Asymptotic behaviour of instantons on Cylinder Manifolds

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Abstract

In this article, we study the instanton equation on the cylinder over a closed manifold $X$ admits non-zero smooth 3-form $P$ and 4-from $Q$. Our results are (i) if $X$ is a good manifold, i.e., $P, Q$ satisfying $d^*_{X} P = d^*_{X} Q = 0$, then the instanton with integrable curvature decays exponentially at the ends, and, (2) if $X$ is a real Killing spinor manifold, i.e., $P, Q$ satisfying $d P = 4Q$ and $d^*_{X} Q = (n - 3) *_{X} P$, we prove the solution of instanton equation is trivial solution under some mild conditions on instanton.

Keywords. instantons, asymptotic behaviour, special holonomy

1 Introduction

Let $X$ be an oriented smooth $n$-dimensional Riemannian manifold, $G$ be a compact Lie group and $P$ be a principal $G$-bundle on $X$, $g_{P}$ be the adjoint bundle of $P$. Let $A$ denote a connection on $P$ with the curvature $F_{A}$. The instanton equation on $X$ can be introduced as follows. Assume there is a 4-form $\Omega$ on $X$, then an $(n - 4)$-form $* \Omega$ exists, where $*$ is the Hodge operator on $X$. A connection $A$ is called an anti-self-dual instanton, when it satisfies the instanton equation

$$
* F_{A} + * \Omega \wedge F_{A} = 0
$$

(1.1)

When $n > 4$, these equations can be defined on the manifold $X$ with a special holonomy group, i.e., the holonomy group $G$ of the Levi-Civita connection on the tangent bundle.
$TX$ is a subgroup of the group $SO(n)$. Each solution of equation (1.1) satisfies the Yang-Mills equation. The instanton equation (1.1) is also well-defined on a manifold $X$ with non-integrable $G$-structures, but equation (1.1) implies the Yang-Mills equation will have torsion.

Instantons on the higher dimension, proposed in [3] and studied in [2, 6, 5, 10, 26], are important both in mathematics [5, 6] and string theory [7, 8]. In mathematics, the articles of Donaldson-Thomas [6] and Donaldson-Segal [5] inspired a considerable amount of work related to gauge theory in higher dimensional. Sá Earp and Walpuski focus on the study of gauge theory on $G_2$-manifolds, they construct $G_2$-instnaton over some $G_2$-manifolds [19, 20, 25]. In string theory, the solutions of the instanton equations on cylinders over nearly Kähler 6-manifolds and nearly parallel $G_2$-manifold had been constructed [1, 9, 15, 16]. In [16] Section 4, they confirm that the standard Yang-Mills functional is infinite on their solutions.

In this paper, we consider the instanton $A$ on the cylinder manifold over a closed manifold which admits non-zero smooth 3-form $P$ and 4-form $Q$. The 4-form $\Omega$ on $Z$ can be defined as

$$\Omega = dt \wedge P + Q.$$ 

Therefore the instanton equation on $Z$ can defined as [1, 10],

$$\ast F_A + (\ast_X P + dt \wedge \ast_X Q) \wedge F_A = 0. \tag{1.2}$$

In [4] Chapter 4 introduces same analytic results about the asymptotic behaviour of ASD connection on 4-manifolds with tubular ends aims to given a complete definition of the Floer groups of a homology 3-sphere. We know from analogous Floer-type theories that is the essential property needed to control solutions over infinite tubes. All of the known construction methods of higher dimensional instantons automatically yield exponential decay. One can see the curvature will satisfies the $L^2$-integrability condition. There is a nature question, whether all $L^2$-integrable instantons are exponential decay. We prove that the instanton equation (1.1) on the cylinder over a closed good manifold decays exponentially at the ends, See Theorem 3.6. If the manifolds admits real Killing spinors, i.e., the forms $P, Q$ satisfying $dP = 4Q$ and $d\ast_X Q = (n-3)\ast_X P$. In [12], the autor prove that the non-trivial solutions of instanton equations over the cylinder of the Riemannian manifolds with real Killing Spinors have infinite Yang-Mills energy. We prove that if the energy dense $\rho(A) = 0$, then the instanton is a flat connection, See Theorem 4.3.

**Remark 1.1.** In dimension 4, the instanton equation on a cylinder is the gradient flow for the Chern-Simons functional on the 3-dimensional cross-section. Critical point of this 3-dimensional functional are precisely flat connection. Similarly, the higher dimensional instanton equation on a cylinder can be regarded as the gradient flow of a Chern-Simons
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functionals on $X$. Critical points of the higher dimensional Chern-Simons functionals are however instantons, not necessarily flat connections. In fact, most examples of instantons on asymptotically cylindrical manifolds do not have $L^2$ curvature because the limit connection is not flat.

2 Fundamental preliminaries

2.1 Chern-Simons Functional

Let $t$ be the standard parameter on the factor $\mathbb{R}$ in the $Z := \mathbb{R} \times X$, where $X$ is a closed, oriented $n$-dimensional Riemannian manifold admits a smooth Riemannian metric $g_X$, let $\{x^j\}_{j=1}^n$ be local coordinates of $X$. A connection $A$ over the cylinder $Z$ is given by a local connection matrix $A = A_0 dt + \sum_{i=1}^n A_i dx^i$, where $A_0$ and $A_i$ dependence on all $n + 1$ variable $t, x^1, \ldots, x^n$. We take $A_0 = 0$ (sometimes called a temporal gauge) and denote $A = \sum_{i=1}^n A_i dx^i$. In this situation, the curvature is given by $F_A = F_A + dt \wedge \dot{A}$, where $\dot{A} = \frac{\partial A}{\partial t}$. We denote $\ast_X$ by the $\ast$-operator of $X$. If $\alpha$ is a 1-form on $X$, then for $\ast$-operator defined on $Z$ with respect to the product metric, we have $\ast(dt \wedge \alpha) = \ast_X \alpha$.

We consider a cylinder over a closed manifold $X$ admits smooth non-zero 3-form $P$, 4-form $Q$. The instanton equation (1.2) is equivalent to

$$\ast_X \dot{A} = - \ast_X P \wedge F_A, \ast_X F_A = - \dot{A} \wedge \ast_X P - \ast_X Q \wedge F_A.$$ (2.1)

Let $P$ be a $G$-bundle over $X$, the space $A$ is an affine space modelled on $\Omega^1(X, g_P)$, so fixing a reference connection $A_0 \in A$, we have $A = A_0 + \Omega^1(X, g_P)$. We define the Chern-Simons functional by

$$CS(A) := - \int_M Tr (a \wedge dA_0 a + \frac{2}{3} a \wedge a \wedge a) \wedge \ast_X P,$$

fixing $CS(A_0) = 0$. This functional is obtained by integrating of the Chern-Simons 1-form

$$\Gamma_A(\beta_A) = -2 \int_M Tr (F_A \wedge \beta_A) \wedge \ast_M P.$$

We find $CS$ explicitly by integrating $\Gamma$ over paths $A(t) = A_0 + ta$, from $A_0$ to any $A = A_0 + a$:

$$CS(A) - CS(A_0) = \int_0^1 \Gamma_{A(t)}(\dot{A}(t)) dt = - \int_M Tr (dA_0 a \wedge a + \frac{2}{3} a \wedge a \wedge a) \wedge \ast_M P + C,$$

where $C = C(A_0, a)$ is a constant and vanishes if $A_0$ is an instanton. Suppose that the 3-form $P$ is co-closed, i.e., $d \ast_X P = 0$. The co-closed condition implies that the Chern-Simons 1-form is closed. So it does not depend on the path $A(t)$ [18].
2.2 Non-degenerate flat connections

We denote

$$M(P, g) := \{ \Gamma : F_\Gamma = 0 \}/G_P,$$

by the moduli space of gauge-equivalence class $[\Gamma]$ of flat connection $\Gamma$ on $P$. Following Uhlenbeck strong compactness theorem [23], we know that the moduli space $M(P, g)$ is compact. We now begin to define the least eigenvalue of the self-adjoint operator $\Delta_A := d_A d_A^* + d_A^* d_A$ with respect to connection $A$ on $L^2(X, \Omega^1(g_P))$:

**Definition 2.1.** The least eigenvalue of $\Delta_A$ on $L^2(X, \Omega^1(g_P))$ is

$$\lambda(A) := \inf_{v \in \Omega^1(g_P) \backslash \{ 0 \}} \frac{\langle \Delta_A v, v \rangle_{L^2}}{\|v\|^2}.$$  \hfill (2.2)

In [14] Lemma 3.3, the author showed the function $\lambda[\cdot]$ with respect to the Uhlenbeck topology is a continuous function on the moduli space of flat connections. We recall the definition of *non-degenerate* flat connection, See [4] Definition 2.4.

**Definition 2.2.** A flat connection $\Gamma$ on $P$ over $X$ is called *non-degenerate*, if only if $\ker \Delta_\Gamma|_{\Omega^1(g_P)} = 0$, i.e., $\lambda(A) > 0$.

Combining the compactness of moduli space of flat connections and the function $\lambda[\cdot]$ is continuous under Uhlenbeck topology, we then have

**Proposition 2.3.** ([14] Proposition 3.6) If the flat connections over $X$ are non-degenerate. Then there is constant $\lambda > 0$ such that

$$\lambda(\Gamma) \geq \lambda, \forall \Gamma \in M(P, g).$$

For $A \in \mathcal{A}_P$ and $\delta > 0$, we set

$$T_{A,\delta} = \{ a \in \Omega^1(X, g_P) \mid d_A^* a = 0, \| a \|_{L^2_{\| \cdot \|}(X)} \leq \delta \}.$$ 

A neighbourhood of $[A] \in \mathcal{B} := \mathcal{A}_P/G_P$ can be described as a quotient of $T_{A,\delta}$, for small $\delta$. In [21] Lemma 1.2, Taubes showed the non-degenerate flat connection (under moduli gauge transformation) is isolated on a closed three manifold. We will extend this property to higher dimension manifold.

**Proposition 2.4.** Suppose $A, \Gamma$ are flat connections over $X$. Suppose also that the flat connections are non-degenerate. Then either

$$\inf_{g \in G} \| g^*(A) - \Gamma \|_{L^2_{\| \cdot \|}(X)} \geq \delta,$$

or $[A] = [\Gamma]$ (moduli gauge transformation), where $\delta = \delta(X, g)$ is positive constant.
Proof. We denote $A$, $\Gamma$ by two different flat connection (under moduli gauge transformation). If there is a sufficiently small constant $\delta$ such that
\[
\inf_{g \in G} \| g^* (A) - \Gamma \|_{L^2_1(X)} \leq \delta,
\]
Now we can choose a gauge transformation $g \in G_P$ such that $g^*(A)$ and $\Gamma$ satisfies gauge fixing, i.e.,
\[
d^*_T (g^*(A) - \Gamma) = 0.
\]
For simply, we also denote $g^* (A)$ to $A$. We denote $a := A - \Gamma$. therefore
\[
0 = F_A = F_\Gamma + d_\Gamma a + a \wedge a = d_\Gamma a + a \wedge a,
\]
Following Proposition 2.3, for any $\alpha \in \Omega^1 (X, g_P)$,
\[
\lambda \| \alpha \|^2_{L^2(X)} \leq \| d_\Gamma \alpha \|^2_{L^2(X)} + \| d^*_T \alpha \|^2_{L^2(X)}.
\]
Furthermore, the Weitzenböck formula gives
\[
(d^*_T d_\Gamma + d_\Gamma d^*_T) \alpha = \nabla^*_T \nabla_\Gamma \alpha + Ric \circ \alpha.
\]
Combining the preceding identities yields,
\[
\| \nabla |a| \|^2_{L^2(X)} \leq \| \nabla_\Gamma a \|^2_{L^2(X)}
\]
\[
\leq \| d_\Gamma a \|^2_{L^2(X)} + C \|a\|^2_{L^2(X)}
\]
\[
\leq (1 + C\lambda^{-1}) \| d_\Gamma a \|^2_{L^2(X)}
\]
\[
\leq (1 + C\lambda^{-1}) \|a \wedge a\|^2_{L^2(X)}
\]
\[
\leq (1 + C\lambda^{-1}) \|a\|^2_{L^\infty(X)} \|a\|^2_{L^{2\lambda^{-1}}(X)}
\]
\[
\leq C(1 + \lambda^{-1}) \|a\|^2_{L^\infty(X)} \|a\|^2_{L^{2\lambda^{-1}}(X)}
\]
\[
\leq C(1 + \lambda^{-1})^2 \| \nabla |a| \|^2_{L^2(X)} \|a\|^2_{L^\infty(X)},
\]
where we use Sobolev embedding $L^\infty_1 \hookrightarrow L^\alpha$ and $L^2 \hookrightarrow L^{\frac{2\alpha}{\alpha-2}}$ and Kato inequality $|\nabla |a| \| \leq |\nabla_\Gamma a|$. If we choose $\|a\|_{L^\infty_1(X)}$ sufficiently small to ensure $\|a\|^2_{L^\infty_1(X)} \leq \frac{\lambda^2}{2C(1+\lambda^{-1})}$. Then $a \equiv 0$. It’s contradict to our initial assumption regarding the connections $A, \Gamma$. \qed

For a flat connection $[\Gamma]$ over a $G_2$-manifold or Calabi-Yau manifold has fully holonomy, i.e., $\pi_1(X)$ is finite, we can show that the least eigenvalue of $\Delta_\Gamma$ is positive, i.e., the flat connections are all non-degenerate. One also can see [13] Theorem 1.1.

**Proposition 2.5.** Let $G$ be a compact Lie group, $P$ be a $G$-bundle over a closed, smooth $G_2$-manifold or Calabi-Yau manifold $X$. If $X$ has fully holonomy. Then the flat connections on $X$ are non-degenerate.
Proof. We denote $\Gamma$ by a flat connection on $X$, for any harmonic 1-form $\alpha$ with respect to $\Delta_{\Gamma} := d^*_{\Gamma} d_{\Gamma} + d_{\Gamma} d^*_{\Gamma}$, following the Weitzenböck formula, we have $\nabla_{\Gamma} \alpha = 0$. Here we use the vanishing of the Ricci curvature on $G_2$- or Calabi-Yau manifolds and $\Gamma$ is flat.

Let $R_{ij} dx^i \wedge dx^j$ denote the Riemann curvature tensor viewed as an $ad(T^*M)$ valued 2-form, The vanishing of $\nabla_{\Gamma} \alpha$ implies

$$0 = [\nabla_i, \nabla_j] \alpha = \text{ad}((F_{ij}) + R_{ij}) \alpha,$$

for all $i, j$. Since $F_{ij}$ vanishes, $R_{ij} \alpha = 0$, and the components of $\alpha$ are in the kernel of the Riemann curvature operator. This reduces the Riemannian holonomy group, unless $\alpha = 0$ which implies $\ker \Delta_{\Gamma}|_{\Omega^1(X, g_p)} = 0$. Thus, we have the dichotomy: $\alpha \neq 0$ implies a reduction of the holonomy of $X$, and $\alpha = 0$ implies the connection $A$ is non-degenerate. \hfill$$\blacksquare$$

2.3 A priori estimate for Yang-Mills equation

We will recall the monotonicity formula for Yang-Mills equation [17, 22]. Let $M$ be a compact Riemannian $n$-manifold with a smooth Riemannian metric $g$ and $E$ is a vector bundle over $M$ with compact structure group $G$. Let $p \in M$, let $r_p \leq \text{inj}(M)$ be a positive number with properties: there are normal coordinates $x_1, \cdots, x_n$ in the geodesic $B_{r_p}(p)$ of $(M, g)$, such that $p = (0, \cdots, 0)$ and for some constant $c(p)$:

$$|g_{ij} - \delta_{ij}| \leq c(p)r^2, \quad |dg_{ij}| \leq c(p)r,$$

where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$.

Remark 2.6. The constant $r_p$ and $c(p)$ can be choose depending only on the injective radius at $p$ and the curvature of $g$. Suppose that the manifold $X$ is compact, then the constant $r_p$ has a lower positive bounded constant $r_M$ and $|c(p)|$ has a upper positive constant $c_M$. The constant $r_M$ and $c_M$ are only depend on $X$ and $g$.

We will always denote $O(1)$ a quantity bounded by a constant depending only on $n$.

For any Yang-Mills connection $A$ of $E$, we have

Theorem 2.7. Let $A$ be any Yang-Mills connection of a $G$-bundle over a compact manifold $M$ with smooth Riemannian metric $g$. For any $p \in M$, there are positive constant $r_p$, $c(p)$ and a depend on $M, g$, such that for any $0 < \sigma < \rho < r_p$, we have

$$\rho^{4-n} e^{ar^2} \int_{B_{\rho}(p)} |F_A|^2 dvol_g - \sigma^{4-n} e^{ar^2} \int_{B_{\sigma}(p)} |F_A|^2 dvol_g$$

$$\geq 4 \int_{B_{\rho}(p) \setminus B_{\sigma}(p)} r^{4-n} e^{ar^2} |\frac{\partial}{\partial r} \cdot F_A|^2 dvol_g + \int_{\sigma} (e^{ar^2} \tau^{4-n}(2a\tau - O(1)c(p)\tau) \int_{B_{\tau}(p)} |F_A|^2 dvol_g) d\tau.$$
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Proof. We choose, for any $\tau$ small enough, $\xi(r) = \xi_\tau(r) = \eta(r/\tau)$, where $\eta$ is smooth and satisfies: $\eta(r) = 1$ for $r \in [0, 1]$, $\eta(r) = 0$ for $r \in [1 + \varepsilon, \infty)$, $\varepsilon > 0$ and $\eta'(r) \leq 0$. By taking $\phi = 1$ on Equation (2.1.8), we have

$$
\frac{\partial}{\partial \tau} \left( r^{4-n} e^{ar^2} \int_M \xi_\tau |F_A|^2 dV_g \right)
= 4r^{4-n} e^{ar^2} \left( \frac{\partial}{\partial \tau} \left( \int_M \xi_\tau \frac{\partial}{\partial r} jF_A|^2 dV_g \right) + (-O(1)c(p) + 2a) \right) \int_M \xi_\tau |F_A|^2 dV_g.
$$

Then, by integrating on $\tau$ and letting $\varepsilon$ tends to zero, we complete the proof of this theorem.

We then recall the $\varepsilon$-regular theorem of Yang-Mills equation [22, 23].

**Theorem 2.8.** ([22] Theorem 2.2.1) Let $A$ be any Yang-Mills connection of a $G$-bundle over a compact manifold $M$ with smooth Riemannian metric $g$. Then there exist positive constants $\varepsilon = \varepsilon(M, g, n)$ and $C = C(M, g, n)$ which depend on $M, g, n$, such that for any $p \in M$ and $0 < \rho < r_p$ whenever

$$
\rho^{4-n} \int_{B_\rho(p)} |F_A|^2 dvol_g \leq \varepsilon,
$$

then

$$
|F_A|(p) \leq \frac{C}{\rho^2} \left( \rho^{4-n} \int_{B_\rho(p)} |F_A|^2 dvol_g \right)^{\frac{1}{2}}.
$$

The constant $r_p$ in Theorem 2.7 and Theorem 2.8 is same. We then have a useful $L^\infty$ estimate for Yang-Mills connection $A$ when the $L^2$-norm of curvature $F_A$ is sufficiently small.

**Corollary 2.9.** Let $A$ be any Yang-Mills connection of a principal $G$-bundle over a compact manifold $M$ with smooth Riemannian metric $g$. Then there are positive constants $\varepsilon_0 = \varepsilon_0(M, g, n)$ and $C = C(M, g, n)$ with following significance. If the curvature $F_A$ of connection $A$ obeying

$$
\|F_A\|_{L^2(M)} \leq \varepsilon_0.
$$

Then

$$
\|F_A\|_{L^\infty(M)} \leq C \|F_A\|_{L^2(M)}. \quad (2.3)
$$

Proof. We choose the constant $a$ in Theorem 2.7 sufficiently large to ensure that

$$
a - O(1)c_M > 0.
$$

Following Theorem 2.7 for any $0 < \sigma < \rho < r_M$,

$$
\sigma^{4-n} \int_{B_\rho(p)} |F_A|^2 dvol_g \leq \rho^{4-n} e^{a(\rho^2 - \sigma^2)} \int_{B_\rho(p)} |F_A|^2 dvol_g \leq \rho^{4-n} e^{a \rho^2} \int_M |F_A|^2 dvol_g,
$$

and

$$
\rho^{4-n} \int_{B_\rho(p)} |F_A|^2 dvol_g \leq \rho^{4-n} e^{a \sigma^2} \int_{B_\rho(p)} |F_A|^2 dvol_g \leq \rho^{4-n} e^{a \sigma^2} \int_M |F_A|^2 dvol_g.
$$
We now let $\rho \not\to r_M$, hence
\[
\sigma^{4-n} \int_{B_\sigma(p)} |F_A|^2 dvol_g \leq C \int_M |F_A|^2 dvol_g, \quad \forall \sigma \in (0, r_M),
\] (2.4)
where $C$ is a positive constant depends on $M$ and metric $g$.

For any $p \in M$, in the geodesic ball $B_\sigma(p)$, $0 < \sigma < r_M$, following Equation (2.4), we have
\[
\sigma^{4-n} \int_{B_\sigma(p)} |F_A|^2 dvol_g \leq C\varepsilon_0, \quad \forall \sigma \in (0, r_M),
\]
where $C = C(M, g)$ is a positive constant. We choose $\varepsilon_0$ small enough to ensure $C\varepsilon_0 \leq \varepsilon$, where $\varepsilon$ is the constant in Theorem 2.8. Then following Theorem 2.8,
\[
|F_A|(p) \leq C\sigma^2(\sigma^{4-n} \int_{B_\sigma(p)} |F_A|^2 dvol_g)^{\frac{1}{2}} \leq \sigma^{-\frac{n}{2}} C\|F_A\|_{L^2(X)}, \quad \forall \sigma \in (0, r_M).
\]

We now let $\sigma \not\to r_M$, thus
\[
|F_A|(p) \leq r_M^{-\frac{n}{2}} C\|F_A\|_{L^2(X)}, \quad \forall p \in M.
\]
We complete this proof. □

3 Asymptotic Behavior

**Definition 3.1.** Let $X$ be a closed, smooth manifold of dimension $n \geq 4$ with a Riemannian metric $g_X$. We call $X$ a good manifold, if $X$ admits non-zero, smooth 3-form $P$ and 4-form $Q$ satisfying $d \ast_X P = d \ast_X Q = 0$.

**Example 3.2.** There are many manifolds are good in the sense of Definition 2.2. For example: (1) $X$ is a Calabi-Yau 3-fold. It is defined as a manifold with a Kähler $(1,1)$-form $\omega$ and a holomorphic form $\Omega \in \Omega^{3,0}$. We can construct a $G_2$-structure on $Z$:
\[
\phi = dt \wedge \omega + \text{Im}\Omega.
\]
We denote $(P, Q) := (\text{Re}\Omega, \frac{1}{2}\omega^2)$, one can see the instanton equation (1.2) is a $G_2$-instanton.

(2) $X$ is a parallel $G_2$-manifold. It is defined as a manifold with a $G_2$-structure 3-form $\phi$. We can construct a $\text{Spin}(7)$-structure on $Z$:
\[
\Phi = dt \wedge \phi + \ast_X \phi.
\]
We denote $(P, Q) := (\phi, \ast\phi)$, one also can see the instanton equation (1.2) is a $\text{Spin}(7)$-instanton.
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Taking the exterior derivative of (1.2) and using the Bianchi identity, the fact \((P, Q)\) is co-closed, it’s easy to see the solution of instanton equation (1.2) also satisfies Yang-Mills equation. In this section, we denote \(X\) by a closed **good** manifold. We begin to study the decay of instantons over tubular ends. At first, we consider a family of bands \(B_T = [T, T + 1] \times X\) which we identify with the model \(B = [0, 1] \times X\) by translation. So the integrability of \(|F_A|^2\) over the end implies that \(\int_{[T, T + 1] \times X} |F_A|^2 \to 0\) as \(T \to \infty\).

On the compact manifold \(B_T\), we can choose a positive constant \(r_T\) by the similar way in above Section 2.3 (Remark 2.6). Since \(B_T\) is identity with \(B_0\) by translation, it’s easy to see \(r_T\) is not dependent on \(T\). For the constant \(\varepsilon = \varepsilon(X, n)\) which satisfies the hypothesis in Theorem 2.8, we can choose a large enough constant \(T\) to ensure that

\[
\rho^{3-n} \int_{B_t} |F_A|^2 \leq C \int_{B_t} |F_A|^2 \leq \varepsilon, \quad \forall \ t \geq T, 0 < \rho < r_T.
\]

Following Corollary 2.9 and Uhlenbeck compactness theorem, we have

**Proposition 3.3.** Suppose \(A\) is an instanton over \(Z := R \times X\) with \(L^2\)-curvature \(F_A\). Then at the end of \(Z\), there is a flat connection \(\Gamma\) over \(X\) such that \(A\) converges to \(\Gamma\), i.e. the restriction \(A|_{X \times \{T\}}\) converges (modulo gauge equivalence) to \(\Gamma\) in \(C^\infty\) over \(X\) as \(T \to \infty\).

By Proposition 3.3 the moduli space \(\mathcal{M}\) of finite energy instantons on \(Z\) is the disjoint union of its subsets \(\mathcal{M}(A_{-\infty}, A_{+\infty})\), where \(A_{-\infty}\) and \(A_{+\infty}\) run over all components of the space of flat connection on \(X\) and \(\mathcal{M}(A_{-\infty}, A_{+\infty})\) is the subset of \(\mathcal{M}\) consisting of instantons with limits in \(A_{-\infty}\) and in \(A_{+\infty}\) over the two ends respectively. If we also suppose the flat connections over a closed manifold \(X\) are isolated, then the limit is unique–independent of the sequence and subsequence chosen. We apply a key result due to Uhlenbeck for the connections with \(L^p\)-small curvature ([24] Corollary 4.3) to prove that

**Corollary 3.4.** Suppose also that the flat connections on \(X\) are non-degenerate. Then there exists positive constants \(C, T\) with following significance. If \(t > T\), there exist a family gauge transformation \(g(t)\) such that

\[
\|g(t)^*(A(t)) - \Gamma\|_{L^p(X)} \leq C\|F_{A(t)}\|_{L^p(X)}, \quad \forall \ 2p > n.
\]

**Proof.** We denote \(B_t = [t, t + 1] \times X\) and \(2p > n\). For the constant \(\varepsilon_0\) which satisfies the hypothesis in Corollary 2.9 we can choose a large enough \(T\) to ensure that

\[
\|F_A\|_{L^2(B_t)} \leq \varepsilon_0, \quad \forall t > T.
\]

Therefore following the estimate in Corollary 2.9 we have

\[
\|F_{A(t)}\|_{L^p(X)} \leq (\text{Vol}(X))^{\frac{1}{p}} \|F_{A(t)}\|_{L^\infty(X)} \leq (\text{Vol}(X))^{\frac{1}{p}} \|F_A\|_{L^\infty(B_t)} \leq C\|F_A\|_{L^2(B_t)},
\]
where $C = C(X, g, p)$ is a positive constant. We can choose $C\|F_A\|_{L^2(B_\epsilon)} \leq \epsilon$, where constant $\epsilon$ satisfies the hypothesis in [24] Corollary 4.3. Then there exist a flat connection $\Gamma(t)$ and a gauge transformation $g(t)$ such that

$$\|g(t)^*(A(t)) - \Gamma(t)\|_{L^1_t(X)} \leq C(p)\|F_A(t)\|_{L^p(X)}, \forall 2p > n + 1.$$  

Since the flat connection over $X$ are isolated, the limit is unique–independent of the sequence and subsequence chosen, then $A(t) \to \Gamma$ in $C^\infty$ under moduli gauge transformation, i.e., for any small enough positive constant $\epsilon_0$, there is a large enough constant $t$ such that,

$$\inf_{g \in G} \|g^*(A(t)) - \Gamma\|_{L^1_t(X)} \leq \epsilon.$$  

Therefore

$$\inf_{g \in G} \|\Gamma(t) - \Gamma\|_{L^1_t(X)} \leq \inf_{g \in G} \|g^*(A(t)) - \Gamma\|_{L^1_t(X)} + \inf_{g \in G} \|g^*(A(t)) - \Gamma(t)\|_{L^1_t(X)}$$

$$\leq \epsilon_0 + C\|F_A(t)\|_{L^2(X)}.$$  

We can choose $T$ sufficiently large to ensure that $\epsilon + C\|F_A(t)\|_{L^2(X)} < \delta$, where $\delta$ satisfies the hypothesis in Proposition [24]. Thus $[\Gamma(t)] = [\Gamma]$. We complete the proof of this corollary. 

We begin to study the decay of instantons over the ends. We prove a useful lemma:

**Lemma 3.5.** If $A$ is an instanton with $L^2$-curvature over $Z := \mathbb{R} \times X$. Then

$$CS(A(T)) - CS(A_\infty) = -\int_{[T, \infty) \times X} Tr(F_A \wedge *F_A). \quad (3.1)$$  

**Proof.** Following the definition of $CS$, we then have

$$CS(A(T')) - CS(A(T)) = -2 \int_{[T, T'] \times X} Tr(F_A(t) \wedge \dot{A}(t)) \wedge *X P$$

$$= -\int_{[T, T'] \times X} Tr(F_A \wedge F_A) \wedge *X + \int_T^{T'} \left(\int_X (Tr(F_A(t) \wedge F_A(t)) \wedge *X Q) dt\right)$$

Following Proposition [3, 3], there exist a flat connection $\Gamma$ over $X$ such that, $A|_{X \times \{T\}}$ converges to $\Gamma$ in $C^\infty$ after suitable gauge transformations. Since $Q$ is co-closed, following Chern-Weil theory, we have

$$\int_X Tr(F_A(t) \wedge F_A(t)) \wedge *X Q = \lim_{t \to \infty} \int_X Tr(F_A(t) \wedge F_A(t)) \wedge *X Q = \int_X Tr(F_\Gamma \wedge F_\Gamma) \wedge *X Q = 0.$$  

Taking the limit over finite tubes $(T, T') \times X$ with $T' \to +\infty$ and $A$ satisfies the instanton equation, we prove the identity (3.1).
Theorem 3.6. Suppose \( A \) is a smooth solution of instanton (1.2) with \( L^2 \)-curvature \( F_A \). Suppose also that the flat connections on \( X \) are non-degenerate. Then there are positive constants \( C', C'' \) such that
\[
|F_A| \leq C'' e^{-C'|t|},
\]
for sufficiently large \(|t|\).

Proof. We prove this using a differential inequality derived from the instanton on \( Z \). Our proof here is similar to Donaldson’s arguments in [4] Section 4.2 for ASD connection. For \( T > 0 \), we set
\[
J(T) = \int_T^\infty \|F_A\|_{L^2(X)}^2 = - \int_{[T,\infty) \times X} Tr(F_A \wedge \ast F_A) = \int_{[T,\infty) \times X} Tr(F_A \wedge F_A) \wedge \ast \Omega.
\]
Following Lemma 3.5 we have
\[
J(T) = CS(A(T)) - CS(A_\infty) \quad (3.2)
\]
where \( A(T) \) is the connection over \( X \) obtained by restriction to \( X \times \{T\} \). Following (3.2), we obtain the \( T \) derivative of \( J \) as
\[
\frac{d}{dT} J(T) = \frac{d}{dT} (CS(A(T)) - CS(A_\infty)). \quad (3.3)
\]
On the other hand, the \( T \) derivative of \( J(T) \) can be expressed as minus the integration over \( X \times \{T\} \) of the curvature density \( |F_A|^2 \), and this is exactly the \( n \)-dimensional curvature density \( |F_{A(T)}|^2 \) plus the density \( |\dot{A}|^2 \). By the relation Equation (2.1) between the two components of the curvature for an instanton, we have \( \|F_{A(T)}\|_{L^2(X)}^2 = \|\dot{A}(T)\|_{L^2(X)}^2 \). Thus
\[
\frac{d}{dT} J(T) = -2 \|F_{A(T)}\|_{L^2(X)}^2 \quad (3.4)
\]
From (3.3) and (3.4), we have
\[
\frac{d}{dT} (CS(A(T)) - CS(A_\infty)) = -2 \|F_{A(T)}\|_{L^2(X)}^2
\]
To these two observations we establish an inequality between the Chern-Simon function \( CS(A(T)) \) and \( \|F_{A(T)}\|_{L^2(X)} \), valid for any connection over \( X \) which is close to \( A_\infty \). We write, for fixed large \( T \),
\[
A(T) = A_\infty + a,
\]
where \( A_\infty \) is a flat connection over \( X \), so we may suppose that \( a \) is small as we please in \( C^\infty \). Also, we may suppose that \( a \) satisfies the Coulomb gauge fixing:
\[
d^*_a a = 0.
\]
Now, we have
\[
CS(A(T)) - CS(A_\infty) = - \int_X Tr(d_{A_\infty} a \wedge a + \frac{2}{3} a \wedge a \wedge a) \wedge_X P
\]
\[
= - \int_X Tr\left( \frac{1}{3} d_{A_\infty} a \wedge a + \frac{2}{3} F_{A(T)} \wedge a \right) \wedge_X P.
\]
We use the fact that the kernel of \(d_{A_\infty} + \lambda^2\) in \(\Omega^1\) is trivial, so following Proposition 2.3 there exist a positive constant \(\lambda\) such that
\[
\|a\|_{L^2(X)} \leq \lambda \|d_{A_\infty} a\|_{L^2(X)}.
\]
We observe that
\[
|\int_X Tr(d_{A_\infty} a \wedge a) \wedge_X P| \leq \|a\|_{L^2(X)} \|d_{A_\infty} a\|_{L^2(X)} \max_X |P|
\leq C\lambda \|d_{A_\infty} a\|^2_{L^2(X)},
\]
and
\[
|\int_X Tr(F_{A(T)} \wedge a) \wedge_X Q| \leq \|F_{A(T)}\|_{L^2(X)} \|a\|_{L^2(X)} \max_X |Q|
\leq C\lambda \|d_{A_\infty} a\|_{L^2(X)} \|F_{A(T)}\|_{L^2(X)},
\]
where \(C = C(X, P, Q)\) is a positive constant. Hence, we get
\[
CS(A(T)) - CS(A_\infty) \leq C\lambda(\|d_{A_\infty} a\|^2_{L^2(X)} + \|d_{A_\infty} a\|_{L^2(X)} \|F_{A(T)}\|_{L^2(X)}).
\]
We denote \(B_t := [t, t+1] \times X\). Following Corollary 3.4, for large enough \(T\), we have
\[
\|a(t)\|_{L^\infty(X)} \leq C\|a(t)\|_{L^p(X)} \leq C\|F_{A(t)}\|_{L^p} \leq C\|F_A\|_{L^p(B_t)} \leq C\|F_A\|_{L^2(B_t)}.
\]
On the other hand \(F_{A(T)} = F_{A_{\infty} + a} = d_{A_\infty} a + a \wedge a\). Therefore
\[
\|F_{A(T)}\|_{L^2(X)} \geq \|d_{A_\infty} a\|_{L^2(X)} - \|a \wedge a\|_{L^2(X)} \geq \|d_{A_\infty} a\|_{L^2(X)} - \|a\|_{L^\infty(X)} \|a\|_{L^2(X)}.
\]
Combining the preceding inequalities (3.6) and (3.7) yields,
\[
\|F_{A(T)}\|_{L^2(X)} \geq \|d_{A_\infty} a\|_{L^2(X)} - C\lambda \|F_A\|_{L^2(B_t)} \|d_{A_\infty} a\|_{L^2(X)}.
\]
Provided \(C\lambda \|F_A\|_{L^2(B_t)} \leq 1/2\), rearrangement gives
\[
\|F_{A(T)}\|_{L^2(X)} \geq 1/2 \|d_{A_\infty} a\|_{L^2(X)}.
\]
Combining the preceding inequalities (3.5) and (3.8) yields,
\[
CS(A(T)) - CS(A_\infty) \leq C\|F_{A(T)}\|^2_{L^2(X)}.
\]
Putting all this together, we get a differential inequality, when $T$ is large enough,

$$J(T) \leq -C \frac{d}{dT} J(T).$$

It is easy to see that this implies that $J$ decays exponentially,

$$J(T) \leq C'' e^{-C'T},$$

where $C' = (C)^{-1}$ and $C'' = J(T_0) e^{-CT_0}$. Finally, having obtained the exponential decay of $J$ we deduce that of the curvature density itself via elliptic estimates on the model band.

We observe that the flat connections on a $G$-bundle over a closed Calabi-Yau 3-fold or $G_2$-manifold with fully holonomy, i.e., the manifold has finite fundamental group, are all non-degenerate. As a simply application, we have

**Corollary 3.7.** Let $X$ be a closed Calabi-Yau 3-fold (or $G_2$-manifold) with fully holonomy, $P$ be a principal $G$-bundle over cylinder $Z := \mathbb{R} \times X$ with $G$ being a compact Lie group. If $A$ is a $G_2$- (or $Spin(7)$-) instanton with $L^2$-curvature $F_A$. Then there exist positive constants $C', C''$ such that

$$|F_A| \leq C'' e^{-C'|t|},$$

for sufficiently large $|t|$.

### 4 Real Killing spinor manifold

Let $(X, g)$ be a real Killing spinor compact manifold of dimension $n$, i.e., there are 3-form $P$ and 4-form $Q$ which satisfy

$$dP = 4Q, \quad d*_X Q = (n-3)*_XP,$$

where $*_X$ is the Hodge star operator on $X$. For $n > 3$, the Chern-Simons functional can then be written as

$$CS(A) = -\frac{1}{2(n-3)} \int_X Tr(F_A \wedge F_A) \wedge *_X Q, \quad (4.1)$$

which is gauge-invariant. We consider the cylinder $Z := \mathbb{R} \times X$ over $X$. The instanton equation (1.2) on the cylinder splits into the two equations (2.1). The gradient flow of Chern-Simons functional (4.1) is not-equivalent flow is equivalent to the instanton equation.
Proposition 4.1. (Energy inequality). Let $A$ be a smooth solution to the heat flow equation (4.1). Then, for any $t_1 < t_2$, we have

$$\text{CS}(A(t_2)) - \text{CS}(A(t_1)) = -\int_{t_1}^{t_2} \| F_A(t) \wedge *_X P \|_{L^2(X)}^2. \quad (4.2)$$

Proof. It is easy to check that

$$dtr(F_A \wedge F_A \ ^* X Q) = -(n-3)tr(F_A \wedge F_A) \wedge *_X P).$$

We then have

$$\frac{\partial}{\partial t} \text{CS}(A(t)) = -\frac{1}{n-3} \int_X Tr\left(\frac{\partial F_A}{\partial t} \wedge F_A \wedge *_X Q\right)$$

$$= -\frac{1}{n-3} \int_X Tr d(A) \wedge F_A \wedge *_X Q$$

$$= -\frac{1}{n-3} \int_X dTr\left(\frac{\partial A}{\partial t} \wedge F_A \wedge *_X Q\right) - \int_X Tr\left(\frac{\partial A}{\partial t} \wedge F_A \wedge *_X P\right)$$

$$= -\| F_A \wedge *_X P \|^2_{L^2(X)}.$$

Equality (4.2) follows from integrating the above identity on $[t_1, t_2]$.

We define the energy density $\rho(A)$ by

$$\rho(A) := \lim_{T \to \infty} \frac{1}{2T} \int_{(-T,T) \times X} |F_A|^2 dvol dt. \quad (4.3)$$

We write $\alpha \lesssim \beta$ to mean that $\alpha \leq C\beta$ for some positive constant $C$ independent of certain parameters on which $\alpha$ and $\beta$ depend. The parameters on which $C$ is independent will be clear or specified at each occurrence. We also use $\beta \lesssim \alpha$ and $\alpha \approx \beta$ analogously.

Lemma 4.2. Let $X$ be a complete manifold of dimension $n$ with a $d$-bounded $k$-form $\omega$, i.e., there exist a $(k-1)$-form $\theta$ such that $\omega = d\theta$, $\alpha$ be a closed form of degree $n-k$. If $\alpha$ satisfies

$$\lim_{r \to \infty} \frac{1}{r} \int_{B_r(x_0)} |\alpha| dvol = 0, \quad (4.4)$$

where $x_0$ is a point on $X$, $B_r(x_0)$ is a geodesic ball. Then there exists a sequence $\{j_i\}_{i \geq 1}$ such that

$$\lim_{i \to \infty} \int_{B_{j_i}(x_0)} \alpha \wedge \omega = 0.$$

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}$ be smooth, $0 \leq \eta \leq 1$,

$$\eta(t) = \begin{cases} 1, & t \leq 0 \\ 0, & t \geq 1 \end{cases}$$
and consider the compactly supported function

$$j(x) = \eta(\rho(x_0, x) - j),$$

where $j$ is a positive integer.

We consider the form $\beta := \alpha \wedge \omega = d(\alpha \wedge \theta)$. We have $f_j \beta = d(f_j \alpha \wedge \theta) - df_j \wedge (\alpha \wedge \theta)$.

By Stokes formula, we obtain

$$| \int_{B_{j+1} \setminus B_j} f_j \beta | = | \int_X df_j \wedge (\alpha \wedge \theta) | \lesssim \int_{B_{j+1} \setminus B_j} |\alpha|$$

and

$$| \int_{B_j} \beta | \leq | \int_X f_j \beta | + \int_{B_{j+1} \setminus B_j} |\beta | \lesssim | \int_X f_j \beta | + \int_{B_{j+1} \setminus B_j} |\alpha|$$

Thus

$$| \int_{B_j} \beta | \lesssim \int_{B_{j+1} \setminus B_j} |\alpha|.$$  \hspace{1cm} (4.5)

By the hypothesis (4.4), there exists a subsequence $\{j_i\}_{i \geq 1}$ such that

$$\lim_{i \to \infty} \int_{B_{j_i+1} \setminus B_{j_i}} |\alpha| = 0. \hspace{1cm} (4.6)$$

It now follow (4.5)–(4.6) that $\lim_{i \to \infty} \int_{B_{j_i}(x_0)} \alpha \wedge \omega = 0.$ \hspace{1cm} \(\square\)

We denote $*$ by the Hodge star operator on $Z$, $D$ by the exterior derivative on $T^*(Z)$. We also denote $\tilde{P} = dt \wedge P$, $\tilde{Q} = dt \wedge Q$. Then the forms $\tilde{P}$, $\tilde{Q}$ satisfy

$$*\tilde{P} = *_X P, \hspace{0.5cm} *\tilde{Q} = *_X Q,$$

and

$$D\tilde{P} = 4\tilde{Q}, \hspace{0.5cm} D * \tilde{Q} = (n-3) * \tilde{P}.$$

**Theorem 4.3.** Let $Z$ be the cylinder over a compact real Killing spinor manifold, $A$ be a solution of instanton equation. If $\rho(A) = 0$, then $A$ is a flat connection.

**Proof.** The Yang-Mills energy function is

$$YM(A) := \| F_A \|_{L^2(Z)}^2 = -\int_{\mathbb{R} \times X} Tr(F_A \wedge F_A) \wedge *\Omega$$

$$= -\int_{\mathbb{R} \times X} Tr(F_A^2) \wedge *\tilde{P} - \int_{\mathbb{R} \times X} Tr(F_A^2) \wedge *_X Q \wedge dt.$$ 

We observe that

$$-\int_{\mathbb{R} \times X} Tr(F_A^2) \wedge *\tilde{P} = -\frac{1}{n-3} \int_{\mathbb{R} \times X} Tr(F_A^2) \wedge D * \tilde{Q}.$$
Since $Tr(F_A^2)$ is a closed $L^1$ form on $Z$ and $\ast \tilde{P}$ is $D$-bounded, then following Lemma 4.2 there exist a sequence $\{j_i\}_{i \geq 1}$ such that
\[
\lim_{i \to \infty} \int_{(-j_i, j_i) \times X} Tr(F_A^2) \wedge \ast \tilde{P} = 0.
\] (4.7)

Following equation (2.1), we then have
\[
- Tr(F_A^2) \wedge \ast \tilde{P} = -2Tr(\frac{\partial A}{\partial t} \wedge F_A) \wedge \ast X = 2|\frac{\partial A}{\partial t}|^2 dt \wedge dvol
\]
Thus
\[
- \int_{(-j_i, j_i) \times X} Tr(F_A^2) \wedge \ast \tilde{P} = 2 \int_{(-j_i, j_i) \times X} |\frac{\partial A}{\partial t}|^2 dt \wedge dvol.
\] (4.8)

It now follows (4.7), (4.8) that
\[
\lim_{i \to \infty} \int_{(-j_i, j_i) \times X} |\frac{\partial A}{\partial t}|^2 dt \wedge dvol = 0,
\]
i.e., $\frac{\partial A}{\partial t} = 0$. The connection $A$ is not dependence on parameter $t$. Thus
\[
\rho(A) = \int_X |F_A|^2 dvol,
\]
by the hypothesis, we obtain that $F_A = 0$. We complete this proof. □

**Corollary 4.4.** Let $Z$ be the cylinder over a compact real Killing spinor manifold, $A$ be a solution of instanton equation (1.2). If the curvature $F_A$ is in $L^p$, $p \geq 2$. Then $A$ is a flat connection.

**Proof.** We denote $B_T = (-T, T) \times X$. For $p = 2$, it is easy to see $\rho(A) = 0$. For $p > 2$, we use the H"older inequality,
\[
\|F_A\|_{L^2(B_T)} \leq \|F_A\|_{L^p(B_T)} (2TVol(X))^{1-\frac{2}{p}}.
\]
Thus $\rho(A) = 0$. Following Theorem 4.3, $A$ is flat. □

We define $\mathcal{M}_d$ as the space of the gauge equivalence classes of instantons $A$ on $P$ satisfying
\[
\|F_A\|_{L^\infty(X)} \leq d.
\]
The space $\mathcal{M}_d$ is endowed with the topology of $C^\infty$ convergence over compact subsets: the sequence $\{|A_i|\}$ in $\mathcal{M}_d$ converges to $[A]$ if and only if there exist gauge transformations $g_i$ satisfying $g_i^*(A_i) \to A$ in $C^\infty$ over every compact subset of $X$. The space $\mathcal{M}_d$ is compact by the Uhlenbeck compactness theorem. We denote by $\rho(d)$ the value of $\rho(A)$ over $[A] \in \mathcal{M}_d$. At the end of $Z$, there are connections $\Gamma_\pm$ over $X$ such that $A$ converges to $\Gamma_\pm$, i.e. the restriction $A|_{X \times \{T\}}$ converges (modulo gauge equivalence) to $\Gamma_\pm$ in $C^\infty$ over $X$ as $T \to \pm \infty$. 
Proposition 4.5. Suppose $A$ is an instanton over $Z := \mathbb{R} \times X$. Then
\[ \rho(d) = \frac{\rho_- + \rho_+}{2}, \]
where $\rho_\pm = \int_X Tr(F_{T \pm}^2 \wedge \ast_X Q)$.

Proof. Following equations (1.2), we have
\[ F_A^2 \wedge \ast \Omega = F_A^2 \wedge dt \wedge \ast_X Q + 2 \frac{\partial A}{\partial t} \wedge dt \wedge F_A \wedge \ast_X P. \]

Therefore, following energy identity in Proposition [4.1]
\[ \int_{(T,T') \times X} |F_A|^2 d\text{vol}_g dt = \int_{(T,T') \times X} Tr(F_A^2 \wedge \ast_X Q) \wedge dt + 2(CS(A(T)) - CS(A(T'))). \]

Since $[A] \in \mathcal{M}_d$, we have $|CS(A(t))| \lesssim d^2$, for all $t \in \mathbb{R}$. Thus
\[ \rho(d) = \lim_{t \to \infty} \frac{1}{2T} \int_{(-T,T) \times X} Tr(F_A^2 \wedge \ast_X Q) = \frac{\rho_- + \rho_+}{2}. \]

We complete this proof. \hfill \square

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References

[1] Bauer, I., Ivanova, T.A., Lechtenfeld, O., Lubbe, F.: Yang-Mills instantons and dyons on homogeneous $G_2$-manifolds. JHEP. **2010**(10), 1–27 (2010)
[2] Carrión, R.R.: A generalization of the notion of instanton, Diff.Geom.Appl. **8**(1), 1–20 (1998)
[3] Corrigan, E., Devchand, C., Fairlie, D.B., Nuysts, J.: First order equations for gauge fields in spaces of dimension great than four, Nucl.Phys.B, **214**(3), 452–464 (1983)
[4] Donaldson, S.K.: Floer homology groups in Yang-Mills theory, Cambridge University Press, (2002)
[5] Donaldson, S.K., Segal, E.: Gauge theory in higher dimensions, II. arXiv:0902.3239, (2009)
[6] Donaldson, S.K., Thomas R.P.: Gauge theory in higher dimensions, The Geometric Universe, Oxford, 31–47 (1998)
[7] Grañá, M.: Flux compactifications in string theory: A comprehensive review. Phys.Rept. **423**(3), 91–158 (2006)
[8] Green, M.B., Schwarz, J.H., Witten, E.: Superstring theory, Cambridge University Press, (1987)
[9] Harland, D., Ivanova, T.A., Lechtenfeld, O., Popov, A.D.: Yang-Mills flows on nearly Kahler manifolds and $G_2$-instantons. Comm.Math.Phys. **300**(1), 185–204 (2010)
[10] Harland, D., Nölle C.: Instantons and Killing spinors, JHEP. 3, 1–38 (2012)

[11] Haupt, A.S.: Yang-Mills solutions and Spin(7)-instantons on cylinders over coset spaces with G₂-structure. JHEP. 3, 1–53 (2016)

[12] Huang, T.: Instanton on Cylindrical Manifolds. Ann. Henri Poincaré 18(2), 623–641 (2017)

[13] Huang, T.: Stable Yang-Mills connections on special holonomy manifolds. J. Geom. Phys. 116, 271–280 (2017)

[14] Huang, T.: An energy gap for complex Yang-Mills equations. SIGMA Symmetry Integrability Geom. Methods Appl. 13 (2017), Paper No. 061, 15 pages.

[15] Ivanova, T.A., Popov, A.D.: Instantons on special holonomy manifolds. Phys.Rev.D 85(10) (2012)

[16] Ivanova, T.A., Lechtenfeld, O., Popov, A.D., Rahn, T.: Instantons and Yang-Mills flows on coset spaces. Lett. Math. Phys. 89 (3), 231–247 (2009)

[17] Price, P.: A monotonicity formula for Yang-Mills fields. Manuscripta Math. 43 131–166 (1983)

[18] Sá Earp, H.N.: Generalised Chern-Simons Theory and G₂-Instantons over Associative Fibrations. SIGMA, 10:083 (2014)

[19] Sá Earp, H.N.: G₂-instantons over asymptotically cylindrical manifolds. Geom. Topol. 19, 61–111 (2015)

[20] Sá Earp, H.N., Walpuski, T.: G₂-instantons over twisted connected sums. Geom. Topol. 19, 1263–1285 (2015)

[21] Taubes, C. H.: Casson’s invariant and gauge theory. J. Diff. Geom. 31(2), 547-599 (1990)

[22] Tian, G.: Gauge theory and calibrated geometry. I. Ann. Math. 151(1), 193–268 (2000)

[23] Uhlenbeck, K. K.: Connections with L_p bounds on curvature, Comm. Math. Phys. 83, 31–42 (1982)

[24] Uhlenbeck, K. K.: The Chern classes of Sobolev connections, Comm. Math. Phys. 101, 445–457 (1985)

[25] Walpuski, T.: G₂-instantons on generalised Kummer constructions. Geom. Topol. 17, 2345–2388 (2013)

[26] Ward, R.S.: Completely solvable gauge field equations in dimension great than four. Nucl.Phys.B 236(2), 381–396 (1984)