Topological lasers are of growing interest as a way to achieve disorder-robust single mode lasing using arrays of coupled resonators. We study lasing in a two-dimensional coupled resonator lattice exhibiting transitions between trivial and topological phases, which allows us to systematically characterize the lasing modes throughout a topological phase. We show that, unlike conventional topological robustness that requires a sufficiently large bulk band gap, bifurcations in topological edge mode lasing can occur even when the band gap is maximized. We show that linear mode bifurcations from single mode to multi-mode lasing can occur deep within the topological phase, sensitive to both the pump shape and lattice geometry. We suggest ways to suppress these bifurcations and preserve single edge mode lasing.

We consider a recently-demonstrated ring resonator lattice with tunable topological properties [20, 21]. Its unit cell is shown in Fig. 1(a), consisting of two sublattices (a, b) whose resonant frequencies can be independently tuned. Each resonator is coupled to its nearest neighbors (shown with diagonal lines) as well as next-nearest neighbors (horizontal and vertical lines). Coupling is mediated by anti-resonant link rings, resulting in the same nearest and next-nearest neighbor coupling strengths, as well as a direction-dependent hopping phase for the nearest neighbor coupling. Considering a single circulation direction within the resonant rings (anti-clockwise), and assuming negligible backscattering between clockwise and anit-clockwise modes, the linear op-
where \( I \) to 1 without loss of generality. We will start by analyzing detuning, \( \phi \) is strength of the linear tight binding Hamiltonian with symmetric gain/loss (population) operators for the \( a \) and \( b \) sublattice detuning, \( P_{a,b} \) and \( P_{b,a} \), and detunings \( \pm M \) with respect to a reference frequency. (b) Phase diagram of the Bloch band structure of the linear tight binding Hamiltonian with symmetric gain/loss (\( P_{a,x,y} = 2\gamma \), \( P_{b,x,y} = 0 \)), as a function of detuning and \( \gamma \). (c-e) Bloch band structures for \( M = 0 \) and \( \gamma = 0.1J \) (c), \( 1.5J \) (d) and \( 3.5J \) (e).

FIG. 1. (a) Unit cell of the lattice, consisting of two sublattices indicated with blue and red discs. Each sublattice connect the nearest and the next nearest node with hopping strength \( J \). The hopping phase between the two sublattices is \( \phi = \pm \pi/4 \). The two sublattices have loss \( \gamma \), pump rates \( P_{a,x,y} \) and \( P_{b,x,y} \), and detunings \( \pm M \) with respect to a reference frequency. (b) Phase diagram of the Bloch band structure of the linear tight binding Hamiltonian with symmetric gain/loss (\( P_{a,x,y} = 2\gamma \), \( P_{b,x,y} = 0 \)), as a function of detuning and \( \gamma \). (c-e) Bloch band structures for \( M = 0 \) and \( \gamma = 0.1J \) (c), \( 1.5J \) (d) and \( 3.5J \) (e).

tical field dynamics in this lattice are well-approximated by the tight binding Hamiltonian

\[
\hat{H} = \sum_{x,y} (\hat{H}_a + \hat{H}_b + \hat{H}_{ab} + \hat{H}_{ab}^\dagger),
\]

\[
\hat{H}_a = \hat{a}_x^\dagger (M + i P_{a,x,y} - \gamma) \hat{a}_{x,y} + J_0 \sum_{x,y} \hat{a}_{x,y} \hat{b}_{x,y} + \hat{a}_{x,y}^\dagger \hat{b}_{x,y},
\]

\[
\hat{H}_b = \hat{b}_x^\dagger (-M + i P_{b,x,y} - \gamma) \hat{b}_{x,y} + J_0 \sum_{x,y} \hat{b}_{x,y} \hat{a}_{x,y} + \hat{b}_{x,y}^\dagger \hat{a}_{x,y},
\]

\[
\hat{H}_{ab} = J e^{i\pi/4} \left( \hat{a}_x^\dagger (\hat{b}_y + \hat{b}_{y+1,y+1}) + \hat{b}_x \hat{a}_{x-1,y} + \hat{b}_{x,y}^\dagger (\hat{a}_{x,y} + \hat{a}_{x,y+1}) \right),
\]

where \( \hat{a}_x^\dagger (\hat{a}_{x,y}) \) and \( \hat{b}_x^\dagger (\hat{b}_{x,y}) \) are photon creation (annihilation) operators for the \( a \) and \( b \) sublattice sites in unit cell \((x,y)\), \( J \) is the coupling strength, \( M \) is the sublattice detuning, \( P_{a,b} \) is the normalized local pump rate, and \( \gamma \) accounts for scattering losses. As \( \hat{H} \) is non-Hermitian, its eigenvalues \( \omega \) are in general complex, with the real and imaginary parts describing the modal frequencies and small signal gain, respectively. We use a class A laser model to describe the saturation of gain at high intensities, such that classical optical field amplitudes evolve according to

\[
i \partial_t |\psi_{x,y}|^2 = \left( \hat{H} - i P_{a,b} |\psi_{x,y}|^2 / I_{\text{sat}} \right) |\psi_{x,y}|^2, \quad |\psi_{x,y}|^2 = a, b
\]

where \( I_{\text{sat}} \) is the saturation intensity, which we normalize to 1 without loss of generality. We will start by analyzing (2) in the linear limit, which describes the initial build up of the lasing modes before the onset of gain saturation.

First we consider a bulk pumping configuration similar to that previously used for the Su-Schrieffer-Heeger model \([9, 11, 14, 15]\), where all sites belonging to one sublattice are pumped and we assign an equal amount of loss to the other sublattice, i.e. setting \( P_{a,x,y} = 2\gamma \) and \( P_{b,x,y} = 0 \). We compute the bulk spectrum by Fourier transforming the tight binding Hamiltonian into momentum \( k = (k_x, k_y) \) space, yielding the Bloch Hamiltonian

\[
\hat{H}(k) = \begin{pmatrix} H_a & H_{ab} \\ H_{ab}^* & H_b \end{pmatrix},
\]

\[
H_a = 2J \cos k_y + M + i\gamma,
\]

\[
H_b = 2J \cos k_x - M - i\gamma,
\]

\[
H_{ab} = J e^{i\pi/4} (1 + e^{i(k_x+ky)}) + Je^{-i\pi/4} (e^{i k_x} + e^{i k_y}).
\]

The eigenvalues of \( \hat{H}(k) \) form two bands, which are classified in Fig. 1(b) as a function of the gain/loss contrast \( \gamma \) and the sublattice detuning \( M \). Increasing the gain/loss contrast induces a narrowing of the bulk band gap. At strong pump rates the coupling \( J \) becomes negligible and the modes of each band are strongly localized to a single sublattice, corresponding to a gap in the imaginary part of the eigenvalue spectrum. If the lattice is initially in the topological phase \( M < 2J \), at moderate pump rates there is an intermediate phase in which both the real and imaginary parts of the energy eigenvalues are gapless, exhibiting non-Hermitian degeneracies at critical moments, similar to the Su-Schrieffer-Heeger model \([14, 15]\).

Representative examples of the band structures in the different phases are shown in Fig. 1(c-e), revealing that the dominant modes, i.e. those with the highest linear gain, are insensitive to the opening and closing of the band gap and always occur at the X points \( k_0 = (0, \pi) \) and \( (\pi, 0) \), where \( H_{ab} = 0 \). More generally in two-dimensional topological models, regardless of sublattice polarization of a uniform pump there will always be some bulk Bloch modes which overlap perfectly with the pump and have the highest gain. Thus, additional spatial structuring of the pump profile is necessary to achieve edge mode lasing.

Next, we consider the spectrum of moderately-sized finite lattices where only the edge resonators are pumped, which can support either bulk or edge mode lasing, i.e. \( P_{a/b,x,y} = P_{\text{edge}} \), where \( x, y \) denote edge sites. Two different edge configurations are of interest: the edges can be terminated always with elements of the same sublattice \((a \text{ or } b)\), or such that the \( C_4 \) discrete rotational symmetry of the lattice is preserved, corresponding to the horizontal edges being terminated with \( a \) sites, and the vertical edges with \( b \) sites. In this work, we focus on the latter case shown in Fig. 2(a), as the single sublattice termination typically exhibits strong competition between bulk and non-topological corner modes. We characterize the dominant lasing mode as a function of the detuning and pump rate in Figs 2(b,c) via their frequency and partic-
To the lasing modes' frequency approaching the bulk band as a function of detuning. (d) Frequency bifurcations of the dominant mode delocalize into the bulk. NE: non-topological (trivial) edge state lasing. TD: topological edge modes starting to lase with single dominant mode. TT: two mode topological edge modes. Different lasing regimes are shown; TS: topological edge mode lasing with single dominant mode. TT: two mode topological edge state lasing. TD: topological edge modes starting to delocalize into the bulk. NE: non-topological (trivial) edge modes. For small pump rates and detuning, topological edge modes have the highest gain [region marked TS in Fig. 2(b,c)]. A representative mode profile in this region is shown in Fig. 3(a). As the detuning increases, these modes gradually delocalize and penetrate into the bulk [TD, Fig. 3(c)], corresponding to a low participation number. Interestingly, this delocalization starts to occur even before the bulk topological transition at $M = 2$, due to the lasing modes’ frequency approaching the bulk band edge. Above $M = 2$ non-topological edge modes lase first [NE, Fig. 3(d)], before bulk modes become dominant at higher detunings [Fig. 3(e)]. When both detuning and pump rate are high, the dominant modes are confined to the corners, corresponding to the maximal participation number [Fig. 3(f)].

Before merging into bulk modes, the dominant topological edge modes also exhibit frequency bifurcations. This bifurcated region is shown with label TT in Figs. 2(b,c). In this regime, two topological edge modes share the same maximum gain. Fig. 2(d) shows the frequency of the dominant mode as a function of detuning for $P = 0.5J$ and $2J$. The dominant edge mode at low detuning has zero frequency, while in the bifurcated regime, the frequency splits. Interestingly, this bifurcation occurs close to the critical detuning where the bulk band gap is maximized, corresponding to the strongest robustness of the edge modes to disorder. Evidently, this protection against disorder does not translate to protection of single edge mode lasing. For moderate gain $P_{\text{edge}} = 0.5J$, above $M = 2J$, the transition to the bulk mode lasing is also apparent as a discontinuity in the lasing frequency as a function of $M$.

We can understand this bifurcation by inspecting the lasing mode profiles in Fig. 3. Before the bifurcation the modal profile shown in Fig. 3(a) resides on both sublattices, experiencing net gain along the entire edge. Small detunings $M$ redistribute the intensity between the sublattices while preserving the mid-gap lasing frequency $\omega = 0$. After the bifurcation, the lasing modes become asymmetric, see e.g. Fig. 3(b). The positive $\omega$ mode preferentially excites the $a$ sublattice, resulting in net gain while propagating along the top and bottom edges and net loss along the left and right edges. The negative $\omega$ mode, preferentially localized to the $b$ sublattice, exhibits the opposite behaviour. With increasing pump rate and detuning these profiles continuously evolve either into corner-like modes similar to Fig. 3(b) or they delocalize into the bulk.

To verify the stability of the dominant lasing modes and the ability to observe these bifurcations in experiment, we perform numerical simulations of the full nonlinear equations of motion (2) using the 4th order Runge-Kutta method, initializing the field at time $t = 0$ with random noise. Fig. 4(a,b) shows the initial dynamics of the optical intensity at one edge resonator, while Fig. 4(c) plots the field’s frequency spectrum in the time interval [100/J, 200/J], after the initial transients have diminished. At low detuning the intensity reaches a steady state, with a single peak in its frequency spectrum centred at zero detuning. Approaching the onset of the bifurcation ($M = 0.6J$), relaxation times get longer due to many modes sharing the same gain, as shown in Fig.
observing in persistent intensity oscillations, Fig. 4(b). These observations are all consistent with the above analysis of the linear spectrum.

There are various ways one could suppress this bifurcation and preserve single edge mode lasing. First, the bifurcation emerges as a transition from net gain along the entire edge, to net gain only in localized regions. Therefore, one of the modes can be suppressed by pumping only one pair of edges, at the expense of increasing the lasing threshold. Alternatively, one could consider spatial modulation of the sublattice detuning \( M \), keeping \( M \) small at the edge sites to maintain strong overlap between the edge mode and the pumped sublattice, and using larger values of \( M \) in the bulk to maximize the band gap and more strongly localize the topological modes to the edge, reducing the lasing threshold. Finally, for sufficiently large detunings \( M \) a narrow band gain medium may be sufficient to isolate one of the modes.

In summary, we have analyzed the lasing modes of a tunable two-dimensional topological resonator lattice. Depending on the frequency detuning between its two sublattices and the pump strength, the lattice’s dominant lasing modes can vary from edge modes to bulk and corner modes. Even in the topological phase, the topological edge state lasing can exhibit bifurcations between single mode lasing at the middle of the band gap, and two mode lasing at frequencies symmetric about the gap center, arising due to the interplay between the sublattice detuning and the pumping profile. Simulations of a class A laser model close to its lasing threshold are consistent with our analysis of the system’s linear modes.

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**FIG. 4.** (a) Intensity at a selected edge site as a function of time for detunings \( M = 0, 0.6 \) and 1.5. (b) Zoomed-in intensity of the \( M = 1.5 \) case. (c) Intensity of the Fourier transform of the field amplitude at each lattice site, for various detunings. The bulk band gap is shaded in blue. We have used an edge pump rate \( P_{\text{edge}} = J \) and uniform losses \( \gamma = 0.5J \).
[21] S. Mittal, V. V. Orre, D. Leykam, Y. D. Chong, and M. Hafezi, Phys. Rev. Lett. 123, 043201 (2019)