A DYNAMICAL MECHANISM FOR THE SELECTION OF PHYSICAL STATES IN ‘GEOMETRIC QUANTIZATION SCHEMES’

P. Maraner
Center for Theoretical Physics,
Laboratory for Nuclear Science and INFN,
Massachusetts Institute of Technology,
Cambridge, MA 02139-4307, USA

Abstract
Geometric quantization procedures go usually through an extension of the original theory (pre-quantization) and a subsequent reduction (selection of the physical states). In this context we describe a full geometrical mechanism which provides dynamically the desired reduction.

1. THE STANDARD VIEWPOINT ON QUANTIZATION: STATES, OBSERVABLES AND TIME EVOLUTION

The usual way to think of a physical system proceeds in two steps. First: kinematics, that is the specification of the possible states and of the observable quantities. Second: dynamics, that is the description of the time evolution of the representative point of the system over the space of all the possible states. As a very typical example we may think of Hamiltonian mechanics, where the states of a system with \( n \) degrees of freedom are specified by the canonical variables \( q^\mu, p_\mu, \mu = 1, \ldots, n \), while observables are identified with smooth functions of \( q \) and \( p \). Dynamics is obtained by pointing out a privileged observable, namely the energy of the system \( h(q, p) \), by means of the canonical flow generated on the phase space. Let us note that still at this classical level, it turns out that a very few of all the possible observables of the theory play a concrete role in the physical description of the system. Furthermore, the selection of relevant observables goes typically through dynamical considerations, making the sharp separation of kinematics and dynamics an artificial one.

Nevertheless the standard way to look at quantization moves from this viewpoint seeking a correspondence between the formal structures of classical and quantum mechanics: states, observables and time evolution. In order to make the quantization procedure a sensible one, that is capable to reproduce standard quantum mechanics, four conditions are usually required: (Q1) the correspondence is asked to be linear; (Q2) the constant function 1 has to be mapped on to the identity operator; (Q3) the Poisson brackets should become \( i \) times the commutators; and (Q4) the canonical variables \( q \) and \( p \) should act irreducibly on the quantum Hilbert space. There are of course many critiques that can be moved to such an approach. As a matter of fact, it results that it is impossible to quantize the whole algebra of classical observables without violating at
least one of these conditions. The quantization program fails already at the kinematical level. Thought different viewpoints have been expressed in the literature it is the common believe that the weak point of the above construction is that of requiring that such only locally defined objects as the canonical coordinates have to be promoted to globally well defined operators acting irreducibly on the quantum Hilbert space.

Once condition \((Q4)\) is lifted, it is in fact possible to proceed in a very general and elegant manner to the construction of the desired correspondence: Kostant’s pre-quantization scheme \([1]\). The whole huge algebra of smooth functions on the phase space is mapped in the algebra of formally self-adjoint operators on a suitable Hilbert space in such a way that conditions \((Q1), (Q2)\) and \((Q3)\) are fulfilled. The problem is that the pre-quantum Hilbert space is too large for physics. In maintaining such an approach it is therefore necessary to introduce a mechanism capable of selecting the subspace of physical states. Thought many different approaches have been suggested, the general viewpoint is that of picking out a real or complex polarization on the classical phase space and requiring that the physical states are the one preserving the polarization. Such a prescription—definitely of kinematical character—works well as long as the quantization of systems with a high degree of symmetry is concerned \([2]\), but it appears more and more problematic as soon as the dynamics of systems with less symmetry or no symmetry at all is considered. There is in fact no longer guarantee that time evolution respects the polarization, and physical states may evolve in non physical ones.

It is the aim of this paper to present a slightly different approach to the problem focusing more on the dynamical aspects rather than on the kinematical ones \([3]\). Working in a coordinate free manner we will be facing the problem of directly defining the quantum dynamics without going through the quantization of the whole algebra of classical observables. This yields a dynamical mechanism that produces the selection of the right set of physical quantum states.

2. COORDINATE FREE QUANTIZATION

A sensible quantization scheme should not depend on the choice of coordinates. Before discussing quantization let us therefore briefly recall how is possible to formulate Hamiltonian mechanics in a coordinate free manner.

Coordinate Free Formulation of Hamiltonian Mechanics \([4]\): It is convenient to denote phase space coordinates by means of a single variable \(\xi = (q^1,...,q^n,p_1,...,p_n)\). In this canonical coordinate frame we introduce the skew-symmetric two-tensors \(\omega_{ij}\),

\[
\omega_{ij} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},
\]

\(I\) is the \(n\)-dimensional identity matrix, and \(\omega^{ij}\) defined by the relation \(\omega_{ik}\omega^{kj} = \delta^j_i\). The fundamental Poisson bracket may so be recasted in the covariant form \(\{\xi^i,\xi^j\} = \omega^{ij}\). We note that a canonical transformation does not affect the form of \(\omega_{ij}\). Furthermore, the information on the canonical structure being contained in \(\omega_{ij}\), we are now free to introduce arbitrary coordinate frames, not necessarily preserving \(\omega^{ij}\). The phase space of a Hamiltonian system may so be identified with a symplectic manifold, that is a \(2n\)-dimensional manifold \(\mathcal{M}\) equipped with a closed nondegenerate two-form, the symplectic form \(\omega_{ij}\). For a Lagrangian system with configuration space \(Q\) the symplectic manifold \(\mathcal{M}\) have to be identified with the cotangent bundle \(T^*Q\), but this is not the most general case. Many system of physical interest are not included in this class and the discussion of more general phase spaces is necessary. In order to give a coordinate
free formulation of the dynamics of the system we have to introduce the canonical one-
form $\theta_i$, defined by the relation $\omega_{ij} = \partial_i \theta_j - \partial_j \theta_i$. $\theta_i$ is defined up to the total derivative of an arbitrary phase space function, $\theta_i \rightarrow \theta_i + \partial \chi$ (note the formal equivalence of the canonical one and two-forms with a vector potential and a magnetic field!). Dynamics is then defined by means of Hamilton's principle

$$\delta \int (\theta_i \dot{\xi}^i - h(\xi))dt = 0 \quad (2.2)$$

The Geometrical Background of (pre-)Quantization: In a coordinate free
language the problem of pre-quantization may therefore be recasted in the following terms.
For every symplectic manifold $\mathcal{M}$ construct a Hilbert Space $H(\mathcal{M})$ such that it is possible to exhibit a map form the algebra of smooth function on $\mathcal{M}$ into that of the formally self-adjoint operators on $H(\mathcal{M})$ satisfying conditions (Q1), (Q2) and (Q3). This is achieved by identifying $H(\mathcal{M})$ with the Hilbert space of the square integrable sections of the line bundle $L$ on $\mathcal{M}$ having the symplectic two-form $\omega_{ij}$ as curvature form. The correspondence between classical and quantum observables may then be constructed in terms of the covariant derivative on the line bundle. Without going too much into details we recall that for $\mathcal{M} = T^*Q$, in a canonical coordinate frame and fixed the gauge $\theta = (0,...,0, -q^1,...,-q^n)$ the general rule yields

$$p_\mu \rightarrow -i\hbar \frac{\partial}{\partial q^\mu}, \quad (2.3)$$

$$q^\mu \rightarrow i\hbar \frac{\partial}{\partial p_\mu} + q^\mu, \quad (2.4)$$

whereas $H(T^*Q)$ roughly corresponds with the space of square integrable functions of $q$ and $p$, $\psi(q,p)$. In this simple case the selection of the right set of physical states is achieved by requiring the wave functions to be constant in the $p_\mu$ directions, that is $\partial \psi / \partial p_\mu = 0$ for $\mu = 1,...,n$. This is of course a very simple case. Nevertheless it somehow suggest to think of the phase space $\mathcal{M}$ as a sort of configuration space on which the operators $-i\hbar \frac{\partial}{\partial \xi} = (-i\hbar \frac{\partial}{\partial q^\mu}, -i\hbar \frac{\partial}{\partial p_\mu})$ play the role of the canonical momenta conjugate to $\xi = (q^\mu, p_\mu)$. We note that operators $2.3$ and $2.4$ may then be identified with the kinematical momenta of a charged particle moving on $\mathcal{M}$ in the magnetic field $\omega_{ij}$ represented by the vector potential $\theta_i$.

3. A SLIGHTLY DIFFERENT VIEWPOINT:

DYNAMICS AND QUANTIZATION ON $T^*\mathcal{M}$

A slightly different way to look at the pre-quantization scheme may infact be that of considering an extension of the mechanical system from the original phase space $\mathcal{M}$ to the enlarged phase space $T^*\mathcal{M}$ (note: the cotangent bundle of the phase space!). Thought this introduces many ambiguities it has the advantage that the quantization of a cotangent bundle is definitely simpler than the one of an arbitrary symplectic manifold. Ambiguities arise both in the extension and in the subsequent reduction of the system and many different mechanisms may be thought of to achieve the aim. From this perspective Kostant’s pre-quantization represents one of the possible extension schemes and the selection of physical states by means of polarizations is only one of the possible reduction procedures. This way of facing quantization appears as a promising one and has been adopted by many authors (see [3, 4, 5, 6, 7] for a few recent examples). In this context, we present a somehow peculiar quantization procedure constructed in the following way. We first extend classical dynamics from the phase space $\mathcal{M}$
to its cotangent bundle $T^*\mathcal{M}$ by constructing a full geometrical theory depending on the parameter $\hbar$. In the regime of small values of the parameter the theory reduces dynamically to Hamiltonian mechanics. We then proceed to the quantization on the extended theory and note how the same dynamical mechanism provides the reduction to the physical sector of the quantum theory.

**A Full Geometrical Extension of Hamiltonian Mechanics:** We start with a Hamiltonian system with phase space $\mathcal{M}$ and Hamiltonian $h(\xi)$. Thought our theory is covariant in character it is useful to parameterize $\mathcal{M}$ by means of a canonical atlas, so that in every coordinate frame the symplectic structure $\omega_{ij}$ appears in the canonical form [2,7]. We now introduce a metric structure on $\mathcal{M}$ requiring that in every canonical frame the metric determinant satisfies the condition $g(\xi) = h^{-2n}(\xi)$. We finally extend our mechanical system to $T^*\mathcal{M}$ by defining dynamics by means of the variational principle

$$\delta \int \left(\frac{1}{2}\hbar g_{ij} \dot{\xi}^i \dot{\xi}^j + \theta_i \dot{\xi}^i\right) dt = 0. \tag{3.1}$$

$\hbar$ is Plank’s constant over $2\pi$. We claim that the phase space trajectories produced by (3.1) differ from the one produced by Hamilton’s principle only over scales of order $\hbar$. Hamiltonian mechanics may therefore be regarded as the effective theory describing the small $\hbar$ regime of the geometrical theory (3.1). Observe that although $\hbar$ appears into the theory, we are not claiming that (3.1) describes quantum mechanics. We just find it very useful to incorporate Plank’s constant in the extension of the theory to $T^*\mathcal{M}$ in such a way that this parameter controls dynamically the reduction of the extended theory to the original one.

Before proceeding in the demonstration of our claim let us note that the variational principle (3.1) is formally equivalent to that describing the free motion of a particle of mass $\hbar$ on the metric manifold $\mathcal{M}$ in the universal magnetic field represented by the symplectic form $\omega_{ij}$. It is therefore possible to visualize the mechanism responsible for the reduction of our theory by thinking of a particle of mass $m$ and charge $e$ moving in a plane under the influence of a magnetic field of magnitude $B$ normal to the plane. In the analogy the plane represents the phase space of a one-dimensional system while the magnetic field its symplectic structure. The regime of a small mass corresponds to that of a strong magnetic field, or equivalently, to that of a nearly homogeneous one. This problem, sometimes called the guiding center problem, has been extensively discussed in the literature [10]. As long as the magnetic field may be considered as homogeneous the particle follows a circular orbit of radius $r_m = mc|\vec{v}|/eB$ the center of which is motionless. For a very small mass the circle is so narrow that the particle appears at rest. However, as soon as a weak inhomogeneity is introduced the center of the orbit—usually called guiding center—starts drifting on the plane. Moreover, the guiding center motion is Hamiltonian. We shall identify the guiding center motion with the motion of our original system while the rapid rotation around the effective trajectory with the degrees of freedom suppressed by the reduction.

Having this picture in mind we now sketch a formal demonstration. Starting from the Lagrangian $\mathcal{L}(\xi, \dot{\xi}) = \frac{1}{2}\hbar g_{ij} \dot{\xi}^i \dot{\xi}^j + \theta_i \dot{\xi}^i$ of the extended system we proceed to the construction of the relative Hamiltonian formalism by introducing the canonical momenta $p^\xi_i = \partial \mathcal{L}/\partial \dot{\xi}^i$ conjugate to the variables $\xi^i$. The Hamiltonian describing the dynamics of the extended system yields

$$\mathcal{H} = \frac{1}{2\hbar} g^{ij}(\xi)(p^\xi_i - \theta_i)(p^\xi_j - \theta_j), \tag{3.2}$$

where $g^{ij}$ denotes the inverse of the metric tensor. In order to discuss the small $\hbar$ regime of the theory it is very convenient to replace the set of canonical variable $\xi^i, p^\xi_i; i = \ldots$
1, ..., 2n, with the gauge covariant kinematical momenta and guiding center coordinates

\[ \Pi_i = \frac{1}{\hbar^{1/2}} (p_i^\xi - \theta_i) \quad \text{and} \quad X^i = \xi^i + h^{1/2}\omega^{ij}\Pi_j. \]

The new set of coordinates is canonical: \( \Pi_\mu \) is conjugate to \( \Pi_{n+\mu} \) and \( X^{n+\mu} \) to \( X^\mu \), \( \mu = 1, ..., n \). Furthermore, in the new set of variables \( 3.2 \) appears as the Hamiltonian of an \( n \)-dimensional harmonic oscillator with masses and frequencies depending on the parameters \( X^i \) and weakly on the ‘positions’ and ‘velocities’ \( \Pi_i \). Since we are only interested in the small \( \hbar \) regime of the theory it appears natural to expand \( g^{ij} \) in powers of \( \hbar^{1/2} \),

\[ \mathcal{H} = \frac{1}{2} g^{ij}(X)\Pi_i\Pi_j + \mathcal{O}(\hbar^{1/2}) \]

(3.3)

which makes clear that only the dependence on the \( X^i \) is relevant. A further analysis of the commutation relation makes it clear that the \( X^i \) may be regarded as slow parameter of the system so that it is possible to perform a second canonical transformation (see \( 3 \) for details) bringing \( 3.3 \) into the form

\[ \mathcal{H} = h(X) \frac{1}{2} \sum_i \Pi_i \Pi_i + \mathcal{O}(\hbar^{1/2}). \]

(3.4)

The condition \( g = h^{-2n} \) has been used. The guiding center motion described by the set of canonical coordinates \( X^i \) and the rapid rotation of the system around the guiding center trajectory are separated up to terms of order \( \hbar^{1/2} \). Our demonstration is completed by observing that the radius of the circular phase space trajectory described by the \( \Pi_i \) is of order \( \hbar \). The effective dynamics produced by the full geometrical variational principle \( 3 \) corresponds therefore to Hamiltonian dynamics.

Quantizing Free Dynamics on \( T^*\mathcal{M} \): The quantization of the extended dynamical system \( 3.1 \) proceeds in a straightforward manner. It is in fact equivalent to the quantization of a particle moving on a curved manifold in an external magnetic field. Thought affected by ordering ambiguities arising from the non trivial geometry the solution of this problem has been extensively discussed in the literature \( 11 \). A little care has to be taken when the topology of the problem is non trivial. The Hilbert space of the system has to be constructed as that of square integrable sections of a line bundle \( L \) over the ‘configuration space’ \( \mathcal{M} \) having the magnetic field \( \omega_{ij} \) as curvature form. The same mathematical framework of pre-quantization is therefore recovered by means of the analogy of our theory with a magnetic system. As an example Kostant’s quantization condition ensuring the existence of the line bundle \( L \)

\[ \int_\Sigma \omega = 2\pi n, \]

\( \Sigma \) an arbitrary compact surface in \( \mathcal{M} \) and \( n \) an integer, reappears as the Dirac’s condition on monopole charge. In any coordinate frame the quantum Hamiltonian describing the theory is given by

\[ \mathcal{H} = \frac{1}{g^{1/2}} \Pi_ig^{ij}g^{1/2}\Pi_j + h\mathcal{I}_1 + h^2\mathcal{I}_2 + ... \]

(3.5)

where the kinematical momenta \( \Pi_i = -i\hbar^{1/2}\partial_i - \theta_i/h^{1/2} \) have been introduced and \( \mathcal{I}_1, \mathcal{I}_2, ... \) are ‘optional’ invariants reflecting the ordering ambiguities inherent the quantization procedure. It is worthwhile to stress that \( \mathcal{H} \) is a globally well defined operator on the Hilbert space of the theory.
Once quantization has been performed the same mechanism producing the reduction of the classical theory to Hamiltonian mechanics provides dynamically the reduction to the physical sector of the quantum theory. Depending only on the canonical formalism the argument goes exactly as the one in classical case and will not be repeated. By introducing the guiding center operators 

$$X^i = \xi^i + \frac{\hbar}{2} \tilde{\omega}^{ij} \Pi_j$$

we obtain a set of operators fulfilling the canonical commutation relations 

$$[X^\mu, X^{\nu+n}] = i\hbar \delta^{\mu
u}$$

and 

$$[\Pi_\mu, \Pi_{\nu+n}] = -i\delta_{\mu\nu}, \mu, \nu = 1, \ldots, n.$$ 

Up to irrelevant terms the dynamics of the physical variables \(X^i\) separates from that of the \(\Pi_i\) and Hamiltonian \(\mathcal{H}\) decomposes as in \(\mathcal{H}\). The energy necessary to induce a transition in the spectrum of the fast variables \(\Pi\) being of order \(1\)—to be compared with \(\hbar\)—the system behaves as frozen in one of the harmonic oscillator eigenstates of 

$$\frac{1}{2} \sum_i \Pi_i \Pi_i$$

and dynamics is effectively reduced to the physical sector described by the \(X\).

References

[1] N. Woodhouse, “Geometric Quantization”, Clarendon Press, Oxford (1980)
[2] E. Onofri, J. Math. Phys. 17, 401, (1976)
[3] The present review is based on J. R. Klauder and P. Maraner, Ann. Phys. 253, 356, (1997)
[4] R. Abraham and J. E. Marsden, “Foundation of Mechanics”, Benjamin, London (1978); V. I. Arnold, “Mathematical Methods of Classical Mechanics”, Springer-Verlag, New York (1978)
[5] J. R. Klauder, Ann. Phys. 188, 120, (1988); J. R. Klauder and E. Onofri, Int. J. Mod. Phys. A4, 3930, (1989); P. Maraner, Mod. Phys. Lett. A7, 2555, (1992); R. Alicki, J. R. Klauder and J. Lewandowski, Phys. Rev. A48, 2538 (1993)
[6] B. Fedosov, J. Diff. Geo. 40, 213, (1994)
[7] E. S. Fradkin and V. Ya. Linetsky, Nuc. Phys. B413, 569, (1994);
[8] E. Gozzi, Phys. Lett. A202, 330, (1995); UTS-DFT-96-03 (1996)
[9] G. Jorjadze, hep-th/9606162 (1996);
[10] T. G. Northrop, “The Adiabatic Motion of Charged Particles”, Interscience, New York (1963); C. S. Gardner, Phys. Rev. 115, 791, (1959); E. Witten, Ann. Phys. 120, 72, (1979); R. G. Littlejohn, J. Math. Phys. 12, 2445, (1979); R. G. Littlejohn, in: “Contemporary Mathematics Vol.28”, J. E. Marsden, ed., American Mathematical Society (1984); P. Maraner, J. Phys. A: Math. Gen. 29, 2199, (1996)
[11] T. T. Wu and C. N. Yang, Phys. Rev. D12, 3845, (1975); C. N. Yang, in: “Understanding the Fundamental Constituents of Matter”, A. Zichichi, ed., Plenum, New York (1978); M. Daniel and C. M. Viallet, Rev. Mod. Phys. 52, 175, (1980)