The off-shell behaviour of propagators and the Goldstone field in higher spin gauge theory on $AdS_{d+1}$ space

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Abstract

A detailed analysis of the structure and gauge dependence of the bulk-to-bulk propagators for the higher spin gauge fields in $AdS$ space is performed. The possible freedom in the construction of the propagators is investigated and fixed by the correct boundary behaviour and correspondence to the representation theory results for the $AdS$ space isometry group. The classical origin of the Goldstone mode and its connection with the gauge fixing procedure is considered.

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1 Introduction

The $\text{AdS}_4/C\text{FT}_3$ correspondence of the critical $O(N)$ sigma model and four dimensional higher spin gauge theory in anti de Sitter space (HS(4)) proposed in \cite{1} increased the interest in the old problems of quantization and interaction of the higher spin gauge theories in AdS space \cite{2,3}. This case of the general $\text{AdS}_{d+1}/C\text{FT}_d$ \cite{4} correspondence is interesting also in view of the investigation of gauge symmetry breaking and mass generation on the higher spin side which are related to well explored properties of the corresponding CFT boundary theory at next-to-leading order in $\frac{1}{N}$. This allows us to compare the result for the anomalous dimensions of conserved currents of the critical $O(N)$ model with the the loop corrections to the bulk-to-bulk propagator of the HS(4) following from the possible special interaction between gauge and conformal scalar fields in AdS. This was done in the last papers of the authors \cite{5,6} where the mass correction for the HS(4) field was evaluated. The analysis of the mass generation for the HS(4) gauge fields and its comparison with the anomalous dimensions of the conformal currents at large $N$ limit is important not only for better understanding of the Higgs phenomenon for HS(4) theory \cite{7,8} but also for getting an answer to a more general question: Does AdS/CFT work correctly on the level of loop diagrams in the general case and is it possible to use this correspondence for real reconstruction of unknown local interacting theories on the bulk from more or less well known conformal field theories on the boundary side \cite{9,10}? For this purpose a better understanding of the structure of the bulk-to-bulk propagators of the higher spin gauge fields becomes very important just as an investigation of the possible forms of local interactions between HS gauge and scalar fields \cite{11,12}.

In this article we perform a precise analysis of the gauge fixing procedure in the Fronsdal theory of massless higher spin fields, and analyze its quantization constructing the correct bulk-to-bulk propagator of the higher spin theory. We describe also on the classical level the origin of the Goldstone mode responsible for the quantum mass generation mechanism considered in the previous paper \cite{6}.
2 De Donder gauge and Goldstone mode

We will use Euclidian $AdS_{d+1}$ with conformal flat metric, curvature and covariant derivatives satisfying

$$ds^2 = g_{\mu\nu}(z)dz^\mu dz^\nu = \frac{L^2}{(z^0)^2} \delta_{\mu\nu} dz^\mu dz^\nu, \quad \sqrt{g} = \frac{L^{d+1}}{(z^0)^{d+1}},$$

$[\nabla_\mu, \nabla_\nu] V^\rho_\lambda = R_{\mu\lambda}^\sigma V^\rho_\sigma - R_{\mu\nu}^\rho V^\sigma_\lambda$,

$R_{\mu\nu}^\rho = -\frac{1}{(z^0)^2} \left( \delta_{\mu\lambda} \delta_\nu^\rho - \delta_{\nu\lambda} \delta_\mu^\rho \right) = -\frac{1}{L^2} \left( g_{\mu\lambda}(z) \delta_\nu^\rho - g_{\nu\lambda}(z) \delta_\mu^\rho \right),$

$R_{\mu\nu} = -\frac{d}{(z^0)^2} \delta_{\mu\nu} = -\frac{d}{L^2} g_{\mu\nu}(z), \quad R = -\frac{d(d+1)}{L^2}.$

For shortening the notation and calculation we contract all rank $\ell$ symmetric tensors with the $\ell$-fold tensor product of a vector $a^\mu$. In this notation Fronsdal’s equation of motion $[2]$ for the double traceless spin $\ell$ field is (from now on we put $L = 1$)

$$F(h^{(\ell)}(z; a)) = \Box h^{(\ell)}(z; a) - (a \nabla) \nabla^\mu \frac{\partial}{\partial a^\mu} h^{(\ell)} + \frac{1}{2} (a \nabla)^2 \Box_a h^{(\ell)}(z; a)$$

$$- \left( \ell^2 + \ell(d - 5) - 2(d - 2) \right) h^{(\ell)} - a^2 \Box_a h^{(\ell - 2)}(z; a) = 0, \quad (1)$$

$$\Box_a \Box_a h^{(\ell)} = 0, \quad (2)$$

$$\Box = \nabla^\mu \nabla_\mu, \quad \Box_a = g^{\mu\nu} \frac{\partial^2}{\partial a^\mu \partial a^\nu}, \quad (a \nabla) = a^\mu \nabla_\mu, \quad a^2 = g_{\mu\nu}(z) a^\mu a^\nu. \quad (3)$$

The basic property of this equation is higher spin gauge invariance with the traceless parameter $\epsilon^{(\ell-1)}(z; a)$,

$$\delta h^{(\ell)}(z; a) = (a \nabla) \epsilon^{(\ell-1)}(z; a), \quad \Box_a \epsilon^{(\ell-1)}(z; a) = 0, \quad \delta F(h^{(\ell)}(z; a)) = 0. \quad (4)$$

The equation $[1]$ can be simplified by gauge fixing. It is easy to see that in the so called de Donder gauge

$$D^{(\ell-1)}(h^{(\ell)}) = \nabla^\mu \frac{\partial}{\partial a^\mu} h^{(\ell)} - \frac{1}{2} (a \nabla) \Box_a h^{(\ell)} = 0, \quad (5)$$

Fronsdal’s equation simplifies to

$$F^{dD}(h^{(\ell)}) = \Box h^{(\ell)} - \left( \ell^2 + \ell(d - 5) - 2(d - 2) \right) h^{(\ell)} - a^2 \Box_a h^{(\ell - 2)} = 0. \quad (6)$$

It was shown (see for example $[3]$) that in the de Donder gauge the residual gauge symmetry leads to the tracelessness of the on-shell fields. So we can define our massless physical spin $\ell$ modes as traceless and transverse symmetric tensor fields satisfying the equation $[6]$

$$[\Box + \ell] h^{(\ell)} = \Delta_\ell (\Delta_\ell - d) h^{(\ell)}, \quad (7)$$

$$\Box_a h^{(\ell)} = \nabla^\mu \frac{\partial}{\partial a^\mu} h^{(\ell)} = 0, \quad (8)$$

$$\Delta_\ell = \ell + d - 2. \quad (9)$$
Note that equation (7) for $\ell = 0$ coincides with the equation for the massless conformal coupled scalar only for $d = 3$.

In a similar way we can describe the massive higher spin modes using the same set of constraints on the general symmetric tensor field $\phi(\ell)(z, a)$ but with the independent conformal weight $\Delta$ (dimension) of the corresponding massive (in means of $AdS$ field) representation of the $SO(d + 1, 1)$ isometry group. This general representation with two independent quantum numbers $[\Delta, \ell]$ under the maximal compact subgroup goes, after imposing a shortening condition $\Delta = \Delta_{\ell} = \ell + d - 2$, to the massless higher spin case (7)-(9) with the following decomposition [5, 7, 8]

$$\lim_{\Delta \to \ell + d - 2} [\Delta, \ell] = [\ell + d - 2, \ell] \oplus [\ell + d - 1, \ell - 1].$$

The additional massive representation $[\ell + d - 1, \ell - 1]$ is the Goldstone field. Reading this decomposition from the opposite side, we can interpret it as swallowing of the massive spin $\ell - 1$ Goldstone field by the massless spin $\ell$ field with generation of a mass for the latter one [6]. For better understanding of this phenomenon we need a more careful investigation of the gauge invariant equation (1) in more general gauges.

First of all note that the gauge parameter $\epsilon^{(\ell - 1)}$ is a traceless rank $\ell - 1$ tensor and therefore in any off-shell consideration (quantization, propagator and perturbation theory) we can use only gauge conditions with the same number of degrees of freedom. The de Donder gauge (5) is just such a type of the gauge due to the tracelessness of the $D^{(\ell - 1)}(h^{(\ell)})$. Nevertheless for on-shell states we can impose more restrictive gauges. Here we consider a one-parameter family of gauge fixing conditions

$$G_{a}^{(\ell - 1)}(h^{(\ell)}) = \nabla_{\mu} \frac{\partial}{\partial a^{\mu}} h^{(\ell)} - \frac{1}{\alpha} (a \nabla) \Box_{a} h^{(\ell)} = 0 \quad (11)$$

This gauge condition coincides with the traceless de Donder gauge if $\alpha = 2$ ($\Box_{a} G_{2}^{(\ell - 1)} = \Box_{a} D^{(\ell - 1)} = 0$). Then we can write our double traceless field $h^{(\ell)}(z; a)$ as a sum of the two traceless spin $\ell$ and $\ell - 2$ fields $\psi^{(\ell)}(z; a)$ and $\theta^{(\ell - 2)}(z; a)$

$$h^{(\ell)}(z; a) = \psi^{(\ell)}(z; a) + \frac{a^{2}}{2\alpha_{0}} \theta^{(\ell - 2)}(z; a), \quad (12)$$

$$\Box_{a} h^{(\ell)} = \theta^{(\ell - 2)}, \quad \Box_{a} \psi^{(\ell)} = \Box_{a} \theta^{(\ell - 2)} = 0, \quad (13)$$

$$\alpha_{0} = d + 2\ell - 3. \quad (14)$$

In this parametrization Fronsdal’s equation of motion with the gauge condition (11) can be written in the form of the following system of equations for the two
independent traceless fields $\psi^{(\ell)}$ and $\theta^{(\ell-2)}$

\[
\nabla^\mu \frac{\partial}{\partial a^\mu} \psi^{(\ell)} + \frac{a^2}{2\alpha_0} \nabla^\mu \frac{\partial}{\partial a^\mu} \theta^{(\ell-2)} = \frac{\alpha_0 - \alpha}{\alpha \alpha_0} (a \nabla) \theta^{(\ell-2)},
\]

\[(\Box + \ell) \psi^{(\ell)} + \frac{\alpha - 2}{2\alpha} \left( (a \nabla)^2 \theta^{(\ell-2)} - a^2 \frac{\alpha(a_0 - 1)}{\alpha_0 (\alpha - 1)} \theta^{(\ell-2)} \right) = \Delta_\ell (\Delta_\ell - d) \psi^{(\ell)},\]

\[(\Box + \ell - 2) \theta^{(\ell-2)} = \left[ \Delta_\theta (\Delta_\theta - d) + \frac{\alpha_0 - \alpha}{\alpha - 1} \right] \theta^{(\ell-2)},\]

\[\Delta_\ell = d + \ell - 2, \quad \Delta_\theta = d + \ell - 1.\]

Now we are ready to discuss different gauge conditions. First of all we see that the de Donder gauge ($\alpha = 2$) leads to the complete separation of the equations of motion for $\psi^{(\ell)}$ and $\theta^{(\ell-2)}$ fields. On the other hand the gauge condition (15) becomes just traceless for $\alpha = 2$ and keeps on to connect the divergence of $\psi^{(\ell)}$ and the traceless part of the gradient of $\theta^{(\ell-2)}$

\[
\nabla^\mu \frac{\partial}{\partial a^\mu} \psi^{(\ell)}(z; a) = \frac{\alpha_0 - 2}{2\alpha_0} G^{(\ell-1)}(z; a),
\]

\[G^{(\ell-1)}(z; a) = (a \nabla) \theta^{(\ell-2)}(z; a) - \frac{a^2}{\alpha_0 - 2} \nabla^\mu \frac{\partial}{\partial a^\mu} \theta^{(\ell-2)}(z; a).\]

Here $G^{(\ell-1)}$ corresponds to the Goldstone representation. Indeed using the equations of motion (16) and (17) with $\alpha = 2$ one can derive that the $G^{(\ell-1)}$ field obeys the following on-shell equation

\[(\Box + \ell - 1) G^{(\ell-1)}(z; a) = \Delta_\theta (\Delta_\theta - d) G^{(\ell-1)}(z; a)\]

(21)

corresponding to the Goldstone representation $[\Delta_\theta = \ell + d - 1, \ell - 1]$ arising in (16). This mode can be gauged away on the classical level together with the trace $\theta^{(\ell-2)}$ but only on-shell. Therefore on the quantum level this mode can arise in loop diagrams and will play the crucial role in the mechanism of mass generation for the higher spin gauge fields as it was shown in our previous paper [6].

Now we return to (15)-(18) and consider the next interesting gauge $\alpha = d + 2\ell - 3$. This is a generalization for the higher spin case of the so-called “Landau” gauge considered in [15] for the case of the graviton in $AdS_{d+1}$. But the difference between the higher spin and graviton ($\ell = 2$) cases is essential. For the graviton we can apply this ”Landau” gauge

\[\nabla^\mu h_{\mu\nu} = \frac{1}{d + 1} \partial_\nu h^{\mu}_\mu\]

(22)

off-shell also because the trace is scalar here and this gauge fixes the same number of degrees of freedom as the de Donder gauge. For $\ell > 2, \alpha \neq 2$ it is easy to see
that condition ([15]) after taking the trace forces the trace components \( \theta^{(\ell-2)} \) of our double traceless field \( h^{(\ell)} \) to be transverse

\[
\nabla^\mu \frac{\partial}{\partial a^\mu} \theta^{(\ell-2)} = 0, \quad (23)
\]

\[
\nabla^\mu \frac{\partial}{\partial a^\mu} \psi^{(\ell)} = \frac{\alpha_0 - \alpha}{\alpha \alpha_0} (a \nabla) \theta^{(\ell-2)}. \quad (24)
\]

Moreover in the "Landau" gauge \( (\alpha = \alpha_0) \) the \( \psi^{(\ell)} \) component is also transverse but its equation of motion is not diagonal like in the de Donder gauge. On the other hand the equation of motion for the field \( \theta^{(\ell-2)} \) is simplified and we have for this field the realization of the representation \([\Delta_\theta = \ell + d - 1, \ell - 2]\)

\[
(\Box + \ell - 2) \theta^{(\ell-2)} = \Delta_\theta (\Delta_\theta - d) \theta^{(\ell-2)}. \quad (25)
\]

So we see that only in the de Donder gauge we have a diagonal equation of motion for the physical \( \psi^{(\ell)} \) components but this component is not transversal due to the presence of the \([\ell + d - 1, \ell - 1]\) Goldstone mode \( G^{(\ell-1)} \). This gauge is most suitable for the quantization and construction of the bulk-to-bulk propagator and for the investigation of the \( AdS_4/CFT_3 \) correspondence in the case of the critical conformal \( O(N) \) boundary sigma model.

3 Propagator

Here we perform a precise analysis of the leading terms of the bulk-to-bulk propagator obtained in [16] using boundary integration of the bulk-to-boundary propagators constructed in [18] using representation theory. Here we construct and analyze the general bitensorial ansatz satisfying the de Donder gauge condition ([5]) and the equation of motion ([6]) with the corresponding delta function on the right hand side. This type of analysis can be simplified due to the following two properties of the \( AdS \) space propagators [16, 17]:

- The propagator is a function of only one bilocal invariant variable, the geodesic distance

\[
\zeta(z_1, z_2) = \frac{(z_1^0)^2 + (z_2^0)^2 + (z_1 - z_2)^2}{2z_1^0 z_2^0} = 1 + \frac{(z_1 - z_2)^\mu (z_1 - z_2)_\mu \delta_{\mu\nu}}{2z_1^0 z_2^0} \quad (26)
\]

- The tensorial structure of the bulk-to-bulk propagator can be explored using
the following basis of the independent bitensors \[19, 20, 21, 16\]

\[
\begin{align*}
I_1(a, c) & := (a\partial)_{1}(c\partial)_{2}\zeta(z_1, z_2), \\
I_2(a, c) & := (a\partial)_{1}\zeta(z_1, z_2)(c\partial)_{2}\zeta(z_1, z_2), \\
I_3(a, c) & := a_1^2I_{2c}^2 + c_2^2I_{1a}^2, \\
I_4 & := a_1^2c_2^2, \\
I_{1a} & := (a\partial)_{1}\zeta(z_1, z_2), \quad I_{2c} := (c\partial)_{2}\zeta(z_1, z_2),
\end{align*}
\]

(27) - (33)

Using this basis we can start to construct the general ansatz. First we introduce a special map from set \(\{F_{k}(\zeta)\}_{k=0}^{\ell}\) of the \(\ell + 1\) functions on \(\zeta\) to the space of \(\ell \times \ell\) bitensors

\[
\Psi^{\ell}[F] = \sum_{k=0}^{\ell} I_{1}^{\ell-k}(a, c)I_{2}^{k}(a, c)F_{k}(\zeta).
\]

(34)

The general expansion can be expressed then as

\[
K^{\ell}(a, c) = \Psi^{\ell}[F] + \sum_{n,m; 0 < 2(n+m) < \ell} I_{3}^{n}I_{4}^{m}\Psi^{\ell-2(n+m)}[G^{(n,m)}].
\]

(35)

We call all monomials in the above sum and the corresponding sets of functions \(\{G^{(n,m)}_{k}\}_{k=0}^{\ell-2(n+m)}\) the "trace terms". The trace terms can be analyzed using the computer program \[17\]. We select the first part and the first and second order trace terms of \(35\)

\[
\begin{align*}
K^{\ell}(a, c) & = \Psi^{\ell}[F] + I_{3}\Psi^{\ell-2}[G] + I_{4}\Psi^{\ell-2}[T] + I_{3}^{2}\Psi^{\ell-4}[H] + \ldots , \\
\{G_{k}^{(1,0)}\}_{k=0}^{\ell-2} & = \{G_{k}^{(\ell-2)}\}_{k=0}^{\ell-2}, \{G_{k}^{(1,1)}\}_{k=0}^{\ell-2} = \{T_{k}\}_{k=0}^{\ell-4}, \{G_{k}^{(2,0)}\}_{k=0}^{\ell-4} = \{H_{k}\}_{k=0}^{\ell-4},
\end{align*}
\]

(36) - (37)

and analyze the restrictions coming from the de Donder gauge fixing and equation of motion. In other words we want to derive differential and algebraical recursion relations for the corresponding sets of functions \(17\) following from the equations for the bulk-to-bulk propagator in the de Donder gauge. First let's define the trace map

\[
\begin{align*}
\Box_{a}\Psi^{\ell}[F] & = I_{2c}^{2}\Psi^{\ell-2}[Tr_{c}F] + O(c_{2}^{2}), \\
(Tr_{c}F)_{k} & = (\ell-k)(\ell-k-1)F_{k} + 2(k+1)(\ell-k-1)\zeta F_{k+1} \\
& + (k+2)(k+1)(\zeta^{2} - 1)F_{k+2},
\end{align*}
\]

(38) - (39)
the divergence and gradient maps

$$\nabla^\mu \frac{\partial}{\partial \alpha^\mu} \Psi^\ell[F] = I_{2\epsilon} \Psi^{\ell-1}[Div F] + O(a^2_\epsilon),$$  \hspace{1cm} (40)$$

$$\begin{align*}
(Div F)_k &= (\ell - k)\zeta F'_k + (k + 1)(\zeta^2 - 1) F''_{k+1} \\
+(\ell - k)(\ell + d + k) F_k + (k + 1)(\ell + d + k + 1) \zeta F_{k+1},
\end{align*}$$  \hspace{1cm} (41)$$

$$\begin{align*}
(a\nabla)\Psi^\ell[F] &= I_{1\alpha} \Psi^\ell[Grad F] + O(a^2_\alpha),
\end{align*}$$  \hspace{1cm} (42)$$

$$\begin{align*}
(Grad F)_k &= F'_k + (k+1) F_{k+1}, \quad F''_k := \frac{\partial}{\partial \zeta} F_k(\zeta),
\end{align*}$$  \hspace{1cm} (43)$$

and finally the Laplacian map

$$\begin{align*}
\Box \Psi^\ell[F] &= \Psi^\ell[Lap F] + O(a^2_\alpha, c^2_\alpha), \\
(Lap F)_k &= (\zeta^2 - 1) F''_k + (d + 1 + 4k) \zeta F'_k + \{\ell + k(d + 2\ell - k)\} F_k \\
2\zeta(k+1)^2 F_{k+1} + 2(\ell - k + 1) F'_{k-1},
\end{align*}$$  \hspace{1cm} (44)$$

$$\begin{align*}
\Box F_k(\zeta) &= (\zeta^2 - 1) F''_k + (d + 1) \zeta F'_k.
\end{align*}$$  \hspace{1cm} (45)$$

For the derivation of these maps we used several relations for the derivatives of the bitensors listed in the Appendix A.

Now we start to explore the restrictions on the sets of functions \( F, G \) and \( H \) in (46) given by the following equations for the propagator in de Donder gauge

$$\begin{align*}
\Box \Box a \, K^\ell(a, c; \zeta) &= 0, \\
\nabla^\mu \frac{\partial}{\partial \alpha^\mu} K^\ell(a, c; \zeta) &= \frac{1}{2} (a\nabla) \Box a \, K^\ell(a, c; \zeta), \\
(\Box + \ell) K^\ell(a, c; \zeta) &= a^2 \Box a \, K^\ell(a, c; \zeta) = \Delta_\ell (\Delta_\ell - d) K^\ell(a, c; \zeta), \quad \zeta \neq 1. \quad (47)
\end{align*}$$

Note that we do not write the delta function with the corresponding projector in the right hand side of (49) assuming that we will focus on the solution of the equation of motions with the right normalized delta function singularity at \( \zeta \to 1 \).

Actually we have to watch the tensorial structure of this singularity also. Taking into account that the leading term of the projector always is \( \delta(z_1, z_2)(a^\mu g_{\mu\nu}(z_1) c^\nu) \) we see that the most important and singular function in our propagator is the function \( F_0 \) from \( \Psi^\ell[F] = I_1^\ell F_0 + \ldots \) because \( I_1(a, b; \zeta) \to -a^\mu g_{\mu\nu}(z_1) c^\nu \) when \( \zeta \to 1 \) and \( \ell \) is even. The structure of the other terms in (49) will be fixed automatically after considering the two other conditions (47) and (48). Substituting (36) in the double tracelessness condition (47) using recursively the trace map (38) and neglecting all trace terms, we obtain the following relations

$$\begin{align*}
[a] \, K^\ell &= I_{2\epsilon} \psi^{\ell-4} \Theta + O(a^2_\epsilon, c^2_\epsilon) = 0, \\
\Theta_k &= (Tr_{\ell-2} Tr_{\ell} F)_k + 4(\alpha_0 - 2) (Tr_{\ell-2} G)_k + 8(\alpha_0 - 2) (\alpha_0 - 4) H_k, \quad (49)
\end{align*}$$

$$\begin{align*}
(Tr_{\ell-2} Tr_{\ell} F)_k &= (\ell - k - 2)(\ell - k - 3)(Tr_{\ell} F)_k \\
+ 2(k+1)(\ell - k - 3) \zeta (Tr_{\ell} F)_{k+1} + (k+2)(k+1)(\zeta^2 - 1)(Tr_{\ell} F)_{k+2}. \quad (50)
\end{align*}$$
We see that from the conditions \( \Theta_k(\zeta) = 0 \) we can express \( H_k(\zeta) \) as functions of the \( G_k(\zeta) \) and \( F_k(\zeta) \). The next neglected term of order \( O(c_2^2) \) will express the functions \( T_k \) from the expansion (36) in a similar way. It is clear that due to the double tracelessness condition we have only two free sets of functions \( F_k(\zeta) \) and \( G_k(\zeta) \).

Then we consider the de Donder gauge condition. After insertion of (36) into the Eq. (48) and using (40) and (38) we obtain

\[
\nabla^\mu \frac{\partial}{\partial a^\mu} K^\ell - \frac{1}{2} (a \nabla) \Box_a K^\ell = I_{2c} \Psi^{\ell-1} [M_k - \frac{1}{2} (a \nabla) I_{2c}^2 \Psi^{\ell-2} [N_k] + O(a_1^2, c_2^2)],
\]

(53)

\[
M_k(\zeta) = (\text{Div}_\ell F)_k + 2(k + 2)G_k + 2G'_{k-1}, \quad \text{for } \zeta \neq 1
\]

(54)

\[
N_k(\zeta) = (\text{Tr}_\ell F)_k + 2\alpha_0 G_k.
\]

(55)

Using the gradient map (42), (43) and the formulas from the Appendix A we can derive the following relation

\[
(a \nabla) I_{2c}^2 \Psi^{\ell-2} [N_k] = I_{2c} \Psi^{\ell-1} [(k + 2)N_k + N'_{k-1}] + O(a_1^2).
\]

(56)

This leads to the final equation

\[
(\text{Div}_\ell F)_k - \frac{1}{2} [(k + 2)(\text{Tr}_\ell F)_k + (\text{Tr}_\ell F)'_{k-1}]
\]

\[
= (\alpha_0 - 2) [(k + 2)G_k + G'_{k-1}].
\]

(57)

From the latter we can express all \( G_k \) as the functions of the unconstrained \( F_k \).

So we understood that double tracelessness and de Donder condition fix the propagator and we have freedom only in the first set of functions \( F_k \) forming the leading term \( \Psi^\ell[F] \) of the propagator \( K^\ell(a, b; \zeta) \). This last free set we can fix only from the dynamical equation of motion (49) using the Laplacian map (44) - (46)

\[
(Lap_\ell F)_k + \ell F_k - \Delta_\ell (\Delta_\ell - d) F_k = 0 \iff
\]

\[
(\zeta^2 - 1) F''_k + (d + 1 + 4k)\zeta F'_k + [2\ell + k(d + 2\ell - k)] F_k
\]

\[
+ 2\zeta(k + 1)^2 F_{k+1} + 2(\ell - k + 1)F'_{k-1} = \Delta_\ell (\Delta_\ell - d) F_k.
\]

(58)

These equations are again recursive and will express the higher functions \( F_k \) through the lower ones.

As an initial condition we use that \( F_0 \) satisfies the wave equation of a scalar field of dimension \( \Delta_\ell \)

\[
(\zeta^2 - 1) F''_0 + (d + 1)\zeta F'_0 - \Delta_\ell (\Delta_\ell - d) F_0 = 0, \quad \zeta \neq 1.
\]

(60)
Then from (59) with \( k = 0 \) follows

\[
F_1(\zeta) = -\frac{\ell}{\zeta} F_0. \tag{61}
\]

This result can be generalized to an ansatz of a "main" term and a "small" (at \( \zeta \to \infty \)) term

\[
F_k = c_k \zeta^{-k} F_0 + f_k, \tag{62}
\]

\[
c_k = (-1)^k \binom{\ell}{k}, \quad f_0 = f_1 = 0. \tag{63}
\]

We introduce the differential operator \( D_k \) to abbreviate (59)

\[
D_k(F_k) + 2\zeta(k+1)^2 F_{k+1} + 2(\ell - k + 1) F_{k-1} = 0 \tag{64}
\]

Inserting only the main part of (62) into (64) we obtain a residual expression

\[
\text{Res}(F_k) = \frac{k c_k}{\zeta^{k+2}} (2\zeta F'_0 - (k + 1) F_0). \tag{65}
\]

It is smaller by \( O(\zeta^{-2}) \) at \( \zeta \to \infty \) than the main term of \( F_k \) and arises in

\[
(\zeta^2 - 1) F_k'' = c_k \zeta^{-k} \left\{ (\zeta^2 - 1) F_0'' - 2k \zeta^{-1}(\zeta^2 - 1) F'_0 + k(k+1) \zeta^{-2}(\zeta^2 - 1) F_0 \right\} \tag{66}
\]

from the two underlined terms. Thus we end up with

\[
D_k(f_k) + 2\zeta(k+1)^2 f_{k+1} + 2(\ell - k + 1) f'_{k-1} = -\text{Res}(F_k) \tag{67}
\]

which can be rewritten as

\[
f_{k+1} = \frac{1}{2(k+1)^2 \zeta} \left\{ -\text{Res}(F_k) - D_k(f_k) - 2(\ell - k + 1) f'_{k-1} \right\}. \tag{68}
\]

The first cases are

\[
f_0 = f_1 = 0, \tag{69}
\]

\[
f_2 = \frac{\ell}{4\zeta^4} (\zeta F'_0 - F_0), \tag{70}
\]

\[
f_3 = -\frac{1}{18\zeta} \left[ \frac{\ell(\ell - 1)}{\zeta^4} (2\zeta F'_0 - 3 F_0) + D_2(f_2) \right]. \tag{71}
\]

The main term of (62) can be summed and gives for the propagator without trace terms

\[
\left( I_1 - \frac{1}{\zeta} I_2 \right)^\ell F_0(\zeta) \tag{72}
\]
In the bulk-to-boundary limit (see the next section for details) this reduces to Dobrev’s propagator \[18\] (without trace terms).

The solution for the wave equation (60) is \[15, 6\]

\[
F_0 = C\zeta^{-\Delta/2}F_1\left(\frac{1}{2}\Delta, \frac{1}{2}(\Delta + 1)\right) = C\zeta^{-\Delta/2}F_1\left(\frac{1}{2}\Delta, \frac{1}{2}(\Delta + 1); \Delta - \mu + 1; \zeta^{-2}\right). \tag{73}
\]

We set the normalization constant \(C\) equal to \[
C = \frac{\Gamma\left(\frac{1}{2}\Delta\right)\Gamma\left(\frac{1}{2}(\Delta + 1)\right)}{(4\pi)^{\mu+\frac{d}{2}}\Gamma(\Delta - \mu + 1)} \tag{74}
\]
and use the fact that for \(\zeta \to 1\)

\[
2F_1\left(\frac{1}{2}\Delta, \frac{1}{2}(\Delta + 1); \Delta - \mu + 1; \zeta^{-2}\right) = \frac{\Gamma(\Delta - \mu + 1)\Gamma(\mu - \frac{1}{2})}{\Gamma\left(\frac{1}{2}\Delta\right)\Gamma\left(\frac{1}{2}(\Delta + 1)\right)}(\zeta^2 - 1)^{-\mu+\frac{1}{2}} + O(1), \quad \Re \mu > \frac{1}{2}, \tag{75}
\]
and

\[
\Box\left(\frac{\mu - \frac{1}{2}}{4\pi}\right)^{-\mu+\frac{1}{2}}(\zeta^2 - 1)^{-\mu+\frac{1}{2}} = -\delta(z_1, z_2) + \text{regular terms}, \tag{76}
\]

So we prove that \(F_0(\zeta)\) appears as the kernel for the inverse wave operator \((-\Box + m^2)\) for the massive scalar field in Euclidian AdS\(_{d+1}\) space with \(m^2 = \Delta\ell(\Delta\ell - d)\).

4 Bulk-to-boundary limit

Now we can take the boundary limit and obtain the spin \(\ell\) bulk-to-boundary propagator from the bulk-to-bulk propagator directly. For this purpose we mention that the boundary of AdS space is approached in the limit \(z^0 \to 0\), \(z^0 \to 0\), which is connected with the limit \(\zeta \to \infty\) due to \[
\lim_{z^0_2 \to 0} 2z^0_1z^0_2\zeta(z_1, z_2) = (z^0_1)^2 + (\bar{z}_1 - \bar{z}_2)^2. \tag{78}
\]

Then following the explanation of the previous section we see that at the boundary only the main term \[72\] survives and we get

\[
\lim_{z^0_2 \to 0, z^0_1 \to 0} (z^0_2)^{-\Delta} \left( I_1 - \frac{1}{\zeta}I_2 \right)^\ell \hat{F}_0(\zeta) = 2^\Delta C \frac{(z^0_1)^{d-2}}{[(z^0_1)^2 + (\bar{z}_1 - \bar{z}_2)^2]^\Delta} [R(a, \bar{c}; z_1 - \bar{z}_2)]^\ell, \tag{79}
\]

\[
R(a, \bar{c}; z_1 - \bar{z}_2) = <\bar{a}, \bar{c}> - 2\frac{(a, z_1) < \bar{z}_1 - \bar{z}_2, \bar{c}>}{(z^0_1)^2 + (\bar{z}_1 - \bar{z}_2)^2}. \tag{80}
\]
Here we introduced the $d + 1$ and $d$ dimensional Euclidian scalar products
\[
(a, z) = \sum_{\mu=0}^{d} a^\mu z_\mu, \quad <\vec{c}, \vec{z}> = \sum_{i=1}^{d} c^i z_i
\] (81)
and the Jacobian tensor
\[
R_{\mu\nu}(z) = \delta_{\mu\nu} - 2 \frac{z_\mu z_\nu}{(z, z)}. \tag{82}
\]

We see that the limit (79) really produces Dobrev’s \[18\] boundary-to-bulk propagator without trace terms.

Actually we need only this leading term because all other trace terms depend on the gauge condition (48) applied to the bulk dependent side of the right hand side of (79). On the other hand we can fix the trace terms by requiring the tensor fields to approach a certain tensor type on the boundary. In the case of irreducible $d$ dimensional CFT currents we have to claim tracelessness with respect to the indices contracted with $\vec{c}$
\[
\Box_c G_{\text{AdS/CFT}}^{(\ell)}(a, \vec{c}; z) = \frac{\partial^2}{\partial \vec{c} \partial \vec{c}} (G_m^{(\ell)}(a, \vec{c}; z) + \text{trace terms}) = 0, \tag{83}
\]
\[
G_m^{(\ell)}(a, \vec{c}; z) = \frac{(z^0)^{d-2}}{(z, z)^\Delta} [R(a, \vec{c}; z)]^\ell. \tag{84}
\]

Here we omit the normalization factor $2^\Delta C$ and put for simplicity $\vec{z}_2 = 0$ and $z_1^\mu = z^\mu$ (we can always restore the right dependence on the boundary coordinate $\vec{z}_2$ using translation invariance in the flat boundary space).

Then considering the boundary limit of the $I_3$ and $I_4$ dependent terms we can easily render the propagator \[83\] traceless on the boundary by the projection\(^\dagger\)
\[
G_{\text{AdS/CFT}}^{(\ell)}(a, \vec{c}; z) = G_m^{(\ell)}(a, \vec{c}; z) - \frac{(a, a) - [R^0(a; z)]^2}{2(a_0 - 1)(z^0)^2} \Box_c G_m^{(\ell)}(a, \vec{c}; z) + O(a^4) + O(c^4). \tag{85}
\]

The complete polynomial expression for $G_{\text{AdS/CFT}}^{(\ell)}(a, \vec{c}; z)$ is presented in the Appendix B, Eqn. (B.10). But here we consider only the first order trace term
\[
\Box_c G_m^{(\ell)}(a, \vec{c}; z) = \ell (\ell - 1) \frac{(z^0)^d}{(z, z)^\Delta} <\vec{c}, \vec{c}> [R(a, \vec{c}; z)]^{\ell-2}, \tag{86}
\]
\[
R^0(a; z) = a^\mu R_\mu^0(z) = a^0 - 2 \frac{z^0(a, z)}{(z, z)}, \quad a_0 = d + 2\ell - 3. \tag{87}
\]

\(^\dagger\)In this section we used the exact expression for Christoffel symbols $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} (\delta^\lambda_{[\mu} \delta_{\nu]} - 2 \delta^\lambda_{(\mu} \delta_{\nu)}$ and the AdS trace rule $\Box_a = (z^0)^2 \delta^{\mu\nu} \frac{\partial^2}{\partial a^\mu \partial a^\nu}$.
The important point of this consideration is the following: The expression (85) is automatically traceless on the AdS side.

\[ \square_a G^{(\ell)}_{\text{AdS/CFT}}(a, \vec{c}; z) = 0, \]  

due to the relations

\[ \delta^{\mu \nu} R^0_\mu(z) R^0_\nu(z) = 1, \quad \delta^{\mu \nu} R^0_\mu(z) R_\nu(\vec{c}; z) = 0 \]  

(89)

This is natural because the original bulk-to-bulk basis \( \{ I_i(a, c; \zeta) \}_{i=1}^4 \) was symmetric with respect to the \( a \leftrightarrow c \) exchange. Then we see that this projection in agreement with de Donder gauge condition (48) (for traceless case) leads to the transverse-traceless bulk-to-boundary propagator (85) for all higher spin fields on AdS side. For proving this we have to calculate several relations in first order of \( (a, a) \) and \( \langle \vec{c}, \vec{c} \rangle \) (see details in Appendix B)

\[ \nabla^\mu \frac{\partial}{\partial a^\mu} G^{(\ell)}_m(a, \vec{c}; z) = \ell(\ell - 1) \frac{(z^0)^{d-1}}{(z, z)^{\Delta}} < \vec{c}, \vec{c} > [R(a, \vec{c}; z)]^{\ell-2} R^0(a; z), \]  

\[ a^\mu \nabla_\mu \square_a G^{(\ell)}_m(a, \vec{c}; z) = \ell(\ell - 1) \frac{(z^0)^{d-1}}{(z, z)^{\Delta}} < \vec{c}, \vec{c} > [R(a, \vec{c}; z)]^{\ell-2} R^0(a; z)(\alpha_0 - 1), \]  

\[ \nabla^\mu \frac{\partial}{\partial a^\mu} \frac{[R^0(a; z)]^2}{(z^0)^2} \square_a G^{(\ell)}_m(a, \vec{c}; z) = 0. \]  

(92)

Putting all together we obtain

\[ \nabla^\mu \frac{\partial}{\partial a^\mu} G^{(\ell)}_{\text{AdS/CFT}}(a, \vec{c}; z) = 0. \]  

(93)

So we see that the Goldstone mode (non transverseness of the propagator) can not be visible after trace projection on the boundary side corresponding to the case of the traceless currents in the large \( N \) limit of the \( O(N) \) sigma model.

The next interesting question which we can ask is the transversal property of the bulk-to-boundary propagator on the boundary side. The answer is negative. The divergence on the boundary side of the traceless bulk-to-boundary propagator is not zero and equals a gauge term (gradient) with respect to the bulk gauge invariance. Using the formulas from Appendix B one can check that

\[ \frac{\partial}{\partial \z} \cdot \frac{\partial}{\partial \vec{c}} G^{(\ell)}_{\text{AdS/CFT}}(a, \vec{c}; z) = a^\mu \nabla_\mu \Lambda^{(\ell-1)}(a, \vec{c}; z), \]  

\[ \Lambda^{(\ell-1)}(a, \vec{c}; z) = 2\ell(\alpha_0 - 1)(\ell + d - 1) - 2(\ell - 1) \frac{(z^0)^d}{(z, z)^{\Delta+1}} [R(a, \vec{c}; z)]^{\ell-1}. \]  

(95)

We see that the boundary trace projection generates the bulk gauge term on the boundary side and is equivalent to the residual on-shell gauge fixing preserving the bulk side de Donder off-shell gauge (this property of the bulk-to-boundary propagator was mentioned in [13] and in [22] for the vector field case).
Finalizing our consideration we can define now the CFT propagator from \( \phi^{(85)} \) by \( a^0 = 0 \) and the limit \( z^0 \to 0 \). Due to the vanishing of \( R^0(a; z) \) in this limit we get

\[
G_{CFT}^{(\ell)}(\vec{a}, \vec{c}; \vec{z}) = \lim_{z^0 \to 0} (z^0)^{2-d} G_{AdS/CFT}^{(\ell)}(\vec{a}, \vec{c}; z)
\]

\[
= G_m^{(\ell)}(\vec{a}, \vec{c}; \vec{z}) - \frac{\langle \vec{a}, \vec{a} \rangle}{2(\alpha_0 - 1)} \square \vec{a} G_m^{(\ell)}(\vec{a}, \vec{c}; \vec{z}) + O(\langle \vec{a}, \vec{a} \rangle^2)
\]

Thus the limit (96) defines the correct CFT two point function for traceless conserved\(^†\) currents.

So we prove that the boundary limit of our bulk-to-bulk propagator in the de Donder gauge is in agreement with the bulk-to-boundary propagator obtained from the AdS isometry group representation theory.

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\(^†\)Note that the gradient of the gauge term also vanishes on the boundary because \( a^\mu \nabla_\mu \Lambda^{(\ell-1)}(a, \vec{c}; z) \sim R^0(a; z) \).
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Appendix A

In this article we use the following rules and relations for $\zeta(z,z')$, $I_{1a}$, $I_{2c}$ and the bitensorial basis $\{I_a\}_{a=1}^4$

\[ \Box \zeta = (d+1) \zeta, \quad \nabla_\mu \partial_\nu \zeta = g_{\mu \nu} \zeta, \quad g^{\mu \nu} \partial_\mu \zeta \partial_\nu \zeta = \zeta^2 - 1, \quad (A.1) \]
\[ \partial_\mu \partial_\nu \zeta \nabla^\mu \zeta = \zeta \partial_\nu \zeta, \quad \partial_\mu \partial_\nu \zeta \nabla^\mu \partial_\nu \zeta = g_{\mu \nu} \zeta + \partial_\mu \zeta \partial_\nu \zeta, \quad (A.2) \]
\[ \nabla_\mu \partial_\nu \zeta \nabla^\mu \zeta = \partial_\nu \zeta \partial_\nu \zeta, \quad \nabla_\mu \partial_\nu \zeta \nabla^\mu \zeta = g_{\mu \nu} \partial_\nu \zeta, \quad (A.3) \]
\[ \frac{\partial}{\partial a_\mu} I_{1a} \frac{\partial}{\partial a_\mu} I_{1a} = \zeta^2 - 1, \quad \frac{\partial}{\partial a_\mu} I_1 \frac{\partial}{\partial a_\mu} I_1 = \zeta I_{2c}, \quad (A.4) \]
\[ \frac{\partial}{\partial a_\mu} I_1 \frac{\partial}{\partial a_\mu} I_1 = c_2^2 + I_{2c}^2, \quad \frac{\partial}{\partial a_\mu} I_2 = \zeta I_{2c}, \quad \Box a I_4 = 2(d+1)c_2^2, \quad (A.5) \]
\[ \frac{\partial}{\partial a_\mu} I_2 \frac{\partial}{\partial a_\mu} I_2 = (\zeta^2 - 1) I_{2c}^2, \quad \Box a I_3 = 2(d+1)I_{2c}^2 + 2c_2^2(\zeta^2 - 1), \quad (A.6) \]
\[ \nabla_\mu \frac{\partial}{\partial a_\mu} I_1 = (d+1)I_{2c}, \quad \nabla_\mu \frac{\partial}{\partial a_\mu} I_2 = (d+2)\zeta I_{2c}, \quad \nabla_\mu I_3 \partial_\mu \zeta = I_2, \quad (A.7) \]
\[ \nabla_\mu \frac{\partial}{\partial a_\mu} I_3 = 4I_1 I_{2c} + 2(d+2)\zeta c_2 I_{1a}, \quad \nabla_\mu I_2 \partial_\mu \zeta = 2\zeta I_2, \quad (A.8) \]
\[ \frac{\partial}{\partial a_\mu} I_1 \partial_\mu \zeta = \zeta I_{2c}, \quad \frac{\partial}{\partial a_\mu} I_1 \partial_\mu \zeta = (\zeta^2 - 1) I_{2c}, \quad \frac{\partial}{\partial a_\mu} I_1 \nabla_\mu I_1 = I_1 I_{2c}, \quad (A.9) \]
\[ \frac{\partial}{\partial a_\mu} I_1 \nabla_\mu I_2 = I_{2c} (\zeta I_1 + I_2) + c_2^2 I_{1a}, \quad \frac{\partial}{\partial a_\mu} I_2 \nabla_\mu I_1 = I_{2c} I_2, \quad (A.10) \]
\[ \frac{\partial}{\partial a_\mu} I_2 \nabla_\mu I_2 = 2\zeta I_{2c} I_2, \quad \nabla_\mu I_1 \nabla_\mu I_1 = a_1^2 I_{2c}, \quad \Box I_1 = I_1, \quad (A.11) \]
\[ \nabla_\mu I_1 \nabla_\mu I_2 = I_2 I_1 + a_1^2 \zeta I_{2c}, \quad \Box I_2 = (d+2)I_2 + 2\zeta I_1, \quad (A.12) \]
\[ \nabla_\mu I_2 \nabla_\mu I_2 = I_2^2 + 2\zeta I_1 I_2 + a_1^2 I_{2c}^2 \zeta^2 + c_2^2 I_1^2, \quad \nabla_\mu I_2 \partial_\mu \zeta = 2\zeta I_2, \quad (A.13) \]
\[ a_1^2 \nabla_\mu I_1 = a_1^2 \zeta, \quad a_1^2 \nabla_\mu I_2 = I_1, \quad a_1^2 \nabla_\mu I_1 = a_1^2 I_{2c}, \quad (A.14) \]
\[ a_1^2 \nabla_\mu I_2 = a_1^2 \zeta I_{2c} + I_{1a} I_1, \quad \nabla_\mu I_1 \partial_\mu \zeta = I_2. \quad (A.15) \]

Appendix B

Here we prove the relations (90)–(93). The more transparent way of working with the boundary-to-bulk propagator for higher spins is to introduce two additional objects

\[ \phi^0(z) = \frac{z^0}{(z,z)}, \quad (B.1) \]
\[ \psi(\vec{c},z) = \frac{<\vec{c}, \vec{z}>}{(z,z)}, \quad (B.2) \]
After that the proof of the condition
\[ a^\mu \partial_\mu \phi^0(z) = \frac{R^0(a;z)(z,z)}{(z,z)}, \quad a^\mu \partial_\mu \psi(\vec{c},z) = \frac{R(a,\vec{c};z)}{(z,z)}, \quad (B.3) \]
\[ \square \phi^0(z) = -(d-1)\partial^\mu \phi^0(z), \quad \square \psi(\vec{c},z) = 0, \quad (B.4) \]
\[ a^\mu a^\nu \nabla_\mu \partial_\nu \psi(\vec{c},z) = 2[\phi^0(z)]^{-1}a^\mu \partial_\mu \phi^0(z)a^\nu \partial_\nu \psi(\vec{c},z), \quad (B.5) \]
\[ \nabla_\mu \phi^0(z) \partial_\mu \phi^0(z) = (\phi^0)^2, \quad \nabla_\mu \phi^0(z) \partial_\mu \psi(\vec{c},z) = 0 \quad (B.6) \]
\[ \nabla_\mu \psi(\vec{c},z) \partial_\mu \psi(\vec{c},z) = (\phi^0)^2 <\vec{c},\vec{c}> , \quad (B.7) \]
\[ \square = \nabla_\mu \partial_\mu, \quad \nabla_\mu \left\{ \phi(\vec{c},z) \right\} = g^\mu_\nu \partial_\nu \left\{ \phi(\vec{c},z) \right\}, \quad (B.8) \]
\[ \nabla_\mu \partial_\nu = g^\mu_\nu \left( \partial_\mu \delta_\nu^\lambda - \Gamma_\mu_\nu^\lambda \right) \partial_\lambda, \quad \Gamma_\mu_\nu^\lambda = \frac{1}{2^\lambda} \left( \delta_\mu^\lambda \delta_\nu_\rho - \delta_\mu^\lambda \delta_\nu^\rho - \delta_\nu^\lambda \delta_\mu^\rho \right). \quad (B.9) \]

Then using \[(B.1)-(B.3)\] we can rewrite the AdS/CFT bulk-to-boundary propagator \[(B.5)\] in the following complete form
\[ G^{(\ell)}_{AdS/CFT}(a,\vec{c};z) = (\phi^0(z))^{d-2} \sum_{k=0}^{[\ell/2]} \frac{(-\ell)_{2k}}{2^{2k} k!(1-\alpha_0)_{2k}} [a^\mu \partial_\mu \psi(\vec{c},z)]^{\ell-2k} \]
\[ \times \left[ <\vec{c},\vec{c}> \left( a^\mu a_\mu (\phi^0(z))^2 - [a^\mu \partial_\mu \phi^0(z)]^2 \right) \right]^k. \quad (B.10) \]

After that the proof of the condition
\[ \nabla_\mu \partial_\alpha G^{(\ell)}_{AdS/CFT}(a,\vec{c};z) = 0 \quad (B.11) \]
reduces to the differentiation of the right hand side of with the covariant Leibniz rules and use of the relations \[(B.4)-(B.6)\].

For taking the divergence on the boundary side of \[(B.5)\] or \[(B.10)\] we need the following identities for \(\psi(\vec{c},z)\) and \(\phi^0(z)\)
\[ \vec{\psi}(z) = \frac{\partial}{\partial \vec{c}} \psi(\vec{c},z) = \frac{z}{(z,z)}, \quad \vec{\psi}(z) \cdot \vec{\psi}(z) = \frac{1}{(z,z)} - [\phi^0(z)]^2, \quad (B.12) \]
\[ \partial_\vec{c} \phi^0(z) = -2\phi^0(z)\vec{\psi}(z), \quad a^\mu a_\mu \partial_\vec{c} \cdot \vec{\psi}(z) = a^\mu \partial_\mu \frac{d-2}{(z,z)} + 4\phi^0(z)a^\mu \partial_\mu \phi^0(z), \quad (B.13) \]
\[ \partial_\vec{c} \phi^0(z) \cdot a^\mu a_\mu \vec{\psi}(z) = 2[\phi^0(z)]^2 a^\mu \partial_\mu \phi^0(z) - \phi^0(z)a^\mu \partial_\mu \frac{1}{(z,z)}; \quad (B.14) \]
\[ a^\mu a_\mu \partial_\vec{c} \vec{\psi}(z) = 2\phi^0(z) a^\mu \partial_\mu \phi^0(z) a^\mu \partial_\mu \psi(\vec{c},z) \]
\[ -2\psi(\vec{c},z) \left( [\phi^0(z)]^2 a^\mu a_\mu - [a^\mu \partial_\mu \phi^0(z)]^2 \right) \quad (B.15) \]
Then performing boundary differentiation of \[(B.10)\] and using \[(B.12)-(B.15)\] we obtain
\[ \frac{\partial}{\partial \vec{c}} \cdot \frac{\partial}{\partial \vec{c}} G^{(\ell)}_{AdS/CFT}(a,\vec{c};z) = a^\mu \nabla_\mu \Lambda^{(\ell-1)}(a,\vec{c};z) + O(<\vec{c},\vec{c}>), \quad (B.16) \]
\[ \Lambda^{(\ell-1)}(a,\vec{c};z) = 2^\ell \frac{(\alpha_0 - 1)(\ell + d - 1) - 2(\ell - 1)}{\alpha_0^2 - 1} [\phi^0(z)]^d [a^\mu \partial_\mu \psi(\vec{c},z)]^{\ell-1}. \quad (B.17) \]