Hamilton Operators, Discrete Symmetries, Brute Force and SymbolicC++

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Abstract To find the discrete symmetries of a Hamilton operator $\hat{H}$ is of central
importance in quantum theory. Here we describe and implement a brute force
method to determine the discrete symmetries given by permutation matrices for
Hamilton operators acting in a finite-dimensional Hilbert space. Spin and Fermi
systems are considered as examples. A computer algebra implementation in Sym-

colicC++ is provided.

1 Introduction

In quantum mechanics the system is described by a self-adjoint (Hamilton) opera-
tor $\hat{H}$ acting in a Hilbert space $\mathcal{H}$. Here we consider the finite dimensional Hilbert
space $\mathbb{C}^n$ where $\hat{H}$ is a hermitian matrix [1, 2, 3, 4, 5]. One of the main tasks is to
find the $n \times n$ unitary matrices $U$ such that $U^* \hat{H} U = \hat{H}$, where $U^* = U^{-1}$. The
$n \times n$ unitary matrices form the compact Lie group $U(n)$. Note that if $U^* \hat{H} U = \hat{H}$
and $V^* \hat{H} V = \hat{H}$ then $(UV)^* \hat{H} (UV) = \hat{H}$, where $(UV)^* \equiv V^* U^*$. Thus the set of
matrices that keep $\hat{H}$ invariant form a group themselves [6].

An important finite subgroup of the group $U(n)$ are the $n \times n$ permutation matrices.
The number of $n \times n$ permutation matrices is $n!$. For a given hermitian $n \times n$ matrix
$\hat{H}$ we want to find all the $n \times n$ permutation matrices $P$ such that

$$P^T \hat{H} P = \hat{H}$$

where $P^{-1} = P^T$. These permutation matrices form a subgroup of all $n \times n$ per-
mutation matrices. Obviously the $n \times n$ identity matrix $I_n$ satisfies $I_n \hat{H} I_n = \hat{H}$. 
Here we describe and implement in SymbolicC++ a brute force method to find these permutation matrices.

After this finite group has been found one determines the conjugacy classes. Now for a finite group \( G \) the number of conjugacy classes is equivalent to the number of non-equivalent irreducible matrix representations. From the conjugacy classes and the permutation matrices we can construct projection matrices to decompose the Hilbert space into invariant sub Hilbert spaces [6]. For example, if the permutation matrix \( P \) satisfies \( P^2 = I_n \), then \( \Pi_1 = (I_n + P)/2, \Pi_2 = (I_n - P)/2 \) are projection matrices (with \( \Pi_1 \Pi_2 = 0 \)) which can be utilized to decompose the Hilbert space \( \mathbb{C}^n \) into invariant subspaces.

2 Examples

We consider four examples: two Fermi systems and two spin systems.

Example 1. Let \( c_j^\dagger, c_j \) \((j = 1, 2, 3)\) be Fermi creation and annihilation operators, respectively. Consider the Hamilton operator

\[
\hat{H} = t(c_1^\dagger c_2 + c_2^\dagger c_1 + c_3^\dagger c_3 + c_3^\dagger c_2 + c_2^\dagger c_3 + c_1^\dagger c_1) + k_1c_1^\dagger c_1 + k_2c_2^\dagger c_2 + k_3c_3^\dagger c_3
\]

and the number operator \( \hat{N} = c_1^\dagger c_1 + c_2^\dagger c_2 + c_3^\dagger c_3 \). Then \([\hat{H}, \hat{N}] = 0\). Since \([\hat{H}, \hat{N}] = 0\) we find \( \hat{N} \) is a constant of motion, i.e. the total number of Fermi particles remains constant in the sense that if \( |n\rangle \) is an eigenstate of the number operator \( \hat{N} \) with eigenvalue \( n \) at time 0, then \( |n\rangle(t) = e^{-i\hat{H}t/\hbar}|n\rangle \) remains an eigenstate of \( \hat{N} \) with eigenvalue \( n \) for all times. Given a basis with two Fermi particles

\[
c_1^\dagger c_2^\dagger |0\rangle, \quad c_1^\dagger c_3^\dagger |0\rangle, \quad c_2^\dagger c_3^\dagger |0\rangle.
\]

Then we find the matrix representation of \( \hat{H} \)

\[
\hat{H} = \begin{pmatrix}
  k_1 + k_2 & t & -t \\
t & k_1 + k_3 & t \\
-t & t & k_2 + k_3
\end{pmatrix}
\]

The matrix representation of \( \hat{N} \) is the diagonal matrix \( 2I_3 \), where \( I_3 \) is the \( 3 \times 3 \) identity matrix. For \( k_1 \neq k_2, k_1 \neq k_3, k_2 \neq k_3 \) no non-trivial symmetry is found. Also for \( k_1 \neq k_2, k_1 \neq k_3, k_2 = k_3 \) no non-trivial symmetry is found. For \( k = k_1 = k_2 = k_3 \) we obtain the permutation matrix

\[
P = \begin{pmatrix}
  0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]
Thus using the projection matrices $\Pi_1 = (I_3 - P)/2$, $\Pi_2 = (I_3 - P)/2$ the Hilbert space $\mathbb{C}^3$ can be decomposed into invariant subspaces. For the case $k = k_1 = k_2 = k_3$ we find the eigenvalues $2k - 2t$ and $2k + t$ (twice).

Example 2. Consider the Hamilton operator (two-point Hubbard model)

$$\hat{H} = t(c_{1\uparrow}^\dagger c_{2\uparrow} + c_{1\downarrow}^\dagger c_{2\downarrow} + c_{2\uparrow}^\dagger c_{1\uparrow} + c_{2\downarrow}^\dagger c_{1\downarrow}) + U(n_{1\uparrow}n_{1\downarrow} + n_{2\uparrow}n_{2\downarrow})$$

where $n_{j\uparrow} := c_{j\uparrow}^\dagger c_{j\uparrow}$, $n_{j\downarrow} := c_{j\downarrow}^\dagger c_{j\downarrow}$. The operators $c_{j\uparrow}^\dagger, c_{j\downarrow}^\dagger, c_{j\uparrow}, c_{j\downarrow}$ are Fermi operators. The Hubbard Hamilton operator commutes with the total number operator $\hat{N}$ and the total spin operator $\hat{S}_z$ where

$$\hat{N} := \sum_{j=1}^2 (c_{j\uparrow}^\dagger c_{j\uparrow} + c_{j\downarrow}^\dagger c_{j\downarrow}), \quad \hat{S}_z := \frac{1}{2} \sum_{j=1}^2 (c_{j\uparrow}^\dagger c_{j\downarrow} - c_{j\downarrow}^\dagger c_{j\uparrow}) \cdot$$

We consider the subspace with two particles $N = 2$ and total spin $S_z = 0$. A basis in this four dimensional Hilbert space is given by

$$c_{1\uparrow}^\dagger c_{1\downarrow}^\dagger |0\rangle, \quad c_{1\uparrow}^\dagger c_{2\downarrow}^\dagger |0\rangle, \quad c_{2\uparrow}^\dagger c_{1\downarrow}^\dagger |0\rangle, \quad c_{2\uparrow}^\dagger c_{2\downarrow}^\dagger |0\rangle$$

We find the matrix representation of $\hat{H}$ for this basis. Using the Fermi anticommutation relations and $c_{j\uparrow}^\dagger|0\rangle = 0$, $c_{j\downarrow}|0\rangle = 0$ for $j = 1, 2$ we obtain the matrix representation of $\hat{H}$ with the given basis

$$H = \begin{pmatrix} U & t & t & 0 \\ t & 0 & 0 & t \\ t & 0 & 0 & t \\ 0 & t & t & U \end{pmatrix} \cdot$$

We find the four permutation matrices $P_0 = I_4 = I_2 \star I_2 = I_2 \otimes I_2$,

$$P_1 = I_2 \star \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \star I_2, \quad P_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \star \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot$$

where we define the star product $\star$ of the two $2 \times 2$ matrices $A, B$ as [6]

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \star \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix} \cdot$$

We note that the star product of two $2 \times 2$ permutation matrices is a $4 \times 4$ permutation matrix. Here $P_1$ is the swap gate. The four permutation matrices $P_0, P_1,$
$P_2, P_3$ form a commutative group with $P_j^2 = I_4$ for $j = 0, 1, 2, 3$. If $P$ is an $n \times n$ permutation matrix with $P^2 = I_n$ then

$$
\Pi_1 = \frac{1}{2}(I_n + P), \quad \Pi_2 = \frac{1}{2}(I_n - P)
$$

are projection matrices with $\Pi_1 \Pi_2 = 0_n$, where $0_n$ is the $n \times n$ zero matrix. Using the permutation matrices $P_0$ and $P_3$ (which form a subgroup) these projection operators can now be used to find the invariant subspaces

$$\begin{align*}
&\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}, & \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}.
\end{align*}$$

These four vectors (after normalization) are known in quantum computing as the Bell basis $[1, 2, 3, 4, 5]$. This leads to the two invariant sub Hilbert spaces

$$\begin{align*}
&\left\{ \frac{1}{\sqrt{2}}(c^\dagger_{11}c^\dagger_{11}|0\rangle + c^\dagger_{21}c^\dagger_{22}|0\rangle), \frac{1}{\sqrt{2}}(c^\dagger_{11}c^\dagger_{11}|0\rangle + c^\dagger_{21}c^\dagger_{22}|0\rangle) \right\} \\
&\left\{ \frac{1}{\sqrt{2}}(c^\dagger_{11}c^\dagger_{11}|0\rangle - c^\dagger_{21}c^\dagger_{22}|0\rangle), \frac{1}{\sqrt{2}}(c^\dagger_{11}c^\dagger_{11}|0\rangle - c^\dagger_{21}c^\dagger_{22}|0\rangle) \right\}.
\end{align*}$$

Example 3. Let $\sigma_1, \sigma_2, \sigma_3$ be the Pauli spin matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Consider the Hamilton operators $[7]$ 

$$\hat{H} = \hbar\omega_3 \sigma_3 \otimes I_2 + \hbar\omega_2 I_2 \otimes \sigma_1 + \epsilon (\sigma_3 \otimes \sigma_1)$$

$$\hat{K} = \hbar\omega_3 \sigma_3 \otimes I_2 + \hbar\omega_2 I_2 \otimes \sigma_1 + \epsilon (\sigma_1 \otimes \sigma_3)$$

where for the second Hamilton operator $\hat{K}$ the interaction term is swapped around, i.e. $\sigma_3 \otimes \sigma_1 \rightarrow \sigma_1 \otimes \sigma_3$. This provides symmetry breaking. For the Hamilton operator $\hat{H}$ we find the symmetries (permutation matrices)

$$P_0 = I_4, \quad P_1 = I_2 \oplus \sigma_1, \quad P_2 = \sigma_1 \oplus I_2, \quad P_3 = \sigma_1 \oplus \sigma_1$$

where $\oplus$ denotes the direct sum. The four matrices form a commutative group under matrix multiplication. All satisfy $P_j^2 = I_4$. Thus we can use the projection
matrices $\Pi_1 = (I_4 + P_j)/2, \Pi_2 = (I_4 - P_j)/2$ to decompose the Hilbert space into two invariant subspaces. On the other hand for the Hamilton operator $\hat{K}$ we only find the identity matrix $P_0 = I_4$, i.e. no non-trivial symmetry is admitted.

Example 4. Consider the Hamilton operator for triple spin interaction

$$\hat{H} = \sigma_1 \otimes \sigma_2 \otimes \sigma_3.$$ 

The eigenvalues of this hermitian and unitary $8 \times 8$ matrix are $+1$ (four-fold degenerate) and $-1$ (four-fold degenerate). Owing to this degeneracy one expects a "large" number of symmetries. Applying the SymbolicC++ code we find 24 permutation matrices listed $P_0, P_1, \ldots, P_{23}$ with $P_0 = I_8$. They form a non-commutative group under matrix multiplication and are a subgroup of the group of $8 \times 8$ permutation matrices. We note that the Kronecker product $\otimes$ and the direct sum $\oplus$ of two permutation matrices is again a permutation matrix [8]. Now we can list the ones with $P_j^2 = I_8$. We have

$$P_1 = I_2 \oplus \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \oplus I_2$$

$$P_2 = (I_2 \ast \sigma_1) \oplus (\sigma_1 \ast I_2)$$

$$P_5 = I_2 \otimes I_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sigma_1 \otimes I_2 \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$P_6 = (\sigma_1 \ast I_2) \oplus (I_2 \ast \sigma_1)$$

$$P_8 = I_2 \otimes \sigma_1 \otimes \sigma_1$$

$$P_{13} = I_2 \otimes I_2 \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \sigma_1 \otimes I_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_{15} = \sigma_1 \otimes I_2 \otimes I_2$$

$$P_{23} = \sigma_1 \otimes \sigma_1 \otimes \sigma_1.$$ 

The other ones can be found by multiplication of these permutation matrices, for example $P_3 = P_1P_2$ etc. Thus the 24 matrices form a subgroup of the permutation group of $8 \times 8$ matrices.

Another spin Hamilton operator studied is [9]

$$\hat{H} = a \sum_{j=1}^{4} \sigma_3(j)\sigma_3(j + 1) + b \sum_{j=1}^{4} \sigma_1(j)$$

with cyclic boundary conditions, i.e. $\sigma_3(5) \equiv \sigma_3(1)$. Here $a, b$ are real constants and $\sigma_1, \sigma_2$ and $\sigma_3$ are the Pauli matrices. Thus the underlying Hilbert space is
\( \mathbb{C}^{16} \). Recall that
\[
\sigma_k(1) = \sigma_k \otimes I_2 \otimes I_2 \otimes I_2, \quad \sigma_k(2) = I_2 \otimes \sigma_k \otimes I_2 \otimes I_2
\]
\[
\sigma_k(3) = I_2 \otimes I_2 \otimes \sigma_k \otimes I_2, \quad \sigma_k(4) = I_2 \otimes I_2 \otimes I_2 \otimes \sigma_k
\]
where \( k = 1, 2, 3 \). We obtain the symmetric \( 16 \times 16 \) matrix for \( \hat{H} \)
\[
\begin{pmatrix}
4a & b & b & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 0 & 0 & b & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 0 & 0 & b & 0 & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\
o & b & b & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 & b & b & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
o & b & 0 & 0 & 0 & b & b & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\
o & 0 & 0 & b & 0 & b & b & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\
o & 0 & b & 0 & 0 & 0 & 0 & b & b & 0 & b & 0 & 0 & 0 & 0 & 0 \\
o & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & b & 0 & b & 0 & 0 & 0 & 0 \\
o & 0 & b & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & b & 0 & b & 0 \\
o & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & b & 0 & b \\
o & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & b & 0 & 0 & 0 & b & 0 \\
o & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & b & 0 & b & 0 & 0 \\
o & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & b & 0 & b & 0 & 0 \\
o & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & b & 0 & b & 0 & 0 \\
o & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & b & 0 & b & 0 & 0 \\
o & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & b & 0 & b & 0 & 0 \\
o & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & b & 0 & b & 0 & 0 \\
\end{pmatrix}
\]
The Hamilton operator \( \hat{H} \) admits the \( C_{4v} \) symmetry group. The order of this non-commutative group is 8. One finds the following set of eight symmetries [9]
\[
E : (1, 2, 3, 4) \rightarrow (1, 2, 3, 4) \quad C_2 : (1, 2, 3, 4) \rightarrow (3, 4, 1, 2)
\]
\[
C_4 : (1, 2, 3, 4) \rightarrow (2, 3, 4, 1) \quad C_4^3 : (1, 2, 3, 4) \rightarrow (4, 1, 2, 3)
\]
\[
\sigma_v : (1, 2, 3, 4) \rightarrow (2, 1, 4, 3) \quad \sigma_v' : (1, 2, 3, 4) \rightarrow (4, 3, 2, 1)
\]
\[
\sigma_d : (1, 2, 3, 4) \rightarrow (1, 4, 3, 2) \quad \sigma_d' : (1, 2, 3, 4) \rightarrow (3, 2, 1, 4)
\]
which form a group isomorphic to \( C_{4v} \). The symmetries can be found by calculating the \( 16 \times 16 \) permutation matrices such that \( \hat{H} = P^T \hat{H} P \).

3 Code Description

Algorithms for finding all permutations of a sequence of objects are described by Knuth [10]. For a given \( n \) the permutation matrices are generated with the following algorithm. The algorithm implements the nested loops
For $j_0 = 0, 1, \ldots, n - 1$ do
For $j_1 = 0, 1, \ldots, n - 1$ do
  :
For $j_{n-1} = 0, 1, \ldots, n - 1$ do
  If $j_0 \neq j_1 \neq \cdots \neq j_{n-1}$ then
    use the permutation $(0, 1, 2, \ldots, n - 1) \rightarrow (j_0, j_1, j_2, \ldots, j_{n-1})$.
End loop
  :
End loop
End loop

Algorithm to find all permutation matrices.

1. Create an array $(j_0, j_1, \ldots, j_{n-1})$ of loop variables.
2. Initialize $j_k := -1$ for $k = 0, 1, \ldots, n - 1$.
3. Initialize the loop variable index $i$ to $i := 0$.
4. While $i \geq 0$
   (a) Iterate.
      Set $j_i := j_i + 1$.
   (b) Termination condition.
      If $j_i = n$ terminate this loop:
      i. Set $j_i := -1$.
      ii. Exit the nested loop.
          Set $i := i - 1$.
      iii. Goto 4.
   (c) If $j_k = j_i$ for some $k \in \{0, 1, \ldots, i - 1\}$ then goto 4
   (d) Enter the next nested loop.
      Set $i := i + 1$.
   (e) Innermost loop completes.
      If $i = n$ then use the permutation $(0, 1, 2, \ldots, n-1) \rightarrow (j_0, j_1, j_2, \ldots, j_{n-1})$
      i.e. the permutation matrix $P$ is given by
      \[
      (P)_{uv} = \begin{cases} 
      1 & \text{if } v = j_u \\
      0 & \text{otherwise}
      \end{cases}
      \]
The SymbolicC++ program [1] utilizes the vector class of the Standard Template Library. The Hamilton operator refers to example 2 in the text (two point Hubbard model).

// permutation.cpp
#include <iostream>
#include <vector>
#include "symbolicc++.h"
using namespace std;

int total;
Symbolic H;

void commutes(const Symbolic &P)
{
    if(P*H==H*P) cout << "P[" << total++ << "] = " << P << endl;
}

void find_perm(int n,void (*use)(const Symbolic&))
{
    int i, k;
    Symbolic P;
    vector<int> j(n,-1);
    i = 0; j[0] = -1;
    while(i >= 0)
    {
        if(++j[i]==n) { j[i--] = -1; continue; }
        if(i < 0) break;
        for(k=0;k<i;++k) if(j[k]==j[i]) break;
        if(k!=i) continue;
        ++i;
        if(i==n)
        {
            P = 0*Symbolic("",n,n);
            for(k=0;k<n;k++) P(k,j[k]) = 1;
            use(P);
            --i;
        }
    }
}

int main(void)
{
    using SymbolicConstant::i;
    Symbolic sqrt2 = sqrt(Symbolic(2));
Symbolic U("U"); Symbolic t("t");
H = ((U,t,t,Symbolic(0)),(t,Symbolic(0),Symbolic(0),t),
    (t,Symbolic(0),Symbolic(0),t),(Symbolic(0),t,t,U));
cout << "H = " << H << endl;
total = 0;
find_perm(H.rows(),commutes);
return 0;
}

A Maxima implementation is available from the authors.

4 Conclusion

We applied a brute force method to find all possible permutation matrices that provide symmetries for given Hamilton operators in a finite dimensional Hilbert space. With growing size of the Hamilton operators matrix representation finding the permutation matrices becomes very time-consuming. A more efficient approach would be to find only the generators of the group of permutation matrices that provide symmetries for a given Hamilton operator. Another open question is how this method can be extended to find other classes of symmetries.

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References

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computing and Quantum Information*, Cambridge University Press, Cambridge (2000).

[2] Y. Hardy and W.-H. Steeb, *Classical and Quantum Computing with C++ and Java Simulations*, Birkhauser Verlag, Basel (2002).

[3] M. Hirvensalo, *Quantum Computing*, second edition, Springer, New York (2004).

[4] N. D. Mermin, *Quantum Computer Science*, Cambridge University Press, Cambridge (2007).
[5] W.-H. Steeb and Y. Hardy, *Problems and Solutions in Quantum Computing and Quantum Information*, third edition, World Scientific Publishing (2011).

[6] W.-H. Steeb, I. Tanski and Y. Hardy, *Problems and Solutions for Groups, Lie Groups, Lie Algebras with Applications*, World Scientific, Singapore (2012).

[7] W.-H. Steeb, Y. Hardy and J. de Greef, Z. Naturforsch. **67a**, 551 (2012).

[8] W.-H. Steeb and Y. Hardy, *Matrix Calculus and Kronecker Product*, second edition, World Scientific, Singapore (2011).

[9] W.-H. Steeb, *Problems and Solutions in Theoretical and Mathematical Physics*, Third edition, Volume II: Advanced Level, World Scientific, Singapore (2009).

[10] D. E. Knuth, *The Art of Computer Programming, Volume 4A: Combinatorial Algorithms, Part 1*, Addison-Wesley (2011).

[11] Y. Hardy, Kiat Shi Tan and W.-H. Steeb, *Computer Algebra with SymbolicC++*, World Scientific, Singapore (2008).