The nature of the Milky Way’s halo revealed by the three integrals of motion

DANIELA CAROLLO1 and MASASHI CHIBA2

1INAF - Osservatorio Astronomico di Trieste, I-34143 Trieste, Italy
2Astronomical Institute, Tohoku University, Sendai 980-8578, Japan

ABSTRACT

Based on a new selection method of halo stars in the Milky Way local volume, we explore the general chemo-dynamical structure of the stellar halo. The method relies on the phase-space distribution defined by the three integrals of motion in an axisymmetric Galactic potential, \((E, L_z, I_3)\), where \(I_3\) is the third integral of motion. We analyse large samples of stars from SDSSS-SEGUE DR7 and DR16 (APOGEE) catalogs, matched with Gaia DR2, and found that, in this phase-space, halo stars are distinguished from disk stars by selecting over (1) \(L_z < 1500 \text{ kpc km s}^{-1}\) & \((2I_3)^{1/2} > 1000 \text{ kpc km s}^{-1}\), and (2) \(E < -1.5 \times 10^5 \text{ km}^2 \text{ s}^{-2}\). The range of \(I_3\) in (1) corresponds to a maximum orbital angle from the Galactic plane beyond 20 deg. These selection criteria are free from the kinematical biases introduced by the simple high-velocity cuts adopted in recent literature. The chemo-dynamical analysis confirms that the halo holds several substructures, including the stellar debris named GaiaEnceladus (GE; Helmi et al. 2018), but we find that an averaged phase-space distribution, phrased as a coarse-grained distribution, shows a monotonic exponential decrease with increasing \(E\) and \(I_3\) like the Michie-Bodenheimer models. The inner stellar halo described in Carollo et al. (2007, 2010) is found to comprise a combination of GE debris, lowest-\(E\) stars (likely in-situ stars), as well as metal-poor prograde stars missed by the high velocity cuts selection. The very metal poor outer halo, \([(\text{Fe/H}) < -2.2]\), exhibits both retrograde and prograde rotation, with an asymmetric \(L_z\) distribution towards high retrograde motions, and larger \(\theta_{	ext{orb}}\) than those possessed by the GE dominated inner halo. Implication for the formation of the stellar halo based on our findings are also discussed.

Keywords: Galaxy: structure – stars: Population II – stars: stellar dynamics – Galaxy: simulations – Galaxy: stellar content

1. INTRODUCTION

The Milky Way’s halo is a gold mine of information on the assembly process and chemical enrichment that led to the Galaxy we observe today. A formation scenario of the stellar halo through chaotic mergings of many subsystems was first proposed by Searle & Zinn (1978), and it has been linked with the current standard scenario of galaxy formation through hierarchical clustering of dark matter halos and associated baryonic process (e.g. White & Rees 1978). Since then, many studies based on analysis of halo’s stellar populations have made progress in understanding the complexity of this fundamental component, in particular, its multiple nature along with a number of individual over-densities and stellar debris streams (see reviews by, e.g., Freeman & Bland-Hawthorn 2002; Helmi 2008; Ivezić, Beers & Jurić 2012; Feltzing & Chiba 2013; Bland-Hawthorn & Gerhard 2016).

Evidence that the halo may comprise more than one stellar population was coming from the analysis of its spatial profile (Sommer-Larsen & Zhen 1990; Preston et al. 1991; Zinn et al. 1993; Kinman et al. 1994; Miceli et al. 2008), and the indication of retrograde motion of halo stars (Majewski et al. 1992; Carney et al. 1996; Wilhelm et al. 1996; Kinman et al. 2007; Lee et al. 2007). However, the first clear demonstration that the Milky Way’s halo comprises “at least” two stellar populations with different kinematics, spatial distribution, and chemical composition was described in Carollo et al. (2007, 2010) (see also Beers et al. 2012).

The second release of the Gaia mission, Gaia DR2 (Brown et al. 2018), has provided a large data set with unprecedented high precision astrometric parameters, and a significant number of works made use of these data, in combination with radial velocities and stellar abundances, tackling every component of the Milky Way. These works have revealed an even more complex but detailed picture of the Galaxy’s halo. For instance, Helmi et al. (2018) (see also Belokurov et al. 2018) showed that, at \([\text{Fe/H}] < -1\), the local halo is dominated by debris stars product of a merging event occurred ~ 10 Gyr ago (Gaia Enceladus; GE). Several other works have reported the presence of halo streams possessing both prograde and retrograde motion, relative to the Galactic disk rotation direction, and also found new chemo-dynamical...
properties of the stellar halo, that were unknown before Gaia (e.g., Bonaca et al. 2017; Myeong et al. 2018a,b,c; Koppelman et al. 2019; Matsuno et al. 2019; Belokurov et al. 2020; Yuan et al. 2020; Bonaca et al. 2020; Naidu et al. 2020).

For the correct understanding of the stellar halo and its formation scenario, it is crucial to select halo stars without introducing any biases, while reducing the contamination from the disk components. Such a selection can be partly accomplished by adopting high velocity cuts because halo stars move differently from the Sun, and this selection method has frequently been adopted in recent works (Nissen & Schuster 2010; Bonaca et al. 2017; Helmi et al. 2018). However, important parts of the stellar halo are missed by the adoption of this kinematically biased selection, leading to an incomplete picture of galaxy formation. This was the case of Eggen, Lynden-Bell & Sandage (1962), where the free-falling collapse scenario was brought by the adopted selection of halo stars based on their high-proper motions. Similar cautions should be taken when selecting halo stars based on their high-proper motions. Similar cautions should be taken when selecting halo stars based on their high-proper motions.

In this work, we develop a new selection scheme, or selection criteria, for halo stars, based on their characteristic distribution in phase-space, combined with the chemical abundance information, and this method is intended to be free from kinematical biases associated with simple high-velocity cuts. The phase space is defined by the three integrals of motion in an adopted axisymmetric Galactic potential: the total binding energy, $E$, the vertical angular momentum, $L_z$, and the third integral, $I_3$. We adopt a gravitational potential of Stäckel type, so that $I_3$ can be written in analytical form (e.g., de Zeeuw 1985).

In order to determine this new set of selection criteria in $(E, L_z, I_3)$, we make use of a large sample of SDSS-SEGUE DR7 and APOGEE DR16 catalogs, matched with Gaia DR2. We then explore the most probable ranges of $(E, L_z, I_3)$ for the halo system that allows a separation from rotating disk components, and derive its general properties. In particular, we focus on the so-called “coarse-grained” phase-space distribution of halo stars, where the averaged properties of the halo system, over the phase-space, are close to a dynamically steady state (Binney & Tremaine 2008). The “coarse-grained” distribution differs from the “fine-grained” one, which still contains non-relaxed small-scale substructures.

The paper is organised as follows. Section 2 describes the selection method for halo stars, the adopted mass model and data. In Section 3 we analyze the distribution of stars in the phase-space and its dependence on metallicity, [Fe/H], as well as $\alpha$-elements abundance. Section 4 present an extended discussion, including the halo duality, the connection with the Sagittarius dwarf galaxy and globular clusters, and the in-situ stellar halo. Implications for the formation of the stellar halo are also discussed in Section 5. Section 6 presents the summary of this work and prospects.

2. UNDERSTANDING THE STELLAR HALO: SELECTION METHOD, MASS MODEL AND ADOPTED DATA

2.1. Background

In many recent works, the selection of halo stars has conveniently been made based on high-velocity cuts, such as $V_{\text{rel}} = |V_{\text{star}} - V_{\text{LSR}}| > V_{\text{lim}}$, with $V_{\text{lim}}$ being 180 to 220 km s$^{-1}$, where $V_{\text{star}}$ is the three-dimensional (3D) velocity of a star, and $V_{\text{rel}}$ is its relative velocity with respect to the Local Standard of Rest (LSR), $V_{\text{LSR}}$ (e.g., Nissen & Schuster 2010; Bonaca et al. 2017; Haywood et al. 2018; Helmi et al. 2018). Some studies have set a cut in a tangential motion instead of a 3D velocity, due to the limited availability of line-of-sight velocity information. Such a selection aims to obtain a straightforward removal of stars with disk-like kinematics, characterized by nearly circular orbits, and thus, small velocity difference with respect to $V_{\text{LSR}}$ ($\sim$ 220 km s$^{-1}$).

However, this selection method is highly kinematically biased against those halo stars possessing low $V_{\text{rel}}$, which likely have low orbital eccentricities, $e$. Also, thanks to Gaia DR2 and other recent star catalogs from ground-based observations (e.g., Qiu et al. 2020), it is now possible to analyze the kinematics of stars at larger distances from the Sun. The velocity distributions of such remote stars can differ from those observed in the vicinity of the Sun, and the overlapping fraction of halo/disk populations in velocity space is also a function of the Galactic position. Therefore, the simple velocity cut of $V_{\text{rel}} > V_{\text{lim}}$ for the selection of halo stars is not well-founded.

The kinematics of stars are generally described in the 6D phase space, $(x, v_r, t)$ and, consequently, the velocity distribution of stars depends on positions, $x$, and even time, $t$. Thus, the velocity-based selection criterion that a star belongs to either disk or halo, is a function of its current position, $x$, in the Galactic space. For example, canonical thick-disk stars possessing mean azimuthal velocity of $(V_\phi)$ $\approx$ 200 km s$^{-1}$ near the position of the Sun, show a finite, negative vertical gradient, $\partial (V_\phi)/\partial z$, such that $(V_\phi)$ at $z = 2$ kpc is only about 160 km s$^{-1}$, and the velocity dispersion in the $\phi$ direction, $\sigma_\phi$, also varying with $z$, is about 60 km s$^{-1}$ at this height (e.g., Carollo et al. 2010). Other velocity dispersion components vary with the vertical distance as well. This suggests that the simple velocity cut described above, which should be applied only to stars in the solar neighborhood, can misclassify disk stars as halo stars, or vice versa. One may then adopt the position-dependent kinematic criteria, but such a method de-
depends on the adopted grids of the spatial coordinates and it is thus complex.

A useful solution is to adopt a more generalized method to characterize halo/disk stars that is less sensitive to kinematic selections/biases, and it is also independent from the present locations of stars. In this respect, we emphasize that an orbit-based selection in combination with other information, such as the chemical abundance, offers an ideal solution in assessing whether a star belongs to a disk or a halo component.

2.2. Stars in the integrals of motion space

In general, stars in the Milky Way possess anisotropic velocity distributions and, therefore, their phase-space distribution function depends on three isolating integrals of motion. For axisymmetric dynamical models, these integrals are denoted as $E$, $L_z$, and $I_3$, where $E$ is the orbital energy, $L_z$ is the angular-momentum component parallel to the $z$ axis, and $I_3$ is the third integral of motion. Thus, the orbits of stars in an axisymmetric gravitational potentials are characterized by their distribution in a phase space defined by the integral of motion $(E, L_z, I_3)$, where halo and disk components are expected to have their own distributions.

While $E$ and $L_z$ are classical integrals with an exact mathematical expression, the determination of the third integral, $I_3$, has long been a central subject in Galactic dynamics: there exist no general analytical expressions for $I_3$, and thus, its form has been investigated from numerical techniques (e.g., Richstone 1980, 1984; Levison & Richstone 1985a,b). However, many axisymmetric models for the Milky Way can be approximated and described with a gravitational potential of Stäckel form, where the Hamilton-Jacobi equation separates in ellipsoidal coordinates (de Zeeuw 1985). In this case, $I_3$ can be explicitly given in analytical form and the orbit of each star is described by three integrals of motion, $E$, $I_2 = L_z^2/2$, and $I_3$, which is a generalization of $L^2 L_z^2$ (where $L$ is the total angular momentum) in the limit of spherical symmetry (Dejonghe & de Zeeuw 1988). The analytical expression for $I_3$ allows a fast estimation of this important quantity for a large number of sample stars, and it is particularly advantageous in the current and future big-data era.

2.3. Selection method

In this work, we adopt a Stäckel form for the Galactic potential and estimate $I_3$ for each star. We then investigate the distribution of the adopted samples in the $(E, L_z, I_3)$ space, and its dependence on the metallicity, $[\text{Fe/H}]$. The most likely set of ranges in $(E, L_z, I_3)$ for the kinematic selection of halo and disk stars are then explored. In particular, in addition to the $E$ vs. $L_z$ diagram commonly employed in previous studies, we consider the distributions of stars in the $I_3$ vs. $L_z$, as well as the $I_3$ vs. $E$ diagram.

We note that in the frequently used angular-momentum diagram defined as $L_{\perp} \equiv (L_x^2+L_y^2)^{1/2}$ vs. $L_z$, the stellar positions are changing with time because $L_{\perp}$ is not an integral of motion in a non-spherical case, although this quantity, as well as $L_z$, can be straightforwardly determined without assuming a gravitational potential. The orbital eccentricity, $e$, in the radial direction, $r$, and the maximum vertical distance of an orbit from the Galactic plane, $z_{\text{max}}$, are also frequently used to characterize the orbital properties, but strictly speaking, they are not ideal parameters for the selection of disk/halo stars because each of these depends on a combination of $E$, $L_z$, and $I_3$, and they are related to each other. For instance, low-$e$ stars can have both large and small $z_{\text{max}}$, so that by simply assigning low-$e$ stars to a disk component leads to a contamination from those halo stars that possess low $e$ and/or large $z_{\text{max}}$. High-$e$ stars, say $e > 0.7$, are likely stellar members of a halo component, but thick-disk stars at $z \sim 2$ kpc can have $e$ as large as 0.8 at the $2\sigma$ significance of the $V_{\phi}$ distribution. Thus, the value of $e$ alone cannot be used for the classification of disk/halo stars.

2.4. Mass model and orbital structure

In this subsection, we briefly describe the mass model that leads to the gravitational potential of Stäckel type adopted in this paper and the main properties of the orbital parameters that are relevant to this analysis. More details are given in Appendix A. Although the essence of this type of a Galaxy model has already been discussed in earlier papers, its basic properties are described below for the sake of completeness.

We adopt the axisymmetric Galactic potential of Stäckel type, which was originally introduced by Sommer-Larsen &
This model consists of a disk and dark halo, where both are of Stäckel type. The former is given by a highly flattened, perfect oblate spheroid, resembling a Kuzmin disk, and the latter represents nearly an isothermal sphere given by a slightly flattened, oblate spheroid, which is originally developed by de Zeeuw, Peletier, & Franx (1986). This type of potential is defined in spheroidal coordinates \((\lambda, \phi, \nu)\), where \(\phi\) corresponds to the azimuthal angle in usual cylindrical coordinates \((R, \phi, z)\), and \(\lambda\) and \(\nu\) are representing the surfaces of a spheroid and hyperboloid, respectively, in the meridional plane (See Figure 1). The foci on the \(z\) axis at \(z = \pm \Delta\) fix the coordinate system and \((\lambda, \nu)\) are bounded with \(-\gamma \leq \nu \leq -\alpha \leq \lambda\), where \(\alpha\) and \(\gamma\) are constants. Note that \(\lambda = \text{const}\). is approaching to \(r = \text{const}\). in spherical coordinates at large \(r\) and \(\nu = \text{const}\.\) is nearly \(\theta = \text{const}\., an angle from the Galactic plane (\(z = 0\))^1, but not exactly the same (Figure 1).

The orbits in this potential posses the three integrals of motion, \(E, I_1 = L^2_z/2,\) and \(I_3\). The third integral, \(I_3\), is explicitly written in analytical form:

\[
I_3 = \frac{1}{2} (L^2_\perp + L^2_\parallel) + \Delta^2 \left[ \frac{1}{2} v^2_z - \frac{G(\lambda) - G(\nu)}{\lambda - \nu} \right],
\]

where \(L_\perp\) and \(L_\parallel\) are the angular momentum components in the \(x\) and \(y\) directions, respectively, and \(v_z\) is the velocity component in the \(z\) direction. \(G(\tau)\) with \(\tau = \lambda, \nu\) is an arbitrary function representing the gravitational potential as given in the Appendix. This expression suggests that \(I_3\) becomes \(\frac{1}{2} (L^2_z - L^2_\parallel) = \frac{1}{2} L^2_\perp\) in the limit of \(\Delta \to 0\), i.e., spherical symmetry. This in turn indicates that \(L_\parallel\) is not an integral of motion in a non-spherical potential as stated in the previous subsection: for instance, even if \(L_\perp \equiv (L^2_\perp + L^2_\parallel)^{1/2} = 0, I_3\) generally takes a non-zero value for a star with \(v_z = 0\) and/or \(z \neq 0\). The boundaries of the orbit in the meridional plane are along \(\lambda = \text{const}\., \nu = \text{const}\., -\gamma \leq \nu \leq \nu_\psi\) and \(\lambda_\perp \leq \lambda \leq \lambda_\parallel\), and these boundaries \((\nu_\perp, \lambda_\perp, \lambda_\parallel)\) are a function of \((E, I_2, I_3)\) (Dejonghe & de Zeeuw 1988). The third integral, \(I_3\), is especially important for constraining \(v_+\), or nearly the maximum angle of the orbit with respect to the equatorial plane, in contrast to the spherically symmetric case.

To get more insights into \(I_3\) in terms of more familiar orbital parameters, we plot, in Figure 2, the values of \((2I_3)^{1/2}\) as a function of the maximum orbital angle from the Galactic plane, \(\theta_{\text{orb}}\), which may be estimated by \(\text{arcsin}(z_{\text{max}}/r_{\text{apo}})\), where \(r_{\text{apo}}\) stands for the apocentric Galactocentric radius in spherical coordinates. We consider \(I_3\) in the form of \((2I_3)^{1/2}\) as this is in the dimension of an angular momentum, and the data used here are the SDSS DR7 calibration stars (see below). Inspection of Figure 2 reveals that \((2I_3)^{1/2}\) correlates well with the maximum orbital angle, \(\theta_{\text{orb}}\), up to \((2I_3)^{1/2} < 1000\) kpc km s\(^{-1}\) (corresponding to \(\theta_{\text{orb}} < 15^\circ - 20^\circ\)), but this correlation starts to have a large dispersion beyond this value, because of a finite mismatch between the spheroidal and spherical coordinates due to the non-zero foci at \(z = \pm \Delta\) in the former (see Figure 1).

We note that an alternative choice of integrals for the orbits is represented by the set of action integrals, \(J = (J_\lambda, J_\phi, J_\nu)\) in the current spheroidal coordinates, where each component is calculated over the integration of each angular component, \((p_\lambda, p_\phi, p_\nu)\). While \(J_\phi\) is equivalent to \(L_z\), the estimation of \(J_\lambda\) and \(J_\nu\) requires numerical calculations even in the Stäckel potential. Other action integrals, \(J_R\) and \(J_\nu\), defined in cylindrical coordinates, have been adopted for orbit classification in several recent works. These differ from \(J_\lambda\) and \(J_\nu\), in particular for orbits reaching large heights, \(z_{\text{max}}\), or having large orbital angles, \(\theta_{\text{orb}}\), from the Galactic plane, but possessing low orbital eccentricities along \(r\). Such orbits may be characterized by large \(J_K\) (and \(J_\nu\)) because of the extended ranges of orbital motion along \(R\), however, they have small \(J_\lambda\), because

---

1 Here, for the convenience of the following discussion, we define \(\theta\) as an angle from the equatorial plane instead of a usual polar angle in spherical coordinates.
of the limited ranges of orbital motion along $\lambda$ (Figure 1). In fact, the set of $J_3$ and $I_3$ reflects the orbital properties more exactly than that of $J_R$ and $J_2$, and it is thus preferred for orbital classification, especially for halo stars which have large orbital angles, $\theta_{orb}$.

In this analysis, we use $I_3$, $E$, and $L_z$ to describe the Galactic stellar components, because these integrals contain sufficient information for the classification of stars in our samples. In what follows, we use $I_3$ in the form of $(2I_3)^{1/2}$ as this quantity has the same physical dimension as $L_z$.

2.5. Data: Gaia DR2, SDSS DR7 calibration stars, SDSS DR7 full sample, and SDSS DR16

When selecting halo stars, the use of different star catalogs can sometimes lead to different results. In some cases, this is due to the individual footprint, or sky coverage, associated to each survey. In this analysis we take into account of this issue by employing two distinct data sets across matched with Gaia DR2, namely, the Sloan Digital Sky Survey (SDSS) DR7 calibration stars and the SDSS DR16 (Ahumada et al. 2019), which includes data from the Apache Point Observatory Galaxy Evolution Experiment (APOGEE), where the latter covers the APOGEE footprints over disk regions, which are only partially included in SDSS DR7.

We first make use of the SDSS DR7 Yanny et al. (2009) calibration stars. The sample consists of ~40,000 stars with stellar parameters obtained by employing the SEGUE Stellar Parameter Pipeline (SSPP; Lee et al. 2008a,b). These parameters, as well as the $\alpha$-elements abundance are available in the SDSS archive. The entire sample of SDSS DR7 stars is also considered for comparison, and it consists of 65,500 stars selected by requiring, $S/N > 40$, as well as reliable stellar parameters and $\alpha$-abundances. Finally, we selected 70,000 unique stars from SDSS Data Release 16 by applying a series of cuts to remove stars with unreliable stellar parameters, or elements abundances. The details of these selections are reported in the Appendix B.1.

The samples are cross-matched with the Gaia DR2 database to retrieve accurate positions, trigonometric parallaxes, and proper motions, using the CDS (Centre de Donnes Astronomiques de Strasbourg) X-Match service, and adopting a very small search radius ($0.6^\circ$ - $0.8^\circ$) to avoid duplicates. The match provides positions, parallaxes, and proper motions for all of the stars in both samples. We then select stars with relative parallax errors of $\sigma_\pi/\pi < 0.2$, and derive their distance estimates using the relation $d = 1/\pi$ (and used these distances to select stars with heliocentric distance $d \leq 4$ kpc). This selection reduces the number of stars to 10,820 for the DR7 calibration stars, and 62,060 for the APOGEE sample. In case of SDSS-SEGUE DR7 full sample the number is reduced to 46,500. The majority of stars in this final samples have errors on proper motions below 0.2 mas yr$^{-1}$. We also adopted a parallax zero-point offset of $\delta_\pi = -0.05$ mas (see Appendix B.3 for a discussion).

Radial velocities for stars in the DR7 sample are derived from matches to an external library of high-resolution spectral templates with accurately known velocities, degraded in resolution to match the SDSS spectra (see Lee et al. 2008a). The typical precision is on the order of 5–20 km s$^{-1}$ (depending on the S/N of the spectra). In case of APOGEE stars, initial measurement of the radial velocity for each star is made by cross correlating each spectrum with the best match in a template library, then radial velocities for each visit are derived again when the visit spectra are combined. Typical accuracy is of the order of 0.35 km s$^{-1}$ (Nidever et al. 2015), however, we found that the majority of the stars in our subsample have an accuracy $< 0.2$ km s$^{-1}$.

The full space and orbital motion is derived by combining the observables obtained from Gaia DR2, i.e., positions, distances, and proper motions ($\alpha$, $\delta$, $\pi$, $\mu_\alpha$, $\mu_\delta$), with the radial velocities provided by SDSS. The velocities calculated in the Local Standard Rest (LSR), assumed to be rotating at 220 km s$^{-1}$, are referred to as $(U, V, W)$ which are corrected for the motion of the Sun by adopting the values $(U, V, W) = (-9,12,7)$ km s$^{-1}$ (Mihalas & Binney 1981)$^3$. The velocity component $U$ is taken to be positive in the direction toward the Galactic anti-centre, the $V$ component is positive in the direction toward Galactic rotation, and the $W$ component is positive toward the north Galactic pole.

3. DISTRIBUTIONS OF STARS IN PHASE-SPACE

3.1. SDSS DR7 calibration stars

3.1.1. General properties of the phase-space distribution

Figure 3 and Figure 4 show the distributions in the phase space defined by the three integrals of motion $(E, L_z, I_3)$, and for the six metallicity intervals, $[\text{Fe/H}] > -0.6, -1 < \text{[Fe/H]} < -0.6, -1.4 < \text{[Fe/H]} < -1$ (Figure 3), and $-1.8 < \text{[Fe/H]} < -1.4, -2.2 < \text{[Fe/H]} < -1.8, \text{[Fe/H]} < -2.2$ (Figure 4), for the SDSS DR7 calibration stars. The grey dots show the entire sample, while the color-coded symbols represent subsamples in the various ranges of metallicity, as indicated in the legends of each panel. In these figures, the left, middle, and right panels show the $(2I_3)^{1/2}$ vs. $L_z$, $E$ vs. $L_z$, and $E$ vs. $(2I_3)^{1/2}$ diagrams, respectively. These panels show interesting global characteristics of the phase-space distribution and their dependence on the metallicity.

$^3$ More recent evaluations of the LSR and solar are available, however we adopt these values for consistency with the (Carollo et al. 2007, 2010) analyses.
As widely investigated and recognized in previous papers, the distribution of stars in the $E$ vs. $L_z$ diagram (middle panels in these figures) are bounded in a parabola shape with the bottom (the lowest $E$) located at $L_z = 0$. This parabola is densely populated by stars on circular orbits in the region of high $|L_z|$ and $E$, say, $L_z > 1000$ kpc km s$^{-1}$ and $E > -1.5 \times 10^5$ km$^2$ s$^{-2}$, whereas the lower $|L_z|$ and $E$ ranges are populated by stars on eccentric halo orbits inside the position of the Sun, as discussed in Section 4.4.2.

In the $E$ vs. $L_z$ diagram the elongated feature around $L_z = 0$ is the Gaia-Enceladus (GE) debris (Helmi et al. 2018) or Sausage structures (Belokurov et al. 2018). The $E$ vs. $L_z$ diagram is also characterized by a large number of highly retrograde ($L_z < 0$), and high energy stars.

It is important to note that the $E$ vs. $(2I_3)^{1/2}$ diagram also shows a parabola-shape boundary (right panels in Figures 3 and 4), however the bottom of this parabola (the lowest $E$) is located at a non-zero value of $(2I_3)^{1/2}$, of the order of $\sim 700$ kpc km s$^{-1}$, in contrast to the $E$ vs. $L_z$ diagram, where the bottom of the parabola exhibits a value of $L_z = 0$. In fact, stars with the lowest range of energy, $E < -1.5 \times 10^5$ km$^2$ s$^{-2}$, and typically at $[\text{Fe/H}] < -1$, tend to populate the interval $400 \lesssim (2I_3)^{1/2} \lesssim 1200$ kpc km s$^{-1}$. These values of $(2I_3)^{1/2}$ correspond to the orbital angles in the range of $5 \deg \lesssim \theta_{\text{orb}} \lesssim 20$–$30\ deg$, and $\theta_{\text{orb}} \approx 10\ deg$ for the lowest $E$ (See Figure 2).

The non-zero $(2I_3)^{1/2}$ at the lowest $E$ is partly due to the second term in Equation (1) where $I_3$ is defined: even when $L_z = 0$, $I_3$ is non-zero for a finite vertical motion $v_z$ and/or position $z$ in the current non-spherical case, whereas $I_3 = 0$ in a spherical limit of $\Delta = 0$. Another possible reason for the non-zero $(2I_3)^{1/2}$ at the lowest $E$ (and $L_z = 0$), is the fan shape of the SDSS footprints, which lack of stars at small radii, $R$, ($R < R_\odot$), and low $z$ (See Figure 15 in Appendix). This implies that stars with small orbital angle, $\theta_{\text{orb}}$, at small $R$, are outside the sampling volume of the DR7 survey.

To get insights into the effect of the survey’s footprints, we also consider the DR16 (APOGEE) catalog, whose footprint includes small $R$ and low $z$ (Figure 15 in Appendix B.3). In Appendix B.4, Figure 18 and 19, show the phase-space distribution of APOGEE stars. The right panels of Figure 18 ($E$ vs. $(2I_3)^{1/2}$ diagram) show that stars with $[\text{Fe/H}] > -0.6$ have $(2I_3)^{1/2} = 0$ kpc km s$^{-1}$ in the range of energy, $-1.5 \times 10^5$ km$^2$ s$^{-2} < E < -1.1 \times 10^5$ km$^2$ s$^{-2}$. These are disk stars with nearly circular orbit in the Galactic plane, and orbiting at small $R$. Such stars are not present in the SDSS DR7 footprint, as can be inferred by examining Figure 3. We also note
that metal-poor stars appear to have a non-zero \((2I_3)^{1/2}\) value at the lowest \(E\), although the paucity of halo samples in the APOGEE survey prevents from investigating in more details this property. The non-zero \((2I_3)^{1/2}\) implies that metal-poor halo stars at smaller \(R\) have indeed a finite orbital angle, \(\theta_{\text{orb}}\), as seen in case of the SDSS DR7 sample.

Examination of the \((2I_3)^{1/2}\) vs. \(L_z\) diagram (left panels in Figures 3 and 4) reveals that stars with \(L_z \approx 2000\) kpc km s\(^{-1}\) possess low values of the third integral of motion, \((2I_3)^{1/2} \lesssim 400\) kpc km s\(^{-1}\), while stars with angular momentum, \(L_z < 2000\) kpc km s\(^{-1}\), have always \((2I_3)^{1/2}\) above \(\sim 400\) kpc km s\(^{-1}\). In particular, as already mentioned above, there are no stars having both \(L_z \approx 0\) and \((2I_3)^{1/2}\) (or \(I_3\) \(\approx 0\)); the shape of the SDSS’s footprints leads to a finite value of \((2I_3)^{1/2}\) at \(L_z = 0\).

### 3.1.2. Dependence on metallicity

In the highest metallicity range ([Fe/H]\(> -0.6\), top panels of Figure 3, dark yellow symbols) many stars have large rotational velocity (high values of \(L_z\)), and possess values of \((2I_3)^{1/2}\) below \(500\) kpc km s\(^{-1}\), but all of them have \((2I_3)^{1/2}\) below \(1000\) kpc km s\(^{-1}\), and \(E\) below \(-1.2 \times 10^5\) km\(^2\)s\(^{-2}\). These stars, are distributed along a parabola with nearly circular orbits in the \(E\) vs. \(L_z\) diagram, and are dominated by stellar members of the thin- and thick-disk components.

As the metallicity decreases to \(-1 <\text{[Fe/H]} < -0.6\) (middle panels in Figure 3, orange symbols), the number of stars having \((2I_3)^{1/2}\) around \(1000\) kpc km s\(^{-1}\), and beyond, increases, as well as those with lower \(L_z\), including some stars with \(L_z\) as small as \(0\). In this range of metallicity, the overlapping thick disk and metal-weak thick disk (MWTD; Carollo et al. 2019), which are predominantly in circular orbits, dominate the distribution, with some halo stars contamination.

In the two intermediate ranges of metallicity represented by \(-1.4 <\text{[Fe/H]} < -1\) (bottom panels in Figure 3, green symbols) and \(-1.8 <\text{[Fe/H]} < -1.4\) (top panels in Figure 4, pink symbols), stars exhibit progressively lower \(L_z\), and there are almost no stars with \((2I_3)^{1/2} < 500\) kpc km s\(^{-1}\). In these metallicity intervals, many stars have \((2I_3)^{1/2} > 1000\) kpc km s\(^{-1}\), reaching values up to \(\sim 3000\) kpc km s\(^{-1}\), in contrast to the higher metallicity ranges ([Fe/H] \(> -0.6\)) where almost no stars possess \((2I_3)^{1/2} > 1000\) kpc km s\(^{-1}\). Also, while the number of stars with large \(L_z\) and low \((2I_3)^{1/2}\) decreases, many stars with lower \(L_z\) tend to populate the elongated feature in the center of the \(E\) vs. \(L_z\) diagram, over the range of \(-500 < L_z < 500\) kpc km s\(^{-1}\), and possessing an extended distribution in both \(E\) and \((2I_3)^{1/2}\). This feature is dominated by the GE debris stars (Helmi et al. 2018), or Sausage structure (Belokurov et al. 2018). Note that the range of metallicity, \(-1.8 <\text{[Fe/H]} < -1.4\), matches with that of the inner halo stellar population discussed in Carollo et al. (2007, 2010), whose metallicity peak is [Fe/H] \(\sim -1.6\). In these intermedi
ate metallicity intervals, there are also many stars with retrograde motion and high energy.

It is interesting to notice that in the \((2I_3)^{1/2}\) vs. \(L_c\) distribution, and in the metallicity range of \(-1.4 < \langle \text{[Fe/H]} \rangle < -1\) (bottom left panel in Figure 3, green symbols), at 700 < \(L_c\) < 1500 kpc km s\(^{-1}\), there exists a distinct distribution of stars, which contains both the MWTD and halo stars. The MWTD has a peak of metallicity of \([\text{Fe/H}] \sim -1.0\) and \(\langle \text{[Fe/H]} \rangle \sim 1200\) kpc km s\(^{-1}\) (Carollo et al. 2019); this feature becomes slightly weaker as the metallicity decreases to \(-1.8 < \langle \text{[Fe/H]} \rangle < -1.4\).

Over the range of \(-2.2 < \langle \text{[Fe/H]} \rangle < -1.4\), another notable feature is clearly present in the \((2I_3)^{1/2}\) vs. \(L_c\) diagram: the elongated distribution of stars from \((L_c, (2I_3)^{1/2}) \approx (1800, 400)\) kpc km s\(^{-1}\) to \((1000, 3500)\) kpc km s\(^{-1}\). Such a feature was originally identified and called a ‘trail’ feature in the \(L_c\) vs. \(L\) diagram by Chiba & Beers (2000) (see their Figure 15). We will discuss the origin of this interval in Section 4.

At lower metallicity, \(-2.2 < \langle \text{[Fe/H]} \rangle < -1.8\) and \([\text{Fe/H}] < -2.2\) (middle and bottom panels of Figure 4, blue and red colors, respectively), the majority of the stars are characterized by \(L_c < 1500\) kpc km s\(^{-1}\), and \((2I_3)^{1/2} > 500\) kpc km s\(^{-1}\), we note in particular the rather sharp boundary at \(L_c \approx 1500\) kpc km s\(^{-1}\), and \((2I_3)^{1/2} \approx 500\) kpc km s\(^{-1}\), in the distributions of these low metallicity stars, which are dominated by halo stars. The distribution below \(L_c = 1500\) kpc km s\(^{-1}\) is extended over large ranges of \(E\) and \((2I_3)^{1/2}\), and there are stars with retrograde motion and high energy, reaching up to \(E = -0.4 \times 10^5\) km\(^2\) s\(^{-2}\), as quantified below (Figure 7).

It is worth noting that stars possessing retrograde motion were already identified and described as members of ‘the outer halo’ component in the original sample employed by Carollo et al. (2007, 2010). The retrograde motion detected in that sample is confirmed when adopting the more precise Gaia DR2 parameters.

### 3.1.3. Comparison with the \([\alpha/\text{Fe}]\) vs. \([\text{Fe/H}]\) diagram

To further investigate the phase-space distribution of the SDSS DR7 calibration stars, we select stars in specific regions of \([\alpha/\text{Fe}]\) vs. \([\text{Fe/H}]\) diagram, each of which represent likely thin disk, thick-disk, MWTD and GE stars, and analyze their locations in the phase-space diagram. In this exercise we use only the \((2I_3)^{1/2}\) vs. \(L_c\) and \(E\) vs. \(L_c\) diagrams, as the \(E\) vs. \((2I_3)^{1/2}\) diagram has less evident features.

In Figure 5 and Figure 6, the bottom-left panel shows the \([\alpha/\text{Fe}]\) vs. \([\text{Fe/H}]\) diagram, where cyan and pink dots represent candidate thin disk and thick disk stars. Red and yellow dots denote the MWTD and GE, respectively, while the dark blue dots show the transition region between the metal-poor thin disk and GE. The rest of the panels in Figure 5 and Figure 6 show the corresponding distributions of the color-coded stars in the \((2I_3)^{1/2}\) vs. \(L_c\), and \(E\) vs. \(L_c\) diagrams, respectively.

Stars with cyan symbol have large \(L_c\) and low values of \((2I_3)^{1/2}\) and they are members of the thin disk stellar population. The color-coded pink and red stars pick up well the thick-disk and MWTD components, and possess larger \((2I_3)^{1/2}\) than thin-disk stars. Comparison between the top right and middle panels reveals that the MWTD includes more stars with large values of \((2I_3)^{1/2}\) than the thick disk, which suggests that stars in the MWTD component have larger orbital angles, \(\theta_{\text{orb}}\), than those in the thick disk. MWTD stars exhibit, on average, lower \(L_c\) than thick disk stars. The MWTD has indeed a lower mean rotational velocity, \(\langle V_\phi \rangle \approx 150\) km s\(^{-1}\), than the thick disk, \(\langle V_\phi \rangle \approx 180\) km s\(^{-1}\), as described in Carollo et al. (2019).

It is interesting to notice that stars with dark blue symbols are located in the same metallicity range as those with red symbols \((-1.3 < \langle \text{[Fe/H]} \rangle < -0.9\), but populate different intervals of \([\alpha/\text{Fe}]\), dark blue, +0.14 < \([\alpha/\text{Fe}]\) < +0.22, and red, +0.23 < \([\alpha/\text{Fe}]\) < +0.4). The middle-bottom panels of Figure 5 and Figure 6 show that stars color-coded with dark blue symbols separate in two different distributions, one having large \(L_c\) (even larger than the cyan symbols) and low \((2I_3)^{1/2}\), and one having \(L_c\) around 0 and extending towards large values of \(E\) and \((2I_3)^{1/2}\). Thus, in the region of the \([\alpha/\text{Fe}]\) vs. \([\text{Fe/H}]\) diagram color-coded with dark-blue symbols, there exist both rapidly-rotating, metal-poor stars, with values of \((2I_3)^{1/2}\) and \(L_c\) comparable to those of the thin disk stellar population, and likely GE debris stars. The rapidly-rotating disk-like stars may have migrated from larger radii in the disk (more metal poor) to the solar neighborhood, where they acquired larger rotational velocities due to the angular momentum conservation, and inducing a velocity-metallicity gradient, \(\Delta(V_\phi)/\Delta([\text{Fe/H}]) < 0\), as observed for the thin disk sequence (Lee et al. 2011b; Han et al. 2020). The dark-blue area in the abundance plane has been discussed in other recent works (see for example, Hayes et al. 2018; Das, Hawkins & Jofré 2020), suggesting the dominance of a non-rotating, accreted population of stars associated with the GE debris structure.
3.1.4. “Coarse-grained” phase-space distribution of halo stars

The halo system contains several sub-structures in the form of stellar streams/over-densities recognizable in spatial distributions, or in the form of long-lived sub-structures identified in phase space. The former are relics of relatively recent accretion/merging events of stellar systems and the latter are associated with such events but at much earlier epochs (e.g., Helmi & White 1999; Freeman & Bland-Hawthorn 2002; Feltzing & Chiba 2013). The halo might be entirely made of substructures (Naidu et al. 2020), and it may still on its way of a dynamically relaxed state. However, while the halo system continues to be in the course of relaxation in a “fine-grained” snap-shot of the phase space, such time varying features will be somehow smoothed out and closer to a steady state in the “coarse-grained” phase space (Binney & Tremaine 2008). Here, we attempt to quantify such a “coarse-grained”, general distribution of halo stars in the \((E, L_z, I_3)\) space by neglecting several fine sub-structures in it.

For this purpose, we show, in Figure 7, the distributions of \(L_z\) (left), \(2I_3^{1/2}\) (middle) and \(E\) (right panels) for the six ranges of metallicity defined earlier in the paper.

At metallicity \([\text{Fe/H}] > -1\), the \(L_z\) distribution (left panels) is dominated by disk-like kinematics components (thin- and thick-disk), while at lower metallicity, \(-1.4 < [\text{Fe/H}] < -1\), the distribution splits in two components, one with peak at \(L_z \sim 1200\) kpc km s\(^{-1}\) and dominated by the MWTD, and one with peak at \(L_z \sim 0\), which represents mainly GE debris stars. As the metallicity decreases \((-1.8 < [\text{Fe/H}] < -1.4\), fourth panel), the \(L_z\) distribution is still dominated by the GE debris, while the MWTD peak becomes weaker, and stars with retrograde motion start to appear. Thus, the average properties of the halo in these two last metallicity ranges are basically governed by two main structures, the MWTD and the GE debris stream. In the lower metallicity intervals, \(-2.2 < [\text{Fe/H}] < -1.8\) and \([\text{Fe/H}] < -2.2\), the \(L_z\) distribution shows a large fraction of stars with \(L_z \sim 0\) or slightly prograde, and a significant fraction of stars with highly retrograde motion. In these metal-poor ranges, the distribution of halo stars may be simply approximated such as \(f(L_z) \propto \exp(-L_z^2)\), where the positive \(L_z\) side shows a rapid decrease, truncated at \(L_z \approx 1500\) kpc km s\(^{-1}\), while the negative \(L_z\) side shows an extended tail originated by the presence of a significant number of stars with large retrograde motions.

In the \((2I_3)^{1/2}\) and \(E\) distributions (middle and right panels), the disk-like components are visible in the two top panels of higher metallicity intervals. At \([\text{Fe/H}] < -1\) (from the third panel), the MWTD feature is absorbed by the general halo distributions, which exhibit an exponential decrease.
with increasing \((2I_3)^{1/2}\) and \(E\), starting from \((2I_3)^{1/2} \approx 700 - 900\) kpc km s\(^{-1}\), and \(E \sim -1.5 - -1.3 \times 10^5\) km\(^2\)s\(^{-2}\), respectively. In fact, by considering \(I\) in the spherical limit, the function \(f(E, I)\) may be approximated by the so-called Michie-Bodenheimer type model (see e.g., Richstone & Tremaine 1984; Sommer-Larsen 1987; Binney & Tremaine 2008),

\[
f(E, I) \approx f(I) \propto \exp(-E/\sigma^2) - 1\exp(-I/r_a^2\sigma^2)
\]

for \(E \leq 0\), where \(\sigma\) is a 1d velocity dispersion and \(r_a\) stands for the anisotropic radius beyond which the velocity dispersion is anisotropic\(^4\).

We adopt this model to describe the \((2I_3)^{1/2}\) and \(E\) distributions for the metallicity ranges of \([-1.8 < \text{[Fe/H]} < -1.4\) and \(-2.2 < \text{[Fe/H]} < -1.8\), and by selecting stars with \((2I_3)^{1/2} > 700 - 800\) kpc km s\(^{-1}\) and \(E > -1.4 \times 10^5\) km\(^2\)s\(^{-2}\) (fourth- and fifth- middle panels of Figure 7). The exponential regression model provides, \((\sigma, r_a) = (122\) km s\(^{-1}\), 4.0 kpc) and \((116\) km s\(^{-1}\), 4.3 kpc), and reproduces very well the dependence of the function \(f(E, I)\) on \(E\) and \(I_3\).

In the most metal-poor range, \([\text{Fe/H}] < -2.2\), the \(E\) and \(I_3\) distributions exhibit a large extent toward high values (middle and right bottom panels), thus, the model requires a systematically larger velocity dispersion to obtain a good exponential fit. We found, \(\sigma = 144\) km s\(^{-1}\), while the anisotropic radius remains the same, \(r_a = 4.2\) kpc.

3.1.5. SDSS DR7 full sample

The entire SDSS-SEGUE DR7 data set is also adopted for comparison with the SDSS-SEGUE DR7 calibration stars sample. After the application of the selection criteria, the full sample contains a larger number of stars with respect the calibration stars sample \((N_{\text{full}} = 46,500)\), however, such increased number didn’t add much more information on the properties of the stellar halo, already obtained with the SDSS DR7 calibration stars. The similarity of the distributions in \(E - L_z\) and \((2I_3)^{1/2} - L_z\) as a function of the metallicity for the two samples can be assessed by comparing Figure 5 and 6 of the main section with Figure 16 and 17 in the Appendix. In case of the full data set, the sub-samples obtained for the various cuts of metallicity, contain a larger number of stars. This is valid, in particular, for the thin- and thick-disk stellar populations (cyan and pink, respectively). Nonetheless, the main features and properties of the recognized stellar populations and debris-streams, and described in section 3, remain the same.

3.2. SDSS DR16

Figure 8 and Figure 9 show the distributions of SDSS DR16 stars in the \((2I_3)^{1/2}\) vs. \(L_z\) and \(E\) vs. \(L_z\) diagrams,

---

\(^4\) In the spherical limit, the Michie-Bodenheimer type model corresponds to \(f(E, L) \propto \exp(-E/\sigma^2) - 1\exp(-L^2/2r_a^2\sigma^2)\) for \(E \leq 0\).
Figure 7. $L_z$ (left panels), $(2I_3)^{1/2}$ (middle panels), and $E$ (right panels) histograms in the six metallicity ranges, for the DR7 calibration stars. The adopted bins are, 0.2 ($10^3$ kpc km s$^{-1}$), 0.07 ($10^3$ kpc km s$^{-1}$), and 0.03 ($10^{5}$ km$^2$ s$^{-2}$), for $L_z$, $(2I_3)^{1/2}$, and $E$, respectively.

respectively. The grey dots show the entire sample. As for the DR7 calibration star sample, various components and features are selected by considering fiducial values for their metallicity and $\alpha$-elements abundance, shown with color-coded dot symbols in the bottom-left panel, in a similar fashion as in Figure 5 and 6. Cyan and pink dots represent likely thin disk and thick disk stars located in the most metal-rich range. Red and yellow dots denote the MWTD and GE, respectively, and the blue dots show the transition region between the metal-poor thin disk and GE.

The DR16 sample shows a broad distribution of stars located at $L_z \sim 2000$ kpc km s$^{-1}$, and $(2I_3)^{1/2} \sim 0$ kpc km s$^{-1}$ (Figure 8). This is caused by the APOGEE footprint (Figure 15 in the Appendix B.3), which includes extended regions along the Galactic plane, and covers large ranges of the Galactocentric radius, $3 \leq R \leq 14$ kpc. This wide footprint includes stars with large values of energy and vertical angular momentum for circular orbits up to $E \approx -1.0 \times 10^5$ km$^2$ s$^{-2}$, at $L_z \approx 3000$ kpc km s$^{-1}$ (see Figure 9, bottom-right panel). We also notice that at $L_z \approx 0$ kpc km s$^{-1}$, APOGEE stars possess slightly lower energy values than SDSS DR7 calibration stars ($E < -1.65 \times 10^5$ km$^2$ s$^{-2}$; see Figure 9). Also, the values of $(2I_3)^{1/2}$ at $L_z \approx 0$ (and lowest $E$) are somewhat smaller than those for the SDSS DR7 calibration stars, suggesting lower orbital angles, $\theta_{orb}$ (compare Figure 5 and Figure 8). This difference is caused by the APOGEE footprint in which more gravitationally bound stars, located at smaller $R$ and $\theta_{orb}$, are sampled (Figure 15, Appendix B.3).

While the APOGEE survey samples very well the thin- and thick-disk stellar populations, the more metal-poor halo stars are under-represented, in particular below [Fe/H]~ −2. On the contrary, in the SDSS DR7 calibration stars sample the
thin disk is under-represented, whereas the halo system is very well sampled. We thus focus on more metal-rich stars covering the disk and disk/halo overlapping regions, which are highlighted with cyan and pink color-coded symbols.

Inspection of the bottom-right panels reveals that thin-disk stars possess large $L_z$ and small values of $(2I_3)^{1/2}$, while thick-disk-like stars (top-right panels) show larger $(2I_3)^{1/2}$ and smaller $L_z$. This suggests, as expected, that the thick disk possesses lower mean rotational velocity and larger orbital angles from the Galactic plane, than the thin disk. The MWTD stars (red symbols, top-middle panel) exhibit large values of $(2I_3)^{1/2} \geq 500$ kpc km s$^{-1}$ (average value $(2I_3)^{1/2} \sim 850$ kpc km s$^{-1}$), which implies that MWTD stars possess orbits that form angles with the Galactic plane of $\theta_{\text{orb}} > 7$ deg (average value $\theta_{\text{orb}} \sim 10$ deg), and systematically larger than those possessed by the thick disk (average
value $\theta_{\text{orb}} \sim 7$ deg). The peak of the $L_z$ distribution for the MWTD is lower than that of the thick disk, showing that the MWTD lags behind the thick disk, in agreement with Carollo et al. (2019). Some stars selected as MWTD members have $L_z \sim 0$ kpc km s$^{-1}$ and $E \sim -1.6 \times 10^5$ km$^2$ s$^{-2}$, which may be halo-star contaminants.

The bottom-middle panels in Figure 8 and 9 show the transition region between the thin disk and GE in the [a/Fe] vs [Fe/H] diagram. Some stars fall in the areas dominated by the disk(s) populations, while most of the stars are characterized by $L_z \sim 0$ kpc km s$^{-1}$, and extended values of $E$ and $(2L_3)^{1/2}$, matching the distribution of stars likely members of the GE debris (yellow dots in the top-left panels). Note that the distribution of stars selected as likely GE members in the abundance diagram exhibits a wide range of vertical angular momentum larger than what expected for the GE debris (i.e. Koppelman et al. (2019)), in both prograde and retrograde motion. This is likely due to contamination from other, smoother parts of the stellar halo. These properties are consistent with what found for the SDSS DR7 calibration stars sample, and can be then considered as general.

4. DISCUSSION

4.1. The stellar halo defined in the phase space

Mapping our stellar halo while living inside it is not a straightforward task, because the information available for each star is incomplete, even in the Gaia era. Indeed, the precise trigonometric parallaxes and proper motions are available only in the confined local volume, and the line-of-sight velocities and metal abundances, derived from the spectroscopic data (mostly provided by ground-based observations), are limited to the relatively bright stars in the targeted regions of the sky. However, the advantageous position of the Sun in the Milky Way allows us to capture and analyse most types of stellar orbits passing near the Sun. These orbits represent the basic ingredients of the stellar halo (May & Binney 1986). Thus, unless the full-depth, full-sky astrometric and spectroscopic data of stars over the very extended halo region are available, fundamental insights into this old and very important Galaxy component, can be obtained primarily from the orbit-based selection in the phase space defined by the integrals of motion, $(E, L_z, I_3)$.

Our analyses of the SDSS-SEGUE DR7 and APOGEE DR16 data sets suggest that halo stars are characterized by two distinctive properties in the phase space: large values of the third integral of motion, $I_3$ [expressed in terms of $(2I_3)^{1/2}$], such that $(2I_3)^{1/2} \gtrsim 1000$ kpc km s$^{-1}$, and vertical angular momentum, $L_z \lesssim 1500$ kpc km s$^{-1}$. Additionally, as will be discussed later in Section 4.4.2, an important portion of halo stars populate the energy range below $-1.5 \times 10^5$ km$^2$ s$^{-2}$. This energy threshold coincides (nearly) with the gravitational binding energy at the position of the Sun, $\Phi(R = R_\odot)$, as estimated from the observed escape velocity of nearby stars of $V_{\text{esc}} \sim 550$ km s$^{-1}$ (e.g. Sakamoto et al. 2003; Koppelman & Helmi 2020), so that $\Phi(R_w) = -V^2_{\text{esc}}/2 \sim -1.5 \times 10^5$ km$^2$ s$^{-2}$. These stars possess values of $I_3$ in the range of $500 < (2I_3)^{1/2} < 1000$ kpc km s$^{-1}$, which corresponds to $\theta_{\text{orb}} \sim 5$-15 deg and, at the lower end of this interval of orbital angles (5 deg), there exist contamination from the MWTD.

These general properties suggest that the selection of pure halo stars can be done by adopting the fiducial limits of (1) $(2I_3)^{1/2} > 1000$ kpc km s$^{-1}$ ($\theta_{\text{orb}} > 15$ deg) & $L_z < 1500$ kpc km s$^{-1}$, and (2) $E < -1.5 \times 10^5$ km$^2$ s$^{-2}$. Note that $\theta_{\text{orb}} \sim 15$ deg corresponds to $z \gtrsim 2.3$ kpc at the solar radius. It is also important to remark that the suggested range of $L_z$ for halo stars ($L_z < 1500$ kpc km s$^{-1}$), in the intermediate metallicity interval of $-1.5 < [\text{Fe/H}] < -1$, contains also a fraction of MWTD stars.

To highlight the significance of these selection criteria for halo stars, we show, in Figure 10, the comparison between two $E$ vs. $L_z$ diagrams obtained by adopting the high-velocity cut selection (left panel: $|V_{\text{star}} - V_{\text{LSR}}| > 200$ km s$^{-1}$), and the new orbits-based selection criteria in the $(E, L_z, I_3)$ phase-space (right panel), for the case of the SDSS DR7 calibration stars. The grey dots show the distribution for the entire sample, while the color-coded symbols represent stars selected by employing the two methods. Examination of this figure reveals that the high-velocity cut selection for halo stars misses a significant portion of the stellar halo represented by prograde stars with both, low and intermediate energy, over the range of $0 \leq L_z \leq 1500$ kpc km s$^{-1}$ and $E \lesssim -1.2 \times 10^5$ km$^2$ s$^{-2}$, whereas our selection criteria include these stars (right panel). This clearly demonstrates the importance of an orbit-based selection, rather than a simple high-velocity cut, to avoid a biased understanding of the Milky Way’s stellar halo.

4.2. Metallicity distribution and the halo duality

Before Gaia, Carollo et al. (2007, 2010) provided evidence of the Milky Way’s halo complexity by demonstrating that it comprises at least two components, namely, the inner and outer halo. The inner halo is characterized by a mean Galactocentric rotational velocity $\sim 0$, and a metallicity distribution function (MDF) with peak at $[\text{Fe/H}] \sim -1.6$, while the outer halo exhibits highly retrograde motion and it is very metal poor, with an MDF having peak at $[\text{Fe/H}] \sim -2.2$. The astrometric parameters used in the above analyses were not as accurate as those provided by Gaia DR2, implying that it was possible to capture mainly the “coarse-grained” properties of halo stars.

To get further insights into this topic, we show, in Figure 11, the MDFs for the SDSS-SEGUE DR7 calibration stars determined for different values of $L_z$. The black his-
tograms represent the MDFs for the sub-samples of stars selected in various ranges of \( L_z \), starting from \( L_z = 1500 \) kpc km s\(^{-1}\) and reaching large distances from the Galactic plane during their orbits, i.e. \((2I_3)^{1/2} > 1000\) kpc km s\(^{-1}\).

In the range of angular momentum, \( 1000 < L_z < 1500\) kpc km s\(^{-1}\) (top panel), the black histogram is dominated by the MWTD stellar population, with metallicity peak at [Fe/H] \(-0.9\) (\( L_z \sim 1200\) kpc km s\(^{-1}\) for the MWTD, Carollo et al. 2019), and some thick disk contamination. On the contrary, the red histogram shows lack of stars with disk-like kinematics, due to the selection over larger values of \((2I_3)^{1/2}, (2I_3)^{1/2} > 1000\) kpc km s\(^{-1}\), and exhibits a metallicity peak of [Fe/H] \(-1.3\). It is important to note that at larger \((2I_3)^{1/2}\), there exist a finite fraction of metal-rich halo stars with [Fe/H] \(-1\): these stars are characterized by large \( L_z, I_3\), and also \( E\), and therefore high orbital eccentricities (see also, Bonaca et al. 2017; Naidu et al. 2020). Thus, this sub-sample (red color) contains also a fraction of metal-rich stars with halo-like kinematics, represented by the portion at the right-end of the MDF.

As \( L_z \) decreases, in the range \( 500 < L_z < 1000\) kpc km s\(^{-1}\), the peak of the MDF represented by the black histogram is shifted to a more metal-poor value of [Fe/H] \(-1.5\), while the fraction of MWTD stars decreases. Below \( L_z = 500\) kpc km s\(^{-1}\) the MDF shows a peak at [Fe/H] \(-1.5\), with a long tails toward lower metallicities. At high \((2I_3)^{1/2}\), the MDF shows a dominant peak at [Fe/H] \(-1.5\), lack of stars with disk kinematic, and a large fraction of very metal-poor stars (second and third panel). In the ranges, \(-500 < L_z < 0\) kpc km s\(^{-1}\), and \(-1000 < L_z < -500\) kpc km s\(^{-1}\), the MDF is still dominated by stars with metallicity, [Fe/H] \(-1.5\) both, at low and high ranges of \((2I_3)^{1/2}\).

Note that, if we consider the mean rotational velocity and its dispersion determined for the inner halo component in Carollo et al. (2010), i.e. \( V_\phi \sim 0\) km s\(^{-1}\), and \( \sigma_{V_r} = 90\) km s\(^{-1}\), which correspond to \( L_z \sim 0\) kpc km s\(^{-1}\), and \( \sigma_{L_z} = 800\) kpc km s\(^{-1}\), then the range \(-1000 < L_z < 1000\) kpc km s\(^{-1}\), and high \((2I_3)^{1/2}\), samples very well the inner halo MDF (panels 2-4). Thus, the inner halo component includes the GE structure (dominant, around the peak of [Fe/H] \(-1.5\)), low energy stars (\( E \leq -1.5 \times 10^5\) km s\(^{-2}\)), at [Fe/H] \(-1\), and metal-poor prograde stars, including the more metal rich, which are missed by the cuts adopted in Helmi et al. (2018). When the stellar orbits become more retrograde, with \( L_z < -1000\) kpc km s\(^{-1}\), the MDF progressively includes more stars with [Fe/H] \(-2\). At high \((2I_3)^{1/2}\), the peak at [Fe/H] \(-1.5\) weakens, and shifts toward the very metal poor zone, at \(-2\). Such distribution is dominated by the outer halo population described in Carollo et al. (2007, 2010).

At what Galactic location the outer halo component starts to dominate?

Figure 12 shows the ratio between the number of stars in the metallicity interval of [Fe/H] \(-2.2\), and that in \(-2 < [\text{Fe/H}] < -1.4\), \( F = N([\text{Fe/H}] < -2.2)/N([\text{Fe/H}] < -1.4)\), in each bin of \((2I_3)^{1/2}\) (left), \( E\) (middle), and \( \theta_{\text{orb}}\) (right), for the SDSS-SEGUE DR7 calibration stars. The first range of metallicity represents very metal-
This orbital angle corresponds to \( z \). The ratio, \( F \), of the very metal-poor stars increases discontinuously near \( \theta \approx 50 \) deg, where \( F \) is defined as \( \theta = 0 \). The middle panel shows that outer halo stars possess very high energy, beyond \( E \approx -0.8 \times 10^3 \) km\(^2\) s\(^{-2}\).

It is important to note that, while GE is still present in the metal-poor outer halo, it does not represent a significant fraction of it (bottom panel in Figure 11). Some of the retrograde sub-structures reported in recent literature, such as Sequoia (Myeong et al. 2019), Arjuna and I’itoi (Naidu et al. 2020), and the dynamically tagged groups (DTG) identified in (Yuan et al. 2020), represent “fine-grained” elements of the outer halo defined in Carollo et al. (2007, 2010). Thus, it is likely that the outer halo component is made of a superposition of several sub-structures, relics of past accretion events. This can be easily recognizable in Figure 6, where the retrograde halo in the \( E \) vs. \( L_\theta \) diagram, resides in the area where several substructures have been identified, around \( L_\theta \sim -1000 \) kpc km s\(^{-1}\) (see also Naidu et al. 2020).

In addition, the SDSS-SEGUE calibration stars are selected to have the best atmospheric parameters and metallicity. This implies that the MDFs of these stars represent the true distributions, in particular, at low metallicity, and for stars outside the disk populations, and it is definitely not biased.

Also, in case of the inner halo stellar population, the anisotropy parameter was found to be, \( \beta = 0.6^5 \) in Carollo et al. (2010), implying that the velocity ellipsoid is radially elongated. This is in agreement with the radial anisotropy of the velocity ellipsoid determined for GE in Belokurov et al. (2018), but this velocity anisotropy is now more significant, \( \beta \sim 0.9 \), after \( Gaia \).

We note that the asymmetry in the \( E \) vs. \( L_\theta \) distribution of halo stars, with the presence of more high-\( E \) retrograde than prograde stars, as seen in our and other local samples (Helmi et al. 2017; Myeong et al. 2018a,b; Yuan et al. 2020), is not caused by a selection effect that avoid contamination from stars with disk-like kinematics, mentioned by Naidu et al. (2020). In our analysis, we have not adopted such an artificial selection method for halo stars and, as we have also stated in Section 2: kinematically selected halo stars are generally biased in favor of high relative velocities with respect to the Sun, giving asymmetry in the \( E \) vs. \( L_\theta \) distribution, but this is not the case in our work.

Also, as seen in Figure 4, largely retrograde stars with \( L_\theta \leq -2000 \) kpc km s\(^{-1}\) and high \( E \) of, say \( E > -0.8 \times 10^3 \) km\(^2\) s\(^{-2}\), possess values of the third integral of motion, \( (2l_\beta)^{1/2} \approx 1500 \) kpc km s\(^{-1}\) (or \( \theta_{\text{orb}} \leq 45 \) deg). These stars may not be present in the H3 survey areas of Naidu et al. (2020), where stars are selected at high Galactic latitudes \( |b| > 40 \) deg (see

\[^5\ beta \ is defined as \( \beta = 1 - (\sigma_r / \sigma_t)^2 \), where \( \sigma_r \) and \( \sigma_t \) are the radial and tangential velocity dispersions, respectively.

Figure 11. Metallicity distributions (MDFs) for the SDSS-SEGUE DR7 calibration stars in different ranges of \( L_\theta \). The black histograms represent the MDFs for the subsamples of stars selected in the various ranges of \( L_\theta \), starting from the top, \( 1000 < L_\theta = 1500 \) kpc km s\(^{-1}\), to the bottom, \( L_\theta < -1000 \) kpc km s\(^{-1}\), while the red histograms show the MDFs in the same ranges of \( L_\theta \), but reaching large distances from the Galactic plane during their orbits, i.e., \( (2l_\beta)^{1/2} > 1000 \) kpc km s\(^{-1}\). Poor halo stars, where the outer halo is dominant with respect to the inner halo, which is well represented in the second metallicity range ([Fe/H]\(_{\text{peak}} \approx -1.6\)). Visual inspection of the left and right panels reveals that the relative fraction of the very metal-poor stars increases discontinuously near \( (2l_\beta)^{1/2} \approx 2000 \) kpc km s\(^{-1}\) and \( \theta_{\text{orb}} \approx 50 \) deg, where \( F = 0.5 \). This orbital angle corresponds to \( z \approx 10 \) kpc at the solar radius. We remind here that \( \theta_{\text{orb}} \) represents the maximum orbital angle from the Galactic plane. The middle panel shows that outer halo stars possess very high energy, beyond \( E \approx -0.8 \times 10^3 \) km\(^2\) s\(^{-2}\).
their Figure 1). This can explain the lack of asymmetry in the distribution of their sample. Moreover, while stars with such high negative \(L_z\) possess large negative \(V_\phi\) in the solar neighborhood, they have small \(|V_\phi|\) at large apocentric radii, due to angular-momentum conservation, thus their rotational signals expressed in terms of \(V_\phi\) is weaker in the outer-halo region (at large distance). For instance, stars that possess \(V_\phi \sim -200\) kpc km s\(^{-1}\) at the location of the Sun will have \(V_\phi \sim -20\) kpc km s\(^{-1}\), at \(R = 100\) kpc.

4.3. On the relation with the Sagittarius dwarf galaxy, its stream, and the Milky Way’s globular clusters

As clearly shown in Figure 4 (left panels), the \(L_z\) distribution of halo stars in the metal-poor ranges of \([Fe/H]< -1.4\) is sharply bounded at \(L_z \approx 1500\) kpc km s\(^{-1}\). This appears to be the case for both SDSS DR7 and DR16, and therefore, the truncation of \(L_z\) at this value can be considered a universal property for candidate halo stars in the local volume. Also, as mentioned in Section 3.1.2., in the SDSS DR7, where the halo system is better represented than in the APOGEE DR16, there exist an elongated structure in the \((2I_3)^{1/2}\) vs. \(L_z\) diagram for the metallicity ranges of \(-2.2 < [Fe/H] < -1.4\), namely from \((L_z, (2I_3)^{1/2}) \approx (1500 – 1800, 400)\) kpc km s\(^{-1}\) to \((1000, 3500)\) kpc km s\(^{-1}\). We note that the truncation value of the vertical angular momentum below \([Fe/H] < -1.4\), \(L_z \approx 1500\) kpc km s\(^{-1}\), is just adjacent to the lower-end of this elongated structure located at \(L_z \approx 1500 – 1800\) kpc km s\(^{-1}\), thus, these two features may be related to each other.

To get more insights into this issue, we explore a possible connection between the sharp cut in \(L_z\) of halo stars with the Sagittarius dwarf spheroidal galaxy (Sgr dSph), its associated stream (Sgr stream), and the Milky Way’s globular clusters (GCs), by inspecting their phase-space distribution.

This is motivated by the fact that the Sgr stream contains metal-poor stars over the range of \(-1.9 < [Fe/H] < -1\), with mildly peak at around \([Fe/H] = -1.4\) (e.g., de Boer, Belokurov & Kroupa 2015; Hayes et al. 2020). Such metallicity interval overlaps with the range where the elongated features is spotted in the left panels of Figure 4 ((2I_3)^{1/2} vs. \(L_z\) diagram).

For the Sgr dSph, we adopt the recent compilation reported in Pawlowski & Kroupa (2020) (see their Table 1) based on the works by Ibata et al. (1997), Dinescu et al. (2005) and the Gaia Collaboration (2018). For the Milky Way’s GCs, we adopt the recent data collection described in Baumgardt et al. (2019), where Gaia DR2 proper motions are included. Figure 13 shows the \((2I_3)^{1/2}\) vs. \(L_z\) and \(E\) vs. \(L_z\) diagrams for the Sgr dSph and the Milky Way’s GCs superimposed with the distribution of the SDSS-SEGUE DR7 calibration stars (grey dots) in the range of metallicity, \(-1.8 < [Fe/H] < -1\), where the elongated feature is more evident.

Examination of Figure 13 reveals that the GCs M 54, Ter 7, Ter 8, and Arp 2 are very close to Sgr dSph in the phase space \((E, L_z, I_3)\), indicating that they are associated to the dwarf galaxy. This is in agreement with previous results reported in (e.g., Law & Majewski 2010; Bellazzini et al. 2020), where the association was based on the GCs position and velocity. Figure 13 also shows that Whiting 1 and Pal 12, are also close to the location of the Sgr dSph in the \((E, L_z, I_3)\) phase space. These GCs are actually found in the Sgr trailing
Figure 13. Sagittarius dwarf galaxy (pink star) and the Milky Way globular clusters (pink circles with names labelled, and blue circles), together with the SDSS DR7 calibration stars in $-1.8 < \text{[Fe/H]} < -1$ (grey dots), in the $(2I_3)^{1/2}$ vs. $L_z$ (left panel), and $E$ vs. $L_z$ diagrams (right panel).

NGC 5634, NGC 4147 and NGC 2419, which may be associated with more ancient wraps of the stream (Bellazzini et al. 2020), have very different $I_3$ values (except NGC 2419), with respect to the Sgr dSph, perhaps caused by the large deviation of the stream from the galaxy’s orbit. We also notice that the positions of the Sgr dSph and its associated GCs in the $E$ vs. $L_z$ diagram are in good agreement with the Sgr-associated substructures of field stars found in Naidu et al. (2020). It is important to remark that the $L_z$ value of the Sgr dSph ($\simeq 1200$ kpc km s$^{-1}$) is close to the sharp bound found in the $(2I_3)^{1/2}$ vs. $L_z$ distribution for $\text{[Fe/H]} \lesssim -1.4$, at $L_z \sim 1400 - 1500$ kpc km s$^{-1}$, and the mentioned elongated structure of stars in the $(2I_3)^{1/2}$ vs. $L_z$ diagram is somehow oriented from this truncation region. Moreover, the Sgr dSph and some GCs are distributed along this structure. A similar dynamical effect from the Large Magellanic Cloud (LMC) was recently examined by Garavito-Camargo et al. (2019), and Cunningham et al. (2020), motivated by a recent determination of the LMC mass, of the order of $10^{11} M_\odot$ (e.g., Erkal et al. 2019). This massive LMC can produce a dynamical effect on both dark-matter and stellar halos of the Milky Way. In such a scenario, the infalling LMC to the Milky Way generates a pronounced wake to the distributions of both dark-matter particles and halo stars. The wake is decomposed in transient and collective responses, where the former (latter) effect is local (global), and provides distinct kinematic patterns in both dark-matter and stellar halos. Similarly to the LMC wake-effect, we suggest that an equivalent but weaker wake is induced by the Sgr dSph in the local halo stars, although further theoretical investigation is needed to confirm this hypothesis.

The Sgr dSph is also thought to affect the structure and dynamics of the Milky Way disk (e.g., Parcell et al. 2011; Laporte et al. 2018). In particular, the galaxy can produce a wobbly structure in the outer parts of the disk (Laporte et al. 2018) as found in recent works (Li et al. 2017; Bergemann, et al. 2018). Thus, the Sgr dSph may play an important role in forging the dynamical structures in the Milky Way, and in its star formation history (Ruiz-Lara et al. 2020).

It is also interesting to note in Figure 13 that some GCs occupy the phase-space region of the GE debris stars, around $L_z = 0$, in both the $(2I_3)^{1/2}$ vs. $L_z$ and $E$ vs. $L_z$ diagrams. This suggest that these GCs, and field halo stars, in this phase-space region, originate from the same progenitor galaxy that formed the GE structure. Also, a fraction of field halo stars in this phase-space region may originate from disrupted GCs (See, e.g., Massari et al. 2019; Kruijssen et al. 2019; Myeong et al. 2019; Kruijssen et al. 2020; Forbes 2020).

4.4. What is the in situ stellar halo?

4.4.1. Formation process of dark-matter and stellar halos

The definition of the so-called in-situ stellar halo, compared to the ex-situ stellar halo, and other halo populations, is somewhat different in various literatures, in particular, among observational studies (e.g., Sheffield et al. 2012; Hayes et al. 2018; Haywood et al. 2018; Di Matteo et al. 2019; Gallart et al. 2019; Conroy et al. 2019; Montalbán et al. 2020; Belokurov et al. 2020; Naidu et al. 2020). This is due to the numerous sources of halo stars generated during the complex galaxy formation process, and the difficulty in identifying such stars in observational studies. Theoretical
predictions suggest that in-situ stars formed inside a parent halo within the virial radius (Zolotov et al. 2010; Tissera et al. 2012, 2013).

While dark halos are entirely formed through hierarchical assembly and merging process, stellar components in dark halos originate from multiple sources, in particular, from star formation within cooled gas originally present in parent halos, in gas stripped from merging satellites and subsequently cooled, or stars are supplied by merging/accreting halos, that disappear after tidal interaction, or survive as luminous satellites, (e.g., Bekki & Chiba 2001; Bullock & Johnston 2005; Zolotov et al. 2009; Purcell et al. 2010; Font et al. 2011; McCarthy et al. 2012; Tissera et al. 2013; Cooper et al. 2015; Rodriguez-Gomez et al. 2016; Deason et al. 2016; D’Souza & Bell 2018a; Monachesi et al. 2019; Fattahi et al. 2020). The question is whether we can really distinguish stars formed inside parent halos from those supplied from outside, when using observational data alone.

A brief review of the halos formation process in the expanding Universe can provide some insights into this issue.

Formation of dark halos in the expanding Universe is totally hierarchical (White & Rees 1978): in the standard structure formation scenario based on a Λ-dominated cold dark matter (CDM) models, small, dense dark halos collapsed first while the Universe was dense, and they merged together leading to larger, more massive halos. Subsequent collapsed dark halos are generally less dense as they formed in a less dense Universe. The merging process between the early high density small halos, and the less dense large halos, results in a more massive halo, where its central compact part is dominated by the denser halo progenitor, while its outer and lower-density section, is made of the larger halo progenitor debris. In this hierarchical process, the main progenitor halo can be defined as the most massive parent halo, when following backward the branching of a merger tree for a current host halo (e.g., Mo, van den Bosch & White 2010).

The formation and evolution of stellar halos differ from those of dark halos. Indeed, stars are formed from cooled interstellar gas, supplied after radiative cooling of virialized hot gas, mostly at the bottom of each dark halo (Rees & Ostriker 1977; Fall & Efstathiou 1980), or from already cold gas directly flowing from intergalactic space (Dekel et al. 2009). Gas in each halo is also supplied by merging/accreting satellites carrying cold gas cores, and hot halo gas (e.g., Kauffmann et al. 1993). Each dark halo grows through merging/accretion with other dark halos, and stellar populations within the product of this merging, are then composed of pre-existing stars formed inside the main progenitor halo, and merged/accreted stars in other halos. The former stellar population formed inside a main parent halo can be regarded as an in situ stellar halo, which consists of stars forged from cooled gas inside it, and those dynamically heated from an already formed stellar disk in the merging process of subhalos/satellites (e.g., Zolotov et al. 2009; Purcell et al. 2010; Font et al. 2011; McCarthy et al. 2012; Tissera et al. 2013; Cooper et al. 2015; Rodriguez-Gomez et al. 2016; Monachesi et al. 2019). Detailed simulation studies for the formation of stellar halos suggest that the fraction of in-situ relative to ex-situ stellar halos increases with the increasing total mass of a main progenitor halo (Rodriguez-Gomez et al. 2016; Monachesi et al. 2019).

4.4.2. Lowest binding energy stars as a part of the in situ stellar halo

How do we select stars belonging to the in-situ stellar halo, from observational data in the local volume? This has been a central theme in recent related works (e.g., Hayes et al. 2018; Haywood et al. 2018; Di Matteo et al. 2019; Gallart et al. 2019; Conroy et al. 2019; Montalbán et al. 2020; Belokurov et al. 2020; Naidu et al. 2020).

A finite fraction of stars, formed from cooled gas in the bottom of a main progenitor halo, or from merged/accreted cold gas supplied by other halos, are most tightly bound to the gravitational potential of that progenitor halo after dissipative cooling. In comparison, other in-situ stars, which are product of a pre-existing dynamically heated disk by merging events, and ex-situ stars supplied by merging/accretion events from outside, are less bound, because of their larger binding energies. Then, candidates in-situ halo stars can be defined as those being most tightly bound to the Milky Way gravitational potential, or, in other words, those having the lowest binding energy, \(E\). As mentioned above, another possible source of in-situ halo stars, are those kicked out from a pre-existing, metal-rich high-\(\alpha\) disk populations, driven by merging events with satellites or subhalos and are expected to have higher \(E\) after dynamical heating (e.g., Bonaca et al. 2017; Belokurov et al. 2020).

Motivated by these scenarios, we consider stars with \(E < -1.5 \times 10^5 \text{km}^2 \text{s}^{-2}\) as candidate in-situ halo stars. This choice can be understood by examining the properties of halo stars in the DR7 sample. The bottom-middle panel of Figure 6 and 9 shows that the extension of candidate GE debris stars, color-coded with dark blue, have energy above \(-1.5 \times 10^5 \text{km}^2 \text{s}^{-2}\) at \(L_z \sim 0\). Thus, stars selected below this energy value should be less contaminated by those associated with the GE structure. This is in agreement with the results of Naidu et al. (2020), where the GE debris stars, selected in terms of orbital eccentricities \((e > 0.7)\), occupy only a small fraction at energies, \(E < -1.5 \times 10^5 \text{km}^2 \text{s}^{-2}\), in the \(E\) vs. \(L_z\) diagram (see their Figure 11). A natural explanation for this effect reside in the formation mechanism of the GE debris, which was likely formed by the merging of a dwarf galaxy with associated dynamical heating of a pre-existing stellar disk. It is expected that these stars possess high binding energy after the merging event.
As discussed in Section 3.1.1, stars with the lowest $E$ range are distributed around $L_e = 0$, and populate the third integral of motion in the range of $400 \lesssim (2I_3)^{1/2} \lesssim 1200$ kpc km s$^{-1}$. Stars with the lowest $E$ have $(2I_3)^{1/2} \approx 700$ kpc km s$^{-1}$ (see right panels in Figure 3 and 4). These value of $(2I_3)^{1/2}$ correspond to orbital angles in the range of $5^\circ \leq \theta_{\text{orb}} \leq 20^\circ - 30^\circ$, and $\theta_{\text{orb}} \approx 10^\circ$ for the lowest $E$. When expressed in terms of the maximum distance from the Galactic plane, $z_{\text{max}}$, these stars have $z_{\text{max}} < 5$ kpc.

Figure 14 shows the apocentric radii, $r_{\text{apo}}$, as a function of the orbital eccentricity, for the DR7 calibration stars sample (grey dots). The red dots represent stars with $E < -1.5 \times 10^5$ km$^2$ s$^{-2}$, and their $r_{\text{apo}}$ are near but below the solar position, $R_\odot$, and their orbital eccentricities are distributed over the range, $e > 0.3$. A fraction of these stars may be contaminated with GE debris, at $e > 0.7$, although such high $e$ stars can actually be in-situ halo stars in the inner region of the Milky Way. The right panel of Figure 14 shows the [$\alpha$/Fe] vs. [Fe/H] diagram for the DR7 calibration stars (grey), and the candidate in-situ stars (red). The majority of the low-energy stars occupy the region at $[\text{Fe/H}] < -1$, but some of them are also in the area dominated by the MWTD, with small contamination from the thick disk ($[\text{Fe/H}] \leq -0.8$ and [$\alpha$/Fe]$ > 0.25$). As can be seen in the top-middle and top-right panels of Figure 6, disk-like stars are located just around the adopted threshold of $E = -1.5 \times 10^5$ km$^2$ s$^{-2}$, in the $E$ vs. $L_z$ diagram, therefore, the contamination from these stars can be removed by adopting a slightly smaller value of $E$.

Thus, candidate in-situ halo stars can be selected by adopting $E < -1.5 \times 10^5$ km$^2$ s$^{-2}$, and [Fe/H]$ < -1$.

We argue that it is unlikely that these stars originate from the dynamical heating of a pre-existing high-[$\alpha$/Fe] stellar disk driven by merging of subhalos or satellites at early epoch, nor from the debris of a merging event of a dwarf galaxy giving rise to the GE structure. Instead, these low-$E$ stellar populations may have been formed from early dissipation processes supplied by cold-gas stream (Dekel et al. 2009), where efficient star formation and chemical evolution are at work in the bottom region of a host dark halo (e.g., Brook et al. 2020).

4.4.3. Other possible candidates for the in-situ stellar halo

In-situ stars that are formed from infalling cooled gas within a dark halo are subject to chemical evolution, accompanying not only gaseous infall but also outflow from the system, and the interplay between these hydrodynamical processes yields a specific metallicity distribution of stars, where a finite fraction is metal-poor, say [Fe/H]$ < -2$ (e.g., Chiappini et al. 1997, 2001; Haywood et al. 2013; Toyouchi & Chiba 2018). In particular, early infalling, pristine gas would have been settled into an equatorial plane of a progenitor dark halo, in the presence of an initial angular momentum (e.g., Katz & Gunn 1991). From this gas, a fractions of very metal-poor stars, with low $L_z$, and relatively high $L_\perp$ may have been formed and left there.

Do such metal-poor stars exist in the Milky Way?
In the sample of Chiba & Beers (2000), which was assembled from several stellar sources in combination with the Hipparcos satellite data, and ground-based observations (Beers et al. 2000), a finite fraction of metal-poor halo stars possess low $J_1$ (low $L_\perp$), low $J_z$, and high $L_z$, as shown in their Figure 16. Note that these stars are located at relatively low Galactic latitudes (see Figure 1 of Beers et al. (2000)), and therefore, stars with the above orbits are indeed present in a local sample. Also, in a recent work based on data from the Pristine Survey (Starkenburg et al. 2017), Sestito et al. (2020) report the discovery of several metal-poor stars, with metallicity, $\text{[Fe/H]} = -2.5$, low orbital eccentricities, and low vertical actions, $J_z$. These stars show both prograde and retrograde rotations, where the former appears to occupy a larger fraction. The idea is that such stars formed, possibly, in the early stages of the Milky Way assembly process, and may still be in the Galactic plane.

Some, or many of these stars, may acquire vertical actions through early dynamical processes, and thus they exhibit a finite $J_1$ integral of motion, leaving a fan shape in the $E$ vs. $(2J_1)^{1/2}$ diagram, as seen in the metal-poor intervals of Figure 4. Although some of these metal-poor stars have been identified in recent surveys, their global properties are still not well understood.

In connection to this issue, it is interesting to remark that recent high-redshift surveys of star-forming galaxies, based on carbon monoxide (CO) or Hα spectroscopy, have identified rotating gaseous disks at redshifts $z \sim 2$ (Förster Schreiber et al. 2009; Price et al. 2016; Genzel et al. 2017). Furthermore, in most recent studies, based on the Atacama Large Millimetre/submillimetre Array (ALMA), galaxies selected through their [C II] emission spectroscopy of H I absorption, show that such gaseous disks on a galaxy scale are already present at redshifts of 4 to 5, or 12 Gyr ago (Neelam et al. 2019; Neeleman et al. 2020). The existence of such disks at high redshift, suggests that the accretion of cold gas onto a dark halo is at work (Dekel et al. 2009), and explains the very early formation of rapidly rotating gaseous disks at $z \sim 5$. The rotational properties of these disks are characterized by a flat rotation curve, thus suggesting the presence of dark halos behind them. The existence of rotating gaseous disks naturally implies star formation activity, leading to the formation of metal-poor stars with disk-like kinematics, like those found in the Pristine Survey (Starkenburg et al. 2017; Sestito et al. 2020).

Note also that the rotational direction of such early gaseous disks, formed from cold accretion flow, is not necessarily the same as that of the currently observed thin/thick disks at $z = 0$. This implies that ancient metal-poor halo stars formed in this way can be rotating in either way (prograde or retrograde) compared to the current thin/thick disks, as observed in Sestito et al. (2020). These stars may occupy regions of the phase-space located along a parabola in the $E$ vs. $L_z$ diagram (see Figure 4). We suggest that many of the in-situ halo stars originated from the above process can be present in the local volume, however such stars are likely outside the survey volume of SDSS.

5. IMPLICATIONS FOR THE FORMATION OF THE STELLAR HALO

5.1. A brief overview prior and subsequent to Gaia

Since Eggen, Lynden-Bell & Sandage (1962) first presented their view of Galaxy formation based on the analysis of nearby stars with various metallicities and orbits, our picture of how a galaxy like our home formed has developed and evolved significantly, initially driven by the advent of newly calibrated data sets (Searle & Zinn 1978; Yoshii & Saio 1979; Norris et al. 1985; Beers & Sommer-Larsen 1995; Carney et al. 1996). The monolithic, infalling galaxy and collapse scenarios, and dissipative radiative cooling of a primordial gas cloud, was proposed by Eggen et al., and was guided by the properties of a kinematically biased sample of halo stars, in which metal-poor stars showed predominantly high-$e$ orbits. The subsequent works based on non-kinematically selected halo samples showed the existence of low-$e$ metal-poor stars, and therefore, challenged the above view of Galaxy formation (Yoshii & Saio 1979; Norris et al. 1985). Modern stellar catalogs, such as those based on the first astrometric satellite, Hipparcos, supported the latter view (Chiba & Yoshii 1998), and even showed the first signature of the hierarchical assembly of the stellar halo (Helmi et al. 1999). In fact, both dissipative and dissipationless processes appear to be at work in the formation of the stellar halo, as showed by the comparison between the observed chemo-dynamical properties of halo stars, and the predictions of CDM-based galaxy formation models Bekki & Chiba (2001).

Additional compelling evidence of such “mixed” formation scenario was later shown by Carollo et al. (2007, 2010), based on their finding of a multiple-halo structure, namely, the inner- and outer-halo. The inner halo, which corresponds to the flattened part of the halo originally found by Sommer-Larsen & Zhen (1990), was suggested to form through the dissipative merging of massive clumps, leading to stars with high eccentric orbits. Part of this halo may have been formed in-situ from the rapid collapse of infalling gas. On the contrary, the distinctive properties of the outer halo stellar population, in particular, its retrograde-rotation and very metal-poor MDF, implied a different origin, possibly through a dissipational caotic merging of low-mass subsystems. Later numerical simulations of hierarchical galaxy formation produce these dual-halo systems, which exhibit many of the observed properties of the Milky Way, as well as M31 (Zolotov et al. 2009; Font et al. 2011; McCarthy et al. 2012; Tissera...
The halo age-structure described in Santucci et al. (2015), Carollo et al. (2016), and Whitten et al. (2019), have also added significant evidence for the outside-in assembly of the halo, with the contribution of the latest merger events to the outskirts. Interestingly, high-resolution numerical simulations in ΛCDM are able to reproduce the observed halo age gradient slopes by considering mainly the contribution of the accreted stars, with a minimal input of in-situ stars (Carollo et al. 2018).

After Gaia, this view of hierarchical clustering scenario for the stellar halo formation has been much strengthened and detailed, owing to the precision of astrometric data and, more importantly, the associated dramatic increase of stellar samples, of the order of a billion. Then, the radially anisotropic velocity field of halo stars in the intermediate metallicity range of $-1.6 < \text{[Fe/H]} < -1$, which corresponds to high-$e$ stars first interpreted as evidence of free-fall Galaxy collapse by Eggen et al., and also a significant portion of the inner halo described in Carollo et al., is actually found to be due to a coherent stellar population in the $E$ vs. $L_z$ diagram (Gaia-Enceladus: Helmi et al. 2018) as well as an elongated feature in the velocity space (Sausage: Belokurov et al. 2018), quantified as $\beta \sim 0.9$. This structure can be a stellar remnant of an infalling, massive dwarf galaxy (Helmi et al. 2018; Kopelman et al. 2018; Deason et al. 2018; Belokurov et al. 2018; Haywood et al. 2018; Myeong et al. 2018), probably driven by the decay of its orbital energy through long-range gravitational interaction with surrounding dark-matter particles.

5.2. Implications from this work

The analysis of a large sample of stars in the phase-space defined by $(E, L_z, I_3)$, and over the metallicity ranges that include the GE structure (Figure 7), suggests that metal-poor halo stars show a nearly exponentially decreasing distribution with increasing $E$ and $I_3$, at $E > -1.4 \times 10^5$ km$^2$ s$^{-2}$, and $(2I_3)^{1/2} > 900$ kpc km s$^{-1}$. This exponential dependence of a distribution function with $E$ is achieved in lower isothermal models, i.e., a family of King models for stellar systems in dynamic equilibrium. There exist also a family of anisotropic spherical models with an exponential dependence on $L_z$, which corresponds to $I_3$ in the current non-spherical case (Binney & Tremaine 2008). As one of such anisotropic models, we have chosen here the Michel-Bodenheimer type distribution shown in Equation (3) (e.g., Sommer-Larsen & Zhen 1990; Chiba & Beers 2000; Carollo et al. 2007, 2010; Belokurov et al. 2018). The fact that the very metal poor outer halo possesses large $(2I_3)^{1/2}$ values, and exhibit a random spin (prograde and retrograde stars; left-bottom panel of Figure 4), suggests that it was likely originated from a chaotic, dissipationless, and random merging, and accretion, of less massive dwarf satellite galaxies. Candidate surviving of such low-mass systems are the ultra-faint dwarfs (UFDs), as envisaged in Carollo et al. (2007).

Thus, the outer layer of the halo is accretion dominated. Indeed, many of such low-mass satellites have probably fallen
into the Milky Way during the very early stage of Galaxy formation, as suggested by their long-term orbital evolution using Gaia DR2’s proper motions (Miyoshi & Chiba 2020). Some of the low-mass satellite galaxies have high \( E \) and large negative (retrograde) \( L_z \) (Myeong et al. 2018d), and exhibit lower [\( \alpha/\text{Fe} \)] ratios than the GE structure (Matsumo et al. 2019). This implies that they have a longer star formation timescale and thus contain younger stars than those comprising the more flattened part of the halo, dominated by the GE structure. Thus, the outer halo may comprise different layers as well: one made of accreted very old and very metal-poor low-mass galaxies (stars are in a more relaxed state, smooth outer halo), and one made of recent accretion events of dwarf satellite galaxies, that are not relaxed yet. It is interesting to note that the latter layer of the outer halo is in agreement with the reported spatial dependence of the ages of globular clusters, where those in the outer parts of the halo, say at \( r > 20 \) kpc, are systematically younger than in the inner parts (Zinn 1985; Mackey & Gilmore 2004). The existence of distinct layers in the outer halo is also confirmed by the age-structure of the Milky Way’s stellar halo described in Santucci et al. (2015) and Carollo et al. (2016). In the later analysis it was inferred a negative age gradient of \(-25 \) Myr kpc\(^{-1}\) (in the radial distance direction), and it was shown, in addition to the existence of a very old central region up to 10-15 kpc, the presence of several younger structures in the outskirts of the halo.

6. SUMMARY OF THE MAIN RESULTS AND PROSPECTS

In this paper we explored the general properties of the Milky Way’s halo system inferred from the analysis of a large sample of stars in the phase-space defined by the three integrals of motion, \((E, L_z, I_3)\), in combination with the chemical abundances. \( I_3 \) is the so-called third integral of motion, which is analytically defined in a Stäkel-form gravitational potential. We have especially focused on the “coarse-grained” phase-space distribution of halo stars, namely, their averaged properties over the phase space being close to a dynamically steady state, rather than the “fine-grained” phase-space distribution, which includes not-yet relaxed small-scale substructures. The main results can be summarized as follows:

- We propose to select halo stars by using an orbit-based selection method, which does not introduce selection bias associated with high velocity cuts. It is found that halo stars can be distinguished from disk stars by selecting over the following two domains in the \((E, L_z, I_3)\) phase-space:

\[
\begin{align*}
(1) \quad & L_z < 1500 \text{ kpc km s}^{-1} \\
& (2) \quad I_3^{1/2} > 1000 \text{ kpc km s}^{-1} \\
& (2) \quad E < -1.5 \times 10^5 \text{ km}^2 \text{ s}^{-2}.
\end{align*}
\]

The threshold value of \((2I_3)^{1/2}\approx 1000 \text{ kpc km s}^{-1}\) in (1) corresponds to a maximum orbital angle from the Galactic plane of \(\theta_{\text{orb}} = 15 - 20 \) deg. The value of \(E = -1.5 \times 10^5 \text{ km}^2 \text{ s}^{-2}\) in (2) nearly coincides with the gravitational binding energy at the position of the Sun.

- Stars with the lowest binding energy have a non-zero third integral of motion, expressed as \((2I_3)^{1/2} \approx 700 \text{ kpc km s}^{-1}\) and \(\theta_{\text{orb}} \approx 10 \) deg. These stars have \(L_z = 0\).

- Stars with a large vertical angular momentum \(L_z \approx 2000 \text{ kpc km s}^{-1}\), which are mainly thin disk stars, possess low \(I_3\), expressed as \((2I_3)^{1/2} \approx 400 \text{ kpc km s}^{-1}\), while stars with \(L_z \leq 2000 \text{ kpc km s}^{-1}\) have always \((2I_3)^{1/2} > 400 \text{ kpc km s}^{-1}\). There are no stars having both \(L_z \approx 0\) and \((2I_3)^{1/2} \approx 0\).

- The metal-weak thick stars possess \((2I_3)^{1/2}\) values larger than the thick disk, which implies larger orbital angles. The average orbital angles result, \(< \theta_{\text{orb}} > 10 \) deg, and \(< \theta_{\text{orb}} > \sim 7 \) deg, for the MWTD, and thick disk, respectively.

- The “coarse-grained” distribution of halo stars in the \((E, L_z, I_3)\) phase-space shows the Michie-Bodenheimer type functional dependence on \(E\) and \(I_3\), such as \(f(E, I_3) \propto \exp(-E/\sigma^2) \exp(-I_3/r_a^2\sigma^2)\), where \(\sigma\) is the 1d velocity dispersion and \(r_a\) is the radius beyond which the velocity dispersion is anisotropic. In the metallicity intervals of \(-1.8 < [\text{Fe/H}] < -1.4\) and \(-2.2 < [\text{Fe/H}] < -1.8\), respectively, we obtain \((\sigma, r_a) = (122 \text{ km s}^{-1}, 40 \text{ kpc})\) and \((116 \text{ km s}^{-1}, 4.3 \text{ kpc})\). In the most metal-poor interval of \([\text{Fe/H}] < -2.2\), we have found \((\sigma, r_a) = (144 \text{ km s}^{-1}, 4.3 \text{ kpc})\), i.e., the distributions of very metal-poor stars are systematically more extended along \(E\) and \(I_3\) than those in the higher metallicity ranges.

- The inner halo stellar population, described in Carollo et al. (2007, 2010), includes a combination of GE debris stars, low-energy in-situ stars, and a significant number of metal-poor prograde stars missed by the high-velocity cuts selection of halo stars.

- Several of the retrograde substructures found in recent literature represent the “fine-grained” elements of the outer halo defined in Carollo et al. (2007, 2010), and it is likely that the outer halo is made of a superposition of substructures, relics of past accretion events.

- The fraction of very metal-poor stars with \([\text{Fe/H}] < -2.2\), comprising the outer halo, relative to more
metal halo, increases discontinuously at 
\((2I_3)^{1/2} \approx 2000\) kpc km s\(^{-1}\), or \(\theta_{\text{orb}} \approx 50\) deg, corresponding to \(z \approx 10\) kpc at the solar radius.

- The \(L_z\) distribution of halo stars shows a notable truncation in the prograde high-\(L_z\) side, at \(L_z \sim 1500\) kpc km s\(^{-1}\). This truncation appears to be connected to the "trail" feature identified in the (2\(I_3\))\(^{1/2}\) vs. \(L_z\) diagram, in the direction of the Sgr dSph, and in the same phase space. This connection implies that the Sgr dSph galaxy is inducing a long-range dynamical effect on local halo stars, and generating a distinct kinematical patterns.

- Candidate in-situ stars, selected with binding energies of \(E < -1.5 \times 10^5\) km\(^2\) s\(^{-2}\), have orbital motions confined inside the solar radius, and orbital angles, \(S^0 \leq \theta_{\text{orb}} \leq 20^\circ - 30^\circ\) (\(z_{\text{max}} < 5\) kpc). Notably, these stars are spread out in the [\(\alpha/\text{Fe}\)] vs. [\(\text{Fe/H}\)] diagram, but the majority of them have [\(\text{Fe/H}\)] < -1 (See Figure 14).

The criteria for the selection of halo stars given in Equation (4) and (5) can be used for sample stars residing at any spatial locations, while this is not the case for the frequently adopted method of \(|V_{\text{star}} - V_{L\text{SR}}| > V_{\text{lim}}\) km s\(^{-1}\), because of the unknown spatial dependence of halo's velocity distribution.

As an example, let's take a nearby star located at \((x, y, z) = (R_\odot, 0, 0)\). The condition, \(L_z < 1500\) kpc km s\(^{-1}\), suggests \(v_\phi \lesssim 180\) km s\(^{-1}\). At this solar position, \(L_x = 0\) and \(L_y = -R_\odot v_z\) for any values of \(v_z\), so that \(I_3\) is simply written as \(I_3 = \frac{1}{2} v_z^2 (R_\odot^2 + \Delta^2)\). Then the condition of \((2I_3)^{1/2} > 1000\) kpc km s\(^{-1}\) for halo stars implies \(v_z (R_\odot^2 + \Delta^2)^{1/2} > 1000\) kpc km s\(^{-1}\), and if \(\Delta = 4\) kpc is adopted, we obtain \(v_z \gtrsim 110\) km s\(^{-1}\). It is worth emphasizing that the extreme values of \(v_\phi = 180\) km s\(^{-1}\) and \(v_z = 110\) km s\(^{-1}\) (while \(v_z\) takes any values), can locate a star inside the high-velocity cut of \(|V_{\text{star}} - V_{L\text{SR}}| = 200\) km s\(^{-1}\). Thus, the simple high-velocity cut misses many halo stars in prograde rotation, and leads to an incomplete picture of the stellar halo. Also, in the case of an arbitrary position along the Galactic plane at \((x, y, z) = (x, 0, 0)\), we obtain \(v_z > 1000/(x^2 + \Delta^2)^{1/2}\) km s\(^{-1}\), therefore the condition on \(v_z\) generally depends on the position, \(x\). Thus, our orbit-based selection criteria for halo stars, given in Equation (5), correspond to a spatially dependent cut in velocity space, in contrast to the constant velocity cut applied to stars at all locations.

Our understanding of the Milky Way's stellar halo based on the analysis of local stellar samples, like those currently adopted, is yet incomplete, due to the limited sampling generated by the survey footprints (Figure 15). Such local samples miss stars with orbits whose apocentric radii are much below the solar radius (Sommer-Larsen & Zhen 1990; Chiba & Beers 2000), i.e., those having very low \(E\), including the candidate in-situ stars identified in this analysis. Also, local stellar samples miss stars with nearly circular orbits in the outer parts of the halo, i.e., those showing tangentially anisotropic velocities with \(\beta < 0\) (e.g., Kafle et al. 2012; Bird et al. 2020). Furthermore, current local samples lack metal-poor stars with low-\(e\) orbits near the Galactic plane, which have been identified in Chiba & Beers (2000) and Sestito et al. (2020).

The limited chemical-abundance information in several regions of the stellar halo is also challenging the understanding of its true nature. The planned large spectroscopic surveys, over wide areas of the halo with WEAVE (Dalton et al. 2012) and SDSS-V (Kollmeier et al. 2017), and over narrow but more distant halo regions with Subaru/PFS (Takada et al. 2014; Tamura et al. 2016), in the near future, will significantly expand and deepen our knowledge of the stellar halo, and thus, its formation process. Moreover, large ground-based photometric surveys designed to search for very metal-poor stars, like the Pristine Survey (Starkenburg et al. 2017), and to measure proper motions of very distant halo stars using 8-10 m telescopes (Qiu et al. 2020) will be essential, combined with further Gaia data releases. Such surveys will bring us closer to obtain a complete picture of the chemodynamical structure of the Milky Way’s halo.

ACKNOWLEDGMENTS

M.C acknowledge partial support from the JSPS and MEXT Grant-in-Aid for Scientific Research on Innovative Areas (No. 17H01101, 18H04434 and 18H05437). Funding for the SDSS and SDSS-II has been provided by the Alfred P. Sloan Foundation, the Participating Institutions, the National Science Foundation, the U.S. Department of Energy, the National Aeronautics and Space Administration, the Japanese Monbukagakusho, the Max Planck Society, and the Higher Education Funding Council for England. The SDSS Web Site is http://www.sdss.org/.

Funding for the Sloan Digital Sky Survey IV has been provided by the Alfred P. Sloan Foundation, the U.S. Department of Energy Office of Science, and the Participating Institutions. SDSS acknowledges support and resources from the Center for High-Performance Computing at the University of Utah.

The SDSS is managed by the Astrophysical Research Consortium for the Participating Institutions. The Participating Institutions are the American Museum of Natural History, Astrophysical Institute Potsdam, University of Basel, University of Cambridge, Case Western Reserve University, University of Chicago, Drexel University, Fermilab, the Institute for Advanced Study, the Japan Participation Group, Johns Hopkins University, the Joint Institute for Nuclear Astrophysics, the Kavli Institute for Particle Astrophysics and Cosmol-
ogy, the Korean Scientist Group, the Chinese Academy of Sciences (LAMOST), Los Alamos National Laboratory, the Max-Planck-Institute for Astronomy (MPIA), the Max-Planck-Institute for Astrophysics (MPA), New Mexico State University, Ohio State University, University of Pittsburgh, University of Portsmouth, Princeton University, the United States Naval Observatory, and the University of Washington.

REFERENCES

Abazajian, K.N., Adelman-McCarthy, J.K., Agueros, M.A., et al., 2009, ApJS, 182, 543
Abolfathi B., et al., 2018, ApJS, 235, 42
Adelman-McCarthy, J. K. et al., 2007, Astrophys. J., Suppl. 172, 634
Aguado, D.S., Ahumada, Romina, Almeida, A., et al. 2019, 240, 23
Ahumada, R., Allende Prieto, C., Almeida, A., et al., 2019, arXiv:1912.02905
Alam, S., et al., 2015, Astrophys. J. Suppl. Ser., 219, 12
Allende Prieto, C., Beers, T.C., Sivarani, T., et al., 2007, AJ, 136, 2070
Allende Prieto, C., Fernandez-Alvar, E., Schlesinger, K. J., et al., 2014, å, 568, 7
An, D., Beers, T. C., Johnson, J. A., et al. 2013, ApJ, 763, 65
An, D. et al., 2015, ApJ, 813
Baumgardt, H., Hilker, M., Sollima, A., et al. 2014, MNRAS, 447, 5138
Beers, T. C., & Sommer-Larsen, J. 1995, ApJS, 96, 175
Beers, T.C., Chiba, M., Yoshii, Y., et al., 2000, AJ, 119, 2866
Beers, T. C., Lee, Y. S., Sivarani, T., et al., 2006, Mem. S.A.It.,77,1171
Beers, T.C., Norris, J.E., Placco, V.M., et al., 2014, ApJ, 794, 58
Bekki, K. & Chiba, M. 2001, ApJ, 558, 666
Bellazzini, M., Ibata, R., Malhan, K., et al. 2020, A&A, 636, 1A107
Carollo, D., Beers, T.C., Placco, V., et al., 2016, Nature Physics, 12,1170
Castelli, F. & Kurucz, R. L., 2003, Modelling of Stellar Atmospheres. 210th IAU Symposium, Edited by N. Piskunov, W.W. Weiss, and D.F. Gray, 210, A20
Cooper, A. P., Cole, S., Frenk, C. S., et al. 2010, MNRAS, 406, 744
Cooper, A. P., D’Souza, R., Kauffmann, G., et al. 2013, MNRAS, 434, 3348
Cooper, A. P. et al. 2015, MNRAS, 454, 3185
Das, P., Hawkins, K., Jofré, P., et al., 2020, MNRAS, 493, 5195
Dawson, K. S., Schlegel, D. J., Ahn, C. P., et al., 2013, AJ, 145, 10
D’Souza, R., & Bell, E. F. 2018, MNRAS, 474, 5300
Deason, A. J., Belokurov, V., Koposov, S. E., et al. 2018, ApJ, 821, 5
Deason, A. J., Belokurov, V., Koposov, S. E., et al. 2018, ApJL, 862, L1
de Boer, T. J. L., Belokurov, V., Koposov, S., et al., 2015, MNRAS, 451, 3489
Brook, C., Kawata, D., Gibson, G., et al. 2020, MNRAS, 495, 2645
Brown, A., et al., 2018, å, 616, A1
Dawson, K. S., & Johnston, K. V. 2005, ApJ, 635, 931
Carollo, B. W., Laird, J. B., Latham, D. W., & Aguilar, L. A. 1996, AJ, 112, 668
Carollo, D., Beers, T. C., Lee, Y. S., et al., 2007, Nature, 450, 1020
Carollo, D., Beers, T. C., Chiba, M., et al. 2010, ApJ, 712, 692
Carollo, D., Beers, T.C., Placco, V., et al., 2016, Nature Physics, 12,1170
Cheng, J.Y., Rockosi C.M., Morrison, F., et al., 2012, ApJ, 752, 51
Chiappini, C., Matteucci, F., & Gratton, R. 1997, ApJ, 477, 765
Chiappini, C., Matteucci, F., & Romano, D. 2001, ApJ, 554, 1044
Chiba, M. & Yoshii, Y. 1998, AJ, 115, 168
Chiba, M. & Beers, T.C., 2000, AJ, 119, 2843
Conroy, C., Naidu, R. P., Zaritsky, D., et al. 2019, ApJ, 887, 237
Cooper, M. C., Griffith, R. L., Newman, J. A., et al. 2012, MNRAS, 419, 3018
Cooper, A. P., D’Souza, R., Kauffmann, G., et al. 2013, MNRAS, 434, 3348
Cooper, A. P. et al. 2015, MNRAS, 454, 3185
Das, P., Williams, A. & Binney, J., 2016, MNRAS, 463, 3169
Das, P., Hawkins, K., Jofré, P., 2020, MNRAS, 493, 5195
Dawson, K. S., Schlegel, D. J., Ahn, C. P., et al., 2013, AJ, 145, 10
D’Souza, R., & Bell, E. F. 2018, MNRAS, 474, 5300
D’Souza, R., & Bell, E. F. 2018, Nature Astronomy, 2, 737
Deason, A. J., Mao, Y.-Y., & Wechsler, R. H. 2016, ApJ, 821, 5
Deason, A. J., Belokurov, V., Koposov, S. E., et al. 2018, ApJL, 862, L1
de Boer, T. J. L., Belokurov, V., Koposov, S., et al., 2015, MNRAS, 451, 3489
Thakar, A. R., Alex Szalay, A., Fekete, G., Gray, J., 2008, Computing in Science & Engineering, 10, 30
Tian, H., Liu, C., Xu, Y., Xue, X., 2019, ApJ, 871, 184
Tian, H., Liu, C., Wang, Y., et al., 2020, arXiv:2007.00852
Tissera, P.B., White, S.D.M., Pedrosa, S., Scannapieco, C., 2010, MNRAS, 406, 922
Tissera P. B., White S. D. M., Scannapieco C., 2012, MNRAS, 420, 255
Tissera P. B., Scannapieco C., Beers T. C., Carollo D., 2013, MNRAS, 432, 3391
Tissera, P.B., Beers, T.C., Carollo, D., Scannapieco, C., 2014, MNRAS, 439, 3128
Toyouchi, T. & Chiba, M. 2018, ApJ, 455, 104
Unavane, M., Wyse, R. F. G., & Gilmore, G. 1996, MNRAS, 278, 727
Villalobos, A. & Helmi, A., 2008, MNRAS, 391, 1806
Wilhelm, R., et al., 1996, ASP Conf. Ser. 92, 171-174, Astronomical Society of the Pacific, San Francisco, 1996, eds. Morrison, H. & Sarajedini, A.)
White, S. D. M., & Rees, M. J. 1978, MNRAS, 183, 341
Whitten, D.D., Beers, T.C., Placco, V., et al., 2019, ApJ, 884, 67
Williams A. A., Belokurov V., Casey A. R., Evans N. W., 2017, MNRAS, 468, 235
Yanny, B., Rockosi, C., Newberg, H.J., et al., 2009, AJ, 137, 4377
York, D. G. et al., 2000, AJ, 120, 1579
Yoshii, Y. & Saio, H. 1979, PASJ, 31, 339
Yuan, Z., Myeong, G.C., Beers, T., et al., 2020, ApJ, 891, 39
Zamora, O., Garcia-Hernandez, D.A., Allende-Prieto, C., et al., 2015, AJ, 149, 181
Zinn, R. 1985, ApJ, 293, 424
Zinn, R., 1993, ASP Conf. Ser. 48, Astronomical Society of the Pacific, San Francisco, eds Smith, G. H. & Brodie, J. P., 38-47
Zinn, J.C., Pinsonneault, M.H., Huber, D., Dennis Stello, 2019, ApJ, 878, 2
Zolotov A., Willman B., Brooks A. M., et al., 2009, ApJ, 702, 1058
Zolotov, A., Willman, B., Brooks, A. M., et al. 2010, ApJ, 721, 738
Zuo, W., Du, C., Jing, Y., 2017, ApJ, 841, 59
APPENDIX

A. MASS MODEL AND ORBITAL MOTION

Here, we describe our adopted Galactic potential of Stäckel type based on the model originally proposed by Sommer-Larsen & Zhen (1990). The complete description of a Stäckel potential is given in de Zeeuw (1985) and Dejonghe & de Zeeuw (1988).

We adopt the axisymmetric potential defined in spheroidal coordinates $(\lambda, \phi, \nu)$, where $\phi$ is the azimuthal angle in cylindrical coordinates of $(R, \phi, z)$ and $(\lambda, \nu)$ are the roots for $\tau$ of

$$\frac{R^2}{\tau + \alpha} + \frac{z^2}{\tau + \gamma} = 1,$$

(A1)

where $\alpha$ and $\gamma$ are constants, constraining the range of $\lambda$ and $\nu$ as $-\gamma \leq \nu \leq -\alpha \leq \lambda$. The coordinates are defined with the focal distance $\Delta = \sqrt{\gamma - \alpha}$ along the $z$ axis and their grids are displayed in Figure 1. The relation between $(\lambda, \nu)$ and $(R, z)$ is then written as

$$R^2 = \frac{(\lambda + \alpha)(\nu + \alpha)}{\alpha - \gamma}, \quad z^2 = \frac{(\lambda + \gamma)(\nu + \gamma)}{\gamma - \alpha}.$$

(A2)

The gravitational potential of this type, $\psi(\lambda, \nu)$, is generally given as

$$\psi(\lambda, \nu) = -\frac{(\lambda + \gamma)G(\lambda) - (\nu + \gamma)G(\nu)}{\lambda - \nu},$$

(A3)

where $G(\tau)$ with $\tau = \lambda, \nu$ is an arbitrary function. In this work, the Galactic potential is represented by the combination of a disk, $G_D(\tau)$, and a dark halo $G_H(\tau)$, thus given as $G(\tau) = G_D(\tau) + G_H(\tau)$. Following Sommer-Larsen & Zhen (1990), the density distribution of a disk is given by a highly flattened, perfect oblate spheroid, resembling a Kuzmin disk, and that of a dark halo is modelled with a slightly flattened, oblate spheroid, resembling an isothermal sphere, originally developed by de Zeeuw, Peletier & Franx (1986) (their $s = 2$ model). Then, $G_D$ and $G_H$, respectively, are written as (with the gravitational constant, $G$)

$$G_D(\tau) = \frac{2G_M}{\pi} \frac{D_D}{\sqrt{\tau + \gamma}} \arctan \frac{\tau + \gamma}{-\gamma}$$

(A4)

and

$$G_H(\tau) = -\frac{4\pi G_M \rho_0 (\gamma - b)}{\pi}$$

$$\times \left[ \ln \frac{\Delta^2 - \gamma + b}{-\gamma + b} - \ln \frac{\tau + \gamma + \Delta^2}{2(\tau + \gamma)} - \ln \frac{\tau + \gamma - b}{-\gamma + b} \right] + \frac{\Delta^2 + \gamma - b}{\sqrt{\tau + \gamma} - \gamma + b} \arctan \frac{\Delta}{\sqrt{\tau + \gamma} - \gamma + b},$$

(A5)

where $M_D$ is the mass of a disk and $\rho_0$ and $b$ are parameters for a dark halo\(^\text{6}\). These parameters are chosen so as to reproduce the observed constraints for the Galactic potential. Here we adopt the set of these parameters in Sommer-Larsen & Zhen (1990), where the rotation curve is flat beyond $R \approx 4$ kpc with a rotation velocity of $223$ km s$^{-1}$ at the position of the Sun $R = 8.5$ kpc (see their Figure 2): $\Delta = \sqrt{\gamma - \alpha} = 4$ kpc, $\sqrt{\gamma} = 0.125$ kpc, $\sqrt{\beta - \gamma} = 6$ kpc, $M_D = 9.0 \times 10^{10} M_\odot$ and $\rho_0 = 2.45 \times 10^7 M_\odot$ kpc$^{-3}$.

We further set $G = G_D + G_H = 0$ at $R_t = 200$ kpc in the Galactic plane, i.e., at $\Delta = R_t - \alpha$, by adding the corresponding constant to $G$, thereby giving $\psi = 0$ at this kind of tidal radius. This adjustment in $G$ is required to bound the sample stars in this potential.

The orbital motion in this potential has the Hamiltonian $H$,

$$H = \frac{p_\lambda^2}{2\rho^2} + \frac{p_\phi^2}{2Q^2} + \frac{p_\nu^2}{2R^2} + \psi(\lambda, \nu),$$

(A6)

with

$$p^2 = \frac{\lambda - \nu}{4(\lambda + \alpha)(\lambda + \gamma)}, \quad Q^2 = -\frac{\lambda - \nu}{4(\nu + \alpha)(\nu + \gamma)}.$$

(A7)

The momentum components conjugate to $(\lambda, \phi, \nu), (p_\lambda, p_\phi, p_\nu)$, are given as

$$p_\lambda = P^2 \lambda, p_\phi = R^2 \phi, p_\nu = Q^2 \nu,$$

(A8)

\(^6\) There is a typo in equation (9) for $G_H$ in the Sommer-Larsen & Zhen (1990) paper, but their calculations adopted the correct one as given here.
where \( \dot{p}_{z} \) denotes the time derivative and \( p_{z} = L_{z} \). The equations of motion are then written as

\[
\begin{align*}
\dot{r}^2 &= \frac{1}{2(\tau + \alpha)} \left[ G(\tau) - \frac{I_2}{\tau + \alpha} - \frac{I_3}{\tau + \gamma} - |E| \right], \\
\dot{\tau}^2 &= L_{z}^2 = 2I_2,
\end{align*}
\]

where \( E(= H) \), \( I_2 = L_{z}^2/2 \) and \( I_3 \) are the three isolating integrals of motion. \( I_3 \) is explicitly written as

\[
I_3 = \frac{1}{2}(L_{z}^2 + L_{\gamma}^2) + \Delta z^2 \left[ \frac{1}{2} \dot{z}^2 - z^2 \frac{G(\lambda) - G(\nu)}{\lambda - \nu} \right],
\]

where \( L_{x} \) and \( L_{y} \) are the angular momentum components in the \( x \) and \( y \) directions, respectively, and \( v_z \) is the velocity component in the \( z \) direction.

### B. DESCRIPTION OF THE DATA

#### B.1. SDSS DR7 Calibration Stars

SDSS is now arrived to its fourth generation and its 16th data release (Ahumada et al. 2019, and reference therein). The SDSS and its extensions uses a dedicated 2.5 m telescope (Gunn et al. 2006) located at the Apache Point Observatory in New Mexico. The telescope is equipped with an imaging camera and a pair of spectrographs, each of which is capable of simultaneously collecting 320 medium-resolution \((R \sim 2000)\) spectra over its 7 deg\(^2\) field of view, so that on the order of 600 individual target spectra and, roughly, 40 calibration star and sky spectra are obtained on a given spectroscopic plugplate (York et al. 2000). The SEGUE sub-survey, carried out as part of SDSS-II, ran from 2005 July to 2008 July. SEGUE obtained some 240,000 medium-resolution spectra of stars in the Galaxy, selected to explore the nature of stellar populations from 0.5 kpc to 100 kpc (Yanny et al. 2009). SDSS and SEGUE obtained spectroscopy and photometry for a small number (16) of calibration stars obtained for each spectroscopic plug plate, chosen for two primary reasons. The first set of these objects, the spectrophotometric calibration stars, includes stars that are selected to approximately remove the distortions of the observed flux of stars and galaxies arising from the wavelength response of the Astrophysical Research Consortium (ARC) 2.5 m telescope and the SDSS spectrographs, as well as the distortions imposed on the observed spectra by Earth's atmosphere. The spectrophotometric calibration stars cover the apparent magnitude range 15.5 < \( g_{0} < 17.0 \), and satisfy the color ranges 0.6 < \( (u - g)_{0} < 1.2; 0.0 < (g - r)_{0} < 0.6 \). The second set of stars, the telluric calibration stars, is used to calibrate and remove night-sky emission and absorption features from SDSS spectra. The telluric calibration stars cover the same color ranges as the spectrophotometric calibration stars, but at fainter apparent magnitudes, in the range 17.0 < \( g_{0} < 18.5 \). Stellar parameters for the calibration stars were obtained by employing the SEGUE Stellar Parameter Pipeline (SSPP; Lee et al. 2008a,b, 2011a) which processes the wavelength- and flux-calibrated spectra generated by the standard SDSS spectroscopic reduction pipeline, obtains equivalent widths and/or line indices for about 80 atomic or molecular absorption lines, and estimates the effective temperature, \( T_{\text{eff}} \), surface gravity, log \( g \), and metallicity, [Fe/H], for a given star through the application various approaches. The internal errors for stars in the temperature range that applies to the calibration stars are \( \sigma_{T_{\text{eff}}} \sim 100 \) K to \( \sim 125 \) K, \( \sigma_{\log g} \sim 0.25 \) dex, and \( \sigma_{[\text{Fe/H}]} \sim 0.20 \) dex.

The SSPP pipeline derives also the \( \alpha \)-elements abundances which are available in the SDSS Catalog Archive Server or CAS (Thakar et al. 2008), only for the SDSS-DR7. The pipeline adopts of a pre-existing grid of synthentic spectra (NEWODF; Castelli & Kurucz 2003), with no enhancement in \( \alpha \)-element abundances, and creates a fine (steps of 0.2 dex for log \( g \) and 0.2 dex for [Fe/H]) grid of spectra by interpolation between the wider model grids (steps of 0.5 dex). The wavelength range is 4500 – 5500 Å, chosen because it contains a large set of metallic lines, but avoids the CH G-band feature \((\sim 4300 \) Å, which can be strong in metal-poor stars) and the Ca II K \((\sim 3933 \) Å) and H \((\sim 3968 \) Å) lines, which can saturate for cool metal-rich stars. The final grid covers 4000 K < \( T_{\text{eff}} < 8000 \) K, in steps of 250 K, 0.0 < log \( g < 5.0 \), in steps of 0.2 dex, and \(-4.0 < [\text{Fe/H}] < -0.4 \), in steps of 0.2 dex. The range in \([\alpha/\text{Fe}]\) introduced for the spectral synthesis covers \(+0.1 < [\alpha/\text{Fe}] < +0.6 \), in steps of 0.1 dex, at each node of \( T_{\text{eff}} \), log \( g \), and [Fe/H]. After creation of the full set of synthetic spectra, they are degraded to SEGUE resolution \((R \sim 2000)\) and re-sampled to 1 Å wide linear pixels (during SSPP processing, the SEGUE spectra are also linearly re-binned to 1 Å per pixel). In the SSPP pipeline, the notation \([\alpha/\text{Fe}]\) denotes an average of the abundance ratios for individual \( \alpha \)-elements weighted by their line strengths in synthesized spectra. In the selected spectral range the dominant features are the magnesium (Mg) and titanium (Ti) lines, which are the primary contributors to the determination of \([\alpha/\text{Fe}]\), with some influence from silicon (Si) and calcium (Ca). In the adopted wavelength range, and at the SDSS spectral resolution, oxygen (O) has no strong detectable features, and it is excluded in the computation of the overall \( \alpha \)-element abundance. The \([\alpha/\text{Fe}]\) measurements were validated with the stars in other external sources such as the ELODIE (Prugniel & Soubiran 2001, and later releases) spectral library, and compared with those.
obtained by analyzing a large sample of SEGUE stars observed at high spectral resolution. The SSPP provides $[\alpha/\text{Fe}]$ abundance for SDSS/SEGUE spectra with a precision of $\sim$ 0.06 dex at $S/N > 50$ and $< 0.1$ dex at $S/N = 20$. A detailed description can be found in Lee et al. (2011a).

Radial velocities for stars in our sample are derived from matches to an external library of high-resolution spectral templates with accurately known velocities (Allende Prieto et al. 2007; Yanny et al. 2009), degraded in resolution to that of the SDSS spectra. The typical precision of the resulting radial velocities are on the order of 3-20 km s$^{-1}$, depending on the $S/N$ of the spectra, with zero-point errors of no more than 3 km s$^{-1}$, based on a comparison of the subset of stars in our sample with radial velocities obtained from the high-resolution spectra taken for testing and validation of the SSPP. The initial sample employed in this analysis consists of $\sim$ 32,000 unique stars selected in the temperature range $4500 \text{ K} < T_{\text{eff}} < 7000 \text{ K}$, where the SSPP pipeline provides the highest accuracy for the derived atmospheric parameters.

B.1.1. SDSS DR7 full sample

In the SDSS archive we selected only stars with high signal-to-noise ($\text{SNR} > 40$), and with critical and cautionary flags provided by the SSPP pipeline set to normal. Stars are selected in the temperature range of $4500 ^\circ \text{K} < T_{\text{eff}} < 7000 ^\circ \text{K}$ where the SSPP pipeline provides the best estimate for the stellar parameters and abundances. This selection provides $N = 65,678$ stars.

B.2. SDSS DR16

In this paper we also make use of data from Data Release 16 of the Apache Point Observatory Galactic Evolution Experiments (APOGEE; Majewski et al. 2017). The APOGEE survey is obtaining high resolution spectroscopy across the entire Milky Way Galaxy with two near-infrared, high-resolution multiplexed spectrographs, the 2.5m Sloan Foundation Telescope at APO, in the northern hemisphere, and the 2.5m Irene du Pont Telescope at LCO (Las Campanas Observatory), in the southern hemisphere. DR16 (Ahumada et al. 2019) is the fourth data release of SDSS-IV (Eisenstein et al. 2011), containing all SDSS-III APOGEE-1 data and SDSS-IV APOGEE-2 data acquired with both instruments through August 2018. DR16 contains information for 437,485 unique stars, including reduced spectra, radial velocities, atmospheric parameters, and individual elements abundances for 15 chemical elements with the ASPCAP pipeline (C, N, O, Na,Mg, Al, Si, S, K, Ca, Ti, V, Mn, Fe, and Ni). (Holtzman et al. 2015; Zamora et al. 2015; Garcia Perez et al. 2016)). We refer the reader to (Holtzman et al. 2018, Jonsson et al. 2020) for a complete and detailed description of the APOGEE data, namely DR14 and DR16. We selected the APOGEE stellar sample from the CASJobs archive service by using the flags provided in the data files, and described in Holtzman et al. (2015). In particular, we rejected stars in the database with unreliable stellar parameters and abundances, such that, VERY_BRIGHT_NEIGHBOR, PERSIST_HIGH, and SNR_BAD are set in STARFLAGS, and STAR_BAD, METAL_WARN, ROTATION_WARN, and METALS_BAD are set in the ASPCAPFLAGS. In addition, we applied other cuts in $T_{\text{eff}}$ and log(g) in order to avoid some issues described in (Holtzman et al. 2015), and we only consider stars with $T_{\text{eff}} > 4000$ K, and $1.0 < \text{log}(\text{g}) < 3.5$. The restriction in $T_{\text{eff}}$ is due to the low quality of the ASPCAP fitting at cooler temperatures, while that in log(g) is applied because stars with $\text{log}(\text{g}) \geq 4.0$ have strong deviation from asteroseismic gravities (Holtzman et al. 2015). Finally, we selected stars with signal-to-noise, $S/N > 80$. By applying the Apogee flags, stellar parameters cuts, and $S/N$ selection we obtained a subsample of $N = 70,130$ stars. In our analysis we have considered the calibrated $[\text{M/H}]$ as the indicator of the global metallicity, because it is corrected for the overestimation on the metal-poor side, while for the $[\alpha/\text{Fe}]$ indicator we choose $[\text{Mg/Fe}]$. Note that it is advisable the use of $[\text{Fe/H}]$ and $[\text{Mg/Fe}]$ because some suspicious values have been removed from these quantities, and there exist also a curious thin $\alpha$-finger in the $[\alpha/\text{M}]$ vs. $[\text{M/H}]$ diagram that is not present at all in the $[\text{Mg/Fe}]$ vs. $[\text{Fe/H}]$ diagram (Jonsson et al. 2020).

B.3. Distances, proper motions, and footprints of these catalogs

The SDSS DR7 calibration stars, the DR7 full sample, and APOGEE DR16 were cross-matched with the Gaia DR2 database (Brown et al. 2018) to retrieve accurate astrometric positions, trigonometric parallaxes, and proper motions, using the CDS (Centre de Donnes Astronomiques de Strasbourg) X-Match service, and adopting a very small search radius (0.8" for DR7 and 0.6" for APOGEE) to avoid duplicates. The match provides positions, parallaxes, and proper motions for all of the stars in the sample. We select stars with relative parallax errors of $\sigma_{\pi}/\pi < 0.2$, and derive their distance estimates using the relation $d = 1/\pi$ (and used these distances to select stars with heliocentric distance $d \leq 4$ kpc). This selection reduces the number of stars to 10,820 for the DR7 calibration stars ($\sim 1/3$rd of the initial sample), and 62,060 for the Apogee sample. The majority of stars in this final sample have errors on proper motions below 0.2 mas yr$^{-1}$. We also select stars with Galactocentric distance in the range 6 kpc $< R < 12$ kpc to include extended regions of the galactic disk. The final number of stars after this selection is 10,814, and 58,912, for DR7 calibration stars and APOGEE DR16, respectively. In case of the full SDSS SEGUE DR7 sample, the same selection provides a final number of stars of $N = 46,500$. 

In this analysis the parallaxes have been corrected for the parallax zero-point offset Lindegren et al. (2018). In Carollo et al. (2019) it was established that the addition of a constant offset model improves the agreement between the distances derived from the inverse of parallax with those inferred with the Bailer-Jones’s method. Therefore, we have corrected the parallaxes with a constant value of 0.054 mas which can be justified by following the results reported in Leung & Bovy (2019), where it is shown that the constant offset model is in very good agreement with the multi-variate offset model up to 10 kpc for a sample of ∼ 265,000 stars in common between APOGEE DR14 (Holtzman et al. 2015; Garcia Perez et al. 2016; Abolfathi et al. 2018) and Gaia DR2.

B.4. The phase space distributions of the DR7 full sample and DR16

In Section 3 of the main paper, we present the phase-space distributions of the SDSS DR7 calibration stars for each interval of [Fe/H], as well as for the selected box areas in the [$\alpha$/Fe] vs. [Fe/H] diagram. We also show the phase-space distributions in these [$\alpha$/Fe] vs. [Fe/H] box areas for the SDSS DR16. Here, for the sake of comparison and also completeness, we show the phase-space distributions for the SDSS DR7 full sample, in the same box areas (Figure 16 and 17), while in Figure 18 and 19 it is shown the SDSS DR16 APOGEE phase-space as a function of the metallicity. The SDSS DR7 full sample shows basically the same phase-space distributions as the DR7 calibration stars sample, and actually, enhances the main properties found in section 3.1 of the main paper, because of the larger number of stars in each bin of metallicity. The phase-space distribution of the DR16 APOgee differs with respect that of the DR7 samples, because of the footprints (Figure 15). In particular, DR16 has a larger coverage in $R$ ($3 \leq R \leq 14$ kpc), which lead to a broader distribution of stars around $L_z \sim 2000$ kpc km s$^{-1}$ and $(2I_2)^{1/2} \sim 0$ kpc km s$^{-1}$, in the metal-rich range, [Fe/H] > −0.6. Also, at $L_z \simeq 0$ kpc km s$^{-1}$, the DR16 APOGEE stars show somewhat lower energy values than the DR7 stellar samples. This is likely due to the fact that some stars in APOGEE are located at smaller $R$, and thus have smaller binding energies than DR7 stars.
Figure 16. The same as Figure 5 but for the SDSS DR7 full sample.

Figure 17. The same as Figure 6 but for the SDSS DR7 full sample.
Figure 18. The same as Figure 3 but for the SDSS DR16 APOGEE stars.

Figure 19. The same as Figure 4 but for the SDSS DR16 APOGEE stars.