TOTALLY GEODESIC HYPERSURFACES OF HOMOGENEOUS SPACES

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Abstract. We show that a simply connected Riemannian homogeneous space $M$ which admits a totally geodesic hypersurface $F$ is isometric to either (a) the Riemannian product of a space of constant curvature and a homogeneous space, or (b) the warped product of the Euclidean space and a homogeneous space, or (c) the twisted product of the line and a homogeneous space (with the warping/twisting function given explicitly). In the first case, $F$ is also a Riemannian product; in the last two cases, it is a leaf of a totally geodesic homogeneous fibration. Case (c) can alternatively be characterised by the fact that $M$ admits a Riemannian submersion onto the universal cover of the group $SL(2)$ equipped with a particular left-invariant metric, and $F$ is the preimage of the two-dimensional solvable totally geodesic subgroup.

1. Introduction

The study of totally geodesic submanifolds of homogeneous spaces dates back to the classical result of Élie Cartan from 1927 ([C] or [Hel, IV, §7]), which says that a totally geodesic submanifold of a symmetric space is the exponent of a Lie triple system. Homogeneous totally geodesic submanifolds of nilpotent Lie groups have been extensively studied in [Ebe, KP, CHN1, CHN2]. The classification of totally geodesic submanifolds of nonsingular two-step nilpotent Lie groups is given in [Ebe].

In the last two decades, a remarkable progress has been achieved in the study of one-dimensional totally geodesic submanifolds — homogeneous geodesics (the geodesics which are the orbits of a one-dimensional isometry group); this includes the deep existence results [Kai, KS, Dus] and the investigation of the g.o. spaces — homogeneous spaces all of whose geodesics are homogeneous (see e.g. [Gor, AN]).

In this paper we investigate the other extremity — totally geodesic hypersurfaces (not necessarily homogeneous) of homogeneous spaces. As one may expect, the existence of such a hypersurface imposes strong restrictions on the ambient space. In particular, if a homogeneous space admits a totally geodesic hypersurface, then it must be a space of constant curvature, provided it belongs to one of the following classes: irreducible symmetric spaces [CN], normal homogeneous spaces [To2], and more generally, naturally reductive homogeneous spaces [Ts1, To1]. Totally geodesic hypersurfaces and extrinsic hyperspheres in manifolds with special holonomy have been recently studied in [JMS]. By [CHN1, Proposition 5], if a nilmanifold admits a totally geodesic homogeneous hypersurface $F$, then its metric Lie algebra is the direct orthogonal sum of a one-dimensional ideal and the ideal tangent to $F$. We prove the following classification theorem.

Theorem 1. Suppose $M$ is a simply connected, connected Riemannian homogeneous space and $F \subset M$ is a complete connected totally geodesic hypersurface. Then one of the following holds.

(a) $M = M_1(c) \times M_2$, the Riemannian product of a space $M_1(c)$ of constant curvature $c$ and a homogeneous space $M_2$. The hypersurface $F$ is the product $F_1(c) \times M_2$, where $F_1(c) \subset M_1(c)$ is totally geodesic.

(b) $M = \mathbb{R}^m_f \times M_2$, the warped product of $\mathbb{R}^m$, $m > 0$, and a homogeneous space $M_2 = G/H$, with the warping function $f : M_2 \to \mathbb{R}$ defined by $f(gH) = \chi(g)$, where $\chi : G \to (\mathbb{R}^+, \cdot)$ is a nontrivial homomorphism with $\chi(H) = 1$. The hypersurface $F$ is the Cartesian product of a hyperplane $\mathbb{R}^{m-1} \subset \mathbb{R}^m$ and $M_2$.

(c) $M = \mathbb{R}_f \times M_2$, the twisted product of $\mathbb{R}$ and a homogeneous space $M_2$. The hypersurface $F$ is a leaf of the totally geodesic fibration $\{t\} \times M_2$, $t \in \mathbb{R}$.

2010 Mathematics Subject Classification. Primary: 53C30, 53C40, secondary: 53B25.

Key words and phrases. totally geodesic hypersurface, homogeneous space.
Moreover, the curves $\mathbb{R} \times \{x\}$, $x \in M_2$, are congruent helices of order two with the curvature $k$ and the torsion $\kappa \neq 0$. With a particular choice of local coordinates $t$ on $\mathbb{R}$ and $u$ on $M_2$, the twisting function is given by $f(t, u) = (\sinh(\alpha(u)) \cos(kt + \beta(u)) + \cosh(\alpha(u)))^{-2}$, where locally $\alpha, \beta : M_2 \to \mathbb{R}$ satisfy $\|\nabla \alpha\|^2 = \sinh(\alpha)^2 \|\nabla \beta\|^2 = k^2$.

The warped (the twisted) product $M_1 \times M_2$ of Riemannian manifolds $(M_1, ds_1^2)$ and $(M_2, ds_2^2)$, with the warping function $f : M_2 \to \mathbb{R}^+$ (respectively, with the twisting function $f : M_1 \times M_2 \to \mathbb{R}^+$), is the Cartesian product $M_1 \times M_2$ equipped with the metric $f ds_1^2 + ds_2^2$. A smooth curve in a Riemannian space is called a helix of order $p \geq 0$, if its first $p$ Frenet curvatures are nonzero constants and the $(p + 1)$-st Frenet curvature vanishes (by analogy with curves in $\mathbb{R}^3$, for helices of order two, we call the first two nonzero curvatures the curvature and the torsion, respectively). Note that we impose the assumption of completeness of $F$ only for convenience; any open portion of a totally geodesic hypersurface of $M$ can be extended to a complete hypersurface by extending all the geodesics.

It follows from Theorem 1 that apart from Case (a), a totally geodesic hypersurface $F$ is a leaf of a totally geodesic fibration of codimension one.

Theorem 1 is intentionally stated in a purely “Riemannian” language (except for a small amount of algebra in Case (b)) avoiding the choice of a particular presentation of $M$ as $G/H$. An important question in the theory of totally geodesic submanifolds of homogeneous spaces is when such a submanifold is homogeneous (that is, is the orbit of a subgroup of $G$). From Theorem 2 below (or from the proof of Theorem 1 given in Section 2) one can deduce that in Case (c) of Theorem 1, the hypersurface $F$ is homogeneous relative to any choice of a connected transitive group $G$ of isometries of $M$. The answer in the other two cases depends on a particular presentation. In Case (a) it can easily be in negative: the group $SU(2)$ with a metric of constant positive curvature contains no two-dimensional subgroups. An example of a non-homogeneous totally geodesic hypersurface from Case (b) is given in Section 2. Note however that in all the cases, the Riemannian manifold $F$ is homogeneous relative to the induced metric.

Theorem 1 has the following obvious but useful corollary.

**Corollary.** A compact, simply connected, connected Riemannian homogeneous space that admits a totally geodesic hypersurface $F$ is the Riemannian product of a standard sphere $S^m$, $m \geq 2$, and a compact homogeneous space $M_2$; then $F$ is (a domain of) the product of a great hypersphere $S^{m-1}$ and $M_2$.

One can give an alternative, more algebraic description of the totally geodesic hypersurface from Case (c) of Theorem 1. The “smallest” example of such a hypersurface is constructed as follows.

**Example 1.** Take $M = SL(2)$, the universal cover of the group $SL(2)$. Denote $\mathfrak{g} = sl(2)$. Let $\mathfrak{f} \subset \mathfrak{g}$ be a two-dimensional subalgebra and let $N \subset \mathfrak{g}$ span the one-dimensional subalgebra $so(2) \subset \mathfrak{g}$. Up to automorphism and scaling one can choose, in the defining representation of $\mathfrak{g}$,

$$N = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathfrak{f} = \left\{ \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$ 

Introduce an inner product on $\mathfrak{g}$ by requiring that $N \perp \mathfrak{f}$ and by specifying it further on $\mathfrak{f}$ in such a way that the operator $\pi_1 \text{ad}_N \pi_1$ is skew-symmetric. Explicitly, choose arbitrary nonzero $a, b \in \mathbb{R}$ and define the inner product $\langle \cdot, \cdot \rangle$ in such a way that the following basis is orthonormal:

$$E_1 = aN = a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_2 = 2b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = b \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Then $\mathfrak{f}$ is a totally geodesic subalgebra of the metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ [Ts2, Theorem 7.2], and so the subgroup $F_1$ tangent to $\mathfrak{f}$ is a totally geodesic hypersurface of $M = SL(2)$ equipped with the left-invariant metric obtained from the inner product (1). Note that $F_1$ is isometric to the hyperbolic space and the functions $\alpha, \beta$ from Case (c) are, up to scaling, the polar coordinates on $F_1$.

**Theorem 2.** Under the assumptions of Theorem 1, either the pair $(M = G/H, F)$ belongs to one of the cases (a), (b), or otherwise there exists a normal subgroup $N \subset G$ such that $H \subset N$, $G/N \simeq SL(2)$, and the projection $\pi : M \to SL(2)$ (where the metric on $SL(2)$ is constructed as in Example 1) is a Riemannian submersion, and $F = \pi^{-1}F_1$. 


2. Proofs

Let $M = G/H$ be a simply connected, connected Riemannian homogeneous space, with $G$ a simply connected, closed, connected transitive group of isometries acting on $M$ from the left and $H$ the (connected) isotropy subgroup of a point $o \in M$. Let $\pi : G \to M$ be the natural projection with $\pi(e) = o$. Denote $(\cdot, \cdot), \nabla$ and $R$ the metric, the Levi-Civita connection and the curvature tensor of $M$ respectively. For vector fields $X, Y \in TM$ we define the operator $X \wedge Y \in \mathfrak{so}(TM)$ by $(X \wedge Y)Z = \langle X, Z \rangle Y - \langle Y, Z \rangle X$. Denote $\mathcal{R} \in \text{Sym}(\mathfrak{so}(TM))$ the curvature operator, the symmetric operator defined by $\langle \mathcal{R}(X \wedge Y), Z \wedge V \rangle = \langle R(X, Y)Z, V \rangle$, where the inner product on the left-hand side is the natural inner product on $\mathfrak{so}(TM)$. For a vector $X$ and a subspace $V$ we denote $X \wedge V$ the subspace $\text{Span}(X \wedge Y : Y \in V)$.

Let $F \subseteq M$ be a connected totally geodesic hypersurface. Without loss of generality we can assume that $o \in F$. Moreover, as $M$ is an analytic Riemannian manifold and as $F$ is totally geodesic, hence minimal, it is an analytic submanifold of $M$. Therefore we can (and will) replace $F$ by a small open disc of $F$ containing $o$. Let $\xi$ be a continuous unit vector field normal to $F$. Consider the Gauss image of $F$ defined by $\Gamma(F) = \{dg^{-1}\xi(x) : x \in F, g \in G, g(o) = x\}$. The set of pairs $(x, g) \in F \times G$ such that $g(o) = x$ is (locally) diffeomorphic to $\pi^{-1}F \simeq F \times H$, so $\Gamma(F)$ is the image of a continuous (in fact, analytic) “Gauss map” $\Phi : \pi^{-1}F \to S_o(1)$, where $S_o(1)$ is the unit sphere of $T_oM$. As $H$ is connected, $\Gamma(F)$ is also connected. Moreover, $\Gamma(F)$ is $H$-left-invariant. Then the subspace $D_o = \text{Span}(\Gamma(F)) \subseteq T_oM$ is $H$-left-invariant, as also is its orthogonal complement $D_o^\perp$. Hence we can define two orthogonal complementary $G$-left-invariant distributions $D$ and $D^\perp$ on $M$ such that $D(o) = D_o$ and $D^\perp(o) = D_o^\perp$. Denote $m = \dim D$.

**Lemma 1.** In the above notation we have:

1. The distribution $D^\perp$ is integrable with totally geodesic leaves. The leaf of $D^\perp$ passing through $o$ locally lies in $F$.

2. The distribution $D$ is integrable with totally umbilical leaves.

3. $D \wedge D$ lies in an eigenspace of $\mathcal{R}$, so that there exists $\lambda \in \mathbb{R}$ such that $\mathcal{R}(X \wedge Y) = \lambda X \wedge Y$, for all $X, Y \in D$.

**Proof.** 1. First note that if $X$ is tangent to $D^\perp$ at some point $x \in F$ and $x = g(o)$, then $dg^{-1}X \in D_o^\perp$ (as $D^\perp$ is $G$-left-invariant), hence $dg^{-1}X \perp dg^{-1}\xi(x)$, so $X \perp \xi(x)$. Thus $D^\perp$ is tangent to $F$.

Now let $X, Y$ be two vector fields tangent to $D^\perp$ in a neighbourhood of $o$. They must be tangent to $F$ at the points of $F$. As $F$ is totally geodesic, we have $(\nabla_X Y)o \perp \xi(o)$. Moreover, for any $x \in F$ and any $g \in G$ such that $g(o) = x$, the vector field $dgX$ and $dgY$ are tangent to $D$ and to $F$ (at the points of $F$), so $(\nabla_{dgX} dgY)(x) \perp \xi(x)$, hence $(\nabla_X Y)o \perp dg^{-1}\xi(x)$. It follows that $(\nabla_X Y)o \in D_o^\perp$. As $D^\perp$ is $G$-left-invariant it follows that everywhere on $G$ we have $\nabla_X Y \in D^\perp$, for any vector fields $X, Y \in D^\perp$. Therefore $[D^\perp, D^\perp] \subseteq D^\perp$, and the leaves tangent to $D^\perp$ are totally geodesic submanifolds of $M$.

2. Let $\eta \in \Gamma(F)$ and let $g \in G$ and $x \in F$ be chosen in such a way that $\eta = dg^{-1}\xi(x)$. Let $Z' \in T_xF \cap D(x)$ and let $X' \in D^\perp$ be a vector field in a neighbourhood of $x$. Then $(\nabla_{Z'} X')_x \perp \xi(x)$. Acting by $dg^{-1}$ we obtain that $(\nabla_{Z} X)_o \perp \eta$ for any $Z \in D_o \cap \eta^\perp$ and for any vector field $X \in D$ in a neighbourhood of $o$. It follows that every $\eta \in \Gamma(F)$ is an eigenvector of the linear operator $L_X$ on $D_o$ defined by $\langle LX_1, N_2 \rangle = \langle (\nabla_{Z_1} X)_o, N_2 \rangle$ ($L_X$ is the adjoint to the Nomizu operator of $X$). As $\Gamma(F)$ is a connected subset of the unit sphere of $T_oM$ spanning $D_o$ we obtain that $L_X$ is proportional to the identity, so that the bilinear form on $D_o \times D_o$ defined by $(Z_1, Z_2) \mapsto \langle (\nabla_{Z_1} X)_o, Z_2 \rangle$ vanishes for all $Z_1 \perp Z_2$, $Z_1, Z_2 \in D_o$. Let $N_1, N_2 \in D$ be orthogonal vector fields in a neighbourhood of $o$ (if $m(= \dim D) = 1$, the claim of the assertion is trivial) and let $X \in D^\perp$ be a vector field in a neighbourhood of $o$. At the point $o$ we have $\langle (\nabla_{N_1} N_2, X) = -\langle (\nabla_{N_1} X, N_2) = 0$. It follows that $[N_1, N_2] \in D$ for any two orthogonal vector fields $N_1, N_2 \in D$, hence for any such $N_1, N_2$. Then $D$ is integrable and the second fundamental form of the leaves vanishes on any pair of orthogonal vectors. It follows that the second fundamental form (in every direction from $D^\perp$) is proportional to the induced inner product on $D$, hence the leaves are totally umbilical.
3. Let \( g \in \pi^{-1} F \) with \( x = g(o) \in F \). From the Codazzi equation at \( x \) we have \( \langle R(X)Y,Z,\xi \rangle = 0 \), for all \( X,Y,Z \in T_xF \). From the symmetries of the curvature tensor it follows that \( \langle R(\xi)X,Y,Z \rangle = \langle R(X,Y)\xi,Z \rangle - \langle R(X,Y)\xi,Z \rangle \) for all \( X,Y,Z \in T_xM \), where \( R : T_xM \to T_xM \) is the Jacobi operator defined by \( R_x = R(\xi,X)\xi \). Then \( R(\xi \wedge X) = (R_x)X \wedge \xi \). As \( R_x \) is symmetric, there exists an orthonormal basis \( e_i, i = 1, \ldots, n-1 \), for \( T_xF \) such that \( R(\xi \wedge e_i) = c_i \xi \wedge e_i \), so that the elements \( \xi \wedge e_i \in \mathfrak{so}(T_xM) \) are the eigenvectors of \( R \in \operatorname{Sym}(\mathfrak{so}(T_xM)) \) [Ts2, Proposition 4.7]. Acting by \( dg^{-1} \), we obtain that for every \( g \in \pi^{-1} F \) there is a direct orthogonal decomposition \( T_xM = \mathbb{R}\Phi(g) \oplus \mathbb{R}F(g) \) (where \( \Phi \) is the Gauss map and \( \Phi(g) = \Gamma(F) \)) such that every subspace \( \Phi(g) \land L_g \subset \mathfrak{so}(T_xM) \) lies in the eigenspace of \( R \in \operatorname{Sym}(\mathfrak{so}(T_xM)) \) with the eigenvalue \( \lambda(g) \) (here \( \lambda(g) \)'s are the \( c_i \)'s without repetitions). Let \( \mathfrak{so}(T_oM) = \oplus_{n=1}^N V_n \) be the orthogonal decomposition of \( \mathfrak{so}(T_oM) \) on the eigenspaces of \( R \), with \( \mu_n \) the corresponding eigenvalues. Then every \( \lambda(g) \) equals to one of the constants \( \mu_n \). As the Jacobi operator \( R_x \) depends continuously (in fact, analytically) on \( g \in \pi^{-1} F \) and all its eigenvalues belong to the finite set \( \{ \mu_n \} \) we obtain that the number of eigenvalues \( \rho(g) = p \) is constant and up to relabelling, every subspace \( \Phi(g) \land L_g \) lies in \( V_s \). Moreover, the dimensions \( m_s = \dim L_s(g) \) are constant and the maps \( g \mapsto L_s(g) \) are analytic maps from \( \pi^{-1} F \) to the Grassmanians \( G(m_s,T_oM) \). It follows that for any \( g,h \in \pi^{-1} F \) and for any \( s \neq l \), we have \( \Phi(g) \land L_s(g) \land L_l(h) = 0 \), so \( \Phi(g) \land L_l(g) \land L_s(h) \land L_l(h) \). Therefore \( \Phi(g) \land L_s(g) \Phi(h) \land L_s(h) \land L_l(h) \). Now if \( \Phi(g) \land L_s(g) \Phi(h) \land L_s(h) \), then \( \Phi(g) \land L_s(g) \Phi(h) \land L_s(h) \). The dimension \( m_s \) is the same as the dimension of \( L_s(h) \). So there exists a small enough neighbourhood \( U \subset \pi^{-1} F \) of \( c \) such that for all \( g,h \in U \) and all \( s = 1, \ldots, p \), we have \( \Phi(g) \land L_s(g) \Phi(h) = L_s(h) \), hence \( L_s(h) \subset \mathbb{R}\Phi(g) \oplus L_s(g) \). Let \( N_s = \text{Span}(L_s(h) : h \in U) \). Then dim \( N_s \geq m_s \) as \( \dim L_s(h) = m_s \), and moreover, since \( N_s \subset \mathbb{R}\Phi(g) \oplus L_s(g) \), for all \( g \in U \), we have \( \dim N_s \leq m_s + 1 \). So we have two possibilities: either \( \dim N_s = m_s \), in which case the subspaces \( L_s(h) \) do not depend on \( h \): \( L_s(h) = N_s \), for all \( h \in U \); or \( \dim N_s = m_s + 1 \), in which case the subspaces \( \mathbb{R}\Phi(g) \oplus L_s(g) \) do not depend on \( g \): \( \mathbb{R}\Phi(g) \oplus L_s(g) = N_s \), for all \( g \in U \). But the latter case occurs for no more than one \( s = 1, \ldots, p \). Indeed, if we suppose that \( \mathbb{R}\Phi(g) \oplus L_s(g) = N_s \) and \( \mathbb{R}\Phi(g) \oplus L_l(g) = N_l \), for all \( g \in U \) and for some \( s \neq l \), then, as \( \Phi(g) \land L_s(g) \land L_l(g) \) are mutually orthogonal, we obtain \( \mathbb{R}\Phi(g) = N_s \land N_l \). It follows that \( \Phi(g) \) is constant, and so the subspaces \( L_s(g) = N_s \land (\Phi(g)) \) do not depend on \( g \). But then \( L_s(g) = \text{Span}(L_s(h) : h \in U) = N_s \), so \( \dim N_s = m_s \), a contradiction. So \( \dim N_s = m_s + 1 \) for no more than one \( s = 1, \ldots, p \), and \( \dim N_s \) is \( m_s \) for all the other \( s \).

Now if \( \dim N_s = m_s \) for all \( s = 1, \ldots, p \), then the vector \( \Phi(g) \) and all the subspaces \( L_s(g) \) are constant: \( \Phi(g) = \Phi \) and \( L_s(g) = N_s \), for all \( g \in U \), hence for all \( g \in \pi^{-1} F \), by analyticity. Then the distribution \( D \) is one dimensional and the claim follows trivially.

Otherwise, suppose that \( \dim N_1 = m_1 + 1 \). Then again \( L_s(g) = N_s \), for all \( g \in \pi^{-1} F \) and for all \( s \geq 2 \). We also have \( D_o = \text{Span}(\Phi(g) : g \in \pi^{-1} F) = \text{Span}(\Phi(g) : g \in U) \) by analyticity, so \( D_o \subset N_1 \). But for any \( g \in \pi^{-1} F \), we have \( \Phi(g) \land E_1(g) = \Phi(g) \land N_1 \subset V_1 \) and \( V_1 \) is the eigenspace of \( R \) with the eigenvalue \( \lambda_1 \). It follows that \( \Phi(g) \land D_o \subset V_1 \), for all \( g \in \pi^{-1} F \), hence \( D_o \land D_o \subset V_1 \), as required. \( \square \)

**Proof of Theorem 1.** Let \( u \in D^1 \) be the mean curvature vector field of the totally umbilical foliation on \( M \) defined by \( D \). As \( D \) is \( G \)-left-invariant, \( u \) is also \( G \)-left-invariant.

We consider two cases for \( m = \dim D \).

**Case 1.** Suppose that \( m (= \dim D) \geq 1 \). Then from Codazzi equation and from Lemma 1(3) we obtain that the \( D^1 \) component of the vector field \( (Z_1, Z_3) \nabla Z_2 \nu - (Z_2, Z_3) \nabla Z_1 \nu \) vanishes, for any \( Z_1, Z_2 \) and \( Z_3 \) tangent to \( D \). It follows that \( \nabla Z_2 \nu \) tangent to \( D \), for any \( Z \) tangent to \( D \), hence the leaves of the foliation defined by \( D \) are extrinsic spheres.

We can introduce analytic local coordinates \( u^1, \ldots, u^m, u^{m+1}, \ldots, u^{n-m} \) in a neighbourhood of any point \( x \in M \) in such a way that \( D = \text{Span}(\partial/\partial u^\alpha : \alpha = 1, \ldots, m), D^1 = \text{Span}(\partial/\partial u^i : i = 1, \ldots, n-m) \). The metric of \( M \) is given by \( ds^2 = A_{ij}(u,v)du^i du^j + B_{ij}(u,v)du^i dv^j \). As \( D^1 \) is totally geodesic, we obtain that \( B_{ij} = B_{ij}(u) \). From the fact that \( D \) is totally umbilical we get \( A_{ij}(u,v) = f(u,v)A_{ij}(v) \) for some positive analytic function \( f \). Then \( \nu = -\frac{1}{2} B_{ij} \partial ln f/\partial u^i \partial/\partial u^j \) and the fact that the leaves tangent to \( D \) are extrinsic spheres gives \( \partial^2 ln f/\partial u^i \partial/\partial u^j = 0 \). It follows that \( f \) is a product of a function of the \( u \)'s by a function of the \( v \)'s, so (with a slight change of notation) \( ds^2 = f(u)A_{ij}(v)du^i du^j + B_{ij}(u)du^i dv^j \), hence \( M \) is locally a warped product. Moreover, from Gauss equation and from Lemma 1(3), every leaf...
Suppose that $\chi$ is a Killing vector field at a single point. This gives Case (a) of Theorem 1.

Now suppose that $f$ is not a constant. From the above and by [BH, Theorem A] the manifold $M$ is a global warped product, $M = \mathbb{R}^m \times M_2$, where $M_2$ is the leaf tangent to $D^\perp$ passing through $o$ and $f : M_2 \to \mathbb{R}^+$. The isotropy subgroup $G_2 \subset G$ of $M_2$ acts transitively and isometrically on $M_2$, so $M_2$ is the homogeneous space $G_2/H$. Moreover, every $g \in G_2$ acts on the $\mathbb{R}^m$ fibers by the homothecy with the coefficient $f(g)$. As this ratio must not depend on $u \in M_2$ we obtain that $f(g) = \chi(g)$, where $\chi : G_2 \to (\mathbb{R}^+, \cdot)$ is a homomorphism with $\chi(H) = 1$. Let $\mathbb{R}^{m-1}$ be the hyperplane of $\mathbb{R}^m$ passing through $o$ and orthogonal to $\xi(o)$. Then the hypersurface $F \subset M_1$, the (Cartesian) product of $\mathbb{R}^{m-1}$ and $M_2$ is totally geodesic and is (the unique complete) totally geodesic hypersurface of $M$ whose normal vector at $o$ is $\xi(o)$. This gives Case (b) of Theorem 1.

Case 2. Suppose that $m = \dim D = 1$. Let $\tau$ be a unit vector field on $M$ which spans $D$ (so that $\gamma(\tau(o)) = \xi(o)$). By construction, $\tau$ is $G$-left-invariant. Moreover, from [BH, Theorem A] the manifold $M$ is diffeomorphic to $\mathbb{R} \times M_2$, where $M_2$ is the leaf of $D^\perp$ passing through $o$. The leaf $M_2$ is a totally geodesic hypersurface and is an open connected subset of $M_2$. Let $G_1 \subset G$ be the connected isotropy subgroup of $M_2$. Then $H \subset G_1$ and $G_1$ has codimension one in $G$. It follows that $F$ is a homogeneous totally geodesic hypersurface.

Now, if the vector field $\tau$ is geodesic, then we get back to case Case (a), with $M_1(c)$ a Euclidean line. Furthermore, if $\tau$ is not geodesic, but the leaves of $D$ are “circles” (one-dimensional extrinsic spheres), that is, if $\nabla_\tau \nu = -\|
u\|^2 \tau$, then repeating the above arguments we get to Case (b), with $m = 1$.

Suppose that $\nabla_\tau \nu \not\parallel \tau$. Consider the Frenet frame $\tau, \nu_1, \nu_2, \ldots$ of the one-dimensional leaves of $D$. We have $\nabla_\tau \tau = k_1 \nu_1 (= \nu)$, $\nabla_\tau \nu_1 = -k_1 \tau + k_2 \nu_2$. By the $G$-left-invariance, all the Frenet curvatures $k_1, k_2, \ldots$ are constant; by our assumption, at least the first two of them, $k_1$ and $k_2$, are nonzero.

Passing to the level of Lie algebras, we need to exercise a certain caution, as the standard identification procedure is carried out via Killing vector fields, however the vector fields $\tau, \nu_1, \nu_2, \ldots$ are not in general Killing. Denote their values at $o$ by the corresponding Roman letters, so that $T = \tau(o)$, $N_1 = \nu_1(o)$, $N_2 = \nu_2(o)$, etc. Note that the spans of the vectors $T, N_1, N_2, \ldots$ are one-dimensional $H$-submodules of $\mathfrak{t}_o M$. Let $\mathfrak{g}, \mathfrak{g}_1$ and $\mathfrak{h}$ be the Lie algebras of $G, G_1$ and $H$ respectively. We have $\mathfrak{h} \subset \mathfrak{g}_1 \subset \mathfrak{g}$, with $\mathfrak{g}_1$ a subalgebra of codimension one in $\mathfrak{g}$. Choose and fix an $\mathfrak{ad}(H)$-invariant complement $\mathfrak{f}$ to $\mathfrak{h}$ in $\mathfrak{g}_1$. The corresponding Killing vector fields are tangent to $D^\perp$, and we can identify $f$ with $D^\perp(o)$. As the inner product is $\mathfrak{ad}(H)$-invariant, we can find a one-dimensional $\mathfrak{ad}(H)$-invariant complement to $\mathfrak{g}_1$ in $\mathfrak{g}$ spanned by an element whose corresponding Killing vector field at $o$ equals $T$. We can identify that element with $T$, and the space $\mathfrak{m} = \mathbb{R}T \oplus f \subset \mathfrak{g}$, with $\mathfrak{t}_o M$. Then we obtain

$$\mathfrak{m} = \mathbb{R}T \oplus f, \quad T \perp f, \quad \mathfrak{g}_1 = f \oplus \mathfrak{h}, \quad [T, h] = 0, \quad [h, f] \subset f, \quad [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_1.$$

We have the following lemma:

**Lemma 2.**

(a) The leaves of $D$ are helices of order two: their first and second Frenet curvatures are nonzero constants, and the third Frenet curvature is zero.

(b) Denote $I = \text{Span}(T, N_1, N_2)$, $S = \text{Span}(N_1, N_2)$ and $\mathcal{J} = (\mathfrak{m} \cap I^\perp) \oplus \mathfrak{h}$. Then

(i) $\mathcal{J}$ is an ideal of $\mathfrak{g}$ containing $\mathfrak{h}$, with $\mathfrak{g}/\mathfrak{J} \simeq \mathfrak{sl}(2)$.

(ii) $\mathfrak{g}_1 = S \oplus \mathcal{J}$.

(iii) $[I, \mathcal{J}] = 0$. 

(c) The subspace \( I \subset \mathfrak{g} \), with the induced inner product and with the Lie algebra structure of \( \mathfrak{g}/I \), is isometrically isomorphic to \( \mathfrak{s}(2) \) with the metric (1), with \( \mathfrak{s} \subset I \) the totally geodesic solvable subalgebra defined by \( \mathfrak{s} = \text{Span}(E_2, E_3) \).

**Proof.** (a and b) From the fact that \( \mathfrak{f} \) is tangent to a totally geodesic hypersurface (and that \( \mathfrak{g}_1 \subset \mathfrak{g} \) is a subalgebra) we obtain that for all \( X, Y \in \mathfrak{f} \),

\[
\langle [T, X]_m, Y \rangle + \langle [T, Y]_m, X \rangle = 0,
\]

where the subscript \( m \) denotes the \( m \)-component. To compute the Frenet frame we shall use the following fact: if \( \tilde{Y} \) and \( \tilde{Z} \) are \( G \)-left-invariant vector fields on \( M \) with \( \tilde{Y}(o) = Y \), \( \tilde{Z}(o) = Z \), then

\[
\langle \nabla_{\tilde{Y}} \tilde{Z}(o) \rangle = \frac{1}{2}[Y, Z]_m + U(Y, Z),
\]

where \( U(Y, Z) \) is \( \mathfrak{g} \)-isometrized to \( \mathfrak{g} \) and \( \langle \nabla_{\tilde{Y}} \tilde{Z}(o) \rangle = \langle U(Y, Z) \rangle \) for all \( X \in \mathfrak{m} \). Note that the first term on the right-hand side of (4) differs by the sign from that in the standard formula for the covariant derivative of Killing vector fields (e.g. [Bes, Proposition 7.28]); equation (4) easily follows from that formula and the fact that the Lie bracket of a \((G\text{-})\)Killing vector field and a \( G \)-left-invariant vector field (as of vector fields on \( M \)) vanishes.

The elements \( N_1, N_2, \ldots \) of the Frenet frame at \( o \) are orthonormal unit vectors in \( \mathfrak{f} \). From (3), (4) and the fact that \( k_1 \nu_1 = \nabla_{\nu} \tau \) we have

\[
\langle [T, [X, T]]_m \rangle = k_1 \langle N_1, X \rangle,
\]

for all \( X \in \mathfrak{m} \). From the Frenet equations we have \( \nu_2 = k_2^{-1}k_1 \tau + k_2^{-1}\nabla_{\nu_1} \tau \). Then for any \( X \in \mathfrak{f} \), we obtain \( \langle N_2, X \rangle = \frac{1}{k_2}k_1^{-1}(\langle [T, N_1]_m, X \rangle + \langle [T, [X, N_1]]_m \rangle + \langle N_1, [X, T]_m \rangle) = k_2^{-1}(\langle [T, N_1]_m, X \rangle \) since \( \mathfrak{g}_1 \) is a subalgebra and by (2), (3), (4). As \( \langle \tau, \nu_2 \rangle = 0 \) we get from (5)

\[
N_2 = k_2^{-1}[\langle N_1, X \rangle] + k_2^{-1}k_1 T.
\]

By a classical result [Lie, T, Hof], a subalgebra \( \mathfrak{g}_1 \subset \mathfrak{g} \) of codimension one must contain the kernel \( i \) of a homomorphism from \( \mathfrak{g} \) to \( \mathfrak{s}(2) \). Denote \( i_i \) the (linear) projection of \( i \) to \( \mathfrak{m} \). As \( i_i \subset \mathfrak{g}_1 \), we have \( \nu_i \subset \mathfrak{f} \), so \( T \subseteq i_i \). Moreover, since \( i_i \subset i + \mathfrak{h} \) we get by (2) \( T, i_i \subset [T, i] \subset i \), as is an ideal. It follows that \( T, i_i \subset i_i \subset i \). From the above, \( T \subseteq i_i \) and \( N_i \not\supseteq T \).

Therefore the codimension of \( i \) in \( \mathfrak{g} \) is at least three (since \( i \subset i_i \subset \mathfrak{m} \), hence it exactly three, with \( \mathfrak{g}/i \cong \mathfrak{s}(2) \) and \( i_i \subset \mathfrak{m} \)). In particular, \( \mathfrak{h} \subset i_i \). This proves assertion (b)(i), with \( \mathfrak{J} = i_i \) and assertion (b)(ii) in view of (2).

Now from (3), (4) we obtain \( \langle \nabla_{\nu_2} \rangle = \langle -[T, X]_m, N_2 \rangle - \frac{1}{2} \langle [N_2, X]_m, T \rangle \) for all \( X \in \mathfrak{m} \). The right-hand side vanishes for all \( X \in i_i \), and also for \( X = N_2 \) and \( X = T \) from (5) or from Frenet equations. It follows that \( \langle \nabla_{\nu_2} \rangle = -k_2 N_1 \), hence \( \nabla_{\nu_2} \rangle = -k_2 \nu_1 \), which proves assertion (a).

Note that for any \( X, Y \in \mathfrak{m} \) with \( [X, h] = [Y, h] = 0 \) we have \( \nabla_X Y, h] = 0 \). Indeed, for an orthonormal basis \( e_i \) for \( \mathfrak{m} \) we have \( \nabla_X Y = \frac{1}{2} \sum_i (\langle [X, Y]_m, e_i \rangle + \langle X, [e_i, Y]_m \rangle + \langle Y, [e_i, X]_m \rangle) e_i \), so \( \langle Z, \nabla_X Y \rangle = \frac{1}{2}[Z, [X, Y]_m] + \frac{1}{2} \sum_i (\langle X, [e_i, Y]_m \rangle + \langle Y, [e_i, X]_m \rangle) \langle Z, e_i \rangle \). For the first term on the right-hand side we have: \( [Z, [X, Y]_m] = [Z, [X, Y]_m] = 0 \), where the first equality follows from the fact that both \( \mathfrak{m} \) and \( \mathfrak{h} \) are \( ad_h \)-invariant, and the second, from the Jacobi identity. As \( [Z, \mathfrak{m}] \subset \mathfrak{m} \) and as \( ad_z \) is skew-symmetric on \( \mathfrak{m} \), we obtain for the second term on the right-hand side (the third one is treated similarly): \( \sum_i X, [e_i, Y]_m \langle Z, e_i \rangle = \sum_i X, [e_i, Y]_m \langle Z, e_i \rangle = -\sum_i X, [e_i, Y]_m \langle Z, e_i \rangle = \langle Y, [e_i, X]_m \rangle \rangle \langle Z, e_i \rangle \rangle = \langle Y, [e_i, X]_m \rangle \rangle \langle Z, e_i \rangle \rangle \rangle = -\sum_i \langle X, [e_i, Y]_m \rangle \langle Z, e_i \rangle \rangle \rangle = \langle -\langle [e_i, Y]_m, X, Z \rangle, h] \rangle \rangle = 0 \). So \( \nabla_X Y, h] = 0 \).

Therefore, as \( T \) commutes with \( h \), the vectors \( N_1 \) and \( N_2 \) also do. This proves assertion (b)(iii).
isometric isomorphism between the metric Lie algebras $I$ and $\mathfrak{sl}(2)$, with the inner product (1), is given by the correspondence $T = E_1$, $N_1 = -E_3$, $N_2 = E - 2$ and $k_1 = 2b$, $k_2 = 2a$.

As the leaves tangent to $D$ are congruent helices of the second order, we will use a more conventional notation for their curvature and torsion: $k = k_1$ and $\kappa = k_2$ respectively.

We can now introduce analytic local coordinates $t, u^1, \ldots, u^{n-1}$ in a neighbourhood of any point $x \in M$ in such a way that $D = \text{Span}(\partial/\partial t)$, $D^\perp = \text{Span}(\partial/\partial u^i : i = 1, \ldots, n - 1)$, $t(x) = u_1^i(x) = 0$, and the leaf of $D$ passing through $x$ is parametrised by the arclength. Then the metric of $M$ in a neighbourhood of $x$ is given by $ds^2 = e^{2\phi(u,t)}dt^2 + B_{ij}(u,t)du^i du^j$, where $\phi$ is an analytic function with $\phi(0,t) = 0$ and $B$ is analytic and positively definite. We have

$$
\begin{align*}
\tau &= e^{-\phi} \frac{\partial}{\partial t}, \quad k\nu_1 = \nabla_\tau r = -B^{ij} \frac{\partial \phi}{\partial u^i} \frac{\partial}{\partial u^j}, \quad k^2 = B^{ij} \frac{\partial \phi}{\partial u^i} \frac{\partial \phi}{\partial u^j} = \text{const} \neq 0, \\
k\kappa\nu_2 &= \nabla_\tau (k\nu_1) + k^2 \tau = -e^{-\phi}B^{ij} \frac{\partial^2 \phi}{\partial u^i \partial u^j} \frac{\partial}{\partial u^j}, \\
\nabla_\tau (k\kappa\nu_2) &= -e^{-\phi}B^{ij} \frac{\partial}{\partial t} \left( e^{-\phi} \frac{\partial^2 \phi}{\partial u^i \partial u^j} \right) \frac{\partial}{\partial u^j}.
\end{align*}
$$

(7)

As the leaves tangent to $D$ are helices of order two, we have from Frenet equations: $\nabla_\tau (k\kappa\nu_2) = -k\kappa^2 \nu_1$, so by (7), $\frac{\partial}{\partial t} (e^{-\phi} \frac{\partial^2 \phi}{\partial u^i \partial u^j}) = -k^2 e^\phi \frac{\partial \phi}{\partial u^i}$, for all $i = 1, \ldots, n - 1$, which gives $\frac{\partial}{\partial t} \left( \frac{\partial^2 \phi}{\partial u^i \partial u^j} - \frac{1}{2} \frac{\partial \phi}{\partial u^i} \frac{\partial \phi}{\partial u^j} \right) + \frac{1}{2} k^2 e^\phi = 0$. It follows that $\frac{\partial^2 \phi}{\partial u^i \partial u^j} - \frac{1}{2} \frac{\partial \phi}{\partial u^i} \frac{\partial \phi}{\partial u^j} + \frac{1}{2} k^2 e^\phi = h(t)$, for some analytic function $h$. But $\phi(0,t) = 0$, so $h(t) = \frac{1}{2} k^2$. This gives the equation $\frac{\partial}{\partial t} \left( e^{-\phi} \left( \frac{\partial^2 \phi}{\partial u^i \partial u^j} + k^2 e^\phi - e^\phi \right) \right) = 0$. Solving this equation we get $e^{-\phi} = \sinh(\alpha(u)) \cos(st + \beta(u)) + \cosh(\alpha(u))$ for some analytic functions $\alpha, \beta$ on a neighbourhood of $x \in M_2$. This gives the required expression for the twisting function in Case (c) of Theorem 1. A direct calculation using the fact that $k^2 = B^{ij} \frac{\partial \phi}{\partial u^i} \frac{\partial \phi}{\partial u^j}$ from (7) shows that $\|\nabla \phi\|^2 = \sinh(\alpha)^2 \|\nabla \nu\|^2 = k^2$, as required.

As we can see from the proof, in Case (c), the subgroup $G_1 \subset G$ acts transitively on $F$, so $F$ is a homogeneous totally geodesic hypersurface. To some surprise, there exist non-homogeneous totally geodesic hypersurfaces belonging to Case (b), as the following example shows.

Example 2. Consider the metric solvable Lie algebra with an orthonormal basis $Z, X_1, X_2, Y$ and with the nonzero brackets $[Z, X_1] = X_1 + X_2$, $[Z, X_2] = -X_1 + X_2$, $[Z, Y] = 2Y$. The corresponding left-invariant metric on $\mathbb{R}^4$ is given by $ds^2 = dx_1^2 + e^{2y}dx_2^2 + e^{2z}(dx_3^2 + dx_4^2)$, with $Z = -\frac{\partial}{\partial x_1}$, $Y = e^{2x} \frac{\partial}{\partial x_2}$, $X_1 = e^{-z} \cos z \frac{\partial}{\partial x_1} - e^{-z} \sin z \frac{\partial}{\partial x_2}$, $X_2 = e^{-z} \sin z \frac{\partial}{\partial x_1} + e^{-z} \cos z \frac{\partial}{\partial x_2}$. The resulting homogeneous space indeed belongs to Case (b), with $M_2 = \{x_1 = x_2 = 0\}$, the hyperbolic space. Moreover, the hypersurface $F = \{x_1 = 0\}$ is totally geodesic, but $\text{Span}(Z, Y, X_2)$ is not a subalgebra.

Proof of Theorem 2. Suppose the pair $(M = G/H, F)$ belongs to Case (c) of Theorem 1. In the notation of Lemma 2, let $N$ be the connected (normal) subgroup of $G$ tangent to the ideal $\mathfrak{g}$. Then by Lemma 2 (b) (i) $N \supset H$ and $G/N = \mathbb{SL}(2)$ (as the Lie group). Moreover, from Lemma 2 (c) it follows that the projection $\pi : M \to \mathbb{SL}(2)$ (defined by $\pi(gH) = gN$ for $g \in G$), where $\mathbb{SL}(2)$ is equipped with the left-invariant metric defined by the inner product (1) (with some specific choice of the constants $a$ and $b$), is a Riemannian submersion. Then the projection of $F$ to $\mathbb{SL}(2)$ is the connected Lie subgroup $G_1/N$ whose Lie algebra is spanned by $\mathfrak{g}$.

From Theorem 2 (or from Lemma 2) it follows that Case (c) of Theorem 1 may only occur if the semisimple part of the Levi-Mal’cev decomposition of the Lie algebra of (any) transitive group of isometries of $M$ contains an ideal isomorphic to $\mathfrak{sl}(2)$.

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