EXPOSITION OF MOTIVIC MEASURES

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Abstract. In this short note we establish some properties of all those motivic measures which can be exponentiated. As a first application, we show that the rationality of Kapranov’s zeta function is stable under products. As a second application, we give an elementary proof of a result of Totaro.

1. Motivic measures

Let \( k \) be an arbitrary base field and \( \text{Var}(k) \) the category of varieties, i.e. reduced separated \( k \)-schemes of finite type. The Grothendieck ring of varieties \( K_0 \text{Var}(k) \) is defined as the quotient of the free abelian group on the set of isomorphism classes of varieties \( [X] \) by the relations \( [X] = [Y] + [X/Y] \), where \( Y \) is a closed subvariety of \( X \). The multiplication is induced by the product over \( \text{Spec}(k) \). When \( k \) is of positive characteristic, one needs also to impose the relation \( [X] = [Y] \) for every surjective radicial morphism \( X \to Y \); see Mustaţă [19, Page 78]. Let \( L := [\mathbb{A}^1] \).

The structure of the Grothendieck ring of varieties is quite mysterious; see Poonen [21] for instance. In order to capture some of its flavor several motivic measures, i.e. ring homomorphisms \( \mu : K_0 \text{Var}(k) \to R \), have been built. Examples include the counting measure \( \mu_# \) (see [19, Ex. 7.7]); the Euler characteristic measure \( \chi_c \) (see [19, Ex. 7.8]); the Hodge characteristic measure \( \mu_H \) (see [14, §4.1]); the Poincaré characteristic measure \( \mu_P \) with values in \( \mathbb{Z}[u] \) (see [14, §4.1]); the Larsen-Lunts “exotic” measure \( \mu_{LL} \) (see [13]); the Albanese measure \( \mu_{Alb} \) with values in the semigroup ring of isogeny classes of abelian varieties (see [19, Thm. 7.21]); the Gillet-Soulé measure \( \mu_{GS} \) with values in the Grothendieck ring \( K_0(\text{Chow}(k)_Q) \) of Chow motives (see [6]); and the measure \( \mu_{NC} \) with values in the Grothendieck ring of noncommutative Chow motives (see [23]). There exist several relations between the above motivic measures. For example, \( \chi_c, \mu_H, \mu_P, \mu_{NC}, \) factor through \( \mu_{GS} \).

2. Kapranov’s zeta function

As explained in [19, Prop. 7.27], in the construction of the Grothendieck ring of varieties we can restrict ourselves to quasi-projective varieties. Given a motivic measure \( \mu \), Kapranov introduced in [11] the associated zeta function

\[
\zeta_\mu(X; t) := \sum_{n=0}^\infty \mu([S^n(X)])t^n \in (1 + R[t])[t],
\]

where \( S^n(X) \) stands for the \( n \)-th symmetric product of the quasi-projective variety \( X \). In the particular case of the counting measure, (2.0.1) agrees with the classical Weil zeta function. Here are some other computations (with \( X \) smooth projective)

\[
\zeta_{\chi_c}(X; t) = (1 - t)^{-\chi_c(X)} \quad \zeta_P(X; t) = \prod_{r \geq 0}(1 - u^r)(-1)^{br} \quad \zeta_{\text{Alb}}(X; t) = \frac{[\text{Alb}(X)]t}{1 - t},
\]

where \( b_r := \dim_{\mathbb{C}}H^r_{dR}(X) \) and \( \text{Alb}(X) \) is the Albanese variety of \( X \); see [22, §3].

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Given a commutative ring $R$, recall from Bloch [2, Page 192] the construction of the big Witt ring $W(R)$. As an additive group, $W(R)$ is equal to $(1 + R[t])^\times, \times).$ Let us write $+_W$ for the addition in $W(R)$ and $1 = 1 + 0t + \cdots$ for the zero element. The multiplication $\ast_W$ in $W(R)$ is uniquely determined by the following requirements:

(i) The equality $(1 - at)^{-1} \ast_W (1 - bt)^{-1} = (1 - abt)^{-1}$ holds for every $a, b \in R$;

(ii) The assignment $R \mapsto W(R)$ is an endofunctor of commutative rings.

The unit element is $(1 - t)^{-1}$. We have also a (multiplicative) Teichmüller map

$$R \longrightarrow W(R) \quad a \mapsto [a] := (1 - at)^{-1}$$

such that $g(t) \ast_W [a] = g(at)$ for every $a \in R$ and $g(t) \in W(R)$; see [2, Page 193].

**Definition 3.1.** Elements of the form $p(t) - W q(t) \in W(R)$, with $p(t), q(t) \in R[t]$ and $p(0) = q(0) = 1 \in R$, are called rational functions.

Let $W_{\text{rat}}(R)$ be the subset of rational elements. As proved by Naumann in [20, Prop. 6], $W_{\text{rat}}(R)$ is a subring of $W(R)$. Moreover, $R \mapsto W_{\text{rat}}(R)$ is an endofunctor of commutative rings. Recall also the construction of the commutative ring $\Lambda(R)$. As an additive group, $\Lambda(R)$ is equal to $W(R)$. The multiplication is uniquely determined by the requirement that the involution group isomorphism $\iota : \Lambda(R) \to W(R), g(t) \mapsto g(-t)^{-1}$, is a ring isomorphism. The unit element is $1 + t$.

4. Exponentiation

Let $\mu$ be a motivic measure. As explained by Mustaţă in [19, Prop. 7.28], the assignment $X \mapsto \zeta_\mu(X; t)$ gives rise to a group homomorphism

$$\zeta_\mu(-; t) : K_0 \text{Var}(k) \longrightarrow W(R).$$

**Definition 4.1.** ([22, §3]) A motivic measure $\mu$ can be exponentiated\(^1\) if the above group homomorphism (4.0.2) is a ring homomorphism.

**Corollary 4.2.** Given a motivic measure $\mu$ as in Definition 4.1, the following holds:

(i) The ring homomorphism (4.0.2) is a new motivic measure;

(ii) Any motivic measure which factors through $\mu$ can also be exponentiated.

This class of motivic measures is well-behaved with respect to rationality:

**Proposition 4.3.** Let $\mu$ be a motivic measure as in Definition 4.1. If $\zeta_\mu(X; t)$ and $\zeta_\mu(Y; t)$ are rational functions, then $\zeta_\mu(X \times Y; t)$ is also a rational function.

**Proof.** It follows automatically from the fact that $W_{\text{rat}}(R)$ is a subring of $W(R)$. \(\square\)

As proved by Naumann in [20, Prop. 8] (see also [22, Thm. 2.1]), the counting measure $\mu_\#$ can be exponentiated. On the other hand, Larsen-Lunts “exotic” measure $\mu_{\text{LL}}$ cannot be exponentiated! This would imply, in particular, that

$$\zeta_{\mu_{\text{LL}}}(C_1 \times C_2; t) = \zeta_{\mu_{\text{LL}}}(C_1; t) \ast_{\mu_{\text{LL}}} (C_2; t)$$

for any two smooth projective curves $C_1$ and $C_2$. As proved by Kapranov in [11] (see also [19, Thm. 7.33]), $\zeta_\mu(C; t)$ is a rational function for every smooth projective curve.

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\(^1\)Note that Kapranov’s zeta function is similar to the exponential function $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The product $X^n$ corresponds to $x^n$ and the symmetric product $S^n(X)$ corresponds to $\frac{x^n}{n!}$ since $n!$ is the size of the symmetric group on $n$ letters.
curve $C$ and motivic measure $\mu$. Using Proposition 4.3, this hence implies that the right-hand side of (4.0.3) is also a rational function. On the other hand, as proved by Larsen-Lunts in [13, Thm. 7.6], the left-hand side of (4.0.3) is not a rational function whenever $C_1$ and $C_2$ have positive genus. We hence obtain a contradiction.

At this point, it is natural to ask which motivic measures can be exponentiated? We now provide a general answer to this question using the notion of $\lambda$-ring. Recall that a $\lambda$-ring $R$ consists of a commutative ring equipped with a sequence of maps $\lambda^n : A \to A, n \geq 0$, such that $\lambda^0(a) = 1$, $\lambda^1(a) = a$, and $\lambda^n(a + b) = \sum_{i+j=n} \lambda^i(a)\lambda^j(b)$ for every $a, b \in R$. In other words, the map
\[
\lambda_t : R \to \Lambda(R) \quad a \mapsto \lambda_t(a) := \sum_n \lambda^n(a)t^n
\]
is a group homomorphism. Equivalently, the composed map
\[
(4.0.4) \quad \sigma_t : R \xrightarrow{\lambda} \Lambda(R) \xrightarrow{\zeta} W(R) \quad a \mapsto \sigma_t(a) := \lambda_{-t}(a)^{-1}
\]
is a group homomorphism. This homomorphism is called the opposite $\lambda$-structure.

**Proposition 4.4.** Let $\mu$ be a motivic measure and $R$ a $\lambda$-ring such that:

(i) The above group homomorphism (4.0.4) is a ring homomorphism;
(ii) We have $\mu([S^n(X)]) = \sigma^n(\mu([X]))$ for every quasi-projective variety $X$.

Under these conditions, the motivic measure $\mu$ can be exponentiated.

**Proof.** Consider the following composed ring homomorphism
\[
(4.0.5) \quad K_0\text{Var}(k) \xrightarrow{\mu} R \xrightarrow{\sigma} W(R).
\]
The equalities $\mu([S^n(X)]) = \sigma^n(\mu([X]))$ allow us to conclude that (4.0.5) agrees with the group homomorphism $\zeta_t(-; t)$. This achieves the proof. \qed

**Remark 4.5.** Let $\mathcal{C}$ be a $\mathbb{Q}$-linear additive idempotent complete symmetric monoidal category. As proved by Heinloth in [9, Lem. 4.1], the exterior powers give rise to a special $\lambda$-structure on the Grothendieck ring $K_0(\mathcal{C})$, with opposite $\lambda$-structure given by the symmetric powers $\text{Sym}^n$. In this case, (4.0.4) is a ring homomorphism.

**Remark 4.6.** Let $\mathcal{T}'$ be a $\mathbb{Q}$-linear thick triangulated monoidal subcategory of compact objects in the homotopy category $\mathcal{T} = \text{Ho}(\mathcal{C})$ of a simplicial symmetric monoidal model category $\mathcal{C}$. As proved by Guletskii in [8, Thm. 1], the exterior powers give rise to a special $\lambda$-structure on $K_0(\mathcal{T}')$, with opposite $\lambda$-structure given by the symmetric powers $\text{Sym}^n$. In the case, (4.0.4) is a ring homomorphism.

**Remark 4.7.** Assume that $k$ is of characteristic zero. Thanks to Heinloth’s presentation of the Grothendieck group of varieties (see [10, Thm. 3.1]), it suffices to verify the equality $\mu([S^n(X)]) = \sigma^n(\mu([X]))$ for every smooth projective variety $X$.

As an application of the above Proposition 4.4, we obtain the following result:

**Proposition 4.8.** The Gillet-Soulé motivic measure $\mu_{GS}$ can be exponentiated.

**Proof.** Recall from [6] that $\mu_{GS}$ is induced by the symmetric monoidal functor
\[
(4.0.6) \quad \mathfrak{h} : \text{SmProj}(k) \to \text{Chow}(k)_{\mathbb{Q}}
\]
from the category of smooth projective varieties to the category of Chow motives. Since the latter category is $\mathbb{Q}$-linear, additive, idempotent complete, and symmetric monoidal, Remark 4.5 implies that the Grothendieck ring $K_0(\text{Chow}(k)_{\mathbb{Q}})$ satisfies
condition (i) of Proposition 4.4. As proved by del Baño-Aznar in [4, Cor. 2.4], we have $h(S^n(X)) \simeq \text{Sym}^n h(X)$ for every smooth projective variety $X$. Using Remark 4.7, this hence implies that condition (ii) of Proposition 4.4 is also satisfied. □

Remark 4.9. Thanks to Corollary 4.2(ii), all the motivic measures which factor through $\mu_{\text{GS}}$ (e.g. $\chi_{c}, \mu_{H}, \mu_{P}, \mu_{NC}$) can also be exponentiated.

5. Application I: rationality of zeta functions

By combining Propositions 4.3 and 4.8, we obtain the following result:

Corollary 5.1. Let $X, Y$ be two varieties. If $\zeta_{\mu_{\text{GS}}}(X; t)$ and $\zeta_{\mu_{\text{GS}}}(Y; t)$ are rational functions, then $\zeta_{\mu_{\text{GS}}}(X \times Y; t)$ is also a rational function.

Remark 5.2. Corollary 5.1 was independently obtained by Heinloth [9, Prop. 6.1] in the particular case of smooth projective varieties and under the extra assumption that $\zeta_{\mu_{\text{GS}}}(X; t)$ and $\zeta_{\mu_{\text{GS}}}(Y; t)$ satisfy a certain functional equation.

Example 5.3. Let $X, Y$ be smooth projective varieties (e.g. abelian varieties) for which $h(X), h(Y)$ are Kimura-finite; see [12, §3]. Consider the ring homomorphism

$$\sigma_t : K_0(\text{Chow}(k)_Q) \longrightarrow W(K_0(\text{Chow}(k)_Q)).$$

As proved by Andrè in [1, Prop. 4.6], $\sigma_t([h(X)])$ and $\sigma_t([h(Y)])$ are rational functions. Since $\zeta_{\mu_{\text{GS}}}(-; t)$ agrees with the composition of $\mu_{\text{GS}}$ with (5.0.7), these latter functions are equal to $\zeta_{\mu_{\text{GS}}}(X; t)$ and $\zeta_{\mu_{\text{GS}}}(Y; t)$, respectively. Using Corollary 5.1, we hence conclude that $\zeta_{\mu_{\text{GS}}}(X \times Y; t)$ is also a rational function.

Recall from Voevodsky [24, §2.2] the construction of the functor $M^c : \text{Var}(k)^P \longrightarrow \text{DM}_{\text{gm}}(k)_Q$ from the category of varieties and proper morphisms to the triangulated category of geometric motives. As proved in [24, Prop. 4.1.7], the functor (5.0.8) is symmetric monoidal. Moreover, given a variety $X$ and a closed subvariety $Y \subset X$, we have

$$M^c(Y) \longrightarrow M^c(X) \longrightarrow M^c(X \setminus Y) \longrightarrow M^c(Y)[1];$$

see [24, Prop. 4.1.5]. Consequently, we obtain the following motivic measure

$$(5.0.9) \quad K_0\text{Var}(k) \longrightarrow K_0(\text{DM}_{\text{gm}}(k)_Q) \quad [X] \mapsto [M^c(X)].$$

Proposition 5.4. The above motivic measure (5.0.9) agrees with $\mu_{\text{GS}}$.

Proof. As proved by Voevodsky in [24, Prop. 2.1.4], there exists a $Q$-linear additive fully-faithful symmetric monoidal functor

$$(5.0.10) \quad \text{Chow}(k)_Q \longrightarrow \text{DM}_{\text{gm}}(k)_Q$$

such that (5.0.10) $\circ h(X) \simeq M^c(X)$ for every smooth projective variety. Thanks to the work of Bondarko [3, Cor. 6.4.3 and Rk. 6.4.4], the above functor (5.0.10) induces a ring isomorphism $K_0(\text{Chow}(k)_Q) \simeq K_0(\text{DM}_{\text{gm}}(k)_Q)$. Therefore, the proof follows from Heinloth’s presentation of the Grothendieck ring of varieties in terms of smooth projective varieties; see [10, Thm. 3.1]. □

Thanks to Proposition 5.4, Example 5.3 admits the following generalization:

Example 5.5. Let $X, Y$ be varieties for which $M^c(X), M^c(Y)$ are Kimura-finite. Similarly to Example 5.3, $\zeta_{\mu_{\text{GS}}}(X \times Y; t)$ is then a rational function.
In the above Examples 5.3 and 5.5, the rationality of \( \zeta_{\mu_{\GS}}(X \times Y; t) \) can alternatively be deduced from the stability of Kimura-finiteness under tensor products; see [12, §5]. Thanks to the work of O’Sullivan-Mazza [18, §5.1] and Guletskii [8], the above Corollary 5.1 can also be applied to non Kimura-finite situations.

**Proposition 5.6.** Let \( X_0 \) be a connected smooth projective surface over an algebraically closed field \( k_0 \) such that \( q = 0 \) and \( p_g > 0 \), \( k := k_0(X_0) \) the function field of \( X_0 \), \( x_0 \) a \( k_0 \)-point of \( X_0 \), \( z \) the zero-cycle which is the pull-back of the cycle \( \Delta(X_0) - (x_0 \times X) \) along \( X_0 \times k \to X_0 \times X_0 \), \( Z \) the support of \( z \), and finally \( U \) the complement of \( Z \) in \( X = X_0 \times k \). Under these notations, the following holds:

(i) The geometric motive \( M^c(U) \) is not Kimura-finite;

(ii) Kapranov’s zeta function \( \zeta_{\mu_{\GS}}(U; t) \) is rational.

**Proof.** As proved by O’Sullivan-Mazza in [18, Thm. 5.18], \( M(U) \) is not Kimura-finite. Since the surface \( U \) is smooth, we have \( M^c(U) \simeq M(U)^*(2)[4] \); see [24, Thm. 4.3.7]. Using the fact that \(-2)[4] \) is an auto-equivalence of the category \( \DM_{\gm}(k)_{\Q} \) and that \( M(U)^* \) is Kimura-finite if and only if \( M(U) \) is Kimura-finite (see Deligne [5, Prop. 1.18]), we conclude that \( M^c(U) \) also is not Kimura-finite.

We now prove item (ii). As proved by Guletskii in [8, §3], the category \( \DM_{\gm}(k)_{\Q} \) satisfies the conditions of Remark 4.6. Consequently, we have a ring homomorphism

\[
\sigma_t : \text{K}_0(\DM_{\gm}(k)_{\Q}) \to W(\text{K}_0(\DM_{\gm}(k)_{\Q})).
\]

As explained by Guletskii in [8, Ex. 5], \( \sigma_t([M(U)]) \) is a rational function. Thanks to Lemma 5.7 below, we hence conclude that \( \sigma_t([M^c(U)]) \) is also a rational function. The proof follows now from the fact that \( \zeta_{\mu_{\GS}}(-; t) \) agrees with the composition of the ring homomorphisms (5.0.9) and (5.0.11). \( \square \)

**Lemma 5.7.** Given a smooth variety \( X \) of dimension \( d \), we have the equality

\[
\sigma_t([M^c(X)]) = \sigma_{\mu_{\GS}(\L^d)}([-d; t]).
\]

**Proof.** The proof is given by the following identifications

\[
\begin{align*}
\sigma_t([M^c(X)]) &= \sigma_t([M(X)^*(d)[2d]]) \\
&= \sigma_t([M(X)^*]_{\mu_{\GS}(\L^d)}) \\
&= \sigma_t([M(X)^*]) * \zeta_{\mu_{\GS}}(\L^d; t) \\
&= \sigma_t([M(X)]) * \zeta_{\mu_{\GS}}(\L^d; t) \\
&= \sigma_{\mu_{\GS}(\L^d)}([-d; t]),
\end{align*}
\]

where (5.0.12) follows from [24, Thm. 4.3.7], (5.0.13) from [5, Lem. 1.18], and (5.0.14) from Remark 6.2 below with \( \mu := \mu_{\GS} \) and \( g(t) := \sigma_t([M(X)]) \). \( \square \)

**Example 5.8.** Let \( U_1, U_2 \) be two surfaces as in Proposition 5.6. Thanks to the above Corollary 5.1, we hence conclude that \( \zeta_{\mu_{\GS}}(U_1 \times U_2; t) \) is a rational function.

**Remark 5.9.** Thanks to Corollary 4.2(ii), the above Examples 5.3, 5.5, and 5.8, hold mutatis mutandis for any motivic measure which factors through \( \mu_{\GS} \).

6. **Application II: Totaro’s result**

The following result plays a central role in the study of the zeta functions.

**Proposition 6.1** (Totaro). The equality \( \zeta_{\mu}(X \times \A^n; t) = \zeta_{\mu}(X; \mu(\L)^n t) \) holds for every variety \( X \) and motivic measure \( \mu \).
Its proof (see [7, Lem. 4.4][19, Prop. 7.32]) is non-trivial and based on a stratification of the symmetric products of \( X \times \mathbb{A}^n \). In all the cases where the motivic measure \( \mu \) can be exponentiated, this result admits the following elementary proof:

**Proof.** Since \([X \times \mathbb{A}^n] = [X][\mathbb{A}^n]\) in the Grothendieck ring of varieties and the motivic measure \( \mu \) can be exponentiated, the proof is given by the identifications

\[
\zeta_\mu(X \times \mathbb{A}^n; t) = \zeta_\mu(X; t) \ast \zeta_\mu(\mathbb{A}^n; t) = \zeta_\mu(X; t) \ast \zeta_\mu(L; t)^{*n} = \zeta_\mu(X; t) \ast (1 + \mu(L)t + \mu(L)t^2 + \cdots)^{*n} = \zeta_\mu(X; t) \ast [\mu(L)]^{*n} = \zeta_\mu(X; t) \ast [\mu(L)^n] = \zeta_\mu(X; \mu(L)^n t),
\]

where (6.0.15) follows from [19, Ex. 7.23] and \([\mu(L)]\) stands for the image of \( \mu(L) \in R \) under the multiplicative Teichmüller map \( R \rightarrow W(R) \).

**Remark 6.2.** The above proof shows more generally that \( g(t) \ast \zeta_\mu(L^n; t) = g(\mu(L)^n t) \) for every \( g(t) \in W(R) \) and motivic measure \( \mu \) which can be exponentiated.

**Remark 6.3.** (Fiber bundles) Given a fiber bundle \( E \rightarrow X \) of rank \( n \), we have \([E] = [X][\mathbb{A}^n]\) in the Grothendieck ring of varieties; see [19, Prop. 7.4]. Therefore, the above proof, with \( X \) replaced by \( E \), shows that \( \zeta_\mu(E; t) = \zeta_\mu(X; \mu(L)^n t) \).

**Remark 6.4.** (\( \mathbb{P}^n \)-bundles) Given a \( \mathbb{P}^n \)-bundle \( E \rightarrow X \), we have \([E] = [X][\mathbb{P}^n]\) in the Grothendieck ring of varieties; see [19, Ex. 7.5]. Therefore, by combining the equality \([\mathbb{P}^n] = 1 + L + \cdots + L^n\) with the above proof, we conclude that

\[
\zeta_\mu(E; t) = \zeta_\mu(X; t) + W_1 \zeta_\mu(X; \mu(L)t) + W_2 \cdots + W_n \zeta_\mu(X; \mu(L)^n t).
\]

7. **G-Varieies**

Let \( G \) be a finite group and \( \text{Var}^G(k) \) the category of \( G \)-varieties, i.e. varieties \( X \) equipped with a \( G \)-action \( \lambda : G \times X \rightarrow X \) such that every orbit is contained in an affine open set. The Grothendieck ring of \( G \)-varieties \( \text{Ker}^G(k) \) is defined as the quotient of the free abelian group on the set of isomorphism classes of \( G \)-varieties \([X, \lambda]\) by the relations \([X, \lambda] = [Y, \tau] + [X \setminus Y, \lambda] \), where \((Y, \tau)\) is a closed \( G \)-invariant subvariety of \((X, \lambda)\). The multiplication is induced by the product of varieties. A motivic measure is a ring homomorphism \( \mu^G : \text{Ker}^G(k) \rightarrow R \). As mentioned in [15, §5], the above measures \( \chi_c, \mu_H, \mu_P \) admit \( G \)-extensions \( \chi^G_c, \mu^G_H, \mu^G_P \).

**Notation 7.1.** Let \( \text{Chow}^G(k) \) be the category of functors from the group \( G \) (considered as a category with a single object) to the category \( \text{Chow}(k) \).

Note that \( \text{Chow}^G(k) \) is still a \( \mathbb{Q} \)-linear additive idempotent complete symmetric monoidal category and that (4.0.6) extends to a symmetric monoidal functor

\[
\mathcal{F}^G : \text{SmProj}^G(k) \rightarrow \text{Chow}^G(k).
\]

Note also that the \( n \text{th} \) symmetric product of a \( G \)-variety is still a \( G \)-variety. Therefore, the notion of exponentiation makes sense in this generality. Gillet-Soulé’s motivic measure \( \mu_{GS} \) admits the following \( G \)-extension:
Proposition 7.2. The above functor (7.0.16) gives rise to a motivic measure
\[ \mu^G_{GS} : K_0 \text{Var}^G(k) \to K_0(\text{Chow}^G(k)_\mathbb{Q}) \]
which can be exponentiated.

Proof. Given a smooth projective variety \( X \) and a closed subvariety \( Y \), let us denote by \( \text{Bl}_Y(X) \) the blow-up of \( X \) along \( Y \) and by \( E \) the associated exceptional divisor. As proved by Manin in [16, §9], we have a natural isomorphism \( \mathfrak{h}(\text{Bl}_Y(X)) \cong \mathfrak{h}(X) \oplus \mathfrak{h}(E) \) in \( \text{Chow}(k)_\mathbb{Q} \). Since this isomorphism is natural, it also holds in \( \text{Chow}^G(k)_\mathbb{Q} \) when \( X \) is replaced by a smooth projective \( G \)-variety \((X, \lambda)\) and \( Y \) by a closed \( G \)-invariant subvariety \((Y, \tau)\). Therefore, thanks to Heinloth’s presentation of the Grothendieck ring of \( G \)-varieties in terms of smooth projective \( G \)-varieties (see [10, Lem. 7.1]), the assignment \( X \mapsto \mathfrak{h}^G(X) \) gives rise to a (unique) motivic measure \( \mu^G_{GS} \). The proof of Proposition 4.8, with (4.0.6) replaced by (7.0.16), shows that this motivic measure \( \mu^G_{GS} \) can be exponentiated. ❑

Remark 7.3. Thanks to Corollary 4.2(ii), all the motivic measures which factor through \( \mu^G_{GS} \) (e.g. \( \chi^G_{C, c}, \mu^G_{H}, \mu^G_{P} \)) can also be exponentiated.

Proposition 4.3 admits the following \( G \)-extension:

Proposition 7.4. Let \( \mu^G \) be a motivic measure which can be exponentiated and \( (X, \lambda), (Y, \tau) \) two \( G \)-varieties. If \( \zeta^G_{\mu}(\cdot; t) \) and \( \zeta^G_{\tau}(\cdot; t) \) are rational functions, then \( \zeta^G_{\mu}(X \times Y; \lambda \times \tau; t) \) is also a rational function.

Example 7.5. Assume that the group \( G \) (of order \( r \)) is abelian and that the base field \( k \) is algebraically closed of characteristic zero or of positive characteristic \( p \) with \( p \nmid r \). Under these assumptions, Mazur proved in [17, Thm. 1.1] that \( \zeta^G_{\mu}(C; \lambda; t) \) is a rational function for every smooth projective \( G \)-curve \((C, \lambda)\) and motivic measure \( \mu^G \). Thanks to Proposition 7.4, we hence conclude that \( \zeta^G_{\mu}(C_1 \times C_2; \lambda_1 \times \lambda_2; t) \) is still a rational function for every motivic measure \( \mu^G \) which can be exponentiated and for any two smooth projective \( G \)-curves \((C_1, \lambda_1)\) and \((C_2, \lambda_2)\).

Finally, Totaro’s result admits the following \( G \)-extension:

Proposition 7.6. Let \( \mu^G \) be a motivic measure which can be exponentiated and \( (X, \lambda), (\mathbb{A}^n, \tau) \) two \( G \)-varieties. When \( G \) (of order \( r \)) is abelian and \( k \) is algebraically closed, Kapranov’s zeta function \( \zeta^G_{\mu}(\cdot; t) \) agrees with

\[ \zeta^G_{\mu}(\cdot; t) + w \zeta^G_{\mu}(\cdot; t) = \left( \prod_{i=0}^{r-1} \prod_{j=1}^{n} \mu^G([\mathbb{A}^1, \tau^i, \ldots, \mathbb{A}^1, \tau^n])t^j \right), \]

where \([\mathbb{A}^n, \tau] = [\mathbb{A}^1, \tau^i] \cdots [\mathbb{A}^1, \tau^n] \).

Proof. Since \([X \times \mathbb{A}^n, \lambda \times \tau] = [X, \lambda][\mathbb{A}^n, \tau] \) in the Grothendieck ring of \( G \)-varieties and the motivic measure \( \mu^G \) can be exponentiated, we have the equality

\[ \zeta^G_{\mu}(\cdot; t) = \zeta^G_{\mu}(\cdot; t) \ast \zeta^G_{\mu}(\cdot; t). \]

Moreover, as explained in [17, Page 1383], we have the following computation

\[ \zeta^G_{\mu}(\cdot; t) = \frac{1}{1 - \mu^G(S^r(\mathbb{A}^n, \tau))t} \left( \prod_{i=0}^{r-1} \prod_{j=1}^{n} \mu^G([\mathbb{A}^1, \tau^i, \ldots, \mathbb{A}^1, \tau^n])t^j \right). \]

Therefore, since \((1 - \mu^G(S^r(\mathbb{A}^n, \tau))t)^{-1} \) is the Teichmüller class \([\mu^G(S^r(\mathbb{A}^n, \tau))])\), the proof follows from the combination of the above equalities. ❑
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