1. INTRODUCTION

A reasonable agreement between HERA data [1–3] and the next-to-leading-order (NLO) approximation of perturbative Quantum Chromodynamics (QCD) has been observed for \( Q^2 \geq 2 \text{ GeV}^2 \) (see reviews in [4] and references therein), which gives us a reason to believe that perturbative QCD is capable of describing the evolution of the structure function (SF) \( F^2 \) and its derivatives down to very low \( Q^2 \) values, where all the strong interactions are conventionally considered to be soft processes.

A standard way to study the \( x \) behavior of quarks and gluons is to compare the data with the numerical solution to the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equations [5] by fitting the parameters of \( x \)-profile of partons at some initial \( Q^2_0 \) and the QCD energy scale \( \Lambda \) [6, 7]. However, for the purpose of analyzing exclusively the small-\( x \) region, there is the alternative of doing a simpler analysis by using some of the existing analytical solutions to DGLAP equations in the small-\( x \) limit [8, 9].

The ZEUS and H1 Collaborations have presented the new precise combined data [10] for the SF \( F^2 \). The aim of this short paper is to compare the combined H1 and ZEUS data with the predictions obtained by using the so-called doubled asymptotic scaling (DAS) approach [9].

To improve the analysis at low \( Q^2 \) values, it is important to consider the well-known infrared modifications of the strong coupling constant. We will use the “frozen” and analytic versions (see, [11] and references therein).

2. PARTON DISTRIBUTIONS AND THE STRUCTURE FUNCTION \( F^2 \)

Here, for simplicity we consider only the leading order (LO) approximation. The structure function \( F^2 \) has the form

\[
F^2(x, Q^2) = [\sigma(x, Q^2) + f^+(x, Q^2)] [\sigma(x, Q^2) + f^-(x, Q^2)],
\]

where \( e = \left( \sum_{i} e_i^2 \right) / f \) is an average charge squared.

The small-\( x \) asymptotic expressions for parton densities \( f^\pm \) look like

\[
f^+_q(x, Q^2) = \left( A_q + \frac{4}{9} A_g \right) I_0(\sigma)e^{-2s} + O(\rho),
\]

\[
f^-_q(x, Q^2) = \frac{4}{9} A_q e^{-2s} + O(x),
\]

where \( I_\nu (\nu = 0, 1) \) are the modified Bessel functions,

\[
s = \ln \left( \frac{a_s(Q^2)}{a_s(Q^2)} \right), \quad \sigma = \sqrt{\ln \left( \frac{1}{x} \right)},
\]

\[
\rho = \frac{\sigma}{2 \ln(1/x)}, \quad a_s(Q^2) = \frac{\alpha_s(Q^2)}{4\pi}
\]
and

\[ \hat{d}_+ = -\frac{12}{\beta_0}, \quad \hat{d}_- = 1 + \frac{20f}{27\beta_0}, \quad d_- = \frac{16f}{27\beta_0} \quad (4) \]

denote singular and regular parts of the anomalous dimensions \( d_+(n) \) and \( d_-(n) \), respectively, in the limit \( n \to 1 \). Here \( n \) is a variable in the Mellin space.

3 We denote the singular and regular parts of a given quantity \( k(n) \) in the limit \( n \to 1 \) by \( \hat{k} / (n - 1) \) and \( \hat{k} \), respectively.

### 3. “FROZEN” AND ANALYTIC COUPLING CONSTANTS

In order to improve an agreement at low \( Q^2 \) values, the QCD coupling constant is modified in the infrared region. We consider two modifications that effectively increase the argument of the coupling constant at low \( Q^2 \) values (see [12]).

In the first case, which is more phenomenological, we introduce freezing of the coupling constant by changing its argument \( Q^2 \to Q^2 + M_\rho^2 \), where \( M_\rho \) is the \( \rho \)-meson mass (see [11] and discussions therein).
The results of LO and NLO fits to H1 and ZEUS data [10], with various lower cuts on $Q^2$, in the fits the number of flavors $f$ is fixed to 4.

| $Q^2 > 5$ GeV$^2$ | $A_g$ | $A_q$ | $Q_0^2$, GeV$^2$ | $\chi^2$/n.d.f. |
|-------------------|------|------|----------------|-----------------|
| LO                | 0.623 ± 0.055 | 1.204 ± 0.093 | 0.437 ± 0.022 | 1.00 |
| LO and an.        | 0.796 ± 0.059 | 1.103 ± 0.095 | 0.494 ± 0.024 | 0.85 |
| LO and fr.        | 0.782 ± 0.058 | 1.110 ± 0.094 | 0.485 ± 0.024 | 0.82 |
| NLO               | −0.252 ± 0.041 | 1.335 ± 0.100 | 0.700 ± 0.044 | 1.05 |
| NLO and an.       | 0.102 ± 0.046 | 1.029 ± 0.106 | 1.017 ± 0.060 | 0.74 |
| NLO and fr.       | −0.132 ± 0.043 | 1.219 ± 0.102 | 0.793 ± 0.049 | 0.86 |

| $Q^2 > 3.5$ GeV$^2$ | $A_g$ | $A_q$ | $Q_0^2$, GeV$^2$ | $\chi^2$/n.d.f. |
|-------------------|------|------|----------------|-----------------|
| LO                | 0.542 ± 0.028 | 1.089 ± 0.055 | 0.369 ± 0.011 | 1.73 |
| LO and an.        | 0.758 ± 0.031 | 0.962 ± 0.056 | 0.433 ± 0.013 | 1.32 |
| LO and fr.        | 0.775 ± 0.031 | 0.950 ± 0.056 | 0.432 ± 0.013 | 1.23 |
| NLO               | −0.310 ± 0.021 | 1.246 ± 0.058 | 0.556 ± 0.023 | 1.82 |
| NLO and an.       | 0.116 ± 0.024 | 0.867 ± 0.064 | 0.909 ± 0.330 | 1.04 |
| NLO and fr.       | −0.135 ± 0.022 | 1.067 ± 0.061 | 0.678 ± 0.026 | 1.27 |

| $Q^2 > 2.5$ GeV$^2$ | $A_g$ | $A_q$ | $Q_0^2$, GeV$^2$ | $\chi^2$/n.d.f. |
|-------------------|------|------|----------------|-----------------|
| LO                | 0.526 ± 0.023 | 1.049 ± 0.045 | 0.352 ± 0.009 | 1.87 |
| LO and an.        | 0.761 ± 0.025 | 0.919 ± 0.046 | 0.422 ± 0.010 | 1.38 |
| LO and fr.        | 0.794 ± 0.025 | 0.900 ± 0.047 | 0.425 ± 0.010 | 1.30 |
| NLO               | −0.322 ± 0.017 | 1.212 ± 0.048 | 0.517 ± 0.018 | 2.00 |
| NLO and an.       | 0.132 ± 0.020 | 0.825 ± 0.053 | 0.898 ± 0.026 | 1.09 |
| NLO and fr.       | −0.123 ± 0.018 | 1.016 ± 0.051 | 0.658 ± 0.021 | 1.31 |

| $Q^2 > 0.5$ GeV$^2$ | $A_g$ | $A_q$ | $Q_0^2$, GeV$^2$ | $\chi^2$/n.d.f. |
|-------------------|------|------|----------------|-----------------|
| LO                | 0.366 ± 0.011 | 1.052 ± 0.016 | 0.295 ± 0.005 | 5.74 |
| LO and an.        | 0.665 ± 0.012 | 0.804 ± 0.019 | 0.356 ± 0.006 | 3.13 |
| LO and fr.        | 0.874 ± 0.012 | 0.575 ± 0.021 | 0.368 ± 0.006 | 2.96 |
| NLO               | −0.443 ± 0.008 | 1.260 ± 0.012 | 0.387 ± 0.010 | 6.62 |
| NLO and an.       | 0.121 ± 0.008 | 0.656 ± 0.024 | 0.764 ± 0.015 | 1.84 |
| NLO and fr.       | −0.071 ± 0.007 | 0.712 ± 0.023 | 0.529 ± 0.011 | 2.79 |

Thus, in the formulae of Section 2 we have to carry out the following replacement:

$$a_s(Q^2) \rightarrow a_s(Q^2) = a_s(Q^2 + M_F^2). \tag{5}$$

The second possibility follows the Shirkov–Solovtsov idea [13] concerning the analyticity of the coupling constant that leads to additional power dependence of the latter. Then, in the formulae of the previous section the coupling constant $a_s(Q^2)$ should be replaced as follows:

$$a_s^{\text{LO}}(Q^2) = a_s(Q^2) - \frac{\Lambda_{\text{LO}}^2}{\beta_0 Q^2 - \Lambda_{\text{LO}}^2}. \tag{6}$$

in the LO approximation and

$$a_s^{\text{NLO}}(Q^2) = a_s(Q^2) - \frac{\Lambda^2}{\beta_0 Q^2 - \Lambda^2} + \ldots. \tag{7}$$

in the NLO approximation. Here the symbol ... stands for the terms that provide negligible contributions when $Q^2 \geq 1$ GeV [13].

Note that the perturbative coupling constant $a_s(Q^2)$ is different in the LO and NLO approximations. Indeed, from the renormalization group equation we can obtain the following equations for the coupling constant

$$\frac{1}{a_s^{\text{LO}}(Q^2)} = \beta_0 \ln \left( \frac{Q^2}{\Lambda_{\text{LO}}^2} \right) \tag{8}$$

in the LO approximation and

$$\frac{1}{a_s^{\text{NLO}}(Q^2)} + \beta_1 \ln \left[ \frac{\beta_0 a_s(Q^2)}{\beta_0 + \beta_1 a_s(Q^2)} \right] = \beta_0 \ln \left( \frac{Q^2}{\Lambda^2} \right) \tag{9}$$

in the NLO approximation. Usually at the NLO level $\overline{\text{MS}}$-scheme is used; therefore, below we apply $\Lambda = \Lambda_{\text{\overline{MS}}}$.  

4. COMPARISON WITH EXPERIMENTAL DATA

By using the results of the previous section we have analyzed H1 and ZEUS data for $F_2$ [10]. In order to keep the analysis as simple as possible, we fix $f = 4$ and $a_s(M_F^2) = 0.1168$ (i.e., $\Lambda^{(4)} = 284$ MeV) in agreement with more recent ZEUS results given in [1].

As can be seen from figure 1 and table, the twist-two approximation is reasonable for $Q^2 \geq 4$ GeV$^2$. At
lower $Q^2$ we observe that the fits in the cases with “frozen” and analytic strong coupling constants are very similar (see also [14, 11]) and describe the data in the low $Q^2$ region significantly better than the standard fit. Nevertheless, for $Q^2 \geq 0.5$ GeV$^2$ there is still some disagreement with the data, which needs to be additionally studied. In particular, the Balitsky–Fadin–Kuraev–Lipatov (BFKL) resummation [15] may be important here [16]. It can be added in the generalized DAS approach according to the discussion in Ref. [17].

5. CONCLUSIONS

We have studied the $Q^2$-dependence of the structure function $F_2$ at small-$x$ values within the framework of perturbative QCD. Our twist-two results are well consistent with precise H1 and ZEUS data [10] in the region of $Q^2 \geq 4$ GeV$^2$, where perturbative theory is thought to be applicable. The usage of “frozen” and analytic modifications of the strong coupling constant, $\alpha_s(Q^2)$ and $\alpha_{an}(Q^2)$, is seen to improve an agreement with experiment at low $Q^2$ values, $Q^2 \geq 0.5$ GeV$^2$.

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REFERENCES

1. C. Adloff et al., “H1 Collab.,” Nucl. Phys. B 497, 3 (1997); Eur. Phys. J. C 21, 33 (2001); S. Chekanov et al., “ZEUS Collab.,” Eur. Phys. J. C 21, 443 (2001).

2. C. Adloff et al., “H1 Collab.,” Phys. Lett. B 520, 183 (2001).

3. T. Lastovicka, H1 Collaboration, Acta Phys. Polon. B 33, 2835; B. Surrow, ZEUS Collaboration, hep-ph/0201025.

4. A. M. Cooper-Sarkar, R. C. E. Devenish, and A. De Roeck, Int. J. Mod. Phys. A 13, 3385 (1998); A. V. Kotikov, Phys. Part. Nucl. 38, 1 (2007) [Erratum-Phys. Part. Nucl. 38, 828 (2007)].

5. V. N. Gribov and L. N. Lipatov, Sov. J. Nucl. Phys. 15, 438 (1972); V. N. Gribov and L. N. Lipatov, Sov. J. Nucl. Phys. 15, 675 (1972); L. N. Lipatov, Sov. J. Nucl. Phys. 20, 94 (1975); G. Altarelli and G. Parisi, Nucl. Phys. B 126, 298 (1977); Yu. L. Dokshitzer, Sov. Phys. JETP 46, 641 (1977).

6. A. D. Martin, W. J. Stirling, R. S. Thorne, and G. Watt, Eur. Phys. J. C 64, 653 (2009); H.-L. Lai, J. Huston, Z. Li, P. Nadolsky, J. Pumplin, D. Stump, and C.-P. Yuan, Phys. Rev. D 82, 054021 (2010); S. Alekhin, J. Blumlein, and S. Moch, Phys. Rev. D 86, 054009 (2012); P. Jimenez-Delgado and E. Reya, Phys. Rev. D 79, 074023 (2009).

7. A. V. Kotikov, G. Parente, and J. Sanchez Guillen, Z. Phys. C 58, 465 (1993); G. Parente, A. V. Kotikov, and V. G. Krivokhizhin, Phys. Lett. B 333, 190 (1994); A. L. Kataev, A. V. Kotikov, G. Parente, and A. V. Sidorov, Phys. Lett. B 388, 179 (1996); 417, 374 (1998); L. Kataev, G. Parente, and A. V. Sidorov, Nucl. Phys. B 573, 405 (2000); A. V. Kotikov and V. G. Krivokhizhin, Phys. At. Nucl. 68, 1873 (2005); B. G. Shaikhatdenov, A. V. Kotikov, V. G. Krivokhizhin, and G. Parente, Phys. Rev. D 81, 034008 (2010) [Erratum-ibid. 81, 079904].

8. A. De Ru’jula, S. L. Glashow, H. D. Politzer, S. B. Treiman, F. Wilczek, and A. Zee, Phys. Rev. D 10, 1649 (1974); R. D. Ball and S. Forte, Phys. Lett. B 336, 77 (1994); L. Mankiewicz, A. Saalfeld, and T. Weigl, Phys. Lett. B 393, 175 (1997).

9. A. V. Kotikov and G. Parente, Nucl. Phys. B 549, 242 (1999); A. Yu. Illarionov, A. V. Kotikov, and G. Parente, Phys. Part. Nucl. 39, 307 (2008).

10. F. D. Aaron et al., “H1 and ZEUS Collaboration,” JHEP 1001, 109 (2010).

11. G. Cvetic, A. Yu. Illarionov, B. A. Kniehl, and A. V. Kotikov, Phys. Lett. B 679, 350 (2009).

12. Yu. L. Dokshitzer and D. V. Shirkov, Z. Phys. C 67, 449 (1995); A. V. Kotikov, Phys. Lett. B 338, 349 (1994); W. K. Wong, Phys. Rev. D 54, 1094 (1996); S. J. Brodsky, V. S. Fadin, V. T. Kim, L. N. Lipatov, and G. B. Pivovarov, JETP Lett. 70, 153 (1999); M. Ciafaloni, D. Colferai, and G. P. Salam, Phys. Rev. D 60, 114036 (1999); JHEP 07, 054 (2000); G. Altarelli, R. D. Ball, and S. Forte, Nucl. Phys. B., V. 621, P. 359 (2002); Andersson Bo et al., Eur. Phys. J. C 25, 77 (2002).

13. D. V. Shirkov and I. L. Solovtsov, Phys. Rev. Lett. 79, 1209 (1997).

14. A. V. Kotikov, A. V. Lipatov, and N. P. Zotov, J. Exp. Theor. Phys. 101, 811 (2005); A. V. Kotikov, V. G. Krivokhizhin, and B. G. Shaikhatdenov, Phys. Atom. Nucl. 75, 507 (2012).

15. L. N. Lipatov, Sov. J. Nucl. Phys. 23, 338 (1976); E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, Phys. Lett. B 60, 50 (1975); Sov. Phys. JETP 44, 443 (1976); 45, 199 (1977); Ya. Ya. Balitzki and L. N. Lipatov, Sov. J. Nucl. Phys. 28, 822 (1978); L. N. Lipatov, Sov. Phys. JETP 63, 904 (1986).

16. H. Kowalski, L. N. Lipatov, and D. A. Ross, BFKL Evolution as a Communicator Between Small and Large Energy Scales, arXiv:1205.6713 [hep-ph].

17. A. V. Kotikov, “Small x behavior of parton distributions. Analytical and ‘frozen’ coupling constants. BFKL corrections,” arXiv:1212.3733 [hep-ph].