Enumeration of (16,4,16,4) Relative Difference Sets

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Abstract

A complete enumeration of relative difference sets (RDS) with parameters (16,4,16,4) in a group of order 64 with a normal subgroup $N$ of order 4 is given. If $N = Z_4$, three of the eleven abelian groups of order 64, and 23 of the 256 nonabelian groups of order 64 contain (16,4,16,4) RDSs. If $N = Z_2 \times Z_2$, six of the abelian groups and 194 of the non-abelian groups of order 64 contain (16,4,16,4) RDSs.

Keywords: Relative difference set; symmetric net.

1 Introduction

A relative difference set (RDS) with parameters $(m,n,k,\lambda)$ in a finite group $G$ of order $mn$ relative to a normal subgroup $N$ of order $n$ is a $k$-subset $R$ of $G$ such that every element of $g \in G \setminus N$ appears exactly $\lambda$ times in the multiset $S = \{ab^{-1} \mid a, b \in R, a \neq b\}$, and no element of $N$ appears in $S$ [1]. An RDS is called abelian if $G$ is abelian, and nonabelian otherwise.

Relative difference sets are closely related to difference sets, group-divisible designs, generalized Hadamard matrices, symmetric nets, and finite geometry [1], [4], [6]. A comprehensive survey on RDS is the paper by Pott [5]. The existence problem of $(p^a, p^b, p^a, p^{a-b})$ RDSs is considered to be one of the most important questions concerning RDSs [5].

In [7], Schmidt studied the existence of abelian $(p^a, p^b, p^a, p^{a-b})$ RDS, and settled the existence problem of abelian (16,4,16,4) RDS completely.

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In this paper, a complete enumeration of $(16, 4, 16, 4)$ RDSs is given, for all groups, abelian and nonabelian, of order 64. In summary, RDSs exist in 6 of the 11 abelian groups of order 64, as well as in 195 of the 256 nonabelian groups of order 64. If $N = Z_4$, three of the eleven abelian groups of order 64, and 23 of the 256 non-abelian groups of order 64 contain $(16, 4, 16, 4)$ RDSs. If $N = Z_2 \times Z_2$, six of the abelian groups and 194 of the non-abelian groups of order 64 contain $(16, 4, 16, 4)$ RDSs. The computer algebra package Magma [2] was used in the computations.

2 RDS and symmetric nets

Our approach to the enumeration of $(16, 4, 16, 4)$ RDSs is based on their link to incidence structures known as symmetric $(4, 4)$-nets.

A symmetric $(4, 4)$-net\(^1\) is an incidence structure $I = (X, B)$ consisting of a set $X$ of 64 points and a collection $B$ of 64 blocks, each block being a subset of 16 points of $X$, having the following properties:

- Each point belongs to 16 blocks.
- There exists a partition $P$ of the point set $X$ into 16 subsets of size 4, called groups, so that every two points belonging to different groups appear together in exactly 4 blocks, while every two points belonging to the same group do not appear together in any block.
- The 64 blocks are partitioned into 16 parallel classes, each class consisting of 4 pairwise disjoint blocks, so that every two blocks belonging to different parallel classes share exactly 4 points.

Other terms used for a structure with the above properties are group-divisible design, or a transversal design [1].

An automorphism of an incidence structure $I$ is any permutation of the point set which preserves the collection of blocks. The set of all automorphisms of $I$ form a group, called the full automorphism group, $Aut(I)$, of $I$. The subgroups of $Aut(I)$ are called automorphism groups.

A symmetric $(4, 4)$-net is class-regular if it admits an automorphism group $N$ of order 4 which acts transitively on each group of points and each parallel class of blocks. The group $N$ is then called a group of bitranslations.

If $R$ is a $(16, 4, 16, 4)$ RDS in a group $G$ of order 64, relative to a normal subgroup $N \leq G$ of order 4, one can associate with $R$ a class-regular $(4, 4)$-net $I$ with point set $G$ and blocks being the subsets $B_g \subseteq G$ of the form

$$B_g = \{Rg \mid g \in G\}.$$

\(^1\)More generally, a net is defined as a resolvable 1-design, and a symmetric net is a net with equal number of points and blocks. For more definitions concerning designs see [1].
The partition $\mathcal{P}$ of the points into subsets of size 4 is defined as the partition of $G$ into cosets of $N$. Consequently, $G$ acts as an automorphism group of $I$, and the subgroup $N$ acts transitively on each point group and each parallel class.

Thus, any $(16, 4, 16, 4)$ RDS corresponds to a class-regular symmetric $(4, 4)$-net which admits a regular automorphism group.

All nonisomorphic class-regular symmetric $(4, 4)$-nets were enumerated by Harada, Lam and Tonchev in [4], and, implicitly, by Gibbons and Mathon in [3] (two incidence structures are isomorphic if there is an incidence preserving bijection between their point sets). Up to isomorphism, there are exactly 226 nets with group of bitranslations $N = Z_2 \times Z_2$, and 13 nets with $N = Z_4$.

These results reduce the enumeration of $(16, 4, 16, 4)$ RDSs to finding sharply transitive regular subgroups $G$ of the full automorphism groups of those class-regular symmetric $(4, 4)$-nets which admit automorphism groups acting transitively on the points, such that $N$ is a normal subgroup of $G$. We used Magma to find the conjugacy classes of sharply transitive regular subgroups.

There are 267 groups of order 64, of which 11 are abelian and 256 are nonabelian. Among the 226 nets with group of bitranslations $Z_2 \times Z_2$, only 200 nonisomorphic regular subgroups of order 64 appeared within the automorphism groups of those nets, of which 6 were abelian and 194 were nonabelian. Among the 13 nets with group of bitranslations $Z_4$, only 26 nonisomorphic regular subgroups of order 64 appeared within the automorphism groups of those nets, of which 3 were abelian and 23 were nonabelian.

3 The results

Tables 1 and 2 list the nets with automorphism groups which admit regular subgroups with normal subgroup $N$. Each entry is as follows:

- $#$: The index of the net within the list of nets with a group of bitranslations $N = Z_2 \times Z_2$ available at [http://www.math.mtu.edu/~tonchev/Z2Z2nets](http://www.math.mtu.edu/~tonchev/Z2Z2nets) and at [http://www.math.mtu.edu/~tonchev/Z4nets](http://www.math.mtu.edu/~tonchev/Z4nets) for the nets with $N = Z_4$. Missing indices indicate that the corresponding nets do not have transitive automorphism groups.

- **Order**: The order of the automorphism group.

- **2-Rank**: The 2-rank of the incidence matrix of the net.

- **Total**: In the format $x/y$, $x$ indicates the total number of conjugacy classes of regular subgroups containing the group of bitranslations, found within the automorphism group of each net, while $y$ is the number of nonisomorphic regular subgroups.

- **Abelian** and **Nonabelian**: In the format $x/y$, $x$ indicates the number of conjugacy classes of each type of subgroup found within the net’s automorphism group, while $y$ is the number of nonisomorphic subgroups.
• **List of indices:** Below the previous data is a list of the indices of the regular subgroups of order 64 found in each automorphism group according to Magma’s list of the groups of order 64. Entries of the form \(x(y)\) indicate that groups isomorphic to group \(x\) appeared in \(y\) distinct conjugacy classes. Entries marked with an asterisk are abelian, and all others are nonabelian.

Tables 3 and 4 give details about the structure of the regular abelian subgroups of order 64 found. Only such subgroups containing the relevant group \(N\) of bitranslations were considered. Groups with the following indices had a single regular abelian subgroup isomorphic to \(\mathbb{Z}_2^6\), and are not listed in Table 3: 6, 99, 100, 103, 104, 105, 107, 111, 113, 120, 121, 127, 128, 131, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 181, 182, 192, 193, 194, 195, 196, 197, 198, 212, 214, 221, 224, 225, 226.

Tables 5 and 6 summarize the structures of the regular abelian subgroups of order 64 found within the nets.

Table 7 gives the structures of all abelian groups of order 64 which do not appear as a regular subgroup in any net. Consequently, those are the groups which do not support any \((16, 4, 16, 4)\) RDS.

| #  | Order  | 2-Rank | Total | Abelian | Nonabelian |
|----|--------|--------|-------|---------|------------|
| 1  | 1105920| 16     | 173/57| 6/2     | 167/55     |
|    |        |        |       |         |            |
|    | 4(2), 9(2), 18(2), 20(2), 23(5), 25(2), 32(6), 33(2), 34, 35(3), 36, 37(3), 56(8), 60(2), 74, 77, 80, 88(4), 90(5), 92(2), 100(2), 102(2), 132, 165, 192(4)*, 193(4), 194, 198(2), 199(3), 200(2), 202(3), 206(3), 207(3), 210(4), 214(3), 215(3), 217(3), 219(4), 223(8), 224(5), 226(2), 227(4), 229(2), 230(2), 232(10), 236, 237(6), 239(4), 240, 241(2), 242(9), 243, 244(5), 245, 262(3), 264(2), 267(2)* |
| 2  | 13824  | 18     | 8/7   | 2/2     | 6/5        |
|    |        |        |       |         |            |
|    | 60(2), 88, 132, 192*, 193, 262, 267* |
| 4  | 4608   | 18     | 8/5   | 0/0     | 8/5        |
|    |        |        |       |         |            |
|    | 56(2), 88(2), 100, 132, 262(2) |
| 5  | 18432  | 16     | 61/21 | 6/2     | 55/19      |
|    |        |        |       |         |            |
|    | 4(2), 9(2), 20, 23, 32(4), 35(2), 56(14), 60(2), 74(3), 88(3), 90, 92, 100, 132(2), 192(5)*, 193(5), 202, 207(2), 242(5), 262(3), 267* |
| 6  | 4608   | 20     | 28/19 | 1/1     | 27/18      |
|    |        |        |       |         |            |
|    | 20(2), 23(2), 58, 60(6), 69(2), 71, 74, 75, 78, 90, 92, 100, 109, 193, 202(2), 204, 207, 247, 267* |
| 8  | 384    | 20     | 4/3   | 0/0     | 4/3        |
|    |        |        |       |         |            |
|    | 88, 132(2), 262 |
| 9  | 4608   | 18     | 8/7   | 2/2     | 6/5        |
|    |        |        |       |         |            |
|    | 60(2), 88, 132, 192*, 193, 262, 267* |
| #  | Order | 2-Rank | Total | Abelian | Nonabelian |
|----|-------|--------|-------|---------|------------|
| 22 | 6144  | 16     | 84/25 | 6/2     | 78/23      |
|    |        |        | 4(2), 9(2), 20, 23, 32(2), 56(29), 60, 66, 67(2), 69(3), 71(2), 73(2), 75(3), 78(2), 88(3), 90, 92, 109, 131, 164, 192(5)*, 193(8), 194(8), 260*, 261 |
| 24 | 1536  | 20     | 48/35 | 0/0     | 48/35      |
|    |        |        | 18, 22, 25(2), 88(2), 93(2), 97, 100(5), 102, 111, 114, 116, 117, 120, 122, 132(5), 133, 143, 145, 149, 151, 156, 158, 163, 166(3), 168, 170, 200, 206, 217, 229, 230, 249, 252, 255, 262 |
| 25 | 1536  | 18     | 9/5   | 0/0     | 9/5        |
|    |        |        | 56(3), 88(2), 100, 132, 262(2) |
| 30 | 1536  | 20     | 50/42 | 0/0     | 50/42      |
|    |        |        | 18, 22, 25(2), 33, 36, 91, 93(2), 97(2), 99, 100(2), 105, 111, 119, 120, 121, 122, 129, 130, 131, 132(2), 133, 142, 143, 144, 145(3), 146, 148, 149, 151, 156, 158, 166(2), 168, 170, 176, 178, 200, 249, 251, 252, 254, 255 |
| 31 | 1536  | 20     | 45/26 | 1/1     | 44/25      |
|    |        |        | 23, 24, 32(6), 35(4), 58, 59, 60, 61(2), 66, 67(2), 69, 72(2), 74, 75, 78, 85, 90(2), 101(2), 136(3), 138(2), 202(2), 207, 212, 242(2), 255(2), 260* |
| 32 | 128   | 20     | 4/4   | 1/1     | 3/3        |
|    |        |        | 146, 148, 206, 246* |
| 33 | 512   | 20     | 66/44 | 1/1     | 65/43      |
|    |        |        | 18, 25, 32(4), 33(4), 35(2), 86(2), 88, 91(2), 97(2), 99, 100, 102(2), 105, 108(3), 115, 116, 119, 122, 132, 133, 135, 138, 139(3), 145, 146(2), 148(2), 149, 151, 155, 158(2), 163, 166(3), 170, 179, 206(2), 217, 223(2), 227, 232, 237, 242, 246*, 251, 254 |
| 36 | 73728 | 16     | 1158/85 | 6/3 | 1152/82 |
|    |        |        | 4(2), 9(2), 18(6), 20(6), 23(13), 25(6), 32(14), 33(10), 34(5), 35(7), 36(5), 37(7), 56(13), 60(3), 62, 74(4), 77(2), 80, 88(6), 90(11), 92(3), 93, 99(2), 100(4), 102(6), 132(2), 192(4)*, 193(6), 194(2), 195(5), 196(5), 197(3), 198(13), 199(15), 200(5), 201(6), 202(7), 203(6), 204(6), 205(7), 206(29), 207(7), 209(8), 210(52), 211(3), 212(3), 213(12), 214(17), 215(16), 216(12), 217(19), 218(9), 219(56), 220(44), 221(18), 222(26), 223(68), 224(12), 225(13), 226(15), 227(58), 228(19), 229(22), 230(11), 231(5), 232(69), 233(46), 234(38), 235(23), 236(20), 237(33), 238(9), 239(5), 240(18), 241(27), 242(15), 243(24), 244(31), 260*, 261, 262(4), 263, 264(5), 265, 267* |
| 78 | 6144  | 20     | 28/24 | 0/0     | 28/24      |
|    |        |        | 4, 5, 9(4), 57, 62, 67, 68, 69(2), 70, 73, 74, 77, 78, 79, 81, 82, 87, 112, 131, 132, 164, 165, 208, 263 |
Table 1: (16, 4, 16, 4) RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

| #  | Order       | 2-Rank | Total  | Abelian | Nonabelian |
|----|-------------|--------|--------|---------|------------|
| 99 | 1152        | 20     | 4/2    | 1/1     | 3/1        |
|    | 60(3), 267* |        |        |         |        |
| 100| 1152        | 22     | 4/2    | 1/1     | 3/1        |
|    | 60(3), 267* |        |        |         |        |
| 103| 384         | 20     | 4/2    | 1/1     | 3/1        |
|    | 60(3), 267* |        |        |         |        |
| 104| 384         | 22     | 4/2    | 1/1     | 3/1        |
|    | 60(3), 267* |        |        |         |        |
| 105| 1152        | 18     | 4/2    | 1/1     | 3/1        |
|    | 60(3), 267* |        |        |         |        |
| 107| 6912        | 20     | 5/3    | 1/1     | 4/2        |
|    | 60(3), 193, 267* | |        |         |   |
| 109| 384         | 22     | 4/4    | 0/0     | 4/4        |
|    | 100, 109, 204, 247 | |        |         |   |
| 111| 512         | 20     | 52/19  | 1/1     | 51/18      |
|    | 20(2), 23(2), 58(2), 60(10), 69(6), 71(2), 74(3), 75(3), 78(3), 90, 92(2), 100, 109, 193(2), 202(6), 204(2), 207(2), 247, 267* | |        |         |   |
| 113| 1536        | 20     | 48/19  | 1/1     | 47/18      |
|    | 20(2), 23(2), 58(2), 60(10), 69(6), 71(2), 74(3), 75(3), 78(3), 90, 92, 100, 109, 193(2), 202(3), 204(2), 207(2), 247, 267* | |        |         |   |
| 115| 256         | 20     | 4/2    | 0/0     | 4/2        |
|    | 9(2), 132(2) |       |        |         |   |
| 116| 768         | 20     | 4/2    | 0/0     | 4/2        |
|    | 9(2), 132(2) |       |        |         |   |
| 117| 128         | 21     | 0/0    | 0/0     | 0/0        |
| 120| 1536        | 20     | 31/19  | 1/1     | 30/18      |
|    | 20(2), 23(2), 58, 60(6), 69(2), 71, 74, 75, 78, 90, 92(2), 100, 109, 193, 202(4), 204, 207, 247, 267* | |        |         |   |
| 121| 384         | 22     | 4/2    | 1/1     | 3/1        |
|    | 60(3), 267* |       |        |         |   |
| 122| 128         | 22     | 4/4    | 0/0     | 4/4        |
|    | 100, 109, 204, 247 | |        |         |   |
| 123| 192         | 22     | 1/1    | 0/0     | 1/1        |
|    | 20 |       |        |         |   |
| 124| 64          | 22     | 1/1    | 0/0     | 1/1        |
|    | 23 |       |        |         |   |
Table 1: (16, 4, 16, 4) RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

| #    | Order | 2-Rank | Total | Abelian | Nonabelian |
|------|-------|--------|-------|---------|------------|
| 125  | 64    | 22     | 1/1   | 0/0     | 1/1        |
| 132  |       |        |       |         |            |
| 126  | 192   | 22     | 1/1   | 0/0     | 1/1        |
| 132  |       |        |       |         |            |
| 127  | 768   | 20     | 6/3   | 1/1     | 5/2        |
|      | 60(4), 193, 267* |     |       |         |            |
| 128  | 256   | 20     | 7/3   | 1/1     | 6/2        |
|      | 60(4), 193(2), 267* |     |       |         |            |
| 131  | 768   | 20     | 6/3   | 1/1     | 5/2        |
|      | 60(4), 193, 267* |     |       |         |            |
| 132  | 512   | 20     | 52/32 | 1/1     | 51/31      |
|      | 23, 24, 32, 33(2), 58, 59(2), 61, 62, 63, 65(2), 66, 67(3), 68(3), 69(7), 70, 71, 72, 75, 77(3), 78(2), 80, 81(2), 85, 90(2), 98(2), 101(2), 139, 192*, 197, 204, 205, 212 |
| 134  | 256   | 20     | 64/21 | 0/0     | 64/21      |
|      | 58, 59(2), 61, 63, 65, 68(2), 69(10), 70(5), 71(4), 72, 73(2), 75(9), 77(4), 78(7), 79(3), 80(3), 81(2), 195(2), 196, 197, 204(2) |     |       |         |            |
| 136  | 512   | 20     | 52/28 | 1/1     | 51/27      |
|      | 23, 24, 32, 35(2), 55*, 58(2), 59, 61, 63(3), 66(2), 67(4), 69(8), 71, 72(4), 74(3), 75, 76, 78(2), 81(2), 90(2), 98, 101(2), 104, 136, 197, 208, 255, 262 |
| 137  | 768   | 20     | 24/13 | 0/0     | 24/13      |
|      | 57, 59, 69(4), 70(3), 72, 77(3), 78(2), 79(2), 80(3), 81, 82, 197, 212 |
| 138  | 1536  | 20     | 45/30 | 1/1     | 44/29      |
|      | 23, 24, 32(2), 33(4), 34(4), 55*, 58, 61, 62(2), 65, 67, 69, 70, 71, 75, 77, 79, 90(2), 98, 101(2), 104, 143(2), 139(3), 194, 196, 203, 205(2), 209, 241(2), 254 |
| 139  | 512   | 21     | 52/40 | 0/0     | 52/40      |
|      | 18, 20, 25(2), 89, 91(2), 92, 95, 97, 98, 100(3), 102(2), 105, 112, 115, 117, 119, 120, 129, 131(2), 132, 133(2), 145(2), 148, 151, 159, 160, 163, 164, 165, 166, 169(2), 198, 200, 207(2), 217(2), 227(2), 229, 252, 255, 264 |
| 140  | 512   | 21     | 66/44 | 0/0     | 66/44      |
|      | 18, 25, 32(2), 33(6), 35(2), 86(2), 88, 91(2), 97, 100(3), 102(2), 105, 108(2), 109, 116, 117, 120, 122, 132(2), 137, 138, 139(3), 143, 148(2), 151(4), 156, 158, 160, 165, 166(3), 168, 182, 204, 206, 217, 222, 225, 230, 232, 233, 244, 247, 252, 255 |
Table 1: \((16,4,16,4)\) RDS with \(N = \mathbb{Z}_2 \times \mathbb{Z}_2\)

| \#  | Order | 2-Rank | Total | Abelian | Nonabelian |
|-----|-------|--------|-------|---------|------------|
| 142 | 512   | 19     | \(52/19\) | 1/1     | \(51/18\)  |
|     |       |        |       |         |            |
|     |       |        |       |         | 20(2), 23(2), 32(2), 60(10), 66(2), 67(2), 69(4), 71(3), 77(4), 78(4), 88, 90, 92, 109(2), 193(2), 195(2), 202(5), 205(2), 267* |
| 143 | 6144  | 20     | \(45/33\) | 1/1     | \(44/32\)  |
|     |       |        |       |         |            |
|     |       |        |       |         | 5, 9(2), 20, 23(2), 32, 57, 58, 60(5), 61, 62, 66(2), 67, 68, 69(2), 70, 71, 77(2), 78, 79, 81, 88, 90(4), 92, 109, 112, 164, 165, 193, 195, 202, 205, 208, 267* |
| 144 | 128   | 20     | \(4/2\) | 1/1     | 3/1        |
|     |       |        |       |         |            |
|     |       |        |       |         | 60(3), 267* |
| 145 | 256   | 20     | \(6/6\) | 1/1     | 63/5       |
|     |       |        |       |         |            |
|     |       |        |       |         | 60(21), 69(21), 77(7), 80(7), 209(7), 267* |
| 146 | 64    | 22     | 1/1   | 1/1     | 0/0        |
|     |       |        |       |         |            |
|     |       |        |       |         | 267* |
| 147 | 128   | 20     | \(4/2\) | 1/1     | 3/1        |
|     |       |        |       |         |            |
|     |       |        |       |         | 60(3), 267* |
| 148 | 128   | 22     | \(4/2\) | 1/1     | 3/1        |
|     |       |        |       |         |            |
|     |       |        |       |         | 60(3), 267* |
| 149 | 128   | 22     | \(4/2\) | 1/1     | 3/1        |
|     |       |        |       |         |            |
|     |       |        |       |         | 60(3), 267* |
| 150 | 384   | 20     | \(4/2\) | 1/1     | 3/1        |
|     |       |        |       |         |            |
|     |       |        |       |         | 60(3), 267* |
| 151 | 1152  | 21     | \(2/2\) | 1/1     | 1/1        |
|     |       |        |       |         |            |
|     |       |        |       |         | 60, 267* |
| 152 | 512   | 21     | \(66/43\) | 0/0     | \(66/43\)  |
|     |       |        |       |         |            |
|     |       |        |       |         | 18(2), 25(2), 32(2), 33(2), 34(3), 35(3), 91(6), 97(2), 99, 100(3), 105(2), 115(2), 117(2), 118, 119(2), 120, 129, 130, 132, 133(3), 144, 145, 146, 148, 149, 151, 159, 160, 164, 165, 200(2), 227, 229, 232, 233, 234, 236, 241, 244, 251, 252, 254, 255 |
| 153 | 512   | 21     | \(66/43\) | 0/0     | \(66/43\)  |
|     |       |        |       |         |            |
|     |       |        |       |         | 18(2), 25(2), 32(2), 33(2), 34(3), 35(3), 91(6), 97(2), 99, 100(3), 105(2), 115(2), 117(2), 118, 119(2), 120, 129, 130, 132, 133(3), 144, 145, 146, 148, 149, 151, 159, 160, 164, 165, 200(2), 227, 229, 232, 233, 234, 236, 241, 244, 251, 252, 254, 255 |
| 154 | 512   | 21     | \(66/43\) | 0/0     | \(66/43\)  |
|     |       |        |       |         |            |
|     |       |        |       |         | 18(2), 25(2), 32(2), 33(2), 34(3), 35(3), 91(6), 97(2), 99, 100(3), 105(2), 115(2), 117(2), 118, 119(2), 120, 129, 130, 132, 133(3), 144, 145, 146, 148, 149, 151, 159, 160, 164, 165, 200(2), 227, 229, 232, 233, 234, 236, 241, 244, 251, 252, 254, 255 |
| 155 | 512   | 21     | \(66/43\) | 0/0     | \(66/43\)  |
|     |       |        |       |         |            |
|     |       |        |       |         | 18(2), 25(2), 32(2), 33(2), 34(3), 35(3), 91(6), 97(2), 99, 100(3), 105(2), 115(2), 117(2), 118, 119(2), 120, 129, 130, 132, 133(3), 144, 145, 146, 148, 149, 151, 159, 160, 164, 165, 200(2), 227, 229, 232, 233, 234, 236, 241, 244, 251, 252, 254, 255 |
| 156 | 256   | 25     | \(8/7\) | 0/0     | \(8/7\)   |
|     |       |        |       |         |            |
|     |       |        |       |         | 35, 84, 91, 97, 170, 172(2), 214 |
| 157 | 256   | 21     | \(68/48\) | 0/0     | \(68/48\)  |
|     |       |        |       |         |            |
|     |       |        |       |         | 86, 89, 95, 97, 98, 99, 105, 109(3), 116(2), 117(2), 118, 119, 122, 123, 128, 130, 133(2), 141, 144, 146(4), 147(2), 149(3), 150(2), 151, 155, 157(2), 159, 160(2), 162, 163(4), 164(2), 166, 170, 171, 179, 180, 216, 218, 219, 221(2), 223, 226, 227(2), 228, 232, 233, 248, 254 |
Table 1: (16, 4, 16, 4) RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

| #  | Order | 2-Rank | Total  | Abelian | Nonabelian |
|----|-------|--------|--------|---------|------------|
| 158| 256   | 21     | 68/47  | 0/0     | 68/47      |
|    | 85, 88, 96(2), 97, 100, 104, 106, 108, 109, 115(2), 116(2), 120(2), 121, 122, 129, 132(2), 133, 143, 145, 148(2), 149(3), 151(7), 158, 159, 160(4), 162, 163, 165(2), 166(4), 169, 172, 180, 182, 201, 204, 206, 210, 213, 220, 228, 229, 230, 232, 233, 235, 247, 252 |
| 166| 1024  | 20     | 52/33  | 1/1     | 51/32      |
|    | 9(2), 20(2), 23(2), 33, 35, 58, 62(3), 63, 65, 66, 67(3), 68(2), 69(2), 72(3), 74, 75(2), 77, 85, 88(2), 90(2), 92(2), 93(4), 100, 104, 109, 114, 192*, 195(2), 204, 205, 212, 217, 262 |
| 167| 512   | 21     | 52/40  | 1/1     | 51/39      |
|    | 18(3), 20(2), 22, 25(2), 33, 36, 83*, 85, 92, 100, 105(3), 111, 112, 114, 119, 120, 121, 122, 131, 132, 145(2), 146, 148, 149, 151, 155, 156(2), 157, 158(2), 166(2), 169(2), 180(2), 199, 200, 207, 210, 214, 223, 232, 237 |
| 168| 512   | 19     | 66/33  | 0/0     | 66/33      |
|    | 20(2), 23(2), 32(6), 33(2), 35(2), 58, 59, 62(2), 66(5), 67(4), 68(3), 69(2), 70(2), 71, 72, 75(3), 76, 78(2), 79(2), 90, 91(2), 93, 94(2), 100, 109, 194, 195(2), 196, 204(2), 205(2), 212, 232(4), 247 |
| 169| 512   | 20     | 48/21  | 0/0     | 48/21      |
|    | 5(2), 9(4), 59(5), 63, 68(2), 69(8), 70, 72(2), 74, 76, 77(2), 78(4), 79(4), 80, 81, 113(2), 132, 164(2), 165, 197(2), 212 |
| 175| 512   | 20     | 2/2    | 0/0     | 2/2        |
|    | 4, 87 |
| 177| 64    | 22     | 1/1    | 0/0     | 1/1        |
|    | 132   |
| 178| 64    | 22     | 1/1    | 0/0     | 1/1        |
|    | 233   |
| 181| 384   | 21     | 4/2    | 1/1     | 3/1        |
|    | 60(3), 267* |
| 182| 1536  | 19     | 76/29  | 1/1     | 75/28      |
|    | 20(2), 23(2), 32(4), 33(4), 34(2), 35(2), 60(3), 88, 90, 92, 99, 100, 193(2), 195, 196, 202(7), 203, 204, 205, 207, 209(2), 211, 216(4), 219(8), 227(8), 232(6), 241(6), 263, 267* |
| 183| 512   | 21     | 52/39  | 0/0     | 52/39      |
|    | 25(2), 86(2), 91(2), 99, 100(3), 102(2), 108(2), 119, 120, 121, 122, 129, 131, 132(2), 135, 137, 139(2), 142, 143, 145(2), 146, 148, 149, 151, 156, 158, 159, 160, 163, 166(3), 168, 169, 178, 182, 217(2), 251, 252, 254, 255 |
Table 1: (16, 4, 16, 4) RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

| #  | Order   | 2-Rank | Total | Abelian | Nonabelian |
|----|---------|--------|-------|---------|------------|
| 184| 256     | 21     | 64/38 | 0/0     | 64/38      |
|    | 96(2), 98, 99, 106, 109, 119(2), 120(2), 121(2), 122, 123, 129(3), 131(2), 132, 133(2), 142, 143, 144(2), 145(4), 146(2), 147(2), 151(4), 157, 158, 159(2), 160(4), 161(2), 162, 165(3), 166(2), 169(2), 170, 172, 176, 178, 180, 182, 252, 254 |
| 185| 512     | 19     | 52/32 | 0/0     | 52/32      |
|    | 20(2), 23(2), 33, 35, 58, 59, 62(2), 66(3), 67(3), 68(3), 69(4), 70(2), 71(2), 72, 74, 75(5), 76, 77(2), 79(2), 80, 90, 91, 93, 94, 100, 109, 194, 195, 204, 205, 212, 247 |
| 186| 512     | 21     | 52/31 | 1/1     | 51/30      |
|    | 18, 19, 83*, 85, 86(2), 88(2), 89(3), 91, 94, 97, 101, 102, 104(2), 105(2), 108, 110, 112(3), 114(3), 115(2), 116(4), 117(6), 126(2), 127(2), 135, 137, 138, 139, 206, 247, 248, 256 |
| 187| 512     | 20     | 48/23 | 0/0     | 48/23      |
|    | 5(2), 9(4), 58, 59, 61, 62(2), 63, 66(4), 67(2), 69(2), 70(3), 72(2), 73, 75(5), 77, 78(3), 79(4), 113(2), 131, 164(3), 196, 204, 212 |
| 190| 1536    | 20     | 28/11 | 0/0     | 28/11      |
|    | 4(2), 9(4), 62(3), 67(4), 73(2), 74(3), 82(2), 87(2), 131(2), 132(2), 263(2) |
| 192| 128     | 21     | 4/2   | 1/1     | 3/1        |
|    | 60(3), 267* |
| 193| 128     | 22     | 4/2   | 1/1     | 3/1        |
|    | 60(3), 267* |
| 194| 384     | 20     | 4/2   | 1/1     | 3/1        |
|    | 60(3), 267* |
| 195| 128     | 21     | 4/2   | 1/1     | 3/1        |
|    | 60(3), 267* |
| 196| 640     | 22     | 4/2   | 1/1     | 3/1        |
|    | 60(3), 267* |
| 197| 384     | 22     | 2/2   | 1/1     | 1/1        |
|    | 60, 267* |
| 198| 384     | 20     | 4/2   | 1/1     | 3/1        |
|    | 60(3), 267* |
| 201| 64      | 22     | 1/1   | 0/0     | 1/1        |
|    | 20 |
| 202| 64      | 22     | 1/1   | 0/0     | 1/1        |
|    | 23 |
| 203| 128     | 22     | 4/4   | 0/0     | 4/4        |
|    | 97, 108, 206, 247 |
Table 1: \((16, 4, 16, 4)\) RDS with \(N = \mathbb{Z}_2 \times \mathbb{Z}_2\)

| #    | Order | 2-Rank | Total  | Abelian | Nonabelian |
|------|-------|--------|--------|---------|------------|
| 204  | 64    | 22     | 1/1    | 0/0     | 1/1        |
| 206  | 64    | 22     | 1/1    | 0/0     | 1/1        |
| 207  | 128   | 22     | 2/1    | 0/0     | 2/1        |
| 208  | 64    | 22     | 1/1    | 0/0     | 1/1        |
| 209  | 64    | 22     | 1/1    | 0/0     | 1/1        |
| 211  | 30720 | 18     | 2316/78| 2/2     | 2314/76    |
| 212  | 2048  | 20     | 92/38  | 1/1     | 91/37      |
| 213  | 2048  | 20     | 48/27  | 0/0     | 48/27      |
| 214  | 1024  | 20     | 32/23  | 1/1     | 31/22      |
| 221  | 768   | 19     | 24/6   | 1/1     | 23/5       |
| 222  | 256   | 22     | 8/8    | 0/0     | 8/8        |

18(4), 20(4), 23(4), 25(4), 32(8), 33(8), 34(4), 35(4), 36(4), 37(4), 60, 62, 88(2), 90(2), 92, 93, 99(2), 100(2), 102(4), 192*, 193(2), 194, 195(9), 196(17), 197(5), 198(22), 199(17), 200(7), 201(22), 202(7), 203(17), 204(15), 205(17), 206(62), 207(7), 208(9), 209(10), 210(128), 211(2), 212(4), 213(32), 214(34), 215(29), 216(32), 217(36), 218(31), 219(132), 220(124), 221(66), 222(66), 223(116), 224(19), 225(31), 226(34), 227(122), 228(65), 229(46), 230(19), 231(19), 232(135), 233(122), 234(122), 235(65), 236(34), 237(65), 238(19), 239(11), 240(34), 241(47), 242(15), 243(63), 244(66), 245(5), 262(2), 263(4), 264(6), 265(2), 267*

212 2048 20 92/38 1/1 91/37
5, 9(2), 20, 23(4), 32, 58(3), 59, 60(10), 61(3), 62(3), 64, 66(6), 67(3), 68(5), 69(6), 70, 71(2), 74, 76, 77(4), 78(3), 79(2), 80, 81, 88, 90(8), 92, 109, 112, 132, 164, 193(2), 195(2), 197, 202(2), 205(3), 208, 267*

213 2048 20 48/27 0/0 48/27
4, 5, 9(4), 59, 62(2), 64, 67(3), 68(5), 69(6), 70, 73, 74(3), 76, 77, 78(3), 79(2), 80, 81, 82, 87, 112, 131, 132(2), 164, 197, 208, 263

214 1024 20 32/23 1/1 31/22
4, 9(2), 24(2), 55*, 57, 59, 66(2), 68, 69(2), 71(3), 72, 75(2), 77, 80, 87, 88, 89, 197, 207, 208(2), 212(2), 222, 247

221 768 19 24/6 1/1 23/5
60(7), 69(7), 77(3), 80(3), 209(3), 267*

222 256 22 8/8 0/0 8/8
35, 90, 114, 131, 145, 169, 178, 222
### Table 1: (16, 4, 16, 4) RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

| #  | Order | 2-Rank | Total  | Abelian | Nonabelian |
|----|-------|--------|--------|---------|-----------|
| 223| 192   | 22     | 1/1    | 0/0     | 1/1       |
|    | 20    |        |        |         |           |
| 224| 768   | 20     | 24/6   | 1/1     | 23/5      |
|    | 60(7), 69(7), 77(3), 80(3), 209(3), 267* | | | | |
| 225| 256   | 20     | 64/6   | 1/1     | 63/5      |
|    | 60(21), 69(21), 77(7), 80(7), 209(7), 267* | | | | |
| 226| 320   | 21     | 1/1    | 1/1     | 0/0       |
|    | 267*  |        |        |         |           |

### Table 2: (16, 4, 16, 4) RDS with $N = \mathbb{Z}_4$

| #  | Order | 2-Rank | Total   | Abelian | Nonabelian |
|----|-------|--------|---------|---------|------------|
| 1  | 512   | 22     | 14/9    | 0/0     | 14/9       |
|    | 58, 59(2), 61(2), 72(4), 85, 175, 208, 212, 262 | | | | |
| 5  | 256   | 22     | 16/10   | 0/0     | 16/10      |
|    | 59, 61, 63, 65, 70(4), 72(2), 195, 197(2), 204(2), 212 | | | | |
| 6  | 512   | 22     | 14/13   | 0/0     | 14/13      |
|    | 58, 59, 61, 63, 65, 70(2), 72, 85, 168, 194, 197, 204, 212 | | | | |
| 8  | 512   | 22     | 14/10   | 2/2     | 12/8       |
|    | 55*, 58(2), 59(3), 61, 70, 104, 142, 192*, 195, 196(2) | | | | |
| 9  | 7680  | 22     | 9/8     | 2/2     | 7/6        |
|    | 55*, 57, 58(2), 61, 104, 143, 197, 260* | | | | |
| 10 | 128   | 22     | 4/4     | 0/0     | 4/4        |
|    | 59, 72, 197, 212 | | | | |
| 11 | 384   | 22     | 2/2     | 1/1     | 1/1        |
|    | 55*, 70 | | | | |
| 12 | 64    | 24     | 1/1     | 0/0     | 1/1        |
|    | 238   |        |         |         |            |
| 13 | 384   | 22     | 2/2     | 0/0     | 2/2        |
|    | 57, 212 | | | | |
Table 3: Structures of abelian regular subgroups in $\mathbb{Z}_2 \times \mathbb{Z}_2$ nets

| #  | Order      | Abelian | Group structure                  |
|----|------------|---------|----------------------------------|
| 1  | 1105920    | 6/2     | $\mathbb{Z}_2^2 \times \mathbb{Z}_4^4(4), \mathbb{Z}_2(2)$ |
| 2  | 13824      | 2/2     | $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2, \mathbb{Z}_2^6$ |
| 5  | 18432      | 6/2     | $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2(5), \mathbb{Z}_2^6$ |
| 9  | 4608       | 2/2     | $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2, \mathbb{Z}_2^6$ |
| 22 | 6144       | 6/2     | $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2(5), \mathbb{Z}_2^4 \times \mathbb{Z}_4$ |
| 31 | 1536       | 1/1     | $\mathbb{Z}_2^3 \times \mathbb{Z}_4$ |
| 32 | 128        | 1/1     | $\mathbb{Z}_2^3 \times \mathbb{Z}_8$ |
| 33 | 512        | 1/1     | $\mathbb{Z}_2^3 \times \mathbb{Z}_8$ |
| 36 | 73728      | 6/3     | $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2(4), \mathbb{Z}_2^2 \times \mathbb{Z}_4, \mathbb{Z}_2^6$ |
| 132| 512        | 1/1     | $\mathbb{Z}_2^4 \times \mathbb{Z}_4$ |
| 136| 512        | 1/1     | $\mathbb{Z}_2^3$ |
| 138| 1536       | 1/1     | $\mathbb{Z}_2^3$ |
| 166| 1024       | 1/1     | $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$ |
| 167| 512        | 1/1     | $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ |
| 186| 512        | 1/1     | $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$ |
| 211| 30720      | 2/2     | $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2, \mathbb{Z}_2^6$ |
| 214| 1024       | 1/1     | $\mathbb{Z}_2^3$ |

Table 4: Structures of abelian regular subgroups in $\mathbb{Z}_4$ nets.

| #  | Order | Abelian | Group structure |
|----|-------|---------|-----------------|
| 8  | 512   | 2/2     | $\mathbb{Z}_4^1, \mathbb{Z}_2^2 \times \mathbb{Z}_4^2$ |
| 9  | 7680  | 2/2     | $\mathbb{Z}_4^1, \mathbb{Z}_2 \times \mathbb{Z}_4$ |
| 11 | 384   | 1/1     | $\mathbb{Z}_4^1$ |

Table 5: All regular abelian subgroups of order 64 appearing in GDDs with bitranslation group $\mathbb{Z}_4$.

| Group Structure |
|-----------------|
| 55 $\mathbb{Z}_4^3$ |
| 192 $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$ |
| 260 $\mathbb{Z}_2^4 \times \mathbb{Z}_4$ |

Table 6: All regular abelian subgroups of order 64 appearing in GDDs with bi-translation group $\mathbb{Z}_2 \times \mathbb{Z}_2$. 

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Table 7: Abelian groups which do not contain any (16, 4, 16, 4) RDS

| Group | Structure               |
|-------|-------------------------|
| 55    | $\mathbb{Z}_4^3$        |
| 83    | $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$ |
| 192   | $\mathbb{Z}_2^2 \times \mathbb{Z}_4^7$ |
| 246   | $\mathbb{Z}_2^3 \times \mathbb{Z}_8$ |
| 260   | $\mathbb{Z}_2^4 \times \mathbb{Z}_4$ |
| 267   | $\mathbb{Z}_2^6$        |

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