Permutation Symmetry of the Scattering Equations

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Abstract

Closed formulas for tree amplitudes of \( n \)-particle scatterings of gluon, graviton, and massless scalar particles have been proposed by Cachazo, He, and Yuan. It depends on \((n - 3)\) quantities \(\sigma_\alpha\) which satisfy a set of coupled scattering equations, with momentum dot products as input coefficients. These equations are known to have \((n - 3)!\) solutions, hence each \(\sigma_\alpha\) is believed to satisfy a single polynomial equation of degree \((n - 3)!\). In this article, we derive the transformation properties of \(\sigma_\alpha\) under momentum permutation, and verify them with known solutions at low \(n\), and with exact solutions at any \(n\) for special momentum configurations. For momentum configurations not invariant under a certain momentum permutation, new solutions can be obtained for the permuted configuration from these symmetry relations. These symmetry relations for \(\sigma_\alpha\) lead to symmetry relations for the \((n - 3)! + 1\) coefficients of the single-variable polynomials, whose correctness are checked with the known cases at low \(n\). The extent to which the coefficient symmetry relations can determine the coefficients is discussed.
I. INTRODUCTION

The number of Feynman tree diagrams for \( n \)-gluon scattering grows rapidly with \( n \). There are 4 diagrams for \( n = 4 \), 25 diagrams for \( n = 5 \), and 220 diagrams for \( n = 6 \). By the time one gets to \( n = 12 \), the number exceeds five billion. It is therefore highly remarkable that if all the gluon helicities are the same, or only one of them is different, the resulting tree amplitude sums up to be zero whatever \( n \) is. If all but two are the same, then the result of the sum consists of only one term, given by the Parke-Taylor formula [1]. Recently, Cachazo, He, and Yuan (CHY) [2] were able to generalize this formula to any helicity configuration, any spacetime dimension \( D \), not only for gluon scattering, but also to graviton and massless scalar scattering amplitudes. Later on similar expressions for the scattering of massive scalar particles were also obtained [3].

The formula consists of a sum of \((n - 3)!\) terms, each of which is associated with a solution of the scattering equations

\[
\sum_{b \neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0 \quad (a = 1, 2, \cdots, n)
\]

in the unknown variables \( \sigma_a \), where \( k_a \) are the incoming momenta. Though the number \((n - 3)!\) is still very large for large \( n \), nevertheless it is very much smaller than the number of Feynman diagrams. For example, these numbers are 1, 2, 6 respectively for \( n = 4, 5, 6 \), to be compared with the numbers 4, 25, 220 of Feynman diagrams. For \( n = 12 \), the number \( 9! = 362,880 \) is much smaller than 5 billion.

The scattering equations possess a modular invariance, so three of the \( n \) variables \( \sigma_\alpha \) can be fixed. By multiplying through with the product of denominators, the scattering equations can be turned into \( n - 3 \) coupled homogeneous polynomial equations of degree \( n - 3 \) each. These equations are known to have \((n - 3)!\) solutions [2], hence the equation to determine a single variable \( \sigma_\alpha \) is expected to be a polynomial equation of degree \((n - 3)!\). It is generally not easy to obtain even the single-variable polynomials, not to speak of their solutions. Though there is no problem to get \( n = 4 \) and \( n = 5 \), special technique is required for \( n \geq 6 \) [4, 5]. However, there are special momentum configurations for which solutions can be obtained for any \( n \) [5, 6].

Analytic solution for \( \sigma_\alpha \) is impossible to obtain except for low \( n \), and for special momentum configurations. However, since they are central to the CHY scattering formulas, it is
desirable to find out as much about them as possible. In this article we will discuss how \( \sigma_\alpha \) transforms under momentum permutations. A set of \textit{symmetry relations} is derived in the next section, and verified against the known solutions of \( n = 4 \) and \( n = 5 \). These relations will also be checked against the exact solutions in certain momentum configurations. For momentum permutations which alter the special momentum configurations, new exact solutions can be obtained for the altered configurations from these symmetry relations.

Let \( A_p \) (\( 0 \leq p \leq (n - 3)! \)) be the coefficients of the polynomial equation for a single \( \sigma_\alpha \). Symmetry relations for \( \sigma_\alpha \) lead to symmetry relations for \( A_p \) which will be worked out in Sec. III. These relations are checked using the known coefficients for \( n = 4 \), \( n = 5 \), and \( n = 6 \). In Sec. IV, we discuss the amount of constraints put on \( A_p \) just by their symmetry relations. The cases for \( n = 4 \) and \( n = 5 \) are worked out in detail to illustrate the general discussion. Up to an overall normalization, the symmetry relations determine all the four parameters controlling the \( n = 4 \) equation, and 36 of the 45 parameters involved in the \( n = 5 \) single-particle polynomial equation.

\section{Permutation Symmetry}

Let us use modular invariance to fix \( \sigma_1 = 0 \), \( \sigma_2 = 1 \), and \( \sigma_n = \infty \). Then the scattering equations (1) for \( \sigma_\alpha \) (\( 3 \leq \alpha \leq n - 1 \)) are

\begin{align*}
0 & = k_1 \cdot k_2 + \frac{k_1 \cdot k_3}{\sigma_3} + \frac{k_1 \cdot k_4}{\sigma_4} + \cdots + \frac{k_1 \cdot k_{n-1}}{\sigma_{n-1}} \quad (2) \\
0 & = k_2 \cdot k_1 + \frac{k_2 \cdot k_3}{\sigma_3 - 1} + \frac{k_2 \cdot k_4}{\sigma_4 - \sigma_3} + \cdots + \frac{k_2 \cdot k_{n-1}}{1 - \sigma_3} \quad (3) \\
0 & = \frac{k_3 \cdot k_1}{\sigma_3} + \frac{k_3 \cdot k_2}{\sigma_3 - 1} + \frac{k_3 \cdot k_4}{\sigma_4 - \sigma_3} + \cdots + \frac{k_3 \cdot k_n}{\sigma_n - \sigma_3} \quad (4) \\
0 & = \frac{k_4 \cdot k_1}{\sigma_4} + \frac{k_4 \cdot k_2}{\sigma_4 - 1} + \frac{k_4 \cdot k_3}{\sigma_3 - \sigma_4} + \cdots + \frac{k_4 \cdot k_{n-1}}{\sigma_{n-1} - \sigma_4} \quad (5) \\
& \quad \cdots \cdots \\
0 & = \frac{k_{n-1} \cdot k_1}{\sigma_{n-1}} + \frac{k_{n-1} \cdot k_2}{\sigma_{n-1} - 1} + \frac{k_{n-1} \cdot k_3}{\sigma_{n-1} - \sigma_3} + \cdots + \frac{k_{n-1} \cdot k_{n-2}}{\sigma_{n-1} - \sigma_{n-2}} \quad (6)
\end{align*}

The solutions \( \sigma_\alpha \) depend on \( k_1, k_2, \cdots, k_n \), but we will skip the arguments and write \( \sigma_\alpha(k_1, k_2, \cdots, k_n) \) simply as \( \sigma_\alpha \). If \( s \) is a permutation of \( n \) objects, then \( \sigma_\alpha(k_{s(1)}, k_{s(2)}, \cdots, k_{s(n)}) \) will be written as \( \sigma_\alpha(s) \). In particular, if \( s = (jk) \) is a transposition, then we will also write \( \sigma_\alpha(s) \) as \( \sigma_\alpha(jk) \), rather than the more cumbersome notation \( \sigma_\alpha((jk)) \).
A. Symmetry relations

The purpose of this subsection is to obtain the following relations between \(\sigma_\alpha(jk)\) and \(\sigma_\beta\) (\(3 \leq \alpha, \beta \leq n - 1\)):

\[
\begin{align*}
\sigma_\alpha(12) &= 1 - \sigma_\alpha \\
\sigma_\alpha(2\alpha) &= \frac{1}{\sigma_\alpha} \\
\sigma_\alpha(2\beta) &= \frac{\sigma_\alpha}{\sigma_\beta}, \quad (\beta \neq \alpha),
\end{align*}
\]

as well as

\[
\begin{align*}
\sigma_\alpha(1\alpha) &= \frac{\sigma_\alpha}{\sigma_\alpha - 1} \\
\sigma_\alpha(1\beta) &= \frac{\sigma_\beta - \sigma_\alpha}{\sigma_\beta - 1}, \quad (\beta \neq \alpha) \\
\sigma_\alpha(\alpha\beta) &= \sigma_\beta, \quad (\beta \neq \alpha) \\
\sigma_\alpha(\beta\gamma) &= \sigma_\alpha, \quad (\alpha, \beta, \gamma \text{ different}),
\end{align*}
\]

and

\[
\begin{align*}
\sigma_\alpha(1n) &= \frac{1}{\sigma_\alpha} \\
\sigma_\alpha(2n) &= \frac{\sigma_\alpha}{\sigma_\alpha - 1} \\
\sigma_\alpha(\alpha n) &= 1 - \sigma_\alpha \\
\sigma_\alpha(\beta n) &= \frac{\sigma_\alpha(1 - \sigma_\beta)}{\sigma_\alpha - \sigma_\beta}, \quad (\beta \neq \alpha).
\end{align*}
\]

To prove these results, let us start from (3). Interchange \(k_2\) with \(k_1\) and compare the resulting equation with (2), we get (7). Next, exchange \(k_3\) with \(k_1\) in (2) and compare the result with (4), we get (10) and (11) for \(\alpha = 3\). The proof for \(\alpha > 3\) is identical. Similarly, exchange \(k_3\) with \(k_2\) in (3) and compare the result with (4), we get (8) and (9) for \(\alpha = 3\). Similar proof works for \(\alpha > 3\) as well. Now interchange \(k_4\) with \(k_3\) in (4), compare the result with (5), and generalize the result to other \(\alpha, \beta, \gamma\), we get (12) and (13).

The proof for relations involving \(k_n\) is a little more complicated, as \(k_n\) does not appear in the scattering equations above. We must use momentum conservation to introduce \(k_n\), and then go through procedures similar to those adopted above. For example, if we replace \(k_1\) by \(- \sum_{i=2}^n k_i\) in (3), then it becomes

\[
0 = k_2 \cdot k_n + \sum_{\beta=3}^{n-1} k_2 \cdot k_\beta \left( 1 - \frac{1}{1 - \sigma_\beta} \right) = k_2 \cdot k_n + \sum_{\beta=3}^{n-1} k_2 \cdot k_\beta \frac{\sigma_\beta}{\sigma_\beta - 1}.
\]
Now interchange $k_n$ with $k_1$ and compare the result with (3), we get (14). Similarly, replace $k_2$ in (2) by momentum conservation to get

$$0 = k_1 \cdot k_n + \sum_{\alpha=3}^{n-1} (1 - \frac{1}{\sigma_\alpha}).$$

(19)

Now interchange $k_n$ with $k_2$ and compare the result with (2), we get (15). Finally, fix a $\alpha \geq 3$, and replace $k_\alpha$ in (2) using momentum conservation, then we get

$$0 = \frac{k_1 \cdot k_n}{\sigma_\alpha} + \sum_{j=2, j \neq \alpha}^{n-1} k_1 \cdot k_j \left( \frac{1}{\sigma_\alpha} - \frac{1}{\sigma_j} \right),$$

(20)

which can be written as

$$0 = k_1 \cdot k_n + \sum_{j=2, j \neq \alpha}^{n-1} k_1 \cdot k_j \left( 1 - \frac{\sigma_\alpha}{\sigma_j} \right).$$

(21)

Exchange $k_n$ with $k_\beta$ and compare with (2). If we set $j = 2$, we get (16). If we set $j = \beta \neq \alpha$, then we get (17) with $\alpha$ and $\beta$ reversed. This completes the proof of the symmetry relations (7) to (13).

Here are some supplemental remarks about the symmetry equations.

1. The symmetry relations are obtained by comparing equations that $\sigma_\alpha$ satisfy, not the solutions themselves. As a result, the $\sigma_\alpha$ appearing on the right of these symmetry relations may be a different solution than the $\sigma_\alpha$ appearing on the left.

2. Since the permutation relation $(ij) = (ik)(kj)(ik)$ is true for any $i, j, k$, the symmetry relations obtained above are not all independent. For momenta not involving $k_n$, we can derive (10) to (13) from (7) to (9). For relations involving $k_n$, another relation (14) is needed to obtain everything else. Details are given in Appendix A.

3. It follows from (7) and (16), (15) and (10), (14) and (8) that

$$\sigma_\alpha = \sigma_\alpha((12)(\alpha n)) = \sigma_\alpha((1n)(2\alpha)) = \sigma_\alpha((2n)(1\alpha))$$

(22)

for every $3 \leq \alpha \leq n - 1$. Since $(12)(\alpha n)$, $(1n)(2\alpha)$, $(2n)(1\alpha)$, together with the identity permutation, form the Klein group $Z_2 \times Z_2$, we shall refer to this identity as the *Klein-group identity.*
4. If we know a single $\sigma_\alpha$, then we can compute every other $\sigma_\beta$ from (12), so the hard work is to find the solutions for a single $\alpha$.

5. Using momentum conservation, $k_n$ can be eliminated so $\sigma_\alpha$ may be considered as a function of $k_1, k_2, \ldots, k_{n-1}$. In that form, equations (14) to (17) will not be useful, so that the only independent relations needed to be considered are (7), (8), and (9).

6. Equation (9) differs from the other two fundamental symmetry relations in that two different $\sigma$’s appear on the right hand side. However, using (12), we can get rid of $\sigma_\beta$. Using also (8), equation (9) can be written in the form

$$
\sigma_\alpha(2\alpha)\sigma_\alpha(2\beta)\sigma_\alpha(\alpha\beta) = 1.
$$

7. With that, it is amusing to note that the three relations (7), (8), (23) can be written as linear, quadratic, and cubic relations, respectively:

$$
\sigma_\alpha + \sigma_\alpha(12) = 1,
$$

$$
\sigma_\alpha\sigma_\alpha(2\alpha) = 1,
$$

$$
\sigma_\alpha(2\alpha)\sigma_\alpha(2\beta)\sigma_\alpha(\alpha\beta) = 1.
$$

B. Direct verification

In this subsection the symmetry relations will be verified directly for $n = 4$ and $n = 5$, where analytic solutions of the scattering equations are available. For four-dimensional spacetime, $n = 6$ solutions are also available [4], but they are lengthy and their verification is not be carried out here.

1. $n = 4$

There is only one nontrivial $\sigma_\alpha$, which is $\sigma_3$. It depends on the Mandelstam variables [7] $s = k_1 \cdot k_2 = k_3 \cdot k_4$, $t = k_1 \cdot k_3 = k_2 \cdot k_4$, and $u = k_1 \cdot k_4 = k_2 \cdot k_3$. These three variables are subject to the constraint $s + t + u = 0$ because all the particles are massless. The scattering equation obtained from (2) is

$$
s + \frac{t}{\sigma_3} = 0,
$$

(25)
whose solution is \( \sigma_3 = -t/s \).

Under momentum permutation (12), \( s \leftrightarrow s \) and \( t \leftrightarrow u \). Thus \( \sigma_3(12) = -u/s = (s + t)/s = 1 - \sigma_3 \), verifying (7). Under momentum permutation (23), \( u \leftrightarrow u \) and \( s \leftrightarrow t \), so \( \sigma_3(23) = -s/t = 1/\sigma_3 \), verifying (8). Equation (9) is irrelevant for \( n = 4 \), thus all the independent symmetry relations have been explicitly verified.

2. \( n = 5 \)

\( \sigma_3 \) and \( \sigma_4 \) depend on six scalar products, \( s = k_1 \cdot k_2 \), \( t_1 = k_1 \cdot k_3 \), \( t_2 = k_1 \cdot k_4 \), \( u_1 = k_2 \cdot k_3 \), \( u_2 = k_2 \cdot k_4 \), and \( v = k_3 \cdot k_4 \). These six variables sum up to zero because \( k_5^2 = (k_1 + k_2 + k_3 + k_4)^2 = 0 \), hence only five of them are independent. In what follows we shall take them to be \( s, t_1, t_2, u_1, u_2 \). Scalar products involving \( k_5 \) can be obtained by momentum conservation.

After multiplying by the product of denominators, eqs. (2) and (3) become

\[
0 = s\sigma_3\sigma_4 + t_1\sigma_4 + t_2\sigma_3, \tag{26}
\]
\[
0 = s(1 - \sigma_3)(1 - \sigma_4) + u_1(1 - \sigma_4) + u_2(1 - \sigma_3). \tag{27}
\]

A linear equation is obtained by subtracting the two,

\[
(s + t_1 + u_1)\sigma_4 + (s + t_2 + u_2)\sigma_3 = u_1 + u_2, \tag{28}
\]

which can be used to eliminate either \( \sigma_3 \) or \( \sigma_4 \). Substituting the result back into (26), we get a quadratic equation determining \( \sigma_3 \) or \( \sigma_4 \):

\[
0 = s(s + t_2 + u_2)\sigma_3^2 + (s(-s + t_1 - t_2 - u_1 - u_2) + t_1u_2 - t_2u_1)\sigma_3 - t_1(s + u_1 + u_2)
:= a\sigma_3^2 + b\sigma_3 + c, \tag{29}
\]
\[
0 = s(s + t_1 + u_1)\sigma_4^2 + (s(-s + t_1 + t_2 - u_1 - u_2) - t_1u_2 + t_2u_1)\sigma_4 - t_2(s + u_1 + u_2)
:= a'\sigma_4^2 + b'\sigma_4 + c', \tag{30}
\]

where

\[
a = s(s + t_2 + u_2),
\]
\[
b = s(-s + t_1 - t_2 - u_1 - u_2) + t_1u_2 - t_2u_1,
\]
\[
c = -t_1(s + u_1 + u_2),
\]
a' = s(s + t_1 + u_1),

b' = s(-s - t_1 + t_2 - u_1 - u_2) - t_1 u_2 + t_2 u_1,

c' = -t_2(s + u_1 + u_2). \tag{31}

The solutions of these quadratic equations are

\[ \sigma_3 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{32} \]
\[ \sigma_4 = \frac{-b' \pm \sqrt{b'^2 - 4a'c'}}{2a'}. \tag{33} \]

We will use \( \sigma_{3+} \) and \( \sigma_{3-} \) to denote solution (32) with the upper and the lower sign, and \( \sigma_{4+} \) and \( \sigma_{4-} \) to denote solution (33) with the upper and the lower sign.

With these explicit solutions, we are now ready to verify the independent symmetry relations (7) to (9). To do so, we need to know how the variables change under the momentum permutation (12), (23), and (24). The result is listed in Table 1. Please remember that \( v = k_3 \cdot k_4 = -(s + t_1 + t_2 + u_1 + u_2) \) is not independent, but its change is also listed in Table 1. Also, the relations obtained from the permutation (34) is not independent either, but it will be convenient to list it as well.

|    | s  | t_1 | t_2 | u_1 | u_2 | v   |
|----|----|-----|-----|-----|-----|-----|
| (12) | s  | u_1 | u_2 | t_1 | t_2 | v   |
| (23) | t_1 | s   | t_2 | u_1 | v   | u_2 |
| (24) | t_2 | t_1 | s   | v   | u_2 | u_1 |
| (34) | s  | t_2 | t_1 | u_2 | u_1 | v   |

Table 1. Transformation of dot products under momentum permutations

Let us now consider each of the symmetry relations separately.

**Eq. (7).** Because of the opposite sign in front of the square roots for solutions + and − in (32) and (33), clearly (7) must be interpreted to mean

\[ \sigma_{\alpha \pm}(12) = 1 - \sigma_{\alpha \mp}, \quad (\alpha = 3, 4). \tag{34} \]

In order for that to be true, we must have

\[ a(12) = a, \quad b(12) = -(2a + b), \quad b(12)^2 - 4a(12)c(12) = b^2 - 4ac, \tag{35} \]
and similarly with \(a, b, c\) replaced by \(a', b', c'\). These equalities are equivalent to

\[
a(12) = a, \quad b(12) = -(2a + b), \quad c(12) = a + b + c, \tag{36}
\]

and similarly with \(a, b, c\) replaced by \(a', b', c'\). Using the explicit expressions in (31), and the (12) row of Table 1, it is easily seen that these identities are true.

\textbf{Eq. (8).} Again, because of the opposite signs in front of the square roots, we must interpret (8) to mean

\[
\sigma_{3\pm}(23) = \frac{1}{\sigma_{3\mp}}, \quad \sigma_{4\pm}(23) = \frac{1}{\sigma_{4\mp}}, \tag{37}
\]

In order for those to be true, we must have

\[
a(23) = c, \quad b(23) = b, \quad c(23) = a \tag{38}
\]

\[
a'(24) = c', \quad b'(24) = b, \quad c'(24) = a'. \tag{39}
\]

These relations can be verified explicitly from (31) and Table 1.

\textbf{Eq. (9).} The verification of this is more complicated, because it involves the ratio of two different \(\sigma\)'s. (9) seems quite impossible unless the square roots in (32) and (33) are identical. The expressions for \(b^2 - 4ac\) and \(b'^2 - 4a'c'\) are both rather lengthy, but straight-forward computation shows that they are indeed equal. I shall use the letter \(d\) to denote them.

Let us compute \(\sigma_{3+}/\sigma_4\). A priori we do not know whether to use \(\sigma_{4+}\) or \(\sigma_{4-}\), but detailed calculation shows that it is \(\sigma_{4-}\). In that case,

\[
\frac{\sigma_{3+}}{\sigma_{4-}} = \frac{a'(-b + \sqrt{d})}{a(-b' - \sqrt{d})} = \frac{a'(-b + \sqrt{d})(-b' + \sqrt{d})}{a(b^2 - d)} = \frac{d + bb' - (b' + b)\sqrt{d}}{4ac'}. \tag{40}
\]

According to (9), this should be either \(\sigma_{3+}(24)\) or \(\sigma_{3-}(24)\). Again, detailed calculation shows that it is the latter, namely, \(\sigma_{3-}(24) = \left(-b(24) - \sqrt{d(24)}\right)/2a(24)\). In order for that to be true, it is necessary to have \(d(24) = d\), which can be verified to be true. Moreover, we need to have

\[
-b(24)/a(24) = (d + bb')/2ac', \quad 1/a(24) = (b' + b)/2ac', \tag{41}
\]

or equivalently,

\[
a(24) = 2ac'/(b + b'), \quad b(24) = -(d + bb')/(b + b'). \tag{42}
\]
Explicit substitution shows that these are indeed true, hence (9), in the form \( \sigma_3^-(24) = \sigma_3^+ / \sigma_4^- \), is verified. Similarly, one can also verify \( \sigma_3^+(24) = \sigma_3^- / \sigma_4^+ \) and \( \sigma_4^\pm(23) = \sigma_4^\pm / \sigma_3^\mp \) to be true.

Equation (9) is the most intriguing of the three independent symmetry relations: it involves two different \( \sigma \)'s on the right. As a result, the symmetry constraints must involve parameters for both \( \sigma_\alpha \) and \( \sigma_\beta \). Furthermore, they must appear nonlinearly, as in (42).

Clearly, for (42) to be true, \( a, b, c \) and \( a', b', c' \) must be closely related. Indeed, equation (12) tells us what their relations are. We will verify that directly below.

\textbf{Eq. (12).} This equation says \( \sigma_3(34) = \sigma_4, \sigma_4(34) = \sigma_3 \). From the sign of the square roots, we now expect \( \sigma_{3\pm}(34) = \sigma_{4\pm} \), which demands

\[
\begin{align*}
    a(34) &= a', & b(34) &= b', & c(34) &= c'.
\end{align*}
\]

These relations can also be explicitly verified to be true.

This completes the verification of the symmetry relations for \( n = 5 \).

\textbf{C. Special configurations}

For certain special momentum configurations, exact solutions for \( \sigma_\alpha \) can be obtained for any \( n \). We discuss some of these found in the literature [5, 6] in this subsection, to show that they either obey the symmetry relations (7), (8), and (9), or these relations can be used to produce exact solutions for permuted momentum configurations. In this connection please recall that the \( \sigma \)'s that appear on the right of (7) to (9) may be a different solution than what appears on the left.

\textit{1.}

Consider the special configuration

\[
\begin{align*}
    k_2 \cdot k_\alpha &= k_\alpha \cdot k_\beta = \mu, \quad (3 \leq \alpha \neq \beta \leq n - 1), \\
    k_1 \cdot k_2 &= k_1 \cdot k_\alpha = \nu,
\end{align*}
\]
where \( \mu \) is arbitrary and \( \nu \) is chosen so that \( k_n^2 = (\sum_{i=1}^{n-1} k_i)^2 = 0 \), namely, \( \nu = -(n-3)\mu/2 \). Dolan and Goddard [5] showed that the solutions of the scattering equations are

\[
\sigma_2 = 1, \quad \sigma_\alpha = \omega_\alpha, \quad (45)
\]

where \( \omega_\alpha \) are \((n-2)\)th roots of unity, different for different \( \alpha \), and none of them equal to unity. The \((n-3)!\) possibilities of arranging such distinct \( \omega_\alpha \)'s constitute the \((n-3)!\) solutions of the scattering equations.

Symmetry relation (8) is obeyed because \( \omega^{-1}_\alpha \) is another \((n-2)\)th root of unity, so it is just another solution of \( \sigma_\alpha \). Symmetry relation (9) is also obeyed because \( \sigma_\alpha/\sigma_\beta \) is another \((n-2)\)th root, not equal to unity if \( \beta \neq \alpha \). Note that neither of the momentum permutations \((2\alpha)\) and \((2\beta)\) changes the momentum configurations in (44).

With momentum permutation (12), the configuration (44) changes into the configuration

\[
k_1 \cdot k_\alpha = k_\alpha \cdot k_\beta = \mu, \quad (3 \leq \alpha, \beta \leq n-1),
\]

\[
k_1 \cdot k_2 = k_2 \cdot k_\alpha = \nu, \quad (46)
\]

whose solutions, according to (7), are

\[
\sigma_2 = 1, \quad \sigma_\alpha = 1 - \omega_\alpha. \quad (47)
\]

This is a new solution for the new configuration (46).

2.

A more general configuration is considered by Kalousios [6], in which \( k_2 \cdot k_\alpha \) is different from \( k_\alpha \cdot k_\beta \):

\[
k_1 \cdot k_\alpha = (1 + \nu)/2, \quad k_2 \cdot k_\alpha = (1 + \mu)/2, \quad k_\alpha \cdot k_\beta = 1, \quad (3 \leq \alpha, \beta \leq n-1). \quad (48)
\]

\( k_1 \cdot k_2 \) is determined by the requirement of \( k_n^2 = 0 \) to be

\[
k_1 \cdot k_2 = -(n-3)(n + \mu + \nu - 2)/2. \quad (49)
\]

According to Kalousios, the solution for \( \sigma'_\alpha = 2\sigma_\alpha - 1 \) is given by the \((n-3)\) roots of the Jacobi polynomial \( P_{n-3}^{(\mu,\nu)}(\sigma'_\alpha) \). In the \( \sigma' \) space, these roots lie in the interval \([-1,1]\), and they are known to reverse their sign when \( \mu \) and \( \nu \) are interchanged. This property agrees
with (7), which when expressed in the \(\sigma'\) variable, says \(\sigma'_\alpha (12) = -\sigma'_\alpha\). Furthermore, in this configuration, the momentum permutation (12) is equivalent to the interchange of \(\mu\) and \(\nu\).

The momentum configuration (48) is \textit{not} invariant under momentum permutation (2\(\alpha\)). Thus this permutation produces a new momentum configuration, and new solutions by using (8) and (9).

III. POLYNOMIAL EQUATIONS

The scattering equations (1) are equivalent to the polynomial equations [5] [8]

\[
0 = \sum_{a_1, a_2, \ldots, a_m \in A'} s_{a_1 a_2 \cdots a_m n} \sigma_{a_1} \sigma_{a_2} \cdots \sigma_{a_m} := h_m, \quad (1 \leq m \leq n - 3),
\]

where \(A' = \{2, 3, 4, \ldots, n - 1\}\), \(\sigma_2 = 1\), and \(s_{a_1 a_2 \cdots a_m n} = \frac{1}{2} (k_{a_1} + k_{a_2} + \cdots + k_{a_m} + k_n)^2\). The sum is taken over all distinct subsets of \(A'\) with \(m\) elements. Using momentum conservation, we can get rid of \(k_n\) in favor of \(k_1\) to write \(s_{a_1 a_2 \cdots a_m n}\) as \(\frac{1}{2} (k_1 + k_{a_1} + \cdots + k_{\bar{a}_{n-2-m}})^2\), where \(\bar{a}_i\) are the complementary indices of \(a_i\) in the set \(A'\).

The advantage of these equations is that each \(\sigma_i\) enters at most linearly. That makes it easier to eliminate all other variables to obtain a polynomial equation for a single variable. A single-variable polynomial can be easily obtained directly from the scattering equations for \(n = 4\) and \(n = 5\), but using these polynomial equations, a sixth order single-variable polynomial is also derived for \(n = 6\) in Ref. [5].

Since there are \((n - 3)!\) solutions, it is natural to assume such single-variable polynomials to be of degree \((n - 3)!\). Accordingly, let us write them in the form

\[
0 = \sum_{p=0}^{\nu} A_p^{(\alpha)} \sigma_{\alpha}^p,
\]

where \(A_p^{(\alpha)}\) is a function of \(k_1, \ldots, k_{n-1}\), and \(\nu = (n - 3)!\).

A. Symmetry of \(A_p^{(\alpha)}\)

Under a permutation of the momenta \(k_i\), \(\sigma_\alpha\) transforms according to the symmetry relations (7), (8), (10), and (13). They induce corresponding relations between the polynomial coefficients \(A_p^{(\alpha)}\) which we will work out in this subsection. The other symmetry relations
involves another variable $\sigma_\beta$ so they will not be immediately useful in determining the symmetry relations of $A_p^{(\alpha)}$.

Under $\sigma \rightarrow 1 - \sigma$, (51) changes into $0 = \sum_{p=0}^{\nu} \tilde{A}_p \sigma^p$, with

$$(-)^{p-\nu} \tilde{A}_p = \sum_{q=p}^{\nu} \frac{q!}{p!(q-p)!} A_q$$

$$= A_p + (p+1)A_{p+1} + \frac{(p+1)(p+2)}{2!} A_{p+2} + \cdots + \frac{(p+1)(p+2)\cdots \nu}{(\nu-p)!} A_\nu, \quad (52)$$

where an overall constant $(-)^\nu$ has been inserted to make $\tilde{A}_\nu = A_\nu$.

Under $\sigma \rightarrow 1/\sigma$, (51) changes into $0 = \sum_{p=0}^{\nu} \bar{A}_p \sigma^p$, with

$$\hat{A}_p = A_{\nu-p}, \quad (53)$$

Under $\sigma \rightarrow \sigma/(\sigma - 1)$, (51) changes into $0 = \sum_{p=0}^{\nu} \bar{A}_p \sigma^p$, with

$$(-)^{\nu-p} \tilde{A}_p = \sum_{q=0}^{p} \frac{(\nu-p+q)!}{q!(\nu-p)!} A_{p-q}$$

$$= A_p + (\nu+1-p)A_{p-1} + \frac{(\nu+1-p)(\nu+2-p)}{2!} A_{p-2} + \cdots$$

$$+ \frac{(\nu+1-p)(\nu+2-p)\cdots \nu}{p!} A_0. \quad (54)$$

Therefore (7), (8), and (10) translate into the relations

$$A_p^{(\alpha)}(1\beta) = \tilde{A}_p^{(\alpha)}, \quad A_p^{(\alpha)}(2\alpha) = \hat{A}_p^{(\alpha)}, \quad A_p^{(\alpha)}(1\alpha) = \bar{A}_p^{(\alpha)}.$$ \quad (55)

The third one can be derived from the first two so it will be ignored from now on. In addition, (12) and (13) imply

$$A_p^{(\alpha)}(\beta\gamma) = A_p^{(\alpha)}, \quad (56)$$

$$A_p^{(\alpha)}(\alpha\beta) = A_p^{(\beta)}.$$ \quad (57)

These relations will be verified directly for $n = 4, 5, 6$.

B. Direct verification for $n = 4, 5, 6$

1. $n = 4$

The single-variable linear polynomial is given in (25), with $A_1 = s$ and $A_0 = t$. Thus

$$A_1(12) = s = A_1, \quad A_0(12) = u = -(s+t) = -(A_1 + A_0), \quad (58)$$

which agrees with (55).
2. \( n = 5 \)

Equation (55) requires

\[
A_2(12) = A_2, \quad A_1(12) = -(A_1 + 2A_2), \quad A_0(12) = A_0 + A_1 + A_2.
\] (59)

In the notation of (29), \( A_2 = a, A_1 = b, A_0 = c \), so this relation is simply (36), which has been verified. Similarly we can verify it for \( \sigma_4 \). It also requires

\[
A_2(23) = A_0, \quad A_1(23) = A_1, \quad A_0(23) = A_2,
\] (60)

which is just (38) and that has also been verified. Similarly, one can verify the identities for \( \sigma_4 \) as well.

3. \( n = 6 \)

Single-variable polynomial equations have been obtained for \( n = 6 \) [4, 5]. Following the recipe given in eq. (3.9) of Ref. [5], with \( x = \sigma_3 \) and \( y = \sigma_4 \) in that equation, a sixth degree polynomial for \( \sigma_5 \) can be obtained. Its \( A_0 \) and \( A_6 \) coefficients are given by

\[
A_0 = (k_4 \cdot k_5 + k_3 \cdot k_5 + k_4 \cdot k_3 + k_1 \cdot k_5 + k_1 \cdot k_4 + k_1 \cdot k_3)^2(k_4 \cdot k_5 + k_1 \cdot k_5 + k_1 \cdot k_4) \\
\quad \times (k_1 \cdot k_5)^2(k_3 \cdot k_5 + k_1 \cdot k_5 + k_1 \cdot k_3),
\]

\[
A_6 = (k_4 \cdot k_3 + k_4 \cdot k_2 + k_2 \cdot k_3 + k_1 \cdot k_4 + k_1 \cdot k_3 + k_1 \cdot k_2)^2(k_4 \cdot k_2 + k_1 \cdot k_4 + k_1 \cdot k_2) \\
\quad \times (k_1 \cdot k_2)^2(k_2 \cdot k_3 + k_1 \cdot k_3 + k_1 \cdot k_2),
\] (61)

and the other coefficients are much too long to write down. According to (55), we should have \( A_6(12) = A_6 \), and \( A_6(25) = A_0 \), which can be seen from (61) to be true.

IV. SYMMETRY CONSTRAINTS

We found the symmetry relations for the coefficients \( A_p^{(\alpha)} \) of the single-variable polynomials in the last section. In this section we investigate the inverse, and ask to what extent the symmetry relations determine the coefficients \( A_p^{(\alpha)} \), and hence the single-variable equations.

Note that the symmetry relations (55) for \( A_p^{(\alpha)} \), derived from (7) and (8), are weaker than the symmetry relations for \( \sigma_\alpha \), as equation (9) was not used. Since (9) involves two \( \sigma \)’s, it is
difficult to translate it into equations for $A^{(a)}_p$. In the special case $n = 5$, we do know what it is. It is given by (42), which is nonlinear and very complicated, but we do not know how to generalize that to a larger $n$. As a result, the symmetry relations for $A^{(a)}_p$ are certainly ‘incomplete’, so we do not expect it to be able to yield the complete expressions for $A^{(a)}_p$, save for the case $n = 4$ where (9) is irrelevant. Still, we would like to see how far they can take us.

More precisely, pretend that we do not know about the scattering equations, either in the original form (1), or in its polynomial form (50), but we assume that we do know the following. Each $\sigma_\alpha$ satisfies a $\nu = (n - 3)!$th degree polynomial equation, whose coefficients $A^{(a)}_p$ are homogeneous polynomials of the scalar products $k_i \cdot k_j$ ($1 \leq i, j \leq n - 1$), also of degree $\nu$. Moreover, $A^{(a)}_p$ satisfy the symmetry relations (55) and (56). The task is to find out to what extent these relations determine $A^{(a)}_p$.

For most of this section we will be discussing a single $\alpha$, so this script will be dropped. To be definite, we will take $\alpha = 3$, though the treatment is identical for the other $\alpha$’s.

The symmetry equations under permutation (12) are

$$
+A_\nu(12) = A_\nu,
$$
$$
-A_{\nu-1}(12) = A_{\nu-1} + \nu A_\nu,
$$
$$
+A_{\nu-2}(12) = A_{\nu-2} + (\nu - 1)A_{\nu-1} + \frac{1}{2!}(\nu - 1)\nu A_\nu,
$$
$$
-A_{\nu-3}(12) = A_{\nu-3} + (\nu - 2)A_{\nu-2} + \frac{1}{2!}(\nu - 2)(\nu - 1)A_{\nu-1} + \frac{1}{3!}(\nu - 2)(\nu - 1)\nu A_\nu,
$$
$$
\vdots
$$
$$
(-)^{\nu}A_0(12) = A_0 + A_1 + \cdots + A_\nu. \tag{62}
$$

Let $A_p = B_p + C_p$, where $B_p$ is even under permutation (12) and $C_p$ is odd. Then (62) is equivalent to the following set of more transparent equations,

$$
C_\nu = 0,
$$
$$
-2B_{\nu-1} = \nu B_\nu,
$$
$$
2C_{\nu-2} = (\nu - 1)C_{\nu-1},
$$
$$
-2B_{\nu-3} = (\nu - 2)B_{\nu-2} + \frac{1}{2!}(\nu - 2)(\nu - 1)B_{\nu-1} + \frac{1}{3!}(\nu - 2)(\nu - 1)\nu B_\nu, \tag{63}
$$

etc. In other words, $C_\nu = 0$, but $B_\nu, C_{\nu-1}, B_{\nu-2}, C_{\nu-3}, \cdots$ are arbitrary. Moreover, the equations give alternately constraints on $B$ and on the next $C$. 

15
The remaining symmetry relations state that \( A_p(23) = A_{\nu-p} \). To see how they can be exploited, we need to know the explicit forms for \( B_p \) and \( C_p \) as a function of \( k_i \cdot k_j \).

In a scattering process involving \( n \) massless particles, the number of scalar products that can be constructed from momenta \( k_1, \cdots, k_{n-1} \) is \( (n-1)(n-2)/2 \). Since \( 0 = k_n^2 = (\sum_{i=1}^{n-1} k_i)^2 \), these scalar products sum up to zero, so there are only \( \mu := (n-1)(n-2)/2 - 1 = n(n-3)/2 \) different ones. The functions \( A_p \) are \( \nu \)th degree homogeneous polynomials in these \( \mu \) variables. As such, it contains \( \lambda := (\nu + \mu - 1)!/(\mu - 1)!\nu! \) terms, and requires \( \lambda \) coefficients to fix. Since there are \( \nu + 1 \) \( A_p \)'s, the total number of parameters needed to determine the coefficients \( A_p \) of a single-variable polynomial is \( \kappa := (\nu + 1)\lambda \). For \( n = 4,5,6 \), we have respectively \( \nu = 1,2,6, \mu = 2,5,9, \lambda = 2,15,3003, \) and \( \kappa = 4,45,21021 \). One of these parameters is an overall normalization constant which can never be determined. The question we ask is how many of these \( \kappa \) parameters can be determined by the symmetry relations (55), and how to calculate them.

We will describe the general procedure to be followed for any \( n \), then proceed to carry out the explicit calculations for \( n = 4 \) and \( n = 5 \). For \( n = 4 \), up to normalization, all the \( \kappa = 4 \) parameters can be determined. For \( n = 5 \), out of the \( \kappa = 45 \) parameters, 36 can be determined, leaving behind 9 arbitrary parameters.

To discuss the general procedure, it is more convenient to use variables that are either even or odd under the momentum exchange (12). The variables \( k_1.k_i + k_2.k_i \), for \( 3 \leq i \leq n-1 \), are even under the exchange, the variables \( k_1.k_i - k_2.k_i \) are odd, and all the other scalar products are even. We shall use \( x_i \) to denote these odd/even variables. Since \( B_p \) is even under momentum permutation (12), it must contain an even number of the odd-variables in every term. Similarly \( C_p \) must contain an odd number of the odd-variables in every term.

For \( n \geq 5 \), \( \nu \) is even, so \( \frac{n}{2} \) is an integer. That allows us to start from the most convenient relation, \( A_{\frac{n}{2}}(23) = A_{\frac{n}{2}} \), which is self conjugate. Let \( x_i \) become \( x'_i = \sum_{j=1}^{\mu} M_{ij} x_j \) under the momentum exchange (23). It is easy to work out what the matrix \( M \) is for any \( n \). The relation \( A_{\frac{n}{2}}(x'_i) = A_{\frac{n}{2}}(x_i) \) therefore provides \( \lambda \) linear equations to determine the \( \lambda \) coefficients of the \( x \)-monomials in \( A_{\frac{n}{2}} \). These linear equations are homogeneous, so an overall normalization can never be determined. In addition, these equations may not be independent, so the solution may contain arbitrary parameters. In the case \( n = 5 \), out of 15 parameters, 6 are determined, leaving behind 9 free ones.

Once \( A_{\frac{n}{2}} \), and hence \( B_{\frac{n}{2}} \) and \( C_{\frac{n}{2}} \), are known, (62) can be used to relate it to \( A_{\frac{n}{2}+1} \) and
then the relation \( A_{2^{n-1}}(23) = A_{2^{n-1}} \) provides further linear equations to determine more unknown parameters, and so on. Continuing this way, we can find out about all the parameters.

We proceed now to illustrate this general procedure with the specific examples \( n = 4 \) and \( n = 5 \).

**A. \( n = 4 \)**

For \( n = 4 \), the Mandelstam variables 
\[ s = k_1k_2 = k_3k_4, \quad t = k_1k_3 = k_2k_4, \quad u = k_1k_4 = k_2k_3 \]
add up to zero, so there is only one even variable \( x_1 = s \), and one odd variable \( x_2 = t - u \).

The functions \( A_1 = p_1x_1 + q_1x_2 \) and \( A_0 = p_0x_1 + q_0x_2 \) are determined by four parameters, \( p_1, q_1, p_0, q_0 \). Symmetry relations (63) tell us \( q_1 = 0 \) and \( 2p_0 = -p_1 \). Under (23) permutation, 
\[ x_1 = s \rightarrow t = (x_2 - x_1)/2 = x'_1 \] and 
\[ x_2 = t - u \rightarrow s - u = (3x_1 + x_2)/2 = x'_2. \]

The relation \( A_1(23) = A_0 \) gives rise to \( p_1x'_1 = p_1(x_2 - x_1)/2 = p_0x_1 + q_0x_2 \), thus \( 2q_0 = p_1 \). Consequently, \( A_1 = p_1x_1 = p_1s \) and \( A_0 = p_1(-x_1 + x_2)/2 = p_1t. \) The scattering equation is then completely determined by the symmetry relations to be \( A_1\sigma_3 + A_0 = 0 = s\sigma_3 + t \), the same as equation (25).

**B. \( n = 5 \)**

The single-variable polynomial is of degree 2. Each \( A_p \) is quadratic in the 5 Mandelstam variables defined in Sec. IIB, with \( \lambda = 15 \) terms and hence 15 coefficients for each of \( p = 0,1,2 \). The even/odd variables are \( x_1 = s, \quad x_2 = t_1 + u_1, \quad x_3 = t_2 + u_2, \quad x_4 = t_1 - u_1, \quad x_5 = t_2 - u_2 \), with the first three even and the latter two odd. Together they form a column vector \( x = (x_1, x_2, x_3, x_4, x_5)^T \). Let \( \xi_p \) be \( 5 \times 5 \) symmetric matrices whose elements are the unknown coefficients, so that \( A_p = x^T\xi_p x \). Parametrize \( \xi_1 \) as follows,

\[
\begin{pmatrix} 
  p_{11} & p_{12} & p_{13} & q_{11} & q_{12} \\
  p_{12} & p_{22} & p_{23} & q_{21} & q_{22} \\
  p_{13} & p_{23} & p_{33} & q_{31} & q_{32} \\
  q_{11} & q_{21} & q_{31} & r_{11} & r_{12} \\
  q_{12} & q_{22} & q_{32} & r_{12} & r_{22} 
\end{pmatrix}, \quad (64)
\]

then \( p_{ij} \) and \( r_{ij} \) contribute to \( B_1 \) and \( q_{ij} \) contribute to \( C_1 \).
From Table 1 we see that under a (23) momentum permutation, \( x \rightarrow x' = Mx \), with

\[
M = \frac{1}{2} \begin{pmatrix}
0 & 1 & 0 & 1 & 0 \\
2 & 1 & 0 & -1 & 0 \\
-2 & -2 & -1 & 0 & 1 \\
2 & -1 & 0 & 1 & 0 \\
2 & 2 & 3 & 0 & 1
\end{pmatrix}.
\]  
(65)

The relation \( A_1(23) = A_1 \) is equivalent to \( M^T \xi_1 M = \xi_1 \). This gives rise to a set of 15 linear equations, whose solution leaves 9 parameters (\( r_{11}, r_{12}, q_{11}, q_{21}, q_{22}, q_{31}, q_{32}, p_{22} \)) free. The rest are given by

\[
\begin{align*}
 p_{11} &= 2q_{21} - 4q_{31} + 4r_{12} + p_{22} + r_{11}, \\
p_{12} &= p_{22} - 2q_{31} + 2r_{12} + q_{11} - r_{11}, \\
p_{13} &= -q_{31} + 2r_{12} + q_{22} - 2q_{32} + 2r_{22}, \\
p_{23} &= q_{31} - r_{12} + q_{22} - 2q_{32} + 2r_{22}, \\
p_{33} &= -2q_{32} + 3r_{22}, \\
q_{12} &= q_{31} + q_{22}.
\end{align*}
\]  
(66)

Using (62), we get

\[
\xi_2 = -\begin{pmatrix}
p_{11} & p_{12} & p_{13} & 0 & 0 \\
p_{12} & p_{22} & p_{23} & 0 & 0 \\
p_{13} & p_{23} & p_{33} & 0 & 0 \\
0 & 0 & 0 & r_{11} & r_{12} \\
0 & 0 & 0 & r_{12} & r_{22}
\end{pmatrix},
\]  
(67)

and from \( A_0 = A_2(23) \), we can compute

\[
\xi_0 = M^T \xi_2 M.
\]

(68)

Thus, the 45 coefficients a priori need to determine \( A_2, A_1, A_0 \) are now reduced to only 9 arbitrary coefficients. The single-variable polynomial equation derived from the scattering equation is given in (29). It corresponds to the special case

\[
(r_{11}, r_{12}, r_{22}, q_{11}, q_{21}, q_{22}, q_{31}, q_{32}, p_{22}) = \left(0, 0, 0, \frac{1}{2}, 0, -\frac{1}{4}, \frac{1}{4}, 0, 0\right).
\]

(69)
Appendix A: Group property of the symmetry relations

Here are the details how (10) to (13) can be derived from (7) to (9):

\[
\sigma_\alpha (1\alpha) = \sigma_\alpha ((12)(2\alpha)(12)) = 1 - \frac{1}{\sigma_\alpha (12)} = \frac{\sigma_\alpha}{\sigma_\alpha - 1}
\]

\[
\sigma_\alpha (1\beta) = \sigma_\alpha ((12)(2\beta)(12)) = 1 - \frac{\sigma_\alpha (12)}{\sigma_\beta (12)} = \frac{\sigma_\alpha - \sigma_\beta}{1 - \sigma_\beta}
\]

\[
\sigma_\alpha (\alpha\beta) = \sigma_\alpha ((2\alpha)(2\beta)(2\alpha)) = \frac{1}{\sigma_\alpha ((2\beta)(2\alpha))} = \frac{\sigma_\beta (2\alpha)}{\sigma_\alpha (2\alpha)} = \sigma_\beta
\]

\[
\sigma_\alpha (\beta\gamma) = \sigma_\alpha ((2\beta)(2\gamma)(2\beta)) = \frac{\sigma_\alpha ((2\gamma)(2\beta))}{\sigma_\beta ((2\gamma)(2\beta))} = \frac{\sigma_\alpha (2\beta)}{\sigma_\beta (2\beta)} = \sigma_\alpha.
\]

We need in addition (14) to show (15) to (17):

\[
\sigma_\alpha (2n) = \sigma_\alpha ((12)(1n)(12)) = 1 - \frac{1}{\sigma_\alpha (12)} = \frac{\sigma_\alpha}{\sigma_\alpha - 1}
\]

\[
\sigma_\alpha (\alpha n) = \sigma_\alpha ((2\alpha)(2n)(2\alpha)) = \frac{1}{\sigma_\alpha ((2n)(2\alpha))} = \frac{\sigma_\alpha (2\alpha) - 1}{\sigma_\alpha (2\alpha)} = 1 - \sigma_\alpha
\]

\[
\sigma_\alpha (\beta n) = \sigma_\alpha ((2\beta)(2n)(2\beta)) = \frac{\sigma_\alpha ((2n)(2\beta))}{\sigma_\beta ((2n)(2\beta))} = \frac{\sigma_\alpha (2\beta)}{\sigma_\beta ((2\beta)(2\beta))} = \frac{\sigma_\alpha (2\beta)/(\sigma_\alpha (2\beta) - 1)}{\sigma_\beta (2\beta)/(\sigma_\beta (2\beta) - 1)}
\]

\[
= \frac{\sigma_\alpha / (\sigma_\alpha - \sigma_\beta)}{1/(1 - \sigma_\beta)} = \frac{\sigma_\alpha (1 - \sigma_\beta)}{\sigma_\alpha - \sigma_\beta}.
\]

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[3] L. Dolan, P. Goddard, arXiv:1311.5200.
[4] S. Weinzierl, arXiv:1402.2516.
[5] L. Dolan, P. Goddard, arXiv:1402.7374.
[6] C. Kalousios, arXiv:1312.7743.
[7] In order not to carry a factor of 2 around, in this paper we define all the (generalized) Mandelstam variables with an additional factor of 1/2, to be \((\sum k_i)^2/2\).
[8] \(\sigma_i\) here is \(1/z_i\) in Ref. [5]. This inverts the role of \(k_1\) and \(k_n\) which is why the generalized Mandelstam variables \(s...\) here involve a \(k_n\) rather than a \(k_1\). We have also defined the Mandelstam variables with an additional factor \(1/2\) to avoid carrying the factor 2 everywhere in the dot products.