Curvature in causal BD-type inflationary cosmology.

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We study a closed model of the universe filled with viscous fluid and quintessence matter components in a Brans-Dicke type cosmological model. The dynamical equations imply that the universe may look like an accelerated flat Friedmann-Robertson-Walker universe at low redshift. We consider here dissipative processes which follow a causal thermodynamics. The theory is applied to viscous fluid inflation, where accepted values for the total entropy in the observable universe is obtained.

I. INTRODUCTION

Recent observation of the Hubble diagram for supernovae Ia indicates that the expansion of the universe is accelerating at the present epoch\textsuperscript{1,2}. This acceleration is attributed to a dark energy residing in space itself, which also balances the kinetic energy of expansion so as to give the universe zero spatial curvature, as deduced from the cosmic microwave background radiation\textsuperscript{3}.

Initial interpretation was to consider that in the universe there exists an important matter
component that, in its most simple description, has the characteristic of the cosmological constant \( \Lambda \), i.e. a vacuum energy density which contributes to a large component of negative pressure, and thus accelerates rather than decelerates the expansion of the universe. An alternative interpretation is to consider quintessence dark energy, in the form of a scalar field with self interacting potential \([4]\).

On the other hand, in recent years an important attention has received cosmological models in which bulk viscosity is considered. For example, it was shown that the introduction of this kind of viscosity into cosmological models can avoid the big bang singularities \([5, 6, 7]\) and any contribution from particle production may be modelled as an effective bulk viscosity \([8]\). The bulk viscosity arises typically in mixtures of different species (as in a radiative fluid) or of the same (but with different energies) fluids. The dissipation due to bulk viscosity converts kinetic energy of the particles into heat, and thus one expects it to reduce the effective pressure in an expanding fluid. This fact may play a crucial role in the inflationary era of the universe since it is interesting to know whether dissipative effects could be strong enough to do a large negative effective pressure leading to inflation.

Many, probably most, of the inflationary cosmological models considered are of flat Friedmann-Robertson-Walker (FRW) type. In this kind of models the velocity gradients causing shear viscosity and temperature gradients leading to heat transport are absent. Thus, any dissipation in FRW universe may be modelled as a bulk viscosity within a thermodynamical approach. These dissipative processes, that consider the bulk viscosity \([9, 10]\), are compatible with the homogeneity and isotropy assumptions for the universe, and may play an important role in the early universe, specially before nucleosynthesis \([11]\).

Curvature models have been studied with an important matter component whose equation of state is given by \( p = -\rho/3 \). Here, the universe expands at a constant speed \([12]\). On the other hand, a flat decelerating universe model has been described \([13]\). Also, following a similar idea in accelerating models, the dark energy potential \([14]\) and the location of the first Doppler peak in the Cosmic Microwave Background spectrum \([15]\) have been studied. In this way, we could consider models where the starting geometry were other than that corresponding to the critical geometry, but at low redshift these are indistinguishable from flat geometries.

In this paper we want to study universe models in which we use the causal thermodynamical theory for processes out of equilibrium \([16]\). The stable and causal thermodynamics
of Israel and Stewart replaces satisfactorily the unstable and non-causal theory of Eckart and Landau and Lifshitz. The main goal in this paper is to study not vanishing curvature causal inflationary universe models. We pretend to do this in Brans-Dicke (BD)-type theory. Here the theory is characterized by a scalar field $\Phi$ and a constant coupling function $\omega_0$ together with a scalar potential associated to $\Phi$. From now on the subscript zero will represent the actual values.

II. THE FIELD EQUATIONS

We consider the effective action to be

$$S = \int d^4x \sqrt{-g}\left\{ \frac{1}{16\pi}(\Phi R - \omega_0 (\partial_{\mu} \Phi)^2 + V(\Phi)) + \frac{1}{2}(\partial_{\mu} Q)^2 + V(Q) + L_M \right\}$$  \hspace{1cm} (1)

where $\Phi$ is the BD scalar field, $V(\Phi)$ is a scalar potential associated to the BD field, $\omega_0$, is the BD parameter, $Q$ the quintessence scalar field with associated potential $V(Q)$, $R$ the scalar curvature and $L_M$ represents the matter contributions other than the $Q$ component.

We will obtain the field equations for a cosmological viscous model where the energy-momentum tensor of the fluid with a bulk viscosity is taken to be

$$T_{\mu\nu} = (\rho_M + p_M + \Pi) v_\mu v_\nu + (p_M + \Pi) g_{\mu\nu}, \hspace{1cm} (2)$$

with $\rho_M, p_M$ the thermodynamical density, the pressure of the fluid, and $\Pi$ is the bulk viscous pressure, respectively. In the co-movil reference system we take $v^\mu v_\mu = -1$ and $v^\mu = \delta^\mu_0$.

The contribution to the energy-momentum tensor related to the quintessence scalar field, becomes given by:

$$T_{\mu\nu} = (\rho_Q + p_Q) v_\mu v_\nu + g_{\mu\nu} p_Q, \hspace{1cm} (3)$$

We can get the field equations of the universe from the action $S$, to the FRW metric.

$$H^2 + H \dot{\Phi} \frac{\dot{\Phi}}{\Phi} = \frac{\omega_0}{6} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 + \frac{8\pi}{3\Phi} (\rho_M + \rho_Q) - \frac{k}{a^2} + \frac{1}{6\Phi} V(\Phi)$$  \hspace{1cm} (4)

$$2 \dot{H} + 3H^2 + \frac{k}{a^2} = -\frac{\omega_0}{2} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 - \frac{1}{\Phi} (\ddot{\Phi} + 2H \dot{\Phi}) + \frac{V(\Phi)}{2\Phi} - \frac{8\pi}{\Phi} (p_M + p_Q + \Pi)$$  \hspace{1cm} (5)
\[
\ddot{\Phi} + 3H\dot{\Phi} + \frac{\Phi^3}{2\omega_0 + 3 \frac{d}{d\Phi} \left( \frac{V(\Phi)}{\Phi^2} \right)} = \frac{8\pi}{2\omega_0 + 3} \left( \rho_M - 3p_M + \rho_Q - 3P_Q - 3\Pi \right) 
\]
where \(a(t)\) is the scale factor, and \(k\) is the curvature parameter that is equal to 1 (closed), 0 (flat) or -1 (open).

The matter conservation equation is given by
\[
\dot{\rho}_M + 3H(\rho_M + p_M + \Pi) = 0 
\]
where the matter has a barotropic equation of state \(p_M = (\gamma - 1)\rho_M\) where the parameter \(\gamma\) is in the range \(0 \leq \gamma \leq 3\). The energy density associated to the quintessence scalar field is given by,
\[
\rho_Q = \frac{1}{2} \dot{Q}^2 + V(Q) 
\]
and the pressure
\[
p_Q = \frac{1}{2} \dot{Q}^2 - V(Q). 
\]

This quintessence scalar field has a equation of state defined by \(p_Q = w_Q\rho_Q\), where \(w_Q\) is the eq. of it parameter, and similar to the matter fluid, it satisfies the conservation equation
\[
\dot{\rho}_Q + 3H(\rho_Q + p_Q) = 0. 
\]

III. CHARACTERISTICS OF THE MODEL

In order to mimic a flat universe model, we impose the following conditions
\[
\frac{8\pi}{3\Phi} \rho_Q - \frac{k}{a^2} + \frac{1}{6\Phi} V(\Phi) = 0 
\]
and
\[
\rho_Q = \frac{\Phi^3}{8\pi(1 - 3\omega_Q)} \frac{d}{d\Phi} \left( \frac{V(\Phi)}{\Phi^2} \right). 
\]
In the following we will assume a power law potential for the BD field, \( V(\Phi) = V_0 \left( \frac{\Phi}{\Phi_0} \right)^\beta \), where \( V_0 \) and \( \beta \) are constant. We will see that \( \beta \) becomes determined entirely by the equation 

\[
\frac{k}{a^2} = \left( \frac{2\beta - 3(1 + w_Q)}{6(1 - 3w_Q)} \right) \frac{V_0}{\Phi_0} \left( \frac{\Phi}{\Phi_0} \right)^{\beta - 1},
\]

where for \( V_0 > 0 \) and \( \Phi_0 > 0 \), we obtain that \( \beta > \frac{3}{2} (1 + w_Q) \) if \( k > 0 \), \( \beta = \frac{3}{2} (1 + w_Q) \) if \( k = 0 \) and \( \beta < \frac{3}{2} (1 + w_Q) \) if \( k < 0 \). Since we are interested in power law inflationary universe models, then we take the scale factor to be given by

\[
a(t) = a_0 \left( \frac{t}{t_0} \right)^N.
\]

with \( N > 1 \). Therefore, we obtain for the BD field

\[
\Phi(t) = \Phi_0 \left( \frac{t}{t_0} \right)^{\frac{2N}{\beta}}
\]

and \( V_0 = \frac{6k(1-3w_Q)\Phi_0}{2\beta-3(1+w_Q)_a^\beta} \). With these solutions, the quintessence energy density results in

\[
\rho_Q = \frac{8\pi(2\beta-3(1+w_Q))}{6k(\beta-2)} \frac{\Phi_0}{a^2} \left( \frac{t}{t_0} \right)^{\frac{2N\beta}{\beta_1}}
\]

Since \( N > 1 \), then we need \( \beta < 0 \) or \( \beta > 1 \) in order to satisfy the energy conservation equation. From the \( Q \) field eq. (11) this gives \( w_Q = -\frac{3}{2} \frac{1-\beta}{1+\beta} \), or equivalently \( \beta = 3 \frac{1 + w_Q}{1 + 3w_Q} \), which becomes determined in terms of the \( w_Q \) parameter as was mentioned above. We should note that since \(-1 < w_Q < -1/3\), then \( \beta < 0 \), therefore the possibility of satisfying \( \beta > 1 \) is not possible. This implies that the BD scalar potential is an inverse power law potential of the BD field.

The field solutions together with the constraint equations yield to the following expression for the quintessence field:

\[
Q(t) = Q_0 \left( \frac{t}{t_0} \right)^{\frac{1+(N-1)\beta}{1-\beta}}
\]

where \( Q_0 = \sqrt{\frac{6k(1+w_Q)(\beta-2)(1-\beta)^2\Phi_0 t_0^2}{8\pi(2\beta-3(1+w_Q))(1+(N-1)\beta)^2a_0^2}} \). From expressions (9) and (10), together with the equation of state for the quintessence field yields to

\[
V(Q) = V(Q_0) \left( \frac{Q}{Q_0} \right)^{\frac{2N\beta}{1+(N-1)\beta}}
\]

where \( V(Q_0) = \frac{6k(\beta-2)(1-w_Q)\Phi_0}{16\pi(2\beta-3(1+w_Q))a_0^2} \). Notice that, since \( \beta < 0 \), then we need just only to satisfy \( N > 2 \) in order to get an usual quintessence scalar potential, i.e., an inverse power law scalar potential.
IV. TRANSPORT EQUATION FOR BULK VISCOSITY

At this point we would like to describe dissipation due to bulk viscosity $\xi$, via Israel-Stewart theory. The bulk viscous pressure $\Pi$ is given by the transport equation (linear in $\Pi$)

$$\tau \dot{\Pi} + \Pi = -3H\xi - \frac{1}{2}\tau \Pi \left( 3H + \frac{\dot{\tau}}{\tau} - \frac{\ddot{\xi}}{\xi} - \frac{\dot{T}}{T} \right),$$

(20)

where $\tau$ is the relaxation time (which removes the problem of infinite propagation speeds) and $T$ is the temperature of the fluid. In the non-causal formulation $\tau = 0$ and then Eq.(20) has a simple form $\Pi = -3H\xi$.

Following refs.[9, 20] we will take the thermodynamic quantities to be simple power functions of the matter density $\rho_M$,

$$\xi = \alpha \rho_M^m, \quad T = \mu \rho_M^r, \quad \tau = \frac{\xi}{\rho_M} = \alpha \rho_M^{m-1}$$

(21)

with $\alpha, \mu, m$ and $r$ greater than zero. The expression for $\tau$ is used as a simple procedure to ensure that the speed of viscous pulses does not exceed the speed of light. For an expanding cosmological model the constant $m$ should be positive and we should satisfy

$$\tau > H^{-1},$$

(22)

in order to have a proper physical behavior for $\xi$ and $\tau$[9, 21].

From the expansion law[13] and from eqs.(8), (20) and (21) (with the standard relation for the temperature of a barotropic fluid in which $r = (\gamma - 1)/\gamma$) we obtain

$$\dot{\rho}_M + (3N + 1) \frac{\dot{\rho}_M}{t} + \frac{1}{\alpha} \rho_M^{1-m} \ddot{\rho}_M + \frac{3N\gamma}{\alpha t} \rho_M^{2-m}$$

$$-9N^2 \left( 1 - \frac{\gamma}{2} \right) \frac{\rho_M}{t^2} - \frac{(2\gamma - 1)}{2\gamma} \frac{\rho_M^2}{\rho_M} = 0.$$  

(23)

This differential eq. may be solve for some specific values of the $m$ parameter. Taking the following anzats[21]

$$\rho_M = \rho_M^0 \left( \frac{t}{t_0} \right)^{\left( \frac{1-m}{t-M} \right)}$$

(24)

with $m < 1$ in order that $\rho_M$ decreases when $t$ increases, we obtain from eq.(23) the constant
\( \rho_0^\prime \) becomes given by

\[
\rho_0^\prime = \left( \frac{\alpha}{t_0(1-m)} \right) \times \\
\left( \frac{(2\gamma)^{-1} - 3N(1-m) - 9N^2(1-\gamma/2)(1-m)^2}{(1 - 3N\gamma(1-m))} \right)^{1/(1-m)}. 
\]

From this we can find \( \alpha \) as a function of the parameters \( \gamma, N, m \) and \( \rho_0^\prime \). The pressure \( p_M \) is obtained using the equation of state \( p_M = (\gamma - 1)\rho_M \).

Now, from eq. (8) and the barotropic eq. of state for the matter fluid we obtain for the bulk viscous pressure

\[
\Pi = -\frac{\dot{\rho}_M}{3H} - \gamma \rho_M,
\]
and substituting eq. (24) in this latter eq. we get

\[
\Pi(t) = -\left( \gamma - \frac{1}{3N(1-m)} \right) \rho_0^\prime \left( \frac{t}{t_0} \right)^{-1/(1-m)}. 
\]

We see that \( \Pi < 0 \) if \( \gamma > 1/(3N(1-m)) \), \( \Pi > 0 \) if \( \gamma < 1/(3N(1-m)) \) and \( \Pi = 0 \) if \( m = 1 - 1/(3N\gamma) \).

In order to satisfy the field eqs. of motion the parameters should follow the equality

\[
\frac{1}{1-m} = 2 - \frac{2N}{1-\beta}, \text{ with } \beta \text{ expressed in terms of } w_Q, \text{ thus } m \text{ becomes a function of the parameter } N \text{ and } w_Q,
\]

\[
m = 1 - \frac{1 - \beta}{2(1 - \beta - N)},
\]
where \( \beta \) is a function of the \( w_Q \) parameter.

At local equilibrium, the entropy satisfy the Gibbs equation

\[
TdS = (\rho_M + p_M) d\left( \frac{1}{n} \right) + \frac{1}{n} d\rho_M,
\]
where \( n \) is the number density and satisfy the conservation equation

\[
\dot{n} + 3Hn = 0,
\]
from which we get

\[
n(t) = n_0 \left( \frac{t}{t_0} \right)^{-3N}
\]
or equivalently \( n(t)a^3(t) = \text{const.} \).
Using (29) and (31) we obtain the well known evolution equation for the entropy (neglecting the heat flux and the shear viscosity)

\[ \dot{S} = -\frac{3H\Pi}{nT}, \]  

(32)

thus from eqs. (10), (21), (24), (27) and (31) we get

\[ \dot{S} = \frac{3n\gamma - 1/(1-m)}{n_0\mu t_0} (\rho_M^0)^{1/\gamma} \left( \frac{t}{t_0} \right)^{(3N-1)\gamma/(1(1-m))}. \]

(33)

For proper physical behaviour we must satisfy \( \dot{S} > 0 \), which implies the condition \( m < 1 - 1/\gamma(3N - 1) \). This inequality together with eq. (28) give an inequality for \( \gamma \) as function of the parameters \( N \) and \( w_Q \),

\[ \gamma > \frac{(1 - \beta - N)}{(1 - \beta)(3N - 1)}. \]

Thus, this parameter becomes bounded from below.

The total entropy \( \Sigma \) in a comoving volume is defined by means \( \Sigma = Sna^3(t) \). Then, by taking Eq. (31) and (32) we may write the growth of total nondimensional comoving entropy over a proper time interval \( t_f \) and \( t_i \) as

\[ \Sigma_f - \Sigma_i = -\frac{3}{k_B} \int_{t_i}^{t_f} \frac{\Pi a^3(t)}{T} dt. \]

(34)

Now taking \( t_i \) and \( t_f \) as the beginning and the exit time for the inflation period of the universe respectively and from eqs. (15), (21), (24), (27), and (34) we obtain for the increase the total nondimensional entropy in the comoving volume \( a^3(t) \) the following expression

\[ \Sigma_f - \Sigma_i = \frac{\gamma a_0^3 (\rho_M^0)^{1/\gamma}}{k_B \mu} \times \]

\[ \left[ \left( \frac{t_f}{t_0} \right)^{3N - \frac{1}{\gamma(1-m)}} - \left( \frac{t_i}{t_0} \right)^{3N - \frac{1}{\gamma(1-m)}} \right]. \]

(35)

We take the values for the beginning and ending time of inflation as \( t_i \approx 10^{-35} \) s and \( t_f \approx 10^{-32} \) s, respectively. In the following numerical calculation we take the reference time \( t_0 \) equal to \( t_f \). Considering that the universe at the end of the period inflation exits to the radiation era, we can constraint some of the constants of integration of the above formulae. Effectively, it is known that the temperature of the universe at the beginning of the radiation era is approximately \( T \sim 10^{14} \text{GeV} \approx 1.16 \times 10^{27} \text{ K} \). Thus, the temperature
at the end of inflation must be \( T_f = 1.16 \times 10^{27} \text{ K} \). On the other hand, we know that in this period \( \rho = a_r T^4 \), where \( a_r = \frac{\pi^2 k^4}{15c^3 h^2} \approx 7.56 \times 10^{-15} \text{ J m}^{-3} K^{-4} \). From here we conclude that at the end of inflation any period (or at the beginning of the radiation era) we have \( \rho_f \approx 10^{93} \text{ J m}^{-3} \). This implies that \( r = 1/4 \) for the exponent in the temperature \( T_i \), i.e. \( \gamma = 4/3 \), as we can see from eq. (21).

Thus the typical values for the inflationary period are:

\[
\begin{align*}
  t_i &\approx 10^{-35} \text{ s}; \quad t_f \approx 10^{-32} \text{ s}; \quad a_i \approx c t_i; \\
  T_f &\approx 10^{27} \text{K}; \quad \gamma = 4/3; \quad \rho \approx \times 10^{93} \text{J/m}^3.
\end{align*}
\] (36)

The e-folding parameter \( Z = \ln[a(t_f)/a(t_i)] \) for the power law inflation (15) takes the form

\[
Z = n \ln \left( \frac{t_f}{t_i} \right).
\] (37)

It is well known that for solving most the problems of the standard model in cosmology we must have \( Z \approx 60 - 70 \). Thus from (21), (35), (36) and (37) we have

\[
\Sigma_f - \Sigma_i \approx \frac{4a_i e^{3Z} (\rho_M^f)^{3/4}}{3k_B \mu} \left( 1 - (10^{-3})^{3N - \frac{3}{a(1-m)}} \right),
\] (38)

where we have used the relation \( a_f^3 = a_i^3 e^{3Z} \), following from the definition of \( Z \), eq (37). We see from Eq. (38) that if the inequality \( 3N \gamma (1-m) >> 1 \) is satisfied (in this case \( \Sigma_f >> \Sigma_i \)) we obtain the accepted value for the total entropy in the observable universe (12)

\[
\Sigma \approx \times 10^{88}.
\] (39)

Then this model can account for the generally accepted entropy production via causal dissipative inflation, without reheating.

Finally, for complete, we will see what happen when \( m = 1 \). For this value and considering eq. (23), which gives the solution (21)

\[
\rho_M = \rho_M^0 e^{-2t/\alpha} \left( \frac{t}{t_0} \right)^{-\gamma (3N+2)},
\] (40)

where \( N^2 = 2\gamma/9 \) and \( \rho_M^0 \) is an arbitrary constant. Unfortunately this solution implies a decreasing entropy

\[
S(t) = \frac{\gamma}{\mu_0 n_0} \left( \rho_M^0 \right)^{1/\gamma} \left( \frac{t_0}{t} \right)^2 e^{-2t/\alpha + \text{const}}
\] (41)

which certainly we should neglect in order to finish with a reasonable physical model.
V. CONCLUSIONS

To conclude, on the light of the above results we may say that there exist an intimate relation between causal dissipative inflation and reheating. We expect to come back to this point of research in the near future, where extensions to more general universe models will be applied.

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