Co-filters in Γ-semigroups ordered under co-order

Daniel A. Romano

Abstract. As a generalization of a semigroup, Sen, in 1981, introduced the concept of Γ-semigroups. In our paper (D. A. Romano, Γ-semigroups with apartness. Bull. Allahabad Math. Soc., 34 (2019), no. 1, 71-83), we introduced and analyzed the concept of Γ-semigroups with apartness in the Bishop’s constructive framework. In this paper, as a continuation of the previous one, we introduce and analyze the concept of Γ-semigroups ordered under co-order. In addition, we introduce and analyze the concept of co-filters in such Γ-semigroups.

Keywords. Bishop’s constructive algebra · Intuitionistic logic · Γ-semigroup with apartness · ordered Γ-semigroup under co-order · co-filter in ordered Γ-semigroup

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1 Introduction

Let \((S, w)\) be a grupoid, where the set \(S\) is the carrier of this structure and \(w : S \times S \rightarrow S\) is a total function. If \(w\) is associative, then the structure is \((S, w)\) a semigroup. Let us suppose that the logical environment in which we analyze this algebraic structure is the Intuitionistic logic \(\text{IL}\) [23], [14]. This assumption implies that the axiom 'Principe TND' (tertium non datur - the logical principle of 'the exclusion of the third') is not valid in this setting. In this logic, the diversity is a fundamental concept equal to the concept of equality in classical logic. Commitments under which we will obey in this text is the Bishop’s principled-philosophical orientation \textbf{Bish} (see, for example: [1], [2], [12]). Now, we look at the carrier \(S\) as a relational system \((S, =, \neq)\), where '=' is the standard equality, and ' \neq ' is an apartness [13]:

\[
\begin{align*}
(\forall x, y \in S)(x \neq y \implies \neg(x = y)) & \quad \text{(consistency);} \\
(\forall x, y \in S)(x \neq y \implies y \neq x) & \quad \text{(symmetry);} \\
(\forall x, y, z \in S)(x \neq z \implies (x \neq y \lor y \neq z)) & \quad \text{(co-transitivity);} \\
\end{align*}
\]
This relation is extensive with respect to the equality in the standard way

\[ = \circ \neq \subseteq \neq \circ \neq \subseteq \neq \]

where \( \circ \) is the standard composition of relations. In addition, any relation \( R \) on \( S \), any functions \( f \) between such sets and any operation \( w \) in \( S \) appearing in this article are strongly extensional relative to the apartness (see, for example: [13], [18]). For a strongly extensional mapping we will hereafter briefly write ‘se-mapping’. Because of the specificity of \( IL \), for some subsets of the \( S \), their strictly extensive doubles will appears. For example, a (strongly extensional) subset \( K \) of a semigroup \( S \) with apartness is a co-ideal of \( S \) if holds

\[(\forall x, y \in S)(xy \in K \implies x \in K \lor y \in K).\]

It is not difficult to show that a strong complement

\[ K^\triangleleft = \{x \in S : (\forall s \in K)(x \neq s)\} \]

of co-ideal \( K \) is an ideal in \( S \). Conversely, in the general case, it does not have to be valid. Thus, the observed structure \( S = ((S, =, \neq), w) \) is a semigroup with apartness. In the last 40 years, this author alone, or in collaboration with other authors, has investigated structures of various types of semigroups with apartnesses [4], [5], [16], [17], [19]. In the article [3], Cherubini and Frigeri introduce the concept of 'inverse semigroups with apartness'. One of the main problems in these researches was "How to find and describe the duals of classical algebraic concepts?" in well-known algebraic structures.

In this text we interested in \( \Gamma \)-semigroups with apartness. Also, we will find and analyze some doubles of substructures of these semigroups. Our investigation of the concept of \( \Gamma \)-semigroups with apartness consists of the observation of specificities that arise by placing the classically defined algebraic structure of \( \Gamma \)-semigroups ([21], [22]) into a different logical environment and using specific Bishop’s constructive algebra tools. The rest of the paper is organized as follows: In Section 2, the concept of \( \Gamma \)-semigroup with apartness (Definition 2.1) and the concept of \( \Gamma \)-cosubsemigroup with apartness (Definition 2.2) are given. In Section 3, the notion of ordered \( \Gamma \)-semigroup with apartness ordered under a co-order relation is given (Definition 3.1). Further, the concept of (left, right) co-filters on such semigroups is introduced (Definition 3.2). In addition, some important properties of these substructures in such algebraic structures have been proven.

2 Preliminaries

To explain the notions and notations used in this article, which but not previously described, we instruct the reader to look at the articles [6], [7], [8],
Here we will introduce some specific substructures of this semigroups that appear only in the Bish version.

**Definition 2.1** ([17], Definition 2.1) Let \((S, =, \neq)\) and \((\Gamma, =, \neq)\) be two non-empty sets with apartness. Then \(S\) is called a \(\Gamma\)-semigroup with apartness if there exist a strongly extensional mapping from \(S \times \Gamma \times S\) \∋ \((x, a, y) \mapsto \rightarrow xay \in S\) satisfying the condition
\[
(\forall x, y, z \in S)(\forall a, b \in \Gamma)((xay)bz = xa(ybz)).
\]
We recognize immediately that the following implications
\[
(\forall x, y, u, v \in S)(\forall a, b \in \Gamma)(xay \neq ubv \Rightarrow (x \neq u \lor a \neq b \lor y \neq v)),
\]
\[
(\forall x, y \in S)(\forall a, b \in \Gamma)(xay \neq xby \Rightarrow a \neq b)
\]
are valid, because \(f\) is a strongly extensional function.

**Definition 2.2** ([17], Definition 2.2) Let \(S\) be a \(\Gamma\)-semigroup with apartness. A subset \(T\) of \(S\) is said to be a \(\Gamma\)-cosubsemigroup of \(S\) if the following holds
\[
(\forall x, y \in S)(\forall a \in \Gamma)(xay \in T \Rightarrow (x \in T \lor y \in T)).
\]
We will assume that the empty set \(\emptyset\) is a \(\Gamma\)-cosubsemigroup of a \(\Gamma\)-semigroup \(S\) by definition.

Our first proposition in this section is the following:

**Proposition 2.3** ([17], Proposition 2.1) If \(T\) is a \(\Gamma\)-cosubsemigroup of a \(\Gamma\)-semigroup with apartness \(S\), then the set \(T^\alpha\) is a \(\Gamma\)-subsemigroup of \(S\).

3 The main results

3.1 The concept of co-ordered \(\Gamma\)-semigroup with apartness

The relation \(\alpha\) is said to be a co-order on the set \(X\) with apartness if it is consistent \(\alpha \subseteq \neq\), co-transitive \(\alpha \subseteq \alpha \ast \alpha\), i.e.
\[
(\forall c, y, z \in X)((x, z) \in \alpha \Rightarrow ((x, y) \in \alpha \lor (y, z) \in \alpha))
\]
and linear in the following sense: \(\neq \subseteq \alpha \cup \alpha^{-1}\). \(\alpha\) is said to be a co-quasiorder on \(X\) if it is a consistent and co-transitive relation on \(X\). An interested reader can find more information about this type of relations in sets (in algebraic structures) in one of our following texts: [15], [16], [18], [19]. A brief recapitulation of a number of algebraic structures ordered by co-quasiorder relation is presented in the article [18].

In the following definition we introduce the concept of co-order relations in \(\Gamma\)-semigroup with apartness.

1 We will not write different apartness relations in different sets on different way, unless it is needed. From the context and to the use of different variables of marking of elements of different sets, it will be clear what type of apartness relation is involved.
Definition 3.1 Let $S$ be a $\Gamma$-semigroup with apartness. A co-order relation $\not\leq$ on $S$ is compatible with the semigroup operations in $S$ if the following holds

$$(\forall x, y, z \in S)(\forall a \in \Gamma)((xaz \not\leq yaz \lor zax \not\leq zay) \implies x \not\leq y).$$

In this case it is said that $S$ is a $\Gamma$-semigroup ordered under co-order $\not\leq$ or it is co-ordered $\Gamma$-semigroup.

Speaking by the language of classical algebra, relation $\not\leq$ is compatible with the internal operation in $S$ if the operation is right and left cancellative with respect to the co-order relation $\not\leq$.

Example 3.1 Let $S$ be a set of all reverse isotone semapping from an ordered set $(P, =, \neq)$ under a co-order $\not\leq$ into another ordered set $(Q, =, \neq)$ under the co-order $\not\leq$ and let $\Gamma$ be set of all reverse isotone semapping from $Q$ to $P$. In both cases, apartness is defined as follows $f \neq g \iff (\exists x)(f(x) \neq g(x))$. For $f, g \in S$ and $\alpha \in \Gamma$ put $f \circ g = g \circ \alpha \circ f$, where $\circ \circ'$ be mark for standard composition between relations. Then $S$ is a $\Gamma$-semigroup. A co-order $' \not\leq' \quad$ relation on $S$ can defined by $f \not\leq g \iff (\exists x \in P)(f(x) \not\leq g(x))$. Then $S$ is a co-ordered $\Gamma$-semigroup.

3.2 The concept of co-filters in ordered $\Gamma$-semigroup

Now we introduce the notion of left, right and both side co-filters in co-ordered $\Gamma$-semigroup $S$.

Definition 3.2 Let $S$ be a $\Gamma$-semigroup with apartness ordered under co-order $\not\leq$.

(a) A $\Gamma$-cosubsemigroup $G$ of $S$ is called a left co-filter of $S$ if

1. $$(\forall x, y \in S)(\forall a \in \Gamma)(x \in G \implies xay \in G),$$ and
2. $$(\forall x, y \in S)(y \in G \implies (x \in G \lor x \not\leq y)).$$

(b) A $\Gamma$-cosubsemigroup $G$ of $S$ is called a right co-filter of $S$ if

3. $$(\forall x, y \in S)(\forall a \in \Gamma)(y \in G \implies xay \in G),$$ and
4. $$(\forall x, y \in S)(y \in G \implies (x \in G \lor x \not\leq y)).$$

(c) A $\Gamma$-cosubsemigroup $G$ of $S$ is called a co-filter of $S$ if $G$ is a left and right co-filter of $S$.

Example 3.2 Let $S$ be a co-ordered $\Gamma$-semigroup and $x, y \in S$ be elements. Let us put $[x]_\not\leq = \{y \in S : x \not\leq y\}$ and $[y]_\not\leq = \{x \in S : x \not\leq y\}$. It is obvious that the set $[y]_\not\leq$ satisfies the condition (2).

Statement 3.1 Condition (1) is equivalent to condition

$$(1') GFS \subseteq G.$$ 

Statement 3.2 Condition (3) is equivalent to condition

$$(3') SFG \subseteq G.$$
Statement 3.3 Condition (2) is equivalent to condition

\[(2') \ (\forall y \in S)(y \in G \implies S \subseteq G \cup \langle y \rangle).\]

The following proposition immediately follows from the previous definition.

Proposition 3.3 Let \(G\) be a (left, right) co-filter in a \(\Gamma\)-semigroup \(S\) with apartness ordered under co-order \(\preceq\). Then \(G\) is a strongly extensional subset in \(S\).

Proof. The claim of this proposition follows from statement (2) in the previous definition and consistency of the relation \(\preceq\).

\(\square\)

Our next statement is related to the concept of co-filters in \(\Gamma\)-semigroup.

Proposition 3.4 Let \(G\) be a left co-filter in a \(\Gamma\)-semigroup \(S\) with apartness ordered under co-order \(\preceq\). Then \(G\) is a left filter in \(S\).

Proof. Since \(G\) is a \(\Gamma\)-cosubsemigroup of \(S\), then \(G\) is a \(\Gamma\)-subsemigroup in \(S\) according to Proposition 2.3.

Let us prove the condition

(i) \((\forall x, y \in S)(\forall a \in \Gamma)(xay \in G \implies x \in G)\).

Let \(x, y, u \in S\) and \(a \in \Gamma\) be arbitrary elements such that \(xay \in G\) and \(u \in G\). From \(u \in G\) follows \(x \in G\) or \(x \not\preceq u\) by (2). Since the first option is \(x \in G\), gives \(xay \in G\) by (1), which is in contradiction with the hypothesis \(xay \in G\), must be \(x \not= u \in G\) by consistency of relation \(\preceq\). So, \(x \in G\).

Let us now prove

(ii) \((\forall x, y \in S)((x \not\preceq y \land x \in G) \implies y \in G)\).

Let \(x, y, u \in S\) be arbitrary elements such that \(x \not\preceq y\), \(x \in G\) and \(u \in G\). From \(u \in G\) follows \(y \not\preceq u\) or \(y \in G\). From the second option \(y \in G\) follows \(x \not\preceq y \lor x \in G\) by (2). As the result obtained contradicts the hypothesis, it must be \(y \not= u \in G\) by the consistency of \(\preceq\). So, \(y \in G\).

Because \(\Gamma\)-subsemigroup \(G\) satisfies conditions (4.1) and (4.2) in Definition 4 in article [11], we have proved that \(G\) is a left filter in \(S\).

\(\square\)

The following two claims can be proved analogously to the previous one, so we will omit their evidence.

Proposition 3.5 Let \(G\) be a right co-filter in a \(\Gamma\)-semigroup \(S\) with apartness ordered under co-order \(\preceq\). Then \(G\) is a right filter in \(S\).

Proposition 3.6 Let \(G\) be a co-filter in a \(\Gamma\)-semigroup \(S\) with apartness ordered under co-order \(\preceq\). Then \(G\) is a filter in \(S\).
3.3 Some important statements

**Theorem 3.7** The union of \( \bigcup_{i \in I} G_i \) a family \( \{G_i\}_{i \in I} \) of left co-filters of a co-ordered \( \Gamma \)-semigroup \( S \) is also a left co-filter in \( S \).

**Proof.** Let \( \{G_i\}_{i \in I} \) be a family of left co-filters of a \( \Gamma \)-semigroup with apartness ordered under co-order \( \not\leq \). Since \( G_i \) is a \( \Gamma \)-cosubsemigroup of ordered \( \Gamma \)-semigroup \( S \) for any \( i \in I \), then \( \bigcup_{i \in I} G_i \) is a \( \Gamma \)-cosubsemigroup of \( S \) by Theorem 2.1 in article [17].

Let us check conditions (1) and (2) in Definition 3.2(a). First, let \( x, y, a \in S \) and \( a \in \Gamma \) be arbitrary elements such that \( x \in \bigcup_{i \in I} G_i \). Then there is an index \( j \in I \) such that \( x \in G_j \). Thus \( xay \in G_j \) by (1) because \( G_j \) is a left co-filter in \( S \). So, \( xay \in \bigcup_{i \in I} G_i \). Second, let \( x, y \in S \) be arbitrary elements such that \( y \in \bigcup_{i \in I} G_i \). Then there exists an index \( j \in I \) such that \( y \in G_j \). Thus \( x \in G_j \lor x \not\leq y \) because \( G_j \) is a left co-filter in \( S \). So, \( x \not\leq y \lor x \in \bigcup_{i \in I} G_i \). Therefore, \( \bigcup_{i \in I} G_i \) is a left co-filter in \( S \) also.

\[ \square \]

As in the case of the preceding propositions, the claims of the following two theorems can be proved by analogy with the previous theorem, so we will omit their proofs.

**Theorem 3.8** The union of \( \bigcup_{i \in I} G_i \) a family \( \{G_i\}_{i \in I} \) of right co-filters of a co-ordered \( \Gamma \)-semigroup \( S \) is also a right co-filter in \( S \).

**Theorem 3.9** The union of \( \bigcup_{i \in I} G_i \) a family \( \{G_i\}_{i \in I} \) of co-filters of a co-ordered \( \Gamma \)-semigroup \( S \) is also a co-filter in \( S \).

The following theorem is about the left co-filter in the \( \Gamma \)-semigroup \( S \) generated by a single element \( x \) in \( S \).

**Theorem 3.10** Let \( S \) be a \( \Gamma \)-semigroup with apartness ordered under a co-order \( \not\leq \) and let \( x \in S \) be an arbitrary element. Then there exists the maximal left co-filter \( L_x \) in \( S \) such that \( x \not\leq L_x \).

**Proof.** Let \( B \) be the family of all left co-filters in \( S \) included in set \( S \setminus \{x\} \). Then \( G_x = \bigcup B \) is the maximal left co-filter in \( S \) such that \( x \not\leq G_x \) by Theorem 3.7.

\[ \square \]

The following two theorems can be proved by analogy with the previous theorem and we will omit their proofs.

**Theorem 3.11** Let \( S \) be a \( \Gamma \)-semigroup with apartness ordered under a co-order \( \not\leq \) and let \( x \in S \) be an arbitrary element. Then there exists the maximal right co-filter \( R_x \) in \( S \) such that \( x \not\leq R_x \).

**Theorem 3.12** Let \( S \) be a \( \Gamma \)-semigroup with apartness ordered under a co-order \( \not\leq \) and let \( x \in S \) be an arbitrary element. Then there exists the maximal co-filter \( G_x \) in \( S \) such that \( x \not\leq G_x \).
4 Final observation

This report is a continuation of our previously published research [17] on \( \Gamma \)-semigroups with apartness. In the mentioned article, the concept of \( \Gamma \)-semigroups with apartness was introduced and analyzed. In addition, the notion of co-ideals in such semigroups was introduced. Several significant claims have been proven.

The problems we treat as primary in observing and analyzing ordered \( \Gamma \)-semi-groups with apartness into the Bishop’s constructive framework are not like the problems encountered in the classical case. Within this aspect of observations, analyzing and classification of the properties of the algebraic structure of \( \Gamma \)-semigroup are much more significant questions about the articulation and existence of classes of substructures in a \( \Gamma \)-semigroup than their interrelations. First, the concept of co-order relation in such algebraic structures is introduced in this article. Then, in such co-ordered \( \Gamma \)-semigroup with apartness is introduced and analyzed the concept of (left, right, both side) co-filters, which is the counterpart of the concept of filters in the classic case. Finally, it is shown that for each element \( x \) in a \( \Gamma \)-semigroup \( S \) there exists the maximum co-filter \( G_x \) in \( S \) such that \( x \prec G_x \).

Thus, the principled-philosophical orientation to existence of duality of objects and properties of algebraic structures within the Bishop’s constructive framework was a fundamental landmark in analyzing the properties of \( \Gamma \)-semigroups with apartness ordered under co-order. So in this article we continue our research of specific algebraic structures within Bishop’s constructive aspect.

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AUTHOR

DANIEL A. ROMANO,
International Mathematical Virtual Institute,
78000 Banja Luka,
Bosnia and Herzegovina,
E-mail: bato49@hotmail.com