Topological order and absence of band insulators at integer filling in non-symmorphic crystals

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Band insulators appear in a crystalline system only when the filling—the number of electrons per unit cell and spin projection—is an integer. At fractional filling, an insulating phase that preserves all symmetries is a Mott insulator; that is, it is either gapless or, if gapped, exhibits fractionalized excitations and topological order. We raise the inverse question—at an integer filling is a band insulator always possible? Here we show that lattice symmetries may forbid a band insulator even at certain integer fillings, if the crystal is non-symmorphic—a property shared by most three-dimensional crystal structures. In these cases, one may infer the existence of topological order if the ground state is gapped and fully symmetric. This is demonstrated using a non-perturbative flux-threading argument, which has immediate applications to quantum spin systems and bosonic insulators in addition to electronic band structures in the absence of spin–orbit interactions.

Fundamental to the study of crystalline materials is the existence of band insulators, and the fact that they occur only when the electronic filling is commensurate with the lattice. Defining the filling as one-half the number of electrons in a unit cell—which accounts for spin degeneracy in the absence of spin–orbit coupling—band insulators occur only if the filling is an integer. At fractional filling a material must be metallic within the free-electron approach. However, interactions could lead to a distinct insulating phase. For fermionic systems, this would have to be a Mott insulator, strictly defined as an insulator that breaks no symmetries but is distinct, that is, separated by a phase transition, from a band insulator. In fact, a far stronger result is true: even interactions cannot cause a system at fractional filling, whether of bosons or fermions, to enter a trivial insulating state. There are only two choices if all symmetries are to be preserved: the system must remain gapless, or develop topological order, which we take here to mean a ground-state degeneracy that is not associated with a broken symmetry, as exemplified by fractional quantum Hall states and gapped quantum spin liquids. These gapped states feature long-range quantum entanglement, leading to excitations with anyonic statistics and ground-state degeneracy that is sensitive to the spatial topology. A concrete example is the one-dimensional \((d = 1)\) Hubbard model at half-filling, with repulsive interactions whose ground state is insulating. The spin excitations in the insulator form a gapless Luttinger liquid, as expected of a fully symmetric state in \(d = 1\), where a gapped state with topological order is forbidden. This result was demonstrated for arbitrary spatial dimensions \(^1\), \(^2\), \(^3\), \(^4\), \(^5\), \(^6\), by extending the \(d = 1\) Lieb–Schultz–Mattis \(^4\) theorem to \(d > 1\) using an elegant flux-threading argument \(^5\), \(^6\).

However, surprisingly little is known about the inverse question: given an integer filling, is it always possible to find a trivial insulating state, such as a band insulator, that is not required to have these exotic properties? Here we show that for many crystals, trivial insulators can be forbidden even when the filling is an integer. For bosons or fermions, if the insulating states preserve symmetry, they must either be topologically ordered or host gapless excitations. We demonstrate a simple crystallographic criterion for this to hold: namely, that the space group of the crystal is non-symmorphic. A crystal is non-symmorphic if there is no choice of origin about which all its symmetries can be decomposed into a product of a lattice translation and a point-group symmetry element. Our key result is succinctly stated: there are integer fillings at which insulators with unique ground states are impossible if the space group is non-symmorphic. For such crystals we show that there is a minimal integer filling, strictly greater than unity, at which trivial insulators are allowed. We call this the non-symmorphic rank, denoted by \(S \geq 1\). Trivial insulators exist only at fillings that are integer multiples of \(S\).

We may rationalize this unexpected result by observing that in non-symmorphic crystals, a fractional lattice translation acts in concert with another transformation to leave the crystal invariant. This, loosely speaking, renders an integer filling equivalent to a fractional one, thus forbidding a trivial insulator. Similar conclusions can be drawn for any system where it is possible to identify a conserved charge. These include spin systems with a conserved spin component, and lattice bosons whose number is conserved. Note that 157 of the 230 distinct space groups in three dimensions are non-symmorphic and have \(S > 1\). These include the ubiquitous hexagonal close-packed (hcp) structure and the diamond-lattice space group relevant to many semiconductors and pyrochlore materials. Table 1 has further examples. The implications therefore are manifold; a few applications will be discussed below.

Before we turn to the main results, it is worth making two observations. First, we make no assumptions beyond lattice symmetry. In particular we do not limit ourselves to tight-binding models where further restrictions may emerge \(^7\), \(^8\). Indeed, the construction of trivial interacting insulating states in those cases relies precisely on the fact that they do not violate the conditions we establish here. Second, when we discuss electronic band structures, we assume that the electron’s spin is purely a spectator; that is, there is no spin–orbit coupling. This allows us to address each spin species independently, as though the electrons were spinless. Moreover, we will connect our key results to extinguished Bragg peaks (for example, see Fig. 1b), which assumes a scalar coupling between electrons and the crystal lattice. Incorporating electronic spin–orbit coupling is an important extension that we leave to future work.
Applications

Band theory. The fact that band insulators are forbidden unless the filling is a multiple of the non-symmorphic rank $S$ strongly constrains the structure of the bulk energy dispersion: it is impossible to detach a set of fewer than $S$ bands so that they touch no other bands, without breaking the crystal symmetry. Non-symmorphic space groups often also describe the symmetry of photonic crystals; hence, their photonic band structures obey similar constraints. For instance, the hcp structure has $S = 2$; we show a tight-binding band structure for the hcp lattice in Fig. 2 where the enforced contacts are explicit. Although the subject of band touchings in crystals has a long history, and the ubiquity of such degeneracies in non-symmorphic crystals has been noted, the connection to a minimum filling for band-insulating behaviour has not been made previously. More importantly, these previous results apply only to non-interacting systems, in contrast to the non-perturbative approach taken here, which allows us to determine the nature of the interacting insulating ground state. Thus, for instance, our arguments also forbid interacting fragile Mott insulators, which have unique ground states.

Spin systems and bosonic insulators. A parallel set of conclusions can be drawn for spin systems, where the filling is related to the total magnetization, and an insulator now corresponds to a phase with a spin gap. We demand at least a U(1) spin-rotation symmetry, although the conclusions also apply to a larger symmetry such as SU(2), as long as it contains a U(1) subgroup. The analogue of the band insulator is a trivial paramagnet, which has gapped excitations, and has neither conventional nor topological order. We conclude that such a trivial paramagnet is disallowed in an SU(2) symmetric spin-1/2 system on the diamond lattice, which is a common non-symmorphic lattice with $S = 2$. In contrast, the pyrochlore lattice has the same space group but twice as many sites per unit cell as diamond, and a trivial quantum paramagnet is not forbidden by our arguments. Another application is to magnetization plateaus in an applied Zeeman field. For example, half-magnetization plateaux of spin-1/2 moments on the two-dimensional (2D) non-symmetric Shastry–Sutherland lattice, which cannot be trivial paramagnets. Applications to SrCu$_2$(BO$_3$)$_2$ (SCBO), a material realizing the SSL where a half-magnetization plateau is observed, will be discussed below. Yet another application is to bosonic Mott insulators. Our arguments demonstrate that on non-symmorphic lattices, Mott insulators at fillings that are not a multiple of $S$ must be topologically ordered, if they are gapped and respect all symmetries.

Flux-threading argument and non-symmorphic rank

In the rest of this paper we outline our argument, and demonstrate its use in specific examples by substantiating the claims made above. For clarity and to fix notation we digress briefly to review some relevant crystallography. We will consider crystalline systems with a given space group, $G$. This has two ingredients: the subgroup of translations $T$, generated by the set of primitive translations $t_i : r \rightarrow r + a_i, i = 1, 2, 3$, where the $a_i$ generate a specific Brillouin zone, and a point group $P$ consisting of discrete rotations, inversions and reflections. By combining the 14 Bravais lattices with the 32 crystallographic point groups, we obtain the 230 space groups. Of these, 73 are symmorphic: there exists at least one point that is left invariant by all of the symmetries, up to translations by a lattice vector. The remaining 157 space groups for which no such point exists are non-symmorphic. Intuitively, a non-symmorphic space group contains one or more essential non-symmorphic operations (glide mirrors or screw rotations) that combine a point-group operation with a fractional lattice translation, and that cannot by any change of origin be rewritten as the product of a point-group operation and an ordinary translation. The latter caveat is important to distinguish these from trivial glides/screws, which can occur even in symmorphic crystals. There are however two exceptional space groups, which are non-symmorphic despite the absence of essential glide or screw operations (see Supplementary Section S1 for further details).

In the remainder, we focus on the 135 non-symmorphic groups that have glide mirrors or screw rotations. Under a non-symmetric symmetry $G$ consisting of a point-group operation $g$ and non-lattice translation $\tau$, the Fourier components of a scalar field (such as the density) transform as $\eta_k \rightarrow n_k e^{i\alpha_k}$, where $\alpha_k$ is an integer multiple of $2\pi$, $n_k = 0$: the associated Fourier component vanishes. An essential glide or screw always has an infinite set of such reciprocal lattice vectors $k$. Thus, non-symmetry (barring two exceptions) has a pronounced experimental manifestation: there are systematic absences in the diffraction pattern where the (scalar) Bragg intensity vanishes.

We assume that the systems we study are described by a Hamiltonian $\hat{H}$ that preserves all of the symmetries of $G$ and in addition that there is a conserved U(1) charge, $Q$ (assumed to take a fixed integer eigenvalue and thus replaceable by a $c$-number throughout) with $[\hat{H}, \hat{Q}] = 0$. We make no assumptions as to the origins of the conserved charge, so for instance the systems we consider could be built out of: spinless fermions or bosons, where $Q$ is just the conserved particle number; spinful fermions with SU(2) spin symmetry, in which case $Q$ is one-half the fermion number (because the two spin components may be treated independently); or lattice spins with (at least) U(1) spin-rotational invariance, in which case we may take the charge on lattice site $r$ to be $S + \hat{S}_r$, where $S$ and $\hat{S}_r$ are its total spin and magnetization, and define $Q$ accordingly. Considering a finite system with $N_c = N^3$ unit cells, we may then define the filling to be the charge per unit cell, that is, $\nu = Q/N_c$, which will be held fixed in the thermodynamic limit. We impose periodic boundary conditions that identify $r$ and $r + Na_i$. 

### Table 1 | Some non-symmorphic groups and their ranks, colloquial structure names and representative materials.

| $d$ | Name                          | Examples                  | Space group | $S$ |
|-----|-------------------------------|---------------------------|-------------|-----|
| 2   | Shastry–Sutherland            | SrCu$_2$(BO$_3$)$_2$      | $p4g$       | 2   |
| 3   | hcp                           | Be, Mg, Zn                | $P6_3/mmc$  | 2   |
| 3   | Diamond                       | C, Si                     | $Fd\bar{3}m$| 2   |
| 3   | Pyrochlore                    | Dy$_2$Ti$_2$O$_7$ (spin ice) | $Fd\bar{3}m$ | 2   |
| 3   | Pyrochlore                    | $\alpha$-SiO$_2$, GeO$_2$ | $P3_121$    | 3   |
| 3   | Pyrochlore                    | Cr$_5$I$_2$               | $P6_222$    | 3   |
| 3   | Pyrochlore                    | $Pr_2$Si$_2$O$_7$, La$_2$Si$_2$O$_7$ | $P4_1$ | 4   |
| 3   | Pyrochlore                    | CsCuCl$_3$               | $P6_1$      | 6   |

### Figure 1 | Flux threading on a non-symmorphic lattice. 

- **a**: A 2D non-symmorphic lattice, with $p4g$ space group. An essential glide symmetry is shown, consisting of a mirror ($\hat{d}_z$) and half-translation ($\hat{f}a_{(2,1)}$).
- **b**: Flux threading changes ground-state momentum from $P_1$ to $P_2$. At odd integer filling, $P_1$ is an extinguished reciprocal lattice vector, denoted by open circles, implying a distinct ground state. At even filling, $P_2$ lands on an allowed reciprocal lattice vector, allowing for a unique ground state.
Our argument is based on the response of the system to the insertion of a gauge flux that couples minimally to the conserved charge $Q$. We first give a heuristic argument, and then a more formal one. The strategy is as follows: we wish to show that if the system is an insulator, the ground state cannot be unique in the thermodynamic limit. Hence, the system cannot be a trivial insulator that respects all symmetries, which should have a unique ground state. A degenerate ground state implies either a broken symmetry or a gapless phase or topological order. That is, it is topologically ordered if it is gapped and breaks no symmetries. To show this, we begin with a ground state $|\Psi\rangle$ and thread a flux quantum through a periodic direction, which, by gauge invariance, returns us to the original Hamiltonian. This procedure produces an eigenstate $|\Psi\rangle$. Earlier work\cite{1,2,19,20} has argued that for an insulator, $|\Psi\rangle$ must be a low-energy state; that is, its energy approaches that of the ground state in the thermodynamic limit. Although rigorous energy bounds can be given only for a different—but gauge-equivalent—flux insertion, for pedagogical reasons we keep a simpler choice with the understanding that the arguments of ref. 2 can be applied, mutatis mutandis, to non-symmorphic symmetries. The key step is to show that $|\Psi\rangle$ is distinct from $|\Psi\rangle$, which would then establish ground-state degeneracy. In the case of fractional filling, these states differ in crystal momentum\cite{19,20}. For integer filling, crystal momentum fails to differentiate between them. However, we will show that on non-symmorphic lattices, one can still distinguish these states using the quantum numbers of the non-symmorphic operations.

Consider threading flux through the system by introducing a vector potential $A = k/N$, where $k$ is a reciprocal lattice vector; this is the most general vector potential that is pure gauge so that $e^{-iA^a\partial^a} = 1$ for any loop $C$ that threads the system. The change in energy on inserting flux is proportional to the total current and is thus bounded in an insulator, where this procedure produces a state degenerate with the ground state in the thermodynamic limit. If the flux threading changes the quantum numbers, the final state is distinct from the initial one and thus the ground state is degenerate.

We use units where $\hbar = c = 1$, in which the flux quantum $\phi_0 = 2\pi$. In the case of spinful fermions, we consider coupling to a single spin species, justified by the fact that spin is assumed to be a spectator.

To determine the change in quantum numbers of the symmetry operators, it is useful to first ignore the lattice potential, in which case it is straightforward to identify the change of the centre-of-mass momentum by computing the force imparted to each charge as the flux is adiabatically switched on. Faraday’s law gives $F_i = \int d\tau$, so that

$$\Delta P = \sum_{i=1}^{Q} \int F_i d\tau = \frac{Q}{N} k = N^2 v k$$

We now reintroduce the lattice. If $\Delta P$ is not in the reciprocal lattice, then it is observable even within the reduced symmetry of the crystal, and we have succeeded in producing a distinct degenerate ground state. For a fractional filling $v = p/q$, it is clear that this is the case as long as we choose $N$ relatively prime to $q$. In other words, the state following flux insertion has a distinct crystal momentum, which means that the quantum number associated with translational symmetry has changed. This is the essence of the Hastings–Oshikawa argument\cite{1,3}. At integer filling, it is clear that no choice of $N$ allows us to distinguish the initial and final states on the basis of lattice translational symmetry. For symmorphic crystals, the argument ends here: no more information about the insulating phase can be obtained purely from lattice symmetries.

For non-symmorphic crystals, in contrast, we demonstrate below that if we can choose $\Delta P$ to correspond to an extinguished Bragg peak, the initial and final states will have different quantum numbers for the non-symmorphic operation responsible for that extinction. We show that such a choice is always possible at unit-filling, implying that an insulator with a unique ground state is forbidden. Intuitively, the absence of Bragg peaks at special crystal momenta indicates that they can serve as a good quantum number to distinguish initial and final states, in the presence of a crystalline potential. In general, as long as $v$ is not a multiple of the non-symmorphic rank $S$, a unique ground state is forbidden (see below and Supplementary Section S3). This result may also be understood by the exercise of trying to isolate a single Bloch band in a non-symmetric crystal without breaking symmetry. Were such a band to exist, its exponentially localized Wannier orbitals would define centres of electronic charge in each unit cell. One now encounters an obstruction: owing to non-symmmorphicity, it is impossible to define a charge centre that is invariant, modulo translations, under all crystal symmetries.

We now bolster this intuitive picture with a more formal argument. Consider a crystal that has a non-symmetric space group $G$, which contains a non-symmetric operation $\bar{G}$. This comprises a point-group operation $g$ followed by a fractional translation $\tau$ in a direction left invariant by $g$; that is,

$$\hat{G} : r \rightarrow gr + \tau$$
where \( \tau \) is not a lattice vector, and \( g \tau = \tau \). We begin with a ground state \( |\psi\rangle \), and assume that it is an eigenstate of all the crystal symmetries, including \( \hat{G} \); that is,

\[
\hat{G} |\psi\rangle = e^{i\phi} |\psi\rangle
\]

We consider the smallest reciprocal lattice vector \( k \) left invariant by \( g \), so that \( g k = k \) and \( k \) generates the invariant sublattice along \( k \). We now thread flux by introducing a vector potential \( A = k/N \) as before, in the process of which \( |\psi\rangle \) evolves to a state \( |\psi'\rangle \) that is degenerate with it. To compare \( |\psi'\rangle \) to \( |\psi\rangle \), we must return to the original gauge, which can be accomplished by the unitary transformation \( |\psi'\rangle \rightarrow U_k |\psi'\rangle = |\psi\rangle \), where

\[
U_k = \exp \left\{ i \int \frac{d^d r}{N} k \cdot \rho(r) \right\}
\]

removes the inserted flux, and \( \rho(r) \) is the density operator corresponding to the conserved charge \( Q \). As \( A \) is left invariant by \( \hat{G} \), threading flux does not alter \( \hat{G} \) eigenvalues, so \( |\psi\rangle \) and \( |\psi'\rangle \) have the same quantum number under \( \hat{G} \); however, on acting with \( U_k \), the eigenvalue changes, as can be computed from the equation:

\[
\hat{G}^{-1} U_k \hat{G} = U_k e^{2\pi i Q/k |Q/N}
\]

where we have defined the phase factor \( \phi_\parallel(k) = \tau \cdot k/2\pi \), and \( Q = vN^3 \) is the total charge. It may be readily verified that because \( g k = k \), \( \phi_\parallel(k) \) is unchanged by a shift in real-space origin. For a non-symmorphic symmetry operation \( \hat{G} \), this phase \( \phi_\parallel(k) \) must be a fraction. This follows because \( \tau \) is a fractional translation. (If a lattice translation had the same projection onto \( k \) as \( \tau \), this would yield an integer phase factor \( \parallel \).

However, this would render the screw/glide removable, that is, reduced to point-group element \( \times \) translation by change of origin.) Thus, for \( \hat{G} \) non-symmorphic, \( \phi_\parallel(k) = p/|S_1| \), with \( p \), \( S_1 \) relatively prime. From (1) we conclude that \( |\psi\rangle \) and \( |\psi'\rangle \) have distinct \( \hat{G} \) eigenvalues whenever \( \phi_\parallel(k)Q/N = pN^2 v/|S_1| \) is a fraction. As we may always choose \( N \) relatively prime to the \( S_1 \), the result of flux insertion is a state distinguished from the original state by its \( \hat{G} \) eigenvalue, unless the filling is a multiple of \( S_1 \). For a glide \( S_1 = 2 \), whereas for a screw \( S_1 \) is the number of times it must be applied before it becomes removable (demonstrated in Supplementary Section S3).

**Non-symmorphic rank.** Recall that trivial insulators are allowed only at fillings that are multiples of an integer \( S \), which we call the non-symmorphic rank. Each non-symmorphic operation \( \hat{G} \) is individually associated with an integer \( S \) that leads to a degeneracy unless it divides the filling \( v \). Hence, the non-symmorphic rank \( S \) is divisible by the least common multiple of the \( S_i \). A tighter bound on the non-symmorphic rank, although subtle to prove (and hence deferred to the Supplementary Section S3) is easy to state: it is the smallest integer \( S \) such that some point is invariant under all of the elements of \( \hat{G} \) up to \( 1/S \) times Bravais lattice vectors.

**Examples.** The statements made earlier about specific examples follow immediately from an examination of the space-group symmetry (detailed in Supplementary Section S4). The hcp crystal (space group \( P6_3/mmc \)) possesses a sixfold screw rotation about its \( c \) axis, two applications of which result in a pure translation; as the only other non-symmorphic operation is a glide also of rank 2, we conclude \( S = 2 \). A model band structure, with enforced contacts between pairs of bands is shown (Fig. 2); breaking the symmetry explicitly lifts these degeneracies, depicted by the broken red lines. Diamond and pyrochlore (space group \( Fd3m \)) possess a fourfold screw axis, but applying this twice yields a removable twofold screw (Fig. 3a). Once again, all remaining non-symmorphic operations have rank 2, so that \( S = 2 \) for both these crystals. In a field, SCBO exhibits a magnetization plateau at half the saturation value\(^{16,17}\). As this corresponds to \( v = 1 \), unlike other plateaux that are at fractional filling\(^{21-24}\), Hastings–Onishikawa arguments cannot be applied to it. However, the 2D SSL that characterizes a CuBO\(_3\) layer in SCBO (Fig. 3b) has the space group \( p4g \) with a glide symmetry (Fig. 1).

In \( d = 2 \), screw axes are impossible so \( S = 2 \). As \( v = 1 < S \), the magnetization plateau must be topologically ordered, or else break symmetry as in some proposed candidates\(^{25}\).

**Discussion.** Our goal has been to shed light on the construction of an insulating ground state at a given integer filling. We showed that a trivial symmetric insulating phase in non-symmorphic crystals is impossible, except at special integer fillings. That such a general impossibility theorem exists in the presence of interactions is striking. Indeed, we obtain a stronger result: for all fillings that are not a multiple of the non-symmorphic rank \( S \), we have shown that any insulating phase—which can emerge only from
interactions—must either break symmetry, remain gapless or exhibit topological order. A corollary is that energy bands in non-interacting, non-symmorphic crystals cannot be detached into groups comprised of fewer than \( s \) bands.

We note that our argument makes reference only to the crystalline space group. One may wish to study specific models of a crystalline lattice, for example, a tight-binding model; this, however, requires further specification of the representations of the space group. Of course, our general arguments apply in such cases as well, but stronger results may be possible for specific models. For instance, free fermions at unit-filling on the honeycomb lattice are always gapless, as evinced by their Dirac dispersion in graphene. However, this result does not follow from our general arguments because the honeycomb lattice is symmorphic. Indeed, for other representations of the same space-group symmetry (for example, the triangular lattice comprising the centres of the hexagons), a trivial insulator can be constructed at the same filling. Thus, the honeycomb lattice result is purely a consequence of further restrictions requiring particles to move on a fixed set of tight-binding sites. These constraints can sometimes be circumvented in interacting systems where a unique symmetric ground state may arise, and hence they seem less robust than those imposed by non-symmorphicity. Whether stronger statements than ours can be made given a specific space-group representation is an interesting open question, although perhaps requires a rather different approach from the one adopted here. Similarly, we observe that our arguments are silent on the nature of the touchings in the free case. A precise determination of the loci of band contacts and the energy dispersion proximate to them typically requires knowledge of specific group representations encoded by the Bloch bands—information that must usually be computed on a case-by-case basis. We also note that fragile Mott insulators, are, like band insulators, allowed only at fillings that are integer multiples of the non-symmorphic rank.

We have noted conditions under which topological order is inevitable, if a gapped insulator preserves all symmetries. It is useful to note that one can construct specific models where such topologically ordered, featureless ground states arise. Heuristically, this occurs through fractionalization—the charges split into fractionally charged partons that are at the appropriate filling to form a uniform insulator. Indeed, this is the intuition behind a featureless insulating state at fractional filling (see ref. 26 for an exactly soluble example). Such a model may be constructed for a featureless insulating state of spinless bosons at half-filling on the square lattice, which necessarily possesses topological order. If we now deform the square lattice into the non-symmorphic lattice of Fig. 1a, by splitting each site into a pair (while requiring 1/2 a boson per site) the resulting topologically ordered insulator is fully symmetric and gapped, at unit-filling on the non-symmorphic SSL.

We have demonstrated that the textbook wisdom of fillings at which band insulators may occur must be sharpened in the context of non-symmorphic lattices. There, the minimum band-insulating filling is not two (spinful) electrons per unit cell, but a larger integer multiple. Thus, non-symmorphic lattices a fully symmetric insulator with two electrons per unit cell immediately implies an exotic ground state, and provides guidance to the search for new phases of matter. We leave extensions of these ideas to systems with spin–orbit coupling as well as to quasicrystals, to future work.

Note added in proof: Subsequent to the submission of this manuscript, some of the results presented here were reproduced using a different approach\(^7^7\).

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References

1. Hastings, M. B. Lieb–Schultz–Mattis in higher dimensions. Phys. Rev. B 69, 104431 (2004).
2. Hastings, M. B. Sufficient conditions for topological order in insulators. Europhys. Lett. 70, 824–830 (2005).
3. Oshikawa, M. Commensurability, excitation gap, and topology in quantum many-particle systems on a periodic lattice. Phys. Rev. Lett. 84, 1535–1538 (2000).
4. Lieb, E., Schultz, T. & Mattis, D. Two soluble models of an antiferromagnetic chain. Ann. Phys. 16, 407–466 (1961).
5. Altman, E. & Auerbach, A. Haldane gap and fractional oscillations in gapped Josephson arrays. Phys. Rev. Lett. 81, 4484–4487 (1998).
6. Oshikawa, M., Yamanaka, M. &affleck, I. Magnetization plateaus in spin chains: Haldane gap for half-integer spins. Phys. Rev. Lett. 78, 1984–1987 (1997).
7. Yao, H. & Kivelson, S. A. Fragile Mott insulators. Phys. Rev. Lett. 105, 164602 (2010).
8. Kimchi, I., Parameswaran, S. A., Turner, A. M. & Vishwanath, A. Featureless and non-fractionalized Bose insulator on the honeycomb lattice at 1/2 site-filling. Preprint at http://arxiv.org/abs/1207.0498 (2012).
9. Yablonovitch, E. Inhibited spontaneous emission in solid-state physics and electronics. Phys. Rev. Lett. 58, 2059–2062 (1987).
10. John, S. Strong localization of photons in certain disordered dielectric superlattices. Phys. Rev. Lett. 58, 2486–2489 (1987).
11. Herring, C. Character tables for two space groups. J. Franklin. Inst. 233, 525–543 (1942).
12. Herring, C. Effect of time-reversal symmetry on energy bands of crystals. Phys. Rev. 52, 361–365 (1937).
13. König, A. & Mermin, N. D. Electronic level degeneracy in nonsymmetric periodic or apriori nonsymmorphic lattices. Phys. Rev. B 56, 13607–13619 (1997).
14. Michel, L. & Zak, J. Connectivity of energy bands in crystals. Phys. Rev. B 59, 5998–6001 (1999).
15. Shastry, B. S. & Sutherland, B. Exact ground state of a quantum-mechanical antiferromagnet. Physica 108B, 1069–1070 (1981).
16. Sebastian, S. E. et al. Fractionation drives crystalline states in a frustrated spin system. Proc. Natl Acad. Sci. USA 105, 20157–20160 (2008).
17. Jaime, M. et al. Magnetostriiction and magnetic texture to 100.75 Tesla in frustrated SrCu2(BO3)2. Proc. Natl Acad. Sci. USA 109, 12404–12407 (2012).
18. König, A. & Mermin, N. D. Screw rotations and glide mirrors: Crystallography in Fourier space. Proc. Natl Acad. Sci. USA 96, 3502–3506 (1999).
19. Oshikawa, M. Insulator, conductor, and commensurability: A topological approach. Phys. Rev. Lett. 90, 236401 (2003).
20. Paramekanti, A. & Vishwanath, A. Extending Luttinger’s theorem to \( s \) fractionalized phases of matter. Phys. Rev. B 70, 245118 (2004).
21. Onizuka, K. et al. 1/3 magnetization plateau in SrCu2(BO3)2—strike order of excited triplets. J. Phys. Soc. Jpn 69, 1016–1018 (2000).
22. Kodama, K. et al. Magnetic superstructure in the two-dimensional quantum antiferromagnet SrCu2(BO3)2. Science 298, 395–399 (2002).
23. Abendschein, A. & Capponi, S. Effective theory of magnetization plateaus in the Shastry–Sutherland lattice. Phys. Rev. Lett. 101, 227201 (2008).
24. Dorier, J., Schmidt, K. P. & Mila, F. Theory of magnetization plateaus in the Shastry–Sutherland model. Phys. Rev. Lett. 101, 250402 (2008).
25. Momoi, T. & Totsuka, K. Magnetization plateaus of the Shastry–Sutherland model for SrCu2(BO3)2: Spin-density wave, supersolid, and bound states. Phys. Rev. B 62, 15067–15078 (2000).
26. Balents, L., Fisher, M. P. A. & Girvin, S. M. Fractionalization in an easy-axis Kagome antiferromagnet. Phys. Rev. B 65, 224412–224419 (2002).
27. Roy, R. Space group symmetries and low lying excitations of many-body systems at integer fillings. Preprint at http://arxiv.org/abs/1212.2944 (2012).

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Author contributions

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Additional information

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Competing financial interests

The authors declare no competing financial interests.