THE 2-ADIC VALUATION OF GENERALIZED FIBONACCI
SEQUENCES WITH AN APPLICATION TO CERTAIN
DIOPHANTINE EQUATIONS.

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ABSTRACT. In this paper we focus on finding all the factorials express-
able as a product of a fixed number of 2k-nacci numbers with $k \geq 2$.
We derive the 2-adic valuation of the 2k-nacci sequence and use it to
establish bounds on the solutions of the initial equation. In addition,
we specify a more general family of sequences, for which we can perform
a similar procedure. We also investigate a possible connection of these
results with $p$-regular sequences.

1. INTRODUCTION

For a fixed integer $r \geq 2$ define the generalized Fibonacci ($r$-nacci) se-
quency $\{t_n\}_{n \geq 0}$ as follows:

$$t_n = \begin{cases} 
0 & \text{for } n = 0, \\
1 & \text{for } 1 \leq n \leq r - 1, \\
\sum_{i=1}^{r} t_{n-i} & \text{for } n \geq r - 1.
\end{cases}$$

Notice, that for $r = 2$ we obtain the usual Fibonacci sequence, which has
already been studied extensively. In this paper we will mostly restrict our-
selves to the case of even $r \geq 4$ and write $r = 2k$ for some $k \geq 2$. The main
motivation for our considerations is to completely solve the equation

$$m! = \prod_{i=1}^{d} t_{n_i}$$

in positive integers $m, n_1, ..., n_d$.

For $p$ prime define the $p$-adic valuation of a non-zero integer $s$ as $\nu_p(s) = \max\{l \geq 0 : p^l | s\}$ and $\nu_p(0) = \infty$. Equation (2) for the case of $r = 3$ and $d = 1$ was solved by Lengyel and Marques in [5] by means of computing $\nu_2(t_n)$
and then applying this result to obtain an effective upper bound on $m$ and
$n_1$. In this paper we will follow a similar procedure for 2k-nacci sequences
with $k \geq 2$.

To begin with, in Theorem 2 we specify a more general family of integer
sequences $\{s_n\}_{n \geq 0}$ for which we are able to solve equation (2) and show a
general procedure to achieve this goal. Informally speaking, we need the
term $s_n$ to grow at least exponentially and $\nu_p(s_n)$ – at most polynomially
with an exponent less than 1, for some $p$ prime.

Theorem 3 provides a simple expression for $\nu_2(t_n)$ when $r = 2k \geq 4$ and the
subsequent corollary shows that the sequence $\{t_n\}_{n \geq 0}$ satisfies the conditions

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of Theorem\textsuperscript{2}. We then find all the solutions of equation (2) for small values of $k$ and $d$.

We also briefly discuss how our results are related to $p$-regular sequences. Recall that a sequence $\{s_n\}_{n \geq 0}$ with rational values is $p$-regular iff its $p$-kernel

$$\mathcal{N}_p(a) = \{s_{p^j} : l \geq 0, 0 \leq j < p^l\}$$

is contained in a finitely generated $\mathbb{Z}$-module. More details on regular sequences can be found in \textsuperscript{1} and \textsuperscript{2}. As we note later, the formula given in Theorem\textsuperscript{3} implies 2-regularity of $\nu_2(t_n)$. However, it turns out that exponential growth of a sequence $\{s_n\}_{n \geq 0}$ and $p$-regularity of $\{\nu_p(s_n)\}_{n \geq 0}$ still do not guarantee that the assumptions of Theorem\textsuperscript{2} are met. In this case we cannot determine, using the shown method, whether the equation (2) has only a finite number of solutions.

2. Main results

As we mentioned before, we start with describing a general situation in which equation (2) can be completely solved. For two sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ we denote $a_n = O(b_n)$ if there exists a positive constant $K$ such that $|a_n| \leq K|b_n|$ for sufficiently large $n$. Similarly, we write $a_n = \Omega(b_n)$ if there exists a positive constant $K$ such that $|a_n| \geq K|b_n|$ for sufficiently large $n$. First, we give an auxiliary lemma, also used in \textsuperscript{5}, which is an easy corollary from Legendre’s formula for $\nu_p(m!)$.

**Lemma 1.** For any integer $m \geq 1$ and prime $p$, we have

$$\frac{m}{p-1} - \frac{\log m}{\log p} - 1 \leq \nu_p(m!) \leq \frac{m-1}{p-1}.$$  

**Theorem 2.** Let $\{s_n\}_{n \geq 0}$ be a sequence of positive integers such that

$$\log s_n = \Omega(n).$$

Let $p$ be a prime. Assume that

$$\nu_p(s_n) = O\left(n^C\right)$$

for some constant $C < 1$. Then for each fixed positive integer $d$ the equation

$$m! = \prod_{i=1}^{d} s_n$$

has only a finite number of solutions in $m, n_1, n_2, \ldots, n_d$.

**Proof.** We adjust the method used in \textsuperscript{3} to a more general setting. Roughly speaking, we will show that if (5) is satisfied and we let both sides grow, then the $p$-adic valuation of the right hand side increases slower than $\nu_p(m!)$. For each value of $p$ we proceed in the same way, so for simplicity assume that $p = 2$. By our assumptions, there exist some positive constants $K_1, K_2$ and an integer $n_0 \geq 0$ such that $\nu_2(s_n) \leq K_1 n^C$ and $\log_2 s_n \geq K_2 n$ for $n \geq n_0$.

Suppose that $n_i \geq n_0$ for $i = 1, 2, \ldots, d$. There is only a finite number of solutions with $m < 6$ because $s_n$ grows at least exponentially. By Lemma\textsuperscript{1} for $p = 2$ we get

$$\frac{1}{2} m \leq m - \lfloor \log_2 m \rfloor - 1 \leq \nu_2(m!),$$

for $m! = \prod_{i=1}^{d} s_n$ in integers $m, n_1, n_2, \ldots, n_d$. 

where the leftmost inequality is true for \( m \geq 6 \). On the other hand,

\[
\nu_2 \left( \prod_{i=1}^{d} s_{n_i} \right) = \sum_{i=1}^{d} \nu_2(s_{n_i}) \leq K_1 \sum_{i=1}^{d} n_i^C \leq dK_1 \left( \max_{1 \leq i \leq d} n_i \right)^C.
\]

Hence, for \( m \geq 6 \)

\[
(7) \quad m \leq 2dK_1 \left( \max_{1 \leq i \leq d} n_i \right)^C.
\]

We need another inequality with \( n_i \) bounded from above by \( m \). By the assumption \( (3) \)

\[
\log_2 \left( \prod_{i=1}^{d} s_{n_i} \right) = \sum_{i=1}^{d} \log_2 s_{n_i} \geq K_2 \sum_{i=1}^{d} n_i \geq K_2 \max_{1 \leq i \leq d} n_i.
\]

Furthermore, for \( m \geq 5 \) the following inequality holds:

\[
(8) \quad m! < \left( \frac{m}{2} \right)^m,
\]

and hence

\[
(9) \quad m \log_2 \frac{m}{2} > K_2 \max_{1 \leq i \leq d} n_i.
\]

Combining inequalities \( (7) \) and \( (9) \) yields

\[
(10) \quad \log_2 \frac{m}{2} > K m^{\frac{1}{C-1}},
\]

where \( K = K_2/(2dK_1)^{1/C} \). But \( C < 1 \) implies that the exponent in \( (10) \) is strictly greater than 0, so \( (10) \) holds only for finitely many \( m \). Thus, by inequality \( (9) \) we obtain an upper bound for all \( n_i \).

Now assume without loss of generality that \( n_i < n_0 \) for \( i = j+1, j+2, \ldots, d \). Observe that for each fixed \( n_{j+1}, n_{j+2}, \ldots, n_d \) the problem is equivalent to solving the equation \( (5) \) with \( d = j \) and the left hand side divided by a positive integer constant \( M \). Then in \( (6) \) and \( (8) \) we need to replace \( m! \) with \( m!/M \) which only changes the set of \( m \) for which both of those inequalities hold. But this leads to the same conclusion as before. \( \square \)

**Remark 1.** Observe that the method of Theorem 2 does not work if we let \( d \) be unbounded. Indeed, the constant \( K \) in \( (10) \) becomes arbitrarily small as \( d \) increases, so we cannot use the subsequent argument. Informally speaking, the \( p \)-adic valuation of the expression \( \prod_{i=1}^{d} s_{n_i} \) might grow too fast for the method to work.

**Remark 2.** The condition \( (3) \) is satisfied for sequences expressible in Binet form. Hence, for linear recurrence sequences, we usually need to check only the condition \( (4) \) for some \( p \). One might also ask whether we can replace it with some other assumption. Shu and Yao proved in \( [7] \) a condition on a binary recurrence sequence, which guarantees \( p \)-regularity of \( \{\nu_p(s_n)\}_{n \geq 0} \), and mentioned a possible generalization to recurrences of higher order. It is known that \( p \)-regular sequences grow at most polynomially, which is a result by Allouche and Shallit \( [1] \). Unfortunately, this does not give a bound on \( C \) in \( (4) \) and the proof of Theorem 2 fails if \( C \geq 1 \). Therefore, some additional information besides regularity needs to be known about \( \{\nu_p(s_n)\}_{n \geq 0} \) in order to put the theorem to use.
The reasoning in Theorem 2 provides an upper bound on the solutions of the equation (5) if we are able to find the values of \(C, K_1, K_2\) and \(n_0\). We will show that it is indeed the case for our sequence \(\{t_n\}_{n \geq 0}\). We start by determining the 2-adic valuation of each \(t_n\). A similar characterization of \(\{v_2(t_n)\}_{n \geq 0}\) for \(r = 2\) is given by Lengyel in [3] and [4] (by a different method).

**Theorem 3.** If \(k \geq 2\), then the sequence \(\{v_2(t_n)\}_{n \geq 0}\) satisfies the following conditions:

\[
\begin{align*}
v_2(t_n) &= 0 & \text{for } n \equiv 1, 2, \ldots, 2k \pmod{2k + 1}, \\
v_2(t_n) &= 1 & \text{for } n \equiv 2k + 1 \pmod{2(2k + 1)}, \\
v_2(t_n) &= v_2(n) + v_2(k - 1) + 2 & \text{for } n \equiv 0 \pmod{2(2k + 1)}.
\end{align*}
\]

The proof of the theorem is presented in Section 3.

**Remark 3.** Theorem 3 implies that \(\{v_2(t_n)\}_{n \geq 0}\) is a 2-regular sequence. Note that this conclusion does not follow directly from the results of [7], as we consider recurrence of any even order.

Now we proceed to show that Theorem 2 can be applied to our sequence \(\{t_n\}_{n \geq 0}\). The following lemma establishes a lower bound on \(t_n\).

**Lemma 4.** For all \(n \geq 1\) we have

\[
t_n \geq \phi^{n-r-1},
\]

where \(\phi\) is the unique real root of the equation \(x^r = x^{r-1} + \ldots + x + 1\) lying in the interval \((1, 2)\).

**Proof.** By lemma 3.6 in [8] there is indeed exactly one such \(\phi\), which in addition lies in the interval \((2(1 - 2^{-r}), 2)\). For \(n = 1, 2, \ldots, r\) the inequality (12) follows from starting conditions for \(t_n\) and the fact that \(t_r = 2(r - 1) \geq 2 > \phi\). Then we proceed easily by induction. \(\square\)

**Corollary 5.** If \(r = 2k \geq 4\) then the equation (2) has only a finite number of solutions in \(m, n_1, n_2, \ldots, n_d\) and this number can be effectively bounded from above.

**Proof.** By theorem 3 we have

\[
v_2(t_n) \leq v_2(n) + v_2(k - 1) + 2 \leq \log_2(n) + v_2(k - 1) + 2 \leq \sqrt{n},
\]

where the last inequality is true for example for \(n \geq 2^{2 \max(2, v_2(k - 1))}\). By lemma 4

\[
\log t_n \geq (n - 2k - 1) \log \phi \geq \frac{1}{2} n \log \phi,
\]

for \(n \geq 2(2k + 1)\), because \(\phi > 1\). Hence, we can take \(n_0 = \max\{2(2k + 1), 2^{2 \max(2, v_2(k - 1))}\}\) and apply the method used in Theorem 2 to find an upper bound on \(m, n_1, n_2, \ldots, n_d\). \(\square\)

We could apply Corollary 5 to find all non-trivial solutions (with \(t_{n_i} > 1\) for each \(i\)) of the equation (2) for given \(k\) and \(d\). However, using the explicit form of \(v_2(t_n)\), one can make the bounds much more precise. We will once again follow the approach shown in [5]. The computations are quite similar to those in Theorem 2 so we omit the details.
By Lemma 1 and Theorem 3 we obtain

\[ m - \lfloor \log_2 m \rfloor - 1 \leq \nu_2(m!) = \sum_{i=1}^{d} \nu_2(t_{n_i}) \leq \sum_{i=1}^{d} [\nu_2(n_i) + \nu_2(k-1) + 2] \]

\[ = d[\nu_2(k-1) + 2] + \nu_2 \left( \prod_{i=1}^{d} n_i \right) \]

(12)

\[ \leq d[\nu_2(k-1) + 2] + \log_2 \left( \prod_{i=1}^{d} n_i \right) . \]

On the other hand, from Lemma 4 and inequality (8), we get for \( m \geq 5 \)

(13) \[ \left[ \sum_{i=1}^{d} n_i - d(2k-1) \right] \log_2 \phi_k \leq \sum_{i=1}^{d} \log_2(t_{n_i}) = \log_2(m!) < m(\log_2 m - 1) , \]

where \( \phi_k \) is the value of \( \phi \) in Lemma 4 corresponding to \( r = 2k \). The AM–GM inequality applied to all \( n_i \), together with (12) and (13), yields

(14) \[ m - \lfloor \log_2 m \rfloor - 1 - d[\nu_2(k-1) + 2] < d \log_2 \left[ \frac{m}{d \log_2 \phi_k} (\log_2 m - 1) + (2k-1) \right] . \]

This gives an upper bound on \( m \) and, consequently, on each \( n_i \). As an example, in the table below we give the upper bound on \( m \) obtained for 2k-nacci sequences with \( 2 \leq k \leq 5 \) and \( 1 \leq d \leq 10 \).

| \( d \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 11 | 19 | 27 | 35 | 43 | 51 | 59 | 67 | 75 | 84 |
| 3 | 13 | 22 | 31 | 40 | 49 | 58 | 67 | 76 | 85 | 94 |
| 4 | 11 | 19 | 28 | 36 | 44 | 52 | 60 | 68 | 76 | 84 |
| 5 | 14 | 25 | 35 | 46 | 56 | 67 | 77 | 88 | 98 | 109 |

Using this result we find that the only non–trivial solution of the equation (2) with \( 2 \leq k \leq 5 \) and \( 1 \leq d \leq 10 \) appears in the 4-nacci sequence and is the single term \( t_5 = 3! \).

3. Proof of Theorem 3

In order to study the 2-adic valuation of \( t_n \) it is enough to focus on \( n \) divisible by \( 2k+1 \) as all the other terms are odd. In this case we can write \( n = s2^l(2k+1) \) for \( s \) odd and \( l \geq 0 \). We will divide our proof into two main parts.

First, we will show by induction on \( l \) and \( s \) that \( 2k \) consecutive terms of the sequence \( \{t_n\}_{n \geq 0} \), starting with \( ts2^l(2k+1) \), satisfy a particular system of congruences, given in Lemma 7. However, as it will turn out, this argument works only for \( l \geq l_0 \), where \( l_0 \) depends on \( \nu_2(k-1) \). Moreover, the initial system of congruences for \( l = l_0 \) involves some constants, which need to be computed.

Therefore, we will have to employ another method for \( l \leq l_0 \). We will show how to obtain the values of \( t_{n+2k+1}, \ldots, t_{n+4k} \) in terms of \( t_n, \ldots, t_{n+2k-1} \) for any \( n \) and then proceed by induction. As a result, we will be able to express \( t_n \) in quite concrete form, given in Lemma 10, involving binomial coefficients weighted by powers of 2.
For simplicity, we introduce the following matrix notation:

$$T_n = \begin{bmatrix} t_n \\ t_{n+1} \\ \vdots \\ t_{n+2k-1} \end{bmatrix}, \quad B_n = \begin{bmatrix} t_n & t_{n+1} & \cdots & t_{n+2k-1} \\ t_{n+1} & t_{n+2} & \cdots & t_{n+2k} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n+2k-1} & t_{n+2k} & \cdots & t_{n+4k-2} \end{bmatrix},$$

where $n \geq 0$. By $C$ we will denote the companion matrix of $t_n$ which has the form

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

(15)

where the entries above the diagonal and in the bottom row are equal to 1 and all other entries are zero. It is easy to check that $C T_n = T_{n+1}$ and $C B_n = B_{n+1}$, so for any positive integers $n$ and $w$ we have

$$C^n T_w = T_{n+w},$$

(16)

$$C^n B_w = B_{n+w}. $$

(17)

First, we state an identity involving the terms of the sequence \{t_n\}_{n \geq 0}.

**Lemma 6.** The matrix $B_0$ is invertible and its determinant is odd. Moreover, for all positive integers $n$, $w$ we have the formula

$$t_{n+w} = T_n^T B_0^{-1} T_w.$$  

(18)

**Proof.** It is easy to see that $t_n$ is even if $n$ is divisible by $2k + 1$. Therefore,

$$\det B_0 \equiv \det \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & \vdots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \end{bmatrix} \pmod{2},$$

where zeros in the latter matrix appear only at positions corresponding to $t_0$ and $t_{2k+1}$ in $B_0$, that is, at $(1,1)$ and $(i,j)$ such that $i + j = 2k + 2$. By subtracting the first row from all the others, we easily obtain $\det B_0 \equiv 1 \pmod{2}$, which proves the first part of the statement.

From (16) and (17), we get

$$T_{n+w} = C^n T_w = B_n B_0^{-1} T_w.$$  

The first coordinate gives us the formula for $t_{n+w}$. \[ \square \]

The identity (18) might seem difficult to apply without an explicit expression for $B_0^{-1}$. However, it plays a major role in deriving the congruence relations in the following lemma.

**Lemma 7.** For any $l \geq 0$ the following congruence relation holds:

$$T_{2^l(2k+1)} \equiv T_0 \pmod{2^{l+1}}.$$ 

(19)

Moreover, if a column vector $A \in \mathbb{Z}^{2k}$ satisfies

$$T_{2^l(2k+1)} \equiv 2^{l_0+1} A + T_0 \pmod{2^{l_0+\nu_2(k-1)+3}},$$

$$T_{2^l(2k+1)} \equiv 2^{l_0+1} A + T_0 \pmod{2^{l_0+\nu_2(k-1)+3}},$$

where $l_0$ is the largest integer such that $\nu_2(k-1)+3 \leq 2^{l_0+1}$, and $\nu_2(k-1)+3 \leq 2^{l_0+1}$. 


where \( l_0 = \nu_2(k - 1) + 2 \), then for any \( l \geq l_0 \) and \( s \geq 1 \) we also have
\[
T_{2^l(2k+1)} \equiv s2^{l+1}A + T_0 \pmod{2^{l+\nu_2(k-1)+3}}.
\]

**Proof.** Obviously (19) is true for \( l = 0 \). Now assume that (19) holds for some \( l \geq 0 \). We can write
\[
t_{2^l(2k+1)+j} = 2^{l+1}a_{l,j} + t_j,
\]
where \( a_{l,j} \) are some positive integers for \( j = 0, 1, \ldots, 2k - 1 \). Define also \( a_{l,2k}, a_{l,2k+1}, \ldots, a_{l,4k-2} \) by the same recurrence as \( \{t_n\}_{n \geq 0} \). Then (21) is satisfied for \( j = 0, 1, \ldots, 4k - 2 \). For convenience denote by \( e_j \in \mathbb{Z}^{2k} \) the vector with 1 on the \( j \)-th position (counting from 0) and 0 on the other positions, and additionally define
\[
A_{l,j} = \begin{bmatrix} a_{l,j} & a_{l,j+1} & \cdots & a_{l,j+2k-1} \end{bmatrix}^T
\]
for \( j = 0, 1, \ldots, 2k - 1 \). It follows from the definition of \( B_0 \) that \( B_0^{-1}T_j = e_j \).

Fix any \( 0 \leq i \leq 2k - 1 \). The formula (18) yields
\[
t_{2^{l+1}(2k+1)+i} = T_{2^l(2k+1)}^T B_0^{-1} T_{2^l(2k+1)+i}
\]
Therefore, using (21) we get
\[
t_{2^{l+1}(2k+1)+i} = 2^{2l+2}A_{l,0}^T B_0^{-1} A_{l,i} + 2^{l+1} \left( A_{l,0}^T e_i + e_0^T A_{l,i} \right) + T_0^T e_i
\]
for some rational \( c_{l,i} \) such that \( \det(B_0)c_{l,i} \) is an integer. In Lemma 6 however, we showed that \( \det B_0 \) is odd which means that \( c_{l,i} \) must be an integer. Thus,
\[
2^{2l+2}|t_{2^{l+1}(2k+1)+i} - t_i|
\]
from which (19) follows. If we choose \( l \geq l_0 = \nu_2(k - 1) + 2 \) then the term \( 2^{2l+2}c_{l,i} \) in (22) is reduced modulo \( 2^{l+\nu_2(k-1)+4} \). We can take \( A = A_{l,0,0} \) to complete the proof of (20) for \( s = 1 \).

To proceed by induction on \( s \) notice that the index \( s2^l(2k + 1) + i \) can be expressed as a sum of indices in the following way:
\[
s2^l(2k + 1) + i = (2^l(2k + 1) + i) + (s - 1)2^l(2k + 1).
\]
We can then perform a similar computation as in (22) to get the desired result. \( \Box \)

Our specific choice of the divisor in (20) equal to \( 2^{l+\nu_2(k-1)+3} \) is based on the observation of \( t_n \) and is indeed effective in proving the formula for \( \nu_2(t_n) \). The numbers \( a_{l,j} \) in (21) are determined uniquely modulo \( 2^{\nu_2(k-1)+2} \). We are particularly interested in finding the value of \( a_{l,0,0} \) which will directly give us the 2-adic valuation of \( t_{2^l(2k+1)} \) for \( l \geq l_0 = \nu_2(k - 1) + 2 \), provided that \( a_{l,0,0} \leq \nu_2(k - 1) + 1 \). However, as mentioned before, we need to develop another method to analyze the case when \( l \leq l_0 \).

We start with deriving a formula for expressing \( t_{n+2k+1}, \ldots, t_{n+4k} \) in terms of \( t_n, \ldots, t_{n+2k-1} \).

**Lemma 8.** Define \( C \) as in (12). Then
\[
C^{2k+1} = 2 \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2^{2k-1} & 2^{2k-1} & \cdots & 2^{2k-1} \end{bmatrix} - \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2^{2k-2} & 2^{2k-3} & \cdots & 1 & 0 \\ 2^{2k-1} & 2^{2k-2} & \cdots & 2 & 1 \end{bmatrix}
\]

where \( l_0 = \nu_2(k - 1) + 2 \), then for any \( l \geq l_0 \) and \( s \geq 1 \) we also have
\[
T_{2^l(2k+1)} \equiv s2^{l+1}A + T_0 \pmod{2^{l+\nu_2(k-1)+3}}.
\]
Proof. Using the identity \( t_{n+2k+1} = 2t_{n+2k} - t_n \), one can show by induction that for any \( 0 \leq i \leq 2k - 1 \)

\[
(23) \quad t_{n+2k+1+i} = 2^{i+1}t_{n+2k} - \sum_{j=0}^{i} 2^{i-j}t_{n+j} = 2 \cdot 2^j \sum_{j=0}^{2k-1} t_{n+j} - \sum_{j=0}^{i} 2^{i-j}t_{n+j}.
\]

We know that \( D = C^{2k+1} \) is the only matrix satisfying \( T_{n+2k+1} = DT_n \) for all \( n \geq 0 \). Thus, for each \( 0 \leq i \leq 2k - 1 \) the coefficients at \( t_{n+j} \) in (23) correspond to the \( i \)-th row of \( C^{2k+1} \) (counting from 0). \( \square \)

We are also going to need two standard identities involving binomial coefficients. For the convenience of the reader we include the proof.

**Lemma 9.** For all positive integers \( m, w \) we have

(a) \( \sum_{i=0}^{w} \binom{m+i}{m} = \binom{m+w+1}{m+1} \),

(b) \( \sum_{i=0}^{w} (m+i)2^i = (-1)^{m+1} + 2^{w+1} \sum_{j=0}^{m} \binom{m+w+1}{m-j}(-2)^j \).

**Proof.** For any fixed \( m \geq 0 \) the formula (a) follows easily from induction on \( w \).

To prove (b) take any \( w, m \geq 0 \) and consider the function

\[
f(x) = \frac{1}{m!} \sum_{i=0}^{w} x^{i+m} = \frac{1}{m!} \frac{x^{m+w+1} - x^m}{x - 1}
\]

for \( x \neq 1 \). It is easy to see that the left side of (b) is equal to \( f^{(m)}(x) \) evaluated at \( x = 2 \). Applying the Leibniz formula we get

\[
f^{(m)}(x) = \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} \left( \frac{1}{x - 1} \right)^{(j)} (x^{m+w+1} - x^m)^{(m-j)}
\]

\[
= \sum_{j=0}^{m} \frac{(-x)^j}{(x-1)^{j+1}} \left[ \binom{m+w+1}{m-j} x^{w+1-n} - \binom{m}{j} \right]
\]

\[
= x^{w+1} \sum_{j=0}^{m} \binom{m+w+1}{m-j} \frac{(-x)^j}{(x-1)^{j+1}} + \frac{1}{x-1} - (x-1)^{m+1}.
\]

Substituting \( x = 2 \) we get the desired result. \( \square \)

The following lemma gives us an easily computable expression for \( 2k \) subsequent terms \( t_n \).

**Lemma 10.** Define the following column vectors in \( \mathbb{N}^{2k} \):

\[
w = [1 \quad 1 \quad \ldots \quad 1]^T, \quad v_m = \left[ \binom{m}{0} \quad \binom{m}{1} \cdot 2^1 \quad \binom{m}{2} \cdot 2^2 \quad \ldots \quad \binom{m}{m} \cdot 2^{2k-1} \right]^T
\]

for \( m \geq 0 \). Then for any \( m \geq 1 \) we have

\[
(24) \quad T_{m(2k+1)} \equiv w + (-1)^{m+1} \cdot 4(k-1) \sum_{i=0}^{m-1} v_i + (-1)^{m+1} v_{m-1} \quad (\text{mod } 2^{2k+1}).
\]

**Proof.** Using the form of \( C^{2k+1} \) given in Lemma 8 it is easy to see that

\[
(25) \quad C^{2k+1}w = 2 \cdot 2kv_0 - (2v_0 - w) = w + 2(2k-1)v_0.
\]

Now fix \( m \geq 0 \). Applying Lemma 9 to each coordinate gives us

\[
(26) \quad C^{2k+1}v_m = 2 \cdot (-1)^{m+1}v_0 - v_{m+1} \quad (\text{mod } 2^{2k+1}).
\]
One can check that
\[ C^{2k+1}T_0 = w + 2(2k-1)v_0 - v_0 = w + 4(k-1)v_0 + v_0, \]
so (24) is true for \( m = 1 \). Now assume that (23) holds for some \( m \geq 1 \). Using (25) and (26) we get
\[ T_{(m+1)(2k+1)} = C^{2k+1}T_m(2k+1) \]
\[ \equiv C^{2k+1}\left[w + (-1)^{m+1} \cdot 4(k-1) \sum_{i=0}^{m-1} v_i + (-1)^{m+1}v_{m-1}\right] \]
\[ \equiv w + 2(2k-1)v_0 + (-1)^{m+1} \cdot 4(k-1) \sum_{i=0}^{m-1} [2 \cdot (-1)^{i+1}v_0 - v_{i+1}] \]
\[ + (-1)^{m+1} \cdot 2 \cdot (-1)^m v_0 - v_m \]
\[ \equiv w + [4(k-1) - 8(k-1)\epsilon_m]v_0 + (-1)^m \cdot 4(k-1) \sum_{i=1}^{m} v_i \]
\[ + (-1)^m v_m \pmod{2^{2k+1}}, \]
where \( \epsilon_m \) is equal to \( m \) modulo 2. Thus, the coefficient at \( v_0 \) is equal to \( (-1)^m \cdot 4(k-1) \), so we can incorporate it into the sum. Finally, we obtain
\[ T_{(m+1)(2k+1)} \equiv w + (-1)^m \cdot 4(k-1) \sum_{i=0}^{m} v_i + (-1)^m v_m \pmod{2^{2k+1}}. \]
\[ \Box \]

We are now ready to prove Theorem 3.

Proof (of Theorem 3). The term \( t_n \) is even iff \( n \) is divisible by \( 2k+1 \), which proves that \( \nu_2(t_n) = 0 \) for \( n \equiv 1, 2, ..., 2k \pmod{2k+1} \). Observe that if \( k \geq 2 \), then \( 2\nu_2(k-1) + 5 \leq 2k+1 \), so by Lemma 6 for \( m \geq 1 \) we have
\[ T_m(2k+1) \equiv w + (-1)^{m+1} \cdot 4(k-1) \sum_{i=0}^{m-1} v_i + (-1)^{m+1}v_{m-1} \pmod{2^{2\nu_2(k-1)+5}}. \]
Looking at the first entry of this vector, we obtain
\[ (27) \quad t_{m(2k+1)} \equiv 1 + (-1)^{m+1} \cdot 4m(k-1) + (-1)^{m+1} \pmod{2^{2\nu_2(k-1)+5}}. \]
Thus, for odd \( m \) we get \( t_{m(2k+1)} \equiv 2 \pmod{4} \), hence \( \nu_2(t_n) = 1 \) for \( n \equiv 2k+1 \pmod{2(2k+1)} \).

Now let \( n = s(2k+1)2^l \) for odd \( s \) and \( l \geq 1 \), so that \( n \equiv 0 \pmod{2(2k+1)} \). We will further split the third case into two subcases, depending whether \( l \leq l_0 = \nu_2(k-1) + 2 \) or \( l > l_0 \). If \( l \leq l_0 \) then from (27) we obtain
\[ (28) \quad t_n \equiv s2^{l+2}(k-1) \pmod{2^{2\nu_2(k-1)+5}}, \]
so \( \nu_2(t_n) = l + 2 + \nu_2(k-1) = \nu_2(n) + \nu_2(k-1) + 2 \).

We cannot extend the same argument to \( l > l_0 \) because we only know the congruence modulo \( 2^{2\nu_2(k-1)+5} \). However, substituting \( s = 1 \) and \( l = l_0 \) in (28) gives us a possible value \( a_{l_0,0} = 2(k-1) \), as defined in Lemma 7. Using Lemma 7 for any \( l > l_0 \), we get in the first coordinate
\[ t_n \equiv 2^{l+1}a_{l_0,0} \equiv 2^{l+2}(k-1) \pmod{2^{l+\nu_2(k-1)+3}}, \]
which again yields \( \nu_2(t_n) = \nu_2(n) + \nu_2(k-1) + 2 \). \[ \Box \]
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