\{V(Y_t^\mu)^2 + \mu_t(V)^2\}dt + dM_t $$

for some martingale $M_t$. By the condition on $V$ and $|X_t^\mu - Y_t^\mu| \leq \frac{1}{2}$, we find a constant $C > 1$ such that

$$C^{-1}V(X_t^\mu) \leq V(Y_t^\mu) \leq CV(X_t^\mu), $$

so that (3.3) implies that for some constant $c_2 > 0$

$$\mathbb{E}(V(X_t^\mu)^2|X_0^\mu) \leq C^2 e^{c_1 t} V(X_0^\mu)^2 + C^2 c_1 \int_0^t e^{c_1(t-s)} \mu_s(V)^2 ds $$

$$\leq c_2 V(X_0^\mu)^2 + c_2 e^{c_2 t} \{N(1 + \gamma(V))^2 \int_0^t e^{(2N-c_1)s} ds \leq c_2 V(X_0^\mu)^2 + \frac{c_2}{1 - K} e^{c_2 t} \{N(1 + \gamma(V))^2 \frac{1}{2N - c_1} e^{(2N-c_1)t}, \quad t \in [0, T], \quad \mu \in \mathcal{C}_N^\gamma. $$

Thus, for any $N \geq N_0 := c_2 + 2\sqrt{c_2}$, we have

$$\sup_{t \in [0, T]} \{\Phi_t(\mu)\}V e^{-Nt} \leq (1 + \gamma(V))\{\sqrt{c_2} + \sqrt{c_2 N} \leq N(1 + \gamma(V))$. 

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Thus, $\Phi_{N}^{\gamma} \subset \mathcal{C}_{N}^{\gamma}$ for $N \geq N_0$.

(b) Let $N \geq N_0$. We prove that $\Phi$ has a unique fixed point in $\mathcal{C}_{N}^{\gamma}$, and hence it has a unique fixed point in $\mathcal{C}^{\gamma}$ as desired. Consider the following complete metric on $\mathcal{C}_{N}^{\gamma}$:

$$\rho_{\lambda}(\mu, \nu) := \sup_{t \in [0, T]} e^{-\lambda t} \| \mu_t - \nu_t \|_V.$$ 

Let

$$\xi_s := \{\sigma_s (\sigma_s \nu_s) [b_s(X^\mu_s, \nu_s) - b_s(X^\mu_s, \mu_s)] (X^\mu_s), \ s \in [0, T].$$

By (1.3),

$$N_t := e^{\int_0^T \xi_s dW_s} - \frac{1}{2} \int_0^T |\xi_s|^2 ds$$

is a martingale, such that

$$\tilde{W}_r := W_r - \int_0^r \xi_s ds, \ r \in [0, t]$$

is a Brownian motion under the probability $Q_t := R_t \mathbb{P}$. Reformulate (3.1) as

$$dX^\mu_t = b_t(X^\mu_t, \nu_t) dt + \sigma_t(X^\mu_t) d\tilde{W}_t, \ X^\mu_0 = X_0, \ r \in [0, t].$$

By the uniqueness we obtain

$$\Phi_t(\nu) = \mathcal{L}_{X^\mu_t | Q_t},$$

where $\mathcal{L}_{X^\mu_t | Q_t}$ stands for the distribution of $X^\mu_t$ under $Q_t$. Then by (3.5), we find a constant $c_1(N) > 0$

$$\| \Phi_t(\mu) - \Phi_t(\nu) \|_V = \sup_{|f| \leq V} |\mathbb{E} [f(\Phi_t(\mu)) - f(\Phi_t(\nu))]|$$

$$\leq \mathbb{E} \left[ \left\{ \mathbb{E}(V(X^\mu_t)^2 | X^\mu_0) \right\} ^{\frac{1}{2}} \left\{ \mathbb{E} \left[ |R_t - 1|^2 |X^\mu_0\right] \right\} ^{\frac{1}{2}} \right]$$

$$\leq c_1(N) \mathbb{E} \left[ V(X_0) \left\{ \mathbb{E}[R^2_t - 1 | X_0] \right\} ^{\frac{1}{2}} \right].$$

Since $\mu \in \mathcal{C}_{N}^{\gamma}$, by (1.3) we find a constant $c_2(N) > 0$ such that

$$|\xi_s|^2 \leq c_2(N) (1 \wedge \| \mu_s - \nu_s \|^2_V), \ s \in [0, T],$$

so that for some constant $c_3(N) > 0$

$$\mathbb{E}[R^2_t - 1 | X_0] \leq \mathbb{E} \left[ e^{2 \int_0^t \xi_s dW_s} - 2 e^{\int_0^t \xi_s dW_s} + \int_0^t |\xi_s|^2 ds \right] \leq \mathbb{E} \left[ e^{2 \int_0^t |\xi_s|^2 ds} | X_0 \right] - 1$$

$$\leq c_3(N) \int_0^t \| \mu_s - \nu_s \|^2_V ds.$$ 

Combining this with (3.7), we find a constant $c_4(N) > 0$ such that

$$\rho_{\lambda}(\Phi(\mu), \Phi(\nu)) = \sup_{t \in [0, T]} e^{-\lambda t} \| \Phi_t(\mu) - \Phi_t(\nu) \|_V$$

$$\leq c_4(N) \int_0^t \| \mu_s - \nu_s \|^2_V ds.$$
\[
\leq c_4(N)E[V(X_0)]\rho_\lambda(\mu, \nu) \sup_{t \in [0, T]} \left( \int_0^t e^{-2\lambda(t-s)}ds \right)^{\frac{1}{2}}
\leq \frac{c_4(N)E[V(X_0)]}{\sqrt{2\lambda}} \rho_\lambda(\mu, \nu), \quad \mu, \nu \in \mathcal{G}_N.
\]

Therefore, when \( \lambda > 0 \) is large enough, \( \Phi \) is contractive under \( \rho_\lambda \) so that it has a unique fixed point in \( \mathcal{G}_N \) as desired.

(c) Proof of (1.4). Let \( X_t \) solve (1.1) with \( \mathcal{L}_{X_0} \in \mathcal{P}_V \), and denote \( \mu_t = \mathcal{L}_{X_t} \). We have \( \sup_{t \in [0, T]} \mu_t(V) < \infty \). By (H1)(3) and Itô’s formula, we find a constant \( c_1 > 0 \) such that

\[
\text{ER1} \quad (3.8) \quad dV(Y_t^\mu) \leq c_1 \{ V(Y_t^\mu) + \mu_t(V) \} dt + dM_t
\]

for some martingale \( M_t \) with

\[
\text{ER*} \quad (3.9) \quad d\langle M \rangle_t \leq c_1^2 V(X_t)^2 dt.
\]

By this and (3.4), we find a constant \( c_2 > 0 \) such that

\[
\text{VES} \quad (3.10) \quad \mu_t(V) = E[V(X_t)] \leq c_2 \int_0^t \mu_s(V) ds, \quad t \in [0, T],
\]

so that by Gronwall’s inequality,

\[
\text{ERR} \quad (3.11) \quad E[V(X_t)] \leq e^{c_2 t} E[V(X_0)], \quad t \in [0, T].
\]

Combining this with (3.8) and applying Itô’s formula, for any \( p \geq 1 \) we find a constant \( c_1(p) > 0 \) such that

\[
dV(Y_t^\mu)^p \leq c_1(p) \{ V(Y_t^\mu)^p + \mu_t(V)^p \} dt + pV(Y_t^\mu)^p-1 dM_t.
\]

By (3.9), (3.4) -and BDG’s inequality, we find a constant \( c_2(p) > 0 \) such that

\[
\xi_t := E \left[ \sup_{s \leq t} V(X_s)^p \bigg| X_0 \right], \quad t \in [0, T]
\]

satisfies

\[
\xi_{t \wedge \tau_n} \leq V(X_0)^p + c_2(p)E[V(X_0)]^p + c_2(p) \int_0^t \xi_{s \wedge \tau_n} ds + c_2(p)E \left[ \left( \int_0^{t \wedge \tau_n} V(X_s)^2 ds \right)^{\frac{1}{2}} \bigg| X_0 \right]
\]
\[
\leq V(X_0)^p + c_2(p)E[V(X_0)] + c_2(p) \int_0^t \xi_{s \wedge \tau_n} ds + \frac{1}{2} \xi_{t \wedge \tau_n} + \frac{c_2(p)^2}{2} \int_0^t \xi_{s \wedge \tau_n} ds, \quad t \in [0, T].
\]

So that for \( c_3(p) := 2c_2(p) + c_2^2(p) \) we obtain

\[
\xi_{t \wedge \tau_n} \leq 2 \{ V(X_0)^p + c_2(p)E[V(X_0)]^p \} e^{c_3(p)t}, \quad t \in [0, T], n \geq 1.
\]

Letting \( n \to \infty \) we derive (1.4) for some constant \( c(p) > 0 \).
Proof of Theorem 1.1(2). Let $\hat{P}_t$ be the Markov semigroup of $X_t^\mu$ solving (3.1) for $\mu := P_t^* \mu$, so that
\begin{equation}
(3.12)
P_t^* \mu = \hat{P}_t^* \mu, \ t \in [0, T].
\end{equation}
By Theorem 2.1 we have
\[ \lim_{y \to x} \| \hat{P}_t^* \delta_x - \hat{P}_t^* \delta_y \|_{\text{var}} = 0, \ x \in \mathbb{R}^d. \]
Since $\mu_n \to \mu$ weakly, we may construct random variables $\{\xi_n\}$ and $\xi$ such that $\mathcal{L}_{\xi_n} = \mu_n$, $\mathcal{L}_{\xi} = \mu$ and $\xi_n \to \xi$ a.s. Thus, by the dominated convergence theorem we obtain
\begin{equation}
(3.13)
\lim_{n \to \infty} \| \hat{P}_t^* \mu_n - \hat{P}_t^* \mu \|_{\text{var}} = \lim_{n \to \infty} \| \mathbb{E}[\hat{P}_t^* \delta_{\xi_n} - \hat{P}_t^* \delta_{\xi}] \|_{\text{var}} \\
\leq \lim_{n \to \infty} \mathbb{E}[\| \hat{P}_t^* \delta_{\xi_n} - \hat{P}_t^* \delta_{\xi} \|_{\text{var}}] = 0.
\end{equation}
Hence,
\begin{equation}
(3.14)
\limsup_{n \to \infty} \| \hat{P}_t^* \mu_n - \hat{P}_t^* \mu \|_V \\
\leq \limsup_{n \to \infty} \left\{ \sup_{|f| \leq N} \| (\hat{P}_t^* \mu_n)(f) - (\hat{P}_t^* \mu)(f) \| + \| \hat{P}_t^* \mu_n - \hat{P}_t^* \mu \|_V \right\} \\
\leq N \limsup_{n \to \infty} \| \hat{P}_t^* \mu_n - \hat{P}_t^* \mu \|_{\text{var}} + \sup_{n \geq 1} \int_{\mathbb{R}^d} \hat{P}_t(V - N)^+ d(\mu_n + \mu) \\
= \sup_{n \geq 1} \left\{ \hat{P}_t^*(\mu_n + \mu) \right\} ((V - N)^+), \ N \geq 1.
\end{equation}
Since $\mu_n(V^p)$ is bounded for some $p \in (1, 2]$, (3.5) implies that
\begin{equation}
(3.15)
\sup_{n \geq 1, t \in [0, T]} (\hat{P}_t^* \mu_n)(V^p) < \infty,
\end{equation}
so that letting $m \to \infty$ in (3.14) we prove
\begin{equation}
(3.16)
\limsup_{n \to \infty} \| \hat{P}_t^* \mu_n - \hat{P}_t^* \mu \|_V = 0.
\end{equation}
On the other hand, by the Girsanov transform in step (b) above for $\mu_n$ replacing $\nu$, we find a constant $c > 0$ such that
\[ \| P_t^* \mu_n - \hat{P}_t^* \mu_n \|_V^2 \leq c \int_0^t \| \mu_s - \nu_s \|_V^2 ds, \ t \in [0, T]. \]
Combining this with (3.16) and Fatou’s lemma due to (3.15), we derive
\[ \limsup_{n \to \infty} \| P_t^* \mu_n - \hat{P}_t^* \mu \|_V^2 \leq 2 \limsup_{n \to \infty} \left\{ \| \hat{P}_t^* \mu_n - \hat{P}_t^* \mu \|_V + \| P_t^* \mu_n - \hat{P}_t^* \mu_n \|_V \right\} \]
\[ \leq \limsup_{n \to \infty} \int_0^t \| P_s^* \mu_n - \hat{P}_s^* \mu_n \|_V^2 ds < \infty, \ t \in [0, T], \]
By Gronwall’s inequality and (3.12), we obtain
\[ \limsup_{n \to \infty} \| P_t^* \mu_n - P_t^* \mu \|_V = \limsup_{n \to \infty} \| P_t^* \mu_n - \hat{P}_t^* \mu \|_V = 0. \]
This implies (1.5).
Proof of Theorem 1.1(3). By (1.6), \(R_t\) in (3.6) is a martingale with

\[ |\xi_s|^2 \leq c \|\mu_s - \nu_s\|_{\text{var}}^2 \]

for some constant \(c > 0\). Then

\[ \|\Phi_t(\mu) - \Phi_t(\nu)\|_{\text{var}} = \sup_{|f| \leq 1} |E[f(X^\mu_t)(R_t - 1)]| \leq E[|R_t - 1|]. \]

By Pinsker’s inequality, we obtain

\[ (E[|R_t - 1|])^2 \leq 2E[R_t \log R_t] = 2E_Q \log R_t = E_Q \int_0^t |\xi_s|^2 ds \leq c^2 \int_0^t \|\mu_s - \nu_s\|_{\text{var}}^2 ds. \]

Combining this with (3.17), as shown in the proof of (1) we see that when \(\lambda > 0\) is large enough, \(\Phi\) is contractive in \(C^{\gamma}\) under the metric

\[ \tilde{\rho}_\lambda(\mu, \nu) := \sup_{t \in [0, T]} e^{-\lambda t} \|\mu_t - \nu_t\|_{\text{var}}. \]

Hence, by Theorem 2.1, (1.1) is well-posed and (1.4) holds.

Let \(\mu_t = P^*_t \mu\) as in the proof of Theorem 1.1(3). As shown above that (1.6), Girsanov’s theorem and Pinsker’s inequality imply

\[ \|P^*_t \mu_n - \hat{P}^*_t \mu_n\|^2_{\text{var}} \leq c \int_0^t \|P^*_s \mu_n - P^*_s \mu\|^2_{\text{var}} ds \]

for some constant \(c > 0\). Thus, by the same reason leading to (1.5), (1.7) follows from (3.13).

4 Proof of Theorem 1.2

Noting that conditions in Theorem 1.2 imply those in Theorem 1.1 and when \(V\) is bounded we have

\[ \|\cdot\|_{\text{var}} \leq \|\cdot\| \leq \|V\|_{\infty} \cdot \|\cdot\|_{\text{var}}, \]

so the first assertion follows. It remains to verify (1.11) and (1.12).

(1) Let \(\hat{P}_t\) be associated to solutions of (2.1) for \(b(\cdot, \delta_0)\) replacing \(b\). By Theorem 2.2 there exist constants \(c', p' > 1\) such that

\[ |\hat{P}_t f(y)|^{p'} \leq e^{c' + \frac{\rho |x-y|^2}{2}} \hat{P}_t |f|^{p'}(x), \quad x, y \in \mathbb{R}^d, t \in (0, T], f \in \mathcal{B}_b(\mathbb{R}^d). \]

Consequently,

\[ |\hat{P}_t f(\mu)|^{2p'} \leq C(t, \mu, \nu) \hat{P}_t |f|^{2p'}(\nu), \quad x, y \in \mathbb{R}^d, t \in (0, T], f \in \mathcal{B}_b(\mathbb{R}^d) \]

holds for

\[ C(t, \mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{2c' + \frac{\rho |x-y|^2}{2}} \pi(dx, dy). \]
Next, let $\tilde{X}_t$ solve (2.1) for $b(\cdot, \delta_0)$ replacing $b$ with initial distribution $\gamma \in \mathcal{P}_\nu$, and denote
\[ \xi_t := \{\sigma_t^* (\sigma_t^*)^{-1}[b_t(\tilde{X}_t, P_t^* \gamma) - b_t(\tilde{X}_t, \delta_0)]\}, \quad t \in [0, T]. \]
By (1.10),
\[ R_t := e^{\int_0^t (\xi_s, dW_s) - \frac{1}{2} \int_0^t |\xi_s|^2 ds}, \quad t \in [0, T] \]
is a martingale, and by Girsanov’s theorem, for any $q > 1$ we find a constant $c(q) > 0$ such that
\[ |P_t f(\gamma)|^q = |\mathbb{E}[R_t f(\tilde{X}_t)]|^q \leq \hat{P}_t |f|^q(\gamma) e^{c(q)(\gamma^2(\nu^2) + V(0)) t}, \quad t \in [0, T]. \]
Similarly,
\[ |\hat{P}_t f(\gamma)|^q \leq P_t |f|^q(\gamma) e^{c(q)(\gamma^2(\nu^2) + V(0)) t}, \quad t \in [0, T]. \]
Combining these with (1.1), we derive (1.11) for any $p > p'$ and some constant $c > 0$.
(2) When $\Phi$ is bounded, (1.11) implies
\[ \sup_{\mu \in \mathcal{P}} ||b(\cdot, \mu) - b(\cdot, \delta_0)||_\infty < \infty. \]
Let $P_t^\mu$ be the Markov semigroup for solutions to (2.1) for $b_t(\cdot, P_t^* \mu)$ replacing $b_t$, by [21, Theorem 4.1], there exists a constant $c' > 0$ such that $P_t^\mu$ satisfies the log-Harnack inequality
\[ P_t^\mu \log f(x) \leq \log P_t^\mu f(y) + \frac{c'|x - y|^2}{t}, \quad x, y \in \mathbb{R}^d, t \in (0, T], 0 < f \in \mathcal{B}_b(\mathbb{R}^d). \]
Consequently,
\[ P_t^\mu \log f(\mu) \leq \log P_t^\mu f(\nu) + \frac{c' \mathbb{W}_2(\mu, \nu)^2}{t}, \quad \mu, \nu \in \mathcal{P}, t \in (0, T], 0 < f \in \mathcal{B}_b(\mathbb{R}^d). \]
Since $(P_t^\mu)^* \mu = P_t^* \mu$, this and Pinsker’s inequality imply
\[ \|P_t^* \mu - (P_t^\mu)^* \nu\|^2_{\text{var}} \leq 2 \text{Ent}((P_t^\mu)^* \mu \| (P_t^\mu)^* \nu) = 2 \sup_{P_t^\mu f(\nu) \leq 1} P_t^\mu \log f(\mu) \leq \frac{2c' \mathbb{W}_2(\mu, \nu)^2}{t}. \]
Since $\|\cdot\|_{\text{var}} \leq 2$, this is equivalent to
\[ (4.2) \quad \|P_t^* \mu - (P_t^\mu)^* \nu\|^2_{\text{var}} \leq \alpha_t := \min \left\{ 4, \frac{2c' \mathbb{W}_2(\mu, \nu)^2}{t} \right\}, \quad t \in (0, T]. \]
On the other hand, let $X_t^{\mu, \nu}$ solve (2.1) with $b_t(\cdot, \mu_t)$ replacing $b_t$ and $\mathcal{L}_{X_0^{\mu, \nu}} = \nu$. Let
\[ \xi_t := \{\sigma_t^* (\sigma_t^*)^{-1}[b_t(X_t^{\mu, \nu}, P_t^* \nu) - b_t(X_t^{\mu, \nu}, P_t^* \mu)]\}, \quad t \in [0, T]. \]
By (1.10) for bounded $\Phi$, we find a constant $K > 0$ such that
\[ \|\xi_t\|^2 \leq K \|P_t^* \mu - P_t^* \nu\|^2_{\text{var}}. \]
So,
\[ R_t := e^{\int_0^t (\xi_s, dW_s) - \frac{1}{2} \int_0^t |\xi_s|^2 ds}, \quad t \in [0, T] \]
is a martingale, and by Girsanov's theorem and Pinsker's inequality, we obtain

\[ \| (P_t^\mu)^* \nu - P_t^\nu \|_{\text{var}}^2 \leq 2 \mathbb{E}_{R_t \nu} [\log R_t] \leq K \int_0^t \| P_s^\mu - P_s^\nu \|_{\text{var}}^2 \, ds. \]

Combining this with (4.2), and we derive

\[ \| P_t^\mu - P_t^\nu \|_{\text{var}}^2 \leq 2 \| P_t^\mu - (P_t^\mu)^* \nu \|_{\text{var}}^2 + 2 \| (P_t^\mu)^* \nu - P_t^\nu \|_{\text{var}}^2 \]

\[ \leq 2 \alpha_t + 2K \int_0^t \| P_s^\mu - P_s^\nu \|_{\text{var}}^2 \, ds. \]

By Gronwall’s inequality, we find a constant \( c > 0 \) such that

\[ \| P_t^\mu - P_t^\nu \|_{\text{var}}^2 \leq \alpha_t + 2K \int_0^t \alpha_s e^{2K(t-s)} \, ds \leq c(t^{-1} - \log[1 \wedge \mathbb{W}_2(\mu, \nu)]) \mathbb{W}_2(\mu, \nu)^2. \]

References

[1] R. Carmona, F. Delarue, *Probabilistic Theory of Mean Field Games with Applications I*, Springer 2019.

[2] S. Fang, T. Zhang, *A study of a class of stochastic differential equations with non-Lipschitzian coefficients*, Probab. Theory Related Fields 85(2006), 580–597.

[3] X. Huang, P. Ren, F.-Y. Wang, *Distribution dependent stochastic differential equations*, Front. Math. China 16(2021), 257–301.

[4] X. Huang, F.-Y. Wang, *Distribution dependent SDEs with singular coefficients*, Stoch. Proc. Appl. 129(2019), 4747–4770.

[5] X. Huang, F.-Y. Wang, *McKean-Vlasov SDEs with drifts discontinuous under Wasserstein distance*, Disc. Cont. Dyn. Syst. Ser. A. 4(2021), 1667–1679.

[6] X. Huang, F.-Y. Wang, *Derivative estimates on distributions of McKean-Vlasov SDEs*, Elect. J. Probab. 26(2021), 1–12.

[7] X. Huang, F.-Y. Wang, *Well-posedness for singular McKean-Vlasov stochastic differential equations*, [arXiv:2012.05014](https://arxiv.org/abs/2012.05014).

[8] N.V. Krylov, *Controlled diffusion processes*, Translated from the Russian by A. B. Aries. Applications of Mathematics, 14. Springer-Verlag, New York-Berlin, 1980.

[9] N.V. Krylov, M. Röckner, *Strong solutions of stochastic equations with singular time dependent drift*, Probab. Theory Relat. Fields 131(2005), 154–196.

[10] M. Röckner, X. Zhang, *Well-posedness of distribution dependent SDEs with singular drifts*, [arXiv:1809.02216](https://arxiv.org/abs/1809.02216), to appear in Bernoulli.
[11] J. Shao, *Harnack inequalities and heat kernel estimates for SDEs with singular drifts*, Bull. Sci. Math. 137(2013), 589–610.

[12] A.-S. Sznitman, *Topics in propagations of chaos*, Lecture notes in Math. Vol. 1464, pp. 165–251, Springer, Berlin, 1991.

[13] A. J. Veretennikov, *On strong solutions and explicit formulas for solutions of stochastic integral equations*, Sbornik: Mathematics 39(1981), 387-403.

[14] C. Villani, *Optimal Transport, Old and New*, Springer-Verlag, 2009.

[15] F.-Y. Wang, *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds*, Probab. Theory Related Fields, 109(1997), 417–424.

[16] F.-Y. Wang, *Harnack inequality for SDE with multiplicative noise and extension to Neumann semigroup on nonconvex manifolds*, Ann. Probab. 39(2011), 1449–1467

[17] F.-Y. Wang, *Harnack Inequalities and Applications for Stochastic Partial Differential Equations*, Springer, 2013, Berlin.

[18] F.-Y. Wang, *Distribution dependent SDEs for Landau type equations*, Stoch. Proc. Appl. 128(2018), 595–621.

[19] F.-Y. Wang, *Distribution dependent reflecting stochastic differential equations*, arXiv:2106.12737.

[20] P. Xia, L. Xie, X. Zhang, G. Zhao, *$L^q(L^p)$-theory of stochastic differential equations*, Stoch. Proc. Appl. 130(2020), 5188–5211.

[21] C. Yuan, S.-Q. Zhang, *A study on Zvonkin’s transformation for stochastic differential equations with singular drift and related applications*, to appear in J. Diff. Equat. arXiv:1910.05903.

[22] X. Zhang, *Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients*, Electr. J. Probab. 16(2011), 1096–1116.

[23] G. Zhao, *On distribution dependent SDEs with singular drifts*, arXiv:2003.04829v3.

[24] A. K. Zvonkin, *A transformation of the phase space of a diffusion process that will remove the drift*, (Russian) Mat. Sb. (N.S.) 93(135)(1974), 129–149, 152.