A target field approach to optimize cylindrical shim coils for superconducting MRI system

Kaihong Wu\textsuperscript{1,2}, Yi Shi\textsuperscript{1}, Yu Wu\textsuperscript{1}, Aihua Xu\textsuperscript{1,3}, Qiangwang Hao\textsuperscript{1}, Chao Dai\textsuperscript{1}, Yuanyuan Ma\textsuperscript{1}, Yongliang Zhang\textsuperscript{1,2} and Muhammad Talib Hussain\textsuperscript{1,2}

\textsuperscript{1} Institute of Plasma Physics, Chinese Academy of Sciences, Hefei, Anhui 230031, China
\textsuperscript{2} University of Science and Technology of China, Hefei, Anhui 230026, China
\textsuperscript{3} Changzhou Vocational Institute of Mechatronic Technology, Changzhou 213164, China

Email: xuah@ipp.ac.cn

Abstract. Target field method is used to design a specific form of the magnetic field which has been verified in many works of literature. And to restrict the current distribution on a rectangular or cylindrical surface with finite size, the current density is usually expanded by Fourier series with some unknown coefficients, which can be determined by setting the target-field points over the imaging region of interest, and then comparing the calculated field values from the Biot–Savart law and the ideal ones. The stream-function method is adopted to discrete the current density to generate winding patterns. But some design results can’t match the predetermined ones. Based on it, this paper proposed a new strategy for cylindrical shim coil design. Finally it can make the final design values meet the ideal fields with limited errors. This method can also be used to design gradient coils for high field superconducting MRI systems.

1. Introduction
Cylindrical shim coils are important parts of high field superconducting MRI system. They are designed to produce different spherical harmonic components over the volume of interest (VOI) with high efficiency to compensate the field deviation caused by impurities of the background magnetic field. One shim coil is usually designed to provide one typical order of such harmonics, so as corresponding to each harmonic of the impurities of the background field and to achieve high homogeneity design goals. And due to the complex structure of these kinds of coils, it is difficult to get their precise winding patterns by a series of forwarding electromagnetic iterative steps. Based on this, Turner [1] proposed a target field method to obtain the distribution of current density from magnetic field, and finally determine the coils’ windings and current values by stream function. The whole series of process called Inversed Electromagnetic Analysis (IEA) has been widely accepted and used. But the original Turner method did not confine the current in a limited region, although some current density truncations and smoothing processing [2] were performed later on, the change and transformation were relatively complex.

For this reason, Forbes et al [3] and Forbes and Crozier [4] proposed the target field method with coils of finite-length. Instead of employing Fourier transforms, this method decomposes the current density into Fourier series, and then the magnetic field of the target region (VOI) is calculated by Biot-Savart’s theorem. Finally, the design purpose is achieved by minimizing the integral sum of the differences between the calculated magnetic fields and ideal ones along the length direction. The calculated equations belong to the typical Fredholm integral equations of the first kind which are so
ill-posed that some regularizations [5] are necessary to solve it. Besides this designed method often contains extra constant components. Based on this, we also apply the Fourier series expressions of current density. And in order to make the designed magnetic field closer to the ideal ones without increasing extra constant value, the field points in the target region are sampled. By minimizing the differences between the calculated fields of the sampling points and the ideal ones, the design purpose of the higher purity of shim coils is achieved. In addition, the sample method is further complemented by linear programming [6-7] to increase its adaptability in practical problems. And the results show that the newly designed sampling method can get precise values and meet the requirements, and the improved method of linear programming may be applied to higher precise problems in the next design stage. This design theory can not only be used to design shim coils but also be helpful to design gradient coils for superconducting MRI system.

2. Theory

2.1. Sample method

The shim coils are located on the cylindrical surface as shown in figure 1 of radius a and length 2L, with an arbitrary source point S (a, θ', z') and the axis is the z-direction. The internal target region is a cylinder of radius c and length 2l along z-direction where one arbitrary target point P (c, θ, z) locates on its cylindrical surface. The current density on the cylindrical surface is decomposed into the Fourier series as follows [8-9]:

$$I_z(\theta, z) = \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{2mL}{n\pi a} [P_{nm} \sin(m\theta) - Q_{nm} \cos(m\theta)] \cdot \sin \left( \frac{n\pi(z + L)}{2L} \right)$$

$$I_\theta(\theta, z) = \sum_{n=1}^{N} P_{n0} \sin \left( \frac{n\pi(z + L)}{2L} \right) + \sum_{n=1}^{N} \sum_{m=1}^{M} [P_{nm} \cos(m\theta) + Q_{nm} \sin(m\theta)] \cdot \cos \left( \frac{n\pi(z + L)}{2L} \right)$$

![Figure 1](image1.png)  **Figure 1.** A cylindrical coil of radius a and length 2L with an internal target region of radius c and length 2l

![Figure 2](image2.png)  **Figure 2.** The distribution of coil region and the target region (in shim coil region, a red square represents one current loop)

where $I_z$ and $I_\theta$ are the components of the current density along the direction of z and \( \theta \) respectively. And $P_{n0}$, $P_{nm}$ and $Q_{nm}$ are the unknown expansion coefficients of $I_z$ and $I_\theta$ and the matrices of nx1, nxm and nxm respectively. The design purpose of this theory is just to determine these coefficients. N and M are positive integers. The higher these values are, the more difficult the calculation is and more complex the winding patterns of coils become.

According to Biot-Savart’s theorem $\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}) \times \mathbf{R}}{R^2} \cdot ds'$, The axial component of the magnetic field on the target region can be expressed as:
\begin{align}
B_z(c, \theta, z) &= \sum_{n=1}^{N} p_{n0} u_{n0} + \sum_{n=1}^{N} \sum_{m=1}^{M} (p_{nm} \cos(m\theta) + q_{nm} \sin(m\theta)) u_{nm} \\
U_{n0} &= -\frac{\mu_0 a}{2\pi} \int_{-L}^{L} \int_{-\pi}^{\pi} \left[ \frac{c \cos \beta - a}{c^2 + a^2 - 2 a c \cos \beta + (z - z')^2} \right] dz' d\beta \\
U_{nm} &= -\frac{\mu_0 a}{2\pi} \int_{-L}^{L} \int_{-\pi}^{\pi} \left[ \frac{c \cos \beta - a}{c^2 + a^2 - 2 a c \cos \beta + (z - z')^2} \right] \cos(m\beta) dz' d\beta
\end{align}

where \( \beta = \theta' - \theta \). The ideal field expression of the target region is expanded by Fourier series as follows:

\[ B_T(c, \theta, z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} B_T(c, \beta, z) d\beta \]

\begin{align}
&+ \sum_{m=1}^{M} \left( \frac{1}{\pi} \int_{-\pi}^{\pi} B_T(c, \beta, z) \cos(m\beta) d\beta \right) \cos(m\theta) \\
&+ \sum_{m=1}^{M} \left( \frac{1}{\pi} \int_{-\pi}^{\pi} B_T(c, \beta, z) \sin(m\beta) d\beta \right) \sin(m\theta)
\end{align}

The expression of error function \( E \) is used to estimate the difference between the calculated magnetic field (equation 3) and the ideal one (equation 4) in the target region:

\[ E = \left( \sum_{k=1}^{K} M_{0,k} - \sum_{n=1}^{N} p_{n0} u_{n0,k} \right)^2 + \left( \sum_{k=1}^{K} \sum_{m=1}^{M} M_{km} - \sum_{n=1}^{N} p_{nm} u_{nm,k} \right)^2 + \left( \sum_{k=1}^{K} \sum_{m=1}^{M} N_{km} - \sum_{n=1}^{N} q_{nm} u_{nm,k} \right)^2
\]

where:

\[ M_{0,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} B_T(c, \beta, z(k)) d\beta \]

\[ M_{km} = \frac{1}{\pi} \int_{-\pi}^{\pi} B_T(c, \beta, z(k)) \cos(m\beta) d\beta \]

\[ N_{km} = \frac{1}{\pi} \int_{-\pi}^{\pi} B_T(c, \beta, z(k)) \sin(m\beta) d\beta \]

\[ K \] is the total number of \( z \) coordinates of the sampling points on the target cylindrical surface, The unknown coefficients \( p_{n0}, p_{nm} \) and \( q_{nm} \) can be obtained by solving the minimum problems of the error function \( E \) that can be converted to the problem of zero derivatives of \( E \) with respect to these three parameters. Since this equation is ill-posed, the stored magnetic energy is taken into account to construct the form of regularization to deal with this problem.

\[ \text{Min } G = E + \lambda F = E + \lambda \left( a \int_{-L}^{L} \int_{-\pi}^{\pi} \left[ f_2^2(\theta, z) + f_2^2(\theta, z) \right] d\theta dz \right) \]

\[ = E + \frac{\pi a L}{2} \left( \sum_{n=1}^{N} p_{n0}^2 + \sum_{n=1}^{N} \sum_{m=1}^{M} \left( 1 + \left( \frac{2mL}{n\pi a} \right)^2 \right) (p_{nm}^2 + q_{nm}^2) \right) \]

where \( F \) is the function of stored magnetic energy, weighted by the penalty factor \( \lambda \) and together with the error function \( E \) to constitute the regularization form \( G \). The new objective function \( G \) can then be rewritten in a simple matrix form as follows:

\[ G = (U_{n0,k}^T p_{n0} - M_{0,k})^2 + (U_{nm,k}^T p_{nm} - M_{km})^2 + (U_{nm,k}^T q_{nm} - N_{km})^2 + \frac{\pi a L}{2} \left( C_1 p_{n0} + C_2 p_{nm}^T + C_2 p_{nm} + C_2 q_{nm} \right)^2 
\]

\[ C_1 \text{ and } C_2 \text{ are diagonal matrices of } n0 \times n0 \text{ and } nm \times nm \text{ respectively, and the diagonal elements are } 
\]

\[ C_{n0,n0} = 2 \pi a L \quad C_{nm,nm} = \pi a L \left[ 1 + \left( \frac{2mL}{n\pi a} \right)^2 \right]. \]

In order to minimize \( G \), we value it to zero. Then obtain the three unknown coefficients of current density which can be expressed as:
After determining the exact value of $P_{no}$, $P_{nm}$ and $Q_{nm}$, the surface current density is discretized to generate winding patterns by stream function method $\nabla \times \Psi = I$. Then we have the stream function $\Psi$:

$$
\Psi(r) = -\sum_{n=1}^{N} \frac{2L}{n\pi} P_{no} \cos \left[ \frac{n\pi(z + L)}{2L} \right] + \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{2L}{n\pi} \left[ P_{nm} \cos(m\theta) + Q_{nm} \sin(m\theta) \right] \cdot \sin \left[ \frac{n\pi(z + L)}{2L} \right]
$$

(9)

The contours of the stream function are the current paths of coils, so taking a series of scattered contours of $\Psi$ to produce winding patterns. $\Delta i = (\Psi_{max} - \Psi_{min})/Nc$. $\Psi = \Psi_{min} + (k - 0.5)\Delta i$. Here $Nc$ refers to the number of designed coil turns, $\Delta i$ represents the current of each turn. While $\Psi$ is the location for each winding line. In this way, the winding pattern and current value can be obtained.

2.2. Linear Programming

In some cases, we need to design the zonal harmonics of some shim coils with high precision and good linearity, and the magnetic field deviation is less than 1%. For this purpose, Crozier and Doddrell [10] proposed random algorithm (simulated annealing) method to design a sequence of circular current-carrying loops and get the desired target field by determining the location and current for each loop. This method has higher precision and stronger robustness. In this paper, we adopt a more efficient local optimization strategy -- linear programming algorithm [6-7] to reasonably arrange the shim coils at peak positions of current values. This design method has faster convergence speed and more accurate calculation results. Figure 2 shows the spatial distribution of coil position and target position, and in the coil region one red square represents one current loop of shim coils. According to the literature [11], the zonal harmonics’ formula of order $(n = 0 - 7)$ is established as follows:

$$
W_{n,m} = \sin \theta_m p_{n+1}^1(\cos \theta_m) \left( \frac{1}{a_m} \right)^{n+1}, \quad n = 0, 1 \ldots 7 \quad m = 1, 2 \ldots K
$$

(10)

$n$ represents the order of the zonal harmonics, where $n=0$ implies the constant field. $K$ refers to the total number of current loops, and $I_m$ is $m$-th cell coil’s current. In this way, matrix of the relation of coil source term to each order of zonal harmonics in the central target region can be established:

$$
A_{n,m} = \sin \theta_m p_{n+1}^1(\cos \theta_m) \left( \frac{1}{a_m} \right)^{n+1}
$$

(11)

The formulation of linear programming can be obtained as follows: where $W$ is the objective function to describe by minimizing the volume of the superconducting wires.

$$
\text{Min} \quad W = \frac{2\pi}{J} \sum_{m=1}^{k} r_m |l_m|
$$

Subject to:

$$
\begin{align*}
A_{n,m} l_m &\leq \varepsilon_n \quad \text{except for } n = i \\
-A_{n,m} l_m &\leq \varepsilon_n \quad \text{except for } n = i \\
-A_{i,m} l_m &\leq -\varepsilon_i \\
|l_m| &\leq l_{max}
\end{align*}
$$

(12)

where $r_m$ is the $m$-th cell coil’s radius. $\varepsilon_n$ is the limit values of $n$-th zonal harmonics, where $n$ exclude $i$. For $i$, it refers to the order of harmonic which we intend to design.

3. Implementations and Results

3.1. z2 shim coil design (zonal harmonics)

In this section, the sampling method is used to design the zonal harmonics of $z2$ shim coils. Due to the higher precise requirements, the Linear Programming (LP) is also adopted to improve the results. And all simulations are conducted in Matlab platform. The shim coils are placed on the cylindrical surface...
of radius $a=0.4m$ and half-length $L=0.5m$, the target region of radius $c=0.11m$ and half-length $l=0.15m$ is located in the inner centre of the coils. We choose $N=M=4$ and $\lambda=10^{-14}$ for this calculation. And the ideal expression of the magnetic field for $z^2$ shim coils can be simplified as:

$$B_T(c, \theta, z) = B_{max}(2z^2 - c^2), \quad B_{max} = 0.04$$

(13)

A stack plot for the $z^2$ shim coils of the magnetic field by sample method is given in figure 3 (a), and the other one by LP method shown in (b). The finite coil size (red) and the target region (green) are indicated with dashed lines. The quadratic relations about $z$ of the field in the target region is also clearly evident from these pictures.

![Figure 3](image)

**Figure 3.** A stack plot for the $z^2$ shim coils of the magnetic field, the finite coil size (red) and the target region (green) are indicated with dashed lines. (a) sample method, (b) linear programming

To further demonstrate the effective accuracy of the designed magnetic field on the cylindrical surface. The field deviation $\varepsilon$ at each target point is defined as:

$$\varepsilon = \frac{B_z(c, \theta, z) - B_T(c, \theta, z)}{B_T(c, \theta, z)}$$

(14)

where $B_z(c, \theta, z)$ refers to the calculated magnetic fields and $B_T(c, \theta, z)$ is the ideal ones. Both of these two methods show good results of field deviation $\varepsilon$ at target points on the cylindrical surface (radius $r=0.11m$, length $l=0.15m$) shown in figure 4. But LP gives a better precise with $\varepsilon$ less than 1%.

![Figure 4](image)

**Figure 4.** The Field Deviation $\xi$ at target points on the cylindrical surface (radius $r=0.11m$, length $l=0.15m$) (a) sample method, (b) linear programming
In order to evaluate the field accuracy of Sampling points along z-direction, comparing the computed mean-field values by Sampling method (blue scatter points) and Linear Programming (black scatter points) with the ideal ones (red line) shown in figure 5(a) -0.3<z<0.3 and (b) -0.15<z<0.15. The yellow dashed circle in figure 5(a) is expanded and vividly displayed in the right figure of (b). Though exceeding the limited target length (l=0.15), LP still gives a good fitted field values shown in the figure 5(a). And within the target length shown in the right figure (b). LP also has better accuracy than the Sampling method as the green blocks show some field deviations of the Sampling method.

Figure 5. The mean magnetic field values along z-direction (a) -0.3<z<0.3, (b) -0.15<z<0.15 by Sampling method (blue scatter points) and Linear Programming (black scatter points)

3.2. z2x shim coil design (tesseral harmonics)

The ideal expression of the magnetic field for z2x shim coils can be simplified as:

\[ B_T(c, \theta, z) = B_{max} (4z^2 - c^2) c \cdot \cos \theta \]  \hspace{1cm} (15)

In this section, the Sampling method is adopted to design the tesseral harmonics of z2x shim coil. And the winding patterns of the coils located on the cylindrical surface of radius a (1.2m) and half-length L (1m) are shown in figure 6(a). The dashed curve gives an inverted current to that of the solid one. And the target region parameters are equal to that of section 3.1. Based on this coil structure, figure 6(b) gives the linear relations of the calculated field values (red circle) with the variation of formula \((4z^2 - c^2)c \cdot \cos \theta\). And the ideal field values are also shown as a skew black line for comparison. And it can be clearly seen that the design results show good coincidence with the ideal ones which demonstrates the high availability of the Sampling theory.

Figure 6.(a)The winding patterns of z2x shim coil of radius a=1.2m and half-length l=1m, (b) The
variation of ideal fields (solid line) and design fields (red circle) with the changes of formula $(4z^2-c^2)c\cdot\cos\theta$.

4. Conclusion
In this paper, a target field approach to optimize cylindrical shim coils for superconducting MRI system is proposed. This theory is applying Sampling strategy over the target region with good calculation precise. And the shim coil’s geometry and size can be obtained according to different requirements. Besides, this method is further improved by Linear Programming in the design of zonal harmonics to reduce its field deviation from 5% to 1%. This method may also be helpful in superconducting MRI systems for designing gradient coils.

Acknowledgments
This work was supported in part by the Strategic Priority Research Program (Type-B) of Chinese Academy of Science (CAS) under Grant XDB25000000 and in part by the Natural Science Foundation of the Jiangsu Higher Education Institutions of China under Grant 19KJB470011.

Reference
[1] R. Turner 1986 *J. Phys. D: Appl. Phys.* **19** 147-51
[2] Jin J 1999 Electromagnetic analysis and design in magnetic resonance imaging *Biomedical Engineering Series* (Boca Raton: CRC Press)
[3] Forbes L K, Crozier S and Doddrell D M 2000 Asymmetric zonal shim coils *Australian Provisional Patent Application* PQ9787
[4] Forbes L K and Crozier S 2001 Asymmetric zonal shim coils for magnetic resonance applications *Med. Phys.* **28** 1644–51
[5] Z. Rui, X. Jing, F. Youyi, L. Yangjing, H. Kefu, Z. Jue and F. Jing 2011 *Meas. Sci. Technol.* **22** 125505.
[6] W. Chunzhong, W. Qiuliang and Z. Quan 2010 *IEEE Trans. Appl. Supercond.* **20** 706-709.
[7] W. Qiuliang, X. Guoxin, D. Yinming, Z. Baozhi, Y. Luguang and K. Keeman 2009 *IEEE Trans. Appl. Supercond.* **19** 2289-2292.
[8] K. F. Lawrence and C. Stuart 2001 *J. Phys. D: Appl. Phys.* **34** 3447.
[9] K. F. Lawrence and C. Stuart 2002 *J. Phys. D: Appl. Phys.* **35** 839.
[10] Crozier S and Doddrell D M 1993 *J. Magn. Reson.* **A103** 354–7
[11] A. K. Kalafala 1990 *IEEE Trans. Magn.* **26** 1181-1188.