Fractional Periodic Processes: Properties and an Application of Polymer Form Factors

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Abstract

In this paper we introduce and study three classes of fractional periodic processes. An application to ring polymers is investigated. We obtain a closed analytic expressions for the form factors, the Debye functions and their asymptotic decay. The relation between the end-to-halftime and radius of gyration is computed for these classes of periodic processes.
1 Introduction

Stochastic processes with a periodicity in time have been used e.g. for the modelling of stochastic ring structures. In particular in polymer science ring polymers were based on a periodic random walk. We shall consider processes on a half-open interval $[0, L)$, with stationary increments depending only on the geodesic distance along the circle of length $L$, as in eq. (2) below, thus ensuring rotational invariance. It is worth pointing out that the standard Brownian bridge on the interval $[0, L]$ does not have this property. However it is possible to define a Brownian version of these processes as we see below.

In the case of fractional Brownian motion this has been done by Istas [1] for the case when the Hurst parameter $H$ is less or equal than the Brownian threshold $H \leq 1/2$. Above this limit the resulting covariance matrix of the process would no longer be positive semi-definite. There exists a recently developed relation between a class of fractional processes named generalized grey Brownian motion and the class of fractional Brownian motions which is given by multiplying the latter with a certain time-independent random variable. We can thus define a periodic generalized grey Brownian motion using this relation.

The paper is organized as follows. In Section 2 we define three classes of real-valued periodic processes. We note that the extension to $\mathbb{R}^d$-versions valued of these classes is straightforward. Section 3 is dedicated to the form factors of these processes. As mentioned before concrete applications are in long range coupled polymer models. The Debye function and the form factors are well known quantities from scattering theory and polymer physics. They give a deeper insight into the scaling behavior of observables linked with the underlying processes. We derive, for all three classes, analytic expressions of the form factors. In the appendix we recall some special functions used in this paper.

2 Classes of Periodic Processes

In this section we introduce the classes of processes used in this paper. They are periodic processes with “time” parameter $t$ varying on the circle $S_L$ of length $L > 0$. We parametrize the points on the circle $S_L$ by their angles $\theta \in [0, 2\pi)$. Fixing the length $L$ of the circle $S_L$, then we may parametrize the points on $S_L$ as $t \in [0, L)$.
We assume given a complete probability space \((\Omega, \mathcal{F}, P)\) for any of these processes.

### 2.1 Periodic fractional Brownian motion

A periodic fractional Brownian motion (pfBm for short) \(B^H_p\) with Hurst parameter \(0 < H \leq 1/2\) is a centered Gaussian process indexed by \(S_L\) with covariance function \(R^H\) given, for any \(0 \leq s,t < L\), by

\[
R^H(t,s;L) := \frac{1}{2}(d_H(t;L) + d_H(s;L) - d_H(t-s;L)),
\]

where

\[
d_H(\tau;L) := \min \left\{ |\tau|^{2H}, (L - |\tau|)^{2H} \right\}.
\]

The existence of these Gaussian processes is based on the positive semi-definiteness of the above covariance matrix, as argued by Istas [1] and [2, Chap. VIII], note however [3] concerning the restriction to \(0 < H \leq 0.5\).

**Remark 1.** For any \(0 \leq s,t < L\), the variance of the increment \(B^H_p(t) - B^H_p(s)\) follows from (1) and (2) and we have

\[
\mathbb{E}\left((B^H_p(t) - B^H_p(s))^2\right) = d_H(t-s;L).
\]

In addition, the characteristic function of \(B^H_p(t) - B^H_p(s)\) is

\[
\mathbb{E}\left(e^{ik(B^H_p(t) - B^H_p(s))}\right) = \exp\left(-\frac{k^2}{2}d_H(t-s;L)\right).
\]

**Proposition 2.**

1. The pfBm process is \(H\)-self-similar with stationary increments.

2. The pfBm process has a continuous modification. For any \(\gamma \in (0,H)\) this modification is \(\gamma\)-Hölder continuous on each finite interval.

**Proof.**

1. The \(H\)-self-similarity is expressed as the following equality in finite-dimensional distribution, for any \(a > 0\) and any \(t \in (0,L)\) we have

\[
B^H_p(at) = a^H B^H_p(t).
\]
This equality can be translated in terms of the covariance function $R^H$, more precisely if $0 \leq s, t < L$ it holds

$$R^H(at, as; aL) = a^{2H} R^H(t, s; L).$$

Hence, we have

$$R^H(at, as; aL) = \frac{1}{2}(d_H(at; aL) + d_H(as; aL) - d_H(at - as; aL))$$

and it is easy to see from (2) that

$$d_H(at; aL) = a^{2H}d_H(t; L), \quad d_H(as; aL) = a^{2H}d_H(s; L), \quad d_H(at - as; aL) = a^{2H}d_H(t - s; L).$$

Then the $H$-self-similarity of $B^H_p$ follows easily. To prove the stationarity of the Gaussian process $B^H_p$ it is sufficient to show that

$$\mathbb{E} \left( (B^H_p(t) - B^H_p(s))^2 \right) = \mathbb{E} \left( (B^H_p(t - s))^2 \right). \quad (5)$$

Equality (5) is a consequence of (3) and the definition of $d_H$, more precisely we have

$$\mathbb{E} \left( (B^H_p(t) - B^H_p(s))^2 \right) = d_H(t - s; L) = \mathbb{E} \left( (B^H_p(t - s))^2 \right).$$

2. Since $B^H_p(t) - B^H_p(s)$ is a centered Gaussian random variable with variance $d_H(|t - s|; L)$ we have

$$\mathbb{E}(|B^H_p(t) - B^H_p(s)|^p) = (d_H(|t - s|; L))^{p/2} \leq |t - s|^{pH}.$$  

Thus, if we take $p > \frac{1}{H}$ we obtain the existence of a continuous modification via the Kolmogorov-Chentsov continuity theorem. For the Hölder exponent one obtain $\gamma \in (0, H - \frac{1}{p})$. \hfill \square

From here on we work with the continuous modification of $B^H_p$ preserving the same notation.
2.2 Periodic Grey Brownian Motion

Grey Brownian motion was introduced by W. Schneider in [4, 5] in order to study the fractional Feynman-Kac formula. Here we are interested in the periodic version of this process which is represented in terms of the pfBm process $B^H_p$ as

$$X^H_p(t) := \sqrt{Y_{2H}B^H_p(t)}, \quad 0 \leq t < L, \ 0 < H \leq \frac{1}{2},$$

where $Y_{2H}$ is the positive random variable, independent of $B^H_p$, with density given via the $M$-Wright function $M^H_{2H}$, see Appendix A-(18). We call this process periodic grey Brownian motion (pgBm for short), see Remark 3 for more details.

It is easy to compute the characteristic function of the increment $X^H_p(t) - X^H_p(s)$, $0 \leq s, t < L$, namely,

$$\mathbb{E}\left(e^{ik(X^H_p(t) - X^H_p(s))}\right) = \int_0^{\infty} M^H_{2H}(\tau)\mathbb{E}\left(e^{ik\sqrt{\tau}(B^H_p(t) - B^H_p(s))}\right) d\tau$$

and using equality (4) we obtain

$$\mathbb{E}\left(e^{ik(X^H_p(t) - X^H_p(s))}\right) = \int_0^{\infty} M^H_{2H}(\tau)e^{-\frac{k^2}{2}\tau d_H(t-s;L)} d\tau.$$ 

Finally, using the Laplace transform of the density $M^H_{2H}$, see (20), we arrive at

$$\mathbb{E}\left(e^{ik(X^H_p(t) - X^H_p(s))}\right) = E^H_{2H}\left(-\frac{k^2}{2}d_H(t - s;L)\right). \quad (6)$$

Here $E^H_{2H}$ is the Mittag-Leffler function defined in (13). It follows from (6) that

$$\mathbb{E}((X^H_p(t) - X^H_p(s))^2) = \frac{d_H(t - s;L)}{\Gamma(2H + 1)}. \quad (7)$$

Remark 3. For $\beta = 2H$ the process $X^\beta_p$ has characteristic function, for any $\lambda \in \mathbb{R}^n$ and $0 \leq t_1 < t_2 < \ldots < t_n < L$

$$\mathbb{E}\left(\exp\left(i \sum_{k=1}^{n} \lambda_k X^\beta_p(t_k)\right)\right) = E^\beta_\beta\left(-\frac{1}{2}(\lambda, \Sigma \lambda)\right),$$

5
where $\Sigma = (a_{kj})_{1 \leq k,j \leq n}$, $a_{kj} = \mathbb{E}(X_p^{\frac{\beta}{2}}(t_k)X_p^{\frac{\beta}{2}}(t_j))$ is the covariance matrix and $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^n$. When the parameter $t$ is interpreted as time, that is $t \in \mathbb{R}_+$ or a subset $I \subset \mathbb{R}_+$, the process $\sqrt{\beta}B_p^H(t)$, $t \in \mathbb{R}_+$ is the one of W. Schneider [5, 4]. A systematic study of this class of processes and its generalization was realized by F. Mainardi and his collaborators, see [6] and references therein.

**Remark 4.** Actually we could start by giving the characteristic functional as in (6) and show the conditions of Minlos-Sazonov’s theorem. This approach leads to the Mittag-Leffler analysis (see [7]) where the law of the process is a probability measure on a space of generalized functions.

**Proposition 5.** The process $X_p^H$ is $H$-self-similar with stationary increments and continuous paths.

**Proof.** The proof follows as in Proposition 2. \qed

### 2.3 Periodic Generalized Grey Brownian Motion

We finally introduce the most general class of periodic processes used in this paper. More precisely, let $X_p^{\beta,H}$ denote the process defined by

$$X_p^{\beta,H}(t) := \sqrt{\beta}B_p^H(t), \quad t \geq 0, \ 0 < \beta \leq 1, \ 0 < H \leq \frac{1}{2},$$

where $B_p^H$ is the pfBm and $Y_\beta$ is the positive random variable, independent of $B_p^H$, with density $M_\beta$. We call this process **periodic generalized grey Brownian motion**. The characteristic function of the increment $X_p^{\beta,H}(t) - X_p^{\beta,H}(s)$, $0 \leq s, t < L$ may be computed as

$$\mathbb{E}\left(e^{ik(X_p^{\beta,H}(t)-X_p^{\beta,H}(s))}\right) = \int_0^\infty M_\beta(\tau)\mathbb{E}\left(e^{ik\sqrt{\tau}(B_p^H(t)-B_p^H(s))}\right) d\tau$$

and using equality (4) we obtain

$$\mathbb{E}\left(e^{ik(X_p^{\beta,H}(t)-X_p^{\beta,H}(s))}\right) = \int_0^\infty M_\beta(\tau)e^{-\frac{k^2}{2}d_H(t-s;L)} d\tau.$$ 

Finally, using the Laplace transform of the density $M_\beta$, see (20), we obtain

$$\mathbb{E}\left(e^{ik(X_p^{\beta,H}(t)-X_p^{\beta,H}(s))}\right) = E_\beta\left(-\frac{k^2}{2}d_H(t-s;L)\right). \quad (8)$$
It follows from (8) that
\[ \mathbb{E}((X^\beta,H_p(t) - X^\beta,H_p(s))^2) = \frac{d_H(t-s; L)}{\Gamma(\beta + 1)}. \] (9)

**Proposition 6.** The process $X^\beta,H_p$ is $H$-self-similar with stationary increments and continuous paths.

**Proof.** The proof is similar to that of Proposition 2. \qed

### 3 Form Factors for Periodic Processes

In this section we explore the form factors and the corresponding Debye functions for the classes of periodic processes introduced above. Explicit analytic expressions are computed for all three classes of periodic processes. The relation between the radius of gyration and end-to-halftime length is also shown.

#### 3.1 Form Factors for Periodic Fractional Brownian Motion

To begin with we note that, given a $d$-dimensional stochastic process $X$, the form factor associated to $X$ is the function defined by
\[ S^X(k) := \frac{1}{n^2} \int_0^n dt \int_0^n ds \mathbb{E}(e^{i(k,X(t) - X(s))}), \quad k \in \mathbb{R}^d, \; n \in \mathbb{N}, \text{ s.e.e.g.} \] (10)

which, in case $X$ is $\nu$-self-similar, simplifies to
\[ S^X(k) = \int_0^1 dt \int_0^1 ds \mathbb{E}(e^{in\nu(k,X(t) - X(s))}). \] (11)

This function encodes in particular, to lowest order in $k$, the *root-mean-square radius of gyration* (or simply *radius of gyration*) of $X$, defined by
\[ (R_g^X)^2 := \frac{1}{2} \frac{1}{n^2} \int_0^n dt \int_0^n ds \mathbb{E}(|X(t) - X(s)|^2) \]

which plays an important role in the study of random path conformations.
Hence, for the class of pfBm, denoting the form factor by $S_{\text{pfBm}}$, for any $k \in \mathbb{R}$, we have

$$S_{\text{pfBm}}(k) := \frac{2}{L^2} \int_0^L \int_0^t E \left( e^{ik(B^H_p(t) - B^H_p(s))} \right) ds \, dt$$

$$= \frac{2}{L^2} \int_0^L \int_0^t e^{-\frac{1}{2}k^2d_H(t-s;L)} ds \, dt.$$ 

Making the change of variable $\tau = t - s$ we obtain

$$S_{\text{pfBm}}(k) = \frac{2}{L^2} \int_0^L (L - \tau)e^{-\frac{1}{2}k^2d_H(\tau;L)} d\tau$$

$$= \frac{2}{L^2} \left( \int_0^{L/2} (L - \tau)e^{-\frac{1}{2}k^2(\tau)^2} d\tau + \int_{L/2}^L (L - \tau)e^{-\frac{1}{2}k^2(L-\tau)^2} d\tau \right)$$

$$= \frac{2}{L^2} \int_0^{L/2} e^{-\frac{1}{2}k^2\tau^2} d\tau$$

$$= \frac{(\frac{1}{2}k^2)^{-\frac{1}{2H}}}{L^H} \left[ \Gamma \left( \frac{1}{2H} \right) - \Gamma \left( \frac{1}{2H}, \frac{1}{2}k^2 \left( \frac{L}{2} \right)^{2H} \right) \right],$$

where $\Gamma(a, z) = \int_z^{\infty} t^{a-1} e^{-t} dt$ is the upper incomplete gamma function, see [10].

Denoting $y^2 := \frac{k^2}{2} \left( \frac{L}{2} \right)^{2H}$ the Debye function for pfBm has the explicit expression

$$S_{\text{pfBm}}(k) = f_{\text{pfBm}}(y; H) := \frac{1}{2Hy^{2H}} \left[ \Gamma \left( \frac{1}{2H} \right) - \Gamma \left( \frac{1}{2H}, y^2 \right) \right].$$

In Figure 1 we plot the Debye function $f_{\text{pfBm}}$ corresponding to the pfBm $B^H_p$ for different Hurst parameters $H$. The asymptotic of the Debye function $f_{\text{pfBm}}$ is given by

$$f_{\text{pfBm}}(y; H) \sim \Gamma \left( \frac{1}{2H} \right) \frac{y^{-1/H}}{2H}, \quad y \to \infty$$

and for $H = \frac{1}{2}$, i.e. in the case of periodic Brownian motion, $f_{\text{pfBm}}$ decays as $y^{-2}$. In general $f_{\text{pfBm}}$ decays as $y^{-\frac{1}{H}}$, see Figure 1(b) where the lines are getting steeper for smaller $H$.

For the the illustration of the dependence on large $y$, Figure 2 shows the Kratky plot for pfBm. Here the asymptotical behavior is very well visible.

The radius of gyration for pfBm is obtained by expanding the form factor $S_{\text{pfBm}}$ to lowest order. We obtain

$$\left( R_y^{\text{pfBm}}(L) \right)^2 = \frac{L^{2H}}{(2H + 1)2^{2H+1}}.$$
Figure 1: Debye function for the pfBm process $B^H_p$ for $H = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$.

Figure 2: Kratky plot for the pfBm process $X^H_p$ for $H = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$. 
to be compared with the linear case with time parameter \( t \in [0, l] \)

\[
(R^{{pfBm}}_g(l))^2 = \frac{l^{2H}}{(2H + 1)(2H + 2)}.
\]

Note that

\[
(R^{{pfBm}}_g \left( \frac{L}{2} \right))^2 = \frac{(R^{{pfBm}}_g(L))^2}{H + 1}.
\]

Computing the end-to-halftime length with time parameter \( t \in [0, \frac{L}{2}] \) gives

\[
\left( R^{{pfBm}}_e \left( \frac{L}{2} \right) \right)^2 = \mathbb{E} \left( \left( D^H_p \left( \frac{L}{2} \right) \right)^2 \right) = d_H \left( \frac{L}{2}, L \right) = \left( \frac{L}{2} \right)^{2H}
\]

which implies the following relation

\[
\frac{(R^{{pfBm}}_e \left( \frac{L}{2} \right))^2}{2(2H + 1)} = (R^{{pfBm}}_g(L))^2.
\]

### 3.2 Form Factors for Periodic Grey Brownian Motion

The form factor of the pgBm process \( X^H_p \) introduced in Subsection 2.2 is given for any \( k \in \mathbb{R} \) by

\[
S^{{pgBm}}(k) = \frac{2}{L^2} \int_0^L \int_0^t \mathbb{E} \left( e^{ik\sqrt{2H} (B^H_p(t) - B^H_p(s))} \right) dsdt
\]

\[
= \frac{2}{L^2} \int_0^L \int_0^\infty M_{2H}(\tau) \mathbb{E} \left( e^{-\frac{1}{2}k^2\sqrt{\tau} (B^H_p(t) - B^H_p(s))} \right) d\tau ds dt.
\]

Applying the Fubini theorem and making the change of variables \( r = t - s \), yields

\[
S^{{pgBm}}(k) = \frac{2}{L^2} \int_0^\infty M_{2H}(\tau) \int_0^L \int_0^t e^{-\frac{1}{2}k^2\tau d_H(t-s;L)} d\tau ds dt
\]

\[
= \frac{2}{L^2} \int_0^\infty M_{2H}(\tau) \int_0^L (L - r) e^{-\frac{1}{2}k^2\tau d_H(r;L)} dr dr.
\]

Once more Fubini’s theorem gives

\[
S^{{pgBm}}(k) = \frac{2}{L^2} \int_0^L (L - r) \int_0^\infty M_{2H}(\tau) e^{-\frac{1}{2}k^2\tau d_H(r)} dr dr
\]

\[
= \frac{2}{L^2} \int_0^L (L - r) E_{2H} \left( -\frac{k^2}{2} d_H(r;L) \right) dr
\]

\[
= \int_0^L E_{2H} \left( -\frac{k^2}{2} \left( \frac{L}{2} \right)^{2H} r^{2H} \right) dr
\]

\[
= E_{2H,2} \left( -\frac{k^2}{2} \left( \frac{L}{2} \right)^{2H} \right).
\]
In the last equality we have used formula (17) with $\rho = \sigma = \alpha = 1$, $\gamma = 2H$ and $E_{\beta,\alpha}$ is the generalized Mittag-Leffler, see [14]. Defining $y^2 := \frac{k^2}{2} \left( \frac{L}{2} \right)^{2H}$ it follows that the Debye function associated to the pgBm process $X^H_p$ is explicitly given by

$$f_{\text{pgBm}}(y; H) = E_{2H,2}(-y^2).$$

The asymptotic of $f_{\text{pgBm}}(\cdot; H)$ follows from Proposition 8.2 such that

$$f_{\text{pgBm}}(y; H) \sim \frac{1}{\Gamma(2H + 2)} \frac{1}{y^2}, \quad y \to \infty. \quad (12)$$

Figure 3: Debye functions for the pgBm process $X^H_p$ for $H = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$.

In Figure 3 we show the Debye function of the pgBm process $X^H_p$ for different values of the parameter $H$, in linear scale Figure 3(a) and LogLog scale Figure 3(b). The asymptotic of the Debye function $f_{\text{pgBm}}(\cdot; H)$ given in (12) is reflected in Figure 3(b) where the slope of the lines for $y$ big being the same. In other words, the lines are parallel.

For the illustration of the dependence on large $y$, Figure 4 shows the Kratky plot for pgBm. Here the asymptotical behavior is very well visible.

We have the following relation between the end–to-halftime length and the radius of gyration for pgBm

$$\frac{\left( R^\text{pgBm} \left( \frac{L}{2} \right) \right)^2}{2(2H + 1)} = \left( R^g_B(L) \right)^2.$$
3.3 Form Factors for Periodic Generalized Grey Brownian Motion

In this subsection we compute the form factor for the general class of pggBm process $X^p_{\beta,H}$ introduced in Subsection 2.3. It corresponds to a generalization of the results obtained above for the pfBm and pgBm processes.

The form factor for pggBm is given, for any $k \in \mathbb{R}$, by

$$S_{pggBm}(k) = \frac{2}{L^2} \int_0^L \int_0^t \mathbb{E} \left( e^{ik(X^p_{\beta,H}(t) - X^p_{\beta,H}(s))} \right) ds dt$$

$$= \frac{2}{L^2} \int_0^L \int_0^L \int_0^\infty M_\beta(\tau) \mathbb{E} \left( e^{-\frac{1}{2}k^2 \sqrt{\tau}(B^p_{\beta,H}(t) - B^p_{\beta,H}(s))} \right) d\tau ds dt.$$

Applying the Fubini theorem and making the change of variables $r = t - s$, yields

$$S_{pggBm}(k) = \frac{2}{L^2} \int_0^\infty M_\beta(\tau) \int_0^L \int_0^t e^{-\frac{1}{2}k^2 \tau d_{H}(t-s;L)} d\tau ds dt$$

$$= \frac{2}{L^2} \int_0^\infty M_\beta(\tau) \int_0^L (L - r) e^{-\frac{1}{2}k^2 \tau d_{H}(r;L)} dr d\tau.$$

Once more Fubini’s theorem yields

$$S_{pggBm}(k) = \frac{2}{L^2} \int_0^L (L - r) \int_0^\infty M_\beta(\tau) e^{-\frac{1}{2}k^2 \tau d_{H}(r;L)} dr d\tau$$

$$= \frac{2}{L^2} \int_0^L (L - r) E_\beta \left( -\frac{k^2}{2} d_{H}(r;L) \right) dr$$

$$= \int_0^1 E_\beta \left( -\frac{k^2}{2} \left( \frac{L}{2} \right)^{2H} r^{2H} \right) dr.$$
Using the equality (16) with $\rho = \sigma = \alpha = 1$ and $\gamma = 2H$ we obtain

\[
S_{\text{pggBm}}(k) = 2\Psi_2 \left( \frac{(1,2H)}{(1,\beta)}, \frac{(1,1)}{(2,2H)} \right) - \frac{k^2}{2} \left( \frac{L}{2} \right)^{2H} \\
= \sum_{n=0}^{\infty} \frac{1}{(1+2Hn)\Gamma(1+\beta n)} \left( -\frac{k^2}{2} \left( \frac{L}{2} \right)^{2H} \right)^n.
\]

The Debye function for $X^\beta_H$ is obtained, denoting $y^2 = \frac{k^2}{2} \left( \frac{L}{2} \right)^{2H}$, as

\[
f_{\text{pggBm}}(y; \beta, H) = \sum_{n=0}^{\infty} \frac{(-y^2)^n}{(1+2Hn)\Gamma(1+\beta n)}.
\]

In Figure 5, we plot the (truncated at $n = 700$) Debye function of $X^\beta_H$ for $\beta = \frac{1}{2}$ and $H = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ in linear scale Figure 5(a) and LogLog scale Figure 5(b).

The radius of gyration for pggBm is obtained by expanding the form factor to lower order

\[
(R^\beta_H(L))^2 = \frac{L^{2H}}{2^{2H+1}(2H+1)\Gamma(\beta + 1)}
\]

and the end-to-halftime length with time parameter $t \in [0, \frac{L}{2}]$ may be computed using (9). We obtain

\[
\left( R^\beta_H \left( \frac{L}{2} \right) \right)^2 = \mathbb{E} \left( \left( X^\beta_H \left( \frac{L}{2} \right) \right)^2 \right) = \frac{d_H \left( \frac{L}{2}, L \right)}{\Gamma(\beta + 1)} = \frac{L^{2H}}{\Gamma(\beta + 1)2^{2H}}.
\]

As a result, the following relation holds

\[
\frac{(R_{\text{pggBm}}^\beta \left( \frac{L}{2} \right))^2}{2(2H+1)} = (R^\beta_H(L))^2.
\]

A The Mittag-Leffler and M-Wright functions

In this appendix we introduce two families of functions which are used in this paper. They are the family of generalized Mittag-Leffler functions $E_{\beta,\alpha}$, $0 < \beta \leq 1$, $\alpha \in \mathbb{C}$ and the family of M-Wright functions $M_\beta$, $0 < \beta \leq 1$ which is a special case of the Wright functions $W_{\lambda,\mu}$, $\lambda > -1$, $\mu \in \mathbb{C}$. More details and these classes of functions may found in [9] and references therein.

The Mittag-Leffler function was introduced by G. Mittag-Leffler in a series of papers [13, 14, 15].
Figure 5: Debye functions for the pggBm process $X^{β,H}_p$ for $β = \frac{1}{2}$ and $H = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$.

**Definition 7** (Mittag-Leffler function). 1. For $β > 0$ the Mittag-Leffler function $E_β$ is defined as an entire function by the following series representation

$$E_β(z) := \sum_{n=0}^{∞} \frac{z^n}{\Gamma(βn+1)}, \quad z \in \mathbb{C}, \quad (13)$$

where $\Gamma$ denotes the gamma function.

2. For any $\rho \in \mathbb{C}$ the generalized Mittag-Leffler function is an entire function defined by its power series

$$E_{β,ρ}(z) := \sum_{n=0}^{∞} \frac{z^n}{\Gamma(βn+ρ)}, \quad z \in \mathbb{C}. \quad (14)$$

Note the relation $E_{β,1}(z) = E_β(z)$ and $E_1(z) = e^z$ for any $z \in \mathbb{C}$.

We have the following asymptotic for the generalized Mittag-Leffler function $E_{β,α}$.

**Proposition 8** (cf. [9, Section 4.7]). Let $0 < β < 2$, $α \in \mathbb{C}$ and $δ$ be such that

$$\frac{βπ}{2} < δ < \min\{π, βπ\}.$$

Then, for any $m \in \mathbb{N}$, the following asymptotic formulas hold:
1. If $|\arg(z)| \leq \delta$, then

$$E_{\beta,\alpha}(z) = \frac{1}{\beta} z^{(1-\alpha)/\beta} \exp(z^{1/\beta}) - \sum_{n=1}^{m} \frac{z^{-n}}{\Gamma(\alpha - \beta n)} + O(|z|^{-m-1}), \quad |z| \to \infty.$$  

2. If $\delta \leq |\arg(z)| \leq \pi$, then

$$E_{\beta,\alpha}(z) = -\sum_{n=1}^{m} \frac{z^{-n}}{\Gamma(\alpha - \beta n)} + O(|z|^{-m-1}), \quad |z| \to \infty.$$  

(15)

The Euler integral transform of the MLf may be used to compute the following integral, with real parts $\Re(\alpha), \Re(\beta), \Re(\sigma) > 0, \rho \in \mathbb{C}$ and $\gamma > 0$, cf. [12, eq. (2.2.13)]

$$\int_{0}^{1} t^{\rho-1}(1-t)^{\sigma-1} E_{\beta,\alpha}(xt^{\gamma}) \, dt = \Gamma(\sigma)_{2}\Psi_{2}\left(\begin{array}{c}(\rho, \gamma), (1, 1) \\ (\alpha, \beta), (\sigma + \rho, \gamma) \end{array} \bigg| x\right), \quad (16)$$

where $2\Psi_{2}$ is the Fox-Wright function (also called generalized Wright function [11, 9, Appendix F, eq. (F.2.14)] and [16]) given for $x, a_i, c_i \in \mathbb{C}$ and $b_i, d_i \in \mathbb{R}$ by

$$\begin{array}{c}2\Psi_{2}\left(\begin{array}{c}(a_{1}, b_{1}), (a_{2}, b_{2}) \\ (c_{1}, d_{1}), (c_{2}, d_{2}) \end{array} \bigg| x\right) = \sum_{n=0}^{\infty} \frac{\Gamma(a_{1} + b_{1} n) \Gamma(a_{2} + b_{2} n) x^{n}}{\Gamma(c_{1} + d_{1} n) \Gamma(c_{2} + d_{2} n) n!}.\end{array}$$

In particular, when $\rho = \alpha$ and $\gamma = \beta$, eq. (16) simplifies to

$$\int_{0}^{1} t^{\alpha-1}(1-t)^{\sigma-1} E_{\beta,\alpha}(xt^{\beta}) \, dt = \Gamma(\sigma)E_{\beta,\alpha+\sigma}(x). \quad (17)$$

Both integrals (16) and (17) appear in computing the form factors in Section 3.

The Wright function is defined by the following series representation which converges in the whole complex $z$-plane

$$W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^{n}}{n!\Gamma(\lambda n + \mu)}, \quad \lambda > -1, \mu \in \mathbb{C}.$$  

An important particular case of the Wright function is the so called $M$-Wright function $M_{\beta}$, $0 < \beta \leq 1$ (in one variable) defined by

$$M_{\beta}(z) := W_{-\beta,1-\beta}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^{n}}{n!\Gamma(-\beta n + 1 - \beta)}. \quad (18)$$

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For the choice $\beta = \frac{1}{2}$ the corresponding $M$-Wright function reduces to the Gaussian density

$$M_{\frac{1}{2}}(z) = \frac{1}{\sqrt{\pi}} \exp \left( -\frac{z^2}{4} \right).$$  \hspace{1cm} (19)

The MLF $E_{\beta}$ and the $M$-Wright are related through the Laplace transform

$$\int_0^\infty e^{-s\tau} M_{\beta}(\tau) \, d\tau = E_{\beta}(-s).$$  \hspace{1cm} (20)

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