ON SPARSE PERFECT POWERS

A. MOSCARIELLO

Abstract. This work is devoted to proving that, given an integer \( x \geq 2 \), there are infinitely many perfect powers, coprime with \( x \), having exactly \( k \geq 3 \) non-zero digits in their base \( x \) representation, except for the case \( x = 2, k = 4 \), for which a known finiteness result by Corvaja and Zannier holds.

Introduction

Let \( k \) and \( x \) be positive integers, with \( x \geq 2 \). In this work, we will study perfect powers having exactly \( k \) non-zero digits in their representation in a given basis \( x \). These perfect powers are exactly (up to dividing by a suitable factor) the set solutions of the Diophantine equation

\[
y^d = c_0 + \sum_{i=1}^{k-1} c_i x^{m_i},
\]

with \( y, d \) positive integers greater than 1, and \( c_0, c_1, \ldots, c_{k-1} \in \{1, \ldots, x - 1\} \) and \( m_1 < \cdots < m_{k-1} \) positive integers. We call perfect powers having a fixed number of non-zero digits \textit{sparse}, borrowing the terminology used for polynomials (a \textit{sparse} polynomial is a polynomial having \textit{relatively few} non-zero terms, compared to its degree) Special cases of this innocent problem have been widely studied in the literature, and its appearance is quite deceiving: for instance, the lowest case, obtained with the positions \( k = 2, c_0 = c_1 = 1 \), is the well-known Catalan’s conjecture, first proposed in 1844, which stood open for nearly 150 years before being proved by Mihailescu (cf. [10]) in the case \( x = 2 \). Furthermore, the case \( k = 2, x > 2 \) (i.e., perfect powers having exactly two digits in their base \( x > 2 \) representation) is still open (cf. [11, §4.4.3]), and is related to the well-known ABC conjecture.

This class of problems also presents some ties to algebraic geometry. In fact, Corvaja and Zannier showed in [5] that solutions of an equation of the form (1) are associated with \( S \)-integral points on certain projective varieties. For instance, assume for the sake of simplicity that \( x = p \) is a prime number, and that \( k, d \) are fixed and \( c_0 = c_1 = \cdots = c_{k-1} = 1 \) in equation (1). Consider, in the projective space \( \mathbb{P}_k \), the variety \( \mathbb{P}_k \setminus D \), where \( D \) denotes the divisor consisting of the \( k - 1 \) lines \( X_i = 0 \), for \( i = 0, \ldots, k - 2 \), and the hypersurface

\[
X_{k-1}^d = X_0^d + \sum_{i=1}^{k-2} X_0^{d-1} X_i,
\]

and let \( S = \{\infty, p\} \). Then, \( S \)-integral points of this variety are such that the values \( y_i = \frac{X_i}{X_0} \), where \( i = 1, \ldots, k - 2 \), and \( y_{k-1} = \left( \frac{X_{k-1}}{X_0} \right)^d - 1 - \sum_{i=1}^{k-2} \frac{X_i}{X_0} \) are all

2020 Mathematics Subject Classification. 11D41, 11P99.

Key words and phrases. base representation, sparse powers.
S-units. Also, the elements \( y_i \) all have the form \( \pm p^{m_i} \) and are such that \( 1 + y_1 + \cdots + y_{k-1} \) is a \( d \)th perfect power, and are thus solutions of equation (1). Now, the study of these points, and their distribution, can also be seen as a particular instance of a conjecture by Lang and Vojta (see [9]); in our context, this conjecture would imply that the set of \( S \)-integral points on \( \mathbb{P}_k \setminus D \) is not Zariski dense.

Besides Mihăilescu’s Theorem, the more general case \( k = 2 \) is still open; however, there is some evidence suggesting that there may be only a finite number of perfect powers having exactly two non-zero digits in any given base \( x \). The case \( k = 3 \) has been studied recently (cf. [2], [7]); in particular, Corvaja and Zannier developed in [7] an approach using \( v \)-adic convergence of analytic series at \( S \)-unit points to reduce this problem to the study of polynomial identities involving lacunary polynomial powers (i.e. polynomial powers \( P(T)^d \) having a fixed number \( k \) of terms). This method allowed them to provide a classification of perfect powers having exactly three non-zero digits.

Specifically, for \( x = 2 \) they obtained the following characterization.

**Theorem 1** ([4]). For \( d \geq 2 \) integer, the perfect \( d \)th powers in \( \mathbb{N} \) having at most three non-zero digits in the binary scale form the union of finitely many sets of the shape \( \{ q2^{md} \mid m \in \mathbb{N} \} \) and, if \( d = 2 \), also the set \( \{(2^a + 2^b)^2 \mid a, b \in \mathbb{N}\} \).

In the same work, the authors comment that their method can be used to obtain results equivalent to Theorem 1 for any given base \( x \). Actually, Theorem 1 states that if \( k = 3 \), \( x = 2 \) there are only a finite number of exceptional solutions, and the infinite family \( y = (2^a + 1) \), \( d = 2 \), corresponding to the polynomial identity \((T + 1)^2 = T^2 + 2T + 1\).

Intuitively, one might expect that as the number of terms \( k \) increases, the number of polynomial powers \( P(T)^d \) having exactly \( k \) terms increases as well. Moreover, since Corvaja and Zannier’s method can be adjusted to study perfect powers with \( k \geq 3 \) non-zero digits, under certain assumption, we might infer that there is an increasing number of infinite families of solutions to equation (1).

However, this is not necessarily the case. In fact, while studying the case \( k = 4 \), Corvaja and Zannier obtained families of lacunary polynomial powers having exactly 4 terms that are not related to solutions of the Diophantine equation \( y^d = c_0 + c_12^{m_1} + c_22^{m_2} + c_32^{m_3} \).

Actually, they proved that this Diophantine equation has only finitely many solutions.

**Theorem 2** ([4, Theorem 1.1]). There are only finitely many odd perfect powers in \( \mathbb{N} \) having precisely four non-zero digits in their representation in the binary scale.

In this work, we prove that these results are exceptional. Namely, we show that it is possible to obtain infinite families of perfect powers (coprime with \( x \)) having exactly \( k \geq 3 \) non-zero digits in their base \( x \geq 2 \) representation (moreover, we will show that we can almost always provide infinite families of perfect squares) for all values of \( x \) and \( k \), except for the case \( x = 2, k = 4 \) studied by Corvaja and Zannier (Theorem 2).

1. **Main result**

Consider the equation

\[
y^d = c_0 + \sum_{i=1}^{k-1} c_i x^{m_i}.
\]
In this work we want to determine whether the Diophantine equation (1) admits infinitely many solutions, for given values of $x$ and $k$. Arguing that some solutions can be induced from polynomial identities, and since intuitively, as the number of terms $k$ increase, we can guess that there are more and more polynomial powers $P(T)^d$ having exactly $k$ non-zero terms, our expectation is that, as $k$ increases, it is easier to find infinite families of perfect powers with exactly $k$ non-zero digits; our approach will focus on finding such families in some specific setting. Actually, we will see that finiteness results can only be obtained in the cases $k = 2$ and $k = 4, x = 2$.

First, notice that the natural expansion of $(1 + X_1 + \cdots + X_{p-1})^d \in \mathbb{C}[X_1, \ldots, X_{p-1}]$ has exactly $(p-1+d)_d$ distinct terms. Therefore, we can choose a suitable specialization $X_i = x^{\alpha_i}$, with positive integers $\alpha_i$ such that different terms of the expansion yield different powers of $x$; under the assumption that $x$ is greater than all coefficients of this expansion, we can obtain a correspondence between the terms of this expansion and the digits of our desired perfect power, and thus obtain perfect powers whose base $x$ representation has exactly $(p-1+d)_d$ non-zero digits. Similarly, under the same assumptions, we can choose a set of exponents $\alpha_i$ such that there are exactly $\beta$ equalities among those terms, for relatively small values of $\beta$, thus obtaining perfect powers having exactly $(p+d)_d - \beta$ non-zero digits in their base $x$ representation (where $\beta$ hopefully takes all values between 0 and $(p-1+d)_d - (p^{-2+d}_d - 1)$).

From this argument it is possible to obtain, for a fixed value of $d$, families of infinite perfect powers having exactly $(p+d)_d - \beta$ non-zero digits in their base $x$ representation; such a construction can be done with some work (with some modifications on the arguments we will use in the next parts of this paper), remembering that $x$ has to be larger than any coefficient appearing in the expansion $(1 + X_1 + \cdots + X_{p-1})^d \in \mathbb{C}[X_1, \ldots, X_{p-1}]$ and making sure to find suitable constructions for all values of $\beta \in [0, \ldots, (p+d)_d - (p^{-d+1}_d)]$.

This simple idea naturally directs us to the best case: the integers $(\frac{1}{2})$ form a sequence of relatively small intervals partitioning $\mathbb{N}$, and the coefficients of the expansion of $(1 + X_1 + \cdots + X_{p-1})^2$ are all either 1 or 2. For $p \geq 1$ and $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{p-1}$ we can expand $(1 + X_1 + \cdots + X_{p-1})^2$ in the following way:

\[
(x^{\alpha_0} + x^{\alpha_1} + \cdots + x^{\alpha_{p-1}})^2 = x^{2\alpha_0} + (2x^{\alpha_0+\alpha_1}) + x^{2\alpha_1} + (2x^{\alpha_2+\alpha_1} + 2x^{\alpha_2+\alpha_0}) + x^{2\alpha_2} + \cdots + x^{2\alpha_{p-3}} + \left(\sum_{i=0}^{p-3} 2x^{\alpha_{p-2}+\alpha_i}\right) + x^{2\alpha_{p-2}} + \left(\sum_{i=0}^{p-2} 2x^{\alpha_{p-1}+\alpha_i}\right) + x^{2\alpha_{p-1}}.
\]

Clearly $x$ is always not less than all the coefficients, and if $x > 2$, this expression can be used as a starting point to yield a representation. However, if $x = 2$, this expression needs to be slightly adjusted to become a binary representation, and for this motive we might have to slightly alter our construction; thus we will discuss the case $x = 2$ separately from the rest.

1.1. **Perfect powers with arbitrary number of binary digits.** Clearly, the only admissible digits in the binary scale are 0 and 1, thus, in base 2, equation (1) becomes

\[
y^d = 1 + 2^{\alpha_1} + \cdots + 2^{\alpha_{k-1}}.
\]

The case $k \leq 4$ has been widely studied in the literature. A well-known Theorem by Mihailescu states that there is only one odd perfect power having exactly two non-zero digits, that is, $3^2 = 1 + 2^3$. Recently, Szalay (see [12]) completely solved the equation
$y^2 = 2^a + 2^b + 1$. Further, the equation $y^n = 2^a + 2^b + 1$, with $n \geq 2$ has been completely solved by Bennett et al. in [1], thus completing the study of perfect powers having exactly 3 non-zero binary digits. In this context, it is worth noticing that the expansion $(1 + 2^{\alpha_1})^2$ (which is a trivial case of our argument) yields an infinite family of perfect squares with this property - see also Theorem [4].

In the same work [1], the authors also solved completely the equation $y^n = 2^a + 2^b + 2^c + 1$ for $n \geq 5$, dealing with perfect powers having 4 non-zero binary digits. In this context, Theorem [2] states that there are only finitely many such perfect powers not divisible by 2.

In this work, we will then focus on the remaining cases, assuming $k \geq 5$. Clearly, Equation (1) can be adjusted to obtain the following binary representation (remember that $\alpha$ contains pairwise distinct terms, ranging between $2^\alpha$ and $2^\beta$)

$$
\begin{align*}
\text{(x)} & \quad (2^{\alpha_0} + 2^{\alpha_1} + \ldots + 2^{\alpha_{p-1}})^2 = 2^{2\alpha_0} + (2^{\alpha_0+\alpha_1+1}) + 2^{2\alpha_1} + (2^{\alpha_2+\alpha_1+1} + 2^{\alpha_2+\alpha_0+1}) + 2^{2\alpha_2} \\
& \quad \quad \quad + \ldots + 2^{2\alpha_{p-1}} + \left(\sum_{i=0}^{p-2} 2^{\alpha_{p-2}+\alpha_1+1}\right) + 2^{2\alpha_{p-2}} + \left(\sum_{i=0}^{p-2} 2^{\alpha_p+\alpha_1+1}\right) + 2^{2\alpha_{p-1}}.
\end{align*}
$$

We rearranged the expression in this way since, for $i = 1, \ldots, p - 1$ the $i$th bracket contains pairwise distinct terms, ranging between $2^{\alpha_i+\alpha_0+1} = 2^{\alpha_i+1}$ and $2^{\alpha_i+\alpha_{i-1}+1}$. Thus if $\alpha_i \geq \alpha_{i-1} + 2$ every term of the $i$th bracket is strictly lower than $2^{2\alpha_i}$, while if $\alpha_i \geq 2\alpha_{i-1} - 1$ then all terms of that bracket are larger than $2^{2\alpha_i-1}$, with equality happening if and only if $2^{\alpha_i+\alpha_0+1} = 2^{2\alpha_i-1}$, that is, if and only if $\alpha_i = 2\alpha_{i-1} - 1$. Hence, if $\alpha_i \geq 2\alpha_{i-1} - 1$, equation (x) yields a perfect square having $(\frac{p+1}{2})$ terms, with at most $p - 2$ coincident terms, given by the number of indexes such that $\alpha_i = 2\alpha_{i-1} - 1$.

Therefore, we can easily prove the following.

**Lemma 3.** Let $k$ be a positive integer greater than 4 not of the form $(\frac{p}{2}) + 1$, for a positive integer $p$. Then there exist infinitely many odd perfect squares having exactly $k$ non-zero digits in their representation in the binary scale.

**Proof.** Write $k$ as $k = (\frac{p+1}{2}) - \beta$, with $\beta \in \{0, \ldots, p - 2\}$. Define a sequence $(\alpha_1, \ldots, \alpha_{p-1})$ of positive integers such that

$$
\begin{cases}
\alpha_1 \geq 3, \\
\alpha_i = 2\alpha_{i-1} - 1 \text{ for } i = 2, \ldots, \beta + 1, \\
\alpha_i > 2\alpha_{i-1} - 1 \text{ for } i > \beta + 2.
\end{cases}
$$

Then, arguing as in the previous paragraphs, we can show that there are exactly $\beta$ coincident terms in the expansion (x); moreover, those coincident terms are of the form $2^{2\alpha_{i-1}}$ and $2^{\alpha_i+\alpha_0+1}$, which then form the term $2^{2\alpha_{i-1} + 2^{\alpha_i+\alpha_0+1}} = 2^{\alpha_i+\alpha_0+2} < 2^{\alpha_i+\alpha_1+1}$ (since $\alpha_1 \geq 3$); thus the positive integer $y = (1 + 2^{\alpha_1} + \cdots + 2^{\alpha_{p-1}})$ is such that $y^2$ has exactly $(\frac{p+1}{2}) - \beta = k$ non-zero digits in its representation in the binary scale.

Notice that if $k = (\frac{p}{2}) + 1$ (i.e. $\beta = p - 1$) this method would not work. Thus we have to prove this case in a slightly different way.

**Lemma 4.** Let $k$ be a positive integer greater than 4 of the form $(\frac{p}{2}) + 1$, with $p$ a positive integer. Then there are infinitely many odd perfect squares having exactly $k$ non-zero digits in their binary representation.
Proof. Notice that the binary representation of \((1 + 2^{\alpha_1} + 2^{\alpha_1+1} + 2^{\alpha_1+2})^2\) is given by
\[
(1 + 2^{\alpha_1} + 2^{\alpha_1+1} + 2^{\alpha_1+2})^2 = 1 + 2^{\alpha_1+1} + 2^{\alpha_1+2} + 2^{\alpha_1+3} + 2^{2\alpha_1} + 2^{2\alpha_1+4} + 2^{2\alpha_1+5},
\]
hence it has exactly \(7 = \binom{4}{2} + 1\) non-zero digits; while, if \(k \geq 11\) define as before an infinite sequence \((\alpha_1, \ldots, \alpha_{p-1})\) of positive integers such that
\[
\begin{cases}
\alpha_1 \geq 4, \\
\alpha_i = \alpha_1 + i - 1 \text{ for } i = 2, 3, \\
\alpha_4 = 2\alpha_1 + 4, \\
\alpha_i = 2\alpha_{i-1} - 1 \text{ for } i > 4.
\end{cases}
\]
Let \(y = 1 + 2^{\alpha_1} + 2^{\alpha_2} + \cdots + 2^{\alpha_{p-1}}\). Then the expansion \((\square)\) of \(y^2\) has \(\binom{p+1}{2}\) terms; let us count how many equalities there are between those terms:
- There are 3 equalities depending on \(\alpha_1, \alpha_2, \alpha_3\) only, which we deduce from the binary representation of \((1 + 2^{\alpha_1} + 2^{\alpha_2} + 2^{\alpha_3})^2\) (which has \(\binom{3}{2} - 3 = 7\) non-zero digits);
- There are \(p - 4\) equalities, one for each of the \(\alpha_i\), with \(i > 4\); these \(\alpha_i\) are chosen so that every term of the form \(2^{2\alpha_i}\) is equal to the maximum term preceding it in the expansion \((\square)\).

Therefore there are exactly \(p - 1\) equalities, and since each of the terms obtained by adding these coincident terms is distinct from any other term of the expansion since \(\alpha_1 \geq 4\), we deduce that \(y^2\) has exactly \(\binom{p+1}{2} - (p - 1) = \binom{p}{2} + 1 = k\) non-zero digits in its representation in the binary scale. \(\square\)

Combining the last two results, we obtain the following result.

**Theorem 5.** Let \(k \geq 2\) be an integer.

- (1) If \(k \in \{2, 4\}\), then there are only finitely many odd perfect powers in \(\mathbb{N}\) having precisely \(k\) non-zero digits in their representation in the binary scale.
- (2) If \(k \notin \{2, 4\}\), then there are infinitely many odd perfect squares in \(\mathbb{N}\) having precisely \(k\) non-zero digits in their representation in the binary scale.

**1.2. Perfect powers with arbitrary number of base \(x \geq 3\) digits.** Let \(x \geq 3\). Determining whether the Diophantine equation \(y^d = c_1x^{m_1} + c_2\) admits finitely or infinitely many solution is a very challenging open problem, studied by several authors (see for instance [I] §4.4.3 for results concerning this class of Diophantine equations); however, it is known that, for fixed \(x \geq 2\), this equation has at most finitely many solutions in integers \(0 \leq c_1, c_2 < x\), \(y\) coprime to \(x\) and \(d \geq 2\), and thus, given a fixed scale \(x \geq 3\), there are at most finitely many perfect powers having exactly \(k = 2\) non-zero digits in their base \(x\) representation.

The case \(k = 3\) has been studied by Bennet and Scheerer (see [B]) for certain values of \(x\) (namely \(x \in \{3, 4, 5, 8, 16\}\)). For our purposes, it suffices to consider the expansion \((x^a + 1)^2 = x^{2a} + 2x^a + 1\) to conclude that there are infinitely many perfect squares not divisible by \(x\) which base \(x\) representation has exactly three non-zero digits.

Similarly, it is easy to see that the perfect cube \((x^a + 1)^3 = x^{3a} + 3x^{2a} + 3x^a + 1\) has exactly four non-zero digits in its base \(x\) representation; thus implying that there are infinitely many perfect cubes having exactly four non-zero digits in their base \(x\) representation.
However, the examples used in the two cases $k = 3, 4$ cannot be used in the general case; in fact, for larger values of $d$, the coefficients of the expansion $(x^a + 1)^d$ become very large, and since we need that $x$ is larger than all of these coefficients, for increasingly many values of $x$ this construction would not yield a base $x$ representation (as each coefficient could be associated with more than one digit).

We approach this case similarly to the case $x = 2$. Consider the expansion (fix $\alpha_0 = 0$)

\[
(x^{\alpha_0} + x^{\alpha_1} + \cdots + x^{\alpha_{p-1}})^2 = x^{2\alpha_0} + (2x^{\alpha_0+\alpha_1}) + x^{2\alpha_1} + (2x^{\alpha_2+\alpha_1} + 2x^{\alpha_2+\alpha_0}) + x^{2\alpha_2} + \cdots + x^{2\alpha_{p-3}} + \left(\sum_{i=0}^{p-3} 2x^{\alpha_{p-2}+\alpha_i}\right) + x^{2\alpha_{p-2}} + \left(\sum_{i=0}^{p-2} 2x^{\alpha_{p-1}+\alpha_i}\right) + x^{2\alpha_{p-1}}.
\]

As before, for $i = 1, \ldots, p - 1$ the $i$th bracket contains pairwise distinct terms, ranging between $x^{\alpha_i+\alpha_0} = x^{\alpha_i}$ and $x^{\alpha_i+\alpha_{i-1}}$. Thus if $\alpha_i \geq \alpha_{i-1} + 1$ all these terms are strictly lower than $x^{2\alpha_i}$, while if $\alpha_i \geq 2\alpha_{i-1}$ we have $\alpha_i + \alpha_{i-1} > \ldots > \alpha_i + \alpha_0 = \alpha_0 \geq 2\alpha_{i-1}$, hence all the terms are strictly larger than $x^{2\alpha_{i-1}}$, with equality happening if and only if $\alpha_i = 2\alpha_{i-1}$, which would imply $x^{\alpha_i+\alpha_{i+1}} = x^{2\alpha_{i-1}}$. Hence, if $\alpha_i \geq 2\alpha_{i-1}$, the equation (*) gives a perfect square having exactly $(\frac{p+1}{2})$ terms, and, just like we did in the case $x = 2$, we can fiddle with our exponents in order to obtain the desired number of equalities (between 0 and $p - 2$). Therefore, the following result is very straightforward.

**Lemma 6.** Let $k$ be a positive integer greater than four not of the form $(\frac{p}{2}) + 1$, with $p$ positive integer, and let $x \geq 3$ be an integer. Then there exist infinitely many perfect squares, not divisible by $x$, having exactly $k$ non-zero digits in their base $x$ representation.

*Proof.* Write $k$ as $k = (\frac{p+1}{2}) - \beta$, with $\beta \in \{0, \ldots, p - 2\}$. Define a sequence $(\alpha_1, \ldots, \alpha_{p-1})$ of positive integers (depending on $\alpha_1$) satisfying the following conditions:

\[
\begin{align*}
\alpha_1 &\geq 3, \\
\alpha_i &\geq 2\alpha_{i-1} \text{ for } i = 2, \ldots, \beta + 1, \\
\alpha_i &> 2\alpha_{i-1} \text{ for } i > \beta + 2.
\end{align*}
\]

Then it is straightforward (arguing as in Lemma 3) to prove that the integer $y = (1 + x^{\alpha_1} + \cdots + x^{\alpha_{p-1}})$ is such that $y^2$ has exactly $(\frac{p+1}{2}) - \beta = k$ non-zero digits in its base $x$ representation. \hfill $\square$

As in the previous Section, the remaining case $k = (\frac{p}{2}) + 1$ is not covered by the previous construction, but requires some slight adjustements to be made, according to the value of $x$; here, we will need to split this case in three subcases.

**Lemma 7.** Let $k \geq 7$ be an integer of the form $(\frac{p}{2}) + 1$, for some positive integer $p$. Then there are infinitely many perfect squares not divisible by 3 having exactly $k$ non-zero digits in their base 3 representation.

*Proof.* First, we consider some special cases:

- The perfect square $(1 + 3^{\alpha_1} + 3^{\alpha_1+1} + 3^{\alpha_1+2})^2$ has exactly 7 non-zero digits in its base 3 representation.
- The expansion $(1 + 3^{\alpha_1} + 3^{\alpha_1+1} + 3^{2\alpha_1} + 3^{2\alpha_1+1})^2$ yields perfect squares having exactly $11 = (\frac{p}{2}) + 1$ non-zero digits in their base 3 representation.
For \( k > 11 \), consider a sequence of positive integers \((\alpha_1, \ldots, \alpha_{p-1})\) such that
\[
\begin{align*}
\alpha_1 &\geq 4, \\
\alpha_2 &\equiv \alpha_1 + 1, \\
\alpha_i &\equiv 2\alpha_1 + i - 3 \text{ for } i = 3, 4, \\
\alpha_i &\equiv 2\alpha_{i-1} \text{ for } i \geq 5.
\end{align*}
\]

Then, by taking the integer \( y = 1 + 3^{\alpha_1} + 3^{\alpha_2} + \cdots + 3^{\alpha_p} \), notice that, for the expansion \((*)\) of \( y^2 \), the following hold:

- There are exactly four equalities between terms of \((*)\) depending on our choice of \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \), which follow from the expansion of \((1 + 3^{\alpha_1} + 3^{\alpha_2} + 3^{\alpha_3} + 3^{\alpha_4})^2 \) (which has exactly 11 non-zero digits in its base 3 representation).
- There are \( p - 5 \) equalities, one for each \( \alpha_i \), with \( i = 5, 6, \ldots, p - 1 \), following from the condition \( \alpha_i = 2\alpha_{i-1} \).

As before, these equalities are such that the terms obtained are distinct from any other term in \((*)\) and that each term of the expansion yields a digit in the base 3 representation of \( y^2 \), which then contains exactly \( (p+1) - 4 - (p-5) = \binom{p}{2} + 1 = k \) non-zero digits.

\[\square\]

**Lemma 8.** Let \( k \geq 4 \) be an integer of the form \( \binom{p}{2} + 1 \), for a positive integer \( p \).

1. There are infinitely many perfect squares not divisible by 4 having exactly \( k \) non-zero digits in their base 4 representation.
2. There are infinitely many perfect squares not divisible by 5 having exactly \( k \) non-zero digits in their base 5 representation.

**Proof.** (1) Fix \( \alpha_1 \geq 2 \), and define a sequence \((\alpha_1, \ldots, \alpha_{p-2})\) of positive integers such that \( \alpha_i > 2\alpha_{i-1} \) for every \( i = 2, \ldots, p-2 \). Take now the integer \( y = 3 \cdot 4^{\alpha_{p-2}} + 2 \left( \sum_{i=0}^{p-3} 4^{\alpha_i} \right) \), with \( \alpha_0 = 0 \) (remember that \( p \geq 3 \)). Then clearly
\[
y^2 = 9 \cdot 4^{2\alpha_{p-2}} + 3 \left( \sum_{i=0}^{p-3} 4^{\alpha_{p-2}+\alpha_i+1} \right) + 4 \left( \sum_{i=0}^{p-3} 4^{\alpha_i} \right)^2.
\]

Now, examining the base 4 representation associated to the right-hand side, the first term yields exactly two non-zero digits, the second one has \( p - 2 \) non-zero digits, while the last bracket gives exactly \( \binom{p-1}{2} \) non-zero digits (by expanding the square and remembering the conditions on \( \alpha_i \)); further, our conditions are such that all terms appearing on the right-hand side are pairwise distinct. Thus the base 4 representation of \( y^2 \) has exactly \( \binom{p-1}{2} + (p - 2) + 2 = \binom{p}{2} + 1 = k \) non-zero digits.

(2) Similarly, for \( \alpha_1 \geq 2 \), define a sequence \((\alpha_1, \ldots, \alpha_{p-2})\) of positive integers such that \( \alpha_i > 2\alpha_{i-1} \) for any \( i = 2, \ldots, p - 2 \), and take \( y = 2 \cdot 5^{\alpha_{p-2}} + 2 \cdot 5^{\alpha_{p-3}} + \left( \sum_{i=0}^{p-4} 5^{\alpha_i} \right) \), with \( \alpha_0 = 0 \). Then
\[
y^2 = 4 \cdot 5^{2\alpha_{p-2}} + 8 \cdot 5^{\alpha_{p-2}+\alpha_{p-3}} + 4 \cdot 5^{2\alpha_{p-3}} + \cdots.
\]
our construction does not work in the case
that it is impossible to impose more than one equality among the exponents of \( 2 \) at least the four terms \( 1 = 3 \) and that in the general expansion

The previous result affirms that the known finiteness results of Mihailescu (for \( k = 2 \)) and Corvaja-Zannier (if \( k = 4 \) and \( x = 2 \)) are the only exceptions to the general rule. However, our construction does not work in the case \( x = 3, k = 4 \); in fact, in that case it is easy to see that it is impossible to impose more than one equality among the exponents of

\[
(1 + 3^{\alpha_1} + 3^{\alpha_2})^2 = 1 + 2 \cdot 3^{\alpha_1} + 3^{2\alpha_1} + (2 \cdot 3^{\alpha_2} + 2 \cdot 3^{\alpha_2 + \alpha_1}) + 3^{2\alpha_2},
\]

and that in the general expansion

\[
(3^{\alpha_0} + 3^{\alpha_1} + \cdots + 3^{\alpha_{p-1}})^2 = 3^{2\alpha_0} + (2 \cdot 3^{\alpha_0 + \alpha_1}) + 3^{2\alpha_1} + (2 \cdot 3^{\alpha_2 + \alpha_1} + 2 \cdot 3^{\alpha_2 + \alpha_0}) + 3^{2\alpha_2}
\]

\[
+ \cdots + 3^{2\alpha_{p-3}} + \left( \sum_{i=0}^{p-3} 2 \cdot 3^{\alpha_{p-2} + \alpha_i} \right) + 3^{2\alpha_{p-2}} + \left( \sum_{i=0}^{p-2} 2 \cdot 3^{\alpha_{p-1} + \alpha_i} \right) + 3^{2\alpha_{p-1}}
\]

at least the four terms \( 1 = 3^{2\alpha_0}, 2 \cdot 3^{\alpha_1}, 2 \cdot 3^{\alpha_{p-1} + \alpha_{p-2}}, 3^{2\alpha_{p-1}} \) have different exponents from the others, and thus are very hard to remove from the final base 3 representation that will

Lemma 9. Let \( x \geq 6 \) and \( k \geq 4 \) be integers, with \( k \) having the form \( \left( \begin{array}{c} p \end{array} \right) + 1 \), for some positive integer \( p \). Then there are infinitely many perfect squares not divisible by \( x \) having exactly \( k \) non-zero digits in their base \( x \) representation.

Proof. Let \( \sigma = \lceil \sqrt{x+1} \rceil \). Since \( x \geq 6 \), clearly \( 2\sigma \leq x \) and \( x < \sigma^2 < 2x \); now, for \( \alpha_1 \geq 2 \), define a sequence \( (\alpha_1, \ldots, \alpha_{p-2}) \) of positive integers such that \( \alpha_i > 2\alpha_{i-1} \) for all \( i = 2, \ldots, p-2 \), and take \( y = \sigma x^{\alpha_{p-2}} + x^{\alpha_{p-3}} + \cdots + x^{\alpha_1} + 1 \). Clearly, fixing \( \alpha_0 = 0 \), we have

\[
y^2 = \sigma^2 x^{2\alpha_{p-2}} + \left( \sum_{i=0}^{p-3} 2\sigma x^{\alpha_{p-2} + \alpha_i} \right) + \left( \sum_{i=0}^{p-3} x^{\alpha_i} \right)^2.
\]

Our choice of \( \sigma \) is such that the first term of the right-hand side has exactly 2 non-zero digits in its base \( x \) representation, while the second one has exactly \( p-2 \) non-zero digits, and the third one has exactly \( \left( \begin{array}{c} p-1 \end{array} \right) \); since all powers of \( x \) appearing in this expansion have distinct exponents, we immediately deduce that the base \( x \) representation of \( y^2 \) has exactly \( \left( \begin{array}{c} p-1 \end{array} \right) + p = \left( \begin{array}{c} p \end{array} \right) + 1 = k \) non-zero digits.

We can combine all the results of this section to achieve the desired result:

Theorem 10. Let \( x \geq 2 \) and \( k \geq 3 \) be integers with \( (x, k) \notin \{(2, 4), (3, 4)\} \). Then there exist infinitely many perfect squares not divisible by \( x \) having exactly \( k \) non-zero digits in their base \( x \) representation.
derive from this expansion; also, from a short computation, the only perfect squares $y^2$ having exactly four non-zero digits in their base 3 representation, for $y \leq 10^7$ not divisible by 3, are obtained for $y \in \{7, 14, 16, 17, 26, 35, 47, 68, 350, 3788\}$.

Therefore, while we were not able to reach a conclusion in this case, we think it might be interesting to ask this Question, with which we finish this work.

**Question 11.** Determine if there are infinitely many squares not divisible by 3 having exactly 4 non-zero digits in their base 3 representation.

**Acknowledgements**

This work is part of my PhD thesis. I would like to thank my advisers, Professors Roberto Dvornicich and Umberto Zannier for their supervision, and for helpful discussions. I would also like to thank the referee for his helpful remarks and suggestions.

**References**

1. M. A. Bennett, Y. Bugeaud and M. Mignotte, Perfect powers with few binary digits and related Diophantine problems, II, *Math. Proc. Cambridge Philos. Soc.* 153 (2012), 525–540.
2. M. A. Bennett, Y. Bugeaud, M. Mignotte, Perfect powers with few binary digits and related Diophantine problems, *Ann. Sc. Norm. Super. Pisa - Cl. sci.* 12, 4 (2013), p. 941-953.
3. M. A. Bennett, AM. Scheerer, Squares with Three Nonzero Digits, In: Elsholtz C., Grabner P. (eds), *Number Theory - Diophantine Problems, Uniform Distribution and Applications* Springer, Cham. (2017), p. 83-108.
4. P. Corvaja, U. Zannier, Finiteness of odd perfect powers with four nonzero binary digits, *Ann. Inst. Fourier* 63, 2 (2013), p. 715-731.
5. P. Corvaja, U. Zannier, Application of the Subspace Theorem to certain Diophantine problems, In: Diophantine Approximation, H. E. Schlickewei et al, Editors, Springer-Verlag (2008), p. 161-174.
6. P. Corvaja, U. Zannier, $S$-unit points on analytic hypersurfaces, *Ann. Sci. École Norm. Sup.* 38, 4 (2005) no. 1, p. 76-92.
7. P. Corvaja, U. Zannier, On the Diophantine equation $f(a^m, y) = b^n$, *Acta Arith.* 94 (2000), p. 25-40.
8. P. Corvaja, U. Zannier, Diophantine equations with power sums and Universal Hilbert Sets, *Indag. Mathem. N. S.* 9 (1998) no. 3, p. 317-332.
9. M. Hindry, J.H. Silverman: *Diophantine Geometry*. Springer, Heidelberg (2000).
10. P. Mihăilescu, Primary cyclotomic units and a proof of Catalan’s conjecture, *J. Reine Angew. Math.* 572 (2004), p. 167-195.
11. W. Narkiewicz, *Rational number theory in the 20th Century : from PNT to FLT*, Springer Monographs in Mathematics, Springer (2012).
12. L. Szalay, The equation $2^n \pm 2^m \pm 2^l = z^2$, *Indag. Math.* 13 (2002), p. 131-142.

**Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy.**

*Email address: moscariello@mail.dm.unipi.it*