A remark on constrained von Kármán theories

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Abstract

We derive the Euler-Lagrange equation corresponding to a variant of ‘non-Euclidean’ constrained von Kármán theories.

1 Introduction

Föppl-von-Kármán theories arise as asymptotic theories modelling the behaviour of thin elastic films, in an energy regime allowing only for very small deformations. The elastic energy of such deformations (with respect to the thickness of the film) is therefore much lower than that of generic nonlinear bending deformations. The asymptotic behaviour of the latter is modelled by the fully nonlinear Kirchhoff plate theory. We refer to [3, 4] for a derivation and thorough discussion of these theories, cf. also [1].

More precisely, the asymptotic behaviour of thin film deformations whose elastic energy lies in a regime just below the nonlinear bending regime is captured by so-called constrained von Kármán theories, cf. [4]. Their behaviour is essentially fully described by their out-of-plane displacement $v : S \rightarrow \mathbb{R}$, where $S \subset \mathbb{R}^2$ is the reference configuration of the sample. The asymptotic elastic energy of such a displacement $v$ is then given by

$$\frac{1}{24} \int_S Q_2 (\nabla^2 v(x)) \ dx + \int_S f \cdot v \ dx,$$

subject to the constraint

$$\det(\nabla^2 v) = 0.$$ 

Here $Q_2$ is the quadratic form of linearised elasticity and $f$ models applied forces.

Motivated by applications in non-Euclidean (or pre-strained) elasticity (cf. e.g. [2] and [13]), we consider variants of functionals as in (1) by allowing a nonzero right-hand side in (2). For simplicity, we restrict to the isotropic case when $Q_2 = | \cdot |^2$ and we do not consider forces. More general situations can be handled in the same way, as our main focus is on the constraint

$$\det \nabla^2 v = k$$

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itself. The problem is therefore to understand, on a bounded domain $S \subset \mathbb{R}^2$, and for given $k : S \rightarrow \mathbb{R}$, the functional

$$W_k(v) = \begin{cases} \int_S |\nabla^2 v(x)|^2 \, dx & \text{if } v \in W^{2,2}_k(S) \\ +\infty & \text{otherwise.} \end{cases}$$

Here

$$W^{2,2}_k = \{ v \in W^{2,2}(S) : \det \nabla^2 v = k \text{ pointwise almost everywhere } \}.$$

We use a similar notation for other function spaces, such as $C^{2,\alpha}_k(S)$ or $C^2_k(S)$. The Monge-Ampère equation $\det \nabla^2 v = k$ has been studied extensively over the last decades. We refer to the book [5] for a list of references on the topic.

The functionals $W_k$ are scalar variants of the functionals studied in [9, 8]. The purpose of this note is to show how the approach developed in those papers can be adapted to the simpler situation considered here. In passing, we provide here a classical functional analytic framework for this sort of problems. Our main focus is on the elliptic case ($k > 0$), which is the simplest one. The methods are, therefore, very basic. Indeed, in this situation, soft arguments readily yield the desired Euler-Lagrange equation.

At the end of the note we discuss the cases when $k$ is constant.

## 2 Main results

For simplicity, we assume throughout this note that $S \subset \mathbb{R}^2$ is a simply connected, bounded domain with a smooth boundary, and we let $k \in C^\infty(\overline{S})$.

### 2.1 Existence of minimisers

As in [9], existence of minimisers can be proven by a robust and straightforward argument.

**Proposition 2.1** The functional $W_k$ attains a minimum in the space $W^{2,2}_k(S)$.

**Proof.** We only need to consider the case when the infimum of $W_k$ is finite. But then the result is a straightforward application of the direct method on the space

$$X = \{ v \in W^{2,2} : \int_S v = 0 \text{ and } \int_S \nabla v = 0 \}.$$

In fact, $W_k$ is obviously $W^{2,2}$-coercive and lower semicontinuous under weak $W^{2,2}$-convergence. But the constraint is stable under weak $W^{2,2}$-convergence, because the determinant is continuous under weak $W^{2,2}$-convergence. Applying Poincaré’s inequality, we obtain the existence of a minimiser in $X$. \qed
It is clear that the same proof also works for general domain dimensions, other energy densities, additional force terms, boundary conditions, etc. As in [8], when \( k > 0 \) then one has a better existence result:

**Proposition 2.2** Assume that \( k > 0 \) on \( \overline{S} \). Then the functional \( W_k \) attains a minimum on the set

\[
W_k^{2,2}(S) \cap C^\infty(S).
\]  

**Proof.** First note that functions \( v \) belonging to the set (4) are either uniformly convex or uniformly concave, and that the infimum of \( W_k \) is the same on both of these components of the set (4). So we will prove that the minimum is attained on the set

\[
X = \{ v \in W_k^{2,2}(S) \cap C^\infty(S) : v \text{ is convex} \}.
\]

In fact, by interior regularity for convex Monge-Ampère equations we have

\[
X = \{ v \in W_k^{2,2}(S) : v \text{ is convex} \}.
\]

Hence the space \( X \) is closed under weak \( W^{2,2}\)-convergence and therefore we can find a minimiser in this space by the same arguments as in proof of Proposition 2.1. □

### 2.2 Lagrange multiplier rule for the elliptic case

The formal Lagrange multiplier rule asserts that critical points of \( W_k \) are critical for the functional

\[
v \mapsto \int_S |\nabla^2 v|^2 - \lambda \det \nabla^2 v \quad (5)
\]

without additional constraints, cf. e.g. [6] for a related situation. Here \( \lambda \) is some Lagrange multiplier. The Euler-Lagrange equation corresponding to (5) is

\[
\text{div} \text{div} (\nabla^2 v - \lambda \cof \nabla^2 v) = 0,
\]

or, since \( \text{div} \cof \nabla = 0 \),

\[
\Delta^2 v - \cof \nabla^2 v : \nabla^2 \lambda = 0.
\]

We will show that, under suitable regularity assumptions, this formal Lagrange multiplier rule can be justified by means of very soft functional analytic arguments.

For a rigorous approach we introduce the following notions, which are variants of those introduced in [9]: A function \( v \in W_k^{2,2}(S) \) is said to be stationary for \( W_k \) if

\[
\frac{d}{dt} \bigg|_{t=0} \int_S |
abla^2 u(t)|^2 = 0
\]
for all (strongly $W^{2,2}$-continuous, say) maps $t \mapsto u(t)$ from a neighbourhood of zero in $\mathbb{R}$ into $W^{2,2}_k(S)$ such that $u(0) = v$ and such that the derivative $u'(0)$ exists.

A function $v \in W^{2,2}_k(S)$ is said to be formally stationary for $W_k$ if

$$\int_S \nabla^2 v : \nabla^2 h = 0 \text{ for all } h \in W^{2,2}(S) \text{ with } \text{cof} \nabla^2 v : \nabla^2 h = 0 \text{ a.e. in } S.$$  

Our main results for the elliptic case are the following two remarks.

**Proposition 2.3** Let $\alpha \in (0,1)$ and let $k > 0$ on $\overline{S}$. Then the set

$$C^{2,\alpha}_k(S) := \{ u \in C^{2,\alpha}(S) : \det \nabla^2 u = k \text{ in } S \}$$

is a $C^\infty$-submanifold of $C^{2,\alpha}(S)$.

**Proposition 2.4** Let $\alpha \in (0,1)$ and let $k > 0$ on $\overline{S}$. If $v \in C^{2,\alpha}_k(S)$ is stationary for $W_k$, then there exists a unique Lagrange multiplier $\lambda \in (C^\infty \cap L^2)(S)$ such that

$$\text{divdiv} \left( \chi_S \left( \nabla^2 v + \lambda \text{cof} \nabla^2 v \right) \right) = 0 \text{ in } D'(\mathbb{R}^2).$$

In particular,

$$\Delta^2 v + \text{cof} \nabla^2 v : \nabla^2 \lambda = 0 \text{ in the classical sense on } S.$$  

3 The elliptic case

3.1 Functional analysis background

In this section, $X$ and $Y$ denote real Banach spaces. Recall that a closed subspace $E$ of $X$ is said to split $X$ if $E$ has a closed complement, i.e., there exists a closed subspace $F$ of $X$ such that $X = E \oplus F$. For a linear operator $G : X \to Y$ we denote by $N(G)$ its kernel and by $R(G)$ its range. The proof of the following result is straightforward.

**Lemma 3.1** Let $G : X \to Y$ and $F : X \to \mathbb{R}$ be bounded linear operators, and assume that the range of $G$ is closed. Then

$$Fh = 0 \text{ for all } h \in X \text{ with } Gh = 0$$

if and only if there exists $\Lambda \in Y'$ such that $F = \Lambda \circ G$. If, moreover, $R(G) = Y$ then $\Lambda$ is unique.

Let $M \subset X$. A vector $h \in X$ is called a tangent vector to $M$ at $v \in M$ provided there exists a map $u$ from a neighbourhood of zero in $\mathbb{R}$ into $M$ such that $u(0) = v$, and such that the derivative $u'(0)$ at 0 exists and equals $h$. We denote the set of all tangent vectors $h \in X$ at $v$ by $T_v M$. We recall the following basic result.
Lemma 3.2 Let $G : X \to Y$ be continuously Fréchet differentiable on $X$ and set
\[ M = \{ u \in X : G(u) = 0 \}. \]
(7)
Assume that, for all $v \in M$, the derivative $G'(v) : X \to Y$ is surjective and the kernel $N(G'(v))$ splits $X$. Then $M$ is a $C^1$-manifold.
More precisely, for all $v \in M$ we have $T_v M = N(G'(v))$ and there exists a continuously Fréchet differentiable homeomorphism $\varphi$ from a neighbourhood of zero in $T_v M$ onto an open neighbourhood of $v$ in $M$ that satisfies
\[ \varphi(h) = v + h + o(\|h\|_X) \text{ as } h \to 0 \text{ in } T_v M. \]
If $G$ is $C^m$ on $X$ then $M$ is a $C^m$-manifold.

Proof. This result is classical, cf. [15]. For the reader’s convenience we recall the proof of the existence of $\varphi$.
Let $E \subset X$ be a closed complement of $N(G'(v))$ in $X$. Define $H : N(G'(v)) \times E \to Y$ by setting $H(h, z) = G(v + h + z)$. Since the partial derivative $D_2 H(0, 0)$ is just the restriction of $G'(v)$ to $E$, the hypotheses show that we can apply the implicit function theorem to $H$. This yields a $C^1$-map $\varphi$ as in the statement, because $D_1 H(0, 0) = 0$. □

Remark. In the context of surfaces, the existence of $\varphi$ as in the conclusion of Lemma 3.2 amounts to the so-called continuation of infinitesimal bendings. We refer to [11] for the elliptic case (cf. also [16] and [14]), and to [10] for the intrinsically flat case.

3.2 Linear elliptic operators

In this section we recall some basic functional analytic properties of linear elliptic operators of the form
\[ Lu := A : \nabla^2 u + B \cdot \nabla u + Cu \]
on a bounded $C^{2,\alpha}$ domain $\Omega \subset \mathbb{R}^n$ for some $\alpha \in (0,1)$. Here $A \in C^0(\overline{\Omega}, \mathbb{R}^{n \times n}_{\text{sym}})$, $B \in L^\infty(\Omega, \mathbb{R}^n)$ and $C \in L^\infty(\Omega)$. We assume $A$ to be strictly elliptic, i.e., there exists $c > 0$ such that
\[ A(x) : (\xi \otimes \xi) \geq c|\xi|^2 \text{ for all } x \in S, \xi \in \mathbb{R}^n. \]
For simplicity, we only consider the case when $C \leq 0$.

Lemma 3.3 Let $p \in (1, \infty)$, let $A \in C^0(\overline{\Omega})$, $B, C \in L^\infty(\Omega)$, assume that $C \leq 0$ and define $G : W^{2,p}(\Omega) \to L^p(\Omega)$ by $Gu = Lu$. Then $G$ is surjective and $N(G)$ splits $W^{2,p}(\Omega)$. 

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Proof. Under the above hypotheses on the coefficients, and for any \( f \in L^p(\Omega) \), the Dirichlet problem

\[
Lu = f \\
u \in W^{1,p}_0(\Omega)
\]  
(8)

has a unique solution \( u \in W^{2,p}(\Omega) \), cf. [5, Theorem 9.15]. Hence \( G \) is surjective.

For each \( f \in L^p(\Omega) \) denote by \( T f \) the solution \( u \) of (8). Then \( T : L^p(\Omega) \to W^{2,p}(\Omega) \) is bounded. And obviously it is a right inverse of \( G \). Since \( G \) is surjective and admits a bounded right inverse, we conclude that \( N(G) \) splits \( W^{2,p}(\Omega) \).

\[ \square \]

Similarly, this time using Schauder theory, one proves the following lemma:

**Lemma 3.4** Assume \( A, B, C \in C^{0,\alpha}((\overline{\Omega})) \), assume that \( C \leq 0 \) and define \( G : C^{2,\alpha}(\overline{\Omega}) \to C^{0,\alpha}(\overline{\Omega}) \) by \( Gu = Lu \). Then \( G \) is surjective and \( N(G) \) splits \( C^{2,\alpha}(\overline{\Omega}) \).

**Lemma 3.5** Assume \( A, B, C \in C^{0,\alpha}((\overline{\Omega})) \), assume that \( C \leq 0 \) and define \( G : W^{2,2}(\Omega) \to L^2(\Omega) \) by \( Gu = Lu \). Then \( C^{2,\alpha}(\overline{\Omega}) \cap N(G) \) is strongly \( W^{2,2} \)-dense in \( N(G) \).

**Proof.** Let \( u \in N(G) \) and let \( u_n \in C^{2,\alpha}(\overline{\Omega}) \) be such that \( u_n \to u \) in \( W^{2,2}(\Omega) \). Then by continuity

\[
Lu_n \to 0 \text{ in } L^2(\Omega).
\]  
(9)

As in the proof of Lemma 3.3, there exists a unique solution \( \rho_n \in (W^{2,2} \cap W^{1,2}_0)(\Omega) \) of

\[
L\rho_n = -Lu_n \text{ in } \Omega.
\]

Moreover, \( \rho_n \in C^{2,\alpha}(\overline{\Omega}) \) because \( Lu_n \in C^{0,\alpha}(\overline{\Omega}) \), and \( \rho_n \to 0 \) in \( W^{2,2}(\Omega) \) by (9). Thus \( u_n + \rho_n \in N(G) \cap C^{2,\alpha}(\overline{\Omega}) \) and \( u_n + \rho_n \to u \) in \( W^{2,2}(\Omega) \).  

\[ \square \]

### 3.3 Proofs of the main results

**Proof of Proposition 2.3.** Let \( m \in \mathbb{N} \) and set \( X = C^{2,\alpha}(\overline{S}) \) and \( Y = C^{0,\alpha}(\overline{S}) \). We must show that the map \( G : X \to Y \) defined by

\[
G(u) = \det \nabla^2 u - k
\]

satisfies the hypotheses of Lemma 3.2 with \( M \) given by (7). But of course \( G \) is in \( C^m \), because it is quadratic. More precisely, for all \( h \in X \) we have

\[
G(v + h) = G(v) + \text{cof } \nabla^2 v : \nabla^2 h + \det \nabla^2 h.
\]
Since
\[ \| \det \nabla^2 h \|_{C^0,\alpha} \leq \| \nabla^2 h \|_{C^0,\alpha} \leq \| h \|_{C^2,\alpha}^2, \]
we have
\[ G'(v)h = \text{cof} \nabla^2 v : \nabla^2 h \quad \text{for all} \quad h \in X. \]
Next we claim that \( G'(v) : X \to Y \) is surjective for all \( v \in M \). But since \( \det \nabla^2 v = k \), by the assumptions on \( k \) we see that \( v \) is either strictly convex or concave. Hence \( \text{cof} \nabla^2 v \) is strictly elliptic. So Lemma 3.4 shows that \( G'(v) \) is surjective, and that \( N(G'(v)) \) splits \( X \).

\[ \text{Lemma 3.6} \quad \text{Assume that} \quad k > 0 \quad \text{on} \quad \overline{S}. \quad \text{If} \quad v \in C_{2,\alpha}^2(\overline{S}) \quad \text{is stationary for} \quad W_k, \quad \text{then} \quad v \quad \text{is formally stationary for} \quad W_k. \]

\[ \text{Proof.} \quad \text{Define} \quad G : W^{2,2}(S) \to L^2(S) \quad \text{by} \quad Gu = \text{cof} \nabla^2 v : \nabla^2 u \quad \text{and} \quad \tilde{G} : C^{2,\alpha}(\overline{S}) \to C^{0,\alpha}(\overline{S}) \quad \text{by} \quad \tilde{G}u = \text{cof} \nabla^2 v : \nabla^2 u, \quad \text{and define} \quad F : W^{2,2}(S) \to \mathbb{R} \quad \text{by} \quad F(v) = \int_S |\nabla^2 v|^2. \]

Since \( F \) is continuously Fréchet differentiable, the fact that \( v \in C_{2,\alpha}^2(\overline{S}) \) is stationary for \( W_k \), combined with Proposition 2.3 (in particular with the existence of \( \varphi \) as in the conclusion of Lemma 3.2), implies that
\[ F'(v)h = 0 \quad \text{for all} \quad h \in N(\tilde{G}). \quad (10) \]
Now \( N(\tilde{G}) = N(G) \cap C^{2,\alpha}(\overline{S}) \). Lemma 3.5 implies that \( N(\tilde{G}) \) is strongly \( W^{2,2} \)-dense in \( N(G) \). Thus by continuity of \( F'(v) \), formula (10) is in fact equivalent to
\[ F'(v)h = 0 \quad \text{for all} \quad h \in N(G). \]
And this means that \( v \) is formally stationary for \( W_k \).

For formal stationary points we use the basic Lagrange multiplier rule to prove the following lemma:

\[ \text{Lemma 3.7} \quad \text{Assume that} \quad k > 0 \quad \text{on} \quad \overline{S}. \quad \text{If} \quad v \in C_{2,\alpha}^2(\overline{S}) \quad \text{is formally stationary for} \quad W_k, \quad \text{then there exists a Lagrange multiplier} \quad \lambda \in L^2(S) \quad \text{such that} \quad (6) \quad \text{holds}. \]

\[ \text{Proof.} \quad \text{Define} \quad F : W^{2,2}(S) \to \mathbb{R} \quad \text{by} \quad Fh = \int_S \nabla^2 v : \nabla^2 h \quad \text{and} \quad G : W^{2,2}(S) \to L^2(S) \quad \text{by} \quad Gh = \text{cof} \nabla^2 v : \nabla^2 h. \quad \text{By hypothesis we know} \quad Fh = 0 \quad \text{for all} \quad h \in N(G). \quad \text{Since} \quad \text{cof} \nabla^2 v \in C^0(\overline{S}), \quad \text{we can apply Lemma 3.3 to see that} \quad G \quad \text{is surjective. Now Lemma 3.1 implies that there exists a unique} \quad \Lambda \quad \text{in the dual of} \quad L^2(S) \quad \text{such that} \quad F = \Lambda \circ G, \quad \text{i.e., there exists} \quad \lambda \in L^2(S) \quad \text{such that} \]
\[ \int_S \nabla^2 v : \nabla^2 h = \int_S \lambda \text{cof} \nabla^2 v : \nabla^2 h \quad \text{for all} \quad h \in W^{2,2}(S). \]

\[ \square \]
Proof of Proposition 2.4. Combine Lemma 3.7 with Lemma 3.6, and observe that (6) implies that
\[ \text{divdiv}(\lambda \cof \nabla^2 v : \nabla^2 h) = \Delta^2 v \text{ in } \mathcal{D}'(S), \]
which by standard elliptic regularity proves that \( \lambda \in C^\infty(S) \) because \( v \in C^\infty(S) \). \( \square \)

4 The case of constant \( k \)

For \( A \in \mathbb{R}^{2 \times 2}_{\text{sym}} \) we have \( |A|^2 = (\text{Tr} A)^2 - 2 \det A \) and \( |A|^2 = 2|A^o|^2 + 2 \det A \), where \( A^o = A - \frac{1}{2}(\text{Tr} A)I \) denotes the trace-free part of \( A \). Thus
\[
|\nabla^2 v|^2 = (\Delta v)^2 - 2k
\]
and
\[
|\nabla^2 v|^2 = 2|\nabla^2 v - \frac{1}{2}\Delta v I|^2 + 2k.
\]

And so
\[
W_k(v) = \int_S ((\Delta v)^2 - 2k) = 2 \int_S \left( |\nabla^2 v - \frac{1}{2}\Delta v I|^2 + k \right).
\]

So an absolute minimum of \( W_k \) is attained if \( \nabla^2 v - \frac{1}{2}\Delta v I \) vanishes identically, i.e.,
\[
\nabla^2 v = \begin{cases} 
\sqrt{k}I & \text{if } k \geq 0 \\
\sqrt{|k|}\text{diag}(1,-1) & \text{if } k < 0.
\end{cases}
\]

If \( k \) is constant then this is the case for
\[
v(x) = \begin{cases} 
\frac{\sqrt{k}}{2}|x|^2 & \text{if } k \geq 0 \\
\frac{\sqrt{|k|}}{2}(x_1^2 - x_2^2) & \text{if } k < 0.
\end{cases}
\]

Remarks.

(i) The above computations are standard in the context of surfaces, cf. [12]. With obvious changes, these arguments also apply to the case of isometric immersions when the Gauss curvature of the reference metric is constant, cf. [9].

(ii) When the energy density \( Q_2 \) is not isotropic, then one can still argue similarly, and one obtains solutions with two unequal constant principal curvatures.
(iii) A nontrivial problem for the case \( k = 0 \) results if one imposes boundary conditions or includes force terms. This situation is covered by the results in \([7, 10]\).

Indeed, the problem addressed there was to study minimisers of the Willmore functional

\[
u \mapsto \int_S |A|^2
\]

among all \( W^{2,2} \) isometric immersions \( u \) of \((S, \delta)\) into \( \mathbb{R}^3 \), where \( \delta \) denotes the standard flat metric in \( \mathbb{R}^2 \) and \( A \) denotes the second fundamental form of \( u \). But by the Gauss-Codazzi-Mainardi equations \( A \) is a possible second fundamental form for such an isometric immersion \( u \) if and only if

\[
A \in \{ \nabla^2 v : v \in W^{2,2}(S) \text{ with } \det \nabla^2 v = 0 \text{ a.e. in } S \}.
\]

Related to this, if \( u \in W^{2,2}(S, \mathbb{R}^3) \), then \( u \) is an isometric immersion if and only if the function \( v = u \cdot e \) satisfies the Darboux equation \( \det \nabla^2 v = 0 \) for any constant \( e \in S^2 \).

References

[1] P. G. Ciarlet. *Mathematical elasticity. Vol. II*, volume 27 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1997. Theory of plates.

[2] E. Efrati, E. Sharon, and R. Kupferman. Elastic theory of unconstrained non-euclidean plates. *J. Mech. Phys. Solids*, 57:762–775, 2009.

[3] G. Friesecke, R. D. James, and S. Müller. A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Comm. Pure Appl. Math.*, 55(11):1461–1506, 2002.

[4] G. Friesecke, R. D. James, and S. Müller. A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence. *Arch. Ration. Mech. Anal.*, 180(2):183–236, 2006.

[5] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[6] J. Guven and M. M. Müller. How paper folds: bending with local constraints. *J. Phys. A*, 41(5):055203, 15, 2008.
[7] P. Hornung. Euler-Lagrange equation and regularity for flat minimizers of the Willmore functional. *Comm. Pure Appl. Math.*, 64(3):367–441, 2011.

[8] P. Hornung. Constrained Willmore equation for disks with positive Gauss curvature. *MIS MPG Preprint*, 2012.

[9] P. Hornung. The Willmore functional on isometric immersions. *MIS MPG Preprint*, 2012.

[10] P. Hornung. Continuation of infinitesimal bendings on developable surfaces and equilibrium equations for nonlinear bending theory of plates. *Comm. PDE*, to appear.

[11] S. B. Klimentov. Extension of higher-order infinitesimal bendings of a simply connected surface of positive curvature. *Mat. Zametki*, 36(3):393–403, 1984.

[12] E. Kuwert and R. Schätzle. Removability of point singularities of Willmore surfaces. *Ann. of Math. (2)*, 160(1):315–357, 2004.

[13] M. Lewicka, L. Mahadevan, and M. R. Pakzad. Models for elastic shells with incompatible strains. *Preprint*, 2012.

[14] M. Lewicka, M. G. Mora, and M. R. Pakzad. The matching property of infinitesimal isometries on elliptic surfaces and elasticity of thin shells. *Arch. Ration. Mech. Anal.*, 200(3):1023–1050, 2011.

[15] L. Ljusternik. On constrained extrema of functionals. *Mat. Sb.*, 41(11):390–401, 1934.

[16] L. Nirenberg. The Weyl and Minkowski problems in differential geometry in the large. *Comm. Pure Appl. Math.*, 6:337–394, 1953.