On Summatory Totient Functions

Leonid G. Fel
Department of Civil Engineering, Technion, Haifa 32000, Israel

e-mail: lfel@tx.technion.ac.il

February 11, 2008

Abstract

The lower and upper bounds are found for the leading term of summatory totient function
\[ \sum_{k \leq N} k^u \phi^v(k) \] in various ranges of \( u \in \mathbb{R} \) and \( v \in \mathbb{Z} \).

Keywords: Summatory totient functions, Asymptotic analysis.

2000 Mathematics Subject Classification: 11N37.

1. We study the summatory totient function associated with the Euler function \( \phi(k) \),

\[ F[k^u \phi^v, N] = \sum_{k \leq N} k^u \phi^v(k) , \quad u \in \mathbb{R} , \quad v \in \mathbb{Z} . \] (1)

The function \( F[k^u \phi^v, N] \) has been the subject of intensive study for the last century and is classically known \[2\] for \( u \leq 0 , \ v = 1 \). The other results include \( u = 0 , \ v = -1 \) \[8\], \( v = -u > 0 \) \[4\], \[3\] and references therein, \( v \geq 0 , \ u = 1 , \ v = 1 \) \[10\], \[15\], \( u = v = -1 \) \[9\], \[17\]. The leading and error terms for \( u = 0 , \ v \in \mathbb{Z}_+ \), were calculated in \[4\] and \[3\], respectively.

An extensive survey on the number-theoretical properties of \( \phi(k) \) and the leading and error terms of some summatory functions (1) is presented in \[14\]. In this article we give the lower and upper bounds for the leading term of \( F[k^u \phi^v, N] \) in various ranges of \( u , \ v \).

For this purpose put the following notations,

\[ \lim_{N \to \infty} \frac{F[k^u \phi^v, N]}{N^{u+v+1}} = A(u, v) , \quad \lim_{N \to \infty} \frac{F[k^u \phi^v, N]}{\ln N} = B(u, v) , \quad \lim_{N \to \infty} F[k^u \phi^v, N] = C(u, v) , \] (2)

and note that for \( v = 0 \) these asymptotics read

\[ A(u, 0) = (u + 1)^{-1} , \quad u > -1 ; \quad B(u, 0) = 1 , \quad u = -1 ; \quad C(u, 0) = \zeta(-u) , \quad u < -1 . \]

Here \( \zeta(s) \) stands for the Riemann zeta function.
2. Start with auxiliary summatory function \( F[k^u J_v, N] = \sum_{k \leq N} k^u J_v(k) \) which is associated with the Jordan totient function \( J_v(k) \),

\[
J_v(k) = k^v \prod_{p^j \mid k} \left( 1 - \frac{1}{p^j} \right) = \sum_{d \mid k} \mu(d) \left( \frac{k}{d} \right)^v, \quad v \in \mathbb{Z}_+, \quad J_v(p^v) = p^{vr} \left( 1 - \frac{1}{p^v} \right),
\]

(3)

where \( \mu(d) \) denotes the M"obius function. The leading term of \( F[k^u J_v, N] \) can be calculated exactly in the different ranges \( u + v > -1 \), \( u + v = -1 \) and \( u + v < -1 \). Making worth of standard analytic methods \([2]\) (see also \([1]\)), we get

\[
\lim_{N \to \infty} \frac{\sum_{k \leq N} k^u J_v(k)}{N^{u+v+1}} = \lim_{N \to \infty} \frac{1}{N^{u+v+1}} \sum_{k \leq N} k^u \sum_{d \mid k} \mu(d) \left( \frac{k}{d} \right)^v = \lim_{N \to \infty} \frac{1}{N^{u+v+1}} \sum_{k_1 \leq N} \frac{\mu(k_1)}{k_1^v} \sum_{k_2 \leq N/k_1} \frac{1}{k_2} = \sum_{k_1=1}^{\infty} \frac{\mu(k_1)}{k_1^{v+1}} = \frac{1}{\zeta(v+1)}, \quad u + v > -1,
\]

(4)

\[
\lim_{N \to \infty} \frac{\sum_{k \leq N} k^{-v-1} J_v(k)}{\ln N} = \lim_{N \to \infty} \frac{1}{\ln N} \sum_{k \leq N} \frac{1}{k^{v+1}} \sum_{d \mid k} \mu(d) \left( \frac{k}{d} \right)^v = \lim_{N \to \infty} \frac{1}{\ln N} \sum_{k_1 \leq N} \frac{\mu(k_1)}{k_1^v} \sum_{k_2 \leq N/k_1} \frac{1}{k_2} = \sum_{k_1=1}^{\infty} \frac{\mu(k_1)}{k_1^{v+1}} = \frac{1}{\zeta(v+1)}, \quad u + v = -1,
\]

(5)

\[
\lim_{N \to \infty} \sum_{k \leq N} \frac{J_v(k)}{k^{-u}} = \lim_{N \to \infty} \sum_{k \leq N} \frac{1}{k} \sum_{d \mid k} \mu(d) \left( \frac{k}{d} \right)^v = \lim_{N \to \infty} \sum_{k_1 \leq N} \frac{\mu(k_1)}{k_1^v} \sum_{k_2 \leq N/k_1} \frac{1}{(k_1 k_2)^{-(u+v)}} = \zeta(-u-v) \cdot \sum_{k_1=1}^{\infty} \frac{\mu(k_1)}{k_1^{-v}} = \frac{\zeta(-u-v)}{\zeta(-u)}, \quad u + v < -1.
\]

(6)

Hence follow the bounds for \( A(u, v) \), \( B(u, v) \) and \( C(u, v) \) in the case \( v \in \mathbb{Z}_+ \).

**Lemma 1**

For \( v \in \mathbb{Z}_+ \) the following asymptotics hold

If \( u + v > -1 \), then 
\[
0 < A(u, v) \leq \frac{(u + v + 1)^{-1}}{\zeta(v + 1)},
\]

If \( u + v = -1 \), then 
\[
0 < B(u, v) \leq \frac{1}{\zeta(v + 1)},
\]

If \( u + v < -1 \), then 
\[
0 < C(u, v) \leq \frac{\zeta(-u-v)}{\zeta(-u)},
\]

where the upper bounds are attained iff \( v = 1 \).
Proof Observe that the following inequality holds
\[
\phi^v(k) = k^v \prod_{p_j \mid k} \left( 1 - \frac{1}{p_j^v} \right) \leq k^v \prod_{p_j \mid k} \left( 1 - \frac{1}{p_j^v} \right) = J_v(k), \quad v \in \mathbb{Z}_+, \quad (7)
\]
since \((1 - x^v)/(1 - x)^v = (1 + x + \ldots + x^{v-1})/(1 - x)^{v-1} \geq 1\) if \(x < 1\). The last inequality becomes rigorous if and only if \(v > 1\). Combining now (7) with (4), (5) and (6) we arrive at the proof of Lemma. \(\square\)

Illustrate Lemma 1 by three known examples taken from [2], p. 71,
\[
A(-\alpha, 1) = \frac{1}{(2 - \alpha)\zeta(2)}, \quad \alpha < 2; \quad B(-2, 1) = \frac{1}{\zeta(2)}; \quad C(-\alpha, 1) = \frac{\zeta(\alpha - 1)}{\zeta(\alpha)}, \quad \alpha > 2.
\]
Two other examples are taken from [4],
\[
A(0, 2) = \frac{1}{3} \prod_p \left( 1 - \frac{1}{p^2} + \frac{1}{p^3} \right) < \frac{1}{3} \prod_p \left( 1 - \frac{1}{p^3} \right) = \frac{1}{3\zeta(3)}, \quad (8)
\]
\[
A(-v, v) = \prod_p \left( 1 - \frac{1}{p^v} \right) \leq \prod_p \left( 1 - \frac{1}{p^{v+1}} \right) = \frac{1}{\zeta(v + 1)},
\]
where inequality becomes rigorous iff \(v > 1\). The last example is taken from [6],
\[
C(-v - s, v) = \zeta(s) \prod_p \left( 1 - \frac{1}{p^v} \left( 1 - \frac{1}{p} \right) \right) < \zeta(s) \prod_p \left( 1 - \frac{1}{p^{v+s}} \right) = \frac{\zeta(s)}{\zeta(v + s)}, \quad s > 1.
\]
3. In the case \(v \in \mathbb{Z}_-\) we represent the function \(F[k^v \phi^v; N]\) as follows,
\[
F[k^v \phi^v; N] = \sum_{k=1}^{N} k^{u+v} \prod_{p_j \mid k} \left( 1 - \frac{1}{p_j^v} \right)^{|u|} \prod_{p_j \mid k} \left( 1 - \frac{1}{p_j^v} \right)^{|v|} > 1, \quad v \in \mathbb{Z}_-, \quad (9)
\]
and prove Lemma on lower bounds.

Lemma 2
For \(v \in \mathbb{Z}_-\) the following asymptotics hold
\[
A(u, v) > \frac{1}{u + v + 1}, \quad u + v > -1; \quad B(u, v) > 1, \quad u + v = -1; \quad C(u, v) > \zeta(-u - v), \quad u + v < -1.
\]
Proof In accordance with definition (2) and inequality (9) calculate the lower bound for different signs of \(u + v + 1\),
1) \(u + v > -1\), \(A(u, v) > \lim_{N \to \infty} \frac{1}{N^{u+v+1}} \sum_{k=1}^{N} k^{u+v} = \frac{1}{u + v + 1}\),
2) \(u + v = -1\), \(B(u, v) > \lim_{N \to \infty} \frac{1}{\ln N} \sum_{k=1}^{N} \frac{1}{k} = 1\),
3) \(u + v < -1\), \(C(u, v) > \sum_{k=1}^{N} \frac{1}{k^{(u+v)}} = \zeta(-u - v)\).
There are different ways to find the bounds applying the Tauberian theorem to the corresponding Dirichlet series or making use of inequalities for arithmetic functions. In this article we follow the refined proof of the Landau theorem \([8]\) given in \([5]\).

**Lemma 3**

For \(v \in \mathbb{Z}_-\) the following asymptotics hold

\[
\begin{align*}
\text{If } u + v & > -1, \quad \text{then} \quad A(u, v) < 2\frac{|v|}{|u|} \cdot D_\infty(v, 1) \cdot (u + v + 1)^{-1}, \\
\text{If } u + v & = -1, \quad \text{then} \quad B(u, v) < 2\frac{|v|}{|u|} \cdot D_\infty(v, 1), \\
\text{If } u + v & < -1, \quad \text{then} \quad C(u, v) < 2\frac{|v|}{|u|} \cdot D_\infty(v, -u - v) \cdot \zeta(-u - v),
\end{align*}
\]

where \(D_\infty(v, s) = \prod_{r=1}^{[\frac{|v|}{|u|}]} \zeta(s + r/2)\).

**Proof**

Consider a summatory function

\[
F\left[ \frac{f(k)}{\phi^m(k)} ; N \right] = \sum_{k \leq N} \frac{f(k)}{\phi^m(k)},
\]

where \(f(k)\) is completely multiplicative function. Notice \([2]\) that

\[
\frac{k}{\phi(k)} = \sum_{d \mid k} \frac{\mu^2(d)}{\phi(d)},
\]

where a sum is taken over all divisors \(d\) of \(k\). Make use of \((11)\) in summation identity \([5]\)

\[
F\left[ \frac{f(k)}{\phi^m(k)} ; N \right] = \sum_{k_1 \leq N} \frac{f(k_1)}{k_1 \phi^{m-1}(k_1)} \sum_{d \mid k_1} \frac{\mu^2(d)}{\phi(d)} = \sum_{k_1, k_2 \leq N} \frac{\mu^2(k_1)}{\phi(k_1)} \cdot \frac{f(k_1 k_2)}{k_1 k_2 \phi^{m-1}(k_1 k_2)}
\]

\[
= \sum_{k_1 \leq N} \frac{\mu^2(k_1)}{\phi(k_1)} \sum_{k_2 \leq N/k_1} \frac{f(k_1 k_2)}{k_1 k_2 \phi^{m-1}(k_1 k_2)},
\]

and perform a multiple summation in the last equality \(m\) times

\[
F\left[ \frac{f(k)}{\phi^m(k)} ; N \right] = \sum_{k_1 \leq N} \frac{\mu^2(k_1)}{\phi(k_1)} \left\{ \sum_{k_1 k_2 \leq N} \frac{\mu^2(k_1 k_2)}{\phi(k_1 k_2)} \left\{ \sum_{k_1 k_2 k_3 \leq N} \frac{\mu^2(k_1 k_2 k_3)}{\phi(k_1 k_2 k_3)} \cdots \right\} \right\} \cdots
\]

\[
\left\{ \sum_{k_m+1 \leq N/\Pi_m} \frac{\mu^2(\prod_{i=1}^m k_i)}{\phi(\Pi_{i=1}^m k_i)} \left\{ \sum_{k_{m+1} \leq N/\Pi_m} \frac{f(\prod_{i=1}^{m+1} k_i)}{\prod_{i=1}^{m+1} k_i} \right\} \cdots \right\} \right\}
\]

\[
\text{---Based on the Tauberian theorem Z. Rudnick} \ [13] \ \text{gave an elegant proof of convergence of summatory function} \ F(u, v; N) / \ln N, \ u + v = -1, \ \text{and calculated its leading term. As for the 2nd approach, in Section 5 we give another proof of convergence of summatory function} \ F(u, v; N), \ u + v < -1, \ \text{based on two inequalities for the Euler totient} \ \phi(k) \ \text{and divisor} \ \sigma(k) \ \text{functions.}
\]
where \( \Pi_m = \prod_{i=1}^{m} k_i \). Denote by \( \Theta [k^u \phi^v(k), N] \) the last sum in (12) and consider it for \( f(k) = k^u \) and \( m = -v, v \in \mathbb{Z}_+ \),

\[
\Theta [k^u \phi^v(k), N] = \sum_{k_{|v|+1} \leq N/\Pi_{|v|}} \prod_{i=1}^{|v|+1} k_i^{u_{i-1} + v_{i}} = \prod_{i=1}^{|v|+1} k_i^{u_{i-1} + v_{i}} \sum_{k_{|v|+1} \leq N/\Pi_{|v|}} k_i^{u_{i-1} + v_{i}}.
\]

Thus, for different signs of \( u + v + 1 \) we have

\[
\Theta [k^u \phi^v(k), N] = (\Pi_{|v|})^{u+v} \sum_{k_{|v|+1} \leq N/\Pi_{|v|}} k_i^{u_{i-1} + v_{i}} < \frac{N_{u+v+1}}{u+v+1} \frac{1}{\Pi_{|v|}}, \quad \text{if} \quad u + v > -1 , \quad (13)
\]

\[
\Theta [k^u \phi^v(k), N] = \frac{1}{\Pi_{|v|}} \sum_{k_{|v|+1} \leq N/\Pi_{|v|}} \frac{1}{k_i^{u_{i-1} + v_{i}}} < \left( \ln \frac{N}{\Pi_{|v|}} + \gamma \right) \frac{1}{\Pi_{|v|}}, \quad \text{if} \quad u + v = -1 , \quad (14)
\]

\[
\Theta [k^u \phi^v(k), N] = \frac{1}{\Pi_{|v|}}^{u+v} \sum_{k_{|v|+1} \leq N/\Pi_{|v|}} \frac{1}{k_i^{u_{i-1} + v_{i}}} < \frac{\zeta(-u-v)}{(\Pi_{|v|})^{u+v}}, \quad \text{if} \quad u + v < -1 . \quad (15)
\]

where \( \gamma \) is the Euler-Mascheroni constant. Denote by \( D_N(v, s) \) the multiple sum

\[
D_N(v, s) = \sum_{k_i \leq N} \frac{1}{\mu^2(k_1)} \left\{ \frac{1}{k_2^{s}} \phi(k_1) \right\} \left\{ \frac{1}{k_3^{s}} \phi(k_1 k_2) \right\} \ldots \left\{ \frac{1}{\Pi_{i=1}^{m} k_i^{s}} \phi(\Pi_{i=1}^{m} k_i) \right\} \ldots \quad (16)
\]

Substitute (13), (14) and (15) into (12) and take in mind (16). Thus, we get

\[
F [k^u \phi^v(k); N] < \begin{cases} 
D_N(v, 1) \cdot (u + v + 1)^{-1} \cdot N^{u+v+1}, & u > -v - 1, \\
D_N(v, 1) \cdot \ln N, & u = -v - 1, \\
D_N(v, -u - v) \cdot \zeta(-u - v), & u < -v - 1.
\end{cases} \quad (17)
\]

Consider the function \( D_N(v, s) \) and make worth of elementary inequalities for the M"obius function \( \mu^2(k) \leq 1 \) and for the Euler function \( \phi(k) \)

\[
\phi(k) \geq \sqrt{k}, \quad \text{if} \quad k \neq 2, 6. \quad (18)
\]

There are two ways how to exploit (18) in order to get the upper bound for \( D_N(v, s) \). One of them is to calculate two separate terms for \( k = 2 \) and \( k = 6 \) in every sum of (16) and to apply \( \phi(k) \geq \sqrt{k} \) to the rest of the terms. This way can provide with very tight bounds, however it needs a lot of arithmetics and gives cumbersome formulas (see Section 4). More sympathetic is a way to make (18) less strong but more universal

\[
\sqrt{2} \cdot \phi(k) \geq \sqrt{k}, \quad k \geq 1. \quad (19)
\]

This leads to the simple expression of the bounds and is sufficient to prove a convergence of the
multiple sum in (14). Indeed, we have

\[
D_N(v, s) < \sum_{k_1 \leq N} \frac{\sqrt{2}}{k_1^2 \sqrt{k_1}} \left\{ \sum_{k_1 \leq N} \frac{\sqrt{2}}{k_2^2 \sqrt{k_1 \sqrt{k_2}}} \left\{ \ldots \left\{ \sum_{k_1 \leq N} \frac{\sqrt{2}}{\prod_{i=1}^{k_1 \leq N} k_i \sqrt{1 \sqrt{k_i}}} \right\} \ldots \right\} \right\}
\]

\[
< 2^{\frac{|v|}{2}} \sum_{k_1 = 1}^{N} k_1^{-(s + \frac{|v|}{2})} \cdot \sum_{k_2 = 1}^{N} k_2^{-(s + \frac{|v| - 1}{2})} \cdot \ldots \cdot \sum_{k_1 \leq N} k_1^{-(s + \frac{|v|}{2})} < 2^{\frac{|v|}{2}} \cdot D_\infty(v, s),
\]  
(20)

where

\[
D_\infty(v, s) = \prod_{r=1}^{\infty} \zeta(s + \frac{r}{2}).
\]  
(21)

Combining now (20) and (17) and taking the limit \( N \to \infty \) in the latter we arrive at the upper bounds for any value of \( u + v + 1 \). □

We illustrate Lemma 3 by three known examples taken from [10], [8] and [16], seq. A065483, respectively,

\[
A(1, -1) = \frac{\zeta(2) \zeta(3)}{\zeta(6)}, \quad B(0, -1) = \frac{\zeta(2) \zeta(3)}{\zeta(6)}, \quad C(-1, -1) = g \zeta(2),
\]  
(22)

where \( g = \prod_p \left[ 1 + p^{-2} (p - 1)^{-1} \right] \simeq 1.3398 \) and \( \zeta(2) \zeta(3) / \zeta(6) \simeq 1.9436 \). All three constants satisfy quite well Lemma 3.

\( 1.9436 < \sqrt{2} D_\infty(-1, 1) = \sqrt{2} \zeta \left( \frac{3}{2} \right) \simeq 3.694, \quad \text{1.3398} < \sqrt{2} D_\infty(-1, 2) = \sqrt{2} \zeta \left( \frac{5}{2} \right) \simeq 1.897. \)  
(23)

4. In this Section we derive the upper bound for \( D_N(v, s) \) defined in (16) in the case \( v = -1 \) and show that one can improve (20) significantly. Indeed, we have

\[
D_N(-1, s) = \sum_{k \leq N} \frac{1}{k^s \phi(k)} < \sum_{k \leq N} \frac{1}{k^s \phi(k)} = \frac{1}{2^s} + \frac{1}{2 \cdot 6^s} + \sum_{k \leq N \atop k \neq 2, 6} \frac{1}{k^s \phi(k)}.
\]  
(24)

Applying inequality (18) to the last sum in (24) we get

\[
D_N(-1, s) < \frac{1}{2^s} + \frac{1}{2 \cdot 6^s} + \sum_{k \leq N \atop k \neq 2, 6} k^{-(s + \frac{1}{2})} = \frac{1}{2^s} \left( 1 - \frac{1}{\sqrt{2}} \right) + \frac{1}{6^s} \left( \frac{1}{2} - \frac{1}{\sqrt{6}} \right) + \zeta \left( s + \frac{1}{2} \right).
\]  
(25)

One can verify that the upper bound (25) is stronger than \( \sqrt{2} \zeta(s + \frac{1}{2}) \) which follows by (20). Indeed, return to (22) and write new upper bounds in accordance with (25),

\[
1.9436 < \frac{1}{2} \left( 1 - \frac{1}{\sqrt{2}} \right) + \frac{1}{6} \left( \frac{1}{2} - \frac{1}{\sqrt{6}} \right) + \zeta \left( 1 + \frac{1}{2} \right) = 2.774,
\]

\[
1.3398 < \frac{1}{2^2} \left( 1 - \frac{1}{\sqrt{2}} \right) + \frac{1}{6^2} \left( \frac{1}{2} - \frac{1}{\sqrt{6}} \right) + \zeta \left( 2 + \frac{1}{2} \right) = 1.417.
\]  
(26)
that is much better then 3.694 and 1.897 found in [23].

However, further evaluation of the upper bounds in the case $v < -1$ leads to extremely long and sophisticated formulas which always can be calculated for any given negative integer $v$.

5. In this Section we give the upper bound for the summatory function $\sum_{k \leq N} k^u \phi^v(k)$, $v < 0$, $u + v < -1$, making worth of the Robin’s theorem [11] for the divisor function $\sigma(k)$.

**Theorem 1**

If $v < 0$ and $u + v < -1$ then

$$C(u, v) < E_m(u, v, \eta) + e^{\gamma |v|} \cdot \zeta^{[v]}(2) \cdot \sum_{r=0}^{\lfloor |v| \rfloor} (-1)^r \frac{|v|!}{r!} \left( \frac{|v|!}{r!} \right)^{-r} d^{r} \zeta(s) \left| \frac{s}{-u-v} \right|,$$

where $\eta = 2.8651$ and

$$E_m(u, v, \eta) = \sum_{k=1}^{m-1} \left( k^u \phi^v(k) - e^{\gamma |v|} \cdot \zeta^{[v]}(2) \cdot \frac{\eta + \ln k}{k^{-u-v}} \right), \quad m \geq 3.$$  

**Proof** Start with known inequality [2]

$$\frac{k^2}{\zeta(2)} < \phi(k) \sigma(k) < k^2,$$

where $\sigma(k)$ denotes the divisor function and satisfies the Robin’s theorem [11]

$$\frac{\sigma(k)}{k} < e^{\gamma \ln k + \frac{D}{\ln \ln k}}, \quad k \geq 3, \quad D = 0.6482 \ldots .$$

Making use of elementary inequalities

$$0 < \ln \ln k - \ln \ln 3 < \ln k - \ln 3, \quad k \geq 3,$$

we combine both inequalities (28) and (29) which give together

$$e^{-\gamma} \frac{1}{\zeta(2)} \frac{1}{\phi(k)} < \ln \ln k - \frac{\ln k - \ln 3}{\ln k} < \frac{1}{\ln 3} \ln k - \frac{\ln k - \ln 3}{\ln k} + \frac{D e^{-\gamma}}{k \ln \ln 3} = \frac{\ln k + \eta}{k},$$

where $\beta = \ln 3 - \ln \ln 3 = 1.00456$ and $\eta = D e^{-\gamma} / \ln \ln 3 - \beta = 2.8651$. Then we have

$$\lim_{N \rightarrow \infty} F(k^u \phi^v; N) < \sum_{k=1}^{m-1} k^u \phi^v(k) + \lim_{N \rightarrow \infty} e^{\gamma |v|} \zeta^{[v]}(2) \sum_{k=m}^{N} \frac{(\eta + \ln k)^{|v|}}{k^{-u-v}} = E_m(u, v, \eta) + e^{\gamma |v|} \zeta^{[v]}(2) \lim_{N \rightarrow \infty} \sum_{k=1}^{N} \frac{(\eta + \ln k)^{|v|}}{k^{-u-v}},$$

There is another similar inequality [12], $k/\phi(k) < e^{\gamma \ln k + 2.50637 / \ln \ln k}$, $k \geq 3$, which can be used for estimation of $C(u, v)$ by the same procedure with a similar precision.
where \( m \geq 3 \) and \( E_m(u, v, \eta) \) is given by

\[
E_m(u, v, \eta) = \sum_{k=1}^{m-1} k^n \phi^v(k) - e^{\gamma|v|} \zeta|v|(2) \sum_{k=1}^{m-1} \frac{(\eta + \ln k)^{|v|}}{k^{-u-v}}.
\]

Consider the sum in (31),

\[
\lim_{N \to \infty} \sum_{k=1}^{N-1} |v| \sum_{r=0}^{\infty} \left( \frac{|v|}{r} \right) \frac{(\ln k)^r \eta^{-r}}{k^{-u-v}} = \sum_{r=0}^{\infty} \left( \frac{|v|}{r} \right) \eta^{-r} \lim_{N \to \infty} \sum_{k=1}^{N} \frac{(\ln k)^r}{k^{-u-v}} ,
\]

and make use of the \( r \)-th derivative of the Riemann zeta function for \( \Re[s] > 1 \) given by

\[
d_r \zeta(s) ds = (-1)^r \sum_{k=1}^{\infty} \frac{(\ln k)^r}{k^w} , \quad d_0 \zeta(s) ds = \zeta(w) .
\]

Thus, we get

\[
C(u, v) < E_m(u, v, \eta) + e^{\gamma|v|} \zeta|v|(2) \sum_{r=0}^{\infty} (-1)^r \left( \frac{|v|}{r} \right) \eta^{-r} \frac{d_r \zeta(s)}{ds^r} s=-u-v ,
\]

that proves Theorem. \( \square \)

In the case \( u = v = -1 \) we have by Theorem [1]

\[
C(-1, -1) < E_m(-1, -1, \eta) + e^{\gamma} \zeta(2) \left( \eta \zeta(2) - \zeta'(2) \right) ,
\]

where according to [16], seq. A073002, the derivative \( \zeta'(2) \) is given by

\[
\zeta'(2) = \zeta(2) \cdot (\gamma + \ln(2\pi) - 12 \ln A_{GK}) = -0.937548 ,
\]

and \( A_{GK} = 1.282427 \) stands for the Glaisher-Kinkelin constant [16], seq. A074962.

Keeping in mind (22) and (35) rewrite (34) in the form

\[
g < \frac{E_m(-1, -1, \eta)}{\zeta(2)} + 10.064 ,
\]

and compare this upper bound with (23) and (26). The numerical calculations show that (36) is stronger than (23) and (26) for \( m \geq 20 \) and \( m \geq 195 \), respectively.

**Acknowledgement**

The useful discussion with Z. Rudnick is highly appreciated.
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