The influence of strong magnetic fields and instantons on the phase structure of the two-flavor NJL model

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Both in heavy-ion collisions as in magnetars very strong magnetic fields are produced, which has its influence on the phases of matter involved. In this paper we investigate the effect of strong magnetic fields \( (B \sim 5m^2_\pi/e = 1.7 \times 10^{19} \text{G}) \) on the chiral symmetry restoring phase transition using the Nambu-Jona-Lasinio model. It is observed that the pattern of phase transitions depends on the relative magnitude of the magnetic field and the instanton interaction strength. We study two specific regimes in the phase diagram, high chemical potential and zero temperature and vice versa, which are of relevance for neutron stars and heavy-ion collisions respectively. In order to shed light on the behavior of the phase transitions we study the dependence of the minima of the effective potential on the occupation of Landau levels. We observe a near-degeneracy of multiple minima with differing occupation numbers, of which some become the global minimum upon changing the magnetic field or the chemical potential. These minima differ considerably in the amount of chiral symmetry breaking and in some cases also of isospin breaking.

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I. INTRODUCTION

Recently it has been noted that very strong magnetic fields can be produced in heavy-ion collisions \[1,3\]. Estimates are that at RHIC magnetic fields are created of magnitude \( 5.3m^2_\pi/e = 1.8 \times 10^{19} \text{G} \) and at LHC of \( 6m^2_\pi/e = 2 \times 10^{19} \text{G} \), and there are even higher estimates \[4\]. Also, certain neutron stars called magnetars exhibit strong magnetic fields, between \( 10^{14} - 10^{15} \text{G} \) \[5, 6\]. These fields occur at the surface, probably in the much denser interior even higher fields are present. Using the virial theorem it can be derived that the maximal strength is \( 10^{18} - 10^{19} \text{G} \) \[7\]. If one assumes that the star is bound by the strong interaction instead of by gravitation, this limit can be even higher.

In both neutron stars and in heavy-ion collisions it is expected that quark matter plays a role. Therefore it is interesting to study how this form of matter behaves in a strong magnetic field. Two different regions in the QCD phase diagram are of relevance here. Heavy-ion collisions probe the low chemical potential and high temperature regime, for neutron stars it is the other way around. In this paper the effect of very strong magnetic fields will be investigated in both regimes.

Much work has been done on how an external magnetic field changes nuclear matter, for a review see Ref. \[8\]. The behavior of ordinary quark matter has been studied using the Nambu-Jona-Lasinio (NJL) model, see for example Refs. \[9,18\] and recently also in the linear sigma model coupled to quarks \[19\]. Most studies investigate the one and two flavor cases, but recently also the three-flavor case has been investigated \[20,21\]. At high quark chemical potential, it is believed that the ground state is a color superconducting phase. The effects of an external magnetic field on such a phase are discussed in Refs. \[22,28\]. Here color superconductivity will not be considered.

In this paper we study the chiral symmetry restoring phase transition, which is strongly influenced by an external magnetic field. From studies in the NJL model it is known that a magnetic field enhances the chiral symmetry breaking \[9\], this is related to the phenomenon of magnetic catalysis of chiral symmetry breaking, introduced in Refs. \[29,31\], further studied for the NJL model in Refs. \[11,10\] and for QED in e.g. Refs. \[32,30\], where also the generation of an anomalous magnetic moment was pointed out \[33,30\]. This enhancement of chiral symmetry breaking can be understood as follows, the B-field anti-aligns the helicities of the quarks and antiquarks, which are then more strongly bound by the NJL-interaction \[9\]. The phenomenon of magnetic catalysis of chiral symmetry breaking leads to interesting behavior, since it allows for phases with broken chiral symmetry and nonzero nuclear density for a range of chemical potentials and magnetic fields \[14,15,17\]. In such a phase nonperiodic magnetic oscillations occur, which means that the constituent quark masses are strongly dependent on the magnetic field, and consequently also other thermodynamic parameters.

In all studies of the influence of magnetic fields on chiral symmetry breaking up to now, the effects of instantons have not been studied explicitly, i.e. as a function of instanton interaction strength. Magnetic fields and instantons
can lead to combined effects. In Ref. [8] it is shown that variations in topological charge, which induce variations of net chirality, in a strong magnetic field give rise to an electrical current. This effect is known as the chiral magnetic effect and could perhaps be observed in heavy-ion collisions. Variations of topological charge can for instance be created by instantons.

Here a related study will be performed. We will investigate the combined effect of instantons and a strong magnetic field on quark matter using the NJL model. In this model instantons induce an extra interaction, the ‘t Hooft determinant interaction, which leads to a mixing between the different quark flavors. Following the analysis of Refs. [37, 38], the strength of the instanton interaction is set by the dimensionless parameter $c$. For $c = 0$ there is no contribution and the quarks are fully independent. Because of the difference in charge of the quarks, the phase transitions are decoupled. The other extreme case is $c = 1/2$, which is actually the most studied case. The quarks are then fully mixed, the constituent quark masses are equal and the phase transitions will always coincide. In Ref. [14] this case is studied in the chiral limit. It is observed that for a range of typical value for the coupling constant, phases with broken chiral symmetry and nonzero nuclear density arise.

In general there is a competition between the magnetic field, which tends to differentiate the constituent quark masses for different flavors, and the instanton interaction which favors equal constituent quark masses. In this work this competition is studied. Apart from studying the ground state as function of magnetic field and chemical potential for various characteristic values of $c$, we also look at the local minima of the effective potential and the corresponding occupation of Landau levels. It is found that in the neighborhood of the chiral phase transition the phase diagram develops metastable phases, differing in the number of filled Landau levels. Some of these local minima become the global one upon increasing the magnetic field or chemical potential, but not all of them do. These phases can have rather different values for the constituent quark masses, in other words, display significantly different amounts of chiral symmetry breaking. Unlike in the case of $c = 1/2$ which is isospin symmetric, in these phases the values of the two constituent quark masses can be very distinct, which corresponds to large isospin violation. Furthermore, we find that for a realistic choice of parameters, appearance of phases of broken chiral symmetry and nonzero nuclear density requires not too large instanton interaction strength, i.e. $c \lesssim 0.1$

As mentioned, we also investigate the role of nonzero temperature at zero chemical potential, which is of relevance for heavy-ion collisions. Without magnetic field the chiral symmetry restoring phase transition at finite temperature is a crossover. In the linear sigma model coupled to quarks it has been observed that the magnetic field turns it into a first order transition [19]. We will see that this is not the case for the NJL model.

This paper is organized as follows. First we derive the effective potential of the NJL model in the mean-field approximation in a magnetic background. Then we discuss the phase diagram as a function of chemical potential, concentrating on the phase with nonzero nuclear density and chiral symmetry breaking. We continue with discussing the temperature dependence and end with conclusions.

II. EFFECTIVE POTENTIAL OF THE NJL MODEL WITH A MAGNETIC FIELD

In this section the effective potential of the NJL model in a strong magnetic background is derived, this analysis is based on Refs. [18, 19]. The effective potential derived in this way is equal to the one in Refs. [14, 16] using the Fock-Schwinger proper time method.

First to set our notation, we briefly review the NJL model introduced in Refs. [39, 40]. It is a quark model with four-point interactions, the gluons are ‘integrated out’. In this paper the following form of the NJL model is used, in the notation of Ref. [38]

$$\mathcal{L}_{\text{NJL}} = \bar{\psi} (i \gamma^\mu \partial_\mu - m - \mu \gamma_0) \psi + \mathcal{L}_{\bar{q}q} + \mathcal{L}_{\text{det}},$$

where $m = m_u = m_d$ is the current quark mass and $\mu = \mu_u = \mu_d$ is the quark chemical potential. Note that in contrast to Ref. [38], here the current quark masses and both chemical potentials are taken equal. Furthermore,

$$\mathcal{L}_{\bar{q}q} = G_1 \left[ (\bar{\psi} \tau_\alpha \psi)^2 + (\bar{\psi} \tau_\alpha i \gamma_5 \psi)^2 \right],$$

is the attractive part of the $\bar{q}q$ channel of the Fierz transformed color current-current interaction [41] and

$$\mathcal{L}_{\text{det}} = 8G_2 \det (\bar{\psi}_R \psi_L) + \text{h.c.},$$

is the ‘t Hooft determinant interaction which describes the effects of instantons [42, 43]. Note that this interaction is flavor mixing. In the literature $G_1$ and $G_2$ are often taken equal, which means that the low energy spectrum consists of $\sigma$ and $\pi$ fields only, but here we will allow them to be different. We will restrict to the two flavor case, using $\tau_a$ with $a = 0, ..., 3$ as generators of U(2).
The symmetry structure of the NJL model is very similar to that of QCD. In the absence of quark masses and the instanton interaction, there is a global SU(3) × SU(2) × SU(2) -symmetry. The instanton interaction breaks it to SU(3) × SU(2) × SU(2) × U(1)_B. For nonzero, but equal quark masses this symmetry is reduced to SU(3) × SU(2) × U(1). If a magnetic field is turned on, the symmetry is reduced to SU(3) × U(1)^2 due to the differences in charge.

We choose the parameters as in Refs. [37, 38]. This means we write
\[ G_1 = (1 - c)G_0, \quad G_2 = cG_0, \] (4)
where the parameter \( c \) controls the instanton interaction strength, while the value for the quark condensate (which is determined by the combination \( G_1 + G_2 \)) is kept fixed. For our numerical studies we will use the following values for the parameters: \( m = 6 \text{ MeV} \), a three-dimensional momentum UV cut-off \( \Lambda = 590 \text{ MeV} / c \) and \( G_0 \Lambda^2 = 2.435 \). These values lead to a pion mass of 140.2 MeV, a pion decay constant of 92.6 MeV and finally, a quark condensate \( \langle \bar{u}u \rangle = \langle \bar{d}d \rangle = (-241.5 \text{ MeV})^3 \) [37, all in reasonable agreement with experimental determinations.

To calculate the ground state of the theory, the effective potential has to be minimized. In this section the effective potential is calculated in the mean-field approximation. We will assume that only the charge neutral condensates \( \langle \bar{u}u \rangle \) and \( \langle \bar{d}d \rangle \) can become nonzero.

To obtain the effective potential in the mean-field approximation, first the interaction terms are “linearized” in the presence of the \( \langle \bar{u}u \rangle \) and \( \langle \bar{d}d \rangle \) condensates (this is equivalent to the procedure with a Hubbard-Stratonovich transformation used in Ref. [38])
\[ (\bar{\psi}\tau_0 \psi)^2 \simeq 2 \langle \bar{\psi}\tau_0 \psi \rangle \bar{\psi}\tau_0 \psi - \langle \bar{\psi}\tau_0 \psi \rangle^2, \] (5)
leading to
\[ \mathcal{L}_{NJL} = \bar{\psi} (i \gamma^\mu \partial_\mu - \mathcal{M} - \mu \gamma_0) \psi - \frac{(M_0 - m)^2}{4G_0} - \frac{M_f^2}{4(1 - 2c)G_0}, \] (6)
where \( \mathcal{M} = M_0 \tau_0 + M_3 \tau_3 \) and
\[ M_0 = m - 2G_0 \langle \bar{\psi}\tau_0 \psi \rangle, \quad M_3 = -2(1 - 2c)G_0 \langle \bar{\psi}\tau_3 \psi \rangle. \] (7)

Now the Lagrangian is quadratic in the quark fields, so we can integrate over them. After going to imaginary time, this results in the following effective potential in the mean-field approximation [44]
\[ \mathcal{V} = \frac{(M_0 - m)^2}{4G_0} + \frac{M_3^2}{4(1 - 2c)G_0} - TN \sum_{f=a}^{d} \sum_{p_0=(2n+1)\pi T} \int \frac{d^3p}{(2\pi)^3} \ln \det [i \gamma_0 p_0 + \gamma_i p_i - M_f - \gamma_0 \mu], \] (8)
where we have introduced the constituent quark masses \( M_a = M_0 + M_3 \) and \( M_d = M_0 - M_3 \).

As we have mentioned earlier, often \( G_1 \) is taken equal to \( G_2 \), which is the case \( c = 1 / 2 \). This choice implies that \( M_3 \) is then always equal to 0, i.e., the constituent quark masses are equal. Note that the reverse is not true. If the assumption is made that the constituent quark masses are equal, \( c = 1 / 2 \) or \( \langle \bar{\psi}\tau_3 \psi \rangle = 0 \) or both. However, for \( M_3 = 0 \) changes in \( c \) cannot be noticed because only the combination \( G_1 + G_2 \) occurs. Hence, the conclusion is that \( \langle \bar{\psi}\tau_3 \psi \rangle = 0 \) and \( c \) can be any value. Instanton effect are nevertheless present (\( G_2 \) can be nonzero after all) and \( \eta \) and \( a_0 \) mesons can still be present in the spectrum. The ratio \( G_2 / G_1 \) can simply not be determined if \( M_3 = 0 \). Since magnetic fields affect the two flavors differently because of the difference in charge, isospin breaking effects are expected and it is unnatural to choose \( M_3 = 0 \) from the start or to take \( c = 1 / 2 \). Using the strange quark condensate it was argued in Ref. [37] that a realistic value of \( c \) would be around 0.2.

A. Including a magnetic field

Now we include a magnetic field, which changes the dispersion relation for the quarks in the following way
\[ p_{0n}^2 = p_z^2 + M^2 + (2n + 1 - \sigma)|q|B, \] (9)
where \( n \) is the quantum number labelling the discrete orbits, \( \sigma \) the spin of the quark, and \( q \) is its charge. The integral over the three-momentum is modified as
\[ \int \frac{d^3p}{(2\pi)^3} \to \frac{|q|B}{2\pi} \sum_{n=0}^{\infty} \int dp_z \frac{2\pi}{2\pi}. \] (10)
Performing the sum over the Matsubara frequencies gives the following effective potential \[14, 15, 18\]
\[
\mathcal{V} = \frac{(M_0 - m)^2}{4G_0} + \frac{M_3^2}{4(1 - 2c)G_0} - \frac{N_c}{2\pi} \sum_{\sigma, n, f} |q_f| B \int \frac{dp_z}{2\pi} E_{p, f}(B) - \frac{N_c}{2\pi} \sum_{\sigma, n, f} |q_f| B \int \frac{dp_z}{2\pi} \left\{ T \ln \left[ 1 + e^{-\sqrt{E_{p, f}(B) + \mu}/T} \right] \right\}.
\]
\[\text{with}\]
\[
\mathcal{V}_0 = \frac{(M_0 - m)^2}{4G_0} + \frac{M_3^2}{4(1 - 2c)G_0} - 2N_c \sum_{f = u} \frac{d}{(2\pi)^3} \sqrt{p^2 + M_f^2},
\]
\[
\mathcal{V}_1(B) = -\frac{N_c}{2\pi} \sum_{f = u} \frac{d}{(2\pi)^3} \sqrt{p^2 + M_f^2} \left\{ \zeta'(-1, x_f) - \frac{1}{2}(x_f^2 - x_f) \ln x_f + \frac{x_f^2}{4} \right\},
\]
\[
\mathcal{V}_2(B, \mu, T) = -\frac{N_c}{2\pi} \sum_{\sigma, n, f} |q_f| B \int \frac{dp_z}{2\pi} \left\{ T \ln \left[ 1 + e^{-\sqrt{E_{p, f}(B) + \mu}/T} \right] \right\}.
\]

where \(E_{p, f}(B) = \sqrt{p^2 + (2n + 1 - \sigma)|q_f|B + M_f^2}\). Following the analysis of Ref. \[18\] this potential can be split in three pieces, a part that is independent of external parameters, a part that only depends on the magnetic field and a part that depends on the magnetic field, chemical potential and temperature.

The term \(\mathcal{V}_0\) is divergent and needs to be regularized. Here a conventional three-momentum UV cut-off is used, yielding the expression

\[
\mathcal{V}_0 = \frac{(M_0 - m)^2}{4G_0} + \frac{M_3^2}{4(1 - 2c)G_0} - \frac{N_c}{8\pi^2} \sum_{f = u} |M_f| \left( M_f^2 \ln \left( \frac{\Lambda}{M_f} + \sqrt{1 + \frac{\Lambda^2}{M_f^2}} \right) - \Lambda \left( M_f^2 + 2\Lambda^2 \right) \sqrt{\frac{\Lambda^2}{M_f^2} + 1} \right).
\]

The expression \(\zeta'(-1, x_f)\) in \(\mathcal{V}_1(B)\) can be written in a more convenient form by differentiating and integrating the function with respect to \(x_f\):

\[
\zeta'(-1, x_f) = \zeta'(-1, 0) + \frac{x_f^2}{2} - x_f \frac{\log(2\pi) + \psi(-2)(x_f)}{2},
\]

where \(\psi^{(m)}(x_f)\) is the \(m\)-th polygamma function. The term \(\zeta'(-1, 0)\) is independent of \(x_f\) and will therefore not be taken into account. The remaining expression is amenable to numerical evaluation.

The summation over \(\sigma\) and \(n\) in \(\mathcal{V}_2(B, \mu, T)\) can be rewritten as

\[
\mathcal{V}_2(B, \mu, T) = -\frac{N_c}{2\pi} \sum_{k, f}(2 - \delta_{k0}) |q_f| B \int \frac{dp_z}{2\pi} \left\{ T \ln \left[ 1 + e^{-\sqrt{E_{p, k}(B) + \mu}/T} \right] \right\} + T \ln \left[ 1 + e^{-\sqrt{E_{p, k}(B) - \mu}/T} \right],
\]

where \(E_{p, k} = \sqrt{p^2 + M_f^2 + 2k|q_f|B}\) and \(k\) denotes the Landau level, which has degeneracy \((2 - \delta_{k0})\).

At zero temperature, \(\mathcal{V}_2\) can be simplified to

\[
\mathcal{V}_2(B, \mu, 0) = -\frac{N_c}{2\pi} \sum_{k, f}(2 - \delta_{k0}) \int \frac{dp_z}{2\pi} \theta(\mu - E_{p, k}) [\mu - E_{p, k}]
\]
\[
= \frac{d}{2\pi} \sum_{j = u} \sum_{k = 0}^{k_{\text{max}}} (2 - \delta_{k0}) \theta(\mu - s_f(k, B)) \frac{|q_f| BN_c}{4\pi^2}
\]
\[
\times \left\{\mu \sqrt{\mu^2 - s_f^2(k, B)} - s_f^2(k, B) \ln \left[ \frac{\mu + \sqrt{\mu^2 - s_f^2(k, B)}}{s_f(k, B)} \right] \right\},
\]

\[\text{for } \mu > s_f^2(k, B)\].
where \( s_f(k, B) = \sqrt{M_f^2 + 2|q_f|Bk} \) and \( k_{f,\text{max}} \) is the upper Landau level, defined as

\[
k_{f,\text{max}} = \left\lfloor \frac{\mu^2 - M_f^2}{2|q_f|B} \right\rfloor,
\]

where the brackets indicate the floor of the enclosed quantity.

We will now use these expressions in a numerical study of the minima of the effective potential, performed along the lines discussed in Refs. [38, 44].

III. RESULTS

We start with considering the case of \( \mu = 0, T = 0, \) and \( c = 0. \) Fig. 1 shows the results for this unmixed case. The magnetic field enhances \( M_u \) and \( M_d, \) which are proportional to \( \langle \bar{u}u \rangle \) and \( \langle \bar{d}d \rangle, \) respectively, consequently the chiral symmetry breaking is enhanced [9]. Because of the charge difference of the quarks, the \( B \)-dependence of the constituent quark masses is not equal. Nonzero \( c \) will cause mixing and will bring the masses closer to each other. As discussed, at \( c = 1/2 \) the constituent quark masses are exactly equal.

![Graph showing the dependence of constituent quark masses on magnetic field](image)

**FIG. 1:** The dependence of the constituent quark masses \( M_u \) and \( M_d \) on the magnetic field \( B. \)

A. Nonzero chemical potential

In this section we turn to the phase structure near the phase transition at nonzero chemical potential and zero temperature. From Refs. [10, 16] it is known that when the isospin chemical potential is nonzero, it is possible to have two phase transitions at low temperature and high baryon chemical potential. Here we study a similar case, instead of nonzero isospin chemical potential, we allow for nonzero magnetic field, we will see that also here the possibility of separate phase transitions for the two quarks arises. We will take equal chemical potentials for the quarks, but the magnetic field acts effectively like a nonzero isospin chemical potential due to the difference in charge of the quarks. Instantons cause mixing between the quarks, if the mixing is strong enough, the two separate phase transitions merge into one. This was extensively investigated in Ref. [10] for the nonzero isospin chemical potential case.

From Ref. [14, 15, 17], where the NJL model in the chiral limit was studied, it is known that Landau quantization induces a more complex phase structure. Apart from the usual phase of broken chiral symmetry with zero nuclear density, there is also the possibility of such a phase with nonzero nuclear density. Here we perform a more detailed study of this case, which is a characteristic phenomenon at nonzero chemical potential and sufficiently strong magnetic fields (cf. Eq. (19) below).
1. The $c = 0$ case

When the determinant interaction is turned off, the up and down quarks are decoupled. This leads to the possibility of separate phase transitions for the quarks. In Figs. 2 and 3 we show the constituent quark mass of the up and down quark respectively as a function of quark chemical potential and magnetic field. As expected, the two quarks have decoupled behavior.

Let us first discuss the behavior of the up quark. At low chemical potential we have the “standard” chiral symmetry breaking NJL ground state with empty Landau levels (LL). Following the nomenclature of Refs. [14, 15] where the $c = 1/2$ case was studied in the chiral limit, this is called phase $B$. Note that this phase always has zero nuclear density. At high chemical potential chiral symmetry is restored, up to the explicit breaking. In this approximate decoupled behavior of separate phase transitions for the quarks. In Figs. 2 and 3 we show the constituent quark mass of the up and down quark.

In the chiral limit the constituent quark masses vanish in the $A_i$ phases. The oscillations are due to the de Haas-van Alphen effect, which in QED and in the two-flavor NJL model for $c = 1/2$ in the chiral limit lead to second order transitions between the $A_i$ phases [14]. However, with our choice of parameters the transitions are weakly first order. In the chiral limit they become second order, like for $c = 1/2$, as can for instance be seen in the nuclear density. For completeness, we mention that in the color superconducting case of Ref. [27, 28] the oscillations in the gap parameter are seen to be continuous, but second order transitions can occur when neutrality conditions are imposed.

For $B$ larger than $4.5m^2/e$ an interesting intermediate phase arises, where the up-quark jumps as a function of $\mu$ first to a phase with a still rather large constituent mass and then to phase $A_1$. This intermediate phase is called $C_0$ in the language of Refs. [14, 15] and corresponds to a phase of broken chiral symmetry having nonzero nuclear density and a filled zeroth LL. So the essential difference between this phase and $A_0$ is the breaking of chiral symmetry. For smaller values of the coupling constant $G_0$ also the phases $C_k$ with $k > 0$ (which are similar to $C_0$ but with more occupied LL) occur. The transitions between the $C_k$ are first order, furthermore, they are nonperiodic in the sense that the difference between the transitions is $B$-dependent as the constituent mass strongly depends on $B$ [14]. If we are in this phase $C_0$ and increase the magnetic field, the constituent quark mass decreases, eventually becoming almost zero, this can be interpreted as a crossover to $A_0$. In the chiral limit the crossover becomes a second order transition. Finally, we would like to note that already at $B = 4m^2/e$ the phase $C_0$ exists as a metastable phase (we will discuss this in more detail later).

The qualitative behavior of the down quark is very similar, as the quarks only differ in charge, consequently Fig. 3 can be directly obtained from Fig. 2 by multiplying $B$ by 2, for ease of comparison we show both figures. If one compares the two figures, one can immediately see that there are large regions where the constituent quark masses are considerably different. This is equivalent to a large nonzero $\langle \psi^\dagger\tau_3\psi \rangle$ condensate, i.e., spontaneous isospin breaking. This will influence the behavior of the mesons accordingly, for example the masses.

Eventually, if one keeps increasing the magnetic field, the phase transitions of the quarks will take place at (almost) the same chemical potential and there will be no spontaneous isospin breaking.

2. The $c \neq 0$ case

In this section the consequences of the instanton interaction is studied, i.e., the parameter $c$ is varied. Increasing $c$ will cause mixing between the constituent quarks, which tends to bring the constituent quark masses together. Around the phase transition there is a competition between the effect of the magnetic field and the instanton interaction.

The competition is illustrated in Fig. 4, where the constituent quark masses are plotted as a function of the quark chemical potential and the magnetic field. The qualitative behavior for different values of the magnetic field is similar. One can see that when $c \neq 0$, the phase transitions are indeed coupled. Furthermore, one observes that the two phase transitions merge into one when $c$ is increased and that the phase $C_0$ disappears. Qualitatively the behavior is similar to the case of nonzero isospin chemical potential studied in Ref. [16], but in that case the phase $C_0$ does not exist.

When the coupling constant $G_0$ is lowered, it is possible to have $C_k$ phases at $c = 1/2$, as in Ref. [14]. Compared to the chiral limit studied there, the region of the phase diagram with $C_k$ phases increases for $m \neq 0$.

More insight into the phase structure and phase transitions is obtained by looking at the behavior of local minima of the effective potential. Near the phase transition at these (large) magnetic fields, metastable phases arise. These
FIG. 2: The dependence of the constituent up-quark mass on $B$ and $\mu$. $A_i$, $B$ and $C_0$ denote the different phases using the scheme of Refs. [14, 15]. Chiral symmetry is broken in phases $B$ and $C_0$, phases $A_0$ and $C_0$ have nonzero nuclear density.

FIG. 3: Same as Fig. 2 now for the down quark.

phases differ in the number of filled LL. Let us take as an example the $c = 1/2$ case, which is the easiest to discuss, as the effective potential is then only a function of $M_u = M_d = M$. In Fig. 5 we show the effective potential as a function of $M$ with $\mu = 378$ MeV and $B = 5m^2_\pi/e$. At these values four minima can be seen, the global minimum is the phase in which the chiral symmetry breaking is largest, i.e. minimum 4. When $\mu$ is increased, minimum 1 will take over, which is $A_1$ for the up quark and $A_2$ for the down quark. The other two local minima never become the global one for our choice of $G_0$, but as they are almost degenerate with the other minima (also for other values of $c$), they are nevertheless important. These local minima correspond to $C_k$ phases and can become the global minimum when $G_0$ is lowered.

Similar results hold for $c \neq 1/2$, also then metastable phases exists with different fillings of LL. In this cases some of the $C_k$ phases can become the global minimum, as we have seen for $c = 0$. Like before, the number of such states depends on the choice of the other parameters.
As the metastable phases differ in the values of the $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$ condensates at small $c$, they again represent rather large broken isospin and will lead to different meson masses. Whenever the system is passing the phase transition, it could be trapped in one of those metastable phases for some time and consequences from the changing meson masses can arise, for example, enhancing or suppressing certain decays.

B. Nonzero temperature

In this section the temperature dependence of the ground state is investigated at zero chemical potential, but with a magnetic field. As the instanton interaction does not influence the temperature dependence much, we only consider $c = 1/2$ for simplicity. Ref. [19] found in the linear sigma model coupled to quarks, the usual crossover becomes a first order transition at very high magnetic fields. However, we find that this is not the case in the NJL model.

In Ref. [19] only the lowest Landau level was taken into account. Here more Landau levels are included, so the effect of the higher Landau levels can be investigated in the NJL model. Since the levels with large $k$ are exponentially suppressed, the summation can be truncated in Eq. (16), we will denote the largest $k$ with $k_{\text{trunc}}$. The value of $k_{\text{trunc}}$ depends on the temperature, constituent quark mass, chemical potential and magnetic field considered. If $M$ and $T$ are increased or if $B$ is decreased, $k_{\text{trunc}}$ has to be increased.

In Fig. 6 we show the temperature dependence of the constituent quark mass at $B = 15m_\pi^2/e$ for four different values of $k_{\text{trunc}}$. The 13 levels case is chosen such that the error is less than 1 percent at $M = 450$ MeV, $T = 450$ MeV. From the figure it can be inferred that taking more Landau levels into account, makes the crossover sharper. Also, there is a significant difference between including the zeroth and first Landau level. It is clear that including more Landau levels, influences the details of the transition. However, the qualitative aspects of the phase transition are not

FIG. 4: The dependence of the constituent quark masses on the quark chemical potential for $B = 5m_\pi^2/e$ and various $c$ values. Solid lines denote the up quarks, the dashed lines the down quark.

FIG. 5: The normalized effective potential at the values $B = 5m_\pi^2/e$, $\mu = 378$ MeV and $c = 1/2$. There are four minima. The numbers below the minima denote the LL occupied for each quark. Note that the minima are almost degenerate.
changed.

FIG. 6: The temperature dependence of the constituent quark mass for strong magnetic field \( (B = 15m^2_\pi/e) \) and various \( k_{\text{trunc}} \) values.

In Fig. 7 the temperature dependence of the constituent quark mass for different values of the external magnetic field is shown. The phase transition remains a crossover, in contrast to the results in the linear sigma model coupled to quarks. This difference is important, as a first order phase transition allows for meta-stable states, whereas a crossover does not.

In Fig. 8 the results in the chiral limit are shown, where the transition remains a second order phase transition, like it is the case at zero magnetic field and confirms the results of [17] who calculated the phase diagram in a strong magnetic field in the chiral limit using the Fock-Schwinger proper time method. Note that the critical temperature increases slightly with increasing magnetic field.

| \( M (\text{MeV}) \) | \( T (\text{MeV}) \) |
|-------------------|-----------------|
| 13 LL             | 23 LL           |
| 0 LL              | 16 LL           |
| 1 LL              | 13 LL           |
| 2 LL              | 11 LL           |

FIG. 7: The temperature dependence of the constituent quark mass for various strong magnetic fields. We also indicate \( k_{\text{trunc}} \).

IV. CONCLUSIONS

The effect of a strong magnetic field on quark matter has been investigated in the NJL model in two regimes, zero temperature and finite chemical potential and vice versa. The first regime is of relevance for (the interior of) magnetars and the second for heavy-ion collisions.

At very high magnetic fields, when \( M \approx 2|q|B \approx \mu \), the phase structure shows a variety of phases and phase transitions due to Landau quantization. As a function of chemical potential, more phase transitions occur, corresponding to Landau levels filling up successively. Due to the difference in charge, this pattern is different for the two quark flavors. When there is no mixing in the absence of the instanton interaction, the two patterns are uncoupled. This generally leads to rather different constituent quark masses, or equivalently, spontaneous isospin breaking \( \langle \bar{\psi} \gamma_3 \psi \rangle \neq 0 \). This affects the mesons inside the medium, for example their masses. It was found that for a realistic choice of parameters
in the NJL model such a phase of broken chiral and isospin symmetry arises around \( B = 4.5 m_s^2/e \), but it is already present as a metastable phase for lower magnetic fields.

When the instanton interaction is included, a competition occurs between the strength \( c \) of this interaction and the magnetic field. This reduces the region in the phase diagram with large \( \langle \bar{\psi}_3 \gamma_3 \psi \rangle \). For \( c \) sufficiently large it disappears entirely, leaving only one phase transition. However, around this transition the phase structure is still rather complex regarding metastable phases, which are characterized by different fillings of Landau levels and which differ only slightly in energy, but much in the amount of chiral symmetry breaking. For lower values of \( c \) some of these near-degenerate minima can also differ considerably in the amount of isospin breaking.

Finally the role of temperature was studied at zero chemical potential. In Ref. [19] it was found in the linear sigma model coupled to quarks that a strong magnetic field changes the usual crossover as a function of temperature into a first order transition. In the NJL model it was found that the crossover remains a crossover. Also it was found that including higher Landau levels in the calculation of the effective potential changes the details of the crossover, it becomes sharper, albeit the qualitative aspects of the transition are not changed. The difference between the two models is important, as the first order transition allows for metastable phases, while a crossover does not.

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