Enhanced Principal Component Analysis under A Collaborative-Robust Framework
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Abstract—Principal component analysis (PCA) frequently suffers from the disturbance of outliers and thus a spectrum of robust extensions and variations of PCA have been developed. However, existing extensions of PCA treat all samples equally even those with large noise. In this paper, we first introduce a general collaborative-robust weight learning framework that combines weight learning and robust loss in a non-trivial way. More significantly, under the proposed framework, only a part of well-fitting samples are activated which indicates more importance during training, and others, whose errors are large, will not be ignored. In particular, the negative effects of inactivated samples are alleviated by the robust loss function. Then we furthermore develop an enhanced PCA which adopts a point-wise $\sigma$-loss function that interpolates between $\ell_2$-norm and squared Frobenius-norm and meanwhile retains the rotational invariance property. Extensive experiments are conducted on occluded datasets from two aspects including reconstructed errors and clustering accuracy. The experimental results prove the superiority and effectiveness of our model.

Index Terms—Dimensionality reduction, principal component analysis, collaborative-robust learning.

I. INTRODUCTION

Principal component analysis (PCA) [1]–[3] as an unsupervised approach has been widely applied in diverse fields such as astronomy, biology, economics, and computer vision. The intention of conventional PCA can be stated from different perspectives. On the one hand, PCA can be interpreted as to project raw data into low-dimensional space which remains the largest variance of transformed data. Therefore, PCA is conventionally viewed as an unsupervised learning algorithm of dimensionality reduction. Since the dimensionality of data in many real applications could be extremely high, learning algorithms frequently suffer from the dimension curse. Consequently, PCA, owing to no need for supervised information, is often employed to reduce dimension during the preprocessing phase. On the other hand, the motivation of PCA is to find a low-rank approximation of data points. In other words, PCA intends to find a subspace with the definite rank whose error is as small as possible. According to this, PCA is also employed to compress data and reconstruct data points from polluted ones. In particular, plenty of 2DPCA [4]–[7] models have been proposed to process spatial data like images.

In practice, datasets always contain a quantity of noises and outliers. Unfortunately, the classical PCA is sensitive to outliers, which leads to poor performance. To address this problem, a spectrum of robust PCA algorithms [2], [7]–[13] have been developed by applying different loss functions. For instance, [2] simply replaces squared $\ell_2$-norm with $\ell_1$-norm. R1PCA [10] utilizes the robust $\ell_{2,1}$-norm, i.e., non-squared $\ell_2$-norm, and thus retains the rotational invariance as well. L1PCA [11] modifies the objective and applies $\ell_1$-norm such that it has rotational invariance. Furthermore, the algorithm proposed in [7] promotes the robustness via capped models [14], [15]. The robust PCA with optimal mean (PCA-OM) [13] takes the mean into account as well as the projection matrix, and therefore gains impressive performance.

However, the existing PCA methods still have several drawbacks. Firstly, the robustness of their losses is limited. Specifically speaking, the $\ell_{2,1}$-norm is sensitive to small noises and the capped models [7] intensively depend on the choice of $\epsilon$ which varies dramatically on different datasets and is hard to set an appropriate value. Secondly, the highly polluted samples and well-fitting samples are treated equally which may do harm to the final result. More precisely, the existing algorithms only consider how to alleviate the errors caused by outliers but not improve the impact of well-fitting samples. As a result, the mentioned limitations hinder the performance of the existing robust PCA techniques.

This paper is a deep extension of our conference paper [16] which intends to reduce dimension via detecting clean and occluded samples and learn their importance automatically. Considering the motivations of all important variations of PCA and our previous work, we further propose a novel model, Enhanced Principal Component Analysis (EPCA), in this paper. The motivation of EPCA is shown in Fig. 1. The merits of this paper are listed as follows:

1. We proposed a novel framework named as collaborative-robust weight learning which utilizes the robust loss to reduce the errors caused by outliers and novel weight learning to enhance the effect of well-fitting data simultaneously. In particular, only $k$ best-fitting samples are activated. Therefore, the robustness of models is enhanced intensively.

2. A novel robust loss, $\sigma$-loss, is designed to address the limitation of loss functions in existing PCA approaches. Moreover, the point-wise $\sigma$-loss is further developed to adapt the collaborative-robust framework.

3. Under the framework, a novel model, Enhanced Principal Component Analysis (EPCA), is proposed. EPCA can learn weights for all samples to augment the effect of well-fitting samples which cooperates with the robust loss to promote the robustness of our model.

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Notations: In this paper, all uppercase words denote matrices while vectors are written in bold lowercases. For a vector \( \mathbf{a} \), \( \| \mathbf{a} \|_1 \) and \( \| \mathbf{a} \|_2 \) denote the \( \ell_1 \) and \( \ell_2 \)-norm respectively. For a matrix \( A \), the \( i \)-th column is represented as \( \mathbf{a}_i \), the \( (i,j) \)-th entry is \( a_{ij} \), and the transpose is denoted by \( M^T \). \( \| A \|_F^2 = \sum_{i,j} a_{ij}^2 \) represents the squared Frobenius-norm while \( \| \mathbf{a} \|_{2,1} = \sum_i \| \mathbf{a}_i \|_2 \) denotes the \( \ell_{2,1} \)-norm. When \( A \) is a squared matrix, \( \text{tr}(A) \) is the trace operator where \( \text{tr}(A) = \sum_i m_{ii} \). \( R(A) \) denote the linear space spanned by its column vectors. \( \nabla f(\mathbf{x}) \) represents the gradient of \( f(\mathbf{x}) \) w.r.t. \( \mathbf{x} \), i.e., \( \nabla f(\mathbf{x}) = \partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_2, \cdots, \partial f/\partial x_n \). To keep the organization clear, all proofs are shown in Section \( \nabla \).

II. RELATED WORK

As a well-known unsupervised dimensionality reduction method, the classical PCA [1] attempts to project the raw features into a low-dimensional space via a linear transformation. Formally speaking, given a dataset \( X \in \mathbb{R}^{d \times n} \) where the \( i \)-th column \( x_i \in \mathbb{R}^d \) represents the \( i \)-th sample, the goal of PCA is to find a \( c \)-dimensional representation for \( X \), which remains the information and structure of the original features. The standard procedure of PCA consists of two steps: 1) Preprocessing: Centralize \( X \) via \( X - \frac{1}{n} \sum_{i=1}^n x_i 1^T \); 2) Eigenvalue decomposition (or singular value decomposition for acceleration): Find a linearly transformed space \( U^T X \) that maximizes the variance of \( n \) samples. As classical PCA is sensitive to outliers and practical datasets are always occluded by noises more or less, plenty of methods [2, 9-12, 17] have been proposed to promote the robustness of PCA. For instance, L1PCA [2, 11] utilizes \( \ell_1 \)-norm to measure the errors, while LpPCA [17] employs \( \ell_p \)-norm to further improve the scalability. Besides, the models proposed in [3, 10, 12, 13] also intend to promote the robustness via replacing the measurement to alleviate the impacts of polluted samples. Overall, all these methods get stuck in finding a robust loss to cope with noises and outliers such that the performance hits a bottleneck. They fail to exploit the importance of different data points. In other words, since there are only a small amount of data points that are occluded severely, an ideal dimensionality reduction model should be able to distinguish clean samples and polluted samples automatically and then treat them differently during training. Accordingly, our conference paper [16] proposes an extension of PCA that depends on no robust loss but introduces a point-wise importance learning mechanisms into PCA. The innovation brings impressive improvement in performance. Considering the intentions of these existing methods, we further propose a novel extension of PCA under a general learning framework.

III. A COLLABORATIVE-ROBUST FRAMEWORK FOR WEIGHT LEARNING

Our previous conference paper [16] focuses on filtering out outliers automatically via a sparse weight learning framework. The framework detects outliers and noises according to the loss of each sample. More specifically, the samples with large loss will be assigned small weights, or even be neglected by setting the corresponding weights as 0 automatically. Through treating data points differently during training, the weight learning mechanism is able to improve the robustness intensively.

The classical thought to enhance the robustness is to utilize a robust loss to replace the conventional \( \ell_2 \)-norm and Frobenius-norm. The underlying motivation is to alleviate the impact of outliers on the loss function since the polluted points usually result in large losses. An ideal robust loss should be able to avoid producing exaggerated values when encountering outliers.

Intuitively, combining the two methods is a feasible way to further improve the performance. However, do we really need both weight learning and robust loss meanwhile?

Since both of the two mechanisms promote the robustness by reducing the effects caused by occluded points, it is redundant to employ both of them. In practice, we may need only one of them to enhance the models due to the same motivation. Along with this insight, we extend our conference work via developing a framework, which incorporates them elegantly. Under this framework, the weight learning acts on clean data while the robust loss focuses on polluted data, such that they work collaboratively.

A. A Collaborative-Robust Framework for Weight Learning

As we emphasize in the last subsection, the purposes of weight learning and robust loss are duplicative. In this section, we propose a general framework which can employ both adaptive weights and a robust loss naturally. All existing methods only concentrate on decreasing errors of polluted samples via robust loss functions but fail to promote the impact of well-fitting ones. Here we propose a novel framework, named as collaborative-robust (co-robust) framework, that has the following merits:

1) The adaptive weight learning attempts to augment the impact of well-fitting samples, i.e., activated samples;
2) the robust function intends to alleviate disturbance of samples with large noises, i.e., inactivated samples.

1) Formulation: The proposed co-robust weight learning framework is formulated as

\[
\min_{w_i, \theta} \sum_{i=1}^n \frac{1}{1-w_i} f(x_i; \theta), \quad s.t. \sum_{i=1}^n w_i = 1, 0 \leq w_i < 1. \quad (1)
\]

where \( w_i \) is the corresponding weight, \( f(x_i; \theta) \) measures the error of the sample \( x_i \), \( \theta \) represents the parameters to learn. Formally, we can further formulate \( f(x_i; \theta) \) as

\[
f(x_i) = h(g(x_i; \theta)). \quad (2)
\]
\(g(x_i; \theta)\) is the a vector-output, or matrix-output function that is defined to calculated the loss of \(x_i\), \(h(\cdot)\) is the loss function. For instance, in the least-squares regression [18], \(g(x_i) = \theta^T x_i - y_i\) and \(h(\cdot) = \| \cdot \|_2^2\) where \(y_i\) denotes the target of \(x_i\). In practice, many machine learning methods aim to improve the performance by employing a robust loss function such as \(\ell_1\)-norm, \(\ell_2,1\)-norm, capped norm, etc. In the following discussion, we will use \(f(\cdot)\) rather than \(h(g(x_i; \theta))\) to keep notations uncluttered if unnecessary.

Here we give some insights about the above elegant framework. Let \(\tilde{w}_i = \frac{1}{1-w_i}\) be the direct weight of \(x_i\). The direct weight is a quantity that is positive correlated with weight. It implies the real importance of \(x_i\) to the final learner. From the intuitive aspect, samples with less reconstruction errors will be assigned to larger weights to minimize the global loss. The direct weight, positive correlated with \(w_i\), implies the real importance of \(x_i\) during training of the model. Unlike sparse models, samples whose weights are assigned to 0 are still retained in the model since \(\tilde{w}_i = \frac{1}{1-0} = 1\). In this case, samples with \(w_i = 0\), the negative impact will be alleviated by the robust loss \(h(\cdot)\). On the contrary, if \(x_i\) is assigned to a large weight (i.e., \(w_i \rightarrow 1\), then the direct weight will approach to infinity (i.e., \(\tilde{w}_i \rightarrow \infty\)) such that the samples will be important ones to the model. Therefore, our model is equivalent to augment the importance of samples with less reconstruction errors, and alleviate the negative effects of the highly polluted samples meanwhile.

From a mathematical perspective, we have the following formulation via Taylor’s theorem

\[
g(w) = \frac{1}{1 - w} = \sum_{j=0}^{\infty} g^{(j)}(0) k! w^j = \sum_{j=0}^{\infty} w^j, \quad (3)
\]

where \(g^{(j)}(0)\) is the \(k\)-order derivative. Consequently, Eq. (1) can be rewritten as

\[
\min_{w_i, \theta} \sum_{i=1}^{n} w_i^j f(x_i; \theta), \quad s.t. \sum_{i=1}^{n} w_i = 1, 0 \leq w_i < 1. \quad (4)
\]

Note that every item \(\sum_{i=1}^{n} w_i^j f(x_i; \theta)\) is the adaptive weighted learning with the exponential regularization which is widely applied in weight learning tasks. So the above objective defined in Eq. (1) could be viewed as a fusion of weighted learning with different \(j\). The sparsity of the exponential regularized weight learning is not guaranteed. Surprisingly, the solution of the co-robust framework can be proved as a sparse solution, which will be elaborated in the next part. Apart from the sparsity, the designed weight learning is parameter-free, which is another attractive property.

2) Optimization: Suppose \(\forall i, f_i > 0\). To learn \(w_i\) adaptively, we fix \(x_i\) and thus the Lagrangian of problem (1) is given as

\[
\mathcal{L} = \sum_{i=1}^{n} \frac{f_i}{1 - w_i} + \lambda (1 - \sum_{i=1}^{n} w_i) + \sum_{i=1}^{n} \gamma_i (-w_i), \quad (5)
\]

\begin{algorithm}
\textbf{Algorithm 1} Algorithm to optimize problem (1)
\textbf{Input:} Arbitrary initial weights \(\{w_i\}_{i=1}^{n}\) and variables to optimize \(\{x_i\}_{i=1}^{n}\) with proper initialization.
\textbf{repeat}
\hspace{1cm} Fix \(w_i\) and update \(x_i\).
\textbf{repeat}
\hspace{1.5cm} Find valid \(k\) via certain strategy.
\hspace{1cm} until \(k\) satisfies constraint (12).
\hspace{1cm} Update \(w_i\) by Eq. (11).
\textbf{until} convergence
\textbf{Output:} Weights \(\{w_i\}_{i=1}^{n}\) and variables \(\{x_i\}_{i=1}^{n}\).
\end{algorithm}

where \(\lambda, \gamma_i\) are Lagrangian multipliers and \(f_i\) represents \(f(x_i)\) for simplicity. The KKT conditions are listed as follows

\[
\begin{cases}
\frac{f_i}{(1-w_i)^2} - \lambda - \gamma_i = 0, \\
\gamma_i w_i = 0, \\
\gamma_i \geq 0, \\
\sum_{i=1}^{n} w_i = 1.
\end{cases} \quad (6)
\]

According to the third condition, we obtain

\[
\begin{align*}
w_i \neq 0, \gamma_i = 0 \Rightarrow w_i &= 1 - \sqrt{\frac{f_i}{\lambda}} > 0, \\
w_i = 0, \gamma_i \geq 0 \Rightarrow 1 - \sqrt{\frac{f_i}{\lambda}} \leq 0.
\end{align*} \quad (7)
\]

Accordingly, the formulation to update \(w_i\) could be written as

\[
w_i = (1 - \sqrt{\frac{f_i}{\lambda}})_{+}. \quad (8)
\]

where \((x)_+ = \max(0, x)\). Without loss of generality, suppose \(f_1 \leq f_2 \leq \cdots \leq f_n\), which means \(w_1 \geq w_2 \geq \cdots \geq w_n\). Without loss of generality, assume that

\[
\sqrt{f_k} < \sqrt{\lambda} \leq \sqrt{f_{k+1}},
\]

then \(w_i = 0\) for any \(i > k\). Combining with the forth formula of Eq. (6), we have

\[
\sum_{i=1}^{n} w_i = k + \sum_{i=1}^{k} \sqrt{\frac{f_i}{\lambda}} = 1 \Rightarrow \sqrt{\lambda} = \frac{1}{k-1} \sum_{i=1}^{k} \sqrt{f_i}. \quad (9)
\]

Substitute Eq. (10) into Eq. (8) and we obtain

\[
w_i = (1 - (k-1)\sqrt{f_i})_{+}. \quad (10)
\]

However, \(k\) is not a hyper-parameter, \(i.e.,\), not an arbitrary integer, since \(k\) has to satisfy the following constraint

\[
\sqrt{f_k} < \sqrt{\lambda} \leq \sqrt{f_{k+1}} \Rightarrow \frac{\sum_{i=1}^{k} \sqrt{f_i}}{\sqrt{f_{k+1}}} + 1 \leq k < \frac{\sum_{i=1}^{k} \sqrt{f_i}}{\sqrt{f_k}} + 1, \quad (12)
\]

which is derived from Eq. (9). Thus the optimal \(w_i\), whose \(i\)-th element is \(w_i\), only has \(k\) non-zero entries provided that \(f_i > 0\) for any \(i\). Since we assume that \(f_i > 0, \frac{\sum_{i=1}^{k} \sqrt{f_i}}{\sqrt{f_{k+1}}} > 0, \frac{1}{\sqrt{f_k}} > 0,\)
which indicates \( k \geq 2 \). Accordingly, our method will not get stuck in the common trivial solution, where only one weight is non-zero, of weight learning. Furthermore, the following theorem proves that \( k \) is unique in this case.

**Theorem 1.** The number of activated samples, \( k \), is unique if \( f_i > 0 \) holds where \( i = 1, 2, \ldots, n \).

**Remark 1.** Assume that \( \exists f_i, f_i = 0 \). Let \( D = \{ i \mid f_i = 0 \} \) and \( |D| \) represents the size of \( D \). When \( |D| > 1 \), it is obvious that \( k \) can be any integers lying in \([2, |D|]\) and \( w_i \in \emptyset \). In this case, we can simply set the corresponding weights \( w_i = \frac{1}{|D|} \). If \( |D| = 1 \), we can simply add a tiny value to \( f_i \) such that \( k \geq 2 \) holds and the trivial solution can be avoided.

In practical scenarios, loss of each sample is usually positive such that the condition of Theorem 1 holds. The concrete procedure of optimization based on \( f_i > 0 \) is summarized in Algorithm 1.

### B. Extension of Co-Robust Weight Learning

As shown in Eq. (14), the framework is equivalent to the fusion of weight learning under the exponential regularization. To increase (or decrease) the number of the adopted items, the proposed co-robust weight learning can be extended as

\[
\min_{w_i, \theta} \sum_{j=1}^{\infty} \sum_{i=1}^{n} w_i^{-\gamma} f(x_i; \theta), \quad \text{s.t.} \quad \sum_{i=1}^{n} w_i = 1, 0 \leq w_i < 1.
\]

(13)

where \( \gamma > 0 \) is a hyper-parameter to control the density of fused items. If \( \gamma < 1 \), the density of fused items increases; if \( \gamma > 1 \), the density of fused items becomes more sparse. To keep simplicity, the above formulation can be rewritten as

\[
\min_{w_i, \theta} \sum_{i=1}^{n} \frac{1}{1 - w_i} f(x_i; \theta), \quad \text{s.t.} \quad \sum_{i=1}^{n} w_i = 1, 0 \leq w_i < 1.
\]

(14)

Furthermore, the uniqueness of \( \{w_i\}_{i=1}^{n} \) still holds provided that \( f_i > 0 \) for any \( i \). The proof is similar with the one of Theorem 1. Unfortunately, the solution of Eq. (14) is not similar with the one in Eq. (14) any more. In this paper, we just pay more attentions on the special case when \( \gamma = 1 \), i.e., Eq. (14). Extensive experiments demonstrate that even the simple setting of \( \gamma \) can lead to impressive performance.

### C. An Example of Co-Robust Framework with A Scalable Robust Loss: \( \sigma \)-Loss

According to the above discussions, the weight learning of the proposed framework is parameter-free. Therefore, a robust loss with hyper-parameters to tune the robustness is recommended. The potential motivation to use a robust loss with hyper-parameters is to ensure the scalability. It should be emphasized that one can employ robust losses without extra parameters (e.g., \( \ell_1 \)-norm, \( \ell_2,1 \)-norm, etc.) if a parameter-free model is required.

1) **\( \sigma \)-Loss:** Here we develop a robust loss, \( \sigma \)-loss, that interpolates between \( \ell_2,1 \)-norm and squared Frobenius-norm and can be extended into point-wise form easily. The motivation of \( \sigma \)-loss is that the objective functions of many learning algorithms (e.g., least squares regression, \( k \)-means, PCA, etc.) can be viewed as the squared Frobenius norm which is too sensitive to outliers. Consequently, a quantity of algorithms, like variations of PCA [10], [15], have been developed to promote the robustness of model by applying the more robust loss, \( \ell_2,1 \)-norm. However, the drawback of \( \ell_2,1 \)-norm is the sensitivity to small perturbations compared with squared Frobenius-norm. Contrarily, the value of squared Frobenius-norm increases quadratically with the increase of input, but it is robust to small perturbations. Therefore, we attempt to design a loss that balances the two losses. The definition of \( \sigma \)-loss for matrix \( A \in \mathbb{R}^{m \times n} \) is given as follows

\[
\|A\|_{\sigma} = \sum_{i=1}^{m} \frac{(1+\sigma)\|a_i\|_2^2}{\|a_i\|_2 + \sigma},
\]

(15)

where \( \sigma \) controls robustness to different type outliers. To further understand the motivation of \( \sigma \)-loss, we give some insights as follows:

1. \( \|A\|_{\sigma} \) is non-negative, twice differential and convex such that it is appropriate as a loss function.
2. When \( \sigma \to 0 \), then \( \|A\|_{\sigma} \to \|A\|_{\ell_2} \).
3. When \( \sigma \to \infty \), then \( \|A\|_{\sigma} \to \|A\|_{\ell_1} \).

Fig. 2 demonstrates the effect of \( \sigma \)-loss function when \( A \in \mathbb{R}^{1 \times 1} \). Clearly, \( \sigma \)-loss function interpolates between \( \ell_2,1 \)-norm and squared Frobenius-norm.

**Remark 2.** \( \sigma \)-loss is not a valid norm since \( \forall k \geq 0, k\|A\|_{\sigma} \neq k\|A\|_{\sigma} \).

2) A Specific Example for Co-Robust Framework with Point-Wise \( \sigma \)-Loss: To apply \( \sigma \)-loss into the proposed co-robust framework, Eq. (14) is extended into the point-wise form. To simply the discussion, we define

\[
\|a\|_{\sigma} = \frac{(1+\sigma)\|a\|_2^2}{\|a\|_2 + \sigma}.
\]

(16)

Accordingly, let \( h(\cdot) = \| \cdot \|_{\sigma} \) and the **collaborative-robust framework with point-wise** \( \sigma \)-loss is formulated as

\[
\min_{w_i, \theta} \sum_{i=1}^{n} \frac{1}{1 - w_i} \|g(x_i; \theta)\|_{\sigma}.
\]

(17)

Therefore, \( \sigma \)-loss and weight learning promote the robustness collaboratively. A vital obstacle of applying a robust loss is...
Algorithm 2 Algorithm to Solve Problem (18)

Input: Data vector \( x \) and \( s \) whose \( i \)-th element is \( s_i \)

repeat

1. Calculate \( d_i = (1 + \sigma) \sum_i \frac{\|g(x_i; \theta)\|_2^2 + 2\sigma}{2(\|g(x_i; \theta)\|_2^2 + \sigma)^2} \);
2. Update \( \theta \) by solving Problem (22);

until convergence

Output: Solution \( \theta \).

A. Enhanced Principal Component Analysis

Consider the space spanned by the data points (i.e., the column space of matrix \( X \)), and then we attempt to find a \( c \)-rank subspace that approximates \( X \) as accurate as possible. The standard process of PCA and its variations is to centralize the data at first. With the preprocessed data, the objective can be formulated as

\[
\min_{\text{rank}(Z) = c} \|X - Z\|_F^2.
\]

Instead of preprocessing data mechanically, we intend to incorporate the pre-processing into model. In the following part, we will find that the co-robust framework generates a specific processing of data.

To understand our motivation, it is necessary to reconsider the aim of the centralization. If the data points scatter in an affine space, it is hard to find a well-fitting low-rank subspace. Fortunately, an affine space can be transformed into a linear space via a proper translation, which can be formulated as

\[
\min_{m, \text{rank}(Z) = c} \|X - m 1^T - Z\|_F^2.
\]

where \( m \) represents the translation vector. Specially, the classical PCA and its many variations are equivalent to fix \( m \) as \( \frac{1}{2} \sum_{i=1}^n x_i \).

To apply the proposed co-robust framework to PCA model, Eq (23) can be rewritten as the point-wise formulation,

\[
\min_{m, \text{rank}(Z) = c} \sum_{i=1}^n \|x_i - m 1^T - z_i\|_2^2.
\]

Combining with Eq. (17), the co-robust framework can be applied and the objective of Enhanced Principal Component Analysis (EPCA) is formulated as

\[
\min_{m, \alpha \in [1, \text{rank}(Z) = c]} \sum_{i=1}^n \frac{1}{1 - \alpha_i} \|x_i - m 1^T - z_i\|_\sigma^2.
\]

As \( \text{rank}(Z) = c \), \( R(Z) \) can be spanned by \( c \) orthonormal vectors \( \{w_1\}_{1}^{c} \) and \( z_i \) can be formulated as \( z = W v_i \) where \( W \in \mathbb{R}^{d \times c} \). Note that \( v_i \) is the coordinate of the \( i \)-th sample under \( \{w_1\}_{1}^{c} \). Accordingly, the original problem can be transformed into

\[
\min_{m, \alpha \in [1, \text{rank}(Z) = c]} \sum_{i=1}^n \frac{1}{1 - \alpha_i} \|x_i - m - W v_i\|_\sigma.
\]

An attractive property of EPCA is its rotational invariance, which is unavailable in a large proportion of PCA methods [2, 11, 17]. Here, we give the definition of the rotational invariance.

Definition 1. Let \( R \) be an orthogonal matrix, \( X \) be the dataset, and \( \mathcal{F}(X) \) denote a representation learner that returns a new representation of \( X \). If \( \mathcal{F}(X) = \mathcal{F}(RX) \) holds for any valid \( R \) and \( X \), then \( \mathcal{F}(\cdot) \) is rotational invariant.

Theorem 3. Given an orthogonal matrix \( R \) subjected to \( R^T R = RR^T = I \), EPCA on \( RX \) and \( X \) will obtain the same low-rank representation, \( V \). In other words, EPCA is rotational invariant.

IV. ENHANCED PRINCIPAL COMPONENT ANALYSIS UNDER THE CO-ROBUST FRAMEWORK

Based on the above discussions, we propose an unsupervised dimensionality reduction method, namely Enhance Principal Component Analysis (EPCA), via the co-robust framework proposed in Section III.
Algorithm 3: Algorithm to solve problem (27)

**Input:** The tradeoff parameter \(\sigma\), initialized \(\{\alpha_i\}_{i=1}^n\) and dataset \(X = [x_1, x_2, \cdots, x_n]\).

repeat

Update \(d_i\) by Eq. (28).
Update \(v_i\) by Eq. (31).
Update \(m\) by Eq. (34).
Update \(W\) by Eq. (38).

repeat

Find a new \(k\).

until \(k\) satisfies constraint (41).

Update \(\alpha_i\) by Eq. (40).

until convergence

**Output:** Orthonormal basis \(W\) of and translation vector \(m\).

B. Optimization of EPCA

The key part of EPCA is how to solve its objective efficiently, which is shown as follows:

**Evaluate** \(d_i\): According to Algorithm 2, the point-wise \(\sigma\)-loss can be optimized by solving a weighted \(\ell_2\)-norm problem. Specifically speaking, \(d_i\) is evaluated by

\[
d_i = (1 + \sigma) \frac{\|x_i - m - Wv_i\|_2 + 2\sigma}{2(\|x_i - m - Wv_i\|_2 + \sigma)^2},
\]

and problem (27) is converted into

\[
\min_{m, W} \sum_{i=1}^n \frac{d_i}{1 - \alpha_i} \|x_i - m - Wv_i\|_2^2.
\]

Optimize \(V\): Let \(J_i = \frac{d_i}{1 - \alpha_i} \|x_i - m - Wv_i\|_2^2\). Take the derivative regarding \(v_i\) and set it to 0,

\[
\frac{\partial J_i}{\partial v_i} = \frac{2d_i}{1 - \alpha_i} (W^TWv_i - W^T(x_i - m)) = 0.
\]

Therefore, the optimal \(v_i\) can be calculated as

\[
v_i = W^T(x_i - m).
\]

Optimize \(m\): Since \(v_i\) can be calculated directly and \(\alpha_i\) is viewed as constant temporarily, problem (29) can be rewritten as

\[
\min_{W} \sum_{i=1}^n \frac{d_i}{1 - \alpha_i} \|x_i - m - Wv_i\|_2^2.
\]

To keep notations uncluttered, let \(\eta_i = \frac{d_i}{1 - \alpha_i}\). Taking the derivative w.r.t. \(m\) and setting it to 0, we obtain

\[
\sum_{i=1}^n 2\eta_i(I - WW^T)^2m - 2\eta_i(I - WW^T)^2x_i = 0.
\]

Note that \((I - WW^T)\) is idempotent, i.e., \((I - WW^T)^2 = (I - WW^T)\). Let

\[
\mu = \frac{\sum_{i=1}^n \eta_i x_i}{\sum_{i=1}^n \eta_i},
\]

and we can derive that

\[
(I - WW^T)m = (I - WW^T)\mu.
\]

The following theorem gives the formulation of its solutions.

**Theorem 4.** The solution set of \(m\) can be formulated as

\[
\{m|m = \mu + W\beta\},
\]

where \(\beta \in \mathbb{R}^c\) denotes an arbitrary vector and will not affect the value of the objective function.

In practice, \(W\) is unavailable before problem (23) is completely solved. Since any \(\beta\) results in the same objective value, we can simply use the particular solution (i.e., \(\beta = 0\), \(m = \mu\)).

Optimize \(W\): Based on the closed-form solution of \(v_i\), Eq. (29) can be rewritten as

\[
\sum_{i=1}^n \frac{d_i}{1 - \alpha_i} \|x_i - m - Wv_i\|_2^2
\]

\[
= \sum_{i=1}^n \eta_i[tr(\bar{x}_i^T\bar{x}_i) - 2tr(\bar{x}_i^TWW^T\bar{x}_i)] + tr(W^TW\bar{x}_i^TWW^T)
\]

\[
= \sum_{i=1}^n \eta_i[tr(\bar{x}_i^T\bar{x}_i) - tr(x_i^TWW^T\bar{x}_i)].
\]

where \(\bar{x}_i = x_i - m\). As \(m\) and \(\alpha_i\) are fixed as constants, problem (29) is equivalent to

\[
\max_{W^TW=I} \sum_{i=1}^n tr(W^TQW),
\]

where

\[
Q = \sum_{i=1}^n \eta_i(\bar{x}_i\bar{x}_i^T) = \sum_{i=1}^n \eta_i(x_i - m)(x_i - m)^T.
\]

Consequently, the optimal \(W\) can be obtained by calculating \(c\) eigenvectors of \(c\) largest eigenvalues.

**Update** \(\alpha_i\): Substitute \(g_i = \|(I - WW^T)(x_i - m)\|_2\) into Eq. (11) and we have

\[
\alpha_i = (1 - \frac{k - 1}{\sqrt{g_i}}),
\]

where \(k\) has to satisfy

\[
\sum_{i=1}^{k} \sqrt{g_i} + 1 \leq k < \sum_{i=1}^{k+1} \sqrt{g_i} + 1.
\]

The optimization procedure of problem (27) is summarized in Algorithm 3. In our implementation, the strategy to find \(k\) is to search it from 1 to \(n\) simply.

V. PROOFS OF THEOREMS

A. Proof of Theorem 2

**Proof.** Without loss of generality, suppose that \(0 \leq f_1 \leq f_2 \leq \cdots \leq f_n\). Let \(J = \sum_{i=1}^n \frac{f_i}{1 - w_i}\). As \(\sum w_i = 1\), we have

\[
J = \frac{f_1}{\sum_{i=2}^{n} w_i} + \sum_{i=2}^{n} \frac{f_i}{1 - w_i}.
\]
Proof.
For any following inequality holds:

\[ \frac{\partial^2 \mathcal{J}}{\partial w_i \partial w_j} = \begin{cases} \frac{2f_i}{\sum_{i=2}^{n} w_i^3}, & i = j, \\ \frac{2f_i}{\sum_{i=2}^{n} w_i^3}, & i \neq j. \end{cases} \]

and furthermore, the Hessian matrix is given as

\[ H_{ij} = \frac{\partial^2 \mathcal{J}}{\partial w_i \partial w_j} \]

Accordingly, the Hessian matrix \( H \) can be represented as

\[ H = \frac{2f_i}{\sum_{i=2}^{n} w_i^3} \mathbf{I} + \text{diag}(\frac{2f_i}{\sum_{i=2}^{n} w_i^3}). \]

It is not hard to realize that \( \forall x \in \mathbb{R}^n \), we have \( x^T H x > 0 \). Clearly, the problem \( (1) \) is convex and the optimal \( w \) is unique. Note that the function

\[ g_i(\lambda) = w_i = (1 - \sqrt{f_i/\lambda})_+, \]

will not decrease with \( \lambda \) goes larger. As \( k \) is at least 2 and \( \sum_{i=1}^{n} w_i = 1 \), for any valid \( \{ w_i \}_{i=1}^{n} \) there is only one \( \lambda \). Hence, \( \lambda \) is unique as well. Utilize the fact

\[ \sqrt{f_k} < \sqrt{\lambda} \leq \sqrt{f_{k+1}}. \]

which completes the proof. \( \square \)

With the help of the above lemma, we give the proof of Theorem \( \square \) as follows.

**Proof of Theorem \( \square \)**
Assume that we fix \( d_i \) and calculate \( \tilde{\theta} \) after step 2 of Algorithm \( \square \) then we have:

\[ \sum_i d_i s_i \| g(x_i; \tilde{\theta}) \|_2^2 \leq \sum_i d_i s_i \| g(x_i; \theta) \|_2^2. \]  

(49)

Due to the definition of \( d_i \) according to Eq. (28), we have:

\[ (1 + \sigma) \sum_i \| g(x_i; \theta) \|_2 + 2\sigma \| g(x_i; \tilde{\theta}) \|_2 \leq (1 + \sigma) \sum_i \| g(x_i; \theta) \|_2 + 2\sigma \| g(x_i; \theta) \|_2. \]

(50)

And based on Lemma \( \square \) we have following inequality:

\[ \frac{\| g(x_i; \tilde{\theta}) \|_2}{\| g(x_i; \theta) \|_2} \leq \frac{\| g(x_i; \theta) \|_2 + 2\sigma}{\| g(x_i; \tilde{\theta}) \|_2 + 2\sigma}. \]

(51)

Through multiplying \((1 + \sigma)s_i \) on both sides of (51) and adding to (50), we thus have

\[ \sum_i s_i (1 + \sigma) \| g(x_i; \tilde{\theta}) \|_2^2 \leq \sum_i s_i \frac{(1 + \sigma) \| g(x_i; \theta) \|_2^2}{\| g(x_i; \theta) \|_2 + 2\sigma}. \]

(52)

Therefore, Algorithm \( \square \) decreases Eq. (18) monotonically in each iteration. \( \square \)

**C. Proof of Theorem \( \square \)**

**Proof.** If \( R \in \mathbb{R}^{d \times d} \) is an orthogonal matrix, then for any vector \( \beta \in \mathbb{R}^d \),

\[ \| R \beta \|_2^2 = \text{tr}(x^T R^T R x) = \text{tr}(x^T x) = \| x \|_2^2. \]

(53)

Accordingly, we have

\[ \sum_{i=1}^{n} \frac{1}{1 - \alpha_i} \| x_i - m - W v_i \|_2^2 \\ = \sum_{i=1}^{n} \frac{1}{1 - \alpha_i} \cdot \frac{(1 + \sigma) \| x_i - m - W v_i \|_2^2}{\| x_i - m - W v_i \|_2 + 2\sigma} \\ \leq \sum_{i=1}^{n} \frac{1}{1 - \alpha_i} \| R x_i - R m - R W v_i \|_2^2. \]

(54)

Therefore, after a rotational operation for \( X \), the optimal low-dimensional representation, \( V \), of the original features is also the optimal solution of the transformed features. \( \square \)
D. Proof of Theorem 4

Proof. Clearly, \( \mu \) is a particular solution for problem (23). Let \( W^\perp \in \mathbb{R}^{d \times (d-c)} \) represent orthogonal complement of \( \{w_i\}_{i=1} \) and \( W_+ = [W; W^\perp] \). Therefore, we have

\[
WW^T = W_+ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W_+^T.
\]

(55)

Accordingly, the eigenvalues of \( (I - WW^T) \) are all 0 and 1. More specifically, \( c \) of them are 0, which means the rank of the nullspace of \( (I - WW^T) \). Note that

\[
(I - WW^T)W = 0.
\]

(56)

Therefore, the feasible solution of \( m \) in Eq. (55) can be denoted by \( \mu + W\beta \).

VI. EXPERIMENT

In this section, the performance of our algorithms from both reconstruction errors and clustering accuracy obtained from reconstructed data is reported.

A. Experimental Setup

The performance of proposed model is evaluated on five datasets including: JAFFE [19], YALE [20], ORL [21], COIL20 [22] and UMIST [23]. All datasets are utilized to measure reconstruction errors and test clustering accuracy. The concrete information of these datasets are summarized in Table I.

| Dataset   | Size  | Dimensionality | Class |
|-----------|-------|----------------|-------|
| JAFFE     | 213   | 1024           | 10    |
| YALE      | 165   | 1024           | 15    |
| ORL       | 400   | 1024           | 40    |
| COIL20    | 1440  | 1024           | 20    |
| UMIST     | 575   | 1024           | 20    |

To demonstrate the superiority of our model, our model is compared with 8 high related approaches including classical PCA [1], robust PCA using \( \ell_1 \)-norm (L1PCA) [17], \( \ell_2 \)-norm (LpPCA) [17], robust PCA using \( \ell_2 \)-norm with non-greedy strategy (L2PCA-NG) [2], robust PCA using \( \ell_{2,1} \)-norm (R1PCA) [10], robust and sparse PCA (RSPCA) [12], robust PCA with optimal mean (PCA-OM) [13], and our conference work (PCA-AN) [16]. To ensure the fairness, the parameters of all baseline algorithms are determined via the similar approaches recorded in corresponding papers. In particular, we empirically fix \( p = 1.2 \), which can be viewed as a trade-off loss between \( \ell_1 \)-norm and \( \ell_2 \)-norm in Lp-PCA.

To test the ability to reconstruct data of various algorithms, all datasets are polluted by the following strategy: (1) sample 20% data points randomly from all datasets; (2) choose 20% features randomly and then reset them by random values. All algorithms use the occluded data as input to train projection matrix \( W \) and mean vector \( m \). All codes are implemented by Matlab R2019a.

B. Reconstruction Errors

The reconstruction errors are calculated by

\[
\varepsilon = \| X - m1^T - WW^T(X^{\text{occ}} - m1^T) \|^2_F,
\]

(57)

where \( W \) and \( m \) are learned from occluded data, \( X \) is the unpolluted data, and \( X^{\text{occ}} \) is the occluded data. Note that in practical applications, the data to process can be viewed as \( X^{\text{occ}} \) while \( X \) is regarded as the underlying and unknown data without noises.

Table II illustrates results of different \( c \) on all datasets. From Table II we conclude that:

1. EPCA obtains impressive results for different \( c \) on almost all datasets as we expect. PCA-AN also obtains good results on several datasets, which indicates the significance of weight learning for PCA.

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TABLE III
CLUSTERING ACCURACY (%)

| Dataset | c   | PCA | L1PCA | LPCA-NG | R1PCA | RSPCA | L_pPCA | PCA-OM | PCA-AN | EPCA |
|---------|-----|-----|-------|---------|-------|-------|--------|--------|--------|------|
| JAFFE   | 10  | 86.85 | 83.57 | 77.72   | 79.81 | 69.01 | 86.85  | 83.57  | 80.28  | 90.14 |
|         | 30  | 94.37 | 86.38 | 74.23   | 90.14 | 81.22 | 92.02  | 89.67  | 83.57  | 97.65 |
|         | 50  | 88.73 | 83.57 | 73.57   | 90.14 | 73.24 | 77.46  | 91.08  | 94.37  | 96.24 |
| YALE    | 10  | 43.03 | 41.82 | 38.65   | 42.42 | 40.61 | 40.00  | 37.58  | 40.61  | 46.06 |
|         | 30  | 38.79 | 41.21 | 38.51   | 44.85 | 40.00 | 43.03  | 36.36  | 38.79  | 46.67 |
|         | 50  | 33.94 | 38.79 | 38.92   | 43.64 | 38.79 | 41.21  | 38.79  | 38.79  | 44.24 |
| ORL     | 10  | 51.50 | 51.00 | 48.88   | 52.75 | 43.25 | 50.50  | 50.00  | 52.50  | 58.75 |
|         | 30  | 53.75 | 55.75 | 53.09   | 49.75 | 49.75 | 55.50  | 59.25  | 65.50  | 62.00 |
|         | 50  | 47.25 | 53.25 | 53.07   | 48.75 | 55.00 | 54.00  | 57.00  | 57.00  | 57.25 |
| COIL20  | 10  | 62.08 | 63.89 | 61.88   | 59.65 | 62.78 | 60.97  | 60.90  | 61.94  | 64.38 |
|         | 30  | 63.61 | 59.51 | 61.32   | 58.13 | 63.26 | 62.15  | 63.68  | 65.14  | 66.39 |
|         | 50  | 65.56 | 60.62 | 60.98   | 63.75 | 61.88 | 60.83  | 65.28  | 63.96  | 66.39 |
| UMIST   | 10  | 28.00 | 29.74 | 30.85   | 28.52 | 30.43 | 28.00  | 28.70  | 29.22  | 31.13 |
|         | 30  | 31.83 | 29.57 | 30.15   | 30.26 | 30.43 | 31.13  | 30.26  | 27.89  | 33.39 |
|         | 50  | 29.91 | 31.48 | 30.06   | 30.26 | 30.26 | 29.74  | 29.74  | 22.17  | 32.17 |

Fig. 3. Reconstruction errors on 4 different datasets when the rank of reconstruction matrix is fixed as 10. Then x-axis represents $\gamma$ where $\sigma = 2^\gamma$.

better performance due to utilization of robust loss and augmentation to weights of better-fitting samples. For instance, the error given by EPCA is $10^6$ less than the error of PCA-OM on JAFFE when $c = 50$.

3. L_pPCA performs better than L1PCA owing to its rotational invariance and appropriate $p$ but poorer than EPCA owing to the lack of weight learning and non-optimal mean using.

C. Clustering Accuracy

To show the effectiveness of EPCA, the clustering accuracy (ACC) is tested on reconstructed data. Table III reports the clustering results on all datasets. Note that we perform $k$-means 100 times and record the mean on each dataset. Since the datasets are occluded severely, the superiority of EPCA is shown impressively. It should be pointed out that PCA-AN (our conference work) performs well on reconstruction errors but fails on the clustering task. The main reason may be that the sparse weight learning leads to the loss of structure information.

D. Parameter Sensitivity and Activated Samples

In our experiments, the parameter $\sigma$ is determined by the following strategy: (1) search in a wide enough range $[2^{-20}, 2^{-19}, \ldots, 2^{20}]$; (2) fine tune in a narrow scope. Fig. 2 illustrates the influence of different $\sigma$ when $c = 10$.

Fig. 4. The relationship between the number of activated samples $k$ and the rank of reconstructed matrix $c$.

To show the impact of $\sigma$ more intuitively, we only show the coarse search instead of fine tune. Note that the x-axis represents $\gamma$ instead of $\sigma$ where $\sigma = 2^\gamma$. From Fig. 3 we can conclude that even when the optimal $\sigma$ is tiny, i.e., equivalent to $\ell_{2,1}$-norm, EPCA performs better than PCA-OM thanks to the weights $\{\alpha_i\}_{i=1}^{n}$. Additionally, Fig. 4 illustrates the relationship between the number of activated samples (denoted by $k$) and the rank of reconstructed matrix (denoted by $c$).

VII. Conclusion

In this paper, based on our previous work, we first propose a general collaborative-robust framework of weight learning which integrates adaptive weight learning and robust functions
utilization via a non-trivial approach. Furthermore, only k samples are activated, i.e., the corresponding weights are non-zero. The activated samples are augmented by weight learning while the effects of inactivated ones are alleviated by the employed robust loss. Then an enhanced principal component analysis (EPCA) which extends from the proposed framework and applies the adaptive loss function is proposed. Moreover, the EPCA is rotational invariant. Extensive experiments report the superiority of our model from two aspects including reconstruction errors and clustering accuracy. In particular, the experiments on clustering exhibit the improvement of EPCA compared with our conference paper.

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