CYCLIC HOMOLOGY OF $H$-UNITAL (PRO-)ALGEBRAS, LIE ALGEBRA HOMOLOGY OF MATRICES AND A PAPER OF HANLON’S

GUILLERMO CORTÍNAS*

Abstract. We consider algebras over a field $k$ of characteristic zero. The article is concerned with the isomorphism of graded vectorspaces

$$H(gl(A)) \cong \wedge(HC(A)[-1])$$

between the Lie algebra homology of matrices and the free graded commutative algebra on the cyclic homology of the $k$-algebra $A$, shifted down one degree. For unital algebras this isomorphism is a classical result obtained by Loday and Quillen and independently by Tsygan. For $H$-unital algebras, it is known to hold too, as is that the proof follows from results of Hanlon’s. However, to our knowledge, the proof is not immediate, and has not been published. In this paper we fill this gap in the literature by offering a detailed proof. Moreover we establish the isomorphism in the general setting of ($H$-unital) pro-algebras.

1. Introduction

We consider not necessarily unital algebras over a fixed field $k$ of characteristic zero. If $A$ is an algebra, we write $gl_n A$ for the Lie algebra of $n \times n$ matrices, put $gl_A = \bigsqcup_n gl_n A$ and consider the Lie algebra homology $H(gl A)$ and the cyclic homology $HC(A)$. We define a natural map of graded vectorspaces

(1) $\varphi : H(gl A) \to \Lambda(HC(A)[-1])$

Recall $A$ is called $H$-unital if its bar homology vanishes ($\square$). We prove here that $\varphi$ is an isomorphism if $A$ is $H$-unital. We obtain this as a particular case of a more general result concerning pro-algebras. In this paper a pro-algebra is an inverse system

(2) $A = \{A_{n+1} \to A_n\}_n$

of algebras and algebra homomorphisms indexed by the positive integers. Note any algebra $A$ can be regarded as a constant pro-algebra where all the maps (2) are identity maps. A pro-algebra $A$ is $H$-unital if for each $r \geq 1$, $H^{bar}_r A = \{H^{bar}_r A \}_n$ is the zero pro-vectorspace; this means that for each $r \geq 1$ and each $n$ there is an $m = m(n, r) \geq n$ such that the map

$$H^{bar}_r(A_m) \to H^{bar}_r(A_n)$$

is zero. The map (1) induces, in each degree $r$ and each level $n$, a map $\varphi_{r, n} : H_r(gl A_n) \to (\Lambda(HC(A_n)[-1]))_r$. As $n$ varies, we obtain a map of pro-vectorspaces

(*) Partially supported by CONICET and the Ramón y Cajal fellowship and by grants ANPCyT PICT 03-12330 and MTM00958.

1
We show (in Thm. 4.2) that if $A$ is $H$-unital, then (3) is an isomorphism of pro-vectorspaces; this means that the pro-vectorspaces $\{\ker \varphi_{r,n}\}_n$ and $\{\coker \varphi_{r,n}\}_n$ are zero in the sense explained above. In particular, applying this to constant pro-algebras, we get that (1) is an isomorphism for any $H$-unital algebra $A$.

The isomorphism

$$H(glA) \cong \Lambda(HC(A)[-1])$$

for a unital algebra $A$ was proved by Loday and Quillen (9) and independently by Tsygan (see [8] for the announcement and [2] for the proof). For $H$-unital $A$ it is cited in the literature (e.g. in [5, E.10.2.6] and [7, pp 88, line 4]) as following from the results of Hanlon’s paper [3]. Hanlon’s paper deals with (Lie, bar and cyclic) cohomology of finite dimensional $C^*$-algebras. What is immediate from the results of [3] is that if $A$ is finite dimensional and $k = \mathbb{C}$ then there is a map between the dual spaces

$$(\Lambda(HC(A)[-1]))^\vee \to (H(glA))^\vee$$

which is an isomorphism if $A$ is $H$-unital. The transpose of (6) gives an isomorphism $\Lambda(HC(A)[-1]) \cong H(glA)$ for $A$ finite dimensional over $\mathbb{C}$. The isomorphism (4) for $H$-unital algebras of not necessarily finite dimension over any field $k$ of characteristic zero also follows from the material in [3], but the proof is more delicate. The purpose of this paper is to give a detailed proof of (4) in the general case of $H$-unital pro-algebras over any field $k$ of characteristic zero.

Next we explain the main ideas of the proof of (4) in the case of $H$-unital algebras. The case of general pro-algebras is proved in a similar manner; see the proof of Thm 4.2 for details. The first step, developed in Section 1, is an elementary application of representation theory, (mainly of the highest weight theorem [4, Thm 14.13]). The Lie algebra $gl_n(k)$ acts naturally on the Chevalley-Eilenberg complex $C(gl_nA)$; under this action, the latter complex decomposes into isotypic components

$$C(gl_nA) = \bigoplus_\mu C(gl_nA)_\mu$$

Sitting inside each component $C(gl_nA)_\mu$ is the subcomplex of maximal weight vectors

$$M_\mu C(gl_nA) \subset C(gl_nA)_\mu$$

It turns out that $M_\mu C(gl_nA)$ and $H(M_\mu C(gl_nA))$ generate $C(gl_nA)_\mu$ and $H(C(gl_nA))$ as $gl_nk$-modules (see lemmas 2.1, 2.2). Moreover in the particular case of the isotype of the trivial representation, the inclusion (7) is an equality (see 2.1). In Section 2 we construct a chain homomorphism (see Thm 5.1)

$$\phi : M_\mu C(gl_nA) \to \Lambda(C^\Lambda A[-1]) \otimes C^{bar,\mu}A$$

where $C^\Lambda A$ is Connes’ complex for cyclic homology, and $C^{bar,\mu}A$ is a certain complex which depends on $\mu$ and on the bar complex of $A$ (see Thm 5.1 for the full expression of the target of the map (8)). If $\mu$ is the isotype of the trivial representation, $C^{bar,\mu}A$ is just $k[0]$; otherwise it is a complex which is acyclic if $A$ is $H$-unital.
We show (also in Thm 3.1) that for each \( p \) there is an \( n_0 = n_0(p) \) such that for all \( \mu \) and all \( n \geq n_0 \), the map \( \phi_p \) is an isomorphism

\[
\phi_p : M_\mu C_p(\mathfrak{gl}_n, A) \rightarrow (\Lambda(C^\partial A[-1])) \otimes C^{\text{bar}, \mu} A_p
\]

By what we have explained above, this implies that for \( A \) \( H \)-unital, and \( n \geq n_0(p + 1) \), the homology of \( M_\mu C(\mathfrak{gl}_n, A) \), and therefore also that of \( C(\mathfrak{gl}_n, A)_\mu \), vanishes in degree \( p \) for all isotypes but that of the trivial representation, in which case it is isomorphic to \( (\Lambda(HC(A)[-1]))_p \). This proves (4) for algebras.

Hanlon’s results come up in the proof of (8). The proof takes advantage of a certain duality between \( \phi \) and a natural map \( \psi \) in the opposite direction defined by Hanlon in [3, 3.6], which is not a chain map, but is such that \( \psi_p \) is an isomorphism for each \( p \) and each \( n \) sufficiently large. See the proof of (3.3) for details.

The rest of this paper is organized as follows. In Section 2 we apply basic representation theory (essentially the highest weight theorem [4, Thm 14.13]) to prove some elementary facts concerning the decomposition of \( C(\mathfrak{gl}_n, A) \) into isotypic components under the action of \( \mathfrak{gl}_n \). The main result of Section 3 is Theorem 3.1, where the chain map (3) is constructed and the isomorphism (6) proved. In Section 4 we prove the main theorem of the paper (Thm 4.3), which says that (3) is an isomorphism for every \( H \)-unital pro-algebra \( A \).

2. Application of representation theory

Throughout this paper \( k \) will be a fixed field of characteristic zero; tensor products, vector spaces and algebras are over \( k \), as are the various (Lie, cyclic, bar) homologies considered. If \( m \geq 0 \) then by a partition of \( m \) of length \( l \) we understand a nonincreasing sequence \( \alpha_1 \geq \cdots \geq \alpha_l \) of positive integers such that \( \sum \alpha_i = m \). We write \( \emptyset \) for empty partition; it is a partition of 0 of length 0. The set of all partitions of a given \( m \geq 1 \) is denoted \( P(m) \). If \( \alpha \) and \( \beta \) are partitions of \( m \) of lengths \( l_1 \) and \( l_2 \), and \( l_1 + l_2 \leq n \), we put

\[
[\alpha, \beta]_n := (\alpha_1, \ldots, \alpha_{l_1}, 0, \ldots, 0, -\beta_{l_2}, \ldots, -\beta_1) \in \mathbb{Z}^n.
\]

Write \( \mathfrak{gl}_n, k \) for the Lie algebra of \( n \times n \) matrices, and \( U(\mathfrak{gl}_n, k) \) for the universal enveloping algebra. If \( V \) is a \( \mathfrak{gl}_n, k \)-module and \( S \subset V \) a subset, then \( U(\mathfrak{gl}_n) \cdot S \) will denote the \( \mathfrak{gl}_n, k \)-submodule of \( V \) generated by \( S \). For \( \mu \in k^n \), put

\[
\begin{align*}
wx_\mu(V) := & \{ v \in V : e_{pp} \cdot v = \mu pv \ (1 \leq p \leq n) \} \\
M_\mu(V) := & \{ v \in wx_\mu V : e_{ij} \cdot v = 0 \ (1 \leq i < j \leq n) \} \\
V_\mu := & U(\mathfrak{gl}_n, k) \cdot M_\mu(V)
\end{align*}
\]

If \( A \) is any vectorspace and \( n \geq 1 \) we view \( \mathfrak{gl}_n, A = \mathfrak{gl}_n \otimes A \) as the tensor product of the adjoint and the trivial \( \mathfrak{gl}_n, k \)-actions; thus

\[
g \cdot (h \otimes a) = [g, h] \otimes a.
\]

If furthermore \( A \) is equipped with an associative (not necessarily unital) algebra structure, then \( \mathfrak{gl}_n, A \) is a Lie algebra and the induced action on the exterior \( k \)-algebra \( \Lambda^\bullet \mathfrak{gl}_n, A \) compatible with the Chevalley-Eilenberg boundary map \( \partial : \Lambda^\bullet \mathfrak{gl}_n, A \rightarrow \Lambda^{\bullet + 1} \mathfrak{gl}_n, A \), so that the Chevalley-Eilenberg complex \( C(\mathfrak{gl}_n, A) = (\Lambda^\bullet \mathfrak{gl}_n, A, \partial) \) is a complex in the category of \( \mathfrak{gl}_n, k \)-modules. If \( \mathfrak{g} \) is a Lie algebra, we write \( H(\mathfrak{g}) \) for the homology of \( C(\mathfrak{g}) \).

Lemma 2.1.
Lemma 2.2.

i) The Chevalley-Eilenberg complex splits into a direct sum of subcomplexes

\[ C(\mathfrak{gl}_n A) = \bigoplus_{m \geq 0} \bigoplus_{\alpha, \beta \in P(m)} M(\mathfrak{gl}_n A)_{[\alpha, \beta]_n}. \]

ii) \( M_{[\emptyset, \emptyset]}(C(\mathfrak{gl}_n A)) = C(\mathfrak{gl}_n A)_{[\emptyset, \emptyset]_n}. \)

Proof. Let the symmetric group \( \Sigma_m \) act on the \( m \)-tensor power \( T^m(\mathfrak{gl}_n A) \) by permuting the factors. Consider the idempotent element \( \epsilon_m = (1/m!) \sum_{\sigma \in \Sigma_m} sg(\sigma) \sigma \in k[\Sigma_m] \). We have \( \Lambda^m \mathfrak{gl}_n A = \epsilon_m T^m(\mathfrak{gl}_n A) \). Thus, because the actions of \( \Sigma_m \) and \( \mathfrak{gl}_n k \) on \( T^m(\mathfrak{gl}_n A) \) commute, \( \epsilon_m (T^m \mathfrak{gl}_n A) = \Lambda^m \mathfrak{gl}_n A \) for all \( \mu \in k^n \).

We shall show that

\[ T(\mathfrak{gl}_n A) = \bigoplus_{m} \bigoplus_{\alpha, \beta \in P(m)} T(\mathfrak{gl}_n A)_{[\alpha, \beta]_n}. \]

Note that, because by definition, the action of \( \mathfrak{gl}_n k \) on \( T^m A \) is trivial,

\[ T(\mathfrak{gl}_n A)_\mu = \bigoplus_{m} (T^m(\mathfrak{gl}_n k))_\mu \otimes T^m A. \]

for all \( \mu \in k^n \). Hence we may assume \( A = k \). As both sides of (12) commute with extension of the scalar field, a scalar extension–descent argument shows we may further restrict ourselves to the case when \( k \) is algebraically closed, or even more specifically to the case \( k = \mathbb{C} \). Hence we can invoke elementary representation theory of \( \mathfrak{gl}_n k \) for \( k \) algebraically closed—explained in [4, Chap. 14] for the case \( k = \mathbb{C} \)—by which the decomposition (12) is valid with \( T(\mathfrak{gl}_n A) \) replaced by any finite dimensional representation \( V \) of \( \mathfrak{gl}_n k \) on which the identity matrix acts trivially, and is the decomposition of \( V \) into isotypic components. The decomposition of part i) of the lemma is proved; compatibility of the latter with the boundary map follows from the fact that any homomorphism between nonisomorphic irreducible representations is zero (Schur’s lemma). Next note that in part ii) the multiplicative structure of \( A \) does not play any role. Moreover as both terms in the identity to prove commute with scalar extension, and since if \( k \subset K \) are fields then every \( K \)-vectorspace is the scalar extension of a \( k \)-vectorspace, if ii) holds for a particular field of characteristic zero then it holds for all such fields. Hence we may again assume \( k \) is algebraically closed, or, even more specifically, that \( k = \mathbb{C} \). Now ii) is a consequence of the highest weight theorem [4] applied to the trivial representation of \( \mathfrak{gl}_n (\mathbb{C}) \).

\[ \square \]

Lemma 2.2.

\[ H(C(\mathfrak{gl}_n A)_{[\alpha, \beta]_n}) = U(\mathfrak{gl}_n A) \cdot H(M_{[\alpha, \beta]_n} C(\mathfrak{gl}_n A)) \]

Proof. We may assume that \( k \) is algebraically closed. Choose a graded subspace

\[ V \subset M_{[\alpha, \beta]_n} (\ker \partial_{C(\mathfrak{gl}_n A)}) = \ker \partial \cap M_{[\alpha, \beta]_n} (C(\mathfrak{gl}_n A)) \]

such that

\[ V \oplus \partial (M_{[\alpha, \beta]_n} (C(\mathfrak{gl}_n A) [+1]) = M_{[\alpha, \beta]_n} (\ker \partial_{C(\mathfrak{gl}_n A)}) \]

By the highest weight theorem ([4 Prop. 14.13]),

\[ \ker \partial_{C(\mathfrak{gl}_n A)_{[\alpha, \beta]_n}} = U(\mathfrak{gl}_n k) \cdot M_{[\alpha, \beta]_n} (\ker \partial_{C(\mathfrak{gl}_n A)_{[\alpha, \beta]_n}}) = U(\mathfrak{gl}_n k) \cdot V \oplus U(\mathfrak{gl}_n k) \cdot \partial (M_{[\alpha, \beta]_n} (C(\mathfrak{gl}_n A) [+1]) = U(\mathfrak{gl}_n k) \cdot V \oplus \partial ((C(\mathfrak{gl}_n A)_{[\alpha, \beta]_n} [+1]) \]
Thus
\[
H(\mathfrak{gl}_n A) = U(\mathfrak{gl}_n k) \cdot V \\
\cong U(\mathfrak{gl}_n k) \cdot H(M_{[\alpha, \beta]}(C(\mathfrak{gl}_n A))).
\]

In the next lemma and below, we shall write \(V^\vee\) for the (graded) dual of a (graded) vector space \(V\). If \(f: V \to W\) is a linear transformation, then \(f^t: W^\vee \to V^\vee\) is its transpose.

**Lemma 2.3.** Let \(V\) be a finite dimensional vector space. Equip \((\Lambda \mathfrak{gl}_n V)^\vee\) with the following \(\mathfrak{gl}_n k\)-module structure
\[
(g \cdot \chi)(x) := \chi(g^t \cdot x).
\]

Then for \(m \geq 0\) and \(\alpha, \beta \in P(m)\) with \(l(\alpha) + l(\beta) \leq n\), the canonical restriction map
\[
(\Lambda \mathfrak{gl}_n V)^\vee \to (M_{[\alpha, \beta]} \Lambda \mathfrak{gl}_n V)^\vee
\]
induces an isomorphism
\[
(\Lambda \mathfrak{gl}_n V)^\vee \rightarrow (M_{[\alpha, \beta]} \Lambda \mathfrak{gl}_n V)^\vee
\]

**(Proof.)** We may assume that \(k\) is algebraically closed. Then (see \([3]\))
\[
\Lambda \mathfrak{gl}_n V = \bigoplus_{\mu \in k^n} w_\mu(\Lambda \mathfrak{gl}_n V)
\]

It follows from this that for all \(\mu \in k^n\), the restriction map \((\Lambda \mathfrak{gl}_n V)^\vee \to (w_\mu \Lambda \mathfrak{gl}_n V)^\vee\) induces an isomorphism
\[
(\Lambda \mathfrak{gl}_n V)^\vee \rightarrow (w_\mu(\Lambda \mathfrak{gl}_n V))^\vee
\]
of which the inverse is induced by the transpose of the projection of \(\Lambda \mathfrak{gl}_n V\) onto \(w_\mu(\Lambda \mathfrak{gl}_n V)\). In particular this holds for \(\mu = [\alpha, \beta]_n\). Let \(\pi_1\) be the projection of \(\Lambda \mathfrak{gl}_n V\) onto the summand of \([13]\) corresponding to \(\mu = [\alpha, \beta]_n\). Write \(u_n k\) and \(l_n k\) for the Lie subalgebras \(\subset \mathfrak{gl}_n k\) of upper triangular and lower triangular matrices. The map \([15]\) sends \(M_{[\alpha, \beta]}((\Lambda \mathfrak{gl}_n V)^\vee)\) onto
\[
\{ \chi \in (w_\mu(\Lambda \mathfrak{gl}_n V))^\vee : u \cdot \chi = 0 \ \forall u \in u_n k\}
\]

(13)
\[
\{ \chi \in (w_\mu(\Lambda \mathfrak{gl}_n V))^\vee : \chi(lx) = 0 \ \forall l \in l_n k, \ x \in \mathfrak{gl}_n V\}
\]

Consider the decomposition
\[
(14) \quad w_{[\alpha, \beta]}(\Lambda \mathfrak{gl}_n V) = \bigoplus_{m \geq 0} \bigoplus_{\alpha', \beta' \in P(m)} \Lambda \mathfrak{gl}_n V_{[\alpha', \beta']_n} \cap w_{[\alpha, \beta]}(\Lambda \mathfrak{gl}_n V)
\]

Write \(\pi_2\) for the projection of \(w_{[\alpha, \beta]}(\Lambda \mathfrak{gl}_n V)\) onto the summand corresponding to \([\alpha, \beta]_n\). If \((\alpha', \beta') \neq (\alpha, \beta)\) then, by \([4], 14.16\), every element of \(\Lambda \mathfrak{gl}_n V_{[\alpha', \beta']_n} \cap w_{[\alpha, \beta]}(\Lambda \mathfrak{gl}_n V)\) is a linear combination of elements of \(M_{[\alpha', \beta']_n} \Lambda \mathfrak{gl}_n V\) with coefficients which are products in \(U(\mathfrak{gl}_n k)\) of strictly lower triangular matrices. It follows that \(\pi_2^t\) induces an isomorphism between \((M_{[\alpha, \beta]} \Lambda \mathfrak{gl}_n V)^\vee\) and \([13]\). Hence the map of the lemma is an isomorphism with inverse induced by \((\pi_2 \pi_1)^t\).
\[\square\]
3. Stable calculation of $M_{[\alpha,\beta]_m}C_p(\mathfrak{gl}_nA)$.

In preparation for the theorem below, we introduce some notation. If $V$ is a graded vectorspace, each tensor power $T^mV$ will be considered as a $\Sigma_m$-module with the action in which the transposition $(i,j)$ acts as follows

\[(i,j) \cdot v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_m = (-1)^{|v_i||v_j|} v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_m.\]

We regard $\mathbb{Z}_m$ embedded in $\Sigma_m$ through the monomorphism sending the generator 1 to (1, $\ldots$, $m$). If $A$ is an algebra, we write

\[C^\lambda_nA := (T^{n+1}A)_{\mathbb{Z}/n+1}\]

and $C^\lambda A = \bigoplus_n C^\lambda_n A$. The Connes cyclic complex (\ref{bib:5} 2.1.4) of $A$, $(C^\lambda A, b)$, is a chain complex whose underlying graded vectorspace is $C^\lambda A$; its homology is the cyclic homology of $A$, $HC(A)$. Let $T^+A$ be the positive degree part of the tensor algebra. The bar complex of $A$ is a chain complex $C^{\text{bar}}A = (T^+A, b')$ of which the underlying graded vectorspace is $T^+(A[-1])$ (\ref{bib:5} Section 2). Note that in particular (\ref{bib:18}) defines an action of $\Sigma_m$ on $C^{\text{bar}}_m(A)$ for each $m \geq 1$. The homology of $C^{\text{bar}}A$ is called the bar homology of $A$, denoted $H^{\text{bar}}(A)$. Note that, as we do not require our algebras to be unital, any vectorspace $V$ may be regarded as an algebra in our sense with zero multiplication map; hence $C(\mathfrak{gl}_n(V))$, $C^\lambda V$ and $C^{\text{bar}}V$ are defined, and are chain complexes with zero boundary map.

In the next theorem and below, by the canonical inclusion $\mathfrak{gl}_n A \subset \mathfrak{gl}_{n+1} A$ we mean that given by the monomorphism

\[g \mapsto \begin{bmatrix} g & 0 \\ 0 & 0 \end{bmatrix}.\]

Let $\alpha$ be a partition of $m \geq 0$. Recall (\ref{bib:4} pp. 57) that a standard $\alpha$-tableau is a filling of the Young diagram of $\alpha$ with the numbers $1, \ldots, m$ such that all rows and columns are increasing. We write $V^\alpha$ for the Specht $k[\Sigma_m]$-module (\ref{bib:4} 4.47) associated to a partition $\alpha$ of $m$; $V^\alpha$ has a basis with one element for each standard tableau of shape $\alpha$. In what follows, we shall abuse notation and identify each standard tableau with the basis vector it corresponds to. We consider $V^\alpha$ as a graded vectorspace concentrated in degree zero.

If $V$ is a graded vectorspace, the sum of the coinvariant spaces

\[AV := \bigoplus_m (T^m V)_{\Sigma_m}\]

is a graded algebra, the symmetric algebra of the graded vectorspace $V$; it is commutative in the graded sense. Note that if $W$ is a vectorspace, then $AW[0] = SW$, the symmetric algebra and $AW[-1] = AW$, the exterior algebra. This admittedly ambiguous notation is the usual one (see for example (\ref{bib:5})), and should arise no confusion.

If $\phi : V \to W$ is a homomorphism of graded vectorspaces and $p \in \mathbb{Z}$, then by $\phi_p$ we mean the map $\phi|_{V_p} : V_p \to W_p$.

**Theorem 3.1.** Let $A$ be a $k$-algebra, $n \geq 1$, $m \geq 0$, and $\alpha$ and $\beta$ partitions of $m$ with $l(\alpha) + l(\beta) \leq n$. There is a natural homomorphism of chain complexes

\[\phi^\alpha : M_{[\alpha,\beta]_m}(C(\mathfrak{gl}_nA)) \to \Lambda(C^\lambda A[-1]) \otimes (T^m(C^{\text{bar}}A) \otimes V^\alpha \otimes V^\beta)_{\Sigma_m}\]
with the following properties

i) \( \phi_p^n \) is a monomorphism for all \( p \) and an isomorphism for \( n \geq p + l(\alpha) + l(\beta) - m \).
In particular if \( n \geq 2p \) then \( \phi_p^n \) is an isomorphism.

ii) The canonical inclusion \( \mathfrak{gl}_n A \subset \mathfrak{gl}_{n+1} A \) sends \( M_{[\alpha,\beta]}(C(\mathfrak{gl}_n A)) \) into
\( M_{[\alpha,\beta]}(C(\mathfrak{gl}_{n+1} A)) \) and the following diagram commutes

\[
\begin{array}{ccc}
M_{[\alpha,\beta]}(C(\mathfrak{gl}_n A)) & \xrightarrow{\phi^n} & \Lambda(C^\Lambda A[-1]) \\
\downarrow & & \downarrow \\
M_{[\alpha,\beta]}(C(\mathfrak{gl}_{n+1} A)) & \xrightarrow{\phi^{n+1}} & \Lambda(C^\Lambda A[-1])
\end{array}
\]

Proof. We shall define a chain map

\( \phi' : C(\mathfrak{gl}_n A) \to R'(A) := \Lambda(C^\Lambda A[-1]) \otimes T^m(C^b A) \otimes V^\alpha \otimes V^\beta \)

The map of the theorem will be the restriction \( \phi := \phi^n \) of \( \phi' \) to \( M_{[\alpha,\beta]}(\mathfrak{gl}_n A) \). We shall show further below that the image of \( \phi' \) really lies in

\( R(A) := \Lambda(C^\Lambda A[-1]) \otimes (T^m(C^b A) \otimes V^\alpha \otimes V^\beta) \Sigma_m \)

We will find it convenient further on to view \( \Lambda(C^\Lambda A[-1]) \) as a trivial \( \Sigma_m \)-module, so that

\[
R'(A)_\Sigma_m = R(A) \\
R'(A)_\Sigma_m = \Lambda(C^\Lambda A[-1]) \otimes (T^m(C^b A) \otimes V^\alpha \otimes V^\beta) \Sigma_m
\]

To define \( \phi' \) we take advantage of the \( DG \)-coalgebra structure

\( \Delta : C(\mathfrak{gl}_n A) \to C(\mathfrak{gl}_n A) \otimes C(\mathfrak{gl}_n A) = C(\mathfrak{gl}_n k \otimes (A \oplus A)) \)

induced by the diagonal map \( A \to A \oplus A \). We shall define two auxiliary maps

\( \theta : C(\mathfrak{gl}_n A) \to \Lambda(C^\Lambda A[-1]) \) and \( \epsilon : C(\mathfrak{gl}_n A) \to T^m(C^b A) \otimes V^\alpha \otimes V^\beta \), and put

(20) \( \phi' = (\theta \otimes \epsilon) \circ \Delta : C(\mathfrak{gl}_n A) \to R'(A) \)

To define \( \theta \), proceed as follows. First define \( \theta^1 : C(\mathfrak{gl}_n A) \to C^\Lambda A[-1] \) by

\[
\theta^1(\text{g}_1 \wedge \cdots \wedge \text{g}_p) = \sum_{1 \leq i_1, \ldots, i_p \leq n} \sum_{\sigma \in \Sigma_p} \frac{1}{p!} s(\sigma) [(g_{i_1})_{i_1, i_2} \otimes \cdots \otimes (g_{i_p})_{i_p, i_1}]
\]

Here \( [ \ldots ] \) denotes the class in \( C^\Lambda A \); \( \theta^1 \) is a chain homomorphism because it is the composite of the map \( \Sigma (10.2.3.1) \) and the trace map \( \Sigma (1.2) \), both of which are chain homomorphisms. Next note that, because \( C(\mathfrak{gl}_n A) \) is a \( DG \)-coalgebra and \( \partial(C_1(\mathfrak{gl}_n A)) = 0 \), the map \( \tilde{\Delta} : C(\mathfrak{gl}_n A) \to T(C(\mathfrak{gl}_n A)) \),

\[
\tilde{\Delta}_p = \sum_{q=0}^p \Delta_p^{(q)} : C_p(\mathfrak{gl}_n A) \to T(C(\mathfrak{gl}_n A))_p
\]

where \( \Delta^{(q)} \) is the \( q \)-fold comultiplication, is a chain map. Let \( \pi : T(C^\Lambda A) \to \Lambda C^\Lambda A \)
be the projection. Put

\[
\theta = \pi T(\theta^1) \tilde{\Delta}.
\]

Note \( \theta \) is a chain homomorphism, since it is a composite of chain homomorphisms.
In preparation for the definition $\epsilon$ we introduce some notation. Write $M_n A$ for $\mathfrak{gl}_n A$ considered as an associative algebra. If $1 \leq i, j \leq n$ and $g = g_1 \otimes \cdots \otimes g_p \in \mathcal{C}^\text{bar}_p (M_n A)$, put

$$(g)_{ij} := \sum_{1 \leq i_1, \ldots, i_p \leq n} (g_{i_1, i_2} \otimes (g_{i_2, i_3} \otimes \cdots \otimes (g_{i_p, j})))$$

$$(\epsilon_{ij}(g)) := \sum_{\sigma \in \Sigma_p} s g(\sigma) g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(n)})_{ij}$$

One checks that $g \mapsto (g)_{ij}$ is a chain homomorphism $\mathcal{C}^\text{bar} (M_n A) \to \mathcal{C}^\text{bar} A$. As $\epsilon_{ij}$ is the composite of the latter with the antisymmetrization map of $[3, 1.3.4]$, it follows that it is a chain homomorphism. Next if $\gamma$ is a partition of $m$ of length $\leq n$, $z$ a standard tableau of shape $\gamma$, and $1 \leq i \leq m$, then we put $\rho_i (z)$ for the row of $z$ containing $i$. Thus if $\alpha$ and $\beta$ are partitions of $m$ of length $\leq n$ and $x, y$ are standard tableaux of shapes $\alpha$ and $\beta$, then the following is a chain map $\mathcal{C}(\mathfrak{gl}_n A) \to T^m (\mathcal{C}^\text{bar} A)$:

$$\epsilon_{\rho(x), \rho(y)} = (\epsilon_{\rho_1(x), n+1-\rho_1(y)} \otimes \cdots \otimes \epsilon_{\rho_m(x), n+1-\rho_m(y)}) \circ \Delta^{(m)}$$

We define

$$\epsilon := \sum_{x, y} \epsilon_{\rho(x), \rho(y)} \otimes x \otimes y$$

Here the sum runs over all pairs $(x, y)$ of a standard $\alpha$-tableau $x$ and a standard $\beta$-tableau $y$. It is clear from the definitions just given that (20) is a chain homomorphism and that in the case $\alpha = \beta = \emptyset$ it is compatible with the inclusions $\mathfrak{gl}_n A \subset \mathfrak{gl}_{n+1} A$. Thus ii) is proved. We remark that the product structure of $A$ does not play any role in the formulas defining $\phi'$, and that it is equivalent to prove the remaining assertions for $A$ or for the underlying vectorspace of $A$ equipped with the zero multiplication map. Hence in what follows $A$ will be any vectorspace, which we shall implicitly regard as an algebra with trivial multiplication. As any vectorspace is a filtering colimit of finite dimensional ones, and all functors involved preserve such filtering colimits, we may further assume that $A$ is finite dimensional. According to [3] Thms 3.6] there is natural map

$$(21) \quad \psi : (R'(A))_{\Sigma_m} \to M_{(\alpha, \beta)} C(\mathfrak{gl}_n A)$$

such that $\psi_p$ is surjective for all $p$ and an isomorphism for $p \geq l(\alpha) + l(\beta) - m$. We remark that the bound $p \geq l(\alpha) + l(\beta)$ stated in loc. cit. is a tautology; the proof is done for $p \geq l(\alpha) + l(\beta) - m$ (see [3] page 219, line 4]). Hanlon states his theorem for $A$ an algebra over $k = \mathbb{C}$. However $\psi$ (whose construction is recalled below) is defined and natural for algebras over any field $k$, and in particular for vectorspaces $A$ considered as $k$-algebras with zero multiplication map. Moreover, as both its source and its target commute with extension of the scalar field, and since if $k \subset K$ are fields then every $K$-vectorspace is the scalar extension of a $k$-vectorspace, the fact that, for a particular value of $p$, $\psi_p$ is surjective or an isomorphism for vectorspaces over $\mathbb{C}$ implies it is one for vectorspaces over any field $k$ of characteristic zero. Write $\psi'$ for the composite of $\psi$ with the projection $R'(A) \to (R'(A))_{\Sigma_m}$. We shall see that the properties of $\phi$ which remain to be proved follow from Hanlon’s result [3] Thms 3.6] and from a certain duality between $\phi'$ and $\psi'$ which we shall establish. For this we need the explicit description of $\psi'$, which we recall next. The map
\( \psi' : R'(A) \to C(\mathfrak{gl}_n A) \) is a composite of the form
\[
\psi' = \mu \circ (\hat{\theta} \otimes \hat{\epsilon}) : R'(A) \to C(\mathfrak{gl}_n A)
\]
where
\[
\mu : C(\mathfrak{gl}_n A)^{\otimes 2} = C(\mathfrak{gl}_n \otimes (A \oplus A)) \to C(\mathfrak{gl}_n A)
\]
is the multiplication induced by the sum map \( A \oplus A \to A \), and
\[
\hat{\theta} : \Lambda(C^A[-1]) \to C(\mathfrak{gl}_n A)
\]
and
\[
\hat{\epsilon} : T^m(C^{\text{bar}} A) \otimes V^\alpha \otimes V^\beta \to C(\mathfrak{gl}_n A)
\]
shall be defined presently. First we introduce notation. If \( 1 \leq i, j \leq n \) and \( a_1, \ldots, a_p \in A^{\otimes p} \), set
\[
e_{ij}(a_1 \otimes \cdots \otimes a_p) := \sum_{1 \leq l_2, \ldots, l_p \leq n} e_{i,l_2}(a_1) \wedge \cdots \wedge e_{l_p,j}(a_p)
\]
Let \( \hat{\theta} \) be the graded algebra map determined by
\[
\hat{\theta}((a_1 \otimes \cdots \otimes a_p)) = \sum_{1 \leq l_2, \ldots, l_p \leq n} e_{l,l_2}(a_1 \otimes \cdots \otimes a_p)
\]
If \( x \) and \( y \) are standard tableaux of shapes \( \alpha \) and \( \beta \), and \( c_1 \otimes \cdots \otimes c_m \in T^m(C^{\text{bar}} A) \), put
\[
\hat{\epsilon}(c_1 \otimes \cdots \otimes c_m \otimes x \otimes y) := \epsilon_{\rho_1(x), n+1-\rho_1(y)}(c_1) \wedge \cdots \wedge \epsilon_{\rho_m(x), n+1-\rho_m(y)}(c_m)
\]
Thus \( \psi' \) is well-defined and natural. To express the relation between \( \psi' \) and \( \phi' \) we need more notation. Consider the bilinear form
\[
\langle \cdot, \cdot \rangle : \mathfrak{gl}_n k^{\otimes 2} \to k, \quad \langle g, h \rangle := Tr(g^t \cdot h)
\]
Let \( ev : A \to A^{\vee \vee} \) be the canonical evaluation map; as we are assuming \( A \) finite dimensional, it is an isomorphism. Define a vectorspace isomorphism
\[
(22) \quad \nu : \mathfrak{gl}_n A \to \mathfrak{gl}_n (A^{\vee})^\vee = \mathfrak{gl}_n k^{\vee} \otimes A^{\vee \vee}, \quad \nu(g \otimes a) = \langle g, \cdot \rangle \otimes ev(a)
\]
Applying the functor \( \Lambda \) to \( \nu \) and composing with \( \Lambda(\mathfrak{gl}_n (A^{\vee}))^\vee \to (\Lambda(\mathfrak{gl}_n A^{\vee}))^\vee \) we obtain an isomorphism \( C(\mathfrak{gl}_n A) \sim (C(\mathfrak{gl}_n (A^{\vee})))^\vee \) which we shall still call \( \nu \). Similarly \( ev \) induces an isomorphism \( R'(A) \sim (R'(A^{\vee}))^\vee \) which we call \( ev \) as well. The map \( N : C^{\Lambda A} \to C^{\Lambda A} \)
\[
N([a_1 \otimes \cdots \otimes a_n]) = n[a_1 \otimes \cdots \otimes a_n]
\]
extends to an algebra isomorphism \( \Lambda C^{\Lambda A}[-1] \sim \Lambda C^{\Lambda A}[-1] \) which we shall also call \( N \). I claim that the following diagram commutes
\[
(23) \quad \begin{array}{ccc}
C(\mathfrak{gl}_n A) & \xrightarrow{\phi'} & R'(A) \\
\downarrow \nu & & \downarrow ev \\
(C(\mathfrak{gl}_n (A^{\vee})))^\vee & \xrightarrow{(\psi' \circ (N \otimes 1))(\nu)} & (R'(A^{\vee}))^\vee
\end{array}
\]
Here \( \psi' = \psi'_{A^{\vee}} \) is the homomorphism the natural transformation \( \psi' \) assigns to the vectorspace (or algebra with trivial multiplication) \( A^{\vee} \). Assume the claim is true.
Then, because \( \psi' \) descends to \( \Sigma_m \)-coinvariants, it follows that the image of \( \phi' \) is contained in the \( \Sigma_m \)-invariants. One checks that \( \nu \) satisfies the following identity
\[
(24) \quad \nu([g,h])(r) = \nu(h)([g', r]) \quad (g \in \mathfrak{gl}_n, \quad h \in C(\mathfrak{gl}_n, A), \quad r \in C(\mathfrak{gl}_n(A^V)))
\]
It follows from \( \text{[23]} \) that the composite of \( \nu \) with the restriction map \( C(\mathfrak{gl}_n(A^V)))^\vee \to (M_{\alpha, \beta})_{\mathfrak{gl}^n} C(\mathfrak{gl}_n(A^V)))^\vee \) sends \( M_{\alpha, \beta} C(\mathfrak{gl}_n(A^V)) \) isomorphically onto \( (M_{\alpha, \beta})_{\mathfrak{gl}^n} C(\mathfrak{gl}_n(A^V)))^\vee \). Since on the other hand, by \( \text{[3 Thm 3.6]} \), \( \psi'(R'(A)) = M_{\alpha, \beta} C(\mathfrak{gl}_n(A), \mathfrak{gl}_{n+1}) \), and the induced map \( \psi_p : R_p(A)_{\Sigma_m} \to M_{\alpha, \beta} C(\mathfrak{gl}_n(A)) \) is injective for \( n \geq p + l(\alpha) + l(\beta) - m \), it follows that the restriction \( \phi \) of \( \psi' \) to \( M_{\alpha, \beta} C(\mathfrak{gl}_n(A)) \) is an injection, and that \( \phi_p \) is surjective for \( n \geq p + l(\alpha) + l(\beta) - m \). It only remains to prove the claim that \( \text{[25]} \) commutes. To see this one checks first that the following two diagrams commute
\[
(25) \quad C(\mathfrak{gl}_n(A)) \xrightarrow{\theta^\vee} C^\lambda A[-1] \quad C(\mathfrak{gl}_n(A)) \xrightarrow{\epsilon_{ij}} C^\text{bar} A
\]
\[
(\nu \circ \nu)(C(\mathfrak{gl}_n(A^V)))^\vee \xrightarrow{(\tilde{\theta})_{\nu \circ \nu}} (C(\mathfrak{gl}_n(A^V)))^\vee \quad (\nu \circ \nu)(C(\mathfrak{gl}_n(A^V)))^\vee \xrightarrow{\nu \circ \nu} (C(\mathfrak{gl}_n(A^V)))^\vee
\]
Second, one checks that if \( \nu \otimes \nu \) is the composite of \( \nu \otimes \nu \) followed by the natural isomorphism
\[
C(\mathfrak{gl}_n(A^V)))^\vee \otimes C(\mathfrak{gl}_n(A^V)))^\vee \cong (C(\mathfrak{gl}_n(A^V))) \otimes C(\mathfrak{gl}_n(A^V))
\]
then
\[
(26) \quad C(\mathfrak{gl}_n(A)) \xrightarrow{\Delta} C(\mathfrak{gl}_n(A)) \xrightarrow{\nu \otimes \nu} (\nu \otimes \nu)(C(\mathfrak{gl}_n(A^V)))^\vee \quad C(\mathfrak{gl}_n(A)) \xrightarrow{\nu \otimes \nu} (\nu \otimes \nu)(C(\mathfrak{gl}_n(A^V)))^\vee
\]
commutes. Because \( \phi' \) is determined by \( \theta^\lambda \), the \( \epsilon_{ij} \), and the coproduct structure, and \( \psi' \) by \( \tilde{\theta}_{\nu \circ \nu} \), the \( \tilde{\epsilon}_{ij} \), and the product structure, the commutativity of \( \text{[25]} \) follows from that of \( \text{[24]} \) and \( \text{[26]} \).

**Remark 3.2.** The inclusion \( \mathfrak{gl}_n A \subset \mathfrak{gl}_{n+1} A \) sends
\[
M_{\alpha, \beta} C(\mathfrak{gl}_n A) \to w_{\alpha, \beta} C(\mathfrak{gl}_{n+1} A)
\]
As
\[
w_{\alpha, \beta} C(\mathfrak{gl}_{n+1} A) \not\supseteq M_{\alpha, \beta} C(\mathfrak{gl}_{n+1} A),
\]
there is no analogue of part ii) of \( \text{[3.7]} \) for \( \phi' \). 

**Remark 3.3.** The map \( \psi = \psi_A \) of \( \text{[24]} \) is defined for every algebra \( A \), finite dimensional or not. It is not a chain homomorphism. For example if \( (\alpha, \beta) = (0, 0) \) and \( n = 1 \), then, under the identification \( \mathfrak{gl}_1 A = A \), we have
\[
\psi([a_0 \otimes a_1 \otimes a_2]) = \psi(a_0a_1 \otimes a_2 - a_0 \otimes a_1a_2 + a_2a_0 \otimes a_1)
\]
\[
= (a_0a_1) \otimes a_2 - a_0 \otimes a_1a_2 + a_2a_0 \otimes a_1
\]
\[
\neq [a_0, a_1] \otimes a_2 - [a_0, a_2] \otimes a_1 + [a_1, a_2] \otimes a_0
\]
\[
= \partial(a_0 \otimes a_1 \otimes a_2) = \partial \psi(a_0 \otimes a_1 \otimes a_2)
\]
On the other hand, if $A$ is finite dimensional, then a choice of basis for $A$ naturally gives rise to isomorphisms of graded vectorspaces

\[(27) \quad C(gl_n A) \sim (C(gl_n A))^{\vee}, \quad C^A \sim (C^A)^{\vee} \text{ and } C^{bar} A \sim (C^{bar} A)^{\vee}.
\]

Using these isomorphisms one can view $C(gl_n A)$, $C^A$, $C^{bar} A$ and $R'(A)_{\Sigma_n}$ as cochain complexes; Hanlon shows [3, 3.11] that $\psi$ is a cochain homomorphism in this sense. In the infinite dimensional case we no longer have the isomorphisms (27). However the map $\phi : C(gl_n A) \to R(A)$ of (29) is always a chain homomorphism (independently of the dimension of $A$) and therefore its transpose $\phi^t : R(A)^{\vee} \to C(gl_n A)^{\vee}$ is always a cochain homomorphism.

4. Main theorem

**Notation 4.1.** If $C$ is a category, we write pro-$C$ for the category of all inverse systems

\[\{\sigma_{n+1} : C_{n+1} \to C_n : n \in \mathbb{Z}_{\geq 1}\}\]

of objects of $C$. The set of homomorphisms of two pro-objects $C$ and $D$ in pro-$C$ is by definition

\[\text{hom}_{pro-C}(C, D) = \lim_{n} \text{colim}_{m} \text{hom}_{C}(C_m, D_n)\]

Thus a map $C \to D \in \text{pro-C}$ is an equivalence class of maps of inverse systems

\[(28) \quad f : \{C_{m(n)}\}_n \to \{D_n\}_n\]

where $m : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ satisfies $m(n) \geq n$ for all $n$. In the particular case when $m$ is the identity we say that $f$ (and also its equivalence class) is a level map. We remark that mapping each object $C \in C$ to the constant pro-object gives a fully faithful functor $C \to \text{pro-C}$. Because of this, we shall identify the objects of $C$ with the constant pro-objects. Assume $C$ has a zero object. Then we say that a pro-object $X$ is zero if it is isomorphic to 0 in pro-$C$. This means that

\[(29) \quad \forall n \; \exists m \geq n \text{ such that } \sigma_{n+1} \circ \ldots \circ \sigma_m = 0.\]

If $C$ happens to be abelian, pro-$C$ is abelian as well, and if (28) represents a homomorphism $[f] : C \to D \in \text{pro-C}$ then $\{\ker C_{m(n)} \to D_n\}_n$ is a kernel and $\{\text{coker } C_{m(n)} \to D_n\}_n$ a cokernel for $[f]$. In particular $[f]$ is an isomorphism if and only if both $\{\ker C_{m(n)} \to D_n\}_n$ and $\{\text{coker } C_{m(n)} \to D_n\}_n$ are zero in the sense of (29). If $F : C \to D$ is a functor and $C = \{C_n\}_n \in \text{pro-C}$, we write $F(C)$ for the pro-object $\{F(C_n)\}_n$. In what follows we shall consider pro-objects in the categories of $k$-algebras and vectorspaces. We call a pro-algebra $A = \{A_n\}_n H$-unital if for each $r$ the pro-vectorspace $H^{bar}_r(A) = 0$. According to (29), this means that for all $r, n \geq 1$ there is an $m = m(n, r)$ such that the map

\[H^{bar}_r(A_m) \to H^{bar}_r(A_n)\]

is zero.

If $A$ is a $k$-algebra, we write $gl A$ for the Lie algebra of all matrices of finite size

\[gl A = \bigcup_{n \geq 1} gl_n A.\]

**Theorem 4.2.** Let $A = \{A_n\}_n$ be a pro-$k$-algebra. Assume that $A$ is $H$-unital. Then for each $r \geq 1$ there is a natural isomorphism of pro-vectorspaces

\[(30) \quad \varphi_r : H_r(gl A) \sim (\Lambda(HC(A)[-1]))_r.\]
Proof. Consider the composite
\[ \varphi^n_r : H_r(\mathfrak{gl}_n A_1) \to H_r(M_{[0,0]}(\mathfrak{gl}_n A_1)) \to (\Lambda(H\mathcal{C}(A_1)[-1]))_r. \]
By part ii) of 3.1, the \( \varphi^n_r \) pass to the limit with respect to \( n \), so that we do have a levelwise map from the left to the right hand side of (30). We shall show that \( \varphi^n_r \) is an isomorphism for \( n \geq 1 + 2r \). Note that if \( \alpha, \beta \in P(m) \) and \( m > r \), then by part i) of 3.1 we have \( M_{[\alpha,\beta]} C_r(\mathfrak{gl}_n A_1) = 0 \) for all \( l \). Thus by 2.1 and 2.2 we have a levelwise decomposition
\[ (31) \quad H_r(\mathfrak{gl}_n A) = H_r(M_{[0,0]}(\mathfrak{gl}_n A)) \oplus \bigoplus_{1 \leq m \leq r} \bigoplus_{\alpha, \beta \in P(m)} U(\mathfrak{gl}_n k) \cdot H_r(M_{[\alpha,\beta]}(C(\mathfrak{gl}_n A))). \]
If \( n > r \), then by 3.1 \( \varphi^n_r \) is onto and its kernel is the term in the second line of (31). If further \( n \geq 2r + 1 \), it follows from 3.1 and our hypothesis that \( H^b_{p\text{bar}}(A) = 0 \) for all \( p \), that each of the finitely many summands in the second line of (31) is the zero pro-vectorspace. This proves that \( \ker \varphi^n_r = 0 \), whence \( \varphi^n_r \) is an isomorphism as wanted. \( \square \)

Example 4.3. Let \( A \) be an algebra; the inclusions \( \{ A^{n+1} \subset A^n \}_n \) define a pro-algebra \( A^\infty := \{ A^n \}_n \). If \( A \) is isomorphic to an ideal of a tensor algebra \( TV \) (or more generally of any quasi-free algebra) then \( A^\infty \) is \( H \)-unital (see [1, Section 4]). Hence theorem 4.2 applies in this case.

References

[1] G. Cortiñas, The obstruction to excision in K-theory and in cyclic homology. Preprint. Available at [http://arxiv.org/abs/math/0111096](http://arxiv.org/abs/math/0111096)
[2] B. Feigin, B. Tsygan. Additive K-theory in K-theory, arithmetic and geometry. Springer Lect. Notes in Math. 1289 (1987) 97–209.
[3] P. Hanlon, On the complete \( GL(n, \mathbb{C}) \)-decomposition of the stable cohomology of \( \mathfrak{gl}_n \). Trans. Amer. Math. Soc. 308 (1988) 209–225.
[4] W. Fulton, J. Harris, Representation Theory, a first course, Graduate Texts in Mathematics 129, Springer Verlag, New York, 1991.
[5] J. L. Loday, Cyclic homology, 1st ed. Grund. math. Wiss. 301. Springer-Verlag Berlin, Heidelberg 1998.
[6] J.J. Loday, D. Quillen, Cyclic homology and the Lie algebra homology of matrices, Comment. Math. Helvetici 59 (1984) 565–591.
[7] A. Suslin, M. Wodzicki, Excision in algebraic K-theory, Ann. of Math. 136 (1992) 51-122.
[8] B. Tsygan. The homology of matrix Lie algebras over rings and the Hochschild homology (in russian). Uspekhi Mat. Nauk 38 (1983) 217–218.
[9] M. Wodzicki, Excision in cyclic homology and in rational algebraic K-theory, Ann. of Math. 129 (1989) 591-639.

Departamento de Álgebra, Geometría y Topología, Universidad de Valladolid
E-mail address: gcorti@agt.uva.es