CASH SUBADDITIVE RISK MEASURES AND INTEREST RATE AMBIGUITY

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A new class of risk measures called cash subadditive risk measures is introduced to assess the risk of future financial, nonfinancial, and insurance positions. The debated cash additive axiom is relaxed into the cash subadditive axiom to preserve the original difference between the numéraire of the current reserve amounts and future positions. Consequently, cash subadditive risk measures can model stochastic and/or ambiguous interest rates or defaultable contingent claims. Practical examples are presented, and in such contexts cash additive risk measures cannot be used. Several representations of the cash subadditive risk measures are provided. The new risk measures are characterized by penalty functions defined on a set of sublinear probability measures and can be represented using penalty functions associated with cash additive risk measures defined on some extended spaces. The issue of the optimal risk transfer is studied in the new framework using inf-convolution techniques. Examples of dynamic cash subadditive risk measures are provided via BSDEs where the generator can locally depend on the level of the cash subadditive risk measure.

Key Words: risk measures, Fenchel-Legendre transform, model uncertainty, inf-convolution, backward stochastic differential equations.

1. INTRODUCTION

The assessment of financial and nonfinancial risks plays a key role for economic agents when pricing assets or managing their wealths. Consequently, over the last decade several measures of risk have been proposed to assess the riskiness of financial and nonfinancial positions and compute cash reserve amounts for hedging purposes. The axiomatic-based monetary risk measures have been largely investigated because most axioms embed desirable economic properties. Coherent risk measures have been introduced by Artzner et al. (1997, 1999) and further developed by Delbaen (2000, 2002); sublinear risk measures by Frittelli (2000); convex risk measures by Föllmer and Schied (2002a), Föllmer and Schied (2002b), and Frittelli and Rosazza Gianin (2002). Examples of convex risk measures related to pricing and hedging in incomplete markets are provided by, for instance,

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El Karoui and Quenez (1996), Carr, Geman, and Madan (2001), Frittelli and Rosazza Gianin (2004), Staum (2004), Filipovic and Kupper (2008), and Jouini, Schachermayer, and Touzi (2008).

However, while the convexity and the monotonicity axioms have been largely accepted by academics and practitioners, the cash additive axiom has been criticized from an economic viewpoint. A basic reason is that while regulators and financial institutions determine and collect today the reserve amounts to cover future risky positions, the cash additivity requires that risky positions and reserve amounts are expressed in the same numéraire. This is a stringent requirement that limits the applicability of cash additive risk measures. Implicitly, it means that risky positions are discounted before applying the risk measure, assuming that the discounting process does not involve any additional risk. Unfortunately, when the interest rates are stochastic this procedure does not disentangle the risk of the financial position per se and the risk associated to the discounting process. Furthermore, payoff functions on risky assets are a priori and contractually determined by economic agents considering different scenarios for the underlying asset. While this procedure is theoretically framed into cash additive risk measures, the cash additive axiom does not allow to account for ambiguous discount factor. For a correct assessment of the current reserve amount, it is equally important to allow for ambiguity on the underlying asset and on the discount factor. This assessment is achieved by relaxing the cash additive axiom and searching for risk measures that preserve the different numéraires of the current reserve amounts and the future risky positions.

The main contribution of this paper is to propose a new class of risk measures called cash subadditive risk measures that are directly defined on the future risky positions and provide the reserve amounts in terms of the current numéraire. To reconcile the two different numéraires, cash subadditive risk measures relax the cash additive axiom into the cash subadditive axiom. This is the minimal requirement to account for the time value of money. Remarkably, the cash subadditive axiom is enough to characterize measures of risk that can be applied also when the cash additive risk measures cannot—as for instance under ambiguous interest rates or defaultable cash flows. Cash subadditive risk measures turn out to be suitable not only for assessing financial risks but also for insurance and other kinds of risks. For example, the put option premium investigated by Jarrow (2002) as a measure of the firm insolvency risk defines a cash subadditive risk measure. Moreover, similarly to the cash additive risk measures, the cash subadditive risk measures can be represented using penalty functions. In particular, we show that cash subadditive risk measures are characterized by minimal penalty functions, which only depend on finitely additive set functions \( \mu \) such that \( 0 \leq \mu(\Omega) \leq 1 \), which we call finitely additive subprobability measure.

The other contributions of this paper are the following. In the framework of cash additive risk measures when the zero-coupon bond is available for the relevant time horizon, we provide the conditions under which discounting the forward risk measure to obtain current reserve amounts defines risk measures additive with respect to the current numéraire and vice versa (Section 2.4).

In Section 3 we introduce the cash subadditive risk measures that we denote by \( \mathcal{R} \). We provide several examples of these new risk measures that generalize the put option premium and naturally arise when accounting for ambiguous discount factor or insurance.

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1 Disentangling the different risks is crucial when implementing hedging strategies as different risks are hedged on different markets.
risks. These risk measures are obtained composing cash additive risk measures and a specific class of random convex functions. A representation result showing the impact of the ambiguous discount factor/numéraire is given.

In Section 4 we study the dual representation of cash subadditive risk measures. Instead of using convex analysis tools, we extend cash subadditive risk measures to an enlarged space of risky positions where they become cash additive. This approach provides a rich financial interpretation of both cash additive and cash subadditive risk measures and allows to derive properties of $R$ using the classical theory on cash additive risk measures. Using the duality result, a characterization of $R$ in terms of deterministic discount factors is easily obtained where any cash subadditive risk measure can be represented as the worst case scenario of a family of discounted forward risk measures.

In Section 5 two other links between cash subadditive and cash additive risk measures are presented where more involved techniques are required. The first link indicates a possible way to recover a representation of a general cash subadditive risk measure where the ambiguous numéraire is explicitly modeled as a random variable on the original space of definition of $R$. The second link shows that cash subadditive risk measures generated via convex functions are compositions of an unconditional and a conditional cash additive risk measure.

In Section 6 using cash subadditive risk measures, we study the problem of designing the optimal transaction between two economic agents in a general framework allowing for ambiguous discount factors. In particular, we show that the risk transfer problem can be reduced to an inf-convolution of cash subadditive risk measures which is again a cash subadditive risk measure.

Finally, in Section 7 we provide a dynamic example of cash subadditive risk measures which are solutions of backward stochastic differential equations (BSDEs). In contrast to the cash additive risk measures generated via BSDEs, the generator of dynamic cash subadditive risk measures, besides being a function of the martingale part, can also depend on the level of the cash subadditive risk measure, generating recursive risk measures. Section 8 concludes.

2. CASH ADDITIVE RISK MEASURES

In this section we recall some key properties of cash additive risk measures and we discuss the cash additive axiom. The following definitions are consistent with the definitions of monetary risk measure in Föllmer and Schied (2002b).

2.1. Definitions and Properties of Cash Additive Risk Measures

Let $(\Omega, \mathcal{A})$ be a measurable space. The risky positions at the relevant time horizon belong to the linear space of bounded functions including constant functions denoted by $\mathcal{X}$.

**Definition 2.1.** A *cash additive risk measure* is a functional $\rho : \mathcal{X} \to \mathbb{R}$ cash additive, convex, and monotone decreasing, that is,

(a) Convexity: $\forall \lambda \in [0, 1]$, $\rho(\lambda X + (1 - \lambda) Y) \leq \lambda \rho(X) + (1 - \lambda) \rho(Y)$;
(b) Monotonicity: $X \leq Y \Rightarrow \rho(X) \geq \rho(Y)$;
(c) Cash additivity (or cash invariance): $\forall m \in \mathbb{R}$, $\rho(X + m) = \rho(X) - m$. 
A cash additive risk measure is coherent when

(d) Positive homogeneity: \( \forall \lambda \in \mathbb{R}^+, \rho(\lambda X) = \lambda \rho(X) \).

(e) \( \rho \) is normalized when \( \rho(0) = 0 \).

(f) \( \rho \) is continuous from below (from above) when \( X_n \uparrow X \Rightarrow \rho(X_n) \downarrow \rho(X) \), \( \rho \) is continuous from below (from above) when \( X_n \downarrow X \Rightarrow \rho(X_n) \uparrow \rho(X) \).

The convexity axiom translates the natural important fact that diversification should not increase risk. In particular, convex combinations of “admissible” risks should be “admissible.”

To shorten the representation of convex combinations of elements, we use the following notation. We denote the barycenter (or convex combination) of the set \( x_I := \{x_{(1)}, x_{(2)}, \ldots, x_{(I)}\}, I \in \mathbb{N} \),

\[
\text{Bar}[x_I] := \text{Bar}^{\lambda_I}[x_I] := \sum_{i=1}^{I} \lambda_i x_{(i)}, \quad \text{where}
\]

\( \lambda_i \in [0, 1], i = 1, \ldots, I, \quad \text{and} \quad \sum_{i=1}^{I} \lambda_i = 1 \).

In particular, \( f \) is a convex function if and only if \( f(\text{Bar}[x_I]) \leq \text{Bar}[f(x)_I] \). The same definition holds for a set \( X_I \) of random variables.

2.2. Dual Representation of Cash Additive Risk Measures

A key property of cash additive risk measures is the dual representation in terms of normalized finitely additive set functions and minimal penalty function (Föllmer and Schied 2002b, Theorem 4.12). The dual point of view emphasizes the interpretation in terms of a worst case scenario related to the agent’s (or regulator’s) beliefs: the agent does not know the true “probability” measure and uses distorted beliefs from a subjective set of normalized additive set functions. Under the additional assumption that risk measures are continuous from below, the dual representation is in terms of \( \sigma \)-additive probability measures (Föllmer and Schied 2002b, Proposition 4.17).

**Theorem 2.2.**

(a) Let \( \mathcal{M}_{1,f}(A) \) be the set of all finitely additive set functions \( Q \) on \( (\Omega, \mathcal{A}) \) normalized to one, \( Q(\Omega) = 1 \), and \( \alpha \) the minimal penalty function taking values in \( \mathbb{R} \cup \{+\infty\} \):

\[
\forall Q \in \mathcal{M}_{1,f}(A), \quad \alpha(Q) = \sup_{X \in \mathcal{X}} \left\{ \mathbb{E}_Q[-X] - \rho(X) \right\}, \quad (\geq -\rho(0))
\]

\[
\text{Dom}(\alpha) = \{ Q \in \mathcal{M}_{1,f}(A) | \alpha(Q) < +\infty \}.
\]

The Fenchel duality relation holds:

\[
\forall X \in \mathcal{X}, \quad \rho(X) = \sup_{Q \in \mathcal{M}_{1,f}(A)} \left\{ \mathbb{E}_Q[-X] - \alpha(Q) \right\}.
\]

Moreover, for any \( X \in \mathcal{X} \) there exists a \( Q_X \in \mathcal{M}_{1,f}(A) \), such that \( \rho(X) = \mathbb{E}_{Q_X}[-X] - \alpha(Q_X) = \max_{Q \in \mathcal{M}_{1,f}(A)} \{ \mathbb{E}_Q[-X] - \alpha(Q) \} \).
(b) Let $\mathcal{M}_1(\mathcal{A})$ denote the set of all probability measures $\mathcal{Q}$ on $(\Omega, \mathcal{A})$. Let $\rho$ be a monetary risk measure continuous from below and suppose that $\beta$ is any penalty function on $\mathcal{M}_1(\mathcal{A})$ representing $\rho$. Then $\beta$ is concentrated on the class $\mathcal{M}_1(\mathcal{A})$ of probability measures, that is, $\beta(\mathcal{Q}) < \infty$ only if $\mathcal{Q}$ is $\sigma-$additive.

See Krätschmer (2005) for necessary conditions to obtain representation results in terms of probability measures.

The following lemma shows that a cash additive risk measure is linear with respect to the linear subspace generated by a position $Y$ if and only if any $\mathcal{Q}$ in the domain of the penalty function satisfies the calibration constraint: $\mathcal{Q}(−Y) = \rho(Y)$. This lemma will be used to derive the results in Section 2.4.

**Lemma 2.3.** Let $\rho$ be a normalized cash additive risk measure on $\mathcal{X}$ and $\mathcal{W}$ a linear subspace of $\mathcal{X}$ containing the constants. The risk measure $\rho$ is a linear on $\mathcal{W}$, if and only if $\rho(\mathcal{W}) = \mathbb{E}_\mathcal{Q}[-\mathcal{W}]$ for any $\mathcal{Q} \in \text{Dom}(\alpha)$. This implies that the risk measure is invariant with respect to $\mathcal{W}$, that is, $\forall X \in \mathcal{X}, \forall \mathcal{W} \in \mathcal{W}$, $\rho(X + \mathcal{W}) = \rho(X) + \rho(\mathcal{W})$.

**Proof.** The dual representation and the linearity of $\rho$ with respect to $\mathcal{W}$ imply that for any $\mathcal{Q} \in \text{Dom}(\alpha)$, $\lambda \in \mathbb{R}, \lambda \rho(\mathcal{W}) = \rho(\lambda \mathcal{W}) = \mathbb{E}_\mathcal{Q}[\lambda(-\mathcal{W})] - \alpha(\mathcal{Q})$, where $\alpha$ is the minimal penalty of $\rho$. Then $\alpha(\mathcal{Q}) \geq -\lambda(\rho(\mathcal{W}) + \mathbb{E}_\mathcal{Q}[\mathcal{W}])$. As the last inequality holds for any $\lambda \in \mathbb{R}$, $\rho(\mathcal{W}) = -\mathbb{E}_\mathcal{Q}[\mathcal{W}]$, $\forall \mathcal{Q} \in \text{Dom}(\alpha)$. The converse is evident.

If the calibration constraint holds, then $\rho(X + \mathcal{W}) = \sup_{\mathcal{Q} \in \text{Dom}(\alpha)}(\mathbb{E}_\mathcal{Q}[-X - \mathcal{W}] - \alpha(\mathcal{Q})) = \sup_{\mathcal{Q} \in \text{Dom}(\alpha)}(\rho(\mathcal{W}) + \mathbb{E}_\mathcal{Q}[-X] - \alpha(\mathcal{Q})) = \rho(\mathcal{W}) + \rho(X)$, for any $X \in \mathcal{X}, \mathcal{W} \in \mathcal{W}$. ⊗

2.3. Cash Additivity and Discounting

The cash additive axiom is motivated by the interpretation of $\rho(X)$ as capital requirement.² Intuitively, $\rho(X)$ is the amount of cash that has to be added to the risky position $X$ in order to make it acceptable (i.e., with nonpositive measure of risk) by a supervising agency:

$$\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0.$$ 

The cash additive property requires that the risky position and the risk measure are expressed in the same numéraire.³ Hence, cash additive risk measures are defined either on the discounted value of the future position (see, for instance, Delbaen 2000 and Föllmer and Schied 2002b) or on the future position itself and give the forward reserve amount to add to the future position at the future date (see, for instance, Rouge and El Karoui 2000). In the next section, assuming that all the agents use the same discount factor for the maturity of interest and there exists a zero-coupon bond for that maturity, we provide a link between cash additive risk measures on the discounted positions and forward cash additive risk measures.

² See Frittelli and Scandolo (2006) for an extensive study of the axiom of cash additivity and the related concept of capital requirement.

³ See Filipovic (2008) for the impact of risky numéraire on capital requirements.
2.4. Forward and Spot Risk Measures under Stochastic Discount Factor

The risky position, \( X_T \), belongs to \( \mathcal{X} \), the linear space of real-valued bounded random variables including constants on the measurable space \((\Omega, \mathcal{F}_t)\). The riskiness of \( X_T \) is assessed at time \( t = 0 \) and \( 1_T \) denotes one unit of cash available at date \( T \). \( D_T \) is the stochastic (nonambiguous) discount factor for the maturity \( T \) used by all agents in the market, \( \mathcal{F}_T \)-measurable, and \( 0 \leq D_T \leq 1 \). \( B_{0,T} > 0 \) denotes the price at time \( t = 0 \) of a zero-coupon bond available in the market that pays one unit of cash at time \( t = T \). To highlight with respect to which numéraire the risk measures are cash additive, we make the following distinction. We call spot risk measure \( \rho_0 \) the cash additive risk measure defined on the discounted value of the future positions \( D_T X_T \), \( X_T \in \mathcal{X} \), which are cash additive with respect to the cash available at time \( t = 0 \), \( \forall X_T \in \mathcal{X} \).

\[
\forall m \in \mathbb{R}, \quad \rho_0(D_T X_T + m) = \rho_0(D_T X_T) + \rho_0(m) \quad \text{and} \quad \rho_0(m) = m \rho_0(1) = -m.
\]

The monetary risk measure, \( \rho_0 \), is defined by Föllmer and Schied (2002b). It represents the cash amount at \( t = 0 \) to add to the discounted position \( D_T X_T \) to make it acceptable. The spot risk measure does not disentangle the discounting risk from the risk of the financial position per se. Furthermore, to meaningfully consider the discounted future position the discount factor cannot be ambiguous.

We call forward risk measure \( \rho_T \) the cash additive risk measure defined on the future position \( X_T \in \mathcal{X} \), which are cash additive with respect to the cash available at time \( T \), \( \forall X_T \in \mathcal{X} \).

\[
\forall m \in \mathbb{R}, \quad \rho_T(X_T + m 1_T) = \rho_T(X_T) + \rho_T(m 1_T) \quad \text{and} \quad \rho_T(m 1_T) = m \rho_T(1_T) = -m 1_T.
\]

The \( \rho_T \) gives the forward cash amount (evaluated at \( t = 0 \)) to add at \( T \) to the position to make it acceptable. When the zero-coupon bond \( B_{0,T} \) is available, the forward reserve \( \rho_T(X_T) \) can be easily discounted at \( t = 0 \). The following proposition shows that this procedure defines a spot risk measure when \( \rho_T \) satisfies a calibration constraint on \( D_T \) and \( B_{0,T} \). Similarly, the spot risk measure \( \rho_0 \) capitalized by \( B_{0,T}^{-1} \) defines a forward risk measure if \( \rho_0 \) satisfies a similar calibration constraint on \( D_T \) and \( B_{0,T} \). The penalty function of \( \rho_0 \) is equal to the penalty function of \( \rho_T \) discounted by \( B_{0,T} \) and the corresponding additive set functions satisfy the usual spot-forward change of measure.

**Proposition 2.4.**

1. Let \( \rho_0 \) be a normalized spot risk measure with minimal penalty function \( \alpha_0 \). The functional

\[
(2.7) \quad q_T(X_T) := B_{0,T}^{-1} \rho_0(D_T X_T), \quad X_T \in \mathcal{X},
\]

is convex and monotone decreasing with respect to \( X_T \), and forward cash additive if and only if \( \rho_0 \) satisfies the calibration constraint, \( \forall \lambda \in \mathbb{R}, \quad \rho_0(\lambda D_T) = -\lambda B_{0,T} \). In this case, any \( Q_0 \in \text{Dom}(\alpha_0) \) is such that \( E_{Q_0}[D_T] = B_{0,T} \). Moreover, if \( D_T \) is bounded away from 0, \( q_T \) satisfies the calibration constraint, \( \forall \lambda \in \mathbb{R}, \quad q_T(\lambda D_T^{-1}) = B_{0,T}^{-1} \rho_0(\lambda) = -\lambda B_{0,T}^{-1} = \lambda q_T(D_T^{-1}) \), and the minimal penalty function of \( q_T \), \( \alpha_T \), is given by
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\[ \alpha_T(Q_T) = B_{0,T}^{-1} \alpha_0(Q_0), \quad \forall Q_T : dQ_0 = \frac{B_{0,T}}{D_T} dQ_T \in \text{Dom}(\alpha_0), \]

and \( \alpha_T = \infty \) otherwise.

(2.8)

(2) Let the discount factor \( D_T \) be bounded away from 0. Given a normalized forward risk measure \( \rho_T \) with penalty function \( \alpha_T \), the functional

\[ q_0(Y) := B_{0,T} \rho_T(YD_T^{-1}), \quad Y \in \mathcal{X} \]

(2.9)

is convex and monotone decreasing with respect to \( Y \) and satisfies \( \forall \lambda \in \mathbb{R}, \ q_0(\lambda D_T) = B_{0,T} \rho_T(\lambda) = -\lambda B_{0,T} = \lambda q_0(D_T). \) Moreover, \( q_0 \) is a spot risk measure if and only if \( \rho_T \) satisfies \( \rho_T(\lambda D_T^{-1}) = -\lambda B_{0,T}^{-1} = \lambda \rho_T(D_T^{-1}), \forall \lambda \in \mathbb{R}. \)

Proof.

(1) If \( \rho_0 \) satisfies \( \rho_0(\lambda D_T) = -\lambda B_{0,T}, \forall \lambda \in \mathbb{R}, \) the forward cash additivity of \( q_T \) follows directly from Lemma 2.3. Conversely, let \( q_T \) be cash additive. This is equivalent to require that \( \rho_0 \) satisfies

\[ \forall X_T \in \mathcal{X}, \quad \forall \lambda \in \mathbb{R}, \quad \rho_0(D_T X_T + \lambda 1_T D_T) = \rho_0(D_T X_T) - \lambda B_{0,T}. \]

(2.10)

Setting \( X_T = 0 \) in equation (2.10) gives the result. To prove equation (2.8) we observe that if \( q_T \) in equation (2.7) is a spot risk measure with minimal penalty function \( \alpha_T \), the definition of the minimal penalty function and Lemma 2.3 give

\[ \alpha_T(Q_T) = \sup_{X_T} \{ \mathbb{E}_{Q_T}[-X_T] - q_T(X_T) \} \]

\[ = \sup_{X_T} \{ B_{0,T}^{-1} \mathbb{E}_{Q_0}[-D_T X_T] - B_{0,T}^{-1} \rho_0(D_T X_T) \}. \]

Since \( D_T \) is bounded away from 0, the one-to-one correspondence between bounded variables and discounted bounded variables implies \( \alpha_T(Q_T) = B_{0,T}^{-1} \alpha_0(Q_0) \), where \( dQ_0 = D_T^{-1}B_{0,T}dQ_T. \) It follows that \( Q_T \) is in the domain of \( \alpha_T \) if and only if \( Q_0 \) is a set function in the domain of \( \alpha_0 \) and satisfies the calibration constraint in equation (2.8). Conversely, a risk measure with minimal penalty function \( \alpha_T \) satisfying equation (2.8) is of the form \( \rho_T(X_T) = B_{0,T}^{-1} \rho_0(D_T X_T). \)

(2) Similar arguments can be used to prove the converse. \( \square \)

Unfortunately, the procedure of computing current reserve amounts discounting forward risk measures (given by \( q_0 \) in equation (2.9)) is feasible only when the zero-coupon bonds for the relevant maturities are available in the market. In this case, the functional \( q_0 \) in equation (2.9) is an example of the general capital requirement defined by Frittelli and Scandolo (2006).

Next section contains the major contribution of this paper, which is the introduction of a new class of risk measures called cash subadditive risk measures. These risk measures provide reserve amounts, which account for the ambiguity on the discount factor. This result is achieved by simply relaxing the cash additive axiom into the cash subadditive axiom and preserving the original difference in the numéraires of reserves and future positions. This will be illustrated by several examples in the finance and insurance frameworks.
3. CASH SUBADDITIVE RISK MEASURES

The following observation provides the intuition for introducing cash subadditive risk measures. Given the (stochastic) discount factor \( 0 \leq D_T \leq 1 \) and a spot risk measure \( \rho_0 \) satisfying equation (2.5), the convex, nonincreasing functional defined on \( \mathcal{X} \) denoted by \( \mathcal{R}(X_T) = \rho_0(D_TX_T) \) is cash subadditive; that is, it satisfies the following inequality:

\[
\forall m \geq 0, \quad \mathcal{R}(X_T + m1_T) = \rho_0(D_TX_T + D_T m) \geq \rho_0(D_TX_T + m) = \rho_0(D_T X_T) - m = \mathcal{R}(X_T) - m.
\]

This inequality is a simple consequence of the time value of the money, that is, \( D_T m \leq m \).

The functional \( \mathcal{R} \) is expressed in terms of the current numéraire but directly defined on the future position expressed in terms of the future numéraire. The function \( m \in \mathbb{R} \mapsto \mathcal{R}(X_T1_T + m1_T) + m \) is nondecreasing; that is, \( \mathcal{R} \) is cash subadditive. This observation highlights the cash subadditive axiom as the minimal condition (imposed by the time value of the money) that has to be satisfied by risk measures that preserve the two different numéraires of current reserve amounts and future risky positions. Remarkably, replacing the cash additive axiom with the cash subadditive axiom is sufficient to characterize risk measures that can be used also when cash additive risk measures cannot, for instance, under stochastic and/or ambiguous interest rates or assessing the risk of defaultable contingent claims. In the sequel, we formally define the cash subadditive risk measures denoted by \( \mathcal{R} \). Then, we provide several examples showing the different applications of these new risk measures. The previous considerations and the following examples motivate the study of cash subadditive risk measures.

3.1. Definition of Cash Subadditive Risk Measures

**Definition 3.1.** A cash subadditive risk measure \( \mathcal{R} \) is a functional \( \mathcal{R} : \mathcal{X} \to \mathbb{R} \), convex, and nonincreasing satisfying the cash subadditive axiom:

\[
\forall m \in \mathbb{R}, \quad \mathcal{R}(X_T + m1_T) + m \text{ is nondecreasing in } m.
\]

A cash subadditive risk measure \( \mathcal{R} \) is coherent when \( \mathcal{R}(\lambda X) = \lambda \mathcal{R}(X), \; \forall \lambda \geq 0 \). A cash subadditive risk measure \( \mathcal{R} \) is normalized when \( \mathcal{R}(0) = 0 \).

The cash subadditive axiom can also be expressed:

\[
\forall m \in \mathbb{R}, \quad \mathcal{R}(X_T + |m|1_T) \geq \mathcal{R}(X_T) - |m| \quad \text{and} \quad \mathcal{R}(X_T - |m|1_T) \leq \mathcal{R}(X_T) + |m|.
\]

Cash subadditive risk measures naturally account for the time value of money. When \( m \) dollars are added to the future position \( X_T \), \( X_T + m1_T \), the capital requirement at time \( t = 0 \) is reduced by less than \( m \) dollars, that is, \( \mathcal{R}(X_T1_T + m1_T) \geq \mathcal{R}(X_T1_T) - m \).

3.2. Examples of Cash Subadditive Risk Measures

This section provides several examples of cash subadditive risk measures. All these risk measures can be obtained composing cash additive risk measures and convex real (random) functions. The first example arises naturally considering an ambiguous discount factor.
3.2.1. Cash Subadditive Risk Measures under Ambiguous Discount Factors. Consider a regulator assessing the risk of a future payoff $X_T$ when the discount factor $D_T$ is ambiguous and ranges between two positive constants, $0 \leq d_L \leq D_T \leq d_H \leq 1$, according to her beliefs. The regulator is endowed with a spot risk measure $\rho_0$ and adverse to ambiguity on discount factor. Hence, she assesses the risk of $X_T$ in the interest rates worst case scenario,

$$\mathcal{R}^{\rho_0}(X_T) := \sup_{D_T \in \mathcal{D}} \{ \rho_0(D_T X_T) \mid d_L \leq D_T \leq d_H \}. \tag{3.1}$$

**Proposition 3.2.** The functional $\mathcal{R}^{\rho_0}$ in equation (3.1) is a cash subadditive risk measure. $\mathcal{R}^{\rho_0}$ can be rewritten as $\mathcal{R}^{\rho_0}(X_T) = \rho_0(-v(X_T))$, where $v(x) = -(d_L x^+ - d_H x^-)$ is convex decreasing function with left derivative $v_x$ such that $v_x \in [-1, 0]$ and $x^+ = \sup(x, 0), \ x^- = \sup(-x, 0)$.

**Proof.** $\mathcal{R}^{\rho_0}$ is a cash subadditive risk measure as it is the supremum of cash subadditive, convex, and monotone functions with respect to $X_T \in \mathcal{X}$. Moreover, since the $\inf_{D_T \in \mathcal{X}} \{ D_T X_T \mid d_L \leq D_T \leq d_H \}$ is attained by $D_T^* = d_L 1_{[X_T \geq 0]} + d_H 1_{[X_T < 0]}$, then $\sup_{D_T \in \mathcal{X}} \{ \rho_0(D_T X_T) \mid d_L \leq D_T \leq d_H \} = \rho_0(\inf_{D_T \in \mathcal{X}} \{ D_T X_T \mid d_L \leq D_T \leq d_H \}) = \rho_0(d_L X_T^+ - d_H X_T^-), \ \text{where} \ v(x) = -(d_L x^+ - d_H x^-).$ 

**Remark 3.3.** When $D_T$ varies between two random variables $D_L$ and $D_H$ in $\mathcal{X}$, $0 \leq D_L \leq D_T \leq D_H \leq 1$, the functional in equation (3.1) is a cash subadditive risk measure $\mathcal{R}^{\rho_0}(X_T) = \rho_0(-V(X_T))$, where $V$ is the random function $V(\omega, x) = -(D_L(\omega)x^+ - D_H(\omega)x^-)$, convex, decreasing with respect to $x$, $V_x \in [-1, 0]$, for any given $\omega \in \Omega$, and $\mathcal{F}_T$-measurable for any given $x \in \mathbb{R}$. Notice that when $D_L = D_T$, $\mathcal{R}^{\rho_0}(X_T) = \rho_0(0)$.

The next example of cash subadditive risk measure is not related to risky/ambiguous discount factor but to insurance risks. Following Jarrow (2002), the put option premium with zero strike price may be used as a possible measure of the firm insolvency risk. The expected losses are discounted using the risk-free gross return $r \geq 1$.

**Corollary 3.4.** Put premium risk measure. The premium of a put option with strike price zero and maturity $T$,

$$\mathcal{R}_p(X_T) := \frac{1}{r} \mathbb{E}^p[-X_T^+]. \tag{3.2}$$

is a coherent cash subadditive risk measure as a function of the underlying asset price $X_T$.

**Proof.** The cash subadditive risk measure in equation (3.1) coincides with the put option premium $\mathcal{R}_p$ when $\rho_0(\cdot) = \mathbb{E}^p[-(\cdot)]$, $d_L = 0$, and $d_H = 1/r$. 

**Remark 3.5.** For any given strike price $K$ the premium of a put option, $\mathcal{R}_p(X_T) := \frac{1}{r} \mathbb{E}^p[(K - X_T)^+]$ is a cash subadditive risk measure. This follows setting in equation (3.1) $\rho_0$ equals to the nonnormalized risk measure $\rho_0(X_T) = \mathbb{E}^p[K - X_T]$ and $-v(x) = \frac{1}{r} \max(K - x, 0)$.

3.2.2. Composing Cash Additive Risk Measures and Convex Functions. Generalizing the previous examples, we show that $\rho_0(-V)$ is a cash subadditive risk measure, where
$V$ is a random function $V : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $V(\omega, x)$, such that, for any $\omega \in \Omega$, $V(\omega, \cdot)$ is lower-semicontinuous (lsc), decreasing, convex, and $V(\omega, 0) = 0$, $V_x \in [-1, 0]$, and for any $x \in \mathbb{R}$, $V(\cdot, x)$ is $\mathcal{F}_T$-measurable. Moreover, $\rho_0(-V)$ can be represented in terms of finitely additive measures and $\mathcal{F}_T$-measurable “discount factors” over a set of possible scenarios that can be chosen according to the beliefs of the agent/regulator.

From standard results in convex analysis $V(\omega, x) = \sup_{y \in \mathbb{R}} \{xy - \beta_T(\omega, y)\}$, where $\beta_T$ is the random convex Fenchel transform of $V$, $\beta_T(\omega, y) := \sup_{x \in \mathbb{R}} \{xy - V(\omega, x)\}$. Notice that $\beta_T$ is finite only if $y \in [-1, 0]$ as $V_x \geq -1$. For example, the Fenchel transform of $v(x) = -(D_Lx^+ - D_Hx^-)$ is $\beta_T(y) = f^D(y)$, where $f^D$ is the convex indicator function of the set $\mathcal{D} = [D_L, D_H]$, equal to 0 on $\mathcal{D}$ and $\infty$ otherwise. While $V_x \geq -1$ is a necessary condition to obtain a cash subadditive functional, the decreasing monotonicity ($V_x \leq 0$) and convexity of $V$ insure the convexity and decreasing monotonicity of $\rho_0(-V)$.

**Proposition 3.6.** Let $V$ be a random, lsc, decreasing convex function as above and $\beta_T$ the convex Fenchel transform of $V$. Let $\rho_0$ be a cash additive risk measure defined on $\mathcal{X}$ with minimal penalty function $\alpha_0$, $\mathcal{R}^{\rho_0, V}(X_T) := \rho_0(-V(X_T))$ is a cash subadditive risk measure, derived from the spot risk measure $\rho_0$ by assessing discount factors ambiguity through the penalty function $\beta_T$,

$$\mathcal{R}^{\rho_0, V}(X_T) = \sup_{D_T \in \mathcal{X}} \{\rho_0(D_TX_T + \beta_T(-D_T)) \mid 0 \leq D_T \leq 1\}. \quad (3.3)$$

Moreover, $\mathcal{R}^{\rho_0, V}(X_T) = \rho_0(-V(X_T))$ admits the dual representation

$$\mathcal{R}^{\rho_0, V}(X_T) = \sup_{Q_0 \in M_{1, f}, D_T \in \mathcal{X}} \{\mathbb{E}_{Q_0}[-D_TX_T] - \alpha^{\rho_0, V}(Q_0, D_T) \mid 0 \leq D_T \leq 1\}. \quad (3.4)$$

$$\alpha^{\rho_0, V}(Q_0, D_T) := \alpha_0(Q_0) + \mathbb{E}_{Q_0}[\beta_T(-D_T)]. \quad (3.5)$$

For instance, if $\rho_0$ is the coherent worst case risk measure, that is, $\rho_0(X_T) = \rho_{\max}(X_T) = \sup_{Q_0 \in M_1} \mathbb{E}_{Q_0}[-X_T]$, then $\mathcal{R}^{\rho_0, V}(X_T) = \rho_{\max}(-V(X_T)) = |-V(X_T)|_\infty = -V(|X_T|_\infty)$ and $\alpha^{\rho_0, V}(Q_0, D_T) := \mathbb{E}_{Q_0}[\beta(-D_T)]$.

**Remark 3.7.** Representations (3.4)-(3.5) provide a better understanding of the different risks involved in the evaluation of the risky position $X_T$. The scenarios could be exogenously determined, for instance, by some regulatory institution. The penalty function $\alpha^{\rho_0, V}$ depending on the ambiguous model and ambiguous discount factor could be determined by the preferences of the economic agent on $Q_0$ and $D_T$.

**Remark 3.8.** Robust expected utilities and cash subadditive risk measures. By definition, the functional $\mathcal{R}^{\rho_0, V}$ admits a representation in terms of the ambiguous model and the convex function on the risky positions, $\mathcal{R}^{\rho_0, V}(X_T) = \sup_{Q_0 \in M_{1, f}} \{\mathbb{E}_{Q_0}[V(X_T)] - \alpha_0(Q_0)\}$. The opposite of $\mathcal{R}^{\rho_0, V}$ can be viewed as examples of robust expected utilities associated with concave functions $U = -V$ and concave penalty functions $\hat{\alpha}_0 = -\alpha_0$, that is, $-\mathcal{R}^{\rho_0, V}(X_T) = \inf_{Q_0 \in M_{1, f}} \{\mathbb{E}_{Q_0}[U(X_T)] - \hat{\alpha}_0(Q_0)\}$. Notice that $U$ does not satisfy the Inada conditions. For robust expected utilities see, for instance, Schied (2004) and Maccheroni, Marinacci, and Rustichini (2004).
Proof. First, we prove that $\mathcal{R}^{\rho_0, V}$ is a cash subadditive risk measure.

Decreasing monotonicity: The increasing monotonicity of $-V$ and the decreasing monotonicity of $\rho_0$ imply the decreasing monotonicity of $\mathcal{R}^{\rho_0, V}$.

Convexity: The concavity of $-V$, the decreasing monotonicity, and the convexity of $\rho_0$ imply the convexity of $\mathcal{R}^{\rho_0, V}$.

Cash subadditivity: $\mathcal{R}^{\rho_0, V}(X_T + m) + m = \rho_0(-(V(X_T + m1_T)) + m = \rho_0(\rho_0(-V(X_T + m1_T) - m) - m)$ is increasing in $m$ if $V(X_T + m1_T)$ - $m$ is decreasing in $m$. As $V_\chi > -1$ the result follows.

Representations: To prove equation (3.3) we observe that

$$\rho_0(-V(X_T)) = \rho_0\left(\inf_{-1 \leq y \leq 0} \{-X_T y + \beta_T(y)\}\right)$$

$$= \rho_0\left(\inf_{\mathcal{D}_T \in \mathcal{X}} \{DTX_T + \beta_T(-DT) | 0 \leq DT \leq 1\}\right).$$

From the decreasing monotonicity of $\rho_0$, for any $\tilde{D}_T \in \mathcal{X}, 0 \leq \tilde{D}_T \leq 1$ we have

$$\rho_0\left(\inf_{DT \in \mathcal{X}} \{DTX_T + \beta_T(-DT) | 0 \leq DT \leq 1\}\right) \geq \rho_0(\tilde{D}_T X_T + \beta_T(-\tilde{D}_T)). \quad (3.6)$$

The result follows setting $\tilde{D}_T = D_T^*$ in equation (3.6), where $D_T^* \in \mathcal{X}$ is the element achieving the inf$_{\mathcal{D}_T \in \mathcal{X}} \{DTX_T + \beta_T(-DT) | 0 \leq DT \leq 1\}$. Finally, representations (3.4)–(3.5) are obtained from the dual representation of $\rho_0$ and from (3.3). $\square$

The penalty function $\alpha^{\rho_0, V}$ in equation (3.5) is not the minimal one. As any pair $(Q_0, D_T)$ defines a unique additive set function $\mu$ absolutely continuous with respect to $Q_0$, $d\mu: DTdQ_0, 0 \leq \mu(\Omega) \leq 1$, the functional $\mathcal{R}^{\rho_0, V}$ can be rewritten as

$$\mathcal{R}^{\rho_0, V}(X_T) = \sup_{\mu \in \mathcal{M}_1, (\mathcal{F}_T)} \{\mu(-X_T) - \gamma(\mu) | 0 \leq \mu(\Omega) \leq 1\},$$

where $\mu(-X_T) := \int -X_T(\omega) \mu(\omega) d\omega$ and $\gamma(\mu) = \inf_{Q_0 \in \mathcal{M}_1, (\mathcal{F}_T)} \{\alpha_0(Q_0) + E_Q[\beta_T(\frac{d\mu}{d\mu_{Q_0}})]\}$ for any $\mu$ such that $d\mu = DTdQ_0, 0 \leq DT, \mu(\Omega) \leq 1$, and $\gamma = \infty$ otherwise.

The next section gives the dual representation of the cash subadditive risk measures $\mathcal{R}$ in terms of the minimal penalty function.

4. MINIMAL CASH ADDITIVE EXTENSION OF $\mathcal{R}$ AND DUALITY

In this section we study the dual representation of cash subadditive risk measures. To obtain duality results we can either use convex analysis tools (for instance, adapting the techniques for convex risk measures in Krätschmer 2007), or recover cash additive risk measures by enlarging the space of risky positions. We adopt the second approach because of its richer financial interpretation, despite the fact that the first one could be less involved. Our approach provides an interesting interpretation of cash additive and cash subadditive risk measures where default events or stochastic numéraires are taken into account. Taking a classical procedure in credit risk modeling, we consider a minimal enlargement of the sample space $\Omega$ and we extend the cash subadditive risk measure $\mathcal{R}$ into a cash additive risk measure, which is in a one-to-one correspondence with $\mathcal{R}$. This allows to derive properties of $\mathcal{R}$ and the dual representation using classical theory on cash additive risk measures.
Cheridito and Kupper (2006) use a similar procedure to decompose dynamic cash additive risk measures in one-step generators and to provide a dual representation of these generators. Interestingly, these generators are cash subadditive risk measures with opposite sign.4

While in the dual representation of cash additive risk measures the set functions \( Q \) are normalized to one in \( M \), the sequel, we introduce the minimal measurable space where the pair \((X_T, x)\) are normalized to one in \( M \). These generators are cash subadditive risk measures with additive risk measures in one-step generators and to provide a dual representation of \( \mu \) where \( \mu \) measures is based on finite additive set functions \( \tau \). Notice that \( \hat{\mu} \) function defined on the enlarged space \( \hat{X}_T \) is endowed with the \( \sigma \)-algebra \( \hat{F}_T \), generated by the bounded random variables \( \hat{X}_T \). Notice that \( \hat{F}_T \) is not the product \( \sigma \)-algebra. Let \( \hat{X} \) be the linear space of all bounded random variables \( \hat{X}_T \). To denote \( \hat{X}_T \in \hat{X} \) we use its coordinates \( \hat{X}_T = (X_T, x) \). The constant variables are denoted by \( \hat{\theta} = (m, m) \) and \( \hat{m} = m l_{\theta=1} + m l_{\theta=0} = m \). The event \( \{ \theta = 0 \} \) is atomic and all \( \hat{F}_T \)-random variables are constant on this event. The event \( \{ \theta = 1 \} \) models the risk affecting the numéraire \( 1_T \). Intuitively, \( \theta \) is associated with the default time \( \tau \) of the counterpart. The event \( \{ \theta = 1 \} \) is equivalent to \( \{ \tau > T \} \). The choice of the atomic \( \sigma \)-algebra \( \hat{F}_T \) implies a one-to-one correspondence between normalized additive set function \( \hat{Q} \) in \( M_{1,\ell} (\hat{F}_T) \) and subprobability set functions in \( M_{s,\ell} (F_T) \) on \( (\Omega, F_T) \).

Indeed, any \( \hat{Q} \) in \( M_{1,\ell} (\hat{F}_T) \) can be decomposed as follows, \( \forall \hat{X}_T = (X_T, x) \in \hat{X} \):

\[
\hat{Q}(\hat{X}_T) = \hat{Q}(X_T l_{\theta=1}) + x \hat{Q}(l_{\theta=0}) = \mu(X_T) + x(1 - \mu(1)),
\]

where \( \mu(\cdot) := \hat{Q}(l_{\theta=1}) \) is an additive subprobability of \( M_{s,\ell} (F_T) \).

The following proposition shows how to extend the cash subadditive functional \( \mathcal{R} \) into a cash additive risk measure \( \hat{\rho} \) on \( \hat{X} \) and the one-to-one correspondence.

**Proposition 4.1.**

(1) A normalized cash subadditive risk measure \( \mathcal{R} \) on \( \mathcal{X} \) defines a normalized cash additive risk measure \( \hat{\rho} \) on \( \hat{X} \),

\[
\forall \hat{X}_T = (X_T, x) \in \hat{X}, \quad \hat{\rho}(\hat{X}_T) := \hat{\rho}((X_T, x)) := \mathcal{R}(X_T - x1_T) - x.
\]

Notice that \( \hat{\rho}(X_T l_{\theta=1}) = \mathcal{R}(X_T) \).

(2) Any cash additive risk measure on \( \hat{X} \) restricted to the event \( \{ \theta = 1 \} \) defines a cash subadditive risk measure that satisfies equation (4.2).

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4 We thank the referee for highlighting this result.
REMARK 4.2. The cash subadditive risk measure $\mathcal{R}$ can be used to measure the risk of defaultable contingent claims $\hat{X}_T$ when there is no compensation ($\chi = 0$) if the default occurs, $\{\theta = 0\}$.

The proof relies on the cash subadditive property to obtain a monotone decreasing functional.

Proof.

(1) *Cash additive:* Let $\hat{X}_T = (X_T, x) \in \hat{X}$ and $m \in \mathbb{R}$. By definition, $\hat{\rho}(X_T1_{\theta=1} + x1_{\theta=0} + m1_{\theta=1} + m1_{\theta=0}) = \mathcal{R}(X_T + m1_T - (x + m)1_T) - (x + m) = \hat{\rho}(\hat{X}_T) - m$.

*Decreasing monotonicity:* Let $\hat{X}_T = (X_T, x)$ and $\hat{Y}_T = (Y_T, y) \in \hat{X}$ such that $\hat{X}_T \geq \hat{Y}_T$, that is, $X_T \geq Y_T$ and $x \geq y$. From the cash subadditivity and the decreasing monotonicity of $\mathcal{R}$, it follows that $\hat{\rho}(\hat{X}_T) = \mathcal{R}(X_T - x1_T) - x \leq \mathcal{R}(X_T - y1_T) - y \leq \mathcal{R}(Y_T - y1_T) - y = \hat{\rho}(\hat{Y}_T)$.

*Convexity:* We use the notation in equation (2.1). From the convexity of $\mathcal{R}$, $\mathcal{R}(\text{Bar}[X_T]) \leq \text{Bar}[\mathcal{R}(X_T)]$. This implies that $\hat{\rho}(\text{Bar}[\hat{X}_T]) = \mathcal{R}(\text{Bar}[X_T - x_T]) - \text{Bar}[x_T] \leq \text{Bar}[\mathcal{R}(X_T - x_T)] - \text{Bar}[x_T] = \text{Bar}[\hat{\rho}(\hat{X}_T)]$, which shows the convexity of $\hat{\rho}$.

(2) Let $\hat{\rho}$ be a cash additive risk measure on $\hat{X}$. We have to show that $\mathcal{R}^\hat{\rho}(X_T) := \hat{\rho}(X_T1_{\theta=1})$ is a cash subadditive risk measure. The decreasing monotonicity and convexity follow from the definition. The cash subadditive property is verified observing that $\mathcal{R}^\hat{\rho}(X_T + m1_T) + m = \hat{\rho}((X_T + m)1_{\theta=1}) + m = \hat{\rho}(X_T1_{\theta=1} - m1_{\theta=0})$ is increasing in $m$. □

4.2. Dual Representation of Cash Subadditive Risk Measures

In the next proposition we use the one-to-one correspondence in equation (4.2) between $\hat{\rho}$ and $\mathcal{R}$ to characterize cash subadditive risk measures. We show that the minimal penalty function of $\mathcal{R}$ and the minimal penalty function $\hat{\rho}$ coincide and are concentrated on the set of subprobability measures $\mathcal{M}_{\sigma,\mathcal{F}_T}$. Moreover, under the additional assumption of continuity from below of $\mathcal{R}$, the dual representation in terms of $\sigma$-additive subprobability measures is obtained.

**Theorem 4.3.**

(a) Any cash subadditive risk measure $\mathcal{R}$ on $\mathcal{X}$ can be represented in terms of finitely additive subprobability measures,

$$
\mathcal{R}(X_T1_{\mathcal{F}_T}) = \sup_{\mu \in \mathcal{M}_{\sigma,\mathcal{F}_T}} \{\mu(-X_T) - \alpha^R(\mu)\}, \quad \alpha^R(\mu) := \hat{\alpha} (\hat{Q}),
$$

where $\mu(\cdot) = \hat{Q}(1_{\theta=1})$ and $\hat{\alpha}$ is any penalty function representing $\hat{\rho}$. In particular, if $\hat{\alpha}$ is the minimal penalty function for $\hat{\rho}$, then $\alpha^R$ is the minimal penalty function for $\mathcal{R}$ and $\alpha^R(\mu) = \sup_{X_T \in \mathcal{X}} \{\mu(-X_T) - \mathcal{R}(X_T)\}$.

(b) When $\mathcal{R}$ is a cash subadditive risk measure continuous from below, any penalty function $\beta$ representing $\mathcal{R}$ is concentrated on the class $\mathcal{M}_{s,\mathcal{F}_T}$ of $\sigma$-additive subprobability measures; that is, $\beta(\mu) < \infty \Rightarrow \mu$ is $\sigma$-additive.
Proof.

(a) From Proposition 4.1, $\mathcal{R}(X_T1_T) = \hat{\rho}(X_T1_{\theta=1})$. Equation (4.3) is implied by the dual representation of $\hat{\rho}$ and the one-to-one correspondence between $\hat{\Omega}$ and $\mu$: $\hat{\Omega}(1_{\theta=1}) = \mu(\cdot)$. Let $\hat{\alpha}$ be the minimal penalty function of $\hat{\rho}$. By definition of the minimal penalty function,

\begin{equation}
\hat{\alpha}(\hat{\Omega}) = \sup_{\hat{X}_T \in \hat{X}_T} \{\mathbb{E}[\hat{\Omega}[-X_T1_{\theta=1} - x1_{\theta=0} - \hat{\rho}(\hat{X}_T)]
\end{equation}

\begin{equation}
= \sup_{\hat{X}_T \in \hat{X}_T} \{\mathbb{E}[\hat{\Omega}[-(X_T - x)1_{\theta=1}] - x - \mathcal{R}(X_T - x1_T) + x]
\end{equation}

\begin{equation}
= \sup_{X_T \in X_T} \{\mathbb{E}[\hat{\Omega}[-(X_T)1_{\theta=1}] - \mathcal{R}(X_T)), \quad \hat{\Omega} \in M_{1,f}(\hat{\mathcal{F}}_T).
\end{equation}

As $\hat{\Omega}(1_{\theta=1}) = \mu(\cdot)$, from equation (4.4) we have $\alpha^\mathcal{R}(\mu) := \hat{\alpha}(\hat{\Omega}) = \sup_{X_T \in X_T} \{\mu(-X_T) - \mathcal{R}(X_T)\}$, showing that $\alpha^\mathcal{R}$ is the minimal penalty function for $\mathcal{R}$.

(b) If $\mathcal{R}$ is continuous from below, the cash additive $\hat{\rho}$ is continuous from below as a function of $\hat{X}_T = (X_T, x)$. Then from Theorem 2.2 follows that the penalty function of $\hat{\rho}$ is concentrated on the class $M_1(\hat{\mathcal{F}}_T)$. This implies that the penalty function of $\mathcal{R}$ is concentrated on the set of $\sigma$-additive subprobability $M_\sigma(\mathcal{F}_T)$.

The next corollary shows a representation of $\mathcal{R}$, where the penalty function depends on constants $c \in [0, 1]$ and probability measures. Frittelli and Scandolo (2006) provide examples of general capital requirement with similar representations. It is interesting to observe that, among these, the only capital requirement that satisfies the property of cash subadditivity is the one reflecting the agent’s temporal risk aversion, which is related to the uncertainty in the numéraire. For more details see Section 6 in Frittelli and Scandolo (2006).

**COROLLARY 4.4.** Any cash subadditive risk measure $\mathcal{R}$ can be represented as follows:

\begin{equation}
\forall X_T \in \mathcal{X}, \quad \mathcal{R}(X_T1_T) = \sup_{(c, Q_T) \in [0, 1] \times M_{1,f}(\mathcal{F}_T)} \{c \mathbb{E}[Q_T(-X_T) - \alpha^\mathcal{R}(c \cdot Q_T)]\}.
\end{equation}

When $\mathcal{R}$ is continuous from below, the penalty function $\alpha^\mathcal{R}(c \cdot Q_T)$ is concentrated on the set $[0, 1] \times M_1(\mathcal{F}_T)$ where $M_1(\mathcal{F}_T)$ is set of $\sigma$-additive subprobability. If $\inf_{X_T \in \mathcal{X}} \mathcal{R}(X_T) = -\infty$, the constants $c$ in equation (4.5) are strictly positive.

**Proof.** Equation (4.5) follows by normalizing the subprobability in equation (4.3), more precisely defining, for any $\mu \in M_{k,f}$ such that $\mu > 0$, $Q_T(\cdot) = \mu(\cdot)/c$ where $c := \mu(\Omega)$. If $\mu = 0$ then $c = 0$ and $Q_T \in M_{1,f}(\mathcal{F}_T)$ is not uniquely identified. The condition $\inf_{X_T \in \mathcal{X}} \mathcal{R}(X_T) = -\infty$ implies $-\alpha^\mathcal{R}(0) = \inf_{X_T \in \mathcal{X}} \mathcal{R}(X_T) = -\infty$ excluding $\mu = 0$.

The following representation, suggested by an anonymous referee, provides a characterization of cash subadditive risk measures in terms of ambiguous “zero-coupon bond” viewed as a deterministic discount factor. The risky positions are assessed via a family of forward convex risk measures (see Equation 2.6) parameterized by the deterministic discounted factors.\(^5\)

\(^5\) We thank the referee for this stimulating suggestion.
COROLLARY 4.5. Any cash subadditive risk measure $R$ such that $\inf_{X_T \in \mathcal{X}} R(X_T) = -\infty$ is a worst discounted forward risk measure, that is,

\begin{equation}
R(X_T 1_T) = \sup_{c \in [0,1]} c \cdot \rho_{T,c}(-X_T),
\end{equation}

where $(\rho_{T,c})$ is a family of forward cash additive risk measures such that the functional $(X_T, c) \in \mathcal{X} \times (0, 1] \rightarrow c \cdot \rho_{T,c}(-X_T)$ is convex.

**Proof.** Since any functional on the right side of equation (4.6) is a cash subadditive, convex, and monotone functional, their supremum shares the same property.

Conversely, given a cash subadditive risk measure $R$ and its dual representation, as $\inf_{X_T \in \mathcal{X}} R(X_T) = -\infty$, we can define the forward risk measures,

$$
\rho_{T,c}(-X_T) = \sup_{Q_T \in M_1(\mathcal{F}_T)} \left\{ E_{Q_T}(-X_T) - \frac{\alpha^R(c \cdot Q_T)}{c} \right\}.
$$

By definition, the family $c \cdot \rho_{T,c}(-X_T)$ is convex in both arguments $(c, X_T)$ and $R$ can be rewritten as in equation (4.6). $\square$

5. OTHER CASH ADDITIVE EXTENSIONS OF $R$

In this section we provide a representation of a cash subadditive risk measure $R$, where the ambiguous discount factor/numéraire is explicitly modeled as random variables of $\mathcal{X}$. While for the cash subadditive risk measures generated via convex functions (in Section 3.2) these representations were easily obtained, to derive similar results for a generic $R$, new assumptions and more involved techniques are required. To achieve this goal, we apply the same procedure as in Section 4 and we extend $R$ to a larger space that contains $\mathcal{X}$. In this case the extension is not unique and requires the introduction of an auxiliary, a priori cash additive risk measure. Then, for the cash subadditive risk measures generated via convex functions, we propose another extension on the same enlarged space obtained through a conditional risk measure.

5.1. Cash Subadditive Risk Measures and Ambiguous Discounted Factors

To define a linear space that contains $\mathcal{X}$, the $\sigma$-algebra $\tilde{\mathcal{F}}_T$ defined in Section 4 is replaced by the product $\sigma$-algebra $\mathcal{G}_T$. On $(\Omega \times (0, 1], \mathcal{G}_T)$ any bounded $\mathcal{G}_T$-random variable $\tilde{X}_T$ can be represented as $\tilde{X}_T(\omega, \theta) = X^1_T(\omega)1_{\theta=1} + X^0_T(\omega)1_{\theta=0}$ and $X^1_T, X^0_T \in \mathcal{X}$. Let $\tilde{\mathcal{X}}$ be the linear space of all the bounded $\mathcal{G}_T$-random variables $\tilde{X}_T$. We refer to $\tilde{X}_T$ using the short notation $\tilde{X}_T = (X^1_T, X^0_T)$. The diagonal elements $\tilde{X}_T = (X_T, X_T)$ coincide with $X_T$ and the corresponding $\sigma$-algebra with $\mathcal{F}_T$. This identification was not possible for the random variables $\tilde{X} = (X_T, \omega)$ defined in the previous section.

Now, we discuss the probabilistic structure of $(\Omega \times (0, 1], \mathcal{G}_T)$. Notice that in this section we consider probability measures and not finite additive set functions.

**Definition 5.1.** For any given probability measure $\bar{\mathbb{Q}} \in \mathcal{M}_1(\mathcal{G}_T)$, let $\mathbb{Q}$ denote the restriction of $\bar{\mathbb{Q}}$ to $\mathcal{F}_T$. $\mathbb{Q} := \bar{\mathbb{Q}}|_{\mathcal{F}_T}$, and $D_T \in [0, 1]$ the $\mathcal{F}_T$-conditional probability of the event $\{\theta = 1\}$. $D_T := \mathbb{E}_{\bar{\mathbb{Q}}}[1_{\theta=1}|\mathcal{F}_T]$, also called discount factor. We denote $\bar{\mathbb{Q}}$ the
probability measure associated with the restriction of \( \widetilde{Q} \) to the event \( \{ \theta = 0 \} \), which is uniquely determined by \( (Q, D_T) \),

\[
(5.1) \quad Q(X_T) = Q(D_T X_T) + (1 - Q(D_T)) \widetilde{Q}(X_T).
\]

\( \overline{Q} \) is a probability measure absolutely continuous with respect to \( Q \), with Radon-Nikodym density given by \( \Delta_T := \frac{d\overline{Q}}{dQ} = \frac{1-D_T}{1-Q(D_T)}, \) \( 0 \leq \Delta_T \leq 1 \). \( Q(\Delta_T) = 1 \).

For any \( \tilde{X}_T = X^1_T 1_{\theta=1} + X^0_T 1_{\theta=0} \in \tilde{X} \),

\[
(5.2) \quad \widetilde{Q}(X^1_T 1_{\theta=1} + X^0_T 1_{\theta=0}) = Q(X^1_T D_T) + Q(X^0_T (1-D_T)) = Q(X^1_T D_T) + (1 - Q(D_T)) \overline{Q}(X^0_T).
\]

**Remark 5.2.** The interpretation of \( D_T \) in credit risk. In credit risk, \( \theta \) is associated with the default time of the counterpart \( \tau \), where \( \tau \) is a positive random variable non-\( F_T \)-measurable. The event \( \{ \theta = 1 \} \) can be viewed as \( \{ \tau > T \} \) and \( E_Q[1_{\theta=1} | F_T] \) as the conditional survival probability function of \( \tau \) at time \( T \). \( \tilde{X}_T = X^1_T 1_{\theta=1} + X^0_T 1_{\theta=0} \in \tilde{X} \) is a defaultable contingent claim that pays \( X^1_T \) (at time \( T \)) if there is no default \( (\tau > T) \) and \( X^0_T \) otherwise.

In the sequel, we extend \( R \) into a cash additive risk measure \( \rho \) on the enlarged space \( \tilde{X} \). Via the penalty function of \( \rho \), a representation of cash subadditive risk measures will be given in terms of the ambiguous probability measure and the ambiguous discount factor, both on the original space of definition of \( R \). To define this cash additive risk measure on \( \tilde{X} \), we use, as in Section 4.1, the cash additive risk measure \( \rho \) in equation (4.2). In this case, \( \tilde{X}_T = (X^1_T, X^0_T) \in \tilde{X} \) has two risky components and we introduce an a priori risk measure \( \overline{\rho} \) assessing the risk of the second component.

**Definition 5.3.** Let \( R \) be a cash subadditive risk measure and \( \overline{\rho} \) a cash additive risk measure, both normalized and defined on \( X \). The functional on \( \tilde{X} \)

\[
(5.3) \quad \rho(\tilde{X}_T) = \rho(X^1_T, X^0_T) := R(X^1_T + \overline{\rho}(X^0_T) 1_T) + \overline{\rho}(X^0_T) = \rho(X^1_T, -\overline{\rho}(X^0_T)),
\]

and its restriction on \( X \),

\[
(5.4) \quad \rho_{R, \overline{\rho}}(X_T) := R(X_T + \overline{\rho}(X_T) 1_T) + \overline{\rho}(X_T) = \rho(X_T, -\overline{\rho}(X_T)),
\]

are cash additive risk measures. Moreover, \( \rho(X_T 1_{\theta=1}) = R(X_T 1_T) \).

The following theorem shows that \( R \) can be written as a function of probability measures \( Q \in \mathcal{M}_1(F_T) \) and \( F_T \)-measurable discount factors \( D_T \in X \), using the minimal penalty function of the cash additive risk measure \( \rho \). This representation is similar to the dual representation (see equations (3.4) and (3.5)) of cash subadditive risk measures generated by convex functions.

We consider penalty functions concentrated on the class of probabilities measures assuming that \( R \) and \( \overline{\rho} \) are continuous from below. This also implies that \( \rho \) and \( \rho_{R, \overline{\rho}} \) are continuous from below.

**Theorem 5.4.** Assume that the convex functionals \( R \) and \( \overline{\rho} \) are continuous from below. Let \( \alpha^R \) and \( \overline{\alpha} \) be the minimal penalty functions of \( R \) and \( \overline{\rho} \), respectively. Let \( \tilde{\alpha} \) be the
The cash subadditive risk measure $\mathcal{R}$ can be represented as

$$\mathcal{R}(X_{T}^{1}) = \tilde{\rho}(X_{T}^{1}_{\theta=1}) = \sup_{\{\tilde{\alpha}(D_{T} \cdot \tilde{\mathbb{Q})} + (1 - \tilde{\mathbb{Q}}(D_{T}))\tilde{\alpha}(\tilde{\mathbb{Q})} = \tilde{\mathbb{Q}}(D_{T}) + (1 - \tilde{\mathbb{Q}}(D_{T}))\tilde{\alpha}(\tilde{\mathbb{Q})}.$$
(2) To obtain the penalty function $\alpha_{R, \mathcal{P}}$ of $\rho_{R, \mathcal{P}}$, we restrict $\hat{\rho}$ on $\mathcal{F}_T$ and we use equation (5.1),

$$\rho_{R, \mathcal{P}}(X_T) = \sup_{Q \in \mathcal{M}_1(\mathcal{F}_T)} \left\{ \langle -X_T, DT \rangle + (1 - \langle DT, Q \rangle) \overline{\alpha}(Q) - (\alpha^R(D_T, Q) - (1 - \langle DT, \overline{\alpha}(Q) \rangle) \right\}$$

$$\rho_{R, \mathcal{P}}(0) = \sup_{Q \in \mathcal{M}_1(\mathcal{F}_T)} \left\{ \langle -X_T, DT \rangle + (1 - \langle DT, Q \rangle) \overline{\alpha}(Q) \right\}.$$

Observing that for a given $Q \in \mathcal{M}_1(\mathcal{F}_T)$ more than one pair $(D_T, \overline{Q})$, $D_T \in \mathcal{X}$, $D_T \in [0, 1]$, can verify $\langle -X_T, DT \rangle + (1 - \langle DT, Q \rangle) \overline{\alpha}(Q)$ yields equation (5.7). Similar calculations show that $\alpha_{R, \mathcal{P}}$ is the minimal penalty function.

5.2. Conditional Risk Measures and Extensions on $\tilde{\mathcal{X}}$

This section reinterprets the cash subadditive risk measures $\mathcal{R}^\rho, V = (\rho(-V))$ studied in Section 3.2.2 These risk measures are now represented as the composition of the unconditional cash additive risk measure $\rho$ and the conditional cash additive risk measure generated by the random function $V$. We obtain the result introducing a more natural extension of $\mathcal{R}^\rho, V$ called $\check{\rho}^V$ to the enlarged space $\tilde{\mathcal{X}}$. The restriction of $\check{\rho}^V$ to the space $\mathcal{X}$ is $\rho$ itself, and $\check{\rho}^V$ can be obtained composing $\rho$ with a cash additive conditional risk measures. Moreover, we show that any cash additive risk measure on $\tilde{\mathcal{X}}$ generated from $\rho$ via a conditional cash additive risk measure is associated to a cash subadditive risk measure generated by a convex function.

As in Section 3.2.2, in the sequel $\rho$ denotes a normalized cash additive risk measure and $\mathcal{V}(\omega, x)$ an $\mathcal{F}_T$-measurable random functional convex monotone decreasing such that $\mathcal{V}(0) = 0$ and with left derivative $V_x \in [-1, 0]$. From Proposition 3.6, we know that $\mathcal{R}^\rho, V(X_T) := \rho(-V(X_T))$ is a cash subadditive risk measure on $\mathcal{X}$.

**Proposition 5.6.** On the enlarged space $\tilde{\mathcal{X}}$, any cash additive risk measure $\rho$ and any random function $V$ define a cash additive risk measure,

$$\check{\rho}^V(X^1_{T, \theta=1} + X^0_{T, \theta=0}) := \rho(-V(X^1_{T} - X^0_{T}) + X^0_{T}), \quad X^1_{T, \theta=1} + X^0_{T, \theta=0} \in \tilde{\mathcal{X}}.$$

$\check{\rho}^V$ coincides with $\mathcal{R}^\rho, V$ on $\{\theta = 1\}$ and with $\rho$ on $\mathcal{X} \subset \tilde{\mathcal{X}}$:

$$\check{\rho}^V(X^1_{T, \theta=1}) = \rho(-V(X_T)) = \mathcal{R}^\rho, V(X_T) \quad \text{and} \quad \check{\rho}^V((X_T, X_T)) = \rho(X_T).$$

Requiring $V$ decreasing monotone and such that $V_x \in [-1, 0]$ is crucial to obtain $\check{\rho}^V$ decreasing monotone (see proof below).

**Proof.** Decreasing monotonicity: $\check{\rho}^V$ is decreasing monotone if $V(X^1_{T} - X^0_{T}) - X^0_{0}$ is decreasing monotone with respect to $(X^1_{T}, X^0_{T})$. Let $\tilde{X}_T = (X^1_{T}, X^0_{T}) \geq \tilde{Y}_T = (Y^1_{T}, Y^0_{T})$, that is, $X^1_{T} \geq Y^1_{T}$ and $X^0_{T} \geq Y^0_{T}$. As $V(x + m) + m$ is not decreasing in $m$, $V(X^1_{T} - X^0_{T}) - X^0_{0}$ is not increasing in $X^0_{T}$, then $V(X^1_{T} - X^0_{T}) - X^0_{0} \leq V(Y^1_{T} - Y^0_{T}) - Y^0_{0}$, where the last inequality is due to the decreasing monotonicity of $V$.

Cash additivity and convexity follow from the definition of $\check{\rho}^V$. \qed
Now, we recall the definition of conditional risk measures that in our setting\(^6\) reads as follows.

**Definition 5.7.**

(1) A cash additive conditional risk measure on \(\mathcal{F}_T\) is a monotone decreasing convex functional, \(\tilde{\rho}_{\mathcal{F}_T} : \tilde{\mathcal{X}} \rightarrow \mathcal{X}\), which satisfies the \(\mathcal{F}_T\)-cash additive axiom, that is,

\[
\forall \tilde{X} \in \tilde{\mathcal{X}}, \quad \forall Y \in \mathcal{X}, \quad \tilde{\rho}_{\mathcal{F}_T}(\tilde{X} + Y) = \tilde{\rho}_{\mathcal{F}_T}(\tilde{X}) - Y.
\]

(2) \(\tilde{\rho}_{\mathcal{F}_T}\) is regular if for any \(F_T \in \mathcal{F}_T\), \(\tilde{X}_T \in \tilde{\mathcal{X}}\), \(\tilde{\rho}_{\mathcal{F}_T}(1_{F_T} \tilde{X}_T) = 1_{F_T} \tilde{\rho}_{\mathcal{F}_T}(\tilde{X}_T)\).

(3) A cash additive risk measure \(\rho\) on \(\mathcal{X}\) is generated from \(\rho\) via a conditional risk measure if there exists a cash additive conditional risk measure on \(\mathcal{F}_T\), \(\tilde{\rho}_{\mathcal{F}_T}\), such that \(\rho(X_T^1, X_T^0) = \rho(-\tilde{\rho}_{\mathcal{F}_T}(X_T^1, X_T^0))\).

It is easy to see that any conditional risk measure on \(\mathcal{F}_T\) is completely determined by its value on the set \(\{\theta = 1\}\). This observation leads to the following proposition.

**Proposition 5.8.** Any \(\mathcal{F}_T\)-measurable random function \(V\) defines a cash additive conditional risk measure on \(\mathcal{F}_T\), \(\tilde{\rho}^V_{\mathcal{F}_T} : \tilde{\mathcal{X}} \rightarrow \mathcal{X}\), given by

\[
\tilde{\rho}^V_{\mathcal{F}_T}(X_T 1_{\theta=1}) := V(X_T) \quad \text{or equivalently by}
\]

\[
\tilde{\rho}^V_{\mathcal{F}_T}((X_T^1, X_T^0)) := V(X_T^1 - X_T^0) - X_T^0.
\]

Conversely, any regular and continuous from above cash additive conditional risk measure on \(\mathcal{F}_T\), \(\tilde{\rho}_{\mathcal{F}_T} : \tilde{\mathcal{X}} \rightarrow \mathcal{X}\), generates a convex random function \(\tilde{V}^\mathcal{F}_T(\lambda) := \tilde{\rho}_{\mathcal{F}_T}(\lambda 1_{\theta=1})\), which satisfies (5.9).

**Proof.**

Decreasing monotonicity: We refer the reader to the proof of decreasing monotonicity in Proposition 5.6. \(\mathcal{F}_T\)-cash invariance and convexity follow respectively from the definition of \(\tilde{\rho}_V^{\mathcal{F}_T}\) and the convexity of \(V\).

Conversely: Define \(\tilde{V}^\mathcal{F}_T(\omega \lambda) := \tilde{\rho}_{\mathcal{F}_T}(\lambda 1_{\theta=1}(\omega))\). \(\tilde{V}(\omega \lambda)\) is \(\mathcal{F}_T\)-measurable convex and monotone decreasing functional such that \(\tilde{V}^\mathcal{F}_T(0) = 0\) and \(\tilde{V}^\mathcal{F}_T \in [-1, 0]\). For the regularity of \(\tilde{\rho}_{\mathcal{F}_T}\), the previous definition can be extended to all the simple \(\mathcal{F}_T\)-random variables \(\sum \lambda_i 1_{A_i}\), where the sets \(A_i \in \mathcal{F}_T\) and \(\{A_i\}_{i=1}^n\) form a partition of \(\Omega\). Hence, \(\tilde{\rho}_{\mathcal{F}_T}(\sum \lambda_i 1_{A_i}) = \sum 1_{A_i} \tilde{V}^\mathcal{F}_T(\lambda_i)\). The continuity from above of \(\tilde{\rho}_{\mathcal{F}_T}\) allows to extend the definition to positive \(X_T \in \mathcal{X}\) and then to any arbitrary \(X_T \in \mathcal{X}\) using standard analysis tools. \(\square\)

The following theorem states the main result of this section showing that any cash subadditive risk measure of the form \(\mathcal{R}^{\rho, V} = \rho(-V)\) can be extended into a cash additive risk measure, which is generated from \(\rho\) via a conditional risk measure. Conversely, any cash additive risk measure \(\rho\) on \(\tilde{\mathcal{X}}\) generated from \(\rho\) via a conditional risk measure is associated to a cash subadditive risk measure of type \(\mathcal{R}^{\rho, V_{\mathcal{F}_T}}\).

\(^6\) For conditional risk measures, see Bion-Nadal (2004), Detlefsen and Scandolo (2005), and references therein.
Theorem 5.9. The cash additive risk measure $\hat{\rho}^V$ in equation (5.8) is generated from $\rho$ via the conditional risk measure $\tilde{\rho}^V_T$ in equation (5.9) associated with $V$, that is, 

\begin{equation}
\hat{\rho}^V(X_T^1_{\theta=1} + X_T^0_{\theta=0}) = \rho(-V(X_T^1 - X_T^0) + X_0) = \rho(-\tilde{\rho}^V_T(X_T^1_{\theta=1} + X_T^0_{\theta=0})).
\end{equation}

Moreover, 

\begin{equation}
\mathcal{R}^{\rho, V}(X_T) = \hat{\rho}^V(X_T^1_{\theta=1}) = \rho(-\tilde{\rho}^V_T(X_T^1_{\theta=1})).
\end{equation}

Conversely, to any cash additive risk measure $\hat{\rho}(\cdot) = \rho(-\tilde{\rho}^V_T(\cdot))$ on $\tilde{X}$ generated by a cash additive conditional risk measure $\tilde{\rho}^V_T$ on $\mathcal{F}_T$ is associated a cash subadditive risk measure of the following form: $\mathcal{R}^{\rho, \hat{\rho}^V_T}(X_T) = \rho(-\hat{\rho}^V_T(X_T))$, where $\hat{\rho}^V_T(X_T) = \tilde{\rho}^V_T(X_T^1_{\theta=1})$.

Proof. The proof follows easily from the previous considerations. \hfill \square

Equation (5.11) suggests that the risk of the future position $X_T$ depends on the risk/ambiguity on the underlying asset model (the unconditional risk measure $\rho$) and on the risk/ambiguity on interest rates (the conditional risk measure $\tilde{\rho}^V_T$) or more in general on the risk affecting the numéraire.

6. OPTIMAL DERIVATIVE DESIGN AND INF-CONVOLUTION

The problem of designing the optimal transaction between two economic agents has been largely investigated both in the insurance and in the financial literature. The risk transfer between the agents takes place through the exchange of a derivative contract and the optimal transaction is determined by a choice criterion. For example, in Barrieu and El Karoui (2006) the choice criterion is given by the minimization of the risk of the agent’s exposure and the risk is assessed using forward cash additive risk measures. Using cash subadditive risk measures, we study this problem in a general framework that allows for ambiguous discount rates. We focus on the problem of the risk transfer between two agents who determine today the reserve to hedge the future exposure when the discount factor for the maturity of interest is ambiguous. To account for this ambiguity, the agents collect the reserve using cash subadditive risk measures and the decision criterion is the minimization of their reserves.

6.1. Transaction Feasibility and Optimization Program

Let $A$ and $B$ be the two agents and suppose that they are evolving in an uncertain universe modeled by the probability space $(\Omega, \mathcal{F}_T)$. Agent $A$ is exposed toward a non-tradable risk that will impact her wealth $X_T^A \in \mathcal{X}$ at the future date $T$. To reduce her risk exposure and the reserve associated, $A$ aims at issuing a derivative contract $H_T \in \mathcal{X}$ with maturity $T$ and selling it to the agent $B$ for a price $\pi_0$. Agent $B$ will enter the transaction only if this transaction reduces or leaves unchanged the reserve that she has to put aside to hedge her future exposure $X_T^B \in \mathcal{X}$. The objective is to find the optimal structure $(H_T, \pi_0)$ according to the decision criterion of the agents given by their cash subadditive risk measures $\mathcal{R}_A$ and $\mathcal{R}_B$.

If the agents agree on the transaction, at time zero $B$ pays $\pi_0$ to $A$. At time $T$ the terminal wealths of the agents $A$ and $B$ are $X_T^A - H_T$ and $X_T^B + H_T$, respectively. $A$
aims at minimizing the current reserve $R_A(X^A_T - H_T)$ for the future exposure $X^A_T - H_T$, knowing that today she receives $\pi_0$ from $B$,

$$\inf_{H_T \in \mathcal{X}, \pi_0} R_A(X^A_T - H_T) - \pi_0.$$  

(6.1)

The constraint to the optimization program (6.1) is that $B$ enters the transaction. This happens when buying $H_T$ for $\pi_0$ reduces or leaves unchanged the reserve $R_B(X^B_T)$ that $B$ would collect not entering the transaction,

$$R_B(X^B_T + H_T) + \pi_0 \leq R_B(X^B_T).$$  

(6.2)

The pricing rule of the $H_T$-structure is fully determined by the buyer $B$ simply binding the constraint at the optimum in equation (6.2),

$$\pi_0^* = \pi_0^*(H_T) = R_B(X^B_T) - R_B(X^B_T + H_T).$$

This price $\pi_0^*$ corresponds to an “indifference” pricing rule from the point of view of the agent $B$ as $\pi_0^*$ gives the maximum amount that agent $B$ is ready to pay to enter the transaction. Given $\pi_0^*$, the optimization program in equation (6.1) becomes

$$R_{A,B}(X^A_T, X^B_T) := \inf_{H_T \in \mathcal{X}} R_A(X^A_T - H_T) + R_B(X^B_T + H_T),$$

(6.3)

where the optimal transaction $H^*_T$ attains the infimum.

### 6.2. Optimal Transaction and Inf-Convolution

The risk transfer problem in equation (6.3) can be rewritten as an inf-convolution of cash subadditive risk measures on $\mathcal{X}$. Indeed, defining $F_T := X^B_T + H_T \in \mathcal{X}$ we have

$$R_{A,B}(X^A_T, X^B_T) = \inf_{F_T \in \mathcal{X}} \{ R_A(X^A_T + X^B_T - F_T) + R_B(F_T) \} =: \mathcal{R}_A \square R_B(X^A_T + X^B_T),$$

(6.4)

where $\square$ denotes the inf-convolution. The value of $R_{A,B}(X^A_T, X^B_T)$ can be interpreted as the residual measure of risk after the transaction $F_T$ has occurred. This residual measure of risk depends on the initial exposures $X^A_T$ and $X^B_T$. The transaction induces an optimal redistribution of the risks of the agents. In the following we show that $R_A \square R_B$ is a cash subadditive risk measure completely characterized by $\mathcal{R}_A$ and $\mathcal{R}_B$ and we provide its dual representation. Also in this case, instead of using convex analysis tools to prove these results, we exploit the one-to-one correspondence between $\mathcal{R}$ and the cash additive risk measure $\hat{\rho}(\hat{X}_T) = \mathcal{R}(X_T - x1_{T}) - x$ defined on $\hat{\mathcal{X}}$ and given in equation (4.2).

We show that the inf-convolution of cash subadditive risk measures on $\mathcal{X}$ is equal to the inf-convolution of their corresponding cash additive risk measures $\hat{\rho}$ on $\hat{\mathcal{X}}$.

**Lemma 6.1.** The inf-convolution of $R_A$ and $R_B$ on $\mathcal{X}$ in equation (6.4) corresponds to the inf-convolution of the cash additive extensions of $R_A$ and $R_B$ on $\hat{\mathcal{X}}$,

$$R_A \square R_B(X^A_T + X^B_T) = \hat{\rho}_A \square \hat{\rho}_B(\hat{X}^A_T + \hat{X}^B_T),$$

(6.5)

where $\hat{X}^A_T := X^A_T1_{\theta=1}$, $\hat{X}^B_T := X^B_T1_{\theta=1}$.

$R_A \square R_B(X^A_T + X^B_T)$ is the infimum on $F_T \in \mathcal{X}$, while $\hat{\rho}_A \square \hat{\rho}_B(\hat{X}^A_T + \hat{X}^B_T)$ is the infimum on the pairs $(F_T, x) \in \hat{\mathcal{X}}$. 

\[ \]
Proof. The result follows observing that any $F_T \in \mathcal{X}$ can be rewritten as $F_T = G_T - x_1 T$, for some $G_T \in \mathcal{X}$ and $x \in \mathbb{R}$, and

$$R_A \square R_B(X_T^A + X_T^B) = \inf_{F_T \in \mathcal{X}} \left\{ R_A(X_T^A + X_T^B - F_T) + R_B(F_T) \right\}$$

$$= \inf_{(G_T, x) \in \mathcal{X} \times \mathbb{R}} \left\{ R_A(X_T^A + X_T^B - (G_T - x_1 T)) + R_B(G_T - x_1 T) \right\}$$

$$= \inf_{G_T \in \mathcal{X}} \left\{ \hat{\rho}_A((X_T^A + X_T^B)_{\theta=1} - \hat{G}_T) + \hat{\rho}_B(\hat{G}_T) \right\}$$

$$= \hat{\rho}_A \square \hat{\rho}_B x(\hat{X}_T^A + \hat{X}_T^B).$$

Barrieu and El Karoui (2006, Theorem 3.3) show that the inf-convolution of cash additive risk measures is a cash additive risk measure. We apply this result to $\hat{X}_T^A / \Omega_1$, $\alpha$ functions of the future position $X_T^A$. When $\hat{\rho}_A \square \hat{\rho}_B(0) > -\infty$, the inf-convolution $\hat{X} \in \hat{\mathcal{X}} \mapsto \hat{\rho}_A \square \hat{\rho}_B(\hat{X})$ is a cash additive risk measure, continuous from below if one of the two risk measures is continuous from below, and with penalty function the sum of the penalties of $\hat{\rho}_A$ and $\hat{\rho}_B$. We showed that any $\hat{\rho}$ constrained to the event $\theta = 1$ defines a cash subadditive risk measure with the same penalty function (Proposition 4.1). Then $R_A \square R_B$ in equation (6.5) is a cash subadditive risk measure. We collect all the previous results in the following theorem.

**Theorem 6.2.** Let $R_A$ and $R_B$ be two cash subadditive risk measures with penalty functions $\alpha_A$ and $\alpha_B$, respectively. Let $R_{A,B}$ be the inf-convolution of $R_A$ and $R_B$

$$\Psi \to R_{A,B}(\Psi) := R_A \square R_B(\Psi) = \inf_{H \in \mathcal{X}} \{ R_A(\Psi - H) + R_B(H) \},$$

and assume that $R_{A,B}(0) > -\infty$. Then

1. $R_{A,B}$ is a cash subadditive risk measure, which is finite for all $\Psi \in \mathcal{X}$.
2. The associated penalty function is given by $\forall \mu \in \mathcal{M}_{\alpha, f}(\mathcal{F}_T), \quad \alpha_{A,B}(\mu) = \alpha_A(\mu) + \alpha_B(\mu)$.
3. $R_{A,B}$ is continuous from below when this property holds for $R_A$ and/or $R_B$.
4. The optimal derivative contract is $H^* = F^* - X_T^B$, where $F^*$ attains the infimum in equation (6.4).

7 For the interpretation of the condition $R_A \square R_B(0) > -\infty$, see Theorem 3.3 in Barrieu and El Karoui (2006).

7. Dynamic Infinitesimal Cash Subadditive Risk Measures

The cash subadditive risk measures considered so far are static measures, assessing the risk of the future position $X_T$ at a given time $t$. In this section, we give an example of dynamic cash subadditive risk measure on the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, where $(\mathcal{F}_t)_{t \in [0, T]}$ is the augmented filtration associated to the $d$-dimensional Brownian motion $W = (W_t)_{t \in [0, T]}$. At any time $t \in [0, T]$, the risk measure assesses the riskiness of the future position $X_T$ taking into account the information available, $\mathcal{F}_t$. In particular, following Peng (2004), El Karoui, Peng, and Quenez (1997), Barrieu and El Karoui (2006), and Rosazza Gianin (2006), who link BSDEs and risk measures, we show that BSDEs with suitable coefficients are cash subadditive risk measures. The main difference with cash additive risk measures generated by BSDEs is that cash subadditive risk measures are now recursive risk measures; that is, the generator can locally depend...
on the level of the cash subadditive risk measure. When the dual representation exists, the penalty function of dynamic cash subadditive risk measures generalizes the penalty function of the static cash subadditive risk measures in Section 3.2.

Dynamic risk measures not based on BSDEs have been recently studied by several authors such as Cvitanic and Karatzas (1999), Wang (1999), Artzner et al. (2004), Cheridito, Delbaen, and Kupper (2004), Frittelli and Rosazza Gianin (2004), Riedel (2004), Frittelli and Scandolo (2006), Cheridito, Delbaen, and Kupper (2006), Weber (2006), and Kloeppel and Schweizer (2006). Here, we consider cash subadditive risk measures generated by BSDEs.

7.1. Some Results on BSDEs

Let $X_T \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$ and $g(t, y, z)$ be a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable coefficient, where $\mathcal{P}$ is the $\sigma$-algebra of real-valued progressively measurable events. Consider the pair of squared-integrable progressively measurable processes $(Y, Z)$ of the following BSDE associated to $(g, X_T)$,

$$-dY_t = g(t, Y_t, Z_t) \, dt - (Z_t, dW_t), \quad Y_T = X_T.$$ 

The existence and the uniqueness of the solution $(Y_t, Z_t)_{t \in [0, T]}$ depend on the properties of the coefficient $g$. Pardoux and Peng (1990) prove that the solution exists and is unique when $g$ is uniformly Lipschitz continuous with respect to $(y, z)$. In this case $g$ is called standard coefficient. When, for any given $t \in [0, T]$, $g$ is continuous with respect to $(y, z) \mathbb{P}$-a.s. and $|g(t, y, z)| \leq C(1 + z^2 + y), \forall (t, y, z) \mathbb{P}$-a.s. ($g$ with linear-quadratic growth, in the sequel), Kobylanski (2000) and Lepeltier and San Martin (1998) show that the BSDE associated with $(g, X_T)$ has a maximal and minimal solution. Uniqueness holds under some additional assumptions.

The following theorem, called comparison theorem, is a crucial tool in the study of one-dimensional BSDEs and corresponding dynamic measures of risk.

**Theorem 7.1.** Let $X^1_T$ and $X^2_T \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$ and $g^1$ and $g^2$ both standard (or both with linear-quadratic growth) coefficients. Let $(Y^1, Z^1)$ and $(Y^2, Z^2)$ be the (maximal) solutions associated to $(g^1, X^1_T)$ and $(g^2, X^2_T)$, respectively. If $X^1_T \geq X^2_T$, $\mathbb{P}$-a.s., and $g^1(t, Y^2_t, Z^2_t) \geq g^2(t, Y^2_t, Z^2_t) \, d\mathbb{P} \times dt$-a.s., then $Y^1_t \geq Y^2_t$ a.s. $\forall t \in [0, T]$. In particular, the maximal solution is still monotone with respect to the terminal condition.

The comparison theorem and the existence of the maximal solution ensure that, if the coefficient $g$ is convex, the solution $Y_t$ of the BSDE $(g, -X_T)$ is also convex when $Y_t$ is considered as a functional of its terminal condition $-X_T$. Moreover, the existence of the maximal solution ensures the time consistency of $(Y_t)_{t \in [0, T]}$; that is, $\forall 0 \leq t_1 \leq t_2 \leq T, \ Y_{t_1}(X_{t_2}) = Y_{t_1}(-Y_{t_2}(X_{t_2}))$. For the derivations of this result see, for instance, El Karoui et al. (1997), Peng (2004), Barrieu and El Karoui (2006), and Rosazza Gianin (2006).

7.2. BSDEs and Cash Subadditive Risk Measures

The link between measures of risk and BSDEs is particularly interesting because it enhances interpretation and tractability of risk measures. Barrieu and El Karoui (2006) point out that the coefficient $g$ of BSDEs can be interpreted as infinitesimal risk measure.
over a time interval \([t, t + dt]\) as \(\mathbb{E}_t[dY_t] = -g(t, Y_t, Z_t) dt\), where \(Z_t\) is the local volatility of the conditional risk measure, \(\mathcal{V}(dY_t) = |Z_t|^2 dt\). Choosing carefully the coefficient \(g\) enables to generate \(g\)-conditional risk measures that are locally compatible with the different agent’s beliefs.

**Example 7.2.** Ambiguous interest rates. Assume that locally \(\mathbb{E}_t[−dY_t] = \mathcal{F}_t\) is driven by the worst case scenario generated by an ambiguous discount rate \(\beta = (\beta_t)_{t \in [0, T]}\), where \(\beta\) is an adapted process ranging between two adapted and bounded processes \((r_t)_{t \in [0, T]}\) and \((R_t)_{t \in [0, T]}\), that is,

\[
\mathbb{E}_t[−dY_{t, R} \mid \mathcal{F}_t] = \sup_{0 \leq r_t \leq \beta_t \leq R_t} (−\beta_t Y_{t, R}) dt.
\]

\(Y_{t, R}\) is the first component solution of the BSDE

\[-dY_t = −(r_t Y_t^+ − R_t Y_t^-) dt − \langle Z_t, dW_t \rangle, \quad Y_T = −X_T,\]

where \(y^+ = \sup(y, 0)\) and \(y^- = \sup(−y, 0)\). More precisely, since \((r_t)_{t \in [0, T]}\) and \((R_t)_{t \in [0, T]}\) are assumed to be bounded, \((Y_{t, R}, Z_{t, R})\) is the unique solution of the standard BSDE with convex Lipschitz coefficient

\[
g_{t, R}(t, y) = R_t y^- − r_t y^+ = \sup_{r_t \leq \beta_t \leq R_t} (−\beta_t y).
\]

Notice that \(y \mapsto g_{t, R}(t, y)\) is a monotone nonincreasing function. To provide the intuition on this risk measure, we apply the comparison theorem to the coefficients \(g_{t, R}(t, y)\) and \(g(t, y) = (−\beta_t y)\), \(\beta_t \in [r_t, R_t]\), with the same terminal condition \(-X_T\). Since \(g_{t, R}(t, y) \geq (−\beta_t)\), \(Y_{t, R}^\beta \geq Y_t^\beta\), where \(Y^\beta\) is the solution of the linear BSDE

\[-dY_t = −\beta_t Y_t dt − \langle Z_t, dW_t \rangle, \quad Y_T = −X_T,\]

and it can be represented as \(Y^\beta_t = \mathbb{E}_t[e^{−\int_t^T \beta_s ds} (−X_T) \mid \mathcal{F}_t]\), \(\forall t \in [0, T]\). Then it follows that \(Y_{t, R}^\beta \geq \sup_{0 \leq r_t \leq \beta_t \leq R_t} Y_{t, R}^\beta\). As the process \(\overline{\beta}_t = R_t 1_{Y_t^\beta < 0} + r_t 1_{Y_t^\beta \geq 0}\) achieves the maximum of \(\sup_{0 \leq r_t \leq \beta_t \leq R_t} (−\beta_t)\), \(Y_{t, R}^\beta = −\overline{\beta}_t Y_{t, R}^\beta\), then the equality \(Y_{t, R}^\beta = Y_{t, R}^{\overline{\beta}}\) holds. Thus, the dual representation of \(Y_{t, R}^\beta\) follows

\[Y_{t, R}^\beta = \mathbb{E}_t[e^{−\int_t^T \overline{\beta}_s ds} (−X_T) \mid \mathcal{F}_t].\]

Notice that, for any \(t \in [0, T]\), \(Y_{t, R}^\beta\) is dominated, but in general not equal to the conditional risk measure \(\mathcal{R}_{t, T}^{\beta, D^*)}\) associated with the worst case discounted factors \(D_{t, T} = D_{t, T}^{-}\) which is the local volatility of the conditional risk measure, \(\mathcal{V}(dY_t) = |Z_t|^2 dt\). Choosing carefully the coefficient \(g\) enables to generate \(g\)-conditional risk measures that are locally compatible with the different agent’s beliefs.

In the sequel, we consider risk measures generated by BSDEs that generalize Example 7.2. For the remaining part of the paper \(g(t, y, z)\) denotes a convex generator in \((y, z)\), standard or with linear growth with respect to \(y\) and quadratic growth in \(z\). The comparison theorem ensures that the (maximal) solution \((Y, Z)\) associated with a \((g, −X_T)\) exists and, for any \(t \in [0, T]\), \(Y_t\) is convex and decreasing with respect to the final condition \(-X_T\).
The coefficient $g^{c,R}(t, y)$ in equation (7.1) depends on $y$ in a convex decreasing way. As observed by Peng (2004) and Barrieu and El Karoui (2006), this is never the case for conditional cash additive risk measures generated by BSDEs. Under some mild additional assumptions, Peng (2004) shows that, for any $t \in [0, T]$, the (maximal) solution $Y_t$ associated with $(g, -X_T)$ is cash additive as functional of its terminal condition if and only if $g$ does not depend on $y$ for any $t \in [0, T]$. Barrieu and El Karoui (2006) study these cash additive solutions as a dynamic risk measure ($\rho$ originally studied by Peng (2004); see also Rosazza Gianin (2006).

A straightforward generalization of their results has been derived in Barrieu and El Karoui (2006). The next result only if $Y_t$ is a conditional cash subadditive risk measure generated by BSDEs. Under some mild additional assumptions, Peng (2004) shows that, for any $t \in [0, T]$, $Y_t$ is a conditional cash subadditive risk measure, $G$ is a time consistent cash subadditive risk measure. We call $G$-conditional cash subadditive risk measure.

Proof. For the convexity and the decreasing monotonicity of $Y_t$ with respect to the terminal condition see, for instance, El Karoui and Quenez (1996) and Peng (1997).

Cash subadditivity: Consider the BSDE satisfied by $R^g_t(X_T + m 1_T) + m = Y^m_T$, $-d Y^m_t = g^m(t, Y^m_t, Z^m_t)dt - \langle Z^m_t, dW_t \rangle$, $Y^m_T = -X_T$.

Since $g^m(t, y, z) = g(t, y - m, z)$, then $g^m(t, y, z)$ is increasing in $m$ (as $g$ is decreasing in $y$). From the comparison theorem it follows that $R^g_t(X_T + m 1_T) + m = Y^m_t$ is increasing in $m$.

7.3. Dual Representation

In this section we focus on a dual representation for $g$-conditional cash subadditive risk measures $R^g$ as in the static case. For the cash additive $g$-conditional risk measures such a representation has been derived in Barrieu and El Karoui (2006). The next result is a straightforward generalization of their results.

The key tool to obtain dual representations is the Legendre transform of the generator $g$ defined by

$$G(t, \beta, \mu) := \sup_{(y, z) \in \mathbb{R} \times \mathbb{R}^d} \{-\beta y - \langle \mu, z \rangle - g(t, y, z)\}.$$ 

The following lemma summarizes the properties of $G$ and $g$.

**Lemma 7.4.** Let $g$ be a continuous convex function on $\mathbb{R} \times \mathbb{R}^d$ satisfying the growth control: there exist two positive constants $C > 0$ and $k > 0$ such that $|g(t, y, z)| \leq |g(t, 0, 0)| + C|y| + \frac{k}{2}|z|^2$.

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8 If $g(t, 0) = 0$ for any $t \in [0, T]$, the $g$-conditional risk measures coincide with the nonlinear expectation originally studied by Peng (2004); see also Rosazza Gianin (2006).
(i) Then the Legendre transform of $g$, $G(t, \beta, \mu)$, takes infinite values if $\beta \not\in [0, \ C]$. Moreover,

\[(7.3) \quad G(t, \beta, \mu) \geq -|g(t, 0, 0)| + \frac{1}{2k} |\mu|^2.\]

(ii) Since $g$ is continuous, for any $t \in [0, T]$, $g(t, Y_t, Z_t) = \sup_{\beta, \mu} \{ -\beta Y_t - \langle \mu, Z_t \rangle - G(t, \beta, \mu) \}$. The maximum is achieved at $(\beta_t, \mu_t)$. The inequality (7.3), we observe that

To show the inequality in (ii), we choose a constant $\varepsilon$ such that $0 < \varepsilon < \frac{1}{2k}$ and we use the inequality (7.3),

\[
\left( \frac{1}{2k} - \varepsilon \right) |\mu_t|^2 \leq |g(t, 0, 0)| + G(t, \beta_t, \mu_t) - \varepsilon |\mu_t|^2 \leq 2|g(t, 0, 0)| + 2C |Y_t| + \frac{k}{2} |Z_t|^2 + \sup_{\mu_t} \{ |\mu_t|, -|Z_t| - \varepsilon |\mu_t|^2 \}.\]

As $\max_{\mu_t, \epsilon} \{ |\mu_t|, -|Z_t| - \varepsilon |\mu_t|^2 \} = \frac{|\mu|^2}{2k}$, then $\left( \frac{1}{2k} - \varepsilon \right) |\mu_t|^2 \leq 2|g(t, 0, 0)| + 2C |Y_t| + (\frac{k}{2} + \frac{1}{4k}) |Z_t|^2$, which proves the inequality.

Proof.

(i) $G(t, \beta, \mu) \geq -\beta y - g(t, y, 0) \geq -\beta y - |g(t, 0, 0)| - C |y|$. Then, if $|\beta| > C$, $\sup_{y \in \mathbb{R}} \{ -\beta y - C|y| \} = +\infty$. Moreover, since $g(t, y, z)$ is monotone decreasing with respect to $y$, $-g(t, y, 0) \geq - g(t, 0, 0), \forall y > 0$ and $G(t, \beta, \mu) \geq -\beta y - g(t, 0, 0), \forall y > 0$. Then $G(t, \beta, \mu) = +\infty$ if $\beta < 0$. To prove the inequality (7.3), we observe that $G(t, \beta, \mu) \geq |\mu_t - z| - g(t, 0, z) \geq |\mu_t - z| - |g(t, 0, 0)| - \frac{k}{2} |z|^2$. As $\max_{z \in \mathbb{R}} \{ |\mu_t - z| - \frac{k}{2} |z|^2 \} = \frac{1}{2k} |\mu|^2$ the result follows.

(ii) Standard results in convex analysis show that, since $g$ is continuous, the duality between $g$ and $G$ holds true and the maximum is achieved.

To show the inequality in (ii), we choose a constant $\varepsilon$ such that $0 < \varepsilon < \frac{1}{2k}$ and we use the inequality (7.3),

\[
\left( \frac{1}{2k} - \varepsilon \right) |\mu_t|^2 \leq |g(t, 0, 0)| + G(t, \beta_t, \mu_t) - \varepsilon |\mu_t|^2 \leq 2|g(t, 0, 0)| + 2C |Y_t| + \frac{k}{2} |Z_t|^2 + \sup_{\mu_t} \{ |\mu_t|, -|Z_t| - \varepsilon |\mu_t|^2 \}.\]

As $\max_{\mu_t, \epsilon} \{ |\mu_t|, -|Z_t| - \varepsilon |\mu_t|^2 \} = \frac{|\mu|^2}{2k}$, then $\left( \frac{1}{2k} - \varepsilon \right) |\mu_t|^2 \leq 2|g(t, 0, 0)| + 2C |Y_t| + (\frac{k}{2} + \frac{1}{4k}) |Z_t|^2$, which proves the inequality.

Now we introduce the class of probability measures that appears in the dual representation. As in Barrieu and El Karoui (2006) the reference is the Girsanov theorem for the BMO-exponential martingales such as defined in Kazamaki (1994),

\[
\Gamma^\mu_t = \mathcal{E}(M_t^\mu) = \exp \left( -\int_0^t \mu_s dW_s - \frac{1}{2} \int_0^t |\mu_s|^2 ds \right),
\]

where $M_t^\mu = \int_0^t \mu_s dW_s$ is a BMO($\mathbb{P}$)-martingale, that is, $\mu$ belongs to BMO($\mathbb{P}$),

\[
\text{BMO}(\mathbb{P}) := \left\{ \psi \in \mathcal{H}^2 \text{ such that } \exists C > 0 : \mathbb{E}_\mathbb{P} \left[ \int_0^T \psi_s^2 ds \mid \mathcal{F}_t \right] \leq C \ a.s., \forall t \in [0, T] \right\},
\]

where $\mathcal{H}^2 = \{ \psi \in \mathcal{P}_1 \text{ such that } \mathbb{E}[\int_0^T \psi_s^2 ds] < \infty \}$. Using (Kazamaki 1994, Section 3.3), $\Gamma^\mu_t$ is the likelihood of an equivalent probability measure on $\mathcal{F}_T$ with respect to $\mathbb{P}$ defined by $dQ^\mu = \Gamma^\mu_t d\mathbb{P}$. Moreover, if $\nu \in \text{BMO}(\mathbb{P})$ then $\nu \in \text{BMO}(Q^\mu)$. Recall that $\Gamma^\mu_t$ is the solution of the forward stochastic differential equation
Now we establish the duality theorem.

**Theorem 7.5.** Let $g$ be a convex coefficient, decreasing with respect to $y$ and with growth $|g(t, y, z)| \leq |g(t, 0, 0)| + C|y| + \frac{K}{2}z^2$. Moreover, assume that there exists a constant $K > 0$ such that $E[\int_t^T |g(s, 0, 0)| ds |\mathcal{F}_t] \leq K$, $\forall t \in [0, T]$. Then the (maximal) solution $(Y, Z)$ of the BSDE

$$-dY_t = g(t, Y_t, Z_t) - \langle Z_t, dW_t \rangle, \quad Y_T = -X_T, \quad X_T \in L^\infty(\mathbb{P}),$$

is bounded and $Z$ is in $\text{BMO}(\mathbb{P})$. Let $G(t, y, z)$ be the Fenchel transform of $g$ and

$$A := \{ (\beta_t, \mu_t)_{t \in [0, T]} | G(t, \beta_t, \mu_t) < + \infty, 0 \leq \beta_t \leq C, \forall t \in [0, T] \text{ and } \mu \in \text{BMO}(\mathbb{P}) \}.$$

Then, the $g$-conditional cash subadditive risk measure $R^g = (R^g_t(X_T))_{t \in [0, T]}$, $R^g_t(X_T) = Y_t$, has the following dual representation:

$$R^g_t(X_T) = \text{ess sup}_{(\beta_t, \mu_t) \in A} \mathbb{E}^Q_{\mathcal{F}_t} \left[ e^{-\int_t^T \beta_s ds} (-X_T) - \int_t^T e^{-\int_t^u \beta_s du} G(s, \beta_s, \mu_s) ds \right].$$

**Remark 7.6.** The dual representation of $R^g$ in equation (7.4) is similar to the dual representation of static cash subadditive risk measures. Here, the subprobability measures are replaced by the $\mathcal{F}_t$-conditional subprobability measures $R^{\beta, \mu}$,

$$\frac{dR^{\beta, \mu}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} := \exp \left( -\int_t^T \mu_s dW_s - \frac{1}{2} \int_t^T \mu_s^2 ds - \int_t^T \beta_s ds \right),$$

and the penalty function becomes

$$\alpha_t(R^{\beta, \mu}) := R^{\beta, \mu} \left( \int_t^T e^{-\int_t^u \beta_s du} G(s, \beta_s, \mu_s) ds \right).$$

**Proof.**

(i) To show that $Z \in \text{BMO}(\mathbb{P})$ we refer the reader to the proof in Barrieu and El Karoui (2006).

(ii) From the Girsanov theorem for BMO-martingales, we know that for any $0 \leq \beta_t \leq C$, $\mu \in \text{BMO}(\mathbb{P})$, $dW^\mu_t = dW_t + \mu_t dt$ is a $\mathbb{Q}^{\mu}$-Brownian motion and

$$-dY_t = g(t, Y_t, Z_t) - \langle Z_t, dW_t \rangle = \left[ g(t, Y_t, Z_t) + \beta_t Y_t + \langle \mu_t, Z_t \rangle \right] dt - \beta_t Y_t dt - \langle Z_t, dW^\mu_t \rangle.$$

Then it follows

$$Y_t(-X_T) = \mathbb{E}^Q_{\mathcal{F}_t} \left[ e^{-\int_t^T \beta_s ds} (-X_T) + \int_t^T e^{-\int_t^u \beta_s du} [g(s, Y_s, Z_s) + \beta_s Y_s + \langle \mu_s, Z_s \rangle] ds \right]$$

$$\geq \mathbb{E}^Q_{\mathcal{F}_t} \left[ e^{-\int_t^T \beta_s ds} (-X_T) - \int_t^T e^{-\int_t^u \beta_s du} G(s, \beta_s, \mu_s) ds \right].$$
To prove the last equality in equation (7.5) at the optimal control \((\bar{\beta}, \bar{\mu})\),

\[
G(t, \bar{\beta}, \bar{\mu}) = -\bar{\beta}_t Y_t - \langle \bar{\mu}_t, Z_t \rangle - g(t, Y_t, Z_t), \quad \forall t \in [0, T],
\]

we need to verify that \((\bar{\beta}, \bar{\mu})\) is admissible. Since \(0 \leq \bar{\beta}_t \leq C\), we only need to verify that \(\bar{\mu}\) is in \(\text{BMO}(\mathbb{P})\). We use the inequality in Lemma 7.4,

\[
|\mu_t|^2 \leq A\big|g(t, 0, 0)\big| + c|Y_t| + B|Z_t|^2.
\]

Since \(|g(t, 0, 0)|^{1/2}\) belongs to \(\text{BMO}(\mathbb{P})\), \(Y\) is bounded and \(Z \in \text{BMO}(\mathbb{P})\), then \(\bar{\mu} \in \text{BMO}(\mathbb{P})\),

\[
Y_t(-X_T) = R^g(X_T) = \mathbb{E}_{\mathbb{Q}}\left[ e^{\int_0^T \beta_s ds} (-X_T) - \int_t^T e^{-\int_s^T \beta_u du} G(s, \beta_s, \mu_s) ds \bigg| \mathcal{F}_t \right],
\]

and this establishes the dual representation. □

8. CONCLUSION

We propose a new class of risk measures called cash subadditive risk measures that accounts for the risk/ambiguity on interest rates when assessing the risk of future financial, nonfinancial, and insurance positions. This goal is achieved by relaxing the debated cash additive axiom into the cash subadditive axiom. We provide several examples of the new risk measures in the static and the dynamic frameworks, such as the put options premium and the robust expected utilities. In the dynamic framework cash subadditive risk measures are generated by BSDEs, enhancing their tractability and interpretability. Cash subadditive risk measures represent a promising research area as these risk measures overcome the issues arising from the cash additive axiom.

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