Leveraging Two-Stage Adaptive Robust Optimization for Power Flexibility Aggregation

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Abstract—To effectively harness the significant flexibility from massive distributed energy resources (DERs) for transmission-distribution interaction, power flexibility aggregation is performed for a distribution system to compute the feasible region of the exchanged power at the substation. Based on the adaptive robust optimization (ARO) framework, this paper proposes a novel methodology for aggregating system-level power flexibility, considering heterogeneous DER facilities, network operational constraints, and unbalanced power flow model. In particular, two power flexibility aggregation models with two-stage optimization are developed for application: one focuses on aggregating active power and computes its optimal feasible intervals over multiple periods, while the other solves the optimal elliptical feasible region for the aggregate active-reactive power. By leveraging ARO technique, the disaggregation feasibility of the obtained feasible regions is guaranteed with optimality. The numerical simulations conducted on a real-world distribution feeder with 126 multi-phase nodes demonstrate the effectiveness of the proposed method.

Index Terms—Power aggregation, distributed energy resources, adaptive robust optimization.

NOMENCLATURE

A. Parameters

| Symbol | Description |
|--------|-------------|
| $N, L$ | Number of buses (except the substation bus), number of distribution lines. |
| $\tilde{v}, \tilde{\psi}$ | Upper, lower limits of the three-phase nodal voltage magnitudes for all buses. |
| $\tilde{i}, \tilde{\psi}$ | Upper, lower limits of the three-phase line current magnitudes for all distribution lines. |
| $p_{\tilde{e}, \psi}^{\tilde{v}}, p_{\tilde{e}, \psi}^{\tilde{v}}$ | Upper, lower limits of active PV power generation in phase $\psi$ of bus $i$ at time $t$. |
| $\tilde{s}_{\tilde{e}, \psi}^{\tilde{v}}$ | Apparent power capacity of PV units in phase $\psi$ of bus $i$ at time $t$. |
| $p_{\tilde{e}, \psi}^{\tilde{c}, \psi}, p_{\tilde{e}, \psi}^{\tilde{c}, \psi}$ | Upper, lower limits of active power output of ES devices in phase $\psi$ of bus $i$ at time $t$. |
| $\tilde{s}_{\tilde{e}, \psi}^{\tilde{c}, \psi}$ | Apparent power capacity of ES devices in phase $\psi$ of bus $i$ at time $t$. |
| $\tilde{E}_{i, \psi}, \tilde{E}_{i, \psi}$ | Upper, lower limits for state of charge of ES devices at bus $i$. |
| $p_{\tilde{e}, \psi}^{\tilde{d}, \psi}, p_{\tilde{e}, \psi}^{\tilde{d}, \psi}$ | Upper, lower limits for controllable active loads in phase $\psi$ of bus $i$ at time $t$. |
| $F_{\text{out}}, F_{\text{out}}$ | Outside temperature for HVAC systems at bus $i$ at time $t$. |
| $\tilde{F}_{i, \psi}, \tilde{F}_{i, \psi}$ | Upper, lower limits of comfortable temperature zone for HVAC systems at bus $i$. |

$\Delta t$ | Length of each time slot under discretized time horizon.

B. Variables

| Symbol | Description |
|--------|-------------|
| $v_i \in \mathbb{R}^{3N}$ | Column vector collecting the three-phase nodal voltage magnitudes for all buses at time $t$. |
| $i_i \in \mathbb{R}^{3L}$ | Column vector collecting the three-phase line current magnitudes for all lines at time $t$. |
| $p_{0,t}, q_{0,t} \in \mathbb{R}^{3}$ | Column vector of the total three-phase net active, reactive power injection at the substation. |
| $p_{0,t}, q_{0,t}$ | Total net active, reactive power injection at the substation at time $t$. |
| $p^{\tilde{e}, \psi}_{i,t}, q^{\tilde{e}, \psi}_{i,t}$ | Active, reactive PV power generation in phase $\psi$ of bus $i$ at time $t$. |
| $p^{\tilde{e}, \psi}_{i,t}, q^{\tilde{e}, \psi}_{i,t}$ | Active, reactive power output of ES devices in phase $\psi$ of bus $i$ at time $t$. |
| $E_{i,t}$ | State of charge of ES devices at bus $i$ at time $t$. |
| $p_{i,t}, q_{i,t}$ | Active, reactive controlled loads in phase $\psi$ of bus $i$ at time $t$. |
| $p_{i,t}, q_{i,t}$ | Active, reactive HVAC loads in phase $\psi$ of bus $i$ at time $t$. |
| $F_{\text{in}}^{\tilde{c}, \psi}_{i,t}, F_{\text{in}}^{\tilde{c}, \psi}_{i,t}$ | Indoor temperature for HVAC systems at bus $i$ at time $t$. |

Note: the same notations without superscript $\psi$ denote the corresponding summation over phases, e.g. $\tilde{p}_{i,t}^{\tilde{v}} := \sum_{\psi} \tilde{p}_{i,t}^{\psi}$.

I. INTRODUCTION

With deepening penetration of renewable generation, such as wind power and solar energy, power systems confront increasing volatility of generation and challenges to maintain power balance, which necessitates additional sources of power flexibility to warrant secure and efficient system operation. On the other hand, a rapid proliferation of distributed energy resources (DERs) has been witnessed in distribution systems, including energy storage (ES) devices, dispatchable photovoltaic (PV) units, electric vehicles (EVs), thermostatically controlled loads (TCLs) and etc. The coordinated dispatch of ubiquitous DERs is envisioned to provide significant flexibility and enable the active interaction between transmission systems and distribution systems [1]. In [2]–[4], decomposition techniques, such as master-slave-splitting method, Benders decomposition, and dual decomposition, are employed to co-optimize transmission and distribution in economic dispatch, reactive power optimization, and market clearing, which require frequent boundary information exchanges under a number of iterations.
To avoid convergence issues in decomposition methods and enable scalable application, power flexibility aggregation is a promising alternative to harness collective DER flexibility for transmission-distribution interaction. Specifically, for a distribution system, power flexibility aggregation is to characterize the feasible region of its exchanged power at the substation interface with the transmission system. This feasible region is essentially determined by internal DER operational conditions and power network constraints. In this way, the complex configurations and states of a distribution grid with massive DERs are represented in a concise and compact form, and the transmission-distribution interaction can be achieved through a hierarchical coordination framework: each distribution grid performs as a virtual power plant and reports its own feasible region of aggregate power, then the transmission system optimizes a holistic dispatch scheme and sends the power regulation signals to each distribution grid for execution.

With massive DER facilities and multiple time periods, procuring the exact feasible region of the aggregate power is computationally intensive and impractical, therefore most researches focus on approximation approaches. In [7–9], the polytope set is employed to describe the admissible power profile of an individual flexible load, then the aggregate flexibility is computed as the Minkowski sum of all these polytope sets. Reference [10] further proposes to use zonotope, a subclass of polytope, to depict the feasible region of DER power, which enables an efficient Minkowski summation. In [11], the aggregate flexibility of heterogeneous deferrable loads is computed via polytopic projection, which formulates an approximate optimization problem for tractable solution. References [12, 13] establish single-stage robust models to schedule the reserve capacities of heating, ventilation and air-conditioning (HVAC) systems, considering the uncertainties of forecast errors and power tracking signals.

Most researches above only focus on a single type of DER, which may not be able to handle a variety of DERs with heterogeneous operational conditions. Moreover, the underlying distribution network and the associated power flow constraints are not taken into account. In [14, 15], Monte-Carlo simulation based methods are proposed to estimate the flexibility range of aggregate active and reactive power (P-Q) with a large number of sampling scenarios. Reference [16] models the feasible P-Q region with time-varying ellipsoids, and computes the corresponding parameters by a data-driven system identification procedure. However, this method cannot guarantee disaggregation feasibility, i.e., any aggregate power trajectory within the obtained feasible region can be realized by appropriately dispatching DERs without violating network or DER operational constraints. In our previous work [17], some heuristic constraints associated with the upper and lower power trajectories are added to ensure disaggregation feasibility, which are conservative and may result in sub-optimal solutions.

Robust optimization (RO) [18] is a well-known technique for dealing with data uncertainty in optimization problems, and adaptive RO (ARO) [19] is further developed to reduce the conservativeness for the uncertain parameters containing adaptive variables. Interestingly, the ARO framework can be borrowed to address some special problems without uncertain parameters, such as the power flexibility aggregation studied in this paper. The intuition is that the feasible region (of the aggregate power) is analogous to the uncertainty set (of the uncertain variables) in ARO. And the disaggregation feasibility property can be exactly interpreted as the adaptive robust constraint: there should exist a corresponding feasible solution for any realization of the uncertain variables. Once modeling power aggregation as an ARO problem, mature ARO analysis and solution techniques can be directly applied.

In this paper, based on the ARO framework, we propose a novel method for system-level power flexibility aggregation, which incorporates a variety of DER facilities, network operational constraints, and a multi-phase unbalanced power flow model. Specifically, the exact feasible region of aggregate power over time is innerly approximated with a parameterized set, and two-stage ARO models are established to obtain the largest inner approximation. The main contributions of this paper are summarized as follows:

1) A system-level power aggregation method based on the two-stage ARO framework is proposed. Since the disaggregation feasibility is modelled with the exact ARO formulation instead of additional heuristic constraints, the optimality of the obtained feasible regions is guaranteed.

2) Two concrete power aggregation models with two-stage optimization are developed for application: one solves the optimal feasible intervals for the aggregate active power over time, and the other solves the optimal elliptical feasible regions for the time-variant aggregate P-Q domain.

The remainder of this paper is organized as follows: Section II introduces the multi-phase network and DER models. Section III presents the two-stage power aggregation method. Section IV develops the solution algorithm. Numerical tests are performed on a real feeder system in Section V, and conclusions are drawn in Section VI.

II. DISTRIBUTED ENERGY RESOURCE AND NETWORK MODELS

Consider a multi-phase unbalanced distribution network described by the graph $G(N_0, \mathcal{E})$, where $N_0$ denotes the set of buses and $\mathcal{E} \subset N_0 \times N_0$ denotes the set of distribution lines connecting the buses. Let $N_0' := \{0\} \cup N$ with $\mathcal{N} := \{1, 2, \ldots, N\}$ and bus 0 denotes the substation interface that exchanges power with the transmission system. As shown in Figure 1 each electric device can be multi-phase wye-connected or delta-connected to the network [20]. Denote $\phi_{\mathcal{V}} := \{a, b, c\}$ and $\phi_{\Delta} := \{ab, bc, ca\}$. Then we use notation $\psi$ to describe the concrete connection manner of an electric device with either $\psi \subseteq \phi_{\mathcal{V}}$ or $\psi \subseteq \phi_{\Delta}$. For instance, $\psi = \{a\}$ if the device is wye-connected in phase A and only has the complex power injection $s^a := p^a + j q^a$, while $\psi = \{ab, bc\}$ if it is delta-connected in phase AB and BC with the complex power injection $s^{ab} := p^{ab} + j q^{ab}$ and $s^{bc} := p^{bc} + j q^{bc}$.
A. Distributed Energy Resource Model

Given a discrete-time horizon $T := \{1, 2, \ldots, T\}$, we consider several typical DERs, including dispatchable PV units, ES devices, directly controllable loads, and HVAC systems. Based on references [21–23], the DER operational models are established as follows, where $N[\text{pv, es, ch, hv}]$ denotes the set of buses connected with the corresponding DER devices.

1) Dispatchable PV Units: $\forall i \in N[\text{pv}], t \in T$

$$P_{i,t}^{\text{pv, } i} \leq P_{i,t}^\alpha \leq P_{i,t}^\beta$$

$$P_{i,t}^{\text{pv, } i}^2 + Q_{i,t}^{\text{pv, } i}^2 \leq (s_{i,t}^{\text{pv, } i})^2$$

2) Energy Storage Devices: $\forall i \in N[\text{es}], t \in T$

$$P_{i,t}^{\text{es, } i} \leq P_{i,t}^\alpha \leq P_{i,t}^\beta$$

$$P_{i,t}^{\text{es, } i}^2 + Q_{i,t}^{\text{es, } i}^2 \leq (s_{i,t}^{\text{es, } i})^2$$

$$E_{i,t} = E_{i,t-1} - \Delta t \cdot p_{i,t}$$

$$E_{i,t} \leq E_{i,t} \leq E_{i,0}$$

Here, the active ES power output $p_{i,t}^{\text{es, } i}$ can be either positive (charging) or negative (discharging). In [22], $\eta_s \in (0,1]$ is the storage efficiency factor that models the energy loss over time, and $E_{i,0}$ denotes the initial state of charge (SOC). We assume 100% charging and discharging energy conversion efficiency for simplicity, i.e., no power loss in the charging or discharging process. Constraint (2d) imposes the SOC limits and requires that the final SOC $E_{i,T}$ recovers the initial value for sustainability.

3) Directly Controllable Loads: $\forall i \in N[\text{dl}], t \in T$

$$P_{i,t}^{\text{dl, } i} \leq P_{i,t}^\alpha \leq P_{i,t}^\beta$$

$$Q_{i,t}^{\text{dl, } i} = \eta_i \cdot P_{i,t}^{\text{dl, } i}$$

In [3], we assume fixed power factors with constant $\eta_i^d$.

4) HVAC Systems: $\forall i \in N[\text{hv}], t \in T$

$$0 \leq P_{i,t}^{\text{hv, } i} \leq P_{i,t}^\alpha, \quad Q_{i,t}^{\text{hv, } i} = \eta_i \cdot P_{i,t}^{\text{hv, } i}$$

$$F_{i,t} \leq F_{i,\text{in}} + \Delta t \cdot \beta_i \cdot p_{i,t}^{\text{hv, } i}$$

$$F_{i,t} = F_{i,t-1} + \alpha_i \cdot (F_{i,\text{out}} - F_{i,t-1}) + \Delta t \cdot \beta_i \cdot p_{i,t}^{\text{hv, } i}$$

In (4a), we assume fixed power factors with constant $\eta_i^d$. Equation (4c) depicts the indoor temperature dynamics, where $\alpha_i \in (0,1)$ and $\beta_i$ are the parameters specifying the thermal characteristics of the buildings and the environment. A positive (negative) $\beta_i$ indicates that the HVAC appliances work in the heater (cooler) mode, and $F_{i,\text{in}}$ is the initial indoor temperature. See [23] for detailed explanations.

B. Power Flow and Network Model

For compact expression, we stack all the three-phase controllable power injection at time $t$ into a long vector as

$$x_t := \left( \begin{array}{c} \mathbf{p}_{i,t}^{\text{pv, } i}, \mathbf{q}_{i,t}^{\text{pv, } i} \\ \vdots \\ \mathbf{p}_{i,t}^{\text{es, } i}, \mathbf{q}_{i,t}^{\text{es, } i} \\ \mathbf{p}_{i,t}^{\text{dl, } i}, \mathbf{q}_{i,t}^{\text{dl, } i} \\ \mathbf{p}_{i,t}^{\text{hv, } i}, \mathbf{q}_{i,t}^{\text{hv, } i} \end{array} \right)$$

(5)

With the fixed-point linearization method introduced in [24], we can derive the linear multi-phase power flow model [6] based on a given operational point.

$$\mathbf{v}_t = \mathbf{A} x_t + \mathbf{a}_t$$

(6a)

$$i_t = \mathbf{B} x_t + b_t$$

(6b)

$$p_{0,t} = d^T x_t + g_t$$

(6c)

$$q_{0,t} = f^T x_t + h_t$$

(6d)

Here, matrices $\mathbf{A}, \mathbf{B}$, vectors $\mathbf{a}_t, b_t, d, f$ and scalar $g_t, h_t$ are all system parameters, whose definitions are provided in [25]. Note that $x_t$ only contains the controllable DER power injection variables, while the time-varying uncontrollable loads and non-dispatchable power generations are treated as given system parameters and captured by $a_t, b_t, g_t, h_t$. In essence, the used power flow linearization method can be viewed as a linear interpolation between two power flow solutions: the given operational point and a known operational point with no power injection. As a result, the resulted linear power flow model [6] has better global approximation accuracy comparing with the standard linearized models based on local first-order Taylor expansion. Hence, the linear model [6] provides an accurate approximation of unbalanced power flow, which is applicable to both meshed and radial networks.

Accordingly, the network constraints can be formulated as

$$\mathbf{v}_t \leq \mathbf{v}_t \leq \hat{\mathbf{v}}$$

(7a)

$$\mathbf{i}_t \leq \mathbf{i}_t \leq \hat{\mathbf{i}}$$

(7b)

which involve the voltage limit constraints (7a) and the line thermal constraints (7b).

C. Comprehensive System Model

To facilitate the subsequent modelling with robust optimization, the linearization method introduced in [26] is applied to approximate the circular apparent power capacity constraint (1b) [26] with several concentric square constraints. As shown in Figure 2, the linear approximation achieves higher accuracy with more square constraints. Generally, two square constraints with the rotation angle of 45° are sufficiently accurate for practical applications. Hence, constraint (1b) of PV units can be approximated by

$$-s_{i,t}^{\text{pv, } i} \leq p_{i,t}^{\text{pv, } i} \leq s_{i,t}^{\text{pv, } i}$$

(8a)

$$-s_{i,t}^{\text{pv, } i} \leq q_{i,t}^{\text{pv, } i} \leq s_{i,t}^{\text{pv, } i}$$

(8b)

$$-\sqrt{2} s_{i,t}^{\text{pv, } i} \leq \sqrt{2} p_{i,t}^{\text{pv, } i} \leq \sqrt{2} s_{i,t}^{\text{pv, } i}$$

(8c)

$$-\sqrt{2} s_{i,t}^{\text{pv, } i} \leq \sqrt{2} q_{i,t}^{\text{pv, } i} \leq \sqrt{2} s_{i,t}^{\text{pv, } i}$$

(8d)

which is similar for constraint (2b) of ES devices.

Define $x := (x_t)_{t \in \mathcal{T}}, p_0 := (p_{0,t})_{t \in \mathcal{T}}, q_0 := (q_{0,t})_{t \in \mathcal{T}}$. After the linearization of circular constraints, the comprehensive system model, including multi-period DER models [1–4]
and network model (9), can be rewritten as the following compact linear form (9):

\[ p_0 = Dx + g \]  \hspace{1cm} (9a)
\[ q_0 = Fx + h \]  \hspace{1cm} (9b)
\[ Wx \leq w \]  \hspace{1cm} (9c)

Here, equation (9a) and (9b) are the stacks of equation (6c) and (6d) for all time periods \( t \in T \) respectively. Equation (9c) captures the DER models and the remaining network constraints, where equalities are reformulated in an equivalent unified form as inequalities. Matrices \( D, F, W \) and vectors \( g, h, w \) are the corresponding system parameters.

III. POWER AGGREGATION METHODOLOGY VIA TWO-STAGE ROBUST OPTIMIZATION

In this section, we interpret the power flexibility aggregation problem as the formulation in the ARO language, and propose two-stage ARO models to aggregate active and reactive power for a distribution system.

A. Power Aggregation Modelling via ARO

Essentially, power flexibility aggregation can be regarded as the projection of high-dimensional network and DER operational constraints onto the feasible region of the aggregate power \( (p_0, q_0) \). However, due to a huge amount of DERs and multi-period power flow relation, the exact feasible region is complex and intractable to procure or use. Instead, an inner approximation is generally performed to obtain a concise and efficient representative of the exact feasible region. There are two desired properties of the approximate feasible region \( \mathcal{D} \):

1) Aggregation optimality: \( \mathcal{D} \) is the optimal inner approximation of the exact feasible region with largest size.
2) Disaggregation feasibility: any aggregate power trajectory \( (p_0, q_0) \) within \( \mathcal{D} \) can be fulfilled by the dispatch of DERs without violating operational constraints.

To achieve these two significant properties, the ARO framework can be leveraged to to formulate the power aggregation problem as model (10):

\[
\begin{align*}
\text{Obj.} \quad & \max_{\mathcal{D}} \text{size}(\mathcal{D}) & (10a) \\
\text{s.t.} \quad & \forall (p_0, q_0) \in \mathcal{D}, \exists x(p_0, q_0) \\
\quad & p_0 = Dx(p_0, q_0) + g \\
\quad & q_0 = Fx(p_0, q_0) + h \\
\quad & Wx(p_0, q_0) \leq w
\end{align*}
\]

where objective (10a) aims to find the largest feasible region of the aggregate power to fully extract the DER flexibility. Constraint (10b) guarantees the disaggregation feasibility in an exact manner through the ARO modelling. In the ARO language, the (approximate) feasible region \( \mathcal{D} \) is regarded as the uncertainty set, and \( (p_0, q_0) \) is treated as the uncertainty variable subject to the uncertainty set \( \mathcal{D} \). The DER dispatch scheme \( x(p_0, q_0) \) refers to the adaptive variable that can be determined after the reveal of the uncertainty variable, thus it is functional on \( p_0 \) and \( q_0 \). Instead of using a static variable \( x \), the introduce of adaptive variables can significantly enhance the optimality of the robust solutions [19].

Model (10) is a general formulation of the power flexibility aggregation problem. In the follows, two concrete aggregation models with two-stage optimization are developed: one computes the optimal feasible intervals for the aggregate active power, while the other solves the optimal elliptical feasible regions for the aggregate P-Q domain.

B. Active Power Aggregation Model

This subsection focuses on characterizing the feasible region of the exchanged active power at the substation interface. To avoid computational burden and facilitate the high-level application, we use the time-decoupling feasible intervals \( \mathbb{D}_p \) to depict the feasible region of the aggregate active power

\[
\mathbb{D}_p := [p_{0,1}^\nu, p_{0,1}^\omega] \times [p_{0,2}^\nu, p_{0,2}^\omega] \times \cdots \times [p_{0,T}^\nu, p_{0,T}^\omega]
\]

where the notations with the superscript "\( \nu \)" and "\( \omega \)" denote the upper and lower power bounds respectively. Accordingly, the possible exchanged active power at substation over time is restrained by the upper power trajectory \( p_0^\nu := (p_0^\nu(t))_{t \in T} \) and the lower power trajectory \( p_0^\omega := (p_0^\omega(t))_{t \in T} \).

To obtain the optimal \( \mathbb{D}_p^* \) with maximal flexibility, based on the general ARO formulation (10), the active power aggregation (APA) model (12) is developed as follows:

\[
\begin{align*}
\text{Obj.} \quad & \max_{\mathbb{D}_p^*} \min_{\xi} \max_{p_0^\nu, p_0^\omega} \mathbf{1}^T (p_0^\nu - p_0^\omega) & (12a) \\
\text{s.t.} \quad & p_0^\nu \leq p_0^\omega \leq p_0^\nu \forall (p_0, q_0) \in \mathbb{U}_1 \quad (12b) \\
\quad & p_0^\nu + \xi \circ (p_0^\omega - p_0^\nu) = Dx(\xi) + g, \forall \xi \in \mathbb{U}_1 \quad (12c) \\
\quad & Wx(\xi) \leq w, \forall \xi \in \mathbb{U}_1 \quad (12d)
\end{align*}
\]

where the notation "\( \circ \)" denotes entry-wise multiplication. We introduce \( \xi := (\xi(t))_{t \in T} \in \mathbb{R}^T \) as the uncertainty variable, and normalize \( \mathbb{D}_p^* \) as the box uncertainty set \( \mathbb{U}_1 \):

\[
\mathbb{U}_1 := \{ \xi | 0 \leq \xi_t \leq 1, \forall t \in T \}. \quad (13)
\]

The objective (12a) is in the form of two-stage optimization. In the first stage, \( (p_0^\nu, p_0^\omega) \) is the "here-and-now" decision that maximizes the total aggregate flexibility. In the second stage, the DER power dispatch scheme \( x(\xi) \) denotes the "wait-and-see" adaptive decision that can be made after the uncertainty variable \( \xi \) is revealed. Through the minimax optimization, it ensures that for the worst-case scenarios in \( \mathbb{U}_1 \), there exists corresponding feasible \( x(\xi) \) to fulfill them. In this manner, the disaggregation feasibility of the solved feasible intervals is guaranteed with optimality.
Besides, the APA model \(12\) can be modified with different objectives and settings to meet the power aggregation goals. For example, by adding a base power trajectory, it can formulate an economic dispatch model that optimally schedule the flexibility reserve and DER power for the distribution system. See the economic power aggregation model in \[17\] for details.

**Remark 1.** Comparing with our previous work \[17\], the APA model \(12\) guarantees the disaggregation flexibility through the adaptive robust constraints \((12c) \quad (12d)\) in an exact way, instead of using some conservative heuristic constraints \((14b) \quad (14c)\), thus the optimality of solutions is enhanced and more power flexibility can be exploited. The numerical comparisons between this proposed method and work \[17\] are also carried out in Section \[\ref{sec:5}\]. In addition, it is not clear how to generalize the heuristic constraints in \[17\] to aggregate both active and reactive power, while the proposed ARO method is clearly applicable to the P-Q aggregation problem, which is elaborated in the next subsection.

### C. Active-Reactive Power Aggregation Model

In power networks, reactive power plays a pivotal role in maintaining voltage security and reducing network loss, and the inverter-based DERs have the capability to independently control active and reactive power \[27\]. Hence, the flexibility of reactive power from massive DERs has the potential to be exploited to support the bulk transmission system. Since active and reactive power are highly coupled in both the network constraints and DER operational constraints, it necessitates a joint P-Q flexibility aggregation scheme.

As investigated in \[14\], \[16\], the elliptical feasible region is a good choice to depict the snapshot of the aggregate P-Q flexibility at each time \(t\). Hence, we parameterize the feasible region \(D_{pq}\) with time-decoupling ellipses:

\[
D_{pq} := \left\{ \left[ p_{0,t}, q_{0,t} \right] + Y_t \cdot \xi_t : ||\xi_t||_2 \leq 1 \right\}
\]

where the parameter \((p_{0,t}, q_{0,t})\) denotes the center and the \(2 \times 2\)-dimension positive semidefinite matrix \(Y_t\) describes the rotation and stretch transformation for the ellipse \[28\].

Accordingly, the active-reactive power aggregation (ARPA) model \(15\) is built to optimally aggregate the P-Q flexibility:

\[
\text{Obj.} \quad \max \min \max \sum \log(\det(Y_t)) \quad \text{subject to}\]

\[
Y_t \geq 0, \quad \xi_t \in \mathbb{R}^2 \forall t \in T
\]

\[
\left[ p_{0,t}, q_{0,t} \right] + Y_t \cdot \xi_t = \left[ d^T, f^T \right] x_t(\xi) + \left[ g_t \right] \quad \forall t \in T, \forall \xi \in U_2
\]

\[
W x(\xi) \leq w, \quad \forall \xi \in U_2
\]

where \(x(\xi)\) denotes the uncertainty variable subject to the uncertainty set \(U_2\):

\[
U_2 := \{ \xi | ||\xi||_2 \leq 1, \forall t \in T \}
\]

The time-variant elliptical feasible regions with parameters \((p_{0,t}, q_{0,t}, Y_t)\) are the “here-and-now” decisions, while the DER power dispatch scheme \(x(\xi)\) denotes the “wait-and-see” adaptive variable that can be determined after the uncertainty variable \(\xi\) is revealed.

In objective \((15a)\), \(\det(Y_t)\) denotes the determinant of matrix \(Y_t\), which equals to the area of the ellipse \[28\] at time \(t\). Thus the objective \((15a)\) aims to maximize the total aggregate flexibility of active and reactive power over \(T\) time periods. Using the ARO constraints \((15c) \quad (15d)\), the disaggregation feasibility is guaranteed with exactness. Due to the semi-definite constraint in \((15b)\), the ARPA model \(15\) is formulated as a two-stage semi-definite programming (SDP) problem.

### IV. Solution Algorithm

Since both the APA model \(12\) and ARPA model \(15\) are two-stage ARO problems, this paper employs the wildly-used column-and-constraint generation (CCG) algorithm \[29\] to solve them. According to the different decision-makings in the two stages, the original ARO model is decomposed as a master problem and a sub-problem, then a master-sub iterative process is performed to obtain the optimal solution. Taking the APA model \(12\) for example, the CCG solution algorithm is presented as follows.

#### A. Master Problem

Following the decomposition structure of CCG algorithm, the master problem is developed as \[17\], which corresponds to the first-stage decision making in the APA model \(12\).

\[
\text{Obj. } f_M = \max \sum_{(p^0,p^\uparrow) \in \mathcal{S}} 1^T (p^0 - p^\uparrow) \quad \text{subject to}\]

\[
\text{Eq. } \sum_{k=1}^{K} (p^0 - p^k) = \mathbf{D} x(\xi)^k + \mathbf{g}, \forall k = 1, 2, \ldots, K
\]

\[
W x(\xi)^k \leq w, \forall k = 1, 2, \ldots, K
\]

where \((\xi^k)_{k=1,2,\ldots,K}\) are given as known parameters.

Essentially, the master problem \[17\] can be regarded as a multi-scenario relaxation of the original two-stage APA model \(12\). In particular, the uncertainty set \(U_1\) in \(12\) is replaced by \(K\) enumerated scenarios \((\xi^k)_{k=1,2,\ldots,K}\) within \(U_1\), and each scenario is assigned a corresponding \(x(\xi)^k\) for adaptivity. Since finite enumerations are used in \[17\] instead of the entire uncertainty set, the objective value \(f_M\) offers an upper bound for the original APA model \(12\). As more and more scenarios and constraints are added, \(f_M\) is expected to decrease to the optimal objective value of the APA model \(12\).

The master problem \(12\) of APA model is a linear programming (LP). Using the same decomposition method, the master problem of ARPA model \(15\) becomes a SDP, whose detailed formulation is omitted here.

#### B. Sub-Problem

The sub-problem \(18\) is associated with the second-stage decision making in the APA model \(12\), which optimizes \(\xi\) and \(x(\xi)\) with given \((\hat{p}_{0,t}, \hat{p}_{0,t})\):

\[
\text{Obj. } f_S = \min \max \sum_{\xi \in U_1} 0 \quad \text{subject to}\]

\[
\sum_{k=1}^{K} (\hat{p}_{0,t} - p^k) = \mathbf{D} x(\xi)^k + \mathbf{g}, \forall k = 1, 2, \ldots, K
\]

\[
W x(\xi)^k \leq w, \forall k = 1, 2, \ldots, K
\]
s.t. $p_0^\gamma + \xi \odot (p_0^\Delta - p_0^\gamma) = Dx(\xi) + g$ \hspace{1cm} (18b)

\[ Wx(\xi) \leq w \hspace{1cm} (18c) \]

With the given $(p_0^\gamma, p_0^\Delta)$, if there exists a certain extreme scenario $\xi \in \mathbb{U}_1$ such that no corresponding feasible $x$ satisfies constraints (18b) (18c), then the optimal objective value $f_S$ is $-\infty$; otherwise $f_S = 0$. Hence, the sub-problem (18) serves as a judge to determine whether the optimal feasible interval $[p_0^\gamma, p_0^\Delta]$ generated by the master problem (17) guarantees the disaggregation feasibility. In addition, its optimal solution $\xi_t$ is regarded as the worst-case scenario that jeopardizes the disaggregation feasibility, thus can be added to the master problem (17) as the enumerated scenario and improve the master solution.

The sub-problem of the ARPA model is similar to (18), with constraint (18b) replaced by the associated constraint of (15c).

### C. Solution Method for Sub-Problem

The sub-problem (18) is minimax bi-level optimization with linear coupling constraints, which is nonconvex and generally hard to solve. To tackle this issue, the following reformulation technique is used to obtain a tractable optimization model.

Firstly, through strong duality on the inner maximization, the sub-problem (18) can be reformulated as the monolithic optimization form (19):

\[
\text{Obj. } \min_{\xi \in \mathbb{U}_1, \mu, \lambda} \left( p_0^\gamma - g \right) \trans \mu + w \trans \lambda + \delta \trans (\mu \odot \xi) \hspace{1cm} (19a)
\]

\[
\text{s.t. } W \trans \lambda + D \trans (\mu - \xi) = 0, \quad \lambda \geq 0 \hspace{1cm} (19b)
\]

where $\delta := p_0^\Delta - p_0^\gamma$, and $\mu$ and $\lambda$ are the dual variables associated with the equality constraint (18b) and the inequality constraint (18c) respectively. In the objective (19a), there exists a bilinear term $\mu \odot \xi$ that complicates the solution. Due to the linear formulation, as proved in (20), the worst-case scenarios must be the extreme points of the box uncertainty set $\mathbb{U}_1$, i.e., the optimal $\xi_t \in \{0, 1\}$ for all $t$.

Therefore, to address the bilinearity issue, the uncertainty variable $\xi$ is forced to be binary without loss of optimality, and the uncertainty set $\mathbb{U}_1$ is replaced with the finite discrete set $\mathbb{U}_1 := \{0, 1\}^T$. Then we define non-negative variables $(\mu^+, \mu^-)$ to substitute $\mu$ with $\mu = \mu^+ - \mu^-$, and introduce new variables $\nu^+, \nu^-$ to substitute the resultant products $\mu^+ \odot \xi, \mu^- \odot \xi$ respectively. The big-M method and constraints (20k) (20d) are added to make this substitution equivalent. As a consequence, the dual sub-problem (19) is equivalently reformulated as (20):

\[
\text{Obj. } f_S = \min_{\mu, \lambda, \nu, \xi} \left( p_0^\gamma - g \right) \trans (\mu^+ - \mu^-) + w \trans \lambda + \delta \trans (\nu^+ - \nu^-) \hspace{1cm} (20a)
\]

\[
\text{s.t. } W \trans \lambda + D \trans (\mu^+ - \mu^-) = 0 \hspace{1cm} (20b)
\]

\[
0 \leq \nu^+ \leq \mu^+, \quad \mu^+ - M (1 - \xi) \leq \nu^+ \leq M \xi \hspace{1cm} (20c)
\]

\[
0 \leq \nu^- \leq \mu^-, \quad \mu^- - M (1 - \xi) \leq \nu^- \leq -M \xi \hspace{1cm} (20d)
\]

\[
\lambda, \mu^+, \mu^- \geq 0, \quad \xi \in \{0, 1\}^T \hspace{1cm} (20e)
\]

where $M$ is a sufficiently large positive number.

The reformulated dual sub-problem (20) is a mixed integer linear programming (MILP). Notice that the dimension of the binary variable $\xi$ is $T$, i.e., the number of time periods, which is independent of the power network and DERs. Hence, the computation complexity caused by the binary variables does not scale up much with the size of the distribution system.

In terms of the sub-problem of the ARPA model (15), the uncertainty set $\mathbb{U}_2$ (16) is a time-decoupling circular region, which has infinite extreme points. Thus the circular constraint linearization method introduced in Section II-C is used to approximate $\mathbb{U}_2$ with the polyhedral uncertainty set $\hat{\mathbb{U}}_2$:

\[
\hat{\mathbb{U}}_2 := \{ \xi_t = (\xi_t^\rho, \xi_t^\nu) | -1 \leq \xi_t^\rho \leq 1, -\sqrt{2} \leq \xi_t^\nu + \xi_t^\rho \leq \sqrt{2}, -1 \leq \xi_t^\rho \leq 1, -\sqrt{2} \leq \xi_t^\rho - \xi_t^\nu \leq \sqrt{2}, \forall t \in T \}
\]

The use of more square constraints can enhance the linearization accuracy, and $\hat{\mathbb{U}}_2$ is an outer (safe) approximation of $\mathbb{U}_2$. Then the worst-case scenario $\xi$ can be parameterized by

\[
\xi_t = \sum_{i=1}^l z_{i,t}e_i, \quad z_{i,t} \in \{0, 1\}, \quad \sum_{i=1}^l z_{i,t} = 1, \quad \forall t \in T
\]

where $e_i \in \mathbb{R}^2$ is one of the extreme points of the approximate polyhedron, and $l = 8$ for the case of $\hat{\mathbb{U}}_2$. As a result, the same reformulation technique above can be applied to obtain a tractable optimization model for the sub-problem of ARPA model (15), which is also a MILP.

### D. Column and Constraint Generation Algorithm

Based on the master-sub decomposition, the CCG algorithm [25] for solving the APA model (12) is presented as Algorithm 1. The master problems of the ARPA model and ARPA model are LP and SDP respectively, and their corresponding sub-problems are both MILP. These optimization problems can be solved efficiently with many available optimizers, such as IBM CPLEX and Gurobi. According to [29, Proposition 2], the CCG algorithm is guaranteed to generate the optimal solution within a finite number of iterations in the order of $O(L)$, where $L$ is the number of extreme points in the uncertainty set.

**Algorithm 1 Column and Constraint Generation Algorithm**

1: **Initialization**: Set $K = 1$ and tolerance $\epsilon > 0$. Initialize $\xi$ with an appropriate value, e.g., $\xi^1 = 1$ or $\xi^0 = 0$, and $f_S$ with a large value.

2: while $|f_S| \geq \epsilon$ do

3:   - **Solve Master Problem (17)** to obtain the optimal power aggregation solution $(p_0^\gamma, p_0^\Delta)$.

4:   - **Solve Sub Problem (20)** with given $(p_0^\gamma, p_0^\Delta)$ to obtain the optimal $\xi^{K+1}_t$ and the objective value $f_S$.

5:   - Generate new variables $x^{K+1}$ and add new constraints (21) to the master problem (17).

6: end while

7: **Output** the final feasible intervals $(p_0^\gamma, p_0^\Delta)$.
V. NUMERICAL SIMULATION

Numerical tests are carried out on a real distribution feeder located within the territory of Southern California Edison (SCE). This feeder contains 126 multi-phase buses with a total of 366 single-phase connections. The nominal voltage at the substation is 12kV (1 p.u.), and we set the upper and lower limits of voltage magnitude as 1.02 p.u. and 0.98 p.u. Dispatchable DERs include 33 PV units, 28 ES devices and 5 HVAC systems. The real load data and real solar irradiance profiles are applied. The total amounts of uncontrollable loads and PV available power from 9:00 to 16:00 are presented as Figure 3 For PV units, we set the lower bound of power generation as zero and take the PV available power in Figure 3 as the upper bound. We set the initial SOC of the ES devices to 50% and the storage efficiency factor \( \kappa_i \) to 0.95. The simulation time is discretized with the granularity of 30 minutes. Detailed configurations and parameters of this feeder system are provided in [31].

Numerical simulations are performed in a computing environment with Intel(R) Core(TM) i7-7660U CPUs running at 2.50 GHz and with 8-GB RAM. All the programmings are implemented in Matlab 2018b. We use the CVX package [32] to model the convex programs, while solve SDP with SDPT3 solver [33] and solve (MI)LP with Gurobi optimizer [34].

A. Implementation of Active Power Aggregation

We implemented the APA model (12) to evaluate the maximal active power flexibility of the test system, and compared the results with our previous method in [17] (Method 1). The feasible intervals of aggregate active power obtained via the APA model and Method 1 are illustrated as Figure 4 Define the aggregate flexibility as \( E_{af} = \sum_{t \in T} (p_{0,t}^c - p_{0,t}^v) \Delta t \). Then the aggregate flexibility values associated with the APA model and Method 1 are 35.42 MWh and 32.97 MWh respectively, i.e., 2.45 MWh more flexibility can be extracted by using the APA scheme. That is because Method 1 imposes conservative heuristic constraints on ES power and HVAC power to ensure the disaggregation feasibility, while the APA scheme guarantees this property with optimality by leveraging the ARO modelling technique. Besides, we tuned the total ES capacity in the test system, and compared the performances of the two methods above. The aggregate flexibility obtained via the APA model and Method 1 is shown in Table I. The results further validates that Method 1 does not fully exploit the ES flexibility due to the conservative constraints, and the superiority of the APA scheme is more significant in the distribution system with higher penetration of ES (or HVAC) facilities.

In terms of the computational efficiency, the average solution times for the master problem (17) and the sub-problem (20) are 6.63s and 174.6s respectively. With the initial uncertainty scenarios \( \xi^1 = 1 \) and \( \xi^2 = 0 \), the CCG algorithm usually converges within one or two iterations.

B. Implementation of Active-Reactive Power Aggregation

We further implemented the ARPA model (15) to compute the elliptical feasible regions for the aggregate P-Q domain. The simulation time horizon is selected from 9:00 to 14:00 with the granularity of 1 hour. The P-Q flexibility aggregation results are illustrated as Figure 5, where the red dot is the center \((p_{0,t}^c, q_{0,t}^c)\) and the blue areas represent the feasible regions of aggregate P-Q over time.

It takes 95.4s and 473.6s on average to solve the associated master problem and sub-problem of the ARPA model [15].

VI. CONCLUSION

In this paper, based on the ARO framework, we propose a novel methodology to perform power flexibility aggregation for unbalanced distribution systems with heterogeneous DER facilities. Two concrete aggregation models (APA and ARPA) with two-stage optimization are developed for different implementation goals. The APA model aggregates purely active power and computes its optimal feasible intervals over time, while the ARPA model characterizes both active and reactive power flexibility and solves the time-variant elliptical feasible regions of the aggregate P-Q domain. Lastly, the numerical tests on a real distribution feeder validate that both the aggregation optimality and disaggregation feasibility are guaranteed with the proposed method. Future work is to develop time-coupling formulations of the aggregate feasible region to fully exploit the DER flexibility.

![Figure 3](image-url) - Total PV power available and uncontrollable loads from 9:00 to 16:00 with the granularity of 30 minutes.

![Figure 4](image-url) - The feasible intervals \([p_{0,t}^0, p_{0,t}^\vee]\) of aggregate active power from 9:00 to 16:00 obtained by APA model and Method 1.

| Total ES capacity/MWh | APA | Method 1 |
|-----------------------|-----|----------|
| 2.72                  | 34.0| 32.80    |
| 5.44                  | 35.42| 32.97    |
| 8.17                  | 36.79| 33.07    |
| 10.89                 | 38.03| 33.15    |

| \(E_{af}/\text{MWh}\) | APA | Method 1 |
|-----------------------|-----|----------|
| 34.0                  | 32.80| 32.97    |
| 35.42                 | 32.97| 33.07    |
| 36.79                 | 33.07| 33.15    |
| 38.03                 | 33.15|          |

![Table I](image-url) - The aggregate flexibility comparison between APA and Method 1 with different ES capacities.
Fig. 5. The time-variant elliptical feasible regions for the aggregate active-reactive power domain from 9:00 to 14:00.

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