ADAPTIVE LEAST-SQUARES FINITE ELEMENT METHODS
FOR LINEAR TRANSPORT EQUATIONS
BASED ON AN H(DIV) FLUX REFORMULATION

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Abstract. In this paper, we study the least-squares finite element methods for the linear hyperbolic transport equations. The linear transport equation naturally allows discontinuous solutions, while the normal component of the flux across the mesh faces needs to be continuous. Traditional least-squares finite element methods using continuous finite element approximations will introduce unnecessary extra error for discontinuous solutions. In order to separate the continuity requirements, a new flux variable is introduced. With this reformulation, the continuities of the flux and the solution can be handled separately and in natural $H(\text{div}; \Omega) \times L^2(\Omega)$ conforming finite element spaces. Several variants of the methods are developed to handle the inflow boundary condition strongly or weakly. With the reformulation, the least-squares finite element methods can handle discontinuous solutions much better than the traditional continuous polynomial approximations. With natural least-squares functionals as a posteriori error estimators, the methods can resolve the discontinuity even when the mesh is not aligned with discontinuity. The smearing and overshooting phenomena are also very mild with adaptive methods. Existence and uniqueness of the solutions and a priori and a posteriori error estimates are established for the proposed methods. Extensive numerical tests are performed to show the effectiveness of the methods developed in the paper.

Key words. least-squares finite element methods, linear hyperbolic equation, linear transport equation, a priori error estimate, a posteriori error estimate, discontinuous solution

AMS subject classifications. 65N15, 65N30, 65N55

1. Introduction. In this paper, we consider the following linear transport equation in the conservative form. It is a scalar linear partial differential equation of hyperbolic type, which is also called the linear advection equation:

\begin{equation}
\nabla \cdot (\beta u) + \gamma u = f \quad \text{in } \Omega,
\end{equation}

\begin{equation}
\quad u = g \quad \text{on } \Gamma_-, \nonumber
\end{equation}

with $\beta$ an advection field and $\Gamma_-$ the inflow boundary. Detailed descriptions of the equation can be found in Section 2.

When developing a finite element method for a linear problem, it is essential to have two important properties [AFW10]: one is the numerical stability, i.e., the resulting discrete problem is well-posed; the other one is the consistency. In the case of the conforming finite element, which means that the finite element space is a finite dimensional subspace of the abstract space that the weak(true) solution of the variational problem belongs to, the consistency is then a result of the combination of the continuity of the weak problem and the approximation error.

For some classical functional spaces, their corresponding conforming finite element spaces are well-known. For example, the piecewise discontinuous polynomial space for...
$L^2(\Omega)$, the global continuous piecewise polynomial space for $H^1(\Omega)$, Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) spaces for $H(\text{div}; \Omega)$, Nédélec edges spaces for $H(\text{curl}; \Omega)$. When these spaces are constructed, it is essential to study the continuity requirements of the corresponding abstract spaces: $H^1(\Omega)$ requires the discrete space to be continuous, $H(\text{div}; \Omega)$ needs the discrete space to be continuous in the normal direction on the element faces (3D) or edges (2D), while $H(\text{curl}; \Omega)$ requires the discrete space to be continuous in the tangential direction on the element edges.

A wrongly chosen space will introduce unnecessary numerical errors on both a priori and a posteriori stages. For example, for the vector Laplacian on a nonconvex polygon, if continuous piecewise linear vector functions which belong to a subspace of $H^1(\Omega) \cap H(\text{div}; \Omega)$ are used to approximate the solution, the numerical solution will converge to a wrong function. The reason is that for nonconvex polyhedron, $H^1(\Omega) \cap H(\text{div}; \Omega)$ is only a proper closed subspace of $H(\text{curl}; \Omega)$ where the true solution belongs to, see [Cos91, AFW10]. The other example is an example in recovery based a posteriori error estimator we discussed in [CZ09]. For the diffusion equation $-\text{div}(\alpha \nabla u) = f$, the flux $-\alpha \nabla u$ is in $H(\text{div}; \Omega)$, but neither the gradient $\nabla u$ nor the flux $-\alpha \nabla u$ belongs to $H^1(\Omega)$. If continuous piecewise linear vector functions are used to recover the gradient or the flux as in the standard ZZ error estimator, it will cause unnecessary refinements in many places. Thus, it is extremely important to choose the right approximation space with respect to the abstract solution space. Nothing more, nothing less.

As pointed in almost all partial differential equation books, it is crucial to realize that unlike the elliptic or parabolic equations where the solution is generally smooth, hyperbolic equations commonly have discontinuous solutions. When choosing a finite element approximation space for hyperbolic equations, we should pay special attentions.

First, we review some known abstract variational formulations for the linear transport equation. We modified the formulations to the conservative settings here. Let $W = \{v \in L^2(\Omega) : \nabla \cdot (\beta v) \in L^2(\Omega)\}$.

(Variational formulation 1) [DPE12] Find $u \in W$, such that

$$\langle \nabla \cdot (\beta u), v \rangle + \langle \gamma u, v \rangle + \langle \beta \cdot n u, v \rangle_{\Gamma^-} = \langle f, v \rangle + \langle \beta \cdot n g, v \rangle_{\Gamma^-}, \quad \forall v \in L^2(\Omega).$$

(Variational formulation 2) [DHSW12] Find $u \in L^2(\Omega)$, such that

$$\langle u, -\beta \cdot \nabla v + \gamma v \rangle = \langle f, v \rangle - \langle g, \beta \cdot n v \rangle_{\Gamma^-}, \quad \forall v \in Y,$$

with $Y = \{v : v \in L^2(\Omega), \beta \cdot \nabla v \in L^2(\Omega), v|_{\Gamma^+} = 0\}$.

(Variational formulation 3 (least-squares)) [CJ88, Jia98, BC01a, BC01b, DSMM04, BG09, BG16] Find $u \in W$, such that

$$\langle \nabla \cdot (\beta u) + \gamma u, \nabla \cdot (\beta v) + \gamma v \rangle + \langle \beta \cdot n u, v \rangle_{\Gamma^-} = \langle f, \nabla \cdot (\beta v) + \gamma v \rangle + \langle \beta \cdot n g, v \rangle_{\Gamma^-}, \quad \forall v \in W.$$

In formulations 1 and 3, the trial space is $W$, in the formulation 2, the test space is $Y$. It is well-known that if the standard $C^0$ piecewise polynomial space is used as trial and test spaces for the variational formulation 1 (1.2), the method frequently does not give reasonable results in contrast to the elliptic and parabolic cases [Joh87]. It is also true for the least-squares variational formulations (1.4), for the simplest piecewise constant discontinuity problem (see the numerical example 7.1), continuous
finite element approximations will introduce unnecessary error, since even the simplest piecewise constant solution is not in the approximation space. If an adaptive method is used, the error indicator will always indicate unnecessary big errors for those elements on the discontinuous region, even when the mesh is fine enough to almost match the discontinuity. Also, Gibbs phenomena like spurious over-shootings are very possible near the discontinuity. The reason is pretty simple, continuous finite element space $\subset H^1(\Omega) \subset W$ is not good for approximating discontinuous functions.

The method in [DHSW12] uses $L^2(\Omega)$ as the trial space, so the standard discontinuous piecewise polynomial space can be used as the discrete trial space, but the test space $Y$ is essentially as complicated as $W$ and needs a very dedicated and complicated construction.

On the other hand, a close look at the space $W$ will find that simple piecewise discontinuous polynomial space is not its subspace since it needs another continuity requirement. For a true solution $u \in W$, the condition of $\nabla \cdot (\beta u) \in L^2(\Omega)$ essentially means

$$u \in L^2(\Omega) \quad \text{and} \quad \beta u \in H(\text{div}; \Omega).$$

Thus the continuity in the normal direction of $\beta u$ needs to be enforced in a strong or weak way. This is probably the reason why continuous finite element spaces are used in [CJ88, Jia98, BC01a, BC01b, DSMM04, BG09, BG16], since continuous finite element space is a subspace of $W$. The only problem is that it requires too much continuity: $u$ may not be continuous at all. In this sense, the famous discontinuous Galerkin method is a right method [RH73, LR74, BMS04] while dealing with the approximation space. In DG methods, the solution is approximated in piecewise discontinuous polynomial space, while the continuity of the normal component of $\beta u$ is enforced weakly.

In this paper, we propose new variational formulations with flux reformulation. Introduce the flux $\sigma = \beta u$, then we have a first order system with appropriate boundary conditions:

$$(1.5) \quad \sigma - \beta u = 0 \quad \text{and} \quad \nabla \cdot \sigma + \gamma u = f.$$  

With the solution $(\sigma, u) \in H(\text{div}; \Omega) \times L^2(\Omega)$, in order to develop a variational formulation, we also need the test spaces and their discrete subspaces, and make sure that the discrete problem is well-posed. One way to set up a variational problem for a first order system is developing a mixed type of formulation. But the equation here is unusual and non-symmetric, the stability of the mixed formulation is not clear. The other way of developing a numerical method is using a Petrov-Galerkin formulation as in [DHSW12, DG10, DG11], where special test functions are constructed. In this paper, we will use the brute-force method by introducing an artificial, externally defined energy-type variational principle, the least-squares variational principle.

Traditionally, new unknowns are introduced in the least-squares finite element method in order to decrease the order of problem, e.g., changing the problem from a second order equation into a first order system so that the resulting discrete problem can use low order finite elements and has a reasonable condition number. For the linear transport equation we study here, it seems unnecessary to introduce new unknowns since the problem is already a first order equation. The reason we introduce the new flux $\sigma$ is that different continuity requirements can be handled separately. In (1.5), the space requirements for the unknowns are two standard spaces: $\sigma \in H(\text{div}; \Omega)$ and $u \in L^2(\Omega)$. Standard Raviart-Thomas $RT_k$ space and the piecewise discontinuous space $P_k$ can be used to approximate them.
For the inflow boundary condition, since the space for $u$ is now only $L^2$, we enforce it on $\sigma$. It can be handled strongly or weakly, thus several closely related least-squares finite element methods are developed here. We call the methods LSFEM and LSFEM-B to denote the method that enforces the boundary condition in the space or by a penalty term, separately. Different weights can be chosen to handle the inflow boundary condition weakly, which lead to two different versions of LSFEM-B methods.

The least-squares finite element methods have several attractive properties: the linear system it produced is symmetric positive definite, and it has a natural and sharp a posteriori error estimator that can be used in adaptive mesh refinements. Because the discrete system is naturally SPD, it opens doors for advanced discrete solves like algebraic multigrid [DSMMO04]. For the a posteriori error estimator for the linear hyperbolic equation, although there are several researches in this direction, the results are less satisfactory compared with the elliptic equations. Normally, only the upper reliability bound is developed, the lower efficiency bound is often not proved [GHM14] or only proved under a saturation assumption [Bur09]. In our methods, the least-squares functional is a natural and sharp error indicator. With respect to the least-squares norms, the error indicator is exact with effectivity constant 1. It is also the best one can get from a posteriori estimator: the numerical methods minimize the least-squares energy, the error indicators estimate exactly the error in least-squares energy norms and point out the bad approximated elements.

Because of the reformulation, the methods developed in the paper can use the lowest order finite element approximation spaces: $RT_0$ and $P_0$. For discontinuous solutions with unaligned meshes, $P_0$ approximations can reduce the over/under shootings. Combined with the adaptive mesh refinements, we show numerically that both over/undershooting effects and the smearing effects can be reduced to almost invisible in "the eye-ball norm".

The computational cost of our method is also comparable to the standard discontinuous Galerkin method. For standard discontinuous Galerkin method [BMS04, DPE12], the lowest element can be used is the linear elements, and there is a half order loss in the convergence rate even when the solution is globally continuous. Even though an extra unknown, the flux, is introduced, because we can use the lowest order elements $RT_0$ and $P_0$, the overall degrees of freedom are of similar size. And when the mesh is aligned with the discontinuity, the convergence rate of errors in the least-squares norms is always optimal. For the case the solution is globally continuous, numerically, we find that the $L^2$ norm of the solution error is also of optimal order.

Besides the LSFEMs with problematic continuous approximations, the nonconforming LSFEM in [DHSW12] and the similar method in [MY18] use discontinuous approximations. The continuity of the normal component of the flux $\beta u$ is weakly enforced by adding a jump term into the discrete formulation. Compared with these methods, the advantages of our methods are that no jump terms on inter-elements faces/edges are needed, which simplifies the implementation. Besides, it is still not very clear what are the right or optimal weight and form of those inter-element jumps, see [DHSW12, MY18].

The paper is organized as follows. Section 2 describes the model linear hyperbolic transport problem. Based on a flux reformulation, a least-squares variational problem with strong enforced inflow boundary condition is presented in section 3. Corresponding LSFEM is developed in Section 4, a priori and a posteriori error estimates are established. Sections 5 and 6 develop two versions of least-squares variational formulations and corresponding finite element methods with weakly enforced boundary
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Section 7 provides numerical results for many test problems. In Section 8, we make some concluding remarks.

2. Model Linear Hyperbolic Transport Equation. Let $\Omega$ be a bounded polyhedral domain in $\mathbb{R}^d$ with Lipschitz boundary. We assume the advective velocity field $\beta = (\beta_1, \cdots, \beta_d)^T$ is a vector-valued function defined on $\bar{\Omega}$ with $\beta \in C^1(\Omega)^d$ for simplicity. We also assume $\gamma \in L^\infty(\Omega)$ satisfying:

$$\gamma + \frac{1}{2} \nabla \cdot \beta \geq 0.$$ 

Note when $\nabla \cdot \beta = 0$ (for example, $\beta$ is a constant vector), $\gamma$ can be 0.

We define the inflow and outflow parts of $\partial \Omega$ in the usual fashion:

$$\Gamma_- = \{ x \in \partial \Omega : \beta(x) \cdot n(x) < 0 \} = \text{inflow},$$

$$\Gamma_+ = \{ x \in \partial \Omega : \beta(x) \cdot n(x) > 0 \} = \text{outflow},$$

where $n(x)$ denotes the unit outward normal vector to $\partial \Omega$ at $x \in \partial \Omega$.

(Assumptions of $\beta$ and $\gamma$) We assume that one of the following assumptions on the coefficients is true:

(i) $0 < |\beta| < C$. For every $\tilde{x} \in \Gamma_-$, let $x(r)$ be a streamline of $\beta$ with initial condition $x(r_0) = \tilde{x}$. Assume that there exists a transformation to a coordinate system such that the streamlines are lined up with the $r$ coordinates direction and the Jacobian of the transformation is bounded. We also assume that every streamline connects $\Gamma_-$ and $\Gamma_+$ with a finite length $\ell(\tilde{x})$ for $\tilde{x} \in \Gamma_-$. Note that this case includes the case $\beta$ is a nonzero constant vector.

(ii) There exists a positive $\gamma_0$, such that

$$\gamma + \frac{1}{2} \nabla \cdot \beta \geq \gamma_0 > 0 \quad \text{in } \Omega.$$ 

We also assume that the inflow and outflow boundaries are well-separated.

Note that this case does not include an important case that $\beta$ is a constant vector and $\gamma = 0$.

Define the following trace space

$$L^2(|\beta \cdot n|; \Gamma_-) := \{ v \text{ is measurable on } \partial \Omega : \int_{\Gamma_-} |\beta \cdot n|v^2 < \infty \}.$$ 

For the inhomogeneous boundary condition $u = g$ on $\Gamma_-$, we assume $g \in L^2(|\beta \cdot n|; \Gamma_-)$.

Theorem 2.1. (Existence and uniqueness of the solution of the linear transport equation) For $g \in L^2(|\beta \cdot n|; \Gamma_-)$, the linear transport equation (1.1) has a unique solution in $W$ assuming one of assumptions of $\beta$ and $\gamma$ is true.

The proof of the theorem with the assumption (i) is based on standard ODE theory, and can be found in [DHSW12, DSMMO04]. For the case with the assumption (ii), the proof can be found in [DHSW12] and Chapter 2 of [DPE12].

Remark 2.2. In [DSMMO04], it is showed that the existence and uniqueness still hold if the requirement of the inflow boundary condition $g$ is relaxed to

$$\int_{\Gamma_-} g^2 \ell(x(s))|\beta \cdot n|/|\beta|ds < \infty,$$

where $\ell(x)$ is the length of the streamline defined by $\beta$ connecting the inflow boundary to the outflow boundary.
Define the following spaces:

\[ \nabla \cdot (\beta u + \gamma v) = f \text{ in } \Omega, \]
\[ u = g \text{ on } \Gamma_-, \]
with \( \mu = \gamma + \nabla \cdot \beta. \)

All the methods developed in this paper can be applied to this form of equation by changing it to the conservative formulation.

3. Least-Squares Variational Problem Based on Flux Reformulation.
In this section, a least-squares variational problem based on flux reformulation is introduced. The boundary condition is strongly enforced in the trial space. The existence and uniqueness of the formulation is discussed.

3.1. Least-squares problem. Introduce the flux \( \sigma = \beta u, \) then

\[ \sigma - \beta u = 0 \quad \text{and} \quad \nabla \cdot \sigma + \gamma u = f. \]

And since \( \nabla \cdot \sigma = f - \gamma u \in L^2(\Omega), \) the flux \( \sigma \in H(\text{div}; \Omega). \)

The inflow boundary condition \( u = g \) on \( \Gamma_- \) can also be written as

\[ \sigma \cdot n = (\beta \cdot n)g, \quad \text{on } \Gamma_. \]

Define the following spaces:

\[ H_{0,-}(\text{div}; \Omega) := \{ \tau \in H(\text{div}; \Omega) : \tau \cdot n = (\beta \cdot n)g \text{ on } \Gamma_- \}, \]
\[ H_{0,-}(\text{div}; \Omega) := \{ \tau \in H(\text{div}; \Omega) : \tau \cdot n = 0 \text{ on } \Gamma_- \}. \]

Then the least-squares variational problem is: Seek solutions \( (\sigma, u) \in H_{g,-}(\text{div}; \Omega) \times L^2(\Omega), \) such that

\[ J(\sigma, u; f, g) = \inf_{(\tau, v) \in H_{0,-}(\text{div}; \Omega) \times L^2(\Omega)} J(\tau, v; f, g), \]

with the least-squares functional \( J \) defined as

\[ J(\tau, v; f, g) := \|\tau - \beta v\|^2_0 + \|\nabla \cdot \tau + \gamma v - f\|^2_0, \quad \forall (\tau, v) \in H_{g,-}(\text{div}; \Omega) \times L^2(\Omega). \]

Its corresponding Euler-Lagrange formulation is: Find \( (\sigma, u) \in H_{g,-}(\text{div}; \Omega) \times L^2(\Omega), \) such that

\[ a(\sigma, u; \tau, v) = (f, \nabla \cdot \tau + \gamma v), \quad \forall (\tau, v) \in H_{0,-}(\text{div}; \Omega) \times L^2(\Omega), \]

where for all \( (\tau, v), (\rho, w) \in H(\text{div}; \Omega) \times L^2(\Omega), \) the bilinear form is defined as

\[ a(\tau, v; \rho, w) = (\tau - \beta v, \rho - \beta w) + (\nabla \cdot \tau + \gamma v, \nabla \cdot \rho + \gamma w). \]

Lemma 3.1. Assuming that the coefficients \( \beta \) and \( \gamma \) satisfy one of the assumptions (i) or (ii), the following defines a norm for \( (\tau, v) \in H_{0,-}(\text{div}; \Omega) \times L^2(\Omega): \)

\[ \|(\tau, v)\| := (\|\tau - \beta v\|^2_0 + \|\nabla \cdot \tau + \gamma v\|^2_0)^{1/2} = a(\tau, v; \tau, v)^{1/2}. \]

Proof. The linearity and the triangle inequality are obvious for \( \|(\tau, v)\| \). Now if \( \|(\tau, v)\| = 0, \) it follows

\[ \tau = \beta v \quad \text{and} \quad \nabla \cdot \tau + \gamma v = 0. \]

Thus, \( \nabla \cdot (\beta v) + \gamma v = 0. \) From the facts \( \tau = \beta v \) and \( \tau \cdot n = 0 \) on \( \Gamma_- \), we get \( \beta \cdot n v = 0 \) on \( \Gamma_- \). Since \( \beta \cdot n \neq 0 \) on \( \Gamma_- \), \( v = 0 \) on \( \Gamma_- \). By Theorem 2.1, \( v = 0 \) is the only solution, thus \( \tau = 0. \) The norm \( \|\cdot\| \) is well defined. \( \square \)
Remark 3.2. It is also clear that
\[ \| (\tau, v) \|_K := (\| \tau - \beta v \|_{0,K}^2 + \| \nabla \cdot \tau + \gamma v \|_{K,0}^2)^{1/2} \]
is a semi-norm on an element \( K \in \mathcal{T} \).

Now, we show the existence and uniqueness of solutions of the least-squares problem by an indirect proof.

Theorem 3.3. The least-squares problem (3.1) has a unique solution \((\sigma, u) \in H_{g,-}(\text{div}; \Omega) \times L^2(\Omega)\) with the assumption \( g \in L^2(\|\beta \cdot n\|; \Gamma_-) \) and the data \( \beta \) and \( \gamma \) satisfying the assumptions (i) or (ii).

Proof. For the existence, with the assumption of \( g \in L^2(\|\beta \cdot n\|; \Gamma_-) \), by the existence Theorem 2.1, there exists a \( u_g \in W \subset L^2(\Omega) \), such that \( u_g = g \) on \( \Gamma_- \) satisfying (1.1). Let \( \sigma_g = \beta u_g \), then
\[ \| \sigma_g \|_0 \leq \| \beta \|_\infty \| u_g \|_0, \quad \| \nabla \cdot \sigma_g \|_0 = \| f - \gamma u_g \|_0 \leq \| f \|_0 + \| \gamma \|_\infty \| u_g \|_0. \]
Also, on the inflow boundary, \( \sigma_g \cdot n = \beta \cdot n u_g = (\beta \cdot n) g \). Thus \( \sigma_g \in H_{g,-}(\text{div}; \Omega) \). That is, the minimization problem has a minimizer \((\sigma_g, u_g) \in H_{g,-}(\text{div}; \Omega) \times L^2(\Omega)\) with \( J(\sigma_g, u_g; f, g) = 0 \).

For the proof of uniqueness, let \((\sigma_1, u_1) \in H_{g,-}(\text{div}; \Omega) \times L^2(\Omega)\) and \((\sigma_2, u_2) \in H_{g,-}(\text{div}; \Omega) \times L^2(\Omega)\) be two solutions of (3.1) or (3.3), and let
\[ E = \sigma_1 - \sigma_2 \quad \text{and} \quad e = u_1 - u_2. \]
It follows that
\[ a(E, e; E, e) = a(\sigma_1, u_1; E, e) - a(\sigma_2, u_2; E, e) = (f, \nabla \cdot E + \gamma e) - (f, \nabla \cdot E + \gamma e) = 0. \]
So \( \| (E, e) \| = 0 \), thus \( E = 0 \) and \( e = 0 \). The uniqueness is then proved.

Remark 3.4. From the proofs of the above lemma and theorem, we can even further reduce the requirements of \( \beta \) and \( \gamma \), as long as they ensure the existence and uniqueness of the solution.

4. Least-Squares Finite Element Method Based on Flux Reformulation. In this section, we develop an LSFEM based on the least-squares variational problem developed in the previous section and derive the a priori and a posteriori error estimates.

4.1. Least-squares finite element method. Let \( \mathcal{T} = \{ K \} \) be a triangulation of \( \Omega \) using simplicial elements. The mesh \( \mathcal{T} \) is assumed to be regular. Also, we denote the set of edges/faces of the triangulation \( \mathcal{T} \) on inflow boundary \( \Gamma_- \) by \( \mathcal{E}_- \). For an element \( K \in \mathcal{T} \) and integer \( k \geq 0 \), let \( P_k(K) \) be the space of polynomials with degrees less than or equal to \( k \). Define the finite element spaces \( RT_k \) and \( P_k \) as follows:
\[ RT_k := \{ \tau \in H(\text{div}; \Omega) : \tau|_K \in P_k(K)^d + xP_k(K), \ \forall K \in \mathcal{T} \}, \]
and
\[ P_k := \{ v \in L^2(\Omega) : v|_K \in P_k(K), \ \forall K \in \mathcal{T} \}. \]
(Assumption on the boundary data) For simplicity, we assume \((\beta \cdot n) g\) on \( \Gamma_- \) can be approximated exactly by the trace of \( RT_k \) space on \( \Gamma_- \), i.e., \( g|_F \in P_k(F) \), for all faces/edges \( F \in \mathcal{E}_- \). Note that this assumption still allows the discontinuous
boundary condition, but it does require that the boundary mesh is aligned with the discontinuity. For an arbitrary \( g \), we need to first interpolate or project \((\beta \cdot \mathbf{n})g\) to the piecewise polynomial space.

Define
\[
RT_{k,g,-} = \{ \tau \in RT_k : \tau \cdot \mathbf{n} = (\beta \cdot \mathbf{n})g \text{ on } \Gamma_-, \}
\]
than our discrete LSFEM problem is:

**LSFEM Problem** We seek solutions \((\sigma_h, u_h) \in RT_{k,g,-} \times P_k\), such that
\[
J(\sigma_h, u_h; f, g) = \inf_{(\tau, v) \in \mathcal{P}} J(\tau, v; f, g).
\]

Or equivalently, find \((\sigma_h, u_h) \in RT_{k,g,-} \times P_k\), such that
\[
a(\sigma_h, u_h; \tau, v) = (f, \nabla \cdot \tau + \gamma v), \quad \forall (\tau, v) \in RT_{k,0,-} \times P_k.
\]

### 4.2. Interpolations and their properties

In order to derive a priori error estimates, we introduce some interpolations and their properties. Note that all properties here are local.

Denote by \(\pi_k : L^2(\Omega) \rightarrow P_k\) the \(L^2\)-projection onto \(P_k\), we have: for \(v \in H^s(K)\), \(s > 0\),
\[
\|v - \pi_kv\|_{0,K} \leq C h^{\min\{s,k+1\}}|v|_{s,K}, \quad \forall K \in \mathcal{T}.
\]

For \(s > 0\), denote by \(I^s_k : H(\text{div}; \Omega) \cap [H^s(\Omega)]^d \mapsto RT_k\) the standard \(RT\) interpolation operator [BBF13]. It satisfies the following approximation property: for \(\tau \in H^s(K)^d\), \(s > 0\),
\[
\|\tau - I^s_k\tau\|_{0,K} \leq C h^{\min\{s,k+1\}}|\tau|_{s,K}, \quad \forall K \in \mathcal{T}.
\]

(The estimate in (4.4) is standard for \(s \geq 1\) and may be proved by the average Taylor series developed in [DS80] and the standard reference element technique with Piola transformation for \(0 < s < 1\).) The following commutativity property is well-known:
\[
\nabla \cdot (I^s_k\tau) = \pi_k \nabla \cdot \tau, \quad \forall \tau \in H(\text{div}; \Omega) \cap H^s(\Omega)^d \text{ with } s > 0.
\]

Thus the following approximation property holds: for \(\tau \in H^s(K)^d\) and \(\nabla \cdot \tau \in H^s(K)\), \(s > 0\),
\[
\|\nabla \cdot \tau - \pi_k(\nabla \cdot \tau)\|_{0,K} \leq C h^{\min\{s,k+1\}}|\nabla \cdot \tau|_{s,K}, \quad \forall K \in \mathcal{T}.
\]

**Remark 4.1.** We use \(H(\text{div}; \Omega) \cap [H^s(\Omega)]^d\) instead of the choice \(\{\tau \in L^p(\Omega)^d \text{ and } \nabla \cdot \tau \in L^2(\Omega)\}\) for \(p > 2\) or \(W^{1,t}(K)\) for \(t > 2d/(d+2)\) in [BBF13] because this Hilbert space based version is more suitable for our analysis.

We also have the following approximation property on edges(2D)/faces(3D) \(F\) of \(K\): for \(\tau \in H^s(K)^d\) and \(\nabla \cdot \tau \in H^s(K)\),
\[
\|\tau - I^s_k\tau\|_{0,F} \leq C h^{\min\{s,k+1\}}|\tau|_{s,K} + h^{1/2} h^{1/2}|\nabla \cdot \tau|_{s,K} \quad \forall K \in \mathcal{T}.
\]

**Proof.** The result follows by approximation properties (4.4) and (4.6) and the following trace inequality: For all \(\tau \in \{\tau \in H(\text{div}; K) : \tau \cdot \mathbf{n} \in L^2(F)\}\),
\[
\|\tau \cdot \mathbf{n}\|_{0,F} \leq C h^{-1/2} |\tau|_{0,K} + h^{1/2} |\nabla \cdot \tau|_{0,K}.
\]
4.3. A priori error estimation.

**Theorem 4.2.** (Cea’s lemma type of result) Let \((\sigma, u)\) be the solution of least-squares variational problem (3.1), and \((\sigma_h, u_h)\) be the solution of LSFEM problem (4.1) with the assumption on the boundary data, the following best approximation result holds:

\[
\|\sigma - \sigma_h, u - u_h\| \leq \inf_{(\tau_h, v_h) \in RT_k,0,- \times P_k} \|\sigma - \tau_h, u - v_h\|. \tag{4.9}
\]

**Proof.** Let \((\tau_h, v_h) \in RT_k,0,- \times P_k\), the following error equation holds:

\[
a(\tau - \tau_h, u - v) = 0, \quad \forall (\tau_h, v_h) \in RT_k,0,- \times P_k.
\]

From the definition of the norm \(\|\cdot\|\), the error equation, and Cauchy-Schwarz inequality, we have

\[
\|\sigma - \sigma_h, u - u_h\|^2 = a(\sigma - \sigma_h, u - u_h; \sigma - \sigma_h, u - u_h) = a(\sigma - \sigma_h, u - u_h; \tau - \tau_h, u - v_h) \leq \|\sigma - \sigma_h, u - u_h\|\|\tau - \tau_h, u - v_h\|,
\]

so \(\|\sigma - \sigma_h, u - u_h\| \leq \|\sigma - \tau_h, u - v_h\|\). Since \((\tau_h, v_h)\) is chosen arbitrarily, the theorem is proved. \(\square\)

Define the following piecewise function space on the triangulation \(T\),

\[
H^s(T) = \{v \in L^2(\Omega) : v|_K \in H^{s,K}(K) \quad \forall K \in T\},
\]

\[
H^s(\text{div}; T) = \{\tau \in (L^2(\Omega))^d : \tau|_K \in (H^{s,K}(K))^d, \nabla \cdot \tau|_K \in H^{s,K}(K) \quad \forall K \in T\},
\]

with \(s\) is a piecewisely defined function, \(s|_K = s_K > 0\).

**Theorem 4.3.** Assume the solution \((\sigma, u)\) \(\in H^s(\text{div}; T) \times H^s(T)\), for \(s > 0\) defined piecewisely, and \((\sigma_h, u_h)\) \(\in RT_k \times P_k\) is the solution of the LSFEM problem (4.1) with the assumption on the boundary data, then there exists a constant \(C > 0\) independent of the mesh size \(h\), such that

\[
\|\sigma - \sigma_h, u - u_h\| \leq C \sum_{K \in T} h_K^{\min(k+1,s_K)} \left(\|u\|_{s_K,K} + \|\sigma\|_{s_K,K} + \|\nabla \cdot \sigma\|_{s_K,K}\right). \tag{4.10}
\]

**Proof.** By the triangle inequality, it is easy to see that

\[
\|\tau, v\|_K \leq \|\tau\|_{0,K} + \|\nabla \cdot \tau\|_{0,K} + (\|\beta\|_{\infty,K} + \|\gamma\|_{\infty,K})\|v\|_{0,K}, \quad \forall K \in T.
\]

Then the theorem follows immediately after Theorem 4.2, the triangle inequality, and the approximation properties (4.3), (4.4), and (4.6). \(\square\)

**Remark 4.4.** 1. Similar as we did in [CHZ17] for elliptic problems, the above a priori result is local with respect to regularities. It establishes the “equip-distribution of errors” foundation of adaptive mesh refinement algorithms. With different local regularities and different local sizes of the solution in respected \(s_K\) norms, the mesh size \(h_K\) can be modified to ensure an almost equal-distribution of the error.
2. Assume that \( \beta, \gamma, \) and \( f \) are sufficiently smooth in an element \( K \), if \( u|_K \in H^{s_K}(K) \), then
\[
\sigma|_K = (\beta u)|_K \in (H^{s_K}(K))^d \quad \text{and} \quad \nabla \cdot \sigma|_K = (f - \gamma u)|_K \in H^{s_K}(K),
\]
so we can safely assume that \( \sigma|_K \) and \( \nabla \cdot \sigma|_K \) have the same smoothness under the condition of the sufficiently smoothness of the data in each element.

3. For piecewise smooth solutions, the above theorem covers two cases. For the case that the mesh is aligned with discontinuity, the solution is still smooth in each element \( K \) with some \( s_K \geq 1 \), we can get optimal convergence result in least-squares norms with respect to the local regularity \( s_K \).

For the more general case that the finite element mesh is not aligned with discontinuity, \( u|_K \) belongs to \( H^{1/2+\epsilon}(K) \) for those elements \( K \) with a passing though discontinuity for some \( \epsilon > 0 \) as pointed out in [DSM04]. This means that we cannot get order 1 on those discontinuous elements. Also \( RT_0 \times P_0 \) should be used on those elements since higher order elements will not contribute more. And it suggests that there will be many mesh refinements along the discontinuity when an adaptive algorithm is used.

4. It is also clear that we should use \( RT_k \times P_k \) pair to ensure the same order of approximation. For the \( BDM_k \times P_k \) or \( BDM_{k+1} \times P_k \), the approximation order will not be balanced and suboptimal like the mixed case with the non-zero diffusion [Dem02].

5. For the extreme case that no smoothness is assumed, i.e., the exact solutions satisfy \( u \in L^2(\Omega) \) and \( \sigma \in H(\text{div}; \Omega) \) only, we can still prove the convergence without an order by the standard density argument. Introduce a smooth \( \sigma_\epsilon \in H_{0,-}(\text{div}; \Omega) \cap C^\infty(\Omega)^d \) and a smooth \( u_\epsilon \in C^\infty(\Omega) \) such that \( \| \sigma - \sigma_\epsilon, u - u_\epsilon \| \leq \epsilon \) for an arbitrary small \( \epsilon > 0 \). The smooth \( (\sigma_\epsilon, u_\epsilon) \) can be well-approximated with a small \( h \). Thus we can show
\[
\| (\sigma - \sigma_h, u - u_h) \| \rightarrow 0, \quad \text{as} \quad h \rightarrow 0.
\]

This analysis can also be localized element-wisely as above.

6. In the theorem, the a priori error estimate is derived for the least-squares energy norm \( \| (\cdot, \cdot) \| \). Our numerical test will also disprove the possibility of a coercivity with respect to the standard norm:
\[
\| (\tau, v) \|^2 \geq C \left( \| \tau \|^2_{H(\text{div}; \Omega)} + \| v \|_0^2 \right), \quad \forall (\tau, v) \in H_{0,-}(\text{div}; \Omega) \times L^2(\Omega),
\]

or the weak discrete version with an \( h \)-independent \( C > 0 \),
\[
\| (\tau, v) \|^2 \geq C \left( \| \tau \|^2_{H(\text{div}; \Omega)} + \| v \|_0^2 \right), \quad \forall (\tau, v) \in RT_{k,0,-} \times P_k.
\]

Because if one of such coercivity results hold, one can show that the error measured in \( H(\text{div}; \Omega) \times L^2(\Omega) \) norm will be optimal for piecewise smooth solutions with discontinuity aligned mesh, which is not the case in our numerical test 7.3.

4.4. A posteriori error estimation. The least-squares functional can be used to define the following fully computable a posteriori local indicator and global error estimator:
\[
\eta^2_{K} := \| \sigma_h - \beta u_h \|^2_{0,K} + \| \nabla \cdot \sigma_h + \gamma u_h - f \|^2_0, \quad \forall K \in \mathcal{T},
\]
and
\[ \eta^2 := \sum_{K \in \mathcal{T}} \eta^2_K = \| \sigma_h - \beta u_h \|_0^2 + \| \nabla \cdot \sigma_h + \gamma u_h - f \|_0^2. \]

**Theorem 4.5.** The a posteriori error estimator \( \eta \) is exact with respect to the least-squares norm \( \left\|( \cdot, \cdot ) \right\| \):
\[ \eta = \left\| (\sigma - \sigma_h, u - u_h) \right\| \quad \text{and} \quad \eta_K = \left\| (\sigma - \sigma_h, u - u_h) \right\|_K. \]

The following local efficiency bound is also true with \( C > 0 \) independent of the mesh size \( h \):
\[ C \eta_K \leq \| \sigma - \sigma_h \|_{H(\text{div};K)} + \| u - u_h \|_{0,K}, \quad \forall K \in \mathcal{T}. \]

**Proof.** Note that the exact solutions satisfy \( \sigma = \beta u \) and \( f = \nabla \cdot \sigma + \gamma u \), so
\[ \eta^2 = \| \sigma_h - \beta u_h \|_0^2 + \| \nabla \cdot \sigma_h + \gamma u_h - f \|_0^2 = \| \sigma - \sigma_h + \beta(u - u_h) \|_0^2 + \| \nabla \cdot (\sigma - \sigma_h) + \gamma(u - u_h) \|_0^2 = \left\| (\sigma - \sigma_h, u - u_h) \right\|_2^2. \]

The proof of the local exactness is identical.

With the triangle inequality, the local efficiency bound for the standard norms can be easily proved. \( \square \)

**Remark 4.6.** Due to the fact that the least-squares functional norm is not equivalent to the standard \( H(\text{div})-L^2 \) norm, it is impossible to get the corresponding reliability result w.r.t. the \( H(\text{div})-L^2 \) norm.

**5. Least-Squares Variational Problems with Boundary Functional.** In this section, in stead of treating the inflow boundary condition as an essential condition, we develop a least-squares method with boundary functional in the free space.

In this section, we assume the inflow boundary condition is not degenerate,
\[ |\beta \cdot n| \geq c > 0 \text{ on } \Gamma_- . \]

**Remark 5.1.** This assumption is essential to guarantee the optimal convergence rate, see the proof of Theorem 6.2.

Define a weight-dependent inner product and its corresponding norm:
\[ (v, w)_{\omega, \Gamma_-} := \sum_{F \in \mathcal{E}_-} \int_F \frac{\omega}{|\beta \cdot n|} v w dx \quad \text{and} \quad \| v \|_{\omega, \Gamma_-} := (v, v)_{\omega, \Gamma_-}^{1/2}. \]

We use two choices here:
\[ \omega_1 = 1 \quad \text{and} \quad \omega_2 = \alpha_F h_F. \]

The following notation is also used to denote the norm on an edge(2D)/face(3D) of an element \( K \):
\[ \| v \|_{\omega, F} := \left( \int_F \frac{\omega}{|\beta \cdot n|^2} v^2 dx \right)^{1/2}. \]
Remark 5.2. Here, $\alpha_F > 0$ is a big enough but $h$-independent constant to ensure the balance of terms. In general, $\alpha_F$ can be chosen depending on $F$ and possibly also depending on the coefficients. The constant $\alpha_F$ comes from the constant that appears in the trace inequality (4.8). In some extreme cases, we find it is necessary to choose $\alpha_F$ to be some constant large enough (10 is large enough in our numerical tests) to ensure that the boundary condition is not too weakly enforced. See detailed discussion in our numerical test 7.6.1.

In this paper, the choice $\alpha_F = 10$ is suggested and used in numerical tests.

The choice of $\omega_1 = 1$ does not have the above issues, but the convergence order is less optimal near the inflow boundary, see our discussion in the a priori error estimates Theorem 6.2.

Let

$$
\Sigma := \{ \tau \in H(\text{div}; \Omega) : \tau \cdot n \in L^2(\Gamma_-) \}
$$

with the weight-dependent norm

$$
\|	au\|_{\Sigma}^2 := \|
abla \cdot \tau\|^2_{\Omega} + \|
abla \cdot \tau\|^2_{\partial \Omega} + \|	au \cdot n\|^2_{\omega_1, \Gamma_-}.
$$

Define the following least-squares functional $L$ for all $(\tau, v) \in \Sigma \times L^2(\Omega)$,

$$
L_i(\tau, v; f, g) := \|	au - \beta v\|^2_{\partial \Omega} + \|
abla \cdot \tau + \gamma u - f\|^2_{\Omega} + \|	au \cdot n - \beta \cdot n g\|^2_{\omega_1, \Gamma_-}, i = 1, 2.
$$

(Least-Squares Problems with Boundary Functional) We seek solutions $(\sigma, u) \in \Sigma \times L^2(\Omega)$, such that

$$
L_i(\sigma, u; f, g) = \inf_{(\tau, v) \in \Sigma \times L^2(\Omega)} L_i(\tau, v; f, g), \quad i = 1, 2.
$$

Its corresponding Euler-Lagrange formulation is: Find $(\sigma, u) \in \Sigma \times L^2(\Omega)$, such that

$$
b_i(\sigma, u; \tau, v) = (f, \nabla \cdot \tau + \gamma v) + (\beta \cdot n g, \tau \cdot n)_{\omega_1, \Gamma_-}, \quad \forall (\tau, v) \in \Sigma \times L^2(\Omega), i = 1, 2,
$$

where, for all $(\tau, v), (\rho, w) \in \Sigma \times L^2(\Omega)$, the bilinear form is:

$$
b_i(\tau, v; \rho, w) := (\tau - \beta v, \rho - \beta w) + (\nabla \cdot \tau + \gamma v, \nabla \cdot \rho + \gamma w) + (\sigma \cdot n, \tau \cdot n)_{\omega_1, \Gamma_-}, i = 1, 2.
$$

Note that for $F \in \mathcal{E}_-$, $\beta \cdot n < 0$, so

$$
(\beta \cdot n g, \tau \cdot n)_{\omega_1, \Gamma_-} = \sum_{F \in \mathcal{E}_-} \int_F \frac{\omega}{|\beta \cdot n|} (\beta \cdot n g)(\tau \cdot n)dx = - \sum_{F \in \mathcal{E}_-} \omega \int_F \tau \cdot n gdx.
$$

Lemma 5.3. Assuming that the data $\beta$ and $\gamma$ satisfy the assumptions (i) or (ii), the following defines a norm for $(\tau, v) \in \Sigma \times L^2(\Omega)$:

$$
\|(\tau, v)\|_B := (\|	au - \beta v\|^2_{\partial \Omega} + \|
abla \cdot \tau + \gamma v\|^2_{\Omega} + \|	au \cdot n\|^2_{\omega_1, \Gamma_-})^{1/2}.
$$

Proof. The proof of the lemma is almost identical to that of Lemma 3.1 by realizing that if $\|(\tau, v)\|_B = 0$, we have

$$
\tau = \beta v \quad \text{and} \quad \nabla \cdot \tau + \gamma v = 0 \text{ in } \Omega, \quad \tau \cdot n = 0 \text{ on } \Gamma_-.
Remark 5.4. Similarly,
\[
\| (\tau, v) \|_{B,K} := \left( \| \tau - \beta v \|_{0,K}^2 + \| \nabla \cdot \tau + \gamma v \|_{0,K}^2 + \sum_{F \in \partial K \cap \Sigma} \| \tau \cdot n \|_{\omega,F}^2 \right)^{1/2}
\]
is a semi-norm on an element \( K \in T \).

Notations \( \| (\tau, v) \|_{B,i} \) and \( \| (\tau, v) \|_{B,i,K} \) with \( i = 1 \) or \( 2 \) are used to denote the (semi-)norms with weights \( \omega = \omega_i \), \( i = 1 \) or \( 2 \).

Theorem 5.5. The least-squares problem (5.2) has a unique solution \((\sigma, u) \in \Sigma \times L^2(\Omega)\) with the assumption \( g \in L^2(|\beta \cdot n|; \Gamma_-) \) and the data \( \beta \) and \( \gamma \) satisfying the assumptions (i) or (ii).

Proof. The proof of existence and uniqueness is very similar to that of Theorem 3.3 and thus we omit it here. \( \square \)

6. LSFEMs with Boundary Functional. In this section, we develop LSFEMs based on the least-squares variational problems with boundary functional developed in the previous section and derive the a priori and a posteriori error estimates.

6.1. LSFEM-B problems. We seek solutions \((\sigma_h, u_h) \in RT_k \times P_k\), such that
\[
\mathcal{L}_i(\sigma_h, u_h; f, g) = \inf_{(\tau, v) \in RT_k \times P_k} \mathcal{L}_i(\tau, v; f, g), \quad i = 1, 2.
\]
Or equivalently, find \((\sigma_h, u_h) \in RT_k \times P_k\), such that
\[
b_i(\sigma_h, u_h; \tau, v) = (f, \nabla \cdot \tau + \gamma v) + (\beta \cdot n g, \tau \cdot n)_{\omega_i, \Gamma_-}, \quad \forall (\tau, v) \in RT_k \times P_k, i = 1, 2.
\]

6.2. A priori error estimation.

Theorem 6.1. (Cea’s lemma type of result) Let \((\sigma, u)\) be the solution of least-squares variational problem with boundary term (5.2), and \((\sigma_h, u_h)\) be the solution of LSFEM problem (6.1), the following best approximation result holds:
\[
\| (\sigma - \sigma_h, u - u_h) \|_B \leq C \inf_{(\tau, v) \in RT_k \times P_k} \| (\sigma - \tau, u - v) \|_B
\]
Proof. The proof is identical to that of Theorem 4.2. \( \square \)

Define the collections of elements with edges(2D)/faces(3D) on the inflow boundary as:
\[
\mathcal{T}_- = \{ K : K \in T, \partial K \cap \Gamma_- \neq \emptyset \}.
\]

Theorem 6.2. Assume the exact solution \((\sigma, u) \in H^s(\text{div}; \mathcal{T}) \times H^s(\mathcal{T})\), for \( s > 0 \) defined piecewisely. Assume \((\sigma_{h,i}, u_{h,i}) \in RT_k \times P_k\) is the solution of LSFEM-B problem (6.1) with weight \( \omega_i \), \( i = 1 \) or \( 2 \), then there exists a constant \( C > 0 \) independent of the mesh size \( h \), such that
\[
\| (\sigma - \sigma_{h,1}, u - u_{h,1}) \|_{B,1} \leq C \sum_{K \in T} h_K^{\min(k+1,s_K)} \left( \| u \|_{s_K,K} + \| \sigma \|_{s_K,K} + \| \nabla \cdot \sigma \|_{s_K,K} \right)
+ C \sum_{K \in \mathcal{T}_-} h_K^{\min(k+1,s_K)-1/2} \| \sigma \|_{s_K,K},
\]
\[
\| (\sigma - \sigma_{h,2}, u - u_{h,2}) \|_{B,2} \leq C \sum_{K \in T} h_K^{\min(k+1,s_K)} \left( \| u \|_{s_K,K} + \| \sigma \|_{s_K,K} + \| \nabla \cdot \sigma \|_{s_K,K} \right).
\]
Proof. We only need to handle the boundary term, the rest of terms are identical to that of Theorem 4.3.

Let \( \mathbf{\tau}_h = I_h^t \mathbf{\sigma} \), by the trace inequality (4.8) and approximation property (4.7), we have

\[
\begin{align*}
\| (\mathbf{\sigma} - \mathbf{\tau}_h) \cdot \mathbf{n} \|_{0,F} & \leq h_F^{-1/2} (\| \mathbf{\sigma} - \mathbf{\tau}_h \|_{0,K} + h_K^{1/2} \| \nabla \cdot (\mathbf{\sigma} - \mathbf{\tau}_h) \|_{0,K}) \\
& \leq C h_K^{\min(k+1,s_K)-1/2} (\| \mathbf{\sigma} \|_{s_K,K} + h_K^{1/2} \| \nabla \cdot \mathbf{\sigma} \|_{s_K,K}).
\end{align*}
\]

By our assumption on \( \mathbf{\beta} \cdot \mathbf{n} \), there exits a constant \( C > 0 \) independent of the mesh size \( h \),

\[
\| (\mathbf{\sigma} - \mathbf{\tau}_h) \cdot \mathbf{n} \|_{\omega_1,F}^2 = \sum_{F \in \mathcal{E}_-} \| (\mathbf{\sigma} - \mathbf{\tau}_h) \cdot \mathbf{n} \|_{\omega_1,F}^2 \leq C \sum_{F \in \mathcal{E}_-} \| (\mathbf{\sigma} - \mathbf{\tau}_h) \cdot \mathbf{n} \|_{0,F}^2
\]

\[
\leq C \sum_{K \in \mathcal{T}_-} h_K^{2 \min(k+1,s_K) - 1} \| \mathbf{\sigma} \|_{s_K,K}^2 + \sum_{K \in \mathcal{T}_-} h_K \| \nabla \cdot \mathbf{\sigma} \|_{s_K,K}^2.
\]

Combined with interior terms, we proved (6.4).

By our assumptions on \( \alpha_F \) and \( \mathbf{\beta} \cdot \mathbf{n} \), there exits a constant \( C > 0 \) independent of the mesh size \( h \),

\[
\| (\mathbf{\sigma} - \mathbf{\tau}_h) \cdot \mathbf{n} \|_{\omega_2,F}^2 = \sum_{F \in \mathcal{E}_-} \| (\mathbf{\sigma} - \mathbf{\tau}_h) \cdot \mathbf{n} \|_{\omega_2,F}^2 \leq C \sum_{F \in \mathcal{E}_-} h_F \| (\mathbf{\sigma} - \mathbf{\tau}_h) \cdot \mathbf{n} \|_{0,F}^2
\]

\[
\leq C \sum_{K \in \mathcal{T}} h_K^{2 \min(k+1,s_K)} \| \mathbf{\sigma} \|_{s_K,K}^2 + h_K \| \nabla \cdot \mathbf{\sigma} \|_{s_K,K}^2.
\]

Combined with interior terms, we proved (6.5).

Remark 6.3. For the case the weight \( \omega = 1 \), we see there is a half-order loss in the error analysis for those elements in \( \mathcal{T}_- \). Compared with the number of elements in \( \mathcal{T} \), the number of elements in \( \mathcal{T}_- \) is small and such sub-optimality often is non-observable in our numerical tests.

For the case the weight \( \omega = \omega_2 \), even though the convergence order is optimal, we do add an uncertainty of choosing \( \alpha_F \). A too small \( \alpha_F \) will lead to imbalance of terms and will cause the boundary condition un-resolved, which will make the adaptive algorithms fail, see our numerical test 7.6.1. For the case that the mesh is not aligned with the discontinuity, which probably is the interesting case, the elements with discontinuity are the major source of the error, and will dominate the inflow half order loss since we can always make sure the mesh on the inflow boundary condition is aligned. In this case, the simple choice \( \omega = 1 \) is probably the better choice.

The discussions in Remark 4.4 are also true for the methods in this section.

6.3. A posteriori error estimation. The least-squares functional can be used to define the following fully computable a posteriori local indicator and global error estimator:

\[
\xi_K^2 := \| \mathbf{\sigma}_h - \mathbf{\beta} \mathbf{u}_h \|_{0,K}^2 + \| \nabla \cdot \mathbf{\sigma}_h + \gamma \mathbf{u}_h - f \|_{0,K}^2 + \sum_{F \in \partial K \cap \mathcal{E}_-} \| \mathbf{\sigma}_h \cdot \mathbf{n} - \mathbf{\beta} \cdot \mathbf{n} \|_{0,F}^2, \forall K \in \mathcal{T}.
\]

and

\[
\xi^2 := \sum_{K \in \mathcal{T}} \xi_K^2 = \| \mathbf{\sigma}_h - \mathbf{\beta} \mathbf{u}_h \|_{0}^2 + \| \nabla \cdot \mathbf{\sigma}_h + \gamma \mathbf{u}_h - f \|_{0}^2 + \| \mathbf{\sigma}_h \cdot \mathbf{n} - \mathbf{\beta} \cdot \mathbf{n} \|_{0,\Gamma_-}^2.
\]
Theorem 6.4. The a posteriori error estimator $\eta$ is exact with respect to $\|\cdot\|_B$-norm:

$$\xi = \|(\sigma - \sigma_h, u - u_h)\|_B \quad \text{and} \quad \xi_K = \|(\sigma - \sigma_h, u - u_h)\|_{B,K}.$$ 

The following local efficiency bounds are also true with a constant $C > 0$ independent of the mesh size $h$. For the method and indicators with $\omega = \omega_1 = 1$,

$$C \xi_K \leq \|\sigma - \sigma_h\|_{H(\text{div};K)} + \|u - u_h\|_{0,K}, \quad \forall K \in \mathcal{T} \setminus \mathcal{T}_-,$$

$$C \xi_K \leq h_K^{-1/2} \|\sigma - \sigma_h\|_{0,K} + \|\nabla \cdot (\sigma - \sigma_h)\|_{0,K} + \|u - u_h\|_{0,K}, \quad \forall K \in \mathcal{T}_-,$$

and for the method and indicators with $\omega = \omega_2$,

$$C \xi_K \leq \|\sigma - \sigma_h\|_{H(\text{div};K)} + \|u - u_h\|_{0,K}, \quad \forall K \in \mathcal{T}.$$

Proof. The local and global exactness results are trivial as the case without the boundary functional. For the local efficiency bounds, the result follows from the triangle inequality and the trace inequality (4.8) if the element belongs to $\mathcal{T}_-$. 

Remark 6.5. In our LSFEM-B method with weight $\omega_2$, the boundary condition of $\sigma$ is treated by mesh size weighting to ensure the optimal convergence order. The more complicated $-1/2$ norm version similar to that in [Sta06] can also be developed.

7. Computational Examples. In all our numerical examples, the lowest order approximations are used, i.e., $P_0$ for $u$ and $RT_0$ for the flux $\sigma$.

We use the name LSFEM to denote the methods we developed in Section 4, and use LSFEM-B1 and LSFEM-B2 to denote the methods with weight $\omega_1$ and $\omega_2$ developed in Section 6, separately. If not stated explicitly, $\alpha_F = 10$ is used in LSFEM-B2 in our numerical tests.

In the adaptive mesh refinement algorithm, the D"{o}fler’s bulk marking strategy with $\theta = 0.5$ is used and the algorithm is stopped when the total number of DOFs reaches $10^6$. All refinements are based on the longest edge bisection algorithm.

For all the numerical examples with domain $(0,1)^2$, the mesh shown in Fig. 1 is used as an initial mesh.

![Initial mesh for all examples with a $(0,1)^2$ domain](image)

Although we have three methods, in our numerical experiments, we find they have almost identical performance. We only show the figures of all three methods in
Examples 7.2 and 7.3. For all other test problems, only LSFEM are shown unless stated explicitly.

7.1. An example with a constant advection field and a piecewise constant solution, matching grid. In this example, we only need do the thought experiment, although the actual computation does confirm our result.

Consider the following problem: \( \Omega = (0,1)^2 \) with \( \beta = (1/\sqrt{2},1/\sqrt{2})^T \). The inflow boundary is \( \{x = 0, y \in (0,1)\} \cup \{x \in (0,1), y = 0\} \), i.e., the west and south boundaries of the domain. Let \( \gamma = 1 \) and choose \( g \) and \( f \) such that the exact solution \( u \) is

\[
\begin{align*}
u &= \begin{cases} 1 & \text{in } y > x, \\ 0 & \text{in } y < x. \end{cases}
\end{align*}
\]

If we choose the mesh aligned with the discontinuity, for example, any refinements of the mesh in Fig. 1. Note that \( u \in P_0 \) and \( \sigma \in RT_0 \). By the best approximation properties Theorems 4.2 and 6.1, the numerical solutions \( u_h \) and \( \sigma_h \) are identical to the exact solution. So no further refinements are needed. This is not true when \( C^0 \) finite elements are used to approximate the discontinuous \( u \) as in [BC01a, DSMMO04, BG09, BG16], many unnecessary refinements are needed.

7.2. An example with a global smooth solution. Consider the following simple problem: \( \Omega = (0,1)^2 \) with \( \beta = (1,1)^T \). The inflow boundary is \( \{x = 0, y \in (0,1)\} \cup \{x \in (0,1), y = 0\} \), i.e., the west and south boundaries of the domain. Let \( \gamma = 1 \). Choose \( f \) and \( g \) such that the exact solution is \( u = \sin(x+y) \).

In Fig. 2, the convergence histories on uniformly refined meshes are shown. Errors measured in least-squares norms and \( \|u - u_h\|_0 \) are all of order 1. The optimal convergence order in \( \|\cdot\|_{B,1} \) norm suggests that the half order loss on those inflow boundary elements is negligible.

The \( L^2 \) errors \( \|u - u_h\|_0 \) for all cases are of order 1, this suggests that for **globally smooth solutions**, we may expect the norm equivalence (or at least in discrete spaces):

\[
\| (\tau, v) \| \approx \| \tau \|_{H(div; \Omega)} + \| v \|_0 \quad \forall (\tau, v) \in H_{0,-}(\text{div}; \Omega) \times L^2(\Omega).
\]

Similar results should also be true for the LSFEM-B2 formulation.

![Convergence histories for the global smooth solution on uniformly refined meshes (LSFEM(left), LSFEM-B1(center), LSFEM-B2(right))]
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the domain. Let \( \gamma = 1 \). Choose \( g \) and \( f \) such that the exact solution \( u \) is

\[
  u = \begin{cases} 
  \sin(x + y) & \text{if } y > x, \\
  \cos(x + y) & \text{if } y < x.
  \end{cases}
\]

We choose an initial mesh that matches the discontinuity (Fig. 1) and uniformly refine it for 8 times. In Fig. 3, we show the convergence histories. For all three formulations, the convergence order of the errors in their corresponding least-squares norms is 1, which matches the optimal convergence theory. The order of \( \|u - u_h\|_0 \) is less than 1 (about 0.6 at late stages). This suggests that the norm equivalence (or in discrete sub-spaces):

\[
  \|(\tau, v)\| \approx \|\tau\|_{H(\text{div}; \Omega)} + \|v\|_0 \quad \forall (\tau, v) \in H_0(\text{div}; \Omega) \times L^2(\Omega),
\]

should not be true for the discontinuous solutions. Similar results should also hold for the LSFEM-B2 formulation.

![Fig. 3. Convergence histories for the piecewise smooth solution with a matching mesh on uniformly refined meshes (LSFEM(left), LSFEM-B1(center), LSFEM-B2(right))](image)

7.4. An example with a piecewise constant solution, non-matching grid. In this example, we discuss the smearing and over/undershootings of the solution when the mesh is not matched with discontinuity.

Consider the problem: \( \Omega = (0, 2) \times (0, 1) \) with \( \beta = (0, 1)^T \). The inflow boundary is \( \{x \in (0, 1), y = 0\} \), i.e., the south boundary of the domain. Let \( \gamma = 0 \) and \( f = 0 \). Choose the inflow boundary condition such that the exact solution is

\[
  u(x, y) = \begin{cases} 
  0 & \text{if } x < \pi/3, \\
  1 & \text{if } x > \pi/3.
  \end{cases}
\]

We set the initial mesh to be as shown on the left of Fig. 4. The bottom central node is \((\pi/3, 0)\) and the top central node is \((1, 1)\). So the inflow boundary mesh is matched with the inflow boundary condition while the mesh is not aligned with the discontinuity in general and will never match with it if bisection mesh refinement is used.

On the right of Fig. 4, we show the solution computed by LSFEM on a mesh after 8 uniform refinements of the initial mesh. Since it essentially is a 1D problem, we project the graph of the solution onto the plane \( y = 0 \), that is, we plot the numerical solution value at the midpoint of \( x \)-axis of each elements. The smearing of the solution is very small, but we do see the under/overshootings. The maximum of \( u_h \) is 1.0629 and the minimum of \( u_h \) is \(-0.0339\).

On the left of Fig. 5, we plot the convergence results of uniform refinements. The decay rate of the error measured in the least-squares norm is about 0.7. The
reason that the rate is less than 1 is that the discontinuity is cutting through those interface elements so that \( u, \sigma, \) and \( \nabla \cdot \sigma \) are not of \( H^1 \) in those elements. But the rate is apparently better than \( 1/2 - \epsilon \), even though all those true solutions are only in \( H^{1/2-\epsilon}(K) \) for those interface elements. The possible reason for the better rate can be that the Sobolev space \( H^{1/2-\epsilon} \) may not be the best space to characterize the piecewisely discontinuous function space. The order of \( \| u - u_h \|_0 \) is about 1/2.

We then test the problem by adaptive mesh refinements. On the center of Fig. 5, adaptive refined meshes after some iterations are shown. Clearly, the refinements are along the discontinuity. On the right of Fig. 5, we show the convergence histories. The error measured in the LS norm is optimal with order 1, while the order of \( \| u - u_h \|_0 \) is about 1/2, which is about the same order as the uniform refinement.

On the left of Fig. 6, we show the decreasing of the overshooting values by adaptive mesh refinements. Here, the overshooting value is defined as \( \max(\max(u_h - 1), - \min(u_h)) \). We clearly see after the mesh is reasonably fine (when the mesh is coarse, the overshooting is actually not very severe since we approximate \( u \) by \( P_0 \)), the overshooting value begins to decrease.

On the right of Fig. 6, we show a projected solution. It is clear that when the mesh is fine, both the overshooting and smearing phenomena are almost negligible. From this example, we can see that adaptive LSFEM formulations can handle the discontinuity and smearing very well.

7.5. An example with a piecewise smooth solution, non-matching grid. Consider the following simple problem: \( \Omega = (0, 1)^2 \) with \( \beta = (\cos(1/8), \sin(1/8))^T \).
The inflow boundary is \( \{x = 0, y \in (0,1)\} \cup \{x \in (0,1), y = 0\} \), i.e., the west and south boundaries of the domain. Let \( \gamma = 1 \). Choose \( g \) and \( f \) such that the exact solution \( u \) is
\[
    u = \begin{cases} 
        \sin(x + y) & \text{if } y > \tan(1/8)x, \\
        \cos(x + y) & \text{if } y < \tan(1/8)x.
    \end{cases}
\]

Note that with an initial mesh as in Fig. 1, any refinement of it will never match the discontinuity.

We show the uniform convergence result on the left of Fig. 7. The convergence order in LS norms is about 0.8. Similar to the piecewise constant solution on non-matching grids, it is worse than order 1 but better than order 1/2. The convergence order for \( \|u - u_h\|_0 \) is about 0.3, which is worse than the piecewise constant non-matching case.

On the center Fig. 7, an adaptive mesh by LSFEM is shown. Many refinements are generated near the discontinuity. On the right of Fig. 7, convergence history of adaptive LSFEM is shown. The rate of convergence of error in the LS norm is about order 1, and \( \|u - u_h\|_0 \) is about order 0.5.

7.6. Curved transport examples.

7.6.1. Zero-one example. We consider an example similar to Example in 4.4.2 of [Gue04]. Consider the problem on the half disk \( \Omega = \{(x, y): x^2 + y^2 < 1; y > 0\} \). Let the inflow boundary be \( \{-1 < x < 0; y = 0\} \). Choose the advection field \( \beta = \)
\[(\sin \theta, -\cos \theta)^T = (y/\sqrt{x^2 + y^2}, -x/\sqrt{x^2 + y^2})^T,\] with \(\theta\) is the polar angle. Let \(\gamma = 0, f = 0\), and the inflow condition and the exact solution be

\[g = \begin{cases} 1 & \text{if } -1 < x < -0.5, \\ 0 & \text{if } -0.5 < x < 0, \end{cases}\]

and

\[u = \begin{cases} 1 & \text{if } x^2 + y^2 > 0.25, \\ 0 & \text{otherwise}. \end{cases}\]

We choose an initial mesh to be as shown on the left of Fig. 8. We choose the bottom central node to be \((0, 0)\) and the node left of it to be \((-0.5, 0)\). So the inflow boundary mesh is matched with the inflow boundary condition. Since the advection field is curved and so is the discontinuity, the mesh will never be aligned with the discontinuity even after refinements. Since the boundary is a half circle, when the mesh refinement is performed, an extra step is taken to map those boundary nodes on the circle to the right positions.

We show the numerical solution computed by LSFEM on a mesh after 8 uniform refinements of the initial mesh on the right of Fig. 8 (LSFEM-B solutions are similar). Small overshootings can be observed near the discontinuity. Along the radius, the solution is essentially one dimensional, we project the graph of the solution onto the radius, see the left of Fig. 9. We do see the smearing and under and overshootings. The maximum and minimum values of numerical solution \(u_h\) are 1.0401 and -0.0381, respectively.

With uniform refinements, the convergence rate of the error in the least-squares norm is about 0.81 and the rate of \(\|u - u_h\|_0\) is about 0.25, see the right of Fig. 9. Since the mesh is not aligned with the discontinuity, the convergence order of the LS energy norm is smaller than 1.

On the left of Fig. 10, we show the adaptive mesh generated by LSFEM after several iterations. We see many refinements along the discontinuity which is very natural. Also, almost uniform refinements can be found in the half ring where \(u = 1\). The reason is that even \(u\) is a constant 1, the flux \(\sigma = \beta\) is not a constant vector and has approximation errors. On the other hand, in the region where \(u = 0\), the flux is also a zero vector and can be exactly computed. So no refinement is needed in the inner half circle.

On the right of Fig. 10, we show the convergence history of the adaptive method. With AFEM, the convergence order of the error in the LS norm is about 1 and is optimal, and the rate of \(\|u - u_h\|_0\) is about 0.5.
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Fig. 9. Curved transport problem: projected numerical solutions on an almost uniform mesh with $h \approx 0.002$ (left), convergence behaviors on uniform refined meshes (right).

Fig. 10. Curved transport problem: adaptive refined meshes after several iterations (left), convergence behaviors on adaptive refined meshes (right).

On the left of Fig. 11, we show the reduction of overshooting values of LSFEM. After the initial stages, the overshooting values are decreasing with refined meshes along the discontinuity (although not strictly monotonically).

On the right of Fig. 11, the projected solution is shown on the final mesh. We can see that the smearings and overshootings are very small compared with the uniform refinements. Thus Gibbs phenomena is not observed.

Fig. 11. Curved transport problem: reduction of overshootings of AFEM (left), projection solutions on adaptive refined meshes (right).
If we choose $\alpha_F = 1$ in the LSFEM-B2 formulation, the numerical computation is not right for this problem. On the left of Fig. 12, the refined mesh generated by LSFEM-B2 and error estimator $\xi$ is shown. Many unnecessary refinements along the inflow boundary are seen. On the right of Fig. 12, we show the convergence histories of the error. For the error measured in the LS norm $\| \cdot \|_B$ the order is optimal, but $\| u - u_h \|_0$ is not decreasing. On Fig. 13, the numerical solution and its projected version are shown. It is very clear the solution is not accurate under this mesh and LSFEM-B with $\alpha_F = 1$.

These all suggest that if we simply choose $\alpha_F = 1$ in LSFEM-B2, the $\| \cdot \|_B$ norm is not well balanced, the weight on the boundary term is too weak.

**Fig. 12.** Refined mesh generated by LSFEM-B2 (left) and convergence histories (right) for the curved problem with $\alpha_F = 1$

**Fig. 13.** Numerical solution (left) and its projected version (right) generated by LSFEM-B2 on a refined mesh for the curved problem with $\alpha_F = 1$

### 7.6.2. Negative-one-one example.

We modify the previous example by letting the inflow condition and the exact solution be

$$g = \begin{cases} 
1 & \text{if} \ -1 < x < -0.5, \\
-1 & \text{if} \ -0.5 < x < 0,
\end{cases} \quad \text{and} \quad u = \begin{cases} 
1 & \text{if} \ x^2 + y^2 > 0.25, \\
-1 & \text{otherwise}.
\end{cases}$$

Note that even the solution $u$ in the inner half disk $\{ x^2 + y^2 < 0.25, y > 0 \}$ is still a constant vector, the flux $\sigma = \beta u = -\beta$ is not. At the origin $(0, 0)$, the flux is singular, so it is expected that many refinements around the origin.
The left of Fig. 14 is a refined mesh. It is clear that the mesh is refined around the origin and the discontinuities. The right of Fig. 14 shows the convergence history. The order of LS energy norm is 1 and that of $\|u - u_h\|_0$ is 1/2.

![Fig. 14. A refined mesh after several iterations (left) and convergence behaviors (right) for the $-1/\epsilon$ curved transport problem.](image1)

In Fig. 15, we present the numerical solutions. It is clear that there is a singularity at the origin, while the solution near the discontinuity behaves similarly as the previous 0-1 example.

![Fig. 15. Numerical solution for the $-1/\epsilon$ curved transport problem on adaptively refined meshes (projected on the left and non-projected on the right).](image2)

### 7.7. A smooth example with a sharp transient layer.

Consider the following problem: $\Omega = (0, 1)^2$ with $\beta = (y + 1, -x)^T / \sqrt{x^2 + (y + 1)^2}$, $\gamma = 0.1$, and $f = 0$. The inflow boundary is $\{x = 1, y \in (0, 1)\} \cup \{x \in (0, 1), y = 0\}$, i.e., the west and north boundaries of the domain. Choose $g$ such that the exact solution $u$ is

$$u = \frac{1}{4} \exp \left( \gamma r \arcsin \left( \frac{y + 1}{r} \right) \right) \arctan \left( \frac{r - 1.5}{\epsilon} \right), \quad \text{with} \quad r = \sqrt{x^2 + (y + 1)^2}.$$

When $\epsilon = 0.01$, the layer can be fully resolved, see Fig. 16. When $\epsilon = 10^{-10}$, the layer is never fully resolved in our experiments and can be viewed as discontinuous, see Fig. 17.

When $\epsilon = 0.01$, we show the numerical results in Fig. 18. The behaviors of the methods are very similar to the global continuous solution case.
When $\epsilon = 10^{-10}$, we show the numerical results in Fig. 19. The behaviors of the methods are very similar to the piecewise smooth solution with non-matching grid case, the example 7.6. The order of convergence of $\|u - u_h\|_0$ is about 0.12. The contours of the solution on the right of Fig. 19 show that the smearings and overshooting are negligible when the mesh is fine enough.

**7.8. General comments about the numerical experiments.** In our numerical tests, we found that all three formulations have almost identical results. The half-order loss on the inflow boundary elements for LSFEM-B1 is neglectable/non-observable, thus for the methods with weakly enforced inflow boundary conditions,
we prefer LSFEM-B1 over LSFEM-B2, since for LSFEM-B2, the current choice of the weight $\alpha_F = 10$ is based on our numerical experience only, we do face the possibility of a too small choice to make the boundary condition too weakly enforced and the whole method unbalanced.

In Tables 1-3, we summarize convergence rates with respect to the smoothness and alignments of the mesh. Here, we assume the lowest order pair $RT_0 \times P_0$ is used.

When the solution is globally smooth, the convergence rates for both the LS norms and $\|u - u_h\|_0$ is of optimal order 1 even with uniform refinements.

When the solution is only piecewisely smooth, if the mesh is aligned with discontinuity, with uniform refinements, the convergence rate for the LS norms is 1, while the rate for $\|u - u_h\|_0$ is between 0.5 and 1. The difference of order between global smooth and piecewise smooth solutions suggests that a mesh-independent norm equivalence between LS norms and the standard norms of $u$ and $\sigma$ does not hold, or at least, we should discuss this equivalence for these two different situations.

When adaptive refinements are used, for non-aligned grids, we can have optimal convergence order 1 in LS norms. For $\|u - u_h\|_0$, we can have order 0.5, which is half-order less than the LS norms.

| Table 1 | Convergence rates for global smooth solutions |
|---------|---------------------------------------------|
| $\|u - u_h\|_0$ | uniform 1 |

| Table 2 | Convergence rates for piecewise smooth solutions - aligned grid |
|---------|---------------------------------------------------------------|
| $\|u - u_h\|_0$ | uniform 1, between 0.5 and 1 |

8. Concluding Remarks. In this paper, several LSFEMs for the linear hyperbolic transport problem are developed based on the flux reformulation of the problem. The new methods can separate two continuity requirements of the solution with the flux in $H(\text{div})$ and the solution in $L^2$. Thus, simple and natural $H(\text{div}) \times L^2$ conforming finite element spaces can be used to approximate the flux and solution. Several variants of the methods are developed to handle the inflow boundary condition strongly or weakly. With the reformulation, the least-squares finite element methods can handle discontinuous solutions much better than the traditional continuous poly-
nominal approximations. With least-squares functionals as natural a posteriori error estimators, the methods can eventually resolve the discontinuity even when the mesh is not aligned with discontinuity by mesh refinements. The smearing and overshooting phenomena are also very mild with adaptive methods. Existence, uniqueness, a priori and a posteriori error estimates are established for the proposed methods. Extensive numerical tests are done to show the effectiveness of the methods developed in the paper.

There are several future research directions. The first is flux-reformulated LSFEMs based on $L^1$-minimization similar to that of [Gue04]. With $L^1$-minimization, the methods have potential to handle the discontinuity better with smaller smearing and overshooting effects. Flux-reformulated LSFEMs based on adaptively weighted $L^2$ norms can also be developed to handle the discontinuity better [Jia98, BG16]. New algorithms are needed to combine the mesh and weight adaptivities.

It is also very natural to generalize the flux-reformulated LSFEMs to neutron transport equations [MR98, MRS00]. With flux reformulation, the methods have potentials to handle rougher solutions.

One of the advantages of the discontinuous Galerkin method is that the system can be solved by successive elimination starting from the inflow boundary, which makes the method semi-explicit, see [RH73, Joh87]. Modifying our methods to develop a similar implementation is an on-going work, and we will apply these methods to the time-dependent problems.

It is always more changeling when apply numerical methods to nonlinear problems. In [DSMM05], flux-reformulated LSFEMs are already suggests for the Burgers equation. But there are many open questions left, for example, how to ensure the numerical solution is the physical meaningful solution, what is the right continuous and discrete space settings, and how to guarantee the existence and uniqueness of the numerical solution? Developing LSFEMs that can answer these questions is also one of our ongoing work.

In [KMM18], $LL^*$-type of least-squares methods are developed for linear hyperbolic equations. In [DHSW12], Petrov-Galerkin methods are developed. These methods have advantages that they try to approximate the solution in the $L^2$-optimal sense. Thus, they have the potential to handle rougher solutions. Applying the similar ideas to flux-reformulations are also possible, and probably can have better results since $H(\text{div})$-norm of the flux contains the information of directional derivatives.

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