Topological phases of parafermionic chains with symmetries

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We study the topological classification of parafermionic chains in the presence of a modified time reversal symmetry that satisfies $T^2 = 1$. Such chains can be realized in one dimensional structures embedded in fractionalized two dimensional states of matter, e.g. at the edges of a fractional quantum spin Hall system, where counter propagating modes may be gapped either by back-scattering or by coupling to a superconductor. In the absence of any additional symmetries, a chain of $Z_m$ parafermions can belong to one of several distinct phases. We find that when the modified time reversal symmetry is imposed, the classification becomes richer. If $m$ is odd, each of the phases splits into two subclasses. We identify the symmetry protected phase as a Haldane phase that carries a Kramers doublet at each end. When $m$ is even, each phase splits into four subclasses. The origin of this split is in the emergent Majorana fermions associated with even values of $m$. We demonstrate the appearance of such emergent Majorana zero modes in a system where the constituents particles are either fermions or bosons.

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I. INTRODUCTION

Much of the extensive research of topological states of matter in the last few years has focused on topological classification of gapped systems$^{1-3}$. The question addressed in such classification is “For a given dimension $d$ and symmetry $S$, what topologically distinct classes of Hamiltonians exist, where two Hamiltonians are topologically distinct if they cannot be deformed to one another without a closure of the energy gap?” This question was mostly studied for systems of non-interacting electrons and for systems which are adiabatically connected to non-interacting electrons$^{4-6}$. Here, we study this question in a setup in which interactions between electrons create two dimensional (2D) fractionalized states that are not adiabatically connected to systems of non-interacting electrons. We classify the gapped states that may be created in effectively one dimensional (1D) systems embedded in such systems. The 1D system may either exist at the edge of the host 2D phase, or along a line cutting through it.

One dimensional fermionic systems on a lattice may always be described in terms of coupled Majorana fermions. For non-interacting electrons it was realized that in the absence of any symmetry these systems fall in two topologically distinct phases$^{7-10}$. The topologically non-trivial phase is characterised by Majorana zero modes at its ends, which are potentially useful for protected quantum information processing. Interactions between electrons do not affect this classification$^{11-14}$. A richer structure emerges in one dimensional systems of the BDI class, composed of systems that conserve fermion parity and that are symmetric to time reversal symmetry $T$, with $T^2 = +1$. For these systems, in the absence of interactions the classifying group is $Z$, while interactions reduce the classification to $Z_2$,$^4,^6$. Hamiltonians of different classes differ by the number of Majorana zero modes at the systems’ ends, and the reduction from $Z$ to $Z_2$ is a consequence of the fact that eight Majorana modes may be gapped by a local quartic interaction that does not violate the BDI symmetries. Similar reductions induced by interactions occur also for other symmetries and dimensions, and were classified in Refs. 15–23.

In this work we go “half a dimension higher”. We study the classification of one dimensional systems that may occur only on edges of fractionalized two dimensional states of matter. An example of such a setup is the edge of a fractional quantum spin Hall system$^{24,25}$, i.e., a 2D system in which electrons of one spin direction form a fractional quantum Hall state of filling $\nu$ while electrons of the other spin direction form a FQH state of filling $-\nu$. An edge structure that is equally good for our purpose forms also in more mundane systems - along a trench cut in a bulk of a fractional quantum Hall state, or along an edge of a double layer electron-hole systems where the electrons and the holes are at Landau filling fractions of equal magnitude and opposite sign. For concreteness, we will focus on the hypothetical construction of a fractional quantum spin Hall system.

Being interested in the classification of gapped systems, we note that such edges may be gapped either by (possibly spin-flipping) back-scattering between counter-propagating modes or by coupling of the counter-propagating modes to (either singlet or spin-polarized) superconductor. At interfaces between two semi-infinite regions of these two types of gapping mechanisms topologically protected zero modes occur. When the gapped edge modes are of integer quantum Hall states the zero modes are Fu-Kane Majorana states$^{26}$. When the gapped edges are of fractional quantum Hall states, the zero modes are “fractionalized Majorana modes,”$^{27-30}$ frequently referred to as “parafermionic zero modes”. These zero modes are localized unitary operators $\xi$ that com-
Parafermions statistics are unitary operators that satisfy \( \chi \) with the Hamiltonian and satisfy \( \xi^m = 1 \). The value of \( m \) depends on the case at hand and will be discussed in detail below.

A central example is schematically depicted in Fig. 1. The edge of a fractional quantum spin Hall state is gapped by alternating back-scattering (B) and superconducting (S) regions. The long B stretch on the outer edge can be thought of as the vacuum, leaving the rest of the outer edge as an open 1D system.

The structure of the paper is the following: In Sec. II we summarize our main results with a focus on the underlying physical picture. The derivation based on the fractionalization of symmetries is presented in Sec. III. The gapped phases identified in Sec. III are then explicitly constructed in Sec. IV, and Sec. V contains concluding remarks.

II. PHYSICAL PICTURE AND SUMMARY OF RESULTS

A. A review of the classification of parafermionic chains without symmetries

The topological classification of Hamiltonians of coupled \( \mathbb{Z}_N \) parafermions in the absence of any additional symmetries was obtained in Refs. 38–40. The \( \mathbb{Z}_N \) parafermions may be deconstructed according to the decomposition of the cyclic group \( \mathbb{Z}_N = \mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_l^{r_l}} \), where \( N = \prod_{i=1}^{l} p_i^{r_i} \) is the decomposition of \( N \) into primes \( p_i \). The \( \mathbb{Z}_N \) parafermion then breaks into \( j \) different parafermions that act on different Hilbert spaces. In the absence of any additional symmetry a chain of \( \mathbb{Z}_N \) parafermions can be in several distinct classes corresponding to the presence or absence of \( \mathbb{Z}_{p_i^{r_i}} \) parafermions at its end. In addition, if \( r_i > 1 \), groups of parafermions whose self statistics is bosonic may condense throughout the entire edge. When this happens, the condensated parafermions form a smaller cyclic group. The overall number of distinct phases is then \( N_{\text{phases}} = (r_1 + 1) \cdots (r_l + 1) \), equal to the number of subgroups of \( \mathbb{Z}_N \).

Below we will extend this classification to a chain of \( \mathbb{Z}_N \) parafermions with an effective time reversal symmetry. Before we do so, we will briefly describe the origin of such an emergent time reversal symmetry in our fractionalized system.
B. Time reversal symmetry in the fractionalized case

The fractional quantum spin Hall insulator is symmetric to time reversal $\mathcal{T}$ with $\mathcal{T}^2 = (-1)^n_F$, where $n_F$ is the number of electrons in the system. Time reversal transforms $\psi^\dagger \rightarrow \psi^\dagger$ and $\psi \rightarrow -\psi$ (with the $\psi^\dagger$ being electron creation operators, and the subscripts denoting spin directions). For $\nu = 1/m$, the backscattering term necessarily violates time reversal symmetry, since it corresponds to a Zeeman coupling to a magnetic field. However, when the magnetic field is purely in the $x-y$ plane the Zeeman coupling is of the form $\lambda \psi^\dagger \psi + h.c.$ and is symmetric under a modified time reversal symmetry $\mathcal{T}$, which is given by $\psi^\dagger \rightarrow \psi^\dagger, \psi \rightarrow \psi$. Physically, this symmetry operation corresponds to time reversal followed by a rotation by $\pi$ around the $z$ axis. Remarkably, $\mathcal{T}^2 = 1$. Thus, under these conditions the problem we consider may be regarded as a fractionalized generalization of the one dimensional systems of class BDI.

For more complicated fractional quantum spin Hall states of $\pm \nu$, the edge may be gapped in a way that does not break the original time reversal symmetry with $\mathcal{T}^2 = -1$. This is allowed when the ratio $\nu/e^*$ is an even number, with $e^*$ being the charge of the smallest charge quasi-particle$^{24}$. In that case the entire system belongs to the one dimensional DIII class.

C. A short review of the non-fractionalized case ($m = 1$)

The $m = 1$ case is topologically equivalent to a one-dimensional spinless fermionic system on a lattice. In the absence of symmetry, a pair of Majorana zero modes in close proximity may be gapped by their mutual coupling. Thus, there are two phases that are topologically distinct from one another, with zero or one Majorana mode at each end.

In the BDI class, with time reversal symmetry that satisfies $\mathcal{T}^2 = 1$, each of the two phases splits into four. This is so since a coupling of two neighboring Majorana fermions may violate time reversal symmetry, but a quartic coupling generally does not. With the most general coupling allowed by the symmetry, the class with zero Majorana modes splits into four subclasses: One trivial subclass with no end modes ($k = 0$); two subclasses denoted by $k = \pm 2$ with two Majorana modes at each end, for which the fermion parity at each end is flipped by time reversal symmetry (a property for which they are termed anomalous); and one subclass with four Majorana modes at each end (denoted by $k = 4$). The $k = 4$ subclass is topologically equivalent$^{34}$ to a Haldane spin-1 chain$^{35,37}$, and carries a Kramers’ doublet at each end.

The class with a single Majorana end mode splits into four subclasses as well: two subclasses with one mode at each end ($k = \pm 1$) and two subclasses with three Majorana modes at each end ($k = \pm 3$). Here $k$ and $-k$ are inverse of each other, in the sense that when combined, the phase and its inverse form a trivial phase. The phase and its inverse have similar physical properties.

Altogether, then, Hamiltonians of a chain of $\mathbb{Z}_2$ parafermions are classified by a $\mathbb{Z}_2$ group in the absence of symmetries, and by $\mathbb{Z}_8$ group in the BDI case. There are several ways by which this $\mathbb{Z}_8$ classification may be derived$^{34-36}$. Most useful to our context is the derivation based on the fractionalization of symmetries$^5$. It is reviewed and generalized in Section III.

In the following section we generalize this classification to a chain of $\mathbb{Z}_N$ parafermions. We note, however, that unlike its one dimensional analog, there is no obvious way to define an addition of gapped phases for a system confined to the edge of a fractionalized bulk. In particular the notion of stacking is ill-defined. Hence, in our analysis below we restrict ourselves to counting the number of distinct phases and do not address the group structure.

D. Classification of parafermionic chains with time reversal symmetry: the general principle

The results derived in Sec. III can be summarized in a very concise way: In the absence of any symmetry, a chain of $\mathbb{Z}_N$ parafermions falls into several topological classes, corresponding to the number of subgroups of $\mathbb{Z}_N$ (equal to the number of distinct divisors of $N$). With modified time reversal symmetry $\mathcal{T}^2 = 1$ (corresponding to symmetry class BDI), each of these phases splits into four distinct subclasses if $N$ is even, and to two subclasses if $N$ is odd.

When the parafermionic chain is symmetric to time reversal symmetry $\mathcal{T}^2 = -1$, the system belongs to DIII class. In this case the zero modes decompose into interface operators which are Kramers’ pairs of Majorana modes and domain operators which are fractional. The $\mathcal{T}^2 = -1$ time reversal symmetry has no effect on the fractional part. The parafermionic chain can be in several distinct classes corresponding to the presence or absence of Kramers’ pairs of Majoranas or of $\mathbb{Z}_{2j}$ parafermions zero modes at its end.

Once the general classification procedure is laid out, the problem reduces to the task of identifying the interface $\mathbb{Z}_N$ parafermions for the physical problem at hand. In subsection II E we will introduce general considerations regarding zero modes in fractionalized edges. Then, in subsection (II F) we give examples for fermionic (II F 1) and bosonic (II F 2) systems.

E. Ground state degeneracy and composition of zero modes

We consider the system depicted in Fig. 1, where the edge of a fractional quantum spin Hall state is gapped by alternating back-scattering ($B$) and superconducting ($S$)
regions. As we now show, the parafermions at the interfaces between $B$ and $S$ regions may generally be understood as a combination of interface Majorana operators with domain operators that originate from the $B$ and the $S$ regions. The topological properties of the latter are determined by the 2D bulk.

When the $S$ and $B$ segments are very large, the ground state is multiply degenerate. The degeneracy of each gapped segment determines the composition of the parafermionic zero modes. Coupling to a superconductor breaks charge conservation to a lower symmetry group. If the constituent particles are fermions, fermion parity is conserved in addition to the fractional part of the charge. Thus for $\nu = \pm 1/m$ the resulting degeneracy is $2m$ per segment. In this case, the zero modes are composed of a $\mathbb{Z}_2$ parafermion (Majorana fermion) $\gamma$ and a $\mathbb{Z}_m$ parafermion which we denote as $\eta$: $\chi_i = \gamma_i \eta_i$. Conversely, if the constituent particles are bosons, charge conservation is fully broken since bosons can be created and annihilated at the domain wall, and fractional charge is conserved up to a charge of a single boson. The zero modes in these phases do not have a fermionic part, and $\chi_i = \eta_i$.

The distinction between interface zero modes and domain zero modes may be highlighted by considering an annulus slab of a fractional topological insulator where the inner and outer edge of the annulus are covered by alternating $S$ and $B$ regions, such that the interfaces that pin parafermionic zero modes on the two edges are kept far from one another, see Fig. 2. In the thin annulus limit, the system becomes one dimensional, and must conform to the classification of one dimensional systems. As such, the ground state degeneracy must be reduced and the only zero modes must be composed of Majorana fermions.$^{5,6}$ Since the interfaces between the gapped sections on the inner and outer edges of the annulus stay macroscopically distant even when the annulus becomes thin, there must be a way for the ground states to couple through domain operators rather than interface operators. Put differently, the removal of the non-Majorana ground state degeneracy is a result of the proximity of the gapped regions to one another, and there must be operators that act within the gapped regions and have matrix elements between ground states. Those “domain operators” would be responsible for the removal of the ground state degeneracy that characterizes the annulus when both edges are entirely gapped by a single mechanism. When the gapping is by back-scattering, the domain operators transfer a $\pm e^*$ dipole between gapped regions, while when the gapping is by a superconductor, the transferred object has a charge $e^*$ in each spin direction. The interface zero modes may then be expressed as composed of Majorana modes which are localized at the interfaces and remain stable in the thin annulus limit, and two domain operators, one in the $S$ region and one in the $B$ region. The Majorana operators commute with the domain operators (as with any fractional excitation.)$^{35}$ Hence, the two operators act in disconnected Hilbert spaces, and thus we may consider them separately.

The low energy subspace of the gapped edge may be described in terms of the charge degree of freedom of the $S$ regions or the spin degree of freedom of the $B$ regions. The charge on each superconducting region may take the values $q \mod 2 = j/m$ with $j = 0, 1, \ldots, (2m-1)$. The interface Majorana operators change this charge by one. The domain operators acting within the superconductor change it by $2/m$. The interface operator $\eta$ is a combination of the latter two, and may be defined as an operators that adds two quasi-particles of one spin direction to an interface. As such, it changes the charge on the $S$ region and the spin in the $B$ region that neighbor with the interface on which it operates. For odd $m$, there is only one combination of Majorana and domain operators for every charge $\Delta j$ added to the superconductor. The same description holds also for the spins in the $B$ region. This is in consistency with the observation that the $\mathbb{Z}_{2m} = \mathbb{Z}_2 \times \mathbb{Z}_m$ for odd $m$.

Although this discussion was presented for $\nu = \pm 1/m$, some parts of it are general. For all the classification questions that we defined here, the first step is to identify the interface parafermionic zero modes. Once those are identified, the edge is described as a chain of coupled parafermion zero modes.
F. Examples

1. Fermionic systems

Let us first consider the case of a prime $m$, with $\nu = \pm 1/3$ being the prominent example. As explained above, for $\nu = 1/m$ the edge system we consider may be mapped to two disconnected subsystems - a one dimensional chain of coupled Majorana modes, and a one dimensional chain of coupled $\mathbb{Z}_m$ parafermionic modes. The classification of the Majorana chain is reviewed in subsection (IIIC). In the absence of additional symmetries except fermion parity conservation, it has two topological classes, with a $\mathbb{Z}_2$ structure. The two classes have obvious analogs for the parafermions: a trivial phase is created if the edge becomes an insulator. The topological phase is classified by $\nu = \pm 1/m$ for the parafermions, except fermion parity conservation, it has two topological classes, with a $\mathbb{Z}_2$ structure. The two classes have obvious analogs for the parafermions: a trivial phase is created if the edge becomes an insulator. The topological phase is classified by $\nu = \pm 1/m$ for the parafermions, except fermion parity is replaced by the fractional part of the electron charge, which has $m$ possible values. For odd $m$, it is then impossible to have a $\mathbb{Z}_m$-symmetric state of affairs where time reversal changes this number and yet brings it back to its original value when operated twice.

This demonstrates that a chain of $\mathbb{Z}_m$ parafermions with $m$ odd has only one symmetry protected phase. As a result, when the cyclic group of the parafermionic chain $\mathbb{Z}_N$ has no $\mathbb{Z}_2$ subgroup (i.e., for $N$ odd), in the presence of modified time reversal symmetry $T^2 = 1$ each of its gapless phases splits into two distinct phases.

For $\nu \neq \pm 1/m$, characterizing the topological properties of the state requires more than the filling factor. Abelian states are described by the integer-valued $K$-matrix and charge vector $t$. In particular, the degeneracy of the ground state on a torus (which is also the number of topologically distinct types of quasi-particles) is $\det K$, the filling factor is $t^T K^{-1} t$, and $e^*$, the charge of the smallest charge quasi-particle, is the minimum over all integer vectors $l$ of $l^T K^{-1} l$ (the charge is given in units of the electron charge). The electron charge is always an integer multiple of $e^*$'s.

We consider a fractional topological insulator in which electrons with spin $\uparrow$ form a state characterized by $K$, and electrons with spin $\downarrow$ are in a state characterized by $-K$. When $\frac{1}{2} = \det K \equiv N$, there are $N$ types of quasi-particles which may be created by combining $j = 1 \ldots N$ quasi-particles of charge $e^*$. Consequently, each of the different quasi particles has a different charge. Furthermore, in that case $N$ is odd. Thus, independent of the precise way in which the edge is gapped in a $B$ region (there generally is more than one possible way), the domain operators will add or remove $\pm je^*$ dipoles to the edge. Similarly, in the superconducting regions all domain operators will add charges of $je^*$ to each spin direction. As a consequence, the interface consists of a $\mathbb{Z}_2$ parafermion and a $\mathbb{Z}_N$ parafermion.

Once the composition of the $\mathbb{Z}_N$ parafermions is determined, the discussion above shows that in the presence of time reversal symmetry, each of the phases of the $\mathbb{Z}_2 \times \mathbb{Z}_N$ parafermionic chain splits into four subgroups when time reversal symmetry is present.

When $\det Ke^* \neq 1$ there are distinct quasi-particles carrying the same charge. For example, when $\det Ke^* = 2$, there are two topologically distinct quasi-particles of any charge $je^*$. Thus, several sets of dipoles may be created, and there may be different charge-conserving gapping mechanisms that condense different sets of dipoles. At the interfaces between regions gapped by different charge conserving mechanisms, neutral parafermionic modes are localized. This route allows for the construction of parafermionic interface modes in systems where charge is conserved.

In Sec. IVB2 we analyse a simple example for this state of affairs, that arises in the $(311)$ quantum Hall state. The topological classification of the chains follows the same principle that it follows in the cases described above, where each topological phase splits into four subgroups in the presence of modified time reversal symmetry.
2. Bosonic systems

One can consider a case where the system is composed of bosons, rather than fermions. Fractional topological insulating states can be constructed in analogy with the fermionic case, by stacking fractional quantum Hall states of bosons with an opposite chirality. There is an important difference in the case where the fractional topological insulator is made of bosons, rather than fermions; this difference is easiest to exemplify on the $\nu = 1/m$ case, with $m$ being even. In that case the boson number in the $S$ region on the edge is conserved only modulo one, rather than two in the fermionic case. This difference stems from the non-conservation of the number of bosons when the edge is coupled to a superfluid. As a consequence, the dimension of the low energy subspace is reduced from $2m$ per $S$ region to $m$ per $S$ region. In fact, it is further reduced to $m/2$ per $S$ region since terms that create and annihilate single bosons at interfaces are also able to measure the parity of the $e^*$ charges on the segment (where, as before, $e^*$ is the charge of the elementary quasiparticle, in units of the charge of a boson). The interface parafermions are then $\mathbb{Z}_{m/2}$ parafermions. Once this difference is taken into account, however, the classification of bosonic phases follows the same route as that of the fermionic ones.

We therefore identify $2^4 = 8$ distinct gapped phases which can be distinguished by the local symmetry operations in the low energy subspace. These eight phases obey a $\mathbb{Z}_8$ group structure.\(^{3-6}\)

B. Parafermion chain

We now turn to the case of a chain of parafermions with an even number, $L$, of zero modes $\chi_j$, that satisfy $\chi_i \chi_j = \chi_j \chi_i e^{i\frac{2\pi}{N} \text{sgn}(i-j)}$, with $N \in \mathbb{Z}$ (i.e., $\mathbb{Z}_N$ parafermions). Note that, compared to Eq. (1), here we suppress the spin ($s$) index - for our purpose, it suffices to treat only parafermion operators of one spin species (e.g., up spin).

Phases of a one dimensional array of $\mathbb{Z}_N$ parafermion zero modes without time reversal symmetry, were classified by Motruk et al. \cite{Motruk2017}. It was found that if the factorization of $N$ into primes is of the form $N = p_1^{r_1} \cdots p_l^{r_l}$, then the overall number of distinct phases is $N_{\text{phases}} = (r_1 + 1) \cdots (r_l + 1)$, equal to the number of subgroups of $\mathbb{Z}_N$.

Here we consider a chain of parafermionic zero modes with time reversal that satisfies $T^2 = 1$, and a $\mathbb{Z}_N$ symmetry that corresponds either to the total charge, $Q = e^{i\pi Q}$, or spin, $S = e^{i\pi S}$. We denote the $\mathbb{Z}_N$ symmetry by $U = S$ or $Q$; in either case, it satisfies $U^N = 1$. Physically, these symmetries arise from the conservation of the fractional part of the spin or charge on the edge; e.g., on an FTI edge coupled to a superconductor, the total charge is conserved mod(2e), the charge of a Cooper pair. Whether the $\mathbb{Z}_N$ symmetry of the parafermion chain corresponds to $Q$ or $S$ depends on the physical realization. If the parafermion modes are formed at the ends of $S$ regions surrounded by a large B region, as in Fig. 1, the Hamiltonian commutes with $Q$, the total fractional charge of the $S$ regions. In a “dual” system where the $B$ and $S$ regions are interchanged, the symmetry is $S$, the total fractional spin of the $B$ regions.

The charge (spin) operator is even (odd) under time reversal, respectively; hence, the total charge and spin satisfy the following relations with time reversal:

$$QT = QT^1, \quad (3)$$
$$ST = TS. \quad (4)$$

In the ground state subspace of a gapped phase, the symmetry operator can be written as a product of quasi-local operators that act near the left and right ends of the system: $U = U_L U_R$. These operators can have exchange statistics corresponding to any number of parafermion zero modes operators:

$$U_L U_R = e^{i\frac{2\pi}{N} U_R U_L}. \quad (5)$$

Similarly, the time reversal operation can be “fractionalized” into a product of quasi-local unitaries that act near the left or right ends, times complex conjugation: $T = O_L O_R K$. We can define local time reversal operators according to $T_{R,L} = O_{R,L} K$. 

III. FRACTIONALIZATION OF SYMMETRIES

A. Review: Majorana chain

We briefly review the results for the case of fermions with a time reversal symmetry $T^2 = 1$. Gapped phases of Fermions in one dimension are classified by a $\mathbb{Z}_8$ index. This classification follows from the fractionalization of the two symmetry operators: time reversal $T$ and fermion parity $P$ in the low energy sector.\(^{4-6}\) When projected on the low energy subspace, any symmetry operation can be represented as a product of two local operators that act near the left and right ends. This allows us to define local time reversal and local fermion parity operators: $T = O_L O_R K$ and $P = P_L P_R$. Here, $P_R$, $O_R$, $P_L$, $O_L$ are unitary operators with support near the right (left) ends of the system, respectively, and $K$ is complex conjugation. While total time reversal $T^2 = 1$ commutes with total parity, $[T, P] = 0$, and $P$ is a bosonic operator, their local constituents obey:

1. The local parity operator may be bosonic, $[P_L, P_R] = 0$ or fermionic, $[P_L, P_R] = 0$.

2. The local time reversal operator, $T_L \equiv O_L K$, may square to $T_L^2 = \pm 1$.

3. The local time reversal operator may commute, $[P_L, T_L] = 0$, or anticommute, $[P_L, T_L] = 0$, with the local fermion parity operator.
The different possible phases of the system are distinguished by the relations between the local symmetry operators; each phase corresponds to a unique projective representation of the symmetry at the ends (or “symmetry fractionalization”). Eq. (5) is an example for such a relation. For $N$ prime, there are two possible distinct phases: $n = 0$ and $n = 1$. (In this case, any value $1 < n < N$ is equivalent to $n = 1$, since one can always redefine the symmetry operator as $U \rightarrow U^\alpha$.) In addition, in the presence of time reversal with $\mathcal{T}^2 = 1$, there is the possibility that $\mathcal{T}_{RL}^2 = \pm 1$, corresponding to distinct phases. The phase with $\mathcal{T}_{RL}^2 = -1$ is analogous to the Haldane phase of integer spin chains\cite{35}, where there is a Kramers pair of zero modes at every end of the system, even though the “microscopic” time reversal operator (that acts on a single site) does not support a Kramers’ degeneracy\cite{36}. In analogy with the Majorana chain, could there be additional non-trivial phases, resulting from a non-trivial exchange relation between time reversal and $U_{RL,L}$?

To address this question, let us analyze separately the cases $U = Q$ and $U = S$, as follows:

1. $U = Q$

Let us write the charge operator as $Q = Q_LQ_R$. We assume that

$$Q_L\mathcal{T} = e^{i\alpha}\mathcal{T}Q_L^\dagger, \quad Q_R\mathcal{T} = e^{i2\pi n/ N}e^{-i\alpha}\mathcal{T}Q_R^\dagger. \quad (6)$$

Here, $\alpha$ is a phase that we aim to find. The second equality follows from the requirement of Eq. (3). In order to fix the overall phase of $Q_{RL}$, we note that

$$Q_N^N = (Q_LQ_R)^N = e^{i2\pi n N/ N}Q_L^NQ_R^N = 1. \quad (7)$$

Hence, we can choose $Q_N^N = 1$, $\left(e^{i\pi(N-1)/ N}Q_R\right)^N = 1$. By commuting $\mathcal{T}$ with $Q_L^N = 1$, we get [using Eq. (6)]

$$Q_L^NT = e^{iN\alpha}\mathcal{T}^NQ_L^N. \quad (8)$$

Hence, we conclude that $e^{iN\alpha} = 1$. We write $\alpha = \frac{2\pi k}{N}$, where $0 \leq k < N$.

Now, let us examine the consequences of Eq. (6). In particular, consider an eigenstate of $Q_L$, $|q_L\rangle$, such that $Q_L|q_L\rangle = e^{i\frac{2\pi k}{N}}|q_L\rangle$ (where $0 \leq q_L < N$). Then,

$$Q_L\mathcal{T}|q_L\rangle = e^{i\alpha}\mathcal{T}Q_L^\dagger|q_L\rangle = e^{i\alpha}\mathcal{T}e^{-i\frac{2\pi q_L}{N}}|q_L\rangle = e^{i\frac{2\pi (k+q_L)}{N}}\mathcal{T}|q_L\rangle. \quad (9)$$

Therefore, $\mathcal{T}|q_L\rangle \propto |q_L + k\rangle$. Iterating this relation twice, we get

$$\mathcal{T}^2|q_L\rangle = |q_L + 2k\rangle \propto |q_L\rangle. \quad (10)$$

Therefore, we conclude that $2k = 0 \mod (N)$. If $N$ is even, this leaves us with the possibilities of $k = 0, N/2$; for $N$ odd, only $k = 0$ is possible.

2. $U = S$

In this case, we can retrace the steps in Eq. (6-8) for $S_{LR}$, obtaining

$$S_L\mathcal{T} = e^{i\alpha}\mathcal{T}S_L, \quad S_R\mathcal{T} = e^{-i\alpha}\mathcal{T}S_R, \quad (11)$$

where $\alpha = \frac{2\pi k}{N}$, and $(S_L)^N = 1$. We are free to redefine the left and right spin operators as $S_L = e^{i2\pi p}S_L, S_R = e^{-i\frac{2\pi p}{N}}S_R$. Inserting $S_L$ into Eq. (11), we find that it has a modified exchange relation with $\mathcal{T}$:

$$\tilde{S}_L\mathcal{T} = e^{i\frac{2\pi (k-2p)}{N}}\mathcal{T}\tilde{S}_L. \quad (12)$$

Hence, we should identify any two values of $k$ that differ by an even integer as two gauge-related descriptions of the same phase. If $N$ is odd, we can always choose $p$ such that $k - 2p = 0 \mod (N)$. For $N$ even, on the other hand, there are two distinct values of $k$: $k = 0$ or $1$, corresponding to two distinct phases.

Summarizing this section, we find that in the presence of time-reversal symmetry that squares to +1, the classification of the gapped phases of $\mathbb{Z}_N$ parafermion chains becomes enriched. If $N$ is odd, each one of the distinct phases in the absence of time reversal symmetry now comes in two different flavors, depending on whether the local time time reversal operation squares to +1 or −1. If $N$ is even, there is an additional binary distinction, depending on the relation between the local charge or spin operations and time reversal: this distinction corresponds to the phase $\alpha$ in Eqs. (6) or (11) (corresponding to cases where the total fractional charge or spin of the chain is conserved, respectively) taking the value 0 or $\pi/ N$. Thus, in the even $N$ case, each of the distinct phases in the absence of time reversal “spits” into four distinct phases when time reversal symmetry is imposed.

IV. CONSTRUCTION OF GAPPED PHASES

Below, we exemplify the symmetry fractionalization arguments described in Sec. III by constructing the different possible phases in different physical examples.

We first review the gapped phases of $\mathbb{Z}_m$ parafermions that are distinct even without any additional symmetry. For $m$ prime, there are two distinct phases that correspond to the two ways in which the parafermionic modes may dimerize in pairs. Coupling a pair of zero modes annihilates the domain trapped between them and connects the two external domains, thus reducing the number of both superconducting and backscattering domains by one. This reduces the overall degeneracy from $m^N$ for $N$ superconducting segments to $m^{(N-1)}$.

The “trivial” phase is constructed by coupling pairs of parafermions across the superconducting segments (namely, by making the superconducting segments shorter). In the open chain geometry portrayed in Fig. 1...
the Hamiltonian then reads:
\[ H_{z=0} = -t \sum_{j=1}^{L} \eta_{2j-1}^{\dagger} \eta_{2j}, \]
where \( t > 0 \) is a coupling constant. The resulting system has a single backscattering segment on the outer edge and a non degenerate ground state. The non trivial phase is constructed by making all backscattering segments shorter:
\[ H_{z=1} = -t \sum_{j=1}^{L} \eta_{2j}^{\dagger} \eta_{2j+1}, \]
leaving the outer edge of the FTI divided into one backscattering segment and one superconducting segment. The two interfaces between the different domains host a pair parafermionic zero modes.

A. Topological phases of \( \mathbb{Z}_m \) parafermionic modes with \( m \) odd and prime and \( \mathcal{T}^2 = 1 \)

In the presence of a modified time reversal symmetry \( \mathcal{T}^2 = 1 \), a richer phase structure emerges, which depends on the parity of \( m \). For concreteness we consider the case of filling fraction \( m = 3 \), although a similar construction can be generalized to other values of \( m \). Our system is composed of alternating S and B domains, surrounded by a large S domain. The low-energy states are labelled by the values of the fractional spin in each B domain, that take the values \( s = -2/3, 0, 2/3 \). We define the following pseudo-spin 1 operators that act on the B domains:
\[ \Sigma^y_j = P^y_{2j/3} - P^y_{2j/3}, \]
\[ \Sigma^x_j \Sigma^x_{j+1} = \left[ 1 - P^y_{2j/3} \right] \left[ 1 - P^y_{2j+1/3} \right] e^{i2\pi \hat{Q}_j}, \]
where:
\[ P^y_j = \frac{1}{3} \left[ 1 + e^{i2\pi \hat{S}_j} + e^{-i2\pi \hat{S}_j} \right]. \]
Here \( e^{i\pi \hat{Q}_j} \) and \( e^{i\pi \hat{S}_j} \) are the local charge and spin operators in the S and B \( j \)th domain, respectively. We then construct the following Hamiltonian:
\[ H = \sum_i J_i \Sigma^x_i \Sigma^x_{i+1} + J_z \Sigma^z_i \Sigma^z_{i+1} \]
In this basis time reversal acts as \( \mathcal{T} = e^{i\pi \hat{S}_y} K \). Since this Hamiltonian only involves the operators \( e^{i\pi \hat{Q}_i} \) and \( e^{i2\pi \hat{S}_i} \), it commutes with the total charge/spin \( [H, e^{i2\pi \hat{Q}_i}] = 0 \), and thus does not lift the global \( \mathbb{Z}_3 \) symmetry. For small enough \( J_z \) this model is in the Haldane phase \(^{35,37} \) which supports a pseudospin-1/2 degree of freedom at its two ends. When the spin projection \( \Sigma_z \) is a good quantum number, this local Kramers’ doublet corresponds to eigenvectors of the local spin operator with eigenvalues \( \pm 1/3 \):
\[ S_L | \pm 1/3 \rangle = e^{\pm i\pi/3} | \pm 1/3 \rangle. \]

Is the composite Haldane phase constructed above topologically equivalent to the standard Haldane phase of a spin 1 chain \((k = 4)\)? To address this question we first note that the standard Haldane phase is its own inverse. This implies that a pair of chains in the Haldane phase are topologically equivalent to a trivial phase, and that their corresponding edge states can be coupled to a spin singlet state without breaking time reversal symmetry. However, the local antiferromagnetic term that couples the two spin 1/2 end states into a non degenerate spin singlet, includes ‘flip flop’ terms which change the edge spin on each chain. For the case of the composite Haldane phase constructed above, flipping the state of the pseudospin doublet in Eq. (19) requires a transfer of a fractional spin of 2/3 which cannot be achieved by coupling the system to a standard spin 1 chain. Hence the only allowed coupling terms between this composite Haldane phase and a standard Haldane phase are of the form \( s_z \Sigma_z \), which leave the ground state of the two chain system four fold degenerate.

There is however a way to circumvent this restriction. We may couple the Kramers’ doublet that appears on the end of the composite Haldane phase to a single B segment with a 3-fold ground states degeneracy, \(| S = 0, 2/3, -2/3 \rangle \). The three ground states of the additional B segment are eigenvectors of the local spin operator:
\[ S_L | S \rangle = e^{\pm i\pi S_z / 3} | S \rangle. \]

We couple the additional B segment to the pseudospin 1/2 degree of freedom at the end through the local Hamiltonian: \( H = -JS_S L \) with \( J > 0 \). The coupled system has two degenerate ground states corresponding to \( |\pm \rangle = | S = \pm 2/3 \rangle \otimes | S_L = \pm 1/3 \rangle \), at the left end of the open chain. One can readily verify that the two states form a Kramers’ pair, and that they both have the same spin eigenvalue:
\[ S_L | \pm \rangle = e^{i\pi (2S + S_L)} | \pm \rangle = | \pm \rangle. \]

The resulting coupled system is therefore the analog of the \( k = 4 \) Fermionic phase with \( [T_L, S_L] = 0 \). In particular, the edge states of this system can be coupled by coupling it to an ordinary spin 1 chain in the Haldane phase.

B. Topological phases of \( \mathbb{Z}_m \) parafermionic modes with \( m \) even

1. Emergent fermions in bosonic systems

We next construct the topological phases of parafermionic modes at the edge of a bosonic fractional topological insulator. We will focus on the case
of a $\nu = 1/m$ Laughlin state for “spin up” bosons and a $\nu = -1/m$ for “spin down,” with $m$ even. The considerations of Sec. III suggest that in the presence of time reversal symmetry, each of the distinct gapped phases on the edge should split into four classes. Here, we construct explicit Hamiltonians that place the system in each of these phases. The new phases that appear in the presence of time reversal symmetry can be traced back to the emergence of zero modes with fermionic self-statistics at the edge. The new phases at the edge of a bosonic FTI are analogous to the phases of a Majorana chain with time reversal symmetry (class BDI).

(a) Coupling the respective forms: We henceforth analyze the case $m = 8$, which is a less trivial example that illustrates the general principle. In the absence of any additional symmetries except the inherent $\mathbb{Z}_4$ symmetry, we can construct two topologically distinct phases corresponding to the two ways in which zero modes can be dimerized. For convenience, we define the “vacuum” or trivial phase as the phase in which the $\chi_j$ zero modes are coupled across the superconducting regions (see Fig. 3); we will also refer to this phase as the $B$ phase. The “non trivial” phase $S$, in which each pair of $\chi_j$ modes across a backscattering region are coupled, hosts a $\mathbb{Z}_4$ parafermionic zero mode at each interface with the trivial phase.

For the case of $m = 8$ there is a third possible phase. To construct this phase, we note that we can write the $\mathbb{Z}_4$ zero mode operators in the following way:

\[ \chi_j \sim \tilde{\gamma}_j \gamma_j, \]

where $\tilde{\gamma}_j$ creates a right/left moving boson, and the bosonic fields satisfy the commutation relations

\[ [\phi(x), \theta(x')] = \frac{i\pi}{m} \Theta(x' - x). \]

As in the fermionic $\nu = 1/m$ case, the edge states of a bosonic $\nu = 1/m$ FTI can be gapped either by backscattering between the edges, or by coupling the edges to a bosonic superfluid. The pair tunneling and backscattering potentials that gap the edge take the following forms:

\begin{align}
H_S &\propto \psi_R^\dagger \psi_L^\dagger + h.c. \sim \cos 2m\phi \\
H_B &\propto \psi_R^\dagger \psi_L^\dagger + h.c. \sim \cos 2m\theta.
\end{align}

where $\psi_R^\dagger \sim e^{-im(\phi \pm \theta)}$ creates a right/left moving boson, and the bosonic fields satisfy the commutation relations

\[ [\phi(x), \theta(x')] = \frac{i\pi}{m} \Theta(x' - x). \]

In an edge with alternating backscattering and superfluid domains (see Fig. 3), either $e^{2i\phi(x_j)}$ or $e^{2i\theta(x_j)}$ obtain a non-zero expectation value in the ground state manifold, where $x_j$ is the coordinate of a point inside the corresponding domain. The two domain operators satisfy the exchange relations:

\[ e^{2i\phi(x_j)} e^{2i\theta(x_{j'})} = e^{i\frac{4\pi}{m} \Theta(x_j - x_{j'})} e^{2i\theta(x_{j'})} e^{2i\phi(x_j)}. \]

At each interface between the two domains a zero mode appears. The zero modes satisfy a $\mathbb{Z}_{m/2}$ exchange rule:

\[ \chi_{2j-1} = e^{2i\phi(x_{j})+2i\theta(x_{j})}, \]
\[ \chi_{2j} = e^{2i\phi(x_{j})+2i\theta(x_{j+1})}. \]

Hence, there must be $m/2$ degenerate ground states per superfluid (or backscattering) domain.

The simplest bosonic FTI state, $m = 2$, has no ground state degeneracy at its edge. For $m = 4$, there is a two-fold ground state degeneracy per superfluid domain, and the $\chi_j$ zero mode operators obey self-fermionic statistics: $\{ \chi_j, \chi_j \} = 0$. In this case, the classification of possible phases can be read off from the case of Majorana chains. There are two topologically distinct phases in the presence of time reversal symmetry, and an interface between them carries a single Majorana zero mode. In the presence of time reversal that satisfies $\mathcal{T}^2 = 1$, each of the two phases splits into four distinct phases, as expected from the considerations of Sec. III.

We stress the difference between the phases at the edge of a $\nu = 1/4$ bosonic FTI and the phases of an ordinary Majorana chain: although the mathematical description of these two systems is identical, their physical nature is different. For example, even though the $\chi_j$ zero mode operators at the edge of a bosonic FTI satisfy self-fermionic statistics, they are not locally related to any microscopic degrees of freedom, which are bosonic in this problem. As such, they cannot be detected by tunneling particles from outside the FTI system. Instead, the $\chi_j$ modes absorb or emit fractional quasi-particles of the bosonic FTI at zero energy. Their fermionic statistics is an emergent, rather than a microscopic, property.

We henceforth analyze the case $m = 8$, which is a less trivial example that illustrates the general principle. In the absence of any additional symmetries except the inherent $\mathbb{Z}_4$ symmetry, we can construct two topologically distinct phases corresponding to the two ways in which zero modes can be dimerized. For convenience, we define the “vacuum” or trivial phase as the phase in which the $\chi_j$ zero modes are coupled across the superconducting regions (see Fig. 3); we will also refer to this phase as the $B$ phase. The “non trivial” phase $S$, in which each pair of $\chi_j$ modes across a backscattering region are coupled, hosts a $\mathbb{Z}_4$ parafermionic zero mode at each interface with the trivial phase.

For the case of $m = 8$ there is a third possible phase. To construct this phase, we note that we can write the $\mathbb{Z}_4$ zero mode operators in the following way:

\[ \chi_j \sim \tilde{\gamma}_j \gamma_j, \]
where
\[
\gamma_{2j-1} = i e^{i \phi(x_j) + 2i \theta(x_j)} , \quad \gamma_{2j} = i e^{2i \phi(x_j) + 4i \theta(x_j)},
\]
\[
\hat{\gamma}_{2j-1} = i e^{2i \phi(x_j) + 4i \theta(x_j)} , \quad \hat{\gamma}_{2j} = e^{2i \phi(x_j) + 4i \theta(x_j+1)}.
\]

While the two sets of operators \( \gamma \) and \( \hat{\gamma} \) do not commute, each of them obeys fermionic self-statistics: \( \{ \gamma_j, \gamma_{j'} \} = 0 \) and \( \{ \hat{\gamma}_j, \hat{\gamma}_{j'} \} = 0 \). Coupling \( \gamma \) operators across a backscattering domain by turning on the following Hamiltonian:
\[
H_{2j} = i t \gamma_{2j} \gamma_{2j+1},
\]
where \( t \) is the coupling strength, commutes with the \( \gamma_{2j} \), \( \hat{\gamma}_{2j+1} \) zero modes at each interface and thus leaves these two operators as zero modes (see Fig 3b). Alternatively, a perturbation that couples \( \gamma \) across a superconducting domain
\[
H_{2j-1} = i t \gamma_{2j-1} \hat{\gamma}_{2j}
\]
commutes with the remaining \( \gamma \) zero modes, see Fig 3c. In either cases, coupling the self-fermionic modes across the respective domain results in a gapped segment in which the commuting operators \( e^{i \phi} \) and \( e^{i \theta} \) acquire a non-zero expectation value in the ground state manifold (i.e., these operators “condense”). This phase has been dubbed a parafermion condensate phase (PC)\(^{39} \), as it is characterized by the parafermion operator \( \chi^2 \sim e^{i \phi + 4i \theta} \) having a non-zero expectation value everywhere in the bulk. (Note that \( \chi^2 \) is a self-boson, and hence it can condense).

Consequently, in the absence of any additional symmetries we find three distinct gapped phases of a \( \mathbb{Z}_4 \) parafermionic chain, and correspondingly three boundary zero modes at interfaces between them:
\[
\chi_{B-S} = \gamma \hat{\gamma},
\]
\[
\chi_{B-PC} = \gamma,
\]
\[
\chi_{S-PC} = \hat{\gamma}.
\]

We next turn to the classification of these systems with time reversal symmetry. Based on the general arguments presented above we expect each phase to split into 4 distinct phases. Below we will construct these phases explicitly. We start from an open chain of zero modes formed by an array of alternating backscattering and parafermion condensate domains (Fig. 3), with one long backscattering domain acting as our vacuum. Each domain wall between B and PC has a zero mode \( \chi_{B-PC} = \gamma \). We define a parity operator
\[
P = -i \gamma_{2j-1} \hat{\gamma}_{2j} = e^{2i(\theta(x_{j+1}) - \theta(x_j))}.
\]
Since \([\mathcal{T}, P] = 0\), the two zero modes transform differently under time reversal: \( \mathcal{T}^{-1} \gamma_{2j-1} \mathcal{T} = \gamma_{2j-1} \) and \( \mathcal{T}^{-1} \hat{\gamma}_{2j} = -\hat{\gamma}_{2j} \) (where we have fixed an overall phase of the operator to \( \gamma_{2j-1} = \gamma_{2j} = 1 \)). It follows that chain of \( \gamma \) zero modes constructed by alternating segments of B and PC realizes a Majorana chain in class BDI.

By coupling the \( \gamma_n \) Majorana-like modes to each other, we can realize 8 distinct phases, with \( N_5 = 0...7 \) protected zero modes at end of the open chain. We denote the phases with an even number of \( \gamma \) zero modes at each end, \( B_0, B_2, B_4, B_6 \) and the phases with odd number of zero modes \( C_1, C_3, C_5, C_7 \), where the subscript counts the number of Majorana-like zero modes at each end of the open chain. Classifying the phases as \( B_i \) or \( C_i \) follows from the effect of time reversal symmetry breaking terms, where the boundary between phases with an even number of Majorana modes and the vacuum (the large B region) can be gapped by breaking time reversal resulting in a trivial phase, while phases with an odd number of Majorana modes at each end cannot be fully gapped, and would correspond to a PC wire with a single Majorana-like mode at each end.

An alternative way to realize a Majorana chain is by constructing alternating segments of S and PC (Fig. 3) with one long backscattering domain acting as our vacuum. For concreteness we consider the outer segments to be S domains. Each interface between S and PC segments hosts a \( \gamma \) zero mode obeying fermionic self-statistics, while the boundary domain walls between the outer S domains and the long B segment host a \( \chi_{B-S} \) mode. The construction of a parity operator \( P = -i \gamma_{2j-1} \hat{\gamma}_{2j+1} \) readily follows the case of the \( \gamma \) chain. By coupling the \( \gamma_5 \) Majorana-like modes to each other, we can realize 8 distinct phases, with a single \( \chi \) zero mode and \( N_\chi = 0...7 \) protected Majorana-like modes at end of the open chain. We denote the phases with an even number of \( \gamma \) zero modes at each end, \( S_0, S_2, S_4, S_6 \) and the phases with odd number of zero modes \( C_1, C_3, C_5, C_7 \). An even number of \( \gamma \) can be gapped by a time reversal symmetry breaking perturbation, resulting in a S chain with a single \( \chi \) zero mode at each end.

We will next show that the \( \tilde{C}_i \) phases are topologically equivalent to the \( C_i \) phases. In particular, this requires that the zero modes that appear at the interface between the two phases can be gapped by a local perturbation without breaking time reversal symmetry. We will demonstrate this explicitly for the case of an interface between \( C_1 \) and \( C_1 \), see Fig. 4. A similar construction can be made for the other three \( C_i \) classes. The \( \chi_2 \) mode at the interface can be split into two Majorana-like modes: \( \gamma_{2n} \) and \( \hat{\gamma}_{2n} \), by nucleating a narrow PC region at the interface between the B and S segments. The resulting boundary system had \( N_\chi = 2, N_\gamma = 2 \) and \( N_\chi = 0 \). We then note that the pair of \( \gamma \) modes located at the two ends of the B segment transform differently under time reversal operation. Hence by coupling the \( \gamma \) modes across the B segment:
\[
\delta H_\gamma = i t \gamma_{2n-1} \gamma_{2n-1}
\]
we can lift the degeneracy associated with them, without violating time reversal symmetry. Similarly, the following time reversal symmetric term:
\[
\delta H_\tilde{\gamma} = i \tilde{\gamma}_{2n-1} \tilde{\gamma}_{2n+1},
\]

lifts the degeneracy associated with the S segment. These local perturbations gap up all interface zero modes while leaving the time reversal symmetry intact. This indicates that the two phases are topologically equivalent.

![Diagram](image)

**FIG. 4.** Local perturbation can gap up the zero modes that appear at the interface between $C_1$ and $C_1$ segments. (a) The dashed line marks the interface between the $C_1$ phase with a single $\gamma_{2n-1}$ at its end, and the $C_1$ phase with one $\chi_{2n}$ and one $\gamma_{2n+1}$ at its end. (b) The $\chi_{2n}$ mode can be split into two Majorana like modes: $\gamma_{2n}$ and $\gamma_{2n}$, by nucleating a narrow PC region in between. (c) Coupling each pair of $\gamma$ ($\tilde{\gamma}$) modes across the B (S) domain gap up the interface zero modes without violating time reversal symmetry.

We therefore conclude in the absence of any additional symmetries, a chain of $\mathbb{Z}_4$ parafermions realises 3 topological phases: B, S, PC. When the system is time reversal symmetric each phase splits into four, resulting in a total of 12 distinct gapped phases: $B_0, B_2, B_4, B_6, S_0, S_2, S_4, S_6$ and $C_1, C_3, C_5, C_7$. According to the arguments of Sec. III, we expect each of the three phases that exist in the absence of time reversal to split into four distinct phases when time reversal is imposed. Thus, the 12 phases exhaust the list of possible phases with time reversal symmetry.

2. **Parafermions with no superconductors**

In this subsection we show that $T^2 = 1$ time-reversal symmetric parafermion chains may be constructed even in a setting in which charge is conserved, i.e., without resorting to coupling the edges to superconductors. This may occur in fractional systems where the smallest fractional charge $e^*$ and the degeneracy (given by the determinant of the $K$ matrix) satisfy $e^* \det K \neq 1$. In this case, there are several different quasi-particles that carry the same charge. It is then possible to have different charge conserving mechanisms that condense topologically distinct sets of dipoles. The interfaces between domains with topologically distinct gaps host neutral parafermionic modes. We focus on a particular example where the neutral zero modes are self fermions that form a $\mathbb{Z}_2$ group.

We consider the (331) quantum Hall state, where the edge theory is governed by the Lagrangian:

$$\mathcal{L} = \frac{1}{4\pi} (\partial_\nu \Phi^T K \partial^\nu \Phi - \partial_\nu \Phi^T V \partial^\nu \Phi) + \frac{1}{2\pi} e^{i\nu T} \partial_\nu \Phi A_\nu.$$  \hspace{1cm} (33)

Here $V$ is a velocity matrix, $t = (1, 1, 1, 1)^T$ is the charge vector, and the $K$-matrix is given by:

$$K = \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix}$$  \hspace{1cm} (34)

with

$$\kappa = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}. \hspace{1cm} (35)$$

The bosonic fields satisfy the commutation relation:

$$[\partial_\nu \Phi_1(x), \Phi_j(x')] = 2\pi \delta(x - x') K^{-1}_{ij}$$  \hspace{1cm} (36)

Local operators composed of products of electron creation and annihilation operators take the form: $e^{i\Lambda^T \Phi}$, where $\Lambda$ is a vector of integers (this follows from the requirement that their mutual statistics with any quasi-particle must be trivial). Charge conserving local operators satisfy $\Lambda^T t = 0$. In accordance with Ref. 47, The edge states can be gapped by two topologically distinct charge conserving perturbations corresponding to:

$$H_1 = \lambda_{1a} \cos (\Lambda_{1a}^T K \Phi) + \lambda_{1b} \cos (\Lambda_{1b}^T K \Phi)$$  \hspace{1cm} (37)

$$H_2 = \lambda_{2a} \cos (\Lambda_{2a}^T K \Phi) + \lambda_{2b} \cos (\Lambda_{2b}^T K \Phi)$$  \hspace{1cm} (38)

where $D = 1, 2$ are the two domains, and $\Lambda_{1a}^T = (1, 0, 1, 0)$, $\Lambda_{1b}^T = (0, 1, 0, 1)$, $\Lambda_{2a}^T = (0, 1, 1, 0)$ $\Lambda_{2b}^T = (1, 0, 0, 1)$.

Each domain has two domain operators $e^{i\Lambda_{a,b}^T \Phi}$, which commute with the respective Hamiltonian. Using products of domain operators in the two neighboring domains, we can construct the following domain wall operator:

$$\chi_n = e^{i(0,1,0,0) \Phi_{n} - i(0,1,1,0) \Phi_{n+1}}$$  \hspace{1cm} (39)

(Different choices of pairs of domain operators result in operators which are related by a neutral boson). These charge neutral operators satisfy fermionic exchange statistics with respect to each other:

$$\{\chi_n, \chi_{n'}\} = 2\delta_{n,n'},$$  \hspace{1cm} (40)

and thus behave as Majorana fermions. The construction of gapped phases follows the usual one dimensional BDI fermionic chain giving rise to 8 classes in the presence of an effective time reversal symmetry.

Note that, as in the bosonic case described above, although the $\chi_n$ operators are formally similar to ordinary Majorana zero modes, they have different physical properties. E.g., a physical electron cannot be absorbed at zero energy at an interface between the two types of domains. Nevertheless, the structure of the possible phases at the edge in the presence of time reversal is the same as that of an ordinary Majorana Majorana.
V. CONCLUSION

We studied gapped phases of $\mathbb{Z}_N$ parafermionic chains in the presence of a modified time reversal symmetry $\mathcal{T} = 1$. Without time reversal, there are $N_{\text{phases}} = (r_1 + 1) \ldots (r_k + 1)$ distinct phases, where $N = p_1^{r_1} \ldots p_k^{r_k}$ is the decomposition of $N$ to prime factors. When time reversal symmetry is imposed, the resulting phase structure depends on the parity of $N$. If $N$ is odd, each of the phases splits into two subclasses. This is since the class with no parafermionic end states splits into a trivial phase and a symmetry protected Haldane phase. Conversely, when $N$ is even, each phase splits into four subclasses. The origin of this split is in the emergent Majorana fermions associated with even values of $N$. Akin to the classification of Majorana chains, the trivial class then splits into four symmetry protected subclasses: the trivial subclass, two anomalous subclasses with two Majorana modes at each end, and the Haldane phase which corresponds to four Majorana modes at each end of the chain. We demonstrate the appearance of such emergent Majorana zero modes in a system where the constituents particles are bosons as well as in the absence of any form of pairing.

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1. A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B 78, 195125 (2008).
2. A. Kitaev, AIP Conf. Proc. 1134, 22 (2009).
3. S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, New J. Phys. 12, 065010 (2010).
4. L. Fidkowski, A. Kitaev, Phys. Rev. B 81, 134509 (2010).
5. A. M. Turner, F. Pollmann, E. Berg, Phys. Rev. B 83, 075102 (2011).
6. L. Fidkowski, A. Kitaev, Phys. Rev. B 83, 075103 (2011).
7. M. Z. Hasan and C. L. Kane, Rev. Mod. Phys 82 3045 (2010).
8. X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83 1057 (2011).
9. J. Alicea , Rep. Prog. Phys. 75, 076501 (2012).
10. T. D. Stanescu and S. Tewari, J. Phys.: Condens. Matter 25, 233201 (2013).
11. S. Gangadharaiaiah, B. Braunecker, P. Simon, and D. Loss, Phys. Rev. Lett. 107, 036801 (2011).
12. L. Fidkowski, R. M. Lutchyn, C. Nayak, and M. P. A. Fisher, Phys. Rev. B 84, 195436 (2011).
13. E. M. Stoudenmire, J. Alicea, O. A. Starykh, and M. P. A. Fisher, Phys. Rev. B 84, 014503 (2011).
14. E. Sela, A. Altland, and A. Rosch, Phys. Rev. B 84, 085114 (2011).
15. E. Tang and X. G. Wen, Phys. Rev. Lett. 109, 096403 (2012).
16. H. Yao and S. Ryu, Phys. Rev. B 88, 064507 (2013).
17. X.-L. Qi, New J. Phys. 15, 65002 (2013).
18. A. Vishwanath and T. Senthil, Phys. Rev. X 3, 011016(2013).
19. Z. C. Gu and M. Levin, Phys. Rev. B 89, 201113 (2014).
20. C. Wang and T. Senthil, Phys. Rev. B 89, 195124 (2014).
21. T. Senthil, Annu. Rev. Condens. Matter Phys. 6, 299 (2015).
22. T. Morimoto, A. Furusaki, and C. Mudry, Phys. Rev. B 92,125104 (2015).
23. Queiroz, Raquel and Khalaf, Esam and Stern, Ady, Phys. Rev. Lett. 117, 206405 (2016).
24. Michael Levin and Ady Stern, Phys. Rev. Lett. 103, 196803 (2009).
25. Ady Stern, Annu. Rev. Condens. Matter Phys. 7, 349 (2016).
26. L. Fu and C. L. Kane, Phys. Rev. Lett. 100, 096407 (2008).
27. Netanel H. Lindner, Erez Berg, Gil Refael, Ady Stern, Phys. Rev. X 2, 041002 (2012).
28. David J. Clarke, Jason Alicea, Kirill Shtengel, Nature Commun. 4, 1348 (2013).
29. Meng Cheng, Phys. Rev. B 86, 195126 (2012).
30. Maissam Barkeshli, Chao-Ming Jian, and Xiao-Liang Qi, Phys. Rev. B 88, 235103 (2013).
31. E. H. Fradkin and L. P. Kadanoff, Nucl. Phys. B 170, 1 (1980).
32. P. Fendley, J. Stat. Mech. 1211, P11020 (2012).
33. This can be seen from the following construction:

\[
\chi_{\uparrow\downarrow}^{\gamma} = \left(\chi_{\uparrow\downarrow}^{1}\right)^{m} \equiv \gamma^\dagger \eta_{\uparrow\downarrow}^{\gamma} \tag{41}
\]

where $\gamma^\dagger = \eta^\dagger$. Then noting that for odd filling fractions $m$:

\[
\gamma^\dagger \eta_{\uparrow\downarrow}^{\gamma} = \eta_{\uparrow\downarrow}^{\gamma} \gamma^\dagger
\]

34. Dganit Meidan, Alessandro Romito, and Piet W. Brouwer Phys. Rev. Lett. 113, 057003 (2014).
35. F. D. M. Haldane, Phys. Lett. A 93A, 464 (1983).
36. F. Pollmann, E. Berg, A. Turner, and M. Oshikawa Phys. Rev. B 85, 075125 (2012).
37. F. D. M. Haldane, Phys. Rev. Lett. 50, 1153 (1983).
38. Johannes Motruk, Ari M. Turner, Erez Berg, Frank Pollmann Phys. Rev. B 88, 085115 (2013).
39. Roberto Bondesan and Thomas Quella, J. Stat. Mech. 2013 10024 (2013).
40. A. Alexandradinata, N. Regnault, Chen Fang, Matthew J. Gilbert, and B. Andrei Bernevig Phys. Rev. B 94, 125103 (2016).
41. M. F. Maghrebi, S. Ganeshan, D. J. Clarke, A. V. Gorshkov, and J. D. Sau Phys. Rev. Lett. 115, 065301 (2015).
42. The determinant of $K$ is odd when $\text{det} Ke^* = 1$. This may be seen by noting that $N$ quasi-particles of charge $e^*$
form an electron, which must be a fermion. If the vector corresponding to an $e^*$ quasi-particle is $l$, the requirement that the electron is a fermion amounts to the requirement that $N^2 l^T K^{-1} l = 1$. Since the elements of $l, K$ are integers, and since $K^{-1}$ can be written as $1/\det K$ times an integer-valued matrix, this requires $\det K = N$ to be odd.

43. B. I. Halperin, Helv. Phys. Acta 56, 75 (1983).
44. X. G. Wen, Phys. Rev. B 41, 12838 (1990).
45. Dung-Hai Lee and Xiao-Gang Wen, Phys. Rev. Lett. 66, 1765 (1991).
46. Michael Levin, Ady Stern, Phys. Rev. B 86, 115131 (2012).
47. Michael Levin, Phys. Rev. X 3, 021009 (2013).