Quantization of the Particle Motion on the \(n\)-Dimensional Sphere

Petre Diţă

*Institute of Atomic Physics, P O Box MG6, Bucharest, Romania*

email: dita@theor1.ifa.ro

We develop here a simple formalism that converts the second-class constraints into first-class ones for a particle moving on the \(n\)-dimensional sphere. The Poisson algebra generated by the Hamiltonian and the constraints closes and by quantization transforms into a Lie algebra. The observable of the theory is given by the Casimir operator of this algebra and coincides with the square of the angular momentum.

I. INTRODUCTION

The quantization of classical Hamiltonians, when the canonical coordinates are not completely independent, is a long-standing problem in quantum mechanics. The constraints, i.e. a set of functions

\[ \varphi_i(q,p) = 0 , \quad i = 1,2,\ldots,p \quad (1.1) \]

restrict the motion of the classical system to a manifold embedded in the initial Euclidean phase space. This has in consequence that the canonical quantization rules

\[ [q_i, p_j] = i\hbar\delta_{ij} \]

are no more sufficient for the quantum description of the physical system.

When the manifold is a proper subspace of an Euclidean space Podolski \[5\] gave a solution by postulating that the Euclidean Laplacian should be replaced by the Laplace-Beltrami operator acting on this manifold. Applied to the motion of a point particle on a \(n\)-dimensional sphere \(S^n\) of radius \(R\) this gives for the Laplace-Beltrami operator the result \(L^2/2R^2\), where \(L\) is the angular momentum of the particle.

The eigenvalues of the Casimir operator \(L^2\) are \(l(l+n-1), \quad l = 0,1,\ldots,\) and in deriving this result one makes use of the Lie algebra of the angular momentum forgetting completely about the canonical variables initially entering in the problem, avoiding in this way any trouble which could appear.

Doubts concerning the correctness of this spectrum have been raised by people who derive the Schrödinger equation by Feynman’s path integral method; see for example \[\text{\cite{5,6}}\]. They found an extra energy term proportional to the Riemann scalar curvature of the manifold, but they do not agree upon the value of the proportionality factor.

Another type of doubt has been raised recently \[\text{\cite{7}}\], namely that the Dirac’s quantization method \[\text{\cite{6}}\] has to be rejected because, at least for this problem, the resulting energy spectrum is physically incorrect.

The last statement \[\text{\cite{6}}\] comes from a misunderstanding of the sublety of the problem: the constraints, like (1), reduce the dimension of the original phase space. The mechanism found by Dirac was the introduction of a new symplectic structure to handle the second-class constraints. The first-class constraints are used to dropping of some pairs of dinamical variables \((q_i, p_i)\) from the naive Hamiltonian, whose effect is that the nonphysical degrees of freedom are eliminated. See ref. \[\text{\cite{7}}\] for a treatment of the rigid rotator in the Dirac-bracket quantization formalism.

The purpose of this paper is to look at the Dirac formalism from a slightly modified point of view and to show that the new proposal leads to correct results. In fact we propose a new method for converting the second-class constraints into first-class constraints.

The Dirac quantized theory \[\text{\cite{6}}\] is patterned after the corresponding classical theory: the observables representing constraints *must* have zero expectation values. This requirement is inconsistent with the fact that there are dynamical variables whose Poisson brackets with the constraints fail to vanish. To solve the problem Dirac has constructed a new type of bracket, the Dirac-bracket, which vanishes whenever one of the two factors is a second-class constraint.

We develop here a formalism that converts the second-class constraints into first-class ones and which leads directly to group properties of the Poisson brackets. By quantization the Poisson algebra goes into a Lie algebra. The *observables* of the theory will be the *Casimir operators* of this algebra and the operators generated by the constraints will commute with the observables in the whole Hilbert space.

With this interpretation the Dirac’s theory of constrained systems gives correct results, and in the particular case of a point particle on \(S^n\) it confirms the Podolski result.

Our idea is to separate all the constants terms which may appear on the left side of Eq. (1.1) and push them on the right side. Thus we prefer to write (1.1) in the new form

\[ \varphi_i(q,p) = a_i , \quad i = 1,2,\ldots,p \quad (1.1') \]

where \(a_i\) are some complex constants.

A reason is that the Poisson bracket structure does not discriminate between \(\varphi\) and \(\varphi + C\), with \(C\) a constant. The main reason is that it now becomes possible to write the Poisson algebra in a closed form like

\[ \{\varphi_i(q,p), \varphi_j(q,p)\} = C^k_{ij} \varphi_k(q,p) \]

\[ \text{for some complex constants} \]

\[ \text{where} \]

\[ C_{ij} = \text{some complex constants} \]
where $H(q,p)$ is the Hamiltonian, $\varphi_i(q,p)$ are the constraints appearing in (1.1') and $C_{ij}^k$ and $C_i^j$ are constant structure coefficients.

Almost all that is found in the Dirac book \[3\], the novelty being only the redefinition of the dynamical variables $\varphi_i$. But this new form has the advantage of transforming at least a part of second-class constraints into first-class constraints as we will show in the next section.

In our opinion the Poisson algebra (1.2) generated by the Hamiltonian and the constraints is the best of the Dirac method.

The Poisson algebra, by the quantization procedure, transforms into a Lie algebra. The true observables of the physical system are given by the Casimir operators of the corresponding Lie algebras. In other words no one of the initial operators does transform into a veritable observable. The observables are given, at least, by quadratic functions of the old operators: Hamiltonian plus constraints together.

The quantic description of the physical model is given by a representation of this algebra onto a Hilbert space. In this way it becomes possible to avoid the canonical variables, which appear also in the Dirac formalism and, sometimes, cause problems since their Dirac brackets are not always canonical; in this sense see the treatment of the three-dimensional rotator in ref. \[7\].

An other consequence of the above idea is a solution of the embarrassing situation of forcing the operators generated by the constraints to vanish on the whole Hilbert space, as Dirac dixit. Because now the constraints are no more observables the above problem disappears. What we can say is that there exists a representation of the Lie algebra into an operator algebra acting on the Hilbert space of the associated physical system such that the operators $\hat{\varphi}_i$, generated by the constraints, should have the numbers $a_i$ in their spectra.

Of course there are cases when the above procedure does not work. An example of such a constraint is

$$\chi_1 = \sum_i c_i q_i = a$$

which is linear in coordinates. Its Poisson bracket with a quadratic free Hamiltonian gives the secondary constraint

$$\chi_2 = \{\chi_1, H\} = \sum_i c_i p_i$$

which is linear in momenta. The Poisson bracket of these two constraints is a constant $\sum c_i^2$, hence the linear constraints in coordinates and/or momenta generate another type of algebras. In these cases the relations (1.2) are supplemented with, at least, a few of the form

$$\{\chi_i, \chi_j\} = a_{ij}$$

where $a_{ij}$ are constants. These cases have to be treated by the Dirac formalism.

With this mild interpretation of Dirac theory the difficulties are overcome and the new theory is ready for applications.

In Sec.2 we treat the point particle on the $n$-dimensional sphere showing that the "Hamiltonian" of the problem is the square of the angular momentum and is obtained as the Casimir operator of the Lie algebra obtained by quantization of the Poisson algebra generated by the Hamiltonian and constraints. In Sec. 3 we consider a related problem: a $(n + 1)$-dimensional harmonic oscillator constrained to move on a hypersurface. The paper ends with Conclusion.

**II. POINT PARTICLE ON $S^N$ SPHERE**

We shall consider a point particle moving on the $n$-dimensional sphere $S^n$ whose equation is

$$r^2 = (q, q) = \sum_{i=1}^{n+1} q_i^2$$

where $(q, q)$ denotes the Euclidean product in the $n + 1$-dimensional space, i.e. we view the particle moving into a subspace of the $n + 1$-dimensional Euclidean space. Phase space degrees of freedom $(q_i, p_i)$ take values over the entire real axis and possess canonical Poisson bracket structure.

The primary constraint is usually written as

$$\varphi = r^2 - R^2 = 0$$

and the Hamiltonian has the form

$$H = \frac{1}{2}(p, p) = \frac{p^2}{2}$$

We define $U = r^2$ as new dynamical variable and by taking the Poisson brackets we get

$$\{U, H\} = 2 \sum q_i p_i = 2(q, p) = 2V$$

$$\{V, H\} = p^2 = 2H$$

(2.1)

$$\{U, V\} = 2U$$

If we use $\varphi$ as dynamical variable instead of $U$, as it is usually done \[3\], we find

$$\{\varphi, H\} = 2V$$

$$\{V, H\} = 2H$$

$$\{\varphi, V\} = 2r^2 = 2(\varphi + R^2)$$

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The last relation is usually written as \( \{ \varphi, V \} = 2R = 2R^2 \) which has consequence that after quantization one gets the commutation relation \( [\varphi, V] = 2i\hbar R^2 \). Here we use the same notation for operators on Hilbert space and for dynamical variables on phase space. The last relation is in conflict with the conditions for dynamical variables on phase space. The last relation will be seen later.

The relation \( \{ \varphi, V \} = 2(\varphi + R^2) \) suggested us the introduction of \( U \) as dynamical variable because

\[
\{ \varphi, V \} = \{ \varphi + R^2, V \} = 2(\varphi + R^2)
\]

Taking \( U = \varphi + R^2 = r^2 \) as dynamical variable has the advantage of closing the algebra as the relations (2.1) show and, more important, by this procedure both \( U \) and \( V \) become first-class constraints.

In the sequel we shall take \( \hbar = 1 \).

By the correspondence principle we obtain from Eqs. (2.1) the commutation relations

\[
[U, H] = 2iV
\]
\[
[V, H] = 2iH
\]
\[
[U, V] = 2iU
\]

In order to solve the problem we state our first postulate: \textit{all the relevant physics concerning the problem is contained in the Lie algebra (2.2)}.

This algebra remind us the known Lie algebra of the \( SU(2) \) group, so we can proceed like in that case. The single observable is the Casimir operator which is easily seen to be

\[
C = V^2 - UH - HU
\]

Indeed by a trivial calculation we obtain that

\[
[C, H] = [C, U] = [C, V] = 0
\]

\( C \) is the true observable of the theory and it commutes with the operators generated by the classical constraints and the classical Hamiltonian.

Like in the \( SU(2) \) case we can look for a common basis of eigenvectors for \( C \) and one of the operators \( H, U, \) or \( V \). We choose \( V \) as the ”third” component because it is singlet out by the algebra (2.2) as we shall see later.

From Eq.(2.3), \( V = (q, p) \) is the scalar product of \( q \) and \( p \) giving the projection of the momentum along the radius. The classical requirement \( V = 0 \) means that the motion of the point particle has to be such that there should be no energy or momentum flow across the sphere surface.

If we change from Euclidean to spherical coordinates \( V \) has the form

\[
V = \frac{1}{i} r \frac{\partial}{\partial r}
\]

where \( \partial / \partial r \) is the normal derivative or the gradient at the point of the sphere determined by its spherical coordinates.

Usually one imposes \( V = 0 \) as an operator equation on the Hilbert space. In our approach \( V \) is not an observable so the equation \( V = 0 \) is senseless. Our point of view is expressed by the second postulate: \textit{in the Hilbert space associated to our physical problem there does exist one vector which is annihilated by \( V \)}.

Thus the eigenvector of \( C \) and \( V \) has to satisfy the equation

\[
r \frac{\partial \Psi (r, \Omega)}{\partial r} |_{r=R} = 0
\]

where by \( \Omega \) we denote the angular variables. The last relation tell us that the eigenvector \( \Psi \) may depend on the radial variable \( r \), but its dependence is such that \( \Psi \) is stationary at \( r = R \). If we require that Eq.(2.4) should be valid for an arbitrary value of \( R \) we get that \( \Psi \) does not depend on \( r \).

By using the commutation relations (2.2) we find that

\[
C = V^2 + 2iV - 2UH
\]

On the other side, in spherical coordinates, \( H \) has the form

\[
2H = -(\frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} L^2)
\]

where \( L^2 \) is the Laplace-Beltrami operator on the unit Sphere \( S^n \) and coincides with the Casimir operator of the orthogonal group \( SO(n + 1) \).

By putting together the previous information we find that \( C \) has the form

\[
C = L^2 + (n + 1) r \frac{\partial}{\partial r}
\]

Taking into account Eq.(2.4) we find that

\[
C \Psi = (L^2 + (n + 1) r \frac{\partial}{\partial r}) \Psi |_{r=R} = L^2 \Psi = l(l + n - 1) \Psi
\]

Although \( C \) is not a Hermitean operator, because \( V \) is not, the action of both \( C \) and \( V \) on the eigenvector \( \Psi \) reduces to the action of the Hermitean operator \( L^2 \). In this way the eigenvalue problem is quantum mechanically well posed and the spectrum is \( l(l + n - 1), l = 0, 1, \ldots \).

Thus the Dirac formalism in the new interpretation tell us that the observable of a particle moving on the \( n \)-dimensional sphere is its angular momentum, a result which everybody expected to be so. This result can be tested directly at the classical and by way of consequence at quantic level.

For simplicity we take \( n = 2 \) and in this case the components of the angular momentum are

\[
L_1 = q_2 p_3 - q_3 p_2, \quad L_2 = q_3 p_1 - q_1 p_3, \quad L_3 = q_1 p_2 - q_2 p_1
\]
Let us introduce its projection on an arbitrary ray

\[ S = q_1 L_1 + q_2 L_2 + q_3 L_3 = l R \]

where \( l \) is a constant, the length of the projection, and consider \( S \) as a new constraint.

Taking into account the Poisson bracket relations

\[ \{ L_i, p_j \} = \epsilon_{ijk} p_k, \quad \{ L_i, q_j \} = \epsilon_{ijk} q_k \]

\[ \{ L_i, L_j \} = \epsilon_{ijk} L_k, \quad \{ x_i, p_j \} = \delta_{ij}, \quad i, j, k = 1, 2, 3 \]

we find that

\[ \{ S, H \} = \{ S, U \} = \{ S, V \} = 0 \]

relation which shows that \( S \) is a conserved quantity at the classical level since it commutes with the Hamiltonian and both constraints! At quantic level this relation says that \( S = f(C) \), i.e. it is a function of the Casimir operator of the Lie algebra.

We think that the above correct quantization of this simplest non-Euclidean system will have a fundamental theoretical interest showing us the route to follow in much more complicated cases.

Now we want to show that \( V \) is the analog of \( L_3 \) of the \( SU(2) \) group. We make the notation

\[ \{ L_{ij}, p_k \} = \epsilon_{ijk} p_k, \quad \{ L_{ij}, q_k \} = \epsilon_{ijk} q_k \]

\[ \{ L_{ij}, L_k \} = \epsilon_{ijk} L_k, \quad \{ x_i, p_j \} = \delta_{ij}, \quad i, j, k = 1, 2, 3 \]

we find that

\[ \{ S, H \} = \{ S, U \} = \{ S, V \} = 0 \]

and the commutation relations (2.2) take the form

\[ [H_1, E_+] = E_+ \]

\[ [H_1, E_-] = -E_- \]  \hspace{1cm} (2.6)

\[ [E_+, E_-] = H_1 \]

i.e. the well known Cartan-Chevalley form of the most simple Lie algebra. From (2.6) we see that \( E_\pm \) are the analog of \( L_\pm \) and \( H_1 \) is the analog of \( L_3 \), where \( L_\pm \) and \( L_3 \) are the usual generators of the \( SU(2) \) group.

III. CONSTRAINED HARMONIC OSCILLATOR

In the following we give a new argument in favour of our interpretation. For this we will consider another simple system: namely the quantization of the \( \ell \)-dimensional oscillator whose Hamiltonian is

\[ H_0 = \frac{1}{2} \sum_{i=1}^{\ell} (p_i^2 + q_i^2) \]

We suppose that its movement is confined to the hypersurface given by the constraint

\[ V_0 = \sum_{i=1}^{\ell} q_i p_i = a \]  \hspace{1cm} (3.1)

where \( a \) is a constant.

If we proceed as above we find the Poisson algebra

\[ \{ H_0, V_0 \} = \sum_{i=1}^{\ell} (q_i^2 - p_i^2) = 2U_0 \]

\[ \{ U_0, V_0 \} = 2H_0 \]

\[ \{ U_0, H_0 \} = 2V_0 \]

After quantization we find the Casimir operator which commutes with \( H_0, U_0, V_0 \)

\[ C_0 = H_0^2 - V_0^2 - U_0^2 \]

We denote by \( L_{ij} = q_i p_j - q_j p_i, \ i < j, \ i, j = 1, 2, \ldots, \ell \) the components of the angular momentum and by using the above expressions for \( H_0, U_0 \) and \( V_0 \) we find that

\[ C = \sum_{i<j} L_{ij}^2 = L^2 \]

i.e. the true Hamiltonian of the problem is again the square of the angular momentum. In fact this problem is a reformulation of the previous one. Indeed \( H_0, U_0 \) and \( V_0 \) are related to \( H, U, V \) by the relations

\[ H_0 = \frac{1}{2} U + H \]

\[ U_0 = \frac{1}{2} U - H \]

\[ V_0 = V \]

Our procedure can be applied whenever the Poisson algebra closes after a finite number of secondary constraints. This may not be the usual situation as a "small perturbation" of the previous problem shows. We deform the constraint (3.1) to

\[ V_1 = \sum_{i=1}^{\ell} c_i q_i p_i = a \]

where \( c_i \) are constants. Then one easily finds that the Poisson algebra never closes. If we define the sequence of "Hamiltonians"

\[ H_n = \frac{1}{2} \sum_{i=1}^{\ell} c_i^{2n} (q_i^2 + p_i^2) \]

\[ n = 0, 1, \ldots \]

and constraints

\[ V_n = \sum_{i=1}^{\ell} c_i^{2n-1} q_i p_i = 0 \]

\[ n = 2, 3, \ldots \]
\[ \mathcal{U}_n = \frac{1}{2} \sum_i \dot{q}_i^{2n-1}(q_i^2 - p_i^2) = 0 \quad n = 1, 2, \ldots \]

after quantization we find an infinite-dimensional algebra. Its commutation relations are

\[ [\mathcal{U}_m, \mathcal{H}_n] = 2i \mathcal{V}_{m+n}, \quad m = 1, 2, \ldots \quad n = 0, 1, \ldots \]

\[ [\mathcal{H}_m, \mathcal{V}_n] = 2i \mathcal{U}_{m+n}, \quad m = 0, 1, \ldots \quad n = 1, 2, \ldots \]

\[ [\mathcal{U}_m, \mathcal{V}_n] = 2i \mathcal{H}_{m+n-1} \quad m, n = 1, 2, \ldots \quad (3.2) \]

\[ [\mathcal{H}_m, \mathcal{H}_n] = 0 \]

\[ [\mathcal{U}_m, \mathcal{U}_n] = 0 \]

\[ [\mathcal{V}_m, \mathcal{V}_n] = 0 \]

where in the last three equations the indices of \( \mathcal{H}_m \) take the values \( m = 0, 1, 2, \ldots \), the range of the others being \( m = 1, 2, \ldots \), according with the notation of the first three equations.

A similar algebra is obtained by deforming the sphere \((q, q) = R^2\) into an ellipsoid by the change \( q_i \rightarrow q_i/a_i \). This shows that the problem of quantization with constraints is not a simple one, its natural place being the representation theory of infinite-dimensional algebras.

### IV. CONCLUSION

The main difficulty appearing in quantization of constrained systems is caused by second-class constraints. To overcome it Dirac invented a new symplectic structure, the Dirac bracket. However its use is not straightforward and we have to take care of when using it. This was a sufficient reason to looking for new methods of quantization.

The best known one is the method of abelian conversion that transforms a second-class constrained system into an abelian gauge theory \([11]\). The idea is to extend the original phase space by introducing new canonical coordinates and to convert the original Hamiltonian into a new one obtained by solving some equations. Upon quantization all non-physical degrees of freedom are removed by a restriction of the Hilbert space to a physical subspace formed by gauge invariant states.

In this paper we observed that there are some cases when the conversion of second-class constraints into first-class ones is very simple, namely we have seen that the obstacle was caused by the presence of constant terms within the functions defining the constraints. Because the Poisson bracket of a dynamical variable with a constant vanishes this opens the possibility to rewrite the original Poisson brackets into a new form by a simple redefinition of some of the dynamical variables. In this way it becomes possible to write the Poisson brackets into the form of Poisson algebra (1.2), that, after quantization, transforms into a Lie algebra. Once obtained this algebra we can use the known powerful machinery of representation theory to find the observables of the physical theory formalized by this algebra and to obtain the spectra of the physically relevant operators.

The lesson to be learned is that for constrained systems no one of the initial dynamical variables transforms into an observable. The observables are given by Casimir operators of Lie algebras, i.e. at least by quadratic functions of dynamical variables. By way of consequence we are no more constrained to impose operators equations on the Hilbert space, like in our case \( \varphi = V = 0 \), because the constraints do not become observables of the quantum theory.

It will be an interesting exercise to find a physical interesting constrained system for which the Cartan subalgebra of the corresponding Lie algebra is two-dimensional, because in this case it will be possible to obtain two observables of the system.

It seems to be a big challenge the construction of an operator representation of the infinite-dimensional algebra (3.2) on a Hilbert space that will solve, for example, the quantization of the motion of a point particle on an ellipsoid.

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