On string density at the origin

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Abstract. In [V. Barcilon Explicit solution of the inverse problem for a vibrating string. J. Math. Anal. Appl. 93 (1983) 222-234] two boundary value problems were considered generated by the differential equation of a string

$$y'' + \lambda p(x)y = 0, \quad 0 \leq x \leq L < +\infty$$

with continuous real function $p(x)$ (density of the string) and the boundary conditions $y(0) = y(L) = 0$ the first problem and $y'(0) = y(L) = 0$ the second one. In the above paper the following formula was stated

$$p(0) = \frac{1}{L^2 \mu_1} \prod_{n=1}^{\infty} \frac{\lambda_n^2}{\mu_n \mu_{n+1}}$$

where $\{\lambda_k\}_{k=1}^{\infty}$ is the spectrum of the first boundary value problem and $\{\mu_k\}_{k=1}^{\infty}$ of the second one. Rigorous proof of (**) was given in [C.-L. Shen On the Barcilon formula for the string equation with a piecewise continuous density function. Inverse Problems 21, (2005) 635–655] under more restrictive conditions of piecewise continuity of $p'(x)$.

In this paper (**) was deduced using

$$p(0) = \lim_{\lambda \to +\infty} \left( \frac{\phi(L, -\lambda)}{\lambda^2 \psi(L, -\lambda)} \right)^2$$

where $\phi(x, \lambda)$ is the solution of (*) which satisfies the boundary conditions $\phi(0) - 1 = \phi'(0) = 0$ and $\psi(x, \lambda)$ is the solution of (*) which satisfies $\psi(0) = \psi'(0) - 1 = 0$.

In our paper we prove that (**) is true for the so-called M.G. Krein’s string which may have any nondecreasing mass distribution function $M(x)$ with finite nonzero $M'(0)$. Also we show that (**) is true for a wide class of strings including those for which $M(x)$ is a singular function, i.e. $M'(x) \equiv 0$.

Keywords: spectral function, Dirichlet boundary condition, Neumann boundary condition, regular string, singular string, Krein’s string.

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1. Introduction

This paper is stimulated by series of papers [1], [2], [3] on the so-called Barcilon formula. In [1] V. Barcilon considered the differential equation
\[ y'' + \lambda p(x)y = 0, \quad 0 \leq x \leq L < +\infty, \]  
(1.1)
where \( \lambda \) is the spectral parameter, the function \( p(x) \) is continuous and \( p(x) > 0 \) on \([0, L]\). This equation describes small transversal vibrations of a string of linear density \( p(x) \) stretched by a unite force, \( L \) is the length of the string. In [1] the spectrum of the boundary value problem generated by (1.1) and the boundary conditions
\[ y'(0) = y(L) = 0 \]
was denoted by \( \{\mu_n\}_{n=1}^\infty \) and the spectrum of the boundary value problem generated by (1.1) and the boundary conditions
\[ y(0) = y(L) = 0 \]
was denoted by \( \{\lambda_n\}_{n=1}^\infty \). There a method of recovering \( p(x) \) using \( L, \{\mu_n\}_{n=1}^\infty \) and \( \{\lambda_n\}_{n=1}^\infty \) was proposed under a strange condition that the given data are '... such that the solution \( p(x) \) of the inverse problem for the vibrating string exists and is continuous and bounded away from the zero...' (see Theorem 2 in [1]). In particular, there was established a formula (following C.-L. Shen [2] we call it Barcilon formula) expressing \( p(0) \) via \( L, \{\mu_n\}_{n=1}^\infty \) and \( \{\lambda_n\}_{n=1}^\infty \):
\[ p(0) = \frac{1}{L^2 \mu_1} \prod_{n=1}^\infty \frac{\lambda_n^2}{\mu_n \mu_{n+1}}. \] (1.2)
The proof in [1] is not correct. Rigorous proof of this formula was given by C.-L. Shen [2] under more restrictive conditions of piecewise differentiability of \( p(x) \) with \( p'(x) \) having finite number of discontinuities.

In the context of the proof of (1.2) in [2] a formula was obtained equivalent to the following one
\[ p(0) = \lim_{z \to +\infty} \left( \frac{\phi(L, -z)}{z^2 \psi(L, -z)} \right)^2, \]
where \( \phi(x, \lambda) \) and \( \psi(x, \lambda) \) \((0 \leq x \leq L)\) denote solutions of a differential equation more general than (1.1) (see (3.2) below) which satisfy the conditions \( \phi(0, \lambda) = \psi_+(0, \lambda) = 1, \phi_+(0, \lambda) = \psi(0, \lambda) = 0 \).

The aim of this paper is to express the value at \( x = 0 \) of the right derivative of the mass distribution function \( M(x) \) of the string via the same data \( L, \{\mu_n\}_{n=1}^\infty \) and \( \{\lambda_n\}_{n=1}^\infty \) not imposing any restrictions of continuity on the density \( p(x) = M'(x) \). Moreover, for a wide class of strings we prove Barcilon formula (1.2) despite \( \frac{dM}{dx} \equiv 0 \) on \([0, L]\) (see Theorem 5.2 and 5.3). Since \( M(x) \neq \text{const} \), this means that \( M(x) \) is not continuous. In this purpose we need first to describe some results on the spectral theory of the string with a regular left end (M. G. Krein’s string) and, for this, some results on the so-called \( R \)-functions.
2. Classes of functions \((R), (S)\) and \((S^{-1})\)

2.1. Class \((R)\)

According to the terminology of [4] a function \(f(z)\) of a complex variable \(z\) is said to be an \(R\)-function or to belong to the class \((R)\) if

1) it is defined and holomorphic in each of the half-planes \(\text{Im} z > 0\) and \(\text{Im} z < 0\),
2) \(f(\overline{z}) = f(z)\) \((\text{Im} z \neq 0)\),
3) \(\text{Im} \, f(z) \geq 0\) \((\text{Im} z \neq 0)\).

Such a function is also often called \(Nevanlinna\) function.

It is easy to see that an \(R\)-function \(f(z)\) can attain a real value for nonreal \(z\) if and only if it is a real constant. Such an \(R\)-function is said to be degenerate. A nondegenerate \(R\)-function maps the open upper (lower) half-plane into itself.

A function \(f\) is an \(R\)-function if and only if it can be represented as (see [4])

\[
f(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\tau(\lambda), \tag{2.1}
\]

where \(\beta \geq 0\), \(\alpha \in \mathbb{R}\) and \(\tau\) is a nondecreasing function such that

\[
\int_{-\infty}^{\infty} \frac{d\tau(\lambda)}{1 + \lambda^2} < \infty,
\]

what guarantees absolute convergence of the integral in (2.1). We will normalize \(\tau\) in representation (2.1) of an \(R\)-function (or a function belonging to a subclass of \((R)\), see below) as follows

\[
\tau(\lambda) = \frac{1}{2}(\tau(\lambda + 0) + \tau(\lambda - 0)) \quad \forall \lambda \in \mathbb{R}, \quad \tau(0) = 0.
\]

Under such normalization \(\tau(\lambda)\) which we will call the spectral function for \(R\)-function \(f(z)\) is uniquely determined by the Stietjes inversion formula

\[
\tau(\lambda) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{0}^{\lambda} \text{Im} f(t + i\epsilon) dt \quad \forall \ \lambda \in \mathbb{R}.
\]

The constant \(\beta\) can be obtained as

\[
\beta = \lim_{\eta \to +\infty} \frac{\text{Im} f(i\eta)}{\eta}.
\]

Each \(R\)-function consists of two functions one of which (upper part) is holomorphic in the open upper half-plane while the other (lower part) is holomorphic in the lower half-plane. In general case the lower part is not an analytic continuation of the upper part. However, if there exists an interval \((a, b)\) where the spectral function \(\tau(\lambda)\) is constant then the integral in the right-hand side of (2.1) exists not only for nonreal \(z\) but also for \(z \in (a, b)\) and attains real values there. In this case the lower part of the \(R\)-function is the analytic continuation of the upper part and this continuation attains real values on \((a, b)\). In the sequel we will deal only with \(R\)-functions defined not only...
On strig density at the origin

for \( \mathbb{C} \setminus \mathbb{R} \) but also on intervals of \( \mathbb{R} \) where \( \tau(\lambda) \) is constant. It should be mentioned that each nondegenerate \( R \)-function is monotonically increasing on the intervals where its spectral function is constant. Also it is known (see \[4\], Sec. 2, Subsec. 3) that if in all points \( \lambda \in (a, b) \subset \mathbb{R} \)

\[
\lim_{\epsilon \to +0} \frac{1}{\pi} \text{Im} f(\lambda + i\epsilon) = g(\lambda)
\]

and the function \( g(\lambda) \) is bounded on \( (a, b) \) then the spectral function \( \tau(\lambda) \) of the \( R \)-function \( f(\lambda) \) is absolutely continuous on \( (a, b) \) and \( \tau'(\lambda) \overset{a.e.}{=} g(\lambda) \).

2.2. Class \((S)\)

A function \( f \) is said to be an \( S \)-function or to belong to the class \((S)\) if

1) \( f \in (R) \),
2) \( f \) is holomorphic in \( \text{Ext}[0, \infty) (= \mathbb{C} \setminus [0, \infty)) \),
3) \( f(z) \geq 0 \) for all \( z \in (-\infty, 0) \).

As it follows from Subsection 2.1 the spectral function \( \tau(\lambda) \) of an \( S \)-function \( f(\lambda) \) is constant on \( (-\infty, 0) \). Moreover, it is known (\[4\], Sec.5, Theorem S.1.5.1) that

\[
\int_{-\infty}^{\infty} \frac{d\tau(\lambda)}{1+\lambda} < \infty,
\]

and representation (2.1) can be reduced to

\[
f(z) = \gamma + \int_{-\infty}^{\infty} \frac{d\tau(\lambda)}{\lambda - z}
\]

(2.2)

where \( \gamma \geq 0 \).

2.3. Class \((S^{-1})\)

A function \( f \) is said to be an \( S^{-1} \)-function or to belong to the class \((S^{-1})\) if

1) \( f \in (R) \),
2) \( f \) is holomorphic in \( \text{Ext}[0, \infty) (= \mathbb{C} \setminus [0, \infty)) \),
3) \( f(z) < 0 \) for all \( z \in (\infty, 0) \).

As it follows from Subsection 2.1 the spectral function \( \tau(\lambda) \) of an \( S^{-1} \)-function \( f(z) \) is constant on \( (-\infty, 0) \) and Theorem S.1.5.2 in \[4\] (\[4\], Sec.5) implies \( \tau(+0) = \tau(-0) \) and

\[
\int_{-\infty}^{\infty} \frac{d\tau(\lambda)}{\lambda + \lambda^2} < \infty.
\]

Representation (2.1) can be reduced to

\[
f(z) = \alpha + \beta z + \int_{+0}^{\infty} \left( \frac{1}{\lambda - z} - \frac{1}{\lambda} \right) d\tau(\lambda)
\]

where \( \alpha \leq 0 \). It was shown in proof of Theorem S.1.5.2 in \[4\] that if \( f(z) \) is holomorphic at \( z = 0 \) then \( \alpha < 0 \) if and only if \( f(0) \neq 0 \).
3. Main theses of the spectral theory of strings

3.1. Sting and classification of its ends

Let $I$ be an interval of one of the kinds $(a, b)$, $(a, b]$, $[a, b)$ or $[a, b]$ where $-\infty \leq a < b \leq \infty$ (in the case of $a \in I$ or $b \in I$ we have $a > -\infty$ and $b < +\infty$, respectively). Let $M(x)$ ($M(x) < \infty$ for all $x \in I$) be a nondecreasing function on $I$ which can have jumps, intervals of constant value, absolutely continuous, singular continuous parts. We set $a_0 =: \inf \mathcal{F}_M$, $b_0 =: \sup \mathcal{F}_M$, where $\mathcal{F}_M$ is the set of points of growth of $M(x)$. We assume that $\mathcal{F}_M$ is an infinite set of points. Let us associate with $I$ and $M$ a string $S(I, M)$ with the mass distribution described by $M(x)$ in sense that $M(x_2 + 0) - M(x_1 - 0)$ is the mass of the part of the string located on $[x_1, x_2]$ for each $x_1, x_2 \in I$ and $x_1 \leq x_2$ (here we assume $M(a - 0) = M(a)$ if $a \in I$ and $M(b + 0) = M(b)$ if $b \in I$). The left (right) end of the string $S(I, M)$ is said to be regular if $a_0 > -\infty$, $M(a) > -\infty$ ($b_0 < \infty$, $M(b) < \infty$). In the opposite case the end is said to be singular. A regular end is said to be completely regular if $a \in I$ ($b \in I$). A string with both ends regular is said to be a regular string. In the opposite case it is said to be a singular string. If $a \notin I$ ($b \notin I$) we set

$$M(a) = \lim_{x \to a+0} M(x), \quad M(b) = \lim_{x \to b-0} M(x).$$

3.2. Differential operation and differential equation of a sting

In this paper we are dealing with strings each completely regular end of which does not bear a point mass. Therefore, we use a definition of the differential operation $l_{I, M}$ of the string $S(I, M)$ which fits to this situation.

**Remark** In the general case one should introduce a notion of associate derivatives: the 'left derivative' $f^l_-(a)$ if the left end of $I$ is completely regular and the 'right derivative' $f^l_+(b)$ if the right end is completely regular.

**Definition 3.1** Let $\mathcal{D}_{I, M}$ be the set of all functions $f(x)$ defined on $I$ such that 1) $f$ is locally absolutely continuous on $I$, 2) there exist finite left $f_+(x)$ and right $f_-(x)$ derivatives at each interior point $x$ of $I$, 3) there exists $M$-measurable function $g$ such that for any two points $x_1, x_2 \in I$ ($x_1 \leq x_2$)

$$f^l_+(x_2) - f^l_-(x_1) = - \int_{x_1+0}^{x_2+0} g(x) dM(x) \quad (3.1)$$

for each of four combinations of signs the same in both sides of (3.1) for the same $x_j$ ($j = 1, 2$).

For a function $f \in \mathcal{D}_{I, M}$ we set $l_{M, I}[f](x) = g(x)$ where $g$ is the function involved in (3.1).

**Remark** We have defined $l_{M, I}[f](x)$ up to equivalence with respect to the $M$-measure. It is clear that for $f \in \mathcal{D}_{I, M}$

$$l_{I, M}[f](x) = - \frac{d}{(d)M(x)} f_+(x) = - \frac{d}{(d)M(x)} f_-(x)$$
at $M$-almost all $x$. Here $\frac{d}{(d)M(x)}$ is the symbol of the symmetric derivative with respect to $M$.

We call differential equation of the string $S(I, M)$

$$l_{I, M}[y] - \lambda y = 0 \quad (x \in I),$$

(3.2)

where $\lambda$ is the spectral parameter. A function $u \in D_{I, M}$ is said to be a solution of (3.2) if $l_{I, M}[u](x) - \lambda u(x) = 0$ for $M$-almost all $x \in I$.

3.3. M. G. Krein’s strings $S_1(I, M)$ and $S_0(I, M)$

We deal with strings $S(I, M)$ the left ends of which are completely regular (for convenience we place them at $x = 0$) while the right ends $x = L$ are either regular and then completely regular with $L < +\infty$ or singular and then $L \leq +\infty$ and $L \notin I$. In present paper we assume that $\inf \mathcal{F}_M = 0$ and, as it was mentioned above, completely regular ends of strings do not bear point masses.

A string $S(I, M)$ is said to be $S_1(I, M)$ if its left end is free to move without friction in the direction orthogonal to $x$-axis, i.e. to the equilibrium position of the string. By $S_0(I, M)$ we denote a string $S(I, M)$ with the left end fixed. We assume that if the right end of a string $S_1(I, M)$ or $S_0(I, M)$ is (completely) regular then it is fixed.

We define fundamental functions $\phi(x, \lambda)$ and $\psi(x, \lambda)$ of strings $S_1(I, M)$ and $S_0(I, M)$, respectively, as the solutions of equation (3.2) which satisfy the initial conditions $\phi(0, \lambda) = 1$, $\phi'_+(0, \lambda) = 0$ and $\psi(0, \lambda) = 0$, $\psi'_+(0, \lambda) = 1$.

It is known (see [5], Sec. 2) that for any fixed $x \in I$ the functions $\phi(x, \lambda)$, $\psi(x, \lambda)$, $\phi'_-(x, \lambda)$, $\psi'_-(x, \lambda)$, $\phi'_+(x, \lambda)$ and $\psi'_+(x, \lambda)$ are entire real functions of $\lambda$ of order not more than $1/2$.

**Remark** A meromorphic in $\mathbb{C}$ or an entire function is said to be real if it attains real values for real values of variable.

Since $\phi(x, 0) = 1$ and $\psi(x, 0) = x$, for each fixed $x \in I$ the following representations are valid:

$$\phi(x, \lambda) = \prod_j \left(1 - \frac{\lambda}{\mu_j(x)}\right), \quad \psi(x, \lambda) = x \prod_j \left(1 - \frac{\lambda}{\lambda_j(x)}\right),$$

where $\mu_j(x)$, $\lambda_j(x)$, $j = 1, 2, ...$ are zeros of entire in $\lambda$ functions $\phi(x, \lambda)$ and $\psi(x, \lambda)$, respectively.

Usually, the set of squares of frequencies of free vibrations of a regular string is called its spectrum. The spectrum depends on the mass distribution and on the ways of connection of its ends. Therefore, we mean by spectra of strings $S_1(I, M)$ and $S_0(I, M)$ the sets of eigenvalues of the boundary value problems

$$l_{I, M}[y] - \lambda y = 0, \quad y'_+(0) = y(L) = 0,$$

(3.3)

$$l_{I, M}[y] - \lambda y = 0, \quad y(0) = y(L) = 0,$$

(3.4)

respectively. It is easy to see that the spectrum $\{\mu_j\}_{j=1}^{\infty}$ of a completely regular string $S_1(I, M)$ is the set of zeros of the entire function $\phi(L, \lambda)$ and the spectrum $\{\lambda_j\}_{j=1}^{\infty}$ of
the completely regular string $S_0(I, M)$ is the set of zeros of $\psi(L, \lambda)$ with the same $I$ and $M$, respectively. This is in accordance with a general definition of $S_0(I, M)$ and $S_1(I, M)$ strings spectra given below.

3.4. Spectral functions of strings $S_1(I, M)$ and the coefficient of dynamic compliance

Let us denote by $L^2_M(I)$ the set of $M$-measurable functions $f(x)$ such that

$$\|f\|^2_M := \int_I |f(x)|^2 dM(x) < \infty.$$ 

We denote by $\hat{L}^2_M(I)$ the set of functions $f \in L^2_M(I)$ which are identically zero in some neighborhood of $x = L$ if the right end is singular. If the right end is regular then $\hat{L}^2_M(I) = L^2_M(I)$.

**Definition 3.1** A nondecreasing on $(-\infty, \infty)$ function $\tau(\lambda)$ normalized by the conditions

$$\tau(\lambda) = \frac{1}{2}(\tau(\lambda + 0) + \tau(\lambda - 0)) \forall \lambda \in (-\infty, \infty), \quad \tau(0) = 0,$$

is said to be a spectral function of the string $S_1(I, M)$ ($S_0(I, M)$) if the mapping $U : f \rightarrow F$ where $f \in \hat{L}^2_M(I)$ and

$$F(\lambda) = \int_0^L f(x)\phi(x, \lambda) dM(x) \quad (F(\lambda) = \int_0^L f(x)\psi(x, \lambda) dM(x))$$

maps isometrically $\hat{L}^2_M(I)$ into $L^2(\tau, -\infty, \infty)$, i.e. if for each function $f \in \hat{L}^2_M(I)$ the 'Parceval identity' is true:

$$\int_{-\infty}^{\infty} |F(\lambda)|^2 d\tau(\lambda) = \int_I |f(x)|^2 dM(x),$$

where $F = Uf$. A spectral function is said to be orthogonal if $U$ maps $\hat{L}^2_M(I)$ into a dense part of $L^2(\tau, -\infty, \infty)$. The set of points of growth of a spectral function is said to be the spectrum of it.

The function

$$T(z) := \lim_{x \to L^-} \frac{\psi(x, z)}{\phi(x, z)}, \quad z \in (\mathbb{C}\setminus[0, +\infty))$$

is said to be the coefficient of dynamic compliance of the string $S_1(I, M)$. If the right end $x = L$ is completely regular then $T(z) = \frac{\psi(L, z)}{\phi(L, z)}$ is a meromorphic function. In any case $T(z)$ is an $S$-function.

Being an $S$-function $T(z)$ has the spectral function $\tau^{(1)}(\lambda)$ which is constant on $(-\infty, 0)$. Since $\inf \mathcal{F}_M = 0$, we have (see [4], Sec.5, [5], Sec.10) instead of general representation (2.2):

$$T(z) = \int_{-\infty}^{+\infty} \frac{d\tau^{(1)}(\lambda)}{\lambda - z}, \quad z \in (\mathbb{C}\setminus[0, +\infty)),$$
On string density at the origin

We keep the norming
\[ \tau^{(1)}(\lambda) = \tau^{(1)}(-0) \quad \text{for} \quad \lambda < 0, \]
\[ \tau^{(1)}(\lambda) = \frac{1}{2}(\tau^{(1)}(\lambda + 0) + \tau^{(1)}(\lambda - 0)) \quad \forall \lambda \in \mathbb{R}, \quad \tau^{(1)}(0) = 0, \quad (3.7) \]
usual for \( R \)-functions (remind that \( (S) \subset (R) \)). Being normed this way the function \( \tau^{(1)}(\lambda) \) is a spectral function of the string \( S_1(I, M) \) (see [5], Sec.3 Main Theorem and Sec.10, Theorem 10.1). This spectral function of the string \( S_1(I, M) \) is called main spectral function. The main spectral function is orthogonal. Notice that
\[
\int_{-\infty}^{+\infty} \frac{d\tau^{(1)}(\lambda)}{1 + \lambda} < \infty.
\]

In case of singular string, \( \tau^{(1)}(\lambda) \) is its unique spectral function with nonnegative spectrum. The spectrum of the main spectral function of a string \( S_1(I, M) \) is said to be the spectrum of this string.

3.5. The main spectral function of the string \( S_0(I, M) \)

The function \( -\frac{1}{T(z)} \) is an \( S^{-1} \)-function. In our case it has a unique representation of the form ([5], Sec. 12)
\[
-\frac{1}{T(z)} = -\frac{1}{L} + \int_{0}^{+\infty} \left( \frac{1}{\lambda - z} - \frac{1}{\lambda} \right) d\tau^{(0)}(\lambda)
\]
where \( \tau^{(0)}(\lambda) \) is a nondecreasing function, normalized by (3.7) with \( \tau^{(0)}(\lambda) \) instead of \( \tau^{(1)}(\lambda) \). This function is a spectral function of the string \( S_0(I, M) \) generated by the same string \( S(I, M) \). It is said to be the main spectral function of this string \( S_0(I, M) \) and its spectrum to be the spectrum of this string.

3.6. Kasahara’s theorem

It was shown in [6] that if a string \( S_1(I, M) \) is such that for some fixed \( \alpha > 0 \) there exists a nonzero finite limit
\[
\lim_{x \to +0} \frac{M(x)}{x^{\alpha}}
\]
then there exist the limits
\[
\lim_{z \to +\infty} T(-z)z^{\frac{\alpha}{\alpha + 1}}, \quad \lim_{\lambda \to +\infty} \tau^{(1)}(\lambda)\lambda^{-\frac{\alpha}{\alpha + 1}}
\]
and the following is true
\[
\lim_{x \to +0} \frac{M(x)}{x^{\alpha}} = \left( B(\alpha) \Gamma\left(\frac{1}{\alpha + 1}\right) \Gamma\left(\frac{2\alpha + 1}{\alpha + 1}\right) \right)^{\alpha + 1} \left( \lim_{z \to +\infty} T(-z)z^{\frac{\alpha}{\alpha + 1}} \right)^{-(\alpha + 1)}.
\]
\[
= \left( B^{-1}(\alpha) \lim_{\lambda \to +\infty} \tau^{(1)}(\lambda)\lambda^{-\frac{\alpha}{\alpha + 1}} \right)^{-(\alpha + 1)}.
\]
where $\Gamma$ is Euler’s gamma-function and
\[
B(\alpha) = \left( \frac{\alpha}{(\alpha+1)^2} \right)^{\mu+1} \Gamma^{-2} \left( \frac{2\alpha+1}{\alpha+1} \right).
\] (3.11)

For $\alpha = 1$ equations (3.10) can be reduced to
\[
\lim_{x \to +0} \frac{M(x)}{x} = \left( \lim_{z \to +\infty} (T(-z)z^{1/2}) \right)^{-2} = \left( \frac{\pi}{2} \lim_{\lambda \to +\infty} \tau^{(1)}(\lambda)\lambda^{-1/2} \right)^{-2}. \tag{3.12}
\]

**Remark** Equations (3.10) remain true if $\tau^{(1)}(\lambda)$ is changed for any other spectral function of the same string (see [6], Lemma 6).

In the case of a string $S(I, M)$ with a regular right end we have $T(z) = \frac{\psi'(L,z)}{\theta(L,z)}$. Therefore, the first equation in (3.12) is an analogue of the one which is also called Barcilon in [3].

Some years after [6], a paper by Kasahara [7] appeared where the results of [6] were generalized and inverted. In particular, Theorem 2 in [7] implies existence and being finite and nonzero of any of the limits in (3.9) guarantees existence of another limit in (3.8) and also validity of (3.10). For $\alpha = 1$ it means that if one of the three limits in (3.12) exists and is finite and nonzero then the two other also exist and (3.12) is true.

It should be noticed that in contradistinction to [1], [2], [3] the results in [6] and [7] were obtained without any assumption of continuity or piecewise differentiability of the density of the string. By the way, the first limit in (3.12), i.e. the right derivative of the function $M(x)$ at $x = 0$, i.e. the density of the string at $x = 0$ can exist and be finite and nonzero even in the case where $M(x)$ is a pure jump function. Example of such a function can be easily constructed.

**3.7. Main spectral function of the string $S_1(I, M)$ and its length (length of the interval $I$)**

If $\tau^{(1)}(\lambda)$ is the main spectral function of a string $S_1(I, M)$ (regular or singular) then
\[
\int_{-0}^{+\infty} \frac{d\tau^{(1)}(\lambda)}{\lambda} = L \tag{3.13}
\]
in both cases of finite and infinite $L$. This result was obtained by M.G. Krein [8].

It should be mentioned that if for some $\epsilon > 0$ the interval $[0, \epsilon)$ has zero $\tau^{(1)}$-measure then the integral in (3.13) is finite and (3.6) implies $T(0) = L < \infty$.

**4. Relation between the discrete spectra of the strings $S_1(I, M)$ and $S_0(I, M)$ generated by the same string $S(I, M)$ and behavior of $M(x)$ at $x \to +0$**

**4.1. Spectral function via two spectra**

Let a string $S(I, M)$ have the length $L < \infty$, the string $S_1(I, M)$ generated by $S(I, M)$ have discrete spectrum $\{\mu_k\}_{k=1}^{\infty}$ where $0 < \mu_1 < \mu_2 < ...$ and let $\{\lambda_k\}_{k=1}^{\infty}$ where
On string density at the origin

\( \lambda_1 < \lambda_2 < \ldots \) be the spectrum of the string \( S_0(I, M) \) generated by the same string \( S(I, M) \). It is known that these spectra interlace:

\[
0 < \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots
\]

Kasahara’s theorem mentioned above shows that information about the behavior of \( M(x) \) at \( x \to +0 \) can be extracted from behavior of the main spectral function \( \tau^{(1)}(\lambda) \) of the string \( S_1(I, M) \) at \( \lambda \to +\infty \). To express \( \tau^{(1)}(\lambda) \) through \( L \), \( \{\mu_k\}_{k=1}^\infty \) and \( \{\lambda_k\}_{k=1}^\infty \) we will use the following theorem:

**Theorem 4.1** The function \( T(z) \) defined by (3.6) admits representation

\[
T(z) = L \prod_{k=1}^{\infty} \frac{1 - \frac{z}{\lambda_k}}{1 - \frac{z}{\mu_k}}, \quad z \in (C \setminus \{\mu_k\}_{k=1}^\infty)
\]

independently of whether the string is regular or not.

**Proof** First of all we notice that (3.6) attains the form

\[
T(z) = \sum_{k=1}^{\infty} \frac{\rho_k}{\mu_k - z}
\]

where \( \rho_j > 0 \) are the jumps of \( \tau^{(1)}(\lambda) \) at \( \mu_j \) \((j = 1, 2, \ldots)\). This implies that the coefficient of dynamic compliance \( T(z) \) of the string \( S_1(I, M) \) which in general is defined on \( C \setminus [0, +\infty) \), in our case can be extended by continuity onto the intervals \((\mu_j, \mu_{j+1}) \). After this \( T(z) \) appears to be holomorphic in the domain \( C \setminus \{\mu_k\}_{k=1}^\infty \) (see [4] or Subsection 2.1).

It follows from Subsections 3.4 and 3.5 that the set of points of the spectrum of the string \( S_0(I, M) \) coincides with the set of zeros of the function \( T(z) \). We notice that according to (3.6), (3.13) and (4.2)

\[
L = \int_{0}^{+\infty} \frac{d\tau^{(1)}(\lambda)}{\lambda} = \sum_{j=1}^{\infty} \frac{\rho_j}{\mu_j} = T(0),
\]

and \( 0 < T(z) < L \forall z \in (-\infty, 0) \). Moreover, \( T(z) \) is continuous and monotonically increasing on \((-\infty, \mu_1)\). On each of the intervals \((\mu_j, \mu_{j+1}) \) the function \( T(z) \) is continuous and monotonically increases from \(-\infty\) to \(+\infty\). It is clear that \( T(z) < 0 \) for \( z \in (\mu_j, \lambda_j) \) and \( T(z) > 0 \) for \( z \in (\lambda_j, \mu_{j+1}) \), \( j = 1, 2, \ldots \). Let us set

\[
Q(z) := \frac{1}{L} T(z).
\]

It is clear that \( Q(z) \) is also an \( S \)-function which is holomorphic in \( C \setminus \{\mu_j\}_{j=1}^\infty \), attains real values of the same sign as \( T(z) \) on \((-\infty, \mu_1), (\mu_j, \mu_{j+1}) \) \((j = 1, 2, \ldots)\). Also it is clear that \( 0 < Q(z) < 1 \) for \( z \in (-\infty, 0) \) and \( Q(0) = 1 \).

Let us consider the function

\[
U(z) = \log Q(z),
\]

where by \( \log \) we mean the branch of \( w = \log v \) defined in \( C \setminus (-\infty, 0] \) which attains real values for \( v \in (0, +\infty) \). Since \( \log v = \log |v| + i \arg v \) this means that \( \arg v = 0 \) for
\( v \in (0, +\infty) \) and due to continuity of \( \log v \) we have \( 0 < \arg v < \pi \) for \( \text{Im} v > 0 \) and \(-\pi < \arg v < 0 \) for \( \text{Im} v < 0 \). Therefore, for the considered branch of \( \log v \) the inequality \( \text{Im} v \text{ Im} \log v > 0 \) is true for \( \text{Im} v \neq 0 \) and \( \log v = \frac{\text{Re} v}{\text{Im} v} \). Thus, our branch of \( \log v \) is an \( R \)-function.

Since \( 0 < Q(z) < Q(0) = 1 \) for \( z < 0 \), this implies \( U(z) < 0 \) for \( z < 0 \). Thus, \( U(z) \) is an \( S^{-1} \)-function and admits the representation

\[
U(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left( \frac{1}{\lambda - z} - \frac{1}{\lambda} \right) \, d\sigma(\lambda)
\]

(4.5)

where \( \alpha \leq 0 \), \( \beta \geq 0 \) and \( \sigma(\lambda) \) being the spectral function of an \( R \)-function is nondecreasing on \((0, +\infty)\) and \( \beta = \lim_{\eta \to +\infty} \frac{\text{Im} U(i\eta)}{\eta} \). In our case \( 0 < \text{Im} U(i\eta) = \arg Q(i\eta) < \pi \) for all \( \eta > 0 \) and, therefore, \( \beta = 0 \).

As it was mentioned in Subsection 2.3 \( \alpha \neq 0 \) if and only if \( U(0) \neq 0 \). However, in our case \( Q(0) = 1 \) and this implies \( U(0) = 0 \) and therefore \( \alpha = 0 \). Now it remains to clarify what \( \sigma(\lambda) \) in (4.5) is in our case.

As it was mentioned in Subsection 2.1 if an \( R \)-function (and consequently an \( S^{-1} \)-function) \( f(z) \) is such that for each \( \lambda \in (a, b) \): \( \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{1}{\pi} \lim_{\epsilon \to 0} f(\lambda + i\epsilon) = g(\lambda) \) and \( g(\lambda) \) is bounded on \((a, b)\) then the spectral function of \( f(z) \) is absolutely continuous and its derivative equals \( g(\lambda) \) almost everywhere on \((a, b)\).

If \( \lambda \in (\lambda_j, \mu_{j+1}) \) then \( Q(\lambda) > 0 \) and \( \arg Q(\lambda) = 0 \). Thus, \( \text{Im} U(\lambda) = 0 \forall \lambda \in (\lambda_j, \mu_{j+1}) \) and it follows from the above mentioned that \( \sigma(\lambda) \) is absolutely continuous on \((\lambda_j, \mu_{j+1}) \) \((j = 1, 2, \ldots)\) and \( \sigma'(\lambda) \equiv 0 \) and, consequently, \( \sigma'(\lambda) = 0 \) for all \( \lambda \in (\lambda_j, \mu_{j+1}) \).

For \( \lambda \in (\mu_j, \lambda_j) \) we have \( Q(\lambda) < 0 \) and, consequently, \( \lim_{\epsilon \to 0} \arg Q(\lambda + i\epsilon) = \pi \forall \lambda \in (\mu_j, \lambda_j) \). Thus, \( \lim_{\epsilon \to 0} \text{Im} U(\lambda + i\epsilon) = \pi \forall \lambda \in (\mu_j, \lambda_j) \). Therefore, \( \sigma(\lambda) \) is absolutely continuous on \((\mu_j, \lambda_j) \) and \( \sigma'(\lambda) \equiv 1 \) on \((\mu_j, \lambda_j) \). This implies \( \sigma'(\lambda) = 1 \forall \lambda \in (\mu_j, \lambda_j) \), \((j = 1, 2, \ldots)\).

Thus, equation (4.5) attains the form

\[
U(z) = \sum_{j=1}^{\infty} \int_{\mu_j}^{\lambda_j} \left( \frac{1}{\lambda - z} - \frac{1}{\lambda} \right) \, d\lambda.
\]

(4.6)

The series in (4.6) converge for all nonreal \( z \) because the integral in (4.5) converges.

It is clear that

\[
\int_{\mu_j}^{\lambda_j} \left( \frac{1}{\lambda - z} - \frac{1}{\lambda} \right) \, d\lambda = \log \frac{1 - \frac{z}{\lambda_j}}{1 - \frac{z}{\mu_j}}.
\]

Substituting it into (4.6) we obtain

\[
U(z) = \sum_{j=1}^{\infty} \log \frac{1 - \frac{z}{\lambda_j}}{1 - \frac{z}{\mu_j}}
\]

and due to (4.3) and (4.4)

\[
T(z) = LQ(z) = Le^{U(z)} = L \exp \sum_{j=1}^{\infty} \log \frac{1 - \frac{z}{\lambda_j}}{1 - \frac{z}{\mu_j}} =
\]
\[
L \prod_{j=1}^\infty \exp \left( \log \frac{1 - \frac{\lambda_j}{\mu_j}}{1 - \frac{\lambda_j}{\mu_j}} \right) = L \prod_{j=1}^\infty \frac{1 - \frac{\lambda_j}{\mu_j}}{1 - \frac{\lambda_j}{\mu_j}}.
\]

Theorem is proved.

Equation (4.11) implies \( \rho_j = - \text{res}_{\mu_j} T(z) \) and using (4.12) we obtain

\[
\rho_j = L (\lambda_j - \mu_j) \frac{\mu_j}{\lambda_j} \prod_{k \neq j} \frac{1 - \frac{\mu_j}{\lambda_k}}{1 - \frac{\mu_j}{\mu_k}}.
\]

Thus,

\[
\tau^{(1)}(\lambda + 0) = \sum_{\mu_j \leq \lambda} \rho_j = L \sum_{\mu_j \leq \lambda} (\lambda_j - \mu_j) \frac{\mu_j}{\lambda_j} \prod_{k \neq j} \frac{1 - \frac{\mu_j}{\lambda_k}}{1 - \frac{\mu_j}{\mu_k}}.
\] (4.7)

**Theorem 4.2** Let two sequences \( \{\mu_n\}_{n=1}^\infty \) and \( \{\lambda_n\}_{n=1}^\infty \) interlace:

\[
0 < \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots
\] (4.8)

Then there exists a unique string \( S(I, M) \) of a given finite length \( L \) such that the spectrum of the string \( S_1(I, M) \) generated by \( S(I, M) \) coincides with \( \{\mu_n\}_{n=1}^\infty \) and the spectrum of the string \( S_0(I, M) \) generated by \( S(I, M) \) coincides with \( \{\lambda_n\}_{n=1}^\infty \).

**Proof** It is enough to repeat arguments in proof of Theorem 1 of [9], Chapter 7 to show that

\[
\prod_{n=1}^\infty \frac{1 - \frac{\lambda_n}{\mu_n}}{1 - \frac{\lambda_n}{\mu_n}} := q(z)
\]

converges for \( z \in \text{Ext}\{\mu_k\}_{k=1}^\infty \) and, moreover, \( q(z) \) is an \( R \)-function. Since \( q(z) > 0 \) for \( z \in (-\infty, 0) \) the function \( q(z) \) is an \( S \)-function. Therefore, \( Lq(z) \) is also an \( S \)-function.

According to M.G. Krein’s theorem (Theorem 11.2 in [5], see [10], page 252 for the proof) there exists a unique string \( S_1(I, M) \) for which \( Lq(z) \) is the coefficient of dynamic compliance. Then \( \{\mu_k\}_{k=1}^\infty \) is the spectrum of this string \( S_1(I, M) \) while \( \{\lambda_k\}_{k=1}^\infty \) is the spectrum of the string \( S_0(I, M) \) with the same \( I \) and \( M(x) \). Equation (4.3) implies that the length of this string is \( Lq(0) = L \). Theorem is proved.

In what follows we will say that the string \( S_1(I, M) \) existence of which is proved in Theorem 4.2. **corresponds to the data** \( L, \{\mu_k\}_{k=1}^\infty, \{\lambda_k\}_{k=1}^\infty \).

4.2. **Necessary and sufficient conditions for existence of the limit** \( \lim_{x \to 0} \frac{M(x)}{x^\alpha} \) and its calculation via two spectra

Let limit (3.8) exist and be finite and nonzero. Then (see Subsection 3.6) the second limit in (3.9) also exists and (3.10) is true, where \( B(\alpha) \) is defined by (3.11). It follows from (3.10) that

\[
\lim_{x \to 0} x^{-\alpha} M(x) = \left( \frac{1}{B(\alpha)} \lim_{k \to \infty} \mu_k^{-\alpha+1} \tau^{(1)}(\mu_k + 0) \right)^{-\alpha+1} = \left( \frac{1}{B(\alpha)} \lim_{k \to \infty} \mu_{k+1}^{-\alpha+1} \tau^{(1)}(\mu_{k+1} - 0) \right)^{-\alpha+1}.
\] (4.9)
Therefore,
\[
\lim_{k \to \infty} \mu_k^{-\frac{\alpha}{\alpha+1}} \tau^{(1)}(\mu_k + 0) = \lim_{k \to \infty} \mu_{k+1}^{-\frac{\alpha}{\alpha+1}} \tau^{(1)}(\mu_{k+1} - 0). \tag{4.10}
\]
Due to (4.7) for each \( n \in \mathbb{N} \) we have \( \tau^{(1)}(\mu_n + 0) = \tau^{(1)}(\mu_{n+1} - 0) \). Then (4.10) implies that
\[
\lim_{k \to \infty} \frac{\mu_{k+1}}{\mu_k} = 1 \tag{4.11}
\]
independently of the behavior of \( M(x) \) in the exterior of any right neighborhood of \( x = 0 \).

This means that existence of a finite nonzero limit in the left-hand side of (4.10) and validity of (4.11) are necessary conditions for existence of a finite and nonzero \( \lim_{x \to +0} \frac{M(x)}{x^\alpha} \) and of validity of the first of equations (4.9). Now we will show that these conditions are sufficient.

First, these conditions imply existence of the limit in the right-hand side of (4.10) and validity of equation (4.10). Next, since \( \tau^{(1)}(\mu_k + 0) = \tau^{(1)}(\lambda) = \tau^{(1)}(\mu_{k+1} - 0) \), in each point \( \lambda \in (\mu_k, \mu_{k+1}) \), the following inequalities are valid:
\[
\mu_k^{-\frac{\alpha}{\alpha+1}} \tau^{(1)}(\mu_k + 0) > \lambda^{-\frac{\alpha}{\alpha+1}} \tau^{(1)}(\lambda) > \mu_{k+1}^{-\frac{\alpha}{\alpha+1}} \tau^{(1)}(\mu_{k+1} - 0). \tag{4.12}
\]
Since the limits in (4.10) are finite and nonzero, inequalities (4.12) together with (4.10) imply that
\[
\lim_{\lambda \to +\infty} \lambda^{-\frac{\alpha}{\alpha+1}} \tau^{(1)}(\lambda)
\]
exists and is finite and nonzero. Then according to Kasahara’s theorem limit (3.8) exists and is finite and nonzero. Now we are able to give necessary and sufficient conditions of existence of a nonzero limit (3.8) in terms of the spectra:

**Theorem 4.3** Let \( M(x) \) be the mass distribution function of a string \( S(I, M) \) corresponding to the data \( L, \{\mu_n\}_{n=1}^\infty \) and \( \{\lambda_n\}_{n=1}^\infty \). Then limit (3.8) exists and is finite and nonzero for some \( \alpha \in (0, \infty) \) if and only if

1) the limit
\[
\lim_{n \to \infty} \left( \mu_n^{-\frac{\alpha}{\alpha+1}} \tau^{(1)}(\mu_n + 0) \right) \tag{4.13}
\]
where \( \tau^{(1)}(\lambda) \) is given by (4.7), exists and is finite and nonzero

2) (4.11) is valid.

Theorem 4.3 and the first of equations (4.9) make possible to find the limit (3.8) using the data \( L, \{\mu_n\}_{n=1}^\infty \) and \( \{\lambda_n\}_{n=1}^\infty \).

**Theorem 4.4** Let \( L > 0 \) and two sequences of numbers \( \{\mu_n\}_{n=1}^\infty \) and \( \{\lambda_n\}_{n=1}^\infty \) satisfy conditions (4.8) and behave asymptotically as follows
\[
\lambda_n = \frac{\pi^2 n^2}{b^2} + O(n^\beta),
\]
\[
\mu_n = \frac{\pi^2 (n - 1/2)^2}{b^2} + O(n^\beta),
\]
where \( \beta \in [0, 1) \), \( b \in (0, \infty) \) and let \( M(x) \) be the mass distribution function of a string \( S(I, M) \) corresponding to these data.

Then
\[
\lim_{x \to +0} M(x)x^{-1} = \frac{1}{L^2 \mu_1} \prod_{n=1}^{\infty} \frac{\lambda_n^2}{\mu_n \mu_{n+1}}.
\] (4.14)

**Proof** It is clear that the product in the right-hand side of (4.14) converges and is finite and not zero. Since the conditions of Theorem 4.2 are satisfied, there exists a unique string \( S(I, M) \) of finite length \( L \) with the spectrum of problem (3.3) \( \{\mu_n\}_{n=1}^{\infty} \) and the spectrum of problem (3.4) \( \{\lambda_n\}_{n=1}^{\infty} \). Due to Theorem 4.1 the coefficient of dynamic compliance \( T(z) \) of this string is given by (4.11). Thus,
\[
\left( \sqrt{z} T(-z) \right)^{-2} = \left( \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\mu_n}}{\lambda_n} \right)^2 \left( \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\mu_n}}{(1 + \frac{z}{\lambda_n})} \right)
\]

Let us evaluate
\[
\left( \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\mu_n}}{\lambda_n} \right)^2 \left( \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\mu_n}}{(1 + \frac{z}{\lambda_n})} \right)^{-2} = \frac{1}{L^2 \mu_1} \prod_{n=1}^{\infty} \frac{\mu_n^2}{\mu_n \mu_{n+1}} \prod_{n=1}^{\infty} \frac{\mu_n}{\mu_n (0)} \frac{\mu_n}{\mu_n (0) + z} \frac{\mu_n}{\mu_n (0) + z}
\]

where \( \mu_n (0) = \frac{\pi^2 (n-1/2)^2}{b^2} \) and \( \lambda_n (0) = \frac{\pi^2 n^2}{b^2} \).

Since
\[
\sqrt{z} \prod_{n=1}^{\infty} \frac{\lambda_n (0) + z}{\lambda_n (0)} = \frac{i \sin \sqrt{-z} b}{b}, \quad \prod_{n=1}^{\infty} \frac{\mu_n (0) + z}{\mu_n (0)} = \cos \sqrt{-z} b,
\]
we obtain
\[
-\frac{1}{L^2 \mu_1} \prod_{n=1}^{\infty} \frac{\lambda_n^2}{\mu_n \mu_{n+1}} \prod_{n=1}^{\infty} \frac{\mu_n}{\mu_n (0) + z} \left( \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\mu_n}}{\mu_n (0) + z} \right)^2 \left( \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda_n (0)}{\lambda_n (0) + z} \right)^{-2}
\]

It is clear that
\[
\lim_{z \to +\infty} (-i \cotan \sqrt{-z} b)^2 = 1,
\]

Since the series
\[
\sum_{n=1}^{\infty} \frac{\lambda_n - \lambda_n (0)}{\lambda_n (0) + z}, \quad \sum_{n=1}^{\infty} \frac{\mu_n - \mu_n (0)}{\mu_n (0) + z}
\]
On string density at the origin

converge absolutely and uniformly in the neighborhood of \( z = +\infty \), we obtain

\[
\lim_{z \to +\infty} \prod_{n=1}^{\infty} \left( 1 + \frac{\mu_n - \mu_n^{(0)}}{\mu_n^{(0)} + z} \right) = \lim_{z \to +\infty} \prod_{n=1}^{\infty} \left( 1 + \frac{\lambda_n - \lambda_n^{(0)}}{\lambda_n^{(0)} + z} \right) = 1.
\]

We have \( \mu_1^{(0)} = \frac{\pi^2}{6b^2} \) and by Wallis formula

\[
\prod_{n=1}^{\infty} \frac{\lambda_n^{(0)}^2}{\mu_n^{(0)} \mu_{n+1}^{(0)}} = \frac{\pi^2}{4}.
\]

Finally we arrive at

\[
\lim_{z \to +\infty} \left( \frac{\prod_{n=1}^{\infty} (1 + \frac{z}{\mu_n})}{L\sqrt{z} \prod_{n=1}^{\infty} (1 + \frac{z}{\lambda_n})} \right)^2 = \frac{1}{L^2 \mu_1} \prod_{n=1}^{\infty} \frac{\lambda_n^2}{\mu_n \mu_{n+1}}.
\]

Now by Kasahara’s theorem for \( \alpha = 1 \) with account of Theorem 4.1 we arrive at (4.14). Theorem is proved.

5. Strings with fast growth of spectra

In Theorem 4.4 like in [1], [2], [3] the sequences \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\mu_n\}_{n=1}^{\infty} \) grow roughly speaking as \( n^2 \). Now let us consider validity of ‘Barcilon formula’ (1.2) in case of more rapid growth of \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\mu_n\}_{n=1}^{\infty} \).

First we recall a remarkable M.G. Krein’s theorem stated in [11]:

**Theorem 5.1** If a string \( S_1(I,M) \) is regular and \( \{\mu_n\}_{n=1}^{\infty} \) is its spectrum then

\[
\lim_{n \to \infty} \frac{n}{\sqrt{\mu_n}} = \frac{1}{\pi} \int_{0}^{L} M'(x)dx.
\]

A proof of this theorem can be found in [12] (Section VI, Theorem 8.1).

This theorem we use to prove the following result.

**Theorem 5.2** If a string \( S_1(I,M) \) of length \( L < \infty \) has the spectrum \( \{\mu_n\}_{n=1}^{\infty} \) which satisfies

\[
\lim_{n \to \infty} \frac{n}{\sqrt{\mu_n}} = 0 \tag{5.1}
\]

then \( M'(x) = 0 \) almost everywhere on \( I \).

**Proof** If the string \( S_1(I,M) \) is regular the statement of Theorem 5.2 immediately follows from Theorem 5.1.

Now let our string \( S_1(I,M) \) be singular. Assume that in contrary to the statement of our theorem the Lebesgue measure of the set \( W := \{x \in I : M'(x) > 0\} \) is positive. Then there exists \( l \in (0, L) \) such that the Lebesgue measure of the set \( \hat{W} := [0, l] \cup W \) is also positive.
Let us consider the string $S_1(\hat{I}, \hat{M})$ where $\hat{I} = [0, l]$ and $\hat{M}(x) = M(x) \forall x \in \hat{I}$. In other words $S_1(\hat{I}, \hat{M})$ is the part of the string $S_1(I, M)$ located on $[0, l]$. Let $\{\hat{\mu}_n\}_{n=1}^\infty$ $(0 < \hat{\mu}_1 < \hat{\mu}_2 < \ldots)$ denote the spectrum of $S_1(\hat{I}, \hat{M})$, i.e. the set of points of growth of the main spectral function $\hat{\tau}^{(1)}(\lambda)$ of this string. The string $S_1(\hat{I}, \hat{M})$ is regular, therefore, due to Theorem 5.1

$$\lim_{n \to \infty} \frac{n}{\sqrt{\hat{\mu}_n}} = \frac{1}{\pi} \int_0^l \sqrt{\hat{M}'(x)}dx = \frac{1}{\pi} \int_W \sqrt{M'(x)}dx > 0. \quad (5.2)$$

From the definition of a spectral function it follows that any of spectral functions of the string $S_1(I, M)$ is a spectral function of the truncated string $S_1(\hat{I}, \hat{M})$. In particular, the main spectral function $\tau^{(1)}(\lambda)$ of $S_1(I, M)$ is a spectral function of $S_1(\hat{I}, \hat{M})$. It is enough to compare (5.1) with (5.2) to clarify that $\hat{\tau}^{(1)}(\lambda)$ do not coincide with $\tau^{(1)}(\lambda)$.

According to Theorem 5 of [13] (see also (14))

$$\tau^{(1)}(\hat{\mu}_{n-1} + 0) < \hat{\tau}^{(1)}(\hat{\mu}_{n-1} + 0) \quad n = 2, 3, \ldots \quad (5.3)$$

and according to Theorem 4 of [13]

$$\tau^{(1)}(\hat{\mu}_n - 0) > \hat{\tau}^{(1)}(\hat{\mu}_n - 0) \quad n = 1, 2, \ldots \quad (5.4)$$

Since $\tau^{(1)}(\hat{\mu}_n - 0) = \hat{\tau}^{(1)}(\hat{\mu}_{n-1} + 0)$ $n = 2, 3, \ldots$, we conclude using (5.3) and (5.4) that

$$\tau^{(1)}(\hat{\mu}_n - 0) > \tau^{(1)}(\hat{\mu}_{n-1} + 0) \quad n = 2, 3, \ldots$$

This means that there exists at least one point of growth of function $\tau^{(1)}(\lambda)$, i.e. at least one of points $\{\mu_j\}_{j=2}^\infty$ on each interval $(\hat{\mu}_{n-1}, \hat{\mu}_n)$.

For $n = 1$ (5.1) gives $\tau^{(1)}(\hat{\mu}_1 - 0) > \hat{\tau}^{(1)}(\hat{\mu}_1 - 0) = \hat{\tau}^{(1)}(0) = 0 = \tau^{(1)}(0)$, i.e. there is at least one point of growth of $\tau^{(1)}(\lambda)$ on the interval $(0, \hat{\mu}_1)$. Thus, for each $n \in \mathbb{N}$ the interval $(0, \hat{\mu}_n)$ contains at least $n$ of points of $\{\mu_n\}_{n=1}^\infty$. Consequently, $\mu_n < \hat{\mu}_n \forall n \in \mathbb{N}$ and according to (5.2)

$$\lim_{n \to \infty} \frac{n}{\sqrt{\mu_n}} > 0,$$

what contradicts (5.1). Thus, our assumption that $W$ has positive Lebesgue measure is false. Theorem is proved.

**Remark** A similar result is true for the spectrum $\{\lambda_k\}_{k=1}^\infty$ of a string $S_0(I, M)$. It follows from the strict interlacing of the spectra $\{\lambda_k\}_{k=1}^\infty$ and $\{\mu_k\}_{k=1}^\infty$ of the strings $S_1(I, M)$ and $S_0(I, M)$, respectively, with the same $I$ and $M$.

When a nonconstant nondecreasing on $I$ function $M(x)$ is such that $M'(x) = 0$ almost everywhere the condition of continuity of $M'(x)$ used in [1] and [2] to prove (4.14) is not satisfied. However, Theorem 5.2 together with the following theorem show that there exists a wide class of strings for which (4.14) is true despite $M'(x)$ is not continuous.

**Theorem 5.3** Let $L > 0$ and two sequences of positive numbers $\{\mu_n\}_{n=1}^\infty$ and $\{\lambda_n\}_{n=1}^\infty$ satisfy conditions (4.8) and

$$\lambda_n = \frac{\pi^4 n^4}{b^4} + O(n^\beta),$$
On strig density at the origin

\[ \mu_n = \frac{\pi^4 (n - 1/2)^4}{b^4} + O(n^\beta), \]

where \( b \in (0, \infty), \beta \in [0, 3). \)

Then (4.14) is true.

**Proof** In our case (4.15) remains true with \( \mu_n^{(0)} = \frac{\pi^4 (n-1/2)^4}{b^4} \) and \( \lambda_n^{(0)} = \frac{\pi^4 n^4}{b^4} \). Let us evaluate substituting \( \tau = \sqrt{-z} \)

\[
\sqrt{z} \prod_{n=1}^{\infty} \frac{\lambda_n^{(0)} + z}{\lambda_n^{(0)}} = i\sqrt{\tau} \prod_{n=1}^{\infty} \frac{\pi^2 n^2/b^2 - \tau}{\pi^2 n^2/b^2} \cos \frac{\sqrt{\tau} \sin \sqrt{-\tau} b}{b^2},
\]

\[
\prod_{n=1}^{\infty} \frac{\mu_n^{(0)} + z}{\mu_n^{(0)}} = \prod_{n=1}^{\infty} \frac{\pi^2(n - 1/2)^2/b^2 - \tau}{\pi^2(n - 1/2)^2/b^2} \prod_{n=1}^{\infty} \frac{\pi^2(n - 1/2)^2/b^2 + \tau}{(n - 1/2)^2} = \cos \sqrt{\tau} b \cos \sqrt{-\tau} b.
\]

Using these formulae we obtain from (4.15)

\[
L \left( \prod_{n=1}^{\infty} \left( 1 + \frac{z}{\mu_n} \right) \right)^2 = \frac{-\mu_1^{(0)} L}{\sqrt{\tau} \prod_{n=1}^{\infty} (\mu_n^{(0)} + z)} \prod_{n=1}^{\infty} \frac{\mu_n^{(0)} \mu_{n+1}^{(0)}}{(\lambda_n^{(0)})^2} (b^2 \cotan \sqrt{\tau} b \cotan \sqrt{-\tau} b)^2 \left( \prod_{n=1}^{\infty} \left( 1 - \frac{\mu_n^{(0)} - \mu_n^{(0)}}{\mu_n^{(0)} + z} \right) \right)^2 \left( \prod_{n=1}^{\infty} \left( 1 - \frac{\lambda_n^{(0)} - \lambda_n^{(0)}}{\lambda_n^{(0)} + z} \right) \right)^{-2}
\]

It is clear that

\[ \lim_{\tau \to \pm \infty} \left( \cotan \sqrt{\tau} b \cotan \sqrt{-\tau} b \right)^2 = 1. \]

Since the series

\[ \sum_{n=1}^{\infty} \frac{\lambda_n^{(0)} - \lambda_n^{(0)}}{\lambda_n^{(0)} + z}, \sum_{n=1}^{\infty} \frac{\mu_n^{(0)} - \mu_n^{(0)}}{\mu_n^{(0)} + z} \]

converge absolutely and uniformly in the neighborhood of \( z = +\infty \)

\[ \lim_{z \to +\infty} \prod_{n=1}^{\infty} \left( 1 - \frac{\mu_n^{(0)} - \mu_n^{(0)}}{\mu_n^{(0)} + z} \right) = \lim_{z \to +\infty} \prod_{n=1}^{\infty} \left( 1 - \frac{\lambda_n^{(0)} - \lambda_n^{(0)}}{\lambda_n^{(0)} + z} \right) = 1 \]

and

\[ \prod_{n=1}^{\infty} \frac{(\lambda_n^{(0)})^2}{\mu_n^{(0)} \mu_{n+1}^{(0)}} = \frac{\pi^4}{16}, \quad \mu_1^{(0)} = \frac{\pi^4}{16b^4}. \]

Therefore, we have

\[ \lim_{z \to +\infty} \left( \frac{L \prod_{n=1}^{\infty} (1 + \frac{z}{\mu_n})}{\sqrt{\tau} \prod_{n=1}^{\infty} (1 + \frac{z}{\lambda_n})} \right)^2 = \frac{1}{L^2 \mu_1} \prod_{n=1}^{\infty} \frac{\lambda_n^2}{\mu_n \mu_{n+1}}. \]

Theorem is proved.
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