SOLUTIONS TO RESONANT BOUNDARY VALUE PROBLEM WITH BOUNDARY CONDITIONS INVOLVING RIEMANN-STIELTJES INTEGRALS

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ABSTRACT. We study the nonlinear boundary value problem consisting of a system of second order differential equations and boundary conditions involving a Riemann-Stieltjes integrals. Our proofs are based on the generalized Miranda Theorem.

1. Introduction. Let $f : [0,1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ be a given continuous function and $g = \text{diag}(g_1, \ldots, g_k)$, $g_i : [0,1] \to \mathbb{R}$, be a given function of bounded variation.

We are concerned with the boundary value problem consisting of the equations

$$ x'' = f(t, x, x'), $$

the initial conditions

$$ x'(0) = 0, $$

and the non-local boundary conditions

$$ x(1) = \frac{1}{0} \int x(s) \, dg(s), $$

where the integrals $\frac{1}{0} \int x_i(s) \, dg_i(s)$ are meant in the sense of Riemann-Stieltjes, $i = 1, \ldots, k$.

We shall assume that $\frac{1}{0} \int dg_i(s) = 1$ for $i = 1, \ldots, k$. In this case the problem (1)–(3) is resonant, since the corresponding homogeneous linear problem has nontrivial solutions: $x(t) = a \in \mathbb{R}^k$.

The existence of solutions to the problem (1)–(3) with multi-point boundary conditions have been studied extensively (see, for instance, [1, 4, 8, 9] and the references therein).

Riemann-Stieltjes integral formulations of the multi-point boundary conditions have also been considered, notably by Infante and Webb [10, 16, 17], Webb and Zima [19] and Franco, Infante and Zima [5].

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The nonlocal resonant problem (1)–(3) was studied in particular in [19], where authors established the existence and multiplicity of positive solutions for non-perturbed boundary value problem at resonance by considering equivalent non-resonant perturbed problem with the same boundary conditions. In [5], authors, using a Leggett-Williams norm-type theorem, showed, in particular, that the problem under consideration has positive solutions. In the papers mentioned above the boundary value problems were scalar and the function $f$ did not depend upon $x'$. The methods used in the papers are not the same as our ones and the assumptions are of completely different kind.

In this paper, we apply the ideas from [14]. The paper is organized as follows. The next section presents some notations, definitions and the generalized Miranda Theorem (see Theorem 2.1). In Section 3 we define an auxiliary problem which allows us to apply Theorem 2.1 and prove the existence of solutions to the problem (1)–(3). We impose on the function $f$ standard growth and sign conditions and assume that for each $i = 1,\ldots,k$ functions $g_i$ are nondecreasing. Moreover, the main theorem can also be applied to the problem (1)–(3) in the case when function $f$ does not depend upon $x'$ (comp. Corollary 1). The paper ends with two examples, the second one concerns the application of our results to a model of thermostat.

2. **Some preliminaries.** In this section, we shall present some basic concepts and results for later use.

Let $X$ and $Y$ be topological spaces.

A space $X$ is said to be contractible, if the identity map of $X$ is homotopic to some constant map of $X$ to itself [13].

A set-valued map $\Theta: X \to Y$ is upper semicontinuous (written usc), if, given an open set $V \subset Y$, the set $\{x \in X : \Theta(x) \subset V\}$ is open [6].

A compact space $X$ is an $R_\delta$-set (we write $X \in R_\delta$), if there is a decreasing sequence $X_n$ of compact contractible spaces such that $X = \bigcap_{n=1}^{\infty} X_n$ [6].

We say that $\Theta: X \to Y$ is an $R_\delta$-map, if it is usc and, for each $x \in X$, $\Theta(x) \in R_\delta$.

We shall discuss the existence of solutions to the problem (1)–(3) by using the following theorem (comp. [14]):

**Theorem 2.1 (The generalization of the Miranda Theorem).** Let $M_i > 0$, $i = 1,\ldots,k$, and $\Psi$ be an admissible map from $\prod_{i=1}^{k} [-M_i, M_i]$ to $\mathbb{R}^k$, i.e. there exist a Banach space $E$, $\dim E \geq k$, a linear, bounded and surjective map $\theta: E \to \mathbb{R}^k$ and $R_\delta$-map $\Theta$ from $\prod_{i=1}^{k} [-M_i, M_i]$ to $E$ such that $\Psi = \theta \circ \Theta$. If for any $i = 1,\ldots,k$ and every $y \in \Psi(x)$, where $|x_i| = M_i$, we have

$$x_i \cdot y_i \geq 0, \quad (4)$$

then there exists $x$ such that $0 \in \Psi(x)$.

For more abstract results with using tangency conditions expressed in terms of tangent cones see [2].

3. **An auxiliary problem.** Consider the following auxiliary initial value problem

$$x'' = f(t, x, x'), \quad x(0) = \alpha, \quad x'(0) = 0, \quad (5)$$

where $t \in [0, 1]$ and $\alpha \in \mathbb{R}^k$ is fixed.
The following assumptions upon $f$ will be needed throughout the paper:

(i) $f : [0, 1] \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous;
(ii) $|f(t, x, y)| \leq a_1|x| + a_2|y| + a_3$, where $a_1, a_2, a_3 > 0$.

The following result is standard. We give its proof for completeness.

**Lemma 3.1.** If assumptions (i), (ii) hold, the problem (5) has at least one global solution for every fixed $\alpha \in \mathbb{R}^k$.

**Proof.** Let $\alpha \in \mathbb{R}^k$ be fixed. The existence of at least one local solution to the problem (5) follows from the assumption (i). Using the theorem on a priori bounds [11], we will show that every such solution is a global one, i.e., any possible solution can be extended to the interval $[0, 1]$.

Let $x$ be a local solution to (5) and observe that then

$$x(t) = \alpha + t \int_0^t x'(s) \, ds. \tag{6}$$

Now, notice that (5) is equivalent to

$$x(t) = \alpha + \int_0^t (t - s) f(s, x(s), x'(s)) \, ds. \tag{7}$$

Moreover, one has

$$x'(t) = \int_0^t f(s, x(s), x'(s)) \, ds. \tag{8}$$

Consequently, from (8) and (ii), we obtain

$$|x'(t)| \leq \int_0^t \left( a_1 |\alpha| + \int_0^s |x'(u)| \, du \right) + a_2|x'(s)| + a_3 \right) \, ds$$

$$\leq \int_0^t \left( a_1 |\alpha| + a_1 s \max_{u \in [0,s]} |x'(u)| + a_2|x'(s)| + a_3 \right) \, ds.$$

Putting $\psi(t) = \max_{s \in [0,t]} |x'(s)|$, we have

$$\psi(t) \leq \int_0^t (a_1 s + a_2) \psi(s) \, ds + \int_0^1 (|\alpha|a_1 + a_3) \, ds$$

$$= \int_0^t (a_1 s + a_2) \psi(s) \, ds + C_\alpha$$

and by Gronwall’s Inequality

$$\psi(t) \leq C_\alpha \exp \left( \int_0^t (a_1 s + a_2) \, ds \right).$$

Consequently, we have

$$|x'(t)| \leq C_\alpha \exp D < \infty,$$

and, by (6), $|x(t)|$ is also bounded on $[0, 1]$, which completes the proof. \qed
Denote by $C^1([0,1], \mathbb{R}^k)$ the Banach space of all continuous functions $x : [0, 1] \to \mathbb{R}^k$ which have continuous first derivatives and by $C^2([0,1], \mathbb{R}^k)$ the Banach space of twice continuously differentiable functions, with the usual norms.

Define the mapping $F : C^1([0,1], \mathbb{R}^k) \times \mathbb{R}^k \to C^1([0,1], \mathbb{R}^k)$ by

$$F(x, \alpha)(t) := \alpha + \int_{0}^{t} (t - s) f(s, x(s), x'(s)) \, ds$$

and observe that by the assumptions $(i)$ and $(ii)$ the operator $F$ is completely continuous.

Note that for each $\alpha \in \mathbb{R}^k$ a function $x$ is a solution to the problem (5) if and only if $x$ is a fixed point of the operator $F(\cdot, \alpha)$.

### 4. The existence of solutions.

In this section, considering the family of all initial value problems (5), we shall show that there is an $\alpha \in \mathbb{R}^k$ such that for the $\alpha$ a solution to the problem (5) is also a solution to the problem (1)-(3), i.e., there exists an $\alpha$ such that a solution to the problem (5) satisfies the nonlocal boundary condition (3). A function $x : [0, 1] \to \mathbb{R}^k$ is said to be a solution of (1)-(3) if $x \in C^2([0,1], \mathbb{R}^k)$ satisfying (1) in $(0, 1)$ and conditions (2), (3).

Denote by $\text{Fix} F(\cdot, \alpha)$ the set of all fixed points of the operator $F(\cdot, \alpha)$. Consider a multifunction $\Theta : \mathbb{R}^k \rightrightarrows C^1([0,1], \mathbb{R}^k)$ given by

$$\Theta(\alpha) := \text{Fix} F(\cdot, \alpha).$$

**Lemma 4.1.** Let the assumptions $(i)$ and $(ii)$ be satisfied. The multifunction $\Theta : \mathbb{R}^k \rightrightarrows C^1([0,1], \mathbb{R}^k)$ is well-defined and usc with compact values. Moreover, $\Theta$ is an $R_\delta$-map.

**Proof.** Since the operator $\text{Fix} F(\cdot, \alpha)$ is completely continuous, one can show that $\Theta$ is usc with compact values ([14], Lemma 2). The fact that $\Theta$ is an $R_\delta$-map follows from the assumption $(ii)$. Indeed, it is well known that if $f$ has a linear growth then the set of all solutions of the problem (5) is an $R_\delta$-set ([7], p. 162).

Now, let us introduce the following assumption:

$(iii)$ for every $i = 1, \ldots, k$ the function $g_i$ is nondecreasing, $\int_{0}^{1} d g_i(s) = 1$ and there is an $s_0 \in (0, 1)$ such that $g_i(s_0) > g_i(0)$.

Define $\theta : C^1([0,1], \mathbb{R}^k) \to \mathbb{R}^k$ as follows

$$\theta(x) := x(1) - \int_{0}^{1} x(s) \, dg(s)$$

and observe that $\theta$ is well-defined.

**Lemma 4.2.** Under the assumption $(iii)$ the mapping $\theta$ is linear, bounded and surjective.

**Proof.** The linearity and continuity of $\theta$ is obvious. We shall show that $\theta$ is surjective. Let $d = (d_1, \ldots, d_k) \in \mathbb{R}^k$. Observe that one can always find a function $\pi \in C^1([0,1], \mathbb{R}^k)$ such that $\pi_1(1) = 0$ and $\int_{0}^{1} \pi_i(s) \, dg_i(s) \neq 0$, $i = 1, \ldots, k$. Let
\[ h_i := \int_0^1 \pi_i(s) \, dg_i(s) \] and set

\[ x_i(t) := -\frac{d_i}{h_i} \pi_i(t). \]

Consequently, for each \( d \) there is an \( x \) such that \( \theta(x) = d \).

The lemma below follows from Lemmas 4.1 and 4.2.

**Lemma 4.3.** Let the assumptions (i)–(iii) be satisfied. The set-valued function \( \Psi : \mathbb{R}^k \to \mathbb{R}^k \) given by \( \Psi := \theta \circ \Theta \), i.e.,

\[ \Psi(\alpha) := \left\{ x(1) - \int_0^1 x(s) \, dg(s) : x \in \text{Fix } F(\cdot, \alpha) \right\}. \]

is admissible.

Now, the following assumption will be needed:

(iv) there exist \( M_i > 0 \) such that for every \( t \in [0, 1] \), \( x \in \mathbb{R}^k \) and \( y \in \mathbb{R}^k \) we have \( x_i f_i(t, x, y) > 0 \) if \( |x_i| \geq M_i, i = 1, \ldots, k \).

**Theorem 4.4.** Under assumptions (i)–(iv) the problem (1)–(3) has at least one solution.

**Proof.** To prove the existence of solutions to the problem (1)–(3), since \( \Psi \) is admissible, it is sufficient to show that the condition (4) of Theorem 2.1 holds.

We shall show that for any \( i = 1, \ldots, k \) and every \( y \in \Psi(\alpha) \), we have \( \alpha_i \cdot y_i > 0 \) for \( |\alpha_i| = M_i \), where \( M_i \) is as in the assumption (iv). First let us consider the case when \( \alpha_i = M_i \). Let \( x \) be a solution to the problem (5) with \( \alpha_i = M_i, i = 1, \ldots, k \), and observe that \( x \in C^2([0, 1], \mathbb{R}^k) \). First, we shall prove that \( x_i(t) < x_i(1) \) for \( t \in [0, 1], i = 1, \ldots, k \).

Observe that \( x_i(0) = \alpha_i \) and \( x_i'(0) = 0 \). Moreover, by (iv), we have

\[ x_i(0) x_i''(0) = \alpha_i f_i(0, x(0), x'(0)) > 0. \]

Consequently, \( x_i \) has a local minimum at 0 and there exists an \( \varepsilon > 0 \) such that \( x_i(t) > \alpha_i \) for \( t \in (0, \varepsilon) \).

Now, assume that for some \( t \in (\varepsilon, 1] \) we have \( x_i(t) < \alpha_i \). Then there exists \( \bar{t} := \inf \{ t : x_i(t) < \alpha_i \} \) such that \( x(\bar{t}) = \alpha_i \) and \( x_i(t) \geq \alpha_i \) for \( t \leq \bar{t} \). Hence, there exists \( t_0 \in (0, \bar{t}] \) such that \( x_i \) has a local maximum at \( t_0 \) greater than \( \alpha_i \). By the assumption (iv), we reach a contradiction. Indeed, we have

\[ 0 \geq x_i(t_0) x_i''(t_0) = x_i(t_0) f_i(t_0, x(t_0), x'(t_0)) > 0. \]

We have proved that \( x_i(t) \geq \alpha_i \) for \( t \in [0, 1] \). Hence, by (iv), we obtain

\[ x_i(t) x_i''(t) = x_i(t) f_i(t, x(t), x'(t)) > 0, \]

which implies that \( x_i''(t) > 0 \) for \( t \in [0, 1] \). Consequently, \( x_i' \) is increasing on \((0, 1]\) and \( x_i'(0) > 0 \) for \( t \in (0, 1) \), since \( x_i'(0) = 0 \). Hence, one has

\[ x_i(1) \geq x_i(t), \quad t \in [0, 1]. \]

Integrating this inequality over \([0, 1]\) with respect to \( g_i \) and using the assumption (iii), we obtain

\[ x_i(1) = \int_0^1 x_i(1) \, dg_i(s) > \int_0^1 x_i(s) \, dg_i(s). \]
and consequently
\[ \alpha_i \left( x_i(1) - \int_0^1 x_i(s) \, dg_i(s) \right) > 0, \]
which means that in this case \( \alpha_i \cdot y_i > 0 \) with \( y \in \Psi(\alpha) \).
By similar arguments, one can show that when \( \alpha_i = -M_i \) we have \( x_i(1) < x_i(t) \), \( t \in (0, 1) \).
Finally, we get that there is an \( \alpha \in \mathbb{R}^k \) such that \( 0 \in \Psi(\alpha) \). This finishes the proof. \( \square \)

Now, consider the case when the function \( f \) does not depend upon \( x' \) and assume that the following conditions hold:
(i)' \( f: [0, 1] \times \mathbb{R}^k \to \mathbb{R}^k \) is continuous;
(ii)' \( |f(t, x)| \leq a_1|x| + a_2 \), where \( a_1, a_2 > 0 \);
(iii)' for every \( i = 1, \ldots, k \) there exist \( M_i > 0 \) such that for every \( t \in [0, 1] \) and \( x \in \mathbb{R}^k \) we have \( x_i f_i(t, x) > 0 \) if \( |x_i| \geq M_i \).

Theorem 4.4 in the special case where \( f \) does not depend upon \( x' \) immediately leads to the following result.

**Corollary 1.** Let the assumptions (i)', (ii)', (iii) and (iv)' be satisfied. Then the problem (1)–(3) has at least one solution.

**Example 4.1.** Let \( k = 2, x = (x_1, x_2), y = (y_1, y_2) \) and let \( g \) satisfy the assumption (iii). Consider the problem (1)–(3) with the function \( f = (f_1, f_2) \) given by
\[
\begin{align*}
  f_1(t, x, y) &= b_1(t, x, y)(x_1 + \cos^2 y_2), \\
  f_2(t, x, y) &= b_2(t, x, y)(x_2 + \sin^2 y_1 + 2),
\end{align*}
\]
where \( b_1, b_2 : [0, 1] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) are continuous, positive and bounded functions. Observe, that the problem (1)–(3) with the function \( f \) given above has at least one nontrivial solutions. Indeed, note that \( f \) satisfies the assumptions (i) and (ii). Moreover, we have
\[
x_2 f_2(t, x, y) = b_2(t, x, y)(x_2^2 + x_2 \sin^2 y_2 + 2x_2) \\
\geq b_2(t, x, y)(x_2^2 - |x_2| \sin^2 y_2 - 2|x_2|) \\
\geq b_2(t, x, y)(x_2^2 - 3|x_2|).
\]
Hence, setting \( M_2 > 3 \), we obtain that \( x_2 f_2(t, x, y) \) for \( |x_2| \geq M_2 \). Let \( M_1 > 2 \). By similar arguments one can show that \( x_1 f_1(t, x, y) > 0 \) when \( |x_1| \geq M_1 \). Consequently, the assumption (iv) of Theorem 4.4 is satisfied and the problem (1)–(3) with \( f \) defined above has at least one nontrivial solution.

**Example 4.2.** In [15], a three-point second-order boundary value problem has been used to model the temperature on a heated bar with a controller at the right end of the bar which adds or eliminates heat depending on the temperature detected by sensors at intermediate points. Solutions of such problem are stationary solutions for a one-dimensional heat equation. Recently, the study of nonlocal boundary value problems for second-order equations has been shown to be effective to the modeling of a thermostat with sensors expressed as linear functionals, see [18].

The problem (1)–(3) can also be interpreted as a thermostat model. In our case sensors give feedback to the endpoints where controllers add or remove heat according to that feedback. In addition, we generalize the equation considering the
heat equation with a nonlinear gradient source terms that vary in time (for the heat equation with a gradient source see, for instance, [3], [12] and papers cited therein). Moreover, now, the heated bar with a controller at 1 adds or removes heat depending on the temperature detected by a sensor put at any points of the bar (it depends on the function $g$). One can control the heat at 0 and 1, depending on what happens over the entire length of the bar. Theorems presented in this paper provide information on the existence of solutions to such problems.

REFERENCES

[1] C. Bai and J. Fang, Existence of positive solutions for three-point boundary value problems at resonance, J. Math. Anal. Appl., 291 (2004), 538–549.
[2] H. Ben-El-Mechaiekh and W. Kryszewski, Equilibria of set-valued maps on nonconvex domains, Trans. Amer. Math. Soc., 349 (1997), 4159–4179.
[3] J. T. Ding and B. Z. Guo, Blow-up and global existence for nonlinear parabolic equations with Neumann boundary conditions, Comput. Math. Appl., 60 (2010), 670–679.
[4] W. Feng, On an M-point boundary value problem, Nonlinear Anal., 30 (1997), 5369–5374.
[5] D. Franco, G. Infante and M. Zima, Second order nonlocal boundary value problems at resonance, Math. Nachr., 284 (2011), 875–884.
[6] L. Górniewicz, Topological Fixed Point Theory of Multivalued Mappings, Mathematics and its Applications, 495. Kluwer Academic Publishers, Dordrecht, 1999.
[7] A. Granas and M. Frigon, Topological Methods in Differential Equations and Inclusions, Kluwer Academic Publishers, 1995.
[8] C. P. Gupta, A generalized multi-point boundary value problem for second order ordinary differential equations, Appl. Math. Comput., 89 (1998), 133–146.
[9] X. Han, Positive solutions for a three-point boundary value problem at resonance, J. Math. Anal. Appl., 336 (2007), 556–568.
[10] G. Infante and J. R. L. Webb, Positive solutions of some nonlocal boundary value problems, Abstr. Appl. Anal., 18 (2003), 1047–1060.
[11] L. C. Piccininni, G. Stampacchia and G. Vidossich, Ordinary Differential Equations in $\mathbb{R}^n$, Translated from the Italian by A. LoBello. Applied Mathematical Sciences, 39. Springer-Verlag, New York, 1984.
[12] P. Souplet and F. B. Weissler, Self-similar subsolutions and blowup for nonlinear parabolic equations, J. Math. Anal. Appl., 212 (1997), 60–74.
[13] E. H. Spanier, Algebraic Topology, Corrected reprint of the 1966 original. Springer-Verlag, New York, [1995?].
[14] K. Szymańska-Dȩbowska, On a generalization of the Miranda Theorem and its application to boundary value problems, J. Differential Equations, 258 (2015), 2686–2700.
[15] J. R. L. Webb, Optimal constants in a nonlocal boundary value problem, Nonlinear Anal., 63 (2005), 672–685.
[16] J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: A unified approach, J. London Math. Soc., (2) 74 (2006), 673–693.
[17] J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems involving integral conditions, NoDEA Nonlinear Differential Equations Appl., 15 (2008), 45–67.
[18] J. R. L. Webb, Existence of positive solutions for a thermostat model, Nonlinear Anal. RWA, 13 (2012), 923–938.
[19] J. R. L. Webb and M. Zima, Multiple positive solutions of resonant and non-resonant nonlocal boundary value problems, Nonlinear Anal., 71 (2009), 1369–1378.

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