A Linear Algebraic Approach to Subfield Subcodes of GRS Codes

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Abstract—The problem of finding subfield subcodes of generalized Reed–Solomon (GRS) codes (i.e., alternant codes) is considered. A pure linear algebraic approach is taken in order to derive message constraints that generalize the well known conjugacy constraints for cyclic GRS codes and their Bose–Chaudhuri–Hocquenghem (BCH) subfield subcodes. It is shown that the presented technique can be used for finding nested subfield subcodes with increasing design distance.

I. INTRODUCTION

Generalized Reed–Solomon (GRS) codes are among the most well-researched classes of error-correcting codes. Efficient decoders up to and beyond half their minimum distance are widely available. One shortcoming of GRS codes is, that their length cannot exceed the size $Q$ of the finite field $\mathbb{F}_Q$ over which they are defined. It was shown by Delsarte [1] that restricting the codeword components of a GRS code to a subfield $\mathbb{F}_q \subset \mathbb{F}_Q$ delivers an alternant code as introduced by Helgert [2, 3]. The latter include Bose–Chaudhuri–Hocquenghem (BCH) [4], Goppa [5], and Srivastava codes as special cases. These codes are defined over the small field $\mathbb{F}_q$ but their length is only restricted by the size $Q$ of the big field. It is obvious that subfield subcodes can be decoded using decoders for their GRS parent code.

Our contribution in this paper is a new (up to our knowledge), purely linear algebraic approach for obtaining subfield subcodes of GRS codes via message constraints that generalize the well known conjugacy constraints for cyclic GRS codes and their BCH subfield subcodes. Moreover, we show that our approach can be used in order to find nested subfield subcodes with increasing distance. Such nested codes are important building blocks of generalized concatenated codes [6].

II. GRS CODES AND THEIR SUBFIELD SUBCODES

Let $\mathbb{F}_Q$ be the finite field with $Q$ elements. For fixed positive integers $n$ and $k$ with $k \leq n \leq Q$, let $\mathbb{F}_Q[x]^{\leq k}$ denote the vector space of polynomials in variable $x$ over $\mathbb{F}_Q$ with degree less than $k$.

For $n$-tuples $A = (a_0, \ldots, a_{n-1})$ and $B = (b_0, \ldots, b_{n-1})$ over $\mathbb{F}_Q$, in which the components of $A$ are distinct and nonzero, the set $\mathcal{C} = \{(b_0 f(a_0), \ldots, b_{n-1} f(a_{n-1})) : f(x) \in \mathbb{F}_Q[x]^{\leq k}\}$ (1) represents the codewords of a GRS code (called modified RS code in [1]) over $\mathbb{F}_Q$ with locators $A$, column multipliers $B$, length $n$, dimension $k$, and minimum distance $d = n - k + 1$, (MDS property, see, e.g., [7]). Note that $f(x) = \sum_{i=0}^{k-1} f_i x^i \in \mathbb{F}_Q[x]^{\leq k}$ is the message polynomial, its coefficients can be chosen freely from $\mathbb{F}_Q$.

Several important classes of polynomial evaluation codes emerge by imposing constraints on $A$, $B$, and $f(x)$. Constraints on $f(x)$ are referred to as message constraints. For example, if $\mathbb{F}_Q$ contains a primitive $n$th root of unity, i.e., an element $\alpha$ of multiplicative order $n$, then a cyclic GRS code of length $n$ is obtained from locators $a_i = \alpha^i$ and column multipliers $b_i = \alpha^{i\delta}$ for some integer parameter $\delta$, where $i = 0, 1, \ldots, n - 1$. Note that in this case (due to Lagrange’s Theorem), $n$ must be a divisor of $Q - 1$, which, in particular, implies that $n$ and $Q$ are coprime. This is a restriction on the possible code length that depends on the field $\mathbb{F}_Q$.

If $\mathbb{F}_q$ is a proper subfield of $\mathbb{F}_Q$ (so that $Q = q^m$ for some integer $m > 1$) then the set $\mathcal{C}' = \mathcal{C} \cap \mathbb{F}_q^n$ is called a subfield subcode of $\mathcal{C}$ over $\mathbb{F}_q$, cf. [7]. It was observed in [1] that subfield subcodes of GRS codes are in fact alternant codes as introduced in [2, 3].

Note that $\mathcal{C}'$ has the same block length as $\mathcal{C}$, but its dimension $k'$ is generally smaller than the dimension $k$ of $\mathcal{C}$. It is not generally true that the design distance $d'$ of $\mathcal{C}'$ is larger than that of $\mathcal{C}$, but certainly $d' \geq d$. We are of course interested in the cases where $d' > d$.

The subfield subcodes of cyclic GRS codes are the BCH codes. They can be obtained directly from (1) by choosing locators and column multipliers of a cyclic GRS code and additionally making the coefficients of the message polynomial $f(x)$ satisfy the message constraint given by the conjugacy constraints

$$f_{\pi s[i]} = f_i^q, \quad i = 0, \ldots, n-1,$$

where

$$\pi s[z] = q i + (q - 1) \delta \mod n,$$

The zero locator is usually allowed in the definition of GRS codes. It is excluded in this paper because of the column multiplier update in Proposition 1.
which holds for canonical encoding as in \( \textbf{4} \) if and only if \( C \subseteq \mathbb{F}_q^m \), cf. \( \textbf{8}, \textbf{9} \). The message constraint restricts the possible choices of the coefficients of \( f(x) \), which is the reason for \( k' < k \).

The choice of \( \delta \) has a huge influence on the design distance \( d' \) of the subfield subcode. For example the \( \mathbb{F}_2 \)-subfield subcode of the cyclic GRS code over \( \mathbb{F}_{64} \) with length \( n = 63 \), dimension \( k = 51 \), minimum distance \( n - k + 1 = 13 \), and parameter \( \delta = 0 \) is the BCH code with dimension \( k' = 10 \) and design distance \( d' = 13 \). Choosing \( \delta = 24 \) instead results in a BCH code with the same dimension but much larger design distance \( d' = 27 \). The latter is almost optimal considering the currently known upper bounds on the minimum distance of linear codes (which allow minimum distance at most 28 for the parameters at hand). This example is elaborated in \( \textbf{9} \).

It is important to note that any decoder that can correct \( t[d] \) errors in the original \( \mathbb{F}_q \)-code can be used to decode up to \( t[d'] \) errors in the subfield subcode. Practically most relevant example of the error correcting radius \( t[1] \) is \( t[x] = \lceil x^{-1}/2 \rceil \).

Our goal in the following two sections is to derive message polynomial multiplication coincides with discrete convolution of the coefficient vectors \( u, v \in \mathbb{F}_q^m \), and the latter can be realized by multiplying \( u \) with the right from a Toeplitz matrix \( T_u \in \mathbb{F}_q^{m \times (2m - 1)} \), whose first row consists of the components of \( v \) followed by \( m - 1 \) zeros. The intermediate result after multiplication is \( (u_0, \ldots, u_{m-1}, w_m, \ldots, w_{2m-2}) = uT_v \), which is obviously twice as long as \( u \) and \( v \) and therefore not an element of \( \mathbb{F}_q^m \). This must be fixed by modular reduction.

### III. Translating Modular Polynomial Multiplication into Vector/Matrix Domain

It is well known that for \( Q = q^m \) with \( q \geq 2 \) either a prime or a power of a prime and an integer \( m \geq 1 \) the finite field \( \mathbb{F}_q \) is given (up to isomorphy) by the quotient

\[
\left( \mathbb{F}_q[x]/(p(x)) \right) ^{\oplus},
\]

where the defining polynomial \( p(x) \in \mathbb{F}_q[x] \) is irreducible over \( \mathbb{F}_q \) and \( \deg[p(x)] = m \). W.l.o.g. we assume \( p(x) \) to be monic. The field operations \( + \) and \( \cdot \) are given by polynomial addition and polynomial multiplication modulo \( p(x) \), respectively. In this work, we exploit \( \mathbb{F}_q[x]/(p(x)) \cong \mathbb{F}_q^m \), which is obtained by identifying polynomials from the quotient with their coefficient vectors (row vectors, zero-padded to length \( m \) if necessary). It is clear that polynomial addition in the quotient turns into componentwise addition in \( \mathbb{F}_q^m \).

Modular polynomial multiplication

\[
u(x) \circ \mu(x) = \mu(x)\nu(x) \mod p(x)
\]
is slightly more complicated to translate into the vector domain. It is instrumental to separate polynomial multiplication \( u(x) = u(x)\nu(x) \mod p(x) \) and modular reduction \( u(x) \mod p(x) \).

Polynomial multiplication coincides with discrete convolution of the coefficient vectors \( u, v \in \mathbb{F}_q^m \), and the latter can be realized by multiplying \( u \) with the right from a Toeplitz matrix \( T_v \in \mathbb{F}_q^{m \times (2m - 1)} \), whose first row consists of the components of \( v \) followed by \( m - 1 \) zeros. The intermediate result after multiplication is \( (u_0, \ldots, u_{m-1}, w_m, \ldots, w_{2m-2}) = uT_v \), which is obviously twice as long as \( u \) and \( v \) and therefore not an element of \( \mathbb{F}_q^m \). This must be fixed by modular reduction.

Modular reduction with respect to the defining polynomial

\[ p(x) = x^m + \sum_{i=0}^{m-1} p_i x^i \]

This allows annihilation of the coefficients \( w_{2m-2}, \ldots, w_m \) of \( u(x) = \sum_{i=0}^{2m-2} w_i x^i \) one after the other (starting from the most significant one) by subtracting

\[ w_jx^j + \sum_{i=0}^{m-1} p_i x^{i+j-m} \]

from \( w(x) \) for \( j = 2m - 2, \ldots, m \) (in that particular order).

How does this translate into the vector/matrix domain? Each step annihilates the most significant coefficient of the respective intermediate result, thereby reducing the possible length of its coefficient vector by one. This means that \( \textbf{3} \) is realized by multiplication with a \( (j+1) \times j \) matrix \( R_j \) over \( \mathbb{F}_q \).

\[
R_{\nu,\mu} = \begin{cases} 
1, & \text{if } \nu = \mu \\
-p_{m-j+\mu}, & \text{if } \nu = j \text{ and } \mu \geq j - m \\
0, & \text{else} 
\end{cases}
\]

\( R_j \) consists of an \( (j-m) \times (j-m) \) identity matrix \( I_{j-m} \), a (row) unit vector \( e \) of length \( m \) and the transposed \( m \times m \) companion matrix \( C^T[p(x)] \) of \( p(x) \). That is,

\[
C^T[p(x)]
\]
Steps $j = 2m - 2, \ldots, m$ of modular reduction of the intermediate result $uT_{v}$ can be performed by multiplication (from the right) with 

$$R = \prod_{j=2m-2}^{m} R_j \in \mathbb{F}_q^{(2m-1) \times m}.$$  

Note that $R$ is independent of the operands $u$ and $v$ and can thus be precomputed.

Altogether, modular polynomial multiplication $\odot$ translates to 

$$\odot : \left\{ \begin{array}{l} \mathbb{F}_q^m \times \mathbb{F}_q^m \\ u, v \end{array} \mapsto uT_{v}R \right\}.$$  

With the usual vector addition $+$ over $\mathbb{F}_q$ as additive field operation we have 

$$\left( \mathbb{F}_q[x]/p(x) \right) \oplus \left( \mathbb{F}_q, +, \odot \right) \cong \left( \mathbb{F}_q, +, \odot \right).$$  

An element of $\mathbb{F}_q$ is an element of the subfield $\mathbb{F}_q \subset \mathbb{F}_q$ if and only if it is constant (polynomial domain) or if all its components except for the leftmost (least significant) one are zero (vector/matrix domain).

### IV. MESSAGE CONSTRAINTS FOR ARBITRARY GRS CODES

Recall that encoding of GRS codes can be accomplished by polynomial evaluation as in (1). It is well known that the latter can be realized by multiplying the coefficient vector $f$ of the message polynomial $f(x) \in \mathbb{F}_q[x]$ (zero-padded to length $k$ if necessary) with a canonical generator matrix given by

$$G = \begin{bmatrix} G_{1,j} \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \\ b_0a_0 & b_1a_1 & \cdots & b_{n-1}a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_0a_0^{k-1} & b_1a_1^{k-1} & \cdots & b_{n-1}a_{n-1}^{k-1} \end{bmatrix} \in \mathbb{F}_q^{k \times n},$$

where $A = (a_0, \ldots, a_{n-1})$ are the locators and $B = (b_0, \ldots, b_{n-1})$ the column multipliers of the code $C$. Recall that $n$ is the length, $k$ the dimension, and $d = n - k + 1$ the minimum distance of $C$.

**Proposition 1** Let $s, t \in \mathbb{N}$ with $s + t \leq k$. If the coefficients $f_i, i \in \{0, \ldots, s-1\} \cup \{k-1-t, \ldots, k-1\}$ of every message polynomial $f(x)$ are known to be zero then the resulting codewords constitute an auxiliary GRS code $C_{\Pi}$: with length $n_{\Pi} = n$, dimension $k_{\Pi} = k - s - t$ and minimum distance $d_{\Pi} = d + s + t$, locators $A_{\Pi} = A$, and column multipliers $B_{\Pi} = \{b_0a_0, \ldots, b_{n-1}a_{n-1}\}$.

The part about the most significant coefficients is trivial to see: if the coefficients $k - 1 - t, \ldots, k - 1$ of every message polynomial are known to be zero then the last $t$ rows of $G$ can be ignored and the resulting GRS code has length $n$, dimension $k - t$, and minimum distance $d + t$.

If additionally the $s$ least significant coefficients of every message polynomial are zero, then the first $s$ rows of $G$ are superfluous. The resulting codewords can be considered as codewords from an auxiliary GRS code of length $n$, dimension $k - s - t$ and minimum distance $d + s + t$. The auxiliary GRS code $C_{\Pi}$ has locators $A$ and column multipliers $(b_0a_0, \ldots, b_{n-1}a_{n-1})$, i.e., its $(k - s - t) \times n$ canonical generator matrix $G_{\Pi}$ is exactly $G$ with its first $s$ and last $t$ rows deleted.

We now ask the following question: which constraint on a message $f \in \mathbb{F}_q^k$ has to hold such that encoding leads to a codeword $c = fG$ from the “small” vector space $\mathbb{F}_q^n$ instead of the “big” space $\mathbb{F}_q^m$? Answering this question will provide us with a precise characterization of the subfield subcode $C' = C \cap \mathbb{F}_q^n$, i.e., a generalization of the conjugacy constraints from (1) for arbitrary GRS codes (and their potentially non-BCH subfield subcodes).

Let us consider encoding with field operations in the vector/matrix domain as elaborated in Section IV Vector-matrix multiplication $(c_0, \ldots, c_{n-1}) = fG$ means calculating 

$$c_j = \sum_{i=0}^{k-1} f_i G_{i,j}, \quad j = 0, \ldots, n - 1,$$

which, over $\mathbb{F}_q^m$, becomes

$$\sum_{i=0}^{k-1} f_i \otimes G_{i,j} = \sum_{i=0}^{k-1} f_i T_{G_{i,j}} R, \quad j = 0, \ldots, n - 1.$$

Consequently, if $f \in \mathbb{F}_q^k$ is interpreted as $\tilde{f} \in \mathbb{F}_q^{mk}$, then the generator matrix becomes

$$\tilde{G} = \begin{bmatrix} T_{G_{1,j}} R \\ \vdots \\ T_{G_{k,j}} R \end{bmatrix} = \begin{bmatrix} T_{b_0R} & T_{b_1R} & \cdots & T_{b_{n-1}R} \\ T_{b_0a_0R} & T_{b_1a_1R} & \cdots & T_{b_{n-1}a_{n-1}R} \\ \vdots & \vdots & \ddots & \vdots \\ T_{b_0a_0^{k-1}R} & T_{b_1a_1^{k-1}R} & \cdots & T_{b_{n-1}a_{n-1}^{k-1}R} \end{bmatrix} \in \mathbb{F}_q^{mk \times mn},$$

i.e., a matrix of matrices. Now when is the codeword $\tilde{c} = (\tilde{c}_0, \ldots, \tilde{c}_{n-1}) = \tilde{f}\tilde{G}$ from $\mathbb{F}_q^m$? As stated at the end of Section III this is the case if and only if all the components except for the first one of all $\tilde{c}_j$ (when interpreted as a vectors from $\mathbb{F}_q^n$, $j = 0, \ldots, n - 1$, are zero. This can be enforced by restricting messages $\tilde{f}$ to the span of a certain matrix $\tilde{\Gamma}$.

Let $h[.]$ be the function that discards the first column of a matrix. Then a basis matrix $\tilde{\Gamma}$ of the vector space of all $\tilde{f}$ that are encoded into a codewords $\tilde{c} \in \mathbb{F}_q^m$ is given by a basis matrix of the kernel of a submatrix of $\tilde{G}$. It can be obtained by solving the homogeneous linear system

$$\begin{bmatrix} h[T_{b_0R}] & h[T_{b_0a_0R}] & \cdots & h[T_{b_0a_0^{k-1}R}] & 0 \\ h[T_{b_1R}] & h[T_{b_1a_1R}] & \cdots & h[T_{b_1a_1^{k-1}R}] & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h[T_{b_{n-1}R}] & h[T_{b_{n-1}a_{n-1}R}] & \cdots & h[T_{b_{n-1}a_{n-1}^{k-1}R}] & 0 \end{bmatrix} = 0.$$  

(5)
over $\mathbb{F}_q$ with $(m-1)n$ equations in $mk$ unknowns. The existence of a non-trivial solution is not guaranteed and depends on the actual choice of locators $A$ and column multipliers $B$. This will be elaborated in the upcoming example.

Before we start with the example let us provide the answer to our question:

$$c = fG \in \mathbb{F}_q^n \iff f \in \text{span}(\Gamma),$$

where $\Gamma \in \mathbb{F}_q^{k \times k}$ is simply $\hat{\Gamma} \in \mathbb{F}_q^{k \times mk}$ interpreted as a $k \times k$ matrix over $\mathbb{F}_q$. We have the following theorem:

\begin{theorem}
If $C$ is a GRS code over $\mathbb{F}_q^m$ with length $n$, dimension $k$, minimum distance $d$, locators $A = (a_0, \ldots, a_{n-1})$, and column multipliers $B = (b_0, \ldots, b_{n-1})$, then its subfield subcode $C' = C \cap \mathbb{F}_q^m$ has generator matrix $G' = \Gamma G \in \mathbb{F}_q^{k \times n}$, where $\Gamma$ is obtained as basis matrix of the solution space of (5). The dimension of $C'$ is $k' = \text{rank}(\Gamma)$ and the design distance is $d' = d^\Gamma$, where $d^\Gamma$ is obtained using Proposition 7.
\end{theorem}

Note that $d^\Gamma$ depends on $s$, $t$ (and thereby also on $\Gamma$), which is not reflected in the $\hat{\Gamma}$ notation.

\begin{example}
Let $C$ be the cyclic GRS code of length $n = 7$, dimension $k = 5$ and minimum distance $d = 3$ over $\mathbb{F}_7^2$ (defining polynomial $p(x) = x^3 + x + 1$, $m = 3$) with parameter $\delta = 0$ (cf. Section 7). The corresponding generator matrix $\hat{G}$ is

$$\hat{G} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

where the $T_{G'_r} R$ blocks are marked by dotted blue boxes and the $h[T_{G'_r} R]$ blocks by red boxes for clarity.

Setting up and solving the linear system (5) delivers the $4 \times (3 \cdot 5)$ basis matrix

$$\hat{\Gamma} = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

over $\mathbb{F}_2$. Thus, the dimension of the subfield subcode in the case $\delta = 0$ is $k' = \text{rank}(\hat{\Gamma}) = 4$. There are no zero columns neither on the left nor on the right of the matrix, meaning that $C^{\hat{\Gamma}} = C$ and consequently the design distance is $d' = d^{\hat{\Gamma}} = d = 3$.

Choosing parameter $\delta = 1$ instead results in the $3 \times (3 \cdot 5)$ basis matrix

$$\hat{\Gamma} = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
\end{bmatrix},$$

and in that case $k' = 3$. The $m = 3$ rightmost columns of this matrix are zero, which, when translated into polynomial domain, means that the $t = [\frac{3}{3}] = 1$ most significant coefficients of every message polynomial fulfilling the constraint are zero. There are no zero columns on the left ($s$ is zero). We can apply Proposition 7 in order to obtain design distance $d' = d^{\hat{\Gamma}} = 4 \geq d$.

Choosing $\delta = 4$ gives the $1 \times (3 \cdot 5)$ basis matrix

$$\hat{\Gamma} = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},$$

and we have $k' = 1$. The 9 leftmost columns and the 5 rightmost columns are zero. Thus, we have $s = [\frac{9}{3}] = 3$ and $t = [\frac{5}{3}] = 1$ and, again using Proposition 7 design distance $d' = d^{\hat{\Gamma}} = 7 \geq d$.

Note that the fact that $C$ is cyclic is not required for obtaining the subfield subcodes. A cyclic code was chosen for the example because tables of BCH codes are widely available for comparison, e.g. in [1].

The example suggests using Proposition 7 in order to obtain subcodes of subfield subcodes with increasing design distance. This is subject of the following section.

V. NESTED SUBFIELD SUBCODES

Finding nested subcodes with increasing distance is conceptually simple. Nevertheless, a full algorithmic description is very technical. Due to space restrictions we can only give a coarse overview of the procedure in this section and refer to the upcoming full paper.

Consider the basis matrix $\hat{\Gamma} \in \mathbb{F}_q^{k' \times mk}$ and, w.l.o.g., assume it is in reduced row echelon form. We are free to remove $u$ rows from $\hat{\Gamma}$, leading (cf. Theorem 1) to dimension $k' - u$.

How does this affect the design distance? In the general case not at all. Consider for example $\hat{\Gamma}$ from (8) and remove its first row. This reduces the dimension of the subfield subcode to $k' - 1 = 2$, but the design distance stays at $d' = d^{\hat{\Gamma}} = 4$. Removing the row only makes the code worse.

If on the other hand we remove the first row of $\hat{\Gamma}$ from (7) we obtain

$$\hat{\Gamma} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.$$

Without doubt, the dimension becomes $k' - 1 = 3$. But what about the design distance? Note the block of 3 zero columns on the left! They allow us to invoke Proposition 1 in order to get an auxiliary code $C^{\hat{\Gamma}}$ with $d' = d^{\hat{\Gamma}} + [\frac{3}{3}] = 4$, increasing the design distance to $d' = 4$. The striking property of this
code is, that it is a true subcode of $C'$ and that its design distance is larger than that of $C'$.

In order to find all nested subfield subcodes for given locators and column multipliers, we have to start with the largest possible GRS code, namely the trivial one with dimension $k = n$ and minimum distance $d = 1$.

**Example 2** Let $C$ be the cyclic GRS code of length $n = 7$, dimension $k = 7$ and minimum distance $d = 1$ over $F_{2^3}$ (defining polynomial $p(x) = x^3 + x + 1$, $m = 3$) with parameter $\delta = 0$ (cf. Section 11).

Setting up and solving the linear system (5) delivers the $7 \times (3 \cdot 7)$ basis matrix shown in (10). The resulting subfield subcode has dimension $k' = 7$ and design distance $d' = d = 1$. With reference to Proposition 11 leading and trailing groups of $m = 3$ zeros in each row are separated from the center of the matrix by an $s$ and $t$ trajectory, respectively.

Removing the last three rows results in the two nested codes from the beginning of this section (dimension 4, design distance 3 and dimension 3, design distance 4, respectively. Removing the first four rows results in a subfield subcode with $k' = 3$ and design distance $d' = 4$.

In general, the procedure for finding nested subcodes with increasing design distance for arbitrary but fixed locators $A$ and column multipliers $B$ can be outlined as follows.

1. Calculate $\tilde{\Gamma}$ for the GRS code with $k = n$.
2. Pick any submatrix $M$ with $k'$ rows, such that
   - (a) its first row is bounded by the $s$ trajectory on the left,
   - (b) its last row is bounded by the $t$ trajectory on the right,
   - (c) $\tilde{\Gamma}$ must not have any nonzero components left of the $s$ and right of the $t$ trajectory in the rows from which $M$ is taken.
3. Starting from the top, remove rows from $M$. Every row removed decreases $k'$ of the nested subcode by one. The design distance of the current nested code coincides with the number of columns located completely outside of the $s$ and $t$ trajectories (counted in groups of $m$).

**VI. CONCLUSION**

Constructing BCH codes (subcodes of cyclic GRS codes) based on minimal polynomials and calculating their design distance based on consecutive zeros in their generator polynomials is textbook knowledge. We provide a more general approach, which can deal with arbitrary GRS codes and their alternant subfield subcodes. Our approach requires nothing else than linear algebra over finite fields, which we believe is an advantage in its own right.

Searching for “good” nested subfield subcodes is particularly simple using an algorithm based on the $s$ and $t$ trajectories from Section 11. An upcoming paper will provide tables of such codes for practically relevant code parameters.

Even though we restricted ourselves to cyclic GRS codes and canonical generator matrices in the examples (since the resulting codes are well known), this is not a restriction of the approach itself. It can also be applied to, e.g., systematic generator matrices and we can hope for finding nested subfield subcodes with systematic encoders that way. One track of ongoing research is further generalization wrt. the representation of $F_{2^p}$, another one is applying the approach to locally recoverable codes (LRC) codes, which can also be interpreted as GRS codes with message constraints.

**REFERENCES**

[1] P. Delsarte, “On subfield subcodes of modified Reed–Solomon codes,” IEEE Trans. Inf. Theory, vol. 21, no. 5, pp. 575–576, Sep. 1975, doi: 10.1109/tit.1975.1055435.

[2] H. J. Helgert, “Noncyclic generalizations of BCH and Srivastava codes,” Inform. Contr., vol. 21, pp. 280–290, Oct. 1972.

[3] ———, “Alternant codes,” Inform. Contr., vol. 26, pp. 369–380, Dec. 1974.

[4] R. C. Bose and D. K. Ray-Chaudhuri, “On a class of error correcting group codes,” Inform. Contr., vol. 3, no. 1, pp. 68–79, Mar. 1960, doi: 10.1016/0019-9958(60)90287-4.

[5] E. Berlekamp, “Goppa codes,” IEEE Trans. Inf. Theory, vol. 19, no. 5, pp. 590–592, Sep. 1973, doi: 10.1109/tit.1973.1055088.

[6] V. Zybakhov, S. Shalgulidze, and M. Bossert, “An Introduction to Generalized Concatenated Codes,” Eur. Trans. Telecomm., vol. 10, no. 6, pp. 609–622, Nov. 1999, doi: 10.1002/ett.4460100606.

[7] F. J. MacWilliams and N. J. A. Sloane, The Theory of Error-Correcting Codes, ser. North-Holland Mathematical Library. North Holland Publishing Co., Jun. 1988. [Online]. Available: http://www.worldcat.org/isbn/0444851933

[8] R. E. Blahut, Algebraic Methods for Signal Processing and Communications Coding (Signal Processing and Digital Filtering), softcover reprint of the original 1st ed. 1992 ed. Springer, Oct. 2011. [Online]. Available: http://www.worldcat.org/isbn/1461427987

[9] C. Senger and F. R. Kschischang, “Syndrome-based Decoding of Polynomial Evaluation Codes without Chien Search,” in 28th Biennial Symposium on Communications (BSC 2016), Jun. 2016.