Impact of order three cycles in complex network spectra

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The asymptotic behaviour of dynamical processes in networks can be expressed as a function of spectral properties of the Adjacency and Laplacian matrices. Although many theoretical results are known for the spectra of traditional configuration models, networks generated through these models fail to describe many topological features of real-world networks, in particular non-null values for the clustering coefficient. Here we study the effects of cycles or order three (triangles) in network spectra. By using recent advances in random matrix theory, we determine the spectrum distribution of the network Adjacency matrix as a function of the average number of triangles attached to each node for networks without modular structure and degree-degree correlations. Furthermore we show that cycles of order three have a weak influence on the Laplacian eigenvalues, fact that explains the recent controversy on the dynamics of clustered networks. Our findings can shed light in the study of how particular kinds of subgraphs influence network dynamics.

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Introduction. — Many works have been devoted to the study of how intrinsic topological properties of networks interfere in the performance of a given dynamical process. Examples of such properties are the presence of triangles [1-3], degree-degree correlations [4-8] or modular organization [9-11]. The main motivation of these studies relies on the fact that the majority of dynamical processes studied are analytically treatable in networks generated through the configuration model. These networks present different properties from real-world maps, mainly in the limit of a large number of nodes. To overcome this discrepancy between random networks and real-world structures, many network models have been proposed in order to create random networks that can mimic properties observed in real-world topologies. For instance, recent random models are able to generate networks presenting transitivity [12-15], assortativity [14] and distributions of pre-defined subgraphs [15, 16].

Recent advances on these new random network models have naturally motivated the study of well known dynamical processes in these structures, such as in percolation and epidemic spreading [12, 15], cascade failure [2, 17] and synchronization [18]. Since the standard configuration model generates networks with locally tree-like structures, i.e., without loops, one of the most frequent question addressed is how dynamics is affected by the presence of triangles (cycles composed by three vertices) [11, 12, 14]. Although many works attempted to study dynamics on clustered networks, there are still crucial questions unsolved. In particular, many works claim that indeed the presence of triangles can influence dramatically network synchronization by suppressing the collective behavior between oscillators [1, 18, 21], whereas in [18] the authors verified analytically and numerically that the presence of triangles does not influence network synchronization. Therefore, previous theoretical results for locally tree-like networks are accurate in describing the synchronization of clustered topologies. This accuracy of tree-based analysis was verified in other dynamical processes, such as bond and k-core percolation and epidemic spreading [22].

In order to analyze the source of these different conclusions on the same subject, one should look at the network models used. For instance, in [11] the authors used the stochastic rewiring algorithm proposed by Kim [23] in order to study synchronization of clustered networks. However, as the authors remark, although the degree distribution is kept fixed, other network properties are changed during the process [11]. Moreover, Ma et al. [19] analyzed the spectrum of clustered scale-free networks using the same procedure as in [11, 23]. They verified that even for small variations in the network clustering coefficient, the smallest nonzero Laplacian eigenvalue is significantly decreased. Note that this quantity determines the synchronization ability of a network [24, 25]. The authors also pointed out that the generated clustered networks present community organization with high-order loops and with similar eigenvalues as ring networks with the same number of nodes [19]. Similar conclusions were drawn in [20, 21] where the Laplacian spectrum of clustered networks generated through stochastic rewiring algorithms was analyzed [20, 21]. In this case, the authors studied the influence of triangles on network synchronization by parametrizing the phase based on Lapla-
cian eigenvectors. They verified that for higher values of network clustering, synchronization is inhibited due to groups of low-lying modes that fail to lock [20 21].

The consideration of stochastic rewiring algorithms suggests that indeed higher triangles can influence network dynamics. However, as mentioned before, many other network properties are significantly changed when the networks are generated. In this way, the influence of triangles on network dynamics is still an open problem. In order to investigate deeply the influence of clustering on dynamical processes, we study the spectrum of clustered networks disentangling it from other network properties. In particular, we analyze the spectrum of the Adjacency and Laplacian matrices of networks presenting a non-vanishing clustering coefficient, since many asymptotic properties of critical exponents of several dynamical processes in networks depend on such eigenvalues [25 26]. Here we focus on the model proposed independently by Newman and Miller [12 13], which allows generating clustered networks with low assortativity and no modular structure. In this way, we can disentangle the effect of correlated topological properties, which is present in rewiring based algorithms [1] [19 21].

Random clustered networks. — The model proposed in [12 13] consists of setting a sequence of single edges \(\{s_i\}\) and a sequence of triangles attached to each node \(\{t_i\}\), for \(i = 1, \ldots, N\). By randomly connecting the stubs of each sequence, we obtain a network in which the conventional degree of a node \(i\) is given by \(k_i = s_i + 2t_i\), since a triangle contributes with two edges to the node degree. With the joint sequence \(\{s_i, t_i\}\) it is possible to define the joint probability distribution \(p_{st}\), which is the probability that a randomly selected node is attached to \(s\) single edges and to \(t\) triangles. Moreover, the joint probability distribution \(p_{st}\) is related to the conventional degree distribution \(p_k\) according to

\[
p_k = \sum_{s, t} p_{st} \delta_{k, s+2t},
\]

where \(\delta_{i,j}\) is the Kronecker delta.

Having defined the distribution of single edges and triangles, it is possible to calculate the network transitivity \(T\) as a function of the moments of the distribution \(p_{st}\). The transitivity \(T\) is defined as

\[
T = \frac{3 \times \text{(number of triangles in the network)}}{\text{(number of connected triples)}} = \frac{3N\Delta}{N_3}.
\]

For the random model of clustered networks, \(3N\Delta = N \sum_{st} t p_{st}\) and \(N_3 = N \sum_k \binom{k}{2} p_k\), which allows calculating the value for the transitivity from Eq. 2.

Spectra of clustered random networks. — It is possible to calculate the spectrum of a random symmetric matrix \(\mathcal{X}\) by using the Stieltjes transform of its average resolvent

\[
\rho(\lambda) = -\frac{1}{N\pi} \text{ImTr}((\lambda I - \mathcal{X})^{-1}),
\]

with \(\lambda\) approaching the real line from above. Furthermore, for a matrix \(\mathbf{X}\) with zero-mean off-diagonal elements, it has been shown that the diagonal elements of \(\mathbf{X}^{-1}\), for sufficiently large entries, satisfy [27 28]

\[
\langle \mathbf{X}^{-1} \rangle = \sum_{i} \frac{P^i(X_{ii})}{X_{ii} - (a^T \mathbf{X}^{-1} a)}
\]

and \(\langle \mathbf{X}^{-1} \rangle = 0\) for \(i \neq j\), where \(a\) is the \(i\)-th column of \(\mathbf{X}\) with diagonal elements \(X_{ii}\) distributed according

![FIG. 1: (Color online) Spectral density \(\rho(\lambda)\) for clustered random network with doubly Poisson distribution with \(\langle k \rangle = 0.9\) for \(\langle k \rangle = 60\) and \(N = 1000\). Each point is an average over 100 different networks. The solid curve is the analytical solution derived from Eq. 7. The red dashed curve corresponds to the standard semicircle law for random graphs, obtained when setting \(t = 0\) in Eq. 7.](image1)

![FIG. 2: (Color online) (Left) smallest eigenvalue \(\lambda^A_1\) of the adjacency matrix and (Right) the second largest eigenvalue \(\lambda^{st}_{N-1}\) of the same matrix for doubly Poisson degree distributions (Eq. 5). For each curve, the network average degree \(\langle k \rangle\) is kept fixed and the balance between \(s\) and \(t\) is varied. Each dot is an averaged over 100 different network realizations. The lines correspond the theoretical values calculated for each average degree using Eq. 1.](image2)
The Laplacian graph is defined as \( L = D - A \), where \( D \) is the diagonal matrix with the nodes degree \( k_i \) and \( A \) the adjacency matrix. The eigenspectrum of \( L \) is known to play an important role in the analysis of several dynamical processes in networks. In the context of the Master Stability Function (MSF) in network synchronization, the ratio between the largest and second eigenvalues \( \lambda_N/\lambda_2 \) of \( L \) indicates whether a given network has a stable synchronous state \([24, 30]\). Therefore the knowledge of its eigenspectrum can provide valuable information about the dynamics of clustered networks. In this way, for the matrix \( L \) we shall write \( \lambda' = D - (A - (A)) \). Also, since the diagonal elements are distributed according to the degree distribution the functions \( h(\lambda) \) and \( g(\lambda) \) are now given by

\[
h(\lambda) = \frac{1}{z - (k - h(\lambda)) - \frac{2(t)}{(k)}}, \quad \text{and} \quad g(\lambda) = \frac{1}{z - (k - h(\lambda))},
\]

which is a generalization of the famous semicircle law for networks with varying clustering. Note that the number of triangles naturally adds a shift term in the range of the continuous band of eigenvalues. In Fig. 1, we show the plot of the spectral density in Eq. 3. The result is in good agreement with the simulated networks. In addition, the distribution presents higher skewness in comparison with the traditional semicircle law, agreeing with the known result from graph theory that the skewness \( s_3 \) of the spectral distribution \( \rho(\lambda) \) should increase with the number of triangles, \( s_3 = 3N\Delta/L\sqrt{k} \), where \( L \) is the number of edges in the graph \([29]\). Furthermore, in Fig. 2, we show the smallest \( \lambda_1 \) and the second largest \( \lambda_{N-1} \) eigenvalue of the adjacency matrix as a function of the average degree due to triangles \( t \), where we observe clearly a linear increasing as \( t \) increases, with good agreement with Eq. 7.

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where \( P_κ \) is the Poisson distribution of \( κ \) with mean \( \langle k \rangle \) for random networks and \( P^i_κ = κ^{-γ}/∑ κ^{-γ} \) for scale-free networks. Solving Eq. 8, 9 with \( ρ(λ) = −1/\sqrt{N} \text{Im} \ g(λ) \) provides the spectrum of the Laplacian matrix \( L \). In Fig. 3 we show the dependency of the second smallest \( λ_2 \) and the largest eigenvalue \( λ_N \) of \( L \) as a function of the average number of triangles \( ⟨τ⟩ \). As we can see, in contrast to the spectrum of the adjacency matrix, \( λ_2 \) and \( λ_N \) are kept constant as a function of \( ⟨τ⟩ \). This result is in sharp contrast to previous works \([19] \,[21]\), where the algebraic connectivity of the networks was found to decrease as a function of the clustering coefficient. However, in these works, the authors remark that the algorithms to generate clustered networks do not guarantee that degree-degree correlations are preserved. Moreover, some works pointed out the emergence of communities and chains in the topology, leading to networks with eigenvalues similar to ring networks \([19]\). In this way, the results observed in Fig. 3 suggest that the conclusions reported in the literature are not due to presence of cycles of order three. In other words, our results suggest that increasing transitivity or clustering coefficient has no impact on the Laplacian eigenvalues, once the assorativity and the network modularity are kept close to zero. This is also in agreement with previous findings that increasing clustering coefficient does not influence the performance of several dynamical process placed on the top of networks \([2] \,[18] \,[22] \,[31] \,[32]\). Furthermore, the algorithm to generate clustered networks is extremely important in order to have a precise comparison between the eigenvalues for different values of \( ⟨τ⟩ \). As in the configuration model, the networks are constructed by drawing two independent degree sequences \( s = \{s_1, s_2, ..., s_N\} \) and \( t = \{t_1, t_2, ..., t_N\} \), standing for single edges and number of triangles attached to each node, respectively. It is important to stress that the networks must have an equal number of edges, since the second smallest eigenvalue \( λ_2 \) of a given network \( G \) in respect of the addition of an edge \( e \) is bounded as \( λ_2(G) \leq λ_2(G + \{e\}) \leq λ_2(G) + 2 \,[29] \).

Subsequently, we consider networks with doubly power law joint degree distribution as

\[
p_{st} ∝ s^{−ξ_s} t^{−ξ_t},
\]

where for \( ξ_s = ξ_t = 3 \), without loss of generality. In Fig. 4 we show the extremal eigenvalues of the Laplacian matrix \( L \) for networks constructed with a joint probability distribution according to Eq. 10. Similarly as in Fig. 3 we again do not observe significant changes in the spectrum as a function of \( ⟨τ⟩ \). Moreover, note in particular that it was already shown that even small values of clustering can induce drops in the values of \( λ_2 \) for scale-free networks \([19]\), suggesting again that solely increasing the presence of triangles does not alter the Laplacian eigenvalues.

Conclusions. — In summary here we have generalized recent results based on random matrix theory for networks with varying number of triangles. More specifically, we have demonstrated how cycles of order three change the network spectrum in the absence of degree-degree correlations and modular structure. In particular, for the Adjacency matrix we have shown that the spectrum distribution \( ρ(λ) \) presents positive skewness, in agreement with known results of classical graph theory. Furthermore, the extreme eigenvalues of the Laplacian matrix were observed to not depend on the number of triangles, a fact that can be related to the accuracy of tree-based theories to describe the dynamics of clustered networks \([2] \,[18] \,[22] \,[31] \,[32]\). The ideas presented here together with previous results on spectral graph theory can shed light in the direction to uncover the spectrum of networks with more sophisticated subgraph structures. These studies will not just provide a better comprehension of network structure, but will contribute to the study of dynamical processes in networks, where the knowledge of the spectrum density is required, e.g., in stability analysis.

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