Entanglement entropy in fermionic Laughlin states

Masudul Haque,1 Oleksandr Zozulya,2 and Kareljan Schoutens2

1Institute for Theoretical Physics, Utrecht University, the Netherlands
2Institute for Theoretical Physics, University of Amsterdam, the Netherlands

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We present analytic and numerical calculations on the bipartite entanglement entropy in fractional quantum Hall states of the fermionic Laughlin sequence. The partitioning of the system is done both by dividing Landau level orbitals and by grouping the fermions themselves. For the case of orbital partitioning, our results can be related to spatial partitioning, enabling us to extract a topological quantity (the ‘total quantum dimension’) characterizing the Laughlin states. For particle partitioning we prove a very close upper bound for the entanglement entropy of a subset of the particles with the rest, and provide an interpretation in terms of exclusion statistics.

Introduction — While the renewed attention in quantum entanglement has mostly been in the context of quantum computing, there is now also growing interest in using entanglement measures for characterizing quantum many-particle states. The measure we focus on is the entanglement entropy, which is defined by partitioning the system under question into two blocks A and B, and using the reduced density matrix of one part (e.g., \( \rho_A = \text{tr}_B \rho \)) obtained by tracing over B degrees of freedom to calculate the von Neumann entropy \( S_A = -\text{tr}[\rho_A \ln \rho_A] \). Generally speaking, the entanglement entropy scales with the size of the boundary between the partitions \( \mathcal{A} \). Prefactors, logarithmic corrections and subleading terms in this basic relationship can all contain important physical information concerning the many-body state.

By now the entanglement entropy has been studied extensively for one-dimensional spin systems, where the variation of \( S_A \) as a function of the size of block A is a sensitive probe into the nature of the ground state \( \mathcal{A} \). Entanglement measures have also been shown to contain revealing behavior in the vicinity of quantum critical points \( \mathcal{A} \) and shown to have fundamental implications for the efficiency of numerical simulation methods for quantum many-body states \( \mathcal{A} \).

Studies of entanglement in higher-dimensional and itinerant systems has been less thorough but have now also started to attract significant attention (e.g., \( \mathcal{A}, \mathcal{A} \)). The case of two dimensions is particularly intriguing because of the known existence (and further unconfirmed possibilities) of topologically ordered phases. It is natural to speculate that entanglement might give a handle on topological properties of quantum states, because the basic intuition about topologically ordered phases is the presence of intricate correlations not easily captured by local observables or traditional correlation functions. Indeed, Refs. 8 and 9 have recently presented the following theorem concerning topologically ordered states: if \( L \) is the boundary between the two blocks, the entanglement entropy scales as \( S_A = \alpha L - \gamma + O(L^{-1}) \). As usual the scaling law applies to situations where \( A \) is large and the total system is infinite. The quantity \( \gamma \) (the topological entanglement entropy) is the logarithm of a quantity known as the total quantum dimension. For a state with anyonic excitations, the quantum dimensions characterize the growth rate of the Hilbert space with anyon number. For fermionic fractional quantum Hall states in the Laughlin sequence with filling fraction \( \nu = 1/m \), the topological entanglement entropy is \( \gamma = \frac{1}{2} \ln m \).

In this Letter, we present a detailed study of the entropy of entanglement in the fermionic Laughlin states. These are the best-known many-particle states with topological order. We carefully choose useful methods of dividing quantum Hall states into A and B blocks. We provide results on the entropy defined by partitioning the Landau-level orbitals in the system into blocks, as well as that obtained by dividing the particles themselves into A and B subsets. The first method allows us, upon extrapolation to the thermodynamic limit, to extract the \( \gamma \) parameter of Refs. 8, 9. To our best knowledge, this is the first example where the results of Refs. 8, 9 are used to compute the total quantum dimension of an experimentally realized topologically ordered medium, directly from a microscopic description.

Very restricted cases of entanglement in Laughlin states, involving one-orbital or one-particle blocks, have appeared previously in the literature, in Refs. 10, 11. The entanglement entropy in some other topologically ordered states has been studied in Ref. 12.

Orbital versus particle entanglement — We first explain our choice of methods used to partition the Laughlin system into blocks A and B. The issue is non-trivial in the case of itinerant systems 10, 11, where, unlike spin systems, the particles are not each fixed in its own site. Although this point has not been stressed in the literature, conformal field theory results on entanglement scaling 12 actually pertain to the blocking of space rather than the particles or spins themselves. For quantum Hall systems, we find a description of the system in terms of the magnetic orbitals more natural than a spatial description. Thus we will partition the orbitals into two sets A and B and calculate the entanglement between them. In fact, there is a close relationship between orbital and spatial partitioning, since the choice of the first \( l_A \) orbitals...
as the $A$ block corresponds approximately to choosing a disk-shaped $A$ block with radius proportional to $\sqrt{l_A}$ in real space. The scaling law of Refs. [5, 6] thus translates to $S_{l_A} = c_1 \sqrt{l_A} - \gamma + \mathcal{O}(1/l_A)$.

In addition, we also consider the perhaps more obvious method of partitioning the particles themselves. In this case, the subsets $A$ and $B$ no longer correspond to connected regions in space. We find that the analysis of particle entanglement also reveals subtle correlation effects.

Quantum Hall states on a sphere — To describe the fractional quantum Hall states, we use the representation of Refs. [13, 14] in which the fermions are placed on a sphere containing a magnetic monopole. The magnetic orbitals in the lowest Landau level are then represented as angular momentum orbitals; for $N$ particles in the Laughlin state $\nu = 1/m$, the total angular momentum is half the number of flux quanta, $L = \frac{N}{2} N_\phi$ with $N_\phi = m(N - 1)$. The $N_\phi + 1$ orbitals are labeled either $l = 0$ to $N_\phi$ or $L_z = -L$ to $+L$. The “filling” acquires the usual meaning $\nu = N/N_\phi$ only in the thermodynamic limit. The orbitals are each localized around a “circle of latitude” on the sphere, with the $l = 0$ orbital localized near one “pole.” For orbital partitioning, we define block $A$ to be the first $l_A$ orbitals, extending spatially from one pole out to some latitude. In the thermodynamic limit, this is equivalent to having a disk-shaped block $A$; since each orbital $l$ is associated with a wavefunction of the form $z^l e^{-|z|^2}$ in usual complex coordinate language, a disk with $l_A$ orbitals has radius $\sqrt{l_A}$.

In this representation the Laughlin wavefunctions are expressed in terms of Schwinger boson operators $a_i^\dagger$, $b_i^\dagger$ as $\prod_{i<j} (a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger)^m |0\rangle$. Each term in the expansion can be interpreted in terms of orbital occupations. For example, in the expansion for the $N = 3$, $m = 3$ wavefunction, the term involving $(a_1^\dagger)^2 a_2^\dagger a_3^\dagger$ corresponds to having the three particles in orbitals 1, 2, and 6 respectively. Together with its permutations, this state reads in “orbital” notation $[0,1,1,0,0,0,1,1]$. Analytic results for orbital partitioning — We first consider the simplest Laughlin state, $m = 3$. With only the $l = 0$ orbital included in partition $A$ ($l_A = 1$), the reduced density matrix $\rho_A$ is $2 \times 2$ and diagonal, and has eigenvalues $N/(3N - 2)$ and $(2N - 2)/(3N - 2)$. The entanglement entropy converges in the large-$N$ limit as $S_{l_A=1} = \ln(3/2^{2/3}) + \ln 2^{2/3}/N + \mathcal{O}(N^{-2})$.

The case of the first two orbitals in the block ($l_A = 2$) is also exactly solvable. The basis for $\rho_A$ is the set $[0,0], [1,0], [0,1], [1,1]$; however the last one can be dropped because the two lowest orbitals are never simultaneously occupied in the $m = 3$ wavefunctions [13]. The reduced density matrix is still diagonal, with a pair of eigenvalues $N/(3N - 2)$ and a single $(N - 2)/(3N - 2)$. The thermodynamic limit is $S_{l_A=2} = \ln 3 + 4/(9N^2) + \mathcal{O}(N^{-3})$.

The case of $l_A = 3$ is already more difficult. The reduced density matrix, now of size $5 \times 5$, is still diagonal; the elements are of the form (up to normalization) $N$, $\alpha_\nu$, $\alpha_\nu (N-\alpha_\nu)$, and $(N-2-\alpha_\nu)$. We find $\alpha_3 = 1$, $\alpha_4 = 92/51$ and $\alpha_5 = 505/203$, etc., but a general form is not obvious; thus we lack an analytic expression for $S_{l_A=3}$ in the thermodynamic limit.

We now turn to a general Laughlin state of filling $\nu = 1/m$. For this state there is at most one fermion in the first $\frac{1}{2}(m+1)$ orbitals [13]; this makes $S_{l_A}$ analytically tractable for $l_A \leq \frac{1}{2}(m+1)$, as was the case for $l_A \leq 2$ for the $m = 3$ case. For $l_A \leq \frac{1}{2}(m+1)$, the basis state is $[0,0,0,\ldots,0], [1,0,0,\ldots,0], [0,1,0,\ldots,0], \ldots [0,0,0,\ldots,1]$. The reduced density matrix is diagonal. In the thermodynamic limit the eigenvalues are $(m - l_A)/m$ for the first state (no particles in $A$) and $1/m$ for each of the rest. The entanglement entropy is

$$S_{l_A} = -\left( \frac{m - l_A}{m} \right) \log \left( \frac{m - l_A}{m} \right) - l_A \left( \frac{1}{m} \log \frac{1}{m} \right)$$

which takes the form $S_{l_A} \sim (\log m/m) l_A$ for $l_A \ll m$. Thus for large $m$ the $S_{l_A}$ versus $\sqrt{l_A}$ curve starts out quadratic, and only later displays the asymptotic linear behavior, with a crossover presumably around $l_A \sim m$.

Orbital partitioning; numerical results — We now present numerical results for entanglement entropy with orbital partitioning. The numerical calculation is based on Laughlin wavefunctions evaluated exactly using algebraic methods; some details are given later in the article.

Fig. 1 shows numerically calculated orbital entanglements ($S_{l_A}$) for several of our wavefunctions, as a function of the number of orbitals ($l_A$) in the $A$ partition. The partition flip symmetry of the entanglement entropy ($S_A = S_B$) implies that $S_{l_A}$ is arc-shaped with a maximum at $l_A^* = (N_\phi + 1)/2$. The initial increasing parts of these curves reflect physics of the macroscopic state, while the downward curvature is a finite-size effect. Scaling information is not immediately obvious because the curves bend down before reaching large $l_A$. However, we can extract some information about the thermodynamic limit by plotting the maximum $S_{l_A}^*$ of the arc-shaped

FIG. 1: (Color online.) Entanglement entropies with orbital partitioning ($S_{l_A}$), for various $N$ and $m$. Inset: maximum $S_{l_A}$ values plotted against square root of positions $l_A^*$ of the maximum, for $m = 3$ wavefunctions of various sizes.
curves against the position of the maxima, \( l_A' \). While the detailed finite-size effects are difficult to analyze, it is reasonable to assume that the macroscopic functional dependence of \( S_A \) (e.g., algebraic or logarithmic dependence) will also appear in the dependence of the maximum. The inset to Fig. 1 shows such a plot for \( m = 3 \) states, with \( l_A' \) 's plotted in a square root scale. The linear behavior is manifest. We have observed the same feature for \( m = 5 \) wavefunctions. We have thus already verified the boundary law \( S_A \propto \sqrt{l_A} \).

A more quantitative approach to the thermodynamic limit is to do a numerical \( N \rightarrow \infty \) extrapolation of the \( S_A \) data for each \( l_A \) (Fig. 2). We fit the \( S_A(N^{-1}) \) data points (inset to Fig. 2) to functions like \( c_0 + c_1/N^{\alpha_1} + c_2/N^{\alpha_2} \), noting that \( S_A = 1, 2 \) indeed have expansions for this form. We use various sets \( \{a_i\} \) of small integers, dropping combinations that give extrapolation functions with pathological features at small \( N^{-1} \). The set of values obtained by this procedure yields the estimates and errors of Fig. 2.

**Extracting the topological entropy** — We note that the \( -\gamma + c_1 \sqrt{l_A} \) behavior \( \gamma \) is expected only for \( l_A \geq 3 \). For extracting \( \gamma \), we thus drop two or more of the lowest points from the available \( S_A \) values. It is not feasible to use only the largest values because the extrapolation uncertainty is largest there (Fig. 2). Dropping 2 to 5 of the lowest points, and giving various relative weights to the higher or lower values, we extract various estimates for \( \gamma \) by fitting \( S_A = -\gamma + c_1 \sqrt{l_A} \). The resulting estimate for the topological entanglement entropy of the \( m = 3 \) state is \( \gamma = -0.60 \pm 0.13 \). This is consistent with the theoretical value of \(-0.55 \). However, larger wavefunctions are probably required for precisely extracting \( \gamma \) for models where it is not known a priori.

**Particle Entanglement** — We now turn to partitioning the Laughlin states into two sets of particles rather than two blocks of orbitals. Noting that the entanglement entropy \( S_A \) is largest and equal to the logarithm of the number of nonzero eigenvalues when \( \rho_A \) has equal nonzero eigenvalues, we can set a naive upper bound for \( S_A \). When \( A \) contains \( n_A \) particles out of a total of \( N \) particles in the \( \nu = 1/m \) Laughlin state, the size of the density matrix is set in first instance by the number of possible ways in which the \( n_A \) particles can be distributed among \( N_0 = 1 + m(N -1) + 1 \) orbitals; thus

\[
S_A \leq \log \left( \frac{N_0 + 1}{n_A} \right).
\]

This bound becomes an exact result for the integer quantum Hall state \( m = 1 \), i.e., the noninteracting case where the Laughlin wavefunction reduces to a single Slater determinant. For the fractional quantum Hall states \( (m \geq 3) \), the bound is sharp for \( n_A = 1 \) but not for \( n_A > 1 \) (Fig. 3).

We note that the absence of terms in the Laughlin wavefunctions with adjacent orbital occupancies near the poles implies that, for \( n_A > 1 \), some of the eigenvalues of the reduced density matrix are zero, leading to a tighter upper bound than above. This idea becomes much more effective with the additional observation that the eigenvalues of \( \rho_A \) are organized in a \( SU(2) \) multiplet structure, due to \( \rho_A \) commuting with the total angular momentum and \( z \)-component \( (L^2_n)_{N \lambda_n} \) and \( L^{\lambda_n} \) of the \( n_A \) particles in \( A \). When any eigenvalue of \( \rho_A \) vanishes due to the absence of corresponding states in the Laughlin wavefunction, the multiplet structure implies that every eigenvalue in the same multiplet must also vanish.

For \( n_A = 2 \), \( m = 3 \), one observes that the vanishing states are part of the \( L_2 = m(N -1) - 1 = N_\phi - 1 \) representation of the \( SU(2) \) symmetry algebra. This reduces by \( 2L_2 + 1 = 2N_\phi - 1 \) the number of 2-particles states contributing to the entropy. For \( n_A = 2 \) and general \( m \), there are \( (m -1)/2 \) multiplets contributing zero eigenvalues, leading to a total of \( 1/(m-1)(2N_\phi - m+2) \) zeros. The number of nonzero eigenvalues is

\[
\left(\frac{N_\phi + 1}{2}\right)^2 - \frac{1}{2}(m-1)(2N_\phi - m+2) = \left(\frac{N_\phi + 1 - (m-1)}{2}\right)
\]

and the logarithm gives a much better upper bound than the \( n_A = 2 \) entanglement entropies (Fig. 3). This bound is still not sharp because the eigenvalues are not all equal. The eigenvalue distribution is itself interesting and may be discussed in future work; however, the effect on \( S_A \) is small. In Fig. 3 we show that the exact values for \( S_{n_A = 2} \) rapidly converge to our improved upper bound.

Generalizing to \( n_A > 2 \), we have rigorously established the following upper bound

\[
S_A \leq \log \left( \frac{N_\phi + 1 - (m -1)(n_A -1)}{n_A} \right).
\]

We claim that the entropy \( S_A \) will be close to this bound for \( n_A \ll N \). The derivation of Eq. (2) relies on a result of Ref. 18 for counting degeneracies of Laughlin states.
on the sphere in the presence of excess flux, that is, of a number of quasi-holes. The combinatorial factor in Eq. (2) has the following interpretation: it is the number of ways \( n_A \) particles can be put into \( N_\phi + 1 \) orbitals, such that between two occupied orbitals there are at least \( m - 1 \) that are empty. Clearly, this counting represents a version of Haldane’s notion of exclusion statistics \( \lambda \).

**Numerical method** — Since it is not known which terms in Laughlin wavefunctions are more important for various entanglement entropies, we avoided truncation schemes and calculated complete wavefunctions. Such calculations are in the spirit of combinatorial considerations of Refs. 15, 16, rather than that of “exact diagonalization” studies. The enumeration of Laughlin wavefunction terms and calculation of coefficients are combinatorially explosive problems 15, 16; symbolic expansion is not feasible beyond more that 5 or 6 particles. Instead, we made use of results in Ref. 15 where the coefficients are expressed as expansions in characters of the symmetric group, \( a(\mu) = \sum c(\lambda) \chi^A(\mu) \). Here (\( \mu \))’s are integer partitions representing terms in the Laughlin wavefunction. The conjugacy classes (\( \lambda \)) and the coefficients of expansion \( c(\lambda) \) are both found by expanding out a Hankel determinant symbolically 15, 16. Characters of the symmetric group \( S_7 \) were calculated using publicly available group theory code while the Hankel determinant expansion was done with a symbolic manipulation routine coded from scratch. The admissible partitions (\( \mu \)) for Laughlin state terms were generated by combinatorial rules 15, 16. With some effort (50+ cpu-days for the largest wavefunctions), we calculated \( m = 3 \) wavefunctions up to \( N = 10 \) fermions and \( m = 5 \) wavefunctions up to \( N = 6 \) fermions.

**Concluding remarks** — We conclude that the entanglement entropy pertaining to the two types of partitioning (spatial and particle) reveal different aspects of the intricate topological order of the Laughlin states. This beautifully illustrates that, more generally, entanglement entropy is a valuable probe for quantum order. For the case of particle partitioning we remark the following. Trusting that eq. (2) gives a close bound to \( S_A \), we have for \( n_A < N \)

\[
S_A - \log \left( \frac{N_\phi + 1}{n_A} \right) \sim - \frac{1}{N} m - 1 \frac{n_A (n_A - 1)}{n^2} + O(1/N^2)
\]

We propose that, in general, the difference between \( S_A \) and the naive expression 11 based on the number of 1-particle states allows a \( 1/N \) expansion that reveals correlations and quantum order in a state.

Our work opens up a number of questions. It would be interesting to obtain numeric data for \( n_A \geq 3 \) particle entanglement (not available at present) to investigate how well our bound 22 works. With further effort, some larger wavefunctions may also be obtained, perhaps leading to more precise extraction procedures for the total quantum dimension. Obviously, other quantum Hall states, e.g., those with paired or clustered structure and non-abelian statistics, deserve attention, and further work in this direction is planned.

While writing up our results, we learned of parallel work 19 that has overlap with some issues we consider.

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