ON REPRESENTATIONS OF CLASSICAL GROUPS OVER PRINCIPAL IDEAL LOCAL RINGS OF LENGTH TWO

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ABSTRACT. We study the complex irreducible representations of special linear, symplectic, orthogonal and unitary groups over principal ideal local rings of length two. We construct a canonical correspondence between the irreducible representations of all such groups that preserves dimensions. The case for general linear groups has already been proved by author.

1. Introduction

Let $F$ be a non-Archimedean local field with ring of integers $\mathcal{O}$. Let $\varphi$ be the unique maximal ideal of $\mathcal{O}$ and $\pi$ be a fixed uniformizer of $\varphi$. Assume that the residue field $\mathcal{O}/\varphi$ has odd characteristic $p$. We denote by $\mathcal{O}_\ell$ the reduction of $\mathcal{O}$ modulo $\varphi^\ell$, i.e., $\mathcal{O}_\ell = \mathcal{O}/\varphi^\ell$. Therefore $\mathcal{O}_1$ will denote the residue field of $\mathcal{O}$.

The representation theory of classical groups over the rings $\mathcal{O}$ and the finite rings $\mathcal{O}_\ell$ has attracted attention of many mathematicians. See Singla [7] for the history of this problem for the General and Special linear groups over the rings $\mathcal{O}$ and their finite quotients $\mathcal{O}_\ell$. In the direction of the classical groups, Lusztig [5] constructed several irreducible representations of reductive groups over finite rings and Jaikin-Zapirain [4] looked at the problem of constructing irreducible representations of compact pro-p groups. But the knowledge of all irreducible representations of classical groups over $\mathcal{O}$ and the finite rings $\mathcal{O}_\ell$ is still far from complete.

In Singla [7], we gave a method to construct irreducible representations of general linear groups $GL_n(\mathcal{O}_2)$. In this article, we use it to construct irreducible representations of other classical groups over the rings $\mathcal{O}_2$. The questions of this article are also motivated by a conjecture of Onn [6, Conjecture 1.2], which says that the isomorphism type of the group algebra of automorphism group of finite $\mathcal{O}$-modules depend on the ring of integers only through the cardinality of its residue field. More generally one can ask this question for the other classical groups over $\mathcal{O}_\ell$ as well. In Singla [7], we proved Onn’s conjecture for groups $GL_n(\mathcal{O}_2)$.

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1.1. Main Results. The classical groups over ring $\mathcal{O}_2$ are defined as:

1. Special Linear Group: $\text{SL}_n(\mathcal{O}_2) = \{g \in \text{GL}_n(\mathcal{O}_2) \mid \det(g) = 1\}$.

2. Symplectic Group: $\text{Sp}_n(\mathcal{O}_2) = \{g \in \text{GL}_{2n}(\mathcal{O}_2) \mid g^t J g = J\}$, where $g^t$ denotes transpose of $g$ and $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

3. Orthogonal Group: $\text{O}_n(\mathcal{O}_2) = \{g \in \text{GL}_n(\mathcal{O}_2) \mid g^t g = I_n\}$.

4. Unitary Group: Let $\tilde{\mathcal{F}}$ be a degree two unramified extension of $\mathcal{F}$ and $\sigma$ be the unique nontrivial Galois automorphism of $\tilde{\mathcal{F}}$. Let $\tilde{\mathcal{O}}_2$ be the corresponding unramified extension of $\mathcal{O}_2$, then $\sigma$ restricts to an automorphism of $\tilde{\mathcal{O}}_2$ (denoted again as $\sigma$). Applying $\sigma$ entry wise we obtain an automorphism of $\text{GL}_n(\tilde{\mathcal{O}}_2)$. For any $g \in \text{GL}_n(\tilde{\mathcal{O}}_2)$, let $g^* = (g^\sigma)^t$. Then the unitary group is defined as $U_n(\mathcal{O}_2) = \{g \in \text{GL}_n(\tilde{\mathcal{O}}_2) \mid gg^* = I_n\}$.

We prove that the number and dimensions of irreducible representations of these classical groups over $\mathcal{O}_2$ depend on $\mathcal{O}$ only through the cardinality of residue field. More precisely we prove the following.

Let $\mathcal{F}$ and $\mathcal{F}'$ be local fields with rings of integers $\mathcal{O}$ and $\mathcal{O}'$, respectively, such that their residue fields are finite and isomorphic (with a fixed isomorphism). Let $\mathfrak{q}$ and $\mathfrak{q}'$ be the maximal ideals of $\mathcal{O}$ and $\mathcal{O}'$ respectively. As described earlier, $\mathcal{O}_2$ and $\mathcal{O}_2'$ denote the rings $\mathcal{O}/\mathfrak{q}^2$ and $\mathcal{O}'/\mathfrak{q}'^2$, respectively. For a ring $R$, we use $C(R)$ as collective notation for any of the classical groups $\text{SL}_n(R)$, $\text{Sp}_n(R)$, $\text{O}_n(R)$, or $U_n(R)$ over $R$.

**Theorem 1.1.** There exists a canonical bijection between the irreducible representations of $C(\mathcal{O}_2)$ and those of $C(\mathcal{O}_2')$, which preserves dimensions.

By the equivalence between the number of conjugacy classes and distinct irreducible representations, we also obtain that the number of conjugacy classes of the classical groups over $\mathcal{O}_2$ depend on $\mathcal{O}$ only through $|\mathcal{O}_1|$. We remark that very little is known about the conjugacy classes of classical group $C(\mathcal{O}_l)$ (See [2, 1, 8]).

2. Proof of Theorem 1.1

First of all, we set up few notations. By character we shall always mean one-dimensional representation. For an abelian group $A$, the set of its characters is denoted by $\hat{A}$. For any group $G$, the set of inequivalent irreducible representations.
is denoted by $\text{Irr}G$. Let

$$\kappa : \text{GL}_n(O_2) \to \text{GL}_n(O_1) \text{ and } \bar{\kappa} : C(O_2) \to C(O_1)$$

be the natural quotient maps with $K = \text{ker}(\kappa)$ and $L(C) = \text{ker} (\bar{\kappa})$. We shall use the following results of Clifford theory.

**Theorem 2.1** (Clifford Theory). Let $G$ be a finite group and $N$ be a normal subgroup. Let $\rho$ be an irreducible representation of $N$ and $T(\rho) = \{g \in G \mid \rho^g = \rho\}$ be the stabilizer of $\rho$. Then the following hold

1. If $\pi$ is an irreducible representation of $G$ such that $\langle \pi|_N, \rho \rangle \neq 0$, then $\pi|_N = e(\oplus_{\rho \in \Omega} \rho)$ where $\Omega$ is an orbit of irreducible representations of $N$ under the action of $G$, and $e$ is a positive integer.

2. Let $A = \{\theta \in \text{Irr}(T(\rho)) \mid \langle \text{Res}^{T(\rho)}_N \theta, \rho \rangle \neq 0\}$ and $B = \{\pi \in \text{Irr}G \mid \langle \text{Res}^G_N \pi, \rho \rangle \neq 0\}$. Then

   $$\theta \to \text{Ind}^{G}_{T(\rho)}(\theta)$$

   is a bijection of $A$ onto $B$.

3. Let $H$ be a subgroup of $G$ containing $N$, and suppose that $\rho$ has an extension $\tilde{\rho}$ to $H$ (i.e., $\tilde{\rho}|_N = \rho$). Then the representations $\chi \otimes \tilde{\rho}$ for $\chi \in \text{Irr}(H/N)$ are irreducible, distinct for distinct $\chi$, and

   $$\text{Ind}^H_N(\rho) = \oplus_{\chi \in \text{Irr}(H/N)} \chi \otimes \tilde{\rho}.$$  

Let $\hat{L(C)}$ denote the set of characters of $L(C)$. The group $C(O_2)$ acts on $\hat{L(C)}$ by conjugation. That is, if $\alpha \in C(O_2)$ and $\phi \in \hat{L(C)}$ then $\phi^\alpha(x) = \phi(\alpha x \alpha^{-1})$ for $x \in L(C)$. For any $\phi \in \hat{L(C)}$, let $T_C(\phi) = \{\alpha \in C(O_2) \mid \phi^\alpha = \phi\}$ be the stabilizer of $\phi$ in $C(O_2)$.

**Proposition 2.2.** For any $\phi \in \hat{L(C)}$,

1. There exists a canonical character $\chi_\phi$ of $T_C(\phi)$ such that $\chi_\phi|_{L(C)} = \phi$.
2. The group $T_C(\phi)/K$ depends on $O$ only through $|O_1|$.
3. The cardinality $|C(O_2)/T_C(\phi)|$ depends on $O$ only through $|O_1|$.

We postpone the proof of this proposition to §3 and 4. Assuming this, we complete the proof of Theorem 1.1.
Proof of Theorem 1.1. Let $S_C$ denote the set of $C(O_2)$-orbits in $\hat{L}(C)$ under conjugation. Therefore, by Clifford Theory, there exists a bijection (also canonical by proposition 2.2) between the sets

$$\bigoplus_{\phi \in S_C} \{\text{Irr}(T_C(\phi)/K)\} \leftrightarrow \text{Irr}(C(O_2)),$$

given by,

$$\delta \mapsto \text{Ind}^{C(O_2)}_{T_C(\phi)}(\chi_\phi \otimes \tilde{\delta}),$$

where $\tilde{\delta}$ is a representation of $T_C(\phi)$ obtained by composing $\delta$ with the natural projection map $T_C(\phi) \to T_C(\phi)/K$. Since the left side of (2.1) depends on $O$ only through the cardinality of $O_1$, theorem follows.

$$\square$$

3. The groups $O_n(O_2)$, $Sp_n(O_2)$, $U_n(O_2)$, and $SL_n(O_2)$ ($p \nmid n$)

Fix a nontrivial additive character $\psi : O_1 \to \mathbb{C}^\times$. For each $A \in M_n(O_1)$ the character $\psi_A : K \to \mathbb{C}^\times$ is defined by

$$\psi_A(I + \pi X) = \psi(\text{Tr}(AX)).$$

The bilinear form $\langle.,.\rangle : M_n(O_1) \times M_n(O_1) \to O_1$, defined by $(A, B) \mapsto \text{Tr}(AB)$ is non-degenerate therefore the assignment $A \mapsto \psi_A$ defines an isomorphism $M_n(O_1) \cong \tilde{K}$. Let $T_G(\psi_A) = \{g \in \text{GL}_n(O_2) \mid \psi_A^g = \psi_A\}$ (in place of $T(\psi_A)$ of Singla [7] to remove any ambiguity) denote the stabilizer of $\psi_A$ in $\text{GL}_n(O_2)$. In this section we prove Proposition 2.2 for Orthogonal, Unitary, Symplectic and Special linear groups $(p \nmid n)$. For this section $C(O_2)$ denotes either $O_n(O_2)$, $U_n(O_2)$, $Sp_n(O_2)$ or $SL_n(O_2)$ $(p \nmid n)$. For any classical group $C(O_2)$, let $M_C$ be the subgroup of $M_n(O_1)$ such that $X \mapsto I + \pi X$ defines an isomorphism $M_C \cong L(C)$. By restricting $\langle.,.\rangle$ to $M_C$, we obtain a bilinear form on $M_C$ as well. Observe that if this restriction is non-degenerate then,

1. The map $A \mapsto \psi_A|_{L(C)}$ defines an isomorphism, $M_C \cong \tilde{L}(C).
2. T_C(\psi_A|_{L(C)}) = T_G(\psi_A) \cap C(O_2)$.

We recall the following result of Singla [7]

Proposition 3.1. For any $\phi \in \tilde{K}$, there exists a canonical character $\chi^G_\phi$ of $T_C(\phi)$ such that $\chi^G_\phi|_K = \phi$ (such a character $\chi^G_\phi$ is called an extension of $\phi$).

Proof. For proof see Proposition 2.2 and Section 5 of Singla [7].

$\square$
Let $\phi = \psi_A|_{L(C)}$ for some $A \in M_C$, then part (a) of Proposition 2.2 follows by taking $\chi_{\phi} = \chi_{G|TC(\phi)}$. Parts (b) and (c) of Proposition 2.2 follow by the following facts:

1. $T_C(\phi)/L(C) \cong \mathbb{Z}_{GL_n(O_1)}(A) \cap C(O_1)$.
2. $|C(O_2)/T_C(\phi)| = |C(O_1)/\mathbb{Z}_{GL_n(O_1)}(A) \cap C(O_1)|$.

These follow easily by definitions of $T_C(\phi)$, $L(C)$ and Corollary 5.3 of Singla [7].

We shall show that the bilinear form $\langle . , . \rangle|_{MC}$ is non-degenerate for Symplectic, Unitary, Orthogonal, and Special linear ($p \nmid n$) groups. Further in §4 we show that it is not true for Special linear group with $p \mid n$, which makes this case bit harder. We shall use a completely different method to solve $SL_n(O_2)(p \mid n)$.

3.1. $O_n(O_2)$. The kernel $L(O)$ of the natural projection map $O_n(O_2) \to O_n(O_1)$ is isomorphic to

$$M_O = \{X \in M_n(O_1) \mid X + X^t = 0\}$$

by $I + \pi X \mapsto X$. For any $A \in M_O$, since $\langle . , . \rangle$ is non-degenerate on $M_n(O_1)$, there exists $Y \in M_n(O_1)$ such that $\text{Tr}(AY) \neq 0$. Take $B = Y - Y^t \in M_O$. By using additivity and $\text{Tr}(Z) = \text{Tr}(Z^t)$ for any $Z \in M_n(O_1)$, we obtain

$$\text{Tr}(AB) = \text{Tr}(A(Y - Y^t)) = \text{Tr}(AY + A^tY^t) = 2\text{Tr}(AY) \neq 0.$$ 

Here we have also used the fact that the residue field is of odd characteristic. Hence $\langle . , . \rangle|_{M_O}$ is non-degenerate.

3.2. $U_n(O_2)$. The kernel $L(U)$ of the natural projection map $U_n(O_2) \to U_n(O_1)$ is isomorphic to the set

$$M_U = \{X \in M_n(O_1) \mid X + X^* = 0\}$$

by $I + \pi X \mapsto X$. The rest of the argument for this follows similar to orthogonal group case, by replacing $Y - Y^t$ with $Y - Y^*$.

3.3. $Sp_n(O_2)$. The kernel $L(Sp)$ of the natural projection map $Sp_n(O_2) \to Sp_n(O_1)$ consists of matrices $I + \pi X$, $X \in M_{2n}(O_2)$ such that

$$(I + \pi X)^tJ(I + \pi X) = J,$$  where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

Let

$$M_{Sp} = \{X \in M_{2n}(O_1) \mid X^tJ + JX = 0\}$$
then $X \mapsto I + \pi X$ is easily seen to give an isomorphism between $M_{Sp}$ and $L(\text{Sp})$. Also observe that,

$$M_{Sp} = \left\{ \begin{bmatrix} U & V \\ W & -U^t \end{bmatrix} \mid U, V, W \in M_n(\mathcal{O}_1), V = -V', W = W' \right\}.$$ 

For any $A = \begin{bmatrix} U & V \\ W & -U^t \end{bmatrix}$, $X = \begin{bmatrix} U' & V' \\ W' & -U'^t \end{bmatrix} \in M_{Sp}$,

$$\text{Tr}(AX) = 2\text{Tr}(UU') + \text{Tr}(VW') + \text{Tr}(WV').$$

By using the fact that $(A, B) \mapsto \text{Tr}(AB)$ is a non-degenerate bilinear form on the set of matrices, symmetric and skew-symmetric matrices over $\mathcal{O}_1$. We obtain that for any nonzero $A \in M_{Sp}$ there exists $X \in M_{Sp}$ such that $\text{Tr}(AX) \neq 0$. Therefore $\langle \cdot, \cdot \rangle|_{M_{Sp}}$ is non-degenerate.

3.4. $\text{SL}_n(\mathcal{O}_2)$, $p \nmid n$: Let $L(SL)$ denote the kernel of the natural projection map $\text{SL}_n(\mathcal{O}_2) \rightarrow \text{SL}_n(\mathcal{O}_1)$. Identify the set $L(SL)$ with $M_{SL} = \{ X \in M_n(\mathcal{O}_1) \mid \text{Tr}(X) = 0 \}$, by $I + \pi X \mapsto X$. We prove that the bilinear form $(A, B) \mapsto \text{Tr}(AB)$ is non-degenerate on $M_{SL}$.

For any non-diagonal matrix $A = (a_{ij}) \in \text{sl}_n(\mathcal{O}_1)$, there exists pair $(i_0, j_0)$ such that $i_0 \neq j_0$ and $a_{i_0j_0} \neq 0$. Take $B = (b_{ij}) \in M_{SL}$ such that $b_{j_0i_0} = 1$ and zeros everywhere else. Then $\text{Tr}(AB) \neq 0$. On the other hand if $A = (a_{ij}) \in M_{SL}$ is a non-zero diagonal matrix, then $p \nmid n$ implies there exists $i_0 \neq j_0$ such that $a_{i_0i_0} \neq a_{j_0j_0}$. Then by taking $B = (b_{ij}) \in M_{SL}$ to be the matrix satisfying $b_{i_0i_0} = 1$, $b_{j_0j_0} = -1$ and $b_{ij} = 0$ for all $(i, j) \notin \{(i_0, i_0), (j_0, j_0)\}$, we obtain that $\text{Tr}(AB) \neq 0$. This proves the assertion.

4. Special Linear Group $\text{SL}_n(\mathcal{O}_2)$, $p \mid n$

Let $L(SL)$ denote the kernel of the natural projection map $\text{SL}_n(\mathcal{O}_2) \rightarrow \text{SL}_n(\mathcal{O}_1)$. As mentioned in §3.4. the set $L(SL)$ can be identified with $M_{SL} = \{ X \in M_n(\mathcal{O}_1) \mid \text{Tr}(X) = 0 \}$, by $I + \pi X \mapsto X$. We firstly show that $\langle \cdot, \cdot \rangle|_{M_{SL}}$ is not non-degenerate by showing that the scalar matrices lie in its radical. Let $A = aI_n \in M_n(\mathcal{O}_1)$, then $p \mid n$ implies $A \in M_{SL}$. Hence for any $X \in M_{SL}$ we obtain $\text{Tr}(AX) = a\text{Tr}(X) = 0$. This implies that $\langle \cdot, \cdot \rangle|_{M_{SL}}$ is not non-degenerate. Hence the method discussed in the last section does not work in this case.

Define an equivalence relation on $M_n(\mathcal{O}_1)$ by $A \sim B$ if there exists a scalar $x \in \mathcal{O}_1$ such that $A = xI + B$. Denote the equivalence class of $A$ under this relation
There exists a character $\psi_{[A]} : L(SL) \to \mathbb{C}^\times$ by
$$\psi_{[A]}(I + \pi X) = \psi(\text{Tr}(AX)).$$

Then $\psi_{[A]}$ is well defined character of $L(SL)$ and $[A] \mapsto \psi_{[A]}$ gives an isomorphism $\mathfrak{L} \to \tilde{L}(SL)$. The group $GL_n(O_2)$ acts on $\mathfrak{L}$ by conjugation via its quotient $GL_n(O_1)$, and therefore on $\tilde{L}(SL)$. For $\alpha \in GL_n(O_2)$ and $\psi_{[A]} \in \tilde{L}(SL)$, we obtain,
$$\psi^\alpha_{[A]}(I + \pi X) = \psi(\text{Tr}(A\kappa(\alpha)X\kappa(\alpha)^{-1})) = \psi_{\kappa(\alpha)^{-1}[A]\kappa(\alpha)}(I + \pi X).$$

Let $T_G(\psi_{[A]}) = \{\alpha \in GL_n(O_2) \mid \psi^\alpha_{[A]} = \psi_{[A]}\}$. Observe that $L(SL)$ is a subgroup of $I + \pi M_n(O_2)$ and the character $\psi_A \in \hat{K}$ restricts to $\psi_{[A]}$ on $L(SL)$. By definitions it follows that $T_G(\psi_A) = \{\alpha \in GL_n(O_2) \mid \psi^\alpha_A = \psi_A\}$ is a subgroup of $T_G(\psi_{[A]})$. Let $T_{SL}(\psi_{[A]})$ be the stabilizer of $\psi_{[A]}$ in $SL_n(O_2)$, then $T_{SL}(\psi_{[A]}) = T_G(\psi_{[A]}) \cap SL_n(O_2)$. We subdivide our further discussion to two cases.

4.1. **The case $T_G(\psi_A) = T_G(\psi_{[A]})$:** The condition $T_G(\psi_A) = T_G(\psi_{[A]})$ implies
$$T_{SL}(\psi_{[A]}) = T_G(\psi_A) \cap SL_n(O_2).$$

Then define $\chi_{\psi_{[A]}} = \chi_{\psi_A}^G |_{T_G(\psi_A) \cap SL_n(O_2)}$, where $\chi_{\psi_A}^G$ is as obtained from Proposition 3.1. Then $\chi_{\psi_{[A]}}|_{L(SL)} = \psi_{[A]}$. This proves the existence of canonical extension, that is Proposition 2.2(a), in this case.

4.2. **The case $T_G(\psi_{[A]}) \neq T_G(\psi_A)$:** Let $T_{SL}(\psi_A)$ denote the set $T_G(\psi_A) \cap SL_n(O_2)$. For this case, we follow the following steps to obtain the character $\chi_{\psi_{[A]}}$ of $T_{SL}(\psi_{[A]})$.

**Step 1:** We show that the group $T_{SL}(\psi_A)$ is normal in $T_{SL}(\psi_{[A]})$ and the group $T_{SL}(\psi_{[A]})/T_{SL}(\psi_A)$ is abelian.

**Step 2:** There exists an abelian group $\mathcal{S}$ such that
$$T_{SL}(\psi_{[A]}) = T_{SL}(\psi_A)\mathcal{S},$$
and the intersection $\mathcal{S} \cap T_{SL}(\psi_A)$ is trivial.

**Step 3** There exists a character $\chi_A$ of $T_{SL}(\psi_A)$ that is invariant in $T_{SL}(\psi_{[A]})$ and hence extends to give required character $\chi_{\psi_{[A]}}$.

By definition of the action of $GL_n(O_2)$ on $M_n(O_1)$ and the set $\{[X] \mid X \in M_n(O_1)\}$ of equivalence classes, we obtain $T_G(\psi_{[A]}) = \{g \in GL_n(O_2) \mid g[A]g^{-1} = [A]\}$ and $T_G(\psi_A) = \{g \in GL_n(O_2) \mid gAg^{-1} = A\}$. Let $g \in T_G(\psi_{[A]})$ be such that $gAg^{-1} = A + xI$ for some $x \in O_1$, and let $z \in T(\psi_A)$. Then,
$$(gzg^{-1}A(gzg^{-1})^{-1} = gzg^{-1}Agz^{-1}g^{-1} = gz(A - xI)z^{-1}g^{-1} = A.$$
This implies that $T_G(ψ_A)(T_{SL}(ψ_A))$ is a normal subgroup of $T_G(ψ_{[A]})(T_{SL}(ψ_{[A]}))$. Further the quotient $T_G(ψ_{[A]})/T_G(ψ_A)(T_{SL}(ψ_{[A]})/T_{SL}(ψ_A))$ is abelian because
\[ g_1g_2A(g_1g_2)^{-1} = (g_2g_1)A(g_2g_1)^{-1}. \]

This completes the proof of Step 1.

We can assume that matrix $A$ has all its eigenvalues in the field $O_1$, for if not we can apply the argument to an extension field of $F$. Hence for further discussion, we shall assume that all the eigenvalues of $A$ lie in the field $O_1$.

The assumption $T_G(ψ_{[A]}) ≠ T_G(ψ_A)$ implies there exists a nonzero scalar $x ∈ O_1$ such that $A$ is conjugate to $xI + A$. Therefore, if $a$ is an eigenvalue of $A$ then so is $a + x$. We arrange the distinct eigenvalues of $A$ in the following order,
\[ a_{11}, a_{12}, \ldots, a_{1p}, a_{21}, a_{22}, \ldots, a_{2p}, \ldots, a_{r1}, a_{r2}, \ldots, a_{rp}, \]

where $a_{ij} = a_{i(j-1)} + x$, $a_{ip} + x = a_{i1}$ for all $1 ≤ i ≤ p, 2 ≤ j ≤ p$, and for $i ≠ i'$, $a_{ij} - a_{i'j} ∉ (x)$ (the additive space generated by $x$). Assume that $A$ is in its Jordan Canonical form (see Theorem 3.5 of Singla [7]), that is
\[ (4.1) \quad A = \oplus_{i=1}^{r} \oplus_{j=1}^{p} A_{ij}, \]

where each $A_{ij}$ is further direct sum of Jordan blocks with unique eigenvalue $a_{ij}$.

Every element of $T_G(ψ_{[A]})$ modulo the group $T_G(ψ_A)$ just permutes the matrices $A_{ij}$ among each other. Therefore every element of $T_G(ψ_{[A]})$ can be written as product of an element of the permutation matrix and an element of the group $T_G(ψ_A)$. The cosets of $T_{SL}(ψ_A)$ in $T_{SL}(ψ_{[A]})$ can be parametrized by the permutation matrices consisting of $pr × pr$ blocks with each $(i, j)^{th}$ block of size equal to size of matrix $A_{ij}$ and with the property that each block is either identity or zero matrix. Let $S$ be the collection of the permutation matrices corresponding to the quotient $T_{SL}(ψ_{[A]})/T_{SL}(ψ_A)$, then $S$ is an abelian group (by Step 1) satisfying,

1. $T_{SL} = ST_{SL}(ψ_A)$.
2. The intersection $S ∩ T_{SL}(ψ_A)$ is trivial.

This proves Step 2. For step 3, first of all we briefly recall the construction of character $χ_{ψ_A}^G$ of $T_G(ψ_A)$ that extends $ψ_A$, from Singla [7].

Let $s : O_1^\times → O_2^\times$ be the unique multiplicative section of the natural projection map $O_2^\times → O_1^\times$. By defining $s(0) = 0$, we obtain a section $s : O_1 → O_2$ of the natural projection map $O_2 → O_1$. By extending it entry wise, we obtain a map from $GL_n(O_1) → GL_n(O_2)$, denoted again by $s$. 


For a matrix $A$ as given in (11), let $Z_{GL_n(O_2)}(s(A)) = \{ g \in GL_n(O_2) \mid gs(A) = s(A)g \}$ and $m_{ij}$ be the size of each matrix $A_{ij}$. Then

$$Z_{GL_n(O_2)}(s(A)) = \prod_{i=1}^{r} \prod_{j=1}^{p} Z_{GL_{m_{ij}}(O_2)}(s(A_{ij})),$$

and $T_{G}(\psi_A) = K.Z_{GL_n(O_2)}(s(A))$ (see Lemma 5.1 and Corollary 5.3 of Singla [7]).

For any $a \in O_1$ define a character $\psi_a : 1 + \pi O_2 \to C^\times$ by $\psi_a(1 + \pi x) = \psi(ax)$. Since $O_2^\times$ is direct product of $O_1^\times$ and $1 + \pi O_2$, the characters $\psi_a$ extend trivially to $O_2^\times$, denote this by $\chi_a$. Define a character $\chi$ of $Z_{GL_n(O_2)}(s(A)) = \prod_{i=1}^{r} \prod_{j=1}^{p} Z_{GL_{m_{ij}}(O_2)}(s(A_{ij}))$ by

$$\chi(\prod_{i=1}^{r} \prod_{j=1}^{p} X_{ij}) = \chi_{a_{11}}(det(X_{11})) \cdots \chi_{a_{rp}}(det(X_{rp})).$$

Then the character $\chi_{\psi_a}^G = \psi_A.\chi$ of $T_{G}(\psi_A)$, defined by $\psi_A.\chi(uv) = \psi_A(u)\chi(v)$ for all $u \in K$ and $v \in Z_{GL_n(O_2)}(s(A))$, satisfies $\chi_{\psi_a}^G|_K = \psi_A$. Let $\chi_A = \chi_{\psi_a}^G|_{SL}(\psi_A)$. Then,

**Lemma 4.1.** The one dimensional representation $\chi_A$ of $T_{SL}(\psi_A)$ is fixed by $T_{SL}(\psi_{[A]})$.

**Proof.** To prove that $\chi_A$ is fixed by $T_{SL}(\psi_A)$, it is sufficient to prove that the restriction of $\chi$ to $Z_{GL_n(O_2)}(s(A)) \cap SL_n(O_2)$ is invariant under the action of elements of the group $S$, as $\psi_A|_{K \cap SL_n(O_2)} = \psi_{[A]}$ and $\psi_{[A]}$ is fixed by $T_{SL}(\psi_{[A]})$ by definition of $T_{SL}(\psi_{[A]})$.

Observe that any permutation matrix $s \in S$ such that $sAs^{-1} = A + x'I$ permutes the matrices $A_{ij}$ and $A_{ij'}$ among each other only if both $A_{ij}$ and $A_{ij'}$ have the same block decomposition and $a_{ij} - a_{ij'} \in (x')$, where $(x')$ denotes the additive space generated by $x'$. Let $X \in (Z_{GL_n(O_2)}(s(A)) \cap SL_n(O_2))$. Then the structure of $Z_{GL_n(O_2)}(s(A))$, we obtain that $X = \prod_{i=1}^{r} \prod_{j=1}^{p} X_{ij}$. Let $det(X_{ij}) = \beta_{ij}(1 + \pi \alpha_{ij})$, where $\beta_{ij} \in O_1^\times$ and $1 + \pi \alpha_{ij} \in 1 + \pi O_2$. Here we have again used the fact that $O_2^\times$ is direct product of $O_1^\times$ and $1 + \pi O_2$. Then $X \in SL_n(O_2)$ implies that $\prod_{i=1}^{r} \prod_{j=1}^{p} \beta_{ij} = 1$ and therefore $\sum_{i=1}^{r} \sum_{j=1}^{p} \alpha_{ij} = 0$. By definition of $\chi$, we obtain

$$\chi(X) = \psi(\sum_{i=1}^{r} \sum_{j=1}^{p} (a_{ij} + (j - 1)x)(\alpha_{ij}))$$

and

$$\chi(sXs^{-1}) = \psi(\sum_{i=1}^{r} \sum_{j=1}^{p} (a_{ij} + x' + (j - 1)x)(\alpha_{ij})).$$

but then $\sum_{i=1}^{r} \sum_{j=1}^{p} \alpha_{ij} = 0$ implies

$$\sum_{i=1}^{r} \sum_{j=1}^{p} (a_{ij} + (j - 1)x)(\alpha_{ij}) = \sum_{i=1}^{r} \sum_{j=1}^{p} (a_{ij} + x' + (j - 1)x)(\alpha_{ij}).$$
Therefore $\chi(X) = \chi(sXs^{-1})$. This proves the lemma.

Hence by Singla [7, Lemma 5.4] (also see Isaacs [3, Exercise 6.18]) $\chi_A$ extends to $T_{SL}(\psi[A])$. Furthermore, as $S$ is an abelian group with a trivial intersection with $T_{SL}(\psi_A)$, we can define extension $\chi_{\psi[A]}$ canonically such that $\chi_{\psi[A]}|_S$ is trivial. This completes the proof of Proposition 2.2(a) for $\text{SL}_n(\mathcal{O}_2)(p \mid n)$.

Let $Z_{GL_n(\mathcal{O}_1)}([A])$ be the subgroup of $GL_n(\mathcal{O}_1)$ consisting of matrices $g \in GL_n(\mathcal{O}_1)$ satisfying $g[A]g^{-1} = [A]$. Parts (b) and (c) of Proposition 2.2 follow by observing,

1. The group $T_{SL}(\psi[A])/L(SL)$ is isomorphic to $Z_{GL_n(\mathcal{O}_1)}([A]) \cap SL_n(\mathcal{O}_1)$.
2. $|SL_n(\mathcal{O}_2)/T_{SL}(\psi[A])| = |SL_n(\mathcal{O}_1)/(Z_{GL_n(\mathcal{O}_1)}([A]) \cap SL_n(\mathcal{O}_1))|$.

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**References**

[1] N. Avni, U. Onn, A. Prasad, and L. Vaserstein. Similarity classes of $3 \times 3$ matrices over a local principal ideal ring. *Comm. Algebra*, 37(8):2601–2615, 2009.

[2] M. Berman, J. Derakhshan, U. Onn, and P. Paajanen. Uniform cell decomposition and applications to chevalley groups. In preparation.

[3] I. M. Isaacs. *Character theory of finite groups*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. Pure and Applied Mathematics, No. 69.

[4] A. Jaikin-Zapirain. Zeta function of representations of compact $p$-adic analytic groups. *J. Amer. Math. Soc.*, 19(1):91–118 (electronic), 2006.

[5] G. Lusztig. Representations of reductive groups over finite rings. *Represent. Theory*, 8:1–14 (electronic), 2004.

[6] U. Onn. Representations of automorphism groups of finite $\sigma$-modules of rank two. *Adv. Math.*, 219(6):2058–2085, 2008.

[7] P. Singla. On representations of general linear groups over principal ideal local rings of length two. *J. Algebra*, 324(9):2543–2563, 2010.

[8] P. Singla. *Representations and Conjugacy Classes of Classical Groups over Finite Local Rings of Length Two*. PhD thesis, Institute of Mathematical Sciences, Chennai, India, http://www.math.bgu.ac.il/~pooja/thesis.pdf, 2010.

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