Energy-Momentum Tensors and Motion in Special Relativity

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Abstract

The notions of “motion” and “conserved quantities”, if applied to extended objects, are already quite non-trivial in Special Relativity. This contribution is meant to remind us on all the relevant mathematical structures and constructions that underlie these concepts, which we will review in some detail. Next to the prerequisites from Special Relativity, like Minkowski space and its automorphism group, this will include the notion of a body in Minkowski space, the momentum map, a characterisation of the habitat of globally conserved quantities associated with Poincaré symmetry – so called Poincaré charges –, the frame-dependent decomposition of global angular momentum into Spin and an orbital part, and, last not least, the likewise frame-dependent notion of centre of mass together with a geometric description of the Møller Radius, of which we also list some typical values. Two Appendices present some mathematical background material on Hodge duality and group actions on manifolds. This is a contribution to the book: Equations of Motion in Relativistic Gravity, edited by Dirk Pützfeld and Claus Lämmerzahl, to be published by Springer Verlag.

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Contents

1 Introduction 3
2 Minkowski space and Poincaré group 7
3 The momentum map and the natural habitat of globally conserved Poincaré charges 22
4 Supplementary conditions and mass centres 27
5 Typical Møller radii 34

Appendices
A Exterior products and Hodge duality 36
B Group actions on manifolds 40

References 49
1 Introduction

This contribution deals with the “problem of motion” in Special Relativity. Thus we work entirely in Minkowski space M (to be defined below) and represent a material system by an energy-momentum tensor $T$ the support of which is to be identified with the set of events (points) in Minkowski space where matter “exists”:

$$\text{supp}(T) := \{ p \in M \mid T(p) \neq 0 \}. \quad (1)$$

A central assumption will be that the material system is spatially well-localised, which here shall mean that $\text{supp}(T)$ has compact intersection with any Cauchy hypersurface in M. Note that Cauchy hypersurfaces end at spatial infinity $I_0$ and that $\text{supp}(T)$ need not have compact intersection with asymptotically hyperboloidal spacelike hypersurfaces which tend to lightlike rather than spacelike infinity. This is depicted in Figure 1.

Definition 1. We say that an energy-momentum tensor $T$ describes a body iff the intersection of $\text{supp}(T)$ with any Cauchy hypersurface in Minkowski space is compact.

Hence we identify the event-set of a body with $\text{supp}(T)$, which, in the sense made precise above, is of finite spatial extent, though it clearly will extend to timelike infinity. This is visualised as the a tubular neighbourhood stretching all the way from past-timelike to future-timelike infinity, as indicated by the shaded vertical tube in Figure 1. It is also clear from Figure 1 that we generally cannot require compact support of $T$ on spacelike hypersurfaces which are not Cauchy, like $L$. In fact, if the body radiated in the finite past, given by the lighter-shaded part of the tubular region in the lower half of Figure 1, the radiation will propagate to $\mathcal{I}_+$ and cover a neighbourhood in $L$ of its 2-sphere of intersection with $\mathcal{J}_+$, which is of non-compact closure. This can be avoided for spacelike hypersurfaces ending at $I_0$ if we require a neighbourhood of $\mathcal{I}_-$ to be free of radiation. This means that the body started to radiate a finite time in the past and that there is no incoming radiation from $\mathcal{J}_-$ arbitrarily close to $I_0$. In fact, describing a quasi-isolated body would presumably mean to exclude incoming radiation altogether. This explains our motivation for Definition 1.
Figure 1: History of a compact object in the conformal compactification of Minkowski space (Penrose Diagram). The five asymptotic regions of Minkowski space are future/past-timelike infinity $I_{\pm}$ (each a single point), future/past-lightlike infinity $J_{\pm}$ (each a three-dimensional lightlike manifold of topology $\mathbb{R} \times S^2$), and spacelike infinity $I_0$ (a single point). The representation is not quite faithful because spacelike infinity, here represented by two points, is really just a single point. A faithful representation is obtained by wrapping the diamond-shaped 2-dimensional figure around a cylinder ($\mathbb{R} \times S^1$), so as to identify both points $I_0$ of the diagram to a single one. $S$ and $L$ are both spacelike hypersurfaces stretching out to “infinity”. But only $S$, which stretches out to spacelike infinity, is a Cauchy surface, i.e., covers all of spacetime in its domain of dependence.
A body should possess globally conserved quantities like linear and angular momentum. These are usually written down in formulae like

\[ P^a = \int_\Sigma T^a_b u^b \, d\mu, \]  
\[ J^{ab}[z] = \int_\Sigma \left[ (x^a - z^a)T^b_c - (x^b - z^b)T^a_c \right] u^c \, d\mu, \]  

where \( \Sigma \) is a Cauchy surface, \( u^a \) are the components of its future-pointing normal, and \( d\mu \) is the measure on \( \Sigma \) induced from the ambient spacetime. See [5] for a conceptually exceptionally clear discussion.

The problem with these expressions is that, on face value, they do not make any sense. For one thing, the integrands are vector/tensor valued, and adding them at different points does not result in anything with an obvious meaning. If we wish to interpret \( P^a \) as the \( a \)-th (covariant) component of the vector of total linear momentum, we should characterise the vector space of which \( P \) is an element. And, moreover, what does it mean to say that total linear (four-)momentum transforms like a four-vector (here covariant)? Likewise, we wish to interpret \( J^{ab}[z] \) as the \( ab \)-th (contravariant) component of the antisymmetric 2nd-rank tensor of angular momentum with respect to the centre \( z \). Again it is unclear what tensor space this \( J[z] \) is an element of and what is meant by stating its representation property under Poincaré transformations. Are these spaces defined at points in spacetime, perhaps at “infinity”, or in an abstract vector/tensor space globally associated to (but not in) spacetime? Also, the difference \( (x^a - z^a) \) that appears in (2b) also makes no immediate sense. Is it supposed to be the \( a \)-th component of some “difference function” on spacetime? Is it supposed to make sense in all coordinate systems, or just special ones; and if the latter holds, what selects these special ones?

Clearly, all these questions do have answers, but these answers delicately depend on the precise mathematical structures with which spacetime is endowed. In our (highly idealised) case of Minkowski space, it is the high degree of symmetry of spacetime that allows us to naturally interpret (2) so as to make unambiguous mathematical and physical sense. Removing or weakening these structures and pretending the expressions (2) to still make sense without further qualifications means to commit a mathematical and conceptual sin. This does not mean that (2) cannot be meaningfully generalised, but these generalisations will generally not be natural in a mathematical sense, that is, they will depend on additional structures and constructions to be imposed or selected “by hand”. The physical interpretation of what is then actually represented by the integrals (2) will delicately depend on these
Figure 2: As emphasised by this conference logo, a central problem is to associate a timelike curve \( S \) to the energy-momentum tensor \( T \). One would expect the line \( S \) to lie in the “convex hull” of the support of \( T \), here represented by the extended tube \( \Sigma \).

by-hand additions. It is therefore the aim of this introductory exposition to clarify the mathematical and physical meaning of (2) in the simplest case, i.e. in Special Relativity. My strategy will be to fully display all the ingredients that go into the proper definition of (2). This, hopefully, will help to distinguish the generic difficulties of the gravitational case from those merely inherited from Special Relativity.

Related to the issue of giving proper meaning to (2) is the definition of “centre of mass” of an extended object. As you can see from its logo, this is a central concern of this conference (see Figure 2). If “motion” is the change of position in time, we need to be clear about how to define “position” in the first place. The issue of how to define position observables in any special-relativistic theory, classical and quantum, is notorious. See, e.g., [6] for a good account. In my contribution I will give a derivation of the Møller radius which represents the ambiguity of defining position for systems with “spin”, i.e., “intrinsic angular momentum”, a notion also to be defined. So let us start at the beginning, asking for the reader’s patience!
2 Minkowski space and Poincaré group

In this section we wish to recall the definitions of Minkowski space and its automorphism group, despite the fact that this is generally considered a commonplace. But we think that there are some subtleties, in particular concerning the characterisation of its automorphism group, the Poincaré group, that deserve to be said more than once. We start with

Definition 2. Minkowski space is a quadruple \((M, V, \eta, +)\), consisting of:

1. A set, \(M\), the elements of which are called spacetime points or events.
2. A real 4-dimensional vector space \(V\).
3. A simply transitive action of \(V\), considered as a group, on \(M\), denoted by +, i.e.,
   \[ M \times V \to M, \quad (p, v) \mapsto p + v. \] (3)

4. A non-degenerate symmetric bilinear form \(\eta \in V^* \otimes V^*\) of signature \((+1, -1, -1, -1)\). \(\square\)

Remark 3. Every non-degenerate bilinear form \(\eta : V \times V \to \mathbb{R}\) on a vector space \(V\) defines an isomorphism \(\eta_i : V \to V^*\) to its dual space \(V^*\) via the requirement \(\eta_i(v)(w) := \eta(v, w)\) for all \(v, w \in V\); in short, \(v \mapsto \eta_i(v) := \eta(v, \cdot)\). Its inverse map is \(\eta^i : V^* \to V\), \(\eta^i := (\eta_i)^{-1}\), which in turn defines a non-degenerate bilinear form on the dual space, \(\eta^{-1} : V^* \times V^* \to \mathbb{R}\), via the requirement \(\eta^{-1}(\alpha, \beta) := \alpha(\eta^i(\beta))\) for all \(\alpha, \beta \in V^*\). On component-level this reads as follows: Let \(\{e_a\mid 1 \leq a \leq n\}\) be a basis of \(V\) and \(\{\theta^a\mid 1 \leq a \leq n\}\) its dual basis of \(V^*\), so that \(\theta^a(e_b) = \delta^a_b\). Then, writing \(v = v^a e_a\) and \(w = w^b \theta^b\) with \(v_a := v^a \eta_{ab}\) and \(\eta_{ab} := \eta(e_a, e_b)\). Similarly, writing \(\alpha = \alpha_a \theta^a\) we get \(\eta_i(\alpha) = \alpha_a e_a\) with \(\alpha^a := \eta^{ab} \alpha_b\) and \(\eta^{ab} := \eta^{-1}(\theta^a, \theta^b)\). This implies \(\delta^a_b = \eta^{ac} \eta_{bc} = \eta^{ca} \eta_{cb} = \delta^a_b\) and, in particular, \(\eta^{ab} = \eta^{ac} \eta^{bd} \eta_{cd}\) and \(\eta_{ab} = \eta^{ad} \eta_{ca} \eta_{db}\). This explains why \(\eta_i\) and \(\eta_i\) are called the operations of “index-raising” and “index lowering”. Sometimes the images of \(\eta_i\) and \(\eta_i\) are indicated by the musical symbols \(\dagger\) (sharp) and \(\flat\) (flat) respectively, i.e., one writes \(\eta_i(\alpha) = \alpha^\sharp\) and \(\eta_i(v) = v^\flat\), which makes sense as long as the bilinear form \(\eta\) with respect to which these maps are defined is self understood. We shall also employ this notation. Note that so far we did not assume \(\eta\) to be symmetric, so that all formulae apply generally. However, from now on, and for the rest of this paper, the symbol \(\eta\) shall always denote the Minkowski metric, which specialises the general case by symmetry and signature. Once \(\eta\) is fixed, the isomorphisms between \(V\) and \(V^*\) as well as its extensions to tensor products is clear from the context and it is sufficient and useful to use shorthand notations, like \(v \cdot w := \eta(v, w) = v^\flat(w)\),
\[ v^2 := v \cdot v, \text{ and } \|v\| := \sqrt{|v \cdot v|}. \]

Given \( J = J^{ab} e_a \otimes e_b \in V \otimes V \) and \( v \in V \), we shall also write \( J \cdot v \) or \( v \cdot J \) for the application of \( J^{ab} e_a \otimes \eta_i(e_b) = J^{a}_b e_a \otimes \theta^b \in \text{End}(V) \) or \( J^{ab} e_b \otimes \eta_i(e_a) = J^{b}_a e_b \otimes \theta^a \in \text{End}(V) \), respectively, to \( v \). The inner products on \( V \) and \( V^* \) can be used to define inner products on any space built by taking tensor products of \( V \) and \( V^* \) just by slotwise contraction. However, in certain circumstances of high symmetry, e.g., for totally antisymmetric tensor products, it is more convenient to renormalise the slotwise inner product by combinatorial factors; like in formula (133) of the Appendix. Finally we recall that the transposed of a general linear map \( A: V \to W \) between vector spaces \( V \) and \( W \) is the linear map \( A^\top : W^* \to V^* \), defined by \( A^\top(\alpha) := \alpha \circ A \) for all \( \alpha \in W^* \). There is a natural isomorphism between a vector space \( V \) and its double dual \( V^{**} \), so that we may identify these spaces without explicit mention. Symmetry of \( \eta \) is then equivalent to \( \eta^\top = \eta \) and symmetry of \( \eta^{-1} \) to \( \eta^{-1} = \eta \).

### 2.1 Affine spaces

Note that 1.-3. define the notion of an affine space. Minkowski space is thus just a real 4-dimensional affine space, the associated vector space of which carries a Lorentz metric. Any vector space \( V \) is a group under addition, with group identity being given by the zero vector and the inverse of \( v \in V \) being \( -v \). It is customary to use the same symbol, +, for the addition of vectors in \( V \) and the action of \( V \) on \( M \). This allows to write the action property in the intuitive form (compare Appendix [B] for the general definition of a group action on a set)

\[ p + (v + w) = (p + v) + w =: p + v + w. \] (4)

But note the different meanings of + in this equation. Moreover, we define the subtraction of a vector by the addition of the inverse:

\[ p - v := p + (-v). \] (5)

This allows one more simplifying notation: Since \( V \) acts simply transitive, there exists a unique \( v \in V \) for any given pair \( (p, q) \in M \times M \) so that \( p = q + v \). We write

\[ v = p - q. \] (6)

Hence the minus sign should be understood as difference map \( M \times M \to V, \)

\[ (p, q) \to p - q, \] defined through \( p = q + (p - q) \). Simple transitivity then implies

\[ (p - o) + (o - q) = p - q, \] (7)
which is equivalent to
\[ p + (q - o) = q + (p - o). \] (8)

2.2 Linear and affine frames

**Definition 4.** A frame \( F \) for an affine space \((M, V, +)\) consists of a tuple \( F = (o, f) \), where \( o \in M \) and \( f \in \text{Lin}(\mathbb{R}^n, V) \) is a frame of the vector space \( V \). Recall that a frame \( f \) of an \( n \)-dimensional real vector space \( V \) is an isomorphism from \( \mathbb{R}^n \) to \( V \). This is equivalent to choosing \( n \) linear independent vectors \( \{e_1, \ldots, e_n\} \subset V \), the images under \( f \) of the canonical basis of \( \mathbb{R}^n \). The map \( f \) is then defined by linear extension:
\[ f(r_1, \ldots, r_n) = \sum_{a=1}^{n} r_a e_a. \] (9)

The inverse map is
\[ F^{-1} : M \to \mathbb{R}^n, \quad p \mapsto F^{-1}(p) : = f^{-1}(p - o) = (\theta^1(p - o), \ldots, \theta^n(p - o)). \] (10)

Given two frames \( F = (o, f) \) and \( F' = (o', f') \), they are related by \( F = F' \circ (F'^{-1} \circ F) \), where
\[ F'^{-1} \circ F : \mathbb{R}^n \to \mathbb{R}^n, \quad (r, \ldots, r_n) \mapsto (r'^1, \ldots, r'^n) \] (11a)
with
\[ r'^a(r^1, \ldots, r^n) = \theta'^a(o - o') + \sum_{b=1}^{n} \theta'^a(e_b) r'^b. \] (11b)

We denote the set of all affine frames of \( M \) by \( \mathcal{F}_M \).

**Remark 5.** Affine spaces naturally inherit a topology from \( \mathbb{R}^n \). It is defined to be the unique topology on \( M \) for which all frame maps (9) are homeomorphisms, i.e., \( F \) and \( F^{-1} \) are continuous (hence \( F \) is an open map). Note
that if a particular \( F' \) is a homeomorphism, than so is any other \( F \), for \( F = F' \circ (F'^{-1} \circ F) \) and \( F'^{-1} \circ F : \mathbb{R}^n \to \mathbb{R}^n \), given by (11b), is clearly a homeomorphism. Hence the open sets in \( M \) are precisely the images of open sets in \( \mathbb{R}^n \) under any \( F \). Moreover, affine frames endow \( M \) with the structure of a smooth \( (C^\infty, \text{or even analytic}) \) manifold since each frame defines a global chart with analytic transition functions (11b) between those charts.

**Definition 6.** Affine frames define special, globally defined coordinates which are called *affine coordinates* or, in a physical context, *inertial coordinates*. Using these we may regard affine spaces as smooth \( (C^\infty, \text{or even analytic}) \) manifolds, as explained in Remark 5.

Recall that the algebra of all linear self-maps of a vector space \( V \) onto itself is denoted by \( \text{End}(V) \) (endomorphisms). The subset of all invertible elements in \( \text{End}(V) \) is called \( \text{GL}(V) \); it forms a group, the general-linear group (of self-isomorphisms, or Automorphisms regarding its structure as vector space) of \( V \). Accordingly, \( \text{End}(\mathbb{R}^n) \) is just given by the algebra of all real \( n \times n \) matrices and \( \text{GL}(\mathbb{R}^n) \) by the group of all \( n \times n \) matrices with non-vanishing determinant.

A frame of \( V \) defines an isomorphism of algebras \( \text{End}(V) \to \text{End}(\mathbb{R}^n) \) through \( A \mapsto A' := f \circ A \circ f^{-1} \). Its restriction to \( \text{GL}(V) \) defines an isomorphism of groups \( \text{GL}(V) \to \text{GL}(\mathbb{R}^n) \). Let us denote by \( \mathcal{F}_V \) the set of all frames of \( V \). There are two natural left actions of groups on \( \mathcal{F}_V \): \( \text{GL}(V) \) acts on the left according to \( (A,f) \mapsto A \circ f \) and \( \text{GL}(\mathbb{R}^n) \) also acts on the left according to \( (B,f) \mapsto f \circ B^{-1} \). Note that \( (B,f) \mapsto f \circ B \) would be a right action; see Appendix (2) for a general discussions of group actions. Both actions commute and are each simply transitive. A combined left action of \( \text{GL}(V) \times \text{GL}(\mathbb{R}^n) \) on \( \mathcal{F}_V \) according to \( ((A,B),f) \mapsto A \circ f \circ B^{-1} \) results. The action of \( \text{GL}(\mathbb{R}^n) \) is sometimes called *passive* since it merely moves the labels (coordinates) in label-space \( \mathbb{R}^n \), whereas \( \text{GL}(V) \)'s action is called *active* since it really moves the points in the space \( V \). Note that these adjectives refer to *different* groups, which are isomorphic but not naturally so since picking any isomorphism requires extra choices to be made. For example, picking a frame \( f \), an \( f \)-dependent isomorphism \( \text{GL}(\mathbb{R}^n) \to \text{GL}(V) \) is defined through the stabiliser subgroup in \( \text{GL}(V) \times \text{GL}(\mathbb{R}^n) \) that fixes \( f \) under the common left action just described. This isomorphism then simply reads \( \text{GL}(\mathbb{R}^n) \ni B \mapsto A := f \circ B \circ f^{-1} \in \text{GL}(V) \) (so that \( A \circ f \circ B^{-1} = f \)), which then also defines a frame-dependent left action of \( \text{GL}(\mathbb{R}^n) \) on \( V \). With respect to the fixed frame \( f \) the latter can then be used to define “active” transformations on \( V \) by means of what previously had been interpreted as
mere label (coordinate) transformations. Failing to clearly state the groups, their domains of action, and the structures to be considered fixed is often the source of considerable confusion regarding the distinction of “active” and “passive” actions.

2.3 Affine groups

Definition 7. Let \((M, V, +)\) be an \(n\)-dimensional real affine space. The affine group, denoted by \(\text{Aff}(M)\), is the group of automorphisms of \((M, V, +)\). This means that \(\text{Aff}(M)\) is the subgroup of bijections of \(M\) preserving the simply transitive action \(V\) on \(M\). The word “preserving” means that for each \(H \in \text{Aff}(M)\) there exists a unique \(h \in \text{Aut}(V)\) so that \(H(p + v) = H(p) + h(v)\) for all \(p \in M\) and all \(v \in V\). Here \(\text{Aut}(V)\) is the automorphism group of \(V\), which is \(\text{GL}(V)\) if we consider its structure as vector space or as topological group, i.e., \(\text{GL}(V)\) are the continuous automorphisms of the topological group \(V\).

\[
\text{Aff}(M) := \{ H : M \to M \mid H(p + v) = H(p) + h(v), h \in \text{GL}(V), \forall v \in V \}.
\]

(12)

Note that this definition makes sense, for if \(p' + v' = p + v\), or \((p' - p) + v' = v\), we have \(H(p' + v') = H(p') + h(v') = H(p + (p' - p)) + h(v') = H(p) + h((p' - p) + v') = H(p) + h(v) = H(p + v)\).

Remark 8. We said that \(\text{Aut}(V)\) is \(\text{GL}(V)\) if we consider \(V\) either as vector space or as topological group, comprising all the continuous automorphisms in the latter case. This qualification is indeed necessary, for if we considered \(V\) merely as algebraic group, as it might seem sufficient at this point, \(\text{Aut}(V)\) would indeed be very much larger than \(\text{GL}(V)\) in that it will also contain all the wildly discontinuous automorphisms that \(V\) inherits from the likewise wildly discontinuous automorphisms of the algebraic group \((\mathbb{R}, +)\). The latter are the discontinuous solutions \(f : \mathbb{R} \to \mathbb{R}\) to the so-called Cauchy functional equation, \(f(x + y) = f(x) + f(y)\), which are also bijections. It is elementary to show that all its solutions necessarily satisfy \(f(qr) = qf(r)\) for all \(q \in \mathbb{Q}\) and all \(r \in \mathbb{R}\). This implies that \(f(q) = qf(1)\), i.e., that \(f\) is linear with slope \(c := f(1)\) on all rational numbers, and hence linear with slope \(c\) on all real numbers if \(f\) were required to be continuous (requiring continuity at one point is sufficient). Without requiring continuity we can only conclude that for fixed \(r \in \mathbb{R}\) and all \(q \in \mathbb{Q}\) we must have \(f(rq) = rq(f(r)/r)\), i.e., that \(f\) is again linear on the \(r\)-multiples of the rationals, but now with possibly \(r\)-dependent slope \(c(r) := f(r)/r\). Indeed, plenty of such discontinuous
solutions exist and can be constructed as follows [8]: Consider $\mathbb{R}$ as vector space over $\mathbb{Q}$ and let $B \subset \mathbb{R}$ be a (Hamel) basis, i.e., for each $r \in \mathbb{R}$ there exists a unique finite subset $\{e_1, \ldots, e_n\} \subset B$ and unique $(q_1, \ldots, q_n) \in \mathbb{Q}^n$, such that $r = \sum_{i=1}^n q_i e_i$. As was shown in [8], the existence of such a basis follows from the well-ordering theorem, though the cardinality of $B$ is that of $\mathbb{R}$, i.e., the basis is uncountable. Now, any bijection $f : B \to B$ gives rise to an element of $\text{Aut}(\mathbb{R}, +)$ by uniquely extending $f$ from $B \subset \mathbb{R}$ to $\mathbb{R}$ in a $\mathbb{Q}$-linear fashion, i.e., by setting $f(\sum q_i e_i) := \sum q_i f(e_i)$ for all finite linear combinations of elements in $B$ over $\mathbb{Q}$. Moreover, if the initial permutation $f : B \to B$ is not linear, i.e., if the function $B \ni e \mapsto f(e) / e$ is not constant, the automorphism $f : \mathbb{R} \to \mathbb{R}$ so defined is “wildly” discontinuous, in the sense that its graph $\{(x, f(x)) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$ is dense! In particular, given any $x \in \mathbb{R}$, the image of any intervall containing $x$ under $f$ is dense in $\mathbb{R}$, no matter how small the intervall was chosen to be. To see this, consider $e_1, e_2 \in B$ so that $f(e_1) / e_1 \neq f(e_2) / e_2$. Given any $(x, y) \in \mathbb{R}^2$ we can uniquely solve the two equations $x = r_1 e_1 + r_2 e_2$ and $y = r_1 f(e_1) + r_2 f(e_2)$, i.e., the single linear equation,

$$\begin{pmatrix} e_1 & e_2 \\ f(e_1) & f(e_2) \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

(13)

for $(r_1, r_2) \in \mathbb{R}^2$ by rational operations, since the $2 \times 2$ matrix in (13) is invertible. In particular, $(x, y)$ depends continuously on $(r_1, r_2)$ so that with rational $(q_1, q_2) \in \mathbb{Q}^2$ in a neighbourhood of $(r_1, r_2)$ we get arbitrarily close to $(x, y)$, as was to be proven. All this implies that the usual abelian group structure underlying vector addition cannot be uniquely specified without requiring continuity. Interestingly this problem was first encountered in analytical mechanics in connection with attempts to mathematically characterise the law for the composition of forces [4] and only later recognised as essential for general axiomatic formulations of vector addition; see, e.g., [11]. For us all this means that we cannot avoid invoking a continuity hypothesis and that we must regard the abelian groups whose simply transitive action we require in the definition of affine spaces as topological groups acting continuously on affine space with its natural topology inherited from $\mathbb{R}^n$; compare Remark 5. One might think that one gets away without continuity requirements if one defines $\text{Aff}(M)$ as that subgroup of the group of bijections (no continuity required here) of $M$ which maps straight lines (physically: inertial trajectories) into straight lines (collinear sets of points into collinear sets would also suffice). A classic result in affine geometry then tells us that such transformations necessarily coincide with the standard continuous affine transformations; see, e.g., [1]. However, here a continuity
requirement has tacitly slipped into the notion of “straight line” (inertial trajectory), which in affine space is defined to be the orbit of a continuous one-parameter subgroup of $V$.

Coming back to the group of affine automorphisms as defined above, we see that Hence an element $H \in \text{Aff}(M)$ is uniquely specified by an ordered pair of points $(p, q) \in M \times M$ and an element $h \in \text{GL}(V)$. The second point $q$ is regarded as the image of the first point $p$ under the map in question, whose definition is now given by $H(p+v) := q + h(v)$. Two such maps, $H$ and $H'$, characterised by $(p, q, h)$ and $(p', q', h')$, respectively, are easily seen to be the same iff $h = h'$ and $q' - q = h(q' - p)$. This defines an equivalence relation on the set $M \times M \times \text{GL}(V)$, the equivalence classes of which are

$$[p, q, h] = \bigcup_{v \in V} \left(p + v, q + h(v), h\right).$$

Hence we may identify $\text{Aff}(M)$ with this quotient space and write $H = [p, q, h]$ for any $H \in \text{Aff}(M)$. The composition of two maps $H = [p, q, h]$ and $H' = [p', q', h']$ can then be calculated

$$H' \circ H(p+v) = H'(q + h(v)) = H'(p' + (q - p') + h(v)) = q' + h'(q - p') + h' \circ h(v).$$

In other words

$$[p', q', h'] \circ [p, q, h] = [p, q' + h'(q - p'), h' \circ h].$$

The first thing to note is that the equivalence class on the right-hand side is unchanged if we replace $(p, q, h)$ with $(p + v, q + h(v), h)$ or $(p', q', h')$ with $(p' + v', q + h'(v'), h')$, which means that this prescription written down in terms of representatives defines indeed a multiplication of equivalence classes. Note that the neutral element is $[p, p, \text{id}_V]$ and the inverse of $[p, q, h]$ is

$$[p, q, h]^{-1} = [p, p - h^{-1}(q - p), h^{-1}].$$

Furthermore, it is easy to check that (16) is associative and hence defines a group multiplication.

An obvious subgroup in $\text{Aff}(M)$ is given by the following subset

$$\text{Trans}(M) := \{[p, q, h] \in \text{Aff}(M) \mid h = \text{id}_V\}.$$
This subgroup is abelian,
\[
[p', q', \text{id}_V] \circ [p, q, \text{id}_V] = \left[ p + (q - p'), \text{id}_V \right] = \left[ p' + (p - p'), q' + (q - p) \right] = \left[ p', q + (q - p) \right],
\]
(19)
(\text{using (7) and (8) at the fourth and fifth equality}) and normal,
\[
[p, q, h] \circ [p', q', \text{id}_V] \circ [p, q, h]^{-1} = \left[ p + h(q' - p'), (p - q) \right],
\]
(20)
It is called the subgroup of translations. It is the kernel of the projection homomorphism
\[
\pi : \text{Aff}(M) \to \text{GL}(V), \quad [p, q, h] \mapsto \pi([p, q, h]) := h.
\]
(21)
If we denote the embedding (injective homomorphism) of Trans(M) into Aff(M) by \( i \), we have the short sequence of groups and maps
\[
\{1\} \longrightarrow \text{Trans}(M) \xrightarrow{i} \text{Aff}(M) \xrightarrow{\pi} \text{GL}(V) \longrightarrow \{1\}
\]
(22)
Here \( \{1\} \) stands for the trivial group with unique group homomorphisms from and to any other group. The tailed and double-headed arrows indicate injective and surjective homomorphisms respectively. This may be briefly summarised by saying that the short sequence is exact, where exactness means that at each group the image of the arriving map is the kernel of the departing one.

Moreover, our sequence (22) is not only exact but it also splits. By this is meant that there are also group embeddings (injective homomorphisms) \( j : \text{GL}(V) \rightarrow \text{Aff}(M) \) so that \( \pi \circ j = \text{id}_{\text{GL}(V)} \). To see this, choose a point \( o \in M \) and define (indicating the dependence of \( j \) on \( o \) by a subscript)
\[
j_o : \text{GL}(V) \rightarrow \text{Aff}(M), \quad h \mapsto j_o(h) := [o, o, h].
\]
(23)
Since $[o, o, h'] \circ [o, o, h] = [o, o + h'(o - o), h'h] = [o, o, h'h]$ one has indeed $i_o(h')i_o(h) = i_o(h'h)$ and $i_o(\text{id}_{\text{GL}(V)}) = \text{id}_{\text{Aff}(M)}$, that is, $i_o$ is a group homomorphism. But note that we needed to select a point $o \in M$ to define the embedding. Two embeddings corresponding to different choices $o$ and $o'$ are related by conjugation with the translation from $o$ to $o'$. Indeed, using that according to (17) we have $[o, o', \text{id}_V]^{-1} = [o, o - (o' - o), \text{id}_V] = [o', o, \text{id}_V]$, we have for all $h \in \text{GL}(V)$

$$[o, o', \text{id}_V] \circ i_o(h) \circ [o, o', \text{id}_V]^{-1} = [o, o', \text{id}_V] \circ [o, o, h] \circ [o', o, \text{id}_V]$$
$$= [o, o', \text{id}_V] \circ [o', o, h]$$
$$= [o', o', h]$$
$$= i_{o'}(h).$$

The relation between the three groups Trans(M), Aff(M), and GL(V) can then be compactly expressed by completing the short exact sequence (22) by a splitting homomorphism $j_o$:

$$\{1\} \longrightarrow \text{Trans}(M) \xrightarrow{i} \text{Aff}(M) \xrightarrow{\pi} \text{GL}(V) \longrightarrow \{1\}$$

(25)

This characterisation in terms of a split exact-sequence is the most natural in view of the homogeneity of $M$. The usual characterisation by means of a semi-direct product $V \rtimes \text{GL}(V)$ is unnatural insofar as the GL(V) subgroup in Aff(M) depends on the choice of a point $o \in M$, violating homogeneity. What one may say is that Aff(M) is isomorphic to $V \rtimes \text{GL}(V)$, but the isomorphism depends on the selection of a point. Only after the point is selected can we locate a linear subgroup in Aff(M) isomorphic to GL(V), namely the image of GL(V) under the embedding $j_o$ (24). Once one agrees to select a point $o \in M$, we may write the general element of Aff(M) in the form $[o, q, h]$. Group multiplication according to (16) then becomes

$$[o, q', h'] \circ [o, q, h] = [o, q' + h'(q - o), h' \circ h] = [o, o + (q' - o) + h'(q - o), h' \circ h].$$

(26)

Having selected $o$ we may identify $M$ with $V$ via $p \mapsto p - o$ (sometimes called the “vectorialisation” of $M$ at $o$) and the group Aff(M) with the set $V \times \text{GL}(V)$. A general group element may then be written $[o, o+v, h] \mapsto (v, h)$ and (26) becomes

$$(v', h') \circ (v, h) = (v' + h'(v), h' \circ h),$$

(27)

which is just the product structure of a semi-direct product $V \times \text{GL}(V)$ with respect to the homomorphism $\text{GL}(V) \rightarrow \text{Aut}(V)$ that is given by the defining representation of GL(V).
Remark 9. The proper statement regarding the structure of the affine group Aff(M) is that it is a downward splitting extension of GL(V) by Trans(M), as summarised by (25). To be a downward extension means that Trans(M) is a normal (or “invariant”) subgroup of Aff(M) so that the quotient Aff(M)/Trans(M) is isomorphic to GL(V). To be “splitting” means that GL(V) may be identified with a subgroup in Aff(M) whose intersection with Trans(M) is merely the group identity. In our case there exist many such splitting embeddings of GL(V) into Aff(M), so that there is no unique way to regard GL(V) as subgroup of Aff(M). The ambiguity is faithfully labelled by the points in M (the point that is fixed under the action of the embedded copy of GL(V) in Aff(M) on M). Given such a splitting, Aff(M) becomes isomorphic to the corresponding semi-direct product V ⋊ GL(V). But this isomorphism depends on the choice of a point in M. If one says that Aff(M) is isomorphic to V ⋊ GL(V) one should add that this isomorphism is not “natural”, since by the very homogeneity of M there is clearly no preferred choice of a point in M.

2.4 Poincaré group

Definition 10. Given Definition 2 of Minkowski space, we define the Poincaré group, Poin(M), to be its group of automorphisms. This means that it must consists of affine transformations including all elements in Trans(M), such that

\[ \text{Aff}(M)/\text{Trans}(M) = \text{Lor}(V) \subset \text{GL}(V), \]

where

\[ \text{Lor}(V) := \left\{ h \in \text{GL}(V) \mid \eta(hv, hw) = \eta(v, w), \forall v, w \in V \right\}. \]  (28)

Hence we have

\[ \text{Poin}(M) := \left\{ H : M \to M \mid H(p + v) = H(p) + h(v), h \in \text{Lor}(V), \forall v \in V \right\}. \]  (29)

\[ \square \]

1 Here we recall that the usual terminology regarding extensions of groups is not quite uniform and hence ambiguous. Suppose three groups H, E and G are related by an exact sequence

\[ 1 \to H \to E \to G \to 1, \]

i.e. that H is a normal (or “invariant”) subgroup of E with quotient E/H isomorphic to G. Then this state of affairs is usually simply expressed by either saying that E is “an extension” of G by H, or of H by G. This ambiguity arises because views differ as to whether one likes to regard the extending or the extended group to be that one which becomes normal in the extension. To avoid such ambiguities the following refined terminology has been proposed in [3]: E is called an upward extension of H by G, or a downward extension of G by H.
Totally analogous to (25), this leads to the splitting exact sequence

\[
\{1\} \longrightarrow \text{Trans}(M) \xrightarrow{i} \text{Poin}(M) \xrightarrow{\pi} \text{Lor}(V) \longrightarrow \{1\}
\]  

(30)

and to the \( o \in M \) dependent(!) isomorphism

\[
\text{Poin}(M) \cong V \rtimes \text{Lor}(V).
\]

(31)

If we complete \( o \) to a full affine frame \( F = (o, f) \), where \( f \in \text{Lin}(\mathbb{R}^n, V) \), and if in addition we require \( f \) to map the standard basis of \( \mathbb{R}^n \) to the orthonormal basis of \( V \) with respect to \( \eta \), i.e., \( \eta_{ab} := \eta(e_a, e_b) = \pm \delta_{ab} \) with one plus and \( n - 1 \) minus signs), we may identify \( M \) with \( \mathbb{R}^n \), and then have

\[
\text{Poin}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \text{Lor}(\mathbb{R}^n).
\]

(32)

where

\[
\text{Lor}(\mathbb{R}^n) := \left\{ L \in \text{GL}(\mathbb{R}^n) \mid \eta_{ab} L^a_c L^b_d = \eta_{cd} \right\}.
\]

(33)

Now, \( \text{Poin}(M) \) is a Lie group. The structure of a differentiable manifold with respect to which all group operations become smooth are again obtained by its isomorphism (non-naturalness is irrelevant here) with the matrix group just described. Note that the semi-direct product (32) can itself be embedded (i.e. mapped by an injective homomorphism) into the group \( \text{GL}(\mathbb{R}^{n+1}) \), via

\[
\mathbb{R}^n \rtimes \text{Lor}(\mathbb{R}^n) \ni (v, L) \mapsto \begin{pmatrix} 1 & 0 \\ v & L \end{pmatrix} \in \text{GL}(\mathbb{R}^{n+1})
\]

(34)

which endows it with the differentiable structure inherited from \( \text{GL}(\mathbb{R}^{n+1}) \). All this is using the preferred affine (or inertial) coordinates of \( M \); compare Definition[6]. Note also that the group multiplication in \( \text{Aff}(M) \) has been explained simply by composition of maps (\( \text{Aff}(M) \) was defined to consists of special bijections of \( M \)). This defines a left action of \( \text{Aff}(M) \) on \( M \) and hence, by simple restriction, a left action of \( \text{Poin}(M) \) on \( M \). This, in turn, defines an anti-homomorphism between the Lie-algebra of \( \text{Poin}(M) \) and the Lie algebra of vector fields on \( M \) (considered as differentiable manifold), where the Lie-algebra structure of the latter is defined by the commutator of vector fields. The reason why we have an anti- rather than a proper homomorphism of Lie algebras is explained in detail in Appendix[3], in which we also review in some detail the notion of Lie-group actions on manifolds.
We recall that the Lie algebra, \( \text{lor}(V) \), of \( \text{Lor}(V) \) is the linear space of endomorphisms \( A \in \text{End}(V) \) which are antisymmetric with respect to the Minkowski inner product \( \eta \), i.e., satisfy \( \eta(Av, w) = -\eta(v, Aw) \) for all \( v, w \in V \). Using the \( \eta \)-induced isomorphism \( \eta : V \to V^* \) and its inverse \( \eta^{-1} \) (compare Remark 3), we can then write down the projection operators \( P_S, P_A : \text{End}(V) \to \text{End}(V) \), which project onto the \( \eta \)-symmetric and \( \eta \)-antisymmetric endomorphisms:

\[
P_S(M) := \frac{1}{2} (M + \eta \circ M^\top \circ \eta), \quad (35a)
\]
\[
P_A(M) := \frac{1}{2} (M - \eta \circ M^\top \circ \eta). \quad (35b)
\]

Hence \( \text{lor}(V) \) can be either characterised as the kernel of \( P_S \) or the image of \( P_A \) in \( \text{End}(V) \). Using the first option we may write

\[
\text{lor}(V) = \text{Ker}(P_S) = \{ A \in \text{End}(V) \mid A = -\eta \circ A^\top \circ \eta \}. \quad (36)
\]

Using the point-dependent isomorphism (31), the Lie algebra of \( \text{Poin}(M) \), denoted by \( \text{poin}(M) \), is the semi-direct product of the Lie algebras \( V \) and \( \text{lor}(V) \). Note that \( V \), considered as abelian group, has a Lie algebra which is isomorphic (as vector space) to \( V \) with trivial Lie product (i.e. all Lie products are zero). Then we get the, likewise point-dependent, isomorphism

\[
\text{poin}(M) \cong V \rtimes \text{lor}(V) = \left\{ (v, M) \in V \times \text{lor}(V) \mid [(v, M), (w, N)] = (Mw - Nv, [M, N]) \right\}. \quad (37)
\]

Here \( Mw \) is the action of \( M \in \text{End}(V) \) on \( w \in V \) and \( [M, N] \) is the commutator, which turns \( \text{End}(V) \), considered as associative algebra, into a Lie algebra. An easy way to see that (37) does indeed give the right Lie product is to use the embedding (34), which induces an embedding

\[
\mathbb{R}^n \rtimes \text{lor}(\mathbb{R}^n) \ni (v, M) \mapsto \begin{pmatrix} 0 & 0 \\ v & M \end{pmatrix} \in \text{End}(\mathbb{R}^{n+1}). \quad (38)
\]

The Lie product of the images of \( (v, M) \) and \( (w, N) \) is then just their commutator, which is immediately seen to be the image of \( (Mw - Nv, [M, N]) \).

Now, as already mentioned above, the left action

\[
\Phi : \text{Poin}(M) \times M \to M, \quad (g, m) \mapsto \Phi(g, m) \equiv \Phi_g(m) \quad (39)
\]
of \( \text{Poin}(M) \) on \( M \) induces a linear map from \( \text{poin}(M) \) to the linear space of vector fields on \( M \), denoted by \( \text{Vec}(M) \) (smooth sections in \( TM \)). This
map is just the differential of \( \Phi \) with respect to the first (group valued) argument evaluated at the group identity. This is explained in all detail in Appendix B; compare (154). Since \( \text{Vec}(M) \) is itself a Lie algebra, where the Lie product is defined to be the commutator of vector fields. With respect to these two Lie structures, the linear map \( \text{poin}(M) \ni X \mapsto V^X \in \text{Vec}(M) \) is a Lie anti-homomorphism. Again we refer to the Appendix B for details; compare (172b). Hence we have

\[
[V^X, V^Y] = -V^{[X,Y]},
\]

where the “anti” is reflected by the minus-sign on the right-hand side.

Moreover, as the left action of \( \text{Poin}(M) \) on \( M \) lifts by push-forward (differential of \( \Phi \) with respect to second (\( M \)-valued) argument) to a left action on \( TM \) and hence \( \text{Vec}(M) \), we can ask for the result of acting with \( g \in \text{Poin}(M) \) on the special vector field \( V^X \). The result is (see equation (173a) of Appendix B)

\[
\Phi_g^* V^X = V^{\text{Ad}_g(X)} \circ \Phi_g.
\]

where \( \text{Ad} \) denotes the adjoint representation of \( \text{Poin}(M) \) on \( \text{poin}(M) \).

Let us at this point say a few words about the adjoint and co-adjoint representation; the latter will become important in what is to follow. An easy way to calculate the adjoint representation is again to identify \( \text{Poin}(M) \) and \( \text{poin}(M) \) according to (32) and (37), respectively, and perform the easy conjugation-calculation using the embeddings (34) and (38). The result is

\[
\text{Ad}_{(a,L)}(v,M) = (L v - L M L^{-1} v, L M L^{-1}).
\]

The co-adjoint representation is the usual representation induced by \( \text{Ad} \) on the dual space, that is, the inverse transposed. As a vector space, \( \text{poin}(M) \) is isomorphic to a linear subspace of \( V \oplus \text{End}(V) \), namely the image of \( \text{id}_V \oplus \mathcal{P}_A \). Note that \( V \oplus \text{End}(V) \) may be identified with \( V \oplus (V \otimes V^*) \). The dual of the vector space \( \text{poin}(M) \) is then isomorphic to a subspace of the dual to \( V \oplus (V \otimes V^*) \), i.e., a subspace of \( V^* \oplus (V^* \otimes V) \). This subspace is the image of \( \text{id}_{V^*} \oplus \mathcal{P}_A^T \). It is called the dual of the Lie algebra \( \text{poin}(M) \), denoted by \( \text{poin}^*(M) \). It is merely considered as a vector space, not a Lie algebra. The natural paring between \((p,J) \in \text{poin}^*(M) \) and \((v,M) \in \text{poin}(M) \) is

\[
[(p,J)](v,M) = p(v) + \frac{1}{2} \text{Tr}(J^T \circ M).
\]

The factor \( 1/2 \) in the second term is introduced because \( M \) obeys the condition \( \mathcal{P}_S(M) = 0 \) and each independent component of \( M \) contributes twice to the trace. We have, by definition of the transposed map, \( \text{Tr}(J^T \circ \mathcal{P}_A M) =: \)
\[
\text{Tr}((P_J^T J)^T \circ M) \quad \text{and likewise for } P_S^T, \text{ which immediately leads to the expressions}
\]

\[
P_S^T (J) := \frac{1}{2} (J + \eta \circ J^T \circ \eta), \quad (44a)
\]

\[
P_J^T (J) := \frac{1}{2} (J - \eta \circ J^T \circ \eta). \quad (44b)
\]

Hence we may characterise \( \text{lor}^*(V) \) by:

\[
\text{lor}^*(V) = \{ J \in \text{End}(V^*) | J = -\eta \circ J^T \circ \eta \}, \quad (45)
\]

and furthermore (as vector spaces)

\[
\text{poin}^*(M) \cong V^* \times \text{lor}^*(V). \quad (46)
\]

As already said, the co-adjoint representation, \( \text{Ad}^* \) of \( \text{Poin}(M) \) on \( \text{poin}^*(M) \) is defined to be the inverse-transposed:

\[
\text{Ad}^* : \text{Poin}(M) \times \text{poin}^*(M) \to \text{poin}^*(M)
\]

\[
((a, L), (p, J)) \mapsto \text{Ad}^*_{(a, L)}(p, J) := (p, J) \circ \text{Ad}^{-1}_{(a, L)}. \quad (47)
\]

Note that the inverse is necessary to get a left action, i.e., \( \text{Ad}^*_{(a, L)} \circ \text{Ad}^*_{(a', L')} = \text{Ad}^*_{(a, L')(a', L')} \). Using

\[
\text{Ad}^{-1}_{(a, L)}(v, M) = (L^{-1}v + L^{-1}Ma, L^{-1}ML), \quad (48)
\]

a straightforward calculation gives, writing \( \tilde{L} := (L^T)^{-1} \) and using the identity \( p(w) = \text{Tr}(w \otimes p) \), valid for any \( w \in V \) and \( p \in V^* \),

\[
[\text{Ad}^*_{(a, L)}(p, J)](v, M) = \tilde{L}p(v) + \frac{1}{2} \text{Tr} \left( [2 \tilde{L}p \otimes a + \tilde{L}J\tilde{L}^{-1}]^T M \right). \quad (49)
\]

This implies

\[
\text{Ad}^*_{(a, L)}(p, J) = (\tilde{L}p, \tilde{L}J\tilde{L}^{-1} + P_A^T (2 \tilde{L}p \otimes a) \right.
\]

\[
= (\tilde{L}p, \tilde{L}J\tilde{L}^{-1} + \tilde{L}p \otimes a - a^b \otimes (\tilde{L}p)^2). \quad (50)
\]

For what follows it is important to compare the adjoint representation (42) of \( \text{Poin}(M) \) on \( \text{poin}(M) \) with the co-adjoint representation (50) of the same group on \( \text{poin}^*(M) \). This is not quite straightforward since the representation spaces are different and hence it is not entirely obvious how to best appreciate their difference. However, it is true that, as vector spaces, \( \text{poin}(M) \) and \( \text{poin}^*(M) \) are isomorphic, though not naturally so. We need
an extra structure to select a specific isomorphism, which in our case is already given to us by the inner product \( \eta \), which was already seen to give an isomorphism \( \eta : V \to V^* \); compare Remark 3. This structure can clearly also be used to define an isomorphisms \( \text{poin}(M) \to \text{poin}^*(M) \). However, it is more convenient to define isomorphisms between each of these vector spaces and \( V \oplus \wedge^2 V \), where \( \wedge^2 V := V \wedge V \) is the antisymmetric tensor product:

\[
\text{poin}(M) \cong V \oplus P_A(V \otimes V^*) \cong V \oplus (V \wedge V) \cong V^* \oplus P_A^*(V^* \otimes V) \cong \text{poin}^*(M).
\]

Indeed, note that under this isomorphism \( \text{Isr}(V) \) gets mapped isomorphically onto the antisymmetric subspace \( \wedge^2 V \subset V \otimes V \). The corresponding representations on \( V \oplus \wedge^2 V \), which are equivalent to \( \text{Ad} \) and \( \text{Ad}^* \) under these isomorphisms, are respectively given by

\[
\text{Ad}_{(a,L)}(v, M) = (Lv - (L \otimes LM) \cdot a, L \otimes LM),
\]

\[
\text{Ad}_{(a,L)}^*(p, J) = (Lp, L \otimes LJ - a \wedge Lp),
\]

where the dot now abbreviates the inner product \( \eta \) in \( V \), as explained in Remark 3. So for \( a, b, c \in V \) we write

\[
(a \wedge b) \cdot c = (a \otimes b - b \otimes a) \cdot c = a \eta(b, c) - b \eta(a, c) =: a(b \cdot c) - b(a \cdot c).
\]

These are now two inequivalent representations of the same group on the same vector space. It is the second, co-adjoint representation that will be physically relevant. It differs from the adjoint representation on how it implements the normal subgroup of translations. Let us, for clarity, just display the two representations if restricted to the subgroup \( \text{Trans}(M) \):

\[
\text{Ad}_{(a, id)}(v, M) = (v - M \cdot a, M),
\]

\[
\text{Ad}_{(a, id)}^*(p, J) = (p, J - a \wedge p).
\]

The obvious difference is that under the adjoint representation translations act non-trivially only on the first summand in \( V \oplus (V \wedge V) \) under the adjoint- , and non-trivially only on the second summand under the co-adjoint representation. As we will see below, the latter corresponds to the familiar origin-dependence of angular momentum and origin-independence of linear momentum.

Finally, using the identification \( \text{LiePoin}(M) \cong V \rtimes (V \wedge V) \), let us explicitly write down the Lie algebra in terms of a basis. Let \( \{ e_a \mid 1 \leq a \leq n \} \) be a basis of \( V \), such that \( \eta(e_a, e_b) = e_a \cdot e_b = \eta_{ab} \) then \( \{ m_{ab} \mid 1 \leq a < b \leq n \} \)
is a basis of $V \wedge V$, where $m_{ab} := e_a \wedge e_b = (e_a \otimes e_b - e_b \otimes e_a)$. Then the Lie products in (37) become

\begin{align}
[e_a, e_b] &= 0, \\
[e_a, m_{bc}] &= \eta_{ab} e_c - \eta_{ac} e_b, \\
[m_{ab}, m_{cd}] &= \eta_{ad} m_{bc} + \eta_{bc} m_{ad} - \eta_{ac} m_{bd} - \eta_{bd} m_{ac}.
\end{align}

**3 The momentum map and the natural habitat of globally conserved Poincaré charges**

We now regard Minkowski space as a Semi-Riemannian manifold $(M, g)$ with Lorentzian metric $g$, which in affine/inertial coordinates (compare Definition 6) is of the form (in four spacetime-dimensions)

\[ g = \eta_{ab} \, dx^a \otimes dx^b, \quad \{\eta_{ab}\} = \text{diag}(1, -1, -1, -1). \]

As discussed above, and in more detail in the Appendix B for each $X \in \text{poin}(M)$ we have a vector field $V^X \in \text{Vec}(M)$ that represents the “infinitesimal” left group-action of Poin(M) on M through an anti Lie-homomorphism $\text{poin}(M) \to \text{Vec}(M), \, X \mapsto V^X$, satisfying (40). Since Poin(M) acts on M by isometries, the Lie derivative of $g$ with respect to each $V^X$ is zero:

\[ L_{V^X} g = 0. \]

In other words, each $V^X \in \text{Vec}(M)$ is a Killing vector-field.

Now, suppose we have an energy-momentum tensor

\[ T = T_{ab} \, dx^a \otimes dx^b \]

which is divergence free with respect to the Levi-Civita covariant derivative determined by $g$. In components with respect to arbitrary coordinate systems this reads

\[ \nabla_a T^{ab} = \partial_a T^{ab} + \Gamma^a_{ac} T^{cb} + \Gamma^b_{ac} T^{ac} = 0. \]

If the coordinates are affine/inertial, the $\Gamma$-coefficients are all zero.

Another way to look at $T$ is to regard it as a co-vector valued 3-form. This is achieved by Hodge dualising the second tensor factor in (58):

\[ \mathcal{T} = T_{ab} \, dx^a \otimes (\ast dx^b), \]

22
where $\star$ is the Hodge duality map the definition of which, together with our conventions, are summarised in Appendix A. Now comes the important point in the whole construction: using the vector fields $V^X$, we can, for each $X \in \text{poin}(M)$ turn (60) into a 3-form that linearly depends on $X$ via

$$T_X := \star i_{V^X} = (V^X)^a T_{ab} (\star dx^b) = (V^X)^a g_{ab} T_{bc} \frac{1}{3!} \varepsilon_{cdef} dx^d \wedge dx^e \wedge dx^f.$$  

(61)

where $i_V$ denotes the map of inserting $V$ into the first co-vector factor of the tensor it is applied to and $\varepsilon_{abcd}$ are the components of the measure 4-form induced by $g$. The zero-divergence condition (59) implies, in view of (57), that each $T_X$ is closed:

$$dT_X = 0.$$  

(62)

This means that to each $X \in \text{poin}(M)$ we can produce a number by integrating $T_X$ over a 3-dimensional hypersurface:

$$\mathcal{M}[F,S](X) := \int_S T_X[F] = \int_S \star \circ i_{V^X} \circ T[F].$$  

(63)

Here we wrote the integrand as a composition of three maps. The first ($T$) maps the field configuration $F$ to a symmetric tensor, the second $i_{V^X}$ contracts this tensor with the vector field $V^X$ and turns it into a one form, and the last ($\star$) turns this one form into an $n-1$ form (a three-form in four dimensions). The last map to be applied in order to get a number is to integrate this form over a hypersurface $S$. This number will depend on three arguments: The fields $F$ on which $T$ depends, the surface $S$ over which we integrate, and the Lie algebra element $X$ which we use to build $V^X$ to contract $T$ with. The value $\mathcal{M}$ takes on all these arguments is called the corresponding momentum.

Suppose now that the fields $F$ on which $T$ depends carry a representation (not necessarily a linear one) of Poin(M). That is, we assume there is a left action $D$ of Poin(M) on the space (not necessarily a vector space) of fields. We assume that the geometric object $T$ is built entirely out of such fields, and that there is no dependence on any other geometric structure not included in our $F$. Then we have the covariance property

$$T[D_h F] = \Phi_{h*} T[F]$$  

(64)

This covariance property, which is crucial for the right representation-theoretic properties of the global charges, is hardly ever stated explicitly. A notable exception, more in words than in formulae, is Fock’s book [7] §31, where it is referred to as “physical principle”. 

2
where $\Phi$ is as in (39) and $\Phi_h^*$ denotes the push-forward of the diffeomorphism $\Phi_h : M \to M$. If $F$ denote standard scalar, vector, and tensor fields, then (64) merely says that the energy-momentum distribution of the pushed-forward fields is just the push-forward of the energy-momentum distribution of the original fields.

Now we are interested in how the momentum changes if we act on the fields $F$ by a Poincaré transformation, leaving the arguments $S, X$ untouched for the moment. We get:

$$\mathfrak{M} [D_h(F), S](X) = \int_S \ast \circ i_{V,X} \circ T \circ D_h \ [F]$$

$$= \frac{1}{S} \int \ast \circ i_{V,X} \circ \Phi_h^* \circ T \ [F]$$

$$= \int_S \ast \circ \Phi_h^* \circ i_{\Phi_h^{-1}V,X} \circ T \ [F]$$

$$= \int_S \ast \circ \Phi_h^* \circ i_{\Phi_h^{-1}V,X} \circ T \ [F]$$

$$= \int_S \Phi_h^{-1}(S) \left( \ast \circ i_{\Phi_h^{-1}V,X} \circ T \ [F] \right)$$

$$= \mathfrak{M} [F, \Phi_h^{-1}(S)](\text{Ad}^{-1}_h(X))$$

$$= \text{Ad}_h^* (\mathfrak{M}) [F, \Phi_h^{-1}(S)](X)$$

Here we broke up the derivation into seven steps, each one showing what happens as we commute the action of $\text{Poin}(M)$ from right to left through the various maps connected by the $\circ$ symbols. At the first step we use (64), at the second step we just use the obvious commutation property of push-forwards with the vector-insertion map, at the third step we use property (41), at the fourth step we use the covariance (intertwining property) of the Hodge map and the definition of the push-forward of a form as the pull-back by the inverse map, in the fifth step we use the elementary property of integrals, sometimes referred to as the “change-of-variables-formula”, in the sixth step we just use the definition (63), and in the seventh and last step we use the definition (47) of the co-adjoint representation.

Now, if $S$ is a Cauchy surface and the support conditions discussed initially are satisfies, we are ensured that the integral converges and the momentum actually exists. Moreover, if $T$ is divergence free, as we assume
here, the momentum does not depend on the particular Cauchy surface chosen, as follows from (62) and Gauss’ theorem. Hence we may delete $S$ as an argument of $\mathfrak{M}$. Since equation (65) is valid for all $X \in \text{poin}(M)$, we may also delete the dependence on $X$, which is linear. We can then and regard (65) as an equation between elements in the dual of the Lie algebra depending merely on $F$ and expressing the fact that they transform under the co-adjoint representation.

**Theorem 11.** A divergence-free energy-momentum tensor describing a body in the sense of Definition 1 and depending on fields which carry a (not necessarily linear) representation $D$ of $\text{Poin}(M)$ defines a map from the space of field configurations to $\text{poin}^*(M)$, called *momentum map*, given by

$$
\mathfrak{M}(X) := \int_S \mathcal{T}_X[F],
$$

(66)

where $S$ is any Cauchy surface. The map is Ad*-equivariant in the sense that

$$
\mathfrak{M} \circ D_h = \text{Ad}^*_h \circ \mathfrak{M},
$$

(67)

for all $h \in \text{Poin}(M)$.

Let us finally see how, and in what sense, the general formula (66) implies the naive expressions (2). For this we express $V^X$ in affine/inertial coordinates and choose for $X$ basis elements of $\text{poin}(M)$ that are adapted to the decomposition of $\text{poin}(M)$ as semi-direct product $V \rtimes \text{lor}(V)$. But here comes the point stressed above: there is no natural identification of $V \rtimes \text{lor}(V)$ with $\text{poin}(M)$. Any such identification is equivalent to the choice of a point $o \in M$. Only with respect to the choice of such a point does it make sense to speak of Lor(V) as a subgroup of Poin(M) and of $\text{lor}(V)$ as a Lie subalgebra of $\text{poin}(M)$.

Let us now choose a system $x^a$ of affine/inertial coordinates so that the vector fields $\partial/\partial x^a$ are orthonormal (i.e. the Minkowski metric $g$ takes the standard form (56)). The coordinate values of the preferred point $o$ is denoted by $z^a := x^a(o)$. Then $V^X$ for $X = (v, M) \in V \oplus (V \otimes V)$ is

$$
V^{(v, M)}(z) = v^a \partial/\partial x^a + \frac{1}{2} M^{ac} \eta_{cb} [(x^a - z^a) \partial/\partial x^b - (x^b - z^b) \partial/\partial x^a].
$$

(68)

Note that $x^a : M \to \mathbb{R}$ are coordinate functions on the manifold whereas $z^a = x^a(o)$ are fixed numbers (constant functions on $M$). The corresponding momentum is then

$$
\mathfrak{M} \left[ X = (v, M) \right] = \eta_{ab} v^a P^b + \frac{1}{2} \eta_{ac} \eta_{bd} M^{ab} J^{cd}[z]
$$

(69)
where, just as in (2),

\[ P^a = \int_S T^a_b \, u^b \, d\mu, \quad (70a) \]

\[ J^{ab}[z] = \int_S [(x^a - z^a)T^b_c - (x^b - z^b)T^a_c] \, u^c \, d\mu. \quad (70b) \]

Here \( u \) is the unit timelike normal to \( S \) and \( d\mu = \ast u^\flat \) (the Hodge dual of the one-form \( u^\flat := \eta(\nabla) \)) is the induced measure (3-form) on \( S \). Note that only the \( J \)'s depend on \( z \) because only they refer to the non-natural (i.e. \( o \)-dependent) embedding of the Lorentz group into the Poincaré group. In contrast, the translation group \( \text{Trans}(M) \) is normal and hence has a natural place in the Poincaré group. Correspondingly, the linear momenta \( P^a \) are natural and do not depend on any arbitrary choices. Note that it immediately follows from (70b) that

\[ J[z + a] = J[z] = a \wedge P \quad (71) \]

which is just the co-adjoint representation of translations stated in (54b).

**Remark 12.** The discussion up to this point answers all the questions posed initially in connection with (70) in the case of Special Relativity. Globally conserved quantities (charges) in connection with Poincaré symmetry are valued in the vector space dual to the Lie algebra and transform according to the co-adjoint representation under Poincaré transformations of the fields to which these quantities belong. The splitting of the space in which the charges take their values into a “translational part” and a “homogeneous part” is not natural as far as the latter is concerned. Therefore the charges of the homogeneous (Lorentz-) part has an additional dependence on a spacetime point whose choice fixes the embedding of the Lorentz group into the Poincaré group. The very notion of, say, angular momentum depends on the choice of this point.
4 Supplementary conditions and mass centres

The $z$ dependence of $J$ may be used to put further more or less physically motivated conditions on $J[z]$ to restrict the choices of $z$. Conditions of that sort are known as supplementary conditions whose aim is to narrow down the choices of $z$ to a one-parameter family $z(\lambda)$ which is timelike and somehow interpreted as the worldline of the body. This line has many names depending on what supplementary conditions one uses. It can be “centre-of-mass”, “centre-of-inertia”, “centre-of-gravity”, “centre-of-spin”, “centre-of-motion”, “centroid”, etc. Early discussions of some of these concepts in Special Relativity were given in [7] and [2]. For comprehensive discussions see [10] and in particular [6].

If $u \in V_1 := \{ v \in V \mid \eta(u, u) = 1 \}$ is a unit timelike vector characterising an inertial frame of reference, we may, e.g., consider the supplementary condition (recall that a dot indicates a contraction using the Minkowski metric)

$$J[z + a] \cdot u = 0 \Leftrightarrow J[z] \cdot u - (P \cdot u) a + (a \cdot u) P = 0. \quad (72)$$

This is equivalent to a linear inhomogeneous equation for $a$

$$\Pi(a) = \frac{J[z] \cdot u}{P \cdot u}, \quad (73a)$$

where

$$\Pi = \text{id} - \frac{P \otimes u^\flat}{P \cdot u} \quad (73b)$$

is the projector onto $u^\perp := \{ v \in V \mid v \cdot u = 0 \}$ parallel to $P$ (caution: not parallel to $u$). Hence the solution space is one-dimensional timelike line in $V$ parallel to $P$:

$$a(z, u; \lambda) = \frac{J[z] \cdot u}{P \cdot u} + \lambda P, \quad \lambda \in \mathbb{R}. \quad (74)$$

Its dependence on $z$ immediately follows from (74) and (71):

$$a(z + b, u; \lambda) = a(z, u; \lambda + (u \cdot b)/(u \cdot P)) - b. \quad (75)$$

Equation (74) is a timelike line in $V$ that represents the worldline of the centre-of-mass in $M$ relative to the origin $z$. The wordline in $M$ clearly does not depend on $z$ (up to reparametrisation) and is simply given by

$$\gamma(u; \lambda) = z + a(z, u; \lambda). \quad (76)$$
Definition 13. The curve $\lambda \mapsto \gamma(u; \lambda)$ is called the *centre-of-mass wordline* relative to the inertial observer $u$.

The body’s angular momentum with respect to this centre-of-mass is

$$S(u) := J[\gamma(u; \lambda)] := J[z + a(z, u; \lambda)].$$

(77)

The right-hand side clearly does not depend on $\lambda$ since shifting $\lambda$ moves $a(z, u; \lambda)$ in the direction of $P$ according to (74) and hence leaves $J$ unchanged according to (71). It then also follows immediately from (75) that the right-hand side of (77) does not depend on $z$. Hence, as indicated, $S$ only depends on $u$.

Definition 14. $S(u)$ is called the body’s *spin* with respect to the inertial observer $u$.

Except for its dependence on $u$, this definition meets standard Newtonian intuition. Indeed, according to this intuition we would call

$$L(z, u) := a(z, u; \lambda) \wedge P$$

(78)

the orbital angular momentum relative to $z$ and $u$ (there is again no $\lambda$-dependence due to $P \wedge P = 0$). Equation (71) then just tells us that the total angular momentum is the sum of the spin and orbital parts:

$$J[z] = L[z + a(z, u; \lambda)] + a(z, u; \lambda) \wedge P = S(u) + L(z, u).$$

(79)

As in Newtonian mechanics, the $z$-dependence of angular momentum resides exclusively in the orbital part. But $S$ and $L$ each also depend on $u$, though in such a way that their sum is independent of $u$. This gives rise to the following

Remark 15. Unlike in Newtonian Mechanics, the splitting of the total angular momentum into a spin ($z$-independent) and an orbital ($z$-dependent) part depends on the inertial frame, here represented by $u$.

Finally, using the expression (74) for $a$, we get the following expression for the spin part,

$$S(u) := J[z] - a(z, u; \lambda) \wedge P = u \cdot \left( P \wedge \frac{J[z]}{P \cdot u} \right),$$

(80a)

which explicitly displays its $u$-dependence. Again note that the $z$-dependence of $J$ (given by (72)) drops out due to the wedge product with $P$. The expression on the right-hand side of (80a) has a simple geometric interpretation,
namely that of the (tensor-factor wise) projection of $J[Z]$ parallel to $P$ onto $u\perp$, we may also write

$$S(u) = \Pi \otimes \Pi (J[z]),$$

where $\Pi$ is as in (73a). Note that application of $\Pi \otimes \Pi$ cancels the $z$-dependence of $J$ and, in exchange, introduces a $u$-dependence. From both expressions (80) the defining equation (72) for the centre-of-mass,

$$S(u) \cdot u = -u \cdot S(u) = 0$$

follows trivially. In (75) we already stated the obvious dependence of the line $\lambda \mapsto \gamma(z, u; \lambda)$ in $\mathbb{V}$ on $z$ (which is just like in Newtonian physics). More interesting, and purely special-relativistic in nature, is its dependence on $u$. It is clear from (74) that any normal timelike vector $u \in \mathbb{V}_1$ in equation (74) yields a worldline $\lambda \mapsto \gamma(u; \lambda)$ in $\mathcal{M}$ parallel to $P$. As $u$ varies over the 3-dimensional hyperbola $\mathbb{V}_1 \subset \mathbb{V}$ we obtain a bundle of straight lines (geodesics) in $\mathcal{M}$ parallel to $P$:

$$B = \bigcup_{u \in \mathbb{V}_1} \bigcup_{\lambda \in \mathbb{R}} \{\gamma(u; \lambda)\}.$$  

In that bundle a particular line $\gamma = \gamma_*$ is distinguished, namely that for which $u \propto P$, i.e.

$$u = u_* := P/\|P\|.$$  

Here we use the notation $\|P\| := \sqrt{P \cdot P}$. This is the only timelike direction the body determines by itself.

**Definition 16.** The inertial frame for which $u \propto P$ is called the body’s rest frame and

$$M_0 := (u_* \cdot P)/c$$

the body’s rest mass. The line $\lambda \mapsto \gamma(u_*; \lambda)$, i.e. the centre-of-mass in the body’s rest frame, is called its centroid, or worldline of the centre-of-inertia [6].

Using (80) we can immediately write down the body’s spin relative to its rest frame,

$$S_* := u_* \cdot (J[z] \wedge u_*),$$

which is clearly independent of $z$. With respect to the body’s centroid, the bundle (82) of wordlines of mass-centres has a simple geometric description:

---

3 Here we assume that $P$ is timelike, which essentially means that we assume the energy-momentum tensor to satisfy the condition of energy dominance.
**Theorem 17.** The intersection of the bundle $\mathcal{B}$ with the hyperplane

$$
\Sigma(u_*, \sigma) := \{ x \in M \mid (x - z) \cdot u_* = \sigma \}
$$

is a 2-disc in perpendicular to the axis of rotation and with radius radius is

$$
R_M = \frac{\| S_* \|}{\| P \|} = \frac{\| S_* \|}{M_0 c}.
$$

(87)

**Definition 18.** The radius (87) is called the Møller radius, first defined in [9] and also discussed in, e.g., [5] and [12]. It measures the degree to which different inertial observers disagree on the spatial location of the centre-of-mass perpendicular to the axis of rotation. Typical orders of magnitude for Møller radii will be given below.

**Proof of Theorem 17.** Note first that $\Sigma(u_*, \sigma)$ is the hyperplane with normal $u_* \propto P$ and timelike distance $\sigma$ from the point $z$. As we may choose any convenient $z$, we take it to lie on the centroid. The hyperplane through $z$ is then

$$
\Sigma(u_*, \sigma = 0) = \{ x \in M \mid (x - z) \cdot u_* = 0 \}.
$$

(88)

Relative to that choice of $z$ (on the centroid) all other mass centres have worldlines

$$
\gamma(u; \lambda) := z + \frac{S_* \cdot u}{P \cdot u} + \lambda P,
$$

(89)

with $\lambda$ parametrising the individual worldline and $u \in V_1$ the different mass-centres. Since $P \cdot S_* = 0$ the second and third term on the right-hand side are perpendicular, so that the worldline $\gamma(u; \lambda)$ intersects $\Sigma(u_*, \sigma = 0)$ at $\lambda = 0$. Hence

$$
\mathcal{B} \cap \Sigma(u_*, \sigma = 0) = \left\{ z + \frac{S_* \cdot u}{P \cdot u} \mid u \in V_1 \right\}
$$

(90)

The claim is that this is a 2-dimensional disc of radius (87) centred at $z$ which lies in the plane perpendicular to the axis of rotation. To see this, we parametrise $u$ by its boost-parameters relative to $u_*$, i.e., by its rapidity $\rho \in [0, \infty)$ and spatial direction $n \in u_*^\perp$, $n^2 = 1$, so that

$$
u = \cosh(\rho) u_* + \sinh(\rho) n.
$$

(91)

Then, assuming $\|S_*\| \neq 0$,

$$
\frac{S_* \cdot u}{P \cdot u} = \frac{\|S_*\|}{\|P\|} \frac{S_* \cdot n}{\|S_*\|} \tanh(\rho).
$$

(92)
Note that \( n \to \frac{S \cdot n}{\|S\|} \) maps \( u^\perp \) into itself. Since it is a non-zero antisymmetric endomorphism of the 3-dimensional vector space \( u^\perp \), it necessarily has a one-dimensional kernel, which is the rotation axis (the common fixed-point set of the rotations generated by the Lie-algebra element \( S \)) and maps the plane perpendicular to that axis into itself. In fact, since we divided by \( \|S\| \), the map in the plane perpendicular to the rotation axis is a rotation by \( \pi/2 \). Hence, as \( n \) runs over the unit 2-sphere in \( u^\perp \) and \( \tanh(\rho) \) over the interval \([0,1]\), the image of the map \( u \to \frac{S \cdot u}{\|S\|} \) becomes the unit 2-disc in \( u^\perp \). □

**Remark 19.** The condition \( u \cdot S = 0 \) makes \( S \) effectively a tensor in the antisymmetric tensor product of the 3-dimensional space \( u^\perp \). Since \( u^\perp \) as well as its antisymmetric tensor product are 3-dimensional, there exists an isomorphism relating them. A preferred one is that of the 3-dimensional Hodge duality map, \( \tilde{\star} \), which is obtained from the full (4-dimensional) Hodge duality map, denoted by \( \star \), by first applying \( \star \) followed by left contraction with \( u \), i.e., \( \star T := u \cdot \star T = \star (T \land u) \); compare (146) of Appendix A. In this way we can uniquely associate a spin vector \( \vec{S} \) with the spin-tensor \( S \) as follows:

\[
\vec{S}_s := -u_s \cdot \star S_s = -\star (S_s \land u_s), \tag{93a}
\]
\[
S_s = -u_s \cdot \tilde{\star} \vec{S}_s = -\star (\vec{S}_s \land u_s). \tag{93b}
\]

Equation (93a) can be seen as definition of \( \vec{S}_s \) and (93b) as its inverse relation. The latter can be obtained from taking the \( \star \) of the first and using the fact that \( \star \circ \star \) is the identity on antisymmetric tensors of odd degree in even dimensions and Lorentzian signature, which follows from combining formulae (140) and (145) of Appendix A. This gives

\[
\star \vec{S}_s = -S_s \land u_s. \tag{94}
\]

Subsequent contraction with \( u_s \), using \( u_s \cdot S_s = 0 \), yields (93b). In passing we also note that the component versions of (93) are

\[
\vec{S}_s^a = -\frac{1}{2} \varepsilon_{abcd} \eta^{dn} S^b_s u^c_s, \tag{95a}
\]
\[
S_s^{cm} = -\varepsilon_{abcd} \eta^{om} \eta^{dn} \vec{S}_s^a u^b_s. \tag{95b}
\]

We note from (93b) that

\[
\vec{S}_s \cdot S_s = \star (\vec{S}_s \land \vec{S}_s \land u_s) = 0, \tag{96}
\]

which means that \( \vec{S}_s \) lies in the intersection of \( u^\perp \) with the kernel of \( S_s \). In other words, \( \vec{S}_s \) points along the axis of rotation. Finally we note that

\[
\eta(\vec{S}_s, \vec{S}_s) = \eta_{ab} \vec{S}^a_s \vec{S}^b_s = -\frac{1}{2} \eta_{ac} \eta_{bd} S^a_s S^c_s = -\frac{1}{2} \eta \otimes \eta(S_s, S_s). \tag{97}
\]

31
By the definition of the normalised inner product on antisymmetric tensors (i.e. dividing by $1/p!$ the $p$-fold tensor products of $\eta$ on antisymmetric $p$-tensors) and setting 

$$
\|S_*\| := \sqrt{|\langle S_*, S_* \rangle_{\text{norm}}|}
$$

(98)

we have (recall $\|\vec{S}_*\| := \sqrt{\eta(\vec{S}_*, \vec{S}_*)}$)

$$
\|S_*\| = \|\vec{S}_*\|.
$$

(99)

This justifies calling $\vec{S}_*$ the Spin vector, which is associated to the (Lie-algebra valued) spin tensor $S_*$. \qed

We end this section by justifying the terminology centre-of-mass. For this we recall that given an energy-momentum tensor $T$ and a unit timelike direction $u$, then $T(u,u)$ is the spatial energy-density in the rest frame of the inertial observer represented by $u$. More precisely, let us foliate the affine space $M$ by affine hyperplanes 

$$
\Sigma(u,\sigma) := \{x \in M \mid (x - z) \cdot u = \sigma\}
$$

(100)

for some given $u \in V_1$ and $z \in M$. Each $\Sigma(u,\sigma)$ is a spacelike hyperplane of Einstein-simultaneity in the inertial frame characterised by $u$. It is clearly also a Cauchy surface in Minkowski space. The 3-form representing the spatial energy-density of $T$ on $\Sigma(u,\sigma)$ is then 

$$
\mathcal{E}(u,\sigma) = T(u,u) \star u^b|_{\Sigma(u,\sigma)},
$$

(101)

where $\star u^b$ is the measure 3-form on $\Sigma(u,\sigma)$ (the Hodge dual to the 1-form $u^b := \eta_i(u) := \eta(u, \cdot)$). The first moment of this energy distribution with respect to $z$ is 

$$
m(z,u;\sigma) := \int_{\Sigma(u,\sigma)} (x - z)\mathcal{E}(u,\sigma) / \int_{\Sigma(u,\sigma)} \mathcal{E}(u,\sigma),
$$

(102)

where we explicitly indicated all dependencies on $z$, $u$, and $\sigma$ and separated the latter by a semicolon to emphasise the special meaning of $\sigma$ as “time-parameter” labelling the different leaves of the foliation orthogonal to $u$. The dependence on $z$ is rather trivial: $m(z + b, u;\sigma) = m(z, u;\sigma) - b$ so that the set of points 

$$
\gamma(u;\sigma) = z + m(z, u;\sigma)
$$

(103)

is independent of $z$. Moreover, from (102) it is obvious that $(\gamma(u,\sigma) - z) \cdot u = \sigma$ so that $\gamma(u,\sigma) \in \Sigma(u,\sigma)$. Note that the construction of the “first moment”
refers to the affine structure of $M$. Given that $T$ satisfies the weak energy-condition we have $T(u, u) \geq 0$, so that $\gamma(u, \sigma)$ lies in the convex hull of $\text{supp}(T) \cap \Sigma(u, \sigma)$.

Now let us calculate the right-hand side of (102). The denominator is, in view of (70a),

$$\int_{\Sigma(u, \sigma)} T(u, u) \ast u^b = u \cdot P$$

independent of $\sigma$ because $P$ is independent of the Cauchy surface the integral is taken over. The $a$-th component of the numerator can be transformed as follows (calling $\ast u^b = d\mu$ and using component language)

$$\int_{\Sigma(u, \sigma)} (x - z)^a \mathcal{E}(u; \sigma) = \int_{\Sigma(u, \sigma)} (x - z)^a T^{bc} u_b u_c \, d\mu$$

$$= \int_{\Sigma(u, \sigma)} 2(x - z)^a T^{bc} u_b u_c \, d\mu$$

$$+ \int_{\Sigma(u, \sigma)} (x - z)^b T^{ac} u_b u_c \, d\mu$$

$$= J^{ab}[z] u_b + \sigma P^a$$

where we used that $(x - z)^a u_a = \sigma$ for $x \in \Sigma(u, \sigma)$. In total we get

$$\gamma(u; \sigma) = z + J[z] \cdot u + \frac{P \cdot u}{P \cdot u} \sigma,$$

which, upon using the new parameter $\lambda := \sigma/(P \cdot u)$, just turns into (76)(74). This justifies the term “centre-of-mass” in Definition 13, where “mass” is to be understood as proportional to energy. For a system of point particles this means dynamical mass, not rest mass.$^4$ We emphasise again that the essential use of the affine structure in this construction. In fact, the very notion of “first”, “second”, etc. “moments” of a distributions presuppose such a structure.

$^4$ This definition of centre-of-mass, using the first moment of the dynamical-mass distribution, corresponds to cases (c) (for arbitrary $u$) and (d) (for $u = u_*$) in [10]. See this reference for a brief historical account of other definitions, e.g., based on the first moment of the rest-mass distribution, and a discussion of their partly peculiar properties, like moving mass centres in the zero momentum frame. There is also the issue of the Poisson structure for the coordinates of mass-centres, linear momentum, and spin, which for the mass centres based on dynamical mass where first discussed in [2]. Again we refer to [6] for a comprehensive discussion.
5 Typical Møller radii

The ambiguity expressed in (87) only exists for bodies with spin. The formula suggest that for elementary particles it may well be of the order of magnitude of other radii, but that for laboratory-size or astrophysical bodies it is likely to be completely negligible. Let us therefore compute a few examples.

A spin-1/2 particle has $\|S\| = \hbar/2$ and thus

$$R_M = \frac{\hbar}{2M_0c} = \frac{1}{4\pi} \frac{\hbar}{M_0c} = \frac{1}{4\pi} \lambda_C$$  \hspace{1cm} (107)

where $\lambda_C$ is the particle’s Compton wavelength. If the particle is electrically charged it has a classical charge-radius $R_{\text{classical}}$ determined by

$$e^2 = \frac{8\pi\varepsilon_0 R_{\text{classical}}}{M_0c^2}.$$ \hspace{1cm} (108)

Hence we have

$$R_M = R_{\text{classical}}/\alpha \approx 137 R_{\text{classical}}$$ \hspace{1cm} (109)

Let’s look at the Proton: Its experimentally determined “proton radius” (CODATA 2010) is

$$R_{\text{charge}}^{(\text{Proton})} = 0.87 \cdot 10^{-15} \text{ m}.$$ \hspace{1cm} (110)

Its Compton wavelength is

$$\lambda_{\text{Proton}} = 1.32 \cdot 10^{-15} \text{ m},$$ \hspace{1cm} (111)

and its Møller radius is

$$R_M^{(\text{Proton})} = \frac{\lambda_{\text{Proton}}}{4\pi} = 1.05 \cdot 10^{-15} \text{ m} \approx \frac{1}{8} R_{\text{charge}}^{(\text{Proton})}.$$ \hspace{1cm} (112)

In comparison, a homogeneous rigid body of mass $M$ and Radius $R$, rigidly spinning at angular frequency $\omega$, has spin angular-momentum equal to

$$S = \frac{2}{5} MR^2 \omega$$ \hspace{1cm} (113)

Hence the ratio of its Møller radius to its geometric radius is

$$\frac{R_M}{R} = \frac{S}{McR} = \frac{2}{5} \left( \frac{R\omega}{c} \right),$$ \hspace{1cm} (114)
which shows that this ratio is of the order of magnitude of the circumferential velocity in units of the velocity of light. Applying this to Earth and Moon (somewhat idealised) gives

\[
R_M^{(\text{Earth})} = 4 \text{ m}, \quad (115a)
\]
\[
R_M^{(\text{Moon})} = 1.1 \text{ cm}. \quad (115b)
\]

Note that Lunar Laser Ranging also locates the moon’s “position” within accuracy of centimeters. Hence the Møller radius is not as ridiculously small as one might have anticipated it to be for astronomical bodies. In fact, let’s take the fast spinning Pulsar PSR J1748-2446ad, whose frequency is 716 Hz corresponding to a period of 1.4 milliseconds, for which we get \( R \omega / c \approx 0.24 \). Hence the ratio of its Møller radius to its geometric radius is

\[
\left( \frac{R_M}{R} \right)_{\text{Pulsar}} \approx 0.1, \quad (116)
\]

which is the typical ratio of relativistic effects for neutron stars.
Appendices

A Exterior products and Hodge duality

Let $V$ be a real $n$-dimensional vector space, $V^*$ its dual space and $T^p V^*$ its $p$-fold tensor product. $T^p V^*$ carries a representation $\pi_p$ of $S_p$, the symmetric group (permutation group) of $p$ objects, given by

$$\pi_p : S_p \to \text{End}(T^p V^*), \quad \pi_p(\sigma)(\alpha_1 \otimes \cdots \otimes \alpha_p) := \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(p)} \quad (117)$$

and linear extension to sums of tensor products. On $T^p V^*$ we define the linear operator of antisymmetrisation by

$$\text{Alt}_p := \frac{1}{p!} \sum_{\sigma \in S_p} \text{sign}(\sigma) \pi_p,$$  

where $\text{sign} : S_p \to \{1, -1\} \cong \mathbb{Z}_2$ is the sign-homomorphism. This linear operator is idempotent (i.e. a projection operator) and its image of $T^p V^*$ under $\text{Alt}_p$ is the subspace of totally antisymmetric tensor-products. We write

$$\pi_p(T^p V^*) =: \bigwedge^p V^*.$$  

Clearly

$$\dim \left( \bigwedge^p V^* \right) = \begin{cases} \binom{n}{p} & \text{for } p \leq n, \\ 0 & \text{for } p > n. \end{cases} \quad (120)$$

We set

$$\bigwedge V^* := \bigoplus_{p=0}^{n} \bigwedge^p V^*. \quad (121)$$

Let $\alpha \in \bigwedge^p V^*$ and $\beta \in \bigwedge^q V^*$, then we define their antisymmetric tensor product

$$\alpha \wedge \beta := \frac{(p+q)!}{p! q!} \text{Alt}_{p+q}(\alpha \otimes \beta) \in \bigwedge^{p+q} V^*. \quad (122)$$

One easily sees that

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha. \quad (123)$$

\footnote{We follow standard tradition to define forms, i.e. the antisymmetric tensor product on the dual vector space $V^*$ rather than on $V$. Clearly, all constructions that are to follow could likewise be made in terms if $V$ rather than $V^*$.}
Bilinear extension of $\wedge$ to all of $\bigwedge V^*$ endows it with the structure of a real $2^n$-dimensional associative algebra, the so-called exterior algebra over $V^*$. If $\alpha_1, \ldots, \alpha_p$ are in $V^*$, we have
\[
\alpha_1 \wedge \cdots \wedge \alpha_p = \sum_{\sigma \in S_p} \text{sign}(\sigma) \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(p)},
\] as one easily shows from (122) and (123) using induction.

If $\{\theta^1, \ldots, \theta^n\}$ is a basis of $V^*$, a basis of $\bigwedge^p V^*$ is given by the following $\binom{n}{p}$ vectors
\[
\{\theta^{a_1} \wedge \cdots \wedge \theta^{a_p} \mid 1 \leq a_1 < a_2 < \cdots < a_p \leq n\}.
\] An expansion of $\alpha \in \bigwedge^p V^*$ in this basis is written as follows
\[
\alpha =: \frac{1}{p!} \alpha_{a_1 \cdots a_p} \theta^{a_1} \wedge \cdots \wedge \theta^{a_p},
\] using standard summation convention and where the coefficients $\alpha_{a_1 \cdots a_p}$ are totally antisymmetric in all indices. On the level of coefficients, (122) reads
\[
(\alpha \wedge \beta)_{a_1 \cdots a_{p+q}} = \frac{(p+q)!}{p!q!} \alpha_{a_1 \cdots a_p} \beta_{a_{p+1} \cdots a_{p+q}},
\] where square brackets denote total antisymmetrisation in all indices enclosed:
\[
\alpha_{[a_1 \cdots a_p]} := \frac{1}{p!} \sum_{\sigma \in S_p} \text{sign}(\sigma) \alpha_{\sigma(a_1) \cdots \sigma(a_p)}. \tag{128}
\]

Suppose there is an inner product (non-degenerate symmetric bilinear form) $\eta$ on $V$ and the associated dual inner product $\eta^{-1}$ on $V^*$ (compare Remark 3). The latter extends to an inner product on each $T^p V^*$ by
\[
\langle \alpha_1 \otimes \cdots \otimes \alpha_p, \beta_1 \otimes \cdots \otimes \beta_p \rangle := \prod_{a=1}^p \eta^{-1}(\alpha_a, \beta_a) \tag{129}
\] and bilinear extension:
\[
\langle \alpha_{a_1 \cdots a_p}, \theta^{a_1} \otimes \cdots \otimes \theta^{a_p}, \beta_{b_1 \cdots b_p} \theta^{b_1} \otimes \cdots \otimes \theta^{b_p} \rangle = \alpha_{a_1 \cdots a_p} \beta_{a_1 \cdots a_p}. \tag{130}
\] In particular, it extends to each subspace $\bigwedge^p V^* \subset T^p V^*$. We have
\[
\langle \alpha_1 \wedge \cdots \wedge \alpha_p, \beta_1 \wedge \cdots \wedge \beta_p \rangle := p! \sum_{\sigma \in S_p} \text{sign}(\sigma) \prod_{a=1}^p \eta(\alpha_a, \beta_{\sigma(a)}). \tag{131}
\]
and hence
\[ \langle \frac{1}{p!} \alpha_{a_1 \ldots a_p} \theta^{a_1} \wedge \cdots \wedge \theta^{a_p}, \frac{1}{p!} \beta_{b_1 \ldots b_p} \theta^{b_1} \wedge \cdots \wedge \theta^{b_p} \rangle = \alpha_{a_1 \ldots a_p} \beta^{a_1 \ldots a_p} \cdot (132) \]

In the totally antisymmetric case it is more convenient to renormalise this product in a \( p \)-dependent fashion. One sets
\[ \langle \cdot, \cdot \rangle_{\text{norm}} \big|_{\bigwedge^p V^*} := \frac{1}{p!} \langle \cdot, \cdot \rangle \big|_{\bigwedge^p V^*} \]
so that
\[ \langle \frac{1}{p!} \alpha_{a_1 \ldots a_p} \theta^{a_1} \wedge \cdots \wedge \theta^{a_p}, \frac{1}{p!} \beta_{b_1 \ldots b_p} \theta^{b_1} \wedge \cdots \wedge \theta^{b_p} \rangle_{\text{norm}} = \frac{1}{p!} \alpha_{a_1 \ldots a_p} \beta^{a_1 \ldots a_p} \cdot (134) \]

Given a choice \( o \) of an orientation of \( V^* \) (e.g. induced by an orientation of \( V \)), there is a unique top-form \( \varepsilon \in \bigwedge^n V^* \) (i.e. a volume form for \( V \)), associated with the triple \((V^*, \eta^{-1}, o)\), given by
\[ \varepsilon := \theta^1 \wedge \cdots \wedge \theta^n, \]
where \( \{\theta^1, \ldots, \theta^n\} \) is any \( \eta^{-1}\)-orthonormal Basis of \( V^* \) in the orientation class \( o \). The Hodge duality map at level \( 0 \leq p \leq n \) is a linear isomorphism
\[ \star_p : \bigwedge^p V^* \to \bigwedge^{n-p} V^*, \]
defined implicitly by
\[ \alpha \wedge \star_p \beta = \varepsilon \langle \alpha, \beta \rangle_{\text{norm}}. \]
This means that the image of \( \beta \in \bigwedge^p V^* \) under \( \star_p \) in \( \bigwedge^{n-p} V^* \) is defined by the requirement that \( (136b) \) holds true for all \( \alpha \in \bigwedge^p V^* \). Linearity is immediate and uniqueness of \( \star_p \) follows from the fact that if \( \lambda \in \bigwedge^{n-p} V^* \) and \( \alpha \wedge \lambda = 0 \) for all \( \alpha \in \bigwedge^p V^* \), then \( \lambda = 0 \). To show existence it is sufficient to define \( \star_p \) on basis vectors. Since \( (136b) \) is also linear in \( \alpha \) it is sufficient to verify \( (136b) \) if \( \alpha \) runs through all basis vectors.

From now on we shall follow standard practice and drop the subscript \( p \) on \( \star \), supposing that this will not cause confusion.

Let \( \{e_1, \ldots, e_n\} \) be a basis of \( V \) and \( \{\theta^1, \ldots, \theta^n\} \) its dual basis of \( V^* \); i.e. \( \theta^a(e_b) = \delta^a_b \). Let further \( \{\theta_1, \ldots, \theta_n\} \) be the basis of \( V^* \) given by the image of \( \{e_1, \ldots, e_n\} \) under \( \eta \) (compare Remark 3), i.e. \( \theta_a = \eta_{ab} \theta^b \). Then, on the basis \( \{\theta_{a_1} \wedge \cdots \wedge \theta_{a_p} \mid 1 \leq a_1 < a_2 < \cdots < a_p \leq n\} \) of \( \bigwedge^p V^* \) the map \( \star \) has the simple form
\[ \star (\theta_{b_1} \wedge \cdots \wedge \theta_{b_p}) = \frac{1}{(n-p)!} \varepsilon_{b_1 \ldots b_p a_{p+1} \ldots a_n} \theta^{a_{p+1}} \wedge \cdots \wedge \theta^{a_n} \cdot (137) \]
This is proven by merely checking (136b) for \( \alpha = \theta^{a_1} \wedge \cdots \wedge \theta^{a_p} \) and \( \beta = \theta^{b_1} \wedge \cdots \wedge \theta^{b_p} \). Instead of (137) we can write

\[
\star(\theta^{a_1} \wedge \cdots \wedge \theta^{a_p}) = \frac{1}{(n-p)!} \epsilon^{a_1 a_{p+1} \cdots a_n} b_1 \cdots b_{p+1} \wedge \cdots \wedge b_n \theta^{b_{p+1}} \wedge \cdots \wedge \theta^{b_n},
\]

which makes explicit the dependence on \( \epsilon \) and \( \eta \).

If \( \alpha = \frac{1}{p!} \alpha_{a_1 \cdots a_p} \theta^{a_1} \wedge \cdots \wedge \theta^{a_p} \), then \( \star \alpha = \frac{1}{(n-p)!} \langle \star \alpha \rangle_{b_1 \cdots b_{n-p}} \theta^{b_1} \wedge \cdots \wedge \theta^{b_{n-p}} \), where

\[
\langle \star \alpha \rangle_{b_1 \cdots b_{n-p}} = \frac{1}{p!} \alpha_{a_1 \cdots a_p} \epsilon^{a_1 \cdots a_{p+1} \cdots a_n} b_1 \cdots b_{p+1} \cdots b_n \theta^{b_{p+1}} \wedge \cdots \wedge \theta^{b_n}.
\]

This gives the familiar expression of Hodge duality in component language. Note that on component level the first (rather than last) \( p \) indices are contracted.

Applying \( \star \) twice (i.e. actually \( \star_{(n-p)} \circ \star_p \)) leads to the following self-map of \( \wedge^p V^\ast \):

\[
\star \left( \star(\theta^{a_1} \wedge \cdots \wedge \theta^{a_p}) \right) = \frac{1}{p!(n-p)!} \epsilon^{a_1 \cdots a_p a_{p+1} \cdots a_{n+1}} b_1 \cdots b_{p+1} \cdots b_{n-p} \wedge \cdots \wedge b_n \theta^{b_{p+1}} \wedge \cdots \wedge \theta^{b_n},
\]

(140)

Note that

\[
\langle \epsilon, \epsilon \rangle_{\text{norm}} = \frac{1}{n!} \eta^{a_1 b_1} \cdots \eta^{a_n b_n} \epsilon_{a_1 \cdots a_n} \epsilon_{b_1 \cdots b_n} = (\epsilon_{12 \cdots n})^2 / \det\{\eta(e_a, e_b)\}.
\]

(141)

This formula holds for any volume form \( \epsilon \) in the definition (136b), independent of whether or not it is related to \( \eta \).

Since the right-hand side of (136b) is symmetric under the exchange \( \alpha \leftrightarrow \beta \), so must be the left-hand side. Using (140) we get

\[
\langle \alpha, \beta \rangle_{\text{norm}} \epsilon = \alpha \wedge \star \beta = \beta \wedge \star \alpha = (-1)^{p(n-p)} \star \alpha \wedge \beta
\]

(142)

\[
= \langle \epsilon, \epsilon \rangle_{\text{norm}} \star \alpha \wedge \star \beta = \langle \epsilon, \epsilon \rangle_{\text{norm}} \langle \star \alpha, \star \beta \rangle_{\text{norm}} \epsilon,
\]

hence

\[
\langle \star \alpha, \star \beta \rangle_{\text{norm}} = \langle \epsilon, \epsilon \rangle_{\text{norm}} \langle \alpha, \beta \rangle_{\text{norm}}.
\]

(143)

From this and (140) it follows for \( \alpha \in \wedge^p V^\ast \) and \( \beta \in \wedge^{n-p} V^\ast \), that

\[
\langle \alpha, \star \beta \rangle_{\text{norm}} = \langle \epsilon, \epsilon \rangle_{\text{norm}}^{-1} \langle \star \alpha, \star \beta \rangle_{\text{norm}} = (-1)^{p(n-p)} \langle \star \alpha, \beta \rangle_{\text{norm}}.
\]

(144)

39
This shows that the adjoint map of $\star$ relative to $\langle \cdot, \cdot \rangle_{\text{norm}}$ is $(-1)^{p(n-p)} \star$.

Formulae (140), (142), (143), and (144) are valid for general $\varepsilon$ in the definition (136b). If we chose $\varepsilon$ in the way we did, namely as the unique volume form that assigns unit volume to an oriented orthonormal frame, as does (135), then we have

$$\langle \varepsilon, \varepsilon \rangle_{\text{norm}} = (-1)^{n-} \quad (145)$$

where $n_-$ is the maximal dimension of subspaces in $V$ restricted to which $\eta$ is negative definite; i.e. $\eta$ is of signature $(n_+, n_-)$. Equation (143) then shows that $\star$ is an isometry for even $n_-$ and an anti-isometry for odd $n_-$ (as for Lorentzian $\eta$ in any dimension).

Finally we note the following useful formula: If $v \in V$ let $i_v : T^p V^* \to T^{p-1} V^*$ the map which inserts $v$ into the first tensor factor. It restricts to a map $i_v : \bigwedge^p V^* \to \bigwedge^{p-1} V^*$. Then, for any $\alpha \in \bigwedge^p V^*$, we have

$$i_v \star \alpha = \star (\alpha \wedge v^\flat) \quad (146)$$

where $v^\flat := \eta_t(v)$ (compare Remark 3). It suffices to prove this for basis elements $v = e_a$ of $V$ and $\alpha = \theta^{a_1} \wedge \cdots \wedge \theta^{a_p}$ of $\bigwedge^p V^*$, which is almost immediate using (138).

### B Group actions on manifolds

Let $G$ be a group and $M$ a set. An action of $G$ of $M$ is a map

$$\Phi : G \times M \to M \quad (147)$$

such that, for all $m \in M$ and $e \in G$ the neutral element,

$$\Phi(e, m) = m, \quad (148)$$

and where, in addition, one of the following two conditions hold:

$$\Phi(g, \Phi(h, m)) = \Phi(gh, m), \quad (149a)$$
$$\Phi(g, \Phi(h, m)) = \Phi(hg, m). \quad (149b)$$

If (147), (148) and (149a) hold we speak of a left action. A right action satisfies (147), (148) and (149b). For a left action we also write

$$\Phi(g, m) := g \cdot m \quad (150a)$$
and for a right action

\[ \Phi(g, m) =: m \cdot g. \quad (150b) \]

Equations (149) then simply become (group multiplication is denoted by juxtaposition without a dot)

\[
\begin{align*}
g \cdot (h \cdot m) &= (gh) \cdot m, \quad (151a) \\
(m \cdot h) \cdot g &= m \cdot (hg). \quad (151b)
\end{align*}
\]

Holding either of the two arguments of \( \Phi \) fixed we obtain the families of maps

\[
\Phi_g : M \to M \\
m \mapsto \Phi(g, m)
\]

for each \( g \in G \), or

\[
\Phi_m : G \to M \\
g \mapsto \Phi(g, m)
\]

for each \( m \in M \). Note that (148) and (149) imply that \( \Phi_{g^{-1}} = (\Phi_g)^{-1} \). Hence each \( \Phi_g \) is a bijection of \( M \). The set of bijections of \( M \) will be denoted by \( \text{Bij}(M) \). It is naturally a group with group multiplication being given by composition of maps and the neutral element being given by the identity map. Conditions (148) and (149a) are then equivalent to the statement that the map \( G \to \text{Bij}(M) \), given by \( g \mapsto \Phi_g \), is a group homomorphism. Likewise, (148) and (149b) is equivalent to the statement that this map is a group anti-homomorphism.

The following terminology is standard: The set \( \text{Stab}(m) := \{g \in G \mid \Phi(g, m) = m\} \subset G \) is called the stabiliser of \( m \). It is easily proven to be a normal subgroup of \( G \) satisfying \( \text{Stab}(g \cdot m) = g(\text{Stab}(m))g^{-1} \) for left and \( \text{Stab}(m \cdot g) = g^{-1}(\text{Stab}(m))g \) for right actions. The orbit of \( G \) through \( m \in M \) is the set \( \text{Orb}(m) := \{\Phi(g, m) \mid g \in G\} =: \Phi(G, m) \) (also written \( G \cdot m \) for left and \( m \cdot G \) for right action). It is easy to see that two orbits are either disjoint or identical. Hence the orbits partition \( M \). A point \( m \in M \) is called a fixed point of the action \( \Phi \) iff \( \text{Stab}(m) = G \). An action \( \Phi \) is called effective iff \( \Phi(g, m) = m \) for all \( m \in M \) implies \( g = e \); i.e., “only the group identity moves nothing”. Alternatively, we may say that effectiveness is equivalent to the map \( G \to \text{Bij}(M), g \mapsto \Phi_g \), being injective; i.e., \( \Phi_g = \text{id}_M \) implies \( g = e \). The action \( \Phi \) is called free iff \( \Phi(g, m) = m \) for some \( m \in M \).
implies \( g = e \); i.e., “no \( g \neq e \) fixes a point”. This is equivalent to the injectivity of all maps \( \Phi_m : G \to M, \; g \mapsto \Phi(g,m) \), which can be expressed by saying that all orbits of \( G \) in \( M \) are faithful images of \( G \).

Here we are interested in smooth actions. For this we need to assume that \( G \) is a Lie group, that \( M \) a differentiable manifold, and that the map \( \Phi \) is smooth. We denote by \( \exp : T_eG \to G \) the exponential map. For each \( X \in T_eG \) there is a vector field \( V^X \) on \( M \), given by

\[
V^X(m) = \frac{d}{dt} \bigg|_{t=0} \Phi(\exp(tX), m) = \Phi_{m*}(X). \tag{154}
\]

Here \( \Phi_{m*} \) denotes the differential of the map \( \Phi_m \) evaluated at \( e \in G \). \( V^X \) is also called the fundamental vector field on \( M \) associated to the action \( \Phi \) of \( G \) and to \( X \in T_eG \). (We will later write Lie(\( G \)) for \( T_eG \), after we have discussed which Lie structure on \( T_eG \) we choose.)

In passing we note that from (154) it already follows that the flow map of \( V^X \) is given by

\[
\text{Fl}^V_X(m) = \Phi(\exp(tX), m). \tag{155}
\]

This follows from \( \exp(sX) \exp(tX) = \exp((s+t)X) \) and (149) (any of them), which imply

\[
\text{Fl}^V_X \circ \text{Fl}^V_X = \text{Fl}^V_{s+t}. \tag{156}
\]

on the domain of \( M \) where all three maps appearing in (156) are defined. Uniqueness of flow maps for vector fields then suffice to show that (155) is indeed the flow of \( V^X \).

Before we continue with the general case, we have a closer look at the special cases where \( M = G \) and \( \Phi \) is either the left translation of \( G \) on \( G \), \( \Phi(g,h) = L_g(h) := gh \), or the right translation, \( \Phi(g,h) = R_g(h) := hg \). The corresponding fundamental vector fields (154) are denoted by \( V^X_R \) and \( V^X_L \) respectively:

\[
V^X_R(h) = \frac{d}{dt} \bigg|_{t=0} \left( \exp(tX) h \right), \tag{157a}
\]

\[
V^X_L(h) = \frac{d}{dt} \bigg|_{t=0} \left( h \exp(tX) \right). \tag{157b}
\]

The seemingly paradoxical labeling of \( R \) for left and \( L \) for right translation finds its explanation in the fact that \( V^X_R \) is right and \( V^X_L \) is left invariant, i.e., \( R_g V^X_R = V^X_R \) and \( L_g V^X_L = V^X_L \). Recall that the latter two equations
are shorthands for
\[ R_{gh} V_X^R(h) = V_R^X(hg), \quad (158a) \]
\[ L_{gh} V_X^L(h) = V_L^X(gh). \quad (158b) \]

The proofs of (158a) only uses (157a) and the chain rule:
\[
R_{gh} V_X^R(h) = R_{gh} \left. \frac{d}{dt} \right|_{t=0} \left( \exp(tX) h \right) \\
= \left. \frac{d}{dt} \right|_{t=0} R_g \left( \exp(tX) h \right) \\
= \left. \frac{d}{dt} \right|_{t=0} \left( \exp(tX) hg \right) \\
= V_R^X(hg). \quad (159a)
\]

Similarly, the proof of (158b) starts from (157b):
\[
L_{gh} V_X^L(h) = L_{gh} \left. \frac{d}{dt} \right|_{t=0} \left( h \exp(tX) \right) \\
= \left. \frac{d}{dt} \right|_{t=0} L_g \left( h \exp(tX) \right) \\
= \left. \frac{d}{dt} \right|_{t=0} \left( gh \exp(tX) \right) \\
= V_L^X(gh). \quad (159b)
\]

In particular, we have
\[
V_X^R(g) = R_{ge} V_X^L(e) = R_{ge} X, \quad (160a) \\
V_X^L(g) = L_{ge} V_X^R(e) = L_{ge} X, \quad (160b)
\]
showing that the vector spaces of right/left invariant vector fields on \( G \) are isomorphic to \( T_e G \). Moreover, the vector spaces of right/left invariant vector fields on \( G \) are Lie algebras, the Lie product being their ordinary commutator (as vector fields). This is true because the operation of commuting vector fields commutes with push-forward maps of diffeomorphisms: \( \phi_*[V,W] = [\phi_*V, \phi_*W] \). This implies that the commutator of right/left invariant vector fields is again right/left invariant. Hence the isomorphisms can be used to turn \( T_e G \) into a Lie algebra, identifying it either with the Lie algebra of right- or left-invariant vector fields. The standard convention is to choose the latter. Hence, for any \( X, Y \in \text{Lie}(G) \), one defines
\[
[X,Y] := [V_X^L, V_Y^L](e). \quad (161)
\]
$T_eG$ endowed with that structure is called $\text{Lie}(G)$. Clearly, this turns $V_L : \text{Lie}(G) \to \text{Vec}(G)$, $X \mapsto V_L^X$, into a Lie homomorphism:

$$V_L^{[X,Y]} = [V_L^X, V_L^Y].$$

As a consequence, $V_R : \text{Lie}(G) \to \text{Vec}(G)$, $X \mapsto V_R^X$, now turns out to be an anti Lie isomorphism, i.e., to contain an extra minus sign:

$$V_R^{[X,Y]} := -[V_R^X, V_R^Y].$$

This can be proven directly but will also follow from the more general considerations below.

On $G$ consider the map

$$C : G \times G \to G$$

$$(h, g) \mapsto hgh^{-1}. \quad (164)$$

For fixed $h$ this map, $C_h : G \to G$, $g \mapsto C_h(g) = hgh^{-1}$, is an automorphism (i.e., self-isomorphism) of $G$. Automorphisms of $G$ form a group (multiplication being composition of maps) which we denote by $\text{Aut}(G)$. It is immediate that the map $C \to \text{Aut}(G)$, $h \mapsto C_h$, is a homomorphism of groups; i.e.,

$$C_e = \text{id}_G, \quad (165a)$$

$$C_h \circ C_k = C_{hk}. \quad (165b)$$

Taking the differential at $e \in G$ of $C_h$ we obtain a linear self-map of $T_eG$, which we call $\text{Ad}_h$:

$$\text{Ad}_h := C_{h*e} : T_eG \to T_eG. \quad (166a)$$

Differentiating both sides of both equations $\textcolor{red}{(165)}$ at $e \in G$, using the chain rule together with $C_k(e) = e$ for the second, we infer that

$$\text{Ad}_e = \text{id}_{T_eG}, \quad (166b)$$

$$\text{Ad}_h \circ \text{Ad}_k = \text{Ad}_{hk}. \quad (166c)$$

This implies, firstly, that each linear map $\textcolor{red}{(166a)}$ is invertible, i.e. an element of the general linear group $\text{GL}(T_eG)$ of the vector space $T_eG$, and, secondly, that the map

$$\text{Ad} : G \to \text{GL}(T_eG)$$

$h \mapsto \text{Ad}_h$ \quad (167)
is a group homomorphism. In other words, Ad is a linear representation of $G$ on $T_e G$, called the adjoint representation.

In (158) we saw that $V^X_R$ and $V^X_L$ are invariant under the action of right and left translations respectively (hence their names). But what happens if we act on $V^X_R$ with left and on $V^X_L$ with right translations? The answer is obtained from straightforward computation. In the first case we get:

$$L_{g*h}(V^X_R(h)) = L_{g*h} \left. \frac{d}{dt} \right|_{t=0} \left( \exp(tX) h \right)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \left( g \exp(tX) h \right)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \left( C_g(\exp(tX)) \, gh \right)$$

$$= V^{Ad_g(X)} (gh), \quad (168a)$$

where we used (166) in the last and the definition of $V^X_R$ in the first and last step. Similarly, in the second case we have

$$R_{g*h}(V^X_L(h)) = R_{g*h} \left. \frac{d}{dt} \right|_{t=0} \left( h \exp(tX) g \right)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \left( h \exp(tX) \right)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \left( h \exp(tX) g \right)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \left( h g C_{g^{-1}}(\exp(tX)) \right)$$

$$= V^{Ad_{g^{-1}}(X)} (gh). \quad (168b)$$

Taking the differential of Ad at $e \in G$ we obtain a linear map from $T_e G$ into $\text{End}(T_e G)$, the linear space of endomorphisms of $T_e G$ (linear self-maps of $T_e G$).

$$\text{ad} := \text{Ad}_e : T_e G \rightarrow \text{End}(T_e G)$$

$$X \mapsto \text{ad}_X. \quad (169)$$

Now, we have

$$\text{ad}_X(Y) = [X, Y] \quad (170)$$

where the right-hand side is defined in (161). The proof of (170) starts from the fact that the commutator of two vector fields can be expressed in terms of the Lie derivative of the second with respect the first vector field in the commutator, and the definition of the Lie derivative. We recall from (155).
that the flow of the left invariant vector fields is given by right translation: $\text{Fl}_t^{V^X_L}(g) = g \exp(tX)$. Then we have

\[
[X,Y] = [V^X_L, V^Y_L](e) = (L_{V^X_L} V^Y_L)(e)
\]

\[
= \frac{d}{dt} \bigg|_{t=0} \text{Fl}_t^{V^X_L} \left( V^Y_L \left( \text{Fl}_t^{V^X_L}(e) \right) \right)
\]

\[
= \frac{d}{dt} \bigg|_{t=0} \text{Fl}_t^{V^X_L} \left( \frac{d}{ds} \bigg|_{s=0} \text{Fl}_s^{V^Y_L} \left( \text{Fl}_t^{V^X_L}(e) \right) \right)
\]

\[
= \frac{d}{dt} \bigg|_{t=0} \text{Ad}_{\exp(tX)}(Y)
\]

\[
= \text{ad}X(Y).
\]

\[171a\]

A completely analogous consideration, now using $\text{Fl}_t^{V^X_R}(g) = \exp(tX)g$, allows to compute the commutator of the right-invariant vector fields evaluated at $e \in G$:

\[
[V^X_R, V^Y_R](e) = (L_{V^X_R} V^Y_R)(e)
\]

\[
= \frac{d}{dt} \bigg|_{t=0} \text{Fl}_t^{V^X_R} \left( V^Y_R \left( \text{Fl}_t^{V^X_R}(e) \right) \right)
\]

\[
= \frac{d}{dt} \bigg|_{t=0} \text{Fl}_t^{V^X_R} \left( \frac{d}{ds} \bigg|_{s=0} \text{Fl}_s^{V^Y_R} \left( \text{Fl}_t^{V^X_R}(e) \right) \right)
\]

\[
= \frac{d}{dt} \bigg|_{t=0} \text{Ad}_{\exp(-tX)}(Y)
\]

\[
= -\text{ad}X(Y)
\]

\[
= -[X,Y].
\]

\[171b\]

Equation \(163\) now follows if we act on both sides of $[V^X_R, V^Y_R](e) = -[X,Y]$ with $R_{g*e}$ and use \(158a\).

We now return to the general case where $M$ is any manifold and the vector field $V^X$ is defined by an action $\Phi$ as in \(154\) and whose flow map is

46
given by (155). Now, given that $\Phi$ is a right action, we obtain

$$
[V^X, V^Y](m) = (L_{V^X}V^Y)(m)
$$

$$
= \frac{d}{dt}\bigg|_{t=0} \text{Fl}^{V^X}_{(-t)*}\left(V^Y(\text{Fl}^{V^X}_t(m))\right)
$$

$$
= \frac{d}{dt}\bigg|_{t=0} \text{Fl}^{V^X}_{(-t)*} \frac{d}{ds}\bigg|_{s=0} \text{Fl}^{V^Y}_s \left(\text{Fl}^{V^X}_t(m)\right)
$$

$$
= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} \Phi\left(\exp(tX)\exp(sY)\exp(-tX), m\right)
$$

$$
= \Phi_{m*e}(\text{Ad}_{\exp(tX)}(Y))
$$

$$
= V^{\text{ad}_X(Y)}(m)
$$

$$
= V^{[X,Y]}(m)
$$

(172a)

where we used (155) and (149a) at the fourth and (170) at the last equality.

Similarly, if $\Phi$ is a left action, we have

$$
[V^X, V^Y](m) = (L_{V^X}V^Y)(m)
$$

$$
= \frac{d}{dt}\bigg|_{t=0} \text{Fl}^{V^X}_{(-t)*}\left(V^Y(\text{Fl}^{V^X}_t(m))\right)
$$

$$
= \frac{d}{dt}\bigg|_{t=0} \text{Fl}^{V^X}_{(-t)*} \frac{d}{ds}\bigg|_{s=0} \text{Fl}^{V^Y}_s \left(\text{Fl}^{V^X}_t(m)\right)
$$

$$
= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} \Phi\left(\exp(-tX)\exp(sY)\exp(tX), m\right)
$$

$$
= \Phi_{m*e}(\text{Ad}_{\exp(-tX)}(Y))
$$

$$
= -V^{\text{ad}_X(Y)}(m)
$$

$$
= -V^{[X,Y]}(m)
$$

(172b)

where we used (155) and (149b) at the fourth and again (170) at the last equality.

Finally we derive the analog of (168) in the general case. This corresponds to computing the push-forward of $V^X$ under $\Phi_g$. If $\Phi$ is a left action we will obtain the analog of (168a), and the analog of (168b) if $\Phi$ is a right action. For easier readability we shall also make use of the notation (150).
For a left action we then get

$$
\Phi_{g*m}(V^X(m)) = \Phi_{g*m} \frac{d}{dt} \bigg|_{t=0} \Phi(\exp(tX), m)
$$

$$
= \frac{d}{dt} \bigg|_{t=0} \Phi(g \exp(tX), m)
$$

$$
= \frac{d}{dt} \bigg|_{t=0} \Phi(C_g(\exp(tX)), g \cdot m)
$$

$$
= \Phi_{(g \cdot m)*e} \frac{d}{dt} \bigg|_{t=0} C_g(\exp(tX))
$$

$$
= \Phi_{(g \cdot m)*e}(\text{Ad}_g(X))
$$

$$
= V^\text{Ad}_g(X)(g \cdot m)
$$

$$
= V^\text{Ad}_g(X)(\Phi(g, m)).
$$

Similarly, if $\Phi$ is a right action,

$$
\Phi_{g*m}(V^X(m)) = \Phi_{g*m} \frac{d}{dt} \bigg|_{t=0} \Phi(\exp(tX), m)
$$

$$
= \frac{d}{dt} \bigg|_{t=0} \Phi(\exp(tX) g, m)
$$

$$
= \frac{d}{dt} \bigg|_{t=0} \Phi(C_{g^{-1}}(\exp(tX)), m \cdot g)
$$

$$
= \Phi_{(m \cdot g)*e} \frac{d}{dt} \bigg|_{t=0} C_{g^{-1}}(\exp(tX))
$$

$$
= \Phi_{(m \cdot g)*e}(\text{Ad}_{g^{-1}}(X))
$$

$$
= V^\text{Ad}_{g^{-1}}(X)(m \cdot g)
$$

$$
= V^\text{Ad}_{g^{-1}}(X)(\Phi(g, m)).
$$

(173a) (173b)
References

[1] Marcel Berger. *Geometry*, volume I. Springer Verlag, Berlin, first edition, 1987. Corrected second printing 1994.

[2] Max Born and Klaus Fuchs. The mass centre in relativity. *Nature*, 40(3676):587–587, 1940.

[3] John H. Conway et al. *ATLAS of Finite Groups*. Oxford University Press, Oxford, 1985.

[4] Jean Gaston Darboux. Sur la composition des forces en statique. *Bulletin des sciences mathématiques et astronomiques*, 9:281–288, 1875.

[5] William Graham Dixon. *Special Relativity. The Foundation of Macroscopic Physics*. Cambridge University Press, Cambridge, 1978.

[6] Gordon N. Fleming. Covariant position operators, spin, and locality. *Physical Review*, 137(1 B):B 188–B 197, 1965.

[7] Vladimir Fock. *The Theory of Space Time and Gravitation*. Pergamon Press, London, first english edition, 1959.

[8] Georg Hamel. Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung: \( f(x + y) = f(x) + f(y) \). *Mathematische Annalen*, 60(3):459–462, 1905.

[9] Christian Møller. On the definition of the centre of gravity of an arbitrary closed system in the theory of relativity. *Communication of the Dublin Institute for Advanced Studies*, Series A(5):1–42, 1949.

[10] Maurice Henry Lecorney Pryce. The mass-centre in the restricted theory of relativity and its connexion with the quantum theory of elementary particles. *Proceedings of the Royal Society (London) A*, 195:62–81, 1948.

[11] Rudolf Schimmack. Ueber die axiomatische Begründung der Vektoraddition. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1903:257–278, 1903.

[12] Roman U. Sexl and Helmuth K. Urbantke. *Relativity, Groups, Particles*. Springer Verlag, Wien, first edition, 2001. First english edition, succeeding the 1992 third revised german edition.