EQUISINGULAR DEFORMATIONS OF LEGENDRIAN CURVES

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Abstract. We construct equisingular semiuniversal deformations of Legendrian curves.

1. Introduction

To consider deformations of the parametrization of a Legendrian curve is a good first approach in order to understand Legendrian curves. Unfortunately, this approach cannot be generalized to higher dimensions. On the other hand the obvious definition of deformation has its own problems. First, not all deformations of a Legendrian curve are Legendrian. Second, flat deformations of the conormal of $y^k - x^n = 0$ are all rigid, as we recall in example 5.3, hence there would be too many rigid Legendrian curves.

We pursue here the approach initiated in [3], following the Sophus Lie original approach to contact transformations: to look at [relative] contact transformations as maps that take [deformations of] plane curves into [deformations of] plane curves. We study the category of equisingular deformations of the conormal of a plane curve $Y$ replacing it by an equivalent category $\text{Def}_{\text{es,µ}}^\ast$, a category of equisingular deformations of $Y$ where the isomorphisms do not come only from diffeomorphisms of the plane but also from contact transformations. Here $\mu$ stands for "microlocal", which means "locally" in the cotangent bundle (cf. [9], [10]).

Example 4.4 presents contact transformations that transform a germ of a plane curve $Y$ into the germ of a plane curve $Y^\chi$ such that $Y$ and $Y^\chi$ are not topologically equivalent or are topologically equivalent but not analytically equivalent.

We call a deformation with equisingular plane projection an equisingular deformation of a Legendrian curve. The flatness of the plane projection is a constraint strong enough to avoid the problems related with the use of a naive definition of deformation and loose enough so that we have enough deformations.

In section 6 we use the results of section 5 on equisingular deformations of the parametrization of a Legendrian curve to show that there are semiuniversal equisingular deformations of a Legendrian curve. In particular, we show that the base space of the semiuniversal equisingular deformation is smooth. This argument does not produce a constructive proof of the existence of the semiuniversal deformation in its standard form. In section 7 we
construct a semiuniversal equisingular deformation of a Legendrian curve $L$ when $L$ is the conormal of a Newton non-degenerate plane curve, generalizing the results of [3]. This type of assumption was already necessary when dealing with plane curves (see [6]). This construction is used in [11] to extend the results of [3] and [7], constructing moduli spaces for Legendrian curves that are the conormal of a semi-quasihomogeneous plane curve with a fixed equisingularity class.

In section 2 we recall some basic results on deformations of curves. In sections 3 and 4 we introduce relative contact geometry (see [1], [12] and [13]).

2. Deformations

We will only consider germs of complex spaces, maps and ideals, although sometimes we will chose convenient representatives. We will follow the definitions and notations of [6].

Let $S$ be the germ of a complex space at a point $o$. Let $m_S$ be the maximal ideal of the local ring $O_{S,o}$. Let $T_oS$ be the dual of the vector space $m_S/m_S^2$. Let $X$ be a smooth manifold and $x \in X$. We denote by $i$ the immersion $(S,o) \hookrightarrow (T_oS,0)$.

Let $\widetilde{M}$ be an $O_{T_oS,0}$-module $[\tilde{\alpha}$ be a section of $\widetilde{M}, \tilde{Y}$ be an analytic set of $(T_oS,0)]$. Let $M$ be an $O_{S,o}$-module $[\alpha$ be a section of $M, Y$ be an analytic set of $(S,o)]$. We say that $\widetilde{M} [\tilde{\alpha}, \tilde{Y}]$ is a lifting of $M [\alpha, Y]$ if $i^*\tilde{\alpha} = \alpha, i^*I_{\tilde{Y}} = I_Y$.

Let $Y$ be a reduced analytic set of $(\mathbb{C}^n, 0)$. In order to define a deformation of $Y$ over $S$ we need to choose a section $\sigma$ of the projection $q: \mathbb{C}^n \times S \to S$. We say that a section $\tilde{\sigma} : T_oS \to \mathbb{C}^n \times T_oS$ is a lifting of $\sigma$ if $\tilde{\sigma} \circ i = i \circ \sigma$. Unless we say otherwise we assume $\sigma$ to be trivial. If $S$ is reduced, $\sigma$ is trivial if and only if $\sigma(S) = \{0\} \times S$. In general, $\sigma$ is trivial if and only if it admits a trivial lifting to $T_oS$.

Let $Y$ be an analytic subset of $\mathbb{C}^n \times S$. For each $s \in S$, let $Y_s$ be the fiber of

$$Y \hookrightarrow \mathbb{C}^n \times S \to S.$$ 

Let $i : Y \hookrightarrow Y$ be a morphism of complex spaces that defines an isomorphism of $Y$ into $Y_s$. We say that $Y \hookrightarrow Y$ defines the deformation (1) of $Y$ over $S$ if (1) is flat.

Every deformation is isomorphic to a deformation with trivial section.

Assume that $Y$ is a hypersurface of $\mathbb{C}^n$ and $f$ is a generator of the defining ideal of $Y$. Let $j$ be the immersion $\mathbb{C}^n \to \mathbb{C}^n \times T$ and let $r$ be the projection $\mathbb{C}^n \times T \to \mathbb{C}^n$. There is a generator $F$ of the defining ideal of $Y$ such that $j^*F = f$. We say that $F$ defines a deformation of the equation of $Y$.

Let $Y \hookrightarrow Y_t \hookrightarrow \mathbb{C}^n \times T \to T$ be two deformations of a reduced analytic set $Y$ over $T$. We say that an isomorphism $\chi : \mathbb{C}^n \times T \to \mathbb{C}^n \times T$ is an
isomorphism of deformations if \( q \circ \chi = q, \ r \circ \chi \circ j = id_{C^n} \) and \( \chi \) induces an isomorphism from \( \mathcal{Y}_1 \) onto \( \mathcal{Y}_2 \).

Given a morphism of complex spaces \( f : S \to T \) and a deformation \( \mathcal{Y} \) of \( Y \) over \( T \), \( f^* \mathcal{Y} = S \times_T \mathcal{Y} \) defines a deformation of \( Y \) over \( S \).

We say that a deformation \( \mathcal{Y} \) of \( Y \) over \( T \) is a **versal deformation** of \( Y \) if given

- a closed embedding of complex space germs \( f : T'' \hookrightarrow T' \),
- a morphism \( g : T'' \to T \),
- a deformation \( \mathcal{Y}' \) of \( Y \) over \( T' \) such that \( f^* \mathcal{Y}' \cong g^* \mathcal{Y} \),

there is a morphism of complex analytic space germs \( h : T' \to T \) such that

\[
 h \circ f = g \quad \text{and} \quad h^* \mathcal{Y} \cong \mathcal{Y}'.
\]

If \( \mathcal{Y} \) is versal and for each \( \mathcal{Y}' \) the tangent map \( T(h) : T_{Y'} \to T_T \) is determined by \( \mathcal{Y}' \), \( \mathcal{Y} \) is called a **semian universal deformation** of \( Y \).

We will now introduce deformations of a parametrization.

Assume the curve \( Y \) has irreducible components \( Y_1, \ldots, Y_r \). Set \( \bar{C} = \bigsqcup_{i=1}^r \bar{C}_i \) where each \( \bar{C}_i \) is a copy of \( C \). Let \( \varphi_i \) be a parametrization of \( Y_i \), \( 1 \leq i \leq r \). The map \( \varphi : \bar{C} \to C^n \) such that \( \varphi|_{\bar{C}_i} = \varphi_i, 1 \leq i \leq r \) is called a **parametrization** of \( Y \).

Let \( \iota_n \) denote the inclusions \( \bar{C} \hookrightarrow \bar{C} \times T, C^n \hookrightarrow C^n \times T \). Let \( \overline{q} \) denote the projection \( \bar{C} \times T \to T \). We say that a morphism of complex spaces \( \Phi : \bar{C} \times T \to C^n \times T \) is a deformation of \( \varphi \) over \( T \) if \( \iota_n \circ \varphi = \Phi \circ \iota \) and \( q_n \circ \Phi = \overline{q} \).

We denote by \( \Phi_i \) the composition \( \bar{C}_i \times T \hookrightarrow \bar{C} \times T \to C^n \times T \to C^n, 1 \leq i \leq r \). The maps \( \Phi_i, 1 \leq i \leq r \), determine \( \Phi \). Let \( \Phi \) be a deformation of \( \varphi \) over \( T \). Let \( f : S \to T \) be a morphism of complex spaces. We denote by \( f^* \Phi \) the deformation of \( \varphi \) over \( S \) given by

\[
 (f^* \Phi)_i = \Phi_i \circ (id_{\bar{C}_i} \times f).
\]

Let \( \Phi' : \bar{C} \times T \to C^n \times T \) be another deformation of \( \varphi \) over \( T \). A morphism from \( \Phi' \) into \( \Phi \) is a pair \( (\chi, \xi) \) where \( \chi : C^n \times T \to C^n \times T \) and \( \xi : \bar{C} \times T \to \bar{C} \times T \) are isomorphisms of complex spaces such that the diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\chi} & \bar{C} \times T & \xrightarrow{\Phi} & C^n \times T & \xrightarrow{id_T} & T \\
\downarrow{id_T} & & \downarrow{\varphi} & & \downarrow{\varphi} & & \downarrow{id_T} \\
T & \xrightarrow{\Phi'} & \bar{C} \times T & \xrightarrow{\Phi'} & C^n \times T & \xrightarrow{id_T} & T
\end{array}
\]

commutes.

Let \( \Phi' \) be a deformation of \( \varphi \) over \( S \) and \( f : S \to T \) a morphism of complex spaces. A morphism of \( \Phi' \) into \( \Phi \) over \( f \) is a morphism from \( \Phi' \) into \( f^* \Phi \).
There is a functor $p$ that associates $T$ to a deformation $\Psi$ over $T$ and $f$ to a morphism of deformations over $f$.

Given a parametrization $\varphi$ of a plane curve $Y$ and a deformation $\Phi$ of $\varphi$, $\Phi$ is the parametrization of a hypersurface $\mathcal{Y}$ of $\mathbb{C}^2 \times T$ that defines a deformation of (the equation of) $Y$.

Let $Y, Z$ be two germs of plane curves of $(\mathbb{C}^2, 0)$.

**Definition 2.1.** Two plane curves $Y, Z$ are *equisingular* if there are neighborhoods $V, W$ of 0 and an homeomorphism $\varphi : V \to W$ such that $\varphi(Y \cap V) = Z \cap W$.

**Theorem 2.2.** Let $(Y_i)_{i \in I} \cup (Z_j)_{j \in J}$ be the set of branches $Y[Z]$. The curves $Y_i, Z_j$ are equisingular if and only if there is a bijection $\varphi : I \to J$ such that $Y_i$ and $Z_{\varphi(i)}$ have the same Puiseux exponents for each $i \in I$ and the contact orders $o(Y_i, Y_j), o(Z_{\varphi(i)}, Z_{\varphi(j)})$ are equal, for each $i, j \in I, i \neq j$.

The definition of *equisingular deformation* of the parametrization [equation] of a plane curve over a complex space is very long and technical. We will omit it. See definitions 2.36 and 2.6 of [6]. We will now present the main properties of equisingular deformations, which characterize them completely.

**Theorem 2.3.** (Theorem 2.64 of [6]) Let $Y$ be a reduced plane curve. Let $\varphi$ be a parametrization of $Y$. Let $f$ be an equation of $Y$. Every equisingular deformation of $\varphi$ induces a unique equisingular deformation of $f$. Every equisingular deformation of $f$ comes from a deformation of $\varphi$.

**Theorem 2.4.** (Corollary 2.68 of [6]) A deformation of the equation of a reduced plane curve $Y$ over a reduced complex space is equisingular if and only if the topology of the fibers does not change.

**Theorem 2.5.** Let $S \hookrightarrow (\mathbb{C}^k, 0)$ be an immersion of complex spaces. Let $\varphi$ be a parametrization of a reduced plane curve. A deformation of $\varphi$ over $S$ is equisingular if and only if it admits a lifting to an equisingular deformation of $\varphi$ over $(\mathbb{C}^k, 0)$.

**Proof.** It follows from Theorem 2.38 of [6].

**Proposition 2.6.** (Proposition 2.11 of [6]) Assume $f_1, ..., f_\ell$ define germs of reduced irreducible curves of $(\mathbb{C}^2, 0)$ and $F$ defines an equisingular deformation over a germ of complex space $S$ of the curve defined by $f_1 \cdots f_\ell$. Then $F = F_1 \cdots F_\ell$, where each $F_i$ defines an equisingular deformation of $f_i$ over $S$.

### 3. Relative contact geometry

We usually identify a subset of $\mathbb{P}^{n-1}$ with a conic subset of $\mathbb{C}^n$. Given a manifold $M$ we will also identify a subset of the projective cotangent bundle $\mathbb{P}^*M$ with a conic subset of the cotangent bundle $T^*M$ (for the canonical $\mathbb{C}^*$-action of $T^*M$).
Let \( q : X \to S \) be a morphism of complex spaces. Let \( p_i, i = 1, 2 \) be the canonical projections from \( X \times_S X \) to \( X \). Let \( \Delta \) denote the diagonal of \( X \to X \times_S X \) and the diagonal immersion \( X \hookrightarrow X \times_S X \). Let \( I_\Delta \) be the defining ideal of the diagonal of \( X \times_S X \). We say that the coherent \( \mathcal{O}_X \)-module \( \Omega^1_{X/S} = \Delta^*(I_\Delta/I^2_\Delta) \) is the sheaf of relative differential forms of \( X \to S \) (see [3]).

Given a local section \( f \) of \( \mathcal{O}_X \) set \( f_i = f \circ p_i, i = 1, 2 \). Consider the morphism \( d : \mathcal{O}_X \to \Omega^1_{X/S} \) given by

\[
    f \mapsto f_1 - f_2 \mod I^2_\Delta.
\]

Notice that, given an open set \( U \) of \( X \) and \( f, g \in \mathcal{O}_X(U) \), \( \varphi \in q^{-1}\mathcal{O}_S \),

\[
    d(fg) = fdg + gdf, \quad \text{and} \quad d(\varphi f) = \varphi df
\]

If \( x_1, \ldots, x_n \in \mathcal{O}_X(U) \) are such that \( \Omega^1_{X/S}|_U \cong \bigoplus_{i=1}^n \mathcal{O}_U dx_i \), we say that \( (x_1, \ldots, x_n) \) is a partial system of local coordinates on \( U \) of \( X \to S \).

Notice that \( (x_1, \ldots, x_n) \) is a partial system of local coordinates of \( X \to S \) on \( U \) if and only if \( \Omega^1_{X/S}|_U = \mathcal{O}_U dx_1 \wedge \cdots \wedge dx_n \).

If \( (x^1, \ldots, x^n) \) is a partial system of local coordinates on \( U \) of \( X \to S \), \( x^i - x^j, i = 1, \ldots, n \), generate \( I_\Delta|_U \). Given \( f \in \mathcal{O}_X(U) \), there are \( a_i \in \mathcal{O}_X(U) \) such that \( df = \sum_{i=1}^n a_i dx^i \). We set

\[
    \frac{\partial f}{\partial x^i} = a_i, \quad i = 1, \ldots, n.
\]

When \( M, S \) are manifolds, \( X = M \times S \) and \( q \) is the projection \( M \times S \to S \) this definition of partial derivative coincides with the usual one because of [3]. When \( S \) is a point, \( \Omega^1_{X/S} \) equals the sheaf of differential forms \( \Omega^1_X \).

If \( \Omega^1_{X/S} \) is a locally free \( \mathcal{O}_X \)-module, we denote by \( \pi = \pi_{X/S} : T^*(X/S) \to X \) the vector bundle with sheaf of sections \( \Omega^1_{X/S} \). Whenever it is reasonable we will write \( \pi \) instead of \( \tau_{X/S} \). We denote by \( \tau_{X/S} : T(X/S) \to X \) the dual vector bundle of \( T^*(X/S) \). We say that \( T(X/S) \) \( [T^*(X/S)] \) is the relative tangent bundle [cotangent bundle] of \( X \to S \).

Let \( \varphi : X_1 \to X_2 \), \( q_1 \circ q_1 \to S \) be morphisms of complex spaces such that \( q_2 \varphi = q_1 \). Let \( \Delta_i : X_i \to X_i \times_S X_i \) be the diagonal map, \( i = 1, 2 \). If we denote by \( \varphi_S \) the canonical map from \( X_1 \times_S X_1 \) to \( X_2 \times_S X_2 \), \( \varphi_S : I_{\Delta_2} \to I_{\Delta_1} \) induces a morphism \( \varphi^* : \Omega^1_{X_2/S} \to \Omega^1_{X_1/S} \) that generalizes the pullback of differential forms. Moreover, \( \varphi^* \) induces a morphism of \( \mathcal{O}_X \)-modules

\[
    \tilde{\rho}_\varphi : \varphi^* \Omega^1_{X_2/S} = \mathcal{O}_{X_1} \otimes_{\varphi^{-1}\mathcal{O}_{X_2}} \varphi^{-1}\Omega^1_{X_2/S} \to \Omega^1_{X_1/S}.
\]

If \( \Omega^1_{X_i/S}, i = 1, 2 \), and the kernel and cokernel of \( \tilde{\rho}_\varphi \) are locally free, we have a morphism of vector bundles

\[
    \rho_\varphi : X_1 \times X_2, T^*(X_2/S) \to T^*(X_1/S).
\]

If \( \varphi \) is an inclusion map, we say that the kernel of \( \tilde{\rho}_\varphi \) and its projectivization, are the conormal bundle of \( X_1 \) relative to \( S \). We will denote by
$T^*_X(X/S)$ or $\mathbb{P}^*_X(X/S)$ the conormal bundle of $X$ relative to $S$. We denote by

$$\varpi_\varphi : T(X_1/S) \to X_1 \times X_2 T(X_2/S)$$

the dual morphism of $\rho_\varphi$. We say that $\varpi_\varphi$ is the relative tangent morphism of $\varphi$ over $S$. These are straightforward generalizations of the constructions of [9].

If $(x_1, ..., x_n)$ is a partial system of local coordinates of $X \to Y$ and $(y_1, ..., y_m)$ is a system of local coordinates of a manifold $Y$, $(x_1, ..., x_n, y_1, ..., y_m)$ is a partial system of local coordinates of $X \times Y \to X \to S$. Hence $\Omega^1_{X/S}$ locally free implies $\Omega^1_{X \times Y/S}$ locally free. Moreover, if $\Omega^1_{X/S}$ is locally free and $E \to X$ is a vector bundle, $\Omega^1_{E/S}$ is locally free.

Let $(x_1, ..., x_n)$ be a partial system of local coordinates of $X \to Y$ on an open set $U$ of $X$. Set $V = \pi^{-1}_X(U)$. There are $\xi_1, ..., \xi_n \in \mathcal{O}_{T^*(X/S)}(V)$ such that, for each $\sigma \in V$,

$$\sigma = \sum_{i=1}^n \xi_i(\sigma) dx_i.$$

Notice that $(x_1, ..., x_n, \xi_1, ..., \xi_n)$ is a partial system of local coordinates of $T^*(X/S) \to S$. Let $o \in X$, $u \in T^*_o T^*(X/S)$. Let

$$\varpi_\pi(\sigma) : T_\sigma(T^*(X/S))/S \to T_o(X/S)$$

be the relative tangent morphism of $\pi$ over $S$ at $\sigma$. There is one and only one $\theta \in \Omega^1_{T^*(X/S)/S}$ such that,

$$\theta(\sigma)(u) = \sigma(\varpi_\pi(\sigma)(u)),$$

for each $o \in X$, each $\sigma \in T^*_o(T^*(X/S))$ and each $u \in T_\sigma(T^*(X/S)/S)$. Given a partial system of local coordinates $(x_1, ..., x_n)$ of $X \to S$ on an open set $U$,

$$\theta|_{\pi^{-1}(U)} = \sum_{i=1}^n \xi_i dx_i.$$

We say that $\theta_{X/S} = \theta$ is the canonical 1-form of $T^*(X/S)$.

Notice that $(d\theta)(\sigma)$ is a symplectic form of $T_\sigma(T^*(X/S)/S)$, for each $\sigma \in T^*(X/S)$. We say that $(x_1, ..., x_n, \xi_1, ..., \xi_n)$ is a partial system of symplectic coordinates of $T^*(X/S)$ (associated to $(x_1, ..., x_n)$).

Assume $M$ is a manifold. When $q$ is the projection $M \times S \to S$ we will replace "$M \times S/S$" by "$M[S]$". Let $r$ be the projection $M \times S \to M$. Notice that $\Omega^1_{M[S]} \to \mathcal{O}_{M \times S} \otimes_{r^{-1} \mathcal{O}_M} r^{-1} \Omega^1_M$ is a locally free $\mathcal{O}_{M \times S}$-module. Moreover, $T^*(M[S]) = T^*M \times_M (M \times S)$. If $i$ is the inclusion $T^*(M[S]) \hookrightarrow T^*(M \times S)$, $i^* \theta_{M \times S} = \theta_{M[S]}$. A system of local coordinates of $M$ is a partial system of local coordinates of $M \times S \to S$.

We say that $\Omega^1_{M[S]}$ is the sheaf of relative differential forms of $M$ over $S$. We say that $T^*(M[S])$ is the relative cotangent bundle of $M$ over $S$.

Let $N$ be a complex manifold of dimension $2n - 1$. Let $S$ be a complex space. We say that a section $\omega$ of $\Omega^1_{N[S]}$ is a relative contact form of $N$ over $S$ if $\omega \wedge d\omega^{n-1}$ is a local generator of $\Omega^2_{N[S]}$. Let $\mathcal{C}$ be a locally free subsheaf of $\Omega^1_{N[S]}$. We say that $\mathcal{C}$ is a structure of relative contact manifold on $N$ over
$S$ if $\mathcal{C}$ is locally generated by a relative contact form of $N$ over $S$. We say that $(N \times S, \mathcal{C})$ is a relative contact manifold over $S$. When $S$ is a point we obtain the usual notion of contact manifold.

Let $(N_1 \times S, \mathcal{C}_1), (N_2 \times S, \mathcal{C}_2)$ be relative contact manifolds over $S$. Let $\chi$ be a morphism from $N_1 \times S$ into $N_2 \times S$ such that $q_{N_2} \circ \chi = q_{N_1}$. We say that $\chi$ is a relative contact transformation of $(N_1 \times S, \mathcal{C}_1)$ into $(N_2 \times S, \mathcal{C}_2)$ if the pull-back by $\chi$ of each local generator of $\mathcal{C}_2$ is a local generator of $\mathcal{C}_1$.

We say that the projectivization $\pi_{X/S} : \mathbb{P}^{*}(X/S) \to X$ of the vector bundle $T^{*}(X/S)$ is the projective cotangent bundle of $X \to S$.

Let $(x_1, ..., x_n)$ be a partial system of local coordinates on an open set $U$ of $X$. Let $(x_1, ..., x_n, \xi_1, ..., \xi_n)$ be the associated partial system of symplectic coordinates of $T^{*}(X/S)$ on $V = \pi^{-1}(U)$. Set $p_{i,j} = \xi_i \xi_j^{-1}$, $i \neq j$,

$$V_i = \{(x, \xi) \in V : \xi_i \neq 0\}, \quad \omega_i = \xi_i^{-1}\theta, \quad i = 1, ..., n.$$

each $\omega_i$ defines a relative contact form $dx_j - \sum_{i \neq j} p_{i,j} dx_i$ on $\mathbb{P}^{*}(X/S)$, endowing $\mathbb{P}^{*}(X/S)$ with a structure of relative contact manifold over $S$.

Let $\omega$ be a germ at $(x, o)$ of a relative contact form of $\mathcal{C}$. A lifting $\tilde{\omega}$ of $\omega$ defines a germ $\tilde{\mathcal{C}}$ of a relative contact structure of $N \times T_oS \to T_oS$. Moreover, $\tilde{\mathcal{C}}$ is a lifting of the germ of $\omega$ at $o$ of $\mathcal{C}$.

Let $(N \times S, \mathcal{C})$ be a relative contact manifold over a complex manifold $S$. Assume $N$ has dimension $2n - 1$ and $S$ has dimension $\ell$. Let $\mathcal{L}$ be a reduced analytic set of $N \times S$ of pure dimension $n + \ell - 1$. We say that $\mathcal{L}$ is a relative Legendrian variety of $N \times S$ over $S$ if for each section $\omega$ of $\mathcal{C}$, $\omega$ vanishes on the regular part of $\mathcal{L}$. When $S$ is a point, we say that $\mathcal{L}$ is a Legendrian variety of $N$.

Let $\mathcal{L}$ be an analytic set of $N \times S$. Let $(x, o) \in \mathcal{L}$. Assume $S$ is an irreducible germ of a complex space at $o$. We say that $\mathcal{L}$ is a relative Legendrian variety of $N$ over $S$ at $(x, o)$ if there is a relative Legendrian variety $\tilde{\mathcal{L}}$ of $(N, x)$ over $(T_oS, 0)$ that is a lifting of the germ of $\mathcal{L}$ at $(x, o)$. Assume $S$ is a germ of a complex space at $o$ with irreducible components $S_i$, $i \in I$. We say that $\mathcal{L}$ is a relative Legendrian variety of $N$ over $S$ at $(x, o)$ if $S_i \times S \mathcal{L}$ is a relative Legendrian variety of $S_i \times S N$ over $S_i$ at $(x, o)$, for each $i \in I$.

We say that $\mathcal{L}$ is a relative Legendrian variety of $N \times S$ if $\mathcal{L}$ is a relative Legendrian variety of $N \times S$ at each $(x, o) \in \mathcal{L}$.

The main problem of defining relative Legendrian variety over a complex space $S$ comes from the fact that $S$ does not have to be pure dimensional, hence we cannot assign a pure dimension to the Legendrian variety.

**Lemma 3.1.** Let $\chi$ be a relative contact transformation from $(N_1 \times S, \mathcal{C}_1)$ into $(N_2 \times S, \mathcal{C}_2)$. Let $\mathcal{L}_1$ be a relative Legendrian curve of $(N_1 \times S, \mathcal{C}_1)$. If $\mathcal{L}_2$ is the analytic subset of $N_2 \times S$ defined by the pull back by $\chi^{-1}$ of the defining ideal of $\mathcal{L}_1$, $\mathcal{L}_2$ is a relative Legendrian variety of $(N_2 \times S, \mathcal{C}_2)$.

**Proof.** Let $\chi : (N_1 \times S, \mathcal{C}_1) \to (N_2 \times S, \mathcal{C}_2)$ be a relative contact transformation over $S$. Let $(x_1, o)$ be a point of $N_1 \times S$. Set $(x_2, o) = \chi(x_1, o)$. There
is a morphism of germs of complex spaces
\[ \tilde{\chi} : (N_1 \times T_o S, (x_1, o)) \to (N_2 \times T_o S, (x_2, o)) \]
such that \( \tilde{\chi} \circ \iota_{N_1} = \iota_{N_2} \circ \chi \). We say that such a morphism is a lifting of \( \chi \). Let \( \tilde{\mathcal{C}}_2 \) be a lifting of \( \mathcal{C}_2 \) at \((x_2, o)\). Then \( \tilde{\mathcal{C}}_1 = \tilde{\chi}^* \tilde{\mathcal{C}}_2 \) is a lifting of \( \mathcal{C}_1 \) at \((x_1, o)\). Moreover, \( \tilde{\chi} \) is a germ of a relative contact transformation.

Let \( \mathcal{L}_1 \) be a germ of a relative Legendrian variety at \((x_1, o)\). There is a lifting \( \tilde{\mathcal{L}}_1 \) of \( \mathcal{L}_1 \) that is a germ of relative Legendrian variety of \( N_1 \times T_o S \). Hence \( \tilde{\chi}(\mathcal{L}_1) \) is a germ of a relative Legendrian variety of \( N_2 \times T_o S \) and \( \tilde{\chi}(\mathcal{L}_1) \) is a lifting of \( \mathcal{L}_2 \) at \((x_2, o)\).

Let \( Y \) be a reduced analytic set of \( M \). Let \( \mathcal{Y} \) be a flat deformation of \( Y \) over \( S \). Set \( X = M \times S \setminus \mathcal{Y}_{\text{sing}} \). We say that the Zariski closure of \( \mathbb{P}^*_X(X/S) \) in \( \mathbb{P}^*(M|S) \) is the conormal \( \mathbb{P}^*_X(M|S) \) of \( \mathcal{Y} \) over \( S \).

**Theorem 3.2.** The conormal of \( \mathcal{Y} \) over \( S \) is a relative Legendrian variety of \( \mathbb{P}^*(M|S) \). If \( \mathcal{Y} \) has irreducible components \( \mathcal{Y}_1, ..., \mathcal{Y}_r \),

\[ \mathbb{P}^*_X(M|S) = \bigcup_{i=1}^r \mathbb{P}^*_X(\mathcal{Y}_i|S) \]

*Proof.* We have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y}_{\text{reg}} & \xleftarrow{i} & X \\
\downarrow{\Delta_{\mathcal{Y}_{\text{reg}}}} & & \downarrow{\Delta_X} \\
\mathcal{Y}_{\text{reg}} \times_S \mathcal{Y}_{\text{reg}} & \xrightarrow{j} & X \times_S X
\end{array}
\]

Since \( I_{\Delta_{\mathcal{Y}_{\text{reg}}}} = j^* (I_{\Delta_X} + I_{Y_{\text{reg}} \times_S Y_{\text{reg}}}/I_{Y_{\text{reg}} \times_S Y_{\text{reg}}}) \),

\[ \Delta_{\mathcal{Y}_{\text{reg}}}^* (I_{\Delta_X}/I_{\mathcal{Y}_{\text{reg}}}) \sim i^* \Delta_{\mathcal{X}}^* (I_{\Delta_X} + I_{Y_{\text{reg}} \times_S Y_{\text{reg}}}/(I_{\Delta_X}^2 + I_{Y_{\text{reg}} \times_S Y_{\text{reg}}})) \]

Let \((x, o) \in \mathcal{Y}_{\text{reg}}\). Let \( \overline{m} \) be the ideal of \( \mathcal{O}_{M \times S, (x, o)} \) generated by \( m_o \). Let \((y_1, ..., y_n)\) be a system of local coordinates of \((M, x)\) such that \( I_{Y, x} = (y_{k+1}, ..., y_n) \). There are \( F_j \in \mathcal{O}_{M \times S, (x, o)} \), \( j = k + 1, ..., n \) such that \( I_{Y, (x, o)} = (F_{k+1}, ..., F_n) \) and \( F_j - y_j \in \overline{m} \). Let \( i = 1, ..., k \), \( x^i = F_i \), \( i = k + 1, ..., n \). Notice that \((x^1, ..., x^n)\) is a partial system of local coordinates of \( X \to S \). Since near \((x, o)\)

\[ I_{\Delta_X} = (x_1 - x_2, ..., x_{n-1} - x_{n-2}) \quad \text{and} \quad I_{Y \times S Y} = (x_1^k + 1, ..., x_1, x_2^k, ..., x_n) \]

it follows from \( [7] \) that \( dx^1, ..., dx^k \) is a local basis of \( \Omega^1_{\mathcal{Y}/S} \), \( dx^1, ..., dx^n \) is a local basis of \( \Omega^*_{\mathcal{Y}/S} \),

\[ \hat{\rho}_j (dx^j) = dx^j, \quad j = 1, ..., k \quad \text{and} \quad \hat{\rho}_j (dx^j) = 0, \quad j = k + 1, ..., n. \]
Hence the kernel of \( \hat{\rho}_i \) at \((x,o)\) equals \( \oplus_{j=k+1}^n \mathbb{C}\{x^1, ..., x^k\}dx^j \). Given the partial system of symplectic coordinates \((x^1, ..., x^n, \xi^1, ..., \xi^n)\), the ideal of the kernel of

\[
\rho_i : \mathcal{Y}_{reg} \times_X T^* (X/S) \to T^* (\mathcal{Y}_{reg}/S)
\]

is generated by \(x^{k+1}, ..., x^n, \xi^1, ..., \xi^k\).

It is enough to prove the second statement when \(S\) is smooth. Its proof relies on the fact that each connected component of \( \mathcal{Y} \) is dense in one of the irreducible components of \( \mathcal{Y} \). \( \Box \)

Let \(q : X \to S\) be a morphism of complex spaces. Let \(y \in Y \subset X\). We say that \(Y\) is a submanifold of \(X \to S\) at \(y\) if there is a partial system of local coordinates \((x_1, ..., x_n)\) of \(X \to S\) near \(y\) and \(1 \leq k \leq n\) such that \(Y = \{x_1 = \cdots = x_k = 0\}\) near \(y\). We say that \(Y\) is a submanifold of \(X \to S\) if \(Y\) is a submanifold of \(X \to S\) at \(y\) for each \(y \in Y\).

Notice that a submanifold of \(X \to S\) is not necessarily a manifold, although it behaves like one in several ways.

Let \(Y \subset X\). Let \(\gamma : \Delta_\varepsilon = \{t \in \mathbb{C} : |t| < \varepsilon\} \to Y\) be a holomorphic curve such that \(\gamma (0) = y\). We associate to \(\gamma\) a tangent vector \(u\) of \(Y\) at \(y\) setting \(u \cdot f = (f \circ \gamma)'(0)\), for each \(f \in \mathcal{O}_{X,y}\). We associate to \(\gamma\) an element \(u\) of \(T_y (X/S)\) setting

\[
(8) \quad u \cdot f = df(y)(\gamma'(0)), \quad f \in \mathcal{O}_{X,y}.
\]

If \(Y\) is a submanifold of \(X \to S\) the set of relative vector fields \(\mathfrak{s}\) define a linear subspace \(T_y (Y/S)\) of \(T_y (X/S)\).

Let us fix a point \(o\) of \(S\). Consider the canonical maps

\[
T^* M \overset{i^*}{\to} T^* (M|S) = (T^* M) \times S \overset{r_*}{\to} T^* M.
\]

Since \(T_\sigma (T^* (M|S)/S) = T_{r(\sigma)} T^* M\) and

\[
(d\theta_{M|S})(\sigma) = (i^* d\theta_M)(r(\sigma)),
\]

\((d\theta_{M|S})(\sigma)\) is a symplectic form of \(T_\sigma (T^* (M|S)/S)\).

The Poisson bracket of \((T^* M)\) induces a Poisson bracket in \(T^* (M|S)\). Let \(f \in \mathcal{O}_{T^* (M|S)}\). Setting \(f_\tau(x,\xi) = f(x,\xi, s)\)

\[
\{f, g\}_{T^* (M|S)}(x,\xi, s) = \{f_\tau, g_\tau\} T^* M(x,\xi).
\]

Let \(W\) be a submanifold of \(T^* (M|S)\). Setting \(W_\tau = \{(x,\xi) \in T^* M : (x,\xi, s) \in W\}\), \(W\) is an involutive submanifold of \(T^* (M|S)\) if and only if \(W_\tau\) is an involutive submanifold of \(T^* M\) for each \(s \in S\). It is well known that \(W_\tau\) is an involutive submanifold of \(T^* M\) if and only if \(T_\sigma W_\tau\) is an involutive linear subspace of \(T_\sigma T^* M\) for each \(\sigma \in W_\tau\).

**Lemma 3.3.** Let \(\mathcal{L}\) be a conic submanifold of \(T^* (M|S)\). The manifold \(\mathcal{L}\) is a Legendrian submanifold of \(\mathbb{P}^* (M|S)\) if and only if \(T_\sigma (\mathcal{L}/S)\) is a linear Lagrangian subspace of \(T_\sigma (T^* (M|S)/S)\) for each \(\sigma \in \mathcal{L}\).
Proof. The submanifold \( W \) considered above is an involutive submanifold of \( T^* (M | S) \) if and only if \( T_\sigma (W | S) \) is a linear involutive subspace of \( T_\sigma (T^* (M | S) | S) \) for each \( \sigma \in W \). The result follows from an argument of dimension. □

**Theorem 3.4.** Let \( \mathcal{L} \) be an irreducible germ of a relative Legendrian analytic set of \( \mathbb{P}^* (M | S) \). If the analytic set \( \pi (\mathcal{L}) \) is a flat deformation over \( S \) of an analytic set of \( M \), \( \mathcal{L} = \mathbb{P}^*_{\pi (\mathcal{L})} (M | S) \).

**Proof.** There is \( s \in S \) such that \( Y \times \{ s \} \subset Y \). Let \( o \) be a smooth point of \( Y \). There is an open set \( U \) of \( Y \) and a system of local coordinates \((y_1, \ldots, y_n)\) on \( U \) such that \( Y \cap U = \{ y_1 = \cdots = y_k = 0 \} \). Since \( Y \) is flat, there is a neighborhood \( V \) of \( s \) and a system of partial symplectic coordinates \((x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)\) on \( \pi^{-1} (U \times V) \) such that

\[
\pi (\mathcal{L}) \cap U \times V = \{ x_1 = \cdots = x_k = 0 \}.
\]

Repeating the argument of Lemma 3.3

\[
\mathcal{L} \cap \pi^{-1} (\pi (\mathcal{L})_{\text{reg}}) = \mathbb{P}^*_{\pi (\mathcal{L})_{\text{reg}}} (M \times S \setminus Y_{\text{sing}} | S).
\]

Since \( \mathcal{L} \) is closed \( \mathbb{P}^*_{\pi (\mathcal{L})} (M | S) \subset \mathcal{L} \). Since \( \mathcal{L} \) is irreducible and both spaces have the same dimension, the inclusion is an equality. □

We present now an alternative construction of the conormal of a flat deformation of a hypersurface. This construction is more suitable to compute the conormal using computer algebra methods. For this purpose it is enough to consider the case where \( S \) is smooth.

Let \( F \) be a generator of the defining ideal of \( Y \). Let \( \mathfrak{I}_{F,(x_i)} \) be the ideal of \( \mathbb{C} \{ c, x, \xi, s \} \) generated by

\[
F, \, \xi_i - cF_{x_i}, \quad i = 1, \ldots, n.
\]

The ideal

\[
\mathfrak{R}_{F,(x_i)} = \mathfrak{I}_{F,(x_i)} \cap \mathbb{C} \{ x, \xi, s \},
\]

defines a conic analytic subset of \( T^* M \times S \), hence it also defines an analytic subset \( \text{Con}_S Y \) of \( \mathbb{P}^* (M | S) \).

**Lemma 3.5.** The ideal \( \mathfrak{R}_{F,(x_i)} \) does not depend on the choice of \( F \) or \( (x_i) \).

**Proof.** Given another system of local coordinates \((y_i)\) there are function \( \eta_i \) such that \( \sum_i \eta_i dy_i = \sum_i \xi_i dx_i \). Since

\[
\sum_i \eta_i dy_i = \sum_i \eta_j \partial_j \sum_i \frac{\partial y_i}{\partial x_j} dx_j = \sum_j \partial_j \sum_i \frac{\partial y_i}{\partial x_j} \eta_i dx_j,
\]

\[
\xi_j - cF_{x_j} = \sum_j \partial_j \sum_i \frac{\partial y_i}{\partial x_j} \eta_i - c \sum_i F_{y_i} \partial_i dx_j = \sum_i \frac{\partial y_i}{\partial x_j} (\eta_i - cF_{y_i}).
\]

Since the Jacobian matrix of the coordinate change is invertible, \( \mathfrak{I}_{F,(x_i)} \) does not depend on \( (x_i) \).

Assume that \( \varphi \) does not vanish. Since \( \xi_i - c(\varphi F)_{x_i} = \xi_i - c\varphi F_{x_i} - cF_{\varphi x_i} \), \( \mathfrak{I}_{\varphi F} \) is generated by

\[
F, \, \xi_i' - cF_{x_i}, \quad i = 1, \ldots, n,
\]

where \( \xi_i' = \varphi^{-1} \xi_i, \, i = 1, \ldots, n \).
Consider the actions of $\mathbb{C}^*$ into $T^*M \times S \times \mathbb{C}$ and $T^*M \times S$ given by

\[ t \cdot ((x_i), (\xi_i), (s_j), c) = ((tx_i), (t\xi_i), (ts_j),tc), \]

\[ t \cdot ((x_i), (\xi_i), (s_j)) = ((tx_i), (t\xi_i), (ts_j)). \]

By (9), the ideals $\mathcal{J}_F [\mathcal{R}_F]$ are generated by homogeneous polynomials on $\xi_1, \ldots, \xi_n, c [\xi_1, \ldots, \xi_n]$. Assume that $\mathcal{R}_F$ is generated by the homogeneous polynomials

\[ P_k(\xi_1, \ldots, \xi_n), k = 1, \ldots, m. \]

It follows from (9) and (10) that $\mathcal{R}_F$ is generated by $P_k(\xi'_1, \ldots, \xi'_n), k = 1, \ldots, m$. If $P_k$ is homogeneous of degree $d_k$, $P_k(\xi'_1, \ldots, \xi'_n) = c^{-d_k} P_k(\xi_1, \ldots, \xi_n)$. Hence $\mathcal{R}_F = \mathcal{R}_F[c^d]$. \hfill \(\square\)

**Theorem 3.6.** If $\mathcal{Y}$ is a flat deformation over $S$ of a hypersurface of $M$, $\mathbb{P}^*(M/S) = \text{Con}_S \mathcal{Y}$. 

**Proof.** If $\mathcal{Y}$ is non singular at a point $o$, there is a partial system of symplectic coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ such that $F = x_1$ in a neighborhood $U$ of $o$. Hence $\mathcal{J}_{F(x_1)}$ is generated by

\[ \xi_1 - c, \xi_2, \ldots, \xi_n, x_1. \]

Therefore $\mathcal{R}_{F(x_1)}$ is generated by $x_1, \xi_2, \ldots, \xi_n$. Hence $\mathbb{P}^*(M/S) = \text{Con}_S \mathcal{Y}$ in $\pi^{-1}(U)$. Therefore $\text{Con}_S \mathcal{Y}$ contains $\mathbb{P}^*(M/S)$. Assume that there is an irreducible component $\Gamma$ of $\text{Con}_S \mathcal{Y}$ that is not contained in $\mathbb{P}^*(M/S)$. Then $\Gamma$ is contained in $\mathcal{Y}_{\text{sing}} \times M \times S$. Hence the set of zeros of $\mathcal{J}_{f,(x_i)}$ contains points of

\[ \mathcal{Y}_{\text{sing}} \times M \times S \times T^* M \times S \times \mathbb{C} \setminus M \times S \times \mathbb{C}. \]

But it follows from (9) that the intersection of the set of zeros of $\mathcal{J}_{F(x_1)}$ with $\mathcal{Y}_{\text{sing}} \times M \times S \times T^* M \times S \times \mathbb{C}$ is contained in $M \times S \times \mathbb{C}$. \hfill \(\square\)

The following Singular routine (see [4]) computes the relative conormal of the hypersurface defined by $z^2 + y^3 + sx^4$ when we assume $\theta = udx + vdy + wdz$ and we look at $s$ as a deformation parameter.

```singular
ring r=0,(c,u,v,w,x,y,z,s),dp;
poly F=z2+y3+sx4;
ideal I=F,u-c*diff(F,x),v-c*diff(F,y),w-c*diff(F,z);
ideal J=eliminate(I,c);
J;
```

If we consider the suitable contact coordinates and choose a different ordering we can reduce substantially the number of equations we obtain.

Let $T_\varepsilon$ be the complex space with local ring $\mathbb{C}\{\varepsilon\}/(\varepsilon^2)$. Let $I, J$ be ideals of the ring $\mathbb{C}\{s_1, \ldots, s_m\}$. Assume $J \subset I$. Let $X, S, T$ be the germs of complex spaces with local rings $\mathbb{C}\{x, y, p\}, \mathbb{C}\{s\}/I, \mathbb{C}\{s\}/J$. Consider the maps $i : X \hookrightarrow X \times S, j : X \times S \hookrightarrow X \times T$ and $q : X \times S \rightarrow S$.

Let $m_X, m_S$ be the maximal ideals of $\mathbb{C}\{x, y, p\}, \mathbb{C}\{s\}/I$. Let $n_S$ be the ideal of $O_{X \times S}$ generated by $m_X m_S$. 

Let $\chi : X \times S \rightarrow X \times S$ be a relative contact transformation. If $\chi$ verifies
\begin{equation}
\chi \circ i = i, \quad q \circ \chi = q \quad \text{and} \quad \chi(0, s) = (0, s)
\end{equation}
for each $s$, there are $\alpha, \beta, \gamma \in \mathfrak{n}_S$ such that
\begin{equation}
(13) \quad \chi(x, y, p, s) = (x + \alpha, y + \beta, p + \gamma, s).
\end{equation}

**Theorem 3.7.** (a) Let $\chi : X \times S \rightarrow X \times S$ be a relative contact transformation that verifies (12). Then $\gamma$ is determined by $\alpha$ and $\beta$. Moreover, there is $\beta_0 \in \mathfrak{n}_S + p\mathcal{O}_{X \times S}$ such that $\beta$ is the solution of the Cauchy problem
\begin{equation}
\beta + p\mathcal{O}_{X \times S} = \beta_0.
\end{equation}
(b) Given $\alpha \in \mathfrak{n}_S$, $\beta_0 \in \mathfrak{n}_S + p\mathcal{O}_{X \times S}$, there is a unique relative contact transformation $\chi$ that verifies (12) and the conditions of statement (a). We denote $\chi$ by $\chi_{\alpha, \beta_0}$.
(c) If $S = T_\varepsilon$ the Cauchy problem (14) simplifies into
\begin{equation}
\frac{\partial \beta}{\partial p} = \frac{\partial \alpha}{\partial p}, \quad \beta + p\mathcal{O}_{X \times T_\varepsilon} = \beta_0.
\end{equation}
(d) Let $\chi = \chi_{\alpha, \beta_0} : X \times T \rightarrow X \times T$ be a relative contact transformation. Then, $\chi$ is a lifting to $T$ of $j^* \chi = \chi_{j^* \alpha, j^* \beta_0} : X \times S \rightarrow X \times S$. If $\chi$ equals
\begin{equation}
(13) \quad \chi(x, y, p, s) = (x + j^* \alpha, y + j^* \beta, p + j^* \gamma, s).
\end{equation}
(e) Assume $\mathcal{O}_T = \mathbb{C}\{s\}$, $\mathcal{O}_{T_0} = \mathbb{C}\{s, \varepsilon\}/(\varepsilon^2, \varepsilon s_1, \ldots, \varepsilon s_m)$. Given a relative contact transformation
\begin{equation}
\chi(x, y, p, s) = (x + A, y + B, p + C, s)
\end{equation}
over $T$ and $\alpha, \beta, \gamma \in \mathfrak{m}_X$,
\begin{equation}
\chi_0(x, y, p, s, \varepsilon) = (x + A + \varepsilon \alpha, y + B + \varepsilon \beta, p + C + \varepsilon \gamma, s, \varepsilon)
\end{equation}
is a relative contact transformation over $T_0$ if and only if
\begin{equation}
(x, y, p, \varepsilon) \mapsto (x + \varepsilon \alpha, y + \varepsilon \beta, p + \varepsilon \gamma)
\end{equation}
is a relative contact transformation over $T_\varepsilon$. Moreover, all liftings of $\chi$ to $T_0$ are of the type (17).

**Proof.** See Theorems 2.4 and 2.6 of [13].

4. Relative Legendrian Curves

Let $\theta = \xi dx + \eta dy$ be the canonical 1-form of $T^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2$. Hence $\pi = \pi_{\mathbb{C}^2} : \mathbb{P}^1 \mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1 \rightarrow \mathbb{C}^2$ is given by $\pi(x, y; \xi) = (x, y)$. Let $U[V]$ be the open subset of $\mathbb{P}^1 \mathbb{C}^2$ defined by $\eta \neq 0, \xi \neq 0$. Then $\theta/\eta, [0/\xi]$ defines a contact form $dy - p dx [dx - qdy]$ on $U[V]$, where $p = -\xi/\eta, q = -\eta/\xi$. Moreover, $dy - p dx$ and $dx - qdy$ define the structure of contact manifold on $\mathbb{P}^1 \mathbb{C}^2$. 

If $L$ is the germ of a Legendrian curve of $\mathbb{P}^*M$ and $L$ is not a fiber of $\pi_M$, $\pi_M(L)$ is the germ of a plane curve with irreducible tangent cone and $L = \pi_M^*(\mathbb{P}^*)^M$.

Let $Y$ be the germ of a plane curve with irreducible tangent cone at a point $o$ of a surface $M$. Let $L$ be the conormal of $Y$. Let $\sigma$ be the only point of $L$ such that $\pi_M(\sigma) = o$. Let $k$ be the multiplicity of $Y$. Let $f$ be a defining function of $Y$. In this situation we will always choose a system of local coordinates $(x, y)$ of $M$ such that the tangent cone $C(Y)$ of $Y$ equals $\{y = 0\}$.

**Lemma 4.1.** The following statements are equivalent:

(a) $\text{mult}_p(L) = \text{mult}_s(Y)$;
(b) $C_\sigma(L) \ni (D\pi(\sigma))^{-1}(0, 0)$;
(c) $f \in (x^2, y)^k$;
(d) if $t \mapsto (x(t), y(t))$ parametrizes a branch of $Y$, $x^2$ divides $y$.

**Proof.** The equivalence of statements holds if and only if it holds for each branch. Assume $Y$ irreducible. Assume $x(t) = t^k$ and $y(t) = t^n\varphi(t) = \tilde{\varphi}(t)$, where $\varphi$ is a unit of $\mathbb{C}\{t\}$. Since $C(Y) = \{y = 0\}$, $n > k$. There is a unit $\psi$ of $\mathbb{C}\{t\}$ such that $p(t) = t^{n-k}\psi(t)$. Statements (a) and (b) hold if and only if $n - k \geq k$. Statement (d) holds if and only if $n \geq 2k$. Remark that $f = \sum_{i=1}^{k} a_i t^{k-1} = \prod_{i=1}^{k} (y - \tilde{\varphi}(\theta^i t))$

where $\theta = \exp(2\pi i/k)$. Since $a_i$ is a homogeneous polynomial of degree $i$ on the $\tilde{\varphi}(\theta^j t)$, $j = 1, \ldots, k$, $a_i \in (x^{[in/k]})$ and $a_k$ generates $(x^n)$. Therefore (c) is verified if and only if $n/k \geq 2$.

We say that a plane curve $Y$ is generic |a Legendrian curve $L$ is in generic position| if it verifies the conditions of Lemma 4.1.

Given a germ of a Legendrian curve $L$ of $U$ at $\sigma$ there is a germ of a contact transformation $\chi : (U, \sigma) \rightarrow (U, \sigma)$ such that $\chi(L)$ is in generic position (see [10] Corollary 1.6.4.).

**Lemma 4.2.** Let $\sigma$ denote the origin of $U$. Assume $L, L_1, L_2$ are germs of Legendrian branches in generic position.

(a) $C_\sigma(L) = \{y = p = 0\}$ if and only if given a parametrization $t \mapsto (x(t), y(t))$ of a branch of $Y$, $x^2 \not\subset (y)$.
(b) $C_\sigma(L_1) \neq C_\sigma(L_2)$ if and only if $\pi(L_1)$ and $\pi(L_2)$ have contact of order 2.

**Proof.** Under the notations of Lemma 4.1 $C_\sigma(L) = \{y = p = 0\}$ if $n \geq 2k + 1$ and $C_\sigma(L) = \{y = p - \psi(0)x = 0\}$ if $n = 2k$. 

Remark that if $Y$ is a germ of a plane curve of $\mathbb{C}^2$ at the origin and $C(Y) = \{y = 0\}$, its conormal is a Legendrian variety contained in $U$. By Darboux’s Theorem each germ of a contact manifold of dimension 3 is isomorphic to the germ of $U$ at $\sigma$, endowed with the contact structure of $U$ defined by $dy - pdx.$
**Definition 4.3.** Let $S$ be a reduced complex space. Let $Y$ be a reduced plane curve. Let $\mathcal{Y}$ be a deformation of $Y$ over $S$. We say that $\mathcal{Y}$ is *generic* if its fibers are generic. If $S$ is a non reduced complex space we say that $\mathcal{Y}$ is *generic* if $\mathcal{Y}$ admits a generic lifting.

Given a flat deformation $\mathcal{Y}$ of a plane curve $Y$ over a complex space $S$ we will denote $\mathbb{P}_{\mathcal{Y}}(\mathbb{C}^2|S)$ by $\text{Con}(\mathcal{Y})$.

Consider the contact transformations from $\mathbb{C}^3$ to $\mathbb{C}^3$ given by

\begin{align*}
(19) \quad & \Phi(x, y, p) = (\lambda x, \lambda y, \mu p), \quad \lambda, \mu \in \mathbb{C}^*, \\
(20) \quad & \Phi(x, y, p) = (ax + bp, y + \frac{ac}{2}x^2 + \frac{bd}{2}p^2 + bcxp + dp, cx + dp), \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1, \\
(21) \quad & \rho_\lambda(x, y, p) = (x, y - \lambda x^2/2, p - \lambda x), \quad \lambda \in \mathbb{C}.
\end{align*}

The contact transformations (20) are called *paraboloidal contact transformations*.

**Example 4.4.** (a) Let $k, n$ be integers such that $(k, n) = 1$ and $0 < k < n$. Let $Y = \{y^k - x^n = 0\}$. Consider the contact transformation $\chi(x, y, p) = (p, y - xp, -x)$. The conormal $L$ of $Y$ is parametrized by

\[ x = t^k, \quad y = t^n, \quad p = \frac{n}{k} t^{n-k}. \]

Therefore, $Y^\chi = \pi(\chi(L))$ admits the equation $(xy/(k-n))^k = x^{n-k}$. We say that $Y^\chi$ is the action of the contact transformation $\chi$ on the plane curve $Y$.

(b) Setting $Y = \{y^3 - x^7 = 0\}$, $\chi(x, y, p) = (x + p, y + p^2/2, p)$, $Y^\chi$ admits a parametrization

\[ x = t^3 + (7/3)t^4, \quad y = t^7 + (49/18)t^8. \]

Changing parameters we get

\[ x = s^3, \quad y = s^7 + \lambda s^8 + h.o.t., \]

with $\lambda \neq 0$. Following [17], $Y^\chi$ and $Y$ have the same topological type but are not analytically equivalent.

**Theorem 4.5.** (See [1] or [12].) Let $\Phi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be the germ of a contact transformation. Then $\Phi = \Phi_1 \Phi_2 \Phi_3$, where $\Phi_1$ is of type (19), $\Phi_2$ is of type (20) and $\Phi_3$ is of type (13), with $\alpha, \beta, \gamma \in \mathbb{C}\{x, y, p\}$. Moreover, there is $\beta_0 \in \mathbb{C}\{x, y\}$ such that $\beta$ verifies the Cauchy problem (14), $\beta - \beta_0 \in (p)$ and

\begin{align*}
(22) \quad & \alpha, \beta, \gamma, \beta_0, \frac{\partial \alpha}{\partial x}, \frac{\partial \beta}{\partial x}, \frac{\partial \beta}{\partial p}, \frac{\partial^2 \beta}{\partial x \partial p} \in (x, y, p).
\end{align*}

If $D\Phi(0)(\{y = p = 0\}) = \{y = p = 0\}$, $\Phi_2 = id_{\mathbb{C}^3}$. 
Lemma 4.6. (Lemma 3.5.4 of [15]) Let \( \alpha, \beta, \gamma \in \mathbb{C}\{t\} \). Assume \( \alpha(0) \neq 0 \).
(a) If \( (t\alpha)^k = t^k\gamma \), \( \alpha \in \mathcal{O}_\Sigma \) if and only if \( \gamma \in \mathcal{O}_\Sigma^* \) and \( \alpha \in \mathcal{O}_\Sigma^* \) if and only if \( \gamma \in \mathcal{O}_\Sigma^* \).
(b) If \( t = s\beta(s) \) solves \( s = t\alpha(t) \), \( \alpha \in \mathcal{O}_\Sigma \) if and only if \( \beta \in \mathcal{O}_\Sigma \) and \( \alpha \in \mathcal{O}_\Sigma^* \) if and only if \( \beta \in \mathcal{O}_\Sigma^* \).

Theorem 4.7 (Theorem 1.3, [3]). Let \( \chi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0) \) be a germ of a contact transformation. Let \( L \) be a germ of a Legendrian curve of \( \mathbb{C}^3 \) at the origin. If \( L \) and \( \chi(L) \) are in generic position, \( \pi(L) \) and \( \pi(\chi(L)) \) are equisingular.

Proof. Assume \( C_\sigma(L) \) is irreducible. Since when \( \chi = \rho_\lambda \) or \( \chi \) is of type \( 19 \), \( \pi(L) \) and \( \pi(\chi(L)) \) are equisingular, we can assume that

\[
C_\sigma(L) = C_\sigma(\chi(L)) = \{ y = p = 0 \}
\]

and \( \chi \) is of type \( 13 \). Let \( L_1, L_2 \) be branches of \( L \). Let \( S[k] \) be the semigroup \([\text{multiplicity}]\) of \( \pi(L_1) \). Let \( S' \) be the semigroup generated by \( (S_0 - k) \cap \mathbb{N} \). There are parametrizations

\[ t \mapsto (x_i(t), y_i(t), p_i(t)) \]

of \( L_i, i = 1, 2 \) such that \( x_1(t) = t^k \), \( y_1 \in \mathcal{O}_S^* \) and \( p_1 \in \mathcal{O}_{S'} \). By \( 22 \) \( \chi(L_1) \) admits a parametrization \( 23 \) with \( x_1(t) = t^k \), unit, \( x_1 \in \mathcal{O}_{S'}, y_1 \in \mathcal{O}_S^* \). By Lemma 4.6 we can assume that, after a reparametrization, \( x_1(t) = t^k \) and \( y_1 \in \mathcal{O}_S^* \). Hence \( \pi(L_1) \) and \( \pi(\chi(L_1)) \) are equisingular.

Assume \( \pi(L_i) \) has multiplicity \( k_i, i = 1, 2 \) and \( k \) is the least common multiple of \( k_1, k_2 \). Assume \( \pi(L_1) \) and \( \pi(L_2) \) have contact of order \( \nu \). Then we can assume that \( x_i(t) = t^{k_i k_j} \), \( \{i, j\} = \{1, 2\} \),

\[
y_2 \equiv y_1 \mod \mathcal{O}_S \text{ and } y_2 \not\equiv y_1 \mod \mathcal{O}_{S'},
\]

where \( S_\ell = \{0\} \cup \ell + \mathbb{N}, S = S_{0k}, S_+ = S_{0k+1} \) and \( S' = S_{0k-k} \). Therefore \( p_2 \equiv p_1 \mod \mathcal{O}_{S'} \). Composing \( \chi \) with \( 23 \) we obtain a parametrization \( 23 \) of \( \chi(L_i) \) such that

\[
x_1 = t^k \text{ \unit}, \ x_2 \equiv x_1 \mod \mathcal{O}_{S'} \text{ and } y_2 \equiv y_1 \mod \mathcal{O}_S, \ i = 1, 2.
\]

By Lemma 4.6, after reparametrization, \( 24 \) holds. The theorem is proven when \( C_\sigma(L) \) is irreducible.

Assume there is \( \lambda_i \) such that \( \pi(L_i) = \{ y = \lambda_i x^2 \}, i = 1, 2 \) and \( \lambda_1 \neq \lambda_2 \). If \( \chi \) is paraboloidal, there are \( \mu_i \) such that \( \pi(\chi(L_i)) = \{ y = \mu_i x^2 \}, i = 1, 2 \) and \( \mu_1 \neq \mu_2 \). By Lemma 4.2 if \( C_\sigma(L_1) \neq C_\sigma(L_2) \), the contact order of \( \pi(L_1) \) and \( \pi(L_2) \) equals 2. Hence the truncation of the Puiseux expansion of \( \pi(L_i) \) equals \( \lambda_i x^2, i = 1, 2 \). Therefore the contact order of \( \pi(\chi(L_1)) \) and \( \pi(\chi(L_2)) \) equals 2. \( \square \)
Definition 4.8. Two Legendrian curves are \textit{equisingular} if their generic plane projections are equisingular.

Lemma 4.9. Assume \( \mathcal{Y} \) is a generic plane curve and \( \mathcal{Y} \hookrightarrow \mathcal{Y} \) defines an equisingular deformation of \( \mathcal{Y} \) with trivial normal cone along its trivial section. Then \( \mathcal{Y} \) is generic.

Proof. By Proposition 2.6 we can assume that \( \mathcal{Y} \) is irreducible. Moreover, we can assume that \( \mathcal{Y} \) is a deformation over a vector space and \( \mathcal{C}_{\{x=y=0\}}(\mathcal{Y}) = \{y = 0\} \). Let \( x = t^k, \ y = t^n + \sum_{i \geq n+1} a_i t^i, \ n \geq 2k \) be a parametrization of \( \mathcal{Y} \). After reparametrization, we can assume that \( \mathcal{Y} \) admits a parametrization of the type

\[
\begin{align*}
x &= t^k, \quad y = \sum_i \alpha_i t^i, \\
p &= \sum_i i\alpha_i t^{i-k}
\end{align*}
\]

define a parametrization of \( \Con(\mathcal{Y}) \),

\[
\mathcal{C}_{\{x=y=0\}}(\Con(\mathcal{Y})) = \{y = p - 2k\alpha_{2k}x = 0\}.
\]

Definition 4.10. Let \( L \) be (the germ of) a Legendrian curve of \( \mathbb{C}^3 \) in generic position. Let \( \mathcal{L} \) be a relative Legendrian curve over (a germ of) a complex space \( S \) at \( o \). We say that an immersion \( i: L \hookrightarrow \mathcal{L} \) defines a deformation

\[
\mathcal{L} \hookrightarrow \mathbb{C}^3 \times S \rightarrow S
\]

of the Legendrian curve \( L \) over \( S \) if \( i \) induces an isomorphism of \( L \) onto \( \mathcal{L}_o \) and there is a generic deformation \( \mathcal{Y} \) of a plane curve \( \hat{Y} \) over \( S \) such that \( \chi(L) \) is isomorphic to \( \Con\mathcal{Y} \) by a relative contact transformation verifying (12).

We say that the deformation (26) is \textit{equisingular} if \( \mathcal{Y} \) is equisingular. We denote by \( \Def^\text{eq}_{\mathcal{L}} \) the category of equisingular deformations of \( L \).

Remark 4.11. We do not demand the flatness of the morphism (26).

Lemma 4.12. Using the notations of definition 4.10, given a section \( \sigma: S \rightarrow \mathcal{L} \) of \( \mathbb{C}^3 \times S \rightarrow S \), there is a relative contact transformation \( \chi \) such that \( \chi \circ \sigma \) is trivial. Hence \( \mathcal{L} \) is isomorphic to a deformation with trivial section.

Proof. We can assume that \( S \) is the germ at the origin of a vector space. Set \( \sigma(s) = (\pi(s), \gamma(s), \rho(s), s) \). Setting \( \chi(x, y, p, s) = (x - \pi(s), y - \gamma(s), p, s) \), we can assume that \( \pi, \gamma \) vanish. Now \( \chi(x, y, p, s) = (x, y - \rho(s)x, p - \rho(s), s) \) trivializes \( \sigma \).

Theorem 4.13. Assume \( \mathcal{Y} \) defines an equisingular deformation of a generic plane curve \( \hat{Y} \) with trivial normal cone along its trivial section. Let \( \chi \) be a relative contact transformation verifying (12). Then \( \mathcal{Y}^\chi = \pi(\mathcal{Y}(\Con\mathcal{Y})) \) is a generic equisingular deformation of \( \hat{Y} \).
Proof. We can assume that $S$ is the germ of a vector space. We only have to prove that (i) $(\mathcal{Y}^s)_s$ is generic and (ii) $(\mathcal{Y}^s)_s$ are equisingular, for small enough $s$. Let $(\mathcal{Y}^s)_{s,i}$ be one branch of $(\mathcal{Y}^s)_s$. Since $(\mathcal{Y}^s)_{s,i}$ is generic its conormal admits a parametrization

$$\psi(t) = (t^k, \epsilon n + \text{h.o.t.}, (n/k)t^{n-k} + \text{h.o.t.}),$$

with $n \geq 2k$ (see Lemma 4.1). By Theorem 4.5 $\chi_s = \Phi_1 \Phi_2 \Phi_3$. Since $\Phi_1$ preserves genericity, we can assume $\Phi_1 = id$. Notice that $(\mathcal{Y}_{s,i})^{\Phi_2}$ is parametrized by

$$(27) \quad t \mapsto (x(t), y(t)),$$

where $x(t) = a t^k + b(n/k)t^{n-k} + \text{h.o.t.}$ and $y(t) \in (t^{2k})$. If $s$ is small enough we can assume $a$ close to 1 and $b$ close to 0. Hence $(x) = (t^k)$. Therefore we can assume $\Phi_2 = id$. Finally $(\mathcal{Y}_{s,i})^{\Phi_3}$ is parametrized by (27), with

$$x(t) = t^k + \psi^*(\alpha), \quad y(t) = t^n + \psi^*(\beta).$$

By (22) $(x) = (t^k)$ and $y \in (t^{2k})$ for small $s$. Now (ii) follows from Theorem 4.7 for $s$ small enough.

5. Deformations of the parametrization

Let $\psi : \tilde{C} \to \mathbb{C}^3$ be the parametrization of a Legendrian curve $L$. We say that a deformation $\Psi$ of $\psi$ is a Legendrian deformation of $\psi$ if the analytic set parametrized by $\Psi$ is a relative Legendrian curve. We say that $(\chi, \xi)$ is an isomorphism of Legendrian deformations if $\chi : \mathbb{C}^3 \times T \to \mathbb{C}^3 \times T$ is a relative contact transformation (see (2)).

**Definition 5.1.** Let $\varphi : \tilde{C} \to \mathbb{C}^2$ be the parametrization of a generic plane curve $\mathcal{Y}$ with tangent cone $\{y = 0\}$. Let $\text{Def}_{\varphi}^{\text{es}}$ be the category of equisingular deformations of $\varphi$. Let $\mathcal{Y}$ be an object of $\text{Def}_{\varphi}^{\text{es}}$. We say that $\mathcal{Y}$ is an object of the full subcategory $\overline{\text{Def}}_{\varphi}^{\text{es}}$ of $\text{Def}_{\varphi}^{\text{es}}$ if $\mathcal{Y}$ is generic and the normal cone of $\mathcal{Y}$ along $\{x = y = 0\}$ equals $\{y = 0\}$.

Let $\psi : \tilde{C} \to \mathbb{C}^3$ be the parametrization of a curve $L$ in generic position. We will denote by $\overline{\text{Def}}_{\psi}^{\text{es}}$, the category of equisingular Legendrian deformations of $\psi$.

**Theorem 5.2.** Let $\varphi : \tilde{C} \to \mathbb{C}^2$ be the parametrization of a generic plane curve $Y$ with tangent cone $\{y = 0\}$. Then a semiuniversal deformation of $\varphi$ in $\text{Def}_{\varphi}^{\text{es}}$ is also a semiuniversal deformation in $\overline{\text{Def}}_{\varphi}^{\text{es}}$.

**Proof.** Assume $\varphi_i(t_i) = (x_i(t_i), y_i(t_i)), \, i = 1, \ldots, r$. Let $I_{\varphi}^{\text{es}}$ be the vector space of the $a \partial_x + b \partial_y$ such that $a = [a_1, \ldots, a_r]^t, \, b = [b_1, \ldots, b_r]^t$, where $a_i, b_i \in \mathbb{C}\{t_i\}t_i$ and

$$t_i \mapsto (x_i(t_i) + \varepsilon a_i(t_i), y_i(t_i) + \varepsilon b_i(t_i)).$$
i = 1, \ldots, r, is an equisingular deformation of \( \varphi \) along the trivial section over \( T_c \). Let \( T_{es}^{\varphi} \) be the quotient of \( I_{es}^{\varphi} \) by the linear subspace of its elements that define trivial deformations. Let
\[
a^i \partial_x + b^j \partial_y, \quad j = 1, \ldots, \ell,
\]
be a family of representatives of a basis of \( T_{es}^{\varphi} \). Set
\[
X_i = x_i + \sum_{j=1}^{\ell} a_i^j s_j, \quad Y_i = y_i + \sum_{j=1}^{\ell} b_i^j s_j
\]
i = 1, \ldots, r. By Theorem 2.38 of [6],
\[
\Phi_i(t_i) = (X_i(t_i), Y_i(t_i)), \quad i = 1, \ldots, r,
\]
defines a semiuniversal deformation of \( \varphi \) in \( \text{Def}^{es}_\varphi \). It is enough to show that \( \Phi_i, i = 1, \ldots, r \) is an element of \( \text{Def}^{es}_\varphi \). Let \( m_i \) be the multiplicity of \( \Phi_i \). Then \( (x_i) = (t_i^{m_i}) \). Since \( \Phi_i \) is equimultiple \( X_i, Y_i \in (t_i^{m_i}) \). Since \( y_i \in (t_i^{2m_i}) \) and \( \Phi_i \) is equisingular
\[
t_i \mapsto (X_i(t_i), Y_i(t_i)/X_i(t_i))
\]
is equimultiple (see [6]). Therefore \( Y_i \in (t_i^{2m_i}) \).

Assume \( \psi \) is a parametrization of the conormal of the curve parametrized by \( \varphi \). Let \( \Phi[\psi] \) be the deformation [Legendrian deformation] of \( \varphi[\psi] \) given by
\[
\Phi_i(t_i, s) = (X_i(t_i, s), Y_i(t_i, s)), \quad [\Psi_i(t_i, s) = (X_i(t_i, s), Y_i(t_i, s), P_i(t_i, s))].
\]
There are functors \( \text{Conf} : \text{Def}^{es}_\varphi \to \text{Def}^{es}_\psi, \quad \pi : \text{Def}^{es}_\psi \to \text{Def}^{es}_\varphi \) given by
\[
(\text{Conf}\Phi)_i = \left( X_i, Y_i, \frac{\partial Y_i}{\partial t} \left( \frac{\partial X_i}{\partial t} \right)^{-1} \right), \quad (\Psi^\pi)_i = (X_i, Y_i).
\]

**Example 5.3.** Let \( \Phi \) be the deformation \( x = t^3, y = t^{10} + st^{11} \) of the plane curve \( Y \) given by the equation \( y^3 - x^{10} \) and parametrized by \( x = t^3, y = t^{10} \). The deformation \( \Phi \) induces the flat deformation given by
\[
y^3 - x^{10} - 3sx^7y - s^3x^{11}.
\]
The conormal \( \Psi \) of \( \Phi \) is given by \( x = t^3, y = t^{10} + st^{11}, 3p = 10t^7 + 11st^8 \).

The semigroup of the conormal curve of \( \{y^3 - x^{10} = 0\} \) equals \( \{3, 6, 7, 9, 10\} \cup \mathbb{N} + 12 \). The semigroup of the conormal of the deformed curve also contains the number 11. Hence the deformation is not flat (see [2]).

It is shown in [3] that each flat deformation of the conormal of \( y^k - x^n = 0 \) is rigid. This result shows that the obvious choice of a definition of deformation of a Legendrian variety is not a very good one. This is the reason to introduce Definitions 4.10 and 5.11.

**Definition 5.4.** Let \( \text{Def}^{es, \mu}_\varphi \) be the category given in the following way: the objects of \( \text{Def}^{es, \mu}_\varphi \) are the objects of \( \text{Def}^{es}_\varphi \), the morphisms of \( \text{Def}^{es, \mu}_\varphi \) are the
pairs $\left(\chi, \xi, \varepsilon\right)$ where $\chi : \mathbb{C}^3 \times T \to \mathbb{C}^3 \times T$ is a relative contact transformation that acts on a deformation $\Phi$ by

$$\left(\chi \cdot \Phi\right)_t = (\chi \circ \text{Con}\Phi_t)^\varepsilon,$$

and leaves invariant the normal cone along $\{x = y = 0\}$ of the image of $\Phi$.

Notice that, by Theorem [4.13] $\chi \cdot \Phi$ defined above is in fact an object of $\mathcal{D}_{ef^{es,\mu}}$.

Let $\mathcal{C}_\varphi$ be a category of deformations of a curve parametrized by $\varphi$. Let $S$ be a complex space. We will denote by $\mathcal{C}_\varphi(S)$ the category of deformations of $\mathcal{C}_\varphi$ over $S$. We will denote by $\mathcal{C}_\varphi(S)$ the set of isomorphism classes of objects of $\mathcal{C}_\varphi(S)$.

The functors $\text{Con} : \mathcal{D}_{ef^{es,\mu}} \to \mathcal{D}_{ef^{es}}$, $\pi : \mathcal{D}_{ef^{es}} \to \mathcal{D}_{ef^{es,\mu}}$ are surjective and define natural equivalences between the functors

$$T \mapsto \mathcal{D}_{ef^{es,\mu}}(T) \quad \text{and} \quad T \mapsto \mathcal{D}_{ef^{es}}(T).$$

Let $\varphi : \mathbb{C} \to \mathbb{C}^2$ be a parametrization of a generic plane curve $Y$ with irreducible components $Y_1, \ldots, Y_r$. Assume $\varphi_i(t) = (x_i(t), y_i(t)), i = 1, \ldots, r$.

We will identify each ideal of $\mathcal{C}_\varphi$ with its image by $\varphi^* : \mathcal{O}_Y \to \mathcal{O}_C$:

$$\mathcal{O}_C = \mathbb{C} \left\{ x_1 \ldots x_r, [y_1 \ldots y_r] \right\} \subset \oplus_{i=1}^r \mathbb{C}\{t_i\} = \mathcal{O}_C.$$

Set $\dot{x} = [\dot{x}_1, \ldots, \dot{x}_r]^t$, where $\dot{x}_i$ is the derivative of $x_i$ with respect to $t_i$, $1 \leq i \leq r$. Let $\dot{\varphi} := \dot{x}\partial_x + \dot{y}\partial_y$ be an element of the free $\mathcal{O}_C$-module $\mathcal{O}_C\partial_x \oplus \mathcal{O}_C\partial_y$, which has a structure of $\mathcal{O}_Y$-module induced by $\varphi^*$.

Let $u_1, \ldots, u_r, v_1, \ldots, v_r \in \mathbb{C}\{t_i\}$. We say that

$$(u_1, \ldots, u_r)\partial_x \oplus (v_1, \ldots, v_r)\partial_y$$

belongs to the equisingularity module $\Sigma^{es}_\varphi$ (see [6]) of $\varphi$ if the deformation $\Phi$ given by $\Phi_i(t, \varepsilon) = (x_i(t), \varepsilon u_i(t), y_i(t) + \varepsilon v_i(t))$ is equisingular and has trivial normal cone along its trivial section.

Let $m_{C,\varphi}$ be the sub $\mathcal{O}_C$-module of $\Sigma^{es}_\varphi$ generated by

$$(a_1, \ldots, a_r)\left(\dot{x}\partial_x + \dot{y}\partial_y\right), \quad a_i \in \mathbb{C}\{t_i\}, \quad 1 \leq i \leq r.$$

For $i = 1, \ldots, r$ set $p_i = \dot{y}_i/\dot{x}_i$. For each $k \geq 0$ set $\mathbf{p}^k = [p_1^k, \ldots, p_r^k]^t$. Let $\mathcal{I}$ be the sub $\mathcal{O}_Y$-module of $\mathcal{O}_C\partial_x \oplus \mathcal{O}_C\partial_y$ generated by $\left(k + 1\right)\mathbf{p}^k\partial_x + k\mathbf{p}^{k+1}\partial_y$, $k \geq 1$.

**Theorem 5.5.** The module $\mathcal{I}$ is contained in $\Sigma^{es}_\varphi$ and

$$\mathcal{D}_{ef^{es,\mu}}(T_x) \simeq \Sigma^{es}_\varphi / \left(m_{C,\varphi} + (x, y)\partial_x \oplus (x^2, y)\partial_y + \mathcal{I}\right).$$

**Proof.** Let $(u_1, \ldots, u_r)\partial_x \oplus (v_1, \ldots, v_r)\partial_y \in \mathcal{I}$ and $\Phi$ be the deformation given by

$$(28) \quad \Phi_i(t, \varepsilon) = (x_i(t), \varepsilon u_i(t), y_i(t) + \varepsilon v_i(t)).$$
We can suppose that for each $i = 1, \ldots, r$

$$u_i = p_i^\ell, \quad v_i = \frac{\ell}{\ell + 1} p_i^{\ell + 1}$$

for some $\ell \geq 1$. Because $Y$ is generic we have that $\text{ord}_{t_i} p_i > \text{ord}_{t_i} x_i$, $2\text{ord}_{t_i} p_i > \text{ord}_{t_i} y_i$ and, by Lemma 4.1, $\Phi$ has generic fibres. The deformation $\Phi$ is the result of the action over the trivial deformation of $Y$ of the relative contact transformation

$$\chi(x, y, p, \varepsilon) = (x + \varepsilon p^\ell, y + \varepsilon \frac{\ell}{\ell + 1} p^{\ell + 1}, p, \varepsilon).$$

As the trivial deformation is equisingular, $\Phi$ is equisingular.

Let $\Phi \in \text{Def}_{\varepsilon, p}^c$ be given as in (28), where $u_i, v_i \in \mathbb{C} \{t_i\}$, $\text{ord}_{t_i} u_i \geq m_i, \text{ord}_{t_i} v_i \geq 2m_i$, $i = 1, \ldots, r$, where $m_i$ is the multiplicity of $Y_i$. We have that $\Phi$ is trivial if and only if there are

$$\xi_i(t_i) = \tilde{t}_i = t_i + \varepsilon h_i,$$

$$\chi(x, y, p, \varepsilon) = (x + \varepsilon \alpha, y + \varepsilon \beta, p + \varepsilon \gamma, \varepsilon),$$

such that $\chi$ is a relative contact transformation, $\xi_i$ is an isomorphism, $\alpha, \beta, \gamma \in (x, y, p) \mathbb{C} \{x, y, p\}$, $h_i \in t_i \mathbb{C} \{t_i\}$, $1 \leq i \leq r$, and

$$x_i(t_i) + \varepsilon u_i(t_i) = x_i(\tilde{t}_i) + \varepsilon \alpha(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)),$$

$$y_i(t_i) + \varepsilon v_i(t_i) = y_i(\tilde{t}_i) + \varepsilon \beta(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)),$$

for $i = 1, \ldots, r$. By Taylor’s formula $x_i(\tilde{t}_i) = x_i(t_i) + \varepsilon x_i(t_i) h_i(t_i), y_i(\tilde{t}_i) = y_i(t_i) + \varepsilon y_i(t_i) h_i(t_i)$ and

$$\varepsilon \alpha(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)) = \varepsilon \alpha(x_i(t_i), y_i(t_i), p_i(t_i)),$$

$$\varepsilon \beta(x_i(\tilde{t}_i), y_i(\tilde{t}_i), p_i(\tilde{t}_i)) = \varepsilon \beta(x_i(t_i), y_i(t_i), p_i(t_i)),$$

for $i = 1, \ldots, r$. Hence $\Phi$ is trivialized by $\chi$ if and only if

(29) \begin{align*}
u_i(t_i) &= \dot{x}_i(t_i) h_i(t_i) + \alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \end{align*}

(30) \begin{align*}v_i(t_i) &= \dot{y}_i(t_i) h_i(t_i) + \beta(x_i(t_i), y_i(t_i), p_i(t_i)), \end{align*}

for $i = 1, \ldots, r$. By Theorem 3.7 (c), (29) and (30) are equivalent to the condition

$$u \partial_x + v \partial_y \in \mathfrak{m} \hat{\phi} + (x, y) \partial_x \oplus (x^2, y) \partial_y + \hat{I}.$$ 

\hfill \Box

**Theorem 5.6.** Set $\ell = \dim \text{Def}_{\varepsilon, p}^c(T_\varepsilon)$. Assume that

(31) \begin{align*}a^i \frac{\partial}{\partial x} + b^j \frac{\partial}{\partial y} &= \begin{bmatrix} a_1^i \\ \vdots \\ a_r^i \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} b_1^j \\ \vdots \\ b_r^j \end{bmatrix} \frac{\partial}{\partial y}, \end{align*}

\[1 \leq j \leq \ell, \text{ represents a basis of } \mathcal{D} \text{ef}^{es,\mu}_\varphi(T_\nu). \] Let \( \Phi : \bar{C} \times \mathbb{C}^k \to \mathbb{C}^2 \times \mathbb{C}^k \) be the deformation of \( \varphi \) given by

\[X_i(t_i, s) = x_i(t_i) + \sum_{j=1}^{\ell} a_i^j(t_i)s_j, \quad Y_i(t_i, s) = y_i(t_i) + \sum_{j=1}^{\ell} b_i^j(t_i)s_j, \]

\( i = 1, \ldots, r. \) Then \( \text{Con} \Phi \) is a semiuniversal deformation of \( \psi \) in \( \mathcal{D} \text{ef}^{es}_\psi. \)

This Theorem is the equivalent for Legendrian curves of Theorem 2.38 of [6] for plane curves.

**Remark 5.7.** Set

\[\bar{M}_\varphi = \Sigma_{\psi}^\text{es} / (m_{\bar{C}} \hat{\varphi} + (x, y)\partial_x \oplus (x^2, y)\partial_y).\]

Then

\[\mathcal{D} \text{ef}^{es}_\varphi(T_\nu) \cong \bar{M}_\varphi.\]

Let \( k = \text{dim} \bar{M}_\varphi \) and assume that (31), \( 1 \leq j \leq k, \) represents a basis of \( \bar{M}_\varphi. \)

Let \( \Phi : \bar{C} \times \mathbb{C}^k \to \mathbb{C}^2 \times \mathbb{C}^k \) be the deformation of \( \varphi \) given by

\[X_i(t_i, s) = x_i(t_i) + \sum_{j=1}^{k} a_i^j(t_i)s_j, \quad Y_i(t_i, s) = y_i(t_i) + \sum_{j=1}^{k} b_i^j(t_i)s_j. \]

Then \( \Phi \) is semiuniversal in \( \mathcal{D} \text{ef}^{es}_\varphi \) (see [6] II Theorem 2.38). If \( \Psi \in \mathcal{D} \text{ef}^{es}_\varphi(T), \) then \( \Psi^\pi \in \mathcal{D} \text{ef}^{es}_\varphi(T). \) Hence there is \( f : T \to \bar{M}_\varphi \) such that \( \Psi^\pi \cong f^* \Phi. \)

Therefore \( \Psi = \text{Con} \Psi^\pi \cong \text{Con} f^* \Phi = f^* \text{Con} \Phi. \) This shows that \( \text{Con} \Phi \) is complete in \( \mathcal{D} \text{ef}^{es}_\varphi. \) It is actually versal and the proof is only technically more complicated.

**Proof.** (of Theorem 5.6) It is enough to show that \( \text{Con} \Phi \) is formally semiuniversal (see remark 5.7 and [5] Satz 5.2).

Let \( \nu : T' \hookrightarrow T \) be a small extension. Let \( \Psi \in \mathcal{D} \text{ef}^{es}_\varphi(T). \) Set \( \Psi' = \nu^* \Psi. \)

Let \( \eta' : T' \to \mathbb{C}^\ell \) be a morphism of complex analytic spaces. Assume that \( (\chi', \xi') \) define an isomorphism

\[\eta'^* \text{Con} \Phi \cong \Psi'.\]

We need to find \( \eta : T \to \mathbb{C}^\ell \) and \( \chi, \xi \) such that \( \eta' = \eta \circ \nu \) and \( \chi, \xi \) define an isomorphism

\[\eta^* \text{Con} \Phi \cong \Psi\]

that extends \( (\chi', \xi'). \)

Let \( A[A'] \) be the local ring of \( T[T']. \) Let \( \delta \) be the generator of \( \text{Ker}(A \to A'). \) We can assume \( A' \cong \mathbb{C}\{z\}/I, \) where \( z = (z_1, \ldots, z_m). \) Set

\[\bar{A}' = \mathbb{C}\{z\} \quad \text{and} \quad \bar{A} = \mathbb{C}\{z, \varepsilon\}/(\varepsilon^2, \varepsilon z_1, \ldots, \varepsilon z_m).\]
Let \( m_A \) be the maximal ideal of \( A \). Since \( m_A \delta = 0 \) and \( \delta \in m_A \), there is a morphism of local analytic algebras from \( \tilde{A} \) onto \( A \) that takes \( \varepsilon \) into \( \delta \) such that the diagram

\[
\begin{array}{ccc}
\tilde{A} & \longrightarrow & A' \\
\downarrow & & \downarrow \\
A & \longrightarrow & A'
\end{array}
\]

commutes. Assume \( \tilde{T}'[\tilde{T}] \) has local ring \( \tilde{A}[\tilde{A}] \). We also denote by \( \tilde{\iota} \) the morphism \( \tilde{T}' \hookrightarrow \tilde{T} \). We denote by \( \kappa \) the morphisms \( T \hookrightarrow \tilde{T} \) and \( T' \hookrightarrow \tilde{T}' \).

Let \( \tilde{\Psi} \in \tilde{D}_{\text{ef}_\psi}(\tilde{T}) \) be a lifting of \( \Psi \).

Consider the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{C} \times \tilde{T}' & \hookrightarrow & \mathbb{C} \times \tilde{T} \\
\downarrow \tilde{\Psi}' & & \downarrow \tilde{\Psi} \\
\mathbb{C}^3 \times \tilde{T}' & \hookrightarrow & \mathbb{C}^3 \times \tilde{T} \\
\downarrow \text{pr} & & \downarrow \text{pr} \\
\tilde{T}' & \longrightarrow & \tilde{T} \\
\end{array}
\]

If \( \text{Con} \Phi \) is given by

\[
X_i(t_i, s), \ Y_i(t_i, s), \ P_i(t_i, s) \in \mathbb{C}\{s, t_i\},
\]

then \( \tilde{\eta}^* \text{Con} \Phi \) is given by

\[
X_i(t_i, \tilde{\eta}'(z)), \ Y_i(t_i, \tilde{\eta}'(z)), \ P_i(t_i, \tilde{\eta}'(z)) \in \tilde{A}'\{t_i\} = \mathbb{C}\{z, t_i\}
\]

for \( i = 1, \ldots, r \). Suppose that \( \tilde{\Psi}' \) is given by

\[
U'_i(t_i, z), \ V'_i(t_i, z), \ W'_i(t_i, z) \in \mathbb{C}\{z, t_i\}.
\]
Then, $\tilde{\Psi}$ must be given by
\[ U_i = U_i' + \varepsilon u_i, \quad V_i = V_i' + \varepsilon v_i, \quad W_i = W_i' + \varepsilon w_i \in \tilde{A}\{t_i\} = \mathbb{C}\{z, t_i\} \oplus \varepsilon \mathbb{C}\{t_i\} \]
with $u_i, v_i, w_i \in \mathbb{C}\{t_i\}$ and $i = 1, \ldots, r$. By definition of deformation we have that, for each $i$,
\[ (U_i, V_i, W_i) = (x(t_i), y(t_i), p_i(t_i)) \mod m_{\tilde{A}}. \]
Suppose $\tilde{\eta}' : \tilde{T}' \to \mathbb{C}^\ell$ is given by $(\tilde{\eta}'_1, \ldots, \tilde{\eta}'_{\ell})$, with $\tilde{\eta}'_i \in \mathbb{C}\{z\}$. Then $\tilde{\eta}$ must be given by $\tilde{\eta} = \tilde{\eta}' + \varepsilon \tilde{\eta}'_0$ for some $\tilde{\eta}'_0 = (\tilde{\eta}'_1^0, \ldots, \tilde{\eta}'_{\ell}^0) \in \mathbb{C}^\ell$. Suppose that $\tilde{\chi}' : \mathbb{C}^3 \times \tilde{T}' \to \mathbb{C}^3 \times \tilde{T}'$ is given at the ring level by
\[ (x, y, p) \mapsto (H'_1, H'_2, H'_3), \]
such that $H' = id \mod m_{\tilde{A}}$, with $H'_i \in (x, y, p)A'(x, y, p)$. Let the automorphism $\tilde{\xi}' : \mathbb{C} \times \tilde{T}' \to \mathbb{C} \times \tilde{T}'$ be given at the ring level by
\[ t_i \mapsto h'_i \]
such that $h' = id \mod m_{\tilde{A}}$, with $h'_i \in (t_i)\mathbb{C}\{z, t_i\}$. Then, from (34) it follows that
\begin{align*}
X_i(t_i, \tilde{\eta}') &= H_1'(U'_i(h'_i), V'_i(h'_i), W'_i(h'_i)), \\
Y_i(t_i, \tilde{\eta}') &= H_2'(U'_i(h'_i), V'_i(h'_i), W'_i(h'_i)), \\
P_i(t_i, \tilde{\eta}') &= H_3'(U'_i(h'_i), V'_i(h'_i), W'_i(h'_i)).
\end{align*}
Now, $\tilde{\eta}'$ must be extended to $\tilde{\eta}$ such that the first two previous equations extend as well. That is, we must have
\begin{align*}
X_i(t_i, \tilde{\eta}) &= (H'_1 + \varepsilon \alpha)(U_i(h'_i + \varepsilon h'_0), V_i(h'_i + \varepsilon h'_0), W_i(h'_i + \varepsilon h'_0)), \\
Y_i(t_i, \tilde{\eta}) &= (H'_2 + \varepsilon \beta)(U_i(h'_i + \varepsilon h'_0), V_i(h'_i + \varepsilon h'_0), W_i(h'_i + \varepsilon h'_0)),
\end{align*}
with $\alpha, \beta \in (x, y, p)\mathbb{C}\{x, y, p\}$, $h'_0 \in (t_i)\mathbb{C}\{t_i\}$ such that
\[ (x, y, p) \mapsto (H'_1 + \varepsilon \alpha, H'_2 + \varepsilon \beta, H'_3 + \varepsilon \gamma) \]
gives a relative contact transformation over $\tilde{T}$ for some $\gamma \in (x, y, p)\mathbb{C}\{x, y, p\}$. The existence of this extended relative contact transformation is guaranteed by Theorem 3.7 (e). Moreover, this extension depends only on the choices of $\alpha$ and $\beta_0$. So, we need only to find $\alpha, \beta_0, \tilde{\eta}'_0$ and $h'_0$ such that (36) holds. Using Taylor’s formula and $\varepsilon^2 = 0$ we see that
\begin{align*}
X_i(t_i, \tilde{\eta}' + \varepsilon \tilde{\eta}'_0) &= X_i(t_i, \tilde{\eta}') + \varepsilon \sum_{j=1}^\ell \partial_{s_j} X_i(t_i, \tilde{\eta}') \tilde{\eta}'_j, \\
Y_i(t_i, \tilde{\eta}' + \varepsilon \tilde{\eta}'_0) &= Y_i(t_i, \tilde{\eta}') + \varepsilon \sum_{j=1}^\ell \partial_{s_j} Y_i(t_i, \tilde{\eta}') \tilde{\eta}'_j.
\end{align*}
Again by Taylor’s formula and noticing that $\varepsilon m_{\tilde{A}} = 0$, $\varepsilon m_{\tilde{A}'} = 0$ in $\tilde{A}$, $h' = id$ mod $m_{\tilde{A}'}$, and $(U_i, V_i) = (x_i(t_i), y_i(t_i))$ mod $m_{\tilde{A}}$ we see that

$$U_i(h_i' + \varepsilon h_i^0) = U_i(h_i' + \varepsilon \hat{U}_i(h_i^0))$$

$$= U_i'(h_i') + \varepsilon(\hat{x}_i h_i^0 + u_i),$$

$$V_i(h_i' + \varepsilon h_i^0) = V_i'(h_i') + \varepsilon(\hat{y}_i h_i^0 + v_i).$$

(38)

Now, $H' = id$ mod $m_{\tilde{A}}$, so

$$\partial_x H'_1 = 1 \mod m_{\tilde{A}}, \quad \partial_y H'_1, \partial_p H'_1 \in m_{\tilde{A}'} A'(x, y, p).$$

In particular,

$$\varepsilon \partial_y H'_1 = \varepsilon \partial_p H'_1 = 0.$$

By this and arguing as in (37) and (38) we see that

$$(H'_1 + \varepsilon \alpha)(U'_i(h_i') + \varepsilon(\hat{x}_i h_i^0 + u_i)), V'_i(h_i') + \varepsilon(\hat{y}_i h_i^0 + v_i), W'_i(h_i') + \varepsilon(\hat{p}_i h_i^0 + w_i))$$

$$= H'_1(U'_i(h_i'), V'_i(h_i'), W'_i(h_i')) + \varepsilon(\alpha(U'_i(h_i'), V'_i(h_i'), W'_i(h_i')) + 1(\hat{x}_i h_i^0 + u_i))$$

$$= H'_1(U'_i(h_i'), V'_i(h_i'), W'_i(h_i')) + \varepsilon(\alpha(x_i, y_i, p_i) + \hat{x}_i h_i^0 + u_i),$$

$$(H'_2 + \varepsilon \beta)(U'_i(h_i') + \varepsilon(\hat{x}_i h_i^0 + u_i)), V'_i(h_i') + \varepsilon(\hat{y}_i h_i^0 + v_i), W'_i(h_i') + \varepsilon(\hat{p}_i h_i^0 + w_i))$$

$$= H'_2(U'_i(h_i'), V'_i(h_i'), W'_i(h_i')) + \varepsilon(\beta(x_i, y_i, p_i) + \hat{y}_i h_i^0 + v_i).$$

Substituting this in (36) and using (35) and (37) we see that we have to find $\tilde{\eta}^0 = (\tilde{\eta}_1^0, \ldots, \tilde{\eta}_k^0) \in C^\ell$, $h_i^0$ such that

$$u_i(t_i), v_i(t_i) = \sum_{j=1}^\ell \tilde{\eta}_j^0 (\partial_{s_j} X_i(t_i, 0), \partial_{s_j} Y_i(t_i, 0)) -$$

$$- h_i^0(t_i)((\hat{x}_i(t_i), \hat{y}_i(t_i)) - (\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \beta(x_i(t_i), y_i(t_i), p_i(t_i)))).$$

Note that, because of Theorem 3.7 (c),

$$(\alpha(x_i(t_i), y_i(t_i), p_i(t_i)), \beta(x_i(t_i), y_i(t_i), p_i(t_i))) \in \tilde{T}$$

for each $i$. Also note that $\tilde{\Psi} \in \mathcal{D} e f_{\tilde{\psi}}^{e, s}$ (T) means that $(u_i, v_i) \in \Sigma_{\tilde{\psi}}^{e, s}$. Then, if the vectors

$$\left(\partial_{s_j} X_i(t_1, 0), \ldots, \partial_{s_j} X_i(t_r, 0)\right) \partial_x + \left(\partial_{s_j} Y_i(t_1, 0), \ldots, \partial_{s_j} Y_i(t_r, 0)\right) \partial_y$$

$$= (a_i^1(t_1), \ldots, a_i^\ell(t_1), \ldots, b_i^1(t_r), \ldots, b_i^\ell(t_r)) \partial_x, \quad j = 1, \ldots, \ell$$

form a basis of $\mathcal{D} e f_{\tilde{\psi}}^{e, s, \mu}(T_{\tilde{\psi}})$, we can solve (39) with unique $\tilde{\eta}_1^0, \ldots, \tilde{\eta}_k^0$ for all $i = 1, \ldots, r$. This implies that the conormal of $\Phi$ is a formally semiuniversal equisingular deformation of $\psi$ over $C^\ell$. □
6. Deformations of the equation I

Let $Y$ be a generic curve with parametrization $\varphi$ and equation $f$. Let $L$ be the conormal of $Y$.

**Definition 6.1.** We will denote by $\mathcal{D}e\mathcal{f}_{f}^{es}$ (or $\mathcal{D}e\mathcal{f}_{Y}^{es}$) the full subcategory of generic equisingular deformations of the plane curve $Y$ such that its normal cone along $\{x = y = 0\}$ equals $\{y = 0\}$.

Let $T$ be a complex space. We associate to a deformation $\Phi$ of $\varphi$ the deformation $Y$ defined by the kernel of $\Phi^{*}: \mathcal{O}_{\mathbb{C}^{2} \times T} \rightarrow \mathcal{O}_{\mathbb{C} \times T}$. We obtain in this way a functor

$$\vartheta: \mathcal{D}e\mathcal{f}_{\varphi}^{es} \rightarrow \mathcal{D}e\mathcal{f}_{f}^{es}.$$  

**Theorem 6.2.** The functor $\vartheta$ is surjective and induces a natural equivalence between the functors $T \mapsto \mathcal{D}e\mathcal{f}_{\varphi}^{es}(T)$ and $T \mapsto \mathcal{D}e\mathcal{f}_{f}^{es}(T)$.

Given a morphism of complex spaces $\sigma: T \rightarrow S$ and $\Phi \in \mathcal{D}e\mathcal{f}_{\varphi}^{es}(S)$,

$$\sigma^{*}\vartheta(\Phi) = \vartheta(\sigma^{*}\Phi).$$

**Proof.** See Theorem 2.64 of [6].

Let $\mathcal{Y}$ be an object of $\mathcal{D}e\mathcal{f}_{\varphi}^{es}$. Since the normal cone of $\mathcal{Y}$ along $\{x = y = 0\}$ equals $\{y = 0\}$, $\text{Con}(\mathcal{Y}) \subset U \times T$.

Let $\psi$ be the parametrization of the conormal of $\varphi$. Let $\Phi \in \mathcal{D}e\mathcal{f}_{\psi}^{es}(T)$. Let $\hat{\vartheta}(\Psi)$ denote the image of $\Psi$. By Theorem 6.4

$$\hat{\vartheta}(\Psi) = \text{Con}(\vartheta(\Psi^{\pi})).$$

**Lemma 6.3.** The functor $\hat{\vartheta}$ is surjective and induces a natural equivalence between the functors $T \mapsto \mathcal{D}e\mathcal{f}_{\psi}^{es}(T)$ and $T \mapsto \mathcal{D}e\mathcal{f}_{L}^{es}(T)$.

Given a morphism of complex spaces $\sigma: T \rightarrow S$ and $\Psi \in \mathcal{D}e\mathcal{f}_{\psi}^{es}(S)$,

$$\sigma^{*}\hat{\vartheta}(\Psi) = \hat{\vartheta}(\sigma^{*}\Psi).$$

**Proof.** If $\mathcal{L}$ is in $\mathcal{D}e\mathcal{f}_{L}^{es}(T)$, $\mathcal{L}^{\pi}$ is in $\mathcal{D}e\mathcal{f}_{f}^{es}(T)$. Therefore $\mathcal{L}^{\pi} = \vartheta(\Phi)$, for some $\Phi \in \mathcal{D}e\mathcal{f}_{\varphi}^{es}(T)$. Setting $\Psi = \text{Con}(\Phi)$, $\hat{\vartheta}(\Psi) = \mathcal{L}$.

By Theorem 6.2 and [40], $\hat{\vartheta}$ induces a natural equivalence and (41) holds.

**Theorem 6.4.** For each Legendrian curve $L$ there is a semiuniversal deformation $\mathcal{L}$ of $L$ in the category $\mathcal{D}e\mathcal{f}_{L}^{es}$. Moreover, $\mathcal{L}$ is defined over a smooth analytic manifold.

**Proof.** Let $\Psi$ be the semiuniversal deformation of the parametrization $\psi$ of $L$ in the category $\mathcal{D}e\mathcal{f}_{\psi}^{es}$. By Lemma 6.3, we can take $\mathcal{L} = \hat{\vartheta}(\Psi)$.  

□
7. Deformations of the equation II

Definition 7.1. Let $\text{Def}^{f_{Y}^{\mathcal{L}}}(T)$ be the category given in the following way: the objects of $\text{Def}^{f_{Y}^{\mathcal{L}}} = \text{Def}^{f_{Y}^{\mathcal{L}}}$ are the objects of $\text{Def}^{f_{Y}}$; two objects $Y, Z$ of $\text{Def}^{f_{Y}^{\mathcal{L}}}(T)$ are isomorphic if there is a relative contact transformation $\chi$ over $T$ such that $Z = Y^\chi$.

Lemma 7.2. Assume $f \in \mathbb{C}[x, y]$ is the defining function of a generic plane curve $Y$. Let $L$ be the conormal of $Y$. For each $\ell \geq 1$ there is $h_\ell \in \mathbb{C}[x, y]$ such that

$$(\ell + 1)p^\ell f_x + \ell p^{\ell+1}f_y \equiv h_\ell \mod I_L.$$ Moreover, $h_\ell$ is unique modulo $I_Y$.

Proof. Let $\Delta$ be the germ of $\mathbb{C}$ at the origin. Let $k_\tau$ be the multiplicity [the conductor] of the branch $Y_\tau$ of $Y$, $\tau = 1, ..., n$. Let $\sigma_\tau : \Delta \to L_\tau$ be the normalization of the conormal $L_\tau$ of $Y_\tau$, $\tau = 1, ..., n$. Let $v_\tau$ be the valuation of $\mathbb{C}[x, y, p]$ associated to $\sigma_\tau$, $\tau = 1, ..., n$. The restriction of $v_\tau$ to $\mathbb{C}[x, y]$ defines the valuation of $\mathbb{C}[x, y]$ associated to the normalization of $Y_\tau$, $\tau = 1, ..., n$. By [17], Section I.2

$$v_\tau(f_{\tau,y}) = c_\tau + k_\tau - 1, \quad v_\tau(xf_{\tau,x}) = v_\tau(yf_{\tau,y}),$$

for $\tau = 1, ..., n$. By (42) and [17] there is $a_{\tau,\ell} \in \mathbb{C}[x, y]$ such that $v_\tau(\ell p^{\ell+1}f_{\tau,y} - a_{\tau,\ell}) = +\infty$, $\tau = 1, ..., n$, for each $\ell \geq 1$. Setting $a_\ell = \sum_{\tau=1}^n a_{\tau,\ell} \prod_{j \neq \tau} f_j$, $v_\tau(\ell p^{\ell+1}f_y - a_\ell) = +\infty$, for $\ell \geq 1$, $\tau = 1, ..., n$.

A similar reasoning shows there are $b_\ell \in \mathbb{C}[x, y]$ such that

$$v_\tau((\ell + 1)p^\ell f_x - b_\ell) = +\infty, \quad \text{for } \ell \geq 1, \quad \tau = 1, ..., n.$$ \hfill \qed

Remark 7.3. Assume $Y$ is irreducible with multiplicity $\nu$. Suppose $Y \in \text{Def}^{f_{Y}^{\mathcal{L}}}(T)$, where $T$ is a reduced complex space and let $\mathcal{L}$ be the relative conormal of $Y$. Let $\Phi$ be the deformation of the parametrization of $Y$ such that $\partial(\Phi) = \mathcal{L}$. Let $\Psi$ be the conormal of $\Phi$. There $A_i \in \mathcal{O}_T$ such that

$$\Psi^*x = t^\nu, \quad \Psi^*y = t^n + \sum_{i \geq n+1} A_it^i \quad \text{and} \quad \Psi^*p = \frac{n}{\nu}t^{n-\nu} + \sum_{i \geq n+1} \frac{i}{\nu} A_it^{i-\nu}.$$ Given $f \in \mathcal{O}_T\{x, y, p\}$, $f \in I_\mathcal{L}$ if and only if $\Psi^*f = 0$.

Theorem 7.4. Let $Y$ be a generic curve. Let $T$ be a complex space. Let $\nu_0 : T \to T_0$ be a small extension and $\chi_0$ be a relative contact transformation over $T_0$. Let $Y_0 \in \text{Def}^{f_{Y}^{\mathcal{L}}}(T_0)$, $Y = y_0^*Y_0$ and $\chi = y_0^*\chi_0$. Assume $\chi_0$ equals [17] and $Y$ $[Y_0, Y^\chi, Y^{\chi_0}]$ are defined by $F [F_0, F^X, F^{\chi_0}_0]$, where $F_0 = F + \varepsilon g$, $g \in \mathbb{C}[x, y]$, and $F^X$ is a lifting of $f$. Then, if $F^{\chi_0}_0$ is a lifting of $F^X$,

$$F^{\chi_0}_0 = F^X + \varepsilon g + \varepsilon \alpha_0 f_x + \varepsilon \beta_0 f_y + \varepsilon \sum_{k \geq 1} \frac{\alpha_k}{k+1} h_k.$$
Proof. Remark that if \( \chi \) equals \([16]\) and \( I_Y \) is generated by \( F \), \( I_{\chi Y} \) is generated by \( F^x \in \mathcal{O}_{C^x \times S} \) such that
\[
F^x(x, y, s) \equiv F(x + A, y + B, s) \mod I_{\chi(L)}.
\]
Let \( L \) denote the conormal of \( Y \). Let \( \mathcal{L}[L_0] \) denote the relative conormal of \( \mathcal{Y}[Y_0] \). We can assume \( s = (s_1, ..., s_m) \),
\[
\mathcal{O}_T = \mathbb{C}\{s\}, \quad \mathcal{O}_{T_0} = \mathbb{C}\{s, \varepsilon\}/n_\varepsilon, \quad n_\varepsilon = (s_1 \varepsilon, ..., s_m \varepsilon, \varepsilon^2).
\]
Since \( I_{\chi_0(L_0)} = I_{\chi(L)} + \varepsilon \mathcal{O}_{C^x \times T_0} \cap I_{\chi_0(L_0)} = I_{\chi(L)} + \varepsilon I_L \) we have the following congruences modulo \( I_{\chi_0(L_0)}^2 \):
\[
F_0^x \equiv F_0(x + A + \varepsilon \alpha y + B + \varepsilon \beta, s, \varepsilon)
\equiv F(x + A + \varepsilon \alpha y + B + \varepsilon \beta, s) \equiv F(x + A, y + B, s) \equiv F + \alpha_0 f_x + \beta_0 f_y + \varepsilon \sum_{k \geq 1} \frac{\alpha_k}{k} h_k.
\]

Corollary 7.5. Let \( F = f + \varepsilon g \) be a defining function of a deformation \( \mathcal{Y} \in \text{Def}_{\mathcal{O}}^{e_\mathcal{Y}}(T_\varepsilon) \). Let \( \chi_{\alpha, \beta_0} \) be a contact transformation over \( T_\varepsilon \). Then
\[
f + \varepsilon g + \varepsilon \alpha_0 f_x + \varepsilon \beta_0 f_y + \varepsilon \sum_{k \geq 1} \frac{\alpha_k}{k+1} h_k
\]
defines the action of \( \chi_{\alpha, \beta_0} \) on \( \mathcal{Y} \).

Definition 7.6. Let \( f \) be a generic plane curve with tangent cone \( \{ y = 0 \} \). We will denote by \( I_f \) the ideal of \( \mathbb{C}\{x, y\} \) generated by the functions \( g \) such that \( f + \varepsilon g \) is equisingular over \( T_\varepsilon \) and has trivial normal cone along its trivial section. We call \( I_f \) the equisingularity ideal of \( f \).

We will denote by \( I_f^\mu \) the ideal of \( \mathbb{C}\{x, y\} \) generated by \( f, (x, y) f_x, (x^2, y) f_y \) and \( h_{k, \ell}, \ell \geq 1 \).

Let \( f = \sum_{k, \ell} a_{k, \ell} \) be a convergent power series. Let \( u, v, d \) be positive integers. Assume \( u, v \) coprime. If \( a_{k, \ell} \neq 0 \) implies \( uk + vl \geq d \) and there are \( k_1, \ell_1, k_2, \ell_2 \) such that \( (k_1, \ell_1) \neq (k_2, \ell_2) \) and \( a_{k_i, \ell_i} \neq 0, i = 1, 2 \), we call
\[
f_{u, v, d}(x, y) = \sum_{uk + vl = d} a_{k, \ell} x^k y^l
\]
a face of \( f \). We say that \( f \) is semiquasihomogeneous (SQH) of type \( (u, v; d) \) if \( f_{u, v, d} \) is a face of \( f \) and \( f_{u, v, d} \) has isolated singularities. We say that \( f \) is Newton non-degenerate (NND) if \( x, y \) do not divide \( f \) and the singular locus of each face of \( f \) is contained in \( \{ xy = 0 \} \).

Lemma 7.7. If \( f \) is generic, \( I_f^\mu \subset I_f \).

Proof. Let \( \alpha \in (x, y), \beta \in (x^2, y) \). Set \( \chi = \chi_{\alpha, 0} [\chi = \chi_{0, \beta}, \chi = \chi_{\mu, 0}] \). By Lemma 7.4, \( f^\chi \) equals
\[
f + \varepsilon \alpha f_x, \quad [f + \varepsilon \beta f_y, \quad f + \varepsilon h_{\ell}/(\ell + 1)].
\]
By Lemma 4.13, $f^\chi$ is equisingular. Since the derivative of $\chi$ leaves invariant \{y = 0\}, then \((x, y)f_x, (x^2, y)f_y \subset I_f\) and $h_\ell \in I_f$, for each $\ell \geq 1$. □

**Theorem 7.8.** If $f$ is generic,

$$D_{ef} f^{es, \mu}(T_\varepsilon) \simeq I_f/I_f^\mu.$$

*Proof.* Let $G \in D_{ef} f^{es, \mu}(T_\varepsilon)$. There is $g \in I_f$ such that $G = f + \varepsilon g$. The deformation $f + \varepsilon g$ is trivial in $D_{ef} f^{es, \mu}(T_\varepsilon)$ if and only if there are $h \in \mathbb{C}\{x, y\}$ and a contact transformation (13) such that

$$G(x + \alpha, y + \beta, \varepsilon) = (1 + \varepsilon h) f \mod \varepsilon I_L.$$

By Corollary 7.5, (45) holds if and only if

$$g + \alpha_0 f_x + \beta_0 f_y + \sum_\ell \frac{\alpha_\ell}{\ell + 1} h_\ell = hf \mod (f).$$

Hence $G$ is trivial if and only if $g \in I_f^\mu$. □

**Remark 7.9.** Each equisingular deformation $F$ of a SQH or NND plane curve $f$ is isomorphic to a deformation $\tilde{F}$, such that $\tilde{F}$ is equisingular via trivial sections (see [16] and [6]). This means that, in the SQH or NND case, if $A \rightarrow A'$ is a small extension with kernel $\varepsilon$ such that $Y' \in D_{ef} f^{es, \mu}(A')$, $Y \in D_{ef} f^{es, \mu}(A)$ defined by $F'$, respectively $F = F' + \varepsilon a(x, y)$, then $f + \varepsilon a(x, y)$ defines a deformation in $D_{ef} f^{es, \mu}(T_\varepsilon)$ (see Theorem 8.2 of [16]).

**Theorem 7.10.** Assume $Y$ is a generic plane curve with conormal $L$, defined by a power series $f$. Assume $f$ is SQH or $f$ is NND. If $g_1, ..., g_n \in I_f$ represent a basis of $I_f/I_f^\mu$ with Newton order $\geq 1$, the deformation $\mathcal{G}$ defined by

$$(46)\quad G(x, y, s_1, ..., s_n) = f(x, y) + \sum_{i=1}^n s_i g_i$$

is a semiumiversal deformation of $f$ in $D_{ef} f^{es, \mu}$.

*Proof.* The choice of $g_1, ..., g_n$ identifies $I_f/I_f^\mu$ with $\mathbb{C}^n$. It is enough to show that (46) is a formally versal deformation of $f$ in $D_{ef} f^{es, \mu}$ and there is a versal deformation of $f$ in $D_{ef} f^{es, \mu}$ (see [5] Satz 5.2). The second requirement follows from Theorem 6.4. Let us prove that the first requirement is fulfilled. We will follow the terminology of the proof of Theorem 7.4. Let $\eta : T \rightarrow \mathbb{C}^n$ be a morphism of complex spaces and let $\chi$ be a relative contact transformation over $T$ such that $\eta^* \mathcal{G} = \mathcal{Y}^\chi$. It is enough to show that there is a unique pair $(\eta_0, \chi_0)$ where $\eta_0$ is a morphism from $T_0$ to $\mathbb{C}^n$ and $\chi_0$ is a relative contact transformation over $T_0$ such that

$$(47)\quad \eta_0 \circ \iota_0 = \eta \quad \text{and} \quad \eta_0^* \mathcal{G} = \mathcal{Y}_0^\chi.$$ 

Because $\eta^* \mathcal{G} = \mathcal{Y}^\chi$ there is $h \in (s)\mathcal{O}_{\mathbb{C}^2 \times T}$ such that

$$(1 + h)\eta^* G = F^\chi.$$
In order for (47) to hold, we need to find \(a \in \mathbb{C}^n\), \(\sigma \in \mathcal{O}_{\mathbb{C}^2}\) and \(\chi_0\) such that
\[
\eta^0 = \eta + \varepsilon a, \quad \text{and} \quad (1 + h + \varepsilon \sigma)\eta_0^*G = F_0^{\chi_0}.
\]
By Theorem 3.7 there are \(A, B_0\) such that
\[
\chi = \chi_{A,B_0}
\]
and \(\chi_0\) exists if and only if there are \(\alpha, \beta_0\) such that
\[
\chi_0 = \chi_{A,B_0 + \varepsilon \beta_0}.
\]
By Theorem 7.4, \(F_0^{\chi_0}\) equals (43). Moreover,
\[
(1 + h + \varepsilon \sigma)\eta_0^*G = (1 + h)\eta^*G + \varepsilon \sigma \eta^*G + \varepsilon(1 + h)\sum_{i=1}^n a_i g_i
\]
(48)
Hence we need to solve the equation
\[
(49) \quad g(1 + h)^{-1} = \sum_{i=1}^n a_i g_i - (1 + h)^{-1}(\varepsilon \sigma f + \alpha_0 f_x + \beta_0 f_y + \sum_{\ell} \alpha_{\ell+1} h_\ell).
\]
Since, as noted in Remark 7.9, \(g(1 + h)^{-1} \in I_f\) there are unique \(a_1, ..., a_n\) such that
\[
g(1 + h)^{-1} - \sum_{i=1}^n a_i g_i \in I_f^\mu.
\]
Hence there are \(\alpha_\ell, \beta_0, \sigma\) such that (49) holds.

**Corollary 7.11.** The relative conormal of \(\mathcal{G}\) is a semiuniversal deformation of the conormal \(L\) of \(Y\) on \(\mathcal{D}ef_{L^\mu}\).

**Proof.** Suppose \(\iota : T' \hookrightarrow T\) is an embedding of complex spaces, \(L \in \mathcal{D}ef_{L^\mu}(T)\), \(L' = \iota^*L \in \mathcal{D}ef_{L^\mu}(T')\). Let \(\eta' : T' \rightarrow \mathbb{C}^n\) be a morphism of complex spaces and \(\chi'\) a relative contact transformation such that
\[
\chi'(L') = \eta'^*\text{Con}(\mathcal{G}).
\]
Let \(\mathcal{Y}' = \pi(L')\) and \(\mathcal{Y} = \pi(L)\). Equation (50) implies that \(\mathcal{Y}' = \eta'^*\mathcal{G} \in \mathcal{D}ef_{f^\mu}(T')\). Because \(\mathcal{G}\) is semiuniversal, there is \(\eta : T \rightarrow \mathbb{C}^n\) with \(\eta' = \eta \circ \iota\) and \(\chi\) relative contact transformation extending \(\chi'\) such that \(\mathcal{Y}^\chi = \eta^*\mathcal{G}\).
This means that \(\eta^*\text{Con}(\mathcal{G}) = \chi(L)\), hence \(\text{Con}(\mathcal{G})\) is semiuniversal. □

**Figure 1.** Monomial base for \(\mathbb{C}\{x,y\}/(f(x,y)f_x(x^2,y)f_y)\).
Example 7.12. If \( f(x, y) = (y^3 + x^7)(y^3 + x^{10}) \), \( f \) is NND and \( I_f \) is generated by the polynomials \( x^2 f_y, y f_x \) and \( x^i y^j \) such that \( 3i + 7j \geq 42 \) and \( 3i + 10j \geq 51 \) (see Proposition 2.17 of [6]).

A semiuniversal object in \( \text{Def}_{f}^{\text{es}} \) (see Proposition 2.69 and Corollary 2.71 of [6]) is given by:

\[
f(x, y) + s_1 x^3 y^5 + s_2 x^5 y^4 + s_3 x^{11} y^2 + s_4 x^{12} y^2 + s_5 x^{14} y + s_6 x^{15} y + s_7 x^{16} y.
\]

See fig. 1. According to Theorem 7.10, the deformation defined by

\[
f(x, y) + s_1 x^3 y^5 + s_2 x^5 y^4 + s_3 x^{14} y
\]

is a semiuniversal deformation of \( f \) in \( \text{Def}_{f}^{\text{es}} \).

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