Decay properties and asymptotic profiles for elastic waves with Kelvin-Voigt damping in 2D

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Abstract. In this paper we consider elastic waves with Kelvin-Voigt damping in 2D. For the linear problem, applying pointwise estimates of the partial Fourier transform of solutions in the Fourier space and asymptotic expansions of eigenvalues and their eigenprojections, we obtain sharp energy decay estimates with additional $L^m$ regularity and $L^p - L^q$ estimates on the conjugate line. Furthermore, we derive asymptotic profiles of solutions under different assumptions of initial data. For the semilinear problem, we use the derived $L^2 - L^2$ estimates with additional $L^m$ regularity to prove global (in time) existence of small data solutions to the weakly coupled system. Finally, to deal with elastic waves with Kelvin-Voigt damping in 3D, we apply the Helmholtz decomposition.

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1. Introduction

Elastic waves describe particles vibrate in the material holding the property of elasticity. Moreover, when the particles are moving and there exists a force acts on the particles restore them to their original position, elastic waves will be produced. For the sake of briefness, the system governing motion of an infinite, isotropic and homogeneous elastic continuum are given by

\[
\begin{align*}
&u_{tt} - a^2 \Delta u - (b^2 - a^2) \nabla \text{div} u = 0, \quad t > 0, \quad x \in \mathbb{R}^n, \\
&(u, u_t)(0, x) = (u_0, u_1)(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

where the unknowns $u = u(t, x) \in \mathbb{R}^n$ for $n = 2, 3$, denote the elastic displacement, $t$ stands for the time and $x$ stands for the space-variable. The positive constants $a$ and $b$ are related to the Lamé constants satisfying $b > a > 0$. The system of elastic waves satisfies the property of finite speed of propagation given by coefficient $b$, which is the speed of propagation of the longitudinal $P$-wave. The coefficient $a$ is the speed of propagation of the transverse $S$-wave. This property have been discussed in the paper [1].

In general, we cannot expect that the system (1.1) models real-world problems, since there exist several kinds resistance, such as fluid resistance and frictional resistance. People always use damping mechanisms to describe oscillation amplitude, which are reduced through the irreversible removal of the vibratory energy in a mechanical system or a component (c.f. [9]).

We recall some results for elastic waves with different damping mechanisms. In recent years, there are some papers devoted the study of linear elastic waves with friction ($\theta = 0$) or structural damping
\((\theta \in (0, 1])\) as follows:
\[
\begin{aligned}
&\left\{ \begin{array}{ll}
u_{tt} - a^2 \Delta u - (b^2 - a^2) \nabla \text{div} \, u + (-\Delta)^\theta u_t = 0, & t > 0, \ x \in \mathbb{R}^n, \\
(u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n.
\end{array} \right. \\
\end{aligned}
\]

In the paper [16], the authors considered the Cauchy problem (1.2) with \(\theta \in [0, 1]\) and initial data taking from \((H^{s+1}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (H^{s}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))\) and derived almost sharp energy estimates in \(n\) dimensional space, \(n \geq 2\), by using the energy method in the Fourier space. For the Cauchy problem (1.2) in two dimensions with \(\theta \in (0, 1)\), the author of [32] studied qualitative properties of solutions, including Gevrey smoothing if \(\theta \in (0, 1)\), propagation of singularities if \(\theta = 1\) and estimates of higher-order energies. In the recent paper [4], smoothing effect, energy estimates with initial data belonging to different function spaces and diffusion phenomena for solutions to the Cauchy problem (1.2) in three dimensions with \(\theta \in [0, 1]\) are investigated by applying diagonalization procedure (see [18, 33, 41] for further explanations about this method) and the energy method in the Fourier space.

More general structural damping mechanisms are considered in [40]. Especially, they studied energy estimates for the following elastic waves with Kelvin-Voigt damping:
\[
\begin{aligned}
&\left\{ \begin{array}{ll}
u_{tt} - a^2 \Delta u - (b^2 - a^2) \nabla \text{div} \, u + (-a^2 \Delta - (b^2 - a^2) \nabla \text{div} \, u) u_t = 0, & t > 0, \ x \in \mathbb{R}^n, \\
(u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n,
\end{array} \right. \\
\end{aligned}
\]

for \(n \geq 2\), are studied. By applying the Haraux-Komornik inequality and the energy method in the Fourier space, the authors of [40] proved almost sharp decay estimates of the total energy. Nevertheless, sharp estimate of the energy and asymptotic profiles of the solution are still not investigated.

Other studies on the dissipative elastic waves can be found in the literature. We refer to [2, 3, 11] for elastic waves with a classical damping term containing time and spatial variables, [18, 20, 30, 31, 34, 35, 37] for elastic waves with thermal dissipative.

The present paper is devoted to the study of the following linear elastic waves with Kelvin-Voigt damping (c.f. [25, 22]) in 2D:
\[
\begin{aligned}
&\left\{ \begin{array}{ll}
u_{tt} - a^2 \Delta u - (b^2 - a^2) \nabla \text{div} \, u + (-a^2 \Delta - (b^2 - a^2) \nabla \text{div} \, u) u_t = 0, & t > 0, \ x \in \mathbb{R}^2, \\
(u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^2,
\end{array} \right. \\
\end{aligned}
\]

and the weakly coupled system of semilinear elastic waves with Kelvin-Voigt damping in 2D, that is, (1.3) with nonlinear terms \(f(u) := (|u|^{p_1}, |u|^{p_2})^T\) on the right-hand sides (one can see (5.55) later). Considering the case \(a^2 = b^2\) in (1.3), the model will be transferred to wave equation with viscoelastic damping. Then, one can apply partial Fourier transform and derive estimates of solutions and asymptotic profiles of solutions (c.f. [5, 7, 14, 24]). However, due to the Lamé operator \((-\alpha^2 \Delta - (b^2 - \alpha^2) \nabla \text{div} \, u)\) with \(b^2 > \alpha^2 > 0\) appearing in the model, we cannot directly follow the methods from [5, 7, 14, 24] to get the sharp energy estimates of an energy and asymptotic profiles for the solutions. Moreover, in this paper we also deal with the three dimensional elastic waves with Kelvin-Voigt damping by applying the Helmholtz decomposition.

The paper is organized as follows. We first investigate some qualitative properties of solutions to the two dimensional linear Cauchy problem (1.3) from Section 2 to Section 4. More precisely, we prepare the pointwise estimates of the partial Fourier transform of solutions in the Fourier space by using the energy method in Section 2. Then, we obtain energy estimates of solutions to the dissipative system (1.3) with initial data having additional \(L^m\) regularity, and \(L^p - L^q\) estimates on the conjugate line. In Subsection 3.1 we derive the asymptotic expansions of eigenvalues and their eigenprojections to show the sharpness of previous pointwise estimates. After constructing asymptotic representation in Subsection 3.2, the asymptotic profiles of solutions are derived in Section 4, where initial data taking from Bessel potential spaces or weighted \(L^1\) spaces. Then, in Section 5 we prove global (in time) existence of small data solutions to the weakly coupled system of semilinear elastic waves with Kelvin-Voigt damping. Additionally, the three dimensional elastic waves with Kelvin-Voigt damping is treated in Section 6. Finally, in Section 7 some concluding remarks complete the paper.
After the change of variables \( v \)
where
Applying the partial Fourier transformation with respect to spatial variable to (2.4), i.e., \( \hat{\partial} \)
By the above matrices we can directly compute

Let us define the function spaces

\[ L^1 \gamma (\mathbb{R}^n) := \{ g \in L^1 (\mathbb{R}^n) : \| g \|_{L^1 \gamma (\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1 + |x|^\gamma) |g(x)| dx \} . \]

Let us define the function spaces \( \mathcal{A}_{m,s} (\mathbb{R}^n) := (H^{s+1} (\mathbb{R}^n) \cap L^m (\mathbb{R}^n)) \times (H^s (\mathbb{R}^n) \cap L^m (\mathbb{R}^n)) \) for \( s \geq 0 \) and \( m \in [1,2] \) with the corresponding norm

\[ \| (g, h) \|_{\mathcal{A}_{m,s} (\mathbb{R}^n)} := \| g \|_{H^{s+1} (\mathbb{R}^n)} + \| g \|_{L^m (\mathbb{R}^n)} + \| h \|_{H^s (\mathbb{R}^n)} + \| h \|_{L^m (\mathbb{R}^n)} . \]

Let \( \chi_{\text{int}}, \chi_{\text{mid}}, \chi_{\text{ext}} \in \mathcal{C}^\infty (\mathbb{R}^2) \) having their supports in \( Z_{\text{int}} (\varepsilon) := \{ \xi \in \mathbb{R}^2 : |\xi| < \varepsilon \} \), \( Z_{\text{mid}} (\varepsilon) := \{ \xi \in \mathbb{R}^2 : \varepsilon \leq |\xi| \leq \frac{1}{2} \} \) and \( Z_{\text{ext}} (\varepsilon) := \{ \xi \in \mathbb{R}^2 : |\xi| > \frac{1}{2} \} \), respectively, so that \( \chi_{\text{mid}} = 1 - \chi_{\text{int}} - \chi_{\text{ext}} \). Here \( \varepsilon > 0 \) is a small constant. We write \( g \lesssim h \), when there exists a constant \( C > 0 \) such that \( g \leq Ch \).

2. Estimates of the solutions to the linear Cauchy problem

In this section we will derive estimate of solutions to the linear elastic waves with Kelvin-Voigt damping
in two dimensions.

Indeed, the system (1.3) can be rewritten by the following form:

\[ u_{tt} - a^2 \Delta (u + u_t) - (b^2 - a^2) \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) (u + u_t) = 0. \tag{2.4} \]

Applying the partial Fourier transformation with respect to spatial variable to (2.4), i.e., \( \hat{\mathcal{F}} (u(t, \xi)) \)
gives

\[ \hat{u}_{tt} + |\xi|^2 A(\eta) \hat{u}_t + |\xi|^2 A(\eta) \hat{u}_t = 0, \tag{2.5} \]

where \( \eta = \xi / |\xi| \in \mathbb{S}^1 \) and

\[ A(\eta) = \begin{pmatrix} a^2 + (b^2 - a^2) \eta_1^2 & (b^2 - a^2) \eta_1 \eta_2 \\ (b^2 - a^2) \eta_1 \eta_2 & a^2 + (b^2 - a^2) \eta_2^2 \end{pmatrix} . \]

Because of our assumption \( b^2 > a^2 > 0 \), the matrix \( A(\eta) \) is positive definite. The eigenvalues of \( A(\eta) \) are \( b^2 \) and \( a^2 \), respectively. We introduce the matrices

\[ M(\eta) = \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_2 & -\eta_1 \end{pmatrix} \quad \text{and} \quad A_{\text{diag}} (|\xi|) = |\xi|^2 \text{diag} (b^2, a^2) . \]

After the change of variables \( v(t, \xi) := M^{-1}(\eta) \hat{u}(t, \xi) \), we define the new ansatz

\[ W(t, \xi) := \begin{pmatrix} v_t(t, \xi) + iA_{\text{diag}}^{1/2} (|\xi|) v(t, \xi) \\ v_t(t, \xi) - iA_{\text{diag}}^{1/2} (|\xi|) v(t, \xi) \end{pmatrix} . \]

By the above matrices we can directly compute

\[ W_t = \begin{pmatrix} -|\xi|^2 M^{-1}(\eta) A(\eta) M(\eta)(v_t + v) + iA_{\text{diag}}^{1/2} (|\xi|) v_t \\ -|\xi|^2 M^{-1}(\eta) A(\eta) M(\eta)(v_t + v) - iA_{\text{diag}}^{1/2} (|\xi|) v_t \end{pmatrix} = \begin{pmatrix} -A_{\text{diag}} (|\xi|)(v_t + v) + iA_{\text{diag}}^{1/2} (|\xi|) v_t \\ -A_{\text{diag}} (|\xi|)(v_t + v) - iA_{\text{diag}}^{1/2} (|\xi|) v_t \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} A_{\text{diag}} (|\xi|) + iA_{\text{diag}}^{1/2} (|\xi|) \\ -\frac{1}{2} A_{\text{diag}} (|\xi|) - iA_{\text{diag}}^{1/2} (|\xi|) \end{pmatrix} W . \]
Therefore, we derive the first-order system
\[
\begin{aligned}
W_t + \frac{1}{2} |\xi|^2 B_0 W - i |\xi| B_1 W &= 0, \quad t > 0, \ \xi \in \mathbb{R}^2, \\
W(0, \xi) &= W_0(\xi), \quad \xi \in \mathbb{R}^2,
\end{aligned}
\]  
(2.6)
where the coefficient matrices $B_0$ and $B_1$ are given by
\[
B_0 = \begin{pmatrix}
  b^2 & 0 & b^2 & 0 \\
  0 & a^2 & 0 & a^2 \\
  b^2 & 0 & b^2 & 0 \\
  0 & a^2 & 0 & a^2
\end{pmatrix}
\quad \text{and} \quad
B_1 = \text{diag} \left( b, a, -b, -a \right).
\]
The solution to (2.6) is given by
\[
W(t, \xi) = e^{t \hat{\Phi}(\xi)} W_0(\xi),
\]
where
\[
\hat{\Phi}(\xi) = -\frac{1}{2} |\xi|^2 B_0 + i |\xi| B_1
\]
\[
= \begin{pmatrix}
  -\frac{k^2}{2} |\xi|^2 + ib|\xi| & 0 & \frac{k^2}{2} |\xi|^2 & 0 \\
  0 & -\frac{k^2}{2} |\xi|^2 + ia|\xi| & 0 & \frac{k^2}{2} |\xi|^2 \\
  \frac{k^2}{2} |\xi|^2 & 0 & -\frac{k^2}{2} |\xi|^2 - ib|\xi| & 0 \\
  0 & \frac{k^2}{2} |\xi|^2 & 0 & -\frac{k^2}{2} |\xi|^2 - ia|\xi|
\end{pmatrix},
\]  
(2.7)
Moreover, the eigenvalue problem corresponding to (2.6) is
\[
\lambda \phi + \left( \frac{1}{2} |\xi|^2 B_0 - i |\xi| B_1 \right) \phi = 0,
\]  
(2.8)
where $\lambda \in \mathbb{C}$ and $\phi \in \mathbb{C}^4$. The eigenvalue $\lambda = \lambda(|\xi|)$ of the problem (2.6) is the value of $\lambda$ satisfying (2.8) for $\phi \neq 0$.

We now derive energy estimates basing on the pointwise estimates of the partial Fourier transform of solutions in the Fourier space. Thus, using the energy method in the Fourier space we prove the following lemma.

**Lemma 2.1.** The solution $W = W(t, \xi)$ to the Cauchy problem (2.6) satisfies the following pointwise estimates for any $\xi \in \mathbb{R}^2$ and $t \geq 0$:
\[
|W(t, \xi)| \lesssim e^{-c\rho(|\xi|) t} |W_0(\xi)|,
\]  
(2.9)
where $\rho(|\xi|) := \frac{|\xi|^2}{1 + |\xi|^2}$ and $c$ is a positive constant.

**Remark 2.1.** From the asymptotic behavior of eigenvalues $\lambda_j(|\xi|)$ for $j = 1, \ldots, 4$, which will be shown later in (3.31) and (3.35), the dissipative structure of the system (2.6) can be characterized by the property
\[
\text{Re} \lambda_j(|\xi|) \leq -c\rho(|\xi|).
\]
Moreover, according to the asymptotic expansions of eigenvalues for $|\xi| \to 0$ and $|\xi| \to \infty$, our pointwise estimates of the partial Fourier transform of solutions stated in Lemma 2.1 are sharp.
Proof. Let us rewritten the system (2.6) in the following explicit form:

\[
\begin{cases}
W_t^{(1)} + \left( \frac{1}{2} b^2 |\xi|^2 - ib |\xi| \right) W^{(1)} + \frac{1}{2} a^2 |\xi|^2 W^{(3)} = 0,
\end{cases}
\]

(2.10)

\[
\begin{cases}
W_t^{(2)} + \left( \frac{1}{2} a^2 |\xi|^2 + ia |\xi| \right) W^{(2)} + \frac{1}{2} a^2 |\xi|^2 W^{(4)} = 0,
\end{cases}
\]

\[
\begin{cases}
W_t^{(3)} + \frac{1}{2} b^2 |\xi|^2 W^{(1)} + \left( \frac{1}{2} b^2 |\xi|^2 + ib |\xi| \right) W^{(3)} = 0,
\end{cases}
\]

\[
\begin{cases}
W_t^{(4)} + \frac{1}{2} a^2 |\xi|^2 W^{(2)} + \left( \frac{1}{2} b^2 |\xi|^2 + ia |\xi| \right) W^{(4)} = 0.
\end{cases}
\]

We multiply the equations (2.10) by \(\bar{W}^{(1)}\), \(\bar{W}^{(2)}\), \(\bar{W}^{(3)}\) and \(\bar{W}^{(4)}\), respectively. Then, adding the resultant equations and taking the real part of them, we obtain

\[
\frac{\partial}{\partial t} \left( |W^{(1)}|^2 + |W^{(3)}|^2 \right) = - b^2 |\xi|^2 \left( |W^{(1)}|^2 + |W^{(3)}|^2 \right) - b^2 |\xi|^2 \Re \left( \bar{W}^{(1)}W^{(3)} + W^{(1)}\bar{W}^{(3)} \right),
\]

\[
\frac{\partial}{\partial t} \left( |W^{(2)}|^2 + |W^{(4)}|^2 \right) = - a^2 |\xi|^2 \left( |W^{(2)}|^2 + |W^{(4)}|^2 \right) - a^2 |\xi|^2 \Re \left( \bar{W}^{(2)}W^{(4)} + W^{(2)}\bar{W}^{(4)} \right).
\]

First of all, calculating the conjugation of (2.10) and times \(i W^{(3)}\) as well as the conjugation of (2.10) and times \(i \bar{W}^{(1)}\), and taking the real part of them give

\[
\Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right) = - \frac{1}{2} b^2 |\xi|^2 \Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right) + b |\xi| \Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right),
\]

\[
\Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right) = - \frac{1}{2} b^2 |\xi|^2 \Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right) + b |\xi| \Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right).
\]

It yields

\[
\frac{\partial}{\partial t} \Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right) = \Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right) + \Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right)
\]

\[
= - b^2 |\xi|^2 \Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right) + 2b |\xi| \Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right).
\]

By the analogous way that

\[
\Re \left( \frac{W_t^{(1)}}{W_t^{(1)}} W^{(3)} \right) = - \frac{1}{2} b^2 |\xi|^2 |W^{(3)}|^2 - \frac{b^2}{2} |\xi|^2 \Re \left( \frac{W_t^{(1)}}{W_t^{(1)}} W^{(3)} \right) - b |\xi| \Re \left( \frac{W_t^{(1)}}{W_t^{(1)}} W^{(3)} \right),
\]

\[
\Re \left( \frac{W_t^{(1)}}{W_t^{(1)}} W^{(3)} \right) = - \frac{1}{2} b^2 |\xi|^2 |W^{(1)}|^2 - \frac{b^2}{2} |\xi|^2 \Re \left( \frac{W_t^{(1)}}{W_t^{(1)}} W^{(3)} \right) - b |\xi| \Re \left( \frac{W_t^{(1)}}{W_t^{(1)}} W^{(3)} \right),
\]

we can calculate

\[
\frac{\partial}{\partial t} \Re \left( \frac{W_t^{(1)}}{W_t^{(1)}} W^{(3)} \right) = \Re \left( \frac{W_t^{(1)}}{W_t^{(1)}} W^{(3)} \right) + \Re \left( \frac{W_t^{(1)}}{W_t^{(1)}} W^{(3)} \right)
\]

\[
= - \frac{1}{2} b^2 |\xi|^2 \left( |W^{(1)}|^2 + |W^{(3)}|^2 \right) - b^2 |\xi|^2 \Re \left( \frac{W_t^{(1)}}{W_t^{(1)}} W^{(3)} \right) - 2b |\xi| \Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right).
\]

Let us define our Lyapunov function \(E^{(1)} = E^{(1)}(t, \xi)\) by

\[
E^{(1)} := |W^{(1)}|^2 + \frac{\gamma_1 |\xi|^2}{1 + |\xi|^2} \Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right) + \frac{\gamma_2 |\xi|^2}{1 + |\xi|^2} \Re \left( \frac{i}{W_t^{(1)}} W^{(3)} \right),
\]

(2.11)
where the parameters $\gamma_1$ and $\gamma_2$ will be fixed later.

Thus, the combination of above estimates implies

$$
\begin{align*}
\frac{\partial}{\partial t} E^{(1)} &= -|\xi|^2 |W^{(1)}|^2 \left( b^2 + \frac{b^2 \gamma_1 |\xi|^2}{2(1 + |\xi|^2)} \right) - |\xi|^2 |W^{(3)}|^2 \left( b^2 + \frac{b^2 \gamma_1 |\xi|^2}{2(1 + |\xi|^2)} \right) \\
&\quad - |\xi|^2 \frac{\gamma_1 |\xi|^2}{1 + |\xi|^2} \text{Re} \left( \overline{W^{(1)}} W^{(3)} \right) \left( b^2 + \frac{b^2 (1 + |\xi|^2)}{\gamma_1 |\xi|^2} - 2b \right) \\
&\quad - |\xi|^2 \frac{\gamma_2 |\xi|^2}{1 + |\xi|^2} \text{Re} \left( iW^{(1)} \overline{W^{(3)}} \right) \left( b^2 + \frac{2b \gamma_1}{\gamma_2} \right).
\end{align*}
$$

(2.12)

Let us choose $0 < \gamma_1 < 1$ and $\gamma_2 = \frac{1}{\gamma}$ in (2.11). Then, we obtain

$$
\frac{\partial}{\partial t} E^{(1)} \leq -C |\xi|^2 E^{(1)} - |\xi|^2 |W^{(3)}|^2 \frac{b^2 \gamma_1 |\xi|^2}{2(1 + |\xi|^2)}.
$$

(2.13)

We define the Lyapunov function $E^{(3)} = E^{(3)}(t, \xi)$ by

$$
E^{(3)} := |W^{(3)}|^2 + \frac{\gamma_3 |\xi|^2}{1 + |\xi|^2} \text{Re} \left( \overline{W^{(1)}} W^{(3)} \right) + \frac{\gamma_4 |\xi|^2}{1 + |\xi|^2} \text{Re} \left( iW^{(1)} \overline{W^{(3)}} \right),
$$

(2.14)

where the parameters $\gamma_3$ and $\gamma_4$ will be determined later.

Similarly, we have

$$
\begin{align*}
\frac{\partial}{\partial t} E^{(3)} &= -|\xi|^2 |W^{(1)}|^2 \left( b^2 + \frac{b^2 \gamma_3 |\xi|^2}{2(1 + |\xi|^2)} \right) - |\xi|^2 |W^{(3)}|^2 \left( b^2 + \frac{b^2 \gamma_4 |\xi|^2}{2(1 + |\xi|^2)} \right) \\
&\quad - |\xi|^2 \frac{\gamma_3 |\xi|^2}{1 + |\xi|^2} \text{Re} \left( W^{(1)} \overline{W^{(3)}} \right) \left( b^2 + \frac{b^2 (1 + |\xi|^2)}{\gamma_3 |\xi|^2} - 2b \right) \\
&\quad - |\xi|^2 \frac{\gamma_4 |\xi|^2}{1 + |\xi|^2} \text{Re} \left( iW^{(1)} \overline{W^{(3)}} \right) \left( b^2 + \frac{2b \gamma_3}{\gamma_4} \right).
\end{align*}
$$

(2.15)

Choosing $0 < \gamma_3 < \frac{b^2}{2}$ and $\gamma_4 = \frac{1}{2\gamma}$ we derive

$$
\frac{\partial}{\partial t} E^{(3)} \leq -C |\xi|^2 E^{(3)} - |\xi|^2 |W^{(1)}|^2 \frac{b^2 \gamma_3 |\xi|^2}{2(1 + |\xi|^2)}.
$$

(2.16)

The combination of (2.13) with (2.16) shows

$$
\frac{\partial}{\partial t} \left( E^{(1)} + E^{(3)} \right) \leq -C |\xi|^2 \left( E^{(1)} + E^{(3)} \right).
$$

(2.17)

Using Gronwall’s inequality in (2.17) we get

$$
\left( E^{(1)} + E^{(3)} \right)(t, \xi) \leq e^{-C|\xi|^2 t} \left( E^{(1)} + E^{(3)} \right)(0, \xi).
$$

(2.18)

To estimate the left and right hand-sides on (2.18), we apply Cauchy’s inequality to get

$$
\begin{align*}
E^{(1)}(t, \xi) + E^{(3)}(t, \xi) &\geq \left( 1 - \frac{(\gamma_1 + \gamma_3) |\xi|^2 + b |\xi|}{2(1 + |\xi|^2)} \right) \left( |W^{(1)}|^2 + |W^{(3)}|^2 \right), \\
E^{(1)}(0, \xi) + E^{(3)}(0, \xi) &\leq \left( 1 + \frac{(\gamma_1 + \gamma_3) |\xi|^2 + b |\xi|}{2(1 + |\xi|^2)} \right) \left( |W^{(1)}|^2 + |W^{(3)}|^2 \right).
\end{align*}
$$

(2.19)

So, we can choose $\gamma_1 = \gamma_3 = \min \left\{ \frac{1}{32}, \frac{b^2}{16} \right\}$ to get

$$
|W^{(1)}(t, \xi)|^2 + |W^{(3)}(t, \xi)|^2 \leq e^{-C|\xi|^2 t} \left( |W^{(1)}(0, \xi)|^2 + |W^{(3)}(0, \xi)|^2 \right),
$$

with a positive constant $c > 0.$
Next, we define the Lyapunov functions $E^{(2)} = E^{(2)}(t, \xi)$ and $E^{(4)} = E^{(4)}(t, \xi)$, respectively,

$$E^{(2)} := |W^{(2)}|^2 + \frac{\gamma_1|\xi|^2}{1 + |\xi|^2} \text{Re}\left(\frac{W^{(2)}}{|W^{(4)}|}\right) + \frac{\gamma_2|\xi|^2}{1 + |\xi|^2} \text{Re}\left(i\frac{W^{(2)}}{|W^{(4)}|}\right),$$

$$E^{(4)} := |W^{(4)}|^2 + \frac{\gamma_3|\xi|^2}{1 + |\xi|^2} \text{Re}\left(\frac{W^{(2)}}{|W^{(4)}|}\right) + \frac{\gamma_4|\xi|^2}{1 + |\xi|^2} \text{Re}\left(i\frac{W^{(2)}}{|W^{(4)}|}\right),$$

where $\gamma_1 = \gamma_3 = \min\left\{\frac{1}{16}, \frac{2}{16}\right\}$ and $\gamma_2 = -\gamma_4 = \frac{9}{2}$. Then, we can follow the similar procedure of the estimate of $E^{(1)} + E^{(3)}$. Finally, it follows

$$|W^{(2)}(t, \xi)|^2 + |W^{(4)}(t, \xi)|^2 \leq e^{-c\rho(|\xi|)t} \left(|W^{(2)}_0(\xi)|^2 + |W^{(4)}_0(\xi)|^2\right),$$

with a positive constant $c > 0$. Consequently, the proof is complete. \(\square\)

We concentrate on the estimates of the classical energy and higher-order energy of solutions with initial data taking from $H^s(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)$. Because of the proof is quite standard, we only sketch it.

**Theorem 2.1.** Let us consider the Cauchy problem (1.3) with initial data satisfying \(\left(|D|u_0^{(k)}, u_1^{(k)}\right) \in (H^s(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)) \times (H^s(\mathbb{R}^2) \cap L^m(\mathbb{R}^2))\) for $k = 1, 2$, where $s \geq 0$ and $m \in [1, 2]$. Then, we have the following estimates for the energies of higher-order:

$$\left\| |D|^{s+1} u^{(k)}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} + \left\| |D|^s u^{(k)}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} \lesssim (1 + t)^{-\frac{2-m+s}{2m}} \sum_{k=1}^2 \left\| (|D|u_0^{(k)}, u_1^{(k)}) \right\|_{(H^s(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)) \times (H^s(\mathbb{R}^2) \cap L^m(\mathbb{R}^2))}.$$  

**Proof.** We can derive the following estimates by the Parseval-Plancherel theorem and the pointwise estimates of the partial Fourier transform of solutions in the Fourier space in Lemma 2.1:

$$\left\| |D|^s \mathcal{F}^{-1}_{\xi \rightarrow x} (W)(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} = \left\| |\xi|^s \mathcal{F} (W)(t, \xi) \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| |\xi|^s e^{-c\rho(|\xi|)t} W_0(\xi) \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| \chi_{\text{int}}(\xi) |\xi|^s e^{-c|\xi|^2 t} W_0(\xi) \right\|_{L^2(\mathbb{R}^2)} + \left\| (\chi_{\text{mid}}(\xi) + \chi_{\text{ext}}(\xi)) |\xi|^s e^{-c|\xi|^2 t} W_0(\xi) \right\|_{L^2(\mathbb{R}^2)}.$$  

For small frequencies, we apply Hölder’s inequality and the Hausdorff-Young inequality to get

$$\left\| \chi_{\text{int}}(\xi) |\xi|^s e^{-c|\xi|^2 t} W_0(\xi) \right\|_{L^2(\mathbb{R}^2)} \lesssim (1 + t)^{-\frac{2-m+s}{2m}} \left\| \mathcal{F}^{-1}(W_0) \right\|_{L^m(\mathbb{R}^2)}.$$  

For middle and large frequencies, we immediately obtain

$$\left\| (\chi_{\text{mid}}(\xi) + \chi_{\text{ext}}(\xi)) |\xi|^s e^{-c|\xi|^2 t} W_0(\xi) \right\|_{L^2(\mathbb{R}^2)} \lesssim e^{-ct} \left\| \mathcal{F}^{-1}(W_0) \right\|_{H^s(\mathbb{R}^2)}.$$  

Together these complete the proof. \(\square\)

**Remark 2.2.** To derive the estimate of the solution itself to (1.3) with \(\left(|D|u_0^{(k)}, u_1^{(k)}\right) \in (L^2(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)) \times (L^2(\mathbb{R}^2) \cap L^m(\mathbb{R}^2))\), we need to estimate \(\left\| \chi_{\text{int}}(\xi) |\xi|^{-1} W(t, \xi) \right\|_{L^2(\mathbb{R}^2)}\). Then, using Hölder’s inequality, we have to derive the estimate the following term:

$$\left\| \chi_{\text{int}}(\xi) |\xi|^{-1} e^{-c|\xi|^2 t} \right\|_{L^{\frac{2m}{2m-1}}(\mathbb{R}^2)}.$$  

Nevertheless, due to a strong influence of the singularity for $|\xi| \to +0$, i.e., non integrability, the following inequality does not hold for all $t \geq 0$ and $m \in [1, 2]$:

$$\left\| \chi_{\text{int}}(\xi) |\xi|^{-1} e^{-c|\xi|^2 t} \right\|_{L^{\frac{2m}{2m-1}}(\mathbb{R}^2)} < \infty.$$
Remark 2.3. It is not reasonable for us to compare the estimates in Theorem 2.1 with those in [40] due to different assumptions of the data spaces. In the recent paper [40], the authors proved estimates for the classical energy to (1.3) with initial data taking from $(H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)) \times (L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2))$ by using the energy method in the Fourier space and the Haraux-Komornik inequality.

Next, we will establish $L^p - L^q$ estimates on the conjugate line by applying Lemma 2.1.

Theorem 2.2. Let us consider the Cauchy problem (1.3) with initial data satisfying $(|D|u_0^{(k)}, u_1^{(k)}) \in S(\mathbb{R}^2) \times S(\mathbb{R}^2)$ for $k = 1, 2$. Then, the following estimates hold for the derivatives of the solution:

$$
\left\| |D|^{s+1}u^{(k)}(t, \cdot) \right\|_{L^q(\mathbb{R}^2)} + \left\| |D|^{s}u_t^{(k)}(t, \cdot) \right\|_{L^q(\mathbb{R}^2)}
\lesssim (1 + t)^{-\frac{q}{2}(s+2(\frac{1}{p} - \frac{1}{q}))} \sum_{k=1}^{2} \left\| (|D|u_0^{(k)}, u_1^{(k)}) \right\|_{H^s_p \times H^s_{1-p} (\mathbb{R}^2)},
$$

where $s \geq 0$, $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$ and $N_{s,p} > s + 2 \left( \frac{1}{p} - \frac{1}{q} \right)$.

Remark 2.4. If we are interested in the case $p \in (1, 2]$, then we can choose $N_{s,p} = s + 2 \left( \frac{1}{p} - \frac{1}{q} \right)$.

Proof. The proof of $L^p - L^q$ decay estimates on the conjugate line is divided into three parts.

Part 1: $L^p - L^q$ decay estimates for small frequencies.

According to the derived $L^2 - L^2$ estimates in Theorem 2.1 we have

$$
\left\| \chi_\text{int}(D)|D|^s \mathcal{F}^{-1}(W)(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} \lesssim (1 + t)^{-\frac{q}{2}} \left\| \mathcal{F}^{-1}(W) \right\|_{L^2(\mathbb{R}^2)},
$$

(2.21)

According to Lemma 2.1 we give a $L^1 - L^\infty$ estimate by applying the Hausdorff-Young inequality

$$
\left\| \chi_\text{int}(D)|D|^s \mathcal{F}^{-1}(W)(t, \cdot) \right\|_{L^\infty(\mathbb{R}^2)} \lesssim \left\| \chi_\text{int}(\xi) |\xi|^se^{-c|\xi|^2t} \right\|_{L^1(\mathbb{R}^2)} \left\| W \right\|_{L^\infty(\mathbb{R}^2)}
\lesssim \left\| \chi_\text{int}(\xi) |\xi|^se^{-c|\xi|^2t} \right\|_{L^1(\mathbb{R}^2)} \left\| \mathcal{F}^{-1}(W) \right\|_{L^1(\mathbb{R}^2)}
\lesssim \left( \int_0^\infty r^{s+1}e^{-cr^2t}dr \right) \left\| \mathcal{F}^{-1}(W) \right\|_{L^1(\mathbb{R}^2)}
\lesssim (1 + t)^{-\frac{q}{2}} \left\| \mathcal{F}^{-1}(W) \right\|_{L^1(\mathbb{R}^2)},
$$

(2.22)

Combining (2.21) with (2.22) and applying the Riesz-Thorin interpolation theorem, we obtain

$$
\left\| \chi_\text{int}(D)|D|^s \mathcal{F}^{-1}(W)(t, \cdot) \right\|_{L^q(\mathbb{R}^2)} \lesssim (1 + t)^{-\frac{q}{2}(s+2(\frac{1}{p} - \frac{1}{q}))} \left\| \mathcal{F}^{-1}(W) \right\|_{L^p(\mathbb{R}^2)},
$$

(2.23)

where $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Part 2: $L^p - L^q$ decay estimates for large frequencies.

For $L^2 - L^2$ estimates, we have

$$
\left\| \chi_\text{ext}(D)|D|^s \mathcal{F}^{-1}(W)(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| (D)^s \mathcal{F}^{-1}(W) \right\|_{L^2(\mathbb{R}^2)}.
$$

We now derive $L^1 - L^\infty$ estimates. For $t \in (0, 1]$, the following estimate holds:

$$
\left\| \chi_\text{ext}(D)|D|^s \mathcal{F}^{-1}(W)(t, \cdot) \right\|_{L^\infty(\mathbb{R}^2)} \lesssim \left\| \langle \xi \rangle^{-2+\epsilon} \right\|_{L^1(\mathbb{R}^2)} \left\| \langle \xi \rangle^{s+2+\epsilon} W_0 \right\|_{L^\infty(\mathbb{R}^2)}
\lesssim \left\| (D)^{s+2+\epsilon} \mathcal{F}^{-1}(W_0) \right\|_{L^1(\mathbb{R}^2)},
$$

(2.24)

where $\epsilon$ is an arbitrary small positive constant. For $t \in [1, \infty)$, we immediately obtain exponential decay. Thus, combining with (2.24) we derive

$$
\left\| \chi_\text{ext}(D)|D|^s \mathcal{F}^{-1}(W)(t, \cdot) \right\|_{L^\infty(\mathbb{R}^2)} \lesssim e^{-ct} \left\| (D)^{s+2+\epsilon} \mathcal{F}^{-1}(W_0) \right\|_{L^1(\mathbb{R}^2)}.
$$
Using the interpolation theorem again, we obtain

\[
\|\chi_{\text{mid}}(D)|D|^{s}F_{\xi \to x}^{-1}(W)(t, \cdot)\|_{L^{p}(\mathbb{R}^{2})} \lesssim e^{-ct}\|F^{-1}(W_{0})\|_{L^{q}(\mathbb{R}^{2})},
\]

where \(1 \leq p \leq 2\) and \(\frac{1}{p} + \frac{1}{q} = 1\).

Part 3: \(L^{p} - L^{q}\) decay estimates for middle frequencies.

According to the pointwise estimate of the partial Fourier transform of solutions in Lemma 2.1 when \(\varepsilon \leq |\xi| \leq 1\), we derive

\[
\|\chi_{\text{mid}}(D)|D|^{s}F_{\xi \to x}^{-1}(W)(t, \cdot)\|_{L^{2}(\mathbb{R}^{2})} \lesssim e^{-ct}\|F^{-1}(W_{0})\|_{L^{2}(\mathbb{R}^{2})},
\]

Then, we have from interpolation theorem

\[
\|\chi_{\text{mid}}(D)|D|^{s}F_{\xi \to x}^{-1}(W)(t, \cdot)\|_{L^{\infty}(\mathbb{R}^{2})} \lesssim e^{-ct}\|F^{-1}(W_{0})\|_{L^{1}(\mathbb{R}^{2})},
\]

where \(1 \leq p \leq 2\) and \(\frac{1}{p} + \frac{1}{q} = 1\).

Summarizing (2.23), (2.26) and (2.25), we complete the proof. \(\square\)

3. Asymptotic expansions and asymptotic representation

With the aim of determining whether the pointwise estimates in the Fourier space in Lemma 2.1 are sharp or not, we now investigate the asymptotic expansion of eigenvalue of (2.6) for \(|\xi| \to 0\) and \(|\xi| \to \infty\). Moreover, to derive the asymptotic profiles of solutions, we need to get the asymptotic expressions of the propagator \(e^{i\Phi(|\xi|)}\) (see the definition in (2.7)).

3.1. Asymptotic expansions

We denote by \(\lambda_{j} = \lambda_{j}(|\xi|), j = 1, \ldots, 4\), the eigenvalues of matrix \(\hat{\Phi}(|\xi|)\). Thus, these eigenvalues are the solutions to the following characteristic equation:

\[
F(\lambda) = \det \left( \lambda I - \hat{\Phi}(|\xi|) \right) = \begin{vmatrix}
\lambda + \frac{b^{2}}{2}|\xi|^{2} - ib|\xi| & \frac{b^{2}}{2}|\xi|^{2} & 0 & 0 \\
0 & \lambda + \frac{a^{2}}{2}|\xi|^{2} - ia|\xi| & 0 & \frac{a^{2}}{2}|\xi|^{2} \\
\frac{a^{2}}{2}|\xi|^{2} & 0 & \lambda + \frac{b^{2}}{2}|\xi|^{2} + ib|\xi| & 0 \\
0 & \frac{a^{2}}{2}|\xi|^{2} & 0 & \lambda + \frac{a^{2}}{2}|\xi|^{2} + ia|\xi| \\
\end{vmatrix} = \lambda^{4} + (a^{2} + b^{2})|\xi|^{2}\lambda + \left( (a^{2} + b^{2})|\xi|^{2} + a^{2}b^{2}|\xi|^{4} \right) \lambda^{2} + 2a^{2}b^{2}|\xi|^{4},
\]

where \(|\xi|\) is regarded as a parameter. We notice that

\[
\frac{d}{d\lambda}F(\lambda) = 4\lambda^{3} + 3(a^{2} + b^{2})|\xi|^{2}\lambda^{2} + 2 \left( (a^{2} + b^{2})|\xi|^{2} + a^{2}b^{2}|\xi|^{4} \right) \lambda + 2a^{2}b^{2}|\xi|^{4}.
\]

Comparing the polynomials \(F(\lambda)\) and \(\frac{d}{d\lambda}F(\lambda)\), we observe there are non-trivial common divisors at most for value of frequencies \(|\xi|\) in a zero measure set. Thus, we have only simple roots of \(F(\lambda) = 0\) outside of this zero measure set.

Let \(P_{j}(|\xi|)\) be the corresponding eigenprojections, which can be expressed as the way

\[
P_{j}(|\xi|) = \prod_{k \neq j} \frac{\hat{\Phi}(|\xi|) - \lambda_{k}(|\xi|)I}{\lambda_{j}(|\xi|) - \lambda_{k}(|\xi|)},
\]

where \(I\) is a identity matrix of dimensions \(4 \times 4\).

In the next step we distinguish the asymptotic expansions of eigenvalues and their corresponding eigenprojections between two cases: \(|\xi| \to 0\) and \(|\xi| \to \infty\). These expansions essentially determine the asymptotic behavior of solutions.
3.1.1. Asymptotic expansions for $|ξ| \to 0$. We deduce that the eigenvalues $λ_j(|ξ|)$ and their corresponding eigenprojections $P_j(|ξ|)$ have the following asymptotic expansions for $|ξ| \to 0$, respectively:

$$
λ_j(|ξ|) = λ_j^{(0)} + λ_j^{(1)}|ξ| + λ_j^{(2)}|ξ|^2 + \cdots ,
$$
(3.29)

$$
P_j(|ξ|) = P_j^{(0)} + P_j^{(1)}|ξ| + P_j^{(2)}|ξ|^2 + \cdots ,
$$
(3.30)

where $λ_j^{(k)} \in \mathbb{C}$, $P_j^{(k)} \in \mathbb{C}^{4×4}$ for all $k \in \mathbb{N}$.

Then, we substitute $λ = λ_j(|ξ|)$ chosen in (3.29) into the characteristic equation (3.27) and calculate the coefficients $λ_j^{(k)}$. After lengthy but straightforward calculations, the value of pairwise distinct coefficients are given by

$$
\begin{align*}
λ_1^{(0)} &= λ_2^{(0)} = λ_3^{(0)} = λ_4^{(0)} = 0, \\
λ_1^{(1)} &= ib, \quad λ_2^{(1)} = −ib, \quad λ_3^{(1)} = ia, \quad λ_4^{(1)} = −ia, \\
λ_1^{(2)} &= λ_2^{(2)} = −\frac{b^2}{2}, \quad λ_3^{(2)} = λ_4^{(2)} = −\frac{a^2}{2}.
\end{align*}
$$

Consequently, the eigenvalues have the asymptotic behaviors for $|ξ| \to 0$

$$
λ_1(|ξ|) = ib|ξ| − \frac{b^2}{2}|ξ|^2 + O(|ξ|^3), \quad λ_2(|ξ|) = −ib|ξ| − \frac{b^2}{2}|ξ|^2 + O(|ξ|^3),
$$
(3.31)

$$
λ_3(|ξ|) = i|ξ| − \frac{a^2}{2}|ξ|^2 + O(|ξ|^3), \quad λ_4(|ξ|) = −i|ξ| − \frac{a^2}{2}|ξ|^2 + O(|ξ|^3).
$$

By the pairwise distinct eigenvalues given in (3.31) and the matrix $\hat{Φ}(|ξ|)$ given in (2.7), we employ (3.28) to calculate $P_j^{(0)}$ that

$$
\begin{align*}
P_1^{(0)} &= \text{diag}(1, 0, 0, 0), & P_2^{(0)} &= \text{diag}(0, 0, 1, 0), \\
P_3^{(0)} &= \text{diag}(0, 1, 0, 0), & P_4^{(0)} &= \text{diag}(0, 0, 0, 1).
\end{align*}
$$
(3.32)

Thus, we get $P_j(|ξ|) − P_j^{(0)} = O(|ξ|)$ for $|ξ| \to 0$.

3.1.2. Asymptotic expansions for $|ξ| \to ∞$. Similarly, the eigenvalues $λ_j(|ξ|)$ and their corresponding eigenprojections $P_j(|ξ|)$ have the following asymptotic expansions for $|ξ| \to ∞$, respectively:

$$
λ_j(|ξ|) = λ_j^{(0)}|ξ|^2 + λ_j^{(1)}|ξ| + λ_j^{(2)}|ξ|^{-1} + λ_j^{(4)}|ξ|^{-2} + \cdots ,
$$
(3.33)

$$
P_j(|ξ|) = P_j^{(0)}|ξ|^2 + P_j^{(1)}|ξ| + P_j^{(2)}|ξ|^{-1} + P_j^{(4)}|ξ|^{-2} + \cdots ,
$$
(3.34)

where $λ_j^{(k)} \in \mathbb{C}$ and $P_j^{(k)} \in \mathbb{C}^{4×4}$ for all $k \in \mathbb{N}$.

We plug $λ = λ_j(|ξ|)$ chosen in (3.33) into the characteristic equation (3.27) and to obtain the pairwise distinct value of coefficients $λ_j^{(k)}$

$$
\begin{align*}
λ_1^{(0)} &= λ_2^{(0)} = 0, \quad λ_3^{(0)} = −b^2, \quad λ_4^{(0)} = −a^2, \\
λ_1^{(1)} &= λ_2^{(1)} = λ_3^{(1)} = λ_4^{(1)} = 0, \\
λ_1^{(2)} &= λ_2^{(2)} = −1, \quad λ_3^{(2)} = λ_4^{(2)} = 1, \\
λ_1^{(3)} &= λ_2^{(3)} = λ_3^{(3)} = λ_4^{(3)} = 0, \\
λ_1^{(4)} &= −\frac{1}{b^2}, \quad λ_2^{(4)} = −\frac{1}{a^2}, \quad λ_3^{(4)} = λ_4^{(4)} = \nu_0, \quad λ_4^{(4)} = \nu_a.
\end{align*}
$$

where $ν_0$ is the solutions of the algebraic equation $a^2b^2ν_0^2 + (5a^2 − 7b^2) ν_0 + 1 = 0$, and $ν_a$ is the solutions of the algebraic equation $a^2b^2ν_a^2 + (5b^2 − 7a^2) ν_a + 1 = 0$. 
Consequently, we investigate the following asymptotic behaviors of pairwise distinct eigenvalues \( \lambda_j(|\xi|) \), \( j = 1, \ldots, 4 \) for \( |\xi| \to \infty \):

\[
\begin{align*}
\lambda_1(|\xi|) &= -1 - \frac{1}{b^2} |\xi|^{-2} + O(|\xi|^{-3}), \\
\lambda_2(|\xi|) &= -1 - \frac{1}{a^2} |\xi|^{-2} + O(|\xi|^{-3}), \\
\lambda_3(|\xi|) &= -b^2 |\xi|^2 + 1 + \nu_b |\xi|^{-2} + O(|\xi|^{-3}), \\
\lambda_4(|\xi|) &= -a^2 |\xi|^2 + 1 + \nu_a |\xi|^{-2} + O(|\xi|^{-3}).
\end{align*}
\]

(3.35)

Also, by straightforward computations, we find that

\[
P_j^{(0)} = P_j^{(1)} = 0_{4 \times 4} \quad \text{for} \quad j = 1, 2, 3, 4,
\]

\[
P_1^{(2)} = \begin{pmatrix}
1 - \frac{b^2 \nu_a}{2} & 0 & \frac{b^2(a^2 + b^2) \nu_a}{2(b^2 - a^2)} & 0 \\
0 & -\frac{1}{2} + a^2 \nu_a & 0 & -\frac{1}{2} + a^2 \nu_a \\
\frac{b^2(a^2 + b^2) \nu_a}{2(b^2 - a^2)} & 0 & 1 + \frac{b^2 \nu_a}{2} & 0 \\
0 & -\frac{1}{2} + a^2 \nu_a & 0 & \frac{1}{2} + a^2 \nu_a
\end{pmatrix},
\]

\[
P_2^{(2)} = \begin{pmatrix}
-\frac{1}{2} + \frac{b^2 \nu_a}{2} & 0 & -\frac{1}{2} + \frac{b^2 \nu_a}{2} & 0 \\
0 & 1 - a^2 \nu_a & 0 & -a^2 \nu_a \\
-\frac{1}{2} + \frac{b^2 \nu_a}{2} & 0 & -\frac{1}{2} + \frac{b^2 \nu_a}{2} & 0 \\
0 & -a^2 \nu_a & 0 & 1 - a^2 \nu_a
\end{pmatrix},
\]

\[
P_3^{(2)} = \frac{1}{2} \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
P_4^{(2)} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix}.
\]

It leads to \( P_j(|\xi|) - P_j^{(0)} |\xi|^2 - P_j^{(1)} |\xi| - P_j^{(2)} = 0(|\xi|^{-1}) \) for \( |\xi| \to \infty \).

### 3.2. Asymptotic representation

In the last subsection we have calculated the asymptotic expansions of eigenvalues \( \lambda_j(|\xi|) \) and their corresponding eigenprojections \( P_j(|\xi|) \) for \( |\xi| \to 0 \) and \( |\xi| \to \infty \). In order to give the representation of solution \( W(t, \xi) = e^{i \hat{\Phi}(|\xi|)} W_0(|\xi|) \), motivated by \([12]\), in this part we will give asymptotic expressions of the propagator \( e^{i \hat{\Phi}(|\xi|)} \) for \( |\xi| \to 0 \) and \( |\xi| \to \infty \).

The propagator \( e^{i \hat{\Phi}(|\xi|)} \) has the following spectral decomposition:

\[
e^{i \hat{\Phi}(|\xi|)} = \sum_{j=1}^{4} e^{\lambda_j(|\xi|)^t} P_j(|\xi|),
\]

(3.37)

where \( \lambda_j(|\xi|) \) are the eigenvalues of \( \hat{\Phi}(|\xi|) \) and \( P_j(|\xi|) \) are their corresponding eigenprojections. Now, we distinguish between two cases: \( |\xi| \to 0 \) and \( |\xi| \to \infty \), respectively, to discuss the asymptotic expressions of (3.37).

#### 3.2.1. Asymptotic expansion of \( e^{i \hat{\Phi}(|\xi|)} \) for \( |\xi| \to 0 \)

We first define the matrix \( \hat{S}_0(t, \xi) \) by

\[
\hat{S}_0(t, \xi) := \sum_{j=1}^{4} e^{\lambda_j^0(|\xi|)^t} P_j^{(0)},
\]

(3.38)

where

\[
\begin{align*}
\lambda_1^0(|\xi|) &= ib|\xi| - \frac{b^2}{2} |\xi|^2, \\
\lambda_2^0(|\xi|) &= -ib|\xi| - \frac{b^2}{2} |\xi|^2, \\
\lambda_3^0(|\xi|) &= ia|\xi| - \frac{a^2}{2} |\xi|^2, \\
\lambda_4^0(|\xi|) &= -ia|\xi| - \frac{a^2}{2} |\xi|^2,
\end{align*}
\]

(3.39)

and \( P_j^{(0)} \) are given in (3.32). We then rewrite the propagator for \( |\xi| \to 0 \) by

\[
e^{i \hat{\Phi}(|\xi|)} = \hat{S}_0(t, \xi) + \hat{R}_0(t, \xi).
\]

(3.40)
Let us derive the pointwise estimate for the remainder $\hat{R}_0(t, \xi)$ for $|\xi| \to 0$.

**Lemma 3.1.** We have the following estimate of remainder:

$$\left| \hat{R}_0(t, \xi) \right| \lesssim |\xi| e^{-c|\xi|^2 t},$$  \hspace{1cm} (3.41)

where $0 < |\xi| \leq \varepsilon$ and $c$ is a positive constant.

**Proof.** From (3.37), (3.38) and (3.40), we can rewrite the remainder by

\[
\hat{R}_0(t, \xi) = \sum_{j=1}^{4} e^{\lambda_j(|\xi|) t} P_j(|\xi|) - \sum_{j=1}^{4} e^{\lambda_0(|\xi|) t} P_j^{(0)}
\]

\[
= \sum_{j=1}^{4} e^{\lambda_j(|\xi|) t} \left( P_j(|\xi|) - P_j^{(0)} \right) + \sum_{j=1}^{4} e^{\lambda_0(|\xi|) t} \left( e^{\lambda_j(|\xi|) t - \lambda_0(|\xi|) t} - 1 \right) P_j^{(0)}
\]

\[
=: \hat{R}_{0,1}(t, \xi) + \hat{R}_{0,2}(t, \xi).
\]

Due to the facts that $P_j(|\xi|) - P_j^{(0)} = O(|\xi|)$ and $e^{\lambda_j(|\xi|) t} \lesssim e^{-c|\xi|^2 t}$ for $|\xi| \to 0$, we immediately obtain the estimate

$$\left| \hat{R}_{0,1}(t, \xi) \right| \lesssim |\xi| e^{-c|\xi|^2 t} \text{ for } |\xi| \to 0.$$

By the similar way, because $\lambda_j(|\xi|) - \lambda_0(|\xi|) = O(|\xi|)$ and

$$e^{c|\xi|^2 t} - 1 = t|\xi|^2 \int_{0}^{1} e^{c|\xi|^2 t \nu} \, d\nu,$$

we have

$$\left| \hat{R}_{0,2}(t, \xi) \right| \lesssim \left| e^{\lambda_j(|\xi|) t} - \lambda_0(|\xi|) t - 1 \right| e^{-c|\xi|^2 t} \lesssim t|\xi|^3 e^{-c|\xi|^2 t} \lesssim |\xi| e^{-c|\xi|^2 t} \text{ for } |\xi| \to 0.$$  

Summarizing above estimates, we complete the proof. \hfill \square

### 3.2.2. Asymptotic expansion of $e^{t\Phi(|\xi|)}$ for $|\xi| \to \infty$.

From (3.33) and (3.35), we now define

$$\hat{S}_\infty(t, \xi) := \sum_{j=1}^{4} e^{\lambda_j^\infty(|\xi|) t} \left( P_j^{(0)} |\xi|^2 + P_j^{(1)} |\xi| + P_j^{(2)} \right),$$  \hspace{1cm} (3.42)

where

$$\lambda_j^\infty(|\xi|) = -1 - \frac{1}{b^2} |\xi|^{-2}, \quad \lambda_0^\infty(|\xi|) = -b^2 |\xi|^2 + 1 + \nu_b |\xi|^{-2},$$

$$\lambda_j^\infty(|\xi|) = -1 - \frac{1}{a^2} |\xi|^{-2}, \quad \lambda_0^\infty(|\xi|) = -a^2 |\xi|^2 + 1 + \nu_a |\xi|^{-2},$$  \hspace{1cm} (3.43)

and $P_j^{(0)}$, $P_j^{(1)}$, $P_j^{(2)}$ are given in (3.36). We write the propagator for $|\xi| \to \infty$

$$e^{t\Phi(|\xi|)} = \hat{S}_\infty(t, \xi) + \hat{R}_\infty(t, \xi).$$ \hspace{1cm} (3.44)

Then, we can derive the following pointwise estimate for the remainder $\hat{R}_\infty(t, \xi)$ for $|\xi| \to \infty$.

**Lemma 3.2.** We have the following estimate of remainder:

$$\left| \hat{R}_\infty(t, \xi) \right| \lesssim e^{-ct},$$  \hspace{1cm} (3.45)

where $|\xi| \geq \frac{1}{\varepsilon}$ and $c$ is a positive constant.
Proof. From (3.42) and (3.44), we obtain
\[ \hat{R}_\infty(t, \xi) = \sum_{j=1}^{4} e^{\lambda_j(|\xi|)t} P_j(|\xi|) - \sum_{j=1}^{4} e^{\lambda_j^\infty(|\xi|)t} \left( P_j^{(0)}|\xi|^2 + P_j^{(1)}|\xi| + P_j^{(2)} \right) \]
\[ = \sum_{j=1}^{4} e^{\lambda_j(|\xi|)t} \left( P_j(|\xi|) - P_j^{(2)} \right) + \sum_{j=1}^{4} e^{\lambda_j^\infty(|\xi|)t} \left( e^{\lambda_j(|\xi|)t} - \lambda_j^\infty(|\xi|)t - 1 \right) P_j^{(2)} \]
\[ =: \hat{R}_{\infty,1}(t, \xi) + \hat{R}_{\infty,2}(t, \xi). \]
Using the facts that \( P_j(|\xi|) - P_j^{(2)} = O(|\xi|^{-1}) \) and \( e^{\lambda_j(|\xi|)t} \lesssim e^{-ct} \) for \(|\xi| \to \infty\), we immediately obtain the estimate
\[ \left| \hat{R}_{\infty,1}(t, \xi) \right| \lesssim |\xi|^{-1} e^{-ct} \leq e^{-ct} \text{ for } |\xi| \to \infty. \]
Since \( \lambda_j(|\xi|) - \lambda_j^\infty(|\xi|) = O(|\xi|^{-3}) \) we have
\[ \left| \hat{R}_{\infty,2}(t, \xi) \right| \lesssim |\xi|^{3} e^{-ct} \leq e^{-ct} \text{ for } |\xi| \to \infty. \]
Thus, the proof is completed.

3.2.3. Conclusion for asymptotic representation. Finally, combining of the estimate of \( \hat{R}_0(t, \xi) \) and \( \hat{R}_\infty(t, \xi) \), we can immediately prove the following statement for the asymptotic expansions of the propagator \( e^{it\Phi(|\xi|)} \). The proof of the next theorem strictly follows the proof of Lemma 4.3 from the paper [12].

**Theorem 3.1.** We have the following asymptotic expansions:
\[ e^{it\tilde{\Phi}(|\xi|)} = \hat{S}_0(t, \xi) + \hat{S}_\infty(t, \xi) + \hat{R}(t, \xi), \]
where the remainder \( \hat{R}(t, \xi) \) satisfies the estimates
\[ \left| \hat{R}(t, \xi) \right| \lesssim \begin{cases} |\xi| e^{-c|\xi|^2t} & \text{for } |\xi| \leq \epsilon, \\ e^{-ct} & \text{for } |\xi| \geq \epsilon. \end{cases} \]

4. Asymptotic profiles

The purpose of this section is to investigate the asymptotic profiles of solutions to (2.6). According to Theorems 2.1 and 2.2 in the previous section, we observe that the decay rate of these estimates is dominated by the behavior of the eigenvalues for \(|\xi| \to 0\). For frequencies in the bounded and large zones, they imply exponential decay providing that we assume a suitable regularity of initial data. Thus, we only explain the asymptotic profiles of solutions for the case \(|\xi| \to 0\).

To begin with, let us introduce the following reference system:
\[ \begin{cases} \ddot{u} - \frac{1}{2} \tilde{M}^2 \Delta \tilde{u} + i \tilde{M} (-\Delta)^{1/2} \tilde{u} = 0, & t > 0, \ x \in \mathbb{R}^2, \\ \tilde{u}(0, x) = \tilde{u}_0(x) := \mathcal{F}^{-1}(W_0)(x), & x \in \mathbb{R}^2, \end{cases} \]
where \( \tilde{u} = (\tilde{u}^{(1)}, \tilde{u}^{(2)}, \tilde{u}^{(3)}, \tilde{u}^{(4)})^T \) and \( \tilde{M} = \text{diag}(-b, -a, b, a) \).
This reference system is consisted of two different evolution equations as follows:
\[ \text{heat equation: } \ 
\ddot{u}_+ - \Delta \tilde{u}^+ = 0, \\
\text{half-wave equation: } \ 
\ddot{u}_- + i (-\Delta)^{1/2} \tilde{u}^- = 0, \]
with the suitable initial data.
Remark 4.1. We point out that the influence of friction and Kelvin-Voigt damping to the asymptotic profiles of solution. Considering the linear dissipative elastic waves

\[ u_{tt} - a^2 \Delta u - (b^2 - a^2) \nabla \operatorname{div} u + u_t = 0, \quad (4.48) \]
\[ u_{tt} - a^2 \Delta u - (b^2 - a^2) \nabla \operatorname{div} u + (-a^2 \Delta - (b^2 - a^2) \nabla \operatorname{div}) u_t = 0, \quad (4.49) \]

with \( b^2 > a^2 > 0 \). From the recent papers \([32, 4]\), we observe that the reference systems to (4.48) can be described by heat equations and heat equations with positive mass. However, when friction replaced by Kelvin-Voigt damping, i.e. dissipative system (4.49), the reference system is consisted of heat equations and half-wave equations.

Taking partial Fourier transform such that \( \hat{W}(t, \xi) = \mathcal{F}_{x \to \xi} \hat{u}(t, x) \) to (4.47), we obtain

\[
\begin{cases}
\hat{W}_t + \frac{1}{2} |\xi|^2 \hat{M}^2 \hat{W} + i |\xi| \hat{M} \hat{W} = 0, \quad t > 0, \ \xi \in \mathbb{R}^2, \\
\hat{W}(0, \xi) = W_0(\xi), \quad \xi \in \mathbb{R}^2.
\end{cases}
\]

(4.50)

According to (3.38), (3.31) and (3.32), we can explicitly express \( \hat{S}_0(t, \xi) \) by

\[
\hat{S}_0(t, \xi) = \begin{cases}
\operatorname{diag} \left( e^{\lambda^1(t)}(\xi)t, e^{\lambda^2(t)}(\xi)t, e^{\lambda^3(t)}(\xi)t, e^{\lambda^4(t)}(\xi)t \right), \\
= \begin{cases}
\operatorname{diag} \left( e^{i b |\xi| t - \frac{\xi^2}{2} |\xi|^2 t, e^{i a |\xi| t - \frac{\xi^2}{2} |\xi|^2 t, e^{-i b |\xi| t - \frac{\xi^2}{2} |\xi|^2 t}, e^{-i a |\xi| t - \frac{\xi^2}{2} |\xi|^2 t} \right) \right) \quad (4.51)
\end{cases}
\]

Then, we know by direct computation that \( \hat{S}_0(t, \xi) \) is the solution of the first-order system (4.50).

For one thing, considering initial data satisfying \( |D| u_0^{(k)}, u_1^{(k)} \in L^m(\mathbb{R}^2) \times L^m(\mathbb{R}^2) \) with \( m \in [1, 2] \) for \( k = 1, 2 \), we state our first result.

Theorem 4.1. Let us consider the Cauchy problem (1.3) with initial data satisfying \( |D| u_0^{(k)}, u_1^{(k)} \in L^m(\mathbb{R}^2) \times L^m(\mathbb{R}^2) \) for \( k = 1, 2 \). Then, we obtain for the solution \( W = W(t, \xi) \) to the Cauchy problem (2.6) the refinement estimate

\[ \left\| \chi_{\text{int}}(D) \mathcal{F}_{\xi \to x} (W - \hat{S}_0 W_0)(t, \cdot) \right\|_{H^s(\mathbb{R}^2)} \lesssim (1 + t)^{-\frac{2-m+s}{2m}} \sum_{k=1}^{2} \left\| (|D| u_0^{(k)}, u_1^{(k)}) \right\|_{L^m(\mathbb{R}^2) \times L^m(\mathbb{R}^2)}, \]

where \( s \geq 0 \) and \( m \in [1, 2] \).

Proof. According to (3.40) and Lemma 3.1 we can estimate

\[
\left| \chi_{\text{int}}(\xi) |\xi|^s (W(t, \xi) - \hat{S}_0(t, \xi) W_0(\xi)) \right| = \left| \chi_{\text{int}}(\xi) |\xi|^s (e^{t \hat{S}_0(t, \xi) W_0(\xi)} - \hat{S}_0(t, \xi) W_0(\xi)) \right|
\]
\[ \lesssim \chi_{\text{int}}(\xi) |\xi|^{s+1} e^{-c |\xi|^2 t} |W_0(\xi)|. \]

Then, applying the Parseval-Plancherel theorem we derive

\[ \left\| \chi_{\text{int}}(D) \mathcal{F}_{\xi \to x} (W - \hat{S}_0 W_0)(t, \cdot) \right\|_{H^s(\mathbb{R}^2)} \lesssim \left\| \chi_{\text{int}}(\xi) |\xi|^{s+1} e^{-c |\xi|^2 t} W_0(\xi) \right\|_{L^2(\mathbb{R}^2)}. \]

Following the procedure of the proof of Theorem 2.1 we immediately complete this proof. \( \square \)

For another, we consider the estimates basing on the \( L^q \) norm for \( q \in [2, \infty] \) in the next statement.

Theorem 4.2. Let us consider the Cauchy problem (1.3) with initial data satisfying \( |D| u_0^{(k)}, u_1^{(k)} \in S(\mathbb{R}^2) \times S(\mathbb{R}^2) \) for \( k = 1, 2 \). Then, we obtain for the solution \( W = W(t, \xi) \) to the Cauchy problem
(2.6) the refinement estimate
\[
\left\| \chi_{\text{int}}(D)\mathcal{F}_{\xi \to x}^{-1}(W - \tilde{S}_0 W_0)(t, \cdot) \right\|_{H^s(\mathbb{R}^2)} \lesssim (1 + t)^{-\frac{s}{2} + \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \sum_{k=1}^{2} \left\| \left( |D|u_0^{(k)}, u_1^{(k)} \right) \right\|_{L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2)},
\]
where \( s \geq 0 \) and \( 1 \leq p \leq 2 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof.** Combining the proof of Theorem 2.2 and (4.52), the proof is complete. \( \square \)

**Remark 4.2.** Comparing Theorems 2.1 and 2.2 with Theorems 4.1 and 4.2 we observe that by subtracting \( \tilde{S}_0(t, \xi)W_0(\xi) \) in the estimates, the decay rate \((1 + t)^{-\frac{s}{2}}\) can be gained.

Before we prove the asymptotic profiles with initial data belonging to weighted \( L^1 \) spaces, we recall the useful tool that Lemma 2.1 stated in [13].

**Lemma 4.1.** Let \( \gamma \in [0, 1] \) and \( g \in L^{1, \gamma}(\mathbb{R}^n) \). Then, the following estimate holds:
\[
|\hat{g}(\xi)| \leq C_\gamma |\xi|^\gamma \|g\|_{L^{1, \gamma}(\mathbb{R}^n)} + \left| \int_{\mathbb{R}^n} g(x) \, dx \right|,
\]
with some constant \( C_\gamma > 0 \).

**Theorem 4.3.** Let us consider the Cauchy problem (1.3) with initial data satisfying \( \left( |D|u_0^{(k)}, u_1^{(k)} \right) \in L^{1, \gamma}(\mathbb{R}^2) \times L^{1, \gamma}(\mathbb{R}^2) \) for \( k = 1, 2 \). Then, we obtain for the solution \( W = W(t, \xi) \) to the Cauchy problem (2.6) the refinement estimate
\[
\left\| \chi_{\text{int}}(D)\mathcal{F}_{\xi \to x}^{-1}(W - \tilde{S}_0 W_0)(t, \cdot) \right\|_{H^s(\mathbb{R}^2)} \lesssim (1 + t)^{-\frac{s}{2} + \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \sum_{k=1}^{2} \left\| \left( |D|u_0^{(k)}, u_1^{(k)} \right) \right\|_{L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)}
+ (1 + t)^{-\frac{s}{2} + \frac{1}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \sum_{k=1}^{2} \left| \int_{\mathbb{R}^2} \left( |D|u_0^{(k)}(x) + u_1^{(k)}(x) \right) \, dx \right|,
\]
where \( s \geq 0 \) and \( \gamma \in [0, 1] \).

**Remark 4.3.** Theorem 4.3 shows that if we take initial data satisfying
\[
\left| \int_{\mathbb{R}^2} \left( |D|u_0^{(k)}(x) + u_1^{(k)}(x) \right) \, dx \right| = 0 \quad \text{for} \quad k = 1, 2,
\]
then the decay rates given in Theorem 4.1 when \( m = 1 \) can be improved by \((1 + t)^{-\frac{\gamma}{2}}\) for \( \gamma \in [0, 1] \).

**Proof.** From our derived estimate (4.52), we get
\[
\left| \chi_{\text{int}}(\xi)|\xi|^s(W(t, \xi) - \tilde{S}_0(t, \xi)W_0(\xi)) \right| 
\lesssim \chi_{\text{int}}(\xi)|\xi|^{s+1}e^{-c|\xi|^2t}|W_0(\xi)|
\lesssim \chi_{\text{int}}(\xi)|\xi|^{s+1+\gamma}e^{-c|\xi|^2t} \sum_{k=1}^{2} \left\| \left( |D|u_0^{(k)}, u_1^{(k)} \right) \right\|_{L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)}
+ \chi_{\text{int}}(\xi)|\xi|^{s+1}e^{-c|\xi|^2t} \sum_{k=1}^{2} \left| \int_{\mathbb{R}^2} \left( |D|u_0^{(k)}(x) + u_1^{(k)}(x) \right) \, dx \right|.
\]
Next, using the Parseval-Plancherel theorem we immediately complete the proof. \( \square \)
5. Weakly coupled system of semilinear elastic waves with Kelvin-Voigt damping in 2D

One of our goal in this paper is to develop sharp energy estimates of solutions to the linear elastic waves with Kelvin-Voigt damping in 2D with initial data \((u_0^{(k)}, u_1^{(k)}) \in \mathcal{A}_{m,s}(\mathbb{R}^2)\) for \(k = 1, 2\), where \(s \geq 0\) and \(m \in [1, 2]\). If initial data is supposed to \(\mathcal{A}_{m,s}(\mathbb{R}^2)\) for \(s \geq 0\) and \(m \in [1, 2]\), one can obtain the next theorem.

**Theorem 5.1.** Let us consider the Cauchy problem (1.3) with initial data satisfying \((u_0^{(k)}, u_1^{(k)}) \in \mathcal{A}_{m,s}(\mathbb{R}^2)\) for \(k = 1, 2\), where \(s \geq 0\) and \(m \in [1, 2]\). Then, we have the following estimates:

\[
\left\| u^{(k)}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} \lesssim (1 + t)^{-\frac{2-m}{2m}} \left( \sum_{k=1}^{2} \left\| (u_0^{(k)}, u_1^{(k)}) \right\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)} \right),
\]

\[
\left\| D u^{(k)}(t, \cdot) \right\|_{H^s(\mathbb{R}^2)} + \left\| u_t^{(k)}(t, \cdot) \right\|_{H^s(\mathbb{R}^2)} \lesssim (1 + t)^{-\frac{2-m+m}{2m}} \left( \sum_{k=1}^{2} \left\| (u_0^{(k)}, u_1^{(k)}) \right\|_{\mathcal{A}_{m,s}(\mathbb{R}^2)} \right).
\]

**Proof.** To get the estimate (5.54), we recall the following estimate in the Fourier space from Lemma 2.4 in the recent paper [40]:

\[
|\xi|^2 |\hat{u}(t, \xi)|^2 + |\hat{u}_t(t, \xi)|^2 \lesssim e^{-c\rho_{\varepsilon_1}(|\xi|) t} \left( |\xi|^2 |\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2 \right),
\]

where

\[
\rho_{\varepsilon_1}(|\xi|) := \begin{cases} \varepsilon_1 |\xi|^2 & \text{for } |\xi| \leq 1, \\
\varepsilon_1 & \text{for } |\xi| > 1,
\end{cases}
\]

with \(\varepsilon_1 > 0\). Then, the application of Hölder inequality and the Hausdorff-Young inequality immediately implies (5.53).

To develop the estimate of the solution itself, we only need to combine the following integral formula:

\[
u^{(k)}(t, x) = \int_0^t u^{(k)}(\tau, x) d\tau + u_0^{(k)}(x)
\]

and the estimate for \(\left\| u_t^{(k)}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)}\) in (5.54) to complete (5.53). \qed

Let us consider the following weakly coupled system of semilinear elastic waves with Kelvin-Voigt damping in two dimensional space:

\[
\begin{cases}
u_{tt} - a^2 \Delta u - (b^2 - a^2) \nabla \operatorname{div} u + (-a^2 \Delta - (b^2 - a^2) \nabla \operatorname{div} u) u_t = f(u), & t > 0, \ x \in \mathbb{R}^2, \\
(u, u_t)(0, x) = (u_0, u_1)(x), & \ x \in \mathbb{R}^2,
\end{cases}
\]

where \(b^2 > a^2 > 0\) and the nonlinear terms on the right-hand sides are

\[
f(u) := \left( |u^{(2)}|^{p_1} + |u^{(1)}|^{p_2} \right)^T \text{ with } p_1, p_2 > 1.
\]

Using estimates (5.53), (5.54), Duhamel’s principle and some tools in Harmonic Analysis (e.g. the Gagliardo-Nirenberg inequality), one can prove the global (in time) existence of small data Sobolev solutions to (5.55).

Before stating our result for the global (in time) existence of small data energy solution, we introduce the balanced exponent \(p_{bal}(m)\) by

\[
p_{bal}(m) := \frac{2(m+2)}{2-m} \text{ with } m \in [1, 2],
\]

and the balanced parameters

\[
\alpha_k(m) := \frac{2(1+m) + (3m+2)p_k + mp_1p_2}{2(p_1p_2 - 1)} \text{ with } m \in [1, 2] \text{ for } k = 1, 2.
\]
Remark 5.1. We observe the relation between the balanced exponent (5.56) and balanced parameters (5.57). For one thing, if we consider the condition $\alpha_1(m) < 1$, it also can be rewritten by
\[
p_1\left(p_2 + 1 - p_{\text{bal}}(m)\right) > p_{\text{bal}}(m).
\]

For another, if we consider the condition $\alpha_2(m) < 1$, it also can be rewritten by
\[
p_2\left(p_1 + 1 - p_{\text{bal}}(m)\right) > p_{\text{bal}}(m).
\]

Theorem 5.2. Let $b^2 > a^2 > 0$ in (5.55). Let us assume $p_1, p_2 > 1$, satisfying one of the following conditions:
1. we assume $p_{\text{bal}}(m) < \min\{p_1; p_2\}$;
2. we assume $\alpha_1(m) < 1$ if $\frac{2}{m} \leq p_2 \leq p_{\text{bal}}(m) < p_1$;
3. we assume $\alpha_2(m) < 1$ if $\frac{2}{m} \leq p_1 \leq p_{\text{bal}}(m) < p_2$.

Then, there exists a constant $\varepsilon_0 > 0$ such that for any initial data $\left(u_0^{(k)}, u_1^{(k)}\right) \in \mathcal{A}_{m,0}(\mathbb{R}^2)$ for $k = 1, 2$, and $m \in [1, 2]$ with
\[
\left\|\left(u_0^{(1)}, u_1^{(1)}\right)\right\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)} + \left\|\left(u_0^{(2)}, u_1^{(2)}\right)\right\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)} \leq \varepsilon_0,
\]
there is uniquely determined energy solution $u \in \left(\mathbf{C}([0, \infty), H^1(\mathbb{R}^2)) \cap \mathbf{C}^1([0, \infty), L^2(\mathbb{R}^2))\right)^2$ to (5.55). Moreover, the solutions satisfies the following estimates:
\[
\left\|u^{(k)}(t, \cdot)\right\|_{L^2(\mathbb{R}^2)} \lesssim (1 + t)^{1 - \frac{2 - m}{2m} + \ell_k} + 2 \sum_{k=1}^{2} \left\|u_0^{(k)}, u_1^{(k)}\right\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)},
\]

\[
\left\|D|u^{(k)}(t, \cdot)|\right\|_{L^2(\mathbb{R}^2)} + \left\|u^{(k)}(t, \cdot)\right\|_{L^2(\mathbb{R}^2)} \lesssim (1 + t)^{-\frac{2 - m}{2m} + \ell_k} + 2 \sum_{k=1}^{2} \left\|u_0^{(k)}, u_1^{(k)}\right\|_{\mathcal{A}_{m,0}(\mathbb{R}^2)},
\]

where
\[
0 \leq \ell_k = \ell_k(m, p_k) := \begin{cases} 0 & \text{if } p_k > p_{\text{bal}}(m), \\ \varepsilon_0 & \text{if } p_k = p_{\text{bal}}(m), \\ \frac{2 - m}{2m} (p_{\text{bal}}(m) - p_k) & \text{if } p_k < p_{\text{bal}}(m), \end{cases}
\]

represent the (no) loss of decay in comparison with the corresponding decay estimates for the solution to the linear Cauchy problem (1.3) (see Theorem 5.1), with $\varepsilon_0 > 0$ being an arbitrary small constant in the limit cases that $p_k = p_{\text{bal}}(m)$ for $k = 1, 2$.

Remark 5.2. Let us recall the weakly coupled system of semilinear damped wave equations
\[
\begin{aligned}
&u_{tt} - \Delta u + u_t = |u|^{p_1}, & t > 0, & x \in \mathbb{R}^n, \\
v_{tt} - \Delta v + v_t = |u|^{p_2}, & t > 0, & x \in \mathbb{R}^n, \\
(u, u_0, v, v_0)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{aligned}
\]

with $n \in \mathbb{N}$ and $p_1, p_2 > 1$. The papers [26, 27, 39] proved the condition for the existence of global (in time) Sobolev solutions to (5.59), which can be described by
\[
\frac{1 + \max\{p_1; p_2\}}{p_1p_2 - 1} < \frac{n}{2}, \quad \text{and especially in two dimensional case} \quad \frac{1 + \max\{p_1; p_2\}}{p_1p_2 - 1} < 1.
\]

For the system (5.55), we interpret the term $(-a^2 \Delta - (b^2 - a^2) \nabla \text{div}) u_t$ as a damping term for the elastic waves. Thus, we point out the condition for the global (in time) existence of small data energy solution to (5.55) is
\[
\alpha_{\text{max}}(m) = \max\{\alpha_1(m); \alpha_2(m)\} = \frac{1 + m + \frac{2m+2}{2} \max\{p_1; p_2\} + \frac{n}{2} p_1 p_2}{p_1 p_2 - 1} < 1.
\]
Proof. First of all, by Duhamel’s principle, we can reduce to consider the linear problem (1.3). In the following, we denote by \( K_0 = K_0(t, x) \) and \( K_1 = K_1(t, x) \) the fundamental solutions to the linear problem, corresponding to initial data, namely,

\[
u(t, x) = K_0(t, x) *_{(x)} u_0(x) + K_1(t, x) *_{(x)} u_1(x)
\]

is the solution to (1.3).

Let us define for \( T > 0 \) the spaces of solutions \( X(T) \) by

\[
X(T) := \left( \mathcal{C} \left( [0, T], H^1(\mathbb{R}^2) \right) \cap \mathcal{C}^1 \left( [0, T], L^2(\mathbb{R}^2) \right) \right)^2
\]

with the corresponding norm

\[
\|u\|_{X(T)} := \sup_{t \in [0, T]} \left( (1 + t)^{-\ell_1} M_1 \left( t; u^{(1)} \right) + (1 + t)^{-\ell_2} M_2 \left( t; u^{(2)} \right) \right),
\]

where

\[
M_k \left( t; u^{(k)} \right) := (1 + t)^{-1 - \frac{2m}{m - 2}} \left\| u^{(k)}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)}
\]

\[+ (1 + t)^{\frac{2m}{m - 2}} \left( \left\| |D| u^{(k)}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} + \left\| u^{(k)}_t(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} \right),
\]

and the parameters in the loss of decay \( (\ell_k > 0) \) and no loss of decay \( (\ell_k = 0) \) are defined in (5.58).

Next, we consider the integral operator \( N : X(T) \to X(T) \), which is defined by

\[
Nu(t, x) := u_{\text{lin}}(t, x) + u_{\text{non}}^{(1)}(t, x) + u_{\text{non}}^{(2)}(t, x),
\]

where

\[
u_{\text{lin}}(t, x) = K_0(t, x) *_{(x)} u_0(x) + K_1(t, x) *_{(x)} u_1(x),
\]

\[
u_{\text{non}}^{(1)}(t, x) = \int_0^t K_1(t - \tau, x) *_{(x)} \left( |u^{(2)}(\tau, x)|_{p_1}, 0 \right)^T d\tau,
\]

\[
u_{\text{non}}^{(2)}(t, x) = \int_0^t K_1(t - \tau, x) *_{(x)} \left( 0, |u^{(1)}(\tau, x)|_{p_2} \right)^T d\tau.
\]

From Theorem 5.1 and \( \ell_k \geq 0 \) for \( k = 1, 2 \), we can get the following estimates:

\[
\|u_{\text{lin}}\|_{X(T)} \lesssim \sum_{k=1}^{2} \left\| \left( u_{0}^{(k)}, u_{1}^{(k)} \right) \right\|_{d_{m, 0}(\mathbb{R}^2)}.
\]

In the next step we should estimate these terms \( \left\| \partial^j_{\tau} |D|^l u_{\text{non}}^{(1)}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} \) and \( \left\| \partial^j_{\tau} |D|^l u_{\text{non}}^{(2)}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} \) for \( j + l = 0, 1 \) and \( j, l \in \mathbb{N} \).

Applying the classical Gagliardo-Nirenberg inequality, we may obtain

\[
\left\| |u^{(2)}(\tau, \cdot)|_{p_1} \right\|_{L^m(\mathbb{R}^2)} \lesssim (1 + \tau)^{-\frac{2m}{m} + \ell_2 + \frac{2}{m} \left\| u \right\|_{X(\tau)}},
\]

\[
\left\| |u^{(1)}(\tau, \cdot)|_{p_2} \right\|_{L^m(\mathbb{R}^2)} \lesssim (1 + \tau)^{-\frac{2m}{m} + \ell_1 + \frac{2}{m} \left\| u \right\|_{X(\tau)}},
\]

where we use our assumption \( \frac{2}{m} \leq \min \{p_1; p_2 \} \) for \( m \in [1, 2] \).

For one thing, in order to estimate of \( u^{(k)} \) for \( k = 1, 2 \), we apply the derived \( (L^2 \cap L^m) - L^2 \) estimate in \( [0, t] \). For another, we use the derived \( (L^2 \cap L^m) - L^2 \) estimate in \( [0, t/2] \) and the derive \( L^2 - L^2 \) estimate in \( [t/2, t] \) to estimate \( \partial^j_{\tau} |D|^l u^{(k)} \) for \( j + l = 1 \) and \( k = 1, 2 \). Therefore, we obtain the following
estimates for \(j + l = 0, 1\) and \(j, l \in \mathbb{N}\):

\[
(1 + t)^{j+l-1+\frac{2m}{m+1}-\ell_1} \left\| \partial_t^j \left[ D \right] u^{(1)}_{\text{non}}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} \\
\lesssim (1 + t)^{-\ell_1} \|u\|_{X(t)}^p \left( \int_0^{1/2} (1 + \tau)(\frac{2m}{m+1} + \ell_2)p_1 + \frac{2}{m} d\tau + (1 + t)(\frac{2m}{m+1} + \ell_2)p_1 + \frac{2}{m} + 1 \right),
\]

\[
(1 + t)^{j+l-1+\frac{2m}{m+1}-\ell_2} \left\| \partial_t^j \left[ D \right] u^{(2)}_{\text{non}}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} \\
\lesssim (1 + t)^{-\ell_2} \|u\|_{X(t)}^p \left( \int_0^{1/2} (1 + \tau)(\frac{2m}{m+1} + \ell_1)p_2 + \frac{2}{m} d\tau + (1 + t)(\frac{2m}{m+1} + \ell_1)p_2 + \frac{2}{m} + 1 \right).
\]

We now need to distinguish between three cases. Without loss of generality, we only give the proof for the case \(p_1 > p_2\).

Case 1: We assume \(p_{\text{bal}}(m) < \min\{p_1; p_2\}\).

In this case it allows us to assume no loss of decay, i.e., \(\ell_1 = \ell_2 = 0\). Our assumption \(p_{\text{bal}}(m) < \min\{p_1; p_2\}\) immediately leads to

\[-\frac{2 - m}{2m}p_1 + \frac{2}{m} < -1 \quad \text{and} \quad -\frac{2 - m}{2m}p_2 + \frac{2}{m} < -1.\]

Hence, we have estimates for \(j + l = 0, 1\) and \(j, l \in \mathbb{N}\)

\[
(1 + t)^{j+l-1+\frac{2m}{m+1}} \left( \left\| \partial_t^j \left[ D \right] u^{(1)}_{\text{non}}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} + \left\| \partial_t^j \left[ D \right] u^{(2)}_{\text{non}}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} \right) \lesssim \|u\|_{X(t)}^p + \|u\|_{X(t)}^{p_2}.
\]

Case 2: We assume \(\alpha_1(m) < 1\) if \(\frac{2}{m} \leq p_2 \leq p_{\text{bal}}(m) < p_1\).

In this case it allows us to assume loss of decay only for the second component and its derivatives with respect to \(x\) and \(t\), i.e., \(\ell_1 = 0\) and

\[
\ell_2 = \begin{cases} 
\epsilon_0 & \text{if } p_2 = p_{\text{bal}}(m), \\
\frac{2 - m}{2m} (p_{\text{bal}}(m) - p_2) & \text{if } p_2 < p_{\text{bal}}(m).
\end{cases}
\]

Due to the assumption \(\frac{2}{m} \leq p_2 \leq p_{\text{bal}}(m)\), the parameter \(\ell_2\) chosen in (5.61) is positive. Moreover, the assumption \(\alpha_1(m) < 1\) implies that

\[
1 + \frac{2}{m} - \frac{2 - m}{2m}p_1 + \frac{2 - m}{2m} \left( \frac{2m + 2}{2 - m} - p_2 \right) p_1 < 0.
\]

We can get these estimates from the combination of the parameter \(\ell_2\) chosen in (5.61), our assumptions (5.62) and \(\frac{2}{m} \leq p_2 \leq p_{\text{bal}}(m) < p_1\)

\[-\frac{2 - m}{2m}p_2 + \frac{2}{m} + 1 - \ell_2 \leq 0 \quad \text{and} \quad -\frac{2 - m}{2m} + \ell_2)p_1 + \frac{2}{m} + 1 < 0.\]

Then, the following estimates hold for \(j + l = 0, 1\) and \(j, l \in \mathbb{N}\):

\[
(1 + t)^{j+l-1+\frac{2m}{m+1}-\ell_1} \left\| \partial_t^j \left[ D \right] u^{(1)}_{\text{non}}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} \lesssim \|u\|_{X(t)}^{p_1},
\]

\[
(1 + t)^{j+l-1+\frac{2m}{m+1}-\ell_2} \left\| \partial_t^j \left[ D \right] u^{(2)}_{\text{non}}(t, \cdot) \right\|_{L^2(\mathbb{R}^2)} \lesssim \|u\|_{X(t)}^{p_2}.
\]

Finally, combining all of the derived estimate, we can prove

\[
\|N u\|_{X(T)} \lesssim \sum_{k=1}^2 \left\| \left( u_0^{(k)}, u_1^{(k)} \right) \right\|_{\mathcal{M}_{m,0}(\mathbb{R}^2)} + \sum_{k=1}^2 \|u\|_{X(T)}^{p_k},
\]

uniform with respect to \(T \in [0, \infty)\).
Thus, we can decompose the solution \( u \) with small constant \( \varepsilon \) where the space \( H^1(\mathbb{R}^3) \) and the vector unknown \( \Delta u \) where the vector fields with curl zero (c.f. [21]).

These derived estimates (5.63) and (5.64) show the mapping \( N : X(T) \to X(T) \) is a contraction for initial data satisfying

\[
\| (u^{(1)}_0, u^{(1)}_1) \|_{\mathcal{A}_{m,0}(\mathbb{R}^2)} + \| (u^{(2)}_0, u^{(2)}_1) \|_{\mathcal{A}_{m,0}(\mathbb{R}^2)} \leq \varepsilon_0,
\]

with small constant \( \varepsilon_0 > 0 \). According to Banach’s fixed-point theorem, we complete the proof. \( \square \)

6. Treatment of elastic waves with Kelvin-Voigt damping in 3D

In this section we consider the following Cauchy problem for weakly coupled system of semilinear elastic waves with Kelvin-Voigt damping in 3D:

\[
\begin{aligned}
&u_{tt} - a^2 \Delta u - (b^2 - a^2) \nabla \text{div} u + (-a^2 \Delta - (b^2 - a^2) \nabla \text{div}) u_t = f(u), \quad t > 0, \; x \in \mathbb{R}^3, \\
&(u, u_t)(0, x) = (u_0, u_1)(x), \quad x \in \mathbb{R}^3,
\end{aligned}
\]

where \( b^2 > a^2 > 0 \) and the nonlinear terms on the right-hand sides are

\[
f(u) := \begin{pmatrix} |u|^{p_1}, |u^{(1)}|^{p_2}, |u^{(2)}|^{p_3} \end{pmatrix}^T \quad \text{with} \quad p_1, p_2, p_3 > 1.
\]

For the corresponding linearized problem

\[
\begin{aligned}
&u_{tt} - a^2 \Delta u - (b^2 - a^2) \nabla \text{div} u + (-a^2 \Delta - (b^2 - a^2) \nabla \text{div}) u_t = 0, \quad t > 0, \; x \in \mathbb{R}^3, \\
&(u, u_t)(0, x) = (u_0, u_1)(x), \quad x \in \mathbb{R}^3,
\end{aligned}
\]

it allows us to use the Helmholtz decomposition.

To begin with, let us recall the following orthogonal decomposition:

\[
(L^2(\mathbb{R}^3))^3 = \nabla H^1(\mathbb{R}^3) \oplus \mathcal{D}_0(\mathbb{R}^3),
\]

where the space \( \nabla H^1(\mathbb{R}^3) \) denotes the vector fields with divergence zero and \( \mathcal{D}_0(\mathbb{R}^3) \) denotes the vector fields with curl zero (c.f. [21]).

Thus, we can decompose the solution \( u = u(t, x) \) into the linearized problem (6.66) into a potential and a solenoidal part

\[
u = u^{p_0} + u^{s_0},
\]

where the vector unknown \( u^{p_0} = u^{p_0}(t, x) \) stands for rotation-free and the vector unknown \( u^{s_0} = u^{s_0}(t, x) \) stands for divergence-free in a weak sense.

Taking account into the relation \( \nabla \text{div} u = \nabla \times (\nabla \times u) + \Delta u \) in three dimensions, we can decoupled the system (6.66) into two viscoelastic damped wave equations with different propagation speeds \( a \) as well as \( b \), respectively,

\[
\begin{aligned}
&u_{tt}^{s_0} - a^2 \Delta u^{s_0} - a^2 \Delta u^{s_0}_t = 0, \quad t > 0, \; x \in \mathbb{R}^3, \\
&(u^{s_0}, u^{s_0}_t)(0, x) = (u_0^{s_0}, u^{s_0}_1)(x), \quad x \in \mathbb{R}^3,
\end{aligned}
\]

and

\[
\begin{aligned}
&u_{tt}^{p_0} - b^2 \Delta u^{p_0} - b^2 \Delta u^{p_0}_t = 0, \quad t > 0, \; x \in \mathbb{R}^3, \\
&(u^{p_0}, u^{p_0}_t)(0, x) = (u_0^{p_0}, u^{p_0}_1)(x), \quad x \in \mathbb{R}^3.
\end{aligned}
\]

The well-posedness of weak solutions to (6.67) and (6.68) have been studied in [15], and the well-posedness of distribution solutions have been investigated in [7]. The \( L^2 - L^2 \) estimates and \( (L^2 \cap L^m) - L^2 \) estimates of the solution with \( m \in [1, 2] \) also have been developed in [7, 5]. Furthermore, \( L^p - L^q \)
estimates not necessary on the conjugate line of solution to the Cauchy problems (6.67) or (6.68) has been investigated in [29, 38]. Lastly, we mention that the asymptotic profiles of solution with initial data taking from $L^1 \gamma (\mathbb{R}^3)$ with $\gamma \in [0, 1]$ has been studied in [14, 23, 24].

To study the semilinear problem (6.65), we need to derive $(L^2 \cap L^m) - L^2$ estimates and $L^2 - L^2$ estimates of solution to the linearized problem (6.66). According to the paper [40], one can obtain the next estimates.

**Theorem 6.1.** Let us consider the Cauchy problem (6.66) with initial data satisfying $(u_0^{(k)}, u_1^{(k)}) \in \mathcal{A}_{m,s}(\mathbb{R}^3)$ for $k = 1, 2, 3$, where $s \geq 0$ and $m \in [1, 2]$. Then, we have the following estimates:

$$
\| u^{(k)}(t, \cdot) \|_{L^2(\mathbb{R}^3)} \lesssim (1 + t)^{-\frac{6 + 5m}{4m}} \sum_{k=1}^{2} \| (u_0^{(k)}, u_1^{(k)}) \|_{\mathcal{A}_{m,0}(\mathbb{R}^3)} \quad \text{if } m \in \left[ 1, \frac{6}{5} \right],
$$

$$
\| D(u^{(k)}(t, \cdot)) \|_{H^s(\mathbb{R}^3)} + \| u_t^{(k)}(t, \cdot) \|_{H^s(\mathbb{R}^3)} \lesssim (1 + t)^{-\frac{6 + 3m + 2m}{4m}} \sum_{k=1}^{2} \| (u_0^{(k)}, u_1^{(k)}) \|_{\mathcal{A}_{m,0}(\mathbb{R}^3)}.
$$

Now, we state our theorem for the global (in time) existence of small data energy solution to (6.65). To begin with, let us introduce the balanced parameter $\tilde{p}_{bal}(m), \tilde{\alpha}_1(m)$ and $\tilde{\alpha}_1(m)$ for $m \in [1, \frac{6}{5}]$ by

$$
\tilde{p}_{bal}(m) := \frac{3 + 2m}{3 - m},
$$

(6.69)

$$
\tilde{\alpha}_1(m) := \frac{m(2 + 3p_2 + p_1p_2)}{2(p_1p_2 - 1)},
$$

(6.70)

$$
\tilde{\alpha}_1(m) := \frac{m(2 + 3(p_2 + 1)p_3 + p_1p_2p_3)}{2(p_1p_2p_3 - 1)}.
$$

(6.71)

**Remark 6.1.** Here we point out the relation between these parameters. If we consider the condition $\tilde{\alpha}_1(m) < \frac{3}{2}$, it also can be rewritten by

$$
p_2(p_1 + 1 - \tilde{p}_{bal}(m)) > \tilde{p}_{bal}(m).
$$

If we consider the condition $\tilde{\alpha}_1(m) < \frac{3}{2}$, it also can be rewritten by

$$
p_3(p_2(p_1 + 1 - \tilde{p}_{bal}(m)) + 1 - \tilde{p}_{bal}(m)) > \tilde{p}_{bal}(m).
$$

**Remark 6.2.** From the recent paper [4], we remark that the balanced exponent shown in (6.69) and the balanced parameters shown in (6.70) as well as (6.71) correspond to the balanced parameters to the weakly coupled system of semilinear viscoelastic elastic waves in 3D.

One can follow the procedure of the proof of Theorem 5.5 in [4] to obtain the following theorem. Without loss of generality, we assume $p_1 < p_2 < p_3$.

**Theorem 6.2.** Let $b^2 > a^2 > 0$ in (6.65) and $m \in [1, \frac{6}{5})$. Let us assume $1 < p_1 < p_2 < p_3$, satisfying one of the following conditions:

1. we assume $\tilde{p}_{bal}(m) < p_1 < p_2 < p_3 \leq 3$;
2. we assume $\tilde{\alpha}_1(m) < \frac{3}{2}$ if $\frac{6}{5} \leq p_1 \leq \tilde{p}_{bal}(m) < p_2 < p_3 \leq 3$;
3. we assume $\tilde{\alpha}_1(m) < \frac{3}{2}$ if $\frac{6}{5} \leq p_1 < p_2 < \tilde{p}_{bal}(m) < p_3 \leq 3$.

Then, there exists a constant $\varepsilon_0 > 0$ such that for any initial data $(u^{(k)}_0, u^{(k)}_1) \in \mathcal{A}_{m,0}(\mathbb{R}^3)$ for $k = 1, 2, 3$, with

$$
\| (u^{(1)}_0, u^{(1)}_1) \|_{\mathcal{A}_{m,0}(\mathbb{R}^3)} + \| (u^{(2)}_0, u^{(2)}_1) \|_{\mathcal{A}_{m,0}(\mathbb{R}^3)} + \| (u^{(3)}_0, u^{(3)}_1) \|_{\mathcal{A}_{m,0}(\mathbb{R}^3)} \leq \varepsilon_0,
$$

there is uniquely determined energy solution

$$
u \in (C([0, \infty), H^1(\mathbb{R}^3)) \cap C^1([0, \infty), L^2(\mathbb{R}^3)))^3$$
to (6.65). Moreover, the solutions satisfies the following estimates:

\[
\left\| u^{(k)}(t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \lesssim (1 + t)^{-\frac{6-5m}{4m} + \hat{\epsilon}_k} \sum_{k=1}^{3} \left\| \left( u^{(k)}_0, u^{(k)}_1 \right) \right\|_{d_{m,0}(\mathbb{R}^3)},
\]

\[
\left\| D[u^{(k)}(t, \cdot)] \right\|_{L^2(\mathbb{R}^3)} + \left\| u^{(k)}(t, \cdot) \right\|_{L^2(\mathbb{R}^3)} \lesssim (1 + t)^{-\frac{6-3m}{4m} + \hat{\epsilon}_k} \sum_{k=1}^{3} \left\| \left( u^{(k)}_0, u^{(k)}_1 \right) \right\|_{d_{m,0}(\mathbb{R}^3)},
\]

where

\[
0 \leq \hat{\epsilon}_1 = \hat{\epsilon}_1(m, p_1) := \begin{cases} 
0 & \text{if } p_1 > \bar{p}_{bal}(m), \\
\epsilon_0 & \text{if } p_1 = \bar{p}_{bal}(m), \\
\frac{3-m}{2m} (\bar{p}_{bal}(m) - p_1) & \text{if } p_1 < \bar{p}_{bal}(m),
\end{cases}
\]

\[
0 \leq \hat{\epsilon}_2 = \hat{\epsilon}_2(m, p_2) := \begin{cases} 
0 & \text{if } p_2 > \bar{p}_{bal}(m), \\
\epsilon_0 & \text{if } p_2 = \bar{p}_{bal}(m), \\
\frac{3-m}{2m} ( (\bar{p}_{bal}(m) - p_1)p_2 + (\bar{p}_{bal}(m) - p_2)) & \text{if } p_2 < \bar{p}_{bal}(m),
\end{cases}
\]

and \( \hat{\epsilon}_3 = 0 \), represent the (no) loss of decay in comparison with the corresponding decay estimates for the solution to the linear Cauchy problem (6.66) (see Theorem 6.1), with \( \epsilon_0 > 0 \) being an arbitrary small constant in the limit cases that \( p_k = \bar{p}_{bal}(m) \) for \( k = 1, 2, 3 \).

7. Concluding remarks

**Remark 7.1.** In Section 5 we have proved the global (in time) existence of small energy solution to (5.55). One also can prove the global (in time) existence of small data Sobolev solution

\[
u \in \left( C \left( [0, \infty), H^{s+1}(\mathbb{R}^2) \right) \right) \cap C^1 \left( [0, \infty), H^s(\mathbb{R}^2) \right),
\]

to (5.55) with initial data taking from \( \mathcal{A}_{m,s}(\mathbb{R}^2) \) for \( s > 0 \) and \( m \in [1, 2) \) by following the next strategy.

For the high regular data and even not embedded in \( L^\infty(\mathbb{R}^2) \) (i.e. \( 0 < s < 1 \)), we can apply the fractional Gagliardo-Nirenberg inequality, the fractional chain rule, the fractional Leibniz rule (c.f [36, 10, 8, 28]). More precisely, we apply the fractional chain rule to get the estimate for the nonlinear term in Riesz potential spaces \( H^s(\mathbb{R}^2) \) with \( s \in (0, 1) \). For example,

\[
\left\| u^{(2)}(\tau, \cdot) \right\|_{H^s(\mathbb{R}^2)} \lesssim \left\| u^{(2)}(\tau, \cdot) \right\|_{L^1(\mathbb{R}^2)}^{\frac{p_1}{4s}} \left\| u^{(2)}(\tau, \cdot) \right\|_{H^s(\mathbb{R}^2)}^{\frac{4s-p_1}{4s}}
\]

where \( \frac{p_1-1}{q_1} + \frac{1}{qs} = \frac{1}{2} \) and \( p_1 > [s] \). Here, \([s]\) denotes the smallest integer large than a given number, \([s] := \min \{ \tilde{s} \in \mathbb{Z} : s \leq \tilde{s} \} \). Then, one can estimate the terms on the right-hand sides by the fractional Gagliardo-Nirenberg inequality.

To estimates the difference between the nonlinearities, we set \( g(u^{(2)}) = u^{(2)}|u^{(2)}|^{p_1-2} \) to get

\[
\left\| u^{(2)}(\tau, x)\right\|^{p_1} - \left\| \tilde{u}^{(2)}(\tau, x)\right\|^{p_1} = p_1 \int_{0}^{1} \left( u^{(2)}(\tau, x) - \tilde{u}^{(2)}(\tau, x) \right) \nu u^{(2)}(\tau, x) \left( (1 - \nu)\tilde{u}^{(2)}(\tau, x) \right) dv.
\]

Then, applying the fractional Leibniz rule we obtain

\[
\left\| u^{(2)}(\tau, \cdot) - \tilde{u}^{(2)}(\tau, \cdot) \right\|_{H^s(\mathbb{R}^2)} \lesssim \int_{0}^{1} \left\| u^{(2)}(\tau, \cdot) - \tilde{u}^{(2)}(\tau, \cdot) \right\|_{H^s_{1}(\mathbb{R}^2)} \left\| g \left( \nu u^{(2)}(\tau, \cdot) + (1 - \nu)\tilde{u}^{(2)}(\tau, \cdot) \right) \right\|_{L^2(\mathbb{R}^2)} dv
\]

\[
+ \int_{0}^{1} \left\| u^{(2)}(\tau, \cdot) - \tilde{u}^{(2)}(\tau, \cdot) \right\|_{L^3(\mathbb{R}^2)} \left\| g \left( \nu u^{(2)}(\tau, \cdot) + (1 - \nu)\tilde{u}^{(2)}(\tau, \cdot) \right) \right\|_{\dot{H}_{3}^{1}(\mathbb{R}^2)} dv.
\]
where \( \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4} = \frac{1}{2} \). We next use the fractional Gagliardo-Nirenberg inequality again to estimate all terms on the right-hand side. Thus, after choosing suitable parameters \( q_1, q_2, r_1, r_2, r_3, r_4 \), a new lower bound \( 1 + \lfloor s \rfloor \) for the exponent \( p_1 \) comes.

For the large regular initial data with \( s > 1 \), it allows us to use the fractional powers (c.f. [6]) and the continuous embedding \( H^s(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \). At this time, we need to give a new lower bound \( 1 + s \) for the exponents.

**Remark 7.2.** In Theorem 6.2 we only show the global existence result for the energy solution to (6.65) with initial data belonging to \( \mathcal{A}_{m,0}(\mathbb{R}^3) \) for \( m \in \left[ 1, \frac{6}{5} \right] \). If one is interested in initial data taking from \( \mathcal{A}_{m,s}(\mathbb{R}^3) \) for all \( m \in [1, 2) \) and \( s \geq 0 \), one can read Section 5 of the recent paper [4].

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Elastic waves with Kelvin-Voigt damping

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