FUNCTIONAL MODEL FOR BOUNDARY VALUE PROBLEMS
AND ITS APPLICATION TO THE SPECTRAL ANALYSIS OF
TRANSMISSION PROBLEMS

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Abstract

We develop a functional model for operators arising in the study of boundary-value
problems of materials science and mathematical physics. We provide explicit formulae
for the resolvents of the associated extensions of symmetric operators in terms
of the associated generalised Dirichlet-to-Neumann maps, which can be utilised in
the analysis of the properties of parameter-dependent problems as well as in the
study of their spectra.

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1. Introduction

The need to understand and quantify the behaviour of solutions to problems of mathematical physics has been central in driving the development of theoretical tools for the analysis of boundary-value problems (BVP). On the other hand, the second part of the last century witnessed several substantial advances in the abstract methods of spectral theory in Hilbert spaces, stemming from the groundbreaking achievement of John von Neumann in laying the mathematical foundations of quantum mechanics. Some of these advances have made their way into the broader context of mathematical physics [32, 18, 40]. In spite of these obvious successes of spectral theory applied to concrete problems, the operator-theoretic understanding of BVP has been lacking. However, in models of short-range interactions, the idea of replacing the original complex system by an explicitly solvable one, with a potential of zero radius (possibly with an internal structure), has proved to be highly valuable [5, 44, 11], [6, 30, 31], [35]. This facilitated an influx of methods of the theory of extensions (both self-adjoint and non-selfadjoint) of symmetric operators to problems of mathematical physics, culminating in the theory of boundary triples.

The theory of boundary triples introduced in [26, 27] has been successfully applied to the spectral analysis of BVP for ordinary differential operators and related setups, e.g. that of finite “quantum graphs”, where the Dirichlet-to-Neumann maps act on finite-dimensional “boundary” spaces, see [16] and references therein. However, in its original form it is not suited for dealing with BVP of partial differential equations (PDE), see [9, Section 7] for a relevant discussion, the key obstacle being the lack of boundary traces $\Gamma_0 u$ and $\Gamma_1 u$ for functions $u$ in the domain of the operator $A$ entering the Green identity

$$\langle Au, v \rangle_{L^2(\Omega)} - \langle u, Av \rangle_{L^2(\Omega)} = \langle \Gamma_1 u, \Gamma_0 v \rangle_{L^2(\partial\Omega)} - \langle \Gamma_0 u, \Gamma_1 v \rangle_{L^2(\partial\Omega)}, \quad u, v \in \text{dom}(A),$$

in other words $\text{dom}(A) \not\subseteq \text{dom}(\Gamma_0) \cap \text{dom}(\Gamma_1)$. Still, despite very productive efforts of [25] and later work of Grubb and Agranovich, this approach could not be fully transferred to the general BVP setup up until very recently, when the works [4, 48, 9] started to appear.

In all cases mentioned above, one can see the fundamental rôle of a certain Herglotz operator-valued analytic function, which in problems where a boundary is present (and sometimes even without an explicit boundary [2]) turns out to be a natural generalisation of the classical notion of the Dirichlet-to-Neumann map. The emergence of this object yields the possibility to apply advanced methods of complex analysis in conjunction with abstract methods of operator and spectral theory, which in turn sheds a light on the intrinsic interplay between the mentioned abstract frameworks and concrete problems of interest in modern mathematical physics.

The present paper is a development of the recent activity [13, 12, 14, 17] aimed at implementing the above strategy in the context of problems of materials science and wave propagation in inhomogeneous media. Our recent papers [15, 16] have shown that the language of boundary triples is particularly fitting for direct and inverse scattering problems on quantum graphs, as one of the key difficulties in their analysis stems from the presence of interfaces through which energy exchange between different components of the medium takes place. In the present work we continue the research initiated in these papers, adapting the technology so that BVP, especially those stemming from materials sciences, are within reach. As in [15, 16], the ideology of the functional model of [43, 35] allows one to efficiently incorporate the information about the energy exchange, by employing a suitable Dirichlet-to-Neumann map.

In our analysis of BVP, we are motivated by models of wave propagation in inhomogeneous media. Here we adopt the approach to the operator-theoretic treatment of BVP suggested by [48], which appears to be particularly convenient for obtaining sharp quantitative information about the scattering properties of the medium, cf. e.g. [17], where this same approach is used as a framework for the asymptotic analysis of homogenisation problems in resonant composite media.

We next outline the structure of the paper. In Section 2 we recall the main points of the
abstract construction of [48] and introduce the key objects for the analysis we carry out later on, such as the dissipative operator $L$ at the centre of the functional model. In Section 3 we construct the minimal dilation of $L$, based on the earlier ideas of [47] in the context of extensions of symmetric operators. Using the functional framework thus developed, in Section 4 we develop a new version of Pavlov’s “three-component” functional model for the dilation [42] and pass to his “two-component”, or “symmetric”, model [43] (see also [35, 47]), based on the notion of the characteristic function for $L$, which is computed explicitly in terms of the $M$-operator introduced in Section 2. In Section 5 we develop formulae for the resolvents of boundary-value operators for a range of boundary conditions $\alpha \Gamma_0 u + \beta \Gamma_1 u = 0$, with $\alpha, \beta$ from a wide class of operators in $L^2(\partial \Omega)$ including those relevant to applications. The last two sections are devoted to the applications of the framework: based on the derived formulae for the resolvents, in Section 6 we establish the functional model for the boundary-value problems from the class discussed earlier, and in Section 7 we prove an interlacing property for the eigenvalues of a transmission problem, relevant to a variety of problems in wave propagation and mechanics of inhomogeneous media.

2. Ryzhov triples for BVP

In this section we follow [48] in developing an operator framework suitable for dealing with boundary value problems. The starting point is a self-adjoint operator $A_0$ in a separable Hilbert space $\mathcal{H}$ with $0 \in \rho(A_0)$, where $\rho(A_0)$, as usual, denotes the resolvent set of $A_0$. Alongside $\mathcal{H}$, we consider an auxiliary Hilbert space $\mathcal{E}$ and a bounded operator $\Pi : \mathcal{E} \to \mathcal{H}$ such that

$$\text{dom}(A_0) \cap \text{ran}(\Pi) = \{0\} \quad \text{and} \quad \ker(\Pi) = \{0\}. \quad (2.1)$$

Since $\Pi$ has a trivial kernel, there is an inverse $\Pi^{-1}$ such that $\Pi^{-1} \Pi = I_{\mathcal{E}}$ (i.e. $\Pi^{-1}$ is the left inverse of $\Pi$).

We define

$$\text{dom}(A) := \text{dom}(A_0) \cap \text{ran}(\Pi), \quad A : A_0^{-1} f + \Pi \phi \mapsto f, \quad f \in \mathcal{H}, \phi \in \mathcal{E}, \quad (2.2)$$

$$\text{dom}(\Gamma_0) := \text{dom}(A_0) \cap \text{ran}(\Pi), \quad \Gamma_0 : A_0^{-1} f + \Pi \phi \mapsto \phi, \quad f \in \mathcal{H}, \phi \in \mathcal{E}, \quad (2.3)$$

where neither $A$ nor $\Gamma$ is assumed closed or indeed closable. The operator given in (2.2) is the null extension of $A_0$, while (2.3) is the null extension of $\Pi^{-1}$. Note that

$$\ker(\Gamma_0) = \text{dom}(A_0). \quad (2.4)$$

For $z \in \rho(A_0)$, consider the abstract spectral boundary value problem

$$\begin{cases}
Au = zu, \\
\Gamma_0 u = \phi, 
\end{cases} \quad \phi \in \mathcal{E}, \quad (2.5)$$

where the second equation is seen as a boundary condition. As it is asserted in [48, Thm. 3.2], there is a unique solution $u$ of the boundary value problem (2.5) for any $\phi \in \mathcal{E}$. Thus, there is an operator (clearly linear) which assigns to any $\phi \in \mathcal{E}$ a vector $u$ being a solution to (2.5). This operator is called the solution operator and is denoted by $\gamma(z)$ (this function is sometimes referred to as the $\gamma$-field). An explicit expression for the solution operator in terms of $A_0$ and $\Pi$ can be obtained as follows. Let $z \in \rho(\Lambda_0)$. One can show, using the fact that $A \supset A_0$, that (see [48, Prop.
Also, according to (3.4),

\[ u = \Pi \phi + z(A_0 - zI)^{-1} \Pi \phi \in \ker(A - zI) \quad \forall \phi \in \mathcal{E}. \]

Moreover, \( \Gamma_0 u = \phi \). Hence, for any \( z \in \rho(A_0) \), the solution operator for (2.5) is given by

\[ \gamma(z) : \phi \mapsto (I + z(A_0 - zI)^{-1}) \Pi \phi. \quad (2.6) \]

Note that

\[ I + z(A_0 - zI)^{-1} = (I - zA_0^{-1})^{-1} \quad (2.7) \]

and that (2.3) and (2.6) immediately imply

\[ \Gamma_0 \gamma(z) = I_{\mathcal{E}}. \quad (2.8) \]

By (2.6), one has \( \text{ran}(\gamma(z)) \subset \ker(A - zI) \), but the inverse inclusion also takes place. Indeed, taking a vector \( u \in \ker(A - zI) \) and writing it in the form \( u = A_0^{-1} f + \Pi \phi \), one obtains

\[ 0 = (A - zI)(A_0^{-1} f + \Pi \phi) = (I - zA_0^{-1}) f - z \Pi \phi, \]

which yields \( f = z(I - zA_0^{-1})^{-1} \Pi \phi \). Thus,

\[ u = A_0^{-1} f + \Pi \phi = \left[ zA_0^{-1}(I - zA_0^{-1})^{-1} + I \right] \Pi \phi = (I - zA_0^{-1})^{-1} \Pi \phi. \]

In view of (2.7), the last expression shows that \( u \in \text{ran}(\gamma(z)) \). Putting together the above, one arrives at

\[ \text{ran}(\gamma(z)) = \ker(A - zI). \quad (2.9) \]

We remark that, since \( A \) is not required to be closed, \( \text{ran}(\gamma(z)) \) is not necessarily a subspace (closed linear set).

In what follows, we consider (abstract) BVP of the form (2.5) associated with the operator \( A \), but with different boundary conditions. To this end, define

\[ \text{dom}(\Gamma_1) := \text{dom}(A_0) + \Pi \text{dom}(A), \]

\[ \Gamma_1 : A_0^{-1} f + \Pi \phi \mapsto \Pi^* f + \Lambda \phi, \quad f \in \mathcal{H}, \phi \in \text{dom}(A), \quad (2.10) \]

where \( \Lambda \) is a self-adjoint operator in \( \mathcal{E} \) which can be seen as a parameter for \( \Gamma_1 \).

On the basis of (2.6), one obtains from (2.10) (see [48, Eq.3.6]) that

\[ \gamma(z)^* = \Gamma_1(A_0 - zI)^{-1}. \quad (2.11) \]

Also, according to [48, Thm.3.6], the so-called Green’s formula is satisfied, namely,

\[ \langle Au, v \rangle_{\mathcal{H}} - \langle u, Av \rangle_{\mathcal{H}} = \langle \Gamma_1 u, \Gamma_0 v \rangle_{\mathcal{E}} - \langle \Gamma_0 u, \Gamma_1 v \rangle_{\mathcal{E}}, \quad u, v \in \text{dom}(\Gamma_1). \quad (2.12) \]

Henceforth, we refer to the triple \( (A_0, \Lambda, \Pi) \) as the Ryzhov triple, or simply “triple”, for the spectral BVP (2.5). This setup stems from the Birman-Krein-Vishik theory [6, 30, 31, 56], rather than the theory of boundary triples [24].

**Definition 1.** For a given triple \( (A_0, \Lambda, \Pi) \), define the operator-valued function \( M \) associated with \( A_0 \) so that, for any \( z \in \rho(A_0) \), \( M(z) \) in \( \mathcal{E} \) on the domain \( \text{dom}(M(z)) := \text{dom}(\Lambda) \) given by

\[ M(z) : \phi \mapsto \Gamma_1 \gamma(z) \phi. \]

The above abstract framework is illustrated (see [48] for details) by the classical setup in which \( A_0 \) is the Dirichlet Laplacian in a bounded domain \( \Omega \) with smooth boundary \( \partial \Omega \), which is self-
adjoint on the domain $H^2(\Omega) \cap H_0^1(\Omega)$. In this case $\Pi$ is simply the Poisson operator of harmonic lift, its left inverse is the operator of boundary trace for harmonic functions and $\Gamma_0$ is the null extension of the latter to $[H^2(\Omega) \cap H_0^1(\Omega)] + \Pi L^2(\partial \Omega)$. Furthermore, $\Lambda$ can be chosen as the Dirichlet-to-Neumann map$^1$ which maps any function $\phi \in H^1(\Omega) =: \text{dom}(\Lambda)$ into $-\frac{\partial u}{\partial n} \mid_{\partial \Omega}$, where $u$ is the solution of the boundary value problem

$$\begin{cases}
\Delta u = 0 \\
u \mid_{\partial \Omega} = \phi
\end{cases}$$

(see [53]). Due to the choice of $\Lambda$, it follows from (2.10) that

$$\text{dom}(\Gamma_1) = [H^2(\Omega) \cap H_0^1(\Omega)] + \Pi H^1(\partial \Omega)$$

$$\Gamma_1 u = -\frac{\partial u}{\partial n} \mid_{\partial \Omega}.$$  

(2.13)

Note that (2.13) follows from the fact that $\Pi^* f = -\frac{\partial f}{\partial n} \mid_{\partial \Omega}$ for $u = A_0^{-1} f$. For the above classical setup, one therefore has $z \in \rho(A_0)$, $M(z)$ is the Dirichlet-to-Neumann map $\phi \mapsto -\frac{\partial u}{\partial n} \mid_{\partial \Omega}$ of the spectral boundary problem (2.5), i.e. $u \in (H^2(\Omega) \cap H_0^1(\Omega)) \oplus \Pi L^2(\partial \Omega)$ is a solution of

$$\begin{cases}
\Delta u = z u \\
u \mid_{\partial \Omega} = \phi,
\end{cases}$$

where $\phi$ belongs to $L^2(\partial \Omega)$, and $M(z)$ is understood as an unbounded operator (which is a sum of an unbounded self-adjoint operator and a bounded one), defined on the domain $H^1(\partial \Omega)$.

This example shows how all the classical objects of BVP appear naturally from the three operators appearing in the triple. In particular, it is worth noting how the energy-dependent Dirichlet-to-Neumann map $M(z)$ is “grown” from its “germ” $\Lambda$ at $z = 0$.

Returning to the abstract setting and taking into account (2.10), one concludes from Definition 1 that

$$M(z) = \Lambda + z \Pi^* (I - z A_0^{-1})^{-1} \Pi,$$  

(2.14)

From this equality, one directly verifies that

$$M(z) - M(w) = \Pi^* [z(I - z A_0^{-1})^{-1} - w(I - w A_0^{-1})^{-1}] \Pi$$

$$= (z - w) \gamma(\overline{z})^* \gamma(w).$$  

(2.15)

Also, due to the self-adjointness of $\Lambda$, one has

$$M^*(z) = M(\overline{z}).$$  

(2.16)

The properties (2.15) and (2.16) together imply that $M$ is an unbounded operator-valued Herglotz function, i.e., $M(z) - M(0)$ is analytic and $3 M(z) \geq 0$ whenever $z \in \mathbb{C}_+$. It is shown in [48, Thm. 3.11(4)] that

$$M(z) \Gamma_0 u_z = \Gamma_1 u_z$$  

(2.17)

for any $u_z \in \ker(A - z I) \cap \text{dom}(\Gamma_1)$. In this work we consider extensions (self-adjoint and non-selfadjoint alike) of the operator

$$\widetilde{A} := A_0 \mid_{\ker(\Gamma_1)}$$  

(2.18)

$^1$For convenience, we define the Dirichlet-to-Neumann map via $-\frac{\partial u}{\partial n} \mid_{\partial \Omega}$ instead of the more common $\frac{\partial u}{\partial n} \mid_{\partial \Omega}$. As a side note, we mention that this is obviously not the only choice for the operator $\Lambda$. In particular, the trivial choice $\Lambda = 0$ is always possible. Our choice of $\Lambda$ is motivated by our interest in the analysis of classical boundary conditions.
which are restrictions of \( A \). It is proven in [48, Sec. 5] that \( \tilde{A} \) is symmetric with equal deficiency indices. Moreover, [48, Prop. 5.2] asserts that
\[
\text{dom}(\tilde{A}) = A^{-1}_0[\text{ran}(\Pi)^⊥],
\]
so \( \tilde{A} \) does not depend on the parameter operator \( \Lambda \), contrary to what could be surmised from (2.18).

Still following [48], we let \( \alpha \) and \( \beta \) be linear operators in the Hilbert space \( \mathcal{E} \) such that \( \text{dom}(\alpha) \supset \text{dom}(\Lambda) \) and \( \beta \) is bounded on \( \mathcal{E} \). Additionally, assume that \( \alpha + \beta \Lambda \) is closable and denote \( \tilde{\beta} := \alpha + \beta \Lambda \). Consider the linear set
\[
\{ A_0^{-1} f + \Pi \phi : f \in \mathcal{H}, \phi \in \text{dom}(\tilde{\beta}) \}
\] (2.19)
Following [48, Lem. 4.3],
\[
(\alpha \Gamma_0 + \beta \Gamma_1)(A_0^{-1} f + \Pi \phi) = \beta \Pi^* f + (\alpha + \beta \Lambda)\phi \quad f \in \mathcal{H}, \phi \in \text{dom}(\Lambda)
\]
implies that \( \alpha \Gamma_0 + \beta \Gamma_1 \) is naturally defined on \( \text{dom}(A_0) + \Pi \text{dom}(\Lambda) \). The assumption that \( \alpha + \beta \Lambda \) is closable is used to extend the domain of definition of \( \alpha \Gamma_0 + \beta \Gamma_1 \) to the set (2.19). Moreover, one shows that \( \mathcal{H}_B := A_0^{-1} \mathcal{H} + \Pi \text{dom}(\tilde{\beta}) \) is a Hilbert space with respect to the norm
\[
\| u \|^2_B := \| f \|^2_\mathcal{H} + \| \phi \|^2_\mathcal{E} + \| \tilde{\beta} \phi \|^2_\mathcal{E}, \quad u = A_0^{-1} f + \Pi \phi.
\]
Then \( \alpha \Gamma_0 + \beta \Gamma_1 \) extends to a bounded operator from \( \mathcal{H}_B \) to \( \mathcal{E} \).

According to [48, Thm. 4.5], if the operator \( \alpha + \beta M(z) \) is boundedly invertible for \( z \in \rho(A_0) \), the spectral boundary value problem
\[
\begin{aligned}
(A - zI) u &= f, \\
(\alpha \Gamma_0 + \beta \Gamma_1) u &= \phi,
\end{aligned} \quad f \in \mathcal{H}, \phi \in \mathcal{E},
\] (2.20)
has a unique solution \( u \in \mathcal{H}_B \). Under this same hypothesis of the operator \( \alpha + \beta M(z) \) being boundedly invertible for \( z \in \rho(A_0) \), [48, Thm. 5.5] asserts that the function
\[
(A_0 - zI)^{-1} - (I - zA_0^{-1})^{-1}\Pi[\alpha + \beta M(z)]^{-1}\beta \Pi^* (I - zA_0^{-1})^{-1}
\] (2.21)
is the resolvent of a closed operator \( A_{\alpha\beta} \) densely defined in \( \mathcal{H} \). Moreover, \( \tilde{A} \subset A_{\alpha\beta} \subset A \) and \( \text{dom}(A_{\alpha\beta}) \) is contained in \( \{ u \in \mathcal{H}_B : (\alpha \Gamma_0 + \beta \Gamma_1) u = 0 \} \).

Among the extensions \( A_{\alpha\beta} \) of \( \tilde{A} \), we single out the operator
\[
L := A - iI,
\] (2.22)
that is, \( \alpha = -iI \) and \( \beta = I \). Since in this case \( \alpha \) and \( \beta \) are scalar operators, and \( \text{dom}(\Gamma_1) \subset \text{dom}(\Gamma_0) \), one has
\[
\text{dom}(L) \subset \text{dom}(\Gamma_1).
\] (2.23)
According to the definition of the domain of \( L \), for any \( h \in \mathcal{H} \), one has
\[
0 = (\Gamma_1 - i\Gamma_0)(L - zI)^{-1}h
= \Gamma_1(L - zI)^{-1}h - i\Gamma_0[(L - zI)^{-1} - (A_0 - zI)^{-1}]h
= M(z)\Gamma_0[(L - zI)^{-1} - (A_0 - zI)^{-1}]h + \Gamma_1(A_0 - zI)^{-1}h - i\Gamma_0[(L - zI)^{-1} - (A_0 - zI)^{-1}]h
\]
since, by (2.4) and the fact that \( L, A_0 \subseteq A \),
\[
(A_0 - zI)^{-1}h \in \ker(\Gamma_0) \quad \text{and} \quad [(L - zI)^{-1} - (A_0 - zI)^{-1}]h \in \ker(A - zI).
\]
Thus
\[
\begin{align*}
\Gamma_0(L - zI)^{-1} &= -(M(z) - iI)^{-1}\Gamma_1(A_0 - zI)^{-1}, \quad z \in \mathbb{C}_-, \\
\Gamma_0(L^* - zI)^{-1} &= -(M(z) + iI)^{-1}\Gamma_1(A_0 - zI)^{-1}, \quad z \in \mathbb{C}_+,
\end{align*}
\]  
(2.24)
where the second equality is deduced in the same way as the first one. The following relations, which are obtained by combining (2.11) and (2.24), will be of use to us:
\[
\begin{align*}
\Gamma_0(L - zI)^{-1} &= -(M(z) - iI)^{-1}\gamma(\mathbb{C})^*, \quad z \in \mathbb{C}_-, \\
\Gamma_0(L^* - zI)^{-1} &= -(M(z) + iI)^{-1}\gamma(\mathbb{C})^*, \quad z \in \mathbb{C}_+.
\end{align*}
\]  
(2.25)

It is proven in [48, Thm. 6.1] that the operator \( L \) of formula (2.22) is dissipative and boundedly invertible (hence maximal). We recall that a densely defined operator \( L \) in \( H \) is called dissipative if
\[
\text{Im}(Lf, f) \geq 0 \quad \forall f \in \text{dom}(L). \tag{2.26}
\]
A dissipative operator \( L \) is maximal if \( \mathbb{C}_- \) is contained in its resolvent set, that is, \( \mathbb{C}_- \subseteq \rho(L) \). Maximal dissipative operators are closed and any dissipative operator admits a maximal extension. When a dissipative operator does not have reducing self-adjoint parts, it is called completely non-selfadjoint.

Furthermore, the function
\[
S(z) := (M(z) - iI)(M(z) + iI)^{-1} = I - 2i(M(z) + iI)^{-1}, \quad z \in \mathbb{C}_+, \tag{2.27}
\]
is the characteristic function of \( L \), see [33, 52]. Since \( M \) is a Herglotz function (see (2.16)), one has the following formula valid for any \( z \in \mathbb{C}_- \):
\[
S^*(\mathbb{C}) := [S(\mathbb{C})]^* = I + 2i(M^*(\mathbb{C}) - iI)^{-1} = I + 2i(M(z) - iI)^{-1}. \tag{2.28}
\]
We remark that the function \( S \) is analytic in \( \mathbb{C}_+ \) and, for each \( z \in \mathbb{C}_+ \), the mapping \( S(z) : \mathcal{E} \to \mathcal{E} \) is a contraction. Therefore, \( S \) has nontangential limits almost everywhere on the real line in the strong operator topology [49].

Recall that a closed operator \( L \) is said to be \textit{completely non-selfadjoint} if there is no subspace reducing \( L \) such that the part of \( L \) in this subspace is self-adjoint. A completely non-selfadjoint symmetric operator is often referred to as \textit{simple}.

\textbf{Proposition 2.1.} \textit{If the symmetric operator} \( \tilde{A} \) \textit{is completely non-selfadjoint, then the dissipative operator} \( L \) \textit{is completely non-selfadjoint.}

\textit{Proof.} Suppose that \( L \) has a reducing subspace \( \mathcal{H}_1 \) such that \( L |_{\mathcal{H}_1} \) is self-adjoint. Take \( w \in \text{dom}(L) \cap \mathcal{H}_1 \). Then (2.12) and (2.23) imply
\[
0 = \langle \Gamma_1 w, \Gamma_0 w \rangle_{\mathcal{E}} - \langle \Gamma_0 w, \Gamma_1 w \rangle_{\mathcal{E}}.
\]
Since \( w \in \ker(\Gamma_1 - i\Gamma_0) \), one obtains from the last equality that \( \|\Gamma_0 w\| = 0 \). Therefore, \( w \in \ker(\Gamma_0) \cap \ker(\Gamma_1) \), which means that \( w \in \text{dom}(\tilde{A}) \). In view of the fact that \( \mathcal{H}_1 \) is an invariant subspace of \( L \) and \( L \supset \tilde{A} \), \( \mathcal{H}_1 \) is an invariant subspace of \( \tilde{A} \). Finally, since \( \tilde{A} \) is symmetric, \( \mathcal{H}_1 \) is actually a reducing subspace of \( \tilde{A} \). Clearly \( \tilde{A} \) is self-adjoint in \( \mathcal{H}_1 \). \( \square \)
3. Self-adjoint dilations for operators of BVP

Any completely non-selfadjoint dissipative operator $L$ admits a self-adjoint dilation [49], which is unique up to a unitary transformation if it is further assumed to be minimal. There is a number of approaches to an explicit construction of the named dilation [10, 35, 38, 39, 47]. In applications, one is compelled to seek a realisation corresponding to a particular setup. In the present paper we develop a way of constructing dilations of dissipative operators convenient in the context of BVP for PDE.

Recall that for any maximal dissipative operator $L$, the self-adjoint operator $A$ in a larger Hilbert space $\mathcal{H} \supset \mathcal{H}$ has the property

$$P_\mathcal{H}(A - zI)^{-1} |_{\mathcal{H}} = (L - zI)^{-1}, \quad z \in \mathbb{C}_-. \tag{3.1}$$

The dilation $A$ is referred to as minimal if

$$\text{span}\{ (A - zI)^{-1}\mathcal{H} \} = \mathcal{H}. \tag{3.2}$$

We start by constructing a minimal dilation of $L$, following a procedure similar to the one used in [41, 42]. Let

$$\mathcal{H} := L^2(\mathbb{R}_-, \mathcal{E}) \oplus H \oplus L^2(\mathbb{R}_+, \mathcal{E}). \tag{3.3}$$

In this Hilbert space, the operator $A$ is defined as follows. Its domain $\text{dom}(A)$ is given by

$$\text{dom}(A) := \left\{ \begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix} \in \mathcal{H} : v_\pm \in W^2_2(\mathbb{R}_\pm, \mathcal{E}), u \in \text{dom}(\Gamma_1) : \begin{array}{c} \Gamma_1 u - i\Gamma_0 u = \sqrt{2}v_- (0) \\
\Gamma_1 u + i\Gamma_0 u = \sqrt{2}v_+ (0) \end{array} \right\},$$

where $W^2_2(\mathbb{R}_+, \mathcal{E})$ and $W^2_2(\mathbb{R}_-, \mathcal{E})$ are the Sobolev spaces [1] of functions defined on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively, and taking values in the Hilbert space $\mathcal{E}$. We remark that the results of the previous section imply that in our case $\mathcal{H}_0 = \text{dom}(\Gamma_1)$. On this domain, the operator $A$ acts according to the rule

$$A : \begin{pmatrix} v_- \\ u \\ v_+ \end{pmatrix} \mapsto \begin{pmatrix} iv_- \\ Au \\ iv_+ \end{pmatrix}. \tag{3.4}$$

**Theorem 3.1.** In the dilated space $\mathcal{H}$, the operator $A$ is self-adjoint and is a self-adjoint dilation (or, in other words, out-of-space extension) of $L$.

**Proof.** The fact that $A$ is an extension of $L$ is rather obvious since $A_0 \subset L \subset A$. Let us establish the self-adjointness of $A$. Abbreviate $u = (v_-, u, v_+) \in \text{dom}(A)$. Then

$$\langle Au, u \rangle - \langle u, Au \rangle = \langle iv_-, v_- \rangle + \langle Au, u \rangle + \langle iv_+, v_+ \rangle - \langle v_-, iv_- \rangle - \langle u, Au \rangle - \langle v_+, iv_+ \rangle$$

$$= i \int_{\mathbb{R}_-} (iv_- \varphi_- + v_- \varphi_- v_-) + i \int_{\mathbb{R}_-} (v_+ \varphi_+ + v_- \varphi_+ v_+) + \langle Au, u \rangle - \langle u, Au \rangle$$

$$= i \|v_-(0)\| - i \|v_+(0)\| = \langle \Gamma_1 u, \Gamma_0 u \rangle - \langle \Gamma_0 u, \Gamma_1 u \rangle$$

But, taking into account the conditions defining $\text{dom}(A)$, one obtains

$$\langle \Gamma_1 u, \Gamma_0 u \rangle - \langle \Gamma_0 u, \Gamma_1 u \rangle = \langle v_-(0) + i\Gamma_0 u, \Gamma_0 u \rangle - \langle \Gamma_0 u, v_+(0) - i\Gamma_0 u \rangle$$

$$= i \langle v_-(0), v_+(0) - v_-(0) \rangle + i \langle v_+(0) - v_-(0), v_+(0) \rangle$$

$$= -i \|v_-(0)\| + i \|v_+(0)\|.$$
We also have where to obtain the first equality we use (3.6). Thus $A$ is symmetric. To complete the proof, it suffices to show that $\text{ran}(A - zI) = \mathcal{H}$ for any $z$ in $\mathbb{C} \setminus \mathbb{R}$.

Consider the operators $\partial_{\pm}$ and $\partial_{\pm}^0$ in $L_2(\mathbb{R}_\pm, \mathcal{E})$ given by

\[
\begin{aligned}
dom(\partial_{\pm}) &:= W^1_2(\mathbb{R}_\pm, \mathcal{E}) \\
\partial_{\pm} : y_{\pm} &\mapsto iy'_{\pm}
\end{aligned}
\]

\[
\begin{aligned}
dom(\partial_{\pm}^0) &:= W^1_2(\mathbb{R}_\pm, \mathcal{E}) \\
\partial_{\pm}^0 : y_{\pm} &\mapsto iy'_{\pm}.
\end{aligned}
\]

Here, $W^1_2(\mathbb{R}_\pm, \mathcal{E})$ is the closure in $W^1_2(\mathbb{R}_\pm, \mathcal{E})$ of the set of smooth functions with compact support in $\mathbb{R}_\pm$ (the space $W^1_2(\mathbb{R}_\pm, \mathcal{E})$ was mentioned above). The operators $\partial_{\pm}^0$ and $\partial_{\pm}^0$ are symmetric operators with deficiency indices $(0, 1)$ and $(1, 0)$, respectively. Also, $\partial_{\pm}^0 = \partial_{\pm}^0$ (see [7, Chap. 4 Sec. 8.4]). Therefore $\rho(\partial_{\pm}) = \mathbb{C}_\pm$ and $\rho(\partial_{\pm}^0) = \mathbb{C}_\mp$.

Take any $z \in \mathbb{C}_-$ and $(h_-, h, h_+) \in \mathcal{H}$. It turns out that the vector $(f_-, f, f_+)$, defined from $(h_-, h, h_+)$ by

\[
\begin{aligned}
f_- &:= (\partial_{-} - zI)^{-1}h_- \\
f &:= (L - zI)^{-1}h + \sqrt{2\gamma}(z)(M(z) - iI)^{-1}f_-\langle 0 \rangle \\
f_+ &:= (\partial_{+}^0 - zI)^{-1}h_+ + e^{-iz} \left[i\sqrt{2\gamma_0}(L - zI)^{-1}h + S^*(\overline{\gamma})f_-\langle 0 \rangle\right]
\end{aligned}
\]

is in $\text{dom}(A)$. Indeed,

\[
(\Gamma_1 - i\Gamma_0) \left\{(L - zI)^{-1}h + \sqrt{2\gamma}(z)(M(z) - iI)^{-1}f_-\langle 0 \rangle\right\} = (\Gamma_1 - i\Gamma_0)\sqrt{2\gamma}(z)(M(z) - iI)^{-1}f_-\langle 0 \rangle
\]

\[
= \sqrt{2}(M(z) - iI)(M(z) - iI)^{-1}f_-\langle 0 \rangle
\]

\[
= \sqrt{2}f_-\langle 0 \rangle,
\]

where to obtain the first equality we use (2.22) and for the second equality we recur to (2.8) and Definition 1. Thus

\[
(\Gamma_1 - i\Gamma_0) f = \sqrt{2}f_-\langle 0 \rangle.
\]

We also have

\[
(\Gamma_1 + i\Gamma_0) f = (\Gamma_1 - i\Gamma_0) f + 2i\Gamma_0 f = \sqrt{2}f_-\langle 0 \rangle + 2i\Gamma_0 \left\{(L - zI)^{-1}h + \sqrt{2\gamma}(z)(M(z) - iI)^{-1}f_-\langle 0 \rangle\right\}
\]

\[
= \sqrt{2}f_-\langle 0 \rangle + 2i\Gamma_0(L - zI)^{-1}h + 2i\sqrt{2}(M(z) - iI)^{-1}f_-\langle 0 \rangle
\]

\[
= 2i\Gamma_0(L - zI)^{-1}h + \sqrt{2} [I + 2i(M(z) - iI)^{-1}] f_-\langle 0 \rangle
\]

\[
= 2i\Gamma_0(L - zI)^{-1}h + \sqrt{2}S^*(\overline{\gamma})f_-\langle 0 \rangle,
\]

where we have used (3.6) to obtain the second equality, (2.8) for the third, and (2.28) for the fourth equality. Hence, we have shown that

\[
(\Gamma_1 + i\Gamma_0) f = \sqrt{2}f_+\langle 0 \rangle.
\]

The equalities (3.6) and (3.7) indicate that $(f_-, f, f_+)^\top \in \text{dom}(A)$.

Now we show that, for $z \in \mathbb{C}_-$,

\[
(A - zI) \begin{pmatrix} f_- \\ f \\ f_+ \end{pmatrix} = \begin{pmatrix} h_- \\ h \\ h_+ \end{pmatrix}.
\]

(3.8)
Recall that \((h_-, h, h_+)\) is a fixed, arbitrary element in \(\mathcal{K}\). On the one hand, it follows from (3.5) that
\[
h_\pm = (\partial_\pm - zI)f_\pm
\]
since \(f_+ = (\partial^0_+ - zI)^{-1}h_+ + e^{-iz\cdot}f_+(0)\). On the other hand, due to the fact that \(L \subset A\) and (2.9) holds, one has
\[
(A - zI) \left[ (L - zI)^{-1}h + \sqrt{2} \Gamma(z)(M(z) - iI)^{-1}f_-(0) \right] = h.
\] (3.10)
In conformity with (3.4), the identities (3.9) and (3.10) show that (3.8) holds. Thus we have shown that \(\text{ran}(A - zI) = \mathcal{K}\) for \(z \in \mathbb{C}_-\).

Now fix an arbitrary \(z \in \mathbb{C}_+\). For any \((h_-, h, h_+)\) \(\in \mathcal{K}\), let us redefine
\[
f_- := (\partial_-^0 - zI)^{-1}h_- + e^{-iz\cdot} \left[ -i \sqrt{2} \Gamma_0(L^* - zI)^{-1}h + S(z)f_+(0) \right]
f := (L^* - zI)^{-1}h + \sqrt{2} \Gamma(z)(M(z) + iI)^{-1}f_+(0)
f_+ := (\partial_+ - zI)^{-1}h_+.
\] (3.11)
In the same way as above, it can be shown that \((f_-, f, f_+)\) \(\in A\) and
\[
(A - zI) \begin{pmatrix} f_- \\ f \\ f_+ \end{pmatrix} = \begin{pmatrix} h_- \\ h \\ h_+ \end{pmatrix}.
\]

The proof is complete.

**Remark 1.** In the proof of Theorem 3.1, we have obtained the following formulae for the resolvent of \(A\). For any \((h_-, h, h_+)\) \(\in \mathcal{K}\), the vector
\[
(A - zI)^{-1} \begin{pmatrix} h_- \\ h \\ h_+ \end{pmatrix} = \begin{pmatrix} f_- \\ f \\ f_+ \end{pmatrix},
\]
where \((f_-, f, f_+)\) is given by (3.5) for \(z \in \mathbb{C}_-\) and by (3.11) for \(z \in \mathbb{C}_+\).

**Theorem 3.2.** The operator \(A\) is a minimal self-adjoint dilation of \(L\).

**Proof.** By Theorem 3.1 one only needs to check minimality. It follows from Remark 1 that, relative to the orthogonal decomposition (3.3), one has
\[
\text{span} \{ (A - zI)^{-1} \mathcal{H} \} = \text{span} \{ e^{-iz^\cdot} \Gamma_0(L^* - zI)^{-1} \mathcal{H} \}
\]
\[
\oplus \text{span} \{ (L - zI)^{-1} \mathcal{H} \} + \text{span} \{ (L^* - zI)^{-1} \mathcal{H} \}
\]
\[
\oplus \text{span} \{ e^{-iz\cdot} \Gamma_0(L - zI)^{-1} \mathcal{H} \}.
\]
Clearly,
\[
\text{span} \{ (L - zI)^{-1} \mathcal{H} \} + \text{span} \{ (L^* - zI)^{-1} \mathcal{H} \} = \mathcal{H}
\]
since \(L\) is densely defined. We next show that
\[
\text{span} \{ e^{-iz\cdot} \Gamma_0(L - zI)^{-1} \mathcal{H} \} = L_2(\mathbb{R}_+, \mathcal{E}).
\] (3.12)
To this end, fix $z \in \mathbb{C}_-$ and $h \in \mathcal{H}$. Assume that $g \in L_2(\mathbb{R}_+, \mathcal{E})$ is such that
\[
0 = \langle e^{-iz_0(L - zI)^{-1}}h, g \rangle_{L_2(\mathbb{R}_+, \mathcal{E})} = \int_{\mathbb{R}_+} e^{-i\xi \operatorname{Re} z} e^{i\xi \operatorname{Im} z} \langle \Gamma_0(L - zI)^{-1}h, g(\xi) \rangle_{\mathcal{E}} \, d\xi.
\]
Therefore, for almost all $\xi \in \mathbb{R}_+$,
\[
\langle \Gamma_0(L - zI)^{-1}h, g(\xi) \rangle_{\mathcal{E}} = 0.
\]
According to (2.24), when $h$ runs through $\mathcal{H}$, the vector $\Gamma_0(L - zI)^{-1}h$ runs through the domain of $M(z)$ which by Definition 1 is a dense set in $\mathcal{E}$ since $\Lambda$ is self-adjoint. Thus, $g(\xi) = 0$ for almost all $\xi \in \mathbb{R}_+$. The equality (3.12) has been established. Repeating this reasoning, one shows that
\[
\overline{\text{span}} \{e^{-iz_0(L^* - zI)^{-1}}h \mid h \in \mathcal{H} \} = L_2(\mathbb{R}_-, \mathcal{E}),
\]
as required. \hfill \Box

As a matter of convenience, we introduce the following family of sets. For any $z_+ \in \mathbb{C}_+$ and $z_- \in \mathbb{C}_-$, define
\[
\mathcal{Y}(z_+, z_-) := (A - z_+I)^{-1} \begin{pmatrix} 0 & L_2(\mathbb{R}_-, \mathcal{E}) \\ L_2(\mathbb{R}_+, \mathcal{E}) & 0 \end{pmatrix} + (A - z_-I)^{-1} \begin{pmatrix} L_2(\mathbb{R}_-, \mathcal{E}) & 0 \\ 0 & 0 \end{pmatrix},
\]
(3.13)
\[
\mathcal{G}(z_+, z_-) := P_{\mathcal{H}} \mathcal{Y}(z_+, z_-),
\]
(3.14)

**Lemma 3.3.** If $L$ is completely non-selfadjoint, then
\[
\overline{\text{span}} \left\{ \mathcal{Y}(z_+, z_-) \mid z_+ \in \mathbb{C}_+, z_- \in \mathbb{C}_- \right\}
\]
is dense in $\mathcal{H}$.

**Proof.** The proof follows closely the one of [47, Prop.2.5]. To simplify the writing, denote by $Y$ the closure of the set given by (3.15). It follows from Remark 1 that
\[
\begin{pmatrix} 0 & L_2(\mathbb{R}_-, \mathcal{E}) \\ L_2(\mathbb{R}_+, \mathcal{E}) & 0 \end{pmatrix} \subset Y \quad \text{and} \quad \begin{pmatrix} L_2(\mathbb{R}_-, \mathcal{E}) & 0 \\ 0 & 0 \end{pmatrix} \subset Y.
\]
(3.16)

Indeed, since $z \in \rho(\mathcal{D}_+)$, if one puts $h_- = h = 0$ in (3.11), then
\[
(A - z_+I)^{-1} \begin{pmatrix} 0 & L_2(\mathbb{R}_-, \mathcal{E}) \\ L_2(\mathbb{R}_+, \mathcal{E}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & L_2(\mathbb{R}_+, \mathcal{E}) \end{pmatrix}.
\]
From this the first inclusion in (3.16) follows. Similarly, if one puts $h_+ = h = 0$ in (3.5), then one obtains the second inclusion in (3.16) since now $z \in \rho(\mathcal{D}_-)$. Hence the orthogonal complement of $Y$ should be in $\mathcal{H}$. On the other hand, it follows from the definition of $Y$ that, for a fixed $z \in \mathbb{C} \setminus \mathbb{R}$, the set $Y$ is in $(A - zI)Y$. Thus, $(A - zI)^{-1}Y \subset Y$. This means that $Y^\bot$ is an invariant subspace for the resolvent of $A$. Since $A$ is self-adjoint, $Y^\bot$ is a reducing subspace for $A$. But then $Y^\bot$ should be trivial since the self-adjoint operator $A \mid _{Y^\bot}$ coincides with $L \mid _{Y^\bot}$. \hfill \Box
Remark 2. It is relevant for what follows to note that Lemma 3.3 yields
\[ \text{span}\left\{ \mathcal{G}(z_+, z_-) \right\} = \mathcal{H}. \]
Moreover, using the formulae for the resolvent of the dilation (see Remark 1), one verifies that
\[ \mathcal{G}(z_+, z_-) = \gamma(z_+)(M(z_+) + iI)^{-1}\mathcal{E} + \gamma(z_-)(M(z_-) - iI)^{-1}\mathcal{E}. \] (3.17)
and, in view of the fact that
\[ \begin{pmatrix} 0 & L_2(\mathbb{R}_-, \mathcal{E}) \\ 0 & 0 \end{pmatrix} \perp \begin{pmatrix} L_2(\mathbb{R}_+, \mathcal{E}) \\ 0 \end{pmatrix} \]
one concludes from Lemma 3.3 that the terms in (3.17) are linearly independent.

4. Spectral form of functional model

Following [35] we introduce a Hilbert space in which we construct a functional model for the operator family \( A_{\alpha, \beta} \), in the spirit of Szőkefalvi-Nagy and Foiaș [49]. The functional model for completely non-selfadjoint maximal dissipative operators that can be represented as additive perturbations of self-adjoint operators was constructed in [42, 41, 43] and further developed in [35]. In the context of boundary triples an analogous construction was carried out in [47]. In the most general setting to date, namely the setting of adjoint operator pairs, a three-component model akin to the one we presented in the previous section was constructed in [10], which however stops short of constructing a functional model for the operators considered. In this section we do precisely that in our setup, which is tailored to study operators of BVP, in the case when symbol of the operator is formally self-adjoint (but the operator itself can be non-selfadjoint due to the boundary conditions). Next we recall the concepts related to the construction of [35]. In the formulae below, we use the subscript “\( \pm \)” to indicate two different versions of the same formula in which the subscripts “+” and “−” are taken individually.

A function \( f \) analytic on \( \mathbb{C}_\pm \) and taking values in \( \mathcal{E} \) is said to be in the Hardy class \( H^2_\pm(\mathcal{E}) \) when
\[ \sup_{y > 0} \int_{\mathbb{R}} \|f(x \pm iy)\|^2_{\mathcal{E}} dx < +\infty \]
(cf. [46, Sec. 4.8]). If \( f \in H^2_\pm(\mathcal{E}) \), then the left-hand side of the above inequality defines \( \|f\|_{H^2_\pm(\mathcal{E})}^2 \).
We use the notation \( H^2_+ \) and \( H^2_- \) for the usual Hardy spaces of \( \mathbb{C} \)-valued functions.

Any element in \( H^2_\pm(\mathcal{E}) \) can be associated with its boundary values existing almost everywhere on the real line. It will cause no confusion if we use the same notation, \( H^2_\pm(\mathcal{E}) \), to denote the spaces of boundary functions. By [46, Sec. 4.8, Thm. B]), \( H^2_\pm(\mathcal{E}) \) are subspaces of \( L^2(\mathbb{R}, \mathcal{E}) \). Also, due to the Paley-Wiener theorem [46, Sec. 4.8, Thm. E]), one verifies that these subspaces are the orthogonal complements of each other (i.e., \( L^2(\mathbb{R}, \mathcal{E}) = H^2_+(\mathcal{E}) \oplus H^2_-(\mathcal{E}) \)).

We now return to the setup of Section 2.

Lemma 4.1. Let the operators \( \Gamma_0 \) and \( L \) be defined by (2.3) and (2.22), respectively. For any \( h \in \mathcal{H} \), \( \Gamma_0(L - \cdot)^{-1}h \) is in \( H^2_+(\mathcal{E}) \) and \( \Gamma_0(L^* - \cdot)^{-1}h \) is in \( H^2_-(\mathcal{E}) \). Moreover,
\[ \|\Gamma_0(L - \cdot)^{-1}h\|_{H^2_+(\mathcal{E})}, \|\Gamma_0(L^* - \cdot)^{-1}h\|_{H^2_-(\mathcal{E})} \leq \sqrt{\pi} \|h\|_{\mathcal{H}}. \]

Proof. The reasoning goes along the lines of the proof of [47, Lem. 2.4] which in turn is based on the one of [35, Thm. 1].
In the next equalities, one uses Green’s formula and the fact that \( L \subset A \) to obtain, for \( z \in \mathbb{C}_- \),
\[
2i \| \Gamma_0(L - zI)^{-1}h \|^2 = \langle i\Gamma_0(L - zI)^{-1}h, i\Gamma_0(L - zI)^{-1}h \rangle - \langle \Gamma_0(L - zI)^{-1}h, \Gamma_0(L - zI)^{-1}h \rangle \\
= \langle \Gamma_1(L - zI)^{-1}h, \Gamma_0(L - zI)^{-1}h \rangle - \langle \Gamma_0(L - zI)^{-1}h, \Gamma_1(L - zI)^{-1}h \rangle \\
= \langle (L - zI)^{-1}h, (L - zI)^{-1}h \rangle - \langle (L - zI)^{-1}h, (L - zI)^{-1}h \rangle \\
= \langle h, (L - zI)^{-1}h \rangle - \langle (L - zI)^{-1}h, h \rangle + (z - \overline{z}) \| (L - zI)^{-1}h \|^2.
\]
Since \( L \) is maximal dissipative, it admits [49] a self-adjoint dilation \( \mathcal{A} \), see (3.1). One concludes, by recurring to the resolvent identity, that
\[
\| \Gamma_0(L - zI)^{-1}h \|^2 = \frac{1}{2i} \left\{ \langle (\mathcal{A} - \overline{\mathcal{A}})^{-1} - (\mathcal{A} - zI)^{-1} \rangle h, h \rangle - (z - \overline{z}) \| (L - zI)^{-1}h \|^2 \right\} \\
= \frac{1}{2i} \left\{ (z - \overline{z}) \| (\mathcal{A} - zI)^{-1}h \|^2 - (z - \overline{z}) \| (L - zI)^{-1}h \|^2 \right\}.
\]
Let \( E(t), t \in \mathbb{R} \), be the resolution of identity [7, Chapter 6] for \( \mathcal{A} \) and set \( z = k - \iota \varepsilon \), \( k \in \mathbb{R} \) and \( \varepsilon > 0 \). Thus
\[
\| \Gamma_0(L - (k - \iota \varepsilon)I)^{-1}h \|^2 = \varepsilon \int_{\mathbb{R}} \frac{d \langle E(t)h, h \rangle}{(t - k)^2 + \varepsilon^2} - \varepsilon \| (L - (k - \iota \varepsilon)I)^{-1}h \|^2.
\]
Now, using Fubini’s theorem,
\[
\int_{\mathbb{R}} \| \Gamma_0(L - (k - \iota \varepsilon)I)^{-1}h \|^2 \, dk = \int_{\mathbb{R}} \left( \varepsilon \int_{\mathbb{R}} \frac{d \langle E(t)h, h \rangle}{(t - k)^2 + \varepsilon^2} \right) \, dk + \varepsilon \int_{\mathbb{R}} \| (L - (k - \iota \varepsilon)I)^{-1}h \|^2 \, dk \\
= \int_{\mathbb{R}} \left( \varepsilon \int_{\mathbb{R}} \frac{dk}{(t - k)^2 + \varepsilon^2} \right) \, d \langle E(t)h, h \rangle + \varepsilon \int_{\mathbb{R}} \| (L - (k - \iota \varepsilon)I)^{-1}h \|^2 \, dk \\
= \pi \| h \|^2 - \varepsilon \int_{\mathbb{R}} \| (L - (k - \iota \varepsilon)I)^{-1}h \|^2 \, dk.
\]
It follows that
\[
\| \Gamma_0(L - zI)^{-1}h \|^2_{H^2(\varepsilon)} \leq \pi \| h \|^2.
\]
The second inequality of the lemma is proven in the same way. \( \square \)

As mentioned in Section 2, the characteristic function \( S \), given in (2.27), has nontangential limits almost everywhere on the real line in the strong topology. Thus, for a two-component vector function \((\tilde{g}, g)\) taking values in \( E \oplus \mathcal{E} \), the integral
\[
\int_{\mathbb{R}} \left\langle \begin{pmatrix} I & S^*(s) \\ S(s) & I \end{pmatrix}, \begin{pmatrix} \tilde{g}(s) \\ g(s) \end{pmatrix} \right\rangle_{E \oplus \mathcal{E}} \, ds, \tag{4.1}
\]
makes sense and is nonnegative due to the contractive properties of \( S \). The space\(^2\)
\[
\mathcal{H} := L^2 \left( E \oplus \mathcal{E}; \begin{pmatrix} I & S^* \\ S & I \end{pmatrix} \right) \tag{4.2}
\]
\(^2\)This is in fact the same construction as proposed by [43] and further developed by [35]. Henceforth in this section we follow closely the analysis of the named two papers, facilitated by the fact that essentially this way to construct the functional model only relies upon the characteristic function \( S \) of the maximal dissipative operator and an estimate of the type claimed in Lemma 4.1 above. A similar argument for extensions of symmetric operators, based on the theory of boundary triples, was developed in [47], [15].
is the completion of the linear set of two-component vector functions \( \tilde{g} : \mathbb{R} \rightarrow \mathcal{E} \) in the norm (4.1), factored with respect to vectors of zero norm. Naturally, not every element of the set can be identified with a pair \( \tilde{g} \) of two independent functions. Notwithstanding, we keep the notation \( \tilde{g} \) for the elements of this space.

Another consequence of the contractive properties of the characteristic function \( S \) is that the inequalities

\[
\| \tilde{g} + S^*g \|_{L^2(\mathbb{R}, \mathcal{E})}, \quad \| S\tilde{g} + g \|_{L^2(\mathbb{R}, \mathcal{E})} \leq \left\| \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \right\|_{\mathcal{H}}
\]

hold for \( \tilde{g}, g \in L^2(\mathbb{R}, \mathcal{E}) \). Thus, for every sequence \( \{ \tilde{g}_n \} \in \mathcal{H} \) being Cauchy with respect to the \( \mathcal{H} \)-topology and such that \( \tilde{g}_n, g_n \in L^2(\mathbb{R}, \mathcal{E}) \) for all \( n \in \mathbb{N} \), the limits of \( \tilde{g}_n + S^*g_n \) and \( S\tilde{g}_n + g_n \) exist in \( L^2(\mathbb{R}, \mathcal{E}) \), so that the objects \( \tilde{g} + S^*g \) and \( S\tilde{g} + g \) can always be treated as \( L^2(\mathbb{R}, \mathcal{E}) \) functions.

Consider the following subspaces of \( \mathcal{H} \)

\[
\mathcal{D}_- := \begin{pmatrix} 0 \\ H^2_+(\mathcal{E}) \end{pmatrix}, \quad \mathcal{D}_+ := \begin{pmatrix} H^2_+(\mathcal{E}) \\ 0 \end{pmatrix},
\]

where we use the Hardy spaces, \( H^2_+(\mathcal{E}) \), which are identified with subspaces \( L^2(\mathbb{R}, \mathcal{E}) \), see above for details. It is easily seen [43] that the spaces \( \mathcal{D}_- \) and \( \mathcal{D}_+ \) are mutually orthogonal in \( \mathcal{H} \).

Define the subspace

\[
K := \mathcal{H} \ominus (\mathcal{D}_- \oplus \mathcal{D}_+),
\]

which is characterised as follows (see [41, 43]):

\[
K = \left\{ \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} : \tilde{g} + S^*g \in H^2(\mathcal{E}), S\tilde{g} + g \in H^2_+(\mathcal{E}) \right\}.
\]

The orthogonal projection \( P_K \) onto the subspace \( K \) is given by (see e.g. [34])

\[
P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = \begin{pmatrix} \tilde{g} - P_+ (\tilde{g} + S^*g) \\ g - P_- (S\tilde{g} + g) \end{pmatrix},
\]

where \( P_\pm \) are the orthogonal Riesz projections in \( L^2(\mathcal{E}) \) onto \( H^2_\pm(\mathcal{E}) \).

**Definition 2** ([47]). Define the mappings \( \mathcal{F}_ \) : \( \mathcal{H} \rightarrow L^2(\mathbb{R}, \mathcal{E}) \) by

\[
\mathcal{F}_+ \begin{pmatrix} h_- \\ h \end{pmatrix} := -\frac{1}{\sqrt{\pi}} \Gamma_0 (L - (\cdot - i0)I)^{-1} h + S^*(\cdot)\hat{h}_- + \hat{h}_+;
\]

and

\[
\mathcal{F}_- \begin{pmatrix} h_- \\ h \end{pmatrix} := -\frac{1}{\sqrt{\pi}} \Gamma_0 (L^* - (\cdot + i0)I)^{-1} h + \hat{h}_- + S(\cdot)\hat{h}_+.
\]

**Lemma 4.2.** Fix \( z_+ \in \mathbb{C}_+ \) and \( z_- \in \mathbb{C}_- \). Consider the map \( \Phi \), defined on

\[
\begin{pmatrix} L^2(\mathbb{R}_-, \mathcal{E}) \\ \mathcal{G}(z_+, z_-) \\ L^2(\mathbb{R}_+, \mathcal{E}) \end{pmatrix}
\]

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by the formula

$$\Phi \left( \begin{array}{c} v_- \\ v \\ v_+ \\ \end{array} \right) = \left( \begin{array}{c} \hat{v}_+ + \frac{i}{\sqrt{2\pi}} \left[ \frac{1}{\cdot - z_-} S^*(\bar{z}_-) w_- - \frac{1}{\cdot - z_+} w_+ \right] \\ \hat{v}_- - \frac{i}{\sqrt{2\pi}} \left[ \frac{1}{\cdot - z_-} w_- - \frac{1}{\cdot - z_+} S(z_+) w_+ \right] \\ \end{array} \right)$$

where \( w_+, w_- \) are determined

$$v = \sqrt{2}\gamma(z_+)(M(z_+) + iI)^{-1}w_+ + \sqrt{2}\gamma(z_-)(M(z_-) - iI)^{-1}w_- \quad (4.7)$$

uniquely by Remark 2.

The map \( \Phi \) satisfies

$$\left( \begin{array}{cc} I & S^* \\ S & I \end{array} \right) \Phi \left( \begin{array}{c} v_- \\ v \\ v_+ \end{array} \right) = \left( \begin{array}{c} \mathcal{F}_v (v_- \\ v \\ v_+) \\ \mathcal{F}_v (v_- \\ v \\ v_+) \end{array} \right) \quad (4.8)$$

Proof. Due to the linear independence of the terms on the right-hand side of (3.17) (see Remark 2), if \( v \) vanishes, then

$$0 = \gamma(z_+)(M(z_+) + iI)^{-1}w_+ = \gamma(z_-)(M(z_-) - iI)^{-1}w_-,$$

where \( w_+, w_- \in \mathcal{E} \). Then, in view of (2.1), (2.6), and (2.7), the vectors \( w_+ \) and \( w_- \) also should vanish. Thus, taking into account Definition 2, one verifies that (4.8) holds when \( v = 0 \). Therefore it only remains to prove the assertion when \( v = 0 \). Under this assumption, consider the first row in the vector equality (4.8):

$$\frac{i}{\sqrt{2\pi}} \left[ \frac{1}{\cdot - z_-} S^*(\bar{z}_-) w_- - \frac{1}{\cdot - z_+} w_+ \right] - \frac{i}{\sqrt{2\pi}} S^*(\cdot) \left[ \frac{1}{\cdot - z_-} w_- - \frac{1}{\cdot - z_+} S(z_+) w_+ \right]
= - \sqrt{2\pi} \Gamma_0(L - (\cdot - i0)I)^{-1} \left[ \gamma(z_+)(M(z_+) + iI)^{-1}w_+ + \gamma(z_-)(M(z_-) - iI)^{-1}w_- \right]$$

This inequality is satisfied whenever

$$\frac{i}{\cdot - z_-} [S^*(\cdot) - S^*(\bar{z}_-)] w_- = 2\Gamma_0(L - (\cdot - i0)I)^{-1} \gamma(z_-)(M(z_-) - iI)^{-1}w_- \quad (4.9)$$

and

$$\frac{i}{\cdot - z_+} [I - S^*(\cdot) S(z_+)] w_+ = 2\Gamma_0(L - (\cdot - i0)I)^{-1} \gamma(z_+)(M(z_+) + iI)^{-1}w_+ \quad (4.10)$$

hold. To verify these equalities consider \( z \in \mathbb{C}_- \). Using the second resolvent identity, it follows from (2.28) that

$$\frac{1}{z - z_-} [S^*(\bar{z}) - S^*(\bar{z}_-)] = \frac{2i}{z - z_-} ((M(z) - iI)^{-1} - (M(z_-) - iI)^{-1})
= \frac{2i}{z - z_-} (M(z) - M(z_-)) (M(z) - iI)^{-1} \quad (4.11)$$
Thus, by (2.15), (2.11) and (2.24), one has
\[
\frac{1}{z - z_-} [S^*(z) - S^*(z_-)] = 2i(M(z) - iI)^{-1} \gamma(z) (M(z_-) - M(z) - iI)^{-1}
\]
\[= 2i(M(z) - iI)^{-1} \Gamma_1 (A_0 - zI)^{-1} \gamma(z_-) (M(z) - iI)^{-1} = -2i \Gamma_0 (L - zI)^{-1} \gamma(z_-) (M(z) - iI)^{-1}. \tag{4.12}
\]

By straightforward calculations, one has
\[
\frac{1}{z - z_+} [I - S^*(z) S(z_+)] = -\frac{2i}{z - z_+} [(M(z) - iI)^{-1} - (M(z) + iI)^{-1} - 2i(M(z) - iI)^{-1} (M(z) + iI)^{-1}]
\]
\[= -\frac{2i}{z - z_+} [(M(z) - iI)^{-1} (M(z) + iI) - (M(z) - iI) - 2iI (M(z) + iI)^{-1}
\]
\[= \frac{2i}{z - z_+} (M(z) - iI)^{-1} [M(z) - M(z_+)] (M(z) + iI)^{-1}
\]
By comparing the last expression with (4.11) and taking into account (4.12), one arrives at
\[
\frac{1}{z - z_+} (I - S^*(z) S(z_+)) = -2i \Gamma_0 (L - zI)^{-1} \gamma(z_+) (M(z) - iI)^{-1},
\]
which shows that (4.10) holds for any \( w_+ \in \mathcal{E} \).

The second entry of the vector equality (4.8) is proven in a similar way. \hfill \square

**Lemma 4.3.** The mapping \( \Phi \), given in Lemma 4.2, is an isometry from
\[
\begin{pmatrix}
L_2(\mathbb{R}_-, \mathcal{E}) \\
0 \\
L_2(\mathbb{R}_+, \mathcal{E})
\end{pmatrix}
\]
on the space \( \mathcal{D}_- \oplus \mathcal{D}_+ \).

**Proof.** For any \( v_- \in L_2(\mathbb{R}_-, \mathcal{E}) \) and \( v_+ \in L_2(\mathbb{R}_+, \mathcal{E}) \), one has
\[
\| \Phi \begin{pmatrix} v_- \\ 0 \\ 0 \end{pmatrix} \|_{\mathcal{B}_0} = \left\| \begin{pmatrix} 0 \\ \tilde{v}_- \end{pmatrix} \right\|_{\mathcal{B}_0} \quad \text{and} \quad \| \Phi \begin{pmatrix} 0 \\ 0 \\ v_+ \end{pmatrix} \|_{\mathcal{B}_0} = \left\| \begin{pmatrix} \tilde{v}_+ \\ 0 \end{pmatrix} \right\|_{\mathcal{B}_0}
\]
Thus, taking into account that the spaces \( \mathcal{D}_- \) and \( \mathcal{D}_+ \) are orthogonal (see the discussion following the formula (4.3)), one has
\[
\| \Phi \begin{pmatrix} v_- \\ 0 \\ v_+ \end{pmatrix} \|^2_{\mathcal{B}_0} = \left\| \begin{pmatrix} 0 \\ \tilde{v}_- \end{pmatrix} \right\|^2_{\mathcal{B}_0} + \left\| \begin{pmatrix} \tilde{v}_+ \\ 0 \end{pmatrix} \right\|^2_{\mathcal{B}_0}.
\]
Finally note that
\[
\left\| \begin{pmatrix} 0 \\ \tilde{v}_- \end{pmatrix} \right\|_{\mathcal{B}_0} = \| \tilde{v}_- \|_{L_2(\mathbb{R}, \mathcal{E})} = \| v_- \|_{L_2(\mathbb{R}_-, \mathcal{E})}
\]
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and
\[ \left\| \left( \hat{v}_+^0 \right) \right\|_{\mathcal{H}} = \left\| \hat{v}_+ \right\|_{L^2(\mathbb{R}, \mathcal{E})} = \left\| v_+ \right\|_{L^2(\mathbb{R}_+, \mathcal{E})}. \]

The surjectivity of the mapping follows from the fact that the Fourier transform is a unitary mapping between \( L^2(\mathbb{R}_+, \mathcal{E}) \) and \( H^2_+^+(\mathcal{E}) \), by the Paley-Wiener theorem.

**Lemma 4.4.** The mapping \( \Phi \), given in Lemma 4.2 and extended by linearity to
\[
\begin{pmatrix}
L^2(\mathbb{R}_-, \mathcal{E}) \\
\text{span}_{z_+ \in \mathbb{C}_+} \mathcal{G}(z_+, z_-) \\
\text{span}_{z_- \in \mathbb{C}_-} \mathcal{G}(z_+, z_-) \\
L^2(\mathbb{R}_+, \mathcal{E})
\end{pmatrix}
\]
(4.13)
is an isometry from the set (4.13) to \( \mathcal{H} \).

**Proof.** Due to (4.4) and Lemma 4.3, the assertion will be proved if one shows first that
\[
\Phi \begin{pmatrix}
0 \\
\text{span}_{z_+ \in \mathbb{C}_+} \mathcal{G}(z_+, z_-) \\
0
\end{pmatrix} \subset K
\]
(4.14)
and, second, that for any \( z_\pm \in \mathbb{C}_\pm \) and \( v \) chosen in accordance with (4.7), one has
\[
\left\| \Phi \begin{pmatrix}
0 \\
v \\
0
\end{pmatrix} \right\|_{\mathcal{H}} = \left\| v \right\|_{\mathcal{H}}.
\]
(4.15)

For establishing (4.14), one has to verify that the vectors
\[
\frac{1}{\cdot - z_-} \begin{pmatrix}
S^*(z_-)w_- \\
v_- 
\end{pmatrix}
\]
and
\[
\frac{1}{\cdot - z_+} \begin{pmatrix}
w_+ \\
S(z_+)w_+
\end{pmatrix}
\]
are orthogonal to \( \mathcal{D}_- \oplus \mathcal{D}_+ \). Assume that \( h_\pm \) is in \( H^2_+^+ \). Taking into account that
\[
\frac{1}{\cdot - z_-} \in H^2_+^+(\mathcal{E})
\]
(4.16)
and analytically continuing the function \( S^* \) to the lower-half plane, one obtains
\[
\left\langle \frac{1}{\cdot - z_-} \begin{pmatrix} S^*(z_-)w_- \\ v_+ \end{pmatrix}, \begin{pmatrix} h_+ \\ h_- \end{pmatrix} \right\rangle_{\mathcal{H}} = \left\langle \frac{1}{\cdot - z_-} S^*(z_-)w_- + S^*h_- \right\rangle_{L^2(\mathcal{E})} - \left\langle \frac{1}{\cdot - z_-} \begin{pmatrix} w_- \\ Sh_+ + h_- \end{pmatrix} \right\rangle_{L^2(\mathcal{E})} - \left\langle \frac{1}{\cdot - z_-} \begin{pmatrix} w_+ \\ Sh_+ \end{pmatrix} \right\rangle_{L^2(\mathcal{E})} = 0.
\]

In the same way, using the fact that
\[
\frac{1}{\cdot - z_+} \in H^2_-(\mathcal{E})
\]
(4.17)
and, analytically continuing the function $S$ to the upper-half plane, one concludes that

$$\left\langle \frac{1}{\cdot - z} \begin{pmatrix} w_+ \\ -S(z)w_+ \end{pmatrix}, \begin{pmatrix} h_+ \\ h_- \end{pmatrix} \right\rangle = \left\langle S - S(z)w_+, h_- \right\rangle_{L_2(\mathcal{E})} = 0$$

It remains to prove (4.15). In view of Lemma 4.2 and Definition 2, one has

$$\left\| \Phi \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\|_{\mathcal{B}}^2 = \left\langle \begin{pmatrix} I \\ S \end{pmatrix}^* \Phi \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Phi \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\rangle_{L_2(\mathcal{E})} = \left\langle \begin{pmatrix} \mathcal{F}_+ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \mathcal{F}_- \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}, \Phi \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\rangle_{L_2(\mathcal{E})}$$

$$= \left\langle \begin{pmatrix} -\frac{1}{\sqrt{\pi}} \Gamma_0(L - (\cdot - i0)I)^{-1}v \cdot -\frac{1}{\sqrt{2\pi}} S^*(\pi_0)w_- - \frac{1}{\cdot - z} w_+ \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2\pi}} \Gamma_0(L^* - (\cdot + i0)I)^{-1}v \cdot -\frac{1}{\sqrt{2\pi}} S(z_+ w_+) \end{pmatrix} \right\rangle_{L_2(\mathcal{E})}.$$

By Lemma 4.1, one knows that $\Gamma_0(L - \cdot)^{-1}v$ is in $H^2(\mathcal{E})$ and $\Gamma_0(L^* - \cdot)^{-1}v$ is in $H^2(\mathcal{E})$. Thus, in view of (4.16) and (4.17), one obtains

$$\left\| \Phi \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\|_{\mathcal{B}}^2 = \left\langle \begin{pmatrix} -\frac{1}{\sqrt{\pi}} \Gamma_0(L - (\cdot - i0)I)^{-1}v \cdot -\frac{1}{\sqrt{2\pi}} S^*(\pi_0)w_- - \frac{1}{\cdot - z} w_+ \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2\pi}} \Gamma_0(L^* - (\cdot + i0)I)^{-1}v \cdot -\frac{1}{\sqrt{2\pi}} S(z_+ w_+) \end{pmatrix} \right\rangle_{L_2(\mathcal{E})}$$

Now, using (2.25), one has

$$\left\| \Phi \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\|_{\mathcal{B}}^2 = \left\langle \begin{pmatrix} -\frac{1}{\sqrt{\pi}} \Gamma_0(L - (\cdot - i0)I)^{-1}v \cdot -\frac{1}{\sqrt{2\pi}} S^*(\pi_0)w_- - \frac{1}{\cdot - z} w_+ \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2\pi}} \Gamma_0(L^* - (\cdot + i0)I)^{-1}v \cdot -\frac{1}{\sqrt{2\pi}} S(z_+ w_+) \end{pmatrix} \right\rangle_{L_2(\mathcal{E})}$$

Due to Remark 2 and Lemma 4.4, the mapping $\Phi$ can be extended by continuity to the whole space $\mathcal{H}$. We will use same notation $\Phi$ for this extension.

**Lemma 4.5.** For any nonreal $z$, one has

$$\Phi(A - zI)^{-1} = \frac{1}{\cdot - z} \Phi.$$

**Proof.** We prove the statement for $z \in \mathbb{C}_+$, as the case $z \in \mathbb{C}_-$ is established in a similar way. Consider an arbitrary $(h_-, h, h_+)^T \in \mathcal{H}$ and let $(f_-, f, f_+)^T$ be the vector defined by (3.11). It follows from (3.9) that

$$(-z) \tilde{f}_\pm = \tilde{h}_\pm \pm \frac{i}{\sqrt{2\pi}} f_\pm(0).$$

(4.18)
Recall that $\hat{h}_\pm$ and $\hat{f}_\pm$ are the Fourier transforms of $h_\pm$ and $f_\pm$, respectively. According to Definition 2 and (3.11), one has

$$
\mathcal{F}_- \left( \begin{pmatrix} f_- \\ f_+ \end{pmatrix} \right) = -\frac{1}{\sqrt{\pi}} \Gamma_0 (L^* - (\cdot + i0)I)^{-1} f + \hat{f}_- + S(\hat{f}_+) 
$$

$$
= -\frac{1}{\sqrt{\pi}} \Gamma_0 (L^* - (\cdot + i0)I)^{-1} \left[ (L^* - zI)^{-1} h + \sqrt{2} \gamma(z)(M(z) - iI)^{-1} f_+(0) \right] 
$$

$$
+ \frac{1}{\sqrt{\pi} (\cdot - z)} \left[ \hat{h}_- + S\hat{h}_+ \right] 
+ \frac{i}{\sqrt{2\pi}} (Sf_+(0) - f_-(0)) 
$$

where to obtain the expression in the second square brackets use (4.18). Thus,

$$
\mathcal{F}_- \left( \begin{pmatrix} f_- \\ f_+ \end{pmatrix} \right) = \frac{1}{\sqrt{\pi}} \mathcal{F}_- \left( \begin{pmatrix} h_- \\ h_+ \end{pmatrix} \right) - \frac{1}{\sqrt{\pi}} \Gamma_0 (L^* - (\cdot + i0)I)^{-1} \gamma(z)(M(z) - iI)^{-1} \sqrt{2} f_+(0) 
$$

$$
+ \frac{1}{\sqrt{\pi} (\cdot - z)} \left[ \Gamma_0 (L^* - zI)^{-1} h + \frac{i}{\sqrt{2}} \left[ f_+(0) - f_-(0) - 2i(M(\cdot) + iI)^{-1} f_+(0) \right] \right] 
$$

(4.19)

The function $\circledast$ will be evaluated at $\zeta \in \mathbb{C}_+$ followed by taking the non-tangential limit on the real line. First, one uses the description of $\text{dom}(A)$ to obtain

$$
f_+(0) - f_-(0) = \sqrt{2}\Gamma_0 f.
$$

Substituting this into $\circledast$ yields

$$
\star = \frac{1}{(\cdot - z)\sqrt{\pi}} \left( \Gamma_0 (L^* - zI)^{-1} h + \Gamma_0 f + \sqrt{2} (M(\zeta) + iI)^{-1} f_+(0) \right) 
$$

$$
= \frac{\sqrt{2}}{(\zeta - z)\sqrt{\pi}} \left( -\Gamma_0 \gamma(z)(M(z) + iI)^{-1} f_+(0) + (M(\zeta) + iI)^{-1} f_+(0) \right) 
$$

$$
= \frac{\sqrt{2}}{(\zeta - z)\sqrt{\pi}} \left( (M(\zeta) + iI)^{-1} - (M(z) + iI)^{-1} \right) f_+(0) 
$$

$$
= -\frac{2}{\pi} (M(\zeta) + iI)^{-1} \left( \frac{M(\zeta) - M(z)}{\zeta - z} \right)(M(z) + iI)^{-1} f_+(0) 
$$

$$
= -\frac{2}{\pi} (M(\zeta) + iI)^{-1} \gamma_\ast \gamma(z)(M(z) + iI)^{-1} f_+(0),
$$

where $f$ in the second equality is replaced by (3.11), while in the third and fourth equalities we have used (2.8) and the second resolvent identity, respectively. Finally, we utilize (2.15) to obtain the last equality above. The identities (2.25) now yield

$$
\star = \sqrt{\frac{2}{\pi}} \Gamma_0 (L^* - \zeta I)^{-1} \gamma(z)(M(z) - iI)^{-1} f_+(0).
$$

The second term on the right-hand side of (4.19) and the expression $\star$ mutually cancel out as $z$ goes to the real line. We have therefore shown that

$$
\mathcal{F}_-(A - zI)^{-1} = \frac{1}{\cdot - z} \mathcal{F}_-, \quad z \in \mathbb{C}_+.
$$
In much the same way, one proves that

$$F_+(A - zI)^{-1} = \frac{1}{z} F_+.$$  

In view of Lemma 4.2 the assertion follows. \qed

**Lemma 4.6.** The operator $\Phi$ maps $\mathcal{H}$ onto $\mathfrak{H}$ unitarily.

**Proof.** In view of Lemma 4.4 the mapping $\Phi$ is an isometry defined in the whole space $\mathcal{H}$. It thus suffices to show that the range of $\Phi$ is dense in $\mathfrak{H}$. Denote by $X$ the set (4.13) and assume the existence of a nonzero element $g$ in $H$ such that

$$0 = \langle g, \Phi \begin{pmatrix} v_- \\ v \\ v_+ \end{pmatrix} \rangle_{\mathfrak{H}} \quad \text{with} \quad \begin{pmatrix} v_- \\ v \\ v_+ \end{pmatrix} \in X.$$  

(4.20)

By Lemma 4.3 and the definition of $K$, see (4.4), this is equivalent to the existence of a nonzero $g \in K$ such that (4.20) holds with $v_- = v_+ = 0$. On the other hand, since $\Phi^* g \in \mathcal{H}$, one has

$$0 = \langle g, \Phi \begin{pmatrix} \varepsilon \\ v \\ 0 \end{pmatrix} \rangle_{\mathfrak{H}} \quad \text{with} \quad \begin{pmatrix} \varepsilon \\ v \\ 0 \end{pmatrix} \in X,$$

which yields a contradiction due to Remark 2. \qed

5. Formulae for the (boundary traces of) the resolvents of boundary-value problems

Our aim here is to derive an explicit formula for the solution operators of the spectral boundary value problem (2.20). First we introduce a version of the related characteristic function. To this end, consider the operator (see (2.21), (5.5), cf. [48, Section 5])

$$-(\alpha + \beta M(z))^{-1} \beta,$$

(5.1)

for all $z$ such that $0 \in \rho(\alpha + \beta M(z))$. It is convenient to assume that $\beta$ is boundedly invertible, which we do henceforth, and abbreviate

$$Q_B := -(B + M(z))^{-1},$$

(5.2)

so that the operator in (5.1) equals $Q_{\beta^{-1} \alpha}$. Finally, we denote by $Q_B$ the set of points $z$ such that $0 \in \rho(B + M(z))$.

In the special case when $B$ is self-adjoint (for example, if $B = \beta^{-1} \alpha$ with self-adjoint $\alpha$ and $\beta$), since $M(z)$ is a Herglotz function, one has $Q_B \supset \mathbb{C}_- \cup \mathbb{C}_+$. It follows from [48, Theorem 5.5] that

$$Q_{\beta^{-1} \alpha} \subset \rho(A_{\alpha \beta}).$$

(5.3)

We next discuss the question of whether the operator in (5.2) is well defined on a rich enough set of values $z$ in the half-planes $\mathbb{C}_+$ and $\mathbb{C}_+$. In the present article we focus on the PDE setting, where the standard choice of boundary conditions implies that $\Lambda$ is the Dirichlet-to-Neumann map. Theorem 5.5 of [48] requires the existence of $Q_B(z)$ for at one point $z \in \mathbb{C}$, which in the most general setup cannot be guaranteed. Considering the PDE setup allows one to make some reasonable assumptions that are bound to hold, provided that the boundary in BVP is at least
Lipschitz, so that [48, Theorem 5.5] is applicable, and moreover, the resulting operator $A_{\alpha \beta}$ has discrete spectrum in $\mathbb{C}_- \cup \mathbb{C}_+$.

**Lemma 5.1.** Suppose that $\Lambda$ is the Dirichlet-to-Neumann map of a PDE problem with a finite-dimensional kernel, such that $\Lambda$ is a pseudo-differential operator of order one.\(^3\) Suppose further that $B$ is bounded. Then $BM(z)^{-1} \in \mathcal{S}_\infty$ for at least one $z \in \mathbb{C}_+$ and at least one $z \in \mathbb{C}_-$.

**Proof.** Choose a finite-rank operator $K$ such that $\Lambda + K$ has trivial kernel. Then $(\Lambda + K)^{-1}$ is well defined and compact. Furthermore, by the second Hilbert identity,

$$M(z)^{-1} - (\Lambda + K)^{-1} = (\Lambda + K)^{-1} \Xi M(z)^{-1},$$

where $\Xi$ is a bounded operator, and hence $M(z)^{-1} \in \mathcal{S}_\infty$, from which the claim follows. □

**Lemma 5.2.** If $BM(z)^{-1} \in \mathcal{S}_\infty$ for at least one $z \in \mathbb{C}_+$ and at least one $z \in \mathbb{C}_-$, then

1) The operator $I + BM(z)^{-1}$ is either not invertible for any $z \in \mathbb{C} \setminus \mathbb{R}$ or it is invertible at all $z \in \mathbb{C} \setminus \mathbb{R}$ with the exception of a discrete set of points.

2) For any $z \in \mathbb{C} \setminus \mathbb{R}$ such that $(I + BM(z)^{-1})^{-1}$ exists, one has

$$(B + M(z))^{-1} = M(z)^{-1}(I + BM(z)^{-1})^{-1}.$$  

**Proof.** The proof is a direct consequence of the classical Analytic Fredholm Theorem, see [45, Theorem 8.92]. □

**Theorem 5.3.** Suppose that $BM(z)^{-1} \in \mathcal{S}_\infty$ for at least one $z \in \mathbb{C}_+$ and at least one $z \in \mathbb{C}_-$. If the second alternative in 1) of Lemma 5.2 holds, i.e. $I + BM(z)^{-1}$ is invertible at all $z \in \mathbb{C} \setminus \mathbb{R}$ with the exception of a discrete set of points, then

1) The operator $A_{\alpha \beta}$ has discrete spectrum in $\mathbb{C} \setminus \mathbb{R}$ accumulating at the real line only.

2) One has

$$\rho(A_{\alpha \beta}) \subset Q_{\beta^{-1} \alpha}. \quad (5.4)$$

**Proof.** Lemma 5.3 implies that the “Krein formula”, cf (2.21), holds at all $z \in \mathbb{C} \setminus \mathbb{R}$ with the exception of a discrete set of points:

$$(A_{\alpha \beta} - z)^{-1} = (A_0 - zI)^{-1} - (I - zA_0)^{-1}\Pi(B + M(z))^{-1}\Pi^*, \quad B = \beta^{-1} \alpha, \quad (5.5)$$

and therefore $\rho(A_{\alpha \beta})$ is discrete in $z \in \mathbb{C} \setminus \mathbb{R}$, which proves the first claim.

Furthermore, the right-hand side of (5.5) is analytic whenever its left-hand side is, i.e. on the set $\rho(A_{\alpha \beta})$, which immediately implies the inclusion (5.4). □

It follows from (5.3) and Theorems 5.1, 5.3 that whenever $B$ is bounded and $1 + BM(z)^{-1}$ is invertible for at least one $z \in \mathbb{C}$, one has $Q_{\beta^{-1} \alpha} = \rho(A_{\alpha \beta})$.

The formulae in the next lemma are analogous to [47, Eqs. 2.18 and 2.22].

**Lemma 5.4.** The following identities hold:

$$\Gamma_0(A_{\alpha \beta} - zI)^{-1} = \Theta_{\beta^{-1} \alpha}(z)^{-1}\Gamma_0(L - zI)^{-1} \quad \forall z \in \mathbb{C}_- \cap Q_{\beta^{-1} \alpha}, \quad (5.6)$$

$$\Gamma_0(A_{\alpha \beta} - zI)^{-1} = \tilde{\Theta}_{\beta^{-1} \alpha}(z)^{-1}\Gamma_0(L^* - zI)^{-1} \quad \forall z \in \mathbb{C}_+ \cap Q_{\beta^{-1} \alpha}, \quad (5.7)$$

\(^3\)For this to be the case, it suffices that the operator has a smooth symbol and the boundary is Lipschitz, see [54].
where $\Theta_B$ and $\hat{\Theta}_B$ are defined via their inverses:

$$
\Theta_B(z)^{-1} = 2iQ_B(z)(I - S^*(\overline{z}))^{-1}, \quad z \in \mathbb{C} \cap Q_B, \quad (5.8)
$$

$$
\hat{\Theta}_B(z)^{-1} = 2iQ_B(z)(I - S(z))^{-1}, \quad z \in \mathbb{C} \cap Q_B. \quad (5.9)
$$

Proof. Fix an arbitrary $h \in \mathcal{H}$ and define

$$
g_{\alpha,\beta} := (A_{\alpha\beta} - zI)^{-1}h, \quad z \in \rho(A_{\alpha\beta}), \quad (5.10)
$$

so that, in particular, $g_{-i\mu,\mu} = (L - zI)^{-1}h$, $z \in \mathbb{C}_+$, and $g_{i\nu,\nu} = (L^* - zI)^{-1}h$, $z \in \mathbb{C}_-$.

In order to prove (5.6), suppose that $z \in \mathbb{C}_- \cap Q_{\beta^{-1}\alpha}$, so that the resolvents $(L - zI)^{-1}$ and $(A_{\alpha\beta} - zI)^{-1}$ are defined on the whole space $\mathcal{H}$. Clearly, the vector

$$
g := g_{-i\mu,\mu} - g_{\alpha,\beta} = ((L - zI)^{-1} - (A_{\alpha\beta} - zI)^{-1})h
$$

is an element of $\ker(A - zI)$. It follows from $g_{-i\mu,\mu} \in \text{dom}(L)$ and $g_{\alpha,\beta} \in \text{dom}(A_{\alpha\beta})$ that $\Gamma_1 g_{-i\mu,\mu} = i\Gamma_0 g_{-i\mu,\mu}$ and $\beta \Gamma_1 g_{\alpha,\beta} = -\alpha \Gamma_0 g_{\alpha,\beta}$, and therefore one has

$$
\begin{align*}
0 &= \beta \Gamma_1 (g + g_{\alpha,\beta}) - i\beta \Gamma_0 (g + g_{\alpha,\beta}) \\
&= \beta \Gamma_1 g - i\beta \Gamma_0 g + \beta \Gamma_1 g_{\alpha,\beta} - i\beta \Gamma_0 g_{\alpha,\beta} \\
&= \beta M(z)\Gamma_0 g - i\beta \Gamma_0 g - \alpha \Gamma_0 g_{\alpha,\beta} - i\beta \Gamma_0 g_{\alpha,\beta},
\end{align*}
$$

(5.11)

where in the last equality we also use the fact that $g \in \ker(A - zI)$, together with Definition 1. Hence, by collecting the terms in the calculation (5.11), one has (cf. (5))

$$
(\alpha + \beta M(z))\Gamma_0 g = (\alpha + i\beta)\Gamma_0 g + g_{\alpha,\beta} = (\alpha + i\beta)\Gamma_0 g_{-i\mu,\mu},
$$

which, in turn, implies that, for $z \in Q_{\beta^{-1}\alpha}$ one has

$$
\left\{I - \left(\beta^{-1}\alpha + M(z)\right)^{-1}(\beta^{-1}\alpha + iI)\right\}\Gamma_0 g_{-i\mu,\mu} = \Gamma_0 g_{\alpha,\beta}. \quad (5.12)
$$

Using a version of the second resolvent identity

$$
(B + iI)^{-1} - (B + M(z))^{-1} = (B + M(z))^{-1}(M(z) - iI)(B + iI)^{-1},
$$

we obtain

$$
I - \left(\beta^{-1}\alpha + M(z)\right)^{-1}(\beta^{-1}\alpha + iI) = \left(\beta^{-1}\alpha + M(z)\right)^{-1}(M(z) - iI) = 2iQ_{\beta^{-1}\alpha}(z)(I - S^*(\overline{z}))^{-1},
$$

where we use the formula (2.28).

The identity (5.7) is proved by an argument similar to the above, where the vector $g_{-i\mu,\mu}$ is replaced with $g_{i\nu,\nu}$, for $z \in \mathbb{C}_-$, and the formula (2.27) is used instead of (2.28).

6. Model for non-necessarily dissipative operators

Lemma 6.1. The following formulae hold:

$$
\Theta_B(z) - \Theta_B(\lambda) = (S^*(\overline{z}) - S^*(\lambda))\chi_B^+, \quad z \in \mathbb{C}_- \cap Q_B,
$$

$$
\hat{\Theta}_B(z) - \hat{\Theta}_B(\lambda) = (S(z) - S(\lambda))\chi_B^-, \quad z \in \mathbb{C}_+ \cap Q_B,
$$

where $\Theta_B$ and $\hat{\Theta}_B$ are defined via their inverses:
where
\[ \chi_B^\pm := \frac{1}{2i}(\pm B + iI). \] (6.1)

**Proof.** By the definition (5.8) and using the representation (2.28), we write, for \( z \in \mathbb{C}_- \cap Q_B, \)
\[
\Theta_B(z) - \Theta_B(\lambda) = -(2i)^{-1}(I - S^*(\overline{\gamma})) (B + M(z)) + (2i)^{-1}(I - S^*(\lambda))(B + M(\lambda)) \\
= (2i)^{-1}(S^*(\overline{\gamma}) - S^*(\lambda))(B + M(z)) + (2i)^{-1}(I - S^*(\lambda))(M(\lambda) - M(z)) \\
= (2i)^{-1}(S^*(\overline{\gamma}) - S^*(\lambda))(B + M(z)) \\
\quad + (iI - M(\lambda))^{-1}[-2i(I - S^*(\lambda))^{-1} + 2i(I - S^*(\overline{\gamma}))^{-1}] \\
= (2i)^{-1}(S^*(\overline{\gamma}) - S^*(\lambda))(B + M(z)) \\
\quad + (iI - M(\lambda))^{-1}(I - S^*(\lambda))^{-1}[-2i(I - S^*(\overline{\gamma})) + 2i(I - S^*(\lambda))](I - S^*(\overline{\gamma}))^{-1} \\
= (2i)^{-1}(S^*(\overline{\gamma}) - S^*(\lambda))(B + M(z)) + (S^*(\overline{\gamma}) - S^*(\lambda))(I - S^*(\overline{\gamma}))^{-1} \\
= (S^*(\overline{\gamma}) - S^*(\lambda))[2i^{-1}(B + M(z)) - (2i)^{-1}(M(z) - iI)] = (S^*(\overline{\gamma}) - S^*(\lambda))\chi_B^+.
\]
as required. Similarly, from the definition (5.9), we obtain, for \( z \in \mathbb{C}_+ \cap Q_B, \)
\[
\widehat{\Theta}_B(z) - \widehat{\Theta}_B(\lambda) = -(2i)^{-1}(I - S(z))(B + M(z)) - (2i)^{-1}(I - S(\lambda))(B + M(\lambda)) \\
= -(2i)^{-1}(S(z) - S(\lambda))(B + M(z)) - (2i)^{-1}(I - S(\lambda))(M(\lambda) - M(z)) \\
= -(2i)^{-1}(S(z) - S(\lambda))(B + M(z)) \\
\quad - (iI + M(\lambda))^{-1}[2i(I - S(\lambda))^{-1} - 2i(I - S(z))^{-1}] \\
= -(2i)^{-1}(S(z) - S(\lambda))(B + M(z)) \\
\quad - (iI + M(\lambda))^{-1}(I - S(\lambda))^{-1}[2i(I - S(z)) - 2i(I - S(\lambda))](I - S(z))^{-1} \\
= -(2i)^{-1}(S(z) - S(\lambda))(B + M(z)) + (S(z) - S(\lambda))(I - S(z))^{-1} \\
= (S(z) - S(\lambda))[-(2i)^{-1}(B + M(z)) + (2i)^{-1}(M(z) + iI)] \\
= (S(z) - S(\lambda))(2i)^{-1}\chi_-.
\]
as required. \( \square \)

**Theorem 6.2.** (i) If \( z \in \mathbb{C}_- \cap Q_{\beta^{-1}\alpha} \) and \( \frac{g}{\bar{g}} \in K, \) then
\[
\Phi(A_{\alpha,\beta} - zI)^{-1}\Phi^* \begin{pmatrix} \bar{g} \\ g \end{pmatrix} = PK_{-\frac{1}{z}} \begin{pmatrix} g - \chi_{\beta^{-1}\alpha}^+ & \bar{g} \\ g \end{pmatrix} (g + S^*g)(z). \] (6.2)

(ii) If \( z \in \mathbb{C}_+ \cap Q_{\beta^{-1}\alpha} \) and \( \frac{g}{\bar{g}} \in K, \) then
\[
\Phi(A_{\alpha,\beta} - zI)^{-1}\Phi^* \begin{pmatrix} \bar{g} \\ g \end{pmatrix} = PK_{-\frac{1}{z}} \begin{pmatrix} \bar{g} - \chi_{\beta^{-1}\alpha}^- & \bar{g} \\ g \end{pmatrix} (S\bar{g} + g)(z). \] (6.3)

Here, \((g + S^*g)(z)\) and \((S\bar{g} + g)(z)\) denote the values at \( z \) of the analytic continuations of the functions \( g + S^*g \in H^2(\mathcal{E}) \) and \( S\bar{g} + g \in H^2(\mathcal{E}) \) into the lower half-plane and upper
half-plane.

Proof. We prove part (i). The proof of part (ii) is carried out along the same lines. For this one should establish the validity of the identities:

\[
\mathcal{F}_\pm(A_{\alpha,\beta} - zI)^{-1}\Phi^{-1}\left(\frac{\tilde{g}}{g}\right) = \mathcal{F}_\pm \Phi^{-1} P_K \left(1 \pm \frac{1}{z} \frac{\tilde{g}}{g} - \chi_{\alpha,\beta}^+ \Theta_{\alpha,\beta}(z)^{-1}(\tilde{g} + S^*g)(z)\right)
\]  

(6.4)

for \(z \in \mathbb{C}_- \cap Q_{\beta-1}\alpha\). First we compute the left-hand-side of (6.4). It follows from Lemma 5.4 that, for \(z, \lambda \in \mathbb{C}_- \cap Q_{\beta-1}\alpha\), and \(h \in \mathcal{H}\),

\[
\Gamma_0(L - zI)^{-1}(A_{\alpha,\beta} - \lambda I)^{-1}h = \Theta_{\beta-1\alpha}(z)\Gamma_0(A_{\alpha,\beta} - zI)^{-1}(A_{\alpha,\beta} - \lambda I)^{-1}h
\]

\[
= \frac{1}{z - \lambda} \Theta_{\beta-1\alpha}(z)\Gamma_0[(A_{\alpha,\beta} - zI)^{-1} - (A_{\alpha,\beta} - \lambda I)^{-1}]h
\]

\[
= \frac{1}{z - \lambda} \left[\Gamma_0(L - zI)^{-1} - \Theta_{\beta-1\alpha}(z)\Gamma_0(A_{\alpha,\beta} - \lambda I)^{-1}\right]h
\]

\[
= \frac{1}{z - \lambda} \left[\Gamma_0(L - zI)^{-1} - \Theta_{\beta-1\alpha}(z)\Theta_{\beta-1\alpha}(\lambda)^{-1}\Gamma_0(L - \lambda I)^{-1}\right]h.
\]

Let \(z = k - i\epsilon\) with \(k \in \mathbb{R}\), then it follows from the above calculation that

\[
\lim_{\epsilon \downarrow 0} \frac{1}{(k - i\epsilon) - \lambda} \left[\Gamma_0(L - (k - i\epsilon)I)^{-1} - \Theta_{\beta-1\alpha}(k - i\epsilon)\Theta_{\beta-1\alpha}(\lambda)^{-1}\Gamma_0(L - \lambda I)^{-1}\right]h.
\]  

(6.5)

Combining the expression for \(\mathcal{F}_+\) from Definition 2 with (6.5) yields

\[
\mathcal{F}_+(A_{\alpha,\beta} - \lambda I)^{-1}h = \frac{1}{-\lambda} \left[\mathcal{F}_+ h - \Theta_{\beta-1\alpha}(\cdot)\Theta_{\beta-1\alpha}(\lambda)^{-1}\mathcal{F}_+ h(\lambda)\right].
\]

Hence, in view of the identity \(\mathcal{F}_+ h = \tilde{g} + S^*g\), which can be obtained from e.g. (4.7), we obtain

\[
\mathcal{F}_+(A_{\alpha,\beta} - \lambda I)^{-1}\Phi^{-1}\left(\frac{\tilde{g}}{g}\right) = \frac{1}{-\lambda} \left[\tilde{g} + S^*g - \Theta_{\beta-1\alpha}(\cdot)\Theta_{\beta-1\alpha}(\lambda)^{-1}(\tilde{g} + S^*g)(\lambda)\right].
\]  

(6.6)

On the basis of Lemma 5.4 and reasoning in the same fashion as was done to obtain (6.6), one verifies

\[
\mathcal{F}_-(A_{\alpha,\beta} - \lambda I)^{-1}\Phi^{-1}\left(\frac{g}{g}\right) = \frac{1}{-\lambda} \left[S\tilde{g} + g - \Theta_{\beta-1\alpha}(\cdot)\Theta_{\beta-1\alpha}(\lambda)^{-1}(\tilde{g} + S^*g)(\lambda)\right].
\]  

(6.7)
Let us focus on the right hand side of (6.4). Note that

\[
P_\alpha \cdot \frac{1}{z} \left( g - \chi_{\beta - 1\alpha}^+ \Theta_{\beta - 1\alpha}(z)^{-1}(\bar{g} + S^*g)(z) \right)
\]

\[
\left( \frac{1}{\cdot - z} - \frac{1}{\cdot - z} \left( \frac{\bar{g}}{\cdot - z} - \frac{1}{\cdot - z} \left[ g + S^*g - S^*\chi_{\beta - 1\alpha}^+ \Theta_{\beta - 1\alpha}(z)^{-1}(\bar{g} + S^*g)(z) \right] \right) \right)
\]

\[
\left( \frac{1}{\cdot - z} \left( \frac{\bar{g}}{\cdot - z} - \frac{1}{\cdot - z} \left[ g - \chi_{\beta - 1\alpha}^+ \Theta_{\beta - 1\alpha}(z)^{-1}(\bar{g} + S^*g)(z) \right] \right) \right)
\]

(6.8)

where (4.6) is used in the first equality and in the second the fact that if \( f \) is a function in \( H^2 \), then, for any \( z \in \mathbb{C}_- \),

\[
P_+ \left( \frac{f}{\cdot - z} \right) = P_+ \left( \frac{f + f(z) - f(z)}{\cdot - z} \right) = P_+ \left( \frac{f(z)}{\cdot - z} \right) = f(z). \quad (6.9)
\]

Now, apply \( \mathcal{T}_+ \Phi^{-1} \) to (6.8) taking into account that \( \mathcal{T}_+ h = \bar{g} + S^*g \) once again:

\[
\mathcal{T}_+ \Phi^{-1} \cdot \frac{1}{\cdot - z} \left( \frac{\bar{g}}{\cdot - z} - \frac{1}{\cdot - z} \left[ \bar{g} + S^*g - (\bar{g} + S^*g)(z) + (S^*(\mathcal{T}) - S^*)\chi_{\beta - 1\alpha}^+ \Theta_{\beta - 1\alpha}(z)^{-1}(\bar{g} + S^*g)(z) \right] \right)
\]

\[
\left( g - \chi_{\beta - 1\alpha}^+ \Theta_{\beta - 1\alpha}(z)^{-1}(\bar{g} + S^*g)(z) \right)
\]

\[
\left( \frac{1}{\cdot - z} \left[ \bar{g} + S^*g - (\bar{g} + S^*g)(z) + (S^*(\mathcal{T}) - S^*)\chi_{\beta - 1\alpha}^+ \Theta_{\beta - 1\alpha}(z)^{-1}(\bar{g} + S^*g)(z) \right] \right)
\]

(6.8)

where (4.6) is used in the first equality and in the second the fact that if \( f \) is a function in \( H^2 \), then, for any \( z \in \mathbb{C}_- \),

\[
P_+ \left( \frac{f}{\cdot - z} \right) = P_+ \left( \frac{f + f(z) - f(z)}{\cdot - z} \right) = P_+ \left( \frac{f(z)}{\cdot - z} \right) = f(z). \quad (6.9)
\]

By combining the last equality with (6.6), we have established the first identity in (6.4).

Now, applying \( \mathcal{T}_- \Phi^{-1} \) to (6.8) and using the identity \( \mathcal{T}_- h = S\bar{g} + g \), we obtain

\[
\mathcal{T}_- \Phi^{-1} \cdot \frac{1}{\cdot - z} \left( \frac{\bar{g}}{\cdot - z} - \frac{1}{\cdot - z} \left[ \bar{g} + S^*g - (\bar{g} + S^*g)(z) + (I - SS^*(\mathcal{T}))\chi_{\beta - 1\alpha}^+ \Theta_{\beta - 1\alpha}(z)^{-1}(\bar{g} + S^*g)(z) \right] \right)
\]

\[
\left( g - \chi_{\beta - 1\alpha}^+ \Theta_{\beta - 1\alpha}(z)^{-1}(\bar{g} + S^*g)(z) \right)
\]

(6.8)

Comparing this last equality with (6.7), we arrive at the second identity in (6.4). \( \Box \)
7. Large-coupling asymptotics for a transmission problem

Suppose that $\Omega$ is a bounded Lipschitz domain, and $\Gamma \subset \Omega$ is a closed Lipschitz curve, so that $\Gamma$ is the common boundary of domains $\Omega_+$ and $\Omega_-$ such that $\Omega_+ \cup \Omega_- = \Omega$. For $a > 0$, $z \in \mathbb{C}$ we consider the “transmission” eigenvalue problem (cf. [50])

\[
\begin{aligned}
-\Delta u_+ &= zu_+ &\text{in } \Omega_+, \\
-a\Delta u_- &= zu_- &\text{in } \Omega_-, \\
u_+ &= u_- &\text{on } \Gamma, \\
\frac{\partial u_+}{\partial n_+} + a \frac{\partial u_-}{\partial n_-} &= 0 &\text{on } \Gamma, \\
\frac{\partial u_+}{\partial n_+} &= 0 &\text{on } \partial \Omega,
\end{aligned}
\]

where $n_\pm$ denotes the exterior normal (defined a.e.) to the corresponding part of the boundary.\(^4\)

The above problem can be understood in the strong sense, i.e. $u_\pm \in H^2(\Omega_\pm)$, the Laplacian differential expression $\Delta$ is the corresponding combinations of weak derivatives of second order, and the boundary values of $u_\pm$ and their normal derivatives are understood in the sense of traces according to the embeddings of $H^2(\Omega_\pm)$ into $H^s(\Gamma)$, $H^s(\partial \Omega)$, $s = 1/2, 3/2$.

We define the Dirichlet-to-Neumann map (see e.g. [53])

\[
\Lambda : \phi \mapsto \frac{\partial u_+}{\partial n_+} + a \frac{\partial u_-}{\partial n_-}, \quad \phi \in H^1(\Gamma),
\]

where $u_\pm$ are the harmonic functions in $\Omega_\pm$ subject to the above boundary condition on $\partial \Omega$ and the Dirichlet condition $u = \phi$ on $\Gamma$. Clearly $\Lambda = \Lambda^+ + a \Lambda^-$, where $\Lambda^+, \Lambda^-$ are the (signed) Dirichlet-to-Neumann maps on each side of the interface $\Gamma$, which are self-adjoint operators in $L^2(\Gamma)$, with domain $H^1(\Gamma)$, so that $\Lambda$ has the same properties as $\Lambda_+, \Lambda_-$, see [17, Lemma 2.1].

According to the framework developed in the previous sections, the problem (7.1) is written in the form

\[
Au = zu, \quad \alpha \Gamma_0 u + \beta \Gamma_1 u = 0,
\]

where $A$ is defined by (2.2), $\Gamma_1$ is defined by (2.10), and $\alpha = 0, \beta = I$. Then the operator $M(z)$ of Definition (1) is the mapping

\[
M(z) : \phi \mapsto \frac{\partial u_+}{\partial n_+} + a \frac{\partial u_-}{\partial n_-}, \quad \phi \in H^1(\Gamma),
\]

where $u_+$ and $u_-$ solve

\[
\begin{aligned}
-\Delta u_+ &= zu_+ &\text{in } \Omega_+,
-a\Delta u_- &= zu_- &\text{in } \Omega_-, \\
u_+ &= u_- &\text{on } \Gamma, \\
\frac{\partial u_+}{\partial n_+} &= 0 &\text{on } \partial \Omega,
\end{aligned}
\]

and the formula (2.14) expresses $M(z)$ in terms of $\Lambda$ and the “Dirichlet decoupling” $A_0^+ \oplus A_0^-$, where $A_0^+$ is the Laplace operator with Dirichlet condition on $\Gamma$ and Neumann condition on $\partial \Omega$, and $A_0^-$ is the Dirichlet Laplacian on $\Omega_-$. Similarly, we define the $M$-operators $M^\pm$ on the components

\(^4\)The Dirichlet boundary condition on $\partial \Omega$ can be replaced by any Robin condition without affecting the analysis of this section.
\[ \Omega_\pm, \text{ so that} \]
\[ M^+(z) : \phi \mapsto \frac{\partial u_+}{\partial n_+}, \quad M^-(z) : \phi \mapsto a \frac{\partial u_-}{\partial n_-}, \quad \phi \in H^1(\Gamma), \]
where \( u_+ \) and \( u_- \) are as above.

As discussed in Section 5, see also [48, Theorem 3.11, Theorem 5.5], the following statement holds, cf. [9].

**Proposition 7.1.** The spectrum of (7.1) coincides with the set of \( z \) for which \( \alpha + \beta M(z) \equiv M(z) \), is not boundedly invertible, equivalently, zero is an eigenvalue of \( M(z) \).

The representation (2.14) applied to \( M^-(z) \) implies that
\[ M^-(z) = a\Lambda^- + z\Pi_+^\ast (I - a^{-1}z(A_0^-)^{-1})^{-1}\Pi_- = a\Lambda^- + z\Pi_+^\ast\Pi_- + O(a^{-1}), \quad (7.2) \]
with a uniform estimate on the remainder term, where \( A_0^- \) is the Dirichlet Laplacian on \( \Omega_- \).

In what follows we analyse the resolvent of the operator \( A_a \) of the transmission problem (7.1), which coincides with the operator \( A_{0I} \), in terms of the notation of Section 2. In particular, the spectrum of \( A_a \) provides the spectrum of (7.1). Our approach is based on the use of the Krein formula 5.5 with \( \alpha = 0, \beta = I \), where for the asymptotic analysis of \( M(z)^{-1} \) we employ (7.2) and separate the singular and non-singular parts of \( \Lambda^- \).

To this end, note first that the spectrum of \( \Lambda^- \) consists of the values \( \mu \) (“Steklov eigenvalues”) such that the problem
\[ \begin{align*}
\Delta u &= 0, \quad u \in H^2(\Omega_-), \\
\frac{\partial u}{\partial n} &= \mu u \quad \text{on} \quad \Gamma,
\end{align*} \]
has a non-trivial solution. The least (by absolute value) Steklov eigenvalue is zero, and the associated normalised eigenfunction (“Steklov eigenvector”) is \( \psi_* = |\Gamma|^{-1/2} \mathbb{1} \in \mathcal{H} \). Introduce the corresponding orthogonal projection \( P := \langle \cdot, \psi_* \rangle_\mathcal{H} \psi_* \), which is a spectral projection relative to \( \Lambda^- \), and decompose the boundary space \( \mathcal{H} \):
\[ \mathcal{H} = P\mathcal{H} \oplus P^\perp \mathcal{H}, \quad (7.4) \]
where \( P^\perp := I - P \). This yields the following matrix representation for \( \Lambda^- \):
\[ \Lambda^- = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_{\perp}\end{pmatrix}, \]
where \( \Lambda_{\perp} := P^\perp \Lambda^- P^\perp \) is treated as a self-adjoint operator in \( P^\perp \mathcal{H} \).

We write the operator \( M(z) \) as a block-operator matrix relative to the decomposition (7.4), followed by an application of the Schur-Frobenius inversion formula, see [55]. To this end, notice that for all \( \psi \in \text{dom} \Lambda \) one has \( P\psi \in \text{dom} \Lambda \), and therefore \( P^\perp \Lambda P \) is well defined on \( \text{dom} \Lambda \). Similarly, \( P^\perp \psi = \psi - P\psi \in \text{dom} \Lambda \), and \( \Lambda P^\perp \) is also well defined. Furthermore, by the self-adjointness of \( \Lambda \), one has \( \Lambda P^\perp \psi = \langle P^\perp \psi, \Lambda \psi_* \rangle \psi_* \), and therefore \( \| \Lambda P^\perp \|_{\mathcal{H} \to \mathcal{H}} \leq \| \Lambda \psi_* \|_{\mathcal{H}} \). It follows that \( \Lambda P^\perp \) is extendable to a bounded mapping on \( P^\perp \mathcal{H} \). A similar calculation applied to \( P^\perp \Lambda P \) and \( \Lambda P \) shows that these are extendable to bounded mappings on \( P\mathcal{H} \). Therefore, for all \( z \in \rho(A_0^+) \cap \rho(A_0^-) \) the operator \( M(z) \) admits the representation
\[ M(z) = \begin{pmatrix} A & B \\ E & D \end{pmatrix}, \quad A, B, E \text{ bounded.} \quad (7.5) \]
For evaluating $M(z)^{-1}$ we use the Schur-Frobenius inversion formula \cite[Theorem 2.3.3]{55}

$$
\begin{pmatrix}
A & B \\
E & D
\end{pmatrix}^{-1} = \begin{pmatrix}
A^{-1} + A^{-1}BS^{-1}EA^{-1} & -A^{-1}BS^{-1} \\
-S^{-1}EA^{-1} & S^{-1}
\end{pmatrix}, \quad S := D - EA^{-1}B. \quad (7.6)
$$

Using the fact that $S^{-1} = (I - D^{-1}EA^{-1}B)^{-1}D^{-1}$, where $\|D^{-1}\| \leq Ca^{-1}$, and therefore $S$ is boundedly invertible with a uniformly small bound, we obtain

$$
M(z)^{-1} = \begin{pmatrix} A & B \\ E & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix} + O(a^{-1}), \quad (7.7)
$$

with a uniform estimate for the remainder term.

**Theorem 7.2.** Fix $\sigma > 0$ and a compact set $K \subset \mathbb{C}$, and denote $K_\sigma := \{ z \in K : \text{dist}(z, \mathbb{R}) \geq \sigma \}$.

There exists $C > 0$ such that for all $z \in K_\sigma$, $a \geq a_0$ one has

$$
\|(A_a - z)^{-1} - (A_{P^\perp} - z)^{-1}\|_{L^2(\Omega) \to L^2(\Omega)} \leq Ca^{-1}.
$$

**Proof.** We use (5.5) with $\alpha = 0$, $\beta = I$ for the resolvent $(A_a - z)^{-1}$ and with $\beta_0 = P^\perp$, $\beta_1 = P$ for $(A_{P^\perp} - z)^{-1}$. In the former case we use (7.7) and in the latter case we write $(P^\perp + PM(z))^{-1}P = P(\sigma)P^{-1}P$ by the Schur-Frobenius inversion formula of \cite[see (7.6)]{55} The claim follows by comparing the two expressions. \hfill \Box

We now rewrite the result of Theorem 7.2 in a block-matrix form relative to the decomposition $\mathcal{H} = P_+ \mathcal{H} \oplus P_- \mathcal{H} = L^2(\Omega_-) \oplus L^2(\Omega_+)$.

This allows us to express the asymptotics of $(A_a - z)^{-1}$ in terms of the generalised resolvent $R_a(z) := P_+(A_a - z)^{-1}P_+$, analysed next.

**Proposition 7.3.** One has

$$
R_a(z) = (A_0^+ - z)^{-1} - S_+^+(M^+(z) + M^-(z))^{-1}(S_+^+)^*.
$$

**Proof.** By the definition of $M^+$, $M^-$ and a direct application of (5.5). \hfill \Box

**Corollary 7.4.** The generalised resolvent $R_a(z)$ is the solution operator for

$$
-\Delta u - zu = f, \quad f \in L^2(\Omega_+), \\
\Gamma^+_1 u = -M^-(z)\Gamma^+_0 u.
$$

(7.9)

Theorem 7.2 now implies a uniform asymptotics for the generalised resolvents $R_a$ as $a \to \infty$.

**Theorem 7.5.** For all $z \in K_\sigma$ the operator $R_a(z)$ admits the asymptotics $R_a(z) = R_{\text{eff}}(z) + O(a^{-1})$, in the operator-norm topology, where $R_{\text{eff}}(z)$ is the solution operator for

$$
-\Delta u - zu = f, \quad f \in L^2(\Omega_+), \\
\alpha(z)\Gamma^+_1 u + \beta \Gamma^+_1 u = 0,
$$

(7.10)

with $\alpha(z) = P^\perp - PB(z)P$ and $\beta = P$.

\footnote{We remark that $P^\perp + PM(z)$ is triangular ($\mathcal{A} = PM(z)P$, $\mathcal{B} = PM(z)P^\perp$, $\mathcal{E} = 0$, $\mathcal{D} = I$ in (7.6)) with respect to the decomposition $\mathcal{H} = P\mathcal{H} \oplus P^\perp \mathcal{H}$.}
Proof. On the one hand, by Theorem 7.2, the resolvent \((A_a - z)^{-1}\) is \(O(a^{-1})\)-close to
\[
(A_{P_{\perp}} - z)^{-1} = (A_0 - z)^{-1} - S_z(P_{\perp} + \overline{PM}(z))^{-1} PS_z^*,
\]
and therefore
\[
R_a(z) = P_+(A_0 - z)^{-1} P_+ - P_+ S_z(P_{\perp} + \overline{PM}(z))^{-1} PS_z^* P_+ + O(a^{-1})
= (A_0^+ - z)^{-1} - S_z^+(P_{\perp} + \overline{PM}(z))^{-1} P(S_z^*)^* + O(a^{-1})
= (A_0^+ - z)^{-1} - S_z^+ P(\overline{PM}(z)P^{-1} P(S_z^*)^* + O(a^{-1})
= (A_0^+ - z)^{-1} - S_z^+ P(\overline{PM}(z)P - PB(z)P)^{-1} P(S_z^*)^* + O(a^{-1}).
\]

(7.11)

On the other hand, by (5.5) and applying the inversion formula (7.6), we obtain
\[
R_{\text{eff}}(z) = (\tilde{A}_0^+ - z)^{-1} - S_z^+(\tilde{P}_{\perp} - \overline{PB(\gamma)}(z)P + \overline{PM}(z))^{-1} P(S_z^*)^*
= (\tilde{A}_0^+ - z)^{-1} - S_z^+ P(\overline{PM}(z)P - PB(z)P)^{-1} P(S_z^*)^*,
\]

(7.12)

Comparing the right-hand sides of (7.11) and (7.12) completes the proof.

Theorem 7.5 can be further clarified by considering the “truncated” boundary space\(^6\) \(\tilde{\mathcal{H}} := PH\). Introduce the truncated harmonic lift by \(\tilde{\Pi}_+ := \Pi_+|_{\tilde{\mathcal{H}}}\) and Dirichlet-to-Neumann map \(\tilde{\Lambda}^+ := PA^+|_{\tilde{\mathcal{H}}}\).

**Theorem 7.6.** 1. The formula
\[
R_{\text{eff}}(z) = (\tilde{A}_0^+ - z)^{-1} - S_z^+(\tilde{M}^+(z) + \overline{PM}^-(z)P)^{-1} (S_z^*)^*
\]
holds, where \(\tilde{S}_z^+\) is the solution operator of the problem
\[
- \Delta u_{\phi} - z u_{\phi} = 0, \quad u_{\phi} \in \text{dom} \ A_0^+ + \text{ran} \, \tilde{\Pi}_+,
\]
\[
\Gamma_0^+ u_{\phi} = \phi, \quad \phi \in \tilde{\mathcal{H}},
\]
and \(\tilde{M}^+ = \overline{PM}^+(z)P|_{\tilde{\mathcal{H}}} = \tilde{\Lambda}^+ + z \Pi_+^+ (1 - z(A_0^+)^{-1})^{-1} \tilde{\Pi}_+\).

*Proof.* By the definition of \(S_z^+\), one has \(S_z^+ = (I - z(A_0^+)^{-1})^{-1} \Pi_+\), and therefore \(\tilde{S}_z^+ = S_z^+ P|_{\tilde{\mathcal{H}}}\). It follows that (7.10) is equivalent to (7.13).

**Corollary 7.7.** The resolvent \(R_{\text{eff}}(z)\) is the generalised resolvent of the problem
\[
- \Delta u - z u = f, \quad f \in L^2(\Omega_+), \quad u \in \text{dom} \ A_0^+ + \text{ran} \, \tilde{\Pi}_+,
\]
\[
\Pi_1^+ u = -\overline{PM}^-(z)P|_{\Pi_1},
\]

(7.14)

Equipped with Theorems 7.5 and 7.6, we provide a convenient representation for the asymptotics of \((A_a - z)^{-1}\) obtained in Theorem 7.2.

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\(^6\)In what follows we consistently supply the (finite-dimensional) “truncated” spaces and operators pertaining to them by the breve overscript.
\textbf{Theorem 7.8.} For the resolvent \((A_a - z)^{-1}\) one has
\[(A_a - z)^{-1} = \begin{pmatrix} R_{\text{eff}}(z) & (\mathcal{R}_z [R_{\text{eff}}(z) - (A^+_0 - z)^{-1}])^\ast \Pi^+ \\ \Pi_- \mathcal{R}_z [R_{\text{eff}}(z) - (A^+_0 - z)^{-1}] & \Pi_- \mathcal{R}_z (\mathcal{R}_z [R_{\text{eff}}(z) - (A^+_0 - z)^{-1}])^\ast \Pi^+ \end{pmatrix} + O(a^{-1}),\]
where \(\mathcal{R}_z := \Gamma^+_0 \mid_{\text{ran}(S^+_0 P)}\), \(z \in \mathbb{C}_\pm\).

\textit{Proof.} First, we note that since \(\text{ran}(S^+_0 P)\) is one-dimensional, the operator \(\mathcal{R}_z\) is well defined as a bounded linear operator from \(\text{ran}(S^+_0 P)\) to \(\mathcal{H}\), where the former is equipped with the standard norm of \(L^2(\Omega)\). We proceed by representing the operator \((A_{p^+} - z)^{-1}\), see Theorem 7.2, in a block-operator matrix form relative to the orthogonal decomposition \(H = L^2(\Omega^+) \oplus L^2(\Omega^-)\). We compare the norm-resolvent asymptotics \((A_{p^+} - z)^{-1}\), provided by Theorem 7.2, with \(R_{\text{eff}}(z)\), which is \(O(a^{-1})\)-close to \(P_+(A_{p^+} - z)^{-1}P_+\), as established by Theorem 7.5:
\[P_-(A_{p^+} - z)^{-1}P_+ = -S^- P (PM^+(z) P + PM^-(z) P)^{-1} P(S^+_0)^\ast\]
\[= -S^- \Gamma^+_0 S^+_0 P (PM^+(z) P + PM^-(z) P)^{-1} P(S^+_0)^\ast\]
\[= S^- \Gamma^+_0 [R_{\text{eff}}(z) - (A^+_0 - z)^{-1}] S^- \mathcal{R}_z [R_{\text{eff}}(z) - (A^+_0 - z)^{-1}].\]
Here in the second equality we use the fact that \(\Gamma^+_0 S^+_0 = I\), and in the third equality we use (5.5), see also (7.12). Passing over to the top-right entry, we write
\[P_+(A_{p^+} - z)^{-1}P_+ = -S^+_0 P (PM^+(z) P - PB(\tau)(z) P)^{-1} P(S^-_0)^\ast\]
\[= (\mathcal{R}_z [R_{\text{eff}}(z) - (A^+_0 - z)^{-1}])^\ast (S^-_0)^\ast\]
\[= (\mathcal{R}_z [R_{\text{eff}}(z) - (A^+_0 - z)^{-1}])^\ast \Pi^+ (1 - z(A_0^{-1})^{-1})^{-1},\]
and the claim pertaining to the named entry follows by a virtually unchanged argument. Finally, for the bottom-right entry we have
\[P_-(A_{p^+} - z)^{-1}P_+ = (A_0^- - z)^{-1} + S^- \mathcal{R}_z (\mathcal{R}_z [R_{\text{eff}}(z) - (A^+_0 - z)^{-1}])^\ast (S^-_0)^\ast,\]
which completes the proof. \(\square\)

The representation for \(R_{\text{eff}}(z)\) given by Theorem 7.6 allows us to further simplify the asymptotics of \((A_a - z)^{-1}\), using the fact that
\[PM^-(z) P = PA^- P + z P \Pi^+ \Pi_- P + O(a^{-1}) = z \Pi^+ \Pi_- + O(a^{-1}), \quad \Pi_- := \Pi_- |_{\mathcal{H}}.\]
As a result, one has
\[R_{\text{eff}}(z) = \tilde{R}_{\text{eff}}(z) + O(a^{-1}), \quad \tilde{R}_{\text{eff}}(z) := (A_0^- - z)^{-1} - \tilde{S}^+_z (\tilde{M}^+(z) + z \Pi^+ \Pi_-)^{-1} (\tilde{S}^+_z)^\ast, \quad (7.15)\]
and hence the following result holds.

\textbf{Theorem 7.9.} The resolvent \((A_a - z)^{-1}\) has the following asymptotics in the operator-norm topology:
\[(A_a - z)^{-1} = \begin{pmatrix} \tilde{R}_{\text{eff}}(z) & (\mathcal{R}_z [\tilde{R}_{\text{eff}}(z) - (A^+_0 - z)^{-1}])^\ast \Pi^+ \\ \Pi_- \mathcal{R}_z [\tilde{R}_{\text{eff}}(z) - (A^+_0 - z)^{-1}] & \Pi_- \mathcal{R}_z (\mathcal{R}_z [\tilde{R}_{\text{eff}}(z) - (A^+_0 - z)^{-1}])^\ast \Pi^+ \end{pmatrix} + O(a^{-1}).\]
Corollary 7.10. The spectra of the operators $A_n$ converge, as $a \to \infty$, with an order $O(a^{-1})$ error estimate, uniformly on compact subsets of $\mathbb{C}$, to the union of the spectrum of $A_0^+$ and the set of values $z$ for which
\[
\text{Ker}(\tilde{M}^+(z) + z\tilde{\Pi}^+\tilde{\Pi}^-) \neq \{0\}.
\]

(7.16)

An explicit representation for the above limit set can be obtained by looking for functions $u \in H^2(\Omega_+)$ of the form $u = v + c$, where $c \in \mathbb{C}$ and $v$ solves the problem
\[
-\Delta v = z(v + c) \text{ in } \Omega_+, \quad v|_{\Gamma} = 0, \quad \frac{\partial v}{\partial n_+}|_{\partial\Omega} = 0,
\]

(7.17)
or equivalently $v = zc(A_0^+ - z)^{-1}1_{\Omega_+}$, where $1_{\Omega_+}$ is identically equal to unity on $\Omega_+$. Then the condition for $z$ to satisfy (7.16) is equivalent to the existence of $c \neq 0$ such that
\[
z c\left(\frac{\partial}{\partial n_+}[ (A_0^+ - z)^{-1}1_{\Omega_+}], 1_{\Gamma} \right)_{L^2(\Gamma)} 1_{\Gamma} + z c\left(\frac{\partial}{\partial n_-}[ (A_0^-)^{-1}1_{\Omega_+}], 1_{\Gamma} \right)_{L^2(\Gamma)} 1_{\Gamma} = 0,
\]

(7.18)
where $1_{\Gamma}$ is identically equal to unity on $\Gamma$. Denote by $\lambda_j^+, j = 1, 2, \ldots$, and $\phi_j^+, j = 1, 2, \ldots$, the eigenvalues and the corresponding normalised eigenfunctions of the operator $A_0^+$. The condition (7.18) is written as
\[
z \left[|\Omega| + z \sum_{j=1}^{\infty} (\lambda_j^+ - z)^{-1} \left(\int_{\Omega_+} \phi_j^+ \right)^2 \right] = 0.
\]

(7.19)
The problem (7.17), whose spectrum is described by (7.19), is related to the “electrostatic problem” discussed in [58, Lemma 3.4].

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