Abstract. We define refined invariants which “count” nodal curves in sufficiently ample linear systems on surfaces, conjecture that their generating function is multiplicative, and conjecture explicit formulas in the case of K3 and abelian surfaces. We also give a refinement of the Caporaso-Harris recursion, and conjecture that it produces the same invariants in the sufficiently ample setting. The refined recursion specializes at $y = -1$ to the Itenberg-Kharlamov-Shustin recursion for Welschinger invariants. We find similar interactions between refined invariants of individual curves and real invariants of their versal families.

1. Introduction

Given a general elliptic fibration $K3 \to \mathbb{P}^1$, we learn by computing $\chi(K3) = 24$ that there must be 24 nodal fibers. For more general irreducible curve classes on a K3, Yau and Zaslow [YZ] argued that taking the Euler characteristic of the relative compactified Jacobian would again yield the number of maximally degenerate fibers; the arguments were clarified by Beauville [Bea] and by Fantechi, van Straten, and the first author [FGvS].

For more general families of curves, similar arguments may be made in terms of the relative Hilbert schemes of points. Let $C \to B$ be a family of reduced planar curves of arithmetic genus $g$, and let $C^{[n]} \to B$ be the relative Hilbert schemes. Certain string-theoretic ideas of Gopakumar and Vafa [GV] motivate the consideration of the following series, and the following change of variables:

$$\sum_{n=0}^{\infty} q^{n+1-g} \chi(C^{[n]}) = \sum_{i=0}^{\infty} n_{C/B}^i \cdot \left( \frac{q}{(1-q)^2} \right)^{i+1-g}.$$

In the case of a single curve $C \to B = \text{pt}$, we write simply $n_C^i$. Pandharipande and Thomas [PT2] prove that $n_C^0 = 0$ when $C$ has cogenus $\delta(C) < i$. From Macdonald’s calculation of the cohomology of symmetric products of curves, it follows that for $C$ smooth we have $n_C^0 = 1$ and $n_C^i = 0$ for $i > 0$. More generally, whenever the singularities of $C$ are unions of smooth branches, the last nonvanishing term is calculated either from [Bea] or [FGvS] to be $n_C^i = 1$.

In the relative situation, it is helpful to view $n^i$ as a constructible function $n^i : B \to \mathbb{Z}$ given by $b \mapsto n_C^i(b)$; evidently $n_{C/B}^i = \int_B n^i(b) d\chi(b)$.

The $n^i$ have an enumerative interpretation. Suppose that $i$ is the maximum cogenus of any curve in the family, that there are finitely many curves of cogenus $i$, and that all
these curves are immersed. Then $n^i_{C/B}$ is just the number of these curves. One exploits this observation by finding another way to express the Euler characteristics of the relative Hilbert schemes. In particular, the following two results have recently been established:

**Theorem 1.** [Sh] Fix $C$ a reduced plane curve, $\Lambda$ a versal deformation of its singularities, and $\Lambda^\delta \subset \Lambda$ the locus of cogenus $\delta$ curves. Then $n^\delta_C$ is the multiplicity of $\Lambda^\delta$.

**Theorem 2.** [KST]. Let $S$ be a surface, and $L$ a $\delta$-very-ample line bundle. Then the number of $\delta$-nodal curves in a general $P^\delta \subset |L|$ is $n^\delta_{C/P^\delta}$, which moreover is given by a certain explicit combination of integrals of Chern classes of $L$ and $T^*S$.

The significance of the second result is that, due to a theorem of Ellingsrud, Lehn, and the first author [EGL], such integrals depend in a universal way on the Chern classes of $L$ and $S$. Such universality of the counts of nodal curves had previously been conjectured by the first author [Göt], and previously proven by Liu [Tze].

The present article poses the following question: **does replacing the topological Euler characteristic on the left hand side by more sophisticated invariants have an enumerative counterpart on the right?**

We begin in Section 2 by studying the case of a single curve. The arguments of [PT2] may be adapted to show that, for $C$ a locally planar reduced curve of arithmetic genus $g$ and $\widetilde{C}$ its normalization of genus $\tilde{g} = g - \delta$, there exist classes $\tilde{N}^i_C$ in the Grothendieck ring of varieties such that

$$\sum_{n=0}^{\infty} q^{n+1-g}[C[n]]^i = \sum_{i=0}^{\delta} \tilde{N}^i_C \left( \frac{q}{(1-q)(1-q[A^1])} \right)^{i-\delta}$$

The right hand side moreover splits into a product over the singularities of $C$.

According to Theorem 1, the Euler number $\chi(\tilde{N}^i_C)$ gives the multiplicity of a certain locus, and is in particular positive. In examples we see a stronger positivity:

**Conjecture 3.** $\tilde{N}^i_C \in \mathbb{Z}_{\geq 0}[A^1]$.

This conjecture is verified computationally for singularities of the form $x^p = y^q$ where $(p,q) = 1$ and $p < 12$, $q < 20$ using the formulas of [ORS] for the classes of the Hilbert schemes.

The meanings of the $\tilde{N}^i_C$ remain mysterious, but there is some evidence that they may be related to real geometry. We define real analogues $n^i_{C/R}$ of the $n^i$ by using the compactly supported Euler number of the real locus, and show these again vanish for $i > \delta(C)$. These have an interpretation analogous to Theorem 1. Recall that nodes of real curves come in three types: elliptic $(x^2 + y^2 = 0)$, hyperbolic $(x^2 - y^2 = 0)$, and pairs of complex conjugate nodes. Thus in the real deformation, the loci $B^k_+ \subset k$-nodal curves split into components according to the types of the nodes.
Theorem 4. Let $C$ be a real reduced plane curve, and let $C \to B$ be a locally versal deformation of its singularities. Let $B^\delta_{\pm} \subset B$ be the locus of nodal curves with $\delta$ nodes of which $\delta_-$ are hyperbolic. Let $D^j$ be a general real disc of dimension $j$ passing near $[C] \in B$. Then

$$n^j_C = \sum_i (-1)^i D^j \cap B^{j,i}$$

Note in particular that while the individual terms on the RHS of the formula may depend on the location of the disc, the theorem asserts that their sum does not.

For the simple singularities, we give a combinatorial formula for the $\tilde{N}^i_C$ in terms of the Dynkin diagram. (This refines the analogous prescription for the multiplicities given in [Sh].) The formula may be interpreted as the choice of a particular real form and a particular disc $D$ in the above statement so that the coefficient of $L^i$ in $N^j_C$ is $D^j \cap B^{j,i}$. For $j = \delta$, Duco van Straten has conjectured that such a disc may be found for any singularity.

In Section 3, we turn to the case of linear systems of curves on surfaces. From the point of view of the argument in [KST], it is natural to refine the Euler characteristic to Hirzebruch’s $\chi_{-y}$ genus, since the latter both factors through the Grothendieck ring of varieties and may be calculated in terms of Chern classes. By an appropriate change of variable, one can define invariants $N^i_C/\mathbb{P}^q \in \mathbb{Z}[[y]]$ which refine the $n^i_C/\mathbb{P}^q$ of Theorem 2 above.

$$\sum_{n=0}^{\infty} q^{n+1-g} \chi_{-y}(C^n) = \sum_{i=0}^{\infty} N^i_C/\mathbb{P}^q \cdot \left( \frac{q}{(1-q)(1-qy)} \right)^{i+1-g}$$

One virtue of the $\chi_{-y}$ genus is that, for a single curve, the $N^i_C$ vanish for $i > \delta(C)$; the analogous statement does not hold for the virtual Poincaré polynomials\footnote{This does not contradict the statement above that the $\tilde{N}^i_C$ vanish even in the Grothendieck ring of varieties; the point being that the $N^i_C$ and $\tilde{N}^i_C$ have different virtual Poincaré polynomials but the same $\chi_{-y}$ genus. Ultimately this is because the $\chi_{-y}$ genus of an abelian variety vanishes.}

However, unlike in the Euler characteristic setting, one cannot “integrate over the base” to conclude the vanishing for $N^i_C/\mathbb{P}^q$. Nonetheless, the experimental evidence suggests:

Conjecture 5. Let $L$ be a line bundle on a surface $S$, and $\mathbb{P}^q \subset |L|$ a linear subsystem with tautological curve $C \to \mathbb{P}^q$. Assume that $\mathbb{P}^q$ contains no non-reduced curves, and that the total space of the relative Hilbert scheme $C^n$ is smooth for all $n$. Then $N^i_C/\mathbb{P}^q$ vanishes for $i > \delta$.

We are able to show:

Theorem 6. The conjecture holds when $K_S$ is numerically trivial.
We also give some evidence for the general statement. By [EGL], one can reduce the validity of the conjecture to the case of \( \mathbb{P}^2 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \). Here we may calculate in low degrees by equivariant localization.

Regarding the hypotheses, note that when \( \delta = 1 \), Pandharipande and Fantechi [Pan] have found families of smooth curves over a smooth base curve with nonzero invariants \( N_i \) for some \( i > 0 \). The base curves are always of genus \( > 0 \), and indeed no such examples can exist over a simply connected base. We are not sure what further aspects of the geometry of families of curves on surfaces are implicated. A calculation of Migliorini has shown that the vanishing cannot be expected for the analogous expression involving Hodge polynomials, already over a one dimensional base.

In Section 4, we focus on the invariants \( N_{C/P}^\delta \). Assuming Conjecture 5, this is the last nonvanishing \( N_i \), and we show that there is an expression of the form

\[
\sum_{\delta = 0}^{\infty} N_{C/P}^\delta \delta \equiv A_1L^2A_2K_2A_3^2A_4^2 + O(s^{k+1})
\]

where \( L \) is \( k \)-very-ample, and the \( A_i \) are some universal power series in \( \mathbb{Q}[y][[s]] \). To avoid carrying around the \( O(s^{k+1}) \), we simply define \( N_{C/P}^\delta \) to be the coefficient of \( s^\delta \) on the RHS; the brackets in the notation serve to remind us that it depends only on the cobordism class of \((S,L)\).

As in [Göt], it is easiest to express the \( A_i \) after a change of variable. Consider the following series in \( \mathbb{Q}[y,y^{-1}][[q]] \)

\[
\Delta(y,q) := q \prod_{n=1}^{\infty} (1 - q^n)^{20} (1 - yq^n)^2 (1 - y^{-1}q^n)^2,
\]

\[
\overline{DG}_2 := \sum_{m=1}^{\infty} mq^m \sum_{d|m} \frac{[d]^2_y}{d}
\]

and let \( D = \frac{q}{dq} \). Above \([n]_y := \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}}\).

It is also convenient to introduce the notation \( N_{C[S,L]}^\delta := y^{-\delta} N_{C[S,L]}^\delta \); these invariants are symmetric under \( y \to 1/y \).

**Conjecture 7.** There exist power series \( B_1(y,q), B_2(y,q) \) in \( \mathbb{Q}[y,y^{-1}][[q]] \), such that

\[
\sum_{\delta \geq 0} N_{C[S,L]}^\delta(y) \overline{DG}_2^\delta = \frac{(\overline{DG}_2/q)^{x(L)} B_1(y,q)K_2 B_2(y,q)L_{K_2}}{(\Delta(y,q) D\overline{DG}_2/q^2)^{x(O_S)/2}}.
\]

The first 11 terms of the series \( B_1, B_2 \) are given explicitly in Section 4.

Assuming Conjecture 5, the content of the above assertion is exhausted by the case where \( S \) is a K3 surface. Moreover as in [Göt] it may be reformulated without the expansion in powers of \( \overline{DG}_2^\delta \):
Conjecture 8. Let \((S_g, L_g)\) be K3 surfaces of genus \(g\) with irreducible polarizations. Then for any \(k\),
\[
\sum_{g=k}^{\infty} q^{g-k} N_{g-k, [S_g, L_g]}(y) = \frac{\tilde{D}\hat{G}_2(y, q)^k}{\Delta(y, q)}.
\]
Similarly, if \((A_g, L_g)\) are abelian surfaces of genus \(g\) with irreducible polarizations,
\[
\sum_{g=k+2}^{\infty} q^{g-k-2} N_{g-k-2, [A_g, L_g]}(y) = \tilde{D}\hat{G}_2(y, q)^k \tilde{D}\hat{G}_2(y, q).
\]
Because of the existence of K3 and abelian surfaces of all genera, and the multiplicative nature of the formulas, we show that Conjecture 8 would follow from its validity at \(k = 0\).

We have been able to check this case for the K3 surface, but not for the abelian surface. Conjecture 8 would also follow from its validity for all \(k\) for the K3 surface alone. Here we note a remarkable coincidence: the series on the RHS of the above formula appears in the work of [MPT] on computing descendant invariants in the (reduced) Gromov-Witten or stable pairs theory of a K3 surface. This leads to a further reformulation:

Conjecture 9. Let \((S, L)\) be a irreducibly polarized K3 surface of genus \(g\), and let \(H\) be the hyperplane class on \(|L|\). Then for all \(k\),
\[
(y - 2 + y^{-1})^{k-1} N_{g-k, [S, L]} = \sum_{n=0}^{\infty} y^{n+1-g} \int_{c^{[n]}_L} c_{n+g-k} (TC_{[L]}^{[n]}) \cdot \rho^*(H^k)
\]

Thus far we have been discussing curves with a number of nodes small compared to the ampleness of the line bundle \(L\); this is the regime to which the conjectures of [GT] and the arguments of [KST] apply. However, when the surface is \(\mathbb{P}^2\), the recursion of Caporaso and Harris [CH] determines the degrees of all such loci of nodal curves, without any such restriction on the ampleness. Indeed, it determines more: fix a line \(H \subset \mathbb{P}^2\), and sequences \(\alpha, \beta\) of integers specifying respectively fixed and moving tangency conditions to \(H\). Then the Caporaso-Harris recursion determines the degrees \(n^\delta(\alpha, \beta)\) of the loci of curves with \(\delta\) nodes and satisfying the tangency conditions \(\alpha, \beta\).

In Section 5 we study the following formal refinement of the Caporaso-Harris recursion. We take from [CH] the notation \(I\alpha = \sum i\alpha_i\) and \(|\alpha| = \sum \alpha_i\); note that the curves counted by \(n^\delta(\alpha, \beta)\) have degree \(I\alpha + I\beta\).

Definition 10. The polynomials \(N^\delta(\alpha, \beta) \in \mathbb{Z}[y^{1/2}, y^{-1/2}]\) are defined by the following recursion:
\[
N^\delta(\alpha, \beta) = \sum_{k, \beta_k > 0} [k]_y \cdot N^\delta(\alpha + e_k, \beta - e_k) + \sum_{\alpha', \beta', \gamma} \left( \prod_i [i]_y^{\beta_i - \beta_i'} \right) \left( \frac{\alpha'}{\alpha} \right) \left( \frac{\beta'}{\beta} \right) N^\delta(\alpha', \beta')
\]

\(^2\)Vakil has generalized the Caporaso-Harris recursion to the case of rational ruled surfaces. In Section 5 we treat these as well; we have restricted in the introduction to \(\mathbb{P}^2\) just for ease of notation.
The limits on the sum and the initial conditions are the same as for the Caporaso-Harris recursion and are given explicitly in Section 5. The refined recursion immediately specializes to the Caporaso-Harris recursion upon setting \( y = 1 \), so certainly \( n^\delta(\alpha, \beta) = N^\delta(\alpha, \beta)\big|_{y=1} \). On the other hand, we know that for \( d \gg \delta \) that the Severi degrees \( n^\delta((0, 0, \ldots), (d, 0, \ldots)) \) are given by the universal formulas, i.e., equal to the numbers \( n^\delta_{\delta, [P^2, O(d)]} \). We conjecture a refined analogue:

**Conjecture 11.** For \( \delta \leq 2d - 2 \), we have \( N^\delta(\alpha, \beta) = y^{-\delta} N^\delta_{\delta, [P^2, O(d)]} \).

The equality at \( y = 1 \) follows from [KS]. At \( y = 0 \), the recursion simplifies, allowing the RHS to be calculated explicitly; on the other hand, a result of Scala allows the LHS to be calculated as well [Sca]; the answers match. Empirically we have verified the equality for some small \( d, \delta \).

In Section 6, we note a connection to real enumerative geometry and to some ideas from tropical geometry. On a real toric surface \( S \), there are real enumerative invariants, counting real \( \delta \)-nodal curves with suitable signs, the real analogues of the Severi degrees. If \( S \) is an unnodal del Pezzo surface, they coincide with the Welschinger invariants, real analogues of the Gromov-Witten invariants. Mikhalkin [Mik] has shown that the Severi degrees and the real enumerative invariants can be computed via tropical geometry: he introduces tropical Severi degrees and tropical Welschinger invariants by assigning Gromov-Witten and Welschinger multiplicities to tropical curves, and shows that they coincide with the Severi degrees and the real enumerative invariants respectively. The Caporaso-Harris formula has been derived tropically by Gathmann and Markwig [GM], and an analogue for the tropical Welschinger invariants by Itenberg, Kharlamov, and Shustin [IKS]. These are specializations of the above recursion, specialized at 1 and \(-1\) respectively. In particular the refined Severi degrees specialize to the Severi degrees and the tropical Welschinger invariants. In [BG], Block and the first author define and study tropical refined Severi degrees by assigning polynomial multiplicities to the tropical curves which specialize to the Gromov-Witten and Welschinger multiplicities.

On the right hand side of Conjecture 11, the specialization \( y \mapsto -1 \) has an entirely different meaning: we are taking the signatures of the relative Hilbert schemes and rearranging them in a certain way. On the other hand, a signed count of real nodal curves in a general \( \mathbb{P}^d \) is obtained from the \( n^\delta_{\delta, \mathbb{P}^d} \) defined earlier. In order that \( n^\delta_{\delta, \mathbb{P}^d} \) match \( N^\delta_{\mathbb{P}^d}(-1) \), the following property would suffice:

**Conjecture 12.** Let \( \mathbb{P}^d \subset |O_{\mathbb{P}^2}(d)| \) be a linear system determined by a subtropical collection of real points. If \( C^{[n]}/\mathbb{P}^d \) is smooth, then its signature is equal to the Euler characteristic of its real locus.

Here roughly speaking a collection of points in \( (\mathbb{R}^*)^2 \) is called subtropical, if it can be degenerated to a tropical collection of points without crossing walls, for the precise definition see [LM] Lemma 2.7.(3)].
We remark briefly on related work. In the physics literature there is the notion of the refined topological string, which gives in some cases a one-parameter deformation of the various curve counting invariants [IKV]. Notably it does not have a “worldsheet” definition, even in the sense of physics. Mathematically, the refined theory is supposed to correspond [DG] to the motivic DT theory [KoS]; the lack of a worldsheet definition corresponds to the fact that we do not know how to correspondingly refine the Gromov-Witten invariants. There have also been intimations that a specialization of the refined theory is related to real invariants [KW, Sec. 5]. Our approach falls roughly into this paradigm insofar as \[ \frac{y^n/2 - y^{-n/2}}{y^{1/2} - y^{-1/2}} \]
are assembled from \( \chi_{-y} \) genera of relative Hilbert schemes, which under the relevant assumptions on \( \mathbb{P}^d \) are just the same as stable pairs spaces. It might plausibly be hoped that the refined Severi degrees also admit an interpretation in the stable pairs theory [PT1, PT2] or its surface variant [KT].

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Notation.
We denote quantum numbers as
\[ [n]_y := \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}}. \]

By the Hirzebruch genus \( X_{-y} \) we mean the characteristic class which on a bundle \( E \) with Chern roots \( x_i \) takes the value
\[ X_{-y}(E) = \prod x_i(1 - ye^{-x_i(1-y)}) (1 - e^{-x_i(1-y)}) \in 1 + (x_i)\mathbb{Q}[y][[x_i]] \]

Also let
\[ ch_{-y}(E) = \sum e^{x_i(1-y)} \]

Setting by definition
\[ \chi_{-y}(X, E) := \sum (-y)^p \chi(X, \Omega^p \otimes E) \]
we have according to Hirzebruch
\[ \chi_{-y}(X, E) = \int_X ch_{-y}(E)X_{-y}(TX). \]

When \( E = \mathcal{O}_X \) we suppress it from the notation. Note that \( \chi_{-y}(X) = \sum_{p,q} (-1)^{p+q} y^q h^{p,q}(X) \) where \( h^{p,q}(X) \) are the Hodge numbers of \( X \). Note the specializations to topological Euler characteristic \( \chi_{-1}(X) = \chi(X) \), holomorphic Euler characteristic \( \chi_0(X) = \chi(X, \mathcal{O}_X) \), and signature \( \chi_1(X) = \sigma(X) \).
2. Invariants of a single curve

2.1. Refined invariants. Let $C$ be a complete, reduced, Gorenstein, complex curve of genus $g$. We consider the following series with coefficients in the Grothendieck group of varieties $K_0(\text{var}/\mathbb{C})$:

$$Z_C(q) = \sum_{n=0}^{\infty} C^{[n]} q^{n+1-g}$$

Note $(\mathbb{P}^1)^{[n]} = \mathbb{P}^n$. Writing $L$ for the class of the affine line, we have:

$$Z_{\mathbb{P}^1}(q) = \frac{q}{(1-q)(1-qL)}$$

We define classes $N^i(C) \in K_0(\text{var}/\mathbb{C})$ by the following formula:

$$\sum_{n=0}^{\infty} C^{[n]} q^{n+1-g} = \sum_{i=0}^{\infty} N^i_C \cdot Z_{\mathbb{P}^1}^{i+1-g}$$

Recall that the Euler characteristic factors through $K_0(\text{var}/\mathbb{C})$, so it makes sense to write $\chi(N^i(C))$. When $C$ is smooth of genus $g$, it follows from Macdonald’s calculation of the cohomology of symmetric products that $\chi(N^i(C)) = 0$ for $i > 0$. However, it certainly need not be the case that $N^i = 0$. For instance, if $E$ is an elliptic curve, then $E^{[n]}$ is the projectivization of a rank $n$ vector bundle over $E$ for $n \geq 1$, hence $N^0_E = 1$ and $N^1_E = E$, and the higher $N^i$ vanish.

Lemma 13. [Har]. Let $C$ be a Gorenstein curve, and let $F$ be a torsion free sheaf on $C$. Write $F^*$ for $\text{Hom}(F, \mathcal{O}_C)$. Then $\text{Ext}^{\geq 1}(F, \mathcal{O}_C) = 0$ and $F = (F^*)^*$. Serre duality holds in the form $\text{Ext}^i(C, F) = \text{Ext}^{-i}(C, F^* \otimes \omega_C^*)$. For $F$ rank one and torsion free, define its degree $d(F) := \chi(F) - \chi(\mathcal{O}_C)$. This satisfies $d(F) = -d(F^*)$, and, for $L$ any line bundle, $d(F \otimes L) = d(F) + d(L)$.

The following result lifts [PT2, Prop. 3.13] from Euler characteristics to $K_0(\text{var}/\mathbb{C})$; the proof is essentially the same.

Proposition 14. Let $C$ be a Gorenstein curve. Then $N^i_C = 0$ for $i > g$.

Proof. Fix a degree one line bundle $\mathcal{O}(1)$ on $C$. Let $\mathcal{J}^0(C)$ denote the moduli space of rank one, degree zero, torsion free sheaves [AK]. We map $C^{[d]} \to \mathcal{J}^0(C)$ by sending the ideal $I \subset \mathcal{O}_C$ to the sheaf $I^* = \text{Hom}(I, \mathcal{O}_C) \otimes \mathcal{O}(-d)$; the fibre is $\mathbb{P}(H^0(C, I^*))$. For $F \in \mathcal{J}^0(C)$, we write the Hilbert function as $h_F(d) := \dim H^0(C, F \otimes \mathcal{O}(d))$. Stratify $\mathcal{J}^0(C)$ into strata over which $h_F$ is constant. The restriction of $C^{[d]}$ to each stratum is the projectivization of a vector bundle of rank $h_F(d)$. Thus we have the equality

$$\sum_{d=0}^{\infty} q^d [C^{[d]}] = \sum_{h} [\{ F \mid h_F = h \}] \sum_{d=0}^{\infty} q^d [\mathbb{P}^{(h-1)d}]$$

Fix $h = h_F$ for some $F$. Evidently $h$ is supported in $[0, \infty)$, and by Riemann-Roch and Serre duality is equal to $d + 1 - g$ in $(2g - 2, \infty)$. Inside $[0, 2g - 2]$, it either increases
by 0 or 1 at each step. Let $\phi_{\pm}(h) = \{d \mid 2h(d - 1) - h(d - 2) - h(d) = \pm1\}$; evidently $\phi_- \subset [0, 2g]$ and $\phi_+ \subset [1, 2g - 1]$, and

$$(1 - q)(1 - qL)\sum_{d=0}^{\infty} q^d[\mathbb{P}h(d-1)] = \sum_{d \in \phi_-(h)} q^dL^h(d-1) - \sum_{d \in \phi_+(h)} q^dL^h(d-1)$$

This is a polynomial in $q$ of degree at most $2g$.

Now let $G = F^* \otimes \omega_C \otimes \mathcal{O}(2 - 2g)$, and $h^\vee = h_G$. By Serre duality and Riemann-Roch, $h^\vee(d) = h(2g - 2 - d) + d + 1 - g$, so in particular, $d \in \phi_{\pm}(h^\vee) \iff 2g - d \in \phi_{\pm}(h)$. Summing over possible $h$, we learn that $f(q) = (1-q)(1-qL)\sum_d q^d[C^d]$ is a polynomial of degree at most $2g$, satisfying $q^{2g}L^g f(1/qL) = f(q)$. From this the vanishing of the higher $N^i$ follows by trivial algebraic manipulations. □

**Remark 15.** We always have $N^0_C = 1$ and $N^1_C = |C| + (g-1)(1 + L)$. Given the vanishing, only $N^2_C$ contributes large powers of $q$; on the other hand when $n \gg 0$ the map $C[n] \to \mathcal{F}^0(C)$ is a projective bundle. Comparison of these terms reveals $N^2_C = \mathcal{F}^0(C)$.

**Example 16.** Let $\mathbb{P}^1$, $A_1$, $A_2$ be rational curves that are smooth, have one node, and have one cusp respectively.

- $N^0_{\mathbb{P}^1} = 1$.
- $N^0_{A_1} = 1$ and $N^1_{A_1} = L$.
- $N^0_{A_2} = 1$ and $N^1_{A_2} = 1 + L$.

In each case Proposition 14 ensures the higher $N^i$ vanish.

Let $\tilde{C}$ be the normalization of $C$. To avoid incorporating the global geometry of $C$ into the invariants, we define $\tilde{N}^i_C$ by the formula:

$$\frac{Z_C}{Z_C} = \sum_{i=0}^{\infty} \tilde{N}^i_C Z_{\mathbb{P}^1}^{i-\delta}$$

Note when $C$ is rational, $N^i_C = \tilde{N}^i_C$. One sees by a straightforward stratification argument that the series on the left hand side only depends on formal neighborhoods of the singularities of $C$. Each such singularity may be found on a rational curve with no other singularities [Lau, Prop. 2.1.1]. Thus it follows from the previous result that $\tilde{N}^i = 0$ for $i \geq \delta(C)$. Or in other words,

$$(1) \sum_{i=0}^{g} \tilde{N}^i_C Z_{\mathbb{P}^1}^{i+1-g} = \left(\sum_{i=0}^{g-\delta} \tilde{N}^i_C Z_{\mathbb{P}^1}^{i+1-g}\right) \left(\sum_{i=0}^{\delta} \tilde{N}^i_C Z_{\mathbb{P}^1}^{i-\delta}\right)$$

**Conjecture 17.** $\tilde{N}^h_C \in \mathbb{Z}_{\geq 0}[L]$.

**Remark.** Theorem 1 realizes $\chi(\tilde{N}^1_C)$ as the multiplicity of the stratum of curves of cogenus $i$ inside the versal deformation of $C$, whence it follows that $\chi(\tilde{N}^1_C) > 0$ for $i \leq \delta(C)$. It may be hoped that this conjecture indicates a refinement of this geometric structure, i.e., that the coefficients of the $\tilde{N}^i_C$ count something.
For rational \( C \), we have \( \tilde{N}_C^\delta = [\mathcal{J}(C)] \); by \([\text{Lau}]\) these Jacobians are known to receive a bijective morphism from an affine Springer fiber for \( \mathfrak{gl} \). It is known that such affine Springer fibers in other types are not necessarily in \( \mathbb{Z}[L] \); however according to Lusztig the status of the \( \mathfrak{gl} \) affine Springer fibers is unknown. From the work of Piontkowski \([\text{Pio}]\) it follows that that for unibranch singularities with a single Puiseux pair, and for unibranch singularities whose links are two-cablings of links of simple unibranch singularities, one has at least \( \tilde{N}_C^\delta \in \mathbb{Z}[L] \). The stated positivity has been checked for unibranch singularities with a single Puiseux pair (e.g. \( x^m = y^n \)) for \( m < 14 \) and \( n < 20 \) using the explicit formula for \( \mathbb{Z}(C) \) given in \([\text{ORS}, \text{Thm. 5}]\).

From the fact that \( \chi(\mathbb{Z}(C)) = \chi(\mathbb{Z}P_1)^{1-g} \), we see that
\[
\nu_i^C := \chi(\mathbb{N}_C^i) = \chi(\tilde{N}_C^i),
\]
in particular that the \( \nu_i^C \) vanish for \( i > \delta(C) \). This fact was used in \([\text{PT2}]\) as evidence that the \( \nu_i^C \) were in fact the Gopakumar-Vafa invariants, and was exploited in \([\text{KST}]\) to count curves on surfaces. Here we note the Hirzebruch \( \chi-y \) genus has the same property, which suggests that it is a more sensible refinement than working in the ring of varieties.

Lemma 18. For \( C \) a smooth curve of genus \( g \), \( \chi-y(\mathbb{Z}(C)) = \chi-y(\mathbb{Z}P_1)^{1-g} \).

Proof. The Hodge structures of symmetric products are known explicitly \([\text{Mac}]\). □

Corollary 19. \( \chi-y(\mathbb{N}_C^i) = \chi-y(\tilde{N}_C^i) \). In particular, \( \chi-y(\mathbb{N}_C^i) = 0 \) for \( i > \delta(C) \).

Remark 20. According to Conjecture \([\text{17}]\) the \( \tilde{N}_C^i \) may be recovered from \( \chi-y(\mathbb{N}_C^i) \) by \( y \mapsto L \). Conversely, one may separate Conjecture \([\text{17}]\) into two pieces, one asking that the \( \tilde{N}_C^i \in \mathbb{Z}[L] \) and the second asking that \( \chi-y(\mathbb{N}_C^i) \) have positive coefficients.

2.2. Curves with simple singularities. Here we express the refined BPS invariants of curves with simple singularities in terms the associated Dynkin diagram. In Figure 1 we recall the ADE classification of simple singularities.

Lemma 21. We define \( A_\infty \) to be the germ at the origin of the curve cut out by \( y^2 = 0 \). Similarly we define \( D_\infty \) by \( xy^2 = 0 \) and \( E_\infty \) by \( y^3 = 0 \). Then, for \( X = A, D, E \), we have an equality \( X^\mu_\infty \mid X^\mu_\infty \) as subsets of \((\mathbb{C}^2)^i\) for any \( i \) up to the delta invariant of \( X_\mu \).

Proof. Any subscheme of length at most \( \delta \) supported at the origin is annihilated by \((x, y)_\delta \).

In each case, the RHS of the equation of \( X_\mu \) already belongs to this ideal. □

Proposition 22.
\[
\sum [A^{[n]}] q^n = \frac{1}{(1-q)(1-q^2L)} \\
\sum [D^{[n]}] q^n = \frac{1-q+q^2L^2}{(1-q)^2(1-q^2L)} \\
\sum [E^{[n]}] q^n = \frac{1}{(1-q)(1-q^2L)(1-q^3L^2)}
\]
Figure 1. The ADE singularities and associated colored diagrams. The subscript gives the total number of vertices of the diagram, and the Milnor number of the singularity. The number of filled vertices of valence one is the number of analytic local branches (except for $A_1$), and the total number of filled vertices is the delta invariant.

Proof. The cases $A_{\infty}, E_{\infty}$ are special cases of [ORS, Prop. 6]. We treat $D_{\infty}$. An arbitrary element in $\mathbb{C}[x,y]/xy^2$ takes the form $c + \lambda y + xf(x) + xyg(x) + y^2h(y)$, where $c, \lambda$ are constants. Fix an ideal $I$ and consider $I \cap (y^2)$. This is generated by some $y^k$. Now consider $I \cap (y)$. A general element has the form $\lambda y + xyg(x) + y^2h(y)$. If $\lambda \neq 0$, then multiplying by $y$ we find $\lambda y^2 \in I$ hence $k = 2$ and moreover we may take $h(y) = 0$. Multiplying by $(1 + yg(x))^{-1}$ we may eliminate $g(x)$ as well, and we have $I \cap (y) = y$. Such ideals (of finite colength) are of the form $(y, x^k)$. Thus let us assume every element in $I \cap (y)$ is of the form $xyg(x) + y^2h(y)$. Multiplying by invertible elements, we may assume $g, h$ are monomial. Note that if $I$ is of finite colength, some elements with nonzero $g(x)$ must appear. So choose an element $x^{n+1}y + by^k$ with $n$ minimal. We are constrained by $1 < k_2 \leq k - 1$. Thus the possibilities are $I \cap (y) = (x^{n+1}y + by^{k-1}, y^k)$, where if $k = 2$ then $b = 0$. Finally $I = (x^{n_3} + cy^{k-1} + xyg(x), x^{n+1}y + by^{k-1}, y^k)$, where $n_3 \geq n_1 + 1$, the constant $c$ is arbitrary and $g(x)$ should be chosen of degree less than $n$. □
**Theorem 23.** Let $X$ be a curve, smooth away from a simple singularity. Color the dots of the associated Dynkin diagram as in Figure 1. Let $n_{w,b}$ be the number of ways to choose $w$ white dots and $b$ black ones such that no two dots are adjacent. Then

$$\tilde{N}_X^h = \sum_{w+b=h} n_{w,b} L^b$$

**Proof.** $Z_X$ is symmetric and determined by its terms of degree less than $q^2$. By Lemma 21, these are determined by the series for $X_\infty$, which we have computed. Matching to the numbers $n_{w,b}$ is an exercise in combinatorics. \qed

**Example 24.** For $A_9$, the nonvanishing invariants are

$$\begin{align*}
\tilde{N}_4 &= 1 + L + L^2 + L^3 + L^4 \\
\tilde{N}_3 &= 4 + 6L + 6L^2 + 4L^3 \\
\tilde{N}_2 &= 6 + 9L + 6L^2 \\
\tilde{N}_1 &= 4 + 4L \\
\tilde{N}_0 &= 1
\end{align*}$$

The $\tilde{N}^i$ are not generally symmetric.

**Example 25.** For $E_6$, the nonvanishing invariants are

$$\begin{align*}
\tilde{N}_3 &= 1 + L + 2L^2 + L^3 \\
\tilde{N}_2 &= 3 + 4L + 3L^2 \\
\tilde{N}_1 &= 3 + 3L \\
\tilde{N}_0 &= 1
\end{align*}$$

**Example 26.** For $E_8$, the nonvanishing invariants are

$$\begin{align*}
\tilde{N}_4 &= 1 + L + 2L^2 + 2L^3 + L^4 \\
\tilde{N}_3 &= 4 + 6L + 7L^2 + 4L^3 \\
\tilde{N}_2 &= 6 + 9L + 6L^2 \\
\tilde{N}_1 &= 4 + 4L \\
\tilde{N}_0 &= 1
\end{align*}$$

For the simple singularities, we have the following remarkable statement, which may be proven by comparing Theorem 23 to the description of the versal deformation of a simple singularity as the quotient of the hyperplane arrangement of the same name by the Weyl group [AGV].
Theorem 27. Let $X$ be a simple singularity. Then there exists some curve $C$ containing as its unique singularity a real form of $X$, and a real disc $D^j$ in the real locus of the versal deformation of $X$ such that

$$\chi_y(N_C^j) = \sum_i y^i \cdot \#(D^j \cap B_{+}^{j,i})$$

where $B_{+}^{j,i}$ is the locus of real nodal curves with $j$ total nodes of which $i$ are hyperbolic.

This result was known to van Straten when $j = \delta(C)$; he had conjectured in this case that it holds for all singularities (see [vS][Conj. 4.7]) on the evidence of its validity for the simple singularities and its validity at $y = 1$ for all singularities [FGvS]. Theorem 1 asserts the validity of the above statement at $y = 1$ for all singularities, so one might analogously conjecture that the statement of Theorem 27 holds always.

2.3. Real invariants. Now suppose $C$ is defined over $\mathbb{R}$. Let $\chi_R$ denote the compactly supported topological Euler characteristic of the real locus. Note this is additive and multiplicative. By way of examples, $\chi_R(\mathbb{A}^1) = -1$, $\chi_R(\mathbb{P}^2) = 1$, and $\chi_R(\mathbb{P}^2+1) = 0$. Let

$$Z_C^R := \sum_{n=0}^{\infty} q^{n+1-g} \chi_R(C[n])$$

We have $Z_{\mathbb{P}^1}^R = q/(1 - q^2)$, and so we define integers $n_{C}^{i,R}$ by the following formula:

$$(2) \quad \sum_{n=0}^{\infty} \chi_R(C[n]) \ q^{n-g+1} = \sum_{i=0}^{\infty} n_{C}^{i,R} \left( \frac{q}{1 - q^2} \right)^{i-g+1}$$

Lemma 28. For $C$ a smooth real curve of genus $g$, $Z_C^R = (Z_{\mathbb{P}^1}^R)^{1-g}$. Equivalently, $n_{C}^{0,R} = 1$ and $n_{C}^{i,R} = 0$ for $i > 0$.

Proof. Let $\sigma$ be the complex conjugation. We have a stratification

$$\prod q^n(\text{Sym}^n(C))(\mathbb{R}) = \left( \prod q^n(\text{Sym}^n(C(\mathbb{R}))) \right) \left( \prod q^{2n}(\text{Sym}^n((C(\mathbb{C}) \setminus C(\mathbb{R}))) / \sigma) \right)$$

Anything coming from $C(\mathbb{R})$ fibres over circles and never contributes to the Euler characteristic. Therefore

$$\sum q^n \chi_R((\text{Sym}^n(C))(\mathbb{R})) = \sum q^{2n} \chi_{\text{Sym}^n((C(\mathbb{C}) \setminus C(\mathbb{R}))) / \sigma} = (1 - q^2)^{g-1}$$

This establishes the Lemma. \qed

For a point $p$ on a curve $C$, write $(C,p)$ for the germ at $p$, and $(C,p)[n]$ for the locus in the Hilbert scheme of points on $C$ of subschemes set-theoretically supported at $p$. For $p \in C(\mathbb{R})$, we write $rb(p)$ and $cb(p)$ for respectively the number of real and complex points above $p$ in the normalization of $C$.  

We have by stratification
\[
\sum q^{\delta(p)} \frac{\sum q^n \chi_R(C, p)^{[n]}}{\sum q^{\delta(p)} \chi_R(C^{[n]})} = \prod_{p \in C^{\text{sing}}(\mathbb{R})} q^{-\delta(p)} \frac{\sum q^n \chi_R(C, p)^{[n]}}{(1 - q^2)^{-\delta(p)/2}(1 - q)^{r\delta(p)}} \times \prod_{p \in C^{\text{sing}}(\mathbb{C})/\sigma} q^{-2\delta(p)} \frac{\sum q^{2n} \chi_R(C, p)^{[n]}}{(1 - q^2)^{r\delta(p)}}
\]

We now compute \(n_C^{R, \delta}\) for a curve \(C\) with \(\delta\) nodes and no other singularities. By the product formula above, it suffices to do this in three special cases.

**Lemma 29.** The only nonvanishing invariants of a curve \(c_\pm\) of arithmetic genus 1 with a single node analytically of the form \(\mathbb{R}[[x, y]]/(x^2 \pm y^2)\) are \(n_0, R = 1\) and \(n_1, R = 1\).

(Caution: \(c_-\) is the one with a node that looks like +.)

**Lemma 30.** The only nonvanishing invariants of a curve of arithmetic genus two with a pair of complex conjugate nodes are \(n_0, R = 1\) and \(n_1, R = 0\) and \(n_2, R = 1\).

We conclude:

**Proposition 31.** For a nodal curve \(C\) with \(\delta = \delta_+ + \delta_- + 2\delta_0\) nodes, where \(\delta_\pm\) are of the form \(\mathbb{R}[[x, y]]/(x^2 \pm y^2)\) and the \(2\delta_0\) are complex conjugates,
\[
n_C^{R, \delta} = (-1)^{\delta_-} = (-1)^{\delta_+ - \delta}
\]
Moreover, \(n_C^{i, R} = 0\) for \(i > \delta\).

**Theorem 32.** Let \(C\) be a real reduced plane curve, and let \(C \to B\) be a versal deformation of its singularities. Let \(B_{+}^{\delta_-, \delta_+} \subset B(R)\) be the locus of nodal curves with \(\delta\) nodes of which \(\delta_-\) are of the form \(\mathbb{R}[[x, y]]/(x^2 - y^2)\). Let \(D_j^b\) be a general disc of dimension \(j\), preserved by complex conjugation, passing near \([C] \in B\). Then
\[
n_C^{j, R} = \sum_k (-1)^k D(R)^j \cap B_{+}^{j, k}
\]
In particular, \(n_C^{j, R} = 0\) for \(j > \delta(C)\).

**Proof.** View \(n^{j, R}\) as a constructible function on \(B(R)\) taking \(b \mapsto n_{C_b}^{j, R}\).

Let \(D_0^i \subset B\) be a general (complex but preserved by conjugation) disc of dimension \(i\) containing \([C]\). Then according to [FGvS, Sh], the first \(C_{D_0}^{[\leq i]}\) are all smooth, and if \(i \geq \delta\) then all \(C_{D_0}^{[n]}\) are smooth. The real locus of a smooth variety is smooth, so the same holds upon passing to real points. Taking \(D^{\delta+1}\) a disc containing \(D_0^\delta\) and \(D_1^\delta\) a sufficiently nearby slice, we have by smoothness that the spaces \(C_{D_0}^{[n]}(R)\) and \(C_{D_1}^{[n]}(R)\) are diffeomorphic and therefore have the same compactly supported Euler characteristics. Note that the first \(j\) Hilbert schemes suffice to determine \(n^{j, R}\). By additivity it follows that
\[
(3) \quad \int_{D_0(R)} n^{j, R} d\chi = \int_{D_1(R)} n^{i, R} d\chi \quad \text{for any } j \leq i, \text{ and for any } j \text{ at all if } i \geq \delta.
\]
We first show the vanishing of the $n^{j,R}$ for $j > \delta(C)$. Note we have already established it for smooth and nodal curves. We induct on $\delta(C)$. Take $i = \delta$ in Equality 3. By [DH, Tes], the locus of curves of cogenus at least $\delta$ is of codimension $\delta$ and is the closure of the locus of $\delta$-nodal curves. Thus by genericity its only intersection with $D_0^\delta$ is at the central point $[C]$, and its only intersection with $D_1^\delta$ is in finitely many nodal curves. Thus by induction and our explicit verification in the case of nodal curves, the integral on the right vanishes for $i > \delta$, hence so does the integral on the left, which again by induction is equal to $n^{i,R}_{C}$.

Now we consider the remaining $n^j$. Take $i = j$ in the above equality. Then by the same reasoning the only contribution to the integral on the left is $n_{C}^{i,R}$, and the only contribution to the integral on the right is the $j$-nodal curves. By our previous calculation, these contribute as required. □

Remark 33. If $C$ has a real line bundle of degree 1 then the proof of Proposition 14 carries through in $K_0(\text{var} / \mathbb{R})$; we may use this instead to conclude that $n^{l,R}_{C} = 0$ for $i > g(C)$. However we have not found an analogous argument if $C$ has no such bundle.

Remark 34. In [Sh], the proof of Theorem 1 used as an input the vanishing of the $n^i_C$ for $i > \delta$; the argument given above shows this was unnecessary.

Remark 35. Thus a necessary condition for the statement of Theorem 27 to hold for a singularity $X$ is the existence a curve $C$ containing a real form of $X$ with $\chi_1(Z_C) = \mathbb{Z}^2_C$.

3. Refined invariants of linear systems

Let $\pi : C \to B$ be a family of plane curves, and $\pi^{[n]} : C^{[n]} \to B$ the relative Hilbert schemes. Denote by $\text{MHM}(B)$ the mixed Hodge modules [Sai] over $B$. We define:

$$
Z_{C/B} = \sum_{n=0}^{\infty} \pi^{[n]}_{!} Q_{C^{[n]}} q^{n+1-g} \in K_0(\text{MHM}(B))[q]
$$

We define invariants $N_{C/B}^{i} \in K_0(\text{MHM}(B))$ by the formula

$$
\sum_{n=0}^{\infty} \pi^{[n]}_{!} Q_{C^{[n]}} q^{n+1-g} = \sum_{i=0}^{\infty} N_{C/B}^{i} \times Z_{C/B}^{i+1-g}
$$

Proposition 37. If $\pi : C \to B$ is a family of integral plane curves and moreover $C^{[n]}$ is smooth for all $n$, then $N_{C/B}^{i} = 0$ for $i > g$. 

Proof. In fact, according to [MY, MS] we know much more:

$$(1 - q)(1 - qQ_B[2](1)) \sum_{n=0}^{\infty} q^n n_i^{[n]} \mathcal{Q}_{\mathcal{C}^{[n]}}[\dim B + n] = \sum_{n=0}^{2g} q^n \text{IC}(B, \Lambda^i R^1 \pi^* \mathcal{Q}_C[\dim B + g])[-i]$$

Here $\pi$ is the restriction of the map to the locus on the base where it is smooth. We deduce the vanishing of the $\mathcal{N}_{\mathcal{C}/B}^i$ from the symmetry $\Lambda^i R^1 \cong \Lambda^{2g-i} R^1$.

□

Remark 38. One may write the same definition in $K_0(\var/\mathbb{B})$, but then we do not know whether $\mathcal{N}^i$ vanishes for $i > g$. The problem arises already for families of smooth curves; the question here is whether a family of Jacobians is equivalent to its torsors in the Grothendieck group of varieties.

We may take pointwise Euler characteristic to define an integer valued constructible function $n^i := \chi(\mathcal{N}^i) \in \text{Con}(B)$, or global Euler characteristic to define $n_{\mathcal{C}/B}^i \in \mathbb{Z}$. Since $n^i$ is supported on the locus of curves of cogenus $i$, certainly $n_{\mathcal{C}/B}^k$ vanishes if $k$ is greater than the maximum cogenus of any curve in the family. Note that $n_{\mathcal{C}/B}^k$ can be defined directly by the formula

$$\sum_{n=0}^{\infty} \chi(C^{[n]}) q^{n+1-g} = \sum_{i=0}^{\infty} n_{\mathcal{C}/B}^i \left( \frac{q}{(1-q)^2} \right)^{i+1-g}$$

Definition 39. We define $\mathcal{N}_{\mathcal{C}/B}^i(y) := \chi_y((B \to pt) \mathcal{N}^i)$. Equivalently we can apply $\chi_y \circ (B \to pt)$ to both sides of Equation [4] in order to directly define $\mathcal{N}_{\mathcal{C}/B}^i(y)$ by the formula

$$\sum_{n=0}^{\infty} \chi_y(C^{[n]}) q^{n+1-g} = \sum_{i=0}^{\infty} \mathcal{N}_{\mathcal{C}/B}^i(y) \left( \frac{q}{(1-q)(1-qy)} \right)^{i+1-g}$$

From Proposition [37] we see that $\mathcal{N}_{\mathcal{C}/B}^i(y)$ vanishes for $i$ greater than the maximum genus of any curve in a family whose relative Hilbert schemes are nonsingular. But for both the $\chi_y$ invariants of a single curve, and the $\chi$ invariants of families, we had vanishing beyond the maximum cogenus. Thus we may at least plausibly ask whether this holds for $\mathcal{N}_{\mathcal{C}/B}^i$. In fact it need not: Fantechi and Pandharipande observed that this vanishing can fail already for $B$ a curve of positive genus and $\mathcal{C} \to B$ a family of smooth curves. However, empirically, the situation appears to be better for linear systems of curves in surfaces. We have the following conjecture:

Conjecture 40. Let $L$ be a line bundle on a surface $S$, $\mathcal{C} \to |L|$ the tautological family of curves, and $\mathbb{P}^\delta \subset |L|$ be a linear subsystem. Assume the relative Hilbert schemes $\mathcal{C}_{\mathbb{P}^\delta}^{[n]}$ are nonsingular for all $n \geq 0$. Then $\mathcal{N}_{\mathcal{C}/\mathbb{P}^\delta}^i(y) = 0$ for $i > \delta$.

From the smoothness criterion in [FGvS] one may deduce that the maximum cogenus of any curve in any such $\mathbb{P}^\delta$ is $\delta$. The assumption holds in the following situations:

Theorem 41. Let $L$ be a line bundle on a surface $S$, $\mathcal{C} \to |L|$ the tautological family of curves, and $\mathbb{P}^\delta \subset |L|$ a general linear subsystem. Then all relative Hilbert schemes $\mathcal{C}_{\mathbb{P}^\delta}^{[n]}$ are nonsingular in the following situations:
• $S$ is arbitrary and $L$ is $\delta$-very ample $^{[KST, FGvS, Sh]}$

• $S$ is a K3 or abelian surface and $L$ is irreducible $^{[Muk]}$.

• $S$ is a rational surface and $P_δ$ contains no non-reduced curves, and no curves with components which intersect $K_S$ non-negatively $^{[KS]}$. In particular, for a general $P \leq 2d - 2 \subset |O_{P(2d)}|$.

Rather than simply integrate to get $n^i_{C/B} = \int_B n^i d\chi$, we can extract more refined information by taking Chern-Schwarz-Macpherson classes $c^SM_\bullet$. We recall that $c^SM_\bullet$ is the unique map from constructible functions to homology which commutes with pushforward and satisfies the normalization $c^SM_\bullet(1_X) = c(TX) \cap [X]$ when $X$ is smooth projective; its existence was conjectured by Deligne and Grothendieck and established by Macpherson $^{[Mph]}$. Now taking $n^i := c^SM_{\bullet}(n^i) \in H^\bullet(B)$, we see by Macpherson’s theorem that

$$\sum_{n=0}^{\infty} \pi^{|n|}_\bullet(1_X) c(TC^{[n]}(b)) q^{n+1-g} = \sum_{i=0}^{\infty} n^i_{C/B} \left(\frac{q}{1-q^2}\right)^{i+1-g}$$

Of course we are also free to take this as the definition of the $n^i_{C/B}$ and conclude from Macpherson’s theorem that $n^i_{C/B}$ vanishes if $i$ is greater than the maximum cogenus of any curve in the family. In good cases, the constructible function $n^i$ and the class $n^i$ carry singularity-theoretic meaning:

**Theorem 42.** Let $C/B$ be a family of reduced plane curves with all $C^{[n]}_B$ nonsingular. Let $B^i$ be the locus of curves of cogenus $i$, and let $B^i_+$ be the sublocus of curves smooth away from $i$ nodes. Assume $B^j \subset B^i_+$ for all $j \geq i$. Then

$$\text{mult}_b(B^i_+) = n^i(b) = E_{b}(B^i_+).$$

Here $E_b$ is the local Euler obstruction. Moreover, $n^i_{C/B}$ is the Chern-Mather class of $B^i_+$. 

**Proof.** The first equality was asserted in $^{[Sh]}$ in the case of a locally versal family, but in fact the same argument applies in the above generality.

Let $\Sigma$ be any stratum of a Whitney stratification for $B^i$, other than the big open set given by the intersection of $B^i_+$ and the smooth locus of $B^i$. Let $D^{k+1} \subset B$ be a generic small disc transverse to $\Sigma$ at some $\sigma$; note that $k \geq i$. Let $D^k_\sigma \subset D^{k+1}$ be a general linear slice not passing through $\sigma$. Let $\chi^i$ denote the constructible function $b \mapsto \chi(c^{[n]}_b)$. By the result of $^{[Sh]}$ on smoothness of relative Hilbert schemes (note $k \geq i$),

$$\chi^i(\sigma) = \int_{D^k_\sigma} \chi^i d\chi = \int_{D^k} \chi^i d\chi$$

$^{3}$ What is actually observed in $^{[KST]}$ is the (obvious) fact that the first $\delta$ relative Hilbert schemes are nonsingular under this hypothesis, and the less obvious fact that the assumption implies that no non-reduced curves or curves of cogenus $> \delta$ occur in the $\mathbb{P}^\delta$. Consequently the smoothness criteria in $^{[FGvS, Sh]}$ may be used to establish smoothness of the remaining relative Hilbert schemes.
As the \( n^i \) are linear combinations of the \( \chi_i \), they satisfy the same formula:

\[
n^i(\sigma) = \int_{D^i} n^i d\chi
\]

By induction on codimension of the strata this formula determines \( n^i(\sigma) \) uniquely from its constant value 1 on the smooth locus of \( B^i_+ \). But according to Dubson-Brylinksi-Kashiwara [BDK], the Euler obstruction satisfies (and is determined by) the same slice formula\(^4\) The identification of \( n^i \) with the Chern-Mather class of \( B^i_+ \) now follows from Macpherson’s construction of the functorial Chern class.

\[\square\]

**Remark 43.** The assumption on genericity of nodal curves in Theorem 42 holds for a locally versal family by [DH, Tes], for the general \( P^\delta \subset |L| \) when \( L \) is \( \delta \)-very-ample by [KST], for the general \( P^\delta \subset |L| \) when \( L \) is irreducible on a general K3 surface (the genericity of nodal curves in maximal cogenus is [Che]), and it was known classically for the general \( P^\delta \subset |O_P^2| \) containing no nonreduced curves.

**Remark 44.** Aluffi [Alu] has shown that the multiplicity and and Euler obstruction of the discriminant of cubic curves on \( P^2 \) differ at a triple line. The argument above fails here because the total space of the restriction of the universal family to a one dimensional disc passing through this point is necessarily singular. Aluffi has also managed to extract enumerative information about curves with singularities more complicated than nodes from the Chern-Mather and Chern-Schwarz-Macpherson classes of discriminants; perhaps the same can be done with the higher Severi strata.

The map \( c^S_M \) admits a refinement due to Brasselet, Schürmann, and Yokura. We denote it \( X^{BSY}_{-y} : \text{MHM}(\cdot) \to H_*(\cdot)[y] \). It commutes with pushforward and obeys the normalization \( X^{BSY}_{-y}(\mathbb{Q}_M) = X_{-y}(TM) \cap [M] \) for \( M \) proper smooth. Thus we may apply their functor to the \( N^i_{C/B}(y,H) \) and conclude that there are \( N^i_{C/B}(y) = X^{BSY}_{-y}(N^i) \in H_*(B)[y] \) such that \( N^i_{C/B}(y) = 0 \) for \( i \) greater than the arithmetic genus of the curves (in a family of integral curves with nonsingular relative Hilbert schemes), and:

\[
\sum_{n=0}^{\infty} \pi^*[n](X_{-y}(TC^{[n]}) \cap [C^{[n]}])q^{n+1-g} = \sum_{i=0}^{\infty} N^i_{C/B}(y) \left( \frac{q}{(1-q)(1-qy)} \right)^{i+1-g}
\]

In the case of interest when \( B = P^\delta \), we denote by abuse \( N^i_{C/B}(y,H) \in H^*(P^\delta) \) the Poincaré dual class of \( N^i_{C/B}(y) \). Since \( N^i_{C/B}(1) \) is, in good cases, the Chern-Mather class of the (codimension \( i \)) Severi variety of cogenus \( i \) curves, we might expect:

\[\footnote{4}{More generally for any equidimensional map \( f : X \to Y \) between smooth varieties, \( f_* C_X = \sum \lambda_i \text{Eu}(\Delta^i(f)) \) where the \( \lambda_i \) are some constants and the \( \Delta^i \) are the following nonsmoothability loci: \( \Delta^{i+1}(f) := \{ y \in Y | X_{D^i} \text{ is singular for every } i \text{ dimensional disc } D^i \ni y \} \).
}

The relation to the present situation is that the assumption and the smoothness criteria of [Sh] guarantee \( \Delta^i = B^i_+ \). Now the coefficients \( \lambda^i \) may be calculated at a smooth point of \( B^i_+ \).
**Conjecture 45.** Let $L$ be a line bundle on a surface $S$, and let $\mathbb{P}^\delta \subset |L|$ be a linear subsystem of reduced curves over which the relative Hilbert schemes $C_{[n]} \rightarrow \mathbb{P}^\delta$ are nonsingular for all $n \geq 0$. Then $N_{C/\mathbb{P}^\delta}(y, H)$ is a polynomial of minimal degree $i$ in $H$, and in particular vanishes for $i > \delta$.

We will see shortly that this conjecture is in fact equivalent to Conjecture 40. First we recall how the Hirzebruch genera of the relative Hilbert schemes may be computed.

### 3.1. Genera of relative Hilbert schemes.

Following Hirzebruch [Hi], we take a (normalized) genus to mean a natural transformation of contravariant functors $\Phi : K^0(\cdot) \rightarrow H^*(\cdot, \Lambda)$ (where $\Lambda$ is a commutative ring) such that

- For the trivial bundle $\mathbb{C}$, we have $\Phi(\mathbb{C}) = 1$.
- Sums go to products: $\Phi(E \oplus F) = \Phi(E)\Phi(F)$.
- There is a power series $f_\Phi \in 1 + z\Lambda[[z]]$ such that, for a line bundle $L$, we have $\Phi(L) = f_\Phi(c_1(L))$.

In the remainder of the paper we will be concerned only with the Hirzebruch genus $\Phi = X_{-y}$, for which $\Lambda = \mathbb{Q}[[y]]$ and $f(z) = \frac{z(1-y-e^{-y(1-y)})}{(1-e^{-y(1-y)})}$. In any case fix some $\Lambda, \Phi$. We write $\phi(X) := \Phi(TX)$.

Let $S$ be a surface, $L$ a line bundle on it, $\mathbb{P}^\delta \subset |L|$ some linear system, $H = O_{\mathbb{P}^\delta}(1)$. Let $S^{[n]}$ be the Hilbert scheme of $n$ points on $S$, and let $Z_n(S) \subset S \times S^{[n]}$ be the universal family, with the projections $q : Z_n(S) \rightarrow S$, $p : Z_n(S) \rightarrow S^{[n]}$. Let $L^{[n]} := p_* q^* L$. This is a vector bundle of rank $n$ on $S^{[n]}$ with fibre $H^0(Z, L|_Z)$ over $Z \in S^{[n]}$. Let $\pi : C_{p^\delta} \rightarrow \mathbb{P}^\delta$ be the universal curve over $\mathbb{P}^\delta$ and denote by $\pi^{[n]} : C_{p^\delta}^{[n]} \rightarrow \mathbb{P}^\delta$ the relative Hilbert scheme of points. The relative Hilbert scheme always has the expected dimension $\delta + n$ [AIK], and is the scheme theoretic zero locus of a tautological section of $L^{[n]} \boxtimes H$; when $C_{p^\delta}^{[n]}$ is nonsingular this section is transverse.

As a book-keeping device, let $e^x$ denote a trivial line bundle with nontrivial $\mathbb{C}^*$ action giving equivariant first Chern class $x$, i.e., $\Phi(e^x) = f_\Phi(x)$.

**Definition 46.** Let

$$D_n^{S,L,\Phi}(x) := \int_{S^{[n]}} \Phi(TS^{[n]}) \frac{c_n(L^{[n]} \otimes e^x)}{\Phi(L^{[n]} \otimes e^x)} \in \Lambda[[x]]$$

**Proposition 47.** Assume $C_{p^\delta}^{[n]}$ is nonsingular. Then

$$\pi^{[n]}_*(\Phi(TC_{p^\delta}^{[n]}) \cap [C^{[n]}]) = f_\Phi(H)^{\delta+1} D_n^{S,L,\Phi}(H) \cap [\mathbb{P}^\delta]$$

In particular,

$$\phi(C_{p^\delta}^{[n]}) = \text{Res}_{x=0} \left( \frac{f_\Phi(x)}{x} \right)^{\delta+1} D_n^{S,L,\Phi}(x)$$
Proof. Denote \( q : S^{[n]} \to pt \) and \( p : \mathbb{P}^d \to pt \). Then
\[
\pi_*\left( \Phi(TC_{\mathbb{P}^d}) \cap [C^{[n]}] \right)
= \left( q \times 1_{\mathbb{P}^d} \right)_* \left( c_n(L^{[n]} \boxtimes H) \Phi(TS^{[n]}) \Phi(T\mathbb{P}^d[\delta]) \Phi(L^{[n]} \boxtimes H) \right) \cap [S^{[n]}] \cap [\mathbb{P}^d]
= \Phi(H)^{\delta+1} q_* \left( \Phi(TS^{[n]}) c_n(L^{[n]} \otimes e^x) \right) \bigg|_{x=H} \cap [\mathbb{P}^d]
\]

\[\square\]

Remark 48. The formula makes sense without requiring smoothness, if we view it as a virtual fundamental class and a virtual tangent bundle (see e.g. \([FG]\)). Thus independent of the singularities of \( C^{[n]} \), what is computed here is \( \phi(C^{[n]}) \) with this virtual structure.

The \( D^{S,L,\Phi}_n \) enjoy a certain multiplicativity. Introduce the series
\[
D^{S,L,\Phi} = \sum D^{S,L,\Phi}_n q^n
\]

We denote by \([S,L]\) the algebraic cobordism class of the pair of the surface \( S \) and the line bundle \( L \). For us, this is the equivalence class of pairs \((S,L)\), where two such \((S_1,L_1)\), \((S_2,L_2)\) are equivalent if the numbers \( L_i^2, L_iK_S, K^2_S, c_2(S_i) \) coincide for \( i = 1,2 \). These form a group where in particular \([S_1,L_1] + [S_2,L_2] = [S_1 \sqcup S_2,L] \) with \( L \) the line bundle which is \( L_1 \) on \( S_1 \) and \( L_2 \) on \( S_2 \).

Proposition 49. The map \([S,L] \mapsto D^{S,L,\Phi}_n\) is a homomorphism from the cobordism group of surfaces with bundles to the multiplicative group of invertible power series in \( q \) with coefficients in \( \Lambda \). In particular there exist universal power series \( D_1, D_2, D_3, D_4 \in \Lambda[[q]] \) such that
\[
D^{S,L,\Phi}_n = D_1^{L^2} D_2^{L K_S} D_3^{K^2_S} D_4^{c_2(S)}.
\]

Proof. The \( D^{S,L,\Phi}_n(x) \) are defined by a genus applied to \( L^{[n]} \) and \( T_{S^{[n]}} \), so the first statement follows from the arguments in \([EG]\). The second follows formally.

The multiplicativity of Proposition 49 allows \( D^{S,L,\Phi}_n(x) \) (and therefore also \( \phi(C^{[n]}) \) and the BPS invariants) to be computed by localization in the following standard way.

If \( (S_i,L_i) \), \( i = 1, \ldots, 4 \) are such that the 4 vectors \((L_i^2, L_iK_S, K^2_S, c_2(S_i))\) are linearly independent, then for any \( (S,L) \) we can write \([S,L] = a_1[S_1,L_1] + a_2[S_2,L_2] + a_3[S_3,L_3] + a_4[S_4,L_4] \), with \( a_i \in \mathbb{Q} \) and thus
\[
D^{S,L,\Phi} = \prod_{i=1}^4 (D^{S,L_i,\Phi})^{a_i},
\]
thus to compute the \( D^{S,L,\Phi} \) it is enough to compute them for the \( (S_i,L_i) \).

\[\text{For a geometric account of algebraic cobordism of varieties with bundles see \([LP]\), we however do not require any results of this theory and only use “cobordism class” as a convenient shorthand.}\]
We can choose the \((S_i, L_i)\) as \((\mathbb{P}^2, \mathcal{O}), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 0)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, -1))\).

In this case \(S = S_i\) is a toric surface, i.e. it has an action by a torus \((\mathbb{C}^*)^2\) with finitely many fixed points, and \(L = L_i\) has an natural equivariant lifting. The action of \((\mathbb{C}^*)^2\) on \(S\) induces in a natural way an action on \(S^{[n]}\), and the equivariant lifting of \(L\) induces an equivariant lifting of \(L^{[n]}\). Thus we can apply equivariant localization to compute \(D_n^{S,L,Y}(x)\), in terms of the weights of the action on the fibres of \(T_{S^{[n]}}\) and \(L^{[n]}\) at the fixed points. The fixed points are parametrized by tuples of Young diagrams and the weights of the action can be expressed explicitly in terms of this data. For more details in a slightly different situation see e.g. [EG, NY, CO].

From now on we specialize \(\Phi = X_{-y}\) and abbreviate \(D_{n}^{S,L}(y, x) := D_{n}^{S,L,X_{-y}}(x)\), \(D_{n}^{S,L}(y, x, q) := \sum_{n \geq 0} D_{n}^{S,L}(y, x)q^n\). In this case a computer calculation yields the \(D_{n}^{S,L}(y, x)\) for \(n \leq 10\) and modulo \(x^{14}\). The \(\chi_{-y}(\frac{[n]}{x})\) are computed from this by Proposition 47.

3.2. A reformulation of the conjectures, and evidence. Let \(t_i\) be the Chern roots of the tangent bundle of the Hilbert scheme, and let \(l_i\) be the Chern roots of the bundle \(L^{[i]}\). In the previous subsection we introduced the series

\[
D_{n}^{S,L}(y, x, q) := \sum_{i=1}^{2n} q^n \int_{S^{[n]}} \prod_{i=1}^{2n} \left( 1 - ye^{-t_i(1-y)} \right) \prod_{j=1}^{n} \left( 1 - e^{-y(l_j + x)(1-y)} \right) \in \mathbb{Q}[y][[x]][[q]]
\]

For convenience we write \(Q = q/((1 - q)(1 - qy))\).

By Proposition 47 we have

\[
\sum_{i=0}^{\infty} N_{C/\mathbb{P}^2}(y, H)Q^i = \left( \frac{Q}{Q} \right)^{1-y} \left( \frac{H(1 - ye^{-H(1-y)})}{1 - e^{-H(1-y)}} \right)^{\delta + 1} D_{n}^{S,L}(y, H, q)
\]

Conjecture 50. For any surface \(S\) and line bundle \(L\), we have

\[(q/Q)^{1-g(L)} D_{n}^{S,L}(y, x, q) \in \mathbb{Q}[y][[x, xQ]]\]

Proposition 51. Conjectures 44, 45, and 36 are equivalent.

6 We record here that the compositional inverse is given by what are called the Narayana numbers,

\[q(Q) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} Q^n y^{k-1} (-1)^{n-1} \frac{n}{k} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} n \\ k-1 \end{array} \right)\]

which specializes to the following formulas involving Catalan numbers,

\[q(Q)|_{y=1} = \sum_{n=1}^{\infty} Q^n (-1)^{n-1} \frac{2n}{n+1} \left( \begin{array}{c} n \\ n \end{array} \right)\]

\[q(Q)|_{y=0} = \frac{Q}{1+Q}\]

\[q(Q)|_{y=-1} = \sum_{n=0}^{\infty} Q^{2n+1} (-1)^n \frac{4n}{2n+1} \left( \begin{array}{c} 2n \\ n \end{array} \right)\]

Note also that \(Q(q) \in q\mathbb{Z}[q, qy]\) and \(q(Q) \in q\mathbb{Z}[Q, Qy]\).
Proof. Note \( f(x) := X_{\gamma}(e^x) \in 1 + x\mathbb{Q}[g][[x]] \) is invertible. From Equation 7, we see that Conjecture 50 implies Conjecture 40 and Conjecture 45.

Assume Conjecture 40. Now consider some fixed linear system \( \mathbb{P}^s \subset |L| \) on some surface \( S \) such that all the relative Hilbert schemes \( \mathcal{C}^{[n]} \to \mathbb{P}^s \) are nonsingular. Conjecture 45 amounts to the statement that

\[
\left( \frac{Q}{q} \right)^{g-1} f(x)^{\delta+1} D^{S,L}(y, x, q) \in \mathbb{Q}[[x, xQ]] + O(x^{\delta+1})
\]

This obviously holds at \( \delta = 0 \); let us prove it holds at \( \delta \) by induction. If we know this statement holds for some \( \delta = r - 1 \) and wish to check it for \( \delta = r \), since \( f(x) \in \mathbb{Q}[[x]] \) we already know the statement modulo \( x^r \). So we need only check

\[
\deg_q \text{Coeff} \left( \frac{Q}{q} \right)^{g-1} f(x)^{r+1} D^{S,L}(y, x, q) \leq r.
\]

But this is precisely the assertion of Conjecture 40 for \( \mathbb{P}^r \). And since all the relative Hilbert schemes will also be smooth over a general \( \mathbb{P}^r \subset \mathbb{P}^s \) for any \( r \leq \delta \), the hypothesis of Conjecture 40 is satisfied and we may deduce Conjecture 45.

Finally, Conjecture 45 asserts that Conjecture 50 holds modulo \( x^{\delta+1} \). We have an expression \( D^{S,L} = D_1^2 D_2^L K_3^s D_3^k K_3^s D_4^c(S) \) where the \( D_i \) are power series starting with 1. Thus we compare \( D^{S,L} \) for various surfaces and line bundles to conclude the statement of Conjecture 50 for the series \( D_i \), modulo some \( x^k \). Taking \( k \to \infty \) by choosing increasingly ample line bundles recovers the statement for \( D^{S,L} \).

\[\square\]

Remark 52. The above argument implies in particular that Conjecture 50 holds in the Euler characteristic limit \( y = 1 \). From this it follows formally that for any \( \mathbb{P}^s \subset |L| \), with no assumptions on the reducedness or irreducibility of the curves that appear or on the smoothness of the relative Hilbert schemes, there are integers \( n_{c/p}^i \) such that

\[
\sum_{n=0}^{\infty} q^{n+1-g} \int_{[\mathcal{C}^{[n]}]} c_{top}(T^{vir} \mathcal{C}^{[n]}) = \delta \sum_{i=0}^{\delta} n_{c/p}^i \left( \frac{q}{(1-q)^2} \right)^{i+1-g}
\]

Recall that \( D^{S,L} \) can be expressed in four universal power series,

\[
D^{S,L} = D_1^2 D_2^L K_3^s D_3^k K_3^s D_4^c(S).
\]

To avoid writing \( \left( \frac{Q}{q} \right)^{g-1} \) we adjust these series slightly.

Definition 53. We write \( \tilde{D}^{S,L} := (Q/q)^{g-1} D^{S,L} \). We also take \( \tilde{D}_1 := (Q/q)^{1/2} D_1 \), \( \tilde{D}_2 := (Q/q)^{1/2} D_2 \), and \( \tilde{D}_3 = D_3 \), \( \tilde{D}_4 = D_4 \) in order that \( \tilde{D}^{S,L} = \tilde{D}_1^2 \tilde{D}_2^L \tilde{D}_3^k \tilde{D}_3^s \tilde{D}_4^c(S) \).

We have \( \tilde{D}^{S,L} \in 1 + (y, x, Q)\mathbb{Q}[y][[x, Q]] \) for all \( S, L \), hence the same is true for the \( \tilde{D}_i \). Similarly, Conjecture 50 is equivalent to the assertion that \( \tilde{D}_i \in \mathbb{Q}[y][[x, xQ]] \) for all \( i \).

Theorem 54. \( \tilde{D}_1, \tilde{D}_4 \in \mathbb{Q}[y][[x, xQ]] \).
Proof. Let \((A, L)\) be a primitively polarized abelian surface of Picard rank 1. If \(L^2 = 2k+2\) then \(\dim |L| = k\) and the curves in \(|L|\) have arithmetic genus \(k + 2\). Note such \((A, L)\) exist for all \(k\). By [Muk] the relative Hilbert schemes are smooth, and so from Proposition 37 we find that the \(N^i\) vanish beyond the arithmetic genus. By this vanishing and the formula \((7)\) extracting the \(N^i\) from \(D^{S,L}\):

\[
\deg_Q \text{Coeff}_{x^k} f_\Phi(x)^{k+1} \tilde{D}^{2k+2}_1 \leq k + 2
\]

We write \(\xi(x, Q) = f_\Phi(x) \tilde{D}^2_1 \in 1 + (x, Q)\mathbb{C}[y][[x, Q]]\). We want to show \(\tilde{D}_1 \in \mathbb{C}[y][[x, Q]]\); since this evidently holds for \(f_\Phi(x)\) and we may take roots of power series starting with 1, it suffices to show this for \(\xi\). So we have \(\deg_Q \text{Coeff}_{x^k} \xi(x, Q)^{k+1} \leq k + 2\).

The following argument is completely formal and does not involve the geometric meaning of \(\xi\). We write \(d_Q(k) := \deg_Q \text{Coeff}_{x^k} \xi(x, Q)^{k+1}\). Let \(k_1 = \min\{k | d_Q(k) > k\}\), assuming this set is nonempty. Then \(d_Q(k_1) = \deg_Q \text{Coeff}_{x^{k_1}} \xi(x, Q)^{k_1+1}\), since no lower (in \(x\)) degree term can contribute such a high power of \(Q\). There are two cases, \(d_Q(k_1) = k_1 + 2\) or \(d_Q(k_1) = k_1 + 1\). In the first case, consider \(\text{Coeff}_{x^{k_1}} \xi(x, Q)^{2k_1+1}\). There will be a contribution from products of two terms of the form \(Q^{k_1+2}x^{k_1}\), which gives the highest possible power of \(Q\) and thus cannot be canceled. But then \(\deg_Q \text{Coeff}_{x^{k_1}} \xi(x, Q)^{2k_1+1} = 2k_1 + 4\), which is a contradiction. In the second case, consider \(\text{Coeff}_{x^{k_1+k_2}} \xi(x, Q)^{3k_1+1}\). In order that the degree \(3k_1 + 3\) contribution from products of three terms \(Q^{k_1+1}x^{k_1}\) be cancelled, there must be some \(h + h' = 3k_1\) with \(d_Q(h) = h + 2\) and \(d_Q(h') > h'\). By minimality of \(k_1\), we have \(h' > k_1\) hence \(h < 2k_1\). Let \(k_2 = \min\{k | d_Q(k) > k + 1\} \leq h < 2k_1\). Finally consider \(\text{Coeff}_{x^{k_1+k_2}} \xi(x, Q)^{k_1+k_2+1}\). There is a contribution from products of terms the form \(x^{k_1}Q^{k_1+1}\) and \(x^{k_2}Q^{k_2+2}\); since \(k_2 < 2k_1\) this contribution cannot be cancelled. This is a contradiction. So finally we must have \(d_Q(k) \leq k\) for all \(k\), hence \(\xi(x, Q) \in \mathbb{C}[y][[x, Q]]\), hence the same holds for \(\tilde{D}_1\).

Now let \((K, L)\) be a primitively polarized K3 surface of Picard rank 1. If \(L^2 = 2g - 2\) then \(\dim |L| = g\) and the curves in \(|L|\) have genus \(g\). Such \((K, L)\) exist for all \(g\). By vanishing of the \(N^i\) beyond the arithmetic genus we have

\[
\deg_Q \text{Coeff}_{x^g} f_\Phi(x)^{g+1} \tilde{D}^{2g-2}_1 \tilde{D}^2_4 \leq g
\]

Since we know \(f_\Phi(x)\), \(\tilde{D}_1 \in \mathbb{C}[y][[x, Q]]\), we may conclude the same for \(D_4\). \(\square\)

**Corollary 55.** Conjectures 40, 45, and 50 hold for surfaces with numerically trivial canonical class.

**Remark 56.** Note the slightly curious nature of the proof of the theorem and corollary: for geometric reasons, namely smoothness of the relative Hilbert schemes and the near equality of the genus and dimension of the linear system for certain line bundles on K3 and abelian surfaces, we know the conjecture for complete linear systems on K3 surfaces and something close for abelian surfaces. Then by leveraging the universality of the
expressions, and the existence of K3 and abelian surfaces of all genera, we can conclude the result also for not-necessarily-complete linear systems.

This sort of approach was suggested to the authors by Pandharipande [Pan], who further suggested that the other power series may be similarly constrained by finding enough other surfaces with nontrivial canonical class but for which nonetheless the genus and dimension of linear systems are close. However to use these (or any) surfaces for the present purposes, one must establish smoothness of the relative Hilbert schemes, which we do not know how to do.

Using the localization calculation described in Section 3.1, we can give evidence for Conjecture 50 for arbitrary surfaces.

**Proposition 57.** Conjecture 50 holds modulo $q^{11}$ and $x^{14}$. Therefore, if $0 \leq \delta \leq 13$ and $\mathbb{P}^3 \subset |L|$ is a linear system over which the relative Hilbert scheme is smooth, there exist polynomials $N_i^{\delta, [S,L]}(y) \in \mathbb{Z}[y]$, where $i = 0, \ldots, \delta$, such that

$$\sum_{n \geq 0} \chi_{-y}(C^{[n]})q^{n+1-g} \equiv \sum_{l=0}^{\delta} N_{i, \mathbb{P}^3}^{\delta}(y)Q^{l+1-g} \mod O(q^{11+1-g}),$$

and furthermore these polynomials are explicitly computed. If moreover $g < 11$ and all curves are irreducible, then the equality is established to all orders.

For example, Conjecture 40 holds for a general $\mathbb{P}^4$ in $|O_{\mathbb{P}^2}(6)|$.

The relation between the various $n, n_i, N_i, N_i^{\delta, [S,L]}$ and the series $D_{S,L}$ has in this section always been contingent on the smoothness of the relative Hilbert schemes over the appropriate linear subsystem $\mathbb{P}^\delta \subset |L|$. To avoid continually making this hypothesis, we introduce the following:

**Definition 58.** For a surface $S$ and a line bundle $L$, we define $N_i^{\delta, [S,L]}$ by the formula

$$\sum_{i=0}^{\infty} N_i^{\delta, [S,L]} Q^i = \left( H(1 - ye^{-H(1-y)}) \right)^{\delta+1} \tilde{D}_{S,L}(y, H, q)$$

and similarly for the specializations $n, n_i, N_i$.

By comparison with Equation 7 we see that for a linear system $\mathbb{P}^\delta \subset |L|$ containing only reduced curves, and whose relative Hilbert schemes are smooth, $N_i^{\delta, [\mathbb{P}^3 \subset |L|]} = N_i^{\delta, [S,L]}$.

### 4. The Term of the Deepest Stratum

For a linear system $\mathbb{P}^\delta \subset |L|$, the numbers $n_i^{\delta, [\mathbb{P}^3]}$ have the clearest enumerative significance, counting the number of $\delta$-nodal curves in the linear system. Thus we might also hope that the $N_i^{\delta, [\mathbb{P}^3]}(y)$ have an enumerative meaning refining this. In any case, assuming Conjecture 50, the $N_i^{\delta, [S,L]}(y)$ are the easiest to compute and their generating function is multiplicative.
Proposition 59. Assume Conjecture [50] or $K_S = 0$. View $\tilde{D}^{S,L}$ as an element of $\mathbb{Q}[y[[x,s]]$ with $s = xQ$. Then
$$\sum_{\delta \geq 0} N_{\delta, [S,L]}^S \delta = \tilde{D}^{S,L}(y, x = 0, s).$$

Corollary 60. Assume Conjecture [50] or $K_S = 0$. Then there exist series $A_i \in \mathbb{Q}[y[[s]]$ such that
$$\sum_{\delta \geq 0} N_{\delta, [S,L]}^S \delta = A_1 L^2 A_2 L K_S A_3^2 A_4^2(s).$$

Proof. Viewing $\tilde{D}_i \in \mathbb{C}[[x,s]]$ where $s = xQ$, take $A_i := \tilde{D}_i|_{x=0}$. □

In the unrefined ($y = 1$) setting, more explicit formulas were expressed after substituting for $s$ a certain quasimodular form. Specifically, in [Götl, Conj. 2.4], the following expansion was proposed:

$$\sum_{\delta \geq 0} N_{\delta, [S,L]}^S \delta \cdot (DG_2)^\delta = \frac{(DG_2/q)^{\chi(L)} B_1 K_S^2 D_2^{LK_S}}{(\Delta \cdot DDG_2/q^2)^{\chi(O_3)/2}}$$

Here $G_2$ is the Eisenstein series, $\Delta$ is the discriminant:
$$\Delta(q) = q \prod_{n=1}^\infty (1 - q^k)^{24}$$
$$G_2(q) = \frac{1}{24} + \sum_{m=1}^\infty q^m \sum_{d|m} d$$

The series $B_1, B_2 \in 1 + q\mathbb{Q}[[q]]$ are not known explicitly, although their first several coefficients may be computed and are given in [Götl]. We also write $D = \frac{d}{dq}$. The above formula is by now a theorem, since the existence of universal formulas has been established [Lim, Tze, KST], and the case of K3 surface (where $K_S = 0$ hence the $B_i$ do not appear) was solved explicitly [BL].

Notation 61. We write $N_{\delta, [S,L]}^S := N_{\delta, [S,L]}/y^\delta$.

Now we give a conjectural refinement of Equation 8. The series $\Delta, DG_2$ are refined as follows:

$$\tilde{\Delta}(y, q) := q \prod_{n=1}^\infty (1 - q^n)^{20}(1 - yq^n)^2(1 - y^{-1}q^n)^2,$$
$$\tilde{DG}_2 := \sum_{m=1}^\infty mq^m \sum_{d|m} \frac{[d]^2}{d}$$

These functions are related to certain Jacobi forms. Let $q = e^{2\pi i \tau}, y = e^{2\pi iz}$

(1) $\tilde{\Delta}(y, q) = \phi_{10,1}(\tau, z)/(y^{1/2} - y^{-1/2})^2$. Here $\phi_{10,1}(\tau, z) = \eta(\tau)^{18}\theta(\tau, z)^2$ is up to normalization the unique Jacobi cusp form on $Sl_2(\mathbb{Z})$ of weight 10 and index 1.
Conjecture 62. There exist universal power series $B_1(y, q)$, $B_2(y, q)$ in $\mathbb{Q}[y, y^{-1}][[q]]$, such that

\begin{equation}
\sum_{g \geq 0} \mathcal{N}_{g,[S,L]}(y)(\tilde{D}G_2)^g = \frac{(\tilde{D}G_2/q)^{\chi(L)}B_1(y, q)^{K_2}B_2(y, q)^{LK_2}}{\left(\Delta(y, q) \tilde{D}D^2/q^2\right)^{\chi(OS)/2}}
\end{equation}

Here, to make the change of variables, all functions are viewed as elements of $\mathbb{Q}[y, y^{-1}][[q]]$.

This conjecture is again checked modulo $q^{11}$; and we get that

\begin{align*}
B_1(y, q) &= 1 - q - ((y^2 + 3y + 1)/y)q^2 + ((y^4 + 10y^3 + 17y^2 + 10y + 1)/y^2)q^3 - ((18y^4 + 87y^3 + 135y^2 + 87y + 18)/y^3)q^4 + ((12y^6 + 120y^5 + 728y^4 + 1061y^3 + 728y^2 + 210y + 12)/y^4)q^5 \\
&- ((2y^8 + 259y^7 + 2102y^6 + 5952y^5 + 8236y^4 + 5952y^3 + 2102y^2 + 259y + 2)/y^5)q^6 + ((162y^8 + 3606y^7 + 19668y^6 + 48317y^5 + 64253y^4 + 48317y^3 + 19668y^2 + 3606y + 162)/y^6)q^7 \\
&- ((47y^{10} + 3789y^9 + 419999y^8 + 177800y^7 + 392361y^6 + 505678y^5 + 392361y^4 + 177800y^3 + 419999y^2 + 3789y + 47)/y^7)q^8 + ((5y^{12} + 2416y^{11} + 6020y^{10} + 445989y^9 + 1576410y^8 + 3197831y^7 + 4018919y^6 + 3197831y^5 + 1576410y^4 + 445989y^3 + 6020y^2 + 2416y + 5)/y^8)q^9 \\
&- ((896y^{12} + 58504y^{11} + 793194y^{10} + 4483755y^9 + 13818256y^8 + 26192369y^7 + 32243357y^6 + 26192369y^5 + 13818256y^4 + 4483755y^3 + 793194y^2 + 58504y + 896)/y^9)q^{10} + O(q^{11}), \\
B_2(y, q) &= \frac{1}{(1 - y^2)(1 - q/y)}(1 + 3q - ((3y^2 + y + 3)/y)q^2 + ((y^4 + 8y^3 + 18y^2 + 8y + 1)/y^2)q^3 \\
&- ((13y^4 + 53y^3 + 76y^2 + 53y + 13)/y^3)q^4 + ((7y^6 + 100y^5 + 316y^4 + 455y^3 + 316y^2 + 100y + 7)/y^4)q^5 \\
&- ((67y^8 + 1243y^7 + 6129y^6 + 14386y^5 + 18870y^4 + 14386y^3 + 6129y^2 + 1243y + 67)/y^5)q^6 \\
&- ((19y^{10} + 1281y^9 + 12417y^8 + 48879y^7 + 104034y^6 + 132579y^5 + 104034y^4 + 48879y^3 + 12417y^2 + 1281y + 19)/y^6)q^7 \\
&- ((2y^{12} + 822y^{11} + 17542y^{10} + 117829y^9 + 393703y^8 + 775411y^7 + 965540y^6 + 775411y^5 + 393703y^4 + 117829y^3 + 17542y^2 + 822y + 2)/y^7)q^8 \\
&- ((310y^{12} + 17206y^{11} + 207047y^{10} + 1085712y^9 + 3197506y^8 + 5913778y^7 + 722539y^6 + 5913778y^5 + 3197506y^4 + 1085712y^3 + 207047y^2 + 17206y + 310)/y^8)q^{10} + O(q^{11}).
\end{align*}

(2) We can write

\begin{align*}
(y - 2 + y^{-1})\tilde{D}G_2 &= \sum_{m=1}^{\infty} q^m \sum_{d|m} m(d^d - 2 + y^{-d}) \\
&= -2(G_2(\tau) + 1/24) + \sum_{d,e>0} e(y^d - y^{-d})q^{de} \\
&= -\frac{1}{2}D \log \left(\frac{\phi_{10,1}(\tau, z)}{\Delta(\tau)}\right) = -\frac{1}{2}D \log (\phi_{-2,1}(\tau, z))
\end{align*}

Here $\Delta(\tau)$ is the discriminant function and $\phi_{-2,1} = \phi_{10,1}/\Delta$ is the up to normalization unique weak Jacobi cusp form of weight $-2$ and index 1 on $SL_2(\mathbb{Z})$. 

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At $y = 1$, we recover modulo $q^{11}$ the functions $B_1(q)$, $B_2(q)$ of \cite{Göt}.

As in \cite{Göt} Rem. 2.6], the expansion in $\bar{D}G_2$ may be exchanged for an expansion in $q$ while simultaneously trading a sum over varying numbers of point conditions while fixing the line bundle for a sum over line bundles while fixing the point conditions. Note the latter form is more natural from the point of view of the GW/DT/pairs theories, and indeed this is the form in which the K3 is solved in \cite{BL}.

In detail this procedure is as follows. For any power series $f \in R[[q]]$ and $g \in q + q^2 R[[q]]$, we may expand $f$ in terms of $g$ by the residue formula:

$$f(q) = \sum_{l=0}^{\infty} g(q)^l \left[ \frac{f(q)Dg(q)}{g(q)^{l+1}} \right]_{q=0}$$

Conjecture 62 asserts that $N_{\delta,[S,L]}$ is the coefficient of $\bar{D}G_2^\delta$ of a certain expression; taking this coefficient by the above residue formula gives the equivalent formulation:

$$N_{\delta,[S,L]}(y) = \left[ \bar{D}G_2^{-\delta-1} D\bar{D}G_2 \frac{(D\bar{G}_2/q)^{\chi(L)}B_1(y,q)K_S^2B_2(y,q)^{LK_S}}{(\Delta(y,q) D\bar{D}G_2/q^2)^{\chi(O_S)/2}} \right]_{q=0} = \text{Coeff}_{q^{(L^2-LK_S)/2}} \left( \frac{(D\bar{G}_2)^{\chi(L)-1-\delta} D\bar{D}G_2 B_1(y,q)K_S^2B_2(y,q)^{LK_S}}{(\Delta(y,q) D\bar{D}G_2)^{\chi(O_S)/2}} \right).$$

We would now like to collect coefficients of $q$ to write the entire series in the (·) in terms of the $N$. So we must choose some values of $\delta$, $[S,L]$ such that $K_S^2$, $LK_S$, $\chi(O_S)$ and $k := \chi(L) - 1 - \delta$ remain constant, but $(L^2 - LK_S)/2$ assumes every integer value starting from $k + 1 - \chi(O_S)$. In other words, the cobordism class of the surface, the number of point conditions $k = \chi(L) - 1 - \delta$, and $LK_S$ are fixed, and $L^2$ varies. Note that it is not necessarily possible to find a fixed surface $S$ and honest line bundles $L_i$ which realize all the desired values. This causes no difficulties as the $N_{\delta,[S,L]}$ may be viewed as just functions of the four values $L^2$, $LK_S$, $K_S^2$, $c_2(S)$. Making this dependence explicit we write $\mathcal{M}_{k,[S]}((L^2 - LK_S)/2, LK_S) := N_{\chi(L)-1-k,[S,L]}$, where the RHS is viewed just as a function of two integers and is determined by evaluating the LHS on the cobordism class $[S,L]$ with the specified invariants. In terms of the $\mathcal{M}$, we have:

$$\sum_{l=k+1-\chi(O_S)}^{\infty} \mathcal{M}_{k,[S]}(l, LK_S)q^l = \bar{D}G_2(y,q)^k B_1(y,q)K_S^2B_2(y,q)^{LK_S} D\bar{D}G_2(y,q) \frac{\Delta(y,q) D\bar{D}G_2(y,q)^{\chi(O_S)/2}}{(\Delta(y,q) D\bar{D}G_2(y,q)/q^2)^{\chi(O_S)/2}}. \tag{10}$$

We can also express this in a slightly different way, fixing $LK_S$, $k$ and varying $\delta$: write $\mathcal{N}_{k,[S]}(\delta, LK_S) := \mathcal{N}_{\delta,[S,L]}^{\delta}$ with $k = \chi(L) - 1 - \delta$. Then

$$\sum_{\delta=0}^{\infty} \mathcal{N}_{k,[S]}(\delta, LK_S)q^\delta \bar{D}G_2(y,q)^k B_1(y,q)K_S^2B_2(y,q)^{LK_S} D\bar{D}G_2(y,q/q) \frac{\Delta(y,q) D\bar{D}G_2(y,q)/q^2)^{\chi(O_S)/2}}{(\Delta(y,q) D\bar{D}G_2(y,q)/q^2)^{\chi(O_S)/2}}. \tag{11}$$
When $S$ is a K3 surface, the RHS simplifies dramatically. Moreover, on the LHS we may actually choose each term K3 surfaces $S_g$ of genus $g$, with irreducible line bundles $L_g$ giving the polarization. That is, $\overline{M}_{k,[K3]}(g-1,0) = N_{g-k,[S_g,L_g]}$. The relative Hilbert schemes over the general $\mathbb{P}^g \subset |L_g|$ are all smooth \cite{Muk}, so the $N_{g,[S_g,L_g]}$ are equal to the geometric $\overline{N}_{g,g}$.

Summarizing the preceding discussion,

**Conjecture 63.** For any $k$,

$$\sum_{g-k} q^{g-1} N_{g-k,[S_g,L_g]}(y) = \frac{DG_2(y,q)^k}{\Delta(y,q)}$$

**Proposition 64.** Assume Conjecture 50. Then Conjectures 62 and 63 are equivalent.

*Proof.\* According to Proposition 60, from Conjecture 50 we can deduce that the $N$ have a multiplicative generating series. The two of its components which are made explicit in Conjecture 62 are determined by the case when $S$ is a K3 surface, and we have seen that the posited explicit formula is equivalent to that given in Conjecture 63. \hfill \Box

**Proposition 65.** Conjecture 63 is true at $k = 0$.

*Proof.\* The relative Hilbert schemes over the general $\mathbb{P}^g \subset |L_g|$ are smooth, so the quantity $\overline{N}_{g,[S_g,L_g]}$ is the $\chi-y$ genus of the relative compactified Jacobian of the tautological family of curves $C/|L_g|$. In fact, Kawai and Yoshioka compute the Hodge polynomial of this space \cite{KY}. Alternatively, as in \cite{Bea} one may note that the relative compactified Jacobian over $|L_g|$ is birational to the Hilbert scheme of points $S_g^{[g]}$, since both are hyperKähler they are deformation equivalent by \cite{Hu}, and the Hodge polynomial of the Hilbert scheme of points was computed in \cite{GS}. In either case the result is given by the RHS. \hfill \Box

We do not know how to compute the $\chi-y$ genera of relative Hilbert schemes over linear subsystems for K3 surfaces. In fact, while the Euler numbers of these spaces are known, the only known calculation of them (due to Maulik, Pandharipande, and Thomas) is extremely indirect, and involves at least two uses of the Gromov-Witten/Pairs correspondence \cite{MPT}. Rather remarkably, the same series which we have conjectured describes the $\overline{N}_g$ is found in \cite{MPT} to encode all the Euler numbers of relative Hilbert schemes over linear subsystems.

**Theorem 66.** \cite{MPT}\ For each $g$, let $(S_g,L_g)$ be a K3 surface of genus $g$, and assume $L_g$ is irreducible. Let $H$ be the hyperplane class on $|L_g|$. Then for any $k$,

$$(y - 2 + y^{-1})^{k-1} \frac{DG_2}{\Delta} = \sum_{g-k} q^{g-1} \sum_{n=0} y^{n+1-g} \int_{c_{|L_g|}} c_{n+g-k}(T_{c_{|L_g|}}) \cdot \rho^*(H^k)$$

Comparing powers of $q$, we see that given \cite{MPT}, Conjecture 63 is equivalent to the following statement:
Conjecture 67. Let \((S, L)\) be a K3 surface of genus \(g\) with \(L\) irreducible. For all \(k\),
\[
(y - 2 + y^{-1})^{k-1}N_{g-k,[S,L]} = \sum_{n=0}^{\infty} y^{n+1-g} \int_{c_{[L]}} c_{n+g-k}(Tc_{[L]}^{[n]}) \cdot \rho^*(H^k)
\]

Remark 68. The statement of Theorem 66 in [MPT] differs slightly; there are some signs owing to the use of \(\Omega\) rather than \(T\), and it is formulated in terms of the space of “stable pairs” rather than the relative Hilbert scheme. But since \(L_g\) is irreducible, these are the same space as per [PT2, Appendix B]. In the language of [MPT], Conjecture 67 may be viewed as asserting that the \(N\) encode certain descendent integrals in the stable pairs theory, or equivalently in the Gromov-Witten theory.

We may instead specialize to abelian surfaces. Let \((A, L_g)\) denote a primitively polarized abelian surface with \(L_g^2 = 2g - 2\) and hence \(\chi(L) = g - 1\) and \(g(L) = g\); assume \(L\) is irreducible. Note such surfaces exist for all \(g \geq 1\). Equation 10 specializes to
\[
\sum_{g=k+2}^{\infty} N_{g-k-2,[A,L_g]} q^{g-1} = \sum_{l=k+1}^{\infty} M_{k,[A]}(l,0)q^l = \tilde{D}G_2(y,q)^k \tilde{D}\tilde{G}_2(y,q)
\]

The above formula is not equivalent to Conjecture 62, since \(\chi(\mathcal{O}_A) = 0\). Moreover we do not know how to establish it, even at \(k = 0\). However:

Proposition 69. Assume Conjecture 50 or \(K_S = 0\). Conjecture 62 is equivalent to the following two formulas:
\[
\sum_{g=0}^{\infty} N_{g,[K3,L_g]} q^{g-1} = \tilde{\Delta}(y,q)^{-1}
\]
\[
\sum_{g=2}^{\infty} N_{g-2,[A,L_g]} q^{g-1} = \tilde{D}\tilde{G}_2(y,q)
\]

Proof. It is enough to show that the given invariants of the complete linear system suffice to determine the series \(A_1, A_4\) and then apply the residue trick explained above. But \(N_{g-2,[A,L_g]}(y) = \text{Coeff}_{s^{2g-2}} A_1(y,s)^{2g-2}\). Beginning at \(g = 2\) this allows to iteratively determine the coefficients of \(A_1\). Then \(N_{g,[K3,L_g]}(y) = \text{Coeff}_{s^{2g-2}} A_1(y,s)^{2g-2} A_4(y,s)^{24}\); since we now know \(A_1\) this determines \(A_4\). 

As we have established Equation 12, it remains only to compute the invariants for complete linear systems on abelian surfaces. In the future we hope to do so by the methods of [KY].

\[\text{8}\] Similarly a computation of the Euler characteristics of all relative Hilbert schemes for complete linear systems on (K3 and) abelian surfaces would yield the formula [MPT, Thm. 6] for linear subsystems for irreducible line bundles on K3 surfaces; this would provide an entirely sheaf theoretic proof.
5. Refined Severi degrees

The Severi degrees $n^{d,\delta}$ are the numbers of $\delta$-nodal reduced degree $d$ curves in $\mathbb{P}^2$ through $(d^2/2) - 1 - \delta$ general points. The famous Caporaso-Harris formula [CH] gives a recursive method of computing the Severi degrees. The recursion involves the relative Severi degrees $n^{d,\delta}(\alpha, \beta)$ which count delta-nodal curves with tangency conditions along a fixed line in $\mathbb{P}^2$. More generally, for a line bundle $L$ on a surface $S$, one can define the Severi degree $n^{L,\delta}$ as the number of $\delta$-nodal reduced curves in $|L|$ though dim $|L| - \delta$ general points, provided this number is finite.

We begin by a review of the Caporaso-Harris recursion formula; we will use the more general formulation of Vakil [Vak], which also applies to rational ruled surfaces. By a sequence we mean a collection $\alpha = (\alpha_1, \alpha_2, \ldots)$ of nonnegative integers, all of which are zero. We write $d$ for the sequence $(d, 0, 0, \ldots)$ and $e_k$ for the sequence whose $k$-th element is 1 and all other ones 0. For two sequences $\alpha, \beta$ we define $|\alpha| = \sum_i \alpha_i$, $I\alpha = \sum_i i\alpha_i$, $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots)$, $(\alpha)^{\beta} = \prod_i (\alpha_i)^{\beta_i}$. We write $\alpha \leq \beta$ to mean $\alpha_i \leq \beta_i$ for all $i$.

Throughout this section we take $S$ to be $\mathbb{P}^2$ or a rational ruled surface. In case $S = \mathbb{P}^2$, let $E$ be a line in $\mathbb{P}^2$, in case $S$ is a rational ruled surface $\Sigma_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$, let $E$ be the class of the section with $E^2 = -e$. We denote $H$ the hyperplane class on $\mathbb{P}^2$, $F$ the class of a fibre on $\Sigma_e$.

Recursion 70. [CH Vak] Let $L$ be a line bundle on $S$ and let $\alpha, \beta$ be sequences with $I\alpha + I\beta = EL$, and let $\delta \geq 0$ be an integer. Let $\gamma(L, \beta, \delta) = \dim |L| - EL + |\beta| - \delta$. The relative Severi degrees $n^{L,\delta}(\alpha, \beta)$ are recursively given as follows: $n^{L,\delta}(\alpha, \beta) = 0$ if $\gamma(L, \beta, \delta) < 0$. If $\gamma(L, \beta, \delta) > 0$, then

$$n^{L,\delta}(\alpha, \beta) = \sum_{k; \beta_k > 0} k \cdot n^{L,\delta}(\alpha + e_k, \beta - e_k)$$

$$+ \sum_{\alpha', \beta', \delta'} \prod_i i^{\beta_i - \beta_i}(\alpha^{\beta_i})^{\beta'} n^{L - E, \delta'}(\alpha', \beta').$$

Here the second sum runs through all $\alpha', \beta', \delta'$ satisfying the condition

$$\alpha' \leq \alpha, \beta' \geq \beta, \quad I\alpha' + I\beta' = E(L - E),$$

$$\delta' = \delta + g(L - E) - g(L) + |\beta' - \beta| + 1 = \delta - E(L - E) + |\beta' - \beta|.$$

Initial conditions: if $\gamma(L, \beta, \delta) = 0$ we have $n^{L,\delta}(\alpha, \beta) = 0$ unless we are in one of the following cases

1. In case $S = \mathbb{P}^2$ we put $n^{H,0}(1, 0) = 1$,
2. In case $S = \Sigma_e$, let $F'$ be the class of a fibre of the ruling; we put $n^{kF,0}(k, 0) = 1$.

We put $n^{L,\delta} := n^{L,\delta}(0, LE)$. In case $S = \mathbb{P}^2$, we write $n^{d,\delta}(\alpha, \beta) := n^{dH,\delta}(\alpha, \beta)$, and $n^{d,\delta} := n^{dH,\delta}(0, d)$,
5.1. **Refined Severi degrees.** We formally introduce a refinement of this recursion.

**Recursion 71.** With the same notations, assumptions, limits of summation, and initial values as for the relative Severi degrees in Recursion 70, we define the refined relative Severi degrees $N^{L,\delta}(\alpha, \beta)(y)$ for $\gamma(L, \beta, \delta) > 0$ by:

$$
N^{L,\delta}(\alpha, \beta)(y) = \sum_{k; \beta_k > 0} \frac{1 - y_k}{1 - y} \cdot N^{L,\delta}(\alpha + e_k, \beta - e_k)(y)
$$

$$
+ \sum_{\alpha', \beta', \delta'} y^{I\alpha' + I\beta} \prod_i \left( \frac{1 - y^i}{1 - y} \right)^{\beta'_i - \beta_i} \left( \frac{\alpha}{\alpha'} \right) \left( \frac{\beta'}{\beta} \right) N^{L-E,\delta'}(\alpha', \beta')(y).
$$

We abbreviate $N^{L,\delta} := N^{L,\delta}(0, LE)$, and, in case $S = \mathbb{P}^2$, $N^{d,\delta}(\alpha, \beta) := N^{dH,\delta}(\alpha, \beta)$, $N^{d,\delta} := N^{dH,\delta}(0, d)$. As with the refined invariants, we define normalized refined relative Severi degrees which are Laurent polynomials in $y^{1/2}$, symmetric under $y \mapsto 1/y$.

**Definition 72.** The normalized (relative) refined Severi degrees $\overline{N}^{L,\delta}(y)$ are defined by

$$
\overline{N}^{L,\delta}(\alpha, \beta)(y) = \frac{N^{L,\delta}(\alpha, \beta)(y)}{y^{d + (|\beta| - |\beta|)/2}}, \quad \overline{N}^{L,\delta}(y) = \frac{N^{L,\delta}}{y^\delta}.
$$

**Proposition 73.** The $\overline{N}^{L,\delta}(\alpha, \beta)(y)$ are determined by the same initial conditions as the $N^{L,\delta}(\alpha, \beta)(y)$ and the recursion

$$
\overline{N}^{L,\delta}(\alpha, \beta) = \sum_{k; \beta_k > 0} [k]_y \cdot \overline{N}^{L,\delta}(\alpha + e_k, \beta - e_k) + \sum_{\alpha', \beta', \delta'} \left( \prod_i [i]_y^{\beta'_i - \beta_i} \right) \left( \frac{\alpha}{\alpha'} \right) \left( \frac{\beta'}{\beta} \right) \overline{N}^{L-E,\delta'}(\alpha', \beta')(y)
$$

with the same conditions on $\alpha', \beta', \delta'$ as above. In particular $\overline{N}^{L,\delta}(\alpha, \beta)(y)$ is symmetric under $y \mapsto 1/y$.

**Proof.** It is enough to prove that every summand on the right hand side of (16) is obtained from the corresponding summand of (17) by multiplying by $y^{d + (|\beta| - |\beta|)/2}$. Each summand in the first sum is multiplied by $y^m$ with

$$
m = (k - 1 + I(\beta - e_k) - (|\beta| - e_k))/2 + \delta = \delta + (I\beta - |\beta|)/2.
$$

Each summand in the second sum is multiplied by $y^m$ with

$$
m = I\alpha' + I\beta + (I(\beta' - \beta) - |\beta' - \beta|)/2 + \delta' + (I\beta' - |\beta'|)/2 = \delta + (I\beta - |\beta|)/2,
$$

where we use $I\alpha' + I\beta' = E(L - E) = \delta - \delta' + |\beta' - \beta|$. $
$

It is clear that the recursions for the refined Severi degrees specialize at $y = 1$ to the recursion for the usual Severi degrees. Thus,

**Proposition 74.** $N^{L,\delta}(\alpha, \beta)(1) = \overline{N}^{L,\delta}(\alpha, \beta)(1) = n^{L,\delta}(\alpha, \beta)$.

According to [KS], if the general $\mathbb{P}^\delta \subset |L|$ contains no non-reduced curves and no curves containing components with negative self intersection, the Severi degrees are computed by the universal formulas: $n^{L,\delta} = n_{\delta,[S,L]}^{\delta}$. We expect the same for refined Severi degrees.
Conjecture 75. Let S be $\mathbb{P}^2$ or a rational ruled surface, let L be a line bundle, and assume $\mathbb{P}^3 \subset |L|$ contains no non-reduced curves and no curves containing components with negative self intersection. Then the refined Severi degrees are computed by the universal formulas: $N^{L,\delta}_i = N^{\delta}_{i,[S,L]}$. Explicitly,

1. On $\mathbb{P}^2$ we have $N^{d,\delta} = N^{\delta}_{[\mathbb{P}^2],\delta}$, for $\delta \leq 2d - 2$.
2. On $\mathbb{P}^1 \times \mathbb{P}^1$ we have $N^{nF+mG,\delta} = N^{\delta}_{[\mathbb{P}^1 \times \mathbb{P}^1,nF+mG]}$ for $\delta \leq \min(2n, 2m)$.
3. On $\Sigma_e$ with $e > 0$, we have $N^{nF+mE,\delta} = N^{\delta}_{[\Sigma_e,nF+mE]}$ for $\delta \leq \min(2n, n - e\delta)$.

Directly from the defining Recursion 71 we have computed all the $N^{d,\delta}(y)$ for $d \leq 15$ and $\delta \leq 30$. Assuming the vanishing Conjecture 50 and part (1) of Conjecture 75, the refined Severi degrees suffice to determine all the power series in Corollary 60 or equivalently in Conjecture 62. Note that Recursion 71 is much more computationally tractable than equivariant localization. Thus under the above assumption, we have verified Conjecture 62 modulo $q^{29}$ and determined $B_1(y,q)$ and $B_2(y,q)$ modulo $q^{29}$.

5.2. Irreducible refined Severi degrees. Denote by $n_0^{L,\delta}$ the irreducible Severi degrees, i.e. informally the number of irreducible $\delta$-nodal curves in $|L| \neq |E|$ passing though $\dim |L| - \delta$ general points. In [Get] Getzler observes in the case $S = \mathbb{P}^2$ that the $n_0^{d,\delta}$ can be expressed in terms of the Severi degrees $n^{d,\delta}$ by the relation

$$
\sum_{d,\delta} \frac{z^{(d+2)\delta - 1}}{((d+2)\delta - 1)!} q^d n_0^{d,\delta} = \log \left( 1 + \sum_{d,\delta} \frac{z^{(d+2)\delta - 1}}{((d+2)\delta - 1)!} q^d n^{d,\delta} \right).
$$

The generalization of this to $n_0^L$ is in [Vak]. The same formula can be used to define the irreducible normalized refined Severi degrees $N_0^{L,\delta}(y)$ by

$$
\sum_{L,\delta} \frac{z^{\dim |L| - \delta}}{((\dim |L| - \delta)!} v^L N_0^{L,\delta}(y) = \log \left( 1 + \sum_{L,\delta} \frac{z^{\dim |L| - \delta}}{((\dim |L| - \delta)!} v^L N^{L,\delta}(y) \right),
$$

and the irreducible refined Severi degrees by $N_0^{L,\delta}(y) := y^\delta N_0^{L,\delta}(y)$. Here $\{v^L\}_{\text{effective}, L \neq E}$ are elements of the Novikov ring, i.e. $v^{L_1}v^{L_2} = v^{L_1 + L_2}$. Evidently $N_0^{L,\delta}(y)$ is a Laurent polynomial in $y$ invariant under $y \mapsto 1/y$, and $N_0^{L,\delta}(1) = N_0^{L,\delta}(1) = n_0^{L,\delta}$.

Theorem 76. [BG] $N_0^{L,\delta}$ has nonnegative integer coefficients.

From this positivity, one can conclude vanishing results for $N_0^{L,\delta}$ from the analogous (known) results for $n_0^{L,\delta}$. For instance $N_0^{L,\delta}(y) = n_0^{L,\delta} = 0$ for $\delta > g(L)$ since there are no irreducible curves of cogenus greater than $g(L)$. 
We list the first few of the $N^{d,\delta}_0(y)$. Write $N^d_0(y, t) := \sum_{\delta \geq 0} N^{d,\delta}_0(y)t^\delta$. We have computed the $N^{d,\delta}_0(y, t)$ for $d \leq 14$. We have

\[
N^1_0(y, t) = 1, \quad N^2_0(y, t) = 1 + (y^2 + 10y + 1)t,
\]
\[
N^3_0(y, t) = 1 + (3y^2 + 21y + 3)t + (3y^4 + 33y^3 + 153y^2 + 33y + 3)t^2 + (y^6 + 13y^5 + 94y^4 + 404y^3 + 94y^2 + 13y + 1)t^3,
\]
\[
N^5_0(y, t) = 1 + (6y^2 + 36y + 6)t + (15y^4 + 156y^3 + 540y^2 + 156y + 15)t^2 + (20y^6 + 268y^5 + 1555y^4 + 4229y^3 + 1555y^2 + 268y + 20)t^3 + (15y^8 + 228y^7 + 1674y^6 + 7407y^5 + 18207y^4 + 7407y^3 + 1674y^2 + 228y + 15)t^4 + (6y^{10} + 96y^9 + 792y^8 + 4398y^7 + 17190y^6 + 42228y^5 + 792y^2 + 96y + 6)t^5 + (y^{12} + 16y^{11} + 139y^{10} + 867y^9 + 4203y^8 + 16377y^7 + 44098y^6 + 16377y^5 + 4203y^4 + 867y^3 + 139y^2 + 16y + 1)t^6.
\]

On $\mathbb{P}^1 \times \mathbb{P}^1$ we have computed the $N^{nF+mG,\delta}_0$ for $1 \leq n, m \leq 8$. We list the first few; write $N^{n,m,\delta}_0 := \sum_{\delta \geq 0} N^{n,m,\delta}_0(y)t^\delta$. We have

\[
N_0^{1,k}(y, t) = 1, \text{ for all } k, \quad N_0^{2,2}(y, t) = 1 + (y^2 + 10y + 1)t,
\]
\[
N_0^{2,3}(y, t) = 1 + (2y^2 + 16y + 2)t + (y^4 + 12y^3 + 79y^2 + 12y + 1)t^2,
\]
\[
N_0^{2,4}(y, t) = 1 + (3y^2 + 22y + 3)t + (3y^4 + 36y^3 + 174y^2 + 36y + 3)t^2 + (y^6 + 14y^5 + 117y^4 + 596y^3 + 117y^2 + 14y + 1)t^3,
\]
\[
N_0^{3,3}(y, t) = 1 + (4y^2 + 26y + 4)t + (6y^4 + 64y^3 + 256y^2 + 64y + 6)t^2 + (4y^6 + 52y^5 + 332y^4 + 1168y^3 + 332y^2 + 52y + 4)t^3 + (y^8 + 14y^7 + 109y^6 + 636y^5 + 2430y^4 + 636y^3 + 109y^2 + 14y + 1)t^4.
\]

5.3. The refined invariants at $y = 0$. Now we compute the specialization of the refined Severi degrees at $y = 0$.

**Notation 77.** For a sequence $\beta$ we write $(|\beta|) := \prod_{i \in \beta}^{|\beta|} i!$.

**Proposition 78.** $N^{L,\delta}(\alpha, \beta)(0) = \binom{|\beta|}{\beta} \binom{g(L)}{\delta}$. In particular $N^{L,\delta}(0) = \binom{g(L)}{\delta}$.

**Proof.** It is easy to see that the statement holds for the initial values. Setting $y = 0$ in the recursion formula (16) gives the two recursion formulas

\[
N_{0}^{L,\delta}(\alpha, \beta)(0) = \sum_{k \in \beta \geq 0} N_{0}^{L,\delta}(\alpha + e_k, \beta - e_k)(0), \quad \text{if } \beta \neq 0,
\]
\[
N_{0}^{L,\delta}(\alpha, 0)(0) = \sum_{\beta = E(L-E) + |\beta|} N_{0}^{L-E,\delta-E(L-E) + |\beta|}(0, \beta)(0).
\]
The first formula gives
\[ N^{L,\delta}(\alpha, \beta)(0) = \binom{|\beta|}{\beta} N^{L,\delta}(\alpha + \beta, 0)(0). \] (19)

The second formula shows in particular that \( N^{L,\delta}(\alpha, 0)(0) \) is independent of \( \alpha \), thus \( N^{L,\delta}(\alpha, 0)(0) = N^{L,\delta}(LE, 0)(0) = N^{L,\delta}(0) \); the last equality is by \([\text{19}]\). Thus \( N^{L,\delta}(\alpha, \beta)(0) = \binom{|\beta|}{\beta} N^{L,\delta}(0) \).

Finally, writing \( L = L_0 + aE \) with \( L_0E = 0 \), we prove \( N^{L,\delta} = \left( \frac{g(L)}{\delta} \right) \) by induction over \( a \). By induction the second equation of \([\text{18}]\) becomes
\[ N^{L,\delta}(0) = \sum_{I\beta = E(L-E)} g(L - E) \left( \delta - E(L-E) + |\beta| \right) \left( \frac{|\beta|}{\beta} \right). \]

Thus we need to show the identity
\[ (1 + t)^{g(L)} = \sum_{\delta \geq 0} \sum_{I\beta = E(L-E)} \left( g(L - E) \left( \frac{|\beta|}{\beta} \right) \right)^{\delta}. \] (20)

Note that by the multinomial formula we have
\[ \frac{x}{1 - x(1 + t)} = \frac{1}{1 - \frac{x}{1+tx}} - 1 = \sum_{n>0} (x + tx^2 + t^2x^3 + \ldots)^n = \sum_{\beta \neq 0} \left( \frac{|\beta|}{\beta} \right) t^{I\beta - |\beta|} x^{I\beta}. \]

Thus the right hand side of \([\text{20}]\) becomes
\[ \text{Coeff}_{x^{E(L-E)}} \left[ \frac{(1 + t)^{g(L-E)}}{1 - x(1 + t)} \right] = (1 + t)^{g(L)}. \]

Using the tropical interpretation of the \( N^{L,\delta}(\alpha, \beta) \) of \([\text{BG}]\) using refined multiplicities (see also Section \([\text{6}]\)), this result has been generalized in \([\text{IM}]\) to arbitrary toric surfaces.

For \( M \) a line bundle on \( S \), let \( M_n := f^*g_*\left( \bigotimes_{i=1}^n pr_i^* M \right) \in \text{Pic.} S^{[n]} \), where \( f : S^{[n]} \to S^{(n)} \) and \( g : S^n \to S^{(n)} \) are the natural morphisms, and \( pr_i : S^n \to S \) is the \( i^{th} \) projection. It is well-known that \( K_{S^{[n]}} = (K_S)^n \).

**Lemma 79.**
\[ \sum_{n,k \geq 0} \chi(S^{[n]}, \Lambda^k(L^{[n]})^\vee)x^kq^n = \frac{(1 + xq)\chi(L^\vee)}{(1 - q)\chi(O_S)}. \]

**Proof.** This is a corollary to \([\text{Scat}]\) Thm. 5.2.1, which implies for line bundles \( L, M \) on \( S \) that \( \chi(S^{[n]}, \Lambda^k L^{[n]} \otimes M_n) = \left( \chi(L \otimes M) \right)^{(\chi(M)+(n-k-1)).} \) We apply this with \( M = K_S \). Thus we have by applying Serre duality on \( S^{[n]} \) and on \( S \):
\[ \chi(S^{[n]}, \Lambda^k(L^{[n]})^\vee) = \chi(S^{[n]}, \Lambda^k L^{[n]} \otimes (K_S)_n) = \left( \chi(L \otimes K_S) \right)^{(\chi(K_S)+(n-k-1))} = \left( \chi(L^\vee) \right)^{(\chi(O_S)+(n-k-1))}. \]
which is equivalent to the statement of the Lemma. □

**Proposition 80.** \( N_{\delta_{[S,L]}}^l(0) = 0 \) for \( l \neq \delta \), and for all \( 0 \leq l \leq \delta \) we have \( N_{\delta_{[S,L]}}^l(0)(0) = \chi_{l}^{(L)} \). In particular if \( S \) is a rational surface, then \( N_{\delta_{[S,L]}}^\delta(0)(0) = (g(L)) \).

**Proof.** Let \( C_\delta \) be the universal curve over the sublinear system \( \mathbb{P}^\delta \subset |L| \). By Proposition 47, we have

\[
\chi_0(C_\delta[n]) = \res_{x=0} \int_{|S[n]|} \left( \frac{1}{1 - e^{-x}} \right)^{\delta+1} \frac{c_n(L[n] \cdot e^x) td(S[n])}{td(L[n] \cdot e^x)} dx.
\]

Note that by definition \( \frac{c_n(L[n] \cdot e^x)}{td(L[n] \cdot e^x)} = \sum_{k=0}^{n} (-e^{-x})^k \text{ch}(\Lambda^k(L[n])^\vee) \). Thus by Riemann-Roch

\[
\chi_0(C_\delta[n]) = \res_{x=0} \left[ \left( \frac{1}{1 - e^{-x}} \right)^{\delta+1} \sum_{k=0}^{n} (-e^{-x})^k \chi(S[n], \Lambda^k(L[n])^\vee) dx \right].
\]

We put \( T = e^{-x} \) and apply Lemma 79 to obtain

\[
\sum_{n \geq 0} \chi_0(C_\delta[n])q^n = -\res_{T=1} \left[ \left( \frac{1}{1 - T} \right)^{\delta+1} \sum_{n \geq 0} \sum_{k=0}^{n} (-T)^{k} q^n \chi(S[n], \Lambda^k(L[n])^\vee) \frac{dT}{T} \right]
\]

Substituting \( T = 1 - \alpha \) this becomes

\[
\sum_{n \geq 0} \chi_0(C_\delta[n])q^n = \res_{\alpha=0} \left[ \frac{1}{\alpha^{\delta+1}} \frac{(1 - q + \alpha q)^{\chi(L^\vee)}}{(1 - q)^{\chi(O_S)}} \sum_{l \geq 0} \alpha^l d\alpha \right]
\]

\[
= \sum_{l=0}^{\delta} \left( \chi(L^\vee) \right) l^q (1 - q)^{\chi(L^\vee) - 1 - \chi(O_S)} = \sum_{l=0}^{\delta} \left( \chi(L^\vee) \right) l^q (1 - q)^{g(L) - l - 1}.
\]

\[
\square
\]

5.4. **Conjectural generalization to higher powers of \( y \).** Proposition 80 and Proposition 78 can be subsumed in the following statements:

1. For any line bundle \( L \) on a surface \( S \) we have

\[
\sum_{\delta \geq 0} N_{\delta_{[S,L]}}^\delta(0)(0) q^\delta = (1 + q) \chi(L^\vee).
\]

2. If \( L \) is an effective divisor on \( \mathbb{P}^2 \) or a rational ruled surface \( S \), then

\[
\sum_{\delta \geq 0} N_{\delta_{[S,L]}}^\delta(0) N^L,\delta(0) q^\delta = (1 + q)^{g(L)} = \sum_{\delta \geq 0} N_{\delta_{[S,L]}}^\delta(0) q^\delta.
\]

We want to give a conjectural extension of these two statements to higher powers of \( y \). We start with the analogue of (1)
**Proposition 83.** Assuming Conjecture 82, Conjecture 81 holds. Furthermore

\[ (1 + q)^{\chi(L^\vee) - 3i} P^i_L(q). \]

Here \( P^i_L(q) \) is a polynomial in \( q \) of degree at most \( 3i \). In particular if \( \chi(L^\vee) \geq 3i \) then \( \operatorname{Coeff}_{y^i} N_{\delta,[S,L]}(y) = 0 \) for \( \delta > \chi(L^\vee) \).

Assuming Conjecture 82 we get by Remark 60

\[ \sum_{\delta \geq 0} N_{\delta,[S,L]}(y) q^\delta = F_1^{\chi(L^\vee)} F_2^{L K S/2} F_3^{K^2} F_4^{\chi(O_S)}, \]

with \( F_i \in \mathbb{Q}[y][[q]] \). We put \( C_1 = (F_1/(1 + q))|_{y\to y(1+q)^3}, \ C_2 = F_2|_{y\to y(1+q)^3}, \ C_3 = F_3|_{y\to y(1+q)^3}, \ C_4 = F_4|_{y\to y(1+q)^3}. \)

**Conjecture 82.** For \( i = 1, \ldots, 4 \) we have \( C_i \in \mathbb{Q}[y][[y^{1/3}]] \cap \mathbb{Q}[q][[y]]. \)

**Proposition 83.** Assuming Conjecture 82, Conjecture 81 holds. Furthermore

\[ \sum_{i \geq 0} P^i_L(q) y^i = C_1^{\chi(L^\vee)} C_2^{L K S/2} C_3^{K^2} C_4^{\chi(O_S)}. \]

**Proof.** By definition

\[ (1 + q)^{\chi(L^\vee)} C_1^{\chi(L^\vee)} C_2^{L K S} C_3^{K^2} C_4^{\chi(O_S)} = \sum_{\delta \geq 0} N_{\delta,[S,L]}(y(1+q)^3) q^\delta. \]

Therefore

\[ \operatorname{Coeff}_{y^i} \left[ \sum_{\delta \geq 0} N_{\delta,[S,L]}(y) q^\delta \right] = (1 + q)^{\chi(L^\vee) - 3i} \operatorname{Coeff}_{y^i} \left[ C_1^{\chi(L^\vee)} C_2^{L K S/2} C_3^{K^2} C_4^{\chi(O_S)} \right]. \]

As by Conjecture 82 all \( C_i \) are in \( \mathbb{Q}[y][[y^{1/3}]] \), we see that the coefficient of \( y^i \) is a polynomial of degree at most \( 3i \) in \( t \). \( \square \)

**Conjecture 82** has been verified modulo \( q^{11} \). Assuming Conjecture 75 it has been verified modulo \( q^{29} \). We list the power series \( C_1, C_2, C_3, C_4 \) modulo \( y^4 \).

\[
\begin{align*}
C_1 &= 1 + (4q + 2q^2)y + (q - 7q^2 + 12q^3 + 15q^4 + 3q^5)y^2 \\
&\quad + (-6q^2 + 56q^3 - 104q^4 - 112q^5 + 26q^6 + 32q^7 + 4q^8)y^3 + O(y^4) \\
C_2 &= 1 + (-2q - 6q^2 - 2q^3)y + (5q^2 + 48q^3 + 35q^4 + 6q^5 + q^6)y^2 \\
&\quad + (14q^3 - 390q^4 - 286q^5 + 60q^6 + 52q^7)y^3 + O(y^4) \\
C_3 &= 1 + (-q - 3q^2 - q^3)y + (q^2 + 16q^3 + 2q^4 - 6q^5 - q^6)y^2 \\
&\quad + (15q^3 - 130q^4 + 66q^5 + 199q^6 + 65q^7)y^3 + O(y^4) \\
C_4 &= 1 + (6q + 18q^2 + 10q^3)y + (18q^2 + 64q^3 + 219q^4 + 22q^5 + 67q^6)y^2 \\
&\quad + (-44q^3 + 336q^4 + 72q^5 + 952q^6 + 2328q^7 + 1608q^8 + 352q^9)y^3 + O(y^4)
\end{align*}
\]
Now we formulate the conjectural analogue of (2). For simplicity we only deal with the case of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

**Conjecture 84.**

1. Let $S = \mathbb{P}^2$ and assume $d \geq i + 2$, then

$$\text{Coeff}_{y^i} \left[ \sum_{\delta \geq 0} N^{d,\delta}(y)q^\delta \right] = (1 + q)^{g(dH) - 3i} P_{dH}(q).$$

2. Let $S = \mathbb{P}^1 \times \mathbb{P}^1$ and assume $n, m \geq i + 1$, then

$$\text{Coeff}_{y^i} \left[ \sum_{\delta \geq 0} N^{nF+mG,\delta}(y)q^\delta \right] = (1 + q)^{g(nF+mG) - 3i} P_{nF+mG}(q).$$

For $d \leq 14$, and for $n, m \leq 8$ this conjecture has been checked modulo $q^{11}$ and, assuming Conjecture 75, modulo $q^{29}$.

6. Refined, real, and tropical

Mikhalkin [Mik] has shown that the Severi degrees of projective toric surfaces can also be computed using tropical geometry: the Severi degrees $n^{L,\delta}$ count – with multiplicities – simple tropical curves through $\dim |L| - \delta$ points in $\mathbb{R}^2$ in tropical general position. Roughly speaking a simple tropical curve $C$ is a trivalent graph $\Gamma$ immersed in $\mathbb{R}^2$ together with some extra data. From this data, one assigns to each vertex $v$ of $\Gamma$ a multiplicity $m(v) \in \mathbb{Z}_{\geq 0}$ and defines the complex multiplicity $m(C)$ as the product of the $m(v)$ over the vertices of $\Gamma$. In [GM] a proof of the Caporaso-Harris recursion formula is given via tropical geometry.

The analogues of the Gromov-Witten invariants in real algebraic geometry are the Welschinger invariants [Wel]. These were originally defined to count real pseudoholomorphic curves in real symplectic manifolds. We restrict attention to the case that $S$ is a smooth projective toric surface. As toric varieties are defined over $\mathbb{Z}$, they certainly carry a real structure, and we write $\sigma$ for the associated anti-holomorphic involution. A real curve in $S$ is an algebraic curve $C \subset S$ with $C = \sigma(C)$, and the real locus of $C$ is $C^\sigma$. Fix a generic set $\Sigma$ of $\dim |L| - \delta$ general real points on $S$. The real enumerative invariant is $W^{L,\delta}(\Sigma) := \sum_C (-1)^{s(C)}$, where $C$ runs through the possibly reducible real curves $C \in |L|$ of geometric genus $g(L) - \delta$, passing through all the points of $\Sigma$, and $s(C)$ is the number of isolated real nodes of $C$, i.e. the points where $C$ analytically locally has the equation $x^2 + y^2$. We denoted by $W^{L,\delta}_0(\Sigma)$ the corresponding sum for irreducible curves. If $S$ is an unnodal (i.e. it contains no rational curve with self intersection $-n$, with $n \geq 2$) del Pezzo surface then the real enumerative invariants coincide with the Welschinger invariants. In [Wel] it was proven that $W^{L,g(L)}_0(\Sigma)$, i.e. the count of curves

---

9 We are here only considering the so-called totally real Welschinger invariants. More generally one could consider for any $0 \leq l \leq (\dim |L| - \delta)/2$ the numbers $W^{L,\delta,l}(\Sigma)$ which count real curves passing through $\dim |L| - \delta - 2l$ real points and $l$ pairs of complex conjugate points.
of geometric genus 0, is independent of the generic Σ. We will denote it just by \( W_{0}^{L,\sigma(L)} \).
In general \( W_{L,\delta}(\Sigma) \) and \( W_{0,\delta}^{L}(\Sigma) \) will depend on Σ via a system of walls and chambers.

In a sense, we have already seen these invariants. For a family of real curves \( C/B \), let \( n_{C/B}^{i,\mathbb{R}} \) be defined by the same formula as the \( n_{C}^{i,\mathbb{R}} \) introduced for individual curves in Section 2. Then we have:

**Proposition 85.** Let \( L \) be a real line bundle on \( S \), and let \( P_{\delta} \subset |L| \) be a linear subsystem determined by the real point conditions \( \Sigma \). Assume that all curves in \( P_{\delta} \) are reduced, that no curves have cogenus greater than \( \delta \), and that all curves of cogenus \( \delta \) are nodal. Then

\[
(−1)^{\delta} n_{C/P_{\delta}}^{\delta,\mathbb{R}} = W_{L,\delta}(\Sigma)
\]

The real enumerative invariants of toric surfaces can also be computed via tropical geometry [Mik, Thm. 6]. For any real line bundle \( L \) and any \( \delta \geq 0 \), the tropical Welschinger invariant \( W_{L,\delta}^{trop} \) counts simple tropical curves in \( C \) in \( |L| \) passing through \( \dim |L| - \delta \) points in \( \mathbb{R}^{2} \) in tropically general position. Here the tropical curves \( C \) are counted with the Welschinger multiplicity \( r(C) \):

\[
r(C) = \prod_{\text{vertices } v} r(v), \quad r(v) = \begin{cases} (-1)^{(m(v)-1)/2} & \text{if } k \text{ odd,} \\ 0 & \text{if } k \text{ even.} \end{cases}
\]

The irreducible Welschinger invariants \( W_{0,\delta}^{L,\text{trop}} \) are defined by summing over only irreducible curves. It is proven in [IKS] that this is independent of the points as long as they are in tropical general position. Finally [Mik] shows that there exists a set \( \Sigma \) of \( \dim |L| - \delta \) real points of \( S \), so that \( W_{L,\delta}(\Sigma) = W_{L,\delta}^{trop} \), and \( W_{0,\delta}(\Sigma) = W_{0,\delta}^{L,\text{trop}} \).

If \( S \) is \( \mathbb{P}^{2} \) or a rational ruled surface, there is a recursion for the tropical Welschinger invariants [IKS]. We write it in a modified form which makes the close relation to the recursion for the Severi degrees more evident.

**Definition 86.** A sequence \( \alpha = (\alpha_{1}, \alpha_{2}, \ldots) \) is called *odd* if \( \alpha_{i} = 0 \) for all even \( i \).

**Recursion 87.** Let \( L \) be a line bundle on \( S \) and let \( \alpha, \beta \) be odd sequences with \( I\alpha + I\beta = EL \), and a let \( \delta \geq 0 \) be an integer. With the same notations and assumptions and initial values as for the relative Severi degrees in Recursion 70 the relative tropical Welschinger invariants \( W_{\nu_{0,\text{trop}}}^{L,\delta}(\alpha, \beta)(y) \) are given by the following recursion formula: if \( \gamma(L, \beta, \delta) > 0 \),

\[
W_{\nu_{0,\text{trop}}}^{L,\delta}(\alpha, \beta)(y) = \sum_{k \text{ odd; } \beta_{k} > 0} (-1)^{(k-1)/2} W_{\nu_{0,\text{trop}}}^{L,\delta}(\alpha + \epsilon_{k}, \beta - \epsilon_{k})(y)
\]

\[
+ \sum_{\alpha', \beta', \delta' \text{ i odd}} \prod_{(i-1)/2} (\beta_{i}' - \beta_{i}) \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} W_{\nu_{0,\text{trop}}}^{L-E,\delta'}(\alpha', \beta').
\]

Here the second sum runs through all odd sequences \( \alpha', \beta' \) and all \( \delta' \) satisfying (15).

---

\(^{10}\) Also the multiplicity assigned in [IKS] differs from those given above which we have taken from [Mik], but it can be shown they are equivalent.
We put $W^L_\text{trop} := W^L_\text{trop}(0, LE)$, and in the case $S = \mathbb{P}^2$, $W^{d,\delta}_\text{trop}(\alpha, \beta) = W^{dH,\delta}_\text{trop}(\alpha, \beta)$, $W^{d,\delta}_\text{trop} = W^{dH,\delta}_\text{trop}(0, d)$.

Note the following specialization:

\begin{equation}
[k]_{-1} = \left(\frac{y^{k/2} - y^{-k/2}}{y^{1/2} - y^{-1/2}}\right) \bigg|_{y = -1} = \begin{cases} 
(1 - 1)^{k-1/2} & k \text{ odd}, \\
0 & k \text{ even}.
\end{cases}
\end{equation}

In particular, the recursion for the refined Severi degrees interpolates between the Caporaso-Harris recursion for Severi degrees and the Itenberg-Kharlamov-Shustin recursion for tropical Welschinger invariants. Thus,

**Proposition 88.** $N^{L,\delta}_{\text{L},\delta}(\alpha, \beta)(1) = n^{L,\delta}_{\text{L},\delta}(\alpha, \beta)$ and $N^{L,\delta}_{\text{L},\delta}(\alpha, \beta)(-1) = W^{L,\delta}_\text{trop}(\alpha, \beta)$.

In [BG], Block and the first author relate the refined Severi degrees to tropical geometry and study them by the methods of tropical geometry. They introduce the refined multiplicity $M(v) := [m(v)]_y$, which specializes to $m(v)$ at $y = 1$ and to $r(v)$ at $y = -1$. Then the refined tropical Severi degrees $N^{L,\delta}_\text{trop}(\Sigma)$ are defined by counting curves with multiplicity $M(C) = \prod M(v)$. Note this definition applies to any smooth toric surface. It is shown that, for $S = \mathbb{P}^2$ or a rational ruled surface, and $\Sigma$ a “vertically stretched” configuration of points, the $N^{L,\delta}_\text{trop}(\Sigma)$ satisfy the recursion (17). Thus $N^{L,\delta}_\text{trop} = N^{L,\delta}_\text{trop}(\Sigma)$.

Itenberg and Mikhalkin have in the meantime shown in [IM] that $N^{L,\delta}_\text{trop}(\Sigma)$ is independent of $\Sigma$, and so we drop it from the notation. For $S = \mathbb{P}^2$ or a rational ruled surface, Conjecture 75 then implies that the $N^{L,\delta}_\text{trop}$ agree with the $N^{\delta,\Sigma}_\text{S,L}$ when $L$ is $\delta$ very ample. More generally one expects:

**Conjecture 89.** [BG] Let $S$ be a smooth projective toric surface and $L$ a real line bundle on $S$. If $L$ is $\delta$-very ample, then $N^\delta_{\text{S},\text{S},L} = N^\delta_{\text{S},\text{S},L}$.

Using the refined multiplicity, in [BG] the $N^{L,\delta}_\text{trop}(\Sigma)$ are studied using methods similar to those employed in [BLO] for the nonrefined Severi degrees. In particular it is shown that, for $L$ sufficiently ample with respect to $\delta$, they are given by refined node polynomials, and the Conjecture says that these agree with the $N^\delta_{\text{S},\text{S},L}$.

According to Conjecture 89 and Proposition 88, we expect:

**Conjecture 90.** Let $L$ be a $\delta$ very ample real line bundle on a (real) toric surface $S$. Then $N^\delta_{\text{S},\text{S},L}(-1) = W^{\delta,\Sigma}_\text{trop}$.

For convenience we record the corresponding specialization of Conjecture 62 at $y = -1$. Consider $\eta(\tau) := q^{1/24} \prod_{n>0} (1 - q^n)$ the Dirichlet eta function and write

\[
\overline{G}_2(\tau) := G_2(\tau) - G_2(2\tau) = \sum_{n>0} \left( \sum_{d|n, \ d \text{ odd}} \frac{n}{d} \right) q^n.
\]
Conjecture 91. \[ \sum_{\delta \geq 0} N_{\delta,[S,L]}^\delta (-1) \mathcal{G}_2(\tau) = \frac{(\mathcal{G}_2(\tau)/q)^{\chi(L)} B_1(-1,q)^K_B B_2(-1,q)^L K_s}{(\eta(\tau)^{16} \eta(2\tau)^2 D \mathcal{G}_2(\tau)/q^2)^{\chi(O_{\delta})/2}} \]

This conjecture has been checked modulo \( q^{15} \) and the coefficients of \( B_1(-1,q), B_2(-1,q) \) have been determined modulo \( q^{15} \). (These computations are numerically easier than those involving the indeterminate \( y \), thus we get to a higher order in \( q \)). The values of the series \( B_i \) are computed to be:

\[
\begin{align*}
B_1(-1,q) &= 1 - q - q^2 - q^3 + 3q^4 + q^5 - 22q^6 + 67q^7 - 42q^8 - 319q^9 + 1207q^{10} - 1409q^{11} - 3916q^{12} + 20871q^{13} - 34984q^{14} + O(q^{15}) \\
B_2(-1,q) &= 1 + q + 2q^2 - q^3 + 4q^4 + 2q^5 - 11q^6 + 24q^7 + 4q^8 - 122q^9 + 313q^{10} - 162q^{11} - 1314q^{12} + 4532q^{13} - 4746q^{14} + O(q^{15})
\end{align*}
\]

When \( S = \mathbb{P}^2 \) or a rational ruled surface, the Severi degrees \( n_{\delta,L}^\delta \) agree with the universal numbers \( n_{\delta,[S,L]}^\delta \) somewhat beyond the regime where \( L \) is \( \delta \) very ample. Specifically, it is conjectured in [Göt] and proven in [KS] that it suffices for the general \( \mathbb{P}^\delta \subset |L| \) to contain no nonreduced curves, and no curves containing components with negative self intersection. We expect the same to hold for the comparison between refined Severi degrees \( N_{\delta,L}^\delta \) and the universal numbers \( N_{\delta,[S,L]}^\delta \), and a fortiori for the specialization at \(-1\). However for this specialization more seems to be true:

**Conjecture 92.** Assume \( S = \mathbb{P}^2 \) or \( S = \Sigma_e \), and the following subloci of \(|L|\) have codimension more than \( \delta \): (1) the nonreduced curves with a component of multiplicity at least \( 3 \), (2) curves containing a component with negative self intersection. Then \( W_{L,\delta}^{r_L} = N_{\delta,[S,L]}^\delta (-1) \). Explicitly the condition amounts to:

1. On \( \mathbb{P}^2 \) we have \( W_{L,\delta}^{r_L} = N_{dH}^\delta (-1) \) if \( \delta \leq 3d - 3 \).
2. On \( \mathbb{P}^1 \times \mathbb{P}^1 \) we have \( W_{L,\delta}^{r_L} = N_{F+MG}^\delta (-1) \) if \( \delta \leq 3 \min(n,m) \).
3. On \( \Sigma_e \) with \( e > 0 \), we have \( W_{L,\delta}^{r_L} = N_{F+mE}^\delta (-1) \), if \( \delta \leq \min(3m, n - em) \).

Using the recursion formula (21) this conjecture has been checked for \( d, \delta \leq 14 \). Assuming (1) of Conjecture 92 and using the recursion formula (21) the conjecture 91 has been checked modulo \( q^{67} \) and \( B_1(-1,q) \) and \( B_2(-1,q) \) have been determined modulo \( q^{67} \). Note that the recursion for \( W_{L,\delta}^{r_L}(\alpha, \beta) \) is much more efficient than those of the \( N_{\delta,L}^\delta(\alpha, \beta)(y) \) or \( n_{\delta,L}^\delta(\alpha, \beta) \) because only odd sequences \( \alpha \) and \( \beta \) occur.

We have seen \( W_{L,\delta}^{r_L} = N_{\delta,L}^\delta (-1) \). In the sufficiently ample setting, taking a linear system \( \mathbb{P}^\delta \) determined by subtropical point conditions, and assuming all conjectures, this implies \( N_{dH}^\delta (-1) = (-1)^d N_{dH}^\delta (-1) = n_{dH}^\delta \). More generally we conjecture:
Conjecture 93. Let $L$ be a sufficiently ample real line bundle on a real toric surface, and let $P^\delta \subset |L|$ be determined by a subtropical collection of point conditions. Then the signatures of the relative Hilbert schemes agree with the compactly supported Euler characteristics of their real loci. That is, $\chi_1(C_{P^\delta}^{[n]}) = \chi_c(C_{P^\delta}^{[n]}(\mathbb{R}))$. Making the BPS change of variables, it follows that $N^\delta_{C/P^\delta}(-1) = n^\delta_{C/R}.$

More generally one may consider the question:

Question 94. Let $X$ be a smooth real variety. When is $\chi_1(X) = \chi_c(X(\mathbb{R}))$?

This has been the subject of some classical study, one general result being that for a “M-variety”, i.e. one for which the total dimension of the $\mathbb{Z}/2\mathbb{Z}$ cohomology is equal for the real and complex locus, the equality holds modulo 16 [DK].

Evidently the equality holds for any variety whose class in the Grothendieck ring of varieties over $\mathbb{R}$ lies inside $\mathbb{Z}[\mathbb{A}^1_{\mathbb{R}}]$. In particular, $\mathbb{R}\mathbb{P}^n$, toric surfaces, and Hilbert schemes of points on toric surfaces qualify. The relative Hilbert schemes are cut out of a product of these by a section of a vector bundle, and the signature behaves predictably under taking such sections. Thus, to study Conjecture 93 it would also suffice to give criteria for the Euler characteristic of the real locus to exhibit the same behavior. In [BB] it is shown this holds in a tropical sense for complete intersections in toric varieties.

Remark 95. The conjectured relation between the refined invariants at $y = -1$ and the Welschinger invariants is in a sense a global analogue of a conjecture of van Straten [vS, Conj. 4.6, 4.7] (see also Theorems 27 and 32 above and the nearby discussion).

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