LOCAL SOLUTIONS OF HARMONICAL AND BI-HARMONICAL EQUATIONS, UNIVERSAL FIELD EQUATION AND SELF–DUAL CONFIGURATIONS OF YANG–MILLS FIELDS IN FOUR DIMENSIONS

A. N. Leznov

Institute for High Energy Physics, 142284 Protvino, Moscow Region, Russia

Abstract

A general method for the construction of solutions of the d’Alambertian and double d’Alambertian (harmonic and bi–harmonic) equations with local dependence of arbitrary functions upon two independent arguments is proposed. The connection between solutions of this kind and self–dual configurations of gauge fields having no singularities is established.

\[^{1}\text{E-mail: leznov@mx.ihep.su}\]
1 Introduction

The general solution of the d’Alambertian equation in $D$-dimensional space depends upon two arbitrary functions of $(D-1)$ independent arguments. In the space of two dimensions the general solution of d’Alambert’s equation $\phi_{\xi,\bar{\xi}} = 0$ has the form $\phi = f(\xi) + \bar{f}(\bar{\xi})$ and locally depends upon two arbitrary functions $f$ and $\bar{f}$. In spaces of higher dimensions the general solution may be expressed either as an integral transform, or else by using what is much the same thing, a construction using Greens functions where the dependence on arbitrary functions becomes nonlocal. Nevertheless in some problems as for example in the theory of radiation some partial solutions arise with a local dependence of the solution on arbitrary functions (the number of independent arguments being less than is necessary to solve the general Cauchy problem). We demonstrate this situation with the well–known example of in and out–going waves in the case ($D = 4$)

$$\Phi = \frac{f(r+it, \exp(-i\phi) \cot \frac{\theta}{2})}{r} + \frac{\bar{f}(r-it, \exp(-i\phi) \tan \frac{\theta}{2})}{r}$$

where $t = x_4$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $x_1 = r \sin \theta \sin \phi$, $x_2 = r \sin \theta \cos \phi$, $x_3 = r \cos \theta$; $f, \bar{f}$ are arbitrary functions of their two respective arguments. We have exhibited (1) only for illustrative purposes.

The goal of this paper is to explain a method of construction of solutions of this kind for the d’Alambert and double d’Alambert equations in three and four-dimensional spaces.

2 The general construction

We shall employ the following notation; $y = x_1 + ix_2, z = x_3 + ix_4$ ($\bar{y} = y^*, \bar{z} = z^*$);

$\Psi = \Psi_{y,\bar{y}} + \Psi_{z,\bar{z}}$.

Let us introduce three variables $u_1 = y + \lambda \bar{z}, u_2 = z - \lambda \bar{y}, u_3 = \lambda$ and consider the equation $F(u_1, u_2, u_3) = \text{Constant}$ as an implicit definition of the function $\lambda$ as a function of independent arguments $y, \bar{y}, z, \bar{z}$; $F$ is an arbitrary function of its three arguments. From these definitions it follows immediately that:

$$\lambda_y = -\frac{F_1}{F_\lambda}, \quad \lambda_{\bar{z}} = -\lambda \frac{F_1}{F_\lambda}, \quad \lambda_z = -\frac{F_2}{F_\lambda}, \quad \lambda_{\bar{y}} = \lambda \frac{F_2}{F_\lambda}$$

(2)

where $F_i = F_{u_i}, F_\lambda \equiv \bar{z}F_1 - \bar{y}F_2 + F_3$. Or

$$\lambda_{\bar{z}} = \lambda \lambda_y, \quad \lambda_{\bar{y}} = -\lambda \lambda_z$$

(3)

In the case of one pair of variables (say $y, \bar{z}$) (3) is known as equation of Monge [1] who first found its exact solution in implicit form. In recent years the generalisation of this equation on the space of higher dimensions has been intensively investigated in a series of papers by D.Fairlie and his collaborators [2] who call this generalised equation the Universal Field Equation.

It is easy to check by direct calculations that each function $\alpha \equiv \alpha(u_1, u_2, u_3)$ is annihilated by the pair of operators $D_1 \equiv \partial_{\bar{z}} - \lambda \partial_{\bar{y}}, \quad D_2 \equiv \partial_{\bar{y}} + \lambda \partial_{\bar{z}}$. From this fact and definitions above the following proposition results.
Let
\[ \phi^s = \alpha(u_1, u_2, u_3)F^s_\lambda \] (4)
then
\[ (\partial_v - \lambda \partial_y)\phi^s = -s\lambda \phi^s, \quad (\partial_y + \lambda \partial_v)\phi^s = s\lambda \phi^s \] (5)
and
\[ \Box \phi^s = (s + 1)(\lambda \phi^s_y - \lambda \phi^s_z) \] (6)
As a corollary in the case \( s = -1 \) we obtain
\[ \phi^{-1}_v = (\lambda \phi^{-1})_y, \quad \phi^{-1}_y = -(-\lambda \phi^{-1})_z, \quad \Box \phi^{-1} = 0 \] (7)

So with the help of an arbitrary root of the equation \( F(u_1, u_2, u_3) = \text{constant} \) it is possible to construct the function \( \phi^{-1} \) which is an exact the local solution of d’Alambert equation.

### 3 Simple examples

In the case when \( F \) is a polynomial of second order with respect to \( \lambda \) we have \( F - \text{constant} = A\lambda^2 + B\lambda + C = 0 \).

\[ \lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, \quad F_\lambda = 2A\lambda + B, \]
\[ \phi_{1,2}^{-1} = \frac{1}{2A\lambda_{1,2} + B} = \pm \frac{1}{A(\lambda_1 - \lambda_2)} = \pm \frac{1}{\sqrt{B^2 - 4AC}} \]

Let \( F = u_1u_2 + a\lambda^2 + a_0\lambda + \bar{a} = (yz + \bar{a}) + \lambda(z\bar{z} - y\bar{y}) + \lambda^2(z\bar{y} + a) = 0 \) and for the corresponding solution of the d’Alambert equation we obtain
\[ \phi = \frac{1}{\sqrt{yz - y\bar{y} + a_0} + (yz + \bar{a})(z\bar{y} + a)} \]
If \( a = \bar{a} = a_0 = 0 \) then we have the usual \( O(4) \) invariant solution \( \phi = \frac{1}{z\bar{z} + y\bar{y}} \) with singularity in the origin of coordinates. In the general case the singularities of this solution are lying an the curves \( y = -\frac{a}{z}, \bar{y} = -\frac{a}{\bar{z}}, (z\bar{z})^2 + a_0(z\bar{z}) - \bar{a}a = 0 \).

The choice \( F \) in the form \( F = u_1 + u_3u_2 \equiv y + \lambda(z + \bar{z}) - \lambda^2\bar{y} \) exactly corresponds to the solution of (7). In this section we have assumed a reality condition which will be given in explicit form in section 6.

### 4 The three-dimensional case

The solutions of three-dimensional Laplace equation may be obtained from construction described above by special choice of the initial functions \( F \) and \( \alpha \)
\[ F = F(u_1 + u_3u_2, u_3), \quad \alpha(u_1 + u_3u_2, \lambda) \] (8)
( compare with the last example of the previous section). Indeed in this case \( u_1 + u_3u_2 \equiv y + \lambda(z + \bar{z}) - \lambda^2\bar{y} \) and \( \lambda \) become independent of the argument \( z - \bar{z} \). For all functions of the above construction \( \partial_z = \partial_{\bar{z}} \) and the d’Alambertian operator goes into three-dimensional Laplace one.
5 Bi d’Alambertian equation

By techniques of section 2 it is not difficult to prove the following relation

\[ \Box \Box \ln \phi^s = -s \Box \Box \ln F_\lambda = \]

(9)

\[ s\left( \frac{\partial^2 \lambda}{\partial y^2} \frac{\partial^2 \lambda}{\partial z^2} - \left( \frac{\partial^2 \lambda}{\partial y \partial z} \right)^2 \right) = s \frac{D_4(F)}{F^4_\lambda} \]

where

\[ D_4(F) = \text{Det} \begin{pmatrix}
0 & F_1 & F_2 & F_3 \\
F_1 & F_{11} & F_{12} & F_{13} \\
F_2 & F_{21} & F_{22} & F_{23} \\
F_3 & F_{31} & F_{32} & F_{33}
\end{pmatrix} \]

(It follows from the linear equation for \( \phi^0 \) (6) that each function of three (exactly two) arguments \( u_1, u_2, u_3 \) satisfy the bi d’Alambertian equation).

The condition \( \Box \Box \ln F_\lambda = 0 \) means that either \( D_4(F) = 0 \) or it must be divided by \( F \) – constant because \( F = \text{constant} \) is the equation which determines \( \lambda \).

The equation \( D_4(F) = 0 \) is exactly the Universal equation of D.B.Fairlie et al in space of three dimensions. The general solution of it is constructed in implicit form and some explicit solutions are noted in [3].

So if we take \( F \) as the solution of Universal equation in the space of three dimensions then function

\[ \phi = \alpha(u_1, u_2, u_3) + \ln F_\lambda \]

will be a local solution of the Bi d’Alambertian, or bi-harmonic equation.

6 The condition of reality

The reality conditions on the solutions \( \Box \Box \ln F_\lambda \) constructed here give further restrictions on the choice of initial function \( F \). Namely

\[ F^*(u_1, u_2, u_3) = F\left( -\frac{u_2}{u_3}, \frac{u_1}{u_3}, -\frac{1}{u_3} \right) \]

In examples of section 3 we have used this restriction without any special mention of it.

By direct calculations it is easy to convince that the Universal equation in three dimensions is invariant with respect to discrete substitution

\[ u_1 \rightarrow -\frac{u_2}{u_3}, \quad u_2 \rightarrow \frac{u_1}{u_3}, \quad u_3 \rightarrow -\frac{1}{u_3} \]

(this is the partial case of \( SL(4, R) \) transformation with respect to which is invariant Universal equation in three dimensions [3]). From the last invariance it follows that among the solutions of Universal equation there are those for which condition of reality is satisfied.
7 Connection with the self-dual configurations of
Yang-Mills field theory

We shall explain this connection on the well-known example of t’Hoof’s solution. All dy-
amical variables in this case are expressed in terms of single function $\phi$ -the solution of
d’Alamber equation in four dimensions. The topological charge density has the form

$$q = \square \square \phi$$

Let us choose $\phi$ in the form

$$\phi = 1 + \sum \phi^{-1}_\nu$$

where each $\phi^{-1}_\nu$ is the local solution of d’Alamber equation of section 2 (7). If we want that
singularities of $\phi^{-1}_\nu$ (the zeroes of $F\lambda$) will not give a contribution into the topological charge
density it is necessary to demand

$$\square \square \ln F\lambda = 0.$$ 

So $F$ can not be taken arbitrary but only as solution of Universal equation (see section 5).

The well-known $5n$-th parametrical instanton solution arise under the choice $\phi^{-1}_\nu$ in the
pole form $\phi^{-1}_\nu = \frac{1}{(x - a)^2}$. In this case $\square \square \ln (x - a)^2 = 0$. This means that singularities in
each pole may be canceled by appropriate gauge transformation.

The analogue consideration may be applied to self-dual solution which arise by the formalism of discrete transformation [F].

8 Conclusion remarks

The main results of the present note are containing in formulae (7),(9) which give the local
solutions of harmonical and bi-harmonical equations in the terms of arbitrary function of
third variables and its derivatives in the first case and in terms of solution of Universal
equation in three dimensions in the second. The most unexpected is the fact of correlation
between the solutions of Universal and bi-harmonical equations in the problems of self-dual
configuration of Yang-Mills gauge theory.

The author is indebted to D.B.Fairlie for discussion of results of this paper.

References

[1] D. Zwilinger Handbook of differential equations (New York, Acad. Press., 1989).

[2] D.B. Fairlie, J.Govaerts and A. Morzov Nuclear Phys. B 373 (1992), 214-232.

[3] D.B. Fairlie and J. Govaerts Linearisation of Universal Field equations DTP-92/47, NI-92/011,
December 1992.

[4] V.B.Deryagin and A.N. Leznov, Geometrical symmetries of Universal equation,(in preparation)
[5] A.N. Leznov, Nonlinear symmetries of integrable systems *J.of Sov.Lazer. Research* **3-4**, 278-288, (1992)

Ch. Devchand and A.N. Leznov, Backlund transformation for supper-symmetric self-dual theories for semisimple gauge groups and hierarchy of $A_1$ solutions. Preprint IHEP DTP 92-170, (1992) (to be published in Commun. Math. Phys).