Chiral Gauge Theories in the Overlap Formalism

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The overlap formula for the chiral determinant is presented and the realization of gauge anomalies and gauge field topology in this context is discussed. The ability of the overlap formalism to deal with supersymmetric theories and Majorana-Weyl fermions is outlined. Two applications of the overlap formalism are discussed in some detail. One application is the computation of a fermion number violating process in a two dimensional U(1) chiral gauge theory. The second application is a measurement of the probability distribution of the index of the chiral Dirac operator in four dimensional pure SU(2) lattice gauge theory.

1 Introduction

Significant progress has been made in the lattice formalism of chiral gauge theories using the overlap formalism. Using this formalism, a fermion number violating process has been computed on the lattice in a two dimensional abelian chiral model thereby demonstrating the validity of the overlap formula. The overlap formalism was inspired by two independent articles. In the first article, a $d+1$ dimensional Dirac fermion with a mass term incorporating a defect is used to realize a localized chiral fermion in $d$ dimensions. This idea of localization orginally appears in a paper by Callan and Harvey and was later used for realizing chiral fermions in the context of condensed matter physics. The second article provides a Pauli-Villars regularization for the fermion determinant in perturbation theory when the fermion is in the sixteen dimensional Weyl representation of the SO(10) gauge group. There is a need for an infinite number of Pauli-Villars fields to regulate the theory, reminiscent of the need for one extra dimension in the first article to realize a localized chiral fermion.

The need to have an infinite number of degrees of freedom per space time point (one extra dimension in the context of the first article and an infinite number of Pauli-Villars fields in the context of the second article) is not surprising for two closely related reasons:

- Existence of gauge anomalies: An extra infinity is needed to realize the gauge anomaly present when the fermion is in an anomalous representation of the gauge group. Without this extra infinity, the anomaly is forced to come out as a consequence of the ultra-violet regularization of the theory even though the anomaly is independent of the regularization scheme.

- Fermion number violation: In a non-perturbative regularization such as the lattice formalism, the number of space time points are finite and an infinite number of fields is needed to realize the non-trivial index of the chiral Dirac
operator due to the Atiyah-Singer index theorem. The index is a property of the Dirac operator per gauge field configuration and therefore has to be realized on a finite lattice.

The central problem in the non-perturbative formalism of chiral gauge theories is to write down a formula for the fermionic determinant where the fermion is in some complex representation of the gauge group. Let \( C(U) \) denote the chiral Dirac operator coupled to a background gauge field \( U \) in some complex representation. Clearly \( C(U) \) does not have an eigenvalue problem since it is a map between \((0, \frac{1}{2})\) and \((\frac{1}{2}, 0)\) spinors under the Lorentz group. Therefore “\( \text{det} C(U) \)” cannot be represented as a product of eigenvalues and one cannot make use of the zeta function regulator. The existence of gauge anomalies is related to this particular nature of the operator. Because \( C(U) \) is a map between two spaces, the operator can have a non-trivial index, i.e., the infinite matrix \( C(U) \) is sometimes square and sometimes rectangular and the difference between the infinite number of columns and rows of this matrix is a finite number that is dictated by the topology of the gauge field background. This interesting property is responsible for fermion number violation. Any formula for the chiral determinant has to respect the above mentioned properties.

2 Overlap formalism

In this section, the overlap formalism will be presented as a representation of the chiral determinant in the continuum. Then a straightforward lattice regularization of the overlap will be described. Gauge anomaly and gauge field topology in the context of the overlap formalism will then follow. Extention of the overlap formalism to the most basic fermionic object in \( 2 + 8k \) dimensions, namely Majorana-Weyl fermions, will also be presented with the aim of extending the overlap formalism to include supersymmetric theories.

2.1 Chiral determinant as an overlap of two vacua

In Euclidean space, “\( \text{det} C(U) \)” is a complex functional of \( U \) and its phase in general is not gauge invariant and carries the information about the anomaly associated with the particular complex representation of the gauge group. Any formula for “\( \text{det} C(U) \)” should reproduce this anomaly. In addition the “\( \text{det} C(U) \)” should be zero for a large class of gauge fields where the matrix \( C(U) \) is not square. Further the formula for left and right handed chiral fermions in the same representation should be related by complex conjugation. The formula should also be in accordance with the standard discrete symmetries of parity and charge conjugation.

The overlap formula is one that satisfies all the above requirements. In the overlap formalism, “\( \text{det} C(U) \)” is represented as an overlap of two many body states composed of fermions. This representation is arrived at as follows. Let \( v_j \) be a basis for the \((0, \frac{1}{2})\) spinors and let \( u_i \) be a basis for the \((\frac{1}{2}, 0)\) spinors. Formally, “\( \text{det} C = \text{det}_{ij} (u_i|C|v_j) = \text{det}_{ij} (u_i|w_j) \)” with \( |w_j \rangle = C |v_j \rangle \). The numbers of basis vectors \( v_j \) and \( u_i \) are the dimensions of the two spaces that is mapped by \( C \). Each vector has many components and the number of components is dependent on our
specific choice of embedding. Specifically, \( u_i | w_j \rangle = \sum_{\alpha} u_{i\alpha} w_{j\alpha} \) where \( \alpha \) labels the components. The number of components could very well be bigger than the dimension of the two spaces mapped by \( C \) in our choice of embedding and this will indeed be the case. The first step in deriving the overlap formula is to make the following operator identification. With every vector \( u_i \), we associate a single particle fermion creation operator \( \hat{u}_i = u_{i\alpha} a_{\alpha}^\dagger \). \( a_{\alpha}^\dagger \) are cannonical fermion creation operators that obey the following standard commutation relations: \( \{ a_{\alpha}^\dagger, a_{\beta}^\dagger \} = 0; \{ a_{\alpha}, a_{\beta} \} = 0; \{ a_{\alpha}, a_{\beta}^\dagger \} = \delta_{\alpha\beta} \). Having made this operator identification, we now define two many body states as \( |+\rangle = \prod_j \hat{w}_j |0\rangle \) and \( |-\rangle = \prod_i \hat{u}_i |0\rangle \) where \( |0\rangle \) is the state that is annihilated by all the destruction operators \( a_{\alpha} \). Now it is straightforward algebra to show that \( \langle -|+ \rangle = \det_{ij} \langle u_i|w_j \rangle \) and this is the overlap formula.

A few remarks are in order at this time. The number of particles making up the two many body states are equal to the dimensions of the two spaces mapped by \( C \). These two dimensions need not be equal for the formula to hold. If they are not equal the number of bodies in the two many body states are not the same and overlap is zero as expected. The number of fermion creation operators need not be equal to either of the two dimensions. It is a consequence of the specific embedding and is equal to or larger than the larger of the two spaces.

Having represented the chiral determinant as an overlap of two many body states, a procedure to obtain the many body states as ground states of two auxiliary hamiltonians is now described. Let

\[ \mathcal{H}^\pm = a^\dagger \mathcal{H}^\pm a \]  

(1)

be two many body Hamiltonians each describing a set of identical non-interacting fermions. The state \( |-\rangle = \prod_i \hat{u}_i |0\rangle \) is simply a choice of the basis for the \((\frac{1}{2}, 0)\) spinors and this can be obtained as a ground state of \( \mathcal{H}^- \) by choosing

\[ \mathcal{H}^- = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]  

(2)

To obtain \( |+\rangle = \prod_j \hat{w}_j |0\rangle = \prod_j C \hat{v}_j |0\rangle \) as a ground state of \( \mathcal{H}^+ \), a choice for the other single particle Hamiltonian is

\[ \mathcal{H}^+ = \begin{pmatrix} m & C \\ C^\dagger & -m \end{pmatrix} \]  

(3)

with \( m > 0 \). Then,

\[ \det C \leftrightarrow \langle -|+ \rangle \]  

(4)

The \( \leftrightarrow \) above is indicative of several points. The above formula is only formal and will become the definition of the chiral determinant when the right hand side is properly regulated. The above formula is valid in the limit \( m \to \infty \) and the parameter \( m \) is to be thought of as an ultraviolet regulator. Finally there is a irrelevant gauge field independent normalization that depends on \( m \) in the formula and as such the above formula is only valid for ratios of determinants in different gauge fields backgrounds.
Eq. 4 can be shown by first noting that the ground states are obtained by filling the single particle negative energy eigenstates given by

$$H^\pm \psi^\pm_i = \lambda^\pm_i \psi^\pm_i; \quad \lambda^\pm_i < 0$$  \hspace{1cm} (5)

Clearly, $\lambda^i_- = -1$ for all $i$ and the eigenvectors with these eigenvalues are

$$\psi^i_- = \begin{pmatrix} u_i \\ 0 \end{pmatrix}$$  \hspace{1cm} (6)

with the set $\{u_i\}$ forming a unitary matrix. To get $\lambda^+_j$ one needs to solve

$$C^\dagger C v_j = (\left(\lambda^+_j\right)^2 - m^2) v_j$$  \hspace{1cm} (7)

In terms of these eigenfunctions, $\psi^+_j$ are given by

$$\psi^+_j = \frac{1}{\sqrt{N_{jj}}} \left( \frac{1}{\sqrt{C^\dagger C + m^2 + m}} C v_j \right); \quad N_{jj} = \left\langle v_j \left[ \frac{C^\dagger C}{\sqrt{C^\dagger C + m^2 + m}} + 1 \right] v_k \right\rangle$$  \hspace{1cm} (8)

with $N$ being a diagonal matrix. The overlap,

$$\langle -|+ \rangle = \det(\langle \psi^i_- | \psi^+_j \rangle) = \frac{1}{\sqrt{\det N}} \frac{\det(u_i)}{\sqrt{C^\dagger C + m^2 + m}} C |v_j\rangle$$  \hspace{1cm} (9)

where the first equality in the above equation is an identity resulting from the canonical commutation relations of the fermion creation and destruction operators. Therefore, we have a formula for the chiral determinant as an overlap of two vacua in Eq. 4. One of the vacua, namely $|\rangle$ is simply a fixed reference state and the other one carries all the information about the gauge field background. The formula uses the embedding of the chiral Dirac operator in a vector like operator, namely $H^+$, and achieves the desired goal. Regularization of the right hand side of Eq. 4 amounts to a regularization of $H^+$ which can be done in a straightforward manner since it is vector like.

### 2.2 Lattice regularization of the overlap

The overlap formula in Eq. 4 is a proper definition for the chiral determinant only after regularization and a definition of the phase of the two many body states. The single particle Hamiltonians in Eq. 3 is equal to $\gamma_5 (D + m)$ where $D = \gamma_\mu (\partial_\mu + iA_\mu)$ is the massless Dirac operator in Euclidean space. On the lattice, the massless Dirac operator is written as

$$D(x\alpha i; y\beta j) = \sum_\mu \gamma_\mu^{\alpha\beta} \frac{1}{2} \left[ \delta_{y,x+\hat{\mu}} U_\mu^i (x) - \delta_{x,y+\hat{\mu}} (U_\mu^i(y))^{ij} \right]$$  \hspace{1cm} (10)

where $x, y$ are sites on the lattice, $\alpha, \beta$ are spin indices, $i, j$ are color indices and $U_\mu(x)$ is the parallel transporter along the direction $\mu$ from $x$ to $x + \hat{\mu}$. In the free theory, $U = 1$, this operator is diagonal in momentum space and is given by
As is well known, this operator has many unwanted particles arising from the zeros at the edge of the Brillouin zone $p_\mu = \pi$ and these unwanted particles have to be removed. To do this, note that the overlap yields the determinant of $C$ only if the Hamiltonian in Eq. 3 has $m > 0$. It is clear that the overlap will be close to unity if $m$ was a large negative number since $H^+ \rightarrow H^-$. The unwanted particles can be removed by making the mass term momentum dependent in such a way that it is positive when $p_\mu = 0$ but is negative when one or many of the $p_\mu = \pi$. This is achieved by replacing the mass term by the usual Wilson term, $m \rightarrow (m - W)(x_\alpha i, y_\beta j)$

$$m \rightarrow (m - W)(x_\alpha i, y_\beta j) = m\delta_{\alpha\beta}\delta_{ij} - \frac{1}{2}m \sum_{\mu} \left[ 2\delta_{xy}\delta_{ij} - \delta_{y,x+\hat{\mu}}U^{ij}(x) - \delta_{x,y+\hat{\mu}}(U^{\dagger}(y))^{ij} \right]$$

(11)

In the free case, $U = 1$, the mass term is replaced by a momentum dependent mass term given by $m - 2 \sum_{\mu} \sin^2 \frac{p_\mu}{2}$. If we pick $m$ in the range $0 < m < 2$, it is clear that the momentum dependent mass term is positive when $p_\mu = 0$ but is negative when one or more $p_\mu = \pi$. Hence the regulated Hamiltonian on the lattice is

$$H^+ = \gamma_5(D - W + m); \quad 0 < m < 2$$

(12)

with $D$ and $W$ given by Eq. [10] and Eq. [11] respectively. In taking the continuum limit, $m$ should be kept fixed at some value in the range $0 < m < 2$ so that it goes to infinity in physical units as the lattice spacing is taken to zero. All values of $m$ in the range $0 < m < 2$ are expected to yield the same continuum theory unless the theory being defined has some marginally relevant parameters. The cutoff effects as one goes to the continuum limit will depend on the actual value of $m$.

Following arguments similar to the one for the chiral determinant above, it is easy to show that the generating functional for fermions in an arbitrary gauge background is given by

$$Z(\eta, \tilde{\eta}) = \langle -|e^{\eta a^\dagger + \tilde{\eta} a}|+\rangle$$

(13)

Therefore insertion of $a$ and $a^\dagger$ operators at appropriate places inside the overlap result in correlation functions for fermions in a gauge field background.

### 2.3 Phase of the many body states and the gauge anomaly

The regulated single particle Hamiltonian in Eq. [12] is a finite matrix on a finite lattice and therefore $|+\rangle$ is a many body state with a finite number of particles that depends upon the background gauge field. To properly define the overlap it is necessary to define the phase of this state. $|-\rangle$ is a reference state that does not depend on the gauge field and its phase can be fixed once and for all by choosing the set $\{u_i\}$ to be the identity matrix. A phase choice for $|+\rangle$ that is consistent with perturbation theory is the Wigner-Brillouin phase choice. Under this phase choice, the overlap $W_B^U(|+\rangle U)_{WB}$ is forced to be a real and positive quantity. Here $|+\rangle_{WB}$ is the free many body state whose phase is assumed fixed. With respect to this state, the phase of $|+\rangle_{WB}$ for all $U$ is defined by the Wigner-Brillouin convention. The subscript $U$ shows that the state depends on the background gauge field and the
superscript WB shows that it obeys the Wigner-Brillouin convention. One can prove that this phase choice results in the correct transformation properties of the chiral determinant under parity charge conjugation and global gauge transformations.

Gauge anomalies now arise for a simple reason. Let $| + \rangle^\text{WB}_U$ and $| + \rangle^\text{WB}_{Ug}$ be many body states with background gauge fields being $U$ and $U^g$ respectively where $U^g$ is a gauge transformation of $U$. Since the Hamiltonian in Eq. 12 undergoes a unitary rotation under this gauge tranformation, the two states are simply related by a unitary rotation. That is

$$| + \rangle^\text{WB}_{Ug} = \mathcal{G} | + \rangle^\text{WB}_U e^{i\phi(U; g)}$$  \hfill (14)

where $\mathcal{G}$ is the unitary operator and the phase on the right-hand side is chosen so that $| + \rangle^\text{WB}$ indeed obeys the Wigner-Brillouin phase convention. Now the overlap

$$\langle - | + \rangle^\text{WB}_U = \langle - | + \rangle^\text{WB}_U e^{i\phi(U; g)} \prod_x g(x)$$ \hfill (15)

If $g(x)$ is not a global transformation, then the phase $e^{i\phi(U; g)} \prod_x g(x)$ is not unity showing that the chiral determinant is not gauge invariant and the presence of a gauge anomaly. Basically what has happened is the follows. The overlap with the Wigner-Brillouin phase convention defines a proper functional of the gauge field background. The presence of a gauge anomaly is the inability to make a proper functional that is also gauge invariant. Anomaly and other perturbative quantities have been succesfully computed in two dimensions and four dimensions using the overlap.

2.4 Overlap and gauge field topology

The specific nature of the operator $C$ led us to a representation of its determinant involving two many body states. As such we had to define the phase of these states which resulted in a natural explanation of the gauge anomaly. It is well known that the integration of the anomaly equation results in the phenomenon of fermion number violation if the gauge field carries non-trivial topology. Therefore it is natural to expect that topology also arises from the fact that we are dealing with two many body states. This is indeed the case. On a finite lattice, the single particle Hamiltonians are finite matrices of size $2K \times 2K$ with $K = V \times N \times S$ where $V$ is the volume of the lattice, $N$ is the size of the paticular representation of the gauge group and $S$ is the number of components of a Weyl spinor (one in two dimensions and two in four dimensions). Then $| - \rangle$ is made up of $K$ particles. If $| + \rangle^\text{WB}_U$ is also made up of $K$ particles, then the overlap is not zero in the generic case. If the background gauge field is such that there are only $K - Q$ negative energy states for $H^+$ then the overlap is zero. Any small perturbation of the gauge field will not alter this situation. Further the overlap $\langle - | a_{i_1}^\dagger \cdots a_{i_Q}^\dagger + \rangle^\text{WB}_U$ will not be zero in the generic case if the fermion is in the fundamental representation of the gauge group showing that there is a violation of fermion number by $Q$ units. Clearly this results in a classification of gauge fields on a finite lattice into topological classes labelled by $Q$. The topological nature of the gauge fields using the overlap has been investigated both in two and four dimensions.
2.5 Supersymmetry and Majorana-Weyl fermions

The overlap formalism enables one to write down a SU(N) gauge theory on the lattice coupled to a single adjoint multiplet of left-handed Weyl fermions. The continuum limit of this theory should be supersymmetric and no fine tuning is needed to achieve this limit making it more attractive than a previous approach using Wilson fermions on the lattice. Using the overlap, one could go a step further and formulate a gauge theory coupled to Majorana-Weyl fermions, the most basic fermion in 2 + 8k dimensions. One application of this would be the ten dimensional N=1 supersymmetric Yang Mills theory. Another application would be the investigation of possible fermion bilinear condensates in two dimensional non-abelian gauge theories coupled to Majorana-Weyl fermions. The underlying structure seems to a mod(2) index associated with the Majorana-Weyl operator.

The overlap formalism for Majorana-Weyl fermions can be obtained using the factorization of the overlap for Weyl fermions. In the chiral basis,

\[ H^+ = \begin{pmatrix} a_1^\dagger & a_2^\dagger \end{pmatrix} \begin{pmatrix} m - W & C \\ C^\dagger & W - m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \]  

(16)

with

\[ C(x\alpha; y\beta) = \sum_\mu \sigma_\mu^{\alpha\beta} \frac{1}{2} [\delta_{y,x+\hat{\mu}} U^{ij}_\mu(x) - \delta_{x,y+\hat{\mu}} (U^{ij}_\mu(y))] \]  

(17)

If the Weyl fermion is in a real representation of the gauge group in 2 + 8k dimensions, then the Weyl operator is skew-symmetric, i.e., C^t = -C. Further W^t = W. Using the above relations, one can show that under the canonical transformation, \( a_1 = \frac{\xi - i\eta}{\sqrt{2}} \) and \( a_2 = \frac{\xi^\dagger - i\eta^\dagger}{\sqrt{2}} \),

\[ H^+ = \frac{1}{2} (\xi^\dagger \xi) \begin{pmatrix} m - W & C \\ C^\dagger & W - m \end{pmatrix} \begin{pmatrix} \xi \\ \xi^\dagger \end{pmatrix} + \frac{1}{2} (\eta^\dagger \eta) \begin{pmatrix} m - W & C \\ C^\dagger & W - m \end{pmatrix} \begin{pmatrix} \eta \\ \eta^\dagger \end{pmatrix} \]  

(18)

Therefore, the many body Hamiltonian factorizes into two identical pieces each one corresponding to a single Majorana-Weyl fermion.

A mod(2) index naturally arises in the context of Majorana-Weyl fermion operator. Clearly, the many body Hamiltonian for a single Majorana-Weyl fermion in Eq. 18 does not conserve particle number. But it does conserve particle number, modulo two. The ground state \(|+\rangle\) is either a superposition of multiparticle states all composed of even number of particles or all composed of odd number of particles. \(|-\rangle\) is independent of the gauge field background and has an even number of particles. If \(|+\rangle\) is made up of states containing an odd number of particles and the overlap is zero. Such a theory only makes sense if one couples an even number of Majoran-Weyl fermions to the gauge field. If one couples two Majorana-Weyl fermions, there is a potential for a fermion bilinear condensate if the background gauge field posses non-trivial mod(2) index.
3 Monte Carlo evaluation of a fermion number violating observable

The overlap formalism described in the previous section makes it possible to deal with chiral gauge theories on the lattice. The overlap clearly passes all the necessary tests needed of a correct formalism of the chiral determinant in a fixed gauge field background. Physics under both perturbative and non-perturbative gauge fields are correctly reproduced by the overlap in two and four dimensions\cite{1}. To test the overlap formalism including the dynamics of the gauge field is a non-trivial matter for two reasons:

- Feasibility to compute something non-perturbative in some model using the overlap on the lattice using present day computers.
- Existence of a chiral model where some non-perturbative results are known by some other methods. Then one can see if the overlap reproduces these results.

The first reason is quite non-trivial due to the following points:

- Anomaly cancellation between different representations of Weyl fermions occurs only in the continuum. On the lattice with a finite lattice spacing, anomaly cancellation occurs only up to lattice spacing effects. That is to say the fermionic determinant on the lattice will have gauge violations which only vanish as one takes the lattice spacing to zero. The existence of gauge violations on the lattice implies that unphysical gauge degrees of freedom affect the dynamics and if the overlap formalism has to reproduce a chiral gauge theory properly on the lattice including the dynamics of gauge fields, then it has to be shown that the unphysical gauge degrees of freedom do not affect the physics. Lattice gauge invariant theories have been shown to be robust under not too large perturbations by gauge breaking terms\cite{13}. In the overlap formalism the gauge breaking appears only in the phase of the fermionic determinant and further it is small when the theory is anomaly free. The issue then is whether the gauge breaking is quantitatively small for the overlap formalism to result in the correct chiral gauge theory in the continuum limit.

- Difficulty in simulating theories with a complex action. The fermionic determinant in a chiral gauge theory is usually complex and therefore standard Montecarlo simulations are not applicable.

- Computation of the chiral determinant using the overlap involves the diagonalization of $H^+$ which is a large matrix on any reasonable lattice.

With all the above issues in mind, a particular chiral U(1) model in two dimensions has proven to be a useful first testing ground for the overlap. This model can be solved in the continuum following techniques similar to that of the massless Schwinger model. There is a fermion number violating process in this model. With a suitable choice of fermionic boundary conditions on the torus the chiral determinant can be made real in the continuum. In the first subsection, the model is defined and the results from the continuum computation of the fermion number violating process is presented. The computational details involved on the lattice
is then discussed. Results from the numerical simulation of this model using the overlap on the lattice shows that the correct continuum results are reproduced.

3.1 11112 model on a continuum torus

The 11112 model is made up of a U(1) gauge field on a torus coupled to four left handed Weyl fermions of charge $q = 1$ and one left handed Weyl fermion of charge $q = 2$. The action is:

$$S = \frac{1}{4e_0^2} \int d^2 x F_{\mu\nu}^2 - \sum_{f=1}^{4} \int d^2 x \bar{\chi}_f \sigma_\mu (\partial_\mu + i A_\mu) \chi_f - \int d^2 x \bar{\psi} \sigma_\mu (\partial_\mu + 2i A_\mu) \psi$$

where $\sigma_1 = 1$, $\sigma_2 = i$ and $\mu = 1, 2$. The $U(1)$ gauge symmetry is anomaly free by $2^2 = 1^2 + 1^2 + 1^2 + 1^2$. The boundary conditions are:

$$\chi_f(x + l_\mu \hat{\mu}) = e^{2\pi i b^\mu_f x \cdot x} \chi_f(x)$$
$$\psi(x + l_\mu \hat{\mu}) = \psi(x)$$
$$F_{\gamma\nu}(x + l_\mu \hat{\mu}) = F_{\gamma\nu}(x)$$

for $\mu = 1, 2$. $\hat{\mu}$ is a unit vector in the $\mu$ direction. The $\bar{\chi}_f$ and $\bar{\psi}$ fields obey complex conjugate boundary conditions. The $b^\mu_f$ are given by:

$$b^1_1 = 0; \quad b^2_1 = 0; \quad b^3_1 = \frac{1}{2}; \quad b^4_1 = 0; \quad b^2_2 = \frac{1}{2}; \quad b^3_2 = 0; \quad b^4_2 = \frac{1}{2}.$$  

The chiral determinant is real and positive for the above choice of boundary conditions. The model can be solved in the continuum torus following closely the technique for solving the massless Schwinger model on the torus.

The massless sector consists of six left moving Majorana Weyl fermions forming a sextet under the global $SU(4)$ acting on the $f$-index of the $\chi_f$’s. These particles are noninteracting. One can choose interpolating fields for these particles which are neutral objects local in the original fields:

$$\rho_{f_1 f_2} = -\rho_{f_2 f_1} = \frac{\pi^2 e^{-\gamma}}{e_0} [\chi_{f_1} \chi_{f_2} \bar{\psi} - \frac{1}{2} \epsilon_{f_1 f_2 f_3 f_4} \bar{\chi}_{f_3} \bar{\chi}_{f_4} \psi]$$

The prefactor is chosen so that $\rho$ becomes a canonical field at large distances. $\gamma$ is Euler’s constant. The long distance behavior of the correlator is

$$\langle \rho_{f_1 f_2} (0) \rho_{f_3 f_4} (x) \rangle = \epsilon_{f_1 f_2 f_3 f_4} \frac{1}{2\pi \sigma} \cdot x$$

and there are two contributions to the correlator. This is due to the non-trivial topology associated with $U(1)$ gauge fields in 2D. One contribution is from the zero topological sector and is of the form

$$\langle \chi_{f_1} (0) \chi_{f_2} (0) \bar{\psi} (0) \bar{\chi}_{f_1} (x) \chi_{f_2} (x) \psi (x) \rangle.$$  

The second contribution is from the unit topological sector and is of the form

$$\langle \epsilon_{f_1 f_2 f_3 f_4} \chi_{f_1} (0) \bar{\psi} (0) \chi_{f_2} (x) \chi_{f_3} (x) \bar{\psi} (x) \rangle.$$
The second contribution violates fermion number by two units. The exact low energy effective Lagrangian of the model, written in terms of the \( \rho \)-fields, is

\[
\mathcal{L} = \frac{1}{2} \sum_{f_1 > f_2} \rho_{f_1, f_2} \sigma \cdot \partial \rho_{f_1, f_2}
\]  

One of the terms in the effective Lagrangian is a 't Hooft vertex, \( V(x) \), which we choose to define as:

\[
V(x) = \frac{\pi^2}{e_0} \chi_1(x) \chi_2(x) \chi_3(x) \chi_4(x) \bar{\psi}(x) (\sigma \cdot \partial) \psi(x)
\]  

This operator violates fermion number by two units and has a nonzero expectation value. On the finite \( t \times l \) lattice

\[
\langle V \rangle_{t \times l} = \frac{64\pi}{(tm_\gamma)} \exp \left[ -\frac{4\pi}{tm_\gamma} \coth \left( \frac{1}{2} lm_\gamma \right) \right] e^{4F(tm_\gamma) - 8H(tm_\gamma, \frac{l}{2})}
\]

where \( m_\gamma^2 = \frac{4e_0^2}{\pi} \) and the functions \( F(\xi) \) and \( H(\xi, \tau) \) are defined below:

\[
F(\xi) = \sum_{n > 0} \frac{1}{n} \left( \frac{1}{\sqrt{n^2 + (\xi/2\pi)^2}} - 1 \right)
\]

\[
H(\xi, \tau) = \sum_{n > 0} \frac{1}{\sqrt{n^2 + (\xi/2\pi)^2}} e^{-\tau \sqrt{(2\pi n)^2 + \xi^2 - 1}}
\]

The infinite volume limit of this quantity is

\[
\langle V \rangle_\infty = \frac{e^{4\pi}}{4\pi^3} \approx 0.081
\]

At \( t = l = \frac{1}{m_\gamma} \), its value is 0.0389, substantially smaller than the value at infinite volume and we will present results of simulations using the overlap formalism at this finite volume.

### 3.2 Numerical results

To perform the numerical simulation on the lattice \( \mathbb{Z}^4 \) and compute the fermion number violating process, the expectation value \( \langle V \rangle \) is written on the lattice using the overlap as

\[
\langle V \rangle = \frac{\int \left[ dU \right] e^{S_g(U)} \langle -|V|+ \rangle^\text{WB}_U}{\int \left[ dU \right] e^{S_g(U)}}
\]

Gauge fields are generated using the pure gauge action \( S_g(U) \) and \( \langle V \rangle \) is computed as a ratio of two observables in the pure gauge theory, namely \( \langle -|V|+ \rangle^\text{WB}_U \) and \( \langle -|+ \rangle^\text{WB}_U \). The pure gauge action is gauge invariant on the lattice but the fermionic observable is not as mentioned in the beginning of this section. Therefore one has to generate gauge fields configurations according to the gauge invariant action to get points on the gauge orbit and integrate the observable over many points on the
As mentioned in the beginning of this section, the violation of gauge invariance in the overlap is restricted to the phase. One can get a feel for the violation of gauge symmetry by plotting the distribution of the phase of the overlap on a generic gauge orbit. Such a distribution in zero topology and unit topology is shown in Fig. 1 and in Fig. 2 respectively. Both figures show that the distribution is well peaked around a central value showing that the violations of the gauge symmetry are small.

In Fig. 3 the result for the computation of the ’t Hooft vertex on various lattices using gauge averaging on orbits is shown. The data fit well as a function of $1/L^2$ and the continuum extrapolation matches well with the number in the continuum. This gives a clear evidence that the overlap formalism successfully reproduces the fermion number violating process in this model.

In this model, there is a Thirring interaction that is marginal. If such a term is generated, the continuum limit will not match with the number obtained in the continuum without the Thirring term. The strength of the Thirring term is regulator dependent and is expected to depend on $m$ which is a regulator parameter in the overlap formalism. It is conceivable that the Thirring term is small for our particular choice of $m$ but it would be a miracle if it were exactly zero. Evidence for the presence for a Thirring term will be deviations from the $1/L^2$ behavior in Fig. 3 for large enough $L$. This point was investigated by going to $L = 24$ and the data show some deviations in the range of $L = 20$ and $L = 24$ but there is no clear
Figure 2: Phase distribution along a generic orbit carrying unit topological charge. The horizontal axis is in units of $\pi$. The phase is measured relative to the Landau gauge phase and, within errors, the average cancels the Landau phase leaving an almost real answer. The histogram contains 100 points.

Figure 3: Data for $L = 8, 10, 12, 14, 16$ versus $1/L^2 \propto a^2$, a linear fit to the points $L \geq 10$, the continuum result (rhombus at $a = 0$), and our estimate for the continuum result from the data (square with error bar at $a = 0$)
evidence for a Thirring term yet.

4 A numerical test of the continuum index theorem on the lattice

The overlap formalism is capable of probing the topology of gauge fields on the lattice as remarked in section 2.4. Given a gauge field configuration on the lattice, there is an associated Hamiltonian $H^+$ ad given by Eq. 12 which is a $2K \times 2K$ matrix. If this matrix has $K - Q$ negative energy eigenstates then the index of the chiral Dirac operator is $Q$. In particular, $Q$ units of fermions in the fundamental representation of the gauge group are created by this gauge field configuration. For a smooth configuration in the continuum, the Atiyah-Singer index theorem relates this to the topological charge of the gauge field. Apriori, it is not obvious that this relation should hold on a finite lattice away from the continuum since the configurations are not expected to be smooth. It is possible to address this question in the context of pure gauge theory if one has a measurement of the distribution of topological charge and a measurement of the distribution of the index for the same pure gauge action on the same lattice. A recent measurement of the distribution of topological charge in pure SU(2) Yang-Mills theory using the standard Wilson action has been performed using an improved colling method.

On a $12^4$ lattice with $\beta = 2.4$ this method gives a Gaussian distribution with a variance of $\langle Q^2 \rangle = 3.9(5)$. Using the overlap, the distribution of the index was also measured on a $12^4$ lattice with $\beta = 2.4$ using the standard Wilson action and the variance of the distribution was $\langle Q^2 \rangle = 3.3(4)$. These two results show that relation between the index and the topological charge holds quite well on a finite lattice away from the continuum. In the following subsection, the details involved in measuring the index are presented and the results are also provided in some detail.

4.1 Measurement of index and results

The index of the chiral Dirac operator is directly related to the spectrum of $H^+$ as given by Eq. 12. In order to arrive at an efficient algorithm to measure the index it is useful to study the spectral flow of

$$H(\mu) = \gamma_5(D - W + \mu)$$

where $H^+ = H(m)$. In the chiral basis,

$$H(\mu) = \begin{pmatrix} \mu - W & C \\ C^\dagger & W - \mu \end{pmatrix}$$

and the condition for a zero eigenvalue of $H(\mu)$ is

$$\begin{pmatrix} \mu - W & C \\ C^\dagger & W - \mu \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \iff u^\dagger W u + v^\dagger W v = \mu$$

implying that a solution can exist only for $\mu > 0$ since $W$ is a positive definite operator. Therefore if $H^+$ has only $K - Q$ negative eigenvalues, then $H(\mu)$ should have had $Q$ zero eigenvalues for certain values of $\mu$ between 0 and $m$. That is the
Figure 4: Five low lying positive and negative eigenvalues of $H(\mu)$ as a function of $\mu$ for five different gauge field configuration. The configurations are from pure SU(2) gauge theory at $\beta = 2.4$ on a $12^4$ lattice.

spectral flow of $H(\mu)$ should show $Q$ levels crossing the axis from below to above. This implies that one only needs to look at a few low lying eigenvalues of $H(\mu)$ as a function of $\mu$ from 0 to $m$ in order to measure the index of the chiral Dirac operator. Several techniques are available to measure a few low lying eigenvalues. One of them is the Lanczos method\cite{Lanczos}.

In Fig. 4 a few low lying eigenvalues of the $H(\mu)$ as a function of $\mu$ is shown for a few configurations in the gauge field ensemble. The spectrum has a gap for $\mu < 0.7$ and the gap closes around this value. Fig. 5 provides a closer look at the flow of four different configurations around the region in $\mu$ where the gap closes. From this one can read the index associated with the configuration. Clearly all crossings do not happen at some fixed value of $\mu$. The finite spread in $\mu$ is a consequence of finite lattice spacing. For smooth instantons embedded on a finite lattice, one can show that smaller instantons cross later in $\mu$\cite{Lanczos}. The region of $\mu$ where the crossing happen will get closer to $\mu = 0$ as one approaches the continuum limit and the spread in $\mu$ will also shrink\cite{Lanczos}. On a finite lattice, one can use the spread to infer some information of the shape distribution of topological objects on the lattice.

In Table 1 the distribution for the topological charge using improved cooling\cite{improved_cooling} is listed along with the distribution for the index obtained using the overlap\cite{overlap}. The two columns are a result of measurements on a different set of independent configurations. The table is plotted in Fig. 6. The close matching of the two distributions indicate that the connection between the index and the topological charge remains valid on the lattice in a probabilistic sense.
Figure 5: Five low lying positive and negative eigenvalues of $H(\mu)$ as a function of $\mu$ for four gauge field configurations with different values of $Q$. The configurations are from pure SU(2) gauge theory at $\beta = 2.4$ on a $12^4$ lattice.

Figure 6: Comparison of the probability distribution of the index (diamonds) with the probability distribution of the topological charge (squares). The squares have been slightly shifted laterally for visual purposes.
Table 1: Comparison of the probability distribution of the index with the probability distribution of the topological charge.

| $Q$ | $p(Q)(\text{Index})$ | $p(Q)(\text{Topology})$ |
|-----|-----------------------|--------------------------|
| 0   | 0.214(50)             | 0.218(37)                |
| ±1  | 0.186(20)             | 0.168(31)                |
| ±2  | 0.129(19)             | 0.122(23)                |
| ±3  | 0.061(14)             | 0.067(25)                |
| ±4  | 0.011(6)              | 0.025(13)                |
| ±5  | 0.004(3)              | 0.005(4)                 |
| ±6  | 0.004(3)              | 0.005(5)                 |
| $\langle Q^2 \rangle$ | 3.3(4)                 | 3.9(5)                   |

5 Conclusions and future directions

Montecarlo measurement of a fermion number violating process in a two dimensional chiral model and a test of the continuum index theorem on the lattice in a four dimensional gauge theory has shown that the overlap formalism is a valid proposal to deal with chiral fermions in a non-perturbative manner. Future work using the overlap has to be planned by keeping the various points mentioned in the beginning of section 3. Problems in two dimensions such as Majorana-Weyl fermions coupled to non-abelian gauge fields could be done without worrying about new stochastic algorithms for the fermions.

To make the overlap formalism into a viable technique in four dimensions one has to think of new algorithms to deal with the overlap. This has to be done in several steps. The first step has already been taken. This is the measurement of the index in pure gauge theory. Fermion dynamics do not play a part and one can use the well developed techniques to deal with gauge dynamics. The measurement of the index is a computation of a fermionic observable but it is one step simpler than the computation of a fermionic correlator. Efficient techniques to deal with low lying eigenvalues of $H^+$ made it possible to measure the index. It would be useful to obtain the continuum distribution of the index in four dimensional pure gauge theories using the overlap. Without much more effort, one could also compute the associated eigenvectors. These eigenvectors will carry information about localized objects. It is plausible that these low lying eigenvectors carry most of the physics information. This would be the case for instance if physics is driven by instantons. Therefore the second step in four dimensions would be to deal with quenched QCD and use an approximate form of the overlap (keeping a few low lying eigenstates of $H^+$) to measure fermionic correlations. Based on the progress made in the second
step one could push further to deal with massless QCD using the overlap. In the context of QCD, the operator does have an eigenvalue problem and it is possible to reduce the overlap formula to the computation of the determinant of a finite matrix. Only after this could one try to deal with chiral gauge theories in four dimensions since one then has to deal with complex action. This is a long path but the ability to deal with chiral fermions in principle makes it possible to start the hike.

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References

1. R. Narayanan, H. Neuberger, Nucl. Phys. B 443, 305 (1995).
2. R. Narayanan, H. Neuberger, Phys. Lett. B 393, 360 (1997).
3. Y. Kikukawa, R. Narayanan, H. Neuberger, hep-lat/9705006.
4. D.B. Kaplan, Phys. Lett. B 288, 342 (1992).
5. S.A. Frolov, A.A. Slavnov, Phys. Lett. B 309, 344 (1993).
6. C.G. Callan, J. Harvey, Nucl. Phys. B 250, 427 (1985).
7. D. Boyanovsky, E. Dagotto, E. Fradkin, Nucl. Phys. B 285, 340 (1987).
8. A proof of this identity can be found in Appendix B of R. Narayanan, H. Neuberger, Nucl. Phys. B 412, 574 (1994).
9. S. Randjbar-Daemi, J. Strathdee, Phys. Lett. B 402, 134 (1997) Nucl. Phys. B 466, 335 (1996); Nucl. Phys. B 461, 305 (1996); Phys. Rev. D 51, 6617 (1995); Nucl. Phys. B 443, 386 (1995); Phys. Lett. B 348, 543 (1995).
10. S. Randjbar-Daemi, private communication.
11. G. Curci, G. Veneziano, Nucl. Phys. B 268, 179 (1986).
12. P. Huet, R. Narayanan, H. Neuberger, Phys. Lett. B 380, 291 (1996).
13. D. Foerster, H.B. Nielsen, M. Ninomiya, Phys. Lett. B 94, 135 (1980).
14. R.Narayanan, H. Neuberger, Nucl. Phys. B 477, 521 (1996).
15. Y. Kikukawa, R. Narayanan, H. Neuberger, Phys. Lett. B 399, 105 (1997).
16. I. Sachs, A. Wipf, Helv. Phys. Acta 65, 652 (1992).
17. M. Atiyah, I.Singer, Ann. Math. 87, 484 (1968).
18. Ph. de Forcrand, M. Garcia Pérez, I.-O, Stamatescu, hep-lat/9701012.
19. R. Narayanan, P. Vranas, hep-lat/9702005, To appear in Nuclear Physics B.
20. G.H. Golub and C.F. van Loan, Matrix computations, Johns Hopkins University Press, 1989.
21. R. Narayanan, B. Singleton, in preparation.
22. T. Schafer, E.V. Shuryak, hep-ph/9610451; R.C. Brower, T.L. Ivanenko, J.W. Negele, K.N. Orginos, *Nucl. Phys. Proc. Suppl.* 53, 547 (1997).
23. H. Neuberger, hep-lat/9707022.