Observer-Based Robust Control Method for Switched Neutral Systems in the Presence of Interval Time-Varying Delays

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Abstract: In this study, the challenges of the controller design of a class of Uncertain Switched Neutral Systems (USNSs) in the presence of discrete, neutral, and time-varying delays are considered by using a robust observer-based control technique. The cases where the uncertainties are norm-bounded and time-varying are emphasized in this research. The adopted control approach reduces the prescribed level of disturbance input on the controlled output in the closed-loop form and the robust exponential stability of the control system. The challenge of parametric uncertainty in USNSs is solved by designing a robust output observer-based control and applying the Yakubovich lemma. Since the separation principle does not generally hold in this research, the controller and observer cannot be designed separately, sufficient conditions are suggested. These conditions are composed of applying the average dwell time approach and piecewise Lyapunov function technique in terms of linear matrix inequalities, which guarantees robust exponential stability of the observer-based output controller. Finally, two examples are given to determine the effectiveness of the proposed method.

Keywords: neutral systems; switched system; uncertainty; dwell time; observer design; $H_\infty$ control

1. Introduction

Switched systems are a category of hybrid systems consisting of several subsystems activated via switching signals [1–3]. Switched systems are extensively employed in modeling various dynamic systems [4–6], for instance, chaotic systems [7], traffic control [8], singular systems [9], networked nonlinear systems [10], chemical processes [11], and mechanical systems [12]. Time delay exists in several practical and engineering systems, such as chemical processes [13], nonholonomic systems [14,15], electrical circuits [16–18], descriptor systems [19], and biological systems [20]. This phenomenon causes the system to perform poorly and can even lead to instability [21]. In practice, time delays exist not only in the states but also in their derivatives. Time-Delay Neutral Systems (TDNSs) can be found in chemical processes, population dynamics, and water pipes [22]. However, Switched Neutral Systems (SNSs) stability is a challenging problem [23,24]. Several Lyapunov-based methods have been proposed to study the stability of SNSs. Dwell time (DT) and Average Dwell Time (ADT) methods have been applied for studying the stabilization of SNSs [25,26]. The problem of stabilization and robust guaranteed cost control of a switched neutral
system in the presence of uncertain interval time-varying delays via the dynamic output feedback approach has been focused on in [26]. The existing condition of an appropriate controller for stabilization and control of the system is changed to solve an iterative convex optimization problem in the form of LMI [26]. Furthermore, uncertainty terms, such as environmental noises, parametric uncertainties, and external disturbances, are frequently encountered in numerous engineering applications, and the exact mathematical model development is difficult [27–30]. Additionally, the presence of uncertainties can degrade the performance of the closed-loop system and even lead to instability. There are several uncertain sources, such as ambient noise, parameter variations, modeling and measurement errors, linearization approximation, and disturbance, in real-world systems. By considering the uncertainty in SNSs, the obtained systems can be named Uncertain Switched Neutral Systems (USNSs). Thus, investigating the robust control approaches of USNSs is critical in both theory and practice [31,32]. In [33], the robust stabilization of USNSs has been investigated with constant delay and parametric uncertainty. In [34], the problem of exponential stabilization has been studied for USNSs with time-varying uncertainties. In the studied work, the neutral delay is constant. Using the ADT approach and the piecewise Lyapunov functional technique, Reference [22] proposed the exponential stabilization condition of USNSs with norm-bounded uncertainty, nonlinear perturbations, and neutral delays [35]. Though the various research methods have been investigated in the works mentioned above, it appears that the parametric uncertainty should be studied in the state-derivatives matrix. In practice, the state feedback (SF) control fails to guarantee the stability of the system when the state variables cannot be accessed from measurement [36]. Thus, observer-based control designs can be a good option in those cases. In the observer-based control, the output dynamic feedback controller is provided, and the state variables can be estimated from the process. Hence, the observer-based control for Switched Time-Delay Systems (STDSs) with or without a neutral type has been a stimulating subject in the control scheme [37–39]. Event-triggered control is an appropriate control approach that can reduce the volume of communications and provide a suitable closed-loop performance. This strategy and an observer-based output feedback control have been designed for SNSs, considering the mixed time-varying delays [40–43]. Another expectation in many practical systems is $H_\infty$ control. This concept has been presented to decrease the disturbance input effect on the regulated output at a given level and satisfy the stability of the closed-loop system. In recent years, $H_\infty$ control has received extensive attention in switched systems, with or without delays.

This paper designs a robust $H_\infty$ control law for a class of USNSs with interval time-varying delays and norm-bounded uncertainties. Its main contributions are as follows:

- It addresses the stabilization and $H_\infty$ control problem of Uncertain Switched Neutral Systems with interval time-varying delays in the system states and their time-derivatives.
- It solves the parametric uncertainty problem in USNSs by designing a robust observer-based control and applying the Yakubovich lemma.
- It suggests sufficient robust exponential stability conditions using the average dwell time approach and piecewise Lyapunov function technique in terms of a set of linear matrix inequalities.
- When the system state variables cannot be measured, the observer-based control approach applies to the stability guarantee. In this study, the separation principle was not met.
- The problem of the USNSs with interval time-varying delays that existed in the state and its derivatives (neutral) is addressed in this paper.
- Due to this study considers the time-varying and explored uncertainties in the state derivatives matrix, the presented model is closer to the practical situations.
- To notice the decay rates, as an important feature of real cases, the exponential stability or stabilization is considered here.
The upper bound of the discrete and neutral delays and their derivatives are effective in this paper designing procedure, causing the treatment to be more general with less conservatism compared to the literature approaches.

The remainder of the article is organized as follows: Section 2 provides the definitions, lemmas, and description of USNs. In Section 3, the problem of robust output observer-based $H_{\infty}$ control for USNs is given. Section 4 illustrates the performance of the proposed approach via a numerical example. Some concluding remarks are finally given in Section 5.

2. Problem Formulation and Preliminaries

Consider the following class of USNs:

$$
\dot{x}(t) = A_{1,\sigma(t)}(t)x(t) + A_{2,\sigma(t)}(t)x(t-d(t)) + A_{3,\sigma(t)}(t)\dot{x}(t-h(t)) + B_{\sigma(t)}(t)u(t) + H_{\sigma(t)}w(t)
$$

(1)

$$
y(t) = C_{\sigma(t)}x(t)
$$

$$
z(t) = C_{1,\sigma(t)}x(t) + C_{2,\sigma(t)}x(t-d(t)) + C_{3,\sigma(t)}\dot{x}(t-h(t)) + B_{1,\sigma(t)}u(t) + H_{1,\sigma(t)}w(t)
$$

$x(t_0 + \theta) = \varphi(\theta), \ \forall \theta \in [-H, 0], \ H = \max \{d_2, h_2\}$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, and $z(t) \in \mathbb{R}^m$ are the state vector, the control input vector, the measurement output vector, and the controlled output, respectively. For convenience, $\sigma(t) = \sigma$ is considered in this study. $A_{1,\sigma(t)}$, $A_{2,\sigma(t)}$, $A_{3,\sigma(t)}$, $B_{\sigma(t)}$, and $B_{1,\sigma(t)}$ are the system matrices with time-varying uncertainties. These uncertainties are assumed to be

$$
A_{1,\sigma(t)} = A_{1,\sigma} + \Delta A_{1,\sigma(t)},
$$

$$
A_{2,\sigma(t)} = A_{2,\sigma} + \Delta A_{2,\sigma(t)},
$$

$$
A_{3,\sigma(t)} = A_{3,\sigma} + \Delta A_{3,\sigma(t)},
$$

$$
B_{\sigma(t)} = B_{\sigma} + \Delta B_{\sigma(t)},
$$

where $A_{1,\sigma}$, $A_{2,\sigma}$, $A_{3,\sigma}$, and $B_{\sigma}$ are the given constant matrices. $C_{\sigma}$, $C_{1,\sigma}$, $C_{2,\sigma}$, $C_{3,\sigma}$, $H_{\sigma}$, and $H_{1,\sigma}$ are the known constant matrix of appropriate dimensions. $\varphi(.)$ is the continuous vector-valued function specifying the initial state of the system. $\sigma(t): [0, +\infty) \rightarrow L = \{1, 2, \ldots, l\}$ is a switching signal. In addition, $\sigma(t) = i$ means that the $i^{th}$ subsystem is activated. The following observer-based control is proposed for the stabilization of the USNS introduced in (1):

$$
\dot{\hat{x}}(t) = A_{c,\sigma(t)}\hat{x}(t) + A_{d,\sigma(t)}\hat{x}(t-d(t)) + A_{e,\sigma(t)}\hat{x}(t-h(t))
$$

$$
+ B_{\sigma}u(t) + L_{\sigma}(y(t) - \hat{y}(t)),
$$

$$
\hat{y}(t) = C_{\sigma}\hat{x}(t),
$$

$$
u(t) = K_{1,\sigma}\hat{x}(t) + K_{2,\sigma}\hat{x}(t-d(t)),
$$

where $\hat{x}(t)$ is the estimation of the state vector, $\hat{y}(t)$ is the observer output vector, $K_{1,\sigma}$ and $K_{2,\sigma}$ are the controller gains, $L_{\sigma}$ is the observer gain, and $A_{c,\sigma}$, $A_{d,\sigma}$, and $A_{e,\sigma}$ are matrices to be specified. Applying (2) to (1) yields:

$$
\ddot{\hat{x}}(t) = \ddot{A}_{1,\sigma(t)}\hat{x}(t) + \ddot{A}_{2,\sigma(t)}\hat{x}(t-d(t)) + \ddot{A}_{3,\sigma(t)}\hat{x}(t-h(t)) + \ddot{H}_{\sigma(t)}w(t) + \ddot{E}_{\sigma(t)}\Delta(t)
$$

(3)

For the sake of convenience, define $\ddot{x}(t) = \begin{bmatrix} e^T(t) & \ddot{e}^T(t) \end{bmatrix}^T$, where the signal $e(t) = x(t) - \hat{x}(t)$ is defined as the estimated error of the USNS, and

$$
\dddot{A}_{1,\sigma(t)} = \begin{bmatrix} A_{1,\sigma(t)} - L_{\sigma(t)}C_{\sigma(t)} & A_{1,\sigma(t)} - A_{c,\sigma(t)} \\ L_{\sigma(t)}C_{\sigma(t)} & A_{c,\sigma(t)} + B_{\sigma(t)}K_{1,\sigma(t)} \end{bmatrix},
$$

$$
\dddot{A}_{2,\sigma(t)} = \begin{bmatrix} A_{2,\sigma(t)} - A_{d,\sigma(t)} & A_{2,\sigma(t)} - A_{d,\sigma(t)} \\ 0 & A_{d,\sigma(t)} + B_{\sigma(t)}K_{2,\sigma(t)} \end{bmatrix},
$$

$$\dddot{H}_{\sigma} = \begin{bmatrix} H_{\sigma} & 0 \end{bmatrix}.$
\[
\tilde{A}_{3,t}(t) = \begin{bmatrix}
A_{3,j}(t) & A_{3,j}(t) - A_{E,j}(t) \\
0 & A_{E,j}(t)
\end{bmatrix}, \quad \tilde{E}_{j}(t) = \begin{bmatrix}
E \\
0
\end{bmatrix},
\]

\[
\Delta(t) = F(t) \{ N_{i} \tilde{e}(t) + (N_{i} + N_{j} K_{1,i}) \hat{x}(t) + N_{2,i} e(t - d(t)) + N_{2,i} K_{2,i} \hat{x}(t - d(t)) + N_{3,i} \tilde{e}(t - h(t)) + N_{3,i} \hat{x}(t - h(t)) \},
\]

\[
z(t) = C_{1,i}[e(t) + \hat{x}(t)] + C_{2,i} [e(t - d(t)) + \hat{x}(t - d(t))] + C_{3,i} \hat{x}(t - h(t))
\]

\[
\Delta K_{1,i}(t) + K_{2,i} \hat{x}(t - d(t)) + H_{1,i} w(t)
\]

\[
\tilde{C}_{1,i} \tilde{x}(t) + \tilde{C}_{2,i} \tilde{x}(t - d(t)) + \tilde{C}_{3,i} \tilde{x}(t - h(t)) + H_{1,i} w(t)
\]

The following assumptions, definitions, and lemmas are considered in the control design.

**Assumption 1.** The delay \( h(t) \) is the time-varying neutral delay satisfying

\[
0 \leq h_{1} \leq h(t) \leq h_{2} < \infty, \quad \dot{h}(t) \leq \mu_{h} < 1,
\]

and \( d(t) \) is the discrete delay meeting

\[
0 \leq d_{1} \leq d(t) \leq d_{2} < \infty, \quad \dot{d}(t) \leq \mu_{d} < 1.
\]

**Assumption 2.** The time-varying matrices \( \Delta A_{1,i}, \Delta A_{2,i}, \Delta A_{3,i}, \) and \( \Delta B_{i} \) are assumed to be norm-bounded with appropriate dimensions satisfying the following condition:

\[
\begin{bmatrix}
\Delta A_{1,i} & \Delta A_{2,i} & \Delta A_{3,i} & \Delta B_{i}
\end{bmatrix} = EF(t) \begin{bmatrix}
N_{1,i} & N_{2,i} & N_{3,i} & N_{4,i}
\end{bmatrix},
\]

where \( E \in \mathbb{R}^{a \times n} \) and \( N_{1,i}, N_{2,i}, N_{3,i} \in \mathbb{R}^{b \times n} \), and \( N_{4,i} \in \mathbb{R}^{b \times m} \) are the constant matrices, and \( F(t) \in \mathbb{R}^{a \times b} \) is the unknown continuous time-varying matrix function with Lebesgue measurable elements, satisfying:

\[
F^{T}(t) F(t) \leq I.
\]

The following results can be easily obtained using the Equation (7):

\[
\Delta^{T}(t) \Delta(t) \leq Q^{T} N N^{T} Q,
\]

\[
Q = \begin{bmatrix}
e(t) & \tilde{e}(t) \\
\tilde{e}(t - d(t)) & \tilde{e}(t - h(t))
\end{bmatrix}, \quad N = \begin{bmatrix}
N_{1,i}^{T} & (N_{1,i} + N_{4,i} K_{1,i})^{T} \\
N_{2,i}^{T} & (N_{2,i} + N_{4,i} K_{2,i})^{T}
\end{bmatrix}.
\]

**Assumption 3.** Suppose that the matrix \( C_{\sigma(t)} \) is full-row rank. For convenience, the singular value decomposition of \( C_{\sigma(t)} \) is of the form \( C_{i} = U_{i} \begin{bmatrix} S_{i} & 0 \end{bmatrix} V_{i}^{T} \), where \( S_{i} \in \mathbb{R}^{p \times p} \) is a diagonal matrix with positive diagonal elements in decreasing order; \( U_{i} \in \mathbb{R}^{p \times p} \) and \( V_{i} \in \mathbb{R}^{n \times n} \) are unitary matrices.

**Assumption 4.** The external noise signal \( \omega(t) \) is time-varying and satisfies

\[
\int_{0}^{\infty} w^{T}(t) w(t) dt < d, \quad d \geq 0
\]

**Definition 1 ([44]).** For any \( T_{2} > T_{1} \geq 0, \) let \( N_{\sigma(T_{1}, T_{2})} \) denote the number of switching \( \sigma(t) \) at an interval \((T_{1}, T_{2})\). If \( N_{\sigma(T_{1}, T_{2})} \leq N_{0} + \frac{T_{2} - T_{1}}{\tau_{a}} \) holds true for any given \( N_{0} \geq 0, \tau_{a} \geq 0, \) the...
constant $\tau_a$ is called the average dwell time. For the sake of convenience and following the common practice in the literature, we consider $N_0 = 0$.

**Definition 2** ([21]). The USNS (1) is said to be robust and exponentially stable under $\sigma(t)$, if there exist constants $M \geq 0$ and $\lambda > 0$ such that

$$||x(t, \varphi)|| \leq Me^{-\lambda(t-t_0)}||\varphi||, \forall t \geq t_0,$$

(10)

for all admissible uncertainties $F^T(t)F(t) \leq I$, where $x(t_0 + \theta) = \varphi(\theta)$.

**Definition 3** ([45]). For a prescribed level of disturbance attenuation $\gamma > 0$, find an observer-based control (2) satisfying the following conditions:

- With $w(t) = 0$, the USNS (1) with observer-based control (2) is exponentially stabilizable.
- Under zero-initial condition $\varphi(t) = 0, \forall t \in [-H, 0]$, the output $z(t)$ satisfies

$$\int_0^\infty z^T(t)z(t)dt \leq \gamma^2 \int_0^\infty w^T(t)w(t)dt.$$

(11)

**Lemma 1 (Yakubovich lemma)** ([46]). Let $\Omega_0(x)$ and $\Omega_1(x)$ two quadratic matrix functions over $\mathbb{R}^n$, and $\Omega_1(x) \leq 0$ for all $x(t) \in \mathbb{R}^n - \{0\}$. Then, $\Omega_0(x) < 0$ holds true for all $x(t) \in \mathbb{R}^n - \{0\}$, if and only if there exists the constant $\varepsilon \geq 0$ such that

$$\Omega_0(x) - \varepsilon \Omega_1(x) < 0, \quad \forall x(t) \in \mathbb{R}^n - \{0\}.$$

(12)

**Lemma 2.** ([47]) For a given $C \in \mathbb{R}^{p \times n}$ with rank $(C) = p$, assume that $X \in \mathbb{R}^{n \times n}$ is a symmetric matrix, then there exists a matrix $\hat{X} \in \mathbb{R}^{p \times p}$ such that $CX = \hat{X}C$, if and only if

$$X = V \begin{bmatrix} \hat{X}_{11} & 0 \\ 0 & \hat{X}_{22} \end{bmatrix} V^T,$$

(13)

where $\hat{X}_{11} \in \mathbb{R}^{p \times p}$ and $\hat{X}_{22} \in \mathbb{R}^{(n-p) \times (n-p)}$.

3. Observer-Based Robust $H_{\infty}$ Control

The class of USNs given in (1) is considered here. The following theorem proves the stabilization of the USNS (1) via the observer-based control (2) in terms of feasible solutions to a certain set of LMIs.

**Theorem 1.** Consider the System (1). Let $\overline{P}_{1,i} = P_{1,i}^{-1}$, $\overline{P}_{2,i} = P_{2,i}^{-1}$, $\overline{Q}_{1,i} = Q_{1,i}^{-1}$, $\overline{Q}_{2,i} = Q_{2,i}^{-1}$, $\overline{R}_{1,i} = R_{1,i}^{-1}$, $\overline{R}_{2,i} = R_{2,i}^{-1}$. Assume that there exist matrices $P_i > 0$, $Q_i > 0$, $R_i > 0$, $X_{1,i}$, $X_{2,i}$, $Y_{1,i}$, $Y_{2,i}$, $W_{1,i}$, $W_{2,i}$, and $W_{3,i}$ and positive constants $a$ and $\gamma$ such that for $i \in L$,

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ * & \Lambda_{22} & \Lambda_{23} \\ * & * & \Lambda_{33} \end{bmatrix} < 0,$$

(14)

where

$$\Lambda_{11} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & A_{2,i} \overline{Q}_{1,i} - W_{2,i} & A_{3,i} \overline{R}_{1,i} \\ * & \Sigma_{22} & 0 & 0 \\ * & * & \Sigma_{33} & 0 \\ * & * & * & \Sigma_{44} \end{bmatrix},$$

$$\Sigma_{11} =$$
\[ \Lambda_{12} = \begin{bmatrix} A_{3,j} R_{2,j} - W_{3,j} & H_{i} & E & \mathcal{P}_{1,i} A_{1,i}^T - C_{i}^T X_{1,i}^T & C_{i}^T X_{1,i}^T \\ W_{3,j} & 0 & 0 & \mathcal{P}_{2,i} A_{1,i}^T - W_{1,i}^T & W_{1,i}^T + Y_{1,i}^T B_{i}^T \\ 0 & 0 & 0 & \mathcal{Q}_{1,i} A_{2,i}^T - 0 & 0 \\ 0 & 0 & 0 & \mathcal{Q}_{2,i} A_{2,i}^T - W_{2,i}^T & W_{2,i}^T + Y_{2,i}^T B_{i}^T \\ 0 & 0 & 0 & \mathcal{R}_{1,i} A_{3,i}^T & 0 \end{bmatrix}, \]

\[ \Sigma_{11} = (A_{1,j} \mathcal{P}_{1,j} - X_{C,i}) + (A_{1,j} \mathcal{P}_{1,j} - X_{C,i})^T + a \mathcal{P}_{1,j}, \]

\[ \Sigma_{12} = A_{1,j} \mathcal{P}_{2,j} + C_{i}^T X_{i}^T - W_{1,i}, \]

\[ \Sigma_{22} = W_{1,j}^T + W_{1,j}^T + B_{i} Y_{1,i} + Y_{1,i}^T B_{i}^T + a \mathcal{P}_{2,j}, \]

\[ \Sigma_{33} = -(1 - \mu d) e^{-\mu d} \mathcal{Q}_{1,j}, \]

\[ \Sigma_{44} = -(1 - \mu d) e^{-\mu d} \mathcal{Q}_{1,j}, \]

\[ \Sigma_{55} = -(1 - \mu h) e^{-\mu h} R_{1,j}, \]

\[ \Sigma_{66} = -(1 - \mu h) e^{-\mu h} R_{2,j}, \]

\[ \Lambda_{13} = \begin{bmatrix} \mathcal{P}_{1,i} C_{1,j}^T & \mathcal{P}_{1,i} N_{1,j}^T & 0 \\ \mathcal{P}_{2,i} C_{1,j}^T + Y_{1,i}^T B_{i}^T & \mathcal{P}_{2,i} N_{1,j}^T + Y_{1,i}^T N_{1,j}^T & 0 & \mathcal{P}_{1,i} \\ \mathcal{Q}_{1,i} C_{2,j}^T & \mathcal{Q}_{1,i} N_{2,j}^T & 0 & 0 \\ \mathcal{Q}_{2,i} C_{2,j}^T + Y_{2,i}^T B_{i}^T & \mathcal{Q}_{2,i} N_{2,j}^T + Y_{2,i}^T N_{2,j}^T & 0 & 0 \end{bmatrix}, \]

\[ \Lambda_{22} = \begin{bmatrix} \gamma^2 I & 0 & 0 \\ * & -1 & 0 \\ * & * & -R_{1,i} \end{bmatrix}, \]

\[ \Lambda_{23} = \begin{bmatrix} \mathcal{R}_{1,i} C_{3,j}^T & \mathcal{R}_{1,i} N_{3,j}^T & 0 & 0 \\ H_{i}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ \Lambda_{33} = \begin{bmatrix} -I & 0 & 0 \\ * & -1 & 0 \\ * & * & -\mathcal{Q}_{2,i} \end{bmatrix}, \]

\[ P_i = \begin{bmatrix} P_{1,j} & 0 \\ 0 & P_{2,i} \end{bmatrix}, Q_i = \begin{bmatrix} Q_{1,j} & 0 \\ 0 & Q_{2,i} \end{bmatrix}, R_i = \begin{bmatrix} R_{1,j} & 0 \\ 0 & R_{2,i} \end{bmatrix}. \]

then for any switching signal with ADT \( \tau_{d} > \tau_{d}^* = \frac{\ln \mu}{\pi} \), where \( \mu \geq 1 \) satisfies

\[ P_i \leq \mu P_i, Q_i \leq \mu Q_i, R_i \leq \mu R_i, \forall i, j \in L, \]

System (1) is exponentially stable using the observer-based control (2) with

\[ A_{C,i} = W_{1,i} R_{2,i}^{-1}, \quad A_{D,i} = W_{2,i} R_{2,i}^{-1}, \]

\[ A_{E,i} = W_{3,i} R_{2,i}^{-1}, \quad K_{1,i} = Y_{1,i} R_{2,i}^{-1}, \]

\[ K_{2,i} = Y_{1,i} R_{2,i}^{-1}, \quad L_i = X_i U_i S_{i}^{-1} S_{i}^{-1} U_i^T. \]
Furthermore, the estimate of the state decay is given by

\[ ||\eta(t)|| \leq \sqrt{\frac{b}{a}} ||\eta_0|| e^{-\lambda(t-t_0)}, \sqrt{\frac{b}{a}} \geq 1, \]  

(17)

where

\[ \lambda = \frac{1}{2} \left( a - \frac{\ln \mu}{\alpha} \right), \quad a = \min_{i \in L} \lambda_{max}(P_i), \]

\[ b = \max_{i \in L} \lambda_{max}(P_i) + d_2 \max_{i \in L} \lambda_{max}(Q_i) + h_2 \max_{i \in L} \lambda_{max}(R_i). \]  

(18)

**Proof:** See Appendix A. \( \square \)

**Remark 1.** When \( \mu = 1 \), given \( \tau_a > \tau_a^* = \frac{\ln \mu}{\alpha^*} \), we will have \( \tau_a > \tau_a^* = 0 \), which means that the switching signal \( \sigma(t) \) can be arbitrary. Thus, (15) takes the following form

\[ P_i \leq P_j, \quad Q_i \leq Q_j, \quad R_i \leq R_j, \quad \forall i, j \in L, \]  

(19)

Equation (18) in the form of (19) is considered as

\[ P_i = P_j = P, \quad R_i = R_j = R, \quad Q_i = Q_j = Q \]  

(20)

This shows that a common Lyapunov function is required for all subsystems.

**Remark 2.** This study presents a controller design method applying the observer to realize exponential stabilization. The LMI approach is employed for the expression of the existing condition. For more closeness between the model and the real system, uncertainties, delay (time-varying and neutral), and nonlinear perturbations are inserted into the model, leading to the problem complexity. By this approach, the system dimension and, consequently, LMI size is increased due to the computational complexity.

**Remark 3.** The stability is proved, employing the conservative Lyapunov function that guarantees sufficient conditions, not the necessary ones. In this paper, the multiple Lyapunov functions are utilized that are less conservative compared to the common Lyapunov function. For more conservativeness reduction, the ADT approach is used. Besides, the upper bound of the discrete and neutral delays and their derivatives are effective in the designing procedure, causing the treatment to be more general with less conservatism compared to the literature approaches.

**Remark 4.** The main differences between the present study and [26] can be summarized as follows. While [26] uses output dynamic feedback, the observer is designed in the present study. In addition, the cost function of [26] is in the form of guaranteed cost control, but this work focuses on the problem of H_{\infty} control approach. Besides, convex optimization is employed in order to convert the problem to LMI in [26]; this research utilizes SVD lemma and change of variables.

### 4. Numerical Example

In this section, two examples are presented for more illustration of the effectiveness of the proposed method.

**Example 1.** We consider the USNS described by (1) comprised of two subsystems with the following constant values:

\[
A_{1,1} = \begin{bmatrix}
-2.0 & -1.0 & -1.2 \\
0.7 & -1.4 & 0.5 \\
-1.3 & 0.5 & -1.0
\end{bmatrix}, A_{2,1} = \begin{bmatrix}
0.2 & 0.0 & 0.1 \\
0.1 & 0.3 & 0.1 \\
0.3 & 0.1 & 0.2
\end{bmatrix}, A_{3,1} = \begin{bmatrix}
0.2 & 0.0 & 0.1 \\
0.1 & 0.3 & 0.1 \\
0.3 & 0.1 & 0.2
\end{bmatrix}, \\
B_1 = \begin{bmatrix}
1.0 & 0.5 & 2.0
\end{bmatrix}^T, \quad C_1 = \begin{bmatrix}
-1.2 & 0.5 & 0.7
\end{bmatrix}, B_{1,1} = 0.2,
\]
C_{1,1} = \begin{bmatrix} -1.2 & 0.5 & 0.7 \end{bmatrix}, \quad C_{2,1} = \begin{bmatrix} 1.0 & 0.6 & 0.9 \end{bmatrix}, \quad C_{3,1} = \begin{bmatrix} -0.8 & 0.5 & 0.4 \end{bmatrix},

N_{1,1} = \begin{bmatrix} 0.2 & 0.0 & 0.1 \\ 0.0 & 0.1 & 0.1 \\ 0.0 & 0.1 & 0.0 \end{bmatrix}, \quad N_{2,1} = \begin{bmatrix} 0.0 & 0.2 & 0.1 \\ 0.0 & 0.1 & 0.1 \\ 0.0 & 0.1 & 0.0 \end{bmatrix},

N_{3,1} = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.0 & 0.1 & 0.2 \\ 0.1 & -0.1 & 0.1 \end{bmatrix}, \quad N_{4,1} = \begin{bmatrix} 0.1 & 0.0 & 0.1 \end{bmatrix}^T,

A_{1,2} = \begin{bmatrix} -3.0 & -1.2 & -3.0 \\ 0.2 & -1.0 & 0.4 \\ -0.7 & 1.1 & -1.2 \end{bmatrix}, \quad A_{2,2} = \begin{bmatrix} 0.2 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}, \quad A_{3,2} = \begin{bmatrix} 0.2 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix},

B_2 = \begin{bmatrix} 0.9 & 0.4 & 1.8 \end{bmatrix}^T, \quad C_2 = \begin{bmatrix} -1.0 & 0.5 & 0.7 \end{bmatrix}, B_{1,2} = 0.5,

C_{1,2} = \begin{bmatrix} -1 & 0.5 & 0.7 \end{bmatrix}, \quad C_{2,2} = \begin{bmatrix} 1.2 & 0.5 & 0.8 \end{bmatrix},

C_{3,2} = \begin{bmatrix} 1.2 & 0.7 & 0.3 \end{bmatrix}, \quad N_{1,2} = \begin{bmatrix} 0.2 & 0.1 & 0.0 \\ 0.1 & 0.1 & -0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix},

N_{2,2} = \begin{bmatrix} 0.0 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.1 \end{bmatrix}, \quad N_{3,2} = \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0.2 & 0.0 & -0.1 \\ 0.1 & 0.1 & 0.1 \end{bmatrix},

N_{4,2} = \begin{bmatrix} 0.2 & 0.1 & 0.0 \end{bmatrix}^T, \quad H_{1,1} = 0.2, \quad H_{1,2} = 0.1,

H_1 = \begin{bmatrix} 0.4 & 0.5 & 0.6 \end{bmatrix}^T, \quad H_2 = \begin{bmatrix} 0.2 & 0.4 & 0.5 \end{bmatrix}^T,

E = 0.5 \times \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad F(t) = \cos(0.1t) \times I_{3 \times 3} (21)

\text{Considering } d(t) = 0.3 + 0.2 \sin(t), \quad a = 0.01, \quad \text{and } h(t) = 0.1 + 0.1 \sin(t), \quad \text{we obtain}
\mu_d = 0.2, \quad \mu_h = 0.1, \quad d_2 = 0.5, \quad \text{and } h_2 = 0.2. \quad \text{The switching signal is shown in Figure 1. \ A robust output observer-based control } u(t) \text{ in the form of (2) is designed such that system (3) reaches exponential stability. In this section, an output observer-based controller is designed for USNS by setting } \mu = 1.01 \text{ (thus, } \tau_a > \tau_a^* = \frac{\ln \mu}{\alpha} = 0.9950).}

\text{Solving (14)–(16) gives the following feasible solutions:}

\begin{align*}
A_{C,1} &= \begin{bmatrix} -3.7061 & -0.1460 & -0.0663 \\ -2.1033 & 0.0935 & 2.4948 \\ 1.3059 & -1.0545 & -3.0714 \end{bmatrix}, \quad A_{C,2} = \begin{bmatrix} -3.5295 & -0.4767 & -1.6496 \\ 0.6550 & -2.2515 & -1.7738 \\ -0.5361 & 1.3005 & -0.9880 \end{bmatrix},
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{switching_signal.png}
\caption{The switching signal where ADT is } \tau_a = 1.5.\end{figure}
\( A_{D,1} = \begin{bmatrix} 1.0623 & 0.5414 & 0.8907 \\ 1.9652 & 1.4266 & 1.8252 \\ -1.3036 & -0.8201 & -1.2887 \end{bmatrix}, A_{D,2} = \begin{bmatrix} 0.3043 & 0.0418 & 0.0899 \\ 0.1532 & 0.2271 & 0.0916 \\ -0.0232 & 0.1013 & 0.2414 \end{bmatrix}, \)

\( A_{E,1} = \begin{bmatrix} -0.6014 & 0.5003 & 0.4794 \\ -1.5211 & 1.2726 & 0.8944 \\ 1.6894 & -0.6811 & -0.4099 \end{bmatrix}, A_{E,2} = \begin{bmatrix} 0.2046 & -0.0164 & -0.0365 \\ 0.1382 & 0.2743 & 0.2394 \\ 0.1051 & 0.0935 & 0.2862 \end{bmatrix}, \)

\( L_1 = \begin{bmatrix} -0.0269 \\ 0.0013 \\ 0.4726 \end{bmatrix}^T, L_2 = \begin{bmatrix} -1.0741 \\ 1.5364 \\ 0.8094 \end{bmatrix}^T, \)

\( K_{1,1} = \begin{bmatrix} 1.0897 \\ -2.3793 \\ -3.4141 \end{bmatrix}, K_{1,2} = \begin{bmatrix} 0.7733 \\ -3.6634 \\ -5.7249 \end{bmatrix}, \)

\( K_{1,1} = \begin{bmatrix} 1.0897 \\ -2.3793 \\ -3.4141 \end{bmatrix}, K_{1,2} = \begin{bmatrix} 0.7733 \\ -3.6634 \\ -5.7249 \end{bmatrix}, \)

The state trajectories and state estimation trajectories of the USNS are shown in Figures 2–4, with the initial conditions satisfying \( \begin{bmatrix} x(0) \\ \hat{x}(0) \end{bmatrix} = \begin{bmatrix} 0.7 \\ -0.1 \end{bmatrix}^T, \)

\( \hat{x}(0) = \begin{bmatrix} 0.5 \\ -0.3 \\ 0.5 \end{bmatrix}, t \in [-0.5, 0]. \) It can be observed that system (3) is exponentially stable. Figures 5–7 depict the output observer-based control trajectory, the observer output, and the estimated error of the system, respectively.
As can be comprehend from Figures 2–4 that the estimations for all states are reached to their actual values, which means that the estimation errors go to zero in an acceptable time. Furthermore, the control signal is completely feasible, and the observer signal tracks the actual output suitably. These points can be understood from Figures 5–7.

**Example 2.** The model of the water-quality dynamic of the River Nile considering two modes of operation are presented as follows [48,49]:

\[
\dot{x}(t) = A_{1,\sigma(t)}(t)x(t) + A_{2,\sigma(t)}(t)x(t - d(t)) + B_{\sigma(t)}(t)u(t) + H_{\sigma(t)}w(t)
\]

\[
y(t) = C_{\sigma(t)}x(t)
\]

\[
z(t) = C_{1,\sigma(t)}x(t) + C_{2,\sigma(t)}x(t - d(t)) + B_{1,\sigma(t)}u(t) + H_{1,\sigma(t)}w(t)
\]

\[
x(t_0 + \theta) = \varphi(\theta), \forall \theta \in [-H, 0], H = \max\{d_2, h_2\}
where

\[
\begin{align*}
A_{1,1} &= \begin{bmatrix} -1.0 & 0.0 \\
-3.0 & -2.0 \end{bmatrix},
A_{2,1} = \begin{bmatrix} -0.55 & 0.70 \\
-0.25 & -0.30 \end{bmatrix},
B_1 = \begin{bmatrix} 1.4 & 0.0 \\
0.0 & 1.5 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} 1.0 & 1.0 \\
1.0 & 2.0 \end{bmatrix},
C_{1,1} = \begin{bmatrix} 0.1 & 0.1 \\
0.1 & 0.2 \end{bmatrix},
C_{1,2} = \begin{bmatrix} -0.1 & 0.1 \\
0.1 & -0.2 \end{bmatrix},
B_{1,1} = \begin{bmatrix} 0.2 & 0 \\
0 & 0.4 \end{bmatrix}, \\
A_{2,1} &= \begin{bmatrix} 1.0 & 0.0 \\
-3.0 & -2.0 \end{bmatrix},
A_{2,2} = \begin{bmatrix} -0.45 & -0.50 \\
-0.15 & -0.10 \end{bmatrix},
B_2 = \begin{bmatrix} 1.2 & 0.0 \\
0.0 & 1.4 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 1.0 & 1.4 \\
1.5 & 1.0 \end{bmatrix},
C_{2,1} = \begin{bmatrix} 0.2 & 0.1 \\
0.1 & 0.1 \end{bmatrix},
C_{2,2} = \begin{bmatrix} -0.1 & 0.1 \\
0.1 & -0.2 \end{bmatrix},
B_{1,2} = \begin{bmatrix} 0.1 & 0 \\
0 & 0.5 \end{bmatrix}, \\
N_{1,1} &= \begin{bmatrix} 0.5 & -0.3 \\
-0.2 & 0.8 \end{bmatrix},
N_{2,1} = \begin{bmatrix} -0.2 & 0.0 \\
0.0 & -0.1 \end{bmatrix},
N_{4,1} = \begin{bmatrix} -0.2 & 0.1 \\
0.1 & -0.2 \end{bmatrix}, \\
N_{1,2} &= \begin{bmatrix} 0.7 & -0.3 \\
-0.6 & 0.7 \end{bmatrix},
N_{2,2} = \begin{bmatrix} -0.1 & 0.1 \\
0.0 & -0.1 \end{bmatrix},
N_{4,2} = \begin{bmatrix} -0.2 & 0.1 \\
0.1 & -0.1 \end{bmatrix}, \\
H_1 &= \begin{bmatrix} 1 & -1 \\
2 & 1 \end{bmatrix},
H_2 = \begin{bmatrix} -1 & 0 \\
2 & 0.8 \end{bmatrix},
H_{1,1} = \begin{bmatrix} 1 & -1 \\
2 & 1 \end{bmatrix},
H_{1,2} = \begin{bmatrix} 0.8 & 0 \\
1 & -1.0 \end{bmatrix}, \\
E_1 = E_2 = 0.1 \times \begin{bmatrix} 0.1 & 0 \\
0 & 0.3 \end{bmatrix},
F(t) = \begin{bmatrix} \sin(0.1t) & 0 \\
0 & \sin(0.1t) \end{bmatrix}. & (22)
\end{align*}
\]

Defining the disturbance signal as \( d(t) = 0.3 + 0.2 \sin(t) \), and \( \alpha = 0.5 \), the values of \( \mu_d \) and \( d_2 \) can be obtained as 0.2 and 0.5, respectively. Considering the control signal \( (u(t)) \) of a robust output observer-based approach in the form of (2), caused the system (3) to reach exponential stability. By quantifying \( \mu = 1.1 \) (thus, \( \tau_o > \tau_0^* = \frac{\ln \mu}{\alpha} = 0.1906 \)), an output observer-based controller is designed for USNS. The below matrices can be provided as the feasible solutions via solving (14)–(16):

\[
\begin{align*}
A_{C,1} &= 10^7 \begin{bmatrix} -0.4617 & 0.2672 \\
1.0394 & -1.0707 \end{bmatrix},
A_{C,2} = 10^7 \begin{bmatrix} 3.8016 & -1.5925 \\
-1.5922 & 0.6451 \end{bmatrix}, \\
A_{D,1} &= \begin{bmatrix} 0.7600 & -0.0954 \\
0.9121 & 0.7380 \end{bmatrix},
A_{D,2} = \begin{bmatrix} 0.3083 & -0.6657 \\
-0.0178 & -0.2518 \end{bmatrix}, \\
L_1 &= 10^7 \begin{bmatrix} 1.0548 & -0.2817 \\
-0.8875 & 1.4418 \end{bmatrix},
L_2 = 10^5 \begin{bmatrix} -2.0540 & 2.8763 \\
3.0830 & -2.0549 \end{bmatrix}, \gamma = 0.03, \\
K_{1,1} &= \begin{bmatrix} -28.1942 & 20.2509 \\
6.0480 & -5.9665 \end{bmatrix},
K_{1,2} = \begin{bmatrix} 435.1665 & -186.0161 \\
12.5632 & -6.4043 \end{bmatrix}, \\
K_{2,1} &= \begin{bmatrix} -1.0871 & 0.0521 \\
-0.2368 & -0.3336 \end{bmatrix},
K_{2,2} = \begin{bmatrix} -0.8792 & 0.6582 \\
-0.1681 & 0.3803 \end{bmatrix}.
\end{align*}
\]

The trajectories of the states and their estimations for the USNS can be demonstrated in Figures 8 and 9, by selecting the initial conditions as \( x(0) = \begin{bmatrix} -3 & 4 \end{bmatrix}^T \), \( \xi(0) = \begin{bmatrix} 2 & -3 \end{bmatrix}^T \), and \( t \in [-0.5,0] \). As can be seen, System (3) is exponentially stable. Figures 10 and 11 illustrate the estimated error and the output observer-based control trajectory, respectively. The appropriate performance of the designed system is demonstrated in Figure 3, in which the output estimated errors for both states reach zero in a short time.
Figure 8. State trajectory of $x_1$ and state estimation trajectory $\hat{x}_1$.

Figure 9. State trajectory of $x_2$ and state estimation trajectory $\hat{x}_2$.

Figure 10. The estimated error of the system.
Figure 11. Output observer-based control trajectories.

5. Conclusions

This paper investigated the problem of a robust observer-based $H_\infty$ control for USNSs with interval time-varying mixed delays. It considered the systems with norm-bounded time-varying uncertainties. By utilizing the average dwell time approach and the piecewise Lyapunov functional technique, the LMI-based feasibility conditions have been established to ensure that the considered system is exponentially stable with a prescribed level of $H_\infty$ performance. The observer gains are determined by solving a set of LMIs. The uncertainties in USNSs are solved by designing an output observer-based controller and employing the Yakubovich lemma. The proposed observer-based $H_\infty$ control is verified through two numerical simulations. The obtained results confirmed the effectiveness and robust performance of the proposed approach. It is worth noting that most previous studies have only dealt with the stabilization and $H_\infty$ control problem of USNSs without considering uncertainty and time-varying delays. However, the present work is more practically oriented because it considers parametric uncertainties and time-varying delays in the states and their derivatives. The extension of the proposed observer-based robust control technique for switched neutral systems with input saturation and multiple time-varying delays can be the topic of our future research. The subject of mixed $H_2/H_\infty$ control for USNSs may be considered in future studies. Besides, by designing the Lyapunov function and employing the free weight matrices, the system conservation can be decreased in switched neutral systems along with uncertainty and nonlinear perturbation. In addition, designing a finite-time controller for the USNSs stabilization, considering the system constraints, and studying uncertain stochastic systems are the other issues for future works.

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Appendix A

Proof of Theorem 1. A Lyapunov functional candidate is defined as follows:

\[
V(t) \triangleq \tilde{x}^T(t) \left[ \begin{array}{cc} P_{1,i} & 0 \\ 0 & P_{2,i} \end{array} \right] \tilde{x}(t) + \int_{t-d(t)}^{t} e^{\alpha(s-t)} \tilde{x}^T(s) \left[ \begin{array}{cc} Q_{1,i} & 0 \\ 0 & Q_{2,i} \end{array} \right] \tilde{x}(s) ds + \int_{t-b(t)}^{t} e^{\beta(s-t)} \tilde{x}^T(s) \left[ \begin{array}{cc} R_{1,i} & 0 \\ 0 & R_{2,i} \end{array} \right] \tilde{x}(s) ds,
\]

(A1)

where \( P_{i,i} > 0, Q_{i,i} > 0, \) and \( R_{i,i} > 0 \) \((i = 1, 2)\) are to be determined. Taking the derivative of \( V(t) \) with respect to \( t \) along the trajectory of the USNS (1), using (4) and (5), and adding \( aV(t) \) (for the satisfaction of the exponential stability), yields:

\[
\dot{V}(t) + aV(t) \leq a \tilde{x}^T(t)P_{i}\tilde{x}(t)
\]

\[
+ \tilde{x}^T(t) \left[ P_{i} \tilde{A}_{1,i} + \tilde{A}_{1,i}^T P_{i} \right] \tilde{x}(t)
\]

\[
+ \tilde{x}^T(t) P_{i} \tilde{A}_{2,i} \tilde{x}(t - d(t))
\]

\[
+ \tilde{x}^T(t - d(t)) \tilde{A}_{2,i}^T P_{i} \tilde{x}(t)
\]

\[
+ \tilde{x}^T(t) P_{i} \tilde{H}_{i} \tilde{w}(t) + \Delta^T(t) \tilde{E}_{i}^T P_{i} \tilde{x}(t)
\]

\[
+ \tilde{w}^T(t) \tilde{H}_{i}^T P_{i} \tilde{x}(t) + \tilde{x}^T(t) P_{i} \tilde{E}_{i} \Delta(t)
\]

\[
+ \tilde{x}^T(t) P_{i} \tilde{A}_{3,i} \tilde{x}(t - h(t))
\]

\[
+ \tilde{x}^T(t - h(t)) \tilde{A}_{3,i}^T P_{i} \tilde{x}(t)
\]

\[
+ \tilde{x}^T(t) Q_{i,j} \tilde{x}(t) + \tilde{x}^T(t) R_{i,j} \tilde{x}(t)
\]

\[
- (1 - \mu_{d}) e^{-a_{2} \gamma} \tilde{x}^T(t - d(t)) Q_{i,j} \tilde{x}(t - d(t))
\]

\[
- (1 - \mu_{h}) e^{-a_{2} \gamma} \tilde{x}^T(t - h(t)) R_{i,j} \tilde{x}(t - h(t)).
\]

Equation (A2) is rewritten as the following linear inequality by adding \( z^T(t)z(t) - \gamma^2 \tilde{w}^T(t)w(t) \) to (A2):

\[
\dot{V}(t) + aV(t) + z^T(t)z(t) - \gamma^2 \tilde{w}^T(t)w(t) \leq \Sigma^T(t) \Delta \Sigma(t),
\]

(A3)

where

\[
\Sigma(t) \triangleq \begin{bmatrix} \tilde{x}^T(t) & \tilde{x}^T(t - d(t)) & \tilde{x}^T(t - h(t)) & \tilde{w}^T(t) & \Delta^T(t) \end{bmatrix}^T
\]

and

\[
\Delta_i \triangleq \begin{bmatrix} \Pi_{11,i} & P_{i} \tilde{A}_{2,i} & P_{i} \tilde{A}_{3,i} & P_{i} \tilde{H}_{i} & P_{i} E_{i} & \tilde{A}_{1,i}^T \\ * & \Pi_{22,i} & 0 & 0 & \tilde{A}_{2,i}^T \\ * & * & \Pi_{33,i} & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & -E_{i} & 0 \\ * & * & * & * & -R_{i}^{-1} \end{bmatrix} + \begin{bmatrix} C_{i}^T \\ C_{i}^T \\ C_{i}^T \\ C_{i}^T \\ C_{i}^T \\ C_{i}^T \end{bmatrix} \begin{bmatrix} C_{i}^T \\ C_{i}^T \\ C_{i}^T \\ C_{i}^T \\ C_{i}^T \\ C_{i}^T \end{bmatrix}.
\]

(A4)
If one can prove that $\Delta_i < 0$, which implies $\dot{V}(t) + aV(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq 0$, the robust exponential stability and $H_{\infty}$ control of the system is guaranteed. According to the Schur complement lemma [30], Condition (A4) is equal to

$$
\Xi_i = \begin{bmatrix} \Pi_{11,i} & P_i \tilde{A}_{2,i} & P_i \tilde{A}_{3,i} & P_i \tilde{H}_i & P_i \tilde{E}_i & \tilde{A}_{1,i}^T & \tilde{C}_{1,i}^T \\ * & \Pi_{22,i} & 0 & 0 & 0 & \tilde{A}_{2,i}^T & \tilde{C}_{2,i}^T \\ * & * & \Pi_{33,i} & 0 & 0 & \tilde{A}_{3,i}^T & \tilde{C}_{3,i}^T \\ * & * & * & -\gamma^2 I & 0 & H_{1,i}^T & H_{1,i}^T \\ * & * & * & * & 0 & \tilde{E}_{1,i}^T & 0 \\ * & * & * & * & * & -R_{i,-1} & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} \quad (A5)
$$

Thus,

$$
\dot{V}(t) + aV(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq \Sigma_i(t) \Xi_i(t). \quad (A6)
$$

Now, if one can prove $\Xi_i < 0$, which implies $\dot{V}(t) + aV(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq 0$, the robust exponential stability and $H_{\infty}$ control of the system is guaranteed. Substituting $\tilde{A}_{1,i}, \tilde{A}_{2,i}, \tilde{A}_{3,i}, \tilde{E}, \tilde{C}_{1,i}, \tilde{C}_{2,i}, \tilde{C}_{3,i}$, and $\tilde{H}_i$ into $\Xi_i$ gives

$$
\Xi_i = \begin{bmatrix} (1,1) & (1,2) & (1,3) \\ * & (2,2) & (2,3) \\ * & * & (3,3) \end{bmatrix}, \quad (A7)
$$

where

$$
e_{11} = P_{1,i}(A_{1,i} - L_i C_i) + (A_{1,i} - L_i C_i)^T P_{1,i} + Q_{1,i} + aP_{1,i},
$$

$$
e_{12} = P_{1,i}(A_{1,i} - A_{C,i}) + (L_i C_i)^T P_{2,i},
$$

$$
e_{22} = P_{2,i}(A_{C,i} + B_i K_{1,i}) + (A_{C,i} + B_i K_{1,i})^T P_{2,i} + Q_{2,i} + aP_{2,i},
$$

$$
e_{33} = -(1 - \mu_d)e^{-\alpha d_2} \Omega_{1,i},
$$

$$
e_{44} = -(1 - \mu_d)e^{-\alpha d_2} \Omega_{2,i},
$$

$$
e_{55} = -(1 - \mu_h)e^{-\alpha b_2} \Gamma_{1,i},
$$

$$
e_{66} = -(1 - \mu_h)e^{-\alpha b_2} \Gamma_{2,i},
$$

$$
(1,1) = \begin{bmatrix} e_{11} & e_{12} & P_{1,i} A_{2,i} & P_{1,i} \left(A_{2,i} - A_{D,i}\right) \\ * & e_{22} & P_{2,i} \left(A_{D,i} + B_i K_{2,i}\right) & 0 \\ * & * & e_{33} & 0 \\ * & * & * & e_{44} \end{bmatrix},
$$

$$
(1,2) = \begin{bmatrix} P_{1,i} A_{3,i} & P_{1,i} \left(A_{3,i} - A_{E,i}\right) & P_{1,i} H_i & P_{1,i} E \\ 0 & P_{2,i} A_{E,i} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,3) = \begin{bmatrix} \left(A_{1,i} - L_i C_i\right)^T & \left(L_i C_i\right)^T & \tilde{C}_{1,i}^T \\ \left(A_{1,i} - A_{C,i}\right)^T & \left(A_{C,i} + B_i K_{1,i}\right)^T & \tilde{C}_{2,i}^T \\ \tilde{A}_{2,i}^T & 0 & \tilde{C}_{3,i}^T \\ \left(A_{2,i} - A_{D,i}\right)^T & \left(A_{D,i} + B_i K_{2,i}\right)^T & \tilde{C}_{3,i}^T \end{bmatrix},
$$

$$
(1,4) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,5) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,6) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,7) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,8) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,9) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,10) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,11) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,12) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,13) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,14) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,15) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,16) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,17) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,18) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,19) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
(1,20) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\[
(2, 2) = \begin{bmatrix}
\varepsilon_{55} & 0 & 0 \\
* & \varepsilon_{66} & 0 \\
* & * & -\gamma^2 I \\
* & * & * \\
\end{bmatrix},
\]
\[
(3, 3) = \begin{bmatrix}
-R_{1,i,j}^{-1} & 0 & 0 \\
* & -R_{2,i,j}^{-1} & 0 \\
* & * & -I \\
A_{3,i,j}^T & 0 & C_{3,i,j}^T \\
(A_{3,i,j} - A_{E,i,j})^T & A_{E,i,j}^T \\
H_i^T & 0 & 0 \\
E_i^T & 0 & 0 \\
\end{bmatrix},
\]
\[
(2, 3) = \begin{bmatrix}
\varepsilon_{55} & 0 & 0 \\
* & \varepsilon_{66} & 0 \\
* & * & -\gamma^2 I \\
* & * & * \\
\end{bmatrix}.
\]

On the other hand, noting (6) and (7), we will have
\[
\Delta^T(t)\Delta(t) - Q^TNN^TQ \leq 0,
\] (A8)

where \(Q\) and \(N\) have been defined before (See (8)). Now, according to Lemma 1, if
\[
\hat{V}(t) + aV(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq \Sigma(t), \Sigma(t) < \Delta^T(t)\Delta(t) - Q^TNN^TQ,
\] (A9)

Then, system (3) is exponentially stable. Hence, (A9) is rewritten as the following inequality:
\[
\hat{V}(t) + aV(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) - \Delta^T(t)\Delta(t) - Q^TNN^TQ < 0,
\] (A10)

Now, according to Lemma 1, if
\[
\Omega_1|_{\varepsilon=1} = \Delta^T(t)\Delta(t) - Q^TNN^TQ \leq 0,
\] (A11)

the condition \(\Omega_0(x) = \hat{V}(t) + aV(t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0\) holds if and only if
\[
\Omega_0 - \varepsilon\Omega_1 < 0.
\] (A12)

It can be seen that (A12) is equivalent to (A10). Writing Equation (A12) in the matrix form, we will obtain the inequality (A13), which will guarantee the stability of the system.
\[
\Xi_i = \begin{bmatrix}
N_{1,i,j}^T \\
(N_{1,i,j} + N_{4,i,j}K_{1,i,j})^T \\
N_{2,i,j}^T \\
(N_{2,i,j} + N_{4,i,j}K_{2,i,j})^T \\
N_{3,i,j}^T \\
N_{3,i,j}^T \\
0 \\
0 \\
0 \\
0 \\
* \ldots \ast \\
* \ldots \ast \\
\end{bmatrix} < 0,
\] (A13)

and \(\Xi_i\) is defined above (see (A7)). Pre- and post-multiplying the matrix \(\Xi_i\) in (A13) by \(\Lambda^T\) and \(\Lambda\), where
\[
\Lambda = \text{diag}\left(P_{1,i,j}^{-1}, P_{2,i,j}^{-1}, Q_{1,i,j}^{-1}, Q_{2,i,j}^{-1}, R_{1,i,j}^{-1}, R_{2,i,j}^{-1}, I, I, I, I, I\right),
\]
we will have

\[
\begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\
\ast & \Gamma_{22} & \Gamma_{23} \\
\ast & \ast & \Gamma_{33}
\end{bmatrix},
\]

(A14)

where

\[
\Gamma_{11} = \begin{bmatrix}
\theta_{11} & \theta_{12} & A_{2,i}Q_{1,i}^{-1} & (A_{2,i} - A_{D,i})Q_{2,i}^{-1} & A_{3,i}R_{1,i}^{-1} \\
* & \theta_{22} & 0 & (A_{D,i} + B_iK_{2,i})Q_{2,i}^{-1} & 0 \\
* & * & \theta_{33} & 0 & 0 \\
* & * & * & \theta_{44} & 0 \\
* & * & * & * & \theta_{55}
\end{bmatrix},
\]

\[
\Gamma_{12} = \begin{bmatrix}
(A_{3,i} - A_{E,i})R_{2,i}^{-1} & H_i & E & P_{1,i}^{-1}(A_{1,i} - L_iC_i)^T \\
A_{E,i}R_{2,i}^{-1} & 0 & 0 & P_{1,i}^{-1}(A_{1,i} - A_{C,i})^T \\
0 & 0 & 0 & Q_{1,i}^{-1}A_{1,i}^T \\
0 & 0 & 0 & Q_{2,i}^{-1}(A_{2,i} - A_{D,i})^T \\
0 & 0 & 0 & R_{1,i}^{-1}A_{3,i}^T
\end{bmatrix},
\]

\[
\Gamma_{13} = \begin{bmatrix}
P_{2,i}^{-1}(L_iC_i)^T & P_{1,i}^{-1}C_{1,i} & P_{1,i}^{-1}N_{1,i} \\
P_{2,i}^{-1}(A_{C,i} + B_iK_{1,i})^T & P_{2,i}^{-1}(C_{1,i} + B_iK_{2,i}) & P_{2,i}^{-1}(N_{1,i} + N_{4,i}K_{1,i})^T \\
0 & Q_{1,i}^{-1}C_{2,i} & Q_{1,i}^{-1}N_{2,i}^T \\
Q_{2,i}^{-1}(A_{D,i} + B_iK_{2,i})^T & Q_{2,i}^{-1}(C_{2,i} + B_iK_{2,i}) & Q_{2,i}^{-1}(N_{2,i} + N_{4,i}K_{2,i})^T \\
0 & R_{1,i}^{-1}C_{3,i} & R_{1,i}^{-1}N_{3,i}^T
\end{bmatrix},
\]

\[
\Gamma_{22} = \begin{bmatrix}
\theta_{66} & 0 & 0 & R_{2,i}^{-1}(A_{3,i} - A_{E,i})^T \\
* & -\gamma^2 I & 0 & H_i^T \\
* & * & -I & E^T \\
* & * & * & -R_{1,i}^{-1}
\end{bmatrix},
\]

\[
\Gamma_{23} = \begin{bmatrix}
R_{2,i}^{-1}A_{2,i}^T & R_{2,i}^{-1}C_{3,i} & R_{2,i}^{-1}N_{3,i}^T \\
0 & H_i^T & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
\Gamma_{33} = \begin{bmatrix}
-R_{2,i}^{-1} & 0 & 0 \\
* & -I & 0 \\
* & * & -I
\end{bmatrix},
\]

\[
\begin{align*}
\theta_{11} &= (A_{1,i} - L_iC_i)P_{1,i}^{-1} + P_{1,i}^{-1}(A_{1,i} - L_iC_i)^T + P_{1,i}^{-1}Q_{1,i}P_{1,i}^{-1} + \alpha P_{1,i}^{-1}, \\
\theta_{12} &= (A_{1,i} - A_{C,i})P_{2,i}^{-1} + P_{1,i}^{-1}(L_iC_i)^T, \\
\theta_{22} &= (A_{C,i} + B_iK_{1,i})P_{2,i}^{-1} + P_{2,i}^{-1}(A_{C,i} + B_iK_{1,i})^T + P_{2,i}^{-1}Q_{2,i}P_{2,i}^{-1} + \alpha P_{2,i}^{-1}, \\
\theta_{33} &= -(1 - \mu_d)e^{-\alpha d}Q_{1,i}^{-1}, \\
\theta_{44} &= -(1 - \mu_d)e^{-\alpha d}Q_{2,i}^{-1}, \\
\theta_{55} &= -(1 - \mu_h)e^{-\alpha h_2}R_{1,i}^{-1}, \\
\theta_{66} &= -(1 - \mu_h)e^{-\alpha h_2}R_{2,i}^{-1}.
\end{align*}
\]

The Schur complement lemma is used for $P_{1,i}^{-1}Q_{1,i}P_{1,i}^{-1}$ and $P_{2,i}^{-1}Q_{2,i}P_{2,i}^{-1}$. Furthermore, in view of Lemma 2 the conditions $C_iP_{1,i} = P_{1,i}C_i$ holds where $P_{1,i} = V_i \begin{bmatrix} \hat{P}_{11,i} & 0 \\ 0 & \hat{P}_{22,i} \end{bmatrix} V_i^T.$
By setting \( T_{1,i} = P_{1,i}^{-1}, T_{2,j} = P_{2,j}^{-1}, Q_{1,i} = Q_{1,i}^{-1}, Q_{2,j} = Q_{2,j}^{-1}, R_{1,i} = R_{1,i}^{-1}, R_{2,j} = R_{2,j}^{-1}, \)
\( Y_{1,i} = K_{1,i} T_{1,i}, Y_{2,j} = K_{2,j} Q_{2,j}, W_{1,i} = A_{C,i} T_{2,i}, W_{2,j} = A_{D,j} Q_{2,j}, \)\( W_{3,j} = A_{E,j} R_{2,j}, \)
the matrix inequality \((A14)\) is equivalent to \((14)\). Assuming that Equation \((14)\) is satisfied, by considering \((A5)\), we will have
\[
\dot{V}(t) + \mu V(t) \leq 0 \tag{A15}
\]
Integrating both sides of \((A15)\) from \( t_s \) to \( t \), the following inequality holds:
\[
V(t) \leq e^{-\mu (t-t_s)} V(t_s) \tag{A16}
\]
From \((15)\) and \((1.1)\), at the switching moment \( t_s \), we will have
\[
V(t_s) \leq \mu V(t_s) \tag{A17}
\]
Therefore, from \((A16)\) and \((A17)\), for \( t \in [t_s, t_{s+1}] \), and according to Definition 2, we know \( \rho = N_e(t_0, t) \leq \frac{t-t_0}{\sigma_0} \), \( t_0 = 0 \), then
\[
V(t) \leq e^{-\mu (t-t_s)} V(t_s) \leq \mu e^{-\mu (t-t_s)} V(t_s) \leq \mu^2 e^{-\mu (t-t_s-1)} V(t_{s-1}) \leq \ldots \leq \mu^\rho e^{-\mu (t-t_0)} V(t_0) \leq e^{-\mu (\ln t-t_0)} V(t_0) \tag{A18}
\]
Furthermore, given the definition of the Lyapunov function \((A1)\) and its monotonousness, the following inequalities hold:
\[
a ||\eta (t)||^2 \leq V(t) \leq e^{-\mu (\ln t-t_0)} V(t_0) \leq b ||\eta (t_0)||^2 \tag{A19}
\]
where \( a \) and \( b \) are defined in \((17)\). Then, we will have
\[
||\eta (t)||^2 \leq \frac{1}{a} V(t) \leq \frac{b}{a} e^{-\mu (\ln t-t_0)} ||\eta (t_0)||^2 \tag{A20}
\]
Now, we will establish the \(H_\infty\) performance defined in \((11)\) for the system in \((3)\). Considering \((A6)\), we have
\[
\dot{V}(t) \leq -\alpha V(t) + \gamma^2 w^T(t) w(t) - z^T(t) z(t) \tag{A21}
\]
Integrating \((A21)\) from \( t_s \) to \( t \), the following inequality holds:
\[
\dot{V}(t) \leq e^{-\alpha (t-t_s)} V(t_s) + \int_{t_s}^{t} e^{-\alpha (t-s)} (\gamma^2 w^T(s) w(s) - z^T(s) z(s)) ds
\leq e^{-\alpha (t-t_s)} V(t_s) + \int_{t_s}^{t} e^{-\alpha (t-s)} T(s) ds \tag{A22}
\]
Therefore, it follows from \((A17)\) and \((A22)\) and the inequality \( \rho = N_e(0, t) \leq \frac{t-0}{\sigma_0} \) that
\[
V(t) \leq \mu e^{-\mu (t-t_s)} V(t_s) + \int_{t_s}^{t} e^{-\mu (t-s)} T(s) ds \leq \mu^\rho e^{-\mu t} V(t_0) + \mu^\rho \int_{0}^{1} e^{-\mu (s-t)} T(s) ds
+ \mu^{\rho-1} \int_{0}^{1} e^{-\mu (s-t)} T(s) ds + \ldots + \mu^{\rho-\rho} \int_{0}^{1} e^{-\mu (s-t)} T(s) ds = e^{-\alpha t} + N_e(0, t) \mu V(0) \tag{A23}
\]
Under zero initial condition, \((A23)\) implies
\[
\int_{0}^{t} e^{-\alpha (t-s)} + N_e(s, t) \mu w^T(t) w(t) dt \leq \gamma^2 \int_{0}^{t} e^{-\alpha (t-s)} + N_e(s, t) \mu w^T(t) w(t) dt \tag{A24}
\]
Multiplying both sides of \((A24)\) by \( e^{-N_e(0, t) \mu w^T(t) w(t)} \) yields
\[
\int_{0}^{t} e^{-\alpha (t-s)} - N_e(s, t) \mu w^T(t) w(t) dt \leq \gamma^2 \int_{0}^{t} e^{-\alpha (t-s)} - N_e(s, t) \mu w^T(t) w(t) dt \tag{A25}
\]
Notice that $N_C(0,s) \leq \frac{s}{a}$ and $\tau_a > \tau_d^i = \frac{ln\mu}{\omega}$, we will have $N_C(0,s)ln\mu \leq a$. Thus, (A25) implies
\[
\int_0^t e^{-s(t-s)}z^T(t)z(t)dt \leq \gamma^2 \int_0^t e^{-s(t-s)}w^T(t)w(t)dt,
\]
(A26)
Integrating the above inequality from $t = 0$ to $\infty$ yields (11). The proof is complete at this point. \qed

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