Optimizing Type-I ($\alpha$) and Type-II ($\beta$) Error Probabilities by Game-Theoretic Linear Programming for Sequential Sampling Plans in Quality Control

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Abstract—A critical step in hypothesis testing at the computer theory and/or engineering decision-making stage is to optimally compute and use type-I ($\alpha$) and type-II ($\beta$) error probabilities. The article’s first research objective is to optimize $\alpha$ and $\beta$ errors, or producer’s and consumer’s risks, or risks of false positives (FP) and false negatives (FN) by employing the merits of a game-theoretical framework. To achieve this goal, the cross-products of errors and non-errors model is proposed. The second objective is to apply the proposed model to an industrial manufacturing quality control mechanism, i.e. sequential sampling plans (SSP). The article proposes an alternative technique compared to prematurely selecting the conventionally pre-specified type-I and type-II error probabilities. One studies mixed strategy, two-players and zero-sum games’ minimax rule derived by von Neumann and executed by Dantzig’s linear programming (LP) algorithm. Further, one equation for one unknown scenario yielding simple algebraic roots validate the computationally-intensive LP optimal solutions. The cost and utility constants are elicited through company-specific input data management. The contrasts between conventional and proposed results are favorably illustrated by tables, figures, individual and comparative plots, and Venn diagrams in order to modify and improve the traditionally executed SSP’s final decisions.

Index Terms—Cross-products of errors, minimax rule, accept-reject-continue-terminate, cost and utility.

I. INTRODUCTION

A. Motivations of Research Proposal and Outputs

1) The primary innovative motivation behind this article lies in optimizing type-I and type-II error probabilities, $\alpha$ and $\beta$, using a game-theoretic computationally intensive LP algorithm to improve the accuracy and credibility of statistical hypothesis testing outcomes in today’s quality control-conscious and information technology-savvy world. This is an alternative to the traditional assumptions of $\alpha$ and $\beta$, with no prior data-centric knowledge about hypothesis tests governing any design process.

2) The secondary motivation is to introduce and implement the hereby optimized alpha (producer’s) and beta (consumer’s) risks by employing the business costs and utility input data to item-by-item sequential sampling plans in industrial quality control. The goal is to aim for company-specific and data-centric quality-control inspection rather than heuristic or predetermined. This article, therefore, proposes an alternative to assuming ubiquitous producer’s (e.g., $\alpha=0.05$) and consumer’s risk values (e.g., $\beta=0.10$). The implementation interface to SSP is achieved through case studies and input data elicitation by user-friendly, easy-to-reproduce software algorithms with satisfactory outcomes.

B. Literature Survey for Type I & II Errors, Game Theory

Aside from the usual rule-of-thumb or best-guess or judgment-call-based choices of such as 1-out-of-20 or 1-out-of-50 etc., there have been alternative attempts to compute alpha (type-I error probability) by deriving the first and second derivatives of the standard Normal distribution curve. This is performed by determining the second derivative to reach a maximum at $z = \pm 1.732$ which corresponds to a $p$-value of 0.083. An alternative approach has been to find a point where the concavity in the Normal distribution curve is maximal to the first derivative. That is, the maximal curvature $k(z)$ occurs when $z = \pm 1.749$ corresponding to a $p$-value of 0.08. The $p$-value is used to reject $H_0$ for a given alpha. The calculus-based algebraic approaches have been earlier studied by Kelley [1] and Grant [2]. As Kelley quoted [1], “No one therefore has come up with an objective statistically based reasoning behind choosing the now ubiquitous 5% level, although there are objective reasons for levels above and below it. And no one is forcing us to choose 5% either.”

The history of type-I and type-II errors goes back to Neyman and Pearson [3] who discovered the problems associated with deciding whether or not a particular sample may be judged as likely to have been drawn from a specific population. They identified two sources of errors, type-I ($\alpha$) and type-II ($\beta$). They observed that “… If the probability of obtaining a result as extreme as the one obtained, supposing that the null hypothesis were true, is lower than a pre-specified cut-off probability (for example, 5%), then the result is said to be statistically significant and the null hypothesis is rejected.”

Fisher [4] proposed the level $P=0.05$ as a limit of statistical significance where he also originated in his book: “The value for which $P=0.05$ or 1 in 20 is 1.96 or nearly 2; it is convenient to take this point as a limit in judging whether a deviation is to be considered significant or not.” A prominent aspect of Neyman and Pearson’s [3] and Fisher’s [5] findings was that one never fully justified, or rigorously proved, as to why, e.g. $P=0.05$ or else was selected as a pre-specified cut-off probability. Over a century of alpha and beta error–related discussions, e.g. by Salkind [6] who wrote that no game-theory was recorded in plain hypothesis testing, and also by Hedberg [7] who recorded that the central theme for Type-II error, or the power, ($1-\beta$), revolves around a symbolic value of $\beta=0.2$, as in the SAGE Research: “…The convention
of the social sciences is to design studies with a power of at least 80% chance of detecting an effect...”

Game theory is a branch of mathematical sciences devoted to the logic of decision-making in societal, physical or managerial interactions, and concerns the behavior of decision-makers who influence each other for optimal resource allocations at times subject to budget constraints to maximize utility as studied by Sahinoglu et al. [8], Szidarovszky and Luo [9]. The statistical decision theory is a one-person game theory. The LP system of equations will optimize producer’s and consumer’s risks by two-player, zero-sum and mixed-strategy-based minimax rule by von Neumann [10] and von Neumann et al. [11] in 1928 and 1944 respectively. A similar algorithm was adopted in two proceedings by Sahinoglu et al. [12], [13] and a monogram [14] and a textbook by Sahinoglu [15]. The effort continued while minimizing COLLOSS (Column Loss) in the Eco-Risk article, and Oil-Drilling Spill Risk-themed article, respectively, in Sahinoglu et al. [16], [17].

Next comes what lies behind the LP problem by Dantzig [18]. The forward and backward proofs of a general representation theorem (GRT) are given by Lewis [19]. Introduction of game theory to risk analysis is by Cox [20].

C. Summary of Sections I to VI

After introduction of goals, outcomes, and an extensive literature survey in section I, the section II studies the game theory-linked LP methods to achieve the itemized goals via the cross-products of risks and non-risks with related definitions. Section III studies example 1 for statistical sequential sampling plans with an input data management scheme at large. Section IV details example 1 through a thematic Venn diagram in probabilistic terms. Section V verifies and justifies the proposed optimal method via the short-cut algebraic roots to lead to a simple computational scheme at large. Section VI conclusively summarizes the content with further research suggestions. It is time to compare different analytical approaches as follow.

II. CROSS-PRODUCTS OF ERRORS WITH CASES

The cross-products of errors and non-errors will be proposed and utilized to construct the LP algorithm to apply to statistical sequential sampling plans employed in industrial quality control.

A. LP Algorithm with Composite-, Partial- and Non-Risk Errors’ Cross-Products

The issue with the classical approaches to hypothesis testing in terms of alpha and beta errors is that the hand-picked ubiquitous assumptions such as α=0.01 or 0.05, and β=0.10 or 0.20 may be detached from the prevalent data-centric sources. Costs or utility (profit) associated with varying error values: (α and β), or non-error values: (1-α, and 1-β) and their cross-products: [α β], [β (1-α)], [(1- α) β] and [(1-α) (1-β)] in the form of partial producer’s or consumer’s risks, or both, or none, are not hitherto considered. No such errors may have incurred with no whatsoever financial loss for the producers and consumers with a complete market satisfaction due to the error-free pair: (1-β) and (1-α). A common routine as Neyman and Pearson and Fisher practiced, is to select type-I error probability (alpha) by an existing best-judgement call for H₀, and then, given an alternative set of H₁ values, to compute a set of type-II error probabilities (beta).

Note a critical detail here is such that the utility is a negative cost effect working versus the positive cost effect in the opposite direction, or vice-versa. For cost and utility concepts, which date back to Nicholas Bernoulli, a good argument is laid by Singpurwalla and Wilson [21]. However, game-theoretic LP methods have not been studied in hypothesis testing educational curricula. This is mainly because the applications to routine hypothesis tests with pertinent costs associated with type-I (α) and type-II (β) errors and their cross products, including utility or profit with respect to non-errors (i.e. confidence=1-α, and power =1-β), are not up to date rigorously formulated. In hypothesis testing, this article associates a variety of costs (income lost due to errors) or a utility (revenue profited due to non-error) and observes where the optimal α and β will unfold by employing the basic principles of the game-theoretic minimax rule. This is an alternative computational technique to the usual rule-of-thumb choices, such as α = 0.05 or α = 0.08 etc. or those by calculus algebra, as critiqued by Kelley [1] and Grant [2]. The proposed empirical and market-friendlier way is an objective approach compared to the previous subjective rule-of-thumb lacking cost and utility inputs. The hypothesis testing literature as in, e.g. Ostle and Mensing [22], lays two types of errors in Table I and Fig. 1:

1) Type-I error: This is when the analyst rejects a true null hypothesis. The probability of a type-I error is α, the significance level; also known as producer’s risk or false positive risk when H₀: Good quality versus H₁: Bad quality.

2) Type-II error: This is when the analyst fails to reject a false null hypothesis for an identical hypothesis as above, i.e. H₀ vs. H₁. The probability of committing a Type-II error, β, is also known as consumer’s risk or false negative risk.

The truth (reality) vs decision (test) of traditional hypothesis testing is conventionally framed as follows:

| Truth (Reality) vs Decision (Test) Elements of Test | Decision |
|---------------------------------------------------|---------|
| [True Ho, False Ho] | Reject Ho | Accept Ho |
| True Ho | Producer’s Risk: α error=FP | No Error=Confidence=1-α=TN |
| False Ho | No Error=Power=1-β=TP | Consumer’s Risk: β error=FN |

Fig. 1. Hypothesis testing plots of type-I (α) and II (β) errors by Neyman and Pearson [3] and Fisher [4], [5]; false positives (α=FP), false negatives (β=FN), true positives (TP) and true negatives (TN) according to Table I.
A. System of Equations to Optimize Type-I & Type-II Error Probabilities

The following subsections will demonstrate the setup of a LP system of equations to optimize type-I (α) and type-II (β) errors, i.e. producer’s and consumer’s risks, via the game theory as follow:

\[ a = P\{\text{Type-I error}\} = P\{\text{reject } H_0 | H_0 \text{ is true}\} \quad (1) \]

\[ \beta = P\{\text{Type-II error}\} = P\{\text{fail to reject } H_0 | H_0 \text{ is false}\} \quad (2) \]

The probability of not committing Type-I error and Type-II errors are defined as confidence or test specificity, and power or test sensitivity, all respectively. The power is given in (3).

\[ \text{Power} = (1-\beta) = P\{\text{reject } H_0 | H_0 \text{ is false}\} \quad (3) \]

Sharma et al. extensively studied these two test concepts, known as test specificity and test sensitivity [23].

The power of hypothesis testing is also represented as \([1-\beta(\theta)]\), where \(\theta\) denotes the true parameter value, e.g., population mean: \(\mu\) or population proportion: \(P\). The \(\beta(\theta)\), the complement of power, is known as the operating characteristics (OC) function used in quality control. The cross-products of errors and non-errors will be coupled with their associated costs. If the cost is negative, this denotes utility. Let \(P_{11} = a\beta, P_{12} = (1-\beta)\), \(P_{21} = (1-\alpha)\beta, P_{22} = (1-\alpha)(1-\beta)\) where \(a = P_{11} + P_{12}\) and \(\beta = P_{11} + P_{21}\). Note, \(C_{11}, C_{12}, C_{21}\) and \(C_{22}\) are the corresponding costs due to cross-products of errors. \(C_{22}\) is the constant due to non-errors while the cross-products in (4) sums to unity. Let \(a = .05, \beta = .10\) such that \(\text{Confidence} = 1-a = 1-.05 = .95, \text{and Power} = 1-\beta = 1-.10 = .90\). Then it follows:

\[ \{a\beta\} + \{(1-\alpha)\beta\} + \{(1-\alpha)(1-\beta)\} = 1; \ 0 < a, \beta <1 \quad (4) \]

\[ \text{FP}(=a) + \text{TN}(=1-a) + \text{FN}(=\beta) + \text{TP}(=1-\beta) = 2; \ 0 < a, \beta <1 \quad (5) \]

The cross-products of cubicles from Table I sums to unity in (4) as follows here: \(0.5\times(100+.05)+(9.5) = (10.05)\times.95 + .005\times 0.45 + 0.95\times 0.95 = 1\), whereas (5) yields 2. Let the cross-products obtained via Table I in (6) to (9) to be:

Composite riskiness \((CoR)\) = \(P_{11} = a\beta\) \quad (6)

Partial riskiness \((PR)\) due to purely \(a\) error = \(P_{12} = (1-\beta)\) \quad (7)

Partial riskiness \((PR)\) due to purely \(\beta\) error = \(P_{21} = (1-\alpha)\beta\) \quad (8)

Composite non-riskiness \((CoNR)\) = \(P_{22} = (1-\alpha)(1-\beta)\) \quad (9)

Nonlinear implies not necessarily linear but includes such functions by Rapcsak [24] for a smooth optimization.

III. SSP AND QUANTITATIVE EXAMPLE 1

The entwined goals of this article are set (i) to optimize \(alpha\) and \(beta\) errors by von Neumann’s LP-enabled minimax rule to the statistical hypothesis testing of good quality of a given lot versus bad quality, and (ii) to apply the preceeding approach to an attributes-type item-by-item SSP.

The test-statistics algorithm in a sequential sampling plan is different than a single-stage sampling by Montgomery [25] and Jamkhaneh and Gildeh [26] such that:

i) When plotted points stay within the limiting boundaries of a single-stage sampling plan, i.e. \(AQL\) (Acceptable Quality Level) and \(RQL\) (Rejectable Quality Level), continue-sampling decision takes over and another sample must be drawn for continued testing.

ii) When plotted points fall on or above the upper limiting level, \(RQL\), the lot is rejected.

iii) When plotted points fall on or below the lower limiting level, \(AQL\), the lot is accepted.

iv) When a threshold sample size (=3n) is reached, and no accept or reject action taken, and continue-sampling prevails, terminate.

Gaus et al. [27] rather than making an absolute decision of accept or reject, refer to lot acceptance/rejection with a confidence interval. However, statistically, a more popular approach is a sequential sampling plan when the analyst keeps testing items from the batch (or lot) and render a decision to either i) continue sampling after each item is inspected, ii) reject, or iii) accept, or iv) finally terminate SSP. The distinction of multiple sampling from SSP is that the maximum number of samples for SSP is prespecified. With sequential sampling, one could end up conducting 100% inspection on the entire batch. The SSP are truncated after the number inspected reaches three times the count with single sampling plan by Beasley [28]. See Theorem 1: Sequential Probability Ratio Test (SPRT) under Wald’s lemma [29] for the SSP and by Roussas [30]. Graphically, SSP can be plotted in Fig. 5 to 9 in subsections III.B and III.C where the cumulative sample size is \(n\) and the cumulative number of defects is \(X\) in the Engineering Statistics Handbook by NIST [31] and textbook by Montgomery [25].

Acceptance (single-stage) sampling plans were historically first proposed by Dodge and Romig [32]. The producer’s and consumer’s risks occur due to the draw of an unrepresentative sample for either wrongly rejecting a lot containing an acceptably small amount of defectives, or accepting a lot containing an unacceptably large amount of defectives, respectively. Here, single-stage acceptance plans are not in focus, but SSP are. Briefly, type-I error of the producer’s risk (5% is common) is the probability of rejecting a good lot or batch, Type-II error of the consumer’s risk (10% is typical), whereas, is the probability of not-rejecting a bad lot. The 5\% (=\(\alpha\)) or 10\% (=\(\beta\)) cited are subjective takes per standard assumptions adopted by MIL-STD-1916 [33].

A. Example 1: Input Data Management of Cost and Utility Constants, and LOSS Variable

Take an illustrative example 1 regarding a hypothetical \(EAP\) (Electric Auto Production) plant as follows with SSP on attributes. Critically embedded chips for electric cars are purchased on item-by-item sampling with \(n\)(sample size)=100 per batch delivery. Let \(p=aQL=\text{Acceptable Quality Limit}=.01\), and \(p=RQL=\text{Rejectable Quality Limit}=.10\) with \(a=.05\) and \(\beta=.1\) for producer’s and consumer’s risks to revisit in section III.B.

How to collect the \(C_{ij} = [C_{11}, C_{12}, C_{21}, C_{22}]\) costs, or the input constants by an enterprise, poses several challenging
limitations. From the corporate world’s sales accounting logs about this hypothetical example 1, this article suggests practical ways to meet the input data demand challenges. Many of the larger merchandisers will break the returns down into four distinct groups, according to a detailed accounting analysis by Hoare [34] as follows in random order.

1) $C_{ij}$. This first group reflects solely the consumer’s faults or customer-based mistakes. This is attributed to producer’s risk experienced by the producer due to an accrued financial loss, $S C_{ij}$. As a merchantiser, one needs to monitor the growth rate for this group. If this unwanted trend begins to rise, it might be a sign that the sales staff is unethically forcing the wrong product onto the market, hence ending up with the consumers who are clueless of what they actually purchased and how best to use it.

2) $C_{ij}$: The other form of a return is a type of merchandise that is broken, or has a quality issue. That is, it’s not the consumer’s fault but that of the producer. This is attributed to the consumer’s risk experienced by the consumer due to an accrued financial loss, $S C_{ij}$. If the issue is brand-related, the producer or manufacturer may consider discontinuing the brand to substitute it with a higher quality product. Popular examples are recall actions in the automotive industry. Class-actions favoring the consumers have recently become a commonplace event.

3) $C_{ij}$. The two adjustments above in the business world are followed by another elusive described item as allowance, discount or incentive, or occasionally a write-off. These vague adjustments to normal sales reflecting defective items or courtesy calls for failure in delivering the product or service in a timely fashion cause issues. Re-education of the sales representatives may be required if customers’ erroneous returns increase since this relates to a wrong kind of purchase. Consumers may not be educated for what they purchased. They claim, the product is defective but it truly is not. This is both consumer’s ($\beta$) and producer’s ($\alpha$) risks merged and compounded, bearing an accrued financial loss, $S C_{ij}$.

4) $C_{ij}$: This is the uncontroverted utility, not returned, with 100% customer satisfaction and no serious issues intercepted. Revisiting the EAP-themed example 1 with Hoare [34] taken as a guide, where Total Sales subtracted by Adjustments (Returns + Allowances + Discounts) denotes the Net + Other Sales. Expressed otherwise, let’s define elements as follow: $SUM(C_{ij})$ = Total Sales: XX,XXX.XXX. $C_{ij}$ = Customer-based returns (due to consumer’s unjustified faults): SXX,XXX. $C_{ij}$ = Producer-based returns (such as recalls due to company-generated faults): SXX,XXX. $C_{ij}$ = Allowances and Discounts (such as write-offs released by the company after an arbitration process, or else, in case the court case costs more for the ambiguous and non-explicit issues due to consumer bad-debt or vendor’s partially unusable bad merchandise, which may not be worth extra re-shelving or re-stocking costs): SXX,XXX. $C_{ij}$ = Net Sales (trouble-free) + other revenues, like insurance or warranty agreements: $S XX$, where this utility quantity when input into the game-testing software of Table III, is taken as a negative cost (since it is a utility): $-S [XX, XXX + XXX]$. 

Covering LOSS = $5K (or $3K) in example 1, the following arguments are in place: If the LOSS variable constraint is taken as $-LOSS \leq \$5K \text{ (or } \$3K)$, LOSS denotes a tolerance and a company-sponsored minimum indemnity to intercept the damage incurred after deductibles due to each of the risk-related constraints per equations (12) to (15). $P_{ij}^{C_{ij}} < LOSS$ for $i, j = 1, 2$ where each of these four constraints are bound not to exceed LOSS = $5K$ (or $3K$), including equation (15) where the utility constant readily obeys. LOSS, akin to a company-paid compensation, is a variable different than $C_{ij}$. The LOSS variable is minimized by the $LP$’s objective function of Min LOSS.

B. Proposed Method Applied to Attributes-Type Sequential Sampling in Example 1

To recap, let the $C_{ij}$ vector be the most recent averages from the Electric Automobile Production (EAP) plant of subsection III-A as a set of hypothetical input data: Total Sales Revenue = $1,000,000 or $1,000K$ where $K=1,000$. No Adjustments Sale = $800K$. Uncontested and suspicion-free non-returned income.

Adjustments Returns = $150K$. Revenue lost from producers’ wrong-doings (consumer’s risk) and consumers’ non-compliance (producer’s risk) are broken down: Consumer-Based = $110K$. Consumer’s noncompliance may cause civil penalties.

Producer-Based = $40K$. Producer’s wrongdoing causes, e.g., recall or class actions.

Allowances or Write-Offs = $50K$. Consumer’s non-adherence can overlap with producer’s errors leading to an inseparable blend of producer’s and consumer’s risks, yielding arbitration.

Based on this breakdown, one examines what kind of a SSP which the CFO (Company Financial Officer) in charge of managerial finances, will risk while the EAP operates optimally lucrative. Select as preceded, $C_{ij} = [C_{ij} = $50K, $C_{ij} = $110K, $C_{ij} = $40K, $C_{ij} = $800K]$ for EAP’s input set is used in Tables II to VIII. The EAP case study uses LOSS = $5K$ or LOSS = $5K$ after deductibles akin to a company’s indemnities to meet any unexpected or emergency rainy-day funds. Table II displays the action-loss based game-theoretic $LP$ formulation of the SSP:

| Actions Taken by Player1 (Return Policy) | EL for action $a_i$ given $C_{ij}$ incurred on Player2 |
|----------------------------------------|---------------------------------------------------|
| $a_1$ (Actn 1: Ambiguous Fault-based Rtn) | $EL(a_1) = S P_{i_1}C_{ij} \leq LOSS$ |
| $a_2$ (Actn 2: Consumer’s Fault-based Rtn) | $EL(a_2) = S P_{i_2}C_{ij} \leq LOSS$ |
| $a_3$ (Actn 3: Producer’s Fault-based Rtn) | $EL(a_3) = S P_{i_3}C_{ij} \leq LOSS$ |
| $a_4$ (Actn 4: No Fault No Rtn Ideal Sale) | $EL(a_4) = S P_{i_4}C_{ij} \leq LOSS$ |

Table II shows how Player2 can find its optimal mixed strategy. The goal here is to calculate probabilities $P_{ij}$ to minimize the expected loss in the SSP process incurred upon Player2 regardless of the strategy executed by Player1. In essence, Player2 will protect itself from any strategy selected by Player1 by making sure its expected market loss is as small as possible even if Player1 selects its own optimal strategy to maximize gain. Given the probabilities, $P_{ij}$ for $i, j = 1, 2$ and the expected losses in Table II, the game theory assumes that Player1 will select a strategy that causes the maximum expected loss incurred upon Player2 based on equation (10):
However, when Player1 selects its strategy, the value of the game will be the expected maximum gain such that this will maximize Player2’s expected loss as well. On the other hand, Player2 will select its optimal mixed strategy using a minimax strategy to minimize the maximum expected loss based on (11) to yield the objective function for Anderson et al. in [35]:

\[
\text{Min } \{ \text{Max } (EL(a_i), EL(a_2), EL(a_3), EL(a_4)) \}
\]

\[
\text{Max } \{ EL(a_1), EL(a_2), EL(a_3), EL(a_4) \}
\]

Finally, (11) identifies the Neumann’s MINIMAX rule. In case the players are reversed, and GAIN replaces LOSS. Then the MAXIMIN rule will replace the MINIMAX rule. The LP system of equations are governed by an objective function. The following spreadsheets show the input and output with an LP algorithm, whereas (19) denoting total cost ($) units accrued is constrained for a maximum net profit. If the LOSS variable is as such: \( \text{LOSS} \geq 5 \), with (12) to (15), one completes the LP system of equations given the binding constraints to minimize the objective function of \( \text{Min LOSS} \) (or MAX GAIN) subject to constraints of (12) to (19) with a solution vector \( P_y = [P_{11}, P_{12}, P_{21}, P_{22}] \), LOSS variable, \( C_y = [C_{11}, C_{12}, C_{21}, C_{22}] \) as inspired by Table II:

\[
P_{11} C_{11} + P_{12} C_{12} + P_{21} C_{21} + P_{22} C_{22} = 1
\]  

\[
0 \leq P_{ij} \leq 1, i,j=1,2
\]  

\[
\text{LOSS} \geq \text{LOSS}_{\text{min}}
\]  

\[
P_{11} + P_{12} + P_{21} + P_{22} = 1
\]

\[
\sum \{P_y, C_y\} = P_{11} C_{11} + P_{21} C_{21} + P_{12} C_{12} + P_{22} C_{22} \leq 0
\]

Assume \( C_y = [C_{11}=$$50, C_{12}=$$110, C_{21}=$$40, C_{22}=-$$800] \), and observe the input and output for Player2’s optimal mixed strategy in Table III (input), Table IV (\( P_y \) for various \( \text{LOSS} \)), Table V (\( \text{LOSS} \geq 3 \)) and Table VI (\( \text{LOSS} \geq 5 \)). If Player2 uses this optimal mixed strategy, Player2’s expected loss for each Player1 strategy follows in Table VII for \( \text{LOSS} \geq 3 \) and Table VIII for \( \text{LOSS} \geq 5 \) with constraints regarding (12) to (15). The vector \( P_y \) is defined as a minimax mixed-strategy solution.

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**Example 1 for \( C_{ij}, i,j=1,2; \text{LOSS} \geq 3 \) with Excel Solver LP**

![Excel Solver Output](image1)

**Example 1 for \( C_{ij}, i,j=1,2; \text{LOSS} \geq 5 \) with Excel Solver LP**

![Excel Solver Output](image2)

Fig. 2 and Fig. 3 yield the minimax rule-based \( \alpha \) and \( \beta \) errors and expected total cost following Table III to VIII.
Note in the EXCEL SOLVER of Table VII and Table VIII, the (non)linear engine serves for constrained minimization problems with differentiable (where partial derivatives of order k are continuous) nonlinear and convex functions, smooth of order k by Rapcsak [24]. This includes the case where all functions are linear, i.e. the LP problem. Revisit section III.A’s example 1 regarding the previously outlined hypothetical EAP (Electric Auto Production) industrial enterprise, which refers to an item-by-item SSP on attributes. Let \( p_1 \) or AQL=Acceptable Quality Limit=.01, and let \( p_2 \) or RQL=Rejectable Quality Limit=.10, and \( \alpha=.05 \) and \( \beta=.10 \) given for the classical producer’s and consumer’s risks. Wald [29], Roussas [30] and NIST [31] give the SSP equations regarding the SPRT (Sequential Probability Ratio Test) for testing \( H_0: p=p_1 \) vs \( H_1: p=p_2 \). The equations for the limit lines with parameters \( p_1, p_2, \alpha, \beta \) for Exp#1 (Note, Exp#1 short for Experiment#1) follow in (20) to (25). Table IX’s Exp#1 to Exp#5 are plotted individually and pairwise in Fig. 5 to 9. Slope is \( s \) and intercepts are \( h_1 \) and \( h_2 \) of Fig. 4. Enter inputs:

\[
\begin{align*}
\text{k} &= \log\left(\frac{p_3(1-p_1)}{(p_1(1-p_2))}\right) = 1.041 \\
\text{h}_1 (\text{accept}) &= \left(\frac{1}{k}\right)\log\left(\frac{(1-\alpha)}{\beta}\right) = 0.939 = 0.94 \\
\text{h}_2 (\text{reject}) &= \left(\frac{1}{k}\right)\log\left(\frac{(1-\beta)}{\alpha}\right) = 1.206 \approx 1.21 \\
\text{s} &= \frac{1}{k}\log\left(\frac{(1-p_1)}{(1-p_2)}\right) = 0.039747 = 0.04 \\
\text{X}_A (\text{acceptance line}) &= sn - h_1 = 0.04n - 0.939 \\
\text{X}_R (\text{rejection line}) &= sn + h_2 = 0.04n + 1.206
\end{align*}
\]

Apply the solutions to the SSP for Exp#1, Exp#2, Exp#3, Exp#4 and Exp#5 by varying type-I and type-II errors in Table IX from Fig. 2 and Fig. 3 and Table III to Table VIII.

C. Numerical Results of Attributes-Type Sequential Sampling Plans: Experiments #1 to #5

The solution vector for LOSS=$5 based on Tables III, IV, VI, VIII and Fig. 3, as plotted to follow up are, \( \alpha = P_{11} + P_{12} = 1.14+.045=.145 \), \( \beta = P_{11} + P_{22} = .1 + .125 = .225 \) for Exp#2. Also, \( \alpha' \) (disjoint pure alpha) \( = P_{12} = .045 \) and \( \beta' \) (disjoint pure beta) \( = P_{22} = .125 \) for Exp#3. For LOSS=$3 by Tables III, IV, V, VII and Fig. 2, the aggregate \( \alpha = P_{11} + P_{12} = .06 + .027 = .087 \) and the aggregate \( \beta = P_{11} + P_{22} = .06 + .075 = .135 \) are for Exp#4. Also \( \alpha' \) (disjoint pure alpha) \( = P_{12} = .027 \) and \( \beta' \) (disjoint pure beta) \( = P_{22} = .125 \) for Exp#5.

The individually plotted sequential sampling plans in Fig. 5 to Fig. 9, respectively, as revealed by Tables IX to XI such that the number of accepts or rejects when continue-sampling ends at \( n=100 \) is differing from that of the classical Exp#1 in Fig. 5. The proposed Exp#3 and Exp#5 with varying LOSS, such as $5K and $3K in Tables IX to XI show that as LOSS value decreases, the aggregate \( \alpha \) and \( \beta \) while reduced to disjoint \( \alpha' \) and \( \beta' \) lift \( h_1 \) and \( h_2 \) to mark the difference. Observe Table X with \( h_1 = .848 \rightarrow 1.069, h_2 = 1.238 \rightarrow 1.474 \).
The disjoint \((\text{LOSS}=5K)\)'s \(\text{Exp}3\) in Tables IX to XI, #defects turns (+) \(\oplus\) \(n=14\), continue sampling for \(n<14\).

The disjoint \((\text{LOSS}=5K)\)'s \(\text{Exp}3\) in Tables IX to XI, #defects turns (+) \(\oplus\) \(n=22\), continue sampling for \(n<22\).

The disjoint \((\text{LOSS}=3K)\)'s \(\text{Exp}5\) in Tables IX to XI, #defects turns (+) \(\oplus\) \(n=21\), continue sampling for \(n<21\).

### TABLE IX: The Input and Output Parameters for the Individual and Comparative Plots in Fig. 5 to 9 Where \(a\) and \(a' = a - n\) are Aggregate and Disjoint Type-I Errors, and \(\beta\) and \(\beta' = \beta - n\) are Aggregate and Disjoint Type-II Errors, Respectively

| Experiment | Type I | Type II | \(p_1\) | \(p_2\) | \(k\) | \(h_1\) | \(h_2\) | \(s\) |
|------------|--------|---------|---------|---------|------|--------|--------|------|
| 1          |       |         |         |         |      |        |        |      |
| 2          |       |         |         |         |      |        |        |      |
| 3          |       |         |         |         |      |        |        |      |
| 4          |       |         |         |         |      |        |        |      |
| 5          |       |         |         |         |      |        |        |      |
| 6          |       |         |         |         |      |        |        |      |
| 7          |       |         |         |         |      |        |        |      |
| 8          |       |         |         |         |      |        |        |      |
| 9          |       |         |         |         |      |        |        |      |

### TABLE X: Input Entries and Output Values from Table IX are Plotted in Fig. 5 to 9. \(\text{Exp}1\)'s Acceptance Value is the First Integer \(\leq X_0\), 0.04n = 0.94 for \(n=1\) to 100 (Table X's 1st COLUMN for \(h_1\) = 94). Also, the Rejection Value is the Next Integer \(>X_0\), 0.04n + 1.21 (the 6th COLUMN of Table X for \(h_1\) = 121). For \(n=1\), the Acceptance, -1, is Impossible. The Rejection, 2, is Impossible. At Last at \(n=24\), as in Fig. 5 and Table X, \(X_0\) is 0 and \(X_2\) is 3. In Table XI, \(X\) MEANS CONTINUE SAMPLING WHEN NO ACCEPTANCE OR REJECTION OCCURS. \(\{n_{\text{max}}=100, n_{\text{min}}=3, n_{\text{step}}=2\}\) IS FOR THE CONVENTIONAL \(\text{Exp}1\), WITH \(\{n_{\text{max}}=100, n_{\text{min}}=3, n_{\text{step}}=6\}\) IS FOR THE PROPOSED \(\text{Exp}5\) IN TABLES X AND XI. \(\text{Exp}1\) & \(\text{Exp}3\) ARE SAME

| n         | \(h_1\) | \(h_2\) |
|-----------|--------|--------|
| 0         | 0.94   | 0.95   |
| 1         | 1.00   | 1.00   |
| 2         | 1.01   | 1.01   |
| 3         | 1.02   | 1.02   |
| 4         | 1.03   | 1.03   |
| 5         | 1.04   | 1.04   |
| 6         | 1.05   | 1.05   |
| 7         | 1.06   | 1.06   |
| 8         | 1.07   | 1.07   |
| 9         | 1.08   | 1.08   |
| 10        | 1.09   | 1.09   |
| 11        | 1.10   | 1.10   |
| 12        | 1.11   | 1.11   |
| 13        | 1.12   | 1.12   |
| 14        | 1.13   | 1.13   |
| 15        | 1.14   | 1.14   |
| 16        | 1.15   | 1.15   |
| 17        | 1.16   | 1.16   |
| 18        | 1.17   | 1.17   |
| 19        | 1.18   | 1.18   |
| 20        | 1.19   | 1.19   |
| 21        | 1.20   | 1.20   |
| 22        | 1.21   | 1.21   |
| 23        | 1.22   | 1.22   |
| 24        | 1.23   | 1.23   |
| 25        | 1.24   | 1.24   |
| 26        | 1.25   | 1.25   |
| 27        | 1.26   | 1.26   |
| 28        | 1.27   | 1.27   |
| 29        | 1.28   | 1.28   |
| 30        | 1.29   | 1.29   |
| 31        | 1.30   | 1.30   |
| 32        | 1.31   | 1.31   |
| 33        | 1.32   | 1.32   |
| 34        | 1.33   | 1.33   |
| 35        | 1.34   | 1.34   |

### TABLE XI: \(\text{Exp}1\)'s (\(n_{\text{max}}=1\) \(=1\)), \(\text{Exp}3\) (LOSS = 5K) and \(\text{Exp}5\) (LOSS = 3K) By Tables IX and X While Decision-Making Differences Are Marked In Rows 24, 22, 27 for \(\text{Exp}1\), \#3 and \#5 Respectively

| \(n_{\text{max}}\) | \(n_{\text{min}}\) | \(n_{\text{step}}\) | \(\text{Exp}1\) | \(\text{Exp}3\) | \(\text{Exp}5\) |
|-------------------|-------------------|-------------------|-----------------|-----------------|-----------------|
| 1                 | 2                 | 2                 | 1               | 1               | 1               |
| 2                 | 2                 | 2                 | 2               | 2               | 2               |
| 3                 | 3                 | 3                 | 3               | 3               | 3               |
| 4                 | 4                 | 4                 | 4               | 4               | 4               |
| 5                 | 5                 | 5                 | 5               | 5               | 5               |

### TABLE XI: \(\text{Exp}1\)'s (\(n_{\text{max}}=1\) \(=1\)), \(\text{Exp}3\) (LOSS = 5K) and \(\text{Exp}5\) (LOSS = 3K) By Tables IX and X While Decision-Making Differences Are Marked In Rows 24, 22, 27 for \(\text{Exp}1\), \#3 and \#5 Respectively

| \(\text{Exp}1\) | \(\text{Exp}3\) | \(\text{Exp}5\) |
|-----------------|-----------------|-----------------|
| 1               | 1               | 1               |
| 2               | 2               | 2               |
| 3               | 3               | 3               |
| 4               | 4               | 4               |
| 5               | 5               | 5               |
with no erroneous returns) = -$800 K. Disjoint total cost is thus -$712 K = α'Cj2 + β'Cj1 + (αβ')Cj2 + (1 - αβ')Cj2. = .027 * $110 + .075 * $40 + 0 + .898 * -$800 since (αβ') overlap of disjoint α and β changed to 0 in Fig. 10.c (Venn diagram) from a non-zero in Fig. 10.b (Venn diagram).

For LOSS = $55K (in Exp#3), disjoint total cost is .045 * $110 + .125 * $40 + 0 + .83 * (-$800) ≈ -$654 K. See Champerowne [36] on the SSP costings for accept, reject and continue-sampling, and Wü rlander [37] on the SPRT performance such as the average sample size (ASN).

The company-specific input cost data produces the aggregate alpha (α) = .145, and the aggregate beta (β) = .225 for LOSS = $5 in Exp#2 per Tables IX to XI. Likely, the aggregate alpha (α) = .087 and the aggregate beta(β) ≈ .135 are for LOSS = $3 in Exp#4. The proposed α*(disjoint pure alpha) = P12 = .045 and β*(disjoint pure beta) = P21 = .125 in Exp#3 are for LOSS = $55 K. Also, α*(disjoint pure alpha) = P12 = .027 and β*(disjoint pure beta) = P21 = .125 in Exp#5 are for LOSS = $3 K. The deviations between Exp#1 (classical) and Exp#3 (proposed with LOSS = $5), and likely Exp#1 (classical) and Exp#5 (proposed with LOSS = $3) are what the article draws attention to by using Cj, i=1, 2, j=1, 2 and a LOSS variable.

IV. VENN DIAGRAMS TO VERIFY OPTIMIZATION

In Fig. 10, Venn diagrams constituting all four sample sets of V are studied; where V stands for vulnerability, which points out to an erroneous decision-making set. Note that [αβ + α(1-β) + (1-α)β + (1-α)(1-β)] = 1 via (4) and (18). The composite sample V1, which aggregates the common-errors intersection V1∩V2 has elements due to the producer’s risk, such as consumers misusing the vehicle and returning e.g. a hybrid vehicle to the dealership due to customers’ user faults per example 1. The composite sample V2 too contains the common-errors intersection V2∩V1, and denotes the consumer’s risk such as factory-recalls or class actions due to the producer’s faults. The discontent consumer returns e.g. the vehicle, to the vendor as in example 1 of section 3.

D. Optimal Solutions’ Interpretations: Aggregate (Composite) and Disjoint (Pure) Risks

With LOSS = $3 K (in Exp#5) of Table IX if the plotted points stay within the limiting boundaries (AQL and RQL), the sequential sampling plan continues and hence, another sample to be drawn by [$1.0 - (α* = .027) - (β* = .075)] * 100% = 89.8% for the percentage of the continue-sampling decision. This results from example 1’s input in Table III per SSP under scrutiny. The expected total cost ([=alpha * relative cost of alpha error + beta * relative cost of beta error + (1-alpha-beta) * relative utility of no errors]) is -$712 K. Note, Cj2 (relative cost of alpha) = $110 K and Cj1, (relative cost of beta) = $40 K, Cj2 (relative cost of the cross-product or intersection of alpha and beta errors) = $50 K and Cj2 (relative utility of no errors, denoting complete satisfaction

\[
\begin{align*}
5 & \times 2 \quad 5 \times 2 \quad 5 \times 2 \\
6 & \times 2 \quad 6 \times 2 \quad 6 \times 2 \\
7 & \times 2 \quad 7 \times 2 \quad 7 \times 2 \\
8 & \times 2 \quad 8 \times 2 \quad 8 \times 2 \\
9 & \times 2 \quad 9 \times 2 \quad 9 \times 2 \\
10 & \times 2 \quad 10 \times 2 \quad 10 \times 2 \\
11 & \times 2 \quad 11 \times 2 \quad 11 \times 2 \\
12 & \times 2 \quad 12 \times 2 \quad 12 \times 2 \\
13 & \times 2 \quad 13 \times 2 \quad 13 \times 2 \\
14 & \times 2 \quad 14 \times 2 \quad 14 \times 3 \\
15 & \times 2 \quad 15 \times 2 \quad 15 \times 3 \\
16 & \times 2 \quad 16 \times 2 \quad 16 \times 3 \\
17 & \times 2 \quad 17 \times 2 \quad 17 \times 3 \\
18 & \times 2 \quad 18 \times 2 \quad 18 \times 3 \\
19 & \times 2 \quad 19 \times 3 \quad 19 \times 3 \\
20 & \times 3 \quad 20 \times 3 \quad 20 \times 3 \\
21 & \times 3 \quad 21 \times 3 \quad 21 \times 3 \\
22 & \times 3 \quad 22 \times 0 \quad 3 \quad 22 \times 3 \\
23 & \times 3 \quad 23 \times 0 \quad 3 \quad 23 \times 3 \\
24 & 0 \quad 3 \quad 24 \times 0 \quad 3 \quad 24 \times 3 \\
25 & 0 \quad 3 \quad 25 \times 0 \quad 3 \quad 25 \times 3 \\
26 & 0 \quad 3 \quad 26 \times 0 \quad 3 \quad 26 \times 3 \\
27 & 0 \quad 3 \quad 27 \times 0 \quad 3 \quad 27 \times 3 \\
28 & 0 \quad 3 \quad 28 \times 0 \quad 3 \quad 28 \times 3 \\
29 & 0 \quad 3 \quad 29 \times 0 \quad 3 \quad 29 \times 3 \\
30 & 0 \quad 3 \quad 30 \times 0 \quad 3 \quad 30 \times 3 \\
31 & 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\
32 & 95 \quad 2 \quad 6 \quad 95 \quad 2 \quad 6 \quad 95 \quad 2 \quad 6 \\
33 & 96 \quad 2 \quad 6 \quad 96 \quad 2 \quad 6 \quad 96 \quad 2 \quad 6 \\
34 & 97 \quad 2 \quad 6 \quad 97 \quad 3 \quad 6 \quad 97 \quad 2 \quad 6 \\
35 & 98 \quad 2 \quad 6 \quad 98 \quad 3 \quad 6 \quad 98 \quad 2 \quad 6 \\
36 & 99 \quad 3 \quad 6 \quad 99 \quad 3 \quad 6 \quad 99 \quad 2 \quad 6 \\
100 & 3 \quad 6 \quad 100 \quad 3 \quad 6 \quad 100 \quad 2 \quad 6
\end{align*}
\]
One proceeds to \( V_1 \cap V_2 \), the intersection of \( V_1 \) and \( V_2 \) comprising both error regions, \( \alpha \) and \( \beta \), in the realm of an ambiguous or controversial decision described in section III.

For the adjustments option, this was classified as allowances or write-offs, which are explained in-depth in subsection III.A. The Venn diagram’s blank error-free region is \( V_1 \cap V_2 \) for none of \( \alpha \) and \( \beta \) errors involved. \( V_1 \) and \( V_2 \) are complements for \( V_1 \) and \( V_2 \), respectively. See Sahinoglu [15] where \( V_1 \) and \( V_2 \) are independent, i.e., not independent, since \( P(V_i \cap V_j) \neq P(V_i)P(V_j) \). Why? Because \( .12 \neq 145* .225 = .033 \), since in the preceding example 1: \( P(V \cap V_1) = P = P_{1,1} = .145, P(V_2) =.225 \). Thus, \( P(V_i \cap V_j) \neq P(V_i)p(V_j) \), is equivalent to expressing \( \alpha^{*} \beta \neq \alpha \) times \( \beta \).

Conditionally dependent samples \( V_1 \) and \( V_2 \) are not independent, but \( P(V_1 \cap V_2) = P(V_1|V_2)P(V_2) = P(V_2|V_1)P(V_1) \). Fig. 10a, 10b, and 10c are the Venn diagram samples.

Let \( P(FP \land FN) = P(\alpha|\beta) \neq 0; \) let \( P(FP \land FN) = P(\alpha|\beta) = 0, \) and \( d = \) Disjoint. Let the 1.h.s in Fig. 10b, the light-blue \( V_i^{*} = \) Disjoint producer’s risk with \( P(V_i \cap V_j) = .045 = P_{12} = \alpha^{*} \).

Let the r.h.s-light-blue \( V_i^{*} = \) Disjoint consumer’s risk with \( P(V_i | V_j) = .125 = P_{23} = \beta^{*} \). Let the middle-dark-blue (\( V_i \cap V_j = \) Intersection of producer’s and consumer’s risks, and \( P(V_i | V_j) = .1 = P_{11} \). Let the blank \( V_i \cap V_j = \) error-free region with no producer’s and no consumer’s risks for \( P(V_i | V_j) = .73 = P_{22} \.

Note, also, \( P(V_i | V_j) = 1 \) is identical to \( P(V_i^{*} | V_j^{*}) = \alpha^{*} \) and \( P(V_i^{*} | V_j^{*}) = \beta^{*} \) or by (18), \( P_{22} = P_{23} = 1.0 \) or \( P_{11} = 1.0 \) and \( P_{12} = 1.0 \). \( \alpha^{*} \beta \) and \( \alpha^{*} \beta^{*} \) are identical to \( \alpha \beta \).

\( \alpha^{*} \beta^{*} \) is Disjoint and \( P_{11} = 1 = P(\alpha^{*} \beta^{*}) = 0 \). Fig. 10c is related to Exp#2 and Exp#4, while \( P(\alpha^{*} \beta^{*}) = P_{12} = 0 \) in Fig. 10.b is related to Exp#3 and Exp#5. The blank is \( P_{22} = P_{23} = 1 \) or \( \alpha^{*} \beta^{*} \) and \( \alpha^{*} \beta \) as per Fig. 10.b for \( P_{11} = P(\alpha^{*} \beta^{*}) \neq 0; \alpha, \beta \) are aggregates. The blank \( P_{22} = 1 - \alpha^{*} \beta^{*} \) in Fig. 10.c: \( \alpha^{*}, \beta^{*} \) are disjoint and \( P_{11} = 1 = P(\alpha^{*} \beta^{*}) \).

V. ALGEBRAIC ROOTS TO VERIFY LP VECTOR SOLUTION

There exists a favorable shortcut technique to serve as an optimality verification tool without using the software programs so as to validate the LP-based feasible solution vector, \( P_{ij} \) given \( Cij \) and LOSS variable(s). What plays a crucial role here is actually the LOSS variable constraint.

Once the LOSS variable is accurately constrained by the financial analyst in (17), it is a simple algebraic task to compute the \( P_{ij} \) roots. That is, \( P_{ij} = LOSS/Cij \) giving the constant \( Cij \) for all \( i \) and \( j \) excluding \( i=2, j=2 \). Once \( P_{11}, P_{12} \) and \( P_{23} \) are calculated, one finds \( P_{22} = 1 = P_{11} + P_{12} + P_{23} \) by subtraction per equation (18) i.e., \( P_{ij} + P_{12} + P_{23} + P_{22} = 1 \) with \( \alpha \) (aggregate or composite) = \( \alpha^{*} \) and \( \beta \) (aggregate or composite) = \( \beta^{*} \) per Fig. 4 clarifies the 3 disjoint actions.

A. Simple Algebraic Root Solutions Applied to Example 1

Example 1 delineates that given input vector, \( Cij = C_{ij} = [S10, S20, S30, S40, S50] \) of LOSS\$55K and LOSS\$35K, respectively; the solution vectors are \( P_{ij} = [P_{11} = .1, P_{12} = .045, P_{23} = .125, P_{22} = .73] \) for LOSS\$55K and \( P_{ij} = [P_{11} = .06, P_{12} = .022, P_{23} = .075, P_{22} = .84] \) for LOSS\$35K.

Table VIII for LOSS\$55K, displaying the EXCEL Solver input and output shows that the three constraints (#3, #4 and #5) referring to (12) to (14) yield \( \hat{P}_{ij} = LOSS/Cij \). Then, \( \hat{P}_{11} = .5/50 = .027 \) and \( \hat{P}_{23} = .1 \cdot .125 = .25 \) as composite errors in Tables III, IV, VII, VIII and Fig. 3. For LOSS\$35K in Table VIII, \( \hat{P}_{ij} = LOSS/Cij \) and \( \hat{P}_{ij} = .055/50 = .027 \) and \( \hat{P}_{22} = .1 \cdot .125 = .387 \) and \( \hat{P}_{22} = .1 \cdot .125 = .135 \). These concur with the software solution vectors in section III’s Tables III, IV, VII and Fig. 2.

Similarly, \( \hat{a} = \hat{P}_{11} = .087 \) and \( \hat{b} = \hat{P}_{11} + \hat{P}_{21} = .135 \) are the composite errors.

B. The Analytical Verification with Simple Algebraic Roots, and Examples 2 and 3

The LP-based algorithm implemented to SSP demonstrates that the feasible solution produced by the three different software algorithms, i.e. i) Microsoft’s EXCEL SOLVER, ii) Author’s JAVA-coded Game-Testing of Appendix A and iii) LP Software by Anderson et al. [35] are validated by the algebraic roots formulated in the preceding subsection V.A.

The three simple algebraic vectors, \( P_{11}, P_{12}, P_{21} \) were calculated and the fourth, \( P_{22} \), by subtraction of the first three from 1.0 per (18). The algebraic roots verify that \( \hat{P}_{ij} = LOSS/Cij \) are identical to the optimal solutions obtained in section II’s Fig. 2 and Fig. 3 and Tables III to VIII. Shortcut algebraic roots are too, optimally best estimates. Generalizations on LOSS input variables are given in APPENDIX B and C. That is, for all \( Cij \), LOSS can be assigned upon need.

APPENDIX B (i.e., example 2) uses a new set of LOSS\$5, $6, $7, $8 validating Exp#2’s roots, such as \( P_{12} = .5/50 = .01, P_{12} = 5/50 = .105, P_{23} = 5/50 = .105 \), and \( P_{12} = .27 \). Total Cost (disjoint) = .055* $110 + .175* $40 + (1 - .055 - .175) (-$800) = -$603 in Table XII of APPENDIX B.

APPENDIX C (i.e., example 3) replicates Tables III and IV solution for LOSS\$5, i.e, to replicate subsection V.A’s \( \hat{a} = .145, \hat{b} = .225, \hat{\alpha} = .045, \hat{\beta} = .135 \) and Total Cost (disjoint) = .045* $110 + 1.25* $40 + (1 - .045 - .125)(-$800) = -$654 in Table XIII of APPENDIX C.

VI. CONCLUSIVE SUMMARY, FUTURE RESEARCH

This article studies an LP-based, and further, simpler linear root-finding solutions, and pertinent industrial applications so to optimize type-I (alpha) and type-II (beta) error
probabilities in response to employing related cost and utility parameters from input data. These probabilities are otherwise known as producer’s and consumer’s risks, or risks of false positive and false negative. Tables IX to XI and Fig. 5 to Fig. 9 epitomize the tangible differences between the old ubiquitous and newly proposed ways. This is in contrast to the prespecified cut-off values of alpha and beta (e.g. $\alpha=0.05$, $\beta=0.10$) that have been traditionally practiced. Kelly [1] and Grant [2] therefore urged attention to this impasse. The choice for LOSS variable(s) and $C_{ij}$, $i,j=1,2$, i.e., $C_{11}$, $C_{12}$ and $C_{21}$, costs and utility constant $C_{22}$, are dictated by the company-savvy historical data per Sahinoglu [15] to [18] and Hoare [34].

Game theory’s extended and value-added approach serves here to contribute to a feasible output vector solution as detailed in example 1 of subsections III.A to III.D. This is why the optimized alpha and beta errors are objective and data-centric rather than the subjectively popular judgment-call selections, usually prespecified as e.g., $\alpha=0.05$ and $\beta=0.10$. The apparent limitation of this research topic may involve the data-scientific challenge of estimating $C_{ij}$ constants and LOSS variables’ constraints. These essentially econometric parameters can be estimated either through data mining, and time series modelling, or any viable computationally intensive approach by the analyst to reflect the actual market realities for a profitable sequential sampling plan solely specific to that company.

In the SSP, the producer establishes a sequential sampling plan for a continued supply of components with reference to AQL, which represents the acceptable upper limit of quality for the supplier’s process that the consumer would consider acceptable as a process average at one end. The consumer may also be interested at the other end, i.e. RQL, to denote the poorest limit of quality that the consumer is willing to accept with a low probability of acceptance in an individual lot by Montgomery [25]. If both rules do not work, the continue-sampling decisive action is adopted to call for a new sample to test. One terminates the SSP after $3n$ many samples as a rule of thumb. The author reasons that the proposed technique with attributes-type item-by-item sampling is applicable to the variables-type by Roussas [30]. The proposed method is to upgrade the hypothesis tests from a subjective to an objective stance; while improving industrial control-savvy item-by-item sequential sampling plans. Example 1 of the attributes-type item-by-item SSP’s steps are outlined as follow from $i, i')$ to $vii, vii')$ in sequence. Note, e.g. for LOSS=$3K$; $i, i')$ to $vii, vii')$ show tasks and outcomes respectively. The numerical outcomes are subject to change for a new set of the specific firm’s input $C_{ij}$ constants and LOSS variable constrained for each case. The step by step algorithm is as follows:

1. Set $H_0$: $P_1(=AQL)$ vs $H_1$: $P_2(=RQL)$, and $n=$ lot sample size.

2. Select $C_T= [C_{11}, C_{12}, C_{21}, C_{22}]$ and provide LOSS constraint(s) for numerical examples.

3. Optimize the aggregate $\alpha$ and $\beta$, and the aggregate total cost, either by game-theory (section III) or identical algebraic roots (section V).

4. Compute the disjoint pure $a^* = \text{producer’s risk due to } \alpha - \alpha^* \beta$, and the disjoint pure $b^* = \text{consumer’s risk due to } \beta - \alpha^* \beta$ and whatsoever no risk due to $(1-\alpha - \beta^*)^2$.

5. Optimize the disjoint $a'$ and $b'$ to calculate the parameters: $h_1$, $h_2$, $k$, $s$ by (20) to (25) to plot the SSP (Fig. 5 to 9) to accept, reject or continue-sampling by Table IX to Table XI.

6. Mark the $SSP$ decision rules, given: $C_T$, LOSS, AQL, RQL and disjoint $a'$ and $b'$.

7. See Tables IX to XI to compare Exp#3 with Exp#5. Accept if plotted points fall below $X_{AQL}$ in $\{n=100, n_{Accept}=3, n_{Reject}=6\}$ per Exp#1. Reject if plotted points fall above $X_{AQL}$ as in $\{n=100, n_{Accept}=3, n_{Reject}=6\}$ per Exp#5. Table XI marks the differences between the usual or conventional Exp#1 and proposed Exp#3 for LOSS=$3K$.

8. The SSP running cost, i.e. $A' C_{12} + B' C_{21} + (1 - a' - b') C_{22}$, incurred by adopting the proposed algorithm will be authentic based on the nature of inputs, i.e. $C_T= [C_{11}, C_{12}, C_{21}, C_{22}]$ and LOSS constraint. Intersection of pure estimates: $(a')^* (b')^* = (a')^* (b')^* = 0$ in Fig. 10.c.

9. $TC=$Total Cost (disjoint Exp#5 in Table IX) = $0.27 \times 110K + 0.075 \times 40 + (1 - 0.27 - 0.075) \times (-800K) = $712K is authentic for Example 1, whereas the same $TC=613K$ for Example 2 in APPENDIX B and finally $TC=654K$ for Example 3 APPENDIX C. The fact that smaller the running total cost becomes, poses no concern. The author’s take is that the proposed Exp#3 uses its proper SSP’s LP-based alpha and beta, optimized to $\alpha^* \approx 2.7\%$, $\beta^* \approx 7.5\%$; not the prespecified alpha and beta errors for LOSS=$3K$.

VII. Final Remarks

Readers’ take per Table XI is such that in the classical approach with the prespecified alpha=0.05 and beta=0.10 referring to Exp#1 of Table IX at the end of $n=100$ samples; the testing analyst decides to accept 3 and reject 6; i.e. $\{n_{Accept}=100, n_{A}=3, n_{R}=6\}$. Whereas, per Exp#5 of Table IX with an assumed LOSS=$3K$, the testing analyst decides to accept 2 (instead of 3 in the classical approach) and reject 6, saving monetary funds while not accepting one more item out of a lot size of $n=100$, i.e. $\{n_{Accept}=100, n_{A}=2, n_{R}=6\}$. Results may amply change if the analyst varies the input parameters.

APPENDIX A: CYBERRISK SOLVER TO RUN THE GAME TESTING APPLET

1. Click www.areslimited.com and type in the login user name: mehmetsuna, password: Mehpareanne, click OK.
2. Go to DOWNLOAD on www.areslimited.com for 1.h.s. menu’s 4th from the top.
3. Click on the CyberRiskSolver v3.0 in red and download.
the application which a ZIP file. Unzip or extract the downloaded application into C:\myapp folder. See C:\myapp\dist folder. Open a Command Prompt and go to C:\myapp\dist folder and run the following command: //For Cyber Risk Solver, java -jar twcSolver.jar. Use license code: EFE28SEP1986 for twcSolver.jar.

4. Click GAME TESTING Applet (checked). Click Open.

**APPENDIX B**

| Variable | Value | Reduced Costs |
|----------|-------|---------------|
| P1       | 0.100 | 0.000         |
| P2       | 0.045 | 0.000         |
| P3       | 0.125 | 0.000         |
| P4       | 0.730 | 0.000         |
| D05S11   | 5.000 | 0.000         |
| D05S12   | 5.000 | 0.000         |
| D08S21   | 5.000 | 0.000         |
| D08S22   | 5.000 | 0.000         |
| ALPRA    | 0.145 | 0.000         |
| BETA     | 0.225 | 0.000         |

**APPENDIX C**

| Example 3: LOSS1 = LOSS2, LOSS3 = LOSS4 = LOSS5 = $5, C2 = $50, $110, $40, $800; DISCOUNTS: α = 0.045 = P1, β = 1.175 = P2, TOTAL COST = -$654 |
|---------------------------------------------------------------|
| **Optimal Solution**                                         |
| **Objective Function Value** = -26,000                       |
| Variable | Value | Reduced Costs |
|----------|-------|---------------|
| P1       | 0.100 | 0.000         |
| P2       | 0.045 | 0.000         |
| P3       | 0.125 | 0.000         |
| P4       | 0.730 | 0.000         |
| D05S11   | 5.000 | 0.000         |
| D05S12   | 5.000 | 0.000         |
| D08S21   | 5.000 | 0.000         |
| D08S22   | 5.000 | 0.000         |
| ALPRA    | 0.145 | 0.000         |
| BETA     | 0.225 | 0.000         |

**CONFLICT OF INTEREST**

The author declares no conflict of interest.

**AUTHOR CONTRIBUTIONS**

The principal author, M.S. began to develop the theory of the original research findings as of 2015 and continued to develop and finalize it up to date. The co-author S.C. contributed as to where he assisted with the working software, web interface, data engineering and computational statistics. Both authors at the final stage approved the current version.

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