Matrix Product Solution to the Reflection Equation
Associated with a Coideal Subalgebra of \( U_q(A_n^{(1)}) \)

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Abstract

We present a new solution to the reflection equation associated with a coideal subalgebra of \( U_q(A_n^{(1)}) \) in the symmetric tensor representations and their dual. Elements of the \( K \) matrix are expressed by a matrix product formula involving terminating \( q \)-hypergeometric series in \( q \)-boson generators. At \( q = 0 \), our result reproduces a known set-theoretical solution to the reflection equation connected to the crystal base theory.

1. Introduction

Reflection equation \([5][20][11]\) is a characteristic structure in quantum integrable systems in the presence of boundaries. It combines the \( K \) matrix encoding the boundary interaction with the \( R \) matrix, another fundamental object governing the integrability in the bulk \([3]\). A variety of solutions to the reflection equation have been constructed up to now. See for example \([2][16][18][19][15]\) and references therein. In this Letter we present a new solution to the reflection equation having a number of outstanding features described below.

First, it is associated with the Drinfeld-Jimbo quantum affine algebra \( U_q(A_n^{(1)}) \) in the symmetric tensor representation \( V_{l,z} \) and its dual \( V^\ast_{l,z} \) with general degree \( l \in \mathbb{Z}_+ \). Here \( z \) denotes the (multiplicative) spectral parameter and \( q \) is assumed to be generic throughout. Both representations \( V_{l,z}, V^\ast_{l,z} \) have the bases \( \{v_\alpha\}, \{v^\ast_\alpha\} \) labeled with an array \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_n^\ast \) satisfying \( \alpha_1 + \cdots + \alpha_n = l \). They include the vector representation as the simplest case \( V_{1,z} \). Our \( K \) matrix \( K(z) = K^{(l)}(z, q) \) is a linear operator reflecting the “particles” into their duals as \( K(z) : V_{l,z} \rightarrow V^\ast_{l,z} \). As such, there are three kinds of \( R \) matrices \( R(z), R^\ast(z) \) and \( R^{**}(z) \) \([12][14]\) coming naturally into the game. They are all well-understood conceptually, and admit explicit formulas owing to the recent works \([13][4][12]\).

The reflection equation takes the form

\[
K_1(x)R^*((xy)^{-1})K_1(y)R((xy)^{-1}) = R^{**}((xy)^{-1})K_1(y)R^*((xy)^{-1})K_1(x),
\]

where \( K_1(x) = K^{(l)}(x, q) \otimes 1 \) and \( K_1(y) = K^{(m)}(y, q) \otimes 1 \). This is an equality of linear maps from \( V_{l,x} \otimes V_{m,y} \) to \( V^\ast_{l,x-1} \otimes V^\ast_{m,y-1} \), where the pair \((l, m) \in \mathbb{Z}_+^2\) is arbitrary. See \([35]\) and \([36]\) for a more concrete description.

Second, let us write the action of our \( K \) matrix on the basis as \( K(z)v_\alpha = \sum_\beta K(z)^\alpha_\beta v^\ast_\beta \). Then, it is dense in the sense that all the matrix elements \( K(z)^\alpha_\beta \) are nontrivial rational function of \( z \) and \( q \). Put plainly, our \( K(z) \) is trigonometric, dense, and of type \( A \) with general rank \( n \) and general “spin” \( l \). These are distinct features from previous works for type \( A \) which are mostly devoted to diagonal \( K \)’s or to the situation \( \min(n-1, l) = 1 \).

Third, our \( K(z) \) is characterized, up to normalization, as the intertwiner of the coideal subalgebra \( B_q \) of \( U_q(A_n^{(1)}) \) generated by the elements

\[
b_i = -e_i + q^2k_if_i + \frac{q}{1-q}k_i \in U_q(A_n^{(1)}) \quad (i \in \mathbb{Z}_n).
\]

Indeed, it is easy to check the right coideal nature \( \Delta B_q \subset B_q \otimes U_q(A_n^{(1)}) \) by applying the coproduct \( \Delta \) in \([2]\) to \( b_i \). The idea to characterize the spectral parameter dependent \( K \) matrices in terms of coideal subalgebras of quantum affine algebras was proposed long ago in the context of affine Toda field theory with boundaries. See for example \([6]\), more recent \([10][19]\) and references therein. Our result may be

\footnote{There are important exceptions \([14][13]\) related to this work although.}
viewed as a systematic implementation of it for the pair $B_q \subset U_q(A^{(1)}_{n-1})$ and the representations $V_{i,z}, V_{i,z}^*$. We note that the above $b_i$ has also appeared in the generalized $q$-Onsager algebra [14] up to convention.

Last but perhaps most intriguingly, our $K$ matrix has the elements that admit an explicit matrix product formula

$$K(z)^2 = \varrho(z)\text{Tr}(z^{-h}\hat{G}^\beta_1 \cdots \hat{G}^\beta_n)$$

with a scalar $\varrho(z)$. The trace is taken over a $q$-boson Fock space on which $h$ acts as the number operator. In terms of the creation $a^+$, the annihilation $a^-$ and the $q$-counting generator $k = q^h$ of the $q$-boson, the matrix product operator is given as $\hat{G}_i^\beta = q^{-\frac{1}{2}h^2}k^{-i}\hat{G}_i^\beta$ with

$$G_i^\beta = (q^{-i}; q)_\infty (q^{1-i}; q)_\infty 2\varphi_1\left(\frac{q^{-i}, q^{-t} - q^{-t}}{-q^{-s}, q}; q, qk\right)(a^+)^{(j-i)-}, \quad s = i + j, \quad t = \min(i, j),$$

where $2\varphi_1$ denotes the $q$-hypergeometric function and $(m)_+ = \max(m, 0)$. A matrix product solution to the reflection equation of this kind was first obtained in [15]. It covered all the fundamental representations of $U_q(A^{(1)}_{n-1})$ whose simplest case goes back to [17]. According to [15], the matrix product structure is a signal of three dimensional (3D) integrability. It is an interesting open problem to elucidate such features for the solution in this Letter. In this regard we note that all the $R$ matrices appearing in the reflection equations [37] are known to admit a matrix product formula originating in the tetrahedron equation [12].

There are further notable properties in our $K$ matrix $K(z)$. At $z = q^{-t}$, elements of $K^{(t)}(z, q)$ exhibit a neat factorization [59]. Combined with the similar property of the $R$ matrices [13 Th.2], it allows us to merge the spectral parameter to the spins $l, m \in \mathbb{Z}_+$ thereby upgrading the latter to generic parameters. Consequently we get a parametric generalization of the solution to the reflection equation. This achieves a boundary analogue of the result concerning the Yang-Baxter equation [13 sec.2.3]. Another feature of interest occurs at $q = 0$, where our $K$ matrix and reflection equation [81] survive quite nontrivially. In fact they are frozen exactly to the set-theoretical (combinatorial) counterparts introduced in [14] to formulate the box-ball system with reflecting end.

The outline of the Letter is as follows. In the next section we recapitulate the relevant representations of $U_q(A^{(1)}_{n-1})$ and the three kinds of $R$ matrices. In Section 3 we introduce the coideal subalgebra $B_q$ and characterize the $K$ matrix as the intertwiner. The reflection equation is formulated, which corresponds to a twisted one in the terminology of [19]. The proof of uniqueness of the intertwiner and the irreducibility of $V_{i,z} \otimes V_{m,y}$ as a $B_q$ module will be given elsewhere. In Section 3 we present the matrix product solution to the intertwining relation. The proof becomes local in the direction of rank, and reduces to some quadratic relations of (nonterminating) $q$-hypergeometric series. In Section 5 a generalization of integrable spin (degrees of symmetric tensors and their dual) to continuous parameters is described. In Section 6 we present the results in yet another gauge and elucidate the connection to the work [13] at $q = 0$. Section 7 contains a brief summary and an outlook. The associated commuting double row transfer matrices (cf. [20]) are left for future study. We set $\mathbb{Z}_+ = \mathbb{Z}_{\geq 0}$ and use the following notations:

$$[u] = \frac{q^u - q^{-u}}{q - q^{-1}}, \quad (z; q)_m = \prod_{k=1}^m (1 - zq^{k-1}), \quad \binom{l}{m}_q = \frac{(q; q)_l}{(q; q)_{l-m}(q; q)_m},$$

$$\theta(\text{true}) = 1, \quad \theta(\text{false}) = 0, \quad e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n \quad (1 \leq j \leq n).$$

2. $U_q(A^{(1)}_{n-1})$ and relevant $R$ matrices

2.1. $U_q(A^{(1)}_{n-1})$ and relevant representations. Let $U_q = U_q(A^{(1)}_{n-1})$ be the Drinfeld-Jimbo quantum affine algebra (without the derivation operator) generated by $e_i, f_i, k_i^{\pm 1} (i \in \mathbb{Z}_n)$ obeying the relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad [k_i, k_j] = 0, \quad k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}.$$  

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu e_i^{(1-a_{ij}-\nu)} e_j \epsilon_i^{(\nu)} = 0, \quad \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu f_i^{(1-a_{ij}-\nu)} f_j \epsilon_i^{(\nu)} = 0 \quad (i \neq j), \quad \text{(1)}$$

2.2. $U_q(A^{(1)}_{n-1})$ and relevant representations. Let $U_q = U_q(A^{(1)}_{n-1})$ be the Drinfeld-Jimbo quantum affine algebra (without the derivation operator) generated by $e_i, f_i, k_i^{\pm 1} (i \in \mathbb{Z}_n)$ obeying the relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad [k_i, k_j] = 0, \quad k_i e_j k_i^{-1} = q^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}.$$  

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu e_i^{(1-a_{ij}-\nu)} e_j \epsilon_i^{(\nu)} = 0, \quad \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu f_i^{(1-a_{ij}-\nu)} f_j \epsilon_i^{(\nu)} = 0 \quad (i \neq j), \quad \text{(1)}$$
where $\delta_{ij} = \theta(i = j)$, $c_i^{(v)} = e_i^{v} / |v|!$, $f_i^{(v)} = f_i^{v} / |v|!$ and $|m|! = \prod_{j=1}^{m} [j]$. The Cartan matrix $(a_{ij})_{i,j \in \mathbb{Z}_n}$ is given by $a_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}$. We employ the coproduct $\Delta$ and the antipode $S$ of the form

$$
\Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta e_i = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta f_i = f_i \otimes 1 + k_i^{-1} \otimes f_i,
$$

(2)

$$
S(k_i) = k_i^{-1}, \quad S(e_i) = -e_i k_i^{-1}, \quad S(f_i) = -k_i f_i.
$$

(3)

For integer arrays $\alpha = (\alpha_1, \ldots, \alpha_k), \beta = (\beta_1, \ldots, \beta_k) \in \mathbb{Z}^k$ of any length $k$, we use the notation

$$
|\alpha| = \sum_{1 \leq i \leq k} \alpha_i, \quad \alpha = \sum_{1 \leq i \leq k} i \alpha_i, \quad (\alpha, \beta) = \sum_{1 \leq i < j \leq k} \alpha_i \beta_j,
$$

(4)

$$
\sigma(\alpha) = (\alpha_2, \ldots, \alpha_k, \alpha_1), \quad \rho(\alpha) = (\alpha_k, \ldots, \alpha_2, \alpha_1),
$$

(5)

where $\sigma$ is a cyclic shift and $\rho$ is the reverse ordering. We will be concerned with the two irreducible representations of $U_q$ labeled with $l \in \mathbb{Z}_+$:

$$
\pi_{l,z} : U_q \rightarrow \text{End}(V_{l,z}), \quad V_{l,z} = \bigoplus_{\alpha \in B_l} \mathbb{C}(q,z)v_{\alpha},
$$

(6)

$$
\pi_{l,z}^* : U_q \rightarrow \text{End}(V_{l,z}^*), \quad V_{l,z}^* = \bigoplus_{\alpha \in B_l} \mathbb{C}(q,z)v_{\alpha}^*,
$$

(7)

where $B_l$ is a finite set of length $n$ arrays specified as

$$
B_l = \{ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_n^n \mid |\alpha| = l \}.
$$

(8)

The index $i$ of $\alpha = (\alpha_i) \in B_l$ should always be understood as elements of $\mathbb{Z}_n$. Now the representations (6) and (7) are specified as

$$
e_{ij} v_\alpha = z^{\delta_{0j}}[\alpha_{j+1} + 1]v_\alpha + e_i - e_{i+1}, \quad e_j v_\alpha = -z^{\delta_{0j}}[\alpha_{j+1} + 1]q^{-\alpha_j + \alpha_{j+1} + 2} v_\alpha,
$$

(9)

$$
f_{ij} v_\alpha = z^{-\delta_{0j}}[\alpha_{j+1} + 1]v_\alpha - e_j + e_{j+1}, \quad f_j v_\alpha = -z^{-\delta_{0j}}[\alpha_{j+1} + 1]q^{\alpha_j - \alpha_{j+1}} v_\alpha - e_j + e_{j+1},
$$

(10)

$$
k_j v_\alpha = q^\alpha v_\alpha, \quad k_j^* v_\alpha = q^{-\alpha_j + \alpha_{j+1}} v_\alpha,
$$

(11)

where $\pi_{l,z}(g), \pi_{l,z}^*(g)$ with $g \in U_q$ are denoted by $g$ for simplicity. In the RHS, $v_\beta, v_\beta^*$ with $\beta \notin B_l$ should be understood as 0. The representation $\pi_{l,z}$ is the (affinization of) degree $l$ symmetric tensor representation, and $\pi_{l,z}^*$ is its antipode dual. Namely, $(\pi_{l,z}^*(g) v_\alpha, v_\beta) = (v_\alpha, \pi_{l,z}(S(g)) v_\beta)$ holds for any $\alpha, \beta \in B_l$ and $g \in U_q$ with respect to the bilinear pairing $(v_\alpha, v_\beta) = \delta_{\alpha, \beta}$. In terms of the classical part $U_q(A_{n-1})$, they are the irreducible representations labeled with the rectangular Young diagrams of shape $1 \times l$ and $(n-1) \times l$, respectively.

### 2.2. $R$ matrices

For simplicity denote the tensor product representation $(\pi_{l,x} \otimes \pi_{m,y}) \circ \Delta$ just by $\pi_{l,x} \otimes \pi_{m,y}$, etc. Consider the three types of quantum $R$ matrices which are characterized, up to normalization, by the commutativity with $U_q$ as

$$
R(x/y) : V_{l,x} \otimes V_{m,y} \rightarrow V_{m,y} \otimes V_{l,x}, \quad (\pi_{m,y} \otimes \pi_{l,x}) R(x/y) = R(x/y) (\pi_{l,x} \otimes \pi_{m,y}),
$$

(12)

$$
R^*(x/y) : V_{l,x}^* \otimes V_{m,y} \rightarrow V_{m,y}^* \otimes V_{l,x}^*, \quad (\pi_{m,y} \otimes \pi_{l,x}^*) R^*(x/y) = R^*(x/y) (\pi_{l,x}^* \otimes \pi_{m,y}),
$$

(13)

$$
R^{**}(x/y) : V_{l,x}^{**} \otimes V_{m,y}^* \rightarrow V_{m,y}^* \otimes V_{l,x}^{**}, \quad (\pi_{m,y}^* \otimes \pi_{l,x}^{**}) R^{**}(x/y) = R^{**}(x/y) (\pi_{l,x}^{**} \otimes \pi_{m,y}^*).
$$

(14)

Note that dependence on $l, m, q$ is suppressed in the $R$ matrices. We specify the matrix elements by

$$
R(z)(v_\alpha \otimes v_\beta) = \sum_{\gamma \in B_l \delta \in B_m} R(z)^{\gamma,\delta}_{\alpha,\beta} v_\gamma \otimes v_\gamma,
$$

(15)

$$
R^*(z)(v_\alpha^* \otimes v_\beta) = \sum_{\gamma \in B_l \delta \in B_m} R^*(z)^{\gamma,\delta}_{\alpha,\beta} v_\gamma^* \otimes v_\gamma^*,
$$

(16)

$$
R^{**}(z)(v_\alpha^{**} \otimes v_\beta^*) = \sum_{\gamma \in B_l \delta \in B_m} R^{**}(z)^{\gamma,\delta}_{\alpha,\beta} v_\gamma^{**} \otimes v_\gamma^*,
$$

(17)

and the normalization

$$
R(z)^{e_1,m_{e_1}}_{e_1,m_{e_1}} = R^*(z)^{e_1,m_{e_1}}_{e_1,m_{e_1}} = R^{**}(z)^{e_1,m_{e_1}}_{e_1,m_{e_1}} = 1.
$$

(18)

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Note $a_{n-1,0} = a_{0,n-1} = -1$ because of $i, j \in \mathbb{Z}_n$. 

\footnote{Note $a_{n-1,0} = a_{0,n-1} = -1$ because of $i, j \in \mathbb{Z}_n$.}
In order to provide explicit formulas for the $R$ matrices, we prepare their building blocks. For complex parameters $\lambda, \mu$ and arrays $\beta = (\beta_1, \ldots, \beta_k), \gamma = (\gamma_1, \ldots, \gamma_k) \in \mathbb{Z}_+^k$ with any length $k$, define

$$\Phi_q(\gamma | \beta; \lambda, \mu) = q^{(\beta - \gamma)\gamma} \prod_{i=1}^{k} \Phi_q(\gamma_i | \beta_i; \lambda, \mu),$$

$$\overline{\Phi}_q(\gamma | \beta; \lambda, \mu) = \theta(\gamma \leq \beta) \prod_{i=1}^{k} \Phi_q(\gamma_i | \beta_i; \lambda, \mu),$$

where $\theta(\gamma \leq \beta)$ stands for $\prod_{i=1}^{k} \theta(\gamma_i \leq \beta_i)$. The function $\Phi_q(\gamma | \beta; \lambda, \mu)$ was introduced in [13] eq.(19) in the study of a stochastic $R$ matrix for $U_q$. Following [14] we define a quadratic combination of (19) as

$$A(z)^{\gamma, \delta}_{\alpha, \beta} = q^{(\alpha, \beta) - (\delta, \gamma)} \sum_{\xi + \eta = \gamma + \delta} \Phi_q(\xi - \delta \mid \xi; q^{m-l}, q^{l-m}) \overline{\Phi}_q(\eta | \beta; q^{-l-m}, q^{-m}).$$

where $\alpha, \gamma \in B_1$ and $\beta, \delta \in B_m$ and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})$ stands for the truncation of $\alpha = (\alpha_1, \ldots, \alpha_n)$. The sum in (21) extends over $\xi, \eta \in \mathbb{Z}_+^{n-1}$ satisfying $\xi + \eta = \gamma + \delta$. There are finitely many such $\xi$ and $\eta$. The function $A(z)^{\gamma, \delta}_{\alpha, \beta}$ satisfies

$$A(z)^{\gamma, \delta}_{\alpha, \beta} = A(z)^{\rho(\alpha), \rho(\beta)} \prod_{i=1}^{n} \Phi_q(z_i^2; q^{m-l}, q^{l-m}) \Phi_q(z_i; q^{-l-m}, q^{-m}).$$

Now the elements of $R$ matrices are expressed as follows $(\delta^\beta_\alpha = \theta(\alpha = \beta))$:

$$R(z)^{\gamma, \delta}_{\alpha, \beta} = \delta^\beta_\alpha R(z)^{\gamma, \delta}_{\alpha, \beta},$$

$$R^*(z)^{\gamma, \delta}_{\alpha, \beta} = \delta^{\gamma^*}_{\alpha} R(z)^{\gamma, \delta}_{\alpha, \beta},$$

$$R^{**}(z)^{\gamma, \delta}_{\alpha, \beta} = \delta^{\gamma^*}_{\alpha} R(z)^{\gamma, \delta}_{\alpha, \beta},$$

See the comments after [79] for the origin of these formulas. The $R$ matrices satisfy the Yang-Baxter equations [3] reversing the components of the tensor products $V_{l, y} \otimes V_{l_{-2}, z_2} \otimes V_{l_{-3}, z_3}, V_{l, y}^* \otimes V_{l_{-2}, z_2} \otimes V_{l_{-3}, z_3}, V_{l, y}^* \otimes V_{l_{-2}, z_2} \otimes V_{l_{-3}, z_3}, V_{l, y}^* \otimes V_{l_{-2}, z_2} \otimes V_{l_{-3}, z_3}$. In terms of $x = y/z_2, y = z_2/z_3$, they read

$$1 \otimes (R(x))(R(xy) \otimes 1) = (R(y) \otimes 1)(1 \otimes R(xy))(R(x) \otimes 1),$$

$$1 \otimes (R^*(x))(R^*(xy) \otimes 1) = (R^*(y) \otimes 1)(1 \otimes R^*(xy))(R^*(x) \otimes 1),$$

$$1 \otimes (R^{**}(x))(R^{**}(xy) \otimes 1)(1 \otimes R^{**}(y)) = (R^{**}(y) \otimes 1)(1 \otimes R^{**}(xy))(R^{**}(x) \otimes 1),$$

$$1 \otimes (R^{**}(x))(R^{***(xy)} \otimes 1)(1 \otimes R^{***(y)}) = (R^{***(y)} \otimes 1)(1 \otimes R^{***(xy)})(R^{***(x)} \otimes 1).$$

3. A coideal subalgebra and $K$ matrix

Consider the element

$$b_i = -e_i + q^2 k_i f_i + \frac{q}{1 - q} k_i \in U_q \quad (i \in \mathbb{Z}_n)$$

and let $B_q$ be the subalgebra of $U_q$ generated by $\{b_i \mid i \in \mathbb{Z}_n\}$. From $\Delta(b_i) = b_i \otimes 1 + 1 \otimes (-e_i + q^2 k_i f_i)$, we see $\Delta B_q \subset B_q \otimes U_q$, meaning that $B_q$ is a right coideal subalgebra of $U_q$. Consider the operator

$$K(z) : V_{l, y} \rightarrow V_{l_{-1}, y}^*, \quad K(z)v_\gamma = \sum_{\gamma \in B_l} K(z)_{\gamma} v_\gamma^*,$$

which satisfies the intertwining relation

$$K(z)\pi_{l, 1}(b) = \pi_{l_{-1}, 1}(b) K(z) \quad (b \in B_q).$$

It suffices to impose [32] for the generators $b = b_i (i \in \mathbb{Z}_n)$. From [9]-[11], it reads explicitly as

$$- \delta^{\delta \alpha} \delta \gamma \gamma_1 + 1 K(z)^{\gamma}_{\alpha} K(z)^{\gamma_{\alpha}} - \delta^{\delta \alpha} \delta \gamma \gamma_1 + 1 K(z)^{\gamma}_{\alpha} K(z)^{\gamma_{\alpha}},$$

where $|\alpha| = |\gamma| = l$ and $K(z)^{\gamma}_{\alpha} = 0$ unless $\alpha, \gamma \in B_l$.

The essentials for our construction are the following claim.
**Theorem 1.** The solution $K(z)$ to the intertwining relation (32) or equivalently (33) ($\forall i \in \mathbb{Z}_n$) is unique up to normalization. Moreover, $V_{i,x} \otimes V_{m,y}$ is irreducible as a $B_q$ module for generic $x$ and $y$.

We will prove this for a more general setting elsewhere based partly on the existence of the crystal base $[8]$. In what follows, $K(z)$ denotes the unique intertwiner normalized as

$$K(z)|_{e_1} = 1. \quad (34)$$

Consider the intertwiner $V_{l,x} \otimes V_{m,y} \to V_{l,x-1}^* \otimes V_{m,y-1}^*$ of the $B_q$ modules constructed in two ways as

$$V_{l,x} \otimes V_{m,y} \overset{R(xy^{-1})}{\rightarrow} V_{m,y} \otimes V_{l,x} \overset{K_1(y)}{\rightarrow} V_{m,y-1}^* \otimes V_{l,x}^* \overset{R^*((xy)^{-1})}{\rightarrow} V_{l,x-1} \otimes V_{m,y-1}^*; \quad (35)$$

$$V_{l,x} \otimes V_{m,y} \overset{K_1(x)}{\rightarrow} V_{l,x-1}^* \otimes V_{m,y} \overset{R^*((xy)^{-1})}{\rightarrow} V_{m,y} \otimes V_{l,x-1} \overset{K_1(y)}{\rightarrow} V_{m,y-1}^* \otimes V_{l,x-1}^*; \quad (36)$$

where $K_1(x) = K^{(i)}(x, q) \otimes 1$ and $K_1(y) = K^{(m)}(y, q) \otimes 1$. The dependence of each $R$ matrix on $l, m$ should be understood appropriately. The composition of (35) and the inverse of (36) gives a map on $V_{l,x} \otimes V_{m,y}$ commuting with $\Delta B_q$. Then the second assertion in Theorem 1 tells that it must be a scalar multiple of the identity operator. The scalar is 1 due to the normalization (18) and (34). In this way, we obtain the reflection equation

$$K_1(x)R^*((xy)^{-1})K_1(y)R(xy^{-1}) = R^{*}(xy^{-1})K_1(y)R^*((xy)^{-1})K_1(x) \quad (37)$$

of the linear operators $V_{l,x} \otimes V_{m,y} \to V_{l,x-1}^* \otimes V_{m,y-1}^*$ for the intertwiner $K(z)$ characterized by the first assertion in Theorem 1. In short, Theorem 1 achieves linearization; the reflection equation which is quadratic in $K(z)$ becomes a corollary of the linear intertwining relation (32). In terms of matrix elements (37) reads

$$\sum_{a_0, b_0, b_3, b_2, a_1, a_2} K(x)_{a_2}^a_{b_2} R^*((xy)^{-1})_{b_2, b_1}^b K(y)_{b_1} R^*((xy)^{-1})_{a_1, a_0}^a = \sum_{a_0, b_0, b_3, b_2, a_1, a_2} R^{*}(xy^{-1})_{a_2, a_1}^a K(y)_{b_1} R^*((xy)^{-1})_{a_1, b_0}^a K(x)_{a_0}, \quad (38)$$

where $a_0, a_3 \in B_l, b_0, b_3 \in B_m$ and the sums range over $a_1, a_2 \in B_l, b_1, b_2 \in B_m$ on the both sides. On the LHS (resp. RHS), they are to obey the weight conservation $a_1 + b_1 = a_0 + b_0, a_1 - b_2 = a_2 - b_3$ (resp. $a_1 - b_0 = a_2 - b_1, a_2 + b_2 = a_3 + b_3$).

**Remark 2.** For the coideal subalgebra generated by $-c_i + c_i k_i f_i + d_i k_i$ with $c_i d_i \neq 0 (\forall i \in \mathbb{Z}_n)$, a necessary condition for the existence of $K(z): V_{i,z} \to V_{i,w}^*$ with $n \geq 3$ is

$$\prod_{i \in \mathbb{Z}_n} c_i = q^{2n} z w^{-1}, \quad d_i^2 = \frac{c_i}{(1 - q)^2}. \quad (39)$$
Such cases can always be reduced to \(39\) by applying an algebra automorphism \(\omega : e_i \mapsto \mu_i e_i, f_i \mapsto \mu_i^{-1} f_i, k_i^{\pm 1} \mapsto k_i^{\pm 1}\) of \(U_q\) for appropriate constants \(\mu_i\). For \(n = 2\), the intertwiner exists without assuming the right constraint in \(39\). However a matrix product formula for such a case is not known in general.

4. Matrix product construction

Let \(A_q\) be the algebra generated by \(a^+, a^-, k\) obeying the relations

\[
ka^+ = q a^+ k, \quad ka^- = q^{-1} a^- k, \quad a^+ a^- = 1 - k, \quad a^- a^+ = 1 - qk.
\]

The algebra \(A_q\) will be called \(q\)-boson. It is equipped with an anti-algebra automorphism

\[
\iota : a^+ \mapsto a^-, \quad k \mapsto k.
\]

Let \(F_q = \bigoplus_{m \geq 0} \mathbb{C} \langle m \rangle\) and \(F_q^* = \bigoplus_{m \geq 0} \mathbb{C} \langle m \rangle\) be the Fock space and its dual equipped with the bilinear pairing \(\langle m|m' \rangle = (q; q)_m \delta_{m, m'}\). They can be endowed with an \(A_q\) module structure by

\[
\langle m|a^+ = \langle m+1|, \quad \langle m|a^- = \langle m+1|, \quad \langle m|a^+ = \langle m-1|j(1-q^m), \quad \langle m|k = \langle m|q^m.
\]

It satisfies \((\langle m|X|m' \rangle)\). We also use \(h\) acting on the Fock spaces as \(h|m \rangle = m|m \rangle\) and \(\langle m|h = \langle m|m \rangle\). Thus one may set \(k = q^h\). By definition, the trace on \(F_q\) means \(\text{Tr}(w^h X) = \sum_{m \geq 0} w^m \langle m|X|m \rangle\) when convergent. The traces appearing in the sequel are always reduced to and evaluated by \(\text{Tr}(w^h k^r) = \frac{1}{1-w^h}\) for some \(w\) and \(r\) in \(\mathbb{Z}\) by relation \(10\).

For each pair \((i, j) \in \mathbb{Z}_2^2\), define an element \(G^i_j \in A_q\) by

\[
G^i_j = \begin{cases} (-q^i q^{-j}) \phi \left( \begin{array}{cc} q^{-j} & -q^i \end{array} ; q^i \right) \langle a^- \rangle^{j-i} \phi \left( \begin{array}{cc} q^{-i} & -q^i \end{array} ; q^i \right) \langle a^+ \rangle^{j-i} & (i \geq j), \\ (-q^i q^{-j}) \phi \left( \begin{array}{cc} q^{-j} & -q^i \end{array} ; q^i \right) \langle a^- \rangle^{i-j} \phi \left( \begin{array}{cc} q^{-i} & -q^i \end{array} ; q^i \right) \langle a^+ \rangle^{i-j} & (i \leq j), \end{cases}
\]

where \(\phi\) is a shorthand for the \(q\)-hypergeometric series

\[
\phi \left( \begin{array}{cc} a & b \end{array} ; c \right) z = \sum_{m \geq 0} (a; q)_m (b; q)_m c^m z^m.
\]

The RHS of \(42\) is terminating and actually involves finitely many terms. Note the properties

\[
G^i_j = \iota(G^j_i), \quad w^h G^i_j = G^i_j w^{j+i-h}.
\]

**Theorem 3.** The \(K\) matrix characterized by \(26\) and \(28\) has the elements expressed by the matrix product formula:

\[
K(z)_{\alpha \gamma} = \frac{q^{(\gamma, \alpha)}(q^{-i} z^{-1}; q)_{l+1}}{(q^2 ; q^2)_l (1-q z^{-1}; q)_l} \text{Tr} \left( q^i z^{-h} G^{\gamma_1}_{\alpha_1} G^{\gamma_2}_{\alpha_2} \cdots G^{\gamma_n}_{\alpha_n} \right) \tag{45}\]

Due to the right property in \(24\) and \(l^2 = |\alpha|^2 = \sum_{i=1}^n \alpha_i^2 + 2 (\alpha, \alpha)\) for \(\alpha \in B_l\), formula \(45\) is also written as:

\[
K(z)_{\alpha \gamma} = \frac{q^{\frac{1}{2} l^2 (q^{-i} z^{-1}; q)_{l+1}}}{(q^2 ; q^2)_l (1-q z^{-1}; q)_l} \text{Tr} \left( z^{-h} G^{\gamma_1}_{\alpha_1} G^{\gamma_2}_{\alpha_2} \cdots G^{\gamma_n}_{\alpha_n} \right) = q^{-\frac{1}{2} l^2} k^{-1} G^i_j,
\]

where the prefactor of the trace is independent of \(\alpha\) and \(\gamma\). Let us sketch a (rather brute force) proof. Substitute \(15\) into \(33\). Applying the right relation in \(14\) and \(\langle \gamma \pm \epsilon_i \pm \epsilon_{i+1}, \alpha \rangle - \langle \gamma, \alpha \rangle = \pm (\alpha_{i+1} - \delta_{i, 0}),\n\langle \gamma, \alpha \pm \epsilon_i \pm \epsilon_{i+1} \rangle - \langle \gamma, \alpha \rangle = \pm (\alpha - \delta_{i, 0})\), we find that \(35\) follows from the \(\delta_{0, 0}\)-free relation:

\[
q^{-\gamma_1} [a_{\alpha_{\gamma_1}} G^{\gamma_2}_{\alpha_{\gamma_2}} = q^{\gamma_1 + \alpha_{\gamma_1} - \alpha_{\gamma_2}} [a_{\alpha_{\gamma_2}} G^{\gamma_2}_{\alpha_{\gamma_2}} + \frac{q^{\alpha_{\gamma_1} - \alpha_{\gamma_2} + 1}}{1-q} G^{\gamma_2}_{\alpha_{\gamma_2}},
\]

which is \(47\) follows from the \(\delta_{0, 0}\)-free relation:

\[
q^{-\gamma_1} [a_{\alpha_{\gamma_1}} G^{\gamma_2}_{\alpha_{\gamma_2}} = q^{\gamma_1 + \alpha_{\gamma_1} - \alpha_{\gamma_2}} [a_{\alpha_{\gamma_2}} G^{\gamma_2}_{\alpha_{\gamma_2}} + \frac{q^{\alpha_{\gamma_1} - \alpha_{\gamma_2} + 1}}{1-q} G^{\gamma_2}_{\alpha_{\gamma_2}}.
\]

Substitute \(42\) into \(47\) and remove a common factor after applying the \(q\)-commutation relations in \(40\). Regarding integer powers of \(q\) as generic variables, one is left to show quadratic relations of the
$q$-hypergeometric series. Below we illustrate a typical case $\alpha_1 > \gamma_1$ and $\alpha_2 < \gamma_2$. (The invariance of (14) by $i$ in (11) reduces the task in the proof to some extent.) The relevant quadratic relation reads

$$
0 = u_1 (u_2 - u_1^2) (-v_1 - q) (q - u_1^2 v_1^2 w: q) \phi (u_1, -u_1; w) \phi (q u_2, -q u_2; -q v_2; y) \\
+ v_1 u_1 v_1 (-u_1 - v_1) (-v_2 - q) \phi (u_1, -u_1; w) \phi (q^{-1} u_2, -q^{-1} u_2; -q v_2; y) \\
- u_1^{-1} (u_2 u_1^2 - u_2 - u_1 v_2) (-v_1 - q) \phi (q^{-1} u_1, -q^{-1} u_1; -q v_1; w) \phi (u_2, -u_2; -q v_2; y) \\
- u_1 (u_1 - u_1) (-v_1 - q) (q - u_2^2 v_1^2 w; q) \phi (q u_1, -q u_1; -q v_2; w) \phi (u_2, -u_2; y) \\
- (1 + q) u_1 u_2 (-v_1 - v_2) (1 - v_1 - v_2) (1 - q - q^{-1} u_2 v_1^2 w) \phi (u_1, -u_1; w) \phi (u_2, -u_2; -v_2; y)
$$

with $y = u_1^2 v_1 - u_2^2 v_2 w$. Applying Heine’s contiguous relations to the factors $\phi (\phi, \phi; w)$, one can rewrite

the RHS as $A \phi (q - q^{-1} u_1; w) + B \phi (u_1, -q^{-1} u_1; w)$ with $A, B$ being linear combinations in $\phi (\phi, \phi; y)$. Then it is straightforward, though tedious, to check $A = 0, B = 0$ by (43). We remark that all the relations like (48) hold for generic $u_1, v_1$, hence for nonterminating $q$-hypergeometric series.

4.1. Basic properties and examples. From the matrix product formula (45) it is easy to derive

$$
K(z)^\alpha_1 = z^{\alpha_1 - \gamma_1} K(z)^{\sigma(\gamma_1)}_\rho(\gamma_1),
$$

$$
K(z)^\alpha_2 = K(z)^{\gamma(\alpha)}_{\alpha(\gamma)} \text{ if } \alpha_i = \gamma_i = 0.
$$

The array $\alpha^{(i)} \in \mathbb{Z}^{n-1}_+$ is obtained from $\alpha \in \mathbb{Z}^n_+$ by dropping the $i$th component $\alpha_i$. The equality (50) is due to $G_0^T = 1$ and $(\gamma^{(i)}, \alpha^{(i)}) = (\gamma, \alpha)$ when $\alpha_i = \gamma_i = 0$. It implies a reduction with respect to rank $n$ when some components are simultaneously zero. In what follows we present the result of an explicit evaluation of (45) for a few typical cases.

Example 4. Consider $U_q(A_1^{(1)})$ for general $l$. Due to (10), $K(z)^{\gamma_1, \gamma_2}_{\alpha_1, \alpha_2} = z^{\gamma_2 - \alpha_2} K(z)^{\gamma_1, \alpha_2}$ holds. Thus, we present the result assuming $s := \gamma_1 - \alpha_1 = \alpha_2 - \gamma_2 \geq 0$ without loss of generality.

$$
K(z)^{\gamma_1, \gamma_2}_{\alpha_1, \alpha_2} = q^{\gamma_1 z_1 \gamma_2 z_2 - \gamma_1 \gamma_2} (q^{-1} z_1^{-1} q)_{l+1} (q z_1 q)_{s+1} (q^{-1} z_1 q)_{s+1} (q z_1 q)_{s+1} (q^{-1} z_1 q)_{s+1} (q z_1 q)_{s+1} \times \\
\sum_{0 \leq j} \sum_{0 \leq k} q^{z_1 - k} (q^{-2} z_2 - q^2 z_2)_{s+1} (q z_1 q)_{s+1} (q^{-1} z_1 q)_{s+1} (q z_1 q)_{s+1} (q^{-1} z_1 q)_{s+1} (q z_1 q)_{s+1} \times \\
\sum_{0 \leq j} \sum_{0 \leq k} q^{z_1 - k} (q^{-2} z_2 - q^2 z_2)_{s+1} (q z_1 q)_{s+1} (q^{-1} z_1 q)_{s+1} (q z_1 q)_{s+1} (q^{-1} z_1 q)_{s+1} (q z_1 q)_{s+1} \times \\
\sum_{0 \leq j} \sum_{0 \leq k} q^{z_1 - k} (q^{-2} z_2 - q^2 z_2)_{s+1} (q z_1 q)_{s+1} (q^{-1} z_1 q)_{s+1} (q z_1 q)_{s+1} (q^{-1} z_1 q)_{s+1} (q z_1 q)_{s+1}.
$$

Example 5. Consider $U_q(A_{n-1}^{(1)})$ with $l = 1$. The relevant matrix product operators are

$$
G_0^0 = 1, \quad G_0^1 = (1 + q) a^+, \quad G_1^0 = (1 + q) a^- , \quad G_1^1 = (1 + q) (1 + q^2) \left( 1 - \frac{q (1 + q)}{1 + q^2} \right) k.
$$

Thus, formula (45) yields

$$
K(z)^{0 \alpha_1}_{\alpha_1} = \frac{z^{\delta_{ij}} + q}{z + q} z^{\theta(i < j)}.
$$

In fact this is the $l = 1$ case of more general

$$
K(z)^{\delta \alpha_1}_{\alpha_1} = \frac{(-q z^{-\delta_{ij}}; q)_l z^{-\theta(i < j)}}{(-q z^{-1}; q)_l}.
$$

Example 6. Consider $U_q(A_{n-1}^{(1)})$ with $l = 2$. In view of (10) and (50), the matrix elements that are not covered by Examples 4 and 5 are reduced to the following cases of $n = 3$:

$$
K(z)^{000}_{011} = \frac{(1 + q)^2}{(q + z)(q^2 + z)}, \quad K(z)^{101}_{101} = \frac{(1 + q)(1 + q + q^2 + q z^2)}{(1 + q^2)(q + z)(q^2 + z)}, \quad K(z)^{101}_{101} = \frac{(1 + q)(q + z + q^2 + q z^2)(1 + q^2)(q + z)(q^2 + z)}{1 + q^2(q + z)(q^2 + z)}.
$$

Let us close the section with the conjecture

$$
\lim_{q \to 0} K(z)^{\gamma_1}_{\alpha_1} = z^{-Q_0(\gamma, \alpha)},
$$

where $Q_0(\gamma, \alpha)$ is defined after (87). This indicates that the present gauge as well as the one treated in Section 8 also has a curious connection to the crystal theory [8, 9, 14, 17].
5. Parametric Generalization

5.1. Factorization at special point. The function \( \Phi_q(\gamma|\beta; 1, \mu) = \Phi_q(\gamma|\beta; \mu, \mu) = \delta_{\gamma,0} \).

Applying it to (21) and (23)–(25), we get
\[
R(q^{-m+l})_{\alpha,\beta} = \delta^{\gamma+\delta}_{\alpha+\beta} \theta(\delta \leq \alpha) q^{(\gamma,\alpha-\delta)+(\alpha-\delta,\delta)} \left( \frac{l}{m} \right)^{-1} \prod_{i=1}^{n} \left( \alpha_i \delta_i \right)_{q^2} (l \geq m),
\]
\[
R^*(q^{-m+l})_{\alpha,\beta} = \delta^{\gamma-\delta}_{\alpha-\beta} q^{(\gamma,\alpha)+(\gamma,\beta)} \left( \frac{l+m}{m} \right)^{-1} \prod_{i=1}^{n} \left( \alpha_i + \delta_i \right)_{q^2},
\]
\[
R^{**}(q^{-m+l})_{\alpha,\beta} = \delta^{\gamma+\delta}_{\alpha+\beta} \theta(\alpha \leq \delta) q^{(\alpha,\beta-\gamma)+(\beta-\gamma,\gamma)} \left( \frac{m}{l} \right)^{-1} \prod_{i=1}^{n} \left( \delta_i \right)_{q^2} (l \leq m),
\]
where we assume \( \alpha, \gamma \in B_l \) and \( \beta, \delta \in B_m \) in all the cases. Up to an overall factor, (58) is due to [93 Th.2]. By the argument similar to the proof of it there, one can show that the \( K \) matrix also has the factorization
\[
K(q^{-l})_{\alpha,\beta} = \prod_{i=1}^{n} \frac{(-q; q)_{\alpha_i+\gamma_i}}{(-q; q)_l}, \quad K(1)_{\alpha,\beta} = \prod_{i=1}^{n} \frac{(-q; q)_{\alpha_i}}{(-q; q)_l} (\alpha, \gamma \in B_l). \tag{59}
\]

5.2. Upgrading \( \lambda = q^{-l}, \mu = q^{-m} \) to generic parameters. In the reflection equation [57], specialize the spectral parameters to \( x = q^{-l}, y = q^{-m} \). Assuming \( l \geq m \), one finds that all the \( R \) and \( K \) matrices have the factorized elements given in the previous subsection. (Note that (58) should be applied after the exchange \( l \leftrightarrow m \).) Apart from the powers of \( q \), (56)–(58) consist of the \( q^2 \)-multinomial \( (q^2; q^2)^{\alpha_1} \left( \prod_{i=1}^{n} q^2 \right)^{\alpha_1} \), \( (q^2; q^2)^{\beta_1} \left( \prod_{i=1}^{n} q^2 \right)^{\beta_1} \), \( (q^2; q^2)^{\gamma_1} \left( \prod_{i=1}^{n} q^2 \right)^{\gamma_1} \), \( (q^2; q^2)^{\delta_1} \left( \prod_{i=1}^{n} q^2 \right)^{\delta_1} \), for \( \alpha, \beta \in B_l \). Here \( \overline{\alpha} \) is a truncation of \( \alpha \) explained after (21). Similar rewriting is possible also for (59). The powers of \( q \) are handled by \( \langle \alpha, \beta \rangle = \overline{\alpha} + \overline{\beta} [\overline{m} - \overline{\beta}] \) for \( \beta \in B_m \). Then from the argument similar to [93 Sec.2.3], it follows that the reflection equation, as well as the Yang-Baxter equation, holds as an identity of a rational function in which \( \lambda = q^{-l} \) and \( \mu = q^{-m} \) are regarded as generic parameters independent of \( q \). Local spin variables in such a setting range over \( \overline{\alpha} \in \mathbb{Z}^n_{+} \) rather than \( \alpha \in B_l \). Below we describe the resulting \( R \) and \( K \) matrices resetting \( \overline{\alpha} \in \mathbb{Z}^n_{+} \) to a simpler notation \( \alpha \in \mathbb{Z}^n_{+} \).

For \( k \geq 1 \), introduce the infinite dimensional space
\[
W = \bigoplus_{\alpha \in \mathbb{Z}^n_{+}} \mathbb{C}(q, \lambda, \mu)u_{\alpha}. \tag{60}
\]
Consider the linear operators depending on the continuous parameters \( \lambda, \mu \) as
\[
\mathcal{K}(\lambda) \in \text{End}(W), \quad \mathcal{R}(\lambda, \mu), \mathcal{R}^*(\lambda, \mu), \mathcal{R}^{**}(\lambda, \mu) \in \text{End}(W \otimes W), \tag{61}
\]
\[
\mathcal{K}(\lambda)u_{\alpha} = \sum_{\gamma \in \mathbb{Z}^n_{+}} \mathcal{K}(\lambda)_{\alpha,\gamma} u_{\gamma}, \quad \mathcal{Q}(\lambda, \mu)(u_{\alpha} \otimes u_{\beta}) = \sum_{\gamma, \delta \in \mathbb{Z}^n_{+}} \mathcal{Q}(\lambda, \mu)_{\alpha,\beta} u_{\delta} \otimes u_{\gamma}, \tag{62}
\]
where \( \mathcal{Q} = \mathcal{R}, \mathcal{R}^*, \mathcal{R}^{**} \). The matrix elements are defined by
\[
\mathcal{K}(\lambda)_{\alpha,\gamma} = q^{\langle \gamma, \alpha \rangle + \frac{1}{2} \left| \alpha \right| (\left| \alpha \right| - 1) + \frac{1}{2} \left| \gamma \right| (\left| \gamma \right| - 1) - \left| \alpha + \gamma \right|} \left( \frac{1}{\lambda - q^2} \right), \tag{63}
\]
\[
\mathcal{R}^{**}(\lambda, \mu)_{\alpha,\beta} = \mathcal{R}(\lambda, \mu)^{\langle \beta, \delta \rangle} (\rho_{\alpha}^{\beta}) (\rho_{\alpha}^{\beta}) = q^{\langle \gamma, \delta \rangle - \left| \alpha \right| (\left| \alpha \right| - 1) - \left| \gamma \right| (\left| \gamma \right| - 1) - \left| \delta \right| (\left| \delta \right| - 1) - \langle \beta, \delta \rangle} \mathcal{F}(\lambda, \mu) \mathcal{Q}, \tag{64}
\]
\[
\mathcal{R}^*(\lambda, \mu)_{\alpha,\beta} = \delta^{\gamma - \delta}_{\alpha - \beta} q^{\langle \gamma, \delta \rangle + \langle \beta, \delta \rangle + \left| \alpha \right| (\left| \alpha \right| - 1) - \left| \beta \right| (\left| \beta \right| - 1) - \left| \gamma \right| (\left| \gamma \right| - 1)} \mathcal{F}(\lambda, \mu), \tag{65}
\]
where \( \mathcal{F}_{q^2} \) is given by (20). Then, the Yang-Baxter equations and the reflection equation are valid:
\[
(1 \otimes \mathcal{R}(\lambda, \mu))(\mathcal{R}(\lambda, \nu) \otimes 1)(1 \otimes \mathcal{R}(\mu, \nu)) = (\mathcal{R}(\mu, \nu) \otimes 1)(1 \otimes \mathcal{R}(\lambda, \nu))(\mathcal{R}(\lambda, \mu) \otimes 1), \tag{66}
\]
\[
(1 \otimes \mathcal{R}^*(\lambda, \mu))(\mathcal{R}^*(\lambda, \nu) \otimes 1)(1 \otimes \mathcal{R}^*(\mu, \nu)) = (\mathcal{R}^*(\mu, \nu) \otimes 1)(1 \otimes \mathcal{R}^*(\lambda, \nu))(\mathcal{R}^*(\lambda, \mu) \otimes 1), \tag{67}
\]
\[
(1 \otimes \mathcal{R}^{**}(\lambda, \mu))(\mathcal{R}^{**}(\lambda, \nu) \otimes 1)(1 \otimes \mathcal{R}^{*}(\mu, \nu)) = (\mathcal{R}^{*}(\mu, \nu) \otimes 1)(1 \otimes \mathcal{R}^{**}(\lambda, \nu))(\mathcal{R}^{**}(\lambda, \mu) \otimes 1), \tag{68}
\]
\[
(1 \otimes \mathcal{R}^{**}(\lambda, \mu))(\mathcal{R}^{**}(\lambda, \nu) \otimes 1)(1 \otimes \mathcal{R}^{**}(\mu, \nu)) = (\mathcal{R}^{**}(\mu, \nu) \otimes 1)(1 \otimes \mathcal{R}^{**}(\lambda, \nu))(\mathcal{R}^{**}(\lambda, \mu) \otimes 1), \tag{69}
\]
\[
\mathcal{K}_1(\lambda) \mathcal{R}^*(\mu, \lambda) \mathcal{K}_1(\mu) \mathcal{R}(\lambda, \mu) = \mathcal{R}^{**}(\mu, \lambda) \mathcal{K}_1(\mu) \mathcal{R}^*(\mu, \lambda) \mathcal{K}_1(\lambda). \tag{70}
\]
The Yang-Baxter (resp. reflection) equations hold as identities of the operators on $W^\otimes 3$ (resp. $W^\otimes 2$). The result [69] was obtained in [13] Sec.2.3 up to a gauge of $R^\ast(\lambda, \mu)$. Two remarks are in order. 

(i) $K(\lambda)$ and $R^\ast(\lambda, \mu)$ are not locally finite in that the corresponding RHS of [62] contains infinitely many terms. However, the Yang-Baxter and the reflection equations make sense as the identities of matrix elements which are finite for any prescribed transitions $u_{\alpha} \otimes u_{\beta} \otimes u_{\gamma} \mapsto u_{\alpha'} \otimes u_{\beta'} \otimes u_{\gamma'}$ and $u_{\alpha} \otimes u_{\beta} \mapsto u_{\alpha'} \otimes u_{\beta'}$.

(ii) The Yang-Baxter equations (63) - (64) remain valid under the replacement

$$R(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta} \mapsto q^{\varphi_1(\delta, \gamma) - \varphi_1(\alpha, \beta)} (\lambda/\mu)^{\varphi_2(\gamma - \alpha)} R(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta},$$

$$R^\ast(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta} \mapsto q^{\varphi_2(\alpha, \delta) - \varphi_1(\beta, \gamma)} (\lambda/\mu)^{\varphi_3(\gamma - \alpha)} R^\ast(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta},$$

$$R^\ast(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta} \mapsto q^{\varphi_2(\alpha, \delta) - \varphi_1(\beta, \gamma)} (\lambda/\mu)^{\varphi_3(\gamma - \alpha)} R^\ast(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta},$$

where $\varphi_1$ (resp. $\varphi_2, \varphi_3$) is any bilinear (resp. linear) function. This can be utilized to simplify (61) and (65) to some extent. However there is no bilinear function $\varphi(\cdot, \cdot)$ such that the transformation $K(\lambda)_{\alpha} \mapsto q^{\varphi(\gamma, \alpha)} K(\lambda)_{\alpha}$ combined with (61) - (63) preserves the reflection equation.

6. Another gauge

The results in Sections 2 and 3 can also be stated in another gauge which suits the study of the limit $q \to 0$ in relation to the crystal theory [8].

6.1. Representation $\pi_{l, z}^\gamma$ and associated $R$ matrices. Consider the representation ([10] eq.(2), [12] eq.(3.14))

$$\pi_{l, z}^\gamma : U_q \to \text{End}(V_{l, x}^\gamma), \quad V_{l, x}^\gamma = \bigoplus_{\alpha \in B_l} \mathbb{C}(q, z)v_{\alpha}^\gamma,$$

$$c_j v_{\alpha}^\gamma = z^{\delta_\alpha \beta}[\alpha_j]_{l, x} v_{\alpha - e_j + e_{j+1}}^\gamma, \quad \lambda_j v_{\alpha}^\gamma = z^{-\delta_\alpha \beta}[\alpha_j]_{l, x} v_{\alpha + e_j - e_{j+1}}^\gamma, \quad k_j v_{\alpha}^\gamma = q^{-\alpha_j + \alpha_{j+1}} v_{\alpha}^\gamma,$$

where again $\pi_{l, z}(g)$ are abbreviated to $g$. It is easy to see the equivalence

$$\pi_{l, z}^\gamma \simeq \pi_{l, (-q)^n z}^\gamma \quad \text{via the identification} \quad v_{\alpha}^\gamma = (-q)^{\alpha} \prod_{i=1}^n (q^2; q^2)_{\alpha_i} v_{\alpha}^*$$

by means of $\{\alpha \pm (e_i - e_{i+1})\} - \{\alpha\} = \pm (-1 + n\delta_0)$. See [11] for the definition of the symbol $\{\alpha\}$. Denote the counterparts of the $R$ matrices in [13] and [14] by

$$R^\gamma(x/y) : V_{l, x}^\gamma \otimes V_{m, y} \to V_{m, y} \otimes V_{l, x}^\gamma, \quad (\pi_{m, y} \otimes \pi_{l, x}^\gamma) R^\gamma(x/y) = R^\gamma(x/y)(\pi_{l, x}^\gamma \otimes \pi_{m, y}),$$

$$R^\gamma(x/y) : V_{l, x}^\gamma \otimes V_{m, y} \to V_{m, y} \otimes V_{l, x}^\gamma, \quad (\pi_{m, y} \otimes \pi_{l, x}^\gamma) R^\gamma(x/y) = R^\gamma(x/y)(\pi_{l, x}^\gamma \otimes \pi_{m, y}).$$

Under the normalization $R^\gamma(z)_{\{e_i, m_{e_i}\}} = R^\gamma(z)_{\{e_i, m_{e_i}\}} = 1$ as in [13], their matrix elements are given by

$$R^\gamma(z)_{\alpha, \beta}^{\gamma, \delta} = \delta_{\alpha - \beta}^{\gamma - \delta} (-q)^{\beta - \gamma} \prod_{i=1}^n (q^2; q^2)_{\alpha_i} A((-q)^n z^{-1})^{\beta, \gamma}_{\delta, \alpha}, \quad R^\gamma(z)_{\alpha, \beta}^{\gamma, \delta} = \delta_{\alpha + \beta}^{\gamma + \delta} A(z)^{\gamma, \delta}_{\alpha, \beta}.$$

The above formula for $R^\gamma(z)_{\alpha, \beta}^{\gamma, \delta}$ was obtained in [1] extending the result of [13]. The one for $R^\gamma(z)_{\alpha, \beta}^{\gamma, \delta}$ and [27] - [29] can be deduced from it by applying the crossing symmetry and the results in [12] especially eqs.(2.7), (2.42) and Th.3.1 therein. The Yang-Baxter equations [27] - [29] with $\ast$ replaced by $\forall$ are valid.

6.2. $K$ matrix and reflection equation. From now on, we set

$$q = -p^2$$

but allow coexistence of $q$ and $p$ when it eases the presentation. Let $B'_q$ be the right coideal subalgebra of $U_q$ generated by

$$b'_i = e_i + qk_i f_i + \frac{p}{1-q} k_i \in U_q \quad (i \in \mathbb{Z}_n).$$

This is related to $b_i$ in [30] via $b'_i = -p^{-1} \omega(b_i)$ where $\omega$ denotes the automorphism mentioned in Remark 2 with $\forall \mu_i = p$. Let

$$K'(z) : V_{l, x} \to V_{l, x-1}^\gamma, \quad K'(z)v_{\alpha} = \sum_{\gamma \in B_l} K'(z)^{\gamma, \gamma}_{\alpha} v_{\gamma}$$

(81)
be the unique map satisfying the intertwining relation
\[ K'(b)\pi_{l,z}(b) = \pi_{l,z}^{\gamma}(b)K'(z) \quad (b \in B'_q) \]
and the normalization \( K'(z)_{l,e_1}^{e_1} = 1 \). From the construction so far we find that its matrix elements are related to those of \( K(z) \) as
\[ K'(z)^{\alpha}_{\gamma} = p^{(\alpha - \gamma)} \prod_{i=1}^{q^2}(q^2)^{y_i}K(p^\gamma z)^{\alpha}_{\gamma} \quad (\alpha, \gamma \in B_l). \]
Similarly to \( \Box \), it satisfies the reflection equation
\[ K'_1(x)R^\vee((xy^{-1})K'_1(y)R(xy^{-1}) = R^\vee(xy^{-1})K'_1(y)R^\vee((xy^{-1})K'_1(x) \]
as linear operators \( V_{l,x} \otimes V_{m,y} \rightarrow V_{l,x-1}^\vee \otimes V_{m,y-1}^\vee \).

6.3. Combinatorial \( R \) and \( K \) at \( q = 0 \). At \( q = 0 \), the \( R \) matrices survive nontrivially as
\[ \lim_{q \rightarrow 0} R(z)^{\gamma}_{\alpha, \beta} = \theta(R(\beta \otimes \alpha) = \gamma \otimes \delta) z^{-Q_0(\beta, \alpha)}, \]
\[ \lim_{q \rightarrow 0} R^\vee(z)^{\gamma}_{\alpha, \beta}/R(z)^{\delta}_{\epsilon_1, \epsilon_2} = \theta(R(\beta \otimes \alpha) = \gamma \otimes \delta) z^{-P_0(\beta, \alpha)}, \]
\[ \lim_{q \rightarrow 0} R^\vee(z)^{\gamma}_{\alpha, \beta} = \theta(R(\beta \otimes \alpha) = \gamma \otimes \delta) z^{-Q_0(\beta, \alpha)}, \]
where \( P_i(\alpha, \beta) = \min(\alpha_{i+1}, \beta_i) \), \( Q_i(\alpha, \beta) = \min_{1 \leq k \leq n} \left\{ \sum_{1 \leq j < k} \alpha_{i+j} + \sum_{k \leq j \leq \beta_{i+j}} \right\} \). The denominator in the second formula is given by \( R^\vee(z)^{\delta}_{\epsilon_1, \epsilon_2} = (-1)^{m-z}(\pi(\beta \otimes \alpha) = \gamma \otimes \delta) z^{-Q_0(\beta, \alpha)} \) from \( \Box \). In the RHS, we regard \( \alpha, \gamma \in B_l \), \( \beta, \delta \in B_m \) as elements of crystals \( \Box \), and \( R, R^\vee, R^\vee \) denote the classical part of the combinatorial \( R \)’s defined in eqs.(2.1), (2.2) and (2.4) in \( \Box \), respectively. They are nontrivial bijections \( B_m \times B_l \rightarrow B_l \times B_m \) obeying the Yang-Baxter equations \( \Box \) eq.(2.7)]. The quantities \( P_i(\alpha, \beta), Q_i(\alpha, \beta) \) are versions of energy functions and known to play an important role \( \Box \).

As for the \( K \) matrix \( \Box \), it has the following behavior at \( q = -p^2 = 0 \):
\[ \lim_{q \rightarrow 0} K'(z)^{\gamma}_{\alpha} / K'(z)^{\delta}_{\epsilon_1, \epsilon_2} = \theta(\gamma = \sigma(\alpha)) z^{\alpha}. \]
The denominator here can be written down explicitly from \( \Box \) and \( \Box \). The transformation \( \alpha \mapsto \gamma = \sigma(\alpha) \) viewed as a bijection on \( B_l \) essentially reproduces the combinatorial \( K \) matrix introduced in \( \Box \) eq.(2.8)] to formulate the box-ball system with reflecting end. Together with the combinatorial \( R \)’s in the above, it forms a set-theoretical solution to the reflection equation. The latter is known to admit a further generalization to the birational maps \( \Box \) App.A. We conclude that the reflection equation \( \Box \), after exchange of the two components, achieves a \( q \)-melting of the combinatorial reflection equation \( \Box \) eq.(2.13)].

Example 7. Let \( n = 5 \). We denote \( v_{(2,1,0,2,0)} \in V_{5,1} \) by one-row semistandard tableau 11244 and similarly \( v_{(0,1,0,3,1)} \in V_{5,1} \) by 24445, etc. With a proper normalization at \( q = 0 \), the action of the two sides of \( \Box \) on a base vector 12235 \( \otimes \) 24 \( \in \) \( V_{3,1} \) proceed, according to \( \Box \), as follows:
\[ 12235 \otimes 124 \xrightarrow{R} 235 \otimes 11224 \xrightarrow{K_i} 124 \otimes 11224 \xrightarrow{R^\vee} 11235 \otimes 135 \xrightarrow{K_i} 12455 \otimes 135, \]
\[ 12235 \otimes 124 \xrightarrow{K_i} 11245 \otimes 124 \xrightarrow{R^\vee} 135 \otimes 11355 \xrightarrow{K_i} 245 \otimes 11355 \xrightarrow{R^\vee} 12455 \otimes 135. \]
The agreement of the output is an example of the set-theoretical reflection equation \( \Box \).

7. Summary and outlook

In Theorem \( \Box \) we have characterized a \( K \) matrix as the intertwiner of the coideal subalgebra \( B_q \) of \( U_q(\mathfrak{g}) \) generated by \( \Box \). By construction, it satisfies the reflection equation \( \Box \). In Theorem \( \Box \) we have constructed it in a matrix product form in terms of terminating \( q \)-hypergeometric series of \( q \)-boson generators.

At \( q = 0 \), the \( K \) matrix here reproduces one of the set-theoretical \( K \) matrices called “Rotateleft” in \( \Box \) eq.(2.10)]. When \( n \) is even, there are further solutions known as “Switch1n” and “Switch2” \( \Box \) eqs.(2.11), (2.12)] which also admit decent generalizations into geometric versions \( \Box \) App.A. To incorporate them into the framework of this Letter, possibly with some other coideal subalgebra, is a natural problem to be addressed. Another important theme is to explore the 3D aspects of the matrix
product (Theorem 3) from the viewpoint of [15]. It amounts to embedding the relations among the operators $G_j^i$ into some sort of quantized reflection equation. We hope to report on these issues elsewhere.

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