Classical Open String Models in 4-Dim Minkowski Spacetime

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Abstract

Classical bosonic open string models in four-dimensional Minkowski spacetime are discussed. A special attention is paid to the choice of edge conditions, which can follow consistently from the action principle. We consider lagrangians that can depend on second order derivatives of worldsheet coordinates. A revised interpretation of the variational problem for such string theories is given. We derive a general form of a boundary term that can be added to the open string action to control edge conditions and modify conservation laws. An extended boundary problem for minimal surfaces is examined. Following the treatment of this model in the geometric approach, we obtain that classical open string states correspond to solutions of a complex Liouville equation. In contrast to the Nambu-Goto case, the Liouville potential is finite and constant at worldsheet boundaries. The phase part of the potential defines topological sectors of solutions.

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1 Introduction.

There is a common conviction that in order to gain more insight into the dynamical structure of QCD we need most likely to use some string representation of this theory. It is suggested by topological nature of $1/N$ expansion [1], area confinement law found in the strong coupling lattice expansion [2], the success of dual models in description of Regge phenomenology, the existence of flux-line solutions in confining gauge theories [3, 4]. More arguments are presented in recent reviews [5, 6].

In spite of numerous works, there is still a state of confusion about the existence of an exact, or even approximate, stringy reformulation of 4-dim QCD at all distance scales. Even at any specific scale, it is not evident what is the adequate set of string variables and fields and how they correspond to QCD gauge fields. Referring only to long distance scale, we usually adopt the naive, but lucid, picture of flux tubes regime. A pair of quarks in the confining phase is joined by a colour flux concentrated in a thin tube. If these quarks are kept sufficiently far apart, the flux tube behaves like a vibrating string. Using string variables as collective coordinates, one should in principle find flux tube excitations by some quantization of the string action. The question what kind of the string action should be employed to represent the flux tube has yet to be answered. It is conceivable that to the lowest order the action is just given by the Nambu-Goto action, which decribes an infinitely thin relativistic bosonic string with constant energy per unit length. As is well known, we cannot be satisfied with this first approximation because of some unacceptable features of the quantized Nambu-Goto string. Apart from the problems with conformal anomaly outside the critical dimension or tachyons and undesirable massless states in quantum spectra (which are presumably less embarrassing at the long distance scale [4]), all standard quantizations give the incorrect number of degrees of freedom if we confront it with QCD predictions [8].

Basically, there are two ways to modify the 4-dim Nambu-Goto action. In the first approach, keeping the conformal invariance we can place additional fields on the worldsheet [9, 10]. The conformal anomaly can be saturated due to the contribution of new conformal fields. Since we can hardly justify the assumption that only massless degrees of freedom are important at the hadronic scale, so the respecting of conformal symmetry is here rather a compromise to make our theory mathematically tractable. The second kind of modifications of the Nambu-Goto action, advocated in many papers (e.g. [11]), is to introduce new action terms representing interactions between transverse string modes. The fact that Regge trajectories derived directly from fundamental quark models [12, 13, 14] depart somewhat from straight lines is a strong argument that vibrating string modes cannot be considered as free. Next, some couplings between these modes (short-distance interactions) would cause preferring smooth string worldsheets, leading to a well-defined quantum theory. Unfortunately, all such string “self-interaction” terms involve higher order derivatives in their lagrangians. Theories with higher order derivatives usually reveal embarrassing pathological features, like lack of the energy bound, tachyons already on
the classical level, unitarity violation due to the presence of negative norm states. Presumably, it means that one must regard any particular effective theory of this type with a limited range of validity. From technical point of view, such theories of strings are non-linear and cannot be linearized by a suit choice of gauge. Subsequently, a string cannot be described as an infinite set of oscillators and there is no analogue of Virasoro algebra. The conformal symmetry is usually spoiled. All that makes the evaluation of physical observables technically difficult.

In this paper, we discuss possible modifications of the Nambu-Goto model (or any other specific bosonic string model) by the change of boundary conditions for open strings. This aspect is not well explored in literature, even though the choice of boundary conditions can be crucial for defining relevant open string models. Let us give some examples:

- Taking the usual hadronic string picture, we assume that quarks live only at the opposite endpoints of the string and communicate through their couplings to the string between. Then, to some extent the choice of worldsheet boundary conditions determines quarks trajectories. For instance, in the classical Nambu-Goto model they are rather peculiar, being boosted periodic light-like (null) curves. Undoubtedly, the unsolved problem how the quark masses and quantum numbers (spin, color) couple to the string variables, partly lies in the proper specification of string edge conditions.

- It is obvious that any internal symmetry of the worldsheet is necessarily broken when worldsheet boundaries are included. Conformal transformations or the full set of all reparametrizations are examples of that. Correspondingly, conformal field theories defined on surfaces with boundaries are usually endowed with only one copy of Virasoro algebra (instead of two, as for closed surfaces). Recently [10], on the same basis the chiral symmetry breaking mechanism has been included in hadronic string models. This simple observation that the existence of boundaries restricts the group of local worldsheet symmetries, indicates that physical observables can essentially depend on fields or currents evaluated on string boundaries.

- One of the straightforward calculations to test some open string model against QCD expectations is to evaluate the static interquark potential. Asymptotically at the long distances scale, this potential is linear and its slope can be related to the string tension. The first quantum corrections give an universal Coulomb term [15] (Casimir effect), being the function of the number of worldsheet fields and of their boundary conditions. In an approximation of flux-tube action by some conformal
string theory, one can represent boundary conditions by the set of relevant conformal operators inserted at boundaries \([16]\). The physical states are now constructed with the help of both bulk and boundary operators. The Coulomb term depends on the effective conformal anomaly \([17]\), being the total conformal anomaly diminished by the weight of the lowest state. This weight is sensitive to the choice of boundary operators \([9]\).

- The influence of worldsheet boundaries on critical string field theories has been discussed in recent papers (see \([18, 19]\) and references therein). In the framework of BRST formalism in the critical dimension, we can consider either Neumann-type (e.g. standard Nambu-Goto edge equations) or Dirichlet-type boundary conditions imposed on worldsheet coordinates. With Dirichlet conditions, we have no physical open strings, but the closed-string theory is radically modified, particularly the massless spectrum. Instead of characteristic exponential fall-off of fixed-angle scattering amplitudes for string models at high energies, we obtain for Dirichlet strings power-like behaviour, like for parton models. We see here that the special type of worldsheet boundaries, where these boundaries are mapped to single spacetime points, implies that some point-like structure may appear at high energies.

In this work, we restrict ourselves to discuss open bosonic string models defined by local lagrangians densities that can depend on second order derivatives of worldsheet coordinates. In Section 2, we present general formulas suitable to perform classical analysis of such string models. In comparison with earlier works on this subject, a different interpretation of the variational problem for string actions with second order derivatives is given. Moreover, all derived classical formulas are explicitly covariant with respect to reparametrization transformations. In Section 3, we derive a general form of a boundary term that can be added to the action, allowed by requirements of Poincare and reparametrization invariances. Such a term can modify edge conditions for open strings while bulk equations of motion are preserved. Canonical conserved quantities are modified by some edge contributions. Section 4 is devoted to the classical analysis of the string model defined by the Nambu-Goto action with some new boundary terms added. It is argued that such an open string model can be a suitable modification of the Nambu-Goto model as far as hadronic string interpretation is concerned. We carry out the classical analysis using the geometric approach, which is particularly convenient for our purposes. The classical open string configurations that extremize the extended action correspond to solutions of a complex Liouville equation. The relevant edge conditions for a Liouville
field are derived. The edge values are constant and finite there. Some preliminary discussion about physical consequences is made. In Appendix, the notation used throughout the paper is introduced and some basic mathematical definitions and equations of surface theory are collected.

2 String Lagrangians with Second Order Derivatives.

In this section, we introduce some general formulas appertained to the classical analysis of string models defined by lagrangians which depend on second order derivatives of worldsheet radius vector. In comparison with previous papers (e.g. [20, 21]), all formulas presented below are explicitly covariant with respect to the reparametrization, and we care especially with the correct derivation of edge conditions for open strings.

Let us consider the general form of the bosonic string action,

$$
S = \int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma \ L_{\text{string}}.
$$

(1)

It is convenient to represent the lagrangian density as

$$
L_{\text{string}} = \mathcal{L}_{\text{string}}(X_{\mu,a}; X_{\mu,ab}) = \sqrt{-g}\mathcal{L}(g^{ab}; X_{\mu,a}; \nabla_a \nabla_b X_{\mu}),
$$

(2)

where $\mathcal{L}$ is some scalar function made up of its specified arguments. Having the string lagrangian with second order derivatives written down in the above form, we can much easier perform mathematical calculations and keep the explicit reparametrization invariance in all following steps.

To derive the classical equations of motion, we are to evaluate the variation of the string action under the infinitesimal change of the worldsheet. Usually, the following boundary conditions are assumed,

$$
\delta X_\mu(\tau_i, \sigma) = \delta \dot{X}_\mu(\tau_i, \sigma) = 0, \quad i = 1, 2.
$$

(3)

There is some subtle problem at this point. The above requirements suggest the different interpretation of the variational problem in comparison with the usual Nambu-Goto case. In (3), not only initial and final string positions are fixed, but
also the initial and final velocities of string points. Therefore, if we consider some
string at the time \( \tau_1 \), another string at the time \( \tau_2 \) and some string trajectory being
a solution of Euler-Lagrange eqs. which interpolates between them, the solution
does not extremize the string action unless we restrict possible deviations of the
worldsheet to those that do not change its tangent vectors at the initial and final
positions. In other words, the string instant state is specified not only by its position,
but also by its velocities. In fact, this modified interpretation is not true as the
boundary conditions (3) are not quite proper for the string variational problem with
second order derivatives. This point will be clarified below.

The classical equations of motion following from (1) can be presented in the
explicitly covariant form
\[
\sqrt{-g} \nabla_a \Pi^a_\mu = 0 ,
\]
where \( \Pi^a_\mu \) is given by the following formula
\[
\Pi^a_\mu = -L \nabla^a X_\mu - \frac{\partial L}{\partial X^a_\mu} + 2 \frac{\partial L}{\partial g^{ab}} g^{bc} \nabla^c X_\mu + \nabla_b \left[ \frac{\partial L}{\partial (\nabla_a \nabla_b X^\mu)} \right].
\]
(5)

For open strings, the edge conditions at \( \sigma = 0, \pi \) must be satisfied,
\[
\sqrt{-g} \Pi^1 \rightdownarrow + \partial_0 \left[ \sqrt{-g} \frac{\partial L}{\partial (\nabla_0 \nabla_1 X^\mu)} \right] = 0 ,
\]
\[
\sqrt{-g} \frac{\partial L}{\partial (\nabla_1 \nabla_1 X^\mu)} = 0 .
\]
(7)

For the sake of more convenient notation, here and throughout the paper we define
and calculate the variational derivatives of \( L \) with the formal assumption that \( g^{01} \)
and \( g^{10} \), \( \nabla_0 \nabla_1 X^\mu \) and \( \nabla_1 \nabla_0 X^\mu \) are independent variables. Thus, all variational
derivatives on r.h.s. of (5) are tensor objects with respect to the reparametrization
invariance. The covariance of edge conditions becomes easy to check if we remind
that in the presence of the worldsheet boundary any reparametrization transforma-
tion \( \sigma^a \rightarrow \tilde{\sigma}^a(\tau, \sigma) \) must satisfy
\[
\tilde{\sigma}(\tau, 0) = 0 , \quad \tilde{\sigma}(\tau, \pi) = \pi .
\]
(8)

It is necessary in order to preserve the condition that the string parameter \( \sigma \) belongs
to the interval \([0, \pi]\). In other case, performing the variation of the string action we
are forced to implement the variations due to the change of \(\sigma\)-interval, and the fact that the set of allowed reparametrization transformations is restricted for open strings manifests in additional Euler-Lagrange eqs.

The derivation of (1) from the standard Euler-Lagrange variational equations is straightforward, so let us only cite the following identities used in this derivation

\[
\frac{\partial \mathcal{L}}{\partial (\nabla_a \nabla_b X^\mu)} X^\mu_{c} = 0 .
\]  
(9)

To prove the above identities for lagrangians which include only scalar constant parameters, it is enough to notice that the scalar (with respect to both reparametrization and Poincare transformations) function \(\mathcal{L}\) can be composed of the following "building blocks"

\[
g^{ab}, \ e^{\mu\nu\rho\sigma}(\nabla_a \nabla_b X^\mu)(\nabla_c \nabla_d X^\nu)X_{\rho,c}X_{\sigma,d}, \ \nabla_a \nabla_b X^\mu \nabla_c \nabla_d X^\nu ,
\]

and refer to the trivial identities

\[
(\nabla_a \nabla_b X^\mu)X^\mu_{c} = 0 .
\]

In general, the origin of identities (9) lies in the reparametrization invariance of the string action (1). The full set of all Noether identities (see (27,29)) following from the reparametrization invariance of the string action with second order derivatives has been derived in [22].

Let us return to the problem of boundary conditions (3) imposed on the variations of the worldsheet. If we assumed only that

\[
\delta X_{\mu}(\tau_i, \sigma) = 0, \ i = 1, 2 ,
\]  
(10)

then using eqs. of motion (4) together with edge conditions (6,7) we would obtain the following result for the variation of the string action

\[
\delta S = \int_0^\pi d\sigma \ \sqrt{-g} \ \frac{\partial \mathcal{L}}{\partial (\nabla_0 \nabla_0 X^\mu)} \delta \dot{X}^\mu_{\tau_1}^{\tau_2}.
\]  
(11)

If \(g = 0\) or the surface is locally flat then the following term vanishes, else we can choose parametrization in such a way that the four vectors \((\dot{X}_\mu, X'_\mu, \ddot{X}_\mu, \dot{X}'_\mu)\) are
linearly independent at the point of the worldsheet with $\tau = \tau_i$. Then, we can write down the general form of $\delta \dot{X}_\mu$ as the linear combination of these vectors,

$$
\delta \dot{X}_\mu = a_1 \dot{X}_\mu + a_2 X'_\mu + a_3 \ddot{X}_\mu + a_4 \dot{X}'_\mu.
$$

(12)

On the other hand, the variation $\delta \dot{X}_\mu$ induced by the change of parametrization $\sigma^a \rightarrow \sigma^a + \delta \sigma^a$ is given by

$$
\delta \dot{X}_\mu = -\ddot{X}_\mu \delta \sigma^0 - \dot{X}'_\mu \delta \sigma^1.
$$

(13)

It means that the variations of $\dot{X}_\mu$ in the directions of $\ddot{X}_\mu$ and $\dot{X}'_\mu$ are not important, because they can be removed by the change of parametrization. In turn, if we restrict ourselves to the "physical" variations of the worldsheet, then with the help of identities (9) we conclude that the term (11) vanishes.

Therefore, there are two ways to define properly the variational problem for string action functionals which depend on second order derivatives. One way is to assume boundary conditions (10) together with the additional requirements that the variations $\delta \dot{X}_\mu$ in the directions of $\ddot{X}_\mu$ and $\dot{X}'_\mu$ vanish, what in light of (13) means that the choice of the parametrization of the worldsheet is locally fixed at boundary points $\tau = \tau_i$. Other way is to take only the boundary conditions (10), as in the Nambu-Goto case, and together with relevant eqs. of motion and edge conditions we obtain additional equations

$$
\sqrt{-g} \frac{\partial \mathcal{L}}{\partial (\nabla_0 \nabla_0 X^\mu)} = 0 \quad \text{for} \quad \tau = \tau_1, \tau_2,
$$

(14)

which have no dynamical content and impose only some boundary constraints on the choice of worldsheet parametrization. Recapitulating, the interpretation of the variational problem for string actions with second order derivatives is the same as in the usual Nambu-Goto case. To derive the classical dynamics of strings from the variational principle it is just enough to consider the boundary conditions (10), i.e. to assume that the initial and final string positions are fixed. The appearence of the term (11) in the action variation and resulted equations reflect only the fact that the geometrical definitions of the initial and final string positions are not invariant.

One more comment on the derivation of edge conditions should be made. They are an integral part of equations of motion. They arise as in the variational problem for open string worldsheets the whole boundary of the worldsheet is not fixed (like
in an ordinary Plateau problem for two-dimensional surfaces), but only a part of it composed of the initial and final string positions. The other part of the worldsheet boundary, defined by trajectories of string endpoints, is not fixed (the ends of open strings are free). However, we can use another equivalent method for the derivation of edge conditions. In the variational problem we can dispense with considering the edge variations (assuming that the whole worldsheet boundary is fixed), and the edge conditions are produced when we demand that there is no flow of the canonical Noether invariants through the string ends. In distinction with the Nambu-Goto case, for strings with second order derivatives it is not enough to assure only that the canonical momentum is conserved. We must check the same independently for the angular momentum, because of its ”spin part” induced by higher order derivatives. The comment on the latter method of the edge conditions derivation is relevant to the recent work of Boisseau and Letelier [23]. They make use of the internal geometrical description of worldsheets to study models of strings with second order derivatives. In this approach, they gain some new insight into the content of dynamical equations. However, their formalism should be corrected for open strings. The set of edge conditions derived from the conservation of total energy-momentum should be supplemented by additional conditions associated with the total angular momentum conservation. In particular, it changes some results of the work [23]. For example, the prediction that the endpoints of the Polyakov rigid string can travel with a speed less than the velocity of light is not valid. Just taking into account the missing set of edge conditions, we check again that these velocities must be light-like, what agrees with the independent proof of this fact given in [22].

In the last part of this section, we write down formulas for Noether invariants. The total momentum reads

$$P_{\mu} = \int_{0}^{\pi} d\sigma \, p_\mu ,$$

where

$$p_\mu = - \frac{\partial L_{\text{string}}}{\partial X^{\mu}_0} + \partial_0 \left( \frac{\partial L_{\text{string}}}{\partial X^{\mu}_{00}} \right) = \sqrt{-g} \Pi^0_\mu - \partial_1 \left[ \sqrt{-g} \frac{\partial L}{\partial (\nabla_0 \nabla_1 X^\mu)} \right].$$

The total angular momentum can be calculated from the following formula,

$$M_{\mu\nu} = \int_{0}^{\pi} d\sigma \, m_{\mu\nu} ,$$

where
where

$$m_{\mu\nu} = x_{[\mu p_{\nu}] a} X_{\nu|a} =$$

$$= \sqrt{-g} X_{[\mu \Pi^0_{\nu}]} - \sqrt{-g} X_{[\mu, a]} X^0 \partial_{X^\mu} \partial_{X^\nu} X^0 - \partial_1 \left[ \sqrt{-g} X_{[\mu \partial_{X^\mu} \partial_{X^\nu} X^0]} \right].$$

3 Boundary Terms for String Actions.

We discuss the general string action functional with some boundary term added,

$$S = \int d^2 \sigma L_{\text{string}}^{\text{bulk}} - \int d^2 \sigma \partial_a V^a. \quad (17)$$

The stationarity of this action results in some equations for the interior of the string following from $L_{\text{string}}^{\text{bulk}}$, and the role of the second action term is to ensure a more general set of edge conditions for an open string case. Below, we will find the general form of this term allowed by requirements of the locality, Poincare and reparametrization invariance. We restrict ourselves to string lagrangians which depend on not higher than second order derivatives, what implies that

$$\frac{\partial V^a}{\partial X^\mu} X^\mu_{,abc} = 0. \quad (18)$$

The above identities give immediately the following equations

$$\frac{\partial V^0}{\partial X^\mu_{,00}} = \frac{\partial V^0}{\partial X^\mu_{,11}} + 2 \frac{\partial V^1}{\partial X^\mu_{,01}} = \frac{\partial V^1}{\partial X^\mu_{,11}} = \frac{\partial V^1}{\partial X^\mu_{,00}} + 2 \frac{\partial V^0}{\partial X^\mu_{,01}} = 0, \quad (19)$$

and their general solution is of the form

$$V^a = \epsilon^{ab} \tilde{A}^c X^\mu_{,bc} + \tilde{B}^a, \quad (20)$$

where $\tilde{A}^c$ and $\tilde{B}^a$ are some arbitrary functions which depend on $X^\mu$ and their first derivatives. The translational invariance of the action requires that

$$0 = \frac{\partial (\partial_a V^a)}{\partial X^\mu} = \partial_a \left( \frac{\partial V^a}{\partial X^\mu} \right), \quad (21)$$

9
therefore there exists function $\Lambda_\mu(X_\nu;X_\nu,a)$ such that

$$\frac{\partial V^a}{\partial X_\mu} = \epsilon^{ab}\partial_b\Lambda_\mu = \epsilon^{ab}\left(\frac{\partial\Lambda_\mu}{\partial X_\nu}X_\nu^b + \frac{\partial\Lambda_\mu}{\partial X_\nu}X_\nu^{bc}\right).$$

(22)

Comparing (22) with (20) we obtain

$$\frac{\partial\Lambda_\mu}{\partial X_\nu,a} = \frac{\partial\tilde{A}_a^\nu}{\partial X^\mu};$$

(23)

$$\epsilon^{ab}\frac{\partial\Lambda_\mu}{\partial X_\mu}X_\nu^b = \frac{\partial B^a}{\partial X_\nu}.$$  

(24)

The above equations are consistent provided that

$$\frac{\partial\Lambda_\mu}{\partial X_\nu} - \frac{\partial\Lambda_\nu}{\partial X_\mu} = F_{\mu\nu},$$

(25)

where $F_{\mu\nu}$ is some constant antisymmetric tensor. Consequently, there exists a scalar function $\lambda(X_\mu;X_\mu,a)$ such that

$$\Lambda_\mu = \frac{1}{2}F_{\mu\nu}X_\nu + \frac{\partial\lambda}{\partial X_\mu}.$$  

Inserting this result in (23,24), after some straightforward steps we get the general form of $V^a$

$$V^a = \epsilon^{ab}\partial_a\lambda + \frac{1}{2}F_{\mu\nu}X_\mu X_\nu^b + \epsilon^{ab}A^c_\mu X_\nu^{bc} + B^a,$$

(26)

where new arbitrary functions $A^c_\mu$ and $B^a$ depend now only on the first derivatives of $X_\mu$. The first term on r.h.s. of (26) can be omitted, as it does not contribute to $\partial_aV^a$.

The next step is to assure that the string action boundary term in (17) defined with the general functional $V^a$ of the form (26) is reparametrization invariant. For this purpose, it is convenient to use the Noether theorem for strings with second order derivatives, namely that the string action functional is invariant under the reparametrization transformations if and only if the lagrangian satisfies the following set of identities

$$\frac{\partial L_{\text{string}}}{\partial X_{,ab}^\mu}X_{,c} = 0,$$

(27)
\[
\frac{\partial \mathcal{L}_{\text{string}}}{\partial X^\mu_{\alpha}} X^\mu_{\beta} + \frac{\partial \mathcal{L}_{\text{string}}}{\partial X^\mu_{\alpha\beta}} X^\mu_{\beta\alpha} + \frac{\partial \mathcal{L}_{\text{string}}}{\partial X^\mu_{\alpha}} X^\mu_{\alpha} - \mathcal{L}_{\text{string}} \delta^\alpha_\beta = 0 ,
\]

(28)

\[
\left[ \frac{\partial \mathcal{L}_{\text{string}}}{\partial X^\mu} - \partial_d \left( \frac{\partial \mathcal{L}_{\text{string}}}{\partial X^\mu_{d}} \right) + \partial_0^2 \left( \frac{\partial \mathcal{L}_{\text{string}}}{\partial X^\mu_{00}} \right) + \partial_d \partial_1 \left( \frac{\partial \mathcal{L}_{\text{string}}}{\partial X^\mu_{01}} \right) + \partial_1^2 \left( \frac{\partial \mathcal{L}_{\text{string}}}{\partial X^\mu_{11}} \right) \right] X^\mu_{\alpha} = 0 ,
\]

(29)

where fixed indices \( a, b, c \) can take values 0 or 1 while the summation over \( d \) is assumed. Substituting (26) into the above equations, we end up with some final general solution for \( V^\alpha \), which leads to the following general form of the lagrangian density,

\[
\partial_\alpha V^\alpha = \frac{1}{2} \alpha \sqrt{-g} R + \beta \sqrt{-g} N + \mathcal{L}_{\text{ext}} ,
\]

(30)

where \( \alpha \) and \( \beta \) are some dimensionless constants, and \( \mathcal{L}_{\text{ext}} \) stands for boundary lagrangians which include Poincare vector or tensor constants, i.e. describe some open systems with external fields. For such lagrangians we have infinitely many possibilities, let us only give some examples

\[
\frac{1}{2} \varepsilon^{ab} F_{\mu\nu} X^\mu_{\alpha} X^\nu_{\beta} , \\
\sqrt{-g} A^\mu \Delta X_\mu , \\
\sqrt{-g} \nabla_\alpha \left( \frac{A^\rho \nabla^a X_\rho}{\sqrt{1 + (T_{\mu\nu} g^{bc} X^\mu_{\beta} X^\nu_{\epsilon})^2}} \right) , \\
\text{etc.}
\]

The first term can be interpreted as the coupling of the charged string endpoints with the external electromagnetic field \[30\].

There are present only two string self-interaction terms in (30). The relevant coefficients \( B^a \) in (26) for these terms vanish, and the coefficients \( A^a_\alpha \) can be calculated from the equations

\[
\frac{\partial A^a_\alpha}{\partial X^\mu_{\beta}} - \frac{\partial A^b_\beta}{\partial X^\mu_{\alpha}} = \frac{\alpha}{\sqrt{-g}} \varepsilon^{ab} G_{\mu\nu} + \beta g^{ab} \tilde{t}_{\mu\nu} .
\]

(31)
Note that the scalar density requirement on $\partial_a V^a$ does not imply that $V^a$ behaves like a vector density under the reparametrization transformations. The considered two self-interaction terms exemplify the case.

Let us summarize the results of this section. We proved that the generic local term which can be added to any specific string action to modify edge conditions for open strings, provided that bulk equations of motion are preserved, has the form (30). We have obtained this conclusion considering only Poincare and reparametrization invariance, and restricting ourselves to local lagrangians with not higher than second derivatives. We did not presume that this term should be polynomial or analytical in fields as well as no ”power-counting” arguments for renormalizability of the quantized theory were applied. Thus, our result derived from a small set of very fundamental assumptions has a general significance.

Remarkably, the only two self-interaction terms displayed on r.h.s. of (30) are polynomial and well known in literature. In Euclidean four-dimensional space, they are topological and related to Euler characteristics and the numbers of self-intersections of two-dimensional surfaces.

4 Minimal Open String Models.

In this section, we examine the string action functional for minimal time-like surface models, defined by the following lagrangian

$$\mathcal{L}_{\text{string}} = -\gamma \sqrt{-g} - \frac{1}{2} \alpha \sqrt{-g} R - \beta \sqrt{-g} N.$$  \hspace{1cm} (32)

The first term is the Nambu-Goto lagrangian, $\gamma$ stands for string tension. The parameters $\alpha$ and $\beta$ are dimensionless. Let us also introduce an angle parameter $\theta \in [-\pi, \pi]$ defined as

$$\tan \frac{\theta}{2} = \frac{\beta}{\alpha}. \hspace{1cm} (33)$$

According to the discussion in the previous section, the lagrangian (32) defines the most general model for free open strings, which worldsheets represent minimal time-like surfaces of zero mean curvature.

Both new terms displayed on r.h.s. of (32) can be relevant for the definition of the hadronic string action. The first boundary term is related to Euler characteristics in its Euclidean version. The genus factors that appear in Polyakov quantum sum over
surfaces [24] can be interpreted as a result of adding such a term to the string action. On the other hand, we will show in this section that this selfinteraction term acts like "mass" term and prevents string ends from propagating with light-like velocities. It may help to couple consistently quark masses to hadronic strings. The relevance of the second boundary term in (32) to QCD string has been also pointed out in many papers. Polyakov [11] suggested that the inclusion of the term that weights worldsheets according to the number of selfintersections could assure the existence of "smooth" phase of surfaces. In other works [24, 26, 27], this term has been used to reproduce an effect of QCD \( \theta \)-vacua in string models. The exact correspondence between the moduli space of the maps associated with a surface theory and the moduli space of the instanton sector of QCD (or any other Yang-Mills theory) has been elaborated in [28]. Exact instanton solutions in the string model with the selfintersections term have been considered in [29]. Finally, in the paper [27] it is argued that this term is necessary for QCD string also in respect of having quark spins included. Below, we will see on the classical level that the Minkowski version of selfintersections term induces the topological sectors of solutions, what could correspond to the degenerated vacuum.

Equations of motion following from (32) are the same as in the usual Nambu-Goto theory,

\[ \Delta X_\mu = 0 , \]  

but supplementary edge equations for string endpoints \( \sigma = 0, \pi \) are now affected by additional terms, and have the following more general form

\[ \gamma \sqrt{-g} \nabla^1 X_\mu - \alpha \partial_0 \left( \frac{1}{\sqrt{-g}} \nabla_0 \nabla_1 X_\mu \right) - \beta \partial_0 \left( \tilde{t}_{\mu\nu} \nabla_0 \nabla_0 X^\nu \right) = 0 , \]  

\[ \frac{\alpha}{\sqrt{-g}} \nabla_0 \nabla_0 X_\mu - \beta \tilde{t}_{\mu\nu} \nabla^1 \nabla_0 X^\nu = 0 . \]  

We will investigate the string dynamical problem given by the system of equations (34,35,36). The best way is to use the geometrical approach [31,30], i.e. to express the content of these equations in terms of worldsheet curvature coefficients. Then, the differential equations transform into algebraic ones. Eq.(34) says that the mean curvature is zero at any point of the worldsheet, namely

\[ g^{ab} K_{ab} = 0 . \]
Edge conditions (35,36) can be integrated with respect to worldsheet time $\tau$ and, after projections onto tangent and normal planes respectively, they yield

\[
\frac{\alpha}{\sqrt{-g}} K_{00}^i + \beta \epsilon^{ij} K_{0}^j = 0 , \tag{38}
\]

\[
\frac{\alpha}{\sqrt{-g}} K_{01}^i - \beta \epsilon^{ij} K_{0}^0 = w^i , \tag{39}
\]

\[
\gamma \sqrt{-g} - w^i K_{01}^i = 0 , \tag{40}
\]

where $w^i$ are arbitrary functions satisfying

\[
D_0 w^i \equiv \partial_0 w^i - \epsilon^{ij} \omega_0 w^j = 0 . \tag{41}
\]

Let us choose one of the string endpoints, specified by $\sigma = 0$ or $\sigma = \pi$. We have here seven linear algebraic equations for local values of six curvature coefficients $K_{ab}^i$, so we can easily find that the solution exists only if the following condition is satisfied

\[
\alpha w^i w^i = \gamma (\alpha^2 + \beta^2) . \tag{42}
\]

From (41) follows that the expression $w^i w^i$ is time independent, what is compatible with the relation (42). Next, we see that the classical solutions exist only for positive sign of $\alpha$,

\[
\alpha > 0 . \tag{43}
\]

It is also interesting to note that the classical model defined by the action composed only of the Nambu-Goto and "self-interaction" terms ($\gamma, \beta \neq 0; \alpha = 0$) is inconsistent.

If the relation (42) is satisfied, then the edge values of the curvature coefficients are easy calculable from (37-40), namely

\[
K_{00}^i = \frac{\beta g g^{11} \epsilon^{ij} w^j}{\alpha^2 + \beta^2} , \tag{44}
\]

\[
K_{01}^i = \frac{\sqrt{-g} (\alpha w^i + \beta \sqrt{-g} g^{01} \epsilon^{ij} w^j)}{\alpha^2 + \beta^2} , \tag{45}
\]

\[
K_{11}^i = -\frac{2 \alpha \sqrt{-g} g^{01} w^i + \beta [1 + (\sqrt{-g} g^{01})^2] \epsilon^{ij} w^j}{(\alpha^2 + \beta^2) g^{11}} . \tag{46}
\]
One can verify that the formulas (44-46) are covariant with respect to both the worldsheet reparametrization and local orthogonal rotation transformations. The scalar functions \( R \) and \( N \) take the following constant values at the boundary of the worldsheet,

\[
\frac{R}{2} = \frac{\gamma \beta^2 - \alpha^2}{\alpha \alpha^2 + \beta^2} = -\frac{\gamma}{\alpha} \cos \theta ,
\]

\[
N = -\frac{2\beta \gamma}{\alpha^2 + \beta^2} = -\frac{\gamma}{\alpha} \sin \theta .
\]

Using the above results, one can check that the lagrangian density (32) vanishes at the string endpoints, what is a general feature of bosonic open string models.

Now, let us turn into the investigation of classical solutions, satisfying equations of motion together with pertinent edge conditions. As usual, we choose the conformal gauge,

\[
\ddot{X}^2 + \dot{X}^2 = 0 ,
\]

which makes equations (34) linear and the general solution reads

\[
X_\mu(\tau, \sigma) = X^L_\mu(\tau + \sigma) + X^R_\mu(\tau - \sigma) .
\]

Let us denote,

\[
\ddot{X}_L^2 = -\frac{1}{4} q^2_+ , \quad \ddot{X}_R^2 = -\frac{1}{4} q^2_- , \quad q_\pm = q_\pm(\tau \pm \sigma) , \quad q_\pm \geq 0 .
\]

Accordingly,

\[
(\ddot{X} \pm \dot{X}')^2 = -(K_{00}^i \pm K_{01}^i)(K_{00}^i \pm K_{01}^i) = -q^2_\pm .
\]

One can introduce new variables (we follow here [30]),

\[
\sqrt{-g} = e^{-\phi} , \quad \psi = \alpha_+ - \alpha_- ,
\]

\[
K_{00}^1 \pm K_{01}^1 = q_\pm \cos \alpha_\pm , \quad K_{00}^2 \pm K_{01}^2 = q_\pm \sin \alpha_\pm .
\]

In the geometrical approach, the role of dynamical equations play Gauss-Peterson-Codazzi-Ricci equations (see Appendix), being the embedding conditions for the worldsheet embedded in enveloping Minkowski spacetime. Referring to (81,82), one can evaluate

\[
e^\phi(\ddot{\phi} - \phi''') = 2e^{2\phi} q_+ q_- \cos \psi ,
\]

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\[ \dot{\alpha}_\pm + \omega_0 = \pm(\alpha'_\pm + \omega_1), \quad (53) \]
\[ \omega'_0 - \dot{\omega}_1 = -e^{\phi}q_+q_- \sin \psi. \quad (54) \]

Peterson-Codazzi eqs. (53) allow us to eliminate torsion coefficients. Two other equations have a nice geometrical interpretation. Gauss eq. (52) relates the internal curvature scalar \( R \) (l.h.s. of (52)) to the scalar build of the external curvature coefficients (r.h.s. of (52)). The internal curvature scalar is build of the connections \( \Gamma^a_{bc} \), introduced for the tangent reper bundle with defined reparametrization transformations. Thus, Gauss eq. (52) describes an immersion of the tangent bundle. Similarly, l.h.s. of Ricci eq. (54) is a scalar expression build of the connections \( \epsilon^{ij}\omega_a \) defined on the orthogonal reper bundle, endowed with local SO(2) transformations. Looking at the r.h.s. of (54) (up to a constant it is equal to the scalar \( N \)), we can interpret Ricci eq. as the immersion of the orthogonal bundle. We see that Gauss and Ricci eqs. couple "internal" with "external" geometry, describing immersions of tangent and orthogonal two-dimensional reper bundles in 4-dim Minkowski space-time. Remarkably, both scalars \( R \) and \( N \) constructed from disposable connections and displayed in the immersion equations have been used in (32).

After eliminating the extrinsic torsion, Gauss and Ricci equations read
\[ \ddot{\phi} - \phi'' = 2e^{\phi}q_+q_- \cos \psi, \quad (55) \]
\[ \ddot{\psi} - \psi'' = 2e^{\phi}q_+q_- \sin \psi. \quad (56) \]

The above equations can be written as one equation on a complex function \( \Phi \equiv \phi + i\psi \) (see (30)),
\[ \ddot{\Phi} - \Phi'' = 2q_+q_-e^{\Phi}. \quad (57) \]

The gauge choice (47) leaves the residual symmetry,
\[ \tau \pm \sigma \rightarrow h_\pm(\tau \pm \sigma), \quad (58) \]
where \( h_\pm \) are arbitrary monotonic functions. Taking
\[ h_\pm(\tau) = \int_{\tau_0}^{\tau} d\tau' q_\pm(\tau'), \]
\((h_\pm \) are monotonical due to \( q_\pm \geq 0)\), the equation (57) rewritten in the new variables (58) takes the standard form of Liouville equation,
\[ \ddot{\Phi} - \Phi'' = 2e^{\Phi}. \quad (59) \]
As it has been proved, the classical Nambu-Goto dynamics (minimal surface problem) reduces to a complex Liouville equation [54]. The functions \( q_{\pm} \) are arbitrary and their choice saturates the gauge freedom associated with the reparametrization invariance. Unlike other gauge theories, in the minimal string model the gauge can be completely fixed without breaking the Lorentz invariance. Obviously, the simplest gauge choice complementary to (17) is

\[
q_{\pm} = 1 .
\]

Later, we will show that this gauge choice is also allowed when edge conditions for worldsheets with boundaries are taken into account. Here, let us note that if we restrict ourselves to reparametrization transformations which preserve worldsheet boundaries (8), what means that

\[
h_+(\tau) = h_-(\tau) = h_+(\tau - 2\pi) + 2\pi ,
\]

then the gauge choice (60) is possible provided that

\[
q_+(\tau) = q_-(\tau) = q_+(\tau + 2\pi) .
\]

Assuming (60), the general solution of (57) reads

\[
\Phi = \log \left( \frac{-4f'(\tau + \sigma)g'(\tau - \sigma)}{[f(\tau + \sigma) - g(\tau - \sigma)]^2} \right) ,
\]

where \( f \) and \( g \) are arbitrary complex functions (not necessary single-valued, only \( \Phi \) should be single-valued). The function \( \Phi \) is left invariant when \( f \) and \( g \) are changed by a modular transformation,

\[
f \rightarrow \frac{af + b}{cf + d} , \quad g \rightarrow \frac{ag + b}{cg + d} , \quad ad - bc = 1 .
\]

It is helpful to know how to translate a given solution \( \Phi \) of Liouville eq. into the explicit radius-vector representation of the string worldsheet \( X_\mu(\tau, \sigma) \). In order to achieve it, we need to integrate Gauss-Weingarten eqs. (79). For this purpose, it is convenient to introduce the reference system composed of two real \( k_\mu, l_\mu \) and one complex \( a_\mu \) null vectors,

\[
k^2 = l^2 = a^2 = ka = la = 0 , \quad kl = -a\bar{a} = 2 .
\]
As a result of the integration of Gauss-Weingarten eqs. we obtain (function arguments are omitted)

\[
\dot{X}_L^{\mu} = \frac{1}{4\sqrt{|f'|}} (|f|^2 k^{\mu} - f a_{\mu} - \bar{f} a_{\mu} + l_{\mu}),
\]

(65)

\[
\dot{X}_R^{\mu} = \frac{1}{4\sqrt{|g'|}} (|g|^2 k^{\mu} - g a_{\mu} - \bar{g} a_{\mu} + l_{\mu}).
\]

(66)

In particular, we can choose

\[k_{\mu} = (1, 0, 0, 1), \quad l_{\mu} = (1, 0, 0, -1), \quad a_{\mu} = (0, 1, 0, 0).\]

Then,

\[
\dot{X}_L^{\mu} = \frac{1}{4\sqrt{|f'|}} (1 + |f|^2, f + \bar{f}, i(f - \bar{f}), 1 - |f|^2),
\]

(67)

\[
\dot{X}_R^{\mu} = \frac{1}{4\sqrt{|g'|}} (1 + |g|^2, g + \bar{g}, i(g - \bar{g}), 1 - |g|^2).
\]

(68)

As it could be expected, the modular transformations (63) coincide with Lorentz transformations of \(X_{\mu}\). The integration of Gauss-Weingarten eqs. gives also results for \(n_{\mu}^i\) variables, namely (here \(\partial_{\pm} = \partial_0 \pm \partial_1\))

\[n_{\mu}^1 + i n_{\mu}^2 = \]

\[
\frac{i}{e^\phi \sin \theta} \left[ \partial_+ \left( e^\phi \dot{X}_{L\mu} \right) e^{i\alpha_-} - \partial_- \left( e^\phi \dot{X}_{R\mu} \right) e^{i\alpha_+} \right].
\]

Insofar, we have proved that Nambu-Goto eqs. together with complete Poincare-invariant gauge-fixing conditions (47) and (60) are equivalent to the problem defined by a complex Liouville eq. (without any additional constraints). To examine open strings case, let us proceed with the derivation of boundary conditions for Liouville complex field \(\Phi\) equivalent to edge conditions (35,36) for \(X_{\mu}\) following from the lagrangian (32). It is straightforward to convince ourselves that the edge conditions, see (44-46), are satisfied if and only if

\[e^{-\phi} = \sqrt{\frac{\alpha}{\gamma}} q_+, \text{ for } \sigma = 0, \pi,\]

(69)
\psi = \pi - \theta \mod 2\pi \text{, for } \sigma = 0, \pi \text{,} \quad (70)

\psi' = 0 \text{, for } \sigma = 0, \pi \text{,} \quad (71)

q_+(\tau) = q_-(\tau) \text{, } q_+(\tau + 2\pi) = q_+(\tau) \text{.} \quad (72)

We see that the edge eqs. (72) are exactly the same as the conditions (61). It means that the gauge choice (60) is allowed for open strings as well.

Summarizing, the classical open string equations following from the lagrangian (32) are equivalent (in conformal gauge (47) supplemented by complementary conditions (60)) to complex Liouville equation

\ddot{\Phi} - \Phi'' = 2e^\Phi \text{,} \quad (73)

with constant Dirichlet boundary conditions for the real part \( \phi = \text{Re}\Phi \),

\[ e^\phi = \sqrt{\gamma / \alpha} \text{ for } \sigma = 0, \pi \text{,} \quad (74) \]

and periodic boundary conditions for the imaginary part \( \psi = \text{Im}\Phi \),

\[ \psi = \pi - \theta \mod 2\pi \text{, } \psi' = 0 \text{ for } \sigma = 0, \pi \text{.} \quad (75) \]

We have evaluated the general form of boundary conditions which can follow consistently from the string action for isolated open strings with not higher than second derivatives. In this paper, we do not develop the thoroughgoing analysis of classical string states. We restrict ourselves to indicate that some essential differences appear while we are comparing the above defined extended boundary problem for minimal worldsheets with the ordinary Nambu-Goto case.

First, let us consider the case \( \beta = 0 (\theta = 0) \). The lowest state solution of Nambu-Goto model, that corresponds to stationary (soliton) solution for Liouville field, represents the rotating rigid rod,

\[ X_\mu = \frac{1}{\lambda^2}(\lambda \tau, \cos(\lambda \tau) \sin[\lambda(\sigma - \pi 2)], \sin(\lambda \tau) \sin[\lambda(\sigma - \pi 2)], 0) \text{.} \quad (76) \]

It is also a solution of our extended boundary problem (73-75), but now the string end points are no longer forced to travel with light-like velocities. The parameter \( \lambda \) is subject to the following equation (for Nambu-Goto configurations \( \lambda = 1 \)),

\[ \left[ \frac{\lambda}{\cos(\lambda \pi/2)} \right]^2 = \sqrt{\gamma / \alpha} \text{.} \quad (77) \]
For $\beta \neq 0 \ (\theta \neq 0)$, there are no solitonic solutions of the Liouville equation. The imaginary part of Liouville field $\psi$, that is an angle variable (see definition (51)), cannot be trivial (i.e. to be constant everywhere on the string). The mapping $e^{i\psi} : [0, \pi] \to S^1$ provide us with some topological winding number, classifying possible solutions.

In the end of this section, let us make a comment on the possible extension of string action (32) by adding the rigidity term [11],

$$L_{\text{rig}} = \kappa \sqrt{-g} (\Delta X_{\mu})^2 .$$ (78)

Obviously, the extended boundary problem for rigid strings is much more complicated. However, it is nice to note that all classical open string solutions defined by the system (73-75) are still exact solutions when the extended boundary problem is formulated with the rigidity action term taken into account. Moreover, all these solutions carry the same energy and angular momentum in both models (the rigidity term does not influence conservation laws for this class of solutions). Presumably, these are the only open rigid string solutions around which a sensible semiclassical quantization can be performed.

**Appendix**

In the appendix, we introduce notation and gather mathematical equations of surfaces theory used throughout this paper. The string worldsheet is denoted by $X_{\mu}(\sigma^a) = X_{\mu}(\tau, \sigma) \ (\sigma \in [0, \pi])$, its derivatives either by $X_{\mu,a} (a = 0, 1)$ or by the dot and the prime for the derivatives over $\tau$ and $\sigma$ parameters respectively. The following causality conditions are imposed on the worldsheet,

$$\dot{X}^2 \geq 0 \ , \ \dot{X}_0 > 0 \ , \ X'^2 \leq 0 .$$

The induced metric is $g_{ab} = X^\mu_{,a} X^\mu_{,b}$, its determinant $g \ (g \leq 0)$. Christoffel coefficients $\Gamma^a_{bc}$, covariant derivative $\nabla_a$ and raising and lowering indices (denoted by small first roman letters $(a,b,c,...)$) are defined with respect to the induced metric. The Riemann-Christoffel tensor $R_{abcd}$ is also defined as usual,

$$R_{abcd} = \partial_c \Gamma_{bda} - \partial_d \Gamma_{cba} + \Gamma^e_{bc} \Gamma_{ade} - \Gamma^e_{bd} \Gamma_{ace} ;$$

and the internal curvature scalar $R$ is introduced together with the following relation,
\[ R^c_{\, abc} = \frac{1}{2} g_{ab} R \, . \]

At any point of the worldsheet two orthonormal vectors \( n^i_\mu \) \((i = 1, 2)\) can be introduced,

\[ n^i_\mu X^\mu_a = 0 \, , \quad n^i_\mu n^j_\mu = -\delta^{ij} \, . \]

\[ \frac{1}{\sqrt{-g}} \varepsilon_{\mu\nu\rho\sigma} \dot{X}^\mu X^\nu n^1_\rho n^2_\sigma = +1 \, . \]

The last condition fixes the orientation of the local frame. There is still some arbitrariness in a choice of orthonormal vectors \( n^i_\mu \), namely one can perform a local SO(2)-rotation in a normal plane \((M \text{ stands for the rotation matrix about the angle } \phi)\),

\[ n^i_\mu \to M^{ij}(\phi)n^j_\mu \, , \quad \phi = \phi(\tau, \sigma) \, , \]

\[ \omega^a \to \omega^a + \partial_a \phi \, . \]

This freedom can be considered as a local symmetry of the system described in the geometric approach. Therefore, for practical purposes it is convenient to use "double-covariant" derivative \( D_a \), i.e. the derivative covariant with respect to both reparametrization change and local orthogonal rotation. This derivative is defined with the help of respective connections \( \Gamma^c_{ab} \) and \( \varepsilon^{ij} \omega_a \).

The projection operator onto the normal plane is denoted by \( G_{\mu\nu} \),

\[ G_{\mu\nu} = \eta_{\mu\nu} - g^{ab} X_{\mu,a} X_{\nu,b} \, , \]

and antisymmetric tensor \( t_{\mu\nu} \) is introduced as usual,

\[ t_{\mu\nu} = \frac{1}{\sqrt{-g}} \varepsilon^{ab} X_{\mu,a} X_{\nu,b} \, , \quad \tilde{t}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} t_{\rho\sigma} \, . \]

Let us also define covariant tensor \( N_{abcd} \) as

\[ N_{abcd} = \tilde{t}^{\mu\nu}(\nabla_a \nabla_b X_\mu) \nabla_c \nabla_d X_\nu \, . \]
and the scalar function $N$ together with the following relation,

$$N_{acb}^c = \sqrt{-g} \varepsilon_{ab} N.$$

The external curvature $K_{ab}^i$ and torsion $\omega_a$ coefficients are defined with Gauss-Weingarten equations (in parentheses we give their form in "double-covariant" notation),

$$X_{\mu,ab} = \Gamma_{\mu,ab}^c X_{\mu,c} + K_{ab}^i n_{\mu}^i, \quad (D_a D_b X_\mu = K_{a b}^{i \mu} n_i^j ).$$

$$\partial_a n_{\mu}^i = K_{a}^{i b} X_{\mu,b} + \varepsilon^{i j} \omega_{a} n_{\mu}^j, \quad (D_a D_b n_{\mu}^i = K_{a b}^{i \mu} X_{\mu,b} ).$$

Instead of using radius vector coordinates, we can represent the surface (up to Poincaré transformations) by induced metric and external curvature and torsion coefficients, which satisfy the following identities (being the compatibility conditions for Gauss-Weingarten eqs.)

$$R_{abcd} = K_{ad}^{i} K_{bc}^{i} - K_{ac}^{i} K_{bd}^{i}, \quad (\text{Gauss eqs.})$$

$$\nabla_a K_{bc}^{i} - \nabla_b K_{ac}^{i} = \varepsilon^{i j} (\omega_a K_{bc}^{j} - \omega_b K_{ac}^{j}), \quad (\text{Peterson - Codazzi eqs.})$$

$$\partial_a \omega_b - \partial_b \omega_a = \varepsilon^{i j} g^{cd} K_{ac}^{i} K_{bd}^{j}, \quad (\text{Ricci eqs.})$$

All above equations are covariant with respect to both reparametrization change and local orthonormal rotation. The Peterson-Codazzi eqs. and Ricci eqs. in "double-covariant" notation have the following form,

$$D_a K_{bc}^{i} = D_b K_{ac}^{i},$$

$$[D_a, D_b] K_{cd}^{i} = -\sqrt{-g} \varepsilon_{ab} \varepsilon^{ij} NK_{cd}^{j}.$$

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