Abstract

In this short note, we consider a graph process recently introduced by Frieze, Krivelevich and Michaeli. In their model, the edges of the complete graph $K_n$ are ordered uniformly at random and are then revealed consecutively to a player called Builder. At every round, Builder must decide if they accept the edge proposed at this round or not. We prove that, for every $d \geq 2$, Builder can construct a spanning $d$-connected graph after $(1 + o(1))n \log n/2$ rounds by accepting $(1 + o(1))dn/2$ edges with probability converging to 1 as $n \to \infty$. This settles a conjecture of Frieze, Krivelevich and Michaeli.

Keywords: random graph, connectivity, random process, sharp threshold, online algorithm, restricted budget

MSC class: 05C80, 05C40

1 Introduction

The random graph process introduced by Erdős and Rényi [6, 7] is currently one of the most well-understood discrete stochastic processes. Several interesting variations of this process have been studied in the literature: examples include the classic $H$-free process [5, 8, 17, 18] with special focus put on the triangle-free process [3, 4, 16] as well as the recently analysed model of multi-source invasion percolation on the complete graph [1, 14]. Also recently, Frieze, Krivelevich and Michaeli [10] introduced the following version: the edges of the complete graph $K_n$ are ordered uniformly at random and are then proposed one at a time to a player called Builder. At first, Builder starts with an empty graph. At every round, Builder must decide if they want to add the currently proposed edge to their graph based only on the sequence of edges proposed up to now. Builder’s goal is to reach a configuration with some graph property $P$. However, the player is allowed to wait for at most $t$ rounds, and moreover, they have the right to accept at most $b$ edges.

The random graph process on restricted budget: known results. Before presenting the main result of this work, we briefly survey known results on this new model from [2, 10]. To do this, recall that a sequence of events $(E_n)_{n=1}^{\infty}$ holds asymptotically almost surely (which we abbreviate a.a.s.) if $P(E_n) \to 1$ as $n \to \infty$.

Most of the results can be roughly divided into two types. The first type corresponds to results where $t$ is the hitting time of a property $P$ by the Erdős-Rényi graph process and $b$ is a constant factor away from the minimum budget needed to ensure the property $P$ deterministically. Such results include:

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• Theorem 1 from [10], showing that a.a.s. Builder can attain a graph of minimum degree $d \geq 1$ at the hitting time $\tau_d$ for this property under the limitation $b \geq o_dn + \omega(\sqrt{n}\log n)$ for some constant $o_d \in (d/2,3d/4]$. Similar result holds for $d$-connectivity for all $d \geq 3$.

• Theorem 3 from [10], showing that a.a.s. Builder can form a Hamiltonian cycle at the hitting time $\tau_2$ given that the authors of [10] characterise the constant $o$ explicitly and conjecture that a.a.s. Builder cannot construct a graph with minimum degree $k$ at the hitting time if $b \leq (o_d - \varepsilon)n$ for any $\varepsilon > 0$. This conjecture was recently disproved by Katsamaktsis and Letzter [12].

We remark that the authors of [10] characterise the constant $o$ explicitly and conjecture that a.a.s. Builder cannot construct a graph with minimum degree $k$ at the hitting time if $b \leq (o_d - \varepsilon)n$ for any $\varepsilon > 0$. This conjecture was recently disproved by Katsamaktsis and Letzter [12].

The second type of results focus on constructing a graph with a property $P$ where both $t$ and $b$ have asymptotically optimal values.

• Theorem 2 from [10] shows that, for every $\varepsilon > 0$ and $d \geq 1$, a.a.s. Builder can attain a graph of minimum degree $d$ for $t \geq (1 + \varepsilon)n\log n/2$ and $b \geq (1 + \varepsilon)dn/2$.

• Theorem 4 from [10] shows a similar result for the construction of a perfect matching.

• Anastos [2] showed that the same conclusion holds for the construction of a Hamiltonian cycle.

In the light of Anastos’ result [2], it would be interesting to know if the constant $C$ in Theorem 3 from [10] could be improved to a $(1 + o(1))$-factor. Finally, related threshold results were shown in [10] for the construction of trees and cycles of constant size.

**The main result.** We start with some basic terminology allowing us to state our main result. For every $s \geq 0$, we denote the graph of all edges accepted by Builder during the first $s$ rounds by $G_s$. Of course, this graph depends on the choices of Builder during the first $s$ rounds. A strategy of Builder is a (possibly random) function which, given an integer $s \leq \binom{n}{2}$ and an $s$-tuple $(e_1, \ldots, e_s)$ of distinct edges of $K_n$, outputs 1 when Builder accepts the edge $e_s$ at round $s$ given that the edges $e_1, \ldots, e_{s-1}$ were proposed before in this order, and 0 if Builder ignores the edge $e_s$. A $(t,b)$-strategy of Builder is a strategy that, for every $s \leq t$, indicates whether to accept the edge presented at round $s$ based on the edges proposed before under the limitation that $|E(G_s)| \leq b$.

In an earlier version of [10], the authors stated the following conjecture (now appearing as Theorem 6.1 and attributed to the current work).

**Conjecture 1.1** (Conjecture 8 in version 1 of [10]). Fix $\varepsilon > 0$ and a positive integer $d \geq 2$. If $t \geq (1 + \varepsilon)n\log n/2$ and $b \geq (1 + \varepsilon)dn/2$, then there exists a $(t,b)$-strategy of Builder such that $G_t$ is a.a.s. $d$-connected.

Note that the statement of Conjecture 1.1 does not hold for $d = 1$ since Builder needs at least $n - 1$ edges to construct a connected graph. The current work is dedicated to the proof of Conjecture 1.1.

**Theorem 1.2.** Fix a positive integer $d \geq 2$. Then, for every $\varepsilon > 0$, given $t \geq (1 + \varepsilon)n\log n/2$ and $b \geq (1 + \varepsilon)dn/2$, there exists a $(t,b)$-strategy of Builder such that a.a.s. $G_t$ is $d$-connected.

In fact, our proof exhibits an appropriate strategy that does not depend on the value of $\varepsilon$.

**Outline of the proof of Theorem 1.2.** The proof is divided into two parts. The idea when $d \geq 4$ is the following. Firstly, we construct consecutively $d$ almost perfect matchings within $s \ll n\log n$ rounds in a closely related model that allows certain repetitions of the edge proposals. Next, we show that the resulting graph $G$ a.a.s. contains a subgraph $G'$ on $n - o(n)$ vertices and minimum degree $d - 1$ which in a sense is close to being $d$-connected. Then, we use $(1 + o(1))n\log n/2$ more rounds and $o(n)$ more accepted edges to simultaneously build upon the graph $G'$ and transform it into a $d$-connected graph, and connect each of the vertices outside $G'$ to $G'$ itself via $d$ edges.
Unfortunately, this approach fails when \( d \in \{2, 3\} \). In this case, we provide an alternative argument. Firstly, we construct a set of long paths. Then, we construct one cycle of length \( n - o(n) \) (when \( d = 2 \)) and \( 3n/4 - o(n) \) (when \( d = 3 \)). Finally, we use this long cycle as a “skeleton” to which we connect all remaining vertices by 2 edges (when \( d = 2 \)) and by 3 edges (when \( d = 3 \)). This strategy bears a resemblance to the approach in Section 4.2.2 in [10].

Notations. As usual, upper and lower integer parts that are of no importance for the arguments are omitted for better readability. In this paper, \( \omega = \omega(n) \) is a fixed function satisfying \( 1 \ll \log \omega \ll \log \log n \), and \( p = p(n) = \log n/n \). Also, for two sets \( U \subseteq V \), we write \( V \setminus U \) for the set of all elements contained in \( V \) but not in \( U \).

For a graph \( G \), we denote by \( V(G) \) the vertex set of \( G \) and by \( E(G) \) the edge set of \( G \). Moreover, \( \Delta(G) \) stands for the maximum degree of \( G \) and, given a vertex \( v \) in \( G \), we denote by \( \deg_G(v) \) the degree of \( v \) in \( G \). For a graph \( G \) and a subgraph \( H \subseteq G \) or a set of vertices \( U \subseteq V(G) \), we write \( G \setminus H \) (resp. \( G \setminus U \)) to denote the graph obtained from \( G \) by deleting all vertices in \( H \) (resp. in \( U \)) from \( G \). The neighbourhood of a vertex \( v \) in \( G \) is denoted by \( N_G(v) \) and, given a set of vertices \( U \subseteq V(G) \), \( N_G(U) = (\bigcup_{v \in U} N_G(v)) \setminus U \). Note that \( \deg_G \) and \( N_G \) are simply written \( \deg \) and \( N \) in case no ambiguity arises. Moreover, for a set \( S \subseteq V(G) \), we denote by \( G[S] \) the graph induced from \( G \) by the set \( S \). Given a connected graph (or multigraph) \( G \), a cutset is a set of vertices \( U \subseteq V(G) \) satisfying that \( G \setminus U \) is a disconnected graph. Moreover, \( G \) is \( d \)-connected if \( |V(G)| \geq d + 1 \) and there is no cutset of \( G \) of size at most \( d - 1 \). Finally, the vertices of \( K_n \) are denoted \( \{w_1, \ldots, w_n\} \).

2 Preliminaries

First of all, we recall one instance of the well-known Chernoff’s inequality.

Lemma 2.1 (see Theorem 2.1 in [11]). Given a binomial random variable \( X \), for every \( t \geq 0 \),

\[
\mathbb{P}(|X - \mathbb{E}X| \geq t) \leq 2 \exp\left(-\frac{t^2}{2(\mathbb{E}X + t/3)}\right).
\]

Next, recall that for a set \( S \) and a real number \( p \in [0, 1] \), the binomial random subset \( \text{Bin}(S, p) \) is a subset of \( S \) in which all elements of \( S \) appear independently with probability \( p \). The next lemma is a classic comparison result between \( \text{Bin}(S, p) \) and the uniform random \( m \)-element subset of \( S \) denoted \( \text{Bin}(S, m) \).

Lemma 2.2 (see [15] and Theorem 1.4 in [9]). Fix a sequence of sets \( (S_n)_{n \geq 1} \) satisfying \( |S_n| = n \) for all \( n \geq 1 \), and functions \( m = m(n) \) and \( \nu = \nu(n) \) satisfying \( m \to \infty \) and \( m^{-1/2} \nu \to \infty \) as \( n \to \infty \). Set \( p_\text{--} = p_\text{--}(n) = \max(0, (m - \nu)/n) \) and \( p_\text{+} = p_\text{+}(n) = \min((m + \nu)/n, 1) \). Then, there is a coupling of the sets \( \text{Bin}(S_n, p_\text{--}), \text{Bin}(S_n, m) \) and \( \text{Bin}(S_n, p_\text{+}) \) so that a.a.s. \( \text{Bin}(S_n, p_\text{--}) \subseteq \text{Bin}(S_n, m) \subseteq \text{Bin}(S_n, p_\text{+}) \).

Remark 2.3. Note that both [15] and Theorem 1.4 in [9] compare the binomial random graph \( G(n, p) \), where edges appear independently with probability \( p \in [0, 1] \), with the original Erdős-Rényi random graph \( G(n, m) \), which is a uniformly chosen graph among the graphs with \( n \) vertices and \( m \) edges. However, the proof method extends verbatim in the more general setting of Lemma 2.2.

The next lemma states and proves a useful property of the Erdős-Rényi process.

Lemma 2.4. Fix an integer \( k \geq 1 \), a positive function \( \mu = \mu(n) \) satisfying \( \frac{\log \log n}{\log n} \ll \mu \ll 1 \), and positive integer functions \( m = m(n) \) and \( M = M(n) \) satisfying \( m = o(n \log n) \) and \( M - m = (1 + \mu) n \log n/2 \). For every \( i \in [n] \), fix a set \( S_i \subseteq V(K_n) \) (possibly depending on the first \( m \) rounds of the Erdős-Rényi process) of size \( |S_i| \geq (1 - \mu/2)n \). Then, the following event holds a.a.s.: for every \( i \in [n] \), at least \( k \) edges connecting \( w_i \) and \( S_i \) are proposed to Builder during the rounds in the interval \( [m + 1, M] \).
Hence, \( G \) is a connected graph. Moreover, every vertex in \( G \) is adjacent to at least \( d \) vertices in \( G \). Fix an integer \( m \) such that \( |E(G(n,p))| \geq m \).

Proof. Recall that \( p = \log n/n \) and set \( N = \left( \frac{n}{\mu} \right) \). Then, conditionally on the event that the random graph \( G(n,p) \) has at least \( m \) edges, it dominates the graph \( G_m \) stochastically with respect to inclusion. Thus,

\[
\mathbb{P}(\Delta(G_m) \geq 3 \log n) \leq \mathbb{P}(\Delta(G(n,p)) \geq 3 \log n | |E(G(n,p))| \geq m) \\
\leq \frac{\mathbb{P}(\Delta(G(n,p)) \geq 3 \log n)}{\mathbb{P}(|E(G(n,p))| \geq m)} \leq \frac{n \mathbb{P}(\deg_{G(n,p)}(w_i) \geq 3 \log n)}{1 - \mathbb{P}(|E(G(n,p))| < m)},
\]

where the last inequality follows from a union bound over all \( n \) vertices in \( K_n \). Using Chernoff’s inequality for the binomial distributions \( \text{Bin}(n-1,p) \) and \( \text{Bin}(N,p) \) shows that

\[
\mathbb{P}(\Delta(G_m) \geq 3 \log n) \leq (1 + o(1))n \mathbb{P}(\text{Bin}(n-1,p) \geq 3 \log n) \\
\leq (1 + o(1))2n \exp \left( -\frac{(3 \log n)^2}{2((n-1)p + \log n)} \right) = n^{-5/4 + o(1)} = o(1).
\]

We condition on the a.a.s. event \( \{\Delta(G_m) \leq 3 \log n\} \) and expose the graph \( G_m \).

Next, we fix \( i \in [n] \) and consecutively expose the edges added at rounds \( m+1, \ldots, M \) in the Erdős-Rényi process. We estimate the probability that only \( j \in [0, k-1] \) edges connecting \( w_i \) and \( S_i \) have been proposed to Builder during the rounds \( m+1, \ldots, M \). On the one hand, at each of these rounds, the probability that the new edge goes between the vertex \( w_i \) and the set \( S_i \) is bounded from above by \( \frac{\mu}{N} \).

On the other hand, the probability that the edge proposed at round \( \ell \in [m+1, M] \) does not go between \( w_i \) and \( S_i \) is bounded from above by

\[
1 - \frac{(1 - \mu/2)n - j - \Delta(G_m)}{N - \ell + 1} \leq 1 - \frac{(1 - \mu/2)n - j - 3 \log n}{N - \ell + 1}.
\]

Hence, the probability that less than \( k \) edges connecting \( w_i \) and \( S_i \) have been proposed to Builder during the rounds \( m+1, \ldots, M \) is bounded from above by

\[
\sum_{j=0}^{k-1} \sum_{j \leq [m+1,M], |J|=j} \left( \frac{n-1}{N-M} \right)^j \prod_{\ell \in [m+1,M] \setminus J} \left( 1 - \frac{(1 - \mu/2)n - j - 3 \log n}{N - \ell + 1} \right)
\]

\[
= O \left( \sum_{j=0}^{k-1} \binom{M-m}{j} \left( \frac{2}{n} \right)^j \prod_{\ell=m+1}^{M} \left( 1 - \frac{(2 - \mu)n - O(\log n)}{n^2 - O(n \log n)} \right) \right)
\]

\[
= O \left( \log n^{k-1} \exp \left( -\left( 1 + O \left( \frac{\log n}{n} \right) \right) \frac{(M-m)(2 - \mu)}{n} \right) \right)
\]

\[
= O \left( \frac{\exp((k-1) \log \log n - (1 + o(1)) \mu \log n/2)}{n} \right) = o\left( \frac{1}{n} \right),
\]

where for the third equality we used that \( (1 + \mu)(2 - \mu)/2 = 1 + \mu/2 + o(\mu) \). The proof is completed by a union bound over all \( n \) vertices of \( K_n \). 

Our last preliminary result is a simple deterministic lemma providing a way to build \( d \)-connected graphs from smaller \( d \)-connected graphs.

**Lemma 2.5.** Fix an integer \( d \geq 1 \) and two graphs \( H \subseteq G \) such that \( H \) is \( d \)-connected and every vertex in \( G \setminus H \) is adjacent to at least \( d \) vertices in \( H \). Then, \( G \) is also a \( d \)-connected graph.

**Proof.** Fix any set \( U \subseteq V(G) \) of size \( |U| = d-1 \). Then, since \( H \) is a \( d \)-connected graph, \( H \setminus U \) is a connected graph. Moreover, every vertex in \( G \setminus H \) is connected by an edge in \( G \setminus U \) to a vertex in \( V(H) \setminus U \). Hence, \( G \setminus U \) is a connected graph, which finishes the proof. 

\[ \square \]
3 Proof for the case $d \geq 4$

We continue with the proof of Theorem 1.2 in the case $d \geq 4$. To start with, we define an auxiliary process. Next, we describe a strategy of Builder in the auxiliary process that within $o(n \log n)$ rounds constructs a well-connected random graph, which we relate to a configuration model with degree sequence of maximum degree $d$. Then, we couple the above strategy with a valid strategy of Builder in the original model, and thus ensure that Builder can a.a.s. construct a large graph that is almost $d$-connected in a certain sense within $o(n \log n)$ rounds. Finally, by using $(1 + o(1))n \log n/2$ additional rounds, we boost this graph to a $d$-connected graph and thus complete the analysis.

Stage 1: the auxiliary process and Builder’s strategy. ($o(n \log n)$ rounds, $dn/2 + o(n)$ accepted edges)

The auxiliary process is divided into $d$ independent iterations, and every iteration consists of the first $n \log n/\omega$ rounds of an Erdős-Rényi process. At every iteration, we let Builder construct a matching by greedily accepting every edge that does not share a vertex with the ones accepted up to now during the current iteration. Finally, we colour the edges constructed at iteration $i \in [d]$ in colour $i$; note that one edge may be coloured in more than one colour.

Using that the $d$ matchings are independent, by conditioning on the set of vertices of degree 1 at every iteration, we obtain a graph distributed according to a configuration model in which every vertex is incident to at most $d$ half-edges in different colours, and monochromatic half-edges are matched uniformly at random. Our motivation to introduce this auxiliary process is rooted in the fact that the configuration model is quite well-understood and easier to work with.

The inconvenience in using the above auxiliary model is that Builder will not be able to construct some of the edges in the $d$ matchings above in the original model. We call an edge repeated if it was proposed at two or more iterations in the auxiliary model. We note that, in order to compare the strategy of Builder in the auxiliary process and in the original one, it is sufficient to ignore the rounds at which repeated edges are proposed for a second, third, etc. time.

The next lemma provides a convenient way to analyse the set $R$ of repeated edges in the graph $G$ constructed in the auxiliary process: it turns out that this set of edges is dominated by a Bernoulli percolation on $G$ with a suitably chosen parameter. This will be useful in the third stage of our analysis where not only the number but also the distribution of the edges in $R$ will matter.

Lemma 3.1. Conditionally on the graph $G$ constructed during the auxiliary process (where only the edges are revealed but not their colours), one may couple the set $R$ of repeated edges in $G$ with a binomial random graph $\hat{G} \sim G(n, p)$ with $p = p(n) = \log n/n$ so that a.a.s. $R \subseteq E(G) \cap E(\hat{G})$.

Proof. Define $q = q(n) = 3 \log n/\omega n$ and, for every $i \in [d]$, let $S_i$ be the set of edges proposed during the $i$-th stage of the auxiliary process. By Lemma 2.2 and a union bound over $i \in [d]$, one may couple $(S_i)_{i=1}^d$ with an i.i.d. family of random graphs $(\hat{G}_i)_{i=1}^d$ with distribution $G(n, q)$ so that a.a.s. $S_i \subseteq E(\hat{G}_i)$ for all $i \in [d]$. Moreover, for every edge $e \in E(K_n)$, the probability that this edge appears in at least two of $(\hat{G}_i)_{i=1}^d$ given that it appears in at least one of $(\hat{G}_i)_{i=1}^d$ is

$$
\frac{1 - (1 - q)^d - dq(1 - q)^{d-1}}{1 - (1 - q)^d} \leq (d - 1)q \leq p,
$$

which concludes the proof. \hfill \Box

Stage 2: deleting the vertices of degree at most $d - 2$.

Let us first remark that the second stage does not require any additional rounds or accepted edges. Instead, it serves to analyse the graph constructed by Builder in the auxiliary process and prepares the terrain for
the additional boosting at Stage 3. The next lemma roughly says that a.a.s. the greedy construction at each of the \(d\) iterations of the auxiliary process outputs an almost perfect matching.

**Lemma 3.2.** A.a.s. the set of vertices that do not participate in some of the \(d\) matchings in the auxiliary process is of size at most \(\omega^3 n / \log n\).

**Proof.** Let us consider the matching constructed at a fixed iteration. Define \(s = (n - \omega^2 n / \log n) / 2\). For every \(i \in [s]\), denote by \(T_i\) the time needed to extend the partial matching of size \(i - 1\) to a matching of size \(i\). Then, \(T_i\) is stochastically dominated by a geometric distribution with parameter \((n - 2(i - 1)) / \binom{n}{2}\): indeed, the set of \(n - 2(i - 1)\) vertices isolated in the time interval \([T_i-1, T_i-1]\) spans \((n - 2(i - 1)) / 2\) edges and the total number of edges yet to be revealed is less than \(\binom{n}{2}\). Therefore,

\[
\mathbb{E} \left[ \sum_{i=1}^{s} T_i \right] \leq n(n-1) \sum_{i=1}^{s} \frac{1}{(n-2i+2)(n-2i+1)} = (1 + o(1)) n^2 \frac{2s}{1} \frac{1}{(n-j)(n-j-1)} \]

\[
= (1 + o(1)) \frac{n^2}{2} \left( \frac{1}{n} - \frac{1}{n}ight) = (1 + o(1)) \frac{\omega^2 n}{2 \log n}.
\]

Finally, Markov’s inequality implies that, at each of the \(d\) iterations, a.a.s. \(\sum_{i=1}^{s} T_i \leq \omega^3 n / \log n\), which completes the proof. \(\square\)

We remark that an alternative proof of the above lemma uses the fact that a.a.s. the uniform random subgraph of \(K_n\) on \(n \log n / \omega\) edges has maximum independent set of size less than \(\omega^3 n / \log n\).

At this point, we condition on the a.a.s. event of Lemma 3.2 and on the coloured vertex degrees, that is, we associate to every vertex in the \(i\)-th matching a half-edge of colour \(i\). As the \(d\) matchings are chosen uniformly at random and independently of each other, the coloured graph constructed in the auxiliary process is distributed according to a configuration model where the half-edges are paired uniformly at random while respecting the colouring. From this alternative viewpoint, edges with more than one colour are seen as multiedges with different colours.

Next, we describe an algorithm whose purpose it to find an almost-spanning subgraph with minimum degree \(d - 1\) where almost all vertices have degree \(d\). While this graph is not \(d\)-connected, it will turn out that a.a.s. one can add \(o(n)\) edges to make it \(d\)-connected. Thus, it is this well-connected subgraph that will be seen as a “skeleton” to which all remaining vertices will attach via \(d\) edges in the next \((1 + o(1)) n \log n \) stages.

The remainder of this stage focuses on the analysis of an exploration process \((\mathcal{A}_s, \mathcal{P}_s, \mathcal{N}_s)_{s \geq 0}\) gradually revealing the random graph \(G\) sampled from the coloured configuration model introduced above. In this exploration process, \(\mathcal{A}_s\) represents the set of *active* vertices at step \(s\) which are “found” by the process but we still do not know their entire neighbourhood in \(G\); \(\mathcal{P}_s\) represents the set of *passive* vertices whose entire neighbourhood has been revealed before step \(s\) and \(\mathcal{N}_s\) represents the *non-explored* vertices which were not yet “found” by the process. Our goal in the end of the exploration process is to find the \((d - 1)\)-core of \(G\) (that is, the largest subgraph of \(G\) with minimum degree at least \(d - 1\)) without revealing its edges and ensure that only \(o(n)\) vertices in this \((d - 1)\)-core have degree \(d - 1\).

We turn to the definition of the exploration process. In the beginning, we set \(\mathcal{A}_0 = \{v \in V(G) : \deg(v) \leq d - 2\}, \mathcal{P}_0 = \emptyset\) and \(\mathcal{N}_0 = V(G) \setminus \mathcal{A}_0\). At step \(s \geq 1\) of the process, look for a vertex in \(\mathcal{A}_{s-1}\) incident to at most \(d - 2\) edges that are still unrevealed throughout the exploration process; we denote this set of vertices by \(\mathcal{A}_{s-1}\). If \(\mathcal{A}_{s-1} = \emptyset\), terminate the exploration process and set \(H = G[\mathcal{A}_{s-1} \cup \mathcal{P}_{s-1}]\). (In this case, by construction, \(H\) is the \((d - 1)\)-core of \(G\).) Otherwise, pick the vertex \(v\) in \(\mathcal{A}_{s-1}\) with the smallest index (recall that \(V(G) = \{w_1, \ldots, w_n\}\) and explore its neighbours, which amounts to setting

\[
\mathcal{A}_s = (\mathcal{A}_{s-1} \setminus \{v\}) \cup (N(v) \cap \mathcal{N}_{s-1}), \quad \mathcal{P}_s = \mathcal{P}_{s-1} \cup \{v\} \quad \text{and} \quad \mathcal{N}_s = \mathcal{N}_{s-1} \setminus N(v).
\]

Define \(\tau\) as the unique integer when \(\mathcal{A}_{\tau} = \emptyset\). We note that the exploration process formally describes the usual way of obtaining the \((d - 1)\)-core of a graph by iteratively deleting vertices of degree at most
Corollary 3.5. First of all, Lemma 3.2 implies that

\[ d - 2. \] However, since this core must remain a random graph in our case, we carefully keep track of the randomness exposed throughout these iterative deletions.

Lemma 3.3. The exploration process \((A_s, P_s, N_s)_{s \geq 0}\) a.a.s. terminates after at most \(n/\omega\) steps.

Proof. First of all, Lemma 3.2 implies that \(|A_0| = |A_0| \leq \omega^3 n/\log n\). Fix an integer \(s \leq n/\omega\). If \(s \leq \tau - 1\), then \(|A'_{s+1}| \geq |A'_s| - 1\) and equality holds unless the vertex \(v\) explored at step \(s\) connects (via an edge revealed at the current step) to some vertex in \(A_s\) or to some vertex in \(N_s\) of degree at most \(d - 1\). Hence, the probability that the number of active vertices with at most \(d - 2\) unmatched edges decreases is at least

\[
1 - (d - 2)\left|A_s\right| + \left|\left\{v \in V(G) : \deg_G(v) \leq d - 1\right\}\right|/|N_s| \geq 1 - d^{dn/\omega + o(n/\omega)} n - dn/\omega \geq 1 - 2d^2/\omega.
\]

Hence, \(|A'_{\min(s, \tau)}|_{s \geq 0}\) is dominated by the process \((X_{\min(s, \tau)})_{s \geq 0}\) where \((X_s)_{s \geq 0}\) is a random walk that makes a step \(-1\) with probability \(1 - 2d^2/\omega\) and step \(+d\) with probability \(2d^2/\omega\). We conclude that

\[
P\left(|A'_{\min(n/\omega, \tau)}| \geq 1\right) \leq \mathbb{P}(X_{n/\omega} \geq 1) \leq \mathbb{P}\left(\left|\left\{s \in [n/\omega] : X_s - X_{s-1} = d\right\}\right| \geq \frac{n}{2d\omega}\right) = o(1),
\]

where the last equality follows from Markov’s inequality and the fact that, on average, the random walk makes \(O(n/\omega^2)\) steps \(+d\) until step \(n/\omega\).

Now, define \(G' = G[A_\tau \cup N_\tau]\). The next two results tell us more about the structure of \(G'\).

Lemma 3.4. The graph \(G'\) contains only vertices of degree \(d - 1\) and \(d\). Moreover, the number of vertices of degree \(d - 1\) in \(G'\) is at most \(|A_\tau| + \left|\left\{v \in V(G) : \deg_G(v) = d - 1\right\}\right|\).

Proof. First of all, note that the vertices in \(G'\) that do not have the same degrees in \(G\) and in \(G'\) are exactly \(A_\tau\). Assuming the first statement of the lemma, this proves the second statement.

We show the first statement. On the one hand, all vertices of degree at most \(d - 2\) in \(G\) are contained in \(P_\tau\). On the other hand, no edge goes between \(P_\tau\) and \(N_\tau\) and, since \(A'_\tau = \emptyset\) by assumption, all vertices in \(A_\tau\) send at least \(d - 1\) edges towards \(A_\tau \cup N_\tau\). We conclude that all vertices in \(A_\tau\) have degree \(d - 1\) in \(G'\), which completes the proof.

Corollary 3.5. A.a.s., at the end of the exploration process, the graph \(G'\) contains at least \(n - n/\omega\) vertices and at most \((d + 1)n/\omega\) vertices of degree \(d - 1\) (in \(G'\)).

Proof. The first statement is implied by Lemma 3.3 (under which \(|P_\tau| \leq n/\omega|\). The second statement follows by combining Lemma 3.4, the fact that a.a.s. \(|\left\{v \in V(G) : \deg(v) \leq d - 1\right\}| \leq \omega^3 n/\log n \ll n/\omega\) by Lemma 3.2 and the fact that a.a.s. \(|A_\tau| \leq d|P_\tau| + |A_0| \leq dn/\omega + o(n/\omega)\) by Lemmas 3.2 and 3.3

Stage 3: analysis of \(G'\) and boosting to a \(d\)-connected graph. \(((1 + o(1))n \log n/2\) rounds, \(o(n)\) accepted edges)

At this last stage, we first find a (multi-)graph \(G''\) obtained from \(G'\) by deleting only a few edges with the additional property that Builder can construct \(G''\) in the original process up to identifying multiple edges. Then, we show that a.a.s. Builder can boost \(G''\) to a \(d\)-connected graph within \((1 + o(1))n \log n/2\) rounds.

To begin with, recall that the exploration process from Stage 2 does not reveal any edge in \(G'\). Therefore, the graph \(G'\) is distributed as a random (multi-)graph constructed according to the coloured configuration model. The next classic lemma is a technical result saying that the typical local structure of \(G'\) is tree-like.

Lemma 3.6. For every integer \(\ell \geq 1\), a.a.s. the number of cycles of length \(\ell\) in \(G'\) is at most \(\log n\).
Proof. First of all, Corollary 3.5 implies that a.a.s. $\Delta(G') = d$ and $n - o(n)$ of the vertices in $G'$ have degree $d$. We condition on this event. Then, the expected number of $\ell$-cycles is bounded from above by $\binom{n}{\ell} d^\ell \frac{1}{(n-o(n))} \leq 2d^\ell \ll \log n$ (where the factor $d^\ell$ comes from choosing the colours of the $d$ edges along the cycle) and the first statement follows from Markov’s inequality.

Recall that one may couple the auxiliary process with the first (at most) $dn \log n/\omega$ rounds of the original process by simply omitting the rounds at which a repeated edge is proposed for a second, third, etc. time. We use this in combination with Lemma 3.1, which allows us to dominate the set of repeated edges in the auxiliary process by a random set including every edge of $G'$ independently with probability $p = \log n/n$. We say that edges belonging to the latter random set are marked and denote by $G''$ the subgraph of $G'$ containing only the unmarked edges. We note a minor point which is nevertheless a bit subtle: if at least one of several multiple edges between two vertices in $G''$ is marked, each of the edges between these two endpoints cannot be used by Builder in the original process. However, conditionally on the a.a.s. event from Lemma 3.6 with $\ell = 2$, the probability of the above event is $O(p \cdot 2 \log n) = o(1)$, so we ignore it.

Denote $n'' = |V(G'')|$. We recall that a.a.s. $n'' = n - o(n)$ and Corollary 3.5 shows that there are a.a.s. only $O(n/\omega)$ of these vertices of degree $d - 1$. We condition on these events.

Lemma 3.7. For every $d \geq 4$, there are a.a.s. no sets $U \subseteq V(G'')$ of size more than $6$ such that $|N_{G''}(U)| \leq d - 1$. Moreover, the number of vertices of $G''$ in sets $U \subseteq V(G'')$ of size in $[6]$ that satisfy $|N_{G''}(U)| \leq d - 1$ is a.a.s. $o(n)$.

Proof. Condition on the a.a.s. events (ensured by Corollary 3.5) that $n'' \geq n - n/\omega$ and that there are at most $(d+1)n/\omega$ vertices of degree $d - 1$ in $G'$. We divide the proof into two cases according to the size of $U$.

First, we fix a sufficiently small $\varepsilon > 0$, $s \in [\varepsilon n, n''/2]$, a vertex set $U \subseteq V(G'')$ of size $s$ and a vertex set $W \subseteq V(G'') \setminus U$ of size at most $(\log n)^2$. We show that, for any choice of $U$ and $W$ as above, $N_{G'}(U) \not\subseteq W$. To begin with, there are at least $ds - (d+1)n/\omega$ half-edges sticking out of $U$ in the coloured configuration model generating the graph $G'$. To have that $N_{G'}(U) \subseteq W$, each of these half-edges has to connect with another half-edge sticking out of $U \cup W$ in the same colour. Moreover, after $j - 1$ of the edges in a given colour sticking out of $U$ have been formed, at least $n'' - 2(j - 1) - (d+1)n/\omega \geq n - 2(j - 1) - (d+2)n/\omega$ half-edges in that same colour remain unmatched. Letting $s_1, \ldots, s_d$ be the number of vertices in $U$ having a half-edge in colour respectively $1, \ldots, d$ sticking out of them, the probability that $N_{G''}(U) \subseteq W$ is at most

$$\prod_{i=1}^d \prod_{j=1}^{[s_i/2]} \left( \frac{s_i + (\log n)^2 - 2(j - 1)}{n - 2(j - 1) - (d+2)n/\omega} \right) = e^{o(n)} \prod_{i=1}^d \left( \frac{[n/2] - (d+2)n/2\omega}{[s_i/2] + (\log n)^2/2} \right)^{-1} = e^{o(n)} \left( \frac{[n/2]}{[s/2]} \right)^{-d},$$

where we used that $s_i = s - o(n)$ for all $i \in [d]$. A union bound over $s \in [\varepsilon n, n''/2]$, the sets $U$ of size $s$ and the sets $W$ of size $\ell \leq (\log n)^2$ gives that the probability that $N_{G''}(U) \subseteq W$ is bounded from above by

$$\sum_{s=\varepsilon n}^{n''} \sum_{\ell=1}^{(\log n)^2} \binom{n-s}{\ell} e^{o(n)} \left( \frac{[n/2]}{[s/2]} \right)^{-d}.$$

However, for every $s \in [\varepsilon n, n''/2]$, we have that $\binom{n-s}{\ell} = e^{o(n)}$ and

$$\binom{n}{s} \binom{[n/2]}{[s/2]}^{-d} = e^{o(n)} \exp \left( s \log \left( \frac{n}{s} \right) + (n-s) \log \left( \frac{n-s}{n-s} \right) - \frac{ds}{2} \log \left( \frac{n}{s} \right) - \frac{d(n-s)}{2} \log \left( \frac{n}{n-s} \right) \right).$$

Since $d \geq 3$, there is a constant $c = c(\varepsilon) > 0$ such that, for every $s \in [\varepsilon n, n''/2]$, the above expression is bounded from above by $e^{-cn + o(n)}$, which is sufficient to conclude that the expression in (2) tends to zero.
In particular, this means that a.a.s. every set $U$ of size $s \in \left[\varepsilon n, n''/2\right]$ has neighbourhood of size at least $(\log n)^2$. Moreover, Markov’s inequality implies that a.a.s. there are at most $(\log n)^2/2$ marked edges, so every set $U$ as above has $\Omega((\log n)^2) > d - 1$ neighbours in $G''$.

Now, fix $s \in [7, \varepsilon n]$, a vertex set $U \subseteq V(G'')$ of size $s$ and a vertex set $W \subseteq V(G'') \setminus U$ of size at most $d - 1$. Then, there are at least $(d - 1)s$ half-edges sticking out of $U$ in the coloured configuration model generating the graph $G'$. To have that $W = N_{G''}(U)$, each of these half-edges either has to connect with another half-edge sticking out of $U \cup W$ in the same colour or participate in a marked edge. Moreover, after $i - 1$ of the half-edges in a given colour sticking out of $U$ have been matched, at least $n'' - 2(i - 1) - (d + 1)n/\omega \geq n'' - 2(i - 1) - (d + 2)n/\omega$ half-edges in that same colour remain unmatched.

With $s_1, \ldots, s_d$ as above, the probability that $N_{G''}(U) = W$ is at most

$$\prod_{i=1}^d \prod_{j=1}^{[s_i/2]} \left( \frac{s_i + (d - 1) - (j - 1)}{n - 2(j - 1) - (d + 2)n/\omega} + p \right).$$

Moreover, one can easily check that, for all $i \in [d], j \in [[s_i/2]]$ and large $n$,

$$\frac{s_i + (d - 1) - (j - 1)}{n - 2(j - 1) - (d + 2)n/\omega} + p \leq \frac{2(s + \log n)}{n}.$$

Using a union bound over all possible choices for $U$ and $W$ (including $\ell = |W|$) and the fact that $[s_1/2] + \ldots + [s_d/2] \geq (d - 1)s/2$, the probability that such sets with $N_{G''}(U) = W$ exist is at most

$$\sum_{\ell=0}^{d-1} \binom{n''}{s} \binom{n'' - s}{\ell} \left( \frac{2(s + \log n)}{n} \right)^{(d-1)s/2} \leq \binom{n}{s} n^{d-1} \left( \frac{2(s + \log n)}{n} \right)^{(d-1)s/2}.$$

Moreover, for every $s \in [7, n''/2 - 1]$,

$$\binom{n}{s+1} n^{d-1} \left( \frac{2(s+1 + \log n)}{n} \right)^{(d-1)(s+1)/2} = \frac{s + 1}{n - s} \frac{n}{2(s + 1 + \log n)} \frac{n}{s + 1 + \log n} \left( \frac{s + \log n}{s + 1 + \log n} \right)^{(d-1)s/2}.$$

Moreover, as $d \geq 4$, the above expression is larger than 1 and uniformly bounded away from 1 over the interval $s \in [7, \varepsilon n]$ for some sufficiently small but fixed $\varepsilon > 0$. As a result, the sum of the right hand side of (3) over $s \in [7, \varepsilon n]$ is of the same order as the term for $s = 7$. Since $d \geq 4$, the latter is bounded by

$$\left( \frac{n}{7} \right)^{n^{d-1}} \left( \frac{2(7 + \log n)}{n} \right)^{7(d-1)/2} = n^{d+6-7(d-1)/2-o(1)} = o(1).$$

Finally, we recall that there are $o(n)$ vertices of degree less than $d$ in $G'$. Moreover, a.a.s. there are at most $(\log n)^2 = o(n)$ marked edges, and $O(\log n) = o(n)$ vertices in cycles of length at most 14 by Lemma 3.6. In particular, a.a.s. the 7-th neighbourhood of all but $o(n)$ of the vertices in $G''$ is isomorphic to the 7-th neighbourhood of any vertex in a $d$-regular tree. We condition on this event and show that none of these vertices is in a set $U$ of size $s \in [6]$ satisfying $|N_{G''}(U)| \leq d - 1$. Fix such a vertex $v$ and let $U' \subseteq U$ be the connected component of $v$ in $G''[U]$. Then, $U'$ has neighbourhood of size at least $d$; indeed, the vertex $v$ itself has $d$ neighbours and adding the vertices in $U'$ one by one so that the set of all added vertices remains connected at every step shows that the size of the neighbourhood cannot decrease, which completes the proof.

To conclude the proof in the case $d \geq 4$, we condition on the a.a.s. event from Lemma 3.6 and the graph $G''$ satisfying the a.a.s. statement of Lemma 3.7. As already pointed out, by the coupling in Lemma 3.1, Builder has a strategy to construct $G''$ in $o(n \log n)$ rounds up to identifying multiple edges (of which there
are at most $\log n$ pairs). At the same time, connecting every vertex $v \in V(G''')$ in a set $U$ with $|U| \leq 6$ and $|N_{G'''}(U)| \leq d - 1$ to $d + 5$ new neighbours in $G'''$ is sufficient to transform $G'''$ into a $d$-connected graph. Indeed, for every such vertex $v$ and every set $U$ as above that contains it, the size of the neighbourhood of $U$ in this larger graph would be at least $(d + 5) - (|U| - 1) \geq (d + 5) - 5 = d$. Using that there are only $o(n)$ such vertices, and there are a.a.s. $o(n)$ vertices in $G \setminus G'$ as well, combining Lemma 2.4 (applied for $k = d + 5$, $\mu = 4/\omega \geq 2|V(G \setminus G')|/n$ and the sets $S_i = V(G''') \setminus \{w_i\}$ for all $i \in [n]$) with Lemma 2.5 finishes the proof in the case $d \geq 4$.

Before continuing with the proof for $d \in \{2, 3\}$, note that there is a natural reason for the above approach to fail in this case. Indeed, when $d = 2$, a configuration model with vertices of degrees 1 and 2 only is a union of paths and thus has plenty of connected components. Moreover, when $d = 3$, the presence of many vertices of degree 2 ensures long paths, which in turn implies that there will a.a.s. exist sets $U$ of size $|U| \gg 1$ satisfying $|N_{G'''}(U)| = 2$.

4 Proof for the case $d \in \{2, 3\}$

As mentioned in the outline of the proof, we divide the construction into three major stages. At the first stage, we build many long vertex-disjoint paths. At the second stage, we merge almost all paths constructed up to now into a single long cycle. For $d = 2$, the third stage will rely on Lemmas 2.4 and 2.5 while for $d = 3$, a bit of additional work is needed.

**Stage 1: building long paths.** ($o(n \log n)$ rounds, $n + o(n)$ accepted edges for $d = 2$, $\frac{4}{\omega} n + o(n)$ accepted edges for $d = 3$)

In this stage, we use $o(n \log n)$ rounds to construct a set of paths of total length $n - o(n)$ for $d = 2$, and $\frac{4}{\omega} n - o(n)$ for $d = 3$. The proof is given for $d = 2$; we point out a single minor modification in the case $d = 3$ along the way.

Fix $t_1 = \omega^5 n$ and $N = n/2\omega^3$. Starting from $N$ vertices, we iteratively construct $N$ disjoint paths. Initially, each of these paths consists of one vertex. Then, at every round, if the proposed edge connects a vertex outside the $N$ paths to the last added vertex in some of the paths, Builder accepts the edge, otherwise they ignore it, see Figure 1.

**Lemma 4.1.** After round $t_1$, the $N$ paths a.a.s. contain at least $n - n/\omega$ vertices. Moreover, at this point a.a.s. at least $(1 - 2/\omega)n$ of these vertices are in paths of length between $\omega^3$ and $3\omega^3$.

Note that, when $d = 3$, instead of reaching round $t_1$ and then stop extending the paths, we stop when the total number of vertices in the $N$ paths reaches $\frac{4}{\omega} n - 1$. By Lemma 4.1 a.a.s. this moment comes before round $t_1$.

**Proof of Lemma 4.1.** Let $T$ denote the round when there remain exactly $n/\omega$ vertices outside the $N$ paths. Note that, at every round $s \in [0, T - 1]$, the probability that one of the $N$ paths is extended is bounded from below by

$$\left(\frac{n}{2\omega^3}\right) \left(\frac{n}{\omega}\right) \left(\frac{n}{2}\right) \geq \frac{1}{\omega^4}.$$ 

Hence, the number of vertices $X_s$ in the $N$ paths after round $s \leq T - 1$ stochastically dominates the sum of $s$ independent Bernoulli random variables with success probability $1/\omega^4$. By the above comparison and Chernoff’s inequality,

$$\mathbb{P}(T > t_1) = \mathbb{P}(X_{t_1} \leq n - n/\omega) \leq \mathbb{P}(\text{Bin}(t_1, 1/\omega^4) \leq n) = o(1),$$

which proves the first statement.
Figure 1: The figure represents the first 4 rounds of the process. The initial \( N \) vertices are ordered horizontally just below the round number. The transparent edges are the ones proposed at the given round. Thus, Builder accepts the first two proposed edges, ignores the third one since it is not incident to the last vertex that entered the first path, and ignores the fourth one since it is not incident to any of the vertices that are currently in the \( N \) paths.

Let us condition on the event \( \{ T \leq t_1 \} \). Denote by \( Y_1 \) the number of vertices in paths of length at most \( \omega^3 \), and by \( Y_2 \) the number of vertices in paths of length at least 3\( \omega^3 \). We will show that \( \mathbb{E}[Y_1 + Y_2] = o(n/\omega) \), and then conclude by Markov’s inequality.

Note that, at every round when some path is augmented, each of the \( N \) paths has equal probability to be the augmented one. Thus, after \( n - o(n) \) vertices have been included in the \( N \) paths, the length of every path is a binomial random variable with parameters \( n - o(n) \) and 1/\( N \). Since on average every path contains \( (1 - o(1))n/N = (2 - o(1))\omega^3 \) vertices after \( t_1 \) rounds, Chernoff’s inequality again implies that one path contains at most \( \omega^3 \) vertices with probability \( \exp(-\Omega(\omega^3)) \leq 1/\omega^3 \). Hence, \( \mathbb{E}Y_1 \leq \omega^3 N/\omega^3 = N = o(n/\omega) \).

Moreover, for every \( i \geq 0 \), Chernoff’s inequality provides also that

\[
\mathbb{P}\left( \text{Bin}(n - n/\omega, 1/N) \geq 3\omega^3 + i \right) \leq \exp(-\Omega(\omega^3 + i)),
\]

so we conclude that

\[
\mathbb{E}Y_2 = N \sum_{i \geq 0} (3\omega^3 + i) \mathbb{P}(\text{Bin}(n - n/\omega, 1/N) \geq 3\omega^3 + i) \leq N \sum_{i \geq 0} (3\omega^3 + i) \exp(-\Omega(\omega^3 + i)) = o(n/\omega).
\]

Thus, using \( \mathbb{E}[Y_1 + Y_2] = o(n/\omega) \) and Markov’s inequality concludes the proof of the second statement.

Stage 2: merging the paths. \((o(n \log n) \text{ rounds, } o(n) \text{ accepted edges})\)

The reasoning in this stage is given for \( d = 2 \). The adaptation to the case \( d = 3 \) is immediate and requires no additional arguments.

To begin with, condition on the a.a.s. events from Lemma \([4,1]\) and let \( M \) be the number of paths of length between \( \omega^3 \) and 3\( \omega^3 \) after \( t_1 \) rounds. We call these paths typical. At the second stage, we introduce an auxiliary directed graph constructed as follows.

Define \( 2M \) sets \((S'_i)_{i=1}^M \) and \((S''_i)_{i=1}^M \) where \( S'_i \) contains the first \( \omega^2 \) vertices and \( S''_i \) contains the last \( \omega^2 \) vertices in the \( i \)-th typical path. Also, define \( t_2 = t_1 + \omega^3 n \) and let Builder accept the edge \( e \) at round \( s \in [t_1 + 1, t_2] \) if there are \( i, j \in [M] \) such that \( e \) connects a vertex in \( S'_i \) to a vertex in \( S''_j \). Now, the
auxiliary directed graph $H$ with vertex set $[M]$ is constructed iteratively by adding the edge $ij$ if there is a round $s \in [t_1 + 1, t_2]$ at which Builder constructs an edge $S'_i$ and $S''_j$, see Figure 2. Note that, for two vertices $i, j$ in $H$, the edges $ij$ and $ji$ can participate simultaneously in $H$.

**Lemma 4.2.** After round $t_2$, the graph $H$ a.a.s. contains a directed cycle of length at least $M - M/\omega$.

**Proof.** First of all, let $\mathcal{E}$ be the event that, for every pair $i, j \in [M]$, there are at most $\omega^4/2$ of all edges between $S'_i$ and $S''_j$ that have been proposed until round $t_1$. We show that $\mathcal{E}$ holds a.a.s. Indeed, by Remark 2.3 one may couple $G_{t_1}$ and $G(n, p)$ for $p = \log n/n$ so that a.a.s. $G_{t_1} \subseteq G(n, p)$. Moreover, the expected number of balanced bipartite graphs on $2\omega^2$ vertices and at least $\omega^4/2$ edges in $G(n, p)$ is bounded from above by

$$
\binom{n}{2\omega^2} \left(\frac{2\omega^2}{\omega^2}\right) \left(\frac{\omega^4}{\omega^4/2}\right) \left(\frac{\log n}{n}\right)^{\omega^4/2} \leq n^{2\omega^2} 2^{2\omega^2 + \omega^4} \left(\frac{\log n}{n}\right)^{\omega^4/2} = o(1).
$$

Since this is an upper bound for $\mathbb{P}(\mathcal{E})$, $\mathcal{E}$ holds a.a.s. and we condition on this event.

Now, fix $i, j \in [M]$ and denote by $E_{i,j}$ the set of edges between $S'_i$ and $S''_j$ that were not proposed until round $t_1$, and by $\mathcal{E}_{i,j}$ the event that at least one edge in $E_{i,j}$ was constructed at some round in the interval $[t_1 + 1, t_2]$. Then, the probability that $H$ contains the edge $ij$ is given by

$$
\mathbb{P}(\mathcal{E}_{i,j}) = 1 - \prod_{s=t_1+1}^{t_2} \left(1 - \frac{|E_{i,j}|}{n(n-1)/2 - s + 1}\right)
$$

$$
= 1 - \exp\left(-(2 + o(1))\frac{|E_{i,j}|\omega^3}{n}\right) = (2 + o(1))\frac{|E_{i,j}|\omega^3}{n}.
$$

Figure 2: The figure depicts the first 5 typical paths (on the left) as well as the graph, induced from $H$ by the 5 corresponding vertices (on the right). The sets $S'_1, \ldots, S'_5$ are depicted at the top while the sets $S''_1, \ldots, S''_5$ are put at the bottom. Then, the three edges between $S'_1$ and $S''_3$, $S'_3$ and $S''_2$, and $S'_2$ and $S''_1$ result in the edges 13, 32 and 21 in $H$.
Moreover, for every pair \(i_1,j_1, i_2,j_2 \in [M]\) with \((i_1,j_1) \neq (i_2,j_2)\), \(\mathbb{P}(E_{i_1,j_1} \cap E_{i_2,j_2})\) can be expressed as

\[
\sum_{u=t_1+1}^{t_2} \prod_{s=t_1+1}^{u-1} \left(1 - \frac{|E_{i_1,j_1}|}{n(n-1)/2 - s + 1}\right) \frac{|E_{i_1,j_1}|}{n(n-1)/2 - s + 1} \left(1 - \prod_{s=u+1}^{t_2} \left(1 - \frac{|E_{i_2,j_2}|}{n(n-1)/2 - s + 1}\right)\right)
\]

\[
+ \sum_{u=t_1+1}^{t_2} \prod_{s=t_1+1}^{u-1} \left(1 - \frac{|E_{i_1,j_1}| + |E_{i_2,j_2}|}{n(n-1)/2 - s + 1}\right) \frac{|E_{i_2,j_2}|}{n(n-1)/2 - u + 1} \left(1 - \prod_{s=u+1}^{t_2} \left(1 - \frac{|E_{i_1,j_1}|}{n(n-1)/2 - s + 1}\right)\right),
\]

where \(u\) denotes the first moment when an edge between \(S_{i_1}'\) and \(S_{j_1}'\), or between \(S_{i_2}'\) and \(S_{j_2}'\), has been proposed to Builder. Moreover, the first expression computes the probability that this edge goes between \(S_{i_1}'\) and \(S_{j_1}'\), and at least one edge between \(S_{i_2}'\) and \(S_{j_2}'\) is proposed at some of the next \(t_2-u\) rounds, while the second expression computes the probability of the other scenario. Using that for every \(u \in [t_1+1, t_2]\) and \(\ell \in [2]\) we have

\[
\prod_{s=t_1+1}^{u-1} \left(1 - \frac{|E_{i_2,j_2}|}{n(n-1)/2 - s + 1}\right) = 1 - o(1)
\]

\[
\prod_{s=u+1}^{t_2} \left(1 - \frac{|E_{i_2,j_2}|}{n(n-1)/2 - s + 1}\right) = 1 - (1 + o(1))\frac{2(t_2-u)|E_{i_2,j_2}|}{n^2},
\]

the above expression rewrites

\[
(1 + o(1)) \frac{8|E_{i_1,j_1}| |E_{i_2,j_2}|}{n^4} \sum_{u=t_1+1}^{t_2} (t_2 - u) = (1 + o(1)) \frac{4(t_2 - t_1)^2 |E_{i_1,j_1}| |E_{i_2,j_2}|}{n^4},
\]

which by (11) is also equal to \((1 + o(1)) \mathbb{P}(E_{i_1,j_1}) \mathbb{P}(E_{i_2,j_2})\). Thus,

\[
\text{Var} \left[ \sum_{i,j \in [M]} 1_{E_{i,j}} \right] = o \left( \mathbb{E} \left[ \sum_{i,j \in [M]} 1_{E_{i,j}} \right]^2 \right),
\]

which means that a.a.s. \(H\) contains

\[
(1 + o(1)) \mathbb{E} \left[ \sum_{i,j \in [M]} 1_{E_{i,j}} \right] \geq \frac{\omega^7 M^2}{n} \geq \frac{\omega^4 M}{4}
\]

degrees. We conclude by Theorem 1 from [13] implying that the binomial random directed graph on \(M\) vertices and with edge probability \(q = \omega^3/M\) a.a.s. contains a directed cycle covering \(M - M/\omega\) vertices, and the fact that \(H\) a.a.s. dominates such a graph by Remark 2.3. \(\square\)

**Stage 3.1: completing the picture for \(d = 2\).** \((1 + o(1)) n \log n/2\) rounds, \(o(n)\) accepted edges

To conclude the proof for \(d = 2\), note that a cycle containing almost all vertices in \(H\) implies that there is a cycle in \(G_t\) that contains \(n - o(n)\) vertices. Indeed, for every \(s \geq 2\), the existence of the cycle \(i_1, \ldots, i_s\) in \(H\) implies that, for every \(j \in [s]\), a vertex \(u_j \in S_{i_j}'\) was connected by an edge to a vertex in \(v_{j+1} = S_{i_{j+1}}''\) (indices seen modulo \(s\)). Hence, the edges \((u_j, v_{j+1})_{j \in [s]}\) and the subpaths from Stage 1 connecting \(v_j\) to \(u_j\) form a cycle \(C\) containing almost all vertices in the \(s\) used paths. Hence, applying Lemma 2.4 (for \(k = 2\) and \(S_i\) the vertex set of the cycle \(C\)) and Lemma 2.5 (for \(H = C\), which is a 2-connected graph) shows that \((1 + o(1)) n \log n/2\) rounds are sufficient to construct two edges from every vertex outside \(C\) to \(C\) itself, which finishes the proof in the case \(d = 2\).

**Stage 3.2: building an almost spanning almost 3-connected graph over the long cycle for \(d = 3\).** \((o(n \log n)\) rounds, \(\frac{3}{4} n + o(n)\) accepted edges)
Denote by $C$ the cycle containing $\frac{4}{3}n - o(n)$ vertices in the case $d = 3$ and set $t_3 = t_2 + n \log n / \omega$, $t_4 = t_3 + n \log n / \omega$ and $t_5 = t_4 + n \log n / \omega$. Also, let us divide the cycle $C$ into three paths of equal (or almost equal) length whose vertex sets we denote by $V_1$, $V_2$ and $V_3$, and denote by $V_4$ a set of vertices outside the cycle satisfying $|V_4| = \max(|V_1|, |V_2|, |V_3|)$ (recall that $|V(C)| \leq \frac{3}{4}n - 1$, so there are at least $\max(|V_1|, |V_2|, |V_3|)$ vertices outside $C$).

Now, during the rounds in the interval $[t_2 + t_3]$ (respectively $[t_3 + t_4]$ and $[t_4 + t_5]$), we let Builder greedily construct a matching between $V_1$ (respectively $V_2$ and $V_3$) and $V_4$. Note that this way, most of the vertices in $C$ obtain one additional neighbour while most vertices outside $C$ connect to three vertices in $C$. Denote by $\tilde{C}$ the graph obtained by restricting $G_{t_5}$ to the union of $C$ and the vertices with three neighbours of $C$.

**Lemma 4.3.** The set of vertices in $C$ of degree 2 in $\tilde{C}$ is stochastically dominated by a binomial random subset of $V(C)$ in which every element is included independently with probability $\omega^4 / \log n$.

**Proof.** By the same argument as in Lemma 3.2, we know that there are a.a.s. at most $\omega^{3}|V(C)|/ \log n$ vertices in $C$ that do not participate in the matching between $V_1$ (respectively $V_2$ and $V_3$) and $V_4$ constructed during rounds $[t_2 + t_3]$ (respectively $[t_3 + t_4]$ and $[t_4 + t_5]$). As a consequence, there are a.a.s. at least $|V_4| - 3\omega^3|V(C)|/ \log n - 1$ vertices in $V_4$ that have three neighbours in $C$ after round $t_5$.

We condition on this event as well as on the set $S$ of these vertices.

Then, for every $i \in [3]$, the subset of vertices in $V_i$ that remains unmatched to a vertex in $S$ is distributed uniformly among all subsets of $V_i$ of size $|V_i| - |S| \leq 1 + 3\omega^3|V(C)|/ \log n$. Hence, by Lemma 2.2, one may a.a.s. stochastically dominate each of them by a binomial random subset of $V_i$ in which every element is chosen with probability $2^{1 + 3\omega^3|V(C)|/ \log n} \leq \omega^4 / \log n$, as desired. □

**Lemma 4.4.** The graph $\tilde{C}$ is 2-connected. Moreover, all cutsets $U$ in $\tilde{C}$ of size 2 satisfy that $U \subseteq C$ and $\tilde{C} \setminus U$ contains two connected components, one of which is a path in $C$ consisting of vertices of degree 2 in $\tilde{C}$.

**Proof.** The first statement follows from Lemma 2.5 for $H = C$ (which is a 2-connected graph) and $G = \tilde{C}$.

Now, if $\tilde{C} = C$, the second statement holds trivially. Otherwise, let $\{v_1, v_2\}$ be a cutset in $\tilde{C}$. Then, since $C$ itself is a 2-connected graph and all other vertices connect to $C$ by 3 edges, we must have that $\{v_1, v_2\} \subseteq V(C)$. Now, assume without loss of generality that $v_1 \in V_1$ and $v_2 \in V_1 \cup V_2$. Suppose for contradiction that the path between $v_1$ and $v_2$ that is disjoint from $V_3$ contains a vertex $u$ of degree 3 in $\tilde{C}$. Then, there is a vertex in $V_3$ that has a common neighbour with $u$ in $\tilde{C} \setminus C$. Thus, all vertices in $C \setminus \{v_1, v_2\}$ are in the same connected component in $\tilde{C} \setminus \{v_1, v_2\}$, and since all vertices in $\tilde{C} \setminus C$ are connected to $C \setminus \{v_1, v_2\}$ by at least one edge, $C \setminus \{v_1, v_2\}$ must be a connected graph, which is a contradiction with the fact that $\{v_1, v_2\}$ is a cutset. This completes the proof. □

For every vertex $v \in V(C)$, denote by $P(v)$ the unique path in $C$ containing $v$ for which the first and the last vertex of $P(v)$ are of degree 3 in $\tilde{C}$, and all remaining vertices are of degree 2 in $\tilde{C}$.

**Corollary 4.5.** The following holds conditionally on the event $\tilde{C} \setminus C \neq \emptyset$: every graph $\tilde{C}$ constructed from $C$ by connecting (by an edge) every vertex $v \in V(C)$ of degree 2 in $\tilde{C}$ to a vertex in $\tilde{C} \setminus P(v)$, is 3-connected.

**Proof.** Suppose for contradiction that this is not the case. Then, there is a cutset $\{v_1, v_2\}$ of $\tilde{C}$ (which is also a cutset of $\tilde{C}$). As in the proof of Lemma 4.4 assume without loss of generality that $v_1 \in V_1$ and $v_2 \in V_1 \cup V_2$. Then, the path between $v_1$ and $v_2$ that is disjoint from $V_3$ contains only vertices of degree 2 in $\tilde{C}$. Let $v$ be one vertex in this path. Then, by Lemma 4.4, $v_1, v_2$ belong to $P(v)$. Moreover, since $v$ connects by an edge to $\tilde{C} \setminus P(v)$ (which itself is a connected graph because all vertices in $\tilde{C} \setminus C$ are connected by an edge to $V_3$), we conclude that $\tilde{C} \setminus \{v_1, v_2\}$ is also connected graph, which finishes the proof of the lemma. □
Finally, we complete the proof for $d = 3$ by combining Lemmas 2.4 and 2.5 (used in the same way as in the case $d = 2$) with the following lemma.

**Lemma 4.6.** A.a.s. Builder has a strategy to construct a 3-connected supergraph of $\hat{C}$ by accepting $o(n)$ additional edges and waiting for $(1 + o(1))n \log n/2$ more rounds.

**Proof.** Recall that, by Lemma 4.3, a.a.s. one can stochastically dominate the set $\{v \in V(C) : \deg_{\hat{C}}(v) = 2\}$ by a binomial random subset of $V(C)$ with probability $\omega^4/\log n$. We work in the binomial model. Then, by Lemma 4.3 and Markov’s inequality, there are at a.a.s. at most $\omega^5n/\log n$ vertices of degree 2 in $\hat{C}$.

Moreover, for every vertex $v$, $P(|P(v)| \geq \log n) \leq (\omega^4/\log n)\log n = o(n^{-1})$, so by a union bound a.a.s. $|P(v)| \leq \log n$ for every $v \in V(C)$ of degree 2 in $\hat{C}$.

Let us condition on all of the above a.a.s. events. Then, the statement of the lemma is an application of Lemma 2.4 allowing to construct edges between the vertices $v \in \{w_i \in V(C) : \deg_{\hat{C}}(w_i) = 2\}$ to the sets $S_i = V(\hat{C} \setminus P(v))$ and $v \in V(G \setminus \hat{C})$ to the sets $S_i = V(\hat{C})$.

The proof of Theorem 1.2 is completed.

Note added. After the completion of the current note, I was informed that the following asymptotic version of Conjecture 1.1 was also proven independently by the authors of [10]. The proof is to appear shortly in an updated version of [10].

**Theorem 4.7.** For every $\varepsilon > 0$ there is $d_0 = d_0(\varepsilon)$ such that for every $d \geq d_0$, the following holds: if $t \geq (1+\varepsilon)n \log n/2$ and $b \geq (1+\varepsilon)dn/2$, then there exists a $(t, b)$-strategy of Builder such that $G_t$ is a.a.s. $d$-connected.

Acknowledgements. The author is grateful to Ivailo Hartarsky and to the two anonymous referees for many useful comments and suggestions.

References

[1] L. Addario-Berry and J. Barrett. Multisource invasion percolation on the complete graph. *Ann. Probab.* 6 (2023), 2131–2157.
[2] M. Anastos. Constructing Hamilton cycles and perfect matchings efficiently. Preprint (2022). arXiv:2209.09860.
[3] T. Bohman. The triangle-free process. *Adv. Math.* 5 (2009), 1653–1677.
[4] T. Bohman and P. Keevash. Dynamic concentration of the triangle-free process. *Random Struct. Algorithms* 2 (2021), 221–293.
[5] T. Bohman and P. Keevash. The early evolution of the $H$-free process. *Invent. Math.* 2 (2010), 291–336.
[6] P. Erdős and A. Rényi. On random graphs I. *Publ. Math. Debr.* 6 (1959), 290–297.
[7] P. Erdős and A. Rényi. On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci.* 5 (1960), 17–60.
[8] P. Erdős and S. Suen and P. Winkler. On the size of a random maximal graph. *Random Struct. Algorithms* 6 (1995), 309–318.
[9] A. Frieze and M. Karoński. Introduction to random graphs. *Cambridge University Press*. Cambridge (2016).
[10] A. Frieze, M. Krivelevich, and P. Michaeli. Fast construction on a restricted budget. Preprint (2022). arXiv:2207.07251.
[11] S. Janson, T. Luczak, and A. Ruciński. Random graphs. *Wiley-Interscience*. New York (2000).
[12] K. Katsamaktsis and S. Letzter. Building graphs with high minimum degree on a budget. Preprint (2024). arXiv:2401.15812.
[13] M. Krivelevich, E. Lubetzky and B. Sudakov. Longest cycles in sparse random digraphs. *Random Structures & Algorithms* **43** (2013), 1–15.

[14] A. Logan, M. Molloy and P. Prałat. A variant of the Erdős-Rényi random graph process. *J. Graph Theory* **2** (2023), 322–345.

[15] T. Luczak. On the equivalence of two basic models of random graph. *Proceedings of Random graphs* **87** (1990), 151–159.

[16] G. Fiz Pontiveros, S. Griffiths and R. Morris. The triangle-free process and the Ramsey number $R(3, k)$. *Mem. Am. Math. Soc.* **1274** (2020).

[17] D. Osthus and A. Taraz. Random maximal $H$-free graphs. *Random Struct. Algorithms* **1** (2001), 61–82.

[18] L. Warnke. Dense subgraphs in the $H$-free process. *Disc. Math.* **23-24** (2011), 2703–2707.