Robustness of the Entangled States $2 \times N \times M$ Against Qubit Loss

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Abstract

A quantum system may suffer from local noises in experiment, and be disturbed partially or completely. In practice, sometimes the multipartite quantum system lose some parts due to local noises, and in this case it is interesting to know whether the residual system still carry information or not. Robust quantum states have the ability to carry on information after the loss of some partites. In this paper, we study $2 \times N \times M$ class of quantum states and identify all robust and fragile states with respect to loss of qubit. We also show the robustness of the $2 \times N \times M$ robust quantum system after the loss of qubit.

1 Introduction

Entanglement [1] is a key feature of quantum theory, describes a quantum correlation that exhibits nonlocal properties. Due to this phenomenon, a physical system has to consist of subsystems and it is not possible to describe properties of individual subsystems even though the pure quantum state of entire system is completely known. During past decades, intensive research undertaken for better understanding of it's theoretical complexity as well as potential for innovative experimental protocols.

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Experimental investigations of entangled states using quantum dots \cite{2}, nuclear spins \cite{3}, electronic spins \cite{4}, optical lattice \cite{5}, and Josephson junctions \cite{6} show that noise factor cannot be neglected that can partially or completely disturb the experiments. Keeping this in mind, Vidal and Tarrach \cite{7} introduced the concept of robustness of entanglement as a measure to quantify how durable an entangled state is in the presence of local noise. The minimal amount of a separable state $\rho_s$ that can wipe out all the entanglement contained in state $\rho$ gives us relative robustness \cite{7, 8}. But entanglement robustness also defined with respect to partial loss \cite{10, 11}. The concept of entanglement robustness against particle loss makes only sense for a system of at least three particles. A tripartite quantum system that retains entanglement after the loss of one particle is called robust system otherwise called fragile. Maximally entangled states violate Bell inequality \cite{12} maximally and are also maximally fragile \cite{13}. One of the standards three qubits state GHZ is maximally entangled state \cite{29} that’s why if one qubit is traced out the remaining reduced state is completely unentangled. So GHZ is a fragile state. While the three qubit W state shows completely opposite behavior under disposal of any one of the three qubit and is called robust quantum state. Any carelessness by the holder of any one of the qubits, results in an uncontrolled decoherence of that qubit, does not completely destroy the entanglement of the remaining qubits of W state. Thus W state still can carry the information after the removal of one qubit. When a particle leaves tripartite compound system, we get a bipartite mixed state. If a tripartite system is fragile then reduced bipartite state can be written as a convex combination of product states

$$\rho_{AB} = \sum_{i=1}^{L} p_i \rho_i^{(A)} \otimes \rho_i^{(B)},$$

(1)

here $p_i > 0$ with $\sum_{i=1}^{L} p_i = 1$, and $\rho_i^{(A)}$ and $\rho_i^{(B)}$ are local density matrices of the particles $A$ and $B$.

The main aim of this work is to identify robust and fragile quantum states against loss of qubit in a class of $2 \times N \times M$ quantum states and to quantify the the robustness of robust quantum states. Throughout the paper the space dimensions $N$ and $M$ are finite.
2 Tripartite $2 \times N \times M$ Quantum state

By adopting the conventions of [15, 16], an arbitrary state of $2 \times N \times M$ can be written as

$$\left| \Psi_{2 \times N \times M} \right> = \sum_{i,j,k} \gamma_{ijk} |i\rangle_0 |j\rangle_1 |k\rangle_2 ,$$

(2)

here $\gamma_{ijk} \in \mathbb{C}$, are a series of complex numbers called coefficients of quantum state, $\psi_0$ represents the first qubit, $\psi_1$ and $\psi_2$ has the dimension of $N$ and $M$ separately (we assume $N \leq M$ without loss of generalities). By using the coefficients of quantum state we can construct the matrix form of $2 \times N \times M$ state

$$\left( \begin{array}{cccc} \gamma_{111} & \gamma_{112} & \cdots & \gamma_{11M} \\ \gamma_{121} & \gamma_{122} & \cdots & \gamma_{12M} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1M1} & \gamma_{1M2} & \cdots & \gamma_{1NM} \\ \gamma_{211} & \gamma_{212} & \cdots & \gamma_{21M} \\ \gamma_{221} & \gamma_{222} & \cdots & \gamma_{22M} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{2M1} & \gamma_{2M2} & \cdots & \gamma_{2NM} \end{array} \right) = \left( \begin{array}{c} \Gamma_1 \\ \Gamma_2 \end{array} \right) ,$$

(3)

Matrix form of state $\Psi_{2 \times N \times M}$ has two submatrices, $\Gamma_{1(j,k)}$ and $\Gamma_{2(j,k)}$ are $N \times M$ complex matrices. To proceed further, we require the vec operation [17] for our tripartite state

$$\mathcal{V}(\Psi_{2 \times N \times M}) \equiv (\gamma_{111} \cdots \gamma_{1M1} \cdots \gamma_{11M} \cdots \gamma_{1MN} \gamma_{211} \cdots \gamma_{2M1} \cdots \gamma_{21M} \cdots \gamma_{2NM})^T ,$$

(4)

which is a way of turning matrices into vector by “stacking” the columns, here $T$ denotes the transpose. Now we have transformed a high rank tensor to a low rank tensor. $\mathcal{V}(\Psi_{2 \times N \times M})$ has dimension $2NM \times 1$, so we can easily construct the density operator as

$$\rho = \left| \Psi_{2 \times N \times M} \right> \left< \Psi_{2 \times N \times M} \right| .$$

(5)

We can also write the density matrix in the submatrices form
\[ \rho = \begin{pmatrix} \Gamma_1 \otimes \Gamma_1^\dagger & \Gamma_1 \otimes \Gamma_2^\dagger \\ \Gamma_2 \otimes \Gamma_1^\dagger & \Gamma_2 \otimes \Gamma_2^\dagger \end{pmatrix}, \] (6)

here every \( \Gamma_i, \Gamma_i^\dagger \) has dimension \( NM \times NM \), \( 1 \leq i \leq 2 \) and the dimension of \( \rho \) is \( 2NM \times 2NM \).

### 2.1 Quantum States with Respect to loss of Qubit

In a general tripartite state if any of the particles decided not to cooperate with other two particles, the residual state, \( \rho_{XY} \) is a mixed state and is defined as \( \rho_{\neg i} = \text{Tr}_{\psi_i}[|\Psi\rangle\langle \Psi|] \), here \( i \in \{0, 1, 2\} \). An entangled state \( |\Psi\rangle \) is said to be fragile (resp. robust) with respect to the loss of any particle if \( \rho_{\neg i} \) is separable (resp. entangled) [18]. Here we would like to investigate the robustness of the entanglement of a tripartite state \( |\Psi\rangle_{2 \times N \times M} \) against the disposal of qubit. If the qubit decide to leave the tripartite state, we get a mixed state \( \rho_{AB} \). In the case of mixed states we can characterize fragility or robustness from the Bloch representation point of view as follows [19, 20],

\[
\rho_{AB} = \frac{1}{NM} \mathbb{1} \otimes \mathbb{1} + \frac{1}{2M} \vec{a} \cdot \vec{\lambda} \otimes \mathbb{1} + \frac{1}{2N} \mathbb{1} \otimes \vec{b} \cdot \vec{\sigma} + \frac{1}{4} \sum_{\mu=1}^{N^2-1} \sum_{\nu=1}^{M^2-1} T_{\mu\nu} \lambda_\mu \otimes \sigma_\nu. \tag{7}
\]

Here \( \text{Tr}[\rho_{AB}^2] < 1 \), \( \mathbb{1} \) is the identity matrix, \( \vec{a} \) and \( \vec{b} \) have the components of \( a_\mu = \text{Tr}[\rho_{AB} (\lambda_\mu \otimes \mathbb{1})] \) and \( b_\nu = \text{Tr}[\rho_{AB} (\mathbb{1} \otimes \sigma_\nu)] \), and the correlation matrix \( T_{\mu\nu} = \text{Tr}[\rho_{AB} (\lambda_\mu \otimes \sigma_\nu)] \). The vector \( \vec{\lambda} \) in equation (7) is defined to be \( \vec{\lambda} \equiv (\lambda_1, \ldots, \lambda_{N^2-1})^T \) with \( \lambda_\mu \) being the generators of \( \text{SU}(N) \), and \( \vec{\sigma} \) is defined similarly with \( \sigma_\nu \) being the generators of \( \text{SU}(M) \).

For example, in case of qubits the three generators of \( \text{SU}(2) \) are Pauli matrices and for \( N = 3 \) in case of qutrit generators are the eight Gell-Mann matrices. The local density matrices for particles \( A \) and \( B \) are obtained as

\[
\rho_A = \text{Tr}_B[\rho_{AB}] = \frac{1}{N} \mathbb{1} + \frac{1}{2} \vec{a} \cdot \vec{\lambda}, \quad \rho_B = \text{Tr}_A[\rho_{AB}] = \frac{1}{M} \mathbb{1} + \frac{1}{2} \vec{b} \cdot \vec{\sigma}, \tag{8}
\]

here \( \vec{a} \) and \( \vec{b} \) are called the Bloch vectors of the local density matrices of the one-particle quantum states. As reduced density matrices are Hermitian so can be unitarily diagonal-
\[
\rho'_A = U_A \rho_A U_A^\dagger = \text{diag}\{\lambda_1^{(A)}, \ldots, \lambda_n^{(A)}, 0, \ldots, 0\},
\]
\[
\rho'_B = U_B \rho_B U_B^\dagger = \text{diag}\{\lambda_1^{(B)}, \ldots, \lambda_m^{(B)}, 0, \ldots, 0\},
\]
here \(\lambda_i^{(A)}\) and \(\lambda_i^{(B)}\) are positive real numbers and \(n, m\) are local ranks that may not be full local ranks of \(n < N\) and \(n < M\).

**Theorem 2.1** All tripartite quantum states \(|\psi_{2\times N\times M}\rangle\) are robust with respect to loss of qubit if their bipartite quantum states \(\rho_{AB}\) with local ranks \(n < N\) and \(m < M\) are not reducible \(n \times m\) states with full local ranks.

**Proof:** If a quantum state \(|\psi_{2\times N\times M}\rangle\) is robust against loss of qubit then it’s bipartite reduced density matrix \(\rho_{AB}\) should must maintain entanglement. Suppose \(|\psi\rangle_{2\times N\times M}\) is fragile then separable \(\rho_{AB} = \sum_i p_i \rho_i^{(A)} \otimes \rho_i^{(B)}\), and
\[
\rho_A = \sum_i p_i \rho_i^{(A)}, \quad \rho_B = \sum_i p_i \rho_i^{(B)}.
\]
Here \(p_i > 0\) with \(\sum_i p_i = 1\); \(\rho_A\), \(\rho_i^{(A)}\), \(\rho_B\), and \(\rho_i^{(B)}\) are all positive semidefinite matrices. According to equation (9), we notice that \(\rho_i^{(A)}\) can only take the following form
\[
\rho_i^{(A)} = \begin{pmatrix} X_{n \times n} & 0 \\ 0 & 0 \end{pmatrix} \quad N \times N.
\]
This is because the diagonal elements of positive semidefinite matrices must be nonnegative, that is \((\rho_i^{(A)})_{kk} = 0\) for \(k > n\). Furthermore, from the row and column inclusion properties we have: if \((\rho_i^{(A)})_{kk} = 0\), then \((\rho_i^{(A)})_{lk} = (\rho_i^{(A)})_{kl} = 0\) for all \(l \in \{1, \ldots, N\}\). Hence, the Bloch vectors \(\vec{r}_i\) of \(\rho_i^{(A)}\) are
\[
\rho_i^{(A)} = \left(\frac{1}{n} \mathbf{1} + \frac{1}{2} \sum_{\mu=1}^{n-1} r_{i\mu} \lambda_{\mu}\right)_{n \times n} \oplus 0_{(N-n) \times (N-n)},
\]
here \(r_{i\mu}\) are components of \(\vec{r}_i\) which lies in a Bloch vector space of \(\text{SU}(n) \subset \text{SU}(N)\). Similar arguments apply to \(\rho_i^{(B)}\) as well. That means, if bipartite reduced density state \(\rho_{AB}\) is separable then \(\mu \leq n^2 - 1\) and \(\nu \leq m^2 - 1\). So entangled \(\rho_{AB}\) cannot be reduced according to local ranks, \(n\) and \(m\). This complete the proof. Q.E.D.
Now we have to consider the robustness against qubit loss of those tripartite states have $N \times M$ bipartite states reducible to $n \times m$ states with full local ranks. The full local ranks states could be further transformed into normal form \[23\] with maximally mixed subsystems, where normal form is only entangled when the original bipartite form is entangled otherwise is separable. The normal form of bipartite state $\rho_{AB}$ is expressed as

$$
\rho_{AB} \mapsto \tilde{\rho}_{AB} = \frac{1}{NM} \mathbf{1} \otimes \mathbf{1} + \frac{1}{4} \sum_{\mu=1}^{N^2-1} \sum_{\nu=1}^{M^2-1} \tilde{T}_{\mu\nu} \lambda_\mu \otimes \sigma_\nu
$$

Hereafter in this paper, reduced density operators $\rho_{AB}$ are assumed to be in their normal forms. Note that in the literature there are studies about the normal form in the separability problem \[25, 26\]. Reformulating Eq. (1) in term of Bloch representation of $\rho_i^{(A)} = \frac{1}{N} \mathbf{1} + \frac{1}{2} \vec{r}_i \cdot \vec{\lambda}_i$, $\rho_j^{(B)} = \frac{1}{M} \mathbf{1} + \frac{1}{2} \vec{s}_j \cdot \vec{\sigma}_j$, and comparing with Eq. (7), it turns to

$$
\sum_i p_i \vec{r}_i = \vec{a} , \quad \sum_j p_j \vec{s}_j = \vec{b} , \quad \sum_{k=1}^{n} p_k \vec{r}_k \vec{s}_k^T = \mathbf{T} ,
$$

where the subscripts in $\vec{r}_i$, $\vec{s}_j$ label different Bloch vectors rather the components of them, and the correlation matrix $\mathbf{T}$ has the matrix elements of $\mathbf{T}_{\mu\nu}$. As, after the loss of qubit the bipartite reduced density operators $\rho_{AB}$ are implied to be in their normal form, so we have

Observation 1 Let $\vec{r}_i$, $\vec{s}_j$ be Bloch vectors of density matrices and $\vec{p} = (p_1, p_2, \cdots, p_L)^T$, we may define two matrices $M_r \equiv M_r \mathbf{D}_p^{\frac{1}{2}}$ and $M_s \equiv M_s \mathbf{D}_p^{\frac{1}{2}}$, where $M_r = \{ \vec{r}_1, \vec{r}_2, \cdots, \vec{r}_L \}$, $M_s = \{ \vec{s}_1, \vec{s}_2, \cdots, \vec{s}_L \}$, and $D_p = \text{diag}\{p_1, p_2, \cdots, p_L\}$ with $0 < p_i \leq 1$ and $\sum_{i=1}^{L} p_i = 1$. For a tripartite state $|\Psi\rangle_{2 \times N \times M}$ whose qubit is lost, the reduced density oprim $\rho_{AB}$ is separable if and only if there exist a number $L$ such that $\mathbf{T} = M_r \mathbf{M}^T_s$ with $M_r \vec{p} = 0$ and $M_s \vec{p} = 0$.

Observation (1) that is well explained in Ref. \[19\] tells us that after the disposal of qubit if the correlation matrix $\mathbf{T}$ can be decomposed with constraint $M_r \vec{p} = 0$ and $M_s \vec{p} = 0$ then the tripartite state $|\Psi\rangle_{2 \times N \times M}$ is fragile.
Note that there are two possible matrix unfoldings of $T_{\mu \nu}$. The matrix unfolding is called the matrization of the tensor $[24]$. Now we can define the Ky Fan norm of the $N$-order tensor $T^N$ over $N$ matrix unfoldings as

$$\| T^{(N)} \|_{KF} = \max \left\{ \| T^{(N)}_{(n)} \|_{KF} \right\}, n = 1, \ldots, N.$$  

Here, $\| T^{(N)} \|_{KF}$ is the Ky Fan norm of matrix $T^{(N)}_{(n)}$, defined as the sum of singular values of $T^{(N)}_{(n)}$, for $n = 1, \ldots, N$. For the normal form states with maximally mixed subsystems, the typical known results using the Bloch representation may be expressed as follows $[22]$: If the bipartite mixed state is separable then

$$\| T \|_{KF} \leq \sqrt{\frac{MN(M-1)(N-1)}{4}}.$$  

If we define $\lambda_0 \equiv 1$, to be more compactly the equation (17) may be expressed as

$$\rho_{AB} = \frac{1}{4} \sum_{\mu, \nu = 0}^{N^2-1, M^2-1} \tilde{T}_{\mu \nu} \lambda_\mu \lambda_\nu,$$

here the extended correlation matrix $\tilde{T}$ is defined as

$$\tilde{T} = \begin{pmatrix} \frac{1}{NM} & \frac{2}{N} b_1 & \cdots & \frac{2}{N} b_{M^2-1} \\ \frac{2}{N} a_1 & T_{11} & \cdots & T_{1(M^2-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{N} a_{N^2-1} & T_{(N^2-1)1} & \cdots & T_{(N^2-1)(M^2-1)} \end{pmatrix}.$$  

The robustness (resp. fragility) of a multipartite entangled state is an LU-invariant so the state $\rho'_{AB} = (U_A \otimes U_B) \rho_{AB} (U_A^\dagger \otimes U_B^\dagger)$ has the same robustness ( resp. fragility) as $\rho_{AB}$. Now we are able to recognize the the robust and fragile tripartite state against qubit loss with reducible bipartite states.

**Example:**

Consider a $2 \times 3 \times 3$ tripartite quantum system, after the loss of qubit we have a bipartite mixed state $[33]$:

$$\rho = \frac{1}{4} \left( I_9 - \sum_{i=0}^{4} |\psi_i\rangle \langle \psi_i| \right).$$
Here, $|\psi_0\rangle = |0\rangle (|0\rangle - |1\rangle)/\sqrt{2}$, $|\psi_1\rangle = (|0\rangle - |1\rangle)|2\rangle/\sqrt{2}$, $|\psi_2\rangle = |2\rangle (|1\rangle - |2\rangle)/\sqrt{2}$, $|\psi_3\rangle = (|1\rangle - |2\rangle)|0\rangle/\sqrt{2}$, $|\psi_4\rangle = (|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)/3$. So from the Bloch representation we find

$$T = -\frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & \frac{\sqrt{27}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{9}{4} & 0 & -\frac{9}{8} & 0 & 0 & 0 & \frac{\sqrt{27}}{8} & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{9}{4} & 1 & 0 & 1 & 0 & -\frac{\sqrt{27}}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{27}}{4} & 0 & \frac{\sqrt{27}}{8} & 0 & 0 & \frac{\sqrt{27}}{2} & 0 & -\frac{3}{8} \end{pmatrix},$$

(21)

Here $||T||_{KF} \simeq 3.1603$, so according to condition (17) our $3 \times 3$ bipartite mixed state is a part of $2 \times 3 \times 3$ robust tripartite state.

**Observation 2** A tripartite quantum state $|\Psi\rangle_{2 \times N \times M}$ whose reduced density matrix contain $|\psi^\pm\rangle = \alpha|ee\rangle \pm \beta|e'e\rangle$ with a definite positive probability that never be $1/2$ and non zero coefficients $\alpha \neq \beta$ is a robust state. Where $e$ and $e'$ are orthogonal.

**Example:**

Consider the following $2 \times N \times M$ quantum state:

$$|\phi\rangle = \frac{1}{\sqrt{\alpha}}(\beta_1|0.NM\rangle + \beta_2|10.M\rangle + \beta_3|1.N0\rangle).$$

(22)

Where $\alpha$ is a normalized constant. If the qubit decided to leave the other partite then the reduced density operator of final bipartite mixed state have form

$$\rho^{NM} = p|\psi\rangle\langle\psi| + (1 - p)|NM\rangle\langle NM|,$$

(23)

here $|\psi\rangle = \frac{1}{\sqrt{2}}(|0.M\rangle + |N0\rangle)$ is a maximally entangled bipartite state. If the $p \neq 0$ then $\rho^{MN}$ is also an entangled state that can carry the information after the removal of qubit.

**Observation 3** All tripartite quantum states $|\Psi\rangle_{2 \times N \times M}$ are robust with respect to loss of qubit if bipartite reduced density matrix partial transpose contain at least one negative eigen value.
Example:

Now we consider a $2 \times 3 \times 3$ pure quantum state

$$|\psi\rangle = \frac{1}{\sqrt{4}}(|010\rangle + |001\rangle + |112\rangle + |121\rangle).$$

(24)

After the loss of qubit, reduced density matrix

$$\rho^{AB} = |10\rangle\langle 01| + |01\rangle\langle 10| + |12\rangle\langle 21| + |21\rangle\langle 12|.$$  

(25)

The partial transpose of $\rho^{AB}$ with respect to first particle has following matrix form

$$\rho^{T_A} = \frac{1}{4} \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}. $$

(26)

The $\rho^{T_A}$ has eigenvalues $\left\{ -\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0 \right\}$. As the first eigenvalue is negative then according to PPT criterion [27] the bipartite mixed state $\rho^{AB}$ is an entangled state, so our $2 \times 3 \times 3$ tripartite pure state $|\psi\rangle$ is a robust quantum state with respect to loss of qubit.

3 Robustness of Quantum State

Up to now we have identified robust and fragile tripartite states $|\Psi\rangle_{2 \times N \times M}$ with respect to loss of qubit. Their robustness can be estimated by using an entanglement quantifier for the residual mixed state [18]. To investigate robustness of the robust states here considered, we evaluate explicitly the dynamics of the negativity [28] and the concurrence [29].

For pure or mixed states $\rho$ in composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the negativity is defined as the absolute value of the sum of the negative eigenvalues of the partially
transposed (PT) density matrix $\rho^T_{AB}$, that can be defined in terms of the trace norm $\|\rho^T_{AB}\|$, the sum of the moduli of eigenvalues of $\rho^T_{AB}$, as \[28\]
\[
\mathcal{N}(\rho) = \frac{\|\rho^T_{AB}\| - 1}{2}.
\] (27)

Here $\rho^T_{AB}$ is partial transpose of $\rho$ with respect to partite $A$ and $\|\rho\| = Tr(\sqrt{\rho \rho^T})$. But in general, the negativity fails to measure entanglement of some entangled states (those with positive partial transpose) in dimensions higher than six \[30\]. However, for the states considered in the previous sections, the negativity vanishes only when the state is fragile (i.e., null negativity means residual state cannot carry information any more).

The concurrence \[29\] for a bipartite pure state $|\psi\rangle$ is
\[
C(|\psi\rangle) = \sqrt{2(1 - tr\rho_A^2)},
\] (28)

here $\rho_A = tr_B(\rho_{AB})$.

This quantifier can be extended over the mixed states $\rho$ by virtue of a convex roof construction \[31\]
\[
C(\rho) = \inf_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \quad \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad p_i \geq 0
\] (29)
of all possible decomposition into pure states $\psi_i$. Consequently, $C(\rho)$ vanishes if and only if $\rho$ is fragile. Fig. (1) shows the relationship between concurrence and negativity.

4 Conclusions

We identify robust and fragile $2 \times N \times M$ quantum states with respect to loss of qubit and discussed their robustness in terms of negativity and concurrence of bipartite reduced density matrix. We find that after the removal of qubit of $2 \times N \times M$ tripartite quantum states, if the bipartite reduced density matrices are not reducible according to local rank then the tripartite quantum states are robust otherwise quantum state may be fragile. We identified the fragility of this type of quantum states in terms of correlation matrix, $T$ decomposition under some constraints. Robust quantum states have ability to transfer information even after the loss of some parts. Our work can be helpful to experimentalists to identify robust quantum states for successful transfer of information even in the presence of local noise.

Acknowledgments

This work was supported in part by the Ministry of Science and Technology of the Peoples’ Republic of China(2015CB856703); by the Strategic Priority Research Program of the Chinese Academy of Sciences, Grant No.XDB23030100; by the National Natural Science Foundation of China(NSFC) under the Grants 11375200 and 11635009; and by the University of Chinese Academy of Sciences. S.M.Z. is supported in part by CAS-TWAS fellowship.

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[33] This example is taken from ref. [22], we recommend [22] and [32] to clarify basic concepts.