Almost complex manifolds are (almost) complex

Tobias Shin

Abstract

We analyze the differential relation corresponding to integrability of almost complex structures, reformulated as a directed immersion relation by Demailly and Gaussier. We prove that the relation has formal solutions up to complex dimension 77, and that, given a formal solution, there is always a holonomic section approximating it. The latter result implies in particular that, up to complex dimension 77, any almost complex manifold admits a sequence of almost complex structures so that the pointwise supremum norms of the Nijenhuis tensors become arbitrarily small.

1 Introduction

In their paper [3], Demailly and Gaussier construct, for a given complex dimension $n$, a universal space $Z$ with an algebraic distribution $D$ for which all almost complex $n$-manifolds immerse into, transverse to $D$. More precisely, they prove the following theorem.

**Theorem 1.1.** (Demailly, Gaussier) For all integers $n \geq 1$, there exists a complex affine algebraic manifold $Z$ of dimension $N = 38n^2 + 8n$ possessing an anti-holomorphic algebraic involution and an algebraic distribution $D \subset TZ$ of codimension $n$, for which every compact $n$-dimensional almost complex manifold $(X, J)$ admits an embedding $f: X \hookrightarrow Z^\mathbb{R}$ transverse to $D$ and contained in the real part of $Z$, such that $J = J_f$, where $J_f$ denotes the almost complex structure on $TZ/D$ pulled back under $f$.

The space $Z$ they construct is a combination of Grassmannians and twistor spaces, built in such a way that essentially “globalizes” the local picture relating Frobenius integrability with the Nijenhuis tensor, via Whitney embedding. Moreover, they give a criterion for when a given almost complex structure $J$ is integrable, with respect to this setup.

**Theorem 1.2.** (Demailly, Gaussier) For every compact $n$-dimensional integrable complex manifold $(X, J)$, there exists an embedding $X \hookrightarrow Z^\mathbb{R}$ transverse to $D$, contained in the real part of $Z$, such that

1. $J = J_f$ and $\overline{\partial}_f$ is injective;
2. $\text{Im}(\overline{\partial}_f)$ is contained in the isotropic locus $I$ of the torsion operator $\theta$ of $D$, the intrinsically defined algebraic locus in the Grassmannian bundle $Gr(n, D) \to Z$ of complex $n$-dimensional subspaces in $D$ consisting of those subspaces $S$ such that $\theta|_{S \times S} = 0$.

The torsion operator $\theta$ is defined as

$$\theta(X, Y) = [X, Y] \mod D.$$ 

Note that by taking the quotient, we obtain a skew symmetric bilinear tensor; we may view it as a holomorphic section of the bundle $\wedge^2D^* \otimes (TZ/D)$. The inclusion condition (ii) $\text{Im}(\overline{\partial}_f) \subset I$ is actually necessary and sufficient for integrability of $J_f$. Here, $\overline{\partial}_f f = \frac{1}{2}(df + J_Z \circ df \circ J_f)$ where $J_Z$ is the fixed complex structure on $Z$.

In the language of Gromov and the $h$-principle [9], the above gives rise to a natural directed immersion problem. As described in [4], the setup is as follows: let $Gr_n(W)$ be the Grassmannian bundle
of tangent $n$-planes to a manifold $W$ of dimension strictly larger than $n$ and $V$ be an $n$-dimensional manifold. Let $A \subset Gr_n(W)$ be an arbitrary subset. An immersion $f : V \to W$ is said to be an $A$-directed immersion if the induced tangential lift $Gdf$ maps into $A$, where $Gdf$ sends a point $v$ to $df_v(T_vV)$. The corresponding differential relation lies in the first jet space of the trivial fibration $V \times W \to V$. All of the above can similarly be done replacing the word immersion with embedding.

In other words, the question of when an almost complex structure can be moved along a path of almost complex structures into one that is integrable is equivalent to a modified directed immersion problem with respect to $\mathcal{D}$ of immersions into the universal space $Z$. One can then apply the philosophy of the $h$-principle and ask:

(i) Are there formal solutions to this differential relation?

(ii) Assuming (i), are there genuine solutions to this differential relation? Does it satisfy the $h$-principle?

In fact, even if there were no obstructions to a formal solution, we already know that this relation fails the $h$-principle: there exist almost complex manifolds in complex dimension 2 that have no integrable complex structures, as shown classically by Van de Ven [21]. It is an open problem as to whether there exist such manifolds in higher dimensions.

This paper is organized as follows: we first give preliminary information about the differential relation and the corresponding subspace of the Grassmannian bundle. We prove in section 3 the following result.

**Theorem 1.3.** There are always formal solutions to the above differential relation for $n \leq 77$.

In section 4 we prove that there is always a holonomic section into an arbitrarily small open neighborhood of the relation, using the method of holonomic approximation and a microextension trick [4]. This implies in particular:

**Theorem 1.4.** Up to complex dimension 77, an almost complex manifold admits a sequence of almost complex structures so that the pointwise supremum norms of the Nijenhuis tensors become arbitrarily small.

We note that the above statement is independent of the background metric chosen. We then prove as a corollary in section 5 that this implies that the Nijenhuis energy has no positive infimum in the above range of dimension.

**Acknowledgements.** The author would like to first and foremost thank his advisor Dennis Sullivan for constant encouragement and guidance throughout the work of this paper. This project is really a love letter to the Stony Brook math department family. The author would also like to thank Frederik Benirschke, Mark De Cataldo, Nathan Chen, Yoon-Joo Kim, Robert Lazarsfeld, Lisa Marquand, John Sheridan, Andrew Sommese, Jason Starr, Sasha Viktorova, Ben Wu, and Ruijie Yang for teaching the author algebraic geometry; Michael Albanese, Jiahao Hu, and Aleksandar Milivojevic for teaching the author algebraic topology; Corey Bregman, Robert Bryant, Jae Ho Cho, Lisandra Hernandez-Vazquez, Ben McMillan, Jordan Rainone, and Dror Varolin for teaching the author complex and differential geometry; Dahye Cho, Mohamed El Alami, Yasha Eliashberg, Oleg Lazarev, Mark McLean, Tony Phillips, Álvaro del Pino, Ying Hong Tham, and Hang Yuan for teaching the author symplectic topology.
2 The subspace $I$ and the relation $R_I$

In this section we provide some preliminary information about the universal space $Z$, the distribution $D$, the subspace $I$, and the differential relation $R_I$. For the entirety of this paper, we only discuss complex dimension. Let $Gr(r,s)$ be the Grassmannian of complex $r$-planes in $s$-space.

The space $Z$ is constructed in [3] as the set of tuples $(z,S',S'',\Sigma',\Sigma'')$ in $\mathbb{C}^{8n} \times Gr(3n,8n) \times Gr(4n,8n) \times Gr(4n,8n)$ where $S' \subset \Sigma'$, $S'' \subset \Sigma''$, and $\Sigma' \oplus \Sigma'' = \mathbb{C}^{8n}$. The space $Z$ is then a quasiprojective subvariety of this product of Grassmannians. The flag decompositions are equivalent to a choice of complex structure on $\mathbb{C}^{8n}$ and $\Sigma' \oplus \Sigma''$ where $S' \subset \Sigma'$ and $S'' \subset \Sigma''$ correspond to the $+i$ and $-i$ eigenspaces respectively. We could also write $Z$ as a subvariety of $\mathbb{C}^{8n} \times F(3n,4n,8n) \times F(3n,4n,8n)$ where $F(3n,4n,8n)$ denotes the flag variety of $3n$-planes in $4n$-space in $\mathbb{C}^{8n}$. There is a natural transitive group action of $\mathbb{C}^{8n} \times GL(8n,\mathbb{C})$ on $Z$, which is by translation on the euclidean factor and matrix multiplication on the flag factors. Therefore $Z$ is a homogeneous space with its dimension (as computed in [3]) equal to $N = 38n^2 + 8n$. Let $N$ denote this quantity for the rest of the paper.

Remark. In [3], $Z$ is actually taken to be an open affine subset of the above space by removal of an appropriate subvariety, but it is shown that any almost complex manifold $X$ embeds into the above quasiprojective variety anyway.

The distribution $D$ is defined at a point $P = (z,S',S'',\Sigma',\Sigma'')$ as the set of tangent vectors $(v,v',u',w',w'')$ where $v \in S' \oplus \Sigma''$ tautologically, with no other conditions on the other components. This gives $T_P Z / D_P \cong \Sigma' / S'$, so $TZ / D$ is isomorphic to the pullback (by inclusion) of the tautological quotient bundle over the flag variety $F(3n,4n,8n)$, and $D$ is corank $n$.

The embedding of an almost complex manifold $(X,J)$ into $Z$ is constructed by first embedding $X$ into the diagonal of $X \times X$, and then embedding $X \times X$ by the product embedding into euclidean space via Whitney. One then complexifies the normal bundle of this embedding and its complex structure $\tilde{J}$, defined as the direct sum of the complex structures $J$, $-J$, and $J_\Delta$ where $-J$ is the conjugate complex structure and $J_\Delta$ is the complex structure on the double normal bundle. The embedding into $Z$ is then by taking as the four subspaces the $+i$ and $-i$ eigenspaces of $\tilde{J}$ on $\mathbb{C}^{8n}$ and the complexified normal bundle, in such a way that $\Sigma' / S' \cong T^{1,0}X$. Thus $Z$ and $D$ are defined tautologically so that $X$ embeds transverse to $D$, and so that the pullback complex structure agrees with the initial complex structure, by way of a real (which is a posteriori complex) isomorphism $f^*(TZ / D) \cong T^{1,0}X$. The idea is to take the proof originally due to Gauss relating Frobenius integrability for the $+i$-eigenspaces with the integrability of a real analytic complex structure, and make it universal by Whitney embedding. For full proof and construction, see [3].

Lemma 2.1. The distribution $D$ is totally non-integrable, i.e., iterated Lie brackets of vector fields tangent to $D$ span $TZ$.

Proof. Note that there is another transitive action on $Z$, given by the affine group $\text{Aff}(\mathbb{C}^{8n})$, which acts by affine transformations on the euclidean factor and the general linear action on the flag factors. That is, for a tuple $(z,S',S'',\Sigma',\Sigma'')$ and an affine element $(A,b)$, we can act by $(A,b) \cdot (z,S',S'',\Sigma',\Sigma'') = (Az+b, AS', AS'', A\Sigma', A\Sigma'')$. Under this action, $D$ becomes an invariant subspace: for an affine element $g \in \text{Aff}(\mathbb{C}^{8n})$, the differential $dg$ maps $D$ to $D$. Now, at a point $z$, consider the subspace $D_z$ defined as the span (over the ring of smooth $\mathbb{R}$-valued functions) of all iterated Lie brackets of vector fields tangent to $D$ evaluated at $z$. For any other $z' \in Z$, there
exists $g \in \text{Aff}(\mathbb{C}^n)$ such that $g \cdot z = z'$ by homogeneity of the affine group action on $Z$. Since $D$ is invariant and biholomorphisms push forward Lie brackets, we have that $\hat{D} = dg_*(\hat{D}_z)$ independent of the choice of $g$. Thus $\hat{D}$ is a well defined subbundle with $D \subset \hat{D} \subset T\hat{Z}$. Moreover, for holomorphic vector fields $X, Y$, we have that $[JX, Y] = [JX, Y]$ \textup{[22]}. One can also see this from the fact that $[JX, Y] = -\mathcal{L}_Y(JX) = -J\mathcal{L}_YX + (\mathcal{L}_Y J)X = J[X, Y]$, with the Lie derivative of $J$ along $Y$ vanishing since holomorphic vector fields admit flows $\phi_t$ that are holomorphic and therefore satisfy $\phi^*_t J = d\phi_{-t} \circ J \circ d\phi_t = J$, so $\mathcal{L}_Y J = \frac{d}{dt}|_{t=0} \phi^*_t J = \frac{d}{dt}|_{t=0} J = 0$. Therefore $\hat{D}$ is $J\hat{Z}$-invariant, since if $v \in \hat{D}$, then we can write $v$ as in the $\mathbb{R}$-linear span of evaluations of holomorphic vector fields spanning $D$ and their iterated brackets by choosing a local holomorphic frame for $D$, and so $J\hat{Z}v$ is in $\hat{D}$ by the above discussion. We conclude $\hat{D}$ is a complex subbundle in $T\hat{Z}$. If $\hat{D}$ is a proper subbundle, then we obtain a proper quotient bundle $T\hat{Z}/D$ of $T\hat{Z}/D$. After choosing a hermitian metric, this gives us a proper complex subbundle of $T\hat{Z}/D$. However, we claim that this bundle has no proper subbundles.

By universality of the construction of $D$ and $Z$ in \textup{[3]}, we have that for given dimension $n$, every almost complex $n$-manifold $X$ embeds into $Z$ by some map $f$ transverse to $D$ such that $f^*(T\hat{Z}/D) \cong TX$. Therefore, to show that $T\hat{Z}/D$ has no proper complex subbundles, it suffices to exhibit in each dimension $n$ an example of an almost complex manifold whose tangent bundle does not split into complex subbundles. However it is known that $\mathbb{CP}^{2k}$ admits no complex subbundles and that $\mathbb{CP}^{2k+1}$ only admits a complex line subbundle and its complement for all $k$ \textup{[6]}. It remains to find odd dimensional almost complex manifolds whose tangent bundles have no complex line subbundles.

The following observation is due to A. Milivojevic: recall that a manifold $M$ is stably almost complex if $TM \oplus \mathbb{R}^k$ is a complex vector bundle, where $\mathbb{R}^k$ is the trivial rank $k$ real vector bundle. We have that $T(M \# N) \oplus \mathbb{R}^k \cong p^*TM \oplus q^*TN$ for $M, N$ two real $d$-dimensional manifolds, $M \# N$ their connect sum, and $p : M \# N \to M$ and $q : M \# N \to N$ the two collapsing maps (\textup{[7]} Lemma 2.1), and so the connect sum of two stably almost complex manifolds is stably almost complex. We have that $S^5 \times S^5$ is complex \textup{[2]} and so its connect sums are stably almost complex, by the above observation. By Theorem 2 in \textup{[23]}, we have that $\#^{25}S^5 \times S^5$ is almost complex. Moreover, it is 2-connected with nonzero Euler characteristic, and so it has no complex line subbundle in its tangent bundle. Since $S^6$ is also 2-connected with nonzero Euler characteristic, we have that products of $S^6$ with $\#^{25}S^5 \times S^5$ are almost complex manifolds that do not admit any complex line subbundle in their tangent bundle. This gives every dimension of the form $3a + 5b$, which gives every odd dimension except 7. Finally, we have that $\mathbb{HP}^2 \# \mathbb{HP}^2/(S^4 \times S^4)$ is a 2-connected almost complex manifold with nonzero Euler characteristic in dimension 4 (\textup{[10]} Prop. 6). Again by taking a product with $S^6$, we obtain dimension 7. We conclude $\hat{D} = D$ or $\hat{D} = T\hat{Z}$. But if $\hat{D} = D$, then $D$ would be integrable, and every almost complex structure would be integrable. \hfill \Box

\textbf{Question.} The author did not know of any examples of odd dimensional complex manifolds that did not admit a complex line subbundle. Recall that every odd real dimensional manifold admits a line subbundle in its tangent bundle, since its euler class vanishes. We ask, as in the real setting: does every odd dimensional complex manifold admit a complex line subbundle in its tangent bundle? Notice that if this were true, then $S^6$ would have no integrable complex structure.

Let $Gr(n, D)$ denote the Grassmannian bundle of complex (i.e., $J\mathbb{C}$-invariant) $n$-planes in the distribution $D$, with fiber $Gr(n, D_z) \cong Gr(n, N - n)$ over $z$, where $N - n$ is the rank of $D$. Let $I$ denote the isotropy locus, i.e., the set of $n$-planes on which the torsion tensor $\theta$ vanishes. We
can use the vanishing of $\theta$ to realize the fiber $I_z = I \cap \text{Gr}(n, D_z)$ as the zero locus of an algebraic section of a holomorphic vector bundle over $\text{Gr}(n, D_z)$. Consider the holomorphic vector bundle $\wedge^2(\gamma^n)^* \otimes \mathbb{C}^n$ where $\gamma^n$ denotes the tautological bundle of $n$-planes, and the star denotes its dual. The appropriate algebraic section $\sigma_z$ is defined by sending a plane $S \subset D_z$ to the map $\theta|_{S \times S}$. We compute the expected dimension of $I_z$ as $\frac{n^2}{2}(75n + 13)$, since $I_z$ is cut out by $\binom{n}{2}n$ equations. Moreover, the dimension of the Grassmannian fiber is $n^2(38n + 6)$.

Recall that a differential relation is any subspace of a jet bundle. The differential relation $\mathcal{R}_f$ is then a subspace of the first jet bundle $J^1(X, Z)$, which is a vector bundle over $X \times Z$ where each fiber over $(x, z)$ is $\text{Hom}_\mathbb{R}(T_x X, T_z Z)$. More precisely, 1-jets are defined formally as germs of local sections, equivalent if they agree on first derivatives. A representative of a 1-jet over a point $x$ can be thought of as a triple $(x, z, L)$ where $z$ is the value of the section at $x$, and $L$ is a “formal” derivative of the section at $x$. Treated as a generic element of the jet space, however, these three elements in the triple have a priori no relation with each other. For more on jets, see [4], [9], or [12].

The differential relation at hand is not on the nose a directed immersion relation, but a slight modification of it. First, it requires that the differential be injective and transverse to the distribution. This way, one can pull back the complex structure on the quotient bundle to obtain the complex structure on the domain. Second, the relation asks that the antiholomorphic differential be injective, where the antiholomorphic differential is defined as

$$\overline{df} = \overline{d}_f f = \frac{1}{2}(df + J_Z \circ df \circ J_f),$$

with $J_f$ the complex structure obtained by pullback via the isomorphism $df$ with $TX$ and $TZ/D$. A crucial observation is that $\overline{df}$ being injective is independent of $J_f$, as implied by the following.

Lemma 2.2. $\overline{df}$ is injective if and only if the immersion $f$ is totally real, i.e., the differential satisfies $df_x(T_x X) \cap J_Z(df_x(T_x X)) = 0$ for all $x \in X$.

Proof. Suppose $f$ is a totally real immersion. If $\overline{df}(v) = 0$, by definition $df(v) = -J_Z \circ df \circ J_f(v)$. Therefore $df(v)$ is an element of $df(TX)$ and $J_Z \circ df(TX)$, so by totally realness, $df(v) = 0$. Since $f$ is also an immersion, $v = 0$. Suppose conversely that $\overline{df}$ is injective. Let $V = df(TX) \cap J_Z(df(TX))$. Notice that $V$ is $J_Z$-invariant. Complexifying $TX$, $TZ$, and $df$, we have by $J_Z$-invariance (i.e., $V$ being a complex vector space) that there exists a $+i$-eigenvector $v \in V$. Moreover, $v = df(u)$ for some $u \in TX \otimes \mathbb{C}$. Since $J_Zv = iv$, we have that $J_{TZ/D}[v] \mod D = i[v] \mod D$, and therefore, by definition of the pullback complex structure, that $J_fu = iv$ (since $df$ is complex linear with respect to the complexification). Evaluating the complexified $\overline{df}$ on $u$, we have

$$\overline{df}(u) = df(u) + J_Z \circ df \circ J_f(u)
= v + iJ_Z \circ df(u)
= v + iJ_Z(v)
= v - v = 0$$

By injectivity of $\overline{df}$, we have that $u = 0$, so $v = 0$. $\square$

By definition, the pullback complex structure $J_f$ satisfies $df \circ J_f \mod D = J_Z \circ df \mod D$, so by precomposing $\overline{df}$ with $J_f$, we have that $\overline{df}$ maps automatically into $D$. Moreover it sends tangent planes to $J_Z$-invariant planes since $\overline{df}$ anticommutates with the complex structures. The directed immersion relation for $\overline{df}$ mapping tangent planes into the isotropy locus $I$ is then well posed.
The relation $\mathcal{R}_{imm}$ for immersions, $\mathcal{R}_D$ for maps transverse to the distribution, and $\mathcal{R}_{\mathbf{imm}}$ for totally real maps are all open relations. In contrast, the relation $\mathcal{R}_I$ that corresponds to $\partial f$ mapping tangent planes into the isotropy locus $I$ is closed. Thus, the relation for directed immersions we are ultimately interested in is a locally closed subspace, being an intersection of closed and open subsets.

To summarize, our relation is contained in the simultaneous intersection of the following three differential relations $\mathcal{R}_{imm}, \mathcal{R}_D, \mathcal{R}_{\mathbf{imm}}$ where, in terms of a 1-jet $(x,z,L)$,

i. $\mathcal{R}_{imm}$ denotes the immersion relation, i.e., that $L$ is injective;

ii. $\mathcal{R}_D$ denotes the transverse to $D$ relation, i.e., that $L$ is transverse to $D$;

iii. $\mathcal{R}_{\mathbf{imm}}$ denotes the relation that $\partial L$ is injective, where $\partial L = \frac{1}{2}(L + J_Z \circ L \circ J_L)$ depends on the complex structure $J_L$ pulled back by $L$ from $TZ/D$. Equivalently, this denotes the totally real relation, i.e., $L_x(T_xX) \cap J_Z(L_x(T_xX)) = 0$.

Observe that we can make sense of $\partial L$ since all we need for a pullback complex structure is a bundle isomorphism. The relation of interest $\mathcal{R}_I$ is the subset of this three-fold intersection where we additionally require $\partial L(T_xX) \in I$ for all $x \in X$. Let $\mathcal{R}_c$ denote the intersection of the three relations above, which are all open. The relation $\mathcal{R}_I$ is the zero set of the map $(x,z,L) \mapsto \sigma_z(\partial L(T_xX))$ defined on $\mathcal{R}_c$, where $\sigma_z$ is the algebraic section that cuts out $I_z$ defined previously. Thus, $\mathcal{R}_I$ is a subspace with expected codimension $\binom{n}{2}$ in the first jet bundle.

3 Formal integrability

A section $\eta$ of the first jet space $J^1(X,Z)$ defined on $X$ is holonomic if it is the 1-jet $j^1 \sigma$ of some section $\sigma$ of $X \times Z$, where $j^1 \sigma(x) = (x,\sigma(x),d\sigma_x)$. Given any differential relation $\mathcal{R}$ in the first jet space $J^1(X,Z)$, we say a section $\eta$ is a formal solution if it has image in $\mathcal{R}$. Such a section is a genuine solution if it is also holonomic. A necessary condition for a section to be a genuine solution is for it to be a formal solution. The goal then is to first see if there are any purely homotopical obstructions to having a formal solution, before looking for a genuine solution. For our particular case, we can analyze the Grassmannian bundle directly; i.e., given a map into $Z$ and a tangential lift into the Grassmannian bundle, we can try and homotope the tangential lift to a map into the subspace $I$.

**Remark.** Note that if we assume we already have an almost complex structure, then we have a holonomic section to $\mathcal{R}_{imm} \cap \mathcal{R}_D$ for free, and vice versa; it’s worth noting here that any obstructions to having a holonomic section here should correspond to known homotopical obstructions for admitting an almost complex structure, but we have not investigated this point.

To homotope a given tangential lift through a fiberwise homotopy, we must study the relative homotopy groups $\pi_*(\text{Gr}(n,N-n),I_z)$ where $I_z$ denotes the fiber over $z$ of the subspace $I$, and $\text{Gr}(n,N-n)$ is the Grassmannian of $n$-planes in $(N-n)$-space. For dimension reasons, we only need to consider whether there are obstructions up to $* = 2n$, as the obstructions lie in $H^*(X,\pi_{n-1}(\text{Gr}(n,N-n),I_z))$ by classical obstruction theory. However, as $I$ may have possibly singular fibers, the obstruction cocycles may not be a priori well defined. The following theorem of Sommese [13] implies that the obstructions are not only well defined, but actually vanish.
Theorem 3.1. (Sommese) Let $E$ be a $k$-ample vector bundle on a compact complex manifold $W$. Assume that $E$ is globally generated by sections and that $B$ is the zero set of a holomorphic section of $E$. Then we have $\pi_j(W, B) = 0$ for $j \leq \dim W - \text{rank } E - k$.

A proof of the above can also be found in Sommese and Van de Ven [20] (Remark 3.2.1). Here, a holomorphic vector bundle $E$ is globally generated if there exist global holomorphic sections that span the fiber $E_x$ at every point $x$. A holomorphic vector bundle $E$ over a connected projective manifold $X$ is said to be $k$-ample [19] if $(\gamma_E^*)^r$ is globally generated for some $r$ and if the map associated to $H^0(\mathbb{P}E, (\gamma_E^*)^r)$ has fibers of at most dimension $k$, where $\gamma_E = \mathcal{O}_{\mathbb{P}E}(-1)$ is the tautological line bundle on the subspace projectivization $\mathbb{P}E$.

Lemma 3.1. The dual of the tautological rank $r$ subspace bundle $(\gamma^*)^r$ over the Grassmannian $Gr(r, s)$ of $r$-planes in $s$-space is $k$-ample for $k = (r-1)(s-r)$.

Proof. Let $E = (\gamma^*)^r$. Its projectivization $\mathbb{P}E$ is the point-plane incidence correspondence in $Gr(r, s) \times Gr(1, s)$, since one is projectivizing each $r$-plane tautologically. Thus there is a natural projection $\mathbb{P}E \xrightarrow{p} Gr(1, s) = \mathbb{P}^{s-1}$. The claim is that this projection is exactly the map associated to global sections of $\mathcal{O}_{\mathbb{P}E}(1)$. Note that by definition, $p^*\mathcal{O}_{\mathbb{P}^{s-1}}(1) \simeq \mathcal{O}_{\mathbb{P}E}(1)$ since, over a point $(\Lambda, \lambda) \in \mathbb{P}E$ where $\Lambda$ is an $r$-plane and $\lambda$ is a line in $\Lambda$, each bundle has as fiber all linear functionals defined on $\mathbb{C}^s$ restricted to $\lambda \subset \Lambda$. It remains to show that the projection $p$ actually arises as a map from taking all global sections, which is equivalent (see [8]) to it being nondegenerate (i.e., does not map into a hyperplane of $\mathbb{P}^{s-1}$) and having $h^0(\mathbb{P}E, p^*\mathcal{O}_{\mathbb{P}^{s-1}}(1)) = s$. Consider the short exact sequence

$$0 \to \gamma^r \to \varepsilon \to Q \to 0$$

where $\varepsilon$ denotes the $s$-dimensional trivial vector bundle and $Q$ denotes the tautological quotient. Dualizing this sequence, we obtain

$$0 \to Q^* \to \varepsilon^* \to E \to 0.$$

We have that $H^0(Gr(r, s), Q^*) = H^1(Gr(r, s), Q^*) = 0$ since $Q^*$ is a subbundle of the trivial bundle and by a Leray spectral sequence computation. Therefore, by the long exact sequence on cohomology, we have that $H^0(Gr(r, s), E) \simeq (\mathbb{C}^s)^*$. We also have $H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(1)) \simeq H^0(Gr(r, s), E)$ canonically as vector spaces (Ch. II, Prop. 7.11 in [10]). We conclude that $h^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(1)) = h^0(Gr(r, s), E) = s$. The map is nondegenerate, since it is the identity in the projective space component, being a projection. The fiber of this projection is then all $r$-planes in $s$-space containing a given line, which has dimension $k$ as in the statement of the lemma.

In our situation, our bundle is $E = \wedge^2(\gamma^n)^* \otimes \mathbb{C}^n$. Moreover the dual of the tautological is globally generated, being a quotient of the trivial bundle. By global generation, all direct sums, quotients, and tensors are $k$-ample [13] for the same $k$, so our bundle $E$ is globally generated and $k$-ample for $k = 38n^3 - 32n^2 - 6n$. By the above theorem of Sommese, we have the following.

Lemma 3.2. The relative homotopy groups $\pi_j(Gr(n, N-n), I_z)$ vanish for $j \leq \frac{1}{2}(-n^3 + 77n^2 + 12n)$.

Note that as $n$ tends to infinity, the right hand side of the inequality tends to 0; as the codimension increases, it becomes harder to detect the vanishing of relative homotopy groups. One could hope for homotopy obstructions for large $n$ (say after $n > 77$), but it seems unlikely.
Theorem 3.2. For \( n \leq 77 \), there are always formal solutions to the differential relation \( \mathcal{R}_t \).

Proof. It follows from the above theorem of Sommese that one can always homotope the given tangential lift to a map into the isotropy locus up to complex dimension 77. We can then equip \( Z \) with a hermitian metric and consider the fibration \( SU(D) \) over \( Z \), where a fiber over \( z \) is \( SU(D_z) \). We can define a map fiberwise from \( SU(D) \) to \( Gr(n, D)|_X \) where \( X \) is identified with its immersed image in \( Z \), by the quotient map \( SU(N-n) \to SU(N-n)/(SU(n) \times SU(N-2n)) \). Note this quotient map \( SU(D) \to Gr(n, D)|_X \) is only well defined as a fibration over the Grassmannian bundle restricted to \( X \), as otherwise there is no canonical choice of \( n \)-plane so as to take quotients (our choice of \( n \)-plane is the image of the tangent plane of \( X \) under \( \overline{\partial}f \)).

We can then use the homotopy lifting property to lift the tangential homotopy \( X \times I \to Gr(n, D)|_X \) to a homotopy \( X \times I \to SU(D) \), by lifting the initial time of the homotopy to the Identity. By homotopy lifting with respect to the pair \( (Z, X) \) as in [11], we can extend the homotopy \( X \times I \to SU(D) \) to a homotopy \( Z \times I \to SU(D) \). For each time \( t \), we then obtain a section of \( SU(D) \); i.e., we have a family of maps \( \Phi_t : D \to D \) such that each one is a complex bundle automorphism of \( D \) that covers the Identity on \( Z \). We now have a linear map \( \Phi_1 \circ \overline{\partial}f \) that maps the tangent planes of \( X \) into the isotropy locus, but not yet a formal solution.

Using the hermitian metric, choose a \( J_Z \)-invariant complementary subbundle of \( D \). Then we can extend \( \Phi_t \) to be the identity on the complementary subspace and consider \( \Phi_t \circ df \). Since we extended as the Identity on \( D^k \), we have \( \Phi_t \circ df \mod D = df \mod D \), so we also have for the pullback complex structures that \( J_{\Phi_t \circ df} = J_f \). Since \( \Phi_t \) is \( J_Z \)-linear, we conclude \( J_{\Phi_t \circ df} \circ df = \Phi_t \circ J_f \cdot f \). Moreover, \( \Phi_t \circ df \) is still an injective, totally real linear map transverse to \( D \) for all \( t \). □

We conclude that in this range of dimension, there are always formal solutions to the directed immersion problem. The question remains as to when \( \Phi_1 \circ df \) is actually a genuine solution, or at least homotopic to a genuine solution. In fact, \( \Phi_1 \circ df \) as constructed above will never be holonomic unless the initial complex structure was integrable to begin with, since the pullback complex structure by the formal solution is the initial complex structure. There may be no genuine solutions at all, as mentioned in the introduction. Nonetheless, we can try and approximate the formal solution by a holonomic section, and see how close we can get to the isotropy locus.

4 Holonomic approximation of a complex structure

In this section, we prove that we can always find a holonomic section that is \( \varepsilon \)-close to our formal solution, albeit after perturbing our manifold within a thickened neighborhood of itself. If \( A \) is a subset, we denote by \( OpA \) an open neighborhood of \( A \), following Gromov’s notation [9]. We now introduce two technical definitions, which can be found in Eliashberg and Mishachev [4].

Let \( X \to V \) be a fibration and let \( I^k \) denote the \( k \)-dimensional unit cube. A differential relation \( \mathcal{R} \subset X^{(r)} \) is locally integrable if given a map \( h : I^k \to V \), a family of sections

\[
F_p : h(p) \to \mathcal{R}, \ p \in I^k
\]

and a family of local holonomic extensions near \( \partial I^k \)

\[
\tilde{F}_p : Op \circ h(p) \to \mathcal{R}, \quad \tilde{F}_p(h(p)) = F_p(h(p)), \ p \in Op(\partial I^k),
\]

where \( \mathcal{R}_t \) is a family of sections such that \( \mathcal{R}_t \subset X^{(r)} \), for each \( t \in I^k \). We now define two technical definitions, which can be found in Eliashberg and Mishachev [4].

Let \( \mathcal{R}_t \subset X^{(r)} \) be a family of sections such that \( \mathcal{R}_t \subset X^{(r)} \), for each \( t \in I^k \). We now define two technical definitions, which can be found in Eliashberg and Mishachev [4].
there exists a family of local holonomic extensions

$$\tilde{F}_p : Op h(p) \to \mathcal{R},$$

$$\tilde{F}_p(h(p)) = F_p(h(p)), \ p \in I^k$$

such that for \( p \in Op(\partial I^k) \), these new extensions agree with the original extensions over \( Op h(p) \). In other words, \( \mathcal{R} \) is locally integrable if any formal solution over a point can be locally extended to a genuine solution. The parameter \( I^k \) is included to say this local solvability holds parametrically and relatively.

Fix \( n = \dim V \). Let a \( \theta_k \)-pair be any pair \((A, B)\) diffeomorphic to \([[-1, 1]^n, [-1, 1]^k \cup \partial([-1, 1]^n)]\).

A differential relation \( \mathcal{R} \) is \textit{microflexible} if for any sufficiently small open ball \( U \subset V \) and any families smoothly parameterized by \( p \in I^m \) of

- \( \theta_k \)-pairs \((A_p, B_p)\) \( \subset U \),
- holonomic sections \( F_p^0 : Op A_p \to \mathcal{R} \), and
- holonomic homotopies \( F_p^\tau : Op B_p \to \mathcal{R}, \ \tau \in [0, 1] \), of the sections \( F_p^0 \) over \( Op B_p \) which are constant over \( Op(\partial B_p) \) for all \( p \in I^m \) and constant over \( Op B \) for \( p \in Op(\partial I^m) \),

there exists a number \( \sigma > 0 \) and a family of holonomic homotopies

$$F_p^\tau : Op A_p \to \mathcal{R}, \ \tau \in [0, \sigma],$$

which extend the family of homotopies

$$F_p^\tau : Op B_p \to \mathcal{R}, \ \tau \in [0, \sigma],$$

and are constant over \( Op(\partial A_p) \) for all \( p \in I^m \) and constant over \( Op A \) for \( p \in Op(\partial I^m) \). In other words, \( \mathcal{R} \) is microflexible if local deformations of genuine solutions can be extended to global deformations of genuine solutions for small times. This property will be used to glue/interpolate from one holonomic section to another when both are defined on the same open neighborhood.

We can now state the following theorem also in Eliashberg and Mishashev [4].

**Theorem 4.1. (Holonomic \( \mathcal{R} \)-approximation theorem)** Let \( \mathcal{R} \subset X^{(r)} \) be a locally integrable microflexible differential relation. Let \( A \subset V \) be a polyhedron of positive codimension and suppose there is a section \( F : Op A \to \mathcal{R} \). Then for arbitrarily small \( \delta, \epsilon > 0 \), there exists a \( \delta \)-small (in the \( C^0 \)-sense) diffeotopy \( h^\tau : V \to V, \ \tau \in [0, 1] \), and a holonomic section \( \tilde{F} : Op h^1(A) \to \mathcal{R} \) such that

$$\|\tilde{F}(v) - F|_{Op h^1(A)}(v)\|_{C^0} < \epsilon$$

for all \( v \in Op h^1(A) \).

In other words, given a formal solution to a locally integrable microflexible differential relation defined on some positive codimension polyhedron, we can find an \( C^0 \)-approximating holonomic solution \textit{in the \( r \)-th jet space} \( X^{(r)} \) defined on a perturbation of the polyhedron. In fact, in the case of \( V = A \times \mathbb{R} \), we can choose our diffeotopy to be a vertical perturbation (i.e., of the form \((x, t) \mapsto (x, h(x, t))\)). We will need the following exercise of Gromov, found in p. 84 of [9].
Exercise 1. (Gromov) Let $\mathcal{R}_{\text{tang}}$ denote the differential relation of immersions $\mathbb{R} \to Z$ that are tangent to the distribution $D$. If $D$ is totally non-integrable, then $\mathcal{R}_{\text{tang}}$ is microflexible.

The exercise above is actually false as stated; Bryant and Hsu show in [1] that there exist examples of totally non-integrable distributions (e.g., Engel structures) which possess rigid integral curves, i.e., curves tangent to $D$ that cannot be deformed relative to their ends. However, the statement becomes true upon restricting to regular integral curves, as suggested in [4]. Here, a regular integral curve is defined as follows: consider all integral curves nearby $\gamma$ (in the Whitney $C^\infty$ topology) with the same starting point and evaluate their endpoints. This defines a map to $Z$ from an infinite dimensional manifold of curves. Then, $\gamma$ is regular if this map is a submersion at $\gamma$. For a precise definition of a regular integral curve, see Hsu [13]. The correct statement is then the following.

Theorem 4.2. (Martínez-Aguinaga, del Pino) Let $\mathcal{R}_{\text{reg-tang}}$ denote the differential relation of immersions $\mathbb{R} \to Z$ that are tangent to the distribution $D$ and regular. If $D$ is totally non-integrable, then $\mathcal{R}_{\text{reg-tang}}$ is locally integrable and microflexible.

For the proof of the above, see [17]. Note that one does not need totally non-integrable as an assumption on $D$ to prove local integrability when considering all integral curves; the assumption is used if restricting to regular curves, which are also proven to be generic [13], [17]. Given the above result, combined with the fact that our distribution $D$ is totally non-integrable, we can now prove the main theorem of the paper, following the same argument as in [4], [9], [17].

Theorem 4.3. For any almost complex manifold $X$ with a formal solution of $\mathcal{R}_I$ and any $\epsilon > 0$, there exists a holonomic section that is $\epsilon$-close to the formal solution.

Proof. Let $\mathcal{R}_c$ denote the intersection of the three differential relations $\mathcal{R}_{\text{imm}}, \mathcal{R}_{\partial \text{imm}},$ and $\mathcal{R}_D$ as before. We now do the following microextension trick: take the almost complex manifold $X$ and consider $X \times \mathbb{R}$, with the modified differential relation $\mathcal{R}_{c-\text{tang}}$ which corresponds to totally real immersions $X \times \mathbb{R} \to Z$ transverse to $D$, but tangent to $D$ and regular when restricted to each fiber $\{x\} \times \mathbb{R}$. The main point of the argument is then that holonomic approximation applies to $\mathcal{R}_{c-\text{tang}}$.

Since $D$ is totally non-integrable, it follows from [17] that $\mathcal{R}_{\text{reg-tang}}$ is both locally integrable and microflexible. The openness of $\mathcal{R}_c$ and local integrability of $\mathcal{R}_{\text{reg-tang}}$ imply the local integrability of $\mathcal{R}_{c-\text{tang}}$: for a formal solution $F = ((x,t), z, L(x) \oplus V(t))$ of $\mathcal{R}_{c-\text{tang}}$ over a point $(x,t)$, we can find a small contractible neighborhood around $\{x\}$ and a holonomic section $(x,f(x), df_x)$ of $\mathcal{R}_c$ agreeing with $(x,z,L(x))$ over $x$, since open relations are locally integrable. Then we can use the local integrability of $\mathcal{R}_{\text{reg-tang}}$ to find a locally defined regular integral curve $\gamma$ that agrees with $V$ over $t$, nowhere vanishing on the image of $f$, which is contractible. Our holonomic section agreeing with $F$ over $(x,t)$ is then defined locally as $((x,t), f(x), df_x \oplus \gamma'(t))$. The parametric condition on the boundary of a cube also holds, again since local integrability of $\mathcal{R}_c$ gives an extension that agrees with the given boundary data, and then local integrability of $\mathcal{R}_{\text{reg-tang}}$ gives a nowhere vanishing vector field defined on a small contractible open neighborhood that flows the extension for short time, agreeing with the given boundary data. Thus $\mathcal{R}_{c-\text{tang}}$ is locally integrable.

Similarly, the openness of $\mathcal{R}_c$ and microflexibility of $\mathcal{R}_{\text{reg-tang}}$ imply the microflexibility of $\mathcal{R}_{c-\text{tang}}$. Suppose we have an arbitrary $\theta_k$-pair $(A, B) \subset U \subset X \times \mathbb{R}$ where $U$ is a sufficiently small open ball, a holonomic section $F^0$ defined on $Op A$, and a holonomic homotopy $F^r$ defined on $Op B$ and constant on $Op(\partial B)$. The idea is that one uses microflexibility of $\mathcal{R}_{\text{reg-tang}}$ to obtain a family of curves over $A$ such that, when glued together, is the desired holonomic homotopy, since openness
guarantees holonomic in the transverse direction (i.e., we want to foliate the desired holonomic homotopy with our curves). As the holonomic homotopy $F^\tau$ is defined on an open neighborhood of $B$, we may assume without loss of generality that $B$ meets the fiber transversely (in the argument, we will parameterize a family of curves by $B$; without transversality, we would need to parameterize by an open neighborhood of $B$, or small intervals transverse to $B$ within the neighborhood). By taking a sufficiently fine subdivision, we may also assume that $B$ intersects each fiber in at most one point. Restricting $F^\tau$ to $\{x\} \times \mathbb{R}$ for points $(x, t)$ on $A$ gives us a holonomic homotopy over midpoints in the interior of curves $(\text{Op}(\{x\} \times I_x)) \cap B$ in $R_{\text{reg-tang}}$, where $I_x$ denotes a small interval over $x$ in $\text{Op} A$.

Treating the curves $I_x$ and their midpoints as theta pairs in each fiber, we have a collection of theta pairs smoothly parameterized by those $x \in X$ such that $\{x\} \times \mathbb{R}$ intersects $B$ nontrivially (so the parameter space is diffeomorphic to $B$), with holonomic homotopies along the midpoints of each theta pair. By microflexibility of $R_{\text{reg-tang}}$ applied to these theta pairs parameterized by $x$, these holonomic homotopies extend for a small time (uniform over the parameter $x$) over the curves $\{(x) \times I_x\}$. This extension agrees with the homotopy on the midpoints $\{(x) \times \mathbb{R}\} \cap \text{Op} B$ and agrees with $F^0$ on $\text{Op}(\partial B)$. Moreover, we can extend them to the rest of $\{(x) \times \mathbb{R}\} \cap \text{Op} A$ by defining them to be $F^0$ restricted to the fibers. We now have a collection of regular integral homotopies parameterized by $B$ and defined on $\{(x) \times \mathbb{R}\} \cap \text{Op} A$, and which agree with $F^\tau$ when restricted to $\{(x) \times \mathbb{R}\} \cap \text{Op} B$, up to a uniform small time $\sigma$. Moreover, these extended holonomic homotopies are smooth with respect to the parameter $x$, by definition of microflexibility. Define the desired holonomic homotopy $\tilde{F}^\tau$ of $R_{c-tang}$ by gluing these extended holonomic homotopies together, with the uniform time $\sigma$ to be the one obtained by microflexibility of $R_{\text{reg-tang}}$. Since the extended homotopies by the curves are small perturbations interpolating between $F^0$ and $F^\tau$, we have that $\tilde{F}^\tau$ restricted to the transverse direction $X \times \{t\}$ remains a totally real immersion transverse to $D$, by openness of $R_c$. Therefore $\tilde{F}^\tau$ is holonomic and $R_{c-tang}$ is microflexible. The same argument holds for the general parametric case, since microflexibility of $R_{\text{reg-tang}}$ allows for the extensions on the boundary of the parameter cube to be constant.

Moreover, any formal/genuine solution of $R_c$ on a simplex $\Delta \subset X$ can be extended to a formal/genuine local solution of $R_{c-tang}$ on $\Delta \times \mathbb{R} \subset X \times \mathbb{R}$, again since $\Delta$ is contractible, and so there are no obstructions to a nonvanishing vector field tangent to $D$ defined on the image of $\Delta$ in $Z$. Thus, we can flow a formal/genuine local solution of $R_c$ on a simplex $\Delta$ for short time to a formal/genuine solution of $R_{c-tang}$ on $\Delta \times \mathbb{R}$, as we did above in the proof of local integrability.

Take a formal solution of $R_I \subset R_c$ on $X$, which exists as proven in section 3. We take a triangulation of $X$ and proceed by induction over the dimensions of the simplices. For the base case of 0-simplices $\Delta^0$, extend the formal solution of $R_I \subset R_c$ to a formal solution of $R_{c-tang}$ on $\Delta^0 \times \mathbb{R}$. By local integrability, we can then find a holonomic section of $R_{c-tang}$ over each 0-simplex. For the inductive step on $k$-simplices $\Delta^k$, assume we have a holonomic section defined on $\partial \Delta^k$, which is a union of $(k-1)$-simplices. By taking a nowhere vanishing vector field defined on $\Delta^k$ tangent to $D$ as above, we extend the formal solution of $R_I \subset R_c$ to a formal solution of $R_{c-tang}$ on $\Delta^k \times \mathbb{R}$, with the extension agreeing with the inductively defined holonomic extension along the boundaries of the simplices. Then by the above holonomic approximation theorem, we can perturb an open neighborhood of $\Delta^k \times 0$ by a vertical diffeotopy $h$ and obtain a holonomic section defined on an open neighborhood of $h(\Delta^k \times 0)$ that is $\epsilon$-close to our formal solution in $R_I$, again agreeing with the inductively defined holonomic section on the boundary. Restrict the section to $h(\Delta^k \times 0)$ for a holonomic section of $R_c$ which is $\epsilon$-close to the original formal solution of $R_I$. Moreover, the
section is well defined globally on the $k$-skeleton of $X$, as the holonomic extensions agree with the holonomic sections defined on the lower dimensional skeleta. Repeat the argument on $\Delta^{k+1}$. Notice that the vertical diffeotopy $h$ is constructed at each step, along each simplex. Finally, we end with a holonomic section defined on $h(X \times 0)$ that is $\epsilon$-close to the isotropy locus.

In summary, given a formal solution mapping into the isotropy locus, we can apply holonomic approximation and find a holonomic section into an open neighborhood of the isotropy locus. In \cite{3}, an expression for the Nijenhuis tensor $N_J$ in terms of the torsion tensor $\theta$ is given as:

$$N_{J_f}(\zeta, \eta) = 4\theta(\overline{\partial}J_f f(x) \cdot \zeta, \overline{\partial}J_f f(x) \cdot \eta)$$

where $x \in X$ and $\zeta, \eta \in T_xX$. On a compact manifold, we have that any two metrics are equivalent, and similarly for any two pointwise norms these metrics define on any tensor. Then, for any choice of metric and norm, we have that the tangential lift $G\overline{\partial}f$ maps into the isotropy locus $I$ if and only if $\|N_{J_f}\| = 0$. For any metric $g$ on $X$, we can push forward the metric by the diffeotopy constructed above to $h(X)$. By construction of the holonomic section, and the continuity of the norm and variation of the complex structure, we have that $\|N_{J'}\|_{h^*g} < \epsilon$, where $J'$ is the almost complex structure pulled back by the holonomic section. By the above equation relating the Nijenhuis tensor with $\theta$, we have that $N_{h^*J_f} = h^*N_{J_f}$ for any smooth map $h$. This implies that $\|N_{h^*J_f}\|_g = \|h^*N_{J_f}\|_g = \|N_{J'}\|_{h^*g} < \epsilon$. This then gives us an almost complex structure on $X$ which has small Nijenhuis norm, with respect to the arbitrary metric $g$. We can iterate the argument of finding a formal solution and applying holonomic approximation and so, we have the following.

**Corollary 4.1.** For $n \leq 77$, an almost complex $n$-dimensional compact manifold $X$ admits a sequence of almost complex structures so that the pointwise Nijenhuis norms become arbitrarily small.

**Remark.** Alternatively, the relation $R_{f^*-\text{tang}} \subset R_{c^*-\text{tang}}$ of maps whose $\overline{\partial}$ lifts into an $\epsilon$-neighborhood of $I$ when restricted to the horizontal component is invariant under precomposing with vertical diffeomorphisms \cite{4}, so we can pull back the holonomic section over $h(X)$ to a holonomic section over $X$ that still maps into an open neighborhood of $I$. Doing so loses the approximation to the original formal solution, but we do not lose the approximation to $I$; the point is that we holonomically approximate the formal solution, which serves as a guide, to obtain a holonomic tangential map that approximates the isotropy locus. Losing $\epsilon$-closeness to the formal solution also only occurs in the first and higher jet spaces; one still keeps the approximation on the level of 0-jets \cite{4}. Regardless, we obtain a map on $X$ that is $\epsilon$-close to $I$, and therefore by continuity obtain an almost complex structure that is close to integrability. Note we do not obtain an almost complex structure that is “close” to the initial complex structure, since we are only approximating a formal solution.

## 5 Remarks on the Nijenhuis energy

In \cite{5}, Evans considers the following functional:

$$\mathcal{N}_{\text{ij}} : \mathcal{J} \to \mathbb{R}, \quad \mathcal{N}_{\text{ij}}(J) = \int_M |N_J|^2 \omega^n$$

where $N_J$ denotes the Nijenhuis tensor with respect to a given almost complex structure on a compact symplectic manifold $(M, \omega)$ and the norm is taken with respect to a given almost Kähler metric, with the functional defined on the space of compatible almost complex structures.
Note that if the infimum over $J$ of the Nijenhuis energy is ever positive, then it would imply that there is a non-integrable almost complex structure compatible with the symplectic form $\omega$. Le and Wang in [15] compute the infimum as zero for the Kodaira-Thurston manifold (which is a symplectic but non-Kähler real 4-dimensional nilmanifold). Evans extends the result and proves in [5] the following theorem.

**Theorem 5.1.** (Evans) If $(M, \omega)$ is a compact, symplectic manifold such that $[\omega] \in H^2(M, \mathbb{Q})$, then the infimum of $\text{Nij}$ over $J$ is zero.

The question of whether the infimum of the Nijenhuis energy is ever positive is raised in both [15] and [5]. However, from the results of section 4, we have the following corollary:

**Corollary 5.1.** The infimum of the Nijenhuis energy is always zero, for dimensions $n \leq 77$.

Since we can always find smaller neighborhoods of the isotropy locus after fixing a metric, we can iterate the argument above and find almost complex structures such that the Nijenhuis energy tends to zero.

**References**

[1] Bryant, R. L., and Hsu, L., 1993. *Rigidity of integral curves of rank 2 distributions*, Invent. Math. 114, pp. 435-461.

[2] Calabi, E., and Eckmann, B., 1953. *A class of compact, complex manifolds which are not algebraic*, Ann. Math. 58, No. 3, pp. 494-500.

[3] Demailly, J.-P., and Gaussier, H., 2017. *Algebraic embeddings of smooth almost complex structures*, J. Eur. Math. Soc. 19, pp. 3391-3419.

[4] Eliashberg, Y., and Mishachev, N., 2002. *Introduction to the h-principle*, Grad. Stud. in Math., Vol. 48, American Mathematical Society, Providence, R.I., pp. 44-46, 66-67, 129-133, 136-138.

[5] Evans, J. D., 2012. *The infimum of the Nijenhuis energy*, Math. Res. Lett. 19, pp. 383-388.

[6] Glover, H. H., Homer, W. D., Stong, R. E., 1982. *Splitting the tangent bundle of projective space*, Indiana Univ. Math. J. 31, No. 2, pp. 161-166.

[7] Goertsches, O., and Konstantis, P., 2017. *Almost complex structures on connected sums of complex projective spaces*, arXiv preprint, [arXiv:1710.05316](http://arxiv.org/abs/1710.05316).

[8] Griffiths, P., and Harris, J., 1994. *Principles of algebraic geometry*, John Wiley & Sons, Inc., Hoboken, N.J., pp. 137, 176.

[9] Gromov, M., 1986. *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 9, Springer-Verlag, Berlin, Heidelberg.

[10] Hartshorne, R., 1977. *Algebraic geometry*, Springer, New York, N.Y., p 162.

[11] Hatcher, A., 2001. *Algebraic topology*, Cambridge University Press, New York, N.Y., p. 379.

[12] Hirsch, M. W., 1976. *Differential topology*, Springer-Verlag, New York, N.Y., pp. 58-66.

[13] Hsu, L., 1992. *Calculus of variations via the Griffiths formalism*, J. Differential Geometry. 36, pp. 551-589.
[14] Lazarsfeld, R., 2004. *Positivity in algebraic geometry II. Positivity for vector bundles, and multiplier ideals*, Springer-Verlag, Berlin, Heidelberg, pp. 26-27.

[15] Le, H. V., and Wang, G., 2001. *Anti-complexified Ricci flow on compact symplectic manifolds*, J. Reine Angew. Math. **531**, pp. 17-31.

[16] Müller, S., and Geiges, H., 2000. *Almost complex structures on 8-manifolds*, Enseign. Math. **46**, pp. 95-107.

[17] Martínez-Aguinaga, J., and del Pino, Á., 2019. *h-principle for regular curves in bracket-generating distributions*. In preparation.

[18] Sommese, A. J., 1983. *A convexity theorem*, Proceedings of Symposia in Pure Math. **40**, Part 2, pp. 497-505.

[19] Sommese, A. J., 1978. *Submanifolds of abelian varieties to Rebecca*, Math. Ann. **233**, pp. 229-256.

[20] Sommese, A. J., and Van de Ven, A., 1986. *Homotopy groups of pullbacks of varieties*, Nagoya Math. J. **102**, pp. 79-90.

[21] Van de Ven, A., 1966. *On the Chern numbers of certain complex and almost complex manifolds*, Proc. Natl. Acad. Sci. USA. **55**, No. 6, pp. 1624-1627.

[22] Varolin, D., 2001. *The density property for complex manifolds and geometric structures*, J. Geom. Anal. **11**, No. 1, pp. 135-160.

[23] Yang, H., 2012. *Almost complex structures on (n-1)-connected 2n-manifolds*, Topol. Its Appl. **159**, pp. 1361-1368.

Stony Brook University, Department of Mathematics

E-mail address: tobias.shin@stonybrook.edu