Energy Spectrum of Anyons
in a Magnetic Field

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Abstract

For the many-anyon system in external magnetic field, we derive the energy spectrum as an exact solution of the quantum eigenvalue problem with particular topological constraints. Our results agree with the numerical spectra recently obtained for the 3- and the 4-anyon systems.

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1. Introduction

The possibility of arbitrary statistics continuously interpolating between the Bose and the Fermi case in space dimensions less than three was suggested some years ago in a beautiful and pioneering paper [1] by Leinaas and Myrheim. A physical model of particles with arbitrary, or fractional, statistics was put forward by Wilczek [2], who baptized such objects as “anyons”. Anyon dynamics has lately received much attention (for comprehensive recent reviews and guides to the literature see ref. [3]-[5]), as it is thought to be a good candidate for the explanation of such important collective phenomena in condensed matter physics as the fractional quantum Hall effect and high temperature superconductivity. To this end, it is then crucial to investigate the dynamics of the many-anyon systems.

The symmetry group of the multi-anyon state vectors is not the group of permutations, it is the braid group. Due to this fact, even in the non-interacting case the many-anyon problem is not separable, and the exact solution has been obtained (by Leinaas and Myrheim in their original paper [1]) only for the case of $N = 2$ non-interacting anyons in a harmonic well. For the three-anyon problem Wu has obtained a special set of exact solutions [6] which have been recently generalized to the arbitrary $N$-anyon system in a magnetic field by Chou [7] and by Dunne et al. [8]. In fact, the problem of $N$ anyons moving in an external harmonic potential can be easily translated in that of $N$ charged anyons moving in a constant magnetic field. All the energy levels obtained from the special exact solutions [6]-[8] exhibit a strict linear dependence on the statistical parameter $\alpha$, as in the exactly solved 2-anyon problem [1].

Recently, very important experiments have been performed by Sporre, Verbaarschot and Zahed, who have solved numerically the Schroedinger eigenvalue problem for the $N = 3$ and the $N = 4$ anyon systems [9],[10]. The energy spectrum so obtained presents some levels with a slight non-linear dependence on $\alpha$, and some with a strictly linear dependence on $\alpha$. The latter include Wu’s exact eigenvalues.

In coincidence with the numerical investigations, Illuminati, Ravndal and Ruud have introduced a semi-classical approximation to determine the energy spectrum of the 3-
anyon system [11]. The energy levels are again linear in $\alpha$ and their slopes agree with the numerical ones. We have quite recently generalized the semi-classical quantization scheme to the general $N$-anyon system in external harmonic potential [12]. The semi-classical solution is obtained in connection with an exact limit of separability of the general many-anyon Hamiltonian, and it is shown to reproduce and explain all the main features of the numerical results also in the 4-anyon problem.

In this paper we wish to address the many-anyon problem in external magnetic field in the framework of the exactly solvable model introduced in [12]. We will solve the eigenvalue problem from topological considerations specific of the classical trajectories of charged particles in a magnetic field. The energy spectrum so obtained reproduces not only the slopes (as in the case of the harmonic potential) but also the actual values of the numerical energy levels.

\section{Classical Trajectories in a Magnetic Field}

Consider the motion of a classical non-relativistic particle with mass $m$ and charge $e$ in a constant magnetic field of strength $B$. Assume the field directed along the $z$-axis and the particle constrained to the $xy$-plane (e.g. when the velocity along the field is zero). The particle then moves in cyclotron orbits with angular velocity $\omega = eB/m$. Choosing the symmetric gauge $A = \frac{1}{2}B \times r$, the lagrangian equation of motion for the particle has the general solution

$$\mathbf{r}(t) = r_0[(\cos \omega t)\mathbf{i} - (\sin \omega t)\mathbf{j}] + \mathbf{R},$$

where $r_0$ is the radius of the cyclotron orbit, $\mathbf{R}$ is the position vector of the center of the orbit, and $\mathbf{i}, \mathbf{j}$ are the unit vectors along the $x$- and the $y$-axis respectively. The energy $E$ and magnitude $L$ of the angular momentum of the particle are

$$E = \frac{1}{2}m\omega^2 r_0^2,$$
\[ L = \frac{1}{2}m\omega(R^2 - r_0^2). \] (3)

We see from eq.(3) that the classical orbits fall into two different topological classes, whether they enclose the origin of coordinates or not. When the angular momentum is positive the orbits do not enclose the origin of coordinates. Negative values of \( L \) correspond instead to orbits enclosing the origin. This feature of the classical orbits plays an important role in the semi-classical analysis of the many-anyon problem.

3. The two-anyon system

In the symmetric gauge, the Lagrangian for \( N \) anyons in a magnetic field is

\[ L = \frac{1}{2} \sum_{i=1}^{N} (m\dot{r}_i^2 + e\mathbf{\hat{r}}_i \cdot \mathbf{B} \times \mathbf{r}_i) - \alpha \hbar \sum_{i<j} \dot{\phi}_{ij}, \] (4)

where \( \mathbf{r}_i = (x_i, y_i) \), and the azimuthal angle \( \phi_{ij} \) is defined by

\[ \phi_{ij} = \arctan \frac{y_j - y_i}{x_j - x_i}. \] (5)

The statistical parameter \( \alpha \) can be restricted to to take positive values in the interval \([0, 1]\). Bose statistics is recovered for \( \alpha = 0 \) and Fermi statistics for \( \alpha = 1 \). We could choose to reverse the sign of the topological term in the Lagrangian, but then \( \alpha \) should vary in the interval \([-1, 0]\) to obtain the correct statistical influence of the winding numbers on the energy eigenvalue problem. The situation here is inverted respect to the case of the harmonic oscillator: there the topological term enters the Lagrangian with a positive relative sign, if \( \alpha \) is chosen to be positive.

We now specialize to the case \( N = 2 \). Introducing polar coordinates \((\rho, \phi)\), the separation of the center of mass motion is immediate, and the effect of statistics is all contained in the Lagrangian for the relative motion

\[ L_{\text{rel}} = \frac{m}{2}(\dot{\rho}^2 + \rho^2 \dot{\phi}^2) + \left( \frac{e}{2}B\rho^2 - \alpha \hbar \right) \dot{\phi}. \] (6)
The corresponding Hamiltonian is

$$H = \frac{1}{2m} (p_\rho^2 + \frac{1}{\rho^2} (p_\phi + \alpha h)^2 - eB(p_\phi + \alpha h) + \frac{1}{4} e^2 B^2 \rho^2).$$

(7)

Apart from the terms containing $\alpha$, eq. (7) is exactly the Hamiltonian for an ordinary particle in a magnetic field. The eigenvalue problem is readily solved, in the same way as for the two-dimensional oscillator with statistical interaction originally treated in [1]. In terms of the Laguerre polynomials the eigenfunctions of $H$ are

$$\psi_{n,m} = e^{-\frac{1}{2} \xi^2} L_n^{|m+\alpha|}(\xi^2)e^{im\phi} \xi^{m+\alpha},$$

(8)

where $\xi = \rho \sqrt{m \omega / 2 \hbar}$. The corresponding energy eigenvalues (modified Landau levels) are

$$E_{n,m} = (2n + 1 + |m + \alpha| - m - \alpha) \frac{\hbar \omega}{2};$$

(9)

the radial quantum number $n$ is a positive integer, while the angular quantum number must be an even integer (positive or negative) if we model the two anyons as bosons with statistical interaction, and thus require the wave functions to be symmetric. The spectrum is drawn in Fig.1. From eq. (9) it follows that the energy is independent of the angular momentum and of $\alpha$ when $m > 0$. From the discussion of section (2) it corresponds to the classical relative orbit not encircling the origin of coordinates, i.e. to the two absolute classical orbits being separated and not winding around each other (see Fig.2a). In turn, a zero winding number implies the corresponding quantum result that the energy does not depend on $\alpha$. A second class of levels depending linearly on $\alpha$ with slope $-2$ is obtained when $m < 0$. The corresponding classical relative orbit encircles the origin, i.e. the orbit of one particle winds around the orbit of the other (Fig.2b), generating a non-zero winding number which, in turn, makes the corresponding energy eigenvalues depend on $\alpha$.

4. Separation of the many-anyon problem

A semi-classical formula for the energy spectrum of $N$ harmonically bound anyons was
derived in [12]. There the semi-classical result emerges as the exact solution of the quantum problem in a suitable limit of separability of the many-body Hamiltonian; it is the limit of non-crossing winding orbits. The same argument applies to the \( N \)-anyon system in a magnetic field. Here we are motivated by the fact that the exact solution of the two-anyon problem corresponds to classically separated cyclotron orbits. We then assume that this property holds in general for \( N > 2 \). Introducing \( N - 1 \) Jacobi relative coordinates \( \{ \rho_k \} \) in polar form

\[
\rho_k = \frac{1}{\sqrt{k(k-1)}}(r_1 + r_2 + \cdots + r_{k-1} - (k-1)r_k),
\]

\[
\phi_k = \arctan \frac{y_1 + y_2 + \cdots + y_k - ky_{k+1}}{x_1 + x_2 + \cdots + x_k - kx_{k+1}},
\]

\( k = 1, 2, \ldots, N-1 \),

(10)

the separation of the \( N \) cyclotron orbits is expressed by the "clustering" of the Jacobi rays: \( \rho_1 < \rho_2 < \cdots < \rho_{N-1} \). Here \( \rho_1 \) is the relative ray of the two particles with the smallest separation, \( \rho_2 \) is the relative ray of the nearest particle to the first two, and so on.

In the clustering approximation, comparing the definitions (5) and (10) of the azimuthal and the relative Jacobi angles, we have that

\[
\phi_{1j} = \phi_{2j} = \cdots = \phi_{j-1j} = \phi_{j-1},
\]

\( j = 2, 3, \ldots, N \),

(11)

and the Lagrangian (4) for \( N \) anyons expressed in Jacobi coordinates separates. The corresponding Hamiltonian for the relative motion reads

\[
H = \frac{1}{2m} \sum_{k=1}^{N-1} \left( p_{\rho_k}^2 + \frac{1}{\rho_k^2}(p_{\phi_k} + k\alpha \hbar)^2 - eB(p_{\phi_k} + k\alpha \hbar) + \frac{1}{4}e^2 B^2 \rho_k^2 \right).
\]

(12)

The eigenvalue problem is readily solved, and the multi-anyon wave functions \( \Psi_N \) factorize in the products of two-anyon wave functions.
\[ \Psi_N = \prod_{k=1}^{N-1} \psi_{n_k,m_k}, \]  

where

\[ \psi_{n_k,m_k} = e^{-\frac{1}{2} \xi_k^2 L_{n_k}^{|m_k+k\alpha|} (\xi_k^2) e^{im_k \phi_k} \xi_k^{|m_k+k\alpha|}}, \]

with \( \xi_k = \rho_k \sqrt{m\omega/2\hbar}. \) The corresponding energy spectrum is the sum of two-anyon contributions with different statistical parameters

\[ E = \sum_{k=1}^{N-1} (2n_k+1 + |m_k + k\alpha| - m_k - k\alpha) \frac{\hbar \omega}{2}. \]  

The radial quantum numbers \( n_k \) are independent of each other and may take any positive integer value. The angular quantum numbers \( m_k \) can take any positive or negative integer value, but they are however constrained to obey the quantum analogue of the clustering condition on the associated classical orbits. Since \( |m_k| \) gives the magnitude of the corresponding Bohr-Sommerfeld orbit \( \rho_k \), we get the relations

\[ |m_k| > |m_{k-1}|. \]

In addition, from the discussion of the two-anyon problem, \( m_1 \) can take only even integer values.

Eqn.(15) combined with rule (16) gives the spectra drawn in Fig.3 and Fig.4 for the 3-and the 4-anyon systems, as a function of \( \alpha \). Neglecting the slight non-linearity of some of the numerical levels, we see that eqns.(15),(16) exactly reproduce both the slopes and the correct intercepts of the numerical eigenvalues [10]. A quick inspection of formula (15) tells that for a generic \( N \), above a certain level there will be \( N(N-1)/2 \) levels with negative slope.

The crucial ingredient in the solution of the many-anyon problem in a magnetic field appears to be the different topology of the classical orbits, encircling or non-encircling each other, which affects the angular momentum and thus the dynamics of the system. This is not true for the oscillator problem, where the separability of the many-anyon problem.
corresponds only to classically non-crossing trajectories, with no distinction between encircling and non-encircling orbits. The clustering rule (16) must then be relaxed in that case. Thus, for the oscillator problem the exact degeneracy is yet to be assessed. Work is in progress in that direction [13].

If we compare the eigenfunctions (14) with the special solutions obtained in refs. [6]-[8], we see that the latter are included as a subset of our solutions for extreme negative values of the angular momentum. Furthermore, it is easy to show in our semi-classical picture that the zeroes of the wave functions of Wu, Chou, Dunne, Lerda, Sciuto and Trugenberger correspond to non-crossing classical orbits. This explains qualitatively why we get them as a subset of our solutions.

5. Discussion and Conclusions

The approach to the many-anyon problem that we have put forward seems to have some strong justifications from the excellent agreement with the numerical results, from reproducing the exact solutions previously found, and also from the intuitive semi-classical picture.

However, much work is to be done to assess the real connection between our solution and the complete solution of the general many-anyon problem. In particular, it is to be understood the role of the new wave functions that we have derived, which correspond to some of the linear and to all of the slightly curved numerical eigenvalues. This suggests the possibility of building the exact wave functions still missing through some perturbative or algebraic procedure applied to our solutions [14].

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Figure Captions

Fig. 1. Energy spectrum of two anyons in a magnetic field, as a function of the statistical parameter.

Fig. 2. Classical orbits of two particles in a magnetic field when the angular momentum is positive (2a), and when it is negative (2b).

Fig. 3. Semi-classical energy spectrum of three anyons in a magnetic field.

Fig. 4. Semi-classical energy spectrum of four anyons in a magnetic field.