The S-duality transformations in type IIB string theory can be seen as local $U(1)$ transformations in type IIB supergravity. We use this approach to construct the $SL(2,\mathbb{Z})$ multiplets associated to supersymmetric backgrounds of type IIB string theory and the transformation laws of their corresponding Killing spinors.
1. Introduction

The low energy limit of type IIB superstring theory is type IIB supergravity [1,2]. Among the symmetries of type IIB string theory, S-duality is peculiar and has concentrated a lot of work since its discovery [3], since it is a non-perturbative (conjectured quantum exact) symmetry relating the strong and weak coupling regimes of string theory. This S-duality symmetry is encoded, classically, in a $SL(2, R)$ group, which is broken in the quantum theory to $SL(2, Z)$. The existence of this symmetry ensures that any solution of type IIB string theory belongs to a $SL(2, Z)$ multiplet.

The $SL(2, Z)$ multiplets to which the fundamental string and the solitonic five branes belong have been identified in [4] and [5] respectively. In this paper we will study a more general case. Our expressions will apply for any solution of the type IIB string effective equations of motion with at least one non vanishing electric or magnetic charge with respect to the three forms (or their duals) appearing in the theory. We identify the $SL(2, Z)$ multiplet to which a given type IIB string solution of this type belongs. We will also study the behaviour of the Killing spinors associated to supersymmetric solutions under S-duality transformations. This is particularly interesting when studying the world-volume theories of extended objects in IIB backgrounds, since it allows to write the conditions for BPS solutions coming from kappa symmetry.

This paper is organized as follows. First we review quickly the aspects of type IIB supergravity and type IIB superstring theory that will be used through the paper, to fix our conventions. The equations of motion of type IIB supergravity are invariant under a local $U(1)$ transformation. Second, we introduce a map relating the equations of motion of type IIB supergravity and type IIB superstring theory and recognize the local $U(1)$ invariance of type IIB supergravity as S-duality transformations of type IIB string theory. This allows one to identify the $SL(2, Z)$ multiplets associated to a given solution of type IIB string theory, and provide us automatically with the transformation law of the Killing spinors of the theory under S-duality. Then we work out explicitly the cases of magnetically and electrically charged $SL(2, Z)$ multiplets, checking our general expressions with examples previously constructed in the literature. In the last section we present our conclusions and discussion.

2. Quick Review of Type IIB Supergravity and the Type IIB String.

The equations of motion for ten dimensional chiral $N = 2$ supergravity have been known for a long time ([1,2]). They are given by (for the bosonic part of the theory):

$$D^\mu P_\mu = -\frac{1}{24}G^{\mu\nu\rho}G_{\mu\nu\rho}$$

$$D^\rho G_{\mu\nu\rho} = P^\rho G^*_\mu P_\nu - \frac{2}{3}iK_{\mu\nu\rho\lambda\sigma}G^{\rho\lambda\sigma}$$

$$R_{\mu\nu} = P_\mu P^*_\nu + P_\nu P^*_\mu + \frac{1}{8}(G^{\alpha\beta}_{\mu\nu}G_{\nu\alpha\beta} + G^{\alpha\beta}_{\mu\nu}G^*_\nu\alpha\beta) - \frac{1}{6}g_{\mu\nu}G^{\tau\alpha\beta}G^*_{\tau\alpha\beta} + \frac{1}{6}K_{\alpha\beta\gamma\sigma\mu}K^{\alpha\beta\gamma\sigma}$$

$$K = *K$$

(2.1)
where $D_\mu$ stands for the covariant derivative with respect to the $U(1)$ gauge field $Q_\mu$ present in type IIB supergravity ($D_\mu = \nabla_\mu - iqQ_\mu$, with $q = 2, 1$ for $P_\mu$ and $G_{\mu\nu\rho}$ respectively) and $*$ denotes Hodge duality.

These equations have well known invariances. One of them (easy to check from (2.1)) is the local $U(1)$ transformation:

$$
P \to P' = e^{2i\Lambda(x)} P
$$

$$
G \to G' = e^{i\Lambda(x)} G
$$

$$
Q \to Q' = Q + d\Lambda
$$

$$
K \to K' = K
$$

$$
g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu}
$$  

We could consider solutions to the equations (2.1) with excitations of the five form $K$. But, as we can see in (2.2), these excitations are neutral under $U(1)$ transformations and, for this reason, will not enter in the discussions below. We set this five form to zero from now on.

A second symmetry of the equations (2.1) are chiral $N = 2$ supersymmetry transformations. In particular, the “dilatino” and “gravitino” transformations under supersymmetry are given by:

$$
\delta \lambda = i P_\mu \Gamma^\mu \epsilon^* - \frac{i}{24} G_{\mu
u\rho} \Gamma^{\mu\nu\rho} \epsilon
$$

$$
\delta \Psi_\mu = D_\mu \epsilon + \frac{1}{96} (G_{\nu\alpha\beta} \Gamma_\mu \nu\alpha\beta - 9 G_{\mu\alpha\beta} \Gamma^{\alpha\beta}) \epsilon^*.
$$  

In the equation above, $\lambda$ and $\Psi_\mu$ are complex Weyl spinors of opposite chirality, $\Gamma_{11} \lambda = \lambda$, $\Gamma_{11} \Psi_\mu = -\Psi_\mu$. The supersymmetry parameters $\epsilon$ form also a complex Weyl spinor satisfying $\Gamma_{11} \epsilon = -\epsilon$. Notice that these supersymmetric transformations are covariant under the local $U(1)$ transformations of (2.2) (the charges of $\lambda$, $\Psi_\mu$ and $\epsilon$ being 3/2, 1/2 and 1/2 respectively). In particular, BPS states will remain BPS after a local $U(1)$ transformation.

The bosonic sector of the low energy limit of type IIB string theory is described by the action (when the five form is turned off and in the Einstein frame)

$$
S_{IIB} = \frac{1}{\alpha' l_s^2} \int d^{10}x \sqrt{-g} [R - \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} e^{2\Phi} \partial_\mu \chi \partial^\mu \chi - \frac{1}{12} (e^{-\Phi} H^{(1)2} + e^\Phi F^2)]
$$  

where $\alpha' = l_s^2$ is the string tension and we have defined

$$
H^{(1)} = dB,
$$

$$
F = H^{(2)} - \chi H^{(1)} = dC - \chi dB
$$  

(being $B$ and $C$ the NS-NS and R-R 2-form potentials respectively).
3. IIB String S-Duality as Local \( U(1) \) IIB Supergravity Transformations.

The equations of motion of type IIB supergravity (2.1) exactly map to the equations of motion derived from the IIB superstring action (2.4) if we make the following identifications between the string and supergravity fields (we use the language of forms):

\[
P = \frac{1}{2} (d\Phi + ie\Phi d\chi)
\]
\[
Q = -\frac{1}{2} e\Phi d\chi
\]
\[
G = e^{-\frac{i}{2}\Phi} H^{(1)} + ie^{\frac{i}{2}\Phi} F.
\]

This map corresponds to a particular parametrization and gauge choice for the supergravity scalar fields and was also obtained in [6]. It can be partially inverted to obtain:

\[
d\Phi = 2(P + iQ)
\]
\[
d\chi = -2Qe^{-\Phi}
\]
\[
H^{(1)} = e^{\frac{i}{2}\Phi} \text{Re} G
\]
\[
H^{(2)} = e^{-\frac{i}{2}\Phi} \text{Im} G + e^{\frac{i}{2}\Phi} \chi \text{Re} G,
\]

where \( \text{Re} G \) and \( \text{Im} G \) denote the real and imaginary parts of \( G \). The action (2.4) can then be written as:

\[
S_{IIB} = \int d^{10}x \sqrt{-g} [R - 2\|P\|^2 - \frac{1}{12}\|G\|^2],
\]

where the \( U(1) \) invariance (2.2) is obvious.

Now, using (2.2) it is straightforward to obtain from (3.2) the local \( U(1) \) transformation laws of \( d\Phi, d\chi, H^{(1)} \) and \( H^{(2)} \). If we demand \( d\Phi' \) to be a real field, this imposes an additional condition on the allowed local \( U(1) \) transformations. \( \Lambda(x) \) has to satisfy the equation:

\[
d\Lambda(x) = -\sin(\Lambda(x)) \cos(\Lambda(x)) d\Phi + \sin^2(\Lambda(x)) e^\Phi d\chi,
\]

whose solution is:

\[
\Lambda(x) = -\arctan\left(\frac{ke^{-\Phi(x)}}{1 + k\chi(x)}\right),
\]

being \( k \) a constant of integration.

Under local \( U(1) \) transformations with this \( \Lambda(x) \), one gets from (3.2) the transformation laws of \( d\Phi \) and \( d\chi \):

\[
d\Phi' = \cos(2\Lambda(x)) d\Phi - \sin(2\Lambda(x)) e^\Phi d\chi
\]
\[
e^{-\Phi'} d\chi' = \sin(2\Lambda(x)) d\Phi + \cos(2\Lambda(x)) e^\Phi d\chi.
\]
These equations can be easily integrated and one obtains:

\[
e^{\Phi'} = e^{\Phi} \left[ e^2(\chi^2 + e^{-2\Phi}) + 2cd\chi + d^2 \right]
\]

\[
\chi' = \frac{ac(\chi^2 + e^{-2\Phi}) + (ad + bc)\chi + bd}{c^2(\chi^2 + e^{-2\Phi}) + 2cd\chi + d^2}
\]

Now, from (3.2) one can also compute the transformation laws for \(H^{(1)}\) and \(H^{(2)}\). One gets:

\[
H^{(1)'} = dH^{(1)} + cH^{(2)}
\]

\[
H^{(2)'} = bH^{(1)} + aH^{(2)}.
\]

Note that (3.7) and (3.8) are nothing but the usual S-duality transformations of type IIB superstring theory realized as \(U(1)\) local transformations in type IIB supergravity. To make contact with the usual \(SL(2, R)\) language of S-duality we have just redefined the two integration constants coming from (3.6) together with \(k\) in (3.5) into the four constants \(a, b, c, d\) plus the \(SL(2, R)\) constraint \(ad - bc = 1\). After this redefinition, \(k = c/d\) and the local \(U(1)\) gauge transformation function \(\Lambda(x)\) can be determined through its sine and cosine:

\[
\sin \Lambda(x) = -\frac{ce^{-\Phi}}{\sqrt{(d + c\chi)^2 + c^2e^{-2\Phi}}}, \quad \cos \Lambda(x) = \frac{d + c\chi}{\sqrt{(d + c\chi)^2 + c^2e^{-2\Phi}}}.
\]

This automatically gives the transformation law of the Killing spinors under S-duality transformations. The Killing spinors \(\epsilon\) have \(\frac{1}{2} U(1)\) charge, therefore:

\[
\epsilon'(x) = e^{\pm \Lambda(x)} \epsilon(x),
\]

with \(\Lambda(x)\) given in (3.9).\(^3\)

The transformation of the Killing spinors can be understood (formally) as a parallel transport using the \(U(1)\) connections \(Q\) and \(Q'\) as follows. Let us take a ten dimensional space-time point \(x_0\) which is a fixed point of the local \(U(1)\) transformation (i.e., \(\Lambda(x_0) = 0\)). Then, from (2.2), we have that

\[
\Lambda(x) = \int_{x_0}^{x} d\Lambda = \int_{x_0}^{x} (Q' - Q) = \int_{x_0}^{x} Q' + \int_{x}^{x_0} Q.
\]

As a result (see (3.10)), the effect of a S-duality transformation on a spinor can be understood as a parallel transport of the spinor \(\epsilon(x)\) from any point \(x\) of space-time to a fixed point \(x_0\) of the \(U(1)\) transformation with the \(Q\) connection, and then parallel transport the resulting spinor back to the initial point \(x\) with the gauge connection \(Q'\). If we perform a \(SL(2, R)\) transformation with \(c = 0\), any point in space-time is a fixed \(U(1)\) point. We can then take \(x_0 = x\) in (3.11) and the Killing spinor remains invariant.

\(^3\) Similar expressions were derived in [7,8] for specific cases.
There can be also fixed $U(1)$ points where $\Phi = \infty$. These points are singular and the classical solutions to the effective string equations are not reliable (there are strong quantum gravity stringy corrections in a neighborhood of these points of size $\sim l_s$). So the parallel transport (3.11) starts from a point where the classical solution of the effective string action is reliable, goes to a singular point, where quantum gravity effects take place, and comes back to the initial point where the classical solution of the effective string action is reliable again (note that $\Lambda(x)$ is well defined all along the path). In this way the S-duality transformations capture non-perturbative information from the supergravity point of view.

4. Magnetic Multiplets.

Given any solution of the equations of motion $g_{\mu\nu}$, $\Phi$, $\chi$, $B$ and $C$, all the members of the $SL(2, Z)$ multiplet associated to it can be obtained from (3.7) and (3.8) once the right parameters $a$, $b$, $c$ and $d$ connecting them are correctly identified. This is what we will be doing in this (next) section, distinguishing the cases in which at least one of the magnetic (electric) charges are non-vanishing.

Let us first consider the simplest case in which there is at least one non-vanishing magnetic charge in the solution to the equations of motion. Let us define the magnetic charges in the following way:

$$q_i = \int H^{(i)}, \quad p_i = \int H^{(i)}'.$$

where the integration is performed over a region that surrounds the world-volume of the charged object. One can use the four equations (3.7) and (3.8) to obtain the three independent values of the constants $a$, $c$ and $d$ ($b$ is obtained from $ad - bc = 1$) in terms of those charges and the values $\Phi_0$, $\chi_0$, $\Phi'_0$ and $\chi'_0$ of the fields $\Phi$, $\chi$, $\Phi'$ and $\chi'$ evaluated at some point (which is usually taken at infinity). The result is the following:

$$a = \frac{1}{\Delta}[q_1^2(e^{\Phi'_0}\chi'_0 + e^{\Phi_0}\chi_0) - e^{\Phi_0}(p_1p_2 + q_1q_2)]$$

$$b = \frac{1}{\Delta}[(p_1p_2 - q_1q_2)(e^{\Phi'_0}\chi'_0 + e^{\Phi_0}\chi_0) + (e^{\Phi_0}q_2^2 - e^{\Phi'_0}p_2^2)]$$

$$c = \frac{1}{\Delta}(q_1^2 e^{\Phi'_0} - p_1^2 e^{\Phi_0})$$

$$d = \frac{1}{\Delta}[p_1^2(e^{\Phi'_0}\chi'_0 + e^{\Phi_0}\chi_0) - e^{\Phi'_0}(p_1p_2 + q_1q_2)],$$

being $\Delta$ the constant:

$$\Delta \equiv q_1 p_1(e^{\Phi'_0}\chi'_0 + e^{\Phi_0}\chi_0) - (q_1 p_2 e^{\Phi'_0} + q_2 p_1 e^{\Phi_0}).$$

Note that, since we are assuming that $q_1$ and/or $q_2$ do not vanish, the constants (4.2) are generically well defined. It is clear from (4.2) that quantization conditions on the charges $q_1, q_2, p_1, p_2$ break $SL(2, R)$ down to $SL(2, Z)$ [9] (for fixed asymptotic values of the scalar fields). Equation (4.2) provides the $SL(2, Z)$ parameters relating any two components of
the multiplet. Since we started from four equations ((3.7) and (3.8)) and three unknowns ($a$, $c$ and $d$), we also get a constraint equation that $q_i$, $p_i$, $\Phi$, $\chi$, $\Phi'$ and $\chi'$ have to satisfy:

$$e^\Phi (q_2 - q_1 \chi)^2 + e^{-\Phi} q_1^2 = e^{\Phi'} (p_2 - p_1 \chi')^2 + e^{-\Phi'} p_1^2. \quad (4.4)$$

The interpretation of this constraint will be clear in what follows. The Killing spinors are straightforwardly computed by using the equations (3.10), (3.9) and (4.2).

Let us work out a specific example. We will take a BPS NS5-brane in the Einstein frame:

$$dS^2_E = H(r)^{-\frac{4}{3}}(-dt^2 + dX^2_{\parallel}) + H(r)^{\frac{2}{3}}(dr^2 + r^2 d\Omega_3)$$

$$e^{\Phi} = \sqrt{H(r)}, \quad H^{(1)} = 2R^2 \epsilon_3, \quad H^{(2)} = \chi = 0, \quad (4.5)$$

being $H(r)$ the harmonic function:

$$H(r) = 1 + \frac{R^2}{r^2}, \quad (4.6)$$

and construct the $SL(2,Z)$ multiplet to which it belongs, and the corresponding Killing spinors (this multiplet was first constructed in [3]).

We will take, moreover, the asymptotic conditions, $\Phi'_0 \equiv \Phi'(\infty)$ and $\chi'_0 \equiv \chi'(<\infty)$ (note that in (1.6) we have already taken $\Phi_0 \equiv \Phi(\infty) = 0$ and $\chi_0 \equiv \chi(\infty) = 0$).

From equation (4.4) we obtain, for this case:

$$q_1^2 = e^{\Phi'_0} (p_2 - p_1 \chi'_0)^2 + e^{-\Phi'_0} p_1^2 \quad (4.7)$$

which allows to eliminate $q_1$ in equations (4.2) and write them in terms of $p_1$ and $p_2$ alone. This gives:

$$a = \frac{1}{q_1} [e^{\Phi'_0} \chi'_0 (p_1 \chi'_0 - p_2) + p_1 e^{-\Phi'_0}]$$

$$b = \frac{p_2}{q_1}$$

$$c = \frac{1}{q_1} e^{\Phi'_0} (p_1 \chi'_0 - p_2)$$

$$d = \frac{p_1}{q_1}. \quad (4.8)$$

Just plugging this result on (4.7) and (3.8) we get the $SL(2, Z)$ multiplet:

$$e^{\Phi'} = c^2 H(r)^{-\frac{4}{3}} + d^2 H(r)^{\frac{2}{3}}$$

$$\chi' = \frac{ac + bdH(r)}{c^2 + d^2 H(r)}$$

$$H^{(1)'} = dH^{(1)}$$

$$H^{(2)'} = bH^{(1)}. \quad (4.9)$$
The Einstein metric is inert under the $U(1)$ transformations (see equation (2.2)), meaning that the mass measured with it is equal for all the members of the multiplet. The coefficient $R$ of the harmonic form is then invariant. Using (4.7), it can be written in terms of the charges $p_1$, $p_2$:

$$q_1 = \int H^{(1)} = 2R^2 w_3 \Rightarrow R_{(p_1,p_2)}^2 = \frac{\sqrt{e^{\Phi_0}(p_2 - p_1\chi_0')^2 + e^{-\Phi_0}p_1^2}}{2w_3}, \quad (4.10)$$

where we denoted by $w_3$ the volume of the unit $S^3$. Usually, the charges are given in units in which $q_1 = 1$. The NS5-brane and the D5-brane are then just the $(1,0)$ and $(0,1)$ elements of this multiplet of $(p_1,p_2)$ magnetic 5-branes.

Let us compute the Killing spinors. In the present case, the function $\Lambda$ depends only on $r$ and can be written in terms of the charges $p_1$ and $p_2$ from the equations (4.8) as:

$$\sin \Lambda_{(p_1,p_2)}(r) = -\frac{e^{\Phi_0}(p_1\chi_0' - p_2)}{\sqrt{e^{2\Phi_0}(p_1\chi_0' - p_2)^2 + p_1^2H(r)}}$$

$$\cos \Lambda_{(p_1,p_2)}(r) = \frac{p_1[H(r)]^{\frac{1}{2}}}{\sqrt{e^{2\Phi_0}(p_1\chi_0' - p_2)^2 + p_1^2H(r)}}. \quad (4.11)$$

Now, using equation (3.10), the Killing spinor of the $(p_1,p_2)$ magnetic 5-brane can be obtained from that of the NS5-brane as:

$$\epsilon_{(p_1,p_2)} = e^{\frac{i}{2} \Lambda_{(p_1,p_2)}(r)} \epsilon_{NS5}, \quad (4.12)$$

where

$$\epsilon_{NS5} = [H(r)]^{-\frac{1}{16}} e^{-\frac{\theta^3}{4} \Gamma_{a\bar{a}} e^{-\frac{\theta^2}{2} \Gamma_{a\bar{a}} e^{-\frac{\theta^1}{2} \Gamma_{a\bar{a}} \epsilon_0}}. \quad (4.13)$$

being $\epsilon_0$ a constant complex Weyl spinor of negative chirality satisfying:

$$\Gamma_{012345} \epsilon_0 = \epsilon_0. \quad (4.14)$$

In eq. (4.13) the $\Gamma$’s with underlined indices represent flat (constant) ten dimensional Dirac gamma matrices and $\theta^1, \theta^2, \theta^3$ are the angular coordinates of the transverse 3-sphere. Notice that the angular dependent part in (4.13) is just the Killing spinor of $S^3$ ([10]).

We have already seen that the ADM masses, computed with the Einstein metric, are equal for all members of the multiplet. However, if we want to visualize the kind of objects that belong to the multiplets from the string theory point of view (i.e., the dependence of the masses on the string coupling constant), we have to compute the ADM masses with the modified Einstein metric $\tilde{g}_{\mu\nu}$. This metric is related to the Einstein metric by:

$$\tilde{g}_{\mu\nu} \equiv \sqrt{g_s} g_{\mu\nu} = e^{\Phi_0} g_{\mu\nu}. \quad (4.15)$$

The modified Einstein metric is defined such that it coincides with the string metric asymptotically. It is clear from (4.13) that the masses of the members of the multiplet, measured
with $\tilde{g}_{\mu\nu}$, are different (due to the change of the string coupling constant under $SL(2, \mathbb{Z})$ transformations). If we write the action (2.4) in terms of the modified Einstein metric, the Newton constant $G_N$ gets rescaled to $G_N = g_s^2 \alpha'^4$. The ADM mass is now computed from the behaviour of $\tilde{g}_{00}$ at infinity [12]:

$$\tilde{g}_{00} \sim -1 + \frac{G_N M}{3u_3 r^2}. \quad (4.16)$$

In our case, after writing the modified Einstein metric associated to the metric in equation (4.5), we get:

$$\tilde{g}_{00} \sim -1 + \frac{1}{4} \frac{\sqrt{g_s R_p^2}}{r^2}. \quad (4.17)$$

Comparing (4.16) and (4.17), and using the equation (4.10), we arrive at:

$$M = \frac{3}{8\alpha'^4} \sqrt{g_s^{-2}(p_2 - p_1 \chi_0')^2 + g_s^{-4}p_1^2}. \quad (4.18)$$

Note that this ADM mass has units of mass per unit of brane volume $(1/(\text{Length}^6))$, so it corresponds to the tensions of the $(p_1, p_2)$ five branes. Note also that the $p_1$ charged objects “contribute” to the mass with terms of order $p_1/g_s^2$ (as corresponds to the magnetic NS five brane) whereas the $p_2$ charged objects “contribute” to the mass with a term of order $p_2/g_s$ (as corresponds to D-branes solitons). Then, (4.9) describes the background configurations corresponding to bound states of solitonic 5-branes and D5-branes [13].

5. The Electric Multiplets.

Let us now consider the cases in which there is at least one non-vanishing electric charge. In the electric case, the conserved charges are given by the equations of motion for the NS-NS and R-R potentials $B$ and $C$:

$$d * S^{(1)} = d * S^{(2)} = 0, \quad (5.1)$$

where we have defined

$$S^{(1)} \equiv e^{-\Phi} H^{(1)} - e^\Phi \chi F$$

$$S^{(2)} \equiv e^\Phi F. \quad (5.2)$$

Then, we define the electric charges as:

$$\hat{q}_i = \int * S^{(i)}, \quad \hat{p}_i = \int * S^{(i)'}$$

where the integration region surrounds the charged object.

From the equations (3.7) and (3.8) we can work out the transformations laws for $S^{(1)}$ and $S^{(2)}$ under local $U(1)$ transformations. The result is the following:

$$S^{(1)'} = a S^{(1)} - b S^{(2)}$$

$$S^{(2)'} = -c S^{(1)} + d S^{(2)}, \quad (5.4)$$
Note that, compared with (3.8), if the magnetic charges transforms with the $SL(2,Z)$ matrix $K$, then the electric charges transforms with $(K^{-1})^T$. Comparing (5.4) with (3.8) we see that, in the electric case, we have to make the changes $q_1 \to \hat{q}_2$, $q_2 \to -\hat{q}_1$, $p_1 \to \hat{p}_2$ and $p_2 \to -\hat{p}_1$ in the equations (4.1), (4.3) and (4.4). These changes give, for the $SL(2,Z)$ constants, the expressions:

\[
\begin{align*}
    a &= \frac{1}{\hat{\Delta}} [q_2^2(e^{\Phi'_0} \chi_0 + e^{\Phi_0} \chi_0) + e^{\Phi_0}(\hat{p}_2 + \hat{q}_1)] \\
    b &= \frac{1}{\hat{\Delta}} [(\hat{q}_1 \hat{q}_2 - \hat{p}_1 \hat{p}_2)(e^{\Phi'_0} \chi_0 + e^{\Phi_0} \chi_0) + (\hat{q}_1^2 e^{\Phi_0} - \hat{p}_1^2 e^{\Phi'_0})] \\
    c &= \frac{1}{\hat{\Delta}} (\hat{q}_2^2 e^{\Phi'_0} - \hat{p}_2^2 e^{\Phi_0}) \\
    d &= \frac{1}{\hat{\Delta}} [\hat{p}_2^2(e^{\Phi'_0} \chi_0 + e^{\Phi_0} \chi_0) + e^{\Phi_0}(\hat{p}_1 + \hat{q}_2)]
\end{align*}
\]

being $\hat{\Delta}$ given by:

\[
\hat{\Delta} \equiv \hat{q}_2 \hat{p}_2(e^{\Phi'_0} \chi_0 + e^{\Phi_0} \chi_0) + (\hat{q}_2 \hat{p}_1 e^{\Phi'_0} + \hat{q}_1 \hat{p}_2 e^{\Phi_0})
\]

The constraint (4.4) now reads:

\[
e^{\Phi}(\hat{q}_1 + \hat{q}_2 \chi)^2 + e^{-\Phi} \hat{q}_2^2 = e^{\Phi'}(\hat{p}_1 + \hat{p}_2 \chi')^2 + e^{-\Phi'} \hat{p}_2^2.
\]

The Killing spinors are computed by using the equations (3.10), (3.3), and (5.6).

Let us work out again an explicit example (4). We will take a BPS fundamental string in the Einstein frame [14]:

\[
\begin{align*}
    dS^2_E &= H(r)^{-\frac{2}{3}} (-dt^2 + dX_1^2) + H(r)^{\frac{2}{3}} (dr^2 + r^2 d\Omega_7) \\
    e^{-\Phi} &= \sqrt{H(r)}, \quad e^{-\Phi} * H^{(1)} = 6R^6 \epsilon_7, \quad H^{(2)} = \chi = 0
\end{align*}
\]

being $H(r)$ the harmonic function:

\[
H(r) = 1 + \frac{R^6}{r^6}.
\]

We will take, moreover, the asymptotic conditions $\Phi'_0 \equiv \Phi'(\infty)$ and $\chi'_0 \equiv \chi'(\infty)$.

The equation (5.8) gives, for this case:

\[
\hat{q}_1^2 = e^{\Phi'_0}(\hat{p}_1 + \hat{p}_2 \chi'_0)^2 + e^{-\Phi'_0} \hat{p}_2^2.
\]
We can use this equation to eliminate \( \hat{q}_1 \) in equations (5.6) and write (5.6) in terms of \( \hat{p}_1 \) and \( \hat{p}_2 \) alone (\( \hat{q}_1 \) is given in (5.11) in terms of \( \hat{p}_1 \) and \( \hat{p}_2 \)). This gives:

\[
a = \frac{\hat{p}_1}{\hat{q}_1} \\
b = \frac{1}{\hat{q}_1} [e^{\Phi_0} \chi_0' (\hat{p}_2 \chi_0' + \hat{p}_1) + e^{-\Phi_0} \hat{p}_2] \\
c = -\frac{\hat{p}_2}{\hat{q}_1} \\
d = \frac{e^{\Phi_0}}{\hat{q}_1} (\hat{p}_2 \chi_0' + \hat{p}_1). \tag{5.12}
\]

Again, just plugging this background configuration on (3.7) and (3.8) we get the \( SL(2, Z) \) multiplet:

\[
e^{\Phi'} = e^{2 \Lambda(r)} + d^2 H(r) \frac{1}{e^{\Phi_0}} \\
\chi' = \frac{ac H(r) + bd}{e^{2H(r)} + d^2} \\
H^{(1)'} = dH^{(1)} \\
H^{(2)'} = bH^{(1)}. \tag{5.13}
\]

The coefficient \( R \) of the harmonic form is now given, using (5.11), by:

\[
\hat{q}_1 = \int *S^{(1)} = 6R^6 w_7 \Rightarrow R^6_{(\hat{p}_1, \hat{p}_2)} \equiv R^6 = \frac{\sqrt{e^{2\Phi_0}(\hat{p}_1 + \hat{p}_2 \chi_0')^2 + e^{-\Phi_0} \hat{p}_2^2}}{6w_7}, \tag{5.14}
\]

where we denoted by \( w_7 \) the volume of the unit \( S^7 \). The equation (5.8) has to be interpreted, again, as equating the masses (more properly, tensions) of the components of the \( SL(2, Z) \) multiplet when measured with the Einstein metric.

We can obtain the Killing spinor of the \((\hat{p}_1, \hat{p}_2)\) strings from equation (3.10):

\[
\epsilon_{(\hat{p}_1, \hat{p}_2)} = e^{\frac{i}{2} \Lambda_{(\hat{p}_1, \hat{p}_2)}(r)} \epsilon_{F1}, \tag{5.15}
\]

where \( \Lambda_{(\hat{p}_1, \hat{p}_2)}(r) \) is given in this case by:

\[
\sin \Lambda_{(\hat{p}_1, \hat{p}_2)}(r) = \frac{\hat{p}_2 [H(r)]^{\frac{1}{2}}}{\sqrt{e^{2\Phi_0}(\hat{p}_2 \chi_0' + \hat{p}_1)^2 + \hat{p}_2^2 H(r)}} \\
\cos \Lambda_{(\hat{p}_1, \hat{p}_2)}(r) = \frac{e^{\Phi_0}(\hat{p}_2 \chi_0' + \hat{p}_1)}{\sqrt{e^{2\Phi_0}(\hat{p}_2 \chi_0' + \hat{p}_1)^2 + \hat{p}_2^2 H(r)}}, \tag{5.16}
\]
and $\epsilon_{F1}$ is the Killing spinor of the fundamental (1, 0) string:

$$\epsilon_{F1} = [H(r)]^{-\frac{\alpha}{16}} e^{-\frac{\theta^7}{2} \Gamma_7 \theta^7} \prod_{i=1}^{6} e^{-\frac{\theta^{i+1}}{2} \Gamma^i \theta^{i+1}} \epsilon_0. \quad (5.17)$$

In the above equation, $\theta^i$ are the angles of the transverse $S^7$ and the product has to be taken from right to left. $\epsilon_0$ is a constant Weyl spinor satisfying:

$$\Gamma_{01} \epsilon_0 = -\epsilon_0^*. \quad (5.18)$$

Now, we can compute the ADM masses of the electric multiplet in the modified Einstein frame similarly to our calculation of the masses for the magnetic multiplets in the previous section. By comparing

$$\tilde{g}_{00} \sim -1 + \frac{G_N M}{7w_7 r^6}. \quad (5.19)$$

with the asymptotic behaviour of the modified Einstein metric associated to the metric in (5.3)

$$\tilde{g}_{00} \sim -1 + \frac{3}{4} \sqrt{g_s} \tilde{R}_p^6 r^6, \quad (5.20)$$

we obtain:

$$M = \frac{21}{24\alpha'} \sqrt{(\tilde{p}_1 + \tilde{p}_2 \chi_0)^2 + g_s^{-2} \tilde{p}_2^2}. \quad (5.21)$$

Note that this ADM mass has units of mass per unit of brane volume $(1/(\text{Length}^2))$, so it corresponds to the tensions of the $(p_1, p_2)$ strings. Note also that the $p_1$ charged objects “contribute” to the mass with terms of order $p_1$ (as corresponds to the perturbative string modes) whereas the $p_2$ charged objects “contribute” to the mass with term of the order $p_2/g_s$ (as it corresponds to D-brane solitons). Thus, the background (5.13) corresponds to bound states of fundamental strings and D1-branes [13].

6. Conclusions and Discussion.

The equations of motion of type IIB supergravity exactly map the equations of type IIB string theory through the identifications (3.1). The S-duality symmetry of type IIB string theory corresponds to those $U(1)$ local transformations of type IIB supergravity (2.2) maintaining the reality of the gauge (3.1). The $SL(2, Z)$ multiplets associated to the solutions of type IIB string theory that we have considered here, are given in (3.7) and (3.8), where the $SL(2, Z)$ constants are fixed by the expectation values of the scalars and the electric or magnetic charges in (4.2) or (5.6). The Killing spinors of the $SL(2, Z)$ multiplets are then computed by (3.10).

In our approach, the self-duality of the D3 branes is reflected by the fact that the supergravity five-form is neutral under $U(1)$ transformations. Moreover, the Killing spinors
associated to D3 multiplets are related by global rotations (see equations (3.9) and (3.10)), due to the constant nature of the scalars for these solutions.

We have checked our general expressions with the previously constructed $SL(2, Z)$ multiplets to which the solitonic five brane ([5]) and fundamental string ([4]) belong, obtaining the same result. In [5] and [4] a different approach for these particular cases was employed. In particular, the Einstein metric was not invariant under the $SL(2, Z)$ transformation and the harmonic form coefficients (4.10) and (5.14) were found by a rescaling due to charge quantization arguments. In other words, the solution to the effective IIB equations of motion which was used to construct the multiplet was not itself in the multiplet. In our approach, the harmonic form coefficient is fixed by the consistency constraint (4.4) and the Einstein metric is invariant. Moreover, we have provided the relation between the members of the multiplet in (4.2) and (5.6).

Any $SL(2, Z)$ multiplet containing a given representative with at least one non-vanishing charge (electric or magnetic) associated to the NS-NS or R-R three forms can be straightforwardly constructed with our formalism. However, it does not cover exotic solutions as the D-Instantons ([15]) and D7-branes ([16]). These solutions are not charged with respect to any of the three forms (or their duals) appearing in the theory. For the D-Instantons, one should adapt our formalism to the Euclidean space. The D7 case is a very subtle one. This background solution is magnetically charged with respect to the one-form R-R field strength $d\chi$ (which, of course, is globally defined). This means that the field $\chi$ is not single-valued along one-cycles surrounding the brane. We can, in this case, cover the cycle with two charts. Inside any chart $\chi$ is single-valued, so we can apply (3.6) safely. The $SL(2, Z)$ transformations performed in both charts are, however, not independent. We have to demand the equality of $d\Phi'$ and $d\chi'$ in the overlapping regions (i.e., the field strenghts have to be globally defined). It is not difficult from (3.6) to show that the $SL(2, Z)$ transformations preserving the single-valued character of the one form field strenghts associated to the potentials $\Phi$ and $\chi$ are those with $c = 0$, which just change the magnetic charge associated to $d\chi$ and the expectation values of the scalar fields. The transformations with $c \neq 0$ lead to non single-valued field strenghts. This suggests that bound states with D7 branes are absent [13].

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