GEOMETRIC STRUCTURES ON THE COCHAINS OF A MANIFOLD

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Abstract. In this paper we develop several algebraic structures on the simplicial cochains of a triangulated manifold that are analogues of objects in differential geometry. We study a cochain product and prove several statements about its convergence to the wedge product on differential forms. Also, for cochains with an inner product, we define a combinatorial Hodge star operator, and describe some applications, including a combinatorial period matrix for surfaces. We show that for a particularly nice cochain inner product, these combinatorial structures converge to their continuum analogues as the mesh of a triangulation tends to zero.

1. Introduction

In this paper we develop combinatorial analogues of several objects in differential and complex geometry, including the Hodge star operator and the period matrix of a Riemann surface. We define these structures on the appropriate combinatorial analogue of differential forms, namely simplicial cochains.

As we recall in section 3, the two essential ingredients to the smooth Hodge star operator are Poincaré Duality and a metric, or inner product. We’ll define the combinatorial star operator in much the same way, using both an inner product and Poincaré Duality, expressed on cochains in the form of a (graded) commutative product.

Using the inner product introduced in [6], we prove the following:

Theorem 1.1. The combinatorial star operator, defined on the simplicial cochains of a triangulated Riemannian manifold, converges to the smooth Hodge star operator as the mesh of the triangulation tends to zero.

We show in section 7 that, on a closed surface, this combinatorial star operator gives rise to a combinatorial period matrix and prove:

Theorem 1.2. The combinatorial period matrix of a triangulated Riemannian 2-manifold converges to the conformal period matrix of the associated Riemann surface, as the mesh of the triangulation tends to zero.

This suggests a link between statistical mechanics and conformal field theory, where it is known that the partition function may be expressed in terms of theta functions of the conformal period matrix [17], see also [18], [20].

The above convergence statements are made precise by using an embedding of simplicial cochains into differential forms, first introduced by Whitney [32]. This approach was used quite successfully by Dodziuk [6], and later Dodziuk and Patodi [7], to show that cochains provide a good approximation to smooth differential...
forms, and that the combinatorial Laplacian convergences to the smooth Laplacian. This formalism will be reviewed in section 4.

In section 5 we describe the cochain product that will be used in defining the combinatorial star operator. This product is of interest in its own right, and we prove several results concerning its convergence to the wedge product on forms; see also [10], [5]. These results may be of interest in numerical analysis and the modeling of PDE’s, since they give a computable discrete model which approximates the algebra of smooth differential forms. The convergence statements on the cochain product, theorems 5.4 through 5.12, are not needed for later sections.

In section 6 we introduce the combinatorial star operator, and show that many of the interesting relations amongst $\star$, $d$, $\wedge$, and the adjoint $d^*$ of $d$, that hold in the smooth setting, also hold in the combinatorial case. Some of the relations, though, are more illusive and may only be recovered in the limit of a fine triangulation.

In section 7 we study the combinatorial star operator on surfaces, and prove several results on the combinatorial period matrix, as mentioned above.

In the last two sections, 8 and 9, we show how an explicit computation of the combinatorial star operator is related to “summing over weighted paths”, and perform these calculations for the circle.

I would like to thank Dennis Sullivan for his generosity, insight and many useful comments on this paper. I’d also like to thank Jozef Dodziuk for his help with several points in his papers [6], [7], and Ruben Costa-Santos and Barry McCoy for their inspiring work in [18].

2. Background and Acknowledgments

In this section we describe previous results that are related to the contents of this paper. My sincere apologies to anyone whose work I have left out.

The cochain product we discuss in this paper was introduced by Whitney in [32]. It was also studied by Sullivan in the context of rational homotopy theory [27], by DuPont in his study of curvature and characteristic class [9], and by Birmingham and Rakowski as a star product in lattice gauge theory [4].

In connection with our result on the convergence of this cochain product to the wedge product of forms, Kervaire has a related result for the Alexander-Whitney product $\cup$ on cochains [16]. Kervaire states that, for differential forms $A, B$, and the associated cochains $a, b$,

$$\lim_{k \to \infty} a \cup b (S^k c) = \int_c A \wedge B$$

for a convenient choice of subdivisions $S^k c$ of the chain $c$. Cheeger and Simons use this result in the context of cubical cell structures in [5]. There they construct an explicit map $E(A, B)$ satisfying

$$\int A \wedge B - a \cup b = \delta E(A, B)$$

and use it in the development of the theory of differential characters. To the best of our knowledge, our convergence theorems for the commutative cochain product in section 5 are the first to appear in the literature.

Several definitions of a discrete analogue of the Hodge-star operator have been made. In [18], Costa-Santos and McCoy define a discrete star operator for a particular 2-dimensional lattice and study convergence properties as it relates to the
Ising Model. Mercat defines a discrete star operator for surfaces in [19], using a triangulation and its dual, and uses it to study a notion of discrete holomorphy and its relation to Ising criticality.

In [29], Tarhasaari, Kettunen and Bossavit describe how to make explicit computations in electromagnetism using Whitney forms and a star operator defined using the de Rham map from forms to cochains. Teixeira and Chew have also defined Hodge operators on a lattices for the purpose of studying electromagnetic theory.

Adams [1], and also Sen, Sen, Sexton and Adams [23], define two discrete star operators using a triangulation and its dual, and present applications to lattice gauge fields and Chern-Simons theory. de Beaucé and Sen [2] define star operators in a similar way and study applications to chiral Dirac fermions, and de Beaucé and Sen have generalized this to give a discretization scheme for differential geometry [3].

In the approaches using a triangulation and its dual, the star operator(s) are formulated using the duality map between the two cell decompositions. This map yields Poincaré Duality on (co)homology. In this paper, we express Poincaré Duality by a commutative cup product on cochains and combine it with a non-degenerate inner product to define the star operator. Working this way, we obtain a single operator from one complex to itself.

Our convergence statements in section 6 are proven using the inner product introduced in [6], and to the best of our knowledge, these are the first results proving a convergence theorem for a discrete analogue of the Hodge star operator.

In Dodziuk’s paper [6], and in [7] by Dodziuk and Patodi, the authors study a combinatorial Laplacian on the cochains and proved that its eigenvalues converge to the smooth Laplacian. Such discrete notions of a Hodge structure, along with finite element method techniques, were used by Kotiuga [15], and recently by Gross and Kotiuga [12], in the study of computational electromagnetism. Jin has used related techniques in studying electrodynamics [14].

In connection with our application of the combinatorial star operator to surfaces, in particular proving the convergence of our combinatorial period matrix to the conformal period matrix, Mercat has a related result in [20]. As part of his extensive study of what he calls “discrete Riemann surfaces”, he assigns to any such object a “period matrix” of twice the expected size. He shows that there are two submatrices of the appropriate dimension ($g \times g$) satisfying the property that, given what he calls “a refining sequence of critical maps”, they both converge to the continuum period matrix of an associated Riemann surface. This uses his results on discrete holomorphy approximations presented in [21]. As with the star operator above, our approach differs in that there is no “doubling” of complexes or operators.

There is another discussion of discrete period matrices presented in [13]. There Xianfeng Gu and Shing-Tung Yau give explicit algorithms for computing a period matrix for a surface. They point out that these can be implemented on the simplicial cochains by the use of the integration map from piecewise linear forms to simplicial cochains.

3. Smooth Setting

We begin with a brief review of some elementary definitions. Let $M$ be a closed oriented Riemannian $n$-manifold. A Riemannian metric induces an inner product on $\Omega(M) = \bigoplus_j \Omega^j = \bigoplus_j \Gamma(\wedge^j T^*M)$ in the following way: a Riemannian metric

\[ g(x, y) = \sum_i g_i(x_i, y_i) \]

where $g_i$ are the components of the metric tensor $g$. This inner product is used to define the Hodge star operator, which is a linear operator that takes a $k$-form to a $(n-k)$-form, and satisfies $g(\star\alpha, \beta) = g(\alpha, \star\beta)$. In particular, for a closed $1$-form $\alpha$, the Hodge star operator $\star\alpha$ is a $1$-form that is defined by the condition $\int_M \alpha \wedge \star\alpha = \pi$, where $\pi$ is a real number that can be interpreted as the area of a unit circle in the $1$-form $\alpha$. The Hodge star operator is a fundamental tool in the study of differential geometry and topology, and plays a crucial role in the formulation of the Hodge decomposition theorem, which states that any $k$-form on a compact oriented $n$-manifold can be expressed as a sum of an exact $k$-form and a coexact $(n-k)$-form. This theorem provides a powerful tool for studying the topology of the manifold and has numerous applications in areas such as algebraic geometry, string theory, and theoretical physics.
determines an inner product on $T^*M_p$ for all $p$, and hence an inner product for each $j$ on $\wedge^j T^*M_p$ (explicitly, via an orthonormal basis). An inner product $(\cdot, \cdot)$ on $\Omega(M)$ is then obtained by integration over $M$. If we denote the induced norm on $\wedge^j T^*M_p$ by $\|\cdot\|_p$, then the norm $\|\cdot\|$ on $\Omega(M)$ is given by

$$\|\omega\| = \left( \int_M |\omega|^2_p \, dV \right)^{1/2}$$

where $dV$ is the Riemannian volume form on $M$.

Let $L^2_\Omega(M)$ denote the completion of $\Omega(M)$ with respect to this norm. We also use $\|\cdot\|$ to denote the norm on the completion. Let the exterior derivative $d : \Omega^j(M) \to \Omega^{j+1}(M)$ be defined as usual.

**Definition 3.1.** The Poincare-Duality pairing $(\cdot, \cdot) : \Omega^j(M) \otimes \Omega^{n-j}(M) \to \mathbb{R}$ is defined by:

$$(\omega, \eta) = \int_M \omega \wedge \eta.$$  

The pairing $(\cdot, \cdot)$ is bilinear, (graded) skew-symmetric and non-degenerate. It induces an isomorphism $(\cdot, \cdot) : \Omega^j(M) \to (\Omega^{n-j}(M))^*$, where here $*$ denotes linear the dual. The induced map $(\cdot, \cdot) : \Omega^{n-j}(M) \to (\Omega^{n-j}(M))^*$ is also an isomorphism and one may check that the composition $(\cdot, \cdot)^{-1} \circ (\cdot, \cdot)$ equals the following operator:

**Definition 3.2.** The Hodge star operator $\star : \Omega^j(M) \to \Omega^{n-j}(M)$ is defined by:

$$\langle \star \omega, \eta \rangle = (\omega, \eta).$$

One may also define the operator $\star$ using local coordinates, see Spivak [25]. We note that this approach and the former definition give rise to the same operator $\star$ on $L^2_\Omega(M)$. We prefer to emphasize definition 3.2 since it motivates definition 6.1, the combinatorial star operator.

**Definition 3.3.** The adjoint of $d$, denoted by $d^*$, is defined by $\langle d^* \omega, \eta \rangle = (\omega, d\eta)$. 

Note that $d^* : \Omega^j(M) \to \Omega^{j-1}(M)$. The following relations hold among $\star$, $d$ and $d^*$. See Spivak [25].

**Theorem 3.4.** As maps from $\Omega^j(M)$ to their respective ranges:

1. $\star d = (-1)^{j+1} d^* \star$
2. $d^* = (-1)^j d^* d$
3. $\star^2 = (-1)^{j(n-j)} d^2$

**Definition 3.5.** The Laplacian is defined to be $\Delta = d^* d + dd^*$. 

Finally, we state the Hodge decomposition theorem for $\Omega(M)$. Let $\mathcal{H}^j(M) = \{ \omega \in \Omega^j(M) | \Delta \omega = 0 \}$ be the space of harmonic $j$-forms.

**Theorem 3.6.** There is an orthogonal direct sum decomposition

$$\Omega^j(M) \cong d\Omega^{j-1}(M) \oplus \mathcal{H}^j(M) \oplus d^*\Omega^{j+1}(M)$$

and $\mathcal{H}^j(M) \cong \mathcal{H}^j_{DR}(M)$, the De Rham cohomology of $M$ in degree $j$. 

4. Whitney Forms

In his book, ‘Geometric Integration Theory’, Whitney explores the idea of using cochains as integrands [32]. A main result is that such objects provide a reasonable integration theory that in some sense generalize the smooth theory of integration of differential forms. This idea has been made even more precise by the work of Dodziuk [6], who used a linear map of cochains into $L_2$-forms (due to Whitney [32]) to show that cochains provide a good approximation of differential forms. In this section we review some of the results. The techniques involved illustrate a tight (and analytically precise) connection between cochains and forms, and will be used later to give precise meaning to our constructions on cochains. In particular, all of our convergence statements about combinatorial and smooth objects will be cast in a similar way.

Let $M$ be a fixed closed smooth $n$-manifold and $K$ a fixed $C^\infty$ triangulation of $M$. We identify $K$ and $M$ and fix an ordering of the vertices of $K$. Let $C^j$ denote the simplicial cochains of degree $j$ of $K$ with values in $\mathbb{R}$. Given an ordering of the vertices of $K$, we have a coboundary operator $\delta : C^j \to C^{j+1}$. Let $\mu_i$ denote the barycentric coordinate corresponding to the $i$th vertex $p_i$ of $K$. Since $M$ is compact, we may identify the cochains and chains of $K$ and for $c \in C^j$ write $c = \sum_{\tau} c_{\tau} \cdot \tau$ where $c_{\tau} \in \mathbb{R}$ and the sum over all $j$-simplicies $\tau = [p_0, p_1, \ldots, p_j]$ of $K$ whose vertices form an increasing sequence with respect to the ordering of vertices in $K$.

We now define the Whitney embedding of cochains into $L_2$-forms:

Definition 4.1. For $\tau$ as above, we define

$$W \tau = j! \sum_{i=0}^{j} (-1)^i \mu_i \ d\mu_0 \wedge \cdots \wedge \widehat{d\mu_i} \wedge \cdots \wedge d\mu_j.$$ 

$W$ is defined on all of $C^j$ by extending linearly.

Note that the coordinates $\mu_\alpha$ are not even of class $C^1$, but they are $C^\infty$ on the interior of any $n$-simplex of $K$. Hence, $d\mu_\alpha$ is defined and $W \tau$ is a well defined element of $L_2 \Omega^j$. By the same consideration, $dW$ is also well defined. Note both sides of the definition of $W$ are alternating, so this map is well defined for all simplicies regardless of the ordering of vertices.

Several properties of the map $W$ are given below. See [32], [6], [7] for details.

Theorem 4.2. The following hold:

1. $W \tau = 0$ on $M \setminus \text{St}(\tau)$
2. $dW = W\delta$

where $\text{St}$ denotes the open star and $\text{—}$ denotes closure.

One also has a map $R : \Omega^j(M) \to C^j(K)$, the de Rham map, given by integration. Precisely, for any differential form $\omega$ and chain $c$ we have:

$$R \omega(c) = \int_c \omega$$

It is a theorem of de Rham that this map is a quasi-isomorphism (it is a chain map by Stokes Theorem). $RW$ is well defined and one can check that $RW = Id$, see [32], [6], [7].
Before stating Dodziuk and Patodi’s theorem that $WR$ is approximately equal to the identity, we first give some definitions concerning triangulations. They also appear [7].

**Definition 4.3.** Let $K$ be a triangulation of an $n$-dimensional manifold $M$. The mesh $\eta = \eta(K)$ of a triangulation is:

$$\eta = \sup r(p, q),$$

where $r$ means the geodesic distance in $M$ and the supremum is taken over all pairs of vertices $p, q$ of a 1-simplex in $K$.

The fullness $\Theta = \Theta(K)$ of a triangulation $K$ is

$$\Theta(K) = \inf \frac{\text{vol}(\sigma)}{\eta^n},$$

where the inf is taken over all $n$-simplexes $\sigma$ of $K$ and $\text{vol}(\sigma)$ is the Riemannian volume of $\sigma$, as a Riemannian submanifold of $M$.

A Euclidean analogue of the following lemma was proven by Whitney in [32] (IV.14).

**Lemma 4.4.** Let $M$ be a smooth Riemannian $n$-manifold.

1. Let $K$ be a smooth triangulation of $M$. Then there is a positive constant $\Theta_0 > 0$ and a sequence of subdivisions $K_1, K_2, \ldots$ of $K$ such that $\lim_{n \to \infty} \eta(K_n) = 0$ and $\Theta(K_n) \geq \Theta_0$ for all $n$.
2. Let $\Theta_0 > 0$. There exist positive constants $C_1, C_2$ depending on $M$ and $\Theta_0$ such that for all smooth triangulations $K$ of $M$ satisfying $\Theta(K) \geq \Theta_0$, all $n$-simplexes of $\sigma = [p_0, p_1, \ldots, p_n]$ and vertices $p_k$ of $\sigma$,

$$\text{vol}(\sigma) \leq C_1 \cdot \eta^n$$

$$C_2 \cdot \eta \leq r(p_k, \sigma_{p_k}),$$

where $r$ is the Riemannian distance, $\text{vol}(\sigma)$ is the Riemannian volume, and $\sigma_{p_k} = [p_0, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n]$ is the face of $\sigma$ opposite to $p_k$.

Since any two metrics on $M$ are commensurable, the lemma follows from Whitney’s Euclidean result, see also [7].

We consider only those triangulations with fullness bounded below by some positive real constant $\Theta_0$. By the lemma, this guarantees that the volume of a simplex is on the order of it’s mesh raised to the power of its dimension. Geometrically, this means that in a sequence of triangulations, the shapes do not become too thin. (In fact, Whitney’s standard subdivisions yield only finitely many shapes, and can be used to prove the first part of the lemma.) Most of the estimates in this paper depend on $\Theta_0$, as can be seen in the proofs. We’ll not indicate this dependence in the statements.

The following theorems are proved by Dodziuk and Patodi in [7]. They show that for a fine triangulation, $WR$ is approximately equal to the identity. In this sense the theorems give precise meaning to the statement: for a fine triangulation, cochains provide a good approximation to differential forms.

**Theorem 4.5.** Let $\omega$ be a smooth form on $M$, and $\sigma$ be an $n$-simplex of $K$. There exists a constant $C$, independent of $f$, $K$ and $\sigma$, such that

$$|\omega(p) - WR\omega|_p \leq C \cdot \sup \left| \frac{\partial \omega}{\partial x^i} \right| \cdot \eta$$
for all \( p \in \sigma \). The supremum is taken over all \( p \in \sigma \) and \( i = 1, 2, \ldots, n \), and the partial derivatives are taken with respect to a coordinate neighborhood containing \( \sigma \).

**Proof.** Using similar techniques we’ll prove a generalization; see Theorem 5.4 and Remark 5.5.

By integrating the above point-wise and applying a Sobolev inequality, Dodziuk and Patodi [7] obtain the following:

**Corollary 4.6.** There exists a positive constant \( C \) and a positive integer \( m \), independent of \( K \), such that

\[
\| \omega - WR\omega \| \leq C \cdot \| (Id + \Delta)^m \omega \| \cdot \eta
\]

for all \( C^\infty \) \( j \)-forms \( \omega \) on \( M \).

**Proof.** This is a special case of Corollary 5.7.

Now suppose the cochains \( C \) are equipped with a non-degenerate inner product \( \langle , \rangle \) such that, for distinct \( i, j \), \( C^i \) and \( C^j \) are orthogonal. Then one can define further structures on the cochains. In particular we have the following:

**Definition 4.7.** The adjoint of \( \delta \), denoted by \( \delta^* \), is defined by \( \langle \delta^* \sigma, \tau \rangle = \langle \sigma, \delta \tau \rangle \).

Note that \( \delta^* : C^j(K) \rightarrow C^{j-1}(K) \) is also squares to zero. One can also define

**Definition 4.8.** The combinatorial Laplacian is defined to be \( \bigtriangleup = \delta^* \delta + \delta \delta^* \).

Clearly, both \( \delta^* \) and \( \bigtriangleup \) depend upon the choice of inner product. For any choice of non-degenerate inner product, these operators give rise to a combinatorial Hodge theory: the space of harmonic \( j \)-cochains of \( K \) is defined to be

\[
H^{\delta}_C j(K) = \{ a \in C^j | \bigtriangleup a = \delta a = \delta^* a = 0 \}.
\]

The following theorem is due to Eckmann [10]:

**Theorem 4.9.** Let \( C \) be a cochain complex with inner product \( \langle , \rangle \), and induced differentials \( \delta \) and \( \delta^* \) as above. There is an orthogonal direct sum decomposition

\[
C^j(K) \cong \delta C^{j-1}(K) \oplus H^{\delta}_C j(K) \oplus \delta^* C^{j+1}(K)
\]

and \( H^{\delta}_C j(K) \cong H^j(K) \), the cohomology of \((K, \delta)\) in degree \( j \).

**Proof.** We’ll write \( C^j \) for \( C^j(K) \). The second statement of the theorem follows from the first.

Using the fact that \( \delta^* \) is the adjoint of \( \delta \), so \( \delta \delta = \delta^* \delta^* = 0 \), it is easy to check that \( \delta C^{j-1} \), \( H^{\delta}_C j \), and \( \delta^* C^{j+1} \) are orthogonal. Thus, it suffice to show:

\[
\dim(C^j) = \dim(\delta C^{j-1} \oplus H^{\delta}_C j \oplus \delta^* C^{j+1})
\]

Let \( \delta^*_j \) denote \( \delta^* \) restricted to \( C^j \). By orthogonality we have:

\[
\dim(C^j) - \dim(\delta^* C^j) = \dim(\ker(\delta^*_j)) = \dim(H^{\delta}_C j) + \dim(\delta^* C^{j+1}).
\]

The proof is complete by showing \( \dim(\delta^* C^j) = \dim(\delta C^{j-1}) \). This follows since, by the adjoint property, both \( \delta : \delta^* C^j \rightarrow \delta C^{j-1} \) and \( \delta^* : \delta C^{j-1} \rightarrow \delta^* C^j \) are injections of finite dimensional vector spaces.
If $K$ is a triangulation of a Riemannian manifold $M$, then there is a particularly nice inner product on $C(K)$, which we’ll call the Whitney inner product. It is induced by the metric $\langle \cdot, \cdot \rangle$ on $\Omega(M)$ and the Whitney embedding of cochains into $L^2$-forms. We’ll use the same notation $\langle \cdot, \cdot \rangle$ for this pairing on $C$: $\langle \sigma, \tau \rangle = \langle W\sigma, W\tau \rangle$.

It is proven in [6] that the Whitney inner product on $C$ is non-degenerate. Further consideration of this inner product will be given in later sections. For now, following [6] and [7], we describe how the combinatorial Hodge theory, induced by the Whitney inner product, is related to the smooth Hodge theory. Precisely, we have the following theorem due to Dodziuk and Patodi [7], which shows that the approximation $WR \approx Id$ respects the Hodge decompositions of $\Omega(M)$ and $C(K)$.

**Theorem 4.10.** Let $\omega \in \Omega^j(M)$, $R\omega \in C^j(K)$ have Hodge decompositions

$$
\omega = d\omega_1 + \omega_2 + d^*\omega_3
$$

$$
R\omega = \delta a_1 + a_2 + \delta^*a_3
$$

Then,

$$
\|\omega_1 - Wa_1\| \leq \lambda \cdot \| (Id + \Delta)^m \omega \| \cdot \eta
$$

$$
\|d\omega_2 - W\delta a_2\| \leq \lambda \cdot \| (Id + \Delta)^m \omega \| \cdot \eta
$$

$$
\|d^*\omega_3 - W\delta^*a_3\| \leq \lambda \cdot \| (Id + \Delta)^m \omega \| \cdot \eta
$$

where $\lambda$ and $m$ are independent of $\omega$ and $K$.

5. **Cochain Product**

In this section we describe a commutative, but non-associative, cochain product. It is of interest in its own right, and will be used to define the combinatorial star operator.

The product we define is induced by the Whitney embedding and the wedge product on forms, but also has a nice combinatorial description. An easy way to state this is as follows: the product of a $j$-simplex and $k$-simplex is zero unless these simplices are faces of a $(j + k)$-simplex, in which case the product is a rational multiple of this $(j + k)$-simplex. We will prove a convergence theorem for this product, and also show that this product’s deviation from being associative converges to zero for ‘sufficiently smooth’ cochains.

From the point of homotopy theory, it is natural to consider this commutative cochain product as part of a $C$-infinity algebra. We use Sullivan’s local construction of a $C$-infinity algebra [28], and show that this structure converges to the strictly commutative associative algebra given by the wedge product on forms. In particular, all of the higher homotopies of the $C$-infinity algebra converge to zero.

Only definition 5.1 and theorem 5.2 are used in later sections. We begin with the definition of a cochain product on the cochains of a fixed triangulation $K$.

**Definition 5.1.** We define $\cup : C^j(K) \otimes C^k(K) \rightarrow C^{j+k}(K)$ by:

$$
\sigma \cup \tau = R(W\sigma \wedge W\tau)
$$

Since $R$ and $W$ are chain maps with respect to $d$ and $\delta$, it follows that $\delta$ is a derivation of $\cup$, that is, $\delta(\sigma \cup \tau) = \delta\sigma \cup \tau + (-1)^{\deg(\sigma)}\sigma \cup \delta\tau$. Also, since $\wedge$ is graded commutative, $\cup$ is as well: $\sigma \cup \tau = (-1)^{\deg(\tau)\deg(\sigma)}\tau \cup \sigma$. It follows from a theorem of Whitney [33] that the product $\cup$ induces the same map on cohomology as the usual (Alexander-Whitney) simplicial cochain product. We now give a combinatorial description of $\cup$. 

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[6], [7], [28], [33]
Remark 5.3. The constant 0-cochain which evaluates to 1 on all 0-simplices is 4 is since statement may be of computational interest since it shows that using cochains equals 1 \([p_{a_0}, p_{a_1}, \ldots, p_{a_j}]\). Thus, by possibly reordering the vertices of \(K\), \(W{\sigma} \cup W{\tau}\) where \(s\) is determined by:

\[\text{orientation}(\sigma) \cdot \text{orientation}(\tau) = s(\sigma, \tau) \cdot \text{orientation}(v)\]

Proof. Recall that for any simplex \(\alpha, W{\alpha} = 0\ on M/\Sigma\alpha\). So, \(\sigma \cup \tau = R(W{\sigma} \wedge W{\tau})\) is zero if their vertices are disjoint. If \(\sigma\) and \(\tau\) intersect in more than one vertex then \(W{\sigma} \wedge W{\tau} = 0\ since it is a sum of terms containing \(d\mu_\alpha \wedge d\mu_\alpha\) for some \(i\). Thus, by possibly reordering the vertices of \(K\), it suffices to show that for \(\sigma = [p_0, p_1, \ldots, p_j]\) and \(\tau = [p_j, p_{j+1}, \ldots, p_{j+k}]\), we have that \((\sigma \cup \tau)([p_0, p_1, \ldots, p_{j+k}]) = s(\sigma, \tau)\ \frac{j!k!}{(j+k+1)!}\). We calculate

\[R(W{\sigma} \wedge W{\tau})([p_0, p_1, \ldots, p_{j+k}])\]

\[= \int_{\mu=[p_0, p_1, \ldots, p_{j+k}]} W([p_0, p_1, \ldots, p_j]) \wedge W([p_j, p_{j+1}, \ldots, p_{j+k}])\]

\[= j!k! \int \sum_{i=0}^{j+k} (-1)^i \mu_i \mu_j \ d\mu_0 \wedge \ldots \wedge \widehat{d\mu_i} \wedge \ldots \wedge d\mu_{j+k}\]

Now, \(\sum_{i=0}^{j+k} \mu_i = 1\), so \(d\mu_0 = - \sum_{i=0}^{j+k} d\mu_i\), and we have that the last expression

\[= j!k! \int \sum_{i=0}^{j+k} (-1)^i \mu_i \mu_j \ (-d\mu_i) \wedge d\mu_1 \wedge \ldots \wedge \widehat{d\mu_i} \wedge \ldots \wedge d\mu_{j+k}\]

\[= j!k! \int \mu_j \sum_{i=0}^{j+k} \mu_i \ d\mu_1 \wedge \ldots \wedge d\mu_{j+k}\]

\[= j!k! \int \mu_j \ d\mu_1 \wedge \ldots \wedge d\mu_{j+k}\]

Now, \(\int d\mu_1 \wedge \ldots \wedge d\mu_{j+k}\) is the volume of a standard \((j + k)\)-simplex, and thus equals \(\frac{1}{(j+k)!}\). From this it is easy to show that \(\int \mu_j \ d\mu_1 \wedge \ldots \wedge d\mu_{j+k} = \pm \frac{1}{(j+k+1)!}\), with the appropriate sign prescribed by the definition of \(s(\sigma, \tau)\). \(\square\)

A special case of this result was derived by Ranicki and Sullivan \[22\] for \(K\) a triangulation of a 4k-manifold and \(\sigma, \tau\) of complimentary dimension. In that paper, they showed that the pairing given by \(\cup\) restricted to simplicies of complimentary dimension gives rise to a semi-local combinatorial formula for the signature of a 4k-manifold.

Remark 5.3. The constant 0-cochain which evaluates to 1 on all 0-simplices is the unit of the differential graded commutative ring \((C^*, \delta, \cup)\).

We now show that the product \(\cup\) converges to \(\wedge\), which perhaps is not surprising, since \(\cup\) is induced by the Whitney embedding and the wedge product. Still, the statement may be of computational interest since it shows that in using cochains
to approximate differential forms, the product $\cup$ is, in an analytically precise way, an appropriate analogue of the wedge product of forms.

**Theorem 5.4.** Let $\omega_1, \omega_2 \in \Omega(M)$ and $\sigma$ be an $n$-simplex of $K$. Then there exists a constant $C$ independent of $\omega_1, \omega_2, K$ and $\sigma$ such that

$$|W(R\omega_1 \cup R\omega_2)(p) - \omega_1 \wedge \omega_2(p)|_p \leq C \cdot \left( c_1 \cdot \sup \left| \frac{\partial \omega_2}{\partial x^i} \right| + c_2 \cdot \sup \left| \frac{\partial \omega_1}{\partial x^i} \right| \right) \cdot \eta$$

for all $p \in \sigma$, where $c_m = \sup|\omega_m|_p$, the supremums are taken over all $p \in \sigma$ and $i = 1, 2, \ldots, n$ and the partial derivatives are taken with respect to a coordinate neighborhood containing $\sigma$.

**Remark 5.5.** By remark 5.3, theorem 5.4 reduces to theorem 4.5 in the case $\omega_1$ is the constant function 1.

**Proof.** Let $\sigma = [p_0, \ldots, p_n]$ be an $n$-simplex contained in a coordinate neighborhood with coordinate functions $x_1, \ldots, x_n$. Let $\mu_i$ denote the $i^{th}$ barycentric coordinate of $\sigma$. By the triangle inequality, and a possible reordering of the coordinate functions, it suffices to consider the case

$$\omega_1 = f \, d\mu_1 \wedge \cdots \wedge d\mu_j$$
$$\omega_2 = g \, d\mu_\alpha_1 \wedge \cdots \wedge d\mu_\alpha_k$$

We first compute $W(R\omega_1 \cup R\omega_2)$. We’ll use the notation $[p_s, \ldots, p_{s+t}]$ to denote both the simplicial chain and the simplicial cochain taking the value one on this chain and zero elsewhere. Let

$$N = \{0, 1, 2, \ldots, n\}$$
$$J = \{1, 2, \ldots, j\}$$
$$K = \{\alpha_1, \ldots, \alpha_k\}.$$ 

Then

$$R\omega_1 = \sum_{\beta \in N - J} \left( \int_{[p_\beta, p_1, \ldots, p_j]} \omega_1 \right) [p_\beta, p_1, \ldots, p_j]$$
$$R\omega_2 = \sum_{\gamma \in N - K} \left( \int_{[p_\gamma, p_\alpha_1, \ldots, p_\alpha_k]} \omega_2 \right) [p_\gamma, p_\alpha_1, \ldots, p_\alpha_k].$$

Now, to compute $R\omega_1 \cup R\omega_2$, we use theorem 5.2. If the sets $J$ and $K$ intersect in two or more elements then $R\omega_1 \cup R\omega_2 = 0$ since, in this case, all products of simplicies are zero.

Now suppose that $J$ and $K$ intersect in exactly one element. Without loss of generality, let us assume $\alpha_1 = 1$. Then the product

$$[p_\beta, p_1, \ldots, p_j] \cup [p_\gamma, p_\alpha_1, \ldots, p_\alpha_k]$$

is non-zero only if $\beta, \gamma$ are distinct elements of the set $Q = N - (J \cup K)$. Using the abbreviated notation

$$[p_s, p_j, p_K] = [p_s, p_1, \ldots, p_j, p_\alpha_1, \ldots, p_\alpha_k]$$
$$\int_{[s]} \omega_1 = \int_{[p_s, p_1, \ldots, p_j]} \omega_1$$
$$\int_{[s]} \omega_2 = \int_{[p_s, p_\alpha_1, \ldots, p_\alpha_k]} \omega_2$$
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we compute
\[
R\omega_1 \cup R\omega_2 = \frac{j!k!}{(j+k+1)!} \sum_{\beta, \gamma \in Q} \left( \int_{[\beta]} \omega_1 \right) \left( \int_{[\gamma]} \omega_2 \right) [p_\beta, p_\gamma, p_\gamma, p_K].
\]

If all of the coefficients (given by the integrals of \(\omega_1\) and \(\omega_2\)) were equal, the above expression would vanish, since the terms would cancel in pairs (by reversing the roles of \(\beta\) and \(\gamma\)). Of course, this is not the case, but the terms are almost equal. We’ll use some estimation techniques developed by Dodziuk and Patodi [7].

An essential estimate that we’ll need for this case and the next is the following: there is a constant \(c\), independent of \(\omega_1, \omega_2, K\) and \(\sigma\), such that for any \(p \in \sigma\), and \(\beta, \gamma\) as above,
\[(1) \quad j!k! \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - f(p) g(p) \leq c \cdot \left( c_1 \cdot \sup \left| \frac{\partial \omega_2}{\partial x^j} \right| + c_2 \cdot \sup \left| \frac{\partial \omega_1}{\partial x^j} \right| \right) \cdot \eta^{j+k+1}
\]
where \(c_m = \sup |\omega_m|\) and the supremums are taken over all \(p \in \sigma\) and \(i = 1, 2, \ldots n\).

To prove this, first note that by the mean value theorem, for any points \(p, q \in \sigma, |\omega_1(q) - \omega_1(p)| \leq c \cdot \sup |\frac{\partial \omega_1}{\partial x^j}| \cdot \eta.\) (Here we’re using the fact that the Riemannian metric and the flat one induced by pulling back along the coordinates \(x^i\) are com-mensurable.) Similarly for \(\omega_2\). Now, fix \(p \in \sigma\) and let \(dV_\beta\) be the volume element on \([p_\beta, p_1, \ldots, p_j]\), and \(dV_\gamma\) be the volume element on \([p_\gamma, p_1, \ldots, p_j]\). Then
\[
\left| j!k! \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - f(p) g(p) \right| \leq c \cdot \left( c_1 \cdot \sup \left| \frac{\partial \omega_2}{\partial x^j} \right| + c_2 \cdot \sup \left| \frac{\partial \omega_1}{\partial x^j} \right| \right) \cdot \eta^{j+k+1}.
\]
This implies, by the triangle inequality, for any \(\beta, \gamma\)
\[(2) \quad \left| \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - \int_{[\gamma]} \omega_1 \int_{[\beta]} \omega_2 \right| \leq c \cdot \left( c_1 \cdot \sup \left| \frac{\partial \omega_2}{\partial x^j} \right| + c_2 \cdot \sup \left| \frac{\partial \omega_1}{\partial x^j} \right| \right) \cdot \eta^{j+k+1}.
\]

Now that we have estimated the coefficients of \(W(R\omega_1 \cup R\omega_2)\), this case is completed by estimating the the product of the \(d\mu_i\)'s that appear in \(W(R\omega_1 \cup \omega_2)\). As shown in [7],
\[
|d\mu_i|_p \leq \frac{\lambda}{r(p_i, |\sigma_i|)},
\]
where \(\sigma_i = [p_0, \ldots, p_{j-1}, p_{j+1}, \ldots, p_N]\) is the face opposite of \(p\), and \(r\) is the Riemannian geodesic distance. So, by lemma [4.4]
\[
|d\mu_i|_p \leq \lambda' \cdot \eta^{-1}.
\]
for some constant \(\lambda\), and therefore

\[
|d\mu_{i_1} \wedge \cdots \wedge d\mu_{i_{j+k}}|_p \leq |d\mu_{i_1}|_p \cdots |d\mu_{i_{j+k}}|_p \leq \lambda \cdot \eta^{-(j+k)}.
\]

By combining (2) and (3), we finally have, for the case that \(J\) and \(K\) intersect in exactly one element,

\[
|W(R\omega_1 \cup R\omega_2)(p) - \omega_1 \wedge \omega_2(p)|_p = |W(R\omega_1 \cup R\omega_2)(p)|_p \\
\leq C \cdot \left( c_1 \cdot \sup |\frac{\partial \omega_2}{\partial x^i}| + c_2 \cdot \sup |\frac{\partial \omega_1}{\partial x^i}| \right) \cdot \eta
\]

We now consider the case that \(J\) and \(K\) are disjoint. We first note that for any \(\tau \in Q - (J \cup K)\), there are exactly \(j + k + 1\) products

\[
[p_\beta, p_1, \ldots, p_j] \cup [p_\gamma, p_\alpha_1, \ldots, p_\alpha_k]
\]

which equal a nonzero multiple of \([p_\tau, p_J, p_K] = [p_\tau, p_1, \ldots, p_j, p_\alpha_1, \ldots, p_\alpha_k]\). These are given by the three mutually exclusive cases:

\[
\beta = \tau, \ \gamma \in J \\
\gamma = \tau, \ \beta \in K \\
\beta = \gamma = \tau
\]

Using the same notation as the previous case, we compute

\[
R\omega_1 \cup R\omega_2 = \frac{j!k!}{(j + k + 1)!} \left( \sum_{[\beta]} \left( \int_{[\beta]} \omega_1 \right) \left( \int_{[\gamma]} \omega_2 \right) [p_0, p_J, p_K] \right) \\
+ \sum_{\tau \in Q - \{0\}} \sum_{[\tau]} \left( \int_{[\beta]} \omega_1 \right) \left( \int_{[\gamma]} \omega_2 \right) [p_\tau, p_J, p_K]
\]

where the sums labeled \(\sum_{[\tau]}\) are over all \(\beta, \gamma\) such that

\[
[p_\beta, p_1, \ldots, p_j] \cup [p_\gamma, p_\alpha_1, \ldots, p_\alpha_k] = \frac{j!k!}{(j + k + 1)!} [p_\tau, p_J, p_K].
\]

From lemma 5.6, which follows the proof of this theorem,

\[
W([p_0, p_J, p_K]) = (j + k)! \ d\mu_J \wedge d\mu_K - \sum_{\tau \in Q - \{0\}} W([p_\tau, p_J, p_K])
\]

So,

\[
|W(R\omega_1 \cup R\omega_2) - \omega_1 \wedge \omega_2|_p \\
\leq \frac{j!k!}{(j + k + 1)!} \left| \sum_{[\beta]} \left( \int_{[\beta]} \omega_1 \right) \left( \int_{[\gamma]} \omega_2 \right) d\mu_J \wedge d\mu_K - \omega_1 \wedge \omega_2 \right|_p \\
+ \frac{j!k!}{(j + k + 1)!} \sum_{\tau \in Q - \{0\}} \left( \sum_{[\beta]} \left( \int_{[\beta]} \omega_1 \right) \left( \int_{[\gamma]} \omega_2 \right) - \sum_{[\beta]} \left( \int_{[\beta]} \omega_1 \right) \left( \int_{[\gamma]} \omega_2 \right) \right) W([p_\tau, p_J, p_K])_p
\]
By our estimates in (2) and (3), the latter term is bounded appropriately. As for the first term, recall that the sum \( \sum_{[0]} \) consists of \( j+k+1 \) terms. We use (2) again to bound

\[
\sum_{[0]} \left( \int_{[\beta]} \omega_1 \right) \left( \int_{[\gamma]} \omega_2 \right) \left( \int_{[0]} \omega_1 \right) \left( \int_{[0]} \omega_2 \right)
\]

and using (1), for fixed \( p \in \sigma \) we have a bound on

\[
\left| \left( \int_{[0]} \omega_1 \right) \left( \int_{[0]} \omega_2 \right) - f(p)g(p) \right|.
\]

Finally, using the triangle inequality and combining (5) and (6) with (3) we can conclude

\[
\frac{j!k!}{(j+k+1)} \sum_{[0]} \left( \int_{[\beta]} \omega_1 \right) \left( \int_{[\gamma]} \omega_2 \right) d\mu_I \wedge d\mu_K(p) - \omega_1 \wedge \omega_2(p) \left|_p \right.
\]

\[
\leq C \cdot \left( c_1 \cdot \sup \left| \frac{\partial \omega_2}{\partial x^j} \right| + c_2 \cdot \sup \left| \frac{\partial \omega_1}{\partial x^j} \right| \right) \cdot \eta
\]

Lemma 5.6. Let \( \sigma = [p_0, p_1, \ldots, p_n] \), \( N = \{1, 2, \ldots, n\} \) and \( I = \{i_1, \ldots, i_m\} \subset N \). Then

\[
W([p_0, p_{i_1}, \ldots, p_{i_m}]) = m! \cdot d\mu_{i_1} \wedge \cdots \wedge d\mu_{i_m} - \sum_{r \in N-I} W([p_r, p_{i_1}, \ldots, p_{i_m}])
\]

Proof. The proof is a computation. We let

\[
d\mu_I = d\mu_{i_1} \wedge \cdots \wedge d\mu_{i_m}
\]

and compute

\[
\frac{1}{m!} W([p_0, p_{i_1}, \ldots, p_{i_m}]) = \mu_0 \cdot d\mu_I + \sum_{s=1}^{m} (-1)^s \cdot \mu_{i_s} \cdot d\mu_0 \wedge d\mu_I^s
\]

\[
= \left( 1 - \sum_{r=1}^{n} \mu_r \right) \cdot d\mu_I + \sum_{s=1}^{m} (-1)^s \cdot \mu_{i_s} \cdot \left( \sum_{r=1}^{n} d\mu_r \right) \wedge d\mu_I^s
\]

\[
= d\mu_I - \sum_{r=1}^{n} \mu_r \cdot d\mu_I - \sum_{s=1}^{m} (-1)^s \cdot \mu_{i_s} \cdot \left( \sum_{r \in N-I} d\mu_r \right) \wedge d\mu_I^s
\]

\[
= d\mu_I - \sum_{r \in N-I} \mu_r \cdot d\mu_I - \sum_{s=1}^{m} (-1)^s \cdot \mu_{i_s} \cdot \left( \sum_{r \in N-I} d\mu_r \right) \wedge d\mu_I^s
\]

\[
= d\mu_I - \sum_{r \in N-I} \mu_r \cdot d\mu_I - \sum_{s=1}^{m} (-1)^s \cdot \mu_{i_s} \cdot d\mu_r \wedge d\mu_I^s
\]

\[
= d\mu_I - \frac{1}{m!} \sum_{r \in N-I} W([p_r, p_{i_1}, \ldots, p_{i_m}])
\]
Corollary 5.7. There exists a constant $C$ and positive integer $m$, independent of $K$ such that
\[
\|W(R\omega_1 \cup R\omega_2) - \omega_1 \wedge \omega_2\| \leq C \cdot \lambda(\omega_1, \omega_2) \cdot \eta
\]
where
\[
\lambda(\omega_1, \omega_2) = \|\omega_1\|_{\infty} \cdot \|(Id + \Delta)^m \omega_2\| + \|\omega_2\|_{\infty} \cdot \|(Id + \Delta)^m \omega_1\|
\]
for all smooth forms $\omega_1, \omega_2 \in \Omega(M)$, where $\|\|$ is the $L_2$-norm on $M$.

Proof. We integrate the point-wise estimate from theorem 5.4 using the facts that $M$ is compact, $\sup |\omega_k| = \|\omega_k\|_{\infty}$, and the Sobolev-Inequality
\[
\sup |\frac{\partial \omega_k}{\partial x^i}| \leq C \cdot \|\omega_k\|_{2m} = C \cdot \|(Id + \Delta)^m \omega_k\|
\]
for sufficiently large $m$, where $\|\|_{2m}$ is the Sobolev $2m$-norm. □

The convergence of $\cup$ to the associative product $\wedge$ is, a priori, a bit mysterious due to the following:

Example 5.8. The product $\cup$ is not associative. For example, in the figure below, $(a \cup b) \cup e = 0$, since $a$ and $b$ do not span a 0-simplex, but $a \cup (b \cup e) = -\frac{1}{4} e$.

In the above example, the cochains $a$, $b$ and $e$ may be thought of as delta functions, in the sense that they evaluate to one on a single simplex and zero elsewhere. If we work with cochains which are “smoother”, i.e. represented by the integral of a smooth differential form, associativity is almost obtained. In fact, the next theorem shows that for such cochains, the deviation from being associative is bounded by a constant times the mesh of the triangulation. Hence, associativity is recovered in the mesh goes to zero limit.

Theorem 5.9. There exists a constant $C$ and positive integer $m$, independent of $K$ such that
\[
\|((R\omega_1 \cup R\omega_2) \cup R\omega_3) - R\omega_1(R\omega_2 \cup R\omega_3)\| \leq C \cdot \lambda(\omega_1, \omega_2, \omega_3) \cdot \eta
\]
for all $\omega_1, \omega_2, \omega_3 \in \Omega(M)$, where $\|\|$ is the Whitney norm and
\[
\lambda(\omega_1, \omega_2, \omega_3) = \sum \|\omega_r\|_{\infty} \cdot \|\omega_s\|_{\infty} \cdot \|(Id + \Delta)^m \omega_t\|
\]
where the sum is over all cyclic permutations $\{r, s, t\}$ of $\{1, 2, 3\}$.

Proof. We can prove this by first showing each of $(R\omega_1 \cup R\omega_2) \cup R\omega_3$ and $R\omega_1 \cup (R\omega_2 \cup R\omega_3)$ are close to $\omega_1 \wedge \omega_2 \wedge \omega_3$ in the point-wise norm $\|\|_p$. The final result is then obtained by integrating and applying the Sobolev inequality to each point-wise error, then applying the triangle inequality.

Let $A \approx B$ mean
\[
|A - B|_p \leq c \cdot \sum \|\omega_r\|_{\infty} \cdot \|\omega_s\|_{\infty} \cdot \sup |\frac{\partial \omega_t}{\partial x^i}| \cdot \eta
\]
We’ll consider the first case,
\[
W((R\omega_1 \cup R\omega_2) \cup R\omega_3) \approx \omega_1 \wedge \omega_2,
\]
only; the second case is similar.

It suffices to consider the case

\[ \omega_1 = f \, d\mu_1 \wedge \cdots \wedge d\mu_j \]
\[ \omega_2 = g \, d\mu_{\alpha_1} \wedge \cdots \wedge d\mu_{\alpha_k} \]
\[ \omega_3 = h \, d\mu_{\beta_1} \wedge \cdots \wedge d\mu_{\beta_l} . \]

The proof is analogous to that of theorem 5.4, the only differences are that the combinatorics of two cochain products is slightly more complicated, and the estimates now involve coefficients which are triple products of integrals over simplicies.

Let

\[ N = \{1, \ldots, n\} \]
\[ J = \{1, \ldots, j\} \]
\[ K = \{\alpha_1, \ldots, \alpha_k\} \]
\[ L = \{\beta_1, \ldots, \beta_l\} \]
\[ Q = N - (J \cup K \cup L) \]

Let us assume \( J \cap K \cap L = \emptyset \); the other cases are similar. Define \( A \sim B \) by

\[ |A - B| \leq C \cdot \|\omega_r\|_{\infty} \cdot \|\omega_s\|_{\infty} \cdot \sup \left| \frac{\partial^{j+k+l+1}}{\partial x^j} \right| \cdot \eta^{j+k+l+1} \]

Using similar techniques as in the proof of theorem 5.4 for all \( a \in N - J, b \in N - K, c \in N - L \)

\[ \left( \begin{array}{c} \int_{[a]} \omega_1 \\ \int_{[b]} \omega_2 \\ \int_{[c]} \omega_3 \end{array} \right) \sim \left( \begin{array}{c} \int_{[0]} \omega_1 \\ \int_{[0]} \omega_2 \\ \int_{[0]} \omega_3 \end{array} \right) \]

\[ j!k!l! \left( \begin{array}{c} \int_{[0]} \omega_1 \\ \int_{[0]} \omega_2 \\ \int_{[0]} \omega_3 \end{array} \right) \sim f(p)g(p)h(p) \]

For any \( \tau \in Q \), there are exactly

\[ (j + k + 1)(j + k + 1) + (j + k + 1)l = (j + k + 1)(j + k + l + 1) \]

products

\[ [p_a, p_1, \ldots, p_j] \cup [p_b, p\alpha_1, \ldots, p\alpha_k] \cup [p_c, p\beta_1, \ldots, p\beta_l] \]

that equal a non-zero multiple of \([p_r, p_j, p_K, p_L] \). Then

\[ \frac{j!k!l!(j + k + 1)j!k!l!(j + k + 1)(j + k + l + 1)}{(j + k + 1)!(j + k + l + 1)!} = \frac{j!k!l!(j + k + 1)(j + k + l + 1)}{(j + k + l)!} \]

so that, by applying lemma 5.6 and equations (8) and (3),

\[ W((R\omega_1 \cup R\omega_2) \cup R\omega_3) \approx \]

\[ \frac{j!k!l!(j + k + 1)(j + k + l + 1)}{(j + k + l)!} \left( \begin{array}{c} \int_{[0]} \omega_1 \\ \int_{[0]} \omega_2 \\ \int_{[0]} \omega_3 \end{array} \right) W([p_0, p_j, p_K, p_L]) \]

\[ + \sum_{\tau \in Q \setminus \{0\}} \left( \begin{array}{c} \int_{[0]} \omega_1 \\ \int_{[0]} \omega_2 \\ \int_{[0]} \omega_3 \end{array} \right) W([p_r, p_j, p_K, p_L]) \]

\[ \square \]
In the previous theorem, we dealt with the non-associativity of $\cup$ analytically. There is also an algebraic way to deal with this, via an algebraic generalization of commutative, associative algebras, called $C_\infty$-algebras. First we’ll give an abstract definition, and then unravel what it means.

**Definition 5.10.** Let $C$ be a graded vector space, and $C[-1]$ denote the graded vector space $C$ with grading shifted down by one. Let $L(C) = \bigoplus_i L^i(C)$ be the free Lie co-algebra on $C$. A $C_\infty$-algebra on $C$ is a degree 1 co-derivation $D : L(C[-1]) \to L(C[-1])$ such that $D^2 = 0$.

A co-derivation on a free Lie co-algebra is uniquely determined by a collection of maps from $L^i(C)$ to $C$ for each $i \geq 1$. If we let $m_i$ denote the restriction of $D$ to $L^i(C)$, then the equation $D^2 = 0$ is equivalent to a collection of equations:

$$m_2^2 = 0$$
$$m_1 \circ m_2 = m_2 \circ m_1$$
$$m_2 \circ m_2 - m_2 \circ m_2 = m_1 \circ m_3 + m_3 \circ m_1$$
$$\vdots$$

We can regard $m_1$ as a differential and $m_2$ a commutative multiplication on $C$. The second equation states that $m_1$ is a derivation of $m_2$. The third equation states that $m_2$ is associative up to the (co)-chain homotopy $m_3$. Note that, due to the shift of grading, $m_j$ has degree $2 - j$.

The following theorem is due to Sullivan [28]. See also [31] for use of similar techniques.

**Theorem 5.11.** Let $(C, \delta)$ be the simplicial cochains of a triangulated space and $\cup$ be any local commutative (possibly non-associative) cochain multiplication on $C$ such that $\delta$ is a derivation of $\cup$. Then there is a canonical local inductive construction which extends $(C, \delta, \cup)$ to a $C_\infty$-algebra.

In the theorem, local means that the product of a $j$-simplex and a $k$-simplex is zero unless they span a $j + k$-simplex, in which case it is a multiple of this simplex. By theorem 5.12, the commutative product $\cup$ defined at the beginning of this section satisfies this and the other conditions of theorem 5.11.

The next theorem shows that the $C_\infty$-algebra on $C$ converges to the strict commutative and associative algebra given by the wedge product on forms in a sense analogous to the convergence statements we’ve made previously. In particular, all higher homotopies converge to zero as the mesh tends to zero.

**Theorem 5.12.** Let $C$ be the simplicial cochains of a triangulation $K$ of $M$, with mesh $0 \leq \eta \leq 1$. Let $m_1 = \delta, m_2 = \cup, m_3, \ldots$ be the extension of $C, \delta, \cup$ to a $C_\infty$-algebra as in theorem 5.11. Then there exists a constant $\lambda$ independent of $K$ such that, for all $j \geq 3$,

$$\|W(m_j(R\omega_1, \ldots, R\omega_j))\| \leq \lambda \cdot \prod_{i=1}^{j} \|\omega_i\|_{\infty} \cdot \eta$$

for all $\omega_1, \ldots, \omega_k \in \Omega(M)$.

**Proof.** Suppose degree $\omega_1, \ldots, \omega_j$ are of degree $\alpha_1, \ldots, \alpha_j$, respectively. Let $\alpha = \sum \alpha_i$. We need two facts. First, for any $\alpha_i$-simplex $\tau$ of $K$,

$$|R\omega_i(\tau)| \leq c \cdot \|\omega_i\|_{\infty} \cdot \eta^{\alpha_i}.$$
Secondly, if $p$ is a point in an $n$-simplex $\sigma$, and the $r$-simplicies which are faces of $\sigma$ are $\sigma_1^r, \ldots, \sigma_m^r$ then, by equation (3),

\begin{equation}
\left| W\left( \sum_{i=1}^{m} \sigma_i^r \right) \right|_p \leq c' \cdot \eta^{-r}.
\end{equation}

Now, since $m_j$ has degree $2 - j$, $m_j(R\omega_1, \ldots, R\omega_j)$ is a linear combination of $(\alpha + 2 - j)$-simplicies. Combining this with (9) and (10), we have for all $p \in M$ and some $\lambda \geq 0$

\[
|W(m_j(R\omega_1, \ldots, R\omega_j))|_p \leq \lambda \cdot \prod_{i=1}^{j} \|\omega_i\|_{\infty} \cdot \eta^{\alpha} \cdot \eta^{-(\alpha+2-j)}
\]

\[
\leq \lambda \cdot \prod_{i=1}^{j} \|\omega_i\|_{\infty} \cdot \eta
\]

The result is obtained by integrating over $M$. \qed

6. Combinatorial Star Operator

In this section we define the combinatorial star operator $\star$ and prove that it provides a good approximation to the smooth Hodge-star $\star$. We also examine the relations which are expected to hold by analogy with the smooth setting. We find that some hold precisely, while others may only recovered in the mesh goes to zero limit.

**Definition 6.1.** Let $K$ be a triangulation of a closed orientable manifold $M$, with simplicial cochains $C = \bigoplus_j C^j$. Let $\langle,\rangle$ be a non-degenerate positive definite inner product on $C$ such that $C^i \perp C^j$ for $i \neq j$. For $\sigma \in C^j$ we define $\star \sigma \in C^{n-j}$ by:

$$\langle \star \sigma, \tau \rangle = \langle \sigma \cup \tau \rangle [M]$$

where $[M]$ denotes the fundamental class of $M$.

We emphasize that, as motivated by definition 3.2, the essential ingredients of a star operator are Poincaré Duality and a non-degenerate inner product. We can regard the inner product as giving some geometric structure to the space. In particular it gives lengths of, and angles between, edges. As in the smooth setting, the star operator depends on the choice of inner product (or Riemannian metric). See section 8 for the definition of a particularly nice class of inner products that we call geometric inner products.

Here are some elementary properties of $\star$.

**Theorem 6.2.** The following hold:

1. $\star \delta = (-1)^{i+1} \delta \star$, i.e. $\star$ is a chain map.
2. For $\sigma \in C^i$ and $\tau \in C^{n-j}$, $\langle \star \sigma, \tau \rangle = (-1)^{i(n-j)} \langle \sigma, \star \tau \rangle$, i.e. $\star$ is (graded) skew-adjoint.
3. $\star$ induces isomorphisms $\mathcal{H}C^i(K) \to \mathcal{H}C^{n-j}(K)$ on harmonic cochains.

**Proof.** The first two proofs are computational:
(1) For $\sigma, \tau \in C$, we have:

\[
\langle \star \delta \sigma, \tau \rangle = (\delta \sigma \cup \tau)[M]
= (-1)^{j+1}(\sigma \cup \delta \tau)[M]
= (-1)^{j+1}\langle \star \sigma, \delta \tau \rangle
= \langle (-1)^{j+1}\delta \star \star \sigma, \tau \rangle
\]

where we have used that fact that $d$ is a derivation of $\cup$ and $M$ is closed.

(2) We compute:

\[
\langle \star \sigma, \tau \rangle = (\sigma \cup \tau)[M]
= (-1)^{j(n-j)}(\tau \cup \sigma)[M]
= (-1)^{j(n-j)}\langle \star \tau, \sigma \rangle
= (-1)^{j(n-j)}\langle \sigma, \star \tau \rangle
\]

(3) Via the Hodge-decomposition of cochains, $\mathcal{H}C^j(K)$ may be identified with the cohomology $\mathcal{H}^j(K)$. Here $\star$ is the composition of two isomorphisms, Poincare Duality (since $M$ is a manifold) and the inverse of the non-degenerate metric.

We remark here that $\star$ is in general not invertible, since the cochain product does not necessarily give rise to a non-degenerate pairing (on the cochain level!). This implies that $\star$ is not an orthogonal map, and $\star^2 \neq \pm Id$.

For the remainder of this section, we’ll fix the inner product on cochains to be the Whitney inner product, so that $\star$ is the star operator induced by the Whitney inner product. This will be essential in showing that $\star$ converges to the smooth Hodge star $\star$, which is defined using the Riemannian metric. First, a useful lemma. Let $\perp$ denote the orthogonal projection of $\Omega^j(M)$ onto the image of $C^j(K)$ under the Whitney embedding $W$.

**Lemma 6.3.** $W \star = \perp \star W$

**Proof.** Let $a \in C^j(K)$ and $b \in C^{n-j}(K)$. Note that $\star W a$ is an $L_2$-form but in general is not a Whitney form. We compute:

\[
\langle W \star a, W b \rangle = \langle \star a, b \rangle = \int_M W a \wedge W b = \langle \star W a, W b \rangle,
\]

Thus, $W \star a$ and $\star W a$ have the same inner product with all forms in the image of $W$, so $W \star = \perp \star W$. $\square$

Now for our convergence theorem of $\star$:

**Theorem 6.4.** Let $M$ be a Riemannian manifold with triangulation $K$ of mesh $\eta$. There exists a positive constant $C$ and a positive integer $m$, independent of $K$, such that

\[
\| \star \omega - W \star R \omega \| \leq C \cdot \|(Id + \Delta)^m \omega\| \cdot \eta
\]

for all $C^\infty$ differential forms $\omega$ on $M$. 
Proof. We compute and use Theorem 4.5

$$\| \star \omega - W \star R\omega \| = \| \star \omega - \perp \star W R\omega \| \leq \| \star \omega - W R\omega \| + \| W \star R\omega - \perp \star W R\omega \|$$

$$\leq \| \star \omega - W R\omega \| + \| W R\omega - W R \star \omega \|$$

$$\leq \| \omega - W R\omega \| + \| W R\omega - \star \omega \| + \| \star \omega - W R \star \omega \|$$

$$\leq 2 \| \omega - W R\omega \| + \| \star \omega - W R \star \omega \|$$

$$\leq 3 C \cdot \| (I d + \Delta)^m \omega \| \cdot \eta \square$$

The operator $\star$ also respects the Hodge decompositions of $C(K)$ and $\Omega(M)$ in the following sense:

**Theorem 6.5.** Let $M$ be a Riemannian manifold with triangulation $K$ of mesh $\eta$. Let $\omega \in \Omega^j(M)$, $R\omega \in C^j(K)$ have Hodge decompositions

$$\omega = d\omega_1 + \omega_2 + d^*\omega_3$$

$$R\omega = \delta a_1 + a_2 + \delta^* a_3$$

There exists a positive constant $C$ and a positive integer $m$, independent of $K$, such that

$$\| \star \omega_1 - W \star a_1 \| \leq C \cdot \| (I d + \Delta)^m \omega \| \cdot \eta$$

$$\| \star d\omega_2 - W \star \delta a_2 \| \leq C \cdot \| (I d + \Delta)^m \omega \| \cdot \eta$$

$$\| \star d^*\omega_3 - W \star \delta^* a_3 \| \leq C \cdot \| (I d + \Delta)^m \omega \| \cdot \eta$$

**Proof.** The proof is analogous to the proof of theorem 6.4 $\square$.

One might now ask further questions about convergence, say for compositions of the operators $\delta$, $\delta^*$, and $\star$. We now discuss some of these questions.

We first note $\delta$ provides a good approximation of $d$ in the sense that $\| d\omega - W \delta R\omega \|$ is bounded by a constant times the mesh. This follows immediately from theorem 4.5, using the fact that $\delta R = R d$. In the same way, using theorem 6.4, $\star \delta$ provides a good approximation to $\star d$. In summary, we have:

$$\pm \delta^* \star = \star \delta \to \star d = \pm d^*$$

One would also like to know if either of $\delta \star$ or $\star \delta^*$ provide a good approximation of $\pm d^* \star = d \star$. Answers to these questions are seemingly harder to come by.

As a precursor, we point out that there is not a complete answer as to whether or not $\delta^*$ converges to $d^*$. In [24], Smits does prove convergence for the case of 1-cochains on a surface. To the author’s mind, and as can be seen in the work of [24], one difficulty (with the general case) is that the operator $\delta^*$ is not local, since it involves the inverse of the cochain inner product, and as can be seen in the work of [8], a first attempt to understand this inverse is described in section 8.

The issue becomes further complicated when considering the operator $\star \delta^*$. We have no convergence statements about this operator. On the other hand, the operator $\delta \star$, which incidentally does not equal $\pm \star \delta^*$, is a bit less mysterious, and we have weak convergence in the sense that

$$\langle W \delta \star R\omega_1 - d \star \omega_1, \omega_2 \rangle$$

1If the cochain inner product is written as a matrix $M$ with respect to the basis given by the simplicies, then $\delta^* = M^{-1} \partial M$ where $\partial$ is the usual boundary operator on chains.
is bounded by a constant $\lambda$ (depending on $\omega_1$ and $\omega_2$) times the mesh.

Finally, one might ask if $\mathbf{\star}^2$ approaches $\pm Id$ for a fine triangulation. While we have no analytic result to state, our calculations for the circle in section 5 suggest this is the case. One can show that a graded symmetric operator squares to $\pm Id$ if and only if it is orthogonal. Hence one might view $\mathbf{\star}^2 \neq Id$ as the failure of orthogonality, which at least for applications to surfaces in section 7 presents no difficulty.

7. Applications to Surfaces

In this section we study applications of the combinatorial star operator on a triangulated closed surface. As motivation, let us first recall some facts from the analytic setting.

Let $M$ be a Riemann surface. There is a Hodge-star operator in the complex valued 1-forms of $M$, defined in local coordinates by $\mathbf{\star} dx = dy$ and $\mathbf{\star} dy = -dx$ and extended linearly over $\mathbb{C}$. One can check that this is well defined using the Cauchy-Riemann equations for the coordinate interchanges. The Hodge-star operator restricts to an orthogonal automorphism of complex valued 1-forms that squares to $-Id$. Furthermore, the harmonic 1-forms split into an orthogonal sum of holomorphic and anti-holomorphic 1-forms corresponding to the $-i$ and $+i$ eigenspaces of the Hodge-star operator.

Riemann studied how the integrals of holomorphic and anti-holomorphic 1-forms, called periods, are related to the underlying complex structure. He showed that for any fixed homology basis these periods satisfy the so-called bi-linear relations. Furthermore, choosing a particular basis for the holomorphic 1-forms gives rise to a period matrix, which, by Torelli’s theorem, determines the conformal structure of the Riemann surface. These period matrices lie in what is called the Siegel upper half space. (Two references for this material are [26] and [11].) An unsolved problem, called the Schottky problem, is to determine which points in the Siegel upper half space represent the period matrix of a Riemann surface.

In this section we’ll show that the combinatorial Hodge-star operator on a triangulated surface induces similar structures. In particular, given any hermitian inner product on the complex valued simplicial 1-cochains, the harmonic cochains split as holomorphic and anti-holomorphic 1-cochains. We’ll prove analogues of the bilinear relations of Riemann, and show how one obtains a combinatorial period matrix. This construction yields it’s own combinatorial Schottky problem, which won’t discuss in the current paper.

After describing our combinatorial construction, we’ll show that if the complex valued simplicial cochains of a triangulated orientable Riemannian 2-manifold are equipped with a particular inner product induced by the Whitney embedding, then all of these structures provide a good approximation to the their continuum analogues. In particular, the holomorphic and anti-holomorphic 1-cochains converge to the holomorphic and anti-holomorphic 1-forms, and the combinatorial period matrix converges to the conformal period matrix of the associated Riemann surface, as the mesh of the triangulation tends to zero. Hence, every conformal period matrix is a limit point of a sequence of combinatorial period matrices.

These statements may be interpreted as saying that a triangulation of a surface, endowed with an inner product on the associated cochains, determines a conformal
structure. Furthermore, for triangulations of a Riemannian 2-manifold, a conformal structure is recovered (in the limit) from algebraic and combinatorial data. Statements like this have been expressed by physicists for some time in various field theories and statistical mechanics.

We now proceed to describe the construction of combinatorial period matrices. First we need to extend some of our definitions from previous sections to deal with complex valued cochains. Let $\langle \cdot, \cdot \rangle$ be any non-degenerate positive definite hermitian inner product on the complex valued simplicial 1-cochains of a triangulated topological surface $K$. We define the associated combinatorial star operator $\star$ by:

$$\langle \star a, b \rangle = \langle a \cup b \rangle^* M,$$

where the bar denotes complex conjugation and $\cup$ is as in section 5, extended linearly over $\mathbb{C}$. Just as with real coefficients, we have a Hodge decomposition $C^1(K) \cong \delta C^0(K) \oplus H^1(K) \oplus \delta^* C^2(K)$ where $H^1$ is the space of complex valued harmonic 1-cochains. Since $\delta^* \star = \star \delta$, by theorem 6.2, $\star$ induces an isomorphism of $H^1$.

**Definition 7.1.** Let $K$, $\langle \cdot, \cdot \rangle$, and $\star$ be as above. We define subspace of the holomorphic 1-cochains by

$$H^{1,0}(K) = \{ \sigma \in H^1(K) | \star \sigma = -i \lambda \sigma \text{ for some } \lambda \geq 0 \}$$

and the subspace of the anti-holomorphic 1-cochains by

$$H^{0,1}(K) = \{ \sigma \in H^1(K) | \star \sigma = i \lambda \sigma \text{ for some } \lambda \geq 0 \}$$

Since $\star$ is not an orthogonal map, $\lambda$ may not equal one. The following theorem shows that the space of harmonic 1-cochains splits into the subspaces of holomorphic and anti-holomorphic cochains.

**Theorem 7.2.** Let $K$ be a triangulation of a surface $M$ of genus $g$ with with a hermitian inner product on the simplicial 1-cochains of $K$, and the induced operator $\star$. There is an orthogonal direct sum decomposition:

$$H^1(K) \cong H^{1,0} \oplus H^{0,1}$$

and each summand on the right has complex dimension $g$. Furthermore, complex conjugation maps $H^{1,0}$ to $H^{0,1}$ and vice versa.

**Proof.** The last assertion follows since $\star$ is linear over $\mathbb{C}$. To prove the decomposition, we first note that the induced map $\star$ on $\mathcal{H}$ has pure imaginary eigenvalues since it is skew-adjoint: $\langle \star \sigma, \tau \rangle = - \langle \sigma, \star \tau \rangle$. If $\sigma_1 \in H^{1,0}$ and $\sigma_2 \in H^{0,1}$ then for some $\lambda_1, \lambda_2 > 0$ then

$$-i \lambda_1 \langle \sigma_1, \sigma_2 \rangle = \langle \star \sigma_1, \sigma_2 \rangle = - \langle \sigma_1, \star \sigma_2 \rangle = i \lambda_2 \langle \sigma_1, \sigma_2 \rangle$$

so $H^{1,0}$ and $H^{0,1}$ are orthogonal. Finally, $dim(H^{1,0}) = dim(H^{0,1}) = g$ since $dim(\mathcal{H}) = 2g$ and the eigenvalues of $\star$ are all non-zero and occur in conjugate pairs. □

We'll now study further properties of holomorphic and anti-holomorphic 1-cochains. As in the smooth case, there is much to be gained by the analyzing the periods of these cochains. To do this, we first give a brief description of the homology basis we’ll evaluate these cochains on.

Without loss of generality, we assume that closed surface $M$ of genus $g$ is obtained by identifying sides of a $4g$-gon, as in the following figure:
The basis \( \{a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g\} \) for the first homology is classically referred to as the canonical basis \([11], [26]\), since it satisfies the following nice property: the intersection of any two basis elements is non-zero only for \( a_i \) and \( b_i \), in which case it equals one. Of course, this basis is not canonical; nevertheless, we’ll work with it. (As a note, the discussion below is basis independent up to an action of the modular symplectic group; we’ll not go into details here.) We assume our triangulation \( K \) is a subdivision of the cellular decomposition given by the canonical homology basis. For any such subdivision, each element of the canonical homology basis is represented as a sum of the edges into which it is subdivided, as in the following figure:

By evaluating a cochain of \( K \) on an element of the canonical homology basis, we mean evaluating it on this subdivided representative.
**Definition 7.3.** For \( h \in H \), the periods A-periods and B-periods of \( h \) are the following complex numbers:

\[
A_i = h(a_i) \quad B_i = h(b_i) \quad \text{for} \quad 1 \leq i \leq g
\]

**Theorem 7.4.** [Riemann’s Bi-linear relations] If \( \sigma, \sigma' \in H^{1,0} \), then the A-periods and B-periods satisfy:

\[
-i\lambda \langle \sigma, \sigma' \rangle = \sum_{i=1}^{g} (A_i B'_i - B_i A'_i) = 0
\]

where \( \lambda \) is such that \( \star \sigma = -i\lambda \sigma \).

**Proof.** Since \( \sigma' \in H^{1,0}, \sigma' \in H^{0,1} \) it follows that \( \langle \sigma, \sigma' \rangle = 0 \). To show the bi-linear relation we compute:

\[
-i\lambda \langle \sigma, \sigma' \rangle = \langle \star \sigma, \sigma' \rangle = (\sigma \cup \sigma')[M]
\]

where the fundamental class \([M]\) of \( M \) may be represented by the sum of the 2-cells of \( K \) appropriately oriented. Now let \( p : U \to M \) be the universal cover, with \( U \) triangulated so that \( p \) is locally a linear isomorphism onto the triangulation \( K \) of \( M \). Let \( S \) denote a fundamental domain in the triangulation of \( U \) so that the induced map \( p_* \) maps the 2-simplicies of \( S \) isomorphically onto the 2-simplicies of \( K \). Then \( p_*(S) = [M] \), so the last expression equals

\[
\langle \sigma \cup \sigma' \rangle([M]) = (p^*\sigma \cup p^*\sigma')(S)
\]

where \( p^* \) denotes the pull back on cohomology. Since \( \sigma \) is holomorphic, it is closed, as is \( p^*\sigma \). Since \( S \) is contractible to a point, the restriction of \( p^*\sigma \) to \( \overline{S} \) may be written as \( p^*\sigma = \delta f \) for some 0-cochain \( f \). Thus, since \( \delta \sigma' = 0 \) we have:

\[
-i\lambda \langle \sigma, \sigma' \rangle = (\delta f \cup p^*\sigma')(S) = (f \cup p^*\sigma')(\partial S) = \sum_{i=1}^{g} (f \cup p^*\sigma')(a_i + a_i^{-1} + b_i + b_i^{-1})
\]

It remains to show that this last expression equals \( \sum_{i=1}^{g} (A_i B'_i - B_i A'_i) \). To do this, we first derive a simple relation for the values of \( f \) on the 0-simplicies contained in the cycles of the canonical homology basis. Consider the following figure:
The chain \( \alpha \) from \( Q \) to \( Q' \) is a cycle. Since is homologous to the cycle made up of chains from \( Q \) to \( P \), \( P \) to \( P' \) and \( P' \) to \( Q' \), and since the first and third push forward to the same chains on \( K \), we have that

\[
f(Q) - f(Q') = f(\partial \alpha) = \delta f(\alpha) = p^* \sigma(\alpha) = p^* \sigma(b_i) = B_i
\]

which means that for any 1-cochains \( p^* \tau \)

\[
(f \cup \tau)(a_i^{-1}) = -((f + B_i) \cup \tau)(a_i) = -(f \cup \tau)(a_i) - B_i \tau(a_i)
\]

Similarly,

\[
(f \cup \tau)(b_i^{-1}) = -((f - A_i) \cup \tau)(b_i) = -(f \cup \tau)(b_i) + A_i \tau(a_i)
\]

So, we finally have that

\[
-i \lambda \langle \sigma, \sigma' \rangle = \sum_{i=1}^{g} (f \cup p^* \sigma')(a_i + a_i^{-1} + b_i + b_i^{-1})
\]

\[
= \sum_{i=1}^{g} -B_ip^* \sigma'(a_i) + A_i p^* \sigma'(b_i)
\]

\[
= \sum_{i=1}^{g} (A_i B'_i - B_i A'_i)
\]

\( \square \)

Replacing \( \sigma' \) with \( \sigma \) in the previous proof shows if \( \sigma, \sigma' \in \mathcal{H}^{1,0} \) then

\[
-i \lambda \langle \sigma, \sigma' \rangle = \sum_{i=1}^{g} (A_i B'_i - B_i A'_i)
\]

where \( \star \sigma = -i \lambda \sigma \). If we apply this to \( \sigma' = \sigma \) we obtain an expression for the norm of a holomorphic 1-cochain in terms of its periods.
Corollary 7.5. If \( \sigma \in H^{1,0} \) satisfies \( \star \sigma = -i \lambda \sigma \) with periods \( A_i \) and \( B_i \) then

\[
\| \sigma \|^2 = \langle \sigma, \sigma \rangle = \frac{i}{\lambda} \sum_{i=1}^{g} (A_i B_i - B_i A_i) \geq 0
\]

From this corollary we immediately have:

Corollary 7.6. Let \( \sigma \) be a holomorphic 1-cochain.

1. If the \( A \)-periods or \( B \)-periods of \( \sigma \) vanish then \( \sigma = 0 \).
2. If the \( A \)-periods and \( B \)-periods of \( \sigma \) are real then \( \sigma = 0 \).

Now let \( \{ \tau_1, \tau_2, \ldots, \tau_g \} \) be a basis for the space of holomorphic cochains. By the corollary, if all the \( A \)-periods of a linear combination of this basis vanish, then this linear combination is identically zero. This implies we can solve uniquely for coefficients \( c_{i,j} \) such that:

\[
\sum_{i=1}^{g} c_{i,j} \tau_i(a_k) = \delta_{j,k}
\]

We put \( \sigma_j = \sum_{i=1}^{g} c_{i,j} \tau_i \) and we call the basis \( \{ \sigma_1, \sigma_2, \ldots, \sigma_g \} \) the canonical basis for the space of holomorphic 1-cochains. From this we obtain the following array of periods:

\[
\begin{pmatrix}
\sigma_1 & 1 & 0 & \cdots & 0 & \sigma_1(b_1) & \sigma_1(b_2) & \cdots & \sigma_1(b_g) \\
\sigma_2 & 0 & 1 & \cdots & 0 & \sigma_2(b_1) & \sigma_2(b_2) & \cdots & \sigma_2(b_g) \\
\vdots & & & \ddots & & & & & \\
\sigma_g & 0 & 0 & \cdots & 1 & \sigma_g(b_1) & \sigma_g(b_2) & \cdots & \sigma_g(b_g)
\end{pmatrix}
\]

Definition 7.7. Let \( \{ \sigma_1, \sigma_2, \ldots, \sigma_g \} \) be the canonical basis for the space of holomorphic 1-cochains and \( \{a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g\} \) the canonical homology basis, so \( \sigma_i(a_j) = \delta_{i,j} \). We define the period matrix \( \Pi = (\pi_{i,j}) \) to be the \( g \times g \) matrix of \( B \)-periods:

\[
\pi_{i,j} = \sigma_i(b_j)
\]

When we wish to emphasize the dependence of \( \Pi \) on \( K \) or \( \langle , \rangle \) we’ll write \( \Pi_K \) or \( \Pi_{K,\langle \rangle} \).

Remark 7.8. Let \( K \) be fixed. If two inner products on \( C^1(K) \) differ by a constant multiple then the associated period matrices are equal. Hence, the combinatorial period matrix is a “conformal invariant”.

Theorem 7.9. Let \( K \) be a triangulated closed surface with a simplicial cochain inner product. The associated period matrix \( \Pi \) is symmetric and \( \text{Im}(\Pi) \) is positive definite.
Proof. It suffices to show for $1 \leq i, j \leq g$, $\sigma_i(b_j) = \sigma_j(b_i)$ where $\sigma_i$ and $\sigma_j$ are canonical holomorphic cochain basis elements. We apply theorem 7.4 and compute:

\[
0 = -i\lambda \langle \sigma_i, \sigma_j \rangle = \sum_{k=1}^{g} \sigma_i(a_k)\sigma_j(b_k) - \sigma_i(b_k)\sigma_j(a_k)
= \sum_{k=1}^{g} (\delta_{i,k}\sigma_j(b_k) - \delta_{j,k}\sigma_i(b_k)) = \sigma_j(b_i) - \sigma_i(b_j)
\]

To prove the second statement, let $\sigma = \sum_{i=1}^{g} c_i\sigma_i$ be a nontrivial $R$-linear combination of the canonical basis of holomorphic cochains. Then $\sigma(a_k) = c_i$. We show

\[
\langle \sigma, \text{Im}(\Pi) \rangle \cdot \sigma > 0
\]

by using corollary 7.5 and computing:

\[
0 < \frac{i}{\lambda} \sum_{k=1}^{g} \sigma(a_k)\overline{\sigma(b_k)} - \sigma(b_k)\overline{\sigma(a_k)} = \frac{i}{\lambda} \sum_{k=1}^{g} c_k \left( \sum_{i=1}^{g} c_i \sigma_i(a_k) \right) - c_k \left( \sum_{i=1}^{g} c_i \sigma_i(b_k) \right)
= \frac{i}{\lambda} \sum_{k=1}^{g} \sum_{i=1}^{g} c_k c_i \sigma_i(b_k) - c_k c_i \sigma_i(b_k) = \frac{2}{\lambda} \sum_{k=1}^{g} \sum_{i=1}^{g} c_k c_i \text{Im}(\sigma_i(b_k))
= \frac{2}{\lambda} \langle \sigma, \text{Im}(\Pi) \rangle \cdot \sigma
\]

\[
\square
\]

To this point, we have assumed $K$ is a triangulated closed topological surface and $\langle , \rangle$ is a non-degenerate inner product on the simplicial cochains of $K$. As remarked in the beginning of this section, the structures we have uncovered (splitting of harmonics, bilinear relations, period matrix etc.) also appear for 1-forms on a Riemann surface. In fact, all of the statements proven above hold for forms as well [20], except one should set $\lambda = 1$, since in this case the Hodge star operator $\ast$ is an orthogonal transformation.

Now let $M$ be an orientable closed Riemannian 2-manifold. The Riemannian metric induces an operator $\ast$ which squares to $-Id$, and by identifying tangent and cotangent space via the metric, this operator $\ast$ gives an almost complex structure. It is a theorem of Gauss that $M$ admits a unique complex structure, i.e. Riemann surface structure, that is compatible with this almost complex structure. This theorem is, a priori, non-trivial, and involves a transcendental construction of holomorphic coordinates charts. By Torelli’s theorem the resulting complex structure is determined uniquely by the period matrix of the associated Riemann surface $M$.

We can extend the usual $L^2$ inner product on the vector space of real valued 1-forms to a hermitian inner product on the space of complexified 1-forms canonically,
by declaring

\[(\omega_1 \otimes z_1, \omega_2 \otimes z_2) = z_1 \overline{z_2} (\omega_1, \omega_2)\]

for \(\omega_1 \otimes z_1 \in T^* M \otimes \mathbb{C}\). Let \(\| \cdot \|\) denote the induced norm.

Now, for any triangulation \(K\), by embedding complex valued 1-cochains into \(T^* M \otimes \mathbb{C}\) via the Whitney embedding, we obtain the induced Whitney inner product on complex valued 1-cochains. For the remainder of this section, we work only with this Whitney inner product. We remark here that while the approximation theorems from Section 4.1 and 6 (using the Whitney inner product) involved real-valued forms and cochains, the proofs follow verbatim for complex coefficients as well.

First we prove the following:

**Lemma 7.10.** Let \(M\) be a Riemannian 2-manifold with triangulation \(K\) of mesh \(\eta\), and \(\mathfrak{h}\) a complex valued 1-form on \(M\), so \(*\mathfrak{h} = -i\mathfrak{h}\). By the Hodge decomposition of cochains and Theorem 7.2 we may write

\[Rh = \delta g + h_1 + h_2 + \delta^* k\]

uniquely for \(h_1 \in \mathcal{H}^{1,0}\) and \(h_2 \in \mathcal{H}^{0,1}\). Then there exists a positive constant \(C\), independent of \(K\), such that

\[\|Wh_1 - \mathfrak{h}\| \leq C \cdot \eta\]

**Proof.** By Theorems 4.10 and 6.5, there is a positive constant \(C\), independent of \(K\), such that

\[C \cdot \eta \geq \|W \star (h_1 + h_2) - \star \mathfrak{h}\| + \|\mathfrak{h} - W(h_1 + h_2)\|\]

\[= \|W \star (h_1 + h_2) - \star \mathfrak{h}\| + \|\star \mathfrak{h} + iW(h_1 + h_2)\|\]

\[\geq \|W \star h_1 + W \star h_2 + iW(h_1 + h_2)\|\]

\[= \|\star h_1 + \star h_2 + i(h_1 + h_2)\|\]

Since \(h_1 \in \mathcal{H}^{1,0}\) and \(h_2 \in \mathcal{H}^{0,1}\) we may write \(\star h_1 = -i\lambda_1 h_1\) and \(\star h_2 = i\lambda_2 h_2\) for some \(\lambda_1, \lambda_2 > 0\). Using the fact that \(\mathcal{H}^{1,0} \perp \mathcal{H}^{0,1}\) we then have

\[C^2 \cdot \eta^2 \geq \| -i\lambda_1 h_1 + i\lambda_2 h_2 + ih_1 + ih_2\|^2\]

\[= \|(1 - \lambda_1)h_1 + (1 + \lambda_2)h_2\|^2\]

\[= \langle (1 - \lambda_1)h_1, (1 - \lambda_1)h_1 \rangle + \langle (1 + \lambda_2)h_2, (1 + \lambda_2)h_2 \rangle\]

\[= |1 - \lambda_1|^2\|h_1\|^2 + |1 + \lambda_2|^2\|h_2\|^2\]

So, we conclude

\[\|h_2\| \leq \frac{C \cdot \eta}{|1 + \lambda_2|} \leq C \cdot \eta\]

and finally,

\[\|Wh_1 - \mathfrak{h}\| \leq \|W(h_1 + h_2) - \mathfrak{h}\| + \|h_2\| \leq 2C \cdot \eta\]

**Remark 7.11.** A closer examination of the proof shows that, for a fine enough triangulation, \(1 - \lambda_1\) is bounded by a constant times the mesh. There is, of course, an analogous statement for anti-holomorphic 1-forms \(\mathfrak{h}\) and the anti-holomorphic part of the cochain \(Rh\).
One can check that the hermitian inner product on 1-forms of $M$, defined in (11), agrees with the usual inner product on the 1-forms of the Riemann surface associated to $M$, given by

$$\langle \omega, \eta \rangle = \int_M \omega \wedge \star \eta.$$  

It is a peculiarity of working in the middle dimension (here 1) that this inner product, and the Hodge star operator, depend only on the conformal class of the Riemannian metric. This implies that the period matrix of the Riemann surface associated to $M$ can be computed by using the inner product in (11) in the following way: split off the harmonic 1-forms and evaluate the appropriate basis of the $-i$ eigenspace of $\star$ on the canonical homology basis. We remark here that this involves a transcendental procedure in the Hodge decomposition of forms. The point of the following theorem is that the period matrix, and therefore the complex structure, is computable to any desired accuracy, from algebraic and combinatorial data.

**Theorem 7.12.** Let $M$ be a closed orientable Riemannian 2-manifold and let $\Pi$ be the period matrix of the Riemann surface associated to $M$. Let $K_n$ be a sequence of triangulations of $M$ with mesh converging to zero. Then, for each $n$, the induced Whitney inner product on the simplicial 1-cochains of $K_n$ gives rise to a combinatorial period matrix $\Pi_{K_n}$, and

$$\lim_{n \to \infty} \Pi_{K_n} = \Pi.$$  

**Proof.** Let $h_1, \cdots, h_g$ be the canonical basis of holomorphic 1-forms with periods

$$h_i(a_j) = \int_B h_i = \delta_{i,j}$$

$$h_i(b_j) = \int_{\partial B} h_i = \pi_{i,j}$$

for $1 \leq i, j \leq g$, and $\pi_{i,j}$ the $(i,j)$ entry of $\Pi$.

For each $n$, let $\varphi_1^n, \cdots, \varphi_g^n$ be a basis for the holomorphic cochains on $K_n$. Then the periods are

$$\varphi_i^n(a_j) = \delta_{i,j}$$

$$\varphi_i^n(b_j) = \pi_{i,j}^n$$

for $1 \leq i, j \leq g$, and $\pi_{i,j}^n$ the $(i,j)$ entry of $\Pi_{K_n}$. Our goal is to show, for all $1 \leq i, j \leq g$,

$$\lim_{n \to \infty} \varphi_i^n(b_j) = h_i(b_j).$$

Let $R_n$ denote the integration map taking 1-forms to cochains on $K_n$. We define $h_i^n$ holomorphic part of the cochain $R_n h_i$. By the previous lemma, $h_i^n \to h_i$ as $n \to \infty$. Therefore, by evaluating on a cycle $a_j$, we see from the Hodge decomposition of these closed forms that

$$\lim_{n \to \infty} h_i^n(a_j) = h_i(a_j) = \delta_{i,j}$$

For each $n$ and $1 \leq i \leq g$ we may write

$$h_i^n = \sum_{k=1}^{g} c_{i,k}^n \varphi_k^n$$
and by evaluating on the cycle \(a_j\) we see similarly that

\[ c^n_{i,j} = \sum_{k=1}^{g} c^n_{i,k} \varphi^n_k(a_j) = h^n_i(a_j) \]

Combining this with equation (12), we have

\[ \lim_{n \to \infty} c^n_{i,j} = \delta_{i,j} \]

which implies

\[ \lim_{n \to \infty} \| \varphi^n_i - h^n_i \| = 0 \]

By the lemma, \( \| h^n_i \| \to \| h_i \| \), so the sequences \( \| h^n_i \| \) and \( \| \varphi^n_i \| \) are bounded. Finally, we have

\[ \lim_{n \to \infty} \varphi^n_i(b_j) = \lim_{n \to \infty} h^n_i(b_j) = h_i(b_j) \]

\[ \square \]

**Corollary 7.13.** Let \( M \) be a closed Riemann surface with period matrix \( \Pi \). Let \( K_n \) be a sequence of triangulations of \( M \) with mesh converging to zero, and combinatorial period matrices \( \Pi_{K_n} \) induced by the Whitney metric. Then,

\[ \lim_{n \to \infty} \Pi_{K_n} = \Pi. \]

**Proof.** While there isn’t a notion of geodesic length on a Riemann surface, a distance converging to zero is well defined since it depends only on a conformal class of metrics. So the statement of the corollary makes sense. Then one can choose any Riemannian metric on \( M \) in the conformal class of metrics determined by \( M \), and apply the above theorem. \[ \square \]

**Corollary 7.14.** Every conformal period matrix is the limit of a sequence of combinatorial period matrices.

### 8. Inner Products and Their Inverses

In this section we study inner products on cochains, as well as the induced “inverse inner product”. Smits also studied the inverse of inner products in [12], where her proved his results on the convergence of the divergence operator \( d^* \) on a surface.

We start with the following definition:

**Definition 8.1.** A geometric inner product on the simplicial cochains \( C = \bigoplus_j C^j \) of a triangulated space \( K \) is a non-degenerate positive definite inner product \( \langle \cdot, \cdot \rangle \) on \( C \) satisfying:

1. \( C^i \perp C^j \) for \( i \neq j \)
2. locality: \( \langle a, b \rangle \neq 0 \) only if \( \text{St}(a) \cap \text{St}(b) \) is non-empty.

**Remark 8.2.** A geometric inner product restricted to 1-cochains, and its induced norm, gives a notion of lengths of edges and the angles between them. It may be interesting to study the consequences of an inner product of signature other than the one considered here.
We assume in this section that all cochain inner products are geometric in the above sense. Note that the Whitney inner product is geometric.

An inner product on $C^*$ induces an isomorphism from $C^*$ to the linear dual of $C^*$, which we denote by $C_*$, and refer to as the simplicial chains (to be more precise, this is the double dual of chains, but we’ll confuse the two since we’re assuming $K$ is compact). The inverse of the inner product is, by definition, the inverse of the isomorphism $C^* \rightarrow C_*$, and is an isomorphism $C_* \rightarrow C^*$. This gives an inner product on the (simplicial) chains $C_*$ and will be denoted by $\langle \cdot, \cdot \rangle^{-1}$.

If one represents a geometric cochain inner product as a matrix, using the standard basis given by the simplicies, then the locality property roughly states that this matrix is “near diagonal”. Of course, the inverse of a diagonal matrix is diagonal, but the inverse of a near diagonal matrix is not near diagonal. Rather, it can have all entries non-zero; i.e. the inverse inner product on chains is not geometric.\footnote{It is true is that the matrix entries decrease in absolute value as they move from the diagonal, so that the inner product of two chains decays rapidly as a function of “geometric distance”.

In this section, we describe the inner product $\langle \cdot, \cdot \rangle^{-1}$ on chains in a geometric way by showing it can be expressed as a weighted sum of paths in a collection of graphs associated to $K$. This will be useful in the next section for making explicit computations of the combinatorial star operator. We begin with some definitions:

**Definition 8.3.** A graph $\Gamma$ (without loops) consists of a set $S$, called vertices, and a collection of cardinality two subsets of $S$, called edges. Two edges of $\Gamma$ are said to be incident if their intersection (as subsets of $S$) is non-empty. A weighted graph is a graph with an assignment of a real number $w(e)$ to each edge $e$. A path $\gamma$ in a graph is a sequences of edges $\{e_i\}_{i \in I}$ such that for each $i$, $e_i$ and $e_{i+1}$ are incident to a common vertex. The weight $w(\gamma)$ of a path $\gamma$ in a weighted graph is the product of the weight of each edge in $\gamma$. By convention, we say there is a unique path of length zero between any vertex and itself, and the weight of this path is one.

**Definition 8.4.** Let $K$ be the simplicial cochain complex of a triangulated $n$-manifold $M$. We define the graph associated to the $j$-simplicies of $K$, denoted $\Gamma(K,j)$, to be the following graph: The vertices of $\Gamma(K,j)$ are the set $\{\sigma_i\}$ of $j$-simplicies of $K$; two distinct vertices $\sigma_1, \sigma_2$ of $\Gamma(K,j)$ form an edge if and only if they are faces of a common $n$-simplex of $K$ (i.e $\text{St}(\sigma_1) \cap \text{St}(\sigma_2)$ is non-empty).

**Corollary 8.5.** Let $K$ be the simplicial cochain complex of a triangulated $n$-manifold $M$.

1. Paths in $\Gamma(K,j)$ correspond to sequences $\{s_i\}_{i \in I}$ of $j$-simplicies in $K$ such that, for each $i$, $s_i$ and $s_{i+1}$ are faces of a common $n$-simplex.

2. $\Gamma(K,0)$ is isomorphic to $K_1$, the 1-skeleton of $K$ (the union of its vertices and edges).

**Proof.** This follows since $K$ is homeomorphic to a manifold. $\square$

Now suppose the cochains $C^*$ of $K$ are endowed with a geometric inner product $\langle \cdot, \cdot \rangle$. (Our motivating example is the Whitney metric on $C^*$, but other examples arise when considering interactions on simplicial lattices.) In this case we associate to $(C^*, \langle \cdot, \cdot \rangle)$ the following collection of weighted graphs.

**Definition 8.6.** Let $C^*$ be the cochains of a finite triangulation $K$ of a manifold, with geometric cochain inner product $\langle \cdot, \cdot \rangle$. We define the weighted graph associated
to the \( j \)-cochains \( C^j \), denoted \( \Gamma_w(K,j) \), to be the following weighted graph: The underlying graph of \( \Gamma_w(K,j) \) is \( \Gamma(K,j) \) and the weight \( w(e) \) of an edge \( e = \{\sigma_1, \sigma_2\} \) equals

\[
w(e) = \frac{\langle \sigma_1, \sigma_2 \rangle}{\|\sigma_1\| \cdot \|\sigma_2\|}
\]

where \( \|\sigma\| = \sqrt{\langle \sigma, \sigma \rangle} \)

**Remark 8.7.** The appropriate analogue corollary 8.5 for weighted graphs holds as well.

The following describes how the metric \( \langle \cdot, \cdot \rangle^{-1} \) on \( C_j \) can be computed by counting weighted paths in the weighted graph \( \Gamma_w(K,j) \) associated to \( (C_j \langle \cdot, \cdot \rangle) \).

**Theorem 8.8.** For \( \sigma_1, \sigma_2 \in C_j \)

\[
\langle \sigma_1, \sigma_2 \rangle^{-1} = \frac{1}{\|\sigma_1\| \cdot \|\sigma_2\|} \sum_{i \geq 0} (-1)^i \sum_{\gamma_i \in \Gamma_w(K,j)} w(\gamma_i)
\]

where \( \gamma_i \) is a path in \( \Gamma_w(K,j) \) of length \( i \), starting at \( \sigma_1 \) and ending at \( \sigma_2 \).

**Proof.** Let \( M \) be the matrix for \( \langle \cdot, \cdot \rangle \) with respect to a fixed ordering of the basis given by the simplicies of \( K \). Let \( D \) be the diagonal matrix, with respect to the same ordered basis, whose diagonal entries are the norm of a simplex. Let \( |M| = D^{-1}MD^{-1} \). Note that the entries of \( |M| \) are normalized since the entries of \( D^{-1} \) are of the form \( \frac{1}{\|\sigma\|} \). In particular the diagonal entries of \( |M| \) equal 1, so we may write

\[
M^{-1} = D^{-1}|M|^{-1}D^{-1} = D^{-1}(I + A)^{-1}D^{-1}
\]

It is easy to check that \( A \) is precisely the weighted adjacency matrix for the weighted graph \( \Gamma_w(K,j) \). Recall that the \( i^{th} \) power of a weighted adjacency matrix counts the sum of the weights of all paths of length \( i \). Then, by the Cauchy-Schwartz inequality, the formula

\[
(I + A)^{-1} = \sum_{i \geq 0} (-1)^i A^i
\]

may be applied above, and we conclude that

\[
\langle \sigma_1, \sigma_2 \rangle^{-1} = \sigma_1 M^{-1} \sigma_2 = \frac{1}{\|\sigma_1\| \cdot \|\sigma_2\|} \sum_{i \geq 0} (-1)^i \sum_{\gamma_i \in \Gamma_w(K,j)} w(\gamma_i)
\]

□

**Remark 8.9.**

1. The above theorem in the case \( j = 0 \), in light of remark 8.7, shows that for vertices \( p \) and \( q \) of \( K \), \( \langle p, q \rangle^{-1} \) may be expressed as a weighted sum over all paths in the 1-skeleton \( K_1 \subset K \).

2. These expressions for \( \langle \cdot, \cdot \rangle^{-1} \) not only provide a nice geometric interpretation, but are also useful for computations, as we will see in section 9 where we compute \( \star \) for the circle.
9. Computation for $S^1$

In this section we compute the operator $\star$ explicitly for the circle $S^1$. We take $S^1$ to be the unit interval $[0, 1]$ with 0 and 1 identified. We consider a sequence of subdivisions, the $n^{th}$ triangulation being given by vertices at the points $v_i = \frac{i}{n}$ for $0 \leq i \leq n$. We denote the edge from $v_i$ to $v_{i+1}$ by $e_i$ for $0 \leq i \leq n$ and orient this edge from $v_i$ to $v_{i+1}$. See the following figure:

![Figure showing vertices and edges of a triangulation]

All operators will be written as matrices with respect to the ordered basis $\{v_0, \ldots, v_{n-1}, e_0, \ldots, e_{n-1}\}$.

Recall that the operator $\star$ is defined by $\langle \star \sigma, \tau \rangle = (\sigma \cup \tau)[S^1]$ where here $[S^1]$ is the sum of all the edges with their chosen orientations. We’ll use the cochain inner product $\langle \cdot, \cdot \rangle$ induced by the Whitney embedding and the standard metric on $S^1$ (i.e. $\langle dt, dt \rangle = 1$). Let $M$ denote the matrix for the cochain inner product and let $C$ denote the matrix for the pairing given by $(\sigma, \tau) \mapsto (\sigma \cup \tau)[S^1]$. Then $\star = M^{-1} C$.

(We suppress the dependence of these operators on the level of subdivision; the $n^{th}$ level $M$ and $C$ are size $2n \times 2n$.)

By the definition of $\cup$ and our chosen orientations we have that

$$C = \left( \begin{array}{c|c} 0 & A \\ \hline A^t & 0 \end{array} \right)$$

where

$$A = \begin{pmatrix} 1/2 & 0 & 0 & \cdots & 0 & 1/2 \\ 1/2 & 1/2 & 0 & \cdots & \cdots & 0 \\ 0 & 1/2 & 1/2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1/2 & 1/2 \end{pmatrix}$$

and $t$ denotes transpose.

One can compute explicitly:

$$\langle \sigma, \tau \rangle = \begin{cases} \frac{2}{3n} & \sigma = \tau \text{ is a vertex} \\ \frac{1}{6n} & \sigma, \tau \text{ are vertices in the boundary of a common edge} \\ \frac{1}{n} & \sigma = \tau \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

So, in our chosen basis, the matrix for the inner product is given by:

$$M = \left( \begin{array}{c|c} B & 0 \\ \hline 0 & nI \end{array} \right)$$
where $I$ denotes the $n \times n$ identity matrix and

$$B = \begin{pmatrix}
2/3n & 1/6n & 0 & \ldots & 0 & 1/6n \\
1/6n & 2/3n & 1/6n & \ldots & \ldots & 0 \\
0 & 1/6n & 2/3n & \ldots & \ldots & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & \ldots & \ldots & 1/6n \\
1/6n & 0 & \ldots & 0 & 1/6n & 2/3n
\end{pmatrix}.$$ 

We now compute $B^{-1}$. Note that one can write $B = \frac{2}{3n} (\frac{1}{4} D + I)$ where

$$D = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 1 \\
1 & 0 & 1 & \ldots & \ldots & 0 \\
0 & 1 & 0 & \ddots & \ldots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & 1 \\
1 & 0 & \ldots & 0 & 1 & 0
\end{pmatrix}$$

then

$$B^{-1} = \frac{3n}{2} \left( \frac{1}{4} D + I \right)^{-1}$$

$$= \frac{3n}{2} (I - \frac{1}{4} D + \frac{1}{4^2} D^2 - \frac{1}{4^3} D^3 \pm \ldots)$$

$$= \frac{3n}{2} \sum_{k \geq 0} (-1/4)^k D^k$$

Note that $D$ is the adjacency matrix for the graph corresponding to the original triangulation $K$, or rather, $\frac{1}{4} D$ is the weighted adjacency matrix for the weighted graph in the following figure:

As shown in section 8, the matrices $\frac{1}{4^k} D^k$ have a geometric interpretation: the $(i, j)$ entry equals the total weight of all paths from $v_i$ to $v_j$ of length $k$. Since in this case all weights are $\frac{1}{4}$, we’ll simply compute the the $(i, j)$ entry of $D^k$, i.e the total number of paths from $v_i$ to $v_j$ of length $k$.

We first note that for the real line with integer vertices, the number of paths of length $r$ between two vertices distance $s$ apart is the binomial coefficient $\left( \frac{r}{2s} \right)$. By
considering the standard covering of the circle with \( n \) vertices by the line we have

\[
d_{i,j}^k = \sum_{t \in \mathbb{Z}} \binom{k+|i-j|+nt}{k}
\]

where the above binomial coefficient is zero unless \( \frac{k+|i-j|+nt}{2} \) is a non-negative integer less than or equal to \( k \). Hence,

\[
M^{-1} = \begin{pmatrix}
\frac{3n}{2} \sum_{k \geq 0} \left( -\frac{1}{4} \right)^k d_{i,j}^k & 0 \\
0 & \frac{1}{nI}
\end{pmatrix}
\]

We conclude that:

\[
\begin{align*}
\bigstar v_i &= \frac{1}{2n} (e_{i-1} + e_i) \\
\bigstar e_i &= 3n \sum_{0 \leq j \leq n-1} \left( \sum_{k \geq 0} \left( -\frac{1}{4} \right)^k \sum_{t \in \mathbb{Z}} \binom{k+|i-j|+nt}{k} \right) v_j
\end{align*}
\]

In the figures below, we plot \( \bigstar e_{n/2} \) for \( n = 10, 20, 50 \). In each figure, the x-axis denotes the circle, triangulated with black dots as vertices. For fixed \( n \), and each \( 0 \leq i \leq n \), we plot the coefficient of \( v_i \) appearing in \( \bigstar e_{n/2} \). We’ve used a triangle to denote this value. To suggest that the plots are roughly Gaussian and that operator approaches a “delta function”, we have connected consecutive plot points with a line.
The matrices and Gaussian type plots we have encountered are reminiscent of those that appear in the study of discrete differential operators. We emphasize here that this phenomena results from the inner product or metric, in particular its inverse.

From our computation of $\star$ one can easily compute $\star^2$, and it is clear that this operator approximates a delta-type function.

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