SUPERCONNECTIONS AND A FINSLERIAN GAUSS-BONNET-CHERN FORMULA

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Abstract. In this paper, we obtain a Finslerian Gauss-Bonnet-Chern formula for an even dimensional closed Finsler manifold \((M, F)\) by using a Mathai-Quillen type formula of Feng and Zhang in \cite{feng-zhang} on the Euler characteristic. As an application, we will deduce a simple Finslerian Gauss-Bonnet-Chern formula for closed Berwald spaces. Moreover, by using the geometric localization procedure, we also obtain a Bao-Chern type Gauss-Bonnet-Chern formula for closed Berwald spaces.

Contents

Introduction 1
1. Superconnections and the Euler characteristic 3
1.1. Superspaces and superconnections 4
1.2. A Mathai-Quillen type formula on the Euler characteristic 5
2. A Finslerian Gauss-Bonnet-Chern formula 8
2.1. A brief review of Finsler geometry 8
2.2. A Finslerian Gauss-Bonnet-Chern formula 9
3. A special case: Berwald space 17
3.1. A Finslerian GBC-formula for Berwald spaces 17
3.2. A Bao-Chern type GBC-formulae for Berwald spaces 18
References 24

Introduction

In the celebrated paper \cite{chern}, S. S. Chern presented a simple and intrinsic proof of the following famous Gauss-Bonnet-Chern formula (also GBC-formula in short) for a closed and oriented Riemannian manifold \((M, g^{TM})\) of dimension \(2n\):

\[
\chi(M) = \left(\frac{-1}{2\pi}\right)^n \int_M \text{Pf}(R^{TM}),
\]

where the Pfaffian \(\text{Pf}(R^{TM})\) is a well-defined \(2n\)-form on \(M\) constructed from the curvature \(R^{TM}\) of the Levi-Civita connection \(\nabla^{TM}\) associated to the Riemannian metric \(g^{TM}\). With

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With the help of the Poincaré-Hopf theorem and noticing that the tangent unit spheres of have the constant volume, Chern obtained his formula (0.1) by computing the following integral
\[ \sum_{a_1, \ldots, a_{2n}=1}^{2n} \epsilon_{a_1, \ldots, a_{2n}} \Omega_{a_1}^{a_2} \wedge \cdots \wedge \Omega_{a_{2n-1}}^{a_n}, \]
for any vector field \( X \) constructed an analogue differential 2
\( \sum_{a_1, \ldots, a_{2n}=1}^{2n} \epsilon_{a_1, \ldots, a_{2n}} \Omega_{a_1}^{a_2} \wedge \cdots \wedge \Omega_{a_{2n-1}}^{a_n}, \]
(0.2)
where
\( \Omega_{a}^{b} := g^{TM}(R^{TM} e_a, e_b). \)

Chern’s formula (0.1) expresses the Euler characteristic \( \chi(M) \) by the integration of the purely geometric differential form \( Pf(R^{TM}) \) on \( M \) and initiates the geometric theory of characteristic classes—Chern-Weil theory, which plays a very important role in the study of modern geometry and topology. The key point in Chern’s proof is his significant transgression formula
\[ \left( \frac{-1}{2\pi} \right)^n \pi^* Pf(R^{TM}) = \frac{-1}{2\pi} \sum_{a} \epsilon_a Pf(R^{TM}) = \frac{-1}{2\pi} \sum_{a} \epsilon_a Pf(R^{TM}) = [X]^* d^{SM} \Pi = -d^{M}[X]^* \Pi. \]

With the help of the Poincaré-Hopf theorem and noticing that the tangent unit spheres of \( M \) have the constant volume, Chern obtained his formula (0.1) by computing the following integral
\[ \left( \frac{-1}{2\pi} \right)^n \int_M Pf(R^{TM}) = \lim_{\epsilon \to 0} \int_{\partial Z_{\epsilon}(X)} [X]^* \Pi. \]

After Chern’s work, many people try to generalize Chern’s formula (0.1) to the Finsler setting. Inspired by Chern’s work, Lichnerowicz \( [11] \) first established a GBC-formula for some special Finsler manifolds by using the Cartan connection \( \nabla^{Car} \) on \( \pi^* TM \). Realized that almost all Finsler geometric quantities live actually on the unit sphere bundle \( SM \), Lichnerowicz constructed an analogue differential 2n-form \( Pf(R^{Car}) \) on \( SM \) and proved the following transgression formula
\[ \left( \frac{-1}{2\pi} \right)^n Pf(R^{Car}) = \frac{-1}{2\pi} \sum_{a} \epsilon_a Pf(R^{Car}) = \frac{-1}{2\pi} \sum_{a} \epsilon_a Pf(R^{Car}) = [X]^* d^{SM} \Pi^{Car}, \]
for some \( (2n-1) \)-form \( \Pi^{Car} \) on \( SM \), where \( R^{Car} \) denotes the curvature of the Cartan connection on \( \pi^* TM \). Following Chern’s strategy, Lichnerowicz also proceeded the computations
\[ \left( \frac{-1}{2\pi} \right)^n \int_M [X]^* Pf(R^{Car}) := \lim_{\epsilon \to 0} \int_{M_{\epsilon}} [X]^* Pf(R^{Car}) = \lim_{\epsilon \to 0} \int_{\partial Z_{\epsilon}(X)} [X]^* \Pi^{Car}, \]
To get the desired Euler number \( \chi(M) \) from the above computations, Lichnerowicz had to assume that the space \( (M, F) \) should be a Cartan-Berwald space and all Finsler unit spheres \( S_x M = \{ Y \in T_x M \mid F(Y) = 1 \} \) should have the same volume as a Euclidean unit sphere. Note that the later assumption holds automatically for all Cartan-Berwald spaces of dimension larger than 2. Moreover, as mentioned by D. Bao and Z. Shen in \( [3] \) (also in \( [15] \)), when the Finsler metric is reversible, then by a theorem of Brickell, any Cartan-Berwald space of dimension larger than 2 must be Riemannian.
Around fifty years later, Bao and Chern [1] dropped the assumption of the Cartan-Berwald condition of Lichnerowicz by using the Chern connection $\nabla^{Ch}$ proposed in [6] and established the following GBC-formula for all $2n$-dimensional oriented and closed Finsler manifolds with the constant volume of Finsler unit spheres:

$$\left(\frac{-1}{2\pi}\right)^n \int_{M} [X]^*\left[\text{Pf}(\hat{R}^{Ch}) + F\right] = \chi(M) \frac{\text{Vol}(\text{Finsler } S^{2n-1})}{\text{Vol}(S^{2n-1})}$$

by proving the under transgression formula

$$\left(\frac{-1}{2\pi}\right)^n \left[\text{Pf}(\hat{R}^{Ch}) + F\right] = -d\text{SM} \Pi^{Ch},$$

where $\hat{R}^{Ch}$ is the skew-symmetrization of the curvature $R^{Ch}$ of the Chern connection $\nabla^{Ch}$ with respect to the fundamental tensor $g_F$ of $F$, $\text{Pf}(\hat{R}^{Ch})$ is the Pfaffian of $\hat{R}^{Ch}$, $F$ is a correction term and $\Pi^{Ch}$ is the associated transgression form (cf. [1] for details).

Note that the constant volume condition in Bao-Chern’s formula (0.4) is a much strong assumption for a Finsler manifold. To avoid this defect, following Bao-Chern’s approach, Lackey [10] and Z. Shen [16] modified the GBC-integrand terms independently by using the unit sphere volume function $V(x) = \text{Vol}(S_x M)$ and obtained some new types of GBC-formulae via the Chern and Cartan connections respectively, for all oriented and closed Finsler manifolds. In [18], Zhao also got a similar formula as Z. Shen’s one in [16] in a different way.

However, a striking difference from the Chern’s formula (0.1) for Riemannian manifolds, the above-mentioned generalizations in the Finsler setting had to make use of an extra vector field $X$ on $M$ in their GBC-integrands. As a result, all these GBC-formulae look not so intrinsic in the spirit of Chern’s original formula (0.1). In [14], Y. Shen asked explicitly whether there is a Gauss-Bonnet-Chern formula for general Finsler manifolds without using any vector fields.

In this paper, by using a Mathai-Quillen type formula of Feng and Zhang on the Euler characteristic in [9], we obtain a Finslerian Gauss-Bonnet-Chern formula. In particular, an explicit GBC-type integrand on $M$, expressed in the integration along fibres of $\pi : SM \to M$, is given.

This paper is organized as follows. In Section 1, we first introduce briefly the Mathai-Quillen’s superconnection formalism (cf. [13] [12]) for reader’s convenience, and then recall the Mathai-Quillen type formula of Feng and Zhang on the Euler characteristic and prove a slight generalization of it. In Section 2, we work out the main results Theorem 1 and Theorem 2 in this paper. As an application, we investigate some special Finsler manifolds–Berwald spaces in Section 3. By the special curvature property of Berwald spaces, our Finslerian GBC-formula reduces to a simple and elegant form, and from which Chern’s GBC-formula is deduced easily. Moreover, inspired by an idea from Weiping Zhang, we also obtain a Bao-Chern type GBC-formula for Berwald spaces by proceeding a geometric localization procedure.

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## 1. Superconnections and the Euler characteristic

We first review some basic definitions and notations on superspaces and superconnections (cf. [13] [12] and also [4], [17], [19] for more details). Then we recall the Mathai-Quillen type formula of Feng and Zhang on the Euler characteristic in [9] and prove a slight generalization of it.
1.1. **Superspaces and superconnections.** A super vector space \( E \) is a vector space with a \( \mathbb{Z}_2 \)-grading \( E = E_+ \oplus E_- \). Let \( \tau_E \in \text{End}(E) \) such that \( \tau_E|_{E_\pm} = \pm 1 \). Then for any \( B \in \text{End}(E) \), the supertrace \( \text{tr}_s[B] \) is defined by
\[
\text{tr}_s[B] = \text{tr}[\tau_E B].
\]
An element \( B \) in \( \text{End}(E) \) is even (resp. odd) if \( B(E_\pm) \subset E_\pm \) (resp. \( B(E_\pm) \subset E_\mp \)) and the degree \( |B| \) of \( B \) is defined to be 0/1 if \( B \) is even/odd. The bracket operation in \( \text{End}(E) \) for a superspace \( E \) always refers to the superbracket
\[
[B_1, B_2] = B_1 B_2 - (-1)^{|B_1||B_2|} B_2 B_1,
\]
for any \( B_1, B_2 \in \text{End}(E) \). One has
\[
\text{tr}_s[B_1, B_2] = 0.
\]
As an example, for any vector space \( V \) of dimension \( m \), the exterior algebra \( \Lambda^*(V^*) \) generated by \( V \) is a superspace with the natural even/odd \( \mathbb{Z}_2 \)-grading:
\[
\Lambda^*(V^*) = \Lambda^{even}(V^*) \oplus \Lambda^{odd}(V^*).
\]
For any \( B \in \text{End}(V) \), the lifting \( B^2 \) of \( B \) is a derivative acting on \( \Lambda^*(V^*) \), that is, \( B^2 \) is linear, and for any \( k \) and \( v^{*1}, \ldots, v^{*k} \in V^* \),
\[
B^2(v^{*1} \wedge \cdots \wedge v^{*k}) := \sum_l v^{*1} \wedge \cdots \wedge (B^* v^{*l}) \wedge \cdots \wedge v^{*k},
\]
where \((B^* v^*)(v) := -v^*(B v)\) for any \( v \in V \) and \( v^* \in V^* \).
Let \( \{v_1, \ldots, v_m\} \) be any basis of \( V \) and let \( \{v^{*1}, \ldots, v^{*m}\} \) be its dual basis for \( V^* \). Set \( Bv_i = B^2 v_i \). Then
\[
B^2 = -\sum_{i,j} B^2 v_i^{*j} \wedge i_{v_j},
\]
where \( i_v \) is the interior multiplication on \( \Lambda^*(V^*) \) induced by \( v \in V \). One verifies easily that
\[
\text{tr}_s[v^{*1} \wedge i_{v_1} \cdots v^{*m} \wedge i_{v_m}] = (-1)^m,
\]
\[
\text{tr}_s[v^{*i_1} \wedge \cdots v^{*i_k} \wedge i_{v_{j_1}} \cdots i_{v_{j_l}}] = 0,
\]
for any \( 1 \leq i_1 < \cdots < i_k \leq m \) and \( 1 \leq j_1 < \cdots < j_l \leq m \) with \( 0 \leq k + l < 2m \).

Given a Euclidean metric \( g^V \) on \( V \). For any \( v \in V \), let \( v^* \) be the metric dual of \( v \), and set
\[
c(v) = v^* \wedge -i_v, \quad \hat{c}(v) = v^* \wedge +i_v.
\]
The for any \( u, v \in V \), one has
\[
c(u)c(v) + c(v)c(u) = -2g^V(u, v), \quad \hat{c}(u)\hat{c}(v) + \hat{c}(v)\hat{c}(u) = 2g^V(u, v), \quad c(u)\hat{c}(v) = -\hat{c}(v)c(u).
\]
Also from (1.6) and (1.7), one gets for any orthonormal basis \( \{v_1, \ldots, v_m\} \) of \( V \),
\[
\text{tr}_s[\hat{c}(v_1)c(v_1) \cdots \hat{c}(v_m)c(v_m)] = 2^m,
\]
\[
\text{tr}_s[\hat{c}(v_{i_1}) \cdots \hat{c}(v_{i_k})c(v_{j_1}) \cdots c(v_{j_l})] = 0,
\]
for any \( 1 \leq i_1 < \cdots < i_k \leq m \) and \( 1 \leq j_1 < \cdots < j_l \leq m \) with \( 0 \leq k + l < 2m \).
A super vector bundle $E = E_+ \oplus E_-$ over a smooth manifold $M$ is a vector bundle with fibres of super vector spaces. Let $\Omega^*(M, E) = \Gamma(\Lambda^*(T^*M) \otimes E)$, which is, in general, an infinite dimensional super vector space with the natural total $\mathbb{Z}_2$-grading. A superconnection $A$ on $E$ is an odd-parity first-order differential operator

$$A : \Omega^\pm(M, E) \to \Omega^\mp(M, E)$$

verifying the following Leibniz rule: for any $\omega \in \Omega^k(M)$, $s \in \Omega^s(M, E)$,

$$(1.10) \quad A(\omega \wedge s) = d\omega \wedge s + (-1)^k \omega \wedge As.$$

The following two simple identities are crucial in the Chern-Weil theory related to the Mathai-Quillen’s superconnection formalism:

$$[A, A^2] = 0, \quad \text{tr}_s[A, B] = d\text{tr}_s[B]$$

for any superconnection $A$ on $E$ and any $B \in \Omega^s(M, \text{End}(E))$.

### 1.2. A Mathai-Quillen type formula on the Euler characteristic

For an oriented and closed manifold $M$ of dimension $2n$, different from the Chern’s construction of the Pfaffian $\text{Pf}(R_{TM})$ from a metric-preserving connection on the tangent bundle $\pi : TM \to M$, Feng and Zhang in [9] expressed the Euler number $\chi(M)$ through the integral over $TM$ of an integrable top-form on $TM$ constructed from any connection $\nabla$ on $TM$ by applying the Mathai-Quillen’s geometric construction of the Thom class (cf. [12]) to the exterior algebra bundle $\pi^*\Lambda^*(T^*M)$.

Let $\nabla_{TM}$ be any connection on $TM$. Then it induces a connection $\nabla_{\Lambda^*(T^*M)}$ on the exterior algebra bundle $\Lambda^*(T^*M)$, which preserves the even/odd $\mathbb{Z}_2$-grading in $\Lambda^*(T^*M)$. Let $\hat{Y}$ denote the tautological section of the pull-back bundle $\pi^*TM$:

$$\hat{Y}(x, Y) := Y \in (\pi^*TM)|_{(x, Y)},$$

where $(x, Y) \in TM$ with $x \in M$ and $Y \in T_x M$. For any given Euclidean metric $g^{TM}$ on $TM$, let $\hat{Y}^*$ denote the dual of $\hat{Y}$ with respect to the pull-back metric $\pi^*g^{TM}$ on $\pi^*TM$. Then the Clifford action $c(\hat{Y}) = \hat{Y}^* \wedge -i_\hat{Y}$ acts on $\pi^*\Lambda^*(T^*M)$ and exchanges the even/odd grading in $\pi^*\Lambda^*(T^*M)$. Moreover,

$$c(\hat{Y})^2 = -|\hat{Y}|^2_{\pi^*g^{TM}}.$$

For any $T > 0$, Feng and Zhang in [9] used the superconnection

$$(1.14) \quad A_T = \pi^*\nabla^{\Lambda^*(T^*M)} + Tc(\hat{Y})$$

on the bundle $\pi^*\Lambda^*(T^*M)$ and proved the following Mathai-Quillen type formula on the Euler number $\chi(M)$:

$$\chi(M) = \left( \frac{1}{2\pi} \right)^{2n} \int_{TM} \text{tr}_s[\exp(A_T^2)].$$

A key point in the formula (1.15) is that the connection $\nabla_{TM}$ on $TM$ needn’t preserve the metric $g^{TM}$ used to define the Clifford action $c(\hat{Y})$.

For the purpose of this paper, we need to generalize the formula (1.15) slightly. Actually, one can choose any connection $\nabla$ and any Euclidean metric $g$ on the pull-back bundle $\pi^*TM$ to define a superconnection on $\pi^*\Lambda^*(T^*M) \equiv \Lambda^*(\pi^*TM)$; let $\nabla^{\Lambda^*(\pi^*T^*M)}$ denote the lifting
of the connection $\nabla$ on $\Lambda^*(\pi^*TM)$; let $\hat{Y}^*_g$ denote the dual of $\hat{Y}$ with respect to the metric $g$ and set $c_g(\hat{Y}) = \hat{Y}^*_g \wedge -i\hat{Y}$; then for any $T > 0$,

\[(1.16) \quad \tilde{A}_T = \nabla^{\Lambda^*(\pi^*TM)} + Tc_g(\hat{Y})\]

is also a superconnection on $\pi^*\Lambda^*(TM)$. Moreover, by using (1.15) and a transgression argument, one can prove the following slight generalization of (1.15) easily.

**Lemma 1.** Let $M$ be a closed and oriented manifold of dimension $2n$. Then for any connection $\nabla$ and any Euclidean metric $g$ on $\pi^*TM$, if the curvature $R = \nabla^2$ is bounded along fibres of $TM$, then the following formula holds for any $T > 0$:

\[(1.17) \quad \chi(M) = \left(\frac{1}{2\pi}\right)^{2n} \int_{TM} \text{tr}_s[\exp(\tilde{A}_T^2)].\]

**Proof.** Here we would like to give a direct proof of (1.17).

We first check the formula (1.17) for $g = \pi^*g^TM$ and $\nabla = \pi^*\nabla^TM$, where $\nabla^TM$ is the Levi-Civita connection on $TM$ associated to the Riemannian metric $g^TM$ on $M$. Let $\nabla^{\Lambda^*(TM)}$ denote the lifting of $\nabla^TM$ on the exterior algebra bundle $\Lambda^*(T^*M)$ and let $R^{\Lambda^*(TM)}$ be its curvature. So the superconnection defined by (1.16) becomes

\[(1.18) \quad A_T = \pi^*\nabla^{\Lambda^*(T^*M)} + Tc(\hat{Y}),\]

and

\[(1.19) \quad A_T^2 = \left(\pi^*\nabla^{\Lambda^*(T^*M)} + Tc(\hat{Y})\right)^2 = \pi^*R^{\Lambda^*(T^*M)} + Tc\left(\pi^*\nabla^TM\hat{Y}\right) - T^2|Y|^2.\]

Let $\{e_1, \ldots, e_{2n}\}$ be a local oriented orthonormal frame for $TM$ and let $\{e_1^*, \ldots, e_{2n}^*\}$ denote the dual frame for $T^*M$. Then by (0.3) and (1.7), one has

\[(1.20) \quad \pi^*R^{\Lambda^*(T^*M)} = -\sum_{a,b}(\pi^*\Omega_a^b e^{*,b} \wedge i e_a) = -\frac{1}{4} \sum_{a,b} (\pi^*\Omega_a^b)(\hat{c}(e_b) + c(e_b))(\hat{c}(e_a) - c(e_a)).\]

We now compute $\text{tr}_s[\exp(A_T^2)]$ fiberwisely. For the simplicity of computations, we choose a local oriented orthonormal frame field $\{e_1, \ldots, e_{2n}\}$ around each $x \in M$ such that $(\nabla^TM e_a)|_x(x) = 0$, $a = 1, \cdots, 2n$. Moreover, we write $Y = \sum g^a e_a$ around $x$. Then from (1.19), (1.20), (1.8), (1.9), (1.2) and the degree counting of differential forms, we have

\[
\int_{T_x M} \text{tr}_s[\exp(A_T^2)] = \int_{T_x M} e^{-T^2|Y|^2} \text{tr}_s[\exp(\pi^*R^{\Lambda^*(T^*M)} + Tc(\pi^*\nabla^TM\hat{Y}))]
\]

\[
= \int_{T_x M} e^{-T^2|Y|^2} \text{tr}_s \left[ \exp \left( -\frac{1}{4} \pi^*\Omega_b^a(x)(\hat{c}(e_b) + c(e_b))(\hat{c}(e_a) - c(e_a)) + Tdy^a c(e_a) \right) \right]
\]

\[
= \int_{T_x M} e^{-T^2|Y|^2} \text{tr}_s \left[ \exp \left( -\frac{1}{4} \pi^*\Omega_b^a(x)\hat{c}(e_b)\hat{c}(e_a) \right) \exp( Tdy^a c(e_a)) \right]
\]

\[
= \int_{T_x M} \frac{(-1)^n}{2^n} e^{-T^2|Y|^2} \text{tr}_s \left[ \sum_{a,b} \pi^*\Omega_b^a(x)\hat{c}(e_b)\hat{c}(e_a) \right] \cdot \left( \sum_{a,b} \pi^*\Omega_b^a(x)\hat{c}(e_b)\hat{c}(e_a) \right)
\]

\[
= \int_{T_x M} \frac{(-1)^n T^{2n}}{2^n} e^{-T^2|Y|^2} \text{tr}_s \left[ \pi^*\text{Pf}(R^TM)(x)\hat{c}(e_1) \cdots \hat{c}(e_{2n}) \right] \cdot \left( \sum_{a,b} \pi^*\Omega_b^a(x)\hat{c}(e_b)\hat{c}(e_a) \right)
\]
Therefore, by Chern’s formula (0.1), the formula (1.17) holds in the current case.

Now we prove the formula (1.17) for any metric \( g \) and any connection \( \nabla \) on the pull-back bundle \( \pi^*TM \) with the bounded curvature \( R = \nabla^2 \) along fibres of \( TM \). For any \( T > 0 \) and \( t \in [0,1] \), set

\[
(1.21) \quad \omega_T = \tilde{A}_T - A_T = \nabla^{\Lambda^*(\pi^*TM)} - \pi^*\nabla^{\Lambda^*(T^*M)} + T(\hat{Y}_g - \hat{Y}^*) \wedge,
\]

\[
(1.22) \quad A_{T,t} = t\tilde{A}_T + (1-t)A_T = \pi^*\nabla^{\Lambda^*(T^*M)} + t\omega_T + Tc(\hat{Y}),
\]

where the superconnections \( \tilde{A}_T \) and \( A_T \) are defined by (1.16) and (1.18), respectively. Moreover,

\[
(1.23) \quad A_{T,t}^2 = \left(\pi^*\nabla^{\Lambda^*(T^*M)}\right)^2 + \left(\pi^*\nabla^{\Lambda^*(T^*M)}, t\omega_T + Tc(\hat{Y})\right) + t^2\omega_T^2 - T^2|\hat{Y}_g|^2.
\]

We have by using (1.11)

\[
\frac{d}{dt}\text{tr}_s\left[\exp(A_{T,t}^2)\right] = \text{tr}_s\left[\left(\frac{d}{dt}A_{T,t}\right)\exp(A_{T,t}^2)\right]
\]

\[
= \text{tr}_s\left[A_{T,t}, \frac{d}{dt}A_{T,t}\right] \exp(A_{T,t}^2)\] = \text{tr}_s\left[[A_{T,t}, \omega_T \exp(A_{T,t}^2)]\right]
\]

\[
= d^{TM}\text{tr}_s\left[\omega_T \exp(A_{T,t}^2)\right].
\]

Therefore,

\[
\text{tr}_s\left[\exp(\tilde{A}_T^2)\right] - \text{tr}_s\left[\exp(A_T^2)\right] = \int_0^1 \frac{d}{dt}\text{tr}_s\left[\exp(A_{T,t}^2)\right] dt
\]

\[
= \int_0^1 d^{TM}\text{tr}_s\left[\omega_T \exp(A_{T,t}^2)\right] dt = d^{TM}\int_0^1 \text{tr}_s\left[\omega_T \exp(A_{T,t}^2)\right] dt.
\]

From (1.21) and (1.22) and the curvature \( R = \nabla^2 \) is bounded along fibres of \( TM \), one verifies easily from (1.23) that \( t\text{tr}_s\left[\omega_T \exp(A_{T,t}^2)\right] \) is exponentially decay along fibres of \( \pi : TM \to M \), and so \( \int_{TM/M} \int_0^1 \text{tr}_s\left[\omega_T \exp(A_{T,t}^2)\right] dt \) is a well-defined differential form on \( M \). Therefore, we have

\[
\int_{TM/M} \int_0^1 \text{tr}_s\left[\omega_T \exp(A_{T,t}^2)\right] dt = \int_M d^{TM} \int_0^1 \text{tr}_s\left[\omega_T \exp(A_{T,t}^2)\right] dt
\]

\[
= 0,
\]

and from which the lemma follows. \( \square \)
2. A Finslerian Gauss-Bonnet-Chern formula

In this section, after a brief review of some basic definitions and notations of Finsler geometry used in this paper (cf. [2], [5] for more details), we will give a Finslerian Gauss-Bonnet-Chern formula for a closed and oriented Finsler manifold \((M, F)\) of dimension \(2n\) by Lemma 1. In particular, we get a well-defined GBC-type integrand on \(M\) through the integration along the fibers.

2.1. A brief review of Finsler geometry. Let \(M\) be a smooth manifold of dimension \(m\). For any local coordinate chart \((U; (x^1, \ldots, x^m))\) on \(M\), let \((\pi^{-1}(U); (x^1, \ldots, x^m, y^1, \ldots, y^m))\) be the induced local coordinate chart on the total space \(TM\) of the tangent bundle \(\pi: TM \to M\). Let \(O\) denote the zero section of \(TM\) and set \(TM_o = TM \setminus O\).

To distinguish elements in \(\pi^*\Lambda^*(T^*M)\) and \(\pi^*TM\) from \(\Lambda^*(T^*M)\) and \(TM\), we will decorate the elements in \(\pi^*\Lambda^*(T^*M)\) and \(\pi^*TM\) with a \(\hat{\cdot}\) notation for clarity. We will also use the summation convention of Einstein in computations to simplify the notations.

A Finsler metric \(F\) on \(M\) is a non-negative smooth function on \(TM_o\) satisfying that the positively homogeneity

\[
F(x, \lambda Y) = \lambda F(x, Y)
\]

for any \(\lambda > 0\) and the induced fundamental tensor

\[
g_F = \frac{1}{2} [F^2] \frac{\partial}{\partial x^i} \rightleftharpoons \frac{\partial}{\partial x^j}
\]

defines a Euclidean structure on the pull-back bundle \(\pi^*TM \to TM_o\). Set

\[
G^i = \frac{1}{4} g^{il} \left\{ [F^2]_{x^k y^j y^k} - [F^2]_{x^j} \right\}, \quad N_j^i = \frac{\partial G^i}{\partial y^j},
\]

and define

\[
\delta = \frac{\partial}{\partial x^i} - N_j^i \frac{\partial}{\partial y^j}, \quad \delta y^i = dy^i + N_j^i dx^j.
\]

Let \(\nabla^{Ch}\) denote the Chern connection on the pull-back bundle \(\pi^*TM \to TM_o\). With respect to the pull-back frame \(\left\{ \frac{\partial}{\partial x^i} \right\}\) for \(\pi^*TM\), set

\[
\nabla^{Ch} \frac{\partial}{\partial x^j} = \bar{\omega}_j^i \rightleftharpoons \frac{\partial}{\partial x^i}.
\]

It is well-known that \(\bar{\omega} := (\bar{\omega}_j^i)\) is determined uniquely by the following structure equations:

\[
0 = dx^j \wedge \bar{\omega}_j^i;
\]

\[
dg_{ij} = g_{ik} \bar{\omega}_j^k + g_{jk} \bar{\omega}_i^k + 2A_{ijk} \frac{\delta y^k}{F},
\]

where \(A_{ijk} = \frac{F}{4}[F^2]_{y^i y^j y^k}\) is the Cartan tensor. The first and the second equation of (2.2) are often refereed as the torsion-free and the almost metric-preserving conditions of the Chern connection, respectively. A direct consequence of the torsion free condition is that \(\bar{\omega}_j^i\) are horizontal one forms on \(TM_o\), which can be written as:

\[
\bar{\omega}_j^i = \Gamma^i_{jk} dx^k, \quad \text{and} \quad \Gamma^i_{jk} = \Gamma^i_{kj}.
\]

Furthermore, one has \(y^k \Gamma^i_{jk} = N_j^i\) and then

\[
\delta y^i = dy^i + y^k \bar{\omega}_k^i.
\]

One the other hand, the almost metric-preserving condition implies that

\[
\nabla^{Ch, \ast}(\tilde{Y}_{g_F}) = (\nabla^{Ch} \tilde{Y})^\ast_{g_F},
\]
where $\nabla^{Ch,*}$ is the dual connection on $\pi^*TM$ of the Chern connection.

Let $R^{Ch} = (\nabla^{Ch})^2$ be the curvature of the Chern connection $\nabla^{Ch}$, which is an $\text{End}(\pi^*TM)$-valued two-form on $TM_o$. By the torsion-freeness of the Chern connection, $R^{Ch}$ is divided into two parts

$$R^{Ch} = R + P,$$

where $R$ is called the Chern-Riemann curvature of $\nabla^{Ch}$, which is an $\text{End}(\pi^*TM)$-valued horizontal two-form on $TM_o$, and $P$ is called the Chern-Minkowski curvature of $\nabla^{Ch}$, which is an $\text{End}(\pi^*TM)$-valued horizontal-vertical two-form on $TM_o$. Set

$$(2.6) \quad R \frac{\partial}{\partial \hat{x}^j} = R^i_j \otimes \frac{\partial}{\partial \hat{x}^i}, \quad P \frac{\partial}{\partial \hat{x}^j} = P^i_j \otimes \frac{\partial}{\partial \hat{x}^i}.$$

By (2.3), a direct computation shows that (cf. (3.2.2), (3.3.3) in [2])

$$(2.7) \quad P^i_j = P^i_{jk} dx^k \wedge \frac{\delta y^j}{F} = -dx^k \wedge \left( \frac{\partial R^i_{jk}}{\partial y^j} \delta y^l \right).$$

Sometimes, it is more convenient to use the special $g_F$-orthonormal local frame $\{e_1, \cdots, e_{2n}\}$ of $\pi^*TM$, which is orthonormal with respect to $g_F$ and satisfies $e_{2n} = \hat{Y}/F$.

Set

$$(2.8) \quad \nabla^{Ch} e_a = \omega^b_a \otimes e_b, \quad R^{Ch} e_a = \Omega^b_a \otimes e_b = (R^b_a + P^b_a) \otimes e_b.$$

Let $\tilde{R}^{Ch}$ be the skew-symmetrization of $R^{Ch}$ with respect to $g_F$, then

$$(2.9) \quad \tilde{R}^{Ch} e_a = \sum_b \frac{1}{2} (\Omega^b_a - \Omega^b_a) \otimes e_b =: \tilde{\Omega}^b_a \otimes e_b.$$

The Pfaffian $\text{Pf}(\tilde{R}^{Ch})$ of $\tilde{R}^{Ch}$ is given by

$$(2.10) \quad \text{Pf}(\tilde{R}^{Ch}) = \frac{1}{2^n n!} \sum_{a_1, \ldots, a_{2n}=1}^{2n} \epsilon_{a_1, \ldots, a_{2n}} \tilde{\Omega}^{a_2}_{a_1} \wedge \cdots \wedge \tilde{\Omega}^{a_{2n}}_{a_{2n-1}},$$

$$= \frac{1}{2^n n!} \sum_{a_1, \ldots, a_{2n}=1}^{2n} \epsilon_{a_1, \ldots, a_{2n}} \Omega^{a_2}_{a_1} \wedge \cdots \wedge \Omega^{a_{2n}}_{a_{2n-1}}.$$

Noticed that the Finsler metric $F$ on $TM_o$ is homogeneous of degree one, all the geometric data, such as $\nabla^{Ch}$, $R^{Ch}$, $R$ and $P$, can be reduced naturally onto $SM$. In this paper, we will use the same notations to denote their reductions on $SM$.

### 2.2. A Finslerian Gauss-Bonnet-Chern formula

Set for any $r > 0$,

$$D_r M = \{ Y \in TM | F(Y) \leq r \}.$$

Let $\rho$ be a non-negative smooth function on $TM$ with $0 \leq \rho \leq 1$ and $\rho(Y) \equiv 1$ for $F(Y) \leq 1/4$ and $\rho(Y) \equiv 0$ for $F(Y) \geq 1/2$.

For any connection $\nabla$ on $\pi : TM \to M$, we get an extension of the Chern connection:

$$(2.11) \quad \nabla_{\rho} = (1 - \rho) \nabla^{Ch} + \rho \pi^* \nabla$$

on $\pi^*TM$ over the total space $TM$. Clearly, the curvature $(\nabla_{\rho})^2$ is bounded along the fibres of $TM$. Let $\nabla^{\Lambda^*(\pi^*TM)}_{\rho}$ denote the induced connection on $\Lambda^*(\pi^*TM)$ of $\nabla_{\rho}$. 

Superconnections and A Finslerian Gauss-Bonnet-Chern Formula 9
Let \( \tilde{g}_F \) be any Euclidean metric on \( \pi^*TM \to TM \) with \( \tilde{g}_F = g_F \) over \( TM \setminus D_{1/2}(M) \). Now we prove the following Finslerian Gauss-Bonnet-Chern formula by Lemma 1 for the metric \( \tilde{g}_F \) and the connection \( \nabla_\rho \) over \( TM \).

**Theorem 1.** Let \((M,F)\) be a closed and oriented Finsler manifold of dimension \(2n\). Then for any connection \( \nabla \) on \( \pi : TM \to M \), one has

\[
\chi(M) = \frac{1}{(2\pi)^{2n}(2n)!} \left\{ \sum_{k=1}^n C_{2n}^{2k} \int_{SM} tr_s \left[ c(e)c(\nabla^Ch)e^{2k-1}(R^k)k(P^k)2n-2k \right] 
+ \int_{SM} tr_s \left[ \theta^2 \left( P^2 \right)^{2n-1} \right] \right\},
\]

(2.12)

where \( e = \tilde{Y}|_{SM} \) and \( R^k, P^k, \theta^k \) are the natural lifting of \( R, P, \theta \) respectively on \( \Lambda^*(\pi^*TM) \), and \( \theta \) is defined by \( \theta := \nabla^Ch - \pi^\ast\nabla \). In particular, the formula (2.12) is independent of the choice of the connection \( \nabla \) on \( TM \).

**Proof.** For any \( T > 0 \), similar to (1.16), we define the following superconnection

\[
\tilde{A}_{\rho, T} = \tilde{\nabla}^{\Lambda^*(\pi^*TM)}_\rho + Tc\tilde{g}_F(\tilde{Y}).
\]

(2.13)

Noticed that the curvature \((\tilde{\nabla}_\rho)^2\) is bounded along the fibres of \( TM \), we get by (1.17)

\[
\chi(M) = \left( \frac{1}{2\pi} \right)^{2n} \int_{TM} tr_s[\exp(\tilde{A}_{\rho, T}^2)].
\]

(2.14)

Since \( \exp(\tilde{A}_{\rho, T}^2) \) is exponentially decay along fibres of \( TM \) and \( \chi(M) \) does not depend on \( T > 0 \), we get

\[
\chi(M) = \lim_{T \to +\infty} \left( \frac{1}{2\pi} \right)^{2n} \int_{TM} tr_s[\exp(\tilde{A}_{\rho, T}^2)] = \lim_{T \to +\infty} \left( \frac{1}{2\pi} \right)^{2n} \int_{D_{1}M} tr_s[\exp(\tilde{A}_{\rho, T}^2)].
\]

(2.15)

Note that for any connection \( \nabla \) on \( TM \), the curvature \( (\pi^*\nabla^{\Lambda^*(TM)})^2 \) of the connection \( \pi^*\nabla^{\Lambda^*(TM)} \) involves no vertical differential forms on \( TM \), where \( \nabla^{\Lambda^*(TM)} \) is the lifting of \( \nabla \) on \( \Lambda^*(TM) \), we have

\[
\int_{D_{1}M} tr_s\left[ \exp((\pi^*\nabla^{\Lambda^*(TM)})^2) \right] = 0.
\]

Therefore, we have

\[
\int_{D_{1}M} tr_s\left[ \exp(\tilde{A}_{\rho, T}^2) \right] = \int_{D_{1}M} \left( tr_s\left[ \exp(\tilde{A}_{\rho, T}^2) \right] - tr_s\left[ \exp((\tilde{\nabla}_\rho^{\Lambda^*(\pi^*TM)})^2) \right] \right)
+ \int_{D_{1}M} \left( tr_s\left[ \exp((\tilde{\nabla}_\rho^{\Lambda^*(\pi^*TM)})^2) \right] - tr_s\left[ \exp((\pi^*\nabla^{\Lambda^*(TM)})^2) \right] \right).
\]

(2.16)
For the first term on the right hand side of (2.16), we have

\[
\int_{D_1 M} \left( \text{tr}_s \left[ \exp(A_{\rho,T}^2) - \exp((\tilde{\nabla}_\rho^{A^*})^{(\pi^*T^* M)})^2 \right] \right) \\
= \int_{D_1 M} \int_0^1 \frac{\partial}{\partial t} \text{tr}_s \left[ \exp((\tilde{\nabla}_\rho^{A^*})^{(\pi^*T^* M)} + tTc_{\tilde{g}_P}(\hat{Y}))^2 \right] dt \\
= \int_{D_1 M} \int_0^1 d^TM \text{tr}_s \left[ Tc_{\tilde{g}_P}(\hat{Y}) \exp((\tilde{\nabla}_\rho^{A^*})^{(\pi^*T^* M)} + tTc_{\tilde{g}_P}(\hat{Y}))^2 \right] dt \\
= \int_{D_1 M} \int_0^1 i^* \text{tr}_s \left[ Tc_{\tilde{g}_P}(\hat{Y}) \exp((\tilde{\nabla}_\rho^{A^*})^{(\pi^*T^* M)} + tTc_{\tilde{g}_P}(\hat{Y}))^2 \right] dt \\
= \int_{SM} \int_0^1 \text{tr}_s \left[ Tc(e) \exp((\nabla^{Ch^*} + tTc(e))^2) \right] dt \\
= \int_{SM} \int_0^1 e^{-t^2T^2} \text{tr}_s \left[ Tc(e) \exp(R^{Ch^*} + tT[\nabla^{Ch^*}, c(e)]) \right] dt \\
= \int_{SM} \int_0^1 e^{-t^2T^2} \text{tr}_s \left[ Tc(e) \exp(R^{Ch^*} + tTc(\nabla^{Ch^*} e)) \right] dt,
\]

where \( i : SM \hookrightarrow TM \) denotes the natural embedding of the unit sphere bundle \( SM \) into \( TM \), and \( \nabla^{Ch^*} \) is the lifting of \( \nabla^{Ch} \) on \( A^*(\pi^*T^* M) \) and \( R^{Ch^*} = (\nabla^{Ch^*})^2 \), and the last equality in (2.17) comes from (2.5).

Now noticed that the term \( R^{Ch^*} \) is a two-form with two Clifford elements and the term \( c(\nabla^{Ch} e) \) is a one-form with one Clifford element, hence by the property (1.6) or (1.9) of the supertrace and a degree counting, we get from (2.17) that

\[
\lim_{T \to +\infty} \int_{D_1 M} \left( \text{tr}_s \left[ \exp(A_{\rho,T}^2) - \exp((\tilde{\nabla}_\rho^{A^*})^{(\pi^*T^* M)})^2 \right] \right) \\
= \lim_{T \to +\infty} \int_{SM} \int_0^1 e^{-t^2T^2} \text{tr}_s \left[ Tc(e) \exp(R^{Ch^*} + tTc(\nabla^{Ch} e)) \right] dt, \\
= \lim_{T \to +\infty} \int_{SM} \int_0^1 e^{-t^2T^2} \text{tr}_s \left[ Tc(e) \exp(tTc(\nabla^{Ch} e)) \exp(R^{Ch^*}) \right] dt, \\
= \sum_{k=1}^n \int_{SM} \frac{1}{(2k-1)!} \text{tr}_s \left[ c(e)c(\nabla^{Ch} e)^{2k-1} \exp(R^{Ch^*}) \right] \cdot \lim_{T \to +\infty} \int_0^1 e^{-t^2T^2} T^{2k} t^{2k-1} dt, \\
= \sum_{k=1}^n \frac{(k-1)!}{2(2k-1)!k!(2n-2k)!} \int_{SM} \text{tr}_s \left[ c(e)c(\nabla^{Ch} e)^{2k-1}(\hat{R}^\rho)^k(P^\rho)^{2n-2k} \right] \\
= \sum_{k=1}^n \frac{c_{2k}}{(2n)!} \int_{SM} \text{tr}_s \left[ c(e)c(\nabla^{Ch} e)^{2k-1}(\hat{R}^\rho)^k(P^\rho)^{2n-2k} \right].
\]

For the second term on the right hand side of (2.16), by setting

\[
\theta_\rho = \tilde{\nabla}_\rho - \pi^*\nabla,
\]
we have

\[
\int_{D_1 M} \left( \text{tr}_s \left[ \exp((\nabla^*(\pi^* T^* M))^2) \right] - \text{tr}_s \left[ \exp((\pi^* \nabla^*(T^* M))^2) \right] \right) dt
\]

= \int_{D_1 M} \int_0^1 \frac{\partial}{\partial t} \text{tr}_s \left[ \exp \left( (\nabla^*(\pi^* T^* M) - (1 - t)\theta^2) \right) \right] dt

= \int_{D_1 M} d^{TM} \int_0^1 \text{tr}_s \left[ \theta^2 \exp \left( (\nabla^*(\pi^* T^* M) - (1 - t)\theta^2) \right) \right] dt

= \int_{SM} \int_0^1 \text{tr}_s \left[ \theta^2 \exp \left( (\nabla^*(\pi^* T^* M) - (1 - t)\theta^2) \right) \right] dt

= \int_{SM} \int_0^1 \text{tr}_s \left[ \theta^2 \exp \left( R^{Ch,\natural} - (1 - t)[\nabla^{Ch,\natural}, \theta^2] + (1 - t)^2 \theta^2 \wedge \theta^2 \right) \right] dt.

Note that

\[
[\nabla^{Ch,\natural}, \theta^2] = \nabla^{Ch,\natural} - \pi^* \nabla^*(T^* M), \theta^2] + [\pi^* \nabla^*(T^* M), \nabla^{Ch,\natural} - \pi^* \nabla^*(T^* M)]
\]

= [\theta^2, \theta^2] - [\pi^* \nabla^*(T^* M), \pi^* \nabla^*(T^* M)] + [\pi^* \nabla^*(T^* M), \nabla^{Ch,\natural}]

= 2\theta^2 \wedge \theta^2 - 2 \left( \pi^* \nabla^*(T^* M) \right)^2 + [\pi^* \nabla^*(T^* M) - \nabla^{Ch,\natural}, \nabla^{Ch,\natural}] + [\nabla^{Ch,\natural}, \nabla^{Ch,\natural}]

= 2\theta^2 \wedge \theta^2 - 2 \left( \pi^* \nabla^*(T^* M) \right)^2 - [\nabla^{Ch,\natural}, \theta^2] + 2R^{Ch,\natural},

and so

\[
[\nabla^{Ch,\natural}, \theta^2] = \theta^2 \wedge \theta^2 - \left( \pi^* \nabla^*(T^* M) \right)^2 + R^{Ch,\natural}.
\]

Combining (2.19) and (2.20), we get

\[
\int_{D_1 M} \left( \text{tr}_s \left[ \exp((\nabla^*(\pi^* T^* M))^2) \right] - \text{tr}_s \left[ \exp((\pi^* \nabla^*(T^* M))^2) \right] \right) dt
\]

= \int_{SM} \int_0^1 \text{tr}_s \left[ \theta^2 \exp \left( R^{Ch,\natural} + (1 - t) \left( \pi^* \nabla^*(T^* M) \right)^2 - R^{Ch,\natural} - \theta^2 \wedge \theta^2 \right) \right]

+ (1 - t)^2 \theta^2 \wedge \theta^2) \right] dt

= \int_{SM} \int_0^1 \text{tr}_s \left[ \theta^2 \exp \left( tR^{Ch,\natural} + (1 - t)\left( \pi^* \nabla^*(T^* M) \right)^2 - t(1 - t)\theta^2 \wedge \theta^2 \right) \right] dt

= \int_{SM} \int_0^1 \text{tr}_s \left[ \theta^2 \exp \left( tR^{Ch,\natural} + (1 - t)\left( \pi^* \nabla^*(T^* M) \right)^2 - t(1 - t)\theta^2 \wedge \theta^2 \right) \right] dt.

By (2.3), the term \( \theta^2 \) is an \( \text{End}(\Lambda^*(\pi^* T^* M)) \)-valued horizontal one form, and so

\[
tR^{Ch,\natural} + (1 - t)\left( \pi^* \nabla^*(T^* M) \right)^2 - t(1 - t)\theta^2 \wedge \theta^2
\]
is an End(Λ*(π*TM))-valued horizontal two form. Hence from (2.21), we get

\[ (2.22) \quad \int_{D_iM} \left( \tr_s \left[ \exp\left( \nabla^\Lambda_\rho (\pi^*T^*M) \right) \right] - \tr_s \left[ \exp\left( \pi^*\nabla^\Lambda (T^*M) \right) \right] \right) \]

\[ \begin{align*}
= & \int_{SM} \int_0^1 \tr_s \left[ \theta^2 \exp \left( tP_x^2 \right) \right] \, dt \\
= & \int_{SM} \frac{1}{(2n-1)!} \int_0^1 t^{2n-1} \, dt \, \tr_s \left[ \theta^2 (P_x^2)^{2n-1} \right] \\
= & \frac{1}{(2n)!} \int_{SM} \tr_s \left[ \theta^2 (P_x^2)^{2n-1} \right].
\end{align*} \]

By (2.15), (2.18) and (2.22), we complete the proof of Theorem 1.

Remark 1. The Chern connection is essential to get the formula (2.12), in which the first term follows from the almost metric-preserving property, while the second term from the torsion-freeness. Also note that formula (2.12) is independent of the choice of the connection \( \nabla \) on \( TM \), we can replace \( \nabla \) by the Levi-Civita connection of a fixed average Riemannian metric of a Finsler manifold to get a more intrinsic integrand.

In the following, by using the induced homogeneous coordinate charts \( (x^i, y^i) \) on \( SM \), we will workout a local version of the formula (2.12). More precisely, we will give an explicit GBC-integrand on \( M \) through the integration along the fibers, in which no information of the pull-back connection \( \nabla \) are involved.

Now we first compute the term \( \tr_s \left[ c(e)c(\nabla^\Lambda e)(\pi^*TM) \right] \), for \( k = 1, \ldots, n \).

From (2.1), (2.6) and (1.5), with respect to the pull-back frame \( \left\{ \frac{\partial}{\partial x^i} \right\} \) of \( \pi^*TM \), we have

\[ (2.23) \quad \omega^i = -\omega^j d\hat{x}^j \wedge i \frac{\partial}{\partial x^i}, \quad R^i = -R^j_i d\hat{x}^j \wedge i \frac{\partial}{\partial x^i}, \quad P^i = -P^j_i d\hat{x}^j \wedge i \frac{\partial}{\partial x^i}. \]

We have also

\[ (2.24) \quad c(e) = \omega \wedge -i_e = F_{yi} d\hat{x}^i \wedge -\frac{y^i}{F} i \frac{\partial}{\partial x^i}, \]

where \( \omega = e^* = F_{yi} d\hat{x}^i \) is the Hilbert form on \( SM \). Denote that

\[ (2.25) \quad (\nabla^\Lambda e)^i := \frac{\delta y^i}{F} - \frac{y^i}{F} d\log F, \quad (\nabla^{\Lambda, *} \omega)_i := g_{ik} \left( \frac{\delta y^k}{F} - \frac{y^k}{F} d\log F \right), \]

\[ (2.26) \quad \gamma^i_j := (\nabla^{\Lambda, *} \omega)_i \left( \nabla^\Lambda e \right)^j, \quad \Xi^i_j := \left( F_{yi} (\nabla^\Lambda e)^j - \frac{y^j}{F} (\nabla^{\Lambda, *} \omega)_i \right). \]

By (2.5), we have

\[ (2.27) \quad c \left( \nabla^\Lambda e \right) = (\nabla^\Lambda e)^* \wedge -i_{\nabla^\Lambda e} = (\nabla^{\Lambda, *} \omega) \wedge -i_{\nabla^{\Lambda, e}} = (\nabla^{\Lambda, *} \omega)_j d\hat{x}^j \wedge -\left( \nabla^\Lambda e \right)^i \frac{\partial}{\partial x^i}. \]
From (2.23)–(2.27), we get

\[(2.28)\]

\[
\text{tr}_s \left[ c(e)c(\nabla^\text{Ch} e)^{2k-1} (R^e)^k (P^e)^{2n-2k} \right] = \text{tr}_s \left[ (R^e)^k (P^e)^{2n-2k} c(\nabla^\text{Ch} e)^{2k-1} c(e) \right]
\]

\[
= \text{tr}_s \left[ \left( -R^e_i \partial i \hat{x}^i \wedge i \frac{\partial}{\partial x^i} \right)^k \left( -P^e_i \partial i \hat{x}^i \wedge i \frac{\partial}{\partial x^i} \right)^{2n-2k} \left( (\nabla^\text{Ch} \omega)_i \partial i \hat{x}^j \wedge - (\nabla^\text{Ch} e)^j_i \hat{x}^i \frac{\partial}{\partial x^i} \right)^{2k-2} \right]
\]

\[
\left( (\nabla^\text{Ch} \omega)_p \hat{x}^p \wedge - (\nabla^\text{Ch} e)^q_i \frac{\partial}{\partial x^i} \right) \left( F_{ij} \partial i \hat{x}^j \wedge - \frac{\hat{g}^r_i}{F} \frac{\partial}{\partial x^i} \right)
\]

\[
= (-1)^k \text{tr}_s \left[ \left( R^e_{i_1} \cdots R^e_{i_k} \partial i_1 \hat{x}^i_1 \wedge i_1 \frac{\partial}{\partial x^{i_1}} \cdots \partial i_k \hat{x}^i_k \wedge i_k \frac{\partial}{\partial x^{i_k}} \right) \left( P^{e_{s_1}}_{s_1} \cdots P^{e_{s_2n-2k}}_{s_{2n-2k}} \partial s_1 \hat{x}^s_1 \wedge i_1 \frac{\partial}{\partial x^{s_1}} \cdots \partial s_{2n-2k} \hat{x}^s_{2n-2k} \wedge i_k \frac{\partial}{\partial x^{s_k}} \right) \right]
\]

\[
\cdot \left( \nabla^\text{Ch} \omega \right)_{p_1} \left( \nabla^\text{Ch} e \right)_{q_1} \cdots \left( \nabla^\text{Ch} \omega \right)_{p_{k-1}} \left( \nabla^\text{Ch} e \right)_{q_{k-1}}
\]

\[
= (-1)^k C_{2k-2}^{k-1} \text{tr}_s \left[ \left( R^e_{i_1} \cdots R^e_{i_k} \partial i_1 \hat{x}^i_1 \wedge i_1 \frac{\partial}{\partial x^{i_1}} \cdots \partial i_k \hat{x}^i_k \wedge i_k \frac{\partial}{\partial x^{i_k}} \right) \left( P^{e_{s_1}}_{s_1} \cdots P^{e_{s_2n-2k}}_{s_{2n-2k}} \partial s_1 \hat{x}^s_1 \wedge i_1 \frac{\partial}{\partial x^{s_1}} \cdots \partial s_{2n-2k} \hat{x}^s_{2n-2k} \wedge i_k \frac{\partial}{\partial x^{s_k}} \right) \right]
\]

\[
\cdot \left( \nabla^\text{Ch} \omega \right)_{p_1} \left( \nabla^\text{Ch} e \right)_{q_1} \cdots \left( \nabla^\text{Ch} \omega \right)_{p_{k-1}} \left( \nabla^\text{Ch} e \right)_{q_{k-1}}
\]

Now we compute the second term \(\text{tr}_s \left[ \theta^2 (P^2)^{2n-1} \right] \) in (2.24). Recall that \(\theta = \nabla^\text{Ch} - \pi^* \nabla \) in Theorem 1. Let \(\theta = (\vartheta^i_j)\) be the connection matrix of \(\nabla\) with respect to the frame \(\left\{ \frac{\partial}{\partial x^i} \right\}\). Then we get

\[
\theta^2 = - \left( \vartheta^i_j - \pi^* \vartheta^i_j \right) \partial i \hat{x}^j \wedge i \frac{\partial}{\partial x^j}.
\]

Using (1.6), we get

\[(2.29)\]

\[
\text{tr}_s \left[ \theta^2 (P^2)^{2n-1} \right] = \text{tr}_s \left[ - \left( \vartheta^i_j - \pi^* \vartheta^i_j \right) \partial i \hat{x}^j \wedge i \frac{\partial}{\partial x^j} \left( -P^e_k \partial k \hat{x}^l \wedge i \frac{\partial}{\partial x^l} \right)^{2n-1} \right]
\]

\[
= P^{j_1}_{i_1} \cdots P^{j_{2n-1}}_{i_{2n-1}} \left( \vartheta^{j_2}_{i_2} - \pi^* \vartheta^{j_2}_{i_2} \right) \text{tr}_s \left[ \partial j_1 \hat{x}^i_1 \wedge i_1 \frac{\partial}{\partial x^{j_1}} \cdots \partial j_{2n-1} \hat{x}^i_{2n-1} \wedge i_k \frac{\partial}{\partial x^{j_{2n-1}}} \right]
\]

\[
= \delta^{j_1 \cdots j_{2n}} P^{j_1}_{i_1} \cdots P^{j_{2n-1}}_{i_{2n-1}} \left( \vartheta^{j_2}_{i_2} - \pi^* \vartheta^{j_2}_{i_2} \right)
\]
Furthermore, by using (2.19), we get

\[
(2.30) \quad \int_{SM/M} P_{i_1} \cdots P_{i_{2n-1}} (\pi^* \varphi^{i_{2n}}) = \varphi^{i_{2n}} \int_{SM/M} P_{i_1} \cdots P_{i_{2n-1}} = 0.
\]

So combining (2.29) and (2.30), we have

\[
(2.31) \quad \int_{SM/M} \operatorname{tr}_S \left[ \theta^s \left( P^s \right)^{2n-1} \right] = \int_{SM/M} \delta^{i_1 \cdots i_{2n}} P_{i_1} \cdots P_{i_{2n-1}} - \varphi^{i_{2n}}.
\]

Finally, from Theorem 1 (2.28) and (2.31), we get the following local version of the Finslerian Gauss-Bonnet-Chern formula (2.12).

**Theorem 2.** Let \((M, F)\) be a closed and oriented Finsler manifold of dimension \(2n\). Let \(R_{\text{Ch}} = R + P\) be the curvature of the Chern connection \(\nabla_{\text{Ch}}\) on the pull-back bundle \(\pi^*TM\) over \(SM\). Then in the induced homogeneous coordinate charts \((x^i, y^i)\) on \(SM\), one has

\[
\chi(M) = \frac{1}{(2\pi)^{2n}(2n)!} \int_{SM/M} \left\{ \sum_{k=1}^{n} (-1)^k C_{2k}^{2k} C_{k-1}^{k-1} \int_{SM} \delta^{i_1 \cdots i_{2n}} R_{i_1} \cdots R_{i_{2n}} \right. \left. - P_{i_1} \cdots P_{i_{2n-1}} \right\},
\]

(2.32)

where \(\varpi^i, R^i_i, \gamma^i_i\) and \(\Xi^i_i\) are defined by (2.7), (2.6), (2.26), respectively.

**Remark 2.** Note that the terms

\[
\delta^{i_1 \cdots i_{2n}} R_{i_1} \cdots R_{i_{2n}} - P_{i_1} \cdots P_{i_{2n-1}}\gamma^i_i\Xi^i_i
\]

are globally defined differential forms on \(SM\), while the term \(\delta^{i_1 \cdots i_{2n}} P_{i_1} \cdots P_{i_{2n-1}} \varpi^i_i\) is not. However, the vertical exactness property (2.7) of the Chern-Minkowski curvature \(P\) guarantees that the following integral along fibres

\[
\int_{SM/M} \delta^{i_1 \cdots i_{2n}} P_{i_1} \cdots P_{i_{2n-1}} \varpi^i_i
\]

is a well-defined global differential form on \(M\).

As an example of Theorem 2, we will give an explicit GBC-formula for a closed and oriented Finsler surface \((M, F)\). Note that in the induced homogeneous coordinate charts \((x^i, y^i)\) on \(SM\), we get from (2.32)

\[
\chi(M) = \frac{1}{8\pi^2} \int_{SM/M} \delta^{i_1 i_2} R_{i_1} \varpi_{i_2} + \int_{SM/M} \delta^{i_1 i_2} P_{i_1} \varpi_{i_2}.
\]

(2.33)

For further investigation, it is more convenience to rewrite (2.33) with respect to the following special \(gf\)-orthonormal oriented frame \(\{e_1, e_2\}\), where

\[
e_1 := \frac{F y^2}{\sqrt{g}} \frac{\partial}{\partial x^1} - \frac{F y^1}{\sqrt{g}} \frac{\partial}{\partial x^2}, \quad e_2 := \frac{y^1}{F} \frac{\partial}{\partial x^1} + \frac{y^2}{F} \frac{\partial}{\partial x^2}.
\]

In this case, the dual frame \(\{\omega^1, \omega^2\}\) is given by

\[
\omega^1 = \sqrt{g} F y^2 d\hat{x}^1 - \sqrt{g} F y^1 d\hat{x}^2, \quad \omega^2 = F y^1 d\hat{x}^1 + F y^2 d\hat{x}^2.
\]
Set

\[ \omega^3 := \omega_2^1 = -\omega_1^2 = \sqrt{g} \left( y_2^1 \frac{\delta y_1}{F} - y_1^1 \frac{\delta y_2}{F} \right). \]

Then under the special \( g_F \)-orthonormal frame above, the Chern curvature forms are

\[ (R^{Ch})^b_a = R^b_a + P^b_a = R^b_{a12} \omega^1 \omega^2 + P^b_{a11} \omega^1 \omega^3 + P^b_{a21} \omega^2 \omega^3. \]

One now verifies by (2.33) the following corollary easily.

**Corollary 1.** For any closed and oriented Finsler surfaces \((M, F)\), we have

\[
\chi(M) = \frac{1}{(2\pi)^2} \left\{ \int_{SM} R^{12}_{12} \omega^1 \omega^2 - \int_M \int_{SM/M} \frac{1}{F^3} (G_1 y^1 + G_2 y^2) P_{111} \omega^1 \omega^3 \omega^3 - \int_M \int_{SM/M} \frac{1}{F^3} \left( \frac{F y^1}{\sqrt{g}} (\log F)_x - \frac{F y^1}{2} G_1 \right) P_{221} \omega^1 \omega^2 \omega^3 \right\},
\]

where \( G_i := \frac{1}{4} (y^j [F^2]_{y^i x^j} - [F^2]_{x^i}) \).

When the surface \((M, F)\) in Corollary 1 is a Landsberg space, that is, \( P_{221} = 0 \), we get

\[
\chi(M) = \frac{1}{(2\pi)^2} \left\{ \int_{SM} R^{12}_{12} \omega^1 \omega^2 - \int_M \int_{SM/M} \frac{1}{F^3} (G_1 y^1 + G_2 y^2) P_{111} \omega^1 \omega^3 \right\}.
\]

On the other hand, for closed and oriented Landsberg surfaces, Bao and Chern \[1\] obtained that

\[
\int_M -R^{12}_{12} \omega^1 \omega^2 = \chi(M) \text{Vol}(\text{Finsler}S^1).
\]

Note that for a Landsberg surface, the volume function

\[
\int_{SM/M} -\omega^3 = \text{Vol}(\text{Finsler}S^1)
\]

is constant. Moreover, due to an observation of Chern (cf. \[7\]), \( R^{12}_{12} \omega^1 \omega^2 \) in fact lives on \( M \).

As a consequence of (2.35)-(2.37), we get the following corollary

**Corollary 2.** Let \((M, F)\) be a closed and oriented Landsberg surface. Then the following equality holds

\[
\left[ (\text{Vol}(\text{Finsler}S^1))^2 - (2\pi)^2 \right] \chi(M) = \int_M \int_{SM/M} \frac{1}{F^3} (G_1 y^1 + G_2 y^2) P_{111} \omega^1 \omega^2 \omega^3.
\]

Moreover, when \((M, F)\) is Berwald, one has \( \text{Vol}(\text{Finsler}S^1) = 2\pi \) or \( \chi(M) = 0 \).

In fact, by the Szabó’s rigidity theorem (cf. \[2\], p.278) that any Berwald surfaces must be locally Minkowskian or Riemannian, and a closed locally Minkowskian surface has zero Euler number, one also gets easily the assertion of Corollary 2 on Berwald spaces.
3. A special case: Berwald space

In this section, we will investigate the GBC-formulae for Berwald spaces. Recall that a Finsler manifold \((M, F)\) is a Berwald manifold if and only if the Chern-Minkowski curvature \(P\) of the Chern connection \(\nabla_{\text{Ch}}\) vanishes. Moreover, for a Berwald space \((M, F)\), the Chern connection \(\nabla_{\text{Ch}}\) on \(\pi^*TM\) is actually defined over the whole total space \(TM\), not only \(TM_0\). In fact, now the Chern connection is the pull back of the Levi-Civita connection on \(TM \to M\) for some Riemannian metric \(g^{TM}\) on \(M\) (cf. Ch. 10 in [2] for more details).

### 3.1. A Finslerian GBC-formula for Berwald spaces.

Note that when the Chern-Minkowski curvature \(P\) vanishes, that is, \((M, F)\) a Berwald space, the theorem becomes very simple, from which we deduce the following Finslerian GBC-formula for Berwald spaces easily.

**Theorem 3.** Let \((M, F)\) be a closed and oriented Berwald space of dimension \(2n\). Then one has,

\[
\chi(M) = \left(\frac{-1}{2\pi}\right)^n \frac{1}{\text{Vol}(S^{2n-1})} \int_M \int_{SM/M} \text{Pf}(\hat{R}_{\text{Ch}})^{2n} \omega_1 \cdots \omega_{2n-1},
\]

where \(\omega_1 \cdots \omega_{2n-1}\) gives the volume form of the fibre when restricting to a fibre of \(SM\).

**Proof.** Since the Chern-Minkowski curvature \(P\) vanishes for Berwald spaces, then form Theorem [4], we get

\[
\chi(M) = \frac{1}{(2\pi)^{2n}(2n)!} \int_{SM} \text{tr}_s \left[ c(e_1) c(\nabla_{\text{Ch}} e) \right]^{2n-1}(R^2)^n.
\]

Hence, under the special \(g_F\)-orthonormal frames \(\{e_1, \cdots, e_{2n}\}\) of \(\pi^*TM\) with \(e_{2n} = e\), we have

\[
\chi(M) = \frac{1}{(2\pi)^{2n}(2n)!} \int_{SM} \text{tr}_s \left[ c(e_1) c(\nabla_{\text{Ch}} e) \right]^{2n-1}(R^2)^n
\]

\[
= \frac{1}{(2\pi)^{2n}(2n)!} \int_{SM} \text{tr}_s \left[ c(e_2) (\sum_{\gamma=1}^{2n-1} \omega_{2n} c(e_\gamma))^{2n-1}(R^2)^n \right]
\]

\[
= -\frac{1}{(2\pi)^{2n}(2n)!} \int_{SM} \text{tr}_s \left[ \omega_1 \cdots \omega_{2n-1} c(e_1) \cdots c(e_{2n}) (\hat{R}_{\text{Ch}} e_1 c(e_1))^{2n} \right]
\]

\[
= \frac{1}{(2\pi)^{2n}(2n)!} \int_{SM} \text{tr}_s \left[ \epsilon_1 \cdots \epsilon_{2n} \hat{R}_{\text{Ch}}^{a_1} \cdots \hat{R}_{\text{Ch}}^{a_{2n}} e_1 \cdots e_{2n} \right]
\]

\[
= \frac{1}{(2\pi)^{2n}(2n)!} \int_{SM} \text{tr}_s \left[ \epsilon_1 \cdots \epsilon_{2n} \hat{R}_{\text{Ch}}^{a_1} \cdots \hat{R}_{\text{Ch}}^{a_{2n}} e_1 \cdots e_{2n} \right]
\]

\[
= \frac{1}{2n+1}(2\pi)^{2n}(2n)! \int_{SM} \text{Pf}(\hat{R}_{\text{Ch}})^{2n} \omega_1 \cdots \omega_{2n-1}
\]

\[
= \left(\frac{-1}{2\pi}\right)^n \frac{1}{\text{Vol}(S^{2n-1})} \int_M \int_{SM/M} \text{Pf}(\hat{R}_{\text{Ch}})^{2n} \omega_1 \cdots \omega_{2n-1}.
\]

**Remark 3.** One should notice that in [3,1], the date \(\hat{R}_{\text{Ch}}\), defined by (2.9), is the skew-symmetrization of \(R_{\text{Ch}}\) with respect to \(g_F\). Hence the differential form \(\text{Pf}(\hat{R}_{\text{Ch}})\) is dependent on the vertical coordinates \(y^i\). However, if the Finsler metric \(F\) is induced by a Riemannian
metric $g^TM$ on $M$, then $M$ itself is a Riemannian manifold. In this case, $\text{Pf}(\tilde{R}^\text{Ch})$ is exactly the Pfaffian $\text{Pf}(R^TM)$ defined by (3.2), which is constant along fibres of $SM$, and therefore, we recover Chern’s formula (3.1) from the formula (3.1) easily.

3.2. A Bao-Chern type GBC-formulae for Berwald spaces. In this subsection, inspired by an idea from Weiping Zhang, we will work out a Bao-Chern type GBC-formula for Berwald spaces. Let $(M, F)$ be a closed and oriented Finsler manifold. Let $X$ be a vector field on $M$ with isolated zero set $Z(X)$. Define $[X]$ by $[X]_x = X_x/F(x, X)$ for any $x \in M \setminus Z(X)$. For sufficiently small $\epsilon > 0$, let $Z_\epsilon(X)$ be the open $\epsilon$-neighborhood of $Z(X)$ in $M$ with respect to the Finsler metric $F$, and set $M_\epsilon = M \setminus Z_\epsilon(X)$. Then $[X]$ determines a pull-back section $\tilde{[X]}$ of $\pi^*TM \to TM_\epsilon$. Note that near each $x \in M_\epsilon$, $\lim_{\epsilon \to 0} [X]_x = X_x/F(x, X)$.

Consider the following family of superconnections on $\Lambda^*(\pi^*TM) \to TM_\epsilon$ for any $t \in [0, 1]$:

$$\hat{A}_{\rho, T, t} = \tilde{\nabla}_\rho^*\pi^*(\pi^*TM) + Tc_{\tilde{g}_F}(\hat{X}) - tTc_{\tilde{g}_F}(\tilde{[X]}) = \tilde{\nabla}_\rho^*\pi^*(\pi^*TM) + Tc_{\tilde{g}_F}(\hat{Y} - t[\tilde{X}]).$$

Clearly, $\hat{A}_{\rho, T, 0}$ is the extended Chern connection defined in the previous section, and for $\hat{A}_{\rho, T, 1}$, we have the following localization formula for any Finsler manifold.

**Lemma 2.**

$$\lim_{T \to \infty} \int_{TM_\epsilon} \tr_s \left[ \exp \hat{A}_{\rho, T, 1}^2 \right] = (-2\pi)^n \int_{M_\epsilon} [X]^n \text{Pf}(\tilde{R}^\text{Ch}).$$

**Proof.** It is clear that

$$\int_{TM_\epsilon} \tr_s \left[ \exp \left( \tilde{\nabla}_\rho^*\pi^*(\pi^*TM) + Tc_{\tilde{g}_F}(\hat{X}) \right)^2 \right]$$

$$= \int_{M_\epsilon} \int_{TM_\epsilon} e^{-T^2|\hat{Y} - [\tilde{X}]|^2} \tr_s \left[ \exp \left( \tilde{R}_\rho^2 + T[\tilde{\nabla}_\rho^*\pi^*(\pi^*TM), c_{\tilde{g}_F}(\hat{Y} - [\tilde{X}])]) \right) \right],$$

where $|\hat{Y} - [\tilde{X}]|^2 = \tilde{g}_F(\hat{Y} - [\tilde{X}], \hat{Y} - [\tilde{X}])$, and $\tilde{R}_\rho^2 = \tilde{\nabla}_\rho^2$.

Note that the zero set of the section $\hat{Y} - [\tilde{X}]$ in $TM_\epsilon$ is exactly $[X](M_\epsilon)$. For a fixed $\delta > 0$, let $B_\delta([X](M_\epsilon))$ be the open $\delta$-tube neighborhood of $[X](M_\epsilon)$ in $TM_\epsilon$. So when $\delta$ is small enough, one has $\tilde{\nabla}_\rho^*\pi^*(\pi^*TM) = \nabla^\text{Ch},\hat{z}$ and $c_{\tilde{g}_F} = g_F$ on $B_\delta([X](M_\epsilon))$. Now by the exponential decay property of the integral in (3.1) along fibres as $T \to +\infty$, we have

$$\lim_{T \to \infty} \int_{TM_\epsilon} \tr_s \left[ \exp \left( \tilde{\nabla}_\rho^*\pi^*(\pi^*TM) + Tc_{\tilde{g}_F}(\hat{Y} - [\tilde{X}]) \right)^2 \right]$$

$$= \int_{[X](M_\epsilon)} \lim_{T \to \infty} \int_{B_\delta([X](M_\epsilon))/[X](M_\epsilon)} e^{-T^2|\hat{Y} - [\tilde{X}]|^2} \left\{ \tr_s \left[ \exp \left( R^\text{Ch},\hat{z} + T[\nabla^\text{Ch},\hat{z}, c_{\tilde{g}_F}(\hat{Y} - [\tilde{X}])]) \right) \right] \right\}.$$ (4.1)

Note that near each $x \in M_\epsilon$, we have from (2.2) and (2.23)

$$\left[ \nabla^\text{Ch},\hat{z}, c_{\tilde{g}_F} \left( \frac{\partial}{\partial \hat{x}^j} \right) \right] = \left[ d^TM + \omega^j, c_{\tilde{g}_F} \left( \frac{\partial}{\partial \hat{x}^j} \right) \right]$$

$$= (d^TM g_{ij}) d\hat{x}^j \wedge + \left[ -\omega^j_k d\hat{x}^k \wedge i_{\frac{\partial}{\partial \hat{x}^j}}, g_{ik} d\hat{x}^k \wedge -i_{\frac{\partial}{\partial \hat{x}^j}} \right]$$

$$= (g_{ik} \omega^j_k + g_{jk} \omega^j_k + 2F^{-1} A_{ijk} \delta y^k) d\hat{x}^j \wedge - \left( g_{il} \omega^j_k d\hat{x}^k \wedge + \omega^j_k i_{\frac{\partial}{\partial \hat{x}^j}} \right)$$

$$= \omega^j_i c_{\tilde{g}_F} \left( \frac{\partial}{\partial \hat{x}^j} \right) + 2F^{-1} A_{ijk} \delta y^k d\hat{x}^j \wedge.$$
Write \([X] = [X]^i \frac{\partial}{\partial x^i}\) near \(x\) and so \(\hat{Y} - \hat{[X]} = (y^i - [X]^i) \frac{\partial}{\partial x^i}\) on \(B_\delta([X](x))\). Then from (2.4) and (3.6), we get

\[
(3.7) \quad [\nabla \text{Ch}_1, c_{g_F} (\hat{Y} - \hat{[X]}')] = \left[ [\nabla \text{Ch}_1, (y^i - [X]^i)] c_{g_F} \left( \frac{\partial}{\partial x^i} \right) \right]
\]

\[
= dTM(y^i - [X]^i) c_{g_F} \left( \frac{\partial}{\partial x^i} \right) + (y^i - [X]^i) \left[ [\nabla \text{Ch}_1, c_{g_F} \left( \frac{\partial}{\partial x^i} \right) \right]
\]

\[
= dTM(y^i - [X]^i) c_{g_F} \left( \frac{\partial}{\partial x^i} \right) + (y^i - [X]^i) \nabla_i c_{g_F} \left( \frac{\partial}{\partial x^i} \right)
\]

\[
+ 2(y^i - [X]^i) F^{-1} A_{ijk} \delta y^k d\hat{x}^j \wedge
\]

\[
= \delta y^i c_{g_F} \left( \frac{\partial}{\partial x^i} \right) - ([\nabla \text{Ch}[X]]^i c_{g_F} \left( \frac{\partial}{\partial x^i} \right) + 2(y^i - [X]^i) F^{-1} A_{ijk} \delta y^k d\hat{x}^j \wedge,
\]

where \([\nabla \text{Ch}[X]]^i = (\nabla \text{Ch}[X]^i) \frac{\partial}{\partial x^i}\) denotes the covariant differential of the section \([X]\). Moreover, from (2.3), one sees that \((\nabla \text{Ch}[X])^i = d[X]^i + [X]^i \nabla_i\) are purely horizontal one forms. By (3.7) and \(y^i A_{ijk} = 0\), we obtain

\[
(3.8) \quad \lim_{T \to \infty} \int_{B_\delta([X](x))} e^{-T^2 |\hat{Y} - \hat{[X]}|^2} \left\{ \text{tr}_s \left[ \exp \left( R^{\text{Ch}_1} + T \left[ [\nabla \text{Ch}_1, c_{g_F} (\hat{Y} - \hat{[X]}')] \right] \right) \right] \right\} (4n)
\]

\[
= \lim_{T \to \infty} \int_{B_\delta([X](x))} e^{-T^2 |\hat{Y} - \hat{[X]}|^2} \left\{ \text{tr}_s \left[ \exp \left( R + P + T \delta y^i c_{g_F} \left( \frac{\partial}{\partial x^i} \right) \right) \right. \right.
\]

\[
-T [\nabla \text{Ch}[X]]^i c_{g_F} \left( \frac{\partial}{\partial x^i} \right) - 2T [X]^i F^{-1} A_{ijk} \delta y^k d\hat{x}^j \wedge \right\} (4n).
\]

Note that for any bounded smooth function \(f\) on \(T_x M\), one has easily

\[
(3.9) \quad \lim_{T \to \infty} \int_{B_\delta([X](x))} e^{-T^2 |\hat{Y} - \hat{[X]}|^2} f(Y) T^{2n} \sqrt{\text{det}(g_{ij})} dy^1 \wedge \cdots \wedge dy^{2n} = \pi^n f([X](x)).
\]

From (3.8), (3.9) and \(((X)|_x)^i A_{ijk}([X]|_x) = 0\), one sees easily that the term \(2T [X]^i F^{-1} A_{ijk} \delta y^k d\hat{x}^j \wedge\) has no contribution to the final result of (3.8).

Note that

\[
(3.10) \quad \bar{P} := \left( \sum_{k,l} P_{i,k}^j \frac{dx^k}{\eta} \wedge (\nabla \text{Ch}[X])^j \right)
\]

gives a well-defined endomorphism \(\bar{P}\) on \(\pi^* TM \to M \setminus Z(X)\), and by (1.3), its lifting \(\bar{P}^2\) on \(\Lambda^*(\pi^* TM) \to M \setminus Z(X)\) is given by

\[
(3.11) \quad \bar{P}^2 = - \sum_{i,j} P_{i,k}^j \frac{dx^k}{\eta} \wedge (\nabla \text{Ch}[X])^j \frac{d\hat{x}^i}{\eta} \wedge \frac{\partial}{\partial x^j} =: \sum_{i,j} P_{i,k}^j (\nabla \text{Ch}[X])^j \frac{d\hat{x}^i}{\eta} \wedge \frac{\partial}{\partial x^j}.
\]
Therefore, the right hand side of equation (3.8) becomes

\[(3.12)\]

\[
\lim_{T \to \infty} \int_{B_{\delta}(\bar{X}(x))} e^{-T^2|\bar{Y} - \bar{X}|^2} \left\{ \text{tr}_s \left[ \exp \left( R^2 \right) \exp \left( P^2 \right) \exp \left( -T(\nabla^T \Xi)^i c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^i} \right) \right) \right. \right.
\]
\[
\left. \left. \exp \left( T\delta y^i c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^i} \right) \right) \right] \right\}^{(4n)}
\]
\[
= \lim_{T \to \infty} \int_{B_{\delta}(\bar{X}(x))} e^{-T^2|\bar{Y} - \bar{X}|^2} \left\{ \text{tr}_s \left[ \exp \left( R^2 \right) \exp \left( P^2 \right) \prod_{i=1}^{2n} \left( 1 + T\delta y^i c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^i} \right) \right) \right] \right\}^{(4n)}
\]
\[
= \lim_{T \to \infty} \int_{B_{\delta}(\bar{X}(x))} e^{-T^2|\bar{Y} - \bar{X}|^2} \left\{ \text{tr}_s \left[ \exp \left( R^2 \right) \sum_{k=0}^{2n} \exp \left( P^2 \right) \sum_{1 \leq i_1 < \ldots < i_k \leq 2n} (-1)^k T^{2n} (\nabla^T \Xi)^{i_1} c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^{i_1}} \right) \right. \right.
\]
\[
\left. \left. \cdots (\nabla^T \Xi)^{i_k} c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^{i_k}} \right) \delta y^{i_{k+1}} c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^{i_{k+1}}} \right) \cdots \delta y^{i_{2n}} c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^{i_{2n}}} \right) \right] \right\}^{(4n)}
\]
\[
= \lim_{T \to \infty} \int_{B_{\delta}(\bar{X}(x))} e^{-T^2|\bar{Y} - \bar{X}|^2} \left\{ \text{tr}_s \left[ \exp \left( R^2 \right) \sum_{k=0}^{2n} \frac{(-1)^k}{k!} \sum_{j_1 \ldots j_k} P^{j_2}_{j_1 s_1} \delta y^{s_1} \cdots P^{j_{2k}}_{j_{2k-1} s_k} \delta y^{s_k} \right. \right.
\]
\[
\left. \left. \cdots (\nabla^T \Xi)^{i_k} c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^{i_k}} \right) \delta y^{i_{k+1}} c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^{i_{k+1}}} \right) \cdots \delta y^{i_{2n}} c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^{i_{2n}}} \right) \right] \right\}^{(4n)}
\]
\[
= \lim_{T \to \infty} \int_{B_{\delta}(\bar{X}(x))} e^{-T^2|\bar{Y} - \bar{X}|^2} \left\{ \text{tr}_s \left[ \exp \left( R^2 \right) \sum_{k=0}^{2n} \frac{(-1)^k}{k!} \sum_{j_1 \ldots j_k} P^{j_2}_{j_1 s_1} (\nabla^T \Xi)^{s_1} \cdots P^{j_{2k}}_{j_{2k-1} s_k} \right. \right.
\]
\[
\left. \left. \cdots (\nabla^T \Xi)^{s_k} d_{\hat{\xi}^{j_1}} \cdots d_{\hat{\xi}^{j_{2k}}} \cdots i \frac{\partial}{\partial \hat{\xi}^{j_{2k}}} \cdot T^{2n} \delta y^{i} c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^{i}} \right) \right. \right.
\]
\[
\left. \left. \left. \delta y^{i_{2n}} c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^{i_{2n}}} \right) \right] \right\}^{(4n)}
\]
\[
= \lim_{T \to \infty} \int_{B_{\delta}(\bar{X}(x))} e^{-T^2|\bar{Y} - \bar{X}|^2} \left\{ \text{tr}_s \left[ \exp \left( R^2 \right) \exp \left( \tilde{P}^2 \right) \exp \left( T \delta y^i c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^i} \right) \right) \right] \right\}^{(4n)}
\]
\[
= \lim_{T \to \infty} \int_{B_{\delta}(\bar{X}(x))} e^{-T^2|\bar{Y} - \bar{X}|^2} \left\{ \text{tr}_s \left[ \exp \left( \tilde{P}^2 \right) \exp \left( T \delta y^i c_{gf} \left( \frac{\partial}{\partial \hat{\xi}^i} \right) \right) \right] \right\}^{(4n)}
\]

Now we will use the special $g_F$-orthonormal frame field \{e_1, \ldots, e_{2n}\}. Let \{\omega^1, \ldots, \omega^{2n}\} be its dual frame field. Set

\[(3.13)\]
\[e_a = u^j_a \frac{\partial}{\partial \hat{\xi}^j}, \quad \frac{\partial}{\partial \hat{\xi}^i} = v^i_a e_a.\]
From (2.10), (3.5), (3.8), (3.11), (3.12), (3.16) and (3.18), we obtain

\[
(3.18)
\]

By (2.8), (3.10), (3.11), (3.15) and (3.17), we have

\[
(3.17)
\]

Since the map \( X \) have

\[
(3.14)
\]

have

\[(3.13)-(3.15)\]

we have

\[
(3.15)
\]

Thus (3.3) holds. \( \square \)

By (2.10), (3.9) and (3.13)-(3.15), we have

\[
(3.16)
\]

Since the map \([X]: M \setminus Z(X) \to TM\) is given by \([X](x) = (x, [X])\) for any \( x \in M \setminus Z(X)\). We have

\[
[X] \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^j} + \frac{\partial [X]^j}{\partial x^i} \frac{\partial}{\partial y^j},
\]

and then

\[
(3.17)
\]

By (2.8), (3.10), (3.11), (3.15) and (3.17), we have

\[
(3.18)
\]

From (2.10), (3.5), (3.8), (3.11), (3.12), (3.16) and (3.18), we obtain

\[
\lim_{T \to \infty} \int_{TM} \text{tr}_s \left[ \exp \left( \frac{T}{\rho} A^* (\pi^* T M) \right) \right] = (-1)^{n} \pi^{n} \int_{[X]} \sum \epsilon_{a_1 \ldots a_{2n}} \tilde{\Omega}_a^{a_2} \wedge \ldots \wedge \tilde{\Omega}_a^{a_{2n-1}}
\]

Thus (3.3) holds. \( \square \)
We will simply denote \( \lim_{\epsilon \to 0} \int_{M_c} [X]^* \text{Pf}(\tilde{R}^\text{Ch}_\rho) \) by \( \int_{M} [X]^* \text{Pf}(\tilde{R}^\text{Ch}) \). Now we state the following Bao-Chern type GBC-formulae for Berwald spaces.

**Theorem 4.** Let \((M, F)\) be a closed and oriented Berwald manifold of dimension \(2n\). Let \(X\) be a vector field on \(M\) with the isolated zero points. Then

\[
\left( \frac{-1}{2\pi} \right)^n \int_{M} [X]^* \text{Pf}(\tilde{R}^\text{Ch}) = \chi(M).
\]

**Proof.** For the family of superconnections defined by (3.2), we have the following transgression formula

\[
\lim_{T \to \infty} \int_{T M_c} tr_s \left[ \exp \tilde{A}^2_{\rho,T,1} \right] - \lim_{T \to \infty} \int_{T M_c} tr_s \left[ \exp \tilde{A}^2_{\rho,T,0} \right] = \lim_{T \to \infty} \int_{T M_c} tr_s \left[ \exp \left( \tilde{\nabla}^\rho T M (\tilde{Y} - [X]) \right)^2 \right] - \lim_{T \to \infty} \int_{T M_c} tr_s \left[ \exp \left( \tilde{\nabla}^\rho T M (\tilde{Y} - \tilde{X}) \right)^2 \right] = - \lim_{T \to \infty} \int_{T M_c} \int_{0}^{1} tr_s \left[ Tc_{\tilde{g}_F}(\tilde{X}) \exp \left( \tilde{\nabla}^\rho T M (\tilde{Y} - \tilde{X}) \right)^2 \right] dt = \lim_{T \to \infty} \int_{T M_c \partial \mathcal{Z}(X)} \int_{0}^{1} tr_s \left[ Tc_{\tilde{g}_F}(\tilde{X}) \exp \left( \tilde{\nabla}^\rho T M (\tilde{Y} - \tilde{X}) \right)^2 \right] dt.
\]

Now we need to deal with the last equality in (3.19) for Berwald spaces. Note that for Berwald spaces, the Chern connection \( \nabla^\text{Ch} \) is the pull-back of the Levi-Civita connection for some a Riemannian metric \( g^{TM} \) on \( TM \). So the connection components \( \omega^j_i \) of \( \nabla^\text{Ch} \) with respect to a pull-back frame \( \{ \partial / \partial \tilde{x}^i \} \) are now defined over the whole space \( TM \) and independent on the vertical variable \( y \), and

\[
dg^{TM}_{ij} = g_{ik}^T \omega^j_i + g_{jk}^T \omega^k_i.
\]

In this case, we define

\[
\tilde{\nabla}^\rho = \nabla^\text{Ch}, \quad \tilde{g}_F = (1 - \rho) g_F + \rho g^{TM}.
\]

From (3.22), (3.23), (3.21) and (3.21), we have

\[
[\nabla^\text{Ch}, \cdot]_{\tilde{g}_F}(\tilde{Y} - t[\tilde{X}]) = \left[ \nabla^\text{Ch}, (y^i - t[X]^i) c_{\tilde{g}_F} \left( \frac{\partial}{\partial \tilde{x}^i} \right) \right]
= d^{TM}(y^i - t[X]^i) c_{\tilde{g}_F} \left( \frac{\partial}{\partial \tilde{x}^i} \right) + (y^i - t[X]^i) \omega^j_i c_{\tilde{g}_F} \left( \frac{\partial}{\partial \tilde{x}^i} \right)
+ d^{TM} \rho(y^j - t[X]^j)(g_{ij}^{TM} - g_{ij}) \tilde{d} \tilde{x}^j + 2(1 - \rho) F^{-1}(y^i - t[X]^i) A_{ijk} \delta y^k \tilde{d} \tilde{x}^j \wedge
= \delta y^i c_{\tilde{g}_F} \left( \frac{\partial}{\partial \tilde{x}^i} \right) - t(\nabla^\text{Ch}[X]^i) c_{\tilde{g}_F} \left( \frac{\partial}{\partial \tilde{x}^i} \right)
+ (y^j - t[X]^j)(g_{ij}^{TM} - g_{ij}) d^{TM} \rho \tilde{d} \tilde{x}^j \wedge - 2(1 - \rho) F^{-1}[X]^i A_{ijk} \delta y^k \tilde{d} \tilde{x}^j \wedge .
\]
By (3.20), (3.21) and (3.22), we get

\[
\lim_{T \to \infty} \int_{T_x M} \int_0^1 \left\{ \text{tr}_s \left[ Tc_{\xi_F}(\hat{X}) \exp \left( \nabla^*_\rho (\pi T^*M) + TC_{\xi_F}(\hat{Y} - t[\hat{X}]) \right) \right] \right\}^{(4n-1)} \, dt
\]

\[
= \int_0^1 \lim_{T \to \infty} \int_{T_x M} e^{-T^2[\hat{Y} - t[\hat{X}]]} \left\{ \text{tr}_s \left[ Tc_{\xi_F}(\hat{X}) \exp \left( R_{\chi n} + T[\nabla^*\chi, c_{\xi_F}(\hat{Y} - t[\hat{X}])] \right) \right] \right\}^{(4n-1)} \, dt
\]

\[
= \int_0^1 \lim_{T \to \infty} \int_{T_x M} e^{-T^2[\hat{Y} - t[\hat{X}]]} \left\{ \text{tr}_s \left[ T[\hat{X}]c_{\xi_F} \left( \frac{\partial}{\partial \hat{x}^i} \right) \exp \left( R_{\chi n} - t(\nabla^*\chi)[X]c_{\xi_F} \left( \frac{\partial}{\partial \hat{x}^i} \right) \right) \right] \right\}^{(4n-1)} \, dt
\]

\[
+ T \delta y^i c_{\xi_F} \left( \frac{\partial}{\partial \hat{x}^i} \right) + T(y^i - t[\hat{X}]^i)(g_{TM} - g_{ij})dTM \rho d\hat{x}^j \wedge \left\{ \prod_{i=1}^{2n} \left( 1 + T \delta y^i c_{\xi_F} \left( \frac{\partial}{\partial \hat{x}^i} \right) \right) \cdot T(y^i - t[\hat{X}]^i)(g_{TM} - g_{ip})dTM \rho \right\}^{(4n-1)} \, dt.
\]

Note that we can and will choose \( \rho(x, Y) = f(F(x, Y)) \) for any nonnegative function \( f(t) \) on \([0, +\infty)\) such that \( 0 \leq f \leq 1 \) with \( f \equiv 1 \) for \( t \leq \frac{1}{4} \) and \( f \equiv 0 \) for \( t \geq \frac{1}{2} \). Since \( \frac{\delta}{\delta x^i} F = 0 \), we have

\[
\begin{align*}
\left. \quad \right. \nonumber
\int_0^1 \lim_{T \to \infty} \int_{T_x M} e^{-T^2[\hat{Y} - t[\hat{X}]]} \left\{ \text{tr}_s \left[ T[\hat{X}]c_{\xi_F} \left( \frac{\partial}{\partial \hat{x}^i} \right) \exp \left( R_{\chi n} - t(\nabla^*\chi)[X]c_{\xi_F} \left( \frac{\partial}{\partial \hat{x}^i} \right) \right) \right] \right\}^{(4n-1)} \, dt
\end{align*}
\]

\[
\begin{align*}
\left. \quad \right. \nonumber
\int_0^1 \lim_{T \to \infty} \int_{T_x M} e^{-T^2[\hat{Y} - t[\hat{X}]]} \left\{ \text{tr}_s \left[ T[\hat{X}]c_{\xi_F} \left( \frac{\partial}{\partial \hat{x}^i} \right) \exp \left( R_{\chi n} - t(\nabla^*\chi)[X]c_{\xi_F} \left( \frac{\partial}{\partial \hat{x}^i} \right) \right) \right] \right\}^{(4n-1)} \, dt
\end{align*}
\]

\[
\begin{align*}
\left. \quad \right. \nonumber
\int_0^1 \lim_{T \to \infty} \int_{T_x M} e^{-T^2[\hat{Y} - t[\hat{X}]]} \left\{ \text{tr}_s \left[ T[\hat{X}]c_{\xi_F} \left( \frac{\partial}{\partial \hat{x}^i} \right) \exp \left( R_{\chi n} - t(\nabla^*\chi)[X]c_{\xi_F} \left( \frac{\partial}{\partial \hat{x}^i} \right) \right) \right] \right\}^{(4n-1)} \, dt
\end{align*}
\]

\[
= 0.
\]

Notice that \( \partial Z_e(X) \) is \( 4n - 1 \) dimension, by (3.23) and (3.25), we obtain

\[
\lim_{T \to \infty} \int_{TM \cap \partial Z_e(X)} \int_0^1 \text{tr}_s \left[ Tc_{\xi_F}(\hat{X}) \exp \left( \nabla^*_\rho (\pi T^*M) + TC_{\xi_F}(\hat{Y} - t[\hat{X}]) \right) \right] \, dt = 0.
\]
Now from (3.19) and (3.26), we get
\[
\lim_{T \to \infty} \int_{T(M)} tr_s \left[ \exp \hat{A}^2_{\rho, T, 1} \right] - \lim_{T \to \infty} \int_{T(M)} tr_s \left[ \exp \hat{A}^2_{\rho, T, 0} \right] = 0.
\]
Finally, by (2.14), (3.3) and (3.27), we have
\[
\left( -\frac{1}{2\pi} \right)^n \int_M [X]^* Pf(\hat{R}^{Ch}) = \left( -\frac{1}{2\pi} \right)^n \lim_{\epsilon \to 0} \int_{M_\epsilon} [X]^* Pf(\hat{R}^{Ch}) = \chi(M).
\]

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