Condensed groups in product varieties

D. Osin

Abstract

A finitely generated group $G$ is called condensed if its isomorphism class in the space of finitely generated marked groups has no isolated points. We prove that every product variety $U\mathcal{V}$, where $U$ (respectively, $\mathcal{V}$) is a non-abelian (respectively, a non-locally-finite) variety, contains a condensed group. In particular, there exist condensed groups of finite exponent. As an application, we obtain some results on the structure of the isomorphism and elementary equivalence relations on the set of finitely generated groups in $U\mathcal{V}$.

1 Introduction

Let $\mathcal{G}$ denote the space of finitely generated marked groups. Informally, $\mathcal{G}$ is the set of all pairs $(G, A)$, where $G$ is a group and $A$ is an ordered finite generating set of $G$, considered up to a natural equivalence relation and endowed with the topology induced by local convergence of Cayley graphs. Given a finitely generated group $G$, we denote by $[G] \subseteq \mathcal{G}$ its isomorphism class; that is,

$$[G] = \{(H, B) \in \mathcal{G} \mid H \cong G\}.$$ 

The following definition is inspired by connections between the topological properties of isomorphism classes in $\mathcal{G}$ and model theory.

**Definition 1.1.** A finitely generated group $G$ is called condensed if $[G]$ has no isolated points.

As shown in [Osi], condensed groups lead to non-trivial examples of subspaces of $\mathcal{G}$ satisfying a topological zero-one law for $\mathcal{L}_{\omega_1, \omega}$-sentences. In addition, the existence of a condensed group in a closed subset $S \subseteq \mathcal{G}$ has strong consequences for the structure of the isomorphism and elementary equivalence relations on $S$.

Known examples of condensed groups include finitely generated groups isomorphic to their direct square, generic torsion-free lacunary hyperbolic groups [Osi], certain solvable groups [Wil], and the iterated monodromy group of the polynomial $z^2 + i$ [Nek]. On the other hand, condensed groups do not occur among linear groups as well as among finitely presented Hopfian groups. It is unknown whether a condensed group can be finitely presented. For more details, we refer the reader to [Osi] and references therein.

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The goal of this note is to suggest an elementary construction of condensed groups based on wreath products. Namely, we prove the following.

**Theorem 1.2.** Let $B$ be a non-abelian group and let $H$ be a finitely generated infinite group. The unrestricted wreath product $B \operatorname{Wr} H$ contains a condensed subgroup.

This theorem provides new examples of condensed groups satisfying non-trivial laws. Using results of [Osi] we also obtain some information about the structure of the isomorphism and elementary equivalence relations on the set of finitely generated groups in decomposable varieties.

**Corollary 1.3.** Suppose that $U$ (respectively, $V$) is a non-abelian (respectively, non-locally-finite) variety of groups. Then the product variety $UV$ contains a condensed group. In particular, the following hold:

(a) the isomorphism relation on the subspace $\{(G, A) \in G \mid G \in UV\}$ is non-smooth;

(b) $UV$ contains a subset of cardinality $2^{\aleph_0}$ consisting of pairwise non-isomorphic, elementarily equivalent, finitely generated groups.

An equivalence relation $E$ on a topological space $X$ is called smooth if there is a Polish space $P$ and a Borel map $\beta : X \to P$ such that for any $x, y \in X$, we have $xE y$ if and only if $\beta(x) = \beta(y)$. Informally, claim (a) of Corollary 1.3 means that finitely generated groups in $UV$ cannot be “explicitly classified” up to isomorphism using invariants from a Polish space. For more on complexity of Borel equivalence relations in $G$, see [TS].

Recall also that two groups are elementarily equivalent if they satisfy the same first order sentences in the language of groups. Claim (b) of Corollary 1.3 shows that elementary equivalence in $UV$ is much weaker than isomorphism. It is worth noting that finding examples of elementarily equivalent, non-isomorphic, finitely generated groups is a rather non-trivial task since the standard tools for constructing models – ultrapowers, omitting types, and the Löwenheim-Skolem theorem – are not available in this case.

It was previously known that the variety of solvable groups of class 3 contains condensed groups. Indeed, this fact easily follows from a result of Williams [Wil] as explained in [Osi, Example 2.8]. Corollary 1.3 implies that such examples can already be found among (nilpotent of class 2)-by-abelian groups. In contrast, abelian-by-nilpotent groups cannot be condensed [Osi, Proposition 6.2]. It would be interesting to know whether condensed groups exist in some other solvable varieties, e.g., in the variety of center-by-metabelian groups.

Yet another variety to which Corollary 1.3 applies is the Burnside variety $B_n$ of all groups of exponent $n = n_1 n_2$, where $n_1 > 2$ and $n_2$ is any number for which $B_{n_2}$ is not locally finite (e.g., we can take $n = 1995$ [Adi]). In particular, we obtain the following.

**Corollary 1.4.** There exist condensed groups of finite exponent.

In the next section, we collect some preliminary information about the space of finitely generated groups, wreath products, and group varieties. The proof of the main results is given in Section 3.
2 Preliminaries

2.1. The space of finitely generated marked groups. We begin with the definition suggested by Grigorchuck [Gri]. Let $G_n$ denote the set of equivalence classes of all pairs $(G, A)$, where $G$ is a group and $A$ an ordered generating set of $G$ of cardinality $n$, modulo the following equivalence relation:

$$(G, (a_1, \ldots, a_n)) \approx (H, (b_1, \ldots, b_n))$$

if the map $a_1 \mapsto b_1, \ldots, a_n \mapsto b_n$ extends to an isomorphism $G \to H$. To simplify our notation, we write $(G, A)$ for the $\approx$-class of $(G, A)$.

We also write $(G, (a_1, \ldots, a_n)) \approx_r (H, (b_1, \ldots, b_n))$ for some $r \in \mathbb{N}$ if there is an isomorphism (in the category of directed graphs) between the balls of radius $r$ around the identity in the Cayley graphs of $G$ and $H$ with respect to the generating sets $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, respectively, that takes edges labeled by $a_i$ to edges labelled by $b_i$ for all $i$.

The topology on $G_n$ is defined by taking the sets

$$U_{G,A}(r) = \{(H, B) \in G_n \mid (H, B) \approx_r (G, A)\},$$

where $(G, A)$ ranges in $G_n$ and $r \in \mathbb{N}$, as the base of neighborhoods. Thus, a sequence $\{(G_i, A_i)\}_{i \in \mathbb{N}}$ converges to $(G, A)$ in $G_n$ if for every $r \in \mathbb{N}$, $(G_i, A_i)$ and $(G, A)$ are $r$-similar for all sufficiently large $i$. It is easy to see that $\approx$ is the intersection of the equivalence relations $\approx_r$, and hence the topology on $G_n$ is well-defined.

We record an elementary observation, which will be used in the next section.

**Lemma 2.1.** Let $(G, A), (H, B) \in G_n$, where $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$. Given a word $w_A$ in the alphabet $A^{\pm 1}$, let $w_B$ denote the word obtained from $w_A$ by replacing all occurrences of $a_i^{\pm 1}$ with $b_i^{\pm 1}$ for all $i$. Suppose that a word $w_A$ of length at most $2r$ in the alphabet $A^{\pm 1}$ represents 1 in $G$ if and only if the corresponding word $w_B$ in the alphabet $B^{\pm 1}$ represents 1 in $H$. Then $(G, A) \approx_r (H, B)$. 

**Proof.** It is easy to see that the map $w_A \mapsto w_B$ induces the required isomorphism between the balls of radius $r$ around the identity in the Cayley graphs of $(G, A)$ and $(H, B)$. 

Identifying $(G, (a_1, \ldots, a_n))$ with $(G, (a_1, \ldots, a_n, 1))$ gives rise to an embedding $G_n \subseteq G_{n+1}$. The topological union

$$G = \bigcup_{n \in \mathbb{N}} G_n$$

is called the space of marked finitely generated groups.

It turns out that there is a group of homeomorphisms of $G$ whose orbits are exactly the isomorphism classes. One obvious corollary of this fact is the following.

**Proposition 2.2** ([Osi, Corollary 6.1]). For every finitely generated group $G$, the isomorphism class $[G]$ is either discrete or has no isolated points in $G$. 

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2.2. Wreath products and varieties of groups. For any groups $B$ and $H$, we denote by $B^H$ the set of all functions $f: H \to B$. We think of $B^H$ as a group with respect to pointwise multiplication. Thus, $B^H$ is isomorphic to the direct product of $|H|$ copies of $B$.

The (unrestricted) wreath product of $B$ and $H$, denoted $B \Wr H$, is the split extension of $B^H$ by $H$ where the action of $H$ on $B^H$ by conjugation is given by the formula

$$(hf^{-1})(x) = f(h^{-1}x)$$

for all $h, x \in H$ and $f \in B^H$.

Recall that a variety is a class of groups satisfying a fixed system of identities. Alternatively, varieties can be characterized via Birkhoff's theorem: a class of groups is a variety if and only if it is closed under taking homomorphic images, subgroups, and (unrestricted) direct products. For more details on varieties of groups and all unexplained notation, the reader may consult the book [Neu].

Given two group varieties $U$ and $V$, the product variety $UV$ is defined to be the class of all extensions of groups from $U$ by groups from $V$. In particular, for any $B \in U$ and $H \in V$, we have $B \Wr H \in UV$.

3 Condensed subgroups of wreath products

3.1. Auxiliary results from topological dynamics. Given a group $H$, we denote by $2^H$ the space of all subsets of $H$ endowed with the product topology, which coincides with the topology of pointwise convergence of characteristic functions in our case. Thus, $2^H$ is a compact Hausdorff space. The group $H$ acts on $2^H$ by homeomorphisms via left and right multiplication. In what follows, we refer to these actions as the left and right actions of $H$.

An action of a group $H$ on a topological space is said to be topologically transitive if for every non-empty open sets $U$, $V$, there exists $h \in H$ such that $h(U) \cap V \neq \emptyset$. The following results is well-known. We provide a proof for completeness.

**Lemma 3.1.** For every infinite group $H$, both left and right actions of $H$ on $2^H$ are topologically transitive.

**Proof.** We only prove the claim for the left action; the argument for the right action is symmetric. Let $U$, $V$ be any non-empty open subsets of $2^H$ and let $S \subseteq U$, $T \subseteq U$. By the definition of the topology on $2^H$, there exist finite subsets $E$, $F \subseteq H$ such that for any $R \subseteq H$ satisfying $R \cap E = S \cap E$ (respectively, $R \cap F = T \cap F$), we have $R \in U$ (respectively, $R \in V$). Since $H$ is infinite, there is $h \in H$ such that $hE$ and $F$ are disjoint. Let

$$R = (hE \cap hS) \cup (T \cap F).$$

Clearly, $h^{-1}R \cap E = h^{-1}(hR \cap hE) = S \cap E$ and $R \cap F = T \cap F$. Hence, $R \in hU \cap V$. \qed

We say that a subset of $2^H$ is $H$-bi-invariant if it is invariant under both left and right actions of $H$. Let $D_L(H)$ (respectively, $D_R(H)$) denote the set of all subsets $S \subseteq H$ whose orbits with respect to the left (respectively, right) action of $H$ are dense in $2^H$. 


Proposition 3.2. For every countably infinite group $H$, the set $D(H) = D_L(H) \cap D_R(H)$ is non-empty and $H$-bi-invariant.

Proof. Since $H$ is countable, there exists a countable basis of neighborhoods $\{U_i\}_{i \in \mathbb{N}}$ in $2^H$. For $i \in \mathbb{N}$, let

$$V_i = \left( \bigcup_{h \in H} hU_i \right) \cap \left( \bigcup_{h \in H} U_i h \right)$$

and

$$C = \bigcap_{i \in \mathbb{N}} V_i.$$  

Since both left and right actions of $H$ on $2^H$ are topologically transitive, the sets $\bigcup_{h \in H} hU_i$ and $\bigcup_{h \in H} U_i h$ are dense in $2^H$; clearly, both of them are also open. Therefore, $C$ is non-empty by the Baire category theorem. Fix any $S \in C$. For any non-empty open $W \subseteq 2^H$, we have $U_i \subseteq W$ for some $i$. Since $S \in V_i$, there exist $h_1, h_2 \in H$ such that $h_1^{-1} S$ and $S h_2^{-1}$ belong to $U_i \subseteq W$. Thus, the orbits of $S$ with respect to the left and right actions of $H$ are dense in $2^H$. In particular, we have $C \subseteq D(H)$ and hence $D(H) \neq \emptyset$.

It is clear that for every $S \in D(H)$ and every $h \in H$, we have $hS \in D_L(H)$. Further, we note that a subset $D \subseteq 2^H$ is dense if and only if for every finite subsets $E \subseteq F \subseteq H$, there is $Q \in D$ such that $Q \cap F = E$. Consider any $S \in D(H)$, $h \in H$, and any finite $E \subseteq F \subseteq H$. Since $\{Sf \mid f \in H \}$ is dense in $2^H$, there is $q \in H$ such that $Sg \cap h^{-1} F = h^{-1} E$. The latter equation is equivalent to $hSg \cap F = E$. This shows that $hS \in D_R(H)$ and thus we have $hS \in D(H)$. Similarly, we have $Sh \in D(H)$ for every $S \in D(H)$ and $h \in H$. 

Remark 3.3. The use of the Baire category theorem in the proof of Proposition 3.2 can be avoided. In fact, this result can also be proved by an explicit but somewhat longer inductive argument. We leave this as an exercise to the interested reader.

3.2. Proof of the main results. 

Henceforth, let $B$ and $H$ be as in Theorem 1.2 and let

$$W = B \Wr H.$$ 

We fix any $a, b \in B$ such that $ab \neq ba$. Given $S \subseteq H$, let $G_S$ denote the subgroup of $W$ generated by $H$ and the elements $\overline{a}, \overline{b}_S \in B^H$ defined as follows:

$$\overline{a}(x) = \begin{cases} a, & \text{if } x = 1, \\ 1, & \text{if } x \neq 1, \end{cases} \quad \text{and} \quad \overline{b}_S(x) = \begin{cases} b, & \text{if } x \in S, \\ 1, & \text{if } x \not\in S. \end{cases} \quad (3)$$

Let

$$X = \{x_1, \ldots, x_n\}$$

be a fixed finite generating set of $H$. We denote by $\xi: 2^H \to \mathcal{G}$ the map defined by the formula

$$\xi(S) = (G_S, Y_S),$$

where

$$Y_S = (x_1, \ldots, x_n, \overline{a}, \overline{b}_S).$$
Lemma 3.4. The map $\xi$ is injective.

Proof. Suppose that $S, T \subseteq H$ and $S \neq T$. Without loss of generality, there is $s \in S \setminus T$. Using (2), we obtain $(s^{-1}b_T s)(1) = b_T(s) = b$ and hence $s^{-1}b_T s a \neq \pi s^{-1}b_T s$. On the other hand, we have $(s^{-1}b_T s)(1) = b_T(s) = 1$ and therefore $s^{-1}b_T s a = \pi s^{-1}b_T s$. Thus, $(G_S, Y_S) \not\cong (G_T, Y_T)$. \hfill \Box

In general, the map $\xi$ is not continuous. However, its restriction to $D_R(H)$ is. The main step in proving this fact is the following lemma. We denote by $|h|_X$ the length of an element $h \in H$ with respect to the generating set $X$ and let

$$Ball_H(n) = \{h \in H \mid |h|_X \leq n\}.$$ 

For a word $w$ in some alphabet, we denote by $||w||$ its length.

Lemma 3.5. Let $S$ and $T$ be elements of $D_R(H)$ such that

$$S \cap Ball_H(2r) = T \cap Ball_H(2r)$$

for some $r \in \mathbb{N}$. A word $w_S$ of length $||w_S|| \leq r$ in the alphabet $Y_S^{\pm 1}$ represents 1 in $W$ if and only if the word $w_T$ in the alphabet $Y_T^{\pm 1}$ obtained from $w_S$ by replacing each occurrence of $b_T^{\pm 1}$ with $b_T^{\pm 1}$ represents 1 in $W$.

Proof. Throughout this proof, we write $u = v$ for two words $u, v$ in some alphabet $A$ if they represent the same element of the free group with the basis $A$. If $u$ and $v$ are words in the alphabet $Y_S^{\pm 1}$ or $Y_T^{\pm 1}$, we write $u =_W v$ if $u$ and $v$ represent the same element of the group $W = B W R H$.

We will prove the forward implication. That is, we assume that $w_S =_W 1$. In the free group with the basis $Y_S$, we can rewrite the word $w_S$ as $w_S = w'_S t$, where $t$ is a word in the alphabet $X^{\pm 1}$ and

$$w'_S = u_1 a^{\alpha_1} u_1^{-1} \cdot v_1 b_T^{\beta_1} v_1^{-1} \cdot \ldots \cdot u_\ell a^{\alpha_\ell} u_\ell^{-1} \cdot v_\ell b_S^{\beta_\ell} v_\ell^{-1}$$

for some $\alpha_i, \beta_i \in \mathbb{Z}$ and some words $u_i, v_i$ in the alphabet $X^{\pm 1}$ of length $||u_i||, ||v_i|| \leq r$ for all $i$. Similarly, we have $w_T = w'_T t$, where

$$w'_T = u_1 a^{\alpha_1} u_1^{-1} \cdot v_1 b_T^{\beta_1} v_1^{-1} \cdot \ldots \cdot u_\ell a^{\alpha_\ell} u_\ell^{-1} \cdot v_\ell b_T^{\beta_\ell} v_\ell^{-1}$$

in the free group with the basis $Y_T$. Since $w_S =_W 1$, the word $t$ must represent 1 in $H$. Therefore,

$$w'_S =_W 1$$

and to prove the lemma it suffices to show that $w'_T =_W 1$. To this end, we first prove an auxiliary result about the exponents $\beta_1, \ldots, \beta_\ell$.

Let $\sim$ be the equivalence relation on the index set $I = \{1, \ldots, \ell\}$ defined by the rule

$$i \sim j \text{ if and only if } v_i =_H v_j.$$

Let $a_i$ be the equivalence class of $i$ for the relation $\sim$. Then $a_1, \ldots, a_\ell$ are words in the alphabet $X^{\pm 1}$ of length $||a_1||, \ldots, ||a_\ell|| \leq r$ and $a_1 a_2 \cdots a_\ell =_L 1$.
Let
\[ I = J_1 \sqcup \ldots \sqcup J_k \]
be the associated partition into equivalence classes. For \( m \in \{1, \ldots, k\} \), let also
\[
\sigma_m = \sum_{i \in J_m} \beta_i.
\]
We want to show that for all \( m \in \{1, \ldots, k\} \), we have
\[
\sigma^m = 1. \tag{6}
\]

Fix any \( m \in \{1, \ldots, k\} \). For any non-empty open subset \( O \) of \( 2^H \), we can find infinitely many \( h \in H \) such that \( Sh^{-1} \in O \) since \( S \in D_R(H) \). In particular, we can find \( h \notin \{u_1, \ldots, u_\ell\} \) such that \( v_i^{-1} \in Sh^{-1} \) if and only if \( i \in J_m \). By (3), we have \( \overline{\pi}(u_i^{-1}h) = 1 \) for all \( i \in I \) and
\[
\overline{b}_S(v_i^{-1}h) = \begin{cases} b, & \text{if } i \in J_m \\ 1, & \text{otherwise}. \end{cases}
\]

Thinking of \( w_S' \) as an element of \( B^H \) and using (2), we obtain
\[
w_S'(h) = \overline{\pi}(u_1^{-1}h) \cdot \overline{b}_S(v_1^{-1}h) \cdot \ldots \cdot \overline{\alpha}(u_\ell^{-1}h) \cdot \overline{b}_S(v_\ell^{-1}h) = \sigma^m
\]
in the group \( B \). Thus, (6) follows from (5).

We are now ready to prove that \( w_T' \equiv 1 \). Clearly, \( w_T' \) also represents an element of \( B^H \). We will show that \( w_T'(h) = 1 \) for all \( h \in H \). There are two cases to consider.

Case 1. First assume that \( |h|_X \leq r \). Then for every \( i = 1, \ldots, \ell \), we have \( |v_i^{-1}h|_Y \leq 2r \). Using (2), (3), and (4), we obtain
\[
v_i \overline{b}^j_i v_i^{-1}(h) = \overline{b}^j_i(v_i^{-1}h) = \overline{b}_S(v_i^{-1}h) = v_i \overline{b}_S v_i^{-1}(h).
\]
Hence, \( w_T'(h) = w_S'(h) = 1 \) in \( B \).

Case 2. Let \( |h|_X > r \). For every \( i = 1, \ldots, \ell \), we have \( u_i^{-1} \neq 1 \) as \( ||u_i|| \leq r \). Therefore, \( u_i \overline{\alpha}(u_i^{-1}h) = \overline{\alpha}(u_i^{-1}h) = 1 \) and we obtain
\[
w_T'(h) = \overline{b}_T(v_1^{-1}h) \cdot \ldots \cdot \overline{b}_T(v_\ell^{-1}h) = \prod_{m=1}^{k} \overline{b}_T(v_{j_m}^{-1}h),
\]
where \( j_m \) is any representative of the equivalence class \( J_m \). Since \( \overline{b}_T(v^{-1}_m h) \in \{1, b\} \) by the definition of \( \overline{b}_T \), we have \( \overline{b}_T(v_{j_m}^{-1}h) = 1 \) for all \( m \) by (6). Thus, \( w_T'(h) = 1 \) in this case as well. \( \square \)

**Corollary 3.6.** The restriction of \( \xi \) to the subspace \( D_R(H) \) of \( 2^H \) is continuous.
Proof. Note that \( \xi(S) \in \mathcal{G}_{n+2} \) for every \( S \subseteq H \). Thus we can work with \( \mathcal{G}_{n+2} \) instead of \( \mathcal{G} \). Consider any \( S \in \mathcal{D}_R(R) \) and any open neighborhood \( \mathcal{N} \) of \( \xi(S) = (G_S, Y_S) \) in \( \mathcal{G}_{n+2} \). By the definition of the topology on \( \mathcal{G}_{n+2} \), there exists \( r \in \mathbb{N} \) such that \( \mathcal{N} \) contains the set

\[
U_{G_S, Y_S}(r) = \{(G, A) \in \mathcal{G}_{n+2} \mid (G, A) \approx_r (G_S, Y_S)\}.
\]

Suppose that some \( T \in \mathcal{D}_R(H) \) satisfies the condition

\[
S \cap Ball_H(4r) = T \cap Ball_H(4r).
\]  

(7)

Combining Lemma 3.5 and Lemma 2.1, we obtain \( (G_T, Y_T) \approx_r (G_S, Y_S) \). This means that \( \xi(T) \in \mathcal{N} \). Since subsets \( T \subseteq H \) satisfying (7) form an open neighborhood of \( S \) in \( \mathcal{D}_R(H) \), we are done. \( \square \)

We are now ready to prove our main result.

Proof of Theorem 1.2. By Proposition 3.2, there exists \( S \subseteq H \) such that \( hS \in \mathcal{D}(H) \) and \( Sh \in \mathcal{D}(H) \) for all \( h \in H \). In particular, the subset

\[
H(S) = \{hS \mid h \in H\} \subseteq 2^H
\]

is non-discrete and contained in \( \mathcal{D}_R(H) \). By Lemma 3.4 and Corollary 3.6, the image of \( H(S) \) in \( \mathcal{G} \) is also non-discrete. In the notation introduced above, we clearly have \( \overline{hS}(hx) = \overline{s}(x) \) for any \( h, x \in H \). Therefore, \( \overline{hS} = h^{-1}\overline{s}h \) in \( W \). It follows that \( G_S = G_{hS} \) for every \( h \in H \) and hence \( \xi(H(S)) \subseteq [G_S] \). Thus, \( [G_S] \) is non-discrete in \( \mathcal{G} \). It remains to apply Proposition 2.2. \( \square \)

Proof of Corollary 1.3. By our assumption, there exists a non-abelian group \( B \in \mathcal{U} \) and a finitely generated infinite group \( H \in \mathcal{V} \). Applying Theorem 1.2, we obtain a condensed group \( G \leq B \text{Wr} H \in \mathcal{U}\mathcal{V} \). Since varieties of groups are closed under taking subgroups, we have \( G \in \mathcal{U}\mathcal{V} \).

It is well-known (and straightforward to verify) that for every variety \( \mathcal{W} \), the set \( \{(G, A) \in \mathcal{G} \mid G \in \mathcal{W}\} \) is closed in \( \mathcal{G} \). This allows us to apply [Osi, Proposition 2.7], which in our settings can be stated as follows: the isomorphism relation on \( \{(G, A) \in \mathcal{G} \mid G \in \mathcal{W}\} \) is smooth if and only if \( \mathcal{W} \) contains no condensed groups. Thus, claim (a) of the corollary holds. Claim (b) follows from the fact that for every condensed group \( G \), the closure of the isomorphism class \([G] \) in \( \mathcal{G} \) contains \( 2^\aleph_0 \) elementarily equivalent, pairwise non-isomorphic groups by Theorem 2.2 and Proposition 2.3 of [Osi]. \( \square \)

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**Denis Osin:** Department of Mathematics, Vanderbilt University, Nashville 37240, U.S.A.
E-mail: denis.v.osin@vanderbilt.edu