Unbounded Multi-Magnon and Spike

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ABSTRACT

We generalize the one magnon solution in $R \times S^2$ to unbounded $M$ magnon and find the corresponding solitonic string configuration in the string sigma model. This configuration gives rise to the expected dispersion relation obtained from the spin chain model in the large 't Hooft coupling limit. After considering $(M, M)$ multi-magnon or spike on $R \times S^2 \times S^2$ as a subspace of $AdS_5 \times S^5$ or $AdS_4 \times CP^3$, we investigate the dispersion relation and the finite size effect for $(M, M)$ multi-magnon or spike.

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1 Introduction

The AdS/CFT duality \[1\] relates type IIB string theory on AdS\(_5\) \(\times\) S\(_5\) with \(\mathcal{N} = 4\) superconformal Yang-Mills (SYM) theory, and it has been celebrated in the last decade as one of the exact duality between string and gauge theory. Recently there has been a lot of works devoted towards the understanding of the worldvolume dynamics of multiple M2-branes \[2, 3\], where a new class of conformal invariant 2+1 dimensional field theories has been found out.

In the development of AdS\(_5\)/CFT\(_4\) duality, an interesting observation is that the \(\mathcal{N} = 4\) SYM theory in planar limit can be described by an integrable spin chain model where the anomalous dimension of the gauge invariant operators were found \[4, 5, 6, 7, 8, 9\]. It was further noticed that the string theory is also integrable in the semiclassical limit \[10, 11, 12, 13, 14\] and the anomalous dimension of the \(\mathcal{N} = 4\) SYM can be derived from the relation among conserved charges of the rotating string AdS\(_5\) \(\times\) S\(_5\). In this connection, Hofman and Maldacena (HM) \[15\] considered a special limit where the problem of determining the spectrum in both sides becomes rather simple. The spectrum consists of an elementary excitation known as magnon that propagates with a conserved momentum \(\mu\) along the infinitely long \[16, 17\] or the finitely long \[18, 19, 20\] spin chain. In the dual formulation, the most important ingredient is the semiclassical string solution, which can be mapped to long trace operator with large energy and large angular momenta. The integrability of AdS\(_5\)/CFT\(_4\) in the planar limit using the Bethe ansatz brings us the hope that the recently proposed AdS\(_4\)/CFT\(_3\) duality will also be solvable by using a similar ansatz \[22\]. Indeed, in \[23, 24, 25, 26, 27, 28, 29\] this has been investigated and many interesting results were found. The magnon solutions were found in the diagonal SU(2) subsector of \(\mathbb{C}P^3\), which is similar to the result obtained from the AdS\(_5\) \(\times\) S\(_5\) case \[30, 31\].

In the present paper, we first investigate the dispersion relation of one magnon on \(R \times S^2\) corresponding to the diagonal SU(2) subsector. As will be shown, the dispersion relation on \(R \times S^2\) coming from AdS\(_5\) \(\times\) S\(_5\) and AdS\(_4\) \(\times\) CP\(_3\) has the same form except the different string
tension corresponding to the effective 't Hooft coupling. By considering the combination of magnons, we generalize one magnon to the configuration which consists of connected magnons. We will call this configuration as the unbounded multi-magnon or shortly, multi-magnon. We find the dispersion relation, which is consistent with the result obtained from the spin chain model in the large 't Hooft coupling limit. To find the $M$ multi-magnon, we investigate the boundary condition between magnons. In the infinite size case, the multi-magnon configuration is described by infinitely differentiable functions with satisfying the equations of motion. For the finite size case, the multi-magnon configuration is not infinitely differentiable at a joining point of two magnons. Since the multi-magnon configuration should be a solution of the equation of motion represented as a second order differential equation, we require that multi-magnon solution becomes differentiable functions at the second order. This requirement gives rise to a constraint between parameters. We show that the multi-magnon satisfies the equation of motion at a joining point of two magnons. Furthermore, the finite size effect are investigated. 

With the same strategy, multi-spike configuration is also investigated.

Secondly, the multi-magnon or spike solution on $\mathbb{R} \times S^2 \times S^2$, which is subspace of $AdS_5 \times S^5$ will be considered. In this case, we find out the more general multi-magnon or spike solution, which describes the combination of $M$ magnon or spike in the first $S^2$ and $M$ magnon or spike in the second $S^2$ so that we call it $(M,M)$ magnon or spike solution. In Ref. [57], using the algebraic curve, it was found that there exist $(M,N)$ multi-magnon solution. Here, using the string sigma model we find the string configuration corresponding to $(M,M)$ multi-magnon. It is still remaining problem to find the string configuration corresponding to $(M,N)$ multi-magnon. Here, we will leave that problem as the future works, and calculate the dispersion relation for $(M,M)$ multi-magnon and spike in the infinite size case. In addition, the finite size correction following Ref. [47] is also calculated.

The rest of the paper is organized as follows. In section 2, we investigate the universal form of the dispersion relation for a multi-magnon solution on $\mathbb{R} \times S^2$, which is a subspace of $AdS_5 \times S^5$ or $AdS_4 \times \mathbb{C}P^3$. Then, we obtain the dispersion relation for multi-magnon consistent with the spin chain’s result in the large ’t Hooft coupling limit. In section 3, we generalize one magnon to the multi-magnon on $\mathbb{R} \times S^2$. To obtain the multi string configuration satisfying the equation of motion, after considering the boundary condition we find a constraint relation between parameters and the dispersion relation for the multi-magnon consistent with the spin chain’s result in the large ’t Hooft coupling limit. In addition, the finite size effect of the $M$ multi-magnon is also investigated. In section 4, with the same method we find the dispersion relation and the finite size effect for $M$ multi-spike. In section 5, we generalize $M$ multi-magnon and spike on $\mathbb{R} \times S^2$ to $(M,M)$ multi-magnon and spike on $\mathbb{R} \times S^2 \times S^2$ and then find the dispersion relation and the finite size effect. In section 6, we finish our work with discussion.
2 One magnon on $R \times S^2$

According to the AdS/CFT correspondence, the string or the SUGRA (supergravity) theory on $AdS_m \times \mathcal{M}$, where $\mathcal{M}$ is a compact manifold, has a dual SYM theory description on the $AdS$ boundary. If there exists an $S^n$ subspace in the compact manifold $\mathcal{M}$, then the string sigma model describing the solitonic open string moving on $R_t \times S^n$ where $R_t$ and $S^n$ indicate time in the $AdS$ and an $n$-dimensional sphere respectively, is reduced to

$$S = \frac{T}{2} \int d^2 \sigma \sqrt{-\text{det}h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu},$$

(1)

where $h^{\alpha\beta}$ and $G_{\mu\nu}$ are metrics for the string world sheet and the target space $R_t \times S^n$, respectively. It is well known that the above string sigma model is classically integrable. This can be understood by the fact that through the Polmeyer reduction, the string sigma model is reduced to the generalized Sine-Gordon theory, which is integrable. Interestingly, the form of the string action does not depend on the details of the compact manifold $\mathcal{M}$ except that $T$ is related to the radius $R_s$ of $S^n$ as

$$T = \frac{R_s^2}{2\pi}.$$  

(2)

In this section, we will shortly review the universal form of the magnon’s dispersion relation in the $R_t \times S^2$ having the metric

$$ds^2 = -dt'^2 + R_s^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

(3)

After redefining $t' = R_s t$, the Polyakov string action, Eq. (1), in the conformal gauge becomes

$$S = \frac{T}{2} \int d^2 \sigma \left[ (\partial_\tau t)^2 - (\partial_\sigma t)^2 - (\partial_\tau \theta)^2 + (\partial_\sigma \theta)^2 - \sin^2 \theta \left\{ (\partial_\tau \phi)^2 - (\partial_\sigma \phi)^2 \right\} \right],$$

(4)

where the string tension is given in Eq. (2).

We first consider a point particle-like solution corresponding to the vacuum solution of the spin chain model with the following ansatz

$$t = \kappa \tau,$$
$$\theta = \frac{\pi}{2},$$
$$\phi = \nu \tau,$$

(5)

which satisfies all equations of motion for $t$, $\theta$ and $\phi$. In the string sigma model, since the world sheet metric $h_{\alpha\beta}$ is non-dynamical we should solve the Virasoro constraints. Under the ansatz in Eq. (5), $T_{\tau \sigma}$ and $T_{\sigma \sigma}$ vanish automatically. The remaining one is $0 = T_{\tau \tau} = \kappa^2 - \nu^2$. If we consider only positive $\kappa$ and $\nu$, we finally obtain

$$\kappa = \nu.$$  

(6)
Due to the translation and rotation symmetry in $t$ and $\phi$ coordinate, there exist two conserved charges

$$E = T\kappa,$$
$$J = T\nu.$$  \hspace{1cm} (7)

Using Eq. (6), the dispersion relation becomes

$$E - J = 0,$$  \hspace{1cm} (8)

which is interpreted as a BPS ground state of the SU(2) spin chain. According to the AdS/CFT correspondence, the closed string spectrum in the large N limit is dual to a single trace operator. When considering the open string which is the half of the closed string with relaxing the boundary condition, this open string is dual to an operator in the long open spin chain where $J$ corresponds to the length of the open spin chain. Furthermore, the isometry of $S^2$ corresponds to the SU(2) R-symmetry of the boundary gauge theory. The above dispersion relation implies that the anomalous dimension vanishes, which means that the point like open string becomes BPS configuration and describes a ground state of the open spin chain with no impurity or magnon.

To consider the magnon in the open spin chain, we should consider the solitonic open string in the world sheet\textsuperscript{1}. For this, we consider the following ansatz

$$t = \kappa \tau,$$
$$\theta = \theta(y),$$
$$\phi = \nu \tau + g(y),$$  \hspace{1cm} (9)

where $y = a \tau + b \sigma$. Due to the rotational symmetry in the $\phi$ direction, $g'$ is reduced to

$$g' = \frac{1}{b^2 - a^2} \left( \frac{a \nu}{\sin^2 \theta} - \frac{c}{\sin \theta} \right),$$  \hspace{1cm} (10)

where the prime means the derivative with respect to $y$ and $c$ is an integration constant. Here, we choose $b^2 - a^2 > 0$, which describes a magnon. In other case $a^2 - b^2 > 0$, as will be shown, the solitonic string solution becomes spike. The equation of motion for $\theta$ becomes

$$\theta'' = -\frac{b^2 \nu^2}{(b^2 - a^2)^2 \sin^3 \theta} \cos \theta \left( \sin^4 \theta - \frac{c^2}{b^2 \nu^2} \right).$$  \hspace{1cm} (11)

Interestingly, this equation can be rederived from the Virasoro constraints represented by the first order differential equation for $\theta$. So we will solve the Virasoro constraints instead of the second order differential equation in Eq. (11). Here, due to the symmetric property of $h_{\alpha\bar{\beta}}$, only three components of $T_{\alpha\beta}$ are independent. Moreover, the conformal symmetry of the

\textsuperscript{1}Here, the solitonic open string in the world sheet is often called the giant magnon in the target space and the magnon in the dual open spin chain model.
string action, which makes the energy-momentum tensor traceless, reduces the number of the independent components to two. For later convenience, we consider the linear combinations of them

\[
0 = T_{\tau\tau} + T_{\sigma\sigma} - 2T_{\tau\sigma}, \\
0 = T_{\tau\tau} + T_{\sigma\sigma} - \frac{a^2 + b^2}{ab}T_{\tau\sigma}.
\]  

The second Virasoro constraint becomes a relation between parameters

\[
\kappa^2 = \frac{c\nu}{a},
\]

and the first gives a differential equation for \(\theta\)

\[
\theta' = \pm \frac{b\nu}{(b^2 - a^2)\sin\theta}\sqrt{\left(\sin^2 \theta_{\text{max}} - \sin^2 \theta\right)\left(\sin^2 \theta - \sin^2 \theta_{\text{min}}\right)},
\]

where

\[
\sin^2 \theta_{\text{max}} = \frac{c}{a\nu}, \\
\sin^2 \theta_{\text{min}} = \frac{ac}{b^2\nu}.
\]

In Eq. (14), at fixed \(\tau\), the plus (or minus) sign implies that the range of \(\sigma\) becomes \(\sigma_L \leq \sigma \leq \bar{\sigma}\) (or \(\bar{\sigma} \leq \sigma \leq \sigma_R\)), where \(\theta_{\text{min}} = \theta(\bar{\sigma})\) and \(\sigma_L\) or \(\sigma_R\) means the left or right end satisfying \(\theta_{L,\text{max}} = \theta_{\text{max}} = \theta(\sigma_L)\) or \(\theta_{R,\text{max}} = \theta_{\text{max}} = \theta(\sigma_R)\), respectively. Note that the differentiation of Eq. (14) with respect to \(y\) gives rise to the equation of motion for \(\theta\) in Eq. (11). To consider the giant magnon having infinite angular momentum, which is called the infinite size limit and corresponds to the magnon in the infinitely long open spin chain, we should set \(\sin \theta_{\text{max}} = 1\).

Then, from Eq. (13) and Eq. (15), some relations between parameters are given by

\[
a = \frac{c}{\nu}, \\
\kappa = \nu.
\]

In this infinite size limit, Eq. (14) is rewritten as

\[
\theta' = \pm \frac{\nu \cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_{\text{min}}}}{b \cos^2 \theta_{\text{min}} \sin \theta},
\]

with

\[
\sin^2 \theta_{\text{min}} = \frac{c^2}{b^2\nu^2}.
\]

Note that using Eq. (16) \(\sin \theta_{\text{min}}\) can be rewritten as \(\frac{a}{b}\), which implies that the minimum value of \(\theta\) is determined by the ratio between \(a\) and \(b\) only. Using the above results, the conserved charges are given by

\[
E = 2T \int_{\theta_{\text{min}}}^{\pi/2} d\theta \frac{\cos^2 \theta_{\text{min}} \sin \theta}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_{\text{min}}}}, \\
J = 2T \int_{\theta_{\text{min}}}^{\pi/2} d\theta \frac{\sin \theta (\sin^2 \theta - \sin^2 \theta_{\text{min}})}{\cos \theta \sqrt{\sin^2 \theta - \sin^2 \theta_{\text{min}}}}.
\]
Another important quantity is the world sheet momentum $p$, which corresponds to the angle difference $\Delta \phi = p$ in the target space,

$$\Delta \phi \equiv \int d\phi = 2 \int_{\theta_{\text{min}}}^{\pi/2} d\theta \frac{\cos \theta \sin \theta_{\text{min}}}{\sin \theta \sqrt{\sin^2 \theta - \sin^2 \theta_{\text{min}}}}.$$  \hspace{1cm} (20)

Using these, the dispersion relation for one giant magnon corresponding to the solitonic string moving on $R_t \times S^2$ can be universally described by

$$E - J = 2T \left| \sin \frac{p}{2} \right|$$ \hspace{1cm} (21)

in the large 't Hooft coupling limit, where the string tension $T$ is also large. For examples, in the $AdS_5 \times S^5$ case \[16\], since $R_s^2 = \sqrt{\lambda}$, the dispersion relation reduced to

$$E - J = \frac{\sqrt{\lambda}}{\pi} \left| \sin \frac{p}{2} \right|.$$ \hspace{1cm} (22)

For the the $AdS_4 \times \mathbb{CP}^3$ case \[47\], the radius of $S^2$ is $R_s^2 = \pi \sqrt{2\lambda}$, so the dispersion relation becomes

$$E - J = \sqrt{2\lambda} \left| \sin \frac{p}{2} \right|.$$ \hspace{1cm} (23)

In the gauge theory side, the dual description for a giant magnon comes from the integrable spin chain model, in which the dispersion relation for one magnon is given by

$$E - J = \sqrt{1 + 4T^2 \sin^2 \frac{p}{2}}.$$ \hspace{1cm} (24)

In the large 't Hooft coupling limit, this dispersion relation becomes Eq. \[21\].

3 Multi-magnon on $R \times S^2$

From now on, we consider unbounded $M$ multi-magnon, where $M$ is the number of the magnon. For this, we parameterize the $\sigma$ range of the $i$-th magnon as

$$- L + L \sum_{i=1}^{m-1} l_i \leq \sigma_m \leq - L + L \sum_{i=1}^{m} l_i,$$ \hspace{1cm} (25)

where $\sum_{i=1}^{M} l_i = 2$ so that the total distance of $M$ magnon in the $\sigma$ direction is $2L$. The range of $\tau$ is parameterized as $- L < \tau < L$. Then, the $M$ multi-magnon can be described by the same action in the previous section

$$S = \frac{T}{2} \int_{-L}^{L} d\tau \int_{-L}^{L} d\sigma \sqrt{-\det h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}},$$ \hspace{1cm} (26)
with the following ansatz

\[ t = \kappa \tau, \]
\[ \theta = \sum_{m=1}^{M} \theta_m(y_m), \]
\[ \phi = \sum_{m=1}^{M} \phi_m = \sum_{m=1}^{M} \nu_m \tau + g_m(y_m), \]  \hspace{1cm} (27)

where the subscript \( m \) implies the \( m \)-th magnon and \( y_m = a_m \tau + b_m \sigma_m \).

Due to the rotational symmetry in the \( \phi \) direction, the equation of motion for \( \phi_m \) reduces to

\[ g'_m = \frac{1}{b_m^2 - a_m^2} \left( a_m \nu_m - \frac{c_m}{\sin^2 \theta_m} \right), \]  \hspace{1cm} (28)

where \( c_m \) is an integration constant. The energy-momentum tensor for \( m \)-th magnon becomes

\[ T^m_{\tau \tau} + T^m_{\sigma \sigma} = \frac{T}{2} \left[ -\kappa^2 + (a_m^2 + b_m^2)(\theta'_m)^2 + (\nu_m + a_m g'_m)^2 + b_m^2 (g'_m)^2 \right], \]
\[ T^m_{\tau \sigma} = \frac{T}{2} a_m b_m (\theta'_m)^2 + b_m (\nu_m + a_m g'_m) g'_m. \]  \hspace{1cm} (29)

From these, the Virasoro constraints can be rewritten as the more simple form like

\[ 0 = T^m_{\tau \tau} + T^m_{\sigma \sigma} - \frac{a_m^2 + b_m^2}{a_m b_m} T^m_{\tau \sigma}, \]
\[ 0 = T^m_{\tau \tau} + T^m_{\sigma \sigma} - 2T^m_{\tau \sigma}. \]  \hspace{1cm} (30)

Since the energy-momentum tensor vanishes at all \( \tau \) and \( \sigma \), the above Virasoro constraints should be applied to all \( M \) magnon. The first constraint gives rise to

\[ \kappa^2 = \frac{\nu_m c_m}{a_m}, \]  \hspace{1cm} (31)

and the second constraint

\[ \theta'_m = \pm \frac{b_m \nu_m}{(b_m^2 - a_m^2) \sin \theta_m} \sqrt{(\sin^2 \theta_{m,max} - \sin^2 \theta_m)(\sin^2 \theta_{m} - \sin^2 \theta_{m,min})}, \]  \hspace{1cm} (32)

where

\[ \sin^2 \theta_{m,max} = \frac{c_m}{a_m \nu_m}, \]
\[ \sin^2 \theta_{m,min} = \frac{a_m c_m}{b_m^2 \nu_m}. \]  \hspace{1cm} (33)

For the multi-magnon configuration to be a solution, \( m \)-th magnon solution should be smoothly connected to \((m - 1)\)-th and \((m + 1)\)-th magnon. This implies that values of \( \theta \) and \( \phi \) for the right end of \( m \)-th and the left end of \((m + 1)\)-th magnon are smoothly connected. First,
to satisfy this boundary condition, \( \theta_{m,max}^R = \theta_{m,max} \) should be same as \( \theta_{m+1,max}^L = \theta_{m+1,max} \), which implies that \( \theta_m \) for all \( m \)-th magnon has the same maximum value

\[
\theta_{m,max} = \theta_{max} \quad \text{for all } m.
\]

(34)

Using this, \( c_m \) can be rewritten in terms of other parameters

\[
c_m = a_m \nu_m \sin^2 \theta_{max}.
\]

(35)

With Eq. (31), this relation means that \( \nu_m \) is constant like \( \nu_m = \nu \) for all \( m \). Because of the smoothness of \( \theta \) at \( \theta_{max} \), \( \sigma_m \) is represented as an integration with respect to \( \theta \)

\[
\sigma_m - \sigma_m^L = \int_{\theta_m}^{\theta_{max}} \frac{d\theta_m}{b_m \theta'_m},
\]

(36)

where \( \sigma_m^L \) and \( \theta_{m,max}^L \) are the values of \( \sigma_m \) and \( \theta_m \) at the left end of the \( m \)-th magnon.

In Eq. (32), \( \theta' \) always vanishes at \( \theta_{max} \) so that \( \frac{\partial \theta}{\partial \sigma} \) is a smooth function of \( \sigma \) at fixed \( \tau \). To investigate the smoothness for the higher derivatives of \( \theta \), we write several terms by using the chain rule

\[
\begin{align*}
\theta^{(2)} &= \frac{b_m^2}{2} \frac{\partial}{\partial \theta} \theta'^2, \\
\theta^{(3)} &= \frac{b_m^3}{2} \theta' \frac{\partial^2}{\partial \theta^2} \theta'^2, \\
\theta^{(4)} &= \frac{b_m^4}{4} \frac{\partial}{\partial \theta} \theta'^2 \frac{\partial^2}{\partial \theta^2} \theta'^2, \\
\theta^{(5)} &= \frac{b_m^5}{4} \theta' \left( \frac{1}{4} \frac{\partial^2}{\partial \theta^2} \theta'^2 \frac{\partial^2}{\partial \theta^2} \theta'^2 + \frac{3}{4} \frac{\partial}{\partial \theta} \theta'^2 \frac{\partial^3}{\partial \theta^3} \theta'^2 + \frac{1}{2} \theta'^2 \frac{\partial^4}{\partial \theta^4} \theta'^2 \right),
\end{align*}
\]

\[
\ldots
\]

(37)

where \( \theta^{(n)} \) means \( \frac{\partial^n}{\partial \sigma^n} \theta \). Since \( \theta' \) vanishes at \( \theta_{max} \), \( \theta^{(n)} \) for odd \( n \) becomes automatically zero. For even \( n \) case, at \( \theta_{m,max}^R \) where \( m \)-th and \( (m+1) \)-th magnon are joined \( \theta^{(n)} \)'s values of \( m \)-th and \( (m+1) \)-th magnon are different, so in general \( \theta \) is not differentiable. Since multi-magnon configuration should be a solution of the equation of motion represented as a second order differential equation, we have to require that at least \( \theta^{(n)} \) with \( n \leq 2 \) is differentiable. As previously mentioned, when \( \theta_{m,max}^R = \theta_{m+1,max}^L = \theta_{max} \), \( \theta^{(0)} \) is smooth. As will be shown, the smoothness of \( \theta^{(2)} \) determines the minimum value of \( \theta \) in terms of other parameters, \( a_m, b \) and \( \theta_{max} \). This is a story of the multi-magnon with a finite size. In the infinite size case with \( \theta_{max} = \pi/2 \), since \( \theta' \) and \( \frac{\partial^2}{\partial \sigma^2} \theta'^2 \) are proportional to \( \cos \theta \), \( \theta^{(n)} \) for all \( n \) vanishes at \( \theta_{max} \). So \( \theta \) in the infinite size case becomes an infinitely differentiable function.

Before investigating the multi-magnon with finite size, we consider the infinite size case. In this case, Eq. (32) and Eq. (28) are reduced to simpler forms

\[
\begin{align*}
\theta'_m &= \pm \frac{b_m \nu \cos \theta_m}{(b_m^2 - a_m^2) \sin \theta_m} \sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,min}}, \\
g'_m &= \frac{a_m \nu}{b_m^2 - a_m^2} \left( 1 - \frac{1}{\sin^2 \theta_m} \right)
\end{align*}
\]

(38)
with
\[ \kappa = \nu, \sin^2 \theta_{max} = 1, \sin^2 \theta_{m,min} = \frac{a_m^2}{b_m^2}. \]  

(39)

Using these results, the conserved charges for the multi-magnon are given by
\[ E = 2T \sum_{m=1}^{M} \int_{\theta_{m,min}}^{\pi/2} d\theta_m \frac{\cos^2 \theta_{m,min} \sin \theta_m}{\cos \theta_m \sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,min}}}, \]
\[ J = 2T \sum_{m=1}^{M} \int_{\theta_{m,min}}^{\pi/2} d\theta_m \frac{\sin \theta_m (\sin^2 \theta_m - \sin^2 \theta_{m,min})}{\cos \theta_m \sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,min}}}. \]  

(40)

The angle difference corresponding to the world momentum of the \( m \)-th magnon is given by
\[ p_m = 2 \int_{\theta_{m,min}}^{\pi/2} d\theta_m \frac{\cos \theta_m \sin \theta_{m,min}}{\sin \theta_m \sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,min}}}. \]  

(41)

Calculating these integrations, we obtain
\[ \sin \theta_{m,min} = \cos \left( \frac{p_m}{2} \right), \]  

(42)

which can be rewritten as \( p_m = \pi - 2\theta_{m,min} \). This result implies that the considered multi-magnon configuration allows for each magnon to have different world sheet momentum \( p_m \). Therefore, the dispersion relation for \( M \) multi-magnon becomes
\[ E - J = 2T \sum_{m=1}^{M} \left| \sin \frac{p_m}{2} \right|. \]  

(43)

Comparing this result with one obtained from the spin chain model
\[ E - J = \sqrt{1 + 4T^2 \sum_{m=1}^{M} \left| \sin \frac{p_m}{2} \right|^2}, \]  

(44)

in the large 't Hooft coupling limit Eq. (44) becomes Eq. (43).

For the multi-magnon with the finite size, the equations for this string configuration are
\[ \theta'_m = \pm \frac{b_m \nu}{(b_m^2 - a_m^2) \sin \theta_m} \sqrt{(\sin^2 \theta_{max} - \sin^2 \theta_m)(\sin^2 \theta_m - \sin^2 \theta_{m,min})}, \]
\[ g'_m = \frac{a_m \nu}{b_m^2 - a_m^2} \left( 1 - \frac{\sin^2 \theta_{max}}{\sin^2 \theta_m} \right), \]  

(45)

where
\[ \sin^2 \theta_{m,min} = \frac{a_m^2}{b_m^2} \sin^2 \theta_{max}. \]  

(46)
Eq. (31) and Eq. (35) give the following relation

\[ \kappa^2 = \nu^2 \sin^2 \theta_{\text{max}}. \]  

(47)

In this case, \( \theta'_m, g'_m \) and \( g''_m \) are automatically zero at \( \theta_{m,\text{max}} = \theta_{\text{max}} \) where \( \theta \) is smooth. \( \theta''_m \) at \( \theta_{\text{max}} \) becomes

\[ \theta''_m|_{\theta=\theta_{\text{max}}} = -\frac{\nu^2}{b_m^2 - a_m^2} \sin \theta_{\text{max}} \cos \theta_{\text{max}}, \]  

(48)

which satisfy the equations of motion in Eq. (11) with \( a_m \) and \( b_m \) instead of \( a \) and \( b \). Therefore, the differentiability of \( \theta^{(2)} \) requires that \( b_m^2 - a_m^2 \) be a constant independent of \( m \). This proves the existence of the \( M \) multi-magnon solution. Following Ref. [47], the dispersion relation with the finite size effect for the \( M \) multi-magnon is

\[ E - J = 2T \sum_{m=1}^{M} \left( |\sin \frac{P_m}{2}| - 4 \sin \frac{3}{2} \frac{P_m}{2} e^{-E_m/(T |\sin \frac{E_m}{2}|)} \right), \]  

(49)

where \( E_m \) is the energy of the \( m \)-th magnon.

4 Multi-spike on \( R \times S^2 \)

Like the multi-magnon, we can also consider a multi-spike solution, for which we set \( a_m^2 - b_m^2 \) positive and interchange the roles of \( \sin \theta_{m,\text{max}} \) and \( \sin \theta_{m,\text{min}} \) in Eq. (32) and Eq. (33)

\[ \theta'_m = \pm \frac{b_m \nu_m}{(a_m^2 - b_m^2) \sin \theta_m} \sqrt{(\sin^2 \theta_{m,\text{max}} - \sin^2 \theta_{m})(\sin^2 \theta_m - \sin^2 \theta_{m,\text{min}})}, \]

\[ g'_m = \frac{1}{a_m^2 - b_m^2} \left( \frac{c_m}{\sin^2 \theta_m} - a_m \nu_m \right), \]  

(50)

where

\[ \sin^2 \theta_{m,\text{max}} = \frac{a_m c_m}{b_m^2 \nu_m}, \]

\[ \sin^2 \theta_{m,\text{min}} = \frac{c_m}{a_m \nu_m}. \]  

(51)

and the constraint equation becomes

\[ \kappa^2 = \frac{\nu_m c_m}{a_m}. \]  

(52)

For the smooth solution, we set \( \sin \theta_{m,\text{max}} = \sin \theta_{\text{max}} \) for all \( m \). Then, \( \nu_m \) can be rewritten as

\[ \nu_m = \frac{a_m c_m}{b_m^2 \sin^2 \theta_{\text{max}}}, \]  

(53)

where \( c_m = \kappa \sin \theta_{\text{max}} b_m \). Inserting this to the last relation in Eq. (51), \( \sin^2 \theta_{m,\text{min}} \) becomes

\[ \sin^2 \theta_{m,\text{min}} = \frac{b_m^2}{a_m^2} \sin^2 \theta_{\text{max}}. \]  

(54)
Like the multi-magnon case, in the infinite size limit $\theta_m$ and $\phi_m$ are infinitely differential functions and there is no restriction to the value of $\sin \theta_{m,\text{min}}$. For this case, the conserved charges and the angle difference for $M$ multi-spike are given by

$$E = 2T \sum_{m=1}^{M} \int_{\theta_{m,\text{min}}}^{\pi/2} d\theta_m \frac{\cos^2 \theta_{m,\text{min}} \sin \theta_m}{\sin \theta_{m,\text{min}} \cos \theta_m \sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,\text{min}}}} \frac{1}{\sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,\text{min}}}},$$

$$J = 2T \sum_{m=1}^{M} \int_{\theta_{m,\text{min}}}^{\pi/2} d\theta_m \frac{\sin \theta_m \cos \theta_m}{\sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,\text{min}}}},$$

$$\Delta \phi = 2 \sum_{m=1}^{M} \int_{\theta_{m,\text{min}}}^{\pi/2} d\theta_m \frac{\sin^2 \theta_m - \sin^2 \theta_{m,\text{min}}}{\sin \theta_{m,\text{min}} \cos \theta_m \sin \theta_m \sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,\text{min}}}} \frac{1}{\sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,\text{min}}}},$$

(55)

From these, the dispersion relation for the $M$ multi-spike is

$$E - T \Delta \phi = 2T \sum_{m=1}^{M} \tilde{\theta}_m,$$

$$J_m = 2T \sin \tilde{\theta}_m,$$

(56)

where $\Delta \phi = \sum_{m=1}^{M} \Delta \phi_m$ and $\tilde{\theta}_m = \pi/2 - \theta_{m,\text{min}}$.

In the finite size case, the differentiability of $\theta^{(n)}$ for $n \leq 2$ gives a relation for $\theta_{m,\text{min}}$

$$\frac{\sin \theta_{m,\min}}{\sin \theta_{m+1,\min}} = \frac{a_m^2 - b_m^2}{a_{m+1}^2 - b_{m+1}^2},$$

(57)

which guarantees the existence of the solution. In the infinite size case, since there is no this kind of constraint $a_m$ and $b_m$ can have arbitrary values. The dispersion relation with the finite size effect for this $M$ multi-spike can be easily calculated by following Ref. [47]

$$E - T \Delta \phi \approx \sum_{m=1}^{M} \left[ 2T \tilde{\theta}_m - \left\{ 4 - \frac{4}{\sqrt{1 - (J_m/2T)^2}} \right\} J_m + \left( 8 - \frac{8}{\sqrt{1 - (J_m/2T)^2}} + \frac{J_m^2}{T^2} \right) E_m \right] e^{-2E_m/J_m},$$

(58)

where $E_m$ and $J_m$ are the energy and angular momentum for $m$-th spike.

## 5 Multi-magnon and spike on $R \times S^2 \times S^2$

In this section, we will generalize the multi-magnon and spike configuration on $R \times S^2$ to ones on $R \times S^2 \times S^2$, which corresponds to the $SU(2) \times SU(2)$ subsector of $SU(4)$ R-symmetry in the boundary gauge theory. Here, the background space, $R \times S^2 \times S^2$, can be obtained as a subspace of $AdS_5 \times S^5$ or $AdS_4 \times CP^3$. In the later case, the calculation and the result are similar to those on $AdS_5 \times S^5$ with a constraints $\psi = \text{constant}$, $\xi = \pi/4$ (see details Ref. [47]).
To obtain \( R \times S^2 \times S^2 \), we first consider \( R \times S^5 \), which is the subspace of \( AdS_5 \times S^5 \) at the center of \( AdS_5 \). \( S^5 \) can be represented by a hypersurface in the six-dimensional flat space by imposing a constraint

\[
R^2 = \sum_{i=1}^{6} X_i^2. \tag{59}
\]

The parameterization satisfying the above constraint is given by

\[
\begin{align*}
X_1 &= R \sin \xi \sin \theta_1 \sin \phi_1, \\
X_2 &= R \sin \xi \sin \theta_1 \cos \phi_1, \\
X_3 &= R \sin \xi \cos \theta_1, \\
X_4 &= R \cos \xi \sin \theta_2 \sin \phi_2, \\
X_5 &= R \cos \xi \sin \theta_2 \sin \phi_2, \\
X_6 &= R \cos \xi \cos \theta_2.
\end{align*} \tag{60}
\]

Then, the flat metric in the six-dimensional space reduces to one for \( S^5 \)

\[
ds^2 = R^2 \left[ -dt^2 + d\xi^2 + \sin^2 \xi (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \cos^2 \xi (d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) \right], \tag{61}
\]

where we insert a time direction. The string action moving in this target space becomes

\[
S = \frac{T}{2} \int d^2 \sigma \left[ -\partial_\alpha t \partial^\alpha t + \partial_\alpha \xi \partial^\alpha \xi + \sin^2 \xi (\partial_\alpha \theta_1 \partial^\alpha \theta_1 + \sin^2 \theta_1 \partial_\alpha \phi_1 \partial^\alpha \phi_1) \\
+ \cos^2 \xi (\partial_\alpha \theta_2 \partial^\alpha \theta_2 + \sin^2 \theta_2 \partial_\alpha \phi_2 \partial^\alpha \phi_2) \right]. \tag{62}
\]

To obtain the string solution moving in \( R \times S^2 \times S^2 \), we investigate the equation of motion for \( \xi \)

\[
0 = \partial^\alpha \partial_\alpha \xi - \sin \xi \cos \xi \left[ \partial_\alpha \theta_1 \partial^\alpha \theta_1 + \sin^2 \theta_1 \partial_\alpha \phi_1 \partial^\alpha \phi_1 \\
- \partial_\alpha \theta_2 \partial^\alpha \theta_2 - \sin^2 \theta_2 \partial_\alpha \phi_2 \partial^\alpha \phi_2 \right]. \tag{63}
\]

For \( \xi = 0 \) or \( \pi/2 \), the above equation is automatically satisfied and the metric in Eq. \( \text{(61)} \) reduces to \( R \times S^2 \), which has been already investigated in the previous section. To satisfy Eq. \( \text{(63)} \), we will consider the case \( \xi = \text{constant} \) but \( \xi = 0 \) or \( \pi/2 \), in which \( \theta_1 = \theta_2 \) and \( \phi_1 = \phi_2 \) satisfy Eq. \( \text{(63)} \). So the solitonic string with these constraints describes a \( (M,M) \) multi-magnon, which implies \( M \) multi-magnon in each \( S^2 \). Note that in the \( AdS_4 \times \mathbb{CP}^3 \) case, the parameterization in Eq. \( \text{(60)} \) satisfy the \( \mathbb{CP}^3 \) constraint only at \( \xi = \pi/4 \). \[ \text{[17]} \]

Since all calculation is very similar to the previous one, we skip the details. To consider the \( (M,M) \) multi-magnon and spike on \( R \times S^2 \times S^2 \), the appropriate ansatz is given by

\[
\begin{align*}
t &= \sum_{m=1}^{M} \left( \sin^2 \xi + \cos^2 \xi \right)^{1/2} \kappa_m, \\
\theta_{1,m} &= \theta_{2,m} = \theta_m (y_m), \\
\phi_{1,m} &= \phi_{2,m} = \nu_m \tau + f_m (y_m), \tag{64}
\end{align*}
\]
where the subscript 1 and 2 in the last two equations represent the first and second $S^2$ sphere and and \(y_m = a_m \tau + b_m \sigma_m\). Note that in the first line in Eq. (64) we write \(\sin^2 \xi + \cos^2 \xi\) explicitly for later convenience. Then, in the infinite size limit the dispersion relation for each sphere is given by

\[
E_1 - J_1 = 2T \sin^2 \xi \sum_{m=1}^M \left| \sin \frac{p_m}{2} \right|, \\
E_2 - J_2 = 2T \cos^2 \xi \sum_{m=1}^M \left| \sin \frac{p_m}{2} \right|
\]  

(65)

where

\[
E_1 = 2T \sin^2 \xi \sum_{m=1}^M \int_{\theta_{m,\min}}^{\pi/2} d\theta_m \frac{\cos^2 \theta_{m,\min} \sin \theta_m}{\cos \theta_m \sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,\min}}}, \\
E_2 = 2T \cos^2 \xi \sum_{m=1}^M \int_{\theta_{m,\min}}^{\pi/2} d\theta_m \frac{\cos^2 \theta_{m,\min} \sin \theta_m}{\cos \theta_m \sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,\min}}}, \\
J_1 = 2T \sin^2 \xi \sum_{m=1}^N \int_{\theta_{m,\min}}^{\pi/2} d\theta_m \frac{\sin \theta_m (\sin^2 \theta_m - \sin^2 \theta_{m,\min})}{\cos \theta_m \sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,\min}}}, \\
J_2 = 2T \cos^2 \xi \sum_{n=1}^N \int_{\theta_{n,\min}}^{\pi/2} d\theta_n \frac{\sin \theta_n (\sin^2 \theta_n - \sin^2 \theta_{n,\min})}{\cos \theta_n \sqrt{\sin^2 \theta_n - \sin^2 \theta_{n,\min}}}, \\
p_m = 2 \int_{\theta_{m,\min}}^{\pi/2} \frac{d\theta_m}{\sin \theta_m \sqrt{\sin^2 \theta_m - \sin^2 \theta_{m,\min}}}.
\]  

(66)

Therefore, the total dispersion relation for \((M, M)\) multi-magnon in \(AdS_5 \times S^5\) becomes

\[
E - J = 2T \sin^2 \xi \sum_{m=1}^M \left| \sin \frac{p_m}{2} \right| + 2T \cos^2 \xi \sum_{m=1}^M \left| \sin \frac{p_m}{2} \right| 
\]  

(67)

\[
= \sum_{m=1}^M \sqrt{\lambda_1} \left| \sin \frac{p_m}{2} \right| + \sum_{m=1}^M \sqrt{\lambda_2} \left| \sin \frac{p_m}{2} \right|, 
\]  

(68)

where \(\lambda_1 = \sqrt{\lambda \sin^4 \xi}\) and \(\lambda_2 = \sqrt{\lambda \cos^4 \xi}\). Due to the different radius of two \(S^2\), the effective 't Hooft coupling in each sphere has different value. If we consider the diagonal subgroup of \(SU(2) \times SU(2)\), then it reduces to the previous result obtained in the \(R \times S^2\) case. Note that Eq. (67) is a universal form of the dispersion relation for \((M, M)\) multi-magnon on \(R \times S^2 \times S^2\). So to describe the \((M, M)\) multi-magnon moving in the \(AdS_4 \times CP^3\), we should set \(\xi = \pi/4\) and \(T = \sqrt{2\lambda}/2\) so that the dispersion relation becomes

\[
E - J = \sum_{m=1}^M \sqrt{2\lambda} \left| \sin \frac{p_m}{2} \right|.
\]  

(69)
which is the same form obtained from the $R \times S^2$ case. Note that the dispersion relation for the each sphere is

$$E_i - J_i = \sum_{m=1}^{M} \sqrt{2\lambda / 2} \left| \sin \frac{p_m}{2} \right|,$$  \hspace{1cm} (70)

where $i$ means $i$-th sphere.

The finite size effect of $(M, M)$ multi-magnon coming from $AdS_5 \times S^5$ background is given by

$$E - J = \sqrt{\lambda_1 \lambda_2} \sum_{m=1}^{M} \left( \left| \sin \frac{p_m}{2} \right| - 4 \left| \sin^3 \frac{p_m}{2} \right| e^{-2\pi E_m / (\sqrt{\lambda_1 |\sin \frac{p_m}{2}|})} \right)$$

$$+ \sqrt{\lambda_2 \lambda_1} \sum_{m=1}^{M} \left( \left| \sin \frac{p_m}{2} \right| - 4 \left| \sin^3 \frac{p_m}{2} \right| e^{-2\pi E_m / (\sqrt{\lambda_2 |\sin \frac{p_m}{2}|})} \right).$$  \hspace{1cm} (71)

In the $AdS_4 \times CP^3$ case, the finite size effect for the $i$-th sphere is

$$E_i - J_i = \sqrt{2\lambda_1 / 2} \sum_{m=1}^{M} \left( \left| \sin \frac{p_m}{2} \right| - 4 \left| \sin^3 \frac{p_m}{2} \right| e^{-2\pi E_m / (\sqrt{2\lambda_1 |\sin \frac{p_m}{2}|})} \right).$$  \hspace{1cm} (72)

So the total finite size effect becomes twice of Eq. (72).

In the infinite size limit, the dispersion relation of the $(M, M)$ multi-spike on the $i$-th sphere becomes

$$E_i - T_i \sum_{m=1}^{M} \Delta \phi_m = 2T_i \sum_{m=1}^{M} \tilde{\theta}_m,$$

$$J_i = 2T_i \sum_{m=1}^{M} \sin \tilde{\theta}_m,$$  \hspace{1cm} (73)

where $\tilde{\theta}_m = \pi / 2 - \theta_{m, min}$ and $\{T_1, T_2\} = \{T \sin^2 \xi, \ T \cos^2 \xi\}$ with $T = \sqrt{\lambda_1 / 2\pi}$ for $AdS_5 \times S^5$ case and $T = \sqrt{\lambda_2 / 2\pi}$ and $\xi = \pi / 4$ for $AdS_4 \times CP^3$ case. The finite size effect for $(M, M)$ multi-spike on the $i$-th sphere of $AdS_5 \times S^5$ becomes

$$E_i - T_i \Delta \phi \approx \sum_{m=1}^{M} \left[ 2T_i \tilde{\theta}_m - \left\{ \frac{4}{\sqrt{1 - (J_{i,m} / 2T_i)^2}} - \frac{J_{i,m}^2}{2T_i^2} \right\} J_{i,m} \right.$$

$$+ \left. \left\{ \frac{8}{\sqrt{1 - (J_{i,m} / 2T_i)^2}} + \frac{J_{i,m}^2}{T_i^2} \right\} E_{i,m} \right\} e^{-2E_{i,m} / J_{i,m}},$$  \hspace{1cm} (74)

where

$$E_{i,m} = T_i \left( \Delta \phi_m + 2\tilde{\theta}_m \right),$$

$$J_{i,m} = 2T_i \sin \tilde{\theta}_m.$$  \hspace{1cm} (75)
6 Discussion

At first, we found a universal form of the dispersion relation for a magnon solution on \( R \times S^2 \). Only the difference between the magnon moving in \( AdS_5 \times S^5 \) and in \( AdS_4 \times \mathbb{CP}^3 \) is the different radius of \( S^2 \), which is represented as a different string tension corresponding to the ’t Hooft coupling. The universality of the dispersion relation can be easily extended to \( R \times S^n \) case. Here, the universal form on \( R \times S^2 \times S^2 \) was also investigated. For example, Eq. 67 is the universal form of the dispersion relation for the multi-magnon on \( R \times S^2 \times S^2 \).

Secondly, we investigated the multi-magnon configuration consisting of \( M \) magnons. this corresponds to the unbounded multi-magnon or shortly, multi-magnon in spin chain model. In the infinite size limit, the dispersion relation of multi magnon gives the same result obtained from the spin chain model in the large ’t Hooft coupling limit. In the finite size case, though the multi-magnon configuration is not infinitely differentiable at \( \theta_{\max} \), by requiring the differentiability of \( \theta^{(2)} \) we found the multi magnon configuration with the finite size satisfying the equation of motion and calculated the finite size effect. With the same method, we also investigated the string configuration corresponding to the multi-spike and found the dispersion relation and the finite size effect. Furthermore, we generalized the multi-magnon solution on \( R \times S^2 \) to \( (M, M) \) multi-magnon solution on \( R \times S^2 \times S^2 \). For the case of the multi-magnon moving in \( AdS_5 \times S^5 \), the radius of \( S^2 \) depends on the value of \( \xi \) so that \( (M, M) \) multi magnon describes the combination of two \( M \) multi-magnons on each sphere with the effectively different ’t Hooft coupling.

The remaining interesting problem is to find the \( (M, N) \) type multi-magnon or spike configuration from the string sigma model. Due to the constraint in Eq. 63, it is not easy to find \( (M, N) \) multi-magnon configuration. We hope to find those string configuration in the next work.

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