Spherically Symmetric Inflation

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Abstract — It is shown in this letter that in the framework of an inhomogeneous geometry and a massive non self-interacting scalar field with spherical symmetry, one needs a homogeneous patch bigger than a dizaine of horizons in order to start inflation. The results are completely independent of initial conditions on the spatial distribution of the scalar field. The initial condition on the metric parameters are also justified. This is a generalization of the results obtained in Ref. [1], showing that their conclusions are rather robust.

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I. INTRODUCTION

The inflation paradigm [2] is an attempt to solve some of the drawbacks of the Standard Cosmological Model, described by the homogeneous and isotropic Friedmann-Lemaitre-Robertson-Walker (FLRW) geometries,

$$ds^2 = -dt^2 + \frac{a(t)^2}{1 + \frac{kr^2}{2}}[dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)]$$,

where the curvature $k$ of the spacelike hypersurfaces can take the values $0, 1, -1$.

These problems are related to the very special initial conditions associated with this model. The first one is called the flatness problem: the present energy density of the Universe is observed to be very close to the critical density, and it must have been much closer in the past, implying that the observed homogeneous and isotropic space must be almost flat. This problem comes from the classical equation giving the density $\rho(t)$ relative to the critical one, $\rho_c(t) = 3H^2(t)/(8\pi G)$, where $H = \dot{a}/a$ is the Hubble expansion rate, which reads

$$\frac{d[\Omega - 1]}{dt} = -\frac{\dot{\Omega}}{\Omega^3},$$

where $\Omega \equiv \rho/\rho_c$. As $\Omega$ is close to unity now, Eq. [2] implies that it must have been arbitrarily closer to unity in the past in the usual Friedmann model, where one has a decelerated ($\dot{a} < 0$) expansion ($\dot{a} > 0$) since the initial singularity, and hence $|\Omega - 1|$ must have been an ever increasing function of time. In other words, $\Omega = 1$ is an unstable point in this model, and to have our old Universe still near this point implies an incredibly fine tuned value for $\Omega$ around the Plank era, $\Omega = 1 \pm 10^{-50}$.

The second problem is much more involved, and concerns the use of the very special homogeneous and isotropic geometry [1] in order to describe our Universe. Either one has in hands a theory of initial conditions (perhaps quantum cosmology) which selects this particular geometry from the possible infinite many inhomogeneous solutions of the Einstein’s equations, or there was a physical mechanism of homogeneization and isotropization which has taken place in the early Universe leading the spacetime geometry to the form of Eq. (1). In this paper, we will be concerned with the second approach.

The first basic assumption one must take for a physical homogeneization and isotropization of the Universe is the requirement that its observed parts had some causal contact sometime in the past. However, if the Universe had a begining some 14 billion years ago, it can be shown that by the time of recombination, when the cosmic background radiation began to propagate independently from matter, one had approximately 100 regions without causal contact presenting the cosmic radiation already with the same temperature. This is the so called horizon problem. Only after taking care of this issue, one can begin to think about a physical mechanism of homogeneization and isotropization.

The flatness and horizon problems can be solved by inflation. It consists of the idea that the early Universe experienced a brief but violent accelerated expansion, which turned $\Omega$ very close to one at this time (see Eq. (2)), and sufficiently enlarged a small causally connected piece of the Universe to its observed size. Furthermore, it induces a mechanism for the origin of matter fluctuations, which gave rise to structure formation [3].

However, it seems that the homogeneity problem is not solved by inflation. In fact, there are works showing that in order for inflation to start in some region, one needs a homogeneous patch of a few horizons size at this region [1]. In order to circumvent this problem, some people evoke the Anthropic Principle [6]; in the regions where inflation does not happen, it is not possible to have galaxies, stars, and hence, intelligent life. Nevertheless, if one wants to rely on this Principle, one could argue in the same way to justify the FLRW geometries [1] without necessity of any period of inflation: the difference would be the need of a much bigger homogeneous region containing a huge number of horizons to begin with, with the appropriate initial perturbations printed on it. However, in an infinite inhomogeneous Universe, at least one
of such regions would exist, and the Anthropic Principle could select it. Without the Anthropic Principle, the role of inflation would then be to reduce drastically the size of the initial homogeneous region, increasing the number of possible regions which can behave like our Universe. In that case, one would then need a more precise knowledge of the possible cosmological scenarios after the Planck era, and a measure of such regions, which would lead us back to the first approach to solve the homogeneity problem: a theory of initial conditions. In such a situation, inflation could alleviate but it would not solve alone this issue.

The homogeneity problem in the inflationary scenario is usually investigated numerically [2, 4, 5], but there are some analytical studies in the literature [10, 13]. One of them [1], deals with an inhomogeneous geometry and a massive non-self-interacting scalar field with spherical symmetry in order to derive the size of the homogeneous patch necessary to yield sufficient inflation, under the assumption of certain initial conditions for the geometry and scalar field. In this letter, we show that the results obtained in Ref. [1] are independent of the choice of initial conditions for the scalar field, and we try to justify some of the choices related to the spacetime geometry. Hence, the result implying the necessity of a homogeneous patch bigger than a dizaine of horizons in order to start inflation is rather robust in this framework.

Note that we never make any splitting of the geometry on a homogeneous background and small perturbations around it. We deal with full general relativity and all their non-linear equations: deviations from homogeneity are not considered to be small. This is because we are interested on the homogeneity problem itself, and restricting ourselves to fluctuations on a homogeneous geometry is not satisfactory as one would be assuming homogeneity from the beginning. The treatment of small linear fluctuations around inflationary models is made in many other papers in order to study the evolution of structures and the back-reaction problem once inflation has started. Here we are interested in the more basic question concerning how inflation itself begins in a spherically symmetric inhomogeneous geometry.

In the next section we present the model, in section III we discuss the initial conditions, and in section IV we present our results concerning inflation. We end this letter with conclusions and comments.

II. THE MODEL

The model consists of an asymptotically flat universe with open spatial sections and spherical symmetry. We write a spherically symmetric geometry in the following form:

$$ds^2 = e^{2\alpha(\eta, r)}[-d\eta^2 + dr^2 + e^{2\beta(\eta, r)}r^2 d\Omega],$$

where $\eta$ is the time coordinate, and $\alpha$ and $\beta$ are arbitrary dimensionless functions of $\eta$ and $r$. The matter content is a massive scalar field minimally coupled to gravity, whose energy-momentum tensor is given by

$$T_{\mu\nu} = \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} [\Phi^{,\alpha} \Phi_{,\alpha} + m^2 \Phi^2].$$

We shall use units where $16\pi = G = 1$, with $\eta$ and $r$ having dimensions of length, while $m$ and $\Phi$ have dimensions of inverse length. From the expressions above, the Einstein’s equations yield two dynamical equations and two constraint equations concerning the functions $\alpha$ and $\beta$. The constraint equations are the $(00)$ and $(01)$ components of the Einstein tensor, while the dynamical equations involve the $(11)$ and $(22)$ ones. The other components are trivial due to the spherical symmetry of the model, except for the equation involving the $(33)$ component, which is equal to the $(22)$ one. Note that the right-hand-side of the following equations miss a length factor due to our choice of units ($G = 1$). The constraint equations are, therefore,

$$3 \alpha'' + \beta'' - 4 \alpha' \beta' + 2 \alpha'' - 2 \alpha'' - \alpha'' - 3 \beta'' +$$

$$+ 4 \alpha' \beta' + 6 r^{-1} \beta' - 4 r^{-1} \alpha' - r^{-2} (1 - e^{2\beta}) =$$

$$= \frac{1}{4} (\Phi^2 + \Phi'^2 + m^2 e^{2\alpha} \Phi^2)$$

and

$$\alpha' - r^{-1} \beta' - \beta' - \alpha \alpha' + \beta \beta' = \frac{1}{4} \Phi'^2,$$

where the primes and dots denote derivatives with respect to the radial and time coordinates, respectively. We will use combinations of the equations involving the $(22)$ and $(11)$ components as our dynamical equations. The first one is the difference between the $(22)$ and $(11)$ equations, namely

$$r^{-2} (1 - e^{2\beta}) + 2 r^{-1} \alpha' + 2 \alpha'' - 2 \alpha'' +$$

$$+ \beta' + \beta'' - 2 \alpha' \beta' - 2 \beta^2 = \frac{1}{2} \Phi'^2,$$

while the second is the difference between twice the $(22)$ equation and the $(11)$ one:

$$- 2 \alpha' + 4 \alpha'' - \alpha' + \alpha'' - 2 r^{-1} \beta' - 2 \beta'' + \beta^2 +$$

$$\beta'' - r^{-2} (1 - e^{2\beta}) = \frac{1}{4} (\Phi'^2 - 3 \Phi'^2 + m^2 e^{2\alpha} \Phi^2).$$

To complete the set, the Klein-Gordon equation for the scalar field is given by

$$\ddot{\Phi} - \Phi'' + 2 (\ddot{\alpha} - \ddot{\beta}) \Phi - 2 (\alpha' - \beta' - r^{-1}) \Phi' + m^2 e^{2\alpha} \Phi = 0.$$
Note that the time evolution given by the Einstein-Klein-Gordon dynamics will subsequently modify these values.

The initial conditions \( \beta = \beta' = 0 \) are in accordance with the Weyl curvature hypothesis \[1\], which states that the Universe should be initially conformally flat. This condition relies on the assumption that the gravitational entropy should be given by the Weyl tensor, and stating that it was zero in the beginning should be equivalent to say that the Universe began in a state of minimum entropy, as Boltzmann conjecture in order to justify the arrow of time. However, up to now, this is just a speculation.

The choice of the initial condition \( \dot{\Phi} = 0 \) at \( \eta = \eta_0 \) relies on the fact that the kinetic term of a scalar field decays exponentially fast in an expanding geometrical patch, at least in the framework of chaotic inflation (see Ref. \[5\], page 267, and references therein). Furthermore, it seems to us that considering \( \dot{\Phi} \neq 0 \) at \( \eta = \eta_0 \) will make things worst for the occurrence of inflation, as long as it can happen only when the kinetic term is negligible with respect to the potential term. Hence, the final result of our paper should be considered as the minimum requirement for the occurrence of inflation in the framework of spherically symmetric general relativity under the Weyl curvature hypothesis.

Concerning the initial shape of the scalar field, we will try do keep this as general as possible, so we can write the following equation for \( \Phi \) at time \( \eta_0 \)

\[
\Phi(\eta_0, r) = \Phi_0 + f(r), \tag{10}
\]

where \( \Phi_0 \) stands for the homogeneous part of the inflation field and \( f(r) \) is an arbitrary function of \( r \) which is asymptotically null at spatial infinity. With this in hand, in the hypersurface \( \eta = \eta_0 \), equation (10) reduces to

\[
\ddot{\alpha}' - \dot{\alpha}\alpha' = 0. \tag{11}
\]

The solution of this equation is \( \dot{\alpha} = Ce^{\alpha} \). It can be easily shown that the constant \( C \) is fixed by constience requirements between the two constraint equations when we go far from the origin. Hence, we obtain

\[
\dot{\alpha} = \frac{1}{2\sqrt{3}} m \Phi_0 e^{\alpha}. \tag{12}
\]

Putting this result into equation (11), we get the radial evolution for \( \alpha \),

\[
2\alpha'' + \alpha'^2 + 4\alpha'^{-1} \alpha' = -\frac{1}{4} (\Phi'' + m^2 (\Phi^2 - \Phi_0^2) e^{2\alpha}). \tag{13}
\]

Near the spatial origin \( r \to 0 \), equation (13) has a power law solution, which is

\[
\alpha \sim \ddot{\alpha} - \frac{1}{24} \left[ m^2 \Phi_0 e^{2\alpha} f + \frac{1}{2} (f' + f^2) \right] r^2, \tag{14}
\]

where the bars above \( f \) and \( \alpha \) denote the values of these functions at \( r = 0 \). Those are the expressions that we will use from now on. It can be shown [1] that equation (14) has an exact solution, and hence the Cauchy initial data is self-consistent.

### IV. CONDITIONS FOR INFLATION

First of all, we remind the reader that there are conditions on the value of the homogeneous field related to the observations [3]. The fluctuations on the cosmic microwave background puts a limit on \( \Phi_0 \), which in our units is \( m\Phi_0^2 < 10^{-4} \times 8\sqrt{3} \). On the other hand, the requirement of 70 e-folds of sufficient inflation sets a minimum value to the scalar field, namely \( \Phi_0 > \sqrt{561} \). Now we move to the conditions of our model.

From equation (7), dropping the \( \beta \) and \( \beta' \) terms according to our choice of initial conditions, we get, near the origin,

\[
\ddot{\beta} = \frac{1}{2} \Phi'^2 \sim \frac{1}{2} f'^2, \tag{15}
\]

and from this expression we can say that \( \beta \) will be small for physical times around

\[
\tau_\beta \sim \frac{\alpha}{f'}. \tag{16}
\]

The first condition for inflation is that \( \beta \) remains small during one e-fold, i.e., \( \tau_\beta > H^{-1} \). Once inflation has started, any anisotropy will decay according to the “no-hair” conjecture, so \( \beta \) will remain small for the rest of the process. We will see in the following lines that this condition is less restrictive than the others related to the function \( \alpha \) of the metric.

From the evolution equation (5) we have

\[
2\ddot{\alpha} - 4\alpha'' + \ddot{\alpha}^2 + \alpha'^2 = \frac{1}{4} (3\Phi'^2 + m^2 e^{2\alpha} \Phi^2). \tag{17}
\]

Using the radial solution (14) for \( \alpha \), we get the dynamical equation

\[
8\ddot{\alpha} + 4\alpha'' = 3f'^2 + 16\alpha'' + m^2 (\Phi_0 f + f^2) e^{2\alpha} + \Phi_0^2 m^2 e^{2\alpha}\tag{18}
\]

because, according to this solution, the term \( \alpha'^2 \) drops off when we are near the origin. The last term on the r.h.s. is the potential of the homogeneous field. In equation (18), we require that the potential dominates the dynamics over the inhomogeneities given by \( \alpha \) and \( f \). This requirement leads us to the relations near the origin

\[
\Phi_0 e^\alpha > \frac{4\sqrt{\alpha''} m}{f}, \quad \Phi_0 > \sqrt{\Phi_0 f + f^2}
\]

and

\[
\frac{\Phi_0}{f'} e^\alpha > \sqrt{\frac{3}{m}}. \tag{19}
\]

Reminding the reader that, near the origin, these conditions involve only constants, Defining the inhomogeneity physical scale as \( D \sim |\Phi_0 f'| e^\alpha \), we get, from the last inequality above, remembering that \( \Phi' = f' \), that the size of the initial inhomogeneity must be

\[
D > \sqrt{\frac{3}{m}}. \tag{20}
\]
The mass of the field is related to the horizon scale $H^{-1}$ during inflation through the Friedmann equation for inflationary expansion

$$H^2 = \frac{1}{6} V(\Phi_0) \rightarrow \frac{1}{m} \sim \frac{1}{\sqrt{12}} \Phi_0 H^{-1}. \quad (21)$$

Having fulfilled every requirement listed above, we are now able to conclude that, from the conditions for sufficient inflation and from the equations (20) and (21), the initial perturbation must be larger than a few horizons, or more specifically

$$D > 11,83 H^{-1}. \quad (22)$$

We recall here that this result is independent of the initial shape of the inhomogeneous inflaton field. The only requirements for this result are the initial conditions $\dot{\Phi} = \beta = \dot{\beta} = 0$ at the hypersurface $\eta_0$. The condition for $\beta$ is consequently satisfied as, from the definition of $D$ and (16),

$$\tau_{\beta} \sim \frac{D}{\Phi_0}, \quad (23)$$

which gives us $\tau_{\beta} > 3,54 H^{-1}$.

V. CONCLUSION

In the framework of an inhomogeneous spherically symmetric geometry and massive non self-interacting scalar field, we have shown that in order to obtain enough inflation to solve some of the problems of the standard cosmological model, one needs a homogeneous region with at least 11.83 horizons. This result was obtained without any assumption about the initial spatial configuration of the scalar field, except for the fact that it should be null at spatial infinity, and hence it is a generalization of the results obtained in Ref. [1], where it was assumed a particular initial exponential spatial distribution for the scalar field. We had also pointed out that the initial geometric configuration may have a physical justification as long as it satisfies the Weyl curvature hypothesis [14], and that the kinetic term of the scalar field should be negligible at the initial hypersurface. Hence, the necessary conditions for the occurrence of inflation in a spherically symmetric geometry with a massive free scalar field under the Weyl curvature hypothesis are that there should be an homogeneous region of about 12 horizons size, and that the kinetic term of the scalar field should be zero.

Our results shows that the conclusions of Ref. [1] are rather robust and they are in accordance with other results [3]. Note that we have already assumed a special symmetry for the fields, which is spherical symmetry. It is expected that in the general case, without imposing any symmetry at all from the beginning, the situation may become worst, although no calculation has been done in this framework. Hence our results, together with the results of other works [1, 5], indicates that inflation is not sufficient to solve the homogeneity problem: a theory of initial conditions is imperative in order to solve this deep issue of cosmology.

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[1] E. Calzetta and M. Sakellariadou, Phys.Rev. D45 2802 (1992).
[2] A. A. Starobinsky, Pis’ma Zh. Eksp. Teor. Fiz. 30, 719 (1979) [JETP Lett. 30, 682 (1979)]; V. Mukhanov and G. Chibisov, JETP Lett. 33, 532 (1981); A. Guth, Phys. Rev. D23, 347 (1981); A. Linde, Phys. Lett. B 108, 389 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).
[3] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. 215, 203 (1992).
[4] R. Wald, General Relativity, Univ. Chicago Press (1984).
[5] D. S. Goldwirth and T. Piran, Phys.Rep. 214 223 (1992).
[6] A. Linde, "Inflation, Quantum Cosmology and the Anthropic Principle", hep-th/0211048.
[7] P. Laguna, H. Kurki-Suonio and R. A. Matzner, Phys.Rev. D44 3077 (1991).
[8] H. Kurki-Suonio, P. Laguna and R. A. Matzner, Phys.Rev. D48 3611 (1993).
[9] D. Goldwirth and T. Piran, Phys.Rev.Lett. 64 2852 (1990).
[10] J. A. Stein-Schabes, Phys.Rev. D35 2345 (1987).
[11] N. Deruelle and D. S. Goldwirth, Phys. Rev. D51, 1563 (1995).
[12] T. Vachaspati and M. Trodden, Phys.Rev. D61 023502 (1999).
[13] A. Berera and C. Gordon, Phys.Rev. D63 063505 (2001).
[14] R. Penrose, in General Relativity: An Einstein Centenary Survey, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979), p. 581; N. Pelavas and K. Lake, Phys. Rev. D 62, 044009 (2000).