COHEN-MACAULAY CELL COMPLEXES

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Abstract. We show that a finite regular cell complex with the intersection property is a Cohen-Macaulay space iff the top enriched cohomology module is the only nonvanishing one. We prove a comprehensive generalization of Balinski’s theorem on convex polytopes.

An algebraic relation of K.Yanagawa implies that the (algebraic) enriched chain and cochain complexes fit into a natural sixtuple of complexes which in the simplicial case includes the resolution of the Stanley-Reisner ring. A consequence turns out to be that there is no single generalization of the Stanley-Reisner ring to cell complexes, but for Cohen-Macaulay cell complexes there is a generalization of the canonical module of the Stanley-Reisner ring.

Introduction

In [7] we introduced enriched homology and cohomology modules for simplicial complexes. They are modules over the polynomial ring $S = k[x_1, \ldots, x_n]$ where $\{1, 2, \ldots, n\}$ are the vertices of the simplicial complex, and their $S$-module ranks are equal to the $k$-vector space dimensions of the corresponding reduced homology and cohomology groups. We showed that a simplicial complex is Cohen-Macaulay iff there is only one nonvanishing enriched cohomology module, the top one.

In this paper we extended this to a class of cell complexes which behaves nicely from a combinatorial viewpoint and includes polyhedral complexes. This is the class of finite regular cell complexes with the intersection property. Enriched homology and cohomology modules may be defined equally well for this class. For simplicial complexes, being Cohen-Macaulay is a topological property. We show that for a cell complex in the class above, there is only one nonvanishing enriched cohomology module iff it is Cohen-Macaulay as a topological space.

In the class of Cohen-Macaulay simplicial complexes the classes of $l$-Cohen-Macaulay simplicial complexes constitute successively more restricted classes. They correspond geometrically to successively higher connectivity of the simplicial complex. We show that this notion of being $l$-Cohen-Macaulay works well also for the class of cell complexes we consider. We show that such a cell complex is $l$-Cohen-Macaulay iff its top enriched cohomology module can occur as an $l-1$'t syzygy module in an $S$-free resolution. Also,
its codimension $r$ skeleton will be $l+r$-Cohen-Macaulay. This generalizes results of [7] for simplicial complexes. It is also a comprehensive generalization of Balinski’s theorem for convex polytopes which says that the 1-skeleton of a convex polytope of dimension $d$ is $d$-connected. (Note that for a graph, being $d$-connected is the same as being $d$-Cohen-Macaulay.)

In [15], K.Yanagawa introduced the notion of square free modules over the polynomial ring $S$. Square free modules provides a natural setting for doing Stanley-Reisner theory. In [16] he defines two dualities $D$ and $A$ on the category of bounded (algebraic) complexes of free square free modules. He then shows the relation

$$D \circ A \circ D \circ A \circ D(\mathcal{P}) = \mathcal{P}[{-n}]$$

where $\mathcal{P}$ is a cochain complex and $\mathcal{P}[{-n}]$ is the complex shifted $n$ steps to the right. Starting from the enriched chain complex $\mathcal{E}[-1]$ and applying $D$ and $A$ we obtain a hexagon of complexes

\[
\begin{tikzcd}
\mathcal{E}[-1] & \mathcal{E}^\vee[-1] & \mathcal{G}^\vee \\
A[-n] & F & G \\
F^\vee & D & A
\end{tikzcd}
\]

We show that when $\Gamma$ is simplicial the complex $F^\vee$ is the resolution of the Stanley-Reisner ring $k[\Gamma]$. If $\Gamma$ is not simplicial then $F^\vee$ will have more than one nonzero cohomology module (which we describe). Thus one might say that although there is no single generalization of the Stanley-Reisner ring to a cell complex, there is a generalization of its resolution.

In case $\Gamma$ is Cohen-Macaulay and simplicial, the complex $F$ has only one nonvanishing homology module, the canonical module of the Stanley-Reisner ring $k[\Gamma]$. We show that for a Cohen-Macaulay cell complex then $F$ still has only one homology module, which may then be considered as a generalization of the canonical module.

The organization of the paper is as follows. Section 1 consists of preliminaries on cell complexes, enriched homology and cohomology modules, simplicial complexes and posets. Section 2 contains the main results, and Section 3 the main bulk of the proofs of the results in Section 2. They are formulated in the setting of posets. In Section 4 we describe the properties of the complexes in the hexagon above.

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1. Preliminaries

1.1. Cell complexes. A finite regular cell complex $\Gamma$ is a Hausdorff topological space $X$ together with a finite set of closed subsets in $X$, homeomorphic to balls and called cells or faces, such that

i. The interiors of the cells partition $X$.

ii. The boundary of each cell in is a union of other cells.

iii. $\emptyset$ is a cell.

The dimension of a cell is its topological dimension and the dimension of $\Gamma$ is the maximum of the dimension of the cells. We write $X = |\Gamma|$. The zero-dimensional cells are the vertices of $\Gamma$ and we denote by $V$ the set of these.

$\Gamma$ is said to have the intersection property if the intersection of any two faces is also a face. This means that the face poset is a meet-semilattice. Two references for such cell complexes are [3] and [4].

Note. In this paper we shall only consider finite regular cell complexes with the intersection property. To avoid repeating this long phrase over and over, we shall simply use the term cell complex as a short for finite regular cell complex with the intersection property.

If $R$ is a subset of the vertices, we denote by $\Gamma_R$ the sub-complex of $\Gamma$ consisting of all faces whose vertex set is contained in $R$. Also $\Gamma_{V \setminus R}$ is denoted $\Gamma_{-R}$. We denote by $r$ the cardinality of $R$.

1.2. Enriched homology and cohomology modules. The oriented augmented chain complex of $\tilde{\mathcal{C}}(\Gamma; k)$ of $\Gamma$ consists of $\tilde{\mathcal{C}}^i(\Gamma; k)$ equal to the vector space over the field $k$ spanned by the $i$-dimensional faces, $\oplus_{\dim f = i} k f$. The differential is given by

$$ f \mapsto \sum_{\dim f' = i-1} \epsilon(f', f) f' $$

where $\epsilon : \Gamma \times \Gamma \to \{-1, 0, 1\}$ is a suitable incidence function. The homology groups of this complex are the reduced homology groups $\tilde{H}_i(\Gamma; k)$ of $\Gamma$.

Now let $S$ be the polynomial ring $k[(x_v)_{v \in V}]$ in variables indexed by the vertices $V$. For a face $f$ let $m_f$ be the product of the variables indexed by the vertices of $f$. We now define the enriched homology modules of $\Gamma$ in the same way as we did for simplicial complexes in [2]. Attaching the variable $x_v$ to $v$, we may form the cellular complex $\mathcal{E}(\Gamma; k)$, see [2], where $\mathcal{E}_i(\Gamma; k)$ is $\oplus_{\dim f = i} S f$. The differential is given by

$$ f \mapsto \sum_{\dim f' = i-1} \epsilon(f', f) \frac{m_f}{m_{f'}} f'. $$

The enriched homology module $H_i(\Gamma; k)$ (or just $H_i(\Gamma)$) is the $i$'th homology module of this complex. It is a module graded by $NV$. For $b$ in $NV$ and $R$ the support of $b$, i.e. the set of nonzero coordinates, the graded part
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$\mathbf{H}_i(\Gamma)_k$ is the reduced homology $\tilde{H}_i(\Gamma_R)$, see [2]. It follows as in [2] that the $S$-module rank of $\mathbf{H}_i(\Gamma)$ is the dimension of $\tilde{H}_i(\Gamma)$ as a vector space over $k$.

Let $\omega_S$ be the canonical module which is $S(-1)$, i.e. the free $S$-module with a generator in multidegree $1 = (1,1,\ldots,1)$. We define the enriched cohomology module $\mathbf{H}^i(\Gamma; k)$ (or just $\mathbf{H}^i(\Gamma)$) as the $i$'th cohomology module of the dualized complex $E(\Gamma; k)^\vee = \text{Hom}_S(E(\Gamma; k), \omega_S)$.

### 1.3. Simplicial complexes.

A simplicial complex $\Delta$ on $[n] = \{1, 2, \ldots, n\}$ is a family of subsets of $[n]$ such that if $X$ is in $\Delta$ and $Y \subseteq X$ then $Y$ is in $\Delta$. A standard reference is [12]. Via its topological realization it becomes a cell complex.

For a subset $R$ of $[n]$, let $\Delta_R$ denote the restricted simplicial complex, consisting of faces that are subsets of $R$ and denote $\Delta_{[n] \setminus R}$ by $\Delta^{-R}$. Also let the link, $\text{lk}_\Delta R$, be the simplicial complex on $[n] \setminus R$ consisting of faces $Y$ such that $Y \cup R$ is a face of $\Delta$.

We may form the Stanley-Reisner ring $k[\Delta]$ which is the polynomial ring $k[x_1, \ldots, x_n]$ divided by the ideal generated by the square free monomials corresponding to the non-faces of $\Delta$. We say that $\Delta$ is Cohen-Macaulay (CM) if $k[\Delta]$ is Cohen-Macaulay. This is equivalent to the following homological criterion given by Hochster [8]

\begin{equation}
\tilde{H}_p(\Delta^{-R}) = 0, \text{ when } p + r < \dim \Delta.
\end{equation}

In [1], K.Baclawski defined a simplicial complex to be $l$-Cohen-Macaulay if $\Delta^{-l}$ is Cohen-Macaulay of the same dimension as $\Delta$ for each subset $R$ of cardinality less than $l$. By Hochsters criterion (1) this is equivalent to

\begin{equation}
\tilde{H}_p(\Delta^{-R}) = 0, \text{ when } p + r < \dim \Delta + l - 1 \text{ and } p < \dim \Delta.
\end{equation}

If $k[\Delta]$ is a Gorenstein ring and $\Delta$ is not a cone, we say that $\Delta$ is Gorenstein*. This is equivalent to $\Delta$ being 2-Cohen-Macaulay and $\tilde{H}^{\dim \Delta}(\Delta) = k$.

### 1.4. Posets.

A standard reference for posets is [13]. Given a poset $P$ with order relation $\leq$, an order ideal $J$ is a subset of $P$ such that if $x$ is in $J$ and $y \leq x$, then $y$ is in $J$. A filter is subset $F$ such that if $y$ is in $F$ and $y \leq x$, then $x$ is in $F$. If $R$ is subset of $P$, denote by $F(R)$ the filter generated by $R$, consisting of $x$ in $P$ such that $x \geq r$ for some $r$ in $R$, and for an element $x$ in $P$ denote by $P_{<x}$ the ideal of elements less than $x$. An open interval $(x, y)$ consists of all $z$ strictly between $x$ and $y$, $x < z < y$. We denote by $\hat{P}$ the poset $P$ with a bottom element 0 and top element 1 adjoined.

The poset $P$ is graded if every maximal chain in $P$ has the same length (the number of elements in the chain minus one). The length of such a maximal chain is the rank of $P$, $rk_P$. For an element $x$ in $P$ the maximal chains descending down from $x$, have the same length, the rank of $x$, $rk_x$. If $rk_x = 0$, $x$ is called an atom.
The poset $P$ is a lattice if each pair $x$ and $y$ have a supremum $x \lor y$ and an infimum $x \land y$.

For a poset $P$ we may form the order complex $\Delta(P)$, a simplicial complex consisting of all the chains in $P$. All the terminology for simplicial complexes may then be transferred to $P$. For simplicity we shall write $\tilde{H}_p(P_R)$ for $\tilde{H}_p(\Delta(P)_R)$. When $\Gamma$ is a cell complex the nonempty cells form the face poset $P(\Gamma)$ with cells ordered by inclusion. Then $P(\Gamma)$ will be a lattice (since we are assuming the intersection property). It is a well-known fact that $\Delta(P(\Gamma))$ is the complete barycentric subdivision of $\Gamma$ and hence as a topological space is homeomorphic to $|\Gamma|$.

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2.1. Criterion for being Cohen-Macaulay. For a simplicial complex $\Delta$, the property of being Cohen-Macaulay is a topological property, \cite{10}. In fact, letting $X$ be the topological realization $|\Delta|$, then $\Delta$ is CM iff

i. $H_p(X, X - \{x\}; k) = 0$ for $p < \dim X$.
ii. $\tilde{H}_p(X) = 0$ for $p < \dim X$.

Hence $\Delta$ and the face poset $P(\Delta)$ are Cohen-Macaulay at the same time. Now define a cell complex $\Gamma$ to be Cohen-Macaulay if its face poset $P(\Gamma)$ is Cohen-Macaulay. Since $\Delta(P(\Gamma))$ is the complete barycentric subdivision of $\Gamma$, this is a topological property of $\Gamma$. Recall the criterion \cite{11} of Hochster. We shall show that this applies equally well as a criterion for a cell complex to be Cohen-Macaulay.

**Theorem 2.1.** A cell complex $\Gamma$ is Cohen-Macaulay if and only if $\tilde{H}_p(\Gamma_R)$ vanishes whenever $R$ is a subset of the vertices $V$ with $p + r < \dim \Gamma$.

**Proof.** The boundary of each cell $c$ is a sphere and so $P(\Gamma)_c$ is Gorenstein*. The only if direction now follows by applying Proposition \cite[3.3]{5} and noting that $P(\Gamma_R)$ is $P(\Gamma)_{\Gamma_R}$. The if direction follows by Proposition \cite[3.5]{5}. □

**Corollary 2.2.** $\Gamma$ is Cohen-Macaulay iff the enriched cohomology modules $\tilde{H}^p(\Gamma)$ vanish for $p < \dim \Gamma$, or equivalently $E(\Gamma)^\vee$ is a resolution of $\tilde{H}^{\dim \Gamma}(\Gamma)$.

**Proof.** The theorem just proven says $\tilde{H}_{\dim \Gamma - r}(\Gamma_R)$ vanishes for $r < i$. This means that the enriched homology $S$-modules $\tilde{H}_{\dim \Gamma - r}(\Gamma)$ have codimension $\geq i$. It is then a standard exercise in algebra that this is equivalent to the dual complex $E(\Gamma)^\vee$ of $E(\Gamma)$ having no cohomology in (cohomological) degrees $< \dim \Gamma$ by using the following (see \cite[18.4]{5} and its proof).

**Fact.** For an $S$-module $M$ of codimension $r$, $Ext^p_S(M, \omega_S)$ vanishes for $p < r$, it has codimension $r$ if $p = r$, and codimension $\geq p$ for $p \geq r$. □
2.2. \textbf{$l$}-CM cell complexes. Recall the notion in \textsection 1.3 of a simplicial complex being $l$-Cohen-Macaulay. This notion turns out to work well for cell complexes also.

\textit{Definition 2.3.} A cell complex $\Gamma$ is $l$-Cohen-Macaulay if $\Gamma_{-R}$ is Cohen-Macaulay of the same dimension as $\Gamma$ for each subset $R$ of the vertices $V$ of cardinality $< l$.

The following, stated in \cite{7} for simplicial complexes, now holds.

\textbf{Theorem 2.4.} $\Gamma$ is an $l$-CM cell complex iff the top cohomology module $H^{\dim \Gamma}(\Gamma)$ can occur as an $l-1$'th syzygy module in a free resolution.

Proof. Given the criterion of Theorem 2.1 applied to the restrictions $\Gamma_{-R}$ were $r < l$, the proof in \cite{7} carries over. \hfill \Box

Balinski’s theorem for convex polytopes, see \cite{17} says that the 1-skeleton of a $d$-dimensional polytope is $d$-connected. The following gives a comprehensive generalization of this since a polytope, being a ball, is 1-Cohen-Macaulay.

\textbf{Corollary 2.5.} If $\Gamma$ is an $l$-CM cell complex, its codimension $r$ skeleton is $l + r$-CM.

Proof. Let $\Gamma_{\leq \dim \Gamma - r}$ be the codimension $r$ skeleton. Then $\mathcal{E}(\Gamma_{\leq \dim \Gamma - r})\vee$ is the truncation of $\mathcal{E}(\Gamma)\vee$ in cohomological degrees $\leq \dim \Gamma - r$ and so the top cohomology module will be an $l + r - 1$'th syzygy module. \hfill \Box

Now it is known that for a simplicial complex, being 2-CM is a topological property, \cite{14}. (But being $l$-CM for $l \geq 3$ is not.) The following shows the same for cell complexes.

\textbf{Theorem 2.6.} A cell complex $\Gamma$ is 2-CM iff $P(\Gamma)$ is a 2-CM poset.

Proof. This follows by Propositions 3.6 and 3.3. \hfill \Box

2.3. \textbf{Gorenstein* cell complexes.} Recall that a simplicial complex $\Delta$ is Gorenstein* iff it is 2-CM and $\bar{H}_{\dim \Delta}(\Delta)$ is equal to $k$, see for instance the argument in \cite{7} Thm. 3.1. We then define a cell complex $\Gamma$ to be Gorenstein* iff it is 2-CM and has $\bar{H}_{\dim \Gamma}(\Gamma)$ equal to $k$. By the above theorem this is a topological property.

\textbf{Theorem 2.7.} $\Gamma$ is a Gorenstein* cell complex iff $\Gamma$ is Cohen-Macaulay and the top cohomology module is a rank one torsion free $S$-module. It naturally identifies as the ideal in $S$ generated be monomials $m_V/m_f$ where $f$ ranges over the facets of $\Gamma$.

Proof. This is completely analog to the proof of Theorem 3.1 in \cite{7}. \hfill \Box

\textit{Remark 2.8.} If $\Gamma$ is a simplicial complex, the the quotient of $S$ by this ideal is the Stanley-Reisner ring of the Alexander dual simplicial complex of $\Gamma$. 

3. Results on posets

This section contains the essences of the proofs of the theorems given in the previous Section. We formulate the results here in terms of posets, and this will be applied to the face poset of a cell complex. Some ingredients will be used repeatedly in the proofs and we inform on these first.

If $\Delta$ is a simplicial complex and $v$ a vertex in $\Delta$, let $st_{\Delta}\{v\}$ consist of the faces of $\Delta$ containing $v$. Thus $\text{lk}_{\Delta}\{v\}$ is the intersection of $st_{\Delta}\{v\}$ and $\Delta\setminus\{v\}$. Mayer-Vietoris for the pair $st_{\Delta}\{v\}$ and $\Delta\setminus\{v\}$ gives the long exact sequence

$$
\cdots \to \tilde{H}_p(\text{lk}_\Delta\{v\}) \to \tilde{H}_p(\Delta\setminus\{v\}) \to \tilde{H}_p(\Delta) \to \cdots
$$

When $P$ is a poset and $x$ is in $P$, the link $\text{lk}_P\{x\}$ is the join $(0, x) \ast (x, 1)$ consisting of subsets of $P$ which are unions of subsets of these two intervals. The Künneth formula, then gives an isomorphism

$$
\tilde{H}_p(\text{lk}_P\{x\}) = \oplus_{i+j=p-1} \tilde{H}_i((0, x)) \otimes \tilde{H}_j((x, 1)).
$$

A well-known criterion for $P$ to be CM is that for every interval $(x, y)$ in $\hat{P}$, $\tilde{H}_p((x, y))$ vanishes unless $p$ is equal to the rank of the interval, $rk_y - rk_x - 2$. We shall also in the following use the assumption that $P$ is Gorenstein*. This is equivalent to the additional requirement that $\tilde{H}_p((x, y)) = k$ whenever $p$ equals the rank of the interval. In particular every interval $(x, y)$ where $rk_y = rk_x + 2$ contains two elements.

**Lemma 3.1.** Let $P$ be a Gorenstein* poset and $x$ an element. Then $P_{-F(x)}$ is acyclic, i.e. $\tilde{H}_p(P_{-F(x)})$ is zero for each $p$.

**Proof.** First note that if $\Delta$ is a Gorenstein* simplicial complex and $x$ is a vertex in $\Delta$, then $\tilde{H}_p(\Delta\setminus\{x\})$ is zero for every $p$, see for instance the argument in [4] Thm.3.1. Let $\{x\} \subseteq J' \subseteq F(x)$ be an order ideal in $F(x)$. Let $J' = J \cup \{t\}$ where $t$ is a maximal element. We have an exact sequence

$$
(3) \quad \tilde{H}_p(\text{lk}_P\{t\}_J) \to \tilde{H}_p(P_{-J'}) \to \tilde{H}_p(P_{-J})
$$

where $\text{lk}_P\{t\}_J$ is the join of $(0, t)_{-F(x)}$ and $(t, 1)$. By induction on the sizes of $P$ and $J$ the outer groups of (3) vanish and hence the middle one. □

**Lemma 3.2.** Let $P$ be Gorenstein* with $\hat{P}$ a lattice, and let $R$ a subset of $P$ with at least two elements. Then there exists a maximal element $s$ in $P$ such that $s \geq r$ for some $r$ in $R$, but not all $r$ in $R$.

**Proof.** Let $t$ be the join of all elements of $R$ and chose a minimal element $r$ in $R$. We construct successively larger elements $s_i$ which fulfill the conditions above. First let $s_1 = r$. There are at least two elements covering $r$. Not both of them are $\geq t$ since then their meet $s_1 = r$ is $\geq t$. Let $s_2$ be one of them not $\geq t$. Continuing this process we finally arrive at a maximal $s$. □

**Proposition 3.3.** Assume $\hat{P}$ is a graded lattice and $P_{<x}$ is Gorenstein* for each $x$ in $P$. If $P$ is 2-CM, then $\tilde{H}_p(P_{-F(R)})$ is zero when $p + r \leq rk_P$ and $p < rk_P$. 

Proof. We shall prove that \( \tilde{H}_p(P_{-J'}) \) is zero when \( R \subseteq J' \subseteq F(R) \) is an order ideal and \( p + r \leq rkP \) and \( p < rkP \). We use induction on the sizes of \( J' \) and \( R \). Note that the case \( J' = R \) is the hypothesis. So assume \( J' = J \cup \{ t \} \) where \( t \) is a maximal element in \( J' \) not in \( R \). We have an exact sequence

\[
\tilde{H}_{p+1}(P_{-J}) \xrightarrow{\eta} \tilde{H}_p(\text{lkn}_P\{t\}_{-J}) \to \tilde{H}_p(P_{-J'}) \to \tilde{H}_p(P_{-J}).
\]

By induction we may assume that the last term of this sequence is zero. To achieve our goal we shall show that \( \eta \) is surjective. We do this by induction on the sizes of \( R \) and \( J \).

Assume \( R = \{ x \} \) consists of one element. Then \( \text{lkn}_P\{t\}_{-J} \) is the join of \( (0,t)_{-F(R)} \) and \( (t,1) \). Using Lemma 3.1 and the Künneth formula, we get that the second term in (4) is zero and so \( \eta \) is surjective.

If \( R' = R \cap \{ 0,t \} \) contains less than \( R \) elements, then \( \text{lkn}_P\{t\}_{-J} \) is the join of \( (0,t)_{-F(R')} \) and \( (t,1) \). If the second term in (4) is nonzero then by the Künneth formula \( \tilde{H}_i((0,t)_{-F(R')}) \) and \( \tilde{H}_j((t,1)) \) are nonzero for some pair \( i \) and \( j \) with \( i + j = p - 1 \). But then \( j \geq rkP - rkt - 1 \) and by induction \( i + r' \geq rkP \). This contradicts \( p + r' \geq rkP \). Hence the second term in (4) is zero and \( \eta \) surjective.

Now the long exact sequence to which \( \tilde{H}_p(\text{lkn}_P\{s\}_{-J'}) \) vanishes. Since this continues the long exact sequence which \( \beta \) belongs to, \( \beta \) must be surjective.

Now the long exact sequence to which \( \eta' \) belongs to is continued by

\[
\tilde{H}_p(\text{lkn}_P\{s\}_{-J''} \cup \{ t \})).
\]

If nonzero, then by the Künneth formula \( \tilde{H}_i((0,s)_{-F(R')}) \) and \( \tilde{H}_j((s,1)_{-\{t\}}) \) are nonzero for some \( i + j = p - 1 \). This gives \( i + r' \geq rkS \) and, since \( (s,1) \) is 2-CM since \( P \) is (being 2-CM is preserved by links), that \( j + 1 \geq rkP - rks \). Then \( p + r' \geq rkP \), contradicting \( p + r < rkP \). Thus vanishes and so \( \eta' \) is surjective. This completes the proof.

Proposition 3.4. Assume \( \hat{P} \) is a graded lattice and \( P_{<x} \) is Gorenstein* for every \( x \) in \( P \). If \( P \) is Cohen-Macaulay, then \( \tilde{H}_p(P_{-F(R)}) \) is zero when \( p + r < rkP \).
Proof. We shall use induction on the size of an order ideal \( R \subseteq J' \subseteq F(R) \) to prove that \( \tilde{H}_p(P_{-J'}) \) is zero for \( p + r < rkP \). When \( J = R \) this is the hypothesis, so assume \( J' = J \cup \{t\} \) where \( t \) is a maximal element of \( J' \setminus R \). There is an exact sequence

\[
\tilde{H}_p(\text{lk}_P\{t\}_{-J}) \rightarrow \tilde{H}_p(P_{-J'}) \rightarrow \tilde{H}_p(P_{-J})
\]

(6) Letting \( R' \) be \( R \cap (0, t) \), then \( \text{lk}_P\{t\}_{-J} \) is the join of \( (0, t)_{-F(R')} \) and \( (t, 1) \). By Proposition 3.3 we know that \( \tilde{H}_i((0, t)_{-F(R')}) \) nonzero implies \( i + r' \geq rkt \). Also \( \tilde{H}_j((t, 1)) \) nonzero implies \( j \geq rkP - rkt - 1 \). Hence the Künneth formula gives \( \tilde{H}_p(\text{lk}_P\{t\}_{-J}) \) is nonzero only if \( p + r' = i + j + r' + 1 \geq rkP \).

Since we are assuming \( p + r < rkP \), the outer terms in (6) vanish and so also the middle term.

\[\square\]

**Proposition 3.5.** Assume \( \hat{P} \) is a graded lattice with \( P_{< x} \) Gorenstein* for all \( x \) in \( P \). If \( \tilde{H}_p(P_{-F(R)}) \) is zero when \( p + r < rkP \) and \( R \) consists of atoms, then \( P \) is Cohen-Macaulay.

Proof. We first show that \( \tilde{H}_p(P_{-F(R)}) \) is zero for any \( R \) with \( p + r < rkP \). We shall use induction on the rank of a maximal element in \( R \) and the number of elements in \( R \) with this maximal rank. So let \( x \) in \( R \) be of maximal rank in \( R \), supposed to be \( > 0 \). There is thus in \( \hat{P} \) an element \( w \) with \( rk \hat{P}_w = rk \hat{P} + 2 \). Since \( P_{< x} \) is Gorenstein*, there are at least two elements strictly between \( w \) and \( x \), say \( y \) and \( z \) with \( y \land z = w \). Let \( R_0 = R \setminus \{x\} \).

Then

\[
F(R_0 \cup \{x\}) = F(R_0 \cup \{y\}) \cap F(R_0 \cap \{z\}).
\]

Hence the order complex \( \Delta(P)_{-F(R_0 \cup \{x\})} \) is the union of \( \Delta(P)_{-F(R_0 \cup \{y\})} \) and \( \Delta(P)_{-F(R_0 \cup \{z\})} \); and \( \Delta(P)_{-F(R_0 \cup \{y, z\})} \) is the intersection of these two complexes. The Mayer-Vietoris sequence

\[
\tilde{H}_p(P_{-F(R_0 \cup \{y\})}) \oplus \tilde{H}_p(P_{-F(R_0 \cup \{z\})}) \rightarrow \tilde{H}_p(P_{-F(R)}) \rightarrow \tilde{H}_{p-1}(P_{-F(R_0 \cup \{y, z\})})
\]

then gives by induction that the middle term vanishes for \( p + r < rkP \).

In order to show that \( P \) is CM, it will be enough to show that \( \tilde{H}_j((y, 1)) \) vanishes for \( j < rkP - rky - 1 \). Since \( \tilde{H}_i((0, y)) \) is \( k \) if \( i = rky - 1 \) and zero otherwise, this is by the Künneth formula the same as showing that \( \tilde{H}_p(\text{lk}_P\{y\}) \) vanishes for \( p < rkP - 1 \). From the exact sequence

\[
\tilde{H}_{p+1}(P) \rightarrow \tilde{H}_p(\text{lk}_P\{y\}) \rightarrow \tilde{H}_p(P_{-\{y\}}) \rightarrow \tilde{H}_p(P)
\]

it follows that we can show that the third term is zero.

Let \( \{y\} \subseteq J' \subseteq F(y) \) be an order ideal. We show by descending induction on the size of \( J' \) that \( \tilde{H}_p(P_{-J'}) \) is zero. Let \( J' = J \cup \{t\} \) where \( t > y \) is a maximal element in \( J' \). There is an exact sequence

\[
\tilde{H}_p(\text{lk}_P\{t\}_{-J}) \rightarrow \tilde{H}_p(P_{-J'}) \rightarrow \tilde{H}_p(P_{-J}) \rightarrow \tilde{H}_{p-1}(\text{lk}_P\{t\}_{-J})
\]

Here \( \text{lk}_P\{t\} \) is the join of \( (0, t)_{-F(y)} \) which is acyclic by Lemma 3.1 and (t, 1). Hence \( \tilde{H}(P_{-J}) \) vanishes by induction. \( \square \)
Proposition 3.6. Assume \( \hat{P} \) is a graded lattice such that \( P_{<x} \) is Gorenstein* for each \( x \) in \( P \). If \( \bar{H}_p(P_{-F(R)}) \) is zero when \( R \) consists of atoms, \( p+r \leq rkP \) and \( p < rkP \), then \( \bar{P} \) is 2-CM.

Proof. It follows exactly as in Proposition 3.5 that \( \bar{H}_p(P_{-F(R)}) \) is zero for any \( R \) with \( p+r \leq rkR \) and \( p < rkR \).

We shall show that \( \bar{H}_p(P_{-R}) \) is zero when \( p+r \leq rkP \) and \( p < rkP \). By Hochster’s criterion (2) this gives \( P \) is 2-CM. We may assume \( R \) is nonempty. Let \( R \subseteq J \subseteq F(R) \) be an order ideal. We prove by descending induction on the size of \( J \) that \( \bar{H}_p(P_{-J}) \) is zero when \( p+r \leq rkP \) and \( p < rkP \). Assume \( J \subsetneq F(R) \) and let \( J' = J \cup \{t\} \) be a strictly larger order ideal in \( F(R) \).

There is an exact sequence

\[
\bar{H}_p(P_{-J'}) \to \bar{H}_p(P_{-J}) \to \bar{H}_{p-1}(lk_P \{t\}_{-J})
\]

If the last term is nonzero, then \( \bar{H}_i((0,t)_{-F(R)}) \) and \( \bar{H}_j((t,1)) \) are nonzero for some \( i, j \) with \( i+j = p-2 \). By induction \( i+r \geq rk \) and by Proposition 3.5, \( j \geq rkP - rkt - 1 \). Hence \( p-2+r \geq rkP-1 \) or \( p+r \geq rkP+1 \), which is against assumption. Hence the outer terms of (7) vanish and so the middle term.

4. The hexagon of complexes

In this section we show that although there is no single generalization of the Stanley-Reisner ring to cell complexes, the resolution of the Stanley-Reisner ring, a complex of free \( S \)-modules, may be generalized to a complex of free \( S \)-modules associated to any cell complex. This complex of free \( S \)-modules together with the enriched chain and cochain complexes fit into a natural sixtuple of complexes associated to any cell complex and we shall describe the properties of this sixtuple. A consequence is that the canonical module of the Stanley-Reisner ring of a Cohen-Macaulay simplicial complex may be generalized to a module associated to any Cohen-Macaulay cell complex.

4.1. Square free modules and complexes. In [15], Yanagawa introduced the notion of square free modules over the polynomial ring \( S \). An \( \mathbb{N}^v \)-graded module \( M \) over \( S \) is square free if the multiplication map \( M_b \xrightarrow{x_v} M_{b+e_v} \), where \( e_v \) is the \( v \)'th coordinate vector, is a bijection when \( v \) is contained in the support of \( b \), i.e. the set of non-vanishing coordinates.

If \( R \) is a subset of \( V \) we shall, where appropriate identify \( R \) with the multidegree \( b \) which is the characteristic vector of \( R \). For a square free module \( M \), independently T.Römer [11] and E.Miller [9] defined its Alexander dual \( M^* \) as follows. For a subset \( R \) of \( V \), \( (M^*)_R \) is the dual \( \text{Hom}_k(M_{R^c}, k) \). If \( v \) is not in \( R \) the multiplication

\[
(M^*)_R \xrightarrow{x_v} (M^*)_{R \cup \{v\}}
\]
is the dual of the multiplication
\[ M_{(R \cup \{v\})^c} \xrightarrow{2v} M_{R^c}. \]
By obvious extension this defines \((M^*)_b\) for all \(b \in N^V\) and all multiplications.

A free square free module is a module
\[ \bigoplus_{R \subseteq V} S \otimes_k B_R \]
where \(B_R\) is a vector space of multidegree \(R\). In a cochain complex \(P^*\) of free square free modules with term
\[ P^i = \bigoplus_{R \subseteq V} S \otimes_k B_R^i \]
the spaces \(B_R^i\) are called the Betti spaces of \(P^*\). The differentials in such a complex are to be homogeneous of degree zero and the complex is called minimal if the differentials in \(k \otimes_S P^*\) vanish. The complex \(P^*\) may be shifted \(n\) steps to the left (resp. right if \(n < 0\)) to a complex \(P^*[n]\) where \(P^*[n]^i\) is equal to \(P^{i+n}\). We may also consider a cochain complex \(P^*\) as a chain complex \(P\) by letting \(P_i\) be \(P^{-i}\). Note that \(P^*[n]\) then is \(P[−n]\).

On the category of complexes of free square free modules there are two functors \(D\) and \(A\), studied in [16]. The functor \(D\) is given by
\[ D(P^*) = \text{Hom}_S(P^*, \omega_S). \]
The functor \(A\) is given by first taking the Alexander dual complex \(P^*\). Unfortunately this is not a complex of free square free modules. What we do is to take a minimal resolution \(Q^*\) of this complex. Let \(Q^*\) is a minimal complex of free modules together with a morphism \(Q^* \rightarrow P^*\) which induces an isomorphism on cohomology. We then write \(A(P^*) = Q^*\).

When \(P^*\) is a minimal cochain complex, Yanagawa in [16] shows the fundamental isomorphism
\[ (8) \quad D \circ A \circ D \circ A \circ D \circ A(P^*) = P^*[−n]. \]

Remark 4.1. In [16] \(D\) is defined slightly differently. There \(D\) is defined as
\[ D(P^*) = \text{Hom}_S(P^*, \omega_S)[n] \]
which is natural in a general algebraic setting. Then the relation (8) is
\[ D \circ A \circ D \circ A \circ D \circ A(P^*) = P^*[2n]. \]

By starting with the enriched chain complex shifted one step to the left, \(E[−1]\), we get by (8) a hexagon
We consider $\mathcal{E}[-1], \mathcal{F},$ and $\mathcal{G}$ as chain complexes and their duals by $\mathbf{D}$ as cochain complexes. Now we proceed to study these complexes.

4.2. **Homology and Betti spaces.** For a complex $\mathcal{P}$ of free $S$-modules define its $i$’th linear strand $\mathcal{P}_{(i)}$ to be given by

$$
\mathcal{P}_{(i)}^j = \bigoplus_{|R|=i-j} S \otimes_k B_R^j.
$$

For a square free module $M$, one may define a complex $L(M)$ (see [16, p.9] where it is denoted by $F(M)$) by

$$
L^i(M) = \bigoplus_{|R|=i} (M_R)^{\circ} \otimes_k S
$$

where $(M_R)^{\circ}$ is $M_R$ but considered to have multidegree $R^c$. The differential is

$$
m^\circ \otimes s \mapsto \sum_{j \notin R} (-1)^{\alpha(j,R)} (x_j m)^{\circ} \otimes x_j s
$$

where $\alpha(j,R)$ is the number of $i$ in $R$ such that $i < j$ after putting some total order on $V$.

The following is [16, Thm. 3.8].

**Proposition 4.2.** Let $Q$ be $D \circ A(\mathcal{P})$. Then

$$
L(H^i(Q))|_{n-i}
$$

is the $i$’th linear strand of $\mathcal{P}$.

**Corollary 4.3.** Let $Q$ be $A \circ D \circ A(\mathcal{P})$. Then

$$
\text{Hom}_S(L(H^{-i}(Q)), \omega_S)[-i]
$$

is the $i$’th linear strand in $\mathcal{P}$. (And similarly with $\mathcal{P}$ and $Q$ interchanged.)
More informally, for a pair \( P \) and \( Q \) on opposite corners of the hexagon, the Betti spaces of \( P \) corresponds to the homology spaces of \( Q \) and vice versa.

Let \( k^i[\Gamma] \) be the square free \( S \)-module given by \( k^i[\Gamma]_F = kF \) if \( F \) is the vertex set of some face \( f \) where the cardinality of \( F \) is \( \dim f + i + 1 \) and \( k^i[\Gamma]_F = 0 \) if \( F \) is not so. For two faces \( f' \) and \( f \) with vertex sets \( F \cup \{v\} \) and \( F \) the multiplication

\[
k^i[\Gamma]_F \cdot x_v \mapsto k^i[\Gamma]_{F \cup \{v\}}
\]

is given by sending \( F \) to \( F \cup \{v\} \). Note that when \( \Gamma \) is a simplicial complex, \( k^0[\Gamma] \) is the Stanley-Reisner ring and \( k^i[\Gamma] \) is zero for \( i \) not 0.

The main observation in this section is the following which shows that \( F^\vee \) is a generalization of the resolution of the Stanley-Reisner ring of a simplicial complex.

**Theorem 4.4.** \( H^{-i}(F^\vee) = k^i[\Gamma] \)

**Proof.** This follows by Corollary 4.3 together with the fact that the linear strands in \( E[-1] \) are exactly

\[
\text{Hom}_S(\mathcal{L}(k^i[\Gamma]), \omega_S)[i].
\]

\( \square \)

**Corollary 4.5.** When \( \Gamma \) is simplicial then \( F^\vee \) is the resolution of the Stanley-Reisner ring and \( G \) is the resolution of the Stanley-Reisner ideal \( I_{\Gamma^*} \) of the Alexander dual simplicial complex \( \Gamma^* \).

**Proof.** When \( \Gamma \) is simplicial then \( k^i[\Gamma] \) is zero for \( i > 0 \) and \( k^0[\Gamma] \) is the Stanley-Reisner ring. Hence \( F \) is a resolution of the Stanley-Reisner ring. Since the Alexander dual module of \( k[\Gamma] \) is exactly \( I_{\Gamma^*} \), the homology of \( G \) is this ideal in homological degree zero and vanishes elsewhere. So \( G \) is a resolution of this ideal. \( \square \)

Now the Betti spaces of the enriched chain complex are of course given by

\[
B_iF(E[-1]) = kF
\]

when \( F \) is the vertices of a face of \( \Gamma \) of dimension \( i - 1 \). The homology of \( E[-1] \) is, as noted in Section 1.2, given by

\[
H_i(E[-1])_b = \tilde{H}_{i-1}(\Gamma_{\text{supp} b}).
\]

On the other hand the cohomology of the enriched cochain complex \( E^\vee \) is by 7 in the simplicial case given by

\[
H^i(E^\vee[-1]) = \tilde{H}^{i-1-|\text{(supp} b\text{)}|}(\text{lk}_\Gamma(\text{supp} b)^c)\).
\]

Using Proposition 4.2 together with the fact that the homologies of \( P \) and \( A(P) \) are Alexander dual modules, and the Betti spaces of \( P \) and \( D(P) \) are Alexander dual spaces, we get in the simplicial case the following table over the homology and Betti spaces.
4.3. **Linear complexes.** A theorem of Eagon and Reiner [6] says that when $\Gamma$ is simplicial then the resolution $G$ of $I_\Gamma^*$ is linear iff $\Gamma$ is Cohen-Macaulay. The following generalizes this.

**Theorem 4.6.** $G$ is linear iff $\Gamma$ is a Cohen-Macaulay cell complex.

**Proof.** The linear strands of $G$ corresponds to the cohomology of $E^\vee[-1]$. But $\Gamma$ is a Cohen-Macaulay cell complex iff $E^\vee[-1]$ has only one nonvanishing cohomology module and hence $G$ only one linear strand. \(\square\)

We then readily get the following.  

- $E$ and $E^\vee$ are linear iff $\Gamma$ is simplicial.  
- $G$ and $G^\vee$ are linear iff $\Gamma$ is Cohen-Macaulay.  
- $F$ and $F^\vee$ are linear iff $\Gamma$ is a simplex on the vertices it contains.

4.4. **Generalizations of the canonical module.** When $\Gamma$ is simplicial and Cohen-Macaulay the complex $F$ has only one cohomology module, the
canonical module $\omega_{k[\Gamma]}$, and this is the Alexander dual module of the only nonvanishing cohomology module of $E^V[-1]$, the top enriched cohomology module of $\Gamma$. For a Cohen-Macaulay cell complex we may therefore consider the Alexander dual of its top cohomology module, let us denote it by $\omega_\Gamma$, as a generalization of the canonical module.

4.5. Convex polytopes. If $\Gamma$ is a convex polytope with boundary $\partial\Gamma$ then $H^\dim\partial\Gamma(\partial\Gamma)$ identifies as an ideal in $S$ by Theorem 2.7. Taking the Alexander dual of the inclusion $H^\dim\partial\Gamma(\partial\Gamma) \hookrightarrow S$ we get a surjection $\omega_\partial\Gamma \twoheadrightarrow S$ and so $\omega_\partial\Gamma$ identifies as a quotient ring of $S$. When $\Gamma$ is simplicial, the canonical module $\omega_{k[\partial\Gamma]}$ is isomorphic to the Stanley-Reisner ring $k[\partial\Gamma]$. Thus in general for convex polytopes $\Gamma$ one might consider $\omega_\partial\Gamma$ as a generalization of the Stanley-Reisner ring, although I would not consider it as the fully natural viewpoint since one then should consider the complex $F^\vee$ instead of $F$. Theorem 4.4 also gives that the quotient ring $\omega_\partial\Gamma$ will be Cohen-Macaulay only when $\Gamma$ is simplicial.

Concretely the ring $\omega_\partial\Gamma$ may be described as the square free ring such that $(\omega_\partial\Gamma)_F$ is $k$ when $F$ is contained in a facet of $\partial\Gamma$ and 0 when $F$ is not contained in a facet. It is the quotient of $S$ by the ideal generated by the $m_F$ where $F$ is not contained in a facet.

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