Data-driven aggregation in non-parametric density estimation on the real line

SERGIO BRENNER MIGUEL *   JAN JOHANNES *

Ruprecht-Karls-Universität Heidelberg

Abstract

We study non-parametric estimation of an unknown density with support in $\mathbb{R}$ (respectively $\mathbb{R}^+$). The proposed estimation procedure is based on the projection on finite dimensional subspaces spanned by the Hermite (respectively the Laguerre) functions. The focus of this paper is to introduce a data-driven aggregation approach in order to deal with the upcoming bias-variance trade-off. Our novel procedure integrates the usual model selection method as a limit case. We show the oracle- and the minimax-optimality of the data-driven aggregated density estimator and hence its adaptivity. We present results of a simulation study which allow to compare the finite sample performance of the data-driven estimators using model selection compared to the new aggregation.

Keywords: Density estimation, minimax theory, Laguerre functions, Hermite functions, projection estimator, aggregation, adaptation

AMS 2000 subject classifications: Primary 62G05; secondary 62G07, 62C20.

1 Introduction

In this paper we consider the data-driven estimation of an unknown density $f$ with non-compact support in the real line given an independent and identically distributed (i.i.d.) sample $X_1, \ldots, X_n$ from $f$. In the literature, non-parametric density estimation is a well-discussed problem and many estimation strategies based on splines, kernels or wavelets, to name but a few, are considered. For an overview of various methods we refer to Comte [2017], Efro-movich [1999], Silverman [2018] and Tsybakov [2008]. Here, we will focus on the projection of the density $f$ on an orthonormal basis and therefore assume its square integrability. This

*Institut für Angewandte Mathematik, MATHEMATIKON, Im Neuenheimer Feld 205, D-69120 Heidelberg, Germany, e-mail: {brennermiguel|johannes}@math.uni-heidelberg.de
has been studied for densities with compact support (e.g. Massart [2007] and Efroimovich [1999]), with support in \( \mathbb{R} \) using wavelets and Hermite functions (e.g. Juditsky et al. [2004] and Belomestny et al. [2019], respectively) or with support in \( \mathbb{R}^+ \) using Laguerre functions (e.g. Comte and Genon-Catalot [2018]).

Here, we cover the projection on the Hermite functions or the Laguerre functions, i.e. the estimation over a set \( A \subseteq \mathbb{R} \) where in the Hermite case \([H] A = \mathbb{R} \) and in the Laguerre case \([L] A = \mathbb{R}^+ \). In the sequel \( L^2_A \) denotes the set of all square integrable functions over \( A \) endowed with its usual inner product \( \langle \cdot, \cdot \rangle_A \) and norm \( \| \cdot \|_A \). Furthermore, let \( \{ \varphi_j, j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \} \) be in case \([H]\) and \([L]\) the Hermite and Laguerre basis, respectively. Therewith, for each \( f \in L^2_A \) we define the family \( (f_k)_{k \in \mathbb{N}} \) of projections of \( f \) onto the subspaces \( (S_k)_{k \in \mathbb{N}} \) where \( S_k \) is the linear subspace spanned by the first \( k \) basis functions. By replacing the unknown coefficients by their empirical counterparts, we consider for \( f_k, k \in \mathbb{N} \), the unbiased orthogonal series estimator (OSE)

\[
\hat{f}_k := k^{-1} \sum_{j=0}^{k-1} \hat{a}_j(f) \varphi_j \text{ with } \hat{a}_j(f) := n^{-1} \sum_{i=1}^{n} \varphi_j(X_i).
\] (1.1)

To measure the performance of the estimator we discuss its mean integrated squared error (MISE) as risk, we state oracle rates and we derive upper bounds for its maximal risk over Sobolev classes. Further we show that the projection estimator with optimal choice of the dimension parameter is minimax-optimal over Sobolev classes. The proof of the lower bound borrows ideas from Belomestny et al. [2017] and Comte et al. [2019]. In practice, however, the optimal choice of the dimension parameter is not feasible since it depends on characteristics of the unknown density \( f \). Therefore, Comte and Genon-Catalot [2018] consider a model selection approach, inspired by the work of Barron et al. [1999] and extensively described in Massart [2007], to select fully data-driven the dimension parameter in such a way that the bias and variance compromise is automatically reached by the resulting estimator. More precisely, the authors choose the random dimension \( \hat{k} \) as a minimum of \(-\|\hat{f}_k\|_A^2 + \text{pen}_k\) over the set of admissible parameters \([1, M_n] := [1, M_n] \cap \mathbb{N}\) for a given upper bound \( M_n \) and sequence of penalty terms \((\text{pen}_k)_{k \in \mathbb{N}}\).

In this work, we study a different data-driven procedure. Introducing \( M_n \in \mathbb{N} \) and aggregation weights \( w := (w_m)_{m \in [1, M_n]} \in [0, 1]^{M_n} \) with \( \sum_{i=1}^{M_n} w_i = 1 \) we define the aggregated estimator \( \hat{f}_w := \sum_{k=1}^{M_n} w_k \hat{f}_k \). Note that the aggregation is called fully data-driven if the aggregation weights depend on the data only. The model selection procedure can be integrated into this aggregation framework via the model selection weights

\[
\hat{w}_k := \delta_{\hat{k}}(\{k\}), \quad k \in [1, M_n]
\] (1.2)

where \( \delta_x \) denotes the usual Dirac measure in \( x \in \mathbb{R} \). However we suggest as a new data-driven
choice the weights defined by
\[
\tilde{w}_k := \frac{\exp(-\kappa n\{-\|\hat{f}_k\|_A^2 + \text{pen}_k^A\})}{\sum_{j=1}^{M_n} \exp(-\kappa n\{-\|f_j\|_A^2 + \text{pen}_j^A\})}, \quad k \in [1, M_n]
\] (1.3)
where the choice of the penalties \((\text{pen}_k^A)_{k \in \mathbb{N}_0}\), the numerical constant \(\kappa \geq 1\) and the upper bound \(M_n\) will be further discussed in Section 3. We refer to them as Bayesian weights since their particular form takes its inspiration from a-posteriori weights in a Bayesian sequence space model (c.f. Johannes et al. [2015]). In this paper we derive upper bounds for the (maximal) risk of the aggregated estimator using either Bayesian weights or model selection weights where throughout the paper \(\hat{k}\) is chosen as a minimum of \(-\|\hat{f}_k\|_A^2 + \text{pen}_k^A\) over \([1, M_n]\). Note that in this situation the Bayesian weights converge to the corresponding model selection weights as \(\kappa\) tends to infinity, i.e., \(\lim_{\kappa \to \infty} \tilde{w}_k = \hat{w}_k\) for each \(k \in [1, M_n]\). The paper is organised as follows: in Section 2 we introduce our basic assumptions, recall the oracle inequalities and develop the minimax theory. We show, in Section 3, the oracle- and the minimax-optimality of the data-driven aggregated density estimator and hence its adaptivity. In Section 2 and 3 we only present key arguments of the proofs while more technical details are postponed to the appendices B and C, respectively. Finally, results of a simulation study are reported in Section 4 which allow to compare the finite sample performance of the aggregated estimator with model selection weights and Bayesian weights of a density given independent observations. Further we introduce the Laguerre and Hermite functions and recall some of their properties in the appendix A.

2 Minimax theory

Given an orthonormal basis \((\varphi_j)_{j \in \mathbb{N}_0}\), we consider for any function \(f \in \mathbb{L}_A^2\) its expansion \(f = \sum_{j \in \mathbb{N}_0} a_j(f) \varphi_j\) with \(a_j(f) := \langle f, \varphi_j \rangle_A\) and for each \(k \in \mathbb{N}\) the subspace \(S_k\) spanned by the first \(k\) basis functions \(\{\varphi_j, j \in [0, k]\}\), where here and subsequently for real numbers \(a < b\) we write shortly \([a, b] := [a, b) \cap \mathbb{Z}\), \([a, b) := (a, b) \cap \mathbb{Z}\), and so forth. Consequently, the projection of \(f \in \mathbb{L}_A^2\) onto \(S_k\) is given by \(f_k = \sum_{j=0}^{k-1} a_j(f) \varphi_j\). For each \(k \in \mathbb{N}_0\) and density \(f \in \mathbb{L}_A^2\) we define \(b_k^2(f) \in [0, 1]\) as follows \(\|f\|_A^2 b_k^2(f) = \|f - f_k\|_A^2 = \sum_{j > k} |a_j(f)|^2\), where we agree on \(f_0 := 0\) and hence \(b_0^2(f) = 1\). Let \(\mathbb{E}_t\) and \(\mathbb{E}_n\) denote, respectively, the expectation with respect to the marginal and joint distribution \(\mathbb{P}_n\) of the i.i.d. \(n\)-sample \((X_i)_{i \in [1, n]}\).

**Oracle optimality.** Elementary calculations show for each \(k \in \mathbb{N}\) the identity
\[
\mathbb{E}_n^n(\|\hat{f}_k - f\|_A^2) + \frac{1}{n} \|f\|_A^2 = \frac{n+1}{n} \|f - f_k\|_A^2 + \frac{1}{n} V_k \quad \text{with} \quad V_k := \sum_{j=0}^{k-1} \mathbb{E}_t(\varphi_j^2(X_1)). \quad (2.1)
\]
In appendix A we briefly recall elementary properties of Laguerre and Hermite functions. As for example, that they are bounded in the usual uniform norm, precisely, $\sup_{j \in \mathbb{N}_0} \| \varphi_j \|_\infty \leq C$ and, hence $\sup_{k \in \mathbb{N}} \frac{1}{k} V_k \leq C$ in case [L] and [H] with $C = \sqrt{2}$ and $C = 1$, respectively. Moreover, Comte and Genon-Catalot [2018] and Belomestny et al. [2019] have shown sharper upper and lower bounds for the term $V_k$. Precisely, setting

$$d := 1, \quad a := -1/2, \quad \nu_j := \mathbb{E}_f (X^a) + 1 \text{ in case } [L]$$
$$d := 10/12, \quad a := 2/3, \quad \nu_j := \mathbb{E}_f (|X|^a) + 1 \text{ in case } [H]$$  (2.2)

there exists a numerical constant $C \geq 1$ such that for each $k \in \mathbb{N}$ hold

$$\left\| \sum_{j=0}^{k-1} \varphi_j^2 \right\|_\infty \leq C k^d \quad \text{and} \quad \sum_{j=0}^{k-1} \mathbb{E}_f (\varphi_j^2 (X)) \leq C \nu_j k^{1.2}.$$  (2.3)

For a sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers with minimal value in a set $B \subset \mathbb{N}$ we define $\arg \min \{ a_n, n \in B \} := \min \{ m \in B : a_m \leq a_n, \forall n \in B \}$. For $n, k \in \mathbb{N}$ we set

$$\mathcal{R}^k_n (f) := [b_n^2 (f) \lor n^{-1} k^{1/2}], \quad k_n^2 (f) := \arg \min \{ \mathcal{R}^k_n (f), k \in \mathbb{N} \} \quad \text{and} \quad \mathcal{R}^\circ_n (f) := \min \{ \mathcal{R}^k_n (f), k \in \mathbb{N} \}.$$  (2.4)

Here for two real numbers $a, b \in \mathbb{R}$ we define $a \lor b := \max \{ a, b \}$ and $a \land b := \min \{ a, b \}$.

**Remark 2.1.** Note that by construction $\mathcal{R}^\circ_n (f) = \mathcal{R}^{k_n^2 (f)}_n (f)$ and $k_n^2 (f) \in \llbracket 1, n^2 \rrbracket$, since $b_n^2 (f) \leq 1 < (n^2 + 1)^{1/2} n^{-1}$, and hence $\mathcal{R}_n^\circ (f) < \mathcal{R}_n^k (f)$ for all $k \in \llbracket n^2 + 1, \infty \rrbracket$. It is worth stressing out that in compact density estimation the oracle dimension typically satisfies $k_n^2 (f) \in \llbracket 1, n \rrbracket$. Obviously, it follows thus $\mathcal{R}_n^\circ (f) = \min \{ \mathcal{R}_n^k (f), k \in \llbracket 1, n^2 \rrbracket \}$ for all $n \in \mathbb{N}$. Moreover, we shall emphasise that $\mathcal{R}_n^\circ (f) \geq n^{-1}$ for all $n \in \mathbb{N}$, and $\mathcal{R}_n^\circ (f) = o(1)$ as $n \to \infty$. We eventually use those elementary findings in the sequel without further reference. □

Combining (2.1) and (2.3) we immediately obtain

$$\inf \{ \mathbb{E}_f^n \| \hat{f}_k - f \|^2_A, k \in \mathbb{N} \} \leq \mathbb{E}_f^n \| \hat{f}_{\mathcal{R}_n^\circ (f)} - f \|^2_A \leq \frac{n+1}{n} \| f \|^2_A + C \nu_j \mathcal{R}_n^\circ (f).$$  (2.5)

The upper bound (2.3) for the variance term cannot be improved, since under the additional assumption $c_f := \inf_{x \in [a,b]} f(x) > 0$ for some $a, b \in \mathbb{R}$, $a < b$, there exists a constant $C_{f,a,b} > 0$ depending on $a, b$ and $f$ such that for each $k$ holds $V_k \geq C_{f,a,b} \sqrt{k}$ (e.g. Comte and Genon-Catalot [2018]). In this situation, from (2.1) it follows also

$$\inf \{ \mathbb{E}_f^n \| \hat{f}_k - f \|^2_A, k \in \mathbb{N} \} \geq \frac{n+1}{n} \| f \|^2_A + C_{f,a,b} - \frac{\| f \|^2_A}{n \mathcal{R}_n^\circ (f)} \mathcal{R}_n^\circ (f).$$  (2.6)

Consequently, the rate $(\mathcal{R}_n^\circ (f))_{n \in \mathbb{N}}$, the dimension parameters $(k_n^2 (f))_{n \in \mathbb{N}}$ and the OSE’s $(\hat{f}_{\mathcal{R}_n^\circ (f)})_{n \in \mathbb{N}}$, respectively, is an oracle rate, an oracle dimension and oracle optimal (up to a
constant) as soon as the leading factor on the right hand side is positive. Throughout the paper we shall distinguish for the density \( f \) and hence it’s associated sequence \((b_k(f))_{k \in \mathbb{N}}\) the following two cases

(p) there is \( K \in \mathbb{N} \) with \( b_{K-1}(f) > 0 \) (with \( b_0(f) = 1 \)) and \( b_K(f) = 0 \),

(np) for all \( K \in \mathbb{N} \) holds \( b_K(f) > 0 \).

**Remark 2.2.** Note that the expansion of \( f \) is in case (p) finite, i.e., \( f = \sum_{j=0}^{K-1} a_j(f) \varphi_j \) for some \( K \in \mathbb{N} \) while in the opposite case (np), it isn’t. Interestingly, in case (p) the oracle rate is parametric, that is, \( \mathcal{R}_n(f) \) is of order \( n^{-1} \). More precisely, if there is \( K \in \mathbb{N} \) with \( b_{K-1}(f) > 0 \) and \( b_K(f) = 0 \), then setting \( n_f := \frac{K^{1/2}}{b_{K-1}(f)} \), for all \( n \geq n_f \) holds \( b_{K-1}(f) > K^{1/2} n^{-1} \), and hence \( k_n^\alpha(f) = K \) and \( \mathcal{R}_n(f) = K^{1/2} n^{-1} \). On the other hand side, in case (np) the oracle rate is non-parametric, more precisely, it holds \( \lim_{n \to \infty} n \mathcal{R}_n(f) = \infty \). Indeed, since \( b_{2n}(f) \) \( \leq \mathcal{R}_{2n}^{\alpha/(2s)}(f) = \mathcal{R}_n(f) = o(1) \) as \( n \to \infty \) follows \( k_n^\alpha(f) \to \infty \), which implies the claim because \( n \mathcal{R}_n(f) \geq (k_n^\alpha(f))^{1/2} \).

Let us first briefly illustrate the last definitions by stating the order of \( k_n^\alpha(f) \) and \( \mathcal{R}_n(f) \) in case (np) for an often considered behaviour of the sequence \((b_k^2(f))_{k \in \mathbb{N}}\). Here and subsequently, we use for two strictly positive sequences \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) the notation \( a_n \sim b_n \) if the sequence \((a_n/b_n)_{n \in \mathbb{N}}\) is bounded away both from zero and infinity. Let us use \( b_k^2(f) \sim k^{-s} \), \( s > 0 \), as a particular specification. In this situation \( k_n^\alpha(f) \sim n^{2/(2s+1)} \) is the oracle dimension and \( \mathcal{R}_n(f) \sim n^{-2s/(2s+1)} \) is the oracle rate.

**Minimax optimality.** For each \( k \in \mathbb{N} \) let us measure the accuracy of \( \hat{k} \) by its maximal risk over classical Sobolev spaces or ellipsoids, respectively, defined for \( s, L \in \mathbb{R}^+ \) by

\[
\mathcal{W}_s^A := \{ f \in L_A^2 : \| f \|_s^2 := \sum_{k \in \mathbb{N}_0} k^s |a_k(f)|^2 < \infty \} \quad \text{and} \quad \mathcal{W}_{s,L}^A := \{ f \in \mathcal{W}_s^A : \| f \|_s^2 < L \}.
\]

For a more detailed discussion on Sobolev-Laguerre spaces \( \mathcal{W}_s^A \) and Sobolev-Hermite space \( \mathcal{W}_s^H \) we refer to Bongioanni and Torrea [2009] and Bongioanni and Torrea [2006], respectively. For \( a \) as in (2.2) we denote further a corresponding subset of densities with finite \( am \)-th moment, \( m \in \mathbb{R}^+ \), by

\[
\mathcal{D}_{s,L,m}^A := \{ f \in \mathcal{W}_{s,L}^A : f \text{ is a density and } \mathbb{E}_f(\| X \|^{am}) \leq L \}
\]

(2.7) and let \( \mathcal{D}_{s,m}^A := \bigcup_{L > 0} \mathcal{D}_{s,L,m}^A \). We note, that for each \( f \in \mathcal{D}_{s,L,m}^A \) with \( m \geq 1 \) and \( v_f \) as in (2.2) we have \( v_f \leq L + 1 \). Moreover, for each \( k \in \mathbb{N} \) holds \( \| f \|_{2s}^2 b_k^2(f) = \| f - f_k \|_s^2 \leq Lk^{-s} \). Exploiting the upper bound (2.5) there is a constant \( C_{s,L} \) depending on the class \( \mathcal{D}_{s,L,m}^A \) only, such that for each \( n \in \mathbb{N} \) with \( k_n^\alpha \sim n^{2/(2s+1)} \) it holds

\[
\sup \{ \mathbb{E}_f^n \| \hat{f}_{k_n^\alpha} - f \|_s^2 : f \in \mathcal{D}_{s,L,m}^A \} \leq C_{s,L} n^{-2s/(2s+1)}.
\]

(2.8)
Now we provide a lower bound to show that the upper bound in (2.8) is minimax-optimal over the ellipsoid $\mathbb{D}^{s,L,m}_A$. Further the following theorem is formulated for the more general ellipsoids $\mathbb{D}^{s,L,m}_A$ because the data-driven aggregation requires stronger moment assumptions.

**Theorem 2.3.** Let $n, m \in \mathbb{N}$ with $n \geq n_s = 8^{2s+1}$, then there exist constants $C_m, L, s, m > 0$ such that for all $L \geq L, s, m$ and for any estimator $\hat{f}$ of $f$ based on an i.i.d. sample $(X_j)_{j \in [1,n]}$, \[ \sup \{ \mathbb{E} \| \hat{f} - f \|^2_A : f \in \mathbb{D}^{s,L,m}_A \} \geq C_m n^{-2s/(2s+1)}. \]

**Proof of Theorem 2.3.** We outline here the main steps of the proof, while more technical details are deferred to the appendix. We will construct a family of functions in $\mathbb{D}^{s,L,m}_A$ by a perturbation of a density $f_\theta$ with small bumps, such that their $L^2$-distance and their Kullback-Leibler divergence can be bounded from below and above, respectively. The claim follows then by applying Theorem 2.5 in Tsybakov [2008]. In both cases [L] and [H] we use the same construction, which we present first.

Given a function $\psi \in \mathbb{L}^2$ for each $K \in \mathbb{N}$ (to be selected below) and $k \in [0, K]$ we define the bump-functions $\psi_{k,K}(x) := \psi(xK - K - k)$, $x \in \mathbb{R}$. For a density $f_\theta \in \mathbb{L}^2_A$ (specified in Lemmata B.2 and B.3 in the appendix B), a bump-amplitude $\delta > 0$ and a vector $\theta = (\theta_1, \ldots, \theta_K) \in \{0, 1\}^K$ we define

\[ f_\theta(x) = f_\theta(x) + \delta K^{-s} \sum_{k=0}^{K-1} \theta_{k+1} \psi_{k,K}(x). \quad (2.9) \]

The choice of $\psi$ is discussed in appendix B too, however, it ensures that $\int_A \psi(x) dx = 0$, and hence, $f_\theta$ integrates to one, and that the support of $\psi$ is contained in $[0, 1]$. Moreover, $f_\theta$ satisfies $c_{f_\theta} := \inf_{x \in [1,2]} f_\theta > 0$ which in turn for any $\delta \in (0, \delta_{f_\theta, \psi}]$ with $\delta_{f_\theta, \psi} := c_{f_\theta}/\|\psi\|_\infty$ implies $f_\theta(x) \geq 0$ for all $x \in A$. Indeed, on $[1,2]^c$ holds $f_\theta = f_\theta$ and for the non trivial case $x \in [1,2]$ there is $k_\theta \in [0, K]$ such that $x \in [1 + k_\theta/O, 1 + (k_\theta + 1)/K]$ and hence

\[ f_\theta(x) = f_\theta(x) + \theta_{k_\theta+1} \delta K^{-s} \psi(xK - K - k_\theta) \geq c_{f_\theta} - \delta \|\psi\|_\infty K^{-s} \geq 0. \]

Moreover, due to Lemma B.2 and Lemma B.3 for the cases [L] and [H], respectively, $f_\theta$ and the family $\{f_\theta : \theta \in \{0, 1\}^K\}$ belong to $\mathbb{D}^{s,L,m}_A$ for some $L, s, m > 0$. Exploiting Varshamov-Gilbert’s Lemma (see Tsybakov [2008]) in Lemma B.1 we show further that there is $M \in \mathbb{N}$ with $M \geq 2^{K/8}$ and a subset $\{\theta^{(0)}, \ldots, \theta^{(M)}\}$ of $\{0, 1\}^K$ with $\theta^{(0)} = (0, \ldots, 0)$ such that for all $j, l \in [0, M]$, $j \neq l$ the $L^2$-distance and the Kullback-Leibler divergence are bounded:

\[ \|f_{\theta^{(j)}} - f_{\theta^{(l)}}\|_A^2 \geq C_{\psi, \delta}^{(1)} K^{-2s} \quad \text{and} \quad \text{KL}(f_{\theta^{(j)}}, f_{\theta^{(l)}}) \leq C_{\psi, \delta}^{(2)} \log(M) K^{-2s-1} \quad (2.10) \]

where $C_{\psi, \delta}^{(1)} > 0$ and $C_{\psi, \delta}^{(2)} < \infty$ depend on $\psi$ and $\delta$ only. Selecting $K = \lceil n/(2s+1) \rceil$ follows

\[ \frac{1}{M} \sum_{j=1}^M \text{KL}((f_{\theta^{(j)}})^{\otimes n}, (f_{\theta^{(j)}})^{\otimes n}) = \frac{n}{M} \sum_{j=1}^M \text{KL}(f_{\theta^{(j)}}, f_{\theta^{(j)}}) \leq C_{\psi, \delta}^{(2)} \log(M) \]
where $C^{(2)}_{\psi, \delta} < 1/8$ for all $\delta^2 < \frac{\log(2)}{8C_{\psi}}\|\psi\|_A^2$ and $M \geq 2$ for $n \geq n_0 = 8^{2s+1}$. Thereby, we can use Theorem 2.5 of Tsybakov [2008], which in turn for any estimator $\hat{f}$ of $f$ implies

$$\sup_{f \in D_A^{(n,m)}} \mathbb{M}(\|f - \hat{f}\|_A^2) \geq \frac{C^{(1)}_{\psi, \delta}}{2} n^{-2s/(2s+1)} \geq \frac{\sqrt{M}}{1+\sqrt{M}} (1 - 2\sqrt{2C^{(2)}_{\psi, \delta}}) \geq 0.07.$$ 

Note that the constant $C^{(1)}_{\psi, \delta}$ does only depend on $\psi$ and $\delta$, hence implicitly also on $m$, but it is independent of the parameters $s, L$ and $n$. The claim of Theorem 2.3 follows by using Markov’s inequality, which completes the proof. □

## 3 Data-driven aggregation

Given a family $\{\hat{f}_k, k \in [1, M_n]\}$ of orthogonal series estimators as in (1.1) the optimal choice of the dimension parameter $k$ in an oracle or minimax sense, does depend on characteristics of the unknown density. Introducing aggregation weights $w = (w_k)_{k \in [1, M_n]} \in [0, 1]^{M_n}$ with $\sum_{k=1}^{M_n} w_k = 1$ we consider here and subsequently the aggregation $\hat{f}_w = \sum_{k=1}^{M_n} w_k \hat{f}_k$. Note that the aggregation weights define a discrete probability measure $P_w$ on the set $[1, M_n]$ by $P_w(\{k\}) = w_k$. Clearly, the random coefficients $(a_j(\hat{f}_w))_{j \in \mathbb{N}_0}$ of $\hat{f}_w$ satisfy $a_j(\hat{f}_w) = 0$ for $j \geq M_n$ and for any $j \in [0, M_n]$ holds $a_j(\hat{f}_w) = (\sum_{k=j+1}^{M_n} w_k) \times \hat{a}_j(f) = \mathbb{E}_w(\|j, M_n\|) \times \hat{a}_j(f)$. Our aim is to prove an upper bound for its risk $\mathbb{E}_w^a(\|\hat{f}_w - f\|_A^2)$ and its maximal risk

$$\sup \{\mathbb{E}_w^a(\|\hat{f}_w - f\|_A^2) : f \in D_A^{(n,m)}\}.$$ 

For arbitrary aggregation weights and penalty sequence the next lemma establishes an upper bound for the loss of the aggregated estimator. Selecting suitably the weights and penalties this bound provides in the sequel our key argument.

**Lemma 3.1.** Consider an aggregation $\hat{f}_w = \sum_{k=1}^{M_n} w_k \hat{f}_k$ and sequences $(\text{pen}_k^\gamma)_{k \in [1, M_n]}$ and $(\text{pen}_k)_{k \in [1, M_n]}$ of non-negative penalty terms. For any $k_- \in [1, M_n]$ and $k_+ \in [1, M_n]$ holds

$$\|\hat{f}_w - f\|_A^2 \leq \frac{2}{14} \text{pen}_k^\gamma + 2\|f\|_A^2 b_{k_-}(f) + 2\|f\|_A^2 \mathbb{P}_w(\|k|\=1, k\leq|k\= 2\text{pen}_k^\gamma / 14) + \frac{2}{14} \sum_{k=1+k_+}^{M_n} \text{pen}_k^\gamma w_k \mathbb{1}_{\|f - f_k\|_A < \text{pen}_k^\gamma / 7} + \frac{2}{14} \sum_{k=1+k_+}^{M_n} \text{pen}_k^\gamma w_k \mathbb{1}_{\|f - f_k\|_A > \text{pen}_k^\gamma / 7}$$

(3.1)

**Remark 3.2.** Keeping Lemma 3.1 in mind let us outline briefly the principal arguments of our aggregation strategy. Selecting the values of $k_+$ and $k_-$ close to the oracle dimension $k_+(f)$ the first two terms in the upper bound of (3.1) are of the order of the oracle rate. The weights are on the other hand selected such that the third and fourth term on the right hand side in (3.1) are negligible with respect to the oracle rate, while the choice of the penalties allows as usual to bound the deviation of the last two terms by concentration inequalities. □
Risk bounds. We derive bounds for the risk of the aggregated estimator \( \hat{f}_w \) using either Bayesian weights \( w := \hat{\omega} \) as in (1.3) or model selection weights \( w := \hat{\omega} \) as in (1.2). Until now we have not yet specified the sequences of penalty terms. Keeping in mind that the oracle dimension \( k_n^*(f) \) belongs to \([1, n^2]\) we set \( M_n := n^2(600 \log n)^{-4} \) and \( M_n := n^2 \) in case [L] and [H], respectively. We estimate \( \nu_f \) defined in (2.2) by its empirical counterpart \( \hat{\nu}_f := 1 + \frac{1}{n^2} \sum_{i=1}^{n} |X_i|^a \) with \( a = -1/2 \) in case [L] and \( a = 2/3 \) in case [H]. For each \( k \in \mathbb{N} \) and a numerical constant \( \Delta > 0 \) we set

\[
\text{pen}_k := \Delta \nu_f k^{1/2} n^{-1} \quad \text{and} \quad \text{pen}_k^\ast := \Delta \hat{\nu}_f k^{1/2} n^{-1}.
\]

(3.2)

Our theory necessitates a lower bound for the numerical constant \( \Delta \) which is for practical application in general too large. In the simulations we use preliminary experiments to determine a good choice for \( \Delta \) (c.f. Baudry et al. [2012]).

**Theorem 3.3.** Consider an aggregation \( \hat{f}_w \) using either Bayesian weights \( w := \hat{\omega} \) as in (1.3) or model selection weights \( w := \hat{\omega} \) as in (1.2) with \( \Delta \geq 84 \mathcal{C} \) and \( \mathcal{E} \) as in (2.3). Suppose the density \( f \in \mathcal{L}^2 \) satisfies \( \|f\|_\infty < \infty \) and \( \mathbb{E}_f(|X_1|^{2a}) < \infty \).

(p) Assume there is \( K \in \mathbb{N} \) with \( 1 \geq b_{K-1}(f) > 0 \) and \( b_K(f) = 0 \). Then there is a finite constant \( \mathcal{C}_f \) given in (C.14) depending only on \( f \) such that for all \( n \geq 3 \) holds

\[
\mathbb{E}_f^n \|f - \hat{f}_w\|_A^2 \leq \mathcal{C}_f n^{-1}.
\]

(3.3)

(np) If \( b_k(f) > 0 \) for all \( k \in \mathbb{N} \), then there is a numerical constant \( \mathcal{C} \) such that for all \( n \geq 3 \)

\[
\mathbb{E}_f^n \|f - \hat{f}_w\|_A^2 \leq \mathcal{C} \left( (\|f\|_A^2 + \nu_f) \rho_n^\circ(f) + (\|f\|_A^3 + 1) \nu_f + \|f\|_A^2 \mathbb{E}_f(|X_1|^{2a}) n^{-1} \right)
\]

with \( \rho_n^\circ(f) := \min_{k \in [1, M_n]} \left\{ \mathcal{R}_n^\circ(f) \lor \exp \left( -\frac{\mathcal{E}_f}{\mathcal{E}_f^\circ} k^{1/2} \right) \right\} \). (3.4)

Before we proof the main result. Let us state an immediate consequence.

**Corollary 3.4.** Let the assumptions of Theorem 3.3 be satisfied. If in case (np) in addition (A1) there is \( n_f \in \mathbb{N} \) such that \( k_n^*(f) \) and \( \mathcal{R}_n^\circ(f) \) as in (2.4) satisfy \( M_n^{3/2} \geq (k_n^*(f))^{1/2} \geq \frac{1}{\mathcal{E}_f^\circ} \log |\mathcal{R}_n^\circ(f)| \) for all \( n \geq n_f \), then there is a constant \( \mathcal{C}_f \) depending only on \( f \) such that \( \mathbb{E}_f^n \|f - \hat{f}_w\|_A^2 \leq \mathcal{C}_f \mathcal{R}_n^\circ(f) \) for all \( n \in \mathbb{N} \) holds true.

**Proof of Corollary 3.4.** If the additional assumption (A1) is satisfied, then the oracle dimension \( k_n^*(f) \) satisfies trivially \( k_n^*(f) \in [1, M_n] \) and \( \exp \left( -\frac{\mathcal{E}_f}{\mathcal{E}_f^\circ} k_n^*(f)^{1/2} \right) \leq \mathcal{R}_n^\circ(f) \) while for \( a \in [1, n_f] \) we have \( \exp \left( -\frac{\mathcal{E}_f}{\mathcal{E}_f^\circ} k_n^*(f)^{1/2} \right) \leq 1 \leq n \mathcal{R}_n^\circ(f) \leq a \mathcal{R}_n^\circ(f) \). Thereby, from (3.4) with \( \mathcal{R}_n^\circ(f) := \min_{k \in [1, M_n]} \mathcal{R}_n^k(f) \) follows the claim, which completes the proof. \( \Box \)
Remark 3.5. Let us briefly comment on the last results. In case (p) the data-driven aggregation leads to an estimator attaining the parametric oracle rate (see Remark 2.2). On the other hand in case (np) the data-driven aggregation leads to an estimator attaining the oracle rate \( R^c_n(f) \) (see Remark 2.2), if the additional assumption (A1) is satisfied. Otherwise, the upper bound \( \rho^c_n(f) \) in (3.4) faces a deterioration compared to the rate \( R^c_n(f) \). Considering again the particular specification \( b_k^2(f) \sim k^{-s} \), \( s > 0 \), we have seen that \( k^c_n(f) \sim n^{2/(2s+1)} \) and \( R^c_n(f) \sim n^{-2s/(2s+1)} \) is the oracle dimension and rate, respectively. Obviously, in this situation the additional assumption (A1) is satisfied for any \( s > 0 \). Thereby, the data-driven aggregated estimator attains the oracle rate for all \( s > 0 \) and thus it is adaptive. However, if \( b_k^2(f) \sim \exp(-k^{2s}) \), \( s > 0 \), then \( k^c_n(f) \sim (\log n)^{1/s} \) and \( R^c_n(f) \sim (\log n)^{1/(2s)n^{-1}} \) is the oracle dimension and rate, respectively. In this situation the additional assumption (A1) is satisfied only for \( s \in (0, 1/2] \). Hence, for \( s \in (0, 1/2] \) the data-driven aggregated estimator attains the oracle rate. In case \( s > 1/2 \), however, with \( (k^c_n)^{1/2} := \frac{1}{\exp(\log R^c_n(f))} \sim (\log n) \) the upper bound in (3.4) satisfies \( \rho^c_n(f) \leq R^c_n(f) \sim (\log n)n^{-1} \). Thereby, the rate \( \rho^c_n(f) \) of the data-driven estimator \( \hat{f}_w \) features a deterioration at most by a logarithmic factor \( (\log n)^{(1-1/(2s))} \) compared to the oracle rate \( R^c_n(f) \), i.e. \( (\log n)n^{-1} \) versus \( (\log n)^{1/(2s)n^{-1}} \).

**Proof of Theorem 3.3.** We outline here the main steps of the proof, while more technical details are deferred to the appendix C. Given penalties as in (3.2) for \( k \in \mathbb{N} \) holds by construction

\[
R^c_k(f) \geq b_k^2(f) \quad \text{and} \quad \Delta[\hat{v}_f \vee v_f] R^c_k(f) \geq [\text{pen}^*_k \vee \text{pen}^*_k] \quad \text{for all} \, k \in \mathbb{N}.
\]  

(3.5)

For arbitrary \( k^c_+, k^c_- \in [1, M_n] \) (to be chosen suitable below) let us define

\[
k_- := \min \left\{ k \in [1, k^c_-] : \|f\|_A^2 b_k^2(f) \leq \|f\|_A^2 b_{k^c_-}^2(f) + 6 \text{pen}^*_k \right\} \quad \text{and} \quad k_+ := \max \left\{ k \in [k^c_+, M_n] : \text{pen}^*_k \leq \|f\|_A^2 b_{k^c_+}^2(f) + 4 \text{pen}^*_k \right\}
\]  

(3.6)

where the defining set obviously contains \( k^c_- \) and \( k^c_+ \), respectively, and hence, it is not empty. Note that only \( k_+ \) depends on the observations, and hence is random. Consider further the event \( \Omega_f := \{ |\hat{v}_f - v_f| \leq v_f/2 \} \) and its complement \( \Omega^c_f \), where by construction for all \( k \in \mathbb{N} \) holds \( \frac{1}{2} \text{pen}^*_k \mathbb{1}_{\Omega_f} \leq \text{pen}^*_k \mathbb{1}_{\Omega^c_f} \leq \frac{3}{2} \text{pen}^*_k \mathbb{1}_{\Omega_f} \mathbb{1}_{\Omega^c_f} \). Exploiting the last bounds and (3.6) it follows

\[
\text{pen}^*_{k^c_+} \mathbb{1}_{\Omega_f} \leq 2 \text{pen}^*_{k^c_-} \mathbb{1}_{\Omega_f} \leq 2(6\|f\|_A^2 b_{k^c_-}^2(f) + 4 \text{pen}^*_k) \mathbb{1}_{\Omega_f} \leq 2(6\|f\|_A^2 b_{k^c_-}^2(f) + 4 \frac{3}{2} \text{pen}^*_k) = 12(\|f\|_A^2 b_{k^c_-}^2(f) + \text{pen}^*_k)
\]

and with \( \text{pen}^*_k \leq \text{pen}^*_{M_n} \leq \Delta v_f \) for all \( k \in [1, M_n] \) also

\[
\sum_{k=1+k^c_-}^{M_n} w_k \text{pen}^*_k \mathbb{1}_{\{\|\hat{f}_k - f_k\|_A > \text{pen}^*_k / 7\}} \leq \Delta v_f \mathbb{1}_{\Omega^c_f} + \sum_{k=1}^{M_n} \text{pen}^*_k \mathbb{1}_{\{\|\hat{f}_k - f_k\|_A > \text{pen}^*_k / 14\}}.
\]

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Combining the last two bounds and Lemma 3.1 we obtain
\[
\|\hat{f}_w - f\|^2_A \leq \frac{2}{7} \left( \Delta \nu_r \mathbb{1}_{\Omega_f} + 12 \|f\|_A^2 b_{k_+}^2 (f) + 12 \text{pen}_R^* (f) \right) + \frac{2}{7} \text{pen}_R^* (\|f\|_A^2 b_{k_+}^2 (f)) + 2 \|f\|^2_A b_{k_-}^2 (f)
\]
\[
+ 2 \sum_{k=1}^{M_n} (\|\hat{f}_k - f_k\|^2_A - \text{pen}_R^* / 14) + \frac{2}{92} \left( \Delta \nu_r \mathbb{1}_{\Omega_f} + \sum_{k=1}^{M_n} \text{pen}_R^* \mathbb{1}_{\{\|f_k - f\|_A^2 \geq \text{pen}_R^* / 14\}} \right)
\]
\[
+ \frac{2}{92} \sum_{k=1}^{M_n} \text{pen}_R^* w_k \mathbb{1}_{\{\|f_k - f\|_A^2 \geq \text{pen}_R^* / 14\}}. \quad (3.7)
\]

We bound the last two terms on the right hand side considering Bayesian weights \(w := \hat{w}\) as in (1.3) and model selection weights \(w := \hat{w}\) as in (1.2) in Lemma C.2 and Lemma C.3, respectively. Combining those upper bounds and (3.7) we obtain
\[
\|\hat{f}_w - f\|^2_A \leq \frac{12}{7} \|f\|_A^2 b_{k_+}^2 (f) + \frac{12}{7} \text{pen}_R^* + 2 \|f\|_A^2 b_{k_-}^2 (f)
\]
\[
+ \frac{8 \|f\|_A^2 \mathbb{1}_{\{k_+ > 1\}} \exp \left( - \frac{3 \Delta \nu_r k_+ / 4}{28} (k_+ / 2) \right) \right) + n^{-1} \text{pen}_R^* + \frac{192 \|f\|_A^2 b_{k_-}^2 (f)}{7 \Delta \nu_r + 2 \|f\|_A^2 \mathbb{1}_{\Omega_f} + 2 \|f\|_A^2 \mathbb{1}_{\{k_+ > 1\}} \}
\]
\[
+ \left( \frac{7 \Delta \nu_r + 2 \|f\|_A^2 \mathbb{1}_{\Omega_f} + 2 \|f\|_A^2 \mathbb{1}_{\{k_+ > 1\}} \right) \{\|f_k - f\|_A^2 \geq \text{pen}_R^* / 14\}. \quad (3.8)
\]

The deviations of the last four terms on the right hand side in (3.8) we bound in Corollary A.5 by exploiting usual concentration inequalities. Thereby, with \(\Delta \geq 84 \mathcal{C} \geq 1\), \(\mathcal{C}\) as in (2.3), \(\frac{3 \Delta \nu_r}{28} \geq 1\) and \(\kappa \Delta \nu_r \geq 1\) combining (3.8) and Corollary A.5 there is a finite numerical constant \(C > 0\) such that for any sample size \(n \in \mathbb{N}, n \geq 3\), any dimension parameter \(k_+, k_- \in [1, M_n]\) and associated \(k_- \in [1, M_n] \) as defined in (3.6) hold
\[
\mathbb{E}_f |\hat{f}_w - f|^2_A \leq \frac{12}{7} \|f\|_A^2 b_{k_+}^2 (f) + \frac{12}{7} \text{pen}_R^* + 2 \|f\|_A^2 b_{k_-}^2 (f)
\]
\[
+ C \|f\|_A^2 \mathbb{1}_{\{k_+ > 1\}} \exp \left( - \frac{e_{\nu_r}}{1+4000 \|f\|^2_\infty} (k_+ / 2) \right) \right) + C \{\|f\|_A^2 \mathbb{1}_{\Omega_f} \}
\]
\[
+ C \left( \|f\|_A^2 \mathbb{1}_{\{k_+ > 1\}} \right) \nu_r n^{-1} + C \{\|f\|_A^2 \mathbb{1}_{\Omega_f} \} \mathbb{E}_r (|X_1|^{2\nu_r}) n^{-1} \quad (3.9)
\]

We distinguish now the two cases (p) and (np) given in (3.3). The tedious case-by-case analysis for (p) we defer to Lemma C.4 in the appendix. Here we consider the case (np) only. However, in both cases the proof is based on an evaluation of the upper bound (3.9) for a suitable selection of the parameters \(k_+, k_- \in [1, M_n]\). Recall the definition of the oracle dimension and rate \(k_+ := k_+(f) \in [1, n^2]\) and \(R_n^k := R_n^k (f)\), respectively. We select \(k_+ := \arg \min \{R_n^k, k \in [1, M_n]\}\). Further the inequalities (3.5) and the definition (3.6) of \(k_- \) implies \|f\|_A^2 b_{k_-}^2 (f) \leq ||f\|_A^2 + 6 \Delta \nu_r \mathcal{R}_n^{k_-}.\) Keeping the last bound together with \(\mathcal{R}_n^{k_+} \geq \mathcal{R}_n^{k_-} \geq n^{-1}\), which holds for all \(k_+ \in [1, M_n]\), in mind, we evaluate the upper bound (3.9) and obtain the assertion (3.4), which completes the proof of Theorem 3.3. □
Maximal risk bounds. The following assertion shows that the aggregation \( \hat{f}_w \) using either Bayesian weights \( w := \hat{w} \) as in (1.3) or model selection weights \( w := \hat{w} \) as in (1.2) attains the minimax optimal rate over Sobolev-ellipsoids \( \mathbb{D}_s^{s, L, 2} \) as in (2.7).

**Theorem 3.6.** Consider an aggregation \( \hat{f}_w \) using either Bayesian weights \( w := \hat{w} \) as in (1.3) or model selection weights \( w := \hat{w} \) as in (1.2) with \( \Delta \geq 84C \) and \( C \) as in (2.3). For each \( s, L \in \mathbb{R}^+ \) with \( s > 1 \) there is a finite constant \( C_{s, L} \) depending only on the class \( \mathbb{D}_s^{s, L, 2} \) such that for all \( n \geq 3 \) holds

\[
\sup \left\{ E_n^0 \left\| \hat{f}_w - f \right\|^2_2 : f \in \mathbb{D}_s^{s, L, 2} \right\} \leq C_{s, L} n^{-2s/(2s+1)}.
\]

The proof of Theorem 3.6 follows along the lines of the proof of case (np) in Theorem 3.3 where we did not specify the asymptotic behaviour of the sequence \( (b_k(f))_{k \in \mathbb{N}} \). Therefore, rather imposing a specific polynomial decay as implied by a Sobolev ellipsoid we characterise it by a strictly positive sequence \( f = (f_k)_{k \in \mathbb{N}} \). Precisely, let

\[
\mathbb{W}_k^A := \{ f \in L^2_A : |f|^2 = \sum_{k \in \mathbb{N}} (a_k(f))^2/k < \infty \} \quad \text{and} \quad \mathbb{W}_k^{L, 2} := \{ f \in L^2_A : |f|^2 \leq L \}.
\]

Obviously, the Sobolev ellipsoid \( \mathbb{W}_s^{s, L} \) corresponds to the special case \( f = (k^{-s})_{k \in \mathbb{N}} \). Keeping (2.7) in mind we denote further a corresponding subset of densities with finite \( \alpha m \)-th moment, \( m \in \mathbb{R}^+ \), by \( \mathbb{D}_s^{s, L, m} := \{ f \in \mathbb{W}_s^{s, L} : f \text{ is a density and } \mathbb{F}_f(|X|^m) \leq L \} \). For \( n, k \in \mathbb{N} \) we set

\[
\mathcal{R}_n^k(f) := [f_k \vee n^{-1/2}], \quad k^*_n(f) := \arg \min \left\{ \mathcal{R}_n^k(f), k \in \mathbb{N} \right\} \quad \text{and} \quad \mathcal{R}_n^*(f) := \min \left\{ \mathcal{R}_n^k(f), k \in \mathbb{N} \right\}.
\]

Here and subsequently, we impose the following minimal regularity conditions.

**Assumption A2.** The sequence \( f = (f_k)_{k \in \mathbb{N}} \) is strictly positive, monotonically non-increasing with \( f_1 \leq 1 \), \( \lim_{k \to \infty} f_k = 0 \) and there is \( C_{f, L} \in \mathbb{R}^+ \) such that \( 1 \leq L \vee v_{\ell_1} \vee \|f\|_A^2 \vee \|f\|_{\infty}^2 \leq C_{f, L} \) for all \( f \in \mathbb{D}_s^{1, L, 2} \).

**Remark 3.7.** We shall emphasise that for any \( f \in \mathbb{D}_s^{1, L, m} \) hold \( v_{\ell_1} \leq 1 + L \) and \( \|f\|_A^2 \leq L \) for all \( k \in \mathbb{N} \). Keeping further \( \sup_{j \in \mathbb{N}} \|v_j\|_{\infty} \leq \sqrt{2} \) in mind we have \( a_0(f) \leq \|v_0\|_{\infty} \leq \sqrt{2} \) and hence \( \|f\|_A^2 \leq 2 + L \). Moreover, if \( \|f\|_{\ell_1} := \sum_{k \in \mathbb{N}} f_k < \infty \), then uniformly for all \( f \in \mathbb{D}_s^{1, L, m} \) we have \( \|f\|_{\infty} \leq (2 + \sum_{k \in \mathbb{N}} (a_k(f))^2/k) \|v_0\|_{\infty} + \sum_{k \in \mathbb{N}} f_k \|v_k\|_{\infty} \leq (2 + L)2(1 + \|f\|_{\ell_1}) \) by applying the Cauchy-Schwarz inequality. Consequently, if \( \|f\|_{\ell_1} < \infty \) then \( L \vee \|f\|_{\ell_1} \leq \|f\|_A^2 \leq (2 + L)2(1 + \|f\|_{\ell_1}) =: C_{f, L} \) for all \( f \in \mathbb{D}_s^{1, L, m} \). In particular, the Sobolev-ellipsoid \( \mathbb{D}_s^{s, L, 2} \) satisfies Assumption A2 for all \( s > 1 \). Note that, under Assumption A2 hold \( \mathcal{R}_n^*(f) = \mathcal{R}_n^{k^*_n(f)}(f) \) and \( k^*_n(f) \in [1, n^2] \). Moreover, we have \( \mathcal{R}_n^*(f) \geq n^{-1}, \mathcal{R}_n^*(f) = o(1) \) and \( n\mathcal{R}_n^*(f) \to \infty \) as \( n \to \infty \). In this situation the rate \( \mathcal{R}_n^*(f) \) is non-parametric and for any \( f \in \mathbb{D}_s^{1, L, m} \) holds by construction \( (2 + L)\mathcal{R}_n^*(f) \geq \|f\|_A^2 \mathcal{R}_n^*(f) \) for all \( n \in \mathbb{N} \).

Exploiting again the identity (2.1), the upper bound (2.3) and the definition (3.10) under Assumption A2 there is a numerical constant such that for all \( n \in \mathbb{N} \)

\[
\sup \left\{ E_n^0 \left\| \hat{f}_{k^*_n(f)} - f \right\|^2_2 : f \in \mathbb{D}_s^{1, L, 2} \right\} \leq C C_{f, L} \mathcal{R}_n^*(f), \quad (3.11)
\]
By applying Lemma 3.1 we derive next bounds for the maximal risk over ellipsoids $\mathbb{D}_A^{1,L,2}$ of the aggregation $\hat{f}_w$ using either Bayesian weights $w := \tilde{w}$ as in (1.3) or model selection weights $w := \hat{w}$ as in (1.2) based on the penalties $(\text{pen}_k^x)_{k \in \mathbb{M}_n}$ given in (3.2).

**Theorem 3.8.** Consider an aggregation $\hat{f}_w$ using either Bayesian weights $w := \tilde{w}$ as in (1.3) or model selection weights $w := \hat{w}$ as in (1.2). Under Assumption A2 there is a numerical constant $C$ such that for all $n \geq 3$ holds

$$\sup \left\{ \mathbb{E}_n^j \| \hat{f}_w - f \|_A^2 : f \in \mathbb{D}_A^{1,L,2} \right\} \leq C \left( \mathcal{C}_{f,L} \rho_n^e(f) + \mathcal{C}_{f,L}^3 n^{-1} \right)$$

with $\rho_n^e(f) := \min_{k \in [1,M_n]} \left\{ [\mathcal{R}_n^k(f) \vee \exp \left( - \frac{\| f \|_{\mathbb{D}_A^{1,L,2}}^2}{400\exp(\mathcal{C}_{f,L})} \right) ] \right\}$. \hspace{1cm} (3.12)

Before we proof the main result let us state an immediate consequence.

**Corollary 3.9.** Let the assumptions of Theorem 3.8 be satisfied. If in addition (A1') there is $n_{f,L} \in \mathbb{N}$ such that $k_n^s(f)$ and $\mathcal{R}_n^s(f)$ as in (3.10) satisfy $M_n^{1/2} \geq (k_n^s)^{1/2} \geq \frac{400\mathcal{C}_{f,L}}{\varepsilon} | \log \mathcal{R}_n^s(f) |$ for all $n \geq n_f$, then there is a numerical constant $C > 0$ such that for all $n \in \mathbb{N}$ holds

$$\sup \left\{ \mathbb{E}_n^j \| \hat{f}_w - f \|_A^2 : f \in \mathbb{D}_A^{1,L,2} \right\} \leq C \left( n_{f,L} \vee \mathcal{C}_{f,L} \right) \mathcal{R}_n^s(f) + \mathcal{C}_{f,L}^3 n^{-1}.$$  

**Proof of Corollary 3.9.** follows in analogy to Corollary 3.4 and we omit the details. \hspace{1cm} $\Box$

**Remark 3.10.** Let us briefly comment on the last results. The data-driven aggregation leads to an estimator attaining the rate $\mathcal{R}_n^s(f)$ due to Corollary 3.9, if the additional assumption (A1') is satisfied. Otherwise, the upper bound $\rho_n^e(f)$ in (3.12) faces a deterioration compared to the rate $\mathcal{R}_n^s(f)$. Considering the Sobolev ellipsoid $\mathbb{D}_A^{1,L,2}$, i.e., $f = (k_n^s)_{k \in \mathbb{N}}$, $s \in \mathbb{R}^+$, where $k_n^s(f) \sim n^{2/(2s+1)}$ and $\mathcal{R}_n^s(f) \sim n^{-2s/(2s+1)}$, Assumption A2 and (A1') are satisfied for each $s > 1$. Consequently, Theorem 3.6 follows immediately from Corollary 3.9. On the other hand if $f = (\exp(-k_n^s))_{k \in \mathbb{N}}$, $s \in \mathbb{R}^+$, then $k_n^s(f) \sim (\log n)^{1/s}$ and $\mathcal{R}_n^s(f) \sim (\log n)^{1/2s}n^{-1}$. In this situation the additional assumption (A1') is satisfied only for $s \in (0,1/2]$. Hence, for $s \in (0,1/2]$ the data-driven aggregation attains the rate $\mathcal{R}_n^s(f)$. In case $s > 1/2$, however, with $(k_n^s)^{1/2} := \frac{400\mathcal{C}_{f,L}}{\varepsilon} | \log \mathcal{R}_n^s(f) | \sim (\log n)$ we have $\rho_n^e(f) \leq \mathcal{R}_n^s(f) \sim (\log n)n^{-1}$. Thereby, the rate $\rho_n^e(f)$ of the aggregation $\hat{f}_w$ features a deterioration at most by a logarithmic factor $(\log n)^{(1-1/(2s))}$ compared to the rate $\mathcal{R}_n^s(f)$, i.e., $(\log n)n^{-1}$ versus $(\log n)^{1/(2s)}n^{-1}$. \hspace{1cm} $\Box$

**Proof of Theorem 3.8.** We make use of the upper bound (3.8) derived in the proof of Theorem 3.3. We note that uniformly for all $f \in \mathbb{D}_A^{1,L,2}$ under Assumption A2 the definition (3.6) of $k_+$ and $k_-$ implies $\| f \|_A^2 b_{\mathcal{C}_{f,L}}^2(f) + \text{pen}_{\mathbb{M}_n}^x \leq (1 + \Delta)\mathcal{C}_{f,L}\mathcal{R}_n^{k_+}(f)$ and $\| f \|_A^2 b_{\mathcal{C}_{f,L}}^2(f) \leq (1 + 6\Delta)\mathcal{C}_{f,L}\mathcal{R}_n^{k_-}(f)$. Combining (3.8), the last bounds, $\| f \|_A^2 \vee \| f \|_A^2 \vee \frac{\| f \|_A^2}{n^2\Delta_2^2} \vee \frac{192\varepsilon}{n^2\Delta_2^2} \leq \mathcal{C}_{f,L}$ and
In Corollary A.6 in appendix A we bound the last four terms uniformly for all \( f \in \mathbb{D}^{L,2}_A \). Therewith, there exists a finite numerical constant \( C > 0 \) such that for all \( n \in \mathbb{N} \)

\[
\sup \{ \mathbb{E}_n \| \hat{w}_n - f \|_A^2 : f \in \mathbb{D}^{L,2}_A \} \leq C \mathcal{C}_{f,L} (\mathcal{R}^{k_n^*}_n (f) + \mathcal{R}^{k_0}_n (f) + \exp \left( -\frac{6}{400 \mathcal{C}_{f,L} (k_0^*)^{1/2}} \right)) + C \mathcal{C}_{f,L}^3 n^{-1}. \tag{3.13}
\]

For \( k_n^* := k_n^*(f) \in [1,n^2] \) and \( \mathcal{R}^{k}_n (f) \) as in (3.10) set \( k_0^* := \arg\min \{ \mathcal{R}^{k}_n (f), k \in [1,M_n] \} \), then for any \( k_0^* \in [1,M_n] \) we have \( \mathcal{R}^{k_0}_n (f) \geq \mathcal{R}^{k_0^*}_n (f) \geq \mathcal{R}^{k_0^*}_n (f) = \min \{ \mathcal{R}^{k}_n (f), k \in \mathbb{N} \} \). The last bounds together with (3.13) imply (3.12), which completes the proof of Theorem 3.8. □

## 4 Numerical study

Let us illustrate the performance of the aggregated estimator \( \hat{w}_n \) using Bayesian weights \( w = \tilde{w}_n \) (see (1.3)) or model selection weights \( w = \hat{w}_n \) (see (1.2)) in both cases [H] and [L]. For the case of [L] we consider the densities

(i) **Gamma Mixture**: \( f(x) = 0.4 \cdot 3.2 x \exp(-3.2 x) + 0.6 \cdot 6.816 \exp(-6.8 x), \)

(ii) **Gamma Distribution**: \( f(x) = \frac{\alpha^\alpha}{\Gamma(\alpha)} \exp(-\alpha x), \)

(iii) **Beta Distribution**: \( f(x) = \frac{1}{560} (0.5x)^3 (1-0.5x)^4 \mathbb{1}_{[0,1]}(0.5x) \)

(iv) **Weibull Distribution**: \( f(x) = 0.75 x^{-0.25} \exp(-x^{0.75}), \)

In case of [H] we investigate the densities

(i) **Gaussian Mixture**: \( f(x) = 0.6 \cdot 2.5 \exp(-x^2/0.8) + 0.4 \cdot 2.5 \exp(-(x-3)^2/0.8) , \)

(ii) **Finite representation**: \( f(x) = \frac{1}{\sqrt{2\pi}} x^2 \exp(-x^2/2), \)

(iii) **Beta Distribution**: \( f(x) = \frac{1}{560} (0.5x)^3 (1-0.5x)^4 \mathbb{1}_{[0,1]}(0.5x) \)

(iv) **Pareto Distribution**: \( f(x) = \frac{0.75}{x^{7.5}} \mathbb{1}_{[1,\infty]}(x). \)

We consider these four cases for the following reasons. The bias of both densities in (i) has an exponential decay as shown by Belomestny et al. [2019] and Comte and Genon-Catalot.
The densities in (ii) have a finite representation in the Laguerre respectively the Hermite basis. The case (iii) and (iv) illustrates the behaviour of the estimators when firstly the density has a compact support and secondly \( \hat{\varphi}_f \) does not have a finite second moment. By minimising an integrated squared error over a family of histogram densities with randomly drawn partitions and weights we select \( \Delta = 1.02 \) and \( \Delta = 0.95 \) in case [L] and [H], respectively. Furthermore, we chose \( \kappa = 9.8 \) and \( \kappa = 5.2 \) in case [L] and [H], respectively.

Figure 1: Considering case [L] (iii) the estimators are depict for 50 Monte-Carlo simulations using model selection weights (top) and Bayesian weights (bottom) with varying sample size \( n = 200 \) (left), \( n = 1000 \) (middle) and \( n = 2000 \) (right). The true density \( f \) is given by the red curve while the dark blue curve is the point-wise empirical median of the 50 estimates.
Figure 2: Considering Bayesian weights and a sample size \( n = 1000 \) the aggregated estimators are depict for 50 Monte-Carlo simulations using the Laguerre (top) and Hermite (bottom) basis in the cases (i) (left), (ii) (middle) and (iii) (right). The true density \( f \) is given by the red curve while the dark blue curve is the point-wise empirical median of the 50 estimates.

Figure 3: Considering Bayesian weights in case (iv) the aggregated estimators are depict for 50 Monte-Carlo simulations using the Laguerre (top) and Hermite (bottom) basis with varying sample size \( n = 200 \) (left), \( n = 1000 \) (middle) and \( n = 2000 \) (right). The true density \( f \) is given by the red curve while the dark blue curve is the point-wise empirical median of the 50 estimates.
Model selection
Bayesian weights
optimal OSE

| n = | Model selection | Bayesian weights | optimal OSE |
|-----|-----------------|------------------|-------------|
|     | 200             | 1000             | 200         | 1000 | 200 | 1000 | 200 | 1000 | 200 |
| [L] | (i) 0.899       | 0.408            | 0.282       | 0.529 | 0.226 | 0.160 | 0.456 | 0.089 | 0.055 |
|     | (ii) 0.540      | 0.265            | 0.168       | 0.289 | 0.139 | 0.088 | 0.096 | 0.022 | 0.009 |
|     | (iii) 0.755     | 0.323            | 0.233       | 0.374 | 0.180 | 0.131 | 0.265 | 0.075 | 0.040 |
|     | (iv) 1.129      | 0.359            | 0.271       | 0.536 | 0.232 | 0.180 | 0.315 | 0.086 | 0.048 |
| [H] | (i) 0.595       | 0.135            | 0.067       | 0.544 | 0.124 | 0.064 | 0.450 | 0.106 | 0.060 |
|     | (ii) 0.179      | 0.0039           | 0.018       | 0.174 | 0.038 | 0.017 | 0.148 | 0.031 | 0.015 |
|     | (iii) 0.454     | 0.121            | 0.061       | 0.415 | 0.114 | 0.054 | 0.308 | 0.096 | 0.043 |
|     | (iv) 6.411      | 3.316            | 2.520       | 6.357 | 3.232 | 2.471 | 2.968 | 1.552 | 1.230 |

Table 1: Over 50 Monte-Carlo simulations accumulated squared distances between the unknown density \( f \) and the aggregated estimator \( \hat{f}_w \) with model selection weights (left), \( \hat{f}_w \) with Bayesian weights (middle) and the OSE \( \hat{f}_{k_{\text{opt}}} \) (right) are presented where \( k_{\text{opt}} \) minimises in each iteration the squared distance \( \hat{f}_k \) and \( f \) over \([1, M_n]\).

Appendix

A Preliminaries

This section gathers preliminary technical results.

Let us firstly introduce the Laguerre and Hermite basis and secondly briefly argue that the inequalities in (2.3) are fulfilled by both basis. The Laguerre functions are defined by

\[
\ell_j(x) := \sqrt{2} L_j(2x) \exp(-x) 1_{\mathbb{R}^+}(x), \quad L_j(x) := \sum_{k=0}^j (-1)^k \binom{j}{k} \frac{x^k}{k!}, \quad x \in \mathbb{R}^+
\]

where \( L_j \) is the Laguerre polynomial of order \( j \in \mathbb{N}_0 \). As proven in Szegö [1918] the Laguerre polynomials are bounded by \( \exp(x/2) \) and therefore for all \( j \in \mathbb{N}_0 \) the function \( \ell_j \) is bounded by \( \sqrt{2} \), in equal \( \| \ell_j \|_{\infty} \leq \sqrt{2} \). The Hermite polynomial \( (H_j)_{j \in \mathbb{N}_0} \) and the Hermite function \( h_j \) of order \( j \in \mathbb{N}_0 \) is given by

\[
h_j(x) := \frac{1}{(2j)!(\sqrt{\pi})^{1/2}} H_j(x) \exp(-\frac{x^2}{2}), \quad H_j(x) := (-1)^j \exp(x^2) \frac{d^j}{dx^j}(\exp(-x^2)), \quad x \in \mathbb{R},
\]

where for each \( j \in \mathbb{N}_0 \) holds \( \| h_j \|_{\infty} \leq 1 \) (see Olver et al. [2010] p.450). Moreover, due to Szegő [1939] p. 242 there is \( C_\infty \in \mathbb{R}^+ \) such that \( \| h_j \|_{\infty} \leq C_\infty (j+1)^{-1/12} \) for all \( j \in \mathbb{N}_0 \) which implies the first part of (2.3). For the second part of (2.3) we refer to Comte and Genon-Catalot [2018] in case of the Hermite functions while for the Laguerre functions we slightly
alternate their proof. Here we change the upper bound for the integral $I_1$ as follows

$$I_1 \leq \frac{c_2}{2^{p+1}} \int_{0}^{1/\nu} u^p(u^\delta) f(u/2) du \leq \frac{c_2}{\sqrt{2^\nu}} \int_{0}^{1/2\nu} u^{p-1/2} f(u) du \leq \frac{c_2}{\sqrt{2k}} \mathbb{E}_f[X^{p-1/2}].$$

Now following the steps as in Comte and Genon-Catalot [2018] there exists $C_{p,\delta} \in \mathbb{R}^+$ such that $\mathbb{E}_f[X^p \ell_k^{(\delta)}(X)^2] \leq C_{p,\delta} [\mathbb{E}_f[X^{p-1/2}] + \mathbb{E}_f[X^p])]k^{-1/2}$ which with $p = \delta = 0$ shows the first part of (2.3). In the sequel we stick to the unified notation of an orthonormal basis $(\varphi_j)_{j \in \mathbb{N}_0}$ in $L_2^\nu$ where for each $j \in \mathbb{N}_0$ in case [L] and [H] $\varphi_j := \ell_j$ and $\varphi_j := h_j$, respectively. For abbreviation, we denote by $\Pi_k$ and $\Pi_k^\perp$ the orthogonal projections on the linear subspace $S_k$ and its orthogonal complement $S_k^\perp$ in $L_2^\nu$, respectively. The next result can be found in Johannes et al. [2015].

**Lemma A.1.** Given $n \in \mathbb{N}$ and $f, \tilde{f} \in L_2^\nu$ consider the families of orthogonal projections \{ $f_k := \Pi_k f, k \in [1, n]$ \} and \{ $\tilde{f}_k := \Pi_k \tilde{f}, k \in [1, n]$ \}. For any $l \in [1, n]$ hold

(i) $\| f_k \|^2_A - \| f_l \|^2_A \leq \frac{1}{4} \| f_l \|^2_A - \frac{1}{2} \| f_l \|^2_A \{ b_k^2(f) - b_l^2(f) \}$, for all $k \in [1, l]$;

(ii) $\| f_k \|^2_A - \| f_l \|^2_A \leq \frac{1}{2} \| f_l \|^2_A + \frac{3}{2} \| f_l \|^2_A \{ b_k^2(f) - b_l^2(f) \}$, for all $k \in [l, n]$.

The next assertion provides our key arguments in order to control the deviations of the reminder terms. Both inequalities are due to Talagrand [1996], the formulation of the first part can be found for example in Klein and Rio [2005], while the second part is based on equation (5.13) in Corollary 2 in Birgé and Massart [1998] and stated in this form for example in Comte and Merlevede [2002].

**Lemma A.2.** (Talagrand’s inequalities) Let $X_1, \ldots, X_n$ be independent $\mathcal{X}$-valued random variables and let $\overline{\nu}_n = n^{-1} \sum_{i=1}^n [\nu_n(X_i) - \mathbb{E}(\nu_n(X_i))]$ for $\nu_n$ belonging to a countable class \{ $\nu_n, h \in \mathcal{H}$ \} of measurable functions. Then,

$$\mathbb{E}(\sup_{h \in \mathcal{H}} |\overline{\nu}_n|^2 - 6\Psi^2) \leq C \left\{ \frac{\tau}{n} \exp \left( \frac{-n\Psi^2}{6\tau} \right) + \frac{\psi^2}{n^2} \exp \left( \frac{-n\Psi}{100\psi} \right) \right\}; \quad (A.1)$$

$$\mathbb{P}(\sup_{h \in \mathcal{H}} |\overline{\nu}_n|^2 \geq 6\Psi^2) \leq 3 \left\{ \exp \left( \frac{-n\Psi^2}{400\tau} \right) + \exp \left( \frac{-n\Psi}{200\psi} \right) \right\} \quad (A.2)$$

for some numerical constant $C > 0$ and where

$$\sup_{h \in \mathcal{H}} \sup_{z \in \mathcal{Z}} |\nu_n(z)| \leq \psi, \quad \mathbb{E}(\sup_{h \in \mathcal{H}} |\overline{\nu}_n|) \leq \Psi, \quad \mathbb{E} \left( \frac{1}{n} \sum_{i=1}^n \text{Var}(\nu_n(X_i)) \right) \leq \tau. \quad (A.3)$$

**Remark A.3.** For $k \in \mathbb{N}$ define the unit ball $\mathbb{B}_k := \{ h \in S_k : \| h \|_A \leq 1 \}$ contained in the linear subspace $S_k$ spanned by \{ $\varphi_j, j \in [0, k]$ \}. For $h \in \mathbb{B}_k$ we define $\nu_h := \Pi_k h$, where $\mathbb{E}_f(\nu_h(X_1)) = \sum_{j=0}^{k-1} a_j(h) \ell_j(f)$ and $\overline{\nu}_h = \sum_{j=0}^{k-1} (\hat{a}_j(f) - a_j(f)) \ell_j(h)$, thus

$$\| \tilde{f}_k - f_k \|^2_A = \sup_{h \in \mathbb{B}_k} \langle \tilde{f}_k - f_k, h \rangle_A^2 = \sup_{h \in \mathbb{B}_k} \sum_{j=0}^{k-1} (\hat{a}_j(f) - a_j(f)) \ell_j(h)^2 = \sup_{h \in \mathbb{B}_k} |\overline{\nu}_h|^2.$$
The last identity provides the necessary argument to apply Talagrand’s inequality Lemma A.2 in the proof of Lemma A.4. Note that, the unit ball $\mathbb{B}_k$ is not a countable set of functions, however, it contains a countable dense subset, say $\mathcal{H}$, since $L^2_k$ is separable, and it is straightforward to see that $\sup_{h \in \mathbb{B}_k} |\tau_h|^2 = \sup_{h \in \mathcal{H}} |\tau_h|^2$.

**Lemma A.4.** Consider $\mathcal{C} \in \mathbb{R}^+$ and $\nu \geq 1$ as in (2.3). There exists a numerical constant $C$ such that for any density $f \in L^2_k$ with $\|f\|_\infty < \infty$, for all $n \geq 3$ and $k \in [1, M_n]$ hold

(i) $\mathbb{E}^n(\|\hat{f}_k - f_k\|^2_A - 6 \mathcal{C} \nu \|f\|_k^{1/2} n^{-1})_+ \leq C(\|f\|_n \exp \left(\frac{-\mathcal{C} \nu \|f\|_\infty}{6\|f\|_\infty} K^{1/2}\right) + \frac{1}{nM_n})$;

(ii) $\mathbb{E}^n(\|\hat{f}_k - f_k\|^2_A \geq 6 \mathcal{C} \nu f_k^{1/2} n^{-1}) \leq C(\exp \left(\frac{-\mathcal{C} \nu \|f\|_\infty}{400\|f\|_\infty} K^{1/2}\right) + \frac{1}{nM_n})$

where in case $[L] \ M_n := n^2(600 \log n)^{-4}$ and in case $[H] \ M_n := n^2$.

**Proof of Lemma A.4.** For $h \in \mathbb{B}_k$ setting $\nu_h = \Pi_h h = \sum_{j=0}^{k-1} a_j(h) \phi_j$ we observe (see Remark A.3) that $\sup_{h \in \mathbb{B}_k} \|\nu_h\|_\infty = \|\sum_{j=0}^{k-1} \phi_j\|_\infty \leq \mathcal{C} k^d$. Therefore, we compute next quantities $\psi$, $\Psi$, and $\tau$ verifying the three inequalities (A.3) required in Lemma A.2. First, making use of (2.3) we have $\sup_{h \in \mathbb{B}_k} \|\nu_h\|_\infty = \|\sum_{j=0}^{k-1} \phi_j\|_\infty \leq \mathcal{C} k^d$.

Next, find $\psi$. Notice that $\|\hat{f}_k - f_k\|^2_A = \sum_{j=0}^{k-1} |\hat{a}_j(f) - a_j(f)|^2 \leq \frac{1}{n} \mathbb{E}_f(\varphi_j(X))$ and by employing (2.3), we obtain $\mathbb{E}_f(\sup_{h \in \mathbb{B}_k} |\tau_h|^2) \leq \frac{1}{n} \sum_{j=0}^{k-1} \mathbb{E}_f(\varphi_j^2(X)) \leq \mathcal{C} \nu f_k^{1/2} n^{-1} =: \psi^2$.

Finally, we set $\tau := \|f\|_\infty$. Indeed, for each $h \in \mathbb{B}_k$ it holds $\frac{1}{n} \sum_{i=1}^{n} \var{\nu_h(X)} \leq \mathbb{E}_f(|\nu_h(X)|^2) \leq \|f\|_\infty \|\Pi_h h\|^2_A \leq \tau$. Replacing in (A.1) the quantities $\psi$, $\Psi$ and $\tau$ there is a numerical constant $C$ such that (keep in mind that $1 - 2d \leq 0$ and $\nu \geq 1$) it holds

$$\mathbb{E}(\|\hat{f}_k - f_k\|^2_A - 6 \mathcal{C} \nu \|f\|_k^{1/2} n^{-1})_+ \leq C \left(\|f\|_n \exp \left(\frac{-\mathcal{C} \nu \|f\|_\infty}{6\|f\|_\infty} K^{1/2}\right) + \frac{1}{nM_n} \exp \left(\frac{-100}{100} \frac{n^{1/2}}{M_n^{d/2-1/4}}\right)\right)$$ (A.4)

for all $k \in [1, M_n]$. In case $[L]$ with $d = 1$, $M_n = n^2(600 \log n)^{-4}$ and $n \geq 3$ we have

$$\frac{M_n^{d+1}}{n} \exp \left(\frac{-1}{100} \left(\frac{n^2}{M_n^{d/2-1/4}}\right)^{1/4}\right) \leq \exp \left((3 - \frac{600}{100})(\log n)\right) \leq 1,$$

and in case $[H]$ with $d = 10/12$ and $M_n = n^2$ we obtain

$$\frac{M_n^{d+1}}{n} \exp \left(\frac{-1}{100} \left(\frac{n^2}{M_n^{d/2-1/4}}\right)^{1/4}\right) \leq n^{8/3} \exp \left(-\frac{1}{100} n^{1/6}\right) \leq \left(\frac{1600}{e}\right)^6.$$

Consequently, combining the bounds for the cases $[L]$ and $[H]$ we obtain the assertion (i). Analogously, replacing in (A.2) the quantities $\psi$, $\Psi$ and $\tau$ we obtain (ii), and we omit the details, which completes the proof.

**Corollary A.5.** Under the assumptions of Lemma A.4 for $(\text{pen}^\psi_k)_{k \in \mathbb{N}}$ as in (3.2) with $\Delta \geq 84 \mathcal{C}$ there is a finite numerical constant $C > 0$ such that for all $n \geq 3$ and $k^2 \in [1, M_n]$ hold
(i) $\sum_{k=1}^{M_n} \mathbb{E}_n^f(\|\hat{j}_k - f_k\|_A^2 - \text{pen}_k^c / 14)_+ \leq C (\|f\|_\infty^3 \vee 1) n^{-1}$;
(ii) $\sum_{k=1}^{M_n} \text{pen}_k^c \mathcal{M}(\|\hat{j}_k - f_k\|_A^2 \geq \text{pen}_k^c / 14) \leq C \mathcal{V}_f (\|f\|_\infty^3 \vee 1) n^{-1}$;
(iii) $\mathcal{M}(\|\hat{j}_k^c - f_k^c\|_A^2 \geq \text{pen}_k^c / 14) \leq C (\exp(-\varepsilon_{\psi} / 400n f_\infty^2 (k^c_\infty)^{1/2}) + n^{-1})$.
Consider $\nu_j$, $\nu_j^c$ and $\Omega_j^c = \{\nu_j - \nu_j^c \leq \nu_j / 2\}$, then for all $n \in \mathbb{N}$ holds
(iv) $\mathcal{M}(\Omega_j^c) \leq 4 \frac{p}{\exp(\varepsilon_{\psi} (|X_1|^{2\alpha}))} n^{-1}$ with $p \in \mathbb{R}$ as in (2.2).

**Proof of Corollary A.5.** Consider (i). Exploiting the elementary bounds $\sum_{k \in \mathbb{N}} \exp\left(-\lambda k^{1/2}\right) \leq \lambda^{-2}$ for all $\lambda > 0$, $\text{pen}_k^c / 14 \geq 6 \mathcal{V}_f k^{1/2} n^{-1}$ for all $k \in [1, M_n]$ and $\nu_j \geq 1$ from Lemma A.4 (ii) together with (iii). We obtain (iv), and we omit the details. The assertion (ii) follows immediately from Lemma A.4 (ii). It remains to show (iv). Analogously, from Lemma A.4 (ii) together with $\sum_{k \in \mathbb{N}} k^{1/2} \exp\left(-\lambda k^{1/2}\right) \leq 3 \lambda^{-3}$ for all $\lambda > 0$ we obtain (ii), and we omit the details. The assertion (iii) follows immediately from Lemma A.4 (ii). It remains to show (iv). Recall that $\nu_j - \nu_j^c = \frac{1}{n} \sum_{i=1}^{n} (|X_i| - \mathbb{E}(|X_i|))$ with $a$ as in (2.2), where $Y_i := |X_i| - \mathbb{E}(|X_i|)$, $i \in [1, n]$, are independent, identically distributed and centred. Thereby, we have $\mathbb{E}_n^f(\nu_j - \nu_j^c)^2 = n^{-1} \mathbb{E}_f^r (|Y_1|)^2 \leq n^{-1} \mathbb{E}_f^r (|X_1|^{2\alpha})$. By using Tchebychev’s inequality we deduce the assertion (iv), which completes the proof. □

**Corollary A.6.** For $(\text{pen}_k^c)_{k \in \mathbb{N}}$ as in (3.2) with $\Delta \geq 84 \mathcal{C}$ there is under Assumption A2 a finite numerical constant $C > 0$ such that for all $n \geq 3$ and $k^c_\infty \in [1, M_n]$ hold
(i) $\sup_{f \in \mathbb{D}_A^{L,m}} \sum_{k=1}^{M_n} \mathbb{E}_n^f(\|\hat{j}_k - f_k\|_A^2 - \text{pen}_k^c / 14)_+ \leq C c_{i,L}^2 n^{-1}$,
(ii) $\sup_{f \in \mathbb{D}_A^{L,m}} \sum_{k=1}^{M_n} \text{pen}_k^c \mathcal{M}(\|\hat{j}_k - f_k\|_A^2 \geq \text{pen}_k^c / 14) \leq C \mathcal{C}_{i,L}^2 n^{-1}$,
(iii) $\sup_{f \in \mathbb{D}_A^{L,m}} \mathcal{M}(\|\hat{j}_k^c - f_k^c\|_A^2 \geq \text{pen}_k^c / 14) \leq C (\exp(-c_{n_0,L} (k^c_\infty)^{1/2}) + n^{-1})$.
Consider $\nu_j$, $\nu_j^c$ and $\Omega_j^c = \{\nu_j - \nu_j^c \leq \nu_j / 2\}$, then for all $n \in \mathbb{N}$ holds
(iv) $\sup_{f \in \mathbb{D}_A^{L,m}} \mathcal{M}(\Omega_j^c) \leq 4L n^{-1}$.

**Proof of Corollary A.6.** The assertions follow immediately from (i)-(iv) in Corollary A.5, respectively, by using that for all $f \in \mathbb{D}_A^{L,m}$ and $k \in \mathbb{N}$ hold $1 \leq \nu_j \vee \|f\|_A^2 \vee \|f\|_\infty^2 \leq \mathcal{C}_{i,L}$ and $\mathbb{E}_f(|X_1|^{2\alpha}) \leq L$, and we omit the details. □

**B Proof of Theorem 2.3.**

Due to the construction (2.9) of the functions $\psi_{k,K}$ we easily see that the function $\psi_{k,K}$ has support on $[1 + k / K, 1 + (k + 1) / K]$ which lead to $\psi_{k,K}$ and $\psi_{L,K}$ having disjoint support if $k \neq
l. Further we will choose $\psi \in C_c^\infty(\mathbb{R})$, the set of all smooth functions with compact support in $\mathbb{R}$, which implies that $\|\psi\|_\infty < \infty$. For instance we can choose $\psi(x) := \sin(2\pi x)g(x)$ where $g(x) := \exp\left(-\frac{1}{1-(2\pi)^2}\right)1_{(0,1)}(x)$. The function $g \in C_c^\infty(\mathbb{R})$ is a often use bump function and it holds for all $x \in \mathbb{R}$ that $g(1/2 + x) = g(1/2 - x)$ which implies that $\int_0^1 \psi(x)dx = 0$. Keep in mind that $f_o$ as in (2.9) is a density for each $\theta \in \{0, 1\}^K$ and $\delta \leq \delta_{f_o, \psi}$ and the density $f_o$ satisfies $c_{f_o} := \inf_{x \in [1,2]} f_o > 0$.

**Lemma B.1.** For $K \geq 8$ there is a subset $\{\theta^{(0)}, \ldots, \theta^{(M)}\}$ of $\{0, 1\}^K$ with $\theta^{(0)} = (0, \ldots, 0)$ such that $M \geq 2^{K/8}$ and for all $j, l \in [0, M]; j \neq l$ holds $\|f_{\theta^{(j)}} - f_{\theta^{(l)}}\|_A^2 \geq \|\psi\|_A^2 \delta^2 K^{-2s}$ and $KL(f_{\theta^{(j)}}, f_{\theta^{(l)}}) \leq \frac{\|\psi\|_A^2}{c_{f_o} \log(2)} \delta^2 \log(M) K^{-2s-1}$ where $KL$ is the Kullback-Leibler-divergence.

**Proof of Lemma B.1.** Since $(\psi_{k,K})_{k \in [0,K]}$ have disjoint support for $\theta, \theta' \in \{0, 1\}^K$ holds

$$\|f_{\theta} - f_{\theta'}\|_A^2 = \delta^2 K^{-2s} \sum_{k=0}^{K-1} (\theta_{k+1} - \theta'_{k+1}) \psi_{k,K}^2 = \delta^2 K^{-2s} \rho(\theta, \theta') \|\psi\|_A^2,$$

where $\rho(\theta, \theta') := \sum_{j=0}^{K-1} 1_{\theta_{j+1} = \theta'_{j+1}}$ is the usual Hamming distance. Applying a change of variables $v = xK - K - k$ we conclude $\|f_{\theta} - f_{\theta'}\|_A^2 = \delta^2 \|\psi\|_A^2 K^{-2s-1} \rho(\theta, \theta')$. Due to the VARSHAMOV-GILBERT LEMMA (see Tsybakov [2008]) for $K \geq 8$ there is a subset $\{\theta^{(0)}, \ldots, \theta^{(M)}\}$ of $\{0, 1\}^K$ with $\theta^{(0)} = (0, \ldots, 0)$ such that $\rho(\theta^{(j)}, \theta^{(k)}) \geq K/8$ for all $j, k \in [0, M], j \neq k$ and $M \geq 2^{K/8}$ implying the first claim $\|f_{\theta^{(j)}} - f_{\theta^{(l)}}\|_A^2 \geq \|\psi\|_A^2 \delta^2 K^{-2s}$. For the second part, since $f_o = f_{\theta^{(0)}}$ and $KL(f_{\theta}, f_o) \leq \chi^2(f_{\theta}, f_o) = \int_1^2 (f_{\theta}(x) - f_{o}(x))^2/f_{o}(x)dx$ it is sufficient to bound the $\chi$-squared divergence where by construction

$$\chi^2(f_{\theta}, f_o) \leq c_{f_o}^{-1} \|f_{\theta} - f_o\|_A^2 = c_{f_o}^{-1} \|\psi\|_A^2 \delta^2 K^{-2s-1} \rho(\theta, \theta^{(0)}) \leq c_{f_o}^{-1} \|\psi\|_A^2 \delta^2 K^{-2s}.$$ 

Using $M \geq 2^K$ follows the second claim $KL(f_{\theta^{(j)}}, f_{\theta^{(l)}}) \leq \frac{\|\psi\|_A^2}{c_{f_o} \log(2)} \delta^2 \log(M) K^{-2s}$ which completes the proof.

It remains to show that $f_o, \theta \in \{0, 1\}^K$, as in (2.9) are elements of the classes $D_{s,L}^{s,m}$. We will consider the cases [L] and [H] separately starting with the case [L]. A similar result was proven by Belomestny et al. [2017] without the additionally moment condition. For the sake of simplicity we denote by $\psi_{k,K}^{(j)}$ the $j$-th derivative of $\psi_{k,K}$ and define the finite constant $C_{j,\infty} := \max(\|\psi_{k,K}^{(j)}\|_\infty, l \in [0, j])$. Here we remark that due to the definition of $\psi_{k,K}$ for any $j \in \mathbb{N}$ the functions $\psi_{k,K}^{(j)}$ have also disjoint support for different values of the index $k$.

**Lemma B.2.** Let $s, m \in \mathbb{N}$ and $f_o(x) := \frac{c}{m!} \exp(-x), x \in \mathbb{R}^+$. Then, there is $L_{s,m,\delta} > 0$ such that $f_o$ and any $f_\theta$ as in (2.9) with $\theta \in \{0, 1\}^K$, $K \in \mathbb{N}$, belong to $D_{s,L}^{s,m,\delta}$.

**Proof of Lemma B.2.** Our proof starts with the observation that $a_j(f_o) = 0$ for $j \geq m + 1$ and hence $|f_o|^2 = \sum_{j=0}^{m} j^s a_j(f_o)^2 \leq 2m^{s+1}$. On the other hand we use Lemma 7.1. in
define the operator $H$ of Belomestny et al. [2016] to bound $|f_\theta - f_o|_s$. Precisely, there exists a constant $A(s) > 0$ such that $|f_\theta - f_o|_s^2 \leq A(s) ||f_\theta - f_o||_s^2$ with $||f_\theta - f_o||_s^2 := \sum_{j=0}^s \|f_\theta - f_o\|_j^2$ and for $j \in [0, s]$

$$\|f_\theta - f_o\|_j^2 := \delta^2 K^{-2s} \int_{\mathbb{R}^+} \left( x^{j/2} \sum_{l=0}^{j} \binom{j}{l} \sum_{k=0}^{K-1} \Theta_{l+1} K^l \psi(l)(xK - k) \right)^2 dx.$$  

Applying Jensen inequality and using disjoint support and boundness of the derivatives implies

$$\|f_\theta - f_o\|_j^2 \leq \delta^2 2^j K^{-2s} \sum_{l=0}^{j} \binom{j}{l} K^l \sum_{k=0}^{K-1} \int_{1+k/K}^{1+(k+1)/K} x^j \psi(l)(xK - k) \right)^2 dx.$$

$$\leq \delta^2 2^j C_{\infty,s} \sum_{l=0}^{j} \binom{j}{l} K^l \sum_{k=0}^{K-1} \int_{1+k/K}^{1+(k+1)/K} x^j dx \leq \delta^2 2^j C_{\infty,s}^2.$$  

It follows $|f_\theta - f_o|_s^2 \leq A(s) ||f_\theta - f_o||_s^2 \leq (s + 1)C_{\infty,s} A(s) \delta^{2s} 2^s$ and hence $|f_\theta|_s^2 \leq 2(|f_\theta - f_o|_s^2 + |f_o|_s^2) \leq (s + 1)C_{\infty,s} \delta^{2s} 2^s + 4m^{s+1}$. Since $E_{\theta,f_o} [X^{-m/2}] = 2 + \delta K^{-s+1} C_{\infty,0} \leq 2 + \delta C_{\infty,0}$

Lemma B.2 holds true with $L_{s,m,\delta} := ((s + 1)C_{\infty,s} \delta^{2s} 2^s + 4m^{s+1}) \vee (2 + \delta C_{\infty,0}).$
Let \( \mathcal{S}(\mathbb{R}) := \{ f \in C_c^\infty(\mathbb{R}) | \forall \alpha, \beta \in \mathbb{N} : \sup_{x \in \mathbb{R}} |x^\alpha f^{(\beta)}(x)| < \infty \} \) be the Schwartz class. Introducing the Bessel potential operator \((-\Delta + \text{Id})^{s/2}\) we use for any \( f \in \mathcal{S}(\mathbb{R}) \) the identity \((-\Delta + \text{Id})^{s/2} f = G_{s/2} \ast f \) where \( \mathcal{F}(G_{s/2}) = (1 + |\xi|^2)^{s/2} \) for \( \xi \in \mathbb{R} \). Here \( \mathcal{F}(f)(\xi) := \int_\mathbb{R} f(x) \exp(-ix\xi)dx \) denotes the usual Fourier transform of \( f \) evaluated at \( \xi \in \mathbb{R} \) (e.g. Adams and Hedberg [2012]). In the sequel we use \( \mathcal{F}(G_{s/2})\mathcal{F}(G_{s/2}) = \mathcal{F}(G_s) \) and \( \Psi, \psi_{k,K} \in C_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}) \) which together with Plancherel’s and the convolution theorem imply \( \|(-\Delta + \text{Id})^{s/2}\Psi_K\|_2^2 = \|\mathcal{F}((-\Delta + \text{Id})^{s/2}\Psi_K)\|_2^2 = \|\mathcal{F}(G_{s/2})\mathcal{F}(\Psi_k)\|_2^2 \) and

\[
\|(-\Delta + \text{Id})^{s/2}\Psi_K\|_2^2 = \langle \mathcal{F}(\Psi_K), \mathcal{F}(G_s)\mathcal{F}(\Psi_K) \rangle = \int_\mathbb{R} \Psi_k(x)(-\Delta + \text{Id})^s\Psi_k(x)dx.
\]

Keeping \( s \in \mathbb{N} \) in mind for any \( f \in C_c^\infty \) we have \((-\Delta + \text{Id})^s f = \sum_{j=0}^s \binom{s}{j} f^{(2j)} \). Since the derivatives \( \psi_{k,K} \) have disjoint supports for different values of the index \( k \) follows

\[
\|(-\Delta + \text{Id})^{s/2}\Psi_K\|_2^2 = \sum_{k=0}^{K-1} \int_{1+k/K}^{1+(k+1)/K} \psi_k(x)(-\Delta + \text{Id})^s\psi_k(x)dx \leq \sum_{k=0}^{K-1} \int_{1+k/K}^{1+(k+1)/K} |\psi_k(x)(-\Delta + \text{Id})^s\psi_k(x)| dx.
\]

Note that \( \|(-\Delta + \text{Id})^s\psi_k\|_\infty \leq \sum_{j=0}^s \binom{s}{j} K^{2j} \|\psi^{(2j)}\|_\infty \leq 2^s K^{2s} C_{\infty,2s} = C_s K^{2s} \). From \( \int_{-\infty}^\infty |\psi_k(x)|dx \leq C_{\infty,0} K^{-1} \) follows \( \|(-\Delta + \text{Id})^{s/2}\Psi_K\|_2^2 \leq C_s K^{2s} \) and whence \( |f_{\theta}^2 - |f_{\theta}|^2| \leq C_s \delta^2 \). Since \( \mathbb{E}_{\|\cdot\|_p}[|X|^{2m}] = \mathbb{E}_{\|\cdot\|_p}[|X|^{2m}]^{1/3} \) and \( \mathbb{E}_{\|\cdot\|_p}[|X|^{2m}] = \int_{-\infty}^\infty x^{2m} f_\theta(x)dx \leq \delta K^{-s} \sum_{k=0}^{K-1} \int_1^2 x^{2m}\psi_k(x)dx \leq (2m)! + 4^m \delta C_{\infty,0} \) Lemma B.3 is satisfied with \( L_{s,m,\delta} := (C_s \delta^2) \vee ((2m)! + 4^m \delta C_{\infty,0})^{1/3} \), which completes the proof. \( \square \)

### C Proofs of section 3

**Proof of Lemma 3.1.** We start the proof with the observation that

\[
\alpha_j(\widehat{f_w}) - \alpha_j(f) = (\widehat{\alpha_j(f)} - \alpha_j(f))\mathbb{P}_w(\| j, M_n \|) - \alpha_j(f)\mathbb{P}_w(\|1, j\|) \text{ for all } j \in [0, M_n]\]

and \( \alpha_j(\widehat{f_w}) - \alpha_j(f) = -\alpha_j(f) \) for all \( j \geq M_n \).

Consequently, we have

\[
\|\widehat{f_w} - f\|_A^2 \leq 2 \sum_{j \in [0, M_n]} |(\widehat{\alpha_j(f)} - \alpha_j(f))\mathbb{P}_w(\| j, M_n \|)|
\]

\[
+ 2 \sum_{j \in [1, M_n]} |\alpha_j(f)|^2\mathbb{P}_w(\|1, j\|) + \sum_{j \geq M_n} |\alpha_j(f)|^2, \quad (C.1)
\]
where we bound the first and the two other terms on the right hand side separately. Considering the first term we split the sum into two parts. Precisely, for \( k_+ \in [1, M_n] \) holds

\[
\sum_{j \in [0, M_n]} (\hat{a}_j(f) - a_j(f))^2 \mathbb{P}_n(\{j, M_n\}) \leq \|\hat{f}_{k_+} - f_{k_+}\|_A^2 + \sum_{l \in [k_+, M_n]} w_l \|\hat{f}_{l} - f_{l}\|_A^2 \\
\leq \frac{1}{14} \text{pen}^y_{k_+} + \sum_{l \in [k_+, M_n]} (\|\hat{f}_{l} - f_{l}\|_A^2 - \text{pen}^y_{l} / 14) + \\
+ \frac{1}{14} \sum_{l \in [k_+, M_n]} w_l \text{pen}^y_{l} 1_{\{\|\hat{f}_l - f_l\|_A^2 < \text{pen}^y_{l} / 7\}}
\]

(C.2)

Consider the second and third term in (C.1) we split the first sum into two parts and obtain

\[
\sum_{j \in [1, M_n]} |a_j(f)|^2 \mathbb{P}_n(\{1, j\}) + \sum_{j \in [1, M_n]} |a_j(f)|^2 \\
\leq \sum_{j \in [1, k_+]} |a_j(f)|^2 \mathbb{P}_n(\{1, j\}) + \sum_{j \in [k_+, M_n]} |a_j(f)|^2 \\
\leq \|f\|_A^2 \{\mathbb{P}_n(\{1, k_+\}] + \text{pen}^2_{k_+}(f)\}
\]

(C.3)

Combining (C.1) and (C.2), (C.3) we obtain the assertion, which completes the proof.

\[\square\]

### C.1 Technical assertions used in the proof of Theorem 3.3

Below we state and proof the technical Lemmata C.2 to C.4 used in the proof of Theorem 3.3.

The proof of Lemma C.2 is based on Lemma C.1 given first.

**Lemma C.1.** Considering Bayesian weights \( \hat{w} \) as in (1.3) for any \( l \in [1, M_n] \) hold

(i) for all \( k \in [1, l] \) we have

\[
\hat{w}_k 1_{\{\|\hat{f}_l - f_l\|_A^2 < \text{pen}^y_{l} / 7\}} \leq \exp (\kappa n \{ \frac{25}{14} \text{pen}^y_l + \frac{1}{2} \|f\|_A^2 \text{b}^2(l)(f) - \frac{1}{2} \|f\|_A^2 \text{b}^2(k)(f) - \text{pen}^y_{k}\})
\]

(ii) for all \( k \in [l, M_n] \) we have

\[
\hat{w}_k 1_{\{\|\hat{f}_k - f_k\|_A^2 < \text{pen}^y_{k} / 7\}} \leq \exp (\kappa n \{ - \frac{1}{2} \text{pen}^y_{k} + \frac{3}{2} \|f\|_A^2 \text{b}^2(l)(f) + \text{pen}^y_{l}\}).
\]

**Proof of Lemma C.1.** Given \( k, l \in [1, M_n] \) and an event \( \Omega_{kl} \) (to be specified below) it follows

\[
\hat{w}_k 1_{\Omega_{kl}} = \frac{\exp(-\kappa n \{ - \|\hat{f}_k\|_A^2 + \text{pen}^y_{k}\})}{\sum_{l \in [1, n]} \exp(-\kappa n \{ - \|\hat{f}_l\|_A^2 + \text{pen}^y_{l}\})} 1_{\Omega_{kl}} \\
\leq \exp (\kappa n \{ - \|\hat{f}_k\|_A^2 + \|\hat{f}_l\|_A^2 + (\text{pen}^y_{l} - \text{pen}^y_{k})\}) 1_{\Omega_{kl}}
\]

(C.4)
We distinguish the two cases (i) \( k \in [1, l] \) and (ii) \( k \in \]l, M_n]. Consider first (i) \( k \in [1, l] \). From (C.4) and (i) in Lemma C.1 (with \( \widehat{f} := \widehat{f}_n \)) follows
\[
\bar{w}_k \mathbb{1}_{\Omega_{kl}} \leq \exp \left( \kappa n \left\{ \frac{1}{2} \| \widehat{f}_l - f_l \|^2_A + \frac{\| f \|^2_A (b^2_k(f) - b^2_l(f)) + (\text{pen}_k^r - \text{pen}_l^r)}{\| f \|^2_A b^2_k(f)} \right\} \right) \mathbb{1}_{\Omega_{kl}}.
\]
Setting \( \Omega_{kl} := \{ \| \widehat{f}_l - f_l \|^2_A < \text{pen}_k^r / 7 \} \) the last bound implies the assertion (i). Consider secondly (ii) \( k \in \]l, M_n]. From (ii) in Lemma C.1 (with \( \widehat{f} := \widehat{f}_n \)) and (C.4) follows
\[
\bar{w}_k \mathbb{1}_{\Omega_{tk}} \leq \exp \left( \kappa n \left\{ \frac{1}{2} \| \widehat{f}_l - f_l \|^2_A + \frac{3}{2} \| f \|^2_A (b^2_k(f) - b^2_l(f)) + (\text{pen}_k^r - \text{pen}_l^r) \right\} \right) \mathbb{1}_{\Omega_{tk}}.
\]
Setting \( \Omega_{tk} := \{ \| \widehat{f}_l - f_l \|^2_A < \text{pen}_k^r / 7 \} \) and exploiting \( b^2_k(f) \geq 0 \) we obtain (ii), which completes the proof.

**Lemma C.2.** Consider Bayesian weights \( \bar{w} \) as in (1.3). For any \( k^\omega, k^\omega_\nu \in [1, M_n] \) and associated \( k^\omega_\nu, k^\omega_\nu \in [1, M_n] \) as defined in (3.6) hold
(i) \( \mathbb{P}_\nu (\{ 1, k^- \} \mathbb{1}_{\Omega_j} \leq \frac{4}{\kappa^2 \Delta \nu^2} \mathbb{1}_{\{ k^- > 1 \}} \exp \left( -\frac{3 \Delta \nu}{28} (k^\omega_\nu)^{1/2} + \mathbb{1}_{\{ k^- > 1 \}} \right) \{ \| \widehat{f}_\omega - f^\omega_\nu \|_A \geq \text{pen}^\omega_\nu / 14 \} \}
\]
(ii) \( \sum_{k \in [k^\omega_\nu, M_n]} \text{pen}^\nu_k \bar{w}_k \mathbb{1}_{\{ \| \widehat{f}_\nu - f^\nu \|_A < \text{pen}^\nu_k / 7 \}} \leq n^{-1} \frac{192 \nu}{\kappa^2 \Delta \nu} \leq n^{-1} \frac{192 \nu}{\kappa^2 \Delta \nu} \) (using \( \widehat{w}_j \geq 1 \)).

**Proof of Lemma C.2.** Consider (i). Let \( k^- \in [1, k^\omega_\nu] \) as in (3.6). For the non trivial case \( k^- > 1 \) from Lemma C.1 (i) with \( l = k^\omega_\nu \) follows for all \( k < k^- \leq k^\omega_\nu \)
\[
\bar{w}_k \mathbb{1}_{\{ \| \widehat{f}_\nu - f^\nu \|_A < \text{pen}^\nu_k / 7 \}} \leq \exp \left( \kappa n \left\{ -\frac{1}{2} \| f \|^2_A b^2_k(f) + \left( \frac{3}{14} \text{pen}^\nu_k + \frac{1}{2} \| f \|^2_A b^2_k(f) \right) \text{pen}^\nu_k \right\} \right).
\]
By using \( \frac{1}{2} \text{pen}^\nu_k \mathbb{1}_{\Omega} \leq \text{pen}^\nu_k \mathbb{1}_{\Omega} \leq \frac{3}{2} \text{pen}^\nu_k \) and the definition (3.6) \( k^- \) satisfies \( \| f \|^2_A b^2_k \geq \| f \|^2_A b^2_{k^-} \), hence \( \| f \|^2_A b^2_{k^-} \geq \| f \|^2_A b^2_{k^-} + 6 \text{pen}^\nu_k \geq \| f \|^2_A b^2_{k^-} \) \( + 4 \text{pen}^\nu_k \mathbb{1}_{\Omega_j} \), which implies
\[
\bar{w}_k \mathbb{1}_{\{ \| \widehat{f}_\nu - f^\nu \|_A < \text{pen}^\nu_k / 7 \}} \mathbb{1}_{\Omega_j} \leq \exp \left( -\frac{3}{28} \kappa \text{pen}^\nu_k - \frac{1}{2} \kappa n \text{pen}^\nu_k \right), \quad \forall k \in [1, k^-].
\]
The last bound, \( \text{pen}^\nu_k = \Delta \nu k^- 1/2 n^{-1}, \kappa \Delta \nu > 0, \) and \( \sum_{k \in \mathbb{N}} \exp(-\lambda k^{-1/2}) \leq \lambda^{-2} \) for any \( \lambda > 0 \) imply together (i), that is,
\[
\mathbb{P}_\nu (\{ 1, k^- \} \mathbb{1}_{\Omega_j} \leq \exp \left( -\frac{3 \Delta \nu}{28} n \text{pen}^\nu_k \right) \sum_{k=1}^{k^- - 1} \exp \left( -\frac{\kappa \Delta \nu}{2} k^{-1/2} \right) + \mathbb{1}_{\{ \| \widehat{f}_\omega - f^\omega_\nu \|_A \geq \text{pen}^\omega_\nu / 14 \}} \leq \frac{4}{\kappa^2 \Delta \nu} \exp \left( -\frac{3 \Delta \nu}{28} (k^\omega_\nu)^{1/2} + \mathbb{1}_{\{ \| \widehat{f}_\omega - f^\omega_\nu \|_A \geq \text{pen}^\omega_\nu / 14 \}} \right).
\]
Consider (ii). Let \( k^\nu_+ \in [k^\omega_\nu, M_n] \) as in (3.6). For the non trivial case \( k^\nu_- < M_n \) from Lemma C.1 (ii) with \( l = k^\nu_+ \) follows for all \( k > k^\nu_+ \)
\[
\bar{w}_k \mathbb{1}_{\{ \| \widehat{f}_\nu - f^\nu \|_A < \text{pen}^\nu_k / 7 \}} \leq \exp \left( \kappa n \left\{ -\frac{1}{2} \text{pen}^\nu_k + \frac{3}{2} \| f \|^2_A b^2_k(f) + \text{pen}^\nu_k \right\} \right).
\]
Thereby, for $k_+$ as in (3.6) satisfying $\frac{1}{4} \text{pen}_k^\Omega \geq \frac{1}{4} \text{pen}_{(k_+ + 1)}^\Omega > \frac{3}{2} \| f \|^2_A b_{k_+}^2 (f) + \text{pen}_{k_+}^\Omega$ holds

$$\tilde{w}_k \mathbb{I}_{\{\| f_k - f_{k+1} \|^2_A < \text{pen}_{k_+}^\Omega / 7\}} \leq \exp (\kappa n \{ - \frac{1}{4} \text{pen}_k^\Omega \}), \quad \forall k \in [k_+, M_n].$$

The last upper bound, $\text{pen}_k^\Omega = \Delta \tilde{w}_f k^{1/2} n^{-1}$ and $\text{pen}_k^\Omega = \Delta v_j k^{1/2} n^{-1}$ imply

$$\sum_{k \in [k_+, M_n]} \text{pen}_k^\Omega \tilde{w}_k \mathbb{I}_{\{\| f_k - f_{k+1} \|^2_A < \text{pen}_{k_+}^\Omega / 7\}} \leq \Delta v_j n^{-1} \sum_{k \in [k_+, M_n]} k^{1/2} \exp \left( - \frac{n \Delta \tilde{w}_f}{4} k^{1/2} \right),$$

which together with $\sum_{k \in \mathbb{N}} k^{1/2} \exp(-\lambda k^{1/2}) \leq 3\lambda^{-3}$ for any $\lambda > 0$ implies the assertion (ii) and completes the proof.

The next result can be directly deduced from Lemma C.2 by letting $\kappa \to \infty$. However, we think the following direct proof provides an interesting illustration of the values $k_+, k_- \in [1, M_n]$ as defined in (3.6).

**Lemma C.3.** Consider model selection weights $\hat{w}$ as in (1.2). For any $k_+, k_- \in [1, M_n]$ and associated $k_+, k_- \in [1, M_n]$ as in (3.6) hold

(i) $\mathbb{P}_\omega ([1, k_- \mathbb{]} \Omega_f \leq 1_{\{k_- > 1\}} \mathbb{I}_{\{\| f_k - f_{k_-} \|^2_A < \text{pen}_{k_-}^\Omega / 14\}}$;

(ii) $\sum_{k \in [k_+, M_n]} \text{pen}_k^\Omega \tilde{w}_k \mathbb{I}_{\{\| f_k - f_{k+1} \|^2_A < \text{pen}_{k_+}^\Omega / 7\}} = 0.$

**Proof of Lemma C.3.** By definition of $\hat{k}$ it holds $-\| f_k \|^2_A + \text{pen}_k^\Omega \leq -\| f_{k_-} \|^2_A + \text{pen}_{k_-}^\Omega$ for all $k \in [1, M_n]$, and hence

$$\| f_k \|^2_A - \| f_{k_-} \|^2_A \geq \text{pen}_k^\Omega - \text{pen}_{k_-}^\Omega \quad \text{for all } k \in [1, M_n]. \tag{C.5}$$

Consider (i). Let $k_- \in [1, k_- \mathbb{]}$ as in (3.6). For the non trivial case $k_- > 1$ it is sufficient to show, that on the event $\Omega_f \mathbb{=} \{ \| \tilde{v} - v_j \| \leq \| v_j / 2\}$, where $\frac{1}{2} \text{pen}_{k_-}^\Omega \leq \text{pen}_k^\Omega \leq \frac{3}{2} \text{pen}_{k_-}^\Omega$, holds $\{ \hat{k} \in [1, k_- \mathbb{]} \} \subseteq \{ \| f_{k_-} - f_{k_-} \|^2_A \geq \text{pen}_{k_-}^\Omega / 14\}$. Indeed, on $\Omega_f$ if $\hat{k} \in [1, k_- \mathbb{]}$, then the definition (3.6) of $k_-$ implies

$$\| f \|^2_A b_{k_-}^2 (f) > \| f \|^2_A b_{k_-}^2 (f) + 6 \text{pen}_{k_-}^\Omega \geq \| f \|^2_A b_{k_-}^2 (f) + 4 \text{pen}_{k_-}^\Omega. \tag{C.6}$$

On the other hand from Lemma A.1 (i) (with $\bar{f} := \hat{f}_n$) follows

$$\| \hat{f}_k \|^2_A - \| \hat{f}_{k_-} \|^2_A \leq \frac{11}{2} \| \hat{f}_{k_-} - f_{k_-} \|^2_A - \frac{1}{2} \| f \|^2_A \{ b_{k_-}^2 (f) - b_{k_-}^2 (f) \}. \tag{C.7}$$

Combining, (C.5) and (C.7) we conclude

$$\frac{11}{2} \| \hat{f}_{k_-} - f_{k_-} \|^2_A \geq \text{pen}_{k_-}^\Omega - \text{pen}_{k_-}^\Omega + \frac{1}{2} \| f \|^2_A \{ b_{k_-}^2 (f) - b_{k_-}^2 (f) \},$$

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which together with (C.6) and $\text{pen}_{\hat{k}}^e > 0$ implies

$$\frac{1}{2} \| \hat{f}_{\hat{k}} - f_k \|^2_A \geq \frac{1}{2} \| f \|^2_A \tilde{b}_k^2(f) - \frac{1}{2} \| f \|^2_A \hat{b}_k^2(f) - \text{pen}_{\hat{k}}^e \geq \frac{1}{2} \| f \|^2_A \hat{b}_k^2(f) \geq \frac{1}{14} \text{pen}_{\hat{k}}^e.$$  

Consequently, on $\Omega_f$ holds $\{ \hat{k} \in [1, k-] \} \subseteq \{ \| \hat{f}_k - f_k \|^2_A \geq \text{pen}_{\hat{k}}^e / 7 \} \subseteq \{ \| \hat{f}_k - f_k \|^2_A \geq \text{pen}_{\hat{k}}^e / 14 \}$, which shows (i). Consider (ii). Let $k_+ \in [k_+^e, M_n]$ as in (3.6). For the non trivial case $k_+ < M_n$ it is sufficient to show that, $\{ \hat{k} \in \| k_+, M_n \} \subseteq \{ \| \hat{f}_k - f_k \|^2_A \geq \text{pen}_{\hat{k}}^e / 7 \}$. If $\hat{k} \in \| k_+, M_n \}$ then the definition (3.6) of $k_+$ implies

$$\text{pen}_{\hat{k}}^e \geq \text{pen}_{\hat{k}+1}^e > 6 \| f \|^2_A \tilde{b}_{k_+}^2(f) + 4 \text{pen}_{\hat{k}_+}^e$$  (C.8)

and due to Lemma A.1 (ii) (with $\tilde{f} := \hat{f}_n$) also

$$\| \hat{f}_k \|^2_A - \| \hat{f}_{k_+} \|^2_A \leq \frac{7}{2} \| \hat{f}_k - f_k \|^2_A + \frac{1}{2} \| f \|^2_A \{ \tilde{b}_{k_+}^2(f) - \tilde{b}_{k_+}^2(f) \}.  \text{ (C.9)}$$

Combining, (C.5) and (C.9) it follows that

$$\frac{7}{2} \| \hat{f}_k - f_k \|^2_A \geq \text{pen}_{\hat{k}}^e - \text{pen}_{\hat{k}_+}^e - \frac{3}{2} \| f \|^2_A \{ \tilde{b}_{k_+}^2(f) - \tilde{b}_{k_+}^2(f) \}$$

which with $\tilde{b}_{k_+}^2(f) > 0$ and (C.8) implies $\{ \hat{k} \in \| k_+, M_n \} \subseteq \{ 7 \| \hat{f}_k - f_k \|^2_A \geq \text{pen}_{\hat{k}}^e \}$, that is

$$\frac{7}{2} \| \hat{f}_k - f_k \|^2_A \geq \frac{1}{2} + \frac{1}{2} \text{pen}_{\hat{k}}^e - \text{pen}_{\hat{k}_+}^e - \frac{3}{2} \| f \|^2_A \tilde{b}_{k_+}^2(f) \geq \frac{1}{2} \text{pen}_{\hat{k}}^e - \frac{1}{2} \| f \|^2_A \tilde{b}_{k_+}^2(f) \geq \frac{1}{2} \text{pen}_{\hat{k}}^e.$$

Thereby, we have shown (ii) and completed the proof.

\[ \square \]

**Lemma C.4.** Let the assumptions of Theorem 3.3 be satisfied. Considering (p) there is a finite constant $C_f$ given in (C.14) such that for all $j \in \mathbb{N}$ holds $\mathbb{E}_f \| \hat{f}_w - f \|^2_A \leq C_f n^{-1}$.

**Proof of Lemma C.4.** The proof is based on an evaluation of the upper bound (3.9) for a suitable selection of the parameters $k_\omega, k_\omega^e \in [1, M_n]$. Considering (p) there is $K \in \mathbb{N}$ with $1 > b_{[K-1]}(f) > 0$ and $b_k(f) = 0$ for all $k > K$. Let $c_f := \frac{6 \Delta \nu}{\| \hat{f}_W, \hat{b}_{[K-1]}(f) \| > 0}$ and $n_f := \min \{ n \in \mathbb{N} : n > c_f K^{1/2} \wedge M_n \geq K \} \in \mathbb{N}$. We distinguish for $n \in \mathbb{N}$ the following two cases, (a) $n \in [1, n_f]$ and (b) $n > n_f$. Firstly, consider (a) with $n \in [1, n_f]$, then setting $k_\omega := 1, k_\omega^e := 1$ we have $k_- = 1, b_1(f) \leq 1 \leq n_f n^{-1}$ and $\text{pen}_1^e \leq \Delta \nu n^{-1}$. Thereby, from (3.9) follows

$$\mathbb{E}_f \| \hat{f}_w - f \|^2_A \leq C \left( \| f \|^2_A n_f + \| f \|^2_{\infty} \vee 1 \right) \nu_f + \| f \|^2_A \nu_f + 1 \mathbb{E}_f (\| X \|^2) n^{-1} \text{ (C.10)}$$

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Secondly, consider (b), i.e., $n > n_f$ and thus $K \in [1, M_n]$. Setting $k^*_n := K$, it follows $b_{k^*_n}(f) = 0$ and $\text{pen}_{k^*_n} \leq \Delta v_f(K)^{1/2}n^{-1}$. From (3.9) follows for all $n > n_f$ thus

$$
\mathbb{E}_f^n\|\hat{f}_n - f\|^2_A \leq 2\|f\|^2_A b_{k^*_n}^n(f) + C\|f\|^2_A (k^*_n > 1) \exp\left(\frac{-\varepsilon_{v_f}}{1 + 400\|f\|_\infty} (k^*_n)^{1/2}\right) \\
+ C\left(\nu_f K^{1/2} + \|f\|^3_\infty \vee 1\right) v_f + \|f\|^2_A \mathbb{E}_f(|X|^{2n})^{-1} \) n^{-1} 
$$

The defining set of $k^*_n := \max\{k \in [K, M_n] : n > c_f k^{1/2}\}$ is not empty, since it contains $K$ by construction for all $n > n_f$. Consequently, $k^*_n \geq K$ and, hence $b_{k^*_n}(f) = 0$, and $(k^*_n)^{1/2}n^{-1} < c_f^{-1} = \frac{\|f\|^2_A b_{k^*_n}^n(f)}{6\Delta v_f(k^*_n)^{1/2}n^{-1}}$. It follows $\|f\|^2_A b_{k^*_n}^n(f) > 6\Delta v_f(k^*_n)^{1/2}n^{-1} = 6\text{pen}_{k^*_n} + \|f\|^2_A b_{k^*_n}^n(f)$ and trivially $\|f\|^2_A b_{k^*_n}^n(f) = 0 < 6\text{pen}_{k^*_n} + \|f\|^2_A b_{k^*_n}^n(f)$. Therefore, setting $k^*_n := k^*_n$ the definition (3.6) of $k_-$ implies $k_- = K$ and hence $b_{k^*_n}^n(f) = b_K^2(f) = 0$.

From (11) for all $n > n_f$ follows now

$$
\mathbb{E}_f^n\|\hat{f}_n - f\|^2_A \leq C\|f\|^2_A \exp\left(\frac{-\varepsilon_{v_f}}{1 + 400\|f\|_\infty} (k^*_n)^{1/2}\right) \\
+ C\left(\nu_f K^{1/2} + \|f\|^3_\infty \vee 1\right) v_f + \|f\|^2_A \mathbb{E}_f(|X|^{2n})^{-1} \) n^{-1} 
$$

Setting $C_f := \frac{1 + 400\|f\|_\infty}{\varepsilon_{v_f}} \vee 1$, $k^*_n := \lfloor C_f^2 n^2 (\log n)^{-6}\rfloor$ for all $n > \exp(c_f^{1/3}C_f^3/3) \vee \exp(C_f)$ holds $(k^*_n)^{1/2}n^{-1} < C_f (\log n)^{-3} < c_f^{-1}$ and $C_f^2 (\log n)^{-2} < 1$, thus $k^*_n \leq [n^2(\log n)^{-4}] \leq M_n$. Thereby, $k^*_n \geq k_+ > C_f(n^{n^2(\log n)^{-6}} - 1) = (C_f n(\log n)^{-3} - 1)(C_f n(\log n)^{-3} + 1) \geq (C_f n(\log n)^{-3}) - 2$ and hence

$$
\exp\left(\frac{-\varepsilon_{v_f}}{1 + 400\|f\|_\infty} (k^*_n)^{1/2}\right) \leq \exp\left(-C_f^{-1}(k^*_n)^{1/2}\right) \leq \exp\left(-C_f^{-1}(C_f n(\log n)^{-3} - 1)\right) \\
= \exp(C_f^{-1}) \exp\left(-\frac{n}{(\log n)^{-4}}\right) \leq \exp\left(-\frac{n}{(\log n)^{-4} - 1}\right)
$$

where $n(\log n)^{-4} > 1$ for all $n \geq 5550$. Thereby, for all $n > \lfloor 5550 \vee \exp(c_f^{1/3}C_f^3/3) \vee \exp(C_f)\rfloor$ holds $\exp\left(\frac{-\varepsilon_{v_f}}{1 + 400\|f\|_\infty} (k^*_n)^{1/2}\right) \leq \exp\left(-\frac{n}{(\log n)^{-4}}\right)$ while for $n \leq \lfloor 5550 \vee \exp(c_f^{1/3}C_f^3/3) \vee \exp(C_f)\rfloor$ holds $\exp\left(\frac{-\varepsilon_{v_f}}{1 + 400\|f\|_\infty} (k^*_n)^{1/2}\right) \leq \exp\left(-\frac{n}{(\log n)^{-4} - 1}\right) \exp\left(-\frac{n}{(\log n)^{-4} - 1}\right) n^{-1}$. Combining both bounds and the definition of $c_f$ and $C_f$ there is a numerical constant $C$ such that for all $n \in \mathbb{N}$ holds

$$
\exp\left(\frac{-\varepsilon_{v_f}}{1 + 400\|f\|_\infty} (k^*_n)^{1/2}\right) \leq C\left(\exp\left(\frac{[1\vee \|f\|_\infty]}{1 + 400\|f\|_\infty} (k^*_n)^{1/2}\right)\right) n^{-1}
$$

The last bound together with (12) implies

$$
\mathbb{E}_f^n\|\hat{f}_n - f\|^2_A \leq C\left(\exp\left(\frac{[1\vee \|f\|_\infty]}{1 + 400\|f\|_\infty} (k^*_n)^{1/2}\right)\right) \\
+ C\left(\nu_f K^{1/2} + \|f\|^3_\infty \vee 1\right) v_f + \|f\|^2_A \mathbb{E}_f(|X|^{2n})^{-1} \) n^{-1} 
$$

Combining (11) and (13) for (a) $n \in [1, n_f]$ and (b) $n \geq n_f$, respectively, for all $K \in \mathbb{N}$ and for all $n \in \mathbb{N}$ follows the claim of Theorem 3.3, that is

$$
\mathbb{E}_f^n\|\hat{f}_n - f\|^2_A \leq C\left(\exp\left(\frac{[1\vee \|f\|_\infty]}{1 + 400\|f\|_\infty} (k^*_n)^{1/2}\right)\right) + \|f\|^2_A n_f \\
+ C\left(\nu_f K^{1/2} + \|f\|^3_\infty \vee 1\right) v_f + \|f\|^2_A \mathbb{E}_f(|X|^{2n})^{-1} \) n^{-1},
$$

which completes the proof.
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