ONE LOOP IN ELEVEN DIMENSIONS

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Abstract

Four-graviton scattering in eleven-dimensional supergravity is considered at one loop compactified on one, two and three-dimensional tori. The dependence on the toroidal geometry determines the known perturbative and non-perturbative terms in the corresponding processes in type II superstring theories in ten, nine and eight dimensions. The ultraviolet divergence must be regularized so that it has a precisely determined finite value that is consistent both with T-duality in nine dimensions and with eleven-dimensional supersymmetry.

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1. Introduction

The leading term in the M-theory effective action is the classical eleven-dimensional supergravity of [1]. Although terms of higher dimension must be strongly constrained by the large amount of supersymmetry they have not been systematically investigated. There is known to be an eleven-form, $\int C^{(3)} \wedge X_8$ (where $X_8$ is an eight-form made out of the curvature $R$ and $C^{(3)}$ is the three-form potential), which is necessary for consistency with anomaly cancellation [2,3]. Eleven-dimensional supersymmetry relates this to a particular $R^4$ term [4] as well as a host of other terms and might well determine the complete effective action. Furthermore, the effective action of the compactified theory depends nontrivially on moduli fields associated with the geometry of the compact manifold. This dependence is very strongly restricted by consistency with the duality symmetries of string theory in ten and lower dimensions. For example, the $R^4$ term in M-theory compactified on a two-torus must be consistent with the structure of perturbative and non-perturbative terms in nine-dimensional IIA and IIB superstrings [5]. This provides strong evidence that it has the form [4]

$$S_{R^4} = \frac{1}{\kappa_{11}^{2/9}} \int d^9 x \sqrt{G^{(9)}} h(\Omega, \bar{\Omega}; V_2) t_8 t_8 R^4,$$

(1)

where $G^{(9)}_{\mu\nu}$ is the M-theory metric in the space transverse to the torus, $\Omega = \Omega_1 + i \Omega_2$ is the complex structure of the torus and $V_2$ is its volume in eleven-dimensional Planck units. The parameter $\kappa_{11}$ has dimensions $[\text{length}]^{9/2}$ in arbitrary units (and the volume of the torus is given by $(\kappa_{11})^{4/9} V_2$ in these units). The notation $t_8 t_8 R^4$ (to be defined below) indicates the particular contraction of four Riemann tensors that arises from integration over fermionic zero modes at one loop in superstring theory [3] as well as from integration over fermionic modes on a D-instanton [5]. The function $h$ has to be invariant under the action of the modular group, $Sl(2,\mathbb{Z})$, acting on $\Omega$ and in [3,4] various arguments were given for why it should have the form

$$h(\Omega, \bar{\Omega}; V_2) = V_2^{-1/2} f(\Omega, \bar{\Omega}) + \frac{2\pi^2}{3} V_2.$$  

(2)

The function $f$ is a modular-invariant non-holomorphic Eisenstein series which is uniquely specified by the fact that it is an eigenfunction of the Laplace operator on the fundamental domain of $Sl(2,\mathbb{Z})$ with eigenvalue $3/4$,

$$\Omega_2^2 \partial_\Omega \partial_{\bar{\Omega}} f = \frac{3}{4} f$$

(3)

(i.e. $f = \zeta(3) E_{\frac{3}{4}}$, where $E_s$ is a Maass waveform of eigenvalue $s(s - 1)$ [7]). Significantly, this is the kind of Ward identity that the threshold corrections in lower-dimensional $N = 2$ theories satisfy [8] and it suggests a very stringent set of geometrical constraints.
The expansion of $f$ for large $\Omega^2$ has two power-behaved terms plus an infinite series of exponentially decreasing terms. These have exactly the correct coefficients to be identified with the tree-level and one-loop terms, together with an infinite series of D-instanton contributions in type IIB superstring theory. This identification makes use of T-duality to relate a multiply-charged D-instanton of type IIB with a multiply-wound world-line of a multiply-charged D-particle of type IIA. Indeed, semiclassical quantization around these D-instanton configurations may be carried out by functional integration around the D-particle background, as outlined in [4].

In this paper we will show how the sum of the perturbative and D-instanton contributions to the $t_ST^4$ term are efficiently encoded in M-theory in the expression for the scattering of four gravitons at one loop in eleven-dimensional supergravity perturbation theory. The particles circulating around the loop are the 256 physical states that comprise the massless eleven-dimensional supergraviton. It may seem surprising that perturbation theory is of any significance since supergravity has terrible ultra-violet divergences in eleven dimensions. Furthermore, the absence of any scalar fields means that there is no small dimensionless coupling constant. However, there is strong reason to believe that the one-loop $t_ST^4$ terms are protected from receiving higher-loop contributions by an eleven-dimensional nonrenormalization theorem since they are related by supersymmetry to the $C^{(3)} \wedge X^8$ term. Upon compactification to ten or fewer dimensions the Kaluza–Klein modes of the circulating fields are reinterpreted in terms of the windings of euclidean D-particle world-lines. The massive D-particles reproduce the D-instanton effects while the massless one (the massless ten-dimensional supergraviton) is equivalent to the perturbative one-loop string effects. The $t_ST^4$ term obtained at tree level in string theory arises, somewhat miraculously, from windings of the D-particle world-lines in the eleventh dimension.

The one-loop diagram can, in principle, be obtained using covariant Feynman rules by summing over the contributions of the component fields circulating in the loop — the graviton, gravitino and third-rank antisymmetric tensor fields. Alternatively, it can be expressed in terms of on-shell superfields. In that case the dynamics is defined by superspace quantum mechanics with the massless superparticle action which reduces in a fixed parameterization of the world-line to

$$S_{\text{particle}} = \frac{1}{2} \int d\tau G_{\mu\nu}(\dot{X}^\mu - i\bar{\Theta} \Gamma^\mu \dot{\Theta})(\dot{X}^\nu - i\bar{\Theta} \Gamma^\nu \dot{\Theta}),$$

(4)

where $\Theta$ is a 32 component $SO(10, 1)$ spinor, $\mu = 1, \ldots, 11$ and the reparameterization constraint requires the action density to vanish on physical states. For present purposes it will be sufficient to limit consideration to processes in which the external states do not carry momentum in the eleventh dimension and which are also not polarized in that direction although these are not essential conditions. This loop amplitude can be calculated by
making use of the light-cone description of the super-particle in which the vertex operator for a graviton has the form,

\[ V^{(r)}(\zeta^{(r)}, \tau^{(r)}) = \zeta^{(r)ik}(\dot{X}^i - \frac{1}{4p^+}\theta_{ijk}\theta k^{(r)}_i)(\dot{X}^k - \frac{1}{4p^+}\theta_{jkl}\theta k^{(r)}_l)e^{ik(r)\cdot X}, \]

where \( i, j, \cdots = 3, \cdots, 11, \zeta^{(r)}_{\mu\nu} \) is the graviton wave function with momentum \( k^{(r)}_\mu \) (where \( (k^{(r)})^2 = 0 = k^{(r)\mu}\zeta^{(r)}_{\mu\nu} \), \( \theta^a \) is a \( SO(9) \) spinor in the light-cone gauge defined by \( \Gamma^+\Theta = 0 \) and \( X^+ = p^+\tau \) (where \( V^\pm \equiv V^1 \pm V^2 \) with timelike \( V^1 \)). This vertex operator is attached at a point \( \tau^{(r)} \) on the world-line and is defined in a frame in which \( k^{(r)+} = 0, \zeta^{(r)+\mu} = 0 \). In a canonical treatment of this system the equations of motion determine that \( X^i = p^i\tau + x^i \) and \( \theta^a = S^a/\sqrt{p^+} \) and the (anti)commutation relations are \( [p^i, x^j] = -i\delta^{ij}, \{S^a, S^b\} = \delta^{ab} \) (just as with the zero mode components of the corresponding relations in the ten-dimensional type IIA superstring theory).

The loop amplitude reduced to \( (11 - n) \) dimensions by compactification on an \( n \)-dimensional torus, \( T^n \), has the form,

\[ A_4^{(n)} = \frac{1}{\pi^{3/2}V_n} \text{Tr} \int d^{11-n}p \int_0^\infty \frac{d\tau}{\tau} \left( \prod_{r=1}^4 \int_0^\tau d\tau^{(r)} V^{(r)} \right) \sum_{\{l_i\}} e^{-\pi(p^2 + G^{(n)IJ}l_1l_J)} \]

\[ = \frac{1}{\pi^{3/2}3V_n} \tilde{K} \int_0^\infty \frac{d\tau}{\tau^{n/2-13/2}} \sum_{\{l_i\}} e^{-\pi\tau G^{(n)IJ}l_1l_J} \int_0^\tau \prod_{r=1}^4 d\tau^{(r)} F(\{k^{(r)}, \tau^{(r)}\}), \]

where \( p = p^i \) is the \( (11 - n) \)-dimensional loop momentum transverse to the compact directions and \( G^{(n)}_{IJ} \) \( (I, J = 1, \cdots n) \) is the metric on \( T^n \) which has volume \( V_n = \sqrt{\text{det} G^{(n)}} \).

The kinematic factor, \( \tilde{K} \), in the second line involves eight powers of the external momenta and follows from the trace over the components of \( S^a \). It may also be written as

\[ \tilde{K} \sim \int d^{16}\eta \prod_{r=1}^4 \left( \zeta^{(r)}_{\mu_1, \nu_1} k^{(r)}_{\nu_1} \bar{\eta}\Gamma_{\mu_2, \nu_2, \rho_2} \eta\bar{\eta}\Gamma_{\omega_2, \rho_2, \tau_2} \right), \]

where \( \eta \) is a chiral \( SO(9,1) \) Grassmann spinor. The overall normalization will be chosen so that \( \tilde{K} \) is the linearized approximation to

\[ t_8 t_8 R^4 \equiv t^{\mu_1 \cdots \mu_8}t_{\nu_1 \cdots \nu_8} R^{\nu_1 \nu_2}_{\mu_1 \mu_2} \cdots R^{\nu_7 \nu_8}_{\mu_7 \mu_8} \]

where the tensor \( t^{\mu_1 \cdots \mu_8} \) was defined in [10]. The function \( F \) is a simple function of the external momenta. Since we are interested here in the leading term in the low-energy limit (the \( t_8 t_8 R^4 \) term) we can set the momenta \( k^{(r)} \) to zero in the integrand so that \( \int \prod d\tau^{(r)} F \) is replaced by \( \tau^4 \) giving

\[ A_4^{(n)} = \frac{\pi^{3/2}}{V_n} \tilde{K} \int_0^\infty \frac{d\tau}{\tau^{n/2-5/2}} \sum_{\{l_i\}} e^{-\pi\tau G^{(n)IJ}l_1l_J}. \]
Though this expression was obtained in a special frame we know that there is an \((11 - n)\)-dimensional covariant extension (including the case \(n = 0\)) that would follow directly from the covariant Feynman rules and should be easy to check by an explicitly calculation using the component form of the supergravity field theory action.

The expression for \(A_4^{(n)}\) will contribute to the \(t_8t_8R^4\) terms in the effective action for M-theory compactified on \(T^n\). In order to determine the dependence of the amplitude on the geometry of the torus on which it is compactified it will be important to express \(A_4^{(n)}\) in terms of the winding of the loop around \(T^n\). This could be obtained directly from the definition of the loop amplitude as a functional integral or by performing a Poisson summation on the \(n\) integers, \(l_i\), which amounts to inverting the metric in (9). The result is

\[
A_4^{(n)} = \pi^{3/2} \tilde{K} \int_0^\infty d\hat{\tau} \hat{\tau}^{1/2} \sum_{\{l_i\}} e^{-\pi \hat{\tau} l_i l_j},
\]

where \(\hat{\tau} = \tau^{-1}\).

The ultraviolet divergence of eleven-dimensional supergravity comes from the zero winding number term, \(\{\hat{l}_i\} = 0\), in the limit that the loop shrinks to a point (\(\hat{\tau} \to \infty\)). We will formally write this divergent term as the ill-defined expression, \(C \equiv \int d\hat{\tau} \hat{\tau}^{1/2}\).\(^4\)

The fact that the one-loop supergravity amplitude is infinite is a signal that point-particle dynamics alone cannot determine the short-distance physics of M-theory. A microscopic theory — such as Matrix theory — should determine the correct finite value of \(C\). This is somewhat analogous to the way in which divergent loop amplitudes in ten-dimensional super Yang–Mills are regularized by ten-dimensional string theory (for example, the \(F^4\) terms in the effective action of the heterotic and open string theories \([13,14]\)). Indeed, we will soon see that consistency with the duality symmetries of string theory together with the assumption that the eleven-dimensional theory can be obtained as a limit of the lower-dimensional theories determines the precise finite renormalized value for the constant, \(C\), that is also consistent with eleven-dimensional supersymmetry. It is a challenge to Matrix theory to reproduce this number.

In the following we will associate the integer \(\hat{l}_r\) with the winding number of the loop around a compact dimension of circumference \(R_{12-r}\) (for \(r \geq 1\)). If a single direction is compactified on a circle of circumference \(R_{11} \equiv V_1\), the loop can be expressed as a sum over the winding number of the euclidean supergraviton world-line.

\[
\frac{1}{\pi^{3/2}} A_4^{(1)} = C \tilde{K} + 2\tilde{K} \int_0^\infty d\hat{\tau} \hat{\tau}^{1/2} \sum_{l_1 > 0} e^{-\pi \hat{\tau} l_1^2} R_{11}^2 = C \tilde{K} + \tilde{K} \zeta(3) \frac{1}{\pi \tilde{R}_{11}^3}.
\]

\(^4\) The presence of a cubically divergent \(t_8t_8R^4\) term in eleven-dimensional supergravity was first suggested in \([1]\).
Rather strikingly, the finite $R_{11}$-dependent term gives a term in the effective ten-dimensional action that is precisely that obtained in [15,16] from the tree-level IIA string theory (here written in the M-theory frame). Although the regularized constant, $C$, is still undetermined, we will see later that it must be set equal to the coefficient of the one-loop $t_8t_8R^4$ term of the low energy effective action of string theory. The absence of any further perturbative or nonperturbative terms is in accord with the conjectures in [5,4].

Compactification on a torus ($n = 2$) gives a richer structure. In this case the one-loop amplitude has the form

$$\frac{1}{\pi^{3/2}} V_2 A_4^{(2)} = V_2 C \tilde{K} + V_2^{-1/2} \tilde{K} \sum_{(\hat{l}_1, \hat{l}_2) \neq (0,0)} \int d\hat{\tau} \hat{\tau}^{1/2} e^{-\pi \hat{\tau} / \Omega_2 |\hat{l}_1 + \hat{l}_2\Omega|^2}$$

$$= V_2 C \tilde{K} + \frac{1}{2\pi} V_2^{-1/2} \tilde{K} \sum_{(\hat{l}_1, \hat{l}_2) \neq (0,0)} \frac{\Omega_2^{3/2}}{|\hat{l}_1 + \hat{l}_2\Omega|^3}$$

$$= \frac{1}{2\pi} \tilde{K} \left( 2\pi CV_2 + V_2^{-1/2} f(\Omega, \Omega) \right),$$

where the divergent zero winding term, $\hat{l}_1 = \hat{l}_2 = 0$ has again been separated from the terms with non-zero winding. The function $f$ in this expression is precisely the (finite) $\Omega$-dependent term in (2). In particular, in the limit $V_2 \to 0$ M-theory should reduce to type IIB superstring theory in ten dimensions [17,18] with the complex scalar field, $\rho \equiv C^{(0)} + ie^{\phi_B}$, identified with $\Omega$ (where $C^{(0)}$ is the $R \otimes R$ scalar and $\phi_B$ is the IIB dilaton). More precisely, the correspondence between the parameters in M-theory and in IIB is,

$$V_2 \equiv R_{10}R_{11} = e^{\frac{i}{2} \phi_B} r_B^{-\frac{5}{4}}, \quad \Omega_2 \equiv \frac{R_{10}}{R_{11}} = e^{-\phi_B}$$

(13)

(where $r_B$ is the radius of the tenth dimension expressed in the IIB sigma-model frame). Using the fact that $\sqrt{G^{(9)}(V_2)^{-\frac{1}{2}}t_8t_8R^4} = \sqrt{g^{B(9)}R_B t_8t_8R^4}$ (where $g^{B(d)}$ denotes the determinant of the IIB sigma-model metric in $d$ dimensions) we see that (12) leads, in the ten-dimensional IIB limit ($r_B \to \infty$), to the expression suggested in [4]. This has the property that, when expanded in perturbation theory ($e^{-\phi_B} = \Omega_2 \to \infty$), it exactly reproduces both the tree-level and one-loop $t_8t_8R^4$ terms of the type IIB theory as well as an infinite series of D-instanton terms [4,5]. Importantly, the divergent term in (12) is proportional to $V_2$ and does not contribute in the limit of relevance to ten-dimensional type IIB – thus the eleven-dimensional one-loop calculation reproduces the complete, finite, $t_8t_8R^4$ effective action in the type IIB theory.

As before, the coefficient of the tree-level term in the type IIB superstring perturbation theory is reproduced by the configurations with $\hat{l}_2 = 0$, in which the particle in the loop winds around the eleventh dimension but not the tenth (obviously there is a symmetry under the interchange of these directions so we could equally well consider the terms...
with \( \hat{l}_1 = 0 \). In order to expand (12) systematically for large \( \Omega_2 \) it is necessary to undo the Poisson summation on \( \hat{l}_1 \) for the terms with \( \hat{l}_2 \neq 0 \). These terms are then expressed as a sum of multiply-wound D-particle world-lines where the winding number is \( \hat{l}_2 \) and the D-particle charge is the Kaluza–Klein charge, \( l_1 \). In the limit \( \nu_2 \rightarrow 0 \) the terms with \( l_1 = 0 \) reproduce the one-loop \( t_8 t_8 R^4 \) term of ten-dimensional type IIB while the \( l_1 \neq 0 \) terms give the contribution of the sum of D-instantons. The precise contribution due to the world-line of a particular wrapped massive D-particle (of mass \( l_1 \) and winding \( \hat{l}_2 \)) to this instanton sum is identical to that obtained by considering semiclassical quantization of four-graviton scattering in this background. Supersymmetry causes all quantum corrections to vanish. The additional fact that the one-loop string theory result is equivalent to the sum of windings of a massless D-particle (the supergraviton) is notable [4]. From the point of view of the string calculation this term arises from wrapping the string world-sheet in a degenerate manner around a circle.

We can now use the additional constraint of T-duality to pin down the precise value of \( C \). This is determined by recalling that the one-loop terms in both the IIA and IIB theories are invariant under inversion of the circumference, \( r_A \leftrightarrow r_B^{-1} \). This equates the coefficients of the \( \nu_2 \) term and the \( \Omega_2^{1/2} \nu_2^{-1/2} \) terms in (12), and the result is that the coefficient \( C \) must be set equal to the particular value,

\[
C = \frac{\pi}{3}.
\]  

The fact that the modular function in (2) is a Maass wave form satisfying (3) is easily deduced from the integral representation, (12). Developing a geometrical understanding of the origin of this equation would be of interest.

Upon compactifying on \( T^3 \) new issues arise. The full U-duality group is \( Sl(3, \mathbb{Z}) \times Sl(2, \mathbb{Z}) \). The seven moduli consist of the six moduli associated with the three-torus and \( C_{123}^{(3)} \), the component of the antisymmetric three-form on the torus. The latter couples to the euclidean three-volume of the M-theory two-brane which can wrap around \( T^3 \). The perturbative eleven-dimensional one-loop expression can be expected to reproduce the effects of the Kaluza–Klein modes but not of the wrapped Membrane world-volume. However, these wrapped Membrane effects will be determined in the following by imposing U-duality and making use of the one-loop results for type II string theory compactified on \( T^2 \) [4]. We will write the complete four-graviton amplitude as

\[
\nu_3 A_4^{(3)} = \pi^{3/2} \hat{K} H,
\]  

where the scalar function \( H \) depends on the seven moduli fields. There are several distinct classes of terms that will make separate contributions to the complete function \( H = \sum_i H_i \).

The effects of the Kaluza–Klein modes are obtained from (10) with \( n = 3 \). In order to compare with string theory on \( T^2 \) we will choose \( R_{11} \) to be the special M-theory direction
so that $R_{11} = e^{2\phi^A/3}$, where $\phi^A$ is the IIA dilaton (although the expression obviously has complete symmetry between all three compact directions). The sums over windings will be divided into various groups of terms. Firstly, there is the term with zero winding in all directions which is again divergent but will be set equal to the regularised value given by $C$ in (14), which implies

$$H_1 = \frac{\pi}{3} \mathcal{V}_3 \equiv \frac{\pi}{3} T_2,$$

where $T_2$ is the imaginary part of the Kähler structure of $T^2$. The sum over $\hat{l}_1 \neq 0$ with $\hat{l}_2 = \hat{l}_3 = 0$ once again leads to the correct tree-level string contribution proportional to $\zeta(3)$,

$$H_2 = \zeta(3) \frac{1}{\pi R_{11}^3} = \zeta(3) \frac{1}{\pi} e^{-2\phi^A}.$$

The remaining sum is over all values of $\hat{l}_1$, $\hat{l}_2$ and $\hat{l}_3$ excluding the $\hat{l}_2 = \hat{l}_3 = 0$ terms. This is usefully reexpressed by converting the $\hat{l}_1$ sum to a sum over Kaluza–Klein modes by a Poisson resummation. The sum of these terms is

$$H_3 + H_4 = \sqrt{\frac{\operatorname{det}G}{G_{11}}} \sum_{(\hat{l}_2, \hat{l}_3) \neq (0,0)} \sum_{\hat{l}_1} \int_0^\infty d\hat{\tau} \exp \left[ 2\pi i \hat{l}_1 \hat{l}_2 \frac{G_{12}}{G_{11}} + 2\pi i \hat{l}_1 \hat{l}_3 \frac{G_{13}}{G_{11}} \right. + \left. \pi \hat{l}_1 \frac{1}{\hat{\tau} G_{11}} - \pi \hat{\tau} \left( \hat{l}_2^2 (G_{22} - \frac{G_{12}^2}{G_{11}}) + \hat{l}_3^2 (G_{33} - \frac{G_{13}^2}{G_{11}}) + 2 (G_{23} - \frac{G_{12} G_{13}}{G_{11}}) \hat{l}_2 \hat{l}_3 \right) \right]
$$

$$= \sqrt{\frac{\operatorname{det}G}{G_{11}}} \sum_{(\hat{l}_2, \hat{l}_3) \neq (0,0)} \sum_{\hat{l}_1} \int_0^\infty d\hat{\tau} \exp \left[ -\pi \hat{l}_1^2 e^{-2\phi^A} \frac{1}{\hat{\tau}} + 2\pi i \hat{l}_1 A^{(i)} - \pi \hat{\tau} \hat{l}_2 \hat{l}_3 g^{A}_{ij} \right].$$

In this expression $G_{ij}$ is the metric on $T^3$ in M-theory coordinates with the convention that $i = 12 - \mu$ ($i = 1, 2, 3$) and the components of the IIA string sigma-model metric on the two-torus are given by

$$g^{A}_{ij} = R_{11} \left( G_{ij} - \frac{G_{1i} G_{1j}}{G_{11}} \right),$$

where $i, j = 2, 3$. The components of the $R \otimes R$ one-form potentials in the directions of the two-torus in (18) are defined by

$$A^{(i)} = \frac{G_{1i}}{G_{11}}.$$ 

The expression (18) depends on the $R \otimes R$ one-form, the complex structure of the two-torus,

$$U = \frac{1}{g_{22}^A} (g_{23}^A + i \sqrt{\operatorname{det}g^A})$$

(21)
and the combination $T_2 e^{-2\phi^A}$. But it does not depend separately on the Kähler structure,

$$T = B_{12} + i\sqrt{\det g^A} = C^{(3)}_{123} + i\mathcal{V}_3,$$

(22)

where $\mathcal{V}_3 = R_9 R_{10} R_{11}$. In the last step we have used the usual identification of the $NS \otimes NS$ two-form with the M-theory three-form, $C^{(3)}$, and the fact that $r_2^A r_3^A = R_9 R_{10} R_{11}$, where $r_i^A$ is the circumference of the dimension labelled $i$ in the IIA sigma-model frame.

The expression (18) contains perturbative and non-perturbative contributions to the $t_8 t_8 R^4$ term in the IIA effective action. The perturbative term is obtained by setting $l_1 = 0$. The resulting double sum over $\hat{l}_2$ and $\hat{l}_3$ is logarithmically divergent, just as in the analogous problem considered in [8]. This is a reflection of the fact that the one-loop diagram in eight-dimensional supergravity is logarithmically divergent. As in [8,4], this divergence may be regularized in a unique manner that is consistent with modular invariance by adding a term, $\Sigma = \ln(T_2 U_2^2 / \Lambda^2)$, giving,

$$H_3 = \sum_{(\hat{l}_2, \hat{l}_3) \neq (0,0)} \frac{U_2}{|\hat{l}_2 + \hat{l}_3 U|^2} - \ln(U_2 T_2 / \Lambda^2)$$

(23)

$$= - \left[ \ln(U_2 |\eta(U)|^4) + \ln(T_2) \right],$$

where $\Lambda^2$ is adjusted to cancel the divergence coming from the sum. So we see that the piece of the perturbative string theory one-loop amplitude that depends on $U$ is reproduced by configurations in which a massless particle propagating in the loop has a world-line that winds around the torus. This is the generalization of the way in which the IIB one-loop term was reproduced earlier by windings of a massless D-particle around a circle.

The terms with $l_1 \neq 0$ in (18) consist of a sum of non-perturbative D-instanton contributions,

$$H_4 = 2 U_2 \rho_2^A \sum_{(\hat{l}_2, \hat{l}_3) \neq (0,0)} \frac{|l_1|}{|\hat{l}_2 + \hat{l}_3 U|} K_1 \left( 2 \pi \rho_2^A |\hat{l}_2 + \hat{l}_3 U||l_1| \right) e^{2i\pi l_1 (\hat{l}_2 A^{(1)} + \hat{l}_3 A^{(2)})},$$

(24)

where $\rho_2^A = r_2^A e^{-\phi^A}$. Using the fact that $K_1(z) = \sqrt{\frac{\pi}{2z}} e^{-z} (1 + o(1/z))$ for large $z$ we see that at weak IIA coupling, $e^{-\phi^A} \to \infty$, these terms are exponentially suppressed. The contribution of these instanton terms in the nine-dimensional case described earlier is obtained by letting $r_3^A \to \infty$. In this case $U \to i\infty$ and only the $\hat{l}_3 = 0$ term in (24) survives. The double sum over $l_1$ and $\hat{l}_2$ becomes the nine-dimensional D-instanton sum contained in (12) which was explicitly given in [4].

So far we have ignored the contributions to the $t_8 t_8 R^4$ term arising from configurations in which the world-volume of the M-theory Membrane is wrapped around $T^3$. Such contributions are obviously not contained in the one-loop $D = 11$ supergravity amplitude.
As with the contributions that came from circulating D-particles, the configurations that contribute to the $t_8 t_8 R^4$ term are described by the multiple windings of world-lines of nine-dimensional BPS states in ultra-short (256-dimensional) multiplets. Recall that these nine-dimensional states are winding states of fundamental strings with no momentum or oscillator excitations which are configurations of the wrapped M-theory Membrane with no Kaluza–Klein excitations. Such contributions are therefore labelled by two integers and depend only on the volume of the three-torus, $\det G$, and on $C^{(3)}$ but are independent of the other five components of the metric (i.e., they depend only on $T$ and $\bar{T}$). These configurations of the IIA string world-sheet are just those that enter the functional integral for the $t_8 t_8 R^4$ term at one loop in string perturbation theory. Indeed, as explained in [4] (and in an analogous problem in [13,14]), the piece of the one-loop string amplitude that depends on $T$ and $\bar{T}$ is given by a sum over non-degenerate wrapped world-sheets and contributes,

$$H_5 = 2 \sum_{m,n>0} \frac{1}{n} \left( e^{2\pi i mn T} + e^{-2\pi i mn \bar{T}} \right) = - \left[ \ln(|\eta(T)|^4) + \frac{\pi}{3} T^2 \right],$$

where $m, n$ are the integers that label the windings of the world-sheet. The sum of $H_1$, $H_3$ and $H_5$ reproduces the full one-loop perturbative string theory result. Applying T-duality in one of the toroidal directions transforms this into the one-loop term of the IIB theory. The complete non-perturbative structure of the ten-dimensional $t_8 t_8 R^4$ terms of the IIB theory can then be recovered using the series of dualities described in [4].

The total contribution to the $t_8 t_8 R^4$ terms in the eight-dimensional effective action is given (in IIA string coordinates) by

$$S_{R^4} \sim \int d^8 x \sqrt{g^{A(8)}} r_2^A r_3^A H t_8 t_8 R^4,$$

where $H = \sum_{i=1}^5 H_i$ and we have ignored an overall constant. This expression is invariant under the requisite $SL(3, \mathbb{Z}) \otimes SL(2, \mathbb{Z})$ U-duality symmetry. The particle winding numbers $(\hat{l}_1, \hat{l}_2, \hat{l}_3)$ transform as a 3 of $SL(3, \mathbb{Z})$ while the windings of the Membrane $(m, n)$ transform as a 2 of $SL(2, \mathbb{Z})$. The decoupling of the two factors in the U-duality group arises from the fact that the ultra-short BPS states in nine dimensions do not contain both a wrapped Membrane and Kaluza–Klein charges. Compactification on $T^4$ to seven dimensions is more complicated since the U-duality group is $SL(5, \mathbb{Z})$, which is not a product of two factors. The $t_8 t_8 R^4$ terms in this case depends on the BPS spectrum in eight dimensions, which was discussed in [19].

In this paper we have studied properties of the one-loop amplitude in eleven-dimensional supergravity compactified on tori to lower dimensions. Upon compactifying
to nine dimensions on $T^2$ this amplitude reproduces the complete perturbative and non-perturbative $t_8 t_8 R^4$ terms in the effective actions for the corresponding string theories if the ultra-violet divergence is chosen to have a particular finite regularized value (a value that can presumably be derived from Matrix Theory). This value is also in agreement with that obtained by supersymmetry which relates it to the $C^{(3)} \wedge X_8$ term [4]. In the limit in which the two-torus has zero volume, $V_2 \to 0$, the regularized term does not contribute and the complete $t_8 t_8 R^4$ term of the ten-dimensional IIB theory is reproduced precisely by the one-loop supergravity calculation. It is noteworthy that the $t_8 t_8 R^4$ terms in the IIB theory only get string-theory perturbative contributions at tree-level and one loop, in addition to the non-perturbative D-instanton contributions. This is tantalizingly similar to the structure of the $F^2$ terms in $N = 2$ super Yang–Mills theory in $D = 4$ dimensions.

Upon compactification to eight dimensions on $T^3$, the one-loop eleven-dimensional supergravity amplitude reproduces the $Sl(3, \mathbb{Z})$-symmetric piece of the $t_8 t_8 R^4$ term that is associated with Kaluza–Klein instantons. The remaining piece that arises from the wrapped Membrane is uniquely determined by consistency with the T-duality that relates the IIA and IIB theories, together with one-loop string perturbation theory. We have not addressed the new issues that arise in compactification on manifolds of non-trivial holonomy or compactification to lower dimensions. For example, compactification on $T^6$ requires considerations of the wrapped world-volume of the M-theory five-brane.

In addition to the $R^4$ terms considered here there are many other terms of the same dimension involving the other fields of ten-dimensional string theory and M-theory. In the language of type IIB supergravity some of these terms conserve the R-symmetry charge (as with the $R^4$ term) while some of them violate it in a manner consistent with the instanton effects (such as the $\lambda^{16}$ term described in [3]).

Since there is no scalar field there is no possibility of a well-defined perturbation expansion in powers of a small coupling constant in eleven-dimensional supergravity. Fortuitously, the relation of the $t_8 t_8 R^4$ term to the eleven-form, $C^{(3)} \wedge X_8$, via supersymmetry, implies that the one-loop expression is exact with no corrections from higher-loop diagrams (since the normalization of the eleven-form is fixed by anomaly cancellation). This adds to the ever-increasing body of evidence that the constraints of maximal supergravity are profoundly restrictive.

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