Two Massive and One Massless $Sp(4)$ Monopoles

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Abstract

Starting from Nahm’s equations, we explore BPS magnetic monopoles in the Yang-Mills Higgs theory of gauge group $Sp(4)$ which is broken to $SU(2) \times U(1)$. A family of BPS field configurations with purely Abelian magnetic charge describe two identical massive monopoles and one massless monopole. We construct the field configurations with axial symmetry by employing the ADHMN construction, and find the explicit expression of the metrics for the 12-dimensional moduli space of Nahm data and its submanifolds.

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I. INTRODUCTION

In this paper we consider the Yang-Mills Higgs theory whose gauge symmetry $Sp(4)$ is broken to $SU(2) \times U(1)$. We investigate a family of purely Abelian configurations which describe two identical massive and one massless monopoles. We approach the problem by solving Nahm’s equations under proper boundary and compatibility conditions. By using the Atiyah-Hitchin-Drinfeld-Mannin-Nahm (ADHMN) construction [1,2], we construct the field configurations in spherically and axially symmetric cases. We then calculate the metrics of the 12-dimensional moduli space $M_{12}$ of Nahm data and its submanifolds. Generally it is expected that the moduli space of Nahm data is isometric to the moduli space of the corresponding monopole configurations. We examine the metric of the moduli space in detail and show that it behaves consistently with what is expected from the dynamics of monopoles.

Recently magnetic monopoles have again become a focus of attention as they play a crucial role in study of electromagnetic duality in the supersymmetric Yang-Mills theories. The relevant magnetic monopole solutions are of the Bogomol’nyi-Prasad-Sommerfield (BPS) type such that the static interaction between magnetic monopoles vanishes [3]. The gauge inequivalent field configurations of the BPS monopole solutions are characterized by the moduli parameters associated with the zero modes of the solutions. The metric of the moduli space determines the low energy dynamics of monopoles [4]. The electromagnetic duality has been explored by studying quantum mechanics on the moduli space of the BPS monopoles.

When the gauge group is not maximally broken so that there is an unbroken non-Abelian gauge symmetry, the moduli space dynamics becomes more subtle because of the global color problem [5]. Nevertheless it has been known that the moduli space is well defined when the total magnetic charge is purely Abelian [6]. Recently some of such moduli spaces have been studied by starting from the maximal symmetry breaking case and restoring the broken symmetry partially [7]. From this point of view some magnetic monopoles become massless,
forming a non-Abelian cloud surrounding remaining massive monopoles. The global part of unbroken gauge symmetry becomes the isometry of the moduli space. The meaning of the moduli space coordinates of massless monopoles changes from their positions and phases to the gauge invariant structure parameters for the cloud and gauge orbit parameters. With a inequivalent symmetry breaking $Sp(4) \rightarrow SU(2) \times U(1)$, an Abelian combination is made of one massive and one massless monopoles. This simple case where the field configuration and the moduli space metric are completely known was studied in detail to learn about the non-Abelian cloud [7,8].

The next nontrivial purely Abelian configurations beyond this simple model are made of two massive and one massless monopoles. Two massive monopole can be distinguished as in the example where $SU(4) \rightarrow U(1) \times SU(2) \times U(1)$. In that case, the so-called Taubian-Calabi metric for the moduli space [7,8,10] is be obtained from the massless limit of that of the maximally broken case [11]. Two massive monopoles are identical in the cases where $SU(3), Sp(4), G_2 \rightarrow SU(2) \times U(1)$. (See Table I and II of Ref. [7].) Sometime ago the moduli space in the case $SU(3) \rightarrow SU(2) \times U(1)$ has been found by Dancer by exploring the moduli space of Nahm data [12,13].

Our approach is similar to Dancer’s. We use the embedding procedure to construct $Sp(4)$ configurations from $SU(4)$ configurations. Some of the field configurations are simpler than Dancer’s. Our spherical symmetric solution is just an embedding of the $SU(2)$ solution. A class of our axially symmetric solutions can be obtained from a linear superposition of configurations for two noninteracting monopoles. Our work provides a further illustration of the role of massless monopoles.

Another motivation for studying the moduli space of configurations involving massless monopoles is that it may lead us to some new insight about mesons and baryons in quenched QCD. Even in quenched QCD, nondynamical external quarks are expected to be confined and form mesons and baryons. Suppose that quenched QCD has been supersymmetrized to $N = 4$ so that there is no confinement. (Here we imagine that all supersymmetric partners are very massive initially and then become light.) If the coupling constant is still
strong, the resulting configurations of mesons and baryons cannot be described by Coulomb potentials as the nonlinear gauge interaction is not negligible. The non-Abelian gauge field should somehow form a cloud around external quarks, making the whole configuration to be a gauge singlet, because of the continuity of the configuration with respect to the mass parameter. The shape of this cloud may remember confinement strings which connected the quarks. This can be regarded as the limit where confining string becomes tensionless.

If the electromagnetic duality holds even when the unbroken gauge symmetry is partially non-Abelian [14], mesons and baryons can have their magnetic dual, which are made of massive and massless monopoles. Indeed massive monopoles play the role of external quarks and massless monopoles play that of non-Abelian cloud. Thus Abelian configurations made of two massive and one massless monopoles can be regarded as dual mesons. More interestingly, the moduli space of three massive and three massless monopoles in the $SU(4) \rightarrow SU(3) \times U(1)$ can be regarded as a magnetic dual of baryons [15]. The structure of the dual baryons may hint a shape of the confinement strings connecting three quarks.

The plan of this work is as follows. In Sec. II, we review the method to find Nahm data for the classical group. In Sec. III, we study the symmetry breaking pattern $Sp(4) \rightarrow SU(2) \times U(1)$, and solve Nahm’s equations with relevant boundary conditions. In Sec. IV, we use the ADHMN method to construct the Higgs field configurations in spherically and axially symmetric cases. This leads to a general understanding of the parameter space in terms of the size of non-Abelian cloud and the distance between massive monopoles. In Sec. V, we find the explicit metrics of the moduli space and its submanifolds. In Sec. VI, we conclude with some remarks.

II. NAHM DATA

The Bogomol’nyi equations satisfied by BPS monopoles can be written as self-dual Yang-Mills equations

$$F_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F_{\rho \sigma}$$

(1)
in $R^4$ with coordinates $x_1, x_2, x_3, x_4$. All the fields of BPS monopoles here depend only on $x_1, x_2, x_3$. Instead if everything depends only on the complementary variable $x_4 = t$, then Eq. (1) leads to the so-called Nahm’s equations,

$$\frac{dA_i}{dt} + [A_4, A_i] = \frac{1}{2} \epsilon_{ijk} [A_j, A_k],$$

where $i, j, k = 1, 2, 3$. The solutions of Nahm’s equations satisfying certain boundary condition that will be stated below are called Nahm data. We can always perform a gauge transformation to eliminate $A_4$, so sometimes $A_4$ is not included in Nahm’s equations. Nahm’s equations are much easier to solve than the original self-dual Yang-Mills equations, since they are ordinary differential equations. The relationship between Bogomol’nyi equations (depend on three variables) and Nahm’s equations has been thoroughly investigated especially in $SU(2)$ gauge group case [11]. There is a kind of duality between $d$ and $4 - d$ dimensional self-dual theories [17]. It is also believed in general that the moduli spaces of Nahm data and BPS monopoles are isometric to each other, which has been proven in the $SU(2)$ case [19]. The idea is that Nahm’s equations are regarded as an infinite dimensional moment map and that the hyperkähler quotient [18] of the infinite dimensional flat space will lead to the natural hyperkähler metric for the moduli space of Nahm data [19,20].

The original Nahm’s method of $SU(2)$ monopoles has been generalized into all types of classical groups [21]. Let’s start with the $SU(N)$ case since all other groups can be treated by embedding them into $SU(N)$. Assuming the asymptotic Higgs field is $\phi_\infty = \text{diag}(\mu_1, \ldots, \mu_N)$ with $\mu_1 < \cdots < \mu_N$ along a given direction, then the Nahm data for multi-monopoles carrying charge $(m_1, \ldots, m_{N-1})$ are defined as $N - 1$ triples $(l^iT_1, l^iT_2, l^iT_3)$ ($l = 1, \ldots, N - 1$) satisfying:

1. For each $l$, $l^iT_i$ ($i = 1, 2, 3$) are analytic $u(m_l)$-valued functions satisfying Nahm’s equations in interval $(\mu_l, \mu_{l+1})$, $l = 1, \cdots, N - 1$. 

2. The boundary conditions relating the Nahm data in two adjoint intervals are the following:
(a) If $m_l > m_{l-1}$, then there exist non-singular limit, $\lim_{t \to \mu_l}^T T_i = \mu_l^{-1} S_i$ and the structure of $^T T_i$ near $t = \mu_l$ is

$$
\lim_{t \to \mu_l}^T T_i = \begin{pmatrix}
\mu_l^{-1} S_i & \ast \\
\ast & \frac{R_i}{t - \mu_l}
\end{pmatrix},
$$

where $^T R_i$ form an $(m_l - m_{l-1})$-dimensional irreducible representation of $su(2)$ (unless $m_l - m_{l-1} = 1$ in which case $^T R_i/(t - \mu_l)$ has to be replaced by a non-singular expression), and "\ast" refers to the elements that are not interested in this paper.

(b) If $m_l < m_{l-1}$, the roles of $(\mu_{l-1}, \mu_l)$ and $(\mu_l, \mu_{l+1})$ are reversed.

(c) If $m_l = m_{l-1}$, the condition is more complicated but fortunately we are not going to confront this situation in this paper.

The way to embed the cases of $SO(N)$ and $Sp(N)$ into $SU(N)$ group is described in Table 1. These embedding procedures are obtained by constraining the $SU(N)$ generators further. The generators $T$ of $Sp(N)$ satisfy the condition $T^T J + JT = 0$ such that $JJ^* = -I$. The generators $T$ of $SO(N)$ satisfy the condition $T^T K + KT = 0$ such that $KK^* = I$. The explicit forms of $J, K$ can be deduced from Table 1.

| $G$   | $G$-charge | $\phi_{\infty}$ in $SU(N)$ | $SU(N)$-charge                      |
|-------|------------|-----------------------------|-----------------------------------|
| $Sp(N)$ | $\rho_1, \ldots, \rho_n$ | $\mu_l = -\mu_{2n+1-l}$ | $m_l = m_{2n-l} = \rho_l$ |
| $N = 2n$ |             | $l = 1, \ldots, n$        | $l = 1, \ldots, n$            |
| $SO(N)$ | $\rho_1, \ldots, \rho_{n-2}$ | $\mu_l = -\mu_{2n+1-l}$ | $m_l = m_{2n-l} = \rho_l$ |
| $N = 2n$ | $\rho_+, \rho_-$ | $l = 1, \ldots, n$        | $l = 1, \ldots, n-2$          |
|         |             |                             | $m_{n-1} = m_{n+1} = \rho_+ + \rho_-$ |
|         |             |                             | $m_n = 2\rho_+$               |
| $SO(N)$ | $\rho_1, \ldots, \rho_n$ | $\mu_l = -\mu_{2n+2-l}$ | $m_l = m_{2n+1-l} = \rho_l$ |
| $N = 2n+1$ |             | $l = 1, \ldots, n+1$      | $l = 1, \ldots, n-1$          |
|         |             |                             | $m_n = m_{n+1} = 2\rho_n$     |
Table 1: The embedding of $Sp(N)$, $SO(N)$ in $SU(N)$.

These embedding procedures enable us to get the $SO(N)$ and $Sp(N)$ Nahm data from the $SU(N)$ data with asymptotic Higgs field $\phi_\infty = \text{diag}(\mu_1, \ldots, \mu_N)$ and the charge $\{m_l\}$. What is new is that we now have one more set of conditions connecting the Nahm data between different intervals:

3. There exist matrices $^lC$ ($l = 1, \ldots, N - 1$) satisfying

$$N^{-l}T_i(-t)^T = (^lC)^T t_i(t) (^lC^{-1}),$$  \hspace{1cm} (4)

and compatibility conditions:

(a) $N^{-l}C = ^lC^T$, \hspace{1cm} for $Sp(N)$

(b) $N^{-l}C = -^lC^T$, \hspace{1cm} for $SO(N)$

These compatibility conditions reflect the fact that we are identifying certain $SU(N)$ monopoles to get $SO(N)$ and $Sp(N)$ monopoles.

In the above discussions we have assumed $\mu_1 < \cdots < \mu_n$, which physically means the gauge symmetry is maximally broken. We can also consider the cases with non-Abelian unbroken symmetry so that some $\mu_i$’s are equal, geometrically this is the case when some of the intervals shrink to zero length. The monopole mass is proportional to the size of the corresponding interval and so the shrunken intervals are corresponding to massless monopoles. All the procedures described above remain unchanged even in this case.

III. NAHM DATA IN THE $SP(4)$ CASE

The model we consider is the $Sp(4)$ Yang-Mills theory with a single Higgs field in the adjoint representation and no potential. The vacuum expectation value of the Higgs field is nonzero and the gauge symmetry is spontaneously broken to $SU(2) \times U(1)$. The roots
and coroots of the $Sp(4) = SO(5)$ group is shown in Fig. 1. Note that in our convention $\alpha^* = \alpha/|\alpha|^2 = \alpha$.

![Root Diagram of $Sp(4)$](image)

**Figure 1:** The root diagram of $Sp(4)$

In this paper we consider the symmetry breaking with $\langle \Phi \rangle = h \cdot H$. The simple roots we choose for convenience is $\beta, \alpha$ rather than $\delta, -\alpha$. For any root $\alpha$, there is a corresponding $SU(2)$ subalgebra,

$$
t^1(\alpha) = \frac{1}{\sqrt{2\alpha^2}}(E_\alpha + E_{-\alpha}),
$$

$$
t^2(\alpha) = \frac{-i}{\sqrt{2\alpha^2}}(E_\alpha - E_{-\alpha}),
$$

$$
t^3(\alpha) = \alpha^* \cdot H.
$$

(5)

Using this $SU(2)$ algebra, we can embed the $SU(2)$ single monopole solution along any root. Thus there is a spherically symmetric monopole configuration for any root $\alpha$ such that $\alpha \cdot h \neq 0$. Since $\beta \cdot h > 0$, the monopole with magnetic charge $\beta^*$ is massive. (Here we are dropping the coupling constant $4\pi/e$.) On the other hand $\alpha^* \cdot h = 0$ and so there is no monopole solution corresponding to the root $\alpha$. As argued in the introduction, the zero mode counting can be done consistently only for purely Abelian configurations. In our case the simplest case has the magnetic charge

$$
\gamma^* = 2\beta^* + \alpha^*,
$$

(6)
so that $\gamma^* \cdot \alpha = 0$. The moduli space of this configuration is 12-dimensional and denoted by $M^{12}$. As discussed in Ref [7], we imagine the $h$ as a limit where $h \cdot \alpha$ is positive but becomes infinitesimal. We can regard $\alpha^*$ monopoles as massless, and so the $\gamma^*$ monopole can be thought as a composite of two identical massive $\beta^*$ monopoles and one massless $\alpha^*$ monopole. Here we can see the internal unbroken gauge group should be $SO(3)_g$ rather than $SU(2)$, because all the generators of $Sp(4)$ transforms as vector or singlet representations under the unbroken generators $t(\alpha)$.

If we have chosen the Higgs expectation value as $h'$, the unbroken $SU(2)$ would be associated with $\beta$. The Abelian configuration could have the magnetic charge $\delta^* = \alpha^* + \beta^*$ such that $\delta^* \cdot \beta = 0$. This configuration can be interpreted as a composite of one massive $\alpha^*$ and one massless $\beta^*$ monopoles. The BPS field configuration and 8-dimensional moduli space of this magnetic charge are known explicitly to be flat $R^4$. This is the model which lead to many insights about non-Abelian cloud [4].

As discussed in Sec. II, Nahm data for $Sp(4)$ can be studied by embedding $Sp(4)$ in $SU(4)$. Thus the Higgs field can be written as $4 \times 4$ traceless Hermitian matrix. As shown in Table 1, the Higgs expectation value can be chosen to be $\langle \Phi \rangle = \text{diag}(-\mu_1, -\mu_2, \mu_2, \mu_1)$ with $\mu_1 \geq \mu_2 \geq 0$. Any generator $T$ of the $Sp(4)$ subgroup should be traceless antihermitian and satisfy

$$TJ + JT^T = 0,$$  \hspace{1cm} (7)

where the $Sp(4)$ invariant tensor $J$ is chosen to be

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (8)

This defines the $Sp(4)$ embedding in $SU(4)$ uniquely, which is also consistent with Table 1. A consistent choice of the Cartan subgroup of $Sp(4)$ is $H_1 = \text{diag}(-1, 1, -1, 1)/2$ and $H_2 = \text{diag}(-1, -1, 1, 1)/2$. The two inequivalent symmetry breaking patterns for $Sp(4) \rightarrow SU(2) \times
$U(1)$ in Fig. 1 correspond to $h \cdot H = \text{diag}(-1, -1, 1, 1)$ and $h' \cdot H = (-1, 0, 0, 1) = H_1 + H_2$. Thus our case with $\mu_1 = \mu_2 = 0$ corresponds to the case where $SU(4) \to SU(2) \times U(1) \times SU(2)$.

From Table 1 in Sec. II, we read that our configuration (3) in $Sp(4)$ has the $SU(4)$ magnetic charge $(1, 2, 1)$, that is, two identical massive monopoles and two distinct massless monopoles. This is exactly the configuration considered by Houghton [22], whose focus was on its hyperkähler quotient space. If we have chosen the expectation value $h'$, the simplest Abelian configurations have the magnetic charge $(1, 1, 1)$ in $SU(4)$, that is, two distinct massive monopoles and one massless monopole, whose moduli space metric has been found as the Taubian-Calabi metric [7,9,10].

According to the previous section, Nahm data $T_\mu(t)$ defined on the interval $[-1, 1]$ are anti-Hermitian two-by-two matrices and satisfy Nahm’s equations

$$\frac{dT_i}{dt} + [T_4, T_i] = \frac{1}{2} \varepsilon_{ijk} [T_j, T_k],$$

and the compatibility condition

$$T_\mu(-t)^T = CT_\mu(t)C^{-1}$$

with a symmetric matrix $C$. The Nahm data should be analytic at the end points $t = \pm 1$. The boundary and compatibility conditions (3) and (4) satisfied by the above Nahm data become

$$(T_\mu(-1))_{11} = (T_\mu(1))_{22}$$

A detailed understanding of the boundary condition will be needed in the case where the massless monopole becomes massive.

The space of Nahm data has the following symmetries:

1. Local gauge transformations $G = \{ g(t) \in U(2) \}$ whose transformations are

$$T_4 \to gT_4g^{-1} - \frac{dg}{dt}g^{-1},$$

$$T_i \to gT_ig^{-1}.$$


They should be consistent with the conditions (10) and (11). Its subgroup is $G_0 = \{ g \in G : g(-1) = g(1) = 1 \}$.

2. Spatial translation group $R^3$ with three parameters $\lambda_i$:

$$T_4 \to T_4$$

$$T_j \to T_j - i\lambda_j I.$$  \hspace{1cm} (13)

3. Spatial rotation group $Spin(3) = \{ a_{ij} \in SO(3) \}$:

$$T_i \to \sum_j a_{ij} T_j.$$  \hspace{1cm} (14)

Notice that Eq. (14) is a pure rotation as there is no residues to be fixed at $t = \pm 1$. (This indicates that the rotational group is $SO(3)$ rather than $SU(2)$.)

To solve Nahm’s equations together with the compatibility condition, we use the spatial translation to make $T_\mu$ traceless. This traceless Nahm data is called centered and describes the monopole configuration in the center of mass frame. We can also choose the gauge $T_4 = 0$. Furthermore we use the spatial rotation to set the $t$-independent $\text{tr} (T_1 T_2)$, $\text{tr} (T_1 T_3)$ and $\text{tr} (T_2 T_3)$ to be zero. After a spatial rotation, we get that for each $j = 1, 2, 3$;

$$T_j = \frac{1}{2} f_j \tau_j,$$ \hspace{1cm} (15)

where quaternions $\tau_j$ are chosen to be

$$\tau_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$  \hspace{1cm} (16)

satisfying $\tau_1 \tau_2 = \tau_3$, etc. Then, Nahm’s equations become the well-known Euler top equations:

$$\dot{f}_1 = f_2 f_3,$$  \hspace{1cm} (17)

$$\dot{f}_2 = f_3 f_1,$$  \hspace{1cm} (17)

$$\dot{f}_3 = f_1 f_2.$$  \hspace{1cm} (17)
We note that $f_1^2 - f_2^2$ and $f_2^2 - f_3^2$ are independent of $t$. Hence let us consider the case $f_1^2 \leq f_2^2 \leq f_3^2$. Then the solution to this set of equations is known in terms of Jacobi elliptic functions as

\begin{align*}
    f_1 &= -\frac{D \text{cn}_k[D(t - t_0)]}{\text{sn}_k[D(t - t_0)]}, \\
    f_2 &= -\frac{D \text{dn}_k[D(t - t_0)]}{\text{sn}_k[D(t - t_0)]}, \\
    f_3 &= -\frac{D}{\text{sn}_k[D(t - t_0)]},
\end{align*}

(18)

where $k \in [0, 1]$ is the elliptic modulus and $D, t_0$ are arbitrary. We can change the sign of any two of $f_1, f_2$ and $f_3$ by 180 degree rotations.

On the other hand, the compatibility condition (10) becomes that for every $j$,

\[ f_j(-t)\tau_j^T = f_j(t)C \tau_j C^{-1} \]

(19)

with a symmetric matrix $C$. The boundary condition (11) becomes $f_1(-1) = -f_1(1)$.

Among linear combinations of $\tau_1$ and $\tau_3$, the right choice for $C$ with Nahm data (18) is

\[ C = \tau_3. \]

(20)

This implies that $f_1$ is an odd function and $f_2, f_3$ are even functions.\footnote{If $f_1^2$ is not chosen to be smallest, it would contradict with the boundary condition (3).} This fixes the parameter $t_0$ to satisfy $\text{cn}_k(Dt_0) = 0$. Then our solution for Nahm’s equation is as follows:

\begin{align*}
    f_1 &= D\sqrt{1 - k^2} \frac{\text{sn}_k(Dt)}{\text{cn}_k(Dt)}, \\
    f_2 &= -D\sqrt{1 - k^2} \frac{1}{\text{cn}_k(Dt)}, \\
    f_3 &= -D \frac{\text{dn}_k(Dt)}{\text{cn}_k(Dt)}. \quad (21)
\end{align*}

This Nahm data is regular for $t \in [-1, 1]$. The analyticity of the data requires that $0 \leq k \leq 1$ and $0 \leq D \leq K(k)$ with $4K(k)$ being the period of $\text{sn}_k$. $K$ is also the first complete elliptic integral $K(k) = \int_0^{\pi/2} d\theta (1 - k^2 \sin^2 \theta)^{-1/2}$. Eqs. (15) and (21) are the Nahm data we are looking for.
looking for. (Actually they are the Nahm data on a representative point of the $SO(3) \times SO(3)$ orbit.) Sometimes we will simply call Eq. (21) as Nahm data. There are eight equivalent copies of above Nahm data: we can exchange $f_2$ and $f_3$, and any two of $f_1, f_2$ and $f_3$ can change their sign. The allowed local gauge transformations of Eq. (12) are made of $g(t)$ such that

$$g(t) = e^{\epsilon_j(t) \tau_j/2}$$

with even $\epsilon_1$ and odd $\epsilon_2, \epsilon_3$ functions. This will be crucial in showing the spherically symmetric Nahm data is not invariant under global gauge transformations due to $\epsilon_2, \epsilon_3$.

The moduli space $M^{12}$ of uncentered three monopoles is the space of gauge inequivalent Nahm data with the gauge action $G_0$. Since the center $U(1)$ of $U(2)$ is tri-holomorphic, we can perform a hyperkähler quotient with the momentum map $\mu = -i(\text{tr} T_1, \text{tr} T_2, \text{tr} T_3)$. This gives the eight-dimensional relative moduli space $M^8$ of the centered Nahm data. Further quotient of this manifold by the internal gauge symmetry $SU(2)$ leads to the five dimensional manifold $N^5 = M^8/SU(2)$. The homeomorphic coordinates for $N^5$ are given in terms of gauge-invariant $t$-independent quantities \cite{12},

$$\lambda_1 = \langle T_1, T_1 \rangle - \langle T_2, T_2 \rangle,$$
$$\lambda_2 = \langle T_1, T_1 \rangle - \langle T_3, T_3 \rangle,$$
$$\lambda_3 = \langle T_1, T_2 \rangle,$$
$$\lambda_4 = \langle T_1, T_3 \rangle,$$
$$\lambda_5 = \langle T_2, T_3 \rangle,$$

where

$$< T, T' > = -\int_{-1}^{1} dt \text{tr}(TT').$$

They form a real traceless $3 \times 3$ matrix and realize a 5-dimensional representation of $SO(3)$. The data (21) leads to the coordinates,
\[
\lambda_1 = -(1 - k^2) D^2, \\
\lambda_2 = -D^2, \\
\lambda_3 = \lambda_4 = \lambda_5 = 0,
\]

and \( \lambda_3 = \lambda_4 = \lambda_5 = 0 \), which is invariant under the 180 degree rotations around three cartesian axes. Thus this data has \( Z_2 \times Z_2 \) isotropy group. \( N^5 \) is a five dimensional manifold homeomorphic to \( R^5 \) and admits a non-free rotational \( SO(3) \) action. Further quotient of this manifold by the spatial rotation group \( SO(3) \) leads to a two dimensional surface \( N^5 / SO(3) \), whose eight copies, as we will see in Sec. V, make a geodesic complete manifold \( \mathcal{J}^2 \). There are also two-dimensional surfaces of revolution, which describe axially symmetric configurations.

Since the gauge group \( SU(2) \) is tri-holomorphic, there is another hyperkähler quotient of \( M^8 \). We choose a \( U(1) \) subgroup which fixes \( \tau_1 \). The corresponding moment map is

\[
\mu = (\text{tr} (T_1(1)\tau_1), \text{tr} (T_2(1)\tau_1), \text{tr} (T_3(1)\tau_1)).
\]

The hyperkähler quotient space \( M^4(\zeta) = \mu^{-1}(\zeta)/U(1) \) is a four-dimensional hyperkähler space. The rotational transformation \( Spin(3) = \{a_{ij}\} \) generates a homeomorphic mapping \( M^4(\zeta_i) \) to \( M^4(a_{ij}\zeta_j) \). We will see later that this family interpolates the flat space \( M^4(0) = R^3 \times S^1 \) to the Atiyah-Hitchin space \( M^4(\infty) \). This family can be regarded as deformations of the Atiyah-Hitchin space. Since any hyperkähler space in four dimensions is self-dual and so Ricci flat, \( M^4(\zeta,0,0) \) can be regarded as one-parameter family of gravitational instantons.

### IV. THE ADHMN CONSTRUCTION

Given Nahm data, we can define the differential operator

\[
\Lambda^\dagger(x) = i \frac{d}{dt} - \sum_{i=1}^{3} (iT_j + x_j) \otimes e_j,
\]

where \( e_j \ (j = 1, 2, 3) \) are quaternion units. The dimension of the kernel of \( \Lambda^\dagger \) depends on the boundary conditions involved in defining Nahm data \( T_i \). For our case it turns out to be four. The basis of \( \text{Ker} \ \Lambda^\dagger \) consists of four orthonormal four-component vectors \( v_\mu, \mu = 1, \ldots, 4 \) with the inner products \( \langle v_\mu, v_\nu \rangle = \int_{-1}^{1} dt \ v^\dagger_\mu \cdot v_\nu = \delta_{\mu\nu} \). In terms of the \( 4 \times 4 \) matrix
\[ V = (v_1, v_2, v_3, v_4), \] the ADH Mn construction of monopole solutions in \( R^3 \) goes as follows: the \( 4 \times 4 \) Hermitian matrix-valued fields

\[
\Phi = \int_{-1}^{1} dt \, t V^\dagger V, \tag{28}
\]

\[
A_j = i \int_{-1}^{1} dt \, V^\dagger \frac{\partial V}{\partial x_j}, \tag{29}
\]

form a BPS monopole field configuration. It is really a configuration in \( SU(4) \) gauge theory and may need a further gauge transformation in \( SU(4) \) to be expressed as a proper \( Sp(4) \) configuration.

We express a single four vector as \( v = (w_1, w_2, w_3, w_4)^T \). With the usual convention of quaternion units (namely \( e_1 = \tau_1, e_2 = \tau_2, e_3 = \tau_3 \)), the equation \( \Lambda^\dagger v = 0 \) can be written as

\begin{align*}
\dot{w}_1 - x_1 w_1 - (x_3 - i x_2) w_3 + \frac{1}{2} f_1 w_1 + \frac{1}{2} (f_3 - f_2) w_4 &= 0, \\
\dot{w}_2 - x_1 w_2 - (x_3 - i x_2) w_4 - \frac{1}{2} f_1 w_2 + \frac{1}{2} (f_2 + f_3) w_3 &= 0, \\
\dot{w}_3 + x_1 w_3 - (x_3 + i x_2) w_1 - \frac{1}{2} f_1 w_3 + \frac{1}{2} (f_2 + f_3) w_2 &= 0, \\
\dot{w}_4 + x_1 w_4 - (x_3 + i x_2) w_2 + \frac{1}{2} f_1 w_4 + \frac{1}{2} (f_3 - f_2) w_1 &= 0. \tag{30}
\end{align*}

It is hard to obtain general solution of the above equations. In this section, we would like to work out several special cases in order to check whether the ADH Mn construction leads to the sensible result. This exercise also yields a general understanding of the physical meaning of parameters \( k \) and \( D \) appearing in Nahm data.

The first case we consider is the spherically symmetric solution with \( D = 0 \) and so

\[
f_1 = f_2 = f_3 = 0. \tag{31}
\]

Clearly this Nahm data is invariant under the spatial SO(3) rotation \([14]\). One may wonder whether this Nahm data is invariant under global gauge transformations \([12]\). Clearly this data \( T_i = 0 \) is invariant under the global \( SO(3) \) gauge rotation \([12]\). However, the initial \( T_4 = 0 \) is not necessarily invariant. The reason is that the gauge parameters \( \epsilon_2, \epsilon_2 \) of Eq. \([22]\)
are odd functions and so their time derivative is nonzero for nontrivial transformations. But \( \epsilon_1 \) is even and so can be constant, leaving \( T_0 \) invariant. Thus, one expects a \( S^2 \) gauge orbit space, which leaves the spherically symmetric solution. This two sphere will also appear in the metric of the moduli space in the next section. (In Dancer’s case, the spherically symmetric solution is not invariant for all three generators of \( SU(2) \) gauge rotation.)

The kernel equations (30) can be easily solved for the spherically symmetric solution and give rise to the Higgs field

\[
\Phi = 2H(2r) \hat{r} \cdot \mathbf{t}(\gamma),
\]

where \( r = \sqrt{x_i x_i}, \hat{r}_i = x_i/r \) and \( H(r) = \coth(r) - 1/r \) is the famous single monopole function. This is the well known single monopole solution with \( \Phi_\infty \propto H \) along \( x_1 \) direction. This configuration is the \( SU(2) \) embedded solution along the composite root \( \gamma \). The energy density is maximized at the center. We just argued that the corresponding Nahm data is not invariant under some of the global gauge transformations. To understand this in terms of the field configuration, we deduce form the root diagram in Fig. 1 that the generators \( t^i(\gamma) \) commute with \( t^3(\alpha) \) but not with \( t^{1,2}(\alpha) \). Thus the spherically symmetric field configuration is not invariant under two of \( t^i(\alpha) \), argued before.

We now turn to the next simplest case, the axially symmetric case. Similar as in Dancer’s case, we have two types of axially symmetric cases. The hyperbolic case appears when \( k = 1 \) and \( 0 \leq D < \infty \) so that

\[
f_1 = f_2 = 0, \quad f_3 = D.
\]  

This Nahm data is invariant under rotation around the \( x_3 \)-axis. Although no hyperbolic function is involved here, we have used the same terminology as used as in Ref. [12], because of a similarity in the qualitative behavior. The trigonometric case appears when \( k = 0 \), so that

\[
f_1 = D \tan(Dt), \quad f_2 = f_3 = -D \sec(Dt)
\]  

with \( 0 \leq D < \frac{\pi}{2} \). This data is invariant under the rotation around the \( x_1 \)-axis.
Our hyperbolic case is much simpler than the corresponding case considered in Dancer’s. After solving Eq. (30), we use Eq. (28) and a gauge transformation to obtain the Higgs configuration,

\[ \Phi = 2H(2r_+)\mathbf{r}_+ \cdot \mathbf{t}(\beta) + 2H(2r_-)\mathbf{r}_- \cdot \mathbf{t}(\delta), \tag{35} \]

where \( r_\pm = (x_1, x_2, x_3 \pm D/2) \). We recognize that this configuration describes \( \beta^* \) and \( \delta^* \) monopoles located at the \( x_3 \) axis with \( x_3 = -D/2 \) and \( x_3 = D/2 \), respectively. Since \([t^i(\beta), t^j(\delta)] = 0\), there is no interaction between these two monopoles, and the field configuration (35) is just a superposition of two corresponding configurations. In Dancer’s hyperbolic case, two massive monopoles are interacting.

The above hyperbolic configuration is not invariant under global gauge rotations of \( t(\alpha) \) as it does not commute with \( t(\beta) \) and \( t(\delta) \). Among the dyonic excitations, there is a simple one which is just a superposition of \( \beta \) dyon and \( \delta \) dyon. Once the magnitudes of their electric charges are not identical, their relative charge is nonzero. This corresponds to the excitation due to the \( t^3(\alpha) \) rotation. Clearly this configuration would preserve the axial symmetry.

In the next section, the motion which changes \( D \) and this relative charge will be described by a flat two dimensional surface of revolution. Especially the configuration with relative electric charge is spherically symmetric when \( D = 0 \), which is consistent with the fact that the spherically symmetric solution is not invariant under the global gauge rotation.

On the other hand our trigonometric solution (34) is more complicated. Equation (30) at \((z,0,0)\) becomes

\[ \dot{\hat{w}}_1 - zw_1 + \frac{1}{2}D \tan(Dt)w_1 = 0, \tag{36} \]
\[ \dot{\hat{w}}_2 - zw_2 - \frac{1}{2}D \tan(Dt)w_2 - D \sec(Dt)w_3 = 0, \tag{37} \]
\[ \dot{\hat{w}}_3 + zw_3 - \frac{1}{2}D \tan(Dt)w_3 - D \sec(Dt)w_2 = 0, \tag{38} \]
\[ \dot{\hat{w}}_4 + zw_4 + \frac{1}{2}D \tan(Dt)w_4 = 0. \tag{39} \]

Notice that the first and fourth equations are not coupled with anything else while the second and third equations are only coupled among themselves. Thus, after a \( SU(4) \) gauge
transformation the Higgs field has the form

\[
\Phi = \begin{pmatrix}
* & 0 & 0 & * \\
0 & * & 0 & 0 \\
0 & 0 & * & 0 \\
* & 0 & 0 & *
\end{pmatrix},
\]

(40)

where * indicates nonvanishing entry. Since \( \Phi^T J + J \Phi = 0 \) with \( J \) in Eq. (8), we get \( \Phi_{33} = -\Phi_{22} \) and \( \Phi_{44} = -\Phi_{11} \). From Eq. (36), we can easily obtain

\[
\Phi_{22} = -\frac{f(z) - f(-z)}{g(z) + g(-z)},
\]

(41)

where

\[
f(z) = e^{2z} \left\{ [(2z + 1)D^2 + 4z^2(2z - 1)] \cos D + D[D^2 + 4z(z - 1)] \sin D \right\},
\]

(42)

\[
g(z) = e^{2z}(D^2 + 4z^2)(2z \cos D + D \sin D).
\]

(43)

We are not going to pursue the details for the corner \( 2 \times 2 \) matrix part of \( \Phi \), which describes the non-Abelian part. Like in the case of Ref. [12], we have a reason to believe that the trigonometric data corresponds to the situation when the energy density is maximized on a ring around the axis of symmetry, even though we have not done the numerical computation to check this. When \( D = 0 \), the configuration is spherically symmetric. When \( D \to \pi/2 \), we will see in a moment that our result approaches the Atiyah-Hitchin case. That case, when axially symmetric, has the ring-like energy distribution. Thus symmetry and continuity imply the ring-like energy distribution for the trigonometric case.

At the limit \( D \to \pi/2 \), Eq. (41) becomes

\[
\Phi_{22} = -\left[ \tanh(2z) - \frac{z}{z^2 + \left( \frac{\pi}{4} \right)^2} \right],
\]

(44)

which is exactly the result of two \( SU(2) \) monopoles [24]. Meanwhile Eqs. (37) and (38) lead to \( \Phi_{11} = -\Phi_{44} = -1 \) and \( \Phi_{14} = \Phi_{41} = 0 \) at \( D = \pi/2 \). Thus the Higgs field (40) along the symmetric axis becomes the Higgs field for charge two \( SU(2) \) monopole configuration.
As a general verification of the suggestion made above, let us check whether the three monopole case degenerates into the $SU(2)$ result when $k = 0, D = \pi/2$, or more generally when $D \to K(k)$. From Nahm data [21], we get

\begin{equation}
    f_1, f_2, f_3 \approx -\frac{1}{1 + t},
\end{equation}

near $t = -1$ and

\begin{equation}
    -f_1, f_2, f_3 \approx -\frac{1}{1 - t}
\end{equation}

near $t = 1$. These are exactly the boundary conditions satisfied by Nahm data for two identical monopoles in the $SU(2)$ case [21]. The removal of massless monopole to spatial infinity leaves two identical monopoles and so corresponds to the $SU(2)$ limit $D \to K(k)$.

Now let us try to find the physical meaning of two parameters $k$ and $D$ from the above mentioned solutions. As we know, there is no spherically symmetric solution for two identical monopoles in $SU(2)$ case. Thus, non-Abelian cloud must play a crucial role in our spherically symmetric solution with $D = 0$. Our spherically symmetric solution is the embedding of the $SU(2)$ solution along the $\gamma$ root. Thus the position of the massless monopole is at origin, implying the minimum cloud size. In the opposite limit where $D \to K(k)$, Nahm data becomes that of the $SU(2)$ case, implying non-Abelian cloud has been removed to spatial infinity. Indeed it seems that non-Abelian cloud size increases monotonically with $D$ for a given $k$. (We should add that the exact nature of non-Abelian cloud is still uncertain when the cloud size is comparable with distance between the massive monopoles.)

In the trigonometric case with $k = 0$, the spherically symmetric solution changes to the ring shape, as we take out massless monopole to the spatial infinity by increasing $D$ from zero to $\pi/2$. In the hyperbolic case with $k = 1$, the cloud is always in its minimum size, so that the field configuration becomes effectively that of two noninteracting $\beta^*$ and $\delta^*$ monopoles.

From these analysis we can also see the meaning of $k$. At least in the Atiyah-Hitchin case we know that small $k$ corresponds to small separation and $k \approx 1$ corresponds to wide
separation, in which case the distance goes roughly like $r \approx K(k)$. Even for other $D$ we could expect $k$ to be related to the distance between two massive monopoles in a qualitatively similar way as in Atiyah-Hitchin case. Of course when non-Abelian cloud has finite size, the separation between massive monopoles cannot go beyond the scope of cloud so $k = 1$ would no longer represent the infinite distance. Figure 2 shows the $k - D$ space. The spherically symmetric case corresponds to $D = 0$ and the trigonometric case does to the line $k = 0$ and $0 < D < \pi/2$. The hyperbolic case corresponds to $k = 1$ and the Atiyah-Hitchin case does to the curve $D = K(k)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2}
\caption{The $k - D$ space.}
\end{figure}

V. THE MODULI SPACE METRIC

Now let us turn our attention to the metric of the moduli space. By using centered Nahm data, we work in the center of mass frame of monopoles. The relative moduli space $M^8$ of Nahm data should isometrically correspond to the relative moduli space of the monopole dynamics. The metric for the center of mass motion is flat, and we expect that

$$M^{12} = R^3 \times \frac{S^1 \times M^8}{\Delta}$$

where $\Delta$ is a discrete subgroup, about which we will not concern in here. Our work of finding the moduli space metric is greatly facilitated by the works done by Dancer [12] and Irwin [25]. Their general derivation works equally well with our problem. However our detailed results are different from theirs. For the sake of completeness, we present their derivation as the way applied to our case.
To calculate the metric of the relative moduli space $M^8$, let us define the tangent vectors of $M^8$. A tangent vector $Y = (Y_1, Y_2, Y_3, Y_4)$ must satisfy the linearized Nahm’s equations,

$$
\dot{Y}_i + [Y_4, T_i] + [T_4, Y_i] = \epsilon_{ijk}[T_j, Y_k].
$$

(48)

Since the moduli space $M^8$ is defined by gauge equivalent Nahm data, the tangent vector should be orthogonal to infinitesimal gauge transformations $\delta T_\mu$ in $\mathcal{G}_0$,

$$
\sum_{\mu=1}^{4} \langle Y_\mu, \delta T_\mu \rangle = 0,
$$

(49)

where the orthogonality is defined with the flat metric on the infinite dimensional affine space [19,20],

$$
ds^2(Y, Y') = \sum_\mu \langle Y_\mu, Y'_\mu \rangle.
$$

(50)

Equation (49) can be written in an explicit form,

$$
\dot{Y}_4 + \sum_{\mu=1}^{4} [T_\mu, Y_\mu] = 0.
$$

(51)

The procedure of solving Eqs. (48) and (51) for tangent vectors has been described in Ref. [12]. In general, $Y_\mu$ can be expressed as $Y_\mu = y_{\mu j}(\tau_j / 2)$. Substituting this expression into Eqs. (48) and (51), we get four closed sets of linear differential equations, whose nonsingular solutions can be parametrized by eight real parameters $m_\mu$, $n_\mu$, as follows:

$$
Y_1 = \frac{1}{2} \left[ \dot{f}_1 I_1 \tau_1 + \left( \dot{f}_2 I_2 + \frac{m_2}{f_2} \right) \tau_2 + \left( \dot{f}_3 I_3 + \frac{n_3}{f_3} \right) \tau_3 \right],
$$

$$
Y_2 = \frac{1}{2} \left[ -\dot{f}_1 I_2 \tau_1 + \left( \dot{f}_2 I_1 + \frac{m_1}{f_2} \right) \tau_2 - \left( \dot{f}_3 I_4 + \frac{n_4}{f_3} \right) \tau_3 \right],
$$

$$
Y_3 = \frac{1}{2} \left[ -\dot{f}_1 I_3 \tau_1 + \left( \dot{f}_2 I_4 + \frac{m_4}{f_2} \right) \tau_2 + \left( \dot{f}_3 I_1 + \frac{n_1}{f_3} \right) \tau_3 \right],
$$

$$
Y_4 = \frac{1}{2} \left[ \dot{f}_1 I_4 \tau_1 + \left( \dot{f}_2 I_3 + \frac{m_3}{f_2} \right) \tau_2 - \left( \dot{f}_3 I_2 + \frac{n_2}{f_3} \right) \tau_3 \right],
$$

(52)

where

$$
I_\mu(t) = \int_0^t dt' \left( \frac{m_\mu}{f_2(t')^2} + \frac{n_\mu}{f_3(t')^2} \right).
$$

(53)
The lower bound of $I_\mu(t)$ is chosen so that they are odd functions. This makes $Y_\mu$ to satisfy the compatibility condition $Y_\mu(-t)^T = Cy_\mu(t)C^{-1}$, which is implied from Eq. (10). This is the tangent vector on the representative point (21) of $SO(3) \times SO(3)$ orbit.

The metric on the moduli space $M^8$ is induced from the flat metric (50) on the infinite dimensional affine algebra. With our solutions (52), the general result is

$$ds^2(Y, Y') = \sum_{\mu=1}^{4} [(g_1 + g_1^2 X) m_\mu m'_\mu + (g_2 + g_2^2 X) n_\mu n'_\mu + g_1 g_2 X (m_\mu n'_\mu + n_\mu m'_\mu)],$$  \hspace{1cm} (54)

where

$$X(k, D) = f_1(1)f_2(1)f_3(1),$$

$$g_1(k, D) = \int_0^1 \frac{dt}{f_2(t)^2},$$

$$g_2(k, D) = \int_0^1 \frac{dt}{f_3(t)^2}. \hspace{1cm} (55)$$

We can calculate the metric by finding the tangent vector at a generic point of $M^8$, which can be obtained by the $SO(3) \times SO(3)$ spatial and gauge rotations of Nahm data (21). Due to the $SO(3) \times SO(3)$ symmetry of the metric, the general metric can be found if it is known near the identity. We want to relate the coordinates $m_\mu, n_\mu$ of the tangent space at the specific point to the infinitesimal changes of the parameters $k, D$ and the infinitesimal $SO(3) \times SO(3)$ transformations (25). This corresponds basically the rotation of a rigid body around three principal axes. Similar to the rigid body case, we can find the metric once we know the moment of inertia around each principal axes, which are the coordinate axis for our Nahm data (13) and (18). The kinetic part for the rigid body case is expressed in terms of the left invariant one-forms

$$\sigma_1 = -\sin \psi d\theta + \cos \psi \sin \theta d\varphi,$$

$$\sigma_2 = \cos \psi d\theta + \sin \psi \sin \theta d\varphi,$$

$$\sigma_3 = d\psi + \cos d\varphi, \hspace{1cm} (56)$$

which correspond to the infinitesimal spatial rotations around three principal axes. The corresponding left-invariant one-forms for the gauge rotations are $\tilde{\sigma}_i$. The relations we seek are
\[m_1 = \frac{1}{2}d\lambda_1,\]
\[n_1 = \frac{1}{2}d\lambda_2,\]
\[m_2 = \lambda_1\sigma_3,\]
\[n_2 = \frac{-g_1\lambda_1}{1 + g_2X}\sigma_3 + \frac{1}{\sqrt{g_2 + g_2^2X}} \left\{ b_3\sigma_3 - c_3\left(\frac{f_1(1)}{f_2(1)}\sigma_3 - \bar{\sigma}_3\right) \right\},\]
\[m_3 = \frac{g_2\lambda_2}{1 + g_1X}\sigma_2 - \frac{1}{\sqrt{g_1 + g_1^2X}} \left\{ b_2\sigma_2 - c_2\left(\frac{f_1(1)}{f_3(1)}\sigma_2 - \bar{\sigma}_2\right) \right\},\]
\[n_3 = -\lambda_2\sigma_2,\]
\[m_4 = \frac{-g_2(\lambda_1 - \lambda_2)}{g_1 + g_2}\sigma_1 - \frac{1}{\sqrt{(g_1 + g_2)(1 + (g_1 + g_2)X)}} \left\{ b_1\sigma_1 - c_1\left(\frac{f_3(1)}{f_2(1)}\sigma_1 - \bar{\sigma}_1\right) \right\},\]
\[n_4 = \frac{g_1(\lambda_1 - \lambda_2)}{g_1 + g_2}\sigma_1 - \frac{1}{\sqrt{(g_1 + g_2)(1 + (g_1 + g_2)X)}} \left\{ b_1\sigma_1 - c_1\left(\frac{f_3(1)}{f_2(2)}\sigma_1 - \bar{\sigma}_1\right) \right\},\]
}(57)

where \(\lambda_1, \lambda_2\) are given in Eq. (25),

\[b_1 = k^2D^2\sqrt{\frac{g_1^2}{(g_1 + g_2)(1 + (g_1 + g_2)X)}},\]
\[b_2 = \frac{g_2D^2}{\sqrt{g_1 + g_1^2X}},\]
\[b_3 = \frac{g_1(1 - k^2)D^2}{\sqrt{g_2 + g_2^2X}},\]
}(58)

and

\[c_1 = \frac{f_2(1)f_3(1)}{f_2(1)g_1} \sqrt{\frac{g_1 + g_2}{1 + (g_1 + g_2)X}},\]
\[c_2 = \frac{\sqrt{g_1 + g_1^2X}}{g_1f_2(1)},\]
\[c_3 = \frac{\sqrt{g_2 + g_2^2X}}{f_3(1)g_2}.\]
}(59)

Once we replace the parameterization (57) into the metric (54), we would have got the explicit form for the metric of the moduli space \(M^8\). Rather than doing this, let us study the metric bit by bit. The two dimensional space \(Y^2\) is the geodesically complete space made of \(g\) copies of the \(k-D\) space, \(N^5/Spin(3)\). These eight copies are discussed in the remark after Eq. (21). This space describes the motion of the monopoles with the vanishing \(SU(2)\)
electric charge and zero angular momentum. The metric of this space from Eqs. (25), (54) and (57) is

\[ ds_{32}^2 = \frac{1}{4} \left\{ X (g_1 d\lambda_1 + g_2 d\lambda_2)^2 + g_1 d\lambda_1^2 + g_2 d\lambda_2^2 \right\}. \] (60)

Figure 3 shows this space in terms of two coordinates, which in the shaded region are \( D \) and \( E \equiv \sqrt{1 - k^2} D \). The above metric at origin is smooth with \( D, E \) playing cartesian coordinates. The origin corresponds to the spherically symmetric configuration. Two coordinate axes correspond to two hyperbolic configurations, which are symmetric along real spatial \( x^2, x^3 \) coordinate axes. The diagonal lines do to one trigonometric one, which is symmetric along real spatial \( x^1 \) axis. The boundary curves correspond to the Atiyah-Hitchin configurations, where the massless monopole have been moved to spatial infinity.

![Diagram](image)

**Figure 3**: A sketch of the geodesically complete space in D-E coordinates. The shaded region corresponds to the \( N^5/\text{Spin}(3) \) space.

While we have not studied in detail the geodesic motion on this space, one can see from symmetry that the trigonometric solutions with velocity pointing to the origin will remain trigonometric after the configuration passes through the origin. With the similar velocity, the hyperbolic solutions remain hyperbolic, which is consistent with a picture of noninteracting two monopoles for the hyperbolic case. This contrasts to Dancer’s case where the trigonometric configurations changes to the hyperbolic case, and vice versa. The
configuration with infinite cloud size would remain the Atiyah-Hitchin configurations and
the boundary curve shows the 90 degree scattering of these monopoles.

The metric on $N^5 = M^8/SU(2)$ with the $Z_2 \times Z_2$ isotropic group is

$$ds_{N^5}^2 = ds_{Y^2}^2 + a_1 \sigma_1^2 + a_2 \sigma_2^2 + a_3 \sigma_3^2$$

(61)

with

$$a_1 = k^4 D^4 \frac{g_1 g_2}{g_1 + g_2},$$

$$a_2 = D^4 \left\{ g_2 + \frac{g_2^2 X}{1 + g_2 X} \right\},$$

$$a_3 = (1 - k^2)^2 D^4 \left\{ g_1 + \frac{g_1^2 X}{1 + g_2 X} \right\}.$$  

(62)

Here one uses the orthogonality condition for the tangential vectors of $N^5$ to that of gauge rotation [12,25], which can be found from Eq. (57) by dropping terms depending on $b_i$ and $c_i$. There is no cross term for the invariant one-forms, which is consistent with $Z_2 \times Z_2$ isotropy group of $N^5$. This metric describes the monopole dynamics with zero $SU(2)$ electric charge but perhaps with nonzero orbital angular momentum. Figure 4 shows two massive monopoles (two half doughnuts on the $x_3$ axis) with generic cloud size and three principal axes. In zero cloud size $k = 1$, the metric is symmetric under the rotation around the $x_3$ axis so that $a_1 = a_2$ and $a_3 = 0$. In the trigonometric case $k = 0$, the metric is symmetric under the rotation around the $x_1$ axis so that $a_1 = 0$ and $a_2 = a_3$.

![Figure 4](This figure shows two massive monopoles with finite size cloud. The central doughnut indicates the symmetric axis $x_1$.)

Figure 4: The massive monopole with finite size cloud. The central doughnut indicates the symmetric axis $x_1$. 
From Eqs. (54) and (57), we get the full metric on $M^8$, which is

$$ds^2_{M^8} = \frac{1}{4} \left\{ X(g_1 d\lambda_1 + g_2 d\lambda_2)^2 + g_1 d\lambda_1^2 + g_2 d\lambda_2^2 \right\}$$

$$+ a_1 \sigma_1^2 + a_2 \sigma_2^2 + a_3 \sigma_3^2 + \left\{ b_1 \sigma_1 - c_1 \left( \frac{f_3(1)}{f_2(1)} \sigma_1 - \tilde{\sigma}_1 \right) \right\}^2$$

$$+ \left\{ b_2 \sigma_2 - c_2 \left( \frac{f_1(1)}{f_3(1)} \sigma_2 - \tilde{\sigma}_2 \right) \right\}^2 + \left\{ b_3 \sigma_3 - c_3 \left( \frac{f_1(1)}{f_2(1)} \sigma_3 - \tilde{\sigma}_3 \right) \right\}^2. \quad (63)$$

This metric is hyperkähler. The isometric group is $SO(3) \times SO(3)$. The $SO(3)$ global gauge transformation is tri-holomorphic and the $SO(3)$ spatial rotation rotates three complex structures of the manifold. There are several interesting limits of this metric. When the cloud size is smallest with $k = 1$, its Nahm date is the hyperbolic case (33) and the above metric becomes

$$ds^2_{\text{hyper}} = dD^2 + D^2 \sigma_1^2 + D^2 \sigma_2^2 + D \tanh D \tilde{\sigma}_1^2 + D \coth D \tilde{\sigma}_2^2 + \tilde{\sigma}_3^2. \quad (64)$$

The moments of inertia for internal gauge transformations are nonzero exactly as we argued in the previous section. Especially when spherically symmetric case with $D = 0$, the coefficient of $\tilde{\sigma}_1$ vanishes, and those of $\tilde{\sigma}_2$ and $\tilde{\sigma}_3$ become identical, implying the $S^2$ gauge orbit space. In large separation $D \gg 1$, the inertia for $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ become identical. The inertia for $\tilde{\sigma}_3$ is constant, which corresponds to $t^3(\alpha)$ dyonic excitations discussed in the next paragraph after Eq. (35).

There are two axially symmetric solutions, that is, hyperbolic and trigonometric. When we include internal global gauge rotations which preserve the axial symmetry, we obtain two dimensional surfaces of revolution. The metric for the trigonometric case with $k = 0, 0 < D < \pi/2$ is

$$ds^2_{\text{trig}} = \sec^2 D (1 + D \tan D) \left( 1 + \frac{\sin D \cos D}{D} \right) \left[ dD^2 + \frac{D^2(\sigma_1 - \tilde{\sigma}_1)^2}{(1 + D \tan D)^2} \right], \quad (65)$$

where $\sigma_1 - \tilde{\sigma}_1$ can be put into a rotation $d\alpha$ around internal and angular angles. As $D \to \pi/2$, the metric (33) becomes

$$ds^2 = d\rho^2 + \frac{1}{4} \rho^2 d\alpha^2 \quad (66)$$
where $\rho = 2\sqrt{\pi/(\pi - 2D)}$. In this limit the massless monopole moves out from the localized massive monopoles, and so the non-Abelian cloud is expected to be more and more spherical with the flat $R^4$ moduli space as in Ref. [7]. The above metric is then a section of $R^4$ with a radial variable $\rho$ as we will see in a moment. In the physical space, the massless cloud size is of order $\rho^2$. The non-Abelian component of the gauge field will change its behavior from $1/r$ to $1/r^2$ as one crosses the this radius $\rho^2$. Another axially symmetric case is hyperbolic one with $k = 1, 0 < D < \infty$, whose metric is

$$ds^2_{\text{hyper}} = dD^2 + \tilde{\sigma}_3^2$$  \hspace{1cm} (67)

Clearly this flat metric is a part of the metric (64).

The limit of large cloud size can be found in the region where $K(k) - D << 1$. In the previous section, we argued that Nahm data in this case becomes that for the Atiyah-Hitchin case. In this limit one can show easily the metric (63) becomes

$$ds^2 = d\rho^2 + \frac{\rho^2}{4} \left\{ (\sigma_1 - \bar{\sigma}_1)^2 + (\sigma_2 + \bar{\sigma}_2)^2 + (\sigma_3 + \bar{\sigma}_3)^2 \right\}$$

$$+ \frac{b^2}{K^2} dK^2 + a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2 + O(\rho^{-1})$$  \hspace{1cm} (69)

where $\rho = 2\sqrt{D/(K(k) - D)}$, $K = K(k)$ and

$$a^2 = \frac{K(K - E)(E - (1 - k^2)K)}{E},$$  \hspace{1cm} (70)

$$b^2 = \frac{EK(K - E)}{E - (1 - k^2)K},$$  \hspace{1cm} (71)

$$c^2 = \frac{EK(E - (1 - k^2)K)}{K - E},$$  \hspace{1cm} (72)

where $E$ being the second complete elliptic integral $E(k) = \int_0^{\pi/2} d\theta \sqrt{1 - k^2 \sin^2 \theta}$. This shows that the asymptotic space is a direct product of $R^4$ and the Atiyah-Hitchin space. As in Ref. [7], we expect that the metric of the massless cloud space approaches that of flat $R^4$, which is exactly what the above limit shows. A combination of orbital and gauge angular variables needs to be introduced [23] to make this $R^4$ explicit.

The part of the moduli space metric we can calculate independently from Nahm’s formalism is the asymptotic metric, which is valid when the mutual distance between monopoles is
large. This can be done by studying the interaction between dyons in large separation \cite{26,11} and taking the massless limit. In the center of mass frame, the relative positions between the massive $\beta^*$ monopoles and the massless $\alpha^*$ monopole are $r_1$ and $r_2$ as shown in Fig. 5. The relative position between two massive monopoles is $r = r_1 + r_2$.

Figure 5: The parameters of the asymptotic metric. The center of mass is at the middle of the line connecting two massive $\beta^*$ monopoles.

In terms of the relative positions and the relative angles $\psi_a, a = 1, 2$, the asymptotic form of the metric for the relative moduli space $M^8$ is

$$ds^2 = \sum_{a,b} \left[ G_{ab} dr_a \cdot dr_b + (G^{-1})_{ab} D\psi_a D\psi_b \right],$$

(73)

where

$$G_{ij} = \begin{pmatrix} 1 + \frac{1}{r_1} - \frac{1}{r} & 1 - \frac{1}{r} \\ 1 - \frac{1}{r} & 1 + \frac{1}{r_2} - \frac{1}{r} \end{pmatrix},$$

(74)

$$D\psi_a = d\psi_a + w(r_a) \cdot dr_a - w(r) \cdot dr,$$

(75)

with the Dirac potential $w$ such that $\nabla \times w(r) = \nabla(1/r)$. If we have removed the direct interaction between two identical massive monopoles, the above metric is identical to the Taubian-Calabi metric of the $SU(4) \rightarrow U(1) \times SU(2) \times U(1)$ case. Since the non-Abelian cloud of a massless monopole is independent of the direct interaction, the $SU(2)$ orbit on non-Abelian cloud would again be the three-dimensional ellipsoid defined by $r_1 + r_2 = \text{constant}$ as shown in Fig. 5. This fact can be seen easily by adapting the argument for the $SU(4)$
In the large cloud size limit, one can compare the exact metric (63) and the above metric. We see \( K/(K-D) \approx r_1 + r_2 \) and \( r \approx -\ln \sqrt{1-k^2} \). The condition \( r_1 + r_2 >> r \) for large size cloud becomes \( K(k) - D << 1 \).

Now we are in position to learn more about the four dimensional space \( M^4(\zeta) \) defined by the moment map (26). For Nahm data (15) and (21) we get \( \zeta = (\zeta, 0, 0) \) with
\[
\zeta = D \sqrt{1-k^2} \frac{\text{sn}_k(D)}{\text{cn}_k(D)}
\]
(76)
The general Nahm data is obtained from that in Eq. (21) by spatial and gauge rotations. Thus \( \zeta \) would be a function of rotational and gauge parameters. Now we see that when \( \zeta = 0 \), we have \( k = 1 \). This corresponds to the hyperbolic data (33) with the minimal size of non-Abelian cloud. (In Dancer’s case the hyperbolic data is expected to have the minimum cloud size, and is different from the \( \zeta = 0 \) case except in large separation.) The four dimensional metric for this case can be obtained from the metric (14) and is the flat \( R^3 \times S^1 \),
\[
ds^2 = dD^2 + D^2(\sigma_1^2 + \sigma_2^2) + \sigma_3^2
\]
(77)
Note that the gauge rotation \( \tilde{\sigma}_2 \) changes the value of \( \zeta \) as it transforms the hyperbolic data (33). In the language of Ref. [4], it moves the massless monopole from the origin. When \( \zeta = \infty \), we have \( D = K(k) \), which means that massless monopole has been removed, resulting in the Atiyah-Hitchin metric. Thus we see that \( M^4(\zeta) \) interpolates between the \( M^4(0) = R^3 \times S^1 \) and the Atiyah-Hitchin space \( M^4(\infty) \).

In terms of the asymptotic form of the metric (73), the \( U(1) \) rotation of \( SU(2) \) gauge rotates \( \psi_1 \rightarrow \psi_1 + \epsilon \) and \( \psi_2 \rightarrow \psi_2 - \epsilon \), whose moment map is
\[
\zeta = \frac{r_1 - r_2}{2}
\]
(78)
as shown in Fig. 5. As \( \zeta \) increases from zero to infinity, the size of the non-Abelian cloud increases from zero to infinity, consistent with the picture discussed in the previous paragraph. Also, we can trivially obtain the asymptotic form of the metric for \( M^4(\zeta) \),
\[ ds^2 = G \, dr^2 + G^{-1}(d\psi + W \cdot dr)^2, \quad (79) \]

where

\[ G = 1 + \frac{1}{2|\mathbf{r} + 2\zeta|} + \frac{1}{2|\mathbf{r} - 2\zeta|} - \frac{1}{|\mathbf{r}|} \quad (80) \]

and \( W \) is decided from the relation \( \nabla G = \nabla \times W \). Clearly this hyperkähler quotient can be done by holding the position of the massless monopole at \( \zeta \) relative to the center of mass and let massive monopoles to move around, interacting each other and with the massless monopole. This process breaks not only the rotational \( SO(3) \) symmetry but also the global gauge symmetry \( SO(3) \). We do not think that there is any remaining symmetry on the \( M^4(\zeta) \) for \( 0 < \zeta < \infty \).

VI. CONCLUSION

We have studied a purely Abelian BPS monopole configuration made of two identical massive monopoles and one massless monopole in the theory where the gauge group \( Sp(4) \) is spontaneously broken to \( SU(2) \times U(1) \). We approached this problem by finding the solutions for the corresponding Nahm’s equations under proper boundary and compatibility conditions. We have used the ADHMN construction to get the spherically and axially symmetric field configurations, which are consistent with the field theory picture. From the analysis of the axially symmetric solutions, we have come to understand the role of the non-Abelian cloud and its size. Then the explicit form of the metric on the eight dimensional moduli space of the relative motion is found. By studying the metric in various limit, we

\footnote{Even in the maximally broken case, there is a conserved \( U(1) \) and so the hyperkähler quotient make sense. The mass parameter of \( \alpha^* \) monopole does not lead to any additional parameter on \( M^4(\zeta) \) since it turns out to scale the positions \( \mathbf{r} \) and \( \zeta \), as far as the asymptotic metric (79) is concerned. This is not a view shared by Ref. [22].}
see the metric for the moduli space of the Nahm data is consistent with what is expected from the monopole dynamics.

We have also studied the metric of various submanifold of this space. Our work provide a further support to the idea that the Nahm’s approach for the BPS monopole configurations and their moduli spaces is valid in general. Our work leads also to some insight on the characteristics of the non-Abelian cloud and the gauge orbit. It is interesting to note that the spherically symmetric solution has nonzero inertia for some of unbroken gauge transformations.

From the previous experiences, we now see how in principle one may find the moduli space metric of two identical massive and one massless monopoles in the theory with $G_2 \rightarrow SU(2) \times U(1)$. To get this, one may start from the theory with $SO(8) \rightarrow SU(2)^3 \times U(1)$ with two identical massive and three distinct massless monopoles. If one identifies two massless monopoles, then the configuration would be that of two massive and two distinct massless monopoles in the theory with $SO(7) \rightarrow SU(2)^2 \times U(1)$. After further identification of all massless monopoles, one would get the desired configuration in the theory with $G_2$.

The hyperkähler quotient of the 8-dimensional relative moduli spaces of these configurations is a four dimensional hyperkähler space. To find the $M^4(\zeta = 0)$, one can consider the asymptotic form of the metric for two massive monopoles with minimum size cloud. (We overlap the massless monopole on one of the massive monopole.) They are $R^3 \times S^1$, Taub-NUT or a double covering of Atiyah-Hitchin, depending whether they are associated with the gauge group $Sp(4)$, $G_2$ or $SU(3)$, respectively. When the cloud size becomes infinite, all these three four-dimensional spaces $M^4(\infty)$ approach the Atiyah-Hitchin space.

Another direction to explore is to find the moduli space in the case when massless monopoles become massive so that there are two identical massive and one distinct massive monopoles. We think that the moduli space in the theory where $Sp(4) \rightarrow U(1)^2$ is simpler than the similar problem in the theory with $SU(3) \rightarrow U(1)^2$. Also this moduli space has a role to play in the $N = 2$ S-duality [27]. Finding the moduli space will be a challenge. Finally it would be very interesting to find out some structure of the moduli space of three
massive and three massless monopoles in the theory where $SU(4) \to SU(3) \times U(1)$. We know the asymptotic form of the metric and it may be good enough. As argued in the introduction, these configurations can be regarded as a magnetic dual of baryons and would imply new insight on the baryon structure.

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