We formulate a general variable transformation for existing wall functions that allows for an explicit wall-shear stress term. The proposed transformation aims to enable an explicit expression of wall-shear stress and simplify the implementation of existing wall functions in simulation codes through a simple transformation of variables. The transformation is defined by introducing a new velocity scale allowing the definition of the new wall unit variable $r^+$ and the corresponding normalized viscosity $\eta^+$. We also demonstrate that the law of the wall in new variables is equivalent to the one expressed in wall-normal variables $(y^+_T)$ and $(\nu^+_T)$. The new form of the law is particularly suitable for implementations in computational fluid dynamics codes as it does not require an iterative procedure to evaluate the wall shear stress. We show how to transform several known and often used expressions of wall functions with and without pressure gradients to allow for an explicit wall-shear stress expression. Finally, we illustrate the new form of wall functions by performing illustrative numerical simulations.

Keywords Variable Transformation · Incompressible Turbulent Boundary Layer · Wall-Shear Stress · Near-Wall Units

1 Introduction

The practice of using boundary conditions based on the law of the wall is commonly used in computational fluid dynamics to reduce mesh requirements near no-slip walls. This approach for handling wall boundary conditions is commonly known by the name of wall functions [1,2]. Wall functions rely on the law of the wall to set the correct wall shear stress $\tau_w$, to compute local turbulent viscosity $\mu_t$, and set boundary conditions for the velocity and turbulence quantities. The implementation of wall functions requires knowledge of the friction velocity $u_T = \sqrt{\frac{\tau_w}{\rho}}$ to determine the wall shear stress which is obtained from the law of the wall for the turbulent boundary layer [1,2,3]. In the case of incompressible equilibrium turbulent boundary layers under zero pressure gradient conditions, the law of the wall is given as a functional relationship between non-dimensional quantities $u^+$ and $y^+_T$ [3]:

$$u^+ = \frac{u}{u_T}$$

$$y^+_T = \frac{u_T y}{\nu}$$
The symbol \( u \) in Eq. (1) denotes a streamwise component of the time averaged velocity vector, \( \nu \) denotes molecular kinematic viscosity, and \( y \) denotes distance in the wall-normal direction. In the case of zero gradient of pressure incompressible turbulent boundary layers, the relationship between \( u^+ \) and \( y_+ \) is obtained by either solving the simplified incompressible Reynolds-averaged Navier-Stokes (RANS) equations

\[
(1 + \nu_+^2) \frac{du^+}{dy_+} = 1, \quad \nu_+ = \frac{\nu}{\nu} \quad (3)
\]
or by using analytic expressions that are defined from theoretical considerations \([4, 5, 6, 7, 8]\). The local value of the wall shear stress is readily obtained by computing the local friction velocity from the known local value of \( u^+ \). The procedure of obtaining the wall shear stress in the case of turbulent boundary layers with a pressure gradient is similar with the addition of the normalized pressure \( p^+ \) to the expression of the wall of the law \([9, 10, 11]\).

For the purpose of discussion in this work, we distinguish between various expressions of the law of the wall that do not require and the ones that require an iterative procedure to compute the wall shear stress. The wall function expressions that do not require an iterative procedure to evaluate the wall shear stress are referred to as the explicit wall function. Similarly, wall functions that need an iterative algorithm for the computation of the wall shear stress are called implicit wall functions. The need for an iterative procedure arises because the friction velocity appears on both sides of the law of the wall \( u^+(u_r) = f(y_+(u_r)) \).

The earliest example of implicitly defined wall function in terms of wall units \( y^+ \) and \( u^+ \) was given by Prandtl \([12]\) that was based on the following set of equations:

\[
u^+ = y_r^+, \quad y_r^+ \leq 5.0 \quad (4)
\]

\[
u^+ = \frac{1}{\kappa} \ln(y_r^+) + B, \quad y_r^+ > 10.8, \quad \kappa = 0.41, \quad B = 5.0 \quad (5)
\]

Eq. (4) is obtained by integrating Eq. (3) under the assumption \( \nu_+^2 \to 0 \) for the values of \( y_r^+ \) between 0 and 5. The second expression is obtained by letting \( \nu_+^2 \gg 1 \) and formally integrating Eq. (3) from \( y_r^+ = 0 \) to \( y_r^+ \to \infty \). Ignoring the buffer layer, two expressions are matched for the value of \( y_r^+ = 10.8 \) to produce the law of the wall for all values of \( y_r^+ \) expressed in Eq. (4) and Eq. (5). The constant of integration \( B \) was obtained by fitting the expression in Eq. (5) to experimental data. To set the correct boundary condition, wall shear stress must be obtained from Eq. (4) and Eq. (5) by computing the friction velocity. Given the definition of \( u^+ \) and \( y_r^+ \), it is evident that an iterative procedure is required to obtain the friction velocity from Eq. (5). This requirement becomes obvious if we cast Eq. (5) in the following form

\[
u^+ = \frac{1}{\kappa} \ln \left( \frac{u_r y}{\nu} \right) + B \quad (6)
\]

Since the right-hand side has \( u_r \) under the logarithmic function, it is not possible to solve for \( u_r \) without using an iterative procedure. In the context of computational codes using the finite volume formulation, the iterative procedure has to be performed for each discrete face of the discretized wall boundary surface, thus increasing the overall computational operations count. A similar iterative procedure is performed for the computation of wall shear stress in finite difference and finite element codes. While the wall function defined by Eq. (4) and Eq. (5) is relatively simple and the iterative procedure of computing \( u_r \) converges well, the wall function is not smooth in the buffer region. Therefore, the wall function based on Eq. (4) and Eq. (5) is not accurate for computations for wall meshes that fall into the buffer region of the boundary layer.

Spalding \([4]\) proposed another form of the implicit wall function for zero pressure gradient boundary layers. Spalding’s law of the wall consists of a single analytical expression that is valid in the laminar sublayer, the buffer zone, and the logarithmic region of the boundary layer. Spalding’s function is given in the form \( y^+_r = f(u^+) \) in contrast to usual dependence of \( u^+ \) on \( y^+_r \). Spalding’s function is one of the earliest examples of an analytical expression suitable for a wide range of \( y^+_r \) values, and it has found wide acceptance and implementation in computational fluid dynamics codes. One advantage of the Spalding’s formulation is that only one expression is needed throughout the boundary layer instead of piece-wise formulation as in the case of Prandtl’s formulation. Spalding’s function was derived to satisfy the requirement that the ratio of the total (turbulent and molecular) and molecular kinetic viscosity \( \epsilon^+ = (\nu + \nu_t)/\nu \) grows with at least third power of \( y^+_r \) away from the wall in the wall-normal direction. To satisfy this requirement, Spalding derived the following function

\[
y^+_r = u^+ + 0.1108 \left[ e^{0.4u^+} - 1 - 0.4u^+ - \frac{(0.4u^+)^2}{2!} - \frac{(0.4u^+)^3}{3!} - \frac{(0.4u^+)^4}{4!} \right] \quad (7)
\]

It is clear from expression in Eq. (7) that Spalding’s wall function requires an iterative method to evaluate the wall shear stress. Newton’s iterative procedure is typically used to evaluate the wall shear stress. Iterations are performed
until the set convergence criterion is achieved, or the maximum number of iterations is exceeded. The quadratic rate of convergence characterizes Newton iterations, and the iterative procedure typically converges within several iterations. However, Newton’s method is known to be sensitive to initial guess, and this property may lead to erroneous results in the computation of the wall shear stress if the unsuitable initial guess is used in the iterative procedure. The computational complexity of the implementation of the Spalding’s wall function is high since the iterative procedure is required to be performed for each discrete face of the wall surface. Given the lack of pressure gradient influence in the formulation of Spalding’s wall function and its computational complexity makes the formulation to be computationally expensive in complex three-dimensional flows where adverse and favorable pressure gradient plays a significant role.

Musker [11] proposed an analytical expression for \( u^+ \) as a function of \( y_r^+ \) by rejecting the Spalding’s function in favor of a simpler expression. Musker postulates the following expression for the variation of \( \nu_r^+ \sim (y_r^+)^3 \) in the boundary layer

\[
\frac{1}{\nu_r^+} = \frac{1}{C(y_r^+)^3} + \frac{1}{\kappa y_r^+}, \quad C = 0.001093, \quad \kappa = 0.41
\]  

(8)

Musker’s formulation uses inverse weighting to blend two different functions, one with the correct scaling of turbulent viscosity in laminar sublayer \( f_2(y_r^+) = \kappa y_r^+ \), and the other one suitable in the inertial region of the turbulent boundary layer \( f_1(y_r^+) = C(y_r^+)^3 \). The blending is performed using the reciprocal function thus allowing the continuous law formulation throughout laminar and inertial layers of the turbulent boundary layer. Using the Reynolds averaged equations for the boundary layer without pressure gradient, the dimensionless velocity gradient

\[
\frac{d u^+}{d y_r^+} = \frac{\kappa + C(y_r^+)^2}{\kappa + C(y_r^+)^2 + C(y_r^+)^3}
\]  

(9)

is integrated to yield the following closed-term expression

\[
u^+ = 5.424 \tan^{-1} \left( \frac{2u^+ - 8.15}{16.7} \right) + \ln \left( \frac{(y_r^+ - 10.6)^{1.6}}{(y_r^+ - 8.15 y_r^+ + 86.0)^2} \right) - 3.52
\]  

(10)

Musker’s approach suffers from the same shortcoming of needing to use an iterative procedure to compute the friction velocity since both \( u^+ \) and \( y_r^+ \) depend on friction velocity \( u_r \). Therefore, Newton’s method is used to evaluate friction velocity. Similarly to Spalding’s wall function, the computational complexity of evaluation of the wall shear stress from the Musker’s wall function is high. The iterative procedure must be applied to each facet of the discrete wall boundary surface. Musker’s wall function suffers from similar computational complexity as Spalding’s wall function since the pressure gradient is not included in the formulation. Therefore, the application of the Musker’s wall function is limited to zero pressure gradient boundary layers.

In contrast, Werner and Wengle [6][13] defined an explicit function for the wall shear stress. The Werner and Wengle formulation is based on the following functions

\[
y^+_r = u^+, \quad y^+_r \leq 11.81
\]  

(11)

\[
u^+ = A(y^+_r)^B, \quad A = 8.3, \quad B = \frac{1}{7}, \quad y^+_r > 11.81
\]  

(12)

Werner and Wengle model, Eq. (11) and Eq. (12), is integrated to yield the following expression for the wall shear stress

\[
\frac{|\tau_w|}{\rho} = \nu \left| \frac{u}{y} \right|, \quad u \leq \frac{\nu}{4y} A^{1/2} \eta
\]  

(13)

\[
\frac{\tau_w}{\rho} = \left( \frac{1 - B}{2} A^{1+B} \left( \frac{u}{y} \right)^{1+B} + \frac{1 + B}{A} \left( \frac{u}{y} \right)^B \right)^{1/\eta}, \quad u > \frac{\nu}{4y} A^{1/2} \eta
\]  

(14)

While the expression in Eq. (14) of Werner-Wengle does not require an iterative procedure to compute the wall shear stress, it does not take into account pressure gradient in the formulation and therefore, it is only applicable to flows without pressure gradients. Werner-Wengle law has found application in many computational codes due to its simplicity and ease of implementation, it is not accurate for large values of \( y^+_r \). Additionally, the expression for the wall shear stress in Eq. (14) is not directly extensible to account for the pressure gradient effects. Also, the transition from the viscous sublayer to inertial region is not smooth due to nature of expressions in Eq. (11) and Eq. (12). The lack of smoothness is not desirable as it leads to less accurate results in the buffer region of the turbulent boundary layer.
Another approach to avoiding the iterative procedure in computing the wall shear stress that is commonly in use in conjunction with $k - \epsilon$ turbulence model \cite{14,15,16} is to use the following expression \cite{17}

$$u_\tau = C_{\mu}^{1/4} k^{1/4}, \quad C_{\mu} = 0.09$$

(15)

However, the expression in Eq. (15) cannot be used in turbulence models that do not use the transport equation for turbulent kinetic energy. Furthermore, asymptotic behavior of $k$ as a function of $y^+$ has to be known to make the value $u_\tau$ consistent with the turbulence model. Given that the transport equation for $k$ is heuristically obtained, the asymptotic behavior of $k$ near the wall carries all assumptions that are built in the turbulence transport model thus making the value of $u_\tau$ depend on those assumptions.

To simplify the computation of the wall shear stress, Kalitzin et al. \cite{7} proposed the use of the new variable denoted here by symbol $r^+ = u^+ y^+\nu$ to define the law of the wall. A significant advantage of the new variable is that it is not defined in terms of friction velocity, thus bypassing the need for the iterative procedure to determine wall shear stress. In their work, the new variable $r^+$ was labeled erroneously as the local Reynolds number, and in this work, we show that the new variable is defined as the normalized velocity where the normalization is performed with respect to local velocity scale $\bar{u} = \nu / y$. In the approach proposed by Kalitzin et al., a numerical method is used to solve the simplified RANS equations displayed in Eq. (5), and the obtained values are used to create a table of wall shear stress as a function of local Reynolds number. The resulting tables utilize an interpolation procedure whenever the value of the local variable $r^+$ falls between two tabulated values. Introduction of the velocity scaling simplified the computation of the wall shear stress significantly as it allowed the use of the lookup tables without the need to use an iterative procedure. From the classification point of view adopted in this work, the approach of Kalitzin et al. constitutes an explicit wall function, albeit in tabular form. However, the main drawback of the wall function by Kalitzin et al. is the absence of the pressure gradient from the formulation. This drawback severely limits the use of the defined wall function in practical computations to flows without pressure gradients. A further drawback of Kalitzin et al. approach is that the law of the wall is formulated in the tabular form. Therefore, an interpolation procedure is required to compute the values of the wall shear stress for the values of $r^+$ falling between two values in the table. While using the interpolation procedure does not significantly add to the computational complexity of implementation of the wall function in computational codes, it is more convenient to use analytic expressions whenever possible.

Many other wall functions proposed by von Karman \cite{18}, Deissler \cite{19}, Rannie \cite{20}, and Reichardt \cite{21} fall in the same category of implicitly defined wall functions. This is not a complete list of proposed wall functions, but all listed ones share the property of excluding the pressure gradient effects from their definition in addition to requiring an iterative method to compute the wall shear stress.

Several attempts to generalize the law of the wall in the presence of pressure gradient is evidenced by numerous references \cite{9,10,11,17}. In Kim and Choudhury \cite{17}, the introduction of the pressure gradient to the law of the wall was obtained by the generalization of the two-layer approach by Launder \cite{22}. Kim and Choudhury proposed a logarithmic velocity profile that is based on the kinetic energy budget in the boundary layer. The resulting expression for the wall shear stress requires the knowledge of the turbulent kinetic energy that is assumed to be constant in the turbulent part of the boundary layer. The wall function expression proposed by Kim and Choudhury is generalized here to be applicable to turbulence models that do not compute turbulent kinetic energy transport as follows

$$u^+ = \begin{cases} 
\frac{y^+}{\tau} + \frac{1}{2} (y^+)^2 p^+ & \text{if } y^+ < 10.8 \\
\frac{1}{8} \ln(y^+) + B + \frac{p^+}{\tau x} [y^+ + 10.8 \ln(y^+) + 11.35] & \text{otherwise}
\end{cases}$$

(16)

where $p^+$ is defined as follows

$$p^+ = \frac{\nu}{\rho \mu^2} \frac{dp}{dx}$$

(17)

The expression in Eq. (16) falls in the implicitly defined wall functions as an iterative procedure is required to compute the wall shear stress. Furthermore, the resulting velocity profile deviates significantly from experimental results for large $y^+$ values for moderate to strong pressure gradients. Furthermore, Röber \cite{23} proposed a continuous formulation of the wall function with pressure gradient by blending the analytic solution for the viscous sublayer and a van Driest damping function. The resulting law of the wall requires an iterative procedure to evaluate the wall shear stress, and it deviates from experimental results for large values of the wall distance $y^+$. The common behavior of Kim and Choudhury and Röber wall functions is that they are not bounded functions with increasing $y^+$ values. Therefore, the utility of the proposed wall functions is limited to lower values of $y^+$.

Duprat et al. \cite{24} proposed another approach to a continuous wall functions with pressure gradient. The approach is based on velocity scaling and van Driest damping to produce a continuous wall function \cite{25}. The resulting expression
for the wall shear stress required an iterative procedure to compute the friction velocity. Furthermore, for large values of the wall distance, the proposed wall function have shown a significant discrepancy between experimental and computed results similarly to models of Kim and Choudhury and of Röber.

Popovac and Hanjalić [26] propose the pressure sensitized wall function that includes acceleration terms as follows

\[
  u^+ = \frac{1}{\kappa \psi} \ln(E y^+_	au) \quad (18)
\]

\[
  \psi = 1 - \frac{C_U^+ y^+_	au}{u^+ \kappa} \quad (19)
\]

\[
  C_U^+ = \frac{\nu}{\rho u_*^2} \left[ \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} \right] \quad (20)
\]

where \( E = e^{B_\tau} \) and \( C_U^+ \) is the acceleration term. The proposed wall function is simplified here to read

\[
  u^+ = \frac{1}{\kappa} \ln(y^+_	au) + B + \frac{C_U^+ y^+_	au}{\kappa} \quad (21)
\]

The wall function of Popovac and Hanjalić also belongs to the class of implicitly defined functions. Therefore, an iterative procedure is also required to compute the wall shear stress. Furthermore, the proposed wall functions tends to produce the negative velocities for large values of \( y^+ \) in adverse pressure gradient environment. In favorable pressure gradient environment, the wall function in Eq. (21) tends to produce large values of the velocity in contrast to experimental data. Similarly to all wall functions with pressure gradients discussed thus far, the proposal of Popovac and Hanjalić also produce the unbounded behavior of the wall function for large values of \( y^+ \).

The proposal of Shih et al. [9] is based on the new velocity scale that includes both viscous and pressure gradient effects have demonstrated consistent velocity profiles for the extended range of the values of wall distance under moderate to severe pressure gradients. The model also included the blending of the buffer region to produce continuous wall functions for all \( y^+ \) values. Shih’s model is based on the Lumley’s analytic solution for the boundary layers with pressure gradient [3]. The agreement between Shih’s model and experimental results is satisfactory for the broad range of \( y^+ \) values. Shih’s model requires an iterative procedure, and its implementation can be difficult. However, Shih’s wall function does not exhibit unbounded behavior for large values of \( y^+ \) and it can be used for the computation of the wall shear stress over the broad range of \( y^+ \) values.

Irrespective of the presence of the pressure gradient, it is apparent that an iterative procedure is required to determine the value of the wall shear stress in all formulations except for Werner and Wengle formulation [6][13]. The goal of this work is to define a wall function that does not require an iterative procedure, and that is universally applicable to flows with and without pressure gradient. One of the main obstacles in achieving this goal is the use wall distance \( y^+ \) and normalized velocity \( u^+ \) since both are defined with reference to friction velocity \( u_* \). Therefore, a different choice of variables can remove the need for an iterative procedure. The focus of this work is on defining the law of the wall in such variables. We generalize Shih’s at al. model to be expressed in the new velocity scaling variable \( r^+ \) and we define the explicit expression for the wall shear stress for boundary layers with and without gradient of pressure.

The paper is organized as follows: in section [2], we introduce the variable \( r^+ \) in Shih’s at al. model. We also introduce the new viscosity variable \( \eta^+ \) related to \( r^+ \) and we show the the relationship between \( \eta^+ \) and \( \nu^+ \). We then state the boundary layer equations in terms of \( \eta^+ \), \( r^+ \), \( u^+ \), and \( p^+ \). We show the equivalence of boundary layer equations and consequently, the law of the wall in the new \( (\eta^+, r^+) \) and old variables \( (\nu^+ , y^+) \). We show the equivalence between the new and old formulations by integrating the boundary layer equations for zero pressure gradient flow conditions. We then modify the Shih at al. model to be formulated in terms of new variables to compute the wall shear stress explicitly. In Section [3] we apply the new definition of Shih at al. wall function to the k – \( \omega \)-SST model of turbulence [27]. In Section [4] we perform example calculations for the backward facing step [28] and planar diffuser problem [29] at various mesh resolutions and compare results to experiments. In Section [5] we provide the summary and conclusions.

2 Formulation of the Boundary Layer Equation in the New Wall Units

Following Shih at al. [9], we introduce the incompressible boundary layer equation in wall units

\[
(1 + \nu^+ \frac{du^+}{dy^+}) = 1 + p^+ y^+ \quad (22)
\]
We seek to express the law of the wall in terms of new variables, i.e., with the help of the following identity:

\[ \tau_w \] represents the wall shear stress. Therefore, the law of the wall (25) requires an iterative procedure to compute the friction velocity \( u_r \).

The solution of Eq. (22) is called the law of the wall, and it symbolically was written as the following expression:

\[ u^+ = f(y_r^+) \] (25)

Given the definitions in Eq. (23), it is clear that if the objective of the computation is to determine the value of the wall shear stress, Eq. (25) requires an iterative procedure to compute the friction velocity \( u_r \). Therefore, if some other velocity scale is used, it is in principle possible to avoid the implicit character of the law of the wall. One such scale is introduced in this work, and it is defined as follows:

\[ \hat{u} = \frac{\nu}{y} \] (26)

The local averaged velocity is normalized to introduce a new variable \( r^+ \)

\[ r^+ = \frac{u}{\hat{u}} \] (27)

We seek to express the law of the wall in terms of new variable i.e., \( u^+ = g(r^+) \). To be able to achieve that goal, the boundary layer equation must be written in terms of the new variable. A transformation between old and new variables is needed to cast the law of the wall in terms of new wall units.

To define the transformation, we observe that the new variable \( r^+ \) can be written in terms of \( u^+ \) and \( y^+ \) as follows:

\[ r^+ = \frac{u}{\hat{u}} = u^+ y_r^+ \] (28)

With the help of the following identity:

\[ d(r^+) = d(u^+ y_r^+) = y_r^+ du^+ + u^+ dy_r^+ \] (29)

Eq. (23) is transformed into the following form:

\[ [u^+ (1 + \nu_i^+ ) + y_r^+] du^+ = dr^+ + p^+ r^+ \] (30)

The new variable \( \eta_i^+ \) that represents the transformed normalized turbulent viscosity is defined as:

\[ \eta_i^+ = u^+ (1 + \nu_i^+) + y_r^+ - 1 \] (31)

so that the boundary layer equation takes the following form:

\[ (1 + \eta_i^+) du^+ = dr^+ + p^+ r^+ \] (32)

Eq. (22) and Eq. (32) are equivalent and can be transformed to each other by using the transformation defined in Eq. (28) and Eq. (31).

The main advantage of the new form of the boundary layer equation in new variables is that the evaluation of the wall shear stress can now be obtained without the recourse to an iterative procedure. Therefore, the law of the wall \( u^+ = g(r^+) \) becomes explicit, and the direct computation of \( u^+ \) is possible by performing a simple function evaluation. Moreover, \( y_r^+ \) is readily available once the value of \( u^+ \) is computed by using the relationship:

\[ y_r^+ = \frac{r^+}{u^+} \] (33)

Solution of Eq. (32) represent the desired law of the wall in new variables that is symbolically given as \( u^+ = g(r^+) \). Since the definition of \( r^+ \) does not depend on friction velocity, normalized velocity \( u^+ \) is evaluated directly. Moreover, all that is required for this computation is the local distance from the wall \( y \), local mean velocity \( u \), and local value of molecular viscosity \( \nu \).
2.1 Asymptotic Solutions of Turbulent Boundary Layer without Pressure Gradient

Boundary layer equation for the case of zero pressure gradient \((p^+ = 0)\) reduces to the following equation in wall units

\[
(1 + \nu^+_t) \, du^+ = dy^+_r
\]  

(34)

Similarly, boundary layer equation in new variables is

\[
(1 + \eta^+_t) \, du^+ = dr^+
\]  

(35)

Eq. (34) and Eq. (35) have the same form and they are linked through transformation of variables given by Eq. (28) and Eq. (31). Both equations can be integrated if the boundary conditions and the form of normalized turbulent viscosity are known.

In the case of the laminar sublayer, turbulent fluctuations are negligible as viscous forces dominate the flow. Since \(\nu^+_t \ll 1\), Eq. (34) takes the following form

\[
du^+ = dy^+_r
\]  

(36)

With no slip boundary condition, Eq. (36) is integrated to yield to following expression

\[
u^+_t = y^+_r, \quad y^+_r < 5
\]  

(37)

The extent of the laminar sublayer is the range of \(y^+_r\) values between 0 and 5. In order to demonstrate that Eq. (35) can be integrated in a similar way, we use the condition \(\nu^+_t \ll 1\) to evaluate the viscosity \(\eta^+_t\)

\[
\eta^+_t = u^+_t + y^+_r - 1
\]  

(38)

Substituting expression from Eq. (38) in Eq. (35), we obtain the following expression

\[
(u^+_t + y^+_r) \, du^+ = dr^+
\]  

(39)

Since in viscous sublayer according to Eq. (37) \(u^+_t = y^+_r\), the solution to Eq. (35) in new variables is

\[
u^+_t = \sqrt{r^+}, \quad r^+ < 5
\]  

(40)

The inertial region of the boundary layer is recovered with the help of the Prandtl’s assumption about the normalized turbulent viscosity

\[
\nu^+_t = \kappa y^+_r
\]  

(41)

where \(\kappa = 0.41\) is an experimentally determined constant. In the inertial region the normalized turbulent viscosity is large and it is assumed that \(\nu^+_t \gg 1\). With this assumption, Eq. (34) becomes

\[
u^+_t \, du^+ = dy^+_r
\]  

(42)

With the help of Eq. (41), the integration of Eq. (42) yields the well known logarithmic law

\[
u^+_t = \frac{1}{\kappa} \ln(y^+_r) + B
\]  

(43)

where \(B = 5.0\) is experimentally fitted constant.

Solution for the inertial region of the boundary layer can be expressed in new variables by enforcing the condition \(\nu^+_t \gg 1\) in expression for modified turbulent viscosity

\[
\eta^+_t = u^+_t \nu^+_t + y^+_r - 1
\]  

(44)

Substituting Eq. (44) in Eq. (35), we obtain

\[
(u^+_t \nu^+_t + y^+_r) \, du^+ = dr^+
\]  

(45)

To be able to solve Eq. (45) in closed form in new variables, we introduce and equivalent Prandtl’s assumption as follows

\[
\nu^+_t = \frac{k r^+ - y^+_r}{u^+}
\]  

(46)

where \(k\) is a new function to be defined later. With the definition in Eq. (46), the solution to Eq. (45) becomes

\[
u^+_t = \frac{1}{k} \ln(r^+) + C, \quad r^+ > = 10.8^2
\]  

(47)
Function $k$ is defined by equating two forms of Prandtl’s law i.e., Eq. (41) and Eq. (46):

$$k = \kappa + \frac{1}{u^+}$$  \hspace{1cm} (48)

Clearly, function $k$ is not a constant and it is related to the value of $\kappa$. The contribution of the normalized velocity to values of $k$ is negligible for large $r^+$ values. For smaller values of $r^+$ the contribution of the normalized velocity $u^+$ to values of $k$ function is finite since the validity of the modified Prandtl’s hypothesis extends only to the lower limit of inertial region bounded by $r^+ > 10.8^2$. We approximate $k$ by a constant value determined by the curve fit to minimize error for the range of $r^+$ values. Estimated values values of $k$ and $C$ constants are as follows:

$$k = \frac{1}{2.1108}, \quad C = 0.7576$$  \hspace{1cm} (49)

The comparison between the law of the wall defined by Eq. (43) in the traditional wall distance variables and the new one defined according to Eq. (47) is shown in Fig. 1. It is clear from the graph in Fig. 1 that the two expressions are equivalent. The viscous sublayer solution is extended beyond the buffer region ($y_\tau \geq 5$) to intersect the inertial region at the point $y_\tau = 10.8$ as is often done in practical implementations of the law of the wall in computational fluid dynamics codes. The practice of ignoring the buffer region leads to errors in the buffer region and it is done here only to compare two asymptotic solutions. Fig. 1 indicates that the law of the wall in the new scaling and old wall distance variables produce physically the same values of $u^+$.

Figure 1: Comparison of wall functions in old and new variables: solid line labeled $f(y_\tau^+)$ corresponds to wall function defined with respect to $y_\tau^+$ and given by Eq. (37) and Eq. (40) while dashed line labeled $g(r^+)$ corresponds to wall function defined in terms of $r^+$ and given by Eq. (40) and Eq. (47). The viscous sublayer solution is extended to values $y_\tau \leq 10.8$ for comparison purposes.
Transformation of various profiles that appear in the literature [4, 6, 8] is in principle possible with some effort. The process becomes difficult for more complex expressions of the normalized turbulent viscosity. One such example is the Musker’s law [5, 11]. Musker’s profile represents the continuous function that is applicable to both laminar sublayer, buffer, and inertial region. Moreover, the profile extends to the wake region. We are interested in representing Musker’s function in new variables and instead of integrating the boundary layer equation, a best fit optimization is used to define the new function. The Musker’s function in new variables takes the following form

$$u^+ = \frac{1}{k} \ln \left( \frac{\sqrt{r^+} + a}{a} \right) + \frac{\alpha}{a + 4\alpha} \left\{ (a - 4\alpha) \ln \left[ \frac{a \left( (\sqrt{r^+} - \alpha)^2 + \beta^2 \right)}{2\alpha (\sqrt{r^+} + a)^2} \right] \right\} + \frac{2\alpha (5a - 4\alpha)}{\beta} \left\{ \tan^{-1} \left( \frac{\sqrt{r^+} - \alpha}{\beta} \right) + \tan^{-1} \left( \frac{a}{\beta} \right) \right\}$$

The coefficient definitions of Eq. (50) are as follows:

$$a = l + \frac{1}{9k^2 l} + \frac{1}{3k}$$

$$l = \left( \frac{1}{2} \frac{4s + 27k^3}{ks} + \frac{2s + 27k^3}{54k^3 s} \right)^\frac{1}{3}, \quad k = 0.23, \quad s = 8.347 \times 10^{-4}$$

$$2\alpha = a - \frac{1}{k}, \quad \beta^2 = 2a\alpha - \alpha^2$$

The new form os the Musker’s function is structurally the same as the original function [5, 11] with the modified coefficients. Furthermore, the independent variable that keeps the same form of the original Musker’s function is $\sqrt{r^+}$ instead of $r^+$. The utility of the new variables becomes apparent when we consider the Musker’s function. Since the variable $r^+$ is readily available, computation of $u^+$ requires only one evaluation of Eq. (50). If the traditional wall units $y^+$ are used, an assumed value of friction velocity must be iteratively improved in order to compute $y^+$ and $u^+$. This process requires several evaluations of Musker’s function in the course of iterations.

The comparison between Musker’s function in wall distance and new scaling variables is shown in Fig. 2. It is evident that the agreement between the old and new variables is close with small differences in the buffer region.

### 2.1.1 Asymptotic Solutions of Turbulent Boundary Layer with Pressure Gradient

Asymptotic solutions for boundary layers without pressure gradients have a limited utility in practical computations as virtually all flows involve pressure gradients. Therefore, we are interested in expanding the use of new variables to wall functions that include pressure gradient. We focus on the generalized wall functions introduced by Shih at al. [9, 10]. Shih’s proposal is built on previous work by Lumley [3] in which a new velocity scale is introduced to encompass both viscous and pressure gradient effects

$$u_c = u_\tau + u_p$$

Friction velocity is defined as before

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

The new velocity scale related to pressure effects was defined earlier and we repeat its definition for clarity

$$u_p = \left( \frac{\nu}{\rho} \left| \frac{dp_w}{dx} \right| \right)^\frac{1}{2}$$

The definition of the velocity scale $u_c$ is well posed as it never becomes zero i.e., in the absence of the pressure gradient it takes the value of the friction velocity. The opposite is valid as in the absence of wall shear stress, $u_c$ takes the value of the velocity induced by the pressure gradients $u_p$. The velocity scale $u_c$ is used to scale the linearized $x$-momentum equation for boundary layers

$$\nu \frac{\partial u}{\partial y} - \overline{u'u''} = \frac{\tau_w}{\rho} + \frac{y dp_w}{\rho dx}$$
The asymptotic solution of Eq. (56) was obtained by Shih at al. [9] in the following form:

$$\frac{u}{u_c} = \frac{\tau_w}{\rho u_c^2} \left[ y_c^+ + \frac{d}{dy} \left( \frac{u_p}{u_c} \right) f_2 \left( y_c^+, \frac{u_p}{u_c} \right) \right]$$

The non-dimensional wall distance $y_c^+$ was defined by using the velocity scale $u_c$

$$y_c^+ = \frac{u_c y}{\nu}$$

It can be observed that unlike the previous definition of the wall distance $y^+$ that was defined with the help of the friction velocity only, the new definition involves both friction and pressure related components. The solution in Eq. (57) was obtained by observing that the expression of Eq. (56) is linear thus allowing to separate the expression in two parts [3]

$$u = u_1 + u_2$$

The velocity $u_1$ is related to the wall shear stress while the velocity $u_2$ is related to the pressure gradients. The decomposition of Eq (56) in two parts yields the following two equations [3,9]

$$\nu \frac{\partial u_1}{\partial y} - \overline{u' v'}_1 = \frac{\tau_w}{\rho}$$

$$\nu \frac{\partial u_2}{\partial y} - \frac{\partial}{\partial y} \left( \frac{u' v'}{ \nu} \right) = \frac{y dp_w}{\rho dy}$$
The asymptotic solutions to Eq. (60) and Eq. (61) are
\[
\frac{u_1}{u_*} = \frac{\tau_w}{\rho u_*^2} f_1 (y_+^*) \tag{62}
\]
\[
\frac{u_2}{u_p} = \frac{\nu}{\rho u_p^3} f_2 (y_+^*) \tag{63}
\]
Asymptotic solution in Eq. (63) uses the non-dimensional wall distances \(y_+^*\) defined as follows
\[
y_+^* = \frac{u_p y}{\nu} \tag{64}
\]
whereas \(y_+^*\) was previously defined in Eq. (3). The non-dimensional wall unit \(y_+^*\) takes into account viscous effects, whereas \(y_+^*\) incorporates pressure gradient effects in its definition. Functions \(f_1\) is obtained from the solution of Eq. (60) so that satisfies boundary conditions and represent velocity profile in viscous sublayer, buffer region, and inertial sublayer.

\[\begin{align*}
f_1(y_+^*) &= \begin{cases} 
  y_+^* + 0.01(y_+^*)^2 - 2.9 \times 10^{-3}(y_+^*)^3 & \text{if } y_+^* \leq 5 \\
  -0.872 + 1.465y_+^* - 0.0702(y_+^*)^2 + 0.00166(y_+^*)^3 & \text{if } 5 \leq y_+^* \leq 30 \\
  -1.495 \times 10^{-5}(y_+^*)^4 & \text{if } 30 \leq y_+^* \leq 140 \\
  8.6 + 0.1864y_+^* - 0.002(y_+^*)^2 + 1.144 \times 10^{-5}(y_+^*)^3 & \text{if } 140 \leq y_+^*
\end{cases}
\end{align*}\]  
(65)

Similarly, \(f_2\) function is given by the following expression
\[\begin{align*}
f_2(y_+^*) &= \begin{cases} 
  0.5(y_+^*)^2 - 7.31 \times 10^{-3}(y_+^*)^3 & \text{if } y_+^* \leq 4 \\
  -15.138 + 8.4688y_+^* - 0.81976(y_+^*)^2 + 3.7292 \times 10^{-2}(y_+^*)^3 & \text{if } 4 \leq y_+^* \leq 15 \\
  -6.3866 \times 10^{-4}(y_+^*)^4 & \text{if } 15 \leq y_+^* \leq 30 \\
  11.925 + 0.934y_+^* - 2.7805 \times 10^{-2}(y_+^*)^2 + 4.6262 \times 10^{-4}(y_+^*)^3 & \text{if } 30 \leq y_+^*
\end{cases}
\end{align*}\]  
(66)

The \(\text{sgn}(\text{arg})\) function which evaluates to +1 if \(\text{arg} > 0\) and −1 if \(\text{arg} < 0\) is introduced so that Eq. (57) is written in the following form
\[
\frac{u}{u_c} = \text{sgn}(\tau_w) \frac{u_*}{u_c} f_1 (y_+^*) + \text{sgn} \left( \frac{d\tau_w}{dx} \right) \frac{u_p}{u_c} f_2 (y_+^*) \tag{67}
\]
Algebraic manipulations lead to the following form of Eq. (66)
\[
\frac{u}{u_2} = \text{sgn}(\tau_w) f_1 (y_+^*) + \text{sgn} \left( \frac{d\tau_w}{dx} \right) \frac{u_p}{u_2} f_2 (y_+^*) \tag{68}
\]
Eq. (68) is solved for the friction velocity
\[
u_2 = \frac{u - u_p \text{ sgn} \left( \frac{d\tau_w}{dx} \right) f_2 (y_+^*)}{\text{sgn}(\tau_w) f_1 (y_+^*)} \tag{69}
\]
Combining Eq. (63), Eq. (68), and Eq. (59), the definition of friction velocity becomes
\[u_2 = \text{sgn}(\tau_w) \frac{u_1}{f_1 (y_+^*)} \tag{70}\]
Since the friction velocity $u_*$ is positive by definition, and $f_1(y^+_p)$ is a positive function, we have
\[ \text{sgn}(\tau_w) = \text{sgn}(u_*) \] (71)
Therefore, Eq. (69) becomes
\[ u_\tau = \frac{u - u_p \; \text{sgn}\left(\frac{dp_w}{dr}\right) f_2(y^+_p)}{\text{sgn}(u_1) f_1(y^+_p)} \] (72)
Evaluation of Eq. (72) requires an iterative procedure due to the presence of $u_\tau$ on both sides of equality. As before, we seek to define the the generalized wall function in terms of variable $r^+$. The new velocity scaling variable $r^+$ has a significant advantage that it does not involve either $u^+$ or $y^+_p$ in its definition directly. In other words, the new scaling variable is defined completely in terms of physical quantities. Therefore, if the expression for the friction velocity is defined in terms of $r^+$, computation of the wall shear stress becomes straightforward evaluation of the expression similar to Eq. (72).

To define Eq. (72) as a function of the new scaling variable, we perform the curve fitting procedure for the function $g_1(r^+)$ defined in Eq. (65) with respect to the new scaling variable $r^+$ to obtain the following expression
\[ g_1(r^+) = \begin{cases} \sqrt{r^+} & \text{if } 0 \leq r^+ < 24.44 \\ 2.8957 \ln(r^+) - 4.4958 & \text{if } 24.44 \leq r^+ < 378.3 \\ 2.3513 \ln(r^+) - 1.2777 & \text{if } 378.3 \leq r^+ < 2275 \\ 2.1973 \ln(r^+) - 0.0873 & \text{otherwise} \end{cases} \] (73)

Eq. (73) and Eq. (66) together with the definition of the wall shear stress, friction velocity, and the knowledge of the physical distance and the local velocity magnitude is sufficient to evaluate the wall shear stress. The expression for the wall shear stress in the new variable becomes
\[ \tau_w = \rho \left( \frac{u - u_p \; \text{sgn}\left(\frac{dp_w}{dr}\right) f_2(y^+_p)}{\text{sgn}(u_1) g_1(r^+)} \right)^2 \] (74)
Therefore, the boundary conditions for the momentum equation is easily specified by computing the wall shear stress from Eq. (74). The boundary conditions for transported turbulence quantities are also easily specified by providing the asymptotic behavior of the turbulence model in boundary layer [7,17].

The explicit expression for the wall shear stress in Eq. (68) is easily adapted to work with other formulations of the gradient-free wall functions. All that is required is to define the function $g_1(r^+)$ for each law of the wall by transforming from $y^+$ to $r^+$ units. A simple example of analytic transformation was shown in the case of the Prandl’s hypothesis [12] and the log-law, as shown in Eq. (40) and Eq. (50). In case of more complicated expressions, a curve fit procedure can be used to obtain the function $g_1(r^+)$ as demonstrated in case of Musker’s [5] and Shih’s [9] law of the wall. Curve fit, or a transformation of variables can also be applied to the Werner and Wengle law of the wall, Eq. (11) and Eq. (12). Expression of Eq. (74) represents a unifying framework for several proposed laws of the wall with pressure gradient effect in the new wall units $r^+$. As such, the proposed expression can be used in numerous numerical codes to provide the pressure effects on the wall shear stress. The essential advantage of the proposed law of the wall is that it is obtained in the new wall units that do not require an iterative procedure to evaluate the wall shear stress.

### 3 Sample Applications of Wall Functions in Transformed Variables

In this section we illustrate the application of the generalized wall function in new set of variables using widely used $k - \omega$ SST model [27]. The governing system of equations for incompressible turbulent flow consists of the transport equation for the momentum and turbulence quantities. The following expression gives the momentum equation in Cartesian tensor notation with Einstein summation rule over the repeated indexes:
\[ \frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} \] (75)
Indexes $i$ and $j$ take the values from the set $(x, y, z)$ that labels three Cartesian directions. As before, all quantities in momentum and turbulence transport equations are time averaged. The symbol $\sigma_{ij}$ represents the viscous stress tensor.
consisting of molecular, and Reynolds averaged stress tensor

\[ \sigma_{ij} = (\mu + \mu_t) 2S_{ij} \]  

(76)

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]  

(77)

Since we are considering incompressible turbulent flows, the density \( \rho \) and molecular viscosity \( \mu \) are assumed to be constant. All quantities in Eq. (75) are time averaged to produce the Reynolds averaged Navier-Stokes equations. In addition, all quantities in subsequent expressions are time averaged and no special notation is used to denote time averaging.

To close Eq. (75), turbulent viscosity field \( \mu_t \) must be known. In this work, we compute the turbulent viscosity field by solving the \( k - \omega \) SST transport equations [27]. All time-averaged quantities must be specified on the boundary of the domain to define the well-posed problem. Boundary conditions away from solid walls pose no difficulties, and they are prescribed as in [27]. At the solid wall where no-slip boundary condition is applied, special care must be taken to specify boundary values of \( k \) and \( \omega \) fields. Boundary conditions for \( k \) and \( \omega \) fields are obtained from the asymptotic near-wall behavior of the corresponding transport equations for \( k \) and \( \omega \) [7].

We distinguish between computations that are wall-resolved and not wall-resolved. We consider the computation to be wall-resolved if the first cell of the computational mesh of the wall produces a \( y^+ \) value that less or equal to one. In this work, we are concerned with computational meshes that are not wall-resolved. The values of \( y^+ \) in the first computational point off the wall take much larger values, i.e., \( y^+ \gg 1 \). The wall shear stress for such computational meshes is evaluated from Eq. (72). Additional specification of \( k \) and \( \omega \) fields near the wall must also be specified. The specific dissipation field \( \omega \) in the laminar sublayer of the turbulent boundary layer is specified as follows:

\[ \omega^+_{vis} = \frac{6}{\beta_1 (y^+)^2} \]  

(78)

where \( \omega^+ \) is normalized specific dissipation defined as

\[ \omega^+ = \frac{\omega_\nu}{u_*^+} \]  

(79)

Eq. (78) contains a singularity as \( y^+ \to 0 \). The singularity is removed by evaluating the value of \( \omega^+ \) at the finite distance from the wall. In cell-centered finite volume methods, the boundary condition is obtained by using the value of \( y^+ \) that corresponds to the cell center of the first cell next to the wall.

The following expression gives the value of \( \omega^+ \) in the logarithmic region of the turbulent boundary value [71]:

\[ \omega^+_{log} = \frac{1}{k \sqrt{C_\mu y^+}}, \quad C_\mu = 0.09 \]  

(80)

The intermediate region between laminar sublayer and logarithmic layer is interpolated by the use of the following blending equation:

\[ \omega^+ = \sqrt{\left( \omega^+_{vis} \right)^2 + \left( \omega^+_{log} \right)^2} \]  

(81)

The functional dependence of \( k^+ \) in the boundary layer is obtained from the expression

\[ k^+ = u_*^+ \omega^+ \]  

(82)

The normalized turbulent kinetic energy \( k^+ \) is defined per following expression

\[ k^+ = \frac{k}{u_*^2} \]  

(83)

It is clear from Eq. (79) and Eq. (83) that the knowledge of the friction velocity is required for the boundary conditions for \( k \) and \( \omega \) transport equations. The value of the friction velocity is computed from Eq. (72) making use of the wall functions with pressure gradients.
4 Computational Examples

To illustrate the application of the new generalized wall functions, two computational studies were performed. The first study is the flow over backward-facing step [28], while the second study was concerned with the flow in a diffuser [29]. Both flows exhibit adverse pressure gradient and the flow separation thus making them suitable for the illustration of the application of wall function boundary condition as defined in Eq. (72), Eq. (81), and Eq. (82).

The second-order finite volume solver library Caelus [30, 31] was used in all computations. Second-order upwind discretization for the convective and central discretization of diffusive fluxes was used for momentum and turbulence equations [32]. The computational approach consisted of the successive uniform mesh refinement, and all computations were performed on the progression of four meshes. The main characteristic of the uniform mesh refinement is that it allows for the constant aspect ratio of cells across all computational meshes while at the same time, different values of $y^+$ were obtained. In this way, the performance of the proposed generalized wall functions is assessed in a controlled numerical experiment. Additionally, $k – \omega$ SST transport model [27] in all computations. However, the proposed generalized wall functions are equally applicable to other equilibrium and non-equilibrium turbulence transport models.

4.1 Backward Facing Step Study

The experimental setup of Driver and Seegmiller [28] produces the variable pressure gradient across the channel that affects the reattachment point of the shear layer emanating from the backward-facing step. The experiment was performed with various deflections of the top wall, demonstrating the influence of the pressure gradient on the reattachment point of the jet. In this work, we focus on a zero deflection angle for numerical illustrations. In addition to reattachment point measurements, Driver and Seegmiller provide the detailed measurements on the pressure and friction coefficients along the top and bottom walls as well as velocity profiles downstream of the backward-facing step.

The computational domain was constructed using a hexahedral mesh extending 40 step heights, $h$, upstream of the step and $50h$ downstream of the step to ensure flow development is not influenced by the boundary conditions. Fig. 3 shows a magnified section, centered on the backward step, of the 65,000 cell mesh and relative domain dimensions.

![Figure 3: Magnified section view of Backward facing step 65,000 cell mesh. Domain dimensions are normalized by the step height, $h$.](image)

Boundary conditions used in computations correspond to atmospheric pressure and temperature while the inlet velocity is $44.2 \text{ m/s}$ and turbulence intensity is $0.061\%$. As indicated, four the meshes, with varying grid resolution, were generated so that the effect of $y^+$ variations can be examined. The increasing grid refinement levels are shown in Table 1.

| Grid Refinement Level | $N$   | $y^+$ |
|-----------------------|-------|-------|
| 1                     | 16,000| 30    |
| 2                     | 65,000| 15    |
| 3                     | 261,000| 6     |
| 4                     | 1,046,000| 3   |

Table 1: Range of mesh element count ($N$), wall spacing ($y^+$), and aspect ratio ($AR$) used for the backward facing step simulations. The grids were refined requiring $AR$ consistency over the desired $y^+$ range.

Figure 4 shows the comparison between experimental and computed results for the range of values of $y^+$ at the lower wall for the normalized axial distance ranging from $x/h = -5$ to $x/h = 35$. The friction coefficient for values of $y^+_r = 30$ and $y^+_r = 15$ show the discrepancy between computed and measured results in the region of reattachment ($5 \leq x/h \leq 15$). The error is larger for the mesh with $y^+_r = 30$, and the error becomes progressively smaller as the
mesh is refined. All meshes underpredict the values of friction coefficient for \(0 \leq x/h \leq 5\). Overall, the shape of computed friction coefficient curves is similar to the experimental one for all meshes. Similarly, the comparison of computed and measured pressure coefficient in Fig. 4 is consistent for all computational meshes with underprediction of the pressure coefficient for \(-5 \leq x/h \leq 6\). The pressure coefficient is overpredicted for \(7 \leq x/h \leq 15\).

Figure 4: Channel lower wall coefficient of friction (\(C_f\)) between \(x/h = -5\) and \(x/h = 35\). Solid circles correspond to the experimental measurements of Driver and Seegmiller \[28\] and simulation results are represented by lines with symbols: square \(- y^+_t = 30\), triangle \(- y^+_t = 15\), diamond \(- y^+_t = 6\), and x \(- y^+_t = 3\).

Velocity profiles \(u/U_{ref}\) are in a good agreement with experimental findings for \(x/h = -4\), \(x/h = 1\), and \(x/h = 4\) as shown in Fig. 6 to Fig. 9. The velocity of the first point of the lower wall for the coarsest mesh \(y^+_t = 30\) displays the discrepancy of the axial location where the velocity achieves the value of zero (no slip condition) compared to other meshes and experimental results. All other meshes predict the velocity profiles in close agreement with experiments. The difference in velocity profiles between experiments and computations become progressively larger for streamwise measurement locations of \(x/h = 6\) and larger.

4.2 Asymmetric Plane Diffuser Study

The asymmetric flow in a plane diffuser experiment \[29, 33\] is the recreation of the original experiment performed by Obi and Matsui \[34\]. The fully turbulent flow on the inlet to the diffuser causes separation and reattachment of the flow at the lower wall. The complex flow features in the plane diffuser pose a challenge for many turbulence models. The presence of the separation region is particularly attractive for testing of wall functions.

The computational domain consists of a two-dimensional hexahedral grid, shown in Fig. 10 with an upstream channel height, \(h\), and expansion ratio of 4.7. The flow enters the domain 110 channel heights upstream and exits 55 channel heights downstream of the plane diffuser.

The boundary conditions are prescribed according to those measured experimentally \[29\]. The inlet flow velocity is \(u = 0.3\, \text{m/s}\) with turbulent kinetic energy and specific dissipation of \(k = 0.2945755\, \text{m}^2/\text{s}^2\), and \(\omega = 97.37245\, \text{1/s}\),
respectively. The upper and lower walls satisfy the no-slip condition, and the outlet boundary was modeled using the zero Neumann condition.

The computational approach consisted of the simulation of a sequence of uniformly refined meshes with decreasing \( y^+ \), as shown in Table 2. Again, the uniform grid refinement was performed to obtain a sequence of meshes with consistent cell aspect ratio and varying \( y^+ \).

![Lower Wall](image)

**Figure 5:** Comparison lower wall coefficient of pressure, \( C_p \), measured experimentally - filled circles [28] and simulated - unfilled symbols detailed in figure legend.

| Grid Refinement Level | \( N \)  | \( y^+ \) |
|-----------------------|----------|-----------|
| 1                     | 5,000    | 40        |
| 2                     | 20,000   | 20        |
| 3                     | 82,000   | 10        |
| 4                     | 332,000  | 5         |

Table 2: Range of mesh element count (\( N \)), wall spacing (\( y^+ \)), and aspect ratio (\( AR \)) used for the asymmetric plane diffuser simulations. The grids were refined requiring \( AR \) consistency over the desired \( y^+ \) range.

Figure 11 displays the distribution of \( y^+ \) along the lower wall for four meshes. The range of \( y^+ \) values are between 5 and 40. The comparison between computed and measured velocity profiles in the diffuser are shown in Fig. 13 where a good agreement between computed and measured results can be seen. The mesh that corresponds to the \( y^+ \) value shows the largest discrepancy between computed and measured velocity profile. Also, Fig. 13 shows that with the uniform mesh refinement the discrepancy becomes smaller, indicating the mesh convergence.
Figure 6: Stream-wise velocity profiles at the location $x/h = -4$ with respect to wall distance, $y$, normalized by step height, $h$. The velocity profile is normalized the reference velocity, $U_{ref}$, defined in Ref [28] for each simulation.

5 Summary and Conclusions

A new formulation of wall functions is proposed in the new set of variables that are more convenient for implementation in computational codes. The main advantage of the new formulation is that it allows the computation of the wall shear stress without the need for an iterative procedure. Moreover, the new formulation is written in an explicit form so that it can be used to generalize several proposed laws of the wall that do not include pressure gradient effects. To achieve this goal, the new near-wall variable $r^+$ was introduced through defining the normalization using the local velocity. The introduction of the normalization with respect to the local velocity is the key to recasting the existing laws of the wall in the form that yields an explicit expression for the wall shear stress. Also, it was demonstrated that the law of the wall in the new and old variables leads to the same values of the wall shear stress. Therefore, the newly proposed formulation is consistent with the classic one with the ability to produce explicit expressions for the wall shear stress. Two computational studies involving the flow over backward-facing step and the flow in the diffuser demonstrate that the new form of generalized wall functions performs well while including the gradient of pressure effects. It is also shown that the new formulation converges to physical results obtained by measurements in the grid refinement study.

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Figure 9: Stream-wise velocity profiles at the location $x/h = 6$ with respect to wall distance, $y$, normalized by step height, $h$. The velocity profile is normalized the reference velocity, $U_{ref}$, defined in Ref [28] for each simulation.

Figure 10: Plane diffuser 20,000 cell computational grid structure.

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Figure 11: Lower wall $y_+^+$ variations for each grid resolution.
Figure 12: Coefficient of pressure along the asymmetric plane diffuser lower wall.
Figure 13: Normalized stream-wise velocity profiles at $x/h = -5.8$, $x/h = 2.6$, $x/h = 6.0$, $x/h = 13.5$, $x/h = 20.0$, and $x/h = 27.23$.