On the boundedness of the global solution of anisotropic quasi-geostrophic equations in Sobolev space

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Abstract
In this paper, we show that the global solution of the surface anisotropic two-dimensional quasi-geostrophic equation with fractional horizontal dissipation and vertical thermal diffusion established by the author Ye in (Non-linearity 33(1): 72, 2019) is bounded in Sobolev spaces uniformly with respect to time.

Keywords Surface quasi-geostrophic equation · Anisotropic dissipation · Global regularity

Mathematics Subject Classification 35-XX · 35Q30 · 76N10

1 Introduction

In this paper we deal with the following surface quasi-geostrophic equation with fractional horizontal dissipation and fractional vertical thermal diffusion:

\[
\begin{cases}
\partial_t \theta + u \cdot \nabla \theta + \mu |\partial_1|^{2\alpha} \theta + \nu |\partial_2|^{2\beta} \theta = 0, \\
\theta(x, 0) = \theta^0(x),
\end{cases}
\]

(AQG)
where $\alpha \in (0, 1), \beta \in (0, 1), \mu > 0$ and $\nu > 0$ are real numbers. Here, we denote by $|\partial_1|$ and $|\partial_2|$ the operators given by

$$
\mathcal{F}(|\partial_1|^{2\alpha} f)(\xi) = |\xi_1|^{2\alpha} \mathcal{F}(f)(\xi), \quad \mathcal{F}(|\partial_2|^{2\beta} f)(\xi) = |\xi_2|^{2\beta} \mathcal{F}(f)(\xi), \quad \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,
$$

where $\mathcal{F}(f)$ represents the Fourier transformation of $f$. The variable $\theta$ represents the potential temperature and the velocity $u_{\theta} = (u_1, u_2)$ is determined by $\theta$ via the formula

$$
u(\theta) = \mathcal{R}_{1}^{\perp} \theta = (\mathcal{R}_{2} \theta, \mathcal{R}_{1} \theta) = \left( -\partial_{2}(-\Delta)^{-\frac{1}{2}} \partial_{\theta}, \partial_{1}(-\Delta)^{-\frac{1}{2}} \theta \right),$$

where $\mathcal{R}_{1}, \mathcal{R}_{2}$ are the standard 2D Riesz transforms. Clearly, the velocity $u$ is divergence free, namely

$$\text{div}(u_{\theta}) := \partial_1 u_1 + \partial_2 u_2 = 0.$$

The system $(AQG)$ is deeply related, in the case when $\alpha = \beta$ and $\mu = \nu$, to the classical dissipation $(QG)$ equation, with its form as follows

$$
\begin{aligned}
\partial_t \theta + u_{\theta} \cdot \nabla \theta + \mu (-\Delta)^{2\alpha} \theta &= 0, \quad x \in \mathbb{R}^2, \quad t > 0 \\
\theta(x, 0) &= \theta^0(x).
\end{aligned}
$$

(QG)

The equations $((AQG))$ and $((QG))$ are special cases of the general quasi-geostrophic approximations for atmospheric and oceanic fluid flow with small Rossby and Ekman numbers. The first mathematical studies of this equation was carried out in 1994 by Constantin, Majda and Tabak. For more details and mathematical and physical explanations of this model we can consult [5–7, 11].

The inviscid quasi-geostrophic equation (i.e., $(QG)$ with $\mu = 0$) shares many properties parallel to those of 3D Euler equations such as the vortex stretch mechanism and thus serves as a lower dimensional model of 3D Euler equations.

The first studies of the system $(AQG)$ is in [13] by Ye, who shows that this equation admits a unique global solution $\theta$ in the space $C(\mathbb{R}^+, H^s(\mathbb{R}^2)), s \geq 2$, such that

$$
|\partial_1|^{\alpha} \theta, |\partial_2|^{\beta} \theta \in L^2_{\text{loc}}(\mathbb{R}^+, H^s(\mathbb{R}^2)), \quad \text{(1.1)}
$$

when $\alpha, \beta \in (0, 1)$ satisfies

$$
\beta > \begin{cases} 
\frac{1}{2\alpha + 1}, & 0 < \alpha \leq \frac{1}{2} \\
\frac{1-\alpha}{2\alpha} & \frac{1}{2} < \alpha < 1.
\end{cases} \quad \text{(1.2)}
$$

We also refer to our result in [1], when we established the global regularity in $C(\mathbb{R}^+, H^s(\mathbb{R}^2)), \alpha, \beta \in (1/2, 1)$ and $s \in (2 - 2 \min\{\alpha, \beta\}, 2)$.

In this paper, we will show that this global solution is in $L^\infty(\mathbb{R}^+, H^s(\mathbb{R}^2))$ and

$$
|\partial_1|^{\alpha} \theta, |\partial_2|^{\beta} \theta \in L^2(\mathbb{R}^+, H^s(\mathbb{R}^2)). \quad \text{(1.3)}
$$

For the sake of simplicity, we will set $\mu = \nu = 1$ throughout the paper, $C$ denoted all constants that is a generic constant depending only on the quantities specified in the context and $C(\theta^0)$ represent all constants depending on the norm of the initial condition $\theta^0$.

2 Main theorems

We explain all the details later in the paper, but let us state here the first main result:

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Theorem 2.1 Let $\theta^0 \in H^s(\mathbb{R}^2)$ for $s \geq 2$. If $\alpha, \beta \in (0, 1)$ satisfy (1.2). Then the system ((AQG)) admits a unique global solution $\theta$ such that

$$\theta \in C_b(\mathbb{R}^+, H^s(\mathbb{R}^2)), \quad |\partial_1|^\alpha \theta, |\partial_2|^\beta \theta \in L^2(\mathbb{R}^+, H^s(\mathbb{R}^2)).$$

Moreover, for any $t \geq 0$

$$\|\theta(t)\|_{H^s}^2 + \int_0^t \|\partial_1|^\alpha \theta\|_{H^s}^2 d\tau + \int_0^t \|\partial_2|^\beta \theta\|_{H^s}^2 d\tau \leq C(\theta^0). \quad (2.1)$$

We outline the main ideas in the proof of this Theorem which indicated by the author in [13] as follows:

Theorem 2.2 (See [13]) Let $\theta^0 \in H^s(\mathbb{R}^2)$ for $s \geq 2$. If $\alpha, \beta \in (0, 1)$ satisfy (1.2). Then the system ((AQG)) admits a unique global solution $\theta$ such that for any $T > 0$ we have

$$\theta \in C([0, T], H^s(\mathbb{R}^2)), \quad |\partial_1|^\alpha \theta, |\partial_2|^\beta \theta \in L^2([0, T], H^s(\mathbb{R}^2)). \quad (2.2)$$

Moreover, for any $t \geq 0$

$$\|\theta(t)\|_{L^p} \leq \|\theta^0\|_{L^p}, \quad \forall p \in [2, +\infty), \quad (2.3)$$

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \|\partial_1|^\alpha \theta\|_{L^2}^2 d\tau + \int_0^t \|\partial_2|^\beta \theta\|_{L^2}^2 d\tau \leq \|\theta^0\|_{L^2}^2, \quad (2.4)$$

$$\|\theta(t)\|_{H^1}^2 + \int_0^t \|\partial_1|^\alpha \theta\|_{H^1}^2 d\tau + \int_0^t \|\partial_2|^\beta \theta\|_{H^1}^2 d\tau \leq \|\theta^0\|_{H^1}^2 + C \int_0^t (1 + \|u\|_{L^\infty}) \|\theta\|_{H^1}^2 d\tau,$$

$$\|\theta(t)\|_{H^2}^2 + \int_0^t \|\partial_1|^\alpha \theta\|_{H^2}^2 d\tau + \int_0^t \|\partial_2|^\beta \theta\|_{H^2}^2 d\tau \leq \|\theta^0\|_{H^2}^2 + C \int_0^t \left(1 + \|\theta\|_{H^1}^2\right) \left(1 + \|\partial_1|^\alpha \theta\|_{H^1}^2 + \|\partial_2|^\beta \theta\|_{H^1}^2\right) \|\theta\|_{H^2}^2 d\tau, \quad (2.5)$$

and

$$\|\theta(t)\|_{H^s}^2 + \int_0^t \|\partial_1|^\alpha \theta\|_{H^s}^2 d\tau + \int_0^t \|\partial_2|^\beta \theta\|_{H^s}^2 d\tau \leq \|\theta^0\|_{H^s}^2 + C \int_0^t \left(1 + \|\nabla u\|_{L^\infty} + \|\nabla \theta\|_{L^\infty}\right) \|\theta\|_{H^s}^2 d\tau, \quad (2.6)$$

where

$$\rho = \begin{cases} \frac{2\beta}{(2\alpha+1)\beta-1}, & \text{if } \beta > \frac{1}{2\alpha+1}, \alpha \leq \frac{1}{2}, \\ \max \left\{ \frac{2\alpha}{2\alpha-1}, \frac{2\alpha}{(2\beta+1)\alpha-1} \right\}, & \text{if } \beta > \frac{1}{2\alpha}, \alpha > \frac{1}{2}. \end{cases} \quad (2.8)$$

Remark 1 Recently, the author with Bennameur [1] proved the global existence of solution in the Sobolov space $H^s(\mathbb{R}^2)$, $s \in (2 - 2 \min\{\alpha, \beta\}, 2)$ in case $\alpha, \beta \in (1/2, 1)$. They used the Gevrey-class regularity of the solution in neighborhood of zero.

More specifically
Theorem 2.3 (See [1]) Let $\alpha, \beta \in (1/2, 1)$ and $\theta^0 \in H^s(\mathbb{R}^2)$, for $e \in (2 - 2 \min \{\alpha, \beta\}, 2)$. Then, the system ((AQG)) admits a unique global solution
\[ \theta \in C(\mathbb{R}^+, H^s(\mathbb{R}^2)) \cap C((0, +\infty), H^2(\mathbb{R}^2)); \quad |\partial_1^{\alpha} \theta|, |\partial_2^{\beta} \theta| \in L^2_{loc}(\mathbb{R}^+, H^s(\mathbb{R}^2)). \]

Now, we ready to state our second result:

Theorem 2.4 Let $\alpha, \beta \in (1/2, 1)$ and $\theta^0 \in H^s(\mathbb{R}^2), s \in (2 - 2 \min \{\alpha, \beta\}, 2)$. Then, the system ((AQG)) admits a unique global solution
\[ \theta \in C_b(\mathbb{R}^+, H^s(\mathbb{R}^2)), \quad |\partial_1^{\alpha} \theta|, |\partial_2^{\beta} \theta| \in L^2(\mathbb{R}^+, H^s(\mathbb{R}^2)). \]

3 Notations and preliminary results

In this short section, we collect some notations and definitions that will be used later, and we give some technical lemmas.

3.1 Notations

- The Fourier transformation in $\mathbb{R}^2$
\[ \mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^2. \] (3.1)

The inverse Fourier formula is
\[ \mathcal{F}^{-1}(f)(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} f(\xi) d\xi, \quad x \in \mathbb{R}^2. \] (3.2)

- The convolution product of a suitable pair of function $f$ and $g$ on $\mathbb{R}^2$ is given by
\[ f \ast g(x) = \int_{\mathbb{R}^2} f(x - y) g(y) dy. \] (3.3)

- If $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are two vector fields, we set
\[ f \otimes g := (g_1 f_1, g_2 f_2), \]

and
\[ \text{div}(f \otimes g) := (\text{div}(g_1 f_1), \text{div}(g_2 f_2)). \]

- For $s \in \mathbb{R}$:
  * The Sobolev space:
\[ H^s(\mathbb{R}^2) := \left\{ f \in S'(\mathbb{R}^2); (1 + |\xi|^2)^{s/2} \hat{f} \in L^2(\mathbb{R}^2) \right\}, \]
denotes the usual inhomogeneous Sobolev space on $\mathbb{R}^2$, with the norm
\[ \|f\|_{H^s} = \left( \int_{\mathbb{R}^2} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \]

  * The Sobolev space:
\[ \dot{H}^s(\mathbb{R}^2) := \left\{ f \in S'(\mathbb{R}^2); \hat{f} \in L^1(\mathbb{R}^2) \text{ and } |\xi|^s \hat{f} \in L^2(\mathbb{R}^2) \right\}, \]
denotes the usual homogeneous Sobolev space on $\mathbb{R}^2$, with the norm
\[ \|f\|_{H^s} = \left( \int_{\mathbb{R}^2} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}. \]

- Let $(X, \| \cdot \|_X)$ be a Banach space. The space of bounded continuous functions from $\mathbb{R}^+$ into $X$, denoted $C_b(\mathbb{R}^+, X)$, is defined as follows
\[ C_b(\mathbb{R}^+, X) := C(\mathbb{R}^+, X) \bigcap L^\infty(\mathbb{R}^+, X). \]

### 3.2 Preliminary results

In the first lemma, we recall that the Riesz transformation is a continuous operator in the Lebesgue space

**Lemma 3.1** (See [8, 12, 14]) For any $p \in (1, +\infty)$, there is a constant $C(p) > 0$ such that
\[ \| \mathcal{R}^\perp \theta \|_{L^p} \leq C(p) \| \theta \|_{L^p}. \] (3.4)

The following lemma is called "The logarithmic Sobolev inequality", and it is a type of Brézis-Gallouet inequality (see [3, 4]). More precisely, it is a simple case of the inequality shown in [10].

**Lemma 3.2** Let $\sigma > 1$, then, their exist a constant $C > 0$, such that for any $f \in H^\sigma(\mathbb{R}^2)$
\[ \| \mathcal{R}^\perp f \|_{L^\infty} \leq C \left( 1 + \| f \|_{L^2} + \| f \|_{L^\infty} \ln\left( e + \| \nabla f \|_{L^2} \right) \right). \] (3.6)

**Proof** We have
\[
\| \nabla f \nabla g - f \nabla g \|_{L^p} \leq C(s, \sigma) \left( \| f \|_{H^{\sigma/s}} \| g \|_{H^{1+\sigma}} + \| g \|_{H^{\sigma/s-1}} \| f \|_{H^{2+\sigma}} \right). \] (3.6)

**Lemma 3.3** (See [9]) Let $s \geq 0$ and $p \in (1, +\infty)$. Then, for any $f, g \in S(\mathbb{R}^2)$ we have
\[ \| \mathcal{J}^s(fg) - f \mathcal{J}^s(g) \|_{L^p} \leq C \left( \| \mathcal{J}^s f \|_{L^p} \| g \|_{L^\infty} + \| \nabla f \|_{L^\infty} \| \mathcal{J}^{s-1} g \|_{L^p} \right), \] (3.5)

where $\mathcal{J}^s := (1 - \Delta)^{\frac{s}{2}}$.

In particulate, we have the following lemma

**Lemma 3.4** Let $s > 1$ and $\sigma > 0$, then, there exist $C(s, \sigma) > 0$, such that for any $f, g \in S(\mathbb{R}^2)$ we have
\[ \| \nabla f \nabla g - f \nabla g \|_{L^2} \leq C(s, \sigma) \left( \| f \|_{H^{\sigma/s}} \| g \|_{H^{1+\sigma}} + \| g \|_{H^{\sigma/s-1}} \| f \|_{H^{2+\sigma}} \right). \] (3.6)
we get
\[
\|\nabla|^s (fg) - f|\nabla|^s g\|_{L^2}^2 \leq C(s) \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |\xi - \eta|^s |\hat{f}(\xi - \eta)| |\hat{g}(\eta)| d\eta \right)^2 d\xi
\]
\[
\leq C(s) \left( \|f_1 g_1\|_{L^2}^2 + \|f_2 g_2\|_{L^2}^2 \right),
\]
where
\[
\mathcal{F}(f_1)(\xi) = |\xi|^s |\mathcal{F}(f)(\xi)|, \quad \mathcal{F}(f_2)(\xi) = |\xi| |\mathcal{F}(f)(\xi)|.
\]

Using Hölder inequality and the fact that, for any \(\sigma > 0\), \(H^{1+\sigma}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)\), we have
\[
\|\nabla|^s (fg) - f|\nabla|^s g\|_{L^2} \leq C(s) \left( \|f_1\|_{L^2} \|g\|_{L^\infty} + \|f_2\|_{L^\infty} \|g_1\|_{L^2} \right)
\]
\[
\leq C(s, \sigma) \left( \|f\|_{H^s} \|g\|_{H^{1+\sigma}} + \|f\|_{H^{1+\sigma}} \|g\|_{H^{1-s}} \right),
\]
which finished the proof. \(\square\)

We recall the anisotropic interpolation lemma concerning the Sobolev spaces

**Lemma 3.5** For \(s, s_1, s_2 \in \mathbb{R}\) and \(z \in [0, 1]\), the following anisotropic interpolation inequalities hold true for \(i = 1, 2\):
\[
\|\partial_i |^{s_1+(1-z)s_2} f\|_{H^s} \leq \|\partial_i |^{s_1} f\|_{H^s}, \quad \|\partial_i |^{s_1+(1-z)s_2} f\|_{H^s} \leq \|\partial_i |^{s_1} f\|_{H^s} \|\partial_i |^{s_2} f\|_{H^{1-z}}. \tag{3.7}
\]
\[
\|\partial_i |^{s_1+(1-z)s_2} f\|_{H^s} \leq \|\partial_i |^{s_1} f\|_{H^s}, \quad \|\partial_i |^{s_1+(1-z)s_2} f\|_{H^s} \leq \|\partial_i |^{s_1} f\|_{H^s}. \tag{3.8}
\]

**Proof** It suffices to show (3.7) for \(i = 1\) as \(i = 2\) can be performed as the same manner: So we have
\[
\|\partial_i |^{s_1+(1-z)s_2} f\|_{H^s}^2 = \int_{\mathbb{R}^2} (1 + |\xi|^2)^{(1-z)} |\xi|^{2s_1+(1-z)s_2} |\hat{f}(\xi)|^2 d\xi
\]
\[
= \int_{\mathbb{R}^2} \left( (1 + |\xi|^2)^{(1-z)} |\xi|^{2s_1} |\hat{f}(\xi)|^2 \right)^{1-z} d\xi = \|f_1 \times f_2\|_{L^1},
\]
where
\[
f_1(\xi) = \left( (1 + |\xi|^2)^{1-z} |\xi|^{2s_1} |\hat{f}(\xi)|^2 \right)^{1-z}, \quad f_2(\xi) = \left( (1 + |\xi|^2)^{(1-z)} |\xi|^{2s_2} |\hat{f}(\xi)|^2 \right)^{1-z}.
\]

The fact that
\[
\frac{1}{1} + \frac{1}{1} = 1.
\]
Using the Hölder’s inequality, we get
\[ \| f_1 \times f_2 \|_{L^1} \leq \| f_1 \|_{L^\frac{1}{\alpha}} \| f_2 \|_{L^\frac{1}{\beta}} \]
\[ \leq \left( \int_{\mathbb{R}^2} (1 + |\xi|^2)^{\alpha} |\xi_i|^{2\alpha_1} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{\alpha}} \left( \int_{\mathbb{R}^2} (1 + |\xi|^2)^{\beta} |\xi_i|^{2\beta_1} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{\beta}} \]
which implies the result for case inhomogeneous. Following the proof of (3.7), the estimate (3.8) immediately holds true. This completes the proof of the lemma.

We finish with the following elementary inequality

**Lemma 3.6** Let \( \alpha > 0 \), then, there is a constant \( C(\alpha) > 0 \) such that
\[ \ln(x) \leq C(\alpha)x^\alpha, \quad \forall x \geq 1. \] (3.9)

## 4 Proof of theorem 2.1

Before we begin the proof, we require the following proposition to prove that the global solution of system ((AQG)) provided in the Theorem 2.2 is uniformly bounded in \( H^1 \) and \( H^2 \):

**Proposition 4.1** If \( \theta_0, \alpha \) and \( \beta \) satisfies the assumptions stated in Theorem 2.1 and let \( \theta \) be the corresponding global solution, then, for any \( t \geq 0 \),
\[ \| \theta(t) \|_{H^1}^2 + \int_0^t \| \partial_1 \theta \|_{H^1}^2 d\tau + \int_0^t \| \partial_2 \theta \|_{H^1}^2 d\tau \leq C(\theta_0). \] (4.1)

**Proof** We have the inequality (2.5) given in Theorem 2.2 given as following, for any \( t \geq 0 \),
\[ \| \theta(t) \|_{H^1}^2 + \int_0^t \| \partial_1 \theta \|_{H^1}^2 d\tau + \int_0^t \| \partial_2 \theta \|_{H^1}^2 d\tau \leq \| \theta_0 \|_{H^1}^2 + C \int_0^t (1 + \| u \|_{L^\infty}) \| \theta \|_{H^1}^2 d\tau, \] (4.2)
where \( \rho > 1 \) give in (2.8).

In order to control \( \| u_\theta \|_{L^\infty} = \| R^\bot \theta \|_{L^\infty} \), we need the logarithmic Sobolev interpolation inequality given in Lemma 3.2, so we obtain
\[ \| u_\theta \|_{L^\infty} \leq C \left( 1 + \| \theta \|_{L^2} + \| \theta \|_{L^\infty} \ln (e + \| \nabla |\sigma \theta \|_{L^2}) \right), \]
where \( 1 < \sigma < 1 + \min\{\alpha, \beta\} \). Using (2.3) in the last inequality, we get
\[ \| u_\theta \|_{L^\infty} \leq C(\theta_0) \left( 1 + \ln (e + \| \nabla |\sigma \theta \|_{L^2}) \right). \]

Moreover, by the Lemma 3.6 we have
\[ \ln (e + \| \nabla |\sigma \theta \|_{L^2}) \leq C \left( 1 + \| \nabla |\sigma \theta \|_{L^2}^{\sigma - 1} \right). \]

Finally, we get
\[ \| u_\theta \|_{L^\infty} \leq C(\theta_0) \left( 1 + \| \nabla |\sigma \theta \|_{L^2}^{\sigma - 1} \right). \]
and
\[
\|
\theta(t)\|^2_{H_1^2} + \int_0^t \| \partial_1 \|^2_{H_1^2} d\tau + \int_0^t \| \partial_2 \|^2_{H_1^2} d\tau \leq \|
\theta(0)\|^2_{H_1^2} + C(\theta^0)
\]
\[
\int_0^t \left( 1 + \| \nabla \|^2_{L^2} \right) d\tau
\]
\[
\leq \|
\theta(0)\|^2_{H_1^2} + C(\theta^0) \int_0^t \| \partial_1 \|^2_{H_1^2} d\tau + C(\theta^0) \int_0^t \| \nabla \|^2_{L^2} \| \theta \|^2_{H_1^2} d\tau.
\]

Now, we need to control \( \| \theta \|_{H_1^1} \), so, we apply the interpolation inequality. The fact that
0 < 1 < \sigma, then \( 1 = z \times \sigma + (1 - z) \times 0 \), and we have
\[
\|
\theta \|_{H_1^1} \leq \| \theta \|_{L^2}^{\frac{1}{2}} \| \nabla \|^\sigma \|_{L^2}^{\frac{1}{2}}
\]
\[
\leq \| \theta(0) \|_{L^2}^{\frac{1}{2}} \| \nabla \|^\sigma \|_{L^2}^{\frac{1}{2}}.
\]

Therefore
\[
C(\theta^0) \| \nabla \|^\sigma \|_{L^2}^{\frac{1}{2}} \| \theta \|^2_{H_1^2} \leq C(\theta^0) \| \nabla \|^\sigma \|_{L^2}^{\frac{1}{2}} \| \nabla \|^\sigma \|_{L^2}^{\frac{1}{2}} \| \theta \|^2_{H_1^2}
\]
\[
\leq C(\theta^0) \| \nabla \|^\sigma \|_{L^2} \| \theta \|^2_{H_1^2}
\]
\[
\leq C(\theta^0) \| \theta \|^2_{H_1^2} + \| \nabla \|^\sigma \|_{L^2}^2.
\]

which imply
\[
\|
\theta(t)\|^2_{H_1^2} + \int_0^t \| \partial_1 \|^2_{H_1^2} d\tau
\]
\[
+ \int_0^t \| \partial_2 \|^2_{H_1^2} d\tau \leq \|
\theta(0)\|^2_{H_1^2} + C(\theta^0) \int_0^t \| \theta \|^2_{H_1^2} d\tau + \int_0^t \| \nabla \|^\sigma \|_{L^2}^2 d\tau.
\]

Since \( \alpha < 1 < \sigma < 1 + \alpha \) and \( \beta < 1 < \sigma < 1 + \beta \), then, using the interpolation inequality we get
\[
\| \nabla \|^\sigma \|_{L^2}^2 \leq 2 \| \partial_1 \|^\sigma \|_{L^2}^2 + 2 \| \partial_2 \|^\sigma \|_{L^2}^2
\]
\[
\leq C \left( \| \partial_1 \|^\sigma \|_{L^2}^2 + \| \partial_2 \|^\sigma \|_{L^2}^2 \right) + \frac{1}{4} \left( \| \partial_1 \|^\sigma \|_{H_1^1}^2 + \| \partial_2 \|^\sigma \|_{H_1^1}^2 \right);
\]
\[
\text{(4.4)}
\]

and
\[
C(\theta^0) \| \theta \|^2_{H_1^2} \leq C(\theta^0) \left( \| \partial_1 \|^2_{L^2} + \| \partial_2 \|^2_{L^2} \right)
\]
\[
\leq C(\theta^0) \left( \| \partial_1 \|^2_{H_1^2} + \| \partial_2 \|^2_{H_1^2} \right) + \frac{1}{4} \left( \| \partial_1 \|^2_{H_1^2} + \| \partial_2 \|^2_{H_1^2} \right).
\]
\[
\text{(4.5)}
\]

Collect (4.4) and (4.5) with (4.3), we get
\[
\|
\theta(t)\|^2_{H_1^2} + \int_0^t \| \partial_1 \|^2_{H_1^2} d\tau
\]
\[
+ \int_0^t \| \partial_2 \|^2_{H_1^2} d\tau \leq \|
\theta(0)\|^2_{H_1^2} + C(\theta^0) \int_0^t \left( \| \partial_1 \|^2_{L^2} + \| \partial_2 \|^2_{L^2} \right) d\tau
\]
\[
+ \frac{1}{2} \int_0^t \left( \| \partial_1 \|^2_{H_1^2} + \| \partial_2 \|^2_{H_1^2} \right) d\tau.
\]
Finally, due to the inequality (2.4) we obtain
\[
\|\theta(t)\|_{H^2}^2 + \frac{1}{2} \int_0^t \|\partial_1|^{\alpha}\theta\|_{H^1}^2 \, d\tau + \frac{1}{2} \int_0^t \|\partial_2|^{\beta}\theta\|_{H^1}^2 \, d\tau \leq \|\theta_0\|_{H^2}^2 + C(\theta_0)\|\theta_0\|_{L^2}^2. \tag{4.6}
\]
So we get that
\[
\theta \in C_b(\mathbb{R}^+, H^1(\mathbb{R}^2)), \quad |\partial_1|^{\alpha}\theta, |\partial_2|^{\beta}\theta \in L^2(\mathbb{R}^+, H^1(\mathbb{R}^2)).
\]
\[
\square
\]

**Proposition 4.2** If $\theta_0$, $\alpha$ and $\beta$ satisfies the assumptions stated in Theorem 2.1 and let $\theta$ be the corresponding global solution, then, for any $t \geq 0$,
\[
\|\theta(t)\|_{H^2}^2 + \int_0^t \|\partial_1|^{\alpha}\theta\|_{H^2}^2 \, d\tau + \int_0^t \|\partial_2|^{\beta}\theta\|_{H^2}^2 \, d\tau \leq C(\theta_0). \tag{4.7}
\]

**Proof** Using (2.6) and Proposition 4.1, we get for any $t \geq 0$,
\[
\|\theta(t)\|_{H^2}^2 + \int_0^t \|\partial_1|^{\alpha}\theta\|_{H^2}^2 \, d\tau + \int_0^t \|\partial_2|^{\beta}\theta\|_{H^2}^2 \, d\tau \leq \|\theta_0\|_{H^2}^2 + C(\theta_0) \int_0^t \left( 1 + \|\partial_1|^{\alpha}\theta\|_{H^2}^2 + \|\partial_2|^{\beta}\theta\|_{H^2}^2 \right) \|\theta\|_{H^2}^2 \, d\tau. \tag{4.8}
\]

Moreover, by the interpolation inequality for $\|\theta\|_{H^2}$, where $\alpha < 2 < 2 + \alpha$ and $\beta < 2 < 2 + \beta$, we have
\[
C(\theta_0)\|\theta\|_{H^2}^2 \leq C(\theta_0) \left( \|\partial_1|^{2}\theta\|_{L^2}^2 + \|\partial_2|^{2}\theta\|_{L^2}^2 \right) \leq C(\theta_0) \left( \|\partial_1|^{\alpha}\theta\|_{L^2}^2 + \|\partial_2|^{\beta}\theta\|_{L^2}^2 \right) + \frac{1}{2} \left( \|\partial_1|^{\alpha}\Delta\theta\|_{L^2}^2 + \|\partial_2|^{\beta}\Delta\theta\|_{L^2}^2 \right);
\]
which implies
\[
\|\theta(t)\|_{H^2}^2 + \int_0^t \|\partial_1|^{\alpha}\theta\|_{H^2}^2 \, d\tau + \int_0^t \|\partial_2|^{\beta}\theta\|_{H^2}^2 \, d\tau \leq C(\theta_0)\|\theta_0\|_{H^2}^2 + C(\theta_0) \int_0^t \left( \|\partial_1|^{\alpha}\theta\|_{H^2}^2 + \|\partial_2|^{\beta}\theta\|_{H^2}^2 \right) \|\theta\|_{H^2}^2 \, d\tau. \tag{4.9}
\]

Gronwall lemma implies that, for any $t \geq 0$ we have
\[
\|\theta(t)\|_{H^2}^2 \leq C(\theta_0)\|\theta_0\|_{H^2}^2 \exp \left[ C(\theta_0) \int_0^t \left( \|\partial_1|^{\alpha}\theta\|_{H^2}^2 + \|\partial_2|^{\beta}\theta\|_{H^2}^2 \right) \, d\tau \right] \leq C(\theta_0)\|\theta_0\|_{H^2}^2 \exp \left[ C(\alpha, \beta, \theta_0) \right].
\]

Finally we get
\[
\|\theta(t)\|_{H^2}^2 + \int_0^t \|\partial_1|^{\alpha}\theta\|_{H^2}^2 \, d\tau + \int_0^t \|\partial_2|^{\beta}\theta\|_{H^2}^2 \, d\tau \leq C(\alpha, \beta, \theta_0), \tag{4.10}
\]
which prove the result. \[
\square
\]

Now, we are ready for the proof of our first main result:
Proof of Theorem 2.1. Taking the inner product of \(((AQG))\) with \((I + |\nabla|^{2\alpha})\theta\) and using the divergence-free condition, we infer that
\[
\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{H^s}^2 + ||\partial_1|^{\alpha}\theta||_{H^s}^2 + \|\partial_2|^{\beta}\theta||_{H^s}^2 = (u_\theta \cdot \nabla \theta, \theta)_{L^2} + \|\nabla(\theta) \cdot \nabla \theta, |\nabla|^{4\alpha}\theta\|_{L^2}
\]
\[
= (|\nabla|^{4\alpha}(u_\theta \cdot \nabla \theta) - u_\theta \cdot \nabla|\nabla|^{4\alpha}\theta, |\nabla|^{4\alpha}\theta)_{L^2} + |\nabla_\theta \cdot \nabla|\nabla|^{4\alpha}\theta, |\nabla|^{4\alpha}\theta\|_{L^2}
\]
\[
\leq |||\nabla|^{4\alpha}(u_\theta \cdot \nabla \theta) - u_\theta \cdot |\nabla|^{4\alpha}\theta||_{L^2}||\nabla|^{4\alpha}\theta\|_{L^2}.
\]

After that, using lemma 3.4 with \(f = u_\theta, g = \theta\) and \(\sigma = \frac{\min(\alpha, \beta)}{2}\) to get
\[
||\nabla|^{4\alpha}(u_\theta \cdot \nabla \theta) - u_\theta \cdot |\nabla|^{4\alpha}\theta||_{L^2} \leq C(s, \sigma) \left(||u_\theta||_{H^{1+\sigma}} ||\nabla\theta||_{H^{1+\sigma}} + ||\nabla\theta||_{H^{1+\sigma}-1} ||u_\theta||_{H^{2+\sigma}}\right)
\]
\[
\leq C(s, \sigma) ||\theta||_{H^{2+\sigma}} ||\theta||_{H^{1+\sigma}}.
\]

Moreover
\[
||\theta||_{H^{2+\sigma}} \leq C \left(1 + ||\theta||_{L^2}^2 + ||\partial_1|^{2+\sigma}\theta||_{L^2}^2 + ||\partial_2|^{2+\sigma}\theta||_{L^2}^2\right)
\]
\[
\leq C \left(1 + ||\partial_1|^{2+\sigma}\theta||_{L^2}^2 + ||\partial_2|^{2+\sigma}\theta||_{L^2}^2\right).
\]

The fact that \(\alpha < 2 + \sigma < 2 + \alpha\) and \(\beta < 2 + \sigma < 2 + \beta\), the lemma 3.4 allows us to show that
\[
||\partial_1|^{2+\sigma}\theta||_{L^2} \leq ||\partial_1|^{\alpha}\theta||_{L^2}^{\frac{\alpha-\sigma}{\alpha}} ||\partial_1|^{2+\sigma}\theta||_{L^2}^{\frac{\sigma}{2}} \leq ||\partial_1|^{\alpha}\theta||_{H^{2}}.
\]

Similarly
\[
||\partial_2|^{2+\sigma}\theta||_{L^2} \leq ||\partial_2|^{\beta}\theta||_{H^{2}}.
\]

Finally, we get
\[
||\theta||_{H^{2+\sigma}} \leq C \left(1 + ||\partial_1|^{\alpha}\theta||_{H^{2}}^2 + ||\partial_2|^{\beta}\theta||_{H^{2}}^2\right).
\] (4.12)

Collecting (4.12) with (4.11), we obtain
\[
\frac{1}{2} \frac{d}{dt} ||\theta(t)||_{H^s}^2 + ||\partial_1|^{\alpha}\theta||_{H^s}^2 + ||\partial_2|^{\beta}\theta||_{H^s}^2
\]
\[
\leq C_1 ||\theta||_{H^s}^2 + C \left(||\partial_1|^{\alpha}\theta||_{H^{2}}^2 + ||\partial_2|^{\beta}\theta||_{H^{2}}^2\right) ||\theta||_{H^s}^2.
\] (4.13)

Now, we need to control \(||\theta||_{H^s}\), the idea is to use lemma 3.4, since \(\alpha < s < \alpha + \beta\) and \(\beta < \beta + \beta = \beta\) we have
\[
||\theta||_{H^s}^2 \leq C(s) \left(||\partial_1|^{\alpha}\theta||_{L^2}^2 + ||\partial_2|^{\beta}\theta||_{L^2}^2\right)
\]
\[
\leq C(s) \left(||\partial_1|^{\alpha}\theta||_{L^2}^{\frac{\alpha}{\alpha}} ||\partial_1|^{\alpha+\alpha\theta||_{L^2}^{\frac{\alpha}{\alpha}}} + ||\partial_2|^{\beta}\theta||_{L^2}^{\frac{\beta}{\beta}} ||\partial_2|^{\beta}\theta||_{L^2}^{\frac{\beta}{\beta}}\right)
\]
\[
\leq C(s) \left(||\partial_1|^{\alpha}\theta||_{L^2}^2 + ||\partial_2|^{\beta}\theta||_{L^2}^2\right) + \frac{1}{2C_1} \left(||\partial_1|^{\alpha}\theta||_{H^{2}}^2 + ||\partial_2|^{\beta}\theta||_{H^{2}}^2\right).
\]

Therefore
\[
\frac{d}{dt} ||\theta(t)||_{H^s}^2 + ||\partial_1|^{\alpha}\theta||_{H^s}^2 + ||\partial_2|^{\beta}\theta||_{H^s}^2 \leq C \left(||\partial_1|^{\alpha}\theta||_{L^2}^2 + ||\partial_2|^{\beta}\theta||_{L^2}^2\right)
\]
\[
+ C \left(||\partial_1|^{\alpha}\theta||_{H^s}^2 + ||\partial_2|^{\beta}\theta||_{H^s}^2\right) ||\theta||_{H^s}^2.
\] (4.14)
Integrating with respect to time yields
\[
\|\theta(t)\|_{H^s}^2 + \int_0^t (\|\partial_1^\alpha \theta\|_{H^s}^2 + \|\partial_2^\beta \theta\|_{H^s}^2) \, d\tau \leq \|\theta(0)\|_{H^s}^2
\]
\[
+C \int_0^t (\|\partial_1^\alpha \theta\|_{H^s}^2 + \|\partial_2^\beta \theta\|_{H^s}^2) \, d\tau
\]
\[
+C \int_0^t (\|\partial_1^\alpha \theta\|_{H^s}^2 + \|\partial_2^\beta \theta\|_{H^s}^2) \|\theta\|_{H^s}^2 \, d\tau
\]
\[
\leq C\|\theta(0)\|_{H^s}^2 + C \int_0^t (\|\partial_1^\alpha \theta\|_{H^s}^2 + \|\partial_2^\beta \theta\|_{H^s}^2) \|\theta\|_{H^s}^2 \, d\tau.
\]
where we used inequality (2.4). We apply Gronwall’s lemma to obtain
\[
\|\theta(t)\|_{H^s}^2 \leq C\|\theta(0)\|_{H^s}^2 \exp \left[ C \int_0^t (\|\partial_1^\alpha \theta\|_{H^s}^2 + \|\partial_2^\beta \theta\|_{H^s}^2) \, d\tau \right]
\]
\[
\leq C\|\theta(0)\|_{H^s}^2 \exp \left[ C(\alpha, \beta, \theta(0)) \right].
\]
Therefore, for any \( t \geq 0 \)
\[
\|\theta(t)\|_{H^s}^2 + \int_0^t \|\partial_1^\alpha \theta\|_{H^s}^2 \, d\tau + \int_0^t \|\partial_2^\beta \theta\|_{H^s}^2 \, d\tau \leq C(\alpha, \beta, \theta(0)).
\]
Finally, we get \( \theta \in C_b(\mathbb{R}^+, H^s(\mathbb{R}^2)) \) and \( |\partial_1^\alpha \theta, |\partial_2^\beta \theta \in L^2(\mathbb{R}^+, H^s(\mathbb{R}^2)) \).

\[\square\]

5 Proof of corollary 2.4

Let \( \theta \) the global solution of \((AQG)\), then, \( \theta \in C(\mathbb{R}^+, H^s(\mathbb{R}^2)) \cap C((0, +\infty), H^2(\mathbb{R}^2)) \), moreover, for any \( T > 0 \)
\[
|\partial_1^\alpha \theta, |\partial_2^\beta \theta \in L^2([0, T], H^s(\mathbb{R}^2)).
\]
Taking \( t_0 > 0 \) and considering the following system
\[
\begin{cases}
\partial_t \gamma + u \cdot \nabla \gamma + |\partial_1|^{2\alpha} \gamma + |\partial_2|^{2\beta} \gamma = 0, \\
\gamma(0) = \theta(t_0) \in H^2(\mathbb{R}^2).
\end{cases}
\]
\[\text{(AQG2)}\]

Therefore, by Theorem 2.1, there exists a unique global solution of \((AQG)\):
\[
\gamma \in C_b(\mathbb{R}^+, H^2(\mathbb{R}^2)), \quad |\partial_1|^{\alpha} \gamma, |\partial_2|^{\beta} \gamma \in L^2(\mathbb{R}^+, H^2(\mathbb{R}^2)).
\]

By the uniqueness of solution we have
\[
\theta(t) = \gamma(t - t_0), \quad \forall t \geq t_0.
\]
Which implies that
\[
\|\theta(t)\|_{L^\infty(\mathbb{R}^+, H^s)} \leq \sup_{0 \leq t \leq t_0} \|\theta(t)\|_{H^s} + \sup_{t \geq t_0} \|\theta(t)\|_{H^s}
\]
\[
\leq \sup_{0 \leq t \leq t_0} \|\theta(t)\|_{H^s} + \sup_{t \geq 0} \|\gamma(t)\|_{H^2} < +\infty.
\]
Moreover
\[
\int_0^{+\infty} \|\partial_1^\alpha \theta\|_{H^s}^2 d\tau = \int_0^{t_0} \|\partial_1^\alpha \theta\|_{H^s}^2 d\tau + \int_{t_0}^{+\infty} \|\partial_1^\alpha \theta\|_{H^s}^2 d\tau \\
\leq \int_0^{t_0} \|\partial_1^\alpha \theta\|_{H^s}^2 d\tau + \int_{t_0}^{+\infty} \|\partial_1^\alpha \theta\|_{H^s}^2 d\tau < +\infty,
\]
and by the same technique we have
\[
\int_0^{+\infty} \|\partial_2^\beta \theta\|_{H^s}^2 d\tau \leq \int_0^{t_0} \|\partial_2^\beta \theta\|_{H^s}^2 d\tau + \int_{t_0}^{+\infty} \|\partial_2^\beta \theta\|_{H^s}^2 d\tau < +\infty.
\]
Finally, we get
\[
\theta \in C_b(\mathbb{R}^+, H^s(\mathbb{R}^2)), \quad |\partial_1^\alpha \theta, |\partial_2^\beta \theta \in L^2(\mathbb{R}^+, H^s(\mathbb{R}^2)).
\]

References

1. Amara, M., Benamer, J.: Global solution of anisotropic quasi-geostrophic equations in Sobolev space. J. Math. Anal. Appl. 516(1), 126512 (2022)
2. Bahouri, H., Chemin, J.Y., Danchin, R.: Fourier analysis and nonlinear partial differential equations, p. 523. Springer, Berlin (2011)
3. Brezis, H., Gallouet, T.: Nonlinear Schrödinger evolution equations. WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER (1979)
4. Brézis, H., Wainger, S.: A note on limiting cases of Sobolev embeddings and convolution inequalities. Commun. Partial Differ. Equ. 5(7), 773–789 (1980)
5. Constantin, P., Lax, P.D., Majda, A.: A simple one-dimensional model for the three-dimensional vorticity equation. Commun. pure appl. math. 38(6), 715–724 (1985)
6. Constantin, P., Majda, A.J., Tabak, E.: Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. Nonlinearity 7(6), 1495 (1994)
7. Cordoba, D.: Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation. Annals Math. 148(3), 1135–1152 (1998)
8. Ju, N.: Dissipative 2D Quasi-geostrophic equation: local well-posedness, global regularity and similarity solutions. Indiana Univ. Math. J. 56(1), 187–206 (2007)
9. Kato, T., Ponce, G.: Commutator estimates and the Euler and Navier-Stokes equations. Comm. Pure Appl. Math. 41, 891–907 (1988)
10. Kozono, H., Ogawa, T., Taniuchi, Y.: The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations. Mathematische Zeitschrift 242(2), 251–278 (2002)
11. Pedlosky, J.: Geophysical fluid dynamics, pp. 10–1007. Springer, New York (1987)
12. Stein, E.M.: Singular integrals and differentiability properties of functions. Princeton University Press, New Jersey (1970)
13. Ye, Z.: On the global regularity for the anisotropic dissipative surface quasi-geostrophic equation. Non-linearity 33(1), 72 (2019)
14. Ziemer, W.P.: Weakly differentiable functions. Springer Verlag, New York (1989)

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