A New Determinant Inequality of Positive Semi-Definite Matrices

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Abstract—A new determinant inequality of positive semi-definite matrices is discovered and proved by us. This new inequality is useful for attacking and solving a variety of optimization problems arising from the design of wireless communication systems.

I. A NEW DETERMINANT INEQUALITY

The following notations are used throughout this article. The notations $[.]^T$ and $[.]^H$ stand for transpose and Hermitian transpose, respectively. $\text{tr}(A)$ and $\det(A)$ denote the trace and the determinant of the matrix $A$, respectively. The symbols $\mathbb{R}^{n \times m}$ and $\mathbb{R}^n$ stand for the set of $n \times m$ matrices and the set of $n$-dimensional column vectors with real entries, respectively. $\mathbb{C}^{n \times m}$ and $\mathbb{C}^n$ denote the set of $n \times m$ matrices and the set of $n$-dimensional column vectors with complex entries, respectively.

We introduce the following new determinant inequality.

Theorem 1: Suppose $A \in \mathbb{C}^{N \times N}$ and $B \in \mathbb{C}^{N \times N}$ are positive semi-definite matrices with eigenvalues $\{\lambda_k(A)\}$ and $\{\lambda_k(B)\}$ arranged in descending order, $D \in \mathbb{R}^{N \times N}$ is a diagonal matrix with non-negative diagonal elements $\{d_k\}$ arranged in descending order. Then the following determinant inequality holds

$$\det(D^H A D + B) \leq \prod_{k=1}^{N} (d_k^2 \lambda_k(A) + \lambda_{N+1-k}(B))$$

The above inequality becomes an equality if $A$ and $B$ are diagonal, and the diagonal elements of $A$ and $B$ are sorted in descending order and ascending order, respectively, i.e. $A = \text{diag}(\lambda_N(A), \ldots, \lambda_1(A))$, and $B = \text{diag}(\lambda_N(B), \ldots, \lambda_1(B))$.

Proof: See Appendix A

II. OPTIMIZATION USING THE NEW DETERMINANT INEQUALITY

The new determinant inequality can be used to solve the following optimization problem

$$\max_C \det(G + C^H H^H R_v^{-1} H C)$$

s.t. $\text{tr}(CR_x C^H) \leq P$

where $G$, $R_v$, and $R_x$ are positive definite matrices. Such an optimization arises when we design the precoding matrix associated with a transmit node so as to maximize the overall channel capacity (Details of the formulation are omitted here).

To gain an insight into (2), we reformulate the problem as follows. Let $\bar{C} \equiv CR_x^2$, and $\bar{G} \equiv R_x^2 G R_x^2$, the objective function (2) can be re-expressed as

$$\det(G + C^H H^H R_v^{-1} H C) = \det(R_x^{-1}) \det(\bar{G} + \bar{C}^H H^H R_v^{-1} H \bar{C})$$

To further simplify the problem, we carry out the SVD: $\bar{C} = U_c D_c V_c^H$ and the eigenvalue decomposition (EVD): $T \equiv H^H R_v^{-1} H = U_c D_t U_t^H$, and $\bar{G} = U_g D_g U_g^H$, where $U_c$, $V_c$, $U_t$, and $U_g$ are $p \times p$ unitary matrices, $D_c$, $D_t$ and $D_g$ are diagonal matrices given respectively as

$$D_c \equiv \text{diag}(d_{c,1}, d_{c,2}, \ldots, d_{c,p}) \quad D_t \equiv \text{diag}(\lambda_1(T), \lambda_2(T), \ldots, \lambda_{p}(T)) \quad D_g \equiv \text{diag}(\lambda_1(\bar{G}), \lambda_2(\bar{G}), \ldots, \lambda_{p}(\bar{G}))$$

in which $\lambda_k(T)$ and $\lambda_k(\bar{G})$ denote the $k$-th eigenvalue associated with $T$ and $\bar{G}$, respectively. Without loss of generality, we assume that the diagonal elements of $D_c$, $D_t$ and $D_g$ are arranged in descending order. We can rewrite (3) as

$$\det(R_x^{-1}) \det(\bar{G} + \bar{C}^H H^H R_v^{-1} H \bar{C}) = \det(R_x^{-1}) \det(\bar{V}_c^H \bar{G} \bar{V}_c + D_c^H U_c^H T U_c D_c)$$

$$\equiv \det(R_x^{-1}) \det(\bar{V}_c^H D_g \bar{V}_c + D_c^H U_c^H T D_c U_c)$$

where $\bar{V}_c \equiv U_c^H V_c$, and $\bar{U}_c \equiv U_c^H U_c$. Resorting to (5), the optimization (2) can be transformed into a new optimization that searches for an optimal set $\{\bar{U}_c, D_c, \bar{V}_c\}$, in which $\bar{U}_c$ and $\bar{V}_c$ are also unitary matrices

$$\max_{\{\bar{U}_c, D_c, \bar{V}_c\}} \det(\bar{V}_c^H D_g \bar{V}_c + D_c^H \bar{U}_c^H D_t \bar{U}_c D_c)$$

s.t. $\text{tr}(D_c D_c^H) \leq P$,

$$\bar{U}_c \bar{U}_c^H = I, \quad \bar{V}_c \bar{V}_c^H = I$$

The optimization involves searching for multiple optimization variables. Nevertheless, we can, firstly, find the optimal $\{\bar{U}_c, \bar{V}_c\}$ given that $D_c$ is fixed. Then substituting the derived optimal unitary matrices into (6), we determine the optimal diagonal matrix $D_c$. Optimizing $\{\bar{U}_c, \bar{V}_c\}$ conditional on a given $D_c$ can be formulated as

$$\max_{\{\bar{U}_c, \bar{V}_c\}} \det(\bar{V}_c^H D_g \bar{V}_c + D_c^H \bar{U}_c^H D_t \bar{U}_c D_c)$$

s.t. $\bar{U}_c \bar{U}_c^H = I, \quad \bar{V}_c \bar{V}_c^H = I$
Letting $A \triangleq \bar{U}_c^H D_t \bar{U}_c$, $B \triangleq \bar{V}_c^H D_g \bar{V}_c$, and utilizing Theorem 1, the objective function (7) is upper bounded by
\[
\begin{align*}
\det(\bar{V}_c^H D_g \bar{V}_c + D_g^c \bar{U}_c^H D_t \bar{U}_c D_c) \\
&\leq \prod_{k=1}^p (d_{c,k}^2 \lambda_k(A) + \lambda_{p+1-k}(B)) \\
&= \prod_{k=1}^p (d_{c,k}^2 \lambda_k(T) + \lambda_{p+1-k}(\bar{G}))
\end{align*}
\] (8)

The above inequality becomes an equality when $\bar{U}_c = I$ and $\bar{V}_c = J$, where $J$ is an anti-identity matrix, that is, $J$ has ones along the anti-diagonal and zeros elsewhere. Therefore the optimal solution to (7) is given by
\[
\bar{U}_c = I, \quad \bar{V}_c = J
\] (9)

Substituting the optimal $\{\bar{U}_c, \bar{V}_c\}$ back into (6), we arrive at the following optimization that searches for optimal diagonal elements $\{d_{c,k}\}$
\[
\max_{\{d_{c,k}\}} \prod_{k=1}^p (d_{c,k}^2 \lambda_k(T) + \lambda_{p+1-k}(\bar{G})) \\
\text{s.t.} \quad \sum_{k=1}^p d_{c,k}^2 \leq P, \quad d_{c,k} \geq 0 \quad \forall k
\] (10)

The above optimization (10) can be solved analytically by resorting to the Lagrangian function and KKT conditions, whose details are not elaborated here.

\section*{Appendix A}
\textbf{Proof of Theorem 1}

Define $\Gamma \triangleq D^H A D$, and its eigenvalues $\{\lambda_k(\Gamma)\}$ are arranged in descending order. Then we have
\[
\det(D^H A D + B) \leq \prod_{k=1}^N (\lambda_k(\Gamma) + \lambda_{N+1-k}(B))
\] (11)

The above inequality comes from the following well-known matrix inequality [1]:
\[
\prod_{k=1}^N (\lambda_k(X) + \lambda_k(Y)) \leq \det(X + Y) \\
\leq \prod_{k=1}^N (\lambda_k(X) + \lambda_{N+1-k}(Y))
\] (12)
in which $X$ and $Y$ are positive semidefinite Hermitian matrices, with eigenvalues $\{\lambda_k(X)\}$ and $\{\lambda_k(Y)\}$ arranged in descending order respectively.

To prove (11), we only need to show that the term on the right-hand side of (11) is upper bounded by
\[
\prod_{k=1}^N (\lambda_k(\Gamma) + \lambda_{N+1-k}(B)) \leq \prod_{k=1}^N (d_{c,k}^2 \lambda_k(A) + \lambda_{p+1-k}(B))
\] (13)

Before proceeding to prove (13), we introduce the following inequalities for the two sequences $\{\lambda_k(\Gamma)\}_{k=1}^N$ and $\{d_{c,k}^2 \lambda_k(A)\}_{k=1}^N$,
\[
\prod_{k=1}^K \lambda_k(\Gamma) \leq \prod_{k=1}^K d_{c,k}^2 \lambda_k(A) \quad 1 \leq K < N
\] (14)

The proof of the inequalities (14) is provided in Appendix B. The inequality relations between these two sequences can be characterized by the notion of “multiplicative majorization” (also termed log-majorization). Multiplicative majorization is a notion parallel to the concept of additive majorization. For two vectors $a \in \mathbb{R}_+^N$ and $b \in \mathbb{R}_+^N$ with elements sorted in descending order ($\mathbb{R}_+$ stands for the set of non-negative real numbers), we say that $a$ is multiplicatively majorized by $b$, denoted by $a \prec_\times b$, if
\[
\prod_{k=1}^K a_k \leq \prod_{k=1}^K b_k \quad 1 \leq K < N
\] (15)

Here we use the symbol $\prec_\times$ to differentiate the multiplicative majorization from the conventional additive majorization $\prec$. Another important concept is that closely related to majorization is schur-convex or schur-concave functions. A function $f: \mathbb{R}^N \to \mathbb{R}$ is said to be multiplicatively schur-convex if for $a \prec_\times b$, then $f(a) \leq f(b)$. Clearly, establishing (13) is equivalent to showing the function
\[
f(a) \triangleq \prod_{k=1}^N (a_k + c_{N+1-k})
\] (16)
is multiplicatively schur-convex for elements $c = [c_k] \in \mathbb{R}_+^N$ arranged in descending order. This multiplicatively schur-convex property can also be summarized as follows.

\textbf{Lemma 1:} For vectors $a \in \mathbb{R}_+^N$, $b \in \mathbb{R}_+^N$, and $c \in \mathbb{R}_+^N$, with their elements arranged in descending order, if $a \prec_\times b$, then we have $f(a) \leq f(b)$, i.e.
\[
\prod_{k=1}^N (a_k + c_{N+1-k}) \leq \prod_{k=1}^N (b_k + c_{N+1-k})
\] (17)

\textbf{Proof:} We prove Lemma 1 by induction. For $N = 2$, we have
\[
f(a) - f(b) = [a_1 + c_2][a_2 + c_1] - [b_1 + c_2][b_2 + c_1]
\]
\[
= [a_1 - b_1][c_1 + [a_2 - b_2]c_2]
\]
\[
\leq c_2[a_1 + a_2 - b_1 - b_2] \leq 0
\] (18)

where (a) can be easily derived by noting that $a_1 a_2 = b_1 b_2$; (b) comes from the fact that $a_1 - b_1 \leq 0$ and $c_1 \geq c_2$; (c) is a result of the following inequality: $a_1 + a_2 \leq b_1 + b_2$, that is, for any two non-negative elements, if their product remains constant, then their sum increases as the two elements are further apart.
Now suppose that for $M$-dimensional vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, the inequality (17) holds true. We show that (17) is also valid for $(M+1)$-dimensional vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. From the inequalities (14), we know that $b_i \geq a_i$. For the special case where $b_1 = a_1$, it is easy to verify that the truncated vector $\mathbf{a}_t \triangleq [a_2 \ldots a_{M+1}]$ is multiplicatively majorized by the truncated vector $\mathbf{b}_t \triangleq [b_2 \ldots b_{M+1}]$, i.e. $\mathbf{a}_t \precsim \mathbf{b}_t$. Therefore we have

$$f(\mathbf{a}_t) \leq f(\mathbf{b}_t) \quad (19)$$

and consequently we arrive at $f(\mathbf{a}) \leq f(\mathbf{b})$ given $b_1 = a_1$.

Now consider the general case where $b_1 > a_1$. There must be at least one index such that $b_1 < a_1$ since the overall products of the two sequences $\{a_k\}_{k=1}^{M+1}$ and $\{b_k\}_{k=1}^{M+1}$ are identical. Without loss of generality, let $l_1$ denote the smallest index for which $b_1 < a_1$. We adopt a pairwise transformation to convert the sequence $\{b_k\}_{k=1}^{M+1}$ into a new sequence $\{\beta_k\}_{k=1}^{M+1}$. Specifically, the first and the $l_1$th entries of $\{b_k\}_{k=1}^{M+1}$ are updated as

$$\begin{cases} 
\beta_1 = a_1, \beta_{l_1} = \frac{b_1 b_{l_1}}{a_1} & \text{if } b_1 b_{l_1} \leq a_1 a_{l_1} \\
\beta_1 = \frac{b_1 b_{l_1}}{a_1}, \beta_{l_1} = a_{l_1} & \text{if } b_1 b_{l_1} > a_1 a_{l_1} \quad (20)
\end{cases}$$

whereas other entries remain unaltered, i.e. $\beta_k = b_k, \forall k \neq 1, l_1$. Clearly, the entries $\beta_1$ and $\beta_{l_1}$ satisfy

$$\beta_1 \beta_{l_1} = b_1 b_{l_1} \quad (21)$$

That is, $[\beta_1 \beta_{l_1}]^T \precsim [b_1 b_{l_1}]^T$. By following the same argument of (18) and noting that $\beta_k = b_k, \forall k \neq 1, l_1$, we have

$$f(\beta) \leq f(\mathbf{b}) \quad (22)$$

where $\beta \triangleq [\beta_1 \ldots \beta_{M+1}]$. Our objective now is to show

$$f(\mathbf{a}) \leq f(\beta) \quad (23)$$

It can be easily verified that $\mathbf{a}$ is multiplicatively majorized by $\beta$, i.e. $\mathbf{a} \precsim \beta$, by noting $\beta_l \geq a_l$ for any $l < l_1$ and $\beta_1 \beta_{l_1} = b_1 b_{l_1}$.

Now we proceed to prove (23). Consider two different cases in (20).

- If $b_1 b_{l_1} \leq a_1 a_{l_1}$, then $\beta_1 = a_1$. In this case, it is easy to verify that the truncated vector $\mathbf{a}_t \triangleq [a_2 \ldots a_{M+1}]$ is multiplicatively majorized by the truncated vector $\mathbf{b}_t \triangleq [\beta_2 \ldots \beta_{M+1}]$, i.e. $\mathbf{a}_t \precsim \beta_t$. Therefore we have

$$f(\mathbf{a}_t) \leq f(\beta_t) \quad (24)$$

and consequently $f(\mathbf{a}) \leq f(\beta)$ as we have $\beta_1 = a_1$.

- For the second case where $b_1 b_{l_1} > a_1 a_{l_1}$, we have $\beta_{l_1} = a_{l_1}$. Define two new vectors $\mathbf{a}_p \triangleq [a_1 \ldots a_{l_1-1} a_{l_1+1} \ldots a_{M+1}]$ and $\mathbf{b}_p \triangleq [\beta_1 \ldots \beta_{l_1-1} \beta_{l_1+1} \ldots \beta_{M+1}]$. From $\mathbf{a} \precsim \beta$, we can readily verify that $\mathbf{a}_p$ is multiplicatively majorized by $\beta_p$, i.e. $\mathbf{a}_p \precsim \beta_p$. Therefore we have

$$f(\mathbf{a}_p) \leq f(\beta_p) \quad (25)$$

and consequently $f(\mathbf{a}) \leq f(\beta)$ as we have $\beta_{l_1} = a_{l_1}$.

Combining (22)–(23), we arrive at (17). The proof is completed here.

**APPENDIX B**

**PROOF OF (14)**

Recall the following theorem [2, Chapter 9: Theorem H.1]

**Theorem:** If $\mathbf{X}$ and $\mathbf{Y}$ are $N \times N$ complex matrices, then

$$\prod_{k=1}^{K} \sigma_k(\mathbf{XY}) \leq \prod_{k=1}^{K} \sigma_k(\mathbf{X})\sigma_k(\mathbf{Y}), \quad K = 1, \ldots, N - 1$$

$$\prod_{k=1}^{N} \sigma_k(\mathbf{XY}) = \prod_{k=1}^{N} \sigma_k(\mathbf{X})\sigma_k(\mathbf{Y}) \quad (26)$$

where $\{\sigma_k(\cdot)\}$ are singular values arranged in a descending order.

By utilizing the above results, we have

$$\prod_{k=1}^{K} \lambda_k(\mathbf{I}) = \prod_{k=1}^{K} \sigma_k(\mathbf{I}) \leq \prod_{k=1}^{K} \sigma_k(\mathbf{D})$$

$$= \left( \prod_{k=1}^{K} \sigma_k(\mathbf{D}) \right) \left( \prod_{k=1}^{K} d_k \right)$$

$$\leq \left( \prod_{k=1}^{K} \sigma_k(\mathbf{D}) \right) \left( \prod_{k=1}^{K} d_k \right)$$

$$= \prod_{k=1}^{K} d_k^2 \lambda_k(\mathbf{A}), \quad K = 1, \ldots, N - 1 \quad (27)$$

**REFERENCES**

[1] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[2] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*. Academic Press, 1979.

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1When $\mathbf{a}$ and $\mathbf{b}$ contain zero elements, the overall product is zero. In this case, we may not find an index such that $b_1 < a_1$. Nevertheless, since we have $b_k \geq a_k$ for all $k$, proof of (17) is evident.