Two-population replicator dynamics and number of Nash equilibria in matrix games

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Abstract – We study the connection between the evolutionary replicator dynamics and the number of Nash equilibria in large random bi-matrix games. Using techniques of disordered systems theory we compute the statistical properties of both, the fixed points of the dynamics and the Nash equilibria. Except for the special case of zero-sum games, one finds a transition as a function of the so-called co-operation pressure between a phase in which there is a unique stable fixed point of the dynamics coinciding with a unique Nash equilibrium, and an unstable phase in which there are exponentially many Nash equilibria with statistical properties different from the stationary state of the replicator equations. Our analytical results are confirmed by numerical simulations of the replicator dynamics, and by explicit enumeration of Nash equilibria.

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Introduction. – Replicator equations (RE) describe the evolution of populations of species interacting through co-operation and competition. A fitness is assigned to each species and species fitter than average increase in concentration while the weight of species less fit than average decreases in time. Replicator dynamics (RD) have found widespread applications in game theory and economics as well as in population dynamics where an equivalence to Lotka-Volterra equations of theoretical biology can be established [1].

In the context of evolutionary game theory [2] RE describe games which are played repeatedly and where at any time-step the interaction is between individuals randomly chosen out of populations of agents. Each player follows a pre-programmed strategy which they cannot change. Over time payoffs are accrued, and agents reproduce accordingly, passing on their strategy to their descendents. Prior to the launch of this evolutionary approach to game theory the analysis of games was mostly concerned with static aspects [3], i.e. with strategically optimal actions by fully rational players and the characterization of potential equilibrium points. It is here assumed that the game is played only once. Players choose potentially stochastically from a set of available actions, and payoffs are then paid to each participant depending on the decisions of all involved players. The notion of a Nash equilibrium (NE) refers to a point in strategy space so that no player can increase his payoff by unilaterally deviating from this point. Rigorous studies of NE in matrix games can be found, e.g., in [4].

In this letter we extend existing work on single-population random replicator dynamics [5] in which a transition between stable and unstable regimes has been found as a function of a so-called “co-operation pressure”, and study the case of multi-population models, i.e. interaction between distinct populations of species. These correspond to so-called “asymmetric” games [1] in which there is more than one type of player, such as in the game known as “the battle of the sexes” in which male and female types of agents have different strategy sets at their disposal. In particular we study the stability of two-population replicator systems, and the relation to the number of NE in the corresponding bi-matrix games. We here extend the work of Berg et al. [6], who computed the number and statistics of NE of matrix games with random payoff matrices in the absence of co-operation pressure, and establish a connection with the work of [5].

Model definitions. – We will consider bi-matrix games between two players, one of type X and one of type Y, with strategy sets \( \Sigma_x \) and \( \Sigma_y \), and payoff functions \( \mu_x, \mu_y : \Sigma_x \times \Sigma_y \rightarrow \mathbb{R} \). We will here restrict to
the case in which \(|\Sigma_x| = |\Sigma_y| = N\). Strategies available to players of type \(X\) are labelled by \(i \in \{1, \ldots, N\}\), the ones available to \(Y\) by \(j \in \{1, \ldots, N\}\). The extension to more general cases with different numbers of available strategies is straightforward. Mixed strategies are then probability distributions over \(\Sigma_x\) and \(\Sigma_y\) respectively, described by vectors \(x = (x_1, \ldots, x_N)\) and \(y = (y_1, \ldots, y_N)\), with \(0 \leq x_i \leq 1\) and \(\sum x_i = 1\), and similarly for the \(\{y_j\}\). Pure strategies are recovered as unit vectors. If player \(Y\) plays mixed strategy \(y\) then the expected payoff for \(X\)’s pure strategy \(i\) reads \(\nu^x_i(y) = \sum_j \mu_x(i, j)y_j\), and analogously for \(\nu^y_j\). We will later extend to cases in which the \(\nu^x_i\) and \(\nu^y_j\) depend on both \(x\) and \(y\). If both players play mixed strategies the expected payoff for \(X\) is denoted by \(\nu^x(x, y) = \sum_i x_i \nu^x_i(y)\) and similarly for \(\nu^y(x, y)\). A NE is then a point \((x^*, y^*)\) so that \(\nu^x(x^*, y^*) = \max_x \nu^x(x, y^*)\) and \(\nu^y(x^*, y^*) = \max_y \nu^y(x^*, y)\), and may be characterised by the following conditions:

\[
x^*_i [\nu^x_i(y^*) - \nu^x_i(x^*, y^*)] = 0, \quad (1)
\]

\[
\nu^x_i(y^*) - \nu^x_i(x^*, y^*) \leq 0, \quad \forall i, \quad (2)
\]

\[
y^*_j [\nu^y_j(x^*) - \nu^y_j(x^*, y^*)] = 0, \quad (3)
\]

\[
\nu^y_j(x^*) - \nu^y_j(x^*, y^*) \leq 0, \quad \forall j. \quad (4)
\]

RE of evolutionary game theory assume large populations of \(X\)- and \(Y\)-type players, respectively, with each individual playing a pre-specified pure strategy. The replicators are hence the pure strategies of the game under consideration and are copied without error from parent to child. \(x_i(t)\) denotes the relative concentration of pure strategy \(i\) in the \(X\)-population, and similarly for \(y_j(t)\). The replicator equations describing the evolution of the \(X\) and \(Y\) populations are then as follows:

\[
\dot{x}_i(t) = x_i(t) [\nu^x_i(y(t)) - \kappa_x(t)], \quad (5)
\]

\[
\dot{y}_j(t) = y_j(t) [\nu^y_j(x(t)) - \kappa_y(t)],
\]

where the \(x_i(t)\) and \(y_j(t)\) are now time-dependent variables, and where \(\kappa_x(t) = \sum x_i(t) \nu^x_i(y(t))\), and similarly for \(\kappa_y(t)\) the average payoffs of the \(X\)- and \(Y\)-type players, respectively. The RE preserve the overall normalisation \(\sum x_i(t) = \sum y_j(t) = 1\) in time. In terms of population dynamics the weights \(\{x_i(t)\}\) and \(\{y_j(t)\}\) correspond to relative concentrations of species, and the payoff functions \(\nu^x\) and \(\nu^y\) may be seen as their respective fitnesses. In a biological setting RD thus describe the temporal evolution of populations of species, where the concentrations of species fitter than average grow and where all other species decrease in relative numbers. In the remainder of the paper we will use the game theoretical and the population dynamical language synonymously. Note that in the above two-population system species of type \(X\) interact through random couplings only with species of type \(Y\) and vice versa.

We will in the following be concerned with replicator systems where all entries of the payoff matrices are Gaussian random variables. Following Peschel and Mende [1] as well as [5], we will also introduce the so-called “co-operation pressures” \(u_x \geq 0\) and \(u_y \geq 0\), and will consider payoff functions of the form

\[
\nu^x_i(x, y) = -2ux_i + \sum_j a_{ij}y_j, \quad (6)
\]

\[
\nu^y_j(x, y) = -2uy_jy + \sum_i b_{ji}x_i.
\]

The role of \(u_x, u_y\) will be clarified below. For \(u_x = u_y = 0\) one recovers the cases studied in [6] (with payoff matrices \(a_{ij} = \mu_x(i, j), b_{ji} = \mu_y(i, j)\)). Both the static and dynamical properties of bi-matrix games are invariant under global rescaling and shifts of all payoff matrix elements. Without loss of generality we may therefore assume that the Gaussian variables \(a_{ij}\) and \(b_{ji}\) are drawn from the following statistics:

\[
a_{ij}^2 = b_{ji}^2 = 1/\Sigma, \quad a_{ij}b_{ji} = \delta_{ij}\delta_{ji} \Gamma / \Sigma \quad (7)
\]

(with \(\Sigma\) an average over the distribution of payoffs). The scaling with \(\Sigma\) is here introduced to guarantee a well-defined thermodynamic limit. The parameter \(\Gamma\) characterises the correlations between the payoff matrices for the two different types of players. If \(\Gamma = -1\) then \(a_{ij} = -b_{ji}\) so that the resulting game is a zero-sum game (at vanishing co-operation pressures), corresponding to a prey-predator relation in the population dynamical setting. For \(\Gamma = 0\) one has uncorrelated payoff matrices, and in the fully symmetric case \(\Gamma = 1\) the two interacting players receive equal payoff (at \(u_x = u_y = 0\)). Note also that we will re-scale the \(x_i\) and \(y_j\) such that they obey the normalisations \(\sum x_i(t) = \sum y_j(t) = N\) at all times \(t\). The co-operation pressures \(u_x\) and \(u_y\) finally control the growth of individual species, large values of \(u_x\) drive the configuration \(x\) into the the interior of the simplex defined by \(\sum x_i = N, 0 \leq x_i \leq N\), and similarly for \(u_y\), see the book by Peschel and Mende for further details [1].

Inspection of eqs. (5) for \(u_x, u_y \rightarrow \infty\) shows that the RD leads to the fixed point \(x_i = y_j \equiv 1, \forall i, j\) when started from non-zero concentrations, corresponding to full co-operation and maximal diversity. In a game theoretic setting large values of \(u_x, u_y\) favour the use of mixed strategies as opposed to pure strategies located at the corners of the above simplices. Finally we note that stable fixed points (FP) of the RE are always NE but not necessarily vice versa.

Stationary states of the replicator dynamics. – We here address the dynamics described by the RE, and formulate an effective theory for macroscopic dynamical order parameters from which we compute properties of stationary fixed point states. The starting point of the
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Analysis is the moment generating functional

\[
Z[\psi^x, \psi^y] = \left\langle \exp \left( i \int dt \left( \sum_{i=1}^{N} \psi^x_i(t) x_i(t) + \sum_{j=1}^{N} \psi^y_j(t) y_j(t) \right) \right) \right\rangle,
\]

where \(\langle \ldots \rangle\) denotes an average over trajectories of the system, i.e. over solutions of the replicator equations (5).

As usual for disordered systems a closed, but implicit set of equations describing the temporal evolution of a small number of disorder-averaged macroscopic order parameters can be derived. In the thermodynamic limit these observables turn out to be given by the correlation functions \(C_x(t,t')\), \(C_y(t,t')\), the response functions \(G_x(t,t')\), \(G_y(t,t')\) and the Lagrange parameters \(\kappa_x(t), \kappa_y(t)\). The correlation and response matrices for the \(X\)-population are defined by

\[
C_x(t,t') = \lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} \langle [x_i(t)x_i(t')] \rangle,
\]

\[
G_x(t,t') = \lim_{N \to \infty} N^{-1} \sum_{i=1}^{N} \langle \frac{\delta x_i(t)}{\delta x_i(t')} \rangle,
\]

and analogous definitions for \(C_y, G_y\) apply. These order parameters are to be determined self-consistently as averages \(C_x(t,t') = \langle x(t)x(t') \rangle \), \(G_x(t,t') = \langle \delta x(t)/\delta x(t') \rangle \), (and analogously for \(C_y, G_y\)) over realisations of the following pair of coupled stochastic effective processes:

\[
\dot{x}(t) = -x(t) \left[ 2u_x x(t) - \Gamma \int_{t_0}^{t} dt' \ G_x(t,t')x(t') \right] - \kappa_x(t) + \eta_x(t),
\]

\[
\dot{y}(t) = -y(t) \left[ 2u_y y(t) - \Gamma \int_{t_0}^{t} dt' \ G_x(t,t')y(t') \right] - \kappa_y(t) + \eta_y(t),
\]

with \(t_0\) the starting point of the dynamics. \(\ldots\) refers to an average over the effective process, i.e. over the noise variables \(\{\eta_x(t)\}\) and \(\{\eta_y(t)\}\). The covariances of these noise variables are given by \(\langle \eta_x(t)\eta_y(t') \rangle = C_y(t,t')\), \(\langle \eta_x(t)\eta_y(t') \rangle = C_y(t,t')\) with no correlations between \(\eta_x\) and \(\eta_y\). Finally, the Lagrange multipliers \(\kappa_x(t)\) and \(\kappa_y(t)\) have to be chosen such that the constraints \(\langle x(t) \rangle = 1\) and \(\langle y(t) \rangle = 1\) are fulfilled at any time \(t \geq t_0\).

Further progress can be made by inspecting FP of the replicator equations, i.e. time-independent solutions \(x(t) \equiv x, y(t) \equiv y\) of the effective processes (with constant correlation functions \(C_x(t,t') \equiv q_x, \ C_y(t,t') \equiv q_y\) and stationary response functions \(G_x(t-t'), G_y(t-t')\)). Similarly to [5] one derives the following equations characterising such FP states

\[
\begin{array}{l}
\chi_x (2u_x - \Gamma \chi_y) = g_0 (\Delta_x),
\end{array}
\]

\[
\begin{array}{l}
\chi_y (2u_y - \Gamma \chi_x) = g_0 (\Delta_y),
\end{array}
\]

\[
\begin{array}{l}
q_x^{-1/2} \chi_x (2u_x - \Gamma \chi_y) = g_1 (\Delta_x),
\end{array}
\]

\[
\begin{array}{l}
q_x^{-1/2} (2u_y - \Gamma \chi_x) = g_1 (\Delta_y),
\end{array}
\]

\[
\begin{array}{l}
(q_x/q_y) (2u_x - \Gamma \chi_y)^2 = g_2 (\Delta_x),
\end{array}
\]

\[
\begin{array}{l}
(q_x/q_y) (2u_y - \Gamma \chi_x)^2 = g_2 (\Delta_y)
\end{array}
\]

with \(\Delta_x = \kappa_x q_x^{-1/2}, \Delta_y = \kappa_y q_y^{-1/2}\) and \(g_n (\Delta) = \int_{-\infty}^{\infty} dz \times e^{-z^2/2} (\Delta - z)^n\). \(\chi_x\) is here the integrated response \(\chi_x = \int dx G_x (x)\), and similarly for \(\chi_y\). The probability for a species of type \(X\) to survive, i.e. to attain a fixed point value \(x > 0\) is obtained as \(\phi_x = P(x > 0) = g_0 (\Delta_x)\) and similarly \(\phi_y\). In terms of evolutionary game theory these correspond to the fractions of pure strategies played with non-zero probability. A linear stability analysis shows that such FP become unstable when \((\chi_x \chi_y)^2 > \phi_x \phi_y\), leading to an unstable phase at low cooperation pressures, usually associated with broken ergodicity [7]. For \(u_x = u_y\), which we will mostly consider in the following, one finds \(\chi_x = \chi_y \equiv \chi\) and \(q_x = q_y \equiv q\), and eqs. (12) as well as the stability condition reduce to those of a single-population replicator system studied in [5]. In the unstable phase the solutions of (12) can no longer be expected to describe the stationary states of the RD accurately.

For symmetric couplings \((\Gamma = 1, i.e. a_{ij} = b_{ji})\) the RD (5) can be written in the form \(\dot{x}_i = x_i \times \langle \partial_x H(x,y) - \kappa_x (t) \rangle\) and similarly for \(y_j\), with \(H(x,y) = 1/2 \sum_{i,j=1}^{N} [x_i y_j (a_{ij} + b_{ji})] - u_x \sum_{i=1}^{N} x_i^2 - u_y \sum_{j=1}^{N} y_j^2\) so that the stationary states of the RD correspond to extrema of \(H\). These can be computed straightforwardly by replica methods. We will not report the details but will only note that a replica-symmetric (RS) ansatz leads to a set of equations identical to (12) at zero temperature and in the thermodynamic limit above \(u_c\) (with \(\Gamma = 1\). For \(\Gamma < 1\) no Lyapunov function \(H\) can be found and the replica approach is inapplicable.

Annealed calculation of the number of Nash equilibria. Finally, the number and statistics of the NE of the corresponding bi-matrix game can be computed by direct integration over phase space enforcing conditions (1)-(4) through suitable delta- and step-functions. It is here convenient to set \(\bar{x}_i = x_i\) if \(x_i > 0\) and \(\bar{x}_i = -2u_x x_i + \sum_j a_{ij} y_j - \kappa_x\) if \(x_i = 0\), and similarly for \(\bar{y}_j\) [6]. The above conditions (1,2) can then be written as

\[
I_f (\vec{x}, \vec{y}) \equiv \bar{x}_i \Theta (\neg \bar{x}_i)
\]

\[
- \left( -2u_x \bar{x}_i \Theta (\neg \bar{x}_i) + \sum_j a_{ij} \bar{y}_j \Theta (\neg \bar{y}_j) - \kappa_x \right) = 0
\]
and (3), (4) translate into \( I^\theta_y (\hat{x}, \hat{y}) = 0 \) with an analogous expression \( I^\theta_y (\hat{x}, \hat{y}) \). \( \Theta(\cdot) \) is the step-function. The number of NE at payoffs \( \kappa_x \) and \( \kappa_y \) is then given by

\[
N(\kappa_x, \kappa_y) = \int_0^\infty D\hat{x}D\hat{y} \prod_i \delta(I^G_i(\hat{x}, \hat{y})) \times \prod_j \delta(I^\theta_j(\hat{x}, \hat{y})) | \det D|,
\]

with \( D\hat{x} = \prod_i d\hat{x}_i \delta(\sum_i \hat{x}_i \Theta(\hat{x}_i) - N) \) and similarly for \( D\hat{y} \). \( \det D \) is a normalising determinant. Performing the disorder-average in an annealed approximation, one converts the computation into a saddle-point problem in the thermodynamic limit. We set \( u_x = u_y = u \) for simplicity and find exponential domination at equal payoff \( \kappa_x = \kappa_y = \kappa \) and \( \lim_{\kappa \to \infty} \frac{1}{\kappa} \ln N(\kappa) = S(\kappa) \), where

\[
S(\kappa) = \text{extr}_{(E, R, q, \phi)} \left\{ 2E - \Gamma R^2 + 2\bar{q} + 2\bar{\phi} + 2 \ln \left( \frac{H((2\mu - \Gamma R)\kappa/q + E)}{\sqrt{2(2\mu - \Gamma R)^2 + 2\bar{q}}} \right) \times \exp \left( \frac{-\kappa^2}{2\bar{q}} + \frac{(E + (2\mu - \Gamma R)\kappa/q)^2}{4\bar{q} + 2(2\mu - \Gamma R)^2/q} \right) + g \right\},
\]

\( g \) denotes the contribution from the normalising determinant, which we do not report explicitly. For \( u = 0 \) we recover the result of [6]. Extremisation of (15) leads to an annealed upper bound of the logarithmic number of NE at payoff \( \kappa \).

**Results.** – Our results are summarised in figs. 1 to 4. Figure 1 shows the phase diagram of the bi-population replicator model. We here choose to present results as a function of the co-operation pressures, i.e., we depict the phase diagram in the \((u_x, u_y)\)-plane at different fixed symmetry parameters \( \Gamma \), and focus the discussion on the effects of co-operation pressure on the stability of the system and on the number of NE in the following. Properties of the system as a function of \( \Gamma \) at fixed zero and non-zero \((u_x, u_y)\) can be obtained by a straightforward transformation, and are briefly discussed below. At any \( \Gamma > -1 \) one finds a transition line, separating a stable regime at large co-operation pressures (top-right of the phase diagram) from an unstable regime. In the phase of large co-operation pressure the system reaches a unique, stable fixed point after some equilibration time. Earlier results of [5,7] suggest that FP are numerous, but unstable below the transition at fully symmetric couplings \( \Gamma = 1 \), and that they are exponentially suppressed for \( \Gamma < 1 \) in the unstable phase; hence volatile, possibly chaotic behaviour can be observed in simulations at \( \Gamma < 1 \) in the unstable phase. For \( \Gamma = 1 \) the dynamics converges to FP for all \((u_x, u_y)\), but the latter are locally instable below the transition. For all \( \Gamma > -1 \) non-ergodicity is observed in the unstable phase, i.e. the microscopic stationary state of the system depends on the starting point of the dynamics in this phase [7], whereas no such sensitivity is found in the ergodic phase at large co-operation pressure. For \( \Gamma = -1 \) no transition is found, and there is one stable FP of the dynamics for any (non-negative) \((u_x, u_y)\). If one were to fix values \( u_x, u_y > 0 \) such that the point \((u_x, u_y)\) is located below the line corresponding to \( \Gamma = 1 \) in fig. 1 and to study the system as a function of \( \Gamma \), one would find a stable phase at small values of \( \Gamma \), and an unstable one at large \( \Gamma > \Gamma_c(u_x, u_y) \). At vanishing co-operation pressures \( u_x = u_y = 0 \), the system is always in the unstable phase for any \( \Gamma > -1 \), and marginally stable for \( \Gamma = -1 \).

We now turn to a discussion of the number of NE in the corresponding matrix games, and relate these to the observed dynamical phase transition as described above. We focus on the case \( u_x = u_y = u \), the above transition between the stable and unstable phases then occurs at \( u_c(\Gamma) = (1 + \Gamma)/(2\sqrt{2}) \).

In fig. 2 we depict results \( S(\kappa) \) for the entropy of NE at payoff \( \kappa \), obtained from eq. (15). These curves typically show a maximum \( S_m(u, \Gamma) \) at intermediate values of \( \kappa \).
indicating the number of dominating NE. If \( S_m(u, \Gamma) > 0 \)
a large number of NE is present in the thermodynamic limit at fixed \((u, \Gamma)\), while for \( S_m < 0 \) NE are exponential-
suppressed. If \( S_m = 0 \) a single NE prevails [6]. As

demonstrated for the case \( \Gamma = 0 \) in the main panel of fig. 2,
\( S_m(u, \Gamma = 0) \) is strictly positive at small \( u \) and approaches
zero as \( u \) is increased. A detailed analysis shows that
\( S_m(u, \Gamma > 0) \) for \( u < u_c(\Gamma) \) and \( S_m(u, \Gamma = 0) = 0 \) for \( u > u_c(\Gamma) \),
where \( u_c(\Gamma) \) marks the onset of instability of FP of the RD
as discussed above. Thus the dynamic transition between
stable and unstable regimes coincides with a static one
separating phases with a unique and with many NE,
respectively. This result holds for general \( \Gamma \). We note
that the co-operation pressure \( u \) acts as a negative diagonal
element of the payoff matrices of the corresponding matrix
games, such that the payoff at maximal entropy of the
NE is reduced as \( u \) is increased. The detailed calcu-
lations shows that at the transition \( u = u_c(\Gamma) \), the curves \( S(\kappa, \Gamma) \)
reach their maxima precisely at zero payoff. The inset of
fig. 2 shows numerical data for the entropy \( S(\kappa) \) of NE.

We here use a support enumeration scheme, which we
apply to an equivalent single-population game in order to
reduce the required computing time. Accessible system
sizes are limited to \( N \approx 20 \). While finite-size effects can be
observed and make a direct quantitative comparison of
these spectra of NE with the theoretical results difficult,
the predicted qualitative behaviour of the number of NE
and the separation of the two phases is well confirmed
within the accuracy of the enumeration procedure.

In order to illustrate this transition between a phase
with \( S_m(u, \Gamma) > 0 \) at \( u < u_c(\Gamma) \), and one in which \( S_m = 0 \) at
higher co-operation pressure \( u > u_c(\Gamma) \) further, we depict
the entropy \( S_m \) of the dominating NE as a function of
\( u \) at fixed values of the symmetry parameter \( \Gamma \) in fig. 3.
Comparison of the analytical predictions for the entropy
\( S_m \) of the dominating NE and numerical results from
enumeration reveals a reasonable quantitative agreement.

The saddle point extremisation of (15) allows one to
calculate the statistics of the dominating NE and to
compare with the stationary states of the RE. We here
focus on the diversity parameter \( q^{-1} \) of the eco-system [8].
Large values of \( q^{-1} \) correspond to many surviving species

\[ \frac{1}{q} = \frac{1}{\min_{m=1}^{M} S_m(u, \Gamma)} \]

Fig. 3: (Colour on-line) Entropy \( S_m \) of dominating NE for \( \Gamma = 0 \)
(left curve) and \( \Gamma = 1 \) (right curve). Solid lines from theory,
symbols from enumeration of NE in single-population systems
with \( N = 20 \), averaged over 1000 samples.

Fig. 4: (Colour on-line) Diversity parameter \( q^{-1} \) at \( \Gamma = -1, 0, 1 \)
(left to right). Solid lines are solutions of (12) in the stable
phase, continued into the unstable phase as dashed lines. Open
symbols are from simulations of bi-replicator systems with
\( 2N = 1000 \) species. Dash-dotted curves show theory for Nash
equilibria for \( u < u_c(\Gamma) \), filled symbols are numerical results for
typical NE from iterated Lemke-Howson algorithm applied
to bi-matrix systems of size \( 2N = 100 \), averaged over 200 samples.

(i.e. many pure strategies played with non-zero proba-
bility). Measurements of \( q^{-1} \) in the stationary states of the
RE and numerical results for the NE are shown in fig. 4.
The latter are here obtained using a repeated Lemke-
Howson algorithm [9]. Note that the fixed-point theory
of the RE applies only above \( u_c \) but has been continued
below as dashed lines. Quantitative deviations between
numerical and theoretical results for NE are due to finite-
size or sampling effects and the annealed approximation
of the theory. The analytical and numerical results show that
the diversity of species in the NE coincides with that at
the FP of the RD above \( u_c(\Gamma) \), but that they differ from
each other below the transition\(^1\). Similar results are found
for the fraction of surviving species \( \phi \) as defined above
after eq. (12). The conditions (1)-(4) are necessary but
not sufficient for the stability of FP of the RE so that
stable FP are always NE but not vice versa. Results thus
indicate that NE are statistically distinct from attractors
of the dynamics below \( u_c \) (and more numerous).

Summary and concluding remarks. – Our main
results are the following: for games different from the
zero-sum type (i.e. for \( \Gamma > -1 \) one finds a dynamical
instability of the FP of the two-population replicator
equations at \( u_c(\Gamma) = (1 + \Gamma)/(2\sqrt{2}) \). For \( \Gamma = 1 \) this insta-
bility coincides with an AT-instability of the RS solution
of the statics [10]. Above \( u_c \) all three approaches (dynam-
ics, NE and replica theory where applicable) lead to the

\(^1\)We here note that the analytical theory for the RD, based on
a fixed-point assumption, agrees surprisingly well with numerical
measurements in stationary states of the RE also below \( u_c(\Gamma) \),
where the system does not converge to a stable fixed point, but where,
on the contrary, it may display volatile behaviour for \( \Gamma < 1 \). The
nature of this agreement is at present unclear, but is consistent with
the finding of only weak RSB effects in the model with symmetric
 couplings [10]. One may also speculate whether the volatile system
fluctuates around internal (unstable) fixed-points, and that time-
averaged quantities may hence reasonably well coincide with the
predictions of a fixed-point theory. Further investigations into these
issues seem justified, and are in progress [11].
same order parameters, describing an ergodic state, with a unique NE coinciding with the unique fixed point of the RD (and with the unique extremum of $H$ for $\Gamma = 1$). Below $u_c$ the stationary state is no longer unique, and there is an exponential number of NE. The statistics of the NE differs from those of the stationary states of the dynamics. These observations generalise the results of [6], which were concerned with vanishing co-operation pressure $u$ and where results are studied as a function of the symmetry parameter $\Gamma$. At $u = 0$ the RD is unstable for all $\Gamma > -1$, and marginally stable for zero-sum games. Except for the latter case the system is therefore in the sub-critical phase of the RE, in line with the reported exponential number of NE in [6]. At $\Gamma = -1$ one is precisely at the phase transition and finds a zero-entropy of dominating NE. The introduction of co-operation pressure changes this picture: while the transition occurs at $\Gamma = -1$ at vanishing co-operation pressure, the onset of non-zero entropy of NE and, equivalently, the dynamic instability occurs at values $\Gamma_c(u) < -1$ at fixed finite values of $u > 0$, leading to an extended stable phase (at $-1 < \Gamma < \Gamma_c$) with a unique NE. This phase is absent at zero co-operation pressure.

In conclusion we have investigated two-population replicator systems and matrix games with tools of disordered systems theory, and have computed the number and typical properties of NE and of the fixed points of the corresponding RE. Several extensions of the present work can be considered, including cases of correlations between different rows and/or columns of the payoff matrices, populations with different numbers of strategies at hand and systems with more than two types of players, as well as the study of other stability concepts of evolutionary game theory in the context of random replicator systems.

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