THE BIRKHOFF THEOREM IN MULTIDIMENSIONAL GRAVITY

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ABSTRACT

The validity conditions for the extended Birkhoff theorem in multidimensional gravity with \( n \) internal spaces are formulated, with no restriction on space-time dimensionality and signature. Examples of matter sources and geometries for which the theorem is valid are given. Further generalization of the theorem is discussed.

1 INTRODUCTION

The original Birkhoff theorem [1] states that in general relativity (GR) the spherically symmetric vacuum field is static and is thus reduced to the Schwarzschild solution. From a wider viewpoint, the theorem indicates a case when the field equations induce, under certain circumstances, an additional field system symmetry that was not postulated at the outset. The theorem is closely related to the quadrupole nature of the gravitational field in GR, more precisely, to the absence of monopole gravitational waves. Thus theorems of this sort are able not only to simplify the treatment of certain physically relevant situations but also to provide their better understanding.

After Birkhoff the theorem was extended to spherical systems with a nonzero cosmological constant \( \Lambda \), the Maxwell or Born-Infeld electromagnetic fields ([2, 3] and others), scalar fields and \( \Lambda \neq 0 \) in GR [4] and some scalar-tensor theories of gravity [5]. In Ref. [6] the theorem was extended to planarly and pseudospherically symmetric Einstein-Maxwell fields.

Another approach was suggested in Refs. [7, 8]: the study was aimed at finding out general conditions under which the staticity theorem could be proved. This allowed all the previously found cases of GR and scalar-tensor theories when the extended Birkhoff theorem is valid, to be included, along with many new ones. The theorem was generalized in two respects: to include more types of space-time symmetry (e.g., planar, cylindrical and pseudoplanar) and more kinds of matter (scalar fields, gauge fields, perfect fluid, etc.).

Here we would like to extend the approach of [7, 8] to multidimensional gravity. One may recall that most modern unification theories incorporate more than four dimensions (e.g., that of superstrings [9]); on the other hand, some studies are undertaken in (2+1) and even (1+1) dimensions where certain hard problems simplify and admit a deeper insight. The low energy limit of many theories, actually embracing an enormous range of energy scales, is reduced to the multidimensional Einstein equations

\[
G^B_A \equiv R^B_A - \delta^B_A R^C_C/2 = -T^B_A, \tag{1}
\]

1
where $R^B_A$ is the $D$-dimensional Ricci tensor and $T^B_A$ is the matter energy-momentum tensor (EMT). We will assume the validity of (1) for some dimension $D$ and some kind of matter and find certain general conditions under which these field equations make the system symmetry increase. The consideration essentially follows the lines of [4, 5]. In Section 2 the extended Birkhoff theorem is proved for multidimensional GR. In Section 3 its different special cases are discussed and Section 4 contains some remarks, in particular, on situations excluded by the requirements of the theorem; its further extension to multidimensional scalar-tensor theories is presented.

**2. THEOREM**

Consider a $D$-dimensional Riemannian or pseudo-Riemannian space with the structure

$$V^D = M^2 \times V_1 \times V_2 \times \ldots \times V_n, \quad \dim V_i = N_i, \quad n = 1, 2, \ldots \quad (2)$$

where $M^2$ is an arbitrary two-dimensional subspace parametrized by the coordinates $u$ and $v$ and $V_i$ are subspaces of arbitrary dimension ($N_i$) and signature whose metric depends on $u$ and $v$ only via conformal (scale) factors. Thus with no further loss of generality the $D$-dimensional metric may be written in the form

$$ds^2_d = \eta_u e^{2\alpha} du^2 + \eta_v e^{2\gamma} dv^2 + \sum_{i=1}^n e^{2\beta_i} ds^2_i \quad (3)$$

where $\eta_u = \pm 1$, $\eta_v = \pm 1$; $\alpha$, $\beta_i$ and $\gamma$ are functions of $u$ and $v$ and $ds^2_i$ are the $u$- and $v$-independent metrics of the subspaces. It is meant that $M^2$ along with one (two-dimensional) or two (one-dimensional) subspaces $V_i$ form the conventional physical space-time while the rest $V_i$ correspond to extra (internal) dimensions. For greater generality we would not like to fix the signs $\eta_u$ and $\eta_v$.

Before formulating the theorem let us introduce the quantity

$$\rho(u, v) \equiv \sum_{i=1}^n N_i \beta_i(u, v). \quad (4)$$

and present the nonzero Ricci tensor components for the metric (3):

$$R^u_u = \Box_u \gamma + \Box_v \alpha + \eta_u e^{-2\alpha}(\rho'' - \alpha' \rho' + \sum_{i=1}^n \beta_i'^2) + \eta_v e^{-2\gamma} \rho \hat{\rho}; \quad (5)$$

$$R^v_v = \Box_u \gamma + \Box_v \alpha + \eta_u e^{-2\alpha} \alpha' \rho' + \eta_v e^{-2\gamma} (\hat{\rho} - \gamma \hat{\rho} + \sum_{i=1}^n \beta_i^2); \quad (6)$$

$$R_{n_i} = e^{-2\beta_i} \mathcal{T}_{n_i} + \delta_{n_i} [((\Box_u + \Box_v) \beta_i + \eta_u e^{-2\alpha} \beta_i \rho' + \eta_v e^{-2\gamma} \hat{\beta}_i); \quad (7)$$

$$R_{uv} = \rho' - \gamma' \hat{\rho} - \alpha \rho' + \sum_{i=1}^n \beta_i \beta_i' \quad (8)$$

where primes and dots stand for partial derivatives $\partial_u$ and $\partial_v$, respectively, and

$$\Box_u = e^{-\alpha - \gamma} \partial_u (e^{\gamma - \alpha} \partial_u), \quad \Box_v = e^{-\alpha - \gamma} \partial_v (e^{\alpha - \gamma} \partial_v), \quad (9)$$
The indices \( m_i \) and \( n_i \) belong to the subspace \( V_i \); the Ricci tensors \( R_{n_i}^{m_i} \) correspond to the metrics \( ds_i^2 \) and do not depend on \( u \) and \( v \).

**Theorem 1.** Let there be a Riemannian space \( V^D \) (2) with metric (3) obeying the Einstein equations (1). If

(A) there is a domain \( \Delta \) in \( M^2 \) where

\[
\text{sign } (\rho^A \rho, A) = \eta_u; \tag{10}
\]

(B) each \( \beta_i(u, v) \) in \( \Delta \) is functionally related to \( \rho \) (certain relations \( F_i(\rho, \beta_i) = 0 \) are valid);

(C) in an orthogonal coordinate frame where \( \rho = \rho(u) \) (its existence is guaranteed by (10)) the EMT component \( T_{uv} = 0 \) and there is a combination

\[
T_v^u + \text{const} \cdot T_u^u \tag{11}
\]

independent of \( v \) and \( \gamma \),

then there is an orthogonal coordinate frame \( (u, v) \) in \( \Delta \) such that the metric (3) is \( v \)-independent.

**Proof.** Let us choose an orthogonal coordinate frame where \( \rho = \rho(u) \), which is possible by Condition A. Then by Condition B all \( \dot{\beta}_i = 0 \). By Condition C the mixed EMT component \( T_{uv} = 0 \) (in the conventional case \( \eta_u = -\eta_v \) that means that there is no energy flow in the frame of reference where \( \rho = \rho(u) \)), and the corresponding component of Eqs.(1) yields \( \dot{\alpha} = 0 \). Now only \( \gamma \) may depend on \( v \). To make the last step and to obtain \( \gamma = \gamma(u) \) it is sufficient to find a combination of the Einstein equations having the form \( \gamma' = f(u) \), whence

\[
\gamma = \gamma_1(u) + \gamma_2(v) \tag{12}
\]

and \( \gamma_2 \) may be brought to zero by a coordinate transformation \( v = v(\bar{v}) \). Observing (5-7), one can see that any combination of the form \( G_u^u + \text{const} \cdot G_v^v \) of the left-hand sides of (1) does contain \( \gamma \) but only in the term \( e^{-2\alpha} \rho' \gamma' \). As \( \rho \neq \text{const} \), our problem is solved when the corresponding combination of \( T_{uv}^D \) does not depend on \( \gamma \) and \( v \), exactly what is required in Condition C. This completes the proof.

The theorem generalizes the results of [4, 8] to arbitrary space-time dimension and signature, including multidimensional Kaluza-Klein-type models with a chain of internal spaces each with a scale factor of its own, such as considered in, e.g., [11, 12].

### 3 SPECIAL CASES

In the following examples, unless otherwise indicated, we will adhere to the conventional interpretation of the Birkhoff theorem, i.e., assume that \( v \) is time (\( \eta_v = 1 \) and \( u \) is a space variable (\( \eta_u = -1 \)). Everything may be easily reformulated for coinciding \( \eta_u \) and \( \eta_v \). No assumptions on the signatures of \( V_i \) are made since they do not affect the conclusions.

In general, the following matter sources satisfy the requirements C of Theorem 1 with no further restrictions on the structure of \( V^D \):
(a) Linear or nonlinear, minimally coupled scalar fields with the Lagrangian $L = \varphi^A \varphi_A - V(\varphi)$ where $V(\varphi)$ is an arbitrary function, under the restriction $\varphi = \varphi(u)$:

$$2T^A_B = \delta^A_B V(\varphi) + \eta_v e^{-2\alpha} \varphi^2 \text{diag}(1, -1, \ldots, -1);$$  \hspace{1cm} (13)

Here and henceforth positions in “diag” are ordered by the scheme $(u, v, \ldots)$.

(b) A massless, minimally coupled scalar field ($L = \varphi^A \varphi_A$) under the restriction $\varphi = \varphi(v)$: the EMT does not contain $\varphi$ but only $\dot{\varphi} = \text{const}$ (the so-called cosmological scalar field):

$$2T^A_B = \eta_v e^{-2\gamma} \rightarrow \dot{\varphi}^2 \text{diag}(-1, 1, -1, \ldots, -1).$$  \hspace{1cm} (14)

(c) Abelian gauge fields ($L = -F^{AB} F_{AB}$, $F^{AB} = \partial_A U_B - \partial_B U_A$) under the restriction that the vector potential $U_A$ has a single nonzero component $U_K(u)$, with a fixed coordinate $K \neq v$, so that among $F_{AB}$ only $F_{uK} = -F_{Ku} \neq 0$:

$$T^u_u = T^K_K = -F^{uK} F_{uK}; \quad \text{other } T^A_B = \delta^A_B \varphi^2 F_{uK}. \hspace{1cm} (15)$$

(d) Nonlinear vector fields with Lagrangians of the form $\Phi(I)$, $I = F^{AB} F_{AB}$, where $\Phi$ is an arbitrary function, under the same restriction as that in item (c) but with $K = v$ (an example is the Born-Infeld nonlinear electromagnetic field):

$$T^u_u = T^v_v = 2(d\Phi/dI) F^{uv} F_{uv} - \Phi/2; \quad \text{other } T^A_B = -\delta^A_B \Phi/2. \hspace{1cm} (16)$$

(e) Some kinds of interacting fields: for instance, the system of an Abelian gauge field and a scalar dilaton field ($L = \varphi^A \varphi_A - e^{2\lambda \varphi} F^{AB} F_{AB}$, $\lambda = \text{const}$) under the constraints of items (a) and (c): the EMT structure combines those of (13) and (15). As $\eta_v$ and $\gamma$-independent, evidently the second condition C of Theorem 1 is satisfied by one of the two combinations $T^u_u \pm T^v_v$. This is just the interaction relevant for multidimensional dilatonic black holes [11-13, 16].

The same is true if the expression $e^{2\lambda \varphi}$ in the Lagrangian is replaced by any function of $\varphi$.

(f) The cosmological term $\Lambda \delta^A_B$ may be added to the left-hand side of (1) with no consequences.

One can easily find other forms of matter, as well as combinations of the above forms of matter and other ones, for which Theorem 1 holds.

As for the diversity of space-time structures to which the theorem applies, it is also very wide. In the 4-dimensional case it includes the symmetries mentioned in [3, 8], namely: spherical, planar, pseudospherical, pseudoplanar, cylindrical, toroidal ($V^D = M^2 \times S^2$, $M^2 \times R^2$, $M^2 \times L^2$, $M^2 \times R^1 \times R^1$, $M^2 \times R^1 \times S^1$, $M^2 \times S^1 \times S^1$, respectively, where $L^2$ is the Lobachevsky plane). It applies to both conventional (Lorentzian) GR and its “Euclidean” counterpart, as well as to Kaluza-Klein type models with a chain of internal spaces with $u$-dependent scale factors. Moreover, multidimensional extensions may incorporate generalized
spherical and other symmetries in the spirit of Tangherlini [14], i.e., $S^m$, $R^m$ or $L^m$ with an arbitrary $m > 2$ instead of $S^2$, $R^2$, $L^2$.

For space-times with horizons, such as the black-hole ones, the theorem states the metric independence on different coordinates in different domains of $M^2$: thus, in the conventional Schwarzschild case it fixes the $t$-independence (staticity) in the $R$ domain and $r$-independence (homogeneity) in the $T$ domain. The same applies to multidimensional black holes considered in many papers (e.g., [11-16]).

Another point of interest is the existence of Abelian gauge fields of various directions which satisfy the theorem, see the above item (c). One may recall such evident examples as Coulomb-like fields for spherical and other similar symmetries, radial, longitudinal and azimuthal electric fields for conventional cylindrically symmetric space-times (and their magnetic counterparts); however, there are configurations with $u$-dependent vector potential components directed in extra dimensions, deserving a separate treatment.

4 COMMENTS

4.1. Condition A of Theorem 1 may be weakened if $M^2$ is a proper Riemannian space ($\eta_u = \eta_v$): instead of (10), it is sufficient to assume just $\rho \neq \text{const}$. Indeed, in this case the orthogonal coordinates $u$ and $v$ may be always chosen so that $\rho = \rho(u)$, for instance, one may put just $u = \rho$.

If $M^2$ is pseudo-Riemannian ($\eta_u = -\eta_v$), then the gradient of $\rho(u, v)$ may be either $u$-like, or $v$-like, or null (sign $\left(\rho^A \rho_A^x\right) = \eta_u, \eta_v, 0$, respectively). To extend the theorem to the case when it is $v$-like one may just change the notations of the coordinates, $u \leftrightarrow v$, irrespective of which of them is spacelike. So in both cases one can achieve $\rho = \rho(u)$.

4.2. Cancellation of Condition B of Theorem 1 (possible if $n > 1$) leads to the existence of at least two functionally independent unknowns. The situation is most obviously exemplified by the Einstein-Rosen vacuum cylindrical gravitational waves [10] ($D = 4$, $N_1 = N_2 = 1$, $v = t$, i.e., time).

More generally, extra-dimension scale factors behave like minimally coupled scalar fields in 4 dimensions, so possible monopole waves may be eliminated just at the expense of the additional assumption B. When the latter is removed, these waves may manifest the instability of static configurations (as is the case for many non-black-hole spherically symmetric multidimensional space-times [11, 13].

4.3. The possibility $\rho = \text{const}$, excluded in the theorem, looks somewhat exotic in the spherically symmetric case but is quite natural for, say, planar symmetry. Let us show that it leads to wave solutions to Eqs. (1) taking as an example a 4-dimensional vacuum space-time with a cosmological constant ($T_{\beta}^A = \delta_{\beta}^A \Lambda$), possessing spherical or planar symmetry ($D = 4$, $n = 1$, $N_1 = 2$; $\mathbf{R}_2^2 = \mathbf{R}_3^3 = \epsilon = +1$, $0$, respectively). In this case $\rho = 2\beta_1$ and Eqs. (1) give:

$$ee^{-\rho} = \Lambda; \quad e^{-2\alpha}(\eta_u \alpha'' + \eta_v \dot{\alpha}) = \Lambda \quad (17)$$

where the coordinates are chosen so that the metric of $M^2$ is conformally flat ($\alpha = \gamma$). There are the following variants:
• $\epsilon = 1$, $\eta_u = -\eta_v$; $\Lambda > 0$. The space-time is formed by a congruence of spheres of equal radii and the only nontrivial metric coefficient $\alpha$ obeys a nonlinear wave equation.

• $\epsilon = 1$, $\eta_u = \eta_v$; $\Lambda > 0$. The same but the equation is nonlinear, elliptic type.

• $\epsilon = 0$, $\eta_u = -\eta_v$; $\Lambda = 0$. Linear waves in a planarly symmetric space-time: $\alpha = \gamma = f_1(u + v) + f_2(u - v)$.

• $\epsilon = 0$, $\eta_u = \eta_v$; $\Lambda = 0$. The only nontrivial metric coefficient $\alpha = \gamma$ is a harmonic function of $u$ and $v$.

4.4. Another possibility rejected in Theorem 1 is that $\rho(u, v)$ has a null gradient in a pseudo-Riemannian $M^2$. The condition $\rho^A \rho_{,A} = 0$ in the coordinates such that $\alpha = \gamma$ leads to $\dot{\rho} = \pm \rho'$. Let us choose the plus sign (re-defining $u \rightarrow -u$ if required), so that $\rho = \rho(\xi)$, $\xi = u + v$.

Consider for instance vacuum planarly symmetric space-times of any dimension $D$, so that

$$V^D = M^2 \times R^{D-2}; \quad \rho = (D - 2)\beta; \quad T^A_B = 0. \quad (18)$$

Substituting $\rho = \rho(\xi)$ and $\alpha = \gamma$ for (18) to Eqs. (1), we obtain:

$$\alpha = \alpha_1(\xi) + \alpha_2(\eta); \quad 2\alpha_1(\xi) = \ln |\beta'| + \beta, \quad \beta = \beta(\xi) \quad (19)$$

where $\eta = u - v$; $\beta(\xi)$ and $\alpha_2(\eta)$ are arbitrary functions. This is a planarly symmetric vacuum wave solution to Eqs. (1) for any dimension $D$.

The examples of items 4.3 and 4.4 show that in Theorem 1, establishing the sufficient conditions for staticity, no condition may be omitted or essentially weakened.

4.5. A case of interest is the one when some $\beta_i$ are only $u$-dependent while others linearly depend on $v$, so that $\dot{\beta}_i = \text{const}$. If still $\dot{\rho} = 0$ and the spaces $V_i$ corresponding to $\dot{\beta}_i \neq 0$ are Ricci-flat, then the proof of Theorem 1 may be properly modified to conclude that all the remaining metric coefficients are $v$-independent.

Imagine, e.g., a 4-dimensional, static, spherically symmetric space-time ($u = r$, radial coordinate; $v = t$, time; $V_1 = S^2$, a 2-dimensional sphere) accompanied by internal Ricci-flat spaces of which some are static (in general, $r$-dependent), others exponentially expanding ($\dot{\beta}_i = \text{const} > 0$) and still others exponentially contracting ($\dot{\beta}_i = \text{const} < 0$). The specific forms of all functions are to be found from Eqs. (1) with a relevant choice of matter. However, as $\dot{\rho} = 0$, this class of solutions cannot contain those in which all extra dimensions are contracting. (This certainly does not mean that such solutions cannot exist at all: they are just not covered by the present treatment.)

4.6. The consideration of Section 2 rests on the geometric structure of the space $V^D$, including certain symmetry requirements. However, the theorem contains no requirements to internal symmetries of the subspaces $V_i$ since no constraints on the dependence of $\overline{R}^{m_i}_n$ on the internal coordinates $y^{nu}$ have appeared. By Eqs. (1) this dependence is just the same as that in the EMT.

However, in most applications, as seen from the examples of Section 3, $V_i$ are either Ricci-flat, or constant curvature spaces: as the EMT is independent of the internal coordinates, the
same is true for $V_i$. Moreover, when all the diagonal components $T_{mi}^m$ are equal to each other, Eqs.(1) force the components $R_{mi}^m$ to be equal as well, while the off-diagonal components are zero. Thus it is the EMT symmetry that forces $V_i$ to be constant curvature spaces.

4.7. In [7, 8] the generalized Birkhoff theorem was extended to a broad class of scalar-tensor theories of gravity in 4 dimensions. The same can be done in the multidimensional case. To do that let us consider in $V^D$ with the metric $g_{MN}$ a scalar-tensor theory described by the Lagrangian

$$\sqrt{g}L = \sqrt{g}[A(\phi)R + B(\phi)\phi^M\phi_M - 2\Lambda(\phi) + L_m]$$  (20)

where $g = \det g_{MN} |; A > 0$, $B$ and $\Lambda$ are any smooth functions of $\phi$ and the matter Lagrangian $L_m$ may depend on both $g_{MN}$ and $\phi$. The conformal mapping (suggested by Wagoner [17] for $D = 4$)

$$g_{MN} = A^{-2/(D-2)}g_{MN}$$  (21)

brings (20) to the form (up to a divergence)

$$\sqrt{g}L = \sqrt{\bar{\gamma}}\left\{\bar{T} + \frac{1}{A^2}\left[AB + \frac{D-1}{D-2}\left(\frac{dA}{d\phi}\right)^2\right]g_{MN}^{\phi,\phi,\phi} + A^{-D/(D-2)}[-2\Lambda(\phi) + L_m]\right\}$$  (22)

where $\bar{\gamma} = \det g_{MN}$ and $\bar{\gamma}$ is the scalar curvature corresponding to $\bar{g}_{MN}$. Variation of (22) with respect to $\bar{g}_{MN}$ yields the Einstein equations with an EMT containing the contribution of the (possibly nonlinear) scalar field $\phi$ and that of matter coupled to $\phi$. For our purpose it is essential that the latter contribution $\bar{T}_{MN}$ coincides with the original EMT ($T_{MN} = (\delta/\delta g^{MN})(\sqrt{g}L_m)$ up to a $\phi$-dependent factor. Consequently, if $\phi = \phi(u)$ and $T_{MN}$ satisfies Condition C of Theorem 1, so does $\bar{T}_{MN}$ and Theorem 1 is applicable to the metric $\bar{g}_{MN}$. However, now it is $\bar{\rho} = \rho + \ln A$ that appears instead of $\rho$ in the formulation of the theorem. Therefore Theorem 1 cannot be directly applied to $g_{MN}$ and its formulation should be properly modified:

**Theorem 2.** Consider a field system with the Lagrangian (20) in a Riemannian space $V^D$ (2) with metric (3). Let there be a domain $\Delta$ in $M^2$ where

(i) all $\beta_i$ and the field $\phi$ are functions of $u$;

(ii) $\bar{\rho} = \rho + \ln A \neq \text{const}$ and

(iii) Conditions C of Theorem 1 are valid for the EMT derived from $L_m$.

Then the coordinate $v$ in $\Delta$ may be chosen so that all $g_{MN}$ are $v$-independent.

4.8. It would be of interest to try to extend the theorem to multidimensional models with nonzero off-diagonal metric components such as $g_{ui}$ with $i$ from extra dimensions, as is the case in the original Kaluza-Klein model. This goes beyond the scope of this paper, although probably such a generalization does exist since the new effective vector fields are unlikely to create monopole waves.

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