Robust and Private Learning of Halfspaces

Badih Ghazi  Ravi Kumar  Pasin Manurangsi  Thao Nguyen
Google Research
Mountain View, CA
{badihghazi, ravi.k53}@gmail.com, {pasin, thaotn}@google.com

Abstract

In this work, we study the trade-off between differential privacy and adversarial robustness under $L_2$-perturbations in the context of learning halfspaces. We prove nearly tight bounds on the sample complexity of robust private learning of halfspaces for a large regime of parameters. A highlight of our results is that robust and private learning is harder than robust or private learning alone. We complement our theoretical analysis with experimental results on the MNIST and USPS datasets, for a learning algorithm that is both differentially private and adversarially robust.

1 Introduction

In this work, we study the interplay between two topics at the core of AI ethics and safety: privacy and robustness.

As modern machine learning models are trained on potentially sensitive data, there has been a tremendous interest in privacy-preserving training methods. Differential privacy (DP) [33, 35] has emerged as the gold standard for rigorously tracking the privacy leakage of algorithms in general (see, e.g., 37, 95 and the references therein), and machine learning models in particular (e.g., [1]), resulting in several practical deployments in recent years (e.g., 40, 87, 50, 5, 33, 2).

Another vulnerability of machine learning models that has also been widely studied recently is with respect to adversarial manipulations of their inputs at test time, with the intention of causing classification errors, e.g., [29, 17, 93, 47, 76]. Numerous methods have been proposed with the goal of training models that are robust to such adversarial attacks, e.g., [69, 48, 49, 85], which in turn has led to new attacks being devised in order to fool these models [6, 25, 88]. See [62] for a recent tutorial on this topic.

Some recent work has suggested incorporating mechanisms from DP into neural network training to enhance adversarial robustness [66, 77, 78]. Given this existing interplay between DP and robustness, we seek to answer the following natural question:

**Is achieving privacy and adversarial robustness harder than achieving either criterion alone?**

Recent empirical work has provided mixed response to the question [90, 89, 52], reporting the success rate of membership inference attacks as a heuristic measure of privacy. Instead, using theoretical analysis and the strict guarantees offered by DP, we formally investigate this question in the classic setting of halfspace learning, and arrive at a near-complete picture.

**Background.** In order to present our results, we start by recalling some notions from robust learning in the PAC model. Let $C \subseteq \{0,1\}^X$ be a (Boolean) hypothesis class on an instance space $X \subseteq \mathbb{R}^d$. A perturbation is defined by a function $P : X \rightarrow 2^X$, where $\mathbb{P}(x) \subseteq X$ denotes the set of allowable perturbed instances starting from an instance $x$. The robust risk of a hypothesis $h$ with respect to a distribution $D$ on $X \times \{\pm 1\}$ and perturbation $P$ is defined as $R_P(h,D) = \Pr_{(x,y) \sim D}[\exists z \in P(x), h(z) \neq y]$. A distribution $D$ is said to be realizable (with respect to $C$ and $P$) iff there exists $h^* \in C$ such that $R_P(h^*,D) = 0$. In the adversarially robust PAC learning problem, the learner is given i.i.d. samples from a realizable distribution $D$ on $X \times \{\pm 1\}$, and the goal is to output a hypothesis $h : X \rightarrow \{\pm 1\}$ such that with probability $1-\xi$ it holds that $R_P(h,D) \leq \alpha$. We refer to $\xi$ as the failure probability and $\alpha$ as the accuracy parameter. A learner is said to be proper if the output hypothesis $h$ belongs to $C$; otherwise, it is said to be improper.

We focus our study on the concept class of halfspaces, i.e., $C_{\text{halfspaces}} := \{h_w | w \in \mathbb{R}^d\}$ where $h_w(x) = \text{sgn}(\langle w, x \rangle)$, in this model with respect to $L_2$ perturbations, i.e., $P_\gamma(x) := \{z \in X : \|z - x\|_2 \leq \gamma\}$ for margin parameter $\gamma > 0$. We assume throughout that the do-
main of our functions is bounded in the $d$-dimensional Euclidean unit ball $B_d := \{x \in \mathbb{R}^d \mid \|x\|_2 \leq 1\}$. We also write $R_\gamma$ as a shorthand for $R_{\gamma,\epsilon}$. An algorithm is said to be a $(\gamma, \gamma')$-robust learner if, for any realizable distribution $D$ with respect to $C_{\text{halfspaces}}$ and $P_{\gamma}$, using a certain number of samples, it outputs a hypothesis $h$ such that w.p. $1 - \xi$, we have $R_{\gamma'}(h, D) \leq \alpha$, where $\alpha, \xi > 0$ are sufficiently smaller than some positive constant. We are especially interested in the case where $\gamma'$ is close to $\gamma$; for simplicity, we use $\gamma' = 0.9\gamma$ as a representative setting throughout. In robust learning, the main quantities of interest are the sample complexity, i.e., the minimum number of samples needed to learn, and the running time of the learning algorithm.

We use the standard terminology of DP. Recall that two datasets $X$ and $X'$ are neighbors if $X'$ results from adding or removing a single data point from $X$.

**Definition 1** (Differential Privacy (DP) [36, 35]). Let $\epsilon, \delta \in \mathbb{R}_{\geq 0}$. A randomized algorithm $\mathcal{A}$ taking as input a dataset is said to be $(\epsilon, \delta)$-differentially private (denoted by $(\epsilon, \delta)$-DP) if for any two neighboring datasets $X$ and $X'$, and for any subset $S$ of outputs of $\mathcal{A}$, it is the case that $\Pr[\mathcal{A}(X) \in S] \leq e^\epsilon \cdot \Pr[\mathcal{A}(X') \in S] + \delta$. If $\delta = 0$, $\mathcal{A}$ is said to be $\epsilon$-differentially private (denoted by $\epsilon$-DP).

As usual, $\epsilon$ should be thought of as a small constant, whereas $\delta$ should be negligible in the dataset size. We refer to the case where $\delta = 0$ as pure-DP, and the case where $\delta > 0$ as approximate-DP.

**Our Results.** We assume that $\epsilon \leq O(1)$ unless otherwise stated, and that $\gamma, \alpha, \xi > 0$ are sufficiently smaller than some positive constant. We will not state these assumptions explicitly here for simplicity; interested readers may refer to (the first lines of) the proofs for the exact upper bounds that are imposed.

We first prove that robust learning with pure-DP requires $\Omega(d)$ samples.

**Theorem 2.** Any $\epsilon$-DP $(\gamma, 0.9\gamma)$-robust (possibly improper) learner has sample complexity $\Omega(d/\epsilon)$.

In the private but non-robust setting (i.e., for an $\epsilon$-DP $(\gamma, 0)$-robust learner), Nguyen et al. [74] showed that $O(1/\gamma^2)$ samples suffice. Together with earlier known results showing that $(\gamma, 0.9\gamma)$-robust learning (without privacy) only requires $O(1/\gamma^2)$ samples (e.g., [11, 61]), our result gives a separation between robust private learning and private learning alone or robust learning alone, whenever $d \gg 1/\gamma^2$.

For the case of approximate-DP, we establish a lower bound of $\Omega(\min \{\sqrt{d}/\gamma, d\})$, which holds only against proper learners. As for Theorem 2 in the context of pure-DP, this result implies a similar separation in the approximate-DP proper learning setting.

**Theorem 3.** Let $\epsilon < 1$. Any $(\epsilon, o(1/n))$-DP $(\gamma, 0.9\gamma)$-robust proper learner has sample complexity $n = \Omega(\min \{\sqrt{d}/\gamma, d\})$.

Our proof technique can also be used to improve the lower bound for DP $(\gamma, 0)$-robust learning. Specifically, Nguyen et al. [74] show a $\Omega(1/\gamma^2)$ lower bound for proper $(\gamma, 0)$-robust learners with pure-DP. We extend it to hold even for improper $(\gamma, 0)$-robust learners with approximate-DP:

**Theorem 4.** For any $\epsilon > 0$, there exists $\delta > 0$ such that any $(\epsilon, \delta)$-DP $(\gamma, 0)$-robust (possibly improper) learner has sample complexity $\Omega \left( \frac{1}{\epsilon \gamma^2} \right)$. Moreover, this holds even when $d \geq O(1/\gamma^2)$.

Finally, we provide algorithms with nearly matching upper bounds. For pure-DP, we prove the following, matching the lower bound in Theorem 2 to within a constant factor when $d \geq 1/\gamma^2$.

**Theorem 5.** There is an $\epsilon$-DP $(\gamma, 0.9\gamma)$-robust learner with sample complexity $\tilde{O}_\alpha \left( \frac{1}{\epsilon} \max \{d, \frac{1}{\gamma^2}\} \right)$.

For approximate-DP, it is already possible to achieve a sample complexity$^4$ of $n = \tilde{O}_\alpha(\sqrt{d}/\gamma)$ [24, 13] but the running time is $\Omega(n^d)$.$^5$ We give a faster algorithm with running time $\tilde{O}_\alpha(nd/\gamma)$.

**Theorem 6.** There is an $(\epsilon, \delta)$-DP $(\gamma, 0.9\gamma)$-robust learner with sample complexity $n = \tilde{O}_\alpha \left( \frac{1}{\epsilon \delta} \cdot \max \{\frac{\sqrt{d}}{\gamma}, \frac{1}{\gamma^2}\} \right)$ and running time $\tilde{O}_\alpha \left( nd/\gamma \right)$.

Our theoretical results and those from prior works are summarized in Table I. Notice that the non-private robust setting and the non-robust private setting each requires only $O(1/\gamma^2)$ samples, whereas our results show that the private and robust setting requires either $\Omega(d)$ samples (for pure-DP) or $\Omega(\sqrt{d}/\gamma)$ samples (for approximate-DP). This separation positively answers the question central to our study.

$^4$Here $O_\alpha(\cdot)$ hides a factor of poly(1/\alpha), and $\tilde{O}(\cdot)$ hides a factor of poly log(1/(\epsilon_0 \gamma \delta)).

$^5$This can be achieved by running the DP ERM algorithm of [13] with the hinge loss; see [24] for the analysis.

$^6$Specifically, [24] uses the DP empirical risk minimization algorithm of [13] with the hinge loss; however, the latter requires $\Omega(n^2)$ iterations and each iteration requires $\Omega(d)$ time.
Table 1: Trade-offs between privacy and robustness. The robust column corresponds to \((\gamma, 0.9\gamma)\)-robust learners, whereas the non-robust column corresponds to \((\gamma, 0)\)-robust learners. For simplicity of presentation, we assume \(a, \epsilon, \xi \in (0, 1)\) are sufficiently small constants, \(d \geq 1/\gamma^2\), and for approximate-DP lower bounds, that \(\delta = o(1/n)\). While approximate-DP upper bounds (marked with \(\tilde\)\) can already be derived from previous work, we give a faster algorithm (Theorem \(6\)). For DP, known results are from \([74]\); for the non-private case, the results follow, e.g., from \([11, 61]\).

We complement our theoretical results by empirically evaluating our algorithm (from Theorem \(6\)) on the MNIST \([65]\) and USPS \([54]\) datasets. Our results show that it is possible to achieve both robustness and privacy guarantees while maintaining reasonable performance. We further provide evidence that models trained via our algorithm are more resilient to adversarial noise compared to neural networks trained via DP-SGD \([1]\).

**Organization.** In the two following sections, we describe in detail the ideas behind each of our proofs. We then present our experimental results in Section \(4\). Finally, we discuss additional related work and several open questions in Sections \(5\) and \(6\) respectively. Due to space constraints, all missing proofs and additional experiments are deferred to the Supplementary Material.

## 2 Sample Complexity Lower Bounds

In this section, we explain the high-level ideas behind each of our sample complexity lower bounds. Our pure-DP lower bound is based on a packing framework and our approximate-DP lower bounds are based on fingerprinting codes.

### 2.1 Pure-DP Lower Bound (Theorem \(2\))

We use the packing framework, a DP lower bound proof technique that originated in \([51]\). Roughly speaking, to apply this framework, we have to construct many input distributions for which the sets of valid outputs for each distribution are disjoint (hence the name “packing”). In our context, this means that we would like to construct distributions \(D^{(1)}, \ldots, D^{(K)}\) such that the sets \(G^{(i)}\) of hypotheses with small robust risk on \(D^{(i)}\) are disjoint. Once we have done this, the packing framework immediately gives us a lower bound of \(\Omega(\log K/\epsilon)\) on the sample complexity; below we describe a construction for \(K = 2^{\Omega(d)}\) distributions, which yields the desired \(\Omega(d/\epsilon)\) lower bound in Theorem \(2\).

Our construction proceeds by picking unit vectors \(w^{(1)}, \ldots, w^{(K)}\) that are nearly orthogonal, i.e., \(|\langle w^{(i)}, w^{(j)} \rangle| < 0.01\) for all \(i \neq j\). It is not hard to see (and well-known) that such vectors exist for \(K = 2^{\Omega(d)}\). We then let \(D^{(i)}\) be the uniform distribution on \((1.01 \gamma \cdot w^{(i)}, +1), (-1.01 \gamma \cdot w^{(i)}, -1)\).

Now, let \(G^{(i)}\) denote the set of hypotheses \(h\) for which \(R_{0.9\gamma}(h, D^{(i)}) < 0.5\). Since our distribution \(D^{(i)}\) is uniform on two elements, we must have that \(R_{0.9\gamma}(h, D^{(i)}) = 0\) for all \(h \in G^{(i)}\). To see that \(G^{(i)}\) and \(G^{(j)}\) are disjoint for any \(i \neq j\), notice that, since \(|\langle w^{(i)}, w^{(j)} \rangle| < 0.01\), the point \(1.01 \gamma \cdot w^{(i)}\) is within distance \(1.8\gamma\) from the point \(-1.01 \gamma \cdot w^{(j)}\). This means that any hypothesis \(h\) cannot correctly classify both \((1.01 \gamma \cdot w^{(i)}, +1)\) and \((-1.01 \gamma \cdot w^{(i)}, -1)\) with margin at least \(0.9\gamma\), which implies that \(G^{(i)} \cap G^{(j)} = \emptyset\). This completes our proof sketch.

We end by remarking that the previous work of Nguyen et al. \([74]\) also uses a packing argument; the main differences between our construction and theirs are in the choice of the distributions \(D^{(i)}\) and the proof of disjointness of the \(G^{(i)}\)’s. Our construction of \(D^{(i)}\) is in fact simpler, since our proof of disjointness can rely on the robustness guarantee. These differences are inherent, since our lower bound holds even against improper learners, i.e., when the output hypothesis may not be a halfspace, whereas the lower bound from \([74]\) only holds against the particular case of proper learners.

### 2.2 Approximate-DP Lower Bound (Theorem \(3\))

We reduce from a lower bound from the line of works \([24, 39, 91, 92]\) inspired by fingerprinting codes. Specifically, these works consider the so-called attribute mean problem, where we are given a set of vectors drawn from some (hidden) distribution \(D\) on \(\{\pm1\}^d\) and the goal is to compute the mean. It is known that getting an estimate to within 0.1 of the true mean in each coordinate requires \(\Omega(\sqrt{d})\) samples. In fact, \([92]\) shows that even outputting a vector with a “non-trivial”
The lower bound once again uses the “embedding” we “embed” to get an improved bound for a smaller margin.

### 2.3 Non-Robust DP Learning Lower Bound

To get an improved bound for a smaller margin $\gamma$, we “embed” $\Omega(1/\gamma^2)$ hard instances above in each of $O(\gamma^2 d)$ dimensions. More specifically, let $T = \gamma/\gamma'$; for each $i \in [T^2]$ we create a distribution $D_i$ that is the hard distribution from the previous paragraph in $d' := d/T$ dimensions embedded onto coordinates $d' (i - 1) + 1, \ldots, d' i$. We then let the distribution $D'$ be the (uniform) mixture of $D^{(1)}, \ldots, D^{(T^2)}$. Since each $D^{(i)}$ is realizable with margin $\gamma'$ via some halfspace $w^{(i)}$, we may take $w^* := \frac{1}{T} \sum_{i \in [T^2]} \frac{w^{(i)}}{\|w^{(i)}\|_2}$ to realize the distribution $D'$ with margin $\frac{1}{T} \gamma' = \gamma$ as desired.

Now, to find any $w$ with small $R_{0.9\gamma}(w, D')$, we roughly have to solve (most of) the $T^2$ instances $D^{(i)}$s. Recall that solving each of these instances requires $\Omega(\sqrt{d'})$ samples. Thus, the combined instance requires $\Omega(\sum_{i \in [T^2]} \frac{1}{\|w^{(i)}\|_2})$ samples, thereby yielding $\Omega(\sqrt{d'})$. Hence, the combined instance requires $\Omega(\sqrt{d'})$. Thus, the combined instance requires $\Omega(\sqrt{d'})$. Hence, the combined instance requires $\Omega(\sqrt{d'})$ samples, thereby yielding $\Omega(\sqrt{d'})$.

### 3 Sample-Efficient Algorithms

In this section, we present our algorithms for robust and private learning. Our pure-DP algorithm is based on an improved analysis of the exponential mechanism. Our approximate-DP algorithm is based on a private, batched version of the perceptron algorithm.

In the following discussions, we assume that $w^*$ is an (unknown) optimal halfspace with respect to the input distribution $D$ and $L_2$-perturbations with margin parameter $\gamma$, i.e., that $w^*$ satisfies $R_\gamma(h_{w^*}, D) = 0$. We may assume without loss of generality that $\|w^*\| = 1$.

#### 3.1 Pure-DP Algorithm (Theorem 5)

Theorem 5 is shown via the exponential mechanism (EM) [71]. Our guarantee is an improvement over the “straightforward” analysis of EM on a $(0.1\gamma)$-net of the unit sphere in $\mathbb{R}^d$, which gives an upper bound of $O_\alpha(d \log(1/\gamma)/\epsilon)$ [24]. On the other hand, when $d \geq 1/\gamma^2$, our sample complexity is $O_\alpha(d/\epsilon)$. The intuition behind our improvement is that, if we take a random unit vector $w$ such that $(w, w^*) \geq 0.99$, then it already gives a small robust risk (in expectation) when $d \geq 1/\gamma^2$ because the component of $w$ orthogonal to $w^*$ is a random $(d - 1)$-dimensional vector of norm less than one, meaning that in expectation it only affects the margin by $O(1/\sqrt{d}) \ll 0.1\gamma$. Now, a random unit vector satisfies $(w, w^*) \geq 0.99$ with probability $2^{-O(d)}$, which (roughly speaking) means that EM should only require $O_\alpha(d/\epsilon)$ samples.

#### 3.2 Approximate-DP Algorithm (Theorem 6)

To prove Theorem 6, we use the DP Batch Perceptron algorithm presented in Algorithm 1. DP Batch Perceptron is the batch and privatized version of the so-called margin perceptron algorithm [24, 27]. That is, in each iteration, we randomly sample a batch of samples and for each sample $(x, y)$ in the batch that is not correctly classified with margin $\gamma'$, we add $y \cdot x$ to the current weight of the halfspace. Furthermore, we add some Gaussian noise to the weight vector to make this algorithm private. We also have a “stopping condition” that terminates whenever the number of samples mislabeled at margin $\gamma'$ is sufficiently small. (We add Laplace noise to the number of such samples to make it private.) To get a $(\gamma, 0.9\gamma)$-robust learner, it suffices for us to set, e.g., $\gamma' = 0.95\gamma$; we use this value of $\gamma'$ in the subsequent discussions.

Before we dive into the details of the proof, we remark that our runtime reduction, compared to the generic algorithm from [13], comes from the fact that, in the accuracy analysis, we only need the number of iterations $T$ to be $O_{\alpha}(1/\gamma^2)$, similar to the perceptron algorithm [75]. On the other hand, the generic theorem of [13] requires $n^2 = O_{\alpha}(d^2/\gamma^2)$ iterations.

The accuracy analysis of DP Batch Perceptron follows the blueprint of that of perceptron [75]. Specifically, we keep track of the following two quantities: $(w_t, w^*)$, the dot product between the current halfspace $w_t$ and the “true” halfspace $w^*$, and $\|w_t\|$, the (Euclidean)
Algorithm 1 DP Batch Perceptron

\begin{algorithm}
    \caption{DP Batch Perceptron_{γ∗,p,T,δ,σ}(\{(x_j,y_j)\}_{j\in[n]})}
    \begin{algorithmic}[1]
        \STATE $w_0 \leftarrow 0$
        \FOR {$i = 1, \ldots, T$}
            \STATE $S_i \leftarrow$ a set of samples where each $(x_j, y_j)$ is independently included w.p. $p$
            \STATE $M_i \leftarrow \emptyset$
            \FOR {$(x, y) \in S_i$}
                \STATE if $\text{sgn}\left(\left\langle \frac{w_{i-1}}{\|w_{i-1}\|}, x \right\rangle - y \cdot \gamma'\right) \neq y$
                \STATE $M_i \leftarrow M_i \cup \{(x, y)\}$
            \ENDFOR
            \STATE Sample $v_i \sim \text{Lap}(b)$
            \STATE if $|M_i| + v_i < 0.3omp$
            \STATE \text{return } $w_{i-1}/\|w_{i-1}\|$
            \STATE $u_i \leftarrow \sum_{(x,y)\in M_i} y \cdot x$
            \STATE Sample $g_i \sim \mathcal{N}(0, \sigma^2 \cdot I_{d \times d})$
            \STATE $w_i \leftarrow w_{i-1} + u_i + g_i$
        \ENDFOR
        \STATE \text{return } FAIL
    \end{algorithmic}
\end{algorithm}

The previous paragraphs outlined the analysis for the noiseless case where $\sigma = 0$. Next, we will describe how the noise $\sigma$ affects the analysis and our choices of parameters. Roughly speaking, we would like the inequalities (1) and (2) to “approximately” hold even after adding noise. In particular, this means that we would like the right-hand side of these inequalities to be affected by at most $o(\gamma m)$ by the noise addition, with high probability. This condition will determine our selection of parameters.

For (1), the inclusion of the noise term $g_i$ adds to the right-hand side by $\langle w^*, g_i \rangle$. The expectation of this term is $\|w^*\| \cdot \sigma \leq \gamma$, which means that it suffices to ensure that $m \geq \hat{\omega}(\sigma / \gamma)$. For (2), it turns out that the dominant additional term is $\|g_i\|^2 / \|w_{i-1}\|$ which, under the assumption that $\|w_{i-1}\| \geq 50m / \gamma$, is at most $O(\gamma / \|g_i\|^2 / m)$; this term is $O(\gamma \sigma^2 / m)$ in expectation. Since we would like this term to be $o(\gamma m)$, it suffices to have $m = \hat{\omega}(\sigma \cdot \sqrt{d})$. By combining these two requirements, we may pick $m = \sigma \cdot \hat{\omega}(\sqrt{d} + 1 / \gamma)$. We remark that the number of iterations still remains $T = O(1 / \gamma^2)$, as in the noiseless case above.

While we have so far assumed for simplicity that $|M_i| = m$ in all iterations, in the actual analysis we only require that $|M_i| \geq m$. Furthermore, it is simple to show that, as long as the current hypothesis has robust risk significantly more than $\alpha$, we will have $|M_i| \geq \Omega_{\alpha}(pn)$ with high probability. Combining with the previous paragraph, this gives us the following condition (assuming that $\alpha$ is constant):

$$pm \geq \sigma \cdot \hat{\omega}(\sqrt{d} + 1 / \gamma).$$

This leads us to pick the following set of parameters:

$$n = \tilde{O}\left(\frac{1}{\gamma} \left(\sqrt{d} + \frac{1}{\gamma}\right)\right), \quad p = \tilde{O}(\gamma), \quad \sigma = \tilde{O}(1).$$

The privacy analysis of our algorithm is similar to that of DP-SGD [13]. Specifically, by the choice of $\sigma = \tilde{O}(1)$ and subsampling rate $p = \tilde{O}(\gamma)$, each iteration of the algorithm is $(O(\gamma \epsilon), O(\gamma^2 \delta))$-DP (e.g., [38, 9]). Since the number of iterations is $T = O(1 / \gamma^2)$, advanced composition theorem [38] implies that the entire algorithm is $(O(\sqrt{T} \cdot \epsilon), O(T \cdot \epsilon^2 \delta)) = (\epsilon, \delta)$-DP as desired.

We end by noting that, despite the popularity of perceptron-based algorithms, we are not aware of any work that analyzes the above noised and batched variant. The most closely related analysis we are aware of is that of Blum et al. [19], whose algorithm uses the entire dataset in each iteration. While it is possible to adapt their analysis to the batch setting, it unfortunately does not give an optimal sample complexity. Specifically, their analysis requires the batch size to be $\Omega(\sqrt{d} / \gamma)$.
resulting in sample complexity of $\Omega(\sqrt{d}/\gamma^2)$. On the other hand, our more careful analysis works even with batch size $O(\sqrt{d} + 1/\gamma)$, which results in the desired $O_{\alpha,\epsilon}(\sqrt{d}/\gamma)$ sample complexity when $d \geq 1/\gamma^2$.

4 Experiments

We run our DP Batch Perceptron algorithm on the MNIST [62] and USPS [63] datasets, both of which involve 10-class digit classification. We train a separate halfspace classifier $w^{(y)}$ for each class $y \in \{1, \ldots, 10\}$ for one epoch. To predict on an image $x$, we output a class $y^*$ that maximizes $\langle w^{(y^*)}, x \rangle$. We tune batch size as a hyperparameter with values 1, 10, 50, 100, 500, 1000, and $\gamma$ with values 1, 0.1, 0.01, 0.001, 0.0001. Each set of experiments is repeated for 20 random trials. To reduce the number of hyperparameters, we slightly modify our algorithm so that we do not stop early (i.e., removing Lines 9 and 10) but instead return the weight vector $w_T$ at the end of the $T$th iteration (where $T$ is set in the algorithm).

The standard deviation $\sigma$ of the Gaussian noise added is determined by a fixed $(\epsilon, \delta)$-DP budget, computed using Renyi DP [1, 72]. The calculations for this follow the implementation in the official TensorFlow Privacy repository (https://github.com/tensorflow/privacy). For experiments with varying $\epsilon$ (first column of Figure 1), we fix $\delta$ to $10^{-5}$ for MNIST and $10^{-4}$ for USPS. We observe that despite the robustness and privacy constraints, DP Batch Perceptron still achieves competitive accuracy on both datasets. We also report performance with varying $\delta$ values of $10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$ (second column of Figure 1), while keeping $\epsilon$ fixed at 1.0.

Adversarial Robustness Evaluation. We compare the robustness of our models against those of neural networks trained with DP-SGD. For the latter, we follow the architecture found in the official TensorFlow Privacy tutorial, which consists of two convolutional layers, each followed by a MaxPool operation, and a dense layer that outputs predicted logits. The network is then trained with batch size 250, learning rate 0.15, $L_2$ clipping-norm 1.0, and for 60 epochs. This configuration yields competitive performance on the MNIST dataset.

To evaluate the robustness of the models, we calculate the robust risk on the test dataset for varying values of the perturbation norm (i.e., margin) $\gamma$. Following common practice in the field, we plot the robust accuracy on the test data, which is defined as one minus the robust risk (i.e., $1 - R_\gamma(h, D)$ where $D$ is the test dataset distribution), instead of the robust risk itself. Similarly, our plots use the unnormalized margin, meaning that the images are not re-scaled to have $L_2$ norm equal to 1 before prediction. Note that each of their pixel values is still scaled (i.e., divided by 255 if necessary) to have values in the range $[0, 1]$.

In the case of DP Batch Perceptron, it is known (see, e.g., [23]) that an example $(x, y)$ cannot be perturbed (using $P_y$) to an incorrect label if $\gamma < \min_{y' \neq y} \frac{\langle w^{(y')}, x \rangle - \langle w^{(y)}, x \rangle}{\|w^{(y')} - w^{(y)}\|}$. This formula allows us to exactly calculate the robust risk of our linear classifiers. We stress that this is a provable robustness guarantee, meaning that it holds against all adversarial attacks with perturbation norm (at most) $\gamma$.

We demonstrate the effect of the change in the required privacy level on the robust risk of our linear classifiers in the right most column of Figure 1. The x-axis of the plots represents the parameter $\gamma$ and the y-axis represents the $\gamma$-robust accuracy on the test dataset.

In contrast to linear models, there is no efficiently-computable formula to calculate robust risk for general neural networks. In this case, we use a variant of a popular adversarial robustness attack (outlined below) to estimate the robust risk of DP-SGD-trained neural networks. Unlike the linear classifier case, this method only gives a lower bound on the robust risk, meaning that more sophisticated attacks might result in even more incorrect classifications.

We now briefly summarize the attack we use against DP-SGD-trained neural networks; a (version of) this method was already presented in [23]. Let $M$ denote a trained model; recall that the last layer of our model consists of 10 outputs corresponding to each class and to predict an image we take $y^*$ with the maximum output. Given a sample $(x, y)$, we would like to determine whether there exists a perturbation $\Delta \in \mathbb{R}^d$ with $\|\Delta\| \leq \gamma$ such that $M$ predicts $x + \Delta$ to be some other class $y' \neq y$. Instead of solving this (intractable) problem directly, the attack considers a modified objective of

$$\min_{\|\Delta\| \leq \gamma} \ell(M(x + \Delta), y')$$

where $\ell$ is some loss function. This optimization problem is then solved using Projected Gradient Descent (PGD). We use the cross entropy loss in our attack.

Comparisons of the robust accuracy of models trained via DP Batch Perceptron and those trained via DP-SGD are shown in Figure 2 for $\delta = 10^{-5}$ and $\epsilon = 0.5, 1, 2$. In the case of $\epsilon = 0.5$, while both classifiers have similar test accuracies (without any perturbation, $\gamma = 0$), as $\gamma$ increases, the robust accuracy rapidly degrades for the DP-SGD-trained neural network compared to that of the DP Batch Perceptron model. This overall trend persists for $\epsilon = 1, 2$; in both cases, the neural networks start off with noticeably larger test...
accuracy when $\gamma = 0$ but are eventually surpassed by halfspace classifiers as $\gamma$ increases.

5 Other Related Work

Learning Halfspaces with Margin. $L_2$ robustness is a classical setting closely related to the notion of margin, e.g., [81, 75]. (See the supplementary material for formal definitions.) Known margin-based learning methods include the classic perceptron algorithm [81, 75] and its many generalizations and variants (e.g., [34, 27, 43, 15, 67, 44]), as well as Support Vector Machines (SVMs) [20, 28]. A number of works also explore a closely related agnostic setting, where the distribution $D$ is not guaranteed to be realizable with a margin (e.g., [10, 86, 68, 18, 31, 32]).

Generalization aspects of margin-based learning of halfspaces is also a widely studied topic (e.g., [10, 97, 11, 61, 70, 55]), and it is known that the sample complexity of robust learning of halfspaces is $O(1/(\alpha\gamma^2))$ [11, 61].

To the best of our knowledge, the first work that combines the study of learning halfspaces with margin and DP is [74]; their results are represented in Table 1. Recently, [46] gave alternative proofs for some results of [74] via reductions to clustering problems, but these do not provide any improved sample complexity or running time.

Adversarially Robust Learning. There has been a rapidly growing literature on adversarial robustness. Some of these works have presented evidence that training robust classifiers might be harder than non-
robust ones, e.g., [1] [21] [23] [30]. Other works aim to demonstrate the accuracy cost of robustness, e.g., [94] [79]. Another line of work seeks to determine the right quantity that governs adversarial generalization, e.g., [83] [73] [60] [90] [58].

**Differentially Private Learning.** Private learning has been a popular topic since the early days of differential privacy (e.g., [59]). Apart from the work of Nguyen et al. [74] on privately learning halfspaces with a margin, a line of work closely related to our setting is the study of the sample complexity of learning threshold functions [15] [42] [28] [4] [56] and halfspaces [14] [57] [58]. These works study the setting where the unit ball $B^d$ is discretized so that the domain $X$ is $X^d \cap B^d$ (i.e., each coordinate is an element of $X$). Interestingly, it has been shown that when $X$ is infinite, halfspaces become unlearnable, i.e., the sample complexity becomes unbounded [4]. On the other hand, an $(\epsilon,o(1/n))$-DP learner with sample complexity $\tilde{O}\left(\frac{d^{0.5}}{\alpha^2}\right) \cdot 2^{O(\log^* |X|)}$ exists [57].

While the above setting is not directly comparable to ours, it is possible to reduce between the margin setting and the discretized setting, albeit with some loss. For example, we may use the grid discretization with $|X| = 0.01\gamma/\sqrt{d}$, to obtain a $(\gamma, 0)$-learner with sample complexity $\tilde{O}\left(\frac{d^{0.5}}{\alpha^2}\right) \cdot 2^{O(\log^*(1/\gamma))}$. This is better than the (straightforward) bound of $O(d \cdot \log(1/\gamma))$ obtained by applying the exponential mechanism [71] when $\gamma$ is very small (e.g., $\gamma \leq 2^{-d^{0.5}}$). It remains an interesting open problem to close such a gap for very small values of $\gamma$.

Several works (e.g., [82] [28]) have studied differentially private SVMs. However, to the best of our knowledge, there is no straightforward way to translate their theoretical results to those shown in our paper as the objectives in the two settings are different.

# 6 Conclusions and Future Directions

In this work, we prove new trade-offs, measured in terms of sample complexity, between privacy and robustness — two crucial properties within the domain of AI ethics and safety, for the classic task of halfspace learning. Our theoretical results demonstrate that DP and adversarially robust learning requires a larger number of samples than either DP or adversarially robust learning alone. We then propose a learning algorithm that meets both criteria, and test it on two multi-class classification datasets. We also provide empirical evidence that despite having a slight advantage in terms of test accuracy on the main task, standard neural networks trained with DP-SGD are not as robust as those trained with our algorithm.

We conclude with a few future research directions. First, it would be interesting to close the gap for the sample complexity of improper approximate-DP ($\gamma, 0.9\gamma$)-robust learners; for $d \gg 1/\gamma^2$, the upper bound is $O(\sqrt{d}/\gamma)$ (Theorem 6) but the lower bound is only $\Omega(1/\gamma^2)$ (Theorem 3). This is the only case where there is still a super-polynomial gap for $d \gg 1/\gamma^2$.

Another technical open question is to improve the lower bounds in Theorem 3 to $\Omega(\min\{\sqrt{d}/\gamma, d\})/\epsilon$. Currently, we are missing the $\epsilon$ term because we invoke a lower bound from [92] (Theorem 11) which was specifically proved only for $\epsilon = 1$.

Furthermore, it would be natural to extend our study to $L_p$ perturbations for $p \neq 2$. An especially noteworthy case is when $p = \infty$, which is a well-studied setting in the adversarial robustness literature.

Finally, it would be very interesting to provide a theoretical understanding of private and robust learning beyond halfspaces, to accommodate complex algorithms (e.g., deep neural networks) that are better suited for more challenging tasks.

# References

[1] Martin Abadi, Andy Chu, Ian Goodfellow, H Brendan McMahan, Ilya Mironov, Kunal Talwar, and Li Zhang. Deep learning with differential privacy. In CCS, pages 308–318, 2016.

[2] John M Abowd. The US Census Bureau adopts differential privacy. In KDD, pages 2867–2867, 2018.

[3] Noga Alon, Oded Goldreich, Johan Håstad, and René Peralta. Simple constructions of almost k-wise independent random variables. In FOCS, pages 544–553, 1990.

[4] Noga Alon, Roi Livni, Maryanthe Malliaris, and Shay Moran. Private PAC learning implies finite littlestone dimension. In STOC, pages 852–860, 2019.

[5] Apple Differential Privacy Team. Learning with privacy at scale. Apple Machine Learning Journal, 2017.

[6] Anish Athalye, Nicholas Carlini, and David Wagner. Obfuscated gradients give a false sense of security: Circumventing defenses to adversarial examples. In ICML, pages 274–283, 2018.

[7] Pranjal Awasthi, Abhratanu Dutta, and Aravindan Vijayaraghavan. On robustness to adversarial examples and polynomial optimization. In NeurIPS, pages 13760–13770, 2019.
[8] Pranjal Awasthi, Natalie Frank, and Mehryar Mohri. Adversarial learning guarantees for linear hypotheses and neural networks. In ICML, 2020.

[9] Borja Balle, Gilles Barthe, and Marco Gaboardi. Privacy amplification by subsampling: Tight analyses via couplings and divergences. In NeurIPS, pages 6280–6290, 2018.

[10] Peter L. Bartlett. The sample complexity of pattern classification with neural networks: The size of the weights is more important than the size of the network. IEEE Trans. Inf. Theory, 44(2):525–536, 1998.

[11] Peter L Bartlett and Shahar Mendelson. Rademacher and Gaussian complexities: Risk bounds and structural results. JMLR, 3:463–482, 2002.

[12] Raef Bassily, Vitaly Feldman, Kunal Talwar, and Abhradeep Guha Thakurta. Private stochastic convex optimization with optimal rates. In NeurIPS, pages 11282–11291, 2019.

[13] Raef Bassily, Adam D. Smith, and Abhradeep Thakurta. Private empirical risk minimization: Efficient algorithms and tight error bounds. In FOCS, pages 464–473, 2014.

[14] Amos Beimel, Shay Moran, Kobbi Nissim, and Uri Stemmer. Private center points and learning of halfspaces. In COLT, pages 269–282, 2019.

[15] Amos Beimel, Kobbi Nissim, and Uri Stemmer. Private learning and sanitization: Pure vs. approximate differential privacy. Theory Comput., 12(1):1–61, 2016.

[16] Shai Ben-David and Hans Ulrich Simon. Efficient learning of linear perceptrons. In NIPS, pages 189–195, 2000.

[17] Battista Biggio, Iigino Corona, Davide Maiorca, Blaine Nelson, Nedim Šrndić, Pavel Laskov, Giorgio Giacinto, and Fabio Roli. Evasion attacks against machine learning at test time. In ECML/PKDD, pages 387–402, 2013.

[18] Aharon Birnbaum and Shai Shalev-Shwartz. Learning halfspaces with the zero-one loss: Time-accuracy tradeoffs. In NIPS, pages 935–943, 2012.

[19] Avrim Blum, Cynthia Dwork, Frank McSherry, and Kobbi Nissim. Practical privacy: the sulq framework. In PODS, pages 128–138, 2005.

[20] Bernhard E. Boser, Isabelle Guyon, and Vladimir Vapnik. A training algorithm for optimal margin classifiers. In COLT, pages 144–152, 1992.

[21] Sébastien Bubeck, Yin Tat Lee, Eric Price, and Ilya Razenshteyn. Adversarial examples from computational constraints. In ICML, pages 831–840, 2018.

[22] Sébastien Bubeck, Yin Tat Lee, Eric Price, and Ilya Razenshteyn. Adversarial examples from computational constraints. In ICML, pages 831–840, 2019.

[23] Mark Bun, Kobbi Nissim, Uri Stemmer, and Salil Vadhan. Differentially private release and learning of threshold functions. In FOCS, pages 634–649, 2015.

[24] Mark Bun, Jonathan Ullman, and Salil P. Vadhan. Fingerprinting codes and the price of approximate differential privacy. SIAM J. Comput., 47(5):1888–1938, 2018.

[25] Nicholas Carlini and David Wagner. Audio adversarial examples: Targeted attacks on speech-to-text. In IEEE SPW, pages 1–7, 2018.

[26] Kamalika Chaudhuri, Claire Monteleoni, and Anand D Sarwate. Differentially private empirical risk minimization. JMLR, 12(3):1069–1109, 2011.

[27] Ronan Collobert and Samy Bengio. Links between perceptrons, MLPs and SVMs. In ICML, volume 69, 2004.

[28] Corinna Cortes and Vladimir Vapnik. Support-vector networks. Machine Learning, 20(3):273–297, 1995.

[29] Nilesh Dalvi, Pedro Domingos, Sumit Sanghai, and Deepak Verma. Adversarial classification. In KDD, pages 99–108, 2004.

[30] Akshay Degwekar, Preetum Nakkiran, and Vinod Vaikuntanathan. Computational limitations in robust classification and win-win results. In COLT, pages 994–1028, 2019.

[31] Ilias Diakonikolas, Daniel Kane, and Pasin Manurangsi. Nearly tight bounds for robust proper learning of halfspaces with a margin. In NeurIPS, pages 10473–10484, 2019.

[32] Ilias Diakonikolas, Daniel M. Kane, and Pasin Manurangsi. The complexity of adversarially robust proper learning of halfspaces with agnostic noise. In NeurIPS, 2020. To appear.

[33] Bolin Ding, Janardhan Kulkarni, and Sergey Yekhanin. Collecting telemetry data privately. In NIPS, pages 3571–3580, 2017.
[34] Richard O. Duda and Peter E. Hart. *Pattern Classification and Scene Analysis*. Wiley, 1973.

[35] Cynthia Dwork, Krishnaram Kenthapadi, Frank McSherry, Ilya Mironov, and Moni Naor. Our data, ourselves: Privacy via distributed noise generation. In *EUROCRYPT*, pages 486–503, 2006.

[36] Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam D. Smith. Calibrating noise to sensitivity in private data analysis. In *TCC*, pages 265–284, 2006.

[37] Cynthia Dwork and Aaron Roth. The algorithmic foundations of differential privacy. *Foundations and Trends in Theoretical Computer Science*, 9(3-4):211–407, 2014.

[38] Cynthia Dwork, Guy N. Rothblum, and Salil P. Vadhan. Boosting and differential privacy. In *FOCS*, pages 51–60, 2010.

[39] Cynthia Dwork, Adam D. Smith, Thomas Steinke, Jonathan Ullman, and Salil P. Vadhan. Robust traceability from trace amounts. In *FOCS*, pages 650–669, 2015.

[40] Úlfar Erlingsson, Vasyl Pihur, and Aleksandra Korolova. RAPPOR: Randomized aggregatable privacy-preserving ordinal response. In *CCS*, pages 1054–1067, 2014.

[41] Vitaly Feldman, Tomer Koren, and Kunal Talwar. Private stochastic convex optimization: optimal rates in linear time. In *STOC*, pages 439–449, 2020.

[42] Vitaly Feldman and David Xiao. Sample complexity bounds on differentially private learning via communication complexity. In *COLT*, pages 1000–1019, 2014.

[43] Yoav Freund and Robert E. Schapire. Large margin classification using the perceptron algorithm. *Mach. Learn.*, 37(3):277–296, 1999.

[44] Claudio Gentile. A new approximate maximal margin classification algorithm. *JMLR*, 2:213–242, 2001.

[45] Claudio Gentile and Nick Littlestone. The robustness of the $p$-norm algorithms. In *COLT*, pages 1–11, 1999.

[46] Badih Ghazi, Ravi Kumar, and Pasin Manurangsi. Differentially private clustering: Tight approximation ratios. In *NeurIPS*, 2020. To appear.

[47] Ian J Goodfellow, Jonathon Shlens, and Christian Szegedy. Explaining and harnessing adversarial examples. In *ICLR*, 2015.

[48] Sven Gowal, Krishnamurthy Dvijotham, Robert Stanforth, Rudy Bunel, Chongli Qin, Jonathan Uesato, Relja Arandjelovic, Timothy Mann, and Pushmeet Kohli. On the effectiveness of interval bound propagation for training verifiably robust models. *arXiv Preprint:1810.12715*, 2018.

[49] Sven Gowal, Jonathan Uesato, Chongli Qin, Posen Huang, Timothy Mann, and Pushmeet Kohli. An alternative surrogate loss for PGD-based adversarial testing. *arXiv Preprint: 1910.09338*, 2019.

[50] Andy Greenberg. Apple’s “differential privacy” is about collecting your data – but not your data. *Wired, June*, 13, 2016.

[51] Moritz Hardt and Kunal Talwar. On the geometry of differential privacy. In *STOC*, pages 705–714, 2010.

[52] Jamie Hayes. Provable trade-offs between private & robust machine learning. *arXiv Preprint: 2006.04622*, 2020.

[53] Matthias Hein and Maksym Andriushchenko. Formal guarantees on the robustness of a classifier against adversarial manipulation. In *NeurIPS*, pages 2266–2276, 2017.

[54] Jonathan J. Hull. A database for handwritten text recognition research. *IEEE PAMI*, 16(5):550–554, 1994.

[55] Sham M. Kakade, Karthik Sridharan, and Ambuj Tewari. On the complexity of linear prediction: Risk bounds, margin bounds, and regularization. In *NIPS*, pages 793–800, 2008.

[56] Haim Kaplan, Katrina Ligett, Yishay Mansour, Moni Naor, and Uri Stemmer. Privately learning thresholds: Closing the exponential gap. In *COLT*, pages 2263–2285, 2020.

[57] Haim Kaplan, Yishay Mansour, Uri Stemmer, and Eliad Tsfadia. Private learning of halfspaces: Simplifying the construction and reducing the sample complexity. In *NeurIPS*, 2020. To appear.

[58] Haim Kaplan, Micha Sharir, and Uri Stemmer. How to Find a Point in the Convex Hull Privately. In *SoCG*, pages 52:1–52:15, 2020.

[59] Shiva Prasad Kasiviswanathan, Homik K. Lee, Kobbi Nissim, Sofya Raskhodnikova, and Adam D. Smith. What can we learn privately? In *FOCS*, pages 531–540, 2008.
[60] Justin Khim and Po-Ling Loh. Adversarial risk bounds via function transformation. arXiv Preprint: 1810.09519, 2018.

[61] Vladimir Kolchinskii and Dmitry Panchenko. Empirical margin distributions and bounding the generalization error of combined classifiers. The Annals of Statistics, 30(1):1–50, 2002.

[62] Zico Kolter and Aleksander Madry. Adversarial robustness: Theory and practice. Tutorial at NeurIPS, 2018.

[63] Beatrice Laurent and Pascal Massart. Adaptive estimation of a quadratic functional by model selection. Annals of Statistics, pages 1302–1338, 2000.

[64] Yann Lecun, Léon Bottou, Yoshua Bengio, and Patrick Haffner. Gradient-based learning applied to document recognition. In Proceedings of the IEEE, pages 2278–2324, 1998.

[65] Yann LeCun, Corinna Cortes, and CJ Burges. MNIST handwritten digit database, 2010. http://yann.lecun.com/exdb/mnist.

[66] Mathias Lecuyer, Vaggelis Atlidakis, Roxana Geambasu, Daniel Hsu, and Suman Jana. Certified robustness to adversarial examples with differential privacy. In IEEE S&P, pages 656–672, 2019.

[67] Yi Li and Philip M. Long. The relaxed online maximum margin algorithm. Mach. Learn., 46(1-3):361–387, 2002.

[68] Philip M. Long and Rocco A. Servedio. Learning large-margin halfspaces with more malicious noise. In NIPS, pages 91–99, 2011.

[69] Aleksander Madry, Aleksandar Makelov, Ludwig Schmidt, Dimitris Tsipras, and Adrian Vladu. Towards deep learning models resistant to adversarial attacks. In ICLR, 2018.

[70] David A. McAllester. Simplified pac-bayesian margin bounds. In COLT, pages 203–215, 2003.

[71] Frank McSherry and Kunal Talwar. Mechanism design via differential privacy. In FOCS, pages 94–103, 2007.

[72] Ilya Mironov. Rényi differential privacy. In CSF, pages 263–275, 2017.

[73] Omar Montasser, Steve Hanneke, and Nathan Srebro. VC classes are adversarially robustly learnable, but only improperly. In COLT, pages 2512–2530, 2019.

[74] Huy Le Nguyen, Jonathan Ullman, and Lydia Zakynthinou. Efficient private algorithms for learning large-margin halfspaces. In ALT, pages 704–724, 2020.

[75] Albert B Novikoff. On convergence proofs for perceptrons. Technical report, Stanford Research Institute, Menlo Park, CA, 1963.

[76] Nicolas Papernot, Patrick McDaniel, and Ian Goodfellow. Transferability in machine learning: from phenomena to black-box attacks using adversarial samples. arXiv Preprint: 1605.07277, 2016.

[77] NhatHai Phan, My T Thai, Han Hu, Ruoming Jin, Tong Sun, and Dejing Dou. Scalable differential privacy with certified robustness in adversarial learning. In ICML, 2020.

[78] NhatHai Phan, Minh Vu, Yang Liu, Ruoming Jin, Dejing Dou, Xintao Wu, and My T Thai. Heterogeneous Gaussian mechanism: Preserving differential privacy in deep learning with provable robustness. In IJCAI, pages 4753–4759, 2019.

[79] Aditi Raghunathan, Sang Michael Xie, Fanny Yang, John C Duchi, and Percy Liang. Adversarial training can hurt generalization. arXiv Preprint: 1906.00632, 2019.

[80] Ali Rahimi and Benjamin Recht. Random features for large-scale kernel machines. In NIPS, pages 1177–1184, 2007.

[81] Frank Rosenblatt. The perceptron: a probabilistic model for information storage and organization in the brain. Psychological Review, 65(6):386, 1958.

[82] Benjamin IP Rubinstein, Peter I Bartlett, Ling Huang, and Nina Taft. Learning in a large function space: Privacy-preserving mechanisms for SVM learning. J. Priv. Confidentiality, 4(1), 2012.

[83] Ludwig Schmidt, Shibani Santurkar, Dimitris Tsipras, Kunal Talwar, and Aleksander Madry. Adversarially robust generalization requires more data. In NeurIPS, pages 5014–5026, 2018.

[84] Bernhard Schölkopf, Kah Kay Sung, Christopher J. C. Burges, Federico Girosi, Partha Niyogi, Tomaso A. Poggio, and Vladimir Vapnik. Comparing support vector machines with Gaussian kernels to radial basis function classifiers. IEEE Trans. Signal Process., 45(11):2758–2765, 1997.

[85] Lukas Schott, Jonas Rauber, Matthias Bethge, and Wieland Brendel. Towards the first adversarially robust neural network model on MNIST. In ICLR, 2019.
[86] Shai Shalev-Shwartz, Ohad Shamir, and Karthik Sridharan. Learning kernel-based halfspaces with the zero-one loss. In *COLT*, pages 441–450, 2010.

[87] Stephen Shankland. How Google tricks itself to protect Chrome user privacy. *CNET, October*, 2014.

[88] Yash Sharma and Pin-Yu Chen. Attacking the Madry defense model with $l_1$-based adversarial examples. *arXiv Preprint: 1710.10733*, 2017.

[89] Liwei Song, Reza Shokri, and Prateek Mittal. Membership inference attacks against adversarially robust deep learning models. In *IEEE SPW*, pages 50–56, 2019.

[90] Liwei Song, Reza Shokri, and Prateek Mittal. Privacy risks of securing machine learning models against adversarial examples. In *CCS*, pages 241–257, 2019.

[91] Thomas Steinke and Jonathan Ullman. Between pure and approximate differential privacy. *J. Priv. Confidentiality*, 7(2), 2016.

[92] Thomas Steinke and Jonathan Ullman. Tight lower bounds for differentially private selection. In *FOCS*, pages 552–563, 2017.

[93] Christian Szegedy, Wojciech Zaremba, Ilya Sutskever, Joan Bruna, Dumitru Erhan, Ian Goodfellow, and Rob Fergus. Intriguing properties of neural networks. In *ICLR*, 2014.

[94] Dimitris Tsipras, Shibani Santurkar, Logan Engstrom, Alexander Turner, and Aleksander Madry. Robustness may be at odds with accuracy. In *ICLR*, 2019.

[95] Salil Vadhan. The complexity of differential privacy. In *Tutorials on the Foundations of Cryptography*, pages 347–450. Springer, 2017.

[96] Dong Yin, Ramchandran Kannan, and Peter Bartlett. Rademacher complexity for adversarially robust generalization. In *ICML*, pages 7085–7094, 2019.

[97] Tong Zhang. Covering number bounds of certain regularized linear function classes. *JMLR*, 2:527–550, 2002.
A Preliminaries

For $m \in \mathbb{N}$, we use $[m]$ to denote $\{1, \ldots, m\}$. For a distribution $D$, we write $r \sim D$ to denote a random variable $r$ distributed as $D$. For a randomized algorithm $\mathcal{A}$, we write $\mathcal{A}(X)$ to denote the distribution of the output of $\mathcal{A}$ on input $X$. For a distribution $D$, we write $\mathcal{A}(D)$ to denote the distribution of the output of $\mathcal{A}$ when the input is drawn from $D$. Sometimes we will allow the number of samples drawn by an algorithm to be a random variable. Furthermore, when $D$ is the distribution of each sample, we may write $\mathcal{A}_D$ to denote the distribution of the output when each of $\mathcal{A}$’s samples is drawn from $D$. We use $D_1 \otimes \cdots \otimes D_m$ to denote the product distribution of the distributions $D_1, \ldots, D_m$. Furthermore, we use $D^\otimes m$ to denote the $m$-fold product of the distribution $D$ with itself.

For convenience, we interchangeably refer to a halfspace by $h_w$ or just the weight vector $w$ itself.

A.1 Margin of Halfspaces

Robust learning of halfspaces is intimately related to the notion of margin. For a margin parameter $\gamma > 0$, we say that an example $(x, y) \in \mathbb{R}^d \times \{\pm 1\}$ is correctly classified by $w$ with margin $\gamma$ iff $\text{sgn}(\langle w, x \rangle - y \cdot \gamma) = y$. The $\gamma$-margin error is defined as $\text{err}_\gamma^D(w) = \Pr_{(x,y) \sim D}[(\langle w, x \rangle - y \cdot \gamma) \neq y]$. The connection between robust learning of halfspaces and learning with margin is given through the following (folklore) lemma; its proof can be found, e.g., in [32].

Lemma 7. For any non-zero $w \in \mathbb{R}^d$, $\gamma \geq 0$ and $D$, $\mathcal{R}_\gamma(w, D) = \text{err}_\gamma^D\left(\frac{w}{\|w\|_2}\right)$.

Due to the above lemma, we may refer to the $\gamma$-margin error for halfspaces instead of their robust risk throughout the paper.

A.2 Boosting the Success Probability

Throughout this work, it is often more convenient to prove lower bounds (resp. upper bounds) only for some large (resp. small) failure probability $\xi \in (0, 1)$. We note that this is without loss of generality, since standard techniques can be used to boost the success probability while incurring small loss in the sample complexity. We sketch the argument below.

Observation 8. For any $\xi, \xi' \in (0, 1)$, the following statement holds: If there is a $(\gamma, \gamma')$-robust learner with failure probability $\xi$, accuracy $\alpha$ and sample complexity $m$, then there exists a $(\gamma, \gamma')$-robust learner with failure probability $\xi'$, accuracy $1.1\alpha$ and sample complexity $O_{\xi, \xi'}(m + 1/\alpha^2)$.

Proof Sketch. Let $\mathcal{A}$ be the $(\gamma, \gamma')$-robust learner with failure probability $\xi$, accuracy $\alpha$ and sample complexity $m$. We define an algorithm $\mathcal{B}$ as follows:

- Let $T := \lceil \frac{\log(0.5\xi')}{\log(1-\xi)} \rceil$ and $M := \lceil 16 \log T \rceil$.
- For $i \in [T]$, run $\mathcal{A}$ on $m$ samples to get a halfspace $w_i$.
- Sample $M$ fresh new samples. Then, output $w_i$ that minimizes the $\gamma$-margin error of $w_i$ on the uniform distribution over these $M$ samples.

Clearly, the algorithm $\mathcal{B}$ uses $m \cdot T + M = O_{\xi, \xi'}(m + 1/\alpha^2)$ samples as desired. For the accuracy, with probability $1 - (1 - \xi)^T \geq 1 - 0.5\xi'$ at least one of the $w_i$’s satisfies $\text{err}_\gamma^D(w_i) \leq \alpha$. Conditioned on this, the Chernoff bound ensures that w.p. $1 - 0.5\xi$ we output a $w_i$ s.t. $\text{err}_\gamma^D(w_i) \leq 1.1\alpha$. We can then conclude the proof via the union bound.

B Lower Bound for Robust Learning of Halfspaces: Pure-DP Case

In this section, we prove our lower bound for $\epsilon$-DP robust learning of halfspaces (Theorem 2), which is restated below.
Theorem 2. Any $\epsilon$-DP $(\gamma, 0.9\gamma)$-robust (possibly improper) learner has sample complexity $\Omega(d/\epsilon)$.

We will use the following (well-known) fact; for completeness, we sketch its proof at the end of this section.

**Lemma 9.** There exist $w^{(1)}, \ldots, w^{(K)} \in \mathbb{R}^d$ where $K = 2^{\Omega(d)}$ such that $\|w^{(i)}\|_2 = 1$ for all $i \in [K]$ and $|\langle w^{(i)}, w^{(j)} \rangle| < 0.01$ for all $i \neq j$.

**Proof of Theorem 2.** We will prove the statement for any $\gamma \leq 0.99, \alpha \leq 0.49$ and $\xi \leq 0.9$.

Let $w^{(1)}, \ldots, w^{(K)}$ be the vectors guaranteed by Lemma 9. For each $i \in [K]$, we define $D^{(i)}$ to be the uniform distribution on two elements: $(1.01\gamma \cdot w^{(i)}, +1)$ and $(-1.01\gamma \cdot w^{(i)}, -1)$. Notice that $R_{0.9\gamma}(w^{(i)}, D^{(i)}) = 0$.

Now, let $G^{(i)} = \{h : \mathbb{B}^d \to \{\pm 1\} \mid R_{0.9\gamma}(h, D^{(i)}) \leq \alpha\}$ denote the set of hypotheses which incurs error no more than $\alpha$ on $D^{(i)}$. The main claim is the following:

**Claim 10.** For every $i \neq j$, $G^{(i)} \cap G^{(j)} = \emptyset$.

**Proof.** Suppose for the sake of contradiction that there exists $h \in G^{(i)} \cap G^{(j)}$ for some $i \neq j$.

Since $\alpha \leq 0.49$ and $D^{(i)}$ is a uniform distribution over only two samples, $R_{0.9\gamma}(h, D^{(i)}) \leq \alpha$ implies that $R_{0.9\gamma}(h, D^{(i)}) = 0$. This implies that

$$h(z) = 1 \quad \forall z \in \mathbb{P}_{0.9\gamma}(1.01\gamma \cdot w^{(i)})$$

By an analogous argument, we have

$$h(z) = -1 \quad \forall z \in \mathbb{P}_{0.9\gamma}(-1.01\gamma \cdot w^{(j)})$$

This is a contradiction since $\mathbb{P}_{0.9\gamma}(-1.01\gamma \cdot w^{(j)}) \cap \mathbb{P}_{0.9\gamma}(1.01\gamma \cdot w^{(i)}) \neq \emptyset$; specifically, $|\langle w^{(i)}, w^{(j)} \rangle| < 0.01$ implies that this intersection contains $0.505\gamma \cdot w^{(i)} - 0.505\gamma \cdot w^{(j)}$.

To finish the proof, consider any $\epsilon$-DP $(\gamma, 0.9\gamma)$-robust learner $A$ with $\alpha \leq 0.49$. Suppose that it takes $n$ samples. Notice that, when we feed it $n$ random samples from $D^{(i)}$, the accuracy guarantee ensures that

$$\Pr_{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n) \sim D^{(i)}}[A((\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)) \in G^{(i)}] \geq 1 - \xi.$$  

As a result, since $A$ is $\epsilon$-DP, we have

$$\Pr[A(\emptyset) \in G^{(i)}] \geq (1 - \xi) \cdot e^{-\epsilon n} \geq 0.1 \cdot e^{-\epsilon n}.$$  

From Claim 10 $G^{(1)}, \ldots, G^{(K)}$ are disjoint, which implies

$$1 \geq \sum_{i \in [K]} \Pr[A(\emptyset) \in G^{(i)}] \geq K \cdot 0.1 \cdot e^{-\epsilon n}.$$  

Thus, we have $n \geq \Omega\left(\frac{\log K}{\epsilon}\right) = \Omega(d/\epsilon)$ as desired.

Finally, we briefly sketch the proof of Lemma 9.

**Proof of Lemma 9.** It is well-known that there exist linear error correcting codes over $\mathbb{F}_2$ with constant rate and distance 0.4995. (See e.g. Section 7 for an explanation.) Equivalently, this means that there exists a linear space $V \subseteq \mathbb{F}_2^d$ of dimension $\Omega(d)$ such that $\|v\|_0 \in [0.4995d, 0.5005d]$ for all non-zero $v \in V$ where $\|\cdot\|_0$ denote the Hamming norm (i.e. number of non-zero coordinates).

Let $v^{(1)}, \ldots, v^{(K)}$ denote the elements of $V$ notice that $K = 2^{\dim(V)} = 2^{\Omega(d)}$. Define $w^{(1)}, \ldots, w^{(K)} \in \mathbb{R}^d$ where

$$w^{(i)}_j = \begin{cases} -1/\sqrt{d} & \text{if } v^{(i)}_j = 0 \\ +1/\sqrt{d} & \text{if } v^{(i)}_j = 1. \end{cases}$$

For $i \neq j$, we have

$$|\langle w^{(i)}, w^{(j)} \rangle| = |1 - 2 \cdot \|v^{(i)} - v^{(j)}\|_0/d| \leq 0.01d,$$

where the latter follows from linearity of $V$. This concludes our proof.
C Lower Bound for Robust Learning of Halfspaces: Approximate-DP Case

For our lower bound for approximate-DP proper learners (Theorem [3]), we will reduce from a lower bound of Steinke and Ullman [92]. To state their results, we will need some additional notation. Let \(\mathcal{U}_{[0,1]}\) denote the uniform distribution on \([0,1]\), and let \(\mathcal{B}_q\) denote the distribution that is \(+1/\sqrt{d}\) with probability \(q\) and \(-1/\sqrt{d}\) otherwise. For \(q \in [0,1]^d\), we use \(\mathcal{B}_q\) to denote \(\mathcal{B}_q \otimes \cdots \otimes \mathcal{B}_q\). Steinke and Ullman [92] prove the following theorem:

**Theorem 11 ([92] Theorem 3)**. Let \(\zeta > 0\) and \(n,d \in \mathbb{N}\) be such that \(n < \zeta \sqrt{d}\). Let \(\mathcal{M}\) be any \((1,\zeta/n)\)-DP algorithm whose output belongs to the \(d\)-dimensional unit Euclidean ball. Let \(X = (x_1, \ldots, x_n)\) be such that \(x_i\) is i.i.d. drawn from \(\mathcal{B}_q\). Then,

\[
\mathbb{E}_{q \sim \mathcal{U}^{\otimes d}_{[0,1]}, x \sim \mathcal{M}(\mathcal{B}_q^{\otimes n})} \left[ \sum_{j \in [d]} w_j \cdot (q_j - 0.5) \right] < \zeta \sqrt{d}.
\]

(4)

In the next subsection, we first show a lower bound of \(\Omega(\sqrt{d})\) for any sufficiently small constant \(\gamma\) (Lemma [13]). Then, in Subsection C.2 we use this to prove a lower bound of \(\Omega(\min\{\sqrt{d}/\gamma, d\})\).

C.1 Lower Bound for \(\gamma = \Omega(1)\)

We cannot use the distribution \(\mathcal{B}_q\) directly since it is not realizable with a large margin. To overcome this, we define \(\mathcal{P}_q\) as the distribution of \(x \sim \mathcal{B}_q\) conditioned on \((q', x) \geq 0.01\) where we write \(q'\) as a shorthand for \(\frac{1}{\sqrt{d}} (2q - 1)\). We will require the following bound:

**Lemma 12**. \(\mathbb{E}_{q \sim \mathcal{U}^{\otimes d}_{[0,1]} \mid d_{TV}(\mathcal{B}_q, \mathcal{P}_q)] \leq o(1/d)\).

**Proof.** The Chernoff bound implies that \(\Pr_{q \sim \mathcal{U}_{[0,1]}^{\otimes d}}[\|q\| \geq 0.1] \geq o(1/d)\). For a fixed \(q\) such that \(\|q\| \geq 0.1\), the Chernoff bound again yields that \(\Pr_{x \sim \mathcal{B}_q}[\langle q', x \rangle \geq 0.01] \leq o(1/d)\), which implies that \(d_{TV}(\mathcal{B}_q, \mathcal{P}_q)] \leq o(1/d)\). Combining these, we have \(\mathbb{E}_{q \sim \mathcal{U}^{\otimes d}_{[0,1]} \mid d_{TV}(\mathcal{B}_q, \mathcal{P}_q)] \leq o(1/d)\) as desired.

Let \(\tilde{\mathcal{P}}_q\) denote the distribution of \((x, +1)\) where \(x \sim \mathcal{P}_q\). Similarly, let \(\tilde{\mathcal{B}}_q\) denote the distribution of \((x, +1)\) where \(x \sim \mathcal{B}_q\). We can now prove our \(\Omega(\sqrt{d})\) lower bound for any sufficiently small constant \(\gamma > 0\), which follows almost immediately from the following lemma.

**Lemma 13**. For any constant \(\gamma, \beta \in (0,1)\) such that \(\gamma > 2\beta\), the following holds. Let \(\mathcal{A}\) be any \((1,o(1/n))\)-DP algorithm with sample complexity \(n\) and whose output belongs to the \(d\)-dimensional unit Euclidean ball. If

\[
\mathbb{E}_{q \sim \mathcal{U}^{\otimes d}_{[0,1]} \mid m, \mathcal{A}(\mathcal{P}_q^{\otimes n})\mid \text{err}_\gamma^m(w)] \leq \beta,
\]

then we must have \(n \geq \Omega(\sqrt{d})\).

**Proof.** Suppose for the sake of contradiction that there exists a \((1,o(1/n))\)-DP algorithm \(\mathcal{A}\) with sample complexity \(n = o(\sqrt{d})\) whose output is a \(d\)-dimensional vector of Euclidean norm at most one that satisfies \(\mathbb{E}_{q \sim \mathcal{U}^{\otimes d}_{[0,1]} \mid m, \mathcal{A}(\mathcal{P}_q^{\otimes n})\mid \text{err}_\gamma^m(w)] \leq \beta\).
On input $x_1, \ldots, x_n \in \{\pm 1/\sqrt{d}\}^d$, $M$ simply works as follows: Run $A$ on $(x_1, +1), \ldots, (x_n, +1)$ to obtain a halfspace $w$ and output $w$. Now, we have that

$$
\mathbb{E}_{q \sim \mathcal{U}_{[0,1]^d}, w \sim M(B_q^\otimes n)} \left[ \sum_{j \in [d]} w_j \cdot (q_j - 0.5) \right]
$$

$$
= \mathbb{E}_{q \sim \mathcal{U}_{[0,1]^d}, w \sim \mathcal{A}(B_q^\otimes n)} \left[ \sum_{j \in [d]} w_j \cdot (q_j - 0.5) \right]
$$

$$
\geq \mathbb{E}_{q \sim \mathcal{U}_{[0,1]^d}, w \sim \mathcal{A}(\mathcal{P}_q^\otimes n)} \left[ \sum_{j \in [d]} w_j \cdot (q_j - 0.5) \right] - \mathbb{E}_{q \sim \mathcal{U}_{[0,1]^d}} \left[ d_{TV}(\mathcal{P}_q^\otimes n, B_q^\otimes n) \cdot (0.5\sqrt{d}) \right]
$$

$$
\geq \mathbb{E}_{q \sim \mathcal{U}_{[0,1]^d}, w \sim \mathcal{A}(\mathcal{P}_q^\otimes n)} \left[ \sum_{j \in [d]} w_j \cdot (q_j - 0.5) \right] - 0.5n\sqrt{d} \cdot \mathbb{E}_{q \sim \mathcal{U}_{[0,1]^d}} \left[ d_{TV}(\mathcal{P}_q, B_q) \right]
$$

$$
\geq \mathbb{E}_{q \sim \mathcal{U}_{[0,1]^d}, w \sim \mathcal{A}(\mathcal{P}_q^\otimes n)} \left[ \sum_{j \in [d]} w_j \cdot (q_j - 0.5) \right] - o(n/\sqrt{d})
$$

$$
= \mathbb{E}_{q \sim \mathcal{U}_{[0,1]^d}, w \sim \mathcal{A}(\mathcal{P}_q^\otimes n)} \left[ \sum_{j \in [d]} w_j \cdot (q_j - 0.5) \right] - o(1), \quad (5)
$$

where in the first inequality we use the fact that $\|w\|_2 \leq 1$, which implies that $\|w\|_1 \leq \sqrt{d}$. Notice that we may rearrange the term inside the expectation in (5) as follows:

$$
\sum_{j \in [d]} w_j \cdot (q_j - 0.5) = \frac{\sqrt{d}}{2} \sum_{j \in [d]} w_j \cdot \frac{2q_j - 1}{\sqrt{d}}
$$

$$
= \frac{\sqrt{d}}{2} \langle w, q \rangle
$$

$$
= \frac{\sqrt{d}}{2} \langle w, \mathbb{E}_{x \sim B_q}[x] \rangle
$$

$$
= \frac{\sqrt{d}}{2} \cdot \mathbb{E}_{x \sim B_q}[(w, x)]
$$

$$
\geq \frac{\sqrt{d}}{2} \cdot \mathbb{E}_{x \sim \mathcal{P}_q}[(w, x)] - o(1)
$$

$$
\geq \frac{\sqrt{d}}{2} \left( \gamma \cdot \mathbb{Pr}_{x \sim \mathcal{P}_q}[\langle w, x \rangle \geq \gamma] - 1 \cdot \mathbb{Pr}_{x \sim \mathcal{P}_q}[\langle w, x \rangle < \gamma] \right) - o(1)
$$

$$
\geq \frac{\sqrt{d}}{2} \left( \gamma \cdot (1 - \text{err}_y^\mathcal{P}_q(w)) - \text{err}_y^\mathcal{P}_q(w) \right) - o(1)
$$

$$
\geq \frac{\sqrt{d}}{2} \left( \gamma - 2 \text{err}_y^\mathcal{P}_q(w) \right) - o(1)
$$

Plugging this back into (5), we have that

$$
\mathbb{E}_{q \sim \mathcal{U}_{[0,1]^d}, w \sim M(B_q^\otimes n)} \left[ \sum_{j \in [d]} w_j \cdot (q_j - 0.5) \right]
$$

$$
\geq \frac{\sqrt{d}}{2} \left( \gamma - 2 \text{err}_y^\mathcal{P}_q(w) \right) - o(1)
$$

$$
\geq \frac{\sqrt{d}}{2} \left( \gamma - 2\beta \right) - o(1)
$$
which contradicts Theorem 11. This concludes our proof.

C.2 Lower Bound for Smaller $\gamma$

We will now reduce from the case $\gamma = \Omega(1)$ to get a larger lower bound for smaller $\gamma$. To do this, it will be convenient to have an “expected version” of Lemma 13 which is stated and proved below.

**Lemma 14.** For any constants $\gamma_0, \beta_0 \in (0, 1)$ such that $\gamma_0 > 4\sqrt{2/\beta_0}$, the following holds. Let $\mathbb{B}$ be any $(1, o(1/n))$-DP algorithm that has access to an oracle $\mathcal{O}$ that can sample from $\mathcal{P}_q$ where $q$ is unknown to $\mathbb{B}$. All of the following cannot hold simultaneously:

1. The expected number of samples $\mathbb{B}$ draws from $\mathcal{O}$ is $o(\sqrt{d})$.
2. $E_{q \sim U_{[0,1]}^d, w \sim B_{\beta_0^q}}[\|w\|^2] \leq 1$.
3. $E_{q \sim U_{[0,1]}^d, w \sim B_{\beta_0^q}}[\text{err}_{\gamma_0}^q(w)] \leq \beta_0$.

**Proof.** Suppose for the sake of contradiction that there exists a $(1, o(1/n))$ algorithm $\mathbb{B}$ that draws $o(\sqrt{d})$ samples from $\mathcal{O}$ in expectation, and satisfies $E_{q \sim U_{[0,1]}^d, w \sim B_{\beta_0^q}}[\|w\|^2] \leq 1$ and $E_{q \sim U_{[0,1]}^d, w \sim B_{\beta_0^q}}[\text{err}_{\gamma_0}^q(w)] \leq \beta_0$. We use $\mathbb{B}$ to construct an algorithm $\mathcal{A}$ that will contradict Lemma 13 as follows:

- Run $\mathbb{B}$.
- If $\mathbb{B}$ attempts to take more than $2n/\beta_0$ sample, simply output $0$.
- Otherwise, let $w$ be the output of $\mathbb{B}$, and output $w' = \frac{w}{\|w\|}$.

Notice that $\mathcal{A}$ is $(1, o(1/n))$-DP and the number of samples used is $2n/\beta_0 = o(\sqrt{d})$.

Let $\beta = 2\beta_0$ and $\gamma = \gamma_0\sqrt{\beta_0/2}$. We will next argue that $E_{q \sim U_{[0,1]}^d, w' \sim \mathcal{A}(\mathcal{P}_q^\gamma)}[\text{err}_{\gamma}^q(w')] \leq \beta$. First, since the expected number of samples of $\mathbb{B}$ is $n$, by Markov’s inequality, the probability that $\mathbb{B}$ takes more than $2n/\beta_0$ samples is at most $\beta_0/2$. As a result, we have that

$$E_{q \sim U_{[0,1]}^d, w' \sim \mathcal{A}(\mathcal{P}_q^\gamma)}[\text{err}_{\gamma}^q(w')] \leq E_{q \sim U_{[0,1]}^d, w \sim B_{\beta_0^q}}[\text{err}_{\gamma}^q(w/\|w\|)] + \beta_0/2.$$

Recall also that $E_{q \sim U_{[0,1]}^d}[\|w\|^2] \leq 1$; Markov’s inequality once again implies that $\Pr_{q \sim U_{[0,1]}^d}[\|w\|^2 > 2/\beta_0] \leq \beta_0/2$. Plugging this into the above inequality, we get that

$$E_{q \sim U_{[0,1]}^d, w' \sim \mathcal{A}(\mathcal{P}_q^\gamma)}[\text{err}_{\gamma}^q(w')] \leq E_{q \sim U_{[0,1]}^d, w \sim B_{\beta_0^q}}[\text{err}_{\gamma}^q(w/\|w\|) \cdot 1[\|w\| \leq \sqrt{2/\beta_0}]] + \beta_0.$$

(From our third assumption on $\mathbb{B}$) $\leq 2\beta_0 = \beta$.

which is a contradiction to Lemma 13 since $\gamma > 2\beta$.}

We can now prove our $\Omega(\min\{\sqrt{d}/\gamma, d\})$ lower bound (Theorem 3). Roughly speaking, when $d \geq 1/\gamma^2$, we “embed” $\Theta(1/\gamma^2)$ hard distributions from Lemma 14 into $\Theta(\gamma^2 d)$ dimensions, which results in the $\Omega(\sqrt{\gamma^2 d} \cdot 1/\gamma^2) = \Omega(\sqrt{\gamma^3 d})$ lower bound as desired.

**Theorem 3.** Let $\epsilon < 1$. Any $(\epsilon, o(1/n))$-DP $(\gamma, 0.9\gamma)$-robust proper learner has sample complexity $n = \Omega(\min\{\sqrt{d}/\gamma, d\})$. 


**Proof.** We will prove this lower bound for \( \gamma \leq 0.01, \alpha, \xi \leq 10^{-6} \).

First, notice that, when \( \gamma \leq 1/\sqrt{d} \), a \((1/\sqrt{d}, 0.9\gamma)\)-robust proper learner is also an \((1/\sqrt{d}, 0)\)-robust proper learner. Hence, by Theorem 4.3, we have \( n = \Omega(d) \) as desired. Thus, we can subsequently only focus on the case \( \gamma \geq 1/\sqrt{d} \), for which we will show that \( n = \Omega(\sqrt{d}/\gamma) \).

Suppose for the sake of contradiction that there is a \((1, o(1/n))\)-DP \((\gamma, 0.9\gamma)\)-robust proper learner \( A \) with \( \alpha, \xi \leq 10^{-6} \) that has sample complexity \( n = o(\sqrt{d}/\gamma) \). Let \( T = [0.01/\gamma] \), and \( d' = [d/T^2] \). We will construct an algorithm \( \mathcal{B} \) that contradicts with Lemma 4.1 in \( d' \) dimensions.

We will henceforth assume w.l.o.g. that \( d = d' \cdot T^2 \). This is without loss of generality since the proof below extends to the case \( d > d' \cdot T^2 \) by padding \( d - d' \cdot T^2 \) zeros to each of the samples.

In the following, we view the \( d \)-dimensional space \( \mathbb{R}^d \) as the tensor \( \mathbb{R}^{T^2} \otimes \mathbb{R}^{d'} \). Furthermore, we write \( e_i \) as a shorthand for the \( i \)-th vector in the standard basis of \( \mathbb{R}^{d'} \).

The algorithm \( \mathcal{B} \) with an oracle \( \mathcal{O} \) to sample from \( \hat{P}_q \) where \( q \) is unknown to \( \mathcal{B} \) works as follows:

- Randomly draw \( q_1, \ldots, q_{T^2} \) i.i.d. from \( U_{[0,1]} \), and randomly sample \( i^* \in [T^2] \).
- Draw \( n \) samples \((x_1, y_1), \ldots, (x_n, y_n)\) independently as follows:
  - Draw \( i \in [T^2] \).
  - If \( i \neq i^* \), then draw \((x, y) \sim \hat{P}_q\) and let the sample be \((x \otimes e_i, y)\).
  - If \( i = i^* \), the draw \((x, y) \sim \hat{P}_q\) using \( \mathcal{O} \) and let the sample be \((x \otimes e_i, y)\).
- Run \( A \) on \((x_1, y_1), \ldots, (x_n, y_n)\). Suppose that the output halfspace is \( w \).
- Write \( w \) as \( \sum_{i \in [T^2]} w_i \otimes e_i \) for \( w_1, \ldots, w_{T^2} \in \mathbb{R}^{d'} \). Then, output \( T \cdot w^{i^*} \).

Clearly, \( \mathcal{B} \) is \((1, o(1/n))\)-DP and it takes \( n/T = o(\sqrt{d}) \) samples in expectation from \( \mathcal{P}_q \).

For the ease of presentation, we will write \( Q \) as a shorthand for the mixture of distribution where we draw \( i \sim [T] \), and return \((x \otimes e_i, y)\) where \((x, y) \sim \hat{P}_q\). Moreover, we write \( \tilde{Q} \) as a similar distribution but when \( q_{i^*} \) is replaced by \( q \). Under this notation, we have that

\[
\mathbb{E}_{q \sim U_{[0,1]}^{d/T^2}, w \sim B} \mathbb{P} \left[ \|w\|^2 \right] = \mathbb{E}_{q_1, \ldots, q_{T^2} \sim U_{[0,1]}^{d/T^2}, i^* \sim [T], w \sim A(\tilde{Q})} \left[ \|T \cdot w^{i^*}\|^2 \right] \\
= \mathbb{E}_{q_1, \ldots, q_{T^2} \sim U_{[0,1]}^{d/T^2}, i^* \sim [T], w \sim A(\tilde{Q})} \left[ \|T \cdot w^{i^*}\|^2 \right] \\
= \mathbb{E}_{q_1, \ldots, q_{T^2} \sim U_{[0,1]}^{d/T^2}, w \sim A(\tilde{Q})} \left[ \frac{1}{T^2} \sum_{i^* \in [T^2]} \|T \cdot w^{i^*}\|^2 \right] \\
\leq 1.
\]

Finally, we will argue the accuracy of \( \mathcal{B} \) where \( \gamma_0 = 0.01, \beta_0 = 2 \cdot 10^{-6} \). Once again we rewrite it as

\[
\mathbb{E}_{q \sim U_{[0,1]}^{d/T^2}, w \sim B} \mathbb{P} \left[ \text{err}_{\gamma_0}(w) \right] = \mathbb{E}_{q_1, \ldots, q_{T^2}, q, i^* \sim U_{[0,1]}^{d/T^2}, w \sim A(\tilde{Q})} \left[ \text{err}_{\gamma_0}(T \cdot w^{i^*}) \right] \\
= \mathbb{E}_{q_1, \ldots, q_{T^2}, i^* \sim U_{[0,1]}^{d/T^2}, \tilde{Q}} \left[ \text{err}_{\gamma_0} \left( T \cdot w^{i^*} \right) \right] \\
\text{(Since } \gamma_0/T \leq 0.1\gamma) \leq \mathbb{E}_{q_1, \ldots, q_{T^2}, i^* \sim U_{[0,1]}^{d/T^2}, \tilde{Q}} \left[ \text{err}_{0.1\gamma} \left( w^{i^*} \right) \right] \\
= \mathbb{E}_{q_1, \ldots, q_{T^2}, w \sim \tilde{Q}} \left[ \frac{1}{T^2} \sum_{i^* \in [T^2]} \text{err}_{0.1\gamma} \left( w^{i^*} \right) \right] \\
= \mathbb{E}_{q_1, \ldots, q_{T^2}, w \sim \tilde{Q}} \left[ \text{err}_{0.1\gamma} \left( w \right) \right].
\]
Now, notice that the halfspace \( w^* = \frac{1}{\pi} \sum_{i \in [T^2]} q_i^* \otimes e_i \) (whose Euclidean norm is at most one) correctly classifies each point in \( \text{supp}(Q) \) with margin \( 0.01 \geq \gamma \). As a result, the accuracy guarantee of \( \mathcal{A} \) ensures that \( \mathbb{E}_{w \sim \mathcal{A}(Q)}[\text{err}_{0.1\gamma}^{\mathcal{A}}(w)] \leq \alpha(1 - \xi) + \xi \leq \beta_0 \). Plugging into the above, we have

\[
\mathbb{E}_{q \sim U_{[0,1]^d}, w \sim \mathbb{P}_n}[\text{err}_{\gamma_{\beta_0}}^{\mathcal{A}}(w)] \leq \beta_0,
\]

which contradicts with Lemma 13. \( \square \)

### D Lower Bound for Non-Robust Learning of Halfspaces

In this section, we provide a lower bound of \( \Omega \left( \frac{1}{\epsilon^2} \right) \) on the sample complexity of non-robust learners (Theorem 4). While quantitatively similar, our lower bound significantly strengthens that of [74] in two aspects: (1) our lower bounds hold against even improper learners whereas the lower bound in [74] is only valid against proper learners and (2) our lower bound holds even against \( (\epsilon, \delta) \)-DP algorithms whereas that of [74] is only valid when \( \delta = 0 \).

**Theorem 4.** For any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that any \( (\epsilon, \delta) \)-DP \((\gamma, 0)\)-robust (possibly improper) learner has sample complexity \( \Omega \left( \frac{1}{\epsilon^2} \right) \). Moreover, this holds even when \( d = O(1/\gamma^2) \).

To prove the above, we will require the following simple lemma, which states that the task of outputting an input bit requires \( \Omega(1/\epsilon) \) equal samples in order to gain any non-trivial advantage over random guessing. The proof follows a straightforward packing argument.

**Lemma 15.** For \( s \in \{ \pm 1 \} \), let \( O_s \) denote the distribution which is \( s \) with probability 1. For any \( \epsilon > 0 \), there exists \( \delta = \Omega(1/\epsilon) \) such that the following holds: There is no \( (\epsilon, \delta) \)-DP algorithm that can take at most \( 10^{-5}/\epsilon \) samples in expectation from \( O_s \) for a random \( s \in \{ \pm 1 \} \) and output \( s \) correctly with probability 0.51.

**Proof.** We may assume that \( \epsilon < 1 \) as it is clear that the algorithm needs at least one sample to output \( s \) correctly with probability 0.51. Furthermore, let \( \delta = \frac{0.001}{1-\epsilon}. \)

Suppose for the sake of contradiction that there is an algorithm \( \mathcal{A} \) that takes in at most \( 10^{-5}/\epsilon \) samples in expectation and output \( s \) correctly with probability 0.51. By Markov inequality, with probability 0.999, \( \mathcal{A} \) takes at most \( n = \lfloor 0.01/\epsilon \rfloor \) samples. Let \( \mathcal{B} \) be the modification of \( \mathcal{A} \) where \( \mathcal{B} \) draws \( n \) samples and runs \( \mathcal{A} \) on them but fails whenever \( \mathcal{A} \) attempts to draw more than \( n \) samples. We have that \( \mathcal{B} \) outputs \( s \) correctly with probability 0.509. In other words, we have

\[
\text{Pr}[\mathcal{B}(s^n) = s] \geq 0.509,
\]

where \( s^n \) denote \( n \) inputs all equal to \( s \).

Since \( \mathcal{A} \) is \( (\epsilon, \delta) \)-DP, \( \mathcal{B} \) is also \( (\epsilon, \delta) \)-DP. Suppose without loss of generality that \( \text{Pr}[\mathcal{B}(\emptyset) \neq 1] \geq \text{Pr}[\mathcal{B}(\emptyset) \neq -1] \). This implies that \( \text{Pr}[\mathcal{B}(\emptyset) \neq 1] \geq 0.5 \). From \( (\epsilon, \delta) \)-DP of \( \mathcal{B} \), we have

\[
\text{Pr}[\mathcal{B}(1^n) \neq 1] \geq e^{-\epsilon} \text{Pr}[\mathcal{B}(1^{n-1}) \neq 1] - \delta
\]

\[
\vdots
\]

\[
\geq e^{-n\epsilon} \text{Pr}[\mathcal{B}(\emptyset) \neq 1] - \delta(1 + e^{-\epsilon} + \cdots + e^{-n\epsilon})
\]

\[
\geq e^{-0.01} \cdot 0.5 - 0.001
\]

\[
> 0.491
\]

which contradicts [6]. This concludes our proof. \( \square \)

We can now prove Theorem 4. Roughly speaking, we “embed” the hard problem in Lemma 15 into each of the \( d = 1/\gamma^2 \) dimensions, which results in the \( d \cdot \Omega(1/\epsilon) = \Omega \left( \frac{1}{\epsilon^2} \right) \) lower bound.

**Proof of Theorem 4**. We prove this statement for any \( \gamma < 1, \alpha \leq 0.4 \) and \( \xi \leq 0.0001 \).
Let $\delta$ be the same as in Lemma 15 and let $d = \lfloor 1/\gamma^2 \rfloor$. Suppose for the sake of contradiction that there exists an $(\epsilon, \delta)$-DP $(\gamma, 0.9\gamma)$-robust learner $A$ that takes in at most $n := \lfloor 10^{-5}d/\epsilon \rfloor$ samples and outputs a hypothesis with error at most $\alpha \leq 0.4$ with probability $1 - \xi \geq 0.9999$. We will use $A$ to construct an algorithm $B$ that can solve the problem in Lemma 15.

For every $i \in [d]$ and $s \in \{\pm 1\}$, we use $D_{i,s}$ to denote the uniform distribution on $(e_i, s)$ and $(-e_i, -s)$. Furthermore, for $s \in \{\pm 1\}^d$, we use $D_s$ to denote the mixture $\frac{1}{d} \sum_{i \in [d]} D_{i,s_i}$. Our algorithm $B$ works as follows:

- Randomly sample $s \in \{\pm 1\}^d$ and randomly sample $i^* \in [d]$.
- Draw $n$ samples $(x_1, y_1), \ldots, (x_n, y_n)$ independently as follows:
  - Randomly pick $i \in [d]$.
  - If $i \neq i^*$, then return a sample drawn from $D_{i,s}$.
  - Otherwise, if $i = i^*$, sample $a \sim O_s$. Then return the sample $(e_i, a)$ with probability 0.5; otherwise, return the sample $(-e_i, -a)$.
- Run $A$ on $(x_1, y_1), \ldots, (x_n, y_n)$ to get a hypothesis $h$.
- With probability 0.5, return $h(e_{i^*})$. Otherwise, return $-h(-e_{i^*})$.

It is obvious to see that $B$ is $(\epsilon, \delta)$-DP and that the expected number of samples $B$ draws from $O_s$ is $n/d \leq 10^{-5}/\epsilon$. Hence, we only need to show that $B$ outputs a correct answer with probability 0.51 to get a contradiction with Lemma 15.

Since $s$ is uniformly drawn from $\{\pm 1\}$, the probability that $B$ outputs the incorrect answer is equal to

$$
\mathbb{E}_{s \sim \{\pm 1\}^d, i \in [d], h \sim A(D_s) \cup n} \left[ \frac{1}{2} \mathbb{1} [h(e_i) \neq s_i] + \frac{1}{2} \mathbb{1} [h(-e_i) \neq -s_i] \right]
$$

$$
= \mathbb{E}_{s \sim \{\pm 1\}^d} \left[ \frac{1}{d} \sum_{i \in [d]} \left( \frac{1}{2} \mathbb{1} [h(e_i) \neq s_i] + \frac{1}{2} \mathbb{1} [h(-e_i) \neq -s_i] \right) \right]
$$

$$
= \mathbb{E}_{s \sim \{\pm 1\}^d} \left[ \text{err}_{D_s}(h) \right].
$$

Now, notice that any $(x, y) \in \text{supp}(D_s)$ is correctly classified by the halfspace $z := \frac{1}{\sqrt{d}} \sum_{i \in [d]} e_i$ with margin $1/\sqrt{d} \geq \gamma$. As a result, the accuracy guarantee of $A$ ensures that $\mathbb{E}_{h \sim A(D_s) \cup n} [\text{err}_{D_s}(h)] \leq 1 - 0.0001 + 0.4 \cdot 0.9999 < 0.41$. Thus, we can conclude that $B$ outputs the correct answer with probability at least $1 - 0.41 > 0.59$. This contradicts Lemma 15. 

**E Pure DP Robust Learner**

In this section, we give a pure-DP algorithm for robust learning of halfspaces:

**Theorem 5.** There is an $\epsilon$-DP $(\gamma, 0.9\gamma)$-robust learner with sample complexity $\Omega(\frac{1}{\epsilon} \max\{d, \frac{1}{\gamma^2}\})$.

To prove this result, we will also need the following generalization bound due to Bartlett and Mendelson [11]:

**Lemma 16** (Generalization Bound for Large Margin Halfspaces [11]). Suppose $\gamma, \tilde{\gamma} \in [0, 1]$ and let $D$ be any distribution on $\mathbb{B}^d \times \{\pm 1\}$. If we let $X$ be drawn from $D \otimes n$, then the following holds with probability $1 - \tilde{\gamma}$:

$$
\forall \mathbf{w} \in \mathbb{B}^d, \text{err}_{0.95\gamma}(\mathbf{w}) \leq \text{err}_{X}(\mathbf{w}) + 400 \sqrt{\frac{\ln(1/\tilde{\gamma})}{n\gamma^2}}.
$$

**Proof of Theorem 5.** We will prove this for $\xi = 0.9$. Let $\Lambda = 10^6 \cdot \sqrt{\log(1/\alpha)} \cdot \max\{\sqrt{d}, 1/\gamma\}$, and $n = 10^4\Lambda^2 + \frac{10^4}{\alpha^2 \gamma^2} = O\left(\frac{\log(1/\alpha)}{\alpha^2} \cdot \max\{d, 1/\gamma^2\} + \frac{1}{\alpha^2 \gamma^2}\right)$.

Here $O_\alpha(\cdot)$ hides a factor of $\text{poly}(1/\alpha)$, and $O(\cdot)$ hides a factor of $\text{poly} \log(1/(\alpha \gamma \delta))$. 


We will next argue the accuracy guarantee of the algorithm. Similar to [71], let 
\[X = ((x_1, y_1),\ldots,(x_n, y_n))\], we define the scoring function \(q\) by
\[q(X, w) = -n \cdot \text{err}_{0.95\gamma}(w).\]
Then, we output \(\hat{w}\) drawn from the distribution with density \(\mu'(w) \propto \mu(w) \cdot \exp\left(\frac{1}{2} \cdot q(X, w)\right)\).
We will next argue the accuracy guarantee of the algorithm. Similar to [71], let \(S_t := \{w \mid q(X, w) \geq t\}\). We start by showing that, with probability 0.99, we have \(\hat{w} \in S_{0.5\alpha n}\). To prove this, we will use the following result from [71]:

**Lemma 17.** For any \(t \geq 0\), \(\Pr[\hat{w} \notin S_t] \leq \exp(-\varepsilon t/2)/\mu(S_t)\).

In light of Lemma [17] it suffices for us to provide a lower bound for \(\mu(S_{0.25\alpha n})\). Recall from the realizable assumption that, there exists a unit-norm \(w^*\) such that \(\text{err}_{X}^{X}(w^*) = 0\). Since \(\mu(S_{0.25\alpha n})\) is rotational-invariant, we may assume for notational convenience that \(w^* = e_d\), the \(d\)-th vector in the standard basis. In this notation, a sample \(w \sim \mu\) may be obtained by:

1. Sample \(w_d \sim N(0,1)\),
2. Sample \(w_\perp \sim N(0, I_{(d-1) \times (d-1)})\),
3. Let \(w = \frac{1}{T} (w_\perp \circ w_d)\) where \(T = \sqrt{\|w_\perp\|^2 + w_d^2}\).

Fix \(i \in [n]\). We will now bound the probability \(\Pr[y_i \langle w, x_i \rangle \leq 0.95\gamma \mid w_d \geq \Lambda]\). Let us write \(y_i x_i\) as \(x_\perp \circ x_d\). \(\langle w_\perp, x_\perp \rangle\) is distributed as \(N(0, \|x_\perp\|)\). Since \(\|x_\perp\| \leq 1\), we may apply standard tail bound of Gaussian which gives
\[
\Pr[\|w_\perp\| < -0.01\Lambda/\gamma] \leq \Pr[\|w_\perp\| < 4 \sqrt{\log(1/\alpha)}] \leq 0.1\alpha. \tag{7}
\]

Observe also that \(\|w_\perp\|^2\) is simply distributed as \(\chi^2_{d-1}\). Hence, via standard tail bound (e.g., [63]), we have
\[
\Pr[\|w_\perp\| > 0.01\Lambda] \leq \Pr[\|w_\perp\| > 4 \sqrt{d \log(1/\alpha)}] \leq 0.1\alpha. \tag{8}
\]

Furthermore, notice that when \(w_d \geq \Lambda\), \(\langle w_\perp, x_i \rangle \geq -0.01\Lambda/\gamma\) and \(\|w_\perp\| \leq 0.01\Lambda\), we have \(y_i \langle w, x_i \rangle > 0.95\gamma\).

As a result, a union bound and the independence of \(w_d\) and \(w_\perp\) implies that
\[
\Pr[y_i \langle w, x_i \rangle \leq 0.95\gamma \mid w_d \geq \Lambda] \leq \Pr[\|w_\perp\| < -0.01\Lambda/\gamma] + \Pr[\|w_\perp\| > 0.01\Lambda] \leq 0.2\alpha. \tag{9}
\]

From (9) and from the linearity of the expectation, we have that
\[
\mathbb{E}[\|i \in [n] \mid y_i \langle w, x_i \rangle \leq 0.95\gamma\| \mid w_d \geq \Lambda] \leq 0.2\alpha n.
\]
By Markov’s inequality, we may conclude that
\[
\Pr[\|w \in S_{0.25\alpha n} \mid w_d \geq \Lambda] \geq 0.1.
\]
Finally, recall that \(w_d\) is distributed as \(N(0,1)\), which implies that \(\Pr[w_d \geq \Lambda] \geq 2^{-10\Lambda^2}\). This gives
\[
\mu(S_{0.25\alpha n}) = \Pr[\|w \in S_{0.25\alpha n} \mid w_d \geq \Lambda] \Pr[w_d \geq \Lambda] \geq 0.1 \cdot 2^{-10\Lambda^2} \geq 2^{-20\Lambda^2}. \tag{10}
\]
Hence, applying Lemma [17] we get that
\[
\Pr[\hat{w} \notin S_{0.5\alpha n}] \leq \frac{\exp(-\varepsilon t/2)}{\mu(S_t)} \leq \frac{\exp(-0.125\alpha n)}{2^{-20\Lambda^2}}. \tag{10}
\]
(From our choice of \(n\)) \(\leq 0.99\).

In other words, with probability 0.99, we have \(\text{err}_{X,0.95\gamma}(\hat{w}) \leq 0.5\alpha\). Finally, via the generalization bound (Lemma [16] with \(\gamma = 0.95\gamma\)), we also have \(\text{err}_{D,0.95\gamma}(\hat{w}) \leq \alpha\) with probability 0.9 as desired.
F Approximate-DP Robust Learner

In this section, we describe our approximate-DP learner and prove its guarantee, restated below:

**Theorem 6.** There is an \((\epsilon, \delta)\)-DP \((\gamma, 0.9\gamma)\)-robust learner with sample complexity \(n = \tilde{O}_\alpha \left( \frac{1}{\epsilon} \cdot \max \left\{ \frac{\sqrt{n}}{\gamma}, \frac{1}{\gamma} \right\} \right)\) and running time \(\tilde{O}_\alpha (nd/\gamma)\).

As alluded to earlier, this algorithm is a noised and batch version of the margin perceptron algorithm \([34, 27]\). The algorithm is presented in Algorithm 1.

The rest of this section is organized as follows. In the next subsection, we provide the utility analysis of the algorithm. Then, in Subsection F.2, we analyze its privacy guarantee. Finally, we set the parameters and prove Theorem 6 in Section F.3.

F.1 Utility Analysis

Suppose that there exists \(w^* \in \mathbb{R}^d\) with \(\text{err}_\gamma(w^*) = 0\). Furthermore, let \(\gamma' = 0.95\gamma\), \(\gamma_{\text{gap}} := \gamma - \gamma'\) and \(B := pn\). Throughout the analysis, we will assume that the following “good” events occur:

- **\(E_{\text{batch-size}}\):** For all \(i \in [T]\), \(|S_i| \leq 1.5B\).
- **\(E_{\text{noise-norm}}\):** For all \(i \in [T]\), \(\|g_i\| \leq B\sqrt{\alpha}\).
- **\(E_{\text{parallel}}\):** For all \(i \in [T]\), \(\langle w_{i-1} + u_i, g_i \rangle \leq 0.01\alpha \gamma_{\text{gap}} B \cdot \|w_{i-1} + u_i\|\).
- **\(E_{\text{opt-noise}}\):** For all \(i \in [T]\), \(\langle w^*, g_i \rangle \geq -0.01\alpha \gamma_{\text{gap}} B\).
- **\(E_{\text{mistake-noise}}\):** For all \(i \in [T]\), \(\nu_i \in [-0.1\alpha B, 0.1\alpha B]\).
- **\(E_{\text{sampled-mistake}}\):** For all \(i \in [T]\) such that \(^{13}\) \(\text{err}_\gamma^X(\langle w_{i-1} \rangle_{\|w_{i-1}\|}) > 0.5\alpha\), we have \(|M_i| \geq 0.4\alpha B\).

Later on, we will select the parameters \(p, n, T, b, \sigma\) so that these events happen with high probability.

**Lemma 18.** Let \(T = \lceil \frac{1500}{\alpha \gamma_{\text{gap}}} \rceil\). If the events \(E_{\text{batch-size}}, E_{\text{noise-norm}}, E_{\text{parallel}}, E_{\text{opt-noise}}, E_{\text{mistake-noise}}\) and \(E_{\text{sampled-mistake}}\) all occur, then the algorithm outputs \(w\) such that \(\text{err}_\gamma^X(w) \leq 0.5\alpha\).

**Proof.** We will show that we always execute Line [10] Once this is the case, \(E_{\text{mistake-noise}}\) and \(E_{\text{sampled-mistake}}\) imply that the output \(w_i/\|w_i\|\) satisfies \(\text{err}_\gamma^X(\langle w_i/\|w_i\| \rangle) \leq 0.5\alpha\) as desired.

To prove that we execute Line 10, let us assume for the sake of contradiction that this is not the case, i.e., that the algorithm continues until reaching the end of the \(T\)-th iteration.

From our assumption that \(E_{\text{mistake-noise}}\) occurs and from the fact that Line [10] was not executed, we have \(|M_i| \geq 0.2\alpha B\) for all \(i \in [T]\). Let \(m_i := \sum_{j \in [i]} |M_i|\) denote the number of \(\gamma'\)-margin mistakes seen up until the end of the \(i\)-th iteration; from the previous bound on \(M_i\), we have

\[
m_i \geq 0.2\alpha Bi. \tag{11}\]

Now, notice that

\[
\langle w^*, w_T \rangle = \left\langle w^*, \left( \sum_{i \in [T]} \sum_{(x,y) \in M_i} y \cdot x \right) + \sum_{i \in [T]} g_i \right\rangle = \sum_{i \in [T]} \sum_{(x,y) \in M_i} y \cdot \langle w^*, x \rangle + \sum_{i \in [T]} \langle w^*, g_i \rangle
\]

\(^{13}\)Similar to before, we use \(X\) to denote \((x_1, y_1), \ldots, (x_n, y_n)\).
We can bound \( \text{err}_T^X(w^*) = 0 \) and \( E_{\text{opt-noise}} \),
\[
\text{From } \left( \sum_{i \in [T]} \sum_{(x,y) \in M_i} \gamma \right) + \sum_{i \in [T]} -0.01 \alpha \gamma \text{gap} \bar{B} \\
= m_T \gamma - 0.01 \alpha \gamma \text{gap} \cdot BT \\
\geq m_T (\gamma - 0.05 \gamma \text{gap}).
\]
Furthermore, for every \( i \in [T] \), we have that
\[
\|w_i\|^2 = \|w_{i-1} + u_i + g_i\|^2 \\
= \|w_{i-1} + u_i\|^2 + 2 \langle w_{i-1} + u_i, g_i \rangle + \|g_i\|^2 \\
\text{(From } E_{\text{parallel}}) \leq \|w_{i-1} + u_i\|^2 + 0.02 \alpha \gamma \text{gap} \cdot \|w_{i-1} + u_i\| + \|g_i\|^2 \\
\text{(From } E_{\text{noise-norm}}) \leq \|w_{i-1} + u_i\|^2 + 0.02 \alpha \gamma \text{gap} \cdot \|w_{i-1} + u_i\| + \alpha B^2 \\
\leq \|w_{i-1}\|^2 + 2 \langle w_{i-1}, u_i \rangle + \|u_i\|^2 + 0.02 \alpha \gamma \text{gap} \cdot (\|w_{i-1}\| + \|u_i\|) + \alpha B^2
\]
We can bound \( \langle w_{i-1}, u_i \rangle \) as follows:
\[
\langle w_{i-1}, u_i \rangle = \sum_{(x,y) \in M_i} y \cdot \langle w_{i-1}, x \rangle \leq |M_i| \cdot \gamma' \|w_{i-1}\|,
\]
where the inequality follows from the condition on Line 6.
Furthermore, we also have that
\[
\|u_i\| = \left\| \sum_{(x,y) \in M_i} y \cdot x \right\| \leq \sum_{(x,y) \in M_i} \|x\| \leq |M_i|.
\]
Plugging the above two inequalities into \[13\], we get
\[
\|w_i\|^2 \leq \|w_{i-1}\|^2 + (2|M_i| \gamma' + 0.02 \alpha \gamma \text{gap} \cdot \|w_{i-1}\| + (|M_i|^2 + 0.02 \alpha \gamma \text{gap} |M_i|) + \alpha B^2) \\
\leq \|w_{i-1}\|^2 + (2|M_i| \gamma' + 0.02 \alpha \gamma \text{gap} \cdot \|w_{i-1}\| + (|M_i|^2 + 0.02 \alpha B \cdot |M_i| + \alpha B^2) \\
\leq \|w_{i-1}\|^2 + (2|M_i| \gamma' + 0.02 \alpha \gamma \text{gap} \cdot \|w_{i-1}\| + 2B|M_i| + \alpha B^2,
\]
where in the last inequality we use the fact that \( |M_i| \leq 1.5B \) which follows from \( E_{\text{batch-size}} \).
The above inequality implies that
\[
\|w_i\| \leq \|w_{i-1}\| + (|M_i| \gamma' + 0.01 \alpha \gamma \text{gap} B) + \frac{B|M_i| + 0.5 \alpha B^2}{\|w_{i-1}\|}.
\]
Notice that when \( \|w_{i-1}\| \geq \frac{100B}{\gamma \text{gap}} \), we have that
\[
\|w_i\| \leq \|w_{i-1}\| + (|M_i| \gamma' + 0.01 \alpha \gamma \text{gap} B) + 0.01|M_i| \gamma \text{gap} + 0.01 \alpha \gamma \text{gap} B \\
= \|w_{i-1}\| + 0.02 \alpha \gamma \text{gap} B + (\gamma' + 0.01 \gamma \text{gap}) \cdot |M_i|.
\]
As a result, we get\[12\]
\[
\|w_T\| \leq \frac{200B}{\gamma \text{gap}} + 0.02 \alpha \gamma \text{gap} \cdot BT + (\gamma' + 0.01 \gamma \text{gap}) \cdot \left( \sum_{i \in [T]} |M_i| \right) \\
= \frac{200B}{\gamma \text{gap}} + 0.02 \alpha \gamma \text{gap} \cdot BT + (\gamma' + 0.01 \gamma \text{gap}) \cdot m_T
\]
\[12\text{Note that the first term } \frac{200B}{\gamma \text{gap}} \text{ comes from an observation that if } i_0 \text{ is the smallest index for which } \|w_{i_0}\| \geq \frac{100B}{\gamma \text{gap}}, \text{ then } \[13\text{ implies that } \|w_{i_0}\| \leq \frac{200B}{\gamma \text{gap}}.\]
\[ m_T(\gamma - 0.05\gamma_{\text{gap}}) \leq \frac{200B}{\gamma_{\text{gap}}} + m_T(\gamma' + 0.11\gamma_{\text{gap}}), \tag{14} \]

From \([12]\) and \([14]\), we have

\[ m_T(\gamma - 0.05\gamma_{\text{gap}}) \leq \frac{200B}{\gamma_{\text{gap}}} + m_T(\gamma' + 0.11\gamma_{\text{gap}}), \]

which implies that

\[ m_T \leq \frac{200B}{\gamma_{\text{gap}}} (\gamma - \gamma' - 0.16\gamma_{\text{gap}}) \]

\[ \leq \frac{200B}{0.8\gamma_{\text{gap}}} \]

\[ = 250B/\gamma_{\text{gap}}. \]

which contradicts \([11]\) and our choice of \( T = \lceil \frac{1500}{\alpha\gamma_{\text{gap}}} \rceil \).

\[ \square \]

\section*{F.2 Privacy Analysis}

**Lemma 19.** For any \( \epsilon, \delta \in (0, 1) \) and any \( T \in \mathbb{N} \), let \( p = \frac{1}{\sqrt{T}}, \sigma = \frac{100\ln(T/\delta)}{\epsilon} \) and \( b = \frac{100\sqrt{\ln(T/\delta)}}{\epsilon} \). Then, Algorithm \([7]\) is \((\epsilon, \delta)\)-DP.

To prove this, we require the following results on amplification by subsampling\([9]\) and advanced composition.

**Lemma 20 (Amplification by Subsampling \([9]\)).** Let \( \mathcal{A} \) be any \((\epsilon_0, \delta_0)\)-DP algorithm such that \( \epsilon_0, \delta_0 \in (0, 1) \). Let \( \mathcal{B} \) be an algorithm that independently selects each input sample w.p. \( \mathcal{p} \) and runs \( \mathcal{A} \) on this subsampled input dataset. Then, \( \mathcal{B} \) is \((2p\epsilon_0, p\delta_0)\)-DP.

**Lemma 21 (Advanced Composition \([38]\)).** Suppose that \( \mathcal{B} \) is an algorithm resulting from running an \((\epsilon_0, \delta_0)\)-DP algorithm \( T \) times (possibly adaptively), where \( \epsilon_0, \delta_0 \in (0, 1) \). Then, \( \mathcal{B} \) is \((\epsilon', (T+1)\delta_0)\)-DP where

\[ \epsilon' = \sqrt{2T \ln(1/\delta_0)} \cdot \epsilon_0 + 2T \epsilon_0^2. \]

**Proof of Lemma \([19]\)** Let \( \epsilon_0 = \frac{\epsilon}{20\sqrt{\ln(T/\delta)}} \) and \( \delta_0 = \frac{\delta}{2\sqrt{T}} \). The Gaussian mechanism with noise standard deviation \( \sigma \) is \((0.5\epsilon_0, \delta_0)\)-DP \([8]\), Appendix A) whereas the Laplace mechanism with parameter \( b \) is \(0.5\epsilon_0\)-DP \([26]\, 14\). As a result, without subsampling, each iteration is \((\epsilon_0, \delta_0)\)-DP. With the subsampling, Lemma \([20]\) implies that each iteration is \((2p\epsilon_0, p\delta_0)\)-DP. Finally, we may apply Lemma \([21]\) ensures that the final algorithm is \((\epsilon', \delta')\)-DP for

\[ \epsilon' = \sqrt{2T \ln(1/(p\delta_0))} \cdot (2p\epsilon_0) + 2T(2p\epsilon_0)^2 \leq \epsilon, \]

and

\[ \delta' = (T+1)p\delta_0 \leq 2\sqrt{T}\delta_0 = \delta, \]

which concludes our proof. \( \square \)

\section*{F.3 Putting Things Together}

**Proof of Theorem \([6]\)** Let \( T = \lceil \frac{1500}{\alpha\gamma_{\text{gap}}} \rceil \) be as in Lemma \([18]\) and let \( p = \frac{1}{\sqrt{T}}, \sigma = \frac{100\ln(T/\delta)}{\epsilon} \) and \( b = \frac{100\sqrt{\ln(T/\delta)}}{\epsilon} \) be as in Lemma \([19]\). Finally, let \( n = \lceil \frac{100\sqrt{2}\sigma \log T}{p\sqrt{\alpha}} + \frac{10000\sigma \sqrt{\log T}}{p\alpha^2 \gamma} + \frac{100 \log T}{\alpha} + \frac{10^{10}}{\alpha^2 \gamma} \rceil \). Notice that \( n = O_{\alpha} \left( \frac{1}{\gamma} \left( \sqrt{\sigma} + \frac{\epsilon}{2} \right) \cdot (\log T)^2 \right) \) as claimed.

\[ \text{Amplification by subsampling results are often stated with the new } \epsilon \text{ being } \ln(1 + p(\epsilon' - 1)) \text{ which is no more than } 2p \epsilon \text{ (from Bernoulli’s inequality and from } 1 + x \leq e^x \text{ for all } x \in \mathbb{R}). \]

\[ \text{In both cases, the } \ell_2 \text{ sensitivity and the } \ell_3 \text{ sensitivity respectively are bounded by one. For the former, this is because each sample effects } \mathbf{w} \text{ only by } y \cdot \mathbf{x} \text{ and } \|y \cdot \mathbf{x}\|_2 = \|\mathbf{x}\| \leq 1. \]
From Lemma 19, our algorithm with the above parameters is \((\epsilon, \delta)\)-DP. Furthermore, the expected running time of the algorithm is \(pnT = O(n\sqrt{T}) = O\left(\frac{n}{\sqrt{\gamma}}\right)\). Moreover, it can be verified via standard concentration inequalities that all of the events required in Lemma 18 happen w.p. 0.99, which means that we output a halfspace \(w\) with \(\text{err}_{X^\gamma}(w) \leq 0.5\alpha\). Finally, the generalization bound (Lemma 16 with \(\hat{\gamma} = \gamma'\)) implies that \(\text{err}_{D^\gamma}(w) \leq \alpha\) as desired.

### G Additional Experiments

#### G.1 Adversarial Robustness Evaluation on USPS Dataset

![Graphs showing robust accuracy comparison between DP-SGD-trained Convolutional neural networks and DP Batch Perceptron halfspace classifiers on USPS dataset for a fixed privacy budget.](image)

Figure 3: Robustness accuracy comparison between DP-SGD-trained Convolutional neural networks and DP Batch Perceptron halfspace classifiers on USPS dataset for a fixed privacy budget. In all three plots, \(\delta = 10^{-5}\) but \(\epsilon\) varies from 0.5, 1, and 2.

We compare the robust accuracy of DP Batch Perceptron classifiers and DP-SGD-trained neural networks in Figure 3 for \(\delta = 10^{-4}\) and \(\epsilon = 0.5, 1, 2\). The architecture and parameters follow the same setup described in Section 4. In the case of \(\epsilon = 0.5\), while both classifiers have similar test accuracies (without any perturbation, \(\gamma = 0\)), as \(\gamma\) increases, the robust accuracy rapidly degrades for the DP-SGD-trained neural network compared to that of the DP Batch Perceptron model. This overall trend persists for \(\epsilon = 1\); the CNN starts off with larger test accuracy when \(\gamma = 0\) but is eventually surpassed by the halfspace classifier as \(\gamma\) increases. On the other hand, when \(\epsilon = 2\), the CNN maintains slightly higher robust accuracy for most perturbation norms in consideration.

#### G.2 Experiments with Gaussian Kernel

It is well-known that accuracy of linear classifiers for digit classifications can be significantly improved via kernel methods (see, e.g., [64, 84]). Here we would like to privately train linear classifiers with Gaussian kernels. Recall that the Gaussian kernel is of the form

\[
k(x, x') = \exp\left(-\frac{||x - x'||^2}{2\sigma^2}\right),
\]

where \(\sigma\) is the so-called width parameter.

Unlike the standard (non-kernel) setting, it is unclear in the Gaussian kernel setting how the noise should be added to obtain DP; the kernel space themselves is not of finite dimension, and the classifier is typically only implicitly represented. To handle this, we follow the approach of Rahimi and Recht [80] (also used in DP-SVM [82]). Specifically, [80] shows that the following approximate embedding \(\phi : \mathbb{R}^d \rightarrow \mathbb{B}^{2\hat{d}}\) has a property that \(\langle \phi(x), \phi(x') \rangle\) is close to \(k(x, x')\):

\[
\phi(x) := \frac{1}{\sqrt{d}} \left(\cos(\rho_1, x), \ldots, \cos(\rho_{\hat{d}}, x), \sin(\rho_1, x), \ldots, \sin(\rho_{\hat{d}}, x)\right),
\]

where \(\rho_1, \ldots, \rho_{\hat{d}}\) are i.i.d. sampled from \(\mathcal{N}(0, \frac{1}{\sigma^2} \cdot I_{d \times d})\). Below we write \(\sigma^*\) to denote \(1/\hat{\sigma}\).

To summarize, this approach allows us to train with (approximate) Gaussian kernel as follows (where \(\sigma^*, \hat{d}\) are hyperparameters):

1. Randomly sample \(\rho_1, \ldots, \rho_{\hat{d}}\) i.i.d. from \(\mathcal{N}(0, (\sigma^*)^2 \cdot I_{d \times d})\).
Robust and Private Learning of Halfspaces

2. For each class \( y \), use DP-Batch-Perceptron on \((\hat{\phi}(x_1), y_1), \ldots, (\hat{\phi}(x_n), y_n)\) to train a halfspace \( w^{(y)} \in \mathbb{R}^{2d} \) for the \( y \)-vs-rest classifier.

3. When we would like to predict \( x \in \mathbb{R}^d \), compute \( \text{argmax}_{y \in \{1, \ldots, 10\}} \langle w^{(y)}, \hat{\phi}(x) \rangle \).

Notice here that the DP guarantee (in the second step) is exactly the same as the DP-Batch-Perceptron guarantee for the non-kernel setting. Similar to Figure 1 we report the (non-robust) test accuracy of DP Batch Perceptron algorithm with Gaussian kernel included in Figure 4 across different \( \epsilon \) values (first column) and different \( \delta \) values (middle column). We find that kernel learning helps to boost performance overall, the gain in accuracy is particularly significant in the case of MNIST dataset (top row).

Figure 4: Performance on the MNIST (top row) and USPS (bottom row) datasets with Gaussian kernel. The horizontal dotted line indicates performance when \( \epsilon = \infty \) (no noise). The width of the kernel for both datasets is tuned as a hyperparameter with values 2, 3.5, 5, 7.5, 10.

As for noise addition, the robustness guarantee for such kernel classifiers is also more complicated than the non-kernel linear classifiers. In Section G.2.1 below, we provide a provable robustness guarantee of the kernel classifiers. Using this provable guarantee, the empirical robustness accuracy is shown in the last column of Figure 4 and its comparison to DP-SGD-trained CNNs is shown in Figure 5. Even though the kernel classifiers start off with similar accuracy (at \( \gamma = 0 \)), it quickly drops and becomes worse than CNNs. We remark here that, in addition to the nature of the kernel, this may also be exacerbated by the fact that the provable robust guarantee for the kernel classifiers is not tight (unlike the non-kernel case).

G.2.1 Robustness Guarantee for Multi-Class Perceptron with Kernel

To compute the robust error for the kernel classifiers, we will use the following result, which is a slightly simplified version of Theorem 2.1 from [53].

**Lemma 22.** Let \( M \) be a classifier which, for each class \( y \in \{1, \ldots, k\} \), computes some function \( f^y : \mathbb{R}^d \to \mathbb{R} \) and predicts the class \( y^* \) that minimizes \( f^y \). Then, for every example \( (x, y) \) and any \( \Delta \in \mathbb{R}^d \) such that

\[
\|\Delta\| \leq \min_{y' \neq y} \sup_{x' \in \mathbb{R}^d} \left\| \nabla f^y(x') - \nabla f^{y'}(x') \right\|
\]

the classifier \( M \) predicts \( y \) on \( x + \Delta \).
As a result, for two classes \( y, y' \) we may bound
\[
\|
\begin{align*}
\frac{\partial}{\partial x} y \cdot \sin(\langle \rho_i, x' \rangle) + w_i^{(y)} \cdot \cos(\langle \rho_i, x' \rangle)
\end{align*}
\]
\( \rho_i \).

Note that this lemma is tight for the non-kernel case, leading to the margin formula \( \gamma < \min_{y \neq y'} \frac{\langle w^{(y)}, x - w^{(y')}, x \rangle}{\|w^{(y)} - w^{(y')}\|} \) that we used earlier.

Our kernel classifier is of the form in Lemma 22 with \( f^y(x') := \langle w^{(y)}, \phi_{\rho_1, \ldots, \rho_d}(x') \rangle \). To apply the lemma, we first compute \( \nabla f \):
\[
\nabla f^y(x') = \frac{1}{\sqrt{d}} \sum_{\rho_i} \left( -w_i^{(y)} \cdot \sin(\langle \rho_i, x' \rangle) + w_i^{(y)} \cdot \cos(\langle \rho_i, x' \rangle) \right) \cdot \rho_i.
\]

As a result, for two classes \( y, y' \), we have
\[
\nabla f^y(x') - \nabla f^{y'}(x') = \frac{1}{\sqrt{d}} \sum_{\rho_i} \left( (w_i^{(y)} - w_i^{(y')}) \cdot \sin(\langle \rho_i, x' \rangle) + (w_i^{(y)} - w_i^{(y')}) \cdot \cos(\langle \rho_i, x' \rangle) \right) \cdot \rho_i.
\]

In the following, we will give an upper bound on \( \|\nabla f^y - \nabla f^{y'}\| \). Let \( \Pi \in \mathbb{R}^{d \times d} \) resulting from concatenating \( \rho_1, \ldots, \rho_d \), and let \( p \in \mathbb{R}^d \) denote the vector for which \( p_i = \frac{1}{\sqrt{d}} (w_i^{(y)} - w_i^{(y')}) \cdot \sin(\langle \rho_i, x' \rangle) + (w_i^{(y)} - w_i^{(y')}) \cdot \cos(\langle \rho_i, x' \rangle) \).

First, notice that
\[
\nabla f^y - \nabla f^{y'} = \Pi p.
\]

Now, we may bound \( \|p\| \) by
\[
\|p\| = \frac{1}{\sqrt{d}} \sqrt{\sum_{\rho_i} \left( (w_i^{(y)} - w_i^{(y')}) \cdot \sin(\langle \rho_i, x' \rangle) + (w_i^{(y)} - w_i^{(y')}) \cdot \cos(\langle \rho_i, x' \rangle) \right)^2}
\]
\[
\text{(Cauchy–Schwarz inequality)} \leq \frac{1}{\sqrt{d}} \sqrt{\sum_{\rho_i} \left( (w_i^{(y)} - w_i^{(y')})^2 + (w_i^{(y)} - w_i^{(y')})^2 \right) \left( \sin((\rho_i, x'))^2 + \cos((\rho_i, x'))^2 \right)}
\]
\[
= \frac{1}{\sqrt{d}} \sqrt{\sum_{\rho_i} \left( (w_i^{(y)} - w_i^{(y')})^2 + (w_i^{(y)} - w_i^{(y')})^2 \right)}
\]
\[
= \frac{1}{\sqrt{d}} \cdot \|w^y - w^{y'}\|
\]

As a result, we have
\[
\|\nabla f^y(x') - \nabla f^{y'}(x')\| = \|\Pi p\| \leq \sigma_{\text{max}}(\Pi) \cdot \frac{1}{\sqrt{d}} \cdot \|w^y - w^{y'}\|,
\]

where \(\sigma_{\text{max}}(\Pi)\) denote the largest singular value of \(\Pi\) (i.e. the operator norm of \(\Pi\) with respect to \(L_2\) norm).

Plugging this back into Lemma 22 we can conclude that each example \((x, y)\) remaining correctly classifies up to perturbation norm of
\[
\frac{\sqrt{d}}{\sigma_{\text{max}}(\Pi)} \cdot \min_{y' \neq y} \frac{f^y(x) - f^{y'}(x)}{\|w^y - w^{y'}\|}.
\]

G.3 Comparison with Support Vector Machines (SVM)

Previous work has introduced different approaches to preserving DP for SVM [82], or convex optimization algorithms in general [41, 12]. Our implementation of DP SVM uses DP SGD [1] with the standard hinge loss and \(L_2\) weight regularization. The regularization strength is chosen from 1, 0.1, 0.01, 0.001, 0.0001, 0.00001, the learning rate from 1, 0.1, 0.01, 0.001, 0.0001. After summing gradients from each batch of data, we add appropriately calibrated Gaussian noise (again, based on Renyi DP) to the weights update.

Figures 6 and 7 compare performance of DP Batch Perceptron and DP SVM with and without kernel respectively. For experiments with varying \(\epsilon\) (first column in both figures), we observe that DP Batch Perceptron outperforms DP SVM in most instances and achieves competitive accuracy on both datasets. For different \(\delta\) values (second column) while keeping \(\epsilon\) fixed at 1.0, the trend still holds to a large extent and both algorithms yield very similar test accuracy. The last column compares the robust accuracy of models trained via DP Batch Perceptron and DP SVM at \(\epsilon = 1, 2\). In the case where no kernel is involved, the former yields better results on MNIST dataset but performs worse on USPS dataset. The opposite trend is observed when Gaussian kernel is included.

(a) Accuracy as \(\epsilon\) varies
(b) Accuracy as \(\delta\) varies
(c) Robust accuracy as \(\epsilon\) varies

Figure 6: Comparison of performance of DP Batch Perceptron vs DP SVM halfspace classifiers on the MNIST (top row) and USPS (bottom row) datasets, when no kernel is involved in the learning process.
Figure 7: Comparison of performance of DP Batch Perceptron vs DP SVM halfspace classifiers on the MNIST (top row) and USPS (bottom row) datasets, with Gaussian kernel.