Variational problems on Riemannian manifolds with constrained accelerations

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Abstract. We introduce variational problems on Riemannian manifolds with constrained acceleration and derive necessary conditions for normal extremals in the constrained variational problem. The problem consists on minimizing a higher-order energy functional, among a set of admissible curves defined by a constraint on the covariant acceleration. In addition, we use this framework to address the elastic splines problem with obstacle avoidance in the presence of this type of contraints.

Keywords: optimal control, affine connection control systems, higher-order variational problems, Riemannian manifolds, higher-order constraints.

1 Introduction

Higher-order Variational problems on Riemannian manifolds have been very well studied in the literature in the last three decades (see for instance \([6,11,15]\)). This study comes motivated by the theory of geodesics, presented, for instance, in Milnor \([21]\), showing a deep and rich example of the close relationship between first order variational problems and differential geometry. Motivated by this connection and applications to dynamic interpolation on manifolds \([22]\), Crouch and Silva Leite \([15]\) started the development of an interesting geometric theory of generalized cubic polynomials on a Riemannian manifold \(M\), in particular on compact connected Lie groups endowed with a left-invariant metric. Further extensions appears in the context of obstacle avoidance problems \([9,14]\), regression problems on Lie groups \([3]\), collision avoidance problems \([10]\), and sub-Riemannian geometry, with connections with non-holonomic mechanics and control, studied by Bloch and Crouch \([7,8]\). These sub-Riemannian problems are determined by additional constraints on a non-integrable distribution on \(M\).

In this work we aim to study generalized cubic polynomials subjected to a specific type of higher-order constraints: linear constraints on the covariant accelerations. This variational problem is closely related to the optimal control problems of underactuated mechanical systems which are particular types of affine connection control systems, studied, for instance, in \([1]\).
Problems with nonholonomic constraints have been previously introduced \cite{6,15} where the authors deduce necessary conditions for the existence of normal as well as abnormal extremals. Moreover, in \cite{6}, the authors studied the problem of dynamic interpolation for obstacle avoidance under the presence of constraints on the velocities. Our contribution here is also the introduction of a variational collision avoidance with second-order nonholonomic constraints which are closely related to affine connection control systems.

The remainder of the paper is organized as follows. In Section 2 we review the subject of constraints on velocities in collision avoidance, in Section 3 we give necessary condition for normal extremals of collision avoidance with constraints on the covariant accelerations. In Section 4 we discuss obstacle avoidance in this setting. Examples and simulation results are also presented in Section 4.

2 Variational problems with linear constraints on the velocities

Given a \(n\)-dimensional Riemannian manifold \(Q\) equipped with the metric \(\langle \cdot, \cdot \rangle\), we denote by \(\Omega\) be the set of all (possibly piecewise) smooth curves \(c : [0, T] \rightarrow Q\) satisfying
\[
c(0) = q_0, \quad c(T) = q_T, \quad \dot{c}(0) = v_0, \quad \dot{c}(T) = v_T,
\]
where \(q_0, q_T \in \mathcal{Q}, v_0 \in T_{q_0}Q\) and \(v_T \in T_{q_T}Q\). Note that the tangent space to \(\Omega\) at a curve \(c\) is the set of vector fields \(X : [0, T] \rightarrow TQ\) along \(c\) satisfying
\[
X(0) = 0, \quad X(T) = 0, \quad \frac{DX}{dt}(0) = 0, \quad \frac{DX}{dt}(T) = 0,
\]
where \(D/dt\) is the covariant derivative with respect to the Levi-Civita connection.

Suppose that we want to find minimizers of the functional
\[
\min \int_0^T \frac{1}{2} \left\langle \frac{D^2c}{dt^2}, \frac{D^2c}{dt^2} \right\rangle dt
\]
among curves in \(\Omega\) that also satisfy the following linear constraints on the velocities
\[
\langle \mu^a(c(t)), \dot{c}(t) \rangle = 0, \quad a = 1, \ldots, n - k,
\]
where \(\{\mu^a\}\) are \(n - k\) 1-forms on \(\mathcal{Q}\).

The span of \(\mathcal{D}\) defines the annihilator space of a distribution \(\mathcal{D} = \cup \mathcal{D}_q\), with fiber defined by \(\mathcal{D}_q = \{v_q \in T_q\mathcal{Q} \mid \langle \mu^a(q), v_q \rangle, \ a = 1, \ldots, n - k\} \). The rank of the distribution is \(k = \dim(\mathcal{D}_q)\). If \((q^i)\) is a coordinate chart on \(\mathcal{Q}\), then the linear constraints are locally given by \(\mu^a_i(q^i)q^i = 0\).

The necessary conditions satisfied by solutions of the preceding variational problem are classically obtained using the constrained Lagrangian approach, where one considers an augmented Lagrangian function and then applies the Lagrange multiplier’s theorem to obtain constrained solutions. However, when we have constraints on velocities or higher-order constraints care must be taken...
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because, by following this procedure, we obtain only normal extremals, though abnormal extremals, which do not satisfy the same necessary conditions, may also exist. In this paper, we will restrict ourselves to the study of normal extremals.

If the augmented Lagrangian $\mathcal{L} : T^{(2)}Q \times \mathbb{R}^{n-k} \to \mathbb{R}$ is given by

$$\mathcal{L} = \frac{1}{2} \left( \frac{D^2 c}{dt^2}, \frac{D^2 c}{dt^2} \right) + \lambda_a (\mu^a(c(t)), \dot{c}(t)),$$

the corresponding second-order Euler-Lagrange equations, which are

$$\frac{d^2}{dt^2} \left( \frac{\partial}{\partial q} \right) - \frac{d}{dt} \left( \frac{\partial}{\partial \dot{q}} \right) + \frac{\partial}{\partial q} = 0,$$

might be given, an intrinsic form (i.e., independent of the choice of local coordinates). In order to do so, let $X_a \in \mathfrak{X}(Q)$ be the vector field defined by

$$\langle \mu^a, Y \rangle = \langle X_a, Y \rangle,$$

∀ $Y \in \mathfrak{X}(Q), \quad (3)$

i.e., $b(X_a) = \mu^a$, then it is shown in [15] that Euler-Lagrange equations are rewritten in the following form

$$\frac{D^4 c}{dt^4} + R \left( \frac{D^2 c}{dt^2}, \dot{c} \right) \dot{c} = \dot{\lambda}_a X_a + \lambda_a S(\dot{c}),$$

where $S_a : TQ \to TQ$ is the fiberwise linear bundle operator defined by

$$d\mu^a(X, Y) = \langle S_a(X), Y \rangle, \quad \forall X, Y \in \mathfrak{X}(Q). \quad (4)$$

Remark 1. A generalization of this problem was studied in [5,6], where the authors add a potential into the picture to study dynamical interpolation problems for obstacle avoidance and further used in [2,14,13,16,17,18] to provide necessary and sufficient conditions of collision avoidance of multi-agent systems. There one studies curves in $\Omega$ satisfying the constraints (2) and minimizing the functional

$$\int_0^T \frac{1}{2} \left( \left\langle \frac{D^2 c}{dt^2}, \frac{D^2 c}{dt^2} \right\rangle + \sigma \left\| \frac{dc}{dt} \right\|^2 + V(c(t)) \right) dt$$

with $V : Q \to \mathbb{R}$ a potential function and $\sigma \geq 0$. In this case, the necessary conditions for $c$ to be a normal extremal are given by the following equations

$$\frac{D^4 c}{dt^4} + R \left( \frac{D^2 c}{dt^2}, \dot{c} \right) \dot{c} - \sigma \frac{D^2 c}{dt^2} + \frac{1}{2} \text{grad}V = \dot{\lambda}_a X_a + \lambda_a S(\dot{c}).$$

3 Variational problems with linear constraints on the covariant accelerations

Given a $n$-dimensional Riemannian manifold $Q$ equipped with the metric $\langle \cdot, \cdot \rangle$, we want to find solutions of the minimal acceleration problem

$$\min \int_0^T \frac{1}{2} \left\langle \frac{D^2 c}{dt^2}, \frac{D^2 c}{dt^2} \right\rangle dt \quad (5)$$
but with the covariant accelerations are subjected to the following linear constraints

$$\left< \mu^a(c(t)), \frac{D^2 c}{dt^2} \right> = 0, \quad a = 1, ..., n-k,$$

where \{\mu^a\} are \(n-k\) 1-forms on \(Q\).

The span of \(D\) defines the annihilator space of a distribution \(D = \cup D_q\), with fiber defined by \(D_q = \{v_q \in T_q Q \mid \langle \mu^a(q), v_q \rangle, \ a = 1, ..., n-k\}\). The rank of the distribution is \(k = \text{dim}(D_q)\). If \((q^i)\) is a coordinate chart on \(Q\), then the linear constraints are locally given by

$$\mu^a_k(q)(\ddot{q}^k + \Gamma^k_{ij} \dot{q}^i \dot{q}^j) = 0,$$

where \((\Gamma^k_{ij})\) are the Christoffel symbols of the Levi-Civita connection.

These kind of problems has appeared before in the control literature in a different form. Suppose that an orthonormal reference frame for \(D\) exists and denote it by \(\{Y_1, ..., Y_k\}\). Then this constrained variational problem is equivalent to the following optimal control problem:

$$\min \int_0^T \frac{1}{2} \langle u, u \rangle \, dt$$

subjected to the dynamics \(\dot{c} = V, \frac{DV}{dt} = u^i Y_i, (u^1, ..., u^k) \in \mathbb{R}^k\).

Recall that the covariant acceleration along the curve \(c\) is locally defined by

$$\frac{DV}{dt} = \dot{V} + \Gamma^k_{ij} \dot{q}^i \dot{q}^j \frac{\partial}{\partial q^k},$$

This type of control system is known as **affine connection control system** (see, e.g., [20,19,1]). It is actually a special type of controlled simple mechanical systems with forces (see [10]) which have the following general form:

$$\dot{c} = V, \quad \frac{DV}{dt} = \xi(F(c, V, u)),$$

where \(\xi : T^*Q \to TQ\) is the musical isomorphism associated to the Riemannian metric and \(F : TQ \times \mathbb{R}^k \to T^*Q\) is a force map, possibly depending on the controls. In the case of affine connection control systems, we have no potential or external forces, apart from those associated with the controls.

The relationship between affine connection control systems and the variational problem [5] without constraints was already discussed in [9] for the case where the control system is fully actuated, meaning that \(k = n\). However, in our case, the constrained variational problem is rather related with the **underactuated** control system (\(k < n\)). The necessary conditions satisfied by normal extremals of the preceding problem are obtained from the augmented Lagrangian \(\mathcal{L} : T^{(2)}Q \times \mathbb{R}^{n-k} \to \mathbb{R}\) given by

$$\mathcal{L} = \frac{1}{2} \left< \frac{D^2 c}{dt^2}, \frac{D^2 c}{dt^2} \right> + \lambda_a \left< \mu^a(c(t)), \frac{D^2 c}{dt^2} \right>.$$
Lemma 1. The fiberwise linear operator \( S_a : TQ \rightarrow TQ \) defined by equation (4) satisfies
\[
\langle S_a(X), Y \rangle = \langle \nabla X X_a, Y \rangle - \langle \nabla Y X_a, X \rangle,
\]
for all \( X, Y \in \mathfrak{X}(Q) \).

Proof. Using the definition of \( S_a \) in equation (4) and the definition of the differential on a 1-form we have that
\[
\langle S_a(X), Y \rangle = X \langle \nabla X X_a, Y \rangle - Y \langle \nabla Y X_a, X \rangle - \langle X, \nabla Y X_a - \nabla Y [X, Y] \rangle = \langle \nabla X X_a, Y \rangle - \langle \nabla Y X_a, X \rangle,
\]
using the fact that the Levi-Civita connection is torsionless and compatible with the metric we obtain
\[
\langle S_a(X), Y \rangle = \langle \nabla X X_a, Y \rangle - \langle \nabla Y X_a, X \rangle + \langle X, \nabla \nabla Y X_a - \nabla Y X_a - \nabla Y [X, Y] \rangle = \langle \nabla X X_a, Y \rangle - \langle \nabla Y X_a, X \rangle,
\]
as desired to show. □

Before stating the main theorem in this section we recall the following well-known lemma from the literature.

Lemma 2. The variation \( c : [-\varepsilon, \varepsilon] \times [0, T] \rightarrow Q \) satisfies
\[
\frac{D}{ds} \frac{D^2 c}{dt^2} = \frac{D^2}{dt^2} \frac{\partial c}{\partial s} + R \left( \frac{\partial c}{\partial s}, \frac{\partial c}{\partial t} \right) \frac{\partial c}{\partial t}.
\]

Theorem 1. If a trajectory \( c : I \rightarrow Q \) is a normal extremal of the functional (5) satisfying the constraints given by (6) then \( c \) satisfies
\[
\frac{D^4 c}{dt^4} + R \left( \frac{D^2 c}{dt^2}, \dot{c} \right) \dot{c} + \lambda a X_a + 2\lambda a \frac{DX_a}{dt} + \lambda a \frac{D^2 X_a}{dt^2}
\]
\[
+ \lambda a \left( R(X_a, \dot{c}) \dot{c} + \nabla \frac{D^2 X_a}{dt^2} - S_a \left( \frac{D^2 c}{dt^2} \right) \right) = 0,
\]
where \( R \) is the curvature of the Levi-Civita connection.

Proof. Suppose \( c \) is a minimum trajectory. Then, from Lagrange Multiplier’s Theorem we have that \( c \) must be a critical value of the functional
\[
\mathcal{J}(c, \lambda_a) = \int_0^T \left( \frac{1}{2} \left( \frac{D^2 c}{dt^2}, \frac{D^2 c}{dt^2} \right) + \lambda a \left( \mu^a(c(t)), \frac{D^2 c}{dt^2} \right) \right) dt,
\]
which means that
\[
\frac{d}{ds} \mathcal{J}(c_s, \lambda_{a,s}) \bigg|_{s=0} = 0
\]
for all variations of the curve \((c, \lambda)\), i.e., for all maps \((c_s, \lambda_{a,s})\) satisfying \((c_0, \lambda_{a,0}) = (c, \lambda)\). By computing the critical values of \( \mathcal{J} \), we find that the first term gives
\[
\int_0^T \left( \frac{D^4 c}{dt^4} + R \left( \frac{D^2 c}{dt^2}, \dot{c} \right) \dot{c} \right),
\]
while the second term splits in two sub-terms.

The first one is
\[
\int_0^T \delta \lambda a \left( \mu^a(c(t)), \frac{D^2 c}{dt^2} \right) dt
\]
with \( \delta \lambda := \frac{\partial \lambda_{a,s}}{\partial s} \bigg|_{s=0} \) and the second is
\[
\int_0^T \lambda a \frac{d}{ds} \bigg|_{s=0} \left( \mu^a(c_s(t)), \frac{D^2 c_s}{dt^2} \right) dt,
\]
and in order to deal with it let
us consider again the vector fields $X_a \in \mathfrak{X}(Q)$ defined in equation \((\text{3})\), i.e., $b(X_a) = \mu^a$. Then
\[
\frac{d}{ds}\bigg|_{s=0} \left\langle X_a(c(t)), \frac{D^2c}{dt^2} \right\rangle = \left\langle \frac{DX_a}{ds}(c(t)), \frac{D^2c}{dt^2} \right\rangle + \left\langle X_a(c(t)), \frac{D}{ds} \frac{D^2c}{dt^2} \right\rangle. \tag{8}
\]
Using Lemma 1 with $X = \frac{D^2c}{dt^2}$ and $Y = \delta c$, we have that the first term in equation \((\text{8})\) gives
\[
\left\langle \frac{DX_a}{ds}(c(t)), \frac{D^2c}{dt^2} \right\rangle = \left\langle \nabla_{\frac{D^2}{dt^2}} X_a, \delta c \right\rangle - \left\langle S_a \left( \frac{D^2c}{dt^2} \right), \delta c \right\rangle.
\]
Observe that $\nabla_{\frac{D^2}{dt^2}} X_a$ is well-defined and its value may be defined using an extension of $\frac{D^2}{dt^2}$ to a vector field $Z$ satisfying $Z(c(t)) = \frac{D^2c}{dt^2}$. Then, $\nabla_{\frac{D^2}{dt^2}} X_a = \nabla_Z X_a|_{c(t)}$.

To simplify the second term in equation \((\text{8})\) we first use Lemma 2 obtaining
\[
\left\langle X_a(c(t)), \frac{D^2c}{dt^2} \right\rangle = \left\langle X_a(c(t)), \frac{D^2c}{dt^2} \delta c + R(\delta c, \dot{\delta} c) \right\rangle.
\]
Using the symmetries from the curvature tensor the term involving the curvature reduces to $\left\langle R(X_a, \dot{\delta} c) \right\rangle c, \delta c)$ while the other term is simplified using integration by parts. Indeed we get that
\[
\int_0^T \lambda_a \left\langle X_a, \frac{D^2c}{dt^2} \delta c \right\rangle = - \int_0^T \left( \lambda_a \left\langle X_a, \frac{D}{dt} \delta c \right\rangle + \lambda_a \left\langle \frac{DX_a}{dt}, \frac{D}{dt} \delta c \right\rangle \right) dt + \left[ \lambda_a \left\langle X_a, \frac{D}{dt} \delta c \right\rangle \right]_{t=0}^{t=T}.
\]
Integrating by parts again we get on one hand
\[
\int_0^T \lambda_a \left\langle X_a, \frac{D}{dt} \delta c \right\rangle = - \int_0^T \left( \lambda_a \left\langle X_a, \delta c \right\rangle + \lambda_a \left\langle \frac{DX_a}{dt}, \delta c \right\rangle \right) dt + \left[ \lambda_a \left\langle X_a, \delta c \right\rangle \right]_{t=0}^{t=T},
\]
while on the other hand we get
\[
\int_0^T \lambda_a \left\langle \frac{DX_a}{dt}, \frac{D}{dt} \delta c \right\rangle = - \int_0^T \left( \lambda_a \left\langle \frac{DX_a}{dt}, \delta c \right\rangle + \lambda_a \left\langle \frac{D^2X_a}{dt^2}, \delta c \right\rangle \right) dt + \left[ \lambda_a \left\langle \frac{DX_a}{dt}, \delta c \right\rangle \right]_{t=0}^{t=T}.
\]
Therefore, the curves $(c, \lambda)$ are a critical value of $J$ if the integral
\[
\int_0^T \left( \frac{D^4c}{dt^4} + R \left( \frac{D^2c}{dt^2}, \dot{c} \right) \dot{c} + \lambda_a X_a + 2\lambda_a \frac{DX_a}{dt} + \lambda_a \frac{D^2X_a}{dt^2} + R(X_a, \dot{c}) \dot{c} \right) \right. \\
+ \left. \lambda_a \left\langle \nabla_{\frac{D^2}{dt^2}} X_a - S_a \left( \frac{D^2c}{dt^2} \right), \delta c \right\rangle + \delta \lambda_a \left\langle X_a, \frac{D^2c}{dt^2} \right\rangle \right) dt
\]
vanishes, since all the boundary terms vanish. By applying the fundamental lemma of calculus of variations, we conclude the proof.

□

Example 1. Suppose we have a planar rigid body which is free to move on every direction of the plane. Its configuration space is \( SE(2) \cong \mathbb{R}^2 \times S^1 \) parametrized by \((x, y, \theta)\). Its Riemannian metric is given by \( g = m(dx \otimes dx + dy \otimes dy) + Jd\theta \otimes d\theta \). Hence, the corresponding Levi-Civita connection has vanishing Christoffel symbols. However, this rigid body can only be actuated in some directions. In particular, we have the following dynamic underactuated control system

\[
\begin{align*}
    m\ddot{x} &= \cos \theta \ u_r, \\
    m\ddot{y} &= \sin \theta \ u_r, \\
    J\ddot{\theta} &= u_{\theta}.
\end{align*}
\]

Consider the optimal control problem consisting of the dynamical equations above and by the cost function given by

\[
\int_0^T \frac{1}{2} \left( u_r^2 + u_{\theta}^2 \right) dt.
\]

Notice that the control system implies the following constraints on the covariant accelerations

\[
\mu \left( \frac{D^2 c}{dt^2} : \frac{D^2 c}{dt^2} \right) = \sin \theta \ddot{x} - \cos \theta \ddot{y} = 0 \quad \text{with} \quad \mu = \sin \theta dx - \cos \theta dy.
\]

Thus, the optimal control problem is equivalent to the constrained variational problem

\[
\int_0^T \frac{1}{2} \left( \frac{D^2 c}{dt^2} : \frac{D^2 c}{dt^2} \right) dt \quad \text{and} \quad \left( \mu, \frac{D^2 c}{dt^2} \right) = 0.
\]

Hence, necessary conditions for normal extremals are given by

\[
\begin{align*}
    x^{(4)} + \frac{\dot{\lambda}}{m} \sin \theta + 2 \frac{\dot{\lambda}}{m} \dot{\theta} \cos \theta + \frac{\lambda}{m} (\dot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) &= 0, \\
    y^{(4)} - \frac{\dot{\lambda}}{m} \cos \theta + 2 \frac{\dot{\lambda}}{m} \dot{\theta} \sin \theta + \frac{\lambda}{m} (\dot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) &= 0, \\
    \theta^{(4)} + \frac{\lambda}{J} (\cos \theta \ddot{x} + \sin \theta \ddot{y}) &= 0,
\end{align*}
\]

where we used the fact that \( X = \frac{\sin \theta}{m} \frac{\partial}{\partial x} - \frac{\cos \theta}{m} \frac{\partial}{\partial y} \) and also that \( \nabla_{\frac{D^2 c}{dt^2}} X = \ddot{\theta} \left( \frac{\cos \theta}{m} \frac{\partial}{\partial x} + \frac{\sin \theta}{m} \frac{\partial}{\partial y} \right) \) and

\[
S \left( a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} \right) = \frac{c}{m} \cos \theta \frac{\partial}{\partial x} + \frac{c}{m} \sin \theta \frac{\partial}{\partial y} - \frac{a \cos \theta + b \sin \theta}{J} \frac{\partial}{\partial \theta}.
\]

4 Application to obstacle avoidance with constraints on the accelerations

Now consider the problem of finding a curve \( c \in \Omega \) satisfying the constraints given by \((6)\) and, for \( V : Q \to \mathbb{R} \) a potential function minimize the functional

\[
\int_0^T \frac{1}{2} \left( \left\langle \frac{D^2 c}{dt^2}, \frac{D^2 c}{dt^2} \right\rangle + \sigma \left\| \frac{dc}{dt} \right\|^2 + V(c(t)) \right) dt.
\]

We will now deduce necessary conditions in order for a curve \( c : [0, T] \to Q \) to be an extremal of the constrained variational problem above.
Theorem 2. A necessary condition for a curve $c$ to be a normal extremal of the previous functional subjected to the constraints in (6) is that it satisfies

$$
\frac{D^4c}{dt^4} + R \left( \frac{D^2c}{dt^2}, \dot{c} \right) \dot{c} - \sigma \frac{D^2c}{dt^2} + \lambda_a X_a + 2\lambda_a \frac{DX_a}{dt} + \lambda_a \frac{D^2X_a}{dt^2} \\
+ \lambda_a \left( R(X_a, \dot{c}) \dot{c} + \nabla \frac{D^2c}{dt^2} X_a - S_a \left( \frac{D^2c}{dt^2} \right) \right) + \frac{1}{2} \nabla V = 0,
$$

where $V : Q \rightarrow \mathbb{R}$ is a potential function.

Proof. The result follows directly from the proof of Theorem 1 to which we must add the variation of the terms containing the potential function $V$ and the velocities. Using the notation in that proof, on one hand we have that

$$
\frac{1}{2} \int_0^T \frac{d}{ds} \left| V(c_s(t)) \right| ds = \frac{1}{2} \int_0^T dV(c(t))(\delta c) dt.
$$

By definition, $\nabla V \in \mathfrak{X}(Q)$ is the unique vector field satisfying $dV(q)(Z) = \langle \nabla V(q), Z \rangle$, $\forall \, Z \in \mathfrak{X}(Q)$, from where the term containing $\nabla V$ follows.

On the other hand, we have

$$
\frac{1}{2} \int_0^T \left| \frac{dc}{dt} \right|^2 dt = - \int_0^T \sigma \left( \nabla \frac{D^2c}{dt^2}, \delta c \right) + [\langle \delta c \rangle]_{s=0}^T,
$$

from where the result follows, since the boundary terms vanish. □

Example 2. Suppose we have again the planar rigid body from the last example. Suppose there is a circular shaped obstacle with centre located at $(0,0)$ and radius $r$ in the $xy$-plane. Consider the artificial obstacle avoidance potential given by $V(x,y,\theta) = \frac{\tau}{x^2 + y^2 - r^2}$, penalizing collision with the obstacle, where $\tau > 0$. Normal extremals must satisfy the equations

$$
x^{(4)} - \sigma \ddot{x} + \frac{\lambda}{m} \sin \theta + 2 \frac{\lambda}{m} \dot{\theta} \cos \theta + \frac{\lambda}{m} (\dot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) - \frac{\tau x}{(x^2 + y^2 - r^2)^2} = 0,
$$

$$
y^{(4)} - \sigma \ddot{y} - \frac{\lambda}{m} \cos \theta + 2 \frac{\lambda}{m} \dot{\theta} \sin \theta + \frac{\lambda}{m} (\dot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) - \frac{\tau y}{(x^2 + y^2 - r^2)^2} = 0,
$$

$$
\theta^{(4)} - \sigma \ddot{\theta} \cos \theta + \frac{\lambda}{m} \sin \theta - \sin \theta \dot{\theta} = 0.
$$

In Figure 1 we show a simulation of our method. A shooting method and a fourth order Runge-Kutta method with $h = 0.1$, $N = 55$, $T = Nh$, are used to simulate the boundary value problem. The curve represents a normal extremum avoiding a static obstacle with $r = 0.1$. The parameters for the trajectory used are $m = 1$, $J = 2$, $\sigma = 0.1$, $\tau = 1$. Boundary condition are given by: $q(0) = (-1, -1, 0)$, $q(T) = (1, 1, 0)$, $v(0) = (0.1, 0, 0.2)$, $v(T) = (0, 0, 0)$,
5 Conclusions and future work

We have studied variational problems on Riemannian manifolds with constrained acceleration, derived necessary conditions for normal extrema, and we have also established the close relation of our problem with affine connection control systems.

Interesting questions may arise from the study of variational problem associated with the optimal control of affine connection control system. For instance, it is known that a dynamic control nonholonomic system, might be expressed as an affine connection control system using the nonholonomic connection $\nabla^{nh}$ (see [19]) giving equations of the form: $\nabla_{\dot{c}}^{nh,\dot{c}} = u^i Y_i$, with $\{Y_i\}$ spanning $\mathcal{D}$. It would be interesting to study the corresponding variational problem.

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References

1. Abrunheiro, L., Camarinha, M.: Optimal control of affine connection control systems from the point of view of Lie algebroids. Int. J. Geom. Methods Mod. Phys. 11 (2014), no. 9, 1450038, 8 pp.
2. Assif, M., Banavar, R., Bloch, A., Camarinha, M. and Colombo, L.: Variational collision avoidance problems on Riemannian manifolds. 2018 IEEE Conference on Decision and Control (CDC), pp. 2791-2796 (2018).
3. Batzies, E., Hüpner, K., Machado, L. and Silva Leite, F. Geometric mean and geodesic regression on Grassmannians, Linear Algebra Appl., 466 (2015), 83–101.
4. A. Bloch, L. Colombo, R. Gupta, and D. M. de Diego, A Geometric Approach to the Optimal Control of Nonholonomic Mechanical Systems. Cham: Springer International Publishing, 2015, pp. 35–64.
5. Bloch, A., Camarinha, M., Colombo, L.: Variational obstacle avoidance problem on Riemannian manifolds. in Proceedings of the IEEE International Conference on Decision and Control, 2017, pp. 146-150.
6. Bloch, A., Camarinha, M., Colombo, L. Dynamic interpolation for obstacle avoidance on Riemannian manifolds. International Journal of Control, 94(3), 588-600, 2019.
7. Bloch, A., Crouch, Nonholonomic and vakonomic control systems on Riemannian manifolds, Fields Institute Comm. 1, 25-52, 1993.
8. Bloch A and Crouch P Nonholonomic control systems on Riemannian manifolds, SIAM J. Control Optim., 33 (1995), pp. 1266-148
9. Bloch,A. M., Gupta, R., Kolmanovsky, V. I.: Neighboring extremal optimal control for mechanical systems on Riemannian manifolds. Journal of Geometric Mechanics, 8 (3), 257-272 (2016).
10. Bullo, F., & Lewis, A.: Geometric Control of Mechanical Systems. Modeling, Analysis, and Design for Simple Mechanical Control Systems. Texts in Applied Mathematics. Springer (2004).
11. Camarinha, M., Silva Leite, F., and Crouch, P.: On the geometry of Riemannian cubic polynomials. Differential Geometry and its Applications, 15(2), 107-135 (2001).
12. Camarinha, M., Silva Leite, F., and Crouch, P.: Riemannian cubics close to geodesics at the boundaries. Journal of Geometric Mechanics. OnlineFirst, 2022. doi: 10.3934/jgm.2022003.
13. Chandrasekaran, R., Colombo, L.J., Camarinha, M., Banavar, R.N., & Bloch, A.M.: Variational collision and obstacle avoidance of multi-agent systems on Riemannian manifolds. 2020 European Control Conference (ECC), 1689-1694, (2020).
14. Colombo, L., Goodman, J.: A Decentralized Strategy for Variational Collision Avoidance on Complete Riemannian Manifolds. Portuguese Conference on Automatic Control. Springer, Cham, 2020, p. 363-372.
15. Crouch, P., Silva Leite, F.: The dynamic interpolation problem: On Riemannian manifolds, Lie groups, and symmetric spaces. Journal of Dynamical and Control Systems, 1,177-202 (1995).
16. Goodman, J., Colombo, L. Collision Avoidance of Multiagent Systems on Riemannian Manifolds. SIAM Journal on Control and Optimization, 60(1), 168–188, 2022.
17. Goodman, J., Colombo, L.: Variational Obstacle Avoidance with Applications to Interpolation Problems in Hybrid Systems. Proceedings of the 7th IFAC Workshop on Lagrangian and Hamiltonian Methods in Nonlinear Control, 2021. IFAC-PapersOnLine 54.19 (2021): 82-87.
18. Goodman, J: Local Minimizers for Variational Obstacle Avoidance on Riemannian manifolds, arXiv preprint arXiv:2201.04395 (2022).
19. Lewis, A., Murray, R. Decompositions for control systems on manifolds with an affine connection. Systems & Control Letters, 31 (4),199-205, (1997).
20. Lewis, A. Affine Connection Control systems. Proceedings of the IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control 128-133 (2000).
21. Milnor, J. Morse Theory. Princeton, NJ: Princeton Univ. Press, 2002.
22. Noakes, L., Heinzinger G., Paden, B. Cubic Splines on Curved Spaces, IMA Journal of Math. Control & Inf. 6, (1989), 465–473.