Network Deployment for Maximal Energy Efficiency in Uplink with Multislope Path Loss

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Abstract

This work aims to design the uplink (UL) of a cellular network for maximal energy efficiency (EE). Each base station (BS) is randomly deployed within a given area and is equipped with $M$ antennas to serve $K$ user equipments (UEs). A multislope (distance-dependent) path loss model is considered and linear processing is used, under the assumption that channel state information (CSI) is acquired by using pilot sequences (reused across the network). Within this setting, a lower bound on the UL spectral efficiency and a realistic circuit power consumption model are used to evaluate the network EE. Numerical results are first used to compute the optimal BS density and pilot reuse factor for a Massive multiple-input-multiple-output (MIMO) network (such that $M \gg K \gg 1$) with three different detection schemes, namely, maximum ratio combining (MRC), zero-forcing (ZF) and multicell minimum mean-squared error (M-MMSE). The numerical analysis shows that the EE is a unimodal function of BS density and achieves its maximum for a relatively small BS densification, irrespective of the employed detection scheme. Therefore, we concentrate on ZF and use stochastic geometry to compute a new lower bound on the spectral efficiency, which is then used to optimize, for a given BS density, the pilot reuse factor, number of BS antennas and UEs. Closed-form expressions are computed from which valuable insights into the interplay between the optimization variables, hardware characteristics, and propagation environment can be obtained.

I. INTRODUCTION

Keeping up with the ever-growing demand for higher data throughput is the major ambition of future cellular networks [1]. An important question is how to evolve communication technologies to deliver higher throughput without prohibitively increasing the power consumption [2]. This calls for new design mechanisms that provide the user equipments (UEs) with high spectral efficiency at moderate energy costs. There is a broad consensus that this wireless capacity growth can only be achieved with a substantial network densification [3] [4]. The main driven for this densification are twofold: small-cell networks [5]–[7] and Massive MIMO [8]–[12]. The former relies on a massive...
deployment of small cells that guarantees lower propagation losses [5]–[7]. The latter makes use of a massive number of base station (BS) antennas to simultaneously serve a relatively large number of UEs by means of spatial multiplexing. A combination of both has also received a lot of interest in the research literature (e.g., [13], [14]). Despite being potentially effective in increasing spectral efficiency, both solutions tend to increase the power consumed by the network; small cells increase the number of deployed BSs, whereas Massive MIMO requires more hardware per BS. The aim of this work is to design a cellular network from scratch to achieve maximal EE, without any a priori assumption on the number of BS antennas, UEs, cell reuse or BS density.

A. Main literature

The optimal deployment of cellular networks has received great attention in the literature. The first attempts were based on the simple Wyner model [15] wherein both BSs and UEs are located on a line at fixed positions. Next, more complex 2D symmetric grid-based deployments (e.g., hexagonal lattice) were considered [16]. Both approaches are not suited for modeling and studying networks characterized by a very irregular and dense structure, as envisioned in future cellular networks. To address this problem, advanced mathematical tools based on stochastic geometry have been employed in the last years (e.g. [17]–[19]). Within the stochastic geometry framework, the locations of BSs form a point process in a compact set whose cardinality is a Poisson distributed random variable that is independent among different disjoint sets. The performance of a cellular network can be measured in many different ways such as coverage probability, area spectral efficiency and area energy efficiency (EE) [20]. Earlier works on the design of EE-optimal cellular networks, equipped with multiple antenna BSs, can be found in [9] and [21] where closed-form expressions are derived for a single-cell scenario and numerical results are given for a multicell setting. The EE analysis of a multicell network is developed in [20], [22], [23] by using stochastic geometry. In [22], the optimization is done while satisfying a quality-of-service (QoS) requirement per UE. In [20], [23], the use of small-cells together with sleeping strategies is proved to be a promising solution for increasing the EE. Generally speaking, small-cells leads to higher EE but this gain saturates quickly as the density of small cells increases. In [13], it has been shown that further benefits can be achieved by using Massive MIMO. All the aforementioned works rely on an idealized path loss model where the received power has a uniform exponential decay with the distance from the serving BS. In practice, the path loss exponent is itself an increasing function of distance [24]. In [24]–[26], the authors adopt a multislope path loss model for which totally different conclusions can be drawn. In particular, it turns out that the propagation environment and fading distribution play a key role
in identifying network operating regimes for which an increase, saturation, or decrease of the EE is observed as the network densifies.

B. Contributions and outline

We consider a cellular network in which the BSs are independently and uniformly distributed in a given area according to a homogeneous Poisson point process (H-PPP) of intensity $\lambda$. Each BS is equipped with an arbitrary number $M$ of BS antennas and serves simultaneously $K$ UEs. Statistical channel inversion is employed in the uplink (UL) to achieve a uniform average signal-to-noise ratio (SNR) across all the UEs. A multislope (distance-dependent) path loss model is considered and different linear processing schemes, namely, MRC, ZF and M-MMSE, are used, under the assumption that channel state information is acquired by using pilot sequences, reused across the network with a factor $\zeta$. The EE of the network is computed by using a lower bound on the average UL SE (valid for any combining scheme) as well as a polynomial power consumption model, thoroughly developed in [12]. Numerical results are used to evaluate the impact of both BS density $\lambda$ and pilot reuse factor $\zeta$ on the EE of a Massive MIMO network such that $M \gg K \gg 1$. The results show that the EE with a multislope path loss is a unimodal\(^1\) function of $\lambda$. Irrespective of the employed detection scheme, the optimal EE is achieved for relatively small values of $\lambda$ and $\zeta$. This is in sharp contrast to [13] where the adoption of a single-slope path loss model lead to the conclusion that densification is always beneficial for EE; the EE is shown to be a monotonic increasing function of $\lambda$ in [13]. The results show also that, although the “optimal” M-MMSE combiner provides the highest EE, the three different schemes behave similarly in terms of EE and area throughput as the BS density increases.

Motivated by the above analysis, we concentrate on ZF and compute a new closed-form lower bound on the average UL SE. This lower bound is used to analytically find in closed-form the EE-optimal network configuration with respect to $M$, $K$ and $\zeta$ while satisfying a signal-to-interference-plus-noise ratio (SINR) constraint. The closed-form expressions reveal the fundamental interplay between the three design parameters, which are also illustrated numerically. It turns out that ZF allows a higher densification of the network while using a smaller pilot reuse factor and achieving a higher EE than with MR. Both schemes employ almost the same optimal number of antennas per BS to approximately serve the same number of UEs, with a ratio $M/K$ between 3 and 14.

\(^1\)A function $f(x)$ is unimodal if, for some value $m$, it is monotonically increasing for $x \leq m$ and monotonically decreasing for $x > m$. 
when using ZF and between 3 and 21 for MR depending on the SINR constraint. In addition, ZF is characterized by a smoother EE function, which is more robust to system changes and thus makes it a better choice.

Compared to its preliminary version in [27], this work is substantially different for the following reasons: (i) it provides the EE analysis for MR, ZF and M-MMSE; (ii) it is based on a multislope path loss model and aims at showing its impact on EE when the network is densified; (iii) more details and insights into the effect of network parameters and circuit power model are given.

The remainder of this paper is organized as follows. Next section introduces basic notation and describes the cellular network with the underlying assumptions and transmission protocols. Section III analyzes the EE of MR, ZF, and M-MMSE based on a realistic circuit power model. In Section IV, we consider the ZF scheme and compute a lower bound on the achievable EE, which is then maximized analytically with respect to $M$, $K$ and $\zeta$. The resulting expressions reveal the fundamental interplay between the three design parameters. Numerical results are used in Section V to validate an alternating optimization algorithm, which allows to optimally design the network. Finally, the major conclusions and implications are drawn in Section VI.

II. NETWORK MODEL AND PROBLEM STATEMENT

We consider the UL of a cellular network wherein the BSs are spatially distributed at locations $\{x_i\}$ within a compact geographic area according to a H-PPP $\Phi = \{x_i; \ i \in \mathbb{N}\} \subset \mathbb{R}^2$ of intensity $\lambda [\text{BS/km}^2]$. Let $A$ be the deployment area of interest, the average number of deployed BSs is simply $\mathbb{E}\{\Phi\} = \lambda A$. Each BS has $M$ antennas and serves $K$ single-antenna UEs over a bandwidth of $B_w [\text{MHz}]$. These $K$ UEs are selected at random from a very large set according to some scheduling algorithm. We assume that each UE is connected to the closest BS such that the coverage area of a BS is its Poisson-Voronoi cell (see Fig. 1). The $K$ UEs are assumed to be uniformly distributed in the Poisson-Voronoi cell. Without loss of generality, we assume that the “typical UE”, which is statistically representative for any other UE in the network [28], has an arbitrary index $k$ and is

$^2$Upper (lower) bold face letters are used for matrices (column vectors). $\mathbf{I}_N$ is the $N \times N$ identity matrix. $\mathbf{0}$ is the zero vector. $(\cdot)^T$, $(\cdot)^*$ and $(\cdot)^H$ are the transpose, conjugate and conjugate transpose operators, respectively. We denote by $\text{tr}(\cdot)$ the matrix trace operator, by $\| \cdot \|$ the Euclidean norm vector operator. $\mathbb{E}_n\{\cdot\}$ denotes the expectation operator with respect to the random vector $\mathbf{n}$, whereas $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_n)$ is shorthand for a circularly-symmetric normal distribution with covariance matrix $\mathbf{R}_n$. We use $\mathbb{C}$, $\mathbb{Z}_+^+$ and $\mathbb{R}$ to denote the sets of all complex-valued numbers, all strictly positive integers, all rational numbers and all real-valued numbers, respectively. $\mathbb{R}^n$ and $\mathbb{C}^n$ are the $n$-dimensional real and complex vector spaces. We denote by $\Gamma(s;x) = \int_x^\infty t^{s-1}e^{-t} \, dt$ the upper incomplete gamma function.
connected to an arbitrary BS \( j \). The network operates according to a synchronous TDD protocol. We denote by \( B_c \) [Hz] and \( T_c \) [s] the coherence bandwidth and time, respectively. Then, the coherence block is composed of \( \tau_c = B_c T_c \) [complex samples]. In each coherence block, \( \tau_p \) samples are used for acquiring channel state information by means of UL pilot sequences, whereas \( \tau_u \) and \( \tau_d \) samples (such that \( \tau_c = \tau_p + \tau_u + \tau_d \)) are used for payload transmission in the UL and DL, respectively. We assume that \( \tau_p = \zeta K \) with \( \zeta \geq 1 \) being the pilot reuse factor and \( \tau_u = \xi (\tau_c - \zeta K) \) with \( \xi \leq 1 \) accounting for the payload UL fraction transmission [12].

A. Received Signal and Power Control Policy

We call \( s_{li} \sim \mathcal{N}_C(0, p_{li}) \) the UL payload signal transmitted from UE \( i \) of cell \( l \) to its serving BS \( l \) with power \( p_{li} = \mathbb{E}_s \{ |s_{li}|^2 \} \). The UL signal \( y_j \in \mathbb{C}^M \) received at BS \( j \) is

\[
y_j = \sum_{i=1}^{K} h_{ji}^i s_{ji} + \sum_{l \in \Phi \setminus \{j\}} \sum_{i=1}^{K} h_{li}^i s_{li} + n_j \tag{1}
\]

where \( n_j \sim \mathcal{N}_C(0, \sigma^2 I_M) \) is the additive Gaussian noise, \( h_{li}^j \in \mathbb{C}^M \) is the channel response between UE \( i \) in cell \( l \) and BS \( j \) modeled as uncorrelated Rayleigh fading, i.e., \( h_{li}^j \sim \mathcal{N}_C(0, \beta_{li}^j I_M) \), where \( \beta_{li}^j \) is the large-scale fading coefficient. We call \( d_{li}^j \) the distance of UE \( i \) in cell \( l \) from BS \( j \) and compute \( \beta_{li}^j \) according to a general multislope path loss model, which is given by:

\[
\beta_{li}^j (d_{li}^j) = \gamma_n \left( \frac{d_{li}^j}{d_{ref}} \right)^{-\alpha_n} \tag{2}
\]

for \( d_{li}^j \in [d_n, d_{n+1}], n = \{0, \cdots, N-1\} \). The coefficients \( \{ \gamma_n \}, \{ \alpha_n \}, d_{ref} \) are design parameters. Specifically, \( 0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_N \) are the power decay factors, \( 0 = d_0 < d_1 < \cdots < d_N = \infty \) denote the distances at which a change in the power decadence occurs and \( d_{ref} \) is the reference distance. Setting \( N = 1 \) yields the widely used single-slope path loss model \( \beta_{li}^j = \gamma_1 (d_{li}^j)^{-\alpha_1} \).

Following [29], we assume the UEs use a statistical channel inversion power-control policy such that \( p_{li} = P_0 / \beta_{li}^j \) where \( P_0 \) is a design parameter. This ensures a uniform ergodic received SNR at BS \( l \) to all the UEs, which it is given by \( \mathbb{E}\{ \| h_{li}^j \|^2 p_{li} \} / \sigma^2 = P_0 / \sigma^2 = \text{SNR}_0 \) and it is assumed to be constant over the coherence block.

B. Pilot Reuse Policy and Channel Estimation

We assume that a pilot book \( \Phi \in \mathbb{C}^{\tau_p \times \tau_p} \) of \( \tau_p \) mutually orthogonal UL sequences is used for channel estimation and call \( \phi_{jk} \in \mathbb{C}^{\tau_p} \) the pilot sequence assigned to the typical UE \( k \) in cell \( j \). It is assumed to have normalized UL pilot sequences, to obtain a constant power level, and this implies
Fig. 1: Deployment of a cellular network with BSs drawn from a H-PPP $\Phi_\lambda$, each one serving $K$ randomly located UEs. The “typical” UE $k$ in cell $j$ is highlighted. The UEs of different cells sharing the same pilot subset $\Phi_l$ are depicted with the same (marker, color) pair. For the sake of illustration, we have $A = 1 \text{km}^2$, $\lambda = 16$, $\zeta = 4$ and $K = 30$.

That $\|\phi_{jk}\|^2 = 1$. To avoid cumbersome pilot coordination, we assume that in each coherence block each BS $l$ picks a subset of $K$ different sequences from $\Phi$, uniformly at random and distribute them among its served UEs. Since $\tau_p = \zeta K$, we have that on average $\zeta = \tau_p/K \in \mathbb{R}^{+\infty}$ different cells in the network share the same pilot subset. This is modeled through a Bernoulli stochastic variable $a_{vl} \sim \mathcal{B}(1/\zeta)$ for $l' \neq l$ and $a_{ll} = 1$. Specifically, if $a_{vl} = 1$ all the UEs in cell $l'$ use the same pilot subgroup of those in cell $l$ and thus causes pilot contamination [12]. This occurs with probability $Pr(a_{vl} = 1) = 1/\zeta$. Similarly, $a_{vl} = 0$ indicates that there is no pilot contamination from cell $l'$ to cell $l$ and vice versa, and happens with probability $Pr(a_{vl} = 0) = 1 - 1/\zeta$. To facilitate understanding, Fig. 1 illustrates a pilot allocation snapshot where different pairs (marker, color) identify different pilot subsets $\{\Phi_l\}$. We call $Y^p_j \in \mathbb{C}^{M \times \tau_p}$ the signal received at BS $j$ during pilot transmission. The vector $y^p_{jli} = Y^p_j \phi_{li}^*$ obtained by correlating $Y^p_j$ with $\phi_{li}$ takes the form:

$$y^p_{jli} = \sqrt{\rho} \sqrt{p_{li}} h^j_{li} + \sum_{l' \in \Phi \setminus \{l\}} a_{vl} \sqrt{\rho} \sqrt{p_{l'i}} h^j_{l'i} + N^p_{jli} \phi_{li}^*$$

(3)

where $N^p_{jli} \phi_{li}^* \sim \mathcal{N}(0, \sigma^2 I_M)$ and the power of the UL transmitted payload signal is scaled by a factor $\rho = P_p/P_0 \geq 1$ to compensate for the lack of beamforming gain during channel acquisition.

**Corollary 1** (e.g. [12]). By using $p_{li} = P_0/\beta^l_{li}$, the MMSE estimate of $h^j_{li}$ at BS $j$ based on $y^p_{jli}$ is

$$\hat{h}^j_{li} = \frac{\beta^l_{li} P_p}{\beta^l_{li} + \sum_{l' \in \Phi \setminus \{l\}} a_{vl} \beta^l_{l'i} + \frac{1}{SNR_p}} y^p_{jli}$$

(4)
with $\text{SNR}_p = \frac{P_p}{\sigma^2}$ as a design parameter. The MMSE estimate $\hat{h}_{li}^j$ and error $\tilde{h}_{li}^j$, conditioned on a realization of $a_{l'l}, l' \in \Phi$, are independent and distributed as $\hat{h}_{li}^j \sim \mathcal{N}_C(0, \gamma_{li}^j I_M)$ and $\tilde{h}_{li}^j \sim \mathcal{N}_C(0, (\beta_{li}^j - \gamma_{li}^j) I_M)$ where

$$
\gamma_{li}^j = \frac{\beta_{li}^j}{\beta_{li}^j + \sum_{l' \in \Phi \setminus \{l\}} a_{l'l} \beta_{l'l}^j + \frac{1}{\text{SNR}_p}} \beta_{li}^j.
$$

(5)

For notational convenience, we define $\bar{H}_l^j = \left[\hat{h}_{l1}^j \ldots \hat{h}_{lK}^j\right] \in \mathbb{C}^{M \times K}$ collecting all the estimates in (4) from all UEs in cell $l$ to BS $j$. Note that the channel estimate $\hat{h}_{l'i}^j$ of a UE $i$ in cell $l'$ using the same pilot sequence of a UE $i$ in cell $l$ (i.e. $a_{l'i} = 1$) can be obtained from $\hat{h}_{li}^j$ in (4) as

$$
\hat{h}_{l'i}^j = \sqrt{\frac{\beta_{l'i}^j}{\beta_{l'i}^j + \beta_{li}^j}} \hat{h}_{li}^j
$$

(6)

where $\hat{h}_{l'i}^j \sim \mathcal{N}_C(0, \gamma_{l'i}^j I_M)$ has variance $\gamma_{l'i}^j = \frac{\beta_{l'i}^j}{\beta_{l'i}^j + \beta_{li}^j} \gamma_{li}^j$. The independent estimation error is then $\tilde{h}_{l'i}^j \sim \mathcal{N}_C(0, (\beta_{l'i}^j - \gamma_{l'i}^j) I_M)$. The expression in (6) is responsible of pilot contamination with spatially uncorrelated channels; the inability of BS $j$ to separate UEs that use the same pilot [8].

### III. Energy Efficiency Analysis

The EE is defined as the amount of information reliably transmitted per unit of energy, which is mathematically expressed as the ratio [21]:

$$
\text{EE} = \frac{\text{Area throughput [bit/s/km}^2\text{]}}{\text{Area power consumption [W/km}^2\text{]}} = \frac{B_w [\text{Hz}] \cdot \text{ASE [bit/s/Hz/km}^2\text{]}}{\text{APC [W/km}^2\text{]}}
$$

(7)

which is measured in [bit/Joule] and can be seen as a benefit-cost ratio, where the service quality (area throughput) is compared with the associated cost (area power consumption). In (7), ASE and APC stand for the area spectral efficiency and area power consumption, respectively.

#### A. Area Spectral Efficiency

Since the “typical UE” is statistically representative for any other UE in the network [28], the area spectral efficiency is obtained $\text{ASE} = \lambda K \text{ SE}$ in [bit/Hz/km$^2$] where SE denotes the average UL SE of the typical UE $k$ in cell $j$ and is obtained averaging over different UE positions, pilot allocations and channel realizations. The multiplicative factor $K$ accounts for the sum spectral

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3A considerable number of papers on EE analysis has considered misleading EE metrics measured in bit/Joule/Hz, instead of bit/Joule. This is pointless since one cannot make the EE bandwidth-independent: the transmit power is divided over the bandwidth while the noise power is proportional to the bandwidth.
efficiency of all UEs in cell $j$ and $\lambda$ is the BS density per km$^2$. A lower bound on SE, which holds for any combining scheme, UE positions, pilot allocations, power allocation policies, is as follows.

**Theorem 1 ([12]).** When the channel is obtained through the MMSE estimator in (4), the UL average ergodic channel capacity of the typical UE $k$ in cell $j$ is lower bounded by

$$SE \geq SE' = \xi \left( 1 - \frac{K \zeta}{\tau_c} \right) \mathbb{E}_{\{d,h,a\}} \left\{ \log_2 \left( 1 + \text{SINR}' \right) \right\}$$

where $\text{SINR}'$ is the instantaneous SINR given by

$$\text{SINR}' = \frac{p_{jk} |v_{jk}^H \hat{h}_{jk}^j|^2}{v_{jk}^H \left( \sum_{l \in \Phi_\lambda} \sum_{i=1}^K p_l \hat{h}_{li}^j \hat{h}_{li}^{jH} + \sum_{l \in \Phi_\lambda} \sum_{i=1}^K p_l (\beta_{li}^j - \gamma_{li}^j) I_M + \sigma^2 I_M \right) v_{jk}}$$

and the expectation $\mathbb{E}_{\{d,h,a\}} \{ \cdot \}$ is computed with respect to UE positions, channel realizations and pilot allocations. The pre-log factor accounts for the pilot overhead.

The optimal $v_{jk}$ that maximizes (8) is given as follows.

**Corollary 2 ([12]).** The instantaneous UL SINR in (9) for a typical UE $k$ in cell $j$ is maximized by

$$v_{jk}^{\text{M-MMSE}} = p_{jk} \left( \sum_{l \in \Phi_\lambda} \sum_{i=1}^K p_l (\hat{h}_{li}^j \hat{h}_{li}^{jH} + (\beta_{li}^j - \gamma_{li}^j) I_M + \sigma^2 I_M)^{-1} \hat{h}_{jk}^j \right).$$

**Proof.** The proof can be found in [12] and it is based on the Rayleigh quotient maximization. □

The optimal combining vector in (10) is known as M-MMSE combiner since it can be proved to be the vector $v_{jk}$ that minimizes the conditional MSE, that is $\mathbb{E} \{ s_{jk} - v_{jk}^H y_j \mid \{ \hat{H}_j \}, \{ a_{l',j} \} \}$. Despite its optimality, the M-MMSE combiner has not been used much in the research literature. The majority of works make use of single-cell processing schemes such as single-cell MMSE (S-MMSE), regularized ZF (RZF), ZF and MR, which are all suboptimal. Specifically, they can be obtained as approximations and simplifications of the optimal M-MMSE [12]. For example, S-MMSE can be obtained by considering the intra-cell channel estimates $\hat{H}_j^j$ only, whereas RZF arises by neglecting interference coming from other cells. The MR combiner $V_{jk}^{\text{MR}} = \hat{H}_j^j$ is obtained for low SNR values whereas the ZF combiner

$$V_{jk}^{\text{ZF}} = \hat{H}_j^j \left( (\hat{H}_j^j)^H \hat{H}_j^j \right)^{-1}$$

4 Note that the UL capacity for a network (such as the one under investigation) with imperfect CSI and inter-cell interference modeled as a shot-noise process is not known yet [30]. As a common practice in these circumstances, we resort to a lower bound.

5 BS $j$ uses the first $\tau_p = \zeta K$ samples for acquiring CSI to decode the UL payload in the remaining $\tau_u$ samples of the coherence block.
is obtained for high SNR values. In the sequel, we consider M-MMSE, ZF and MR. M-MMSE provides the highest SE, but using the highest complexity. MR has the lowest complexity, but also the lowest SE. Finally, ZF strikes a good balance between SE and complexity.

B. Area Power Consumption

The area power consumption is obtained from the power consumption PC as

$$\text{APC} = \lambda \left( \frac{1}{\eta} P_{\text{TX}} + P_{\text{CP}} \right) \quad \text{[W/km}^2\text{]} \quad (12)$$

where $P_{\text{TX}}$ accounts for the average ergodic power usage for UL transmission (payload and pilots) in an arbitrary cell $j$ with $\eta \in (0, 1]$ being the power amplifier (PA) efficiency whereas $P_{\text{CP}}$ accounts for the power consumed by circuitry and can be computed as in [13], [21]. Both are evaluated next.

**Corollary 3.** Recalling that $\tau_u$ and $\tau_p$ samples are respectively used in UL for data and pilot transmissions, the average ergodic power usage is given by

$$P_{\text{TX}} = (\tau_u + \tau_p) K U \quad (13)$$

with

$$U = \frac{1}{\eta P_0} \sum_{n=1}^{N} \frac{\Gamma(1 + \alpha_n/2; R_n) - \Gamma(1 + \alpha_n/2; R_{n+1})}{\left(\pi \lambda\right)^{\alpha_n/2}}. \quad (14)$$

**Proof.** Since the BS locations are drawn from a H-PPP $\Phi$ and the UEs are uniformly distributed in the Poisson-Voronoi cells, the distance $d_{jk}$ from the typical UE $k$ to its serving BS $j$ is Rayleigh distributed as $d_{jk} \sim \mathcal{R}(\sigma)$ with $\sigma = 1/\sqrt{2\pi\lambda}$ and its probability density function is given by $f_d(d) = \frac{d}{\sigma^2} e^{-d^2/2\sigma^2}$ for $d > 0$ [30]. Since $\mathbb{E}_s\{\|s_{jk}\|^2\} = p_{jk}$ irrespective of whether $s_{jk}$ is a payload or pilot signal, by setting $p_{jk} = P_0/\beta_{jk}^2$ we obtain $P_{\text{TX}} = \frac{\tau_u + \tau_p}{\eta} K \mathbb{E}_d\{1/\beta_{jk}^2\}$. Finally, using the path loss model in (2) we obtain $\mathbb{E}_d\{1/\beta_{jk}^2\} = \sum_{n=1}^{N} \frac{\Gamma(1 + \alpha_n/2; R_{n+1}) - \Gamma(1 + \alpha_n/2; R_n)}{(\pi \lambda)^{\alpha_n/2}}$ from which (14) follows by using (68) in Appendix B.

The power needed to run the circuitry of an arbitrary BS $j$ can be modeled as follows [13], [21]

$$P_{\text{CP}} = P_{\text{FIX}} + P_{\text{TC}} + P_{\text{C-BH}} + P_{\text{CE}} + P_{\text{LP}} \quad (15)$$

where $P_{\text{FIX}}$ is the power consumed for site-cooling, control signaling and load-independent backhauling, $P_{\text{TC}}$ for the transceiver chain, $P_{\text{C-BH}}$ for coding and load-dependent backhauling cost, $P_{\text{CE}}$ for channel estimation process and $P_{\text{LP}}$ for linear processing. All these terms can be expressed as a function of the system parameters reported in Table I. In particular, we have that
TABLE I: Network and system parameters.

| Parameter                           | Value   | Parameter                           | Value   |
|-------------------------------------|---------|-------------------------------------|---------|
| Fixed power: $P_{\text{FIX}}$       | 5 W     | Far-field path loss exponent: $\alpha$ | 3.76    |
| Power for BS Local Oscillator: $P_{\text{LO}}$ | 0.1 W   | Coherence block length: $\tau_c$    | 200 samples |
| Power per BS antennas: $P_{\text{BS}}$ | 0.2 W   | Propagation loss at distance $d_{\text{ref}}$: $\Upsilon$ | $-148.1$ dB |
| Power for antenna at UE: $P_{\text{UE}}$ | 0.1 W   | Bandwidth: $B_w$                     | 20 MHz  |
| Power for data coding: $P_{\text{COD}}$ | 0.01 W/(Gbit/s) | Deployment area: $A$ | 1 km$^2$ |
| Power for backhaul traffic: $P_{\text{BT}}$ | 0.025 W/(Gbit/s) | UL fraction of payload block: $\xi$ | $1/3$ |
| Power for data decoding: $P_{\text{DEC}}$ | 0.08 W/(Gbit/s) | Noise variance: $\sigma^2$ | $-94$ dBm |
| BS computational efficiency: $L_{\text{BS}}$ | 750 Gflops/W | Signal-to-noise ratio of payload block: $\text{SNR}_0$ | 5 dB |
| HPA efficiency: $\eta$              | 0.5     | Signal-to-noise ratio of pilot block: $\text{SNR}_p$ | 15 dB |

$P_{\text{TC}} = M P_{\text{BS}} + P_{\text{LO}} + K P_{\text{UE}}$ whereas $P_{\text{C-BH}} = B_w K \text{SE} (P_{\text{COD}} + P_{\text{DEC}} + P_{\text{BT}})$. The evaluation of $P_{\text{CE}}$ and $P_{\text{LP}}$ requires first to evaluate the computational complexity of channel estimation and linear processing in terms of flops per coherence block.\(^6\) The MMSE estimation has complexity $K(M \tau_p + M)$ since, for any UE, it requires $M \tau_p$ operations to compute $y^p_{lij} = Y^p_j \phi^*_li$ in (3) and $M$ operations for computing $\hat{h}^j li$ in (4). To transform these figures into consumed power, let $L_{\text{BS}}$ be the computational efficiency of the BS measured in [flops/W] and recall that a complex multiplication requires three real flops. Since there are $B_w/\tau_c$ coherence blocks per second and $\tau_p = \zeta K$, we obtain

$$P_{\text{CE}} = \frac{3}{L_{\text{BS}}} \frac{B_w}{\tau_c} K M (\zeta K + 1) \quad (16)$$

The power $P_{\text{LP}}(M, K)$ consumed by linear processing can be quantified as $P_{\text{LP}} = P_{\text{LP-r}} + P_{\text{LP-c}}$ where $P_{\text{LP-r}}$ accounts for the power consumed by reception of payload samples (i.e., evaluation of $y_{jk} = v^H_{jk} y_j$) whereas $P_{\text{LP-c}}$ is the power required for the computation of the combiner. The former can be quantified as

$$P_{\text{LP-r}} = \frac{3B_w}{\tau_c L_{\text{BS}}} M K \tau_u = \frac{3B_w}{\tau_c L_{\text{BS}}} MK \xi (\tau_c - \zeta K) \quad (17)$$

whereas the latter depends on the combiner $v_{jk}$ used at BS $j$. Table II provides the power consumed by M-MMSE, ZF and MR.

C. Numerical analysis

We now use the developed power model to design the network for maximal EE. To this end, we adopt the parameters listed in Table I [12]. Specifically, we consider a deployment area of $A = 1 \text{ km}^2$.

\(^6\)In this work, we consider only a real multiplication as flop since the cost due to addition is somehow negligible.
### TABLE II: Power consumed by different combining vectors $v_{jk}$.

| Combining scheme | $P_{LP-c,j}$ |
|------------------|--------------|
| M-MMSE           | $\frac{\lambda A (M^2 + 3M)K}{2} + (M^2 - M)K + \frac{MB^2}{3} + 2M + M\zeta K^2 (\zeta - 1)$ |
| ZF               | $\frac{MB}{\sigma^2_L BS} \left( \frac{3K^2 M}{2} + \frac{KM}{2} + \frac{K^3 - K + \zeta K}{3} \right)$ |
| MR               | $\frac{MB}{\sigma^2_L BS} (\zeta K)$ |

---

**Fig. 2:** EE (in Mbit/Joule) as a function of $\lambda$ (in BS/km$^2$) and $\zeta$. Results are obtained from Monte Carlo simulations for MR, ZF and (optimal) M-MMSE combiners within a Massive MIMO setting with $M = 100$ and $K = 10$. The global optimum is star-marked for which the corresponding values assumed by the network deployment variables are also indicated.

wherein $\mathbb{E}\{\Phi_\lambda\} = \lambda A$ BSs are randomly deployed according to a H-PPP. A wrap-around topology is used to simulate the H-PPP in the whole $\mathbb{R}^2$ and keep the translation invariance. Inspired by [24] and [14], a bounded $N = 3$ slopes path loss model is used for the large scale fading in (2) with

$$
\beta_{li}^j (d_{li}) = \begin{cases} 
\beta_{li,1} = \Upsilon_1 \left( \frac{d}{d_{ref}} \right)^{-\alpha_1} & 0 = d_0 < d < d_1 \\
\beta_{li,2} = \Upsilon_2 \left( \frac{d}{d_{ref}} \right)^{-\alpha_2} & d_1 \leq d < d_2 \\
\beta_{li,3} = \Upsilon_3 \left( \frac{d}{d_{ref}} \right)^{-\alpha_3} & d_2 \leq d < d_3 = \infty
\end{cases}
$$

(18)
where \([\alpha_1, \alpha_2, \alpha_3,] = [0, 2, 4], [d_1, d_2] = [10, 446] \, \text{[m]} \) and \([\Upsilon_1, \Upsilon_2, \Upsilon_3] = [d_1 - \alpha_1, d_2 - \alpha_2, \Upsilon]. \) The cut off distance \(d_2\) comes from the usage of a simple 2-ray ground reflection case, with \(d_2 \approx 4h_t h_r/\lambda_c\) where \(h_t = 10\) and \(h_r = 1.65\) are the antenna heights of BS and UE, and \(\lambda_c = c/f_c\) is the operating wavelength at a carrier frequency \(f_c = 2\, \text{GHz}. \) A Massive MIMO setup is considered, which is roughly characterized by an antenna-UE ratio \(M/K\) of 10 in order to meet the channel hardening and favorable propagation conditions [12]. Within this setup, we assume \(M = 100\) and \(K = 10.\)

Fig. 2 shows the EE of MR, ZF and M-MMSE as a function of BS density and pilot reuse factor \(\zeta.\) In particular, we consider \(\lambda \in \{1, 2, \ldots, 10, 20, \ldots, 60\} \) and \(\zeta \in \{1, \ldots, 10\} \) with an average \(\text{SNR}_0 = 5\, \text{dB}.\) We see that with all schemes the EE is a pseudo-concave function and has a unique global optimizer \((\lambda^*, \zeta^*)\) at which maximum EE* Mbit/Joule is achieved. Each point uniquely determines the EE-optimal network deployment configuration for the corresponding scheme. We notice that M-MMSE provides the highest EE, followed by ZF, while MR achieves the lowest EE. M-MMSE does not only provide the highest EE but has also the smoothest EE around the optimum. This makes it more robust to a variation of network settings (e.g., pilot reuse and BS density). From Fig. 2a, we can see that the maximal EE value with M-MMSE is \(\text{EE}^* = 11\, \text{Mbit/Joule}\) and is achieved at \((\zeta^*, \lambda^*) = (3, 5).\) With ZF (see Fig. 2b), we have \(\text{EE}^* = 9.63\, \text{Mbit/Joule}\) at \((\zeta^*, \lambda^*) = (2, 5),\) whereas with MR (see Fig. 2c) we obtain \(\text{EE}^* = 6.47\, \text{Mbit/Joule}\) at \((\zeta^*, \lambda^*) = (2, 7).\) Note that, irrespective to the combining scheme, the optimal EE is achieved for a pilot reuse factor between 2 and 3 and also for a relatively small BS density (e.g., a few BSs per km²). This latter result is in contrast to [13], wherein the EE monotonically increases as \(\lambda\) grows large. This was a consequence of the single-slope path loss model adopted in [13]. Further details on this are given next.

Fig. 3a depicts the EE of MR, ZF and M-MMSE as a function of \(\lambda \in \{1, 2, \ldots, 10, 20, \ldots, 60\}\) for the optimal pilot reuse factors \(\zeta^*,\) provided by Fig. 2. Comparisons are made with the EE achieved with a single-slope path loss model with large scale fading parameters \(\{\alpha_1, \Upsilon_1\}\) as in Table I. As expected, in this latter case, the EE is a monotonic non-decreasing function of \(\lambda\) for all schemes. For completeness, Fig. 3b illustrates the EE with MR, ZF and M-MMSE as a function of the corresponding average Area Throughput, obtained for \(\lambda \in \{1, 2, \ldots, 10, 20, \ldots, 60\}\). We see that there exist operating conditions under which it is possible to jointly increase both the area throughput and EE up to the maximum EE point, but further increases in throughput can only come at a loss in EE. The curves are quite smooth around the maximum EE point; thus, there is a variety of throughput values or, equivalently, BS densities that provide nearly maximum EE with higher area throughput. With M-MMSE, selecting \(\lambda = 9 > \lambda^* = 6\) leads to a 27% increase in
Fig. 3: EE (in Mbit/Joule) as a function of (a) $\lambda$ (in BS/km$^2$) and (b) Area Throughput in (in Gbit/s/km$^2$) for the optimal numerical $\zeta^\star$. Results are obtained from Monte Carlo simulations for MR, ZF and (optimal) M-MMSE combiners within a Massive MIMO setting with $M = 100$ and $K = 10$. In Fig. 3a both single-slope and multislope path loss model are considered while in Fig. 3b only the multislope is of concern for the sake of illustration.

area throughput while the EE is reduced by 12% only. The gain is even higher when considering MR and ZF. Specifically, they allow a reduction of respectively 19% and 10% in EE to achieve 46% and 56% higher area throughput.

To summarize, Figs. 2 and 3 show that the additional computational complexity of M-MMSE processing pays off both in terms of EE and area throughput. Moreover, the analysis shows that, for a Massive MIMO setup, reducing the cell size does not bring benefits in terms of EE; the optimal EE is roughly achieved for the same BS density $\lambda$ for all detection schemes.

IV. ENERGY EFFICIENCY MAXIMIZATION WITH ZF

Monte Carlo simulations were used above to examine the EE of different network configurations for a given pair of $(M, K)$ and detection scheme. In the following, we look at the EE from a different perspective: we design the network from scratch to achieve maximal EE for a given BS density, without any a priori assumption on $M$ and $K$. In doing so, we concentrate on ZF and show how the EE maximization problem can be solved analytically, without the need of heavy Monte Carlo simulations. Moreover, such an analysis exposes fundamental behaviors with respect to network parameters that cannot be easily inferred from the numerical analysis. In particular, we are interested in finding the EE-optimal tuple of parameters $\theta = (\zeta, K, M)$ defined over a set $\Theta = \{\theta : 1 \leq \zeta < \tau_c/K, (M, K) \in \mathbb{Z}_{++} \times \mathbb{Z}_{++}\}$ where the channel coherence block length $\tau_c$ represents the upper limit on the pilot signaling overhead. To this end, we resort to an alternative
lower bound for the average ergodic capacity, which is called Use-and-then-Forget (UatF) bound that is as follows.

**Theorem 2** (e.g. [12]). The UL average ergodic channel capacity of the typical UE $k$ in cell $j$ is lower bounded by $SE \geq \xi_{UL} \left( 1 - \frac{K\zeta}{\tau_c} \right) \mathbb{E}_d \{ \log_2 (1 + \text{SINR}) \}$ where the SINR of the typical UE, conditioned on a realization of UEs locations, is given by

$$\text{SINR} = \frac{p_{jk} \| \mathbb{E}_{(h,a)} \{ v_h^j h^j_{jk} \} \|^2}{\sum_{l \in \Phi \lambda} \sum_{i=1}^K p_l \mathbb{E}_{(h,a)} \{ | v_h^i h^i_{li} |^2 \} - p_{jk} \| \mathbb{E}_{(h,a)} \{ v_h^j h^j_{jk} \} \|^2 + \sigma^2 \mathbb{E}_{(h,a)} \{ \| v_h \|^2 \} }.$$  \hspace{1cm} (19)

The lower bound in Theorem 2 is less tight than the previous bound in Theorem 1 since the instantaneous channel estimates are not utilized during signal detection [11]. However, it does not rely on MMSE estimator, but can be applied along with any channel estimator and also allows to compute a tractable lower bound on the average UL SE of the typical UE when ZF is used.

**Lemma 1.** When the channel is obtained through the MMSE estimator in (4), the UL powers $\{p_{jk}\}$ are chosen as $p_{jk} = p_0 / \beta_j^2 \zeta$ and the ZF combining is chosen as in (11), a lower bound on the UL average ergodic channel capacity of the typical UE $k$ in cell $j$, computed using the UatF bound in (19), is given by $SE = \xi \left( 1 - \frac{K\zeta}{\tau_c} \right) \log_2 (1 + \text{SINR})$ where

$$\text{SINR} = \frac{M - K}{\text{INT} + (M - K)\mu_2 / \zeta}$$  \hspace{1cm} (20)

with

$$\text{INT} = \left( K + \frac{1}{\text{SNR}_0} \right) \left( 1 + \frac{\mu_1}{\zeta} + \frac{1}{\text{SNR}_p} \right) + K \mu_1 \left( 1 + \frac{1}{\text{SNR}_p} \right) + \frac{K}{\zeta} ( \mu_1^2 + \mu_2 ) - K \left( 1 + \frac{\mu_2}{\zeta} \right),$$  \hspace{1cm} (21)

and $\mu_\kappa$ for $\kappa = 1, 2$

$$\mu_\kappa = \sum_{n=1}^N \frac{2}{\kappa \alpha_n - 2} \left( \Gamma \left( 2; \pi \lambda R^{2n}_n \right) - \Gamma \left( 2; \pi \lambda R^{2n}_{n-1} \right) \right)$$

$$\quad + \frac{2e_n(\kappa)}{(\pi \lambda)^{\frac{\kappa \alpha_n - 2}{2}}} \left( \Gamma \left( 1 + \frac{\kappa \alpha_n}{2}; \pi \lambda R^{2n}_{n-1} \right) - \Gamma \left( 1 + \frac{\kappa \alpha_n}{2}; \pi \lambda R^{2n}_n \right) \right)$$  \hspace{1cm} (22)

with $e_n(\kappa) = \frac{-R^{2n-\kappa \alpha_n}}{\kappa \alpha_n - 2} + \sum_{i=n+1}^N \left( \frac{Y_i}{Y_n} \right)^\kappa \frac{R^{2-\kappa \alpha_i}_i R^{2-\kappa \alpha_i}_{i-1}}{\kappa \alpha_i - 2}$.

**Proof.** The proof is available in the Appendices and is articulated in two parts: the first part is given in Appendix A wherein all the expectations in (19) with respect to channel and pilot realizations are computed, while the second part is given in Appendix B and makes use of all the computed terms to evaluate the expectation with respect to both BS nd UE locations. \hfill \Box
Notice that the numerator of $\text{SINR}$ in (20) scales with $M - K$ since each BS sacrifices $K$ degrees of freedom for interference suppression within the cell. The pilot contamination term scales also with $M - K$ and accounts for the coherent interference due to UEs that use the same pilot sequence as the typical UE. Many of the interference terms in INT increase with $K$ since having more UEs lead to both more intra-cell and inter-cell interference due to the imperfect CSI and lack of multicell processing. Comparing the above expressions with those obtained in [13] with MR combining and a single-slope path loss model, it follows that the numerator scales with $M$ rather than $M - K$ since MR combining overcome the interference and noise by amplifying the signal of interest using the full array gain of $M$. The interference from other cells is the same for both schemes except for the extra negative term in (21) given by $K(1 + \mu_2/\zeta)$, which makes ZF combining preferable to MR combining whenever the reduced interference is more substantial than the loss in array gain. The pilot contamination term scales also with $M$ (as the useful signal) rather than $M - K$ as in (20) since MR combining does not benefit from interference suppression.

### A. Problem Statement

The EE maximization problem using Lemma 1 is formulated as

$$\theta^* = \arg\max_{\theta \in \Theta} \frac{\text{EE}(\theta)}{\text{APC}(\theta)} = \frac{B_w \text{ASE}(\theta)}{\text{APC}(\theta)}$$

subject to $\text{SINR}(\theta) = \gamma$ (24)

where $\text{ASE}(\theta) = \lambda K \text{SE}(\theta)$ and $\gamma > 0$ is a design parameter, $\text{SINR}(\theta)$ is given in (20) and $\text{APC}(\theta)$ can be obtained from (12). This constraint is imposed to avoid that the EE optimization may lead to an optimizer with poor SE. More details are given next. By adopting the power consumption model developed in Section III and expanding the contribution due to $P_{\text{TX}}$ in (13) and $P_{\text{CP}}$ in (15)–(17), the PC with ZF combining can be rewritten in the following equivalent form:

$$\text{APC}(\theta) = \lambda \left( C_0 + C_1 K - \zeta C_2 K^2 + C_3 K^3 + D_0 M + D_1 MK + D_2 MK^2 \right) + A B_w \text{ASE}$$

(25)

with $C_0 = P_{\text{FIX}} + P_{\text{LO}}$, $C_1 = P_{\text{UE}} + 5B_w/(\tau_c L_{\text{BS}})$, $C_2 = \mathcal{U}/\tau_c$, $C_3 = B_w/(\tau_c L_{\text{BS}})$, $D_0 = P_{\text{BS}}$, $D_1 = 3B_w(\frac{5}{2} + \tau_c)/(\tau_c L_{\text{BS}})$, $D_2 = 9B_w/(2\tau_c L_{\text{BS}})$ and $A = P_{\text{COD}} + P_{\text{DEC}} + P_{\text{BT}}$. Note that the functional dependence of $\text{APC}(\theta)$ on $\lambda$ is due to the term $\mathcal{U}$ given in (14), which depends on the transmit power. Due to the unavoidable inter-cell interference in cellular networks, the optimization problem (23) is only feasible for some values of $\gamma$. This feasible range is obtained in [13] observing that $\text{SINR}$ is a monotonically increasing function of $M$ and the constraint $\zeta < \tau_c/K$ must be satisfied, which leads to $\gamma < \tau_c/(K\mu_2)$.
B. Optimal Pilot Reuse Factor

We begin by computing the optimal pilot reuse factor $\zeta^*$ when $M$ and $K$ are fixed.

**Lemma 2.** Consider any pair of $(M, K)$ for which the problem (23) is feasible. The SINR equality constraint in (23) is satisfied by selecting

$$\zeta^* = \frac{B_1(M, K)\gamma}{M - K - B_2(K)\gamma}$$

(26)

where $B_1: \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ and $B_2: \mathbb{Z}^+ \to \mathbb{R}$ are given by

$$B_1(M, K) = K\left(\mu_1(1 + \mu_1) - \mu_2\right) + M\mu_2 + \frac{1}{\text{SNR}_0}\mu_1$$

(27)

$$B_2(K) = K\left(\frac{1}{\text{SNR}_p} + \mu_1\left(1 + \frac{1}{\text{SNR}_p}\right)\right) + \frac{1}{\text{SNR}_0}\left(1 + \frac{1}{\text{SNR}_p}\right)$$

(28)

**Proof.** We gather the terms that contain $\zeta$ in (20) and we obtain (26) by solving the equality. 

The above lemma provides insights into how the EE-optimal pilot reuse factor $\zeta^*$ depends on the other system parameters. In particular, it shows that $\zeta^*$ must increase with $K$ to guarantee a certain average SINR equal to $\gamma$. This is intuitive since increasing $\zeta$ leads to better channel estimation which can partially suppress the increased interference due to more UEs. Comparing (26) with the optimal pilot reuse factor

$$\zeta_{\text{MR}}^* = \frac{B_1(M, K)\gamma + 2K\mu_2\gamma}{M - K\gamma - B_2(K)\gamma}$$

(29)

obtained in [13] with MR combining, it follows that with MR the denominator scales with $K\gamma$ rather than $K$ and we have a positive extra term in the numerator. Since usually $\gamma \geq 1$ (to ensure reasonable average SE constraint), it turns out that a smaller pilot reuse factor can be used with ZF due to its interference suppression capabilities. Notice that $\zeta^*$ is a decreasing function of $M$, $\text{SNR}_0$ and $\text{SNR}_p$. This is because all these parameters amplify the desired signal, which, as a consequence, improves the channel estimation and makes the system operate in a less noise limited regime. The pilot reuse factor $\zeta^*$ reduces as the path loss exponents $\{\alpha_n\}$ increase (since $B_1$ and $B_2$ are reduced), which is natural since inter-cell interference decays more quickly. Notice that when $\gamma$ is high enough, setting $\zeta = \zeta^*$ allows to solve also problems with inequality constraints of the form $\text{SINR} \geq \gamma$. Indeed, $\text{SINR}(\zeta)$ is monotonic non-decreasing and thus any $\zeta \geq \zeta^*$ satisfies the inequality constraint. Let us now denote with $\zeta_{\text{opt}}$ the maximizer of the unimodal function $\text{SE}(\zeta)$. Then, if $\gamma$ is high enough, then $\zeta^* \geq \zeta_{\text{opt}}$ and picking $\zeta = \zeta^*$ is the best we can do.
Plugging $\zeta^*$ as in (26) into (23), the EE maximization problem becomes

$$\max_{M,K \in \mathbb{Z}^+} \text{EE}(\zeta^*) = \frac{B_w \text{ASE}(\zeta^*)}{\text{APC}(\zeta^*)}$$ \hspace{1cm} (30)

subject to $1 \leq \zeta^* \leq \tau_c/K$. \hspace{1cm} (31)

Next, we look for the optimal values of $M$ and $K$ solving the above problem.

### C. Optimal Number of Antennas per BS

To find the optimal values for $M$ and $K$, an integer-relaxed version of (30) is considered. The integer-valued solutions are then retrieved from the relaxed ones by projection. For analytic tractability, we replace $M$ with $\bar{c} = M/K$, which is the number of BS antennas per UE. This yields:

$$\max_{c \in \mathbb{R}^+, K \in \mathbb{Z}^+} \text{EE}(\zeta^*) = \frac{B_w \text{ASE}(\zeta^*)}{\text{APC}(\zeta^*)}$$ \hspace{1cm} (32)

subject to $K \tau_c < \frac{\zeta^* K}{\tau_c} \leq 1$ \hspace{1cm} (33)

with

$$\frac{\zeta^* K}{\tau_c} = K \frac{B_1(\bar{c}, K) \gamma / \tau_c}{K(\bar{c} - 1) - B_2(K) \gamma}.$$ \hspace{1cm} (34)

For a given $K$, the EE-maximizing value of $\bar{c}$ is obtained as:

**Lemma 3.** For any fixed $K > 0$ such that (32) is feasible, the EE is maximized by

$$\bar{c}^* = \min \left( \max \left( \bar{c}', \bar{c}_1 \right), \bar{c}_2 \right)$$ \hspace{1cm} (35)

with

$$\bar{c}' = \frac{r_1}{r_0} + \sqrt{-\frac{q_0}{q_2} - \frac{q_1 r_1}{q_2 r_0} + \left( \frac{r_1}{r_0} \right)^2}, \hspace{1cm} \bar{c}_1 = \frac{a_1 + a_3}{a_2 - a_0} \quad \text{and} \quad \bar{c}_2 = \frac{K \bar{c} a_1 + a_3}{a_2 - K a_0},$$ \hspace{1cm} (36)

where we used $r_0 = a_2 - a_0$, $r_1 = a_1 + a_3$, $q_0 = a_1 a_6 + a_3 a_5$, $q_1 = a_3 a_4 + a_0 a_6 - a_2 a_5$ and $q_2 = a_2 a_4$, while all the auxiliary parameters $\{a_i\}$ are listed in Table III.

**Proof.** By using the notation introduced in the lemma, we begin with computing the term $K \zeta^*/\tau_c$ that appears both in $\text{ASE}(\zeta^*)$ and $\text{APC}(\zeta^*)$. Plugging (27) and (28) into (34) yields $K \zeta^*/\tau_c = \frac{a_0 \bar{c} + a_1}{a_2 \bar{c} - a_3}$ such that the objective function in (32) reduces to $\frac{1 - a_0 \bar{c} + a_1}{a_4 \bar{c} + a_5 - a_0 a_6 \bar{c}^2 + a_1 a_3}$ which can be easily shown to be a quasi-concave function of $\bar{c}$. By taking the first derivative and equating to zero we obtain $\bar{c}'$ in (36), which corresponds to the solution to the unconstrained problem. Further, the constraint in (33) can be rewritten as $1 \leq \frac{a_0 \bar{c} + a_1}{a_2 \bar{c} - a_3} \leq \frac{\tau_c}{K}$ which implies $\bar{c}_1 \leq \bar{c} \leq \bar{c}_2$. This yields the desired result. \hfill \square
Notice that the constraint $\bar{c} \geq \bar{c}_1$ is active for $\gamma \leq \tau_c/(\mu^2 K)$. Assuming single-slope pathloss model with typical values: $\alpha_1 = 3.76$, $\tau_c = 200$ symbols and a very high number of UEs (worst case) as $K = 24$, e.g., then we obtain $\gamma \leq 23$, namely a gross SE constraint of 4.6 bit/s/Hz/UE, which implies that this constraint is always active for practical cases. The second constraint $\bar{c} \leq \bar{c}_2$ is instead active for $\gamma \leq \tau_c^2/(\mu^2 K^2)$, which obviously applies even more. This lemma shows how the optimal $\bar{c}$ depends on the other system parameters. In particular, we see that $\bar{c}'$ increases roughly linearly with $K$ and $\gamma$. This is reasonable since the network tends to equip the BSs with more antennas in order to guarantee an increase of the minimum average SINR to each UE. In contrast, the contrary happens with respect to the the circuit power parameters given by $c_0 = P_{\text{FIX}} + P_{\text{SYN}}$ and $c_1 = P_{\text{UE}} + 5B_0/(\tau_c L_{BS}) + U(1 + 1/\tau_c)$. In particular, $\bar{c}'$ is directly proportional to the BS density as $\lambda^{\alpha_0/4}$ (since $U$ in $a_5$ is reduced as $\lambda^{-\alpha_0/2}$); larger antenna arrays must be used if the BS density increases. The same happens with respect to $D_0 = P_{BS}$ since it becomes more costly to have additional antennas when $P_{BS}$ increases. Finally, fewer antennas are needed when SNR$_0$ and SNR$_p$ are increased since the rate requirement is achieved by using a higher transmitted power during either data transfer or channel estimation.

D. Optimal Number of UEs per cell

We now look for the optimal $K$ when $\bar{c}$ is given.

Lemma 4. For any fixed $\bar{c} > 0$ such that the relaxed problem (32) is feasible, the optimal number of UEs is maximized by

$$K^* = \max \left( K_2, \max \left( K_1, \min \left( K', \overline{K}_1 \right) \right) \right)$$

(37)

where $K'$ is the real root of the following quintic equation $p_5 x^5 + p_4 x^4 + p_3 x^3 + p_2 x^2 + p_1 x + p_0 = 0$ where $p_5 = -m_3 n_4$, $p_4 = 2n_4(-m_1 + m_2)$, $p_3 = (-m_1 + m_2)n_3 - m_3 n_2$, $p_2 = n_1(-m_3 + 3m_1)$, $p_1 = n_1(-m_1 + m_2) + 3m_3 n_0$ and $p_0 = 2(-m_1 + m_2)$ and with $K_2 = -b_1 + b_3/\tau_c$ and

$$K_1 = -\frac{(b_1 - b_2) - \sqrt{(b_1 - b_2)^2 - 4b_0 b_3}}{2b_0} \quad \overline{K}_1 = -\frac{(b_1 - b_2) + \sqrt{(b_1 - b_2)^2 - 4b_0 b_3}}{2b_0}$$

(38)

Proof. As already done in Lemma 3, we begin with computing the term $K \zeta^*/\tau_c$, which is given by $K^{b_0 K + b_1}$, being $\{b_i\}$ auxiliary parameters that are listed in Table III. By doing so, the objective function in (32) can be rewritten as

$$\text{EE}(K) = \frac{-m_3 K^3 + m_2 K^2 - m_1 K}{n_4 K^4 + n_3 K^3 + n_2 K^2 + n_1 K - n_0}$$

(39)
with \( m_3 = b_0, m_2 = b_2 - b_1, m_1 = b_3 \) and \( n_4 = b_2b_7, n_3 = b_2b_6 - b_3b_7 - b_0b_8, n_2 = b_2b_5 - b_3b_6 - b_1b_8, n_1 = b_2b_4 - b_3b_5 \) and \( n_0 = b_3b_4 \). Then, this lemma can be easily proved by taking the derivative of (39) with respect to \( K \) and then considering the constraints on \( K\zeta^*/\tau_c \).

In particular, it is trivial to show that constraint \( K_2 \) is active if and only if \( \gamma \leq \gamma_2^{-1} \) with \( \gamma_2 = \frac{1}{\bar{c} - 1} \left( \mu_1^2 + \mu_1 \left( 2 + \frac{\bar{c}}{\text{SNR}_p} \right) + \mu_2 (\bar{c} - 1) + \frac{1}{\text{SNR}_p} \right) \). Notice that there exists no generic closed-form root expression for a quintic equation but solutions can be easily found by means of exhaustive search over the domain set. This can be further speeded up by using for example a bisection method over a feasible set. To gain insights into how \( K^* \) is affected by the system parameters, assume that the power consumption required for linear processing due to combining at the BS is negligible, which implies \( C_3 = D_2 \approx 0 \). This case is relevant as all these terms essentially decrease with the computational efficiency \( L_{BS} \), which is expected to increase rapidly in the future. If \( L_{BS} \) is very large, we can further neglect other terms due to linear processing and channel estimation, i.e. \( D_1 \approx 0 \). For the sake of tractability, we assume also that both the SNR and the BS density are sufficiently large (as \( \lambda \to \infty \), since \( U \) reduces as \( \lambda^{-\alpha_0/2} \) we have \( C_2 \to 0 \) and \( \text{SNR}_0 \gg \gamma \)). Then, the following result is of interest:

**Corollary 4.** Consider the optimization problem (32) where \( \bar{c} = M/K \) and \( K \) are relaxed to be real-valued variables. For any fixed \( \bar{c} > 0 \) such that the relaxed problem is feasible, if we let \( \lambda \to \infty, L_{BS} \to \infty \) and \( \text{SNR}_0 \gg \gamma \), the optimal number of UEs is

\[
K^*_\infty = \frac{C_0}{C_1 + D_0\bar{c}} \left( \sqrt{1 + \frac{b_2 - b_1}{b_0} \frac{C_1 + D_0\bar{c}}{C_0}} - 1 \right) \tag{40}
\]

**Proof.** If \( \bar{c} \) is given, \( C_2 = C_3 = D_1 = D_2 = 0 \) and \( b_3 = 0 \), then (39) reduces to

\[
\frac{\text{EE}_\infty(K)}{K} = K \frac{m_2 - Km_3}{n_1 + Kn_2} \tag{41}
\]

which is a quasi-concave function whose maximum is achieved for (40).

The above result coincides with that in [13] and shows that, under the above circumstances, \( K^* \) decreases with \( \bar{c}^* \) as \( \sqrt{1/\bar{c}}^* \). From (40), it is found that \( K^* \) increases with the static energy consumption \( C_0 \), while it decreases with \( C_1 \) and \( D_0 \). The same behavior is observed for the optimal number of BS antennas. Therefore, we may conclude that more BS antennas and UEs per cell can be supported only if the increase in circuit power has a marginal effect on the consumed power. In addition, we note that \( K^* \) is a decreasing function of \( \gamma \), since the interference increases as more UEs are served. We later show that when using hardware parameters as in Table I and having reasonable
SNR values, the optimal $K$ computed using Corollary 4 achieves practically the same performance of the one in Lemma 4 that requires exhaustive search.

**E. Alternating Optimization**

To summarize, we first showed in Lemma 2 how to compute the optimal pilot reuse factor. Then, Lemma 3 and Lemma 4 (Corollary 4) showed how to optimize the EE separately with respect to $M$ and $K$. To solve the original problem (23) jointly with respect to all the parameters, we propose the following alternating optimization algorithm:

1) Assume that an initial feasible point $\theta_0 = (\zeta^*_0, \bar{c}_0^*, K_0^*)$ is given;

2) Compute the optimal $\zeta^*_t$ by using Lemma 2;

3) Compute the optimal $\bar{c}_t^*$ by using Lemma 3;

4) Compute the optimal $K_t^*$ by using Lemma 4 (Corollary 4);

5) Optimal point at step $t$: $\theta_t = (\zeta_t^*, M_t^* = \bar{c}_t^* K_t^*, K_t^*)$

6) Repeat 2) – 4) until convergence is achieved.

This algorithm converges to the global optimum of the relaxed problem since the EE is pseudo-concave in each variable and bounded above [31]. The convergence of the above algorithm is validated by numerical analysis in the sequel.

**V. NUMERICAL RESULTS**

Numerical results are now used to design the network and validate the theoretical analysis done in Section IV considering ZF combining and MMSE channel estimation. The circuit power parameters as well as the channel parameters are taken from [13] and [21] and listed in Table I.
Fig. 4: EE (in Mbit/Joule) as a function of $\lambda$ (in BS/km$^2$), for fixed $\text{SNR}_0 = 5$ dB and $\text{SNR}_p = 15$ dB. Results are obtained for MR, ZF and (optimal) M-MMSE combining schemes within a Massive MIMO setting with $M = 100$ and $K = 10$, considering multislope path loss model in (2). The optimal pilot reuse factor $\zeta$ is computed numerically. Curves are obtained from Monte Carlo simulations (empty-markers) and using the closed-form expression (filled-markers) in Lemma 1 and (25) for ZF and in [13] for MR extended to multi slope model in (2).

We consider a squared deployment area of 1 km$^2$ with wraparound topology wherein $E\{\Phi_\lambda\} = \lambda A$ BSs are deployed as described in Section II. The transmission bandwidth is $B_w = 20$ MHz and each coherence block consists of $\tau_c = 200$ samples. We assume that $\text{SNR}_p = 15$ dB and $\text{SNR}_0 = 5$ dB.

We start by recalling that in Lemma 1 we extend the result in [13] by computing a closed-form lower bound on the UL average ergodic channel capacity of the typical UE, this time considering ZF combining and multislope pathloss channel model. In Fig. 4, we numerically validate the proposed lower bound by Monte-Carlo simulations.\textsuperscript{7} In particular, the EE is shown as a function of $\lambda$ with $\zeta$ being optimized numerically at each point. The EE is obtained as in (23) using the power model in (12) – (17) and the average ergodic SE either lower bounded as in Lemma 1, i.e. $\text{SE}$, or computed by Monte-Carlo simulations.

Fig. 5 shows the EE lower bound as a function of $K$ and $M > K$ for optimal $\zeta$ with $\gamma = 3$. The pilot reuse is chosen optimally according to (26) and the BS density is fixed as $\lambda = 10$ BS/km$^2$, which is found to be a good compromise between EE and area throughput in Section III. As it is seen, EE is pseudo-concave and has a unique global maximizer, which can be computed by applying the alternating optimization algorithm in Section IV. Here, $K^*_\infty$ computed using Corollary 4 is used in

\textsuperscript{7}Notice that in this work we aim at finding and drawing conclusions on the EE-optimal deployment setup, therefore we are not concerned about ensuring the tightness of the bound as long as the shape of the numerical curve is resembled.
the proposed joint optimization routine. It is worth noting that the heuristic solution $K^*$ resembles the optimal numerical one for $\text{SNR}_0 = 5$ dB and using the hardware parameters as in Table I. Numerical simulations shows that this simplification keeps holding even for relatively small SNR values (it is below the 3% relative error for SNR at the order of $-10$ dB, which is very low). The global maximizer is achieved by the triplet $(M^*, K^*, \zeta^*) = (53, 6, 8.02)$ and gives a maximum of $\text{EE}^* = 3.66 \text{ Mbit/Joule}$. The ratio $M/K$ for the considered setup is slightly more than 8, which leads to a massive MIMO setup. The cell reuse $1/\zeta^* \approx 12\%$ is low enough to ensure robustness against pilot contamination. Compared to [13], the results in Fig. 5 shows that ZF is characterized by a smoother EE function with respect to MR, which makes ZF more robust to system changes and thus a better choice as BS combiner scheme. In Table II, ZF and MR are compared by using $\lambda = 10 \text{ BS/km}^2$ and with $\gamma \in \{1, 3, 7\}$, which corresponds to the average SEs $\log_2(1+\gamma) \in \{1, 2, 3\}$ [bit/s/Hz/UE]. It turns out that, for a given reasonable $\gamma$, both schemes employ almost the same optimal number of antennas at each BS and serve approximately the same number of UEs. With both schemes, the ratio $M/K$ increases almost linearly with $\gamma$ as stated in Lemma 3. ZF achieves a higher EE than MR, on equal $\gamma$, in the case that we study. This is a direct consequence of the ZF capabilities to handle intra-cell interference, which allows the BS to serve more UEs with a smaller number of antennas. This has a dual-positive effect: higher area throughput due to multiplexing gain and lower APC because of smaller arrays (especially when the network operates in a high area...
TABLE IV: Optimal network design parameters and performance achieved for $\lambda = 10$ BS/km$^2$ at different $\gamma$ values.

| Combiner  | $\text{EE}^*$ [Mbit/Joule] | Area throughput$^*$ [Mbit/s/km$^2$] | APC$^*$ [W/km$^2$] | $M^*$ | $K^*$ | $\zeta^*$ | Cell Reuse $1/\zeta^*$ [%] |
|-----------|---------------------------|---------------------------------|------------------|-------|-------|-----------|-----------------------------|
| ZF ($\gamma = 1$) | 3.81 | 672 | 176 | 53 | 13 | 3.4 | 28.98 |
| ZF ($\gamma = 3$) | 3.66 | 607 | 166 | 53 | 6 | 8.02 | 12.47 |
| ZF ($\gamma = 7$) | 2.71 | 453 | 167 | 56 | 3 | 16.34 | 6.12 |
| MR ($\gamma = 1$) | 3.58 | 617 | 172 | 52 | 12 | 3.80 | 26.28 |
| MR ($\gamma = 3$) | 2.96 | 517 | 174 | 58 | 5 | 8.98 | 11.14 |
| MR ($\gamma = 7$) | 2.03 | 446 | 220 | 82 | 3 | 17.08 | 5.86 |

throughput regime, which requires larger antenna array). Notice also that, as claimed in Lemma 4, $K^*$ decreases with $\gamma$ since the interference increases as more UEs are served. The pilot reuse factor is slightly smaller than with MR. This happens because ZF mitigates the intra-cell interference and allows a higher inter-cell interference, though this effect is mitigated by the incoming inter-cell interference that is not handle by either of these combining scheme.

VI. CONCLUSIONS

We designed a cellular network for maximal EE with MR, ZF and M-MMSE under the assumption of imperfect CSI and a multislope path loss model. This was formulated as an optimization problem by using a lower bound on the SE and a state-of-the-art power consumption model. The variables were pilot reuse factor $\zeta$ and BS density $\lambda$ for a Massive MIMO network. The results showed that the additional computational complexity of M-MMSE processing pays off in terms of EE and area throughput, though all the scheme behaves substantially the same with respect to $\lambda$, that is, reducing the cell size does not bring benefits in terms of EE. To get further insights, we concentrated on ZF and formulated the optimization problem by using stochastic geometry and a new lower bound on the SE. The variables were pilot reuse factor, number of BS antennas and UEs per BS. The results showed that ZF allows a higher densification of the network and the use of a smaller pilot reuse factor while achieving a higher EE than with MR combining. Also, it turned out that the EE-optimal configuration resembles a Massive MIMO setup.

APPENDIX A — PROOF OF LEMMA 2

To ease understanding, the proof that allows to obtain the lower bound for the UL average ergodic SE in Lemma 2, when ZF combiner is used for signal detection, is articulated in two steps. The first aims at computing all the inner expectations in (19) with respect to jointly the channel and pilot realizations, whereas the second makes use of all these terms, together with the power allocation
policy in Section II, to evaluate the outer expectation with respect to both the BSs and UEs locations.

A.1. Computation of all the expectations in (19)

Let \( \mathbf{v}_{jk} = \mathbf{V}_{j}^{ZF} \mathbf{e}_k \) be the ZF detector for the typical UE \( k \) in cell \( j \), with \( \mathbf{V}_{j}^{ZF} \) as in (11) and \( \mathbf{e}_k \) being the \( k \)th vector of the standard basis.

Received signal power: The useful term is computed as:

\[
|\mathbb{E}_{\{\mathbf{h},\mathbf{a}\}} \{ \mathbf{v}_{jk}^H \mathbf{h}_{ji}^j \}|^2 = |\mathbb{E}_{\{\mathbf{h},\mathbf{a}\}} \{ \mathbf{v}_{jk}^H \hat{\mathbf{h}}_{ji}^j \}|^2 = 1
\]

where (a) follows from Corollary 1 and (b) comes from the ZF combining.

Noise power: The noise term is obtained as:

\[
\mathbb{E}_{\{\mathbf{h},\mathbf{a}\}} \{ ||\mathbf{v}_{jk}||^2 \} \overset{(a)}{=} \mathbb{E}_{\{\mathbf{h},\mathbf{a}\}} \left\{ \left[ (\mathbf{H}_j^j)^H \mathbf{H}_j^j \right]^{-1} \right\}_{k,k} = \frac{1}{M-K} \mathbb{E}_{\{\mathbf{a}\}} \left\{ \frac{1}{\gamma_{jk}^j} \right\}
\]

where (a) is due to the ZF combining, (b) exploits both the statistics of \( \hat{\mathbf{h}}_{ji}^j \) in Corollary 1, with \( \gamma_{jk}^j \) denoted in (5), and the properties of Wishart matrices (e.g., see [9, Proof of Proposition 3] or [32] for a more general treatment).

Intra-cell interference power: Consider now the cell of interest \( j \) and let us compute the intra-cell interference. Then, for any interferer UE \( i \) in cell \( j \) we have that

\[
\mathbb{E}_{\{\mathbf{h},\mathbf{a}\}} \{ ||\mathbf{v}_{jk}^H \mathbf{h}_{ji}^j ||^2 \} = \mathbb{E}_{\{\mathbf{h},\mathbf{a}\}} \{ \text{tr}(\mathbf{v}_{jk}^H \mathbf{v}_{jk}^H \hat{\mathbf{h}}_{ji}^j (\hat{\mathbf{h}}_{ji}^j)^H) \} = \mathbb{E}_{\{\mathbf{h},\mathbf{a}\}} \{ \text{tr}(\mathbf{v}_{jk}^H \mathbf{v}_{jk}^H \hat{\mathbf{h}}_{ji}^j (\hat{\mathbf{h}}_{ji}^j)^H) \}
\]

\[
= \delta(i-k) + \mathbb{E}_{\{\mathbf{a}\}} \left\{ (\beta_{ji}^j - \gamma_{ji}^j) \mathbb{E}_{\{\mathbf{h}\}} \{ ||\mathbf{v}_{jk}||^2 \} \right\} \overset{(b)}{=} \delta(i-k) + \mathbb{E}_{\{\mathbf{a}\}} \left\{ (\beta_{ji}^j - \gamma_{ji}^j) \mathbb{E}_{\{\mathbf{h}\}} \{ ||\mathbf{v}_{jk}||^2 \} \right\}
\]

with \( \delta(\cdot) \) as the delta Kronecker function and where (a)–(b) follow from Corollary 1 while using ZF combining for the first term and (c) is due to (43).

Inter-cell interference power: The inter-cell interference collected by BS \( j \) from cell \( l \), e.g., depends on whether the UEs in that cell use the same or a different pilot subset \( \Phi_l \) than the one used in cell \( j \), that is \( (l,i) \in \mathcal{P}_{j,k} \) with \( \mathcal{P}_{j,k} = \{(l,i) \in \Phi_{\lambda} \setminus \{j\} \times \{1, \cdots, K\} : a_{lj} = 1\} \). Consider a UE \( i \) that does not cause pilot contamination to UE \( k \) of cell \( j \). In that case, \( (l,i) \notin \mathcal{P}_{j,k} \) and

\[
\mathbb{E}_{\{\mathbf{h},\mathbf{a}\}} \{ ||\mathbf{v}_{jk}^H \hat{\mathbf{h}}_{li}^i ||^2 | a_{lj} = 0 \} = \mathbb{E}_{\{\mathbf{h},\mathbf{a}\}} \{ \text{tr}(\mathbf{v}_{jk}^H \mathbf{v}_{jk}^H \hat{\mathbf{h}}_{li}^i (\hat{\mathbf{h}}_{li}^i)^H) | a_{lj} = 0 \}
\]

\[
= \beta_{li}^j \mathbb{E}_{\{\mathbf{h},\mathbf{a}\}} \{ ||\mathbf{v}_{jk}||^2 | a_{lj} = 0 \} \overset{(a)}{=} \beta_{li}^j \mathbb{E}_{\{\mathbf{h},\mathbf{a}\}} \{ ||\mathbf{v}_{jk}||^2 | a_{lj} = 0 \} \overset{(b)}{=} \beta_{li}^j \mathbb{E}_{\{\mathbf{a}\}} \left\{ \frac{1}{\gamma_{lj}^i} \right\}
\]

where (a) follows from Corollary 1 and the fact that \( \mathbf{v}_{jk} \) is a function of \( \{\hat{\mathbf{h}}_{ji}^j\}_{i=1}^K \), which are statistically independent from \( \{\hat{\mathbf{h}}_{li}^i\}_{i=1}^K \) (and so \( \{\hat{\mathbf{h}}_{ji}^j\}_{i=1}^K \)) in presence of no pilot contamination,
for which (6) does not hold and \((b)\) is due to (43). Consider an interferer UE \(i\) that cause pilot contamination to UE \(k\) of cell \(j\). Then, \((l, i) \in \mathcal{P}_{j, k}\) and using Corollary 1 leads to

\[
E_{\{h, a\}} \{ |v_{jk}^H h_{ji}^j|^2 | a_{lj} = 1 \} = E_{\{h, a\}} \{ |v_{jk}^H \tilde{h}_{ji}^j|^2 | a_{lj} = 1 \} + E_{\{h, a\}} \{ |v_{jk}^H \hat{h}_{ji}^j|^2 | a_{lj} = 1 \}
\]

(46)

Let us now tackle both contributions to the pilot contamination separately. The former is

\[
E_{\{h, a\}} \{ |v_{jk}^H \tilde{h}_{ji}^j|^2 | a_{lj} = 1 \} = \frac{\beta_{ji}^j}{M - K} \mathbb{E}_{\{a\}} \left\{ \frac{1}{\gamma_{jk}} |a_{lj} = 1 \right\} - \frac{1}{M - K} \mathbb{E}_{\{a\}} \left\{ \frac{\gamma_{ji}^j}{\gamma_{jk}} |a_{lj} = 1 \right\}
\]

(47)

where in \((a)\) we make use of (6) that accounts the linearity between channel estimates having the same pilot sequence and \((b)\) follows from the same considerations used for (44) since conditioning has no impact here. The latter term is computed as

\[
E_{\{h, a\}} \{ |v_{jk}^H \hat{h}_{ji}^j|^2 | a_{lj} = 1 \} = \mathbb{E}_{\{h, a\}} \{ \text{tr}(v_{jk}^H v_{jk}^H h_{ji}^j) | a_{lj} = 1 \}
\]

(48)

where \((a)\) follows from Corollary 1 while keeping the conditioning, \((b)\) is due to (43) and \((c)\) follows from (6). The inter-cell interference term can be computed by taking into account the probability of having or not pilot contamination between UE \(i\) of cell \(l\) and the typical UE, that is, \(\Pr(a_{lj} = 1) = 1/\zeta\) and \(\Pr(a_{lj} = 0) = 1 - 1/\zeta\). In particular, from (45) – (48) we obtain

\[
E_{\{h, a\}} \{ |v_{jk}^H h_{ji}^j|^2 \} = \Pr(a_{lj} = 1) E_{\{h, a\}} \{ |v_{jk}^H h_{ji}^j|^2 | a_{lj} = 1 \} + \Pr(a_{lj} = 0) E_{\{h, a\}} \{ |v_{jk}^H h_{ji}^j|^2 | a_{lj} = 0 \}
\]

\[
= \frac{1}{\zeta} \frac{(\beta_{ji}^j)^2}{\beta_{li}^j \beta_{ji}^j} \delta(i - k) - \frac{1}{\zeta} \frac{(\beta_{ji}^j)^2}{M - K} \frac{(\beta_{li}^j)^2}{\beta_{li}^j \beta_{ji}^j} \mathbb{E}_{\{a\}} \left\{ \frac{\gamma_{ji}^j}{\gamma_{jk}} |a_{lj} = 1 \right\} + \frac{\beta_{li}^j}{M - K} \mathbb{E}_{\{a\}} \left\{ \frac{1}{\gamma_{jk}} \right\}
\]

(49)

A.2. Computation of the lower bound on the UL SE

To begin with, let us define the following quantities for \(i = 1, \ldots, K\)

\[
\vartheta_{ji}^{(1)} = \sum_{l \in \Phi_{\lambda \setminus \{j\}}} \frac{\beta_{ji}^j}{\beta_{li}^j}, \quad \vartheta_{ji}^{(2)} = \sum_{l \in \Phi_{\lambda \setminus \{j\}}} \left( \frac{\beta_{ji}^j}{\beta_{li}^j} \right)^2
\]

(50)
which are then used to compute the inner expectations in (43), (44) and (49) with respect to the pilot realizations only. Then, the following terms are of interest

\[
\mathbb{E}_{(a)} \left\{ \frac{1}{\gamma_{jk}} \right\} = \frac{1}{\beta_{jk}} \left( 1 + \frac{\varphi_{jk}}{\zeta} + \frac{1}{\text{SNR}_p} \right) \quad (51)
\]

\[
\mathbb{E}_{(a)} \left\{ \frac{\gamma_{ji}}{\gamma_{jk}} \right\} = \left\{ \begin{array}{ll}
\mathbb{E}_{(a)} \{1\} = 1 & \text{if } i = k \\
\mathbb{E}_{(a)} \left\{ \frac{1}{\gamma_{jk}} \right\} \mathbb{E}_{(a)} \{\gamma_{ji}\} = \frac{\beta_{jk}}{\beta_{ji} + \varphi_{ji}} + \frac{1}{\text{SNR}_p} & \text{if } i \neq k
\end{array} \right. \quad (52)
\]

where (51) follows from (5) with \( \mathbb{E}_{(a)} \{a_{ij}\} = 1/\zeta \) and \( \varphi_{jk} \) that is denoted in (50), while in (52) we exploit the fact that UEs \( i \) and \( k \) in cell \( j \) cannot share the same pilot sequence when \( i \neq k \); and (a) comes directly from jointly using Jensen’s inequality and (51). The noise term in (19) is computed by plugging (51) into (43) as follows

\[
\frac{\beta_{jk}}{\text{SNR}_0} \mathbb{E}_{(h,a)} \{||v_{jk}||^2\} = \frac{1}{M - K} \frac{1}{\text{SNR}_0} \left( 1 + \frac{\varphi_{jk}}{\zeta} + \frac{1}{\text{SNR}_p} \right). \quad (53)
\]

with \( \text{SNR}_0 = P_0/\sigma^2 \). The interference contribution is decomposed as

\[
\sum_{l \in \Phi_x} \sum_{i=1}^K \frac{\beta_{jk}}{\beta_{li}} \mathbb{E}_{(h,a)} \{||v_{jk}^H h_{li}^j||^2\} = \sum_{i=1}^K \frac{\beta_{jk}}{\beta_{ji}} \mathbb{E}_{(h,a)} \{||v_{jk}^H h_{ji}^j||^2\} + \underbrace{\sum_{l \in \Phi_x \setminus \{j\}} \sum_{i=1}^K \frac{\beta_{jk}}{\beta_{li}} \left( 1 - \frac{1}{\zeta} \right) \mathbb{E}_{(h,a)} \{||v_{jk}^H h_{li}^j||^2|a_{l,i} = 0\}}_{\text{Intra-cell interference}} + \underbrace{\frac{1}{\zeta} \mathbb{E}_{(h,a)} \{||v_{jk}^H h_{ji}^j||^2|a_{l,i,j} = 1\}}_{\text{Inter-cell interference (no pilot cont.)}} + \underbrace{\frac{1}{\zeta} \mathbb{E}_{(h,a)} \{||v_{jk}^H h_{ji}^j||^2|a_{l,i,j} = 1\}}_{\text{Inter-cell interference (pilot cont.)}}. \quad (54)
\]

The first term accounts for intra-cell interference from all UEs \( i \) in cell \( j \) and thus jointly using (51) and (52) into (44) becomes

\[
\sum_{i=1}^K \frac{\beta_{jk}}{\beta_{ji}} \mathbb{E}_{(h,a)} \{||v_{jk}^H h_{ji}^j||^2\} \leq 1 + \frac{1}{M - K} \left[ K \left( 1 + \frac{\varphi_{jk}}{\zeta} + \frac{1}{\text{SNR}_p} \right) - \sum_{i=1}^K \frac{1}{\zeta} \left( 1 + \frac{\varphi_{jk}}{\zeta} + \frac{1}{\text{SNR}_p} \right) \right].
\]

The second term accounts for the interference from all UEs \( i \) in cells \( l \neq j \) and thus it can be computed substituting (51) – (52) into (49), which reads

\[
\sum_{l \in \Phi_x \setminus \{j\}} \sum_{i=1}^K \frac{\beta_{jk}}{\beta_{li}} \mathbb{E}_{(h,a)} \{||v_{jk}^H h_{li}^j||^2\} \leq \frac{1}{M - K} \left[ \left( 1 + \frac{\varphi_{jk}}{\zeta} + \frac{1}{\text{SNR}_p} \right) \sum_{i=1}^K \varphi_{ji} + \frac{(M - K)}{\zeta} \varphi_{jk}^2 \right]
\]

\[
- \frac{1}{\zeta} \sum_{i=1}^K \varphi_{ji} \left( 1 + \frac{\varphi_{jk}}{\zeta} + \frac{1}{\text{SNR}_p} \right).
\]

\(^8\)Hereafter we do not consider the conditioning anymore since, as we will see in a while, we lower bound this term when averaging over the BSs and UEs locations in the outer expectation.
Plugging (42) and (53) – (55) together into (19) we have

$$\text{SINR} \geq \frac{M - K}{\left( K + \frac{1}{\text{SNR}_p} + \sum_{i=1}^{K} \varphi_j^{(1)} \right) \left( 1 + \frac{\varphi_j^{(1)}}{\zeta} + \frac{1}{\text{SNR}_p} \right) + \frac{M-K}{\zeta} \varphi_j^{(2)} - \sum_{i=1}^{K} \left( \frac{1+\varphi_j^{(1)}}{1+\varphi_j^{(1)}+\frac{1}{\text{SNR}_p}} \right) \left( 1 + \frac{\varphi_j^{(2)}}{\zeta} \right) } \quad (56)$$

This completes the first part of the proof.

**APPENDIX B — PROOF OF LEMMA 1**

Next, a tractable lower bound on the UL average ergodic SE of the typical UE is computed, where the expectation is taken with respect to the BSs and UEs locations. To begin with, the Jensen’s inequality is applied to move the expectation inside the logarithm and obtain

$$\mathbb{E}_d \left\{ \log_2 \left( 1 + \frac{1}{\text{SINR}^{-1}} \right) \right\} \geq \log_2 \left( 1 + \frac{1}{\mathbb{E}_d \{ \text{SINR}^{-1} \} } \right) \geq \log_2 \left( 1 + \text{SINR} \right). \quad (57)$$

where we denote with $\text{SINR} = \mathbb{E}_d \{ \text{SINR}^{-1} \}$. Before proceeding further, let us observe that

$$\mathbb{E}_d \left\{ \frac{1+\varphi_j^{(1)}}{\zeta} + \frac{1}{\text{SNR}_p} \right\} = \begin{cases} \mathbb{E}_d \{1\} = 1 & \text{if } i = k \\ \mathbb{E}_d \left\{ 1 + \frac{\varphi_j^{(1)}}{\zeta} + \frac{1}{\text{SNR}_p} \right\} \mathbb{E}_d \left\{ \frac{1}{1+\varphi_j^{(1)}+\frac{1}{\text{SNR}_p}} \right\} \geq 1 & \text{if } i \neq k \end{cases} \quad (58)$$

$$\mathbb{E}_d \left\{ \frac{1+\varphi_j^{(2)}}{\zeta} + \frac{1}{\text{SNR}_p} \right\} = \begin{cases} \mathbb{E}_d \left\{ \frac{1+\varphi_j^{(2)}}{\zeta} + \frac{1}{\text{SNR}_p} \right\} \mathbb{E}_d \left\{ \varphi_j^{(2)} \right\} & \text{if } i = k \\ \mathbb{E}_d \left\{ \frac{\varphi_j^{(2)}}{1+\varphi_j^{(1)}+\frac{1}{\text{SNR}_p}} \right\} \mathbb{E}_d \left\{ 1 + \frac{\varphi_j^{(1)}}{\zeta} + \frac{1}{\text{SNR}_p} \right\} \geq \mathbb{E}_d \left\{ \varphi_j^{(2)} \right\} & \text{if } i \neq k \end{cases} \quad (59)$$

where in (58) we use the independence between UEs distance realizations for $i \neq k$ together with the Jensen’s inequality, while in (59) we lower bound the expectation$^9$.

From (56), the expectation of SINR$^{-1}$ can be expanded as follows:

$$\text{SINR} \geq \frac{1}{M - K} \left( \left( K + \frac{1}{\text{SNR}_p} \right) \left( 1 + \frac{1}{\mathbb{E}_d \{ \varphi_j^{(1)} \} } \right) + \left( 1 + \frac{1}{\text{SNR}_p} \right) \sum_{i=1}^{K} \mathbb{E}_d \{ \varphi_j^{(1)} \} + \frac{M - K}{\zeta} \mathbb{E}_d \{ \varphi_j^{(2)} \} - K - \frac{1}{\zeta} \sum_{i=1}^{K} \mathbb{E}_d \{ \varphi_j^{(2)} \} \right) \quad (60)$$

Now, in order to obtain the achievable lower bound in (20) we introduce the following Lemma:

$^9$Let us consider one term of the sum within $\varphi_j^{(1)}$ at a time and denote with $x = \beta_j^{(1)} / \beta_j^{(1)}$. Then we have $\mathbb{E}_x \{ x \} \geq \frac{\mathbb{E}_x \{ x^2 \} }{\mathbb{E}_x^{(1)} \{ x \} } \geq \frac{\mathbb{E}_x \{ x^2 \} }{\mathbb{E}_x^{(1)} \{ x \} }$ by applying Jensen’s inequality first (since $b = 1 + \frac{1}{\text{SNR}_p} > 0$) and Holder’s inequality at second.
Lemma 5. Assume a multislope path-loss model \( \beta^j_{lk}(d^l_{lk}) \) as in (2) and \( d^l_{lk} \in \Phi_{\lambda \setminus \{j\}} \) with \( d^l_{lk} \) being the distance between UE \( k \) in cell \( l \) and the BS in cell \( j \) (where \( \Phi_{\lambda \setminus \{j\}} \) describes the set of BSs distributed as an H-PPP with density \( \lambda \) ) we have

\[
\mathbb{E}_d \left\{ \vartheta_{ji}^{(\kappa)} \right\} = \mathbb{E}_d \left\{ \sum_{l \in \Phi_{\lambda \setminus \{j\}}} \left( \frac{\beta^j_{lk}(d^l_{lk})}{\beta^j_{li}(d^l_{li})} \right)^\kappa \right\} = \mu_\kappa, \quad \forall \kappa = \{1, 2\} \tag{61}
\]

\[
\mathbb{E}_d \left\{ \vartheta_{jk}^{(1)} \vartheta_{ji}^{(1)} \right\} = \mathbb{E}_d \left\{ \sum_{n \in \Phi_{\lambda \setminus \{j\}}} \sum_{l \in \Phi_{\lambda \setminus \{j\}\setminus\{n\}}} \left( \frac{\beta^j_{nk}(d^l_{nk})}{\beta^j_{li}(d^l_{li})} \right) \left( \frac{\beta^j_{lk}(d^l_{lk})}{\beta^j_{li}(d^l_{li})} \right) \right\} \leq \mu_1^2 + \mu_2 \tag{62}
\]

where \( \mu_\kappa \) for \( \kappa = 1, 2 \) is denoted in (22).

**Proof.** We start by considering the BSs distributed in a circular area of finite radius \( r \) and wrap around in the radial domain to keep the translation invariance. Considering the closest BS policy association in Section II and the result in Corollary 3, the average number of inter-cell interferers in that limited circular area are \( \lambda \lambda \) with \( \lambda \lambda (r, \lambda) = \pi(r^2 - \mathbb{E}_d \{ (d^j_{jk})^2 \}) = \pi(r^2 - \frac{1}{\pi \lambda}) \). This is because there are no interfering BSs closer than the one is serving the typical UE \( k \) in cell \( j \), that is BS \( j \). Then, (61) – (62) can be written as [13, Appendix B]

\[
\mathbb{E}_d \left\{ \vartheta_{ji}^{(\kappa)} \right\} = \lambda \lambda (r, \lambda) \mathbb{E}_d \left\{ \left( \frac{\beta^j_{lk}(d^l_{lk})}{\beta^j_{li}(d^l_{li})} \right)^\kappa \right\} \tag{63}
\]

\[
\mathbb{E}_d \left\{ \vartheta_{jk}^{(1)} \vartheta_{ji}^{(1)} \right\} \leq \lambda \lambda (r, \lambda) \left( \lambda \lambda (r, \lambda) - 1 \right) \mathbb{E}_d \left\{ \left( \frac{\beta^j_{lk}(d^l_{lk})}{\beta^j_{li}(d^l_{li})} \right)^2 \right\} + \mathbb{E}_d \left\{ \left( \frac{\beta^j_{lk}(d^l_{lk})}{\beta^j_{li}(d^l_{li})} \right)^2 \right\} \tag{64}
\]

Therefore, (63) – (64) requires computing the following term for \( \kappa = \{1, 2\} \)

\[
\mathbb{E}_d \left\{ \left( \frac{\beta^j_{lk}(d^l_{lk})}{\beta^j_{li}(d^l_{li})} \right)^\kappa \right\} = (a) \mathbb{E}_d \left\{ \beta(d^l_{lk})^{-\kappa} \mathbb{E}_d \{ \beta(d^l_{lk})^{\kappa} | d^l_{lk} \} \right\} \equiv (b) \mathbb{E}_d \left\{ \beta(d^l_{lk})^{-\kappa} \int_{d^l_{lk}}^r \beta(x)^\kappa \frac{2x}{r^2 - d^l_{lk}} dx \right\}
\]

\[
= (c) \sum_{n=1}^N \left( \int_{R_{n-1}}^{R_n} \frac{2 \beta_n(y)^{-\kappa}}{r^2 - y^2} \left( \int_y^r x \beta(x)^\kappa dx \right) f_d(y) dy \right)
\]

\[
= (d) \sum_{n=1}^N \left( \int_{R_{n-1}}^{R_n} \frac{2 \beta_n(y)^{-\kappa}}{r^2 - y^2} \left( \sum_{i=n+1}^N \int_{R_{i-1}}^{R_i} x \beta_i(x)^\kappa dx + \int_{R_n}^{R_n} x \beta_n(x)^\kappa dx \right) f_d(y) dy \right)
\]

where in (a) we use the theorem of total expectation conditioning over \( d^l_{ik} \), (b) is due to the fact that there are no interfering BSs closer than the \( j \)th and change of variable to polar coordinates and in (c)–(d) we use the multislope pathloss model in (2) where \( f_d(d^l_{jk}) \) is the probability density
function of the distance from the typical UE \( k \) to its serving BS \( j \). Then, by using (65) and taking the limit for \( r \to \infty \), to consider wrap around in \( \mathbb{R}^2 \), we obtain\(^{10}\)

\[
\lambda A(r, \lambda) \mathbb{E}_d \left\{ \left( \frac{\beta_{lk}^f(d_{lk})}{\beta_{lk}^l(d_{lk})} \right)^{\kappa} \right\} \to 2\lambda \pi \sum_{n=1}^{N} \left( \frac{1}{\kappa \alpha_n} \frac{1}{2} \int_{R_{n-1}}^{R_n} y^2 f_d(y) \, dy + c_n(\kappa) \int_{R_{n-1}}^{R_n} y^{\kappa \alpha_n} f_d(y) \, dy \right).
\]

(66)

Finally, by using the statistics in Corollary 3 the two integral expectations found into (66) can be computed with some change of variables and standard elements of calculus and are given as follows

\[
\int_{R_{n-1}}^{R_n} y^2 f_d(y) \, dy = \frac{1}{\pi \lambda} \left( \Gamma\left(2; \pi \lambda R_{n-1}^2\right) - \Gamma\left(2; \pi \lambda R_n^2\right) \right)
\]

(67)

\[
\int_{R_{n-1}}^{R_n} y^{\kappa \alpha_n} f_d(y) \, dy = \frac{1}{(\pi \lambda)^{\frac{\kappa \alpha_n}{2}}} \left( \Gamma\left(1 + \frac{\kappa \alpha_n}{2}; \pi \lambda R_{n-1}^2\right) - \Gamma\left(1 + \frac{\kappa \alpha_n}{2}; \pi \lambda R_n^2\right) \right).
\]

(68)

Plugging (65) – (68) into (61) – (62) completes the proof.

\[\square\]

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\(^{10}\)Notice that the function within the integral is a fractional polynomial of second order degree that converges to a bounded real-value as \( r \to \infty \). For this reason, the bounded convergence theorem conditions are satisfied and that operation is allowed.
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