Induced ∗-Representations and $C^*$-Envelopes of Some Quantum ∗-Algebras

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Abstract. We consider three quantum algebras: the $q$-oscillator algebra, the Podleś sphere and the $q$-deformed enveloping algebra of $su(2)$. To each of these ∗-algebras we associate certain partial dynamical system and perform the “Mackey analysis” of ∗-representations developed in Yu. Savchuk and K. Schmüdgen, Unbounded induced representations of ∗-algebras, Algebr. Represent. Theory, DOI: 10.1007/s10468-011-9310-6. As a result we get the description of “standard” irreducible ∗-representations. Further, for each of these examples we show the existence of a “$C^*$-envelope” which is canonically isomorphic to the covariance $C^*$-algebra of the partial dynamical system. Finally, for the $q$-oscillator algebra and the $q$-deformed $U(su(2))$ we show the existence of “bad” representations.

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Introduction and preliminaries

The aim of this paper is to demonstrate a unified approach to the ∗-representation theory of various quantum algebras based on the techniques developed in [13]. Most of the quantum ∗-algebras (especially those related to non-compact quantum groups) possess unbounded ∗-representations. The main problem in the theory of unbounded ∗-representations is to define and classify the “well-behaved” ∗-representations of a given ∗-algebra. We recall two classical examples.

Example. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra, $G$ be the corresponding simply connected Lie group and $U_\mathbb{C}(\mathfrak{g})$ be the complex enveloping ∗-algebra of $\mathfrak{g}$. A ∗-representation $\pi$ of $\mathfrak{g}$ is called integrable if $\pi = dU$ for some unitary representation $U$ of $G$. If $G \neq \mathbb{R}$ there exists a ∗-representation of $\mathfrak{g}$ which is not integrable and, moreover, cannot be extended to an integrable representation even

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in a larger Hilbert space, see [14]. Already in the case \( G = \mathbb{R}^2, \ g = \mathbb{C}[x_1, x_2] \) the category of all \(*\)-representations of \( g \) is in a certain sense “very large” as shown in [16, Section 9].

**Example.** Let \( W_n \) be the \( n \)-dimensional Weyl algebra. That is, \( W_n \) is a complex \(*\)-algebra generated by self-adjoint elements \( p_i, q_i, \ i = 1, \ldots, n \), satisfying \( [p_i, q_j] = -\delta_{ij}i, \ [p_i, p_j] = [q_i, q_j] = 0 \). A self-adjoint \(*\)-representation \( \pi \) of \( W_n \) is called integrable if \( P_i = \pi(p_i), \ Q_j = \pi(q_j), \ i, j = 1, \ldots, n, \) are self-adjoint and the one-parameter unitary groups \( e^{itP_i}, e^{isq_j} \) satisfy the Weyl commutation relations. Already for \( W_1 \) one can show the existence of “bad representations” and show that the category of all \(*\)-representations is again “very large”, whereas the only integrable \(*\)-representations are sums of copies of the Schrödinger representation.

We investigate the following three \(*\)-algebras in detail: the \( q \)-oscillator algebra \( \mathcal{A}_q \) for \( q > 0 \), the \( q \)-deformed enveloping algebra \( \mathcal{U}_q(su(2)) \), \( q > 0 \), and the Podleś spheres \( \mathcal{O}(S^2_{q^r}) \), \( q \in (0, 1) \), \( r \in (0, \infty) \). The algebras \( \mathcal{A}_q \) and \( \mathcal{U}_q(su(2)) \) are deformations of \( W_1 \) and \( \mathcal{U}_C(su(2)) \) respectively, however, for both these algebras the notion of “integrability” cannot be generalized in a direct way. Instead of this we use the approach from [13], which applies to all three algebras \( \mathcal{A}_q, \ \mathcal{O}(S^2_{q^r}), \ \mathcal{U}(su(2)) \) as well as to their classical analogues. Let \( \mathcal{A} \) denote one of these algebras. The basic idea is to find a natural \( \mathbb{Z} \)-grading \( \mathcal{A}_k, \ k \in \mathbb{Z} \), for \( \mathcal{A} \) such that \( \mathcal{A}_0 = \mathcal{B} \) is commutative. Further, we define the “positive” spectrum \( \hat{\mathcal{B}}^+ \) of \( \mathcal{B} \) as the set of those characters \( \chi \in \hat{\mathcal{B}} \) which satisfy \( \chi(a^*a) \geq 0 \) for all \( a \in \mathcal{A} \) such that \( a^*a \in \mathcal{B} \). The group grading of \( \mathcal{A} \) defines a structure of a \(*\)-algebraic bundle in the sense of [5], and there is a canonical partial action \( \alpha \) of \( \mathbb{Z} \) on \( \hat{\mathcal{B}}^+ \).

By means of the partial dynamical system \((\hat{\mathcal{B}}^+, \mathbb{Z}, \alpha)\) we

- define well-behaved \(*\)-representations,
- show that the irreducible ones naturally correspond to the orbits of \((\hat{\mathcal{B}}^+, \mathbb{Z}, \alpha)\),
- construct the dual partial action \( \beta \) on \( C_0(\hat{\mathcal{B}}^+) \) and the partial crossed product \( C^*_\beta \)-algebra \( C_0(\hat{\mathcal{B}}^+) \times_\beta \mathbb{Z} \) in the sense of [3]; using the Woronowicz’s theory of affiliated operators, we establish a Morita equivalence between \( C_0(\hat{\mathcal{B}}^+) \times_\beta \mathbb{Z} \) and \( \mathcal{A} \).

It turns out that every irreducible well-behaved \(*\)-representation of \( \mathcal{A} \) is induced from a one-dimensional representation. This result can be viewed as an analogue of the following theorem by Kirillov (see [6]): Every irreducible unitary representation of a nilpotent Lie group is induced from a one-dimensional representation of a certain subgroup.

In each of three cases the constructed crossed product \( C^*_\beta \)-algebra is of a special kind. Namely, the partial action of \( \mathbb{Z} \) on \( C_0(\hat{\mathcal{B}}^+) \) is generated by a single partial automorphism \( \Theta \), see [3,9] and Section 0.2. In this case the partial crossed product \( C^*_\beta \)-algebra coincides with the covariance \( C^*_\beta \)-algebra of the partial automorphism in the sense of [3, Definition 3.7]. In the case \( \mathcal{A} = \mathcal{O}(S^2_{q^r}) \) all \(*\)-representations are bounded, hence well-behaved, and the crossed product \( C^*_\beta \)-algebra \( C_0(\hat{\mathcal{B}}^+) \times_\alpha \mathbb{Z} \) is isomorphic to the enveloping \( C^*_\beta \)-algebra of \( \mathcal{A} \).

Finally, for \( \mathcal{A}_q \) and for \( \mathcal{U}_q(su(2)) \) we show the existence of “bad” representations. More precisely, we prove the existence of a \(*\)-representation which is not
well-behaved and cannot be extended to a well-behaved $*$-representation even in a larger Hilbert space. It generalizes the well-known results for $W_1$ and $U(su(2))$.

Other examples which can be analyzed in the same spirit include various bounded and unbounded $*$-algebras: quantum group algebras $SU_q(2)$, $SU_q(1,1)$, $q$-deformed $U(su(1,1))$, different deformations of CAR and CCR, AF pre-$C^*$-algebras (see [4]) etc.

0.1. $*$-Algebras and $*$-representations. By a $*$-algebra we mean a complex associative algebra $\mathcal{A}$ equipped with a mapping $a \mapsto a^*$ of $\mathcal{A}$ into itself, called the involution of $\mathcal{A}$, such that $(\lambda a + \mu b)^* = \lambda a^* + \mu b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$. The unit of $\mathcal{A}$ (if it exists) will be denoted by $1_{\mathcal{A}}$ or simply by $1$. For every $*$-algebra $\mathcal{A}$ denote by $\sum \mathcal{A}^2$ the set of finite sums $\sum a_i^*a_i$, $a_i \in \mathcal{A}$.

Throughout this paper we use some terminology and results from unbounded representation theory in Hilbert space (see e.g. [16]). We repeat some basic notions and facts. If $T$ is a Hilbert space operator, $\mathcal{D}(T)$, $\overline{T}$ and $T^*$ denote its domain, its closure and its adjoint, respectively. Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$ with scalar product $(\cdot, \cdot)$. A $*$-representation of a $*$-algebra $\mathcal{A}$ on $\mathcal{D}$ is an algebra homomorphism $\pi$ of $\mathcal{A}$ into the algebra $L(\mathcal{D})$ of linear operators on $\mathcal{D}$ such that $\langle \pi(a)\varphi, \psi \rangle = \langle \varphi, \pi(a^*)\psi \rangle$ for all $\varphi, \psi \in \mathcal{D}$ and $a \in \mathcal{A}$. We call $\mathcal{D}(\pi) := \mathcal{D}$ the domain of $\pi$ and write $\mathcal{H}_\pi := \mathcal{H}$. Two $*$-representations $\pi_1$ and $\pi_2$ of $\mathcal{A}$ are (unitarily) equivalent if there exists an isometric linear mapping $U$ of $\mathcal{D}(\pi_1)$ onto $\mathcal{D}(\pi_2)$ such that $\pi_2(a) = U\pi_1(a)U^{-1}$ for $a \in \mathcal{A}$. The direct sum representation $\pi_1 \oplus \pi_2$ acts on the domain $\mathcal{D}(\pi_1) \oplus \mathcal{D}(\pi_2)$ by $(\pi_1 \oplus \pi_2)(a) = \pi_1(a) \oplus \pi_2(a)$, $a \in \mathcal{A}$. A $*$-representation $\pi$ is called irreducible if a direct sum decomposition $\pi = \pi_1 \oplus \pi_2$ is only possible when $\mathcal{D}(\pi_1) = \{0\}$ or $\mathcal{D}(\pi_2) = \{0\}$. For a $*$-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ we denote by $\text{Res}_{\mathcal{B}} \pi$ its restriction to $\mathcal{B}$. The graph topology of $\pi$ is the locally convex topology on the vector space $\mathcal{D}(\pi)$ defined by the norms $\varphi \mapsto \|\varphi\| + \|\pi(a)\varphi\|$, where $a \in \mathcal{A}$. If $\mathcal{D}(\pi)$ denotes the completion the $\mathcal{D}(\pi)$ in the graph topology of $\pi$, then $\overline{\pi}(a) := \overline{\pi(a)} \mid \mathcal{D}(\pi)$, $a \in \mathcal{A}$, defines a $*$-representation of $\mathcal{A}$ with domain $\mathcal{D}(\pi)$, called the closure of $\pi$. In particular, $\pi$ is closed if and only if $\mathcal{D}(\pi)$ is complete in the graph topology of $\pi$. A $*$-representation $\pi$ is called non-degenerate if $\pi(\mathcal{A})\mathcal{D}(\pi) := \text{lin} \{\pi(a)\varphi \mid a \in \mathcal{A}, \varphi \in \mathcal{D}(\pi)\}$ is dense in $\mathcal{D}(\pi)$ in the graph topology of $\pi$. If $\mathcal{A}$ is unital and $\pi$ is non-degenerate, then we have $\pi(1_{\mathcal{A}})\varphi = \varphi$ for all $\varphi \in \mathcal{D}(\pi)$. We say that $\pi$ is cyclic if there exists a vector $\varphi \in \mathcal{D}(\pi)$ such that $\pi(\mathcal{A})\varphi$ is dense in $\mathcal{D}(\pi)$ in the graph topology of $\pi$. For a $C^*$-algebra $\mathfrak{A}$ and a Hilbert space $\mathcal{H}$, denote by $\text{Rep}(\mathfrak{A}, \mathcal{H})$ the category of non-degenerate $*$-representations of $\mathfrak{A}$ on $\mathcal{H}$. By $\text{Rep}\mathfrak{A}$ denote the category of all non-degenerate $*$-representations of $\mathfrak{A}$.

We recall the induction procedure for $*$-representations of general $*$-algebras developed in [13, Section 2] in a slightly more general context. However, we will not perform this procedure but use Proposition 1.2 to get the explicit formulas. Let $\mathcal{B} \subseteq \mathcal{A}$ be $*$-algebras. A linear map $p : \mathcal{A} \to \mathcal{B}$ is called a bimodule projection if $p(a^*) = p(a)^*$, $p(b_1ab_2) = b_1p(a)b_2$, $p(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, for all $a \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$. Let $\rho$ be a $*$-representation of $\mathcal{B}$. Denote by $\mathcal{A} \otimes_{\mathcal{C}} \mathcal{D}(\rho)$ the quotient of $\mathcal{A} \otimes_{\mathcal{C}} \mathcal{B}$ by the linear span of vectors $ab \otimes \varphi - a \otimes \rho(b)\varphi, a \in \mathcal{A}, b \in \mathcal{B}, \varphi \in \mathcal{D}(\rho)$. We say that $\rho$
is \emph{inducible} from $\mathcal{B}$ to $\mathcal{A}$ via $p$ if the sesquilinear form
\begin{equation}
(\sum_k x_k \otimes \varphi_k, \sum_l y_l \otimes \psi_l)_0 := \sum_{k,l} \langle \rho(\rho(y_l^* x_k)) \varphi_k, \psi_l \rangle,
\end{equation}
is positive semi-definite on $\mathcal{A} \otimes _{\mathcal{B}} \mathcal{D}(\rho)$. Denote by $K_\rho$ the kernel of $\langle \cdot, \cdot \rangle_0$. Then $\mathcal{D}_0 = \mathcal{A} \otimes _{\mathcal{B}} \mathcal{D}(\rho)/K_\rho$ is an inner-product space. Define a $*$-representation $\pi$ on $\mathcal{D}_0$ via
\begin{equation}
\pi(a)(\sum_i [a_i \otimes \varphi_i]) := \sum_i [aa_i \otimes \varphi_i],
\end{equation}
where $\sum_i [a_i \otimes \varphi_i] \in \mathcal{D}_0$ denotes the image of $\sum_i a_i \otimes \varphi_i$ under the quotient mapping. Finally define $\Ind \rho$ to be the closure of $\pi$.

Our major application of the induction procedure will be in the following context. Let $G$ be a discrete group, $e \in G$ be the identity element and $\mathcal{A}$ be a $G$-graded $*$-algebra. That is, $\mathcal{A}$ is a direct sum of vector spaces $\mathcal{A}_g$, $g \in G$, such that
\begin{equation}
\mathcal{A}_g \cdot \mathcal{A}_h \subseteq \mathcal{A}_{gh} \quad \text{and} \quad (\mathcal{A}_g)^* \subseteq \mathcal{A}_{g^{-1}} \quad \text{for} \quad g, h \in G. \tag{0.2}
\end{equation}
The elements of $\cup_{g \in G} \mathcal{A}_g$ are called \emph{homogeneous}. For every subgroup $H \subseteq G$ the sum $\oplus_{g \in H} \mathcal{A}_g =: \mathcal{A}_H$ is a $*$-subalgebra of $\mathcal{A}$ and the canonical projection $p: \mathcal{A} \to \mathcal{A}_H$ is a bimodule projection. If $\mathcal{A}_e$ is commutative then a character $\chi: \mathcal{A}_e \to \mathbb{C}$ is inducible (via $p_e: \mathcal{A} \to \mathcal{A}_e$) if and only if $\chi(a^*a) \geq 0$ for all homogeneous $a \in \mathcal{A}$.

\subsection{Partial actions and partial crossed products.}

The constructions and results of this subsection are taken from [3,9]. A \emph{partial action} of a discrete group $G$ on a set $X$ is a pair
\begin{equation}
\alpha = (\{\mathcal{D}_g\}_{g \in G}, \{\alpha_g\}_{g \in G}),
\end{equation}
where $\mathcal{D}_g \subseteq X$, $g \in G$, are subsets and $\alpha_g: \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$ are bijections such that
\begin{enumerate}
\item $\alpha_g(\mathcal{D}_g \cap \mathcal{D}_h) = \mathcal{D}_{gh} \cap \mathcal{D}_g$, $g, h \in G$,
\item $\alpha_{gh}(x) = \alpha_h(\alpha_g(x))$, $x \in \mathcal{D}_{g^{-1}} \cap \mathcal{D}_{g^{-1} h^{-1}}$,
\item $\mathcal{D}_e = X$, $\alpha_e = \text{Id}_X$.
\end{enumerate}

For a partial action $\alpha = (\{\mathcal{D}_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$ on a topological space $X$ we require in addition that $\mathcal{D}_g$ are open sets and $\alpha_g: \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$, $g \in G$, are homeomorphisms. We call $(X, G, \alpha)$ a \emph{partial dynamical system (p.d.s.)}.

For a partial action $\beta = (\{I_g\}_{g \in G}, \{\beta_g\}_{g \in G})$ of $G$ on a $C^*$-algebra $\mathfrak{B}$ we require in addition that $I_g$, $g \in G$, are closed two-sided ideals and $\beta_g: I_{g^{-1}} \to I_g$ are $*$-isomorphisms. We call $(\mathfrak{B}, G, \beta)$ a \emph{partial $C^*$-dynamical system (C$^*$-p.d.s.)}. For a p.d.s. $(X, G, \alpha)$ where $X$ is a locally compact Hausdorff space we define the \emph{dual C$^*$-p.d.s.} as follows. Put $\mathfrak{B} = C_0(X)$, $I_g = C_0(\mathcal{D}_g)$ and define $\beta_g: I_{g^{-1}} \to I_g$ by
\begin{equation}
(\beta_g(f))(x) = f(\alpha_g^{-1}(x)), \quad x \in \mathcal{D}_g, f \in I_{g^{-1}}, \quad g \in G.
\end{equation}
Direct computations show that $\beta = (\{I_g\}_{g \in G}, \{\beta_g\}_{g \in G})$ is a partial action on $\mathfrak{B}$ and that $(\mathfrak{B}, G, \beta)$ is a C$^*$-p.d.s.
Let \((\mathcal{B}, G, \beta), \ \beta = (\{I_g\}_{g \in G}, \{\beta_g\}_{g \in G})\) be a \(C^*\)-p.d.s. The partial crossed product \(C^*\)-algebra \(\mathfrak{A} = \mathcal{B} \times_\beta G\) is the enveloping \(C^*\)-algebra of the \(*\)-algebra \(\mathcal{B}G\) defined as follows. \(\mathcal{B}G \subseteq \mathcal{B} \otimes \mathbb{C}[G]\) is the linear span of the set \(\{a \otimes g \mid a \in I_g\}\), with multiplication and involution defined by

\[(a \otimes g)(b \otimes h) := \alpha_g(\alpha_{g^{-1}}(a)b) \otimes gh, \ \ (a \otimes g)^* := \alpha_{g^{-1}}(a^*) \otimes g^{-1}.
\]

The examples of \(C^*\)-p.d.s. which appear below are of a special kind. Recall [3], that a partial automorphism of a \(C^*\)-algebra \(\mathfrak{A}\) is a triple \(\Theta = (\theta, I, J)\), where \(I, J \subseteq \mathfrak{A}\) are closed two-sided ideals and \(\theta : I \to J\) is a \(*\)-isomorphism. Set \(I_0 = \mathfrak{A}\) and define \(I_n, \ n \in \mathbb{Z}\), by induction

\[I_{n+1} = \{a \in J \mid \theta^{-1}(a) \in I_n\}, \ \text{for} \ n \geq 0,\]
\[I_{n-1} = \{a \in I \mid \theta(a) \in I_n\}, \ \text{for} \ n \leq 0.\]

In particular, \(I = I_{-1}\) and \(J = I_1\). It can be checked, see [3, Section 3], that the triple \((\mathfrak{A}, \mathbb{Z}, \beta)\), where \(\beta = (\{I_g\}_{g \in \mathbb{Z}}, \{\theta^n\}_{n \in \mathbb{Z}})\) is a \(C^*\)-p.d.s. The partial crossed product algebra \(\mathfrak{A} \times_\beta \mathbb{Z}\) is called the covariance algebra of \((\mathfrak{A}, \Theta)\) and is denoted by \(C^*(\mathfrak{A}, \Theta)\). As in the case of a crossed-product by a \(*\)-automorphism, \(*\)-representations of \(C^*(\mathfrak{A}, \Theta)\) are in one-to-one correspondence with covariant representations of the pair \((\mathfrak{A}, \Theta)\), see [3, Section 5]. In case of the \(C^*\)-p.d.s. defined by \((\mathfrak{A}, \Theta)\) a covariant representation \(\pi \times u\) consists of a \(*\)-representation \(\pi : \mathfrak{A} \to B(H)\) and a partial isometry \(u\), whose initial and final spaces are \(\pi(I)\mathcal{H}\) and \(\pi(J)\mathcal{H}\) respectively, so that

\[\pi(\theta(b)) = u\pi(b)u^*\]holds for every \(b \in I\).

If the latter is satisfied, then \(\pi \times u\) becomes a \(*\)-representation of \(\mathfrak{A}\mathbb{Z}\), hence of \(C^*(\mathfrak{A}, \mathbb{Z})\), via

\[(\pi \times u)(f \otimes k) = \pi(f)u^k\]

for \(f \otimes k \in \mathfrak{A}\mathbb{Z}\), where \(u^{-k} = u^k\) for \(k \in \mathbb{N}\).

### 0.3. Unbounded elements affiliated with \(C^*\)-algebras and \(C^*\)-envelopes.

The theory of unbounded elements affiliated with a \(C^*\)-algebra was developed in [18], see also [8]. Let \(\mathfrak{A}\) be a \(C^*\)-algebra and let \(T\) be a densely defined closed linear operator on \(\mathfrak{A}\). Denote by \(D(T) \subseteq \mathfrak{A}\) its domain.\(^1\) The adjoint operator \(T^*\) is defined as follows. For \(y, z \in \mathfrak{A}\), write \(y \in D(T^*)\), \(T^*y = z\) if \((Tx, y) = (x, z)\) holds for all \(x \in D(T)\). Following [8] we say that \(T\) is affiliated\(^2\) with \(\mathfrak{A}\) and write \(T \eta \mathfrak{A}\), if \(D(T^*)\) and the range of \(1 + T^*T\) are dense in \(\mathfrak{A}\), see [8, Chapter 9].

Every non-degenerate \(*\)-representation of a \(C^*\)-algebra \(\mathfrak{A}\) can be continued to the set \(\mathfrak{A}^\mathcal{D}\) of all operators affiliated with \(\mathfrak{A}\). Namely, for every \(\pi \in \text{Rep}(\mathfrak{A}, \mathcal{H})\) and \(T \eta \mathfrak{A}\), there exists a closed operator \(\pi(T) \eta \pi(\mathfrak{A})\) with a core \(\pi(D(T))\mathcal{H}\) such that

\[\pi(T)(\pi(a)\varphi) = \pi(Ta)\varphi, \ \text{for all} \ \varphi \in \mathcal{H}, \ a \in D(T).\]

Moreover, if \(D_0 \subseteq D(T)\) is a core of \(T\), then \(\pi(D_0)\mathcal{H}\) is a core of \(\pi(T)\).

\(^1\)Recall that \(\mathcal{D}(\cdot)\) is domain of a Hilbert space operator.

\(^2\)In [8] the term regular operator on \(\mathfrak{A}\) is used.
Definition 0.1. Let \( \mathcal{A} \) be a \(*\)-algebra with a given category of \(*\)-representations \( \text{Rep}\mathcal{A} \) and fixed generators \( a_1, \ldots, a_n \). We will say that a \( C^* \)-algebra \( \mathfrak{A} \) is a \( C^* \)-envelope of \( \mathcal{A} \) if there exist affiliated elements \( A_1, \ldots, A_n, \eta \mathfrak{A} \) such that
\[
\pi(A_i) = \overline{\rho(a_i)}, \quad i = 1, \ldots, n.
\] (0.3)
defines an equivalence functor \( \rho \mapsto \pi \) between \( \text{Rep}\mathcal{A} \) and \( \text{Rep}\mathfrak{A} \).

Remarks 1. If every \(*\)-representation of \( \mathcal{A} \) is bounded, then there exists the enveloping \( C^* \)-algebra \( C^*_\text{env}(\mathcal{A}) \), which is obviously a \( C^* \)-envelope of \( \mathcal{A} \).

2. In the last definition, the isomorphism class of \( \mathfrak{A} \) depends a priori on the choice of the generators \( a_i \) and of the category \( \text{Rep}\mathcal{A} \). However, we cannot provide any example, where \( \mathfrak{A} \) would depend on the generators \( a_i \).

1. The orbit method

In this section we recall the orbit method developed in [13]. Throughout the section \( G \) is a countable discrete group and \( \mathcal{A} \) is a \( G \)-graded \(*\)-algebra. We assume that the \(*\)-subalgebra \( \mathcal{B} := \mathcal{A}_e \) is commutative and denote by \( \hat{\mathcal{B}} \) the set of all characters of \( \mathcal{B} \) (i.e. nontrivial \(*\)-homomorphisms \( \chi: \mathcal{B} \to \mathbb{C} \)). Further, we define the "positive" spectrum \( \hat{\mathcal{B}}^+ \subseteq \hat{\mathcal{B}} \) to be the set of all characters \( \chi \in \hat{\mathcal{B}} \) which satisfy\(^3\)
\[
\chi(a^*a) \geq 0 \text{ for all homogeneous elements } a \in \mathcal{A}.
\] (1.1)

Lemma 1.1. Assume that for every \( g \in G \) there exists an element \( a_g \in \mathcal{A}_g \) such that \( \mathcal{A}_g = a_g \mathcal{B} \). Then \( \chi \in \hat{\mathcal{B}} \) belongs to \( \hat{\mathcal{B}}^+ \) if and only if \( \chi(a_g^*a_g) \geq 0 \) for all \( g \in G \).

Proof. The "only if" part is clear. Assume that \( \chi(a_g^*a_g) \geq 0 \) for all \( g \in G \). By assumption, if \( c_g \in \mathcal{A}_g \), then \( c_g = a_g b \) for some \( b \in \mathcal{B} \). Hence
\[
\chi(c_g^*c_g) = \chi(b^*a_g^*a_g b) = \chi(a_g^*a_g) \chi(b^*b) \geq 0.
\]

The set \( \hat{\mathcal{B}}^+ \) consists of those characters which satisfy (0.1), i.e. are inducible from \( \mathcal{B} \) to \( \mathcal{A} \) via \( p_e \).

Definition 1.1. For \( g \in G \) define\(^4\)
\[
\mathcal{D}_{g^{-1}} = \left\{ \chi \in \hat{\mathcal{B}}^+ \mid \chi(a_g^*a_g) \neq 0 \text{ for some } a_g \in \mathcal{A}_g \right\}.
\] (1.2)

If \( \chi \in \mathcal{D}_{g^{-1}} \) and \( \chi(a_g^*a_g) \neq 0 \) set
\[
(\alpha_g(\chi))(b) := \frac{\chi(a_g^*b a_g)}{\chi(a_g^*a_g)} \text{ for } b \in \mathcal{B}.
\] (1.3)

\(^3\)The theory developed in [13] requires the additional condition \( \chi(c^*d)\chi(d^*c) = \chi(c^*e)\chi(d^*d) \) for all \( \chi \in \hat{\mathcal{B}}^+, g \in G, c, d \in \mathcal{A}_g \), which holds automatically. It can be checked using the equation \( (c^*cd^*d)^2 = (c^*cd^*d)(c^*cd^*d) \) which follows by commutativity of \( \mathcal{B} \).

\(^4\)In [13] the notation \( \alpha_g: \mathcal{D}_g \to \mathcal{D}_{g^{-1}} \) was used.
Direct computations (see [13, Proposition 13]) show that the pair \( \alpha = (\{ \alpha_g \}_{g \in G}, \{ D_g \}_{g \in G}) \) is a well-defined partial action of \( G \) on \( \hat{B}^+ \). We also use \( \chi^g \) instead of \( \alpha_g(\chi) \). For a character \( \chi \in \hat{B}^+ \) denote by \( \text{Orb}_\chi \subseteq \hat{B}^+ \) its orbit under the partial action of \( G \).

**Proposition 1.2** (see Proposition 16 in [13]). Let \( \chi \in \hat{B}^+ \) and \( \pi = \text{Ind}_\chi \) be the induced \(*\)-representation. For every \( g \in G \) such that \( \chi \in D_{g^{-1}} \) fix an element \( a_g \in A_g \) such that \( \chi(a_g^* a_g) \neq 0 \). Then there exists an orthonormal base \( \{ e_g \mid \chi \in D_{g^{-1}} \} \) in \( D(\pi) \) such that for \( h \in G \) and \( b_h \in A_h \) we have

\[
\pi(b_h)e_g = \frac{\chi(a_h^* b_h a_g)}{\chi(a_g^* a_g)^{1/2} \chi(a_g^* a_g)^{1/2} \chi b_h} \text{ if } \chi \in D_{g^{-1}h^{-1}}
\]

and \( \pi(b_h)e_g = 0 \) otherwise. In particular, if \( b \in B \), we have \( \pi(b)e_g = \chi^g(b)e_g \).

For an element \( b \in B \) define its "Gel'fand transform"

\[
\hat{b} : \hat{B} \to \mathbb{C}, \quad \hat{b}(\chi) = \chi(b), \quad \chi \in \hat{B}.
\]

The set \( \hat{B} \) is equipped with the weak topology defined by \( \{ \hat{b} \mid b \in B \} \) and the Borel structure generated by the open sets. By definition of \( \hat{B}^+ \) it is a closed subset of \( \hat{B} \). It can be checked, that the partial action of \( G \) is topological. That is, \( D_g, g \in G \), are open sets, and \( \alpha_g : D_{g^{-1}} \to D_g \) are homeomorphisms. Since \( G \) is countable and the one-point sets are closed, the \( G \)-orbits are Borel subsets of \( \hat{B}^+ \).

**Definition 1.2.** A closed \(*\)-representation \( \pi \) of \( A \) is called well-behaved if:

(i) there exists a spectral measure \( E_\pi \) on \( \hat{B}^+ \) such that

\[
\overline{\pi(b)} = \int_{\hat{B}^+} \hat{b}(\chi) dE_\pi(\chi) \text{ for } b \in B,
\]

(ii) for all \( a_g \in A_g, g \in G \), and all Borel subsets \( \Delta \subseteq \hat{B}^+ \), we have

\[
\pi(a_g)E_\pi(\Delta) \supseteq E_\pi(\alpha_g(\Delta \cap D_{g^{-1}})) \pi(a_g).
\]

A well-behaved representation \( \pi \) is associated with an orbit \( \text{Orb}_\chi \) if \( E_\pi \) is supported on the set \( \text{Orb}_\chi \). Denote by \( \text{Rep} \mathcal{A} \) the category of all well-behaved representations.

By [13, Proposition 17], relation (1.4) can be replaced with

\[
u_g \int f(t) dE_\pi(t) \subseteq \int_{D_g} f(\alpha_{g^{-1}}(t)) dE_\pi(t) \cdot u_g.
\]

where \( u_g \) is the partial isometry in the polar decomposition \( \pi(a_g) = u_g c_g \), and \( f \) is any measurable function on \( \hat{B}^+ \). If \( f \) is bounded, then "\( \subseteq \)" becomes an equality.

In the next proposition we collect some basic properties of well-behaved representations. For the proof see Propositions 18, 29 and Theorem 7 in [13].
Proposition 1.3. (i) Every bounded \(\ast\)-representation is well-behaved.

(ii) If the partial action of \(G\) on \(\widehat{\mathcal{B}}^+\) possesses a measurable countably separated section, then every irreducible well-behaved representation is associated with an orbit.

(iii) Condition (i) in Definition 1.2 holds automatically if \(\mathcal{B}\) is countably generated, and the restriction of \(\pi\) on \(\mathcal{B}\) is integrable, that is \(\pi(a)\) is normal for all \(b \in \mathcal{B}\).

A measurable set \(\Gamma\) is countably separated if and only if there exist Borel sets \(B_k, k \in \mathbb{N}, \Gamma \subseteq \bigcup_{k \in \mathbb{N}} B_k\) such that for arbitrary \(x, y \in \Gamma, x \neq y\), we have \(x \in B_{k_0}, y \notin B_{k_0}\) for some \(k_0 \in \mathbb{N}\). A subset \(\Gamma\) containing exactly one point from each orbit is called a section of a partial dynamical system.

Recall, that for a subgroup \(H \subseteq G, \oplus_{g \in H} A_g\) is a \(\ast\)-subalgebra of \(A\) denoted by \(A_H\).

Theorem 1.4 (See Theorem 5 in [13]). Let \(\chi \in \widehat{\mathcal{B}}^+\) be a character and let \(H = \text{St}\chi\) be its stabilizer group. Then the map

\[
\rho \mapsto \text{Ind}_{A_H \uparrow A}(\rho) = \pi
\]

is a bijection from the set of unitary equivalence classes of inducible \(\ast\)-representations \(\rho\) of \(A_H\) for which

\[
\text{Res}_{\mathcal{B}}\rho \text{ corresponds to a multiple of the character } \chi
\]

onto the set of unitary equivalence classes of well-behaved representations \(\pi\) of \(A\) associated with \(\text{Orb}\chi\). A \(\ast\)-representation \(\rho\) satisfying (1.6) is bounded and inducible. Moreover, \(\pi\) is irreducible if and only if \(\rho\) is irreducible.

The last theorem suggests the following algorithm for the description of all irreducible well-behaved representations of \(A\):

- determine \(\mathcal{B}, \mathcal{B}^+\), the partial action of \(G\) on \(\mathcal{B}^+\) and a section \(\Gamma \subseteq \mathcal{B}^+\),

- for each \(\chi \in \Gamma\)

  - if the stabilizer \(\text{St}\chi\) is trivial, compute \(\text{Ind}\chi\),

  - otherwise find all irreducible representations \(\rho\) of \(A_{\text{St}\chi}\) satisfying (1.6) and compute \(\text{Ind}\rho\).

If Proposition 1.3 (ii) applies, then we obtain all irreducible well-behaved representations of \(A\).

2. The \(q\)-oscillator algebra

By the quantum harmonic oscillator (\(q\)-oscillator) we mean the relation

\[
aa^* = 1 + qa^*a, \quad q > 0.
\]
In this section, we use the notation of the \( q \)-calculus \([ [k] ]_q = 1 + q + \ldots + q^{k-1}\). Further, we put
\[
F(t) := 1 + qt.
\]
Clearly \( F([ [k] ]_q) = [ [k + 1] ]_q, \ k \in \mathbb{N}_0 \).

In [1] the authors have obtained the following representations of (2.1) by Hilbert space operators:

- For every \( q > 0 \) the Fock representation \( \pi_F \) acting on the orthonormal base \( \{ e_k \}_{k \in \mathbb{N}_0} \) as
  \[
  \pi_F(a) e_k = [ [k] ]_q^{1/2} e_{k-1}, \ \pi_F(a^*) e_k = [ [k + 1] ]_q^{1/2} e_{k+1}, \ \text{where} \ e_{-1} := 0. \tag{2.2}
  \]
- For \( q \in (0, 1) \) the series of unbounded \( * \)-representations \( \pi_\gamma, \gamma \in (0, 1) \) acting on the orthonormal base \( \{ e_k \}_{k \in \mathbb{Z}} \) as
  \[
  \pi_\gamma(a) e_k = \left( \frac{1 + q^{\gamma + k}}{1 - q} \right)^{1/2} e_{k+1}, \ \pi_\gamma(a^*) e_k = \left( \frac{1 + q^{\gamma + k+1}}{1 - q} \right)^{1/2} e_{k-1}. \tag{2.3}
  \]
- For \( q \in (0, 1) \) the series of one-dimensional \( * \)-representations
  \[
  \pi_\varphi(a) = e^{\varphi r}(1 - q)^{-1/2}, \ \pi_\varphi(a^*) = e^{-\varphi r}(1 - q)^{-1/2}, \ \varphi \in [0, 2\pi). \tag{2.4}
  \]

Using the orbit method described in the previous section, we classify all irreducible well-behaved representations of the \( q \)-oscillator algebra
\[
\mathcal{A} = \mathbb{C} \langle a, a^* \mid aa^* = qa^*a + 1 \rangle, \quad q > 0.
\]
We will see that the formulas for the irreducible well-behaved representations of \( \mathcal{A} \) coincide with (2.2)–(2.4).

We now introduce the ingredients needed for the orbit method. Define the \( \mathbb{Z} \)-grading on \( \mathcal{A} \) by setting \( a \in \mathcal{A}_1 \), \( a^* \in \mathcal{A}_{-1} \) and put \( \mathcal{B} : = \mathcal{A}_0 \). It is easily checked that \( \mathcal{B} = \mathbb{C}[N] \), where \( N = a^* a \), and \( \mathcal{A}_n = a^n \mathcal{B}, \ \mathcal{A}_{-n} = a^{-n} \mathcal{B} \) for every \( n \in \mathbb{N} \).

Using induction on \( k \in \mathbb{N} \) we obtain the relations
\[
a^k a^k = \prod_{j=1}^{k} (q^j N + [ [j] ]_q 1), \ k \in \mathbb{N}. \tag{2.5}
\]
\[
a^k a^k = \prod_{j=0}^{k-1} (q^{-j} N + [ [-j] ]_q 1), \ k \in \mathbb{N}.
\]

Since \( \mathcal{B} = \mathbb{C}[N] \), every character on \( \mathcal{B} \) is of the form \( \chi_t(N) = t \in \mathbb{R} \). In what follows we identify the space of all characters \( \widetilde{\mathcal{B}} \) with \( \mathbb{R} \).

**Proposition 2.1.** (i) \( \widetilde{\mathcal{B}}^+ = \{ [ [k] ]_q \mid k \in \mathbb{N}_0 \} \) for \( q \geq 1 \),
\[
\widetilde{\mathcal{B}}^+ = \{ [ [k] ]_q \mid k \in \mathbb{N}_0 \} \cup [1/(1 - q), +\infty) \quad \text{for} \ q \in (0, 1).
\]

(ii) The partial action \( \alpha = (\{ \mathcal{D}_n \}_{n \in \mathbb{Z}}, \{ \alpha_n \}_{n \in \mathbb{Z}}) \) is given as follows.

\[
\mathcal{D}_n = \{ [ [k] ]_q \mid k \geq n \} \ \text{if} \ q \geq 1,
\]
\[
\mathcal{D}_n = \{ [ [k] ]_q \mid k \geq n \} \cup [1/(1 - q), \infty) \ \text{if} \ q \in (0, 1).
\]

If \( \chi_t \in \mathcal{D}_n \), then \( \chi_t^n = \chi_{F^{-n}(t)} \). In particular, \( \chi^n_{[ [k] ]_q} = \chi_{[ [k - n] ]_q} \) for \( n \leq k \).
Proof. (i) By Lemma 1.1 a character \( \chi \in \hat{B} \) belongs to \( \hat{B}^+ \) if and only if \( \chi(a^ka^k) \geq 0 \), \( \chi(a^ka^k) \geq 0 \) for all \( k \in \mathbb{N} \). Further, (2.5) implies that \( a^na^n = \sum_{j=0}^n \alpha_j N^j \) for some \( \alpha_j \geq 0 \). Hence \( \chi_t \in \hat{B}^+ \) if and only if \( \chi(a^ka^k) \geq 0 \) for all \( k \in \mathbb{N} \). The last system of inequalities is equivalent to
\[
\prod_{j=0}^{k} (t - \lfloor [j] \rfloor q) \geq 0 \quad \text{for all } k \in \mathbb{N}_0. \quad (2.6)
\]
Consider first \( q \geq 1 \). Then \( \lfloor [k] \rfloor q \to \infty \), \( k \to \infty \), and (2.6) is satisfied if and only if \( t = \lfloor [k] \rfloor q \) for some \( k \in \mathbb{N}_0 \). If \( q \in (0, 1) \), then \( \lfloor [k] \rfloor q \to \frac{1}{1-q} \), \( k \to \infty \), and every \( t \geq \frac{1}{1-q} \) satisfies (2.6). For \( t \in \left[ 0, \frac{1}{1-q} \right) \), (2.6) holds if and only if \( t = \lfloor [k] \rfloor q \) for some \( k \in \mathbb{N}_0 \).

(ii) One can verify by induction on \( n \in \mathbb{N} \) that \( F^n(t) = q^n t + \lfloor [n] \rfloor q \) for all \( n \in \mathbb{Z} \). Using (2.5), we obtain
\[
\chi_t^n(N) = \frac{\chi_t(a^n Na^n)}{\chi_t(a^n a^n)} = \chi_t(q^n N + \lfloor [n] \rfloor q) = \chi_{F^{-n}(t)}(N),
\]
for \( \chi_t \in D_{-n} \), \( n \in \mathbb{N} \), and
\[
\chi_t^{-n}(N) = \frac{\chi_t(a^n Na^n)}{\chi_t(a^n a^n)} = q^{-1} \chi_t(q^{n+1} N + \lfloor [n+1] \rfloor q) - q^{-1} = \chi_{F^n(t)}(N).
\]
for \( \chi_t \in D_n \), \( n \in \mathbb{N} \). Inequalities (2.6) imply that for \( q > 0 \) and \( t = \lfloor [k] \rfloor q \), we have \( \chi_t \in D_{-n} \) if and only if \( n \leq k \). In case \( q \in (0, 1) \) and \( t \geq \frac{1}{1-q} \), we have \( \chi_t \in D_{-n} \) for all \( n \in \mathbb{Z} \).

Using Proposition 2.1, we conclude that the stabilizer \( \text{St}_{\chi_t} \) of \( \chi_t \in \hat{B}^+ \) is trivial except for the case \( t = 1/(1-q) \), where the stabilizer is \( \mathbb{Z} \). Define the subset \( \Gamma \subset \hat{B}^+ \) as
\[
\Gamma = \{0\} \cup \left\{ \frac{1}{1-q} \right\} \cup \left\{ \frac{1+q^\gamma}{1-q} \mid \gamma \in (0, 1) \right\}, \quad \text{if } q \in (0, 1),
\]
\[
\Gamma = \{0\}, \quad \text{if } q \geq 1.
\]

Direct computations using Proposition 2.1 show that each orbit under the partial action of \( \mathbb{Z} \) on \( \hat{B}^+ \) intersects \( \Gamma \) in exactly one point, i.e. \( \Gamma \) is a section of the partial action. The topology on \( \hat{B}^+ \) is induced from the standard topology on \( \mathbb{R} \). Hence \( \Gamma \) is countably separated and measurable. By Proposition 1.3(ii) every irreducible well-behaved representation of \( A \) is associated to some \( \text{Orb}_{\chi} \), \( \chi \in \Gamma \). For \( \chi \in \Gamma \) we consider three cases.

(i) Case \( \chi = \chi_0 \). Since the stabilizer of \( \chi \) is trivial, the only irreducible well-behaved representation associated to \( \text{Orb}_{\chi} \) is (up to unitary equivalence)
\[
\pi_F := \text{Ind}_{\chi}. \quad \text{Using Proposition 1.2 and relations (2.5), we calculate the}
\]
theorem 2.2.

\[ \pi_F(a)e_n = \frac{\chi(a^{n-1}a^* a^n)}{\chi(a^{n-1}a^*(n-1))^{1/2} \chi(a^n a^n)^{1/2}} e_{n+1} = \frac{\chi(a^{n-1}a^* a^n)^{1/2}}{\chi(a^{n-1}a^*(n-1))^{1/2}} e_{n+1} \]

\[ = q^{n/2} (\chi(N) - [[-n]] q)^{1/2} e_{n+1} = [[n]] q^{1/2} e_{n+1}, \]

\[ \pi_F(a^*)e_n = \frac{\chi_0(a^{n+1}a^* a^n)}{\chi_0(a^{n+1}a^*(n+1))^{1/2} \chi(a^n a^n)^{1/2}} e_{n-1} = \frac{\chi_0(a^{n+1}a^*(n+1))^{1/2}}{\chi_0(a^n a^n)^{1/2}} e_{n-1} \]

\[ = q^{(n+1)/2} (\chi(N) - [[-n-1]] q)^{1/2} e_{n-1} = [[n+1]] q^{1/2} e_{n-1}, \]

where \( e_1 := 0 \) and \( n \in \mathbb{N}_0 \). It exists for any \( q > 0 \) and is bounded if and only if \( q \in (0, 1) \).

(ii) Case \( \chi = \chi_{1+\frac{\gamma}{q^2}}, \gamma \in (0, 1) \). The stabilizer of \( \chi \) is again trivial, thus \( \pi_{\gamma} := \text{Ind}_{1+\frac{\gamma}{q^2}} \chi \) is the only irreducible well-behaved representation associated to \( \text{Orb}_{\chi} \). We calculate the action of \( \pi_{\gamma}(a) \) respectively \( \pi_{\gamma}(a^*) \) using Proposition 1.2 and relations (2.5). For \( n \in \mathbb{Z} \) we have

\[ \pi_{\gamma}(a)e_n = \frac{\chi(a^{n+1}a a^n)}{\chi(a^{n+1}a^*(n+1))^{1/2} \chi(a^n a^n)^{1/2}} e_{n+1} = \frac{\chi(a^{n+1}a a^n)^{1/2}}{\chi(a^{n+1}a^*(n+1))^{1/2}} e_{n+1} \]

\[ = \left( q^n - \frac{1 + q^\gamma}{1 - q} + \frac{1 - q^{-n}}{1 - q} \right)^{1/2} e_{n+1} = \left( q^\gamma - \frac{1}{1 - q} \right)^{1/2} e_{n+1}. \]

In the same way we obtain

\[ \pi_{\gamma}(a^*)e_n = \left( \frac{1 + q^\gamma - n+1}{1 - q} \right)^{1/2} e_{n-1}, \text{ for } n \in \mathbb{Z}. \]

Note that \( \pi_{\gamma} \) is not bounded for every \( \gamma \in (0, 1) \).

(iii) Case \( \chi = \chi_{1-\frac{\gamma}{q^2}} \). The stabilizer group \( H \) of \( \chi_{1-\frac{\gamma}{q^2}} \) is \( \mathbb{Z} \). Let \( \rho \) be an irreducible *-representation of \( A \) satisfying (1.6). Since \( \chi(aa^* - a^* a) = 0 \), we have \( \rho(a^*)\rho(a^*) = \rho(a^*)\rho(a) \). By Schur’s Lemma \( \rho \) is one-dimensional. For \( \lambda \in \mathbb{C} \) such that \( \lambda = \rho(a) \) we get \( |\lambda|^2 = \rho(aa^*) = \rho(1 + qa^* a) = 1 + q |\lambda|^2 \).

Hence \( \rho = \rho_\varphi \), for some \( \varphi \in [0, 2\pi) \), where \( \rho_\varphi(a) = e^{i\varphi}(1 - q)^{-1/2} \). Since \( \mathcal{A}_H = A, \pi_\varphi := \text{Ind}_{\mathcal{A}_H} \rho_\varphi \) is equivalent to \( \rho_\varphi \) and we have

\[ \pi_\varphi(a) = e^{i\varphi}(1 - q)^{-1/2}, \pi_\varphi(a^*) = e^{-i\varphi}(1 - q)^{-1/2}, \varphi \in [0, 2\pi). \]

By Theorem 1.4 these are all (up to unitary equivalence) irreducible well-behaved representations of \( A \). Moreover, putting \( e_k := e_{-k}, k \in \mathbb{Z} \), we see that the above formulas coincide with (2.2), (2.3) and (2.4) respectively. We have proved the following

**Theorem 2.2.** Every irreducible well-behaved representation of the \( q \)-oscillator algebra, \( q > 0 \), is induced from a one-dimensional *-representation.
2.1. Existence of bad $*$-representations. In this subsection we prove the existence of a $*$-representation $\pi$ of $A$ which is not well-behaved and which cannot be continued to a well-behaved representation in a possibly larger Hilbert space. The idea is similar to the proof of [14, Theorem 4.1].

Lemma 2.3. The polynomial

$$p := (N - 1)(N - (1 + q)) \in \mathbb{C}[N]$$

is positive in every well-behaved $*$-representation of $A$ and $p \notin \sum \mathcal{A}^2$.

Proof. We first show that every element of $\sum \mathcal{A}^2 \cap \mathcal{B}$ is of the form

$$\sum_{k=0}^{n} a^k a^k \cdot p_k p_k, \text{ where } p_k \in \mathbb{C}[N], \quad n \in \mathbb{N}. \quad (2.7)$$

Indeed, an element $b \in \mathcal{B}$ belongs to $\sum \mathcal{A}^2$ if and only if $b = \sum b_j^* b_j$, where $b_j \in \mathcal{A}_{k_j}$, $k_j \in \mathbb{Z}$. Since $\mathcal{A}_k = a^k \cdot \mathcal{B}$ for every $k \in \mathbb{Z}$ (here $a^{-k} = a^k$ for $k > 0$), we obtain $b = \sum a^k a^k \cdot s_k$, where $s_k \in \sum \mathcal{B}^2$. It is a well-known fact, that every positive polynomial in $\mathbb{C}[N]$ is a single square $r^* r$. Hence $s_k = p_k^* p_k$, $p_k \in \mathbb{C}[N]$, for $k \in \mathbb{Z}$. Furthermore, relations (2.5) imply $a^* a^* n \in \sum \mathcal{B}^2 + a^* a \sum \mathcal{B}^2$, which proves (2.7).

Let $\pi$ be a well-behaved representation of $A$ with associated spectral measure $E_\pi$. Since $\text{supp} E_\pi \subseteq \hat{\mathcal{B}}^+$ and $p \geq 0$ on $\hat{\mathcal{B}}^+$, we have $\pi(p) = \int_{\hat{\mathcal{B}}^+} p(\lambda) dE_\pi(\lambda) \geq 0$.

Assume to the contrary that $p \in \sum \mathcal{A}^2$. Since the degree of $p(N)$ in $\mathbb{C}[N]$ is 2, we get by (2.7)

$$p = f^* f + a^* a \cdot g^* g + a^2 a^2 \cdot h^* h = f^* f + N g^* g + \frac{1}{q} N(N - 1) h^* h$$

for some polynomials $f, g, h \in \mathbb{C}[N]$, where $\deg f \leq 1$, $\deg g = 0$ and $\deg h = 0$, that is, $g$ and $h$ are constant. Setting $N := 1$, we obtain $|f(1)|^2 + |g|^2 = 0$, i.e. $g = 0$, $f(1) = 0$. Setting $N := 1 + q$, we get $|f(1 + q)|^2 + (1 + q) |h|^2 = 0$ which implies $h = f(1 + q) = 0$. Since $\deg f \leq 1$, $f \equiv 0$, i.e. $p \equiv 0$, a contradiction. $
$

For the proof of the next theorem we will need the following technical result, see [15, Lemma 2].

Lemma 2.4. Let $\mathcal{A}$ be a unital $*$-algebra which has a faithful $*$-representation $\pi$ (that is, $\pi(a) = 0$ implies that $a = 0$) and is a union of a sequence of finite dimensional subspaces $E_n$, $n \in \mathbb{N}$. Assume that for each $n \in \mathbb{N}$ there exists a number $k_n \in \mathbb{N}$ such that the following is satisfied: If $a \in \sum \mathcal{A}^2$ is in $E_n$, then we can write $a$ as a finite sum $\sum a_j^* a_j$ such that all $a_j$ are in $E_{k_n}$.

Then the cone $\sum \mathcal{A}^2$ is closed in $\mathcal{A}$ with respect to the finest locally convex topology on $\mathcal{A}$.

Theorem 2.5. There exists a $*$-representation $\pi$ of the $q$-oscillator algebra $\mathcal{A}$ which cannot be extended to a well-behaved representation in a possibly larger Hilbert space.
Proposition 2.1 implies that the automorphism \( \Theta = (\pi, \xi, I, J) \) in the sense of Definition 0.1. For let \((C_0(\hat{B}^+)), \mathbb{Z}, \beta)\) be the \(C^\ast\)-p.d.s. dual to \((\hat{B}^+, \mathbb{Z}, \alpha)\). More precisely, define the partial action \( \beta = (\{I_k\}_{k \in \mathbb{Z}}, \{\beta_k\}_{k \in \mathbb{Z}}) \) on \(C_0(\hat{B}^+)) \) by setting \(I_k := C_0(D_k)\) and
\[
(b_k(f))(t) := f(\alpha_k(t)) = f(F_k(t)), \text{ for } f \in I_{-k}, \ t \in D_k.
\]
Proposition 2.1 implies that the \(C^\ast\)-p.d.s. \((C_0(\hat{B}^+)), \mathbb{Z}, \beta)\) is defined by the partial automorphism \( \Theta = (\theta, I, J) \), where
\[
I = I_{-1}, \ J = I_1 = C_0(\hat{B}^+), \ (\theta(f))(t) = (\beta_1(f))(t) = f(1 + qt), \ f \in I_{-1}.
\]
We define \( \mathfrak{A} := C^\ast(C_0(\hat{B}^+), \Theta) = C_0(\hat{B}^+) \times_\beta \mathbb{Z} \).

**Theorem 2.6.** Consider the \(q\)-oscillator algebra \( \mathcal{A} \) with generators \( a, a^\ast \) and the category of well-behaved representations \( \text{Rep}\mathcal{A} \). Then \( \mathfrak{A} \) is a \(C^\ast\)-envelope of \( \mathcal{A} \).

**Proof.** Let \( \mathfrak{A}_0 \) be the linear hull of \[
\{ f \otimes k \in \mathfrak{A} \mid k \in \mathbb{Z}, \ \text{supp}f \subseteq D_k \text{ is compact} \}.
\]
\( \mathfrak{A}_0 \) is obviously dense in \( \mathfrak{A} \). For \( f \otimes k \in \mathfrak{A}_0 \) define
\[
A(f(t) \otimes k) = \sqrt{1 + qt}f(1 + qt) \otimes (k + 1),
\]
and
\[
A^\ast(f(t) \otimes k) = \sqrt{t}f(q^{-1}t - q^{-1}) \otimes (k - 1)
\]
Then \( A \) and \( A^\ast \) are densely defined linear operators on \( \mathfrak{A} \) and their closures, denoted again by \( A \) and \( A^\ast \), are adjoint to each other. For \( f \otimes k \in \mathfrak{A}_0 \) we have
\[
A^\ast A(f(t) \otimes k) = A^\ast(\sqrt{\alpha_{-1}(t)}f(\alpha_{-1}(t)) \otimes (k + 1)) = tf(t) \otimes k.
\]
The last equation shows that the range of \( I + A^\ast A \) is dense in \( \mathfrak{A}_0 \subseteq \mathfrak{A} \), so that \( A \) is affiliated with \( \mathfrak{A} \). By [18, Theorem 1.4.] the adjoint \( A^\ast \) is also affiliated with \( \mathfrak{A} \). We show the correspondence (0.3) between the generators \( a, a^\ast \in \mathcal{A} \) and affiliated elements \( A, A^\ast \in \eta \mathfrak{A} \).

By [3, Theorem 5.6] every \( \ast \)-representation of \( \mathfrak{A} \) is given by a covariant representation \( \pi \times u \) of \((C_0(\hat{B}^+)), \Theta)\). Here \( \pi : C_0(\hat{B}^+) \rightarrow B(\mathcal{H}_\pi) \) is a \(\ast\)-representation of \(C_0(\hat{B}^+)\) and \( u \) is a partial isometry on \( \mathcal{H}_\pi \) satisfying \( \pi(\theta(b)) = u\pi(b)u^\ast \) for every
Indeed, for every $b \in I$. By the spectral theory of commutative $C^*$-algebras, there exists a unique spectral measure $E_{\pi}$ on $\hat{B}^+$ such that

$$\pi(f) = \int_{\hat{B}^+} f(t)dE_{\pi}(t), \; f \in C_0(\hat{B}^+).$$

By definition of $\theta$ for $f \in I_{-1} = C_0(D_{-1})$ we have

$$u \left( \int f dE_{\pi} \right) u^* = u\pi(f)u^* = \pi(\theta(f)) = \int f(1+qt)dE_{\pi}(t).$$

Multiplying the latter by $u$ from the right and remembering that the initial space of $u$ is $\pi(I_{-1})\mathcal{H}_{\pi} = E_{\pi}(D_{-1})\mathcal{H}_{\pi}$ we get

$$u \int f dE_{\pi} = \int f(1+qt)dE_{\pi}(t) \cdot u, \; \text{for} \; f \in I_{-1}. \quad (2.8)$$

The extension of $(\pi \times u)$ to $A$ and $A^*$ is given by

$$(\pi \times u)(A) = u \int \sqrt{i}dE_{\pi}, \; (\pi \times u)(A^*) = \int \sqrt{i}dE_{\pi}(t) \cdot u^* \quad (2.9)$$

Indeed, for every $f \otimes k \in \mathfrak{A}_0$ we have

$$u \int \sqrt{i}dE_{\pi} \cdot ((\pi \times u)(f \otimes k)) = u \int \sqrt{i}dE_{\pi} \cdot \int f dE_{\pi} \cdot u^k$$

$$= u \int \sqrt{i}f(t)dE_{\pi} \cdot u^k = \int \sqrt{1+qt}f(1+qt)dE_{\pi} \cdot u^{k+1} = (\pi \times u)(A(f \otimes k)).$$

Since $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is a core of $A$, $\pi(\mathfrak{A}_0)\mathcal{H}_{\pi}$ is a core of $A$, and we get the first part of (2.9). The second part follows from $(\pi \times u)(A^*) = ((\pi \times u)(A))^*$. Let $\rho$ be a well-behaved representation of $\mathfrak{A}$, and $E_{\rho}$ be the corresponding spectral measure on $\hat{B}^+ \subseteq \mathbb{R}_+$. Further, let $\overline{\rho(a)} = u_1c_1$ be the polar decomposition of $\overline{\rho(a)}$. Since $a^*a$ is the generator of $B$, $E_{\rho}$ coincides with the spectral measure of $\overline{\rho(a^*a)} = \overline{\rho(a)^*\rho(a)}$. Hence

$$\overline{\rho(a)} = u_1 \int \sqrt{i}dE_{\rho}, \; \text{and} \; \overline{\rho(a^*)} = \int \sqrt{i}dE_{\rho} \cdot u_1^*. \quad (2.10)$$

Since $\ker u_1 = \ker c_1$, the initial space of $u_1$ is the range of $E_{\rho}(\hat{B}^+ \setminus \{0\}) = E_{\rho}(D_{-1})$. Further, we have $u_1c_1^2u_1^* = 1 + qc_1^2$, which implies that $\ker u_1^*$ is trivial, so that the final space of $u_1$ is $E_{\rho}(\hat{B}^+) = E_{\rho}(D_{1})$. Applying (1.5) to $f \in C_0(D_{-1})$ we obtain

$$u_1\rho(f)u_1^* = u_1 \int f(t)dE_{\rho}(t) \cdot u_1^* = \int f(\alpha_{-1}(t))dE_{\rho}(t) \cdot u_1^*u_1$$

$$= \int f(1+qt)dE_{\rho}(t) = \rho(\theta(f)), \; \text{i.e.} \; (\rho_B \times u_1), \; \text{where} \; \rho_B \text{ is the restriction of } \rho \text{ to } B, \text{ defines a covariant representation of } (C_0(\hat{B}^+), \theta). \; \text{The correspondence} \; (0.3) \text{ between } \pi \text{ and } \rho \text{ follows now by comparing} \; (2.9) \text{ with} \; (2.10). \; \blacksquare
Remark. In [19, Section 3] the author shows that the operators $p, q$ of the Weyl algebra $W_1$ generate a $C^*$-algebra $\mathfrak{A}$ in the sense of the Definition 3.1 therein, and that $\mathfrak{A}$ is the algebra of compact operators. It corresponds to the fact that the $C^*$-envelope of $q$-CCR with $q = 1$ is isomorphic to the partial crossed product $C_0(\mathbb{N}_0) \times_\alpha \mathbb{Z} \simeq K(l^2(\mathbb{N}_0))$.

3. The Podleś sphere

In this section we investigate $\ast$-representations of the Podleś sphere $O(S^2_{qr})$. We consider only the case $q \in (0, 1), r \in (0, \infty)$. The cases $r = 0, r = \infty$ can be treated similarly. Recall [11] that $\mathcal{A} := O(S^2_\delta)$ is the unital $\ast$-algebra generated by $a = a^\ast, b, b^\ast$ and defining relations

$$ab = q^{-2}ba, \; ab^\ast = q^2b^\ast a, \; b^*b = a - a^2 + r1, \; bb^* = q^2a - q^4a^2 + r1. \tag{3.1}$$

The defining relations imply that every $\ast$-representation of $\mathcal{A}$ is bounded and hence well-behaved by Proposition 1.3 (i). In [11] the following irreducible $\ast$-representations of $\mathcal{A}$ were obtained.

- Two infinite-dimensional $\ast$-representations $\pi_{\pm}$ which act on an orthonormal base $\{e_k\}_{k \in \mathbb{N}_0}$ of the representation space $\mathcal{H}_\pm$ by

$$\pi_{\pm}(a)e_k = q^{2k}\lambda_\pm e_k, \; \pi_{\pm}(b)e_k = (q^{2k}\lambda_\pm - (q^{2k}\lambda_\pm)^2 + r)^{1/2}e_{k-1},$$

$$\pi_{\pm}(b^*)e_k = (q^{2(k+1)}\lambda_\pm - (q^{2(k+1)}\lambda_\pm)^2 + r)^{1/2}e_{k+1}, \; e_{-1} := 0,$$

where $\lambda_\pm := \frac{1}{2} \pm (r + \frac{1}{2})^{1/2}$.

- The series of one-dimensional $\ast$-representations $\pi_\varphi, \; \varphi \in [0, 2\pi)$,

$$\pi_\varphi(a) = 0, \; \pi_\varphi(b) = e^{i\varphi}r^{1/2}, \; \pi_\varphi(b^*) = e^{-i\varphi}r^{-1/2}.$$
Proof. (i) Lemma 1.1 and relations (3.2) imply that \( \chi_t, \ t \in \mathbb{R}, \) belongs to \( \widehat{\mathcal{B}}^+ \) if and only if the following inequalities are satisfied for all \( n \in \mathbb{N}: \)

\[
\chi_t(b^n b^n) = \prod_{k=0}^{n-1} (q^{-2k}t - q^{-4k}t^2 + r) \geq 0, \\
\chi_t(b^n b^* b^n) = \prod_{k=1}^{n} (q^{2k}t - q^{4k}t^2 + r) \geq 0.
\] (3.3)

Assume \( q^{-2k}t - q^{-4k}t^2 + r > 0 \) for all \( k \in \mathbb{N}_0. \) Since \( q^{-2k} \to +\infty, \ k \to +\infty, \) it is possible only if \( t = 0, \) i.e. \( \chi_t = \chi_\infty. \) If \( t \neq 0 \) we get \( q^{-2k}t - q^{-4k}t^2 + r = 0 \) for some \( k \in \mathbb{N}_0, \) whence

\[
t = \frac{-q^{-2k} \pm \sqrt{q^{-4k} + 4q^{-4k}r}}{-2q^{-4k}} = q^{2k}\lambda_\pm.
\]

One can easily check that every \( t = q^{2k}\lambda_\pm, \ k \in \mathbb{N}_0, \) satisfies (3.3).

(ii) Relations (3.3) imply that \( \mathcal{D}_{-n} = \{ \chi_{m,\pm} \mid m \geq n \} \cup \{ \chi_\infty \}. \) Assume that \( \chi_{m,\pm} \in \mathcal{D}_{-n}, \) where \( n \in \mathbb{N}_0. \) Using relations (3.2) we obtain

\[
\chi_{m,\pm}^n(a) = \frac{\chi_{m,\pm}(b^n b^n)}{\chi_{m,\pm}(b^n b^n)} = \frac{\chi_{m,\pm}(b^n b^n)\chi_{m,\pm}(q^{-2n}a)}{\chi_{m,\pm}(b^n b^n)} = \chi_{m-n,\pm}(a).
\]

For \( \chi_\infty \) we have \( \chi_\infty(b^n b^n) = \chi_\infty(b^n b^n) = r^n \neq 0 \) for all \( n \in \mathbb{Z} \) by equations (3.3). Hence \( \chi_\infty \in \mathcal{D}_n \) for all \( n \in \mathbb{Z} \) and \( \chi_\infty^n(a) = 0. \)

Let \( \Gamma \) be the subset \( \{ \chi_{0,+, \chi_{0,-}, \chi_\infty} \} \subseteq \widehat{\mathcal{B}}^+. \) Obviously \( \Gamma \) is a measurable countably separated section of the p. d. s. \( (\widehat{\mathcal{B}}^+, \mathbb{Z}, \alpha). \) We calculate all irreducible *-representations associated with \( \text{Orb}\chi, \ \chi \in \Gamma. \)

(i) Case \( \chi_{0,\pm}. \) The stabilizer of \( \chi_{0,\pm} \) is trivial by Proposition 3.1, (ii). Put \( \pi_\pm := \text{Ind}\chi_{0,\pm}. \) We use Proposition 1.2 to compute the action of \( \pi_\pm \) on the orthonormal base \( \{ e_{-k} \}_{k \in \mathbb{N}_0}. \)

\[
\pi_\pm(b)e_{-k} = \left( q^{2k}\chi_{0,\pm}(a) - q^{-4k} \chi_{0,\pm}(a^2) + r \right)^{1/2} e_{-k+1} = \left( q^{2k}\lambda_\pm - \left( q^{2k}\lambda_\pm \right)^2 + r \right)^{1/2} e_{-k+1},
\]

\[
\pi_\pm(b^*)e_{-k} = \left( q^{2(k+1)}\lambda_\pm - \left( q^{2(k+1)}\lambda_\pm \right)^2 + r \right)^{1/2} e_{-k-1},
\]

\[
\pi_\pm(a)e_{-k} = \chi_{0,\pm}^k(a) = q^{2k}\lambda_\pm e_{-k}.
\]

(ii) Case \( \chi_\infty. \) The stabilizer group \( H \) of \( \chi \) is \( \mathbb{Z}. \) Let \( \rho \) be an irreducible *-representation of \( \mathcal{A}_H \) satisfying (1.6). Since \( \chi(bb^* - b^*b) = 0, \) we have \( \rho(b^*)\rho(b^*) = \rho(b^*)\rho(b). \) By Schur’s Lemma \( \rho \) is one-dimensional. For \( \lambda \in \mathbb{C} \) such that \( \lambda = \rho(b) \) we get \( |\lambda|^2 = \rho(bb^*) = r. \) Hence \( \rho = \rho_\varphi, \) for some \( \varphi \in [0, 2\pi), \) where \( \rho_\varphi(b) = e^{i\varphi}r^{1/2}. \) Since \( \mathcal{A}_H = \mathcal{A}, \) \( \pi_\varphi := \text{Ind}_{\mathcal{A}_H \uparrow \mathcal{A}}\rho_\varphi \) is equivalent to \( \rho_\varphi \) and we get

\[
\pi_\varphi(a) = 0, \pi_\varphi(b) = e^{i\varphi}r^{1/2}, \pi_\varphi(b^*) = e^{-i\varphi}r^{1/2}; \ \varphi \in [0, 2\pi).
\]
By Theorem 1.4 these are all, up to unitary equivalence, irreducible \(*\)-representations of \(A\). Setting \(e_k := e_{-k}, \ k \in \mathbb{N}_0\), we see that these coincide with the ones found in [11]. In particular, we have the following

**Theorem 3.2.** Every irreducible \(*\)-representation of the Podleś sphere \(O(S^2_{qr})\), \(q \in (0, 1), \ r \in [0, \infty]\), is induced from a one-dimensional \(*\)-representation.

In the remaining part of this section we describe the enveloping \(C^*\)-algebra of \(O(S^2_{qr})\). For let \((C_0(\mathbb{B}^+), \mathbb{Z}, \beta)\) be the \(C^*\)-p.d.s. dual to \((\mathbb{B}^+, \mathbb{Z}, \alpha)\) as defined in the Subsection 0.2. Note, that the sets \(D_k, \ k \in \mathbb{Z}\), are compact, hence \(I_k := C(D_k)\). By definition of \(\beta\) we have

\[
(\beta_k(f))(t) = f(\alpha_{-k}(t)) = f(q^{2k}t), \ f \in I_{-k}, k \in \mathbb{Z}.
\]

It is easily seen from the description of \(\alpha\) in the Proposition 3.1 that the partial action \(\beta = (\{I_k\}_{k \in \mathbb{Z}}, \{\beta_k\}_{k \in \mathbb{Z}})\) is defined by the partial automorphism \(\Theta = (\theta, I, J)\), where \(\theta = \beta_1, \ I = I_{-1}, J = I_1 = A\).

**Theorem 3.3.** The enveloping \(C^*\)-algebra \(A\) of \(A\) is isomorphic to the covariance algebra \(C^*(C(\mathbb{B}^+), \Theta) \simeq C(\mathbb{B}^+) \times_{\beta} \mathbb{Z}\).

**Proof.** The proof goes similarly to the proof of the Theorem 2.6 by replacing the \(\eta\)-relation with \(\varepsilon\)-relation.

We first define \(*\)-homomorphism \(\varepsilon : A \to C(\mathbb{B}^+) \times_{\beta} \mathbb{Z}\) by setting

\[
\varepsilon(a) = t \otimes 0, \ \varepsilon(b) = (q^2t - q^4t^2 + r)^{1/2} \otimes 1, \ \varepsilon(b^*) = (t - t^2 + r)^{1/2} \otimes (-1).
\]

Direct computations using (3.4) show that \(\varepsilon(a), \varepsilon(b), \varepsilon(b^*)\) satisfy the defining relations of \(O(S^2_{qr})\), that is, \(\varepsilon\) is well-defined. Every representation \(\pi\) of \(A\) is bounded, hence well-behaved by Proposition 1.3. That is, \(\pi\) gives rise to a covariant representation \(\pi|_\mathcal{B} \times u\), where \(u\) is the partial isometry in the polar decomposition \(\pi(b) = uc\). On the other hand, every \(*\)-representation of \(A\) is given by a covariant representation of the partial automorphism \(\Theta\). This proves the correspondence (0.3) for the representations of \(A\) and \(\mathcal{A}\).

\[\blacksquare\]

4. The quantum algebra \(U_q(su(2))\)

In this section \(A\) is the \(q\)-deformed enveloping \(*\)-algebra \(U_q(su(2))\), \(q > 0, \ q \neq 1\), which is generated by \(E, F, K, K^{-1}\) satisfying the defining relations

\[
KK^{-1} = K^{-1}K = 1, \ KEK^{-1} = q^2E, \ KFK^{-1} = q^{-2}F,
\]

\[
[E, F] = EF - FE = \frac{K - K^{-1}}{q - q^{-1}},
\]

\[
E^* = FK, \ F^* = K^{-1}E, \ K^* = K.
\]

In this section, we use the standard notation \([n] \equiv [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\), where \(n \in \mathbb{Z}\) and \(q \neq 0\). Further \(X^0\) denotes \(1\) if \(X\) is one of the four generators \(E, F, K, K^{-1}\).

Consider the \(*\)-representations \(\pi_{\omega, l}, \ \omega = \pm 1, \ l \in \mathbb{Z}/2\mathbb{N}_0\), of \(U_q(su(2))\), which
act on an orthonormal base \( \{ e_m \}_{m=-l,...,l} \) as follows:
\[
\begin{align*}
\pi_{\omega,l}(K)e_m &= \omega q^{2m}e_m, \\
\pi_{\omega,l}(E)e_m &= q^{m+1}\sqrt{|l-m|}[l+m+1]e_{m+1}, \\
\pi_{\omega,l}(F)e_m &= \omega q^{-m}\sqrt{|l+m|}[l-m+1]e_{m-1},
\end{align*}
\] (4.1)

where we set \( e_{l+1} := 0 \) and \( e_{-l-1} := 0 \), cf. [7, Section 3.2.3] and [17]. We will show that every irreducible well-behaved representation of \( \mathcal{A} \) is unitarily equivalent to \( \pi_{\omega,l} \) for some \( l \in \frac{1}{2}\mathbb{N}_0, \omega = \pm 1 \).

Define a \( \mathbb{Z} \)-grading of \( \mathcal{A} \) by setting \( E \in \mathcal{A}_{1}, F \in \mathcal{A}_{-1} \) and \( K,K^{-1} \in \mathcal{A}_0 \).

Then
\[
B := \mathcal{A}_0 = \text{lin} \{ E^lF^lK^m \mid l \in \mathbb{N}_0, m \in \mathbb{Z} \}.
\]
The \( * \)-subalgebra \( B \subseteq \mathcal{A} \) is commutative and is equal to \( \mathbb{C}[EF,K,K^{-1}] = \mathbb{C}[C_q,K,K^{-1}] \). For \( n \in \mathbb{N}_0 \), we have
\[
\begin{align*}
\mathcal{A}_n &= E^nB = \text{lin}\{E^{n+l}F^lK^m \mid l \in \mathbb{N}_0, m \in \mathbb{Z} \}, \\
\mathcal{A}_{-n} &= F^nB = \text{lin}\{F^{n+l}E^lK^m \mid l \in \mathbb{N}_0, m \in \mathbb{Z} \}.
\end{align*}
\]

One can verify by a direct computation that the quantum Casimir element \( C_q \) is a central element in \( \mathcal{A} \), where
\[
C_q = EF + \frac{q^{-1}K + q K^{-1}}{(q-q^{-1})^2}.
\]

The following lemma can be easily proved by induction.

**Lemma 4.1.** For every \( n \in \mathbb{N} \) we have
\[
\begin{align*}
(i) \ [E,F^n] &\equiv EF^n - F^nE = [n]F^{n-1}[K;1-n], \\
(ii) \ [E^n,F] &\equiv E^nF - FE^n = [n]E^{n-1}[K;n-1],
\end{align*}
\]
where we set \( [K;l] := (q^lK - q^{-l}K^{-1})/(q - q^{-1}) \) for \( l \in \mathbb{Z} \).

This lemma implies the following relations:
\[
\begin{align*}
E^nF^n &= \prod_{j=1}^n (EF + [j-1][K;-j]), \\
F^nE^n &= \prod_{j=1}^n (EF - [j][K;j-1]), \ n \in \mathbb{N}.
\end{align*}
\] (4.2)

Since \( B = \mathbb{C}[C_q,K,K^{-1}] \), every character \( \chi \in \hat{B} \) is equal to some \( \chi_{st} \in \hat{\mathcal{B}} \), \( (s,t) \in \mathbb{R} \times \mathbb{R}\setminus\{0\} \) where
\[
\chi_{st}(C_q) = s, \ \chi_{st}(K) = t.
\]
Proposition 4.2. (i) A character \( \chi_{st} \in \widehat{B} \) belongs to \( \widehat{B}^+ \) if and only if

\[
t = \pm q^{m-n} \quad \text{and} \quad s = \frac{\pm q^{m+n+1} \pm q^{-m-n-1}}{(q-q^{-1})^2}, \quad \text{where} \ m, n \in \mathbb{N}_0.
\]

In particular,

\[
\widehat{B}^+ = \{ \chi_{m,n,+} \mid m, n \in \mathbb{N}_0 \} \cup \{ \chi_{m,n,-} \mid m, n \in \mathbb{N}_0 \},
\]

where \( \chi_{m,n,\pm} = \chi_{st} \) with \( s, t \) from above.

(ii) The partial action \( \alpha = (\{D_n\}_{n \in \mathbb{Z}}, \{\alpha_n\}_{n \in \mathbb{Z}}) \) is given as follows:

\[
D_{-k} = \{ \chi_{m,n,\pm} \mid -m \leq k \leq n \}, \quad \text{and} \ \chi_{m,n,\pm} = \chi_{m+k,n-k,\pm}.
\]

Proof. (i) Lemma 1.1 and equations (4.2) imply that \( \chi \in \widehat{B}^+ \) if and only if the following inequalities are satisfied for arbitrary \( k \in \mathbb{N} \):

\[
\chi(E^kF^kK^{-k}) = \chi(K^{-1})^k \prod_{j=1}^{k} \chi(EF + [j - 1][K; j]) \geq 0, \quad (4.3)
\]

\[
\chi(F^kE^kK^k) = \chi(K)^k \prod_{j=1}^{k} \chi(EF - [j][K; j] - 1) \geq 0. \quad (4.4)
\]

We show that there exist \( m, n \in \mathbb{N}_0 \) such that

\[
\chi(EF + [m][K; -m - 1]) = 0, \quad (4.5)
\]

\[
\chi(EF - [n + 1][K; n]) = 0. \quad (4.6)
\]

Assume the contrary, i.e. \( \chi(EF + [k][K; -k - 1]) \neq 0 \) for all \( k \in \mathbb{N}_0 \). Suppose \( t > 0 \). Then by (4.3) we have

\[
\chi(EF) > -[k] \chi([K; -k - 1]) = \frac{(q^{-2k-1} - q^{-1})t + (q^{2k+1} - q)t^{-1}}{(q-q^{-1})^2}, \quad k \in \mathbb{N}_0.
\]

The right hand side of the last inequality tends to \( +\infty \) as \( k \to \infty \), which is a contradiction. In the same way one obtains a contradiction for \( t < 0 \). Thus, \( \chi(EF + [m][K; -m - 1]) = 0 \) for some \( m \in \mathbb{N}_0 \). Similarly one can prove that \( \chi(EF - [n + 1][K; n]) = 0 \) for some \( n \in \mathbb{N}_0 \), using inequalities (4.4). Subtracting (4.6) from (4.5) yields

\[
[m] \chi([K; -m - 1]) = -[n + 1] \chi([K; n])
\]

\[
\iff (q^{-1} - q^{2m-1})t - (q^{2m+1} - q)t^{-1} = (q^{-1} - q^{2n+1})t - (q^{-2n-1} - q)t^{-1}
\]

\[
\iff t^2 = \frac{q^{2m+1} - q^{-2n-1}}{q^{2n+1} - q^{-2m-1}} = \frac{q^{2m}(q-q^{-2m-2n-1})}{q^{2n}(q-q^{-2m-2n-1})}
\]

\[
\iff t = \pm q^{m-n}.
\]
Further we obtain
\[ \chi(C_q) = [n + 1]\chi([K; n]) + \frac{q^{-1}\chi(K) + q\chi(K^{-1})}{(q - q^{-1})^2} = \pm t + q^{-1} t - q^{-2} t - 1, \]
\[ \frac{\pm q^{m+n+1} + q^{-m-n-1}}{(q - q^{-1})^2}. \]

(ii) Observe that \( \chi_{m,n,\pm}(E^{K}E^{k}K^{-k}) \neq 0 \) if and only if \( k \leq m \) by (4.3) and (4.5). Analogously, \( \chi_{m,n,\pm}(E^{k}E^{k}K^{k}) \neq 0 \) if and only if \( k \leq n \) by (4.4) and (4.6). This implies that \( \chi_{m,n,\pm} \in \mathcal{D}_{-k} \) if and only if \( -m \leq k \leq n \). Now suppose \( k \in \{0, 1, \ldots, n\} \). Since \( C_q \) commutes with \( E, F \), we have
\[ \chi_{m,n,\pm}(K) = \chi_{m,n,\pm}(E^{k}E^{k}K) = \chi_{m,n,\pm}(E^{k}E^{k}q^{2k}K) = q^{2k} \chi_{m,n,\pm}(K), \]
\[ \chi_{m,n,\pm}(C_q) = \chi_{m,n,\pm}(E^{k}C_qE^{k}) = \chi_{m,n,\pm}(C_q). \]

The last equalities also hold for \( k \in \{-m, -m + 1, \ldots, 0\} \). Hence, if \( \chi_{m,n,\pm} \) is defined, then \( \chi_{m,n,\pm}(K) = \pm q^{(m+k)-(n-k)} = \chi_{m+k,n-k,\pm}(K) \) and \( \chi_{m,n,\pm}(C_q) = \chi_{m,n,\pm}(C_q). \)

In particular, the previous proposition implies that for each \( \chi \in \hat{B}^+ \) the stabilizer \( \text{St} \chi \) is trivial. We set
\[ \Gamma := \{ \chi_{0,n,+, n \in \mathbb{N}_0} \cup \{ \chi_{0,n,-, n \in \mathbb{N}_0} \}. \]

As in Section 2, we conclude that \( \Gamma \) is a measurable countably separated section of the partial action. Using Proposition 4.2 we conclude that \( \text{Orb} \chi_{0,n,\pm} \) consists of \( n + 1 \) elements and hence \( \text{Ind} \chi_{0,n,\pm} \) has dimension \( n + 1 \) by Proposition 1.2, where \( \chi_{0,n,\pm} \in \Gamma \). Now put \( l := \frac{n}{2} \) and \( \pi_{\omega,l} := \text{Ind} \chi_{0,n,\pm} \), where \( \omega = \pm 1 \). Let \( \{e_{l+m}\}_{m = -l, -l+1, \ldots, l} \) be an orthonormal base of the representation space \( \mathcal{H}_{\pi_{l,\pm}} \) of \( \pi_{l,\pm} \). For notational convenience, we put \( e_{l+1} := 0 \), and \( e_{-l-1} := 0 \).

Using Proposition 1.2, relations (4.2), Proposition 4.2 and the facts that \( \chi_{0,n,\pm}(EF) = 0 \), \( \chi_{0,n,\pm}([K; l + m]) = -\omega[l - m] \), we obtain the action of \( \pi_{\omega,l} \) on the base vectors \( e_{l+m} \).

\[ \pi_{\omega,l}(K)e_{l+m} = \chi_{0,n,\pm}(E^{l+m})e_{l+m} = \chi_{l+m,n-l-m,\pm}(K)e_{l+m} = \omega q^{2m}e_{l+m}, \]
\[ \pi_{\omega,l}(E)e_{l+m} = \frac{\chi_{0,n,\pm}(E^{l+m}e_{l+m+1})}{\chi_{0,n,\pm}(E^{l+m+1}e_{l+m+1})} \]
\[ = \frac{\chi_{0,n,\pm}(E^{l+m}(e_{l+m+1})^{1/2}e_{l+m+1})}{\chi_{0,n,\pm}(E^{l+m}(e_{l+m+1})^{1/2}e_{l+m+1})} \]
\[ = (q^{l+m+1}(l+m+2)-(l+m)(l+m+1))^{1/2} \]
\[ \times (\chi_{0,n,\pm}(EF - [l + m + 1][K; l + m]) \chi_{0,n,\pm}(K))^{1/2}e_{l+m+1} \]
\[ = q^{m+1} \sqrt{|l - m}[l + m + 1]ezl+m+1, \]
\[ \pi_{\omega,l}(F)e_{l+m} = \pi_{\omega,l}(E^{K^{-1}})e_{l+m} = \omega q^{-2m} \pi_{\omega,l}(E)^{l+m} \]
\[ = \omega q^{-m} \sqrt{|l + m][l - m + 1]ezl+m-1. \]
Putting $e_m := e_{l+m}$, $m = -l, \ldots, l$, we see that every irreducible well-behaved representation of $\mathcal{U}_q(\mathfrak{su}(2))$ is unitarily equivalent to $\pi_{\omega,l}$, given by the formulas (4.1), for some $\omega \in \{-1, +1\}$ and $l \in \frac{1}{2} \mathbb{N}_0$. Summarizing the above discussion, we obtain the following

**Theorem 4.3.** Every irreducible well-behaved representation of $\mathcal{U}_q(\mathfrak{su}(2))$, $q \in \mathbb{R}_+ \setminus \{1\}$, is induced from a one-dimensional $*$-representation.

Similarly to Lemma 2.3 and Theorem 2.5 one can prove the following Lemma and Theorem.

**Lemma 4.4.** The polynomial

$$(EF - [2][K; 1])(EF - [3][K; 2]) \in \mathbb{C}[EF, K, K^{-1}]$$

is positive in every well-behaved representation of $\mathcal{A}$ and is not of the form $\sum_{k=1}^{n} a_k^* a_k$ for $a_k \in \mathcal{A}$.

**Theorem 4.5.** There exists a $*$-representation of $\mathcal{U}_q(\mathfrak{su}(2))$, $q \in \mathbb{R}_+ \setminus \{1\}$, which has no well-behaved extension in a possibly larger Hilbert space.

Let $\text{Rep} \mathcal{A}$ denote the category of well-behaved non-degenerate representations of $\mathcal{A}$. Then $\mathcal{A}$, considered with $\text{Rep} \mathcal{A}$ and generators $E, F, K, K^{-1}$, has a $C^*$-envelope $\mathfrak{A}$ in the sense of Definition 0.1. As in Section 2, let $(C_0(\hat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ be the $C^*$-p.d.s. dual to $(\hat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$. The description of the p.d.s. $(\hat{\mathcal{B}}^+, \mathbb{Z}, \alpha)$ in Proposition 4.2 implies that the $C^*$-p.d.s. $(C_0(\hat{\mathcal{B}}^+), \mathbb{Z}, \beta)$ is defined by the partial automorphism $\Theta = (\theta, I, J)$, where $I = I_{-1}$, $J = I_1$, $(\theta(f))(t) = (\beta_1(f))(t)$, $f \in I_{-1}$.

The proof of the following theorem is completely analogous to the proof of Theorem 2.6.

**Theorem 4.6.** Consider the $q$-deformed enveloping algebra $\mathcal{A} = \mathcal{U}_q(\mathfrak{su}(2))$ with generators $E, F, K, K^{-1}$ and the category of well-behaved representations $\text{Rep} \mathcal{A}$. Then the covariance algebra $\mathfrak{A} := C^*(C_0(\hat{\mathcal{B}}^+), \Theta)$ is a $C^*$-envelope of $\mathcal{A}$.

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