Importance of intrinsic and non-network contribution in PageRank centrality and its effect on PageRank localization

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(Dated: September 2, 2016)

PageRank centrality is used by Google for ranking web-pages to present search result for a user query. Here, we have shown that PageRank value of a vertex also depends on its intrinsic, non-network contribution. If the intrinsic, non-network contributions of the vertices are proportional to their degrees or zeros, then their PageRank centralities become proportional to their degrees. Some simulations and empirical data are used to support our study. In addition, we have shown that localization of PageRank centrality depends upon the same intrinsic, non-network contribution.

I. INTRODUCTION

Complex network tools have been used in different fields from social, technological and biological networks [1][2], to analyze the structure and dynamics of the network. Different centrality measures have been used to find the important vertices in a network e.g., degree centrality, eigenvector centrality, Katz centrality, vertex betweenness centrality, edge betweenness centrality, closeness centrality[2], etc. Among them, the simplest measure is the degree centrality. Degree of a vertex is the number of edges connected to it.

PageRank is another important measure and is used in network tasks like, information retrieval, link prediction and community detection[3][4]. In 1998, Brin and Page [5] developed the PageRank algorithm to rank websites in their search engine. PageRank score, $x_i$, of a vertex $i$ in an undirected and unweighted network, is defined as $x_i = \alpha \sum_j A_{ij} \frac{x_j}{k_j} + \beta_i$, where $0 < \alpha < 1$ is the damping factor, also known as teleportation factor, $A$ is the adjacency matrix corresponding to the network, $k_j$ is the degree of the vertex $j$ and $\beta_i$ is the intrinsic, non-network contribution. Many researchers have studied the significance of $\alpha$ and $\beta_i$ in PageRank [6][10]. However, till now, no study has been taken place for the same for $\beta_i$. In many cases the value of $\beta_i$ has been taken as 1 [6][8][9].

Localization of eigenvector is a common phenomenon in most of the networks and it is frequently suggested that the main cause of localization is due to the presence of high degree vertices [11][12]. Localization means that most of the weight of the centrality accumulates in a few number of vertices. Recently, empirical studies in PageRank suggest that PageRank does not show localization [13].

In this paper, we show that depending upon intrinsic and non-network contribution, PageRank value changes. This article has been organized in the following way: in Chapter II, we show that if the intrinsic and non-network contribution of the vertices are the same or proportional to their degrees, then the PageRanks remain same or proportional to the degree of the vertices. We also show that if the intrinsic and non-network contribution of the vertices is zero then PageRank values become proportional to the degrees. In Chapter III, our study on simulated and empirical networks support our findings. On the contrary to the common belief we show that, for some networks, PageRank centrality can be localized and the value of $\beta_i$ can effect in the localization of the PageRank centrality. Chapter IV concludes with some discussion.

II. PAGERANK CENTRALITY: INTRINSIC AND NON-NETWORK CONTRIBUTION PROPORTIONAL TO DEGREE

Consider a simple undirected connected network $G$ of size $n$, with adjacency matrix $A$. The PageRank centrality $x_i$ of a vertex $i$ in the network $G$ is defined as:

$$x_i = \alpha \sum_{j=1}^{n} A_{ij} \frac{x_j}{k_j} + \beta_i,$$

where $0 < \alpha < 1$ is the damping factor (Google search engine uses $\alpha = 0.85$ [7]), $k_j$ denotes the degree of vertex $j$, and $\beta_i$ is the intrinsic and non-network contribution to the vertex $i$.

In matrix form, the Equation (1) can be written as:

$$x = \alpha AD^{-1}x + \beta,$$

where $D$ is the diagonal matrix with elements $D_{ii} = k_i$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)^T$. Equation (2) can be written as:

$$x = (I - \alpha AD^{-1})^{-1}\beta,$$

Since $||\alpha AD^{-1}||_1 \leq \alpha < 1$, Equation (3) can be written as the sum of an infinite series:

$$x = \sum_{i=0}^{\infty} \alpha^i (AD^{-1})^i \beta.$$
Now $\mathbf{AD}^{-1} = \mathbf{P}$, is a column stochastic matrix. Since $G$ is undirected and connected $\mathbf{P}$ is diagonalizable. One can write $\beta$ as a linear combination of the eigenvectors $\mathbf{w}_i$ of $\mathbf{P}$, as

$$\beta = \sum_{j=1}^{n} c_j \mathbf{w}_j. \tag{5}$$

Then the Equation (4) becomes

$$\mathbf{x} = \sum_{i=0}^{\infty} \alpha^i \mathbf{P}^i \left( \sum_{j=1}^{n} c_j \mathbf{w}_j \right). \tag{6}$$

Let $1 = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$ be the eigenvalues of $\mathbf{P}$ corresponding to the eigenvectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$, respectively. Then

$$\mathbf{x} = \sum_{i=0}^{\infty} \alpha^i \left( \sum_{j=1}^{n} c_j \mu_j^i \mathbf{w}_j \right). \tag{7}$$

After some simple calculation, we get

$$\mathbf{x} = \sum_{j=1}^{n} c_j \left( \sum_{i=0}^{\infty} \alpha \mu_i^j \right) \mathbf{w}_j. \tag{8}$$

Since, $\alpha \mu_i < 1, \forall i = 1, \ldots, n$, we get

$$\mathbf{x} = \sum_{j=1}^{n} c_j \frac{1}{1 - \alpha \mu_j} \mathbf{w}_j. \tag{9}$$

Therefore, the PageRank vector $\mathbf{x}$ of a network is the linear sum of eigenvectors of $\mathbf{P}$.

**Remark 1.** If the value of $\beta$ is proportional to degrees, then Equation (6) becomes:

$$\beta = c_1 \mathbf{w}_1, \tag{10}$$

where $\mathbf{w}_1$ is the leading eigenvector of $\mathbf{P}$. Note that $\mathbf{w}_1$ is proportional to the degree of the network. Therefore, Equation (9) reduces to

$$\mathbf{x} = c_1 \frac{1}{1 - \alpha} \mathbf{w}_1. \tag{11}$$

Hence, PageRank centrality is proportional to the degree of the network.

**Remark 2.** PageRank can also be quantified successively with an initial estimation of $\mathbf{x}$ from Equation (1) or (equivalently Equation (2)). Equation (2) will be,

$$\mathbf{x}_1 = \alpha \mathbf{AD}^{-1} \mathbf{x}_0 + \beta, \tag{12}$$

where $\mathbf{x}_0$ is any initial value of $\mathbf{x}$ usually taken $\mathbf{x} = 0$.

If the intrinsic and non-network contribution is zero, then from Equation (12), we get

$$\mathbf{x}_1 = \alpha \mathbf{AD}^{-1} \mathbf{x}_0. \tag{13}$$

In this case, we take $\mathbf{x}_0$ any non-zero vector as $\mathbf{x} = 0$ will not contribute anything. After $t$ iterative steps we have

$$\mathbf{x}_{t+1} = (\alpha \mathbf{P})^t \mathbf{x}_0. \tag{14}$$

Now, writing $\mathbf{x}_0$ as a linear combination of the eigenvectors $\mathbf{w}_j$ of $\mathbf{P}$, we get

$$\mathbf{x}_0 = \sum_{i=1}^{n} c_i' \mathbf{w}_i, \tag{15}$$

for some appropriate choice of $c_i'$. Then

$$\mathbf{x}_{t+1} = \alpha^t \sum_{i=1}^{n} c_i' \mu_{i}^t \mathbf{w}_i. \tag{16}$$

Since $|\mu_i| < 1$ for all $i \neq 1$, we get $\mathbf{x}_{t+1} \rightarrow c_1' \mathbf{w}_1$ in the limit $t \rightarrow \infty$. As $c_1'$ and $\alpha$ are constants and $\mathbf{w}_1$ is proportional to degree, it follows that PageRank centrality is proportional to the degree centrality.

### III. RESULTS

We explore four different values of intrinsic and non-network contribution, $\beta$.

1. $\beta_i$ is inversely proportional to the degree of vertex $i$.
2. All $\beta_i$ are equal to one.
3. $\beta_i$ is proportional to the degree of vertex $i$.
4. $\beta_i$ is proportional to the square of the degree of vertex $i$.

In all the above cases, we have taken $\alpha = 0.85$.

Our numerical simulation on Erdős-Rényi random network [14], we have observed that if the value of $\beta_i$ is proportional to the inverse of the degree of vertex $i$ then PageRank score increases for a few number of low degree vertices and for other, PageRank is stable (see Figure 1(a)). Here ‘stable’ means, whether the PageRank scores are positively correlated with the degree. If $\beta$ is equal or proportional to the degrees of the vertices then PageRank centrality is proportional to the degree of the vertices (see Figure 1(c)). PageRank values are stable when $\beta$ is either equal to one or proportional to the square of the degree of vertex (see Figure 1(b) and Figure 1(d)). In comparison with random network, small-world and scale-free network are stable with any value of $\beta$ that we have considered (see Figure 2 and Figure 3).
However, in real-world networks, the results are not consistent for the above mentioned cases. In Blog network, we have observed that PageRank score fluctuates when $\beta$ is inversely proportional to the degree of vertex (see Figure 4(a)). When all $\beta_i$ are equal to one, PageRank value fluctuates but less than that in the previous case (see Figure 4(b)). When $\beta_i$ is proportional to the degree of vertex $i$, PageRank score correlates positively (see Figure 4(c)). PageRank values are stable when $\beta_i$ is proportional to the square of the degree of vertex $i$ (see Figure 4(d)).

In P2P (peer-to-peer) network, we have seen high fluctuation of PageRank value for some high degree vertices when $\beta$ is inversely proportional to the degree of a vertex (see Figure 5(a)). Similar observation is also found when all $\beta_i$ are equal to one (see Figure 5(b)). When $\beta_i$ is proportional to the degree of vertex $i$, PageRank score is also proportional to the degree of vertex $i$ (see Figure 5(c)). PageRank score is stable when $\beta_i$ is proportional to the square of the degree of vertex $i$ (see Figure 5(d)).

For Email network, we have observed strong fluctuation of PageRank value when $\beta_i$ is inversely proportional to the degree of vertex $i$ (see Figure 6(a)). When all $\beta_i$ are equal to one, we have seen less fluctuation of PageRank value than that in the first case (see Figure 6(b)). PageRank value also becomes proportional to the vertex degree when $\beta_i$ is proportional to the same (see Figure 6(c)). When $\beta_i$ is proportional to the square of the degree of vertex $i$, higher degree vertices hold higher PageRank values (see Figure 6(d)).

In PGP (Pretty Good Privacy) network, some high degree vertices fluctuate in PageRank values when $\beta_i$ is inversely proportional to the degree of a vertex $i$ (see Figure 7(a)). Similar result has been found when all $\beta_i$ are equal to one (see Figure 7(b)). When $\beta_i$ is equal or proportional to the degree of vertex $i$, PageRank value is proportional to the same (see Figure 7(c)). PageRank value increases with the increment of degree of vertices when $\beta_i$ is proportional to the square of the degree of vertex $i$ (see Figure 7).

Internet network is comparatively stable than other networks when $\beta_i$ is inversely proportional to the degree of a vertex $i$ (see Figure 8(a) and Figure 8(b)). When $\beta_i$ is equal or proportional to the degree of vertex $i$, PageRank value is also proportional to the same (see Figure 8(c)). When $\beta_i$ is proportional to the square of the degree of vertex $i$, high degree vertices take the high PageRank score (see Figure 8(d)).

Localization of PageRank centrality: The extent of localization can be measured by calculating inverse participation ratio (IPR) \[11, 15\]. The IPR of a PageRank vector $x$ is calculated as:

\[ IPR = \frac{\sum_{i=1}^{n} x_i^2}{\left(\sum_{i=1}^{n} x_i^4\right)^{1/2}}. \] (17)

When IPR is of order $O(1)$ as $n \to \infty$, the PageRank vector $x$ is localized. If $IPR \to 0$, $x$ is delocalized \[12, 15\].

We observe that random and small-world network do not show localization of PageRank centrality. In these two networks, the lowest IPRs observe when $\beta$ is inversely proportional to the degree of vertex and the highest IPRs observe when $\beta$ is the square of the degree of the vertices (see Figure 9(A) and 9(B)). However, in scale-free network the PageRank is localized and the highest IPR is observed when $\beta$ is proportional to the degree of a vertex and the lowest IPR is observed when all $\beta_i$ are equal to one (see Figure 9(C)).

In Blogs network, the PageRank is localized and the highest IPR is observed when the value of $\beta$ is inversely proportional to the degree of the vertices. The lowest value of IPR is seen when $\beta$ is proportional to the degree of the vertex (see Figure 9(D)).

The PageRank is not localized in P2P and Email networks. In both of these networks, the highest value of IPRs are observed when $\beta$ is proportional to the square of the degree of vertex and the lowest IPR is observed when all $\beta_i$ are equal to one (see Figure 9(E) and 9(F)).

The PageRank is localized in PGP network and the highest value IPR is observed when $\beta$ is proportional to the degree of the vertex and the lowest IPR is observed when $\beta$ is proportional to the degree of the vertex (see Figure 9(G)).

In Internet network, PageRank centrality is localized. The highest value of IPR is observed when $\beta$ is the square of the degree of the network and the lowest IPR is observed when $\beta$ is proportional to the degree of the vertex (see Figure 9(H)).

IV. DISCUSSION AND CONCLUSION

In this work, we have shown that PageRank centrality depends upon intrinsic, non-network contribution of the vertices. If the intrinsic and non-network contribution of the vertices is proportional to the degree then PageRank score becomes proportional to the degree of the vertices. We have also shown that if the intrinsic and non-network contribution of PageRank centrality is zero, then it also becomes proportional to the degree of the network. Our numerical simulation of three network models (Erdős-Rényi random network, Watts-Strogatz small-world network and Barabási-Albert scale-free network) shows that the PageRank scores are more resilient in the intrinsic and non-network contribution in small-world and scale-free networks than in the random network. Among all real-world networks studied here, Blog, P2P, E-mail and PGP, PageRank score fluctuates heavily when intrinsic and non-network contribution are either inversely proportional to the degree or equal to one. When the value of $\beta$ is proportional to degree, PageRank keeps the same proportional value. High degree vertices possess high PageRank value when the $\beta$ is proportional to the square of the degree. Internet network shows very small fluctuation of PageRank score when $\beta$ is either inversely proportional or equal to one and for other two
values of $\beta$ show similar result as other real networks. On the contrary to the common belief we show that PageRank centrality can show localization for some networks and the extent of localization depends upon the intrinsic and non-network contribution of the PageRank centrality. In synthetic networks, we have seen that the value of IPR increases with the increase in the value of $\beta$. However, in the real-world networks studied here, no such correlation is found. In Blog network, IPR attains the highest value when $\beta$ is inversely proportional to the degree and for other networks the highest value of IPR was observed when $\beta$ is proportion to the square of the degree. In a nutshell, we have observed that the intrinsic and non-network contribution plays an important role in PageRank centrality calculation. Here, we have considered the underlying undirected structure of all the networks. The giant component (i.e., the largest connected sub-network) has been considered when the network is not connected.

Network construction and source:

Three simulated networks: Erdős-Rényi random network is constructed with 1000 vertices and two vertices are connected with probability $p = 0.01$ [14]. In Watts-Strogatz small-world network, the number of vertices is 1000, average degree of initial regular graph is 10 and the rewiring probability is 0.4 [16]. Barabási-Albert scale-free network is generated with 1000 vertices and size of the seed network $m_0 = 5$. A new vertex is added with the existing $m = 5$ vertices [17].

Blogs network: Here vertices are the blogs and an edge represents a hyperlink between two blogs. The number of vertices and edges are 1222 and 16714, respectively. The data were downloaded from KONECT [15] and used in [19].

Gnutella peer-to-peer (P2P) network: Gnutella is a system where individual can exchange files over the Internet directly without going through a Web site. Here, peer-to-peer means persons to persons. The vertices are hosts in the Gnutella network topology and the edges represent connection between them. The number of vertices are 22663 and number of edges are 54693. P2P network data were downloaded from SNAP[20] and used in [21][22].

E-mail network: E-mail network was constructed based on the exchange of mails between the members of the University of Rovira i Virgili (Tarragone). Here the vertices are the users, and two users are connected by an edge if one has sent or received email from other. The number of vertices and edges are 1133 and 5451, respectively. The data were downloaded from [23] and used in [24].

Pretty Good Privacy network: Here the vertices are the users of Pretty Good Privacy (PGP) algorithm and edges are the interactions between them. The number of vertices and edges are 10680 and 24316, respectively. The data were downloaded from [24] and used in [25].

Internet network: The vertices in this network are autonomous systems (collection of computers and routers[2]) and the edges are the routes taken by the data traveling between them. The number of vertices and edges are 22963 and 48436, respectively. Internet data were downloaded from SNAP[20] and used in [26].

Acknowledgments

The author thanks Anirban Banerjee and Abhijit Chakraborty for critical reading and suggestion. Author also thanks Prof. Asok Kumar Nanda for helping to prepare the manuscript. Author gratefully acknowledges the financial support from CSIR, India through a doctoral fellowship.

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FIG. 1: PageRank on random network for different values of $\beta$: (a) inverse of the vertex degree, (b) equal for all vertices, (c) proportional to the vertex degree, (d) square of the vertex degree. Random network model proposed by Erdős-Rényi [14]. Here random network generated with 1000 vertices and the probability of connecting two vertices is 0.01. Simulated result is average over 100 iterations.

FIG. 2: PageRank on small-world network for different values of $\beta$: (a) inverse of the vertex degree, (b) equal for all vertices, (c) proportional to the vertex degree, (d) square of the vertex degree. Small-world network proposed by Watts-Strogatz [16]. Here small-world network generated by rewiring regular ring lattice of size 1000 and average degree 10 with rewiring probability 0.4. Simulated result is average over 100 iterations.
FIG. 3: **PageRank stability on scale-free network for different values of \( \beta \):** (a) inverse of the vertex degree, (b) equal for all vertices, (c) proportional to the vertex degree, (d) square of the vertex degree. Scale-free network proposed by Barabási-Albert [17]. Here scale-free network generated of size 1000 and size of the seed network is \( m_0 = 5 \) and a new vertex added with existing \( m = 5 \) vertices. Simulated results is average over 100 iterations.

FIG. 4: **PageRank on Blogs network for different values of \( \beta \):** (a) inverse of the vertex degree, (b) equal for all vertices, (c) proportional to the vertex degree, (d) square of the vertex degree. In Blog network [19], vertices are the blogs and an edge represents a hyperlink between two blogs.
FIG. 5: PageRank on Gnutella peer-to-peer network for different values of $\beta$: (a) inverse of the vertex degree, (b) equal for all vertices, (c) proportional to the vertex degree, (d) square of the vertex degree. In Gnutella peer-to-peer network [21], the vertices are hosts in the Gnutella network topology and edges are the routes taken by data traveling between them.

FIG. 6: PageRank on Email network for different values of $\beta$: (a) inverse of the vertex degree, (b) equal for all vertices, (c) proportional to the vertex degree, (d) square of the vertex degree. In Email network [24], the vertices are the users and two users are connected by an edge if one sent or received email from other.
FIG. 7: PageRank on PGP network for different values of $\beta$: (a) inverse of the vertex degree, (b) equal for all vertices, (c) proportional to the vertex degree, (d) square of the vertex degree. In PGP network [25], vertices are the user of Pretty Good Privacy (PGP) algorithm and edges are the interactions between them.

FIG. 8: PageRank on Internet network for different values of $\beta$: (a) inverse of the vertex degree, (b) equal for all vertices, (c) proportional to the vertex degree, (d) square of the vertex degree. In Internet network [27], vertices are the autonomous systems (collection of computers and routers) and the edges show the route taken by the data traveling between them.
FIG. 9: Inverse participation ratio of various networks for different values of $\beta$: (a) inverse of the vertex degree, (b) equal for all vertices, (c) proportional to the vertex degree, (d) square of the vertex degree. Networks are (A) Random network, (B) Small-world network, (C) Scale-free network, (D) Blog network, (E) P2P network, (F) Email network, (G) PGP network and (H) Internet network.