Low-energy and low-momentum representation of the virtual Compton scattering amplitude

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Abstract

We perform an expansion of the virtual Compton scattering amplitude for low energies and low momenta and show that this expansion covers the transition from the regime to be investigated in the scheduled photon electroproduction experiments to the real Compton scattering regime. We discuss the relation of the generalized polarizabilities of virtual Compton scattering to the polarizabilities of real Compton scattering.

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I. INTRODUCTION

Over the past few years, the virtual Compton scattering (VCS) reaction $\gamma^* + p \to \gamma + p'$, which can be accessed in the process $ep \to e'p'\gamma$, has received renewed interest \[1\]. In particular, the framework devised by Guichon et al. \[2\] to identify new electromagnetic observables, namely the generalized polarizabilities (GPs), served as the basis of the scheduled VCS experiments at MAMI, Jefferson Lab, and MIT Bates \[1\]. In Ref. \[2\], a kinematical scenario was proposed, for which the final-photon energy $\omega'$ in the $\gamma p'$ c.m. frame is well below the pion production threshold but the three-momentum of the initial virtual photon, $\bar{q} \equiv |\vec{q}|$, is large compared with $\omega'$. Keeping only terms linear in $\omega'$, the regular, structure-dependent part of the VCS amplitude was parametrized in terms of ten GPs which are functions of $\bar{q}^2$.

To some degree these functions can be interpreted as generalizations of the electromagnetic polarizabilities of real Compton scattering (RCS). In a recent publication \[3\], we investigated the regular part of the VCS amplitude on the basis of a covariant approach and found that not all of the ten GPs are independent, once the constraints due to charge conjugation combined with nucleon crossing have been imposed. In fact, four independent relations were found, reducing the number of independent GPs from ten to six. Predictions for the GPs have been obtained in various frameworks \[2,4–11\].

In Refs. \[12,13\], a different kinematic regime was discussed for VCS from a spin-0 target, namely that of small energies and small three-momenta. This regime is of particular interest when studying the transition from VCS to low-energy RCS. In the $\gamma p'$ c.m. frame all kinematic quantities can be expressed in terms of $\omega'$, $\bar{q}$, and $\cos(\theta) = \hat{q} \cdot \hat{q}'$. For example, the virtual-photon energy $\omega$ is given by

$$\omega(\omega', \bar{q}) = \omega' + \sqrt{M^2 + \omega'^2} - \sqrt{M^2 + \bar{q}^2} = \omega' + \frac{\omega'^2}{2M} - \frac{\bar{q}^2}{2M} + O(r^4), \quad r \in \{\omega', \bar{q}\}, \quad (1)$$

which has to be compared with the approximation

$$\omega_0 \equiv \omega(0, \bar{q}) = M - \sqrt{M^2 + \bar{q}^2}, \quad (2)$$

utilized in the framework of Ref. \[2\]. Structure-dependent terms of RCS are of order $\omega^2$ and higher (see, e.g., Refs. \[14,15\]), and thus, in general, beyond the scope of the multipole expansion of Ref. \[2\]. Therefore, if one is interested in $\bar{q}$-dependent effects in the transition region it will not suffice to only take account of terms linear in $\omega'$. Instead, higher-order terms in $\omega'$ competing with $\bar{q}$-dependent effects must also be considered.

Even though it may be difficult to experimentally isolate individual structures in this kinematic region, it will be important to have a complete description, including the low-momentum region, for the following reason. From a theoretical point of view, the GPs are defined for arbitrary $\bar{q}$, whereas the scheduled VCS experiments are designed for large three-momenta ($\bar{q} >> \omega'$). Of course, it is desirable to extract data for various $\bar{q}$ in order to determine the momentum evolution of a given polarizability. As will be shown in Sec. III in a model-independent way, at $\bar{q} = 0$ all GPs—except for one linear combination—either vanish \[3\] or are related to the polarizabilities in RCS. For the low-momentum evolution of the GPs in the transition region, where the initial photon is almost real, a simultaneous expansion in $\omega'$ and $\bar{q}$ is required such that equally important higher-order terms in $\omega'$ are included. This transition region will be discussed in the following.
Our paper is organized as follows: In Sec. II we sketch the derivation of the general structure of the VCS amplitudes for the simultaneous expansion in $\omega'$ and $\bar{q}$. We then introduce an additional $1/M$ expansion and derive explicit expressions for the GPs in terms of the 12 c.m. amplitudes $A_i$. In Sec. III we apply the expansion of Sec. II to RCS and show the relations between the GPs of VCS and the RCS polarizabilities. We also list the multipole expansion of the RCS polarizabilities. In Sec. IV we discuss forward scattering in VCS and RCS. Section V contains a short summary.

II. LOW-ENERGY AND LOW-MOMENTUM EXPANSION FOR VCS

The general amplitude of $\gamma^*(q) + N(p_i) \rightarrow \gamma(q') + N(p_f)$ can be parametrized in terms of 12 independent functions \[12\]. We assume that a division into pole terms and a residual part has been performed such that both pieces are separately gauge invariant and satisfy the appropriate symmetry requirements \[2,17\]. The following we will discuss the residual of 12 independent functions \[16\]. We assume that a division into pole terms and a residual of the 12 c.m. amplitudes $A_i$ depends on $\bar{q}$, $\omega' = |\bar{q}'|$, and $\cos(\theta) = \hat{q} \cdot \bar{q}'$, where $\bar{q}$ and $\bar{q}'$ are the three-momenta of the initial and final photons in the c.m. frame. In Ref. \[2\] a multipole decomposition of the c.m. amplitude was performed, keeping only terms linear in $\omega'$. The result was expressed in terms of ten GPs which are functions of $\bar{q}^2$. Such an expansion is expected to work below pion production threshold for large enough $\bar{q}$.

The connection between the GPs of Ref. \[2\] and the invariant functions $f_i = f_i(q^2, q \cdot q', q \cdot P), i = 1 \ldots 12$ (see Eqs. (7), (A10), and (A11) of Ref. \[3\]) is given by

\[
P^{(01,01)}(q^2) = \frac{2}{3} \left[ \frac{E_i + M}{2E_i} \right] \left\{ f_1(q^2) - 2M \frac{q^2}{\omega_0} f_2(q^2) - 2M \omega_0 \left[ 2f_6(q^2) + f_9(q^2) - f_{12}(q^2) \right] \right\}
\]

\[
= -\frac{4\pi}{e^2} \sqrt{2} \frac{2}{3} \alpha(q^2)
\]

\[
P^{(11,11)}(q^2) = -\frac{8}{3} \left[ \frac{E_i + M}{2E_i} \right] f_1(q^2) = -\frac{4\pi}{e^2} \sqrt{8} \frac{2}{3} \beta(q^2)
\]

\[
\hat{P}^{(01,1)}(q^2) = \frac{4M}{3} \left[ \frac{E_i + M}{2E_i} \right] \left\{ f_2(q^2) + \frac{\omega_0^2}{q^2} \left[ 2f_6(q^2) + f_9(q^2) - f_{12}(q^2) \right] \right\}
\]

\[
P^{(01,12)}(q^2) = \frac{\sqrt{2}}{3} \left[ \frac{E_i + M}{2E_i} \right] M \omega_0 \left[ 8Mf_6(q^2) + f_7(q^2) + 4Mf_9(q^2) + 4f_{11}(q^2) - \omega_0 f_{12}(q^2) \right]
\]

\[
P^{(11,02)}(q^2) = \frac{2\sqrt{2}}{3\sqrt{3}} \left[ \frac{E_i + M}{2E_i} \right] \left[ \frac{\omega_0^2}{2q^2} f_5(q^2) + \frac{1}{2} f_7(q^2) + 2f_{11}(q^2) + \frac{M \omega_0^2}{q^2} f_{12}(q^2) \right]
\]
\[ P^{(01,01)}(q^2) = \frac{1}{3} \sqrt{\frac{E_i + M}{2E_i}} \omega_0 \left[ f_5(q^2) + f_7(q^2) + 4f_{11}(q^2) + 4Mf_{12}(q^2) \right], \] (8)

\[ P^{(11,00)}(q^2) = \frac{2}{3\sqrt{3}} \sqrt{\frac{E_i + M}{2E_i}} \left[ (\omega_0^2 - 3M\omega_0)f_5(q^2) + q^2f_7(q^2) + 4q^2f_{11}(q^2) \\
+ (3Mq^2 - 6M^2\omega_0 + 2M\omega_0^2)f_{12}(q^2) \right], \] (9)

\[ P^{(11,11)}(q^2) = -\frac{2}{3} \sqrt{\frac{E_i + M}{2E_i}} \frac{M\omega_0^2}{q^2} \left[ f_5(q^2) + \omega_0f_{12}(q^2) \right], \] (10)

\[ \hat{P}^{(11,2)}(q^2) = -\frac{\sqrt{2}}{3\sqrt{5}} \sqrt{\frac{E_i + M}{2E_i}} \frac{\omega_0}{q^2} \left[ f_5(q^2) + f_7(q^2) + 4f_{11}(q^2) \right], \] (11)

\[ \hat{P}^{(01,1)}(q^2) = \frac{2}{3\sqrt{6}} \sqrt{\frac{E_i + M}{2E_i}} \frac{\omega_0}{q^2} \left[ (2M - \omega_0)f_5(q^2) + 8M^2f_6(q^2) \\
+ (M - \omega_0)f_7(q^2) + 4M^2f_9(q^2) + 4(M - \omega_0)f_{11}(q^2) \\
- 3M\omega_0f_{12}(q^2) \right], \] (12)

with \( f_i(q^2) \equiv f_i(2M\omega_0, 0, 0) \). These relations result from comparing the expressions for the functions \( A_i \) obtained within the framework of the truncated multipole expansion on the one hand with the covariant approach evaluated in the c.m. frame on the other hand.

Making use of the symmetry properties of the \( f_i \) with respect to photon crossing and charge conjugation combined with nucleon crossing, we distinguish three subclasses according to their different kinematic expansions. Up to and including \( O(k^3) \), \( k \in \{q, q'\} \), the expansions read

\[
\begin{align*}
    f_i &= f_{i,0} + f_{i,2a}q \cdot q' + f_{i,2b}q^2 + f_{i,2c}(q \cdot P)^2 + O(k^4) \quad (i = 1, 2, 5, 6, 11, 12), \\
    f_i &= f_{i,1}q \cdot P + f_{i,3a}q \cdot Pq \cdot q' + f_{i,3b}q \cdot Pq^2 + f_{i,3c}(q \cdot P)^3 + O(k^5) \quad (i = 3, 4, 8, 10), \\
    f_i &= f_{i,2b}q^2 + O(k^4) \quad (i = 7, 9).
\end{align*}
\] (13)

The increment is always \( O(k^2) \) and coefficients with a subscript \( b \) contribute for virtual photons only. According to Eq. (11), the four-momenta \( q \) and \( q' \) are not completely independent. However, a truncation after \( O(k^3) \) is sufficient to generate expressions for the amplitudes \( A_i \) \( (i = 1 \ldots 12) \) up to \( O(r^5) \), \( r \in \{\omega', \bar{q}'\} \). As can be seen below, it will be necessary to include terms of \( O(r^4) \) \([O(r^5)]\) in the spin-independent (spin-dependent) sector in order to incorporate the first non-vanishing effects due to the virtuality of the initial photon. We note that the invariant functions \( f_3, f_5, f_8, \) and \( f_{12} \) do not contribute to RCS since they multiply structures containing either factors of \( q^2 \) or \( q \cdot \epsilon \) (cf. Eqs. (A10) of Ref. [1]).

At \( O(r^2) \) the amplitudes \( A_i \) receive contributions from two different structure constants, \( f_{1,0}, f_{2,0} \).

At \( O(r^3) \) we find six additional structure constants:

\[
    f_{4,1}, f_{5,0}, f_{6,0}, f_{10,1}, f_{11,0}, f_{12,0}. 
\]

Moreover, due to Eq. (11) there are \( O(r^3) \) terms involving \( f_{1,0} \) and \( f_{2,0} \) which are suppressed by one power of \( 1/M \) relative to the corresponding \( O(r^2) \) contributions. At \( O(r^4) \) there are seven new constants,
and again various lower-order structure constants suppressed by factors of $1/M$. Finally, at $\mathcal{O}(r^5)$ one obtains 21 new structure constants,

\begin{align*}
& f_{1,2a}, f_{1,2b}, f_{1,2c}, f_{2,2a}, f_{2,2b}, f_{2,2c}, f_{3,1}, \\
& f_{4,3a}, f_{4,3b}, f_{4,3c}, f_{5,2a}, f_{5,2b}, f_{5,2c}, f_{6,2a}, f_{6,2b}, f_{6,2c}, \\
& f_{7,2a}, f_{7,2b}, f_{8,1}, f_{9,2a}, f_{9,2b}, f_{10,3a}, f_{10,3b}, f_{10,3c}, f_{11,2a}, f_{11,2b}, f_{11,2c}, \\
& f_{12,2a}, f_{12,2b}, f_{12,2c}.
\end{align*}

together with $1/M$-suppressed lower-order constants.

After the simultaneous expansion in terms of $\omega'$ and $\bar{q}$ it is useful to further expand the amplitudes $A_i$ in powers of $1/M$ (see also Ref. [10]). In particular, such a type of expansion is closely related to the power counting used in heavy baryon chiral perturbation theory (HBChPT) or any other theory which can be organized in terms of a $1/M$ expansion. For the spin-independent part of the VCS amplitude, such a HBChPT calculation was carried out in [7] keeping only the contributions of the lowest non-vanishing order in a $1/M$ expansion. The results of this calculation could be mapped onto the leading-order terms of a general structure analysis of the spin-independent VCS amplitudes [12,13]. With the scheme developed in Ref. [3] all prerequisites for performing the $1/M$ expansion of the complete VCS amplitude are available, enabling us to add the spin-dependent part to our previous results [13] and to organize the expansion in powers of $\omega'$ and $\bar{q}$.

Expanding the nucleon spinors,

\begin{align}
& u(-\bar{q}) = \left( 1 + \frac{\bar{\gamma} \cdot \bar{q}}{2M} + \ldots \right) u(0), \\
& \bar{u}(-\bar{q}’) = \bar{u}(0) \left( 1 + \frac{\bar{\gamma} \cdot \bar{q}’}{2M} + \ldots \right),
\end{align}

and using Eqs. (7) and (A10) of Ref. [3] as well as Eq. (13), one obtains the general low-energy parametrization of the structure-dependent amplitudes $A_i$ to $\mathcal{O}(r^5)$ and to leading order in $1/M$. In the following these amplitudes are called $A_i^{HB}$, the superscript $HB$ referring to the $1/M$ expansion. For example, the expression for the spin-independent amplitude $A_i^{HB}$ reads

\begin{equation}
A_i^{HB} = \omega' \bar{q} \cos \theta (-f_{1,0} - 4M^2 f_{2,0}) \\
+ \omega' \bar{q} \cos \theta (f_{1,0}) \\
+ \omega' \bar{q} (f_{1,2a} - f_{1,2b} - 4M^2 f_{1,2c} - 4M^2 f_{2,2a} - 4M^2 f_{2,2b} - 16M^4 f_{2,2c} + 4M^2 f_{3,1}) \\
+ \omega' \bar{q} \cos \theta (2f_{1,2a} + f_{1,2b} + 4M^2 f_{1,2c} + 4M^2 f_{2,2a}) \\
+ \omega' \bar{q}^2 (f_{1,2b} + 4M^2 f_{2,2b} - 4M^2 f_{3,1}) \\
+ \omega' \bar{q}^2 \cos \theta (-f_{1,2a}) \\
+ \omega' \bar{q}^3 \cos \theta (-f_{1,2b}) + \mathcal{O}(r^6).
\end{equation}

As an example of the spin-dependent sector we quote the result for $A_3^{HB}$,
\[ A_3^{HB} = \omega'^3 \left( -8M^2 f_{4,1} - \frac{1}{2} f_{5,0} - 4M f_{10,1} - 4f_{11,0} \right) \\
+ \omega'^2 \bar{q} \cos \theta \left( 4M f_{10,1} + 4f_{11,0} \right) \\
+ \omega' \bar{q}^2 \left( \frac{1}{2} f_{5,0} \right) \\
+ \omega' \left( -8M^2 f_{4,3a} - 8M^2 f_{4,3b} - 32M^4 f_{4,3c} - \frac{1}{2} f_{5,2a} - \frac{1}{2} f_{5,2b} - 2M^2 f_{5,2c} \\
+ M f_{8,1} - 4M f_{10,3a} - 4M f_{10,3b} - 16M^2 f_{10,3c} - 4f_{11,2a} - 4f_{11,2b} - 16M^2 f_{11,2c} \right) \\
+ \omega' \bar{q} \cos \theta \left( 8M^2 f_{4,3a} + \frac{1}{2} f_{5,2a} - M f_{8,1} + 8M f_{10,3a} + 4M f_{10,3b} + 16M^2 f_{10,3c} \\
+ 8f_{11,2a} + 4f_{11,2b} + 16M^2 f_{11,2c} \right) \\
+ \omega'^3 \bar{q}^2 \left( 8M^2 f_{4,3b} + \frac{1}{2} f_{5,2a} + f_{5,2b} + 2M^2 f_{5,2c} - M f_{8,1} \\
+ 4M f_{10,3b} + 4f_{11,2b} \right) \\
+ \omega'^3 \bar{q}^2 \cos^2 \theta (-4M f_{10,3a} - 4f_{11,2a}) \\
+ \omega'^2 \bar{q}^3 \cos \theta \left( -\frac{1}{2} f_{5,2a} + M f_{8,1} - 4M f_{10,3b} - 4f_{11,2b} \right) \\
+ \omega' \bar{q}^4 \left( -\frac{1}{2} f_{5,2b} \right) + \mathcal{O}(r^7) \right). \tag{17} \]

Expressions for the complete set of the 12 amplitudes \( A_i^{HB} \) are listed in Appendix G of Ref. [18].

Let us briefly discuss the implications of Eqs. (16) and (17). In the spin-independent sector, the first non-vanishing contributions to the structure-dependent amplitudes appear at \( \mathcal{O}(r^2) \). However, the \( \mathcal{O}(r^2) \) terms in the structure-dependent part of VCS do not introduce any new constants as compared with RCS [13,17]. This can be seen from the RCS limit \( \bar{q} = \omega' \) of Eq. (13). Additional structures due to the virtuality of the initial-state photon only appear at \( \mathcal{O}(r^4) \). For example, in Eq. (13) the terms proportional to \( f_{1,2b} \) and \( f_{2,2b} \) [see Eq. (13)] only contribute in VCS and disappear for RCS. At leading order in the \( 1/M \) expansion there are no structures with odd powers of \( r \) in the spin-independent sector. This is no longer true at sub-leading orders as can be seen, e.g., from Eq. (14). Regarding the spin-dependent sector, e.g., Eq. (17), one obtains the first non-vanishing contributions at \( \mathcal{O}(r^3) \). Again we find modifications due to the virtuality of the initial photon at two orders higher in \( r \), i.e., there are no coefficients involving a subscript \( b \) at \( \mathcal{O}(r^3) \). Of course, at \( \mathcal{O}(r^5) \) the \( b \) coefficients drop out in RCS. Finally, to leading order in \( 1/M \) the spin-dependent sector does not contain any structures with even powers of \( r \).

It is certainly unrealistic to expect the full set of structure constants to be experimentally determined in the near future. Nevertheless, the parametrization of Eqs. (16) and (17) is valuable in the analysis of special kinematical situations in connection with a model calculation. This parametrization allows for an estimate of terms of higher order in \( \omega' \) which were discarded in the framework of Ref. [4]. Such terms will become important as one approaches the RCS limit \( \bar{q} \to \omega' \).

Using Eqs. (13)–(12) together with the kinematic expansions of Eq. (13) and truncating
the 1/M expansion at leading order, one obtains the ten GPs in terms of the structure constants. We only list the first two non-vanishing terms of the Taylor expansion in $q^2$ to leading order in 1/M:

\[
P_{GH}^{(0,1)}(q^2) = \sqrt{\frac{2}{3}} \left[ f_{1,0} + 4M^2 f_{2,0} + q^2 \left( -f_{1,2b} - 4M^2 f_{2,2b} \right) \right] + O(q^4),
\]

\[
P_{GH}^{(1,0)}(q^2) = -\sqrt{\frac{8}{3}} \left[ f_{1,0} - q^2 f_{1,2b} \right] + O(q^4),
\]

\[
P_{GH}^{(0,1)}(q^2) = \frac{4}{3} M \left[ f_{2,0} - q^2 f_{2,2b} \right] + O(q^4),
\]

\[
P_{GH}^{(0,1)}(q^2) = -\frac{4}{3} \sqrt{2} \left[ \left( M f_{6,0} + \frac{1}{2} f_{11,0} \right) + q^2 \left( -M f_{6,2b} - \frac{1}{8} f_{7,2b} - \frac{1}{2} M f_{9,2b} - \frac{1}{2} f_{11,2b} \right) \right] + O(q^4),
\]

\[
P_{GH}^{(1,0)}(q^2) = \frac{4}{3} \sqrt{2} \left[ f_{11,0} + q^2 \left( -\frac{1}{4} f_{7,2b} - f_{11,2b} \right) \right] + O(q^4),
\]

\[
P_{GH}^{(0,1)}(q^2) = q^2 \left( -\frac{1}{6M} f_{5,0} - \frac{2}{3M} f_{11,0} - \frac{2}{3} f_{12,0} \right) + \frac{1}{2} q^4 \left( \frac{1}{3M} f_{5,2b} + \frac{1}{3M} f_{7,2b} + \frac{4}{3M} f_{11,2b} + \frac{4}{3} f_{12,2b} \right) + O(q^6),
\]

\[
P_{GH}^{(1,0)}(q^2) = \frac{2}{\sqrt{3}} \left[ q^2 \left( \frac{1}{2} f_{5,0} + \frac{4}{3} f_{11,0} + 2M f_{12,0} \right) + q^4 \left( -\frac{1}{2} f_{5,2b} - \frac{1}{3} f_{7,2b} - \frac{4}{3} f_{11,2b} - 2M f_{12,2b} \right) \right] + O(q^6),
\]

\[
P_{GH}^{(1,1)}(q^2) = -\frac{1}{6M} \left[ q^2 f_{5,0} - q^4 f_{5,2b} \right] + O(q^6),
\]

\[
P_{GH}^{(1,2)}(q^2) = -\sqrt{\frac{2}{53}} \left[ -\frac{1}{4M} f_{5,0} - \frac{1}{M} f_{11,0} + q^2 \left( \frac{1}{4M} f_{5,2b} + \frac{1}{4M} f_{7,2b} + \frac{1}{M} f_{11,2b} \right) \right] + O(q^4),
\]

\[
P_{GH}^{(0,1)}(q^2) = \frac{1}{\sqrt{6}} \left[ -\frac{2}{3} f_{5,0} - \frac{8}{3} M f_{6,0} - \frac{4}{3} f_{11,0} + q^2 \left( \frac{2}{3} f_{5,2b} + \frac{8}{3} M f_{6,2b} + \frac{1}{3} f_{7,2b} + \frac{4}{3} M f_{9,2b} + \frac{4}{3} f_{11,2b} \right) \right] + O(q^4).
\]

A comparison between the above equations and the expressions for $A_i^{HB}$ reveals the possibility of directly calculating the GPs to leading order in 1/M from the amplitudes $A_i^{HB}$ of a given model by means of partial derivatives. Utilizing the fact that the $f_i$ taken at $\omega' = 0$ only depend on the variable $\tilde{q}^2$ and performing the Taylor series with respect to $\tilde{q}^2$, one can derive a set of six equations which, taken together with the four relations of Eq. (21) in Ref. [3], allow for a determination of all ten GPs from the $A_i^{HB}$. One possible representation of the set of six equations is
This technique, employing the kinematic expansions of Eqs. (13), avoids the tedious exercise of constructing the VCS amplitudes to quadratic order in \( \omega' \) for arbitrary \( \bar{q}^2 \).

### III. REAL COMPTON SCATTERING

In this section we show that a simultaneous kinematic expansion in terms of \( \omega' \) and \( \bar{q} \) allows for a well-defined transition to the low-energy expansion of the RCS amplitude. Such a transition is beyond the scope of the formalism of Ref. [2] which has been devised for a kinematics implying \( \bar{q} \gg \omega' \) and keeping only terms linear in \( \omega' \).

In the Coulomb gauge, the initial and the final state photons of RCS are purely transverse. Consequently, the invariant matrix element is also purely transverse. Starting with Eq. (13) of Ref. [3] and imposing the constraints of time-reversal invariance \( (A_5 = A_7, A_6 = A_8) \), the invariant matrix element is decomposed into six amplitudes,

\[
\vec{\varepsilon}_T \cdot \vec{M}_T = \varepsilon'^{\ast} \cdot \vec{\varepsilon}_T A_1 + \varepsilon'^{\ast} \cdot \hat{q} \vec{\varepsilon}_T \cdot \hat{q}' A_2 \\
+ i\vec{\sigma} \cdot (\varepsilon'^{\ast} \times \vec{\varepsilon}_T) A_3 + i\vec{\sigma} \cdot (\hat{q}' \times \hat{q}) \varepsilon'^{\ast} \cdot \vec{\varepsilon}_T A_4 \\
+ [i\vec{\sigma} \cdot (\varepsilon'^{\ast} \times \hat{q}) \vec{\varepsilon}_T \cdot \hat{q}' - i\vec{\sigma} \cdot (\vec{\varepsilon}_T \times \hat{q}') \varepsilon'^{\ast} \cdot \hat{q}] A_5 \\
+ [i\vec{\sigma} \cdot (\varepsilon'^{\ast} \times \hat{q}') \vec{\varepsilon}_T \cdot \hat{q}' - i\vec{\sigma} \cdot (\vec{\varepsilon}_T \times \hat{q}) \varepsilon'^{\ast} \cdot \hat{q}] A_6 .
\]

Recently, it has been demonstrated by Ragusa [19] that the structure-dependent part of the RCS amplitude can be parametrized in terms of six polarizabilities if one restricts the expansion of the RCS amplitude to third order in the photon energy. Ragusa’s analysis was performed in the Breit frame. In the c.m. frame \( \omega = \bar{q} = \omega' = |\bar{q}'| \), the RCS amplitudes, expanded to \( \mathcal{O}(\omega^3) \), may be expressed by the coefficients of Eq. (13) as

\[
A_1 = \omega^2 \left[ - (1 - \cos \theta) f_{1,0} - 4M^2 f_{2,0} \right] + \omega^3 \left[ -4M (1 + \cos \theta) f_{2,0} \right] , \\
A_2 = -\omega^2 f_{1,0} + 4M f_{2,0} \omega^3 , \\
A_3 = \omega^3 \left[ -8M^2 f_{4,1} + (1 - \cos \theta) (4M f_{10,1} - 4 f_{11,0}) \right] , \\
A_4 = 4M f_{10,1} \omega^3 ,
\]
\[ A_5 = \omega^3 [-4Mf_{10,1} - 2f_{11,0}], \]
\[ A_6 = \omega^3 [4Mf_{6,0} + 2f_{11,0}]. \]  

(35)

A comparison with Ragusa’s definitions yields the relations

\[
\alpha = \frac{e^2}{4\pi} (-f_{1,0} - 4M^2 f_{2,0}),
\]
\[
\beta = \frac{e^2}{4\pi} f_{1,0},
\]
\[
\gamma_1 = \frac{e^2}{4\pi} (-8M^2 f_{4,1} - 4Mf_{10,1} - 4f_{11,0}),
\]
\[
\gamma_2 = \frac{e^2}{4\pi} 4Mf_{10,1},
\]
\[
\gamma_3 = \frac{e^2}{4\pi} (4Mf_{6,0} + 2f_{11,0}),
\]
\[
\gamma_4 = -\frac{e^2}{4\pi} (4Mf_{10,1} + 2f_{11,0}).
\]

(36)

where \( \alpha \) and \( \beta \) are the conventional electric and magnetic polarizabilities and \( \gamma_1 \) to \( \gamma_4 \) Ragusa’s spin polarizabilities. We note that the Lorentz transformation from the Breit frame to the c.m. frame generates terms of order \( O(\omega^3) \) in the spin-independent part of the amplitude, which are \( 1/M \) suppressed compared with the corresponding \( O(\omega^2) \) terms and do not contain any additional structure constants [cf. the RCS limit of Eq. (16)]. We repeat that the RCS amplitude does not receive any contribution from the invariant functions \( f_3, f_5, f_8, \) and \( f_{12} \).

Using Eqs. (3)–(12) in combination with the expansion of the functions \( f_i \), Eq. (13), one obtains from Eq. (36) the following relations between Ragusa’s polarizabilities and the GPs at \( \bar{q} = 0 \):

\[
\alpha = -\frac{e^2}{4\pi} \sqrt{\frac{3}{2}} P^{(01)0}(0),
\]
\[
\beta = -\frac{e^2}{4\pi} \sqrt{\frac{3}{8}} P^{(11)0}(0),
\]
\[
\gamma_3 = -\frac{e^2}{4\pi} \frac{3}{\sqrt{2}} P^{(01)1}(0),
\]
\[
\gamma_2 + \gamma_4 = -\frac{e^2}{4\pi} \frac{3\sqrt{3}}{2\sqrt{2}} P^{(11)1}(0).
\]

(37)

We now use the fact that only four of the seven spin-dependent GPs are independent (see Eqs. (21) of Ref. [3]) and the result that at \( \bar{q} = 0 \) the model-independent relations (22) and (23) of Ref. [3] hold. By means of Eqs. (18) - (27) and Eq. (36) one then finds, in the limit \( \bar{q} \rightarrow 0 \), that \( f_{5,0} \) is the only VCS term not determined by RCS or model-independent constraints,

\[
f_{5,0} = 3M \sqrt{\frac{5}{2}} \hat{P}^{(11)1}(0) - \frac{3\sqrt{6}}{4} \hat{P}^{(01)1}(0) + \frac{4\pi}{e^2} (\gamma_2 + \gamma_4 - \frac{1}{2} \gamma_3).
\]

(38)
Note that all these results can also be read off the heavy-baryon expansions of Eqs. (18)–(27). Since higher orders in the $1/M$ expansion only affect kinematic terms beyond the leading order, the results for the GPs at $\bar{q} = 0$ obtained in the heavy-baryon framework are true for any order in $1/M$.

The multipole decomposition of the amplitudes $A_i$ reads

$$A_1 = -\left[ \sqrt{\frac{3}{8}} \cos \theta H^{(11,11)0}(\omega) + \sqrt{\frac{3}{8}} H^{(21,12)0}(\omega) \right] + \mathcal{O}(\omega^4),$$

$$A_2 = \sqrt{\frac{3}{8}} H^{(11,11)0}(\omega) + \mathcal{O}(\omega^4),$$

$$A_3 = -\left[ \frac{3}{4} H^{(11,11)1}(\omega) + \frac{3}{2} H^{(11,22)1}(\omega) + \cos \theta \frac{3}{4} H^{(11,11)1}(\omega) + \cos \theta \frac{3}{2} H^{(11,22)1}(\omega) \right] + \mathcal{O}(\omega^4),$$

$$A_4 = -\left[ \frac{3}{4} H^{(11,11)1}(\omega) - \frac{3}{2} H^{(11,22)1}(\omega) \right] + \mathcal{O}(\omega^4),$$

$$A_5 = \frac{3}{4} H^{(11,11)1}(\omega) + \mathcal{O}(\omega^4),$$

$$A_6 = \frac{3}{2} H^{(21,12)1}(\omega) + \mathcal{O}(\omega^4),$$

where we used the notation of Ref. [2] and only kept those multipoles contributing up to $\mathcal{O}(\omega^3)$. Furthermore, we have exploited time-reversal invariance which results in the two relations

$$H^{(21,12)1}(\omega) = H^{(12,21)1}(\omega),$$

$$H^{(11,22)1}(\omega) = H^{(22,11)1}(\omega).$$

Comparing Eqs. (35) with (39) and using the definitions of Eqs. (36), the multipole content of Ragusa’s six polarizabilities is then

$$\alpha = -\frac{e^2}{4\pi} \sqrt{\frac{3}{8} \frac{d^2}{d\omega^2}} H^{(21,12)0}(0),$$

$$\beta = -\frac{e^2}{4\pi} \sqrt{\frac{3}{8} \frac{d^2}{d\omega^2}} H^{(11,11)0}(0),$$

$$\gamma_1 = \frac{e^2}{4\pi} \frac{d^3}{d\omega^3} \left[ H^{(21,12)1}(0) + 2H^{(21,12)1}(0) \right],$$

$$\gamma_2 = \frac{e^2}{4\pi} \frac{d^3}{d\omega^3} \left[ H^{(11,11)1}(0) - 2H^{(11,22)1}(0) \right],$$

$$\gamma_3 = \frac{e^2}{4\pi} \frac{d^3}{d\omega^3} H^{(21,12)1}(0),$$

$$\gamma_4 = \frac{e^2}{4\pi} \frac{d^3}{d\omega^3} H^{(11,11)1}(0).$$

In summary, only two linear combinations of the four spin-dependent RCS polarizabilities as defined by Ragusa [19,20] can be related to the GPs of Ref. [2]. At first sight this behavior is somewhat surprising, since all multipoles appearing at lowest order in $\omega$ in the spin-dependent RCS amplitude, $H^{(11,11)1}$, $H^{(11,22)1}$, $H^{(21,21)1}$, and $H^{(21,12)1}$, are also present.
in the kinematical limit of Ref. [2]. However, one can expect that the spin polarizabilities not related to the GPs of Guichon et al. can be obtained from the VCS multipoles by higher derivatives in $\omega'$.

**IV. FORWARD SCATTERING**

Finally, we consider the special case of forward VCS ($\theta = 0, \hat{q} = \hat{q}'$) for which the amplitude reads (see Eqs. (13) and (14) of Ref. [3])

$$\bar{\epsilon}_T \cdot \bar{M}_T = \bar{\epsilon}'^{ts} \cdot \bar{\epsilon}_T A_1 + i \bar{\sigma} \cdot \bar{\epsilon}_T A_3,$$

$$M_2 = i \bar{\sigma} \cdot \bar{\epsilon}_T' \times \bar{\epsilon}_T A_1.$$

Again, we can express the VCS amplitudes in terms of the functions $f_i$ and, using the relations (21) of Ref. [3], obtain to leading order in $\omega'$

$$A_1(\omega', \bar{q}) = -\sqrt{\frac{3}{8}} \omega' \sqrt{\frac{E_i}{\bar{M}}} (\bar{q} - \omega_0) P^{(11,11)0}(\bar{q}^2) + \mathcal{O}(\omega^2),$$

$$A_3(\omega', \bar{q}) = \frac{3}{2} \omega' \sqrt{\frac{E_i}{\bar{M}}} (\bar{q} - \omega_0) \bar{q} \omega_0 P^{(11,11)1}(\bar{q}^2) + \mathcal{O}(\omega^2),$$

$$A_{11}(\omega', \bar{q}) + A_{12}(\omega', \bar{q}) = \frac{3}{2} \omega' \sqrt{\frac{E_i}{\bar{M}}} (\bar{q} - \omega_0) P^{(01,01)1}(\bar{q}^2) + \mathcal{O}(\omega^2).$$

Finally, we quote a well-known result for the non-Born contribution to RCS. The two terms surviving in forward direction contain the sum of the electric and magnetic polarizability, $\alpha + \beta$, and the forward spin (or vector) polarizability $\gamma$,

$$A_1 = \frac{4\pi}{e^2} \omega^2 (\alpha + \beta) + \mathcal{O}(\omega^3),$$

$$A_3 = \frac{4\pi}{e^2} \omega^3 \gamma + \mathcal{O}(\omega^4),$$

where

$$\gamma = \gamma_1 - \gamma_2 - 2\gamma_4$$

$$= \frac{e^2}{4\pi} (-8M^2 f_{4,1})$$

$$= -\frac{e^2}{4\pi} \frac{1}{8} d^3 \left[ H^{(21,21)1}(0) + H^{(11,11)1}(0) + 2H^{(21,12)1}(0) + 2H^{(11,22)1}(0) \right].$$

We note, in particular, that this $\gamma$ is not contained in the kinematical limit of Ref. [2].

**V. SUMMARY**

We have discussed the structure-dependent, non-pole part of the virtual Compton scattering amplitude to be investigated in $ep \rightarrow e'p'\gamma$ experiments below pion-production threshold [1]. The most general parametrization requires 12 functions of three variables. We
have related the GPs of Guichon et al. — representing a truncated low-energy multipole expansion—to the 12 invariant functions \( f_i \) of the covariant approach recently discussed in Ref. [2]. Based upon photon-crossing symmetry and the combination of nucleon-crossing with charge-conjugation symmetry, we have expanded the invariant functions \( f_i \) through \( \mathcal{O}(k^3) \). We have then performed a low-energy and low-momentum expansion in the \( p/\gamma \) c.m. frame through \( \mathcal{O}(r^4) \) and \( \mathcal{O}(r^5) \) in the spin-independent and spin-dependent sectors, respectively. Such an expansion covers the transition region between real Compton scattering and the regime to be investigated in the scheduled photon electroproduction experiments, including the leading-order effects due to the virtuality of the photon. On top of the kinematic expansion we have expanded the c.m. amplitudes to leading order in \( 1/M \), thus providing a basis for a direct comparison with the predictions of HBChPT.

The combined low-energy and small-momentum expansion allows for a smooth transition from VCS to RCS. In the spin-independent sector and for \( \bar{q} \to 0 \), the two independent GPs are related to the RCS electromagnetic polarizabilities \( \alpha \) and \( \beta \) [2]. In the spin-dependent sector, only two of the four spin polarizabilities of Ragusa [19,20] can be related to spin-dependent GPs in that limit. On the other hand, there exists one combination which remains finite and is not entirely given in terms of the RCS polarizabilities of Refs. [19,20]. Finally, we discussed the forward scattering amplitude and found that there is no relation between the forward spin polarizability \( \gamma \) and the GPs of [2].

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REFERENCES

[1] G. Audit et al., CEBAF Report No. PR 93-050, 1993; J. F. J. van den Brand et al., CEBAF Report No. PR 94-011, 1994; G. Audit et al., MAMI proposal “Nucleon Structure Study by Virtual Compton Scattering,” 1995; J. Shaw et al., MIT Bates proposal 97–03, 1997.

[2] P. A. M. Guichon, G. Q. Liu, and A. W. Thomas, Nucl. Phys. A591, 606 (1995).

[3] D. Drechsel, G. Knöchlein, A. Yu. Korchin, A. Metz, and S. Scherer, Phys. Rev. C 57, 941 (1998).

[4] G. Q. Liu, A. W. Thomas, and P. A. M. Guichon, Aust. J. Phys. 49, 905 (1996).

[5] M. Vanderhaeghen, Phys. Lett. B 368, 13 (1996).

[6] A. Metz and D. Drechsel, Z. Phys. A 356, 351 (1996).

[7] T. R. Hemmert, B. R. Holstein, G. Knöchlein, and S. Scherer, Phys. Rev. D 55, 2630 (1997).

[8] T. R. Hemmert, B. R. Holstein, G. Knöchlein, and S. Scherer, Phys. Rev. Lett. 79, 22 (1997).

[9] A. Metz and D. Drechsel, Z. Phys. A 359, 165 (1997).

[10] M. Kim and D.-P. Min, Seoul National University report SNUTP-97-046, 1997, hep-ph/9704381.

[11] B. Pasquini and G. Salmè, Report No. INFN-ISS 97/17, 1997, nucl-th/9802079.

[12] H. W. Fearing and S. Scherer, TRIUMF report No. TRI-PP-96-28, Mainz report No. MKPH-T-96-18, nucl-th/9607050, 1996, to appear in Few-Body Systems.

[13] D. Drechsel, G. Knöchlein, A. Metz, and S. Scherer, Phys. Rev. C 55, 424 (1997).

[14] B. R. Holstein, Comments Nucl. Part. Phys. 20, 301 (1992).

[15] A. I. L’vov, Int. J. Mod. Phys. A 8, 5267 (1993).

[16] R. A. Berg and C. N. Lindner, Nucl. Phys. 26, 259 (1961).

[17] S. Scherer, A. Yu. Korchin, and J. H. Koch, Phys. Rev. C 54, 904 (1996).

[18] G. Knöchlein, Virtual Compton Scattering off the Nucleon and Chiral Perturbation Theory, PhD thesis (Shaker Verlag, Aachen, 1997).

[19] S. Ragusa, Phys. Rev. D 47, 3757 (1993).

[20] S. Ragusa, Phys. Rev. D 49, 3157 (1994).