FRACTIONAL ORLICZ-SOBOLEV EMBEDDINGS

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Abstract. The optimal Orlicz target space is exhibited for embeddings of fractional-order Orlicz–Sobolev spaces in \( \mathbb{R}^n \). An improved embedding with an Orlicz–Lorentz target space, which is optimal in the broader class of all rearrangement-invariant spaces, is also established. Both spaces of order \( s \in (0,1) \), and higher-order spaces are considered. Related Hardy type inequalities are proposed as well. An extension theorem is proved, that enables us to derive embeddings for spaces defined in Lipschitz domains. Necessary and sufficient conditions for the compactness of fractional Orlicz-Sobolev embeddings are provided.

1. Introduction

The present paper is aimed at offering optimal Sobolev–Poincaré type inequalities and related embeddings for fractional-order Orlicz-Sobolev spaces. These spaces extend the classical fractional Sobolev spaces introduced in \([4, 56, 81]\). Given a number \( s \in (0,1) \) and a Young function \( A : [0, \infty) \to [0, \infty] \), namely a convex function vanishing at 0, the fractional-order Orlicz-Sobolev space \( W^{s,A}(\mathbb{R}^n) \) is defined via a seminorm \( | \cdot |_{s,A,\mathbb{R}^n} \) built upon the functional defined as

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dx \, dy}{|x-y|^n}
\]

for a measurable function \( u \) in \( \mathbb{R}^n \). The definition of the seminorm \( | \cdot |_{s,A,\mathbb{R}^n} \) is given, via the functional (1.1), in analogy with the notion of Luxemburg norm in Orlicz spaces. The bases for a theory of the spaces \( W^{s,A}(\mathbb{R}^n) \), motivated e.g. by the analysis of nonlinear fractional Laplacians with non-polynomial kernels, have recently been laid in \([42, 50]\) under the \( \Delta_2 \)-condition and a sublinear growth condition near zero on the Young function \( A \). Neither of these additional assumptions will be imposed throughout.

The standard Gagliardo functional and the associated seminorm \( | \cdot |_{s,p,\mathbb{R}^n} \), underlying the notion of the fractional Sobolev space \( W^{s,p}(\mathbb{R}^n) \) for \( p \in [1, \infty) \), are recovered by the choice \( A(t) = t^p \). A renewed interest in the area around fractional Sobolev spaces has flourished in the last two decades. This has been favoured by a myriad of investigations on nonlocal equations of elliptic and parabolic type, whose solutions naturally belong to the spaces \( W^{s,p}(\mathbb{R}^n) \). A touch of recent contributions in this connection is furnished by \([7, 8, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 44, 45, 51, 52, 53, 54, 55, 56, 57, 58, 59, 61, 62, 63, 64, 65, 66, 67, 68, 70, 71, 73, 76, 78, 83, 86]\). Comprehensive treatments of the theory of fractional Sobolev spaces, as special instances of the more general Besov spaces, can be found e.g. in \([10, 60]\). A self-contained presentation of their basic properties is provided in \([43]\).

Embeddings for the spaces \( W^{s,p}(\mathbb{R}^n) \) into Lebesgue spaces are classical. In particular, if \( 1 \leq p < \frac{n}{s} \), then there exists a constant \( C \) such that

\[
\|u\|_{L^{\frac{n}{s}}(\mathbb{R}^n)}^{\frac{n}{sp}} \leq C|u|_{s,p,\mathbb{R}^n}
\]

for every measurable function \( u \) decaying to 0 (in a suitable sense) near infinity. An improved version of inequality (1.2) has been established in \([54]\). It asserts that, in fact,

\[
\|u\|_{L^{\frac{n}{s}}(\mathbb{R}^n)}^{\frac{n}{sp}} \leq C|u|_{s,p,\mathbb{R}^n},
\]

for a measurable function \( u \) decaying to 0 (in a suitable sense) near infinity.

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where the Lorentz space \( L^{np/p}_{\infty} (\mathbb{R}^n) \subseteq L^{np/p} (\mathbb{R}^n) \).

Our main results amount to sharp counterparts of inequalities (1.2) and (1.3) for general seminorms \( | \cdot |_{s,A,\mathbb{R}^n} \). Given any \( s \in (0, 1) \) and any Young function \( A \), with subcritical growth corresponding to the assumption \( p < \frac{n}{s} \) in the case of powers, we detect the optimal Orlicz target space \( L^{A_{s,A}} (\mathbb{R}^n) \) such that

\[
\| u \|_{L^{A_{s,A}} (\mathbb{R}^n)} \leq C | u |_{s,A,\mathbb{R}^n},
\]

for some constant \( C \) and every measurable function \( u \) decaying to 0 near infinity. An explicit formula for the Young function \( A_{s,A} \) is provided, which only depends on \( A \) and on the ratio \( \frac{n}{s} \). Here, and in what follows, the expression “optimal target space” referred to an embedding or inequality means “smallest possible” within a specified family, in the sense that if the embedding also holds with the optimal target space in question replaced by another space from the same family, then the former is continuously embedded into the latter.

Inequality (1.4) is derived as a consequence of a stronger inequality

\[
\| u \|_{L(\hat{A}, \frac{np}{s}) (\mathbb{R}^n)} \leq C | u |_{s,A,\mathbb{R}^n},
\]

where \( L(\hat{A}, \frac{np}{s}) (\mathbb{R}^n) \) – a space of Orlicz-Lorentz type depending only on \( A \) and on the ratio \( \frac{np}{s} \) – is optimal in the larger class of all rearrangement-invariant spaces.

In particular, these results reproduce inequalities (1.2) and (1.3) when \( A(t) = t^p \). The latter also provides new information about (1.3), and tells us that the space \( L^{np/p}_{\infty} (\mathbb{R}^n) \) is indeed optimal in (1.3) among all rearrangement-invariant spaces. Let us mention that embeddings for the spaces \( W^{s,A} (\mathbb{R}^n) \), under additional technical assumptions on \( A \) and into non-optimal (in general) target spaces have recently appeared in [6].

Optimal inequalities for Orlicz-Sobolev spaces of fractional-order \( s > 1 \) are also presented. As customary, these spaces are defined on replacing the function \( u \) in (1.1) by \( \nabla [s] u \), the vector of all weak derivatives of \( u \) whose order is the integer part \([s]\) of \( s \). The inequalities in question parallel (1.4) and (1.5). They extend inequalities (1.6) and (1.7) to any \( s \in (0, n) \setminus \mathbb{N} \), and take the form

\[
\| u \|_{L^{A_{s,A}} (\mathbb{R}^n)} \leq C | \nabla [s] u |_{s,A,\mathbb{R}^n},
\]

and

\[
\| u \|_{L(\hat{A}, \frac{np}{s}) (\mathbb{R}^n)} \leq C | \nabla [s] u |_{s,A,\mathbb{R}^n},
\]

respectively, for functions \( u \) all of whose derivatives up to the order \([s]\) decay to 0 near infinity. The sharp spaces \( L^{A_{s,A}} (\mathbb{R}^n) \) and \( L(\hat{A}, \frac{np}{s}) (\mathbb{R}^n) \) appearing in (1.6) and (1.7) are defined exactly via the same formulas as in the case when \( s \in (0, 1) \), save that now are applied for \( s \in (1, n) \setminus \mathbb{N} \). Our conclusions could thus be formulated via statements simultaneously covering the cases when \( s \in (0, 1) \) and \( s \in (1, n) \setminus \mathbb{N} \). We prefer to enucleate the results for \( s \in (0, 1) \) in a separate section for ease of presentation, and also because those for \( s \in (1, n) \setminus \mathbb{N} \) call for a combination of the former and of inequalities for integer-order Orlicz-Sobolev spaces.

The integer-order Orlicz-Sobolev embeddings have been established in [31, 32, 35, 36]. Importantly, these embeddings are exactly matched by the fractional-order embeddings announced above, although the latter rely on a substantially different approach. Indeed, applying our formulas for the optimal spaces \( L^{A_{s,A}} (\mathbb{R}^n) \) and \( L(\hat{A}, \frac{np}{s}) (\mathbb{R}^n) \) with \( s \in \mathbb{N} \) recovers the optimal Orlicz and the optimal rearrangement-invariant space, respectively, in the Orlicz-Sobolev inequality of integer-order \( s \).

Closely related fractional-order Hardy type inequalities in \( \mathbb{R}^n \) are proposed as well. In fact, a crucial step in our approach is a Hardy inequality of order \( s \in (0, 1) \), which extends to the Orlicz realm a result from [65]. The Hardy inequality for \( s \in (1, n) \setminus \mathbb{N} \) is, by contrast, deduced as a consequence of inequality (1.7).

Analogous inequalities and embeddings when \( \mathbb{R}^n \) is replaced by a sufficiently regular bounded subset \( \Omega \) – a bounded Lipschitz domain – are established. In order to treat this variant, we prove an extension theorem for functions in the space \( W^{s,A} (\Omega) \), a generalization of a well-known result for fractional Sobolev spaces that can be found, for instance, in [43, Theorem 5.4].
Compact embeddings for fractional-order Orlicz-Sobolev spaces are characterized as well. A necessary and sufficient condition on a rearrangement-invariant space \( Y(\Omega) \) for the embedding
\[
W^{s,A}(\Omega) \to Y(\Omega)
\]
to be compact is exhibited when \( s \in (0,n) \setminus \mathbb{N} \) and \( \Omega \) is a bounded Lipschitz domain. As a consequence, the embedding
\[
W^{s,A}(\Omega) \to L^B(\Omega)
\]
is shown to be compact for an Orlicz space \( L^B(\Omega) \) if and only if the Young function \( B \) grows essentially more slowly near infinity than the function \( A_n^2 \) appearing in (1.6). Local versions of these compactness results are also provided in the case when \( \Omega = \mathbb{R}^n \). Like the other results of this paper, on setting \( s \in \mathbb{N} \) in their statements, our fractional compact embeddings perfectly tie up with their integer-order counterparts proved in [31, 32, 35, 36].

The material is organized as follows. Section 2 and Section 3 are devoted to notations, definitions and necessary background from the theory of Orlicz and rearrangement-invariant spaces, and the theory of integer and fractional-order Orlicz-Sobolev spaces, respectively. Several sharp one-dimensional Hardy type inequalities in Orlicz and rearrangement-invariant spaces of critical use in the proofs of our main results are collected in Section 4. Some of them are known, but others are new. The main results are exposed in Sections 5–9, whereas Section 7 is concerned with embeddings of arbitrary order. Fractional Orlicz-Sobolev spaces on open subsets of \( \mathbb{R}^n \) and their embeddings are the subject of Section 8. The objective of the final Section 9 are criteria for the compactness of embeddings.

### 2. ORLICZ SPACES AND REARRANGEMENT-INVARIANT SPACES

A function \( A : [0, \infty) \to [0, \infty] \) is called a Young function if it has the form
\[
A(t) = \int_0^t a(\tau)d\tau \quad \text{for } t \geq 0
\]
for some non-decreasing, left-continuous function \( a : [0, \infty) \to [0, \infty] \) which is neither identically equal to 0 nor to \( \infty \). Clearly, any convex (non trivial) function from \([0, \infty)\) into \([0, \infty]\), which is left-continuous and vanishes at 0, is a Young function.

Note that, if \( k \geq 1 \), then
\[
kA(t) \leq A(kt) \quad \text{for } t \geq 0.
\]
The Young conjugate \( \tilde{A} \) of \( A \) is defined by
\[
\tilde{A}(t) = \sup \{\tau t - A(\tau) : \tau \geq 0\} \quad \text{for } t \geq 0.
\]
The following representation formula for \( \tilde{A} \) holds:
\[
\tilde{A}(t) = \int_0^t a^{-1}(\tau)d\tau \quad \text{for } t \geq 0.
\]
Here, \( a^{-1} \) denotes the left-continuous inverse of the function \( a \) appearing in (2.1).

A Young function \( A \) is said to dominate another Young function \( B \) globally if there exists a positive constant \( C \) such that
\[
B(t) \leq A(Ct) \quad \text{for } t \geq 0.
\]
The function \( A \) is said to dominate \( B \) near infinity [resp. near zero] if there exists \( t_0 > 0 \) such that (2.4) holds for \( t \geq t_0 \) [\( t \leq t_0 \)]. The functions \( A \) and \( B \) are called equivalent globally [near infinity] [near zero] if they dominate each other globally [near infinity] [near zero]. Plainly, the function \( A \) dominates [is equivalent to] \( B \)
globally if and only if $A$ dominates [is equivalent to] $B$ near zero and near infinity.
The function $B$ is said to grow essentially more slowly near infinity than $A$ if
\begin{equation}
\lim_{t \to \infty} \frac{B(\lambda t)}{A(t)} = 0
\end{equation}
for every $\lambda > 0$. Note that condition (2.5) is equivalent to
\begin{equation}
\lim_{t \to \infty} \frac{A^{-1}(t)}{B^{-1}(t)} = 0.
\end{equation}

The growth of a Young function $A$ can be compared with that of a power function via its Matuszewska-Orlicz indices. Recall that the upper Matuszewska-Orlicz index $I(A)$ of a finite-valued Young function $A$ is defined as
\begin{equation}
I(A) = \lim_{\lambda \to \infty} \log \left( \frac{\sup_{t>0} A(\lambda t)}{A(t)} \right) / \log \lambda.
\end{equation}
The Matuszewska-Orlicz index $I(\infty)(A)$ of $A$ near infinity is defined analogously, with $\sup_{t>0} A(\lambda t)$ replaced by $\lim_{t \to \infty} A(\lambda t)/A(t)$.

Let $\Omega$ be a measurable subset of $\mathbb{R}^n$, with $n \geq 1$. We set
\begin{equation}
\mathcal{M}(\Omega) = \{ u : \Omega \to \mathbb{R} : u \text{ is measurable} \},
\end{equation}
and
\begin{equation}
\mathcal{M}_+(\Omega) = \{ u \in \mathcal{M}(\Omega) : u \geq 0 \}.
\end{equation}
The notation $\mathcal{M}_d(\Omega)$ is adopted for the subset of $\mathcal{M}(\Omega)$ of those functions $u$ that decay near infinity, in the sense that all their level sets $\{|u| > t\}$ have finite Lebesgue measure for $t > 0$. Namely,
\begin{equation}
\mathcal{M}_d(\Omega) = \{ u \in \mathcal{M}(\Omega) : |\{|u| > t\}| < \infty \text{ for every } t > 0 \},
\end{equation}
where $|E|$ stands for the Lebesgue measure of a set $E \subset \mathbb{R}^n$. Of course, $\mathcal{M}_d(\Omega) = \mathcal{M}(\Omega)$ provided that $|\Omega| < \infty$.

The Orlicz space $L^A(\Omega)$, associated with a Young function $A$, on a measurable subset $\Omega$ of $\mathbb{R}^n$, is the Banach function space of those real-valued measurable functions $u$ in $\Omega$ for which the Luxemburg norm
\begin{equation}
\|u\|_{L^A(\Omega)} = \inf \left\{ \lambda > 0 : \int \limits_{\Omega} A \left( \frac{|u|}{\lambda} \right) \, dx \leq 1 \right\}
\end{equation}
is finite. In particular, $L^A(\Omega) = L^p(\Omega)$ if $A(t) = t^p$ for some $p \in [1, \infty)$, and $L^A(\Omega) = L^\infty(\Omega)$ if $A(t) = 0$ for $t \in [0, 1]$ and $A(t) = \infty$ for $t > 0$.

When convenient for specific choices of $A$, we shall also adopt the notation $A(L)(\Omega)$ to denote the Orlicz space $L^A(\Omega)$.

The Hölder type inequality
\begin{equation}
\int \limits_{\Omega} |uv| \, dx \leq 2\|u\|_{L^A(\Omega)} \|v\|_{L^{-1}(\Omega)}
\end{equation}
holds for every $u \in L^A(\Omega)$ and $v \in L^\infty(\Omega)$.

If $A$ dominates $B$ globally, then
\begin{equation}
\|u\|_{L^B(\Omega)} \leq C\|u\|_{L^A(\Omega)}
\end{equation}
for every $u \in L^A(\Omega)$, where $C$ is the same constant as in (2.4). If $|\Omega| < \infty$ and $A$ dominates $B$ near infinity, then inequality (2.12) continues to hold for some constant $C$ depending also on $A$, $B$ and $|\Omega|$. Thus, if $A$ is globally equivalent to $B$, then $L^A(\Omega) = L^B(\Omega)$, up to equivalent norms. The same is true even if $A$ and $B$ are just equivalent near infinity, provided that $|\Omega| < \infty$.

The Orlicz spaces are members of the more general class of rearrangement-invariant spaces, whose definition is based upon that of decreasing rearrangement of a function.
The decreasing rearrangement $u^*$ of a function $u \in M(\Omega)$ is the (unique) non-increasing, right-continuous function from $[0, \infty)$ into $[0, \infty]$ which is equidistributed with $u$. In formulas,

$$u^*(r) = \inf\{t \geq 0 : \{|x \in \Omega : |u(x)| > t\} \leq r\} \quad \text{for } t \geq 0.$$ 

Moreover, we define the function $u^{**} : (0, \infty) \to [0, \infty]$ as

$$u^{**}(r) = \frac{1}{r} \int_0^r u^*(\rho)d\rho \quad \text{for } r > 0.$$ 

Notice that $u^* \leq u^{**}$. The Hardy-Littlewood inequality states that

$$\int_\Omega |uv| dx \leq \int_0^{\|\Omega\|} u^* v^* dr$$

for all functions $u, v \in M(\Omega)$. As a consequence, one also has that

$$\int_\Omega A(|uv|) dx \leq \int_0^{\|\Omega\|} A(u^* v^*) dr$$

for every Young function $A$.

A Banach function space $X(\Omega)$, in the sense of Luxemburg [9, Chapter 1, Section 1], is called a rearrangement-invariant space if

$$\|u\|_{X(\Omega)} = \|v\|_{X(\Omega)} \quad \text{whenever } u^* = v^*.$$ 

The associate space $X'(\Omega)$ of $X(\Omega)$ is the rearrangement-invariant space of all functions in $M(\Omega)$ for which the norm

$$\|v\|_{X'(\Omega)} = \sup_{u \neq 0} \frac{\int_\Omega |uv| dx}{\|u\|_{X(\Omega)}}$$

is finite. Notice that, given two rearrangement-invariant spaces $X(\Omega)$ and $Y(\Omega)$,

$$X(\Omega) \to Y(\Omega) \quad \text{if and only if} \quad Y'(\Omega) \to X'(\Omega)$$

with the same embedding constants. Here, and in what follows, the arrow “ $\to$ ” stands for continuous embedding.

If $X(\Omega) = L^A(\Omega)$ for some Young function $A$, then

$$(L^A)'(\Omega) = L^{A'}(\Omega),$$

up to equivalent norms, with absolute equivalence constants.

Let $\Omega$ be a measurable set in $\mathbb{R}^n$. With any function $u : \Omega \to \mathbb{R}$, we can associate the function $E_0(u) : \mathbb{R}^n \to \mathbb{R}$ defined as

$$E_0(u)(x) = \begin{cases} 
    u(x) & \text{if } x \in \Omega \\
    0 & \text{if } x \in \mathbb{R}^n \setminus \Omega.
\end{cases}$$

The map $u \mapsto E_0(u)$ plainly defines a linear operator. Given a rearrangement-invariant space $X(\mathbb{R}^n)$, we denote by $X(\Omega)$ the rearrangement-invariant space on $\Omega$ equipped with the norm defined as

$$\|u\|_{X(\Omega)} = \|E_0(u)\|_{X(\mathbb{R}^n)}$$

for every function $u \in M(\Omega)$. Note that, if $X(\mathbb{R}^n) = L^A(\mathbb{R}^n)$ for some Young function $A$, then the space $X(\Omega)$ defined as in (2.20) agrees with the Orlicz space $L^A(\Omega)$.

The representation space $X(0, \|\Omega\|)$ of a rearrangement-invariant space $X(\Omega)$ is the unique rearrangement-invariant space on $(0, \|\Omega\|)$ satisfying

$$\|u\|_{X(\Omega)} = \|u^*\|_{X(0, \|\Omega\|)}$$

for every $u \in X(\Omega)$. 
If $|\Omega| < \infty$, then
\begin{equation}
L^\infty(\Omega) \to X(\Omega) \to L^1(\Omega)
\end{equation}
for every rearrangement-invariant space $X(\Omega)$.

Given any $\lambda > 0$ and $L > 0$, the dilation operator $E_\lambda$, defined at $f \in M(0, L)$ by
\begin{equation}
(E_\lambda f)(t) = \begin{cases} f(t/\lambda) & \text{if } 0 < t/\lambda \leq L \\ 0 & \text{if } L < t/\lambda, \end{cases}
\end{equation}
is bounded on any rearrangement-invariant space $X(0, L)$, with norm not exceeding $\max\{1, 1/\lambda\}$.

Assume that $|\Omega| < \infty$ and let $X(\Omega)$ and $Y(\Omega)$ be rearrangement-invariant spaces. We say that the space $X(\Omega)$ is almost-compactly embedded into $Y(\Omega)$ if
\begin{equation}
\lim_{L \to 0^+} \sup_{\|u\|_{X(\Omega)} \leq 1} \|\chi_{(0,L)}u^*\|_{Y(0,|\Omega|)} = 0.
\end{equation}

Here, and in what follows, $\chi_E$ denotes the characteristic function of a set $E$. By [79, Theorem 3.1], equation (2.24) is equivalent to the following condition:
\begin{equation}
\text{if } \{u_i\} \text{ is a bounded sequence in } X(\Omega) \text{ such that } u_i \to 0 \text{ a.e., then } \lim_{i \to \infty} \|u_i\|_{Y(\Omega)} = 0.
\end{equation}

In the special case of Orlicz spaces $L^A(\Omega)$ and $L^B(\Omega)$, one has that
\begin{equation}
L^A(\Omega) \text{ is almost-compactly embedded into } L^B(\Omega) \text{ if and only if } B \text{ grows essentially more slowly than } A,
\end{equation}
see, e.g., [74, Theorem 4.17.7]).

The Orlicz-Lorentz spaces are a family of function spaces that extends that of the Orlicz spaces. Given a Young function $A$ and a number $q \in \mathbb{R}$, we denote by $L(A,q)(\Omega)$ the Orlicz-Lorentz space of all functions $u \in M(\Omega)$ for which the quantity
\begin{equation}
\|u\|_{L(A,q)(\Omega)} = \|r^{-\frac{1}{q}}u^*(r)\|_{L^A(0,|\Omega|)}
\end{equation}
is finite. Under suitable assumptions on $A$ and $q$, this quantity is a norm, and $L(A,q)(\Omega)$, equipped with this norm, is a (non-trivial) rearrangement-invariant space. This is certainly the case when $q > 1$ and
\begin{equation}
\int_0^\infty \frac{A(t)}{t^{1+q'}} dt < \infty,
\end{equation}
see [35, Proposition 2.1].

The spaces $L(A,q)(\Omega)$ come into play in the description of the associate spaces of another closely related family of Orlicz-Lorentz type spaces. They are denoted by $L[A,q](\Omega)$, and consist of all functions $u \in M(\Omega)$ that make the functional
\begin{equation}
\|u\|_{L[A,q](\Omega)} = \|r^{-\frac{1}{q}}u^{**}(r)\|_{L^A(0,|\Omega|)}
\end{equation}
finite. One can verify that, if $q < -1$, then this functional is a rearrangement-invariant norm that renders $L[A,q](\Omega)$ a rearrangement-invariant space provided that either $|\Omega| < \infty$, or $|\Omega| = \infty$ and
\begin{equation}
\int_0^\infty \frac{A(t)}{t^{1+(-q)'}} dt < \infty,
\end{equation}
where $(-q)' = \frac{q}{q+1}$, the Hölder conjugate of $-q$. For special choices of the function $A$, the space $L(A,q)(\Omega)$ agrees, up to equivalent norms, with customary Lorentz type spaces. Assume, for instance, that $|\Omega| < \infty$ and that
\begin{equation}
A(t) \text{ is equivalent to } t^p(\log t)^\alpha(\log \log t)^\beta \text{ near infinity.}
\end{equation}
for some powers $p, \alpha$ and $\beta$. Then, depending on the relations among $p, q$ and $\alpha$, the space $L(A,q)(\Omega)$ agrees with the Lorentz space $L^{\sigma,p}(\Omega)$, the Lorentz-Zygmund space $L^{\sigma,\gamma}(\Omega)$ or with the generalized Lorentz- Zygmund space $L^{\sigma,\gamma,\delta}(\Omega)$, for a suitable choice of the parameters $\sigma, p \in (0, \infty)$ and $\gamma, \delta \in \mathbb{R}$. Recall that $L^{\sigma,p}(\Omega)$, $L^{\sigma,\gamma}(\Omega)$ and $L^{\sigma,\gamma,\delta}(\Omega)$ are the spaces of those functions $u \in \mathcal{M}(\Omega)$ for which the quantity

\begin{equation}
\| u \|_{L^{\sigma,p}(\Omega)} = \left\| u^*(r)r^{\frac{1}{p} - \frac{1}{\sigma}} \right\|_{L^p(0,|\Omega|)}, \tag{2.31}
\end{equation}

\begin{equation}
\| u \|_{L^{\sigma,\gamma}(\Omega)} = \left\| u^*(r)r^{\frac{1}{\sigma} - \frac{1}{\gamma}}(\log(1 + |\Omega|/r))^{\gamma} \right\|_{L^p(0,|\Omega|)}, \tag{2.32}
\end{equation}

\begin{equation}
\| u \|_{L^{\sigma,\gamma,\delta}(\Omega)} = \left\| u^*(r)r^{\frac{1}{\sigma} - \frac{1}{\gamma}}(\log(1 + |\Omega|/r))^{\gamma}(\log(1 + \log(1 + |\Omega|/r)))^{\delta} \right\|_{L^p(0,|\Omega|)}, \tag{2.33}
\end{equation}

respectively, is finite. Notice that $L^p(\Omega) = L^{\sigma,p}(\Omega)$, $L^{\sigma,\gamma}(\Omega) = L^{\sigma,\gamma}(\Omega)$ and $L^{\sigma,\gamma,\delta}(\Omega) = L^{\sigma,\gamma,\delta}(\Omega)$. The full range of parameters $\sigma, p, \gamma$ for which $L^{\sigma,\gamma}(\Omega)$ is nontrivial is exhibited in [74, Remark 9.10.2(a)]. A characterization of the parameters for which the functional defined by (2.32) is (equivalent to) a norm, and $L^{\sigma,\gamma}(\Omega)$ equipped with this norm is a rearrangement-invariant space, can be found in [74, Theorem 9.10.4]. This will always be the case in our use of the spaces $L^{\sigma,\gamma}(\Omega)$, as well as of that of the spaces $L^{\sigma,\gamma,\delta}(\Omega)$, for which an analogous characterization is stated in [74, Lemma 9.3.1, Remark 9.3.2 and Lemma 9.5.6].

3. Fractional Orlicz-Sobolev spaces

Assume that $\Omega$ is an open subset of $\mathbb{R}^n$. Given $m \in \mathbb{N}$ and a Young function $A$, we denote by $V^{m,A}(\Omega)$ the homogeneous Orlicz-Sobolev space given by

\begin{equation}
V^{m,A}(\Omega) = \{ u \in W^{m,1}_{\text{loc}}(\Omega) : |\nabla^m u| \in L^A(\Omega) \}. \tag{3.1}
\end{equation}

Here, $\nabla^m u$ denotes the vector of all weak derivatives of $u$ of order $m$. If $m = 1$, we also simply write $\nabla u$ instead of $\nabla^1 u$. The notation $W^{m,A}(\Omega)$ is adopted for the classical Orlicz-Sobolev space defined by

\begin{equation}
W^{m,A}(\Omega) = \{ u \in V^{m,A}(\Omega) : \| \nabla^k u \|_{L^A(\Omega)} < \infty, k = 0, \ldots, m - 1 \}, \tag{3.2}
\end{equation}

where $\nabla^0 u$ has to be interpreted as $u$. The space $W^{m,A}(\Omega)$ is a Banach space equipped with the norm

\begin{equation}
\| u \|_{W^{m,A}(\Omega)} = \sum_{k=0}^m \| \nabla^k u \|_{L^A(\Omega)}. \tag{3.3}
\end{equation}

Now, let $s \in (0, 1)$. The seminorm $|u|_{s,A, \Omega}$ of a function $u \in \mathcal{M}(\Omega)$ is given by

\begin{equation}
|u|_{s,A, \Omega} = \inf \left\{ \lambda > 0 : \int_\Omega \int_\Omega A \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n} \leq 1 \right\}. \tag{3.4}
\end{equation}

The homogeneous fractional Orlicz-Sobolev space $V^{s,A}(\Omega)$ is defined as

\begin{equation}
V^{s,A}(\Omega) = \{ u \in \mathcal{M}(\Omega) : |u|_{s,A, \Omega} < \infty \}. \tag{3.5}
\end{equation}

The definitions of the seminorm $|u|_{s,A, \Omega}$ and of the space $V^{s,A}(\Omega)$ carry over to vector-valued functions $u$ just by replacing the absolute value of $u(x) - u(y)$ by the norm of the same expression on the right-hand side of equation (3.3).

The subspace $V^{s,A}(\Omega) \cap \mathcal{M}_d(\Omega)$ of those functions in $V^{s,A}(\Omega)$ that decay near infinity is denoted by $V^{s,A}_d(\Omega)$. Thus,

\begin{equation}
V^{s,A}_d(\Omega) = \{ u \in V^{s,A}(\Omega) : |\{ |u| > t \}| < \infty \text{ for every } t > 0 \}. \tag{3.6}
\end{equation}

The definition of $V^{s,A}(\Omega)$ is extended to all $s \in (0, \infty) \setminus \mathbb{N}$ in a customary way. Denote by $[s]$ the integer part of $s$, and set $\{ s \} = s - [s]$, the fractional part of $s$. Then we set

\begin{equation}
V^{s,A}(\Omega) = \{ W^{[s],1}_{\text{loc}}(\Omega) : \nabla^{[s]} u \in V^{(s),A}(\Omega) \}. \tag{3.7}
\end{equation}
In analogy with (3.5), we extend definition (3.5) to every $s \in (0, \infty) \setminus \mathbb{N}$ on setting
\begin{equation}
V_d^{s,A}(\Omega) = \{ u \in V^{s,A}(\Omega) : |(\nabla^k u)| < \infty \text{ for } k = 0, 1, \ldots, [s], \text{ and for every } t > 0 \}.
\end{equation}
The functional $|\nabla^k u|_{s,A,\Omega}$ defines a norm on the space $V_d^{s,A}(\Omega)$.

If $|\Omega| < \infty$ and $s \in (0, 1)$, we also define the space
\begin{equation}
V_{s,A}^\perp(\Omega) = \{ u \in V^{s,A}(\Omega) : (\nabla^k u)_\Omega = 0, \text{ for } k = 1, \ldots, [s] \},
\end{equation}
where
\[
u_\Omega = \frac{1}{|\Omega|} \int_\Omega u \, dx,
\]
the mean value of $u$ over $\Omega$. Definition (3.8) is extended to any $s \in (0, \infty) \setminus \mathbb{N}$ on setting
\begin{equation}
V_d^{s,A}(\Omega) = \{ u \in V^{s,A}(\Omega) : u_\Omega = 0 \},
\end{equation}
and is a Banach space equipped with the norm
\[
\|u\|_{V^s,\Omega} = \|u\|_{W^{[s],\Omega}} + |\nabla^k u|_{s,A,\Omega}.
\]
Clearly, $W^{s,A}(\Omega) \to V_d^{s,A}(\Omega)$, and, as a consequence of Proposition 8.5, Section 8, $W^{s,A}(\Omega) = V_d^{s,A}(\Omega)$ if $\Omega$ is bounded. The space $V_d^{s,A}(\Omega)$ naturally arises as a natural maximal domain space for various embeddings of ours to hold.

For the sake of completeness, let us recall that inclusion relations hold between integer-order and fractional-order Orlicz-Sobolev spaces. If $s \in (0, 1)$ and $A$ is a Young function, then
\begin{equation}
W^{1,A}(\mathbb{R}^n) \to W^{s,A}(\mathbb{R}^n).
\end{equation}
Moreover, denote by $\overline{A}$ the Young function defined as
\begin{equation}
\overline{A}(t) = \int_0^t \int_{\mathbb{S}^{n-1}} A(|x_1| \tau) \, dH^{n-1}(x) \frac{d\tau}{\tau}
\end{equation}
for $t \geq 0$, where $x = (x_1, \ldots, x_n)$, $\mathbb{S}^{n-1}$ stands for the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$, and $H^{n-1}$ for the $(n-1)$-dimensional Hausdorff measure. Then the function $\overline{A}$ is equivalent to $A$, and if $u \in W^{1,A}(\mathbb{R}^n)$, then there exists $\lambda_0 > 0$ such that
\begin{equation}
\lim_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x) - u(y)|}{\lambda|x-y|^s}\right) \frac{dx \, dy}{|x-y|^n} = \int_{\mathbb{R}^n} \overline{A}\left(\frac{|\nabla u|}{\lambda}\right) \, dx
\end{equation}
for every $\lambda \geq \lambda_0$.
If, in particular, $u$ belongs to the subspace of $W^{1,A}(\mathbb{R}^n)$ of those functions such that
\begin{equation}
\int_{\mathbb{R}^n} A\left(\frac{|u|}{\lambda}\right) \, dx + \int_{\mathbb{R}^n} A\left(\frac{|\nabla u|}{\lambda}\right) \, dx < \infty
\end{equation}
for every $\lambda > 0$, then equation (3.13) also holds for every $\lambda > 0$. Recall that the subspace of functions $u$ fulfilling (3.14) agrees with the closure of $C_0^\infty(\mathbb{R}^n)$ in $W^{1,A}(\mathbb{R}^n)$. It coincides with the whole of $W^{1,A}(\mathbb{R}^n)$ if and only if $A$ fulfills the so-called $\Delta_2$-condition. Embedding (3.11) and an analogue of equation (3.13) hold with $\mathbb{R}^n$ replaced by any bounded Lipschitz domain. As a consequence of embedding (3.11), one also has that
\begin{equation}
W^{[s]+1,A}(\mathbb{R}^n) \to W^{s,A}(\mathbb{R}^n)
\end{equation}
for every $s \in (0, \infty) \setminus \mathbb{N}$.

In the classical case when $A(t) = t^p$ for some $p \geq 1$, embedding (3.11) and equation (3.13) have been established.
in [11]. For functions $A$ satisfying the $\Delta_2$-condition and with the function $A$ in a somewhat implicit form, they are proved in [50]. The present general version can be found in [2].

We conclude this section with a fractional-order Pólya–Szegö principle on the decrease of the functional $A$ under symmetric rearrangement of functions $u$. Recall that the symmetric rearrangement $u^*$ of a function $u \in \mathcal{M}_d(\mathbb{R}^n)$ is defined as

$$u^*(x) = u^*(\omega_n|x|^n) \quad \text{for} \quad x \in \mathbb{R}^n,$$

where $\omega_n$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Thus, $u^*$ is radially decreasing about 0 and is equidistributed with $u$. The fractional Pólya–Szegö principle goes back to [3, 5] in the case when $A$ is a power. The result for Young functions $A$ satisfying the $\Delta_2$-condition and functions $u \in W^{s,A}(\mathbb{R}^n)$ is the subject of [42]. The general version stated in Theorem 3.1 below can be proved via the same route. The necessary variant is sketched after its statement.

**Theorem 3.1. [Fractional Pólya–Szegö principle]** Let $s \in (0,1)$ and let $A$ be a Young function. Assume that $u \in \mathcal{M}_d(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x) - u(y)|}{|x-y|^s}\right) \frac{dx \, dy}{|x-y|^n} \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u^*(x) - u^*(y)|}{|x-y|^s}\right) \frac{dx \, dy}{|x-y|^n}. \tag{3.16}$$

**Sketch of proof.** The proof follows along the same lines as that of [42, Theorem 3.7]. One step of the proof of that theorem requires that $u$ be approximated by a subsequence of polarizations of $u$ that converges to $u^*$ a.e. in $\mathbb{R}^n$. This is guaranteed if $u$ is just nonnegative and belongs to the space $\mathcal{M}_d(\Omega)$. Actually, as observed in [82, Section 4.1], under these assumptions, there exists a sequence of polarizations of $u$ (with respect to a sequence of hyperplanes independent of $u$) that converges to $u^*$ in measure. The assumption that $u$ be nonnegative is not a restriction, since

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x) - u(y)|}{|x-y|^s}\right) \frac{dx \, dy}{|x-y|^n} \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x)| - |u(y)|}{|x-y|^s}\right) \frac{dx \, dy}{|x-y|^n}, \tag{3.17}$$

and $|u|^* = u^*$.

4. **One-dimensional Hardy type inequalities in Orlicz spaces**

The results recalled in the first part of this section concern optimal target norms in inequalities for the integral operator $T_s$ defined, for $n \in \mathbb{N}$, $s \in (0,n)$ and $L \in (0,\infty]$, as

$$T_s f(r) = \int_r^L \varrho^{-1+\frac{s}{n}} f(\varrho) \, d\varrho \quad \text{for} \quad r \in [0,L], \tag{4.1}$$

for $f \in \mathcal{M}_+(0,L)$.

We begin with the optimal Orlicz target space corresponding to an Orlicz domain space $L^A(0,L)$, where $A$ is a Young function such that

$$\int_0^\infty \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} \, dt = \infty \tag{4.2}$$

and

$$\int_0^\infty \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} \, dt < \infty. \tag{4.3}$$

Such an Orlicz target is defined in terms of the Young function $A_{\frac{n}{s}}$ given by

$$A_{\frac{n}{s}}(t) = A(H^{-1}(t)) \quad \text{for} \quad t \geq 0, \tag{4.4}$$

where

$$H(t) = \left(\int_0^t \left(\frac{\tau}{A(\tau)}\right)^{\frac{s}{n-s}} \, d\tau\right)^{\frac{n-s}{s}} \quad \text{for} \quad t \geq 0. \tag{4.5}$$
Theorem A. Let $n \in \mathbb{N}$, $s \in (0, n)$, $L \in (0, \infty]$. Assume that $A$ is a Young function fulfilling conditions (4.2) and (4.3). Let $A_\natural$ be the Young function defined by (4.1). Then there exists a constant $C = C(\frac{n}{s})$ such that
\begin{equation}
\left\| \int_r^L f(q)q^{-1+\frac{s}{n}} \, dq \right\|_{L^A(\hat{A}, L)} \leq C \|f\|_{L^A(0,L)}
\end{equation}
for every function $f \in L^A(0,L)$. Moreover, $L^A(\hat{A}, L)$ is the optimal Orlicz target space in (4.6).

Inequality (4.6) is equivalent to [32] inequality (2.7), with $n$ replaced by $n/s$. The optimality of the space $L^A(\hat{A}, L)$ follows from [31] Lemma 1, where such an optimality is proved with $A_\natural$ replaced by an equivalent Young function. Such an equivalence is shown in [34] Lemma 2.

We next focus an inequality parallel to (4.6), but with a target space which is optimal among all rearrangement-invariant spaces. Let $n$, $s$, $L$ and $A$ be as in Theorem A. Denote by $\hat{A}$ the Young function given by
\begin{equation}
\hat{A}(t) = \int_0^t \hat{a}(\tau) \, d\tau \quad \text{for } t \in [0, \infty),
\end{equation}
where
\begin{equation}
\hat{a}^{-1}(r) = \left( \int_{a^{-1}(r)}^\infty \left( \int_0^t \left( \frac{1}{a(q)} \right)^{\frac{s}{n-s}} \, dq \right)^{-\frac{1}{n}} \, dt \right)^{\frac{1}{s-n}} \quad \text{for } r \geq 0.
\end{equation}
Let $L(\hat{A}, \frac{n}{s})(0, L)$ be the Orlicz-Lorentz space equipped with the norm defined as in (2.27), namely as
\begin{equation}
\|f\|_{L(\hat{A}, \frac{n}{s})(0,L)} = \|r^{-\frac{n}{s}}f^*(r)\|_{L(\hat{A}, L)}
\end{equation}
for $f \in M(0,L)$. Assumption (4.3) ensures that condition (2.28) is certainly fulfilled with $A$ replaced by $\hat{A}$ and $q$ by $\frac{1}{s}$. This is a consequence of [32] Propositions 2.1 and 2.2. Thus, $L(\hat{A}, \frac{n}{s})(0, L)$ is actually a rearrangement-invariant space. By [35] Lemma 2.3, assumption (4.3) is equivalent to condition (2.30), with $A$ replaced by $\hat{A}$ and $q$ by $\frac{n}{s}$. Hence, the space $L[\hat{A}, -\frac{n}{s}](0, L)$, endowed with the norm defined as in (2.30) by
\begin{equation}
\|f\|_{L[\hat{A}, -\frac{n}{s}](0,L)} = \|r^{\frac{n}{s}}f^{**}(r)\|_{L(\hat{A}, L)}
\end{equation}
for $f \in M(0,L)$, is also a rearrangement-invariant space. Moreover, one has that
\begin{equation}
L[\hat{A}, -\frac{n}{s}](0, L) = L[\hat{A}, \frac{n}{s}](0, L),
\end{equation}
up to equivalent norms, where $L(\hat{A}, \frac{n}{s}')(0, L)$ denotes the associate space of $L(\hat{A}, \frac{n}{s})(0, L)$. Property (4.11) is stated and established in [37] Lemma 4.5] for $L < \infty$; the proof in the case when $L = \infty$ is completely analogous.

Theorem B. Let $n \in \mathbb{N}$, $s \in (0, n)$, $L \in (0, \infty]$. Assume that $A$ is a Young function fulfilling conditions (4.2) and (4.3), and let $\hat{A}$ be the Young function defined by (4.7). Then there exists a constant $C = C(\frac{n}{s})$ such that
\begin{equation}
\left\| \int_r^L f(q)q^{-1+\frac{s}{n}} \, dq \right\|_{L(\hat{A}, \hat{A}, L)} \leq C \|f\|_{L^A(0,L)}
\end{equation}
for every function $f \in L^A(0,L)$. Moreover, $L(\hat{A}, \frac{n}{s})(0, L)$ is the optimal rearrangement-invariant target space in (4.12).

Inequality (4.12) agrees with inequality [35] inequality (3.1)], with $n$ replaced with $n/s$. The optimality of the space $L(\hat{A}, \frac{n}{s})(0, L)$ follows from [35] inequalities (4.6)–(4.8)].

Let us notice that the function $A$ always dominates $\hat{A}$. Moreover, $A$ is equivalent to $\hat{A}$ if and only if $A(t)$ grows less than the power $t^{\frac{n}{s}}$ in the sense that its Matuszewska-Orlicz index $I(A)$, defined by (2.7), is smaller.
\( \frac{n}{s} \). An analogous property holds near infinity in connection with the index \( I_\infty(A) \). These assertions are proved in [35] Propositions 5.1 and 5.2], and collected in the following proposition.

**Proposition C.** Let \( n \in \mathbb{N} \) and \( s \in (0, n) \). Assume that \( A \) is a Young function fulfilling conditions \([4.2]\) and \([4.3]\), and let \( A_\frac{n}{s} \) be the Young function defined by \([4.7]\). Then there exists a constant \( c = c(n/s) \) such that
\[
(4.13) \quad \widehat{A}(t) \leq A(ct) \quad \text{for } t > 0.
\]
Moreover, the function \( \widehat{A} \) is equivalent to \( A \) globally [near infinity] if and only if \( I(A) < \frac{n}{s} \) \( I_\infty(A) < \frac{n}{s} \).

As a consequence of Proposition C, the space \( L(\widehat{A}, \frac{n}{s})(0, L) \) reduces to \( L(A, \frac{n}{s})(0, L) \) if \( A(t) \) is subcritical with respect to \( t^\frac{n}{s} \) in the sense of Matuszewska-Orlicz indices.

**Proposition D.** Let \( n \in \mathbb{N} \), \( s \in (0, n) \). Assume that \( A \) is a Young function fulfilling conditions \([4.2]\) and \([4.3]\).

(i) If \( I(A) < \frac{n}{s} \), then \( L(\widehat{A}, \frac{n}{s})(0, \infty) = L(A, \frac{n}{s})(0, \infty) \), up to equivalent norms.

(ii) If \( I_\infty(A) < \frac{n}{s} \), then \( L(\widehat{A}, \frac{n}{s})(0, L) = L(A, \frac{n}{s})(0, L) \), up to equivalent norms, for every \( L > 0 \).

The embedding of the space \( L(\widehat{A}, \frac{n}{s})(0, L) \) into \( L^{A_\frac{n}{s}}(0, L) \) is a trivial consequence of the optimality of the former in inequality \([4.12]\) in the class of all rearrangement-invariant spaces, which includes, in particular, the Orlicz spaces. This fact is stated in Proposition \([4.1]\) below. A direct proof of this proposition is however given, which shows that the norm of the embedding in question is independent of \( A \) and \( L \), a piece of information of use in the proofs of our main results.

**Proposition 4.1.** Let \( n \in \mathbb{N} \), \( s \in (0, n) \) and \( L \in (0, \infty] \). Assume that \( A \) is a Young function fulfilling conditions \([4.2]\) and \([4.3]\). Let \( A_\frac{n}{s} \) and \( \widehat{A} \) be the Young functions defined as in \([4.4]\) and \([4.7]\), respectively. Then
\[
(4.14) \quad L(\widehat{A}, \frac{n}{s})(0, L) \to L^{A_\frac{n}{s}}(0, L).
\]
Moreover, the norm of embedding \([4.14]\) depends only on \( \frac{n}{s} \).

**Proof.** Denote by \( L[\widehat{A}, -\frac{n}{s}](0, L) \) the Orlicz–Lorentz space endowed with the norm defined as in \([2.29]\). Thanks to equation \([4.11]\) and to property \([2.17]\), embedding \([4.14]\) is equivalent to
\[
(4.15) \quad (L^{A_\frac{n}{s}})'(0, L) \to L(\widehat{A}, \frac{n}{s})'(0, L),
\]
with the same embedding constants. Also, by property \([2.18]\),
\[
(4.16) \quad (L^{A_\frac{n}{s}})'(0, L) = L^{\widehat{A}_\frac{n}{s}}(0, L),
\]
up to equivalent norms, with absolute equivalence constants. Owing to equations by \([4.11]\)–\([4.16]\), embedding \([4.14]\) will follow if we show that
\[
(4.17) \quad L^{\widehat{A}_\frac{n}{s}}(0, L) \to L[\widehat{A}, -\frac{n}{s}](0, L).
\]
Embedding \([4.17]\) is equivalent to the inequality
\[
(4.18) \quad \|r^{\frac{n}{s}}f^{**}(r)\|_{L^{\widehat{A}_\frac{n}{s}}(0, L)} \leq C\|f^*\|_{L^{A_\frac{n}{s}}(0, L)}
\]
for some constant \( C \) and for every function \( f \in \mathcal{M}_+(0, L) \). Moreover, the norm of embedding \([4.17]\) equals the optimal constant \( C \) in inequality \([4.18]\). Inequality \([4.18]\) is in its turn equivalent to the inequality
\[
(4.19) \quad \left\| \int_r^L g(\varphi)\varphi^{-1+\frac{n}{s}}d\varphi \right\|_{L^{A_\frac{n}{s}}(0, L)} \leq C'\|g\|_{L^{A}(0, L)}
\]
In the case when $L < \infty$. Theorem 3.5. The proof for $f$ defined by (4.23) $(\ast)$ is nothing but (4.6), and hence the latter holds with a constant $C'$ depending only on $\frac{n}{s}$. 

A characterization of the optimal rearrangement-invariant target space $X_s(0, L)$, corresponding to any given rearrangement-invariant domain space $X(0, L)$, for the operator $T_s$ defined by (4.11) is contained in the next result. Notice that, in view of Theorem B, if $X(0, L) = L^A(0, L)$, then $X_s(0, L) = L(A, s)$.

**Theorem E.** Let $s \in (0, n)$ and $L \in (0, \infty]$. Let $\| \cdot \|_{X(0, L)}$ be a rearrangement-invariant norm. If $L = \infty$, assume, in addition, that

\begin{equation}
(4.20)
\|(1 + r)^{-1+\frac{s}{r}}\|_{X'(0, \infty)} < \infty.
\end{equation}

(i) Define the functional $\| \cdot \|_{Z(0, L)}$ as

\begin{equation}
(4.21)
\|f\|_{Z(0, L)} = \left\| r^{-\frac{s}{r}} f^s(r) \right\|_{X'(0, L)}
\end{equation}

for $f \in M(0, L)$. Then $\| \cdot \|_{Z(0, L)}$ is a rearrangement-invariant norm on $(0, L)$. Denote by $\| \cdot \|_{X_s(0, L)}$ the norm defined by

\begin{equation}
\|f\|_{X_s(0, L)} = \|f\|_{Z(0, L)}
\end{equation}

for $f \in M(0, L)$. Then

\begin{equation}
(4.22)
\left\| \int_r^L g^{-1+\frac{s}{r}} f(q) dq \right\|_{X_s(0, L)} \leq C \|f\|_{X(0, L)}
\end{equation}

for every $f \in X(0, L)$. Moreover, $X_s(0, L)$ is the optimal rearrangement-invariant target space in (4.22). (ii) Let $s_1, s_2 > 0$ be such that $s_1 + s_2 < n$. Then

\begin{equation}
(4.23)
(X_{s_1})_{s_2}(0, L) = X_{s_1+s_2}(0, L).
\end{equation}

In the case when $L < \infty$, Part (i) of Theorem E is established in [48] Theorem 4.5 and Part (ii) in [39] Theorem 3.5]. The proof for $L = \infty$ follows along the same lines; the result of Part (i) is stated in [49] Theorem 4.4] and that of Part (ii) in [66] Proposition 4.3].

**Remark 4.2.** Condition (4.20) is indispensable if $L = \infty$. Indeed, if (4.20) fails, then the functional $\| \cdot \|_{Z(0, L)}$ given by (4.21) is trivial in the sense that $\|f\|_{Z(0, L)} < \infty$ only if $f = 0$. Furthermore, the inequality

\begin{equation}
(4.24)
\left\| \int_r^\infty g^{-1+\frac{s}{r}} f(q) dq \right\|_{Y(0, \infty)} \leq C \|f\|_{X(0, \infty)}
\end{equation}

does not hold for any rearrangement-invariant space $Y(0, \infty)$.

The last two results of this section are new. They amount to Hardy type inequalities in integral form in Orlicz spaces. Of course, they can be equivalently stated in the form of the boundedness of suitable integral operators in the relevant Orlicz spaces.
Lemma 4.3. Let $s \in (0,1)$ and let $L \in (0,\infty)$. Assume that $A$ is any Young function. Then

\begin{equation}
\int_0^L A \left( r^{-s} \int_0^r f(q) \, dq \right) \frac{dr}{r} \leq \int_0^L A \left( \frac{1}{s} r^{1-s} f(r) \right) \frac{dr}{r}
\end{equation}

for every function $f \in \mathcal{M}_+(0,L)$.

Proof. The change of variables $r = e^{-\xi}$ and $q = e^{-\eta}$ turns inequality (4.25) into

\begin{equation}
\int_{\log \frac{1}{r}}^\infty A \left( e^{s\xi} \int_{\xi}^\infty f(e^{-\eta}) e^{-\eta} \, d\eta \right) \, d\xi \leq \int_{\log \frac{1}{r}}^\infty A \left( \frac{1}{s} e^{(s-1)\xi} f(e^{-\xi}) \right) \, d\xi.
\end{equation}

Of course, $\log \frac{1}{r}$ has to be understood as $-\infty$ if $L = \infty$. On setting

\begin{equation}
g(\xi) = e^{(s-1)\xi} f(e^{-\xi}) \quad \text{for } \xi > \log \frac{1}{r},
\end{equation}

the last inequality can be rewritten as

\begin{equation}
\int_{\log \frac{1}{r}}^\infty A \left( e^{s\xi} \int_{\xi}^\infty g(\eta)e^{-\eta} \, d\eta \right) \, d\xi \leq \int_{\log \frac{1}{r}}^\infty A \left( \frac{1}{s} g(\xi) \right) \, d\xi.
\end{equation}

Define the operator

\begin{equation}
Hg(\xi) = e^{s\xi} \int_{\xi}^\infty g(\eta)e^{-\eta} \, d\eta \quad \text{for } \xi > \log \frac{1}{r},
\end{equation}

for $g \in \mathcal{M}_+ \left( \log \frac{1}{r}, \infty \right)$. We claim that $H : L^\infty \left( \log \frac{1}{r}, \infty \right) \to L^\infty \left( \log \frac{1}{r}, \infty \right)$, and

\begin{equation}
\|H\|_{L^\infty \to L^\infty} \leq \frac{1}{s}.
\end{equation}

Indeed,

\begin{align*}
\|Hg\|_{L^\infty \left( \log \frac{1}{r}, \infty \right)} &= \sup_{\xi \in \left( \log \frac{1}{r}, \infty \right)} e^{s\xi} \int_{\xi}^\infty g(\eta)e^{-\eta} \, d\eta \\
&\leq \|g\|_{L^\infty \left( \log \frac{1}{r}, \infty \right)} \sup_{\xi \in \left( \log \frac{1}{r}, \infty \right)} e^{s\xi} \left[ \frac{e^{-\eta s}}{-s} \right]_{\eta=\xi}^{\eta=\infty} = \frac{1}{s} \|g\|_{L^\infty \left( \log \frac{1}{r}, \infty \right)}
\end{align*}

for $g \in L^\infty \left( \log \frac{1}{r}, \infty \right)$. Moreover, $H : L^1 \left( \log \frac{1}{r}, \infty \right) \to L^1 \left( \log \frac{1}{r}, \infty \right)$, and

\begin{equation}
\|H\|_{L^1 \to L^1} \leq \frac{1}{s},
\end{equation}

inasmuch as

\begin{align*}
\|Hg\|_{L^1 \left( \log \frac{1}{r}, \infty \right)} &= \int_{\log \frac{1}{r}}^\infty e^{s\xi} \int_{\xi}^\infty g(\eta)e^{-\eta} \, d\eta \, d\xi = \int_{\log \frac{1}{r}}^\infty g(\eta)e^{-\eta} \int_{\log \frac{1}{r}}^\infty e^{s\xi} \, d\xi \, d\eta \\
&\leq \frac{1}{s} \int_{\log \frac{1}{r}}^\infty g(\eta) \, d\eta = \frac{1}{s} \|g\|_{L^1 \left( \log \frac{1}{r}, \infty \right)}
\end{align*}

for $g \in L^1 \left( \log \frac{1}{r}, \infty \right)$. From (4.27)–(4.28), via an interpolation theorem by Calderón [9, Chapter 3, Theorem 2.12], one can infer that

\begin{equation}
H : L^A \left( \log \frac{1}{r}, \infty \right) \to L^A \left( \log \frac{1}{r}, \infty \right)
\end{equation}

with

\begin{equation}
\|H\|_{L^A \to L^A} \leq \frac{1}{s}.
\end{equation}
Notice that in deducing (4.29) and (4.30) one makes use of the fact that \( L^A (\log \frac{1}{r}, \infty) \) is a rearrangement-invariant space, and hence an exact interpolation space between \( L^1 (\log \frac{1}{r}, \infty) \) and \( L^\infty (\log \frac{1}{r}, \infty) \) (see [9, Chapter 3, Theorem 2.12]). Therefore,

\[
(4.31) \quad \|Hg\|_{L^A (\log \frac{1}{r}, \infty)} \leq \left\| \frac{1}{s} g \right\|_{L^A (\log \frac{1}{r}, \infty)}
\]

for every \( g \in L^A (\log \frac{1}{r}, \infty) \). An application of inequality (4.31) with \( A \) replaced by \( A_M \), defined by \( A_M(t) = \frac{A(t)}{M} \) for \( t > 0 \), where

\[
M = \int_{\log \frac{1}{r}}^\infty A\left(\frac{1}{s} g(r)\right) dr,
\]

yields (4.26) via the definition of Luxemburg norm.

**Lemma 4.4.** Let \( n \in \mathbb{N} \), \( s \in (0, n) \), \( L \in (0, \infty] \). Assume that \( A \) is a Young function fulfilling conditions (1.2) and (1.3), and let \( \hat{A} \) be the Young function defined by (1.7). Then there exists a constant \( C = C(n/s) \) such that

\[
(4.32) \quad \int_0^L \hat{A} \left( r^{-s} \int_0^L f(g) \,dg \right) r^{n-1} \,dr \leq \int_0^L A\left( C r^{1-s} f(r) \right) r^{n-1} \,dr
\]

for every function \( f \in \mathcal{M}_+(0, L) \). Moreover, \( \lim_{s \to s_0} C(n/s) < \infty \) for any \( s_0 \in [0, n) \).

**Proof.** On replacing, if necessary, \( f \) by \( f x_0(L) \), it suffices consider the case when \( L = \infty \). The change of variables \( t = r^n \) and \( \tau = \theta^n \) turns inequality (4.32) into

\[
(4.33) \quad \int_0^\infty \hat{A} \left( \frac{\xi^{-\frac{\tau}{n}}}{n} \int_\xi^\infty f\left(\frac{1}{n} \eta \right) \eta^{-\frac{\tau}{n}} \,d\eta \right) \,d\xi \leq \int_0^\infty A\left( C \xi^{\frac{1-s}{n}} f\left(\xi^\frac{1}{n}\right) \right) \,d\xi.
\]

On setting

\[
g(\xi) = \xi^{-\frac{s}{n}} f\left(\xi^\frac{1}{n}\right) \quad \text{for} \quad \xi > 0,
\]

inequality (4.33) reads

\[
(4.34) \quad \int_0^\infty \hat{A} \left( \frac{\xi^{-\frac{\tau}{n}}}{n} \int_\xi^\infty g(\eta) \eta^{-1+\frac{\tau}{n}} \,d\eta \right) \,d\xi \leq \int_0^\infty A(Cg(\xi)) \,d\xi
\]

for \( g \in \mathcal{M}_+(0, \infty) \). Denote by \( T \) the operator defined as

\[
Tg(\xi) = \xi^{-\frac{s}{n}} \int_\xi^\infty g(\eta) \eta^{-1+\frac{s}{n}} \,d\eta \quad \text{for} \quad \xi > 0,
\]

for \( g \in \mathcal{M}_+(0, \infty) \). We have that \( T \): \( L^1(0, \infty) \to L^1(0, \infty) \), with

\[
(4.35) \quad \|T\|_{L^1(0, \infty)} \leq \frac{n}{n-s},
\]

inasmuch as

\[
\|Tg\|_{L^1(0, \infty)} = \int_0^\infty \xi^{-\frac{s}{n}} \int_\xi^\infty g(\eta) \eta^{-1+\frac{s}{n}} \,d\eta \,d\xi = \int_0^\infty g(\eta) \eta^{-1+\frac{s}{n}} \int_0^\infty \xi^{-\frac{s}{n}} \,d\xi \,d\eta = \frac{n}{n-s} \|g\|_{L^1(0, \infty)}
\]

for \( g \in L^1(0, \infty) \). Also, \( T \): \( L^{\frac{n}{s}}(0, \infty) \to L^{\frac{n}{n-s}}(0, \infty) \), with

\[
(4.36) \quad \|T\|_{L^{\frac{n}{s}} \to L^{\frac{n}{n-s}}} \leq 1.
\]

Actually,

\[
\|Tg\|_{L^{\frac{n}{n-s}}(0, \infty)} = \sup_{\xi \in (0, \infty)} \xi^{\frac{s}{n}} (Tg)^*(\xi) = \sup_{\xi \in (0, \infty)} \xi^{\frac{s}{n}} Tg(\xi) = \int_0^\infty g(\eta) \eta^{-1+\frac{s}{n}} \,d\eta \leq \|g\|_{L^1(0, \infty)}
\]
for $g \in L_{-1}^{\infty}(0, \infty)$, where the second equality follows from the fact that $(Tg)^* = Tg$, for $Tg$ is a decreasing function, and the inequality follows from the Hardy–Littlewood inequality, since the function $\eta \mapsto \eta^{-1+\frac{n}{\alpha}}$ is decreasing. Owing to the boundedness properties (4.35) and (4.36) of the operator $T$, inequality (4.34) follows from [35, Theorem 3.1]. The finiteness of the limit of the constant $C(n/s)$ can be checked via a close inspection of the proof of that theorem.

5. A fractional Hardy type inequality: case $s \in (0, 1)$

This section is devoted to a proof of the Hardy type inequality stated below. Apart from its own interest, it is critical in our approach to the other main results of this paper.

**Theorem 5.1. [Fractional Orlicz–Hardy inequality]** Let $n \in \mathbb{N}$ and $s \in (0, 1)$. Assume that $A$ is a Young function satisfying conditions (4.2) and (4.3) and let $\hat{A}$ be the Young function given by (4.7). Then, there exists a constant $C = C(n, s)$ such that

\[ (5.1) \quad \| u(x) \|_{L^\hat{A}(\mathbb{R}^n)} \leq C\| u \|_{1, n,A,\mathbb{R}^n} \]

for every function $u \in V^A_d(\mathbb{R}^n)$. Moreover, $\lim_{s \to 1^-} C(n, s) < \infty$. In particular,

\[ (5.2) \quad \int_{\mathbb{R}^n} \hat{A} \left( \frac{|u(x)|}{|x|^s} \right) dx \leq (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u(x) - u(y)|}{|x-y|^n} \right) \frac{dx}{|x|} \frac{dy}{|y|} \]

for every function $u \in M_d(\mathbb{R}^n)$.

The following example provides us with an application of Theorem 5.1 to a Young function whose behaviour near zero and near infinity is of power-logarithmic type. Although quite simple, this model Young function enables us to recover the results known until now and to exhibit genuinely new inequalities. This model function will also be called into play in order to illustrate the results of the next sections.

**Example 5.2.** Consider a Young function $A$ such that

\[ (5.3) \quad A(t) \text{ is equivalent to } \begin{cases} t^{p_0}(\log^\frac{1}{s} t)^{\alpha_0} & \text{near zero} \\ t^p(\log t)^\alpha & \text{near infinity,} \end{cases} \]

where either $p_0 > 1$ and $\alpha_0 \in \mathbb{R}$, or $p_0 = 1$ and $\alpha_0 \leq 0$, and either $p > 1$ and $\alpha \in \mathbb{R}$ or $p = 1$ and $\alpha \geq 0$. Here, equivalence is meant in the sense of Young functions.

Let $n \in \mathbb{N}$ and $s \in (0, 1)$. The function $A$ satisfies assumption (4.2) if

\[ (5.4) \quad \text{either } 1 \leq p < \frac{n}{s} \text{ and } \alpha \text{ is as above, or } p = \frac{n}{s} \text{ and } \alpha \leq \frac{n}{s} - 1, \]

and satisfies assumption (4.3) if

\[ (5.5) \quad \text{either } 1 \leq p_0 < \frac{n}{s} \text{ and } \alpha_0 \text{ is as above, or } p_0 = \frac{n}{s} \text{ and } \alpha_0 > \frac{n}{s} - 1. \]

Theorem 5.1 tells us that, under assumptions (5.4) and (5.5), inequalities (5.1) and (5.2) hold if

\[ (5.6) \quad \text{if } 1 \leq p_0 < \frac{n}{s} \text{ and } \alpha_0 \text{ is as above, or } p_0 = \frac{n}{s} \text{ and } \alpha_0 > \frac{n}{s} - 1 \]

and

\[ (5.7) \quad \text{if } p = \frac{n}{s} \text{ and } \alpha < \frac{n}{s} - 1 \]

In particular, the choices $p_0 = p < \frac{n}{s}$ and $\alpha_0 = \alpha = 0$ yield $\hat{A}(t) = t^p$, and inequalities (5.1) and (5.2) recover (apart from the specific form of the constant involved) [65, Inequality (3)].
The following preliminary results will be of use in the proof of Theorem 5.1.

**Proposition 5.3.** Let \( s \in (0, 1) \) and let \( A \) be a Young function. Assume that \( u \in V^{s,A}(\mathbb{R}^n) \) and \( \{|u| > 0\}| < \infty \). Then there exists a constant \( C = C(n, s, \{|u| > 0\}) \) such that

\[
\|u]\|_{L^A(\mathbb{R}^n)} \leq C|u|_{s,A,\mathbb{R}^n}.
\]

**Proof.** Let us set \( U = \{u^* > 0\} \) and \( d = \text{diam}(U) \). By Theorem 3.1, given any \( \lambda > 0 \), we have that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x) - u(y)|}{\lambda|x - y|^s}\right) \frac{dx\,dy}{|x - y|^n} \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{u^*(x) - u^*(y)}{\lambda|x - y|^s}\right) \frac{dx\,dy}{|x - y|^n}
\]

\[
\geq \int_U \int_{\{y \in \mathbb{R}^n: 2(d+1) \geq |x - y| \geq d+1\}} A\left(\frac{u^*(x)}{\lambda|x - y|^s}\right) \frac{dy\,dx}{|x - y|^n}
\]

\[
= \int_U \int_{\{y \in \mathbb{R}^n: 2(d+1) \geq |x - y| \geq d+1\}} A\left(\frac{(2(d+1))^s}{\lambda|x - y|^s}\right) \frac{u^*(x)}{|x - y|^n} \frac{dy\,dx}{|x - y|^n}
\]

\[
\geq \int_U A\left(\frac{u^*(x)}{\lambda(2(d+1))^s}\right) \int_{\{y \in \mathbb{R}^n: 2(d+1) \geq |x - y| \geq d+1\}} (2(d+1))^s \frac{dy\,dx}{|x - y|^{s+n}}
\]

\[
= \frac{c(2^s - 1)}{s} \int_U A\left(\frac{u^*(x)}{\lambda(2(d+1))^s}\right) \frac{dx}{|x - y|^n}
\]

for some positive constant \( c = c(n) \). Note that the third inequality holds, owing to property (2.22). Hence, inequality (5.8) follows. \( \square \)

**Corollary 5.4.** Let \( s \in (0, 1) \) and let \( A \) be a Young function. Assume that \( u \in V^{s,A}_{d}(\mathbb{R}^n) \). Then

\[
\int_{E} |u(x)| dx < \infty
\]

for every set \( E \subset \mathbb{R}^n \) with \( |E| < \infty \). In particular, \( u \in L^1_{\text{loc}}(\mathbb{R}^n) \).

**Proof.** Define the function \( \overline{u} : \mathbb{R}^n \to [0, \infty) \) as

\[
\overline{u} = \left(|u| - 1\right)\chi_{\{|u| > 1\}}.
\]

One can verify that

\[
|u(x) - u(y)| \geq |\overline{u}(x) - \overline{u}(y)| \quad \text{for } x, y \in \mathbb{R}^n.
\]

Thus, \( \overline{u} \in V^{s,A}(\mathbb{R}^n) \), and \( \{|\overline{u} > 0\}| < \infty \). By Proposition 5.3, \( \overline{u} \in L^A(\mathbb{R}^n) \), and since \( \{|\overline{u} > 0\}| < \infty \), the second embedding in (2.22) ensures that \( \overline{u} \in L^1(\mathbb{R}^n) \). On the other hand, \( |u| \leq \overline{u} + 1 \), whence

\[
\int_{E} |u(x)| dx \leq \int_{E} \overline{u} dx + |E| < \infty
\]

for any set \( E \subset \mathbb{R}^n \) with \( |E| < \infty \). Hence, (5.10) follows. \( \square \)

Our proof of Theorem 5.1 makes use of a classical approach in the theory of fractional Sobolev norms of functions in \( \mathbb{R}^n \), which consists in an extension of the relevant functions to \( \mathbb{R}^{n+1} \). In particular, we follow the outline of the proof of [52, Theorem 2]. However, specific ad hoc Orlicz space techniques and sharp one-dimensional Hardy type inequalities in Orlicz spaces, presented in Section 4, have to be exploited in the present setting.

**Proof of Theorem 5.1.** Let \( u \in M_d(\mathbb{R}^n) \). If the right-hand side of inequality (5.1) is infinite for a certain constant \( C \) to be specified later, then the conclusion is trivially true. We may thus assume that it is finite. Hence, in particular, \( u \in V^{s,A}_{d}(\mathbb{R}^n) \). Owing to Theorem 3.1, the integral on the right-hand side of inequality
In particular, \( u \in L^1_{\text{loc}}(\mathbb{R}^n) \). Let \( \psi : \mathbb{R}^n \to [0, \infty) \) be the function defined as
\[
\psi(y) = \frac{(n+1)}{n}(1 - |y|)\chi_{\{|y|>1\}} \quad \text{for } y \in \mathbb{R}^n.
\]
The function \( U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \), given by
\[
U(x,t) = \int_{\mathbb{R}^n} \psi(y)u(x + ty)\,dy \quad \text{for } (x,t) \in \mathbb{R}^n \times [0, \infty),
\]
is thus well defined. One has that
\[
|\nabla U(x,t)| \leq \frac{(n+1)(n+2)}{t \omega_n} \int_{\{|y|<1\}} |u(x + ty) - u(x)|\,dy \quad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0, \infty).
\]
Inequality (5.13) can be verified as follows. We have that
\[
\int_{\{|u|>1\}} |u|\,dx = \int_{\{|\psi|>1\}} u^*\,dx < \infty.
\]
Differentiating the leftmost side of equation (5.14) with respect to \( x \) and with respect to \( t \) yields
\[
\int_{\mathbb{R}^n} \nabla \psi\left(\frac{z-x}{t}\right) \,dz = 0 \quad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0, \infty),
\]
and
\[
\int_{\mathbb{R}^n} \nabla \psi\left(\frac{z-x}{t}\right) \cdot \frac{z-x}{t} \,dz_{t+n+1} + n \int_{\mathbb{R}^n} \psi\left(\frac{z-x}{t}\right) \,dz_{t+n+1} = 0 \quad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0, \infty),
\]
respectively, where “\( \cdot \)” stands for scalar product. Therefore,
\[
\nabla_x U(x,t) = \int_{\mathbb{R}^n} (u(z) - u(x))\nabla \psi\left(\frac{z-x}{t}\right) \,dz_{t+n+1} \quad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0, \infty),
\]
and
\[
U_t(x,t) = \int_{\mathbb{R}^n} (u(z) - u(x))\nabla \psi\left(\frac{z-x}{t}\right) \cdot \frac{z-x}{t} \,dz_{t+n+1} - n \int_{\mathbb{R}^n} (u(z) - u(x))\psi\left(\frac{z-x}{t}\right) \,dz_{t+n+1} \quad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0, \infty),
\]
where \( \nabla_x U \) and \( U_t \) denote the vector of the derivatives of \( U \) with respect to \( x \), and the derivative of \( U \) with respect to \( t \). Since
\[
\psi\left(\frac{z-x}{t}\right) \leq \frac{n+1}{\omega_n} \chi_{\{|z-x|<t\}}(z)
\]
and
\[
\left|\nabla \psi\left(\frac{z-x}{t}\right)\right| \leq \frac{n+1}{\omega_n} \chi_{\{|z-x|<t\}}(z), \quad \left|\nabla \psi\left(\frac{z-x}{t}\right) \cdot \frac{z-x}{t}\right| \leq \frac{n+1}{\omega_n} \chi_{\{|z-x|<t\}}(z),
\]
for a.e. \( (x,t) \in \mathbb{R}^n \times [0, \infty) \), we deduce from inequalities (5.17) and (5.18) that
\[
|\nabla U(x,t)| \leq |\nabla_x U(x,t)| + |U_t(x,t)| \leq \frac{(n+1)(n+2)}{\omega_n} \int_{\{|z-x|<t\}} |u(z) - u(x)| \,dz_{t+n+1} = \frac{(n+1)(n+2)}{t \omega_n} \int_{\{|y|<1\}} |u(x + ty) - u(x)|\,dy \quad \text{for a.e. } (x,t) \in \mathbb{R}^n \times [0, \infty),
\]
namely (5.13). On setting \( K = (n + 1)(n + 2) \), one has that
\[
(5.19) \quad \int_0^\infty \int_{\mathbb{R}^n} t^{-1} A \left( t^{1-s} |\nabla U(x,t)| \right) \, dx \, dt
\]
\[
\leq \int_0^\infty \int_{\mathbb{R}^n} t^{-1} A \left( \frac{K}{t^s \omega_n} \int_{\{|y| < 1\}} |u(x + ty) - u(x)| \, dy \right) \, dx \, dt
\]
\[
\leq \int_0^\infty \int_{\mathbb{R}^n} t^{-1} \frac{1}{\omega_n} \int_{\{|y| < 1\}} A \left( \frac{K|u(x + ty) - u(x)|}{t^s} \right) \, dy \, dx \, dt
\]
\[
= \frac{1}{\omega_n} \int_0^\infty \int_{\mathbb{R}^n} \int_{\{|y| < 1\}} A \left( \frac{K|u(x + z) - u(x)|}{t^s} \right) \, dx \, dy \, dz \, dt
\]
where the first inequality holds by (5.13), the second inequality by Jensen’s inequality (since \( \omega_n = |\{|y| < 1\}| \)), the first equality by Fubini’s theorem, the second equality by the change of variables \( z = ty \), and the last one by Fubini’s theorem again. Now, note that
\[
(5.20) \quad \int_t^\infty \tau^{-n-1} A \left( \frac{r}{\tau} \right) \, d\tau = \frac{r^{n-s}}{s} \int_0^\infty \tau^{n-s} A(\tau) \, d\tau = \frac{1}{s t^n} F \left( \frac{r}{t^s} \right) \quad \text{for } r, t > 0,
\]
where \( F \) is the Young function defined as
\[
F(t) = t^{-\frac{n}{s}} \int_0^t \tau^{-1} A(\tau) \, d\tau \quad \text{for } t > 0.
\]
We claim that
\[
(5.21) \quad F \text{ is equivalent to } A.
\]
Precisely, since the function \( A(t)/t \) is non-decreasing,
\[
(5.22) \quad F(t) \leq t^{-\frac{n}{s}} \frac{A(t)}{t} \int_0^t \tau^{-1} A(\tau) \, d\tau = \frac{s}{n + s} A(\tau) \quad \text{for } t > 0,
\]
and
\[
F(t) \geq t^{-\frac{n}{s}} \frac{A(t)}{t} \int_0^t \tau^{-1} A(\tau) \, d\tau \geq t^{-\frac{n}{s}} \frac{\frac{A(t)}{t}}{\frac{t}{2}} \int_0^t \tau^{-1} A(\tau) \, d\tau = \frac{2s}{n + s} \left( 1 - \frac{1}{2^{\frac{n}{s} + 1}} \right) A(t) \geq \frac{s}{n + s} A(t) \quad \text{for } t > 0.
\]
for \( t > 0 \). From (5.19), (5.20), (5.21) and (5.22) one obtains that
\[
(5.23) \quad \int_0^\infty \int_{\mathbb{R}^n} t^{-1} A \left( t^{1-s} |\nabla U(x,t)| \right) \, dx \, dt \leq \frac{1}{n \omega_n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( K \frac{|u(x) - u(y)|}{|x - y|^{1+s}} \right) \, dx \, dy \, dz
\]
Next,
\[
(5.24) \quad \frac{|u(x)|}{|x|} = \frac{|U(x,0)|}{|x|} \leq \frac{|U(x,t)|}{|x|} + \frac{1}{|x|} \int_0^t \left| \frac{\partial U}{\partial \tau} (x,\tau) \right| \, d\tau \quad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty).
\]
Observe that if \( G \) is a Young function, then
\[
(5.25) \quad \int_0^r \tau^{-1} G \left( \tau^{-1-s} \eta \right) \, d\tau = \frac{1}{1-s} \int_0^{r^{1-s} \eta} \frac{G(\tau)}{\tau} \, d\tau \geq \frac{1}{1-s} G \left( r^{1-s \eta}/2 \right) \quad \text{for } r, \eta > 0,
\]
where the last inequality holds since the function $G(\tau)/\tau$ is non-decreasing. Equation (5.25), applied with $G = \widehat{A}$, $r = |x|$, $\eta = \frac{|u(x)|}{|x|}$, and equation (5.24) yield

\begin{equation}
\widehat{A}\left(\frac{|u(x)|}{2|x|^s}\right) \leq (1-s) \int_0^{|x|} t^{-1} \widehat{A}\left(t^{1-s}\frac{|u(x)|}{|x|}\right) dt \\
\leq (1-s) \int_0^{|x|} t^{-1} \widehat{A}\left(t^{1-s}\frac{|U(x,t)|}{|x|}\right) dt + t^{1-s} \int_0^t \left| \frac{\partial U}{\partial \tau}(x,\tau) \right| d\tau \\
\leq (1-s) \int_0^{|x|} t^{-1} \widehat{A}\left(2t^{1-s}\frac{|U(x,t)|}{|x|}\right) dt \\
+ (1-s) \int_0^{|x|} t^{-1} \widehat{A}\left(2t^{1-s} \int_0^t \left| \frac{\partial U}{\partial \tau}(x,\tau) \right| d\tau \right) dt
\end{equation}

for $(x,t) \in \mathbb{R}^n \times (0, \infty)$.

Thus, owing Lemma 4.4 applied with $A$ replaced by $\widehat{A}$, and Proposition C

\begin{equation}
\int_{\mathbb{R}^n} \widehat{A}\left(\frac{|u(x)|}{2|x|^s}\right) dx \leq (1-s) \int_{\mathbb{R}^n} \int_0^{|x|} t^{-1} \widehat{A}\left(2t^{1-s}\frac{|U(x,t)|}{|x|}\right) dt dx \\
+ (1-s) \int_{\mathbb{R}^n} \int_0^\infty t^{-1} \widehat{A}\left(C t^{1-s} |\nabla U(x,t)|\right) dt dx.
\end{equation}

We now make use of polar coordinates $(\rho, \theta, \varphi)$ in $\mathbb{R}^n \times (0, \infty)$, with $\rho \in (0, \infty)$, $\theta \in (0, \frac{\pi}{2})$, $\varphi \in Q$, where $Q$ is a parallelepiped in $\mathbb{R}^{n-1}$. In particular, $\rho = \sqrt{|x|^2 + t^2}$ and $\cos \theta = t/\rho$. From Lemma 4.4 one infers that

\begin{equation}
\int_{\mathbb{R}^n} \int_0^{|x|} t^{-1} \widehat{A}\left(\frac{t^{1-s} |U(x,t)|}{\sqrt{2|x|^s+t^2}}\right) dt dx \leq \int_{\mathbb{R}^n} \int_0^\infty t^{-1} \widehat{A}\left(\frac{t^{1-s} |U(x,t)|}{\sqrt{|x|^2 + t^2}}\right) dt dx \\
= \int_Q \int_0^\infty \left(\cos \theta\right)^{-1} \rho^{-1} \widehat{A}\left(\cos \theta)^{1-s} \rho^{-s} |\widehat{U}(\rho, \theta, \varphi)|\right) \rho^n J(\theta, \varphi) d\rho d\theta d\varphi \\
\leq \int_Q \int_0^\infty \left(\cos \theta\right)^{-1} \rho^{-1} \widehat{A}\left(C(\cos \theta)^{1-s} \rho^{-1-s} \left| \frac{\partial \widehat{U}}{\partial \rho}(\rho, \theta, \varphi)\right| \right) \rho^n J(\theta, \varphi) d\rho d\theta d\varphi \\
\leq \int_{\mathbb{R}^n} \int_0^\infty t^{-1} A\left(C t^{1-s} |\nabla U(x,t)|\right) dt dx
\end{equation}

for some constant $C = C(n,s)$. Here, $\widehat{U}$ denotes the expression of $U$ in polar coordinates $(\rho, \varphi, \theta)$. Also, observe that the present application of Lemma 4.4 relies upon the equality

\begin{equation}
\widehat{U}(\rho, \theta, \varphi) = \int_\rho^\infty \frac{\partial \widehat{U}}{\partial \tau}(r, \theta, \varphi) dr
\end{equation}

for a.e. $(\theta, \varphi)$. Equality (5.30) holds provided that

\begin{equation}
\lim_{\rho \to \infty} \widehat{U}(\rho, \theta, \varphi) = 0 \quad \text{for } (\theta, \varphi) \in (0, \frac{\pi}{2}) \times Q.
\end{equation}

Equation (5.31) can be verified as follows. We have that

\begin{equation}
|U(x,t)| = \frac{1}{t^n} \int_{\mathbb{R}^n} \psi\left(\frac{y-x}{t}\right) u(y) dy \leq \frac{C}{t^n} \int_{|y-x|<t} |u(y)| dy
\end{equation}

for $(x,t) \in \mathbb{R}^n \times (0, \infty)$. 
for some constant $C = C(n)$. Thus, for each $(\theta, \varphi)$, there exists a constant $C = C(n, \theta)$ such that, for every $\varrho > 0$, there exists a ball $B_\varrho \subset \mathbb{R}^n$ of radius $\varrho$ satisfying

\begin{equation}
|U(\varrho, \theta, \varphi)| \leq \frac{C}{|B_\varrho|} \int_{B_\varrho} |u(y)| \, dy.
\end{equation}

Set

$$\mu(\tau) = |\{ |u| > \tau \}|$$

and observe that $\mu(\tau) < \infty$ for every $\tau > 0$, since $u \in \mathcal{M}_d(\mathbb{R}^n)$. Define the non-decreasing function $g : (0, \infty) \to [0, \infty)$ as

$$g(\tau) = \begin{cases} \frac{1}{\mu(\tau)} & \text{if } \tau \in (0, 1] \\ \frac{1}{\mu(1)} & \text{if } \tau \in (1, \infty). \end{cases}$$

Then the function $G : [0, \infty) \to [0, \infty)$, given by

$$G(\tau) = \int_0^\tau g(r) \, dr \quad \text{for } \tau \geq 0,$$

is a Young function such that $G(\tau) > 0$ for $\tau > 0$. Consequently,

\begin{equation}
\lim_{\tau \to 0} \frac{\tilde{G}(\tau)}{\tau} = 0.
\end{equation}

Now, we claim that $u \in L^G(\mathbb{R}^n)$. To verify this claim, note that

\begin{equation}
\int_{\mathbb{R}^n} G(|u|) \, dx = \int_0^\infty g(\tau) \mu(\tau) \, d\tau = \int_0^1 \tau \, d\tau + \frac{1}{\mu(1)} \int_1^\infty \mu(\tau) \, d\tau \leq 1 + \frac{1}{\mu(1)} \int_{|u| > 1} |u(y)| \, dy < \infty,
\end{equation}

where the last inequality holds by property (5.11). Thus, owing to equations (5.33), (5.11) and (5.34)

\begin{equation}
\lim_{\varrho \to \infty} \frac{1}{|B_\varrho|} \int_{B_\varrho} |u(y)| \, dy \leq \lim_{\varrho \to \infty} \frac{2C}{|B_\varrho|} \|u\|_{L^G(\mathbb{R}^n)} \|1\|_{L^{G^{-1}}(1/|B_\varrho|)} = 0,
\end{equation}

whence (5.31) follows. Inequalities (5.28) and (5.29) imply that

\begin{equation}
\int_{\mathbb{R}^n} \tilde{A} \left( \frac{|u(x)|}{|x|^s} \right) \, dx \leq (1 - s) \int_{\mathbb{R}^n} \int_0^\infty t^{-1} A(Ct^{1-s} |\nabla U(x,t)|) \, \mu(\tau) \, d\tau \, dt \, dx
\end{equation}

for some constant $C = C(n,s)$, with the property that $\lim_{s \to 1^-} C(n,s) < \infty$. Inequality (5.2) is a consequence of equations (5.23) and (5.37).

Inequality (5.1) can be deduced on applying inequality (5.2) with $u$ replaced by $u/\lambda$ for any $\lambda > 0$. \hfill \Box

6. Sobolev embeddings: case $s \in (0,1)$

The Orlicz-Sobolev embedding for the space $V_{d,A}^s(\mathbb{R}^n)$ of order $s \in (0,1)$, with optimal Orlicz target space, reads as follows.

**Theorem 6.1. [Optimal Orlicz target space]** Let $n \in \mathbb{N}$ and $s \in (0,1)$. Assume that $A$ is a Young function satisfying conditions (4.2) and (4.3), and let $A^\frac{1}{s} \tilde{A}$ be the Young function defined as in (4.4). Then,

\begin{equation}
V_{d,A}^s(\mathbb{R}^n) \to L^{A^\frac{1}{s} \tilde{A}}(\mathbb{R}^n),
\end{equation}

and there exists a constant $C = C(n,s)$ such that

\begin{equation}
\|u\|_{L^{A^\frac{1}{s} \tilde{A}}(\mathbb{R}^n)} \leq C|u|_{s,A,\mathbb{R}^n}
\end{equation}

for every function $u \in V_{d,A}^s(\mathbb{R}^n)$. Moreover, $L^{A^\frac{1}{s} \tilde{A}}(\mathbb{R}^n)$ is the optimal target space in inequality (6.2) among all Orlicz spaces.
A counterpart of embedding (6.1), with an improved target space which is optimal in the broader class of all rearrangement-invariant spaces, is stated in the next result.

**Theorem 6.2. [Optimal rearrangement-invariant target space]** Assume that \( n \in \mathbb{N}, s \in (0,1) \) and \( A \) are as in Theorem 6.1. Let \( \hat{A} \) be the Young function given by (4.7) and let \( L(\hat{A}, \frac{n}{s})/(\mathbb{R}^n) \) be the Orlicz-Lorentz space defined as in (2.27). Then

\[
V^{s,A}_{d}(\mathbb{R}^n) \rightarrow L(\hat{A}, \frac{n}{s})(\mathbb{R}^n),
\]

and there exists a constant \( C = C(n,s) \) such that

\[
\|u\|_{L(\hat{A}, \frac{n}{s})(\mathbb{R}^n)} \leq C|u|_{s,A,\mathbb{R}^n}
\]

for every function \( u \in V^{s,A}_{d}(\mathbb{R}^n) \). Moreover, \( L(\hat{A}, \frac{n}{s})(\mathbb{R}^n) \) is the optimal target space in inequality (6.3) among all rearrangement-invariant spaces.

We emphasize that assumption (4.3) on the Young function \( A \), appearing in Theorems 6.1 and 6.2, is necessary for an embedding of the space \( V^{s,A}_{d}(\mathbb{R}^n) \) to hold into any rearrangement-invariant space. This is the content of the following proposition.

**Proposition 6.3.** Let \( n \in \mathbb{N} \) and \( s \in (0,1) \), and let \( A \) be a Young function. Assume that

\[
V^{s,A}_{d}(\mathbb{R}^n) \rightarrow Y(\mathbb{R}^n)
\]

for some rearrangement-invariant space \( Y(\mathbb{R}^n) \). Then condition (4.3) is fulfilled.

**Example 6.4.** Let \( A \) be a Young function as in (5.3). Assume that the parameters \( p, p_0, \alpha \) and \( \alpha_0 \) fulfill conditions (5.4) and (5.5). Theorem 6.1 then tells us that embedding (6.1) and inequality (6.2) hold if

\[
A^{\frac{n}{s}}(t) \text{ is equivalent to } \begin{cases} \frac{n}{p_0} (\log \frac{t}{p_0})^{\frac{n}{n-p_0}} & \text{if } 1 \leq p_0 < \frac{n}{s} \\ e^{-t^{\frac{1}{s}}(\alpha_0+1)-n} & \text{if } p_0 = \frac{n}{s} \text{ and } \alpha_0 > \frac{n}{s} - 1 \end{cases} \text{ near zero,}
\]

and

\[
A^{\frac{n}{s}}(t) \text{ is equivalent to } \begin{cases} \frac{n}{p} (\log t)^{\frac{n}{n-p}} & \text{if } 1 \leq p < \frac{n}{s} \\ e^{-t^{\frac{1}{s}}(\alpha+1)-n} & \text{if } p = \frac{n}{s} \text{ and } \alpha < \frac{n}{s} - 1 \text{ near infinity.}
\]

Moreover, the target space in the resultant embedding and inequality is optimal among all Orlicz spaces.

In the special case when

\[
p = p_0 < \frac{n}{s} \text{ and } \alpha = \alpha_0 = 0,
\]

this recovers inequality (1.2) for the classical fractional space \( W^{s,p}(\mathbb{R}^n) \). In the borderline situation corresponding to

\[
p = p_0 = \frac{n}{s}, \quad \alpha = 0 \text{ and } \alpha_0 > \frac{n}{s} - 1,
\]

a fractional embedding of Pohozaev-Trudinger-Yudovich type [75, 84, 85] is established – see also the recent paper [71] in this connection.

On the other hand, Theorem 6.2 provides us with the optimal embedding (6.3) and inequality (6.4), with a Young function \( \hat{A} \) whose behaviour is described in (6.7) and (6.8). The specific choices (6.8) yield inequality (1.3) – a fractional extension of results of [69, 72] – since the Orlicz-Lorentz target space (6.3) coincides with the standard Lorentz space \( L^{\frac{n}{n-p},p}(\mathbb{R}^n) \) in this case. Also, when the parameters \( p, p_0, \alpha, \alpha_0 \) are as in (6.9), inequality (6.4) takes the form of a fractional inequality of Brezis-Wainger-Hansson type [19, 57].

Lemma 6.5 below is critical in the proof of the optimality of the target spaces in Theorems 6.1 and 6.2 and in the proof of Proposition 6.3.
Lemma 6.5. Let $n \in \mathbb{N}$ and $s \in (0, 1)$. Let $A$ be a Young function and let $Y(\mathbb{R}^n)$ be a rearrangement-invariant space. Assume that there exists a constant $C$ such that

$$
\|u\|_{Y(\mathbb{R}^n)} \leq C|u|_{s,A,\mathbb{R}^n}
$$

for every function $u \in V_d^{s,A}(\mathbb{R}^n)$. Then there exists a constant $C'$ such that

$$
\left\| \int_{\mathbb{R}} f(\varphi) \varphi^{-1+\frac{s}{n}} \, d\varphi \right\|_{Y(0,\infty)} \leq C'\|f\|_{L^A(0,\infty)}
$$

for every function $f \in L^A(0,\infty)$.

Remark 6.6. Under the assumption that the function $A$ fulfills conditions (4.2)–(4.3), a converse of Lemma 6.3 also holds. Namely, inequality (6.11) is a sufficient condition for inequality (6.10). To verify this assertion, recall from Theorem B that the space $L(\overline{\mathbb{A}}, \overline{n})(0, \infty)$ is optimal in inequality (6.4). Thereby, if inequality (6.11) holds, then $L(\overline{\mathbb{A}}, \overline{n})(0, \infty) \rightarrow Y(0, \infty)$. Hence, $L(\overline{\mathbb{A}}, \overline{n})(\mathbb{R}^n) \rightarrow Y(\mathbb{R}^n)$ as well, and inequality (6.10) is thus a consequence of (6.4).

Proof of Lemma 6.5. In what follows, the relation $\lesssim$ between two expressions will be used to denote that the former is bounded by the latter, up to a positive constant depending only on $n$ and $s$. The relation $\approx$ means that the two expressions are bounded by each other, up to positive constants depending only on $n$ and $s$. Assume that inequality (6.10) holds. Owing to [73] Theorem 1.1, inequality (6.11) holds for every function $f \in \mathcal{M}_+(0,\infty)$ if and only if it just holds for every non-increasing function $f : (0, \infty) \rightarrow [0, \infty)$. Thus it suffices to prove inequality (6.11) for this class of functions $f$. Given any $f$ of this kind, define the function $u : \mathbb{R}^n \rightarrow [0, \infty)$ as

$$
u(x) = \int_{\mathbb{R}^n} f(r) r^{-1+\frac{s}{n}} \, dr \quad \text{for } x \in \mathbb{R}^n.
$$

Let $x, y \in \mathbb{R}^n$. Suppose first that $|y| \geq 2|x|$. Then

$$
\frac{|u(x) - u(y)|}{|x - y|^s} = \frac{\int_{|\omega_n|x|^n} f(\varphi) \varphi^{-1+\frac{s}{n}} \, d\varphi - \int_{|\omega_n|y|^n} f(\varphi) \varphi^{-1+\frac{s}{n}} \, d\varphi}{|x - y|^s}
= \frac{\int_{|\omega_n|x|^n} f(\varphi) \varphi^{-1+\frac{s}{n}} \, d\varphi}{|x - y|^s} \lesssim \int_{|\omega_n|x|^n} f(\varphi) \varphi^{-1+\frac{s}{n}} \, d\varphi.
$$

Thus,

$$
(6.12) \quad \int_{\mathbb{R}^n} \int_{|y| \geq 2|x|} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dy}{|y|^n} \lesssim \int_{\mathbb{R}^n} \int_{|y| \geq 2|x|} A \left( C \int_{|\omega_n|x|^n} f(\varphi) \varphi^{-1+\frac{s}{n}} \, d\varphi \right) \frac{dy}{|y|^n} \, dx
\lesssim \int_{\mathbb{R}^n} \int_{|y| \geq 2|x|} A \left( C' f(\varphi) \right) r^{-1+\frac{s}{n}} \, d\varphi \frac{dy}{|y|^{n+s}} \, dx
\lesssim \int_{0}^{\infty} A \left( C' f(\varphi) \right) r^{-1+\frac{s}{n}} \int_{|\omega_n|x| < r} |x|^{-s} \, dx \, dr
\lesssim \int_{0}^{\infty} A \left( C' f(\varphi) \right) r^{-1+\frac{s}{n}} r^{1-\frac{s}{n}} \, dr \approx \int_{0}^{\infty} A \left( C' f(\varphi) \right) \, dr,
$$

for some positive constants $C$ and $C'$ depending on $n$ and $s$. In particular, the second inequality in chain (6.12) relies upon Jensen’s inequality.
Now assume that \(|x| \leq |y| \leq 2|x|\). Thereby,

\[
\frac{|u(x) - u(y)|}{|x - y|^s} \leq \frac{\int_{\omega_n} |f(r)| r^{-1 \frac{n}{s}} \, dr}{|x - y|^s} \lesssim f(\omega_n) \frac{|y|^s - |x|^s}{|x - y|^s} \lesssim f\left(\frac{\omega_n}{2^n} |y|^n\right) \frac{|y|^s - |x|^s}{|x - y|^s} \lesssim f\left(\frac{\omega_n}{2^n} |y|^n\right) \frac{|y|^s - |x|^s}{|x - y|^s} \lesssim f\left(\frac{\omega_n}{2^n} |y|^n\right) \frac{|x - y|}{|y|^{1-s}|x - y|^s} \lesssim f\left(\frac{\omega_n}{2^n} |y|^n\right) |x - y|^{1-s} |y|^{s-1}.
\]

Note that

\(|x - y|^{1-s} |y|^{s-1} \leq (|x|^{1-s} + |y|^{1-s}) |y|^{s-1} \leq 2.
\]

Thus, there exists a constant \(C = C(n, s)\) such that

\[
A\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \leq A\left(2Cf\left(\frac{\omega_n}{2^n} |y|^n\right) \frac{|x - y|}{|y|^{1-s}|x - y|^s} \right) \lesssim |x - y|^{1-s} |y|^{s-1} A\left(2Cf\left(\frac{\omega_n}{2^n} |y|^n\right)\right),
\]

where the last inequality holds thanks to property (2.2). Therefore,

\[
(6.13) \quad \int_{\mathbb{R}^n} \int_{\{x \leq |y| \leq 2|x|\}} A\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \, dx \, dy
\]

\[
\lesssim \int_{\mathbb{R}^n} \int_{\{x \leq |y| \leq 2|x|\}} A\left(2Cf\left(\frac{\omega_n}{2^n} |y|^n\right)\right) |y|^{s-1}|x - y|^{1-s-n} \, dy \, dx
\]

\[
\lesssim \int_{\mathbb{R}^n} A\left(2Cf\left(\frac{\omega_n}{2^n} |y|^n\right)\right) |y|^{s-1} \int_{\{|x| \leq |y|\}} |x - y|^{1-s-n} \, dx \, dy
\]

\[
\lesssim \int_{\mathbb{R}^n} A\left(2Cf\left(\frac{\omega_n}{2^n} |y|^n\right)\right) |y|^{s-1} \int_{\{|x| \leq |y|\}} |x - y|^{1-s-n} \, dx \, dy
\]

\[
\lesssim \int_{\mathbb{R}^n} A\left(2Cf\left(\frac{\omega_n}{2^n} |y|^n\right)\right) |y|^{s-1} |y|^{1-s} dy = \int_{\mathbb{R}^n} A\left(2Cf\left(\frac{\omega_n}{2^n} |y|^n\right)\right) dy
\]

\[
= \int_0^\infty A\left(2Cf\left(\frac{r}{2^n}\right)\right) \, dr \approx \int_0^\infty A\left(2Cf\left(r\right)\right) \, dr.
\]

Coupling inequality (6.12) with (6.13) yields

\[
\int_{\mathbb{R}^n} \int_{\{|y| \geq |x|\}} A\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \, dx \, dy \lesssim \int_0^\infty A\left(Cf\left(r\right)\right) \, dr
\]

for some positive constant \(C = C(n, s)\). Exchanging the roles of \(x\) and \(y\) tells us that

\[
\int_{\mathbb{R}^n} \int_{\{|x| \geq |y|\}} A\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \, dx \, dy \lesssim \int_0^\infty A\left(Cf\left(r\right)\right) \, dr.
\]

Altogether,

\[
(6.14) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \, dx \, dy \lesssim \int_0^\infty A\left(Cf\left(r\right)\right) \, dr
\]

for some constant \(C = C(n, s)\). On replacing \(f\) by \(f/\lambda\) for any \(\lambda > 0\) in inequality (6.14) one deduces that

\[
|u|_{s,A,\mathbb{R}^n} \leq C\|f\|_{L^A(0,\infty)}
\]

for some positive constant \(C = C(n, s)\). On the other hand,

\[
u^*(r) = \int_r^\infty f(\varrho) \varrho^{-1 + \frac{n}{s}} \, d\varrho \quad \text{for } r > 0,
\]

and

\[
\|u\|_{Y(\mathbb{R}^n)} = \|u^*\|_{\overline{Y}(0,\infty)}.
\]

Inequality (6.11) follows from equations (6.15)–(6.17). \(\square\)
Theorem 7.1. [Higher-order optimal Orlicz target space]

Let

\[ \text{Orlicz target space for embeddings of the space } V \]

be a Young function fulfilling conditions (4.2) and (4.3), and let \( A_n \) be the Young function defined as in (1.1) - (1.3). Then

\[ V^{s,A}(\mathbb{R}^n) \rightarrow L^{A_n}(\mathbb{R}^n), \]

and there exists a constant \( C \) such that

\[ \|u\|_{L^{A_n}(\mathbb{R}^n)} \leq C |\nabla^{[s]} u|_{\{s\}, A_n}. \]

Proof of Theorem 6.2. Inequality (3.16) ensures that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx}{|x - y|^n} \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u^*(x) - u^*(y)|}{|x - y|^s} \right) \frac{dx}{|x - y|^n}.
\]

Since

\[
\int_{\mathbb{R}^n} \tilde{A} \left( \frac{|u^*(x)|}{|x|^s} \right) dx = \int_0^\infty \tilde{A} \left( \frac{\omega_{n}^s |u^*(r)|}{r^s} \right) dr,
\]

an application of inequality (5.2) to the function \( u^* \) yields

\[
\int_0^\infty \tilde{A} \left( C \frac{|u^*(r)|}{r^s} \right) dr \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u^*(x) - u^*(y)|}{|x - y|^s} \right) \frac{dx}{|x - y|^n}
\]

for a suitable positive constant \( C = C(n,s) \). From inequalities (6.18) and (6.20) we deduce that

\[
\int_0^\infty \tilde{A} \left( C \frac{|u^*(r)|}{r^s} \right) dr \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx}{|x - y|^n}.
\]

Inequality (6.21) is a version of (6.4) in integral form. Inequality (6.4) follows on applying (6.21) with \( u \) replaced by \( u/\lambda \) for any \( \lambda > 0 \).

Proof of Proposition 6.3. Assume that embedding (6.5) holds for some rearrangement-invariant space \( Y(\mathbb{R}^n) \), namely that inequality (6.10) holds. Then, by Proposition 6.5, inequality (6.11) holds as well. A necessary condition for one-dimensional Hardy type inequalities – a dual version of [37, Lemma 1], see e.g. [33, Lemma 2] – implies that

\[
\sup_{r > 0} \|1\|_{Y(0,r)} \| e^{-1+\frac{s}{n}} \|_{L^{A_n}(r,\infty)} < \infty.
\]

Computations show that the second norm on the left-hand side of equation (6.22) is finite for any \( r > 0 \) if and only if

\[
\int_0^\infty \frac{\tilde{A}(t) dt}{t^{1+\frac{s}{n}} < \infty},
\]

see e.g. [30, Lemma 3]. Condition (6.23) is in its turn equivalent to (6.5) – see [35, Lemma 2.3].

7. Sobolev embeddings: case \( s > 1 \)

The results of the previous section are extended here to any fractional-order power \( s \in (0,n) \). The optimal Orlicz target space for embeddings of the space \( V^s,A(\mathbb{R}^n) \) is exhibited in the following theorem.

Theorem 7.1. [Higher-order optimal Orlicz target space] Let \( n \in \mathbb{N} \) and \( s \in (0,n) \setminus \mathbb{N} \). Assume that \( A \) is a Young function fulfilling conditions (1.2) and (1.3), and let \( A_n \) be the Young function defined as in (1.1) - (1.3). Then

\[
V^s,A(\mathbb{R}^n) \rightarrow L^{A_n}(\mathbb{R}^n),
\]

and there exists a constant \( C \) such that

\[
\|u\|_{L^{A_n}(\mathbb{R}^n)} \leq C |\nabla^{[s]} u|_{\{s\}, A_n}.
\]
for every function \( u \in V_{d}^{s,A}(\mathbb{R}^n) \). Moreover, \( L^A\frac{p}{s}(\mathbb{R}^n) \) is the optimal target space in inequality (7.2) among all Orlicz spaces.

The next result enhances Theorem 7.1 and provides us with the optimal rearrangement-invariant target space for embeddings of \( V_{d}^{s,A}(\mathbb{R}^n) \).

**Theorem 7.2. [Higher-order optimal rearrangement-invariant target space]** Let \( n, s \) and \( A \) be as in Theorem 7.1. Let \( \hat{A} \) be the Young function defined as in (1.7)–(1.8), and let \( L(\hat{A}, \frac{p}{s})(\mathbb{R}^n) \) be the Orlicz-Lorentz space equipped with the norm given by (4.9). Then

\[
V_{d}^{s,A}(\mathbb{R}^n) \to L(\hat{A}, \frac{p}{s})(\mathbb{R}^n),
\]

and there exists a constant \( C \) such that

\[
\|u\|_{L(\hat{A}, \frac{p}{s})(\mathbb{R}^n)} \leq C|\nabla|^{s}|u|_{s,A,\mathbb{R}^n}
\]

for every function \( u \in V_{d}^{s,A}(\mathbb{R}^n) \). Moreover, \( L(\hat{A}, \frac{p}{s})(\mathbb{R}^n) \) is the optimal target space in inequality (7.1) among all rearrangement-invariant spaces.

**Remark 7.3.** Assumption (4.3) on the Young function \( A \) in Theorems 7.1 and 7.2 is necessary for an embedding of the form

\[
V_{d}^{s,A}(\mathbb{R}^n) \to Y(\mathbb{R}^n)
\]

to hold for some rearrangement-invariant space, also if \( s \in (0,n) \setminus \mathbb{N} \). Indeed, Proposition 6.3 continues to hold for \( s \) in this range, with a completely analogous proof which makes use of Lemma 7.6 below, a higher-order analogue of Lemma 6.5.

As a consequence of Theorem 7.2, we can derive a higher-order version of the Hardy type inequality (5.1).

**Theorem 7.4. [Higher-order fractional Orlicz–Hardy inequality]** Let \( n, s \) and \( \hat{A} \) be as in Theorem 7.2. Then there exists a constant \( C \) such that

\[
\left\| \frac{|u(x)|}{|x|^s} \right\|_{L(\hat{A}, \frac{p}{s})(\mathbb{R}^n)} \leq C|\nabla|^{s}|u|_{s,A,\mathbb{R}^n}
\]

for every function \( u \in V_{d}^{s,A}(\mathbb{R}^n) \).}

**Example 7.5.** Assume that \( A \) is a Young function as in (5.3), with parameters \( p, p_0, \alpha \) and \( \alpha_0 \) fulfilling conditions (5.4) and (5.5). Then Theorem 7.1 yields embedding (7.1) and inequality (7.2) for every \( s \in (0,n) \setminus \mathbb{N} \), where the Young function \( A_2 \) obeys equations (6.6) and (6.7). For the same range of exponents \( s \), Theorem 7.2 tells us that embedding (7.3) and inequality (7.4) hold, where the Young function \( \hat{A} \) fulfills (5.6) and (5.7). Moreover, the target spaces in the relevant embeddings and inequalities are optimal in their respective classes.

The special choices of the parameters \( p, p_0, \alpha, \alpha_0 \) as in (6.8) or (6.9) produce higher-order versions of inequalities (1.2) and (1.3), or of their limiting versions in the spirit of Pohozaev-Trudinger-Yudovich and Brezis-Wainger-Hansson, respectively.

The proof of the optimality of the target spaces in Theorems 7.1 and 7.2 relies up the following lemma.

**Lemma 7.6.** Let \( n \in \mathbb{N} \) and \( s \in (0,n) \setminus \mathbb{N} \). Let \( A \) be a Young function and let \( Y(\mathbb{R}^n) \) be a rearrangement-invariant space. Assume that there exists a constant \( C \) such that

\[
\|u\|_{Y(\mathbb{R}^n)} \leq C|\nabla|^{s}|u|_{s,A,\mathbb{R}^n}
\]

for every function \( u \in V_{d}^{s,A}(\mathbb{R}^n) \). Then there exists a constant \( C' \) such that

\[
\left\| \int_{r}^{\infty} f(q)q^{-1+s} \frac{dq}{q} \right\|_{Y(0,\infty)} \leq C'|f|_{L^{A}(0,\infty)}
\]

for every function \( f \in L^{A}(0,\infty) \).
Remark 7.7. The implication of Lemma 7.6 can be reversed if the function $A$ satisfies conditions (1.2) and (4.3). This has been pointed out in Remark 6.6 for the case when $s \in (0, 1)$. The argument supporting this assertion is completely analogous to that presented in that remark.

An algebraic inequality to be used in the proof of Lemma 7.6 is the subject of the next result.

**Lemma 7.8.** Let $x, y \in \mathbb{R}^n$, $n \geq 1$ be such that $0 < |x| \leq |y| \leq 2|x|$. Let $i \in \mathbb{N} \cup \{0\}$ and let $\beta \in \mathbb{R}$ be such that $i \leq n$ and $\beta + i \geq 0$. Assume that $\alpha_1, \alpha_2, \ldots, \alpha_i \in \{1, 2, \ldots, n\}$. Then

$$(7.8) \quad |x_{\alpha_1} \cdots x_{\alpha_i}| |x|^\beta - y_{\alpha_1} \cdots y_{\alpha_i}| |y|^\beta \lesssim |x - y||x|^{\beta + i - 1},$$

up to a multiplicative constant depending on $i$ and $\beta$. Here, the products $x_{\alpha_1} \cdots x_{\alpha_i}$ and $y_{\alpha_1} \cdots y_{\alpha_i}$ have to be interpreted as $1$ if $i = 0$.

**Proof.** Fix $x$ and $y$ as in the statement. If $i = 0$, then inequality (7.8) reads

$$||x|^\beta - |y|^\beta| \lesssim |x - y||x|^{\beta - 1}.$$  

This inequality holds, for instance, as a consequence of the mean value theorem for functions of several variables. Let us now assume that $i \geq 1$. Notice that inequality (7.8) will follow if we show that

$$(7.9) \quad \sum_{\alpha_1=1}^n \cdots \sum_{\alpha_i=1}^n (x_{\alpha_1} \cdots x_{\alpha_i} |x|^\beta - y_{\alpha_1} \cdots y_{\alpha_i} |y|^\beta)^2 \leq C|x - y|^2 |x|^{2(\beta + i - 1)}$$

for some constant $C > 0$ depending on $i$ and $\beta$. Thanks to homogeneity of inequality (7.9), we may assume, without loss of generality, that $|x| = 1$, and hence that $1 \leq |y| \leq 2$. Inequality (7.9) then turns into

$$\sum_{\alpha_1=1}^n \cdots \sum_{\alpha_i=1}^n (x_{\alpha_1} \cdots x_{\alpha_i} - y_{\alpha_1} \cdots y_{\alpha_i} |y|^\beta)^2 \leq C|x - y|^2,$$

which can be rewritten as

$$|x|^{2i} - 2|y|^\beta (x \cdot y)^i + |y|^{2\beta + 2i} \leq C(|x|^2 - 2x \cdot y + |y|^2),$$

namely

$$(7.10) \quad 1 - 2|y|^\beta (x \cdot y)^i + |y|^{2\beta + 2i} \leq C(1 - 2x \cdot y + |y|^2),$$

since we are assuming that $|x| = 1$.

Inequality (7.10) can, in its turn, be rewritten as

$$1 + |y|^{2\beta + 2i} + 2x \cdot y(C - |y|^\beta (x \cdot y)^i - 1) \leq C(1 + |y|^2).$$

Observe that $|x \cdot y| \leq |x||y| = |y| \leq 2$. Furthermore, inasmuch as $[-|y|, |y|] \subseteq [-2, 2]$, the function

$$[-|y|, |y|] \ni r \mapsto r(C - |y|^\beta r^{i - 1})$$

is non-decreasing, provided that $C > i2^{\beta + i - 1}$. Altogether, we deduce that

$$x \cdot y(C - |y|^\beta (x \cdot y)^i - 1) \leq |y|(C - |y|^\beta + i - 1).$$

It thus suffices to show that

$$1 + |y|^{2\beta + 2i} + 2|y|(C - |y|^\beta + i - 1) \leq C(1 + |y|^2),$$

or, equivalently,

$$(|y|^{\beta + i} - 1)^2 \leq C(|y| - 1)^2,$$

namely,

$$|y|^{\beta + i} - 1 \leq \sqrt{C}(|y| - 1).$$

This inequality clearly holds if $1 \leq |y| \leq 2$, provided that $C$ is sufficiently large, depending on $\beta$ and $i$. $\square$
Proof of Lemma 7.6. We focus on the case when \( s \in (1, \infty) \setminus \mathbb{N} \), and hence \( n \geq 2 \), since the result for \( s \in (0, 1) \) has already been proved in Lemma 6.5. Assume that inequality (7.6) holds. By the reason explained in the proof of Lemma 6.5 in connection with inequality (6.11), it suffices to prove inequality (7.7) for every non-increasing function \( f : (0, \infty) \to [0, \infty) \). Moreover, on replacing, if necessary, \( f \) by \( f \chi_{(0,L]} \) for \( L > 0 \), and letting \( L \to \infty \), we may also assume that \( f \) has a bounded support. Passage to the limit as \( L \to \infty \) in inequality (7.7) applied with \( f \) replaced by \( f \chi_{(0,L]} \) is legitimate owing to the Fatou property of rearrangement-invariant norms [3, Chapter 1, Definition 1.1]. Denote, for simplicity, \([s] = m\). Given any function \( f \) as above, define the function \( u : \mathbb{R}^n \to [0, \infty) \) as

\[
(7.11) \quad u(x) = \int_{\omega_n|x|^n}^\infty \cdots \int_{r_m}^\infty f(r_{m+1})r_{m+1}^{-m+1} \frac{dr_{m+1}}{2\omega_n|x|^n} \cdots dr_1 \quad \text{for } x \in \mathbb{R}^n.
\]

It is easily verified that \( u \) is \( m \)-times weakly differentiable, and that \(|\{ |\nabla^k u| > t \}| < \infty\) for every \( k = 0, 1, \ldots, m \) and every \( t > 0 \). From Fubini's theorem, one can deduce that

\[
(7.12) \quad u(x) = \frac{1}{m!} \int_{\omega_n|x|^n}^\infty \cdots \int_{r_m}^\infty f(r)r^{-m+1+\frac{1}{n}}(r - \omega_n|x|^n)^m \, dr \geq \int_{2\omega_n|x|^n}^\infty f(r)r^{-1+\frac{1}{n}} \, dr \quad \text{for } x \in \mathbb{R}^n.
\]

Throughout this proof, the relations \( \geq, \leq \) and \( \approx \) hold up to multiplicative constants depending on \( n, m \) and \( s \). The same dependence concerns all constants appearing explicitly. Inequality (7.12), combined with the boundedness of the dilatation operator on rearrangement-invariant spaces, implies that

\[
(7.13) \quad \|u\|_{Y(\mathbb{R}^n)} \geq \left| \int_{2\omega_n|x|^n}^\infty f(r)r^{-1+\frac{1}{n}} \, dr \right|_{Y(\mathbb{R}^n)} = \left| \int_{2r}^\infty f(r)\rho^{-1+\frac{1}{n}} \, d\rho \right|_{Y(0,\infty)} \geq \left| \int_{r}^\infty f(r)\rho^{-1+\frac{1}{n}} \, d\rho \right|_{Y(0,\infty)}
\]

for any rearrangement-invariant space \( Y(\mathbb{R}^n) \). Let \( g : (0, \infty) \to (0, \infty) \) be the function defined as

\[
g(r) = \frac{1}{m!} \int_{\omega_n|y|^n}^\infty f(\rho)\rho^{-m+1+\frac{1}{n}}(\rho - \omega_n|y|^n)^m \, d\rho \quad \text{for } r > 0.
\]

One can verify (see [1] Proof of Theorem 3.1)] that any \( m \)-th order derivative of \( u \) is a linear combination of terms of the form

\[
\frac{x_{a_1} \cdots x_{a_{k-1}} g^{(k)}(|x|)}{|x|^{m-k+1}},
\]

where \( i = 0, 1, \ldots, m, \ k = 1, \ldots, m \) and \( a_1, \ldots, a_i \in \{1, \ldots, n\} \). Furthermore, \( g^{(k)}(r) \) is a linear combination of functions of the form

\[
r_{j-1} \int_{r_{j-1}}^\infty \cdots \int_{r_m}^\infty \frac{1}{\omega_{n+1}r^n} \cdots \int_{r_m}^\infty \frac{1}{\omega_{n+1}r^n} \cdots \int_{r_m}^\infty f(r_{m+1})r_{m+1}^{-m+1+\frac{1}{n}} \, dr_{m+1} \cdots dr_j.
\]

where \( j = 1, \ldots, k \). Note that, if \( j = k = m \), then the last expression has to be interpreted as

\[
r_m \int_{\omega_{n+1}r^n}^\infty \frac{1}{\omega_{n+1}r^n} \cdots \int_{r_m}^\infty \frac{1}{\omega_{n+1}r^n} \cdots \int_{r_m}^\infty f(r_{m+1})r_{m+1}^{-m+1+\frac{1}{n}} \, dr_{m+1}.
\]

Altogether, we deduce that any \( m \)-th order derivative of \( u \) agrees with a linear combination of terms of the form

\[
x_{a_1} \cdots x_{a_i} |x|^{j-n-i} \int_{\omega_n|x|^n}^\infty \cdots \int_{r_m}^\infty \frac{1}{\omega_{n+1}r^n} \cdots \int_{r_m}^\infty f(r_{m+1})r_{m+1}^{-m+1+\frac{1}{n}} \, dr_{m+1} \cdots dr_j,
\]

where \( i = 0, 1, \ldots, m, \ j = 1, \ldots, m \) and \( a_1, \ldots, a_i \in \{1, \ldots, n\} \).

For a.e. \( x, y \in \mathbb{R}^n \) we have what follows. Assume first that \(|y| \geq 2|x|\). Then

\[
(7.14) \quad |\nabla^m u(x) - \nabla^m u(y)| \leq |\nabla^m u(x)| + |\nabla^m u(y)| \leq \sum_{j=1}^{m} |x|^{j-n-m} \int_{\omega_n|x|^n}^\infty \cdots \int_{r_m}^\infty f(r_{m+1})r_{m+1}^{-m+1+\frac{1}{n}} \, dr_{m+1} \cdots dr_j.
\]
where the third inequality follows via an analogue of equation \[7.12\], and the last one thanks to the monotonicity of \(f\). Thus,

\[
\int_{\mathbb{R}^n} \int_{|y| \geq 2|x|} A \left( \frac{\nabla^m u(x) - \nabla^m u(y)}{|x - y|^{s-m}} \right) \, dx \, dy \\
\leq \int_{\mathbb{R}^n} \int_{|y| \geq 2|x|} A \left( C |x|^{s-m} |y|^{m-s} f(\omega_n |x|^n) \right) \frac{dy}{|y|^n} dx + \int_{\mathbb{R}^n} \int_{|y| \geq 2|x|} A \left( C f(\omega_n |y|^n) \right) dx \, dy
\]

for some constants \(C\) and \(C'\). The change of variables \(t = C'r^{m-s} |x|^{s-m} f(\omega_n |x|^n)\) yields

\[
\int_{|2|x|}^{\infty} A \left( C'r^{m-s} |x|^{s-m} f(\omega_n |x|^n) \right) \frac{dr}{r} dx \approx \int_0^{\infty} A(t) \frac{dt}{t} \leq A \left( C' f(\omega_n |x|^n) \right),
\]

where the last inequality holds due to the monotonicity of the function \(t \mapsto A(t)/t\). Therefore,

\[
\int_{\mathbb{R}^n} \int_{|y| \geq 2|x|} A \left( \frac{\nabla^m u(x) - \nabla^m u(y)}{|x - y|^{s-m}} \right) \, dx \, dy \leq \int_{\mathbb{R}^n} A \left( C f(\omega_n |x|^n) \right) dx = \int_0^{\infty} A(\omega_r) \, dr
\]

for some constant \(C\).

Let us now assume that \(|x| \leq |y| \leq 2|x|\). Then

\[
\frac{|\nabla^m u(x) - \nabla^m u(y)|}{|x - y|^{s-m}}
\]

is dominated by

\[
\sum_{i=0}^{m} \sum_{(a_1, \ldots, a_i) \in \{1, \ldots, n\}^i} \sum_{j=1}^{m} |x_{a_1} \cdots x_{a_i} | |x|^{j_n-m-i} \int_{\omega_n |x|^n} \int_{r_{j+1}}^{\infty} \cdots \int_{r_m}^{\infty} f(r_{m+1}) r_m^{-1 + \frac{s}{n}} dr_{m+1} \cdots dr_j + \frac{m}{2|x|} |x|^{j_n-m-i} \int_{\omega_n |y|^n} \int_{r_{j+1}}^{\infty} \cdots \int_{r_m}^{\infty} f(r_{m+1}) r_m^{-1 + \frac{s}{n}} dr_{m+1} \cdots dr_j
\]

for some constant \(C\).
\[
\sum_{j=1}^{m} |x|^{j_{n-m}} \int_{\omega_{n}|x|^{n}} \int_{r}^{\infty} f(q) q^{-j-2+\frac{n}{s}} dq dr + \sum_{j=1}^{m} \int_{\omega_{n}|x|^{n}} f(r) r^{-j-1+\frac{n}{s}} dr \leq |y-x||x|^{-m+\frac{s}{n}} f(\omega_{n}|x|^{n}).
\]

Observe that the third inequality holds owing to Lemma 7.8, the fourth one by an analogue of equation (7.12), and the last one since \( f \) is non-increasing. Therefore,

\[
\int_{\mathbb{R}^{n}} \int_{|x|\leq|y|\leq2|x|} A \left( \frac{|\nabla^{m} u(x) - \nabla^{m} u(y)|}{|x-y|^{s-m}} \right) \frac{dx dy}{|x-y|^{n}} \leq \int_{\mathbb{R}^{n}} \int_{|x|\leq|y|\leq2|x|} A \left( C |x|^{-s-m} |y-x|^{m-s+1} f(\omega_{n}|x|^{n}) \right) \frac{dy}{|y-x|^{n}} dx
\]

\[
= \int_{\mathbb{R}^{n}} \int_{|x|\leq|z+x|\leq2|x|} A \left( C |x|^{-s-m} |z|^{m-s+1} f(\omega_{n}|x|^{n}) \right) \frac{dz}{|z|^{n}} dx
\]

\[
\leq \int_{\mathbb{R}^{n}} \int_{|x|\leq3|x|} A \left( C |x|^{-s-m} |z|^{m-s+1} f(\omega_{n}|x|^{n}) \right) \frac{dz}{|z|^{n}} dx
\]

\[
\leq \int_{\mathbb{R}^{n}} \int_{0}^{3|x|} A \left( C |x|^{-s-m} r^{m-s+1} f(\omega_{n}|x|^{n}) \right) \frac{dr}{r} dx
\]

for some positive constant \( C \). The change of variables \( t = C |x|^{-s-m} f(\omega_{n}|x|^{n}) r^{m-s+1} \) yields

\[
\int_{0}^{3|x|} A \left( C |x|^{-s-m} r^{m-s+1} f(\omega_{n}|x|^{n}) \right) \frac{dr}{r} \approx \int_{0}^{C f(\omega_{n}|x|^{n})} A(t) \frac{dt}{t} \leq A(C f(\omega_{n}|x|^{n})�\).
\]

Thereby,

\[
(7.16) \int_{\mathbb{R}^{n}} \int_{|x|\leq|y|\leq2|x|} A \left( \frac{|\nabla^{m} u(x) - \nabla^{m} u(y)|}{|x-y|^{s-m}} \right) \frac{dx dy}{|x-y|^{n}} \leq \int_{\mathbb{R}^{n}} A(C f(\omega_{n}|x|^{n})) dx = \int_{0}^{C f(\omega_{n}|x|^{n})} A(C f(\omega_{n}|x|^{n})) \frac{dr}{r} dx.
\]

For some positive constant \( C \). Adding inequality (7.17) to a parallel inequality obtained by exchanging the roles of \( x \) and \( y \), and applying the resultant inequality with \( f \) replaced by \( f/\lambda \) for any \( \lambda > 0 \) yield

\[
\int_{\mathbb{R}^{n}} \int_{|x|\leq|y|} A \left( \frac{|\nabla^{m} u(x) - \nabla^{m} u(y)|}{|x-y|^{s-m}} \right) \frac{dx dy}{|x-y|^{n}} \leq \int_{0}^{C f(\omega_{n}|x|^{n})} A(C f(\omega_{n}|x|^{n})) \frac{dr}{r} dx
\]

for some constant \( C \). Now recall that \( m = [s] \) and \( s - m = \{s\} \) to infer that

\[
(7.18) |\nabla^{[s]} u|_{s,A,\mathbb{R}^{n}} \lesssim \|f\|_{L^{A}(0,\infty)}.
\]

Combining inequalities (7.13) and (7.18), shows that (7.16) implies (7.7).

We have now all the preliminaries at our disposal to accomplish the proof of Theorem 7.2.

**Proof of Theorem 7.2*** In this proof we need to make use of the function \( \widehat{A} \), defined as in (4.7)–(4.8), also with \( s \) replaced by \( \{s\} \). For clarity of notation, we shall denote by \( \widehat{A}_{s} \) and \( \widehat{A}_{\{s\}} \) the functions defined by (4.7)–(4.8) with \( s \) and \( \{s\} \), respectively. Theorem 6.2 applied with \( u \) replaced by \( \nabla^{[s]} u \), and with \( s \) replaced by \( \{s\} \), tells us that

\[
(7.19) \|\nabla^{[s]} u\|_{L^{(\widehat{A}_{s},\{s\})}}(\mathbb{R}^{n}) \leq C |\nabla^{[s]} u|_{s,A,\mathbb{R}^{n}}
\]
for some constant $C$ and for every function $u \in V^s,A_d(\mathbb{R}^n)$. Inequality (7.21) will thus follow if we prove inequality (7.20), we make use of Theorems B and E. By Theorem B,

$$\|T_{\{s\}}f\|_{L(\hat{\Lambda}_s, \frac{n}{m-1})} \leq C \|f\|_{L^A(0, \infty)},$$

for some constant $C$ and for every function $f \in L^A(0, \infty)$, where $T_{\{s\}}$ is the operator defined as in (4.1). Furthermore, $L(\hat{\Lambda}_s, \frac{n}{m-1})$ is the optimal rearrangement-invariant target space in (7.21). The same result also tells us that

$$\|T_{\{s\}}f\|_{L(\hat{\Lambda}_s, \frac{n}{m})} \leq C \|f\|_{L^A(0, \infty)},$$

for some constant $C$ and every function $f \in L^A(0, \infty)$, and that $L(\hat{\Lambda}_s, \frac{n}{m})$ is the optimal rearrangement-invariant target space in the inequality

$$\|T_{\{s\}}f\|_\infty \leq C \|f\|_{L(\hat{\Lambda}_s, \frac{n}{m-1})} \|\nabla f\|_{L^\infty(0, \infty)}$$

for some constant $C$ and for every function $f \in L(\hat{\Lambda}_s, \frac{n}{m-1})$. Hence, since $s = [s] + \{s\}$, from Theorem E, part (ii), one can deduce that

$$X_{\{s\}}(0, \infty) = L(\hat{\Lambda}_s, \frac{n}{m})(0, \infty).$$

Therefore,

$$\|T_{\{s\}}f\|_{L(\hat{\Lambda}_s, \frac{n}{m})} \leq C \|f\|_{L(\hat{\Lambda}_s, \frac{n}{m-1})} \|\nabla f\|_{L^\infty(0, \infty)}$$

for some constant $C$ and for every function $f \in L(\hat{\Lambda}_s, \frac{n}{m-1})$. Owing to the reduction principle for integer-order Sobolev inequalities in the version of Theorem 3.3, inequality (7.25) implies inequality (7.20).

The optimality of the space $L(\hat{\Lambda}_s, \frac{n}{m})(\mathbb{R}^n)$ in inequality (7.4) follows from Lemma 7.6 and Theorem B. □

Proof of Theorem 7.7. Inequality (7.2) follows from Theorem 7.2 and Proposition 4.11. The optimality of the space $L^A(\Omega)$ is a consequence of Lemma 7.6 and Theorem A. □

Proof of Theorem 7.4. Property (2.11) ensures that

$$\|\frac{\mid u(x) \mid}{\mid x \mid^s} \|_{L^\ast} \leq \|\omega^\frac{n}{m} r^{-\frac{n}{m}} u^\ast(r) \|_{L^A(0, \infty)} = \omega^\frac{n}{m} \|u\|_{L(\hat{\Lambda}_s, \frac{n}{m})}.$$ (7.4)

Coupling inequality (7.26) with (7.4) yields (7.5). □

8. Embeddings on Domains

So far, we have been dealing with embeddings and corresponding Sobolev–Poincaré inequalities for functions defined in the whole of $\mathbb{R}^n$. This section is devoted to their counterparts for fractional Orlicz-Sobolev spaces on open subsets of $\mathbb{R}^n$.

The open sets that will be considered are bounded Lipschitz domains according to the following definition. If $n \geq 2$, we make use of the notation $x = (x', x_n)$ for $x \in \mathbb{R}^n$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$, and set $Q = \{ x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < 1, |x_n| < 1 \}$, $Q_+ = \{ x \in Q : x_n > 0 \}$ and $Q_0 = \{ x \in Q : x_n = 0 \}$. A set $\Omega \subset \mathbb{R}^n$ is called a bounded Lipschitz domain if it is a bounded open set and there exists a finite number of balls $\{B_j\}_{j=1}^k$ such that $\partial^c \Omega \subset \bigcup_{j=1}^k B_j$, and corresponding Lipschitz continuous homeomorphisms with Lipschitz continuous inverses $T_j : Q \to B_j$, such that $T_j(Q_+) = B_j \cap \Omega$ and $T_j(Q_0) = B_j \cap \partial^c \Omega$. If $n = 1$, then a bounded Lipschitz domain $\Omega \subset \mathbb{R}$ is just the union of a finite family of bounded intervals at positive distance from each other.
As in the case of embedding in $\mathbb{R}^n$, we premise our results in the basic case of spaces of order $s \in (0, 1)$.

**Theorem 8.1.** [Optimal embeddings of order $s \in (0, 1)$ on domains] Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Assume that $s \in (0, 1)$ and that $A$ is a Young function satisfying conditions (4.2) – (4.3).

(i) One has that

$$W^{s, A}(\Omega) \rightarrow L^{A^{\frac{n}{s}}}(\Omega),$$

and $L^{A^{\frac{n}{s}}}(\Omega)$ is the optimal Orlicz target space in (8.1). Moreover, there exists a constant $C = C(n, s, \Omega)$ such that

$$\|u\|_{L^{A^{\frac{n}{s}}}(\Omega)} \leq C|u|_{s, A, \Omega}$$

for every function $u \in V^{s, A}_\perp(\Omega)$.

(ii) One has that

$$W^{s, A}(\Omega) \rightarrow L(\hat{A}, \frac{n}{s})(\Omega),$$

and $L(\hat{A}, \frac{n}{s})(\Omega)$ is the optimal rearrangement-invariant target space in (8.3). Moreover, there exists a constant $C = C(n, s, \Omega)$ such that

$$\|u\|_{L(\hat{A}, \frac{n}{s})(\Omega)} \leq C|u|_{s, A, \Omega}$$

for every function $u \in V^{s, A}_\perp(\Omega)$.

Optimal arbitrary-order fractional Orlicz-Sobolev embeddings on domains are stated in the next theorem.

**Theorem 8.2.** [Higher-order optimal embeddings on domains] Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Assume that $s \in (0, n) \setminus \mathbb{N}$ and that $A$ is a Young function satisfying conditions (4.2) – (4.3).

(i) One has that

$$W^{s, A}(\Omega) \rightarrow L^{A^{\frac{n}{s}}}(\Omega),$$

and $L^{A^{\frac{n}{s}}}(\Omega)$ is the optimal Orlicz target space in (8.5). Moreover, there exists a constant $C$ such that

$$\|u\|_{L^{A^{\frac{n}{s}}}(\Omega)} \leq C|\nabla^{[s]}u|_{s, A, \Omega}$$

for every $u \in V^{s, A}_\perp(\Omega)$.

(ii) One has that

$$W^{s, A}(\Omega) \rightarrow L(\hat{A}, \frac{n}{s})(\Omega),$$

and $L(\hat{A}, \frac{n}{s})(\Omega)$ is the optimal rearrangement-invariant target space in (8.7). Moreover, there exists a constant $C$ such that

$$\|u\|_{L(\hat{A}, \frac{n}{s})(\Omega)} \leq C|\nabla^{[s]}u|_{s, A, \Omega}$$

for every $u \in V^{s, A}_\perp(\Omega)$.

**Example 8.3.** Assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$. Let $s \in (0, n) \setminus \mathbb{N}$. Consider a Young function $A$ as in (5.3) under assumptions (5.4) and (5.5) on the parameters $p, p_0, \alpha, \alpha_0$. By Theorem 8.2, Part (i), embedding (8.5) holds with $A^{\frac{n}{s}}$ fulfilling condition (6.7). Note that, since $|\Omega| < \infty$, only the behaviour near infinity of the function $A^{\frac{n}{s}}$ is relevant now. Thus, embedding (8.5) reads

$$W^{s, A}(\Omega) \rightarrow \begin{cases} L^{\frac{n}{s-p}}(\log L)^{\frac{np}{s-p}}(\Omega) & \text{if } 1 \leq p < \frac{n}{s} \\
 \exp L^{\frac{n}{s-(\alpha+1)s}}(\Omega) & \text{if } p = \frac{n}{s} \text{ and } \alpha < \frac{n}{s} - 1 \\
 \exp \exp L^{\frac{n}{s-\alpha}}(\Omega) & \text{if } p = \frac{n}{s} \text{ and } \alpha = \frac{n}{s} - 1,\end{cases}$$

and the target spaces are optimal in the class of all Orlicz spaces. Embedding (8.3) reproduces or extends to the fractional case various results scattered in the literature. The case corresponding to (6.8) is classical.
Integer-order Sobolev embeddings parallel to (8.9) are special instances of the general results of [36], which, in their turn, include various borderline cases established in [16, 55, 75, 84, 85]. In fact, the paper [16], and some sequel contributions by the same authors, also deal with fractional embeddings, but defined in terms of potentials instead of difference quotients.

As far as augmented embeddings with sharp rearrangement-invariant target spaces are concerned, Theorem 8.2, Part (ii), tells us that embedding (8.7) holds with $A$ obeying (5.6) and (5.7). In this case, the resultant space $L(A, \frac{n}{s})(\Omega)$ agrees (up to equivalent norms) with a (generalized) Lorentz-Zygmund space. Thus, embedding (8.7) can be written as

\[
W^{s, A}(\Omega) \to \begin{cases} 
L^{\frac{n}{s} - \frac{np}{s} + \frac{np}{n} p}(\Omega) & \text{if } 1 \leq p < \frac{n}{s} \\
L^{\infty, \frac{n}{s}; \frac{n}{s} - 1}(\Omega) & \text{if } p = \frac{n}{s} \text{ and } \alpha < \frac{n}{s} - 1 \\
L^{\infty, \frac{n}{s}; \frac{n}{s} - 1}(\Omega) & \text{if } p = \frac{n}{s} \text{ and } \alpha = \frac{n}{s} - 1,
\end{cases}
\]

all target spaces being optimal among all rearrangement-invariant spaces. Embedding (8.10) is well known in the integer-order case – see [36]. The results of the latter paper encompass, in particular, classical embeddings of $\Omega$ [72] and of $\Omega$ under (6.8) and (6.9), respectively.

Theorems 8.1 and 8.2 rely upon their analogues in $\mathbb{R}^n$ and on the following extension domain for fractional Orlicz-Sobolev spaces on Lipschitz domains.

**Theorem 8.4. [Extension operator for fractional Orlicz-Sobolev spaces]** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, with $n \geq 1$. Assume that $s \in (0, 1)$ and let $A$ be a Young function. Then, there exist a linear extension operator $E : W^{s, A}(\Omega) \to W^{s, A}(\mathbb{R}^n)$ and a constant $C = C(s, \Omega)$ such that

\[
E(u) = u \quad \text{in} \quad \Omega,
\]

and

\[
\|E(u)\|_{W^{s, A}(\mathbb{R}^n)} \leq C\|u\|_{W^{s, A}(\Omega)}
\]

for every $u \in W^{s, A}(\Omega)$.

Moreover, there exists a constant $C = C(s, \Omega)$ such that

\[
|E(u)|_{s, A, \mathbb{R}^n} \leq C|u|_{s, A, \Omega}
\]

for every $u \in V^{s, A}(\Omega)$.

The following Poincaré type inequality, of independent interest, is one ingredient in the proof of Theorem 8.4.

**Proposition 8.5. [Fractional Orlicz–Poincaré inequality]** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, with $n \geq 1$. Assume that $s \in (0, 1)$ and that $A$ is a Young function. If $u \in V^{s, A}(\Omega)$, then $u \in L^A(\Omega)$. Moreover, there exists a constant $C = C(s, \Omega)$ such that

\[
\|u - u_\Omega\|_{L^A(\Omega)} \leq C|u|_{s, A, \Omega}
\]

for every function $u \in V^{s, A}(\Omega)$. In particular,

\[
\int_{\Omega} A(|u(x) - u_\Omega|) \, dx \leq \int_{\Omega} \int_{\Omega} A \left( C|u(x) - u(y)| \right) \frac{dx \, dy}{\ |x - y|^{n}}
\]

for every function $u \in \mathcal{M}_d(\Omega)$.

**Proof.** Let $u \in V^{s, A}(\Omega)$. Suppose, for the time being, that we already know that $u \in L^1(\Omega)$. Hence, $u_\Omega$ is well defined. Since $\Omega$ is bounded, we have that $|x - y| \leq C$ for some constant $C = C(\Omega)$. Thus, there exist constants $C = C(s, \Omega)$ and $C' = C'(s, \Omega)$ such that

\[
\int_{\Omega} A(|u(x) - u_\Omega|) \, dx = \int_{\Omega} A \left( \frac{1}{|\Omega|} \int_{\Omega} (u(x) - u(y)) \, dy \right) \, dx \int_{\Omega} \frac{1}{|\Omega|} \int_{\Omega} A(|u(x) - u(y)|) \, dy \, dx
\]
\[
\int_{\Omega} A(\left| T_t(u)(x) - (T_t(u))_\Omega \right|) \, dx \leq \int_{\Omega} A(\frac{C|u(x) - u(y)|}{|x - y|^s}) \, dx \, dy \leq \int_{\Omega} A(\frac{C'|u(x) - u(y)|}{|x - y|^s}) \, dx \, dy \leq \frac{C}{|\Omega|} \int_{\Omega} \int_{\Omega} A \left( \frac{C|u(x) - u(y)|}{|x - y|^s} \right) \, dx \, dy
\]

Note that the first inequality is due to Jensen inequality and the last one holds owing to property (2.2). This established inequality (8.15). Inequality (8.14) follows on applying (8.16) with \( u \) replaced by \( u/\lambda \) for any \( \lambda > 0 \). It remains to show that, if \( u \) is any function in \( V^{s,A}(\Omega) \), then \( u \in L^1(\Omega) \). Given \( t > 0 \), denote by \( T_t : \mathbb{R} \to \mathbb{R} \) the function defined as \( T_t(r) = \min\{|r|, t|\text{sign}(r)| \} \) for \( r \in \mathbb{R} \). One can verify that
\[
|T_t(u)(x) - T_t(u)(y)| \leq |u(x) - u(y)| \quad \text{for } x, y \in \Omega.
\]

Since \( T_t(u) \in L^\infty(\Omega) \), and hence \( T_t(u) \in L^1(\Omega) \), we may apply inequality (8.16) with \( u \) replaced by \( T_t(u) \) and deduce that
\[
\int_{\Omega} A(\left| T_t(u)(x) - (T_t(u))_\Omega \right|) \, dx \leq \int_{\Omega} A(\frac{C'|u(x) - u(y)|}{|x - y|^s}) \, dx \, dy \leq \int_{\Omega} A(\frac{C|u(x) - u(y)|}{|x - y|^s}) \, dx \, dy
\]
for \( t > 0 \). Next, denote by \( \text{med}(u) \) the median of \( u \) given by \( \text{med}(u) = \inf\{\tau \in \mathbb{R} : |\{u > \tau\}| \leq |\Omega|/2\} \), and observe that
\[
\text{med}(T_t(u)) = \text{med}(u) \quad \text{if } t > |\text{med}(u)|.
\]

Also, there exists a constant \( C = C(|\Omega|) \) such that
\[
\int_{\Omega} A(|v(x) - v(y)|) \, dx \leq \int_{\Omega} A(C|v(x) - v(y)|) \, dx
\]
for every function \( v \in L^1(\Omega) \), see e.g. [67, Lemma 2.1]. From inequalities (8.17) – (8.19) we infer that
\[
\int_{\Omega} A(\left| T_t(u)(x) - \text{med}(u) \right|) \, dx \leq \int_{\Omega} A(\frac{C|u(x) - u(y)|}{|x - y|^s}) \, dx \, dy
\]
for some constant \( C \) and for every \( t > |\text{med}(u)| \). Since \( \lim_{t \to \infty} T_t(u) = u \text{ a.e. in } \Omega \), passing to the limit as \( t \to \infty \) in inequality (8.20) yields, by Fatou's lemma,
\[
\int_{\Omega} A(\left| u(x) - \text{med}(u) \right|) \, dx \leq \int_{\Omega} A(\frac{C|u(x) - u(y)|}{|x - y|^s}) \, dx \, dy.
\]

Hence, given any \( \lambda > 0 \),
\[
\int_{\Omega} A(|u(x)|/\lambda) \, dx \leq \int_{\Omega} A\left( \frac{2|u(x) - \text{med}(u)|}{\lambda} \right) \, dx + |\Omega|A(2|\text{med}(u)|/\lambda)
\]
\[
\leq \int_{\Omega} \int_{\Omega} A\left( \frac{2C|u(x) - u(y)|}{\lambda|x - y|^s} \right) \, dx \, dy + |\Omega|A(2|\text{med}(u)|/\lambda) < \infty.
\]

Since \( u \in V^{s,A}(\Omega) \), the double integral in equation (8.22) is finite provided that \( \lambda \) is sufficiently large. This shows that \( u \in L^A(\Omega) \), and hence, owing to the second embedding in (2.22), \( u \in L^1(\Omega) \).

**Proof of Theorem 8.4.** Inequality (8.13) is a consequence of (8.12) and (8.14). The proof of inequality (8.12) is patterned on that of [13, Theorem 5.4], and is split in steps. We focus the case when \( n \geq 2 \), the one-dimensional case being analogous, and even simpler.

**Step 1.** Let \( E \) be a compact set such that \( E \subset \Omega \). Let \( \mathcal{E}_0 \) be the linear operator defined by (2.19). Then there exists a constant \( C = C(n, s, E, \Omega) \) such that, if \( u \in W^{s,A}(\Omega) \) and \( u = 0 \) in \( \Omega \setminus E \), then \( \mathcal{E}_0(u) \in W^{s,A}(\mathbb{R}^n) \) and
\[
||\mathcal{E}_0(u)||_{W^{s,A}(\mathbb{R}^n)} \leq C||u||_{W^{s,A}(\Omega)}.
\]

Plainly,
\[
||\mathcal{E}_0(u)||_{L^A(\mathbb{R}^n)} = ||u||_{L^A(\Omega)}.
\]
It thus suffices to show that
\begin{equation}
(8.25) \quad |\mathcal{E}_0(u)|_{s,A,\mathbb{R}^n} \leq C\|u\|_{W^{s,A}(\Omega)}
\end{equation}
for some constant $C = C(n, s, E, \Omega)$. Since $\mathcal{E}_0(u)$ vanishes outside $E$,
\begin{equation}
(8.26) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|\mathcal{E}_0(u)(x) - \mathcal{E}_0(u)(y)|}{|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n} = \int_{\Omega} \int_{\Omega} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n}
\end{equation}
\begin{equation}
+ 2 \int_{E} \left( \int_{\mathbb{R}^n \setminus \Omega} A \left( \frac{|u(x)|}{|x - y|^s} \right) \frac{dy}{|x - y|^n} \right) \, dx.
\end{equation}
Set $d_{E,\Omega} = \text{dist}(E, \mathbb{R}^n \setminus \Omega)$. Thereby,
\begin{equation}
(8.27) \quad A \left( \frac{|u(x)|}{|x - y|^s} \right) = A \left( \frac{|u(x)|}{d_{E,\Omega}^s} \right) \frac{d_{E,\Omega}^s}{|x - y|^s} \leq A \left( \frac{|u(x)|}{d_{E,\Omega}^s} \right) \frac{d_{E,\Omega}^s}{|x - y|^s} \quad \text{if} \quad x \in E \quad \text{and} \quad y \in \mathbb{R}^n \setminus \Omega.
\end{equation}
Notice that the last inequality holds by property (2.2), inasmuch as $\frac{d_{E,\Omega}}{|x - y|^s} \leq 1$. Hence,
\begin{equation}
(8.28) \quad \int_{E} \int_{\mathbb{R}^n \setminus \Omega} A \left( \frac{|u(x)|}{|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n} \leq d_{E,\Omega}^s \int_{E} \left( \int_{\mathbb{R}^n \setminus \Omega} \frac{dy}{|x - y|^{n+s}} \right) A \left( \frac{|u(x)|}{d_{E,\Omega}^s} \right) \, dx.
\end{equation}
The last integral over $\mathbb{R}^n \setminus \Omega$ in equation (8.28) is convergent, since $n + s > n$ and $\text{dist}(y, E) \geq d_{E,\Omega} > 0$ if $y \in \mathbb{R}^n \setminus \Omega$. Inequalities (8.26) and (8.28), applied with $u$ replaced by $u/\lambda$, yield
\begin{equation}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|\mathcal{E}_0(u)(x) - \mathcal{E}_0(u)(y)|}{\lambda|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n} \leq \int_{\Omega} \int_{\Omega} A \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n} + C \int_{\Omega} A \left( \frac{|u(x)|}{\lambda d_{E,\Omega}^s} \right) \, dx.
\end{equation}
for some constant $C = C(n, s, E, \Omega)$. Hence, inequality (8.25) follows.

**Step 2.** Assume that $\Omega$ is symmetric about the hyperplane $\{x_n = 0\}$. Set $\Omega_+ = \{x \in \Omega : x_n > 0\}$ and $\Omega_- = \{x \in \Omega : x_n \leq 0\}$. Given any function $u : \Omega_+ \to \mathbb{R}$, define the function $\mathcal{E}_1(u) : \Omega \to \mathbb{R}$ as
\begin{equation}
(8.29) \quad \mathcal{E}_1(u)(x) = \begin{cases} 
2u(x', x_n) & \text{if} \quad x_n \geq 0 \\
2u(x', -x_n) & \text{if} \quad x_n < 0.
\end{cases}
\end{equation}
If $u \in W^{s,A}(\Omega_+)$, then $\mathcal{E}_1(u) \in W^{s,A}(\Omega)$ and
\begin{equation}
(8.30) \quad \|\mathcal{E}_1(u)\|_{W^{s,A}(\Omega)} \leq 4 \|u\|_{W^{s,A}(\Omega_+)}.
\end{equation}
Clearly,
\begin{equation}
(8.31) \quad \|\mathcal{E}_1(u)\|_{L^A(\Omega)} \leq 2 \|u\|_{L^A(\Omega_+)}.
\end{equation}
On the other hand, given $\lambda > 0$,
\begin{equation}
(8.32) \quad \int_{\Omega} \int_{\Omega} A \left( \frac{|\mathcal{E}_1(u)(x) - \mathcal{E}_1(u)(y)|}{\lambda|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n} = \int_{\Omega_+} \int_{\Omega_+} A \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n}
\end{equation}
\begin{equation}
+ 2 \int_{\Omega_+} \int_{\Omega_-} A \left( \frac{|u(x) - u(y', -y_n)|}{\lambda|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n}
\end{equation}
\begin{equation}
+ \int_{\Omega_-} \int_{\Omega_+} A \left( \frac{|u(x', -x_n) - u(y', -y_n)|}{\lambda|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n}
\end{equation}
\begin{equation}
\leq 4 \int_{\Omega_+} \int_{\Omega_+} A \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n},
\end{equation}
where the last inequality is due to the fact that \((x_n - y_n)^2 \geq (x_n + y_n)^2\) if \(x_n \geq 0\) and \(y_n \leq 0\). Inequality (8.32) implies that

\[
|\mathcal{E}_1(u)|_{s,A,\Omega} \leq 4|u|_{s,A,\Omega+}.
\]

Inequality (8.30) follows from (8.31) and (8.33).

**Step 3.** Let \(\zeta : \Omega \rightarrow [0, 1]\) be a Lipschitz continuous function whose Lipschitz constant agrees with \(L\). Then there exists a constant \(C = C(s, L, \Omega)\) such that for every \(u \in W^{s,A}(\Omega)\), one has that \(\zeta \in W^{s,A}(\Omega)\) and

\[
\|\zeta\|_{s,A,\Omega} \leq C \|u\|_{s,A,\Omega}.
\]

Inasmuch as \(0 \leq \zeta \leq 1\),

\[
\|\zeta\|_{L^A(\Omega)} \leq \|u\|_{L^A(\Omega)}.
\]

Moreover,

\[
\int_\Omega \int_\Omega A\left(\frac{|\zeta(x) u(x) - \zeta(y) u(y)|}{|x-y|^n}\right) \frac{dx \, dy}{|x-y|^n} \\
\leq \int_\Omega \int_\Omega A\left(\frac{2|u(y)||\zeta(x) - \zeta(y)|}{|x-y|^n}\right) \frac{dx \, dy}{|x-y|^n} + \int_\Omega \int_\Omega A\left(\frac{2|\zeta(x)||u(x) - u(y)|}{|x-y|^n}\right) \frac{dx \, dy}{|x-y|^n}
\]

\[
\leq \int_\Omega \int_\Omega A\left(\frac{2|u(y)||\zeta(x) - \zeta(y)|}{|x-y|^n}\right) \frac{dx \, dy}{|x-y|^n} + \int_\Omega \int_\Omega A\left(\frac{2|u(x) - u(y)|}{|x-y|^n}\right) \frac{dx \, dy}{|x-y|^n}.
\]

Since \(\zeta\) has Lipschitz constant \(L\),

\[
\int_\Omega \int_\Omega A\left(\frac{2|u(y)||\zeta(x) - \zeta(y)|}{|x-y|^n}\right) \frac{dx \, dy}{|x-y|^n} \\
\leq \int_\Omega \int_{\Omega \cap \{|x-y| \leq 1\}} A(2L|u(y)||x-y|^{-s}) \frac{dx \, dy}{|x-y|^n} + \int_\Omega \int_{\Omega \cap \{|x-y| > 1\}} A\left(\frac{2|u(y)||\zeta(x) - \zeta(y)|}{|x-y|^n}\right) \frac{dx \, dy}{|x-y|^n}
\]

\[
\leq \int_\Omega \int_{\Omega \cap \{|x-y| \leq 1\}} A(2L|u(y)||x-y|^{-s}) \frac{dx \, dy}{|x-y|^n} + \int_\Omega \int_{\Omega \cap \{|x-y| > 1\}} A\left(\frac{2|u(y)|}{|x-y|^n}\right) \frac{dx \, dy}{|x-y|^n}
\]

\[
\leq \int_\Omega \int_{\Omega \cap \{|x-y| \leq 1\}} A(2L|u(y)|) \frac{dx}{|x-y|^{n+s-1}} + \int_\Omega \int_{\Omega \cap \{|x-y| > 1\}} A\left(\frac{2|u(y)|}{|x-y|^{n+s}}\right) \frac{dx \, dy}{|x-y|^n}.
\]

\[
\leq C \int_\Omega A(C' |u(y)|) \frac{dx}{|x-y|^{n+s}} \leq \int_\Omega A(C'' |u(y)|) \frac{dx}{|x-y|^{n+s}}.
\]

for some constants \(C = C(s, \Omega)\), \(C' = C'(L)\) and \(C'' = C''(s, L, \Omega)\). Observe that the second inequality holds since \(|\zeta(x) - \zeta(y)| \leq 1\), the third one holds owing to property (2.22), and the fourth one since \(n + s - 1 < n\) and \(n > n + s\), and the last one by property (2.22) again. From inequalities (8.36) and (8.37), applied with \(u\) replaced by \(u/\lambda\) for any \(\lambda > 0\), we infer that

\[
|\zeta u|_{s,A,\Omega} \leq C \|u\|_{s,A,\Omega}.
\]

for some constant \(C = C(s, L, \Omega)\). Coupling inequality (8.35) with (8.38) yields (8.34).

**Step 4.** Conclusion.

Let \(Q, Q_+\), \(\{B_j\}_{j=1}^k\) and \(\{T_j\}_{j=1}^k\) be as in the definition of bounded Lipschitz domain at the beginning of this section. Since \(\mathbb{R}^n = \bigcup_{j=1}^k B_j \cup (\mathbb{R}^n \setminus \partial \Omega)\), there exists a smooth partition of unity \(\{\zeta_j\}_{j=0}^k\) with respect to this covering of \(\mathbb{R}^n\) such that \(\text{supp} \zeta_0 \subset \mathbb{R}^n \setminus \partial \Omega\), \(\text{supp} \zeta_j \subset B_j\) for \(j = 1, \ldots, k\), \(0 \leq \zeta_j \leq 1\) for \(j = 0, 1, \ldots, k\), and
Given any function $u \in W^{s,A}(\Omega)$, define the function $v_j : Q_+ \to \mathbb{R}$, for $j = 1, ..., k$ as
\[ v_j(\tilde{y}) = u(T_j(\tilde{y})) \quad \text{for } \tilde{y} \in Q_. \]

We claim that $v_j \in W^{s,A}(Q_+)$ for $j = 1, ..., k$, and
\[(8.39) \quad \|v_j\|_{W^{s,A}(Q_+)} \leq C \|u\|_{W^{s,A}(B_j \cap \Omega)} \]
for some constant $C = C(s, \Omega)$. This claim follows from the following chain:
\begin{align*}
(8.40) \quad \int_{Q_+} \int_{Q_+} A \left( \frac{|v_j(\tilde{x}) - v_j(\tilde{y})|}{|\tilde{x} - \tilde{y}|^s} \right) \frac{d\tilde{x} d\tilde{y}}{|\tilde{x} - \tilde{y}|^n} &= \int_{Q_+} \int_{Q_+} A \left( \frac{|u(T_j(\tilde{x})) - u(T_j(\tilde{y}))|}{|\tilde{x} - \tilde{y}|^s} \right) \frac{d\tilde{x} d\tilde{y}}{|\tilde{x} - \tilde{y}|^n} \\
&= \int_{B_j \cap \Omega} \int_{B_j \cap \Omega} A \left( \frac{|u(x) - u(y)|}{|T_j^{-1}(x) - T_j^{-1}(y)|^s} \right) |\det(J(T_j^{-1}))| \frac{dx dy}{|T_j^{-1}(x) - T_j^{-1}(y)|^n} \\
&\leq \int_{B_j \cap \Omega} \int_{B_j \cap \Omega} A \left( \frac{|C|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^n},
\end{align*}
for some constant $C$ depending on the Lipschitz constant of $T_j$ and the Lipschitz constant of $T_j^{-1}$. Here, $\det(J(T_j^{-1}))$ denotes the determinant of the Jacobian of the map $T_j^{-1}$. Note that the last inequality relies upon property [22] as well.

Next, let $\overline{v}_j : Q \to \mathbb{R}$ be the function obtained on extending $v_j$ to $Q$ as in Step 2, namely $\overline{v}_j = \mathcal{E}_1(v_j)$. Therefore, $\overline{v}_j \in W^{s,A}(Q)$, and
\[(8.41) \quad \|\overline{v}_j\|_{W^{s,A}(Q)} \leq 4 \|v_j\|_{W^{s,A}(Q_+)} \]
for $j = 1, ..., k$. Now, let $w_j : B_j \to \mathbb{R}$ be given by
\[(8.42) \quad w_j(x) = \overline{v}_j(T_j^{-1}(x)) \quad \text{for } x \in B_j. \]

A chain analogous to (8.40) ensures that $w_j \in W^{s,A}(B_j)$ and
\[(8.43) \quad \|w_j\|_{W^{s,A}(B_j)} \leq C \|\overline{v}_j\|_{W^{s,A}(Q)} \]
for some constant $C = C(s, \Omega)$. Definition [8.42] immediately tells us that $w_j = u$ on $B_j \cap \Omega$, and hence $\zeta_j w_j = \zeta_j u$ on $B_j \cap \Omega$. By Step 3, $\zeta_j w_j \in W^{s,A}(B_j)$ and
\[(8.44) \quad \|\zeta_j w_j\|_{W^{s,A}(B_j)} \leq C \|w_j\|_{W^{s,A}(B_j)} \]
for $j = 1, ..., k$, for some constant $C = C(s, \Omega)$. On the other hand, $\zeta_j w_j$ has compact support in $B_j$. Hence, the extension $\mathcal{E}_0(\zeta_j w_j) : \mathbb{R}^n \to \mathbb{R}$ of $\zeta_j w_j$ to $\mathbb{R}^n$, defined as in Step 1, is such that $\mathcal{E}_0(\zeta_j w_j) \in W^{s,A}(\mathbb{R}^n)$ and
\[(8.45) \quad ||\mathcal{E}_0(\zeta_j w_j)||_{W^{s,A}(\mathbb{R}^n)} \leq C \|\zeta_j w_j\|_{W^{s,A}(B_j)} \]
for $j = 1, ..., k$, for some constant $C = C(s, \Omega)$. Also, since $\zeta_0 u = 0$ in a neighborhood of $\partial \Omega$, the extension of $\zeta_0 u$ to $\mathbb{R}^n$ given by $\mathcal{E}_0(\zeta_0 u)$ belongs to $W^{s,A}(\mathbb{R}^n)$, and, by Steps 1 and 3,
\[(8.46) \quad ||\mathcal{E}_0(\zeta_0 u)||_{W^{s,A}(\mathbb{R}^n)} \leq C ||\zeta_0 u||_{W^{s,A}(\Omega)} \leq C' \|u\|_{W^{s,A}(\Omega)} \]
for some constants $C = C(s, \Omega)$ and $C' = C'(s, \Omega)$. Finally, consider the extension $\mathcal{E}(u) : \mathbb{R}^n \to \mathbb{R}$ of $u$ to $\mathbb{R}^n$ given by
\[ \mathcal{E}(u) = \mathcal{E}_0(\zeta_0 u) + \sum_{j=1}^{k} \mathcal{E}_0(\zeta_j w_j). \]

Then $\mathcal{E}$ defines a linear operator on $W^{s,A}(\Omega)$ such that, $\mathcal{E}(u) = u$ in $\Omega$ and, owing to inequalities (8.39), (8.41), (8.43), (8.44), (8.45) and (8.46),
\[ ||\mathcal{E}(u)||_{W^{s,A}(\mathbb{R}^n)} \leq C \|u\|_{W^{s,A}(\Omega)} \]
for some constant $C = C(s, \Omega)$ and for every $u \in W^{s,A}(\Omega)$. The proof is complete. $\square$
As in the case of fractional Orlicz-Sobolev spaces in $\mathbb{R}^n$, the validity of an embedding on a domain implies a corresponding one-dimensional Hardy type inequality. This is the content of the following lemma, to be used in the proof of the optimality of the target spaces in the embeddings of Theorems 8.1 and 8.2

**Lemma 8.6.** Let $n \in \mathbb{N}$ and $s \in (0, n) \setminus \mathbb{N}$. Let $A$ be a Young function, let $\Omega$ be an open set in $\mathbb{R}^n$ such that $|\Omega| < \infty$ and let $Y(\Omega)$ be a rearrangement-invariant space. Assume that there exists a constant $C$ such that

\begin{equation}
\|u\|_{Y(\Omega)} \leq C\|u\|_{W^{s,A}(\Omega)}
\end{equation}

for every function $u \in W^{s,A}(\Omega)$. Then there exists a constant $C'$ such that

\begin{equation}
\left\| \int_0^{|\Omega|} f(\varrho)\varrho^{-1+\frac{s}{n}} \, d\varrho \right\|_{Y(0,|\Omega|)} \leq C'\|f\|_{L^A(0,|\Omega|)}
\end{equation}

for every function $f \in L^A(0,|\Omega|)$.

**Proof, sketched.** The proof follows along the same lines as that of Lemma 7.6. One can assume, without loss of generality, that $0 \in \Omega$. Let $B$ be a ball, centered at 0 and with measure $L$, contained in $\Omega$. Consider trial functions $u$ in inequality (8.47) of the form (7.11), with $f$ supported in $(0, L)$. Then an analogous argument as in the proof of Lemma 7.6 tells us that

\begin{equation}
\left\| \int_r^L f(\varrho)\varrho^{-1+\frac{s}{n}} \, d\varrho \right\|_{Y(0,L)} \leq C\left(\|f\|_{L^A(0,L)} + \left\| \int_r^L f(\varrho)\varrho^{-1+\frac{s}{n}} \, d\varrho \right\|_{L^A(0,L)} \right)
\end{equation}

for every function $f \in L^A(0,L)$. On the other hand, (the same proof of) Inequality (4.10) yields

\begin{equation}
\left\| \int_r^L f(\varrho)\varrho^{-1+\frac{s}{n}} \, d\varrho \right\|_{L^A(0,L)} \leq CL\varpi\|f\|_{L^A(0,L)}
\end{equation}

for some constant $C = C(s, n)$ and for every function $f \in L^A(0,L)$. Inequalities (8.49) and (8.50) imply that (8.48) holds with $|\Omega|$ replaced by $L$. Inequality (8.48) holds in its original version, and in fact with $|\Omega|$ replaced by any $L > 0$, as a consequence of the boundedness of the dilation operator, defined as in (2.23), in any Orlicz space.

**Proof of Theorem 8.7.** Inequality (8.2) follows via inequality (8.13) and Theorem 6.1. Embedding (8.1) can be deduced from an application of inequality (8.2) with $u$ replaced by $u - u_0$ for any function $u \in W^{s,A}(\Omega)$. The optimality of the target space in embedding (8.1) is a consequence of Lemma 8.6 and of Theorem A. Inequality (8.3), and hence embedding (8.3), follow via inequality (8.13) and Theorem 6.2. The optimality of the target space in embedding (8.3) is a consequence of Lemma 8.6 and of Theorem B.

**Proof of Theorem 8.2.** Let us begin by proving inequality (8.8). Let $u \in W^{s,A}(\Omega)$. As in the proof of Theorem 7.2, we denote by $\widehat{A}$ and $\widehat{A}(s)$ the functions associated with $A$ as in (4.7)–(4.8) with $s$ and $\{s\}$, respectively. An application of embedding (8.3) to each $[s]$-th order weak derivative of $u$ tells us that

\begin{equation}
\|\nabla^{[s]} u\|_{L^A(\widehat{A}(s),\frac{n}{2})}\leq C\|\nabla^{[s]} u\|_{W^{s,A}(\Omega)}
\end{equation}

for some constant $C = C(s, \Omega)$. On the other hand, owing to inequality (7.25) and to the reduction principle for integer-order Sobolev inequalities of [10, Theorem 6.1],

\begin{equation}
\|u\|_{L^A(\widehat{A},\frac{n}{2})}\leq C\left(\|\nabla^{[s]} u\|_{L^A(\widehat{A}(s),\frac{n}{2})}\right) + \sum_{k=0}^{[s]-1}\|\nabla^k u\|_{L^1(\Omega)}
\end{equation}

for some constant $C = C(s, \Omega)$. Coupling inequalities (8.51) and (8.52), and making use of property (2.12), implies that

\begin{equation}
\|u\|_{L^A(\widehat{A},\frac{n}{2})}\leq C\left(\|\nabla^{[s]} u\|_{W^{s,A}(\Omega)} + \sum_{k=0}^{[s]-1}\|\nabla^k u\|_{L^1(\Omega)}\right) \leq C'\|u\|_{W^{s,A}(\Omega)}
\end{equation}
for some constants $C$ and $C'$ depending on $s$ and $Ω$. This establishes embedding (8.5).

As far as inequality (8.8) is concerned, assume that $u ∈ V_+^{s,A}(Ω)$. Since $(∇^{[s]}u)_Ω = 0$, inequality (8.4), applied to each $[s]$-th order derivative of $u$ tells us that

$\|(∇^{[s]}u)\|_{L(\tilde{A}_s, \frac{n}{(s-n)+})}(Ω) ≤ C\|∇^{[s]}u\|_{\{s\}, A, Ω}$

for some constant $C = C(s, Ω)$. On the other hand, inasmuch as we are also assuming that $(∇^k u)_Ω = 0$ for $k = 0, \ldots, [s] - 1$, by the integer-order result of [49] Theorem 6.1 and inequality (7.25) again,

$\|u\|_{L(\tilde{A}_s, \frac{n}{s})}(Ω) ≤ C\|∇^{[s]}u\|_{L(\tilde{A}_s, \frac{n}{(s-n)+})}(Ω)$

for some constant $C = C(s, Ω)$. Inequality (8.8) follows from (8.54) and (8.55).

In order to deduce embedding (8.7) from inequality (8.8), observe that, for each function $u$ for which $P_n u$ are linear combinations of the components of $∫_Ω |∇^k u| dΩ$, for $k = 0, \ldots, [s]$, with coefficients depending on $[s], k$ and $Ω$. Thus, there exist constants $C = C(s, Ω)$ and $C' = C'(s, Ω)$ such that

$\|u\|_{L(\tilde{A}_s, \frac{n}{s})}(Ω) ≤ \|u - P_n u\|_{L(\tilde{A}_s, \frac{n}{s})}(Ω) + \|P_n u\|_{L(\tilde{A}_s, \frac{n}{s})}(Ω)$

$≤ C\|∇^{[s]}u\|_{L(\tilde{A}_s, \frac{n}{(s-n)+})}(Ω) + C\|1\|_{L(\tilde{A}_s, \frac{n}{s})}(Ω)\sum_{k=0}^{[s]} ∫_Ω |∇^k u| dΩ$

$≤ C\|∇^{[s]}u\|_{L(\tilde{A}_s, \frac{n}{(s-n)+})}(Ω) + 2C\|1\|_{L(\tilde{A}_s, \frac{n}{s})}(Ω)\|1\|_{L(\tilde{A}_s, \frac{n}{s})}(Ω)\|1\|_{L(\tilde{A}_s, \frac{n}{s})}(Ω) ≤ C\|u\|_{W^{s,A}(Ω)}$,

where the second inequality holds owing to (8.54) applied to $u - P_n u$, and the third inequality is due to (2.11). Embedding (8.7) is a consequence of inequality (8.56).

The optimality of the target space in embedding (8.5) is a consequence of Lemma 8.6 and of Theorem B. Embedding (8.5) and inequality (8.6) follow from embedding (8.7) and inequality (8.8), respectively, via Proposition 4.1. Lemma 8.6 again and Theorem A imply that the target space is optimal in embedding (8.5) among all Orlicz spaces.

9. Compact embeddings

We conclude by criteria for the compactness of fractional Orlicz-Sobolev embeddings into Orlicz spaces and, more generally, into rearrangement-invariant spaces.

The results concerning spaces defined in the whole of $\mathbb{R}^n$ have necessarily a local nature, in the following sense. Given any non-integer positive number $s$, a Young function $A$ and a rearrangement-invariant space $Y(\mathbb{R}^n)$, we say that the embedding $V_+^{s,A}(\mathbb{R}^n) → Y_{loc}(\mathbb{R}^n)$ is compact if every bounded sequence in $V_+^{s,A}(\mathbb{R}^n)$ has a subsequence whose restriction to $E$ converges in $Y(E)$ for every bounded measurable set $E$ in $\mathbb{R}^n$. Here, $Y(E)$ denotes the rearrangement-invariant space given by the restriction of $Y(\mathbb{R}^n)$ to $E$, defined as in (2.20).

A necessary and sufficient condition for compact embeddings into an Orlicz space amounts to requiring that the Young function that defines the latter space grows essentially more slowly near infinity (in the sense of (2.25)) than the Young function that defines the optimal Orlicz target for merely continuous embeddings. This is the content of the following theorem.

**Theorem 9.1.** Let $n ∈ \mathbb{N}$ and $s ∈ (0, n) \setminus \mathbb{N}$. Let $A$ be a Young function fulfilling conditions (1.2) and (4.3), and let $A_{\frac{n}{s}}$ be the Young function defined by (4.4). Assume that $B$ is a Young function. The following properties
are equivalent:
(i) $B$ grows essentially more slowly near infinity than $A^n$, namely

\[
\lim_{t \to \infty} \frac{B(\lambda t)}{A^n(t)} = 0
\]

for every $\lambda > 0$.
(ii) The embedding

\[
V_d^{s,A}(\mathbb{R}^n) \to L^B_{\text{loc}}(\mathbb{R}^n)
\]

is compact.
(iii) The embedding

\[
W^{s,A}(\Omega) \to L^B(\Omega)
\]

is compact for every bounded Lipschitz domain $\Omega$ in $\mathbb{R}^n$.

**Example 9.2.** Assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$. Let $s \in (0, n) \setminus \mathbb{N}$ and let $A$ be a Young function obeying (5.3), (5.4) and (5.5). An application of Theorem 9.1 and the use of property (2.6) tell us that the embedding

\[
W^{s,A}(\Omega) \to L^B(\Omega)
\]

is compact if and only if $B$ is a Young function fulfilling

\[
\begin{align*}
\lim_{t \to \infty} \frac{t^{\frac{n-sp}{n}} (\log t)^{-\frac{\alpha}{n}} B^{-1}(t)}{s} & = 0 & \text{if } 1 \leq p < \frac{n}{s} \\
\lim_{t \to \infty} \frac{(\log t)^{\frac{n-s}{n}} B^{-1}(t)}{s} & = 0 & \text{if } p = \frac{n}{s} \text{ and } \alpha < \frac{n}{s} - 1 \\
\lim_{t \to \infty} \frac{\log(t)^{\frac{n-s}{n}} B^{-1}(t)}{s} & = 0 & \text{if } p = \frac{n}{s} \text{ and } \alpha = \frac{n}{s} - 1.
\end{align*}
\]

A parallel conclusion holds, with $A$ and $B$ as above, for the embedding $V_d^{s,A}(\mathbb{R}^n) \to L^B_{\text{loc}}(\mathbb{R}^n)$.

Theorem 9.1 will be deduced via the following characterization of compact embeddings into rearrangement-invariant spaces. Due to the generality of the latter class of function spaces, such a characterization is naturally less explicit than (9.1), but still handy for applications to customary spaces.

**Theorem 9.3.** Let $n \in \mathbb{N}$, $s \in (0, n) \setminus \mathbb{N}$ and let $A$ be a Young function fulfilling conditions (4.2) and (4.3). Assume that $Y(\mathbb{R}^n)$ is a rearrangement-invariant space. The following properties are equivalent:

(i)

\[
\lim_{L \to 0^+} \sup_{\|f\|_{L^A(0,L)} \leq 1} \left\| \int_0^L f(\varrho)^{s-1} \varrho^s \, d\varrho \right\|_{Y(0,L)} = 0.
\]

(ii) The embedding

\[
V_d^{s,A}(\mathbb{R}^n) \to Y_{\text{loc}}(\mathbb{R}^n)
\]

is compact.
(iii) The embedding

\[
W^{s,A}(\Omega) \to Y(\Omega)
\]

is compact for every bounded Lipschitz domain $\Omega$ in $\mathbb{R}^n$. 
Example 9.4. Let $\Omega$, $s$ and $A$ be as in Example 8.2. Assume that $1 \leq r, q \leq \infty$ and $\gamma \in \mathbb{R}$ are such that either $r = q = 1$ and $\gamma \geq 0$, or $1 < r < \infty$, or $r = \infty$, $q < \infty$ and $\gamma + 1/q < 0$, or $r = q = \infty$ and $\gamma \leq 0$ (by [74, Theorem 9.10.4], this assumption ensures that $L^{r,q;\gamma}(\Omega)$ is equivalent to a rearrangement-invariant space). Then the embedding

$$
W^{s,A}(\Omega) \to L^{r,q;\gamma}(\Omega)
$$

is compact if and only if either

$$
1 \leq p < \frac{n}{s}, \quad \alpha \in \mathbb{R} \quad \text{and} \quad \begin{cases} r < \frac{np}{n-sp}, \\ r = \frac{np}{n-sp}, & p \leq q, \quad \frac{\alpha}{p} > \gamma, \\ r = \frac{np}{n-sp}, & p > q, \quad \frac{\alpha}{p} + \frac{1}{p} > \gamma + \frac{1}{q}, \end{cases}
$$

or

$$
p = \frac{n}{s}, \quad \alpha \leq \frac{n}{s} - 1 \quad \text{and} \quad \begin{cases} r < \infty, \\ r = \infty, \quad \frac{\alpha s + 2}{n} - 1 > \gamma + \frac{1}{q}. \end{cases}
$$

If $1 \leq p < n/s$, or $p = n/s$ and $\alpha < n/s - 1$ then this fact follows from a combination of Theorem 9.3 [80], Theorems 4.1 and 4.2 and Proposition 7.2] and of Example 8.3. In the case when $p = n/s$ and $\alpha = n/s - 1$, one needs to additionally observe that the space $L^{\infty,\frac{2}{s} - \frac{n}{n-1}}(\Omega)$ is continuously embedded into $L^{r,q;\gamma}(\Omega)$ if and only if either $r < \infty$, or $r = \infty$ and $\gamma + 1/q < 0$ (see, e.g., [74, Theorem 9.5.14]), and that this embedding is in fact almost-compact thanks to the strict inequality in the last condition.

The embedding $V^{s,A}(\mathbb{R}^n) \to L^{r,q;\gamma}_{\text{loc}}(\mathbb{R}^n)$ is compact under the same conditions on the exponents $r, q; \gamma$ as in (9.9) or (9.10).

Our proof of Theorem 9.3 makes use of the following lemma.

Lemma 9.5. Let $n \in \mathbb{N}$, $s \in (0,n) \setminus \mathbb{N}$ and let $A$ be a Young function fulfilling conditions (4.2) and (4.3). 

(i) Any bounded sequence in $V^{s,A}(\mathbb{R}^n)$ has a subsequence which converges a.e. in $\mathbb{R}^n$.

(ii) Assume that $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$. Then any bounded sequence in $W^{s,A}(\Omega)$ has a subsequence which converges a.e. in $\Omega$.

A first step in the proof Lemma 9.5 in its turn relies upon the next result on the almost-compact embedding (according to the notion recalled in Section 2) of the optimal Orlicz target space in fractional Orlicz-Sobolev embeddings into $L^1$.

Lemma 9.6. Let $n \in \mathbb{N}$, $s \in (0,n)$ and let $A$ be a Young function fulfilling conditions (4.2) and (4.3). Let $A_\Omega$ be the function defined by (4.4). Assume that $E$ is a measurable bounded set in $\mathbb{R}^n$. Then the Orlicz space $L^{\frac{\alpha}{s}, A}(E)$ is almost-compactly embedded into $L^1(E)$.

Proof. Since the function $t \mapsto \frac{t}{A(t)}$ is non-increasing, one has that

$$
t \frac{1}{A(t)} \leq \frac{1}{A(1)} \quad \text{for} \quad t \geq 1.
$$

Hence

$$
H(t) = \int_0^t \frac{A(s)}{A(t)} \frac{1}{s^n} ds + \int_t^1 \left( \frac{\tau}{A(\tau)} \right)^{\frac{s}{n-s}} d\tau \lesssim t \quad \text{for} \quad t > 1,
$$

up to a constant independent of $t$. Thus,

$$
H(t) \lesssim t^{\frac{n}{n-s}} \lesssim A(t) t^{\frac{n}{n-s}} \quad \text{near} \quad t \to \infty,
$$

whence

$$
A_\Omega(t) = A(H^{-1}(t)) \gtrsim t^{\frac{n}{n-s}} \quad \text{near} \quad t \to \infty.
$$
In particular,

\[(9.13) \lim_{t \to \infty} \frac{t}{A_2(t)} = 0.\]

The conclusion hence follows, owing to property (2.26). \(\square\)

Proof of Lemma 9.5. We provide a proof of Part (i), the proof of Part (ii) being analogous. Let \(B\) be an open ball in \(\mathbb{R}^n\). It suffices to show that any bounded sequence \(\{u_i\}\) in \(V_d^{s,A}(\mathbb{R}^n)\) has a subsequence which is convergent in \(L^1(B)\). Assume, for the time being, that \(s \in (0,1)\). By the Riesz-Kolmogorov compactness theorem, this conclusion will follow if we show that \(\{u_i\}\) is a bounded sequence in \(L^1(B)\) and that, for every \(\varepsilon > 0\),

\[(9.14) \text{there exists } \delta > 0 \text{ and a compact set } \widehat{B} \subseteq B \text{ such that } \|u_i\|_{L^1(B,\widehat{B})} < \varepsilon \]

and

\[(9.15) \int_B |u_i(x + h) - u_i(x)| \, dx < \varepsilon \quad \text{if } h \in \mathbb{R}^n, \ |h| < \delta \text{ and } i \in \mathbb{N}.\]

The boundedness of the sequence \(\{u_i\}\) in \(V_d^{s,A}(\mathbb{R}^n)\) amounts to the existence of a positive constant \(C\) such that

\[(9.16) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u_i(x) - u_i(y)|}{C|x-y|^s}\right) \frac{dy \, dx}{|x-y|^n} \leq 1\]

for \(i \in \mathbb{N}\). This piece of information implies, via Theorem 7.1, that \(\|u_i\|_{L^A_B(B)} \leq C\) for some constant \(C\) independent of \(i\). The almost-compact embedding of \(L^A_B(B)\) into \(L^1(B)\) established in Lemma 9.6 then ensures that the sequence \(\{u_i\}\) is bounded in \(L^1(B)\), and that property (9.14) holds. It remains to prove property (9.15). To this purpose, we shall show that

\[(9.17) \int_B |u_i(x + h) - u_i(x)| \, dx \lesssim |h|^s\]

for \(h \in \mathbb{R}^n\) and \(i \in \mathbb{N}\). Here, and in the remaining part of this proof, the relations \(\lesssim\) and \(\approx\) hold up to constants depending on \(s, n, A\) and on the constant \(C\) appearing in equation (9.16). Fix \(h \in \mathbb{R}^n\). We have that

\[(9.18) \int_B |u_i(x + h) - u_i(x)| \, dx = \int_{B \cap \{x : |u_i(x+h) - u_i(x)| \leq 2^{s+1}|h|^s\}} |u_i(x + h) - u_i(x)| \, dx
\]

\[+ \int_{B \cap \{x : |u_i(x+h) - u_i(x)| > 2^{s+1}|h|^s\}} |u_i(x + h) - u_i(x)| \, dx \]

\[\leq 2^{s+1}|h|^s|B| + \int_{B \cap \{x : |u_i(x+h) - u_i(x)| > 2^{s+1}|h|^s\}} |u_i(x + h) - u_i(x)| \, dx\]

for \(i \in \mathbb{N}\). Fix \(i \in \mathbb{N}\), and assume that \(x \in B, y, h \in \mathbb{R}^n\). Since

\[|u_i(x + h) - u_i(x)| \leq |u_i(x + h) - u_i(y)| + |u_i(y) - u_i(x)|,\]

one has that

\[|u_i(x + h) - u_i(y)| \geq \frac{1}{2}|u_i(x + h) - u_i(x)| \quad \text{or} \quad |u_i(y) - u_i(x)| \geq \frac{1}{2}|u_i(x + h) - u_i(x)|.\]

Therefore, either

\[(9.19) \left|\{y \in B_{|h|} : |u_i(x + h) - u_i(y)| \geq \frac{1}{2}|u_i(x + h) - u_i(x)|\}\right| \geq \frac{\omega_n |h|^n}{2},\]

or

\[(9.20) \left|\{y \in B_{|h|} : |u_i(y) - u_i(x)| \geq \frac{1}{2}|u_i(x + h) - u_i(x)|\}\right| \geq \frac{\omega_n |h|^n}{2}.\]
Here, \( B_{|h|}(x) \) denotes the ball in \( \mathbb{R}^n \) centered at \( x \) and having radius \( |h| \). In what follows, we assume that \( (9.19) \) is in force, the argument in the case when \( (9.20) \) holds being completely analogous. Set

\[
S(x, h) = \{ y \in B_{|h|}(x) : |u_i(x + h) - u_i(y)| \geq \frac{1}{2}|u_i(x + h) - u_i(x)| \}.
\]

The following chain holds:

\[
(9.21) \quad \int_{\{x \in B : |u_i(x + h) - u_i(x)| > 2^{i+1}|h|^s\}} |u_i(x + h) - u_i(x)| \, dx
\]

\[
= \int_{\{x \in B : |u_i(x + h) - u_i(x)| > 2^{i+1}|h|^s\}} \frac{1}{|S(x, h)|} \int_{S(x, h)} |u_i(x + h) - u_i(x)| \, dy \, dx
\]

\[
\leq \frac{1}{|h|^n} \int_{\{x \in B : |u_i(x + h) - u_i(x)| > 2^{i+1}|h|^s\}} \int_{S(x, h)} |u_i(x + h) - u_i(y)| \, dy \, dx
\]

\[
\leq \frac{1}{|h|^n} \int_{\{(x, y) : x \in B, y \in B_{|h|}(x), |u_i(x + h) - u_i(y)| > 2^{i+1}|h|^s\}} |u_i(x + h) - u_i(y)| \, dy \, dx
\]

\[
= \frac{1}{|h|^n} \int_{\{(x, y) : x \in B, y \in B_{|h|}(x), |u_i(x + h) - u_i(y)| > 2^{i+1}|h|^s\}} \frac{|u_i(x + h) - u_i(y)|}{|x + h - y|^{n+s}} |x + h - y|^{n+s} \, dy \, dx
\]

\[
\leq |h|^s \int_{\{(x, y) : x \in B, y \in B_{|h|}(x), |u_i(x + h) - u_i(y)| > |x + h - y|^{n+s}\}} \frac{|u_i(x + h) - u_i(y)|}{|x + h - y|^{n+s}} \, dy \, dx
\]

\[
\leq |h|^s \int_{\{x \in B : |u_i(x + h) - u_i(x)| > |x + h - y|^{n+s}\}} \frac{|u_i(x) - u_i(y)|}{|x - y|^{n}} \, dy \, dx
\]

\[
\leq |h|^s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u_i(x) - u_i(y)|}{|x - y|^{n}} \right) \frac{dy \, dx}{|x - y|^{n}} \lesssim |h|^s.
\]

Note that the last inequality holds thanks to property \( (9.11) \). An application of inequalities \( (9.18) \) and \( (9.21) \) with \( u \) replaced by \( u/C \), where \( C \) is the constant appearing in equation \( (9.16) \), yields \( (9.15) \).

Let us next consider the case when \( s \in (1, n) \setminus \mathbb{N} \). The argument above applied with the sequence \( \{u_i\} \) replaced by \( \{\nabla^{|s|} u_i\} \) tells us that there exists a subsequence of \( \nabla^{|s|} u_i \), still indexed by \( i \), which converges in \( L^1(\mathcal{B}) \). Moreover, by Theorem 6.2, the sequence \( \nabla^{|s|} u_i \) is bounded in the space \( L(\hat{\mathcal{A}}_{\{s\}}, \frac{n}{|s|})(\mathbb{R}^n) \), where we are adopting the notation \( \hat{\mathcal{A}}_{\{s\}} \) introduced in the proof of Theorem 7.2. Making use of inequality \( (7.20) \) with \( s \) replaced by \( k + \{s\} \), implies that the sequence \( \{\nabla^{|s|+k} u_i\} \) is bounded in \( L(\hat{\mathcal{A}}_{\{s\}+k}, \frac{n}{|s|+k})(\mathbb{R}^n) \) for \( k = 1, 2, \ldots, |s| \). In particular, the sequence \( \nabla^k u_i \) is bounded in \( L^1(\mathcal{B}) \) for \( k = 0, 1, \ldots, |s| - 1 \). On taking, if necessary, a subsequence we may also assume that the sequence \( \{\int_{\mathcal{B}} \nabla^k u_i \, dx\} \) converges. From an application of the Poincaré inequality in \( W^{1,1}(\mathcal{B}) \), one can infer that the sequence \( \{\nabla^{|s|-1} u_i\} \) converges in \( L^1(\mathcal{B}) \). An iteration of the same argument implies that the sequence \( \{\nabla^k u_i\} \) converges in \( L^1(\mathcal{B}) \) for \( k = 0, 1, \ldots, |s| - 1 \). The convergence of a subsequence of \( \{u_i\} \) in \( L^1(\mathcal{B}) \), hence follows via the choice \( k = 0 \).

The existence of a subsequence of \( \{u_i\} \) that converges a.e. on the whole of \( \mathbb{R}^n \), follows from a diagonal argument, by an iterated application of the result established above to the sequence of balls \( \{\mathcal{B}_j\} \), centered at 0, with radius \( j \in \mathbb{N} \).

We conclude with proofs of the main results of this section.

**Proof of Theorem 9.2.** We begin by proving that property (i) implies (ii). Let \( \{u_i\} \) be a bounded sequence in \( W^{1,1}_d(\mathbb{R}^n) \). By Lemma 9.5 there exists subsequence of \( \{u_i\} \), still denoted by \( \{u_i\} \), which converges a.e. in \( \mathbb{R}^n \) to some function \( u \). Moreover, Theorem 7.2 guarantees that \( \{u_i\} \) is bounded in \( L(\hat{\mathcal{A}}, \frac{n}{|s|})(\mathbb{R}^n) \). By Fatou’s lemma, \( u \) belongs to \( L(\hat{\mathcal{A}}, \frac{n}{s})(\mathbb{R}^n) \) as well. Hence, \( \{u_i - u\} \) is a bounded sequence in \( L(\hat{\mathcal{A}}, \frac{n}{|s|})(\mathbb{R}^n) \). Owing to property \( (4.11) \), to the definition of the Orlicz-Lorentz space \( L[\hat{\mathcal{A}}, \frac{n}{s}](0, L) \) and to the fact that \( L(\hat{\mathcal{A}}, \frac{n}{s})(0, L) \)
(up to equivalent norms), one has that
\begin{equation}
\|f\|_{L_2(\mathbb{R}^n)} \approx \|r^{\frac{1}{2}} f^{**}(r)\|_{L_2(\mathbb{R}^n)}.
\end{equation}

Throughout this proof, the relations \(\lesssim\) and \(\approx\) hold up to constants depending on \(s\), \(n\) and \(A\). Thanks to Theorem 4.2, assumption (9.5) implies that the space \(L_2(\mathbb{R}^n)\) is almost compactly embedded into \(Y(E)\) for any bounded measurable set \(E\) in \(\mathbb{R}^n\). An application of property (2.25) thus tells us that the sequence \(\{u_n - u\chi_E\}\) converges to 0 in \(Y(E)\), and hence the sequence \(\{u_n\} \chi_{E}\) converges to \(u\chi_{E}\) in \(Y(E)\).

Let us next show that, conversely, (ii) implies (i). Property (ii) implies that
\begin{equation}
\|u\|_{Y(\mathbb{R}^n)} \leq C|\nabla u|_{\{s\},A,\mathbb{R}^n}
\end{equation}
for some constant \(C\) and every function \(u \in V_2^{s,A}(\mathbb{R}^n)\). Thus, by Lemma 7.6 the limit in (9.5) is finite. Let \(T_s\) be the operator defined by (1.1), with \(L = \infty\). For each \(i \in \mathbb{N}\), choose a function \(f_i \in \mathcal{M}_+(0, \infty)\), supported in the interval \([0, \frac{1}{2}]\), and such that \(\|f_i\|_{L^A(0, \frac{1}{2})} \leq 1\) and
\begin{equation}
\sup_{\|f\|_{L^A(0, \frac{1}{2})} \leq 1} \left\|T_s(\chi(0, \frac{1}{2}) f)\right\|_{\mathcal{Y}(0, \frac{1}{2})} < \left\|T_s f_i\right\|_{\mathcal{Y}(0, \frac{1}{2})} + \frac{1}{i}.
\end{equation}

Set \(m = [s]\) and, for \(i \in \mathbb{N}\), let \(u_i : \mathbb{R}^n \to [0, \infty)\) be the function defined as
\begin{equation}
u_i(x) = \int_{\omega_n}^\infty \int_{r_1}^\infty \cdots \int_{r_m}^\infty f_i(r_{m+1})r_{m+1}^{-m-1+\frac{s}{n}} dr_{m+1} \ldots dr_1 \qquad \text{for } x \in \mathbb{R}^n.
\end{equation}

Equation (7.18), with \(u\) and \(f\) replaced by \(u_i\) and \(f_i\), tells us that
\begin{equation}
|\nabla^m u_i|_{\{s\},A,\mathbb{R}^n} \lesssim \|f_i\|_{L^A(0, \frac{1}{2})} \leq 1
\end{equation}
for \(i \in \mathbb{N}\). In addition, we have \(|\{\nabla^k u_i > t\}| < \infty\) for every \(t > 0\) and \(k = 0, \ldots, m\) since \(u_i\) is compactly supported. Therefore, the function \(u_i \in V_2^{s,A}(\mathbb{R}^n)\). Since the supports of the functions \(u_i\) are uniformly bounded for \(i \in \mathbb{N}\), assumption (ii) ensures that there exists a subsequence of \(\{u_i\}\), still denoted by \(\{u_i\}\), which is convergent in \(Y(\mathbb{R}^n)\). Thanks to the properties of \(f_i\), one has that \(\lim_{i \to \infty} u_i = 0\), whence
\begin{equation}
\lim_{i \to \infty} \|u_i\|_{Y(\mathbb{R}^n)} = 0.
\end{equation}

On the other hand, by inequality (7.12), with \(u\) and \(f\) replaced by \(u_i\) and \(f_i\),
\begin{equation}
u_i(x) \gtrsim \int_{2\omega_n}^\infty f_i(r)r^{-1+\frac{s}{n}} dr \quad \text{for } x \in \mathbb{R}^n.
\end{equation}

Consequently,
\begin{equation}
\|u_i\|_{Y(\mathbb{R}^n)} \gtrsim \|T_s f_i\|_{\mathcal{Y}(0, \frac{1}{2})}
\end{equation}
for \(i \in \mathbb{N}\). Coupling equation (9.24) with (9.25) yields
\begin{equation}
\lim_{i \to \infty} \|T_s f_i\|_{\mathcal{Y}(0, \frac{1}{2})} = 0.
\end{equation}

Property (i) hence follows, via equation (9.23) and the monotonicity with respect to \(L\) of the expression under the limit.

The proof of the equivalence of properties (i) and (iii) is analogous to that of the equivalence of (i) and (ii), and is omitted for brevity.

\textbf{Proof of Theorem 9.7.} As in the previous proof, we limit ourselves to showing the equivalence of properties (i) and (ii). We first prove that (i) implies (ii). Let \(\{u_n\}\) be a bounded sequence in \(V_2^{s,A}(\mathbb{R}^n)\). By Lemma 9.5 there exists a subsequence of \(\{u_n\}\), still denoted by \(\{u_n\}\), which converges a.e. in \(\mathbb{R}^n\) to some function \(u\). Furthermore, assumption (i), coupled with Theorem 4.1 ensures that \(\{u_n\}\) is a bounded sequence in \(L^{\frac{A}{2}}(\mathbb{R}^n)\). By Fatou’s lemma, the function \(u\) belongs to \(L^{\frac{A}{2}}(\mathbb{R}^n)\), and hence \(\{u_n - u\}\) is a bounded sequence in \(L^{\frac{A}{2}}(\mathbb{R}^n)\). Assumption (9.1), combined with [7] Theorem 4.17.7, tells us that the space \(L^{\frac{A}{2}}(E)\) is almost compactly
embedded into $L^B(E)$ for any bounded measurable set $E$ in $\mathbb{R}^n$. By applying property (2.25), we thus obtain that the sequence $\{u_i - u\}$ converges to 0 in $L^B(E)$, whence the sequence $\{u_i\chi_E\}$ is convergent in $L^B(E)$.

We conclude by proving that (ii) implies (i). Assume that property (ii) holds. Thanks to Theorem 9.3, this piece of information ensures that condition (9.5) holds with $\sum (0, L) = L^B(0, L)$. Owing to Theorem B and Theorem 4.2, this condition in its turn implies that the space $L(\tilde{\mathcal{A}}, \frac{n}{s})(0, 1)$ is almost-compactly embedded into $L^B(0, 1)$. On testing this almost-compact embedding on characteristic functions of intervals of the form $(0, L)$ with $L \in (0, 1)$, one infers that

\begin{equation}
\lim_{L \to 0^+} \frac{\|\chi_{(0,L)}\|_{L^B(0,1)}}{\|\chi_{(0,L)}\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)}} = 0.
\end{equation}

By Theorem B and Theorem E,

$$
\|f\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)} \approx \|\tilde{f}\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)}
$$

for every function $f \in \mathcal{M}(0,1)$. Here, and in what follows, equivalence is up to multiplicative constants depending only on $n$, $s$ and $A$. In particular,

\begin{equation}
\|\chi_{(0,L)}\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)} \approx \|\chi_{(0,L)}\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)} \approx \frac{L}{\|\tilde{\chi}_{s}(0,L)(r)\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)}}
\end{equation}

for $L \in (0, 1)$, up to equivalence constants independent of $L$. Notice that the first equivalence holds thanks to a general property of rearrangement-invariant norms [9, Chapter 2, Theorem 5.2]. On the other hand,

\begin{equation}
\|r^{\frac{s}{n}}\chi_{(0,L)}(r)\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)} \approx \|r^{\frac{s}{n}}\chi_{(0,L)}(r)\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)} + L\|r^{-1+\frac{s}{n}}\chi_{(L,1)}(r)\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)}
\end{equation}

for $L \in (0, 1)$. In particular, if $L \in (0, \frac{1}{2})$, then

\begin{equation}
\|r^{-1+\frac{s}{n}}\chi_{(L,1)}(r)\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)} \geq \|r^{-1+\frac{s}{n}}\chi_{(L,2L)}(r)\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)} \geq \|r^{-1+\frac{s}{n}}\chi_{(L,2L)}(r)\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)} = 2L^{-1+\frac{s}{n}}\|\chi_{(0,L)}(r)\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)} \approx 2^{-1+\frac{s}{n}}L^{-1}\|r^{\frac{s}{n}}\chi_{(0,L)}(r)\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)}.
\end{equation}

Equations (9.27)–(9.29) tell us that

\begin{equation}
\|\chi_{(0,L)}\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)} \approx \frac{1}{\|r^{\frac{s}{n}}\chi_{(L,1)}(r)\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)}}
\end{equation}

as $L \to 0^+$. Furthermore, owing to [34, Lemma 1],

\begin{equation}
\|r^{-1+\frac{s}{n}}\chi_{(L,1)}(r)\|_{L(\tilde{\mathcal{A}}, \frac{n}{s})(0,1)} \approx A^{-1}(1/L)
\end{equation}

as $L \to 0^+$. On the other hand, $\|\chi_{(0,L)}\|_{L^B(0,1)} = \frac{1}{B^{-1}(1/L)}$ for $L \in (0, 1)$. This equality, and equations (9.30) and (9.31) entail that condition (9.26) is equivalent to

\begin{equation}
\lim_{t \to \infty} \frac{A^{-1}(t)}{B^{-1}(t)} = 0,
\end{equation}

and the latter is in its turn equivalent to (9.1). \hfill \square

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