ON THE MODULI SPACE OF QUASI-HOMOGENEOUS FUNCTIONS

LEONARDO MEIRELES CÂMARA AND MARIA APARECIDA SOARES RUAS

Abstract. We relate the moduli space of analytic equivalent germs of reduced quasi-homogeneous functions at \((\mathbb{C}^2, 0)\) with their bi-Lipschitz equivalence classes. We show that any non-degenerate continuous family of (reduced) quasi-homogeneous functions with constant Henry-Parusiński invariant is analytically trivial. Further we show that there are only a finite number of distinct bi-Lipschitz classes among quasi-homogeneous functions with the same Henry-Parusiński invariant providing a maximum quota for this number.

1. Preliminaries

The main goal of this paper is to relate the moduli space of analytically equivalent germs of quasi-homogeneous functions with their bi-Lipschitz classes. In order to turn our statements more precise, and to give appropriate answers, we need to introduce some terminology.

We say that two function germs \(f, g \in \mathcal{O}_2\) are analytically equivalent if there is \(\Phi \in \text{Diff}(\mathbb{C}^2, 0)\) such that \(g = f \circ \Phi\). In this case, \(\Phi\) is said to be an analytic equivalence between \(f\) and \(g\). A germ of holomorphic function \(f\) is said to be quasi-homogeneous if it is analytically equivalent to a quasi-homogeneous polynomial. More precisely, if there is a local system of coordinates in which \(f\) can be written in the form \(f(x, y) = \sum_{a_i + b_j = d} a_{ij} x^i y^j\) where \(a, b, d \in \mathbb{N}\).

Recall that a germ of holomorphic function \(f \in \mathcal{O}_2\) is reduced if it has isolated singularity. Up to a permutation of the variables \(x\) and \(y\), any reduced quasi-homogeneous polynomial \(f\) with weights \((p, q)\) can be (uniquely) written in the form

\[
f(x, y) = x^m y^k \prod_{j=1}^{n} (y^p - \lambda_j x^q)
\]

where \(m, k \in \{0, 1\}, p, q \in \mathbb{Z}_+, p \leq q, \gcd(p, q) = 1,\) and \(\lambda_j \in \mathbb{C}^*\) are pairwise distinct. In particular \(C = f^{-1}(0)\) has \(n + m + k\) distinct branches. The triple \((p, q, n)\) is clearly an analytic invariant of the curve. In case \(m = 1\)

2010 Mathematics Subject Classification. Primary 32S05; Secondary 14J17.

Key words and phrases. bi-Lipschitz moduli, analytic moduli, quasi-homogeneous polynomials.
For $k = 0$, this polynomial is called \textit{commode} \((\text{AI}, \text{CS})\). Up to a local change of coordinates, the normal forms given by the sets I), II) and III) below determine a stratification of the moduli space of analytically equivalent germs of singular quasi-homogeneous functions (cf. \([\text{CS}, \text{K}]\)):

I) $k=0$ and $p=q=1$, i.e., \((1.1)\) reduces to $f(x, y) = x^m \prod_{j=1}^{n} (y - \lambda_j x)$,

II) $k=0$ and $1 = p < q$, i.e., \((1.1)\) reduces to $f(x, y) = x^m \prod_{j=1}^{n} (y - \lambda_j x^q)$,

III) $1 < p < q$, i.e., \((1.1)\) reduces to $f(x, y) = x^m y^k \prod_{j=1}^{n} (y^p - \lambda_j x^q)$.

Therefore, a reduced quasi-homogeneous function is said to be of type I, II, or III respectively in cases I), II), or III) above. Note that each $\lambda_j$ is a root of the polynomial $Q_\lambda(t) = \prod_{j=1}^{n} (t - \lambda_j)$, uniquely determined by $f(x, y)$.

\textbf{Theorem 1.1} \([\text{CS}]\). The analytic moduli space of germs of reduced quasi-homogeneous functions is given respectively by

1) $\frac{\text{Symm}(\mathbb{P}^1_\mathbb{C}(n))}{\text{PSL}(2,\mathbb{C})}$ for function germs of type I;

2) $\mathbb{Z}_2 \times \frac{\text{Symm}(\mathbb{C}_\Delta(n))}{\text{Aff}(\mathbb{C})}$ for function germs of type II;

3) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\text{Symm}(\mathbb{C}_\ast(n))}{\text{GL}(1,\mathbb{C})}$ for function germs of type III,

where for $M = \mathbb{P}^1, \mathbb{C}, \mathbb{C}^\ast$, $\text{Symm}(M_\Delta(n))$ is the quotient space by the action of the symmetric group $S_n$ on $M_\Delta(n) := \{(x_1, \cdots, x_n) \in M^n : x_i \neq x_j \text{ for all } i \neq j\}$ given by $(\sigma, x) \mapsto \sigma \cdot x = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$.

The analytic classification of the non-reduced case can be achieved by a similar reasoning \([\text{CS}]\).

Now let us turn to the bi-Lipschitz equivalence. Two function-germs $f, g \in \mathcal{O}_2$ are called \textit{bi-Lipschitz equivalent} if there exists a bi-Lipschitz map-germ $\phi: (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $f = g \circ \phi$.

In \([\text{HP}]\), Henry and Parusiński showed that the bi-Lipschitz equivalence of analytic function germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ admits continuous moduli. They introduce a new invariant based on the observation that a bi-Lipschitz homeomorphism does not move much the regions around the relative polar curves. For a single germ $f$ defined at $(\mathbb{C}^2, 0)$, the invariant is given in terms of the leading coefficients of the asymptotic expansions of $f$ along the branches of its generic polar curve.
Fernandes and Ruas in [FR2] study the strong bi-Lipschitz triviality: two function germs $f$ and $g$ are strongly bi-Lipschitz equivalent if they can be embedded in a bi-Lipschitz trivial family, whose triviality is given by integrating a Lipschitz vector field. They show that if two quasi-homogeneous (but not homogeneous) function-germs $(\mathbb{C}^2,0) \to (\mathbb{C},0)$ are strongly bi-Lipschitz equivalent then they are analytically equivalent. This result does not hold for families of homogeneous germs with isolated singularities and same degree since they are Lipschitz equivalent ([FR1]).

In some sense, for quasi-homogeneous (but not homogeneous) real function-germs in two variables the problem of bi-Lipschitz classification is quite close to the problem of analytic classification.

The moduli space of bi-Lipschitz equivalence is not completely understood yet, not even in the case of quasi-homogeneous function germs. This is the question we address in this paper. For quasi-homogeneous (but not homogeneous) polynomials in the plane, we ask whether Henry-Parusinski’s invariant characterizes completely their bi-Lipschitz class. In Theorems 4.1 and 4.4 we compare the moduli spaces of analytic and bi-Lipschitz equivalences. Finally, in Theorem 4.5 we prove that a bi-Lipschitz trivial family of quasi-homogeneous (but not homogeneous) function germs in the plane is analytically trivial.

In what follows, we assume that the quasi-homogeneous polynomial $f$ is miniregular in $y$ in the sense that $x = 0$ is not a line in the tangent cone of the curve $(f = 0)$. Finally, we note that two reduced quasi-homogeneous function germs of the form (1.1) are bi-Lipschitz equivalent if and only if their corresponding commode parts ($k = m = 0$) are equivalent ([FR1]).

2. Polar curves and bi-Lipschitz invariants

Taking into account the remark in the last paragraph, it is enough to consider germs of commode quasi-homogeneous functions, i.e., germs of functions as in (1.1), with $k = m = 0$ and $1 \leq p < q$. Here we describe some bi-Lipschitz invariants of these polynomials.

The polynomial $Q_\lambda(t)$. Let

\begin{equation}
(2.1) \quad f_\lambda(x,y) = \prod_{j=1}^{n} (y^p - \lambda_j x^q),
\end{equation}

where $p, q \in \mathbb{Z}_+, 1 \leq p < q, \gcd(p,q) = 1,$ and $\lambda_j \in \mathbb{C}^*$. Let $\partial_y f_\lambda = 0$ be the polar curve with respect to the direction $(0 : 1) \in \mathbb{P}^1$.

Now let $t = y^p/x^q$, then $f_\lambda(x,y) = x^{pq}Q_\lambda(t)$, where $Q_\lambda(t) = \prod_{j=1}^{n} (t - \lambda_j)$. The following hold:
Proposition 2.1.  a) \( f_\lambda(x, y) \) determines uniquely \( Q_\lambda(t) \). Reciprocally, for every pair of natural numbers \( 1 \leq p < q, \gcd(p, q) = 1 \), \( Q_\lambda(t) \) determines a unique polynomial \( f_\lambda(x, y) \);

b) Let \( \kappa = (\kappa_1, \ldots, \kappa_{n-1}) \) be all the (non-necessarily distinct) critical points of \( Q_\lambda(t) \). Then \( Q_\lambda(t) = n \prod_{j=1}^{n-1} (t - \kappa_j) \) and

\[
\partial_y f_\lambda(x, y) = pny^{p-1} \prod_{j=1}^{n-1} (y^p - \kappa_j x^q);
\]

c) Moreover, with \( \kappa = (\kappa_1, \ldots, \kappa_{n-1}) \) one has:

\[
\sigma_\ell(\kappa) = \frac{n - \ell}{n} \sigma_\ell(\lambda), \ell = 0, \ldots, n-1,
\]

where \( \sigma_\ell \) is the elementary symmetric polynomial of degree \( \ell \).

Proof. Condition a) follows immediately from the equation \( f_\lambda(x, y) = x^{na} \cdot Q_\lambda(y^p/x^q) \). Now \( Q_\lambda(t) = \prod_{j=1}^{n} (t - \lambda_j) \) may also be written in terms of the elementary symmetric polynomials \( \sigma_j(\lambda) \) as \( Q_\lambda(t) = \sum_{j=0}^{n} (-1)^j \sigma_j(\lambda) t^{n-j} \), \( \sigma_0(\lambda) = 1 \). Notice that \( Q_\lambda(t) \) is a degree \( n - 1 \) polynomial in the variable \( t \), thus we can write \( Q_\lambda(t) = n \prod_{j=1}^{n-1} (t - \kappa_j) \). The conditions (b) and (c) follow immediately from this.

The Henry-Parusiński invariants. We keep the notation of Proposition 2.1 and order the \( \kappa_i \)'s so that the ones equal to zero appear on places \( r + 1, \ldots, n - 1 \). More precisely, the irreducible branches of the polar curve \( \partial_y f_\lambda(x, y) = 0 \) are given by \( y^p - \kappa_\ell x^q = 0 \), for \( \ell = 1, \ldots, r; \) if \( p > 1 \), one more branch is also given by \( y = 0 \). Their Puiseux parametrisations are given by, respectively, \( \gamma_\ell(s) = (s^p, \alpha_\ell s^q) \), where \( \alpha_\ell = \sqrt[n]{\kappa_\ell} \neq 0 \), and by \( \gamma_0(s) = (s, 0) \).

When \( f \) is as in (2.1), the tangent cone of \( f(x, y) = 0 \) contains only one singular line given by \( y = 0 \). Notice that when \( f \) is a reduced homogeneous polynomial then the tangent cone of \( f(x, y) = 0 \) contains no singular line.

Recall from [HP, p. 225] that the Henry-Parusiński invariants are obtained in the following way. For each polar arc \( \gamma \) tangent to a singular line of the tangent cone of \( f(x, y) = 0 \), we associate two numbers: \( h = h(\gamma) \in \mathbb{Q}_+ \) and \( c = c(\gamma) \in \mathbb{C}^* \) given by the expansions \( f(\gamma(s)) = cs^h + \ldots, \) \( c \neq 0 \). In particular, the Henry-Parusiński invariants are not defined for reduced homogeneous germs.

Since \( f \) is quasi-homogeneous, \( h \) is determined by the weights, so we can omit it in the definition of the invariants. The invariant of Henry-Parusiński of a quasi-homogeneous \( f \) is the set \( \text{Inv}(f) = \left\{ \frac{c_0, c_1, \ldots, c_p}{c^p} \right\} \) for germs of type III (i.e., \( p > 1 \) in (2.1)) and \( \text{Inv}(f) = \left\{ \frac{c_1, \ldots, c_p}{c^p} \right\} \) for germs of type II, (i.e. \( p = 1 \) in (2.1)), where \( c_j \) is the coefficient of the leading term of \( f(\gamma_j(s)) \). If \( p > 1 \),
two sets \( \{c_0, c_1, \ldots, c_r\} \) and \( \{c'_0, c'_1, \ldots, c'_r\} \) define the same Henry-Parusiński invariant if there exists \( \xi \in \mathbb{C}^* \) such that
\[
\{c'_0, c'_1, \ldots, c'_r\} = \{c_0\xi^{q_n}, c_1\xi^{p_qn}, \ldots, c_r\xi^{p_qn}\}.
\]
If \( p = 1 \), two sets \( \{c_1, c_2, \ldots, c_r\} \) and \( \{c'_1, c'_2, \ldots, c'_r\} \) define the same Henry-Parusiński invariant if there exists \( \xi \in \mathbb{C}^* \) such that
\[
\{c'_1, c'_2, \ldots, c'_r\} = \{c_1\xi^{q_n}, c_2\xi^{q_n}, \ldots, c_r\xi^{q_n}\}.
\]
We have \( f \circ \gamma_0(s) = (-1)^n \lambda_1 \cdots \lambda_n s^{nq} \) (i.e., if \( p > 1 \)) and \( f \circ \gamma_\ell(s) = \prod_{j=1}^n (\kappa_\ell - \lambda_j) s^{pqn} \). Thus we define
\[
\rho_0 := (-1)^n \lambda_1 \cdots \lambda_n \text{ and } \rho_\ell := \prod_{j=1}^n (\kappa_\ell - \lambda_j).
\]

**Remark 2.2.** Although the polar arcs are not necessarily reduced, as we shall see in the next paragraph, this is precisely what happens in the generic case.

**The generic polynomial in two variables.** Let \( H^d_{p,q} \) be the set of commode monic and reduced quasi-homogeneous polynomials in two variables with relatively prime weights \((p,q), 1 \leq p < q\), and total degree \(d\). Then \( H^d_{p,q} \) is an affine space of dimension \( n \). From Proposition 2.1 there is an isomorphism \( H^d_{p,q} \cong P^n_1 \), where \( P^n_1 \) is the affine space of monic polynomials of degree \( n \), say \( Q_\sigma(t) = t^n + \sum_{j=1}^n (-1)^j \sigma_j t^{n-j} \). We write \( \psi : H^d_{p,q} \rightarrow P^n_1 \) and \( \pi : P^n_1 \rightarrow \mathbb{C}^n, \psi(f) = Q \) and \( \pi(Q) = \sigma := (\sigma_1, \ldots, \sigma_n) \).

The following result holds true:

**Theorem 2.3.** There is a Zariski open set \( Z \subset H^d_{p,q} \) such that for every \( f \in Z \) the following holds:

a) the polar curve \( \partial_y f_x = 0 \) has \( n-1 \) distinct roots;

b) For \( \ell \neq \ell', \rho_\ell \neq \rho_{\ell'}, 1 \leq \ell \leq n-1 \).

**Proof.** The family \( Q_\sigma := Q_\sigma(t) \) is the versal unfolding of \( Q_0(t) = t^n \). Therefore, as usual, we can define

1. \( B_L := \{ \sigma \in \mathbb{C}^n : \exists t_0 \in \mathbb{C} \text{ such that } Q'_\sigma(t_0) = Q''_\sigma(t_0) = 0 \} \);
2. \( B_G := \{ \sigma \in \mathbb{C}^n : \exists t_1, t_2 \in \mathbb{C} \text{ such that } Q'_\sigma(t_1) = Q'_\sigma(t_2) = 0 \text{ and } Q_\sigma(t_1) = Q_\sigma(t_2) \} \);

the local and semi-local subsets of the total bifurcation set \( B = B_L \cup B_G \) of \( Q_0 \). Since they are proper algebraic sets of \( \mathbb{C}^n \), we can take \( Z_0 = Z_L \cap Z_G \), where \( Z_L \) is the Zariski open set given by the complement \((\pi \circ \psi)^{-1}(B_L)\) and \( Z_G \) is the complement \((\pi \circ \psi)^{-1}(B_G)\). Polynomials in \( Z_L \) satisfy a) and those in \( Z_G \) satisfy b). Then \( Z_0 \) satisfies the required conditions. \( \Box \)
Remark 2.4. Notice that $B \subset \mathbb{C}^n$ has a stratification which induces a stratification in $(\pi \circ \psi)^{-1}(B)$.

3. Analytic moduli

The analytic classification of quasi-homogeneous function-germs at the origin of $\mathbb{C}^2$ was given in [K] and [CS] (cf. Theorem 1.1). We revisit the analytic classification from [CS] under the light of bi-Lipschitz invariants.

Let $M$ be a manifold and $M_{\Delta}(n) := \{(x_1, \cdots, x_n) \in M^n : x_i \neq x_j \text{ for all } i \neq j\}$. Let $S_n$ denote the group of $n$ elements and consider its action on $M_{\Delta}(n)$ given by $(\sigma, \lambda) \mapsto \sigma \cdot \lambda = (\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(n)})$. The quotient space induced by this action is denoted by $\text{Symm}(M_{\Delta}(n))$. Now suppose a Lie group $G$ acts on $M$ and let $G$ act on $M_{\Delta}(n)$ in the natural way $(g, \lambda) = (g \cdot \lambda_1, \cdots, g \cdot \lambda_n)$ for every $\lambda \in M_{\Delta}(n)$. Then the actions of $G$ and $S_n$ on $M_{\Delta}(n)$ commute. Thus we obtain a natural action of $G$ on $\text{Symm}(M_{\Delta}(n))$. Given $\lambda \in M_{\Delta}(n)$, denote its equivalence class in $\text{Symm}(M_{\Delta}(n))/G$ by $[\lambda]$.

Here we present some useful distinct characterizations of the analytic moduli space of reduced quasi-homogeneous function-germs.

The moduli space $\mathcal{M}^0_{[n],1}$. Let $\mathcal{M}^0_{[n],1}$ denote the moduli space of analytically equivalent punctured Riemann spheres with one marked puncture and $n$ unordered punctures, i.e., $\mathcal{M}^0_{[n],1} = \frac{\text{Symm}(\mathcal{C}_{\Delta}(n))}{\text{Aff}(\mathbb{C})}$. From Theorem 1.1 this coincides with the moduli space of quasi-homogeneous functions of type II.

Now we give a suitable description of the space $\mathcal{M}^0_{[n],1}$.

Let $\mathcal{W}^{n-1}_{0} := \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \text{Symm}(\mathcal{C}_{\Delta}(n)) : \lambda_1 + \cdots + \lambda_n = 0\}$ and consider the natural action of $\mathbb{C}^*$ on $\mathbb{C}^n$ given by $(\lambda, z) \mapsto \lambda z = (\lambda z_1, \cdots, \lambda z_n)$. Clearly, this action induces an analogous action on $\mathcal{W}^{n-1}_{0}$. Their cosets induce the following isomorphism.

Lemma 3.1. The moduli space $\mathcal{M}^0_{[n],1}$ is isomorphic to $\mathcal{W}^{n-1}_{0}/\text{GL}(1, \mathbb{C})$.

Proof. First notice that each class $[\lambda] \in \frac{\text{Symm}(\mathcal{C}_{\Delta}(n))}{\text{Aff}(\mathbb{C})}$ has a representative $[\lambda_0] \in \mathcal{W}^{n-1}_{0}$, thus it is enough to show that $[\lambda] = [\lambda'] \in \frac{\text{Symm}(\mathcal{C}_{\Delta}(n))}{\text{Aff}(\mathbb{C})}$ if and only if $[\lambda_0] = [\lambda_0'] \in \frac{\mathcal{W}^{n-1}_{0}}{\text{GL}(1, \mathbb{C})}$. Let $\Phi : \text{Symm}(\mathcal{C}_{\Delta}(n)) \longrightarrow \mathcal{W}^{n-1}_{0}$ be the map given by $\lambda = (\lambda_1, \cdots, \lambda_n) \mapsto \lambda_0 := \lambda - \frac{\lambda_1 + \cdots + \lambda_n}{n}(1, \cdots, 1)$. Then a straightforward calculation shows that $\Phi(a\lambda + b) = a\Phi(\lambda)$, i.e., if $\lambda' = a\lambda + b$, then $\lambda'_0 = a\lambda_0$. Thus $\Phi$ induces a well defined map $\overline{\Phi} : \frac{\text{Symm}(\mathcal{C}_{\Delta}(n))}{\text{Aff}(\mathbb{C})} \longrightarrow \frac{\mathcal{W}^{n-1}_{0}}{\text{GL}(1, \mathbb{C})}$. This map is clearly surjective, thus it suffices to show that it is injective. Now let $\lambda, \lambda' \in \text{Symm}(\mathcal{C}_{\Delta}(n))$ be such that $\Phi(\lambda') = a\Phi(\lambda)$, this implies the existence of $b, b' \in \mathbb{C}$ such that $\lambda' - b'(1, \cdots, 1) = a(\lambda - b(1, \cdots, 1))$. The result then follows. □
Now consider the above defined actions of $\mathbb{Z}_n$ on $\mathbb{C}^m$ and $\text{Symm}(\mathbb{C}^m)$, and let $\mathbb{W}^n_0 \subset \mathbb{W}^{n-1}_0$ be given by $\mathbb{W}^{n-2}_{0,1} = \{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{W}^{n-1}_0 : \lambda_1 \cdots \lambda_n = 1 \}$, then we can reduce $\mathcal{M}^0_{[n],1}$ a bit further.

**Lemma 3.2.** The moduli space $\mathcal{M}^0_{n,[1]}$ is isomorphic to $\mathbb{W}^{n-2}_{0,1}/\mathbb{Z}_n$.

**Proof.** Consider the map $\Psi : \mathbb{W}^n_0 \longrightarrow \mathbb{W}^{n-2}_{0,1}$ given by $\lambda = (\lambda_1, \ldots, \lambda_n) \mapsto \overline{\lambda} = \frac{1}{\lambda_1 \cdots \lambda_n} \lambda$. Similarly, it is enough to show that $[\lambda] = [\lambda'] \in \frac{\mathbb{W}^{n-1}_n}{\text{GL}(1, \mathbb{C})}$ if and only if $[\overline{\lambda}] = [\overline{\lambda}'] \in \frac{\mathbb{W}^{n-2}_{0,1}}{\mathbb{Z}_n}$. If $\lambda' = a\lambda$, then

$$\overline{\lambda'} = \frac{1}{\sqrt[\lambda_1 \cdots \lambda'_n]} \lambda' = \frac{a}{\omega^{-m} \cdot a} \frac{1}{\sqrt[\lambda_1 \cdots \lambda_n]} \lambda = \omega^m \cdot \overline{\lambda},$$

where $\omega = \exp(2\pi i/n)$. Thus $\Psi$ induces a map $\overline{\Psi} : \frac{\mathbb{W}^{n-1}_n}{\text{GL}(1, \mathbb{C})} \longrightarrow \frac{\mathbb{W}^{n-2}_{0,1}}{\mathbb{Z}_n}$. Clearly, $\overline{\Psi}$ is surjective, thus it suffices to prove that it is injective. Now notice that $\overline{\lambda'} = \omega^k \overline{\lambda}$ implies

$$\frac{\sqrt[\lambda_1 \cdots \lambda'_n]}{\sqrt[\lambda_1 \cdots \lambda_n]} = \frac{\omega^k \lambda}{\overline{\lambda}}.$$ 

But this is equivalent to say that $\lambda' = a\lambda$ for some $a \in \mathbb{C}$. \hfill \Box

Now recall the natural action $\mathbb{Z}_n$ on $\mathbb{C}^m$ given by $\zeta^s \mapsto (\zeta^s \cdot \mu_1, \zeta^s \cdot \mu_2, \ldots, \zeta^s \cdot \mu_m)$ with $\zeta = \exp(2\pi i/n)$, whose orbit space is denoted by $\mathbb{C}^m/\mathbb{Z}_n$, then the previous lemma leads to the following isomorphism.

**Lemma 3.3.** The map $\Xi : \mathbb{W}^{n-2}_{0,1} \longrightarrow \mathbb{W}^{n-2}_{0,1}$ given by $\lambda = (\lambda_1, \ldots, \lambda_n) \mapsto \kappa = (\kappa_1, \ldots, \kappa_n)$, where $\sigma_\ell(\kappa) = \frac{n-\ell}{n} \sigma_\ell(\lambda)$ for all $\ell = 1, \ldots, n-1$, induces the following isomorphism:

$$\mathcal{M}^0_{[n],1} \cong \frac{\mathbb{W}^{n-2}_{0,1}}{\mathbb{Z}_n}. \quad (3.1)$$

**Proof.** Since $\frac{n-\ell}{n} \sigma_\ell(\kappa)$, $\ell = 1, \ldots, n-1$, are the coefficients of the monic polynomial $z^n + \frac{n}{n-1} \sigma_1(\kappa) z^{n-1} + \cdots + \frac{n}{n-(n-1)} \sigma_{n-1}(\kappa) z + 1$ having $\{\lambda_1, \ldots, \lambda_n\}$ as roots, then $\Xi$ induces a bijective map $\Xi : \frac{\mathbb{W}^{n-2}_{0,1}}{\mathbb{Z}_n} \longrightarrow \frac{\mathbb{W}^{n-2}_{0,1}}{\mathbb{Z}_n}$. \hfill \Box

**The moduli space $\mathcal{M}^0_{[n],2}$.** Let $\mathcal{M}^0_{[n],2}$ denote the moduli space of analytically equivalent punctured Riemann spheres with two ordered punctures and $n$ unordered punctures, i.e., $\mathcal{M}^0_{[n],2} = \frac{\text{Symm}(\mathbb{C}^n)}{\text{GL}(1, \mathbb{C})}$. From Theorem 1.1, this coincides with the moduli space of quasi-homogeneous functions of type III. Let us give a suitable description of $\mathcal{M}^0_{[n],2}$.

On the analytic variety $\mathbb{V}^{n-1} = \{ \lambda \in \mathbb{C}^n : \lambda_1 \cdots \lambda_n = 1 \}$, we may consider the (effective) action of $\mathbb{Z}_n$ given by the multiplication by the $n$-th roots of unity, i.e., $\zeta^s \mapsto \zeta^s \cdot \lambda$ with $\zeta = \exp(2\pi i/n)$. Therefore, we have the following isomorphism.
**Lemma 3.4.** Let $\mathbb{V}^{n-1} = \{ \lambda \in \text{Symm} \mathbb{C}_\Delta^*(n) : \lambda_1 \cdots \lambda_n = 1 \}$, then

\[ (3.2) \quad \mathcal{M}_{0,2}^0 \cong \frac{\mathbb{V}^{n-1}}{\mathbb{Z}_n}. \]

*Proof.* The map $\Psi : \text{Symm} \mathbb{C}_\Delta^*(n) \to \mathbb{V}^{n-1}$ given by $\lambda = (\lambda_1, \ldots, \lambda_n) \mapsto \overline{\lambda} = \frac{1}{\sqrt[n]{\lambda_1 \cdots \lambda_n}}$ induces a well defined map $\Psi : \frac{\text{Symm} \mathbb{C}_\Delta^*(n)}{\text{GL}(1, \mathbb{C})} \to \frac{\mathbb{V}^{n-1}}{\mathbb{Z}_n}$. In fact, $[\lambda'], [\lambda] \in \mathbb{V}^{n-1}$ are equivalent in $\text{Symm} \mathbb{C}_\Delta^*(n)$ if and only if $\alpha \in \mathbb{C}^*$ such that $\lambda' = \alpha \lambda$. Since $\lambda_1 \cdots \lambda_n = 1 = \lambda_1' \cdots \lambda_n'$, then $\alpha^n = 1$.

As before, it is enough to show that $[\lambda] = [\lambda] \in \frac{\text{Symm} \mathbb{C}_\Delta^*(n)}{\text{GL}(1, \mathbb{C})}$ if and only if $[\lambda] = [\lambda] \in \frac{\mathbb{V}^{n-1}}{\mathbb{Z}_n}$. Since $\Psi$ is clearly surjective, we only have to prove that it is injective. In fact, suppose $[\lambda] = [\lambda] \in \frac{\mathbb{V}^{n-1}}{\mathbb{Z}_n}$, then

\[ \frac{1}{\sqrt[n]{\lambda_1 \cdots \lambda_n}} \lambda' = \overline{\lambda} = \alpha \overline{\lambda} \frac{1}{\sqrt[n]{\lambda_1 \cdots \lambda_n}} \lambda, \quad \alpha^n = 1. \]

Therefore, $\lambda' = a \lambda$ for some $a \in \mathbb{C}$. $\square$

4. **Analytic moduli and bi-Lipschitz invariants**

In this section, we establish the relationship between the analytic moduli space and the Henry-Parusiński invariant.

4.1. **Type II commode functions.** From (2.3) and (2.5), we obtain the correspondence

\[ \kappa = (\kappa_1, \ldots, \kappa_{n-1}) \to \rho = (\rho_1, \ldots, \rho_{n-1}), \]

where $\rho_j$ is given by $\rho_j = \prod_{j=1}^{n-1} (\kappa_j - \lambda_j)$. This correspondence is represented by the system of equations

\[ (4.1) \quad \begin{cases} (\kappa_1)^n + \sum_{\ell=1}^{n-1} (-1)^{\ell} \frac{\sigma_{\ell}(\rho)}{n-\ell} \kappa_1^{n-\ell} + (-1)^n \mu_n = \rho_1, \\ \vdots \\ (\kappa_{n-1})^n + \sum_{\ell=1}^{n-1} (-1)^{\ell} \frac{\sigma_{\ell}(\rho)}{n-\ell} \kappa_{n-1}^{n-\ell} + (-1)^n \mu_n = \rho_{n-1}, \end{cases} \]

where $\mu_n = 1$, and defines a map $\Upsilon : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}, \kappa_j \mapsto \rho_j, j = 1, \ldots, n-1$. It follows from Lemma 3.3 that in order to compare the analytic invariants and the Henry-Parusiński invariants it suffices to consider the restriction of $\Upsilon$ to the hyperplane $\mathbb{W}_0^{n-2} = \{ (\kappa_1, \cdots, \kappa_{n-1}) \in \mathbb{C}^{n-1} : \kappa_1 + \cdots + \kappa_{n-1} = 0 \}$. Notice that $\mathbb{W}_0^{n-2}$ is a manifold admitting a system of coordinates given by $\overline{\rho} = (\kappa_1, \cdots, \kappa_{n-2})$. In particular, the last equation of (4.1) determines immediately $\rho_{n-1}$ in terms of these $n - 2$ parameters. Therefore, $\text{Im}(\Upsilon)$ may be considered as (an algebraic) graph over $\mathbb{C}^{n-2}$ with coordinates given by $\overline{\rho} = (\rho_1, \cdots, \rho_{n-2})$.

Denote the restriction of $\Upsilon$ to $\mathbb{W}_0^{n-2}$ by $\Upsilon_{II} := \Upsilon|_{\mathbb{W}_0^{n-2}}$, then in the above systems of coordinates $\Upsilon_{II} = (\Upsilon_{II,1}, \cdots, \Upsilon_{II,n-2})$ is given by
ON THE MODULI SPACE OF QUASI-HOMOGENEOUS FUNCTIONS

(4.2) \[
\begin{cases}
\Upsilon_{II,1}(\kappa) = (\kappa_1)^n + \sum_{\ell=1}^{n-1}(-1)^{\ell} \frac{n}{n-\ell} \sigma_{\ell}(\kappa)(\kappa_1)^{n-\ell} + (-1)^n, \\
\Upsilon_{II,\kappa-2}(\kappa) = (\kappa_{n-2})^n + \sum_{\ell=1}^{n-1}(-1)^{\ell} \frac{n}{n-\ell} \sigma_{\ell}(\kappa)(\kappa_{n-2})^{n-\ell} + (-1)^n.
\end{cases}
\]

Let $\mathcal{HP}_{II} := \text{Im}(\Upsilon|_{W_0^{n-2}})$, then to each $\rho \in \mathcal{HP}_{II}$ there corresponds a unique class of the Henry-Parusinski invariant of a type II function germ. Besides, since $W_0^{n-2}$ is an affine space of dimension $n - 2$, each generic point in the image of $\Upsilon$ admits $n^{n-2}$ points in its pre-image (Bezout’s theorem).

From (4.1) and Lemma 3.3, $\Upsilon$ induces a map between the corresponding moduli spaces, say $\Upsilon_{II} : \mathcal{M}_{[n,1]}^0 \rightarrow \mathcal{HP}_{II}$. In this case, each point $\rho \in \mathcal{HP}_{II}$ admits precisely $n^{n-3}$ points in its pre-image (counting multiplicities) with respect to $\Upsilon_{II}$.

For any $\rho = (\rho_1, \cdots, \rho_{n-2}) \in \mathbb{C}^{n-2}$, we say that $\Upsilon_{II}^{-1}(\rho)$ is a degenerate fiber of $\Upsilon_{II}$ if $\rho_j = 0$ for some $j \in \{1, \cdots, n-2\}$ or else if the fiber has a multiple root; otherwise, we say that $\Upsilon_{II}^{-1}(\rho)$ is a non-degenerate fiber. We have

**Theorem 4.1.** For type II functions in $H_{pq}^d$, the analytic moduli space with fixed Henry-Parusinski invariant is determined by the equivalence classes in $W_0^{n-2}$ of the non-degenerate fibers of $\Upsilon_{II} : W_0^{n-2} \rightarrow \mathcal{HP}_{II}$. More precisely, for each $\rho \in \text{Im}(\Upsilon_{II})$ there exist $\#\Upsilon_{II}^{-1}(\rho) \leq n^{n-3}$ analytic types of function-germs with the same Henry-Parusinski invariant $\rho$. The equality holds for generic polynomials in $\mathcal{Z} \subset H_{pq}^d$.

**Proof.** From the description of the space $\mathcal{M}_{[n,1]}^0$ (cf. (3.1)) and from (2.4) and (2.5), the map $\Upsilon_{II} : W_0^{n-2} \rightarrow \mathbb{C}^{n-2}$ induces a surjective map $\Upsilon_{II} : \mathcal{M}_{[n,1]}^0 \rightarrow \mathcal{HP}_{II}$. In other words, the correspondence between the Henry-Parusinski invariant and the analytic invariants is determined by the orbits of the action of $\mathbb{Z}_n$ on the fibers over $\text{Im} \Upsilon_{II}$ of the homogeneous map $\Upsilon_{II} : W_0^{n-2} \rightarrow \mathbb{C}^{n-2}$, given by $\Upsilon_{II}(\kappa) = (\Upsilon_{II,1}(\kappa), \cdots, \Upsilon_{II,n-2}(\kappa))$, where

(4.3) \[\Upsilon_{II,j}(\kappa) := (\kappa_j)^n + \sum_{\ell=1}^{n-1}(-1)^{\ell} \frac{n}{n-\ell} \sigma_{\ell}(\kappa)(\kappa_j)^{n-\ell} + (-1)^n, \quad j = 1, \ldots, n-2.\]

Finally, Bezout’s theorem and the action of $\mathbb{Z}_n$ on the fibers over $\text{Im} \Upsilon_{II}$ lead to the desired result \[\square\]

Let us see some examples. Since for $n = 2$ there is trivially just one analytic class (due to classical complex analysis arguments), then we shall only consider $n \geq 3$. 

Example 4.2. Suppose \( n = 3 \), then \( \kappa = (\kappa_1, \kappa_2) \) and \( \mu_j = \sigma_j(\lambda) \). Since \( \kappa \in \mathbb{W}_0^{n-2} \), then \( \Upsilon_{II} \) is given by
\[
\begin{align*}
(\kappa_1)^3 + (-1)^2 \frac{3}{3-2} \sigma_2(\kappa)\kappa_1 &= \rho_1 - (-1)^3, \\
(\kappa_2)^3 + (-1)^2 \frac{3}{3-2} \sigma_2(\kappa)\kappa_2 &= \rho_2 - (-1)^3.
\end{align*}
\]
Since \( \kappa_2 = -\kappa_1 \), then we have
\[
\begin{align*}
-2(\kappa_1)^3 &= (\kappa_1)^3 + 3(\kappa_1\kappa_2)\kappa_1 = \rho_1 + 1 \\
-2(\kappa_2)^3 &= (\kappa_2)^3 + 3(\kappa_1\kappa_2)\kappa_2 = \rho_2 + 1 
\end{align*}
\]
\( \Leftrightarrow \)
\[
\begin{align*}
(\kappa_1)^3 &= -\frac{1+\rho_1}{2}, \\
(\kappa_2)^3 &= -\frac{\rho_2+1}{2}.
\end{align*}
\]
In particular, \( \rho_2 = -\rho_1 - 2 \). If we let \( \alpha = -\sqrt[3]{\frac{1+\rho_1}{2}} \) be one of the cubic roots of \( \frac{1+\rho_1}{2} \) and \( \omega = \exp(\frac{2\pi i}{3}) \), then
\[
\begin{align*}
\kappa_1 &= -\omega^s \alpha, \\
\kappa_2 &= \omega^s \alpha, \quad s = 0, 1, 2
\end{align*}
\]
Therefore, there is just \( 3^{3-2} = 1 \) analytic class corresponding to the same Henry-Parusiński invariant \( \rho = (\rho_1, -2 - \rho_1) \), where \( \rho_1 \in \text{Im} \Upsilon_{II} \).

Example 4.3. For \( n = 4 \), the system \((\text{II})\) assumes the form
\[
\begin{align*}
(\kappa_1)^4 + \frac{4}{4-2} \sigma_2(\kappa)(\kappa_1)^{4-2} - \frac{1}{4-3} \sigma_3(\kappa)(\kappa_1)^{4-3} &= \rho_1 - (-1)^4, \\
(\kappa_2)^4 + \frac{4}{4-2} \sigma_2(\kappa)(\kappa_2)^{4-2} - \frac{1}{4-3} \sigma_3(\kappa)(\kappa_2)^{4-3} &= \rho_2 - (-1)^4, \\
(\kappa_3)^4 + \frac{4}{4-2} \sigma_2(\kappa)(\kappa_3)^{4-2} - \frac{1}{4-3} \sigma_3(\kappa)(\kappa_3)^{4-3} &= \rho_3 - (-1)^4.
\end{align*}
\]
or equivalently
\[
\begin{align*}
(\kappa_1)^4 + 2(\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3)(\kappa_1)^2 - 4\kappa_1\kappa_2\kappa_3(\kappa_1) &= \rho_1 - 1, \\
(\kappa_2)^4 + 2(\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3)(\kappa_2)^2 - 4\kappa_1\kappa_2\kappa_3(\kappa_2) &= \rho_2 - 1, \\
(\kappa_3)^4 + 2(\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3)(\kappa_3)^2 - 4\kappa_1\kappa_2\kappa_3(\kappa_3) &= \rho_3 - 1.
\end{align*}
\]
In other words
\[
\begin{align*}
(\kappa_1)^2[(\kappa_1)^2 + 2(\kappa_1\kappa_2 + \kappa_1\kappa_3 - \kappa_2\kappa_3)] &= \rho_1 - 1, \\
(\kappa_2)^2[(\kappa_2)^2 + 2(\kappa_1\kappa_2 - \kappa_1\kappa_3 + \kappa_2\kappa_3)] &= \rho_2 - 1, \\
(\kappa_3)^2[(\kappa_3)^2 + 2(-\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3)] &= \rho_3 - 1.
\end{align*}
\]
Assuming that \( \kappa_3 = -(\kappa_1 + \kappa_2) \), then we have
\[
\begin{align*}
(\kappa_1)^2[2(\kappa_2)^2 + 2\kappa_1\kappa_2 - (\kappa_1)^2] &= \rho_1 - 1, \\
(\kappa_2)^2[2(\kappa_1)^2 + 2\kappa_1\kappa_2 - (\kappa_2)^2] &= \rho_2 - 1, \\
(\kappa_1 + \kappa_2)^2[3(\kappa_1)^2 + 4\kappa_1\kappa_2 + 3\kappa_2^2] &= \rho_3 - 1.
\end{align*}
\]
For fixed \( (\rho_1, \rho_2) \in \text{Im}(\Upsilon_{II}) \) the classical Bezout’s theorem says that there are precisely \( 4^2 = 16 \) solutions for the above system. Further, the last one tells us which value \( \rho_3 \) must have in order that \( \rho = (\rho_1, \rho_2, \rho_3) \) be the associated Henri-Parusiński invariant. Considering the symmetries, this information tells us that there are \( 4^{4-3} = \frac{16}{4} \) distinct analytic types of quasi-homogeneous functions with the Henry-Parusiński invariant \( \rho \).
4.2. Type III commode functions. From \((2.3)\) and \((2.5)\), we obtain the correspondence

\[
κ = (κ_1, \ldots, κ_{n-1}) \rightarrow ρ = (ρ_1, \ldots, ρ_{n-1}),
\]

where \(ρ_0 = (-1)^nλ_1 \cdots λ_n\) and \(ρ_ℓ = \prod_{j=1}^n(κ_ℓ - λ_j)\) for \(ℓ > 0\). This correspondence is represented by the map \(Υ_{III} : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}, \kappa \mapsto c = (c_1, \ldots, c_{n-1})\), given by

\[
\begin{align*}
(k_1)^n + \sum_{ℓ=1}^{n-1}(-1)^{n-1-ℓ} \frac{n-ℓ}{n-ℓ} σ_ℓ(k_1)(κ_1)^{n-ℓ} + (-1)^n &= (-1)^nc_1, \\
&\vdots \\
(k_{n-1})^n + \sum_{ℓ=1}^{n-1}(-1)^{n-1-ℓ} \frac{n-ℓ}{n-ℓ} σ_ℓ(k_1)(κ_{n-1})^{n-ℓ} + (-1)^n &= (-1)^nc_{n-1},
\end{align*}
\]

where

\[
λ_1 \cdots λ_n = 1 \quad \text{and} \quad c_ℓ := \frac{ρ_ℓ}{ρ_0} = \frac{(κ_ℓ - λ_1) \cdots (κ_ℓ - λ_n)}{(-1)^nλ_1 \cdots λ_n} \neq 0.
\]

From Lemma 3.4, each \(c = (c_1, \ldots, c_{n-1}) \in \mathcal{HP}_{III} = \text{Im}(Υ_{III})\) corresponds to only one Henry-Parusiński invariant and thus induce a map from the analytic moduli space to the Henry-Parusiński invariant, say \(Υ_{III} : M^0_{[n],2} \rightarrow \mathcal{HP}_{III}\).

As before, let \(c \in \mathcal{HP}_{III}\), then we say that \(Υ_{III}^{-1}(c)\) is a degenerate fiber of \(Υ_{III}\) if \(c_j = 0\) for some \(j \in \{1, \ldots, n\}\) or else if the fiber has a multiply root; otherwise we shall say that \(Υ_{III}^{-1}(c)\) is a non-degenerate fiber.

**Theorem 4.4.** For type III functions in \(\mathcal{H}_{pq}^d\), the analytic moduli space with fixed Henry-Parusiński invariant is determined by the images in \(\mathcal{Y}_{III}^{-1}\) of non-degenerate fibers of \(Υ_{III} : \mathcal{V}^{n-1} \rightarrow \mathbb{C}^{n-1}\). More precisely, for each \(c \in \mathcal{HP}_{III}\) there exist \(\#[Υ_{III}^{-1}(c)] \leq n^{n-2}\) analytic types of function-germs with the same Henry-Parusiński invariant \(c\). The equality holds for generic polynomials in \(Z \subset \mathcal{H}_{pq}^d\).

**Proof.** From the description of the space \(M^0_{[n],2}\) (cf. (3.2)) and also from \((2.3)\) and \((2.5)\), the correspondence between the Henry-Parusiński invariant and the analytic invariants is determined by the orbits of the action of \(\mathbb{Z}_n\) on the fibers over \(\mathcal{HP}_{III}\) of the map \(Υ_{III} : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}\) given in coordinates by \(Υ_{III}(κ) = (Υ_1(κ), \ldots, Υ_{n-1}(κ))\), where

\[
(4.4) \quad Υ_j(κ) := (κ_j)^n + \sum_{ℓ=1}^{n-1}(-1)^{n-ℓ} \frac{n-ℓ}{n-ℓ} σ_ℓ(κ)(κ_j)^{n-ℓ}.
\]

□

As a consequence of the above theorems, we are able to generalize the main result in [FR2] as follows.
Theorem 4.5. Let \( f_t \) be a continuous family of germs of quasi-homogeneous (and not homogeneous) functions with isolated singularity and constant Henry-Parusiński invariant then \( f_t \) is analytically trivial.

Proof. First recall from \([FR2]\) that it suffices to consider the commode quasi-homogeneous case. Let \( H_{p,q}^d \) denote the set of commode quasi-homogeneous polynomials with weights \((p, q)\) and quasi-homogeneous degree \(d\). Let \([H_{p,q}^d]\) denote the set of analytic conjugacy classes in \(H_{p,q}^d\) and \(\pi : H_{p,q}^d \longrightarrow [H_{p,q}^d]\) the natural projection, then we have the commutative diagram

\[
\begin{array}{c}
H_{p,q}^d \xrightarrow{\pi} \Upsilon \rightarrow \mathcal{H}P  \\
[0, 1] \xrightarrow{f_t} [H_{p,q}^d] \xrightarrow{\Upsilon} \mathcal{H}P
\end{array}
\]

where \(\Upsilon\) is the map in the statement of Theorems 4.1 and 4.4 and \(\mathcal{H}P\) the space of Henry-Parusiński invariants. Suppose that \(f_t\) is a continuous family with constant Henry-Parusiński invariant \(\rho\). Then \(\pi \circ f_t\) is a continuous family of analytic classes contained in \([\Upsilon]^{-1}(\rho)\). Since \([\Upsilon]\) is a ramified covering, then \([\Upsilon]^{-1}(\rho)\) is discrete and the result follows. \(\square\)

5. Examples

Now let us study some examples.

Example 5.1. Suppose \(n = 2\), then \(\kappa = \kappa_1\) and \(\mu_1 = \sigma_1(\lambda) = \frac{2}{\kappa_1^2}\sigma_1(\kappa) = 2\kappa_1\). Thus

\[
(\kappa_1)^2 + (-1)^1(2\kappa_1)(\kappa_1)^{2-1} = c_1 - 1 \iff (\kappa_1)^2 = 1 - c_1.
\]

Equivalently,

\[
\kappa_1 = \pm \sqrt{1 - c_1}.
\]

Since the square roots of unity are given by \(\pm 1\), then the equivalence (3.2) ensures that both points correspond to just one analytic class \([\lambda]\).

Example 5.2. Suppose \(n = 3\), then \(\kappa = (\kappa_1, \kappa_2)\) and \(\mu_j = \sigma_j(\lambda)\). Hence

\[
\begin{cases}
(\kappa_1)^3 + (-1)^1\frac{3}{3-1}\sigma_1(\kappa)(\kappa_1)^2 + (-1)^2\frac{3}{3-2}\sigma_2(\kappa)\kappa_1 = c_1 - 1, \\
(\kappa_2)^3 + (-1)^1\frac{3}{3-1}\sigma_1(\kappa)(\kappa_2)^2 + (-1)^2\frac{3}{3-2}\sigma_2(\kappa)\kappa_2 = c_2 - 1,
\end{cases}
\]

or equivalently

\[
\begin{cases}
(\kappa_1)^3 - \frac{3}{3-1}(\kappa_1 + \kappa_2)(\kappa_1)^2 + 3(\kappa_1\kappa_2)\kappa_1 = c_1 - 1, \\
(\kappa_2)^3 - \frac{3}{3-2}(\kappa_1 + \kappa_2)(\kappa_2)^2 + 3(\kappa_1\kappa_2)\kappa_2 = c_2 - 1.
\end{cases}
\]

In other words,

\[
\begin{cases}
-\frac{1}{2}(\kappa_1)^3 + \frac{3}{2}\kappa_2(\kappa_1)^2 = c_1 - 1, \\
-\frac{1}{2}(\kappa_2)^3 + \frac{3}{2}\kappa_1(\kappa_2)^2 = c_2 - 1.
\end{cases}
\]
Multiplying the first equation by \(-2\) and summing it to the second one multiplied by 2, we have
\[
(k_1 - k_2)^3 = 2(c_2 - c_1).
\]
Let \(\alpha\) be a cubic root of \(2(c_2 - c_1)\) and \(\omega = \exp(2\pi i/3)\), then the cubic roots of \(2(c_2 - c_1)\) are of the form \(\omega\alpha, \omega^2\alpha, \omega^3\alpha = \alpha\). Hence \(k_1 - k_2 = \omega^s\alpha\), \(s = 0, 1, 2\). Back to the system of equations, we have
\[
2(1 - c_1) = (k_1)^3 - 3(k_1)^2k_2 = (k_1)^2(k_1 - 3k_2) = (k_1)^2[3(k_1 - k_2) - 2k_1]
\]
\[
= (k_1)^2[3\omega^s\alpha - 2k_1] = -2(k_1)^3 + 3\omega^s\alpha(k_1)^2;
\]
i.e.,
\[
(5.1) \quad (k_1)^3 - \frac{3\alpha\omega^s}{2}(k_1)^2 + (1 - c_1) = 0.
\]
In order to be more specific, let us pick \(c_1 := -1/3\) and \(c_2 := 1129/729\). Then
\[
\alpha = \sqrt[3]{2(c_2 + 1/3)} = \sqrt[3]{2 \cdot \frac{1129 + 3}{3^5}} = \sqrt[3]{2 \cdot \frac{1372}{3^5}} = \sqrt[3]{\frac{143}{9^3}} = \frac{14}{9}.
\]
In this case, equations (5.1) assume the form
\[
(k_1)^3 - \frac{7\omega^s}{3}(k_1)^2 + \frac{4}{3} = 0.
\]
Then for \(s = 0\) the possible values of \(k_1\) are given by the solutions of the cubic equation \(z^3 - \frac{7}{3}z^2 + \frac{4}{3} = 0\), which are \(\{2, 1, -2/3\}\). Since \(\kappa = (k_1, k_2) = (k_1, k_1 - \omega^s\alpha)\), then
\[
(s = 0) : \kappa = \begin{cases}
(2, 2 - 14/9) = (2, 4/9); \\
(1, 1 - 14/9) = (1, -5/9); \\
(-2/3, -2/3 - 14/9) = (-2/3, -20/9).
\end{cases}
\]
For each \(s = 1, 2\), the solutions are of the form
\[
(s = 1) : \kappa = \begin{cases}
\left(\frac{2e^{\frac{4\pi i}{3}} + \frac{4}{9}e^{\frac{4\pi i}{3}}}{9}, \frac{-5}{9}e^{\frac{4\pi i}{3}}\right) \\
\left(\frac{2}{3}e^{\frac{2\pi i}{3}}, \frac{-5}{9}e^{\frac{2\pi i}{3}}\right).
\end{cases}
\]
\[
(s = 2) : \kappa = \begin{cases}
\left(\frac{2e^{\frac{4\pi i}{3}} + \frac{4}{9}e^{\frac{4\pi i}{3}}}{9}, \frac{4}{9}e^{\frac{4\pi i}{3}}\right) \\
\left(\frac{-5}{9}e^{\frac{4\pi i}{3}}, \frac{-20}{9}e^{\frac{4\pi i}{3}}\right).
\end{cases}
\]
In order to be more precise, let us write down the explicit expressions of the functions. First recall from (2.2) that
\[
f(x, y) = y^{3p} - \frac{3}{2}(k_1 + k_2)y^{2p}x^q + 3k_1k_2y^{p}x^{2q} - x^{3q}.
\]
Thus for each value of \(\kappa\) we have the following quasi-homogeneous functions in three distinct analytic classes
\[
\kappa = \begin{cases}
(2, 4/9) \quad \Rightarrow \quad f^1(x, y) = y^{3p} - \frac{33}{9}y^{2p}x^q + \frac{24}{9}y^{p}x^{2q} - x^{3q}; \\
(1, -5/9) \quad \Rightarrow \quad f^2(x, y) = y^{3p} - \frac{18}{9}y^{2p}x^q - \frac{15}{9}y^{p}x^{2q} - x^{3q}; \\
(-2/3, -20/9) \quad \Rightarrow \quad f^3(x, y) = y^{3p} + \frac{39}{9}y^{2p}x^q + \frac{40}{9}y^{p}x^{2q} - x^{3q}.
\end{cases}
\]
whose polars are given by

$$
\kappa = \begin{cases} 
(2, 4/9) & \Rightarrow \partial_y f^1(x, y) = 3py^{p-1}(y^{2p} - \frac{22}{9}y^{p+q} + \frac{8}{9}x^{2q}), \\
(1, -5/9) & \Rightarrow \partial_y f^2(x, y) = 3py^{p-1}(y^{2p} - \frac{1}{3}y^{p+q} - \frac{2}{9}x^{2q}), \\
(-2/3, -20/9) & \Rightarrow \partial_y f^3(x, y) = 3py^{p-1}(y^{2p} + \frac{26}{9}y^{p+q} + \frac{40}{27}x^{2q}).
\end{cases}
$$

From (2.2) and (2.5), we have \( \rho_0 = 1 \) and

$$
\rho_j = (\kappa_j - \lambda_1)(\kappa_j - \lambda_2)(\kappa_j - \lambda_3) = \kappa_j^3 - \sigma_1(\lambda)\kappa_j^2 + \sigma_2(\lambda)\kappa_j - 1.
$$

Thus

$$
\kappa = \begin{cases} 
(2, 4/9) & \Rightarrow \begin{cases} 
\rho_1 = \kappa_3^3 - \frac{32}{9}\kappa_1^2 + \frac{24}{9}\kappa_1 - 1 = -21/9, \\
\rho_2 = \kappa_3^3 - \frac{32}{9}\kappa_2^2 + \frac{24}{9}\kappa_2 - 1 = -329/729, \\
\rho_3 = \kappa_3^3 - \frac{32}{9}\kappa_3^2 - \frac{14}{9}\kappa_3 - 1 = -21/9, \\
\rho_4 = \kappa_3^3 - \frac{32}{9}\kappa_4^2 - \frac{14}{9}\kappa_4 - 1 = -329/729, \\
\rho_5 = \kappa_3^3 + \frac{40}{9}\kappa_1^2 + \frac{40}{9}\kappa_1 - 1 = -21/9, \\
\rho_6 = \kappa_3^3 + \frac{40}{9}\kappa_2^2 + \frac{40}{9}\kappa_2 - 1 = -329/729.
\end{cases}
\end{cases}
$$

Therefore, \( f^1, f^2 \) and \( f^3 \) have the same Henry-Parusiński invariant but represent three distinct analytic classes.

References

[Ar] V. I. Arnold, Chapitres supplémentaires de la théorie des équations différentielles ordinaires. Editions Mir, Moscou 1980.

[CS] L. Câmara and B. Scárdua, A comprehensive approach to the moduli space of quasi-homogeneous singularities. (English summary) Singularities and foliations. geometry, topology and applications, 459–487, Springer Proc. Math. Stat., 222, Springer, Cham, 2018.

[FR1] A. Fernandes and M.A.S. Ruas, Bilipschitz determinacy of quasihomogeneous germs. Glasgow Math. J. 46 (2004), 77–82.

[FR2] A. Fernandes and M.A.S. Ruas, Rigidity of bi-Lipschitz equivalence of weighted homogeneous function-germs in the plane, Proc. Amer. Math. Soc. 141 (2013), 1125–1133.

[HP] J.-P. Henry and A. Parusiński, Existence of moduli for bilipschitz equivalence of analytic functions, Comp. Math. 136 (2003), 217–235.

[K] C. Kang, Analytic types of plane curve singularities defined by weighted homogeneous polynomials, Trans. Amer. Math. Soc. 352, 9, (2000), 3995–4006.

[Sa] K. Saito, Quasihomogen isoliere Singularitäten von Hyperflächen, Invent. Math. 14(1971), 123–142.