ALGEBRAS WITH THE SAME (ALGEBRAIC) GEOMETRY

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Abstract. Some basic notions of classical algebraic geometry can be defined in arbitrary varieties of algebras Θ. For every algebra H in Θ one can consider algebraic geometry in Θ over H. Correspondingly, algebras in Θ are considered with the emphasis on equations and geometry. We give examples of geometric properties of algebras in Θ and of geometric relations between them. The main problem considered in the paper is when different \(H_1\) and \(H_2\) have the same geometry.

INTRODUCTION

0.1. Speaking on universal algebraic geometry, we assume that the basic variety is arbitrary or sufficiently arbitrary. Under non-classical algebraic geometry we mean algebraic geometry in various specific (fixed) varieties Θ, i.e., non-classical stands for not necessarily classical. One can consider algebraic geometry in groups, in rings (associative or Lie), and in other structures. All this is united by the general idea of nonclassical algebraic geometry. Hence, there appeared universal problems and problems arising from the peculiarities of a concrete variety Θ.

We distinguish varieties Com-\(P\), Ass-\(P\) and Lie-\(P\). The first one is the variety of all commutative and associative algebras with the unit over the field \(P\). The geometry associated with this variety is regarded as the classical algebraic geometry over \(P\). The second one is the variety of all associative (not necessarily commutative) algebras with the unit over \(P\). Lie-\(P\) is the variety of all Lie algebras over \(P\).

For every algebra \(H \in \Theta\) we have its algebraic structure, its logic and its geometry. The interaction of these three components is the main idea of the theory under consideration. This leads to a number of new problems. For example, when do two algebras \(H_1\) and \(H_2\) have the same geometry, and how
does one understand this fact. “The same algebra” means isomorphism of
algebras, “the same logic” can be treated as the coincidence of elementary
theories.

0.2. With every algebra \( H \in \Theta \) we associate two categories. They are the
category \( K_\Theta(H) \) of algebraic sets over \( H \), and the category of algebraic varieties
\( \tilde{K}_\Theta(H) \). Here, algebraic variety is viewed as an algebraic set, considered up
to isomorphism of algebraic sets. Thus, the category \( \tilde{K}_\Theta(H) \) is the skeleton
of the category \( K_\Theta(H) \). Both categories represent the geometry of \( H \) and are
geometrical invariants of the algebra \( H \). Now we can view geometries in the
algebras \( H_1 \) and \( H_2 \) to be the same if the categories \( K_\Theta(H_1) \) and \( K_\Theta(H_2) \) are
isomorphic or the categories \( \tilde{K}(H_1) \) and \( \tilde{K}(H_2) \) are isomorphic. On the other
hand, in the category theory it is known that two categories have isomorphic
skeletons if and only if these categories are equivalent. Hence, we distinguish
two problems:

1) When are the categories \( K_\Theta(H_1) \) and \( K_\Theta(H_2) \) isomorphic?

2) When are these categories equivalent?

In fact, we consider the formulated problems in respect to special correct iso-
morphism and correct equivalence. This approach reflects the idea of coinci-
dence of geometries. Correctness is inspired by the essence of the matter, as
explained later.

Let us present here two specific results.

The first one relates to classical algebraic geometry over a field \( P \). For an ar-
bitrary extension \( L \) of the field \( P \) denote by \( K_P(L) \) the corresponding category
\( K_\Theta(L) \).

Let \( L_1, L_2 \) be two extensions of the field \( P \). The following conditions are
equivalent:

1) The categories \( K_P(L_1) \) and \( K_P(L_2) \) are correctly isomorphic.

2) These categories are correctly equivalent.

3) There exists an extension \( L \) of the field \( P \) such that \( L_1 \) and \( L \) are
   semi-isomorphic, and \( L_2 \) and \( L \) have the same quasi-identities.
The second, more simple, result relates to groups.

Let $\Theta = \text{Grp}$ be the variety of all groups, $H_1$ and $H_2$ two nonperiodical abelian groups. Then the following three conditions are equivalent:

1) The categories $K_{\Theta}(H_1)$ and $K_{\Theta}(H_2)$ are correctly isomorphic.
2) These categories are correctly equivalent.
3) $H_1$ and $H_2$ have the same quasi-identities.

Other cases of groups and algebras are considered in the same spirit.

Let us note that correctness of the isomorphism of categories $K_{\Theta}(H_1)$ and $K_{\Theta}(H_2)$ is well coordinated with the lattices of algebraic sets in affine spaces.

Any category $K_{\Theta}(H)$, can be considered from the point of view of the possibility to solve systems of equations in the algebra $H$. This category is, in some sense, a measure of algebraic closeness of the given $H$, depending on the structure of algebra $H$.

An important part in the proofs is played by investigation of automorphisms of categories of free algebras of varieties. For any $\Theta$ denote by $\Theta^0$ the category of free in $\Theta$ algebras $W = W(X)$ with $X$ finite. Automorphisms and autoequivalencies of such a category $\Theta^0$ are essentially tied with the geometry in $\Theta$.

0.3. We consider also the category $K_{\Theta}$ of algebraic sets over different $H \in \Theta$. Its skeleton $\tilde{K}_{\Theta}$ is a category of algebraic varieties over different $H$. Both these categories are geometrical invariants of the whole variety $\Theta$. There naturally arise problems on isomorphism and equivalence of different $K_{\Theta_1}$ and $K_{\Theta_2}$. Here $\Theta_1$ and $\Theta_2$ could be subvarieties of some big variety $\Theta$.

0.4. Let us make some notes on the plan of the paper. The paper is organized as follows. At the beginning we recall basic definitions. The second, third and fourth sections are devoted to special notions which play main part in the solution of the problem whether geometries in different algebras are the same.
In the fifth section we present universal theorems on coincidence of geometries. In the seventh section these universal theorems are specialized for varieties Com-$P$, Ass-$P$ and Lie-$P$. The previous sixth section contains preparation material for the final seventh one.

1. Basic definitions

1.1. Algebraic sets and closed congruences. General results about geometry in groups can be found in [2,3,4,16,17,18,36,37]. Fix an arbitrary variety of algebras $\Theta$. Denote by $\Theta^0$ the category of free in $\Theta$ algebras $W = W(X)$ with $X$ fixed. All $X$ are supposed to be subsets of some infinite universum $X^0$. Thus, $\Theta^0$ is a small category.

Fix an algebra $H$ in $\Theta$. Consider the set of homomorphisms $\text{Hom}(W,H)$ as an affine space over $H$. Here points are homomorphisms $\mu : W(X) \to H$. For $X = \{x_1, \ldots, x_n\}$, we have a bijection $\alpha_X : \text{Hom}(W,H) \to G(n)$ by the rule $\alpha_X(\mu) = (\mu(x_1), \ldots, \mu(x_n))$. If, further, $(w, w')$ is a pair of elements in $W$, then the point $\mu : W \to H$ is a solution of the equation $w = w'$ in $H$ if $w^\mu = w'^\mu$, $(w, w') \in \text{Ker} \mu$. The kernel of a homomorphism is a congruence of the algebra $W$.

Let now $A$ be a subset in the affine space $\text{Hom}(W,H)$ (a set of points), and $T$ a binary relation in $W$ (a set of pairs $(w, w'); w, w' \in W$). We set:

$$
\begin{align*}
T' &= T'_H = A = \{ \mu : W \to H \mid T \subset \text{Ker} \mu \} \\
A' &= A'_W = T = \bigcap_{\mu \in A} \text{Ker} \mu
\end{align*}
$$

This gives the Galois correspondence between sets of points and binary relations. We call a set of points $A$ such that $A = T'$ for some $T$ an algebraic (closed) set in $\text{Hom}(W,H)$. A relation $T$ with $T = A'$ for some $A$ is a congruence in $W$. We call such a congruence an $H$-closed one.

For every $A$ we have a closure $A''_H = (A')'_H$ and $T''_H = (T'_H)'_W$ holds for every $T$. It is easy to understand that the congruence $T$ in $W$ is $H$-closed if and only if there is an injection $W/T \to H/I$ for some $I$.

We need some auxiliary notions.
The class of algebras \( \mathfrak{X} \subset \Theta \) is called a \textit{prevariety} if \( \mathfrak{X} \) is closed under cartesian products and subalgebras. For an arbitrary \( \mathfrak{X} \) the corresponding closure up to prevariety is \( SC(\mathfrak{X}) \). Here \( S \) and \( C \) are closure operators on classes of algebras: \( C \) under cartesian products and \( S \) under subalgebras. We can now say that the congruence \( T \) in \( W \) is \( H \)-closed if and only if \( W/T \in SC(H) \). Besides, if \( T \) is an arbitrary binary relation in \( W \), then \( T''_H \) is an intersection of all congruences \( T_\alpha \) with \( T \subset T_\alpha \) and \( W/T_\alpha \in SC(H) \).

Consider further formulas of the form

\[
\left( \bigwedge_{(w,w') \in T} (w \equiv w') \right) \Rightarrow (w_0, w'_0).
\]

We call them \textit{generalized (infinitary) quasi-identities}. If \( T \) is finite, we have an ordinary quasi-identity.

The following Proposition easily follows from the definitions.

\textbf{Proposition 1.} The inclusion \( (w_0, w'_0) \in T''_H \) takes place if and only if the formula (*) holds in the algebra \( H \).

1.2. \textbf{Categories of algebraic sets.} Define a category of affine spaces \( K^0_\Theta(H) \). Objects of this category have the form \( \text{Hom}(W, H) \) where \( W \) is an object of the category \( \Theta^0 \). Morphisms

\[ \tilde{s} : \text{Hom}(W(X), H) \to \text{Hom}(W(Y), H) \]

are given by the morphisms in \( \Theta^0 \)

\[ s : W(Y) \to W(X) \]

by the rule \( \tilde{s}(\nu) = \nu s \) for every \( \nu : W(X) \to H \). We have here a contravariant functor

\[ \Theta^0 \to K^0_\Theta(H). \]

This functor determines duality of categories, if the algebra \( H \) generates the whole variety, i.e., \( \Theta = \text{Var}(H) = \text{QSC}(H) \). Here, \( Q \) is the operator of taking homomorphic images.

Let us now define the category \( K_\Theta(H) \) of all algebraic sets over \( H \). Objects of this category have the form \((X, A)\), where \( A \) is an algebraic set in the affine
space \( \text{Hom}(W(X), H) \). We consider the affine space \( \text{Hom}(W(X), H) \) as an object of the category \( K_{\Theta}(H) \) as well. As an algebraic set it is determined by one equation \( x = x \). Morphisms \( [s] : (X, A) \to (Y, B) \) are determined by homomorphisms \( s : W(Y) \to W(X) \) with the property \( \tilde{s}(\nu) = \nu s \in B \) if \( \nu \in A \). These are exactly those \( s \) for which \( (w^s, w'^s) \in A' \) if \( (w, w') \in B' \).

For such \( s \) we have a homomorphism
\[
\pi : W(Y)/B' \to W(X)/A'.
\]

Simultaneously we have a mapping \( [s] : A \to B \) and consider it as a morphism in the category \( K_{\Theta}(H) \).

It is clear that the category of affine spaces \( K_{\Theta}^0(H) \) is a subcategory of the category \( K_{\Theta}(H) \).

Let us define the category \( C_{\Theta}(H) \). Its objects are of the form \( W(X)/T \), where \( W = W(X) \) is an object of the category \( \Theta^0 \) and \( T \) is an \( H \)-closed congruence in \( W(X) \). Morphisms in \( C_{\Theta}(H) \) are homomorphisms of algebras, and the category \( C_{\Theta}(H) \) is a full subcategory of the category \( \Theta \).

Let us note further that if \( \text{Var}(H) = \Theta \) the transition \( (X, A) \to W(X)/A' \) determines duality of categories \( K_{\Theta}(H) \) and \( C_{\Theta}(H) \).

We denote the corresponding skeletons of categories by \( \tilde{K}_{\Theta}(H) \) and \( \tilde{C}_{\Theta}(H) \). These two categories are also dual. Objects of the category \( \tilde{K}_{\Theta}(H) \) are called algebraic varieties over the algebra \( H \). They are algebraic sets over \( H \) considered up to isomorphisms in \( K_{\Theta}(H) \).

The following proposition \([23]\) takes place.

Let \( \Theta_1 = \text{Var}(H) \). Then the categories \( K_{\Theta}(H) \) and \( K_{\Theta_1}(H) \) are isomorphic. The categories \( C_{\Theta}(H) \) and \( C_{\Theta_1}(H) \) are isomorphic as well.

For definitions of the categories \( K_{\Theta} \) and \( C_{\Theta} \) see \([33], [28]\).

2. Geometrical equivalence of algebras

2.1. Preliminaries.

Definition 1. Algebras \( H_1 \) and \( H_2 \) in \( \Theta \) are called geometrically equivalent if \( T''_{H_1} = T''_{H_2} \) in \( W \) for any \( W = W(X) \) and \( T \) in \( W \).
Proposition 2 follows from the Proposition 1.

**Proposition 2.** The algebras \( H_1 \) and \( H_2 \) are geometrically equivalent if and only if their generalized quasi-identities coincide.

This implies

**Proposition 3.** If \( H_1 \) and \( H_2 \) are geometrically equivalent then they have the same quasi-identities, i.e., the quasivarieties \( q \text{Var}(H_1) \) and \( q \text{Var}(H_2) \) coincide.

The converse statement is not valid in general case (see Theorem 2).

For every class of algebras \( \mathfrak{X} \subset \Theta \), we define a class \( L\mathfrak{X} \) as follows: \( H \in L\mathfrak{X} \) if every finitely-generated subalgebra \( H_0 \) in \( H \) belongs to the class \( \mathfrak{X} \).

The class \( LSC(\mathfrak{X}) \) is a locally-closed prevariety, generated by the class \( \mathfrak{X} \).

A.I. Maltsev [21], [22] proved that if the class \( SC(\mathfrak{X}) \) is axiomatizable, then this class is a quasivariety. In this case, \( LSC(\mathfrak{X}) \) is a quasivariety as well. In the general case, such a class is not axiomatizable. However, the following theorem takes place [34]:

*For any \( \mathfrak{X} \), the class \( LSC(\mathfrak{X}) \) is determined by infinitary quasi-identities of the class \( \mathfrak{X} \).*

Here arises a natural question what are the conditions providing \( LSC(\mathfrak{X}) = q \text{Var}(\mathfrak{X}) \). This question is related to Proposition 3.

We call the class \( \mathfrak{X} \) logically compact (\( q_\omega \)-compact [27]) if each of its infinitary quasi-identity \( \left( \bigwedge_{(w,w') \in T} (w \equiv w') \right) \Rightarrow w_0 \equiv w'_0 \), where \( T \) is a binary relation in \( W = W(\mathfrak{X}) \), reduced to a finite quasi-identity \( \left( \bigwedge_{(w,w') \in T_0} (w \equiv w') \right) \Rightarrow w_0 = w'_0 \) with a finite subset \( T_0 \) in \( T \).

We can now claim that if \( \mathfrak{X} \) is a logically compact class, then

\[ LSC(\mathfrak{X}) = q \text{Var}(\mathfrak{X}). \]

Actually, the opposite is also true (see below).

Let us note also that the problem of coincidence of classes \( SC(\mathfrak{X}) \) and \( q \text{Var}(\mathfrak{X}) \) was solved by V.A. Gorbunov [13].
Proposition 4. Algebras $H_1$ and $H_2$ are geometrically equivalent if and only if

$$LSC(H_1) = LSC(H_2).$$

2.2. Geometrically noetherian algebras. We introduce, first, some new definitions.

Definition 2. An algebra $H \in \Theta$ is called geometrically noetherian if for any $W = W(X)$ and $T$ in $W$ there exists a finite subset $T_0$ of $T$ such that $T''_H = T''_0H$.

The following proposition is proved in a standard way.

Proposition 5. An algebra $H \in \Theta$ is geometrically noetherian if and only if for every $W \in \text{Ob } \Theta^0$ the ascending chain condition for $H$-closed congruences holds.

The equivalent condition is descending chain condition for algebraic sets in $\text{Hom}(W, H)$ for every $W \in \text{Ob } \Theta^0$.

Definition 3. We call a variety $\Theta$ noetherian if every $W \in \text{Ob } \Theta^0$ is noetherian (by congruences).

Obviously, if $\Theta$ is a noetherian variety then every algebra $H \in \Theta$ is geometrically noetherian.

Examples. 1) A classical variety $\text{Com-P}$ is noetherian.

2) All noetherian subvarieties in $\text{Ass-P}$ are described \cite{1}.

3) The variety $\mathfrak{N}_c$ of all nilpotent groups of the nilpotency class $c$ is noetherian.

4) Every variety consisting of locally finite groups is noetherian.

5) A variety of the form $\mathfrak{N}_c \Theta$, where $\Theta$ is a locally finite variety, is noetherian.

6) A free group $F(X)$ with finite $X$ is geometrically noetherian \cite{13}.

7) Finitely-dimensional associative and Lie algebras are geometrically noetherian \cite{3}.
We note that the notion of geometrical noetherianity of an algebra, as well as the notion of geometrically equivalence of algebras, does not depend on the choice of variety containing the algebras under consideration.

Now let us generalize the notion of geometrical noetherianity.

2.3. Logical noetherianity.

**Definition 4.** An algebra $H \in \Theta$ is called locally geometrically noetherian if for every free algebra $W$ and every set $T$ in $W$ and for every pair $(w_0, w_0') \in T''_H$ there exists a finite subset $T_0$ in $T$, depending, generally, on $(w_0, w_0')$, such that $(w_0, w_0') \in T''_0 H$.

Proposition 6 follows directly from above.

**Proposition 6.** The algebra $H \in \Theta$ is locally geometrically noetherian if every infinitary quasi-identity in $H$ is reduced in $H$ to a finite quasi-identity.

This is the reason why locally geometrically noetherian algebras we call also logically noetherian.

**Proposition 7.** The algebra $H$ is logically noetherian if and only if the union of any directed system of $H$-closed congruences is also an $H$-closed congruence for every $W \in \text{Ob } \Theta^0$.

*Proof.* Let the algebra $H$ be logically noetherian and $T$ a union of some directed system of $H$-closed congruences $T_\alpha$, $\alpha \in I$. $T$ is a congruence. We need to check that it is $H$-closed.

Take $T''_H$ and let it contain the pair $(w, w')$. Find a finite subset $T_0$ in $T$ with $(w, w') \in T''_0 H$. We have $T_\alpha$ with $T_0 \subset T_\alpha$. Then $(w, w') \in T''_0 H \subset T''_H = T_\alpha \subset T$. Thus $(w, w') \in T$, $T = T''_H$.

To prove the opposite, assume the condition of directed systems of $H$-closed congruences.

Take an infinite set $T$ in $W$. Consider in $T$ all possible finite subsets $T_\alpha$. All $T''_{\alpha H}$ constitute a directed system of $H$-closed congruences. Let $T_1$ be the union of all congruences of this system. $T \subset T_1 \subset T''_H$. Since $T_1$ is $H$-closed, then
$T_1 = T''_H$. If $(w, w') \in T''_H = T_1$, then $(w, w') \in T''_{\alpha H}$ for some $\alpha$. This means that the algebra $H$ is logically noetherian.

It is clear that geometrical noetherianity of algebras implies its logical noetherianity. Show that the opposite is not true for the case of groups $\Theta = \text{Grp}$. Consider free groups $F = F(X)$, where $X$ are finite subsets in $X^0$, and consider all possible invariant subgroups $U$ in them. Denote by $H$ the discrete direct product of all $F(X)/U$. We have injections $F(X)/U \to H$. Therefore, all invariant subgroups in every $F(X)$ are $H$-closed. From this it follows that the group $H$ is not geometrically noetherian. However, it is logically noetherian by Proposition 7.

Similar examples can be found in the variety $\text{Ass}-P$ and various other cases.

2.4. Logical noetherianity and geometrical equivalence.

**Theorem 1.** [27] The equality $LSC(H) = q \text{Var}(H)$ takes place if and only if the algebra $H$ is logically noetherian.

**Proof.** In [27] the theorem is proved for groups, but similar considerations are valid in the general situation. Note that [27] uses a different term ($q_\omega$-compactness) instead of the term “logical noetherianity.”

We present the proof for arbitrary $\Theta$, taking into account, in particular, the case of associative and Lie algebras.

Note first of all that the algebra $H$ is logically noetherian if the class $\mathfrak{X}$, consisting of one algebra $H$, is logically compact. Thus, if the algebra $H$ is logically noetherian, then $LSC(H) = q \text{Var}(H)$.

We now prove the opposite. Let $LSC(H) = q \text{Var}(H)$ be given. Check that the algebra $H$ is logically noetherian. Take an algebra $W = W(X) \in \text{Ob } \Theta^0$. Take a congruence $T$ in $W$ which is the union of the directed system of $H$-closed congruences $T_\alpha$, $\alpha \in I$. We want to verify that $T$ is $H$-closed as well, i.e., $W/T \in SC(H)$. In our conditions we just need to check that every quasi-identity of the algebra $H$ holds in $W/T$.

Let the quasi-identity

$$w_1 \equiv w'_1 \land \cdots \land w_n \equiv w'_n \rightarrow w_0 \equiv w'_0 \tag{**}$$
be written in elements from $W(Y)$, and let it be fulfilled in the algebra $H$. Check that it holds also in $W/T$.

Take a homomorphism $\mu : W(Y) \rightarrow W(X)/T$, and the corresponding commutative diagram

\[
\begin{array}{ccc}
W(Y) & \xrightarrow{\mu} & W(X) \\
\downarrow{\mu} & & \downarrow{\nu} \\
W(X)/T & &
\end{array}
\]

Here $\nu$ is a natural homomorphism. Besides, for every $\alpha \in I$ consider natural homomorphisms $\nu_\alpha : W(X) \rightarrow W(X)/T_\alpha$. Assume that $w_i^{\mu} = w_i^{\mu \nu}$; $w_i^{\mu \nu \alpha} = w_i^{\mu \nu \alpha}$ holds for every $i = 1, \ldots, n$. We can choose $\alpha \in I$ such that $w_i^{\mu \nu \alpha} = w_i^{\mu \nu \alpha}$. We proceed from the homomorphism $\nu_\alpha \mu_0 : W(Y) \rightarrow W(X)/T_\alpha$.

Since the quasi-identity (23) holds in $W(X)/T_\alpha$, we have also $w_0^{\mu \nu \alpha} = w_0^{\mu \nu \alpha}$. The last formula gives $w_0^{\mu \nu} = w_0^{\mu \nu} ; w_0^{\mu} = w_0^{\mu}$. This means that the quasi-identity (23) holds in the algebra $W(X)/T$ and the congruence $T$ is $H$-closed. Hence the algebra $H$ is logically noetherian. \hfill $\square$

Note that similar arguments can be used in the case when instead of one algebra $H$ we take an arbitrary logically compact class. (see [14]).

Theorem 2 easily follows from the theorem just proved (see [27]).

**Theorem 2.** If the algebra $H = H_1 \in \Theta$ is not logically noetherian, then there exists its ultrapower $H_2$ which is not geometrically equivalent to the algebra $H_1$. Here $H_1$ and $H_2$ have the same elementary theories and, in particular, their quasi-identities coincide.

**Proof.** Since $H$ is not logically noetherian, we have the inequality

\[ LSC(H) \neq q \text{Var}(H). \]

According to [14], we have a presentation $q \text{Var}(H) = SCC_{up}(H)$. Here $C_{up}$ is an operator which takes ultraproducts of algebras.

The class $C_{up}(H_1)$ has an algebra $H_2$ which does not belong to the class $LSC(H_1)$. Therefore, $LSC(H_1) \neq LSC(H_2)$, and the algebras $H_1$ and $H_2$ are not geometrically equivalent. The algebra $H_2$ is an ultrapower of the algebra $H = H_1$. \hfill $\square$
Theorem 3. If the algebras $H_1$ and $H_2$ are logically noetherian, then they are geometrically equivalent if and only if they have the same quasi-identities.

2.5. Examples. Problems. Consider the problem of existing not logically noetherian algebras in $\Theta$.

1. $\Theta = \text{Grp}$. Let us do it for the cases $\Theta = \text{Grp}$ and $\Theta = \text{Ass-P}$, starting with groups. Using [12], consider finitely presented groups in the form $F(X)/U$, where $F(X)$ is a free group over finite $X$, and $U$ is an invariant subgroup in $F = F(X)$ with the finite set of generators (as the invariant subgroup).

Let $H$ be a discrete direct product of all such $F(X)/U$. In the countable group $H$ there is a countable set of finitely generated subgroups.

Show that the group $H$ is not logically noetherian. We use here the known Theorem (see [19]) that there exists a continuum of two-generated simple groups. One of such groups, say $\Gamma$, is not embeddable into the group $H$.

Consider a surjection $\mu : F(x, y) \to \Gamma$. Let $U = \ker \mu$. Take a sequence $u_1, u_2, \ldots, u_n, \ldots$ of all elements of $U$.

Denote by $U_n$ an invariant subgroup in $F = F(X, Y)$, generated by elements $u_1, \ldots, u_n$. The union of all $U_n$ is $U$. Besides, $F(x, y)/U_n$ is embedded injectively in $H$ and, hence, all $U_n$ are $H$-closed. We check that $U$ is not an $H$-closed invariant subgroup.

Assume that $U$ is $H$-closed and $\Gamma \approx F(x, y)/U$ is embedded into $H^I$ for some $I$. We assume that $\Gamma$ is a subgroup in $H^I$. Consider a system of invariant subgroups $U_\alpha$ in $H^I$ with $H^I/U_\alpha \approx H$ and $\bigcap_\alpha U_\alpha = 1$. The image $\Gamma$ in $H^I/U_\alpha$ is isomorphic to $\Gamma / \Gamma \cap U_\alpha$. $\Gamma$ is a simple group. If $\Gamma \cap U_\alpha = \Gamma$ always holds true, then we get a contradiction. Therefore, $\Gamma \cap U_\alpha = 1$ for some $\alpha$ and $\Gamma$ is embedded into $H \approx H^I/U_\alpha$, which contradicts the choice of $\Gamma$. Thus, the invariant subgroup $U$ is not $H$-closed, and the group $H$ is not logically noetherian.

Similar considerations are valid in the case $\Theta = \text{Ass-P}$.

2. $\Theta = \text{Ass-P}$. Let us call an algebra $H \in \Theta$ correct if there exists a surjection $H \to P$. A simple algebra is correct if and only if it coincides with $P$. In the general case $P$ is a subalgebra in $H$. 
Let us note that if $H_\alpha$, $\alpha \in I$, is a family of correct algebras and $H$ is their free product in $\Theta$ then all embeddings $i_\alpha : H_\alpha \to H$ are injective.

Indeed, let us fix $\alpha$ and consider homomorphisms $\nu_\beta : H_\beta \to H_\alpha$ for all $\beta \in I$. If $\beta = \alpha$, then $\nu_\beta = \nu_\alpha : H_\alpha \to H_\alpha$ is an identical isomorphism; if $\beta \neq \alpha$, then $\nu_\beta : H_\beta \to H_\alpha$ is a homomorphism on the subalgebra $P$ in $H_\alpha$. By the definition of a free product there is $\nu : H \to H_\alpha$ such that $\nu i_\alpha = \nu_\alpha$.

If now $i_\alpha$ is not injective then we come to the contradiction with the definition of $\nu_\alpha$.

Consider free algebras $W = W(X)$ in $\Theta = \text{Ass-P}$ with finite $X$. In every such algebra consider finitely-generated ideals $U$, for which the factor-algebra $W/U$ is correct. Take for $H$ a free product of all such $W/U$ for different $X$ and $U$.

Assume further that the field $P$ is countable. Then the algebra $H$ is also countable and in $H$ there exists a countable set of finitely generated subalgebras.

**Theorem 4.** The algebra $H$ is not a logically noetherian algebra.

**Proof.** We want to show that for the given algebra $H$ there exists a free algebra $W = W(X)$ such that the union of the increasing sequence of $H$-closed ideals can be not an $H$-closed ideal.

We plan to show this with the help of the appropriate finitely generated algebra $\Gamma$, which is not embeddable to any Cartesian power of the algebra $H$. In order to find such an algebra we need some observations concerning group algebras of simple groups.

Let $G$ be a simple group and $PG$ be its group algebra. Consider ideals $V \subset PG$. The canonical homomorphism $PG \to PG/V$ implies $\mu : G \to PG/V$. The kernel $\text{Ker} \mu$ consists of the elements $g \in G$ such that $g - 1 \in V$. This kernel either the whole group $G$ or 1. In the first case we have $g - 1$ lies in $V$ for every $g \in G$. Then the ideal $V$ coincides with the augmentation ideal $\Delta$. In the second case $(g - 1) \in V$ implies $g = 1$. In the second case we call the ideal $V$ faithful ideal.
Thus, if $V \neq \Delta$ then $V$ is faithful. The union of the increasing sequence of faithful ideals is again a faithful ideal. Therefore, there are maximal faithful ideals in $PG$. Let $V_0$ be a maximal faithful ideal. If $V_0 \subset V$ and $V$ does not coincide with $PG$ then either $V_0 = V$ or $V = \Delta$.

Take an algebra $\Gamma = PG/V_0$. We have an injection $G \to \Gamma$. There are two possibilities:

1. $\Gamma$ is a simple algebra. 2. $\Gamma$ has a unique proper ideal $\bar{\Delta} = \Delta/V_0$.

Let further the group $G$ be finitely generated. Then this group is simultaneously a finitely generated as a semigroup. Then the group algebra $PG$ is a finitely generated algebra, and $\Gamma$ is also finitely generated.

The algebra $H$ is countable. In such an algebra there exists not more than countable set of finitely generated groups. So, we can find a finitely generated simple group $G$ which is not embeddable to $H$. Then the algebra $\Gamma$ is not embeddable to $H$.

Assume that for $\Gamma$ the second case takes place, that is there exists a unique ideal $\bar{\Delta}$ in $\Gamma$. Suppose that $\Gamma$ is embeddable as a subalgebra to $H^I$. Take a system of ideals $V_\alpha, \alpha \in I$ such that $H^I/V_\alpha \approx H$ and $\cap_{\alpha \in I} V_\alpha = 0$. If $\Gamma \cap V_\alpha = 0$ for some $\alpha$ then $\Gamma$ is embeddable to $H$. Contradiction. If this intersection is not equal to zero, then it always contains $\bar{\Delta}$, which contradicts $\cap_{\alpha \in I} V_\alpha = 0$. Therefore, the algebra $\Gamma$ is not embeddable in a Cartesian power of $H$.

Take now a finitely generated algebra $W(X)$ with the surjection $\mu : W(X) \to \Gamma$. Take $U = \text{Ker } \mu$. Let $u_1, \ldots, u_n, \cdots$ be all elements of algebra $U$. Denote by $U_n$ the ideal generated by the first $n$ elements. Then algebra $W(X)/U_n$ is finitely presented and correct. Such algebra is embeddable to $H$. Hence, every ideal $U_n$ is $H$-closed. However, the union of these ideals, i.e., the ideal $U$ is not $H$-closed since $W(X)/U$ is isomorphic to $\Gamma$ which is not embeddable to any Cartesian power of $H$. This means that in the second case we found an appropriate algebra $W(X)$ which makes $H$ not logically noetherian.

Suppose now that for $G$ and $\Gamma$ the first case holds, i.e., algebra $\Gamma$ is a simple algebra. Consider $P \times \Gamma = \Gamma^*$. There are only two ideals in this algebra, namely $P$ and $\Gamma$. Besides, assume that $G$ is not embeddable also in $H \times H$.
Since $G$ is embeddable in $\Gamma^*$ then $\Gamma^*$ is not embeddable in $H$ and $H \times H$. We show that $\Gamma^*$ is not embeddable in any $H^I$.

As before, consider a system of ideals $V_\alpha$ with $H^I/V_\alpha \approx H$ and $\cap V_\alpha = 0$ in $H^I$. The image $\Gamma^*$ in $H^I/V_\alpha$ is isomorphic to the algebra $\Gamma^*/\Gamma^* \cap V_\alpha$. There are only two proper ideals in the algebra $\Gamma^*$, namely, $P$ and $\Gamma$. If $\Gamma^* \cap V_\alpha = 0$ for some $\alpha$, then $\Gamma^*$ is embedded into $H$. Contradiction. If $\Gamma^* \cap V_\alpha = P$ for some $\alpha$, then already $\Gamma$ is embedded into $H$. Besides, $P \subset H$. Then $\Gamma^* = P \times \Gamma \subset H \times H$. Contradiction. Other cases lead to contradiction with $\cap V_\alpha = 0$. Hence, the embedding $\Gamma^* \to H^I$ is impossible.

The algebra $\Gamma^*$ is finitely generated and correct. Consider the surjection $W(X) \to \Gamma^*$. It remains to repeat the arguments above. We found again the appropriate $W(X)$. So, the algebra $H$ is not logically noetherian. The theorem is proved.

The condition on the field to be countable can be eliminated. Indeed, let $K$ be an arbitrary field and $P$ its countable subfield. According to Theorem 4 construct an algebra $H$ over $P$. It can be proved that by extending scalars to the elements of $K$ we get an algebra over $K$ which is not logically noetherian.

**Problem 1.** Let $W = W(X)$ be a free in Ass-$P$ algebra with $|X| \geq 2$. Is it true that $W$ is not geometrically noetherian, but is logically noetherian?

**Problem 2.** The same question for free Lie algebras (see also [26]).

**Problem 3.** Let $G$ be a group, and $PG = H$ its group algebra. The problem is to find the relationship between the geometrical and logical noetherianity for $G$ and $H$.

This is, indeed, a wide topic related to various problems in the group algebra theory.

**Problem 4.** Is it true that there exists continuum finitely generated simple associative algebras?

---

1When the paper was finished I have been informed that a solution of this problem is contained in the forthcoming paper [20].
Problem 5. Is it true that there exists continuum finitely generated simple Lie algebras?

The positive answer on problem 5 would allow to construct an example of not logically noetherian Lie algebra.

Note that in the paper by V.Bludov and D.Gusev [8] there is an example of the solvable group of class 3 which is not logically noetherian. See also [6], [20],[38].

3. Geometrical Similarity of Algebras

3.1. Isomorphism of functors. If algebras $H_1$ and $H_2$ are geometrically equivalent, then the categories $C_\Theta(H_1)$ and $C_\Theta(H_2)$ coincide, while categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are isomorphic. The notion of geometrical similarity of algebras is related to necessary and sufficient conditions of isomorphism of categories of algebraic sets.

Let us recall the notions of homomorphism and isomorphism of two functors of a category, which we will use in the sequel.

Let two functors $\varphi_1, \varphi_2 : C_1 \to C_2$ of the categories $C_1, C_2$ be given. The homomorphism (natural transformation) of functors $s : \varphi_1 \to \varphi_2$ is a function, relating a morphism in $C_2$, denoted by $s_A : \varphi_1(A) \to \varphi_2(A)$ to every object $A$ of the category $C_1$. For every $\nu : A \to B$ in $C_1$ there is a commutative diagram

$$
\begin{array}{ccc}
\varphi_1(A) & \xrightarrow{s_A} & \varphi_2(A) \\
\varphi_1(\nu) & & \varphi_2(\nu) \\
\varphi_1(B) & \xrightarrow{s_B} & \varphi_2(B)
\end{array}
$$

in the case of covariant $\varphi_1$ and $\varphi_2$.

For contravariant $\varphi_1$ and $\varphi_2$ the corresponding diagram is

$$
\begin{array}{ccc}
\varphi_1(A) & \xleftarrow{s_A} & \varphi_2(A) \\
\varphi_1(\nu) & & \varphi_2(\nu) \\
\varphi_1(B) & \xleftarrow{s_B} & \varphi_2(B)
\end{array}
$$

An invertible $s : \varphi_1 \to \varphi_2$ is isomorphism (natural isomorphism) of functors. The isomorphism property holds if $s_A : \varphi_1(A) \to \varphi_2(A)$ is an isomorphism in $C_2$ for any $A$. 
### 3.2. **Functor** $\text{Cl}_H$

Consider a (contravariant) functor $\text{Cl}_H : \Theta^0 \to \text{Set}$ for every algebra $H \in \Theta$. If $W = W(X)$ is an object of $\Theta^0$, then $\text{Cl}_H(W)$ is the set of all $H$-closed congruences $T$ in $W$. If, further, $s : W(Y) \to W(X)$ is a morphism of $\Theta^0$, then we have a mapping of sets $\text{Cl}_H(s) : \text{Cl}_H(W(X)) \to \text{Cl}_H(W(Y))$. This mapping is defined by the following rule: if $T$ is an $H$-closed congruence in $W(X)$, then $\text{Cl}_H(s)(T) = s^{-1}T$. It is always an $H$-closed congruence in $W(Y)$. Here, $w(s^{-1}T)w'$ if $wTsw'$. For every subvariety $\Theta_1$ in $\Theta$, containing an algebra $H$, we have also $\text{Cl}_H : \Theta_0^1 \to \text{Set}$. These two different $\text{Cl}_H$ are well correlated. If $W = W(X) \in \text{Ob} \Theta^0$, then $W_0 = W_0(X)$ is an object in $\Theta_1^0$ with the natural homomorphism $W \to W_0$. It is easily checked that there is a bijection between the sets $\text{Cl}_H(W)$ and $\text{Cl}_H(W_0)$.

It follows from definitions that the algebras $H_1$ and $H_2$ are geometrically equivalent if and only if the functors $\text{Cl}_{H_1}$ and $\text{Cl}_{H_2}$ coincide. Besides, $\text{Var}(H_1) = \text{Var}(H_2)$. The notion of *geometrical similarity* assumes that the varieties $\Theta_1 = \text{Var}(H_1)$ and $\Theta_2 = \text{Var}(H_2)$ do not necessarily coincide, but there is an isomorphism of categories $\varphi : \Theta_1^0 \to \Theta_2^0$ with the isomorphism of functors $\alpha(\varphi) : \text{Cl}_{H_1} \to \text{Cl}_{H_2} \varphi$ depending on $\varphi$ under one additional condition, described later.

In the commutative diagram

\[
\begin{array}{ccc}
\Theta_1^0 & \xrightarrow{\varphi} & \Theta_2^0 \\
\text{Cl}_{H_1} \downarrow & & \downarrow \text{Cl}_{H_2} \\
\text{Set} & & \\
\end{array}
\]

functors $\text{Cl}_{H_1}$ and $\text{Cl}_{H_2}$ act on the categories $\Theta_1^0$ and $\Theta_2^0$, respectively, and commutativity of the diagram is treated as an isomorphism of functors $\text{Cl}_{H_1}$ and $\text{Cl}_{H_2} \varphi$.

### 3.3. **Function $\beta$ on the category of free algebras**

Given $\Theta$ and the category $\Theta^0$, consider a special function $\beta$. Take two arbitrary objects $W_1$ and $W_2$ in the category $\Theta^0$. Let $T$ be a congruence in $W_2$. Denote by $\beta = \beta_{W_1,W_2}(T)$ a binary relation in $\text{Hom}(W_1, W_2)$. This relation is defined as follows: $s_1 \beta s_2$ holds for $s_1, s_2 : W_1 \to W_2$ if and only if $w^*T w^s$ for every $w \in W_1$. The isomorphism $\alpha = \alpha(\varphi)$ should commute with the functor $\beta$. 
3.4. Geometrical similarity.

**Definition 5.** Let $H_1$ and $H_2$ be algebras in $\Theta$, $\Theta_1 = \text{Var}(H_1)$, $\Theta_2 = \text{Var}(H_2)$. The algebras $H_1$ and $H_2$ are called geometrically similar if

1. There exists an isomorphism $\varphi : \Theta_1^0 \to \Theta_2^0$.
2. There exists a function $\alpha = \alpha(\varphi)$ such that a bijection $\alpha(\varphi)_W : \text{Cl}_{H_1}(W) \to \text{Cl}_{H_2}(\varphi(W))$ holds for every $W \in \text{Ob}\Theta_1^0$.
3. The function $\alpha$ is coordinated with the function $\beta$.

The last condition means that

$$\varphi(\beta_{W_1W_2}(T)) = \beta_{\varphi(W_1)\varphi(W_2)}(\alpha(\varphi)_{W_2}(T)).$$

Here $W_1, W_2$ are objects in $\Theta_1^0$, $T$ is an $H_1$-closed congruence in $W_2$, and for every relation $\rho$ in $\text{Hom}(W_1, W_2)$ the relation $\varphi(\rho)$ is defined by the rule: $s'_1 \varphi(\rho) s'_2$ holds for $s'_1, s'_2 : \varphi(W_1) \to \varphi(W_2)$ if there are $s_1, s_2 : W_1 \to W_2$ such that $\varphi(s_1) = s'_1$, $\varphi(s_2) = s'_2$ and $s_1 \rho s_2$.

Show now that the function $\alpha = \alpha(\varphi)$ is uniquely determined by these conditions, and give the formula for its calculation. To this end, consider a function $\rho$ with $\rho_W = \beta_{W_1W_2}$ for every $W \in \text{Ob}\Theta_1^0$. Besides, define a function $\tau$, such that $\tau_W(\rho) = T$ is a relation in $W$, defined by the rule: $w_1 T w_2$ if there is $w \in W$ with $w'' = w_1$, $w''' = w_2$ and $\nu \rho \nu'$.

It is proved (see [31]) that if $T$ is a congruence in $W$, then $\tau_W \rho_W(T) = T$.

**Proposition 8.** $\alpha(\varphi)_W(T) = \tau_{\varphi(W)}(\varphi(\rho_W(T)))$ holds true.

**Proof.** By the condition of coordination between $\alpha$ and $\beta$, we have $\varphi(\rho_W(T)) = \rho_{\varphi(W)}(\alpha(\varphi)_W(T))$. Here $T$ is an $H_1$-closed congruence in $W$, $\alpha(\varphi)_W(T)$ is an $H_2$-closed congruence in $\varphi(W)$. Let us apply $\tau_{\varphi(W)}$.

$$\tau_{\varphi(W)}(\varphi(\rho_W(T))) = \tau_{\varphi(W)}(\alpha(\varphi)_W(T)) = \alpha(\varphi)_W(T).$$

The proved formula allows to state that for every $W \in \text{Ob}\Theta_1^0$ the mapping $\alpha(\varphi)_W : \text{Cl}_{H_1}(W) \to \text{Cl}_{H_2}(\varphi(W))$ determines the isomorphism of latices of algebraic sets in $\text{Hom}(W, H_1)$ and $\text{Hom}(\varphi(W), H_2)$. 
Finally, it is proved [32] that $\alpha(\varphi)$ gives an isomorphism of functors $\text{Cl}_{H_1} \to \text{Cl}_{H_2} \varphi$. It is easy to understand that the relation of geometrical similarity of algebras in $\Theta$ is reflexive, symmetric and transitive. It is also clear that geometric equivalence is a particular case of geometrical similarity. □

3.5. Inner automorphisms of the category of free algebras. Assume further that $\text{Var}(H_1) = \text{Var}(H_2) = \Theta$ for the algebras $H_1$ and $H_2$. This is a natural condition; it always holds for the variety Com-$P$ if the field $P$ is infinite. In this case similarity of the algebras $H_1$ and $H_2$ is defined by an automorphism $\varphi : \Theta^0 \to \Theta^0$. For various special $\varphi$, similarity to some extent is reduced to geometrical equivalence. Let us consider one of the such cases.

For an arbitrary category $C$, let us call its automorphism $\varphi : C \to C$ an inner if there is an isomorphism of functors $s : 1_C \to \varphi$. Here for every object $A$ we have an isomorphism $s_A : A \to \varphi(A)$ and for every $\nu : A \to B$ the diagram

$$
\begin{array}{c}
A \xrightarrow{s_A} \varphi(A) \\
\downarrow \nu \\
B \xrightarrow{s_B} \varphi(B)
\end{array}
$$

is commutative. Thus, $\varphi(\nu) = s_B \nu s_A^{-1}$. This motivates the word “inner”.

Similarly, one can define an inner endomorphism (endofunctor) of a category: it is an arbitrary $\varphi : C \to C$, isomorphic to a unit automorphism $1_C$.

**Proposition 9.** If similarity of the algebras $H_1$ and $H_2$ is determined by an inner automorphism $\varphi$ of the category $\Theta^0$, then $H_1$ and $H_2$ are geometrically equivalent.

**Proof.** Let an isomorphism $s : 1_{\Theta^0} \to \varphi$, an object $W$ in $\Theta^0$, and a congruence $T$ in $W$ be given. Check that

$$
\alpha(\varphi)_W(T) = s_W T.
$$

Here $s_W : W \to \varphi(W)$ is an isomorphism of objects and $s_W T$ is a congruence in $\varphi(W)$, defined by the rule: $w'_1(s_W T) w'_2$ holds if and only if $w'_1 = s_W(w_1)$, $w'_2 = s_W(w_2)$ and $w_1 T w_2$. Denote $s_W T = T^*$ and check that

$$
\varphi(\rho_W(T)) = \rho_{\varphi(W)}(T^*). \quad (***)
$$
Let \( \mu, \mu' \in \text{End} \varphi(W) \) and \( \mu \varphi(\rho_W(T)) \mu' \) take place. Then: \( \nu, \nu' \in \text{End} W; \) \( \nu \rho_W(T) \nu', \mu = \varphi(\nu), \mu' = \varphi(\nu') \). For every \( w \in W \) we have \( \nu(w) T \nu'(w) \). Thus,

\[
s_W \nu(w) T^* s_W \nu'(w);
\]

\[
s_W \nu s_W^{-1}(s_W w) T^* s_W \nu' s_W^{-1}(s_W(w));
\]

\[
\mu(w_1) T^* \mu'(w_1).
\]

Here, \( w_1 = s_W(w) = w_1 \) is an arbitrary element in \( \varphi(W) \) which gives us \( \mu(\rho_{\varphi(W)}(T^*)) \mu' \).

Let, now, \( \mu(\rho_{\varphi(W)}(T^*)) \mu' \) holds. This means that \( \mu(w_1) T^* \mu'(w_1) \) holds for every \( w_1 \in \varphi(W) \). Take \( \nu \) and \( \nu' \in \text{End} W \) with \( \mu = \varphi(\nu) = s_W \nu s_W^{-1}, \mu' = s_W \nu' s_W^{-1} \) and \( w \) with \( s_W(w) = w_1 \), where \( w \) is an arbitrary element in \( W \). We have \( s_W(\nu(w)) T^* s_W(\nu'(w)) \). This gives \( \nu(w) T \nu'(w), \nu \rho_W(T) \nu' \). Then \( \varphi(\nu) \varphi(\rho_W(T)) \varphi(\nu') \) and \( \mu \varphi(\rho_W(T)) \mu' \). The equality (***) is checked. Now we have

\[
\alpha(\varphi_W(T)) = \tau_{\varphi(W)} \varphi(\rho_W(T)) = \tau_{\varphi(W)} \rho_{\varphi(W)}(T^*) = T^* = s_W T.
\]

Let \( T \) be an \( H_1 \)-closed congruence in \( W \). Then \( s_W T \) is an \( H_2 \)-closed congruence in \( \varphi(W) \) by the definition of similarity. On the other hand, using the isomorphism \( s_W^{-1}: \varphi(W) \to W \) and the fact that isomorphism of objects in \( \Theta^0 \) preserves the \( H \)-closeness condition for every \( H \) (see [29]), we conclude that \( T \) is an \( H_2 \)-closed congruence as well. Hence every \( H_1 \)-closed congruence in \( W \) is \( H_2 \)-closed. Applying \( \varphi^{-1} \), conclude the opposite. Thus, \( H_1 \) and \( H_2 \) are geometrically equivalent.

\[\square\]

Other examples of this kind will be given in the section devoted to the case \( \Theta = \text{Ass-P} \). It is proved for \( \Theta = \text{Grp} \) that all automorphisms of the category \( \Theta^0 \) are inner [23].

4. Geometrical Coordination of Algebras

4.1. Additional information on categories. Coordination of algebras leads to the necessary and sufficient conditions of equivalence of two categories of algebraic sets.
We recall here some required information from category theory [25], [39].

Let the categories \( C_1 \) and \( C_2 \) be given. They are equivalent if there exists a pair of functors \( \varphi : C_1 \to C_2 \) and \( \psi : C_2 \to C_1 \) such that \( \psi \varphi \approx 1_{C_1} \), \( \varphi \psi \approx 1_{C_2} \). Here, \( 1_C \) is a unity functor of a category. The sign \( \approx \) here denotes isomorphism of functors. We say that the pair \( (\varphi, \psi) \) determines equivalence of categories \( C_1 \) and \( C_2 \). If \( \psi \varphi = 1_{C_1} \), \( \varphi \psi = 1_{C_2} \), then the pair \( (\varphi, \psi) \) determines isomorphism of categories and \( \psi = \varphi^{-1} \).

It is proved [39] that if \( (\varphi, \psi) \) is an equivalence, then each of the functors \( \varphi \) and \( \psi \) possesses the following two properties

1. Completeness;
2. Univalency.

For \( \varphi : C_1 \to C_2 \), completeness means that for every object \( B \) of \( C_2 \) there exists an object \( A \) of \( C_1 \), such that \( \varphi(A) \approx B \). Univalency means that for any two objects \( A \) and \( B \) of \( C_1 \), the functor \( \varphi \) induces a bijection \( \varphi_{A,B} : \text{Hom}(A,B) \to \text{Hom}(\varphi(A), \varphi(B)) \). In particular, for every \( A \) this gives an isomorphism \( \varphi_A : \text{End} A \to \text{End} \varphi(A) \). Let us call a functor \( \varphi \) with these two properties a relational isomorphism of categories. If \( \varphi \) is a relational isomorphism, then it has a relational inverse functor \( \psi \) such that the pair \( (\varphi, \psi) \) determines equivalence of categories. There could be many relational inverse functors for \( \varphi \).

If \( \varphi \) is an isomorphism, then there is only one inverse functor \( \varphi^{-1} \), but there are many relational inverse ones.

Let us fix a small category \( C \). Consider endofunctors (endomorphisms) \( \varphi : C \to C \). They constitute a semigroup \( \text{End} C \). Relational automorphisms (autoequivalences) form a subsemigroup in \( \text{End} C \), denoted by \( \tilde{\text{Aut}}(C) \). The group of automorphisms \( \text{Aut}(C) \) is a group of invertible elements in \( \text{End}(C) \).

It is checked that the isomorphism relation \( \approx \) in is a congruence in \( \text{End}(C) \): \( \varphi_1 \approx \varphi_2 \) and \( \psi_1 \approx \psi_2 \) imply \( \varphi_1 \psi_1 \approx \varphi_2 \psi_2 \).

We can now pass to the factor-semigroup \( \text{End}^0(C) = \text{End}(C)/\approx \). Denote a group of invertible elements in \( \text{End}^0(C) \) by \( \text{Aut}^0(C) \). Fix a natural homomorphism \( \delta : \text{End}(C) \to \text{End}^0(C) \). If \( \delta(\varphi) = \overline{\varphi} \) is an invertible element in \( \text{Aut}^0(C) \),
then we take \( \psi = \varphi^{-1} \) and, therefore, \( \varphi\psi = \psi\varphi = \mathbf{1}_C, \varphi\psi = \psi\varphi = \varphi \),
\( \varphi\psi \approx 1_C \approx \psi\varphi \). Thus, \( \varphi \) is an autoequivalence, like \( \psi \). The subsemigroup \( \widetilde{\text{Aut}}(C) \) is a full co-image of the group \( \text{Aut}^0(C) \). A homomorphism \( \delta \) induces the homomorphism \( \delta : \text{Aut}(C) \to \text{Aut}^0(C) \). The latter is surjective if every autoequivalence \( \varphi \) of the category \( C \) is isomorphic to some automorphism \( \psi \),
\( \varphi \approx \psi, \varphi\psi^{-1} = \varphi_0 \approx 1_C, \varphi = \varphi_0\psi \). Here \( \varphi_0 \) is also an autoequivalence, namely it is an inner one. Besides, let us note that if \( \varphi = \varphi_0\psi \), then all relationally inverse functors to \( \varphi \) are of the form \( \psi^{-1}\varphi_1 \), where \( \varphi_1 \) is an arbitrary functor, isomorphic to \( 1_C \).

Every \( \varphi_1 \), isomorphic to \( 1_C \) is simultaneously an autoequivalence of the category.

Note that the kernel of the homomorphism \( \delta : \text{Aut}(C) \to \text{Aut}^0(C) \) is an invariant subgroup \( \text{Int}(C) \) in \( \text{Aut}(C) \), consisting of all inner automorphisms, which is isomorphic to a trivial automorphism.

Given a small category \( C \) and an object \( A \), denote by \([A]\) the class (set) of all objects in \( C \), isomorphic to \( A \). The set of all objects \( \text{Ob}(C) \) is decomposed into such classes.

**Theorem 5. (G. Zhitomirsky [11])** If all classes \([A]\) have the same cardinality pairwise, then every autoequivalence of the category \( C \) is isomorphic to an automorphism.

**Proof.** Let \( \varphi : C \to C \) be an autoequivalence. For every object \( A \) we set:
\( \overline{\varphi}[A] = [\varphi(A)] \).

It follows from the general categorical considerations that \( \overline{\varphi} \) is a substitution on the set of classes of isomorphic objects: its definition does not depend on the choice of the representative \( A \) in the classes of isomorphic objects. In the conditions of the theorem we have a bijection \( \psi_{[A]} : [A] \to \overline{\varphi}[A] \). Fix these bijections. Further, for every object \( A \) we set:
\( \psi(A) = \psi_{[A]}(A) \in \overline{\varphi}[A] = [\varphi(A)] \).

Here \( \psi \) is a substitution on the set \( \text{Ob} C \). Since \( \psi(A) \in [\varphi(A)] \), then \( \psi(A) \) and \( \varphi(A) \) are isomorphic. For every \( A \) fix some isomorphism \( s_A : \varphi(A) \to \psi(A) \).
For any \( \nu : A \to B \) consider a diagram

\[
\begin{array}{ccc}
\varphi(A) & \xrightarrow{s_A} & \psi(A) \\
\varphi(\nu) | & & | \psi(\nu) \\
\varphi(B) & \xrightarrow{s_B} & \psi(B)
\end{array}
\]

Correspondingly, \( \psi(\nu) = s_B \varphi(\nu) s_A^{-1} \). Under such a definition, \( \psi \) is an automorphism of the category \( C \), the diagram is commutative and \( \varphi \) and \( \psi \) are isomorphic. \( \square \)

Let us apply these general facts to the category \( \Theta^0 \), where \( \Theta \) is an arbitrary variety of algebras. Here for every algebra \( W = W(X) \), the class \([W]\) has the same cardinality as the initial universal set \( X^0 \). Hence, we have the following corollary.

**Corollary 1.** Every autoequivalence of the category \( \Theta^0 \) is isomorphic to an automorphism.

For every autoequivalence \( \varphi \) we have an automorphism \( \psi \) with \( \varphi = \varphi_0 \psi \), where \( \varphi_0 \) is an inner autoequivalence. The homomorphism \( \delta : \text{Aut}(\Theta^0) \to \text{Aut}^0(\Theta^0) \) is always surjective.

It is easy to understand that Theorem 5 admits a generalization. Every equivalence of different categories \( \Theta^0_1 \) and \( \Theta^0_2 \) is naturally isomorphic to an isomorphism of these categories. If \( \Theta^0_1 \) and \( \Theta^0_2 \) are equivalent, then they are isomorphic.

Fix an arbitrary object \( A_0 \) in every class \([A]\). This gives the full subcategory in \( C \). Such a subcategory is considered as the skeleton of the category \( C \), denoted by \( \tilde{C} \). The category \( \tilde{C} \) can be represented also as a category of classes \([A]\).

An autoequivalence \( \varphi : C \to C \) is called special if \( \varphi[A] = [\varphi(A)] = [A] \) for any \( A \). This means that the objects \( A \) and \( \varphi(A) \) are always isomorphic.

**Theorem 6.** If \( \varphi \) is a special autoequivalence, then it can be represented as \( \varphi = \varphi_0 \varphi_1 \), where \( \varphi_0 \) is an inner autoequivalence and \( \varphi_1 \) is an automorphism which does not change objects.
Proof. We do not use here the previous theorem and the axiom of choice. First we build an inner autoequivalence \( \varphi_0 \), setting \( \varphi_0(A) = \varphi(A) \) for every object \( A \). Then we fix an isomorphism \( s_A : A \to \varphi(A) = \varphi_0(A) \). For \( \nu : A \to B \) we set

\[
\varphi_0(\nu) = s_B \nu s_A^{-1} : \varphi_0(A) \to \varphi_0(B).
\]

Here \( \varphi_0 \) is a functor and \( \varphi_0 \approx 1_C \). Solving the equation \( \varphi = \varphi_0 \varphi_1 \) with respect to \( \varphi_1 \), we set \( \varphi_1(A) = A \) for every \( A \). For \( \nu : A \to B \) the equality \( \varphi(\nu) = \varphi_0 \varphi_1(\nu) \) should hold; here we have \( \varphi_1(A) = A \) and \( \varphi_0 \varphi_1(\nu) = s_B \varphi_1(\nu) s_A^{-1} \). Setting \( \varphi_1(\nu) = s_B^{-1} \varphi(\nu) s_A \), we find the automorphism \( \varphi_1 \), which solves the equation. \( \square \)

4.2. Geometrical coordination. Let us pass to the notion of geometrical coordination of algebras, generalizing geometrical similarity.

Definition 6. Let the algebras \( H_1 \) and \( H_2 \) be given in \( \Theta_1 = \text{Var}(H_1) \), \( \Theta_2 = \text{Var}(H_2) \). The algebras \( H_1 \) and \( H_2 \) are called coordinated if

1) There exists an equivalence of categories \( \varphi : \Theta_1 \to \Theta_2 \) and \( \psi : \Theta_2 \to \Theta_1 \)

2) For the pair \( (\varphi, \psi) \) there exist embeddings:

\[
\alpha(\varphi)_W : \text{Cl}_{H_1}(W) \to \text{Cl}_{H_2}(\varphi(W)), \quad W \in \text{Ob } \Theta_1;
\]

\[
\alpha(\psi)_W : \text{Cl}_{H_2}(W) \to \text{Cl}_{H_1}(\psi(W)), \quad W \in \text{Ob } \Theta_2.
\]

3) The functions \( \alpha(\varphi) \) and \( \alpha(\psi) \) commute with the corresponding \( \beta \).

It follows from the third condition that, in particular, if \( W \in \text{Ob } \Theta_1 \) and \( T \) is an \( H_1 \)-closed congruence in \( W \), then \( \varphi(\rho_W(T)) = \rho_{\varphi(W)}(\alpha(\varphi)_W(T)) \). As above, we deduce formulas for the corresponding \( W \) and \( T \):

\[
\alpha(\varphi)_W(T) = \tau_{\varphi(W)} \varphi(\rho_W(T)),
\]

\[
\alpha(\psi)_W(T) = \tau_{\psi(W)} \psi(\rho_W(T)).
\]

In the proofs that follow, we sometimes take into account the univalency property of the functors \( \varphi \) and \( \psi \).

Note that from the definition follows that the transitions

\[
\alpha(\varphi) : \text{Cl}_{H_1} \to \text{Cl}_{H_2} \varphi,
\]
\[ \alpha(\psi) : Cl_{H_2} \to Cl_{H_1} \psi. \]

turn out to be natural transformations of functors.

**Proposition 10.** If \( \varphi : \Theta_1 \to \Theta_2 \) is an isomorphism of categories and \( \psi = \varphi^{-1} \), then the coordination of the algebras \( H_1 \) and \( H_2 \) means that these algebras are similar.

**Proof.** We need to check that in the conditions above
\[ \alpha(\varphi)_W : Cl_{H_1}(W) \to Cl_{H_2}(\varphi(W)) \]
is a bijection, and
\[ \alpha(\psi)_{\varphi(W)} : Cl_{H_2}(\varphi(W)) \to Cl_{H_1}(W) \]
is the inverse bijection, \( W \in \Theta_1^0 \).

Take \( W \in \mathrm{Ob} \Theta_1^0 \) and an \( H_1 \)-closed congruence \( T \) in \( W \). Then
\[ \varphi(\rho_W(T)) = \rho_{\varphi(W)}(\alpha(\varphi)_W(T)), \]
where \( \alpha(\varphi)_W(T) \) is \( H_2 \)-closed congruence in \( \varphi(W) \). Applying \( \psi = \varphi^{-1} \), we get
\[ \rho_W(T) = \psi(\rho_{\varphi(W)}(\alpha(\varphi)_W(T))) = \rho_W(\alpha(\psi)_{\varphi(W)}(\alpha(\varphi)_W(T))). \]
Hence, \( T = \alpha(\psi)_{\varphi(W)}(\alpha(\varphi)_W(T)) \).

We get a similar result if we take \( W \in \Theta_2^0 \) and \( T \) is an \( H_2 \)-closed congruence in \( W \).

Evidently, the coordination relation of two algebras is reflexive and symmetric. Transitivity follows from the considerations below.

Let the algebras \( H_1, H_2 \) and \( H_3 \) be given in the variety \( \Theta \). Correspondingly, \( \Theta_1 = \mathrm{Var}(H_1), \Theta_2 = \mathrm{Var}(H_2), \Theta_3 = \mathrm{Var}(H_3) \). Let the pair of functors \( \varphi_1 : \Theta_1^0 \to \Theta_2^0 \) and \( \psi_1 : \Theta_2^0 \to \Theta_1^0 \) determine coordination of the algebras \( H_1 \) and \( H_2 \), and another pair \( \varphi_2 : \Theta_2^0 \to \Theta_3^0, \psi_2 : \Theta_3^0 \to \Theta_2^0 \) determines coordination for \( H_2 \) and \( H_3 \).

We have \( \varphi = \varphi_2 \varphi_1 : \Theta_1^0 \to \Theta_3^0 \) and \( \psi = \psi_1 \psi_2 : \Theta_3^0 \to \Theta_1^0 \). Check that the pair \((\varphi, \psi)\) determines coordination of the algebras \( H_1 \) and \( H_2 \).
Calculate $\alpha(\phi_2\phi_1)$ and $\alpha(\psi_1\psi_2)$. Take $W \in \text{Ob } \Theta_1^0$, and let $T$ be an $H_1$-closed congruence in $W$. Let us consider the congruence

$$\alpha(\phi_2\phi_1)W(T) = \tau_{\phi_2\phi_1}(w)\phi_2\phi_1(\rho_w(T)).$$

We have (compare Proposition 10) $\varphi_1(\rho_w(T)) = \rho_{\varphi_1(w)}(\alpha(\varphi_1)w(T))$, where $\alpha(\varphi_1)w(T)$ is an $H_2$-closed congruence in $\varphi_1(W)$. Further,

$$\varphi_2\varphi_1(\rho_w(T)) = \varphi_2(\rho_{\varphi_1(w)}(\alpha(\varphi_1)w(T))) = \rho_{\varphi_2\varphi_1(w)}(\alpha(\varphi_2)\varphi_1(w)\alpha(\varphi_1)w(T)).$$

Applying $\tau_{\varphi_2\varphi_1(w)}$ we get $\alpha(\varphi_2\varphi_1)w(T) = \alpha(\varphi_2)\varphi_1(w)\alpha(\varphi_1)wT$. Here the congruence $\alpha(\varphi_2\varphi_1)w(T)$ is an $H_3$-closed congruence in $\varphi_2\varphi_1(W)$, since $\alpha(\varphi_1)w(T)$ is an $H_2$-closed congruence in $\varphi_1(W)$. We have an inclusion

$$\alpha(\varphi_2\varphi_1)W : \text{Cl}H_1(W) \to \text{Cl}H_3(\varphi_2\varphi_1(W)).$$

Similarly, we calculate $\alpha(\psi_1\psi_2)W(T)$ for $W \in \text{Ob } \Theta_3^0$, where $T$ is an $H_3$-closed congruence in $W$. This gives an embedding

$$\alpha(\psi_1\psi_2)W : \text{Cl}H_3(W) \to \text{Cl}H_3(\psi_1\psi_2(W)).$$

Commutativity of $\alpha$ and $\beta$ is evident. This gives the corresponding transitivity.

**Proposition 11.** Let $\text{Var}(H_1) = \text{Var}(H_2) = \Theta$, and $(\varphi, \psi)$ be an autoequivalence of the category $\Theta^0$, and $H_1$ and $H_2$ be coordinated algebras in respect to $(\phi, \psi)$. If this autoequivalence is inner, then $H_1$ and $H_2$ are geometrically equivalent.

**Proof.** The proof is similar to that for geometrical similarity. We take into account univalency of the functors $\varphi$ and $\psi$. □

**4.3. Decomposition of similarity and coordination relations.** Let the pair of functors $(\varphi, \psi)$ determine coordination of the algebras $H_1$ and $H_2$, and there is a decomposition $\varphi = \varphi_0\varphi_1$, $\psi = \psi_1\psi_0$. Assume also that there is an algebra $H$ such that the pair $(\varphi_1, \psi_1)$ determines coordination of the algebras $H$ and $H_1$. We want the pair $(\varphi_0, \psi_0)$ to determine coordination of the algebras $H$ and $H_2$. 
We solve this problem of decomposition of coordination relations in the conditions:

1) \( \text{Var}(H_1) = \text{Var}(H) = \text{Var}(H_2) = \Theta. \)
2) \( \varphi_1 = \zeta, \psi_1 = \zeta^{-1}, \) where \( \zeta \) is an automorphism of the category \( \Theta^0, \)
   determining similarity of the algebras \( H_1 \) and \( H. \)
3) The automorphism \( \zeta \) does not change objects.

The next proposition is valid under the conditions (1) – (3).

**Proposition 12.** Let the pair \((\varphi, \psi)\) determine coordination of the algebras \( H_1 \) and \( H_2, \) \( \varphi = \varphi_0 \zeta, \psi = \zeta^{-1} \psi_0, \) and the automorphism \( \zeta \) determine similarity of the algebras \( H_1 \) and \( H. \) Then the pair \((\varphi_0, \psi_0)\) is an autoequivalence of the category \( \Theta^0, \) determining coordination of the algebras \( H \) and \( H_2. \)

**Proof.** Taking into account conditions, consider

\[
\alpha(\varphi)_W = \alpha(\varphi_0 \zeta)_W = \alpha(\varphi_0)_W \alpha(\zeta)_W = \alpha(\varphi_0)_W \cdot \alpha(\zeta)_W.
\]

We have an embedding \( \alpha(\varphi)_W : \text{Cl}_{H_1}(W) \to \text{Cl}_{H_2}(\varphi(W)). \) Besides, \( \varphi(W) = \varphi_0(W). \) We have also a bijection \( \alpha(\zeta)_W : \text{Cl}_{H_1}(W) \to \text{Cl}_{H}(W). \)

Let now \( T^* \) be an arbitrary \( H \)-closed congruence in \( W, T^* \in \text{Cl}_{H}(W). \) We take \( T^* = \alpha(\zeta)_W(T), \ T \in \text{Cl}_{H_1}(W). \) Then

\[
\alpha(\varphi_0)_W(T^*) = \alpha(\varphi_0)_W \alpha(\zeta)_W(T) = \alpha(\varphi)_W(T) \in \text{Cl}_{H_2}(\varphi_0(W)).
\]

Thus, we have an embedding \( \alpha(\varphi_0)_W : \text{Cl}_{H}(W) \to \text{Cl}_{H_2}(\varphi_0(W)). \)

We then work with \( \psi = \zeta^{-1} \psi_0. \)

For every algebra \( W \in \text{Ob} \Theta^0 \) we have an embedding \( \alpha(\psi)_W : \text{Cl}_{H_2}(W) \to \text{Cl}_{H_1}(\psi(W)). \) Here, \( \psi(W) = \psi_0(W), \psi_0 = \zeta \psi. \) Further,

\[
\alpha(\psi_0)_W = \alpha(\zeta)_{\psi(W)} \alpha(\psi)_W = \alpha(\zeta)_{\psi_0(W)} \alpha(\psi)_W.
\]

Let now \( T \in \text{Cl}_{H_2}(W). \) Then \( \alpha(\psi)_W(T) \in \text{Cl}_{H_1}(\psi(W)) = \text{Cl}_{H_1}(\psi_0(W)). \)

We have also a bijection \( \alpha(\zeta)_{\psi_0(W)} : \text{Cl}_{H_1}(\psi_0(W)) \to \text{Cl}_H(\psi_0(W)). \) Hence, \( \alpha(\psi_0)_W(T) = \alpha(\zeta)_{\psi_0(W)} \alpha(\psi)_W(T) \) and \( \alpha(\psi_0)_W(T) \in \text{Cl}_H(\psi_0(W)). \) This means that there exists an embedding \( \alpha(\psi_0)_W : \text{Cl}_{H_2}(W) \to \text{Cl}_H(\psi_0(W)). \)
Note further that the pair \((\varphi_0, \psi_0)\) is an autoequivalence of the category \(\Theta^0\). We have:

\[
\varphi \psi = \varphi_0 \zeta^{-1} \psi_0 = \varphi_0 \psi_0 \approx 1_{\Theta^0}
\]
\[
\psi \varphi = \zeta^{-1} \psi_0 \varphi_0 \zeta \approx 1_{\Theta^0} \quad \text{and} \quad \psi_0 \varphi_0 \approx 1_{\Theta^0}.
\]

Besides, we have checked that there are embeddings

\[
\alpha(\varphi_0)_W : \Cl_H(W) \to \Cl_{H_2}(\varphi_0(W)).
\]
\[
\alpha(\psi_0)_W : \Cl_{H_1}(W) \to \Cl_H(\psi_0(W)).
\]

It is left to check coordination of the mappings \(\alpha(\varphi_0)_W\) and \(\alpha(\psi_0)_W\) with the function \(\beta\). Let \(W_1\) and \(W_2\) be objects in \(\Theta^0\) and \(T\) a congruence in \(W_2\). Denote \(\beta_{W_1,W_2}(T) = \beta\), and let \(\alpha(\varphi)_W(T) = T^*\) be a congruence in \(\varphi(W_2)\). Denote \(\beta^* = \beta_{\varphi(W_1),\varphi(W_2)}(T^*)\).

We consider \(T\) to be an \(H_1\)-closed congruence in \(W_2\). Then \(T^*\) is an \(H_2\)-closed congruence in \(\varphi(W_2)\). Under these conditions \(\varphi(\beta) = \beta^*\). We will repeat the similar calculations for \(\varphi_0\). Proceed from \(\varphi = \varphi_0 \zeta\) and \(\alpha(\varphi)_W = \alpha(\varphi_0)_W \alpha(\zeta)_W\). Let now \(T\) be an \(H\)-closed congruence in \(W_2\), \(T = \alpha(\zeta)_W(T_1)\) where \(T_1\) is an \(H_1\)-closed congruence in \(W_2\). We have \(\alpha(\varphi)_W(T_1) = \alpha(\varphi_0)_W(T)\). Besides, \(\varphi(W_1) = \varphi_0(W_1), \varphi(W_2) = \varphi_0(W_2)\). Then

\[
\beta_{\varphi_0(W_1),\varphi_0(W_2)}(\alpha(\varphi_0)_W(T)) = \beta_{\varphi(W_1),\varphi(W_2)}(\alpha(\varphi)_W(T)) = \varphi(\beta_{W_1,W_2}(T_1)).
\]

We need to check that \(\varphi(\beta_{W_1,W_2}(T_1)) = \varphi_0(\beta_{W_1,W_2}(T)).\) Here \(T = \alpha(\zeta)_W(T_1)\). The functor \(\zeta\) is coordinated with \(\beta\). Hence, \(\zeta(\beta_{W_1,W_2}(T_1)) = \beta_{W_1,W_2}(\alpha(\zeta)_W(T_1))\), and \(T_1\) is an \(H_1\)-closed congruence in \(W_2\). Applying \(\varphi_0\),

\[
\varphi_0(\beta_{W_1,W_2}(T_1)) = \varphi_0(\beta_{W_1,W_2}(\alpha(\zeta)_W(T_1))) = \varphi_0(\beta_{W_1,W_2}(T)).
\]

This gives \(\varphi(\beta_{W_1,W_2}(T_1)) = \varphi_0(\beta_{W_1,W_2}(T))\). Finally, \(\beta_{\varphi_0(W_1),\varphi_0(W_2)}(\alpha(\varphi_0)_W(T)) = \varphi_0(\beta_{W_1,W_2}(T))\). We have checked coordination of \(\varphi_0\) and \(\beta\).

Let us pass to \(\psi_0\) and \(\beta, \psi_0 = \zeta \psi\). Use that the functors \(\zeta\) and \(\psi\) commute with \(\beta\). Take once more the objects \(W_1\) and \(W_2\) in \(\Theta^0\), and let \(T\) be an \(H_2\)-closed congruence in \(W_2\). We have \(\psi(\beta_{W_1,W_2}(T)) = \beta_{\psi(W_1),\psi(W_2)}(\alpha(\psi)_W(T))\). Applying \(\zeta\), we get

\[
\psi_0(\beta_{W_1,W_2}(T)) = \zeta(\beta_{\psi(W_1),\psi(W_2)}(\alpha(\psi)_W(T)) =
\]
\[ = \beta_{\psi_0(w_1),\psi_0(w_2)}(\alpha(\zeta)\psi(w_2)\alpha(\psi)w_2(T)) = \beta_{\psi_0(w_1),\psi_0(w_2)}(\alpha(\psi_0)w_2(T)). \]

We have also checked correspondence of \( \psi_0 \) and \( \beta \). Thus, \( \varphi_0 \) and \( \psi_0 \) determine geometrical coordination of the algebras \( H \) and \( H_2 \). The proposition is proved. \( \square \)

5. Isomorphisms and equivalences of categories of algebraic sets

5.1. Correctness. Define first correct isomorphisms and correct equivalences under the conditions \( \text{Var}(H_1) = \text{Var}(H_2) = \Theta \). Every isomorphism

\[ F : K_\Theta(H_1) \to K_\Theta(H_2) \]

is in one-to-one correspondence with the isomorphism

\[ \Phi : C_\Theta(H_1) \to C_\Theta(H_2). \]

The category of affine spaces \( K^0_\Theta(H) \) is a subcategory of \( K_\Theta(H) \).

Correctness of an isomorphism \( F \) assumes that \( F \) respects the categories of affine spaces, that is \( F \) induces

\[ F^0 : K^0_\Theta(H_1) \to K^0_\Theta(H_2). \]

Correspondingly, \( \Phi \) induces an automorphism of the category \( \Theta^0 \)

\[ \varphi : \Theta^0 \to \Theta^0. \]

Recall that the category \( \Theta^0 \) is a subcategory in \( C_\Theta(H_1) \) and \( C_\Theta(H_2) \). The equality \( F(\text{Hom}(W,H_1)) = \text{Hom}(\varphi(W),H_2) \) always holds true. Besides, suppose that for every object \( (X,A) \) of the category \( K_\Theta(H_1) \), the equality \( F((X,A)) = (Y,B) \), where \( B \) is an algebraic set in the affine space \( \text{Hom}(W(Y),H_2) \), and \( W(Y) = \varphi(W(X)) \) holds.

This definition of correctness of isomorphism is quite natural and in the sequel, isomorphism of categories means correct isomorphism.

Note that it follows from the definition that if \( \mu : W(X) \to W(X)/T \) is a natural homomorphism in the category \( C_\Theta(H_1) \), then a natural homomorphism \( \Phi(\mu) : \varphi(W) \to \varphi(W)/T^* \) in \( C_\Theta(H_2) \) corresponds to this \( \mu \) [33].
Let us now pass to the correct equivalence. We have

\[ F_1 : K_\Theta(H_1) \to K_\Theta(H_2), \]
\[ F_2 : K_\Theta(H_2) \to K_\Theta(H_1). \]

The pair of functors \((F_1, F_2)\) determines the equivalence of categories. Simultaneously, we have an equivalence

\[ \Phi_1 : C_\Theta(H_1) \to C_\Theta(H_2), \]
\[ \Phi_2 : C_\Theta(H_2) \to C_\Theta(H_1). \]

As we have done above, we assume correspondence of the functors with the categories of affine spaces and, subsequently, with \(\Theta^0\). In particular, the pair \((\Phi_1, \Phi_2)\) induces autoequivalence of the category \(\Theta^0\). The functors \(\varphi : \Theta^0 \to \Theta^0\) and \(\psi : \Theta^0 \to \Theta^0\) are relatively mutually inverse. Here, as before, the functors \(\Phi_1\) and \(\Phi_2\) are coordinated with natural homomorphisms.

5.2. Isomorphism and equivalence of categories. The following two theorems are of universal character; they relate to arbitrary varieties \(\Theta\). Their usage assumes knowledge of the structure of automorphisms and autoequivalences of categories \(\Theta^0\) in various special situations.

**Theorem 7.** The categories \(K_\Theta(H_1)\) and \(K_\Theta(H_2)\) are isomorphic if and only if the algebras \(H_1\) and \(H_2\) are geometrically similar.

**Proof.** See [31]. \(\square\)

**Theorem 8.** The categories \(K_\Theta(H_1)\) and \(K_\Theta(H_2)\) are equivalent if and only if the algebras \(H_1\) and \(H_2\) are geometrically coordinated.

**Proof.** Let, first, \(K_\Theta(H_1)\) and \(K_\Theta(H_2)\) be (correctly) equivalent. We use the equivalence

\[ \Phi = \Phi_1 : C_\Theta(H_1) \to C_\Theta(H_2), \]
\[ \Psi = \Phi_2 : C_\Theta(H_2) \to C_\Theta(H_1). \]

This pair induces the autoequivalence

\[ \varphi : \Theta^0 \to \Theta^0, \quad \psi : \Theta^0 \to \Theta^0, \quad \varphi \psi \approx 1_{\Theta^0} \approx \psi \varphi. \]
Check that the pair \((\varphi, \psi)\) determines coordination of the algebras \(H_1\) and \(H_2\). Take functions \(\alpha(\varphi)\) and \(\alpha(\psi)\). Let \(T\) be an \(H_1\)-closed congruence in \(W\), \(W \in \text{Ob } \Theta^0\). Consider a natural homomorphism \(\mu : W \to W/T\). It is a morphism in \(C_\Theta(H_1)\) with a corresponding natural homomorphism \(\Phi(\mu) : \varphi(W) \to \varphi(W)/T^*\). The congruence \(T^*\) is \(H_2\)-closed and uniquely defined.

Setting \(\alpha(\varphi)_W(T) = T^*\), we have \(\alpha(\varphi)_W : \text{Cl}_{H_1}(W) \to \text{Cl}_{H_2}(\varphi(W))\). Similarly, \(\alpha(\psi)_W : \text{Cl}_{H_2}(W) \to \text{Cl}_{H_1}(\psi(W))\).

Now we need to check commutativity with the function \(\beta\). Take \(W_1, W_2 \in \text{Ob } \Theta^0\). Let \(T\) be an \(H_1\)-closed congruence in \(W_2\). Consider a natural homomorphism \(\mu_T : W_2 \to W_2/T\). For \(s_1, s_2 : W_1 \to W_2\) the relation \(s_1 \beta_{W_1, W_2}(T)s_2\) holds if and only if the equality \(\mu_Ts_1 = \mu Ts_2\) takes place. Rewrite in these terms the corresponding commutativity condition. Given \(\mu Ts_1 = \mu Ts_2\), apply \(\Phi\) to the equality above and get \(\Phi(\mu_T)\varphi(s_1) = \Phi(\mu_T)\varphi(s_2)\). Denote \(\Phi(\mu_T) = \mu_{T^*} : \varphi(W_2) \to \varphi(W_2)/T^*\). This is a natural homomorphism with \(T^* = \alpha(\varphi)_{W_2}(T)\).

We have \(\mu_{T^*}\varphi(s_1) = \mu Ts_2\). This is equivalent to

\[
\varphi(s_1)\beta_{\varphi(W_1), \varphi(W_2)}(T^*)\varphi(s_2), \tag{** ** }
\]

and \(s_1 \beta_{W_1, W_2}(T)s_2\) implies

\[
\varphi(s_1)\beta_{\varphi(W_1), \varphi(W_2)}(\alpha(\varphi)_{W_2}(T))\varphi(s_2).
\]

Let now \(\mu_{T^*} s'_1 = \mu_{T^*} s'_2\) hold for \(s'_1, s'_2 : \varphi(W_1) \to \varphi(W_2)\). Using univalency of the functor \(\varphi\), find \(s_1, s_2 : W_1 \to W_2\) with \(\varphi(s_1) = s'_1, \varphi(s_2) = s'_2\). Then

\[
\mu_{T^*}\varphi(s_1) = \mu_{T^*}\varphi(s_2); \\
\Phi(\mu_T)\varphi(s_1) = \Phi(\mu_T)\varphi(s_2); \\
\Phi(\mu Ts_1) = \Phi(\mu Ts_2).
\]

Using univalency of the functor \(\Phi\), we conclude: \(\mu Ts_1 = \mu Ts_2\). Hence, the condition \(s'_1 \beta_{\varphi(W_1), \varphi(W_2)}(T^*)s'_2\) holds if and only if \(s'_1 = \varphi(s_1), s'_2 = \varphi(s_2)\) and \(s_1 \beta_{W_1, W_2}(T)s_2\). This means exactly that

\[
\varphi(\beta_{W_1, W_2}(T)) = \beta_{\varphi(W_1), \varphi(W_2)}(T^*) = \beta_{\varphi(W_1), \varphi(W_2)}(\alpha(\varphi)_{W_2}(T)).
\]

The commutativity condition for \(\alpha\) and \(\beta\) is checked.

The proof for the functor \(\psi : \Theta^0 \to \Theta^0\) is similar.
We have proved that the algebras $H_1$ and $H_2$ are geometrically coordinated. Prove the opposite.

Let the algebras $H_1$ and $H_2$ be geometrically coordinated. Prove that the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly equivalent. It is sufficient to prove this for the categories $C_\Theta(H_1)$ and $C_\Theta(H_2)$.

Proceed from the autoequivalence $\varphi : \Theta^0 \to \Theta^0$, $\psi : \Theta^0 \to \Theta^0$ and the corresponding functions $\alpha(\varphi)$ and $\alpha(\psi)$.

For every $W \in \Theta^0$ we have the mappings
\[
\alpha(\varphi)_W : \text{Cl}_{H_1}(W) \to \text{Cl}_{H_2}(\varphi(W)),
\]
\[
\alpha(\psi)_W : \text{Cl}_{H_2}(W) \to \text{Cl}_{H_1}(\psi(W)).
\]
Define the functors $\Phi : C_\Theta(H_1) \to C_\Theta(H_2)$ and $\Psi : C_\Theta(H_2) \to C_\Theta(H_1)$ with the conditions $\Psi \Phi \approx 1_{C_\Theta(H_1)}$ and $\Phi \Psi \approx 1_{C_\Theta(H_2)}$, determining correct equivalence of the categories $C_\Theta(H_1)$ and $C_\Theta(H_2)$.

Start with the definition of $\Phi$. Take an arbitrary object $W/T$, in $C_\Theta(H_1)$, $T \in \text{Cl}_{H_1}(W)$. Take $T^* = \alpha(\varphi)_W(T)$. It is an $H_2$-closed congruence in $\varphi(W)$. We set: $\Phi(W/T) = \varphi(W)/T^*$. It is an object in the category $C_\Theta(H_2)$. Define further $\Phi$ on the morphisms. Let the morphism $\sigma : W_1/T \to W_2/T_2$ be given in $C_\Theta(H_1)$. This $\sigma$ determines a commutative diagram
\[
\begin{array}{c}
W_1 \xrightarrow{\sigma} W_2 \\
\downarrow \mu_{T_1} \quad \downarrow \mu_{T_2} \\
W_1/T_1 \xrightarrow{\sigma} W_2/T_2
\end{array}
\]
Here $s$ is not determined uniquely by $\sigma$, but it induces $\sigma$, $\bar{s} = \sigma$. $\mu_{T_1}$ and $\mu_{T_2}$ are natural homomorphisms.

Let us consider the diagram
\[
\begin{array}{c}
\varphi(W_1) \xrightarrow{\varphi(s)} \varphi(W_2) \\
\downarrow \mu_{T_1}^* \quad \downarrow \mu_{T_2}^* \\
\varphi(W_1)/T_1^* \xrightarrow{\Phi(\sigma)} \varphi(W_2)/T_2^*
\end{array}
\]
where $\mu_{T_1}^* = \Phi(\mu_{T_1})$, $\mu_{T_2}^* = \Phi(\mu_{T_2})$. We want to define $\Phi(\sigma)$ to make the diagram be commutative.

We want to check that $\varphi(s)$ induces a morphism $\varphi(W_1)/T_1^* \to \varphi(W_2)/T_2^*$. Check first that if $w_1, w_2 \in \varphi(W_1)$ and $w_1 T_1^* w_2$, then $\varphi(s)(w_1) T_2^* \varphi(s)(w_2)$. Take
\[ \rho^* = \rho_{\varphi(W_1)}T_1^* \]. Let \( \mu\rho^*\mu', \mu, \mu' \in \text{End}(\varphi(W_1)) \). Take further \( \nu, \nu' \in \text{End}(W_1) \) with \( \varphi(\nu) = \mu, \varphi(\nu') = \mu' \). As before, we have \( \nu\rho\nu' \), where \( \rho = \rho_{W_1}(T_1) \). For every \( w \in W_1 \), we have \( w^\nu T_1 w^{\nu'} \). This means also that \( \mu T_1 \nu = \mu T_1 \nu' \). Applying the initial diagram, we get \( w^{\nu\rho} T_2 w^{\nu'\rho} \). We use now that \( \alpha \) and \( \beta \) commute. The definition of geometrical coordination of algebras (commutativity of \( \alpha \) and \( \beta \)) implies \( w^{\varphi(s_1)} T_2^* w^{\varphi(s_2)} \) for every \( w \in \varphi(W_2) \). We have:

\[ w^{\mu\varphi(s)} T_2^* w^{\mu'\varphi(s)} \]

\[ \varphi(s)(w^\mu) T_2^* \varphi(s)(w^{\mu'}) \].

We can find \( w \) such that \( w^\mu = w_1, w^{\mu'} = w_2 \), which leads to \( \varphi(s)(w_1) T_2^* \varphi(s)(w_2) \). Hence, \( \varphi(s) \) induces a homomorphism \( \overline{\varphi(s)} : \varphi(W_1)/T_1^* \to \varphi(W_2)/T_2^* \). We set \( \Phi(\sigma) = \overline{\varphi(\sigma)} \).

We need also to check that this definition of \( \Phi(\sigma) \) does not depend on the choice of \( s \) with \( \overline{s} = \sigma \). Take \( \mu T_2 s_1 = \sigma T_1 = \mu T_2 s_2 \). For every \( w \in W_1 \), we have \( \mu T_2 s_1(w) = \mu T_2 s_2(w) \) and \( w^{s_1} T_2 w^{s_2} \). For every \( w \in \varphi(W_1) \) we have \( w^{\varphi(s_1)} T_2^* w^{\varphi(s_2)} \). This follows from the commutativity with the function \( \beta \). Simultaneously, \( \mu T_2 \varphi(s_1) = \mu T_2 \varphi(s_2) \). Take, further, an arbitrary

\[ \overline{w} \in \varphi(W_1)/T_1^*, \overline{w} = w^{\mu T_2}, \ w \in \varphi(W_1). \]

Then

\[ \overline{w^{\varphi(s_1)}} = \mu T_2 w^{\varphi(s_1)} = \mu T_2 w^{\varphi(s_2)} = \overline{w^{\varphi(s_2)}}. \]

Hence, \( \overline{\varphi(s_1)} = \overline{\varphi(s_2)} = \Phi(\sigma) \). We have defined \( \Phi(\sigma) : \Phi(W_1/T_1) \to \Phi(W_2/T_2) \) for an arbitrary \( \sigma : W_1/T_1 \to W_2/T_2 \).

Check that \( \Phi \) carries the multiplication of morphisms.

Let a commutative diagram

\[
\begin{array}{ccc}
W_1 & \xrightarrow{\sigma_1} & W_2 & \xrightarrow{\sigma_2} & W_3 \\
W_1/T_1 & \xrightarrow{\sigma_1=\overline{s_1}} & W_2/T_2 & \xrightarrow{\sigma_2=\overline{s_2}} & W_3/T_3 \\
\end{array}
\]

\[\boxed{}\]
be given in \( C_{\Theta}(H_1) \). Apply \( \Phi \):

\[
\varphi(W_1) \xrightarrow{\varphi(s_1)} \varphi(W_2) \xrightarrow{\varphi(s_2)} \varphi(W_3)
\]

\[
\mu_1^s=\mu_1^* \quad \Phi(\varphi) \quad \mu_2^* \quad \Phi(\varphi) \quad \mu_3^* = \mu_3^s
\]

\[
\varphi(W_1)/T_1 \xrightarrow{\Phi(\pi)} \varphi(W_2)/T_2 \xrightarrow{\Phi(s_2)} \varphi(W_3)/T_3
\]

From the first diagram we have \( s_{1s_2} \equiv s_{1s_2} \); from the second one, we have

\[
\overline{\varphi(s_1)\varphi(s_2)} = \Phi(s_1s_2) = \varphi(s_1s_2) = \Phi(s_1) = \Phi(s_2).
\]

It is also clear that \( \Phi(1) = 1 \). Thus, the functor \( \Phi \) is built and it induces \( \varphi : \Theta^0 \to \Theta^0 \). Similarly, we build \( \Psi : C_{\Theta}(H_2) \to C_{\Theta}(H_1) \) by \( \psi \), which also induces \( \psi : \Theta^0 \to \Theta^0 \). It is left to check that \( \Phi \) and \( \Psi \) give equivalence of categories.

We need to check that the product \( \Psi \Phi = \Phi_0 \) is an inner autoequivalence of the category \( C_{\Theta}(H_1) \), and \( \Phi \Psi = \Psi_0 \) is an inner autoequivalence of the category \( C_{\Theta}(H_2) \).

First fix \( \psi \varphi = \varphi_0 : \Theta^0 \to \Theta^0 \) and \( \varphi \psi = \psi_0 : \Theta^0 \to \Theta^0 \). These are inner autoequivalences. Let \( \varphi_0 \) relate to the isomorphism \( s : 1_{\Theta^0} \to \varphi_0 \) and \( \psi_0 \) is defined by the isomorphism \( s' : 1_{\Theta^0} \to \psi_0 \). Extend these \( s \) and \( s' \) up to \( S : 1_{C_{\omega}(H_1)} \to \Phi_0 \) and \( S' : 1_{C_{\omega}(H_2)} \to \Psi_0 \). Since the autoequivalences \( \varphi_0 \) and \( \psi_0 \) are inner, then for every \( W \in \text{Ob } \Theta^0 \) and the congruence \( T \) in \( W \) we have \( \alpha(\varphi_0)_W(T) = s_W T \), \( \alpha(\psi_0)_W(T) = s'_W T \). The isomorphism \( s_W : W \to \varphi_0(W) \)

now induces the isomorphism

\[
s_W : W/T \to \Phi_0(W/T) = \varphi_0(W)/T^*;
\]

where \( T^* = \alpha(\varphi_0)_W(T) \). We have \( \varphi_0 = \psi \varphi \). Further, use \( \alpha(\psi \varphi)_W(T) = \alpha(\psi) \varphi(W) \alpha(\varphi)_W(T) \).

By definition,

\[
\Phi(W/T) = \varphi(W)/\alpha(\varphi)_W(T);
\]

\[
\Phi_0(W/T) = \Psi \Phi(W/T) = \Psi(\varphi(W)/\alpha(\varphi)_W(T)) = \psi \varphi(W)/\alpha(\psi) \varphi(W) \alpha(\varphi)_W(T)
\]

\[
= \psi \varphi(W)/\alpha(\psi \varphi)_W(T) = \varphi_0(W)/\alpha(\varphi_0)_W(T).
\]

Thus, \( \Psi \Phi(W/T) = \psi \varphi(W)/\alpha(\psi \varphi)_W(T) \) and, simultaneously, we have an isomorphism \( \overline{s_W} : W/T \to \Psi \Phi(W/T) \). Here \( W/T \) is an arbitrary object of the category \( C_{\Theta_1(H_1)} \).
Define now the function $S$ by the rule $S_W := \overline{w_W}$, $W = W/T$. Check that this defines the isomorphism of functors $\overline{S} : 1_{C_\Theta(H_1)} \to \Psi\Phi = \Phi_0$.

Let the morphism $\sigma : W_1/T_1 \to W_2/T_2$ be given in $C_\Theta(H_1)$, with the corresponding commutative diagram

\[
\begin{array}{ccc}
W_1 & \xrightarrow{\nu} & W_2 \\
\downarrow{\mu_{T_1}} & & \downarrow{\mu_{T_2}} \\
W_1/T_1 & \xrightarrow{\sigma = \overline{\nu}} & W_2/T_2
\end{array}
\]

We need to check that $W_1/T_1 \xrightarrow{s_{W_1}} \Psi\Phi(W_1/T_1) \xrightarrow{\psi\varphi(\nu)} \Psi\Phi(W_2) \xrightarrow{\sigma = \overline{\nu}} \Psi\Phi(W_2/T_2)$ holds. Here $W_1 = W_1/T_1$, $W_2 = W_2/T_2$.

Apply the functor $\Psi\Phi$ to the previous diagram.

\[
\begin{array}{ccc}
\Psi\Phi(W_1) & \xrightarrow{\psi\varphi(\nu)} & \Psi\Phi(W_2) \\
\downarrow{\Psi\Phi(\mu_{T_1})} & & \downarrow{\Psi\Phi(\mu_{T_2})} \\
\Psi\Phi(W_1) & \xrightarrow{\Psi\Phi(\sigma)} & \Psi\Phi(W_2)
\end{array}
\]

Here $\Psi\Phi(\sigma) = \overline{\psi\varphi(\nu)}$. It does not depend on the choice of the representative $\nu$. For $\nu : W_1 \to W_2$ we have

\[
\begin{array}{ccc}
W_1 & \xrightarrow{s_{W_1}} & \psi\varphi(W_1) \\
\downarrow{\nu} & & \downarrow{\psi\varphi(\nu)} \\
W_2 & \xrightarrow{s_{W_2}} & \psi\varphi(W_2)
\end{array}
\]

For $T_1$ from $W_1$ and $T_2$ from $W_2$ there hold the following rules:

\[
\overline{\nu} = \sigma : W_1/T_1 \to W_2/T_2,
\]

\[
\overline{s_{W_1}} : W_1/T_1 \to \Psi\Phi(W_1/T_1),
\]

\[
\overline{s_{W_2}} : W_2/T_2 \to \Psi\Phi(W_2/T_2),
\]

\[
\Psi\Phi(\overline{\nu}) = \Psi\Phi(\sigma) = \overline{\psi\varphi(\nu)}.
\]

Now we check commutativity of the diagram

\[
\begin{array}{ccc}
W_1/T_1 & \xrightarrow{\overline{s_{W_1}}} & \psi\varphi(W_1)/\alpha(\psi\varphi)_{W_1}(T_1) \\
\downarrow{\overline{\nu}} & & \downarrow{\overline{\psi\varphi(\nu)}} \\
W_2/T_2 & \xrightarrow{\overline{s_{W_2}}} & \psi\varphi(W_2)/\alpha(\psi\varphi)_{W_2}(T_2)
\end{array}
\]
Rewrite the diagram in the following way:

$$
\begin{array}{ccc}
W_1 & \xrightarrow{s_{W_1}} & \psi\varphi(W_1)/\psi\varphi(W_1)/s_{W_1}T_1 \\
\downarrow & & \downarrow \psi\varphi(\nu) \\
W_2 & \xrightarrow{s_{W_2}} & \psi\varphi(W_2)/\psi\varphi(W_2)/s_{W_2}T_2
\end{array}
$$

But this diagram directly follows from the diagram for $\nu : W_1 \to W_2$. We take $w_1 \in W_1$ and act according to the rules above. Thus, we have checked coordination of the function $S : 1_{C_{\Theta}(H_1)} \to \Psi\Phi$ with the morphisms of the category and $S : 1_{C_{\Theta}(H_1)} \to \Psi\Phi$ is an isomorphism of functors. We repeat the same for $\varphi\psi$ and $s' : 1_{\Theta^0} \to \varphi\psi$, thus coming to the isomorphism $S' : 1_{C_{\Theta}(H_2)} \to \Phi\Psi$. Correctness of the equivalence of the categories $C_{\Theta}(H_1)$ and $C_{\Theta}(H_2)$ follows from the fact that $\Phi$ induces $\varphi$, $\Psi$ induces $\psi$ and, by definition, $\Phi$ and $\Psi$ are coordinated with the natural homomorphisms. The theorem is proved.

We will apply this theorem for the cases of the varieties Com-$P$, Ass-$P$, and Lie-$P$.

6. Automorphisms and Autoequivalences of Categories of Free Algebras of Varieties

6.1. The general problem and relation to main problems. We are interested in automorphisms and autoequivalences of categories of the form $\Theta^0$, where $\Theta^0$ is a variety of algebras. The form of such automorphisms and autoequivalences determines the peculiarities of the similarity and coordination relations. We have already seen that if all automorphisms of the category $\Theta^0$ are inner, then all autoequivalences are inner as well, and, thus for such $\Theta$, the following conditions are equivalent:

1. The algebras $H_1$ and $H_2$ in $\Theta$ are geometrically similar.
2. They are geometrically equivalent.
3. They are geometrically coordinated.

Define further semi-inner automorphisms. We consider them in general situation.
Let $\Theta$ be an arbitrary variety of algebras and $G$ an algebra in $\Theta$. Consider a new variety, denoted by $\Theta^G$. Define first the category $\Theta^G$. Its objects have the form $h : G \to H$, where $H$ is an algebra in $\Theta$ and $h$ is a morphism in $\Theta$. We call such objects $G$-algebras in $\Theta$, and denote them by $(H, h)$. The morphisms in $\Theta^G$ are represented by commutative diagrams in $\Theta$:

\[
\begin{array}{ccc}
G & \xrightarrow{h} & H \\
\downarrow{h'} & & \downarrow{\mu} \\
H' & & 
\end{array}
\]

An algebra $(H, h)$ is called faithful if $h$ is an injection. We consider elements of the algebra $G$ as nullary operations and add them to the signature of the variety $\Theta$, thus gaining the variety $\Theta^G$. For every set $X$ a free algebra $W = W(X)$ in $\Theta^G$ is represented as free product

\[
i_G : G \to G * W_0(X) = W(X),
\]

where $W_0 = W_0(X)$ is a free algebra in $\Theta$ over $X$, $i_G$ is an embedding related to free multiplication. Here $i_G$ turns out to be an injection.

In the category $\Theta^G$, along with its morphisms, consider also semimorphisms. They are represented by diagram

\[
\begin{array}{ccc}
G & \xrightarrow{h} & H \\
\downarrow{\sigma} & & \downarrow{\nu} \\
G & \xrightarrow{h'} & H'
\end{array}
\]

where $\sigma$ is an endomorphism of the algebra $G$. We consider a semimorphism as a pair $(\sigma, \nu)$, while a morphism is a pair $(1, \nu)$. We consider semi-isomorphisms and semi-automorphisms for the objects from $\Theta^G$.

Let us pass to the category $(\Theta^G)^0$ of all free algebras $W = W(X)$ in $\Theta^G$ with finite $X$.

Define semi-inner automorphisms of this category.

**Definition 7.** The automorphism $\varphi : (\Theta^G)^0 \to (\Theta^G)^0$ is called semi-inner if there exists a semi-isomorphism of functors $(\sigma, s) : 1_{(\Theta^G)^0} \to \varphi$ with the automorphism $\sigma$ of the algebra $G$. 
This means that for every object $W$ of the category $(\Theta^G)^0$ a semi-isomorphism $(\sigma, s_W) : W \rightarrow \varphi(W)$ is fixed, and for every morphism $\nu : W_1 \rightarrow W_2$ we have

$$W_1 \xrightarrow{(\sigma, s_{W_1})} \varphi(W_1) \quad \nu \downarrow \quad \varphi(\nu) \quad W_2 \xrightarrow{(\sigma, s_{W_2})} \varphi(W_2)$$

Here $\varphi(\nu) = (\sigma, s_{W_2})(1, \nu)(\sigma^{-1}, s^{-1}_{W_1}) = (1, s_{W_2}\nu s^{-1}_{W_1})$ is a morphism of the category $(\Theta^G)^0$.

All semi-inner automorphisms of the category $(\Theta^G)^0$ constitute a subgroup in $\text{Aut}(\Theta^G)^0$, containing the invariant subgroup $\text{Int}(\Theta^G)^0$.

Varieties of algebras $\text{Ass-P}$ and $\text{Com-P}$ are varieties of $\Theta^G$ type. Here $\Theta$ is the variety of associative rings, with the unit in the first case, and $\Theta$ is the variety of commutative and associative rings with the unit in the second case, where $G = P$ is a field. The first case assumes that embeddings $h : P \rightarrow H$ are embeddings into the center of the ring $H$.

Consider the corresponding semimorphisms $(\sigma, s) : H \rightarrow H'$. Here $s : H \rightarrow H'$ is a homomorphism of rings and $s(\lambda a) = \lambda^s(a)$, $\lambda \in P$, $a \in H$.

We consider semimorphisms also in the category of modules $\text{Mod-K}$ and the category of Lie algebras over a field. These varieties are not varieties of the $\Theta^G$ type. However, semi-inner automorphisms are naturally defined for the categories $(\text{Mod-K})^0$ and $(\text{Lie-P})^0$.

Let us quote results from [23].

1. If $\Theta = \text{Grp}$ is a variety of all groups, then all automorphisms of the category $\Theta^0$ are inner.
2. If $\Theta$ is a variety of all semigroups, then the group $\text{Aut}(\Theta^0)$ is a direct product of the group $\text{Int}(\Theta^0)$ and a cyclic group of order two.
3. All automorphisms of the category $(\text{Com-P})^0$ are semi-inner.
4. If the ring $K$ is left-noetherian, then all automorphisms of the category $(\text{Mod-K})^0$ are semi-inner.
5. If $F$ is a free group of finite rank, then all automorphisms of the category $\text{Grp}^F$ are semi-inner.
This list of results can be accomplished by the result on Lie algebras (see Theorem 10).

Correspondingly, autoequivalences of categories are described in all these cases.

Recall that every autoequivalence \( \varphi \) of the category \( \Theta^0 \) has the form \( \varphi = \varphi_0 \zeta = \zeta \psi_0 \), where \( \zeta \) is an automorphism and \( \varphi_0 \) and \( \psi_0 \) are inner. See also [10], [11].

6.2. Semi-inner automorphisms and autoequivalences. The definitions are already given above; now we consider some details. We return to the situation \( \Theta^G \).

To every automorphism \( \sigma \) of the algebra \( G \), we construct the corresponding semi-inner automorphism \( \hat{\sigma} \) of the category \( (\Theta^G)^0 \). For every \( W = W(X) = G \ast W_0(X) \) we have two embeddings

\[
i_G \sigma : G \to G \ast W_0, \\
i_W : W_0 \to G \ast W_0,
\]

This gives the corresponding endomorphism in \( \Theta \)

\[
\sigma_W : G \ast W_0 \to G \ast W_0.
\]

We have also an inverse endomorphism \( \sigma_W^{-1} \) and, hence, \( \sigma_W \) is an automorphism in \( \Theta \).

It is easy to understand that the commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{i_G} & W \\
\sigma \downarrow & & \downarrow \sigma_W \\
G & \xrightarrow{i_G} & W
\end{array}
\]

takes place, and, thus, the pair \( (\sigma, \sigma_W) \) defines the semi-automorphism of the algebra \( W \).

Let, further, \( (\sigma, s) \) be an arbitrary pair, such that for every \( W \in \text{Ob}(\Theta^G)^0 \) a semi-automorphism \( (\sigma, s_W) : W \to W \) be fixed. The pair \( (\sigma, s) \) defines a semi-inner automorphism \( (\hat{\sigma}, \hat{s}) \) of the category \( (\Theta^G)^0 \). It does not change objects and for every \( \nu : W_1 \to W_2 \) we have \( (\hat{\sigma}, \hat{s})(\nu) = s_{W_2} \nu s_{W_1}^{-1} \). In particular,
if always $s_W = \sigma_W$, then a semi-inner automorphism of the category $(\Theta^G)^0$, denoted by $\hat{\sigma}$, corresponds to the automorphism $\sigma \in \text{Aut}(G)$.

Consideration of the pair $(1, s)$ leads to the inner automorphism $\hat{s}$ of the category $(\Theta^G)^0$.

Let us now treat semi-inner autoequivalences. They are defined exactly in the same way as semi-inner automorphisms. An autoequivalence $\varphi : (\Theta^G)^0 \to (\Theta^G)^0$ is semi-inner if it is given by a semi-isomorphism of functors $(\sigma, s) : 1_{(\Theta^G)^0} \to \varphi$.

Show that every such $\varphi$ can be represented as $\varphi = \hat{\sigma}\varphi_1 = \varphi_0\hat{\sigma}$, where $\varphi_0$ and $\varphi_1$ are inner autoequivalences.

Let a semi-inner autoequivalence $\varphi$ be given by $(\sigma, s) : 1_{(\Theta^G)^0} \to \varphi$. For every $W \in \text{Ob}(\Theta^G)^0$ we have a semi-isomorphism $(\sigma, s_W) : W \to \varphi(W)$. Consider also $(\sigma, \sigma_W) : W \to W$ and $(\sigma^{-1}, \sigma_W^{-1}) : W \to W$.

Take a product

$$(\sigma, s_W)(\sigma^{-1}, \sigma_W^{-1}) = (1, s_W\sigma_W^{-1}) : W \to \varphi(W).$$

Denote $s'_W = s_W\sigma_W^{-1}$. We have an isomorphism $s'_W : W \to \varphi(W)$.

Consider a function $s'$ defined by the rule $s'_W = s_W\sigma_W^{-1}$. The function $s'$ determines the inner autoequivalence $\varphi_0 : (\Theta^G)^0 \to (\Theta^G)^0$ acting on the objects as $\varphi$ does: $\varphi_0(W) = \varphi(W)$. We have $(\sigma, s_W) = (1, s'_W)(\sigma, \sigma_W)$ and, correspondingly, $\varphi = \varphi_0\hat{\sigma}$. Similarly, we define the decomposition $\varphi = \hat{\sigma}\varphi_1$. The same considerations are applicable to automorphisms.

6.3. Application. The following proposition was proved in [7].

**Proposition 13.** If the algebras $H_1$ and $H_2$ are geometrically similar and their similarity is defined by the semi-inner automorphism, then there exists an algebra $H$ which is semi-isomorphic to the algebra $H_1$ and geometrically equivalent to the algebra $H_2$.

The existence of such $H$ means that $H_1$ and $H_2$ are similar.

We want to prove also a similar proposition for the relation of geometrical coordination of algebras, but first let us make an auxiliary remark.
Given a $G$-algebra $(H, h)$ and $\sigma \in \text{Aut}(G)$, build a new $G$-algebra $(H_1, h_1)$, $H_1 = H$, keeping in mind the commutative diagram

\[
\begin{array}{c}
G \xrightarrow{h_1} H_1 \\
\downarrow \sigma \downarrow \quad \downarrow \mu = 1 \\
G \xrightarrow{h} H
\end{array}
\]

Here $h_1 = h\sigma$ and the algebras $(H, h)$ and $(H_1, h_1)$ are semi-isomorphic.

**Proposition 14.** (See [7]). The algebras $(H, h)$ and $(H_1, h_1)$ are geometrically similar, and their similarity is defined by the automorphism $\hat{\sigma} : (\Theta^G)^0 \to (\Theta^G)^0$.

Recall that we consider the situation $\text{Var}(H) = \Theta^G$. In this case, it is easy to check that $\text{Var}(H_1) = \Theta^G$ holds. Besides, the algebras $(H, h)$ and $(H_1, h_1)$ are faithful $G$-algebras.

**Proposition 15.** Let $\text{Var}(H_1) = \text{Var}(H_2) = \Theta^G$ and let the $G$-algebras $H_1$ and $H_2$ be geometrically coordinated by the semi-inner autoequivalence $(\varphi, \psi)$. Then there exists a $G$-algebra $H$, semi-isomorphic to $H_1$ and geometrically equivalent to $H_2$. In particular, $H_1$ and $H_2$ are geometrically similar.

**Proof.** We use Proposition 12. Let $\varphi$ and $\psi$ be related to the automorphism $\sigma$ of the algebra $G$. This means that we can proceed from the decomposition $\varphi = \varphi_0 \hat{\sigma}$, $\psi = \hat{\sigma}^{-1} \psi_0$, where $(\varphi_0, \psi_0)$ is an inner autoequivalence of the category $\Theta^0$ and the automorphism $\hat{\sigma}$ of the category $(\Theta^G)^0$ corresponds to $\sigma$. By the given $\sigma \in \text{Aut}(G)$ take the $G$-algebra $H$, semi-isomorphic to the algebra $H_1$, such that $\hat{\sigma}$ defines similarity of the algebras $H_1$ and $H$ (see Proposition 14). According to Proposition 12, the pair $(\varphi_0, \psi_0)$ defines coordination of the algebras $H$ and $H_2$. Since the pair $(\varphi_0, \psi_0)$ is inner, the algebras $H$ and $H_2$ are geometrically equivalent.

The proposition is proved. □

7. VARIETIES $\text{Com}-P$, $\text{Ass}-P$ AND $\text{Lie}-P$

7.1. $\Theta = \text{Com}-P$. Assume the field $P$ is infinite. The variety $\text{Com}-P$ is noetherian and is generated by each of its algebras. Two algebras $H_1$ and $H_2$
are geometrically equivalent, if they have the same quasi-identities. Besides, semimorhisms are naturally defined in Com-$P$. It is proved in [7] that every automorphism of the category $(\text{Com-}P)^0$ is semi-inner. Then every autoequivalence of this category is semi-inner as well. Now, taking into account the previous considerations, we can formulate

**Theorem 9.** Let $H_1$ and $H_2$ be two algebras in $\Theta = \text{Com-}P$. Then the following conditions are equivalent.

1. The categories $K_{\Theta}(H_1)$ and $K_{\Theta}(H_2)$ are correctly isomorphic.
2. These categories are correctly equivalent.
3. There exists an algebra $H \in \Theta$ such that $H_1$ and $H$ are semi-isomorphic, and $H$ and $H_2$ have the same quasi-identities.

### 7.2. $\Theta = \text{Ass-}P$.

First of all, we are interested in automorphisms of the category $\Theta^0 = (\text{Ass-}P)^0$. In every category we can consider inner automorphisms. This category $\Theta^0$ has also semi-inner automorphisms. They are defined according to the general approach from Section 5. Let us do it directly.

If $H_1$ and $H_2$ are two associative algebras over $P$, then their semimorphism $H_1 \rightarrow H_2$ is given by the pair $(\sigma, \nu)$, where $\sigma \in \text{Aut } P$, and $\nu : H_1 \rightarrow H_2$ is a homomorphism of rings. Here, if $\lambda \in P$ and $a \in H$, then $\nu(\lambda a) = \lambda^\sigma \nu(a)$.

In Section 5 for every $G$-algebra $(H, h)$ and every $\sigma \in \text{Aut}(G)$, we considered the $G$-algebra $(H_1, h_1)$ with $H$ and $H_1$ coinciding in $\Theta$, and $h = h_1 \sigma$. The $G$-algebras $H$ and $H_1$ are semi-isomorphic. Now we reproduce this construction in Ass-$P$.

Let $H$ be an associative algebra over the field $P$. The embedding $h : P \rightarrow H$ is defined by the rule $\lambda^h = \lambda \cdot 1$ for every $\lambda \in P$. Then $\lambda \cdot a = \lambda^h \cdot a$ for every $a \in H$.

Take a new algebra, denoted by $H^\sigma$, for the given $\sigma \in \text{Aut}(P)$. We set: $H$ and $H^\sigma$ coincide as rings, and we change multiplication by a scalar

$$\lambda \circ a = \lambda^{\sigma^{-1}} \cdot a = (\lambda^{\sigma^{-1}})^h \cdot a = h\sigma^{-1}(\lambda) \cdot a = \lambda^{h_1} \cdot a.$$
Thus, the algebra $H^\sigma$ is $(H_1, h_1)$ in the sense of general construction. The identical transformation $H \to H^\sigma$ determines semi-isomorphism of the algebras $H^\sigma$ and $H$. Every semi-isomorphism can be decomposed in such a semi-isomorphism and isomorphism.

We call the algebra $H^\sigma$ a $\sigma$-twisted algebra with respect to $H$. It is checked that if $\text{Var}(H) = \Theta$, then $\text{Var}(H^\sigma) = \Theta$ as well.

Consider further free algebras $W = W(X)$ in $\Theta = \text{AssP}$, cf [9]. Denote by $S(X)$ a free monoid over $X$ and $S_0(X)$ a free semigroup over $X$. The algebra $W = W(X)$ is a semigroup algebra $PS(X)$. Every element of $W(X)$ is uniquely represented in the form

$$w = \lambda_0 + \lambda_1 u_1 + \cdots + \lambda_k u_k, \quad \lambda \in P, \ u \in S_0(X).$$

For every $\sigma \in \text{Aut}(P)$ denote by $\sigma_W : W \to W$ a mapping, defined by the rule

$$\sigma_W(w) = \lambda_0^\sigma + \lambda_1^\sigma u_1 + \cdots + \lambda_k^\sigma u_k.$$

Here $\sigma_W$ is an automorphism of rings and the pair $(\sigma, \sigma_W)$ defines semi-automorphism of the algebra $W$.

This definition corresponds to the general definition given above.

Denote by $\overline{\sigma}$ a function, choosing $\overline{\sigma}_W = \sigma_W$ for every $W$. The pair $(\sigma, \overline{\sigma})$ defines a semi-inner automorphism $\hat{\sigma}$ of the category $\Theta^0$. Here $\hat{\sigma}$ does not change objects, and for every $\nu : W_1 \to W_2$ we have

$$\hat{\sigma}(\nu) = \sigma_{W_2} \cdot \nu \cdot \sigma_{W_1}^{-1} : W_1 \to W_2.$$

An arbitrary semi-inner automorphism $\varphi$ of the category $\Theta^0$ is defined by the semi-isomorphism of the functors

$$(\sigma, s) : 1_{\Theta^0} \to \varphi.$$  

Such a $\varphi$ is represented as $\varphi = \varphi_0 \hat{\sigma} = \hat{\sigma} \varphi_1$, where $\varphi_0$ and $\varphi_1$ are inner automorphisms, and $\varphi_0(W) = \varphi(W) = \varphi_1(W)$ for every $W$.

We have also semi-isomorphism

$$(\sigma, s_W) : W \to \varphi(W).$$

For $\nu : W_1 \to W_2$ we have $\varphi(\nu) = s_{W_2} \cdot \nu \cdot s_{W_1}^{-1}$. 
Consider further a mirror automorphism of the category \( (\text{Ass-}P)^0 \). This notion relates to the idea of antimorphism in the category \( \text{Ass-}P \). First consider antihomomorphisms of semigroups.

A mapping of semigroups \( \mu : S_1 \to S_2 \) is called an antihomomorphism if \( \mu(ab) = \mu(b)\mu(a) \), \( a, b \in S_1 \).

Let now \( S = S(X) \) be a free semigroup. For every \( u = x_{i_1} \cdots x_{i_n} \) in \( S \) take \( \overline{u} = x_{i_n} \cdots x_{i_1} \). Then the transition \( u \to \overline{u} \) is an antiautomorphism of the semigroup \( S \). Indeed, let \( u = x_{i_1} \cdots x_{i_n}, \ v = x_{j_1} \cdots x_{j_m} \). Then

\[
\overline{u} \cdot \overline{v} = x_{i_1} \cdots x_{i_n} x_{j_1} \cdots x_{j_m} = x_{j_m} \cdots x_{j_1} x_{i_n} \cdots x_{i_1} = \overline{v} \cdot \overline{u}.
\]

If now \( H_1, H_2 \) are associative algebras over the field \( P \), then the mapping \( \mu : H_1 \to H_2 \) is an antihomomorphism of algebras if \( \mu \) is correlated with addition and multiplication by a scalar, and \( \mu(ab) = \mu(b) \cdot \mu(a) \) for \( a, b \in H_1 \).

For an arbitrary algebra \( H \) take an opposite algebra \( H^* \). The sets \( H \) and \( H^* \) coincide also as vector spaces, but multiplication in \( H^* \) is defined by the rule \( a \circ b = b \cdot a \). An identical mapping \( H \to H^* \) here is an antiisomorphism of algebras.

Let now \( W = W(X) = PS(X) \) be a free associative algebra.

For every its element \( w = \lambda_0 + \lambda_1 u_1 + \cdots + \lambda_k u_k \) take \( \overline{w} = \lambda_0 + \lambda_1 \overline{u}_1 + \cdots + \lambda_k \overline{u}_k \), and show that the transition \( w \to \overline{w} \) is an antiautomorphism of the algebra \( W \).

Given \( w_1 = \alpha_0 + \alpha_1 u + \cdots + \alpha_k u_k \) and \( w_2 = \beta_0 + \beta_1 v_1 + \cdots + \beta_\ell v_\ell \), we have

\[
w_1 w_2 = \sum_{i,j} \alpha_i \beta_j u_i v_j,
\]

\[
\overline{w_1 w_2} = \sum_{i,j} \alpha_i \beta_j u_i \overline{v}_j = \sum_{i,j} \beta_j \alpha_i \overline{v}_j \cdot \overline{u}_i = \overline{w_2} \cdot \overline{w_1}.
\]

Correlation with addition and multiplication by a scalar are also evident.

Now we consider the mirror automorphism of the category \( \Theta^0 = (\text{Ass-}P,)^0 \) denoted by \( \delta \). This \( \delta \) does not change objects. Let the homomorphism \( \nu : W_1 = W(X) \to W(Y) = W_2 \) be given. Define \( \delta(\nu) : W_1 \to W_2 \) by \( \delta(\nu)(x) = \overline{\nu(x)} \) for every \( x \in X \). Further we need additional calculations.
Let $u = x_{i_1} \ldots x_{i_n} \in S_0(X)$. Consider
$$\delta(\nu)(u) = \delta(\nu)(x_{i_1}) \ldots \delta(\nu)(x_{i_n}) = \overline{\nu(x_{i_1})} \ldots \overline{\nu(x_{i_n})} = \overline{\nu(x_{i_1} \ldots \nu(x_{i_n}) = \nu(\overline{\nu}).}
$$

If now $w = \lambda_0 + \lambda_1 u_1 + \ldots + \lambda_k u_k \in W(X)$, then
$$\delta(\nu)(w) = \lambda_0 + \lambda_1 \delta(\nu)(u_1) + \ldots + \lambda_k \delta(\nu)(u_k) = \lambda_0 + \lambda_1 \overline{\nu(u_1)} + \ldots + \lambda_k \overline{\nu(u_k)}$$
$$= \overline{\nu(\lambda_0 + \lambda_1 \overline{u_1} + \ldots + \lambda_k \overline{u_k})} = \nu(\overline{w}).
$$

Hence $\delta(\nu)(w) = \nu(\overline{w})$. Assume now that $\nu = \nu_1 : W_1 \to W_2$ and $\nu_2 = W_2 \to W_3$ are given. Check that $\delta(\nu_2 \nu_1) = \delta(\nu_2) \delta(\nu_1)$. Take an arbitrary $x \in X$. Then
$$\delta(\nu_2) \cdot \delta(\nu_1)(x) = \delta(\nu_2)(\nu_1(x)) = \overline{\nu_2(\nu_1(x))} = \nu_2 \nu_1(x) = \delta(\nu_2 \nu_1)(x).$$

It is also clear that $\delta(1) = 1$, and, thus, $\delta : \Theta^0 \to \Theta^0$ is a functor. Since $\delta^2 = 1_{\Theta^0}$, then $\delta$ is an automorphism.

Here $\delta$ is not inner and is not semi-inner, but is quasi-inner. Besides, if $\widetilde{\text{Int}}(\Theta^0)$ is a subgroup in $\text{Aut}(\Theta^0)$, consisting of semi-inner automorphisms, then $\delta$ belongs to the normalizer of this subsemigroup.

Denote by $\eta$ a function, giving an antiautomorphism $\eta_W$ of the algebra $W$ by $\eta_W(w) = \overline{w}$ for every $W = W(X)$. Show that $\delta(\nu) = \eta_W \cdot \nu \cdot \eta_W^{-1}$ holds for every $\nu : W_1 \to W_2$. Take an arbitrary $x \in X$, $W_1 = W(X)$. Then
$$\eta_W \cdot \nu \cdot \eta_W^{-1}(x) = \eta_W \cdot \nu(x) = \eta_W(\nu(x)) = \overline{\nu(x)} = \delta(\nu)(x),$$
for every $x \in X$. Hence, $\delta(\nu) = \eta_W \cdot \nu \cdot \eta_W^{-1}$. We checked that $\delta$ is quasi-inner in this sense.

**Proposition 16.** The automorphism $\delta$ belongs to the normalizer of the subgroup in $\text{Aut}(\Theta^0)$, consisting of semiinner automorphisms.

**Proof.** Let, first $\varphi$ be an inner automorphism, defined by the isomorphism of functors $s : 1_{\Theta^0} \to \varphi$. We have $\delta^2 = 1_{\Theta^0}$, $\delta^{-1} = \delta$. Consider $\delta \varphi \delta$ and apply it to $\nu : W_1 \to W_2$. Then
$$\delta \varphi \delta(\nu) = \delta(\varphi(\delta(\nu))) = \delta(s_{W_2} \delta(\nu) s_{W_1}^{-1}) =$$
$$= \delta(s_{W_2}) \delta^2(\nu) \delta(s_{W_1})^{-1} = \delta(s_{W_2}) \nu \delta(s_{W_1})^{-1}.$$

Thus, $\delta \varphi \delta$ is an inner automorphism, defined by the isomorphism $\delta(s) : 1_{\Theta^0} \to \delta \varphi \delta^{-1}$, where $\delta(s)$ is a function defined by $\delta(s)_W = \delta(s_W)$. In the case of
semigroups we have \( \delta(s_W) = s_W \), where \( \delta \varphi \delta = \varphi \). In our situation this is not true, and \( \delta \varphi \delta \neq \varphi \). Indeed, if \( s_W(x) = \lambda_0 + \lambda_1 u_1 + \cdots + \lambda_k u_k \), where all \( u_i \) depend on many variables, then \( \delta(s_W)(x) = \overline{s_W(x)} \neq s_W(x) \).

Let further \( \sigma \in \text{Aut}(P) \). Consider the automorphism \( \hat{\sigma} \) of the category \( \Theta^0 \). Show that \( \hat{\sigma} \) and \( \delta \) commute. Proceed once more from \( \nu : W_1 \to W_2 \), and check that \( \hat{\sigma} \delta(\nu) = \delta \hat{\sigma}(\nu) \). Let \( W_1 = W(X), x \in X \). Then

\[
\hat{\sigma}(\delta(\nu)) = \sigma_{W_2} \delta(\nu) \delta^{-1}_{W_1};
\]

\[
\hat{\sigma}(\delta(\nu))(x) = \sigma_{W_2} \delta(\nu) \sigma^{-1}_{W_1}(x) = \sigma_{W_2} \delta(\nu)(x) = \sigma_{W_2}(\nu(x)).
\]

Let \( \nu(x) = \lambda_0 + \lambda_1 u_1 + \cdots + \lambda_k u_k \). Then

\[
\sigma_{W_2} \delta(\nu)(x) = \sigma_{W_2}(\lambda_0 + \lambda_1 \overline{u}_1 + \cdots + \lambda_k \overline{u}_k) = \lambda_0^\sigma + \lambda_1^\sigma \overline{u}_1 + \cdots + \lambda_k^\sigma \overline{u}_k.
\]

Here all \( u_i \) are elements of \( S_0(Y) \), \( W_2 = W(Y) \). Now

\[
\sigma_{W_2} \delta(\nu)(x) = \delta(\sigma_{W_2} \nu s^{-1}_{W_1})(x) = \sigma_{W_2} \nu \sigma^{-1}_{W_1}(x) = \\
\sigma_{W_2}(\lambda_0 + \lambda_1 u_1 + \cdots + \lambda_k u_k) = \lambda_0^\sigma + \lambda_1^\sigma \overline{u}_1 + \cdots + \lambda_k^\sigma \overline{u}_k.
\]

The proposition is proved. \( \square \)

**Corollary.** If \( \varphi \) belongs to a subgroup generated by semi-inner automorphisms and the automorphism \( \delta \) then \( \varphi \) is either a semi-inner automorphism, or \( \varphi = \varphi_0 \delta \), where \( \varphi_0 \) is a semi-inner automorphism.

**Problem 6.** Whether it is true that the group \( \text{Aut}(\text{Ass-P})^0 \) is generated by semi-inner and mirror automorphisms?

**Problem 7.** Let \( F = F(X) \) be a free non-commutative Lie algebra. Whether it is true that every automorphism of the semigroup \( \text{End } F \) is semi-inner?

The similar result for the category of free Lie algebras is proved.

Let us pass to the geometrical problems. For every free algebra \( W \) consider its antiautomorphism \( \eta_W : W \to W, \eta_W(w) = \overline{w} \). It is clear that if \( T \) is an ideal in \( W \), then its image \( \eta_W(T) = T^* \) is also an ideal, and \( w \in T^* \) if \( \overline{w} \in T \). Check that \( \alpha(\delta)_W(T) = T^* \).
Take $\rho = \rho_W(T)$ and $\rho^* = \rho_W(T^*)$. Verify that $\rho^* = \delta(\rho)$. Let $\nu\nu'$ hold. For every $w \in W$ and $w_1 = \overline{w} \in W$ we have $\nu(w) - \nu'(w) \in T$;

$$\delta(\nu)(w) - \delta(\nu')(w) = \nu(\overline{w}) - \nu'(\overline{w}) = \nu(\overline{w}) - \nu'(\overline{w}) \in T^*.$$  

Therefore, $\delta(\nu)\rho^*\delta(\nu')$.

Let now $\mu \rho^* \mu'$. Take $\mu = \delta(\nu)$, $\mu' = \delta(\nu')$. We have $\delta(\nu)(w) - \delta(\nu')(w) = \nu(\overline{w}) - \nu'(\overline{w}) \in T^*$ for every $w \in W$, in which case $\nu(\overline{w}) - \nu'(\overline{w}) \in T$, $\nu \rho \nu'$. The equality $\delta(\rho) = \rho^*$ is verified. Further,

$$\alpha(\delta)(T) = \tau_W(\delta(\rho_W(T))) = \tau_W(\delta(\rho)) = \tau_W(\rho^*) = T^*.$$  

**Proposition 17.** Let the algebras $H_1$ and $H_2$ be antiisomorphic. Then they are geometrically similar, and similarity is defined by the automorphism $\delta : \Theta^0 \to \Theta^0$.

**Proof.** Let $\mu : H_1 \to H_2$ be an antiisomorphism. Consider the commutative diagram

$$\begin{array}{ccc}
W & \xrightarrow{\eta_W} & W \\
\nu \downarrow & & \downarrow \nu' \\
H^I_1 & \xrightarrow{\pi} & H^I_2
\end{array}$$

where $\pi$ is an antiisomorphism defined by the antiisomorphism $\mu$, and $\nu$, $\nu'$ are one-to-one corresponding homomorphisms of algebras. Prove now that if $T$ is an $H_1$-closed ideal, then $T^*$ is $H_2$-closed, and vice versa.

An injection $W/T \to H^I_1$ can be substituted by a homomorphism $\nu : W \to H^I_1$ with the kernel $T$. It is easy to see that $T$ is Ker$\nu$ if and only if $T^*$ is Ker$\nu'$. Hence the embedding $W/T \to H^I_1$ defines the embedding $W/T^* \to H^I_2$, and vice versa.

It is left to check that $\delta$ and the function $\beta$ commute. It is done in the same way as for $\delta(\rho) = \rho^*$.  

We call the algebras $H_1$ ad $H_2$ almost geometrically equivalent if there exists a sequence $H_1, H, H', H_2$ such that $H_1$ and $H$ are antiisomorphic or isomorphic, $H$ and $H'$ are semi-isomorphic, and $H'$ and $H_2$ are geometrically equivalent.
We can now state that if $\text{Var}(H_1) = \text{Var}(H_2) = \text{Ass}-P$ and Problem 6 about automorphisms of the category $(\text{Ass}-P)^0$ is solved positively, then the following conditions are equivalent:

1. Categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly isomorphic.
2. They are correctly equivalent.
3. $H_1$ and $H_2$ are almost geometrically equivalent.

This is the main conjecture. Let us discuss the statement in more detail.

Let the algebras $H_1$ and $H_2$ be coordinated by an autoequivalence $(\varphi, \psi)$ of the category $\Theta^0$. Assume that $\varphi = \varphi_0 \hat{\sigma}, \psi = \delta^{-1} \hat{\sigma}^{-1} \psi_0$, where $(\varphi_0, \psi_0)$ is an inner autoequivalence. For $\delta$ take an algebra $H$, opposite to $H_1$. The algebras $H$ and $H_1$ are similar in respect to $\delta$, and $H$ and $H_2$ are coordinated with respect to $(\varphi_0 \hat{\sigma}, \hat{\sigma}^{-1} \psi_0)$ (Proposition 12). Take an algebra $H'$ by $H$, which is $\sigma$-twisted with respect to $H$. The algebras $H$ and $H'$ are similar with respect to $\hat{\sigma}$, $H'$ and $H_2$ are coordinated with respect to $(\varphi_0, \psi_0)$. Since $(\varphi_0, \psi_0)$ is an inner autoequivalence, $H'$ and $H_2$ are geometrically equivalent.

7.3. Variety of Lie algebras Lie-$P$. The following theorem is proved in [24].

**Theorem 10.** Every automorphism of the category of free Lie algebras is semi-inner. Every autoequivalence of this category is semi-inner as well.

Consider an application of this theorem.

For every Lie algebra $H$ and every automorphism $\sigma$ of the field $P$ consider a Lie algebra $H'^\sigma$, coinciding with $H$ as a ring, while the multiplication by a scalar is defined the new rule:

$$\lambda \circ a = \lambda^{\sigma^{-1}} \cdot a; \quad \lambda a = \lambda^\sigma \circ a.$$  

The identity mapping $H \to H'^\sigma$ is a semi-isomorphism of algebras.

The following theorem takes place:

**Theorem 11.** Let $\text{Var}(H_1) = \text{Var}(H_2) = \text{Lie}-P$. Then the following conditions are equivalent:

1. The categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are isomorphic.
2. These categories are equivalent.

3. The algebra $H_1^\sigma$ is geometrically equivalent to the algebra $H_2$ for some $\sigma \in \text{Aut}(P)$.

Proof. Prove first that for any algebra $H$ the algebras $H$ and $H^\sigma$ are geometrically similar with respect to an automorphism $\hat{\sigma} : \Theta^0 \rightarrow \Theta^0$; $\Theta = \text{Lie-}P$.

Define the automorphism $\hat{\sigma}$. Let $W = W(X)$ be a free Lie algebra over $P$ with finite $X$. Define for it a semi-automorphism $\sigma_W : W \rightarrow W$. Apply $\sigma_W$ to an element $w \in W$. We define the action of $\sigma_W$ inductively. Set: $\sigma_W(x) = x$ for every $x \in X$. If $w = w_1 \cdot w_2$, then $\sigma_W(w) = \sigma_W(w_1) \cdot \sigma_W(w_2)$. Analogously, if $w = w_1 + w_2$, then $\sigma_W(w) = \sigma_W(w_1) + \sigma_W(w_2)$. If, finally, $w = \lambda w_1$, $\lambda \in P$, then $\sigma_W(w) = \lambda^\sigma \cdot \sigma_W(w)$. It can be verified with the help of the suitable basis in $W$ that this definition is correct. The pair $(\sigma, \sigma_W)$ determines a semi-automorphism of the algebra $W$.

Set further: $\hat{\sigma}(W) = W$ for every $W \in \text{Ob} \Theta^0$ and $\hat{\sigma}(\nu) = \sigma_W \nu^{\sigma_W^{-1}}$ for $\nu : W_1 \rightarrow W_2$. This defines the semi-inner automorphism $\hat{\sigma} : \Theta^0 \rightarrow \Theta^0$.

We could not define here $\hat{\sigma}$ via the general approach, applied to varieties of $\Theta^G$ type, since the variety Lie-\(P\) is not of such type.

Let us now link homomorphisms $W \rightarrow H$ and $W \rightarrow H^\sigma$. Take $\mu = \nu^* : W \rightarrow H^\sigma$ corresponds to $\nu : W \rightarrow H$ by the rule $\mu(\sigma_W(w)) = \nu(w)$, $w \in W$. Here $\mu \sigma_W = \nu$, $\mu = \nu \sigma_W^{-1}$, $\mu(w) = \nu(\sigma_W^{-1}w)$. The mapping $\mu$ is coordinated with the operations of the ring.

Check now that $\mu$ is a homomorphism of algebras. Indeed,

$$\mu(\lambda w) = \nu(\sigma_W^{-1}(\lambda w)) = \nu(\lambda^{\sigma_W^{-1}} \sigma_W^{-1}(w)) = \lambda^{\sigma_W^{-1}} \nu(\sigma_W^{-1}(w)) = \lambda^{\sigma_W^{-1}} \mu(w) = \lambda \circ \mu(w).$$

We have also: $w \in \text{Ker} \nu$ if and only if $\sigma_W(w) \in \text{Ker} \mu$. If $A$ is a set of $H$-points, $A \subset \text{Hom}(W, H)$, then a set $A^*$ of $H^\sigma$-points, $A^* \subset \text{Hom}(W, H^\sigma)$, corresponds to $A$.

Let now $T$ be an ideal in $W$. Denote by $\sigma_W T$ an ideal in $W$, consisting of all $\sigma_W(w)$, $w \in T$. It is clear now that

$$\bigcap_{\nu \in A} \text{Ker} \nu = T \Leftrightarrow \bigcap_{\mu \in A^*} \text{Ker} \mu = \sigma_W T.$$
This means that the ideal $T$ is $H$-closed if and only if the ideal $\sigma_W T$ is $H^\sigma$-closed.

Check that the transition $T \to T^* = \sigma_W T$ is coordinated with the function $\beta$ for $\varphi = \hat{\sigma}$.

Take algebras $W_1$ and $W_2$ in $\text{Ob } \Theta^0$. Let $T$ be an ideal in $W_2$. Denote $\beta = \beta_{W_1,W_2}(T)$, $\beta^* = \beta_{W_1,W_2}(T^*)$ for $T^* = \sigma_W(T)$. We need to check that

$$s \beta s' \Leftrightarrow \hat{\sigma}(s) \beta^* \hat{\sigma}(s')$$

for $s, s' : W_1 \to W_2$. We have

$$\hat{\sigma}(s) = \sigma_{W_2} s \sigma_{W_1}^{-1}, \quad \hat{\sigma}(s') = \sigma_{W_2} s' \sigma_{W_1}^{-1}.$$ 

Take an arbitrary $w \in W_1$ and consider a difference

$$\hat{\sigma}(s)(w) - \hat{\sigma}(s')(w) = \sigma_W(s(\sigma_{W_1}^{-1}(w))) - s'(\sigma_{W_1}^{-1}(w)).$$

An arbitrary element in $w_1 \in W_1$ has the form $w_1 = \sigma_{W_1}^{-1}(w)$. Let now $s \beta s'$ take place. Then $s(w_1) - s'(w_1) \in T$. Hence $\hat{\sigma}(s)(w) - \hat{\sigma}(s')(w) \in T^*$, which gives $\hat{\sigma}(s) \beta^* \hat{\sigma}(s')$.

It is also clear that if $\hat{\sigma}(s)(w) - \hat{\sigma}(s')(w) \in T^*$, then $s(w_1) - s'(w_1) \in T$ and, therefore, $s \beta s'$.

Prove now that

$$\alpha(\hat{\sigma})_W(T) = T^* = \sigma_W T.$$ 

It follows from considerations above that $\hat{\sigma}(\rho_W(T)) = \rho_W(T^*)$. Applying $\tau_W$, we get

$$\alpha(\hat{\sigma})_W(T) = \tau_W(\hat{\sigma}(\rho_W(T))) = \tau_W \rho_W(T^*) = T^*.$$ 

We have checked that there is a bijection

$$\alpha(\hat{\sigma})_W : \text{Cl}_H(W) \to \text{Cl}_{H^*}(W)$$

and the function $\alpha$ commutes with $\beta$. This means that the automorphism $\hat{\sigma}$ determines similarity of algebras $H$ and $H^\sigma$.

Let us now finish the proof of the theorem.

The categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are isomorphic if and only if $H_1$ and $H_2$ are similar.
The similarity of $H_1$ and $H_2$ is determined by some automorphism $\varphi : \Theta^0 \rightarrow \Theta^0$. According to Theorem 10, an automorphism $\varphi$ is semi-inner and it can be represented as $\varphi = \hat{\sigma} \varphi_0$ where $\varphi_0$ is an inner automorphism.

Let us pass to the algebra $H_1^{\sigma}$. The algebras $H_1$ and $H_1^{\sigma}$ are similar in respect to $\hat{\sigma}$. According to the similarity decomposition rule we conclude that $H_1^{\sigma}$ and $H_2$ are similar with respect to $\varphi_0$ and, consequently, they are geometrically equivalent. This leads to the equivalence of the first and the third conditions of Theorem 11. Equivalence of the second and the third connections is checked similarly.

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□

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