Mass Ladder Operators from Spacetime Conformal Symmetry

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Ladder operators can be useful constructs, allowing for unique insight and intuition. In fact, they have played a special role in the development of quantum mechanics and field theory. Here, we introduce a novel type of ladder operators, which map a scalar field onto another massive scalar field. We construct such operators, in arbitrary dimensions, from closed conformal Killing vector fields, eigenvectors of the Ricci tensor. As an example, we explicitly construct these objects in anti-de Sitter spacetime (AdS) and show that they exist for masses above the Breitenlohner-Freedman (BF) bound. Starting from a regular seed solution of the massive Klein-Gordon equation (KGE), mass ladder operators in AdS allow one to build a variety of regular solutions with varying boundary condition at spatial infinity. We also discuss mass ladder operator in the context of spherical harmonics, and the relation between supersymmetric quantum mechanics and so-called Aretakis constants in an extremal black hole.

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I. INTRODUCTION

Exactly solvable systems play a crucial role in understanding the physical content of a given theory and on isolating the main features of certain solutions. Some of such “golden” systems, like the hydrogen atom in quantum mechanics are well-known cornerstones in mathematics, physics and chemistry. In some cases, it is even possible to relate families of solutions of a given problem without a detailed knowledge of any of its solutions. One such remarkable technique, in the context of the Schrödinger equation, consists in using ladder operators. These enable one to construct algebraically all, or part of, the energy eigenvalues and eigenfunctions. However, finding ladder operators in general a challenging task, because their relations to the symmetries of the system remain unclear – usually, therefore, they are called a dynamical symmetry.

Such enterprise is specially relevant within general relativity or the gauge-gravity approach to field theories. In such a framework, test fields on curved spacetimes have been repeatedly studied and found – at least at linearized level – to inherit the background symmetries; they are thus helpful in providing a geometric picture of spacetime. Conversely, some techniques were developed which make explicit use of spacetime symmetries to handle these fields. For example, test fields can be conveniently separated by means of harmonic functions, on spacetimes which are maximally symmetric or contain a maximally symmetric subspace. We will focus our attention on the prototypical example of the KGE,

\begin{equation}
(\Box - m^2)\Phi = 0 ,
\end{equation}

where $\Box \equiv g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$ is the d’Alembertian and $m$ is a mass parameter.

In this paper, we report that if a background spacetime has a particular conformal symmetry (whose precise conditions will be stated below), there exists a differential operator $D$, called a mass ladder operator, which maps a solution to the KGE with mass squared $m^2$ into another solution with different mass squared $m^2 + \delta m^2$, i.e.,

\begin{equation}
(\Box - (m^2 + \delta m^2))D\Phi = 0 ,
\end{equation}

where $\delta m^2$ is the variation of the mass squared. Hence, we provide a geometric interpretation to ladder operators in terms of the symmetry of a background spacetime.

Our formulation can be useful in Riemannian geometry, where the KGE is replaced by the Helmholtz-like equation, i.e., the eigenvalue equation for the Laplacian,

\begin{equation}
(\Delta - \lambda)\Phi = 0 ,
\end{equation}

where $\Delta \equiv g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$ is the Laplacian and $\lambda$ is the eigenvalue of the Laplacian. For example, when we consider the Laplacian on $S^2$, we obtain ladder operators which change the azimuthal quantum number $\ell$ [1, 2]. The reason why such ladder operators exist on $S^2$ has never been explained in terms of conformal symmetry. In our formulation, we can explicitly construct the ladder operators from conformal Killing vectors on $S^2$.

II. MASS LADDER OPERATORS FOR SCALARS

Conformal symmetry of an $n$-dimensional spacetime $(M,g_{\mu\nu})$ is defined by the invariance for a metric $g_{\mu\nu}$ under the conformal transformation $g_{\mu\nu} \rightarrow g'_{\mu\nu} = \exp(2Q)g_{\mu\nu}$, where $Q$ is a function on $M$. The infinitesimal transformation is described by the conformal Killing
equation
\[ \nabla_\mu \zeta^\nu + \nabla_\nu \zeta_\mu = 2Q g_{\mu \nu}, \quad Q = \frac{1}{n} \nabla_\mu \zeta^\mu, \] (4)
where \( \zeta^\mu \) is known as a conformal Killing vector, and \( Q \) is called the associated function. In particular, a conformal Killing vector is said to be closed if it satisfies the condition \( \nabla_\mu \zeta^\nu = 0 \). Hence, \( \zeta^\mu \) is a closed conformal Killing vector (CCKV) if it satisfies the equation
\[ \nabla_\mu \zeta^\nu = Q g_{\mu \nu}, \quad Q = \frac{1}{n} \nabla_\mu \zeta^\mu. \] (5)

Using this equation, we obtain the following result: (the detailed derivation is shown in the next section).

Suppose that an \( n \)-dimensional spacetime \((M, g_{\mu \nu})\) admits a CCKV \( \zeta^\mu \) satisfying (4). If \( \zeta^\mu \) is an eigenvector of the Ricci tensor with a constant eigenvalue, \( R^\mu_\nu \zeta^\nu = \chi (n-1) \zeta^\mu \), then there exists a one-parameter family of mass ladder operators
\[ D_k \equiv \mathcal{L}_\zeta - kQ, \] (6)
where \( \mathcal{L}_\zeta \) denotes the Lie derivative with respect to \( \zeta^\mu \), such that the commutation relation with the \( \text{d'Alembertian} \) \( \Box \equiv \nabla_\mu \nabla^\mu \) is given by
\[ \Box, D_k = \chi (2k + n - 2) D_k + 2Q (\Box + \chi k (k + n - 1)), \] (7)
where \( k \) is a parameter, and the commutator is considered as acting on a scalar field.

Since the action of the commutator on a scalar field \( \Phi \) leads to
\[ (\Box - (m^2 + \delta m^2)) D_k \Phi = D_{k-2} (\Box - m^2) \Phi, \] (8)
together with \( m^2 = -\chi k (k + n - 1) \) and \( \delta m^2 = \chi (2k + n - 2) \), \( D_k \) maps a solution \( \Phi \) to the KGE with mass squared \( m^2 \) into another solution \( D_k \Phi \) with mass squared \( m^2 + \delta m^2 = -\chi (k-1) (k+n-2) \). Therefore, \( D_k \) are mass ladder operators which connect massive solutions to the KGE from the mass squared \( m^2 \) to \( m^2 + \delta m^2 \). This corresponds in terms of \( k \) to shifting \( k \) to \( k-1 \). It is also found that \( D_{k-n+2} \) maps a solution to the KGE into a solution from the mass squared \( m^2 \) to \( m^2 + \delta m^2 \). This corresponds to shifting \( k-1 \) to \( k \). Since \( D_k \) is surjective (or onto) every \( k \), all the solutions with mass squared \( m^2 + \delta m^2 \) can be constructed from the solutions with mass squared \( m^2 \) (see Appendix B). It should be noted that the operators have the index \( k \) and its value must be chosen appropriately depending on the mass of a solution to act.

When \( D_k \) connects solutions to the KGE with two real mass squared \( m^2 \) and \( m^2 + \delta m^2 \), \( k \) is required to be real, and the following inequalities must be satisfied
\[ \frac{1}{4} (n-1)^2 \leq m^2, \quad \chi < 0 \quad \text{or} \quad m^2 \leq \frac{1}{4} (n-1)^2, \quad \chi > 0, \] (9)
where the equality is attained for \( k = -(n-1)/2 \). Thus, mass ladder operators can exist only when the value of the mass squared \( m^2 \) satisfies the above inequalities. We also notice that for a fixed \( m^2 \), there are two mass ladder operators \( D_{k\pm} \), where
\[ k_\pm = \frac{1 - n \pm \sqrt{(n-1)^2 - 4m^2/\chi}}{2}. \] (10)

These two operators become a mass raising or mass lowering operator, depending on the value of the mass squared \( m^2 \). For a negative \( \chi \), if \( m^2 > \chi (n-2)/4 \), they correspond to mass raising and lowering operators, respectively, and otherwise both become mass raising operators. For positive \( \chi \), roles are reversed.

Generally, multiples of the ladder operators can be considered. Since we have \([\text{cf. (5)}]\)
\[ (\Box - (k-s)(k+s-1))D_{k-s} \cdots D_{k-3}D_{k-2}D_k \Phi = D_{k-s-2} \cdots D_{k-3}D_{k-2}(\Box - k(k+1)) \Phi, \]
the multiple operator \( D_{k-s} \cdots D_{k-1}D_k \) can shift the mass squared labeled by \( k \) to the one by \( k-s \).

Physically important examples are maximally symmetric spacetimes. Actually, we can construct mass ladder operators for the KGEs in such spacetimes. If we consider \( AdS_n \) spacetime with a cosmological constant \( \Lambda = \chi(n-1) < 0 \), the first inequality in Eq.(3) coincides with the condition for the mass above BF bound \([1, 3]\). This means we can define mass ladder operator for the massive scalar with the mass above BF bound. In \( dS_n \) spacetime with a cosmological constant \( \Lambda = \chi(n-1) > 0 \), the second inequality in Eq.(3) is the condition for that the solution of KGE does not have oscillation solution for long wave limit (see, e.g., \([6]\)).

### III. Derivation of Mass Ladder Operators

In \( n \geq 2 \) dimensions, one can show the following commutation relation when acting on a scalar
\[ [\Box, \mathcal{L}_\zeta] = 2Q (\Box - (n-2)(\nabla_\mu Q) \nabla_\mu, \] (11)
\[ 3 \] Taking the adjoint of Eq. (6), (see discussion in Ref. [3], we obtain \( D_k^\dagger (\Box - (m^2 + \delta m^2)) = (\Box - m^2) (D_k + 2Q)^\dagger \) where \( \dagger \) means the adjoint operator. Since \( \Box - m^2 \) and \( \Box - (m^2 + \delta m^2) \) are self-adjoint operators, we can see that \( (D_k + 2Q)^\dagger = -D_{k-n+2} \) shifts mass from \( m^2 + \delta m^2 \) to \( m^2 \).

\[ 4 \] General spacetimes admitting mass ladder operators are discussed in Appendix B.
where $\zeta^\mu$ is a conformal Killing vector satisfying Eq. (4). If $\zeta^\mu$ is a CCKV, it satisfies Eq. (5). Differentiating this equation, we obtain
\begin{equation}
\nabla_\mu Q = \frac{1}{1 - n} R^\nu_\mu \zeta^\nu .
\end{equation}
Assuming in addition that $\zeta^\mu$ is an eigenvector of the Ricci tensor,
\begin{equation}
R^\nu_\mu \zeta^\nu = \chi (n - 1) \zeta^\mu ,
\end{equation}
where $\chi$ is constant, we arrive at the condition that the gradient of the function $Q$ is proportional to $\zeta^\mu$,
\begin{equation}
\nabla_\mu Q = - \chi \zeta_\mu .
\end{equation}
Under this condition, Eq. (11) is
\begin{equation}
[\Box, L_\zeta] = 2Q \Box + \chi (n - 2) L_\zeta .
\end{equation}
Furthermore, since Eq. (14) leads to
\begin{equation}
\Box Q + \chi n Q = 0 ,
\end{equation}
we obtain
\begin{equation}
[\Box, Q] = -2 \chi L_\zeta - n \chi Q .
\end{equation}
Given Eqs. (15) and (17), it is easy to calculate the commutation relation between the d’Alembertian and $D_k$ given by Eq. (6) and then obtain Eq. (7).

Suppose there are more than one ladder operators $D_{a,k} = L_\zeta_a - kQ_a (a = 1, 2, \cdots, N)$ for the KGE with mass squared $m^2 = - \chi k (k + n - 1)$. Then it would be natural to compute the commutation relations between them because one expects that they form a Lie algebra. First, it is important to see that the commutator of CCKVs $\zeta^\mu_a$, $\zeta^\mu_b$ is a Killing vector, which satisfies the Killing equation $\nabla^\nu \zeta^\mu_a + \nabla^\nu \zeta^\mu_b = 0$. Here, we have used the condition that $\zeta^\mu_a$ are eigenvectors of the Ricci tensor. Hence we have
\begin{equation}
[H_k, D_{a,k}] = \chi (2k + n - 2) D_{a,k} + 2Q_a \bar{H}_k
\end{equation}
\begin{equation}
[H_k, L_{\zeta_a}] = 0 ,
\end{equation}
where $\bar{H}_k \equiv \Box + \chi k (k + n - 1)$. The first relation shows that, since $Q_a$ is a function, $D_{a,k}$ becomes a ladder operator only for particular scalar fields $\Phi_k$ obeying the equation $\bar{H}_k \Phi_k = 0$. Since we have $\bar{H}_k (D_{a,k} \Phi_k) = 0$, one can construct $N$ solutions to the equation $\bar{H}_k \Phi_k = 0$ from a single $\Phi_k$. The second relation shows that the Lie derivative along the Killing vector $\zeta^\mu_a$ acts on any solution to the equation $\bar{H}_k \Phi_k = 0$ as symmetry. Since it is also shown that
\begin{equation}
- \chi d(\zeta_{a\mu} \zeta^\mu_b) = d(Q_a \Phi_b) ,
\end{equation}
we find
\begin{equation}
- \chi \zeta_{a\mu} \zeta^\mu_b = Q_a \Phi_b + C_{ab}
\end{equation}
with a constant $C_{ab}$. Thus the commutation relations between the ladder operators $D_{a,k}$ and the Killing vectors $\zeta^\mu_{a\mu}$ constructed from CCKVs are calculated as
\begin{equation}
[D_{a,k}, D_{b,k}] = L_{\zeta_{a\mu}} ,
\end{equation}
\begin{equation}
[D_{a,k}, L_{\zeta_{a\mu}}] = C_{ab} D_{c,k} - C_{ac} D_{b,k} ,
\end{equation}
\begin{equation}
[L_{\zeta_{a\mu}}, L_{\zeta_{b\nu}}] = C_{ad} L_{\zeta_{c\nu}} - C_{bc} L_{\zeta_{a\mu}} - C_{ac} L_{\zeta_{d\nu}} + C_{bd} L_{\zeta_{a\nu}} ,
\end{equation}
which form a Lie algebra. This implies that solutions to the equation $\bar{H}_k \Phi_k = 0$ become the representation of this Lie group. As seen later, this is conformal group in a maximally symmetric spacetime.

IV. MASS LADDER OPERATORS IN ADS

The metric of the $n$-dimensional anti-de Sitter spacetime ($AdS_n$) in Poincaré coordinate is
\begin{equation}
ds^2 = \frac{dr^2}{r^2} + r^2 \sum_{A,B=0}^{n-2} \eta_{AB} dA^A dX^B ,
\end{equation}
where $\eta_{AB} = \text{diag}[-1, 1, \cdots, 1]$ is the metric on the $n-1$ dimensional Minkowski spacetime $M^{n-1}$, and $A, B$ run over $0, 1, \cdots, n-2$. The massive KGE on this spacetime is
\begin{equation}
(r^2 \partial^2_r + nr \partial_r + r^2 \Box_{M^{n-1}} - m^2) \Phi = 0 ,
\end{equation}
where $\Box_{M^{n-1}}$ is d’Alembertian on $M^{n-1}$. In the $AdS_n$ spacetime, there are $n + 1$ CCKVs $\zeta^\mu_a$ which satisfy $\nabla^\nu \zeta_{a\nu} = Q_a g_{\mu\nu}$, where $a$ runs over $-1, 0, 1, \cdots, n-1$. Since the Ricci curvature satisfies $R_{\mu\nu} = -(n - 1) g_{\mu\nu}$, we have $\chi = -1$, and all the CCKVs are eigenvectors of the Ricci tensor. Thus, from (9), we obtain $n + 1$ mass

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If a spacetime admits two CCKVs $\zeta^\mu_1$ and $\zeta^\mu_2$ which are respectively eigenvectors of the Ricci tensor with eigenvalues $\chi_1$ and $\chi_2$, a linear combination is also CCKV, but it is not an eigenvector of the Ricci tensor unless $\chi_1 = \chi_2$. 
ladder operator\footnote{The map between $k = 1$ and $k = 0$ in AdS$_2$ case was partially discussed in \cite{Banks:1998dd}.}

\begin{align}
D_{-1,k} &= r^2 \partial_r - kr, \\
D_{A,k} &= x^A \partial_r + r^{-1} \sum_{B=0}^{n-2} \eta^{AB} \partial_B - k x^A r, \\
D_{n-1,k} &= (-1 + r^2 \sum_{A,B=0}^{n-2} \eta_{AB} x^A x^B) \partial_r + 2 r^{-1} \sum_{A=0}^{n-2} x^A \partial_A \\
&- k \left( r^{-1} + n^{-2} \sum_{A,B=0}^{n-2} \eta_{AB} x^A x^B \right). \tag{30}
\end{align}

Using global coordinates, we can confirm that these mass ladder operators are regular beyond the Poincaré horizon. Hence, a regular solution to the KGE can be mapped into another regular solution with different mass (see Appendix D).

For simplicity, we discuss how the mass ladder operators act on the solution to the KGE under separation of variables, $\Phi = \alpha(x^A) \tilde{\Phi}(r)$. Then the KGE reduces to

\begin{align}
(\Box_{\text{M}s} - L^2) \alpha &= 0, \\
\left[ r^2 \partial_r^2 + nr^{-1} \partial_r - m^2 + \frac{L^2}{r^2} \right] \tilde{\Phi}(r) &= 0, \tag{32}
\end{align}

where $L^2$ is the separation constant. Solving Eq. (32) around spatial infinity, we obtain the asymptotic behavior of the solution as

\begin{equation}
\tilde{\Phi}(r) = r^{\Delta_+} \sum_{i=0}^{\ell} c^{(i)}_{\ell} + r^{-\Delta_-} \sum_{i=0}^{\ell-2} c^{(i)}_{\ell-2}, \tag{33}
\end{equation}

where $\Delta_{\pm} = \pm (1 - n \pm \sqrt{(n - 1)^2 + 4m^2})/2$ and $c^{(i)}_{\ell} = (-1)^i L^2 c^{(0)} \prod_{j=1}^{i} ((\pm \Delta_{\pm} + n - 2j - 1)(\pm \Delta_{\pm} - 2j - m^2))^{-1}$ for $i \geq 1$ and $c^{(0)}_{\ell}$ are constant. The two modes with the leading terms $r^{\Delta_+}$, $r^{-\Delta_-}$ are called non-normalizable and normalizable modes, respectively. From (i), we can see $\Delta_{\pm} = \pm k_s$ for the above ladder operators. Acting the ladder operators on $\Phi$, a non-normalizable mode is mapped into a non-normalizable mode, and a normalizable mode is mapped into a normalizable mode unless $m^2$ or $m^2 + \delta m^2$ is between $m^2_{R,F}$ and $m^2_{R,F} + 1$, where $m^2_{R,F} := -(n - 1)^2/4$ is the BF bound mass. If the mass is between the above region then two modes are normalizable. Note that the ladder operators do not necessarily keep the form of separation of variables due to the derivative with respect to $x^A$.

In particular, $D_{a_1, -k - n + 2} D_{a_2, k}$ maps a solution of KGE to another solution of the same KGE, we can obtain variety of solutions from a single seed solution. Since the ladder operators are regular everywhere, if a seed solution $\Phi$ is regular, $D_{a_1, -k - n + 2} D_{a_2, k} \Phi$ is also regular. From the point of view of AdS/CFT correspondence the ratio of the coefficients between normalizable and non-normalizable modes is the expectation value of the operator. If the asymptotic behavior of $D_{a_1, -k - n + 2} D_{a_2, k} \Phi$ is different from $\Phi$, this corresponds to different physical situation. If we use $D_{-1}$, we can show $D_{-1, -k - n + 2} D_{-1, k} \Phi = -L^2 \Phi$ for a solution with the separation of variables form $\Phi = \alpha(x^A) \tilde{\Phi}(r)$. If we use other ladder operators, $D_{a_1, -k - n + 2} D_{a_2, k} \Phi$ is different from $\Phi$.

We comment on massless scalar fields in $\text{AdS}_5 \times S^5$. The massless KGE in $\text{AdS}_5 \times S^5$ reduces to the effective massive KGE in $\text{AdS}_5$

\begin{equation}
(\Box_{\text{AdS}_5} - \Lambda(\ell + 4)) \Phi = 0, \tag{34}
\end{equation}

where $\ell$ denotes the different Kaluza-Klein modes. The mass spectrum corresponds to the masses which can be mapped from massless scalar fields in $\text{AdS}_5$ by using the mass ladder operators. This implies that there is a duality among the zero mode and Kaluza-Klein modes on massless scalar fields in $\text{AdS}_5 \times S^5$.

\section{V. Ladder Operators in Sphere and Spherical Harmonics}

By applying our formulation to the 2-dimensional sphere $S^2$, we obtain three ladder operators

\begin{align}
D_{1,k} &= \cos \theta \cos \phi \partial \theta - \sin \phi \partial \phi + k \sin \theta \cos \phi, \\
D_{-1,k} &= \cos \theta \sin \phi \partial \theta + \cos \phi \partial \phi + k \sin \theta \sin \phi, \\
D_{0,k} &= \sin \theta \partial \theta - k \cos \theta, \tag{37}
\end{align}

where $\theta$ and $\phi$ are spherical coordinates on $S^2$ in which the metric is given by $ds^2_{S^2} = d\theta^2 + \sin^2 \theta d\phi^2$. The ladder operators map solutions of the eigenvalue equation for the Laplacian $\Delta_{S^2}$ with eigenvalues $\lambda = -k(k + 1)$,

\begin{equation}
(\Delta_{S^2} - \lambda) \Phi = 0, \tag{38}
\end{equation}

into solutions with eigenvalues $\lambda = -k(k - 1)$. It should be noted that $k$ is not necessarily integer. If $k$ is an integer $\ell$, spherical harmonics\footnote{The normalized spherical harmonics are given by $Y_{\ell,m} = P^{m}_{\ell} \cos \theta \text{e}^{im\phi}$ the eigenfunctions for $\Delta_{S^2}$ with $\lambda = -\ell(\ell + 1)$, and the ladder operators change the quantum number $\ell$, while the usual ladder operators $L_{\ell}$ constructed from the spherical symmetry change the quantum number $m$. Introducing $D_{\pm,k} = D_{1,k} \pm i D_{-1,k}$, we can reproduce the relations in $S^2$.} $Y_{\ell,m}$ are the eigenfunctions for $\Delta_{S^2}$ with $\lambda = -\ell(\ell + 1)$, and the ladder operators change the quantum number $\ell$, while the usual ladder operators $L_{\ell}$ constructed from the spherical symmetry change the quantum number $m$. Introducing $D_{\pm,k} = D_{1,k} \pm i D_{-1,k}$, we can reproduce the relations in $S^2$.}
mass a conformal transformation our ladder operators are regarded as supercharges [9]. We symmetry of had never been uncovered; we stress that the conformal fact, one can obtain shape invariant potentials [8] in su-
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formalism to higher dimensional spheres
These are useful relations to obtain the entire spectrum of the Laplacian on $S^2$. Their relation to geometry of $S^2$
had never been uncovered; we stress that the conformal symmetry of $S^2$ is crucial for the existence of such ladder
operators. We should note that we can also apply our formalism to higher dimensional spheres $S^n$.

Solutions of [38] which are not spherical harmonics will have a singular behavior. However, it is possible that
the ladder operator can map such singular solution to a regular one [9]. For example, if we consider $\Phi = e^{i \phi} / \tan \theta$
which satisfies $\Delta_s \Phi = 0$ and is singular at the pole, we can show $D_{0,-1} \Phi = Y_{11}$.

VI. RELATION WITH SUPERSYMMETRIC QUANTUM MECHANICS

The concept of ladder operators was developed in the context of exactly solvable systems in quantum mechanics.
One is thus naturally led to inquire whether mass ladder operators can be framed in this context as well. In
fact, one can obtain shape invariant potentials [8] in supersymmetric quantum mechanics from the KGE [9] where
our ladder operators are regarded as supercharges [9]. We make a conformal transformation $g_{\mu \nu} = \Omega^2 g_{\mu \nu}$
with an appropriate conformal factor $\Omega$ such that a CCKV $\xi^\mu$ for $g_{\mu \nu}$ is transformed into a Killing vector for $g_{\mu \nu}$.
Then, the massive KGE $(\Box - \ell^2) \Phi = 0$ for $g_{\mu \nu}$ is written in terms of $\bar{g}_{\mu \nu}$ as

$$[\partial^2_\lambda + \Box - V(\lambda, m^2)] \bar{\Phi} = 0,$$

where $\Phi = \Omega^{(2-n)/2} \Phi$, $\partial_\lambda = \xi^\mu \partial_\mu$ is a Killing vector and $\Box$ is the Laplacian (or d’Alembertian) on an
$(n - 1)$-dimensional space (or spacetime). Thus, with
the separation of variables $\Phi = \psi(\lambda) \Theta(x^\perp)$, we obtain the Schrödinger equation in one dimension,

$$[-\partial_\lambda^2 + V(\lambda, m^2)] \psi = E \psi,$$

where $E$ is the separation constant. The potential $V$ is given, up to a constant, by $1/\cos^2 \lambda$, $1/\cosh^2 \lambda$ or $1/\lambda^2$.
These are known as shape invariant potentials in super-symmetric quantum mechanics, the mass ladder operators
being regarded as supercharges (see Appendix [9] for details).

VII. ARETAKIS CONSTANTS

Mass ladder operators also appear naturally in black hole physics. In Refs. [10–12], it has been shown that an
extreme Reissner-Nordström black hole is linearly unstable. In their analysis, a certain quantity (“Aretakis con-
stant”), conserved only on the horizon, plays an important role. We now show that such constants can be con-
structed from our ladder operator, in four-dimensional extreme Reissner-Nordström black holes (more details
can be found in Appendix [9]).

The near horizon geometry of extreme Reissner-Nordström black holes is described by $AdS_2 \times S^2$, and
massless scalar fields on this spacetime behave as a massive scalar field on $AdS_2$ with an effective mass $m^2 = \ell (\ell + 1)$ where $\ell$ is azimuthal quantum number of the spherical harmonics. Thus, we focus on the KGE on $AdS_2$ with this mass. The metric of $AdS_2$ in ingoing Eddington-Finkelstein coordinate is

$$ds^2 = -r^2 dv^2 + 2 dv dr.$$

Take solutions $\Phi$ of the KGE, $(\Box - m^2) \Phi = 0$, with $m^2 = \ell (\ell + 1)$, $(\ell = 0, 1, \cdots)$ on this spacetime. Then, one can show that

$$\partial_\ell \partial^{\ell + 1} \Phi |_{r=0} = 0.$$

Thus, the quantities $\partial_\ell \partial^{\ell + 1} \Phi |_{r=0}$ are constant on the Poincaré horizon $r = 0$. This is the Aretakis constant
in $AdS_2$ [12].

While the quantity $\partial_\ell \partial^{\ell + 1} \Phi$ is not constant outside of $r = 0$, since $AdS_2$ is maximally symmetric, we may expect the existence of quantities which are constants on every out-going null hypersurface. In fact, we can show

$$(\partial_r + \frac{r^2}{2} \partial_\ell) \left[ \left( \frac{vr}{2} + 1 \right) \partial^{\ell + 1} \Phi = 0. \right]$$

Since $\partial_r + (r^2/2) \partial_\ell$ is an out-going null vector field,

$$A_\ell \equiv \left( \frac{vr}{2} + 1 \right)^{2(\ell+1)} \partial^{\ell + 1} \Phi,$$

is indeed constant on every out-going null hypersurface, and $A_\ell$ coincides with the Aretakis constant on $r = 0$. In
this sense, $A_\ell$ is a generalization of the Aretakis constant.
In AdS$_2$, the operator $D_k$ changes the mass squared from $k(k+1)$ into $(k-1)k$. So, $D_1 D_2 \cdots D_{\ell-1} D_{\ell}$ maps a massive scalar field with $m^2 = \ell(\ell+1)$ into massless scalar field. Since we can solve the two dimensional massless KGE, we can write

$$D_1 D_2 \cdots D_{\ell-1} D_{\ell} \Phi = F(x^+) + G(x^-),$$

(51)

where we used the double null coordinates $(x^+, x^-)$. Thus $\partial_{x^-} D_1 D_2 \cdots D_{\ell-1} D_{\ell} \Phi = G(x^-)$ is constant on every outgoing null hypersurface $x^- = \text{const}$. In fact this coincides with $A_\ell$ up to a function of $x^-$. Note that the choice of CCKVs $\zeta_1, \zeta_0, \zeta_2$ does not affect this conclusion. If we consider Reissner-Nordström black hole spacetime without taking near horizon limit, we can still derive the Aretakis constant on the horizon in a similar way. Since there is a relation between Aretakis constant and Newman-Penrose constant [13], the present analysis suggests that Newman-Penrose constant also can be constructed from our ladder operator.

VIII. DISCUSSION

We developed a mass ladder operator formalism for the massive KGE and explicitly constructed the operators for AdS$_n$ and $S^2$. It is possible, and we showed that this happens on $S^2$, that the ladder operator maps a singular to a regular solution even if CCKVs and the associated functions are regular. Naturally, in the context of AdS/CFT correspondence regular solutions are preferred objects. However, the property above might help in providing a physical interpretation to singular solutions.

The ladder operators on $S^2$ were originally obtained by embedding $S^2$ into three dimensional Euclid space $E^3$ [2] or sphere $S^3$ [1]. The harmonic functions on $E^3$ are known as regular and irregular solid harmonics. According to [2], taking the covariant derivatives of the solid harmonics along $\partial_e, \partial_\theta$ and $\partial_\phi$, yields differential recurrence relations between the solid harmonics with different azimuthal and magnetic quantum numbers. By restricting the recurrence relations onto $S^2$, we obtain the ladder relations for the spherical harmonics. Higuchi [1] also constructed the symmetric tensor harmonics on $S^n$ in the reductive construction, where $S^n$ is embedded into $S^{n+1}$. This suggests the existence of the ladder operators for vector or tensor fields on $S^n$ and also maximally symmetric spacetimes.

Another interesting direction is to consider higher-order operators. For symmetry of the Laplace equation or KGE in a curved spacetime, they have been studied by many authors [14] [17]. While our formulation in this paper focused on first-order mass ladder operators, it would be of great interest to consider higher-order mass ladder operators if there exists a curved spacetime which admits a crucial higher-order operator not reducible to first-order operators.

We also showed the relation between these operators and supersymmetric quantum mechanics potentials having shift shape invariance. If we start from generic 1-dimensional supersymmetric quantum mechanics potential, we can expect to obtain a class of scalar fields with potential which has a ladder structure.

As an application, we constructed Aretakis constant from mass ladder operators on AdS$_2$. If we consider Reissner-Nordström spacetimes without taking the near horizon limit, the Aretakis constant on the horizon can be derived in a similar way. This suggests the intriguing possibility of mass ladder operators being useful constructs also for less symmetric spacetimes, with only approximate conformal symmetry.

In Minkowski spacetime, the existence of mass ladder operators (shown in Appendix C) is not surprising, as there is no scale in the problem other than the mass parameter in the massive KGE. In curved spacetimes however, the different hierarchy (as compared to curvature scale) in the mass of scalar fields is expected to play a fundamental role. Notwithstanding, if we consider the curved spacetimes which admit mass ladder operators (including the maximally symmetric spacetimes), solutions of KGE between different masses are connected. Furthermore, the map induced by the ladder operator is surjective (or onto), so all the solutions with mass squared $m^2 = \delta m^2$ can be constructed from the solutions with mass squared $m^2$. This suggests that the physical properties of KGE with different masses, which are connected by the ladder operator, are very similar contrary to the naive expectation.

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Appendix A: Explicit metric form admitting a mass ladder operator

In [18], all canonical forms for metrics admitting a CCKV, denoted by $\zeta^{\mu}$, were investigated in arbitrary dimensions. In Lorentzian signature, they are classified according to whether $\zeta^{\mu}$ is null or not. In the null case, $\zeta^{\mu}$ becomes a covariantly constant null vector, so that $Q = 0$ and $R^{\mu\nu}\zeta^{\sigma} = 0$. Hence, the operator does not become a ladder operator for the d’Alembertian. In the non-null case, it is possible to introduce a function $\lambda$, called the potential of $\zeta^{\mu}$, such that $d\lambda$ is the 1-form dual to $\zeta^{\mu}$. Using the potential as a coordinate, we can choose a local coordinate system $(x^{\mu}) = (\lambda, x^{i})$. Then, a metric in the case in $n$ dimensions is written as

$$ds^{2} = g_{\mu\nu}dx^{\mu}dx^{\nu} = \frac{1}{f(\lambda)}d\lambda^{2} + f(\lambda)\tilde{g}_{ij}(x)dx^{i}dx^{j},$$ (A1)

where $f(\lambda)$ is an arbitrary function and $\tilde{g}_{ij}$ is an $(n-1)$-dimensional metric. This metric admits a CCKV

$$\zeta = f(\lambda)\frac{\partial}{\partial \lambda}.$$ (A3)

If we impose the condition $\{A\}$ for this spacetime, $f$ takes the form

$$f(\lambda) = -\chi \lambda^{2} + c_{1}\lambda + c_{0},$$ (A4)

where $c_{0}$ and $c_{1}$ are constant. With $f(\lambda)$ given by (A4), the ladder operator is

$$D_{k} = f(\lambda)\partial_{\lambda} - \frac{k}{2}f'(\lambda),$$ (A5)

Appendix B: Surjectivity and kernel

We can show that $D_{k}$ is a surjective (onto) map, i.e., for arbitrary solution of $(\Box - m^{2} - \delta m^{2})\Phi = 0$, we can find a solution of the equations

$$D_{k}\Phi = \Phi \quad (B1) \quad (\Box - m^{2})\Phi = 0, \quad (B2)$$

where $m^{2}$ and $\delta m^{2}$ are given by $m^{2} = -\chi k(k+n-1), m^{2} + \delta m^{2} = -\chi (k-1)(k+n-2)$. The general solution of (B1) is

$$\Phi = f^{k/2}\left(\int d\lambda f^{-1-k/2}\Phi + P(x^{i})\right),$$ (B3)

where $P(x^{i})$ is arbitrary function of $x^{i}$. After a straightforward calculation, we obtain

$$(\Box - m^{2})\Phi = f^{-1+k/2}\left(\Box + k(k+n-2)(c_{1}^{2} + 4c_{0}\lambda)\right)P(x^{i}),$$

where we used Eq. (A4). For $P = 0$ we recover Eq. (B2), showing that $D_{k}$ is a surjective map.

If there exists a non trivial solution of the equation $(\Box + k(k+n-2)(c_{1}^{2} + 4c_{0}\lambda)/4)P(x^{i}) = 0$, such functional degrees of freedom correspond to the kernel of $D_{k}$, i.e., the solutions of both $D_{k}\Phi = 0$ and $(\Box - m^{2})\Phi = 0$. In particular, if $c_{1} = c_{0} = 0$, $P = \text{const}$ is a nontrivial solution, then $\Phi = C f^{k/2}$ becomes a kernel of $D_{k}$.

Appendix C: Another ladder operator for $\chi = 0$, $Q = \text{const}$ case

The operator $D_{k}$ relates scalars of different mass if the eigenvalue of the Ricci tensor $\chi$ is not zero. However, for constant $Q$ a ladder operator can be defined even for $\chi = 0$ case, albeit in a modified way. If $Q = c = \text{const}$, the conformal Killing equation (4) becomes the homothetic Killing equation

$$\nabla_{\mu}\zeta_{\nu} + \nabla_{\nu}\zeta_{\mu} = 2cg_{\mu\nu}.$$ (C1)

The commutation relation (7) with $k = 0$ is

$$[\Box, \mathcal{L}_{\zeta}] = \chi(n-2)\mathcal{L}_{\zeta} + 2c\Box.$$ (C2)

If $\chi$ is zero, we can define another ladder operator $\tilde{D}_{\lambda} := e^{\lambda}\mathcal{L}_{\zeta} = \sum_{j=0}^{\infty}(j!)^{-1}(\lambda\mathcal{L}_{\zeta})^{j}$ with a parameter $\lambda$, which satisfies the commutation relation $[\Box, \tilde{D}_{\lambda}] = (e^{2c\lambda} - 1)\tilde{D}_{\lambda}\Box$. (C3)

Acting on a scalar field $\Phi$, we obtain

$$\Box\tilde{D}_{\lambda}\Phi - e^{2c\lambda}\tilde{D}_{\lambda}\Box\Phi = 0.$$ (C4)

If $\Phi$ satisfies a massive KGE, then (C4) becomes

$$\Box - e^{2c\lambda}m^{2}\tilde{D}_{\lambda}\Phi = 0.$$ (C5)

This shows that $\tilde{D}_{\lambda}$ maps a scalar field with $m^{2}$ to that with $e^{2c\lambda}m^{2}$. Since the parameter $\lambda$ is an arbitrary number, $\tilde{D}_{\lambda}$ can change the mass continuously. Note that $\tilde{D}_{\lambda}$ cannot change the signature of the mass squared, but can change the absolute value. In Minkowski spacetime $g_{\mu\nu} = \eta_{\mu\nu}$, we can explicitly construct the ladder operator as $\tilde{D}_{\lambda} := e^{\lambda}(x^{a}\partial_{a} + \zeta^{i}\partial_{i})$, where $\zeta^{\mu}$ is an arbitrary Killing vector on $\eta_{\mu\nu}$.

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10 If $\zeta^{\mu}$ is timelike, i.e., $f < 0$, $-\tilde{g}_{ij}$ should be a positive definite metric so that the metric $g_{\mu\nu}$ has $[-,+,+,\ldots,+]$ signature.

11 It is possible to show that, in addition to (A1), the metric (A1) can admit a CCKV if $\tilde{g}_{ij}$ admits a CCKV. Actually, the CCKV equation $\nabla_{\mu}\zeta_{\nu} = Qg_{\mu\nu}$ for the metric (A1) can be solved by

$$\zeta = \frac{\tilde{Q}}{2f}(\lambda)\sqrt{f(\lambda)}\frac{\partial}{\partial \lambda} = \frac{1}{\sqrt{f(\lambda)}}\tilde{c}^{i}\frac{\partial}{\partial x^{i}},$$ (A2)

where $\tilde{c}^{i}$ is a CCKV for $\tilde{g}_{ij}$. The associated function $Q$ of $\zeta^{\mu}$ is given by $Q = (\chi/\lambda)\sqrt{f(\lambda)Q}$.

12 We can show this relation by using the equation $[\Box, (\mathcal{L}_{\zeta})^{n}] = ((2c + \chi)n - (\mathcal{L}_{\zeta})^{n})\Box$, where $n$ is a positive integer.
Appendix D: Regularity of ladder operators

To see the regularity of the ladder operator on $AdS_n$ beyond the Poincaré horizon, introduce global coordinates

\[
\begin{align*}
    r &= \frac{\cos \tau - \Omega_{n-1} \sin \rho}{\cos \rho}, \\
    t &= \frac{\sin \tau}{\cos \rho}, \\
    x^i &= \frac{\Omega_i \sin \rho}{\cos \rho}, \quad (i = 1, 2, \ldots, n-2), \\
    s &\equiv \sqrt{\rho^2 + \chi^2}, \quad \rho \equiv (x^i x^i)^{1/2},
\end{align*}
\]

where $\Omega_i$ satisfy the relation $\sum_{i=1}^{n-1} \Omega_i^2 = 1$. In this coordinate system, the metric becomes

\[
ds^2 = \frac{1}{\cos^2 \rho} (-d\tau^2 + d\rho^2 + \sin^2 \rho \sum_{i=1}^{n-1} d\Omega_i^2).
\]

Spatial infinity corresponds to $\rho = \pm \pi/2$. Note that $\sum_{i=1}^{n-1} d\Omega_i^2$ is the metric of a $(n-2)$-dimensional unit sphere. The associated functions of CCKVs $Q_a, (a = -1, 0, 1, \ldots, n-1)$ in these coordinates are

\[
\begin{align*}
    Q_{-1} &= \frac{\cos \tau - \Omega_{n-1} \sin \rho}{\cos \rho}, \\
    Q_0 &= \frac{\sin \tau}{\cos \rho}, \\
    Q_i &= \frac{\Omega_i \sin \rho}{\cos \rho}, \quad (i = 1, 2, \ldots, n-2), \\
    Q_{n-1} &= \frac{\cos \tau + \Omega_{n-1} \sin \rho}{\cos \rho}.
\end{align*}
\]

Thus, $Q_a$ is finite except at spatial infinity. Since the 1-form $d\Omega_a$ is regular (except at the sphere’s pole), the 1-forms $dQ_a$ are also regular in $-\pi/2 < \rho < \pi/2$. In $AdS_n$, $dQ_a = \gamma_{a\mu} dx^\mu$, so CCKVs $\zeta_a^\mu$ and the ladder operators $D_{a,k}$ are regular in $-\pi/2 < \rho < \pi/2$.

Appendix E: Conformal transformation and supersymmetric quantum mechanics

Given a CKV $\zeta^\mu$ for a metric $g_{\mu\nu}$, we can make a conformal transformation $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ under which $\zeta^\mu$ is a Killing vector. We have already seen that if a spacetime admits a CCKV $\zeta^\mu$, the metric and CCKV have the forms \[A1\] and \[A3\], respectively. Hence, by setting $\Omega = 1/\sqrt{f}$, the CCKV $\zeta_a^\mu$ for $\bar{g}_{\mu\nu}$ becomes a Killing vector for $\bar{g}_{\mu\nu}$. Under this conformal transformation, we have \[A1\]

\[
(\Box - m^2)\Phi = \Omega^{(n+2)/2} \left(\Box - V(\lambda, m^2)\right)\Phi,
\]

where $\Phi = \Omega^{(2-n)/2} \Phi$ and

\[
V(\lambda, m^2) = (16m^2 f + (n-2)^2 (f')^2 + 4(n-2)ff')/16.
\]

Hence, massive KGE on $\bar{g}_{\mu\nu}$, $(\Box - m^2)\Phi = 0$ leads to

\[
\Box \Phi - V(\lambda, m^2)\Phi = 0,
\]

where $\Box$ is the d’Alembertian on $\bar{g}_{\mu\nu}$. In addition, if we assume the function $f(\lambda)$ is given by \[A3\], the potential $V$ becomes a quadratic polynomial of $\lambda$,

\[
V = s_0 + s_1 \lambda + s_2 \lambda^2
\]

with the coefficients

\[
\begin{align*}
    s_0 &= c_1^2 (n - 2)^2/16 + c_0 (m^2 + \chi(1 - n/2)), \\
    s_1 &= c_1 (m^2 - (n-2)n\chi/4), \\
    s_2 &= \chi(-4m^2 + (n-2)n\chi)/4.
\end{align*}
\]

Furthermore, we introduce the coordinate $\lambda$ as $\zeta^\mu \partial_\mu = f \partial_\lambda$. Since $\zeta \Phi = f \partial_\lambda [f \partial_\lambda \Phi] + \Box \Phi$, \[E3\] is

\[
\frac{\partial^2}{\partial \lambda^2} \Phi + \Box \Phi + \frac{(c_1^2 + 4c_0\chi)}{16\chi}
\]

\[
\times \left[ \frac{4m^2 - n(n-2)\chi}{\cos^2(\lambda - c_1^2 - 4c_0\chi)/2} + (n-2)^2 \right] \Phi = 0
\]

where $\Box$ is the d’Alembertian on an $(n-1)$-dimensional spacetime. Imposing $[\Box, \partial_\lambda] = 0$ and $[\Box, Q] = 0$, the separation of variables $\Phi = \psi(\lambda) \Theta(x^i)$ leads to the Schrödinger equation in one dimension

\[
H(m^2)\psi \equiv \left[ -\frac{d^2}{dz^2} + V(m^2, z) \right] \psi = E \psi,
\]

where we have introduced the coordinate $z = \lambda \sqrt{-c_1^2 - 4c_0\chi}/2$, in which the potential is given by

\[
V(m^2, z) = \frac{m^2}{\cos^2 z} - \frac{n(n-2)/4}{z^2} + \frac{(n-2)^2}{4},
\]

and $E$ is a separation constant. This is known as a shape invariant potential in supersymmetric quantum mechanics. This form of potential changes to $1/\cosh^2 z$ if the signature of $-c_1^2 - 4c_0\chi$ is negative. We have assumed that either $c_0$ or $c_1$ are non-vanishing. When $c_0 = c_1 = 0$, the potential becomes a quadratic polynomial of $1/\lambda$, as in the problem of the hydrogen atom.

The present conformal transformation transforms the ladder operator $\bar{D}_k$ for $\bar{g}$ into the ladder operator $D_k = \Omega^{(2-n)/2} D_k \Omega^{-(2-n)/2}$ for $\bar{g}$. To be explicit, it is written in the coordinate $z$ as

\[
D_k = \frac{d}{dz} - \left( \frac{k - n}{2} \right) \tan z + \text{const.}
\]

In supersymmetric quantum mechanics, the Hamiltonian $H$ and supercharge $Q$ are related via $Q^2 = H$. In the present case, this can be realized by setting

\[
H = \begin{pmatrix} H(m^2) & 0 \\ 0 & H(m^2 + \delta m^2) \end{pmatrix},
\]

\[
Q = \begin{pmatrix} 0 & -\bar{D}_{k-n+2} \\ \bar{D}_k & 0 \end{pmatrix}.
\]

Thus, our ladder operator corresponds to the supercharge in supersymmetric quantum mechanics.
Appendix F: Aretakis constants

1. Mass ladder operators in $AdS_2$

In double null coordinates, $AdS_2$ metric is given by

$$ds^2_{AdS_2} = -\frac{4|\Lambda|}{(x^+ - x^-)^2} dx^+ dx^- , \quad (F1)$$

where $1/\sqrt{|\Lambda|}$ is the AdS radius. Setting $\Lambda = 1$, the KGE $\mathcal{D}_m \Phi = 0$ is constant on out-going null hypersurfaces $k$.

By acting the mass lowering operators repeatedly on a massive scalar field of $m^2 = k(k+1)$, we can annihilate the mass. Hence, we obtain the operator

$$D^{(k)}_{i_1 i_2 \cdots i_k} = D_{i_1} \cdots D_{i_k} , \quad (F14)$$

which map a scalar field with mass $k(k+1)$ into a massless scalar field. By using $D^{(1)}_{i_1 i_2 \cdots i_k}$ in Eq. (F14), we can construct conserved quantities on every out-going null hypersurface. Since $D^{(1)}_{i_1 i_2 \cdots i_k} \Phi_\ell$ satisfies a two dimensional massless KGE, we can write $D^{(1)}_{i_1 i_2 \cdots i_k} \Phi_\ell = \phi_+ (x^+) + \phi_- (x^-)$, where $\phi_{\pm} (x^\pm)$ are arbitrary functions of $x^\pm$, respectively. Taking the derivative w.r.t. $x^-$ of $D^{(1)}_{i_1 i_2 \cdots i_k} \Phi_\ell$, the quantity

$$\partial_- D^{(1)}_{i_1 i_2 \cdots i_k} \Phi_\ell = \partial_- \phi_- (x^-) \quad (F15)$$

is constant on out-going null hypersurfaces $x^+ = \text{const}$.

and the other ones are CCKVs, which are given by

$$\zeta_{-1} = \partial_+ - \partial_- , \quad (F6)$$

$$\zeta_0 = x^+ \partial_+ - x^- \partial_- , \quad (F7)$$

$$\zeta_1 = (x^+)^2 \partial_+ - (x^-)^2 \partial_- . \quad (F8)$$

Since $AdS_2$ admits three CCKVs, we are able to construct three one-parameter families of mass ladder operators

$$D_{-1,k} = \partial_+ - \partial_- + \frac{2k}{x^+ - x^-} , \quad (F9)$$

$$D_{0,k} = x^+ \partial_+ - x^- \partial_- + \frac{2k(x^+ + x^-)}{x^+ - x^-} , \quad (F10)$$

$$D_{1,k} = (x^+)^2 \partial_+ - (x^-)^2 \partial_- + \frac{2kx^+ x^-}{x^+ - x^-} . \quad (F11)$$

where $k$ is a real parameter. $D_{i,k}$ maps a solution to the KGE on $AdS_2$ with mass squared $k(k+1)$ into another solution with mass squared $k(k-1)$.

It should be emphasized that for any solution satisfying the BF bound, $m^2 \geq -1/4$, two operators $D_{i,k}$ exist for each $i = -1, 0, 1$. For a fixed $m^2$, the corresponding two values for $k$ are given by

$$k_{\pm} = \frac{-1 \pm \sqrt{1+4m^2}}{2} . \quad (F12)$$

Especially in the range $0 \leq m^2$, one of the two operators is a mass raising and the other a mass lowering operator. If $-1/4 \leq m^2 < 0$, both become mass raising operators. If $k$ is a natural number, then $m^2$ is shifted by $D_{i,-k}$ and $D_{i,k}$ as follows:

$$\cdots \xrightarrow{D_{i,k+1}} k(k+1) \xrightarrow{D_{i,-(k+1)}} \cdots \xrightarrow{D_{i,-k}} \frac{k(k-1)k}{D_{i,-(k-1)}} \xrightarrow{D_{i,k}} \cdots \xrightarrow{D_{i,-3}} \frac{6}{D_{i,-2}} \xrightarrow{D_{i,-1}} \frac{2}{D_{i,k}} \xrightarrow{D_{i,-1}} 0 \quad (F13)$$

2. Aretakis constants in $AdS_2$

In ingoing Eddington-Finkelstein coordinates, $AdS_2$ metric is written in the form

$$ds^2 = -r^2 dv^2 + 2dvdr , \quad (F16)$$

where the AdS radius has been already taken to be unit. The Poincaré horizon is located at $r = 0$, which is an out-going null hypersurface. In the present coordinates, the KGE with $m^2 = \ell(\ell+1), (\ell = 0, 1, \cdots)$ is given by

$$2\partial_v \partial_r \Phi_\ell + \partial_r (r^2 \partial_v \Phi_\ell) - \ell(\ell+1)\Phi_\ell = 0 . \quad (F17)$$

From this equation, it follows that

$$\partial_r \partial_v^{\ell+1} \Phi_\ell \bigg|_{r=0} = 0 , \quad (F18)$$

where $\Phi_\ell$ is a solution for $m^2 = \ell(\ell+1)$. The quantities $H_\ell \equiv \partial_v^{\ell+1} \Phi_\ell \bigg|_{r=0}$ are known as Aretakis constants \[2\].
The quantities \( \partial^\ell+1 \Phi_\ell \) are constants on the Poincaré horizon, but not outside. However, since \( AdS_2 \) is maximally symmetric, we expect the existence of quantities which are constants on every out-going null hypersurface. In fact, (F18) can extend to the outside of the Poincaré horizon, and we obtain
\[
\left( \partial_\nu + \frac{r^2}{2} \partial_\tau \right) \left[ \frac{1}{2} \left( \frac{r^2}{2} + 1 \right)^2 \partial^\ell+1 \Phi_\ell \right] = 0. \tag{F19}
\]
Hence, we define the quantity
\[
A_\ell \equiv \left( \frac{r^2}{2} + 1 \right)^2 \partial^\ell+1 \Phi_\ell, \tag{F20}
\]
which coincides with the Aretakis constant \( H_\ell \) at the Poincaré horizon. Since \( \partial_\nu \equiv (r^2/2) \partial_\tau \) is an out-going null vector field, \( A_\ell \) is indeed constant on every out-going null hypersurface. In what follows, we still call these Aretakis constants. For the metric form (F10), we arrange the coordinate transformation \( x^+ = v \) and \( x^- = v + 2/r \) and obtain the metric form (F1). Since we have \( \partial_+ = \partial_\nu + (r^2/2) \partial_\tau \) and \( \partial_- = -(r^2/2) \partial_\nu \), (F19) is
\[
\partial_+ A_\ell = 0, \tag{F21}
\]
which means that \( A_\ell \) is a solution to the massless KGE. In double null coordinates, Eq. (F20) is written as
\[
A_\ell = (x^-)^{2(\ell+1)} L_\ell^{(\ell+1)} \Phi_\ell, \tag{F22}
\]
where
\[
L_\ell^{(\ell+1)} \equiv \frac{1}{(x^+ - x^-)^{2(\ell+1)}} [(x^+ - x^-)^2 \partial_-]^{\ell+1}. \tag{F23}
\]
Since there is symmetry between \( x^+ \) and \( x^- \), we can also define an operator \( L_\ell^{(\ell+1)} \)
\[
L_\ell^{(\ell+1)} \equiv \frac{1}{(x^+ - x^-)^{2(\ell+1)}} [(x^+ - x^-)^2 \partial_+]^{\ell+1}. \tag{F24}
\]
which can give a conserved quantity \( L_\ell^{(\ell+1)} \Phi_\ell \) on every ingoing null hypersurface.

We explicitly checked this quantity is equal to \( A_\ell \) up to some function of \( x^- \) for \( \ell = 0, 1, 2 \). We should note that regardless of the choice of \( \xi_i \), this quantity provides an Aretakis constant. We conjecture that \( \partial_+ D^{(\ell)}_{i_1 \ldots i_\ell} \Phi_\ell \) is related to the Aretakis constant via
\[
A_\ell = W_{i_1 \ldots i_\ell} \partial_- D^{(\ell)}_{i_1 \ldots i_\ell} \Phi_\ell, \tag{F26}
\]
where \( W_{i_1 \ldots i_\ell} (x^-) \) is a function of \( x^- \).

We point out that \( L_\ell^{(\ell+1)} \) are related to the mass annihilation operator \( D^{(\ell)}_{i_1 \ldots i_\ell} \) in Eq. (F14) up to the KGE. For example, \( L_\ell^{(2)} \) are written as
\[
L_\ell^{(2)} = \pm \partial_\pm D_{-1,-1} - \frac{1}{(x^+ - x^-)^2} [□_{AdS_2} - 2] (F27)
\]
with \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). Introducing \( r \equiv \rho - 1 \),
\[
ds^2 = - \left( 1 - \frac{1}{\rho} \right)^2 dv^2 + 2dvdr + \rho^2 d\Omega^2,
\]
with \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). Introducing \( r \equiv \rho - 1 \),
\[
ds^2 = - \left( r^2 - r^3 \frac{(r + 2)}{r + 1} \right) dv^2 + 2dvdr + (r + 1)^2 d\Omega^2.
\]
The leading term in \( v, r \) part is \( AdS_2 \) whose metric is \( ds_{AdS_2} = -r^2 dv^2 + 2dvdr \). By using spherical harmonics on \( S^2 \), the massless KGE on this spacetime is written as
\[
□_{AdS_2} - \ell(\ell + 1) + \partial_\tau + rK \Phi = 0, \tag{F27}
\]
where \( K \) is an operator written in the form
\[
K = f_0 \partial_v + f_2 + r f_3 \partial_r + r^2 f_4 \partial^2_r, \tag{F28}
\]
with certain functions \( f_1, f_2, f_3 \) and \( f_4 \) which are regular on the horizon. In the near horizon limit, i.e., \( v \rightarrow v/\epsilon, r \rightarrow \epsilon r \) and \( \epsilon \rightarrow 0 \), the equation describes a massive scalar on \( AdS_2 \) with an effective mass \( \ell(\ell + 1) \)
\[
□_{AdS_2} - \ell(\ell + 1) \Phi = 0. \tag{F29}
\]
However, when using the ladder operator for this spacetime, we need to discuss the sub-leading terms. If we write \( \Phi = e^{-r/2} \tilde{\Phi} \), then Eq. (F27) becomes
\[
□_{AdS_2} - \ell(\ell + 1) + r \tilde{K} \tilde{\Phi} = 0. \tag{F30}
\]
where $\tilde{K}$ is an operator such that $\tilde{K}\Phi$ is regular on the horizon like $K$ in Eq. (F28). Next we introduce the operator $D_{i,k} \equiv L_{\zeta_i} - kQ_i$ where $\zeta_i$ are the CCKVs on AdS$_2$

$$D_{-1,k} = \nu^2 \partial_r + \partial_v - kr,$$

$$D_{0,k} = r(1 + vr)\partial_r + \partial_v - k(1 + vr),$$

$$D_{1,k} = (v^2r^2 + 2vr + 2)\partial_r + v^2\partial_v - kv(2 + vr).$$

Then we can see that,

$$[\Box_{AdS_2} - \ell(\ell - 1)] D_{i,k}\Phi = D_{i,k-2}(\Box_{AdS_2} - \ell(\ell + 1))\Phi$$

(F31)

If the RHS of this equation vanishes, we can say that $D_{i,k}$ acts as a ladder operator. However, since $\Phi$ satisfies Eq. (F30), the RHS of this equation does not vanish. By using Eq. (F30), the RHS is

$$\text{RHS} = D_{i,k-2}(-r\tilde{K}\Phi).$$

(F32)

If we choose $\zeta_{-1}$ or $\zeta_0$ for $\zeta_i$, the RHS vanishes at $r = 0$ for regular $\Phi$. However, if we choose $\zeta_1$ for $\zeta_i$, the RHS does not vanish on the horizon because $\zeta_1$ contains $\partial_r$ with finite coefficient on the horizon. For this reason, only $D_{-1,k}$ and $D_{0,k}$ can act as ladder operators.

Similar to the case of pure AdS$_2$, acting the ladder operator $\ell$ times, we can show

$$\Box_{AdS_2}(D_{i,1}D_{i,2}\cdots D_{i,\ell}\Phi)$$

$$= D_{i,1-\ell}D_{i,2}\cdots D_{i,\ell}(-r\tilde{K}\Phi).$$

(F33)

If we choose $\zeta_{-1}$ or $\zeta_0$ for $\zeta_i$, RHS vanishes at $r = 0$ for regular $\Phi$. This implies

$$\partial_r D_{i,1}D_{i,2}\cdots D_{i,\ell}(-r\tilde{K}\Phi)\bigg|_{r=0} = \text{const}$$

(F34)

on the horizon because $\Box_{AdS_2} \propto \partial_r\partial_r$ at $r = 0$. If we define $A_\ell \equiv \partial_r D_{i,1}D_{i,2}\cdots D_{i,\ell}(e^{r/2}\Phi)$, $A_\ell$ becomes constant on the horizon.

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