AN APPLICATION OF EXPANDERS TO $\mathcal{B}(\ell_2) \otimes \mathcal{B}(\ell_2)$

NARUTAKA OZAWA

Abstract. With the help of Kirchberg’s and Selberg’s theorems, we prove that the minimal tensor product of $\mathcal{B}(\ell_2)$ with itself does not have the WEP (weak expectation property) of Lance.

1. Introduction

Kirchberg [K] showed a remarkable theorem that there is a unique $C^*$-norm on the tensor product between a $C^*$-algebra with the LLP and a $C^*$-algebra with the WEP. (See [P2] for a simpler proof.) In the same paper, he raised several interesting problems. Among others, he asked if there is a unique $C^*$-norm on the tensor product of $\mathcal{B}(\ell_2)$ with itself. This problem was solved negatively by Junge and Pisier [JP]. Their second approach uses the expanders (see also [Va2]). We refer the reader to a book of Lubotzky [Lu] for the information of expanders. In this paper, we will give another application of expanders (or more precisely, of Selberg’s theorem [Se]) to the tensor product of $\mathcal{B}(\ell_2)$ with itself. Our proof proceeds in the same spirit as that of Junge and Pisier [JP] (and also of Voiculescu [Vo]) to produce an uncountable family of operator spaces inside a separable metric space of operator spaces embeddable into the full group $C^*$-algebra $C^*(F)$ of a free group $F$. See [JP] for the detail.

Theorem 1. The $C^*$-algebra $\mathcal{B}(\ell_2) \otimes_{\min} \mathcal{B}(\ell_2)$ does not have the WEP.

This theorem is a corollary of the following proposition, which is of independent interest.

Proposition 2. There are a set $\Lambda$ and an action $\sigma$ of $\Gamma = \text{PSL}(2, \mathbb{Z})$ on $\Lambda$ (as permutations) such that the corresponding full crossed product $\ell_\infty(\Lambda) \rtimes \Gamma$ does not have the LLP.

We recall that an action $\alpha$ of a discrete group $\Gamma$ on a $C^*$-algebra $A$ is a homomorphism of $\Gamma$ into the group of $*$-automorphisms of $A$. A $C^*$-algebra with an action of $\Gamma$ is called a $\Gamma$-$C^*$-algebra and a map between $\Gamma$-$C^*$-algebras is said to be $\Gamma$-equivariant if the map is compatible with the $\Gamma$-actions. The full crossed product $C^*$-algebra $A \rtimes_\alpha \Gamma$ is then defined as the universal $C^*$-algebra generated by a copy of $A$ and a unitary representation $U$ of $\Gamma$ under the relation $\text{Ad}U(g)(a) = U(g)aU(g)^* = \alpha(g)(a)$ for $g \in \Gamma$ and $a \in A$. We will often omit $\alpha$ and denote the full crossed $C^*$-algebra simply by $A \rtimes \Gamma$. It follows from the $\Gamma$-equivariant Stinespring theorem that a $\Gamma$-equivariant unital completely positive map between $\Gamma$-$C^*$-algebras naturally extends to a unital completely positive map

1991 Mathematics Subject Classification. 46L05, 46L06.
Key words and phrases. expanders, local lifting property, weak expectation property.
Partially supported by JSPS Postdoctoral Fellowships for Research Abroad.
between their full crossed products. This fact will be used in the proof of Lemma 7.

We recall the definitions of the LLP \( \mathbb{E} \) and the WEP \( \mathbb{L} \).

**Definition 3.** A unital C\(^\star\)-algebra \( A \) has the LLP (the local lifting property) if for any unital completely positive map \( \phi \) from \( A \) into a quotient C\(^\star\)-algebra \( B/J \) and any finite dimensional operator subsystem \( E \) in \( A \), there is a unital completely positive lifting of \( \psi : E \to B \) of \( \phi|_E \). A C\(^\star\)-algebra \( A \) has the WEP (the weak expectation property) if for any faithful representation \( A \subseteq \mathbb{B}(\mathcal{H}) \), there is a unital completely positive map \( \Phi \) from \( \mathbb{B}(\mathcal{H}) \) into \( A^{\star\star} \) which is identical on \( A \).

Besides nuclear C\(^\star\)-algebras, a typical example of a C\(^\star\)-algebra with the LLP is the full group C\(^\star\)-algebra \( C^\star(\mathbb{F}_\infty) \) of the free group \( \mathbb{F}_\infty \) on countably many generators and a typical example with the WEP is \( \mathbb{B}(\ell_2) \), the C\(^\star\)-algebra of all bounded linear operators on the separable infinite dimensional Hilbert space \( \ell_2 \). Pisier \([P2]\) showed that the LLP is closed under taking a full free product (see also \([Be]\)). It follows that the full group C\(^\star\)-algebra \( C^\star(\Gamma) \) has the LLP since \( \Gamma \) is isomorphic to the free product \( \langle \mathbb{Z}/2\mathbb{Z} \rangle \ast \langle \mathbb{Z}/3\mathbb{Z} \rangle \). As we mentioned in the beginning, Kirchberg \([K]\) proved the following.

**Theorem 4** (Kirchberg \([K]\)). For C\(^\star\)-algebras \( A \) and \( B \), we have

1. \( A \otimes_{\min} B = A \otimes_{\max} B \) if \( A \) has the LLP and \( B \) has the WEP,
2. \( A \otimes_{\min} \mathbb{B}(\ell_2) = A \otimes_{\max} \mathbb{B}(\ell_2) \) if and only if \( A \) has the LLP,
3. \( C^\star(\mathbb{F}_\infty) \otimes_{\min} B = C^\star(\mathbb{F}_\infty) \otimes_{\max} B \) if and only if \( B \) has the WEP.

We use the following variant of the deep theorem of Selberg \([S]\) which has already been applied to C\(^\star\)-algebras by Bekka \([Be]\) to show that some full group C\(^\star\)-algebras of residually finite groups are not residually finite dimensional C\(^\star\)-algebras. We refer the reader to Sections 4.3 and 4.4 in \([La]\) for this theorem.

**Theorem 5** (Selberg \([S]\)). The trivial representation of \( SL(2,\mathbb{Z}) \) is isolated in the set of all unitary representations which factor through \( SL(2,\mathbb{Z}/m\mathbb{Z}) \) for some \( m \in \mathbb{N} \).

This theorem means that for any finite generating subset \( S \) of \( SL(2,\mathbb{Z}) \), there are a constant \( \kappa_S > 0 \) and a continuous function \( \alpha_S : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) with \( \alpha_S(0) = 0 \) such that the following holds: if \( \pi \) is a unitary representation of \( SL(2,\mathbb{Z}) \) on a Hilbert space \( \mathcal{H} \), which factors through \( SL(2,\mathbb{Z}/m\mathbb{Z}) \) for some \( m \in \mathbb{N} \), and \( \xi \in \mathcal{H} \) is a unit vector with \( \varepsilon = \max_{g \in S} \| \pi_g \xi - \xi \| < \kappa_S \), then there is a unit vector \( \eta \in \mathcal{H} \) such that \( \pi_g \eta = \eta \) for all \( g \in SL(2,\mathbb{Z}) \) and \( \| \xi - \eta \| < \alpha_S(\varepsilon) \). We observe here that the uniform convexity of a Hilbert space implies that, for each \( n \), there is a continuous function \( \beta_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) with \( \beta_n(0) = 0 \), such that the following holds: if \( \xi_1, \ldots, \xi_n \) are vectors in \( \mathcal{H} \) with \( \| \xi_i \| \leq 1 \) such that \( \sum_{i=1}^n \xi_i \| > n(1 - \varepsilon) \), then \( \| \xi_i - \xi_j \| < \beta_n(\varepsilon) \) for all \( i, j \). In particular, if \( u_1, u_2, \ldots, u_n \) are contractions on \( \mathcal{H} \) with \( \sum_{i=1}^n u_i \| > n(1 - \varepsilon) \), then there is a unit vector \( \xi \) such that \( \| u_i \xi - \xi \| < \beta_n(\varepsilon) \) for all \( i \).

**Acknowledgment.** The author thanks Professor Eberhard Kirchberg for pointing out a mistake in an earlier draft, Professor Gilles Pisier for suggesting a clearer presentation and the referee for some simplification of the proof of Theorem \( \Lambda \). A part of this study was done under the support of JSPS Postdoctoral Fellowships for Research Abroad.
2. Proofs

We recall from [21] that $OS_d$ is the set of all $d$-dimensional operator spaces, equipped with the cb Banach Mazur distance topology. By the definition of the LLP, the set of $d$-dimensional operator subspaces of a (not necessarily separable) $C^*$-algebra with the LLP is contained in the set of $d$-dimensional operator subspaces of the separable $C^*$-algebra $C^*(\mathbb{F}_\infty)$, and a fortiori, is separable in $OS_d$. Therefore, to show a $C^*$-algebra $A$ does not have the LLP, it suffices to show that the set of $d$-dimensional operator subspaces of $A$ is not separable for some $d$. This was done by Junge and Pisier [11] for $A = B(\ell_2)$. In this section, we will find an explicit example of an action $\sigma$ of $\Gamma = PSL(2,\mathbb{Z})$ on a set $\Lambda$ and $d$ such that the set of $d$-dimensional operator subspaces of $\ell_\infty(\Lambda) \rtimes \Gamma$ is non-separable.

For each prime number $p$, we let $\Lambda_p$ be the projective space $((\mathbb{Z}_p^2 - \{0\})/\mathbb{Z}_p^2$, where $\mathbb{Z}_p^2$ is the two dimensional vector space over the finite field $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. We observe $|\Lambda_p| = (p^2 - 1)/(p - 1) = p + 1$ and denote by $[t]$ the equivalence class of $(t 1)^T$ for $t \in \mathbb{Z}_p$ and by $[\infty]$ the equivalence class of $(1 0)^T$. The action of $\Gamma$ on $\mathbb{Z}_p^2$ (through linear transformation by $SL(2,\mathbb{Z}_p)$) induces a transitive action $\sigma_p$ of $\Gamma$ on the set $\Lambda_p$. Let $\pi_p$ be the corresponding unitary representation of $\Gamma$ on $\ell_2(\Lambda_p)$. For example, letting

$$h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be elements in $\Gamma$, we have $\pi_p(h)[t] = [t+1]$ and $\pi_p(k)[t] = [t-1]$ for all $t$ in $\mathbb{Z}_p \cup \infty$. Let $\chi_p = |\Lambda_p|^{-1/2} \sum_{x \in \Lambda_p} \delta_x \in \ell_2(\Lambda_p)$ be the constant function of norm 1 and let $z \in B(\ell_2(\Lambda_p))$ be the projection onto the one dimensional subspace $\mathbb{C}\chi_p$. We observe that $z$ is in the center of $C^*(\pi_p(\Gamma))$. Let $\pi^p$ be the subrepresentation of $\pi_p$, which is the restriction to the subspace $(1 - z)\ell_2(\Lambda_p)$.

**Lemma 6.** The representation $\pi^p$ is irreducible for every prime number $p$.

**Proof.** This is well-known (cf. p71–72 in [21]), but we include the proof for the reader’s convenience. We assert that the eigenspace $F$ of $\pi_p(h)$ w.r.t. the eigenvalue 1 is the two dimensional subspace $F'$ spanned by $\chi_p$ and $\delta_{[\infty]}$. Indeed, if $\zeta = \sum_{t \in \mathbb{Z}_p \cup \infty} c(t)\delta_{[t]}$ is in $F$, then it follows from the equation $\pi_p(h)\zeta = \zeta$ that $c(t) = c(0)$ for all $t \in \mathbb{Z}_p$. This shows $F \subset F'$, while the converse inclusion is clear. Therefore, $C^*(\pi^p(\Gamma))$ contains the rank one projection onto the subspace spanned by $(1 - z)\delta_{[\infty]}$ and the irreducibility of $\pi^p$ follows from the transitivity of $\sigma_p$. \qed

Combined with Schur’s lemma, this implies that, for $p \neq q$, any fixed vector for the representation $\overline{\chi}_q \otimes \pi_p$ is a scalar multiple of $\overline{\chi}_q \otimes \chi_p$, where $\overline{\chi}_q$ is the conjugate representation of $\pi_q$ on the conjugate Hilbert space $\overline{\ell_2(\Lambda_q)}$.

Let $\Omega$ be the set of all prime numbers and let $\Lambda = \bigsqcup_{p \in \Omega} \Lambda_p$ be the disjoint union. Then, the collection $(\sigma_p)_{p \in \Omega}$ induces an action $\sigma$ of $\Gamma$ on the set $\Lambda$ and an action $\alpha$ on $\ell_\infty(\Lambda)$. We denote by $U(g)$ the implementing unitary of $g \in \Gamma$ in the full crossed product $\ell_\infty(\Lambda) \rtimes_\alpha \Gamma$. Fixing a faithful representation $C^*(\Gamma) \subset B(H)$ of the full group $C^*$-algebra of $\Gamma$, with $u(g)$ denoting the unitary corresponding to $g \in \Gamma$, we define a covariant representation

$$\rho: \ell_\infty(\Lambda) \rtimes \Gamma \to B(\ell_2(\Lambda) \otimes H)$$

by $\ell_\infty(\Lambda) \ni a \mapsto a \otimes 1 \in B(\ell_2(\Lambda) \otimes H)$ and $\Gamma \ni g \mapsto \pi(g) \otimes u(g) \in B(\ell_2(\Lambda) \otimes H)$, where we put $\pi(g) = \bigoplus_{p \in \Omega} \pi_p(g) \in B(\ell_2(\Lambda))$. 
Lemma 7. The representation $\rho$ is faithful.

Proof. Consider the diagonal embedding of $(\ell_\infty(\Lambda), \alpha)$ into $(B(\ell_2(\Lambda)), \Ad \pi)$. Since $\ell_\infty(\Lambda)$ is the range of a $\Gamma$-equivariant conditional expectation from $B(\ell_2(\Lambda))$, the canonical morphism

$$\ell_\infty(\Lambda) \times_\alpha \Gamma \to B(\ell_2(\Lambda)) \rtimes_\Ad \pi \Gamma$$

is faithful. Since $\Ad \pi$ is inner, we have an isomorphism

$$B(\ell_2(\Lambda)) \rtimes_\Ad \pi \Gamma \cong B(\ell_2(\Lambda)) \otimes_{\text{max}} C^*(\Gamma),$$

where the implementing unitaries $U(g)$ in the left hand side are mapped to $\pi(g) \otimes u(g)$ in the right hand side. It follows from the LLP of $C^*(\Gamma)$ and Theorem 7 that

$$B(\ell_2(\Lambda)) \otimes_{\text{max}} C^*(\Gamma) = B(\ell_2(\Lambda)) \otimes_{\text{min}} C^*(\Gamma) \subset B(\ell_2(\Lambda) \otimes \mathcal{H}).$$

Composing these three morphisms, we obtain the conclusion. $\square$

We fix a unitary $v_p$ in $\ell_\infty(\Lambda_p)$ for each $p$ with $(v_p \chi_p | \chi_p) = 0$ and define an element $v_\omega$ in $\ell_\infty(\Lambda)$, for each subset $\omega$ of $\Omega$, by

$$v_\omega(x) = \begin{cases} v_p(x) & \text{if } x \in \Lambda_p \text{ with } p \in \omega, \\ 0 & \text{if } x \in \Lambda_p \text{ with } p \notin \omega. \end{cases}$$

Let $S = \{I, h, k\}$ be the finite set of generators of $\Gamma$ and let $E_\omega$, for each subset $\omega$ of $\Omega$, be the four dimensional operator space in $\ell_\infty(\Lambda) \rtimes \Gamma$ spanned by $v_\omega$ and $U(g)$, $g \in S$.

Lemma 8. The subset $\{E_\omega : \omega \text{ a subset of } \Omega\}$ of $O_{S_4}$ is non-separable in the cb Banach-Mazur distance topology.

Proof. To prove $\{E_\omega\}_\omega$ is non-separable, suppose the contrary that $\{E_\omega\}_\omega$ is separable. We fix $\varepsilon > 0$. As it was argued in Remark 2.10 in [14], it follows that one can find distinct $\omega$ and $\omega'$, and $q \in \omega' \setminus \omega$ such that there is a complete contraction $\varphi : E_\omega \to E_{\omega'}$ with $\|v_\omega - \varphi(v_\omega)\| < \varepsilon$ and $\|U(g) - \varphi(U(g))\| < \varepsilon$ for $g \in S$. Let $\pi_q : \ell_\infty(\Lambda) \rtimes \Gamma \to B(\ell_2(\Lambda_q))$ be the covariant representation corresponding to the quotient map from $\ell_\infty(\Lambda)$ onto $\ell_\infty(\Lambda_q)$ in $B(\ell_2(\Lambda))$ and the unitary representation $\pi_q$ of $\Gamma$ on $\ell_2(\Lambda_q)$. It follows that $\psi = \pi_q \circ \varphi$ is a complete contraction with $\|v_q - \psi(v_\omega)\| < \varepsilon$ and $\|\pi_q(g) - \psi(U(g))\| < \varepsilon$ for $g \in S$. Hence, we have that

$$\|\overline{\pi_q \otimes v_\omega + \sum_{g \in S} \pi_q(g) \otimes U(g)}\|_{B(\ell_2(\Lambda_q) \otimes_{\min} \ell_\infty(\Lambda) \rtimes \Gamma)} \geq \|\overline{\pi_q \otimes \psi(v_\omega) + \sum_{g \in S} \pi_q(g) \otimes \psi(U(g))}\|_{B(\ell_2(\Lambda_q) \otimes \ell_2(\Lambda_q))}$$

$$\geq \|\overline{\pi_q \otimes v_\omega + \sum_{g \in S} \pi_q(g) \otimes \pi_q(g)}\|_{B(\ell_2(\Lambda_q) \otimes \ell_2(\Lambda_q))} - 4\varepsilon$$

$$= 4(1 - \varepsilon).$$

Combining the above inequality with Lemma 7, we have

$$\|\overline{\pi_q \otimes v_\omega \otimes 1 + \sum_{g \in S} \pi_q(g) \otimes \pi_q(g) \otimes u(g)}\|_{B(\ell_2(\Lambda_q) \otimes \ell_2(\Lambda) \otimes \mathcal{H})} > 4(1 - \varepsilon).$$

By the uniform convexity of Hilbert spaces, we can find a unit vector

$$\zeta = \sum_{(x, y) \in \Lambda_q \times \Lambda} \delta_x \otimes \delta_y \otimes \zeta(x, y) \in \ell_2(\Lambda_q) \otimes \ell_2(\Lambda) \otimes \mathcal{H}$$

\]
such that
$$\|\zeta - (\overline{\pi_q} \otimes u_\omega \otimes 1)\zeta\| < \beta(\varepsilon)$$
and
$$\|\zeta - (\overline{\pi_q} \otimes \pi(\gamma) \otimes u(g))\zeta\| < \beta(\varepsilon)$$
for \(g \in S\), where \(\beta: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) is a continuous function with \(\beta(0) = 0\) (cf. the remarks following Theorem 5). From the first inequality and the fact that \(q \notin \omega\), we may assume that \(\zeta\) is zero on \(\Lambda_q \times \Lambda_q\). We put
$$\xi = \sum_{(x,y) \in \Lambda_q \times \Lambda_q} \delta_x \otimes \delta_y \|\zeta(x,y)\| \in \ell_2(\Lambda_q) \otimes \ell_2(\Lambda).$$
It follows that
$$\beta(\varepsilon) \geq \|\zeta - (\overline{\pi_q}(g) \otimes \pi(\gamma) \otimes u(g))\zeta\| \|\xi\|_{\ell_2(\Lambda_q) \otimes \ell_2(\Lambda)} \geq \|\xi\|_{\ell_2(\Lambda_q) \otimes \ell_2(\Lambda)}$$
for \(g \in S\). We are now in position to employ Selberg’s theorem. Indeed, the unit vector \(\xi\) is zero on \(\Lambda_q \times \Lambda_q\) and, for \(p \neq q\), Lemma 6 implies that any fixed vector for the representation \(\pi_q \otimes \pi_p\) is a scalar multiple of \(\chi_q \otimes \chi_p\). Thus, it follows from Theorem 2 that
$$\|\xi - (\overline{\chi_q} \otimes \eta)\| < \alpha(\beta(\varepsilon))$$
for some unit vector \(\eta \in \ell_2(\Lambda)\), where \(\alpha: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) is a continuous function with \(\alpha(0) = 0\) (cf. the remarks following Theorem 5). Finally, we have
$$\beta(\varepsilon) \geq \|\zeta - (\overline{\chi_q} \otimes \eta)\| - 2\alpha(\beta(\varepsilon))$$
(recall that we chose \(\nu_p\) so that \(\chi_p \perp \nu_p \chi_p\), but this gives a contradiction when \(\varepsilon > 0\) is chosen sufficiently small. This completes the proof.

We now prove Proposition 2 and Theorem 1.

**Proof.** Proposition 2 follows from Lemma 8 as we have explained in the first paragraph of this section. We turn to the proof of Theorem 1. To show that \(B(\ell_2) \otimes_{\min} B(\ell_2)\) does not have the WEP, suppose the contrary that \(B(\ell_2) \otimes_{\min} B(\ell_2)\) has the WEP. We take \(\Lambda\) as in Proposition 2 so that, by Theorem 1,
$$(\ell_\infty(\Lambda) \rtimes \Gamma) \otimes_{\min} B(\ell_2) \neq (\ell_\infty(\Lambda) \rtimes \Gamma) \otimes_{\max} B(\ell_2).$$
By universality, we have the canonical isomorphism
$$(\ell_\infty(\Lambda) \rtimes \Gamma) \otimes_{\max} B(\ell_2) = (\ell_\infty(\Lambda) \otimes B(\ell_2)) \rtimes \Gamma,$$
here \(\Gamma\) acts on \(B(\ell_2)\) trivially. By the assumption that \(B(\ell_2(\Lambda)) \otimes_{\min} B(\ell_2)\) has the WEP, arguing as Lemma 7, one can show that the canonical morphism
$$(\ell_\infty(\Lambda) \otimes B(\ell_2)) \rtimes \Gamma \subset B(\ell_2(\Lambda) \otimes \ell_2 \otimes H)$$
is faithful. Composing these, we have a faithful representation
\[(\ell_\infty(\Lambda) \rtimes \Gamma) \otimes \mathcal{B}(\ell_2) \to \mathcal{B}(\ell_2(\Lambda) \otimes \mathcal{H} \otimes \ell_2)\]
which obviously factors through \((\ell_\infty(\Lambda) \rtimes \Gamma) \otimes_{\min} \mathcal{B}(\ell_2)\). This is absurd. \qed

APPENDIX A. A pathology in equivariant KK-theory

With the help of Wassermann’s construction \([\tilde{M}]\), we prove the following.

**Theorem A.1.** Let \(\Gamma = SL(3, \mathbb{Z})\) and let \(M = \prod_{n=1}^{\infty} M_n\). There is a short exact sequence of separable commutative \(\Gamma\)-\(C^*\)-algebras \(0 \to J \to B \to A \to 0\) such that the corresponding sequence
\[K_0(M \otimes (J \rtimes \Gamma)) \to K_0(M \otimes (B \rtimes \Gamma)) \to K_0(M \otimes (A \rtimes \Gamma))\]
is not exact.

**Corollary A.2.** The six-term exact sequence in \(\Gamma\)-equivariant KK-theory fails to hold for the short exact sequence of separable commutative \(\Gamma\)-\(C^*\)-algebras appearing in Theorem A.1.

This corollary was pointed out by Skandalis and the proof is almost the same as \([3]\). Maghfoûl \([\tilde{M}]\) proved that such a pathology does not occur under a certain \(K\)-theoretical amenability condition on \(\Gamma\). On the other hand, Higson, Lafforgue and Skandalis \([\ref{HLS}]\) found a similar pathology for Gromov’s non-exact group \(\Gamma\). [\ref{K}]

From now on, we denote the group \(SL(3, \mathbb{Z})\) by \(\Gamma\). For each prime number \(p\), we let \(\Lambda_p\) be the projective space \((\mathbb{Z}_p^3 - \{0\})/\mathbb{Z}_p^2\) and observe that \(|\Lambda_p| = (p^3 - 1)/(p - 1) = p^2 + p + 1\). The action of \(\Gamma\) on \(\mathbb{Z}_p^3\) (through linear transformation by \(SL(3, \mathbb{Z}_p)\)) induces a transitive action \(\sigma_p\) of \(\Gamma\) on the set \(\Lambda_p\). Let \(\pi_p\) be the corresponding unitary representation of \(\Gamma\) on \(\ell_2(\Lambda_p)\). Let \(\chi_p = |\Lambda_p|^{-1/2} \sum_{x \in \Lambda_p} \delta_x \in \ell_2(\Lambda_p)\) be the constant function of norm 1 and let \(z \in \mathbb{B}(\ell_2(\Lambda_p))\) be the projection onto the one dimensional subspace \(\mathbb{C} \chi_p\). We observe that \(z\) is in the center of \(C^*(\pi_p(\Gamma))\). Let \(\pi_p^{\sigma}\) be the subrepresentation of \(\pi_p\), which is the restriction to the subspace \((1 - z)\ell_2(\Lambda_p)\). A similar proof to that of Lemma 3 in Section 2 yields the following lemma.

**Lemma A.3.** The representation \(\pi_p^{\sigma}\) is irreducible for every prime number \(p\).

Let \(\Omega\) be an infinite set of odd prime numbers such that \(p, q \in \Omega\) and \(p > q\) implies that \(p \equiv 1 \mod q\) (and in particular \(p > 2q\)). Such an infinite set \(\Omega\) exists by Dirichlet’s theorem. For each \(p \in \Omega\), we define the subset \(X_p \subset \Lambda_p\) by
\[X_p = \{(1 \ a \ b)^T \in \Lambda_p : a = 0, 2, 4, \ldots, p - 1, \ b \in \mathbb{Z}_p\}\]
and observe that \(|\Lambda_p|/3 < |X_p| = (p^2 + p)/2 < |\Lambda_p|/2\). For \(h = I_3 + e_{21} \in \Gamma\), we have that
\[\sigma_p(h^q)X_p \cap X_p \cap \sigma_p(h^{-q})X_p = \emptyset\]
whenever \(p, q \in \Omega\) are distinct. Indeed, this easily follows from the fact that we have either \(q \equiv 1 \mod p\) (and hence \(\sigma_p(h^q) = \sigma_p(h)\)) or \(2q < p\) when \(p, q \in \Omega\) are distinct. The action \(\sigma_p\) induces an action of \(\Gamma\) on \(\ell_\infty(\Lambda_p)\) and on \(\prod_{p \in \Omega} \ell_\infty(\Lambda_p)\). We often identify \(\ell_\infty(\Lambda_p)\) with the diagonal of \(\mathbb{B}(\ell_2(\Lambda_p))\). For each \(p \in \Omega\), let \(e_p \in \ell_\infty(\Lambda_p)\) be the characteristic function of \(X_p\). Let \(e = (e_p)_p \in \prod_{p \in \Omega} \ell_\infty(\Lambda_p)\) be a projection, \(B\) be the unital separable \(\Gamma\)-\(C^*\)-subalgebra of \(\prod_{p \in \Omega} \ell_\infty(\Lambda_p)\) generated by \(e\) and \(J = \bigoplus_{p \in \Omega} \ell_\infty(\Lambda_p)\) and let \(Q\) be the \(\Gamma\)-equivariant quotient from \(B\) onto
Since we have \( (25) \), we denote respectively by \( r \) and \( s \) the eigenspaces of \( \pi(\Omega) \), and \( t \) the eigenspace of \( \pi(\Omega) \) containing the unit, we define a selfadjoint element \( s \) and a projection \( t \) in \( M \otimes_{\text{min}} (B \rtimes \Gamma) \) by

\[
s = \frac{1}{|S|} \sum_{g \in S} \pi(g) \otimes U(g) \quad \text{and} \quad t = \tau \otimes e + (1 - e) \otimes (1 - e),
\]

where \( \pi(g) = (\pi_q(g))_q \in \prod_{p \in \Omega} B(\ell_2(\Lambda_q)). \) Since \( \Gamma = SL(3, \mathbb{Z}) \) has the Kazhdan property [HV] [Va1], there is \( 0 < \varepsilon < 1 \) such that \( \text{Sp}(s) \subset [-1, 1 - \varepsilon] \cup \{1\} \) (cf. the remarks following Theorem 5). We will prove that the spectrum of \( r = s + t \) has a gap around 2. For this reason, we decompose \( r \) into a direct sum. For each \( q \in \Omega \), let \( Q_q \) be the canonical quotient from \( \prod_{p \in \Omega} \ell_\infty(\Lambda_p) \) onto \( \ell_\infty(\Lambda_q) \), and let \( Q_q' \) be the canonical quotient from \( \prod_{p \in \Omega \setminus \{q\}} \ell_\infty(\Lambda_p) \) onto \( \ell_\infty(\Lambda_q) \). We still denote their restriction to \( B \) by \( Q_p \) and \( Q_q' \) and let \( A_q = Q_p(B) = \ell_\infty(\Lambda_q) \) and \( A_q' = Q_q'(B) \). Since we have \( B = A_q \rtimes A_q' \) as a \( \Gamma \)-\( C^* \)-algebra, \( B \rtimes \Gamma = A_q \rtimes \Gamma \rtimes A_q' \rtimes \Gamma \). Therefore, we have

\[
M \otimes (B \rtimes \Gamma) \subset \prod_{q \in \Omega} \left( B(\ell_2(\Lambda_q)) \otimes (A_q \rtimes \Gamma) \oplus B(\ell_2(\Lambda_q')) \otimes (A_q' \rtimes \Gamma) \right).
\]

We denote respectively by \( r_q \), \( s_q \) and \( t_q \) the direct summands of \( r \), \( s \) and \( t \) in \( B(\ell_2(\Lambda_q)) \) and by \( r_q' \), \( s_q' \) and \( t_q' \) the direct summands of \( r \), \( s \) and \( t \) in \( B(\ell_2(\Lambda_q')) \).

Lemma A.4. We have \( 2 \in \text{Sp}(r_q) \subset [-1, 2 - 10^{-4}\varepsilon] \cup \{2\} \) and \( \text{Sp}(r_q') \subset [-1, 2 - 10^{-4}\varepsilon] \).

Proof. Let \( C^*(\Gamma) \subset \mathbb{B}(\mathcal{H}) \) be a faithful representation and denote by \( u(g) \) the unitary in \( C^*(\Gamma) \) corresponding to \( g \in \Gamma \). Since \( A_q = \ell_\infty(\Lambda_q) \) is finite dimensional, it is not hard to see that the representation of \( A_q \rtimes \Gamma \) on \( \ell_2(\Lambda_q) \otimes \mathcal{H} \) given by \( A_q \ni a \mapsto a \otimes 1 \) and \( \Gamma \ni g \mapsto \pi_q(g) \otimes u(g) \) is faithful. Hence, we have

\[
s_q = \frac{1}{|S|} \sum_{g \in S} \pi_q(g) \otimes \pi_q(g) \otimes u(g) \quad \text{and} \quad t_q = \tau_q \otimes e_q \otimes 1 + (1 - e_q) \otimes (1 - e_q) \otimes 1
\]

on \( \ell_2(\Lambda_q) \otimes \ell_2(\Lambda_q) \otimes \mathcal{H} \). We identify the Hilbert space \( \ell_2(\Lambda_q) \otimes \ell_2(\Lambda_q) \otimes \mathcal{H} \) with the space \( \mathcal{K} \) of Hilbert-Schmidt operators from \( \ell_2(\Lambda_q) \) to \( \ell_2(\Lambda_q) \otimes \mathcal{H} \) so that \( \pi_q(g) \otimes \pi_q(g) \otimes u(g) \) acts on \( \mathcal{K} \) by \( \mathcal{K} \ni T \mapsto (\pi_q(g) \otimes u(g))T \pi_q(g)^* \in \mathcal{K} \). Then, it follows from the uniform convexity of a Hilbert space (cf. the remarks following Theorem 5) that the eigenspace \( \mathcal{K}_0 \) of \( s_q \) w.r.t. the eigenvalue 1 is

\[
\mathcal{K}_0 = \{ T \in \mathcal{K} : (\pi_q(g) \otimes u(g))T \pi_q(g)^* = T \text{ for all } g \in \Gamma \}.
\]

For each \( \xi : A_q \to \mathcal{K} \), we associate \( S_\xi \in \mathcal{K} \) defined by \( S_\xi(\delta_x) = \delta_x \otimes \xi_x \) and then we define the subspace \( \mathcal{K}_1 \) of \( \mathcal{K} \) by

\[
\mathcal{K}_1 = \{ S_\xi \in \mathcal{K} : \xi \text{ satisfies } u(g)\xi_x = \xi_{\pi_q(g)x} \text{ for all } x \in \Lambda_q \text{ and } g \in \Gamma \}.
\]

Since \( \mathcal{K} \) contains a non-zero fixed vector for the unitary representation \( u \), it is not too difficult to see that \( \mathcal{K}_1 \) is non-empty and contained in the intersection of eigenspaces of \( s_q \) and \( t_q \) w.r.t. the eigenvalues 1 (and hence \( r_q|_{\mathcal{K}_1} = 2 \)). We claim that \( \text{Sp}(r_q|_{\mathcal{K}_1}) \subset [-1, 2 - 10^{-4}\varepsilon] \). This easily follows if we prove that \( \|t_q(T)\| < (25/36)^{1/2}\|T\| \) for any \( T \in \mathcal{K}_0 \cap \mathcal{K}_1 \). To prove this, we give ourselves \( T \in \mathcal{K}_0 \cap \mathcal{K}_1 \).
of norm 1. Since $T \in \mathcal{K}_0$, we have that $\xi : \Lambda_q \to \mathcal{H}$, given by $\xi_x = (T\delta_x)(x)$, satisfies $u(T)\xi_x = \xi_{\sigma(T)x}$ for every $x \in \Lambda_q$ and $g \in \Gamma$. It follows that $0 = (T, S) = \sum_x \|(T\delta_x)(x)\|^2$ and hence $(T\delta_x)(x) = 0$ for all $x \in \Lambda_q$. We define $\tilde{T} \in \mathcal{B}(l_2(\Lambda_q))$ by $\tilde{T}\delta_x = [T\delta_x]_l$, where for $\zeta \in l_2(\Lambda_q) \otimes \mathcal{H}$, the vector $|\zeta\rangle \in l_2(\Lambda_q)$ is given by $|\zeta(x)\rangle = \|\zeta(x)\|_{\mathcal{H}}$. Since $\tilde{T}$ commutes with $\pi_q(\Gamma)$ and its diagonal entries are zero, it follows from Lemma A.3 that $\tilde{T} = \lambda 1 + \mu z$ with $n\lambda + \mu = 0$, where $n = |\Lambda_q|$. Since $\|\tilde{T}\| = \|T\| = 1$, we obtain $\tilde{T} = (n(n-1))^{-1/2}(nz - 1)$. Therefore, we have
\[
\|t_q(T)\|^2 = \|(e_q \otimes 1) T e_q + ((1 - e_q) \otimes 1) T (1 - e_q)\|_{\mathcal{K}}^2 = \sum_{x \in X_q} \|e_q \tilde{T} (\delta_x)\|^2 + \sum_{x \in \Lambda_q \setminus X_q} \|((1 - e_q) \tilde{T} (\delta_x))\|^2 = (n(n-1))^{-1}(1 + (n - |X_q|)(n - |X_q| - 1)) < (1/2)^2 + (2/3)^2 = 25/36
\]
since $n/3 < |X_q| < n/2$. This completes the proof of the first half.

Take a faithful representation $A'_q \times \Gamma \subset \mathcal{B}(\mathcal{H})$ and denote $e'_q = Q'_q(e) \in A'_q$ and $U'_q(g) \in A'_q \times \Gamma$ the implementing unitary for $g \in \Gamma$. Then, by the construction, we have
\[
f := 3^{-1}(\text{Ad} U'_q(h^q)(e'_q) + e'_q + \text{Ad} U'_q(h^{-q})(e'_q)) \leq 2/3
\]
in $A'_q$ as $\sigma_p(h^q) X_p \cap X_p \cap \sigma_p(h^{-q}) X_p = \emptyset$ for all $p \in \Omega \setminus \{q\}$. We note that
\[
s'_q = \frac{1}{|S|} \sum_{g \in S} \pi_q(g) \otimes U'_q(g) \quad \text{and} \quad t'_q = \pi_q(e'_q) + (1 - e'_q) \otimes (1 - e'_q)
\]
on $l_2(\Lambda_q) \otimes \mathcal{H}$. We identify the Hilbert space $l_2(\Lambda_q) \otimes \mathcal{H}$ with the space $\mathcal{K}$ of Hilbert-Schmidt operators from $l_2(\Lambda_q)$ to $\mathcal{H}$ so that $\pi_q(g) \otimes U'_q(g)$ acts on $\mathcal{K}$ by $\mathcal{K} \ni T \mapsto U'_q(g) T \pi_q(g) \in \mathcal{K}$. Then, it follows from the uniform convexity of a Hilbert space (cf. the remarks following Theorem A.3) that the eigenspace $\mathcal{K}_0$ of $s'_q$ w.r.t. the eigenvalue 1 is
\[
\mathcal{K}_0 = \{ T \in \mathcal{K} : U'_q(g) T \pi_q(g)^* = T \text{ for all } g \in \Gamma \}.
\]
We claim that $\|t'_q(T)\| < (8/9)^{1/2} \|T\|$ for any $T \in \mathcal{K}_0$. Then the second half of this lemma follows. To prove this, we give ourselves $T \in \mathcal{K}_0$ of norm 1. Since $U'_q(h^q) T = T \pi_q(h^q) = T$, we have
\[
\|t_q(T)\|^2 = \text{Tr}(T^* e'_q T e_q + T^* (1 - e'_q) T (1 - e_q)) = \text{Tr}(T^* T e_q + T^* (1 - f) T (1 - e_q)) \leq 1 - 3^{-1} \text{Tr}(T^* T e_q).
\]

Since $T^* T$ commutes with $\pi_q(\Gamma)$, it follows from Lemma A.4 that $T^* T = \lambda 1 + \mu z$ for some real number $\lambda$ and $\mu$ with $|\lambda|_{\Lambda_q} + \mu = 1$. Hence, we have that $\text{Tr}(T^* T e_q) = |X_q|/|\Lambda_q| > 1/3$. This completes the proof.

We now prove Theorem A.1.

**Proof.** Since we have $\text{Sp}(r) \subset [-1, 2 - 10^{-4}] \cup \{2\}$ by Lemma A.4, the spectral projection $d$ of $r$ corresponding to the spectral subset $\{2\}$ is contained in the C*-algebra $M \otimes_{\text{min}} (B \times \Gamma)$. Since
\[
M \otimes_{\text{min}} (A \times \Gamma) \subset \prod_{q \in \Omega} \left( \mathcal{B}(l_2(\Lambda_q)) \otimes (A \times \Gamma) \right)
\]
and the quotient $Q$ from $B$ onto $A$ factors through $A'_q$ for each $q \in \Omega$, we have that $(\id_M \otimes \tilde{Q})(d) = 0$ by Lemma A.4, where $\tilde{Q}$ is the quotient from $B \times \Gamma$ onto $A \times \Gamma$ induced by $Q$. Finally, we observe that the $K_0$-element corresponding to $d$ does not come from $K_0(M \otimes_{\min}(J \times \Gamma))$ as any element from $K_0(M \otimes_{\min}(J \times \Gamma))$ vanishes on $\tau_q$ for all but finitely many $q \in \Omega$, where $\tau_q$ is the tracial state on $B(\ell_2(\Lambda_q)) \otimes_{\min} B(\ell_2(\Lambda_q))$ evaluated through $\tilde{\pi}_q: B \times \Gamma \to B(\ell_2(\Lambda_q))$.

\[ \square \]

References

[Be] M. B. Bekka, On the full $C^*$-algebras of arithmetic groups and the congruence subgroup problem, Forum Math. 11 (1999), 705–715.

[Bo] F. Boca, A note on full free product $C^*$-algebras, lifting and quasidiagonality, Operator theory, operator algebras and related topics (Timisoara, 1996), 51–63, Theta Found., Bucharest, 1997.

[FH] W. Fulton and J. Harris, Representation theory. A first course, Graduate Texts in Mathematics, 129. Springer-Verlag, New York, 1991.

[G] M. Gromov, Random walk in random groups, IHES preprint, January 2002.

[HV] P. de la Harpe and A. Valette, La propriété (T) de Kazhdan pour les groupes localement compacts, With an appendix by M. Burger. Astérisque 175 (1989), 158 pp

[HLS] N. Higson, V. Lafforgue and G. Skandalis, Counterexamples to the Baum-Connes conjecture, Geom. Funct. Anal. 12 (2002), no. 2, 330–354.

[JP] M. Junge and G. Pisier, Bilinear forms on exact operator spaces and $B(H) \otimes B(H)$, Geom. Funct. Anal. 5 (1995), 329–363.

[K] E. Kirchberg, On non-semisplit extensions, tensor products and exactness of group $C^*$-algebras, Invent. Math. 112 (1993), 449–489.

[La] C. Lance, On nuclear $C^*$-algebras, J. Funct. Anal. 12 (1973), 157–176.

[Lu] A. Lubotzky, Discrete groups, expanding graphs and invariant measures, With an appendix by J. D. Rogawski. Progress in Mathematics, 125. Birkhäuser Verlag, Basel, 1994. xii+195 pp.

[M] M. Maghfoûl, Semi-exactitude du bifoncteur de Kasparov équivariant, K-Theory 16 (1999), 245–276.

[P1] G. Pisier, Exact operator spaces, in “Recent Advances in Operator algebras - Orléans 1992”, Astérisque Soc. Math. France 232 (1995), 159–186.

[P2] G. Pisier, A simple proof of a theorem of Kirchberg and related results on $C^*$-norms, J. Op. Theory 35 (1996) 317–335.

[Se] A. Selberg, On the estimation of Fourier coefficients of modular forms, Proc. Sympos. Pure Math., Vol. VIII pp. 1–15 Amer. Math. Soc., Providence, R.I., 1965

[Sk] G. Skandalis, Le bifoncteur de Kasparov n’est pas exact, C. R. Acad. Sci. Paris Sér. I Math. 313 (1991), 939–941.

[Va1] A. Valette, Old and new about Kazhdan’s property (T), Representations of Lie groups and quantum groups (Trento, 1993), 271–333, Pitman Res. Notes Math. Ser., 311, Longman Sci. Tech., Harlow, 1994.

[Va2] A. Valette, An application of Ramanujan graphs to $C^*$-algebra tensor products, Discrete Math. 167/168 (1997), 597–603.

[Vo] D. Voiculescu, Property $T$ and approximation of operators, Bull. London Math. Soc. 22 (1990), 25–30.

[W] S. Wassermann, $C^*$-algebras associated with groups with Kazhdan’s property $T$, Ann. of Math. (2) 134 (1991), 423–431.

Department of Mathematical Science, University of Tokyo, 153-8914, Japan
E-mail address: narutaka@ms.u-tokyo.ac.jp