Can a chaotic solution in the QCD evolution equation restrain high-energy collider physics?

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Abstract

We indicate that the random aperiodic oscillation of the gluon distributions in a modified Balisky–Fadin–Kuraev–Lipatov (BFKL) equation has positive Lyapunov exponents. This first example of chaos in QCD evolution equations, raises the sudden disappearance of the gluon distributions at a critical small value of the Bjorken variable $x$ and may stop the increase of the new particle events in an ultra high energy hadron collider.

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The gluon distributions are important knowledge in research of high energy collider physics. The gluon distributions of the nucleon cannot be extracted directly from the measured structure functions in deep inelastic scattering experiments and they are mainly predicted by using the QCD evolution equations. However, the linear Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP)\textsuperscript{1,2} and Balitsky-Fadin-Kuraev-Lipatov (BFKL)\textsuperscript{3} evolution equations are no longer reliable at ultra higher energy, therefore a series nonlinear evolution equations, for example, the GLR-MQ-ZRS Gribov-Levin-Ryskin, Mueller-Qiu, Zhu-Ruan-Shen (GLR-MQ-ZRS)\textsuperscript{4–6} and Balitsky-Kovchegov (BK)\textsuperscript{7} equations were proposed, in which the corrections of the gluon fusion are considered. An important prediction of these equations is that the gluon distributions approach to a so-called ‘saturation limit’ asymptotically at small Bjorken variable $x$, where gluon splitting is balanced with fusion.

As we well know, the nonlinear dynamics may have a characteristic solution–chaos, which have been observed in many natural phenomena\textsuperscript{8}. Therefore, it is interesting to ask: Does nonlinear QCD evolution equation of gluon distribution also have chaotic solution? How does it impact the gluon distribution? Recently, we have proposed a nonlinear modified BFKL (MD-BFKL) equation in Ref. [9], which describes the corrections of the gluon recombination to the BFKL equation. We found that the unintegrated gluon distribution function $F(x, k^2)$ in the MD-BFKL equation in the fixed coupling approximation begins its smooth evolution under suppression of the gluon fusion likes the solution of the BK equation, but when $x$ comes to a smaller critical $x_c$, $F(x, k^2)$ will oscillate aperiodically near the evolution endpoint $k^2_0$. This behavior has the characteristic feature of chaos: random and sensitive to the initial conditions. Furthermore, with the enhancement of oscillation, the distribution $F(x, k^2)$ will disappear suddenly at $x_c$. This kind of phase transition led by chaotic solutions will certainly call our attention to reconsider the future
of high energy collider physics carefully.

In this letter, we indicate that the above-mentioned oscillate solution in the MD-BFKL equation is really chaos and it still holds in the running coupling case.

Fig. 1: A part of the amplitudes for four related evolution equations. (a) The DGLAP equation; (b) the BFKL equation; (c and d) the GLR-MQ-ZRS and MD-BFKL equations, which contain the leading corrections of the gluon recombination to the DGLAP and BFKL equations, respectively.

We develop the MD-BFKL equation\textsuperscript{9} due to the following considerations: as a leading logarithmic approximation, the DGLAP equation neglects the correlation of initial partons. With the increase of parton densities, more initial gluons should be considered in evolution. By adding initial gluons on the elementary amplitude of the DGLAP equation in Fig. 1(a) step by step, we can reach the amplitudes in Figs. 1(b)-1(d). In a unified theoretical framework and by making use of time ordered perturbation theory (TOPT),\textsuperscript{10}, we derived the well-known the BFKL and GLR-MQ-ZRS equation as well as a new MD-BFKL equation, which reads
\[-x \frac{\partial F(x, k)}{\partial x} \]
\[= \frac{\alpha_s N_c}{\pi^2} \int d^2 k' \frac{k^2}{k'^2 (k' - k)^2} \left[ F(x, k') - \frac{1}{2} F(x, k) \right] \]
\[+ \frac{18 \alpha_s^2}{\pi^2 R_N^2 N_c^2} \int d^2 k' \frac{k^2}{k'^2 (k' - k)^2} \left[ \frac{1}{k'^2} F^2 \left( \frac{x}{2}, k' \right) - \frac{1}{2 k'^2} F^2 \left( \frac{x}{2}, k \right) \right] \]
\[-\frac{36 \alpha_s^2}{\pi^2 R_N^2 N_c^2} \int d^2 k' \frac{k^2}{k'^2 (k' - k)^2} \left[ \frac{1}{k'^2} F^2(x, k') - \frac{1}{2 k'^2} F^2(x, k) \right], \quad (1)\]

where the three terms on the right-hand side are the BFKL kernel of gluon splitting, nonlinear anti-shadowing and shadowing kernels led by gluon fusion, respectively. Each term includes real and virtual parts in order to ensure that the evolution kernels are infrared safe. It should be pointed out that Eq. (1) is different from the BK equation both in their elementary amplitudes and structures. Although the derivation of this equation is tedious, the reason of it can be proven by the consistency among this equation and DGLAP, BFKL and GLR equations. In fact, as we see in Fig. 1, the structures of the MD-BFKL and GLR-MQ-ZRS equations have the similar relationship of those in the BFKL and DGLAP equations: at leading approximation, the elementary amplitudes of the BFKL and DGLAP equations share the same evolution kernel-gluon splitting. However, in the BFKL-initial state, there are two gluons correlated by relative transverse momentum $k'$, therefore the initial state in the BFKL equation can connect with the gluon splitting vertex by two different ways. With simple algebra, we can find that the factor $1/k^2$ in the DGLAP evolution kernel becomes $k^2/k'^2 (k' - k)^2$ in the BFKL equation because of the contributions of the interference diagrams. Similarly, the GLR-MQ-ZRS and MD-BFKL equations have the same gluon recombination kernel, but in the later equation the evolution kernel connect with the initial state also by two ways. Replacing the evolution kernel in the GLR-MQ-ZRS equation in such ways, one can reach the MD-BFKL equation. That is, once the DGLAP, BFKL and GLR-MQ-ZRS equations are
determined, the form of the MD-BFKL equation (1) is also fixed. We should emphasize that a complete evolution equation must include contributions from virtual diagrams for infrared safety, however, they are cancelled in the GLR-MQ-ZRS equation, while cannot be neglected in the MD-BFKL equation.

In this work, we use a running coupling $\alpha_s(k^2)$ in Eq. (1) and so we need to consider the diffusion in $\ln k^2$ with decreasing $x$, which leads to an increasingly large contribution from the infrared region of $k^2$ where the equation is not expected to be valid. For this sake, as Ref. [11] we split the integration region for real gluon emission in Eq. (1) up into two parts: region(A) 0 to $k^2_0$ and region(B) $k^2_0$ to $\infty$. In region(B) the MD-BFKL equation as it stands is taken to hold and in region(A) $F(x, k^2)$ is parameterized as $Ck^2/(k^2 + k_a^2)$ with $k_a^2 = 1$ GeV$^2$, where the parameter $C$ keeps the smooth connection between two parts. For simplicity, we neglect the contributions from the antishadowing effects and take the cylindrically symmetric solution. We use the Runge-Kutta method to compute the evolution equations. Assuming that a symmetric Gaussian input distribution exists at the evolution stating point $x_0 = 10^{-3}$.

\[
F(x_0 = 10^{-3}, k^2)(k^2)^{-1} = \beta \exp \left[ -\frac{\log^2(k^2/1GeV^2))}{40} \right],
\]

where $\beta = 0.1$. The solutions of equation (1) are shown in Fig.2. Firstly, we can see that the distribution $F(x, k^2)$ disappear suddenly at $x = x_c$. Furthermore, curves with different values of $k^2$ turn down at the same $x_c$. If we regard $x$ as an order parameter, Fig. 2 exhibits a phase transition of first kind.
Fig.2: Dependence of the unintegrated gluon distribution on $x$ in the MD-BFKL equation (1) for different values of $k^2$. The results show that $F(x, k^2)$ suddenly drops near a critical value of $x \sim x_c \simeq 3.8 \times 10^{-8}$. The dashed curves are the corresponding solution of the BFKL equation with $k^2 = 50 GeV^2$.

The reason of the sudden disappearance of the gluon distribution can be innovated by the relation of $F(x, k^2)$ and $k^2$ in Fig.3. From the start point $x_0$, gluons diffuse on the transverse momentum space rapidly under the action of the BFKL linear kernel. Because of the gluon fusion, the above diffusion towards low $k^2$ is suppressed obviously. All these are like that the BK equation has predicted. While what is different is that, when $x$ goes to $x_c$, the oscillation of the curve will happen and increase rapidly once it near $k_0^2$. This leads to a huge shadowing effect and the distribution $F(x, k^2)$ disappears at $x_c$. It is interesting that dispersed gluons will gather near $k_0^2$ before disappearance. This kind of oscillation is random because $k$ is not ordered in evolution. Furthermore, the above
aperiodic oscillation is very sensitive to the initial conditions. Especially, the oscillation will be enhanced with the increase of the numerical calculating precision. These features are also observed in other chaos phenomena universally.

Fig.3: Dependence of the unintegrated gluon distribution on $k^2$ for different values of $x$. The results present the oscillations near $x_c$ when $k^2 \to k_0^2$. This leads to a huge shadowing effect and the distribution $F(x, k^2)$ disappears at $x < x_c$.

A standard criterion of chaos is that the system has the positive Lyapunov exponents, which indicates a strong sensitivity to small changes in the initial conditions. We regard $y = \ln 1/x$ as ‘time’ and calculate the Lyapunov exponents $\lambda(k^2)$ in a finite region: $10^{-7} \leq x \leq 0.2 \times 10^{-8}$, where the oscillation of the distribution is obvious. We divide equally the above mentioned $y$-region into $n$ parts with $y_1, y_2, \ldots, y_{n+1}$ and $\tau = (y_{n+1} - y_1)/n$. Assuming that the distribution evolves to $y_1$ from $y_0 = \ln 1/x_0$ and results $F(y_1, k)$. Corresponding to a given value $F(y_1, k)$ at $(y_1, k)$, we perturb it to $F(y_1, k) + \Delta$ with $\Delta \ll 1$. Then we continue the evolutions from $F(y_1, k)$ and $F(y_1, k) + \Delta$ to $y_2$ from $y_1$. 

respectively, and denote the resulting distributions as $F(y_2, k)$ and $\tilde{F}(y_2, k)$. Making the difference $\Delta_2 = |\tilde{F}(y_2, k) - F(y_2, k)|$. In the following step, we repeat the perturbation $F(y_2, k) \to F(y_2, k) + \Delta$ and let the next evolutions from $F(y_2, k)$ and $F(y_2, k) + \Delta$ from $y_2$ to $y_3$ respectively and get the results $\Delta_3 = |\tilde{F}(y_3, k) - F(y_3, k)|$...... (see Fig. 4). The Lyapunov exponents for the image from $y$ to $F(y, k)$ are defined as

$$\lambda(k^2) = \lim_{n \to \infty} \frac{1}{n\tau} \sum_{i=2}^{n+1} \ln \frac{\Delta_i}{\Delta}. \quad (3)$$

The Lyapunov exponent of the gluon distribution in the MD-BFKL equation with the input Eq. (2) are presented in Fig. 5. For comparison, we give the Lyapunov exponents but using the BFKL and BK equations in Fig. 5. The positive values of the Lyapunov exponents clearly show that the oscillation of $F(x, k) \sim k^2$ is chaos of the MD-BFKL equation. Therefore, we conclude that chaos in the MD-BFKL equation lead to the sudden disappearance of the gluon distributions.
Fig. 4: Schematic programs calculating the Lyapunov exponents of the evolution equation. The results are insensitive to the value of $\Delta$.

Fig. 5: Plots of the Lyapunov exponents in the region $10^{-7} \leq x \leq 0.2 \times 10^{-8}$. The
positive Lyapunov exponents show that the corresponding solution of the MD-BFKL equation is chaos.

The important questions are at which scale gluons will disappear and how much it will impact negatively on high energy collider physics. Unfortunately, we cannot predict the start point of the evolution of MD-BFKL equation at this moment. But considering that the MD-BFKL equation works right after GLR-MQ-ZRS and BK equations, gluon disappearance should happen after the saturation phenomena predicted by GLR-MQ-ZRS and BK equations.

A typical process testing new particle with mass $M$ on high energy hadron collider, for example in the gluon fusion model it directly relates to the unintegrated gluon distribution via

$$
\sigma = \int d^2k_1 \int d^2k_2 \int_0^1 dx_1 \int_0^1 dx_2 \delta(x_1x_2 - \tau)\delta^{(2)}(k_1 + k_2 - p_T)F_1(x_1, k_1)F_2(x_2, k_2)d\sigma, \quad (4)
$$

which is the function of $\tau = M^2/s$. The increase of the new particle events with increasing energy $s$ will be stopped due to the gluon disappearance when $\tau \leq x_c$. Thus we need to re-estimate the trials to the new physics in hadron collider physics.

Finally, we emphasize that the MD-BFKL equation is constructed based on a naive partonic picture (Fig. 1) and simple leading QCD corrections. Many higher order corrections are neglected, such as possible mixture of the operators with different twists, the next leading logarithmic (NLL)) and next leading order (NLO) corrections, singularities from non-perturbative parts in the factorization procedure. Of course, the MD-BFKL equation will only be an applicable QCD evolution equation until all the above corrections are considered. An important questions is: will chaos effect we demonstrated in the MD-BFKL equation disappear after further corrections are considered? To answer this question, we
point out that chaos does not appear in the nonlinear GLR equation is related to the fact that its evolution kernel has no singularity. On the other hand, although the evolution kernel in the nonlinear term of the BK equation is also the singular BFKL kernel, these singularities can be absorbed into the re-definition of the scattering amplitudes, like its form in momentum space. Therefore the BK equation has no chaotic solution, either. Thus we suggest the fact that chaos appear in the MD-BFKL equation firstly is related to the singularities in its nonlinear terms. From the experiences in the study of the BFKL kernel, those possible QCD corrections we mentioned to the MD-BFKL will probably make singularities of this nonlinear equation more complicated, instead of removing these singularities completely, since the regularized singular parts of the evolution kernel always dominate the evolution. In this situation, we could expect that more interesting chaos phenomena will appear in the new MD-BFKL equation. These phenomena will be most interesting to high energy physics and nonlinear science.

In summary, a random oscillation of the unintegrated gluon distributions in a modified BFKL equation is indicated as chaos, which has the positive Lyapunov exponents. This first example of chaos in QCD evolution equations, raises the sudden disappearance of the gluon distributions at a critical small value of the Bjorken variable $x$ and may stop the increase of the new particle events in a ultra high energy hadron collider.

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