

2-VERMA MODULES

GRÉGOIRE N AISSE AND PEDRO VAZ

Abstract. We construct a categorification of parabolic Verma modules for symmetrizable Kac–Moody algebras using KLR-like diagrammatic algebras. We show that our construction arises naturally from a dg-enhancement of the cyclotomic quotients of the KLR-algebras. As a consequence, we are able to recover the usual categorification of integrable modules. We also introduce a notion of dg-2-representation for quantum Kac–Moody algebras, and in particular of parabolic 2-Verma modules.

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1. Introduction

The study of categorical actions of (quantum enveloping algebras of) Kac–Moody algebras leads to many interesting results. An impressive example is due to Chuang and Rouquier [11], who introduced categorical actions of $\mathfrak{sl}_2$ to prove the Broué abelian defect group conjecture for symmetric groups. Another interesting result is Webster's construction of homological versions of quantum invariants of links obtained by the Reshetikhin–Turaev machinery [45].

Until recently, only categorifications of integrable representations of quantum Kac–Moody algebras were known. These are given by additive (or abelian) categories, on which the quantum group acts by (exact) endofunctors respecting certain direct sum decompositions, corresponding to the defining relations of the algebra (see for example [15, 19, 25, 26, 38]). In [34], the authors followed a slightly different approach to

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construct a categorification of the universal Verma module $M(\lambda)$ for quantum $\mathfrak{sl}_2$. The construction of [34] is given in the form of an abelian, bigraded (super)category, where the commutator relation takes the form of a (non-split) natural short exact sequence

$$0 \to FE \to EF \to QK \oplus \Pi QK^{-1} \to 0,$$

where $\Pi$ is the parity shift functor, and $Q$ a categorification of $\frac{1}{q-q^{-1}}$ in the form of an infinite direct sum. This category is obtained as a certain category of modules over cohomology rings of infinite Grassmannians and their Koszul duals. Categorification of Verma modules appeared independently in the litterature with a strongly different flavor in [12] and in [5].

Studying the endomorphism ring of $F^k := F \circ \cdots \circ F$ yields a (super)algebra $A_k$ that extends the ubiquitous nilHecke algebra $NH_k$. This superalgebra was studied by the authors in the follow up [35], where it was used to construct an equivalent categorification of Verma modules for quantum $\mathfrak{sl}_2$. The supercenter of $A_k$ was also studied in [4]. The definition of the superalgebra $A_k$ and its supercenter were extended in [37] to the case of a Weyl group of type $B$.

The superalgebra $A_k$ comes equipped with a family of differentials $d_n$ for $n \geq 0$. The corresponding dg-algebras are formal, with homology being isomorphic to the $n$-cyclotomic quotients of the nilHecke algebra. These quotients are known to categorify the irreducible integrable $U_q(\mathfrak{sl}_k)$-representations $V(n)$ of highest weight $n$. We interpret this as a categorification of the universal property of the Verma module $M(\lambda)$, that is there is a surjection $M(\lambda) \twoheadrightarrow V(n)$ for all $n$. This also means the dg-algebra $(A_k, d_n)$ can be seen as a dg-enhancement of the cyclotomic nilHecke algebra $NH_k^n$, and in particular, of categorified $V(n)$.

In [22, 24] and [38], Khovanov–Lauda and Rouquier introduced generalizations of the nilHecke algebra for any Cartan datum. These algebras are presented in the form of braid-like diagrams in [22, 24], with strands labeled by simple roots and decorated with dots. It is proven in [22, 24, 38] that KLR algebras categorify the half quantum group associated with the input Cartan datum. Khovanov and Lauda conjectured that certain quotients of these algebras categorify irreducible, integrable representations of the quantum group. Due to the isomorphism between these quotient algebras and cyclotomic Hecke algebras in type $A$ (see [7, 38]), these quotients have become known as cyclotomic KLR algebras. The corresponding cyclotomic conjecture was first proven in [8, 9, 27] for some special cases, and then for all symmetrizable Kac–Moody algebras by Kang–Kashiwara in [19], and independently by Webster in [45].

In this paper, we introduce a version of KLR algebra associated to a pair $(\mathfrak{p}, \mathfrak{g})$, where $\mathfrak{p}$ is a (standard) parabolic subalgebra of a quantum Kac-Moody algebra $\mathfrak{g}$. This construction generalizes the algebra $A_k$ from [34], which we view as associated to the (standard) Borel subalgebra of $\mathfrak{sl}_2$. The usual KLR algebra is recovered by taking $\mathfrak{p} = \mathfrak{g}$. We prove that certain ‘cyclotomic quotients’ of these $\mathfrak{p}$-KLR algebras categorify parabolic Verma modules induced over the parabolic subalgebra $\mathfrak{p}$, with the cyclotomic quotient depending on the highest weight. The proof goes by showing first that if $\mathfrak{p} = \mathfrak{b}$ is the (standard) Borel subalgebra of $\mathfrak{g}$, then the $\mathfrak{b}$-KLR algebra is equipped with a categorical $\mathfrak{g}$-action similar
to the one constructed in [35]. In particular, it categorifies the universal Verma module of $\mathfrak{g}$. Next, we show that the $\mathfrak{b}$-KLR algebra can be equipped with a family of differentials, turning it into a dg-enhancement of the cyclotomic $\mathfrak{p}$-KLR algebras. This induces a categorical $\mathfrak{g}$-action on the cyclotomic $\mathfrak{p}$-KLR algebra. In particular, we recover the usual categorical action on cyclotomic KLR algebras, and we can reinterpret Kang–Kashiwara’s proof of Khovanov–Lauda’s cyclotomic conjecture in terms of dg-enhanced KLR algebras.

The world of dg-categories also allows to reinterpret the usual categorical $\mathfrak{sl}_2$-commutator relation in terms of mapping cones. More precisely, the derived category of dg-modules over the dg-enhanced KLR algebra comes equipped with functors $E_i$, $F_i$ and an autoequivalence $K_i$ for all simple root $\alpha_i$, that categorifies the action of the Chevalley generators $E_i$, $F_i$ and of the Cartan element $K_i = q_i^{H_i}$. Then, the $\mathfrak{sl}_2$-commutator relation of the categorical action takes the form of a quasi-isomorphism of mapping cones

$$\text{Cone}(F_i E_i \to E_i F_i) \cong \text{Cone}(Q_i K_i \to Q_i K_i^{-1}),$$

where $Q_i$ is a direct sum of grading shift copies of the identity functor that categories $\frac{1}{q_i} - q_i$. Whenever $F_i$ is locally nilpotent, $\text{Cone}(Q_i K_i \to Q_i K_i^{-1})$ is quasi-isomorphic to a finite direct sum of shifted copies of the identity functor, corresponding to the usual notion of an integrable categorical $\mathfrak{g}$-action (as in [19] for example).

Categorification of parabolic Verma modules have found connections with topology in the work of the authors in [33]. In particular, they have constructed Khovanov–Rozansky’s triply graded link homology using parabolic 2-Verma modules of $\mathfrak{gl}_{2k}$. On the decategorified level, the connection between the HOMFPY-PT link polynomial and Verma modules was not known before. We expect to find in the future more connections between categorified Verma modules and low-dimensional topology.

**Outline of the paper.** In Section 2, we recall the basics about quantum groups and their parabolic Verma modules.

In Section 3, we introduce the $\mathfrak{b}$-KLR algebra $R_\mathfrak{b}$ (Definition 3.3) as a diagrammatic algebra over a unital commutative ring $\mathbb{k}$, in the same spirit as Khovanov–Lauda’s [22]. We construct a faithful action on a polynomial ring and exhibit a basis, proving $R_\mathfrak{b}$ is a free $\mathbb{k}$-module.

In Section 4, we introduce the $\mathfrak{p}$-KLR algebra $R_\mathfrak{p}$ for any (standard) parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. We also introduce the corresponding $N$-cyclotomic quotient $R_\mathfrak{p}^N$. We introduce a differential $d_N$ on $R_\mathfrak{b}$, turning it into a dg-enhancement of $R_\mathfrak{p}^N$. In particular, we prove the following theorem:

**Theorem 4.4.** The dg-algebra $(R_\mathfrak{b}(m), d_N)$ is formal with homology

$$H(R_\mathfrak{b}(m), d_N) \cong R_\mathfrak{p}^N(m).$$

In Section 5, we construct a categorical action of $U_q(\mathfrak{g})$ on $R_\mathfrak{b}$, where the action of the Chevalley generators $F_i$ and $E_i$ is given by functors $F_i$ and $E_i$ which are defined in terms of induction and restriction functors for the map that adds a strand labeled $i$. The $\mathfrak{sl}_2$-commutator relation takes the form of a non-split natural short exact sequence. Let
Theorem 5.2. There is a natural short exact sequence
\[ 0 \to F_i E_i \Id_I \to E_i F_i \Id_I \to \bigoplus_{[\gamma_i, \alpha_i]} \Id_I \to 0, \]
for all \( i \in I \), and there is a natural isomorphism
\[ F_i E_j \cong E_j F_i, \]
for all \( i \neq j \in I \).

Fix \( \mathfrak{p} \subset \mathfrak{g} \), and let \( I_f \) be the set of simple roots for which \( F_i \in \mathfrak{p} \). Let \( \bigoplus_{[\gamma_i]} \Id_I \) be a finite direct sum of degree shifts of the identity functor that categorifies the quantum integer \( [n]_q \). The categorical \( \mathfrak{g} \)-action on \( R_b \) lifts to the dg-algebra \( (R_b, d_N) \), and thus to \( R_b^N \) by Theorem 4.4. The short exact sequence of Corollary 5.2 lifts to a short of exact sequence of complexes, inducing a long exact sequence in homology. This allows us to compute the action of the functors of induction \( F_i^N \) and restriction \( E_i^N \) on \( R_b^N \):

Theorem 5.17. For \( i \notin I_f \) there is a natural short exact sequence
\[ 0 \to F_i^N E_i^N \Id_I \to E_i^N F_i^N \Id_I \to \bigoplus_{[\gamma_i, \alpha_i]} \Id_I \to 0, \]
and for \( i \in I_f \) there are natural isomorphisms
\[ E_i^N F_i^N \Id_I \cong F_i^N E_i^N \Id_I \bigoplus_{[\gamma_i, \alpha_i]} \Id_I, \quad \text{if } n_i - \alpha_i \geq 0, \]
\[ F_i^N E_i^N \Id_I \cong E_i^N F_i^N \Id_I \bigoplus_{[\gamma_i, \alpha_i]} \Id_I, \quad \text{if } n_i - \alpha_i \leq 0. \]
Moreover, there is a natural isomorphism
\[ F_i^N E_j^N \cong E_j^N F_i^N, \]
for \( i \neq j \in I \).

In Section 6, we compute the asymptotic Grothendieck group of \((R_b, d_N)\). The asymptotic Grothendieck group is a refined version of Grothendieck group, that was introduced by the first author in [32]. It allows taking in consideration infinite iterated extensions of objects, such as infinite projective resolutions and infinite composition series (see Definition 6.3). Let \( M^p(\Lambda, N) \) be the parabolic Verma module of highest weight \((\Lambda, N)\), and \( M^p(\Lambda, N) \) be the c.b.l.f. derived category of \((R_b, d_N)\) (see Section 6.1).

Theorem 6.14. The asymptotic Grothendieck group \( \mathcal{QK}_0^\Lambda(\mathcal{M}^p(\Lambda, N)) \) is a \( U_q(\mathfrak{g}) \)-weight module, with action of \( E_i, F_i \) given by \([E_i], [F_i]\). Moreover, there is an isomorphism of \( \mathcal{QK} \)-modules
\[ \mathcal{QK}_0^\Lambda(\mathcal{M}^p(\Lambda, N)) \otimes \mathbb{Q} \cong M^p(\Lambda, N). \]

In Section 7, we introduce a notion of categorical dg-action of \( \mathfrak{g} \) on a pretriangulated dg-category (Definition 7.2), and of (parabolic) 2-Verma module (Definition 7.6). In particular, we show that \( \mathcal{M}^p(\Lambda, N) \) admits a dg-enhancement \( \mathcal{M}_d^p(\Lambda, N) \) in the form of a dg-category.
It yields an example of parabolic 2-Verma module, for which Theorem 6.14 takes the following form:

**Corollary 7.8.** For all \( i \in I \) there is a quasi-isomorphism of cones

\[
\text{Cone}(F_{i}^{N}E_{i}^{N}\text{Id}_{\nu} \to E_{i}^{N}F_{i}^{N}\text{Id}_{\nu}) \cong \text{Cone}(Q_{i}\lambda_{i}q_{i}^{-\alpha_{i}^{\vee}(\nu)}\text{Id}_{\nu} \to Q_{i}\lambda_{i}^{-1}q_{i}^{-\alpha_{i}^{\vee}(\nu)}\text{Id}_{\nu}),
\]

in \( \mathfrak{End}_{\mathfrak{Hq}}(\mathcal{D}_{d_{q}}(R_{b}, d_{N})) \).

Finally, in Appendix A we recall the construction of the homotopy category of dg-categories up to quasi-equivalence, based on Toen [42]. We also recall how to compute the (derived) dg-hom-spaces between pretriangulated dg-categories.

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## 2. Quantum Groups and Verma Modules

We recall the basics about quantum groups and their (parabolic) Verma modules. Our presentation is close to [18] and [29], where the proofs can be found. References for classical results about Verma modules are [30] and [17] (and [2] for the quantum case).

### 2.1. Quantum groups. A generalized Cartan matrix is a finite dimensional square matrix \( A = \{a_{ij}\}_{i,j \in I} \in \mathbb{Z}^{I \times I} \) such that

- \( a_{ii} = 2 \) and \( a_{ij} \leq 0 \) for all \( i \neq j \in I \);
- \( a_{ij} = 0 \iff a_{ji} = 0 \).

One says that \( A \) is symmetrizable if there exists a diagonal matrix \( D \) with positive entries \( d_{i} \in \mathbb{Z}_{>0} \) for all \( i \in I \), such that \( DA \) is symmetric. A Cartan datum consists of

- a symmetrizable generalized Cartan matrix \( A \);
- a free abelian group \( Y \) called the weight lattice;
- a set of linearly independent elements \( \Pi = \{\alpha_{i}\}_{i \in I} \subset Y \) called simple roots;
- a dual weight lattice \( Y^{\vee} := \text{Hom}(Y, \mathbb{Z}) \);
- a set of simple coroots \( \Pi^{\vee} = \{\alpha_{i}^{\vee}\}_{i \in I} \subset Y^{\vee} \);

such that

- \( \alpha_{i}^{\vee}(\alpha_{j}) = a_{ij} \);
- for each \( i \in I \) there is a fundamental weight \( \Lambda_{i} \in Y \) such that \( \alpha_{j}^{\vee}(\Lambda_{i}) = \delta_{ij} \) for all \( j \in I \).

The abelian subgroup \( X := \bigoplus_{i} \mathbb{Z}_{\geq 0} \alpha_{i} \subset Y \) is called the root lattice. We also write \( X^{+} := \bigoplus_{i} \mathbb{N} \alpha_{i} \subset X \) for the positive roots. Given a Cartan datum, since \( A \) is symmetrizable with \( d_{i}a_{ij} = d_{j}a_{ji} \), one can construct a symmetric bilinear form

\[
(-,-) : Y \times Y \to \mathbb{Z},
\]

respecting

- \( (\alpha_{i}, \alpha_{i}) = 2d_{i} \in \{2, 4, \ldots\} \);
• \( (\alpha_i|\alpha_j) = \delta_{ij} \) for all \( i \neq j \);
• \( p_i(y) = 2\frac{(\alpha_i|y)}{\alpha_i} \) for all \( y \in Y \).

In the end, a Cartan datum is completely determined by \((I, X, Y, (-|-))\).

**Definition 2.1.** The quantum Kac–Moody algebra \( U_q(g) \) associated to a Cartan datum \((I, X, Y, (-|-))\) is the associative, unital \( \mathbb{Q}(q) \)-algebra generated by the set of elements \( E_i, F_i \) and \( K_\gamma \) for all \( i \in I \) and \( \gamma \in Y^\vee \), with relations for all \( i \in I \) and \( \gamma, \gamma' \in Y^\vee \):

\[
K_0 = 1, \quad K_\gamma K_{\gamma'} = K_{\gamma+\gamma'},
\]

\[
K_\gamma E_i = q^{\gamma(\alpha_i)} E_i K_\gamma, \quad K_\gamma F_i = q^{-\gamma(\alpha_i)} F_i K_\gamma,
\]

One also imposes the \( \mathfrak{sl}_2 \)-commutator relation for all \( i, j \in I \):

\[
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},
\]

where \( q_i := q^{d_i} \) and \( K_\gamma := K_{\alpha_\gamma} \).

Finally, there are the Serre relations for \( i \neq j \in I \):

\[
\sum_{r+s=1-a_{ij}} (-1)^r \left[ 1 - \frac{a_{ij}}{r} \right] q_i E_i^r E_j^s = 0,
\]

\[
\sum_{r+s=1-a_{ij}} (-1)^r \left[ 1 - \frac{a_{ij}}{r} \right] q_i F_i^r F_j^s = 0.
\]

This ends the definition of \( U_q(g) \).

Given a sequence \( \mathbf{i} = i_1 \cdots i_m \) of elements in \( I \), we write \( F_i := F_{i_1} \cdots F_{i_m} \) and \( E_i := E_{i_1} \cdots E_{i_m} \). We write \( \text{Seq}(I) \) for the set of such sequences. Any element of \( U_q(g) \) decomposes as a sum of elements \( F_i K_\gamma E_j \) with \( \mathbf{i}, \mathbf{j} \in \text{Seq}(I) \).

The half quantum group \( U^-_q(g) \) of \( U_q(g) \) is the subalgebra generated by the elements \( \{F_i\}_{i \in I} \). As a \( \mathbb{Q}(q) \)-vector space, it admits a basis given by a subset of \( \{F_i\}_{i \in \text{Seq}(I)} \).

### 2.2 Weight modules.

Let \( M \) be an \( U_q(g) \)-module with ground ring \( R \supseteq \mathbb{Q}(q) \). Consider a \( \mathbb{Z} \)-linear functional

\[
\lambda: Y^\vee \to R^*,
\]

where the group structure on \( R^* \) is the product. For each such \( \lambda \) and \( y \in Y \), we call \((\lambda, y)\)-weight space the set

\[
M_{\lambda, y} := \{v \in M | K_\gamma v = \lambda(\gamma) q^{\gamma(y)} v \text{ for all } \gamma \in Y^\vee \}.
\]

Note that \( E_i M_{\lambda, y} \subseteq M_{\lambda, y + \alpha_i} \) and \( F_i M_{\lambda, y} \subseteq M_{\lambda, y - \alpha_i} \). A weight module is a module that decomposes as a direct sum of weight spaces. A highest weight module is a module \( M \) such that \( M = U_q(g)v_\lambda \) for some \( v_\lambda \in M_{\lambda, 0} \) with \( E_i v_\lambda = 0 \) for all \( i \in I \). In that case, we call \( \lambda \) the highest weight and we have

\[
M \cong \bigoplus_{y \in X^+} M_{\lambda - y}.
\]
as $R$-module.

One says that a $U_q(\mathfrak{g})$-module $M$ is integrable if for each $v \in M$ there exists $k \gg 0$ such that $E_i^k v = 0$ and $F_i^k v = 0$ for all $i \in I$. Any finite dimensional module is integrable, and any integrable module is a weight module with $\lambda(\Pi^\vee) \subset \mathbb{Z}[q]$. We consider only type 1 modules, that is $\lambda(\Pi^\vee) \subset \mathbb{N}[q]$.

Let $M$ be a highest weight module with highest weight vector $v_\lambda \in M_{\lambda,0}$. Then we set $\lambda_i := \lambda(\alpha_i^\vee)$ for each $i \in I$. We are interested in $\lambda$ such that each $\lambda_i$ is either $\lambda_i = q^{n_i}$ for some $n_i \in \mathbb{Z}$ or $\lambda_i$ is formal. In that case, we write it $\lambda_i = q^{\beta_i}$ where we interpret $\beta_i$ as a formal parameter.

2.2.1. Parabolic Verma modules. The (standard) Borel subalgebra $U_q(\mathfrak{b})$ of $U_q(\mathfrak{g})$ is generated by $K_\gamma$ and $E_i$ for all $\gamma \in Y^\vee$ and $i \in I$. A (standard) parabolic subalgebra of $U_q(\mathfrak{g})$ is a subalgebra containing $U_q(\mathfrak{b})$. It is generated by $K_\gamma, E_i$ and $F_j$ for all $\gamma \in Y^\vee, i \in I$ and $j \in I_f$ for some fixed subset $I_f \subset I$. The part given by $K_\gamma, E_j$ and $F_j$ for $j \in I_f$ is called the Levi factor and written $U_q(\mathfrak{l})$. The nilpotent radical $U_q(\mathfrak{n})$ is generated by $E_i$ for all $i \in I_r := I \setminus I_f$. Note that parabolic subalgebras are in bijection with partitions $I = I_f \sqcup I_r$.

Let $U_q(\mathfrak{p})$ be a parabolic subalgebra determined by $I = I_f \sqcup I_r$. For each $i \in I_f$, we choose a weight $n_i \in \mathbb{N}$. For each $j \in I_r$ we choose a weight $\lambda_j \in \{q^{\beta_j}, q^{n_j}\}$. We write $N = \{n_i\}_{i \in I_f}$ and $\Lambda = \{\lambda_j\}_{j \in I_r}$. Let $V(\Lambda, N)$ be the unique (type 1) integrable, irreducible representation of $U_q(\mathfrak{l})$ on the ground ring $R = \mathbb{Q}(q, \Lambda)$, and with highest weight $\lambda$ determined by

$$\lambda(\alpha_k^\vee) = \begin{cases} q^{n_i}, & \text{if } k = i \in I_f, \\ \lambda_j, & \text{if } k = j \in I_r. \end{cases}$$

We extend it to a representation of $U_q(\mathfrak{p})$ by setting $U_q(\mathfrak{n})V(\Lambda, N) = 0$.

**Definition 2.2.** The parabolic Verma module of highest weight $(\Lambda, N)$ associated to $U_q(\mathfrak{p}) \subset U_q(\mathfrak{g})$ is the induced module

$$M^p(\Lambda, N) := U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p})} V(\Lambda, N).$$

Whenever $U_q(\mathfrak{p}) \subsetneq U_q(\mathfrak{g})$, we have that $M^p(\Lambda, N)$ is an infinite dimensional module. Moreover, for all parabolic Verma modules, there is a $\mathbb{Q}(q)$-linear surjection

$$U_q^- (\mathfrak{g}) \otimes_{\mathbb{Q}(q)} R \twoheadrightarrow M^p(\Lambda, N).$$

**Example 2.3.** If $U_q(\mathfrak{p}) = U_q(\mathfrak{b})$, then $N = \emptyset$, and $V(\Lambda, N) \cong \mathbb{Q}(q, \Lambda)v_\Lambda$ is 1-dimensional, and such that

$$E_i v_\Lambda = 0, \quad K_\gamma v_\Lambda = \prod_{j \in I} \lambda_j^{\gamma(\alpha_j)} v_\Lambda.$$ 

In this case, we simply call it Verma module, and denote it $M^p(\Lambda)$. If $\lambda_j = q^{\beta_j}$ is formal for all $j \in I_r$, then we call it the universal Verma module.

**Example 2.4.** If $U_q(\mathfrak{p}) = U_q(\mathfrak{g})$, then $\Lambda = \emptyset$ and $M^p(\Lambda, N) \cong V(N)$ is an integrable, irreducible $U_q(\mathfrak{g})$ representation.
Since $q$ is a generic parameter we can apply Jantzen’s criterion [17, Theorem 9.12], thanks to the results in [2]. We obtain that $M^p(\Lambda, N)$ is irreducible whenever $\lambda_j \notin \{q^n | n \in \mathbb{N}\}$ for all $j \in I_p$. If $\lambda_j = q^{n_j}$ for $n_j \in \mathbb{N}$, then $M^p(\Lambda, N)$ contains a non-trivial, proper submodule, which is isomorphic to $M^p(\Lambda_{-n_j-2}, N)$ for $\Lambda_{-n_j-2}$ given by exchanging $q^{n_j}$ with $q^{-n_j-2}$ in $\Lambda$. Moreover, the quotient

$$\frac{M^p(\Lambda, N)}{M^p(\Lambda_{-n_j-2}, N)} \cong M^{p+j}(\Lambda \setminus \{q^{n_j}\}, N \cup \{n_j\}),$$

is isomorphic to the parabolic Verma module associated to the parabolic subalgebra $p + j$ given by adding $j$ to $I_f$, that is generated by $p$ and $F_j$.

Furthermore, whenever $\lambda_j = q^{\beta_j}$ is formal, there is a surjective map

$$ev_{n_j} : M^p(\Lambda, N) \to M^p(\Lambda_{\beta_j}, N),$$

for all $n_j \in \mathbb{Z}$, given by evaluating $\beta_j = n_j$.

These two facts together allow us to define a partial order on parabolic Verma modules. For this, we say that there is an arrow from $M^p(\Lambda, N)$ to $M^{p'}(\Lambda', N')$ if we have an evaluation map $ev_{n_j}$ such that

$$ev_{n_j}(M^p(\Lambda, N)) \cong M^{p'}(\Lambda', N'),$$

or if there is a short exact sequence

$$0 \to \frac{M^p(\Lambda_{-n_j-2}, N)}{M^p(\Lambda, N)} \to M^p(\Lambda, N) \to M^{p'}(\Lambda', N') \to 0.$$

For parabolic Verma modules $M$ and $M'$ we say that $M$ is bigger than $M'$ if there is a chain of arrows from $M$ to $M'$. In that case, there is an $M''$, which is either trivial or a parabolic Verma module, and a short exact sequence

$$0 \to M'' \to ev(M) \to M' \to 0,$$

where $ev$ is a composition of evaluation maps $ev_{n_j}$. With this partial order, the universal Verma module is a maximal element and each integrable, irreducible module is a minimum. This also means that we can recover any parabolic Verma module from the universal one.

### 2.2.2. The Shapovalov form.

Let $\rho : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})^{\text{op}}$ be the $\mathbb{Q}(q)$-linear algebra antinvolution given by

$$(4) \quad \rho(E_i) := q_i^{-1}K_i^{-1}F_i, \quad \rho(F_i) := q_i^{-1}K_iE_i, \quad \rho(K_{\gamma}) := K_{\gamma},$$

for all $i \in I$ and $\gamma \in Y^\vee$.

**Definition 2.5.** The Shapovalov form

$$(-,-) : M^p(\Lambda, N) \times M^p(\Lambda, N) \to \mathbb{Q}(q, \Lambda),$$

is the unique bilinear form respecting

- $(v_{\Lambda,N}, v_{\Lambda,N}) = 1$, for $v_{\Lambda,N}$ the highest weight vector;
- $(uv, v') = (v, \rho(u)v')$ where $\rho$ is defined in (4);
- $f(v, v') = (fv, v') = (v, fv')$,

for all $v, v' \in M^p(\Lambda, N), u \in U_q(\mathfrak{g})$ and $f \in \mathbb{Q}(q, \Lambda)$. 

2.2.3. **Basis.** Since parabolic Verma modules are highest weight modules, they admit at least one basis given in terms of elements of the form $F_i v_{\Lambda,N}$ for $i \in \text{Seq}(I)$, where $v_{\Lambda,N}$ is a highest weight vector. In particular, as $R$-modules they are all submodules of $U_q^{-}(\mathfrak{g}) \otimes \mathbb{Q}(q) R$, meaning that these basis lives in a subset of $\{F_i v_{\Lambda,N} | i \in \text{Seq}(I)\}$ modded out by the Serre relations. We call such a basis an induced basis and write it $\{v_{\Lambda,N} = m_0, m_1, \ldots\}$. Any element in such basis takes the form $F_i = F_i^{b_1} \cdots F_i^{b_r}$ for some $i_1, \ldots, i_r \in I$ and $b_1, \ldots, b_r \in \mathbb{N}$, with $i_i \neq i_{i+1}$. Replacing each $F_i^{b}$ by the divided power basis $F_i^{(b)} := F_i^{b}/([b]_q!)$ yields another basis $\{v_{\Lambda,N} = m'_0, m'_1, \ldots\}$. Lusztig’s canonical basis [29] is given by a certain choice of such a divided power basis characterized by

$$ (m'_i, m'_j) - 1 \in \mathbb{Z}_+^{2}[q, \Lambda], $$

for any order such that $0 < q < \lambda_i$ (see Section 6 for a definition of $\mathbb{Z}_+^{2}[q, \Lambda]$). Whenever $M^p(\Lambda, N)$ is irreducible, the Shapovalov form is non-degenerate. Therefore, in this case, there is a dual canonical basis uniquely determined by

$$ (m'_i, m'_j) = \delta_{ij}. $$

3. **The $b$-KLR algebras**

Fix once and for all a Cartan datum $(I, X, Y, (-|-))$, and let

$$ d_{ij} := -\alpha_i^\vee (\alpha_j) \in \mathbb{N}. $$

For $\nu \in X^+$ we write

$$ \nu = \sum_{i \in I} \nu_i \cdot \alpha_i, \quad \nu_i \in \mathbb{N}, $$

and we set $|\nu| := \sum_i \nu_i$, and $\text{Supp}(\nu) := \{i | \nu_i \neq 0\}$.

We also fix a choice of scalars in a commutative, unital ring $\mathbb{k}$ as introduced in [39]. Following the conventions in [10], it consists of:

- $t_{ij} \in \mathbb{k}^\times$ for all $i, j \in I$;  
- $s_{ij}^{t} \in \mathbb{k}$ for $i \neq j, 0 \leq t < d_{ij}$ and $0 \leq v < d_{ji}$;  
- $r_i \in \mathbb{k}^\times$ for all $i \in I$,

respecting

- $t_{ii} = 1$;  
- $t_{ij} = t_{ji}$ whenever $d_{ij} = 0$;  
- $s_{ij}^{t} = s_{ji}^{t}$;  
- $s_{ij}^{0} = 0$ whenever $t(\alpha_i | \alpha_i) + v(\alpha_j | \alpha_j) = -2(\alpha_i | \alpha_j)$.

In addition, whenever $t < 0$ or $v < 0$, we put $s_{ij}^{t} := 0$. Thus we have $s_{ij}^{pq} = 0$ for $p > d_{ij}$ or $q > d_{ji}$. We will also write $s_{ij}^{d_{ij}} := t_{ij}$ and $s_{ij}^{0d_{ji}} := t_{ji}$. Hence if $(\alpha_i | \alpha_j) = 0$ we get $s_{ij}^{00} = s_{ji}^{00} = t_{ij} = t_{ji}$.

**Definition 3.1 ([22, 38]).** For $m \in \mathbb{N}$, the Khovanov–Lauda–Rouquier (KLR) algebra $R(m)$ is the $\mathbb{k}$-algebra generated by braid-like diagrams on $m$ strands, read from bottom to top, such that
two strands can intersect transversally, but no triple intersections are allowed;
- strands can be decorated by dots (we use a dot with a label $k$ to denote $k$ consecutive dots on a strand);
- each strand is labeled by a simple root, written $i \in I$, that we (usually) write at the bottom;
- multiplication is given by concatenation of diagrams, which preserves the labeling (i.e. connecting two strands with different labels gives zero);
- diagrams are taken modulo planar isotopies and the following local relations:

$$\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
  \node[circle, fill=black, inner sep=1pt] at (0.5,0.5) {};
\end{tikzpicture}
\end{align*}
\quad \begin{cases}
\begin{array}{ll}
0 & \text{if } i = j, \\
\sum_{t,v} s_{ij}^{tv} \begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
  \node[circle, fill=black, inner sep=1pt] at (0.5,0.5) {};
\end{tikzpicture} & \text{if } i \neq j,
\end{array}
\end{cases}
$$

(5)

for all $i, j \in I$,

$$\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
  \node[circle, fill=black, inner sep=1pt] at (0.5,0.5) {};
\end{tikzpicture}
\end{align*}
\quad \begin{cases}
\begin{array}{ll}
\begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
  \node[circle, fill=black, inner sep=1pt] at (0.5,0.5) {};
\end{tikzpicture} & i = j, \\
\begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
  \node[circle, fill=black, inner sep=1pt] at (0.5,0.5) {};
\end{tikzpicture} & i \neq j,
\end{array}
\end{cases}
$$

(6)

for all $i \neq j \in I$,

$$\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
  \node[circle, fill=black, inner sep=1pt] at (0.5,0.5) {};
\end{tikzpicture}
\end{align*}
\quad \begin{cases}
\begin{array}{ll}
\begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
  \node[circle, fill=black, inner sep=1pt] at (0.5,0.5) {};
\end{tikzpicture} & i = j, \\
\begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
  \node[circle, fill=black, inner sep=1pt] at (0.5,0.5) {};
\end{tikzpicture} & i \neq j,
\end{array}
\end{cases}
$$

(7)

for all $i \neq j \in I$.

$$\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
  \node[circle, fill=black, inner sep=1pt] at (0.5,0.5) {};
\end{tikzpicture}
\end{align*}
\quad \begin{cases}
\begin{array}{ll}
\begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
  \node[circle, fill=black, inner sep=1pt] at (0.5,0.5) {};
\end{tikzpicture} & i = j, \\
\begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
  \node[circle, fill=black, inner sep=1pt] at (0.5,0.5) {};
\end{tikzpicture} & i \neq j,
\end{array}
\end{cases}
$$

(8)

for all $i, j, k \in I$. In addition, $R(m)$ is $\mathbb{Z}$-graded by setting

$$\deg_q \begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
  \node[circle, fill=black, inner sep=1pt] at (0.5,0.5) {};
\end{tikzpicture} := -(\alpha_i|\alpha_j), \quad \deg_q \begin{tikzpicture}[baseline=(current bounding box.center), scale=0.5]
  \draw[thick,blue] (0,0) -- (1,1);
  \draw[thick,blue] (1,0) -- (0,1);
\end{tikzpicture} := (\alpha_i|\alpha_i).$$

**Remark 3.2.** Note that in Eq. (5) and Eq. (8), the sum $\sum_{t,v} s_{ij}^{tv}$ can be restricted to the finite number of pairs $t, v \in \mathbb{N}$ such that $t(\alpha_i|\alpha_i) + v(\alpha_j|\alpha_j) = -2(\alpha_i|\alpha_j)$. Moreover, it contains at least two non-zero elements with invertible coefficients, given by $t = d_{ij}$, $v = 0$ and $t = 0$, $v = d_{ji}$. 
As proven in [22, 24] (see also [38]), these algebras categorify the half quantum group $U_q^\infty(g)$ associated to $(I, X, Y, (-|-))$, as a (twisted) bialgebra. The multiplication and comultiplication are categorified using respectively induction and restriction functors, obtained by putting diagrams side by side.

For each non-negative integral highest weight $N := \{n_i \in \mathbb{N}|i \in I\}$, there is a $N$-cyclotomic quotient $R^N(m)$ of $R(m)$ given by modding out the two-sided ideal generated by all diagrams of the form

$$\begin{vmatrix} n_i \\ i \\ \hline \\ \cdots \\ \\ \hline \\ j \\ k \end{vmatrix} = 0.$$

As first conjectured in [22] and proven in [19] and independently in [45], these cyclotomic quotients categorify the irreducible integrable $U_q(g)$-module of highest weight $N$, where the action of $F_i$ (resp. $E_i$) is given by induction (resp. restriction) along the map $R(m) \hookrightarrow R(m + 1)$ that adds a vertical strand with label $i$, at the right.

3.1. $b$-KLR algebra. Our first goal is to construct a dg-enhancement of the cyclotomic KLR algebras $R^N(m)$, in the same spirit as in [35]. We introduce the following algebra:

**Definition 3.3.** For $m \in \mathbb{N}$, the $b$-KLR algebra $R_b(m)$ is the $k$-algebra generated by braid-like diagrams on $m$ strands, read from bottom to top, such that

- two strands can intersect transversally, but no triple intersections are allowed;
- strands can be decorated by dots;
- regions in-between strands can be decorated by floating dots, which are labeled by a subscript in $I$ and a superscript in $\mathbb{N}$;
- each strand is labeled by a simple root, written $i \in I$;
- multiplication is given by concatenation of diagrams, which preserves the labeling;
- diagrams are taken modulo planar isotopies that preserve the relative height of the floating dots, and modulo the KLR relations Eq. (5–8) and the following local relations:

$$\begin{align*}
\begin{vmatrix} a_i \\ i \\ \hline \\ \cdots \\ \\ \hline \\ b_j \\ j \end{vmatrix} &= -\begin{vmatrix} a_i \\ i \\ \hline \\ \cdots \\ \\ \hline \\ b_j \\ j \end{vmatrix} \\
\begin{vmatrix} a_i \\ i \\ \hline \\ \cdots \\ \\ \hline \\ a_i \\ i \end{vmatrix} &= 0,
\end{align*}$$

meaning floating dots anti-commute with each other for all $i, j \in I$ and $a, b \in \mathbb{N}$,

$$\begin{align*}
\begin{vmatrix} a_i \\ i \\ \hline \\ \cdots \\ \\ \hline \\ a_j \\ j \end{vmatrix} &= \begin{cases} 
\begin{vmatrix} a_i - 1 \\ i \\ \hline \\ \cdots \\ \\ \hline \\ a_i - 1 \\ i \end{vmatrix} & \text{if } i = j \text{ and } a > 0, \\
\sum_{t,v} (-1)^v s_t^v t_i^{a+v} & \text{if } i \neq j,
\end{cases}
\end{align*}$$

where $s_t^v$ is the signed permutation matrix associated to the permutation $(t, v)$. 


Moreover, a floating dot in the left-most region is zero
\[
O_{i}^{a} \mid \cdots \mid j = 0.
\]

Given a diagram, it is sometimes useful to decorate some of its regions with an element
\( K := \sum_{i \in I} k_i \cdot \alpha_i \in X^+ \), where \( k_i \) denotes the number of strands with label \( i \) to the left of the region. The algebra \( R_b \) is \( \mathbb{Z}^{1+|I|} \)-graded (a \( q \)-grading and a \( \lambda_k \)-grading for each \( k \in I \)) with

\[
\deg_{q} \left( \begin{array}{c}
\gamma \\
\ell
\end{array} \right) := (\alpha_i| \alpha_j), \\
\deg_{q} \left( \begin{array}{c}
\bullet \\
i
\end{array} \right) := (\alpha_i| \alpha_i), \\
\deg_{\lambda_k} \left( \begin{array}{c}
\gamma \\
\ell
\end{array} \right) := 0, \\
\deg_{\lambda_k} \left( \begin{array}{c}
\bullet \\
i
\end{array} \right) := 0,
\]

and

\[
\deg_{q} \left( \begin{array}{c}
\gamma \\
K
\end{array} \right) := (1 + a - \alpha_i| \gamma (K) + k_i) (\alpha_i| \alpha_i), \\
\deg_{\lambda_k} \left( \begin{array}{c}
\gamma \\
K
\end{array} \right) := 2 \delta_{ik}.
\]

This ends the definition of \( R_b(m) \).

### 3.2. Tightened basis.

Before going any further, let us introduce some useful notations borrowed from [22]. First, let \( R_b(\nu) \) be the subalgebra of \( R_b(m) \) given by diagrams where there are exactly \( \nu_i \) strands labeled \( i \), for each \( i \in I \). We also denote \( \text{Seq}(\nu) \) the set of all ordered sequences \( i = i_1 i_2 \cdots i_m \) with \( i_k \in I \) and \( i \) appearing \( \nu_i \) times in the sequence. The symmetric group \( S_m \) acts on \( \text{Seq}(\nu) \) with the simple transposition \( \sigma_{i} \in S_m \) acting on \( i = i_1 i_2 \cdots i_m \in \text{Seq}(\nu) \) by permuting \( i_k \) and \( i_{k+1} \). Sometimes, for \( K = \sum_{i \in I} k_i \cdot \alpha_i \in X^+ \), we abuse notation by writing \( \sigma_{K} \) instead of \( \sigma_{|K|} \).

For \( i = i_1 i_2 \cdots i_m \in \text{Seq}(\nu) \), let \( 1_i \in R_b(\nu) \) be the idempotent given by \( m \) vertical strands with labels \( i_1, i_2, \ldots, i_m \), that is

\[
1_i := \left| \begin{array}{c}
i_1 \\
i_2 \\
i_m
\end{array} \right.
\]
We have $1_i 1_j = \delta_{ij}$ for all $i, j \in \text{Seq}(\nu)$, and so there is a decomposition of $\mathbb{k}$-modules

$$R_b(\nu) \cong \bigoplus_{i,j \in \text{Seq}(\nu)} 1_j R_b(\nu) 1_i.$$  

Our goal is to construct a basis of $1_j R_b(\nu) 1_i$ as $\mathbb{k}$-module.

3.2.1. *An action of $R_b(\nu)$ on a polynomial space.* We construct a polynomial representation of $R_b(\nu)$ with a similar flavor as in [22, §2.3]. We fix $\nu \in X^+$ with $|\nu| = m$. For each $i \in I$ we define

$$Q_i := \mathbb{k}[x_{1,i}, \ldots, x_{\nu_i,i}] \otimes \mathcal{O}^{\nu_i} \langle \omega_{1,i}, \ldots, \omega_{v_i,i} \rangle.$$  

We write $Q_I := \bigotimes_{i \in I} Q_i$, where $\otimes$ means the supertensor product in the sense that $\omega_{\ell,i} \omega_{\ell',j} = -\omega_{\ell',j} \omega_{\ell,i}$ for all $i, j \in I$ and $x_{\ell,i}$ commutes with everything. Thus, $Q_I$ is a supercommutative superring. Then, we construct the ring

$$Q_{\nu} := \bigoplus_{i \in \text{Seq}(\nu)} Q_i 1_i,$$

where the elements $1_i$ are central idempotents. It is $\mathbb{Z}^{1+|I|}$-graded by setting

$$\deg_q(x_{\ell,i}) = (\alpha_i | \alpha_i), \quad \deg_q(\omega_{\ell,i}) = (1 - \ell)(\alpha_i | \alpha_i),$$

$$\deg_{\lambda_j}(x_{\ell,i}) = 0, \quad \deg_{\lambda_j}(\omega_{\ell,i}) = 2\delta_{ij}.$$  

We first construct an action of the symmetric group $S_m$ on $Q_{\nu}$ by letting the simple transposition

$$\sigma_k : Q_I 1_i \rightarrow Q_I 1_{\sigma_k i},$$

act by sending

$$x_{p+1,i} 1_i \mapsto \begin{cases} x_{p+1,i} 1_{\sigma_k i}, & \text{if } i_k = i_{k+1} = i \text{ and } p = \# \{ s \leq k | i_s = i \}, \\ x_{p-1,i} 1_{\sigma_k i}, & \text{if } i_k = i_{k+1} = i \text{ and } p = 1 + \# \{ s \leq k | i_s = i \}, \\ x_{p,i} 1_{\sigma_k i}, & \text{otherwise}, \end{cases}$$

for $i \in I$, $p \in \{1, \ldots, \nu_i\}$ and $i = i_1 \ldots i_m$, and by sending

$$\omega_{p,i} 1_i \mapsto \begin{cases} (\omega_{p,i} + (x_{p,i} - x_{p+1,i}) \omega_{p+1,i}) 1_{\sigma_k i}, & \text{if } i_k = i_{k+1} = i \text{ and } p = \# \{ s \leq k | i_s = i \}, \\ \omega_{p,i} 1_{\sigma_k i}, & \text{otherwise}, \end{cases}$$

which we extend to $Q_{\nu}$ by setting $\sigma_k(fg) := \sigma_k(f) \sigma_k(g)$ for all $f, g \in Q_{\nu}$.

**Proposition 3.4.** The procedure described above yields a well-defined action of $S_m$ on $Q_{\nu}$.

**Proof.** The proof is a straightforward computation. We leave the details to the reader. □

Then, we define inductively the element $\omega_{p,j}^a \in Q_I$ for $a \in \mathbb{N}$ as

$$\omega_{p,j}^0 := \omega_{p,j}, \quad \omega_{p,j}^{a+1} := \omega_{p-1,j}^a - x_{p,j} \omega_{p,j}^a.$$
For $K = \sum_{i \in I} k_i \cdot i \in X^+$ such that $k_i \leq \nu_i$, we define $\omega^i_\alpha(K) \in Q_I$ inductively as

$$
\omega^i_\alpha(K) := \begin{cases} 
0, & \text{if } k_j = 0, \\
\omega^i_\alpha , & \text{if } k_i = 0 \text{ for all } i \neq j, \\
\sum_{t,v} (-1)^t s^t v x_k i, \omega^{a+v}(K - i), & \text{otherwise,}
\end{cases}
$$

where $K - i$ is a shorthand for $K - 1 \cdot \alpha_i$.

**Lemma 3.5.** The element $\omega^i_\alpha(K)$ is well-defined.

**Proof.** Take $i \neq i' \neq j \in I$ such that $k_i > 0$ and $k_{i'} > 0$. We can suppose by induction that $\omega^i_\alpha(K - i - i')$ is well-defined for all $b \geq 0$. Then we have

$$
\sum_{t,v} (-1)^t s^t v x_k i, \sum_{i',v'} (-1)^{i'} s^{i'} v' x_{k_{i'}} i', \omega^{a+v}(K - i - i')
$$

$$
= \sum_{t,v} (-1)^{i'} s^{i'} v' x_{k_{i'}}, \sum_{t,v} (-1)^t s^t v x_k i, \omega^{a+v}(K - i' - i),
$$

for all $i \neq i' \neq j \in I$. \qed

It will be useful to give $\omega^i_\alpha(K)$ a non-inductive expression. We write $K^{\cup j} := \sum_{i \neq j} k_i \cdot \alpha_i$. For a given non-negative integer $n_i \in \mathbb{N}$ we define

$$
\varepsilon^j_{n_i,i}(x_{k_j,i}) := \sum_{|V| = n_i} \left( \prod_{\ell=1}^{k_i} s^{v_{\ell} t_{\ell} x_{k_{\ell},i} \ell} \right) \in P_I,
$$

with the sum being over all partitions $V : v_1 + \cdots + v_{k_i} = v_i$ such that $(\alpha_i|\alpha_i)|v_\ell(\alpha_j|\alpha_j)$ for each $\ell \in \{1, \ldots, k_i\}$, and with $t_\ell := -2(\alpha_i|\alpha_i) - v_\ell(\alpha_j|\alpha_j)$. This is a symmetric polynomial of $q$-degree $-2k_i(\alpha_i|\alpha_j) - v_i(\alpha_j|\alpha_j)$ whenever it is non zero. Clearly, we can suppose $v_\ell \leq d_{ji}$, and therefore we can also suppose that $n_i \leq d_{ji} k_i$. For $n \in \mathbb{N}$ we define

$$
\varepsilon^j_{v}(x_K) := \sum_{|V| = n} \left( \prod_{i \neq j} \varepsilon^j_{n_i,i}(x_{k_i,i}) \right) \in P_I,
$$

with the sum being over all partitions $V : \sum_{i \neq j} n_i = n$. Notice that $\varepsilon^j_{v}(x_K)$ is a polynomial of $q$-degree $-\alpha_j(K^{\cup j}) - n(\alpha_j|\alpha_j)$.

**Lemma 3.6.** We have

$$
\omega^i_\alpha(K) = \sum_{n=0} \omega^i_\alpha^{\alpha+n}(K^{\cup j}) (-1)^n \nu^i_\alpha^{a+n} \varepsilon^j_{v}(x_K) \in P_I.
$$

**Proof.** A straightforward computation shows that the RHS of Eq. (14) respects the recursive definition of $\omega^i_\alpha(K)$, which proves the equality. \qed

We now have all the tools we need to define an action of $R_p(\nu)$ on $P_{\nu}$. First, we choose an arbitrary orientation $i \leftarrow j$ or $i \rightarrow j$ for each pair of distinct $i,j \in I$. Then, we let $a \in R_p(\nu)_1$ act as zero on $P_{\nu}1_i$ whenever $j \neq i$. Otherwise, we declare that
• the dot
\[ \bullet \]
acts as multiplication by \( x_{k_i+1,i} 1_i \);

• the floating dot
\[ K \]
acts as multiplication by \( \omega^a_j(K) 1_i \);

• the crossing
\[ i \]
acts as
\[ \text{if } i \neq j, \]
\[ \text{if } i = j. \]

**Proposition 3.7.** The rules above define an action of \( R_b(\nu) \) on \( Q_\nu \).

**Proof.** We have to check the validity of the KLR relations Eq. (5–8) and of the relations involving floating dots Eq. (9–11), as well as the relations coming from regular isotopies.

We start by proving the KLR relations. Clearly Eq. (5), Eq. (6) and Eq. (7) are satisfied. The case \( i \neq k \) of Eq. (8) is also straightforward. For \( i \leftarrow j \) and \( k = i \) we compute the action of the LHS of Eq. (8) on \( f \in Q_\nu \) as

\[
    f \mapsto \left( \sum_{t,v} s_{ji}^t y^t x_1^v \right) \sigma_1 \sigma_2 \sigma_1(f) - \sigma_2 \sigma_1 \left( \sum_{t,v} s_{ji}^t y^t x_2^v \sigma_2(f) \right)
\]

\[
    = \left( \sum_{t,v} s_{ji}^t y^t x_1^v \right) \frac{f - \sigma_1 \sigma_2 \sigma_1(f)}{x_1 - x_2} - \left( \sum_{t,v} s_{ji}^t y^t x_2^v \right) \frac{\sigma_2 \sigma_1 \sigma_2(f)}{x_1 - x_2}
\]

\[
    = \sum_{t,v} s_{ji}^t y^t \frac{x_1^v f - x_2^v}{x_1 - x_2} = \sum_{t,v} s_{ji}^t y^t \frac{x_1^v x_2^r}{x_1^v + \cdots + x_2^s},
\]

where \( x_1, x_2 \) correspond with the \( x_{k_i,i}, x_{k_i+1,i} \) and \( y \) with \( x_{k_j,j} \). What remains coincides with the RHS of Eq. (8). A similar computation applies for the case \( i \rightarrow j \).
For the relations involving floating dots, we remark that Eq. (9) follows from the supercommutativity of $Q_\nu$, and $\omega^a_j(K)$ respects Eq. (10) by construction. For relation Eq. (11), we apply the action of the LHS on some $f_{ij}^t$ and we obtain

$$f \mapsto \left( \sum_{t,v} s^t_{ij} y^t x^v \right) \omega^a_j(K + j) f,$$

and for the RHS we obtain

$$f \mapsto (\omega^a_j(K + i + j) + \sum_{t,v} s^t_{ij} \sum_{u+\ell= v-1} (-1)^u \omega^{a+u}_j(K) x^t y^\ell) f$$

$$= \left( \sum_{t,v} (-1)^v s^t_{ij} x^t \omega^{a+v}_j(K + j) + \sum_{t,v} s^t_{ij} \sum_{u+\ell= v-1} (-1)^u \omega^{a+u}_j(K) x^t y^\ell \right) f.$$

Then we compute

$$\omega^{a+v}_j(K + j) = \left( \sum_{u+\ell= v-1} (-1)^{u-1-v} y^\ell \omega^{a+u}_j(K) \right) + (-1)^v y^v \omega^{a}_j(K + j),$$

which implies that the action of the RHS of Eq. (11) coincides with the one of the LHS.

The only non trivial relation coming from regular isotopies we need to verify is given by the commutation of a floating dot and a crossing at its left. This is a consequence of the fact that Eq. (12) is a symmetric polynomial, which commutes with divided difference operators.

$\square$

3.2.2. Left-adjusted expressions. Recall from [35, §2.2.1] that a reduced expression $\sigma_{i_r} \cdots \sigma_{i_1}$ of $w \in S_n$ is left-adjusted if $i_r + \cdots + i_1$ is minimal. Equivalently, it is left-adjusted if and only if

$$\min_{t \in \{0, \ldots, r\}} \sigma_{i_t} \cdots \sigma_{i_1}(k) \leq \min_{t \in \{0, \ldots, r\}} \sigma_{j_t} \cdots \sigma_{j_1}(k),$$

for all $k \in \{0, \ldots, n\}$ and all other reduced expression $\sigma_{j_r} \cdots \sigma_{j_1} = w$. In this condition, we write

$$\min_w(k) := \min_{t \in \{0, \ldots, r\}} \sigma_{i_t} \cdots \sigma_{i_1}(k).$$

Note that a left adjusted expression always exists and is unique up to distant permutation ( $\sigma_i \sigma_j \leftrightarrow \sigma_j \sigma_i$ for $|i - j| > 1$), so that $\min_w(k)$ is well-defined. In particular, one can obtain a left-adjusted reduced expression for any permutation by taking its representative in the coset decomposition

$$S_n = \bigsqcup_{a=1}^n S_{n-1} \sigma_{n-1} \cdots \sigma_a,$$

applied recursively. If we think of a reduced expression in terms of string diagrams, then it is left-adjusted if all strings are pulled as far as possible to the left.
Example 3.8. The permutation $(1\ 3\ 2\ 4) \in S_4$ admits as left-adjusted reduced expression the word $\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2$ which comes from the summand $S_2\sigma_3\sigma_2$ in the first step of the recursive decomposition (15). Note that $\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2$ is also left-adjusted while $\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2$ and $\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3$ are not. In terms of string diagrams, we consider as example the following reduced expression of the permutation $w = (1\ 4\ 3\ 5\ 2) \in S_5$:

It is not left-adjusted since the 4th strand (read at the bottom) can be pulled to the left. Hence we obtain the following left-adjusted minimal presentation:

Suppose $\sigma_{i_r} \cdots \sigma_{i_1}$ is a left-adjusted reduced expression of $w$. Then we can choose for each $k \in \{1, \ldots, n\}$ an index $t_k \in \{1, \ldots, r\}$ such that

$$\sigma_{i_{t_k}} \cdots \sigma_{i_1}(k) = \min_w(k).$$

Clearly this choice is not necessarily unique and we can have $t_k = t_{k'}$ for $k \neq k'$. However, it defines a partial order $<$ on the set $\{1, \ldots, n\}$ where $k < k'$ whenever $t_k \leq t_{k'}$. We extend this order arbitrarily and we write $<_t$ for it. There is a bijective map $\zeta : \{1, \ldots, n\} \to \{1, \ldots, n\}$ which sends $k < k'$ to $\zeta(k) <_t \zeta(k')$, so that $t_{\zeta(k)} \leq t_{\zeta(k')}$. In terms of string diagrams, the map $\zeta$ tells us in which order the strands attain their (chosen) leftmost position while reading from bottom to top. In particular, $\zeta(k)$ gives the starting point of the strand that attains its leftmost position in $k$th position.

Example 3.9. Consider again the following left-adjusted string diagram:

Both the 1st and 3rd strand attain their leftmost position at the bottom of the diagram, thus we can choose $\zeta(1) = 1$ and $\zeta(2) = 3$. Then both the 2nd and 4th strand attain their leftmost position, thus we can take $\zeta(3) = 4$ and $\zeta(4) = 2$. Finally, the 5th strand attains its leftmost position and we put $\zeta(5) = 5$.

For $k \in \{1, \ldots, n+1\}$, we put

$$w^k := \sigma_{i_{t_{\zeta(k)}}} \cdots \sigma_{i_{t_{\zeta(k-1)}}},$$

where it is understood that $t_{\zeta(0)} := 0$ and $t_{\zeta(n+1)} := r$. It defines a partition of the reduced expression of $\sigma_{i_r} \cdots \sigma_{i_1} = w$. Moreover, it is constructed so that

$$w^k \cdots w^1(\zeta(k)) = \min_w(\zeta(k)),$$

for all $1 \leq k \leq n$. 
Example 3.10. Consider again \( w = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2 \) with \( i_1 = 2, i_2 = 3, i_3 = 1, i_4 = 2, i_5 = 1 \). We can choose for example \( t_1 = 0, t_2 = 0, t_3 = 3 \) and \( t_4 = 5 \). Then we can put \( \zeta(1) = 1 \) (or 2), \( \zeta(2) = 2 \) (or 1), \( \zeta(3) = 3 \) and \( \zeta(4) = 4 \), with \( w^1 = 1, w^2 = 1, w^3 = \sigma_1\sigma_3\sigma_2 \) and \( w^4 = \sigma_1\sigma_2 \).

3.2.3. A generating set. We say that a floating dot is \textit{tight} if it is placed immediately to the right of the left-most strand, and has superscript 0. We can also suppose it has the same subscript as the label of the strand at its left (otherwise it would slide to the left and be zero).

Lemma 3.11. The algebra \( \mathcal{R}_b(\nu) \) is generated by KLR elements (i.e. dots and crossings) and tight floating dots.

Proof. We first compute

\[
\begin{align*}
\sigma_i - \sigma_i \quad \Rightarrow \quad \dot{i} \quad \Rightarrow \quad \dot{i} \quad \Rightarrow \quad \dot{i} \quad \Rightarrow \quad \dot{i}
\end{align*}
\]

for all \( a \geq 0, i \in I \). Eq. (16), together with Eq. (11) and Eq. (10) allows to bring all floating dots to the left. Then applying Eq. (10) recursively allows to transform all floating dots with superscript bigger than zero into dots and tight floating dots.

We write \( \omega \) for a tight floating dot, \( \tau_a \) for a crossing between the \( a \)th and \((a + 1)\)th strands (counting from left), and \( x_a \) for a dot on the \( a \)th strand, where we suppose the label of the strands given by the context, in the form of an idempotent \( 1_i \). We also define the \textit{tightened floating dot} in \( \mathcal{R}_b(\nu) \) as \( \theta_a := \tau_{a-1} \cdots \tau_1 \omega \tau_1 \cdots \tau_{a-1} \), or diagrammatically

\[
\theta_a := \quad \cdots
\]

We also write \( \theta_a^0 := \theta_a \) and \( \theta_a^{-1} := 1 \).

Lemma 3.12. Tightened floating dots anticommute with each others, up to adding terms with a smaller number of crossings, that is

\[
\theta_a \theta_b = -\theta_b \theta_a + R, \quad (\theta_a)^2 = 0 + R',
\]

where \( R \) (resp. \( R' \)) possesses strictly less crossings than \( \theta_a \theta_b \) (resp. \( (\theta_a)^2 \)), for all \( 1 \leq a, b \leq m \).
Proof. We first compute that

\[
(17) \quad \kappa \ell a_i b_j = 0,
\]

for all \(i, j, k, \ell \in I\) and \(a, b \in \mathbb{N}\). Then we obtain

\[
+ R_0 \overset{(17)}{=} - R_1,
\]

where both \(R_0\) and \(R_1\) have less crossings. \(\square\)

Fix \(i, j \in \text{Seq}(\nu)\). Since they are both sequences of the same elements, there is a subset \(jS_i \subset S_m\) of permutations \(w \in S_m\) such that \(i_k = j_{w(k)}\) for all \(k \in \{1, \ldots, m\}\). Given such a permutation \(w \in jS_i\), we can choose a left-adjusted reduced expression. It comes with a partition \(w^{m+1} \ldots w^2 w^1 = w\) and a bijection \(\zeta : \{1, \ldots, m\} \rightarrow \{1, \ldots, m\}\), such that

\[
(18) \quad w^k \cdots w^1(\zeta(k)) = \min_w(\zeta(k)),
\]

for all \(1 \leq k \leq m\). Then, consider the collection of elements

\[
jB_i := \left\{x_m^{a_m} \cdots x_1^{a_1} \tau_{w^{m+1}} \theta_{\min_w(\zeta(m))}^{\ell_m} \tau_{w^{m}} \cdots \theta_{\min_w(\zeta(2))}^{\ell_2} \tau_{w^2} \theta_{\min_w(\zeta(1))}^{\ell_1} \tau_{w^1} \right\}
\]

in \(1_j R_0(\nu) 1_i\). Diagrammatically, elements in \(jB_i\) can be constructed using the following algorithm:

1. choose a permutation \(w \in jS_i\), consider its corresponding string diagram and make it left-adjusted by bringing all strands to the left;
2. for each strand, choose whether we want to add a floating dot. If so, add a tightened floating dot where the strand attains its left-most position by pulling the strand to the far left and adding the floating dot immediately at its right;
3. for each strand, choose a number of dots to add at the top of the diagram.

Example 3.13. Take \(i = i_1i_2i_3i_4i_5\) and \(j = i_2i_5i_4i_1i_3\), and consider the following left-adjusted permutation \(w \in jS_i\):

\[
w = \begin{pmatrix}
 i_1 & i_2 & i_3 & i_4 & i_5 \\
 1 & 2 & 3 & 4 & 5
\end{pmatrix}
\]
Take $\ell_1 = 1, \ell_2 = 0, \ell_3 = 1, \ell_4 = 0$ and $\ell_5 = 0$ (for the same $\zeta$ as in Example 3.9). Then we obtain the following element in $jB_i$:

![Diagram]

**Proposition 3.14.** Elements in $jB_i$ generate $1_jR_b(\nu)1_i$ as a $\mathbb{k}$-vector space.

**Proof.** The proof is an induction on the number of crossings. By Lemma 3.11, we can assume that all floating dots are tight. By Eq. (6) and Eq. (7) we can bring all the dots to the top of any diagram, at the cost of adding diagrams with fewer crossings. Moreover, all braid isotopies hold up to adding terms with a lower amount of crossings thanks to Eq. (5) and Eq. (8).

We claim that we can also assume that there is at most one floating dot at the immediate right of each strand. Indeed, suppose there are two tight floating dots at the right of the same strand. Then we can apply a braid-isotopy to bring it as most as possible to the left, until it is possibly blocked by other tight floating dots. We depict it by the following picture:

![Diagram]

where the dashed strand in red represents the one we want to pull, and the boxes represent other elements in $R_b(\nu)$. If there is no floating dot in-between, then it is zero by Eq. (9). Otherwise, we apply Eq. (17) to jump the bottom floating dot over all the floating dots in-between, until we have two floating dots in the same region at the top, which is zero by Eq. (9). This proves the claim.

Finally, we observe that given a strand with a single tight floating dot, we can tighten it by braid isotopy, until we end up with a tightened floating dot. Since by Lemma 3.12 tightened floating dots anticommute, this concludes the proof.

### 3.2.4. The basis theorem.

**Proposition 3.15.** The action in Proposition 3.7 is faithful.

**Proof.** The proof is inspired by [39, Proposition 3.8] (see also [22, Theorem 2.5] for a different approach). We claim that elements of $jB_i$ act as linearly independent endomorphisms on $P_\nu$. The action yields morphisms

$$ P_\nu 1_i \to P_\nu 1_j, $$
that we will consider as endomorphisms of $P_i$.

First we extend the scalars to $\mathbb{k}(x_{1,i}, \ldots, x_{\nu_i})$ in $P_i$ for all $i \in I$. We claim that different choices of $w \in J_S_i$ and $\ell_i \in \{-1, 0\}$ give linearly independent operators. Notice that since $i, j$ is fixed, $w$ is given by choices of permutations between strands of the same label. Since crossings between strands with different labels act as multiplication by a polynomial, we can ignore them as being multiplication by a scalar. By [35, Corollary 3.9], we know that different choices of permutations and tightened floating dots for strands with label $i$ act as linearly independent operators on $P_i$, hence proving our claim. Finally, taking into account the multiplication by the polynomial given by the choice of the $a_i \in \mathbb{N}$ as in Eq. (18) concludes the proof.

**Theorem 3.16.** The $\mathbb{k}$-module $1_j R_b(\nu) 1_i$ is free with basis $j B_i$.

**Proof.** It follows from Proposition 3.14 and Proposition 3.15.

From this, we also deduce the following:

**Corollary 3.17.** The $\mathfrak{b}$-KLR algebra admits a presentation given by the KLR-generators and tight floating dots, subjected to the KLR-relations Eq. (5–8) together with

\[
\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center),x=0.8cm,y=0.8cm]
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\draw[thick] (0.5,0) -- (0.5,1);
\draw[thick] (0.5,0) .. controls (0.7,0.2) and (0.8,0.7) .. (0.5,1);
\draw[thick] (0.5,0) .. controls (0.7,-0.2) and (0.8,-0.7) .. (0.5,1);
\node at (0.5,0) {\textbullet};
\node at (0.5,1) {\textbullet};
\node at (0,0) {\textbullet};
\node at (1,1) {\textbullet};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\end{tikzpicture} + \begin{tikzpicture}[baseline=(current bounding box.center),x=0.8cm,y=0.8cm]
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\draw[thick] (0.5,0) -- (0.5,1);
\draw[thick] (0.5,0) .. controls (0.7,0.2) and (0.8,0.7) .. (0.5,1);
\draw[thick] (0.5,0) .. controls (0.7,-0.2) and (0.8,-0.7) .. (0.5,1);
\node at (0.5,0) {\textbullet};
\node at (0.5,1) {\textbullet};
\node at (0,0) {\textbullet};
\node at (1,1) {\textbullet};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\end{tikzpicture} = 0,
\end{align*}
\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center),x=0.8cm,y=0.8cm]
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\draw[thick] (0.5,0) -- (0.5,1);
\draw[thick] (0.5,0) .. controls (0.7,0.2) and (0.8,0.7) .. (0.5,1);
\draw[thick] (0.5,0) .. controls (0.7,-0.2) and (0.8,-0.7) .. (0.5,1);
\node at (0.5,0) {\textbullet};
\node at (0.5,1) {\textbullet};
\node at (0,0) {\textbullet};
\node at (1,1) {\textbullet};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\end{tikzpicture} = 0,
\end{align*}
\]

for all $i, j \in I_f$.

4. DG-enhancement

We fix a subset $I_f \subset I$ and consider the associated parabolic subalgebra $U_q(\mathfrak{p}) \subset U_q(\mathfrak{g})$ as defined in Section 2.2. For each $j \in I_f$, we also choose a weight $n_j \in \mathbb{N}$, and write $N := \{n_j\}_{j \in I_f}$.

**Definition 4.1.** The $\mathfrak{p}$-KLR algebra $R_p(m)$ is given by forgetting the $\lambda_j$-degree for each $j \in I_f$ in $R_b(m)$ and modding out by

\[
\begin{tikzpicture}[baseline=(current bounding box.center),x=0.8cm,y=0.8cm]
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\draw[thick] (0.5,0) -- (0.5,1);
\draw[thick] (0.5,0) .. controls (0.7,0.2) and (0.8,0.7) .. (0.5,1);
\draw[thick] (0.5,0) .. controls (0.7,-0.2) and (0.8,-0.7) .. (0.5,1);
\node at (0.5,0) {\textbullet};
\node at (0.5,1) {\textbullet};
\node at (0,0) {\textbullet};
\node at (1,1) {\textbullet};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\end{tikzpicture} = 0,
\]

for all $j \in I_f$. The $N$-cyclotomic quotient $R_p^N(m)$ of $R_p(m)$ is given by modding out by

\[
\begin{tikzpicture}[baseline=(current bounding box.center),x=0.8cm,y=0.8cm]
\draw[thick] (0,0) -- (1,1);
\draw[thick] (0,1) -- (1,0);
\draw[thick] (0.5,0) -- (0.5,1);
\draw[thick] (0.5,0) .. controls (0.7,0.2) and (0.8,0.7) .. (0.5,1);
\draw[thick] (0.5,0) .. controls (0.7,-0.2) and (0.8,-0.7) .. (0.5,1);
\node at (0.5,0) {\textbullet};
\node at (0.5,1) {\textbullet};
\node at (0,0) {\textbullet};
\node at (1,1) {\textbullet};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\node at (0,0) {};\node at (1,1) {};
\end{tikzpicture} = 0,
\]

for all $j \in I_f$. 
In particular, \( R_b(m) \) is the usual KLR algebra \( R(m) \) (see Definition 3.1). Its \( N \)-cyclotomic quotient \( R_b^N(m) \) is also the usual cyclotomic quotient of the KLR algebra. Taking \( I_f = \emptyset \) gives \( p = b \) and we recover Definition 3.3.

We equip \( R_b(m) \) with a homological \( \mathbb{Z} \)-grading, denoted \( h \), by setting
\[
\deg_h \left( \begin{array}{c}
i \\
j \end{array} \right) = 0, \quad \deg_h \left( \begin{array}{c}
i \end{array} \right) = 0, \quad \deg_h \left( \begin{array}{c}a \\
K \end{array} \right) = 1,
\]
for all \( i, j \in I \). Then, we equip \( R_b(m) \) with a differential \( d_N \) by setting
\[
\begin{aligned}
&d_N \left( \begin{array}{c}
i \\
j \end{array} \right) = d_N \left( \begin{array}{c}
i \end{array} \right) = 0, \\
&d_N \left( \begin{array}{ccc}o \\
j \\
i_1 & \cdots & i_{m-1} \end{array} \right) = \\
&\begin{cases} 
0, & \text{if } j \notin I_f, \\
(-1)^{n_j} n_j & \text{if } j \in I_f.
\end{cases}
\end{aligned}
\]
We extend the definition of \( d_N \) to the whole algebra using the graded Leibniz rule \( d_N(xy) = d_N(x)y + (-1)^{\deg_h(x)}xd_N(y) \) and Lemma 3.11. Checking that \( d_N \) is well-defined is straightforward using Corollary 3.17. From this, we derive that for \( j \in I_r \) we have
\[
d_N \left( \begin{array}{c}a \\
K \end{array} \right) = (-1)^{n_j - k_j + 1 + a} \sum_{r=0}^{\alpha_j^c(K^{ij})} \mathfrak{h}_{n_j + a - k_j + 1 + r} (x_{k_j} \varepsilon^j_r(x_K)),
\]
where \( x_{k_j} \) is a dot on the \( k \)th strand with label \( i \), \( \mathfrak{h}_n \) is the \( n \)th complete homogeneous polynomial, and \( \varepsilon^j_r(x_K) \) is defined in Eq. (13).

**Definition 4.2.** We refer to the dg-algebra \((R_b(m), d_N)\) as the *dg-enhanced KLR algebra*.

**Proposition 4.3.** If \( n_j - \nu_j - \alpha_j^c(\nu^{ij}) < 0 \), then \((R_b(\nu), d_N)\) is acyclic.

**Proof.** Taking \( a := -(n_j - \nu_j - \alpha_j^c(\nu^{ij}) + 1) \) and considering the floating dot placed on the far right with subscript \( j \) and superscript \( a \) yields
\[
d_N \left( \begin{array}{c}a \\
\nu \end{array} \right) = (-1)^{\alpha_j^c(\nu^{ij})}.
\]
Thus, \( H(R_b(\nu), d_N) \cong 0 \).

Our goal for the rest of the section will be to prove the following:

**Theorem 4.4.** The dg-algebra \((R_b(m), d_N)\) is formal with homology \( H(R_b(m), d_N) \cong R_b^N(m) \).
4.1. **Proof of Theorem 4.4.** Denote $1_{(m,i)} := \sum_{j \in I^m} 1_{ji}$, or diagrammatically

$$1_{(m,i)} := \sum_{(j_1, \ldots, j_m) \in I^m} \delta_{j_1 i, \ldots, j_m i}.$$  

It is an idempotent of $R_b(m + 1)$. We also define $1_{(\nu,i)} := \sum_{j \in \text{Seq}(\nu)} 1_{ji}$ for $\nu \in X^+$. The algebra $R_b(m)$ acts on $1_{(m,i)} R_b(m + 1)$ by first adding a vertical strand labeled $i$ at the right of $D \in R_b(m)$ and then multiplying on the left in $R_b(m + 1)$.

We now introduce some other diagrammatic notations as in [35, §3.1]. We draw $R_b(m)$ (viewed as $R_b(m)$-$R_b(m)$-bimodule) as a box labeled by $m$

$$R_b(m) = \begin{array}{c} \vdots \\ \cdots \\ m \\ \cdots \\ \vdots \end{array}$$

and $\otimes_m := \otimes R_b(m)$ becomes stacking boxes on top of each other. Moreover, when $R_b(m + 1)$ is viewed as a left $R_b(m)$-module, as a right $R_b(m)$-module or as an $R_b(m)$-$R_b(m)$-bimodule, we draw respectively

$$\begin{array}{c} \vdots \\ \cdots \\ m + 1 \\ \cdots \\ \vdots \end{array}$$

**Lemma 4.5.** As a left $R_b(m)$-module, $1_{(m,i)} R_b(m + 1)$ is free with decomposition

$$\bigoplus_{a=1}^{m+1} (R_b(m) 1_{(m,i)} \tau_m \cdots \tau_a x_\ell \oplus R_b(m) 1_{(m,i)} \tau_m \cdots \tau_a \theta_\ell),$$

where $\theta_\ell := \tau_{a-1} \cdots \tau_1 \omega x_1 \tau_1 \cdots \tau_{a-1}.$

We draw this as

$$\begin{array}{c} \vdots \\ \cdots \\ m + 1 \\ \cdots \\ \vdots \end{array} \quad \begin{array}{c} \vdots \\ \cdots \\ m \\ \cdots \\ \vdots \end{array} \quad \begin{array}{c} \vdots \\ \cdots \\ m + 1 \\ \cdots \\ \vdots \end{array}$$

**Proof.** By Theorem 3.16 we obtain

$$\begin{array}{c} \vdots \\ \cdots \\ m + 1 \\ \cdots \\ \vdots \end{array} \quad \begin{array}{c} \vdots \\ \cdots \\ m \\ \cdots \\ \vdots \end{array} \quad \begin{array}{c} \vdots \\ \cdots \\ m + 1 \\ \cdots \\ \vdots \end{array}$$

We then apply Eq. (6) and Eq. (7) to bring all the dots to the desired position. It is a triangular change of basis, concluding the proof. \qed
From now on, we will draw boxes with label ‘m, d\(_N\)’ to denote the dg-algebra \((R_b(m), d_N)\). Similarly, a box with label \(H(m)\) denotes its homology \(H(R_b(m), d_N)\). Then, the decomposition in Lemma 4.5 lifts directly to a direct sum decomposition of dg-modules whenever \(i \notin I_f\). Otherwise, for \(i \in I_f\), it lifts to the mapping cone

\[
\text{Cone} \left( \begin{array}{ccc}
\bigoplus_{a=1}^{m+1} \oplus \ell_{\ell \geq 0} & \bigoplus_{a=1}^{m+1} \oplus \ell_{\ell \geq 0} & \bigoplus_{a=1}^{m+1} \oplus \ell_{\ell \geq 0} \\
\end{array} \right) \xrightarrow{\partial_N} \left( \begin{array}{ccc}
\bigoplus_{a=1}^{m+1} \oplus \ell_{\ell \geq 0} & \bigoplus_{a=1}^{m+1} \oplus \ell_{\ell \geq 0} & \bigoplus_{a=1}^{m+1} \oplus \ell_{\ell \geq 0} \\
\end{array} \right)
\]

where the map \(\partial_N\) is induced by the differential of \((R_b(m+1), d_N)\).

We will prove Theorem 4.4 using induction on the number of strands \(m\). Therefore, we can assume \((R_b(m), d_N)\) to be formal.

Following [20], recall that for a dg-algebra \((A, d_A)\), we say that a dg-module is a relatively projective module if it is a direct summand of a free module in \((A, d_A)\)-mod. Moreover, an \((A, d_A)\)-module \(Y\) satisfies property (P) if there is an exhaustive filtration of \((A, d_A)\)-modules

\[
0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r \subset F_{r+1} \subset \cdots \subset Y,
\]

such that each \(F_{r+1}/F_r\) is isomorphic in \((A, d_A)\)-mod to a relatively projective module. An \((A, d_A)\)-direct summand of a property (P) module is called cofibrant. Also recall the following result of homological algebra:

**Lemma 4.6.** Let \((A, d_A)\) be a dg-algebra, \((M, d_M)\) be a right \((A, d_A)\)-module, and \((N, d_N)\) a left one. If \((M, d_M)\) is formal and \((N, d_N)\) is cofibrant, then we have

\[
H\left( (M, d_M) \otimes_{(A,d_A)} (N, d_N) \right) \cong H\left( H(M, d_M) \otimes_{(A,d_A)} (N, d_N) \right).
\]

**Proof.** Tensoring with a cofibrant dg-module preserves quasi-isomorphisms. \(\square\)

Therefore we obtain an exact sequence

\[
\begin{array}{ccc}
\bigoplus_{a=1}^{m+1} \oplus \ell_{\ell \geq 0} & \bigoplus_{a=1}^{m+1} \oplus \ell_{\ell \geq 0} & \bigoplus_{a=1}^{m+1} \oplus \ell_{\ell \geq 0} \\
\end{array} \xrightarrow{\partial_N} \left( \begin{array}{ccc}
\bigoplus_{a=1}^{m+1} \oplus \ell_{\ell \geq 0} & \bigoplus_{a=1}^{m+1} \oplus \ell_{\ell \geq 0} & \bigoplus_{a=1}^{m+1} \oplus \ell_{\ell \geq 0} \\
\end{array} \right) \rightarrow \cdots \rightarrow 0,
\end{array}
\]

thanks to Lemma 4.6 and Eq. (19).

**Proposition 4.7.** The exact sequence Eq. (21) is a short exact sequence, with \(\partial_N\) being injective.
Theorem 4.4 above is a direct consequence of Proposition 4.7. Therefore, we now focus on proving this proposition. This is in fact similar to Kang–Kashiwara’s [19, Eq. (4.13)], with basically only a change of basis, and thus we will follow the same ideas. We introduce the equivalent of ‘$g_a$’ from the reference and draw it as an undercrossing:

\[
\begin{align*}
\undercrossing_{i,j} &:= \begin{cases} 
\begin{array}{c}
\begin{array}{c}
\bullet \quad \bullet \\
\quad \downarrow \\
\quad \downarrow 
\end{array}
\end{array} & \text{if } i \neq j, \\
\begin{array}{c}
\begin{array}{c}
\bullet \quad 2 \\
\quad \downarrow \\
\quad \downarrow 
\end{array}
\end{array} & \text{if } i = j.
\end{cases}
\end{align*}
\]

In order to shorten out our diagrams, we introduce the convenient notation

\[\begin{array}{c}
\begin{array}{c}
\bullet \\
\quad \downarrow \\
\quad \downarrow 
\end{array}
\end{array} := \begin{array}{c}
\begin{array}{c}
\bullet \\
\quad \downarrow \\
\quad \downarrow 
\end{array}
\end{array} \quad \text{if } i \neq j.
\]

It respects the relation

\[\frac{}{\undercrossing_{i,j} \cdot \undercrossing_{i,j} \cdot \undercrossing_{i,j}} \quad \text{if } i \neq j.
\]

We also have that

\[\frac{}{\undercrossing_{i,j} \cdot \undercrossing_{i,j} \cdot \undercrossing_{i,j}} \quad \text{if } i \neq j.
\]

**Lemma 4.8.** ([19, Lemma 4.12]) **Undercrossings respect the following relations:**

\[\begin{array}{c}
\begin{array}{c}
\bullet \\
\quad \downarrow \\
\quad \downarrow 
\end{array}
\end{array} \quad \text{if } i \neq j.\]

\[\begin{array}{c}
\begin{array}{c}
\bullet \\
\quad \downarrow \\
\quad \downarrow 
\end{array}
\end{array} \quad \text{if } i \neq j.
\]

for all $i, j, k \in I$.

Still as in [19], in order to construct a “nearly inverse” for $\tilde{d}_N$, we define the map

\[\begin{array}{c}
\begin{array}{c}
\vcenter{\hbox{\includegraphics[width=0.2\textwidth]{fig1.png}}} \\
\vcenter{\hbox{\includegraphics[width=0.2\textwidth]{fig2.png}}}
\end{array}
\end{array} \quad \text{if } i \neq j.
\]
as multiplication on the left (or diagrammatically stacking above) with the element

\[ \tilde{\theta}_{m+1} := \begin{array}{c}
\vdots \\
\vdots \\
i
\end{array} \]

**Lemma 4.9.** The map \( P \) defined above is a map of \( H(R_b(m), d_N) \)-modules.

**Proof.** We need to verify that \( \tilde{\theta}_{m+1} \) commutes with the elements in \( H(R_b(m), d_N) \). Crossings and dots slide over the upper part of the \((m + 1)\)th strand in \( \tilde{\theta}_{m+1} \) at the cost of adding diagrams with fewer crossings. Because there are fewer crossings, we can slide the floating dot coming from \( \tilde{\theta}_{m+1} \) to the part \( H(R_b(m), d_N) \) of the diagram, which gives zero. The crossings and dots in the remaining terms then slide over the lower part thanks to Lemma 4.8. Tight floating dots with subscript \( j \notin I_f \) also slide over \( \tilde{\theta}_{m+1} \) thanks to Eq. (11). \( \square \)

**Lemma 4.10.** The composition \( P \circ \tilde{d}_N \) is given on \( H(R_b(m), d_N) \otimes_R \nu, i R_b(m + 1) \) by multiplication by

\[
\sum_{\nu, i}^{2\nu_{i-\alpha_i}(\nu)} \sum_{p=0}^{n_i+\rho_i} x_{m+1}^i \varepsilon_p^i(x_{\nu}),
\]

where \( \varepsilon_p^i(x_{\nu}) \) is as in Eq. (13).

**Proof.** The proof is similar to [19, Theorem 4.15]. We have

\[
(H(m)) \quad \quad \quad (H(m))
\]

We prove by induction on the number of strands \( m \) that

\[
\equiv \sum_{\nu, i}^{2\nu_{i-\alpha_i}(\nu)} \sum_{p=0}^{n_i+\rho_i} x_{m+1}^i \varepsilon_p^i(x_{\nu}),
\]

where \( \equiv \) means equality up to adding elements killed in the quotient \( H(R_b(m), d_N) \cong R_{p,N}^N(m) \). If \( m = 0 \), then it is trivial. Thus we suppose by induction that it holds for \( m - 1 \). We fix the label of the strands on the diagram above as \( ij \) with \( j = j_1 \cdots j_m \in \text{Seq}(\nu) \), and we consider the different possible cases.
If \( j_m \neq i \), then the result follows by applying Eq. (5) with Lemma 4.8, and using the induction hypothesis.

If \( j_m = i \), we first observe that

\[(26) \quad \xrightarrow{i \ i} = r_i \quad \xrightarrow{i \ i} \]

Then, we need to consider \( j_{m-1} \). If \( m = 1 \), we have that

\[
\begin{align*}
\xrightarrow{i \ i \ i} & = r_i \xrightarrow{i \ i} - r_i \xrightarrow{i \ i} \\
& = r_i \xrightarrow{i \ i} - r_i \xrightarrow{i \ i} n_i + 1 + r_i^2 \xrightarrow{i \ i} - r_i^2 \xrightarrow{i \ i} n_i \\
\end{align*}
\]

Moreover, we observe that

\[
\xrightarrow{i \ i} n_i \equiv 0,
\]

which finishes the case \( m = 1 \). For \( j_{m-1} = i \), we have

\[
\begin{align*}
\xrightarrow{i \ i \ i} & = r_i^2 \xrightarrow{i \ i} - r_i \xrightarrow{i \ i} \\
\end{align*}
\]

using Eq. (26), Eq. (23) and Lemma 4.8. Using Eq. (22) followed by Lemma 4.8 and Eq. (26) we obtain

\[
\begin{align*}
\xrightarrow{i \ i} & = r_i \\
\end{align*}
\]

Keeping in mind Eq. (25), we have

\[
\begin{align*}
\xrightarrow{i \ i \ i} \equiv \xrightarrow{i \ i \ i} \in \xrightarrow{i \ i \ i} H(m) i
\end{align*}
\]
by Eq. (8) and Eq. (7). This means we can apply the induction hypothesis to get

\[
\begin{align*}
\begin{array}{l}
    \mathcal{Z}_i \\ n_i & \equiv r_i^3
\end{array}
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
\begin{array}{l}
    \mathcal{Z}_i \\ n_i & = r_i \\
    \mathcal{Z}_i \\ n_i & = r_i^2 \\
    \mathcal{Z}_i \\ n_i & = r_i^3
\end{array}
\end{align*}
\]

Putting these two results together and using Eq. (7), we obtain

\[
\begin{align*}
\begin{array}{l}
    \mathcal{Z}_i \\ n_i & \equiv r_i^2 \\
    \mathcal{Z}_i \\ n_i & \equiv r_i^2 \\
\end{array}
\end{align*}
\]

which concludes this case.

For the final case \( j_{m-1} = j \neq i \), we compute

\[
\begin{align*}
\begin{array}{l}
    \mathcal{Z}_i \\ i & = r_i \\
    \mathcal{Z}_i \\ i & = r_i \\
    \mathcal{Z}_i \\ i & = r_i \\
\end{array}
\end{align*}
\]

using Eq. (8). Then we obtain for the first term on the RHS of the second equality, using the induction hypothesis together with Eq. (5)
On the other hand, we have for all \( t, v \) that
\[
\sum_{u \neq t} u \overbrace{v}^{t} = \overbrace{t}^{v} - \overbrace{v}^{t}
\]
Putting these results together with the case \( j_m \neq i \) yields
\[
\equiv r_i^2
\]
which concludes the proof. \( \square \)

**Proof of Proposition 4.7.** The polynomial Eq. \((24)\) is monic (up to invertible scalar) with leading terms \( x_{m+1}^{\alpha_i - \alpha_i} \). Therefore, multiplication by Eq. \((24)\) yields an injective map. Thus, Lemma 4.10 tells us that \( P \circ d_N \) is injective, and so is \( d_N \). \( \square \)

As a consequence, this also ends the proof of Theorem 4.4.

5. Categorical action

For each \( i \in I \) there is a (non-unital) inclusion \( R_b(m) \hookrightarrow R_b(m+1)_{1_{(m,i)}} \), given by adding a vertical strand with label \( i \) to the right of a diagram \( D \in R_b(m) \):

\[
\begin{array}{c}
\text{D} \\
\end{array}
\rightarrow
\begin{array}{c}
\text{D} \\
\end{array}
\]

This gives rise to induction and restriction functors
\[
\text{Ind}_m^{m+i}: R_b(m)\text{-mod} \to R_b(m+1)\text{-mod},
\]
\[
\text{Ind}_m^{m+i}(-) \cong R_b(m+1)_{1_{(m,i)}} \otimes_m -,
\]
\[
\text{Res}_m^{m+i}: R_b(m+1)\text{-mod} \to R_b(m)\text{-mod},
\]
\[
\text{Res}_m^{m+i}(-) \cong 1_{(m,i)}R_b(m+1) \otimes_{m+1} -.
\]

which are adjoint.

We write
\[
R_b^{xi}(\nu) := R_b(\nu) \otimes \mathbb{k}[\xi_i] \cong \bigoplus_{\ell \geq 0} q_i^{2\ell} R_b(\nu),
\]
with \( \deg q_i(\xi_i) = (\alpha_i|\alpha_i) \). We will prove the following theorem in the next subsection:
Theorem 5.1. There is a short exact sequence
\[
0 \to q_i^{-2}R_b(\nu)1_{(\nu-i,i)} \otimes_{m-1} 1_{(\nu-i,i)}R_b(\nu) \to 1_{(\nu,i)}R_b(\nu + i)1_{(\nu,i)} \to R_b^i(\nu) \oplus \lambda_i^2q_i^{-2\alpha_i^\nu(\nu)}R_b^i(\nu)[1] \to 0,
\]
of \(R_b(\nu)-R_b(\nu)\)-bimodules for all \(i \in I\). Moreover, there is an isomorphism
\[
q^{-(\alpha_i^\nu(\nu))}R_b(\nu)1_{(\nu-i,i)} \otimes_{m-1} 1_{(\nu'-j,j)}R_b(\nu') \cong 1_{(\nu',j)}R_b(\nu + i)1_{(\nu,i)},
\]
for all \(i \neq j \in I\) and \(\nu' + j = \nu + i\).

As we will see in the proof of Theorem 5.1, we can picture these facts as a short exact sequence of diagrams

\[
\begin{array}{ccc}
\cdots & j & \cdots \\
\downarrow \quad m & \downarrow & \downarrow \\
\cdots & j & \cdots \\
\downarrow \quad m+1 & \downarrow & \downarrow \\
i & \downarrow & i \\
\cdots & \cdots & \cdots \\
\end{array}
\xhookrightarrow{\oplus_{\ell \geq 0} \ell} \xrightarrow{\oplus_{\ell \geq 0} \ell} \begin{array}{ccc}
\cdots & j & \cdots \\
\downarrow \quad m & \downarrow & \downarrow \\
\cdots & j & \cdots \\
\downarrow \quad m & \downarrow & \downarrow \\
i & \downarrow & i \\
\cdots & \cdots & \cdots \\
\end{array}
\]

where the cokernel vanishes whenever \(i \neq j\). We write \(\pi\) for the projection

\[
\begin{array}{ccc}
\cdots & i & \cdots \\
\downarrow \quad m+1 & \downarrow & \downarrow \\
i & \downarrow & i \\
\cdots & \cdots & \cdots \\
\end{array}
\xrightarrow{\oplus_{\ell \geq 0} \ell} \xrightarrow{\oplus_{\ell \geq 0} \ell} \begin{array}{ccc}
\cdots & i & \cdots \\
\downarrow \quad m & \downarrow & \downarrow \\
i & \downarrow & i \\
\cdots & \cdots & \cdots \\
\end{array}
\]

We write \(\text{Id}_\nu := R_b(\nu) \otimes_m (-)\) and we define
\[
F_i := \bigoplus_{m \geq 0} \text{Ind}_m^{m+i}, \quad E_i := \bigoplus_{m \geq 0} \bigoplus_{|\nu| = m} \lambda_i^{-1}q_i^{1+\alpha_i^\nu(\nu)} \text{Res}_m^{m+i} \text{Id}_{\nu+i}.
\]
These are exact functors thanks to Lemma 4.5. Define
\[
(27) \quad \bigoplus_{[\beta_i^\nu(\nu)]_q} \text{Id}_\nu := \bigoplus_{\ell \geq 0} q_i^{1+2\ell} \left( \lambda_i^{-1}q_i^{\alpha_i^\nu(\nu)} \text{Id}_\nu \oplus \lambda_i q_i^{-\alpha_i^\nu(\nu)} \text{Id}_\nu[1] \right).
\]
It is a categorification of the fraction \(\frac{\lambda_i q_i^{-\alpha_i^\nu(\nu)} - \lambda_i^{-1}q_i^{\alpha_i^\nu(\nu)}}{q_i - q_i^{-1}}\). We obtain:

Corollary 5.2. There is a natural short exact sequence
\[
0 \to F_iE_i \text{Id}_\nu \to E_iF_i \text{Id}_\nu \to \bigoplus_{[\beta_i^\nu(\nu)]_q} \text{Id}_\nu \to 0,
\]
for all \(i \in I\), and there is a natural isomorphism
\[
F_iE_j \cong E_jF_i,
\]
for all \(i \neq j \in I\).
Proposition 5.3. For each $i, j \in I$ there is a natural isomorphism

$$
\bigoplus_{a=0}^{[(d_{ij}+1)/2]} \left[ \frac{d_{ij} + 1}{2a} \right] q_i^{2a} F_j F_i^{d_{ij} + 1 - 2a} \cong \bigoplus_{a=0}^{[d_{ij}/2]} \left[ \frac{d_{ij} + 1}{2a + 1} \right] q_i^{2a+1} F_j F_i^{d_{ij} - 2a},
$$

and in particular for $(\alpha_i | \alpha_j) = 0$ we have $F_i F_j 1_{\nu} \cong F_j F_i 1_{\nu}$. By adjunction, the same isomorphism exists for the $\mathcal{E}_i, \mathcal{E}_j$.

Proof. Similarly as in the case of the usual KLR algebras, it follows from Eq. (7) and Eq. (8) (the proof of [24, Proposition 6] can be applied directly).

5.1. Proof of Theorem 5.1. By symmetry along the horizontal axis, we obtain a decomposition of $R_b(m + 1)$ as a right $R_b(m)$-module similar to the one of Lemma 4.5. Note that the left and right decompositions are not compatible, and therefore we do not have a decomposition as a $R_b(m)$-$R_b(m)$-bimodule. However, the surjection $R_b(m + 1) \twoheadrightarrow q^{2\ell} R_b(m)$ that projects on the summand $R_b(m) x_{m+1}^e$, given by taking $a = m + 1$ in Lemma 4.5, is a (left-invertible) map of bimodules.

We define the map

$$
\pi_L^e : 1_{(\nu, i)} R_b(m + 1) 1_{(\nu, i)} \to \lambda_i^2 q_i^{2\ell - 2\alpha_i^\vee} R_b(\nu)[1],
$$

as the projection map on the summand $R_b(m) \theta_{n+1}^e$ in the left decomposition of $R_b(m + 1)$ as $R_b(m)$-module in Lemma 4.5. Similarly, let

$$
\pi_R^e : 1_{(\nu, i)} R_b(m + 1) 1_{(\nu, i)} \to \lambda_i^2 q_i^{2\ell - 2\alpha_i^\vee} R_b(\nu)[1],
$$

be the projection map on $\theta_{n+1}^e R_b(m)$ in the right decomposition.

Lemma 5.4. We have

$$
\pi_L^e(y) = (-1)^{\deg_b(y)} \pi_R^e(y)
$$

for all $y \in R_b(m + 1)$.

Proof. We can suppose $y = \theta_{n+1}^e y'$ with $y' \in R_b(m)$. We want to prove that $y = (-1)^{\deg_b(y)} y' \theta_{n+1}^e + y_0$ for some $y_0 \notin R_b(m) \theta_{n+1}^e$. For this, it is enough to show that $y_1 \theta_{n+1}^e y_2 = (-1)^{\deg_b(z)} y_1 z \theta_{n+1}^e y_2 + z_0$ where $y_1, y_2 \in R_b(m)$, $z_0 \notin R_b(m) \theta_{n+1}^e$ and $z$ is any generator of $R_b(m)$ (i.e. crossing, dot or tight floating dot).
If \( z = x_a \) and is on a strand labeled \( j \neq i \), then it slides freely over \( \theta_{m+1} \) thanks to Eq. (6). If the strand is labeled \( i \), then we compute

\[
\ell 
\begin{array}{c}
\text{(7)}
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
\ell
\end{array}

+ \begin{array}{c}
\tau_i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
l
\end{array}
\begin{array}{c}
l
\end{array}

- r_i^{-1}
\begin{array}{c}
l
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
l
\end{array}

\begin{array}{c}
l
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
l
\end{array}

\begin{array}{c}
l
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
\ell
\end{array}

\begin{array}{c}
\text{(8)}
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
\ell
\end{array}

+ \begin{array}{c}
\tau_i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
l
\end{array}
\begin{array}{c}
l
\end{array}

- r_i^{-1}
\begin{array}{c}
l
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
l
\end{array}

\begin{array}{c}
l
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
\ell
\end{array}

\begin{array}{c}
\text{+ R,}
\end{array}

where the double strands represent multiple parallel strands (the number depending on \( m \) and \( a \)), and \( R \) is a sum of terms of the following form:

\[
\ell
\begin{array}{c}
\text{and its mirror along the horizontal axis. Note that it is implicitly assumed that each of}
\end{array}
\begin{array}{c}
\text{these diagrams have the element } y_1 \text{ at the top and } y_2 \text{ at the bottom. Using Lemma 4.5,}
\end{array}
\begin{array}{c}
\text{we can rewrite the composition of the last three terms in the equation above with } y_2 \text{ as elements in}
\end{array}
\begin{array}{c}
\bigoplus_{a=1}^{n} 
\bigoplus_{p \geq 0} \big( R_b(m-1) \tau_m \tau_{m-1} \cdots \tau_a x_a^p \big) + R_b(m) \theta_{m+1}. \text{ Hence they form the term}
\end{array}
\begin{array}{c}
z_0.\end{array}

If \( z = \tau_i \) is a crossing, then we obtain the desired property by Eq. (8), and applying a similar reasoning as above.

Finally if \( z = \omega \) and is at right of a strand labeled \( j \neq i \), it follows directly from Eq. (17). Otherwise, if the strand is labeled \( i \), we compute

\[
\ell 
\begin{array}{c}
\text{(9,7)}
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
\ell
\end{array}

= \begin{array}{c}
\text{(17)}
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
\ell
\end{array}

- \begin{array}{c}
l
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
l
\end{array}
\begin{array}{c}
\ell
\end{array}

+ r_i^{-1}
\begin{array}{c}
\sum_{r+s \leq \ell-1} r
\end{array}
\begin{array}{c}
r
\end{array}
\begin{array}{c}
s
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
l
\end{array}

Then for all \( r, s \geq 0 \) we compute using Eq. (7) again

\[
\begin{array}{c}
r
\end{array}
\begin{array}{c}
s
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
\text{=}
\end{array}
\begin{array}{c}
s
\end{array}
\begin{array}{c}
r
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
i
\end{array}
\begin{array}{c}
\text{.}
\end{array}
\]
Looking at these elements in the global picture yields
\[ \begin{align*}
&\begin{array}{c}
\text{s} \\
\text{r}
\end{array}
\begin{array}{c}
i \\
j
\end{array}
\begin{array}{c}
m \\
\ldots
\end{array} = \\
\begin{array}{c}
\text{s} \\
\text{r}
\end{array}
\begin{array}{c}
i \\
j
\end{array}
\begin{array}{c}
m \\
\ldots
\end{array} + R
\end{align*}\]
which is an element not contained in \( R_b(m)\theta_{n+1}^\ell \) for the same reasons as before. We see that together they form the element \( z_0 \), concluding the proof. \( \square \)

We now have all the ingredients we need to prove Theorem 5.1.

**Proof of Theorem 5.1.** We first construct an injective map
\[ u_{ij} : q^{-(\alpha_i | \alpha_j)} R_b(\nu)1_{(m-1,i)} \otimes_{m-1} 1_{(m,j)} R_b(\nu) \hookrightarrow 1_{(\nu,j)} R_b(m+1) 1_{(\nu,i)} \]
of \( R_b(m)-R_b(m) \)-bimodules, by setting (as in [19, Proposition 3.3])
\[ u_{ij}(x \otimes_{m-1} y) := x \tau_{m} y. \]

In terms of diagrams, it consists of adding a crossing at the right
\[ \begin{array}{c}
\begin{array}{c}
m \\
\ldots
\end{array} \begin{array}{c}
i \\
j\begin{array}{c}
m \\
\ldots
\end{array} \end{array} \\
\begin{array}{c}
\begin{array}{c}
m \\
\ldots
\end{array} \begin{array}{c}
i \\
j\begin{array}{c}
m \\
\ldots
\end{array} \end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
m \\
\ldots
\end{array} \begin{array}{c}
i \\
j\begin{array}{c}
m \\
\ldots
\end{array} \end{array} \\
\begin{array}{c}
\begin{array}{c}
m \\
\ldots
\end{array} \begin{array}{c}
i \\
j\begin{array}{c}
m \\
\ldots
\end{array} \end{array}
\end{array} \end{array} \]

Then, we construct a surjective map
\[ 1_{(\nu,i)} R_b(m+1) 1_{(\nu,i)} \twoheadrightarrow R_b(\nu)1_{(m-1,i)} \otimes_{m-1} 1_{(m,j)} R_b(\nu) \]
by projecting onto the direct summands
\[ \bigoplus_{\ell \geq 0} x_{m+1}^\ell \bigoplus \bigoplus_{a=1}^m \bigoplus_{\ell \geq 0} R_b(m-1)1_{(m,i)} \tau_{m-1} \cdots \tau_a x_a^\ell 1_{j} \]
and so
\[ u_{ij}(R_b(\nu)1_{(m-1,i)} \otimes_{m-1} 1_{(m,j)} R_b(\nu)) \]
\[ \approx \bigoplus_{a=1}^m \bigoplus_{\ell \geq 0} \bigoplus \bigoplus_{\ell \geq 0} R_b(m-1)1_{(m,i)} \tau_{m} \tau_{m-1} \cdots \tau_a x_a^\ell 1_{j} \]
Finally, exactness follows directly from Lemma 4.5, since
\[ R_b(\nu)1_{(m-1,i)} \otimes_{m-1} 1_{(m,j)} R_b(\nu) \]
and so
\[ u_{ij}(R_b(\nu)1_{(m-1,i)} \otimes_{m-1} 1_{(m,j)} R_b(\nu)) \]
\[ \approx \bigoplus_{a=1}^m \bigoplus_{\ell \geq 0} \bigoplus \bigoplus_{\ell \geq 0} R_b(m-1)1_{(m,i)} \tau_{m} \tau_{m-1} \cdots \tau_a x_a^\ell 1_{j} \]
\[ \oplus R_b(m-1)1_{(m,i)} \tau_{m-1} \cdots \tau_a \theta_a^\ell 1_{j}. \]
We remark that whenever $i \neq j$, we have
\[
\bigoplus_{\ell \geq 0} 1_{(\nu,j)} x_{m+1}^\ell R_b(m) 1_{(\nu,i)} \oplus 1_{(\nu,j)} R_b(m) \theta_{m+1}^\ell 1_{(\nu,i)} = 0,
\]
and thus $u_{ij}$ is an isomorphism, concluding the proof.

5.2. Long exact sequence. We want to lift Theorem 5.1 to the dg-world of $(R_b(m), d_N)$, and study the long exact sequence that it induces. Therefore we define
\[
y_N : \bigoplus_{\ell \geq 0} \to \bigoplus_{\ell \geq 0}
\]
as the $R_b(m)$-$R_b(m)$-bimodule map given by
\[
y_N \left( \begin{array}{c} \vdots \\ a \\ \vdots \end{array} \right) := \pi \left( \begin{array}{c} n_i + a \\ \vdots \end{array} \right) \in \bigoplus_{\ell \geq 0}
\]
whenever $i \in I_f$, and $y_N = 0$ for $i \notin I_f$. Then we define
\[
(R_b(\nu) \oplus \lambda_i^2 \theta_{m+1}^\ell R_b(\nu)[1], d_N)
\]
\[
:= \text{Cone} \left( (\lambda_i^2 \theta_{m+1}^\ell R_b(\nu)[1], d_N) \xrightarrow{y_n} (R_b(\nu), d_N) \right),
\]
and
\[
(R_b(\nu) 1_{(m-1,i)} \otimes_{m-1} 1_{(m-1,j)} R_b(\nu), d_N)
\]
\[
:= (R_b(\nu) 1_{(m-1,i)}, d_N) \otimes (R_b(\nu), d_N) (1_{(m-1,j)} R_b(\nu), d_N).
\]

Proposition 5.5. There is a short exact sequence of dg-bimodules
\[
0 \to q_i^{-(\alpha_i \gamma_i)} (R_b(\nu) 1_{(\nu-i,i)} \otimes_{m-1} 1_{(\nu-i,j)} R_b(\nu), d_N) \to (1_{(\nu,i)} R_b(\nu + i) 1_{(\nu,i)}, d_N)
\]
\[
\to (R_b(\nu) \oplus \lambda_i^2 \theta_{m+1}^\ell R_b(\nu)[1], d_N) \to 0
\]
for all $i \in I$. Moreover, there is an isomorphism
\[
q^{-(\alpha_i \gamma_i)} (R_b(\nu) 1_{(\nu-i,i)} \otimes_{m-1} 1_{(\nu-j,j)} R_b(\nu'), d_N) \cong (1_{(\nu',j)} R_b(\nu + i) 1_{(\nu',i)}, d_N)
\]
for all $i \neq j \in I$ and $\nu + i = \nu' + j$.

Proof. It is a straightforward consequence of Theorem 5.1.

In order to understand the consequences of this short exact sequence in homology, we need to compute the homology
\[
H(R_b(\nu) \oplus \lambda_i^2 \theta_{m+1}^\ell R_b(\nu)[1], d_N),
\]
for all \( i \in I_f \).

Therefore, we want to compute the projection of the element

\[
\bar{\pi} \left( \begin{array}{c}
\vdots \\
p
\vdots \\
\end{array} \right) \in \bigoplus_{\ell \geq 0} \begin{array}{c}
\vdots \\
H(m)
\vdots \\
\end{array}
\]

for all \( p \geq n_i \). Note that we project on the homology of \((R_b(m), d_N)\). This will ease some of the computations we need to do. We write \( \bar{\pi} \) when we take the composite of \( \pi \) with the projection on the homology of \((R_b(m), d_N)\). More precisely, \( \bar{\pi} \) is given by

\[
\bar{\pi} := 1 \otimes \pi : \begin{array}{c}
\vdots \\
H(m)
\vdots \\
\end{array} \rightarrow \bigoplus_{\ell \geq 0} \begin{array}{c}
\vdots \\
H(m)
\vdots \\
\end{array}
\]

Similarly, we write \( \bar{y}_N \).

**Lemma 5.6.** If \( p \geq 2\nu_i \), then

\[
\pi \left( \begin{array}{c}
\vdots \\
p
\vdots \\
\end{array} \right) = \zeta \begin{array}{c}
\vdots \\
p - \alpha_i^\vee(\nu) + \bigoplus_{\ell = 0}^{p - \alpha_i^\vee(\nu) - 1} \begin{array}{c}
\vdots \\
H(m)
\vdots \\
\end{array}
\end{array}
\]

for some invertible element \( \zeta \in \mathbb{k}^\times \). If \( p < 2\nu_i \), then

\[
\pi \left( \begin{array}{c}
\vdots \\
p
\vdots \\
\end{array} \right) \in \bigoplus_{\ell = 0}^{p - \alpha_i^\vee(\nu)} \begin{array}{c}
\vdots \\
H(m)
\vdots \\
\end{array}
\]

**Proof.** The proof is an induction on \( m \). If \( m = 0 \), then it is trivial. Suppose the statement holds for \( m - 1 \). We fix the labels of the strands as the bottom as \( j = j_1 \cdots j_m \in \text{Seq}(\nu) \). If \( j_1 = i \), then we compute

\[
\begin{array}{c}
\vdots \\
p
\vdots \\
\end{array} = \sum_{r = 1}^{r+i} r \sum_{s = p-1}^{r+s} \begin{array}{c}
\vdots \\
p
\vdots \\
\end{array} - r^2 \sum_{s = p-2}^{r+s} (r+1) \begin{array}{c}
\vdots \\
p
\vdots \\
\end{array}
\]

Proof. The proof is an induction on \( m \). If \( m = 0 \), then it is trivial. Suppose the statement holds for \( m - 1 \). We fix the labels of the strands as the bottom as \( j = j_1 \cdots j_m \in \text{Seq}(\nu) \). If \( j_1 = i \), then we compute

\[
\begin{array}{c}
\vdots \\
p
\vdots \\
\end{array} = \sum_{r = 1}^{r+i} r \sum_{s = p-1}^{r+s} \begin{array}{c}
\vdots \\
p
\vdots \\
\end{array} - r^2 \sum_{s = p-2}^{r+s} (r+1) \begin{array}{c}
\vdots \\
p
\vdots \\
\end{array}
\]
Then, using Eq. (8) we have
\[
\sum_{\substack{r+s \\ r+s \leq p-3}} (r+2) \pi \left( \begin{array}{c} r+1 \\ i \\ i \end{array} \right) \in \bigoplus_{\ell=0}^{p-\alpha_s'(\nu)-1} \begin{array}{c} p-\alpha_s'(\nu)-1 \\ m \end{array} \bigg\} \ell \bigg\} \begin{array}{c} \ell \\ i \end{array}
\]
so that, since \( s < d_{ijk} \), we obtain by the induction hypothesis
\[
\pi \left( \begin{array}{c} s \\ i \\ i \end{array} \right) \equiv \zeta' \bigg\} \begin{array}{c} s-\alpha_s'(\nu-i) \\ i \end{array} \bigg\} \begin{array}{c} s-\alpha_s'(\nu-i)-1 \\ m \end{array} \bigg\} \ell \bigg\} \begin{array}{c} \ell \\ i \end{array}
\]
Moreover, still by the induction hypothesis, we have
\[
\sum_{\substack{r+s \\ r+s \leq p-3}} (r+2) \pi \left( \begin{array}{c} r+1 \\ i \\ i \end{array} \right) \in \bigoplus_{\ell=0}^{p-\alpha_s'(\nu)-1} \begin{array}{c} p-\alpha_s'(\nu)-1 \\ m \end{array} \bigg\} \ell \bigg\} \begin{array}{c} \ell \\ i \end{array}
\]
Finally, if \( p \geq 2\nu_s \), by the induction hypothesis we get for \( s = p-2 \),
\[
\pi \left( \begin{array}{c} s \\ i \\ i \end{array} \right) \equiv \zeta' \bigg\} \begin{array}{c} s-\alpha_s'(\nu-i) \\ i \end{array} \bigg\} \begin{array}{c} s-\alpha_s'(\nu-i)-1 \\ m \end{array} \bigg\} \ell \bigg\} \begin{array}{c} \ell \\ i \end{array}
\]
which concludes the case by observing that \( s-\alpha_s'(\nu-i) = p-\alpha_s'(\nu) \), and taking \( \zeta = r_s^2 \zeta' \).

If \( p < 2\nu_s \), the claim is immediate by the induction hypothesis.
For the case \( j_1 = j \neq i \), we use Eq. (5) and then the induction hypothesis to get
\[
\pi \left( \begin{array}{c} p \\ j \\ i \end{array} \right) = \sum_{t,v} \pi \left( \begin{array}{c} v \\ j \\ i \end{array} \right) \bigg\} \begin{array}{c} \ell \\ i \end{array}
\]
where we recall that \( s_{ij}^{d_{ij}0} = t_{ij} \). We conclude by applying the induction hypothesis, observing that \( d_{ij} + p - \alpha_i^\nu (\nu - j) = p - \alpha_i^\nu (\nu) \).

Consider also the following result, which is akin to [19, Lemma 5.4].

**Lemma 5.7.** We have for \( k < k' \) and \( t = k' - k \),

\[
\bar{y}_N \left( \begin{array}{c} k' \\ i \end{array} \right) = \bar{y}_N \left( \xi_i^k \right) + \sum_{\ell=0}^{t-1} \left( H(m) \right)_i \ell
\]

where

\[
\bar{y}_N \left( \xi_i^k \right) = \bar{y}_N \left( \begin{array}{c} k \\ i \end{array} \right)
\]

**Proof.** First we observe that

\[
n_i + k' = n_i + k \in H(R_b(m), d_N).
\]

using Eq. (7) and Eq. (6), and the fact that \( n_i \) dots on the left strand is annihilated in \( H(R_b(m), d_N) \).

Then, using Lemma 4.5 we obtain

\[
n_i + k = \psi_k + \bar{y}_N \left( \xi_i^k \right)
\]

for some \( \varphi_k, \psi_k \in R_b(m) \). We conclude by observing that

\[
\psi_k = \psi_k - r_i \sum_{r+s = i-1}^{r+s} \varphi_k
\]

thanks to Eq. (7).
Proposition 5.8. Putting $\rho_i := n_i - \alpha_i^\vee(\nu)$, we have

$$
\tilde{y}_N \left( \begin{array}{c}
\nu \\
\ldots
\end{array} \right) = \zeta \quad \vdots \quad \left\{ k + \rho_i + \bigoplus_{\ell=0}^{k+\rho_i-1} H(m) \right\}_i
$$

which is 0 whenever $k + \rho_i < 0$, and where $\zeta \in k^\times$.

Proof. If $n_i \geq 2\nu_i$, then the result follows from Lemma 5.6. Otherwise, we take $k' = 2\nu_i - n_i$ and the result follows from Lemma 5.6 for $k \geq k'$. Suppose $k < k'$ and put $t = k' - k$. Then, by Lemma 5.7 we obtain

$$
\tilde{y}_N \left( \begin{array}{c}
k' \\
\ldots
\end{array} \right) = \tilde{y}_N(\xi^k) + \sum_{\ell=0}^{t-1} \bigoplus_{\ell=0}^{\max(k' + \rho_i, t) - 1} H(m) \left\{ \ell \right\}_i
$$

Therefore, we have

$$
\tilde{y}_N(\xi^k) \equiv \zeta \quad \vdots \quad \left\{ k' + \rho_i + \bigoplus_{\ell=0}^{k+\rho_i-1} H(m) \right\}_i
$$

From this, we deduce

$$
\tilde{y}_N(\xi^k) \equiv \zeta \quad \vdots \quad \left\{ k + \rho_i + \bigoplus_{\ell=0}^{k+\rho_i-1} H(m) \right\}_i
$$

which concludes the proof. \[ \Box \]

We now have all the tools we need to compute the homology of the cokernel of the short exact sequence of Proposition 5.5.

Proposition 5.9. There is an isomorphism of $R_p^N(\nu)$-$R_p^N(\nu)$-bimodules

$$
H \left( R_b^N(\nu) \oplus \lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} R_b^N(\nu)[1], d_N \right)
$$

\[ \cong \begin{cases} 
\bigoplus_{\ell=0}^{\rho_i-1} q_i^{2\ell} R_p^N(\nu), & \text{if } \rho_i \geq 0, \\
\lambda_i^2 q_i^{-2\alpha_i^\vee(\nu)} \bigoplus_{\ell=0}^{-\rho_i-1} q_i^{2\ell} R_p^N(\nu)[1], & \text{if } \rho_i \leq 0,
\end{cases} \]

where $\rho_i = n_i - \alpha_i^\vee(\nu)$. 
Proof. First suppose $\rho_i \geq 0$. Then, Proposition 5.8 tells us that $\tilde{y}_N(\xi^k_i)$ is a monic polynomial (up to invertible scalar) with leading terms $\xi^{k+\rho_i}$. This gives us the first case. If $\rho_i \leq 0$, then we have $\tilde{y}_N(\xi^k_i) = 0$ for $k < -\rho_i$. Moreover, $\zeta^{-1} \tilde{y}_N(\xi^{-\rho_i}_i) = 1$, and in general $\tilde{y}_N(\xi^k_i)$ is a monic polynomial with leading term $\xi^{k+\rho_i}$ for $k > -\rho_i$. This concludes the proof. \qed

5.3. Strongly projective dg-modules. The following notions were originally introduced by Moore [31]. We use the presentation given in [41], which is best suited for our notations.

**Definition 5.10** ([41, Definition 8.5]). Let $(R, 0)$ be a ring $R$ viewed as a dg-$\mathbb{Z}$-algebra concentrated in degree zero. An $(R, 0)$-module $(Q, d_Q)$ is strongly projective if $H(Q, d_Q)$ and $\text{im} d_Q$ are both projective $R$-modules.

**Lemma 5.11** ([44, Theorem 9.3.2]). Let $(P, d_P)$ be a strongly projective right $(R, 0)$-module and $(N, d_N)$ any left $(R, 0)$-module, then

$$H\left((P, d_P) \otimes_{(R, 0)} (N, d_N)\right) \cong H(P, d_P) \otimes_R H(N, d_N).$$

**Definition 5.12** ([41, Definition 8.17]). Let $(A, d_A)$ be a dg-$R$-algebra. A left (resp. right) $(A, d_A)$-module $(P, d_P)$ is strongly projective if it is a dg-direct summand of $(A, d_A) \otimes_{(R, 0)} (Q, d_Q)$ (resp. $(Q, d_Q) \otimes_{(R, 0)} (A, d_A)$) for some strongly projective $(R, 0)$-module $(Q, d_Q)$.

**Proposition 5.13** ([41, Lemma 8.23]). If $(P, d_P)$ is a strongly projective right $(A, d_A)$-module and $(N, d_N)$ is any left $(A, d_A)$-module, then

$$H\left((P, d_P) \otimes_{(A, d_A)} (N, d_N)\right) \cong H(P, d_P) \otimes_{H(A, d_A)} H(N, d_N).$$

Note that if $(P, d_P)$ is a strongly projective $(A, d_A)$-module, then $H(P, d_P)$ is a projective $H(A, d_A)$-module. Indeed, we can assume $(P, d_P) = (A, d_A) \otimes_{(R, 0)} (Q, d_Q)$, and we have

$$H(P, d_P) \cong H(A, d_A) \otimes_R H(Q, d_Q).$$

Since $H(Q, d_Q)$ is a projective $R$-module, it is a direct summand of a free $R$-module $F$. Therefore $H(P, d_P)$ is a direct summand of $H(A, d_A) \otimes_R F$, which is a free $H(A, d_A)$-module.

**Remark 5.14.** This result does not hold in general. As a counterexample we can take $(A, d) = (\mathbb{Q}[x], 0)$ and consider the dg-module $(X, d_X) = \text{Cone}(\mathbb{Q}[x] \rightarrow \mathbb{Q}[x])$. In this case we have that $H(X, d_X) \cong \mathbb{Q}$ but $H((X, d_X) \otimes_{(A, d)} (X, d_X)) \cong \mathbb{Q} \oplus \mathbb{Q}[1]$.

5.3.1. Strong projectivity of $R_b(m + 1)$. Our next goal is to show the following:

**Proposition 5.15.** The $(R_b(m), d_N)$-module $(1_{(m, i)} R_b(m + 1), d_N)$ is strongly projective.

It is obvious for $i \notin I_f$ by Lemma 4.5, and thus we can assume $i \in I_f$. We first construct the mapping cone

$$(Q, d_Q) := \text{Cone}\left(\bigoplus_{a=1}^{m+1} R_\ell(m)1_{(\nu, i)} \tau_m \cdots \tau_a \theta_a^e \rightarrow \bigoplus_{a=1}^{m+1} R_\ell(m)1_{(\nu, i)} \tau_m \cdots \tau_a \xi_a^e\right),$$
where we think of $\tau_m \cdots \tau_a \theta_a^\ell$ as a formal symbol that represents a degree shift corresponding to the degree of the element $1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^\ell$ in $R_b(m+1)$. The map $d_Q$ is given by first embedding $R_q(m)$ into $R_b(m+1)$ through the diagrams

$$R_q(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^\ell \hookrightarrow$$

then applying $d_N$ of $(R_b(m+1), d_N)$, then decomposing the image in the left-decomposition

$$\bigoplus_{a=1}^{m+1} \bigoplus_{\ell \geq 0} R_b(m)1_{(\nu,i)}\tau_m \cdots \tau_a x_a^\ell$$

and finally projecting unto the part in homological degree zero of $R_b(m)$, which is trivially isomorphic to $R_q(m)$. Moreover, $(R_b(m), d_N)$ is a (right) module over $(R_q, 0)$ which acts by gluing KLR diagrams on the bottom. Then, we have, as $(R_b(m), d_N)$-modules

$$(R_b(m+1), d_N) \cong (R_b(m), d_N) \otimes_{(R_q(m), 0)} (Q, d_Q).$$

Therefore, we want to show that $(Q, d_Q)$ is strongly projective as $(R_q(m), 0)$-module. We write

$$Q_1[\xi_i] := \bigoplus_{a=1}^{m+1} \bigoplus_{\ell \geq 0} R_q(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^\ell,$$

$$Q_0[\xi_i] := \bigoplus_{a=1}^{m+1} \bigoplus_{\ell \geq 0} R_q(m)1_{(\nu,i)}\tau_m \cdots \tau_a x_a^\ell,$$

where we identify $\xi_i$ with $x_a$ in $Q_0$, and $\xi_i^\ell$ with $x_a^\ell$ in $\theta_a^\ell$. Note that $d_Q$ is not $k[\xi_i]$-linear.

**Lemma 5.16.** The map $$d_Q : Q_1[\xi_i] \to Q_0[\xi_i]$$

defined above is injective.

**Proof.** Recall the map $P$ of Lemma 4.9 given by multiplication by $\tilde{\theta}_{m+1}$. Since floating dots are also annihilated in $R_q(m)$, multiplication by $\tilde{\theta}_{m+1}$ also defines a map

$$(31) \quad P' : Q_0[\xi_i] \to Q_1[\xi_i].$$

We reconsider the proof of Lemma 4.10 to show that $P' \circ d_Q$ is injective. First, we introduce an order on the summands of $Q_1[\xi_i] = \bigoplus_{a=1}^{m+1} \bigoplus_{\ell \geq 0} R_q(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^\ell$ by declaring that

$$R_q(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^\ell < R_q(m)1_{(\nu,i)}\tau_m \cdots \tau_{a'} \theta_{a'}^\ell,$$

$$R_q(m)1_{(\nu,i)}\tau_m \cdots \tau_a \theta_a^\ell < R_q(m)1_{(\nu,i)}\tau_m \cdots \tau_{a} \theta_{a}^\ell.$$  

for all $a > a'$, $\ell < \ell'$, and for all $\ell''$. In other words, if there are more crossings under the floating dot, then the term is smaller. If there is the same amount of crossings, then we consider the amount of dots at the left of the floating dot, and lesser dots meaning a smaller term.
We claim that if $Z \in R_\ell(m)1_{(\nu,i)}\tau_m \cdots \tau_0\theta_\ell$ then
\[ P' \circ d_Q(Z) = r_i^{2\nu_i} \sum_{p=0}^{2\nu_i-\alpha_\ell} Z x_m^{n_i+p+\ell} + H, \]
where $H < Z x_m^{n_i+2\nu_i-\alpha_\ell}$. This implies that $P' \circ d_Q$ is in echelon form (with pivot being invertible scalars), and thus is injective. By consequence, so is $d_Q$.

In order to prove our claim, we need to tweak the proof of Lemma 4.10. We need to keep track of the terms that are annihilated when working over the cyclotomic quotient, and show these appear as lower terms in the order defined above. The case $j_m = i$ remains the same. The case $j_m \neq i$ becomes
\[
\begin{array}{cc}
\text{p} & = r_i^2 - r_i^2 + r_i - r_i \\
\text{i} & \text{i} & \text{i} & \text{i} & \text{i} & \text{i}
\end{array}
\]
where $p = n_i + \ell$. The first term is the leading term. The second term possesses less dots on the left of the floating dot, and so it is smaller. If $a = 0$, then the last two terms possess one more crossing at the bottom of the floating dot, and therefore they are smaller. If $a = 1$, then they are annihilated by Eq. (5). Finally, the two remaining cases $j_{m-1} \neq i$ and $j_{m-1}$ follow from the same arguments as in the proof of Lemma 4.10, with the lower terms in the induction hypothesis only adding lower terms because:
\[
\begin{array}{cc}
\text{i} & \text{i} \\
\text{i} & \text{i} & \text{i} & \text{i}
\end{array}
\]
by (8), and,
\[
\begin{array}{cc}
\text{i} & \text{i} \\
\text{i} & \text{i}
\end{array}
\]
by Eq. (8) and Eq. (5). This concludes the proof of the claim, and therefore of the proposition. \qed

Proof of Proposition 5.15. The proof is a revisitation of the proof of [19, Lemma 4.18] that applies to our particular case.

Recall the map $P''$ from Eq. (31). We know that $P'' \circ d_Q$ is given by multiplying by a monic polynomial with leading term $x_m^{n_i+2\nu_i-\alpha_\ell}$ plus some remaining map giving lower terms. In particular, it is injective and we have a short exact sequence
\[ 0 \to Q_1[\xi_i] \xrightarrow{P'' \circ d_Q} Q_1[\xi_i] \to \text{cok}(P' \circ d_Q) \to 0. \]
Moreover, since $P' \circ d_Q$ is in echelon form, it means that $\cok(P' \circ d_Q)$ is a projective $R_q(m)$-module. Thus, the sequence splits as $R_q(m)$-modules with splitting map $\sigma : Q_1[\xi] \to Q_1[\xi]$, and we get $\sigma \circ P' \circ d_Q = \Id_{Q_1[\xi]}$. Then, the short exact sequence
\[
0 \longrightarrow Q_1[\xi] \xrightarrow{d_Q} Q_0[\xi] \longrightarrow H(Q, d_Q), \longrightarrow 0,
\]

obtained thanks to Lemma 5.16 splits with splitting map given by $\sigma \circ P'$. Since $Q_0[\xi]$ is a projective $R_q(m)$-module, so is $H(Q, d_Q)$. Finally, $d_Q(Q_1[\xi])$ is also projective since $d_Q$ is injective and $Q_1[\xi]$ is projective. \hfill $\square$

5.4. **Functors.** We define for all $i \in I$ the functors
\[
F_i^N (-) := \bigoplus_{m \geq 0} R^N_p (m + 1) 1_{(m, i)} \otimes R^N_p (m) (-),
\]

\[
E_i^N (-) := \bigoplus_{m \geq 0} \bigoplus_{|\nu|=m} \lambda_i^{-1} q_i^{-1 + \alpha_i^\nu} 1_{(\nu, i)} R^N_p (m + 1) \otimes R^N_p (m + 1) (-),
\]

where we interpret $\lambda_i = q^{n_i}$ whenever $i \in I_f$. Thanks to Proposition 5.15, these are exact. For $n \in \mathbb{N}$, we write
\[
\bigoplus_{[n]_{q_i}} \Id_{\nu} := \bigoplus_{\ell=0}^{n-1} q_i^{\ell-n+2\ell} \Id_{\nu},
\]

for the finite direct sum that categorifies $[n]_{q_i}$.

**Theorem 5.17.** For $i \notin I_f$ there is a natural short exact sequence
\[
0 \to F_i^N E_i^N \Id_{\nu} \to E_i^N F_i^N \Id_{\nu} \to \bigoplus_{[\beta_i - \alpha_i^\nu \nu]_{q_i}} \Id_{\nu} \to 0,
\]

and for $i \in I_f$ there are natural isomorphisms
\[
E_i^N F_i^N \Id_{\nu} \cong F_i^N E_i^N \Id_{\nu} \bigoplus_{[n_i - \alpha_i^\nu \nu]_{q_i}} \Id_{\nu}, \quad \text{if } n_i - \alpha_i^\nu (\nu) \geq 0,
\]

\[
E_i^N F_i^N \Id_{\nu} \cong E_i^N F_i^N \Id_{\nu} \bigoplus_{[\alpha_i^\nu \nu - n_i]_{q_i}} \Id_{\nu}, \quad \text{if } n_i - \alpha_i^\nu (\nu) \leq 0.
\]

Moreover, there is a natural isomorphism
\[
F_i^N E_j^N \cong E_j^N F_i^N,
\]

for $i \neq j \in I$.

**Proof.** The short exact sequence Eq. (32) and the isomorphism Eq. (34) are immediate consequences of Proposition 5.5 and Proposition 5.15. For the isomorphisms Eq. (33), Proposition 5.5 and Proposition 5.15 give a long exact sequence of $R^N_p (\nu)-R^N_p (\nu)$-bimodules. By Proposition 5.9 it truncates to a short exact sequence
\[
0 \to F_i^N E_i^N \Id_{\nu} \to E_i^N F_i^N \Id_{\nu} \to \bigoplus_{[\rho_i]} \Id_{\nu} \to 0,
\]

if $\rho_i = n_i - \alpha_i^\nu (\nu) \geq 0$, and a short exact sequence
\[
0 \to \bigoplus_{[-\rho_i]} \Id_{\nu} \to F_i^N E_i^N \Id_{\nu} \to E_i^N F_i^N \Id_{\nu} \to 0,
\]
if $\rho_i = n_i - \alpha_i^\vee (\nu) \leq 0$. In the first case, we can identify

$$
\oplus_{[\rho_i]} \text{Id}_\nu \cong q_i^{1-\rho_i} \bigoplus_{\ell=0}^{\rho_i-1} \nu \bigcup_i \ell
$$

and the map $E_i^N F_i^N \text{Id}_\nu \to \oplus_{[\rho_i]} \text{Id}_\nu$ is induced by the projection $\pi$. Thus the sequence splits with the splitting map $\oplus_{[\rho_i]} \text{Id}_\nu \to E_i^N F_i^N \text{Id}_\nu$, given by the sum of maps $R_p^N (\nu) \xi^\ell \to R_p^N (\nu + i)$ that add a vertical strand labeled $i$ carrying $\ell$ dots at the right of a diagram in $R_p^N (\nu)$. In the second case, we also identify

$$
\oplus_{[-\rho_i]} \text{Id}_\nu \cong q_i^{1+\rho_i} \bigoplus_{\ell=0}^{-\rho_i-1} \nu \bigcup_i \ell
$$

Moreover the map $\oplus_{[-\rho_i]} \text{Id}_\nu \to F_i^N E_i^N \text{Id}_\nu$ is induced by the connecting homorphism $\delta$. Using the notations of Eq. (29) it takes the form

$$
\delta \left( \begin{array}{c}
\ldots
\bigcup_i k
\ldots
\end{array} \right) = u_{ij}^{-1} \left( \begin{array}{c}
\ldots
\bigcup_i k
\ldots
\end{array} \right) = \left( \begin{array}{c}
\psi_k
\ldots
\phi_k
\ldots
\psi_{k+1}
\phi_{k+1}
\end{array} \right)
$$

where $u_{ij}$ is the monomorphism defined in Eq. (28), and $0 \leq k < -\rho_i$. We also note that Eq. (30) tells us that

$$
(35)
$$

Moreover since $g_N (\xi_i^{-\rho_i}) = \zeta$ and $\bar{g}_N (\xi_i^\ell) = 0$ for $\ell < -\rho_i$, we obtain by Eq. (30) again that

$$
(36)
$$
As in [19, Proof of Theorem 5.2], we construct a map $\Phi : F^N_i E^N_i \rightarrow \bigoplus_{-\rho_1} \text{Id}_\nu$ induced by the morphism of bimodules

$$\Phi : \begin{bmatrix} y \\ \vdots \\ x \end{bmatrix} \mapsto \sum_{r+s = -\rho_1 - 1} r \begin{bmatrix} k \\ \vdots \end{bmatrix}_i$$

for all $x, y \in R^N_p(\nu)$.

Then we compute

$$\Phi \circ \delta \left( \begin{bmatrix} \vdots \\ k \end{bmatrix}_i \right) \overset{(35)}{=} \sum_{r+s = -\rho_1 - 1} r \begin{bmatrix} \psi_{k+r} \\ \vdots \\ \phi_{k+r} \end{bmatrix}_i \begin{bmatrix} \vdots \\ k \end{bmatrix}_i + \sum_{r = -\rho_1 - k} -\rho_1 - 1 - r \begin{bmatrix} \psi_{k+r} \\ \vdots \\ \phi_{k+r} \end{bmatrix}_i \begin{bmatrix} \vdots \\ -1 \end{bmatrix}_i.$$

Therefore, $\Phi \circ \delta$ is given by a triangular matrix with invertible elements on the diagonal, and thus is an isomorphism. In particular, $\delta$ is left invertible, concluding the proof. \qed

**Corollary 5.18.** For $i \in I_f$, then $1_\nu E_i$ and $F_i 1_\nu$ are biadjoint (up to shift).

**Proof.** By the results in [6], we know the splitting map $E^N_i F^N_i \text{Id}_\nu \rightarrow F^N_i E^N_i \text{Id}_\nu$ of Theorem 5.17 together with the unit and counit of the adjunction $F_i \dashv E_i$ allow to construct a unit and counit for the adjunction $E_i \dashv F_i$. \qed

**Proposition 5.19.** For each $i, j \in I$ there is a natural isomorphism

$$\bigoplus_{a=0}^{\lfloor (d_{ij}+1)/2 \rfloor} \left[ d_{ij} + \frac{1}{2a} \right]_{q_i} (F^N_i)^{2a} F^N_j (E^N_i)^{d_{ij}+1-2a} \cong \bigoplus_{a=0}^{\lfloor d_{ij}/2 \rfloor} \left[ d_{ij} + \frac{1}{2a + 1} \right]_{q_i} (F^N_i)^{2a+1} F^N_j (F^N_i)^{d_{ij}-2a}.$$

By adjunction, the same isomorphism exists for the $E^N_i, E^N_j$.

**Proof.** This follows from Proposition 5.3. \qed

In particular, there is a strong 2-action of the 2-Kac–Moody algebra of [23, 38] associated to $\langle E_i, F_i, K_i \rangle_{i \in I_f}$ on $\bigoplus_{\nu \in X^*} R^N_p(\nu)$-mod through $F^N_i, E^N_i$. 
5.5. A differential on $R^N_p$. We fix a subset $I_f \subset I'_f \subset I$ and consider the parabolic subalgebras $U_q(p) \subset U_q(p') \subset U_q(\mathfrak{g})$. For each $j \in I'_f \setminus I_f$ we choose a weight $n'_j \in \mathbb{N}$. For $j \in I_f$ we take $n'_j := n_j \in \mathbb{N}$, and we write $N' := \{n'_j \}_{j \in I'_f}$. Then, we equip the cyclotomic $p$-KLR algebra $R^N_p(m)$ with a differential $d^N_{N'}$, which is zero on dots and crossings and

$$d^N_{N'} \left( \begin{array}{cccc} j & 0 & \cdots & \vphantom{0} \\ i_1 & i & \cdots & i_{m-1} \end{array} \right) := \begin{cases} 0, & \text{if } j \notin I'_f, \\ (-1)^{n_j} \cdot n_j & \quad \text{if } j \in I'_f \setminus I_f, \\ \quad \quad \quad \vdots & \quad \quad \quad \vdots & \quad \quad \quad \vdots \\ j & i_1 & \cdots & i_{m-1} \end{cases}$$

As before, we extend using the graded Leibniz rule, and verifying that $d^N_{N'}$ is well-defined is straightforward.

**Theorem 5.20.** The dg-algebra $(R^N_p(m), d^N_{N'})$ is formal with homology

$$H(R^N_p(m), d^N_{N'}) \cong R^N_{p'}(m).$$

**Proof.** We have $R^N_p(m) \cong H(R_b(m), d_N)$ and $R^N_{p'}(m) \cong H(R_b(m), d_{N'})$ by Theorem 4.4. Moreover, $d^N_{N'}$ can be lifted to $R_b(m)$. We split the homological grading of $R_b(m)$ in three: a first one that counts the amount of floating dots with subscript in $I_f$, a second one for the floating dots with subscript in $I'_f \setminus I_f$, and a third one for $I \setminus I'_f$ that we ignore for the moment. Then, we have that $d^N_{N'}$ has degree $(0, -1)$ and $d_N$ has degree $(-1, 0)$, and they commute with each other. Thus we have a (bounded) double complex $(R_b, d_N, d^N_{N'})$ with total complex being $(R_b, d_{N'})$, since $d_{N'} = d_N + d^N_{N'}$. In particular, there is a spectral sequence from $H(R^N_p(m), d^N_{N'})$ to $H(R_b, d_{N'}) \cong R^N_{p'}(m)$. Now, Theorem 4.4 tells us that $H(R_b, d_N)$ is concentrated in homological degree zero (for the first homological grading). Thus, the spectral sequence converges at the second page, and in particular $H(R^N_p(m), d^N_{N'}) \cong R^N_{p'}(m)$. \(\square\)

We interpret this result as a categorical version of the fact that if there is an arrow from a parabolic Verma module $M^p(\Lambda, N)$ to $M^{p'}(\Lambda', N')$ (see Section 2.2), then there is a surjection $M^p(\Lambda, N) \to M^{p'}(\Lambda', N')$. Indeed, in that case there is a surjective quasi-isomorphism $(R^N_p, d_N) \to (R^N_{p'}, 0)$, inducing equivalences of derived categories that commute up to quasi-isomorphism with the categorical actions of $U_q(\mathfrak{g})$.

6. The categorification theorems

Recall that the $\mathbb{k}$-algebra of formal Laurent series $\mathbb{k}((x_1, \ldots, x_n))$ (as constructed in [3], see also [32, §5]) is given by first choosing a total additive order $<$ on $\mathbb{Z}^n$. One says that a cone $C := \{\alpha_1 v_1 + \cdots + \alpha_n v_n | \alpha_i \in \mathbb{R}_{\geq 0} \} \subset \mathbb{R}^n$ is compatible with $<$ whenever $0 < v_i$ for all $i \in \{1, \ldots, n\}$. Then, we set

$$\mathbb{k}((x_1, \ldots, x_n)) := \bigcup_{e \in \mathbb{Z}^n} x^e \mathbb{k}_<[[x_1, \ldots, x_n]],$$

where $\mathbb{k}_<[[x_1, \ldots, x_n]]$ consists of formal Laurent series in $\mathbb{k}[[x_1, \ldots, x_n]]$ such that the terms are contained in a cone compatible with $<$. We will also write $\mathbb{k}^+_<[[x_1, \ldots, x_n]]$ for the
elements in $k[[x_1, \ldots, x_n]]$ with terms contained in a cone without the 0 element (i.e. series with the degree zero term being zero). We obtain a ring by equipping $k((x_1, \ldots, x_n))$ with the usual addition and multiplication of series. Requiring that all series are contained in cones compatible with $<$ ensures that the product of two elements in $k((x_1, \ldots, x_n))$ is well-defined. Indeed, under these conditions, any coefficient in the product can be determined by summing only a finite amount of terms.

6.1. **C.b.l.f. derived category.** We fix an arbitrary additive total order $<$ on $\mathbb{Z}^n$. We say that a $\mathbb{Z}^n$-graded $k$-vector space $M = \bigoplus_{g \in \mathbb{Z}^n} M_g$ is c.b.l.f. (cone bounded, locally finite) dimensional if

- $\dim M_g < \infty$ for all $g \in \mathbb{Z}^n$;
- there exists a cone $C_M \subset \mathbb{R}^n$ compatible with $<$ and $e \in \mathbb{Z}^n$ such that $M_g = 0$ whenever $g - e \notin C_M$.

In other words, $M$ is c.b.l.f. dimensional if and only if

$$\operatorname{gdim}_q M := \sum_{g \in \mathbb{Z}^n} x^g \dim(M_g) \in x^e k[[x_1, \ldots, x_n]].$$

Let $(A, d)$ be a $\mathbb{Z}^n$-graded dg-$k$-algebra, where $A = \bigoplus_{(h, g) \in \mathbb{Z} \times \mathbb{Z}^n} A^h_g$, and $d(A^h_g) \subset A^{h-1}_g$. Suppose that $(A, d)$ is concentrated in non-negative homological degrees, that is $A^h_g = 0$ whenever $h < 0$. Let $\mathcal{D}(A, d)$ be the derived category of $(A, d)$. Let $\mathcal{D}^I(A, d)$ be the full triangulated subcategory of $\mathcal{D}(A, d)$ consisting of $(A, d)$-modules having homology being c.b.l.f. dimensional for the $\mathbb{Z}^n$-grading. We call $\mathcal{D}^I(A, d)$ the c.b.l.f. derived category of $(A, d)$.

**Definition 6.1 ([32]).** We say that $(A, d)$ is a positive c.b.l.f. dimensional dg-algebra if

1. $A$ is c.b.l.f. dimensional for the $\mathbb{Z}^n$-grading;
2. $A$ is non-negative for the homological grading;
3. $A^h_0 = 0$ for $h > 0$;
4. $(A, d)$ decomposes into a direct sum of shifted copies of relatively projective modules $P_i := Ae_i$ for some idempotent $e_i \in A$, such that $P_i$ is non-negative for the $\mathbb{Z}^n$-grading and $A^h_0 P_i$ is semisimple.

**Remark 6.2.** As explained in [32, Remark 9.5], condition (3.) cannot be respected whenever $P_i := Ae_i$ is acyclic. However, in this case there is a quasi-isomorphism $(A, d) \xrightarrow{\sim} (A/Ae_i, A, d)$ and we can weaken hypothesis (3.) so that it is respected only after removing all acyclic $P_i$. This is the case of $(R_b, d_N)$.

6.1.1. **Asymptotic Grothendieck group.** As already observed in [1] (see also [34, Appendix]), one caveat of the usual definition of the Grothendieck group is that it does not allow to take into consideration infinite iterated extensions of objects. We need to introduce new relations in the Grothendieck groups to handle such situations. One solution is to use asymptotic Grothendieck groups, as introduced by the first author in [32].
Let \( \mathcal{C} \) be a triangulated subcategory of some triangulated category \( \mathcal{T} \). Suppose \( \mathcal{T} \) admits countable products and coproducts, and these preserve distinguished triangles. Let \( K_0^\Delta(\mathcal{C}) \) be the triangulated Grothendieck group of \( \mathcal{C} \).

Recall the Milnor colimit \( \text{MColim}_{r \geq 0}(f_r) \) of a collection of arrows \( \{X_r \to X_{r+1}\}_{r \in \mathbb{N}} \) in \( \mathcal{T} \) is the mapping cone fitting inside the following distinguished triangle

\[
\bigoplus_{r \in \mathbb{N}} X_r \xrightarrow{1-f_r} \bigoplus_{r \in \mathbb{N}} X_r \to \text{MColim}_{r \geq 0}(f_r) \to
\]

where the left arrow is given by the infinite matrix

\[
1 - f_* := \begin{pmatrix}
1 & 0 & 0 & 0 & \ldots \\
-f_0 & 1 & 0 & 0 & \ldots \\
0 & -f_1 & 1 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

There is a dual notion of Milnor limit. Consider a collection of arrows \( \{X_{r+1} \to X_r\}_{r \geq 0} \) in \( \mathcal{T} \). The Milnor limit is the object fitting inside the distinguished triangle

\[
\text{MLim}_{r \geq 0}(f_r) \to \bigoplus_{r \geq 0} X_r \xrightarrow{1-f_r} \bigoplus_{r \geq 0} X_r \to
\]

**Definition 6.3.** The asymptotic triangulated Grothendieck group of \( \mathcal{C} \subset \mathcal{T} \) is given by

\[
K_0^\Delta(\mathcal{C}) := K_0^\Delta(\mathcal{C})/T(\mathcal{C}),
\]

where \( T(\mathcal{C}) \) is generated by

\[
[Y] - [X] = \sum_{r \geq 0} [E_r]
\]

whenever both \( \bigoplus_{r \geq 0} \text{Cone}(f_r) \in \mathcal{C} \) and \( \bigoplus_{r \geq 0} E_r \in \mathcal{C} \), and

\[
Y \cong \text{MColim}(X = F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} \cdots),
\]

is a Milnor colimit, or

\[
X \cong \text{MLim}(\cdots \xrightarrow{f_1} F_1 \xrightarrow{f_0} F_0 = Y),
\]

is a Milnor limit, and

\[
[E_r] = [\text{Cone}(f_r)] \in K_0^\Delta(\mathcal{C}),
\]

for all \( r \geq 0 \).

In a \( \mathbb{Z}^n \)-graded triangulated category \( \mathcal{T} \), we define the notion of c.b.l.f. direct sum as follows:

- take a a finite collection of objects \( \{K_1, \ldots, K_m\} \) in \( \mathcal{T} \);
- consider a direct sum of the form

\[
\bigoplus_{g \in \mathbb{Z}^n} x^g(K_{1,g} \oplus \cdots \oplus K_{m,g}), \quad \text{with} \quad K_{i,g} = \bigoplus_{j=1}^{k_{i,g}} K_i[h_{i,j,g}],
\]

where \( k_{i,g} \in \mathbb{N} \) and \( h_{i,j,g} \in \mathbb{Z} \) such that:
• there exists a cone $C$ compatible with $\prec$, and $e \in \mathbb{Z}^n$ such that for all $j$ we have $k_{i_j} = 0$ whenever $g - e \notin C$;
• there exists $h \in \mathbb{Z}$ such that $h_{i,j} \geq h$ for all $i,j,g$.

If $\mathcal{T}$ admits arbitrary c.b.l.f. direct sums, then $K_0(\mathcal{T})$ has a natural structure of $\mathbb{Z}((x_1, \ldots, x_n))$-module with

$$\sum_{g \in C} a_g x^{e+g} [X] := [\bigoplus_{g \in C} x^{g+e} X^{\otimes \alpha_g}],$$

where $X^{\otimes \alpha_g} = \bigoplus_{|g|=\alpha_g} X[\alpha_g]$ and $\alpha_g = 0$ if $a_g > 0$ and $\alpha_g = 1$ if $a_g < 0$.

**Theorem 6.4 ([32, Theorem 9.15]).** Let $(A, d)$ be a positive c.b.l.f. dg-algebra, and let $\{P_j\}_{j \in J}$ be a complete set of indecomposable cofibrant $(A, d)$-modules that are pairwise non-isomorphic (even up to degree shift). Let $\{S_j\}_{j \in J}$ be the set of corresponding simple modules. There is an isomorphism

$$K_0^A(\mathcal{D}^U(A, d)) \cong \bigoplus_{j \in J} \mathbb{Z}((x_1, \ldots, x_i))[P_j],$$

and $K_0^A(\mathcal{D}^U(A, d))$ is also freely generated by the classes of $\{[S_j]\}_{j \in J}$.

**Proposition 6.5 ([32, Proposition 9.18]).** Let $(A, d)$ and $(A', d')$ be two c.b.l.f. positive dg-algebras. Let $B$ be a c.b.l.f. dimensional $(A', d')$-$(A, d)$-bimodule. The derived tensor product functor

$$F : \mathcal{D}^U(A, d) \rightarrow \mathcal{D}^U(A', d'), \quad F(X) := B \otimes^L_{(A, d)} X,$$

induces a continuous map

$$[F] : K_0^A(\mathcal{D}^U(A, d)) \rightarrow K_0^A(\mathcal{D}^U(A', d')).$$

We will need the following definitions in Section 7:

**Definition 6.6.** Let $\{K_1, \ldots, K_m\}$ be a finite collection of objects in $\mathcal{C}$, and let $\{E_r\}_{r \in \mathbb{N}}$ be a family of direct sums of $\{K_1, \ldots, K_m\}$ such that $\bigoplus_{r \in \mathbb{N}} E_r$ is a c.b.l.f. direct sum of $\{K_1, \ldots, K_m\}$. Let $\{M_r\}_{r \in \mathbb{N}}$ be a collection of objects in $\mathcal{C}$ with $M_0 = 0$, such that they fit in distinguished triangles

$$M_r \xrightarrow{f_r} M_{r+1} \rightarrow E_r \rightarrow$$

Then, we say that an object $M \in \mathcal{C}$ such that $M \cong_{\mathcal{T}} \text{Colim}_{r \geq 0}(f_r)$ in $\mathcal{T}$ is a c.b.l.f. iterated extension of $\{K_1, \ldots, K_m\}$.

**Definition 6.7.** We say that $\mathcal{V}$ is c.b.l.f. generated by $\{X_j\}_{j \in J}$ for some collection of elements $X_j \in \mathcal{V}$ if for any object $Y$ in $\mathcal{V}$ we can take a finite set $\{Y_k\}_{k \in K}$ of retract $Y_k \subset X_{j_k}$ such that $Y$ is isomorphic to a c.b.l.f. iterated extension of $\{Y_k\}_{k \in K}$. 
6.2. Categorification. In this section we assume that $R_b(\nu)$ is a $k$-algebra over a field $k$. We also choose an arbitrary order $< \mathrel{\text{for constructing}} \mathbb{Z}((q, \Lambda))$ such that $0 < q < \lambda_i$ for all formal $\lambda_i = q^{\beta_i} \in \Lambda$. We assume that the parabolic Verma module $M^p(\lambda, N)$ is constructed over the ground ring $\mathcal{R} := \mathbb{Q}((\Lambda, q))$ (instead of $\mathbb{Q}(\Lambda, q)$).

Every idempotent of $R_b(\nu)$ is the image of an idempotent of the classical KLR algebra $R_\vartheta(\nu)$ under the obvious inclusion $R_\vartheta(\nu) \hookrightarrow R_b(\nu)$. Thanks to [22, Section 2.5] we know all the idempotents of $R_\vartheta(\nu)$. We define the element

$$e_{i,n} := \tau_{\vartheta_n} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} i_{i-1} \cdots i \in R_\vartheta(n),$$

where $\vartheta_n$ is the longest element in $S_n$. Let $\text{Seqd}(\nu)$ be the set of expressions $i_1^{(m_1)} i_2^{(m_2)} \cdots i_r^{(m_r)}$ for different $r \in \mathbb{N}$ and $m_\ell \in \mathbb{N}$ such that $\sum_{\ell=1}^r m_\ell \cdot \alpha_i = \nu$. For each $i \in \text{Seqd}(\nu)$ we define the idempotent

$$e_i := e_{i_1,m_1} \otimes e_{i_2,m_2} \otimes \cdots \otimes e_{i_r,m_r} \in R_\vartheta(\nu),$$

where $x \otimes y$ means we put the diagram of $x$ at the left of the one of $y$. Identifying $e_i$ with its image in $R_b(\nu)$, as in [22], we define a projective left $R_b(\nu)$-module

$$P_i := R_b(\nu)e_i.$$ 

Then, we put

$$\langle i \rangle := - \sum_{\ell=1}^r \frac{m_\ell (m_\ell - 1)}{2} d_{i_\ell},$$

and we define $\tilde{P}_i := q^{-\langle i \rangle} P_i$.

When writing $\ldots i \ldots$ and $\ldots j \ldots$ we mean we take two sequences $i_1 \ldots i_2$ and $i_1 \ldots j_2$ in $\text{Seqd}(\nu)$ that coincide everywhere except on $i$ and $j$. From the decomposition of the nilHecke algebra [22, §2.2] we get an isomorphism of $R_p^N$-modules

$$\tilde{P}_{\ldots i m \ldots} \cong \bigoplus_{[m]_q \geq 1} \tilde{P}_{\ldots i (m) \ldots}. $$

Mimicking the arguments in [22, Proposition 2.13] and [24, Proposition 6] we have the following:

**Proposition 6.8.** There are isomorphisms

$$\bigoplus_{a=0}^{[d_{ij}+1]/2} \tilde{P}_{\ldots i (2a) j i (d_{ij}+1-2a) \ldots} \cong \bigoplus_{a=0}^{[d_{ij}/2]} \tilde{P}_{\ldots i (2a+1) j i (d_{ij}-2a) \ldots}$$

for all $i \neq j \in I$.

Equipping $R_b(\nu)$ with $d_N$ induces a differential on $\tilde{P}_i$, and Proposition 6.8 holds for the dg-version $(\tilde{P}_i, d_N)$. We put

$$\mathcal{M}^p(\Lambda, N) := \bigoplus_{m \geq 0} \mathcal{D}^B(R_b(m), d_N),$$

with the particular case $\mathcal{M}(\Lambda)$ meaning $p = \mathfrak{b}$ and $N = \emptyset$, and therefore $d_N = 0$. Note that $\mathcal{D}^B(R_b(m), d_N) \cong \mathcal{D}^B(R_p^N(m), 0)$. Let $qK_0^\Delta(-) := K_0^\Delta(-) \otimes_{\mathbb{Z}((q, \Lambda))} \mathbb{Q}((q, \Lambda))$. 


Proposition 6.9. The $\mathbb{Z}^{1+|\Lambda|}$-graded dg-algebra $(R_q(m), d_N)$ is a positive c.b.l.f. dimensional dg-algebra.

Proof. Clearly, $R_q(m)$ is c.b.l.f. dimensional for the $\mathbb{Z}^{1+|\Lambda|}$-grading, and is non-negative for the homological grading. We can also assume we have applied Remark 6.2. Recall that the part in homological degree zero of $R_q(m)$ is isomorphic to the usual KLR algebra $R_q(m)$. As explained in [22, §3.3], for each monomial $f \in U_q^-(g)$, we have a projective $R_q(m)$-module $P_f$ (defined similarly as $P_i$ for $f = F^{(m_1)}_{i_1} \cdots F^{(m_r)}_{i_r}$). Moreover, by [22, Proposition 3.22] extended for any $g$, $P_f$ is indecomposable if and only if $f$ is a canonical basis element. Also, the quadratic form in [29, §14.2] corresponds with the graded dimension of the graded hom-spaces between these projective $R_q(m)$-modules. The same applies for the homological degree zero part of the graded hom-spaces between our $P_i$’s. Then, by [29, Theorem 14.2.3], we obtain that
\[ \text{gdim } HOM_{R_q(m)}(P_i, P_j) - \delta_{i,j} \in \mathbb{Z}^+_\leq \mathbb{Z} \mathbb{L} \mathbb{A}, \]
which concludes the proof. \qed

Proposition 6.10. There is an isomorphism of $\mathbb{Q}(\mathbb{Q}(q, \Lambda))$-modules
\[ U_q^-(g) \otimes_{\mathbb{Q}(q)} \mathbb{Q}((q, \Lambda)) \cong \mathbb{Q} K_0^\Lambda(\mathbb{M}(\Lambda)), \]
and a $\mathbb{Q}(\mathbb{Q}(q, \Lambda))$-linear surjection
\[ U_q^-(g) \otimes_{\mathbb{Q}(q)} \mathbb{Q}((q, \Lambda)) \twoheadrightarrow \mathbb{Q} K_0^\Lambda(\mathbb{M}(\Lambda, N)), \]
both sending $F^{(m_1)}_{i_1} F^{(m_2)}_{i_2} \cdots F^{(m_r)}_{i_r}$ to $[(\tilde{P}_i, d_N)]$ for $i = i_1^{(m_1)} i_2^{(m_2)} \cdots i_r^{(m_r)}$.

Proof. Since projective modules of $R_q(\nu)$ are in bijection with the ones of the classical KLR algebra $R_q(\nu)$ and respect the categorified Serre relations (see Proposition 6.8), both claims are a direct consequence of the main results in [22, 24], together with Theorem 6.4. \qed

Consider the subring $P(\nu)$ of $R_q(\nu)$ consisting of dots on vertical strands (without floating dots). It admits an action of the symmetric group permuting the strands (with labels) and dots on them (not to be confused with the action of $S_m$ on $P_\nu$ from Section 3.2). We write $\text{Sym}(\nu) := P(\nu)^{S_m}$ for the subring of invariants under this action. Clearly it lies in the center of $R_q(\nu)$ but this inclusion is strict (see [35] or [4] for a study of the center in the case of $\mathfrak{sl}_2$).

The supercenter of $R_q(\nu)$ contains $\text{Sym}(\nu) \otimes \bigotimes_{i \in I} \wedge^\cdot <\overline{\omega}_i^{(0)}, \ldots, \overline{\omega}_i^{(n-1)}>$ where $\overline{\omega}_i^a$ is a floating dot with subscript $i$, superscript $a$ and placed in the rightmost region:
\[ \overline{\omega}_i^a := \left\vert\begin{array}{c} \cdots \otimes \overset{a}{o} \end{array}\right. \]
We conjecture that the supercenter contains no other elements.

Conjecture 6.11. There is an isomorphism of rings
\[ Z(R_q(\nu)) \cong \text{Sym}(\nu) \otimes \bigotimes_{i \in I} \wedge^\cdot <\overline{\omega}_i^{(0)}, \ldots, \overline{\omega}_i^{(n-1)}>, \]
where $Z(R_q(\nu))$ is the supercenter of $R_q(\nu)$.
In general $R_{p',\mu}(\nu)$ is not a free module over $\text{Sym}(\nu) \otimes \bigotimes_{i \in I} \wedge^*(\tilde{\omega}_i^0, \ldots, \tilde{\omega}_i^{\nu-1})$, but we have the following.

**Proposition 6.12.** $R_b(\nu)$ is a free module over $\text{Sym}(\nu)$ of rank $2^m(m!)^2$.

**Proof.** It follows from Theorem 3.16 and the fact $P(\nu)$ is a free module of rank $m!$ on $\text{Sym}(\nu)$.

Since $\text{Sym}(\nu)$ lies in the center of $R_b(\nu)$, any simple $R_b(\nu)$-module is annihilated by $\text{Sym}^+(\nu)$, where $\text{Sym}^+(\nu)$ consists of the elements in $\text{Sym}(\nu)$ with non-zero degree. In particular, a simple $R_b(\nu)$-module must be a finite dimensional $R_b(\nu)/\text{Sym}^+(\nu)R_b(\nu)$-module. Since $R_b(\nu)/\text{Sym}^+(\nu)R_b(\nu)$ has finite dimension over $k$, we only have finitely many simple modules, up to shift and isomorphism. For each $i \in \text{Seq}(\nu)$ such that $P_i$ is indecomposable, we let $S_i$ be the unique simple quotient of $P_i$. We put $S_i := q^{-i}S_i$. If $(P_i, d_N)$ is not acyclic, then it lifts automatically to a dg-version $(\tilde{S}_i, 0)$ because of Proposition 6.9.

By Lemma 4.5 and Proposition 5.15 we know that $E_i \text{Id}_\nu$ and $F_i \text{Id}_\nu$ are exact. Moreover, they respect the conditions of Proposition 6.5. Therefore, they induce maps

$$[E_i \text{Id}_\nu] : K_0(\mathcal{D}^f(R_b(\nu), d_N)) \to K_0^\Delta(\mathcal{D}^f(R_b(\nu-i), d_N)),$$

$$[F_i \text{Id}_\nu] : K_0(\mathcal{D}^f(R_b(\nu), d_N)) \to K_0^\Delta(\mathcal{D}^f(R_b(\nu+i), d_N)).$$

Then, Theorem 4.4, Theorem 5.17 and Proposition 5.19 tell us that $qK_0^\Delta(\mathcal{M}^p(\Lambda, N))$ is a $U_q(\mathfrak{g})$-weight module. By Proposition 6.10 we know that $qK_0^\Delta(\mathcal{M}^p(\Lambda, N))$ is cyclic as $U_q(\mathfrak{g})$-module, with highest weight generator given by the class of $(R_b(0), d_N) \cong (k, 0)$. Thus $qK_0^\Delta(\mathcal{M}^p(\Lambda, N))$ is a highest weight module.

As in [22], let $\psi : R_b(\nu) \to R_b(\nu)^\text{op}$ be the map that takes the mirror image of diagrams along the horizontal axis. Given a left $(R_b(\nu), d_N)$-module $M$, we obtain a right $(R_b(\nu), d_N)$-module $M^\psi$ with action given by $m^\psi \cdot r := (-1)^{\text{deg}_h(r)} \text{deg}_h(m) \psi(r) \cdot m$ for $m \in M$ and $r \in R_b(\nu)$. Then, we define the bifunctor

$$(-, -) : \mathcal{M}^p(\Lambda, N) \times \mathcal{M}^p(\Lambda, N) \to \mathcal{D}^f(\mathbb{C}, 0), \quad (M, M') := M^\psi \otimes_{(R_b, d_N)}^L M',$$

where $\otimes^L$ is the derived tensor product.

**Proposition 6.13.** The bifunctor defined above respects:

- $(\text{Ind}_m^{m+i} M, M') \cong (M, \text{Res}_m^{m+i} M')$ for all $M, M' \in \mathcal{M}^p(\Lambda, N)$;
- $(\bigoplus_f M, M') \cong (M, \bigoplus_f M')$ for all $f \in \mathbb{Z}(q, \Lambda)$.

**Proof.** Straightforward.

Comparing Proposition 6.13 with Definition 2.5, we deduce that $(-, -)$ is a categorification of the Shapovalov form on $K_0^\Delta(\mathcal{M}^p(\Lambda, N))$. Moreover, it turns $\tilde{S}_i$ into the dual of $\tilde{P}_i$ for each $i \in \text{Seq}^d(\nu)$ such that $P_i$ is indecomposable. Recall $\mathcal{M}^p(\Lambda, N)$ is the parabolic Verma module, and we assume $\Lambda = \{q^\beta, i \in I_r\}$ contains only formal weights.
Theorem 6.14. The asymptotic Grothendieck group \( \mathcal{K}_0^\Delta(\mathcal{M}^p(\Lambda, N)) \) is a \( U_q(\mathfrak{g}) \)-weight module, with action of \( E_i, F_i \) given by \([E_i], [F_i] \). Moreover, there is an isomorphism of \( U_q(\mathfrak{g}) \)-modules
\[
\mathcal{K}_0^\Delta(\mathcal{M}^p(\Lambda, N)) \cong M^p(\Lambda, N).
\]

Proof. We already proved the first claim above. Because of Proposition 4.3, for \( i \in I_F \), both \([F_i]\) and \([E_i]\) act as locally nilpotent operators. In particular, the \( U_q(\mathfrak{l}) \)-submodule of \( \mathcal{K}_0^\Delta(\mathcal{M}^p(\Lambda, N)) \) given by
\[
U_q(\mathfrak{l}) \otimes_{U_q(\mathfrak{g})} [(R_0, d_N)],
\]
is an integrable module for the Levi factor \( U_q(\mathfrak{l}) \). Since it is an integrable cyclic weight module, it must be isomorphic to \( V(\Lambda, N) \) (see [29]). Therefore, there is a surjective \( U_q(\mathfrak{g}) \)-module morphism
\[
\gamma : M^p(\Lambda, N) \to \mathcal{K}_0^\Delta(\mathcal{M}^p(\Lambda, N)).
\]

Since \( M^p(\Lambda, N) \) is irreducible and \( \gamma \) is non-zero, it must be an isomorphism. \( \square \)

Let \( m_r = F_i v_{\Lambda, N} \) be an induced basis element of \( M^p(\Lambda, N) \) with \( i \in \text{Seq}(\nu) \). Then, the isomorphism of Theorem 6.14 identifies \( m_r \) with the class \([([R_0(\nu), d_N])_1] \). Similarly, let \( m'_s = F_j v_{\Lambda, N} \) for \( j \in \text{Seqd}(\nu) \) be a canonical basis element, and let \( m^s \) be its dual in the dual canonical basis. Then, the isomorphism identifies \( m'_s \) with \([([\tilde{P}_j, d_N]) \) and \( m^s \) with \([([\tilde{S}_j, d_N]) \). Moreover, computing the c.b.l.f. composition series of \( \tilde{P}_i \) (see [32, §7]) or taking a certain cofibrant replacement of \( \tilde{S}_i \) (see [32, §9]) gives a categorical version of the change of basis between canonical and dual canonical basis elements.

7. 2-Verma modules

Let \( k \) be a field of characteristic 0. Let \( \mathcal{V} \in \text{dg-cat}_k \) be a \( \mathbb{Z} \)-graded pretriangulated dg-category (see Definition A.23). Let \( \mathcal{E}nd_{\text{Hqe}}(\mathcal{V}) := \mathcal{H}om_{\text{Hqe}}(\mathcal{V}, \mathcal{V}) \) be the dg-category of quasi-endofunctors on \( \mathcal{V} \) (see Appendix A.5.1).

Remark 7.1. For example, \( \mathcal{V} \) could be the dg-category \( \mathcal{D}_{dg}(R, d) \) of cofibrant dg-modules over a dg-algebra \( (R, d) \) (see Definition A.15). Then, the subcategory of \( \mathcal{E}nd_{\text{Hqe}}(\mathcal{V}) \) consisting of coproduct preserving quasi-functors would be given by the dg-enhanced derived category of dg-bimodules \( \mathcal{D}_{dg}((R, d)^{op} \otimes (R, d)) \) (see Theorem A.21).

Let \( Q_i := \bigoplus_{\ell \geq 0} q_i^{1+2\ell} \text{Id} \). It is a categorification of \( \frac{q_i}{1-q_i} = \frac{1}{q_i-1} \). We start by introducing a notion of dg-categorical action and dg-2-representation.

Definition 7.2. A \emph{weak dg-categorical} \( U_q(\mathfrak{g}) \)-action on \( \mathcal{V} \) is a collection of quasi-endofunctors \( F_i, E_i, K_\gamma \in Z^0(\mathcal{E}nd_{\text{Hqe}}(\mathcal{V})) \) for all \( i \in I \) and \( \gamma \in Y^\vee \) such that
- there are isomorphisms
  \[
  K_0 \cong \text{Id}, \quad K_\gamma K_{\gamma'} \cong K_{\gamma + \gamma'}, \quad K_\gamma E_i \cong q^{\gamma(\alpha_i)} E_i K_\gamma, \quad K_\gamma F_i \cong q^{-\gamma(\alpha_i)} F_i K_\gamma,
  \]
  where \( q \) denotes the shift in the \( q \)-grading;
there is a quasi-isomorphism

\[
\text{(37)} \quad \text{Cone}(F_i E_j \xrightarrow{u_{ij}} E_j F_i) \xrightarrow{\delta_{ij}} \text{Cone}(Q_i K_i \xrightarrow{h} Q_i K_i^{-1}),
\]

where \( K_i := K_{\varphi_i} \);

- there are isomorphisms

\[
\sum_{a=0}^{[(d_{ij}+1)/2]} \left[ \frac{d_{ij} + 1}{2a} \right]_{q_i} F^a_i F_j F^a_i, \\
\sum_{a=0}^{[(d_{ij}+1)/2]} \left[ \frac{d_{ij} + 1}{2a} \right]_{q_i} E^a_i E_j E^a_i
\]

for all \( i \neq j \in I \).

We say a weak dg-categorical \( U_q(\mathfrak{g}) \)-action on \( \mathcal{V} \) is a \textit{dg-categorical action} if in addition

- \( F_i \) is left adjoint to \( q^{-1}_i K_i E_i \) in \( Z^0(\mathfrak{g}nd_{Hq^e}(\mathcal{V})) \);
- there is a map of algebras

\[
R_q(i) \to Z^0(\text{End}(F_i)) := \bigoplus_{\epsilon \in \mathbb{Z}} Z^0(\text{Hom}(F_i, q^\epsilon F_i)),
\]

with \( F_i := F_{i_1} \cdots F_{i_m} \), for all \( i \in \text{Seq}(m) \), inducing a surjection

\[
R_q(i) \otimes_k Z^0(\text{End}_{\mathcal{V}}(M)) \twoheadrightarrow Z^0(\text{End}_{\mathcal{V}}(F_i M)),
\]

for all \( M \in \mathcal{V} \);

- \( \mathcal{V} \) is dg-triangulated (i.e. \( H^0(\mathcal{V}) \) is idempotent complete).

Such a \( \mathcal{V} \) carrying a dg-categorical action is called a \textit{dg-2-representation} of \( U_q(\mathfrak{g}) \).

The following notions are dg-2-categorical lifts of the notions of weight module and integrable module.

**Definition 7.3.** We say that a dg-2-representation \( \mathcal{V} \) is a \textit{weight dg-2-representation} if there is a map

\[
\lambda : Y^\vee \to \mathfrak{g}nd_{Hq^e}(\mathcal{V}),
\]

where \( \lambda(\gamma) \) commutes with the grading shift \( q \) for all \( \gamma \in Y^\vee \) and \( \lambda(\gamma) \circ \lambda(\gamma') \equiv \lambda(\gamma + \gamma') \), such that

\[
\mathcal{V} \cong \bigoplus_{\gamma \in Y} \mathcal{V}_{\lambda, \gamma}, \quad K_i|_{\mathcal{V}_{\lambda, \gamma}}(-) = \lambda(\gamma) q^{(\gamma)}(-).
\]

**Definition 7.4.** We say that a weight dg-2-representation \( \mathcal{V} \) is \textit{i-integrable} if

- \( \lambda(\alpha_i^+) = q^{n_i} \) for some \( n_i \in \mathbb{N} \);
- there is a quasi-isomorphism

\[
\text{Cone}(Q_i K_i \mathcal{V}_{\lambda, \gamma} \xrightarrow{h_i} Q_i K_i^{-1} \mathcal{V}_{\lambda, \gamma}) \xrightarrow{\delta_{ij}} \bigoplus_{\alpha_i} \text{Id},
\]

where \( \bigoplus_{\alpha_i} \text{Id} := \bigoplus_{\alpha_i} \text{Id}[1] \) whenever \( m < 0 \);

- \( F_i \) and \( E_i \) are locally nilpotent.
Under some mild hypothesis, this definition recovers the notion of integrable 2-representation from [38] and [10].

**Proposition 7.5.** Suppose $V$ is $i$-integrable for all $i \in I$. Also suppose that there is some $M \in \mathcal{V}_{\lambda,0}$ such that $E_i M = 0$ for all $i \in I$ and $\text{End}_V(M) \cong (\mathbb{k}, 0)$, and $H^0(\mathcal{V})$ is c.b.l.f. generated by $\{F_i M\}_{i \in \text{Seq}(I)}$. Then, $H^0(\mathcal{V})$ carries an integrable categorical $U_q(\mathfrak{g})$-action in the sense of [38].

**Proof.** First, by adjunction, Eq. (37) and Eq. (38), we have

$$\text{gdim}_q H^0(\text{END}_V(F_i M)) \cong \text{gdim}_q R^N_q(i),$$

for all $i \in \text{Seq}(I)$. In particular, we have that $x_i^n 1_i$ acts by 0 on $H^0(\text{End}_V(F_i M))$ for all $i \in I$, and $x_i 1_i$ acts non-trivially whenever $n_i > 1$. Thus, there is a map

$$\gamma : R^N_q(i) \rightarrow H^0(\text{END}_V(F_i M)).$$

Since $\gamma$ is surjective, we obtain $R^N_q(i) \cong H^0(\text{END}_V(F_i M))$, and the result follows from Theorem 5.17.

For a $\mathbb{Z}^n$-graded dg-algebra $(A, d)$, we put $\mathcal{D}_{dg}(A, d)$ for the dg-category having as objects the one in $\mathcal{D}_{dg}(A, d) \cap \mathcal{D}_{dg}(A, d)$ and the hom-spaces inherited from $\mathcal{D}_{dg}(A, d)$. It is a dg-enhancement of the c.b.l.f. derived category of $(A, d)$.

**Definition 7.6.** A parabolic 2-Verma module for $p$ is a $\mathbb{Z} \times \mathbb{Z}^{|I^+|}$-graded weight dg-2-representation $V$ such that

- the highest weight space $\mathcal{V}_\lambda := \mathcal{V}_{\lambda,0} \cong \mathcal{D}_{dg}^I(\mathbb{k}, 0)$;
- there exists $M \cong (\mathbb{k}, 0) \in \mathcal{V}_\lambda$ such that $\mathcal{V}_{\lambda,y}$ is c.b.l.f. generated by $\{F_i M\}_{i \in \text{Seq}(y)}$ for all $-y \in X^+$, and $\mathcal{V}_{\lambda,y} = 0$ otherwise;
- $V$ is $i$-integrable for all $i \in I$;
- $h_j = 0$ and $\alpha^-_y = \lambda_j$ (the degree shift) for all $j \not\in I$;
- for each $j \not\in I$, $n_j \in \mathbb{N}$ and $i \in \text{Seq}(I)$, after specializing $\lambda_j = q^{n_j}$, there exists a differential $d_{n_j}$ commuting with the differential $d$ of $(\text{End}_V(F_i M), d)$ such that the triangulated dg-category generated by c.b.l.f. iterated extension of the representable modules of $V^{n_j} := \bigoplus_{i \in \text{Seq}(I)} (\text{End}_V(F_i M), d + d_{n_j})$ is $j$-integrable with $\lambda(\alpha^-_y) = q^{n_j}$.

**Proposition 7.7.** Let $V$ be a parabolic 2-Verma module. There is an isomorphism

$$(R^i_q(d_N), d_N) \cong \text{END}_V(F_i M),$$

in $\mathcal{D}(\mathbb{k}, 0)$ for $M \cong (\mathbb{k}, 0) \in \mathcal{V}_\lambda$.

**Proof.** First, by adjunction together with Eq. (37) and Eq. (38) we have

$$\text{END}_V(F_i M) \cong \text{HOM}_V(M, q_i^{-1}K_i E_i M) \cong R^N_q(i),$$

in $\mathcal{D}(\mathbb{k}, 0)$ for all $i \in I$. Also,

$$\text{gdim}_q H^*(\text{END}_V(F_i M)) = \text{gdim}_q R^N_q(i),$$
for all $i \in \text{Seq}(I)$. In particular, there is a relation up to homotopy

$$\alpha + \beta = 0, \quad (41)$$

in $\text{END}_{\mathcal{V}}(F_i F_j M)$ for all $i, j \in I_r$, identifying the diagrams with the image of the KLR elements under the surjection $R_g(ij) \to Z^0(\text{END}_{\mathcal{V}}(F_i F_j M))$, and the floating dot coming from the isomorphism Eq. (39). Then, the existence of $d_{n_i}$ and $d_{n_j}$ forces to have $\alpha = \beta$. Thus, by Corollary 3.17, there is an $A_{\mathcal{X}}$-map

$$(R_b(i), d_N) \to \text{END}_{\mathcal{V}}(F_i M).$$

By Eq. (40), we conclude it is a quasi-isomorphism. Thus, there exists an isomorphism $(R_b(i), d_N) \cong \text{END}_{\mathcal{V}}(F_i M)$ in $\mathcal{D}(k, 0)$. □

Using Theorem A.21 we can think of $F^N_i$ and $E^N_i$ from Section 5.4 as quasi-endofunctors of $\mathcal{D}_{dg}(R_b, d_n)$. By Proposition 5.5 we obtain immediately the following:

**Corollary 7.8.** For all $i \in I$ there is a quasi-isomorphism of cones

$$\text{Cone}(F^N_i E^N_i \text{Id}_\nu \to E^N_i F^N_i \text{Id}_\nu) \cong \text{Cone}(Q_i \lambda_i q^{-\alpha_i^\vee(\nu)}_i \text{Id}_\nu \to Q_i \lambda_i^{-1} q^\alpha_i^\vee(\nu) \text{Id}_\nu),$$

in $\mathcal{E}\text{nd}_{Hqe}(\mathcal{D}_{dg}(R_b, d_N))$.

Together with Proposition 5.19, it means that the dg-enhancement $\mathcal{M}_{dg}^p(\Lambda, N)$ of $\mathcal{M}^p(\Lambda, N)$ (obtained by replacing $\mathcal{D}^I_f(R_b(m), d_N)$ with $\mathcal{D}_{dg}^I(R_b(m), d_N)$) is a weight dg-$2$-representation of $U_q(\mathfrak{g})$, where

$$\lambda(\alpha_i^\vee) := \begin{cases} \lambda_i, & \text{whenever } i \in I_r, \\ q^{n_i}, & \text{whenever } i \in I_f. \end{cases}$$

Then, by Theorem 5.17, we obtain that $\mathcal{M}_{dg}^p(\Lambda, N)$ is a parabolic 2-Verma module.

**Corollary 7.9.** Let $\mathcal{V}$ be a parabolic 2-Verma module. There is a quasi-equivalence

$$\mathcal{M}_{dg}^p(\Lambda, N) \cong \mathcal{V}.$$**

*Proof.* Since $\mathcal{V}_{\lambda,y}$ is c.b.l.f. generated by $\bigoplus_{i \in \text{Seq}(y)} F_i M$, we have that $\mathcal{V}_{\lambda,y}$ is completely determined as dg-category by $\text{END}_{\mathcal{V}}(F_i M)$. Thus, we conclude by using Proposition 7.7. □

**Remark 7.10.** A parabolic 2-Verma module can also be given a ‘2-categorical’ interpretation as an $(\infty, 2)$-category where the hom-spaces are stable $(\infty, 1)$-categories. For this, it is enough to see $\mathcal{D}_{dg}(R_b(\nu), d_N)$ as $0$-cells in the $(\infty, 2)$-category of $A_{\mathcal{X}}$-categories constructed by Faonte [13], and replace $\mathcal{H} \text{om}_{Hqe}$ by the dg-nerve of Lurie [28]. Thanks to [14], we know that this is a stable $(\infty, 1)$-category.
Appendix A. Summary on the homotopy category of dg-categories and pretriangulated dg-categories

We gather some useful results on the homotopy category of dg-categories. References for this section are [20] and [42]. We also suggest [21] and [43] for nice surveys on the subject.

Our goal is to recall how to construct a category of dg-categories up to quasi-equivalence, so that the space of functors between two ‘triangulated categories’ is ‘triangulated’.

A.1. Dg-categories. Recall the definition of a dg-category:

Definition A.1. A dg-category $\mathcal{A}$ is a $k$-linear category such that:

- $\text{Hom}_\mathcal{A}(X,Y)$ is a $\mathbb{Z}$-graded $k$-vector space;
- the composition $\text{Hom}_\mathcal{A}(Y,Z) \otimes_k \text{Hom}_\mathcal{A}(X,Y) \xrightarrow{-\cdot-} \text{Hom}_\mathcal{A}(X,Z)$ preserves the $\mathbb{Z}$-degree;
- there is a differential $d: \text{Hom}_\mathcal{A}(X,Y)^i \to \text{Hom}_\mathcal{A}(X,Y)^{i-1}$ such that
  
  \[
  d^2 = 0, \quad d(f \circ g) = df \circ g + (-1)^{|f|} f \circ dg.
  \]

Remark A.2. We use a differential of degree $-1$ to match the conventions used in the rest of the paper.

Example A.3. Any dg-algebra $(A,d)$ can be seen as a dg-category $BA$ with a single abstract object $\ast$ and $\text{Hom}_{BA}(\ast,\ast) := (A,d)$.

Example A.4. Let $\mathcal{C}$ be an abelian, Grothendieck, $k$-linear category. Consider the category $C(\mathcal{C})$ of complexes in $\mathcal{C}$, and define $C_{dg}(\mathcal{C})$ as the category where

- objects are complexes in $\mathcal{C}$;
- hom-spaces are homogeneous maps of $\mathbb{Z}$-graded modules;
- the differential $d: \text{Hom}_{C_{dg}(\mathcal{C})}(X^\ast, Y^\ast)^i \to \text{Hom}_{C_{dg}(\mathcal{C})}(X^\ast, Y^\ast)^{i-1}$ is given by
  
  \[
  df := d_Y \circ f - (-1)^{|f|} f \circ d_X.
  \]

This data forms a dg-category.

Given a dg-category $\mathcal{A}$, one defines

1. the underlying category $Z^0(\mathcal{A})$ as
   - having the same objects as $\mathcal{A}$;
   - $\text{Hom}_{Z^0(\mathcal{A})}(X,Y) := \ker(\text{Hom}_{\mathcal{A}}(X,Y)^0 \xrightarrow{d} \text{Hom}_{\mathcal{A}}(X,Y)^{-1})$;
2. the homotopy category $H^0(\mathcal{A})$ (or $[\mathcal{A}]$) as
   - having the same objects as $\mathcal{A}$;
   - $\text{Hom}_{H^0(\mathcal{A})}(X,Y) := H^0(\text{Hom}_{\mathcal{A}}(X,Y), d)$.

Example A.5. For $\mathcal{C}$ as in Example A.4, we have $Z^0(C_{dg}(\mathcal{C})) \cong C(\mathcal{C})$ and $H^0(C_{dg}(\mathcal{C})) \cong Kom(\mathcal{C})$ the homotopy category of complexes in $\mathcal{C}$. 
A.2. Category of dg-categories.

**Definition A.6.** A dg-functor $F: \mathcal{A} \to \mathcal{B}$ is a functor between two dg-categories such that $F(d_A f) = d_B(F f)$. We write $[F]: H^0(\mathcal{A}) \to H^0(\mathcal{B})$ for the induced functor.

We write dg-cat for the *category of dg-categories*, where objects are dg-categories and hom-spaces are given by dg-functors.

Let $F, G: \mathcal{A} \to \mathcal{B}$ be a pair of dg-functors between dg-categories. One defines $\mathcal{H}om(F, G)$ as the $\mathbb{Z}$-graded $k$-module of homogeneous natural transformations equipped with the differential induced by $d \in \text{Hom}_k(FX, GX)$ for all $X \in \mathcal{A}$. Then, we put $\text{Hom}(F, G) := Z^0(\mathcal{H}om(F, G))$.

**Definition A.7.** A dg-functor $\mathcal{A} \to \mathcal{B}$ is a quasi-equivalence if

- $F: \text{Hom}_A(X, Y) \xrightarrow{\sim} \text{Hom}_B(FX, FY)$ is a quasi-isomorphism for all $X, Y \in \mathcal{A}$;
- $[F]: H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is essentially surjective (thus an equivalence).

One defines the dg-category $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ of dg-functors between $\mathcal{A}$ and $\mathcal{B}$ as

- objects are dg-functors $\mathcal{A} \to \mathcal{B}$;
- hom-spaces are $\text{Hom}_{\mathcal{H}om(\mathcal{A}, \mathcal{B})}(F, G) := \mathcal{H}om(F, G)$.

There is also a notion of tensor product of dg-categories $\mathcal{A} \otimes \mathcal{B}$ defined as

- objects are pairs $X \otimes Y$ for all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$;
- hom-spaces are $\text{Hom}_{\mathcal{A} \otimes \mathcal{B}}(X \otimes Y, X' \otimes Y') := \text{Hom}_\mathcal{A}(X, X') \otimes_k \text{Hom}_\mathcal{B}(Y, Y')$ with composition

$$(f' \otimes g') \circ (f \otimes g) := (-1)^{|g'|||f|}(f' \circ f) \otimes (g' \circ g);$$

- the differential is $d(f \otimes g) := df \otimes g + (-1)^{|f|}f \otimes dg$.

Then, there is a bijection

$$\text{Hom}_{\text{dg-cat}}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \simeq \text{Hom}_{\text{dg-cat}}(\mathcal{A}, \mathcal{H}om(\mathcal{B}, \mathcal{C})).$$

This defines a symmetric closed monoidal structure on dg-cat. However, the tensor product of dg-categories does not preserve quasi-equivalences.

A.3. Dg-modules. Let $\mathcal{A}$ be a dg-category. The opposite dg-category $\mathcal{A}^{\text{op}}$ is given by

- same objects as in $\mathcal{A}$;
- $\text{Hom}_{\mathcal{A}^{\text{op}}}(X, Y) := \text{Hom}_\mathcal{A}(Y, X)$;
- composition $g \circ_{\mathcal{A}^{\text{op}}} f := (-1)^{|f||g|}f \circ_{\mathcal{A}} g$.

A *left (resp. right)* dg-module $M$ over $\mathcal{A}$ is a dg-functor

$$M: \mathcal{A} \to C_d(k), \quad (\text{resp. } N: \mathcal{A}^{\text{op}} \to C_d),$$

where $C_d(k)$ is the dg-category of $k$-complexes. The *dg-category of (right) dg-modules* is $\mathcal{A}^{\text{op}}\text{-mod} := \mathcal{H}om(\mathcal{A}^{\text{op}}, C_d(k))$. The category of (right) dg-modules is $C(\mathcal{A}) := Z^0(\mathcal{A}\text{-mod})$, and it is an abelian category. The *derived category* $\mathcal{D}(\mathcal{A})$ is the localization of $Z^0(\mathcal{A}^{\text{op}}\text{-mod})$ along quasi-isomorphisms.
Moreover, for any $X \in \mathcal{A}$ there is a right dg-module
\[ X^\wedge := \text{Hom}_{\mathcal{A}}(-, X). \]
One calls such dg-module representable. Any dg-module quasi-isomorphic to a representable dg-module is called quasi-representable. It yields a dg-enriched Yoneda embedding $\mathcal{A} \to \mathcal{A}^{\text{op}}\text{-mod}$.

**Example A.8.** Let $(A, d)$ be a dg-algebra. Then $Z^0(BA)\text{-mod} \cong (A, d)\text{-mod}$ and $\mathcal{D}(BA) \cong \mathcal{D}(A, d)$. The unique representable dg-module $\text{Hom}_{BA}(-, \ast)$ is equivalent to the free module $(A, d)$.

**A.4. Model categories.** We recall the basics of model category theory from [16]. Model category theory is a powerful tool to study localization of categories. For example, we can use it to compute hom-spaces in a derived category. We will mainly use it to describe the homotopy category of dg-categories up to quasi-equivalence.

Let $M$ be a category with limits and colimits.

**Definition A.9.** A model category on $M$ is the data of three classes of morphisms
- the weak equivalences $W$;
- the fibrations $\text{Fib}$;
- the cofibrations $\text{Cof}$;

satisfying
- for $X \xrightarrow{f} Y \xrightarrow{g} Z \in M$, if two out of three terms in \{f, g, g \circ f\} are in $W$, then so is the third;
- stability along retracts: $W, \text{Fib}$ and $\text{Cof}$ are stable along retracts, that is if we have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & X \\
\downarrow{g} & & \downarrow{f} & & \downarrow{g} \\
X' & \xrightarrow{f} & Y' & \xrightarrow{g} & X'
\end{array}
\]

and $f \in W, \text{Fib}$ or $\text{Cof}$ then so is $g$.
- factorization: any $X \xrightarrow{f} Y$ factorizes as $p \circ i$ where $p \in \text{Fib}$ and $i \in \text{Cof} \cap W$ or $p \in \text{Fib} \cap W$ and $i \in \text{Cof}$, and the factorization is functorial in $f$;
- lifting property: given a commutative square diagram

\[
\begin{array}{ccc}
A & \xrightarrow{p \in \text{Fib}} & X \\
\downarrow{Cof \exists \triangleright} & & \downarrow{p \in \text{Fib}} \\
B & \xrightarrow{p \in \text{Fib}} & Y
\end{array}
\]
with \( i \in \text{Cof} \) and \( p \in \text{Fib} \), if either \( i \in W \) or \( p \in W \), then there exists \( h : B \to X \) making the diagram commute.

We tend to think about fibrations as ‘nicely behaved surjections’, and cofibrations as ‘nicely behaved injections’.

The localization \( \text{Ho}(M) := W^{-1}M \) of \( M \) along weak equivalences is called the homotopy category of \( M \). It has a nice description in terms of homotopy classes of maps between fibrant and cofibrant objects.

**Definition A.10.** If \( \emptyset \to X \in \text{Cof} \), then we say \( X \) is cofibrant. If \( Y \to * \in \text{Fib} \), then \( Y \) is fibrant.

One says that \( f \sim g \), that is \( f : X \to Y \) is homotopy equivalent to \( g : X \to Y \), if there is a commutative diagram

\[
\begin{array}{ccc}
  & \downarrow & \\
  X & \xrightarrow{f} & Y \\
  \downarrow & & \downarrow \\
  C(X) & \xrightarrow{h} & Y \\
  \downarrow & & \downarrow \\
  X & \xrightarrow{g} & Y \\
\end{array}
\]

where \( i \sqcup j : X \sqcup X \to C(X) \in \text{Cof} \). One calls \( C(X) \) the cylinder object of \( X \). When \( X \) is cofibrant and \( Y \) fibrant, then \( \sim \) is an equivalence relation on \( \text{Hom}_M(X,Y) \). Moreover, we have

\[
\text{Hom}_{\text{Ho}(M)}(X,Y) \cong \text{Hom}_M(X,Y)/\sim
\]

whenever \( X \) is cofibrant and \( Y \) fibrant. Note that any \( X \in M \) admits a cofibrant replacement \( QX \) since we have a commutative diagram

\[
\begin{array}{ccc}
  \emptyset & \oto & X \\
  \oto & & \oto \\
  \oto & & \oto \\
  \oto & & \oto \\
  QX & \oto & \oto \\
\end{array}
\]

Similarly, any \( Y \in M \) admits a fibrant replacement \( RY \).

Let \( M^{\text{cf}} \) be the full subcategory of \( M \) given by objects that are both fibrant and cofibrant. Let \( M^{\text{cf}}/\sim \) be the quotient of \( M^{\text{cf}} \) by identifying maps that are homotopy equivalent. Then, the localization functor \( M \to \text{Ho}(M) \) restricts to \( M^{\text{cf}} \), inducing an equivalence of categories

\[
M^{\text{cf}}/\sim \xrightarrow{\cong} \text{Ho}(M).
\]
Example A.11. Let $C(k)$ be the category of complexes of $k$-modules. It comes with a model category structure where $W$ is the quasi-isomorphisms, $Fib$ is the surjective maps, and $Cof$ is given by the maps respecting the lifting property. All objects are fibrant and the cofibrant objects are essentially the complexes of projective $k$-modules. Then $Ho(C(k)) \cong \mathcal{D}(k)$.

A model category on $M$ is a $C(k)$-model category if it is (strongly) enriched over $C(k)$, and the models are compatible (see [43, §3.1] for a precise definition). This definition means that we have:

- a tensor product $- \otimes - : C(k) \times M \to M$;
- an enriched dg-hom-space $\mathcal{H}om_M(X,Y) \in C(k)$ for any $X,Y \in M$ compatible with the tensor product:
  $$\text{Hom}_M(E \otimes X, Y) \cong \text{Hom}_{C(k)}(E, \mathcal{H}om_M(X,Y));$$
- $Ho(M)$ is enriched over $\mathcal{D}(k) \cong Ho(C(k))$;
- a derived hom-functor $\mathcal{R}\mathcal{H}om_M(X,Y) := \mathcal{H}om_M(QX, RY) \in \mathcal{D}(k)$, where $QX$ is a cofibrant replacement of $X$, and $RY$ a fibrant replacement of $Y$;
- $\text{Hom}_{Ho(M)}(X,Y) \cong H^0(\mathcal{R}\mathcal{H}om_M(X,Y))$.

Note that in particular for $X,Y \in M^{cf}$ we have $\text{Hom}_{Ho(M)}(X,Y) \cong H^0(\mathcal{H}om(X,Y))$.

Example A.12. Let $\mathcal{A}$ be a dg-category. There is a $C(k)$-model category on $\mathcal{A}$-mod where $W$ is given by the quasi-isomorphisms, $Fib$ are the surjective morphisms, and $Cof$ is given by the maps respecting the lifting property. Then $Ho(\mathcal{A}$-mod) $\cong \mathcal{D}(\mathcal{A})$.

Remark A.13. In the $C(k)$-model category $\mathcal{A}$-mod, all objects are fibrant. Moreover, $P$ is cofibrant if and only if for all surjective quasi-isomorphism $f : L \xrightarrow{\sim} X$ (i.e. map in $W \cap Fib$) then there exists $h : P \to L$ such that the following diagram commutes:

$$\begin{array}{ccc}
\emptyset & \longrightarrow & L \\
\downarrow & & \downarrow \\
\mathbb{Z}h & \searrow & X \\
\downarrow & & \downarrow \\
P & \longrightarrow & X
\end{array}$$

In a practical way, cofibrant dg-modules are quasi-isomorphic to direct summand of dg-modules admitting a (possibly infinite) exhaustive filtration where all the quotients are free dg-modules.

Definition A.14. For $M$ a $C(k)$-model category, let $\underline{M}$ (resp. $\text{Int}(M)$) be the dg-category with

- the same objects as $M$ (resp. $M^{cf}$);
- $\text{Hom}_{\underline{M}}(X,Y) := \mathcal{H}om_M(X,Y)$.

Then, we have $H^0(\text{Int}(M)) \cong Ho(M)$, and we say that $\text{Int}(M)$ is a dg-enhancement of $Ho(M)$.
Definition A.15. We write

$$\mathcal{D}_{dg}(\mathcal{A}) := \text{Int}(\mathcal{A}\text{-mod})$$

for the dg-enhanced derived category of $\mathcal{A}$.

Note that $\mathcal{D}_{dg}(\mathcal{A})$ is a dg-enhancement of $\mathcal{D}(\mathcal{A})$ since we have $H^0(\mathcal{D}_{dg}(\mathcal{A})) \cong \mathcal{D}(\mathcal{A})$.

Example A.16. Let $R$ be a $k$-algebra viewed as a dg-category with trivial differential. Then we have that $\mathcal{D}_{dg}(R)$ is the dg-category of complexes of projective $R$-modules.

A.5. The model category of dg-categories. Let $W$ be the collection of quasi-equivalences in dg-cat. Let $\text{Fib}$ be the collection of dg-functors $F : \mathcal{A} \to \mathcal{B}$ in dg-cat such that

1. $F_{X,Y} : \text{Hom}_\mathcal{A}(X, Y) \to \text{Hom}_\mathcal{B}(FX, FY)$ is surjective;
2. for all isomorphism $v : F(X) \cong Y \in H^0(\mathcal{B})$ there exists an isomorphism $u : X \cong Y_0 \in H^0(\mathcal{A})$ such that $[F](u) = v$.

This defines a model structure on dg-cat where everything is fibrant. One calls

$$\mathcal{H}qe := Ho(\text{dg-cat})$$

the homotopy category of dg-categories (up to quasi-equivalence).

How can we compute $\text{Hom}_{\mathcal{H}qe}(\mathcal{A}, \mathcal{B})$? It appears that constructing a cofibrant replacement for $\mathcal{A}$ is in general a difficult problem. However, we can do the following:

1. replace $\mathcal{A}$ by a $k$-flat quasi-equivalent dg-category $\mathcal{A}'$: meaning it is such that

   $$\text{Hom}_{\mathcal{A}'}(X, Y) \otimes_k -$$

   preserves quasi-isomorphisms (e.g. when $\text{Hom}_{\mathcal{A}'}(X, Y)$ is cofibrant in $C(k)$, i.e. a complex of projective $k$-modules);

2. define $\text{Rep}(\mathcal{A}, \mathcal{B})$ as the subcategory of $\mathcal{D}(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$ with $F \in \text{Rep}(\mathcal{A}, \mathcal{B})$ if and only if for all $X \in \mathcal{A}$ there exists $Y \in \mathcal{B}$ such that

   $$X \otimes \mathcal{L} F \cong_{\mathcal{D}(\mathcal{B})} Y \vee,$$

   (in other words, $F$ is a dg-bimodule sending representable $\mathcal{A}$-modules to quasi-representable $\mathcal{B}$-modules);

3. then

   $$\text{Hom}_{\mathcal{H}qe}(\mathcal{A}, \mathcal{B}) \cong \text{Iso}(\text{Rep}(\mathcal{A}, \mathcal{B})),$$

   where $\text{Iso}$ means the set of objects up to isomorphism.

Remark A.17. Note that whenever $k$ is a field, all dg-categories are $k$-flat.

We refer to elements in $\text{Rep}(\mathcal{A}, \mathcal{B})$ as quasi-functors. Note that a quasi-functor $F : \mathcal{A} \to \mathcal{B}$ induces a functor

$$[F] : H^0(\mathcal{A}) \to H^0(\mathcal{B}).$$

Thus, we can think of $\text{Rep}(\mathcal{A}, \mathcal{B})$ as the category of ‘representations up to homotopy’ of $\mathcal{A}$ in $\mathcal{B}$.
A.5.1. **Closed monoidal structure.** If $\mathcal{A}$ is cofibrant, then $- \otimes \mathcal{A}$ preserves quasi-equivalences and one can define the bifunctor

$$- \otimes^L - : \text{Hqe} \times \text{Hqe} \to \text{Hqe}, \quad \mathcal{A} \otimes^L \mathcal{B} := Q\mathcal{A} \otimes Q\mathcal{B},$$

where $Q\mathcal{A}$ and $Q\mathcal{B}$ are cofibrant replacements. Then, as proven by Toen [42], there exists an internal hom-functor $\mathcal{RHom}_{\text{Hqe}}(-, -)$ such that

$$\text{Hom}_{\text{Hqe}}(\mathcal{A} \otimes^L \mathcal{B}, \mathcal{C}) \cong \text{Hom}_{\text{Hqe}}(\mathcal{A}, \mathcal{RHom}_{\text{Hqe}}(\mathcal{B}, \mathcal{C})).$$

Therefore, Hqe is a symmetric closed monoidal category.

**Remark A.18.** Note that the internal hom can not simply be the derived hom functor (because tensor product of cofibrant dg-categories is not cofibrant in general).

Define the **dg-category of quasi-functors** $\text{Rep}_{\text{dg}}(\mathcal{A}, \mathcal{B})$ as

- the objects in $\text{Rep}(\mathcal{A}, \mathcal{B}) \cap (\mathcal{A}^{\text{op}} \otimes \mathcal{B}^{\text{mod}})_{\text{cf}}$;
- the dg-homs $\mathcal{H}om(X, Y)$ of $\text{Int}(\mathcal{A}^{\text{op}} \otimes \mathcal{B}^{\text{mod}})$.

In other words, $\text{Rep}_{\text{dg}}(\mathcal{A}, \mathcal{B})$ is the full subcategory of quasi-functors in $\mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$, thus of cofibrant dg-bimodules that preserves quasi-representable modules. It is a dg-enhancement of $\text{Rep}(\mathcal{A}, \mathcal{B})$.

If $\mathcal{A}$ is $k$-flat, then

$$\mathcal{RHom}_{\text{Hqe}}(\mathcal{A}, \mathcal{B}) \cong_{\text{Hqe}} \text{Rep}_{\text{dg}}(\mathcal{A}, \mathcal{B}).$$

Thus $H^0(\mathcal{RHom}_{\text{Hqe}}(\mathcal{A}, \mathcal{B})) \cong \text{Hom}_{\text{Hqe}}(\mathcal{A}, \mathcal{B})$.

**Remark A.19.** If $k$ is a field of characteristic 0, then the dg-category $\mathcal{RHom}_{\text{Hqe}}(\mathcal{A}, \mathcal{B})$ is equivalent to the $A_{\infty}$-category of strictly unital $A_{\infty}$-functors [13].

**Example A.20.** We have $\text{Rep}_{\text{dg}}(\mathcal{A}, \text{Int}(C(k))) \cong \text{Int}(\mathcal{A}^{\text{op}} \text{-mod}) \cong \mathcal{D}_{\text{dg}}(\mathcal{A})$.

Recall that classical Morita theory says that for $A$ and $B$ being $k$-algebras, there is an equivalence

$$\text{Hom}^{\text{cop}}(A \text{-mod}, B \text{-mod}) \cong A^{\text{op}} \otimes_k B \text{-mod},$$

where $\text{Hom}^{\text{cop}}$ is given by the functors that preserve coproducts.

Similarly, we put $\text{Rep}^{\text{cop}}_{\text{dg}}(\mathcal{D}_{\text{dg}}(\mathcal{A}), \mathcal{D}_{\text{dg}}(\mathcal{B}))$ for the subcategory of $\text{Rep}_{\text{dg}}(\mathcal{D}_{\text{dg}}(\mathcal{A}), \mathcal{D}_{\text{dg}}(\mathcal{B}))$ where $F \in \text{Rep}^{\text{cop}}_{\text{dg}}(\mathcal{D}_{\text{dg}}(\mathcal{A}), \mathcal{D}_{\text{dg}}(\mathcal{B}))$ if and only if $[F] : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ preserves coproducts.

**Theorem A.21.** If $\mathcal{A}$ is $k$-flat, then we have

$$\mathcal{RHom}^{\text{cop}}_{\text{Hqe}}(\mathcal{D}_{\text{dg}}(\mathcal{A}), \mathcal{D}_{\text{dg}}(\mathcal{B})) := \text{Rep}^{\text{cop}}_{\text{dg}}(\mathcal{D}_{\text{dg}}(\mathcal{A}), \mathcal{D}_{\text{dg}}(\mathcal{B})) \cong_{\text{Hqe}} \mathcal{D}_{\text{dg}}(\mathcal{A}^{\text{op}} \otimes \mathcal{B}).$$

Under the hypothesis of Theorem A.21, the internal composition of dg-quasifunctors preserving coproducts is given by taking a cofibrant replacement of the derived tensor product over $\mathcal{A}$. 

A.6. **Pretriangulated dg-categories.** Basically, a triangulated dg-category is a dg-category such that its homotopy category is canonically triangulated. But before being able to give a precise definition, we need to do a detour through Quillen exact categories, Frobenius categories and stable categories.

A.6.1. **Frobenius structure on** $\mathcal{C}_{pA}$. Recall a Quillen exact category [36] is an additive category with a class of short exact sequences

$$0 \to X \stackrel{f}{\to} Y \stackrel{g}{\to} Z \to 0,$$

called conflations, which are pairs of ker-coker, where $f$ is called an inflation and $g$ a deflation, respecting some axioms:

- the identity is a deflation;
- the composition of deflations is a deflation;
- deflations (resp. inflations) are stable under base (resp. cobase) change.

A Frobenius category is a Quillen exact category having enough injectives and projectives, and where injectives coincide with projectives. The stable category $\mathcal{C}$ of a Frobenius category $\mathcal{C}$ is given by modding out the maps that factor through an injective/projective object. It carries a canonical triangulated structure where:

- the suspension functor $S$ is obtained by taking the target of a conflation

$$0 \to X \to IX \to SX \to 0,$$

where $IA$ is an injective hull of $X$, for all $X \in \mathcal{C}$;

- the distinguished triangles are equivalent to standard triangles

$$X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} SX,$$

obtained from conflations by the following commutative diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & X \\
\downarrow^{Id} & & \downarrow^{g} \\
0 & \longrightarrow & IX \\
\end{array}
\begin{array}{ccc}
& & \longrightarrow \\
& & \downarrow^{h} \\
& & 0 \\
\end{array}
\begin{array}{ccc}
 & \longrightarrow & Y \\
\downarrow^{f} & & \downarrow \\
 & \longrightarrow & Z \\
\end{array}
\begin{array}{ccc}
 & \longrightarrow & SX \\
\downarrow^{h} & & \downarrow \\
 & \longrightarrow & 0 \\
\end{array}
$$

**Example A.22.** Let $\mathcal{A}$ be a small dg-category. One can put a Frobenius structure on $C(\mathcal{A}) := Z^0(\mathcal{A}^{\text{op}}\text{-mod})$ by using split short exact sequences as class of conflations. Then there is an equivalence $C(\mathcal{A}) \cong H^0(\mathcal{A}^{\text{op}}\text{-mod})$, and the suspension functor coincides with the usual homological shift. Moreover, $\mathcal{D}(\mathcal{A})$ inherits the triangulated structure from $H^0(\mathcal{A}^{\text{op}}\text{-mod})$, where distinguished triangles are equivalent to distinguished triangles obtained from all short exact sequences in $C(\mathcal{A})$.

A.6.2. **Pretriangulated dg-categories.** Remark for any dg-category $\mathcal{A}$ there is a Yoneda functor

$$Z^0(\mathcal{A}) \to C(\mathcal{A}), \quad X \mapsto \text{Hom}_{\mathcal{A}}(-, X).$$
**Definition A.23.** A dg-category $\mathcal{T}$ is **pretriangulated** if the image of the Yoneda functor is stable under translations and extensions (for the Quillen exact structure on $C(\mathcal{T})$ described in Example A.22).

This definition implies that
- $Z^0(\mathcal{T})$ is a Frobenius subcategory of $C(\mathcal{T})$;
- $H^0(\mathcal{T})$ inherits a triangulated structure, called **canonical triangulated structure**, from $H^0(\mathcal{T} \text{-mod})$.

**Example A.24.** Let $\mathcal{A}$ be a dg-category. We have that $\mathcal{D}_{dg}(\mathcal{A})$ is pretriangulated with $Z^0(\mathcal{D}_{dg}(\mathcal{A})) \cong C(\mathcal{A})^{cf}$. Moreover, the canonical triangulated structure of $H^0(\mathcal{D}_{dg}(\mathcal{A}))$ coincides with the usual on $\mathcal{D}(\mathcal{A})$.

Then, it is possible to show that

- any dg-category $\mathcal{A}$ admits a pretriangulated hull $\text{pretr}(\mathcal{A})$ such that
  $$\mathcal{R} \mathcal{H} \text{om}_{\mathcal{H}e}(\mathcal{A}, \mathcal{T}) \xrightarrow{\cong} \mathcal{R} \mathcal{H} \text{om}_{\mathcal{H}e}(\text{pretr}(\mathcal{A}), \mathcal{T}),$$
  for all pretriangulated dg-category $\mathcal{T}$;
- $\mathcal{R} \mathcal{H} \text{om}_{\mathcal{H}e}(\mathcal{A}, \mathcal{T})$ is pretriangulated whenever $\mathcal{T}$ is pretriangulated;
- any dg-functor $F : \mathcal{T} \to \mathcal{T}'$ between pretriangulated dg-categories induces a triangulated functor $[F] : H^0(\mathcal{T}) \to H^0(\mathcal{T}')$.

Note that for $\mathcal{A}$ being $k$-flat, the pretriangulated structure of $\mathcal{R} \mathcal{H} \text{om}_{\mathcal{H}e}(\mathcal{D}_{dg}(\mathcal{A}), \mathcal{D}_{dg}(\mathcal{B}))$ restricts to the one of $\mathcal{D}_{dg}(\mathcal{A}^{op} \otimes \mathcal{B})$ (viewed as sub-dg-category). In particular, we obtain distinguished triangles of quasi-functors from short exact sequences of dg-bimodules.

**Definition A.25.** For a morphism $f : X \to Y \in Z^0(\mathcal{T})$ in the underlying category of pretriangulated dg-category $\mathcal{T}$, one calls **mapping cone** an object $\text{Cone}(f) \in \mathcal{T}$ such that

$$\text{Cone}(f)^\wedge \cong \text{Cone}(X^\wedge \xrightarrow{f} Y^\wedge) \in H^0(\mathcal{T} \text{-mod}).$$

A.6.3. **Dg-Morita equivalences.**

**Definition A.26.** A dg-functor $F : \mathcal{A} \to \mathcal{B}$ is a dg-Morita equivalence if it induces an equivalence

$$LF : \mathcal{D}(\mathcal{A}) \xrightarrow{\cong} \mathcal{D}(\mathcal{B}) : X \mapsto F(X),$$

where $QX$ is a cofibrant replacement of $X$.

**Example A.27.** In particular, a quasi-equivalence is a dg-Morita equivalence and the functor that sends dg-categories to their pretriangulated hull $\mathcal{A} \mapsto \text{pretr}(\mathcal{A})$ is a dg-Morita equivalence.

**Theorem A.28** ([40]). There is a model structure $\text{dg-cat}_{\text{mor}}$ on $\text{dg-cat}$ where the weak-equivalences are the dg-Morita equivalences and the fibrations are the same as before.

**Definition A.29.** We say that $\mathcal{T}$ is triangulated if it is fibrant in $\text{dg-cat}_{\text{mor}}$. 
Equivalently, $\mathcal{T}$ is triangulated if and only if the Yoneda functor induces an equivalence $H^0(\mathcal{T}\text{-mod}) \xrightarrow{\sim} \mathcal{D}^c(\mathcal{T})$ (i.e. every compact object is quasi-representable). Also equivalently, $\mathcal{T}$ is triangulated if and only if $\mathcal{T}$ is pretriangulated and $H^0(\mathcal{T}\text{-mod})$ is idempotent complete.

In particular, any category admits a triangulated hull $tr(\mathcal{A})$ (i.e. fibrant replacement). It is given by $tr(\mathcal{A}) := \mathcal{D}_{dg}(\mathcal{A})$, the dg-category of compact objects in $\mathcal{D}_{dg}(\mathcal{A})$.

**Example A.30.** Let $R$ be a $\mathbb{k}$-algebra viewed as a dg-category. Then $\mathcal{D}_{dg}(R)$ is the dg-category of perfect complexes, i.e. bounded complexes of finitely generated projective $R$-modules.

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Max-Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany
Email address: gregoire.naisse@gmail.com

Institut de Recherche en Mathématique et Physique, Université Catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium
Email address: pedro.vaz@uclouvain.be