Algebraic and arithmetic area for \( m \) planar Brownian paths

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Abstract. The leading and next to leading terms of the average arithmetic area \( \langle S(m) \rangle \) enclosed by \( m \to \infty \) independent closed Brownian planar paths, with a given length \( t \) and starting from and ending at the same point, are calculated. The leading term is found to be \( \langle S(m) \rangle \sim (\pi t/2) \ln m \) and the 0-winding sector arithmetic area inside the \( m \) paths is subleading in the asymptotic regime. A closed form expression for the algebraic area distribution is also obtained and discussed.

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1. Introduction

The question of the area, be it algebraic, \( A \), or arithmetic, \( S \), enclosed by a closed Brownian planar path, of a given length \( t \) and starting from and ending at a given point, is already an old subject. It was first raised in the mid-twentieth century, in relation to the well-known Levy law \([1]\) for the probability distribution \( P(A) \) of the algebraic area.

The question of the probability distribution \( P(S) \) of the arithmetic area is a much more difficult issue, as a result of its nonlocal nature. Interestingly, the complete calculation of the first moment of the distribution, \( \langle S \rangle = \pi t / 5 \), has only recently been achieved, by SLE techniques \([2]\). Since on the other hand the average arithmetic area \( \langle S_n \rangle = t / (2\pi n^2) \) of the \( n \neq 0 \)-winding sectors inside the curve has been known for some time \([3]\), thanks to path integral techniques, one can readily deduce from these results that the average arithmetic area \( \langle S_0 \rangle \) of the 0-winding sectors\(^1\) inside the curve is \( \pi t / 30 \). An \( n \)-winding sector is defined as a set of points enclosed \( n \) times by the path and a 0-winding sector is made up of points which are either outside the path, or inside it but enclosed an equal number of times clockwise and anticlockwise, as illustrated in figure 1.

Recently some progress has been made on the geometrical structure of the \( n \)-winding sectors thanks to numerical simulations of the Hausdorff dimension of their fractal perimeter \([4]\). Note finally that, having in mind simpler winding properties, the asymptotic probability distribution at large time of the angle spanned by one open path around a given point is also known \([5]\).

Clearly, the random variables \( S_n \) and \( S_0 \) are such that \( S = \sum_{n=-\infty}^{\infty} S_n \) and \( A = \sum_{n=-\infty}^{\infty} nS_n \). They happen to be the basic objects needed to define quantum mechanical models where random magnetic impurities are modeled by Aharonov–Bohm vortices \([6]\). They also occur in the context of localization: for instance \([7]\) for magnetoconductance computations in weakly disordered media, the correction to the Drude conductivity is

\[
\Delta \sigma = \frac{e^2}{\pi \hbar} \left\langle 1 - \cos \frac{2eBA}{\hbar} \right\rangle \tag{1}
\]

\(^1\) The path integral approach diverges for \( n = 0 \) because it cannot distinguish 0-winding sectors inside the path from the outside of the path, the latter being of infinite area and, trivially, also constituting a 0-winding sector.

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where \( B \) is the magnetic field and the average is over all the Brownian paths that enclose an algebraic area \( A \). Moreover in [8], it is shown that the effect of a magnetic field on the localization length in Anderson insulators can also be expressed through an average similar to (1).

The question addressed in the present work concerns the generalization of \( \langle S \rangle = \pi t/5 \) to the arithmetic area \( \langle S(m) \rangle \) spanned by \( m \) independent closed paths of a given length \( t \) starting from and returning to the same point. More precisely, one would like to have some information on the scaling of \( \langle S(m) \rangle \) when \( m \rightarrow \infty \). Using again path integral techniques along the lines of [3], we will show that it is possible to compute exactly the leading and next to leading asymptotic terms of \( \langle S(m) - S_0(m) \rangle \) where the contribution of the 0-winding sectors inside the \( m \) paths has been subtracted precisely for the reason discussed above in the one-path case. One will find that the leading asymptotic term scales like \( \ln m \), namely \( \langle S(m) - S_0(m) \rangle \sim (\pi t/2) \ln m \) with, as already stated, no information so far on \( \langle S_0(m) \rangle \) inside the paths.

To go a little bit further, one might consider a simplification of the problem by looking at the arithmetic area of the convex envelope of the paths, a simpler geometrical object.
than their actual fractal envelope. It is known [9] that the area of the convex envelope of one opened path is \( \langle S \rangle_{\text{convex}} = \pi t/2 \). Recently it has been found, in [10], that the area of the convex envelope of one closed path is \( \langle S \rangle_{\text{convex}} = \pi t/3 \) whereas the asymptotic behavior of the area for \( m \to \infty \) closed paths is \( \langle S(m) \rangle_{\text{convex}} \sim (\pi t/2) \ln m \). The identical scalings for \( \langle S(m) - S_0(m) \rangle \) and \( \langle S(m) \rangle_{\text{convex}} \) mean that, in geometrical terms, the \( m \) paths tend to fully occupy the area available inside their convex envelope when \( m \) is large. Clearly, as \( \langle S(m) \rangle_{\text{convex}} \) is by construction bigger than the actual \( \langle S(m) \rangle \), the latter lies between \( \langle S(m) - S_0(m) \rangle \) and \( \langle S(m) \rangle_{\text{convex}} \). Now, since the leading asymptotic behaviors of the lower and upper bounds are found to be equal, for the asymptotic regime one concludes necessarily that \( \langle S(m) \rangle \sim (\pi t/2) \ln m \) and that \( \langle S_0(m) \rangle \) is subleading².

2. Algebraic area distribution

As a warm up, let us consider the generalization of Levy’s law to \( m \) independent paths. Let us first start with one path of length \( t \), starting from and ending at a given point \( \vec{r} \), so that \( \vec{r}(0) = \vec{r}(t) = \vec{r} \). The algebraic area enclosed by this path is \( A = \vec{k} \cdot \int_0^t \vec{r}(\tau) \wedge d\vec{r}(\tau)/2 \) where \( \vec{k} \) is the unit vector perpendicular to the plane. The path integral leads to

\[
\langle e^{iBA} \rangle = \frac{G_B(\vec{r}, \vec{r})}{G_0(\vec{r}, \vec{r})}
\]

(2)

where \( G_0(\vec{r}, \vec{r}) = 1/(2\pi t) \) and

\[
G_B(\vec{r}, \vec{r}) = \int_{\vec{r}(0)=\vec{r}}^{\vec{r}(t)=\vec{r}} \mathcal{D}\vec{r}(\tau)e^{-1/2 \int_0^t \vec{r}(\tau) \cdot d\vec{r}(\tau)/2} = \frac{1}{2\pi t \sinh(Bt/2)} e^{Bt/2}.
\]

In (2) the average \( \langle \cdot \rangle \) has been taken over the set \( C \) of all paths of length \( t \) starting from and ending at \( \vec{r} \). As a result, \( G_B \) is the Landau propagator at coinciding points of a charged particle in a uniform magnetic field \( B = B\vec{k} \), and, since this system is translation invariant, its partition function per unit area [11]. Fourier transforming, the probability distribution of \( A \)—Levy’s law—is

\[
P(A) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{Bt/2}{\sinh(Bt/2)} e^{-iBA} dB = \frac{\pi}{2t \cosh^2(\pi A/t)}.
\]

(4)

In the case of \( m \) paths, one should compute instead of (4)

\[
P_m(A) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{Bt/2}{\sinh(Bt/2)} \right)^m e^{-iBA} dB.
\]

(5)

The integration in (5) can be performed by rewriting \( 1/\sinh(Bt/2)^m = 2^m e^{-mBt/2}/(1 - e^{-Bt})^m \) and by expanding, when the integration variable \( B > 0 \), the denominator in powers of \( e^{-Bt} \). One obtains finally

\[
P_m(A) = \frac{m!}{2\pi t} \sum_{k=0}^{\infty} \left( \begin{array}{c} k + m - 1 \\ k \end{array} \right) \frac{1}{(k + m/2 + 1A/t)^{m+1}} + \text{cc}
\]

(6)

² This can be easily understood by noticing that 0-winding sectors tend to disappear when more and more paths overlap as \( m \) increases, which also implies that \( n \)-winding indices tend to increase.
where the complex conjugate term corresponds to the $B < 0$ integration (this amounts to setting $A \rightarrow -A$ in the $B > 0$ integration result).

Noticing that $A$ is the sum of $m$ independent random variables each satisfying Levy’s law, one expects $A$ to be Gaussian when $m$ becomes large. Indeed, one can observe that, when $m \rightarrow \infty$, the main contribution to the integral in (5) comes from small $B$ values. Thus,

$$
\left( \frac{Bt/2}{\sinh(Bt/2)} \right)^m \sim \left( \frac{1}{1 + (B^2t^2/24)} \right)^m \sim e^{-m(B^2t^2/24)}. 
$$

Rescaling the area as $A' = A/(t\sqrt{m})$, (5) leads to

$$
P_\infty(A') = \sqrt{\frac{6}{\pi}} e^{-6A'^2}. \quad (7)
$$

In terms of $A'$, we get, from (6) or from contour integration,

$$
P_2(A') = \frac{\pi \sqrt{2}}{\sinh^2(\pi \sqrt{2}A')} \left( \pi \sqrt{2} A' \coth(\pi \sqrt{2}A') - 1 \right) \quad (8)
$$

$$
P_3(A') = \frac{\pi \sqrt{3}}{2 \cosh^2(\pi \sqrt{3}A')} \left( 3 - 6 \pi \sqrt{3} A' \tanh(\pi \sqrt{3}A') \right.
\left. - \left( (\pi \sqrt{3}A')^2 + \frac{\pi^2}{4} \right) (1 - 3 \tanh^2(\pi \sqrt{3}A')) \right) \quad (9)
$$

Figure 2 clearly shows that $P_m(A')$ converges quickly to the Gaussian $P_\infty(A')$ when $m$ becomes large. In particular $A$ scales like $\sqrt{m}$ when $m \rightarrow \infty$. The scaling will be different for the arithmetic area discussed in section 3.

3. Winding properties and arithmetic area for $m$ Brownian paths

3.1. The case of one path: notation and known results

The arithmetic area enclosed by a planar Brownian path is closely related to its winding properties. Let us again consider a path of length $t$, starting from and ending at $\vec{r}$, and let us set $\theta$ to be the angle wound by the path around a fixed point, say the origin $O$. Again the average $\langle e^{i\alpha \theta} \rangle$ is taken over $C$:

$$
\langle e^{i\alpha \theta} \rangle = \frac{G_\alpha(\vec{r}, \vec{r})}{G_0(\vec{r}, \vec{r})} \quad (10)
$$

where

$$
G_\alpha(\vec{r}, \vec{r}) = \int_{\vec{r}(0)=\vec{r}' \atop \vec{r}(t)=\vec{r}} DR(\tau) e^{-1/2 \int_0^t \dot{\vec{r}}^2(\tau) d\tau + i\alpha \int_0^t \dot{\theta}(\tau) d\tau} = \frac{1}{2\pi t} e^{-r^2/t} \sum_{k=-\infty}^{\infty} I_{|k-\alpha|} \left( \frac{r^2}{t} \right). \quad (11)
$$

By symmetry, the average depends only on $r$ with $G_\alpha$ being the propagator of a charged particle coupled to a vortex at the origin (the functions $I_{|k-\alpha|}$ are modified Bessel functions). Obvious symmetry and periodicity considerations such as $G_\alpha = G_{\alpha+1} = G_{1-\alpha}$

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Algebraic and arithmetic area for \( m \) planar Brownian paths

Figure 2. The distribution \( P_m(A') \) for the rescaled algebraic area \( A' = A/\sqrt{m} \). Starting from the curve with the greatest maximum, \( m = 1 \), \( m = 2 \), \( m = 3 \), \( m = \infty \).

allow us to restrict to \( 0 \leq \alpha \leq 1 \). We also set \( r^2/t \equiv x \) (so that \( 2\pi r \, dr = \pi t \, dx \)) and \( \langle e^{i\alpha \theta} \rangle \equiv G_\alpha(x) \) with

\[
G_\alpha(x) = e^{-x} \sum_{k=-\infty}^{+\infty} I_{|k-\alpha|}(x)
\]  

(12)

so that \( G_0(x) = 1 \).

Integrating over \( \vec{r} \)—while keeping the position of the vortex fixed at the origin—is the same as integrating over the vortex position—while keeping the starting and ending points of the path fixed. Therefore this amounts to counting the arithmetic areas \( S_n \) of the \( n \)-winding sectors, with the 0-winding sector included:

\[
\int_0^\infty \pi t \, dx \, G_\alpha(x) = \sum_{n \neq 0} \langle S_n \rangle e^{i\alpha 2\pi n} + \langle \tilde{S}_0 \rangle.
\]  

(13)

This integral diverges since, as already stressed in the introduction, \( \tilde{S}_0 \) is the sum of \( S_0 \), the arithmetic area of the 0-winding sectors enclosed by the path, and the area outside the path, which is infinite. However, since formally for \( \alpha = 0 \),

\[
\int_0^\infty \pi t \, dx \, G_0(x) = \sum_{n \neq 0} \langle S_n \rangle + \langle \tilde{S}_0 \rangle,
\]  

(14)
then
\[ Z_\alpha \equiv \pi t \int_0^\infty dx \left( 1 - G_\alpha(x) \right) = \sum_{n \neq 0} \langle S_n \rangle \left( 1 - e^{i\alpha 2\pi n} \right) \] (15)
is finite. One deduces
\[ \langle S_n \rangle = -\int_0^1 Z_\alpha e^{-i\alpha 2\pi n} d\alpha \quad n \neq 0 \] (16)
and
\[ \langle S - S_0 \rangle \equiv \sum_{n \neq 0} \langle S_n \rangle = \int_0^1 Z_\alpha d\alpha. \] (17)

Using the Laplace transform of the modified Bessel functions, we readily recover
\[ Z_\alpha = \pi t \alpha (1 - \alpha) \] (18)
\[ \langle S_n \rangle = \frac{t}{2\pi n^2} \quad n \neq 0 \] (19)
\[ \langle S - S_0 \rangle = \frac{\pi t}{6}. \] (20)

3.2. The case of \( m \) independent paths

It is easy to see that \( G_\alpha(\vec{r}, \vec{r})^m \) provides the appropriate measure for counting the sets of \( m \) closed paths of length \( t \) starting from and ending at \( \vec{r} \). Following the same line of reasoning as in section 3.1, we get
\[ Z_\alpha(m) \equiv \pi t \int_0^\infty dx \left( 1 - G_\alpha(x)^m \right) = \sum_{n \neq 0} \langle S_n(m) \rangle \left( 1 - e^{i\alpha 2\pi n} \right) \] (21)
and
\[ \langle S - S_0(m) \rangle \equiv \sum_{n \neq 0} \langle S_n(m) \rangle = \int_0^1 Z_\alpha(m) d\alpha \] (22)
where \( S(m), S_n(m) \) and \( S_0(m) \) stand respectively for the total, \( n \)-winding sector and 0-winding sector arithmetic areas enclosed by the \( m \) paths. For example a sector of points which have been enclosed once by one path in the clockwise direction, enclosed twice by another path in the anticlockwise direction, and not enclosed by the other \( m - 2 \) paths has winding number \( n = -1 + 2 = 1 \).

To find the leading behavior of \( S(m) \) in the large \( m \) limit, one has to evaluate \( G_\alpha(x)^m \) when \( m \to \infty \), and so one needs a tractable expression for \( G_\alpha(x) \). To achieve this goal, one notes from (12) that
\[ G_\alpha(x) = F_\alpha(x) + F_{1-\alpha}(x) \] (23)
with \( F_\alpha(x) = e^{-x} \sum_{k=0}^{+\infty} f_{k+\alpha}(x) \). Using [12] the Laplace transform of \( e^{-x}I_\nu(x) \) one obtains the Laplace transform of \( F_\alpha(x) \):
\[ \frac{1}{\sqrt{2s}} 2^{-\alpha-1/2} \left( \sqrt{s + 2} + \sqrt{s} \right)^{(2\alpha-1)} \] (24)
From the inverse Laplace transform it follows that
\[ F_\alpha(x) = \int_0^x \frac{1}{\sqrt{2\pi(x-t)}} e^{-t} I_{\alpha-1/2}(t) \, dt. \] (25)

Using finally the integral representation
\[ I_{\alpha-1/2}(t) = \frac{(t/2)^{\alpha-1/2}}{\sqrt{\pi} \Gamma(\alpha)} \int_{-1}^1 e^{iu}(1-u^2)^{\alpha-1/2} \, du \] (26)

one gets after integration, term by term,
\[ \frac{dF_\alpha(x)}{dx} = \frac{1}{\sqrt{\pi}} (2x)^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-2x)^k}{k!} \frac{\Gamma(k+\alpha+1/2)}{\Gamma(k+2\alpha)}. \] (27)

Taking also into account \( dF_{1-\alpha}(x)/dx \) and using properties of degenerate hypergeometric functions, one finally gets
\[ \frac{dG_\alpha(x)}{dx} = \frac{2}{\sqrt{\pi}} \sin(\pi \alpha) e^{-2x} (2x)^{\alpha-1} U \left( \alpha - \frac{1}{2}; 2; 2x \right) \] (28)

where \( U \) is the degenerate hypergeometric function [12]
\[ U(a; b; z) = \frac{z^{-a}}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} \left( 1 + \frac{t}{z} \right)^{-b-a-1} \, dt. \] (29)

It is easy to verify that \( G_\alpha(x) = G_{1-\alpha}(x) \), as it should.

From (28),
\[ G_\alpha(\infty) = G_\alpha(0) + \int_0^\infty \frac{dG_\alpha(x)}{dx} \, dx = 1, \] (30)

as it should far away from the origin (we have used that when \( \alpha \neq 0 \), \( G_\alpha(0) = 0 \)). More precisely, since for \( x \) large, \( U(a; b; z) \sim z^{-a} \), one has
\[ 1 - G_\alpha(x) \sim \frac{\sin(\pi \alpha)}{\sqrt{2\pi x}} e^{-2x} \quad \text{when} \quad x \to \infty. \] (31)

On the other hand, for \( 0 < \alpha < 1/2 \),
\[ G_\alpha(x) \sim \frac{(x/2)^\alpha}{\Gamma(\alpha+1)} \quad \text{when} \quad x \to 0. \] (32)

Changing the variable using \( 2x = y \ln m \) in (21),
\[ Z_\alpha(m) = \frac{\pi t}{2} \ln m \int_0^\infty \, dy \left( 1 - \left( G_\alpha \left( \frac{y \ln m}{2} \right) \right)^m \right), \] (33)

one sees that in the limit \( m \to \infty \):
- when \( y = 0 \), that is \( y \ln m = 0 \), \((G_\alpha)^m = 0\);
- when \( y > 0 \), that is \( y \ln m \to \infty \), the asymptotic form (31) can be used, so
\[ \left( G_\alpha \left( \frac{y \ln m}{2} \right) \right)^m \to \left( 1 - \frac{\sin(\pi \alpha)}{\sqrt{\pi y \ln m}} m^{-y} \right)^m \sim e^{-\left( \sin(\pi \alpha)/\sqrt{\pi y \ln m} \right) m^{1-y}}. \] (34)
It follows that
\[
\left( G_\alpha \left( \frac{y \ln m}{2} \right) \right)^m \to_{m \to \infty} \Theta(y - 1)
\] (35)
where \( \Theta \) is the Heaviside function. Finally
\[
Z_\alpha(m) \sim \frac{\pi t}{2} \ln m
\] (36)
and
\[
\langle S(m) - S_0(m) \rangle \sim \frac{\pi t}{2} \ln m.
\] (37)
As already discussed in section 1, this means that since \( \langle S(m) \rangle_{\text{convex}} \sim (\pi t/2) \ln m \), at leading order
\[
\langle S(m) \rangle \sim \frac{\pi t}{2} \ln m
\] (38)
and, necessarily, \( \langle S_0(m) \rangle \) is subleading.

In figure 3, numerical simulations (random walks on a square lattice) for \( \langle S(m) \rangle_{\text{convex}}/t \), \( \langle S(m) \rangle/t \) and \( \langle S(m) - S_0(m) \rangle/t \) are displayed. One sees that if \( \langle S(m) \rangle_{\text{convex}}/t \) converges rapidly to \((\pi/2) \ln m\), this is not the case for \( \langle S(m) \rangle/t \) and \( \langle S(m) - S_0(m) \rangle/t \). This means that subleading corrections are needed. They originate from the fact that, when \( m \) is large but not infinite, \( (G_\alpha)^m \) deviates from a Heaviside function. One should compute
\[
\frac{1}{t} \langle S(m) - S_0(m) \rangle = \frac{\pi}{2} \ln m + \frac{\pi}{2} \ln m(c_1 + c_2)
\] (39)
with
\[
c_1 = \int_0^1 d\alpha \int_0^\infty dy \left( 1 - \left( G_\alpha \left( \frac{y \ln m}{2} \right) \right)^m \right)
\] (40)
\[
c_2 = -\int_0^1 d\alpha \int_0^1 dy \left( G_\alpha \left( \frac{y \ln m}{2} \right) \right)^m.
\] (41)
For \( y > 1 \),
\[
1 - \left( G_\alpha \left( \frac{y \ln m}{2} \right) \right)^m \sim m \frac{\sin(\pi \alpha)}{\sqrt{\pi y \ln m}} m^{-y}
\] (42)
and so
\[
\frac{\pi}{2} \ln mc_1 \sim \frac{1}{\sqrt{\pi \ln m}}
\] (43)
For \( 0 < y \leq 1 \),
\[
\left( G_\alpha \left( \frac{y \ln m}{2} \right) \right)^m \sim e^{-am^{1-y}}
\] (44)
Algebraic and arithmetic area for $m$ planar Brownian paths

Figure 3. The average arithmetic area: (i) Monte Carlo simulations (10000 events) of closed random walks of $N = 10^6$ steps on a square lattice; $m = 4, 8, 16, \ldots, 1024$; +: $\langle S(m) \rangle_{\text{convex}}/t$, *: $\langle S(m) \rangle/t$, ×: $\langle S(m) - S_0(m) \rangle/t$; (ii) analytical results for $\langle S(m) - S_0(m) \rangle/t$; upper continuous line: $\pi \ln(m)/2$ (leading order); lower dotted line: equation (46) which includes the subleading corrections. To generate closed random walks of $N$ steps it is enough to remark that the number of such walks with $M$ steps to the right is $N!/ (M! (N/2 - M)!)^2$. One deduces the probability density for the random variable $M$ in the limit $N, M, N/2 - M \to \infty$ using the Stirling formula. Finally one draws randomly objects from four boxes containing initially $M$, $M$, $N/2 - M$ and $N/2 - M$ objects.

with $a = \sin(\pi \alpha)/\sqrt{\pi y \ln m} \approx \sin(\pi \alpha)/\sqrt{\pi \ln m}$ since when $m$ is large, $y$ is peaked at 1. Changing the variable using $am^{1-y} = z$ and integrating over $z$, then over $\alpha$, leads to

$$\frac{\pi}{2} \ln mc_2 \sim -\frac{\pi}{4} \ln \ln m - \frac{\pi}{2} \left( \ln \sqrt{4\pi - C} \right) - \frac{1}{\sqrt{\pi \ln(m)}} + \text{subleading} \quad (45)$$

where $C$ is the Euler constant. Collecting all terms, we finally obtain

$$\frac{1}{t} \langle S(m) - S_0(m) \rangle = \frac{\pi}{2} \ln m - \frac{\pi}{4} \ln \ln m - \frac{\pi}{2} \left( \ln \sqrt{4\pi - C} \right) + \text{subleading} \quad (46)$$

One sees in figure 3 that the subleading corrections greatly improve the fit: the lower dotted line (46) is indeed not far from the numerical data (the agreement is of course not entirely perfect but further subleading corrections seem hard to obtain).

4. Conclusion

In conclusion, we have established that the leading behavior of the average arithmetic area $\langle S(m) \rangle$ enclosed by $m \to \infty$ independent closed Brownian planar paths is $\frac{\pi}{4} \ln m$. The algebraic area, on the other hand, scales like $t\sqrt{m}$. One should stress that the quite different asymptotic behaviors pertain to the essentially different natures of the areas.
Algebraic and arithmetic area for $m$ planar Brownian paths

considered: the algebraic area is additive, which is not the case for the arithmetic area. On another front, it remains a real challenge to get some information on the subleading asymptotic behavior of the 0-winding sector area $\langle S_0(m) \rangle$. Again path integral techniques are not adapted to this case, whereas SLE machinery should in principle work.

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