Identities for the number of standard Young tableaux in some \((k, \ell)\) hooks

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Abstract: Closed formulas are known for \(S(k, 0; n)\), the number of standard Young tableaux of size \(n\) and with at most \(k\) parts, where \(1 \leq k \leq 5\). Here we study the analogue problem for \(S(k, \ell; n)\), the number of standard Young tableaux of size \(n\) which are contained in the \((k, \ell)\) hook. We deduce some formulas for the cases \(k + \ell \leq 4\).

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1 Introduction

Given a partition \(\lambda\) of \(n\), \(\lambda \vdash n\), let \(\chi^\lambda\) denote the corresponding irreducible \(S_n\) character. Its degree is denoted by \(\deg \chi^\lambda = f^\lambda\) and is equal to the number of Standard Young tableaux (SYT) of shape \(\lambda\) \([7, 8, 12, 14]\). The number \(f^\lambda\) can be calculated for example by the hook formula \([7, \text{Theorem 2.3.21}], [12, \text{Section 3.10}], [14, \text{Corollary 7.21.6}]\). We consider the number of SYT in the \((k, \ell)\) hook. More precisely, given integers \(k, \ell, n \geq 0\) we denote

\[
H(k, \ell; n) = \{\lambda = (\lambda_1, \lambda_2, \ldots) \mid \lambda \vdash n \text{ and } \lambda_{k+1} \leq \ell\}
\]

and

\[
S(k, \ell; n) = \sum_{\lambda \in H(k, \ell; n)} f^\lambda.
\]

1.1 The cases where \(S(k, \ell; n)\) are known

For the "strip" sums \(S(k, 0; n)\) it is known \([10, 14]\) that

\[
S(2, 0; n) = \binom{n}{\lfloor \frac{n}{2} \rfloor} \quad \text{and} \quad S(3, 0; n) = \sum_{j \geq 0} \frac{1}{j+1} \binom{n}{2j} \binom{2j}{j}.
\]

Let \(C_j = \frac{1}{j+1} \binom{2j}{j}\) be the Catalan numbers, then Gouyon-Beauchamps \([6, 14]\) proved that

\[
S(4, 0; n) = C_j_{\lfloor \frac{n}{4} \rfloor} \cdot C_{j+1} \quad \text{and} \quad S(5, 0; n) = 6 \sum_{j = 0}^{\lfloor \frac{n}{5} \rfloor} \binom{n}{2j} \cdot C_j \cdot \frac{(2j + 2)!}{(j + 2)!(j+3)!}.
\]
As for the "hook" sums, until recently only $S(1, 1; n)$ and $S(2, 1; n) = S(1, 2; n)$ have been calculated:

1. It easily follows that $S(1, 1; n) = 2^{n-1}$.
2. The following identity was proved in [11, Theorem 8.1]:

$$S(2, 1; n) = \frac{1}{4} \left( \sum_{r=0}^{n-1} \binom{n-r}{\lfloor \frac{n-r}{2} \rfloor} \binom{n}{r} + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor-1} k! \cdot (k+1)! \cdot (n-2k-2)! \cdot (n-k-1) \cdot (n-k) \right) + 1.$$  

1.2 The main results

In Section 2 we prove Equation (10), which gives (sort of) a closed formula for $S(3, 1; n)$ in terms of the Motzkin-sums function. For the Motzkin-sums function see [13, sequence A005043]. Equation (10) in fact is a "degree" consequence of a formula of $S_n$ characters, of interest on its own, see Equation (9).

In Section 3 we find some intriguing relations between the sums $S(4, 0; n)$ and the "rectangular" sub-sums $S^*(2, 2;, n)$, see below identities (12) and (13).

Finally, in Section 4 we review some cases where the hook-sums $S(k, \ell; n)$ are related, in some rather mysterious ways, to humps calculations on Dyck and on Motzkin paths, see (14), (16), and Theorem 4.1.

As usual, in some of the above identities it is of interest to find bijective proofs, which might explain these identities.

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2 The sums $S(3, 1; n)$ and the characters $\chi(3, 1; n)$

Define the $S_n$ character

$$\chi(k, \ell; n) = \sum_{\lambda \in H(k, \ell; n)} \chi^\lambda \quad \text{so} \quad \deg(\chi(k, \ell; n)) = S(k, \ell; n).$$

2.1 The Motzkin-sums function

Define the $S_n$ character

$$\Psi(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \chi^{(k, k^{1^{n-2k}})} \quad \text{and denote} \quad \deg \Psi(n) = a(n).$$
We call $\Psi(n)$ the Motzkin-sums character. Note that
\[
\deg \chi^{(k,k,1^{n-2k})} = f^{(k,k,1^{n-2k})} = \frac{n!}{(k-1)! \cdot k! \cdot (n-2k)! \cdot (n-k) \cdot (n-k+1)},
\]
hence
\[
a(n) = \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n!}{(k-1)! \cdot k! \cdot (n-2k)! \cdot (n-k) \cdot (n-k+1)}. \tag{4}
\]

By [13, sequence A005043] it follows that $a(n)$ is the Motzkin-sums function. The reader is referred to [13] for various properties of $a(n)$. For example, $a(n) + a(n+1) = M_n$, where $M_n$ are the Motzkin numbers. Also $a(1) = 0$, $a(2) = 1$ and $a(n)$ satisfies the recurrence:
\[
\text{for } n \geq 3 \quad a(n) = \frac{n-1}{n+1} \cdot (2 \cdot a(n-1) + 3 \cdot a(n-2)). \tag{5}
\]

Note also that for $n \geq 2$ Equation (1) can be written as
\[
S(2,1; n) = \frac{1}{4} \left( \sum_{r=0}^{n-1} \binom{n-r}{\frac{n-r}{2}} \binom{n}{r} + a(n) - 1 \right) + 1. \tag{6}
\]

The asymptotic behavior of $a(n)$ can be deduced from that of $M_n$. We deduce it here, even though it is not needed in the sequel.

**Remark 2.1.** As $n$ goes to infinity,
\[
a(n) \approx \frac{\sqrt{3}}{8 \cdot \sqrt{2\pi}} \cdot \frac{1}{n\sqrt{n}} \cdot 3^n \quad \text{and} \quad a(n) \approx \frac{1}{4} \cdot M_n.
\]

**Proof.** By standard techniques it can be shown that $a(n)$ has asymptotic behavior
\[
a(n) \approx c \cdot \left(\frac{1}{n}\right)^g \cdot \alpha^n
\]
for some constants $c, g$ and $\alpha$ – which we now determine. By [10]
\[
M_n \approx \frac{\sqrt{3}}{2\sqrt{2\pi}} \cdot \left(\frac{1}{n}\right)^{3/2} \cdot 3^n.
\]

With
\[
M_n = a(n) + a(n+1) \approx c \cdot (1 + \alpha) \cdot \left(\frac{1}{n}\right)^g \cdot \alpha^n
\]
this implies that $\alpha = 3$, that $g = 3/2$ and that $c = \frac{\sqrt{3}}{8 \cdot \sqrt{2\pi}}$. \hfill \qed
2.2 The outer product of $S_m$ and $S_n$ characters

Given an $S_m$ character $\chi_m$ and an $S_n$ character $\chi_n$, we can form their outer product $\chi_m \hat{\otimes} \chi_n$. The exact decomposition of $\chi_m \hat{\otimes} \chi_n$ is given by the Littlewood-Richardson rule \cite{7}, \cite{8}, \cite{12}, \cite{14}. In the special case that $\chi_n = \chi^{(n)}$, this decomposition is given, below, by Young’s rule. Also

$$\text{deg}(\chi_m \hat{\otimes} \chi^{(n)}) = \text{deg}(\chi_n) \binom{n+m}{n}. \quad (7)$$

**Young’s Rule** \cite{8}: Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash m$ and denote by $\lambda^+ \vdash n$ the following set of partitions of $m+n$:

$$\lambda^+ = \{ \mu \vdash n + m \mid \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \}.$$ 

Then

$$\chi^\lambda \hat{\otimes} \chi^{(n)} = \sum_{\mu \in \lambda^+} \chi^\mu.$$ 

**Example 2.2.** \cite{10}, \cite{14} Given $n$, it follows that

$$\chi^{(\lfloor n/2 \rfloor)} \hat{\otimes} \chi^{(\lceil n/2 \rceil)} = \chi(2,0;n), \quad \text{and by taking degrees, } S(2,0;n) = \binom{n}{\lfloor n/2 \rfloor}. \quad (8)$$

2.3 A character formula for $\chi(3,1;n)$

**Proposition 2.3.** With the notations of (2) and (3),

$$\chi(3,1;n) = \frac{1}{2} \cdot \left[ \chi(2,0,n) + \sum_{j=0}^{n} \Psi(j) \hat{\otimes} \chi^{(n-j)} \right]. \quad (9)$$

By taking degrees, Example 2.2 together with (3) and (7) imply that

$$S(3,1;n) = \frac{1}{2} \cdot \left[ \binom{n}{\lfloor n/2 \rfloor} + \sum_{j=0}^{n} a(j) \cdot \binom{n}{j} \right]. \quad (10)$$

**Proof.** Denote

$$\Omega(n) = \sum_{j=0}^{n} \Psi(j) \hat{\otimes} \chi^{(n-j)}$$

and analyze this $S_n$ character. Young’s rule implies the following:

Let $\mu \vdash n$, then by Young’s rule $\chi^\mu$ has a positive coefficient in $\Omega(n)$ if and only if $\mu \in H(3,1;n)$. Moreover, all these coefficients are either 1 or 2, and such a coefficient equals 1 if and only if $\mu$ is a $\leq 2$ two rows partition $\mu = (\mu_1, \mu_2)$. It follows that

$$\chi(2,0;n) + \Omega(n) = 2 \cdot \sum_{\lambda \in H(3,1;n)} \chi^\lambda. \quad (11)$$

This implies (10) and completes the proof of Proposition 2.3. \qed
3 The sums $S(4,0;n)$ and $S^*(2,2;n)$

Definition 3.1. 1. Let $n = 2m$, $m \geq 2$ and let $H^*(2,2;2m) \subset H(2,2;2m)$ denote the set of partitions $H^*(2,2;2m) = \{(k+2,k+2,2^{m-2-k}) \vdash 2m \mid k = 0, \ldots m-2\}$ (the partitions in the $(2,2)$ hook with both arm and leg being rectangular), then denote

$$S^*(2,2;2m) = \sum_{\lambda \in \mathcal{H}^*(2,2;2m)} f^\lambda.$$ 

2. Let $n = 2m+1$, $m \geq 2$ and let $H^*(2,2;2m+1) \subset H(2,2;2m+1)$ denote the set of partitions $H^*(2,2;2m+1) = \{(k+3,k+2,2^{m-2-k}) \vdash 2m+1 \mid k = 0, \ldots m-2\}$ (the partitions in the $(2,2)$ hook with arm nearly rectangular and leg rectangular), then denote

$$S^*(2,2;2m+1) = \sum_{\lambda \in \mathcal{H}^*(2,2;2m+1)} f^\lambda.$$ 

Recall from Section 1.1 that $S(4,0;2m-1) = C_m^2$ and $S(4,0;2m) = C_m \cdot C_{m+1}$. We have the following intriguing identities.

Proposition 3.2. 1. Let $n = 2m$ then

$$S(4,0;2m-2) = C_{m-1} \cdot C_m = S^*(2,2;2m).$$

Explicitly, we have the following identity:

$$C_{m-1} \cdot C_m = \frac{1}{m \cdot (m+1)} \cdot \left(\begin{array}{c} 2m-2 \\ m-1 \end{array}\right) \cdot \left(\begin{array}{c} 2m \\ m \end{array}\right) = \sum_{k=0}^{m-2} \frac{(2m)!}{k! \cdot (k+1)! \cdot (m-k-2)! \cdot (m-k-1)! \cdot (m-1) \cdot m^2 \cdot (m+1)}. \quad (12)$$

2. Let $n = 2m+1$ then

$$\frac{2m+1}{m+2} \cdot S(4,0;2m-1) = \frac{2m+1}{m+2} \cdot C_m^2 = S^*(2,2;2m+1).$$

Explicitly, we have the following identity:

$$\frac{2m+1}{m+2} \cdot C_m^2 = \frac{1}{(m+1) \cdot (m+2)} \cdot \left(\begin{array}{c} 2m \\ m \end{array}\right) \left(\begin{array}{c} 2m+1 \\ m \end{array}\right) = \sum_{k=0}^{m-2} \frac{(2m+1)! \cdot 2}{k! \cdot (k+2)! \cdot (m-k-2)! \cdot (m-k-1)! \cdot (m-1) \cdot m \cdot (m+1) \cdot (m+2)}. \quad (13)$$

Proof. Equation (12) is the specialization of Gauss’s $2F1(a,b;c;1)$ with $a = 2 - m$, $b = 1 - m$, $c = 2$ [1], and (13) is similar. Alternatively, the identities (12) and (13) can be verified by the WZ method [9], [15].
4 Hook-sums and humps for paths

A Dyck path of length $2n$ is a lattice path, in $\mathbb{Z} \times \mathbb{Z}$, from $(0, 0)$ to $(2n, 0)$, using up-steps $(1, 1)$ and down-steps $(1, -1)$ and never going below the $x$-axis. A hump in a Dyck path is an up-step followed by a down-step.

A Motzkin path of length $n$ is a lattice path from $(0, 0)$ to $(n, 0)$, using flat-steps $(1, 0)$, up-steps $(1, 1)$ and down-steps $(1, -1)$, and never going below the $x$-axis. A hump in a Motzkin path is an up-step followed by zero or more flat-steps followed by a down-step.

We count now humps for Dyck and for Motzkin paths and observe the following intriguing phenomena: The humps-calculations in the Dyck case correspond the $2 \times n$ rectangular shape $\lambda = (n, n)$ to the $(1, 1)$ hook shape $\mu = (n, 1^n)$. And in the Motzkin case we show below that it corresponds the $(3, 0)$ strip shape partitions $H(3, 0; n)$ to the $(2, 1)$ hook shape partitions $H(2, 1; n)$.

4.1 The Dyck case

The Catalan number

$$C_n = \frac{(2n)!}{n!(n+1)!}$$

is the cardinality of a variety of sets [14]; here we are interested in two such sets. First, $C_n = f^{(n,n)}$, the number of SYT of shape $(n, n)$. Second, $C_n$ is the number of Dyck paths of length $2n$. Let $\mathcal{H}D_n$ denote the total number of humps in all the Dyck paths of length $2n$, then

$$\mathcal{H}D_n = \binom{2n-1}{n},$$

see [3], [4], [5]. Since $\binom{2n-1}{n} = f^{(n,1^n)}$, we have

$$C_n = f^{(n,n)} \quad \text{and} \quad \mathcal{H}D_n = f^{(n,1^n)},$$

which we denote by

$$\mathcal{H} : (n, n) \longrightarrow (n, 1^n). \quad (14)$$

4.2 The Motzkin case

Like the Catalan numbers, also the Motzkin numbers $M_n$ are the cardinality of a variety of sets; for example $M_n = S(3, 0; n)$, [10], [14], [13 sequence A001006], which gives the Motzkin numbers a SYT interpretation. Also, $M_n$ is the number of Motzkin paths of length $n$. Let $\mathcal{H}M_n$ denote the total number of humps in all the Motzkin paths of length $n$, then by [13 sequence A097861]

$$\mathcal{H}M_n = \frac{1}{2} \sum_{j \geq 1} \binom{n}{j} \binom{n-j}{j}.$$  \hspace{1cm} (15)
We show below that this implies the intriguing identity $H_M(n) = S(2, 1; n) - 1$, which gives a SYT-interpretation of the numbers $H_M(n)$. Thus the humps-calculations in the Motzkin case corresponds the $(3, 0)$ strip shape partitions $H(3, 0; n)$ to the $(2, 1)$ hook shape partitions $H(2, 1; n)$. We denote this by

$$H: H(3, 0; n) \to H(2, 1; n).$$

(16)

**Theorem 4.1.** The number of humps for the Motzkin paths of length $n$ satisfies

$$H_M(n) = S(2, 1; n) - 1.$$

**Proof.** Combining Equations (11) and (15), the proof of Theorem 4.1 will follow once the following binomial identity – of interest on its own – is proved.

**Lemma 4.2.** For $n \geq 2$

$$2 \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n}{j} \binom{n-j}{j} = \sum_{r=0}^{n-1} \left( \binom{n-r}{\lfloor n-r/2 \rfloor} \binom{n-r}{r} \right) + a(n) - 1 =$$

$$= \sum_{r=0}^{n-1} \left( \binom{n-r}{\lfloor n-r/2 \rfloor} \binom{n-r}{r} \right) + \sum_{k=1}^{\lfloor n/2 \rfloor-1} \frac{n!}{k! \cdot (k+1)! \cdot (n-2k-2)! \cdot (n-k-1) \cdot (n-k)}.$$

Equation (17) was verified by the WZ method. About this method, see [9], [15]. We remark that it would be interesting to find an elementary proof of this identity.

This completes the proof of Theorem 4.1.

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