A multivariate CLT for «typical» weighted sums with rate of convergence of order $O(1/n)$

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Abstract The "typical" asymptotic behavior of the weighted sums of independent random vectors in $k$-dimensional space is considered. It is shown that in this case the rate of convergence in the multivariate central limit theorem is of order $O(1/n)$. This extends the one-dimensional Klartag and Sodin (2011) result.

1 Introduction and Main Result

Let $X, X_1, X_2, \ldots, X_n$ be independent identically distributed random vectors in $\mathbb{R}^k$ with finite third absolute moment $\gamma^3 = \mathbb{E}[|X|^3] < \infty$, zero mean $\mathbb{E}X = 0$ and unit covariance matrix $\text{cov}(X) = I$. Let $Z$ be the standard Gaussian random variable in $\mathbb{R}^k$ with zero mean and unit covariance matrix. Denote by $\mathcal{B}$, the class of all Borel convex sets in $\mathbb{R}^k$.

Sazonov [1] obtained the following error bound of approximation for distribution of the normalized sum of random vectors by the standard multivariate normal law:

$$\sup_{B \in \mathcal{B}} \left| \mathbb{P} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \in B \right) - \mathbb{P}(Z \in B) \right| \leq C(k) \frac{\gamma^3}{\sqrt{n}},$$

where $C(k)$ depends on dimension $k$ only.

The bound (1) is optimal one in general. Moreover, the rate $O(1/\sqrt{n})$ can not be improved under higher order moment assumptions. This is easy to show in one-dimensional case $k = 1$ taking $X$ such that

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\[
\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = 1/2.
\]

However, the situation is different when we consider a weighted sum

\[\theta_1 X_1 + \cdots + \theta_n X_n,\]

where \(\sum_{j=1}^n \theta_j^2 = 1\). If we are interested in the typical behavior of these sums for most of \(\theta\) in the sense of the normalized Lebesgue measure \(\lambda_{n-1}\) on the unit sphere

\[S^{n-1} = \{(\theta_1, \ldots, \theta_n) : \sum_{j=1}^n \theta_j^2 = 1\},\]

then we have to refer to a recent remarkable result due to Klartag and Sodin. In \([2]\) they have showed that in one-dimensional case \(k = 1\) for any \(\rho : 1 > \rho > 0\), there exists a set \(Q \subseteq S^{n-1} : \lambda_{n-1}(Q) > 1 - \rho\), and a constant \(C(\rho)\) depending on \(\rho\) only such that for any \(\theta = (\theta_1, \ldots, \theta_n) \in Q\) one has

\[
\sup_{a, b \in \mathbb{R}, a < b} \left| \mathbb{P}\left(a \leq \sum_{i=1}^n \theta_i X_i \leq b\right) - \int_a^b \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \right| \leq C(\rho) \frac{\delta^4}{n}, \tag{2}
\]

where \(\delta^4 = \mathbb{E}\|X\|^4\) and \(C(\rho) \leq C \log^2(1/\rho)\) with some absolute constant \(C\). It is clear that \(C(\rho) \to \infty\) as \(\rho\) tends to 0. And the case of equal weights, that is when \(\theta_i = 1/\sqrt{n}\) for all \(i = 1, \ldots, n\) is the worst case in the sense of closeness of distribution function of weighted sum to the standard normal distribution function.

Bobkov \([3]\) refined the rates of approximation for distributions of weighted sums \([2]\) up to order \(O(n^{-3/2})\) by using the Edgeworth correction of the fourth order provided \(\mathbb{E}\|X\|^5 < \infty\). In addition, see \([4]\) and \([5]\) for recent approximation results related to weighted sums in one-dimensional case \(k = 1\).

In this paper, we consider the rate of convergence of a weighted sum of independent random vectors to a multivariate standard normal vector. The estimate \([2]\) for weighted sums is generalized to the multidimensional case in the form of the following theorem.

**Theorem 1** Let \(X_1, X_2, \ldots, X_n\) be independent random vectors of dimension \(k\) with zero means \(\mathbb{E}X_j = 0\), unit covariance matrices \(\text{cov}(X_j) = I\) and finite fourth absolute moments \(\delta_j^4 = \mathbb{E}\|X_j\|^4\) for \(j = 1, \ldots, n\). Denote by \(\mathcal{B}\) the class of all convex Borel sets and by \(\Phi\) the multidimensional normal distribution with zero mean and unit covariance matrix. Let \(\delta^4 = \frac{1}{n} \sum_{j=1}^n \delta_j^4\) and \(\lambda_{n-1}\) be the normalized Lebesgue measure on the unit sphere \(S^{n-1} = \{(\theta_1, \theta_2, \ldots, \theta_n) : \sum_{j=1}^n \theta_j^2 = 1\}\). Then, for any \(\rho > 0\), there is a subset \(Q \subseteq S^{n-1}\) with \(\lambda_{n-1}(Q) > 1 - \rho\) and constant \(C(\rho, k)\) such that for any \(\theta = (\theta_1, \ldots, \theta_n) \in Q\), one has

\[
\sup_{B \in \mathcal{B}} \left| \mathbb{P}\left(\sum_{j=1}^n \theta_j X_j \in B\right) - \Phi(B) \right| \leq C(\rho, k) \frac{\delta^4}{n}.
\]
Moreover, \( C(\rho, k) \leq C(k) \ln^2 \left( \frac{1}{\rho} \right) \), where \( C(k) \) is a universal constant depending only on the dimension \( k \).

If we replace the class \( \mathcal{B} \) with a smaller class \( \mathcal{B}_\delta \) of all centered ellipsoids the situation changes noticeably. In this case, the distribution of the normalized sum of i.i.d. random vectors with \( \theta_1 = \cdots = \theta_n = 1/\sqrt{n} \) is approximated by a Gaussian distribution on the class \( \mathcal{B}_\delta \) with an accuracy of the order from \( o(1/\sqrt{n}) \) up to \( O(1/n) \) under the appropriate dimension of space and when the summands satisfy some moment conditions, for example, finiteness of the fourth absolute moment. See, e.g. [6], [7] and [8]. For non-i.i.d. random vectors case see [9].

2 Notation and auxiliary results

Notation In the following, the weight coefficients \( \theta_1, \theta_2, \ldots, \theta_n \), according to the statement of the theorem, will belong to the unit sphere

\[
S^{n-1} = \{ \theta_1, \theta_2, \ldots, \theta_n : \sum_{j=1}^{n} \theta_j^2 = 1 \}.
\]

Define \( \Theta = (\Theta_1, \Theta_2, \ldots, \Theta_n) \) as a random vector uniformly distributed on \( S^{n-1} \), and for a given set \( Q \subseteq S^{n-1} \) denote the normalized Lebesgue measure on the unit sphere as

\[
\lambda_{n-1}(Q) = P(\Theta \in Q).
\]

We denote the absolute fourth-order moment of the random vector \( X_j, j = 1, \ldots, n \), as

\[
\delta_j^4 = \mathbb{E}\|X_j\|^4,
\]

fourth-order weighted absolute moment

\[
\delta^4_\theta = \sum_{j=1}^{n} \theta_j^4 \delta_j^4,
\]

and the averaged absolute moment of the fourth order

\[
\delta^4 = \frac{1}{n} \sum_{j=1}^{n} \delta_j^4.
\]

We introduce truncated random variables \( Y_j \) and \( Z_j, j = 1, \ldots, n \), as

\[
Y_j = X_j \mathbb{1}_X(\|\theta_j X_j\| \leq 1),
\]

\[
Z_j = Y_j - \mathbb{E}Y_j,
\]
where $1_X(A)$ is the indicator function of an event $A$. For these random variables, we define the weighted expectation and the weighted covariance matrix as

$$A_n = \sum_{j=1}^{n} \theta_j \mathbb{E} Y_j,$$

$$D = \sum_{j=1}^{n} \theta_j^2 \text{cov}(Z_j),$$

$$Q^2 = D^{-1}. \tag{7}$$

We also use the notation for the distributions of random vectors appeared in the proof: over all Borel sets $B$

$$F_X(B) = P\left( \sum_{j=1}^{n} \theta_j X_j \in B \right), \tag{9}$$

$$F_Y(B) = P\left( \sum_{j=1}^{n} \theta_j Y_j \in B \right), \tag{10}$$

$$F_Z(B) = P\left( \sum_{j=1}^{n} \theta_j Z_j \in B \right),$$

$$F_{X_j}(B) = P(\theta_j X_j \in B), \; j = 1, \ldots, n, \tag{11}$$

$$F_{Y_j}(B) = P(\theta_j Y_j \in B), \; j = 1, \ldots, n. \tag{12}$$

Also $\Phi_{\alpha, V}$ will denote the distribution of the multidimensional standard normal law with expectation $\alpha$ and covariance matrix $V$. The proof will use the technique of characteristic functions. The characteristic function of the random vector $Z_j$ is defined as

$$\varphi_j(t) = \mathbb{E} \exp(it Z_j), \; j = 1, \ldots, n. \tag{13}$$

We denote $\tilde{F}_Z = \prod_{j=1}^{n} \varphi_j(\theta_j t)$ and $\tilde{\Phi}_{\alpha, V}$ as the corresponding characteristic functions of weighted sums of random vectors $\sum_{j=1}^{n} Z_j \theta_j$, and a random vector with multivariate normal distribution.

The absolute moment of order $s$ of the random vector $Z$ is defined as $\rho_s(Z) = \mathbb{E}\|Z\|^s$. Also, for the coefficients $\theta_1, \theta_2, \ldots, \theta_n$, the weighted absolute moment of order $s$ is determined as

$$\rho_s = \sum_{j=1}^{n} \rho_s(\theta_j Z_j) \tag{14}$$

and

$$\eta_s = \sum_{j=1}^{n} \rho_s(Q \theta_j Z_j), \tag{15}$$
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for 1 ≤ m ≤ s − 1 we have

$$\rho_s(Q\theta_j Z_j) \leq \|Q\|s \rho_s(\theta_j Z_j) = \|Q\|s \rho_s(\theta_j Y_j - \mathbb{E}Y_j)$$

$$\leq \|Q\|s^2 \rho_s(\theta_j Y_j) \leq \|Q\|s^2 \rho_{s-m}(\theta_j X_j). \quad (16)$$

For a given nonnegative vector $\alpha$, we define $\alpha$ the moment of the random vector $Z$ as

$$\mu_\alpha(Z) = \mathbb{E}Z^\alpha, \quad (17)$$

for this value the following inequality holds

$$\mu_\alpha(Z) \leq \rho_{|\alpha|}(Z). \quad (18)$$

Let the random vector $Z_j$ have finite absolute moments of order $m$. Then the characteristic function in a neighborhood of zero satisfies the Taylor expansion

$$\varphi_j(t) = 1 + \sum_{1 \leq |\nu| \leq m} \mu_\nu(Z_j) \frac{(it)^\nu}{\nu!} + o(||t||^m), \quad j = 1 \ldots n,$$

as $t$ tends to zero. Next, we define the logarithm of a nonzero complex number as $z = r \exp(i\xi)$ as

$$\log(z) = \log(r) + i\xi,$$

where

$$r > 0, \quad \xi \in (-\pi, \pi].$$

Thus, we always take the so-called main branch of the logarithm. The characteristic function of the random vector $Z_j$ is continuous and equal to one at zero. Consequently, in a neighborhood of zero, the Taylor expansion takes place

$$\log(\varphi_j(t)) = \sum_{1 \leq |\nu| \leq m} \kappa_\nu(Z_j) \frac{(it)^\nu}{\nu!} + o(||t||^m), \quad j = 1 \ldots n.$$

as $t$ tends to zero. The expansion coefficients of the logarithm of the characteristic function $\kappa_\nu$ are called the cumulants of the random vector $Z_j$. The cumulants $\kappa_\nu$ are explicitly expressed in terms of moments (see Ch.2 Sect.6 in [10]). In particular, the following inequality holds:

$$|\kappa_\nu(Z_j)| \leq c_1(\nu) \rho_{|\nu|}(Z_j), \quad j = 1 \ldots n. \quad (19)$$

We also point out the most important property of the cumulants of the sum of random vectors and denote the cumulants of the weighted sum as

$$\kappa_\nu = \kappa_\nu \left( \sum_{j=1}^n \theta_j Z_j \right) = \sum_{j=1}^n \theta_j^{|\nu|} \kappa_\nu(Z_j) \quad (20)$$

and
\[ \kappa_r(t) = \kappa_r\left(\sum_{j=1}^{n} \theta_j Z_j, t\right) = \sum_{|v|=r} \frac{\kappa_v t^v}{v!}. \]  

Also for the characteristic function of the weighted sum, in a neighborhood of zero, one has

\[ \log \left(\prod_{j=1}^{n} \varphi_j(t)\right) = \sum_{r=1}^{m} \frac{\kappa_r(t)}{r!} t^r + o(|t|^m). \]

Let us define the polynomials \( P_r(t, \kappa_v) \) from the formal expression

\[ 1 + \sum_{r=1}^{\infty} P_r(t, \kappa_v) u^r = \exp \left( \sum_{s=1}^{\infty} \frac{\kappa_{s+2}(t)}{(s+2)!} u^s \right), \]

explicitly, we obtain the expressions

\[
P_0(t, \kappa_v) = 1, \]

\[
P_r(t, \kappa_v) = \sum_{m=1}^{r} \frac{1}{m!} \sum_{i_1, \ldots, i_m, j \geq r} \left[ \sum_{v_1, \ldots, v_m, |v'|=i+j+2} \frac{\kappa_{v_1} \cdots \kappa_{v_m}}{v_1! \cdots v_m!} t^{v_1 + \cdots + v_m}. \right. \]

For a given positive vector \( \alpha \) and a number \( m \), we denote \( D^\alpha \) and \( D_m \) as differential operators

\[ D^\alpha f(t) = \frac{\partial^{\alpha_1 + \alpha_2 + \cdots + \alpha_k} f(t)}{(\partial t_1)^{\alpha_1} (\partial t_2)^{\alpha_2} \cdots (\partial t_k)^{\alpha_k}} \]

and

\[ D_m f(t) = \frac{\partial f(t)}{\partial t_m}. \]

**Lemma 1** Let \( Z_1, Z_2, \ldots, Z_n \) be independent random vectors (nondegenerate at zero) with a finite absolute moment of order \( s \). Then for any \( 2 < r \leq s \)

\[ \left( \frac{\rho_r}{\rho_2^s} \right)^{\frac{1}{r}} \leq \left( \frac{\rho_s}{\rho_2^s} \right)^{\frac{1}{r}}, \]

where \( \rho_s \) is defined in (17). 

**Proof** The proof of Lemma follows the scheme of the proof of Lemma 6.2 [10]. Let us show that \( \log \rho_r \) is a convex function on \([2, s]\), where \( \rho_s = \sum_{j=1}^{n} \rho_s(\theta_j Z_j) = \sum_{j=1}^{n} \mathbb{E}||\theta_j Z_j||^s \) (see [12]). This follows from Holder’s inequality for \( \alpha + \beta = 1 \) and \( r_1, r_2 \in [2, s] \)

\[ \rho_{\alpha r_1 + \beta r_2}(Z_j) \leq \rho_{\alpha r_1}(Z_j) \rho_{\beta r_2}(Z_j), \]

\[ \rho_{\alpha r_1 + \beta r_2} \leq \sum_{j=1}^{n} \rho_{\alpha r_1}(\theta_j Z_j) \rho_{\beta r_2}(\theta_j Z_j) \]
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\[
\left( \sum_{j=1}^{n} \rho_{r_1} (\theta_j Z_j) \right)^\alpha \left( \sum_{j=1}^{n} \rho_{r_2} (\theta_j Z_j) \right)^\beta = \rho_{r_1}^\alpha \rho_{r_2}^\beta.
\]

so

\[
\log(\rho_{\alpha_1+\beta_2}) \leq \alpha \log(\rho_{r_1}) + \beta \log(\rho_{r_2}).
\]

Further, suppose that \(\rho_2 = 1\), then

\[
\log(\rho_{r_2^{-1}}) = \frac{\log(\rho_r) - \log(\rho_2)}{r - 2}
\]

increases due to the fact that it is the slope between the points \((2, \rho_2)\) and \((r, \rho_r)\) of the function \(\rho_r\). In the general case, when \(\rho_2 \neq 1\), it is necessary to consider the random vectors \(\hat{Z}_j = Z_j/\sqrt{\rho_2}, \ j = 1, \ldots, n\).

Lemma 2 Let \(X_1, X_2, \ldots, X_n\) be independent random vectors with zero mean, unit covariance matrix and finite fourth absolute moment. If the following condition holds \(\delta^4_\beta \leq (8k)^{-1}\), then the weighted covariance matrix \(D\) satisfies

\[
\left\langle t, Dt \right\rangle - \|t\|^2 \leq 2k \delta^4_\beta \|t\|^2
\]

and

\[
\|D-I\| \leq \frac{1}{4}, \quad \frac{3}{4} \leq \|D\| \leq \frac{5}{4}, \quad \|D^{-1}\| \leq \frac{4}{3}.
\]

Where \(\delta^4_\beta\) and matrix \(D\) are defined in (2) and (7).

Proof The proof of Lemma follows the scheme of the proof of Corollary 14.2 [10]. First, we prove two auxiliary inequalities for the mathematical expectations of the original and truncated random vectors

\[
\left| \mathbb{E}\theta_j Y_{ji} \right| = \left| \mathbb{E}X_{ji} \theta_j 1_X (\|X_j \theta_j\| > 1) \right|
\]

\[
\leq \mathbb{E}\|X_j \theta_j\| 1_X (\|X_j \theta_j\| > 1) = \theta_j^2 \mathbb{E}\|X_j\|^4 \leq \theta_j^4 \delta_j^4,
\]

and

\[
\theta_j^2 \mathbb{E}X_{ji} X_{j'i} - \mathbb{E}Y_{ji} Y_{j'i} = \theta_j^2 \mathbb{E}X_{ji} X_{j'i} 1_X (\|\theta_j X_{j'i}\| > 1)
\]

\[
\leq \mathbb{E}\|\theta_j X_{j'i}\|^2 1_X (\|\theta_j X_{j'i}\| > 1) \leq \theta_j^4 \mathbb{E}\|X_{j'i}\|^4 = \theta_j^4 \delta_j^4.
\]

Also, note that

\[
\left| \mathbb{E}\theta_j Y_{ji} \right| = \left| \mathbb{E}\theta_j X_{ji} 1_X (\|X_j \theta_j\| \leq 1) \right| \leq 1.
\]

Define Kronecker delta function as \(\delta_{ij} = 1_X (i = j)\), for \(i, j = 1, \ldots, k\). By definition of covariance matrix of the weighted sums \(D = \sum_{j=1}^{n} \theta_j^2 \text{cov}(Z_j)\) (see (7)), one has

\[
\left\langle t, Dt \right\rangle - \left\langle t, t \right\rangle = \sum_{i,l} t_i t_l (d_{il} - \delta_{il}),
\]
wherein
\[ d_{il} - \delta_{il} \leq \sum_{j=1}^{n} \theta_{ji}^2 |\text{cov}(X_{ji}, X_{jl}) - \text{cov}(Y_{ji}, Y_{jl})| \]

\[ \leq \sum_{j=1}^{n} \theta_{ji}^2 \|X_{ji}X_{jl} - \mathbb{E}Y_{ji}Y_{jl} + \mathbb{E}Y_{ji}\mathbb{E}Y_{jl}\| \leq \sum_{j=1}^{n} \left( \theta_{ji}^2 \delta_j^4 + \theta_{ji}^4 \delta_j^4 \right) = 2\delta_{ij}^4, \]

where \( \delta_j^4 = \mathbb{E}\|X_j\|^4, \delta_j^4 = \frac{1}{n} \sum_{j=1}^{n} \delta_j^4 \) (see (4), (5)). Finally we get
\[ \left| \sum_{i,j} t_i t_j (d_{il} - \delta_{il}) \right| \leq 2\delta_{ij}^4 \left( \sum_{i} |t_i|^2 \right)^2 \leq 2k\delta_{ij}^4 \|t\|^2. \]

By the definition of the matrix norm, it follows that
\[ \|D - I\| = \sup_{\|t\| \leq 1} \left| \langle t, (D - I)t \rangle \right| \leq 2k\delta_{ij}^4. \]

Since \( \delta_{ij}^4 < (8k)^{-1} \), then
\[ \|D - I\| \leq \frac{1}{4}, \quad \frac{3}{4} \leq \|D\| \leq \frac{5}{4}. \]

Further,
\[ \langle t, Dt \rangle \geq \|t\|^2 - \frac{1}{4} \|t\|^2 = \frac{3}{4} \|t\|^2. \]

Therefore, for the inverse matrix \( D^{-1} \) one has
\[ \|D^{-1}\| \leq \frac{4}{3}. \]

**Lemma 3** Let \( Z_1, Z_2, \ldots, Z_n \) be independent random vectors with zero mean, non-degenerate covariance matrix \( D \) (see (7)) and finite absolute moment \( \rho_r \) of order \( s \geq 3 \). Then there are constants \( c_1(k, s) \) and \( c_2(k, s) \) such that for any \( |\alpha| \leq s \) and \( \|t\| \leq c_1(k, s) \min \left\{ \eta_s^{-\frac{k}{2}}, \eta_s^{-\frac{k}{2}} \right\} \) one has
\[ \left| D^\alpha \left[ \prod_{j=1}^{n} \varphi_j(\theta_j Qt) - \exp \left( -\frac{1}{2} \|t\|^2 \right) \sum_{r=0}^{s-3} P_r(iQt, \kappa_r) \right] \right| \]
\[ \leq c_2(k, s) \eta_s \left( \|t\|^{s-|\alpha|} + \|t\|^{3(s-2)+|\alpha|} \right) \exp \left( -\frac{1}{4} \|t\|^2 \right), \]
where \( \varphi(t) \) is defined in (13), matrix \( Q \) in (8), \( \eta_s \) in (12) and polynomial \( P_r(t, \kappa_r) \) to (22), (23).

**Proof** The proof follows the scheme of the proof of Theorem 9.11 [10]. First, suppose that the covariance matrix is \( D = I \), then from the definition of weighted
absolute moments \( \rho_s = \sum_{j=1}^{n} \rho_s(\theta_j Z_j) \), \( \eta_s = \sum_{j=1}^{n} \rho_s(Q\theta_j Z_j) \) (see [14](15)), where \( Q^2 = D^{-1} \) (see [16]), it follows that \( \eta_s = \rho_s \). Using Holder’s inequality, we obtain that for the characteristic function of the random vector \( Z_j \) at \( \|t\| \leq \rho_s^{-1} \), the following inequality holds

\[
|\varphi_j(\theta_j t) - 1| \leq \frac{\mathbb{E}(t, \theta_j Z_j)^2}{2} \leq \frac{\|t\|^2 \mathbb{E}||\theta_j Z_j||^2}{2} \leq \frac{\|t\|^2}{2} \left( \mathbb{E}||\theta_j Z_j||^s \right)^{\frac{s}{2}} \leq \frac{\|t\|^2}{2} \rho_s^{\frac{s}{2}} \leq \frac{1}{2},
\]

therefore, the characteristic function of the random vector \( \theta_j Z_j \), defined as \( \varphi_j(t) = \mathbb{E} \exp(itZ_j) \) (see [13]), does not vanish in a given interval, and therefore the following functions can be defined

\[
h_j(t) = \log(\varphi_j(\theta_j t)) - \left( -\theta_j^2||t||^2 + \sum_{r=1}^{s-3} \frac{\kappa_{r+2}(Z_j, it)}{(r+2)!} \right),
\]

\[
h(t) = \sum_{j=1}^{n} h_j(t),
\]

\[
\zeta(t) = \frac{1}{2} ||t||^2 + \sum_{r=1}^{s-3} \frac{\kappa_{r+2}(it)}{(r+2)!},
\]

where \( \kappa_r(t) = \sum_{|v| = r} \frac{\kappa_v}{v!} \) (see [20], [21]). Since the weighted absolute moment of the second order is \( \rho_2 = k \), then by Lemma 1 for \( 2 < r \leq s \),

\[
(\rho_s)^{\frac{1}{s-r}} \leq (\rho_s)^{\frac{1}{r}} (\rho_2)^{\frac{1}{s-2}} = (\rho_s)^{\frac{1}{r}} (k)^{\frac{1}{s-2}} \leq (\rho_s)^{\frac{1}{s-r}} k
\]

and for the cumulants of the distribution \( \kappa_v \), for \( 2 < |v| \leq s \), due to [18], [19]

\[
|\kappa_v|^{\frac{1}{|v|}} \leq (\kappa_1(v) \rho_{|v|})^{\frac{1}{|v|}} \leq c(v) k \rho_s^{\frac{1}{s-v}} \leq \hat{c}_1(s, k) \rho_s^{\frac{1}{s-v}}.
\]

Next, consider the expression

\[
D^a \left[ \prod_{j=1}^{n} \varphi_j(\theta_j t) - \exp(\zeta(t)) \right] = D^a \left[ \exp(h(t) - 1) \exp(\zeta(t)) \right]
\]

\[
= \sum_{0 \leq \alpha \leq a} c(\alpha, \beta) D^\beta \exp(\zeta(t)) D^{a-\beta} (\exp(h(t)) - 1). \tag{24}
\]

Denote

\[
c(s, k) = \sum_{r=0}^{s-3} \sum_{|v|=r+2} \frac{1}{v!}
\]

and notice that for \( \|t\| \leq \left( \hat{c}_1(s, k) 8c(s, k) \rho_s^{\frac{1}{s-v}} \right)^{-1} \), the following chain of inequalities holds...
\[
\left| \sum_{r=0}^{s-3} \sum_{|v|=r+2} \frac{K_v}{v!} t^v \right| \leq \sum_{r=0}^{s-3} \sum_{|v|=r+2} \frac{||r||^r+2 \left( \hat{c}_1(s,k) \rho_s^r \right)^r}{v!} \leq \frac{||r||^2}{\sum_{r=0}^{s-3} \sum_{|v|=r+2} \left( \frac{\hat{c}_1(s,k) \rho_s^r}{v!} \right)^r} \leq \frac{||r||^2}{\sum_{r=0}^{s-3} \sum_{|v|=r+2} \left( \frac{\hat{c}_1(s,k) \rho_s^r}{v!} \right)^r} \leq \frac{||r||^2}{8c(s,k) \sum_{r=0}^{s-3} \sum_{|v|=r+2} \frac{1}{v!}} = \frac{||r||^2}{8}, \quad (25)
\]

Therefore, the module of the function \( \zeta(t) \) is bounded

\[
|\zeta(t)| \leq \frac{||r||^2}{2} + \frac{5||r||^2}{8}.
\]

Similarly, it can be shown that the modulus of the derivative of this function

\[
\left| D^\beta \zeta(t) \right| = \left| D^\beta \sum_{r=0}^{s-3} \sum_{|v|=r+2} \frac{K_v}{v!} t^v \right| = \left| \sum_{r=0}^{s-3} \sum_{|v|=r+2} \frac{K_v}{(v-\beta)!} t^{v-\beta} \right| \leq \sum_{r=0}^{s-3} \sum_{|v|=r+2, v \geq \beta} \frac{\hat{c}_1(s,k) \rho_s^r}{(v-\beta)!} \left| t \right|^{r+2-\beta} \leq c_2(s,k,\beta)||r||^{2-\beta}.
\]

Further, let \( j_1, j_2, \ldots, j_r \) be non-negative numbers \( \beta_1, \beta_2, \ldots, \beta_r \) non-negative vectors satisfying the equality \( \sum_{i=1}^r \beta_i = \beta \), since

\[
||r||^2 \sum_{i=1}^r j_i \leq ||r||^2 + ||r||^{2-|\beta|},
\]

then the derivative \( \zeta(t) \) has the following representation for \( t \neq 0 \)

\[
\left| (D^\beta \zeta(t))^{j_1} \ldots (D^\beta \zeta(t))^{j_r} \right| \leq c_3(s,k) \left( ||r||^2 \right)^{\sum_{i=1}^r j_i (2-|\beta|)} \leq c_3(s,k) \left( \left( ||r||^2 + ||r||^{2-|\beta|} \right)^{\frac{1}{2}} \right),
\]

Lemma 9.2 [10] implies that for \( ||r|| \leq \left( \hat{c}_1(s,k)8c(s,k) \rho_s^r \right)^{-1} \)
\[ D^\beta \exp(\zeta(t)) \leq c_4(s, k)(\|t\|^2 - |\beta|) \exp(\zeta(t)) \]
\[ \leq c_4(s, k)(\|t\|^2 - |\beta|) \exp\left(-\frac{3}{8}\|t\|^2\right). \] (26)

Further, for \( \beta : 0 \leq |\beta| \leq s \), one has
\[ D^\beta (D^\beta h_j)(0) = 0, \quad [0 \leq |\beta| \leq s - |\beta| - 1]. \]

Therefore, applying Corollary 8.3 [10] and the fact that \( g \equiv D^\beta h_j \) we obtain the inequality
\[ D^\beta h_j(t) \leq \sum_{|\beta|\leq s-|\beta|} \frac{|\beta|}{\beta!} \sup \left\{ (D^\beta g)(ut) : 0 \leq u \leq 1 \right\}. \]

If \( |\beta| = s - |\beta| \), then by Lemma 9.4 [10], the following relation holds:
\[ \left| D^\beta g(ut) \right| = \left| D^{\alpha + \beta} h_j(ut) \right| = \left| D^{\alpha + \beta} \log \varphi_j(\theta_j ut) \right| \leq |\theta_j|^x c_2(s) \rho_s(Z_j), \]
so
\[ \left| D^\beta h(t) \right| \leq c_2(s) \rho_s \|t\|^{s-|\beta|} \left( \sum_{|\beta|\leq s-|\beta|} \frac{1}{\beta!} \right) \] (27)

If \( \beta = 0 \), then similarly as in (26) we get that
\[ |h(t)| \leq c_2(s) \left( \sum_{|\beta|\leq s} \frac{1}{\beta!} \rho_s \|t\|^s \right) \leq \frac{\|t\|^2}{8}. \]

If \( \alpha - \beta = 0 \)
\[ \left| D^{\alpha - \beta} (\exp(h(t)) - 1) \right| = \left| \exp(h(t)) - 1 \right| \leq |h(t)| \exp(|h(t)|) \]
\[ \leq c_5(s, k) \rho_s \|t\|^s \exp\left(\frac{\|t\|^2}{8}\right). \] (28)

If \( \alpha > \beta \)
\[ D^{\alpha - \beta} \left( \exp(h(t)) - 1 \right) = D^{\alpha - \beta} \exp(h(t)), \]
then in this case the derivative is represented as a linear combination of the following form
\[ (D^{\beta_1} h(t))^{j_1} \ldots (D^{\beta_c} h(t))^{j_c} \exp(h(t)), \]
where \( \sum_{i=1}^c j_i \beta_i = \alpha - \beta \).

From (27) and the inequality \(|x|^a \leq \|x\|^b + \|x\|^c, \quad 0 \leq b \leq a \leq c\), it follows that
\[ \left| (D^{\beta_1} h(t))^{j_1} \ldots (D^{\beta_c} h(t))^{j_c} \right| \leq c_6(s, k) \rho_s \left( \|t\| \rho_s^{-1} \right)^{(s-2) \sum_{i=1}^c j_i - 1} \|t\|^2 \sum_{i=1}^c j_i \rho_s^{-|\alpha - \beta|}. \]
\[ \leq c_7(s,k)\rho_s(||r||^{|\alpha-\beta|} + ||r||^{|\alpha-\beta|+2}). \]

Therefore, if \( \alpha > \beta \), then

\[ |D^{\alpha-\beta}(\exp(h(t)) - 1)| \leq c_9(s,k)\rho_s(||r||^{|\alpha-\beta|} + ||r||^{|\alpha-\beta|+2}) \exp \left( -\frac{||r||^2}{8} \right). \]

Using (26), (28), (29) in (24), we get

\[ D^\alpha \left[ \prod_{j=1}^{n} \varphi_j(\theta_j t) \right] - \exp \left( -\frac{||r||^2}{2} + \sum_{r=1}^{s-3} \lambda_r + \sum_{r=1}^{s} \lambda_r \right) \]

\[ \leq c_9(s,k)\rho_s(||r||^{|\alpha-\beta|} + ||r||^{|\alpha-\beta|+2}) \exp \left( -\frac{||r||^2}{4} \right). \]

Next, we use Lemma 9.7 \[10\] when setting \( u = 1 \) and the inequality

\[ |\kappa_{rs}|^{\frac{1}{s-2}} \leq \hat{c}_1(s,k)\rho_s^{\frac{s}{s-2}}, \]

for \( 2 < |\nu| \leq s \). Also taking into account that the derivative is represented as a linear combination of terms of the following form

\[ D^\alpha \left[ \prod_{j=1}^{n} \varphi_j(\theta_j t) \right] - \exp \left( -\frac{||r||^2}{2} + \sum_{r=1}^{s-3} \lambda_r + \sum_{r=1}^{s} \lambda_r \right) \]

\[ \leq c_9(s,k)\rho_s(||r||^{|\alpha-\beta|} + ||r||^{|\alpha-\beta|+2}) \exp \left( -\frac{||r||^2}{4} \right). \]

we get that

\[ \left| D^\alpha \left[ \exp \left( -\frac{||r||^2}{2} + \sum_{r=1}^{s-3} \lambda_r (it) + \sum_{r=1}^{s} \lambda_r \right) \right] \right| \]

\[ \leq c(s,k)\rho_s(||r||^{|\alpha|} + ||r||^{3s-2+|\alpha|}) \exp \left( -\frac{1}{4}||r||^2 \right), \]

which completes the proof of the Lemma for the case when the covariance matrix \( D = I \). In order to prove in the general case \( D \neq I \), it is necessary to consider the transformed sequence of random vectors \( QZ_1, QZ_2, \ldots, QZ_n \), then \( \prod_{j=1}^{n} \varphi_j(\theta_j Qt) \) will be the characteristic function of the sum of random vectors \( Q(\sum_{j=1}^{n} Z_j) \), and in this case, the weighted sum of random vectors has a unit covariance matrix, and the Lemma holds for these random vectors.

**Lemma 4** Let \( X_1, X_2, \ldots, X_n \) be independent random vectors with zero mean, unit covariance matrix and finite fourth absolute moment. Let \( l > 0 \) be such that for subset \( U = \{ j : |\theta_j| \leq l/\delta_j \} \), one has \( \sum_{j \in U} \theta_j^2 \geq 1/8 \). If \( \delta_j^4 \leq (8k)^{-1} \), then for
A multivariate CLT for weighted sums

\[ \left\| t \right\| \leq (8\sqrt{\kappa}l)^{-1}, \text{it holds} \]

\[ \left| D^n \prod_{j=1}^n \varphi_j(\theta_j t) \right| \leq c_1(\alpha, k)(1 + \| t \|^{|\alpha|}) \exp \left( -\frac{1}{48}\| t \|^2 \right). \]

Where \( \delta_{\theta}^1 \) is defined in (5) and \( \varphi_j(t) \) in (13).

**Proof** The proof of Lemma follows the scheme of proofs of Lemma 2.2 [2] and Lemma 14.3 [10]. For the random vector \( Z_i = Y_j - \mathbb{E}Y_j \), where \( Y_j = X_j^1_1 \chi_1(||\theta_jX_j|| \leq 1) \), consider the expansion of the characteristic function of a Taylor series as \( t \) tends to zero,

\[ \varphi_j(\theta_j t) = 1 - \frac{1}{2}\theta_j^2\mathbb{E}(Z_j, t)^2 + \frac{1}{6}\theta_j^3\mathbb{E}(Z_j, t)^3\xi, \]

where \( |\xi| \leq 1 \). For \( \| t \| \leq (8\sqrt{\kappa}l)^{-1} \) and \( j \in \mathbb{U} \) one has

\[ \theta_j^2\mathbb{E}(Z_j, t)^2 \leq \theta_j^2\| t \|^2\mathbb{E}\| Z_j \|^2 \leq \frac{\mathbb{E}\| Z_j \|^2}{64k\delta_j^4} \leq \frac{4\mathbb{E}\| X_j \|^2}{64k\delta_j^4} = \frac{k}{16k\delta_j^4} < 1. \]

Also

\[ |\theta_j|^3\mathbb{E}(Z_j, t)^3 \leq |\theta_j^3|\| t \|^3\mathbb{E}\| Z_j \|^3 \leq |\theta_j|^2\| t \|^2 \frac{2^3\mathbb{E}\| X_j \|^3}{8\sqrt{\kappa}\delta_j^2} \]

\[ \leq |\theta_j|^2\| t \|^2 \frac{2^3\sqrt{k}\delta_j^2}{8\sqrt{\kappa}\delta_j^2} \leq |\theta_j|^2\| t \|^2, \]

hence

\[ 1 - \frac{1}{2}\theta_j^2\mathbb{E}(Z_j, t)^2 > 0. \]

And

\[ |\varphi_j(\theta_j t)| \leq 1 - \frac{1}{2}\theta_j^2\mathbb{E}(Z_j, t)^2 + \frac{1}{6}|\theta_j|^3\mathbb{E}(Z_j, t)^3 \]

\[ \leq \exp \left( -\frac{1}{2}\theta_j^2\mathbb{E}(Z_j, t)^2 + \frac{1}{6}|\theta_j|^3\mathbb{E}(Z_j, t)^3 \right) \]

\[ \leq \exp \left( -\frac{1}{2}\theta_j^2\mathbb{E}(Z_j, t)^2 + \frac{1}{6}|\theta_j|^3\| t \|^2 \right). \]

Further, note that

\[ \exp \left( \frac{1}{2}|\theta_j|^3\mathbb{E}(Z_j, t)^3 \right) \leq \exp \left( \frac{1}{2}(\mathbb{E}|\theta_j(\mathbb{E}Z_j, t)^3 \right)^{\frac{1}{4}} \]

\[ \leq \exp \left( \frac{1}{2}(\mathbb{E}|\theta_j(\mathbb{E}Z_j, t)^3 \right)^{\frac{1}{3}} - \frac{1}{6}|\theta_j|^3\mathbb{E}(Z_j, t)^3 \right) \leq \exp \left( \frac{2}{3} \right). \]

Now we denote the subset \( N_r = \{ j_1, j_2, \ldots, j_r \} \) as a subset of \( N = \{ 1, 2, \ldots, n \} \), consisting of \( r \) elements, for \( \| t \| \leq (8\sqrt{\kappa}l)^{-1} \) the following chain of inequalities
holds

\[ | \prod_{j \in \mathcal{N} \setminus \mathcal{N}_r} \varphi_j(\theta_j t) | \leq | \prod_{j \in \mathcal{U} \setminus \mathcal{N}_r} \varphi_j(\theta_j t) | \leq \exp \left( \sum_{j \in \mathcal{U} \setminus \mathcal{N}_r} \left[ - \frac{1}{2} \theta_j^2 \mathbb{E}(Z_j, t)^2 + \frac{1}{6} \theta_j^2 ||t||^2 \right] \right) \]

\[ \exp \left( \sum_{j \in \mathcal{U} \setminus \mathcal{N}_r} \left[ - \frac{1}{2} \theta_j^2 \mathbb{E}(Z_j, t)^2 + \frac{1}{6} \theta_j^2 ||t||^2 \right] - \sum_{j \in \mathcal{U} \setminus \mathcal{N}_r} \left[ - \frac{1}{2} \theta_j^2 \mathbb{E}(Z_j, t)^2 + \frac{1}{6} \theta_j^2 ||t||^2 \right] \right) \]

\[ \leq \exp \left( \sum_{j \in \mathcal{U} \setminus \mathcal{N}_r} \left[ - \frac{1}{2} \theta_j^2 \mathbb{E}(Z_j, t)^2 + \frac{1}{6} \theta_j^2 ||t||^2 \right] \right) \exp \left( \frac{2r}{3} \right) \]

\[ \leq \exp \left( - \frac{1}{2} \sum_{j=1}^{n} \theta_j^2 ||t||^2 + \frac{1}{2} \sum_{j \in \mathcal{U} \setminus \mathcal{N}} \theta_j^2 ||t||^2 + \frac{1}{6} ||t||^2 \right) \exp \left( \frac{2r}{3} \right) \]

\[ = \exp \left( - \frac{1}{2} \mathbb{E}(D_t, t) + \frac{1}{2} \mathbb{E}(D_t, t) - ||t||^2 \sum_{j \in \mathcal{N} \setminus \mathcal{U}} \theta_j^2 \right) \exp \left( \frac{2r}{3} \right) \]

\[ \leq 2k \sum_{j \in \mathcal{U} \setminus \mathcal{N}} \theta_j^2 \exp \left( \frac{2r}{3} \right) \]

\[ \leq 2 \sum_{j \in \mathcal{U} \setminus \mathcal{N}} \theta_j^2 \exp \left( \frac{2r}{3} \right) \]

Note that the norm of the covariance matrix satisfies the estimate \( \|D\| > 3/4 \), also, by definition, the inequality for the sum of the weight coefficients is satisfied \( \sum_{j \in \mathcal{U} \setminus \mathcal{U}} \theta_j^2 \leq 1/8 \). If we denote the matrix \( D_2 \) as the weighted sum of the covariance matrices of the random vectors \( X_j, j \in \mathcal{N} \setminus \mathcal{U} \), then by Lemma 2, we obtain

\[ \left| \prod_{j \in \mathcal{N} \setminus \mathcal{N}_r} \varphi_j(\theta_j t) \right| \leq \exp \left( \left( - \frac{3}{8} \right) \sum_{j \in \mathcal{N} \setminus \mathcal{U}} \theta_j^2 \right) \exp \left( \frac{2r}{3} \right) \]

so

\[ \left| \prod_{j \in \mathcal{N} \setminus \mathcal{N}_r} \varphi_j(\theta_j t) \right| \leq \exp \left( \left( - \frac{3}{8} \right) \sum_{j \in \mathcal{N} \setminus \mathcal{U}} \theta_j^2 \right) \exp \left( \frac{2r}{3} \right) \]

For \( \alpha = 0 \) the statement of the Lemma is proved.

Before proceeding to the proof of the case \( \alpha \neq 0 \), consider the modulus of the derivative of the characteristic function of the random vector \( Z_j \)

\[ |D_m \varphi_j(\theta_j t)| = |\theta_j| \mathbb{E}(Z_j, t) \exp \left( i \theta_j t, Z_j \right) |. \]

(30)

If a positive vector \( \beta \) satisfies the condition \( \|\beta\| = 1 \) then

\[ |D_{\beta} \varphi_j(\theta_j t)| = |\theta_j| \mathbb{E}(\mathbb{E}(Z_j, \beta) \exp \left( i \theta_j t, Z_j \right) - 1) | \]

\[ \leq |\theta_j| \mathbb{E}(Z_j, \beta) \exp \left( i \theta_j t, Z_j \right) | \leq \theta_j^2 \mathbb{E}(Z_j, t) \leq \theta_j^2 \mathbb{E}(Z_j, ||t||). \]
from (30) we also get that for any vector $\beta$ with $|\beta| \geq 2$

$$\left| D^\beta \varphi_j (\theta_j t) \right| \leq |\theta_j|^{|\beta|} \mathbb{E}|Z_j^\beta| \leq 2^{|\beta|} \theta_j^2 \rho_2 (X_j),$$

finally we get that for any non-negative vector $\beta > 0$

$$\left| D^\beta \varphi_j (\theta_j t) \right| \leq c_2 (\alpha, k) \theta_j^2 \rho_2 (X_j) \max \{1, ||t||\}.$$

Now consider the positive vector $\alpha > 0$, according to the rule of differentiation of the product of functions, we obtain that

$$D^\alpha \prod_{j=1}^n \varphi_j (\theta_j t) = \sum_{j \in N \setminus N_r} \prod_{j \in N} \varphi_j (\theta_j t) D^\beta \varphi_j (\theta_j t) \ldots D^\beta \varphi_j (\theta_j t),$$

where $N_r = \{ j_1, \ldots, j_r \}$, $1 \leq r \leq |\alpha|$, $\beta_1, \beta_2, \beta_3, \ldots, \beta_r$ vectors that meet the conditions $|\beta_j| \geq 1$ ($1 \leq j \leq r$) and $\sum_{j=1}^r \beta_j = \alpha$. The number of multiplications in each of the $n^\alpha$ terms of the expression (31) is

$$\frac{\alpha_1! \ldots \alpha_k!}{\prod_{j=1}^r \prod_{i=1}^k \beta_{ji}!},$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta_j = (\beta_{j_1}, \ldots, \beta_{j_k})$, $1 \leq j \leq r$. Each term in the expression (31) is bounded by the value

$$\exp \left( \frac{\alpha^2}{3} - \frac{1}{48} ||t|| \right) \prod_{j \in N_r} b_j,$$

where $b_j = c_2 (\alpha, k) \rho_2 (X_j) \theta_j^2 \max \{1, ||t||\}$, therefore from (31) we obtain

$$\left| D^\alpha \prod_{j=1}^n \varphi_j (\theta_j t) \right| \leq \sum_{1 \leq r \leq |\alpha|} c_2 (\alpha, r) \exp \left( \frac{\alpha^2}{3} - \frac{1}{48} ||t|| \right) \sum_{j \in N_r} b_j,$$

where the outer summation is over all $r$ elements from $N$. It remains to evaluate the expression

$$\sum_{r} \prod_{j \in N_r} b_j \leq \left( \sum_{j=1}^n b_j \right)^r = (c_2 (\alpha, k) \rho_2 \max \{1, ||t||\})^r = (c_2 (\alpha, k) k)^r (1 + ||t||^r),$$

which completes the proof. \(\square\)

**Lemma 5** Let $X_1, X_2, \ldots, X_n$ be independent random vectors with zero mean, unit covariance matrix and finite fourth absolute moment. Let $(\Theta_1, \Theta_2, \ldots, \Theta_n)$ be a random vector uniformly distributed on the unit sphere $S^{n-1}$. Then with probability greater than $1 - C_2 (\alpha, k) \exp \left( -c_2 (\alpha, k) \frac{1}{n^2} \right)$ one has
\[
\int_{\frac{\mu_k}{\delta^k} \leq |r| \leq \frac{\mu_k}{\delta^k}} |D^\alpha \prod_{j=1}^{n} \varphi_j(\Theta_j t)| dt \leq C_3(\alpha, c, k) \frac{\delta^4}{n}.
\]

Where \( \varphi_j(t) \) is defined in (13) and \( \delta^4 \) in (5).

**Proof** The proof of Lemma follows the scheme of the proof of Lemma 3.5 [2]. Let us denote \( g_j(t) \) as the characteristic function of the random vector \( X_j \). First, using the Chebyshev inequality, we obtain the estimate

\[
\mathbb{E} 1_X(||\theta_j X_j|| \geq 1) = P(||\theta_j X_j|| \geq 1) \leq \mathbb{E} ||X_j||^2 \theta_j^2 = k\theta_j^2.
\]

then, applying a number of transformations, we obtain the following chain of inequalities for estimating the characteristic function

\[
\mathbb{E} \exp \left( i \Theta_j \langle t, X_j \rangle \right) \left| \Theta_j \right| = \left| \mathbb{E} \exp \left( i \Theta_j \langle t, X_j \rangle 1_X(\|\Theta_j X_j\| \leq 1) \right) \right| \left| \Theta_j \right|
\]

\[
= \mathbb{E} \left[ \exp \left( i \Theta_j \langle t, X_j \rangle 1_X(\|\Theta_j X_j\| \leq 1) \right) \left( 1_X(\|\Theta_j X_j\| \leq 1) + 1_X(\|\Theta_j X_j\| > 1) \right) \right] \left| \Theta_j \right|
\]

\[
= \mathbb{E} \left[ \exp \left( i \Theta_j \langle t, X_j \rangle \right) \right] \left[ \mathbb{E} 1_X(\|\Theta_j X_j\| > 1) \left( 1 - \exp \left( i \Theta_j \langle t, X_j \rangle \right) \right) \right] \left| \Theta_j \right|
\]

\[
\leq |g_j(\Theta_j t)| + \mathbb{E} 1_X(\|\Theta_j X_j\| > 1) \left( 1 - \exp \left( i \Theta_j \langle t, X_j \rangle \right) \right) \left| \Theta_j \right|
\]

\[
\leq |g_j(\Theta_j t)| + 2\mathbb{E} 1_X(\|\Theta_j X_j\| > 1) \left| \Theta_j \right|
\]

\[
\leq |g_j(\Theta_j t)| + 2k\theta_j^2.
\]

Next, we will show that for any \( r \) one has

\[
\mathbb{E} |g_j(\Theta_j r)|^2 \leq 1 - c_1 \min \left\{ \frac{\|r\|^2}{n}, \frac{1}{\delta^4} \right\}.
\]

Let us denote \( X_j' \) as an independent copy of the random vector \( X_j \), and define a random vector \( X = X_j - X_j' \) with the corresponding distribution \( \hat{F}_X \). Also denote \( f_n \) and \( J_n \) as the distribution density and characteristic function of the component of a random vector uniformly distributed on the unit sphere, \( \Theta_j \). Further, changing the order of integration, we obtain
\[ \mathbb{E}[g_j(\Theta_j \mathbf{r})^2] = \int |g_j(rt)|^2 f_n(t) dt = \int \left[ \int \exp \left( \langle it(t,y) \rangle \right) d\hat{F}_x \right] f_n(t) dt \]

\[ = \int \left[ \int f_n(t) \exp \left( \langle it(t,y) \rangle \right) dt \right] d\hat{F}_x = \int J_n(\langle r, x \rangle) d\hat{F}_x = \mathbb{E}J_n(\langle r, \hat{X} \rangle) \]

Lemma 3.3 \cite{2} implies that the estimate holds for the characteristic function of a random variable uniformly distributed on the unit sphere.

\[ \mathbb{E}J_n \left( \frac{\langle r, \hat{X} \rangle}{\|r\|} \right) \leq 1 - c_3 \mathbb{E} \min \left\{ \frac{\|r\|^2}{n} \left( \frac{\langle r, \hat{X} \rangle}{\|r\|} \right)^2, 1 \right\}. \]

Let us define the random variable \( X'' \) as \( X'' = \frac{\langle r, \hat{X} \rangle^2}{2\|r\|^2} \) and \( \tau = \frac{\|r\|^2}{n} \), then

\[ \mathbb{E}X'' = \sum_{i=1}^{k} r_i ^2 \hat{X}_i^2 + 2 \sum_{1 \leq j < i \leq k} r_j r_i \hat{X}_j \hat{X}_i = \frac{2\|r\|^2}{2\|r\|^2} = 1, \]

\[ \mathbb{E}(X'')^2 = \frac{\langle r, \hat{X} \rangle^4}{4\|r\|^4} \leq \mathbb{E} \frac{\|r\|^4}{4\|r\|^4} \|\hat{X}\|^4 \leq \frac{2\delta_j^4 + 6\delta_j^2}{4} \leq 2\delta_j^4. \]

Let us show that \( \mathbb{E} \min\{\tau X'', 1\} \geq c_4 \mathbb{E} \min\{\tau, \delta_j^{-4}\} \). Since the right-hand side increases with \( \tau \), it suffices to prove the inequality for \( \tau < (10\delta_j^4)^{-1} \). From Lemma 3.1 \cite{2} it follows \( \mathbb{E}1_X(X'' \leq 10\delta_j^4)X'' \geq \frac{4}{5} \), so for \( 0 < \tau \leq (10\delta_j^4)^{-1} \) one has

\[ \mathbb{E} \min\{\tau X'', 1\} \geq \mathbb{E}1_X(X'' \leq 10\delta_j^4) \min\{\tau X'', 1\} = \tau \mathbb{E}1_X(X'' \leq 10\delta_j^4)X'' > \frac{\tau}{2}, \]

hence

\[ \mathbb{E}[g_j(\Theta_j \mathbf{r})]^2 \leq 1 - c_1 \min \left\{ \frac{\|r\|^2}{n}, \frac{1}{\delta_j^4} \right\}. \]

Similarly, consider \( Z_j - Z_j' \), where \( Z_j' \) is an independent copy of \( Z_j \), also note that \( \mathbb{E}\Theta_j = n^{-1} \) and \( \mathbb{E}\Theta_j^4 \leq \hat{C}/n^2 \). And, using the inequality \( \|x\|^2 \leq n^{1/2} \|x\| \), we obtain the estimate

\[ \mathbb{E}[\varphi_j(\Theta_j \mathbf{t})]^2 = \mathbb{E} \left[ \mathbb{E} \exp \left( \langle i\Theta_j(t, Z_j - Z_j') \rangle \right) \right] = \mathbb{E} \mathbb{E} \exp \left( \langle i\Theta_j(t, Y_j - Y_j') \rangle \right) \]

\[ \leq \mathbb{E} \left[ |g_j(\Theta_j t)| + 2k\Theta_j^2 \right] \left( |g_j(-\Theta_j t)| + 2k\Theta_j^2 \right) \leq \mathbb{E}[g_j(\Theta_j \mathbf{t})]^2 + 2k \frac{1}{n} + 4k^2 \frac{\hat{C}}{n^2} \]

\[ \leq 1 - c_1 \min \left\{ \frac{\|r\|^2}{n}, \frac{1}{\delta_j^4} \right\} + 2k \frac{1}{n} + 4k^2 \frac{\hat{C}}{n^2}. \]

Since \( \delta_j^4 \geq k^2 \) and in the region \( \|r\|^2 \geq n/\delta_j^4 \), then the estimate holds.
Finally, we come to the conclusion.

In Lemma 4 it was shown that
\[
|D^\beta \varphi_j(\theta_j|t)| \leq c_2(\alpha, k)\delta_j^2 \rho_2(X_j) \max\{1, ||t||\},
\]
from this it immediately follows
\[
\left( \mathbb{E} |D^\beta \varphi_j(\Theta_j|t)|^2 \right)^{\frac{1}{2}} \leq c_2(\alpha, k)\rho_2(X_j) \max\{1, ||t||\} \left( \mathbb{E} \Theta_j^2 \right)^{\frac{1}{2}} = c_2(\alpha, k)\rho_2(X_j) \max\{1, ||t||\} \frac{\sqrt{\mathcal{C}}}{n}.
\]

Further, using Theorem 1 [14], we obtain
\[
\mathbb{E} \left[ \prod_{j \in \mathcal{N}_r} \varphi_j(\Theta_j|t) \right] D^\beta_1 \varphi_{j_1}(\Theta_{j_1}) \cdots D^\beta_{\nu} \varphi_{j_\nu}(\Theta_{j_\nu}) \leq \left( \prod_{j \in \mathcal{N}_r} \left( \mathbb{E} |\varphi_j(\Theta_j|t)|^2 \right)^{\frac{1}{2}} \right) \left( \mathbb{E} |D^\beta_1 \varphi_{j_1}(\Theta_{j_1})|^2 \right)^{\frac{1}{2}} \cdots \left( \mathbb{E} |D^\beta_{\nu} \varphi_{j_\nu}(\Theta_{j_\nu})|^2 \right)^{\frac{1}{2}}.
\]

Let us denote the subset of indices \( \mathcal{G} = \{ j : \delta_j < 2\delta \} \) and we come to the conclusion that
\[
\delta^4 = \frac{1}{n} \sum_{j=1}^{n} \delta_j^4 \geq \frac{1}{n} \sum_{j \notin \mathcal{G}} \delta_j^4 \geq \frac{n - |\mathcal{G}|}{n} 16\delta^4,
\]
therefore, \( |\mathcal{G}| \geq n/2 \). Due to this, the following chain of inequalities is valid
\[
\mathbb{E} \prod_{j=1}^{n} \varphi_j(\Theta_j|t) \leq \prod_{j=1}^{n} \left( 1 - c_1 \min \left\{ \frac{||t||^2}{n}, \frac{1}{\delta_j^4} \right\} + 2k \frac{1}{n} + 4k\mathcal{C} \frac{1}{n^2} \right)^{\frac{1}{2}} \leq \prod_{j \notin \mathcal{G}} \left( 1 - c_1 \min \left\{ \frac{||t||^2}{n}, \frac{1}{\delta_j^4} \right\} + 2k \frac{1}{n} + 4k^2\mathcal{C} \frac{1}{n^2} \right)^{\frac{1}{2}} \leq \left( 1 - \mathcal{C}_3 \min \left\{ \frac{||t||^2}{n}, \frac{1}{\delta_j^4} \right\} + 2k \frac{1}{n} + 4k^2\mathcal{C} \frac{1}{n^2} \right)^{\frac{1}{2}} \leq \exp \left( \frac{k}{2} + \mathcal{C}k^2 \right) \exp \left( - \mathcal{C}_6 \min \left\{ ||t||^2, \frac{n}{\delta^4} \right\} \right).
\]

Finally, we come to the conclusion
\[
\prod_{j \in \mathcal{N}_r} \left( \mathbb{E} |\varphi_j(\Theta_j|t)|^2 \right)^{\frac{1}{2}} \leq \exp \left( \frac{k}{2} + \mathcal{C}k^2 \right) \left( 1 - \frac{\mathcal{C}}{k^2} \right)^{\frac{1}{r}} \exp \left( - \mathcal{C}_6 \min \left\{ ||t||^2, \frac{n}{\delta^4} \right\} \right),
\]
and similarly, as in the proof of Lemma 4, we obtain the estimate
\[
\mathbb{E} |D^n \prod_{j=1}^{n} \varphi_j(\theta_j t)| \leq c_3(\alpha, k)(1 + \|t\|^{\alpha}) \exp \left( - \overline{\tau}_6 \min \left\{ \|t\|^2, \frac{n}{\delta^4} \right\} \right)
\]
and therefore
\[
\int_{\frac{\|k\|}{\delta^2} \leq \|t\| \leq \frac{\alpha}{\delta^4}} \mathbb{E} |D^n \prod_{j=1}^{n} \varphi_j(\Theta_j t)| dt 
\leq \int_{\frac{\|k\|}{\delta^2} \leq \|t\| \leq \frac{\alpha}{\delta^4}} c_3(\alpha, k)(1 + \|t\|^{\alpha}) \exp \left( - \overline{\tau}_7 \min \left\{ \|t\|^2, \frac{n}{\delta^4} \right\} \right) dt 
\leq \int_{\frac{\|k\|}{\delta^2} \leq \|t\| \leq \frac{\alpha}{\delta^4}} c_3(\alpha, k)(1 + \|t\|^{\alpha}) \exp \left( - \overline{\tau}_8 \frac{n}{2\delta^4} \right) dt 
\leq \overline{C}_7(\alpha, c, k) \exp \left( - \overline{\tau}_9(k) \frac{n}{\delta^4} \right).
\]
Using the Chebyshev inequality,
\[
P \left( \int_{\frac{\|k\|}{\delta^2} \leq \|t\| \leq \frac{\alpha}{\delta^4}} |D^n \prod_{j=1}^{n} \varphi_j(\Theta_j t)| dt \geq \sqrt{\overline{C}_7(\alpha, k) \exp \left( - \overline{\tau}_9(k) \frac{n}{\delta^4} \right)} \right) 
\leq \sqrt{\overline{C}_7(\alpha, k) \exp \left( - \overline{\tau}_9(k) \frac{n}{\delta^4} \right)}.
\]
We see that there is a subset \( Q \subseteq \delta^{n-1} \) on the unit sphere with probability
\[
P(\Theta \in Q) \geq 1 - \overline{C}_2(\alpha, k) \exp \left( - \overline{\tau}_2(k) \frac{n}{\delta^4} \right),
\]
such that for any vector of weight coefficients \( (\theta_1, \theta_2, \ldots, \theta_n) \in Q \) the inequality
\[
\int_{\frac{\|k\|}{\delta^2} \leq \|t\| \leq \frac{\alpha}{\delta^4}} |D^n \prod_{j=1}^{n} \varphi_j(\theta_j t)| dt \leq \sqrt{\overline{C}_7(\alpha, k) \exp \left( - \overline{\tau}_9(k) \frac{n}{\delta^4} \right)} \leq C_3(\alpha, k) \frac{\delta^4}{n}.
\]

**Lemma 6** Let \( Z_1, Z_2, \ldots, Z_n \) be independent random vectors with zero mean and finite absolute moment of order \( r + 2 \), \( r \geq 1 \). If the weighted covariance matrix \( D \) satisfies \( \|D\| > 3/4 \), then for any positive vector \( \alpha \) such that \( |\alpha| \leq 3r \) the following inequalities hold
\[
\int |D^n P_1(it, \kappa) \exp \left( - \frac{1}{2} \langle Dt, t \rangle \right)| dt \leq C_6(\alpha, k) \sum_{|\nu|=3} |\kappa_\nu|,
\]
Lemma 9.3 [10] implies that

for \( r \geq 2 \), also

where \( D \) is defined in (7), \( \rho_{r+2} \) in (14) and the polynomial \( P_r(t, \kappa_\nu) \) in (22), (23).

**Proof** The proof of Lemma follows the scheme of the proof of Lemma 9.5 [10]. Lemma 9.5 [10] implies that

now we show that for any positive vector \( \alpha \) such that \( 0 \leq |\alpha| \leq 3r \) one has

Differentiating the polynomial of the asymptotic expansion, we obtain the following representation

\[
D^\alpha P_r(z, \kappa_\nu) = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{i_1, \ldots, i_m; \sum_{j=1}^m \nu_j = r} \left[ \frac{\kappa_{\nu_{i_1}} \cdots \kappa_{\nu_{i_m}}}{\nu_{i_1}! \cdots \nu_{i_m}!} \right] \\
\times \frac{(\nu_1 + \cdots + \nu_m)!}{((\nu_1 + \cdots + \nu_m - \alpha)!)} z^{\nu_1 + \cdots + \nu_m - \alpha},
\]

where \( \kappa_\nu = \sum_{j=1}^{\infty} \theta_j |\nu_j| \kappa_j(Z_j) \). Estimating the cumulants in this expression through the corresponding absolute moments (see (19)), we come to the conclusion

\[
|\kappa_{\nu_{i_1}} \cdots \kappa_{\nu_{i_m}}| \leq c_1(\nu_1) \rho_{1+i_2+\cdots+i_m+2} \cdots c_1(\nu_m) \rho_{i_m+2} \\
\leq c_1(\nu_1) \cdots c_1(\nu_m) \rho_2^{\nu_1+\cdots+i_m+2} \rho_{1+i_2+\cdots+i_m+2} \cdots \rho_{i_m+2}^{\nu_m+2} \\
\leq c_1(\nu_1) \cdots c_1(\nu_m) \rho_2^{\nu_1+\cdots+i_m+2} \rho_{r+2}^{m-1} = c_1(\nu_1) \cdots c_1(\nu_m) \rho_{r+2}^{m-1},
\]

using the well-known inequality \( ||t||^a \leq ||t||^b + ||t||^c \) for \( 0 \leq b \leq a \leq c \), we come to the fact that

\[
|D^\alpha P_r(z, \kappa_\nu)| \leq C_3(\alpha, r)(1 + \rho_2^{r-1})(1 + \|z\|^{3r-|\alpha|})\rho_{r+2}.
\]
Further, since 
\[ D^\alpha \exp \left( -\frac{1}{2} \langle Dt, t \rangle \right) P_r(it, \kappa_\nu) = \sum_{0 \leq \beta \leq \alpha} D^{\alpha-\beta} \exp \left( -\frac{1}{2} \langle Dt, t \rangle \right) D^\beta P_r(it, \kappa_\nu), \]
we obtain that for any \( r \geq 1 \)
\[ \int |D^\alpha P_r(it, \kappa_\nu) \exp \left( -\frac{1}{2} \langle Dt, t \rangle \right)| dt \leq C(\alpha, k, r)(1 + \rho_{2r}^{-1}) \rho_{2r+2}. \]
Separately, it is necessary to consider the case when \( r = 1 \)
\[ \int \left| \sum_{0 \leq \beta \leq \alpha} D^{\alpha-\beta} P_1(it, \kappa_\nu) D^\beta \exp \left( -\frac{1}{2} \langle Dt, t \rangle \right) \right| dt \]
\[ \leq \sum_{|\nu|=3} |\kappa_\nu| \left( \int \left| \sum_{0 \leq \beta \leq \alpha} D^{\alpha-\beta \frac{\nu(t)^{\nu}}{\nu!}} C(\beta, k)(1 + \|t\|^{\|\beta\|}) \exp \left( -\frac{3}{8} \|t\|^2 \right) \right| dt \]
\[ \leq \sum_{|\nu|=3} C_6(\alpha, \nu, k) |\kappa_\nu| \leq C_6(\alpha, k) \sum_{|\nu|=3} |\kappa_\nu|. \]

**Lemma 7** Let \( \Theta \) be a random vector uniformly distributed on the unit sphere \( S^{\alpha-1} \), \( \nu \) is a positive vector with \( |\nu| = 3 \), then for any \( t > 0 \),
\[ P \left( \left| \sum_{j=1}^n \Theta_j^j \mu_\nu(Z_j) \right| \geq \frac{\delta^3}{n} \right) \leq \hat{C}_3 \exp \left( -\hat{c}_3 t^{\frac{2}{3}} \right) \]
and
\[ P \left( \sum_{j=1}^n \delta^4_j \Theta^4_j \geq \frac{t \delta^4}{n} \right) \leq \hat{C}_4 \exp \left( -\hat{c}_4 \sqrt{t} \right). \]
Where \( \delta^3_j, \delta^4, \mu_\nu(Z_j) \) are defined in (3), (5), (7), respectively.

**Proof** The proof follows the scheme of the proof of Lemma 4.1 [2]. Let \( \Gamma_1, \Gamma_2, \ldots, \Gamma_n \) be a sequence of independent random variables with standard normal distribution and \( Z \) as a random variable independent of \( \Gamma_j \) for \( j = 1, \ldots, n \), which has a chi-square with \( n \) degrees of freedom. Using the properties of the normal distribution, one can show that the representation \( \Theta_j = \Gamma_j/\sqrt{Z} \) is holds for \( j = 1, \ldots, n \).
Applying a number of transformations, we obtain
\[ P\left( \left| \sum_{j=1}^{n} \mu_{\nu}(Z_{j}) \Omega_{j}^{3} \right| \geq \frac{\delta^{4}}{n} \right) = P\left( \left| \sum_{j=1}^{n} \frac{\mu_{\nu}(Z_{j}) \Gamma_{j}^{3}}{Z_{j}^{2}} \right| \geq \frac{t \sqrt{n} \delta^{2}}{4} \right) \]
\[ \leq P\left( \left| \sum_{j=1}^{n} \mu_{\nu}(Z_{j}) \Gamma_{j}^{3} \right| \geq \frac{t \sqrt{n} \delta^{2}}{4} \right) + C_{1} \exp\left( -c_{1}n \right). \]

Due to the fact that \( \mathbb{E} \exp(c_{2} \Gamma_{j}^{2}) \leq 2 \), where \( c_{2} \) is an absolute constant, therefore the random variable \( Y = \sum_{j=1}^{n} \mu_{\nu}(Z_{j}) \Gamma_{j}^{3} \) belongs to class \( \psi_{2/3} \) see Sect. 2 and 3 in [12]. And, therefore, we can apply the inequality for the moments Sect 3 in [12], which asserts that for any \( p \geq 2 \) the following estimate holds
\[ \left( \mathbb{E}|Y|^{p} \right)^{\frac{1}{p}} \leq \sqrt{2} \sqrt{p} \left( \sum_{j=1}^{n} \mu_{\nu}(Z_{j})^{2} \right)^{\frac{1}{2}}. \]

Using the inequality for \( \nu = \nu_{1} + \nu_{2} \) with \( |\nu_{1}| = 1, |\nu_{2}| = 2 \)
\[ (\mu_{\nu}(Z_{j}))^{2} = (\mathbb{E}Z_{j}^{\nu_{1}} Z_{j}^{\nu_{2}}) \leq (\mathbb{E}Z_{j}^{2\nu_{1}})(\mathbb{E}Z_{j}^{2\nu_{2}}) \leq 2^{2} \mathbb{E}X_{j}^{2\nu_{1}} 2^{4} \rho_{4}(X_{j}) = 2^{6} \delta_{j}^{4}, \]
we get that
\[ C_{2} \sqrt{p} \left( \sum_{j=1}^{n} (\mu_{\nu}(Z_{j}))^{2} \right)^{\frac{1}{2}} \leq C_{3} \sqrt{p} \left( \sum_{j=1}^{n} \delta_{j}^{4} \right)^{\frac{1}{2}} = C_{3} \sqrt{p} \sqrt{n} \delta^{2}. \]

Applying the Chebyshev inequality, we come to the conclusion that
\[ P\left( \left| \sum_{j=1}^{n} \mu_{\nu}(X_{j}) \Omega_{j}^{3} \right| \geq t \sqrt{n} \delta^{2} \right) \leq \frac{\mathbb{E}|Y|^{p}}{(t \sqrt{n} \delta^{2})^{p}} \leq \left( \frac{C_{3} \sqrt{p}}{t} \right)^{p}. \]

Put \( p = \left( \frac{t}{C_{3} \sqrt{p}} \right)^{\frac{1}{2}} \), if we consider \( t \geq 2 \sqrt{C_{3}} \), then we get that \( p \geq 2 \). Note that
\[ \left( \frac{C_{3} \sqrt{p}}{t} \right)^{p} = \exp\left( - \left( \frac{t}{C_{3} \sqrt{p}} \right)^{\frac{1}{2}} \log(2) \right), \]
therefore we get that for any \( t \geq C_{3} \)
\[ P\left( \left| \sum_{j=1}^{n} \mu_{\nu}(X_{j}) \Omega_{j}^{3} \right| \geq t \delta^{4} \right) \leq \exp\left( -C_{3} t^{\frac{3}{2}} \right) + C_{1} \exp(-c_{1}n), \]
as
To estimate the first term in the sum, we use the following transformation

\[
\left| \sum_{j=1}^{n} \mu_{\nu}(X_j)\Theta_j^{3} \leq \left( \sum_{j=1}^{n} (\mu_{\nu}(X_j))^2 \Theta_j^{3} \right)^{1/2} \leq \left( \sum_{j=1}^{n} 2^{b_{j}} \delta_{j}^{4} \Theta_{j}^{4} \right)^{1/4} \leq 2^{3} \sqrt{n} \delta^{2}.
\]

therefore, for any \( t \geq 0 \) one has

\[
P\left( \left| \sum_{j=1}^{n} \beta_{j} \Theta_{j}^{3} \right| \geq \frac{t}{n} \right) \leq \tilde{C}_{2} \exp \left( -\tilde{c}_{3} t^{3/2} \right).
\]

Similarly, we come to the conclusion that

\[
\left( \sum_{j=1}^{n} \beta_{j} \Theta_{j}^{3} \right) \leq \tilde{C}_{2} \exp \left( -\tilde{c}_{3} t^{3} \right).
\]

3 Proof of the main theorem

Proof Assume that \( \delta_{0}^{4} \leq (8k)^{-1} \). We split the original integral into several terms, for this we add and subtract the term with the distribution of the sum of truncated random variables defined in (9), (10), then estimate each term separately

\[
\left| \int_{B} d(F_{X} - \Phi) \right| \leq \left| \int_{B} d(F_{X} - F_{Y}) \right| + \left| \int_{B} d(F_{Y} - \Phi) \right|.
\]

To estimate the first term in the sum, we use the following transformation

\[
\left| \int_{B} d(F_{X} - F_{Y}) \right| \leq \left| \int_{B} d\left( \sum_{j=1}^{n} F_{X_{j}} * \cdots * F_{X_{j-1}} \ast (F_{X_{j}} - F_{Y}) \ast F_{Y_{j+1}} * \cdots * F_{Y_{n}} \right) \right|
\]
From this it follows that the norm of the weighted mathematical expectation is bounded by the value

\[ \sum_{j=1}^{n} P(|\theta_j X_j| > 1) \leq \sum_{j=1}^{n} \delta_j^4 \delta_j^4 = \delta_n, \]

where \( F_{X_j}, F_{Y_j} \) are defined (11), (12) and “∗” denotes the function convolution. To estimate the second term, we additionally split this integral into the sum of three integrals, adding and subtracting new terms

\[ \left| \int_{B} d(F_Y - \Phi) \right| = \left| \int_{B_n=B^c \{ A_n \}} d(F_Z - \Phi_{A_n,t}) \right| \]

\[ \leq \left| \int_{B_n} d(F_Z - \Phi_{0,D}) \right| + \left| \int_{B_n} d(\Phi - \Phi_{0,D}) \right| + \left| \int_{B_n} d(\Phi_{A_n,t} - \Phi) \right|. \]

To estimate the last integral, we show that due to \( \mathbb{E} X_j = 0, j = 1, \ldots, n \), the following inequality holds:

\[ \|\mathbb{E} \theta_j Y_j\|^2 = \sum_{i=1}^{k} \left( \mathbb{E} \theta_j (Y_{ji} - X_{ji}) \right)^2 \leq \sum_{i=1}^{k} \left( \mathbb{E} |\theta_j X_j| 1_{X_j} (|\theta_j X_j| > 1) \right)^2 \]

\[ \leq k \left( \mathbb{E} |\theta_j X_j| \right)^2 = k(\theta_j^4 \delta_j^4)^2. \]

From this it follows that the norm of the weighted mathematical expectation is bounded by the value

\[ \|A_n\| \leq \sum_{j=1}^{n} \|\mathbb{E} \theta_j Y_j\| \leq \sqrt{k} \sum_{j=1}^{n} \theta_j^4 \delta_j^4 < \frac{1}{8\sqrt{k}}. \]

From Theorem 4 [13] we obtain that

\[ \left| \int_{B_n} d(\Phi_{A_n,t} - \Phi) \right| \leq \sqrt{\frac{3}{\pi}} \|A_n\| \leq C_1 \sqrt{k} \delta_n. \]

To estimate the next integral from the sum, we use Theorem 3 [13] and Lemma [2]

\[ \left| \int_{B_n} d(\Phi - \Phi_{0,D}) \right| \leq C_2 \|I - D\|_2 = C_2 \left( \sum_{i,l} (d_{il} - v_{il})^2 \right)^{1/2} \]

\[ \leq C_2 \left( \sum_{i,l} (2\delta_j^4)^2 \right)^{1/2} \leq 2kC_2 \delta_n. \]

It remains to estimate the last integral, for this we use the most important inequality from Corollary 11.5 [2]
\[ \int_{B_n} \left| d(F_Z - \Phi_{0,D}) \right| \leq C_3 \int_{B_n} \left| d((F_Z - \Phi_{0,D}) \times K_\epsilon) \right| + \hat{C}_1(k) \epsilon, \]

where \( \epsilon = 4a \sqrt{\frac{K}{\pi}} \) and \( a = 2^{-\frac{1}{2}} k^{\frac{3}{2}} \), and \( K_\epsilon(x) \) is a kernel function (more details 13.8-13.13 in [10]), the most important property of this function is that its characteristic function \( K(t) = 0 \) is equal to zero for \( ||t|| > n/\delta^4 \). By Lemma 11.6 [10], from estimating the difference of distributions, we can go to estimating the difference of the corresponding characteristic functions

\[ \int \left| d(F_Z - \Phi_{0,D}) \times K_\epsilon \right| \leq \hat{C}_2(k) \max_{0 \leq |\alpha + \beta| \leq k+1} \int |D^\alpha (\hat{F}_Z - \Phi_{0,D})(t)D^\beta \hat{K}_\epsilon(t)| dt. \]

Since \( |D^\beta \hat{K}_\epsilon(t)| \leq \hat{c} \), we get that

\[ \int |D^\alpha (\hat{F}_Z - \Phi_{0,D})(t)D^\beta \hat{K}_\epsilon(t)| dt \leq \hat{C}_3(k) \int |D^\alpha (\hat{F}_Z - \Phi_{0,D})(t)| dt. \]

Denote \( E_n = c_1(k, k+3) \min \left\{ \eta_{k+3}^{-\frac{1}{2}}, \eta_{k+3} \right\} \), where \( \eta_{k+3} = \sum_{j=1}^n \rho_{k+3}(Q\theta_j Z_j) \) (see (15)) and \( c_1(k, k+3) \) is a constant from the statement of Lemma [3], further, we add and subtract terms of the asymptotic expansion of the logarithm of the characteristic function. Considering that, by definition, \( \hat{F}_Z = \prod_{j=1}^n \varphi_j(\theta_j t) \) (see (13)), we can split the integral into the sum of several terms by dividing the region of integration into several parts, similarly to inequality (71) [11]

\[ \int_{||t|| \leq \frac{n}{\sqrt{a}}} |D^\alpha (\hat{F}_Z - \Phi_{0,D})(t)| dt \]

\[ \leq \int_{||t|| \leq \frac{n}{\sqrt{a}} E_n} \left| D^\alpha \left[ \prod_{j=1}^n \varphi_j(\theta_j t) - \exp \left( -\frac{1}{2} \langle D t, t \rangle \right) \sum_{r=0}^k P_r(it, \kappa_r) \right] \right| dt \]

\[ + \int_{\sqrt{\frac{n}{160}} E_n \leq ||t|| \leq \frac{n}{\sqrt{a}}} \left| D^\alpha \prod_{j=1}^n \varphi_j(\theta_j t) \right| dt + \int_{\frac{n}{\sqrt{a}} \leq ||t|| \leq \frac{n}{\sqrt{a}}} |D^\alpha \prod_{j=1}^n \varphi_j(\theta_j t)| dt \]

\[ + \int_{\sqrt{\frac{n}{E_n}} \leq ||t||} \left| D^\alpha \exp \left( -\frac{1}{2} \langle D t, t \rangle \right) \right| dt + \int |D^\alpha \sum_{r=1}^k P_r(it, \kappa_r) \exp \left( -\frac{1}{2} \langle D t, t \rangle \right) | dt \]

\[ = I_1 + I_2 + I_3 + I_4 + I_5. \]

Then, by Lemma [4] we obtain
\[
\det Q \leq \|Q^2\|^{\frac{1}{2}} \leq \left(\frac{4}{3}\right)^{\frac{1}{2}},
\]

\[
\|Qt\| \geq \|D\|^{-\frac{1}{2}} \|tr\| \geq \left(\frac{4}{5}\right)^{\frac{1}{2}} \|tr\|,
\]

\[
\left\{\|Qt\| \leq \sqrt{\frac{4}{5} E_n}\right\} \subset \left\{\|tr\| \leq c_1 (k, k + 3) \min \left\{\eta_{k+3}, \eta_{k+3}^{-\frac{1}{2}}\right\}\right\},
\]

where the matrix \(Q\) is defined in (8). Also, taking into account that for any \(s \geq 4\) from (16) it follows

\[
\eta_s \leq 2^s \|Q\|^s \sum_{j=1}^{n} \rho_4(\theta_j X_j) = \|Q\|^s 2^s \delta_0^4 \leq 2^s \left(\frac{4}{3}\right)^{\frac{1}{2}} \delta_0^4.
\]

Substituting \(t = Qt, s = k + 3\) and using Lemma 3, we come to the fact that

\[
I_1 = \int_{\|Qt\| \leq \sqrt{\frac{4}{5} E_n}} \left|D^a\left[\prod_{j=1}^{n} \varphi_j(\theta_j Qt) - \exp\left(-\frac{\|tr\|^2}{2}\right) \sum_{r=0}^{k} P_r(iQt, \kappa_r)\right]\right| \det Q dt
\]

\[
\leq \int_{\|t\| \leq E_n} \left(\frac{4}{3}\right)^{\frac{1}{2}} \left|D^a\left[\prod_{j=1}^{n} \varphi_j(\theta_j Qt) - \exp\left(-\frac{\|tr\|^2}{2}\right) \sum_{r=0}^{k} P_r(iQt, \kappa_r)\right]\right| dt
\]

\[
\leq \int \left(\frac{4}{3}\right)^{\frac{1}{2}} c_2 (k, k + 3) \eta_{k+3} (\|tr\|^{k+3-\alpha} + \|tr\|^{3(k+1)+\alpha}) \exp\left(-\frac{1}{4} \|tr\|^2\right) dt
\]

\[
\leq \mathcal{C}_1(\alpha, k) \eta_{k+3} \leq 2^{k+3} \left(\frac{4}{3}\right)^{\frac{1}{2}} \mathcal{C}_1(\alpha, k) \delta_0^4 \leq C_1(\alpha, k) \delta_0^4.
\]

Then, we denote the subset \(\mathcal{G} = \left\{ j : \delta_j^2 < 5 \delta_0^2 \right\}\), the following holds

\[
\delta_0^4 = \frac{1}{n} \sum_{j=1}^{n} \delta_j^4 \geq \frac{1}{n} \sum_{j \in \mathcal{G}} \delta_j^4 > \frac{n - |\mathcal{G}|}{n} (5 \delta_0^2)^2,
\]

hence \(|\mathcal{G}|/n > 24/25 > 4/5\). Using Lemma 3.2 (23), we obtain that with probability greater than \(1 - \mathcal{C}_1 \exp(-\mathcal{C}_1 n)\) a random vector \(\Theta\) uniformly distributed on the unit sphere \(S^{n-1}\) satisfies the following condition

\[
\sum_{j \in \mathcal{U}} \theta_j^2 > \frac{1}{8} \mathcal{U} = \left\{ j \in \mathcal{G} : |\Theta_j| < \frac{40}{\sqrt{n}} \right\}.
\]

If we set \(l = 200 \delta_0^2 / \sqrt{n}\), then one has \(40 / \sqrt{n} \leq l / \delta_j^2\), for \(j \in \mathcal{G}\). Therefore, using Lemma 4 we obtain that there is a subset of \(\mathcal{Q}_1\) with \(\lambda_{n-1}(\mathcal{Q}_1) \geq 1 - \mathcal{C}_1 \exp(-\mathcal{C}_1 n)\) such that
By Lemma 5 there exists a subset \( Q \) by Lemma 6, A multivariate CLT for weighted sums.

For any vector of weight coefficients \( \alpha \), \( \beta \), \( \gamma \)

\[
\sqrt{E_n} \leq \|t\| \leq \frac{\sqrt{c}}{1000/\sqrt{n}},
\]

\[
\leq \int \left( \|t\|^{-1} \left( \frac{4}{5} \right) \right) \frac{1}{2} c_1(k, k + 3) \min \left\{ \eta_{k+3}, \eta_{k+3}^{-1} \right\}^{-k-3}
\]

\[
\times c_1(\alpha, k)(1 + ||\alpha||) \exp \left( -\frac{1}{48} ||\alpha||^2 \right) dt
\]

\[
\leq \hat{C}_2(\alpha, k) \min \left\{ \eta_{k+3}^{-1}, \eta_{k+3}^{-1} \right\}^{-k-3} = \hat{C}_2(\alpha, k) \eta_{k+3} \max \left\{ 1, \eta_{k+3}^{-1} \right\}
\]

\[
\leq \hat{C}_2(\alpha, k) 2^{k+3} \left( \frac{4}{3} \right) \frac{k+1}{n} \delta_{\theta}^4 \left( 1 + \eta_{k+3}^{-1} \right) \leq C_2(\alpha, k) \delta_{\theta}^4.
\]

By Lemma 5 there exists a subset \( Q_2 \) with \( \lambda_{n-1}(Q_2) \geq 1 - \hat{C}_2(\alpha, k) \exp \left( -\frac{c_2(\alpha)}{\delta_{\theta}^2} \right) \)

such that for any vector of weight coefficients \( (\theta_1, \theta_2, \ldots, \theta_n) \in Q_2 \) one has

\[
I_3 = \int \left| D_\alpha \left[ \prod_{j=1}^n \varphi_j(\theta_j t) \right] \right| dt \leq C_3(\alpha, k) \frac{\delta_{\theta}^4}{n}.
\]

By Lemma 6

\[
I_4 = \int \left| D_\alpha \exp \left( -\frac{1}{2} \langle D, t \rangle \right) \right| dt
\]

\[
\leq \int \left( \|t\|^{-1} \left( \frac{4}{5} \right) \right) \frac{1}{2} c_1(k, k + 3) \min \left\{ \eta_{k+3}, \eta_{k+3}^{-1} \right\}^{-k-3}
\]

\[
\times c_4(\alpha, k)(1 + ||\alpha||) \exp \left( -\frac{3}{8} ||\alpha||^2 \right) dt \leq C_4(\alpha, k) \delta_{\theta}^4.
\]

and

\[
I_5 = \int \left| D_\alpha \sum_{r=1}^k P_r(it, \kappa_r) \exp \left( -\frac{1}{2} \langle D, t \rangle \right) \right| dt
\]

\[
\leq \sum_{r=2}^k C_5(\alpha, k) \rho_{r+2} (1 + \rho_{r+1}^{-1}) + C_6(\alpha, k) \sum_{|\kappa_r| = 3} |\kappa_r|
\]
\[ \leq C_5(\alpha, k)\delta_\theta^4 + C_6(\alpha, k) \sum_{|\nu|=3} \left| \sum_{j=1}^n \theta_j^3 \mu_\nu(Z_j) \right| \]

\[ = C_5(\alpha, k)\delta_\theta^4 + C_6(\alpha, k) \sum_{|\nu|=3} \left| \sum_{j=1}^n \theta_j^3 \mu_\nu(Z_j) \right|. \]

We get that there is a subset of weight coefficients \(Q_1 \cap Q_2\) with measure

\[ \lambda_{n-1}(Q_1 \cap Q_2) \geq 1 - \tilde{\mathcal{C}}_2(k) \exp \left( -\tilde{c}_2(k) \frac{n}{\delta^4} \right) - \tilde{\mathcal{C}}_1 \exp(-\tilde{c}_1 n) \]

\[ \geq 1 - \tilde{\mathcal{C}}_5(k) \exp \left( -\tilde{c}_5(k) \frac{n}{\delta^4} \right), \]

such that for any vector of weight coefficients \((\theta_1, \theta_2, \ldots, \theta_n) \in Q_1 \cap Q_2\) one has

\[ \left| \int_B d(F_X - \Phi) \right| \leq C_7(k) \left( \delta_\theta^4 + \frac{\delta_\theta^4}{n} + \sum_{|\nu|=3} \left| \sum_{j=1}^n \theta_j^3 \mu_\nu(Z_j) \right| \right). \tag{33} \]

Note that if \(\delta_\theta^4 > (8k)^{-1}\), the inequality (33) holds automatically for a certain choice of universal constants. Also, without loss of generality, we can require that

\[ \log^2 \left( \frac{\rho}{2\tilde{\mathcal{C}}_5(k)} \right) \leq \tilde{c}_5(k) \frac{n}{\delta^4}, \]

similarly, otherwise the statement of the Theorem holds for a special choice of the constant in the inequality. Further,

\[ \frac{\rho}{2\tilde{\mathcal{C}}_5(k)} \geq \exp \left( -\tilde{c}_5(k) \left( \frac{n}{\delta^4} \right)^{1/2} \right) \]

and

\[ \lambda_{n-1}(Q_1 \cap Q_2) > 1 - \tilde{\mathcal{C}}_5(k) \frac{\rho}{2\tilde{\mathcal{C}}_5(k)} \geq 1 - \frac{\rho}{2}. \]

By Lemma 7 there exists a subset \(Q_3\) with \(\lambda_{n-1}(Q_3) \geq 1 - \rho/2\), for which

\[ \delta_\theta^4 + \sum_{|\nu|=3} \left| \sum_{j=1}^n \theta_j^3 \mu_\nu(Z_j) \right| \leq \tilde{\mathcal{C}}_7 \left( \log \left( \frac{1}{\rho} \right) \right)^2 \delta_\theta^4 + \tilde{\mathcal{C}}_8(k) \left( \log \left( \frac{1}{\rho} \right) \right)^{1/2} \delta_\theta^4. \]

Finally, for any vector of coefficients \((\theta_1, \theta_2, \ldots, \theta_n) \in Q = Q_1 \cap Q_2 \cap Q_3\) we have

\[ \sup_{B \in \mathcal{B}} \left| \int_B d(F_X - \Phi) \right| \leq \left( \log \left( \frac{1}{\rho} \right) \right)^2 C(k) \frac{\delta_\theta^4}{n} \leq C(\rho, k) \frac{\delta_\theta^4}{n}, \]

moreover

\[ \lambda_{n-1}(Q) \geq 1 - \frac{\rho}{2} - \frac{\rho}{2} = 1 - \rho. \]
A multivariate CLT for weighted sums

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