Study of Navier-Stokes equation by using Iterative Laplace Transform Method (ILTM) involving Caputo- Fabrizio fractional operator

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Abstract. This article arrangement with N-S equation containing the Caputo-Fabrizio differential operator of fractional order. The Iterative Laplace Transform Method (ILTM) has been applied to found numerical solution of time-fractional N-S equation in a tube with unsteady fluid flow in the Caputo-Fabrizio sense. The ILTM is an elegant coupling of transform of the Laplace and new Iterative method (NIM). This scheme provides numerical solution in the terms of power series with easily computable terms. It is observed that the solutions of N-S equations obtained by the ILTM rapidly convergent to exact solutions.

Keywords: Caputo-Fabrizio fractional derivative operator (C-FFDO); Iterative Laplace transform method (ILTM); Time-fractional N-S equations; Analytical solution.

1. Introduction

A branch of applied mathematics, we study of arbitrary order derivatives and integrals, known as fractional calculus. The most important advantages of using fractional derivative model is that we can calculate memory, history, or non-local effects through this, which is difficult to models through integer order derivatives [16, 22, 24]. Motivated by this fact, many researchers has been developed numerous analytic methods containing relatively fractional differential operators, like Riemann-Liouville [1], Caputo [30], Hilfer[9] etc. to find the approximate solutions [3, 7, 14, 28, 29]. But the problem in using these fractional operators is that they have singular kernel which occurs at the end part of an interval of definition. To rectify this problem Caputo-Fabrizio [6] defined a fractional derivative through non-singular kernel called as the Caputo-Fabrizio fractional derivative operator (C-FFDO). The main difference is that Caputo fractional derivative [5] defined as power law, and the Caputo-Fabrizio fractional derivative is defined as exponential decay law. Over time, the applications of the C-FFDO has been developed and analyzed by several researchers [2, 4, 11-13, 15, 21, 27, 31].

The Navier-Stokes equations is elementary problem of various geophysical fluid mechanical problems. The N-S equations has been presented the relation between the pressure and external force acting on a fluid to the reaction of fluid flow. The N-S equation and continuity equation are given as:
\[ \frac{\partial \varphi}{\partial \tau} + (\varphi \nabla) \varphi = -\nu \nabla P + k \nabla^2 \varphi , \quad (1) \]

\[ \nabla \varphi = 0, \quad (2) \]

In eqs. (1) and (2) \( \nu \), \( P \), \( k \), \( \varphi \), and \( \tau \) represents density, pressure, kinematics viscosity, flow velocity, velocity and time respectively. El-Shashed and Salem [8] has been presented generalized N-SE of order \( \zeta \), \( 0 < \zeta \leq 1 \). The generalized N-S equation of order \( \zeta \), \( 0 < \zeta \leq 1 \) in the sense of the Caputo-Fabrizio fractional derivative operator (C-FFDO) is given as:

\[ C^\zeta_F D_\tau^\zeta \varphi + (\varphi \nabla) \varphi = -\nu \nabla P + k \nabla^2 \varphi , \quad (3) \]

Eq. (1) has been handled by numerous researchers like Ganji et al. [10] by HPM, Momani and Odibat [23] by ADM, Kumar et al. by HPTM [18] and MFLDM [19], Amit et al. by [25] q-HATM, Khan et al. by HPM and VIM [20], Ragab et al. by HAM [26]. But it has not been resolved by the ILTM.

In this paper, we get the solution of N-S equations by the ILTM. The ILTM is an elegant coupling of NIM [17] and Laplace transform. The objective of this paper is obtain the solution of the time-fractional N-S equation. The main advantage of this scheme is that it is merging two powerful methods (LTM and NIM) and it can be decreasing the volume of the computational work as compared to the classical methods.

2. Preliminaries

**Definition 2.1** Let \( \varphi \in G^\delta (0,\delta) , \delta > 0, 0 < \zeta < 1 \), then the time fractional Caputo-Fabrizio fractional differential operator (C-FFDO) has been defined by [6] as:

\[ C^\zeta_F D_\tau^\zeta \varphi (\tau) = \frac{2 - \zeta}{2(1 - \zeta)} \Theta (\zeta) \left[ \int_0\exp \left\{ -\frac{\zeta(t-s)}{1-\zeta} \right\} \varphi (\tau_i)d\tau_i \right], \quad \tau \geq 0, 0 < \zeta < 1, \quad (4) \]

In equation (4) \( \Theta (\zeta) \) represents normalization function, which holds the property \( \Theta (0) = \Theta (1) = 1 \).

**Definition 2.2** The C-FFDO of order \( 0 < \zeta < 1 \) has been represented by [4] as:

\[ C^\zeta I_\tau^\zeta \varphi (\tau) = \frac{2(1 - \zeta)}{2 - \zeta} \Theta (\zeta) \varphi (\tau) + \frac{2\zeta}{2 - \zeta} \Theta (\zeta) \int_0^\tau \varphi (\tau_i)d\tau_i , \quad (5) \]

**Definition 2.3** The Laplace transform (LT) of the C-FFDO of order \( 0 < \zeta \leq 1 \) and \( p \in \mathbb{R} \) has been defined by [6] as:

\[ L \left( C^\zeta_F D_\tau^\zeta \varphi (\tau) \right) (s) = \frac{1}{1 - \zeta} L \left( \varphi^{(p+1)} (\tau) \right) L \left\{ \exp \left\{ -\frac{\zeta}{1-\zeta} \right\} \right\} = \frac{s^{p+1} L (\varphi (\tau)) - s^p \varphi (0) - s^{p-1} \varphi ^\prime (0) - \ldots - \varphi ^{(p)} (0)}{s + \zeta \left( 1 - s \right)} , \quad (6) \]
Specific cases:

\[ L \left( c^f D_t^\zeta \varphi (t) \right) (s) = \frac{s L(\varphi (t)) - \varphi (0)}{s + \zeta(1-s)}, \quad p = 0, \]

\[ L \left( c^f D_t^{\zeta+1} \varphi (t) \right) (s) = \frac{s^2 L(\varphi (t)) - s \varphi (0) - \varphi' (0)}{s + \zeta(1-s)}, \quad p = 1. \]

3. Main Result

In this section, we solve the Navier-Stokes equations for one-dimensional motion of a viscous fluid with unsteady flow by using the ILTM. The N-S equation in cylindrical coordinates is given as:

\[ \frac{\partial \varphi}{\partial t} = -\nu \frac{\partial P}{\partial z} + k \left( \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} \right), \quad (7) \]

with the initial condition:

\[ \varphi (r, 0) = \phi (r) \quad (8) \]

If the fractional derivative of order \( \zeta \) is applying on (7) at extant time derivative term, the equation (7) admit the form:

\[ \frac{\partial^\xi \varphi}{\partial r^\xi} = P_i + k \left( \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} \right), \quad 0 < \xi \leq 1, \quad (9) \]

where \( p_i = -\nu^{-1} \frac{\partial P}{\partial z} \). To apply the ILTM, (9) can be written in the form of the C-FFDO as:

\[ c^f D_t^\zeta \varphi (r) = P_i + k \left( \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} \right), \quad 0 < \zeta \leq 1, \quad (10) \]

we use the linear property of the Laplace transform on (10), it yields:

\[ L \left[ c^f D_t^\zeta \varphi (t) \right] = L[P_i] + L \left[ k \left( \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} \right) \right], \quad (11) \]

with the initial condition (8), we have:

\[ L[\varphi (r, t)] = \sigma (r, s) + \left( s + \zeta(1-s) \right) L \left[ \nu \left( \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} \right) \right], \quad (12) \]

where, \( \sigma (r, s) = \frac{\phi (r)}{s} + \frac{s + \zeta(1-s)}{s} L[P_i] \)

By taking inverse Laplace transform on both sides of (12), we have:
\[ \varphi(r, \tau) = \sigma(r, \tau) + L^1 \left[ \left( \frac{s + \zeta(1-s)}{s} \right) L \left[ k \left( \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} \right) \right] \right]. \] (13)

The term \( \sigma(r, \tau) \) gained from the source term. Now, we will use NIM [7], we have the solution as an infinite series represents as:

\[ \varphi(r, \tau) = \sum_{n=0}^{\infty} \varphi_n(r, \tau) \] (14)

Substituting (14) in (13), we have:

\[ \sum_{n=0}^{\infty} \varphi_n(r, \tau) = \sigma(r, \tau) + L^1 \left[ \left( \frac{s + \zeta(1-s)}{s} \right) L \left[ k \left( \frac{\partial^2 \varphi_n}{\partial r^2} \sum_{n=0}^{\infty} \varphi_n(r, \tau) + \frac{1}{r} \frac{\partial}{\partial r} \sum_{n=0}^{\infty} \varphi_n(r, \tau) \right) \right] \right] \] (15)

Comparing the coefficients of \( \varphi_n(r, \tau) \), for \( n=0,1,2,3, \ldots \) to both sides of (15), we get:

\[ \varphi_0(r, \tau) = \sigma(r, \tau) \] (16)

\[ \varphi_1(r, \tau) = L^1 \left[ \left( \frac{s + \zeta(1-s)}{s} \right) L \left[ k \left( \frac{\partial^2 \varphi_0}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_0}{\partial r} \right) \right] \right] \] (17)

\[ \varphi_2(r, \tau) = L^1 \left[ \left( \frac{s + \zeta(1-s)}{s} \right) L \left[ k \left( \frac{\partial^2 \varphi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_1}{\partial r} \right) \right] \right] \] (18)

\[ \varphi_3(r, \tau) = L^1 \left[ \left( \frac{s + \zeta(1-s)}{s} \right) L \left[ k \left( \frac{\partial^2 \varphi_2}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_2}{\partial r} \right) \right] \right] \] (19)

\[ \ldots \]

\[ \ldots \]

Thus the series solution of (7) is:

\[ \varphi(r, \tau) = \sum_{n=0}^{\infty} \varphi_n(r, \tau) \] (20)

4. Numerical problems

By some numerical problems, we establish the efficiency and accuracy of our solution of Navier-Stokes equation in this section.
Example: 4(a)

Time-fractional Navier-Stokes equation is given as:

\[
\frac{c^\alpha}{D^\alpha} \varphi_r = P_1 + \frac{\partial^\alpha \varphi}{\partial r^\alpha} + \frac{1}{r} \frac{\partial \varphi}{\partial r}, \quad 0 < \zeta \leq 1,
\]

with the initial condition:

\[
\varphi(r,0)=1-r^2.
\]

Applying the Laplace transform on equation (21) and used (22), we have:

\[
L\left[ \varphi(r,\tau) \right] = \frac{1}{s} \left(1-r^2\right) + \left(\frac{s+\zeta(1-s)}{s^2}\right) P_1 + \left(\frac{s+\zeta(1-s)}{s}\right) L\left[ \frac{\partial^\alpha \varphi}{\partial r^\alpha} + \frac{1}{r} \frac{\partial \varphi}{\partial r} \right],
\]

By taking inverse of the Laplace transform:

\[
\varphi(r,\tau) = \left(1-r^2\right) + P_1 \left(1+\zeta(\tau-1)\right) + L^{-1}\left[ \frac{s+\zeta(1-s)}{s} \right] L\left[ \frac{\partial^\alpha \varphi}{\partial r^\alpha} + \frac{1}{r} \frac{\partial \varphi}{\partial r} \right],
\]

By using NIM in the above equation, we get:

\[
\sum_{n=0}^{\infty} \varphi_n(r,\tau) = \left(1-r^2\right) + P_1 \left(1+\zeta(\tau-1)\right)
\]

\[
+ L^{-1}\left[ \frac{s+\zeta(1-s)}{s} \right] \left[ \frac{\partial^\alpha \varphi}{\partial r^\alpha} \left(\sum_{n=0}^{\infty} \varphi_n(r,\tau)\right) \right] + \frac{1}{r} \frac{\partial}{\partial r} \left(\sum_{n=0}^{\infty} \varphi_n(r,\tau)\right)
\]

Comparing the coefficients of \(\varphi_n(r,\tau)\), for \(n=0,1,2,3,\ldots\) to both sides of (25), we get:

\[
\varphi_0(r,\tau) = \left(1-r^2\right) + P_1 \left(1+\zeta(\tau-1)\right)
\]

\[
\varphi_1(r,\tau) = L^{-1}\left[ \frac{s+\zeta(1-s)}{s} \right] L\left[ \frac{\partial^\alpha \varphi}{\partial r^\alpha} + \frac{1}{r} \frac{\partial \varphi}{\partial r} \right]
\]

\[
= L^{-1}\left[ \frac{s+\zeta(1-s)}{s} \right] \left( -2 + \frac{1}{r}(-2r) \right)
\]

\[
= 4 \left(1+\zeta(\tau-1)\right)
\]

\[
\varphi_2(r,\tau) = L^{-1}\left[ \frac{s+\zeta(1-s)}{s} \right] L\left[ \frac{\partial^\alpha \varphi}{\partial r^\alpha} + \frac{1}{r} \frac{\partial \varphi}{\partial r} \right]
\]

\[
= 0
\]

\[
\varphi_n(r,\tau) = 0, \quad \forall n \geq 2
\]
Thus, the approximate solution of (21) is acquired as:

\[
\varphi(r, \tau) = \sum_{n=0}^{\infty} \phi_n(r, \tau)
\]

\[
\varphi(r, \tau) = \left(1 - r^2\right) + (P_1 - 4)\left(1 + \zeta(\tau - 1)\right)
\]

(29)

(30)

If \( \zeta = 1 \), then we have the solution of the classical Navier-Stokes equation is:

\[
\varphi(r, \tau) = \left(1 - r^2\right) + (P_1 - 4)\tau
\]

(31)

We observe that for approximate solution of example 4.1 only third-order terms will be used and if \( \zeta = 1 \), then the solution is reduced to the solution of the classical N-S equation given as (31). Figure 1 implicates the numerical results of example 4.1 for values of \( \zeta \) as \( \zeta = \frac{1}{3} \), \( \zeta = \frac{2}{3} \), \( \zeta = 1 \) and eq. (31). In figure 1 we observe that when time raises then the solution of example 4.1 shrinkages rapidly.

![Figure 1](image_url)

**Figure 1.** implicates that the solution \( \varphi(r, \tau) \) w.r.t. \( r \) and \( \tau \) when (a) \( \zeta = \frac{1}{3} \), (b) \( \zeta = \frac{2}{3} \), (c) \( \zeta = 1 \) and (d) solution of eq. (31).
Example: 4(b)

Time-fractional Navier-Stokes equation is given as:

\[ c_F D_t^{\zeta} \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r}, \quad 0 < \zeta \leq 1, \]  

(32)

with the initial condition:

\[ \varphi(r, 0) = r^2, \]  

(33)

In a similar way as above, we have:

\[ \sum_{n=0}^{\infty} \varphi_n(r, \tau) = r + L^1 \left[ \left( \frac{s + \zeta (1-s)}{s} \right) L \left[ \frac{\partial^2 \varphi_n}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_n}{\partial r} \right] \right], \]  

(34)

Comparing the coefficients of \( \psi_n(r, \tau) \) to both sides of eq. (34), we get:

\[ \varphi_0(r, \tau) = r, \quad n = 0, 1, 2, 3, \ldots \]  

(35)

\[ \varphi_1(r, \tau) = L^1 \left[ \left( \frac{s + \zeta (1-s)}{s} \right) L \left[ \frac{\partial^2 \varphi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_1}{\partial r} \right] \right], \]  

(36)

\[ = \frac{1}{r} \left( 1 + (\tau - 1) \zeta \right) \]

\[ \varphi_2(r, \tau) = L^1 \left[ \left( \frac{s + \zeta (1-s)}{s} \right) L \left[ \frac{\partial^2 \varphi_2}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_2}{\partial r} \right] \right], \]  

(37)

\[ = L^1 \left[ \left( \frac{s + \zeta (1-s)}{s} \right) L \left[ \frac{2}{r^3} + \frac{2}{r^3} (\tau - 1) + \frac{1}{r^3} \left( \frac{1}{r^2} - \frac{1}{r^2} \left( \tau - 1 \right) \right) \right] \right], \]

\[ = \frac{(1 - \zeta)^2}{r^3} \frac{2 \zeta (1 - \zeta)}{r^3} \tau + \frac{\zeta^2 \tau^2}{2!} \]

\[ \varphi_3(r, \tau) = L^1 \left[ \left( \frac{s + \zeta (1-s)}{s} \right) L \left[ \frac{\partial^2 \varphi_3}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_3}{\partial r} \right] \right], \]  

(38)

\[ = 9 \frac{(1 - \zeta)^3}{r^3} + 27 \frac{\zeta (1 - \zeta)^2}{r^3} \tau + 27 \frac{\zeta^2 (1 - \zeta) \tau^2}{2!} + 9 \frac{\zeta^3 \tau^3}{3!} \]

Thus the solution obtained as:
\[ \varphi(r, \tau) = r + \frac{1}{r} \left( \frac{1}{1 + \zeta (r - 1)} \right) + \frac{1 - \zeta}{r} + \frac{2 \zeta (1 - \zeta)}{r^3} + \frac{\zeta^2}{r^4} \tau^2 + \frac{\zeta}{r^5} \tau^2 + 9 \frac{(1 - \zeta)}{r^7} + 27 \frac{\zeta (1 - \zeta)}{r^9} + 27 \frac{\zeta^2 (1 - \zeta)}{r^{10}} \tau + 27 \frac{\zeta^3}{r^{11}} \tau^2 + 9 \frac{\zeta^4}{r^{12}} \tau^3 + \ldots \] (39)

If we take \( \zeta = 1 \), then we get the solution of (32) is:

\[ \varphi(r, \tau) = r + \sum_{n=1}^{\infty} \frac{1^2 \times 3^2 \times 5^2 \times \ldots \times (2n-3)^2}{r^{2n-1}} \frac{\tau^n}{n!} \] (40)

Equation (40) gives the same solution as obtained by Momani et al. [28], J. Singh et al. [23].

5. Concluding remarks

The Iterative Laplace transform method (ILTM) in Caputo-Fabrizo sense has been applied to obtained solution of the N-S equation. The main intention of this approach is that numerical as well as analytical solutions has been obtained in the term of convergent series with easily computable term. It is observed that the proposed scheme more powerful and realistic to obtained solutions of N-S equations and the solutions obtained by ILTM rapidly convergent to the exact solutions.

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