Generalized probability and current densities: A field theory approach

M. Izadparast* and S. Habib Mazharimousavi†

Department of Physics, Faculty of Arts and Sciences,
Eastern Mediterranean University, Famagusta,
North Cyprus via Mersin 10, Turkey
(Dated: March 7, 2022)

Abstract

We introduce a generalized Lagrangian density - involving a non-Hermitian kinetic term - for a quantum particle with the generalized momentum operator. Upon variation of the Lagrangian, we obtain the corresponding Schrödinger equation. The extended probability and particle current densities are found which satisfy the continuity equation.

*Electronic address: masoumeh.izadparast@emu.edu.tr
†Electronic address: habib.mazhari@emu.edu.tr
I. INTRODUCTION

The idea of the generalized momentum operator has been extended in our earlier proposal [1, 2], which is subjected to the non-relativistic and non-Hermitian quantum mechanics. The thought of the non-Hermitian Hamiltonian was discussed initially in Ref. [3, 4]. Yet, Bender and Boettcher have initiated $\mathcal{PT}$-symmetric quantum physics by employing a harmonic-oscillator-like $\mathcal{PT}$-symmetric potential as a toy model [5]. They have shown that even if the Hamiltonian is non-Hermitian the eigenvalues can be real, upon which, a $\mathcal{PT}$-symmetric operator is defined. The $\mathcal{PT}$ operator is composed of the parity and the time reversal operators, namely i.e., $\mathcal{P}x\mathcal{P} = -x$ and $\mathcal{T}i\mathcal{T} = -i$ [6]. For a $\mathcal{PT}$-symmetric Hamiltonian - a less general non-Hermitian Hamiltonian - the interaction potential is considered to be complex. Hence, the unitarity for the unbroken $\mathcal{PT}$-symmetry (under which the energy spectrum of the system is real) in a quantum mechanical system has to be conserved, which is one of the prominent principles of quantum mechanics. Accordingly, in Ref. [7] the infinitesimal probability density has been considered and a method introduced to find the path integral in the complex plane $\mathbb{C}$. For further investigations on the presented outcomes in Ref. [5], the approach of field theory, also, is presented in the context of $\mathcal{PT}$-symmetric quantum physics. Bender et. al. in the field approach have discussed the idea of broken symmetry of a separated parity and time reversal operators, although the field is $\mathcal{PT}$-symmetric, in Ref. [8]. Over the latter study, the idea has been expanded in a two-dimensional supersymmetric quantum field theory according to a defined potential $-ig\,(i\phi)^{1+\delta}$ for $\delta > 0$. Besides, in Ref. [9], the authors have applied the technique of truncating the Schwinger-Dyson equations in a set of fields with the non-Hermitian Hamiltonian families, $\frac{g}{N}(i\phi)^N$, provided that $N \geq 2$. In the latter paper, the corresponding solution and renormalization have been obtained and the properties of the scalar quantum field $-g\phi^4$ in four-dimensional space-time discussed. Later on, in Ref. [10], Bagchi et. al. have argued meticulously the $\mathcal{PT}$-symmetric field theory upon the variation of Lagrangian to find the generalized continuity equation, using the Schrödinger equation and its $\mathcal{PT}$-symmetric conjugate. Furthermore, the modified normalization constant has been obtained on the real $x$-axis. A significant study has been carried out in Ref. [11] in which presents a perturbative method to find the $\mathcal{C}$ operator in quantum mechanics, included systems with higher degrees of freedom, and particularly in quantum field theory. The same authors in a short letter in Ref. [12] have presented the successful
physical confirmation of $\mathcal{PT}$-symmetric quantum field theory corresponding to the field $i\phi^3$, and redefined the inner product in Hilbert space using the perturbation method to build $\mathcal{C}$ operator. The Hamiltonian of a free fermionic field theory in Ref. [13] has been investigated and shown that its $\mathcal{PT}$ invariance depends on the corresponding mass term. According to a study in Ref. [13], it has been confirmed that the $\mathcal{PT}$-symmetric massive Thirring and the scalar sine-Gordon models are dual to each other and equivalent to their Hermitian version. Regarding the importance of field $\phi^3$ in the context of $\mathcal{PT}$-symmetric quantum field theory, in Ref. [14] the authors have compared the renormalization-group properties of Hermitian field $g\phi^3$ and $\mathcal{PT}$-symmetric $ig\phi^3$ field theories. However, in Ref. [15], the critical behavior of $\mathcal{PT}$-symmetric $i\phi^3$ quantum field theory has been studied in $6-\epsilon$ dimensions around the exceptional points. Furthermore, the calculation on the critical exponent has been carried out employing the mean-field approximation and the renormalization-group technique.

In Ref. [16], having considered $\mathcal{PT}$-symmetric quantum theory, the logarithmic time-like Liouville Lagrangian has been discussed. Accordingly, the authors have found the energy of a quantum mechanical system assuming the semiclassical limit. Recently, Alexandre et. al. in a remarkable study in Ref. [17] have argued Noether theorem considering complex scalar and fermionic $\mathcal{PT}$-symmetric field theories for Hermitian and Anti-Hermitian mass using the variation of Lagrangian. Very recently, Mazharimousavi rigorously, in Ref. [18] through the approach of classical field theory, has declared that one field is sufficient to find a nonlinear or generalized Schrödinger equation in the context of the position dependent mass (PDM) or deformed momentum operator. Besides, in Ref. [18], it has been indicated that the generated field equations lead to the standard Schrödinger equation. Whereas, the corresponding probability density remains real and positive in both Hermitian and non-Hermitian approaches.

Here in this paper, we introduce a generalized Lagrangian density extending the latter approach, in Ref. [18] through the conventional Hermitian or $\mathcal{PT}$-symmetric field theory using the generalized momentum operator [1, 2]. We study the physical aspects of the Lagrangian density including the corresponding Schrödinger equations and the continuity equation.

The present paper is organized as follows. In Sec. II we introduce the generalized Lagrangian density which corresponds to the extended definition of the generalized momentum operator in the real domain. Besides that, by applying the principle of stationary action
through the Euler-Lagrange equation, we obtain the generalized Hamiltonian and the corresponding Schrödinger equation. Then, the continuity equation is found under the concept of the generalized probability and current densities. In Sec. III and Sec. IV we discuss the Lagrangian densities in the complex domain once for the conventional momentum operator and, again, for the generalized momentum operator, respectively. Finally, we summarize our paper in the Conclusion.

II. LAGRANGIAN DENSITY IN REAL DOMAINE: HERMITIAN SCHÖDINGER EQUATION

We recall the generalized momentum operator presented in Ref. [1, 2] or in the general form in Eq. (10) in Ref. [18], i.e

\[ \hat{p} = -i\hbar \left( A \partial_x + \frac{A'}{2} \right), \] (1)

where the auxiliary function \( A(x) \) is a real function of \( x \). Then, we introduce the generalized Lagrangian density

\[ \mathcal{L} = i\hbar \dot{\Psi} \Psi^* - \frac{\hbar^2}{2m} A^2 \psi^* \psi' - \left[ -\frac{\hbar^2}{4m} A A'' - \frac{\hbar^2}{8m} A'^2 + V(x) \right] \Psi \Psi^*, \] (2)

which a dot and prime stand for the derivative with respect to \( t \) and \( x \), respectively, and \( * \) implies the complex conjugate. By Applying the variation of the action with respect to \( \Psi^* (x,t) \) and \( \Psi (x,t) \) and choosing the interaction potential \( V(x) \) to be real the field equations are expressed as

\[ i\hbar \dot{\Psi} = -\frac{\hbar^2}{2m} \left( (A^2 \psi')' + \frac{A'' A}{2} \psi + \frac{A'^2}{4} \psi \right) + V(x) \Psi \] (3)

and

\[ -i\hbar \dot{\Psi}^* = -\frac{\hbar^2}{2m} \left( (A^2 \psi^*)' + \frac{A'' A}{2} \psi^* + \frac{A'^2}{4} \psi^* \right) + V(x) \Psi^*, \] (4)

respectively. Eqs. (3) and (4) give the generalized Schrödinger equation and its complex conjugate. To find the stationary form of latter equations, we use the field in terms of separable time and spatial functions given by

\[ \Psi (x,t) = \psi (x) \exp \left( -\frac{i E t}{\hbar} \right), \] (5)
in which plugging into the field equation (3) leads to

\[ E\psi = -\frac{\hbar^2}{2m} \left( (A^2\psi')' + \frac{A''A}{2} \psi + \frac{A'^2}{4} \psi \right) + V(x)\psi. \] (6)

Whilst, the Hamiltonian operator \( \hat{H} \) in Ref. [1, 2] is defined to be

\[ \hat{H} = \hat{H}^\dagger = -\frac{\hbar^2}{2m} \left( A^2\partial_x^2 + 2AA'\partial_x + \frac{A''A}{2} + \frac{A'^2}{4} \right) + V(x), \] (7)

which is accordingly admits Eq. (10) in Ref. [18]. Next, we use the Lagrangian density (2) and the definition of Hamiltonian density

\[ \mathcal{H} = \sum \pi_\sigma f_\sigma - \mathcal{L}, \] (8)

where \( f_\sigma \in \{ \Psi, \Psi^* \} \) and \( \pi_\sigma \) is the momentum-density conjugate to \( f_\sigma \), and calculate the explicit form of \( \mathcal{H} \). To do so, let’s calculate \( \pi_\sigma \) as

\[ \pi_\Psi = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = i\hbar \Psi^*, \quad \pi_{\Psi^*} = \frac{\partial \mathcal{L}}{\partial \dot{\Psi^*}} = 0. \] (9)

After the substitution into Eq. (8), one-dimensional Hamiltonian density is obtained to be

\[ \mathcal{H} = \frac{\hbar^2}{2m} A^2\Psi^*\Psi' - \frac{\hbar^2}{4m} AA'\Psi^*\Psi - \frac{\hbar^2}{8m} A'^2\Psi^*\Psi + V(x)\Psi^*\Psi. \] (10)

Having Hamiltonian density, we apply \( E = \int \mathcal{H} dx \) to find the energy of the system. Explicitly, one finds

\[ E = \int \left[ -\frac{\hbar^2}{2m} \Psi^* \left( A^2\partial_x^2 + 2AA'\partial_x + \frac{A''A}{2} + \frac{A'^2}{4} \right) \Psi + \Psi^*V(x)\Psi \right] dx \] (11)

According to Eq. (7) and Eq. (11) provided that \( \Psi(x,t) = \Psi^*(x,t) \), considering that \( \hat{H} \) is Hermitian the two terms become identical and the energy reduces to

\[ E = \int \Psi \hat{H} \Psi dx = \int \Psi^* \hat{H} \Psi dx = \langle \hat{H} \rangle, \] (12)

in which \( \langle \hat{H} \rangle \) is the expectation value of \( \hat{H} \). Next, we utilize (3) and (4) to find the continuity equation. Let’s multiply by \( \Psi \) and \( \Psi^* \) from the left the equations (4) and (3), respectively. Then by subtraction of the two equations, we obtain

\[ i\hbar \partial_t (\Psi\Psi^*) = -\frac{\hbar^2}{2m} \partial_x \left( A^2 (\Psi\Psi'^* - \Psi^*\Psi') \right). \] (13)
This is the continuity equation provided we define

$$\rho = \Psi\Psi^*,$$  \hspace{1cm} (14)

to be the probability density and

$$J_x = \frac{\hbar}{2im} (A^2 (\Psi^*\Psi' - \Psi\Psi'^*) - \Psi\Psi^*'),$$  \hspace{1cm} (15)

to be the particle current density. Hence the continuity equation

$$\partial_t \rho (x,t) + \partial_x j (x,t) = 0$$  \hspace{1cm} (16)

holds. We would like to comment that the present outcomes in (14) and (15) are significant since the conservation of probability density is confirmed.

III. LAGRANGIAN DENSITY IN COMPLEX DOMAINE : NON-HERMITIAN SCHRÖDINGER EQUATION

We expand our discussion into the complex domain such that our proposed momentum and the potential are complex, $V(x), A(x) \in \mathbb{C}$, provided that both functions be $\mathcal{PT}$-symmetric. Thus, the Lagrangian density is given by, applying (1),

$$\mathcal{L} = i\hbar \dot{\Psi}\Psi^* - \frac{\hbar^2}{2m} A^2 \Psi^*\Psi' - \left[ V(x) - \frac{\hbar^2}{4m} AA'' - \frac{\hbar^2}{8m} A^2 \right] \Psi \Psi^*,$$  \hspace{1cm} (17)

where $\Psi^* = \mathcal{PT}\Psi$. Accordingly, we obtain the Schrödinger equations using the Euler-Lagrange equations with respect to $\psi^*$ and $\psi$ given by

$$i\hbar \dot{\psi} = -\frac{\hbar^2}{2m} (A^2 \psi')' + \left[ V(x) - \frac{\hbar^2}{4m} AA'' - \frac{\hbar^2}{8m} A^2 \right] \psi,$$  \hspace{1cm} (18)

and

$$-i\hbar \dot{\psi}^* = -\frac{\hbar^2}{2m} (A^2 \psi^*)' + \left[ V(x) - \frac{\hbar^2}{4m} AA'' - \frac{\hbar^2}{8m} A^2 \right] \psi^*,$$  \hspace{1cm} (19)

respectively. Moreover, the corresponding Hamiltonian is $\mathcal{PT}$-symmetric which is admitted in Eq. (7). Following up the earlier discussion upon the conservation of probability density, one can approve the similar method and represent the generalized continuity equation in

$$\partial_t (\Psi\Psi^*) + \frac{\hbar}{2im} \partial_x (A^2 (\Psi^*\Psi' - \Psi\Psi'^*)) = 0.$$  \hspace{1cm} (20)
According to the generalized continuity equation, the probability density is demonstrated by
\[ \rho = \Psi \Psi^\# \] (21)
and the generalized current density admitted in
\[ J_x = \frac{\hbar}{2im} \left( A^2 (\Psi^\# \Psi' - \Psi \Psi'^\#) \right) . \] (22)

We note that with \( \Psi (x,t) = \Psi^\#(x,t) \), the particle current density vanishes and consequently,
\[ \frac{d}{dt} \int dx \rho = 0, \] (23)
which is the conservation of the total probability on \( x \in \mathbb{R} \) [10]. Having been obtained the Hamiltonian density in accordance with the recent Lagrangian density in
\[ \mathcal{H} = \frac{\hbar^2}{2m} A^2 \Psi^\# \Psi' - \frac{\hbar^2}{4m} A A'' \Psi^\# \Psi - \frac{\hbar^2}{8m} A^2 \Phi^\# \Phi + V(x) \Phi^\# \Phi, \] (24)
where the energy is presented by \( E = \int \mathcal{H} dx \) related to the \( \mathcal{P} \mathcal{T} \)-symmetric field theory found to be
\[ E = \int \Psi^\# \left[ -\frac{\hbar^2}{2m} \left( A^2 \partial_x^2 + 2A' A \partial_x + \frac{AA''}{2} + \frac{A^2}{4} \right) + V(x) \right] \Psi dx. \] (25)

With the situation that \( \Psi (x,t) = \Psi^\#(x,t) \), the energy expectation value expressed by
\[ E = \int \Psi \hat{H} \Psi dx = \int \Psi^\# \hat{H} \Psi dx = \langle \hat{H} \rangle . \] (26)

Now, we suppose that there exist a second field, according to Ref. [18], defined as \( \Phi(x,t) = \frac{\Phi(x,t)}{A} \) and input into Eq. (17) in which \( \Phi(x,t) \) is a \( \mathcal{P} \mathcal{T} \)-symmetric field, then we find the Lagrangian density
\[ \mathcal{L} = i\hbar \frac{\Phi \Phi^\#}{A} - \frac{\hbar^2}{2m} \left[ A \Phi'^\# \Phi' - \frac{1}{2} A' \left( \Phi \Phi'' \right)' \right] - \left[ V(x) - \frac{\hbar^2}{4m} A A'' \right] \frac{\Phi \Phi^\#}{A}. \] (27)
The time-independent Schrödinger equation using the Euler-Lagrange equations is found
\[ i\hbar \dot{\Phi} = -\frac{\hbar^2}{2m} A (A \Phi')' + V(x) \Phi \] (28)
and
\[ -i\hbar \dot{\Phi}^\# = -\frac{\hbar^2}{2m} A (A \Psi'^\#)' + V(x) \Phi^\#, \] (29)
respectively. We multiply Eq. (28) by $\Phi^\#$ and Eq. (29) by $\Phi$ then subtract them to find the generalized continuity equation, and accordingly the probability density

$$\rho = \frac{\Phi\Phi^\#}{A},$$

(30)

and the generalized current density

$$J_x = \frac{\hbar A}{2im} \left( \Phi^\# \Phi' - \Phi \Phi'^\# \right).$$

(31)

Hence, for the non-Hermitian field which is $\mathcal{PT}$-symmetric the probability density is conserved.

IV. CONCLUSION

We have introduced a Lagrangian density in field theory for a quantum particle with a generalized momentum operator [1, 2]. By virtue of the variation of the Lagrangian density, we obtained the Schrödinger equation which describes the behavior of such quantum particles in Hermitian and non-Hermitian systems. Furthermore, we calculated the Hamiltonian density and showed that the energy of the system is the expectation value of the Hamiltonian operator. Besides, the obtained outcomes are significant due to the demonstration of the similar pattern according to the real and complex domains in which the continuity equation is approved for Hermitian and non-Hermitian Lagrangian density. Regarding the non-Hermitian case, the continuity equation is confirmed and shown that it has no contradiction defining one field according to Ref. [18] for a complex non-Hermitian system.

[1] M. Izadparast and S. H. Mazharimousavi, Phys. Scr. 95, 075220 (2020). I, II, II, IV
[2] M. Izadparast and S. H. Mazharimousavi, Phys. Scr. 95, 105216 (2020). I, II, II, IV
[3] Bessis, unpublished (1992). I
[4] E. Caliceti, S. Graffi and M. Maioli, Comm. Math. Phys. 75, 51 (1980);
    G. Alvarez, J. Phys. A: Math. Gen. 27, 4589 (1995). I
[5] C. M. Bender, S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998). I
[6] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007). I
[7] C. M. Bender, D. W. Hook, P. N. Meisinger and Q. H. Wang, Phys. Rev. Lett. 104, 061601 (2010).

[8] C. M. Bender and K. A. Milton, Phys. Rev. D 57, 3595 (1998).

[9] C. M. Bender, K. A. Milton, V. M. Savage, Phys. Rev. D 62, 085001 (2000).

[10] B. Bagchi, C. Quesne, M. Znojil, Mod. Phys. Lett. A 16, 2047 (2001).

[11] C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. D 70, 25001 (2004).

[12] C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. Lett. 93, 251601 (2004).

[13] C. M. Bender, H. F. Jones and R. J. Rivers, Phys. Lett. B 625, 333 (2005).

[14] C. M. Bender, V. Branchina and E. Messina, Phys. Rev. D 85, 085001 (2012).

[15] C. M. Bender, V. Branchina and E. Messina, Phys. Rev. D 87, 085029 (2013).

[16] C. M. Bender, D. W. Hook, N. E. Mavromatos and S. Sarkar, Phys. Rev. Lett. 113, 231605 (2014).

[17] J. Alexandre, P. Millington and D. Seynaeve Phys. Rev. D 96, 065027 (2017); E. Noether, Invariante Variationsprobleme, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse 1918, 235 (1918).

[18] S. H. Mazharimousavi, Eur. Phys. J. Plus 136, 807 (2021).

[19] G. Lévai and M. Znojil, Phys. A: Math. Gen. 33, 7165 (2000).

[20] G. Lévai and M. Znojil, M. Phys. Lett. A, 16, 1973 (2001).

[21] E. J. Weniger, Comput. Phys. 10, 496 (1996).