The Newtonian limit of metric gravity theories with quadratic Lagrangians

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Abstract
The Newtonian limit of fourth-order gravity is worked out discussing its viability with respect to the standard results of general relativity. We investigate the limit in the metric approach which, with respect to the Palatini formulation, has been much less studied in the recent literature, due to the higher order of the field equations. In addition, we refrain from exploiting the formal equivalence of higher-order theories considering the analogy with specific scalar–tensor theories, i.e. we work in the so-called Jordan frame in order to avoid possible misleading interpretations of the results. Explicit solutions are provided for several different types of Lagrangians containing powers of the Ricci scalar as well as combinations of the other curvature invariants. In particular, we develop the Green’s function method for fourth-order theories in order to find out solutions. Finally, the consistency of the results with respect to general relativity is discussed.

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1. Introduction

The study of possible modifications of Einstein’s theory of gravity has a long history which reaches back to the early 1920s [1–3].

 Corrections to the gravitational Lagrangian, leading to higher-order field equations, were already studied by several authors [4–6] shortly after general relativity was proposed. Developments in the 1960s and 1970s [7–11], partly motivated by the quantization schemes proposed at that time, made clear that theories containing only a $R^2$ term in the Lagrangian were not viable with respect to their weak-field behavior. Buchdahl, in 1962 [7], rejected pure $R^2$ theories because of the non-existence of asymptotically flat solutions.
The early proposed amendments of Einstein’s theory were aimed at the unification of gravity with other branches of physics, such as electromagnetism; recently the interest in such modifications comes also from cosmology. For a comprehensive review, see [12]. In order to explain observational data, additional ad-hoc concepts, such as dark energy/matter, are introduced within Einstein’s theory. On the other hand, the emergence of such stopgap measures in a cosmological context could be interpreted as a first signal of the breakdown of general relativity on these scales [13, 14] and led to the proposal of many alternative modifications of the underlying gravity theory (see [15, 16] for reviews).

While it is very natural to extend Einstein’s gravity to theories with additional geometric degrees of freedom, see, for example, [17–19] for some general surveys on this subject as well as [20] for a list of works in a cosmological context, recent attempts focused on the old idea of modifying the gravitational Lagrangian in a purely metric framework, leading to higher-order field equations. Due to the increased complexity of the field equations in this framework, the main body of works dealt with some formally equivalent theories, in which a reduction of the order of the field equations was achieved by considering the metric and the connection as independent objects [21]. In addition, many authors exploited the formal relationship to scalar–tensor theories to make some statements about the weak-field regime, which was already worked out for scalar–tensor theories [22].

In this paper, we shall study the Newtonian limit of fourth-order gravity theories in which extensions of the Hilbert–Einstein Lagrangian are considered. We are going to focus on the weak-field limit within the metric approach. At this point, it is useful to recall that it was already shown in [23] that different variational procedures do not lead to equivalent results in the case of quadratic-order Lagrangians.

In addition, we carry out our analysis in the so-called Jordan frame, i.e. we do not reduce the theory under consideration to a simpler one by means of a conformal transformation. This is due to the fact that it was already shown earlier [24, 25] that nonlinear theories of the kind considered here, with the exception of some special cases, could not be physically equivalent if they have undergone a conformal transformation. The debate on this topic is open as can be seen in the recent literature (see, for example, [26–31]).

By considering some admissible choices for the gravitational Lagrangian with quadratic corrections, we explicitly work out the weak-field limit. Such considerations are developed also in relation to the companion papers [32–34] where we have considered the Newtonian limit explicitly for \( f(R) \) gravity, taking into account the generic analytic functions of the Ricci scalar \( R \), and the spherically symmetric solutions versus the weak-field limit, respectively.

In principle, any alternative or extended theory of gravity should allow us to recover positive results of general relativity, for example in a weak-limit regime, then starting from the Hilbert–Einstein Lagrangian

\[
L_0 = R, \tag{1}
\]

the following terms,

\[
L_1 = R^2, \tag{2}
\]

\[
L_2 = R_{\alpha\beta}R^{\alpha\beta}, \tag{3}
\]

\[
L_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}, \tag{4}
\]

and combinations of them, represent the obvious minimal choices for an extended gravity theory with respect to general relativity. Since the variational derivative of \( L_3 \) can be linearly expressed [3, 35–37] via the variational derivatives of \( L_1 \) and \( L_2 \), one may omit \( L_3 \) in the final Lagrangian of a fourth-order theory without loss of generality. In this paper, we will consider
the Newtonian limit of the combined Lagrangian (1)–(3); see section 2, as a straightforward generalization of the Einstein theory which is obviously recovered in low-curvature regimes.

As we said, several works focused on the cosmological implications of additional terms to the Hilbert–Einstein Lagrangian [38, 39, 41, 42]. In particular, the terms of the form $R^{-n}$ (with positive $n$) have been taken into account to explain the observed accelerated behavior of the Hubble flow. Although such a term may lead to an alternative explanation of the acceleration effect, its singular behavior clearly leads to problems in the low-curvature regime. The explicit exclusion of flat solutions is in contradiction with the basic assumptions of most of the weak-field approximation schemes of general relativity, the most prominent examples being the post-Minkowskian and the post-Newtonian approximation [32]. In addition, the non-validity of flat solutions leads to the paradoxical situation that standard linearization procedures can no longer be based on Minkowskian space in the lowest order and are therefore not well defined in such a framework. While in a purely cosmological context this drawback could be circumvented by basing the analysis on a curved background, in the lowest order, it is not obvious how such a theory could make sense at local scales. In addition, it is not clear how one could have a smooth and well-defined transition to general relativity for this kind of theories. Due to these drawbacks, we will consider here theories which allow us to recover the flat solution.

The plan of this paper is the following. Section 2 is devoted to give the general form of the fourth-order field equations and their approximations at the lowest order in the weak-field limit, i.e. the Newtonian one. Section 3 is devoted to the solutions of the field equations in the Newtonian limit. In section 4, we develop, in details, the method of Green’s functions for a system with spherical symmetry, while in section 5, we exhibit explicit solutions derived using the Green’s function. We draw conclusion in section 6 and present a possible outlook for future developments. In appendix A, we present an alternative approach to solve the field equations, while we summarize the conventions and the dimensions of quantities used throughout the text in appendix B.

2. Gravity with quadratic Lagrangians: the field equations and the Newtonian limit

In this section, we discuss the fourth-order field equations and their Newtonian limit. This higher order with respect to the standard second order of Einstein field equations is due, as well known, to the integration of the boundary terms. These terms disappear in general relativity, thanks to the divergence theorem, but this is not possible for several alternative theories of gravity, as the higher-order ones, and then the derivative order of field equations results augmented.

In this paper, we are interested to achieve the correct Newtonian limit of gravity theories with quadratic Lagrangians in the curvature invariants. This result can be achieved under two main hypotheses: (i) asking for low velocities with respect to the light speed and (ii) asking for week fields. By these requests, the metric tensor is independent of time, and second-order perturbation terms can be discarded in the field equations (see also [32] for details). It is worth stressing that the Newtonian limit of any relativistic theory of gravity is related to such hypotheses and it is a misunderstanding to consider only the recovering of the Newtonian potential. In other words, a more general theory of gravity gives rise, in the Newtonian limit, to gravitational potentials which can be very different from the standard Newtonian one.

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4 As explicitly demanded by some authors, see [43]p 3.
5 We use the term local here for the rather broad range $\sim 10^{-2} - 10^{11}$ m, hence encompassing laboratory as well as solar-system experiments.
In the third and in the fourth part of this section, we shall discuss the field equations in the Newtonian limit to show that the fourth-order contributions to the potential cannot be trivially discarded.

2.1. General form of the field equations

Let us now come back to the choices displayed in (1)–(4), for which the left-hand side of the field equations takes the general form:

\[ 0 H_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \]  
\[ 1 H_{\mu\nu} = 2 R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^2 - 2 R_{;\mu\nu} + 2 g_{\mu\nu} \Box R, \]  
\[ 2 H_{\mu\nu} = 2 R_{\mu}^{\quad \alpha} R_{\nu\alpha} - \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - 2 R_{(\mu}^{\quad \alpha;\nu\beta)} + 2 g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + 4 R^\mu_{\quad \alpha} R^{\alpha\beta} R_{\nu\beta} + 4 R^\mu_{\quad \alpha} R_{\nu\beta} R^{\alpha\beta}, \]  
\[ 3 H_{\mu\nu} = 2 R_{\mu\nu\alpha\beta} R^{\alpha\beta} - \frac{1}{2} g_{\mu\nu} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + 4 R^\mu_{\quad \alpha} R^{\alpha\beta\gamma\delta} R_{\nu\beta\gamma\delta}. \]  

All of the three expressions in (6)–(8) involve fourth-order differential operators. Due to the identity

\[ 1 H_{\mu\nu} - 4^2 H_{\mu\nu} + 3 H_{\mu\nu} = 0, \]  

which holds in a four-dimensional spacetime [37], only two of the expressions in (6)–(8) are independent, and we are free to use any two independent linear combinations in our analysis. This identity gives rise to the well-known Gauss–Bonnet topological invariant which recently acquired a lot of importance in cosmology as a possible source of dark energy [40]. Furthermore, for a Lagrangian comprising a general function of the Ricci scalar, we have

\[ f(R) H_{\mu\nu} = \frac{df}{dR} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f - \left( \frac{df}{dR} \right)_{;\mu\nu} + g_{\mu\nu} \Box \frac{df}{dR}. \]  

Here we denoted the covariant derivatives by a semicolon \(^6\) and the d’Alembert operator by \(\Box\). With these considerations in mind, let us to consider the Newtonian limit of such a theory of gravity.

2.2. The Newtonian limit

Here we are not interested in entering the theoretical discussion on how to formulate a mathematically well-sound Newtonian limit of general relativistic field theories; for this, we point the interested reader to [44–50]. In this section, we provide the explicit form of the field equations for the different admissible choices of Lagrangians collected in the introduction at the lowest, i.e. Newtonian, order. In the language of the post-Newtonian approximation, we are going to consider the field equations up to the order \(O(c^{-2})\), where \(c\) denotes the speed of light. For the verification of our calculations we made use of a modified version of the Procrustes package [51].

We only mention, in passing, that there has also been a discussion of a somewhat alternative way to define the Newtonian limit in higher-order theories in the recent literature; see for example [52]. In this work, the Newtonian limit is identified with the maximal symmetric solution, which is not necessarily Minkowski spacetime in \(f(R)\) theories which could be singular.

\(^6\) We denote partial derivatives with respect to the coordinates by a comma.

\(^7\) \(\Box = \frac{\partial^2 - \nabla \nabla}{c^2}.\)
Let us start from a flat background and work out the corresponding field equations and hydrodynamic equations to the Newtonian order. Our conventions are that $g_{\alpha\beta}$, with $\alpha, \beta = 0, 1, 2, 3$, can be transformed to $\eta_{\alpha\beta} = \text{diag} (1, -1, -1, -1)$ along a given curve. Latin indices $i, j$ run from 1, 2, 3; the coordinates are labeled by $x^a = (c^0, x^1, x^2, x^3) = (ct, x^1, x^2, x^3)$. We start with the following ansatz for the metric:

\begin{align*}
g_{00} &= 1 - \frac{2U}{c^2} + O(c^{-4}), \\
g_{0a} &= \frac{1}{c^3} h_{0a} + O(c^{-2}), \\
g_{ab} &= -\left(1 + \frac{2V}{c^2}\right) \delta_{ab} + O(c^{-4}). \tag{11}
\end{align*}

Apparently, the orders involved in this ansatz for the line element reach beyond the Newtonian order, which we are mainly interested in this work.

On the matter side, i.e. the right-hand side of the field equations, we start with the general definition of the energy–momentum tensor of a perfect fluid:

\begin{equation}
T_{\alpha\beta} = \left(\rho c^2 + \Pi \rho + p\right) u_\alpha u_\beta - p g_{\alpha\beta}, \tag{12}
\end{equation}

where $\Pi$ denotes the internal energy density, $\rho$ the energy density and $p$ the pressure. Following the procedure outlined in [53], we derive the explicit form of the energy momentum as follows:

\begin{align*}
T_{00} &= \rho c^2 + O(c^{-2}), \tag{13} \\
T_{0a} &= c\rho v^a + O(c^{-1}), \tag{14} \\
T_{ab} &= \rho v^a v^b + p \delta_{ab} + O(c^{-2}). \tag{15}
\end{align*}

The general form of the field equations is given by

\begin{equation}
H_{\mu\nu} = 8\pi G c^4 T_{\mu\nu}, \tag{16}
\end{equation}

with a generalized tensor, $H_{\mu\nu}$, being a combination of the expressions specified in (5)–(8), which, in turn, depend on the final form of the Lagrangian.

### 2.3. The quadratic Lagrangians and the Newtonian limit of the field equations

Let us consider now the field equations, in the Newtonian limit, for the possible quadratic Lagrangians which we compare to the Newtonian limit of the standard Hilbert–Einstein Lagrangian. It is important to stress that the field equations, in the Newtonian limit, are considered up to the order $O(c^{-2})$ while the vector component to the order $O(c^{-3})$ is related to the post-Newtonian limit of the theory (see [54] for details). Up to the Newtonian order, the left-hand side of the field equations, i.e. (5)–(8), takes the following form for the metric given in (11):

- The Hilbert–Einstein Lagrangian ($L_0 = R$)

\begin{align*}
0 H_{00} &= -\frac{2}{c^2} \nabla^2 V, \tag{17} \\
0 H_{0a} &= 0, \tag{18}
\end{align*}

\begin{itemize}
  \item Obviously $0 H_{\mu\nu}$ is the Einstein tensor.
  \item Here we made use of the following operator definition $\nabla^2 := \delta^{ab} \frac{\partial^2}{\partial x^a \partial x^b}$, as well as $\nabla^4 := \nabla^2 \nabla^2$.
\end{itemize}
\[ 0 H_{ab} = \frac{1}{c^2} [ (\nabla^2 V - \nabla^2 U) \delta_{ab} - (V - U)_{,ab} ]. \]  
(19)

- The \( R^2 \)-Lagrangian (\( \mathcal{L}_1 = R^2 \))

\[ 1 H_{00} = \frac{4}{c^2} (2 \nabla^4 V - \nabla^4 U), \]  
(20)

\[ 1 H_{0a} = 0, \]  
(21)

\[ 1 H_{ab} = \frac{4}{c^2} [(\nabla^4 U - 2 \nabla^4 V) \delta_{ab} + (2 \nabla^2 V - \nabla^2 U)_{,ab} ]. \]  
(22)

- The \( R_{\alpha\beta} R^{\alpha\beta} \)-Lagrangian (\( \mathcal{L}_2 = R_{\alpha\beta} R^{\alpha\beta} \))

\[ 2 H_{00} = \frac{2}{c^2} (2 \nabla^4 V - \nabla^4 U), \]  
(23)

\[ 2 H_{0a} = 0, \]  
(24)

\[ 2 H_{ab} = \frac{1}{c^2} [(\nabla^4 U - 3 \nabla^4 V) \delta_{ab} + (3 \nabla^2 V - \nabla^2 U)_{,ab} ]. \]  
(25)

- The \( R_{\alpha\beta\gamma \delta} R^{\alpha\beta\gamma \delta} \)-Lagrangian (\( \mathcal{L}_3 = R_{\alpha\beta\gamma \delta} R^{\alpha\beta\gamma \delta} \))

\[ 3 H_{00} = -\frac{4}{c^2} \nabla^4 U, \]  
(26)

\[ 3 H_{0a} = 0, \]  
(27)

\[ 3 H_{ab} = -\frac{4}{c^2} (\nabla^4 V \delta_{ab} - \nabla^2 V_{,ab}). \]  
(28)

2.4. The combined Lagrangian

Let us now combine the different terms of the last section into the same Lagrangian. This combination is the basis for our investigation for the remaining part of the paper. Since terms resulting from \( R^n \) with \( n \geq 3 \) do not contribute at the order \( \mathcal{O} \left( c^{-2} \right) \), the most general choice for the Lagrangian is

\[ L = a_1 R + a_2 R^2 + b_1 R_{\mu\nu} R^{\mu\nu} + c_1 R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}. \]  
(29)

Due to the identity given in (9), it is sufficient to study

\[ L = a_1 R + a_2 R^2 + b_1 R_{\mu\nu} R^{\mu\nu}, \]  
(30)

in four dimensions, where we introduced the constants \( a_1, a_2, b_1 \). If we take into account the results from (13)–(15) as well as (17)–(25) the explicit form of the field equations (16) up to the Newtonian order is

\[ -2a_1 \nabla^2 V + (8a_2 + 2b_1) \nabla^4 V - (4a_2 + 2b_1) \nabla^4 U = 8\pi G \rho, \]  
(31)

\[ [a_1 (\nabla^2 V - \nabla^2 U) - (8a_2 + 3b_1) \nabla^4 V + (4a_2 + b_1) \nabla^4 U] \delta_{ab} + [(8a_2 + 3b_1) \nabla^2 V - (4a_2 + b_1) \nabla^2 U + a_1 (U - V)]_{,ab} = 0, \]  
(32)

the equations which we are going to solve.

\[ ^{10} \text{Note that } [a_1] = [\text{length}]^4, [a_2] = [b_1] = [\text{length}]^2. \]

\[ ^{11} \text{The coefficients of (29) are different from those of (30).} \]
3. Considerations on the field equations in the Newtonian limit

In this section, we are going to formulate the problem to solve the field equations (31)–(32) in the most general way. It is worth noting that the isotropic coordinates for the metric (11) allow us to search for solutions independently of the symmetry of the physical system (which can be spherical, cylindrical, etc). The results which we are going to achieve are completely general since we will search for solutions in terms of Green’s functions. However, being the combined Lagrangian (30) built up by various terms, the field equations strictly depend on the coupling constants. As we will see, the value of such coefficients has a crucial role for the validity of the approach since, from a physical viewpoint, we have to obtain the Newtonian limit of general relativity as soon as the quadratic corrections disappear. This aspect of the problem is not accurately faced in the literature and can lead to wrong conclusions. Our aim is to develop a method which allows us to control, step by step, the Newtonian limit in agreement with the results of general relativity. This is possible at three levels: Lagrangian, field equations and solutions. In the second part of this section, we will analyze the various cases of field equations considering particular values of the coefficients. Specifically, we will take into account the values where the proposed approach fails. It is interesting to note that any time the Hilbert–Einstein term is absent into the Lagrangian, the field equations are fourth order (see table 1). In other words, if the Hilbert–Einstein term is not present, we do not recover the Laplace/Poisson equations.

3.1. The general approach to decouple the field equations

By introducing two new auxiliary functions (A and B), equations (31)–(32) become

\[
\nabla^2 \left\{ \frac{4a_2 + b_1}{2a_2 + b_1} A + \frac{a_1}{2a_2 + b_1} B + \nabla^2 \left[ \frac{2b_1(3a_2 + b_1)}{a_1(2a_2 + b_1)} A - \frac{2a_2}{2a_2 + b_1} B \right] \right\} = 8\pi G \rho, \tag{33}
\]

\[
\nabla^2 (A + \nabla^2 B) \delta_{ab} - (A + \nabla^2 B)_{ab} = 0, \tag{34}
\]

where A and B are linked to U and V via

\[
A := a_1 (V - U), \tag{35}
\]

\[
B := (4a_2 + b_1) V - (8a_2 + 3b_1) U. \tag{36}
\]

Obviously, we must require \(a_1(2a_2 + b_1) \neq 0\), which is the determinant of transformations (35)–(36). Let us introduce a new function \(\Phi\) defined as follows:

\[
\Phi := A + \nabla^2 B. \tag{37}
\]

At this point, we can use the new function \(\Phi\) to decouple the system (33)–(34). In fact, we obtain

\[
-\frac{2b_1(3a_2 + b_1)}{a_1(2a_2 + b_1)} \nabla^6 B - \frac{6a_2 + b_1}{2a_2 + b_1} \nabla^4 B + \frac{a_1}{2a_2 + b_1} \nabla^2 B = 8\pi G \rho - \nabla^2 \tau_I, \tag{38}
\]

\[
\nabla^2 \Phi_{ab} - \Phi_{ab} = 0, \tag{39}
\]

where \(\tau_I := \frac{4\omega e_{ab} + \omega}{\dot{\omega}(2a_2 + b_1)} \Phi + \frac{2b_1(3a_2 + b_1)}{a_1(2a_2 + b_1)} \nabla^2 \Phi\). We are interested in the solution of (38) in terms of the Green’s function \(G_I(x, x')\) defined by

\[
B(x) = Y_I \int d^3 x' G_I(x, x') \sigma_I(x'), \tag{40}
\]
where
\[ \sigma_I(x) := 8\pi G\rho(x) - \nabla^2 \tau_I(x), \]
and \( Y_I \) being a constant, which we introduced for dimensional reasons. Then the set of equations (31)–(32) is equivalent to
\[ \frac{2b_1(3a_2 + b_1)}{a_1(2a_2 + b_1)} \nabla^2_{x'} \tilde{G}_I(x, x') + \frac{6a_2 + b_1}{2a_2 + b_1} \nabla^4_{x'} \tilde{G}_I(x, x') = -\frac{a_1}{2a_2 + b_1} \nabla^2_x \tilde{G}_I(x, x') = -Y_I \delta(x - x'), \]

\[ \nabla^2 \Phi(x) \delta_{ab} - \Phi(x)_{,ab} = 0, \]
where \( \delta(x - x') \) is the three-dimensional Dirac \( \delta \)-function. The general solutions of equations (31)–(32) for \( U(x) \) and \( V(x) \), in terms of the Green’s function \( \tilde{G}_I(x, x') \) and the function \( \Phi(x) \), are
\[ U(x) = Y_I \left( \frac{8a_2 + 3b_1}{2a_1(2a_2 + b_1)} \nabla^2_{x'} \Phi(x') - \frac{4a_2 + b_1}{a_1(2a_2 + b_1)} \nabla^4_{x'} \Phi(x') \right) \int d^3 x' \tilde{G}_I(x, x') \left[ 8\pi G\rho(x') \right. \]
\[ \left. - \frac{2b_1(3a_2 + b_1)}{a_1(2a_2 + b_1)} \nabla^2_{x'} \Phi(x') - \frac{4a_2 + b_1}{a_1(2a_2 + b_1)} \nabla^4_{x'} \Phi(x') \right] - \frac{8a_2 + 3b_1}{2a_1(2a_2 + b_1)} \Phi(x), \]

\[ V(x) = Y_I \left( \frac{4a_2 + b_1}{2a_1(2a_2 + b_1)} \nabla^2_{x'} \Phi(x') - \frac{4a_2 + b_1}{2a_2 + b_1} \nabla^4_{x'} \Phi(x') \right) \int d^3 x' \tilde{G}_I(x, x') \left[ 8\pi G\rho(x') \right. \]
\[ \left. - \frac{2b_1(3a_2 + b_1)}{a_1(2a_2 + b_1)} \nabla^2_{x'} \Phi(x') - \frac{4a_2 + b_1}{a_1(2a_2 + b_1)} \nabla^4_{x'} \Phi(x') \right] - \frac{4a_2 + b_1}{2a_1(2a_2 + b_1)} \Phi(x). \]

Equations (31)–(32) represent a coupled set of fourth-order differential equations. The total number of integration constants is eight. With the substitution (37), it has been possible to decouple the set of equations, but now the differential order is changed. The total differential order is the same; indeed, we have one equation of sixth order (38) and another equation of second order (39), while previously we had two equations of fourth order. Obviously, the number of integration constants is conserved. The possibility of decoupling the field equations (31)–(32) is strictly related to the choice to express the auxiliary field \( A \) in terms of \( B \) by inverting relation (37), deriving equation (31) and reducing (32) to the second order. In appendix A, we discuss a different where the set of equations (31)–(32) remains of fourth order by using the relation between \( B \) and \( A \).

### 3.2. Field equations for particular values of the coupling constants

In this subsection, we want to analyze the behavior of the field equations (31)–(32) for those values of the coupling constants \( a_1, a_2, b_1 \) where the transformations (35)–(36) do not hold. Specifically, in table 1, we display several cases of the field equations (31)–(32), for different choices of the coupling constants, where the determinant of transformations (35)–(36) is zero. First of all, we have to note that, for \( a_2 = b_1 = 0 \) (case i in table 1), we trivially obtain the same result of general relativity in isotropic coordinates. Furthermore, by asking for \( U = V \), the spatial equation is satisfied. It is straightforward to derive the solution of the Newton potential (see section 5.3 for details). Particularly, interesting is also case iv, where both terms \( R_{ab} R^{ab} \) and \( R^2 \) give similar contributions. In other words, we have the same situation of the Lagrangian \( \mathcal{L} = a_1 R + a_2 R_{ab} R^{ab} \) with a redefinition of the couplings. However, this Lagrangian is compatible with transformations (35)–(36) and then the results of section 3.1
hold. In cases (ii, iii, v, vi, vii), the differential operator $\nabla^2$ never appears as a linear term since the invariants $R^2$ and $R_{\mu\nu}R^{\mu\nu}$ give rise to higher-order terms in the field equations. In these cases, the full field equations (not in the weak-field regime) give $g_{\mu\nu} \Box R - R_{\mu\nu}$ for the Lagrangian $R^2$ and $-2R_{(\mu|a}^{(a}|(\nu|b) + \Box R_{\mu\nu} + g_{\mu\nu} R^{\alpha\beta}:_{ab}$ for $R_{\mu\nu}R^{\mu\nu}$ which are fourth-order equations. In the weak-field regime, one obtains the equations reported in table 1. As we will see in section 5.3 devoted to the solutions, all these cases do not present a Newtonian potential.

### Table 1. Explicit form of the field equations for different choices of the coupling constants for which the determinant of transformations (35)–(36) vanishes. Cases i, ii, iii are the Lagrangians introduced in section 2.3 ($R, R^2, R_{\mu\nu}R^{\mu\nu}$).

| Cases | Choices of $a_1, a_2, b_1$ | Corresponding field equations |
|-------|-----------------|-------------------------------|
| i     | $a_2 = 0$       | $\nabla^2 V = -\frac{x^2}{a_1^2} \rho,$ |
|       | $b_1 = 0$       | $\nabla^2 [V - U]_{ab} - [V - U]_{ab} = 0$ |
| ii    | $a_1 = 0$       | $\nabla^2 (2V - U) = 2\frac{x^2}{a_2^2} \rho,$ |
|       | $b_1 = 0$       | $\nabla^2 (\nabla^2 (2V - U))_{ab} - [\nabla^2 (2V - U)]_{ab} = 0$ |
| iii   | $a_1 = 0$       | $\nabla^2 (U - V) = -\frac{4x^2}{a^2} \rho,$ |
|       | $a_2 = 0$       | $\nabla^2 (\nabla^2 (U - 3V))_{ab} - [\nabla^2 (U - 3V)]_{ab} = 0$ |
| iv    | $b_1 = -2a_2$   | $2a_2 \nabla^4 V - a_1 \nabla^2 V = 4\pi G \rho,$ |
|       |                 | $\nabla^2 [a_1 (V - U) - 2a_2 \nabla^2 (V - U)]_{ab}$ |
|       |                 | $-[a_1 (V - U) - 2a_2 \nabla^2 (V - U)]_{ab} = 0$ |
| v     | $a_1 = 0$       | $\nabla^4 U = 2\frac{x^4}{a^4} \rho,$ |
|       | $b_1 = -4a_2$   | $\nabla^2 (\nabla^2 V)_{ab} - [\nabla^2 V]_{ab} = 0$ |
| vi    | $a_1 = 0$       | $\nabla^4 V = 2\frac{x^4}{a^4} \rho,$ |
|       | $b_1 = -2a_2$   | $\nabla^2 (\nabla^2 (V - U))_{ab} - [\nabla^2 (V - U)]_{ab} = 0$ |
| vii   | $a_1 = 0$       | $\nabla^4 (2V + U) = 2\frac{x^4}{a^4} \rho,$ |
|       | $b_1 = -\frac{8\pi}{4}$ | $\nabla^2 (\nabla^2 U)_{ab} - [\nabla^2 U]_{ab} = 0$ |

4. Green’s functions for spherically symmetric systems

We are interested in the solutions of field equations (16) at the order $O(\varepsilon^{-2})$ by using the method of Green’s functions. We have to stress that the method of Green’s functions does not work in the general case since the field equations are nonlinear. However, the Newtonian limit of the theory (based on the hypothesis that metric nonlinear terms can be discarded) allows that also the field equations result linearized. By solving the field equations with the Green’s function method, one obtains, as a first result, the solution in terms of gravitational potential in the point-mass case. Then by using equations (44)–(45), obtained in the weak-field limit and then in the Newtonian linear approximation for a spatial distribution of matter, we obtain, in principle, the gravitational potential for a given density profile. If the matter possesses a spherical symmetry, also the Green’s function has to be spherically symmetric. In this case, the correlation between two points has to be a function of the radial coordinate only, that is, $G(x, x') = G(|x - x'|)$. It is important to stress again the fact that the approach works if and only if we are in the linear approximation, i.e. in the Newtonian limit.
4.1. A general Green’s function for the decoupled field equations

Let us introduce the radial coordinate \( r := |x - x'| \); with this choice, equation (42) for \( r \neq 0 \) becomes

\[
2b_1(3a_2 + b_1)\nabla^2 \mathcal{G}_I(r) + a_1(6a_2 + b_1)\nabla^4 \mathcal{G}_I(r) - a_1^2\nabla^2 \mathcal{G}_I(r) = 0,
\]

where \( \nabla^2 = r^{-2}\partial_r (r^{-2}\partial_r) \) is the radial component of the Laplacian in polar coordinates. The solution of (46) is

\[
\mathcal{G}_I(r) = K_{I,1} - \frac{1}{r} \left[ K_{I,2} + \frac{b_1}{a_1} \left( K_{I,3} e^{-\sqrt{-\frac{a_1}{a_2}}r} + K_{I,4} e^{\sqrt{-\frac{a_1}{a_2}}r} \right) \right] - \frac{2(3a_2 + b_1)}{a_1} \left( K_{I,5} e^{-\sqrt{-\frac{b_1}{b_2}}r} + K_{I,6} e^{\sqrt{-\frac{b_1}{b_2}}r} \right),
\]

(47)

where \( K_{I,1}, K_{I,2}, K_{I,3}, K_{I,4}, K_{I,5}, K_{I,6} \) are constants. The integration constants, \( K_{I,i} \), have to be fixed by imposing the boundary conditions at infinity and in the origin. A physically acceptable solution has to satisfy the condition \( \mathcal{G}(x, x') \to 0 \) if \( |x - x'| \to \infty \), then the constants \( K_{I,1}, K_{I,4}, K_{I,6} \) in equation (47) have to vanish. We note that, if \( a_2 = b_1 = 0 \), the Green’s function of the Newtonian mechanics is found. In this case, we have the complete analogy with the electromagnetism. More precisely, when we do not consider higher-order terms than the Hilbert–Einstein one in the gravitational Lagrangian, we obtain, in the Newtonian limit, a field equation analog to the electromagnetic one for the scalar component (electric potential). This means that we have the same form of the Green’s function [55].

To obtain the conditions on the constants \( K_{I,2}, K_{I,3}, K_{I,5} \) we consider the Fourier transform of \( \mathcal{G}(x, x') \):

\[
\mathcal{G}_I(x, x') = \int \frac{d^3k}{(2\pi)^{3/2}} \mathcal{G}_I(k) e^{i(\mathbf{k} \cdot \mathbf{x} - \mathbf{k} \cdot \mathbf{x}')}.
\]

(48)

\( \mathcal{G}_I(x, x') \) depends on the nature of the poles of \( |k| \) and on the values of the arbitrary constants \( a_1, a_2, b_1 \). If we define two new quantities \( \lambda_1, \lambda_2 \in \mathbb{R} \),

\[
\lambda_1 := \frac{a_1}{b_1}, \quad \lambda_2 := \frac{a_1}{2(3a_2 + b_1)},
\]

(49)

we obtain

\[
\mathcal{G}_I(x, x') = \frac{\pi}{18} \frac{Y^{-1}}{|x - x'|} \left[ \frac{\lambda_2^2 - \lambda_1^2}{\lambda_1^2 \lambda_2^2} \left( e^{-\lambda_1|x-x'|} - e^{-\lambda_2|x-x'|} \right) + \frac{e^{-\lambda_1|x-x'|}}{\lambda_1^2} + \frac{e^{-\lambda_2|x-x'|}}{\lambda_2^2} \right].
\]

(50)

This Green’s function corresponds to that in (47). Obviously, we have three possibilities for the parameters \( \lambda_1 \) and \( \lambda_2 \). In fact, \( \lambda_1 \) and \( \lambda_2 \) are related to the algebraic signs of \( a_1, a_2, b_1 \) and then we can also achieve real values for such parameters. This means that we have three possibilities: both imaginary, one real and one imaginary. In table 2, we provide the complete set of Green’s functions \( \mathcal{G}_I(x, x') \), depending on the choices of the coefficients \( a_2 \) and \( b_1 \) (with a fixed sign of \( a_1 \)). The various modalities in which we obtain the Green’s functions are due to the various sign combinations of the arbitrary constants. In general, the parameters \( \lambda_{1,2} \) indicate characteristic scale lengths where corrections to the Newtonian potential can be appreciated. It is worth noting that, thanks to the forms of the Green’s functions (see table 2), the Newtonian behavior is always asymptotically recovered. When one considers a point-like source, \( \rho \propto \delta(x) \), and by setting \( \Phi(x) = 0 \) the potentials (44)–(45) are proportional to \( \mathcal{G}_I(x, x') \). Without losing generality we have

\[
U(x) = U_0 \frac{e^{-\lambda_1|x|}}{|x|} + U_1 \frac{e^{-\lambda_2|x|}}{|x|} + U_2 \frac{e^{-\lambda_3|x|}}{|x|},
\]

(51)
In fact, if we consider the Fourier transform of the potentials \( U \) and \( V \), the potential sixth order of (42), which depends on the coupled form of the system of equations (31)–(32) at particular values of the parameters where the general approach developed in section 4.1 does not work. In addition, for a correct Newtonian component, we assumed \( a_1 > 0 \). In fact, when \( a_2 = b_1 = 0 \) the field equations (31) and (32) give us the Newtonian theory of gravity if \( a_1 = 1 \).

| Cases | Choices of \( a_2, b_1 \) | Green’s function \( \hat{G}_I(x, x') \) |
|-------|-----------------|---------------------|
| viii  | \( b_1 < 0 \) \( 3a_2 + b_1 > 0 \) | \( \sqrt{\frac{1}{4\pi}} \int \frac{d^3k}{(2\pi)^3/2} \frac{1}{2k^2} e^{i\mathbf{k} \cdot \mathbf{x}} - \frac{e^{-i(3\mathbf{k} \cdot \mathbf{x})}}{a_1} + \frac{e^{-i(3\mathbf{k} \cdot \mathbf{x})}}{a_1} \) |
| ix    | \( b_1 > 0 \) \( 3a_2 + b_1 < 0 \) | \( \sqrt{\frac{1}{4\pi}} \int \frac{d^3k}{(2\pi)^3/2} \frac{1}{2k^2} e^{i\mathbf{k} \cdot \mathbf{x}} + \frac{\cos(3\mathbf{k} \cdot \mathbf{x})}{a_1} - \frac{\cos(3\mathbf{k} \cdot \mathbf{x})}{a_1} \) |
| x     | \( b_1 < 0 \) \( 3a_2 + b_1 < 0 \) | \( \sqrt{\frac{1}{4\pi}} \int \frac{d^3k}{(2\pi)^3/2} \frac{1}{2k^2} e^{i\mathbf{k} \cdot \mathbf{x}} - \frac{e^{-i(3\mathbf{k} \cdot \mathbf{x})}}{a_1} - \frac{e^{-i(3\mathbf{k} \cdot \mathbf{x})}}{a_1} \) |

where \( U_0, U_1, U_2 \) are some integration constants. An analogous behavior is obtained for the potential \( V(x) \). We note that in the vacuum case we found a Yukawa-like correction to Newtonian mechanics but with two scale lengths related to the quadratic corrections in the Lagrangian (30) (see also the above expressions (49)). This behavior is strictly linked to the sixth order of (42), which depends on the coupled form of the system of equations (31)–(32). In fact, if we consider the Fourier transform of the potentials \( U \) and \( V \):

\[
U(x) = \int \frac{d^3k}{(2\pi)^3/2} \hat{U}(k) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad V(x) = \int \frac{d^3k}{(2\pi)^3/2} \hat{V}(k) e^{i\mathbf{k} \cdot \mathbf{x}},
\]

the solutions of equations (31)–(32) are

\[
U(x) = \int \frac{d^3k}{(2\pi)^3/2} \frac{4\pi G[a_1 + (8a_2 + 3b_1)\mathbf{k}^2] \hat{\rho}(k) e^{i\mathbf{k} \cdot \mathbf{x}}}{a_1 - \mathbf{k}^2 [a_1 + 2(3a_2 + b_1)\mathbf{k}^2]^2},
\]

\[
V(x) = \int \frac{d^3k}{(2\pi)^3/2} \frac{4\pi G[a_1 + (4a_2 + b_1)\mathbf{k}^2] \hat{\rho}(k) e^{i\mathbf{k} \cdot \mathbf{x}}}{a_1 - \mathbf{k}^2 [a_1 + 2(3a_2 + b_1)\mathbf{k}^2]^2},
\]

where \( \hat{\rho}(k) \) is the Fourier transform of the matter density. It is possible to show, by applying the Fourier transform to the potentials \( U \) and \( V \), that the poles in equations (53)–(54) are always three.

Finally, if \( \hat{\rho}(k) = \frac{M}{(2\pi)^3} \) (the Fourier transform of a point-like source) solutions (53)–(54) are similar to (51). In fact, if we suppose that \( b_1 \neq 0 \) and \( 3a_2 + b_1 \neq 0 \), solutions (53)–(54) are

\[
U(x) = \frac{GM}{a_1|x|} \left( 1 - \frac{4}{3} e^{-\lambda_1|x|} + \frac{1}{3} e^{-\lambda_2|x|} \right),
\]

\[
V(x) = \frac{GM}{a_1|x|} \left( 1 - \frac{2}{3} e^{-\lambda_1|x|} - \frac{1}{3} e^{-\lambda_2|x|} \right).
\]

4.2. Green’s functions for particular values of the coupling constants

The obtained Green’s functions deserve some comments. First of all, we have to consider the particular values of the parameters where the general approach developed in section 4.1 does
not work. For example, if \( b_1 = 0 \), we have only one Yukawa-like correction. The Green’s function has to satisfy the equation,

\[
3\nabla^4_x G_I(x, x') - \frac{a_1}{2a_2} \nabla^2_x G_I(x, x') = -Y^{-1}_I \delta(x - x'),
\]

obtained from (42) by setting \( b_1 = 0 \). In this case, the Green’s function (Fourier transformed) is

\[
\hat{G}_I(k) = -\frac{2a_2 Y^{-1}_I}{6a_2 k^4 + a_1 k^2},
\]

and the Lagrangian becomes \( L = a_1 R + a_2 R^2 \). Since at the level of the Newtonian limit, as discussed, the powers of the Ricci scalar higher then two do not contribute, we can conclude that (58) is the Green’s function for any \( f(R) \)-theory at the Newtonian order, if \( f(R) \) is some analytical function of the Ricci scalar. The same result is achieved by considering a particular choice of the constants in the theory, e.g. \( b_1 = -2a_2 \). In table 1 (case iv), we provide the field equations for this choice, and the related Green’s function is

\[
\hat{G}_I(2a_2 \nabla^4 - a_1 \nabla^2)(k) \propto \frac{1}{2a_2 k^4 + a_1 k^2}.
\]

The spatial behavior of (58)–(59) is the same, but the coefficients are different since the theories are different. The interpretation of the result is the same as that in section 3.2 since we have to take into account a proper scale length. In fact, equation (58) presents a null pole for \( k^2 = 0 \) which gives the standard Newtonian potential, and a pole in \( k^2 = \lambda^2_2 \), which gives the Yukawa-like correction. Finally, we need the Green’s function for the differential operator \( \nabla^4 \). From table 1, the field equations present always a quadratic Laplacian operator (case i excluded). This means that the equation to solve is

\[
\nabla^4_x G_I(\nabla^4)(x, x') = \delta(x - x').
\]

By introducing the variable \( r = |x - x'| \), we have that equation (60) becomes

\[
\nabla^4_x G_I(\nabla^4)(r) = 0,
\]

with the solution

\[
G_I(\nabla^4)(r) = K_{I,1} + \frac{K_{I,2}}{r} + K_{I,3} r + K_{I,7} r^2,
\]

where \( K_{I,1}, K_{I,2}, K_{I,3}, K_{I,7} \) are generic integration constants.

Let us now consider the fact that the Green’s function has to be null at infinity. The only possible physical choice for the squared Laplacian is

\[
G_I(\nabla^4)(x, x') \propto \frac{1}{|x - x'|}.
\]

Considering the last possibility, we will end up with a force law increasing with distance [10]. In conclusion, we have shown the general approach to find the solutions of the field equations by using the Green’s functions. In particular, the vacuum solutions with a point-like source have been used to find out directly the potentials; however, it remains the most important issue to find out solutions when we consider systems with an extended matter distribution.

5. Solutions by the Green’s functions in a spherically symmetric distribution of matter

In this section, we explicitly determine the gravitational potential in the inner and in the outer region of a spherically symmetric matter distribution. This is a delicate problem since the Gauss theorem is not valid for the gravity theories which we are considering. In fact, in the
Newtonian limit of general relativity, the equation for the gravitational potential, generated by a point-like source,
\[ \nabla^2 G_{\text{New mech}}(\mathbf{x}, \mathbf{x}') = -4\pi \delta(\mathbf{x} - \mathbf{x}'), \tag{64} \]
is not satisfied by the new Green’s functions developed above. If we consider the flux of force lines \( F_{\text{New mech}} \), defined as
\[ F_{\text{New mech}} := -\frac{GM(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} = -GM \nabla_x G_{\text{New mech}}(\mathbf{x}, \mathbf{x}'), \tag{65} \]
we obtain, as standard, the Gauss theorem:
\[ \int_{\Sigma_1} d\Sigma_{\text{New mech}} \cdot \mathbf{n} \propto M, \tag{66} \]
where \( \Sigma_1 \) is a generic two-dimensional surface, and \( \mathbf{n} \) is its surface normal. The flux of the field \( F_{\text{New mech}} \) on the surface \( \Sigma_1 \) is proportional to the matter content \( M \), inside to the surface independently of the particular shape of the surface (Gauss theorem or Newton theorem for the gravitational field \([56]\)). On the other hand, if we consider the flux defined by the new Green’s function, its value is not proportional to the enclosed mass but depends on the particular choice of the surface:
\[ \int_{\Sigma_1} d\Sigma_{\text{New mech}} \cdot \mathbf{n} \propto M/\Sigma_1. \tag{67} \]
Hence \( M/\Sigma_1 \) is a mass function depending on the surface \( \Sigma_1 \). Then we have to find the solution inside/outside the matter distribution by evaluating the quantity
\[ \int d^3\mathbf{x}' G_I(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}'), \tag{68} \]
and by imposing the boundary condition on the separation surface.

5.1. The general solution by the Green’s function \( G_I(\mathbf{x}, \mathbf{x}') \)

By considering expressions (44) and (45) with the Green’s function (50) and by assuming \( \Phi(x) = 0 \), we have
\[ U(\mathbf{x}) = 4\pi G Y_I \left( \frac{8a_2 + 3b_1}{a_1(2a_2 + b_1)} \nabla^2 - a_1 \right) \int d^3\mathbf{x}' G_I(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}'), \tag{69} \]
\[ V(\mathbf{x}) = 4\pi G Y_I \left( \frac{4a_2 + b_1}{a_1(2a_2 + b_1)} \nabla^2 - a_1 \right) \int d^3\mathbf{x}' G_I(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}'). \tag{70} \]

We have to note that the hypothesis, \( \Phi(x) = 0 \), is not particular, since when we considered the Hilbert–Einstein Lagrangian to give the Newtonian solution, we imposed an analogous condition. In fact, considering the spatial components of the Einstein equations and then, by asking for \( \Phi = 0 \), we get the condition that the two metric potential \( U \) and \( V \) have to be equal (see case i in table 1). In the general case, the role of \( \Phi \) is given by \( V - U \). From the time–time component, we can obtain an expression for \( U \). The next step, in principle, is to search for a non-trivial solution \( \Phi \). This task is very difficult in general but can be realized for some particular cases. For example, since the Green’s function can be found under the spherical symmetry hypothesis, also the spatial distribution of matter has to be spherically symmetric. Denoting the radius of the sphere with total mass \( M \) by \( \xi \), we have the matter–density function,
\[ \rho(\mathbf{x}) = \frac{3M}{4\pi \xi^3} \Theta(\xi - |\mathbf{x}|), \tag{71} \]
where \( \Theta(\xi - |x|) \) is the Heaviside function. For the potential \( U(x) \), we obtain the implicit expression

\[
U(x) = \frac{3GMY_I}{|x|^3} (8a_2 + 3b_1) \nabla^2 x - \frac{a_1}{a(2a_2 + b_1)} \int_0^\xi dx' |x'|^2 \int_0^{2\pi} d\phi' \int_0^\pi d\theta' \sin \theta' G_I(x, x'),
\]

(72)

and an analogous relation for \( V(x) \). After some algebra, we get the explicit form

\[
U_{\text{in}}(x) = \frac{(2\pi)^{3/2}}{2a_1} \frac{GM}{|x|} \left[ \frac{\lambda_1^2 (2 + 3\lambda_2^2 \xi^2) - 8\lambda_2^2}{\lambda_1^2 \lambda_2} - |x|^2 + 8 e^{-\lambda_1 \xi} (1 + \lambda_1 \xi) \frac{\sinh(\lambda_1 |x|)}{\lambda_1^3 |x|} \right.
\]

\[
- 2e^{-\lambda_2 \xi} (1 + \lambda_2 \xi) \frac{\sinh(\lambda_2 |x|)}{\lambda_2^3 |x|} \right],
\]

(73)

\[
U_{\text{out}}(x) = \frac{(2\pi)^{3/2}}{a_1} \frac{GM}{|x|} \left[ 4(2\pi)^{3/2} \frac{GM}{\lambda_1^3 \xi^3} (\lambda_1 \xi \cosh(\lambda_1 \xi) - \sinh(\lambda_1 \xi)) e^{-\lambda_1 |x|} \right.
\]

\[
+ \frac{(2\pi)^{3/2}}{a_1} \frac{GM}{\lambda_2^3 \xi^3} [\lambda_2 \xi \cosh(\lambda_2 \xi) - \sinh(\lambda_2 \xi)] e^{-\lambda_2 |x|} |x|. \]

(74)

Relations (73)–(74) give the solutions for the gravitational potential \( U \) inside and outside the constant spherically symmetric matter distribution. A similar relation is found for \( V(x) \). The boundary condition on the surface \(|x| = \xi\) is satisfied:

\[
U_{\text{in}}(\xi) - U_{\text{out}}(\xi) = 0. \quad (75)
\]

We note that the corrections to the Newtonian terms are ruled by \( G_I(x, x') \). In order to achieve the behavior of the potential inside the matter distribution, let us perform a Taylor expansion for \( \lambda_i |x| \ll 1 \). We have

\[
\frac{\sinh(\lambda_1 |x|)}{\lambda_1 |x|} \simeq \text{constant} + |x|^2 + \cdots. \quad (76)
\]

Like for the standard Newtonian potential, this means that the inner solution of the corrected potential is traced by the matter distribution.

The outer solution has to be discussed in detail. For fixed values of the distance \(|x|\), the external potential \( U_{\text{out}}(x) \) depends on the value of the radius \( \xi \), then the Gauss theorem does not work also if the Bianchi identities hold [11]. In other words, since the Green’s function does not scale as the inverse distance but has an exponential behavior, the Gauss theorem (67) does not hold. This means that the potential depends on the total mass and on the matter distribution in the space. In particular, if the matter distribution takes a bigger volume, the potential \(|U_{\text{out}}(x)|\) increases and vice versa. We can write

\[
\lim_{\xi \to \infty} \frac{\lambda \xi ^3 \cosh(\lambda \xi) - \sinh(\lambda \xi)}{\lambda_1^3 \xi^3} = \infty. \quad (77)
\]

Obviously, the limit of \( \xi \) has to be considered up to the maximal value of \(|x|\). The term defined in (77) can be defined as a sort of a geometric factor which takes into account the spatial matter distribution. The limit puts in evidence the dependence on the matter distribution of the outer potential. The spherical symmetry allows us to find out the Green’s functions but, in principle, they can also be achieved without this hypothesis. At this point, it is interesting to consider the physical meaning of the Green’s functions. Equation (50) represents the correlation (i.e. the interaction) between two points in the space, that is, it gives the possibility of calculating the potential in a given point as a function of the ‘charge’ (the point mass) of another point. Such a charge is described by a Dirac \( \delta \)-function which is the source term of the Green’s function. By summing up the contributions of all infinitesimal volume elements, we obtain the potential for
a given matter distribution. This analysis can be concluded with some considerations related to the behavior for $|x| \gg \xi$. This means that we are moving away from the matter distribution. Such a limit can be given also as $\xi \to 0$, that is,

$$\lim_{\xi \to 0} \frac{3}{\lambda^2} \frac{\lambda \xi \sinh(\lambda \xi) - \sin(\lambda \xi)}{\lambda^2 \xi^3} = 1.$$  

(78)

For $U_{\text{out}}(x)$, we have

$$\lim_{\xi \to 0} U_{\text{out}}(x) = \frac{(2\pi)^{3/2} GM}{a_1 |x|} - \frac{4(2\pi)^{3/2} GM}{3a_1} \frac{e^{-\lambda_1 |x|}}{|x|} + \frac{(2\pi)^{3/2} GM}{3a_1} \frac{e^{-\lambda_2 |x|}}{|x|}.$$  

(79)

We can choose $a_1 = (2\pi)^{3/2}$, and then, for a point-like mass, we have

$$\lim_{\xi \to 0} U_{\text{out}}(x) = \frac{GM}{|x|} - \frac{4 GM}{3} \frac{e^{-\lambda_1 |x|}}{|x|} + \frac{1 GM}{3} \frac{e^{-\lambda_2 |x|}}{|x|}.$$  

(80)

The last expression is compatible with the discussion in section 4.1.

5.2. Further solutions by the Green’s functions $G_I(x, x')$

For the sake of completeness, let us derive the expression for the potential $U(x)$ for the other two Green’s functions in table 2. By performing a similar calculation, but now considering case ix in table 2, we obtain

$$U_{\text{in}}(x) = \frac{GM}{2\xi^3} \left\{ \frac{\lambda_1^2 (3\lambda_1^2 \xi^2 - 2) + 8\lambda_2^2}{\lambda_1^2 \lambda_2^2} |x|^2 - \frac{8}{\lambda_1^2} \left[ \cos(\lambda_1 \xi) + \lambda_1 \xi \sin(\lambda_1 \xi) \right] \right\},$$  

$$+ \frac{2}{\lambda_2^2} \left[ \cos(\lambda_2 \xi) + \lambda_2 \xi \sin(\lambda_2 \xi) \right] \frac{\sin(\lambda_2 |x|)}{\lambda_2 |x|},$$  

(81)

$$U_{\text{out}}(x) = \frac{GM}{|x|} - \frac{4(2\pi)^{3/2} GM}{a_1 \lambda_1^2 \xi^3} \left[ \sin(\lambda_1 \xi) \right]$$  

$$- \lambda_1 \xi \cos(\lambda_1 \xi) \left[ \frac{\cos(\lambda_1 |x|)}{|x|} \right] + \frac{(2\pi)^{3/2} GM}{a_1 \lambda_2^2 \xi^3} \left[ \sin(\lambda_2 \xi) \right]$$  

$$- \lambda_2 \xi \cos(\lambda_2 \xi) \left[ \frac{\cos(\lambda_2 |x|)}{|x|} \right].$$  

(82)

Also in this case the boundary conditions (75) are satisfied. The considerations of preceding subsection hold also for solutions (81)–(82). The only difference is that now we have oscillating behaviors instead of exponential behaviors. The correction term to the Newtonian potential in the external solution can be interpreted as the Fourier transform of the matter density $\rho(x)$. In fact, we have

$$\int \frac{d^3x'}{(2\pi)^3} \rho(x') e^{-ik'x'} = \frac{3M}{(2\pi)^{3/2}} \frac{\sin(|k|\xi) - |k|\xi \cos(|k|\xi)}{|k|^3 \xi^3}.$$  

(83)

and in the point-like mass limit, it is

$$\int \frac{d^3x'}{(2\pi)^3} \rho(x') e^{-ik'x'} = \frac{M}{(2\pi)^{3/2}},$$  

(84)

we obtain again the external solution for the point-like source as a limit of (82):

$$\lim_{\xi \to 0} U_{\text{out}}(x) = \frac{GM}{|x|} - \frac{2 GM \cos(\lambda_1 |x|)}{3 |x|} + \frac{1 GM \cos(\lambda_2 |x|)}{6 |x|}.$$  

(85)
Finally, for the last case in table 2, we have

\[
U_{\text{in}}(x) = \frac{GM}{2|x|} \left\{ \frac{\lambda_1^2 (3\lambda_2^2 \xi^2 - 2) - 8\lambda_2}{\lambda_1^2 \lambda_2^2} - |x|^2 + \frac{8}{\lambda_1^2} e^{-\lambda_1 \xi} (1 + \lambda_1 \xi) \frac{\sinh(\lambda_1 |x|)}{\lambda_1 |x|} \right. \\
+ \left. \frac{2}{\lambda_2^2} \left[ \cos(\lambda_2 \xi) + \lambda_2 \xi \sin(\lambda_2 \xi) \right] \frac{\sin(\lambda_2 |x|)}{\lambda_2 |x|} \right\},
\]

(86)

\[
U_{\text{out}}(x) = \frac{GM}{|x|} - \frac{4(2\pi)^{3/2}}{a_1} \frac{GM}{\lambda_1^4 \lambda_2^4} \left[ \lambda_1 \xi \cosh(\lambda_1 \xi) - \sinh(\lambda_1 \xi) \right] \frac{e^{-\lambda_1 |x|}}{|x|} + \frac{(2\pi)^{3/2}}{a_1} \frac{GM}{\lambda_2^2 \xi^3} \\
\times \left[ \sin(\lambda_2 \xi) - \lambda_2 \xi \cos(\lambda_2 \xi) \right] \frac{\cos(\lambda_2 |x|)}{|x|}.
\]

(87)

The limit of the point-like source is valid also in this case, that is,

\[
\lim_{\xi \to 0} U_{\text{out}}(x) = \frac{GM}{|x|} - \frac{4}{3} \frac{GM e^{-\lambda_1 |x|}}{|x|} + \frac{1}{3} \frac{GM \cos(\lambda_2 |x|)}{|x|}.
\]

(88)

Results (80)–(88) mean that, for suitable distance scales, the Gauss theorem is recovered and the theory agrees with the standard Newtonian limit of general relativity.

5.3. Other solutions and their physical consistency

In table 3, we provide solutions, in terms of the Green’s function of the corresponding differential operator, for the field equations shown in table 1. Case i corresponds to the Newtonian theory and the arbitrary constant \(a_1\) can be absorbed in the definition of matter Lagrangian as above. The implicit solution is

\[
U(x) = V(x) = G \int d^3\chi \frac{\rho(\chi)}{|x - \chi|}.
\]

(89)

For case iv, we have

\[
U(x) = V(x) = G \int d^3\chi \left[ \frac{1 - e^{-\sqrt{\frac{GM}{m}}|x - \chi|}}{|x - \chi|} \right] \rho(\chi).
\]

(90)

The solutions make sense only if \(a_1/a_2 > 0\), which gives a scale length. Also in this case, we can have different signatures for \(a_1\) and \(a_2\) which give oscillating corrections to the Newtonian potential. We have to note that both the above cases have been solved with the hypothesis \(\Phi = 0\). These two cases are the only ones which exhibit the standard Newtonian limit (obviously the former). The remaining cases can exhibit divergences and incompatibilities. This is obvious since, as discussed in section 3.2, the absence in the Lagrangians of terms linear in the Ricci curvature scalar gives field equations with higher-order Laplacian operators (see cases ii, iii, v, vi and vii). Precisely, without terms like \(\nabla^2 U + \cdots = \rho\), we could not achieve regular Newtonian-like behaviors. This fact could give problems in comparing inner and outer solutions with respect to matter distributions. In other words, the Newtonian potential is necessary not only to achieve physically interesting situations but also, from a mathematical point of view, to regularize solutions. In fact, case ii presents the incompatibility between the solution obtained from the 00-component and that from the \(ab\)-component. The incompatibility can be removed if we consider, as the Green’s function for the differential operator \(\nabla^4\), the trivial solution \(G_0(x)|_B = \text{const}\). Only with this choice, the arbitrary integration
constant $U_0$ can be interpreted as $GM$. However another problem remains, namely the divergence in the origin and then we can conclude that the solution,

$$2V(x) - U(x) = \frac{GM}{|x - x'|},$$

holds only in vacuum.

Besides, terms like $\int d^3x' \mathcal{G}_{ij}(x, x') \rho(x')$ have to be discussed for the choice (63). The field equation with $\nabla^4$ (see table 1) gives

$$\nabla^4 U(x) \propto \nabla^4 \int d^3x' \frac{\rho(x')}{|x - x'|} = -4\pi \nabla^2 \rho(x) \neq -4\pi \rho(x),$$

which is consistent only if $\rho(x) = 0$. Due to these considerations, also in the remaining cases, we can consistently consider only vacuum solutions.

### 6. Conclusions and outlook

In this paper, we have studied the Newtonian limit of gravitational theories whose action presents quadratic curvature invariants beside the standard Ricci curvature scalar of general relativity. In particular, we have considered the problem to find out solutions of the field equations developed up to the perturbation order $O(c^{-2})$. This is intended as the Newtonian limit while taking into account terms up to $O(c^{-3})$ and beyond is the post-Newtonian approximation (see, for example, [54]).
After deriving the full fourth-order field equations, we have developed the metric and the stress–energy tensors in the Newtonian limit. The main metric quantities, in this limit, are the two gravitational potentials $U$ and $V$ which are the solutions of the field equations both in the presence and in the absence of matter. At the order $c^{-2}$, quadratic curvature invariants give rise to $\nabla^2$ and $\nabla^4$ operators acting on $U$ and $V$ in the field equations. Our task has been to develop an approach to solve such equations and to find out corrections to the Newtonian potential emerging, as standard, from general relativity.

The method consists in searching for suitable combinations of the gravitational potentials $U$ and $V$ by which it is possible to decouple the field equations. After field equations are suitably decoupled, one can define Green’s functions which allow us to obtain the potentials. These potentials, however, strictly depend on the coupling parameters appearing in the Lagrangian of the theory. Conversely, such coupling constants allow us to classify the field equations, and then the solutions, selecting, in particular, some singular cases.

A detailed discussion has been developed for systems presenting spherical symmetry. In this case, the role of corrections to the Newtonian potential is clearly evident. In general, such corrections are oscillating behaviors or Yukawa-like terms. This means that one of the effects to introduce quadratic curvature invariants is to select characteristic scale lengths which could have physical interests as we will discuss below. Besides, such corrections invalidate the Gauss theorem because any matter distribution depends on such scale lengths. Furthermore, if the Newtonian potential term is not present, there could be compatibility problems, and some solutions are physically consistent only in a vacuum. Furthermore, for spherically symmetric distributions of matter, we discussed the inner and the outer solution and the boundary conditions.

From a physical viewpoint, this systematic work is needed in order to fully develop the weak-field limit of such relativistic theories of gravity and then compare them with observations and experiments. In fact, the correct interpretation of data strictly depends on the self-consistency of the theory and, vice versa, data correctly interpreted could definitively confirm or rule out deviations from general relativity [57]. It is worth pointing out that extended or alternative theories of gravity seem good candidates to solve several shortcomings of modern astrophysics and cosmology since they could address several issues of cosmological dynamics without introducing unknown forms of dark matter and dark energy (see, e.g., [13, 38]). Nevertheless, a ‘final’ alternative theory solving all the issues has not been found out up to now and the debate on modifying gravitational sector or adding new (dark) ingredients is still open. Beside this general remark related to the paradigm (extending gravity and/or adding new components), there is a methodological issue to ‘recover’ the standard and well-tested results of general relativity in the framework of these alternative schemes. The recovering of a self-consistent Newtonian limit is the test bed of any theory of gravity which pretends to enlarge or correct the Einstein general relativity.

Taking into account also the results presented in [32, 33], it is clear that only general relativity presents directly the Newtonian potential in the weak-field limit while corrections (e.g. Yukawa-like terms) appear as soon as the theory is nonlinear in the Ricci scalar. This occurrence could be particularly useful to solve the problem of missing matter in large astrophysical systems such as galaxies and clusters of galaxies as discussed in [14, 58]. In fact, dark matter (and dark energy) could be nothing else but the effects that general relativity, experimentally tested only up to solar-system scales, does not work at extragalactic scales and then it has to be corrected. Assuming this alternative point of view, we do not need to search for unknown ingredients, up to now not found at the fundamental level, but we need only to revise the behavior of the gravitational field at infrared scales. These scales could be ruled by corrections to the Newtonian potential, as shown in this paper. In forthcoming researches, we
intend to confront such solutions with experimental data, as done in [14, 58], in order to see if large self-gravitating systems could be modeled by them.

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Appendix A. Alternative approach to solve the field equations

In this appendix, we discuss an alternative approach to solve the field equations where, instead of using the relation $A = \Phi - \nabla^2 B$ as in section 3.1 to obtain solutions (44)–(45), we adopt the inverse relation $\nabla^2 B = \Phi - A$. As noted above, the first relation makes the differential degree of system increase but we have a relation between the solutions $A$ and $B$. In the second case, the differential degree remains the same but we have a non-local relation between the solutions:

$$B(\mathbf{x}) = \frac{1}{4\pi} \int \frac{d^3\mathbf{y}}{|\mathbf{x} - \mathbf{y}|} A(\mathbf{y}) - \Phi(\mathbf{y}). \quad (A.1)$$

In this case, the boundary conditions play a crucial role in the integration process. In general, considering the inverse relation $\nabla^2 B = \Phi - A$, we have a new set of equations:

$$\frac{2b_1(3a_2 + b_1)}{a_1(2a_2 + b_1)} \nabla^4 A + \frac{6a_2 + b_1}{2a_2 + b_1} \nabla^2 A - \frac{a_1}{2a_2 + b_1} A = 8\pi G \rho - \tau_{II}, \quad (A.2)$$

and the function $\Phi$ is a constant having the dimension of a length $^{-1}$. Equations (31)–(32) become

$$\frac{2b_1(3a_2 + b_1)}{a_1(2a_2 + b_1)} \nabla^4 \mathcal{G}_{II}(\mathbf{x}, \mathbf{y}) + \frac{6a_2 + b_1}{2a_2 + b_1} \nabla^2 \mathcal{G}_{II}(\mathbf{x}, \mathbf{y}) - \frac{a_1}{2a_2 + b_1} \mathcal{G}_{II}(\mathbf{x}, \mathbf{y}) = Y^{-1}_2(\mathbf{x} - \mathbf{y}). \quad (A.6)$$

where $\tau_{II} := -\frac{a_1}{2a_2 + b_1} \Phi + \frac{2a_2 + b_1}{2a_2 + b_1} \nabla^2 \Phi$. Now by introducing a new Green’s function $\mathcal{G}_{II}(\mathbf{x}, \mathbf{y})$, we have

$$A(\mathbf{x}) = Y_{II} \int d^3\mathbf{y} \mathcal{G}_{II}(\mathbf{x}, \mathbf{y}) \sigma_{II}(\mathbf{y}), \quad (A.4)$$

and $Y_{II}$ is a constant having the dimension of a length $^{-1}$. The general solutions of equations (31)–(32), by introducing the Green’s function $\mathcal{G}_{II}(\mathbf{x}, \mathbf{y})$ and the function $\Phi(\mathbf{x})$, are

$$U(\mathbf{x}) = -\frac{8a_2 + 3b_1}{2a_1(2a_2 + b_1)} Y_{II} \int d^3\mathbf{y} \mathcal{G}_{II}(\mathbf{x}, \mathbf{y}) \times \left[ 8\pi G \rho(\mathbf{y}) + \frac{a_1}{2a_2 + b_1} \Phi(\mathbf{y}) - \frac{2a_2}{2a_2 + b_1} \nabla^2 \Phi(\mathbf{y}) \right] + \frac{Y_{II}}{8\pi(2a_2 + b_1)} \int d^3\mathbf{y} d^3\mathbf{z} \frac{\mathcal{G}_{II}(\mathbf{y}, \mathbf{z})}{|\mathbf{x} - \mathbf{y}|} \left[ 8\pi G \rho(\mathbf{z}) \right].$$
With the second relation between $A$ and $B$, equation (A.2) is a fourth-order equation, but in this case, the potentials $U$, $V$ are linked to $A$ through repeated integrations of (A.8)–(A.9); for the first choice, we have only the integral (44)–(45). If we consider, instead, equation (A.6), we can find a similar Green’s function $g_{II}(x, x')$ for solutions (A.8)–(A.9). In fact, for $r \neq 0$, we have

$$V(x) = -\frac{(4a_2 + b_1)}{2a_2(2a_2 + b_1)} Y_{II} \int d^3x' g_{II}(x, x') \left[ 8\pi G \rho(x') \right]$$

and its solution is similar to (47):

$$g_{II}(r) = C \int \left[ K_{II,3} e^{-\sqrt{\frac{2}{a_1}} r} + K_{II,4} e^{-\sqrt{\frac{2}{a_1}} r} + K_{II,5} e^{-\sqrt{\frac{2}{a_1}} r} + K_{II,6} e^{-\sqrt{\frac{2}{a_1}} r} \right]$$

where, as above, $K_{II,3}, K_{II,4}, K_{II,5}, K_{II,6}$ are constants. It is worth noting that it is not possible to factorize the Laplacian in (A.6) and, in terms of Fourier transform, a vanishing pole is not present. If the Fourier transformed function has no pole in the origin ($k^2 = 0$), this means that, in the field equation, we do not have all terms containing a Laplacian but, some term can be interpreted as the mass for the field. On the other hand, if it is not possible to factorize a Laplacian in the field equation; this means that the pole $k^2 = 0$ is absent in the Fourier transform of the Green’s function and a Newtonian potential scaling as $r^{-1}$ is not present. Let us remember that a potential scaling like $1/r$ is used in the Green’s function proportional to $\int d^3k k^2 e^{i k \cdot (x-x')}$. Furthermore, the analogy between the two approaches is complete when we consider the link between them being

$$g_{II}(x, x') = \nabla_x^2 \tilde{g}_{II}(x, x').$$

### Appendix B. Conventions and dimensions

In order to fix the notation, we provide two tables with definitions. Table B1 gives an overview of the geometrical quantities used in the paper. We make use of the summation convention over identical upper and lower indices throughout the paper. The dimensions of the different quantities appearing throughout the work are displayed in table B2.
### Table B1. List of conventions and definitions.

| Object | Definition/convention |
|---------|-----------------------|
| Index ranges | $\alpha, \beta = 0, 1, 2, 3; i, j = 1, 2, 3$ |
| Flat metric | $g_{ab} = \text{diag}(1, -1, -1, -1)$ |
| Coordinates | $x^a = (x^0, x^1, x^2, x^3) = (ct, x^1, x^2, x^3)$ |
| Vectors | $\nabla = \left(\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3\right)$ |
| Symmetrization | $T_{\alpha\beta\gamma\delta} = \frac{1}{2}(T_{\alpha\beta\gamma\delta} + T_{\delta\gamma\beta\alpha})$ |
| Kronecker | $\delta^{ij}_\alpha = 1$ if $\alpha = \beta, 0$ else |
| Connection | $\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\sigma'}(g_{\nu\nu'}, g_{\mu\mu'} + g_{\mu\nu'} - g_{\nu\mu'})$ |
| Riemann tensor | $R^{\gamma}_{\mu\nu\rho} = \Gamma^{\gamma}_{\mu\rho,\nu} - \Gamma^{\gamma}_{\mu\nu,\rho} + \Gamma^{\gamma}_{\sigma\nu} \Gamma^{\sigma}_{\rho\mu} - \Gamma^{\gamma}_{\sigma\mu} \Gamma^{\sigma}_{\rho\nu}$ |
| Ricci tensor | $R_{\mu\nu} = R^\rho_{\rho\mu\nu}$ |

### Table B2. Dimensions of the quantities considered in the paper.

| Dimensions of Physical Quantities | $l, m, s$ |
|-----------------------------------|----------------|
| $g_{ab}, \delta_{ab}, G, a_1, K_{I,1}$ | $c, U^1, U^2, h_0, h^0, \varrho, \Lambda^2, \Phi^2, \tau^2$ |
| $R_{\mu\nu}, R^{\mu\nu}$ | $M$ |
| $\delta^{ij}_\alpha$ | $U_0, U_1, U_2$ |
| $\delta^{ij}_\alpha$ | $B$ |
| $\delta^{ij}_\alpha$ | $\varrho$ |
| $\delta^{ij}_\alpha$ | $T_{ab}, P$ |
| $\delta^{ij}_\alpha$ | $G$ |

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