Persistence of random walk records

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Received 10 April 2014, revised 8 May 2014
Accepted for publication 21 May 2014
Published 9 June 2014

Abstract
We study records generated by Brownian particles in one dimension. Specifically, we investigate an ordinary random walk and define the record as the maximal position of the walk. We compare the record of an individual random walk with the mean record, obtained as an average over infinitely many realizations. We term the walk ‘superior’ if the record is always above average, and conversely, the walk is said to be ‘inferior’ if the record is always below average. We find that the fraction of superior walks, S, decays algebraically with time, \( S \sim t^{-\beta} \), in the limit \( t \to \infty \), and that the persistence exponent is nontrivial, \( \beta = 0.382\,258 \ldots \). The fraction of inferior walks, I, also decays as a power law, \( I \sim t^{-\alpha} \), but the persistence exponent is smaller, \( \alpha = 0.241\,608 \ldots \). Both exponents are roots of transcendental equations involving the parabolic cylinder function. To obtain these theoretical results, we analyze the joint density of superior walks with a given record and position, while for inferior walks it suffices to study the density as a function of position.

Keywords: random walk, record, first passage, persistence, nonequilibrium dynamics, data analysis

PACS numbers: 05.40.Fb, 05.40.Jc, 02.50.Cw, 02.50.Ey

(Some figures may appear in colour only in the online journal)

1. Introduction

The record defined as the extremum in a sequence of variables is a useful characteristic of a dataset. Extreme value theory [1–5] and analysis of sequences of uncorrelated random variables [6, 7] provide the basis for understanding record statistics. Records in problems ranging from finance [8–11] and sport [12] to random structures [13, 14] and complex networks [15–17] typically involve sequences of correlated random variables. However, the current theoretical understanding of extreme values of correlated random variables is still far from complete [18–20].
Figure 1. An illustration of a superior walk (left) and an inferior walk (right). The spacetime diagrams ($t$ versus $x$) display the average record (black line), the position (blue line) and the record (thick red line).

Brownian trajectories are prime examples of correlated time series [21–23]. Established record statistics of discrete-time one-dimensional random walks include the distribution of the number of records and the mean duration of the longest record [24, 25]. Records in ensembles of random walks, especially the distribution of the maximum, have also been studied both for independent [26, 27] and for interacting random walks [28, 29]. However, much less is known about random walk records in higher dimensions [30, 31].

Recent studies show that first-passage [32] and persistence properties [33, 34] of records have rich phenomenology [35, 36]. For a sequence of uncorrelated random variables, the probability that all records are above average decays algebraically with sequence length, and this behavior is governed by a nontrivial persistence exponent. Such persistence characteristics were used to analyze earthquake data [35, 36]. In this article, we study similar persistence characteristics of random walk records.

We consider a discrete-time random walk in one dimension. The walk starts at the origin, $x(0) = 0$, and in each time step the walk makes a jump: $x(t + 1) = x(t) + \Delta_t$. The jump lengths $\Delta_t$ are independent random variables chosen from a symmetric distribution with finite variance: $\langle \Delta \rangle = 0$ and $\langle \Delta_i \Delta_j \rangle = \delta_{ij} \langle \Delta^2 \rangle$ with $\langle \Delta^2 \rangle < \infty$.

The record $r(t)$ is defined as the maximal position of the random walk in the time interval $(0, t)$

$$r(t) = \max\{x(0), x(1), x(2), \ldots, x(t)\}. \quad (1)$$

We compare the record with the average record $a(t) = \langle r(t) \rangle$, where the brackets denote an average over all possible realizations of the random process governing the position $x(t)$. As shown in figure 1, we compare the sequence of records $\{r(0), r(1), r(2), \ldots, r(t)\}$ generated by the random walk with the sequence of average records $\{a(0), a(1), a(2), \ldots, a(t)\}$. We call a random walk superior if all records exceed the average, $r(\tau) \geq a(\tau)$, for all $\tau = 0, 1, 2, \ldots, t$. Similarly, we define an inferior walk as one for which all records trail the average, $r(\tau) \leq a(\tau)$, for all $\tau = 0, 1, 2, \ldots, t$.

We now define $S(t)$ and $I(t)$ as the probability that at time $t$, a walk is superior or inferior. Our main result is that the probabilities $S(t)$ and $I(t)$ decay algebraically with time

$$S \sim t^{-\beta} \quad \text{and} \quad I \sim t^{-\alpha} \quad (2)$$
as \( t \to \infty \). The persistence exponents are transcendental numbers \( \beta = 0.382 
{}258 \ldots \) and \( \alpha = 0.241 
{}608 \ldots \). Both exponents are related to roots of the parabolic cylinder function,
\[
D_{2\beta+1}(\sqrt{2/\pi}) = 0 \quad \text{and} \quad D_{2\alpha}(-\sqrt{2/\pi}) = 0. \tag{3}
\]
The asymptotic behaviors (2)–(3) apply as long as the jump length distribution has zero mean and finite variance. Hence, we can restrict our attention to random walks with unit jump length: \( \Delta = 1 \) and \( \Delta = -1 \) are chosen with equal probabilities.

The rest of this paper is organized as follows. In section 2, we briefly summarize the basic properties of the record, including the distribution of records and the mean record. Since the record is coupled to position, we study the joint density of superior walks with given record and position. This distribution obeys the diffusion equation, and we obtain the long-time asymptotic behavior using scaling analysis (section 3). In the complementary case of inferior walks, the analysis simplifies because it suffices to consider only the position (section 4). A few generalizations are mentioned in section 5, and concluding remarks are given in section 6.

2. The average record

We use a simple random walk as a model for Brownian motion in one dimension [23, 37]. The random walk starts at the origin, \( x = 0 \), at time \( t = 0 \), and in each time step, its position changes by a fixed amount
\[
x(t + 1) = \begin{cases} x(t) - 1 & \text{with prob. 1/2;} \\ x(t) + 1 & \text{with prob. 1/2.} \end{cases} \tag{4}
\]
With these jump rules, the average position does not change, \( \langle x(t) \rangle = 0 \), while the mean square displacement equals time \( \langle x^2(t) \rangle = t \).

The record defined in equation (1) equals the maximum position to date. For a simple random walk, the average record grows as the square root of time
\[
a(t) \simeq A \sqrt{t}, \quad \text{with} \quad A = \sqrt{2/\pi}. \tag{5}
\]
This behavior represents the leading asymptotic behavior. Similar behavior holds as long as the jump length distribution has zero mean and a finite variance, and, in general, the ratio between the average record and the mean square displacement approaches a constant, \( a(t)/\sqrt{\langle x^2(t) \rangle} \to A \), in the limit \( t \to \infty \) [21, 22].

Let \( q(r,t) \) be the probability distribution function that the record equals \( r \) at time \( t \). This quantity follows from the probability \( Q(r,t) \) that a random walker starting at the origin never crosses \( r \) during the time interval \( (0,t) \). The quantity \( Q(r,t) \) is well known and can be conveniently expressed using the error function \( Q(r,t) = \text{erf}(r/\sqrt{2t}) \) in the long-time limit [32, 37]. The probability \( q(r,t) \) is then \( Q(r+1,t) - Q(r,t) \), which is asymptotically equivalent to \( dQ(r,t)/dr \). As a result, the probability distribution \( q(r,t) \) is a one-sided Gaussian,
\[
q(r,t) \simeq \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{r^2}{2t}\right) \tag{6}
\]
for \( r \geq 0 \). In particular, we recover the probability that the random walk remains in negative half-space: \( q(0,t) \simeq A/\sqrt{t} \) [32].

Let \( M_n = \int_0^\infty dr r^n q(r,t) \) be the \( n \)th moment of the record distribution. The zeroth moment, \( M_0 = 1 \), reflects that the probability distribution is normalized, and the first moment, \( M_1 \simeq A \sqrt{t} \), gives the average quoted in (5). The second moment \( M_2 \simeq t \) gives the variance,
\[
\langle r^2 \rangle - \langle r \rangle^2 = \left(1 - \frac{2}{\pi}\right)t. \tag{7}
\]
The average (5), the variance (7) and the distribution function (6) show that the record grows as the square-root of time: \( r \sim \sqrt{t} \). Hence, the typical record mimics the behavior of the typical position \( x \sim \sqrt{t} \).
3. Superior walks

We now focus on superior walks, that is, walks for which the record exceeds the average record, \( r(\tau) \geq a(\tau) \) at all times \( \tau \leq t \) (see figure 1). Since record \( r \) is coupled to position \( x \), we have to consider how the pair of coordinates \( (x, r) \) evolves with time. The position changes at each time step. However, the record may or may not change, and there are two possibilities. When \( x < r \), the position changes but the record stays the same (see figure 2)

\[
(x, r) \rightarrow \begin{cases} 
(x-1, r) & \text{with prob. 1/2;} \\
(x+1, r) & \text{with prob. 1/2.}
\end{cases}
\]  

(8)

When \( x = r \), the position changes and, depending on the jump direction, the record may increase,

\[
(r, r) \rightarrow \begin{cases} 
(r-1, r) & \text{with prob. 1/2;} \\
(r+1, r+1) & \text{with prob. 1/2.}
\end{cases}
\]  

(9)

As illustrated in figure 2, the position performs an ordinary random walk in the \( x-r \) plane, and there is also upward ‘slip’ along the diagonal \( x = r \).

Let \( P(x, r, t) \) be the density of superior walks with position \( x \) and record \( r \geq x \) at time \( t \). A sum over all records and positions gives the probability that a random walk is superior:

\[
S(t) = \sum_{r=0}^{t} \sum_{x=0}^{r} P(x, r, t).
\]  

(10)

In the discrete-time formulation, the sums are actually finite, \( 0 \leq r \leq t \) and \( -t \leq x \leq r \).

The jump rules (8)–(9) imply the following recurrence equations that relate the density at time \( t+1 \) to the density at time \( t \),

\[
P(x, r, t+1) = \frac{P(x-1, r, t) + P(x+1, r, t)}{2}
\]  

(11a)

\[
P(r, r, t+1) = \frac{P(r-1, r, t) + P(r-1, r-1, t)}{2}.
\]  

(11b)

Equation (11a) is valid for all \( x < r \) and it corresponds to cases where the record was set before the final step. Equation (11b) describes the evolution of the density along the diagonal \( x = r \), and it contains a contribution from walks in which the record was set at the final step.
For the random walk (4), position \( x \) and record \( r \) are discrete variables. Since we are interested in the long-time asymptotic behavior, we may treat these variables as continuous. The density \( P(x, r, t) \) satisfies the diffusion equation

\[
\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2 P}{\partial x^2} \tag{12}
\]

in the domain \(-\infty < x < r \) and \( r > 0 \). This equation follows from the recurrence equation (11a), and it reflects that the position undergoes an ordinary diffusion process. To obtain (12), we replace the left-hand side in (11a) with a first-order Taylor expansion in time, \( P + \partial P/\partial t \), and, similarly, replace the right-hand side with a second-order expansion in position \( x \). The diffusion equation (12) is subject to the boundary condition

\[
2 \frac{\partial P}{\partial x} + \frac{\partial P}{\partial r} = 0 \tag{13}
\]

on the diagonal \( x = r \). This relation, which properly accounts for the upward slip along the boundary, can be derived from the recurrence equation (11b) by repeating the steps leading to (12).

We now introduce a new variable \( y \), which is a linear combination of record and position:

\[
y = 2r - x. \tag{14}
\]

With this transformation of variables \((r, x) \rightarrow (r, y)\), the diffusion process takes place in the domain \( y \geq r \geq 0 \), and, importantly, the boundary condition (13) simplifies to \( \partial P/\partial r = 0 \), along the diagonal \( y = r \). According to equation (12), the density \( P \equiv P(y, r, t) \) still obeys the diffusion equation

\[
\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \tag{15}
\]

in the domain \( y > r \geq 0 \).

As discussed in section 2, the position and the record both grow as the square root of time, \( x \sim \sqrt{t} \) and \( r \sim \sqrt{t} \), and, consequently, \( y \sim \sqrt{t} \). Hence, the density of superior walks has the scaling form

\[
P(y, r, t) \sim t^{-\beta-1} \Phi \left( \frac{y}{\sqrt{t}}, \frac{r}{\sqrt{t}} \right). \tag{16}
\]

This scaling form is compatible with (10) and the algebraic decay \( S \sim t^{-\beta} \). The scaling function \( \Phi \equiv \Phi(Y, R) \) depends on the variables \( Y = y/\sqrt{t} \) and \( R = r/\sqrt{t} \) corresponding to the scaled position and record, respectively. For superior walks, we have \( y \geq r \geq a \) and, from equation (5), we conclude \( Y > R > A \) with \( A = \sqrt{2/\pi} \). Hence, we have the boundary condition \( \Phi(Y, R) = 0 \) on the line \( R = A \).

The boundary condition \( \partial P/\partial r = 0 \) on the diagonal \( y = r \) implies \( \partial \Phi/\partial R = 0 \) when \( Y = R \). This suggests we should seek a scaling function that depends on the variable \( Y \) alone, \( \Phi \equiv \Phi(Y) \). By substituting the scaling form (16) into the diffusion equation (15), we find that \( \Phi \) obeys the second-order ordinary differential equation

\[
\Phi'' + Y \Phi' + 2(\beta + 1) \Phi = 0, \tag{17}
\]

where the prime denotes differentiation with respect to \( Y \). The boundary condition is \( \Phi(A) \equiv 0 \). The first two terms in (17) imply that \( \Phi \) has a Gaussian tail, \( \Phi \sim \exp(-Y^2/2) \) when \( Y \rightarrow \infty \). Next, we make the transformation \( \Phi(Y) = \phi(Y) \exp(-Y^2/4) \) and arrive at the parabolic cylinder equation with index \( 2\beta + 1 \) [38]

\[
\phi'' + \left( 2\beta + 3 \frac{Y^2}{2} - \frac{3}{4} \right) \phi = 0. \tag{18}
\]
This equation has two independent solutions: $D_{2\beta+1}(Y)$ and $D_{2\beta+1}(-Y)$, where $D_v$ is the parabolic cylinder function of index $v$. Since the density vanishes, $\Phi \to 0$ as $Y \to \infty$, we choose the former solution and, therefore, $\Phi(Y) = D_{2\beta+1}(Y) \exp(-Y^2/4)$ (see figure 3).

The boundary condition $\Phi(A) = 0$ 'selects' the persistence exponent $\beta$ as the smallest root of the transcendental equation

$$D_{2\beta+1}(A) = 0,$$  \hspace{1cm} (19)

in agreement with the announced result (3).

In terms of the original variables $x$ and $r$, the joint density $P(x, r, t)$ has the asymptotic behavior

$$P(x, r, t) \sim t^{-1-\beta} D_{2\beta+1} \left( \frac{2r - x}{\sqrt{t}} \right) \exp \left[ -\frac{(2r - x)^2}{4t} \right].$$

This distribution holds for $r > A\sqrt{t}$ and $-\infty < x < r$. We obtained the density $P(x, r, t)$ up to a prefactor that cannot be determined using scaling analysis alone. Finally, the asymptotic behavior $D_v(z) \sim z^v \exp(-z^2/4)$ shows that the density has a Gaussian tail $P(x, r, t) \sim \exp[-(2r - x)^2/(4t)]$.

It is straightforward to compute various moments of the joint distribution. In the long-time limit, these moments are directly related to moments of the scaling function $\Phi(Y)$, and it is convenient to use the adjusted moments $m_n = \int_{A}^{\infty} dY (Y^n - A^n) \Phi(Y)$. Remarkably, the average position of superior walks $\langle x \rangle_{\sup}$ coincides with the average record, $\langle x \rangle_{\sup} \simeq A\sqrt{t}$. As expected, the average record of superior walks grows faster: $\langle r \rangle_{\sup} \simeq C \sqrt{t}$ with $C = m_2/(2m_1)$ or $C = 1.478591$. Finally, the position and the record are correlated random variables, $\langle xr \rangle_{\sup} \neq \langle x \rangle_{\sup} \langle r \rangle_{\sup}$, as follows from $\langle xr \rangle_{\sup} \simeq ct$ with $c = A^2/2 + m_3/(6m_1)$.

4. Inferior walks

Inferior walks are simpler to analyze because they can be defined in terms of position alone: a walk is inferior if and only if $x(\tau) \leq a(\tau)$ for all $\tau = 0, 1, 2, \ldots, t$. Indeed, if the position exceeds the average record, the record necessarily crosses the average. Conversely, if the position never exceeds the average record, then the record remains below average. Hence, inferior walks map onto diffusion in the presence of a receding trap with a location that grows as square root of time, a problem that was solved in [39].
Since it is not necessary to keep track of the record, we study the distribution of position. Let \( P(x, t) \) be the density of inferior walks with position \( x \) at time \( t \). The probability \( I(t) \) that a walk remains inferior after \( t \) steps is the integral of the density, \( I(t) = \int_{-\infty}^{\infty} dx P(x, t) \). The density of inferior walks obeys the diffusion equation

\[
\frac{\partial P(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2 P(x, t)}{\partial x^2},
\]

in the domain \(-\infty < x < a(t)\), and is subject to the boundary condition \( P(a, t) = 0 \). We anticipate the scaling behavior

\[
P(x, t) \sim t^{-\alpha-1/2} \Psi\left( \frac{x}{\sqrt{t}} \right),
\]

and impose the boundary condition \( \Psi(A) = 0 \). The prefactor in (21) reflects the algebraic decay \( I(t) \sim t^{-\alpha} \).

Substituting the scaling form (21) into the diffusion equation (20), we find that the scaling function obeys

\[
\Psi'' + X \Psi' + (2\alpha + 1) \Psi = 0.
\]

Here, the prime denotes differentiation with respect to the scaling variable \( X = x/\sqrt{t} \). With the transformation \( \Psi(X) = \psi(X) \exp(-X^2/4) \), the function \( \psi(X) \) obeys the parabolic cylinder equation

\[
\psi'' + \left( 2\alpha + 1 - \frac{X^2}{4} \right) \psi = 0.
\]

This equation has two linearly independent solutions: \( D_{2\alpha}(X) \) and \( D_{2\alpha}(-X) \). The density should vanish as \( X \to -\infty \), and this requirement gives \( \psi(X) = D_{2\alpha}(-X) \). The boundary condition \( \Psi(A) = 0 \) leads to the transcendental equation stated in (3),

\[
D_{2\alpha}(-A) = 0.
\]

In terms of the original variables, the density of inferior walks has the asymptotic behavior

\[
P(x, t) \sim t^{-\alpha-1/2} D_{2\alpha}\left( \frac{-x}{\sqrt{t}} \right) \exp\left[ -\frac{x^2}{4t} \right].
\]

In particular, the density has a Gaussian tail \( P(x, t) \sim \exp(-x^2/2t) \) as \( x \to -\infty \).

Thus far, we have considered only the maximal position, but one can also consider the maximal and minimal positions simultaneously. When \( |x(t)| \leq a(t) \) for all \( t = 0, 1, 2, \ldots, t \), the minimal position and the maximal position are both inferior with respect to the average record. The density of such \textit{meek} random walks satisfies the diffusion equation (20) in the growing interval \([-a(t), a(t)]\). We seek a scaling solution \( P(x, t) \sim t^{-\gamma-1/2} \Psi(X) \) and arrive at the same ordinary differential equations (22)–(23), with a new persistence exponent \( \gamma \) replacing \( \alpha \). The density is symmetric, \( \psi(X) = \psi(-X) \), and, thus, \( \psi(X) = D_{2\gamma}(X) + D_{2\gamma}(-X) \). The boundary condition \( \Psi(A) = \Psi(-A) = 0 \) leads to the transcendental equation

\[
D_{2\gamma}(A) + D_{2\gamma}(-A) = 0
\]

that specifies the persistence exponent \( \gamma = 1.698 \, 282 \ldots \). The probability that a walk is meek, \( \int_a^\infty dx P(x, t) \sim t^{-\gamma} \), is therefore much smaller than the probability that the walk is either superior or inferior.
5. Simulations and extensions

Figure 4 shows results of Monte Carlo simulations for the fractions $S$ and $I$ of superior and inferior walks. These results are in excellent agreement with the theoretical predictions. In the computations, we first calculated the average record $a(t)$ by considering $M$ independent realizations of a discrete-time random walk with $t$ steps. We then measured the fraction of all realizations in which the maximal position of the walk is always larger or always smaller than the average record, $r(\tau) \geq a(\tau)$ or $r(\tau) \leq a(\tau)$ for all $\tau \leq t$.

To verify that the asymptotic behavior (2)–(3) is robust (namely, it applies to a broad class of jump length distributions with zero mean and finite variance), we considered two distributions of jump length: (i) a fixed step size: $\Delta = 1$ or $\Delta = -1$ with equal probabilities, and (ii) a variable step size, chosen with uniform probability in the domain $-1 \leq \Delta \leq 1$. The results shown in figure 4 are for the latter case and were obtained using $M = 10^6$ independent realizations.

Our analysis compared the record $r$ with the average record $a \simeq A \sqrt{t}$. We can compare the record with other length scales that grow as the square root of time. We thus define a walk to be $\sigma-$superior (respectively $\sigma-$inferior) if $r(\tau) \geq \sigma \sqrt{\tau}$ (respectively $r(\tau) \leq \sigma \sqrt{\tau}$) for all $\tau = 0, 1, 2, \ldots, t$. A straightforward generalization of the above analysis shows that the persistence exponents $\alpha(\sigma)$ and $\beta(\sigma)$ that govern the abundance of such walks are given by (figure 5)

$$D_{2\alpha}(-\sigma) = 0 \quad \text{and} \quad D_{2\beta+1}(\sigma) = 0. \quad (27)$$

As expected, the persistence exponent $0 < \alpha < \infty$ increases monotonically with $\sigma$, while the exponent $0 < \beta < 1/2$ decreases monotonically. The maximal value $\alpha(0) = 1/2$ follows from the probability that a random walk remains in the negative half space, $S \sim t^{-1/2}$ [32]. The exponent $\alpha$ decays rapidly, $\alpha \simeq (\sigma / \sqrt{8\pi}) e^{-\sigma^2/2}$, while the exponent $\beta$ diverges, $\beta \simeq \sigma^2/8$, in the limit $\sigma \to \infty$ (see [40]).

Figure 5 shows that both $\alpha < \beta$ and $\alpha > \beta$ are possible and that both persistence exponents vary continuously with $\sigma$. We note that for uncorrelated random variables, it was also found that $\alpha < \beta$ and $\alpha > \beta$ are both feasible, and that both exponents are continuous.
functions of some control parameter [35]. However, we do not believe that there is a deeper connection between these two sets of exponents or that it is possible to obtain the persistence characteristics of records by mapping correlated random variables onto uncorrelated random variables.

Another useful by-product of our analysis is the joint distribution $P(x, r, t)$ of position and record for a one-dimensional random walk. If we do not impose any restriction on the record, then $\beta = 0$ in equations (16), (17). The corresponding solution of (17) with $\beta = 0$ is $D_1(Y) \exp(-Y^2/4)$ and, therefore,

$$P(x, r, t) \simeq \sqrt{\frac{2}{\pi t^3}} (2r - x) \exp \left[ -\frac{(2r - x)^2}{2t} \right].$$

Normalization of the probability distribution sets the numerical prefactor. By integrating (28) over position, one recovers the record distribution (6). The joint distribution (28) was originally discovered by Lévy—see [21, 22].

The joint distribution (28) shows that record and position are correlated variables. In particular, $\langle xr \rangle \simeq t/2$ whereas $\langle x \rangle = 0$ and $\langle r \rangle \simeq A\sqrt{t}$. Higher-order moments also reflect that $x$ and $r$ are correlated and, for example, $\langle x^2 r^2 \rangle \simeq 2t^2$ whereas $\langle x^2 \rangle = \langle r^2 \rangle = t$.

6. Conclusions

In conclusion, we used the average record to characterize the motion of a Brownian particle in one dimension. A random walk is said to be superior if its maximal position is always above average and, similarly, it is said to be inferior if the maximal position is always below average. We find that the probability that a walk is superior or inferior decays algebraically with time. This power-law decay is characterized by nontrivial persistence exponents.

For inferior walks, it suffices to keep track of the position of the walk alone and, consequently, the problem reduces to diffusion in the presence of a properly chosen moving trap. For superior walks, it is necessary to keep track of both the record and position, and, consequently, the problem involves diffusion in a two-dimensional space. This random process consists of diffusion in the position coordinate and directional motion when the record and the position are equal. Nevertheless, a linear transformation of variables reduces the problem to
one-dimensional diffusion in the presence of a moving trap as far as the asymptotic behavior is concerned (figure 2).

Our results address the leading asymptotic behavior. However, different implementations of a random walk are not entirely equivalent. For instance, there are corrections to the leading behavior of the average (5), and \( a(t)/\sqrt{\langle x^2(t) \rangle} = A + C t^{-1/2} + O(t^{-1}) \). The constant \( C \) depends on the distribution of jump lengths, and its derivation requires rather intricate analysis [41–43]. It will be interesting to understand how corrections to the leading asymptotic behavior (2)–(3) depend on the distribution of jump lengths.

Finally, we mention that records have been extensively used to analyze empirical data such as earthquake inter-event times [44, 45] and temperature readings [46, 47]. Comparing a sequence of records with a baseline such as the average provides a measure of performance. Specifically, for a given ensemble of datasets representing random variables that are expected to obey the same statistics, it is straightforward to calculate the average record. Then, records for individual datasets can be compared with the measured average to identify instances where the datasets have superior or inferior records. Precisely this approach was used to identify fractions of superior and inferior record sequences for inter-event times between successive earthquakes [35, 36]. Persistence of records is also useful in finance, where it is natural to quantify the performance of a company by comparing its stock price with the stock index for the respective sector [8–10].

Acknowledgments

We thank Satya Majumdar for useful discussions and correspondence and acknowledge DOE grant DE-AC52-06NA25396 for support (EB).

References

[1] Feller W 1968 *An Introduction to Probability Theory and Its Applications* (New York: Wiley)
[2] Leadbetter M R, Lindgren G and Rootzen H 1983 *Extremes and Related Properties of Random Sequences and Processes* (Berlin: Springer)
[3] Gumbel E I 2004 *Statistics of Extremes* (New York: Dover)
[4] Ellis R S 2005 *Entropy, Large Deviations, and Statistical Mechanics* (Berlin: Springer)
[5] Resnick S I 2007 *Extreme Values, Regular Variation and Point Processes* (Berlin: Springer)
[6] Arnold B C, Balakrishnan N and Nagaia H N 1998 *Records* (New York: Wiley-Interscience)
[7] Nevzorov V B 2001 *Records: Mathematical Theory* (Translation of Mathematical Monographs vol 194) (Providence, RI: American Mathematical Society)
[8] Embrechts P, Klüppelberg G and Mikosch T 1997 *Modelling Extremal Events for Insurance and Finance* (Berlin: Springer)
[9] Bouchaud J -P and Potters M 2003 *Theory of Financial Risk and Derivative Pricing* (Cambridge: Cambridge University Press)
[10] Novak S Y 2011 *Extreme Value Methods with Applications to Finance* (London: CRC Press)
[11] Wergen G, Bogner M and Krug J 2011 *Phys. Rev. E* 83 051109
[12] Gembris D, Taylor J G and Suter D 2002 *Nature* 417 506
[13] Ben-Naim E, Krapivsky P L and Majumdar S N 2001 *Phys. Rev. E* 64 R035101
[14] Ben-Naim E and Krapivsky P L 2004 *Europhys. Lett.* 65 151
[15] Krapivsky P L and Redner S 2002 *Phys. Rev. Lett.* 89 258703
[16] Godreche C and Luck J M 2008 *J. Stat. Mech.* P11006
[17] Godreche C, Grandclaude H and Luck J M 2010 *J. Stat. Mech.* P02001
[18] Majumdar S N and Krapivsky P L 2003 *Physica A* 318 161
[19] Krug J 2007 *J. Stat. Mech.* P07001
[20] Wergen G 2013 *J. Phys. A: Math. Theor.* 46 223001
[21] Lévy P 1948 *Processus Stochastiques et Mouvement Brownien* (Paris: Gauthier-Villars)
[22] Itô K and McKean H P 1965 *Diffusion Processes and Their Sample Paths* (Springer: New York)

[23] Möhring P and Peres Y 2010 *Brownian Motion* (Cambridge: Cambridge University Press)

[24] Sparre Andersen E 1953 *Math. Scand.*, 1 263
Sparre Andersen E 1954 *Math. Scand.*, 2 195

[25] Majumdar S N and Ziff R M 2008 *Phys. Rev. Lett.* 101 050601

[26] Krapivsky P L, Majumdar S N and Rosso A 2010 *J. Phys. A: Math. Theor.* 43 315001

[27] Wergen G, Majumdar S N and Scher G 2012 *Phys. Rev. E* 86 011119

[28] Scher G, Majumdar S N, Comtet A and Randon-Furling J 2008 *Phys. Rev. Lett.* 101 150601

[29] Kobayashi N, Izumi M and Katori M 2008 *Phys. Rev. E* 78 051102

[30] Edery Y, Kostinski A and Borkowitz B 2011 *Geophys. Res. Lett.* 389 L16403

[31] Majumdar S N 2010 *Physica A* 389 4299

[32] Redner S 2001 *A Guide to First-Passage Processes* (Cambridge: Cambridge University Press)

[33] Bray A J, Majumdar S N and Schehr G 2013 Persistence and first-passage properties in non-equilibrium systems *Adv. Phys.* 62 225

[34] Derrida B, Hakim V and Pasquier V 1995 *Phys. Rev. Lett.* 75 751

[35] Ben-Naim E and Krapivsky P L 2013 *Phys. Rev. E* 88 022145

[36] Miller P W and Ben-Naim E 2013 *J. Stat. Mech.* P10025

[37] Krapivsky P L, Redner S and Ben-Naim E 2010 *A Kinetic View of Statistical Physics* (Cambridge: Cambridge University Press)

[38] Bender C M and Orszag S A 1978 *Advanced Mathematical Methods for Scientists and Engineers* (New York: McGraw-Hill)

[39] Krapivsky P L and Redner S 1996 *Am. J. Phys.* 64 546

[40] Ben-Naim E and Krapivsky P L 2010 *J. Phys. A: Math. Theor.* 43 495007

Ben-Naim E and Krapivsky P L 2010 *J. Phys. A: Math. Theor.* 43 495008

[41] Coffman E G, Flajolet P, Flato L and Hofri M 1998 *Probab. Eng. Inform. Sci.* 12 373

[42] Comtet A and Majumdar S N 2005 *J. Stat. Mech.* P06013

[43] Sabhapandit S 2011 *Europhys. Lett.* 94 20003

[44] Newman W I, Malamud B D and Turcotte D L 2010 *Phys. Rev. E* 82 066111

[45] Shcherbakov R, Davidsen J and Tiampo K F 2013 *Phys. Rev. E* 87 052811

[46] Eichner J F, Koscielny-Bunde E, Bunde A, Havlin S and Schellnhuber H J 2003 *Phys. Rev. E* 68 046133

[47] Redner S and Petersen M R 2006 *Phys. Rev. E* 74 061114