On a Static solution of Einstein Equations with incoming and outgoing radiation

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Abstract

Einstein equations with $T_{\mu\nu} = k_\mu k_\nu + \ell_\mu \ell_\nu$ where $k, \ell$ are null are considered with spherical symmetry and staticity. The solution has naked singularity and is not asymptotically flat. However, it may be interpreted as an envelope for any static spherical body making it more massive. Such an interpretation and some of its implications are detailed.

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Introduction

Any astrophysical body has a temperature and thus is a source of outgoing radiation. Equally well, every such body also receives an incoming radiation eg. the cosmic background radiation. Depending upon the respective temperatures there will either be a net outgoing or incoming flux of radiation. It is then conceivable that the two rates exactly match and one reaches an equilibrium situation. In such a situation, the stress tensor in the vicinity of such a body may be taken to be of the form

\[ T_{\mu\nu} = k_\mu k_\nu + \ell_\mu \ell_\nu, \quad k^2 = \ell^2 = 0, \quad k.\ell > 0 \quad (1) \]

When the rates are unequal either the \( k \) or the \( \ell \) term may be taken to be dominating and one essentially gets the Vaidya solution (non stationary) \([1]\), or collapsing null fluid shell case. However when the rates are precisely matched, both terms are important and one can look for a static solution. As a first step in this direction of course one can consider the simpler case of spherically symmetric solution.

In the present work we consider such a solution. Some of the salient features of the solution are the following.

Let \( r \) denote the usual Schwarzschild radial coordinate.

1. As \( r \to 0 \) the solution has a curvature singularity which is naked (i.e. no event horizon).

2. As \( r \to \infty \), the metric components go as \( \ln(r) \) and thus the solution is not asymptotically flat.

3. These two features make it difficult to interpret the solution physically. However, one can consider the solution to be valid for \( R \leq r \leq \bar{R} \) range. At \( R \) one can match the solution for a typical interior Schwarzschild solution while at \( \bar{R} \) one can match it with the standard exterior Schwarzschild space-time. The asymptotics of the solution permit such a \((C^0)\) matching. If \( M \) denotes the mass of the interior solution and \( \bar{M} \) denote the mass indicated by the matching at \( \bar{R} \), thus \( \bar{M} > M \) and therefore the mass measured (deduced) from \( r \gg \bar{R} \) is larger than \( M \). One can consider the matching at \( R \) just outside a black hole or even with a negative mass Schwarzschild solution, but \( \bar{M} \) is always positive for sufficiently large \( \bar{R} \).
The paper is organised as follows.

Section 1 contains basic equations which are straightforward to derive. In Section 2, we present analysis of qualitative features of the solution. The asymptotics are discussed and numerical solutions are presented corroborating the qualitative analysis. In Section 3 we discuss some of the possible matchings and summarise our conclusions. The appendix contains a few details of general spherically symmetric and static non-empty space time. A somewhat similar exact solution is also presented and the details of matchings are discussed.

**Section 1: Basic equations**

The basic equations are (Signature + - - - )

\[ R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = 8\pi T_{\mu \nu} \]  

\[ T_{\mu \nu} \equiv \rho k_{\mu} k_{\nu} + \sigma \ell_{\mu} \ell_{\nu}, \quad k^2 = \ell^2 = 0, \quad k.\ell > 0. \]  

where \( k^{\mu}, \ell^{\mu} \) vector fields represent massless radiation outgoing and incoming respectively.

Clearly \( g^{\mu \nu} T_{\mu \nu} = 0 \) and therefore \( R \) term can be dropped. Using spherical symmetry and staticity we write

\[ ds^2 = F(r) dt^2 - G(r) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

The killing vectors are:

\[ \frac{\partial}{\partial t} \rightarrow \text{generating time translations} ; \]

\[ \xi(\alpha, \beta, \gamma) \equiv \xi^\theta \partial_\theta + \xi^\phi \partial_\phi \quad \text{where}, \]

\[ \xi^\theta = \alpha \sin \phi + \beta \cos \phi \]

\[ \xi^\phi = \gamma + \cot \theta (\alpha \cos t\phi - \beta \sin \phi), \]

generating isometries implied by spherical symmetry.
It follows immediately that

\[ \mathcal{L}_\xi T_{\mu\nu} = 0 \iff \mathcal{L}_\xi k_\mu = \mathcal{L}_\xi \ell_\mu = 0 \]  

(6)

Combining with \( k^2 = \ell^2 = 0 \) one gets

\[
\begin{align*}
  k^\mu &= k^0 (1, \sqrt{\frac{F}{G}}, 0, 0), & \ell^\mu &= \ell^0 (1, -\sqrt{\frac{F}{G}}, 0, 0) \\
  k^0, \ell^0 > 0 & & 2k^0\ell^0 F > 0
\end{align*}
\]

(7)

Here \( \mu = 0, 1, 2, 3 \leftrightarrow t, r, \theta, \phi \) respectively. Since \( k^0, \ell^0 \) are arbitrary at this stage, we absorb \( \rho \) and \( \sigma \) in \( k^0, \ell^0 \) respectively.

We can actually solve for \( k^0 \) and \( \ell^0 \) using the conservation equations.

\[
\nabla_\nu T^{\mu\nu} = 0 \Rightarrow (k^\mu \nabla \cdot k + k \cdot \nabla k^\mu) + (\ell^\mu \nabla \cdot \ell + \ell \cdot \nabla \ell^\mu) = 0.
\]

(8)

Evaluating the covariant derivatives etc. shows that each bracket is separately zero implying that \( k^\mu \) and \( \ell^\mu \) integral curves are geodesics (non affinely parametrised). Furthermore, one gets

\[
\begin{align*}
  k^0 &= \frac{B_+}{r F}, & \ell^0 &= \frac{B_-}{r F}, \quad \text{where } B_\pm \text{ are constants.} \\
  k^0, \ell^0 > 0 & & 2k^0\ell^0 F > 0
\end{align*}
\]

(9)

Therefore,

\[
\begin{align*}
  k^\mu &= \frac{B_+}{r F} \left(1, \sqrt{\frac{F}{G}}, 0, 0\right), & \ell^\mu &= \frac{B_-}{r F} \left(1, -\sqrt{\frac{F}{G}}, 0, 0\right)
\end{align*}
\]

(10)

The nonvanishing components of \( T_{\mu\nu} \) then are

\[
\begin{align*}
  T_{00} &= \frac{B_+^2 + B_-^2}{r^2}, & T_{11} &= \frac{G}{F} T_{00}, & T_{01} &= \sqrt{\frac{G}{F}} \left(\frac{B_+^2 - B_-^2}{r^2}\right)
\end{align*}
\]

(11)

For spherically symmetric, static metric \( R_{01} \) is zero and therefore \( B_+^2 = B_-^2 \) and since \( k^0, \ell^0 \) are both positive we take \( B_+ = B_- \equiv B \). Then,

\[
\begin{align*}
  T_{00} &= \frac{A}{r^2}, & T_{11} &= \frac{A}{r^2} \frac{G}{F} \\
  k^0 &= \ell^0 = \frac{B}{r F}, & A &\equiv 2B^2
\end{align*}
\]

(12)

(13)
Note that in the geometrised units we are using $T$ has dimensions of $(\text{length})^{-2}$ and thus $A$ is dimensionless. Since $T_{00} \sim \frac{1}{r^2}$ we see both that we cannot have asymptotic flatness and that there will be a curvature singularity at $r = 0$.

It is straightforward to compute $R_{\mu\nu}$. Out of the 10 Einstein eqns. the eqns for $(\mu\nu) = 02, 03, 12, 13, 23$ are identically satisfied. The 01 equation has already been used to set $B_+ = B_-$. The 22 and 33 equations are identical. This leaves us with 3 nontrivial equations. Setting $f \equiv \sqrt{(F)}$ $g \equiv \sqrt{(G)}$, the equations are:

\[(00) : g^{-1} \left( \frac{L}{g} \right)' + 2r^{-1} \frac{L}{g} = 8\pi f^{-1} \frac{A}{r^2} \]
\[(11) : -f^{-1} \left( \frac{L}{g} \right)' + 2r^{-1} \frac{g'}{g} = 8\pi g f^{-2} \frac{A}{r^2} \]
\[(33) : -f^{-1} \frac{r'}{g} + g^{-1} \frac{g'}{g} - r^{-1} \left( \frac{1}{g^2} - 1 \right) = 0 \]

\[\text{Remarks} \]

1. $A = 0$ gives the standard Schwarzschild case. The 33 eqn is independent of $A$ and is invariant under constant rescaling of $f$.

2. Under $f \rightarrow \lambda f$ the 00 and 11 equations retain their form but with $A \rightarrow A/\lambda^2$.

3. All the equations are also invariant under $r \rightarrow \lambda r$ and therefore there is no intrinsic scale available at the level of the equations. This has to be provided by physical boundary conditions. This is of course true for Schwarzschild case as well (indeed whenever the stress tensor is traceless and has no dimensionful parameters).

4. By suitable combinations at 00 and 11 equations we get one equation which contains $A$ dependent term while another one which does not contain explicit $A$ dependence. The $A$ independent combination is a second order differential equation while $A$ dependent one is a first order equation. The 33 equation is 1st order.

It is straightforward to verify that the second order equation is automatically satisfied if the two first order equations are satisfied.
Defining \( r = \mu_0 e^\xi \), \( \mu_0 \) an arbitrary scale and \( \lambda \equiv 8\pi A \), we write the equations in terms of \( F \) and \( G \), as:

\[
\frac{dF}{d\xi} = \lambda G + F(G-1) \tag{15}
\]

\[
\frac{dG}{d\xi} = \lambda \frac{G^2}{F} - G(G-1) \tag{16}
\]

\[
\frac{2}{F} \frac{d^2 F}{d\xi^2} - \left( \frac{1}{F} \frac{dF}{d\xi} \right)^2 - \frac{dF}{F} \frac{dG}{d\xi} \frac{Gd\xi}{G} - \frac{2}{G} \frac{dG}{d\xi} = 0 \tag{17}
\]

The last equation is the second order equation which is identically satisfied if the first two equations hold. Thus the basic equations to be solved are the first two equations.

To summarise:

\[
ds^2 = F dt^2 - G dr^2 - r^2 d\Omega^2
\]

\[
k^0 = \frac{\sqrt{A/2}}{rF} \quad (= c^0)
\]

\[
k^\mu = k^0(1, \sqrt{\frac{F}{G}}, 0, 0)
\]

\[
l^\mu = k^0(1, -\sqrt{\frac{F}{G}}, 0, 0)
\]

\[
r \equiv \mu_0 e^\xi , \quad \leftrightarrow \frac{d}{d\xi}
\]

\[
F' = \lambda G + F(G-1)
\]

\[
G' = \lambda \frac{G^2}{F} - G(G-1)
\]

The case \( \lambda = 0 \) gives the Schwarzschild solution.

For \( \lambda \neq 0 \), we set \( F' \equiv \lambda \Phi \). The equations for \( G \) and \( \Phi \) then have no \( \lambda \) dependence and are:

\[
\Phi' = G + \Phi(G-1) \quad (A)
\]

\[
G' = \frac{G^2}{\Phi} - G(G-1) \quad (B)
\]

In the next section we will analyse these equations.
Section 2: Analysis of the equations

\[ \Phi > 0, \quad G > 0 \]
\[ \Phi' = G + \Phi(G - 1), \quad G' = \frac{G^2}{\Phi} - G(G - 1) \]  

(18)

Therefore,

\[ G = \frac{\Phi + \Phi'}{\Phi + 1} \]

Eliminating G from the second equation, gives a second order equation for \( \Phi \), namely,

\[ \Phi''\{\Phi(\Phi + 1)\} + \Phi'\{\Phi(\Phi - 2)\} - \Phi^2 - 2\Phi^2 = 0. \]  

(19)

This can be integrated once to give,

\[ \Phi'(1 + \frac{1}{\Phi}) + \Phi - 2\ln(\Phi) - 2\xi = C \]  

(20)

or,

\[ \Phi' = (C - \Phi + 2\ln(\Phi) + 2\xi)\left(\frac{\Phi}{\Phi + 1}\right) \]  

(21)

Substituting in the expression for \( G \), we get the exact equations:

\[ G = \frac{\Phi}{(\Phi + 1)^2}[C + 1 + 2\ln(\Phi) + 2\xi] \quad \text{(solves } G \text{ in term of } \Phi) \]  

(22)

\[ \Phi' = \frac{\Phi}{(\Phi + 1)}[C - \Phi + 2\ln(\Phi) + 2\xi] \quad \text{(1st order equation for } \Phi) \]

The first integral has given us one constant of integrations.

Remarks:

1. \( G \equiv 0, \Phi' = -\Phi \) is an exact solution. This follows from both the original equations for \( \Phi \) and \( G \) and from the above equations. However, this is not an acceptable solution.

2. If \( \Phi' = 0 \) then

\[ C + 2\ln(\Phi) + 2\xi = \Phi \]  

(23)

\[ \Phi'' = \left(\frac{\Phi}{\Phi + 1}\right)' + \left(\frac{\Phi}{\Phi + 1}\right) \left[-\Phi' + \frac{2}{\Phi} \Phi' + 2\right] \]  

(24)

Therefore,

\[ \Phi''|_{\Phi'=0} = 0 + \frac{2\Phi}{\Phi + 1} > 0 \]  

(25)
Thus $\Phi$ has at the most one minimum. The minimum is determined by

$$\Phi(\dot{\xi}) - 2\ln(\Phi(\dot{\xi})) = C + 2\dot{\xi}$$  \hspace{1cm} (26)$$

Therefore if $\Phi(\dot{\xi}) \equiv a$ then $a - 2\ln(a) = C + 2\dot{\xi}$ determines $a$ given $C$ and $\dot{\xi}$.

By adjusting $\mu_0$, we can always choose $\dot{\xi}$ to be zero or any value.

At the minimum of $\Phi$, $\Phi = C + 2\ln(\Phi) + 2\xi$ which implies:

$$G = \frac{\Phi}{(\Phi + 1)^2} [\Phi + 1] = \frac{\Phi}{\Phi + 1} < 1$$

If $\Phi \to 0$ at any finite $\xi_0$ then $\Phi' \to 0$ as $\xi \to \xi_0$ and $G \to 0$ as well. Since at finite $\xi$ we have everything regular, the det $g \neq 0$ and therefore $\Phi$ and $G$ both can not be allowed to go to zero.

Thus, there can be no event horizon at any finite $\xi_0$, and $\Phi$ and $G$ are necessarily $> 0 \ \forall$ finite $\xi$.

Now consider the asymptotics:

Observe that $\Phi$ can not be oscillatory since it can have at the most one extremum. Hence it is either bounded or unbounded as $\xi \to \pm\infty$.

If $\Phi$ has a finite, non-zero limit, then the equation implies that $\Phi' \to \pm\infty$ which is absurd. Therefore $\Phi$ either vanishes or diverges to $\infty$.

If $\Phi \to 0$, then we can approximate the equation for $\Phi'$ as:

$$\Phi' \simeq 2\Phi (\xi + \ln(\Phi)) \Rightarrow$$

$$\ln(\Phi) \simeq #e^{2\xi} - \frac{1}{2} - \xi.$$  \hspace{1cm} (27)

This will go to $-\infty$ provided $#$ is zero and $\xi \to +\infty$. Clearly then as $\xi \to -\infty$, $\Phi$ must diverge to infinity.

As $\xi \to +\infty, \Phi' \to 0$ which in turn implies that $G \to 0$ as well. But $(\Phi G)' = 2G^2$ and therefore $\Phi G$ increases monotonically and hence can not vanish. Thus $\Phi$ it must diverge as $\xi \to +\infty$ as well.

To summarise, $\Phi \to +\infty$ as $\xi \to \pm\infty$ must hold which in turn implies that $\Phi$ must have a minimum.

Consider approximate solution as $\Phi \to \infty (\xi \to \pm\infty)$. Let

$$\eta \equiv \Phi - 2\ln(\Phi) - 2\xi - C.$$  \hspace{1cm} (27)
Therefore
\[ \eta' = \frac{\Phi - 2}{\Phi + 1}(-\eta) - 2 \quad \text{where } \Phi = \Phi(\eta) \quad (28) \]

Expanding in powers of $1/\Phi$,
\[ \eta' = -\eta(1 - \frac{2}{\Phi})(1 - \frac{1}{\Phi} + \frac{1}{\Phi^2} - \frac{1}{\Phi^3} \ldots) - 2 \quad \forall \Phi > 1. \quad (29) \]

For $\Phi >> 1$, $\eta' \approx -\eta - 2 \Rightarrow$
\[ \eta = De^{-\xi} - 2 \quad \Rightarrow \]
\[ \Phi - 2 \ln(\Phi) = C - 2 + 2\xi + De^{-\xi} \quad (30) \]

and this is consistent with $\Phi >> 1$ provided either $\xi \to +\infty$ or $\xi \to -\infty$.

The corresponding asymptotic behaviour for $G$ can be deduced from the behaviour of $\Phi$. The leading behaviours are given below.

For $\xi \to \infty$ : $\Phi \approx 2\xi$
\[ : \quad G \approx 1 + \frac{1}{2\xi} \]

For $\xi \to -\infty$ : $\Phi \approx De^{-\xi}$
\[ : \quad G \approx \frac{D^2}{C + 1 + 2\ln(D)} \]

**Remark:** The $\xi \to -\infty$ behaviour shows that $\Phi G \to constant$. We may choose this constant to be 1 by choosing $D = e^{-C/2}$. The asymptotic form then resembles the form for a negative mass Schwarzschild solution. This is not surprising since as $\xi \to -\infty$, $G \to 0$ and $\Phi \to \infty$, the first terms in the basic equations become negligible and the equations approximate to the standard Schwarzschild equations.

To summarise: The equations can be integrated once exactly to give

1.
\[ G = \frac{\Phi}{(\Phi + 1)^2} \{C + 1 + 2\xi + 2\ln(\Phi)\} \]
\[ \Phi' = \left(\frac{\Phi}{\Phi + 1}\right) \{C + 2\xi - \Phi + 2\ln(\Phi)\} \]

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2. Regularity at finite $\xi \Rightarrow \Phi, G > 0 \forall$ finite $\xi$. Therefore no event horizon is possible.

3. $\Phi$ can have at the most 1 extremum which must be a minimum.

4. $\Phi \to \infty$ as $\xi \to \pm\infty$ and therefore $\Phi$ does have the minimum.

5. $G' > 0$ as $\xi \to -\infty$ while $G' < 0$ as $\xi \to \infty$ and therefore $G'$ must vanish for some finite $\xi$. $G$ then has a unique maximum.

This qualitative picture is corroborated by numerical integration of the $\Phi - G$ equations as shown in the figure below. The specific initial values are chosen for convenience only.

Figure 1: The solution has $G(0) = 1.25, G'(0) = 0.0$

Section 3: Interpretation of the solution

We have four freely specifyable parameters: the arbitrary scale $\mu_0$, the constant of integration $\lambda$ coming from the conservation equation, the constant $C$ and $a \equiv \Phi(0)$ (say). After the substitution $F = \lambda \Phi$, the $\lambda$ drops out of all differential equations. It still appears in the time components of the vector field $k, \ell$ and the 00 component of the stress tensor. It also appears in the metric as a coefficient of the $dt^2$ term. By rescaling time we can remove it
from the metric and from the vector fields $k, \ell$. Clearly then its value cannot have any physical meaning. In effect we take $\lambda = 1$.

The scale $\mu_0$ on the other hand appears non trivially in the following sense. If we take any stationary observer with four velocity $u^\mu = \xi^\mu/\sqrt{\Phi}$, then the energy density measured in his/her rest frame is given by,

$$\rho_u(r) = T_{\mu\nu}u^\mu u^\nu = \frac{\frac{1}{8\pi^2 \Phi(r)}}{\frac{1}{8\pi \mu_0^2 \Phi(\xi)}}$$

(31)

How could a scale be chosen? As noted earlier, our solution has a curvature singularity as $r \to 0$. So a natural approach is to consider the solution to be valid for $r \geq R$ for some $R$. This $R$ then provides a natural scale ($\mu_0 \equiv R$). We also noted earlier that the $\xi$ going to $-\infty$ behaviour resembles that of a negative mass $(-m)$ Schwarzschild solution. The constant $D$ then equals $2m/\mu_0$. The parameter $2m$ then provides a natural scale. In either of the cases, a natural scale $\mu_0$ can be chosen. It remains now to choose $C$ and $a$.

If matching with negative mass Schwarzschild solution is considered then the choice $\mu_0 = 2m$ gives $D = 1$ or $C = 0$. The constant $a$ is left unconstrained (> 0).

A more “realistic” matching is to choose an $R$, the radius of some physical body and match our solution with an interior Schwarzschild solution (perfect fluid case for instance)[3]. For an interior solution, the function $G$ is expressed in terms of a mass function $M(r)$

$$M(r) \equiv 4\pi \int_0^r \rho(r')(r')^2 dr'$$

and the equations are integrated (usually numerically). The $\Phi$ function is trivially determined once $\rho(r)$ is determined (The equation of state gives the pressure $P(r)$). The matching is minimally required to have $F, G$ and $F'$ to be continuous across the matching surfaces (See the appendix for details). The continuity of $\Phi$ across $R$ can always be ensured trivially since for the interior $F$ there is a constant of integration which can always be adjusted.

A physical body provides the following physical data namely, the physical radius $R$ which gives $\mu_0$; the physical mass $M$ (in the absence of the radiation shell) which gives $G(R) = (1 - 2M/R)^{-1}$; and the radiation density measured by a stationary observer in his/her rest frame $\rho_u(R)$ which gives
As discussed in the appendix, \( \rho_u(R) = P(R) \) because of continuity of \( \Phi' \). Thus all the data needed for specifying a particular solution is available.

Thus for interior matching, we take:

\[
\begin{align*}
  b &\equiv G(0) = \frac{1}{1 - \frac{2M}{R}} \\
  a &\equiv \Phi(0) = \frac{8\pi G\rho}{R^2}
\end{align*}
\]  

(32)

Given \( a \) and \( b \), the constant \( C \) is given by,

\[ C = \frac{b(a + 1)^2}{a} - 1 - 2 \ln(a) \]  

(33)

This gives a method of choosing the constants of integration in a given physical context, thus determining the solution appropriate for the context.

The solution so determined is to be evolved up to some finite \( \bar{\xi} \) as we do not have asymptotic flatness. At this point we would like to match our solution to an exterior Schwarzschild solution. As discussed in the appendix, it is not possible to do so while maintaining the continuity of \( \Phi' \) and a “regularising” thin layer must be added on. This can be done. Suppressing the “thickness” of the regularising layer we see that the continuity of \( G \) provides us with an \( \bar{M} \). However, since \( \Phi(\bar{\xi}) \) is not equal to \( G^{-1}(\bar{\xi}) \) the exterior \( F \) function will go to \( \Phi(\bar{\xi}) \) as \( \xi \) goes to infinity. The mass given by the Komar integral will then have a normalization such that this mass is given by the \( \bar{M} \).

Thus for exterior matching at \( \bar{\xi} \), we set

\[ G(\bar{\xi}) = \frac{1}{1 - \frac{2\bar{M}}{R} e^{-\bar{\xi}}} \]  

(34)

It follows then,

\[ \frac{\bar{M}}{M(R)} = \left[ \frac{G(\bar{\xi}) - 1}{G(0) - 1} \frac{G(0)}{G(\bar{\xi})} \right] e^{\bar{\xi}} \]  

(35)

Mass of such a body will be larger by factors, from the mass it would have had in the absence at the radiation shell. (equivalently from the mass determined from the interior dynamics).

To get a feel, let us put in some numbers. Let \( \Lambda_T \) denote the energy density of the background radiation at temperature \( T \). It is given by,

\[ \Lambda_T \sim 10^{-15} \times T^4 \text{ ergs/cm}^3 \]
The \( \rho_u(\xi) \) on the other hand is given in conventional units by,

\[
\rho_u(\xi) = \left( \frac{c^4}{G_{\text{Newton}}} \right) \left( \frac{1}{8\pi R^2} \right) \left( e^{-2\xi} \right)
\]

Therefore,

\[
e^\xi \sim \left( \frac{c^4}{8\pi G_{\text{Newton}} \Lambda_T} \right)^{1/2} \left( \frac{1}{R \sqrt{\Phi T^2}} \right)
\]

Or,

\[
e^\xi \sim (10^{31}) \left( \frac{1}{R \sqrt{\Phi T^2}} \right)
\]

Putting \( \xi = 0 \) gives,

\[
a \sim (10^{62}) \left( \frac{1}{R^2 T^4} \right)
\]

For white dwarf (say), \( b \) is typically about \( 1 + 10^{-4} \) [3]. Typical white dwarf radius is about \( 10^9 \) cm. An astrophysical body such as a white dwarf could not be expected to have been formed in the earlier epochs and thus the background temperature can not be larger that about \( 10^4 \). \( a \) is then about \( 10^{28} \).

Notice that in this case \( a(b-1) \gg b \). As \( \xi \) is increased \( a \) increases and \( b \) decreases, relatively slowly, maintaining the inequality. Thus the first terms in both of the basic equations are negligible. But then the equations approximate the usual Schwarzschild case and \( M(\xi) \) read off from \( G \) will be essentially a constant i.e. \( \bar{M}/M \) will be very close to 1. Numerical corroboration of this is shown in the figure below.

It is also clear that to get significant deviations from the usual Schwarzschild case one must have \( a(b-1)/b \) to be comparable to 1 or less than 1. In any astrophysical context (excluding black holes), the \( T \) would be about the same order while \( b-1 \) continues to be not too small. Only way then to get a deviation is to reduce \( a \) i.e. increase \( R \).

Indeed if we take a spherical galaxy to be the inner body then \( R \sim 10^{22} \) cms. (\( 10^5 \) light years), the mass \( M \) is about \( 10^{12} \) solar mass giving \( b-1 \sim 10^{-4} - 10^{-5} \). \( a \) is about 100 and significant deviations from the usual Schwarzschild solution can be expected. The figure below corroborates this expectation. The ratio \( \bar{M}/M \) is several orders of magnitudes bigger than 1 even for \( \bar{R}/R \) slightly larger than 1 ! The radiation shell will of course be dark since it
will be merged with the background radiation. The shell then seems to be a
candidate for dark matter at least in some cases.

What about the core being a black hole? Clearly we can not match our
solution at the horizon but we could try matching outside the horizon. A
black hole can provide an out going flux only via the Hawking mechanism.
Since we are using stationary observers to get the value of \( a \), the energy den-
sity should be that corresponding to the local temperature. The temperature
is then given by,

\[
T = 10^{-6} \frac{M_\odot}{M} \sqrt{b}
\]

Setting \( \alpha \equiv M/M_\odot \) and \( \beta \equiv R/R_{\text{Schwarzschild}} \), one gets,

\[
a(b-1)/b \sim 10^{76} \alpha^2 \frac{(\beta - 1)^2}{\beta^5}
\]

For \( R \) not too large (\( \beta \) of the order of 10, say) deviations are possible only
for extremely light black holes!

Remark: Quite apart from these numbers, the precise matching of the
two rates is fragile though. As the universe expands, the incoming rate will
decrease and the black hole will begin loosing its mass, thereby increasing its
out going rate and preventing return to equilibrium. The net result of the
earlier equilibrium is perhaps to delay the evaporation process.
The similarity of our solution to a negative mass Schwarzschild solution for \( r \) close to zero suggests a purely speculative possibility of taking the core to be a “negative mass” body. Of course no such body is known! If at all it “exists” one could only imagine a quantum origin. All the scales may then be taken to be Planckian. \( \Phi \) then is of the order of 1. Numerical solution then indicates that the \( \bar{M} \) becomes positive for \( \bar{R}/R \) greater than about 2 - 10.

Though spherical symmetry, staticity and the particular form of the stress tensor are obvious idealizations, one can still observe the following:

1. The stress tensor satisfies all the usual energy conditions and as such is a physically possible/admissible one. The Einstein equations then lead us to a solution discussed above.

2. Apart from the role of providing incoming radiation, the background radiation or its temperature appears explicitly in providing one of the constants of integration, \( \Phi(0) \). The solution then seems best interpreted as a radiation shell near the surface of a spherical static body. This shell however contribute to the mass significantly only for bodies with sizes on the galactic scale. The effective darkness of such bodies indicates a possibility for dark matter at least in some cases.
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Appendix

We collect here a few details of the most general equations for static, spherically symmetric non-empty space times which are useful for discussion of the matchings involved.

Define the orthonormal set of vectors:

\[
\begin{align*}
\epsilon^\mu_0 &= \frac{1}{\sqrt{F}}(1, 0, 0, 0); \\
\epsilon^\mu_1 &= \frac{1}{\sqrt{G}}(0, 1, 0, 0); \\
\epsilon^\mu_2 &= \frac{1}{r}(0, 0, 1, 0); \\
\epsilon^\mu_3 &= \frac{1}{\sin(\theta)}(0, 0, 0, 1).
\end{align*}
\]

Any $T^\mu\nu$ can then be expressed as,

\[
T^\mu\nu = \rho_{ab} \epsilon^\mu_a \epsilon^\nu_b, \quad \rho_{ab} = \rho_{ba}.
\]

Spherical symmetry and staticity ($R_{01} = 0$) implies that,

\[
\rho_{ab} = \text{diag}(\rho_0(r), \rho_1(r), \rho_2(r), \rho_3(r)),
\]

with $\rho_3(r) = \rho_2(r)$.

The conservation equations give a single equation:

\[
\frac{d\rho_1}{dr} + \frac{2(\rho_1 - \rho_2)}{r} + \frac{(\rho_0 + \rho_1) dF}{2F} \frac{dr}{dr} = 0.
\]

Some special cases are:

- **Perfect fluid**: $\rho_0 \equiv \rho$, $\rho_1 = \rho_2 \equiv P$
- **Reissner-Nordstrom**: $\rho_0 = \frac{Q^2}{r^2}$, $\rho_0 = \rho_2 = -\rho_1$
- **Present case**: $\rho_2 = 0$, $\rho_1 = \rho_0 \equiv \rho$
The Einstein equations can be organised as explained in section 1 to get a set of three first order differential equations as:

\[
\frac{dF}{dr} = F(G - 1) + (8\pi r^2 F \rho_1)G \quad \ldots \quad (a)
\]

\[
\frac{dG}{dr} = -G(G - 1) + (8\pi r^2 F \rho_0)G^2 \quad \ldots \quad (b)
\]

\[
\frac{d\rho_1}{dr} = 2(\rho_2 - \rho_1) - \frac{\rho_0 + \rho_1}{2F} \frac{dF}{dr} \quad \ldots \quad (c)
\]

Defining,

\[
\sigma_a \equiv 8\pi r^2 F \rho_a \quad a = 0, 1, 2
\]

and using the dimensionless variable \(\xi \equiv \ln(r/\mu_0)\), we get:

\[
\frac{dF}{d\xi} = F(G - 1) + \sigma_1 G \quad \ldots \quad (a)
\]

\[
\frac{dG}{d\xi} = -G(G - 1) + \sigma_0 G^2 \quad \ldots \quad (b)
\]

\[
\frac{d\sigma_1}{d\xi} = 2\sigma_2 - \frac{\sigma_0 - \sigma_1}{2F} \frac{dF}{d\xi} \quad \ldots \quad (c)
\]

Mathematically we have an underdetermined system of equations with \(\sigma_0\) and \(\sigma_2\) (say) as freely specifiable function. Physically of course the \(\sigma_a\)’s are to be determined by the dynamics of the matter constituents eg. Maxwell equations for the Reissner-Nordstrom case, equation of state for the perfect fluid case and modeling in our case.

Usually one notes that the (6b) equation does not involve \(F\) and solves this equation in terms of,

\[
M(r) \equiv 4\pi \int \rho_0(r')r'^2 dr' , \quad G \equiv (1 - \frac{2M(r)}{r})^{-1}
\]

The (6c) equation which involves only \(\rho_a\)’s is then solved and (6a) equation can then be trivially integrated.

Since in our case equation (6c or 7c) are trivially integrated we were able to reduce the remaining equations to a single first order differential equation.

In fact following our way of organising the equations one can construct the following exact solution.

Taking \(\sigma_1 = \sigma_0 \equiv \sigma\) and \(\sigma_2 = -Ae^{-\xi}\) gives us \(\sigma = B + 2Ae^{-\xi}\) and allows the remaining two equations to be reduced to a single first order equation for
F. $A = 0$ reproduces our equation (22) of section 2. However for $B = 0$ and $A > 0$ (to satisfy energy conditions) further integration gives an exact solution ($F = A\Phi, A > 0$):

\[
\begin{align*}
\Phi + 2e^{-\xi} \ln(\Phi) &= C + (D - 2\xi)e^{-\xi}, \\
G &= \frac{C}{(\Phi + 2e^{-\xi})^2} e^{-3\xi}, \\
\rho &= \frac{1}{4\pi\rho_0^2} e^{-3\xi}, \\
\rho_2 &= -\rho/2
\end{align*}
\]

(8)

This solution again has naked singularity and although $\rho$ falls off faster than before, it is still not fast enough to get asymptotic flatness.

Now we address the issue of matching our solution on the interior to a physical body and on the exterior to an exterior Schwarzschild solution.

It is useful to note that the energy flux, as implicit in the Komor integral, depends on derivative of $F$. The extrinsic curvature for the hypersurfaces $t = i$ constant and $r = \text{constant}$ also depend on the derivative of $F$ apart from on $F, G, r$ etc. None depend on the derivative of $G$ though. If the matching are to ensure continuity of any (or all) of these, then one must demand continuity of $F, G$ and $F'$ across the matching spheres. The equations 6a (or 7a) show immediately that such a matching can not be done on the exterior because $\rho_1$ is not continuous across $\bar{R}$. On the interior though such a matching is possible. Note that the matching requires continuity of $\rho_1$ only and NOT of $\rho_0$.

For an interior body described by perfect fluid stress tensor we have,

\[
\frac{dF}{d\xi}|_{\xi=0} = 2G(0) \frac{M(0) + 4\pi R^3 P(0)}{R}.
\]

Putting $a \equiv F(0), b \equiv G(0), d \equiv F'(0)$ and noting that $b = (1 - 2M(0)/R)^{-1}$ we see that

\[
d = a(b - 1) + 8\pi R^2 abP(0) \quad \text{(for interior body)}
\]

while for our solution

\[
d = a(b - 1) + b
\]

Thus the matching implies,

\[
a = \frac{1}{8\pi R^2 P(0)}
\]
By comparing with equation (31) of section 3, we see that \( P(0) = \rho_u(0) \). This fixes the constant of integration for the interior solution as well as provides an initial condition for our equations.

In the absence of the radiation shell, \( R \) is determined precisely by demanding \( P(R) = 0 \). We can not do so. So our choice must be somewhat smaller than the usual value for \( R \).

Recall from section 3 that we chose \( a \) by equating the energy density measured by a stationary observer to the background energy density. In the conventional units, \( \rho_u(0) \) (and hence \( P(0) \)) is about \( 10^{-15} \times T^4 \) which is small enough so that the value of \( R \) will be very close to that determined in the absence of the radiation shell. So for estimate purposes we can use our earlier estimates. The solution thus determined is essentially the same as in the section 3.

The exterior matching is to be examined now because the estimate of \( \bar{M} \) depends crucially on this.

We can modify the matching at \( \bar{\xi} \) used in section 3 by adding a thin “regularising layer” which will match with our solution at \( \bar{R} \) and match with exterior Schwarzschild solution at slightly farther away. This matching of course is to have continuity of \( F' \) as well.

To describe the thin layer, we consider the general equation (7). We retain \( \sigma_2 = 0 \) condition but allow \( \sigma_1 \neq \sigma_0 \). We will choose \( \sigma_1 \) suitably and solve for \( \sigma_0 \) using eqn(7c), i.e.

\[
\sigma_0 = \sigma_1 - 2\sigma_1' \frac{F}{F'}.
\]

We choose \( \sigma_1 \) such that:

\[
\begin{align*}
\sigma_1(\bar{\xi}) &= 1 ; \quad \sigma_1(\bar{\xi} + 2\epsilon) = 0 ; \\
\sigma_1'(\bar{\xi}) &= 0 ; \quad \sigma_1'(\bar{\xi} + 2\epsilon) = 0 .
\end{align*}
\]

This implies that \( \sigma_0(\bar{\xi}) = 1 \) and \( \sigma_0(\bar{\xi} + 2\epsilon) = 0 \). By choosing \( \sigma_1 \) to be monotonically decreasing we can ensure that energy conditions are satisfied. A simple choice is:

\[
\sigma_1(\xi) = \frac{1}{4} e^{-\frac{1}{2} \nu (\xi - \bar{\xi})^2} \left\{ \left( \frac{\xi - \bar{\xi} - \epsilon}{\epsilon} \right)^3 - 3 \left( \frac{\xi - \bar{\xi} - \epsilon}{\epsilon} \right) + 2 \right\} \tag{10}
\]

By taking \( \nu \) large one can control the decrease in \( \sigma_1 \) while by taking \( \epsilon \) small one can make the layer thin. At \( \bar{\xi} + 2\epsilon \) one can match with the exterior Schwarzschild solution and read off mass from the value of \( G \).
Numerical exploration of this thin layer shows that the qualitative conclusions derived in the section 3 do not change. In particular, for stellar scales the radiation shell contributes negligibly while for galactic scales there is significant enhancement of the mass as detected from far away.

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