ON SUBSPACE CONCENTRATION FOR DUAL CURVATURE MEASURES

KATHARINA ELLER AND MARTIN HENK

Abstract. We study subspace concentration of dual curvature measures of convex bodies $K$ satisfying $\gamma(-K) \subseteq K$ for some $\gamma \in (0, 1]$. We present upper bounds on the subspace concentration depending on $\gamma$, which, in particular, retrieves the known results in the symmetric setting. The proof is based on a unified approach to prove necessary subspace concentration conditions via the divergence theorem.

1. Introduction

Let $K^n$ denote the set of convex bodies in $\mathbb{R}^n$, i.e., the family of all convex and compact subsets $K \subset \mathbb{R}^n$ with non-empty interior. The subfamily of convex bodies containing the origin in their interior, i.e., $0 \in \text{int} K$, is denoted by $K^n_o$ and the subset of origin-symmetric convex bodies, i.e., the sets $K \in K^n$ satisfying $K = -K$, is denoted by $K^n_e$. A convex body $K$ is called centered if its centroid is located at the origin, i.e.,

$$\frac{1}{\text{vol}(K)} \int_K x \, d\mathcal{H}^n(x) = 0,$$

where, in general, $\mathcal{H}^k$ denotes the $k$-dimensional Hausdorff measure, and when referring to the $n$-dimensional volume we will write $\text{vol}$ instead of $\mathcal{H}^n$. The set of all centered convex bodies in $\mathbb{R}^n$ is denoted by $K^n_c$, and, in particular, we have $K^n_e \subset K^n_e \subset K^n_o$.

As usual, for $x, y \in \mathbb{R}^n$ let $\langle x, y \rangle$ denote the standard inner product on $\mathbb{R}^n$, and $|x| = \sqrt{\langle x, x \rangle}$ the Euclidean norm of $x$. We write $B_n$ for the $n$-dimensional Euclidean unit ball, i.e., $B_n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and $S^{n-1} = \partial B_n$, where $\partial A$ is the set of boundary points of a set $A \subset \mathbb{R}^n$.

There are two far-reaching extensions of the classical Brunn-Minkowski theory, the $L_p$-Brunn-Minkowski theory and the dual Brunn-Minkowski theory. Both of them are cornerstones of modern convex geometry and both of them arise, roughly speaking, by studying the volume of the sum of convex bodies, where the usual Minkowski addition for building the sum is replaced by another kind of addition. In the case of the $L_p$-Brunn-Minkowski theory this is the so called $L_p$-addition, introduced by Firey [17] and Lutwak [35, 36, 37] for which we also refer to [42, Section 9.1, 9.2]. In the dual Brunn-Minkowski theory the so called radial addition, introduced by Lutwak [34], is used (see also [42, Section 9.3]).

One of the central problems in classical Brunn-Minkowski theory is the Minkowski-Christoffel problem asking for necessary and sufficient conditions characterizing the surface area measures of a convex body among the finite Borel measures on the sphere. For a definition of these surface area measures
and on the state of the art of the Minkowski-Christoffel problem we refer to [42, Chapter 8].

In the ground-breaking paper [27] by Huang, Lutwak, Yang and Zhang, the missing “dual” counterparts to these surfaces area measures within the dual Brunn-Minkowski theory were introduced. They are called dual curvature measures. In contrast to the surface area measures, they admit an explicit integral representation. To this end, for \( K \in \mathcal{K}_n^{(o)} \) let \( \rho_K \) be the radial function, i.e., for \( x \in \mathbb{R}^n \setminus \{0\} \) let

\[
\rho_K(x) = \max\{\rho > 0 : \rho x \in K\}.
\]

Then for \( q \in \mathbb{R} \), the \( q \)-th dual curvature measure of \( K \) is a finite Borel measure on \( S^{n-1} \) given by

\[
\tilde{C}_{K,q}(\eta) = \frac{1}{n} \int_{\alpha_k^*(\eta)} \rho_K(u)^q \, d\mathcal{H}^{n-1}(u),
\]

where for a Borel set \( \eta \subseteq S^{n-1} \), the set \( \alpha_k^*(\eta) \) consists of all \( u \in S^{n-1} \) such that the boundary point \( \rho_K(u)u \) of \( K \) has an outer unit normal vector in \( \eta \).

In analogy to the above mentioned classical Minkowski-Christoffel problem, the dual Minkowski problem, posed by Huang et al. in [27], asks for necessary and sufficient conditions when a finite Borel measure \( \mu \) on the sphere is the \( q \)-th dual curvature measure of a convex body \( K \in \mathcal{K}_n^{(o)} \).

Among these dual curvature measures there are two particular important measures. The 0-th dual curvature measure coincides up to a constant with Alexandrov’s integral curvature measure of the polar body of \( K \), and the corresponding Minkowski problem, known as Alexandrov problem has been solved by Alexandrov [1]. For extensions to the \( L_p \) setting of the Alexandrov problem we refer to [28, 39] and the references within.

The \( n \)-th dual curvature measure is in fact the cone volume measure \( V_K \) of \( K \) that is

\[
\tilde{C}_{K,n}(\eta) = V_K(\eta) = \frac{1}{n} \int_{\nu_k^{-1}(\eta)} \langle \nu_K(u), u \rangle \, d\mathcal{H}^{n-1}(u),
\]

where \( \nu_K(\cdot) \) is the spherical image map (see Section 2), essentially the Gauss map on the regular boundary points of \( K \). The characterization of the cone volume measure is known as the logarithmic Minkowski problem. It has been studied extensively over the last few years in many different contexts, see, e.g., [3, 4, 5, 6, 7, 8, 10, 14, 25, 32, 43, 44, 46], and for results in the general \( L_p \) setting see, e.g., [2, 15, 23, 29].

Regarding the dual Minkowski problem there is an obvious necessary condition, namely the measure \( \mu \) must not be concentrated on any closed hemisphere of \( S^{n-1} \). For \( q < 0 \) this is surprisingly also sufficient as shown by Yiming Zhao [47]. For positive parameters \( q \) the behaviour seems to be different and a quantitative ”subspace concentration” appears. In order to describe it, we set

\[
I(\mu, L) = \frac{\mu(S^{n-1} \cap L)}{\mu(S^{n-1})},
\]

for a linear subspace \( L \subset \mathbb{R}^n \), \( \dim L \geq 1 \), and a non-zero finite Borel measure \( \mu \) on \( S^{n-1} \). Due to the joint efforts of Böröczky, Henk, Huang, Lutwak,
Pollehn, Yang, Zhang, Zhao, [9, 12, 27, 48], a complete solution of the dual Minkowski problem in the even case is known in the range $q \in (0, n)$.

**Theorem I** (Theorem 1.1, [12]). Let $q \in (0, n)$, and let $\mu$ be an even non-zero finite Borel measure on $\mathbb{S}^{n-1}$. Then there exists a convex body $K \in \mathcal{K}^n_e$ such that $\mu = \tilde{C}_{K,q}$ if and only if

$$I(\mu, L) < \min \left\{ \frac{\dim L}{q}, 1 \right\}$$

for all proper linear subspaces $L \subset \mathbb{R}^n$.

For $q = n$, i.e., for the log-Minkowski problem a complete solution in the even case was given by Böröczky, Lutwak, Yang and Zhang.

**Theorem II** (Theorem 1.1, [11]). Let $\mu$ be an even non-zero finite Borel measure on $\mathbb{S}^{n-1}$. Then there exists a convex body $K \in \mathcal{K}^n_e$ such that $\mu = \tilde{C}_{K,n}$ if and only if

$$I(\mu, L) \leq \frac{\dim L}{n},$$

for all proper linear subspaces $L \subset \mathbb{R}^n$, and whenever equality holds in (1.1) for some $L$ then there exists a complementary subspace $L'$ such that $\mu$ is concentrated on $(L \cup L') \cap \mathbb{S}^{n-1}$.

For $q > n$ the dual (even) Minkowski problem is open, some necessary conditions are known, however, at least for $q > n + 1$.

**Theorem III** (Theorem 1.7, [26]). Let $q > n + 1$ and $K \in \mathcal{K}^n_e$. Then

$$I(\tilde{C}_{K,q}, L) < \frac{\dim L + q - n}{q},$$

for all proper linear subspaces $L \subset \mathbb{R}^n$.

This inequality is best possible and it is likely to be sufficient as well. For $n = 2$, (1.2) holds even true for $q > 2$.

In the non-even case we know only very little for $q > 0$. In fact, only the case $q = n$ (cone-volume measure) has been studied in this respect and even there we do not have matching necessary and sufficient conditions. For centered convex bodies it was shown by Böröczky and Henk [7] (see also [25] for the polytopal case) that (1.1) is also necessary.

**Theorem IV** (Theorem 1.3, [7]). Let $K \in \mathcal{K}^n_c$. Then

$$I(\tilde{C}_{K,n}, L) \leq \frac{\dim L}{n},$$

for all proper linear subspaces $L \subset \mathbb{R}^n$, and whenever equality holds for some $L$ then there exists a complementary subspace $L'$ such that $\mu$ is concentrated on $(L \cup L') \cap \mathbb{S}^{n-1}$.

The proof of the necessity of the inequalities in Theorems I, III, IV are based on three different approaches. The main purpose of this paper is i) to unify these approaches and ii) based on this unification to establish first results on the subspace concentration of the dual curvature measures of arbitrary bodies $K \in \mathcal{K}^{n(o)}$.
Theorem 1.1. Let $K \in \mathcal{K}^n_{c(o)}$, $\gamma \in (0,1]$ such that $\gamma(-K) \subseteq K$. Let $L \subset \mathbb{R}^n$ be a proper subspace and let $q \in \mathbb{R}$ with $q > \dim L + 1$. Then

$$\mathcal{I}(C_{K,q}, L) \leq \begin{cases} \frac{\dim(L) + \frac{n-1}{n}(q-\dim(L))}{q}, q \leq n, \\ \frac{(\dim(L) + q-n) + \frac{1}{n}(n-\dim(L))}{q}, q > n + 1. \end{cases}$$

For $K \in \mathcal{K}^n_{c}$, i.e., $\gamma = 1$, this theorem implies essentially the necessity parts of Theorem I and Theorem III. The additional restriction $q > \dim L + 1$ (instead of $q > \dim L$) in the range $q \leq n$ is caused by our more general approach, but is likely to be not necessary. As for centered convex bodies $K \in \mathcal{K}^n_{c}$, the asymmetry parameter $\gamma$ in the theorem above may be chosen to be at least $1/n$ (cf. [24, 45]) we get as a corollary

Corollary 1.2. Let $K \in \mathcal{K}^n_{c}$. Let $L \subset \mathbb{R}^n$ be a proper subspace and let $q \in \mathbb{R}$ with $q > \dim L + 1$. Then

$$\mathcal{I}(C_{K,q}, L) \leq \begin{cases} \frac{\dim(L) + \frac{n-1}{n}(q-\dim(L))}{q}, q \leq n, \\ \frac{(\dim(L) + q-n) + \frac{1}{n}(n-\dim(L))}{q}, q > n + 1. \end{cases}$$

Numerical results indicate that for $K \in \mathcal{K}^n_{c}$ the same inequalities hold true as in the even case. With our approach, as we will see this amounts to control a certain integral of directional derivatives, which we can handle efficiently only in case $q = n$ leading to Theorem IV.

In order to describe our approach which is based on [7, 9, 26] we need some more notation. For $K \in \mathcal{K}^n_{c(o)}$, $L \subset \mathbb{R}^n$ a proper subspace and $q \in \mathbb{R}$ with $q > \dim L$ let

$$g_{K,L,q}(x) = \int_{K \cap (x+L^\perp)} |z|^{q-n} \, d\mathcal{H}^{n-\dim L}(z)$$

where $x \in K|L$, i.e., $x$ belongs to the orthogonal projection of $K$ onto $L$, and $L^\perp$ is the orthogonal complement of $L$. For $q < n$ the integrand displays a singularity at the origin and is unbounded. However as long as we require $q > \dim L$ the integral exists.

By applying a generalized divergence theorem from [40] and establishing regularity properties of the section function $g_{K,L,q}$ we will show

Theorem 1.3. Let $K \in \mathcal{K}^n_{c(o)}$, $L \subset \mathbb{R}^n$ be a proper subspace and let $q \in \mathbb{R}$ with $q > \dim L + 1$. Then it holds

$$\mathcal{I}(C_{K,q}, L) = \frac{\dim L}{q} + \frac{1}{n C_{K,q}(S^{n-1})} \int_{K|L} \langle \nabla g_{K,L,q}(x), x \rangle \, d\mathcal{H}^{n-\dim L}(x).$$

For $q = n$ the function $g_{K,L,n}$ is log-concave and based on this property it was shown in [7] that $\int_{K|L} \langle \nabla g_{K,L,n}(x), x \rangle \, d\mathcal{H}^{n-\dim L}(x) \leq 0$. This implies Theorem IV except for the range $\dim L + 1 \geq q > \dim L$. For $q \neq n$, however, the slicing function is not log-concave and thus behaves quite differently.

Theorem 1.1 will follow immediately from Theorem 1.3 and the following bounds
Theorem 1.4. Let $K \in K_n(\gamma)$, $\gamma \in (0, 1]$ such that $\gamma(-K) \subseteq K$. Let $L \subseteq \mathbb{R}^n$ be a proper subspace and let $q \in \mathbb{R}$ with $q > \dim L + 1$. Then it holds
\[
\frac{1}{nC_{K,q}(S^{n-1})} \int_{K|L} \langle \nabla g_{K,L,q}(x), x \rangle \, d\mathcal{H}^{\dim L}(x)
\leq \begin{cases} 
\frac{q - \dim(L)}{q} \frac{1 - \gamma}{1 + \gamma}, & q \leq n, \\
\frac{(q-n) + \frac{1}{q} - (n - \dim(L))}{q}, & q > n + 1.
\end{cases}
\]

For results on the dual Minkowski problem in the smooth setting we refer to [13, 30, 33] and the references within. The paper is organized as follows: Necessary notation and preliminaries from Convex Geometry will be given in Section 2. The proof of Theorem 1.3 is presented in Section 3 where we actually prove a result for a slightly larger class of functions than $g_{K,L,q}(x)$ (see Theorem 3.8). Section 4 is devoted to the proof of Theorem 1.4 and thus of Theorem 1.1.

2. Preliminaries and Notation

We begin with a few basic facts about convex bodies and functions for which we refer to [18, 22, 41, 42]. A function $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called quasiconcave (or unimodal) if $f((1 - \lambda)x + \lambda y) \geq \min\{f(x), f(y)\}$ holds for all $x, y \in \mathbb{R}^n$.

As usual, a function $f : A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$, is called Lipschitz continuous or just Lipschitz if there exists a constant $L \geq 0$ such that for all $x, y \in A$
\[|f(x) - f(y)| \leq L |x - y|.
\]
A function $f : A \to \mathbb{R}^m$ will be called locally Lipschitz if for every $x \in A$ there exists an open neighbourhood $U \subseteq A$ such that $f|_U$ is Lipschitz. By a standard compactness argument we have that a locally Lipschitz function $f : A \to \mathbb{R}^m$ is Lipschitz on all compact subsets of $A$.

For $p \in \mathbb{R}$ we denote by $H(p, n)$ the class of all functions $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_{\geq 0}$ which are positively homogeneous of degree $p$, i.e., for all $x \in \mathbb{R}^n \setminus \{0\}$ and $\alpha > 0$ we have
\[f(\alpha x) = \alpha^p f(x).
\]
Observe for $p < 0$ and $f \in H(p, n)$ we must have $\lim_{x \to 0} f(x) = \infty$. The next proposition states the fact that a function $f \in H(p, n)$ which is Lipschitz restricted to the sphere $S^{n-1}$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$. We will state this fact in a rather explicit form for later purpose.

Lemma 2.1. Let $p \in \mathbb{R}$, $f \in H(p, n)$ be Lipschitz on $S^{n-1}$. Then $f$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$. More precisely, let $a \in \mathbb{R}^n \setminus \{0\}$. Then for all $x, y \in a + \frac{1}{2}|a|B_n$ we have
\[|f(x) - f(y)| \leq c_f |a|^{p-1} |x - y|,
\]where $c_f$ is a constant depending only on $f$. 
Proof. First we observe that the power functions of the norm are locally Lipschitz on \( \mathbb{R}^n \setminus \{0\} \). To this end we note that

\[
\max_{z \in a + \frac{1}{2} |a|B_n} |z|^{p-1} = c_p |a|^{p-1}
\]

where \( c_p \) is a constant depending only on \( p \). For \( p \neq 0 \) we get by the mean value theorem for \( x, y \in a + \frac{1}{2} |a|B_n \)

\[
| |x|^p - |y|^p | \leq |p| \left( \max_{z \in a + \frac{1}{2} |a|B_n} |z|^{p-1} \right) |x - y|
\]

(2.1)

for a constant \( \tilde{c}_p \), depending only on \( p \). For \( p = 0 \) we set \( \tilde{c}_p = 0 \) and the inequality is certainly still true.

Now let \( \varpi = z/|z| \in S^{n-1} \) for \( z \in \mathbb{R}^n \setminus \{0\} \) and, moreover, let \( \alpha = \max \{ f(z) : z \in S^{n-1} \} \). By assumption there exists a constant \( L \) such that

\[
|f(\varpi) - f(\eta)| \leq L |\varpi - \eta|
\]

for all \( x, y \in \mathbb{R}^n \setminus \{0\} \). As the convex function \( |t\varpi - \eta|^2 \), \( t \in \mathbb{R} \), is minimal at \( t = \langle \varpi, \eta \rangle = 1 \) we conclude for \( |x| \geq |y| \) that

(2.2)

\[
|f(\varpi) - f(\eta)| \leq L |\varpi - \eta| \leq L \left| \frac{x}{|y|} \varpi - \eta \right| = L \left| \frac{1}{|y|} |x - y| \right|
\]

Then for \( x, y \in a + \frac{1}{2} |a|B_n \), \( |x| \geq |y| \), we may write in view of (2.1) and (2.2)

\[
|f(x) - f(y)| = \left| \left| x \right|^p f(\varpi) - \left| y \right|^p f(\eta) \right| \\
\leq \left| y \right|^p |f(\varpi) - f(\eta)| + |f(\varpi)|| |x|^p - |y|^p | \\
\leq \left| y \right|^{p-1} L |x - y| + f(\varpi) \tilde{c}_p |a|^{p-1} |x - y| \\
\leq (c_p L + \alpha \tilde{c}_p) |a|^{p-1} |x - y|.
\]

With \( c_f = c_p L + \alpha \tilde{c}_p \) the assertion follows. \( \square \)

For a given convex body \( K \in K^n \) the support function \( h_K : \mathbb{R}^n \to \mathbb{R} \) is defined by

\[
h_K(u) = \max_{x \in K} \langle u, x \rangle.
\]

The support function is convex, continuous and, in particular, \( h_K \in H(1, n) \). The hyperplane

\[
H_K(u) = \{ x \in \mathbb{R}^n : \langle u, x \rangle = h_K(u) \}
\]

is a supporting hyperplane of \( K \) and for a boundary point \( v \in \partial K \cap H_K(u) \), the vector \( u \) will be called an outer normal vector. If in addition \( u \in S^{n-1} \) then is an outer unit normal vector. Let \( \partial^* K \subseteq \partial K \) be the set of all boundary points having an unique outer unit normal vector. We remark that the set of boundary points not having an unique outer normal vector has measure zero, that is \( \mathcal{H}^{n-1}(\partial K \setminus \partial^* K) = 0 \).

The spherical image map \( \nu_K : \partial^* K \to S^{n-1} \) maps a point \( x \) to its unique outer unit normal vector.

A kind of dual counterpart to the support function is the radial function \( \rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) for \( K \in K_n^{(n)} \). It is given by

\[
\rho_K(u) = \max \{ \rho > 0 : \rho u \in K \}.
\]
The radial function $\rho_K$ is positive, continuous, Lipschitz on $S^{n-1}$ with respect to the Euclidean metric, quasiconcave and $\rho_K \in H(-1,n)$.

Let $\Omega \subset S^{n-1}$ be the set of all unit vectors such that for $u \in \Omega$ the boundary point $\rho_K(u)u$ has an unique outer normal vector. The map $\alpha_K : \Omega \to S^{n-1}$ with $\alpha_K(u) = \nu_K(\rho_K(u)u)$ is called the radial Gauss map. For $\eta \subseteq S^{n-1}$, the reverse radial Gauss image of $\eta$ is defined by

$$\alpha_K^*(\eta) = \{ u \in S^{n-1} | \rho_K(u)u \in H_K(v) \text{ for some } v \in \eta \}.$$ 

The reverse radial Gauss image of $\eta$ consists of all $u \in S^{n-1}$ such that the boundary point $\rho_K(u)u$ has an outer unit normal vector in $\eta$ (see, e.g., [27]).

The maximal Euclidean distance between two points of $K$, i.e., the diameter of $K$, is denoted by $D(K)$, and for $A, B \subset \mathbb{R}^n$

$$d(A, B) = \inf \{ \delta > 0 | A \subseteq B + \delta B_n \text{ and } B \subseteq A + \delta B_n \}$$

denotes the Hausdorff distance between $A$ and $B$.

### 3. Proof of Theorem 1.3

As mentioned before, our proof strategy for Theorem 1.3 allows for a slightly extended definition of the dual curvature measure depending on a function $\varphi$ satisfying homogeneity and Lipschitz continuity. Inspired by previous generalizations as for example in [38] we present the results of this section in this generalized form.

**Definition 3.1.** Let $K \in K_n^{(o)}$, $q \in \mathbb{R}$, and let $\varphi \in H(q-n,n)$ be Lipschitz continuous on $S^{n-1}$. For a Borel set $\eta \subseteq S^{n-1}$ let

$$\tilde{C}_{K,\varphi,q}(\eta) = \frac{1}{n} \int_{\alpha_K^*(\eta)} \varphi(u)\rho_K(u)^q du.$$ 

Obviously, for $\varphi = | \cdot |^{-n} = \rho_{B_n}^{n-q}$ we get the dual curvature measure. Without assuming homogeneity and Lipschitz continuity we draw the connection to existing definitions: For $\varphi = \rho_Q^{-q}$ the $q$-th dual curvature measure with star body $Q$ introduced in [38] by Lutwak, Yang and Zhang is recovered. For $\varphi = (h_K \circ \alpha_K)^{-q} \rho_Q^{-q}$ the $L_p$ dual curvature measure also introduced in [38] is retrieved. For sake of completeness we mention that there exists an even more general definition namely the general dual Orlicz curvature measure introduced and examined in [19, 20]. However, our definition is tailored to the new approach presented.

In analogy to [9, Lemma 2.1] we can express $\tilde{C}_{K,\varphi,q}(\eta)$ for $q > 0$ as an integral of the function $\varphi(x)$.

**Lemma 3.2.** Let $K \in K_n^{(o)}$, $q > 0$ and $\varphi \in H(q-n,n)$. Then

$$\tilde{C}_{K,\varphi,q}(\eta) = \frac{q}{n} \int_{\{ x \in K : |x| \in \alpha_K^*(\eta) \}} \varphi(x) dH^n(x).$$
Proof. As in [9] one obtains by using spherical coordinates, Definition 3.1 and the homogeneity of $\varphi$:

$$\frac{q}{n} \int_{\left\{ x \in K \colon |x| \in \alpha_k^*(\eta) \right\}} \varphi(x) \, d\mathcal{H}^n(x) = \frac{q}{n} \int_{\alpha_k^*(\eta)} r^{q-1} \varphi(r u) \, dr \, du$$

$$= \frac{q}{n} \int_{\alpha_k^*(\eta)} \varphi(u) r^{q-1+q-n} \, dr \, du$$

$$= \frac{1}{n} \int_{\alpha_k^*(\eta)} \varphi(u) \rho_K(u)^q \, du$$

$$= \tilde{C}_{K,\varphi,q}(\eta). \quad \Box$$

Since we are going to evaluate the integral in (3.1) along slices of $K$ with affine planes we set for $x \in \mathbb{R}^n$, $q > 0$ and $\varphi \in H(q-n,n)$

$$g_{K,L,\varphi,q}(x) = \int_{K \cap (x + L^\perp)} \varphi(z) \, d\mathcal{H}^{n-\dim L}(z),$$

where $L$ is a proper subspace of $\mathbb{R}^n$ with orthogonal complement $L^\perp$. In order for this integral to exist we have to assume $q > \dim L$. Observe that

$$\tilde{C}_{K,\varphi,q}(S^{n-1}) = \int_{K \cap L} g_{K,L,\varphi,q}(x) \, d\mathcal{H}^{\dim L}(x).$$

In the next two lemmas we collect some basic properties of the function which enable us to apply a divergence theorem later on.

**Lemma 3.3.** Let $K \in K_n^{(\alpha)}$, $L \subset \mathbb{R}^n$ be a proper subspace and $q > \dim L$. Let $\varphi \in H(q-n,n)$ be Lipschitz continuous on $S^{n-1}$. Then

i) $g_{K,L,\varphi,q}$ is bounded on $K \cap L$.

ii) $g_{K,L,\varphi,q}$ is upper semicontinuous in $K \cap L$.

iii) For $x \in K \cap L$ it holds

$$\lim_{n \to \infty} g_{K,L,\varphi,q}(e^{-\frac{1}{n}}x) = g_{K,L,\varphi,q}(x).$$

**Proof.** For $x \in K \cap L$ let $K_x = K \cap (x + L^\perp)$ and let $k = \dim L$.

For i) let $R > 0$ such that $K \subseteq RB_n$, and let $\alpha \in \mathbb{R}_{>0}$ such that $\varphi(v) \leq \alpha$ for all $v \in S^{n-1}$. Applying spherical coordinates with respect to an orthonormal basis in $L \cup L^\perp$ we can write

$$g_{K,L,\varphi,q}(x) = \int_{K_x} \varphi(z) \, d\mathcal{H}^{n-k}(z)$$

$$\leq \int_{RB_n \cap L^\perp} \varphi(z + x) \, d\mathcal{H}^{n-k}(z)$$

$$= \int_{RB_n \cap L^\perp} \left| z + x \right|^{q-n} \varphi\left( \frac{z + x}{|z + x|} \right) \, d\mathcal{H}^{n-k}(z)$$

$$\leq \alpha \int_{RB_n \cap L^\perp} \left| z + x \right|^{q-n} \, d\mathcal{H}^{n-k}(z).$$

For $q \geq n$ the integrand is bounded and so it is $g_{K,L,\varphi,q}(x)$. So let $q < n$. As $x$ and $z$ are contained in orthogonal subspaces we have $|z + x| \geq |z|$ and...
so we may write
\[ g_{KL,\varphi,q}(x) \leq \alpha \int_{R \cap L^\perp} |z|^{q-n} \, dH^{n-k}(z). \]

Since \( q > k \), the integral is bounded.

In order to show ii), let \( x \in K|L \) and \( y_m \in K|L, m \in \mathbb{N} \), with \( \lim_{m \to \infty} y_m = x \). By the Blaschke selection theorem we can assume that the sequence \( C_m = K_{y_m} - y_m \subset L^\perp \) converges to a compact convex set \( C \subset L^\perp \) with respect to the Hausdorff distance. Thus \( K_{y_m} \) converges to \( x + C \) with \( x + C \subseteq K \cap (x + L^\perp) = K_x \). Then we obtain by the Lebesgue’s dominated convergence theorem
\[
\lim_{m \to \infty} g_{KL,\varphi,q}(y_m) = \lim_{m \to \infty} \int_{K_{y_m}} \varphi(z) \, dH^{n-k}(z) = \int_{K_x} \varphi(z) \, dH^{n-k}(z) = g_{KL,\varphi,q}(x).
\]

Finally, we come to iii). As \( 0 \in \text{int} \, K \) it holds \( e^{-\frac{1}{m}K_x} \subseteq K_{e^{-\frac{1}{m}x}} \) for \( m \in \mathbb{N}_{\geq 1} \). Thus
\[
g_{KL,\varphi,q}(e^{-\frac{1}{m}x}) &= \int_{K_{e^{-\frac{1}{m}x}}} \varphi(z) \, dH^{n-k}(z) \\
&\geq \int_{e^{-\frac{1}{m}K_x}} \varphi(z) \, dH^{n-k}(z) \\
&= e^{-\frac{k}{m}} \int_{K_x} \varphi(z) \, dH^{n-k}(z) \\
&= e^{-\frac{ak}{m}} g_{KL,\varphi,q}(x).
\]

Hence \( g_{KL,\varphi,q}(x) \leq \lim_{m \to \infty} g_{KL,\varphi,q}(e^{-\frac{1}{m}x}) \) and combined with ii) the claim follows.

Next we want to study Lipschitz continuity and differentiability properties of \( g_{KL,\varphi,q}(x) \). To this end we need the following lemma.

**Lemma 3.4.** Let \( K \in \mathcal{C}^n \), \( L \subset \mathbb{R}^n \) be a proper subspace, \( q > \dim L \) and let \( \varphi \in H(q-n,n) \) be Lipschitz continuous on \( S^{n-1} \). For \( x \in K|L \) let \( K_x = K \cap (x + L^\perp) \) and let \( U(x,\varepsilon) = x + (\varepsilon B_n \cap L) \) for an \( \varepsilon > 0 \).

i) For \( x \in \text{int} \, K|L \) there exists \( \overline{x} > 0 \) and a constant \( \overline{\tau}_x \) depending on \( x \) such that for all \( x_1, x_2 \in U(x,\overline{x}) \) with \( |x_1| \geq |x_2| \)
\[
\left| \int_{K_{x_1}} \varphi(z) \, dH^{n-k}(z) - \int_{K_{x_2} + (x_1 - x_2)} \varphi(z) \, dH^{n-k}(z) \right| \leq \overline{\tau}_x |x_1 - x_2|.
\]
ii) For $x \in \text{int} K [L \setminus \{0\}]$ there exists $\varepsilon_x > 0$ and a constant $c_\varphi$ depending only on $\varphi$ such that for all $x_1, x_2 \in U(x, \varepsilon_x)$

$$\left| \int_{K_{x_2}} \varphi(z) \, d\mathcal{H}^{n-k}(z) - \int_{K_{x_2} + x_1 - x_2} \varphi(z) \, d\mathcal{H}^{n-k}(z) \right|$$

$$(3.3)$$

$$\leq \left( c_\varphi \int_{-x_2 + K_{x_2}} |z + x|^{q-n-1} \, d\mathcal{H}^{n-k}(z) \right) |x_1 - x_2|.$$  

Proof. Let $k = \dim L$. Further, for abbreviation we write $K_i = -x_i + K_{x_i} \subseteq L^\perp$ for $i = 1, 2$. Let $R > 0$ such that $(1/R)B_n \subseteq K \subseteq RB_n$, and let $\alpha \in \mathbb{R}_{\geq 0}$ such that $\varphi(v) \leq \alpha$ for all $v \in S^{n-1}$.

For i) observe that

$$\left| \int_{K_{x_1}} \varphi(z) \, d\mathcal{H}^{n-k}(z) - \int_{K_{x_2} + (x_1 - x_2)} \varphi(z) \, d\mathcal{H}^{n-k}(z) \right|$$

$$= \left| \int_{K_{x_1} + x_1} \varphi(z) \, d\mathcal{H}^{n-k}(z) - \int_{K_{x_2} + x_1} \varphi(z) \, d\mathcal{H}^{n-k}(z) \right|$$

$$\leq \left( \int_{K_{x_1} \setminus K_{x_2} \cup K_{x_2} \setminus K_{x_1}} |z + x_1|^{q-n} \, d\mathcal{H}^{n-k}(z) \right) |x_1 - x_2|$$

$$\leq \alpha \int_{K_{x_1} \setminus K_{x_2} \cup K_{x_2} \setminus K_{x_1}} |z + x_1|^{q-n} \, d\mathcal{H}^{n-k}(z).$$

Now we claim that $|z + x_1|^{q-n} \leq \max\{(2R)^{q-n}, R^{n-q}\}$ for $z \in K_{x_1} \setminus K_{x_2} \cup K_{x_2} \setminus K_{x_1}$:

If $q \geq n$ this follows from $K \subseteq RB_n$. So let $q < n$, and suppose $|z + x_1|^{q-n} \geq R^{n-q}$, i.e., $|z + x_1| \leq 1/R$. Since $z$ and $x_1$ are contained in orthogonal subspaces and since $|x_1| \geq |x_2|$ we also have $|z + x_2| \leq 1/R$. As $(1/R)B_n \subseteq K$ this implies $z + x_i \in K \cap (x_i + L^\perp) = K_{x_i}$, $i = 1, 2$, and we get the contradiction $z \in K_{x_1} \cap K_{x_2}$.

We conclude

$$\left| \int_{K_{x_1} + x_1} \varphi(z) \, d\mathcal{H}^{n-k}(z) - \int_{K_{x_2} + x_1} \varphi(z) \, d\mathcal{H}^{n-k}(z) \right|$$

$$\leq \alpha \max\{(2R)^{q-n}, R^{n-q}\} \text{vol}_{n-k}(K_{x_1} \setminus K_{x_2} \cup K_{x_2} \setminus K_{x_1}).$$

By a result of Groemer [21, Theorem. i)] on comparing different metrics on the space of convex bodies we have

$$\text{vol}_{n-k}(K_{x_1} \setminus K_{x_2} \cup K_{x_2} \setminus K_{x_1}) \leq c(n, K) \, d(K_{x_1}, K_{x_2}),$$

where $c(n, K)$ is a constant depending only on $n$ and $K$. On the other hand, according to [31, Lemma 2.3] the Hausdorff distance of sections of convex bodies is locally Lipschitz continuous, i.e., there exists $\overline{\varepsilon}_x > 0$ and a constant $c_\varepsilon > 0$ such that $d(K_{x_1}, K_{x_2}) \leq c_\varepsilon |x_1 - x_2|$ for all $x_1, x_2 \in U(x, \overline{\varepsilon}_x)$.

As $d(K_{x_1}, K_{x_2}) \leq d(K_{x_1 \setminus K_{x_2}}, K_{x_2 \setminus K_{x_1}})$ we have shown

$$\left| \int_{K_{x_1}} \varphi(z) \, d\mathcal{H}^{n-k}(z) - \int_{K_{x_2} + (x_1 - x_2)} \varphi(z) \, d\mathcal{H}^{n-k}(z) \right| \leq \overline{\varepsilon}_x |x_1 - x_2|$$

for all $x_1, x_2 \in U(x, \overline{\varepsilon}_x)$ and a suitable constant $\overline{\varepsilon}_x$.  

Now we come to ii) and here we assume \( x \in \text{int } K \setminus \{0\} \). First we note that

\[
\left| \int_{K_{x_1}} \varphi(z) \, d\mathcal{H}^{n-k}(z) - \int_{K_{x_2}+(x_1-x_2)} \varphi(z) \, d\mathcal{H}^{n-k}(z) \right|
\]

\[
= \left| \int_{K_2} \varphi(z + x_2) \, d\mathcal{H}^{n-k}(z) - \int_{K_2} \varphi(z + x_1) \, d\mathcal{H}^{n-k}(z) \right|
\]

\[
\leq \int_{K_2} |\varphi(z + x_2) - \varphi(z + x_1)| \, d\mathcal{H}^{n-k}(z).
\]

Let now \( \varepsilon_x = \frac{1}{7} |x| \), and \( x_1, x_2 \in U(x, \varepsilon_x) \). Then for \( z \in L^\perp \) we have \( z + x_1, z + x_2 \in z + x + \frac{1}{2} |x| B_n \subseteq z + x + \frac{1}{2} |z + x| B_n \). In view of Lemma 2.1 we get

\[
\int_{K_2} |\varphi(z + x_2) - \varphi(z + x_1)| \, d\mathcal{H}^{n-k}(z)
\]

\[
\leq c_\varphi |x_1 - x_2| \int_{K_2} |z + x|^{q-n-1} \, d\mathcal{H}^{n-k}(z),
\]

where \( c_\varphi \) is a constant depending on \( \varphi \). \( \square \)

**Proposition 3.5.** Let \( K \in \mathcal{K}^n(o) \), \( L \subset \mathbb{R}^n \) be a proper subspace, \( q > \dim L \), and let \( \varphi \in H(q - n, n) \) be Lipschitz on \( S^{n-1} \).

i) \( g_{K,L,\varphi,q}(x) \) is locally Lipschitz in \( \text{int}(K|L) \setminus \{0\} \).

ii) \( g_{K,L,\varphi,q}(x) \) is almost everywhere differentiable in \( \text{int}(K|L) \).

iii) Let \( q > \dim L + 1 \). Then

\[
\int_{K|L} |\langle \nabla g_{K,L,\varphi,q}(x), x \rangle| \, d\mathcal{H}^{\dim L}(x) < \infty.
\]

**Proof.** The second statement follows directly from i) via Rademacher’s theorem (cf. Theorem 3.1.6 in [16]). In order to verify i) we use Lemma 3.4 and its notation. So let \( x \in \text{int } K|L \setminus \{0\} \), \( \delta_x = \min\{\varepsilon_x, \varepsilon_x\} \) and let \( x_1, x_2 \in U(x, \delta_x) \) and assume \( |x_1| \geq |x_2| \). Then

\[
|g_{K,L,\varphi,q}(x_1) - g_{K,L,\varphi,q}(x_2)|
\]

\[
= \left| \int_{K_{x_1}} \varphi(z) \, d\mathcal{H}^{n-k}(z) - \int_{K_{x_2}} \varphi(z) \, d\mathcal{H}^{n-k}(z) \right|
\]

\[
\leq \int_{K_{x_1}} \varphi(z) \, d\mathcal{H}^{n-k}(z) - \int_{K_{x_2}+(x_1-x_2)} \varphi(z) \, d\mathcal{H}^{n-k}(z)
\]

\[
+ \int_{K_{x_2}} \varphi(z) \, d\mathcal{H}^{n-k}(z) - \int_{K_{x_2}+(x_1-x_2)} \varphi(z) \, d\mathcal{H}^{n-k}(z)
\]

\[
\leq |x_1 - x_2| \left( c_\varphi + c_\varphi \int_{-x_2 + K_{x_2}} |z + x|^{q-n-1} \, d\mathcal{H}^{n-k}(z) \right).
\]
Again assuming that \( K \subseteq RB_n \) we may bound
\[
|g_{K,L,\varphi,q}(x_1) - g_{K,L,\varphi,q}(x_2)|
\leq |x_1 - x_2| \left( \overline{c}_x + c_\varphi \int_{RB_n \cap L^+} |z + x|^{q-n-1} \, d\mathcal{H}^{n-k}(z) \right),
\]
As \( |z + x| \leq 2R \) the last integral is bounded by \( \overline{c} = (2R)^{q-k-1} \text{vol}_{n-k}(B_n \cap L^+) \) if \( q - n - 1 \geq 0 \). If \( q - n - 1 < 0 \) we note that \( |z + x| \geq |x| > 0 \) and so
\[
\int_{RB_n \cap L^+} |z + x|^{q-n-1} \, d\mathcal{H}^{n-k}(z) \leq \overline{c}_x = |x|^{q-n-1} R^{n-k} \text{vol}_{n-k}(B_n \cap L^+).
\]
Altogether we obtain,
\[
|g_{K,L,\varphi,q}(x_1) - g_{K,L,\varphi,q}(x_2)| \leq |x_1 - x_2| \left( \overline{c}_x + c_\varphi \max \{\overline{c}, \overline{c}'\} \right),
\]
which shows i).

In order to verify iii) we will first argue that in the case \( x \in \text{int} e^{-\frac{1}{\varphi}} K \subseteq L \setminus \{0\} \) and \( q > \dim L + 1 \) we can make the constants in (3.4) independent of \( x \).

We start with \( \overline{c}_x \) from (3.4) appearing in the case \( q - n - 1 < 0 \). If \( q > \dim L + 1 \) and so \( q - n - 1 > \dim L - n \) the integral \( \int_{RB_n \cap L^+} |z + x|^{q-n-1} \, d\mathcal{H}^{n-k}(z) \) is bounded from above by a constant \( \overline{c}' \) for any \( x \in K \subseteq L \).

Hence, (3.4) becomes
\[
|g_{K,L,\varphi,q}(x_1) - g_{K,L,\varphi,q}(x_2)| \leq |x_1 - x_2| \left( \overline{c}_x + c_\varphi \max \{\overline{c}, \overline{c}'\} \right),
\]
for all \( x_1, x_2 \in U(x, \delta_x) \). By a standard compactness argument we can bound the constants \( c_x \) for all \( x \in e^{-\frac{1}{\varphi}} K \subseteq L \) by a constant \( \overline{c} \) and so we get
\[
|g_{K,L,\varphi,q}(x_1) - g_{K,L,\varphi,q}(x_2)| \leq |x_1 - x_2| \left( \overline{c} + c_\varphi \max \{\overline{c}, \overline{c}'\} \right),
\]
for all \( x_1, x_2 \in U(x, \delta_x) \), and \( x \in e^{-\frac{1}{\varphi}} K \subseteq L \setminus \{0\} \). Hence, for any \( x \in e^{-\frac{1}{\varphi}} K \subseteq L \setminus \{0\} \) where \( \nabla g_{K,L,\varphi,q}(x) \) exists, it holds
\[
|\langle \nabla g_{K,L,\varphi,q}(x), x \rangle| = \lim_{\varepsilon \to 0} \frac{|g_{K,L,\varphi,q}(x + \varepsilon x) - g_{K,L,\varphi,q}(x)|}{|\varepsilon|} \leq (\overline{c} + c_\varphi \max \{\overline{c}, \overline{c}'\}) \, |x|.
\]
According to ii) the gradient exists almost everywhere in \( K \subseteq L \) and so we have
\[
\int_{K \subseteq L} |\langle \nabla g_{K,L,\varphi,q}(x), x \rangle| \, d\mathcal{H}^{\dim L}(x)
= \lim_{m \to \infty} \int_{e^{-\frac{1}{\varphi}} K \subseteq L} |\langle \nabla g_{K,L,\varphi,q}(x), x \rangle| \, d\mathcal{H}^{\dim L}(x) < \infty. \tag*{□}
\]

Regarding Proposition 3.5 ii) we remark that in general \( g_{K,L,\varphi,q} \) is not locally Lipschitz in \( 0 \), as the following example shows: Let \( k \in \{1, \ldots, n-1\} \) and \( \varphi(\cdot) = |\cdot|^{q-n} \), denote \( K = B_k \times B_{n-k} \subseteq \mathbb{R}^n \) and \( L = \mathbb{R}^k \). Note that the sections of \( K \) are the same up to translation. Therefore the summand in (3.2) is zero and only the term in (3.3) is relevant to decide local Lipschitz
continuity in zero. Let \( k + 1 > q > k \) and \( x \in K \) with \(|x| < 1\). As before we obtain
\[
g_{K,L,q}(x) = \int_{B_{n-k}} |x + z|^{q-n} d\mathcal{H}^{n-k}(z)
\]
\[
= \int_{0}^{1} \int_{S^{n-k-1}} r^{n-k-1} |x + ru|^{q-n} dudr
\]
\[
= (n - k) \text{vol}(B_{n-k}) |x|^{q-k} \int_{0}^{1} s^{n-k-1} \sqrt{s^2 + 1}^{q-n} ds
\]
\[
\leq (n - k) \text{vol}(B_{n-k}) |x|^{q-k} \left( \left( \frac{1}{\sqrt{2}} \right)^{n-q} \int_{0}^{1} s^{q-k-1} ds + \int_{1}^{\infty} s^{q-k-1} ds \right)
\]
\[
\leq (n - k) \text{vol}(B_{n-k}) \frac{1}{q - k} \left( \left( \frac{1}{\sqrt{2}} \right)^{n-q} |x|^{q-k} + 1 - |x|^{q-k} \right).
\]

On the other hand it holds
\[
g_{K,L,q}(0) = \int_{B_{n-k}} |z|^{q-n} d\mathcal{H}^{n-k}(z)
\]
\[
= \int_{0}^{1} \int_{S^{n-k-1}} r^{q-k-1} dudr
\]
\[
= (n - k) \text{vol}(B_{n-k}) \frac{1}{q - k}.
\]

This gives
\[
\frac{|g_{K,L,q}(x) - g_{K,L,q}(0)|}{|x|} \geq (n - k) \text{vol}(B_{n-k}) \frac{1}{q - k} |x|^{q-k-1} \left( 1 - \left( \frac{1}{\sqrt{2}} \right)^{n-q} \right),
\]

which goes to infinity as \(|x|\) goes to zero.

The next lemma which gives a representation of \( \tilde{C}_{K,\varphi,q}(\eta) \) via the inverse Gauss map was proven for \( \varphi = |\cdot|^{q-n} \) in [27] and our slightly more general case can be done analogously.

**Lemma 3.6.** Let \( K \in K_{(o)}^{o} \), \( q > 0 \). Let \( \varphi \in H(q-n,n) \) be Lipschitz continuous on \( S^{n-1} \). Let further \( \eta \subseteq S^{n-1} \) be a Borel set. Then
\[
\tilde{C}_{K,\varphi,q}(\eta) = \frac{1}{n} \int_{\nu_{K}(\eta)} \langle \nu_{K}(x), x \rangle \varphi(x) d\mathcal{H}^{n-1}(x).
\]

**Proof.** See [27, Lemma 3.5].

We aim to express the dual curvature measure on a subspace as a multiple of the dual curvature measure on the whole sphere plus a term depending on the directional derivative of \( g_{K,L,\varphi,q} \). This can be established via a divergence theorem following the approach presented in [8, 25]. The following divergence theorem presented in [40] will be employed.
Theorem 3.7. Let $A \in \mathcal{K}^n$. Let $G : A \to \mathbb{R}^n$ be a bounded vector field, which is locally Lipschitz continuous in $A \setminus \{a\}$ for some $a \in A$. Furthermore suppose that $\text{div} G \in L_1(A)$. Then it holds

$$\int_A \text{div} G(x) \, d\mathcal{H}^n(x) = \int_{\partial^* A} (G(x), \nu_A(x)) \, d\mathcal{H}^n(x).$$

This is a special case of the divergence theorem presented in Proposition 7.4.3 in [40] for sets of bounded variation and admissible vector fields. Locally Lipschitz vector fields and convex bodies satisfy these presumptions. The next theorem is to some extend our main result as it relates $\tilde{C}_{K,\varphi,q}(\mathbb{S}^{n-1})$ to $\tilde{C}_{K,\varphi,q}(\mathbb{S}^{n-1})$ via Theorem 3.7. All further results are based on this.

Theorem 3.8. Let $K \in \mathcal{K}^n_{(o)}$, $L \subset \mathbb{R}^n$ be a proper subspace and $q > \dim L$. Let $\varphi \in H(q - n, n)$ be Lipschitz continuous on $\mathbb{S}^{n-1}$. Furthermore assume that for any $m \in \mathbb{N} \geq 1$

$$\int_{e^{-\frac{1}{m}} K \setminus L} |(\nabla g_{K,L,\varphi,q}(x), x)| \, d\mathcal{H}^{\dim L}(x) < \infty.$$

Then it holds

$$I(\tilde{C}_{K,\varphi,q}, L) = \frac{\dim L}{q} + \frac{1}{n} \frac{1}{C_{K,\varphi,q}(\mathbb{S}^{n-1})} \lim_{m \to \infty} \int_{e^{-1/m} K \setminus L} (\nabla g_{K,L,\varphi,q}(x), x) \, d\mathcal{H}^{\dim L}(x).$$

Proof. We follow the proof outline of Lemma 3.3 in [8]. Let $\dim L = k$, and for $m \in \mathbb{N} \geq 1$ we set

$$E_m = e^{-\frac{1}{m}} K \setminus L.$$

The relative boundary of $E_m$ with respect to the subspace $L$ will be denoted by $\partial E_m$. We define the vector field $G : K \setminus L \to \mathbb{R}^n$ by

$$G(x) = g_{K,L,\varphi,q}(x)x.$$

By Proposition 3.5 ii) $G$ is almost everywhere differentiable on $K \setminus L$ and for its divergence $\text{div} G$ we find

$$\text{div} G(x) = kg_{K,L,\varphi,q}(x) + (\nabla g_{K,L,\varphi,q}(x), x).$$

Hence with (3) it follows

$$\int_{E_m} |\text{div} G(x)| \, d\mathcal{H}^k(x) \leq k \frac{\mu}{q} C_{K,\varphi,q}(\mathbb{S}^{n-1}) + \int_{E_m} |(\nabla g_{K,L,\varphi,q}(x), x)| \, d\mathcal{H}^k(x) < \infty.$$

So it holds $\text{div} G \in L_1(E_m)$. Thus we may apply the divergence theorem, i.e., Theorem 3.7 and get

$$\int_{E_m} \text{div} G(x) \, d\mathcal{H}^k(x) = \int_{\partial E_m} (G(x), \nu_{E_m}(x)) \, d\mathcal{H}^{k-1}(x).$$
First we consider the right-hand side of (3.6). For a regular boundary point $x \in \partial^* K|L$ we have $\nu_{K|L}(x) = \nu_{E_m}(e^{-\frac{1}{m}}x)$ and so we may write

$$\int_{\partial^* E_m} \langle G(x), \nu_{E_m}(x) \rangle \, d\mathcal{H}^{k-1}(x) = e^{-\frac{k-1}{m}} \int_{\partial^* K|L} \langle G(e^{-\frac{1}{m}}x), \nu_{K|L}(x) \rangle \, d\mathcal{H}^{k-1}(x) = e^{-\frac{k-1}{m}} \int_{\partial^* K|L} g_{K,L,\varphi,q}(e^{-\frac{1}{m}}x) \langle x, \nu_{K|L}(x) \rangle \, d\mathcal{H}^{k-1}(x).$$

By Lemma 3.3 iii) we have $g_{K,L,\varphi,q}(e^{-\frac{1}{m}}x) \to g_{K,L,\varphi,q}(x)$ pointwise for $m \to \infty$ and with the Lebesgue dominated convergence theorem and the definition of $g_{K,L,\varphi,q}(x)$ we obtain

$$\lim_{m \to \infty} \int_{\partial^* E_m} \langle G(x), \nu_{E_m}(x) \rangle \, d\mathcal{H}^{k-1}(x) = \int_{\partial^* K|L} g_{K,L,\varphi,q}(x) \langle x, \nu_{K|L}(x) \rangle \, d\mathcal{H}^{k-1}(x) = \int_{\partial^* K|L} \int_{K \cap (x + L^+)} \varphi(z) \langle x, \nu_{K|L}(x) \rangle \, d\mathcal{H}^{n-k}(z) \, d\mathcal{H}^{k-1}(x).$$

Now set $M = \partial K \cap (L^+ + \partial^* K|L)$. Then the set of regular points in $M$ is precisely the set of all regular boundary points of $K$ having their unique outer normal vector in $L \cap S^{n-1}$. In view of Lemma 3.6 we get

$$\lim_{m \to \infty} \int_{\partial^* E_m} \langle G(x), \nu_{E_m}(x) \rangle \, d\mathcal{H}^{k-1}(x) = \int_{M} \varphi(z) \langle z|L, \nu_{K|L}(z|L) \rangle \, d\mathcal{H}^{n-1}(z) = \int_{\nu^{-1}_K(L \cap S^{n-1})} \varphi(z) \langle z, \nu_K(z) \rangle \, d\mathcal{H}^{n-1}(z) = n \tilde{C}_{K,\varphi,q}(L \cap S^{n-1}).$$

Next we turn to the left-hand side of (3.6) which by (3.5) is

$$\int_{E_m} \text{div} G(x) \, d\mathcal{H}^{k}(x) = k \int_{E_m} g_{K,L,\varphi,q}(x) \, d\mathcal{H}^{k}(x) + \int_{E_m} \langle \nabla g_{K,L,\varphi,q}(x), x \rangle \, d\mathcal{H}^{k}(x).$$

Again by the Lebesgue dominated convergence theorem it holds

$$\lim_{m \to \infty} \int_{E_m} g_{K,L,\varphi,q}(x) \, d\mathcal{H}^{k}(x) = \int_{K|L} g_{K,L,\varphi,q}(x) \, d\mathcal{H}^{k}(x) = n \tilde{C}_{K,\varphi,q}(S^{n-1}).$$
as a corollary:

iii) we obtain a slight generalization of Theorem 1.3 as a corollary:

**Corollary 3.9.** Let $K \in \mathcal{K}_{(o)}^n$, $L \subset \mathbb{R}^n$ be a proper subspace and $q > \dim L + 1$. Let $\varphi \in H(q-n,n)$ be Lipschitz continuous on $\mathbb{S}^{n-1}$. Then it holds

$$n\tilde{C}_{K,\varphi,q}(\mathbb{S}^{n-1} \cap L) = \frac{n}{q} \dim(L)\tilde{C}_{K,\varphi,q}(\mathbb{S}^{n-1}) + \int_{K|L} \langle \nabla g_{K,L,\varphi,q}(x), x \rangle d\mathcal{H}^{\dim L}(x).$$

4. Proof of Theorem 1.1 and 1.4

Here we return to the function $\varphi = |\cdot|^{q-n}$ and depending on $q$ we have to distinguish the quasiconcave range $q \leq n$ and the convex range $q \geq n + 1$ of this function. The quasiconvex range $n < q < n + 1$ remains open.

Now let $K \in \mathcal{K}_{(o)}^n$, $\gamma \in [0,1]$ such that $\gamma(-K) \subseteq K$. In order to exploit this fact for our purposes we note that for any $x \in K|L$

$$-\gamma \left(K \cap (x + L^\perp)\right) \subseteq K \cap (-\gamma x + L^\perp).$$

Hence for any $\lambda \in [0,1]$ we have

$$\left(\frac{\gamma + \lambda}{1 + \gamma}\right) K_x + \left(\frac{1 - \lambda}{1 + \gamma}\right) (-K_x) \subseteq K_{\lambda x}$$

where we set $K_y = K \cap (y + L^\perp)$ for $y \in K|L$. Observe that the left-hand side is in general strictly larger than $\lambda K_x$ which is contained in $K_{\lambda x}$ as $0 \in K$.

4.1. The quasiconcave range $q \leq n$. First we state a lemma from [7] in a different but equivalent form.

**Lemma 4.1.** Let $K \in \mathcal{K}^n$ with $\dim(K) = k$. Let $\varphi \in H(p,n)$ be quasiconcave, even and integrable on $k$-dimensional compact convex sets. Then it holds for $\lambda_0, \lambda_1 > 0$

$$\int_{\lambda_0 K + \lambda_1 (-K)} \varphi(z) d\mathcal{H}^k(z) \geq (\lambda_0 + \lambda_1)^{p+k} \int_K \varphi(z) d\mathcal{H}^k(z).$$

**Proof.** Setting $\lambda = \frac{\lambda_0}{\lambda_0 + \lambda_1} \in [0,1]$ we get

$$\int_{\lambda_0 K + \lambda_1 (-K)} \varphi(z) d\mathcal{H}^k(z) = (\lambda_0 + \lambda_1)^{p+k} \int_{\lambda K + (1-\lambda)(-K)} \varphi(z) d\mathcal{H}^k(z) \geq (\lambda_0 + \lambda_1)^{p+k} \int_K \varphi(z) d\mathcal{H}^k(z),$$
Lemma 4.2. Let $K \in K^n_{(0)}$, $\gamma \in [0,1]$ such that $\gamma(-K) \subseteq K$, $L \subseteq \mathbb{R}^n$ be a proper subspace and let $\dim(L) < q \leq n$. Let $x \in \text{int} K\lvert L$ and assume that $\nabla g_{K,L,q}(x)$ exists. Then it holds

$$\langle \nabla g_{K,L,q}(x), x \rangle \leq \frac{1 - \gamma}{\gamma + 1} (q - \dim(L)) g_{K,L,q}(x).$$

Proof. For $y \in K\lvert L$ let $K_y = K \cap (y + L^\perp)$, let $\lambda \in [0,1)$ and set $\lambda_0 = \frac{\gamma}{\gamma + 1}$ and $\lambda_1 = \frac{1 - \gamma}{\gamma + 1}$. With (4.1) we have

$$\lambda_0 K_x + \lambda_1 (-K_x) \subseteq K_{\lambda x}$$

and Lemma 4.1 gives

$$g_{K,L,q}(\lambda x) = \int_{K_{\lambda x}} |z|^{q-n} d\mathcal{H}^{n-\dim L}(z) \geq \int_{\lambda_0 K_x + \lambda_1 (-K_x)} |z|^{q-n} d\mathcal{H}^{n-\dim L}(z) \geq (\lambda_0 + \lambda_1)^{n-\dim(L)+q-n} \int_{K_x} |z|^{q-n} d\mathcal{H}^{n-\dim L}(z) = \left(1 + (1 - \lambda)\frac{\gamma - 1}{\gamma + 1}\right)^{q-\dim(L)} g_{K,L,q}(x).$$

Setting $\lambda = 1 - \varepsilon$ yields

$$\frac{g_{K,L,q}((1 - \varepsilon)x) - g_{K,L,q}(x)}{\varepsilon} \geq f'(0) g_{K,L,q}(x),$$

with $f(t) = \left(\frac{\gamma - 1}{\gamma + 1} t + 1\right)^{q-\dim(L)}$. Hence,

$$\langle \nabla g_{K,L,q}(x), -x \rangle = \lim_{\varepsilon \rightarrow 0} \frac{g_{K,L,q}((1 - \varepsilon)x) - g_{K,L,q}(x)}{\varepsilon} \geq f'(0) g_{K,L,q}(x) = \frac{\gamma - 1}{\gamma + 1} (q - \dim(L)) g_{K,L,q}(x).$$

Remark 4.3. Lemma 4.2 also holds for the section function $g_{K,L,\varphi,q}$ when $\varphi \in H(q - n, n)$ is quasiconcave and even.

4.2. The convex setting in the range $q \geq n + 1$.

Lemma 4.4. Let $K_0, K_1 \in K^n_{(0)}$ with $\dim(K_1) = k$, $\text{vol}(K_0) = \text{vol}(K_1)$ and assume that their affine hulls are parallel. For $\lambda_0, \lambda_1 > 0$ and $p \geq 1$, it holds

$$\int_{\lambda_0 K_0 + \lambda_1 K_1} |z|^p d\mathcal{H}^k(z) + \int_{\lambda_0 K_1 + \lambda_1 K_0} |z|^p d\mathcal{H}^k(z) \geq (\lambda_0 + \lambda_1)^k |\lambda_0 - \lambda_1|^p \left(\int_{K_0} |z|^p d\mathcal{H}^k(z) + \int_{K_1} |z|^p d\mathcal{H}^k(z)\right).$$
Proof. Setting $\lambda = \frac{\lambda_0}{\lambda_0 + \lambda_1}$ and applying Theorem 1.9 in [26], it holds
\[
\int_{\lambda_0 K_0 + \lambda_1 K_1} |z|^p \, d\mathcal{H}^k(z) + \int_{\lambda_0 K_1 + \lambda_1 K_0} |z|^p \, d\mathcal{H}^k(z)
= (\lambda_0 + \lambda_1)^{p+k} \left( \int_{\lambda_0 K_0 + (1-\lambda) K_1} |z|^p \, d\mathcal{H}^k(z) + \int_{\lambda_0 K_1 + (1-\lambda) K_0} |z|^p \, d\mathcal{H}^k(z) \right)
\geq (\lambda_0 + \lambda_1)^{p+k} [2\lambda - 1] \left( \int_{K_0} |z|^p \, d\mathcal{H}^k(z) + \int_{K_1} |z|^p \, d\mathcal{H}^k(z) \right)
= (\lambda_0 + \lambda_1)^k |\lambda_0 - \lambda_1|^p \left( \int_{K_0} |z|^p \, d\mathcal{H}^k(z) + \int_{K_1} |z|^p \, d\mathcal{H}^k(z) \right).
\]
\[\square\]

Lemma 4.5. Let $K \subset K_{(\gamma)}$, $\gamma \in [0,1]$ such that $\gamma(-K) \subseteq K$, $L \subset \mathbb{R}^n$ be a proper subspace and let $q \geq n+1$. Let $x \in \text{int} K \setminus L$ and assume that $\nabla g_{K,L,q}(x)$ exists. Then it holds
\[
\langle \nabla g_{K,L,q}(x), x \rangle \leq \left( (q - n) + \frac{1 - \gamma}{\gamma + 1} (n - \dim(L)) \right) g_{K,L,q}(x).
\]
Proof. For $y \in K \setminus L$ let $K_y = K \cap (y + L^\perp)$, let $\lambda \in [0,1)$ and set $\lambda_0 = \frac{2 + \lambda}{\lambda + 1}$ and $\lambda_1 = \frac{\lambda}{\lambda + 1}$. With (4.1) we have
\[
\lambda_0 K_x + \lambda_1 (-K_x) \subseteq K_{\lambda x}
\]
and Lemma 4.4 gives for $K_0 = K_x$, $K_1 = -K_x$
\[
g_{K,L,q}(\lambda x) = \int_{K_{\lambda x}} |z|^{q-n} \, d\mathcal{H}^{n-\dim L}(z)
\geq \int_{\lambda_0 K_x + \lambda_1 (-K_x)} |z|^{q-n} \, d\mathcal{H}^{n-\dim L}(z)
\geq (\lambda_0 + \lambda_1)^{n-\dim L} |\lambda_0 - \lambda_1|^{q-n} \int_{K_x} |z| \, d\mathcal{H}^{n-\dim L}(z)
= \left( 1 + (1 - \lambda) \frac{\gamma - 1}{\gamma + 1} \right)^{n-\dim L} \lambda^{q-n} g_{K,L,q}(x).
\]
Setting $\lambda = 1 - \varepsilon$ it follows
\[
g_{K,L,q}(\lambda (1 - \varepsilon) x - g_{K,L,q}(x) \geq \frac{f(\varepsilon) - f(0)}{\varepsilon} g_{K,L,q}(x)
\]
with $f(t) = (1 - t)^{q-n} \left( \frac{\gamma - 1}{\gamma + 1} t + 1 \right)^{n-\dim L}$. Hence,
\[
\langle \nabla g_{K,L,q}(x), -x \rangle = \lim_{\varepsilon \to 0} \frac{g_{K,L,q}(\lambda (1 - \varepsilon) x) - g_{K,L,q}(x)}{\varepsilon}
\geq f'(0) g_{K,L,q}(x)
= \left( -(q - n) + \frac{\gamma - 1}{\gamma + 1} (n - \dim(L)) \right) g_{K,L,q}(x).
\]
\[\square\]

Proof of Theorem 1.4. Lemma 3.2 combined with Lemma 4.2 and Lemma 4.5 respectively yields the bounds
\[
\int_{K \setminus L} \langle \nabla g_{K,L,q}(x), x \rangle \, d\mathcal{H}^{\dim L}(x) \leq c_q \int_{K \setminus L} g_{K,L,q}(x) \, d\mathcal{H}^{\dim L}(x) = c_q \frac{n}{q} \hat{C}_{K,q}(S^{n-1})
\]
with
\[
c_q = \begin{cases} 
(q - \dim(L))^{\frac{1+\gamma}{1+\gamma}}, q \leq n, \\
(q - n) + \frac{1}{1+\gamma}(n - \dim(L)), q > n + 1.
\end{cases}
\]

**Proof of Theorem 1.1.** The claim follows as a simple corollary of combining Theorem 1.3 and the bounds for the directional derivative, i.e., Theorem 1.4.

**Acknowledgements**

The authors thank Christian Kipp for his helpful comments and suggestions.

**References**

[1] A. Aleksandrov. Existence and uniqueness of a convex surface with a given integral curvature. In CR (Doklady) Acad. Sci. URSS (NS), volume 35, pages 131–134, 1942.

[2] G. Bianchi, K. J. Böröczky, and A. Colesanti. Smoothness in the $L_p$ Minkowski problem for $p < 1$. J. Geom. Anal., 30(1):680–705, 2020.

[3] K. J. Böröczky. The logarithmic Minkowski conjecture and the $L_p$-Minkowski problem. Oct. 2022.

[4] K. J. Böröczky and A. De. Stable solution of the logarithmic Minkowski problem in the case of hyperplane symmetries. J. Differential Equations, 298:298–322, 2021.

[5] K. J. Böröczky and P. Hegedűs. The cone volume measure of antipodal points. Acta Math. Hungar., 146(2):449–465, 2015.

[6] K. J. Böröczky, P. Hegedűs, and G. Zhu. On the discrete logarithmic Minkowski problem. Int. Math. Res. Not. IMRN, (6):1807–1838, 2016.

[7] K. J. Böröczky and M. Henk. Cone-volume measure of general centered convex bodies. Adv. Math., 286:703–721, 2016.

[8] K. J. Böröczky and M. Henk. Cone-volume measure and stability. Adv. Math., 306:24–50, 2017.

[9] K. J. Böröczky, M. Henk, and H. Pollehn. Subspace concentration of dual curvature measures of symmetric convex bodies. Journal of Differential Geometry, 109(3):411–429, 2018.

[10] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. The logarithmic Minkowski problem. J. Amer. Math. Soc., 26(3):831–852, 2013.

[11] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. The logarithmic Minkowski problem. Journal of the American Mathematical Society, 26(3):831–852, 2013.

[12] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, and Y. Zhao. The dual Minkowski problem for symmetric convex bodies. Adv. Math., 356:1066805, 30, 2019.

[13] S. Chen and Q.-R. Li. On the planar dual Minkowski problem. Adv. Math., 333:87–117, 2018.

[14] S. Chen, Q.-R. Li, and G. Zhu. The logarithmic Minkowski problem for non-symmetric measures. Trans. Amer. Math. Soc., 371(4):2623–2641, 2019.

[15] K.-S. Chou and X.-J. Wang. The $L_p$-Minkowski problem and the Minkowski problem in centroaffine geometry. Advances in Mathematics, 205(1):33–83, 2006.

[16] H. Federer. Geometric measure theory. Springer, 2014.

[17] W. J. Firey. $p$-means of convex bodies. Mathematica Scandinavica, 10:17–24, 1962.

[18] R. J. Gardner. Geometric tomography, volume 6. Cambridge University Press Cambridge, 1995.

[19] R. J. Gardner, D. Hug, W. Weil, S. Xing, and D. Ye. General volumes in the Orlicz–Brunn–Minkowski theory and a related Minkowski problem i. Calculus of Variations and Partial Differential Equations, 58(1):1–35, 2019.

[20] R. J. Gardner, D. Hug, S. Xing, and D. Ye. General volumes in the Orlicz–Brunn–Minkowski theory and a related Minkowski problem ii. Calculus of Variations and Partial Differential Equations, 59(1):1–33, 2020.
[21] H. Groemer. On the symmetric difference metric for convex bodies. *Beiträge zur Algebra und Geometrie*, 41(1):107–114, 2000.
[22] P. M. Gruber. *Convex and discrete geometry*, volume 336. Springer, 2007.
[23] L. Guo, D. Xi, and Y. Zhao. The $L_p$ chord Minkowski problem for $0 \leq p < 1$. Jan. 2023.
[24] P. C. Hammer. The centroid of a convex body. *Proceedings of the American Mathematical Society*, 2(4):522–525, 1951.
[25] M. Henk and E. Linke. Cone-volume measures of polytopes. *Adv. Math.*, 253:50–62, 2014.
[26] M. Henk and H. Pollehn. Necessary subspace concentration conditions for the even dual Minkowski problem. *Adv. Math.*, 323:114–141, 2018.
[27] Y. Huang, E. Lutwak, D. Yang, and G. Zhang. Geometric measures in the dual Brunn–Minkowski theory and their associated Minkowski problems. *Acta Mathematica*, 216(2):325–388, 2016.
[28] Y. Huang, E. Lutwak, D. Yang, and G. Zhang. The $L_p$-Aleksandrov problem for $L_p$-integral curvature. *Journal of Differential Geometry*, 110(1):1–29, 2018.
[29] D. Hug, E. Lutwak, D. Yang, and G. Zhang. On the $L_p$ Minkowski problem for polytopes. *Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science*, 33(4):699–715, 2005.
[30] Y. Jiang and Y. Wu. On the 2-dimensional dual Minkowski problem. *J. Differential Equations*, 263(6):3230–3243, 2017.
[31] V. Klee. Polyhedral sections of convex bodies. *Acta Mathematica*, 103(3-4):243–267, 1960.
[32] A. V. Kolesnikov. Mass transportation functionals on the sphere with applications to the logarithmic Minkowski problem. *Mosc. Math. J.*, 20(1):67–91, 2020.
[33] Q.-R. Li, W. Sheng, and X.-J. Wang. Flow by Gauss curvature to the Aleksandrov and dual Minkowski problems. *J. Eur. Math. Soc. (JEMS)*, 22(3):893–923, 2020.
[34] E. Lutwak. Dual mixed volumes. *Pacific Journal of Mathematics*, 58(2):531–538, 1975.
[35] E. Lutwak. The Brunn–Minkowski–Firey theory i: mixed volumes and the Minkowski problem. *Journal of Differential Geometry*, 38(1):131–150, 1993.
[36] E. Lutwak. Selected affine isoperimetric inequalities. *Handbook of convex geometry*, pages 151–176, 1993.
[37] E. Lutwak. The Brunn–Minkowski–Firey theory ii: affine and geominimal surface areas. *Advances in Mathematics*, 118(2):244–294, 1996.
[38] E. Lutwak, D. Yang, and G. Zhang. $L_p$ dual curvature measures. *Advances in Mathematics*, 329:85–132, 2018.
[39] S. Mui. On the $L^p$ Aleksandrov problem for negative $p$. *Adv. Math.*, 408(part A):Paper No. 105753, 26, 2022.
[40] W. F. Pfeffer. *The divergence theorem and sets of finite perimeter*. CRC Press, 2012.
[41] R. T. Rockafellar. *Convex analysis*, volume 18. Princeton university press, 1970.
[42] R. Schneider. *Convex bodies: The Brunn–Minkowski theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2 edition, 2013.
[43] A. Stancu. The discrete planar $L_0$-Minkowski problem. *Adv. Math.*, 167(1):160–174, 2002.
[44] A. Stancu. On the number of solutions to the discrete two-dimensional $L_0$-Minkowski problem. *Adv. Math.*, 180(1):290–323, 2003.
[45] W. Süss. Über eine Affininvariante von Eibereichen. *Archiv der Mathematik*, 1(2):127–128, 1948.
[46] G. Xiong. Extremum problems for the cone volume functional of convex polytopes. *Adv. Math.*, 225(6):3214–3228, 2010.
[47] Y. Zhao. The dual Minkowski problem for negative indices. *Calculus of Variations and Partial Differential Equations*, 56(2):18, 2017.
[48] Y. Zhao. Existence of solutions to the even dual Minkowski problem. *Journal of Differential Geometry*, 110(3):543–572, 2018.
