PROVING SOME IDENTITIES OF GOSPER ON 
q-TRIGONOMETRIC FUNCTIONS

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ABSTRACT. Gosper introduced the functions \(\sin_q z\) and \(\cos_q z\) as \(q\)-analogues for the trigonometric functions \(\sin z\) and \(\cos z\) respectively. He stated but did not prove a variety of identities involving these two \(q\)-trigonometric functions. In this paper, we shall use the theory of elliptic functions to prove three formulas from the list of Gosper on the functions \(\sin_q z\) and \(\cos_q z\).

1. Introduction

Throughout the paper let \(q = e^{\pi i \tau}\) with \(\operatorname{Im}(\tau) > 0\), let \(\tau' = -\frac{1}{\tau}\), and let \(p = e^{\pi i \tau'}\). Note that the assumption \(\operatorname{Im}(\tau) > 0\) guarantees that \(|q| < 1\) and \(|p| < 1\). For a complex variable \(a\), the \(q\)-shifted factorials are given by

\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad (a; q)_\infty = \lim_{n \to \infty} (a; q)_n
\]

and for brevity let

\[
(a_1, \ldots, a_k; q)_n = (a_1; q)_n \cdots (a_k; q)_n, \quad (a_1, \ldots, a_k; q)_\infty = (a_1; q)_\infty \cdots (a_k; q)_\infty.
\]

The first Jacobi theta function is defined as follows:

\[
\theta_1(z, q) = \theta_1(z \mid \tau) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(2n+1)^2/4} \sin(2n + 1)z,
\]

which has the following very useful infinite product representations

\[
\theta_1(z \mid \tau) = i q^{\frac{1}{4}} e^{-iz} (q^2 e^{-2iz}, e^{2iz}, q^2; q^2)_\infty = 2q^{\frac{1}{4}} (\sin z)(q^2 e^{2iz}, q^2 e^{-2iz}, q^2; q^2)_\infty.
\]

See Whittaker and Watson [12, p. 469] and Gasper and Rahman [4, p. 15]. For the purpose of this work we will need the following basic properties of the function \(\theta_1(z \mid \tau)\) which can be derived straightforwardly by the definitions.

\[
\theta_1(k \pi \mid \tau) = 0 \quad (k \in \mathbb{Z}),
\]

\[
\theta_1\left(\frac{\pi}{2} \mid \tau\right) = 2q^{\frac{1}{4}} (-q^2; q^2)_\infty (q^2; q^2)_\infty,
\]

\[
\theta_1(z + \pi \mid \tau) = -\theta_1(z \mid \tau) = \theta_1(-z \mid \tau),
\]

\[
\theta_1(z + \pi \tau \mid \tau) = -q^{-1} e^{-2iz} \theta_1(z \mid \tau),
\]

\[
\theta_1'(0 \mid \tau) = 2q^{\frac{1}{4}} (q^2; q^2)_3.
\]
Moreover, one can show that for any positive integer \( k \) we have
\[
\theta_1 \left( z + \pi \tau \mid \frac{\tau}{k} \right) = (-1)^k q^{-k} e^{-2kiz} \theta_1 \left( z \mid \frac{\tau}{k} \right).
\] (2)

Further, Jacobi’s imaginary transformation for the function \( \theta_1(z \mid \tau) \) states that
\[
\theta_1(z \mid \tau) = (-i\tau)^{-\frac{1}{2}}(i\tau) e^{\frac{iz^2}{2}} \theta_1(z\tau' \mid \tau').
\] (3)
See [12, p. 475]. Differentiating (3) with respect to \( z \) and combining with the basic properties we find
\[
\theta_1'(0 \mid \tau') = (-i\tau)^{\frac{1}{2}} 2q^{\frac{1}{4}} (q^2; q^2)^3.
\] (4)
Gosper [5] introduced \( q \)-analogues of \( \sin(z) \) and \( \cos(z) \) as follows:
\[
\sin_q(\pi z) = q^{(z-1/2)^2} \prod_{n=1}^{\infty} \frac{1 - q^{2n-2z}}{1 - q^{2n-1}} = q^{(z-\frac{1}{4})^2} \frac{q^{2z}, q^{2-2z}; q^2}{(q; q^2)^2\infty},
\]
\[
\cos_q(\pi z) = q^{z^2} \prod_{n=1}^{\infty} \frac{1 - q^{2n-2z-1}}{1 - q^{2n-1}} = q^{z^2} \frac{q^{1+2z}, q^{1-2z}; q^2}{(q; q^2)^2\infty}.
\]
It is easy to see that \( \cos_q(z) = \sin_q(\pi/2 - z) \). Gosper proved a variety of identities involving these two functions including the following
\[
\sin_q(z) = \frac{\theta_1(z, p)}{\theta_1 \left( \frac{z}{2}, p \right)} \quad \text{where } (\ln p)(\ln q) = \pi^2,
\]
which is readily seen to be equivalent to
\[
\sin_q(z) = \frac{\theta_1(z \mid \tau')}{\theta_1 \left( \frac{z}{\tau'}, \tau' \right)}. \quad (5)
\]
Clearly the formula (5) combined with the identities \( \cos_q(z) = \sin_q(\pi/2 - z) \) and \( \theta_1(z + \pi) = -\theta_1(z) \) yield
\[
\cos_q(z) = \frac{\theta_1 \left( z + \frac{\pi}{2}, p \right)}{\theta_1 \left( \frac{z + \pi}{2}, p \right)} = \frac{\theta_1 \left( z + \frac{\pi}{2} \mid \tau' \right)}{\theta_1 \left( \frac{z + \pi}{2} \mid \tau' \right)}. \quad (6)
\]
See Gosper [5] p. 98]. The author also stated without proof many identities based on a computer algebra facility called MACSYMA. Among the formulas, we find the following which we will mark with the same labels as in Gosper [5].
\[
\sin_q(2z) = \frac{1}{2 \Pi_q} \sqrt{\left( \sin_q z \right)^2 - \left( \sin_q q^2 z \right)^4}, \quad (q-\text{Double})
\]
\[
\cos_q(2z) = (\sin_q z)^2 - (\sin_q q^2 z)^2, \quad (q-\text{Double3})
\]
\[
\sin_q(3z) = \frac{1}{3 \Pi_q} \sin_q z - \left( \frac{1}{3 \Pi_q} \right) (\sin_q q^3 z)^3, \quad (q-\text{Triple})
\]
and
\[
\sin_q(3z) = \Pi_q \left( \cos_q z \right)^2 \sin_q z - \left( \sin_q q^3 z \right)^3, \quad (q-\text{Triple2})
\]
where
\[
\Pi_q = q^{\frac{1}{4}} \left( \frac{q^{2}; q^2}{(q; q^2)_{\infty}} \right).
\] (7)
As it was observed by Gosper, it is easy to verify that \( (q-\text{Double}) \) is equivalent to
\[
\cos_q(2z) = \frac{1}{2 \Pi_q} \sqrt{\left( \cos_q z \right)^2 - \left( \cos_q q^2 z \right)^4} \quad (q-\text{Double4})
\]
and that by a combination of \(q\text{-Double}_1\), \(q\text{-Double}_2\) and \(q\text{-Double}_3\) we have
\[
\cos_q(2z) = (\cos_q z)^4 - (\sin_q z)^4, \quad (q\text{-Double}_5)
\]
See Gosper \([5]\) p. 89–93. We notice that many of Gosper’s formulas are \(q\)-analogues for well-known trigonometric functions. For instance, after introducing the function \(\cos z\) as in \([3]\), Gosper showed that
\[
\sin_q(2z) = \frac{\Pi_q}{\Pi_{q^2}} \sin_{q^2} z \cos_q z, \quad (q\text{-Double}_2)
\]
which is clearly a \(q\)-analogue of the famous formula \(\sin 2z = 2 \cos z \sin z\). Recently, Mező \([9]\) gave a different proof for \(q\text{-Double}_2\). In his proof, Mező among other things analysed the logarithmic derivatives with respect to \(z\) of \(\sin_q(2z)\) and \(\sin_{q^2}(z) \cos_q(z)\) and found that they coincide, implying that the ratio
\[
\frac{\sin_q(2z)}{\sin_{q^2}(z) \cos_q(z)} = C(q), \quad (8)
\]
for a constant \(C(q)\) which he was able to determine as in the formula \(q\text{-Double}_2\). Alternatively, taking into account the relations \([5]\) and \([6]\), formula \(q\text{-Double}_3\) can be written as
\[
\frac{\theta_1(2z + \frac{\pi}{2} | \tau')}{\theta_1(\frac{\pi}{2} | \tau')} = \left( \frac{\theta_1\left( z + \frac{\pi}{2} | \frac{\tau'}{2} \right)}{\theta_1\left( \frac{\pi}{2} | \frac{\tau'}{2} \right)} \right)^2 - \left( \frac{\theta_1\left( \frac{\pi}{2} | \frac{\tau'}{2} \right)}{\theta_1\left( \frac{\pi}{2} | \frac{\tau'}{2} \right)} \right)^2,
\]
which after rearrangement becomes
\[
\theta_1\left( z + \frac{\pi}{2} | \frac{\tau'}{2} \right) \theta_1^2\left( \frac{\pi}{2} | \frac{\tau'}{2} \right) = \theta_1\left( \frac{\pi}{2} | \tau' \right) \theta_1^2\left( z + \frac{\pi}{2} | \frac{\tau'}{2} \right) - \theta_1\left( \frac{\pi}{2} | \tau' \right) \theta_1^2\left( z | \frac{\tau'}{2} \right). \quad (9)
\]
Furthermore, again by virtue of \([5]\) and \([6]\) note that formula \(q\text{-Double}_2\) means
\[
\frac{\theta_1(2z | \tau')}{\theta_1\left( \frac{\pi}{2} | \tau' \right)} = C(q) \frac{\theta_1\left( z | \frac{\tau'}{2} \right)}{\theta_1\left( \frac{\pi}{2} | \frac{\tau'}{2} \right)} \frac{\theta_1\left( z + \frac{\pi}{2} | \frac{\tau'}{2} \right)}{\theta_1\left( \frac{\pi}{2} | \frac{\tau'}{2} \right)},
\]
or equivalently,
\[
\theta_1(2z | \tau') \theta_1^2\left( \frac{\pi}{2} | \frac{\tau'}{2} \right) = C(q) \theta_1\left( \frac{\pi}{2} | \tau' \right) \theta_1\left( z | \frac{\tau'}{2} \right) \theta_1\left( z + \frac{\pi}{2} | \frac{\tau'}{2} \right),
\]
which by observing that
\[
C(q) = \frac{\theta_1^2\left( \frac{\pi}{2} | \frac{\tau'}{2} \right)}{\theta_1^2\left( \frac{\pi}{2} | \frac{\tau'}{2} \right)}
\]
means that
\[
\theta_1\left( z | \frac{\tau'}{2} \right) \theta_1\left( z + \frac{\pi}{2} | \frac{\tau'}{2} \right) = \theta_1\left( 2z | \tau' \right) \theta_1^2\left( \frac{\pi}{2} | \frac{\tau'}{2} \right). \quad (10)
\]
After recognising the equivalent forms \([9]\) and \([10]\) as three-term addition formulas involving theta functions, this author in \([3]\) proved both \(q\text{-Double}_2\) and \(q\text{-Double}_3\) by employing the theory of elliptic functions. Moreover, by following the same steps as before, we can check that each one of Gosper’s identities \(q\text{-Double}_1\), \(q\text{-Triple}_1\), and \(q\text{-Triple}_2\) is a three-term addition formulas involving theta functions. See below Theorem 1, Theorem 2 and Theorem 3. The theory of
elliptic functions proved to be a powerful tool to study this type of addition formulas. For recent papers dealing with addition formulas by means of elliptic functions, we refer to Liu \[7\]. See also Whittaker and Watson \[12\] and Shen \[10\], \[11\] for more additive formulas involving theta functions and applications. For a more direct approach to produce addition formulas involving theta functions through infinite sums manipulations, we refer to the book by Lawden \[6\]. In this paper our goal is to prove Gosper’s formulas \((q\)-Double\), \((q\)-Triple\), and \((q\)-Triple\)) using the theory of elliptic functions. The paper is organized as follows. In Section 2 we state three theorems and prove three corollaries which are the equivalent forms of \((q\)-Double\), \((q\)-Triple\), and \((q\)-Triple\)). In Section 3 we state and prove a general result which we shall need to prove the three theorems in Section 2. Section 4, Section 5, and Section 6 are devoted to proofs for Theorem 1, Theorem 2, and Theorem 3 respectively.

2. Main results

**Theorem 1.** For all complex number \(z\) we have

\[
\theta_1^2(2z \mid \tau') \theta_1^1 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{2} \mid \frac{\tau'}{4} \right) = \left( \frac{1}{2} \frac{\Pi_{q}^q}{\Pi_{q}^{q^3}} \right)^2 \theta_1^2 \left( z \mid \frac{\tau'}{4} \right) \theta_1^1 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{2} \mid \tau' \right)
\]

- \(\theta_1^2 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{2} \mid \frac{\tau'}{4} \right)
\]

**Corollary 1.** For all complex number \(z\) we have

\[
\sin_q(2z) = \frac{1}{2} \frac{\Pi_{q}^q}{\Pi_{q}^{q^3}} \sqrt{(\sin_q z)^2 - (\sin_q^3 z)^4}.
\]

**Proof.** By the relations \((5)\) and \((6)\), we can readily see that \((q\)-Double\), which is the statement of this corollary, is an equivalent form of Theorem 1. \(\square\)

**Theorem 2.** For all complex number \(z\) we have

\[
\theta_1(3z \mid \tau') \theta_1^1 \left( \frac{\pi}{2} \mid \frac{\tau'}{3} \right) \theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{9} \right) = \frac{1}{3} \frac{\Pi_{q}^q}{\Pi_{q}^{q^3}} \theta_1 \left( z \mid \frac{\tau'}{9} \right) \theta_1 \left( \frac{\pi}{2} \mid \tau' \right) \theta_1^3 \left( \frac{\pi}{2} \mid \frac{\tau'}{3} \right)
\]

- \(\theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{3} \right) \theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{9} \right)
\]

**Corollary 2.** For all complex number \(z\) we have

\[
\sin_q(3z) = \frac{1}{3} \frac{\Pi_{q}^q}{\Pi_{q}^{q^3}} \sin_q z - \left( 1 + \frac{1}{3} \frac{\Pi_{q}^q}{\Pi_{q}^{q^3}} \right) (\sin_q^3 z)^3.
\]

**Proof.** By the relations \((3)\) and \((6)\), it is easy to check that the identity \((q\)-Triple\) is equivalent to Theorem 2. \(\square\)

**Theorem 3.** For all complex number \(z\) we have

\[
\theta_1(3z \mid \tau') \theta_1^1 \left( \frac{\pi}{2} \mid \frac{\tau'}{3} \right) \theta_1 \left( z + \frac{\pi}{2} \mid \frac{\tau'}{3} \right) \theta_1 \left( \frac{\pi}{2} \mid \tau' \right)
\]

- \(\theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{3} \right) \theta_1 \left( z + \frac{\pi}{2} \mid \frac{\tau'}{3} \right)
\]
Corollary 3. For all complex number \( z \) we have
\[
\sin_q(3z) = \frac{\Pi_q}{\Pi_3} (\cos_{q_3} z)^2 \sin_{q_3} z - (\sin_{q_3} z)^3.
\]

Proof. By the relations (5) and (10), it is easy to check that the identity \((q^2\text{-Triple}_2)\) is equivalent to Theorem 3. \(\square\)

3. A General Result

We note that Theorem 1, Theorem 2, and Theorem 3 below are consequences of the following result which is an extension of a theorem of Liu [8, Theorem 1].

Theorem 4. Let \( k \) be a positive integer, let \( l \) be a nonnegative integer, and let \( h_1(z) \) and \( h_2(z) \) be entire functions satisfying the following two conditions:
\[
\begin{align*}
\sin_z(\pi) & = 0, \\
\cos_z(\pi) & = 1
\end{align*}
\]

If \( h_1(z) \) and \( h_2(z) \) have a zero of order at least \( l \) at \( z = 0 \), then there is a constant \( C \) such that
\[
(h_1(x) + (-1)^l h_1(-x)) (h_2(y) + (-1)^l h_2(-y))
- (h_2(x) + (-1)^l h_2(-x)) (h_1(y) + (-1)^l h_1(-y))
= C \theta_1^k \left( x \mid \frac{\tau}{k} \right) \theta_1^k \left( y \mid \frac{\tau}{k} \right) \theta_1 \left( x + y \mid \frac{\tau}{k} \right) \theta_1 \left( x - y \mid \frac{\tau}{k} \right)
\]

(11)

Proof. Following the notation of Liu [8], let
\[
H(x) = (h_1(x) + (-1)^l h_1(-x)) (h_2(y) + (-1)^l h_2(-y))
- (h_2(x) + (-1)^l h_2(-x)) (h_1(y) + (-1)^l h_1(-y))
\]
and let
\[
G(x) = \theta_1^k \left( x \mid \frac{\tau}{k} \right) \theta_1 \left( x + y \mid \frac{\tau}{k} \right) \theta_1 \left( x - y \mid \frac{\tau}{k} \right).
\]

Then it is easily checked, with the help on the assumptions on the functions \( h_1 \) and \( h_2 \), that
\[
H(x + \pi) = (-1)^l H(x) \text{ and } H \left( x + \frac{\pi}{k} \right) = (-1)^l q^{-\frac{(2l+1)\pi}{k}} e^{-2(2l+1)ix} H(x).
\]

Moreover, by the basic properties in (11) and the formula (2) we have
\[
G(x + \pi) = (-1)^l G(x) \text{ and } G \left( x + \frac{\pi}{k} \right) = (-1)^l q^{-\frac{(2l+1)\pi}{k}} e^{-2(2l+1)ix} G(x).
\]

Thus the ratio \( \frac{H(x)}{G(x)} \) is an elliptic function with periods \( \pi \) and \( \frac{2\pi}{k} \). Suppose for the moment that \( 0 < y < \pi \) and consider the fundamental period parallelogram with corners \( 0, \pi, \frac{2\pi}{k}, \pi + \frac{2\pi}{k} \). Clearly, in this parallelogram \( G(x) \) has a zero of order \( l \) at \( x = 0 \) and simple zeros at \( x = y \) and \( x = \pi - y \). Next, by the assumptions on \( h_1(x) \) and \( h_2(x) \) we can check that \( H(x) \) has a zero of order at least \( l \) at \( x = 0 \) and zeros at \( x = y \) and \( x = \pi - y \). So, \( \frac{H(x)}{G(x)} \) has no poles in the period parallelogram and therefore \( \frac{H(x)}{G(x)} = C(y) \) where \( C(y) \) is a constant depending possibly only on \( y \).
That is,

\[ (h_1(x) + (-1)^i h_1(-x)) (h_2(y) + (-1)^j h_2(-y)) - (h_2(x) + (-1)^i h_2(-x)) (h_1(y) + (-1)^j h_1(-y)) \]

\[ = C(y) \theta'_1 \left( x \mid \frac{\tau}{k} \right) \theta_1 \left( x + y \mid \frac{\tau}{k} \right) \theta_1 \left( x - y \mid \frac{\tau}{k} \right). \]

Interchanging the roles of \( x \) and \( y \), we get

\[ (h_1(x) + (-1)^i h_1(-x)) (h_2(y) + (-1)^j h_2(-y)) - (h_2(x) + (-1)^i h_2(-x)) (h_1(y) + (-1)^j h_1(-y)) \]

\[ = C(x) \theta'_1 \left( y \mid \frac{\tau}{k} \right) \theta_1 \left( x + y \mid \frac{\tau}{k} \right) \theta_1 \left( x - y \mid \frac{\tau}{k} \right). \]

As the last two identities have equal left-hand-sides, we derive that

\[ \frac{C(y)}{\theta'_1 \left( y \mid \frac{\tau}{k} \right)} = \frac{C(x)}{\theta'_1 \left( x \mid \frac{\tau}{k} \right)}, \]

showing that \( \frac{C(y)}{\theta'_1 \left( y \mid \frac{\tau}{k} \right)} \) is independent of \( y \) as well. Thus we conclude that

\[ C(y) = C \theta'_1 \left( y \mid \frac{\tau}{k} \right). \]

In fact, the result extends to all complex numbers \( x \) and \( y \) by analytic continuation. This completes the proof. \( \square \)

Note that if \( k = 1 \) and \( l = 1 \), then Theorem 4 is [8, Theorem 1]. Note also that the constant \( C \) is obtained by taking the \( l \)-th derivative derivative with respect to \( y \) in (11) as follows:

\[ h_2^{(l)}(0) (h_1(x) + (-1)^i h_1(-x)) - h_1^{(l)}(0) (h_2(x) + (-1)^j h_2(-x)) \]

\[ = C \frac{H}{2} \left( \theta'_1 \left( 0 \mid \frac{\tau'}{4} \right) \right)^l \theta^{2+l} \left( x \mid \frac{\tau}{k} \right). \quad (12) \]

4. PROOF OF THEOREM 1

Let

\[ h_1(z) = \theta'_1 \left( z \mid \frac{\tau'}{2} \right) \quad \text{and} \quad h_2(z) = \theta^2 \left( z \mid \frac{\tau'}{4} \right). \]

It can be checked with the help of the elementary properties in (11) that \( h_1(z) \) and \( h_2(z) \) both satisfy the conditions of Theorem 4 for \( k = l = 2 \). Then with the relation (12) at hand and some obvious simplifications, the constant \( C \) becomes

\[ C = 4 \left( \frac{\theta'_1 \left( 0 \mid \frac{\tau'}{2} \right)}{\theta'_1 \left( 0 \mid \frac{\tau'}{4} \right)} \right)^2. \]

Now putting together in the formula (11) we get

\[ 4 \theta'_1 \left( x \mid \frac{\tau'}{2} \right) \theta^2 \left( y \mid \frac{\tau'}{4} \right) - 4 \theta^2 \left( x \mid \frac{\tau'}{4} \right) \theta'_1 \left( y \mid \frac{\tau'}{2} \right) \]

\[ = 4 \left( \frac{\theta'_1 \left( 0 \mid \frac{\tau'}{2} \right)}{\theta'_1 \left( 0 \mid \frac{\tau'}{4} \right)} \right)^2 \theta'_1 \left( x \mid \frac{\tau'}{2} \right) \theta^2 \left( y \mid \frac{\tau'}{4} \right) \theta_1 \left( x + y \mid \frac{\tau'}{2} \right) \theta_1 \left( x - y \mid \frac{\tau'}{2} \right). \]
By formula (10) we see that the left-hand-side of (13) equals
\[ \theta_1^2 \left( y \mid \frac{\tau'}{2} \right) \theta_1^2 \left( y + \frac{\pi}{2} \mid \frac{\tau'}{2} \right) = \left( \frac{\theta_1^2 \left( 0 \mid \frac{\tau'}{2} \right)}{\theta_1^2 \left( 0 \mid \frac{\pi}{4} \right)} \right)^2 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( y \mid \frac{\tau'}{4} \right) \]
\[ - \left( \frac{\theta_1^2 \left( 0 \mid \frac{\tau'}{2} \right)}{\theta_1^2 \left( 0 \mid \frac{\pi}{4} \right)} \right)^2 \left( \frac{\theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right)}{\theta_1 \left( \frac{\pi}{2} \mid \frac{\pi}{4} \right)} \right)^2 \theta_1^4 \left( y \mid \frac{\tau'}{2} \right). \]

By formula [10] we see that the left-hand-side of (13) equals
\[ \theta_1^2 \left( 2y \mid \tau' \right) \theta_1^4 \left( \frac{\pi}{4} \mid \frac{\tau'}{2} \right) \]
Therefore, identity [10] means
\[ \theta_1^2 \left( 2y \mid \tau' \right) \theta_1^4 \left( \frac{\pi}{4} \mid \frac{\tau'}{2} \right) = \left( \frac{\theta_1^4 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right)}{\theta_1^4 \left( \frac{\pi}{2} \mid \frac{\pi}{4} \right)} \right)^2 \theta_1^2 \left( y \mid \frac{\tau'}{4} \right) \theta_1^2 \left( \frac{\pi}{2} \mid \tau' \right) \]
\[ - \left( \frac{\theta_1^4 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right)}{\theta_1^4 \left( \frac{\pi}{2} \mid \frac{\pi}{4} \right)} \right)^2 \left( \frac{\theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right)}{\theta_1 \left( \frac{\pi}{2} \mid \frac{\pi}{4} \right)} \right)^2 \theta_1^4 \left( y \mid \frac{\tau'}{2} \right) \theta_1^4 \left( \frac{\pi}{2} \mid \tau' \right). \]

Then multiplying both sides of the foregoing formula by \( \frac{\theta_1^4 \left( \frac{\pi}{4} \mid \frac{\tau'}{2} \right)}{\theta_1^4 \left( \frac{\pi}{4} \mid \frac{\pi}{4} \right)} \) yields
\[ \theta_1^2 \left( 2y \mid \tau' \right) \theta_1^4 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{2} \mid \frac{\tau'}{4} \right) \]
\[ = \left( \frac{\theta_1 \left( 0 \mid \frac{\tau'}{2} \right)}{\theta_1 \left( 0 \mid \frac{\pi}{4} \right)} \right)^2 \theta_1^6 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{4} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{2} \mid \tau' \right) \theta_1^2 \left( y \mid \frac{\tau'}{4} \right) \]
\[ - \left( \frac{\theta_1 \left( 0 \mid \frac{\tau'}{2} \right)}{\theta_1 \left( 0 \mid \frac{\pi}{4} \right)} \right)^2 \theta_1^6 \left( \frac{\pi}{4} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{4} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{2} \mid \tau' \right) \theta_1^2 \left( y \mid \frac{\tau'}{4} \right) \]
\[ \quad = \left( \frac{\theta_1 \left( 0 \mid \frac{\tau'}{2} \right)}{\theta_1 \left( 0 \mid \frac{\pi}{4} \right)} \right)^2 \theta_1^6 \left( \frac{\pi}{4} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{4} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{2} \mid \tau' \right) \theta_1^2 \left( y \mid \frac{\tau'}{4} \right) \]
\[ - \left( \frac{\theta_1 \left( 0 \mid \frac{\tau'}{2} \right)}{\theta_1 \left( 0 \mid \frac{\pi}{4} \right)} \right)^2 \theta_1^6 \left( \frac{\pi}{4} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{4} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{2} \mid \tau' \right) \theta_1^2 \left( y \mid \frac{\tau'}{4} \right). \]

To complete the proof, it remains to show that
\[ \left( \frac{\theta_1 \left( 0 \mid \frac{\tau'}{2} \right)}{\theta_1 \left( 0 \mid \frac{\pi}{4} \right)} \right)^2 \theta_1^6 \left( \frac{\pi}{4} \mid \frac{\tau'}{2} \right) \theta_1^2 \left( \frac{\pi}{4} \mid \frac{\tau'}{2} \right) = \frac{1}{4} \left( \frac{\Pi_{\tau}}{\Pi_{\tau'}^4} \right)^2. \]
By virtue of the relations (13) and (14) we get

\[
\frac{\theta_1^2 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right)}{\theta_1^2 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right)} = q^{-\frac{1}{8}} \frac{(q^2; q^4)^{\frac{1}{4}}}{(q^2)^2},
\]

\[
\frac{\theta_1^2 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right)}{\theta_1^2 \left( \frac{\pi}{2} \mid \frac{\tau'}{2} \right)} = 2q^{-\frac{1}{8}} \frac{(q^2; q^4)^{\frac{1}{4}}(q^8; q^8)^{\frac{1}{2}}}{(q^2)^2},
\]

\[
\left( \frac{\theta_1' \left( 0 \mid \frac{\tau'}{2} \right)}{\theta_1' \left( 0 \mid \frac{\tau'}{2} \right)} \right)^2 = \frac{1}{4} q^{-\frac{1}{2}} (q^4; q^4)^{\frac{1}{8}} (q^2; q^8)^{\frac{1}{4}}.
\]

Then by the previous three ratios we see that identity (14) is equivalent to

\[
\frac{1}{8} q^{-\frac{1}{8}} (q^4; q^4)^{\frac{1}{6}} \cdot q^{-\frac{1}{8}} \frac{(q^2; q^4)^{\frac{1}{4}}}{(q^2)^2}. 2q^{-\frac{1}{8}} \frac{(q^2; q^4)^{\frac{1}{4}}(q^8; q^8)^{\frac{1}{2}}}{(q^2)^2} = \frac{1}{4} q^{-\frac{1}{2}} (q^2; q^8)^{\frac{1}{4}},
\]

or equivalently,

\[
\frac{1}{4} q^{-\frac{1}{2}} (q^4; q^4)^{\frac{1}{4}}(q^2; q^4)^{\frac{1}{4}} = \frac{1}{4} q^{-\frac{1}{2}} (q^2; q^8)^{\frac{1}{4}},
\]

which is obviously true. This completes the proof.

5. Proof of Theorem 2

We can check that the functions \( h_1(z) = \theta_1^3 \left( z \mid \frac{\tau'}{3} \right) \) and \( h_2(z) = \theta_1(3z \mid \tau') \) satisfy the conditions of Theorem 4 for \( k = 3 \) and \( l = 1 \). Now letting in Theorem 3 \( x = \frac{\pi}{2} \) and \( y = z \), and using the formula (14) give

\[
4\theta_1^3 \left( \frac{\pi}{2} \mid \frac{\tau'}{3} \right) \theta_1(3z \mid \tau') - 4\theta_1^3 \left( \frac{3\pi}{2} \mid \tau' \right) = 12 \frac{\theta_1' \left( 0 \mid \frac{\tau'}{3} \right)}{\theta_1' \left( 0 \mid \frac{\tau'}{3} \right)} \theta_1 \left( \frac{\tau'}{3} \right) \theta_1 \left( \frac{\pi}{2} \right) + \theta_1^3 \left( \frac{\tau'}{3} \right) \theta_1 \left( \frac{\pi}{2} \right) - \theta_1 \left( \frac{\pi}{2} \right) \theta_1 \left( \frac{\pi}{2} \right) \theta_1 \left( \frac{\pi}{2} \right). (15)
\]

Then multiplying both sides of the foregoing identity by \( \theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{9} \right) \), using the basic properties (11), and simplifying we conclude that the last identity means

\[
\theta_1(3z \mid \tau') \theta_1^3 \left( \frac{\pi}{2} \mid \frac{\tau'}{3} \right) \theta_1 \left( \frac{\tau'}{3} \right) + \theta_1^3 \left( \frac{\tau'}{3} \right) \theta_1 \left( \frac{\pi}{2} \right) \theta_1 \left( \frac{\pi}{2} \right) \theta_1 \left( \frac{\pi}{2} \right) = 3 \frac{\theta_1' \left( 0 \mid \frac{\tau'}{3} \right)}{\theta_1' \left( 0 \mid \frac{\tau'}{3} \right)} \theta_1 \left( \frac{\tau'}{3} \right) \theta_1 \left( \frac{\pi}{2} \right) + \theta_1 \left( \frac{\pi}{2} \right) \theta_1 \left( \frac{\pi}{2} \right) \theta_1 \left( \frac{\pi}{2} \right). (16)
\]

On the other hand, in Theorem 4 let this time letting,

\[
h_1(z) = \theta_1^3 \left( z \mid \frac{\tau'}{3} \right) \quad \text{and} \quad h_2(z) = \theta_1 \left( z \mid \frac{\tau'}{9} \right),
\]
with \( k = 3 \) and \( l = 1 \) and using the formula \( \text{[11]} \) we get

\[
4\theta_1^3 \left( x \mid \frac{\tau'}{3} \right) \theta_1 \left( y \mid \frac{\tau'}{9} \right) - 4\theta_1 \left( x \mid \frac{\tau'}{9} \right) \theta_1^3 \left( y \mid \frac{\tau'}{3} \right) = \frac{4\theta_1 \left( 0 \mid \frac{\tau'}{3} \right)}{\theta_1 \left( 0 \mid \frac{\tau'}{9} \right)} \theta_1 \left( x \mid \frac{\tau'}{3} \right) \theta_1 \left( y \mid \frac{\tau'}{3} \right) \theta_1 \left( x + y \mid \frac{\tau'}{3} \right). \tag{17}
\]

Now make the substitution \( x = \frac{\pi}{2} \) and \( z = y \) in the previous identity and multiply both sides by \( A_q := \frac{\Pi_q}{q^3} \) to obtain

\[
\theta_1 \left( z \mid \frac{\tau'}{9} \right) \theta_1^3 \left( \frac{\pi}{2} \mid \frac{\tau'}{3} \right) A_q - \theta_1^3 \left( z \mid \frac{\tau'}{3} \right) \theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{3} \right) A_q = \frac{\theta_1 \left( 0 \mid \frac{\tau'}{3} \right)}{\theta_1 \left( 0 \mid \frac{\tau'}{9} \right)} \theta_1 \left( z \mid \frac{\tau'}{3} \right) \theta_1^2 \left( z + \frac{\pi}{2} \mid \frac{\tau'}{3} \right) \theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{3} \right) A_q \tag{17}
\]

Clearly, the desired result holds if and only if the right-hand sides of \( \text{[16]} \) and \( \text{[17]} \) are equal. Therefore, we will be done if we derive that

\[
3\theta_1 \left( 0 \mid \tau' \right) \theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{9} \right) = \frac{1}{3} \theta_1 \left( 0 \mid \frac{\tau'}{9} \right) \theta_1 \left( \frac{\pi}{2} \mid \tau' \right) \frac{\Pi_q}{q^3},
\]

or equivalently,

\[
9 \theta_1 \left( 0 \mid \tau' \right) \theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{9} \right) = \frac{\Pi_q}{q^3}. \tag{18}
\]

By an appeal to the relations \( \text{[3]} \) and \( \text{[4]} \) we find

\[
\frac{\theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{9} \right)}{\theta_1^2 \left( \frac{\pi}{2} \mid \tau' \right)} = \sqrt{3} \frac{(q^2; q^2)_{\infty}^3 (q^18^18)_{\infty}^2}{(q; q^2)_{\infty}^2 (q^2; q^2)_{\infty}^2},
\]

\[
\frac{\theta_1 \left( 0 \mid \tau' \right)}{\theta_1 \left( 0 \mid \frac{\tau'}{9} \right)} = \frac{1}{3} \frac{q^2 (q^2; q^2)_{\infty}^3 (q^18^18)_{\infty}^2}{(q^2; q^2)_{\infty}^2 (q^2; q^2)_{\infty}^2},
\]

which yields

\[
9 \theta_1 \left( 0 \mid \tau' \right) \theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{9} \right) = \frac{q^2 (q^2; q^2)_{\infty}^3 (q^18^18)_{\infty}^2}{(q^18^18)_{\infty}^2 (q^2; q^2)_{\infty}^2} = \frac{\Pi_q}{q^3},
\]

as desired in formula \( \text{[18]} \).

6. **Proof of Theorem \( \text{[3]} \)**

By the formula \( \text{[16]} \) we have

\[
\theta_1 \left( 3z \mid \tau' \right) \theta_1^3 \left( \frac{\pi}{2} \mid \frac{\tau'}{3} \right) + \theta_1^3 \left( z \mid \frac{\tau'}{3} \right) \theta_1 \left( \frac{\pi}{2} \mid \tau' \right)
= 3 \theta_1 \left( 0 \mid \frac{\tau'}{3} \right) \theta_1 \left( z \mid \frac{\tau'}{3} \right) \theta_1^2 \left( z + \frac{\pi}{2} \mid \frac{\tau'}{3} \right) \theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{3} \right).
\]

Therefore, to prove the theorem we need only show that

\[
3 \frac{\theta_1 \left( 0 \mid \tau' \right) \theta_1 \left( \frac{\pi}{2} \mid \frac{\tau'}{9} \right)}{\theta_1 \left( 0 \mid \frac{\tau'}{9} \right) \theta_1 \left( \frac{\pi}{2} \mid \tau' \right)} = \frac{\Pi_q}{q^3}. \tag{19}
\]
By the identities (3) and (4) we have
\[
\frac{\theta_1 \left( \frac{\pi}{2} \mid \tau' \right)}{\theta_1^2 \left( \frac{\pi}{2} \mid \tau' \right)} = \sqrt{3} \frac{(q^3; q^6)_\infty^2 (q^6; q^6)_\infty}{(q; q^2)_\infty^3 (q^2; q^2)_\infty},
\]
\[
\frac{\theta_1' \left( 0 \mid \tau' \right)}{\theta_1' \left( 0 \mid \tau \right)} = \frac{1}{3 \sqrt{3}} q^{- \frac{3}{2}} \frac{(q^2; q^2)_\infty^2}{(q^6; q^6)_\infty},
\]
from which we get
\[
3 \frac{\theta_1' \left( 0 \mid \tau' \right)}{\theta_1' \left( 0 \mid \tau \right)} \frac{\frac{\pi}{2}}{\frac{\pi}{2}} \frac{\pi}{2} \frac{\pi}{2} = q^{- \frac{3}{2}} \frac{(q^2; q^2)_\infty^2 (q^3; q^6)_\infty^2}{(q; q^2)_\infty^3 (q^6; q^6)_\infty^2} = \frac{\Pi_{q}}{\Pi_{q^3}},
\]
as desired in formula (19). This completes the proof.

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