Hartogs Type Theorems on Coverings of Stein Manifolds

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Abstract

We prove an analog of the classical Hartogs extension theorem for certain (possibly unbounded) domains on coverings of Stein manifolds.

1. Introduction.

Let \( D \subset \mathbb{C}^n \) \((n > 1)\) be a bounded open set with a connected smooth boundary \( bD \). The classical Hartogs theorem states that any holomorphic function in some neighbourhood of \( bD \) can be extended to a holomorphic function on a neighbourhood of the closure \( \overline{D} \). The first rigorous proof of this theorem was given by Brown in 1936 see, e.g., [F]. In [Bo] Bochner proved a similar extension result for functions defined on the \( bD \) only. In modern language his result says that for a smooth function defined on the \( bD \) and satisfying the tangential Cauchy-Riemann equations there is an extension to a holomorphic function in \( D \) which is smooth on \( \overline{D} \). In fact, this statement follows from Bochner’s proof (under some smoothness conditions). However at that time there was not yet the notion of a \( CR \)-function. Over the years significant contributions to the area of Hartogs theorem were made by many prominent mathematicians, see the history and the references in the paper of Harvey and Lawson [HL, Section 5]. A general Hartogs-Bochner type theorem for bounded domains \( D \) in Stein manifolds is proved by Harvey and Lawson [HL, Theorem 5.1]. The proof relies heavily upon the fact that for \( n \geq 2 \) any \( \overline{\partial} \)-equation with compact support on a Stein manifold has a compactly supported solution. In this paper we present a Hartogs type theorem for certain (possibly unbounded) domains on coverings of Stein manifolds which gives an extension of the above cited result of [HL]. In order to formulate this theorem we first introduce some notation and basic definitions.

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Let \( r : M' \to M \) be an unbranched covering of a Stein manifold \( M \) of complex dimension \( n \geq 2 \). Let \( D \subset M' \) be a domain with a connected piecewise smooth boundary \( bD \) such that \( r(D) \subset M \). Assume that \( M \) is equipped with a Riemannian metric \( g_M \). By \( d \) we denote the path metric on \( M' \) induced by the pullback of \( g_M \).

For a fixed \( o \in D \) we set \( d_o(z) := d(o, z) \), \( z \in M' \).

Also, by \( \overline{D} \subset M' \) we denote the closure of \( D \) and by \( \mathcal{O}(D) \) the space of holomorphic functions on \( D \). Next, recall that a continuous function \( f \) on \( bD \) is called CR if for every smooth \((n, n-2)\)-form \( \omega \) on \( M' \) with a compact support one has

\[
\int_{bD} f \cdot \partial \omega = 0.
\]

If \( f \) and \( bD \) are smooth this is equivalent to \( f \) being a solution of the tangential CR-equations: \( \partial_b f = 0 \) (see, e.g., [KR]).

Suppose that \( f \in C(bD) \) is a CR-function satisfying for some positive numbers \( c_1, c_2, \delta \) the following conditions

(1) \[
|f(z)| \leq e^{c_1 e^{c_2 d_o(z)}} \quad \text{for all} \quad z \in bD;
\]

(2) for any \( z_1, z_2 \in bD \) with \( d(z_1, z_2) \leq \delta \)

\[
|f(z_1) - f(z_2)| \leq e^{c_1 e^{c_2 \max(d_o(z_1), d_o(z_2))}} d(z_1, z_2).
\]

**Theorem 1.1** There is a constant \( c > 0 \) such that for any CR-function \( f \) on \( bD \) satisfying conditions (1) and (2) with \( c_2 < c \) there exists \( \hat{f} \in \mathcal{O}(D) \cap C(\overline{D}) \) such that \( \hat{f}|_{bD} = f \) and \( |\hat{f}(z)| \leq e^{\check{c}_1 e^{c_2 d_o(z)}} \) for all \( z \in D \) with \( \check{c}_1 \) depending on \( c_1, c_2, \delta \) and \( c \).

**Remark 1.2** (A) If, in addition, \( bD \) is smooth of class \( C^k \), \( 1 \leq k \leq \infty \), and \( f \in C^s(bD) \), \( 1 \leq s \leq k \), then the extension \( \hat{f} \) belongs to \( \mathcal{O}(D) \cap C^s(\overline{D}) \). This follows from [HL, Theorem 5.1].

(B) Condition (2) means that \( f \) is locally Lipschitz with local Lipschitz constants growing double exponentially.

**Corollary 1.3** Assume that instead of condition (1) the function \( f \) in Theorem 1.1 satisfies the weaker condition

(1') \[
|f(z)| \leq e^{\phi(z)} \quad \text{for all} \quad z \in bD
\]

where \( \phi : M' \to \mathbb{R} \) is a uniformly continuous function with respect to \( d \).

Then there is a constant \( C \) (depending on \( M, M' \) and \( \phi \) only) and a function \( \hat{f} \in \mathcal{O}(D) \cap C(\overline{D}) \) such that \( \hat{f}|_{bD} = f \) and \( |\hat{f}(z)| \leq C e^{\phi(z)} \) for all \( z \in D \) with \( C = 1 \) for \( \phi \equiv \text{const} \).
2. Proofs.

2.1. Proof of Theorem 1.1. Since \( r(D) \subset \subset M \), there is a strictly pseudoconvex domain \( S \subset \subset M \) such that \( r(D) \subset \subset S \). Let \( S' \) be a connected component of \( r^{-1}(S) \subset M' \) containing \( D \). Then \( r : S' \to S \) is an unbranched covering of \( S \). Further, it follows from [Br1, Theorem 2.1] that there is a function \( g \in \mathcal{O}(S') \cap C(\overline{S'}) \) and a constant \( C = C(S', M') \) such that

\[
|g(z) - d_o(z)| < C \quad \text{and} \quad |dg(z)| < C \quad \text{for all} \quad z \in S'. \tag{2.1}
\]

(Here the norm \( |\omega(z)| \) of a differential form \( \omega \) at \( z \in S' \) is determined with respect to the Riemannian metric pulled back from \( M \).) From the first inequality in (2.1) one obtains, see [Br1, Example 4.3], that there is a constant \( c > 0 \) such that for any \( c_1 > 0 \) and \( 0 < c_2 < c \)

\[
e^{c_1 e^{2d_o(z)}} \leq |e^{c' e^{2g(z)}}| \leq e^{c'' e^{2d_o(z)}} \quad \text{for all} \quad z \in S'; \tag{2.2}
\]

here \( c', c'' \) are positive constants depending on \( c_1, c_2, c \) so that \( c'' \to 0 \) as \( c \to 0 \).

We set

\[
G_{c_1, c_2}(z) := e^{-c_1 e^{2g(z)}}, \quad z \in S'. \tag{2.3}
\]

Let us choose \( c \) in Theorem 1.1 to be the same as in (2.2). Retaining the notation of Theorem 1.1 consider the function

\[
f_1(z) := f(z) \cdot G_{c_1 e^{2\delta}, c_2}(z), \quad z \in bD.
\]

Lemma 2.1 \( f_1 \) is a bounded Lipschitz \( CR \)-function on \( bD \).

Proof. Condition (1) for \( f \) and the definition of \( f_1 \) imply that

\[
|f_1(z)| \leq 1 \quad \text{for all} \quad z \in bD.
\]

Thus to show that \( f_1 \) is Lipschitz it suffices to check the Lipschitz condition for all pairs \( z_1, z_2 \in bD \) with \( d(z_1, z_2) \leq \delta \). For such pairs we have

\[
|f_1(z_1) - f_1(z_2)| \leq |f(z_1) - f(z_2)| \cdot |G_{c_1 e^{2\delta}, c_2}(z_1)| +

|f(z_2)| \cdot |G_{c_1 e^{2\delta}, c_2}(z_1) - G_{c_1 e^{2\delta}, c_2}(z_2)| := I + II.
\]

Using condition (2) for \( f \), (2.2) and the triangle inequality we obtain

\[
I \leq e^{c_1 (e^{2\max(d_o(z_1), d_o(z_2))) - e^{2(d_o(z_1) + 2\delta)})} d(z_1, z_2) \leq d(z_1, z_2). \tag{2.4}
\]

To estimate \( II \) note that according to (2.1) and (2.2) we have

\[
|dG_{c_1, c_2}(z)| = |c' c_2 G_{c_1, c_2}(z) e^{2g(z)} dg(z)| \leq c_1 c_2 C e^{-c_1 e^{2d_o(z)} + c_2 (C + d_o(z))}.
\]

From here we obtain (for some \( c_3 > 0 \))

\[
|dG_{c_1 e^{2\delta}, c_2}(z)| \leq c_3 e^{-c_1 e^{2(d_o(z))}} \quad \text{for all} \quad z \in S'. \tag{2.5}
\]
Further, since \( r(D) \subset S \) is compact, for a sufficiently small \( \delta \) the metric \( d \) is geodesic in any metric ball \( B_\delta \) on \( S' \) of radius \( \delta \) centered at a point \( D \). (This follows from the definition of \( d \).) Without loss of generality we will assume that this \( \delta \) is the same as in Theorem 1.1. Thus integrating inequality (2.5) along geodesics in \( S' \) we get

\[
II \leq e^{c_1 e^{c_2 d_0 (z_2)}} C_3 e^{-c_1 e^{c_2 (\delta + (d_0 (z_2) - \delta))}} d(z_1, z_2) = C_3 d(z_1, z_2). \tag{2.6}
\]

Now the Lipschitz condition for \( f_1 \) follows from inequalities (2.4) and (2.6). The fact that \( f_1 \) is CR follows directly from its definition.

Based on this lemma we reduce the required statement to an extension theorem for the function \( f_1 \). Namely we will show

**Claim.** Under the hypotheses of the lemma there is a function \( \hat{f}_1 \in O(D) \cap C(D) \) such that

\[
\hat{f}_1|_{bD} = f_1 \quad \text{and} \quad \sup_D |\hat{f}_1| = \sup_{bD} |f_1|.
\]

Then the function \( \hat{f} := \hat{f}_1 / G e^{\delta \cdot \partial \hat{f}_1} \) is the required extension of Theorem 1.1.

To establish this claim, first, using the McShane theorem [M] let us extend \( f_1 \) to a Lipschitz function \( \tilde{f}_1 \) on \( S' \) with the same Lipschitz constant as for \( f_1 \). Since locally the metric \( d \) is equivalent to the Euclidean metric, by the Rademacher theorem, see, e.g., [Fe, Section 3.1.6], the function \( \tilde{f}_1 \) is differentiable almost everywhere.

In particular, \( \partial \tilde{f}_1 \) is a \((0, 1)\)-form on \( S' \) whose coefficients in its local coordinate representations are \( L^\infty \)-functions. Let \( \chi_D \) be the characteristic function of \( D \). Consider the \((0, 1)\)-form on \( S' \) defined by

\[
\omega := \chi_D \cdot \partial \tilde{f}_1.
\]

**Lemma 2.2** \( \omega \) is \( \partial \)-closed in the distributional sense.

**Proof.** We must prove that

\[
\int_{S'} \omega \wedge \partial \phi = 0 \tag{2.7}
\]

for every \((2n - 1)\)-form \( \phi \) of class \( C^\infty \) with a compact support on \( S' \) (recall that \( \dim C S' = n \)). Comparing types of the forms in (2.7) we see that, in fact, it suffices to prove the latter identity for \( \phi \) of type \((n, n - 2)\). Since this problem is local, it suffices to prove (2.7) for \( \phi \) supported in a sufficiently small neighbourhood of a point of \( S' \). Further, by the definition of \( \omega \) applying the Stokes formula we get that identity (2.7) is valid for \( \phi \) supported in a sufficiently small neighbourhood of a point of \( D \). Thus it remains to consider the case of \( \phi \) supported in a sufficiently small neighbourhood \( U_z \) of a boundary point \( z \in bD \). (Without loss of generality we may assume that \( U_z \) is a coordinate neighbourhood.) Thus we have

\[
\int_{S'} \omega \wedge \partial \phi = \int_{U_z} \omega \wedge \partial \phi = \int_{U_z \cap D} \partial \tilde{f}_1 \wedge \partial \phi = \int_{U_z \cap D} d(\tilde{f}_1 \cdot \partial \phi) = \int_{U_z \cap bD} f_1 \cdot \partial \phi = 0 .
\]

We used here that \( f_1 \) is CR and the Stokes formula. \( \square \)
Remark 2.3 Normally, the Stokes formula is applied to smooth forms. However, it is also valid for forms with Lipschitz coefficients. To see this we first apply it to sequences of regularized forms obtained from these Lipschitz forms and then pass to the limit as the parameter of the regularization tends to 0. To justify this procedure one uses the fact that for a Lipschitz function \( f \) on a bounded domain \( D \subset \mathbb{R}^k \) the sequence \( \{ f_\epsilon \} \) of regularizations of \( f \) converges uniformly to \( f \) on every compact subset of \( D \) as \( \epsilon \to 0 \). Moreover, the sequence \( \{ df_\epsilon \} \) is uniformly bounded on every compact subset of \( D \) and converges almost everywhere to \( df \) (see, e.g., [Fe, Section 4.1.2]).

Lemma 2.4 There is a bounded continuous function \( \tilde{F} \) on \( S' \) equals 0 on \( S' \setminus D \) such that \( \overline{\partial F} = \omega \) in the distributional sense.

Proof. Let us consider a finite open cover \( U = \{ U_i \}_{i \in I} \) of a neighbourhood \( N \) of \( \overline{S} \) such that each \( U_i \) is relatively compact in a simply connected coordinate chart on \( M \) \( \) and in these local coordinates is identified with an open Euclidean ball in \( \mathbb{C}^n \). By \( \tilde{U} \) we denote the open cover \( \{ r^{-1}(U_i) \}_{i \in I} \) of \( N' := r^{-1}(N) \). In every connected component \( V \) of \( r^{-1}(U_i) \) we introduce the local coordinates obtained by the pullback of the coordinates on \( U_i \). (Note that \( r|_V : V \to U_i \) is biholomorphic.) By the definition of \( \tilde{U} \) in every such \( V \) the metric \( d \) is equivalent to the Euclidean distance on \( \mathbb{C}^n \) with the constants of the equivalence depending on \( U_i \) only. Since \( f_1 \) is Lipschitz, this implies that in \( V \) the form \( \omega \) is written as

\[
\omega(z) := \sum_{j=1}^{n} a_j(z) \overline{z}_j \quad \text{with} \quad \sup_{z \in V; 1 \leq j \leq n} |a_j(z)| \leq C ; \tag{2.8}
\]

here \( z_1, \ldots, z_n \) are the above introduced local coordinates on \( V \) and the constant \( C \) is independent of the choice of \( V \) and \( U_i \). Based on Lemma 2.2 and using (2.8) one can solve the equation \( \overline{\partial} F = \omega \) on \( V \) to obtain a solution \( F_V \) which is an \( L^\infty \)-function on \( V \) satisfying

\[
\sup_{z \in V} |F_V(z)| \leq C', \tag{2.9}
\]

where \( C' \) depends on \( C \) and \( n \) only, see, e.g., [H, Theorem 6.9]. (Here \( F_V \) is the solution in the distributional sense.) Moreover, if we identify \( V \) with the unit Euclidean ball \( B \subset \mathbb{C}^n \) we can find such a solution \( F_V \) by the formula

\[
F_V(z) = \frac{n!}{(2\pi i)^n} \int_{(\xi,\lambda_0) \in B \times [0,1]} \omega(\xi) \times \eta \left( (1 - \lambda_0) \frac{\overline{\xi} - z}{|\xi - z|^2} + \lambda_0 \frac{\overline{\xi} - \overline{z}}{1 - \langle \overline{\xi}, z \rangle} \right) \wedge \eta(\xi) ,
\]

see, e.g., [H, Section 4.2]; here for \( v = (v_1, \ldots, v_n) \) and \( w = (w_1, \ldots, w_n) \)

\[
\eta(v) = dv_1 \wedge \cdots \wedge dv_n , \quad <v,w> = \sum_{j=1}^{n} v_j \cdot w_j \quad \text{and} \quad |v|^2 = \langle v, v \rangle .
\]

Since the coefficients of \( \omega \) are \( L^\infty \)-functions, the above formula implies also that \( F_V \) is continuous on \( V \). Indeed to show that \( F_V \) is continuous at \( z_0 \in V \) consider a sequence \( \{ z_i \} \) convergent to \( z \). Without loss of generality we assume that \( \{ z_i \} \) belongs to
the open Euclidean ball $B_\epsilon(z_0)$ centered at $z_0$ of radius $\epsilon$ for a sufficiently small $\epsilon$. Next, we write $F_V(z) = F_1(z) + F_2(z)$ where $F_1(z)$ is obtained by the integration of the integrand form in the definition of $F_V(z)$ over $B_\epsilon(z_0) \times [0, 1]$ and $F_2(z)$ by the integration of this form over $(B \setminus B_\epsilon(z_0)) \times [0, 1]$. Since the integrand forms for $F_2(z_i)$ are uniformly bounded on $(B \setminus B_\epsilon(z_0)) \times [0, 1]$ and pointwise converge there as $i \to \infty$ to the integrand form for $F_2(z_0)$, $\lim_{i \to \infty} F_2(z_i) = F_2(z_0)$. To estimate $F_1(z_i)$ we use the substitution $w = \xi - z_i$ and pass to polar coordinates in the obtaining integral. Then it is readily seen that for some $c > 0$

$$|F_1(z_i)| \leq cC \cdot \text{diam}(B_\epsilon(z_0)) = 2\epsilon cC, \quad 0 \leq i < \infty.$$  

Therefore $\lim_{i \to \infty} |F_V(z) - F_V(z_i)| \leq 4\epsilon cC$ for any $\epsilon$, that is $F_V$ is continuous at $z_0$.

Further, for connected components $V$ and $W$ of $r^{-1}(U_i)$ and $r^{-1}(U_j)$ such that $U_i \cap U_j \neq \emptyset$ we set

$$F_{VW}(z) = F_V(z) - F_W(z), \quad z \in V \cap W.$$  

Since $\overline{\partial}F_{VW} = 0$ in the distributional sense, $F_{VW} \in \mathcal{O}(V \cap W)$. Thus considering all possible $V$ and $W$ we get a holomorphic 1-cocycle $\{F_{VW}\}$ on the cover $\mathcal{U}$ of $N'$. Moreover, by (2.9) we have

$$\sup_{V, W, z \in V \cap W} |F_{VW}(z)| \leq 2C'.$$

This implies that the direct image of $\{F_{VW}\}$ with respect to $r$ is a holomorphic 1-cocycle on the cover $\mathcal{U}$ with values in a holomorphic Banach vector bundle with the fibre $l_\infty(X)$ where $X$ is the fibre of the covering $r : S' \to S$, see the proof of Theorem 2.1 in [Br] for details. Repeating literally the main argument from the proof of this theorem (based on a Banach valued version of Cartan's B theorem) together with the fact that there is a Stein neighbourhood $N_1$ of $\overline{S}$ such that $N_1 \subset N$ we get holomorphic functions $f_V \in \mathcal{O}(V \cap S')$ such that

1. $f_V(z) - f_W(z) = F_{VW}(z), \quad z \in (V \cap W) \cap S'$, and

2. $$\sup_{z \in V \cap S'} |f_V(z)| \leq \tilde{C}$$

where $\tilde{C}$ depends on $C'$, $N_1$ and the cover $\mathcal{U}$ only.

Let us define a continuous function $F$ on $S'$ by the formula

$$F(z) := F_V(z) - f_V(z), \quad z \in V \cap S'. \quad (2.10)$$

According to (2.9) and condition (2) $F$ is bounded. Also, it satisfies (in the sense of distributions) the equation $\overline{\partial}F = \omega$ on $S'$. Since $\omega \equiv 0$ outside $\overline{D}$, the function $F$ is holomorphic there. Observe that since the boundary of $D$ is connected, the set $S' \setminus \overline{D}$ is connected. Thus the application of Corollary 2.9 of [Br] gives a bounded function
\[ H \in \mathcal{O}(S') \text{ such that } H|_{S' \backslash D} = F. \text{ We set } \tilde{F} := F - H. \text{ Then by the definition } \tilde{F} \text{ is bounded and continuous on } S' \text{ equals } 0 \text{ on } S' \backslash D. \text{ Moreover, } \partial \tilde{F} = \omega. \Box
\]

Using this lemma we define

\[ \hat{f}_1(z) = f_1(z) - \tilde{F}(z), \quad z \in \overline{D}. \]

Then

\[ \hat{f}_1|_{bD} = f_1 \quad \text{and} \quad \partial \hat{f}_1(z) = \partial f_1(z) - \partial \tilde{F}(z) = \omega - \omega = 0 \quad \text{for} \quad z \in D. \]

This shows that \( \hat{f}_1 \in \mathcal{O}(D) \cap C(\overline{D}). \) Thus \( \hat{f}_1 \) is the required holomorphic extension of the function \( f_1. \) To complete the proof of the Claim it suffices to show that

\[ \sup_{D} |\hat{f}_1| = \sup_{bD} |f_1|. \]

To do this let us consider the product \( \hat{f}_1 \cdot G_{c_1,c_2} \) where \( G_{c_1,c_2} \) is the function from (2.3). Since the function \( \hat{f}_1 \) is Lipschitz on \( S' \), it satisfies

\[ |\hat{f}_1(z)| \leq c_1 + c_2 d_o(z), \quad z \in S'. \]

But \( \tilde{F} \) is bounded on \( S' \) and therefore the last inequality implies that

\[ |\hat{f}_1(z)| \leq \tilde{c}_1 + c_2 d_o(z), \quad z \in \overline{D}. \]

This and (2.2) show that for any \( \epsilon > 0 \) there is a positive number \( R \) such that for any \( z \in \overline{D} \) satisfying \( d_o(z) \geq R \) one has

\[ |\hat{f}_1(z) \cdot G_{c_1,c_2}(z)| < \epsilon. \]

In particular, there is an \( R_0 \) such that

\[ \sup_{D} |\hat{f}_1 \cdot G_{c_1,c_2}| = \sup_{B_{R_0} \cap \overline{D}} |\hat{f}_1 \cdot G_{c_1,c_2}| \]

where \( B_{R_0} \) is the open ball on \( S' \) centered at \( o \) of radius \( R_0. \) Since \( B_{R_0} \cap \overline{D} \) is compact and \( \hat{f}_1 \cdot G_{c_1,c_2} \in \mathcal{O}(D) \cap C(\overline{D}), \) the last identity implies that

\[ \sup_{D} |\hat{f}_1 \cdot G_{c_1,c_2}| = \sup_{B_{R_0} \cap bD} |\hat{f}_1 \cdot G_{c_1,c_2}| \leq \sup_{bD} |f_1|. \]

Finally, observe that \( \hat{f}_1 \cdot G_{c_1,c_2} \) converges pointwise to \( \hat{f}_1 \) as \( c_1 \to 0, \) see (2.2). Therefore we have

\[ |\hat{f}_1(z)| = \lim_{c_1 \to 0} |\hat{f}_1(z) \cdot G_{c_1,c_2}(z)| \leq \sup_{bD} |f_1|, \quad z \in D. \]

This implies the required identity and completes the proof of the Claim and therefore of the theorem. \( \Box \)

2.2. Proof of Corollary 1.3. Let \( f \) be a \( CR \)-function satisfying the hypotheses
of Corollary 1.3. Since $\phi$ is uniformly continuous on $M'$ with respect to the path metric $d$, there is a constant $C'$ such that

$$|\phi(z)| \leq C'd_o(z), \quad z \in M'. \quad (2.11)$$

In particular, condition $(1')$ implies condition (1) for $f$. Thus by Theorem 1.1 there exists an extension $\hat{f} \in \mathcal{O}(D) \cap C(\overline{D})$ of $f$.

Since $S \subset \subset M$ in the proof of Theorem 1.1 is strictly pseudoconvex, it follows from [Br, Theorem 2.1] that for every function $\phi : S' \to \mathbb{R}$ uniformly continuous with respect to the metric $d$ on $S'$ there is a holomorphic function $f_\phi \in \mathcal{O}(S')$ and a constant $C = C(\phi, S')$ such that

$$|f_\phi(z) - \phi(z)| < C \quad \text{for all} \quad z \in S'.$$

Let us consider the function

$$\tilde{f}(z) := \hat{f}(z) \cdot e^{-f_\phi(z)}, \quad z \in D.$$

Then by the hypotheses

$$|\tilde{f}(z)| \leq e^C \quad \text{for} \quad z \in bD.$$

From (2.11) and Theorem 1.1 we obtain for some $c' > 0$ (with $c_2 < c$)

$$|\tilde{f}(z)| \leq e^{c'e^{c_2d_o(z)}}, \quad z \in D. \quad (2.12)$$

Let us take $\tilde{c}_2$ such that $c_2 < \tilde{c}_2 < c$ and consider the function $\tilde{f} \cdot G_{c_1, \tilde{c}_2}$ where $G_{c_1, \tilde{c}_2}$ is the function from (2.3). Then from (2.2) and (2.12) it follows that for any $\epsilon > 0$ there is a positive number $R$ such that for any $z \in \overline{D}$ satisfying $d_o(z) \geq R$ one has

$$|\tilde{f}(z) \cdot G_{c_1, \tilde{c}_2}(z)| < \epsilon.$$

Now we apply the same argument as at the end of the proof of Theorem 1.1 to get

$$\sup_D |\tilde{f} \cdot G_{c_1, \tilde{c}_2}| \leq \sup_{bD} |\tilde{f}|.$$

Since $G_{c_1, \tilde{c}_2}$ converges pointwise to 1 as $c_1 \to 0$, from the last inequality we obtain

$$\sup_D |\tilde{f}| \leq \sup_{bD} |\tilde{f}| \leq e^C.$$

From here by the definitions of $\tilde{f}$ and $f_\phi$ we have

$$|\tilde{f}(z)| \leq e^{2C} \cdot e^{\phi(z)}, \quad z \in D.$$

Clearly the above arguments give $C = 0$ for $\phi \equiv \text{const}$.

This completes the proof of the corollary. \hfill \box
3. Concluding Remarks.

Let $r : M' \to M$ be an unbranched covering of a Stein manifold $M$ with $\dim \mathbb{C} M \geq 2$. We equip $M'$ with a path metric $d$ obtained by the pullback of a Riemannian metric on $M$. Assume that $D \subset M$ is a domain with a connected $C^k$-boundary $bD$, $1 \leq k \leq \infty$. We set $D' = r^{-1}(D)$ and $bD' = r^{-1}(bD)$. Let $\psi : M' \to \mathbb{R}_+$ be a function such that $\log \psi$ is uniformly continuous with respect to $d$. For every $x \in M$ we introduce the Banach space $l_{p,\psi,x}(M')$, $1 \leq p \leq \infty$, of functions $g$ on $r^{-1}(x) \subset M'$ with norm

$$
|g|_{p,\psi,x} := \left( \sum_{y \in r^{-1}(x)} |g(y)|^p \psi(y) \right)^{1/p}.
$$

Next, we introduce the Banach space $H_{p,\psi}(D')$, $1 \leq p \leq \infty$, of functions $f$ holomorphic on $D'$ with norm

$$
|f|_{p,\psi} := \sup_{x \in D} |f|_{p,\psi,x}.
$$

In [Br, Theorem 2.7] a sharper version of Corollary 1.3 for domains $D'$ is proved in connection with a certain problem posed by Gromov, Henkin and Shubin [GHS]. Namely it was established that

For every $C^{s}$-function $f \in C^{s}(bD')$, $0 \leq s \leq k$, satisfying

$$
f|_{r^{-1}(x)} \in l_{p,\psi,x}(M') \quad \text{for all} \quad x \in D \quad \text{and} \quad \sup_{x \in bD} |f|_{p,\psi,x} < \infty
$$

there exists a function $f' \in H_{p,\psi}(D') \cap C^{s}(\overline{D'})$ such that $f'|_{bD'} = f$. Moreover, for some $c = c(M', M, \psi, p)$ one has

$$
|f'|_{p,\psi} \leq c \sup_{x \in bD} |f|_{p,\psi,x}.
$$

It would be interesting to formulate and prove an analog of this result for other unbounded domains in $M'$. A possible formulation of such a result is as follows.

Let $D \subset M'$ be an unbounded domain with a connected smooth boundary $bD$ such that $r(D) \subset M$. By $dV_{M'}$ and $dV_{bD}$ we denote the Riemannian volume forms on $M'$ and $bD$, respectively, obtained by the pullback of a Riemannian metric on $M$. Next, by $H^p_\psi(D)$, $1 \leq p \leq \infty$, we denote the Banach space of holomorphic functions $g$ on $D$ with norm

$$
\left( \int_{z \in D} |g(z)|^p dV_{M'}(z) \right)^{1/p}.
$$

Also, $L^p_\psi(bD)$ stands for the Banach space of functions $g$ on $bD$ with norm

$$
\left( \int_{z \in D} |g(z)|^p dV_{bD}(z) \right)^{1/p}.
$$

**Problem.** Let $f \in L^p_\psi(bD) \cap C(bD)$ be a $C^{k}$-function. Under what additional conditions on $f$ and $bD$ does there exist $f' \in H^p_\psi(D) \cap C(\overline{D})$ such that $f'|_{bD} = f$?
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