Quantum key distribution overcoming extreme noise: simultaneous subspace coding using high-dimensional entanglement

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High-dimensional entanglement promises to increase the information capacity of photons and is now routinely generated exploiting spatio-temporal degrees of freedom of single photons. A curious feature of these systems is the possibility to certify entanglement despite strong noise in the data. We show that it is also possible to exploit this noisy entanglement by introducing a protocol that uses multiple subspaces of the high-dimensional system simultaneously. Our protocol can be used to increase key rates in realistic conditions. To that end, we conduct two simulations of our protocol for noise models that apply to the two most commonly used sources of high-dimensional entanglement: time-bins and spatial modes.

Quantum communication is one of the most mature areas of quantum technologies, with Quantum Key Distribution (QKD) [1–5] as its most prominent example. QKD is a cryptographic primitive that allows two parties, Alice and Bob, to securely establish a shared secret key in the presence of an adversary, Eve. QKD protocols can be roughly classified according to the variables used for the encoding (discrete versus continuous [6–8]), the presence or absence of entanglement between the legitimate pairs (prepare and measure [1, 9, 10] versus entanglement-based [2, 11, 12]), the dimension of the encoding states (qubit [1, 2, 9, 11] versus qudit [13–28]) versus continuous variable [7, 29–36], and the level of trust that the legitimate parties have on their devices (trusted devices [1, 9, 11] versus partially device independent [37–44] versus fully device independent [45–49]). In this letter we are focusing on trusted-device high-dimensional discrete-variable entanglement-based protocols which are widely implemented by using temporal (time-bin [50–52]) or spatial encoding [53–57].

The majority of practical QKD implementations use binary encoding of quantum states in photons, such as polarisation [58] or time-bin qubits [59]. Entanglement in multiple degrees of freedom, i.e. high-dimensional entanglement [60, 61], has some obvious advantages. Up to \( \log(d) \) bits can be encoded into a single photon using \( d \) degrees of freedom. This potentially addresses one of the biggest challenges of QKD, namely the low rates compared to classical cryptographic protocols. There is, however, another curious feature when turning to entanglement based quantum communication. High-dimensional entanglement features a noise resistance that increases with the coding dimension. This leads to practical ways of distributing entanglement with high-losses, background light or other sources of noise [62, 63]. One of the open problems here is the question, whether this surviving entanglement can actually be useful. In other words, can we still harness the high-dimensional nature of entanglement in situations where noise dominates the signal and qubit based QKD would be impossible? In this letter we affirmatively answer that question and provide a simultaneous subspace coding protocol for QKD and provide extensive noise models for the two paradigmatic sources of high dimensional entanglement to showcase the suitability of the protocol for practical advantages in QKD. The advantage of projecting onto subspaces is twofold: first, a further increase in visibility through subspace post-selection, second, a decrease in the alphabet implies a smaller toll on the key rate in the error correction phase.

The protocol: The general idea of the protocol is to use a genuine \( d \times d \) dimensional entangled quantum system to perform multiple instances of a QKD protocol dividing the outcomes in different subspaces. To this end let \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \) be the joint Hilbert space of the system shared by Alice and Bob, \( A \) and \( B \), with \( \dim(\mathcal{H}) = d \times d \). In principle the choice of the subspaces is arbitrary, but to simplify the notation we will formulate the protocol using multiple subspaces of size \( k \). To this end let \( d = \ell \cdot k \). Then both \( \mathcal{H}_A \) and \( \mathcal{H}_B \) can be divided into \( \ell \) subspaces of size \( k \) as \( \mathcal{H}_A = \mathcal{H}_{A_0} \oplus \cdots \oplus \mathcal{H}_{A_{\ell-1}} \) and \( \mathcal{H}_B = \mathcal{H}_{B_0} \oplus \cdots \oplus \mathcal{H}_{B_{\ell-1}} \).

The QKD protocol, we present next, requires two measurement settings – the computational basis measurement and a measurement in a basis which is mutually unbiased with respect to the computational basis in each subspace of size \( k \). Therefore, let \( \{A_{1x}^m\}_{x=0}^{d-1} \) and \( \{B_{1y}^n\}_{y=0}^{d-1} \) denote the projectors onto computational basis of \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively, and \( \{A_{2x}^m\}_{x=0}^{d-1} \) and \( \{B_{2y}^n\}_{y=0}^{d-1} \) are sums of mutually unbiased measurements in subspaces of size \( k \). Formally, \( A_2^x = U A_1^x U^\dagger \) and \( B_2^y = U^* B_1^y U^\dagger \), where \( U = \sum_{m=0}^{d-1} \sum_{i,j=0}^{k-1} \omega_{k,ij}^m \{|mk+i\rangle \langle mk+j|\} \) and \( \omega_k = e^{2\pi i / k} \). Alice’s measurement outcome \( x = mk+i \) is interpreted
The asymptotic key rate of the Subspace QKD protocol, (Protocol 1), can be calculated in similar fashion and is given by

$$K_{TOT} \geq \sum_{m=0}^{\ell-1} P(M = m)K_m,$$

where \(P(M = m)\) is the probability that both Alice and Bob obtain an outcome in subspace \(m\) and \(K_m\) is the rate in subspace \(m\), given by \(K_m = H(X|\rho_{ET})_{\gamma^m} - H(X'|Y')_{\gamma^m}\), where \(\rho^m\) is the state effectively shared by the parties in the subspace \(m\). (For a precise definition of the state \(\rho^m\), and detailed proof of this result see Appendix A.I).

In order to compute the key rate \(K_m\) for the subspace \(m\) we lower bound the conditional entropy \(H(X'|\rho_{ET})_{\gamma^m}\) by the conditional min-entropy: \(H(X'|\rho_{ET})_{\gamma^m} \geq H_{\text{min}}(X'|\rho_{ET})_{\gamma^m}\), where \(H_{\text{min}}(X'|\rho_{ET})_{\gamma^m} = -\log P^m_{\gamma}\) and \(P^m_{\gamma}\) is the average guessing probability computed using the effective state \(\tilde{\rho}^m\). To determine \(P^m_{\gamma}\) for a subspace \(m\) from measurement results, we use the correlations of Alice’s and Bob’s outcomes in the second basis, expressed as

$$W^m_k = \sum_{i=0}^{k-1} P(i|22, m),$$

where \(P(i|22, m) = \frac{P(x = mk+i, y = mk+i|22)}{P(M = m)}\) is the probability that Alice and Bob obtain equal outcomes when they measure in the basis \(\{A^m_x\}_{x=0}^{d-1}\) and \(\{B^m_y\}_{y=0}^{d-1}\) and obtain outcomes in the subspace \(m\). In Appendix A.II, we present and solve the optimization problem that allows to show that Eve’s guessing probability for the subspace \(m\) can be expressed as a function of the subspace dimension \(k\) and the correlation \(W^m_k\) by:

$$H_{\text{min}}(X'|\rho_{ET})_{\gamma^m} = -\log_2 \left( \frac{\sqrt{W^m_k} + \sqrt{(k-1)(1-W^m_k)}}{k} \right).$$

The conditional entropy \(H(X'|Y')_{\gamma^m}\) for the subspace \(m\) can be estimated using the probability distribution of outcomes in the basis \(\{A^m_x\}_{x=0}^{d-1}\) and \(\{B^m_y\}_{y=0}^{d-1}\).

**Isotropic state example:** Here, we derive the total key rate obtainable in practical QKD setups, where the prepared state is an isotropic state \(\psi_d^\perp(v) = \sqrt{1-v^2} |d\rangle \otimes |d\rangle\) with visibility \(v\). Let us first calculate the asymptotic key rate in case \(k = d\), that is, Alice and Bob use standard QKD and derive the key from the whole available Hilbert space. Applying the test round measurement of both Alice and Bob to \(\rho_d(v)\) leads to \(W_{d} = v + \frac{1-v}{d^2}\) and thus via equation (3)

$$H(X|\rho_{ET}) \geq -\log_2 \left( \frac{\sqrt{v^2 + 1 - v^2} + (d-1)\sqrt{1-v^2}}{d^2} \right).$$

To determine \(H(X'|Y')\), we observe that, given \(\rho_d(v)\), the probability distribution for Alice obtaining result \(x\) and Bob obtaining result \(y\) in the key rounds is given by \(P_{key}(xy) = \)
\( \rho_{d,v,k} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle\langle ii| \) and \( v \) is the respective conditional probability distribution is \( P_{\rho_{d,v,k}}(x|y) = \delta_{xy} + \frac{1}{d} \).

Similar calculations can be done in case Alice and Bob perform Protocol 1 with subspaces \( \mathcal{H}_{A_k} \otimes \mathcal{H}_{B_k} \) of size \( k \times k \). In such a case they effectively measure the state \( \rho_{d,v}^{(\tilde{v})} \) in each subspace \( m \), which can obtained by projecting \( \rho_d(v) \) onto this subspace. Because of the symmetry of \( \rho_d(v) \), the state \( \rho_{d,v}^{(\tilde{v})} \) is independent of \( m \) and its density matrix is equivalent to \( \rho_{d,v}^{(\tilde{v})} = |\tilde{v}\rangle\langle \tilde{v}| + \frac{1}{d} \mathbb{1}_k \otimes \mathbb{1}_k \), where \( |\tilde{v}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle \) and \( v = v(d,v,k) := vd/(vd + k - v k) \).

For each subspace \( \mathcal{H}(X'|E_T)_{\tilde{p}^{(\tilde{v})}} \) and \( \mathcal{H}(X'|Y')_{\tilde{p}^{(\tilde{v})}} \) we can now set \( \tilde{p}^{(\tilde{v})} = \tilde{p}_k(\tilde{v}) \). Test round measurements of this state lead to \( W_k^{m} = \frac{vd + 1 - v}{vd + k - v k} \), which (using Eq. (3)) results in:

\[
H(X'|E_T)_{\tilde{p}^{(\tilde{v})}} \geq -\log_2 \left( \frac{(vd+1-v)+(k-1)(1/v-1)}{vd+k-vk} \right) \geq -\log_2 \left( \frac{vd+1-v+k-1}{vd+k-vk} \right). \tag{4}
\]

Evaluating also \( H(X'|Y')_{\tilde{p}_k(\tilde{v})} \) and summing over all subspaces leads to:

\[
K^{iso}_{TOT}(d,v,k) \geq \left( \frac{vd + k - v k}{d} \right) \log_2 \left( \frac{k}{(vd+1-v)+(k-1)(1/v-1)} \right)
+ \left( \frac{vd + 1 - v}{d} \right) \log_2 (vd + 1) - (k-1) (1/v - 1). \tag{5}
\]

For each \( d \) and \( v \), which are usually known experimental parameters, one can optimize \( K^{iso}_{TOT}(d,v,k) \) over the subspace size \( k \) to determine the protocol implementation that leads to the optimal key rate. Another interesting property one can investigate is the critical visibility, i.e., at which visibility one cannot obtain a positive key rate anymore. This is generally a rather complicated function of both \( k \) and \( d \). However, considering even \( d \) and \( k = 2 \), it can be shown that positive key rate can be obtained for \( v \geq 1/(1 + d t_b T_B \gamma^{-1}) \).

In particular, in case of constant \( v \), one can always obtain a positive key rate by increasing the global dimension \( d \). This is in accordance with the intuition – the robustness of entanglement in the isotropic state increases with the dimension. In practice, however, the visibility \( v \) is not a constant, but rather a function of the dimension \( d \). Moreover, this function crucially depends on the physical implementation of qubits. To make the discussion complete, in what follows we provide dimension dependent noise models for the two most common ways to produce and control high-dimensional quantum states – temporal and spatial degrees of freedom in photons.

**Realistic noise models:** When it comes to realistic noisy implementations, the different behavior of high-dimensional systems and many copies of a two-dimensional system stems from the way the noise interacts with the information carrier. This interaction depends on the implementation of the high-dimensional system. Our work aims at showcasing that high-dimensional entanglement can be advantageous for QKD purposes, and to this end, we study two different state-of-the-art implementations of our protocol. The first employs
We calculate the visibility to be

\[ V = \frac{\gamma}{e^{\frac{\gamma}{d}} - 1 + d\left[1 - e^{-\left(\mu^A + \frac{\gamma}{d}\right)}\left[1 - e^{-\left(\mu^B + \frac{\gamma}{d}\right)}\right]\right]} \quad \text{and the rate of post-selected rounds is given by} \]

\[ R(d,v) = C e^{-d(\mu^A + \mu^B)} e^{\frac{\gamma}{d}} \left(\frac{\mu^A + \mu^B}{d}\right) \]

\[ \times \left\{ d^2 \left[1 - e^{-\mu^A - \frac{\gamma}{d}}\right]\left[1 - e^{-\mu^B - \frac{\gamma}{d}}\right] + d \left(e^{\frac{\gamma}{d} - 1}\right) \right\} \]

where \( \mu^A/B \) is the average number of dark counts per detector, \( \gamma \) is the average number of uncorrelated photons due to the environment, losses and detector inefficiencies, \( \gamma \) is the average number of detectable correlated photons, and finally \( C \) is a related, also dimension-independent, parameter (for details see the calculations in Appendix A.IV). We can now express the achievable key rate in \( \text{bits}/\text{s} \) as

\[ K = R(d,v)K_{\text{TOT}}^{\text{iso}}(d,v,k) \]

In Figure 2, we plot \( K \) vs the total dimension \( d \) for different choices of the subspace size \( k \). We observe that the key rate increases as the dimension \( d \) increases, but only up to a certain limit, and for different \( d \) the choice of the optimal [71] subspace size is different. Importantly enough, this plot clearly shows the advantage of using subspace instead of full-space encoding: notice that the optimal key rate in the case of full-space encoding is an order of magnitude smaller than the key rate that can be achieved with an appropriate choice of subspace encoding.

Moving on, to photons entangled in spatial degrees of freedom. Due to spatial symmetry, the state produced by the laser source should be of the form \( \Psi = \sum_{l=-\infty}^{\infty} c_l |l\rangle_A |l\rangle_B \), where \( l \) denotes momentum modes and \( c_l \) depends on the source specifications. This state is subsequently projected in a space spanned by a finite subset of modes, \( l \), with cardinality \( d \); our encoding space, hence arises from an effective discretization with respect to the finite resolution of the detectors. For details, see [70] and our Appendix A.IV. In our noise model, we consider noise effects originating from losses, environmental photons, detector inefficiencies and dark counts. For the key rate, Alice and Bob post-select the rounds in which they both obtained one click, and just like in the previous implementation, the visibility \( v \) includes the rounds where the clicks came from a source photon pair. Hence, in this implementation each party needs a detector for each mode, resulting in dark counts contributing the most to noise through more frequent accidental coincidences.

We calculate the visibility to be

\[ V = \frac{\gamma}{e^{\frac{\gamma}{d}} - 1 + d\left[1 - e^{-\left(\mu^A + \frac{\gamma}{d}\right)}\left[1 - e^{-\left(\mu^B + \frac{\gamma}{d}\right)}\right]\right]} \quad \text{and the rate of post-selected rounds is given by} \]

\[ R(d,v) = C e^{-d(\mu^A + \mu^B)} e^{\frac{\gamma}{d}} \left(\frac{\mu^A + \mu^B}{d}\right) \]

\[ \times \left\{ d^2 \left[1 - e^{-\mu^A - \frac{\gamma}{d}}\right]\left[1 - e^{-\mu^B - \frac{\gamma}{d}}\right] + d \left(e^{\frac{\gamma}{d} - 1}\right) \right\} \]

where \( \mu^A/B \) is the average number of dark counts per detector, \( \gamma \) is the average number of uncorrelated photons due to the environment, losses and detector inefficiencies, \( \gamma \) is the average number of detectable correlated photons, and finally \( C \) is a related, also dimension-independent, parameter (for details see the calculations in Appendix A.IV). We can now express the achievable key rate in \( \text{bits}/\text{s} \) as

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Conclusions: We have presented a simultaneous k-dimensional subspace QKD protocol for d-dimensional quantum systems. Using two noise models for the most paradigmatic platforms for photonic high-dimensional entanglement, we showcase that the protocol can indeed provide a viable pathway towards practically improved QKD. Most of the improvement comes from the fact that subspaces of high-dimensional systems are more resilient to physical noise [62], compared to direct coding in a comparable dimension. Surprisingly, the optimal subspace size even in noisy scenarios often goes beyond two dimensions.

The actual value of achievable key rates highly depends on the implementation and specific noise parameters, from dark counts and background to losses and device fidelities. To showcase the capabilities of the proposed protocol, we have chosen two of the most paradigmatic state-of-the-art experimental implementations [62]. For all parameter ranges, high-dimensional encodings led to improved key rates. We believe that the noise models we provide, together with the SDPs for computing key rates of our protocol, will be useful for optimising system parameters for a broad family of future setups.

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APPENDIX

A.I. Key rates of the subspace QKD protocol

In this appendix we prove that the key rate of the Subspace-QKD protocol, Protocol 1, can be computed using the key rates of each subspace, as stated in the following theorem.

Theorem 1. The asymptotic key rate of the subspace-QKD protocol, Protocol 1, is given by

\[
K_{TOT} \geq \sum_{m=0}^{\ell-1} p(M = m) (H(X'|E)_{\tilde{\rho}^m} - H(X'|Y')_{\tilde{\rho}^m}), \tag{A.1}
\]

where the conditional entropies are evaluated on the states \(\tilde{\rho}^m_{X'Y'E}\) given by

\[
\tilde{\rho}^m_{X'Y'E} = (\mathcal{E}_{X'Y'E\leftarrow AB}^M \otimes \id_E)(\tilde{\psi}_{ABE}^m)
\]

and \(\tilde{\psi}_{ABE}^m\) is the purification of the state

\[
\rho_{AB}^m = \frac{\Pi_A^m \otimes \Pi_B^m (\rho_{AB}) \Pi_A^m \otimes \Pi_B^m}{p(M = m)} \tag{A.3}
\]

with

\[
p(M = m) = \text{Tr}(\Pi_A^m \otimes \Pi_B^m \rho_{AB}). \tag{A.4}
\]

Proof. The protocol explores multiple subspaces of size \(k\), where \(d = \ell \cdot k\). Then both \(\mathcal{H}_A\) and \(\mathcal{H}_B\) can be divided into \(\ell\) subspaces of size \(k\) as \(\mathcal{H}_A = \mathcal{H}_{A_0} \oplus \cdots \oplus \mathcal{H}_{A_{\ell-1}}\) and \(\mathcal{H}_B = \mathcal{H}_{B_0} \oplus \cdots \oplus \mathcal{H}_{B_{\ell-1}}\).

The QKD protocol involves two measurement settings: \(\{A^x_i\}_{x=0}^{d-1}\) and \(\{B^y_j\}_{y=0}^{d-1}\) denote the computational bases of \(\mathcal{H}_A\) and \(\mathcal{H}_B\), respectively, and \(\{A_2^x\}_{x=0}^{d-1}\) and \(\{B_2^y\}_{y=0}^{d-1}\) are sums of mutually unbiased measurements in subspaces of size \(k\). Formally, \(A_2^x = U A^x_i U^\dagger\) and \(B_2^y = U^* B^y_j U^\top\), where \(U = \sum_{m=0}^{\ell-1} \sum_{i,j=0}^{k-1} \omega_k^{ij} \ket{mk+i} \bra{mk+j}\) and \(\omega_k = e^{2\pi i/k}\). Note that the two measurements of the parties are block diagonal with blocks of size \(k\). Therefore, Alice’s measurement outcome \(x = mk+i\) is interpreted as outcome \(i\) in the \(m\)-th subspace. Bob’s measurement outcomes are interpreted analogously.

The subspace-QKD protocol, given in Protocol 1, can be divided in the following maps

\[
\mathcal{E}_{KAKBE'\leftarrow ABE}^{Subspace\leftarrow QKD} = \mathcal{E}_{K_AK_BE'\leftarrow ABE}^{IR-P\leftarrow K_AK_BE'} \circ \mathcal{E}_{X'Y'E\leftarrow XYE}^{Sub} \circ (\mathcal{E}_{X'\leftarrow AB}^M \otimes \id_E)(\rho_{ABE}). \tag{A.5}
\]

where \(\mathcal{E}_{X'Y'\leftarrow AB}^{M} \otimes \id_E\) represents the measurements implemented in Step 2, \(\mathcal{E}_{X'Y'E'\leftarrow XYE}^{Sub}\) corresponds to the subspace selection implemented in Step 3, and finally \(\mathcal{E}_{KAKBE'\leftarrow XYE}^{IR-P\leftarrow K_AK_BE'}\) describes the classical post-processing applied to the raw key consisting of information reconciliation and privacy amplification.

The difference between the subspace-QKD protocol, Protocol 1, and a standard high-dimensional QKD protocol is in the subspace selection described in Step 3. The selection of subspaces, Step 3 in Protocol 1, corresponds to Alice and Bob applying a local projection described by \(\{\Pi_{m}^i\}_{m=0}^{\ell-1}\) with elements

\[
\Pi_{m} = \sum_{i=0}^{k-1} \ket{mk+i} \bra{mk+i}, \tag{A.6}
\]

and selecting the cases where both get the same outcome. This step resembles the procedure called advantage distillation that has been studied in the classical setting [72] as well as in QKD protocols [73–75]. Indeed the subspace selection in Step 3 aims to select the rounds in which higher correlations between Alice and Bob are observed. This is exactly the objective of advantage distillation procedures such as the ones studied in [73–75]. However the procedures studied in [73–75] consist of processing several rounds together (corresponding to a joint quantum operations in several copies of the state) while the procedure given by Step 3 only involve one copy of the state (and corresponds to a single copy filter operation).
Now, note that the map implemented in Step 3 is a projection into subspaces, and this operation commutes with both measurements of Alice and Bob. Therefore we can, instead, describe the QKD protocol as first Alice and Bob perform the projection and then proceed with the measurements in the resulting state

\[
\rho_{\text{SubspaceQKD}} = \mathcal{E}_{KAB}^{IR-P A}(\rho_{A B E}) = \mathcal{E}_{KAB}^{IR-P A}(\rho_{A B E}) \circ (\mathcal{E}_{X Y^I \rightarrow A'B'} \otimes i d_E) \circ \mathcal{E}_{\text{Sub}}^{ABE}(\rho_{A B E}).
\]  

(A.7)

So in order to perform the security analysis we will use this alternative description. Our goal is to determine the state shared by the parties after the action of the map \(\mathcal{E}_{\text{Sub}}^{ABE}(\rho_{A B E})\). Initially, Eve distributes a state \(\rho_{A B}\) to Alice and Bob such that she holds a purification of it

\[
\rho_{A B} = \text{Tr}_E |\psi_{A B E}\rangle\langle \psi_{A B E}|.
\]  

(A.8)

Applying the map \(\mathcal{E}_{\text{Sub}}^{ABE}(\psi_{A B E})\) leads to

\[
\mathcal{E}_{\text{Sub}}^{ABE}(\rho_{A B E}) = \rho_{A' B' E M}
\]  

(A.9)

where the register \(M\) records the result of the projection, i.e. \(M\) takes value \(m\) if \(m_A = m_B = m\), and \(M = \perp\) otherwise:

\[
\rho_{A' B' E M} = \sum_{m=0}^{\ell-1} p(M = m) \rho^{m}_{A B E} \otimes |m\rangle \langle m|_M + p(M = \perp) |\perp\rangle \langle \perp|_A \otimes \rho_E \otimes |\perp\rangle \langle \perp|_M
\]  

(A.10)

where

\[
\rho^{m}_{A B E} = \frac{\Pi^m_A \otimes \Pi^m_B \otimes I_E (|\psi_{A B E}\rangle \langle \psi_{A B E}|) \Pi^m_A \otimes \Pi^m_B \otimes I_E}{p(M = m)}
\]  

(A.11)

and \(p(M = m)\) is the probability that Alice and Bob get outcome \(m\) in the projection, given by

\[
p(M = m) = \text{Tr}(\Pi^m_A \otimes \Pi^m_B \otimes I_E |\psi_{A B E}\rangle \langle \psi_{A B E}|).
\]  

(A.12)

Now, Alice and Bob will perform measurements on the state \(\rho_{A' B' E M}\), conditioned on the information available to the eavesdropper is given by

\[
H(X'|EM)_{\rho_{X'EM}} = \sum_{m=0}^{\ell-1} p(M = m) H(X|EM = m)_{\rho_{X'EM}}
\]  

(A.14)

\[
= \sum_{m=0}^{\ell-1} p(M = m) H(X'|E)_{\rho^{m}_{X'E}}
\]  

(A.15)

\[
\geq \sum_{m=0}^{\ell-1} p(M = m) H(X'|E)_{\tilde{\rho}^{m}_{X'E}}
\]  

(A.16)

The first equation follows from the properties of conditional von Neumann entropy for cq-states. In the last step we consider the entropy evaluated on the state that results from Alice and Bob applying the measurements to \(\tilde{\rho}_{A' B' E M}\), where

\[
\tilde{\rho}_{A' B' E M} = \sum_{m=0}^{\ell-1} p(M = m) \tilde{\rho}^{m}_{A B E} \otimes |m\rangle \langle m|_M + p(M = \perp) |\perp\rangle \langle \perp|_A \otimes \rho_E \otimes |\perp\rangle \langle \perp|_M
\]  

(A.17)

and

\[
\tilde{\rho}^{m}_{A B E} = \text{Tr}_E |\psi^{m}_{A B E}\rangle\langle \psi^{m}_{A B E}|.
\]  

(A.18)
where $|\psi_{ABE}^m\rangle$ is the purification of $\rho_{AB}^m$. Giving Eve the purification of $\rho_{AB}^m$ before the measurements only increases her power, which proves the lower bound. Similarly for the required information to be exchanged for information reconciliation

$$H(X'|Y'M)_{\rho_{X'Y'M}} = \sum_{m=0}^{\ell-1} p(M = m) H(X'|Y'M = m)_{\rho_{X'Y'M}}$$  \hspace{1cm} (A.19)

$$= \sum_{m=0}^{\ell-1} p(M = m) H(X'|Y')_{\rho_{X'Y'}^m}$$  \hspace{1cm} (A.20)

$$= \sum_{m=0}^{\ell-1} p(M = m) H(X'|Y')_{\tilde{\rho}_{X'Y'}^m}$$  \hspace{1cm} (A.21)

and the last step follows from the fact that $\tilde{\rho}_{AB}^m = \rho_{AB}^m$.

\[\square\]

### A.II. Solution of the SDP and the choice of $W$

In this appendix, we present in detail the optimization problem for calculating the average guessing probability of Eve, and its solution. The average guessing probability is obtained by maximizing, over all possible tripartite states $\rho_{ABE}$ (recall that Eve holds a purification of $\rho_{ABE}$) and all possible measurements of Eve $\{E^e\}_e$, the probability $P_g$ that Eve’s guesses Alice’s outcomes, and then performing a weighted average of these probabilities:

$$P_g = \max_{\rho_{ABE}, \{E^e\}_e} \sum_{e,y} \text{Tr} (\rho_{ABE} A^e_y \otimes B^e_y)$$

s.t. $\text{Tr} (\hat{W} \rho_{ABE}) = W,$

$\rho_{ABE} \geq 0,$

$\text{Tr} (\rho_{ABE}) = 1,$

$E^e \geq 0 \ \forall e,$

$\sum_e E^e = 1,$

(A.22)

where $A_1, B_1$ stand for the computational basis and $\hat{W}$ is the yet-to-be-defined operator, with $W$ its measured value that constrains the optimization. $W$ will be constructed depending on which target state the experiment is trying to produce. Here, we consider $W$ to be the average value of equal outcomes that Alice and Bob get in the second basis, $W = \sum_{xx} P(xx|22)$, which is the average value of the operator $\hat{W} = \sum_{xx} A^x_1 \otimes B^x_2$. Our goal now is to express the guessing probability as a function of $W$ and $d$, by solving the following optimization problem which we obtain from the previous one by substituting $\rho_e := \text{Tr}_E (\rho_{ABE} E^e)$:

$$P_g(W,d) = \max_{\{\rho_e\}_e} \sum_{e=0}^{d-1} \text{Tr} (\rho_e A^e \otimes B^e)$$

s.t. $\text{Tr} (\hat{W} \sum_{e=0}^{d-1} \rho_e) = W,$

$\rho_e \geq 0 \ \forall e \in \{0, \ldots, d-1\},$

$\text{Tr} (\sum_{e=0}^{d-1} \rho_e) = 1,$

whose dual is

$$\min_{\gamma, S} \gamma + SW,$$

s.t. $\gamma 1_d + S \hat{W} \geq |e\rangle \langle e| \otimes 1_d \ , \forall e.$
The eigenvalues of $\hat{S} - |e\rangle\langle e| \otimes 1_d$, as a function of $S$ and the local dimension $d$, for a given $e$ are $\lambda_{\pm} = \frac{S - 1 \pm \sqrt{(S-1)^2 + 4S(d-1)/d}}{2}$, and the optimization now reads:

$$\min_{\gamma, S} \gamma + SW,$$

s.t. $\gamma + \lambda_{-} \geq 0 \ \forall e,$

$$\gamma + \lambda_{+} \geq 0 \ \forall e.$$

Since $\lambda_{+} \geq \lambda_{-}$, for all $S, d$, and $\lambda_{\pm}$ are the same for all $e$, we can relax the constraints to:

$$\min_{\gamma, S} \gamma + SW,$$

s.t. $\gamma + \lambda_{-} \geq 0$.

We finally solve $\frac{\partial}{\partial S} \lambda_{-} = W$, which gives

$$W = \frac{1}{2} \left( 1 - \frac{2S + 2 - 4/d}{2\sqrt{(S-1)^2 + 4S(d-1)/d}} \right), \quad S = \frac{2}{d} - 1 + \frac{1 - 2W}{d} \sqrt{\frac{d-1}{W(1-W)}}$$

and obtain the form of the guessing probability as a function of $W$ and $d$:

$$P_g(W, d) = -\lambda_{-} + SW = \frac{(\sqrt{W} + \sqrt{(d-1)(1-W)})^2}{d}.$$

### A.III. Implementation with temporal degrees of freedom

We start by considering a hyper-entangled state of the form

$$|\Psi\rangle = |\phi^-\rangle_{AB} \otimes \int dt f(t) |t\rangle_A \otimes |t\rangle_B . \quad (A.23)$$

This is the state of two entangled photons (one for Alice and one for Bob), with two degrees of freedom: the time of arrival $t$ of the photon at the respective labs of Alice and Bob and their polarization. The time of arrival, i.e., the time at which the detector clicks, is a continuous variable, which can be discretized by considering time bins of size $t_b$. Setting a time frame $F$ outside of which a photon is “lost” and taking $F$ to be a multiple of $t_b$ we have effectively a discrete system of dimension $d = \frac{F}{t_b}$.

The probability that at the frame $[0, F]$ exactly $n$ pairs of entangled photons are produced is given by the Poisson distribution:

$$P_F(n) = \frac{(\lambda F)^n e^{-\lambda F}}{n!},$$

with $\lambda$ being the production rate of the photon pairs. Both $F$ and $\lambda$ are tunable parameters, and we assume that $\lambda$ is small enough, such that multi-photon events in the same frame are negligible. In particular, we choose $\lambda$ such that

$$P_F(n \geq 2) = 1 - P_F(0) - P_F(1) = 1 - (1 + \lambda F) e^{-\lambda F} < \epsilon.$$

The average number of photons per frame is $\lambda F$ and for $\lambda F < 0.2$ we get $P_F(n \geq 2) < 0.015$, which is small enough with respect to the noise scale in our model. Note, though, that by decreasing the production rate $\lambda$ we are also decreasing the key rate, see Equation (A.25) at the end of this section, therefore we should tune these parameters carefully.

We consider two types of noise, namely the noise due to the interaction of the photons with the environment before entering the labs of Alice and Bob, and the noise introduced due to the detectors’ inefficiency. Because of its interaction with the environment, a photon can be lost with probability $P_L$. Given $n$ photons, the probability that $n_L$ of them are lost while the rest arrive at the lab is $P_L^{n_L} (1 - P_L)^{n - n_L} \binom{n}{n_L}$.

Moreover, photons from the environment may be introduced in the system. We assume that the environment produces on average $\nu$ photons per second. The number of photons arriving to the frame from the environment will then follow the Poisson distribution, $P_E(n) = (\nu F)^n e^{-\nu F} / (n!)$.
As far as the noise due to the detectors is concerned, each detector has probability \( P_C \) to click when a photon arrives, and probability per second \( \mu \) to click when no photon is there. These events are called dark counts and they also follow the Poisson distribution, \( P_D(n) = (\mu F)^n e^{-\mu F} / (n!) \).

The probability that both Alice and Bob receive in their labs \((i, j)\) photons in a time frame \( F \) is

\[
P(i, j) = \sum_{n=0}^{\infty} \sum_{n_1 = \max\{n-i, 0\}}^{n} \sum_{n_2 = \max\{n-j, 0\}}^{n} P_F(n) P_L^{n_1} (1-P_L)^{n-n_1} \binom{n}{n_1} P_L^{n_2} (1-P_L)^{n-n_2} \binom{n}{n_2} P_E(i-n+n_1) P_E(j-n+n_2).
\]

Given that \( i \) photons enter, the probability of obtaining exactly one click in a frame \( F \) is

\[
P(\text{click}|i) = (1 - P_C)^{-1} (P_C P_D(0)i + (1 - P_C) P_D(1)) = e^{-\mu F} (1 - P_C)^{i+1} \left( \frac{i P_C}{1 - P_C} + \mu F \right),
\]

and the probability that both Alice and Bob get one click is \( P(11) = \sum_{i,j=0}^{\infty} P(\text{click}|i) P(\text{click}|j) P(i, j) \).

After applying the approximation \( P_F(n \geq 2) \approx 0 \), we can calculate \( P(11) \) to be:

\[
P(11) \approx e^{-F[2(\mu + \nu P_C) + \lambda]} F \beta,
\]

with \( \beta := \left[ \lambda \alpha^2 + F(\mu + \nu P_C)^2 \right] \) and \( \alpha := \left[ P_C(1 - P_L) + F(\mu + P_C \nu) P_L + F(\mu + \nu P_C)(1 - P_C)(1 - P_L) \right] \).

In the above expressions note that, if only a photon pair is produced, the probability that it passes and gets detected is \( P_C(1 - P_L) \), the probability that it passes but does not get detected is \( (1 - P_C)(1 - P_L) \), and the probability that it gets lost is \( P_L \), and an environment photon or a dark count is making the click instead with probability \( F(\mu + \nu P_C) \).

The term \( e^{-2(\mu + \nu P_L) F} \) is the probability that all the extra photons of the environment are not detected and there are no dark counts. If there is no pair in the frame, the click must have come from the environment or it is a dark count.

If the setup is asymmetric (one detector is close to the source, the other is far), we can modify the formula to include different parameters for Alice and Bob:

\[
P(11) \approx e^{-\left( \mu^A + \mu^B + \nu \alpha^A P_C^A + \nu \alpha^B P_C^B + \lambda \right) F} \left[ \lambda \alpha^A \alpha^B + F(\mu^A + \nu \alpha^A P_C^A)(\mu^B + \nu \alpha^B P_C^B) \right].
\]

In our noise model, we do not consider finite size effects (the number of rounds is sufficiently large), neither border effects on the frame (\( F \) is sufficiently large, so the error of the clock that decides when the frame begins and ends is negligible), nor errors related to the relaxation time of the detectors (which is the time a detector needs before being able to detect another photon. If the frame were approximately the same size as the relaxation time this effect would be important, but we choose \( F \) to be sufficiently large for this purpose). We also assume that the interaction with the environment can only destroy a photon, and that the photon pairs coming from the environment are uncorrelated. Furthermore, \( F \) and the production rate \( \lambda \) are chosen such that the probability of observing two or more entangled photons during a single frame is negligible. With these assumptions, we have a model that gives us the rate of "valid" rounds per second, as a function of \( F \), which, in turn, is proportional to the local dimension \( d \):

\[
R(d) = P(11)/F = e^{-F[2(\mu + \nu P_C) + \lambda]} \beta.
\]  

For large \( F = dt_b \), we have \( \beta \approx F(\mu + \nu P_C)^2 [(1 + P_C P_L - P_C)^2 + \lambda F] \propto F \).

We can also estimate the visibility, i.e. the probability that – given that both parties had exactly one click – the photons that clicked were the entangled ones coming from the laser source and not the environment or dark counts. First, we calculate the probability of a photon pair to survive and get detected

\[
P_S = P_F(1 - P_L)^2 P_C^2 e^{-2(\mu + \nu P_C)},
\]

which gives the visibility as a function of the dimension \( d \) to be \( v(d) = P_S / P(11) = \lambda(1 - P_L)^2 P_C^2 / \beta \), while for an asymmetric setup we have \( v(d) = \lambda(1 - P_L^A)^2(1 - P_L^B)^2 P_C^A P_C^B / \beta \).

Finally, for large \( d, \lambda \propto 1/F \) and \( \beta \propto F \), thus making the visibility scale as \( d^{-2} \).

In order to take into account multi-photon events, we write \( \alpha \) and \( P(11) \) as

\[
\alpha(n) = (1 - P_C + P_C P_L)^{n-1} \left[ nP_C(1 - P_L) + F(\mu + \nu P_C)(1 - P_C + P_C P_L) \right],
\]

\[
P(11) = e^{-\left( \mu^A + \mu^B + \nu \alpha^A P_C^A + \nu \alpha^B P_C^B \right) F} \sum_{n=0}^{\infty} P_F(n) \alpha^A(n) \alpha^B(n),
\]
and obtain

$$P(11) = e^{-F(T_A + T_B + \gamma)}(F^2T_AT_B + F\gamma) \quad \text{and} \quad R = e^{-F(T_A + T_B + \gamma)}(FT_AT_B + \gamma), \quad (A.25)$$

where $S = 1 - P_C + P_CP_L$, $Q = \mu + \nu P_C$, $T_{A/B} = Q_{A/B} + \lambda S_{B/A}(1 - S_{A/B})$ and $\gamma = \lambda(1 - S^A)(1 - S^B)$.

Accordingly, the generalized success probability becomes

$$P_S = e^{-F(Q^A + Q^B)}\sum_{n=0}^{\infty} P_F(n)(S^A S^B)^n - 1 (1 - S^A)(1 - S^B)n = e^{-F(T_A + T_B + \gamma)\gamma F}.$$ 

The maximum of $P_S$ is for $F = 1/(T_A + T_B + \gamma)$, and the visibility becomes $v = 1/(1 + FT_AT_B\gamma^{-1})$.

A.IV. Implementation with spatial degrees of freedom

A basic parameter of our model is $\Delta t$, the coincidence window in which two events, for Alice and Bob, are considered coincident. Note that multiple clicks in the same coincidence window are treated as a single event. Another parameter is related to the projection of an infinite-dimensional entangled state of the form $|\Psi\rangle = \sum_{l=-\infty}^{\infty} c_l |l\rangle_A \otimes |l\rangle_B$ into a finite dimensional space. This is the probability $P_P(d) := \text{Tr}(1_d^2 |\Psi\rangle\langle\Psi| 1_d^2)$, which we assume to be constant, thus providing lower dimensions with an advantage. One could give an advantage to higher dimensions by dropping this assumption.

![Diagram](https://example.com/diagram.png)

**FIG. A.1.** A schematic representation of the noise model: a laser source (a) produces entangled pairs distributed in time with a Poisson distribution, with $\lambda$ as the average number of photons per second. The pairs are distributed to the parties and suffer from party-dependent losses (b) with probability $P_L$. On top of the entangled photons the parties receive $\nu$ environmental photons per second on average, distributed as well with a Poissonian. For each of the modes that are being measured, there is an associated detector (d), which comes with an average number of dark counts $\mu$ per second and an efficiency $P_C$.

In our model, we consider that a click can come either from the laser or from the environment or from the dark counts. We start with the laser photons, by assuming that they follow a Poisson distribution, factorized by the probability $P_P(d)$ of being within the modes $-d/2$ and $d/2$. The probability that the laser produces $j$ detectable photons given that $n$ in total are emitted is $$(\Delta t \lambda)^n \frac{e^{-\Delta t \lambda}}{n!} P_P^J(d)(1 - P_P(d))^{n-j} \binom{n}{j}.$$ Given $j$ photons produced from
the laser, the probability they produce no click in one of the labs is

\[ P(0|j) = \sum_{r=0}^{j} P_L^{j-r}(1-P_L)^r \left( \begin{array}{c} j \\ r \end{array} \right) (1-P_C)^r \left( \begin{array}{c} r \\ 0 \end{array} \right) = [1-P_C(1-P_L)]^j = (1-T)^j, \] where \( T = P_C(1-P_L) \),

while the probability that they produce one or more clicks in a single detector is

\[ P(1|j) = \sum_{r_1=1}^{j} P_L^{j-r_1}(1-P_L)^{r_1} \left( \begin{array}{c} j \\ r_1 \end{array} \right) \sum_{r_2=1}^{r_1} \frac{(d-1)^{r_1-r_2}}{d^{r_2}} (1-P_C)^{r_1-r_2} \left( \begin{array}{c} r_1 \\ r_2 \end{array} \right) \left( \begin{array}{c} d \\ 1 \end{array} \right) \left( \begin{array}{c} r_2 \\ r_1 \end{array} \right) = d \left[ (1-T + \frac{T}{d})^j - (1-T)^j \right]. \]

In the above, \( P_L \) reflects the losses affecting the entangled photons, and we further assumed that all modes suffer the same losses, therefore we absorbed them in \( P_C \). In case one would like to further refine the noise model, they could consider different losses for different modes. We also calculate the probability that, given \( j \) photons were produced, Alice and Bob both get one click from a laser photon in different detectors, as this way we account for entangled photons. We have

\[ P(\neq j) = d(d-1) \sum_{r_1, r_2, r_3=0}^{j} \sum_{r_0=0}^{j-1} \frac{j!}{r_0!r_1!r_2!r_3!} \times (P_L^AP_L^B)^{r_0}[P_L^B(1-P_L^A)]^{r_1}[P_L^A(1-P_L^B)]^{r_2}[(1-P_L^A)(1-P_L^B)]^{r_3} \times \]

\[ \times \sum_{l_1=0}^{r_1} \left( \frac{d-1}{d} \right)^{r_1-l_1} (1-P_C)^{r_1-l_1} \left( \begin{array}{c} r_1 \\ l_1 \end{array} \right) \left( \begin{array}{c} d-1 \\ d \end{array} \right) \sum_{l_2=0}^{r_2} \left( \frac{d-1}{d} \right)^{r_2-l_2} (1-P_C)^{r_2-l_2} \left( \begin{array}{c} r_2 \\ l_2 \end{array} \right) \left( \begin{array}{c} d-1 \\ d \end{array} \right) \times \]

\[ \times \sum_{s_1+p_1+q_1=0}^{r_3} \sum_{s_2+p_2+q_2=0}^{r_3} \left( \begin{array}{c} d-2 \\ d \end{array} \right)^{q_3} \left( \frac{d}{d} \right)^{s_3+p_3+q_3} (1-P_C)^{(s_3+p_3+q_3)} \times [1-(1-P_C)^{(l_1+s_1+l_2)}[1-(1-P_C)^{(l_2+l_3)}] = \]

\[ = d(d-1) \left[ (1-T^A)(1-T^B)^j + \left( 1-T^A \right) \left( 1-T^B \right) + \frac{1}{d}[T^A(1-T^B) + T^B(1-T^A)]^j \right] - \]

\[ - \left( 1-T^B \right) \left( 1-T^A + T^A/d \right)^j - \left( 1-T^A \right) \left( 1-T^B + T^B/d \right)^j \right]. \]

Similarly, the probability that, given \( j \) photons, Alice and Bob both get one click from a laser photon in the same detector is

\[ P(= j) = d \sum_{r_1, r_2, r_3=0}^{j} \sum_{r_0=0}^{j-1} \frac{j!}{r_0!r_1!r_2!r_3!} \times (P_L^AP_L^B)^{r_0}[P_L^B(1-P_L^A)]^{r_1}[P_L^A(1-P_L^B)]^{r_2}[(1-P_L^A)(1-P_L^B)]^{r_3} \times \]

\[ \times \sum_{l_1=0}^{r_1} \left( \frac{d-1}{d} \right)^{r_1-l_1} (1-P_C)^{r_1-l_1} \left( \begin{array}{c} r_1 \\ l_1 \end{array} \right) \sum_{l_2=0}^{r_2} \left( \frac{d-1}{d} \right)^{r_2-l_2} (1-P_C)^{r_2-l_2} \left( \begin{array}{c} r_2 \\ l_2 \end{array} \right) \left( \begin{array}{c} d-1 \\ d \end{array} \right) \times \]

\[ \times \sum_{l_3=0}^{r_3} \left( \frac{d-1}{d} \right)^{r_3-l_3} \left( \begin{array}{c} l_3 \\ l_3 \end{array} \right) \left( \begin{array}{c} d-1 \\ d \end{array} \right) (1-P_C)^{r_3-l_3} (1-P_C)^{r_3-l_3} \left( \begin{array}{c} r_3 \\ l_3 \end{array} \right) \times [1-(1-P_C)^{(l_1+l_2+l_3)}[1-(1-P_C)^{(l_2+l_3)}] = \]

\[ = d \left[ (1-T^A)(1-T^B)^j + \left( 1-T^A \right) \left( 1-T^B \right) + \frac{1}{d}[T^A(1-T^B) + T^B(1-T^A) + T^A T^B] \right] - \]

\[ - \left( 1-T^A + T^A/d \right) \left( 1-T^B \right)^j - \left( 1-T^A \right) \left( 1-T^B + T^B/d \right)^j \right]. \]

We can now proceed to the clicks due to dark counts. Again, their distribution is Poissonian with multiple clicks in the same detector counting as one. Therefore, the probability of no clicks in a single detector is \( e^{-\Delta t \mu} \), while the probability of one or more clicks in one detector is \( 1 - e^{-\Delta t \mu} \). In total, the probability of \( n \) dark counts in all \( d \)
detectors is \( P_D(n, d) = (e^{-\Delta t\mu})^{d-n}(1 - e^{-\Delta t\mu})^{n}(d) \), which also gives another quantity that we need: the probability that, given that a detector already clicked because of a laser photon, all other detectors do not click because of dark counts. Denoting this probability by \( P(0, d - 1) \), we have

\[
P(0, d - 1) = P_D(1, d) \frac{1}{d} + P_D(0, d) = e^{-(d-1)\Delta t\mu}.
\]

Finally, we consider the last type of clicks that Alice and Bob register, the ones coming from environmental photons. We assume that they are produced according to a Poisson distribution. Given \( r \) photons in the same mode, the probability that at least one of them clicks is \( 1 - (1 - P_C)^r \). Furthermore, the probability that \( r \) out of \( q \) photons go in the same mode, one of them clicks, while all the others do not click is:

\[
\sum_{r=1}^{q} \left( \frac{d-1}{d} \right)^{q-r} (1 - P_C)^{q-r} \left( \frac{1}{d} \right)^r \left( 1 - (1 - P_C)^r \right) \left( \frac{q}{r} \right) = \left[ 1 - \frac{P_C(d-1)}{d} \right]^{q} - (1 - P_C)^q,
\]

and we multiply it with the Poissonian distribution of environmental photons and the number of modes to obtain

\[
P_E(1, d) = d \sum_{q=0}^{\infty} (\nu \Delta t)^q \left[ 1 - \frac{P_C(d-1)}{d} \right]^{q} e^{-\nu \Delta t} \frac{\nu \Delta t}{q!} - d \sum_{q=0}^{\infty} (\nu \Delta t)^q (1 - P_C)^q \frac{e^{-\nu \Delta t}}{q!} = dP_E(0^*, d) \left( 1 - e^{-P_C \nu \Delta t/d} \right),
\]

with \( P_E(0^*, d) = e^{-P_C \nu \Delta t/(d-1)/d} \), which is the probability that, in case a detector already clicked because of a laser photon or a dark count, \( r \) out of \( q \) environmental photons end up in this detector, while the rest \( q - r \) end up in the other detectors and none of them clicks. Note that losses affecting environmental photons are absorbed in \( \nu \).

With all the above in place, we can now calculate the quantities of interest, namely the visibility and the key rate. We start with the probability that, given \( j \) photons locally, a single detector clicks

\[
P(1) = P(1|j)P_D(0, d - 1)P_E(0^*, d) + P(0|j)P_D(1, d)P_E(0^*, d) + P(0|j)P_D(0, d)P_E(1, d)
\]

\[
= dP_D(0, d - 1)P_E(0^*, d) \left[ \left( 1 - T + \frac{T}{d} \right)^j - (1 - T)^j e^{-\Delta t(\mu + P_C \nu/d)} \right],
\]

and we continue with the probability that both parties get a single click

\[
P(11) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} (\Delta t\lambda)^n e^{-\Delta t\lambda} \frac{n^j}{n!} P_P(d)[1 - P_P(d)]^{n-j} \binom{n}{j}
\]

\[
\times \left[\left[ P(1|j)P_D(0, d - 1)P_E(0^*, d) + P(0|n)P_D(1, d)P_E(0^*, d) + P(0|j)P_D(0, d)P_E(1, d) \right]^A \times
\]

\[
\times \left[ P(1|j)P_D(0, d - 1)P_E(0^*, d) + P(0|n)P_D(1, d)P_E(0^*, d) + P(0|j)P_D(0, d)P_E(1, d) \right]^B +
\]

\[
+ \left[ P_D(0, d - 1)P_E(0^*, d) \right]^A \left[ P_D(0, d - 1)P_E(0^*, d) \right]^B [P(\neq |j) + P(= |j) - P^A(1|j)P^B(1|j))] =
\]

\[
d e^{-\Delta t(d-1)(\mu^A + \xi^A/d + \mu^B + \xi^B/d)} e^{-\Delta t\gamma} \left( d \left( 1 - e^{-\Delta t(\mu^A + \xi^A/d)} \right) \left( 1 - e^{-\Delta t(\mu^B + \xi^B/d)} \right) + e^{\Delta t\gamma/d} - 1 \right),
\]

where

\[
\xi^A/B = P_C^A/B \nu^A/B + P_P(d)\lambda P_L^B/A(1 - P_C^A/B)(1 - P_C^A/B + P_L^A/B P_L^A/B) \quad \text{and}
\]

are all experimental constants independent of \( d \). Note that \( \gamma \) is the same as in the previous implementation of temporal encoding, and represents the average number of detectable entangled photons, while \( \xi \) represents the environmental and laser photons that click independently in the labs. We are now able to write that the rate of “valid” rounds per second is

\[
R(d) = \frac{P(11)}{\Delta t} = C e^{-d(\mu^A + \mu^B)} e^{(\xi^A + \xi^B)/d} \left\{ d^2 \left( 1 - e^{-\mu^A + \xi^A/d} \right) \left[ 1 - e^{-\mu^B + \xi^B/d} \right] + d \left( e^{\gamma/d} - 1 \right) \right\},
\]

where \( C = e^{\Delta t(\mu^A + \mu^B - \xi^B - \xi^A - \gamma)/\Delta t} \).
Finally, in order to get the expression for the visibility, we need the probability that an entangled pair clicks on both labs, while all other detectors do not click. However, once the detectors click because of the entangled pair, they might also receive any number of other photons and register dark counts. We can go around this cumbersome calculations, by directly considering the probability that the same detector clicks for both Alice and Bob (which is due to the correlated photons and the noise), and subtract the probability that different detectors click (which is due to the noise only). Essentially, we subtract the $P(\neq |j\rangle)$ contribution in $P(11)$ from its $P(= |j\rangle)$ contribution to obtain

$$P_S = de^{-\Delta t(d-1)(\mu_A^A+\xi^A/d+\mu_B^B+\xi^B/d)}e^{-\Delta t\gamma}\left(e^{\Delta t\gamma/d} - 1\right),$$

which, in turn, gives us the visibility

$$v(d) = \frac{P_S}{P(11)} = \frac{1}{1 + d\left(1 - e^{-\Delta t(\mu_A^A+\xi^A/d)}\right)\left(1 - e^{-\Delta t(\mu_B^B+\xi^B/d)}\right)\left(e^{\Delta t\gamma/d} - 1\right)}.$$  

By re-scaling with $\Delta t$, we can also write

$$v(d) = \frac{e^{\gamma/d} - 1}{e^{\gamma/d} - 1 + d\left[1 - e^{-\left(\mu_A^A+\xi^A/d\right)}\right]\left[1 - e^{-\left(\mu_B^B+\xi^B/d\right)}\right]}.$$  

Note that for large $d$ and small $\Delta t$ the visibility scales as $v(d) \approx \frac{1}{1 + d^2\Delta t(\mu_A^A+\mu_B^B+\gamma)}$. 
