‘Algorithmic cooling’ in a momentum state quantum computer

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We describe a quantum computer based upon the coherent manipulation of two-level atoms between discrete one-dimensional momentum states. Combinations of short laser pulses with kinetic energy dependent free phase evolution can perform the logical invert, exchange, CNOT and Hadamard operations on any qubits in the binary representation of the momentum state, as well as conditional phase inversion. These allow a binary right-rotation, which halves the momentum distribution in a single coherent process. Fields for the coherent control of atomic momenta may thus be designed as quantum algorithms.

Proposed schemes for quantum computation have tended, quite naturally, to focus on quantum analogues of classical binary computing elements. The nuclear spins of a molecule, or of an ensemble of trapped atoms or ions, thus mimic the bits of a conventional computer. In this article, we address a less obvious system, in which information is represented by the momentum of a single atom or molecule, which is manipulated using laser pulses in a one-dimensional geometry that restricts each species to a ladder of equally spaced momentum states. Although any one laser pulse can change the species momentum, through photon absorption and stimulated emission, by only a single photon impulse, we find that sequences of pulses, interspersed with periods of momentum-dependent phase evolution, allow a full suite of quantum computational operations on the qubits comprising the binary representation of the momentum state.

The size of the momentum state quantum computer grows in proportion to the number of quantum states included, where conventional candidates instead scale with the number of qubits representing those states. The number of laser pulses needed to perform each logical operation increases similarly and, although the overall duration proves to be less drastically affected, momentum state systems therefore hold limited promise for real computing. Nonetheless, the scheme outlined here is based upon simple and readily-available elements, albeit in complex combinations, and could thus complement NMR systems as a testbed for experimental studies.

It is, however, in the design of complex fields for coherent control that we foresee the greatest potential, for if momentum-changing operations can form the basis of a quantum computer then the pulse sequences for optical manipulation may be optimized as quantum computational algorithms. In this respect, the momentum state quantum computer is an enthusiastic development of schemes for interferometric cooling and the coherent amplification of laser cooling.

Our scheme is based upon the motion in one dimension (henceforth taken to be vertical) of a sample of two-level atoms, such as an atomic beam interacting with transversely propagating laser beams, as shown in Fig. 1. We shall refer to four coherent operations:

\[ W_+ (\alpha, \phi) \quad \text{a short upward laser pulse} \]
\[ W_-(\alpha, \phi) \quad \text{a short downward laser pulse} \]
\[ F(\omega t) \quad \text{free evolution (electronic energy)} \]
\[ G(t/\tau) \quad \text{free evolution (kinetic energy)} \]

where \( \tau = 2m/(\hbar k^2) \). Here, the short (and therefore spectrally broad) laser pulses couple the upper and lower atomic levels, between which population is transferred through Rabi cycling for the duration of the pulse. Conventionally, we describe the overall effect of the pulse through the phase \( 2\alpha \) of the Rabi cycle incurred, the population being inverted when \( 2\alpha = \pi \) (the so-called ‘\( \pi \) pulse’), and restored when \( 2\alpha = 2\pi \). Other fractions of a \( \pi \) pulse will convert an initially pure state into a superposition. The phases \( \phi \) are determined by the relative optical phases of the laser pulses. The free evolution operations \( F(\omega t) \) and \( G(t/\tau) \) correspond simply to the components of the time-dependent wavefunction phase \( \exp(-iEt/\hbar) \) that correspond to the electronic energy and vertical momentum component respectively. Weitz and Hänisch have shown how the electronic and kinetic energy contributions to the free phase evolution may be separated by appropriate insertion of pairs of \( \pi \)-pulses that invert the atomic population, as we discuss later.

In this one-dimensional geometry, the atom is constrained to a ladder of momentum states that are spaced at intervals of the photon momentum \( \hbar k = \hbar \omega/c \) (\( \omega \) being the frequency of the resonant transition) and alternate between the ground and excited electronic levels \( g \) and \( e \). We label these states according to their momentum components, in units of the photon impulse \( \hbar k \). We initially assume, for our analysis of the momentum state quantum computer, that these momenta take integer values, but this assumption will be relaxed when we later...
consider the consequences for atomic manipulation.

We now convert our representation of the ladder of states to binary, using the notation \( Q_n \ldots Q_2 Q_1 Q_0 \), with the least significant bit on the right. This is the crucial step in our analysis; yet, apart from the correspondence of \( Q_0 \) to the electronic state of the atom, binary representation seems at first rather unpromising, for computational notation usually helps only when the bits themselves can be manipulated. While the momentum-changing laser pulses here move population by at most one state at a time, however, appropriate combinations prove to offer exactly such bit-wise manipulation.

The key, as in the interferometric cooling scheme of Ref. [6], is the dependence of the phase of free evolution upon the momentum. For two levels \( p + \Delta p/2 \) and \( p - \Delta p/2 \) (in units of \( \hbar k \)) and electronic energy difference \( E_{21} = h\omega \), the relative phase \( \psi \) evolves according to

\[
\psi = \frac{E_{21} t}{\hbar} + \frac{(h k)^2 t}{2m\hbar} \left( \left( p + \frac{\Delta p}{2} \right)^2 - \left( p - \frac{\Delta p}{2} \right)^2 \right) \\
= \omega t + \frac{h k^2 t}{m} p \Delta p.
\]

Any pair of momentum states thus incur a relative phase that evolves according to their average momentum \( p \). Cancellation of the electronic contribution to the phase, by inserting \( \pi \) pulse pairs so that the states spend equal times in the ground and excited levels [6], merely changes the average \( \Delta p \) and hence the rate at which this proceeds.

We illustrate the capacity for bit-wise manipulation with the example of a three qubit right rotation,

\[
\{Q_2, Q_1, Q_0\} \rightarrow \{Q_0, Q_2, Q_1\}.
\]

In our largely diagrammatic description, which has its origins in Fig. 1(c), pulses or complete pulse sequences coupling adjacent levels are indicated by \( \bigotimes \) when they produce a superposition (e.g., a \( \pi/2 \) pulse), \( \bigcirc \) when they cause inversion (the \( \pi \) pulse) and \( \bigotimes \) when \( 2\alpha = 0, 2\pi \) and so on. For periods of free evolution, we simply indicate the relative phases introduced between coupled states.

First consider a pair of upward-travelling \( \pi/2 \) pulses, separated by a period of free evolution that introduces between coupled states a relative phase adjusted to give in each case an integer power of \( \exp(-i\pi/2) \). This forms a simple interferometer which, depending upon the original state moment, can invert population, return it to its original state, or leave initially pure states coupled (see Fig. 1(c)). The sequence repeats every \( 8\hbar k \).

![FIG. 2: Bloch-vector representation of the first stage (Eq. (2)) of the right-rotation. The first pulse rotates the four ground states into the horizontal plane; free evolution distributes these around the vertical axis according to their momenta; the second pulse then returns two to pure states, leaving the others in mixed states. Aside from phase corrections, the full right-rotation operation takes 18 \( \pi/2 \) pulses and 26 \( \pi \) pulses.](image-url)
Matrices are in principle infinite, all non-zero terms cluster around the leading diagonal and any element $m_{i,j}$ differs from the diagonally displaced term $m_{i+2n,j+2n}$ only through its momentum dependence, so we may summarize the matrices as $4 \times 4$ elements, given below and derived from the equations of Friedberg and Hartmann [9].

Matrices that we use in practice must merely be expanded to cover the sequence of interactions and the momentum range that we wish to describe. The following matrices act on the states $\{2, 1, 0, -1\}$.

For upward and downward travelling fractional $\pi$ pulses corresponding to Bloch vector rotation through ‘Rabi angle’ $2\alpha$ and optical phase $\phi$, we have

$$W_+(\alpha, \phi) = \begin{pmatrix}
\cos \alpha & 0 & 0 & 0 \\
0 & \cos \alpha & i e^{i\phi} \sin \alpha & 0 \\
0 & i e^{-i\phi} \sin \alpha & \cos \alpha & 0 \\
0 & 0 & 0 & \cos \alpha
\end{pmatrix}$$

and

$$W_- (\alpha, \phi) = \begin{pmatrix}
\cos \alpha & i e^{-i\phi} \sin \alpha & 0 & 0 \\
i e^{i\phi} \sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & \cos \alpha & i e^{-i\phi} \sin \alpha \\
0 & 0 & i e^{i\phi} \sin \alpha & \cos \alpha
\end{pmatrix}.$$
and kinetic energies are respectively

\[
F(\omega t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{-i\omega t} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & e^{-i\omega t}
\end{pmatrix}
\] (7)

and

\[
G\left(\frac{t}{\tau}\right) = \begin{pmatrix}
e^{-i(p_0+2)t/\tau} & 0 & 0 & 0 \\
0 & e^{-i(p_0+1)^2t/\tau} & 0 & 0 \\
0 & 0 & e^{-ip_0^2t/\tau} & 0 \\
0 & 0 & 0 & e^{-i(p_0-1)^2t/\tau}
\end{pmatrix}
\] (8)

where \(\tau = \frac{2m}{\hbar k^2}\). Weitz and Hansch’s sequence for interferometric cooling thus becomes

\[
W_+\left(\frac{\pi}{4}\right) \cdot G\left(\frac{2T-T'}{\tau}\right) \cdot F(\omega (2T-T')) \cdot W_-\left(\frac{\pi}{2}\right) \cdot G\left(\frac{2T'}{\tau}\right) \cdot F(2\omega T) \cdot W_-\left(\frac{\pi}{2}\right) \cdot F(\omega T') \cdot W_+\left(\frac{\pi}{4}\right)
\]

No attempt has yet been made to optimize the sequences given here: the phase corrections often serve only to demonstrate the exact equivalence to a quantum computer, and reductions in the complexity, duration or momentum sensitivity of each operation should be possible. Nor have we examined superpositions involving more than two states \[^{[10]}\] or interactions at more than one wavelength \[^{[11]}\]. Instead of the fractional Rabi coupling of electronic transitions assumed for simplicity, our scheme could be more robustly implemented using Raman transitions \[^{[12]}\] for adiabatic passage \[^{[13]}\] between Zeeman or hyperfine levels, possible even with modulated c.w. lasers \[^{[14]}\]. The scheme, which we think of as a form of ‘algorithmic cooling’ \[^{[15]}\] in its broadest sense, could in principle be extended to three dimensions. Owing to the non-resonant nature of the pulsed interactions, it would also be suitable for molecules, for which the large impulse per coherent cycle would be a particular advantage.

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TABLE I: Basic operations of the momentum state quantum computer. No attempt has been made to optimize the pulse sequences, which run from right to left, and some uncorrected phases remain in the operations marked with an asterisk. For the operation $G$, $p_0$ is taken to be zero (mod 8).

| level | name          | description                                           | sequence                                                                 |
|-------|---------------|-------------------------------------------------------|-------------------------------------------------------------------------|
| basic | $G(t/\tau)$   |                                                       | $W_-(\pi, 0) \cdot FG(t/4\tau) \cdot W_+(\pi, 0) \cdot FG(t/4\tau)$ \cdot $W_-(\pi, 0) \cdot FG(t/4\tau) \cdot W_+(\pi, 0)$ \cdot FG(t/4\tau) \cdot W_+(\pi, 0)$ |
| 1 qubit | NOT(0)       | $Q_0 \rightarrow Q_0$                               | $F\left(\frac{\pi}{2}\right) \cdot W_+\left(\frac{\pi}{2}, 0\right) \cdot F\left(\frac{\pi}{2}\right)$ |
|        | CP1(0)       | if state=0, invert phase                              | $F(\pi) \cdot W_+\left(\pi, 0\right)$                                   |
|        | HAD(0)       | Walsh-Hadamard on $Q_0$                              | $W_+\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \cdot F(\pi) \cdot W_+\left(\pi, 0\right)$ |
| 2 qubit | EX(1,0)      | $\{Q_1, Q_0\} \rightarrow \{Q_0, Q_1\}$             | $F\left(\frac{\pi}{2}\right) \cdot W_-\left(\frac{\pi}{2}, \pi\right) \cdot G\left(\frac{\pi}{2}\right) \cdot W_-\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \cdot F\left(\frac{\pi}{8}\right)$ |
|        | CNOT(1,0)    | $\{Q_1, Q_0\} \rightarrow \{Q_1, Q_1 \oplus Q_0\}$ | $F\left(\frac{\pi}{2}\right) \cdot W_+\left(\frac{\pi}{2}, \pi\right) \cdot G\left(\frac{\pi}{2}\right) \cdot W_+\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \cdot F\left(\frac{\pi}{8}\right)$ |
|        | CNOT(1,0)    | $\{Q_1, Q_0\} \rightarrow \{Q_1, Q_1 \oplus Q_0\}$ | $W_+(\pi, 0) \cdot F\left(\frac{4\pi}{\tau}\right) \cdot W_+(\frac{\pi}{2}, 0) \cdot G\left(\frac{\pi}{2}\right) \cdot W_+(\frac{\pi}{4}, \frac{\pi}{4}) \cdot F\left(\frac{5\pi}{4}\right)$ |
|        | CP2(0)       | if state=0, invert phase                              | $F\left(\frac{8\pi}{\tau}\right) \cdot G\left(\frac{\pi}{2}\right) \cdot W_+\left(\pi, 0\right)$ |
|        | HAD(1,0)     | Walsh-Hadamard on $Q_1, Q_0$                          | $EX(1,0) \cdot HAD(0) \cdot EX(1,0) \cdot HAD(0)$                      |
| 3 qubit | SW3(2,3)*    | swap states 2, 3                                     | $W_+\left(\frac{\pi}{2}, 0\right) \cdot G\left(\frac{\pi}{2}\right) \cdot W_+\left(\frac{\pi}{4}, \frac{4\pi}{8}\right) \cdot F\left(\frac{\pi}{8}\right) \cdot G\left(\frac{\pi}{2}\right)$ |
|        | SW3(2,3)*    | swap states 3, 4                                     | $W_+(\frac{\pi}{2}, 0) \cdot G\left(\frac{\pi}{2}\right) \cdot W_+\left(\frac{\pi}{8}, \frac{4\pi}{8}\right) \cdot F\left(\frac{\pi}{8}\right) \cdot G\left(\frac{\pi}{2}\right)$ |
|        | SW3(2,3)*    | swap states 4, 5                                     | $W_+(\frac{\pi}{2}, 0) \cdot G\left(\frac{\pi}{2}\right) \cdot W_+\left(\frac{\pi}{4}, \frac{4\pi}{8}\right) \cdot F\left(\frac{\pi}{8}\right) \cdot G\left(\frac{\pi}{2}\right)$ |
|        | EX(2,1)      | $\{Q_2, Q_1\} \rightarrow \{Q_2, Q_2\}$            | $W_+(\pi, 0) \cdot CNOT(1,0) \cdot EX(1,0) \cdot G\left(\frac{\pi}{2}\right) \cdot F\left(\frac{13\pi}{8}\right) \cdot EX(1,0) \cdot CNOT(1,0) \cdot SW3(3,4) \cdot NOT(0) \cdot F(\pi) \cdot NOT(0) \cdot SW3(4,5) \cdot NOT(0) \cdot F(\pi) \cdot NOT(0) \cdot SW3(2,3) \cdot SW3(3,4) \cdot G\left(\frac{\pi}{2}\right) \cdot F\left(\frac{\pi}{8}\right)$ |
|        | RR3          | $\{Q_2, Q_1, Q_0\} \rightarrow \{Q_0, Q_2, Q_1\}$    | $EX(2,1) \cdot EX(1,0)$                                                |
|        | RL3          | $\{Q_2, Q_1, Q_0\} \rightarrow \{Q_1, Q_0, Q_2\}$   | $RR3 \cdot RR3$                                                        |
|        | CP3(0)       | if state=0, invert phase                              | $NOT(0) \cdot RL3 \cdot NOT(0) \cdot RL3 \cdot F\left(\frac{5\pi}{8}\right) \cdot G\left(\frac{\pi}{2}\right) \cdot RR3 \cdot SW3(4,5) \cdot F\left(\frac{\pi}{8}\right)$ |
|        |              |                                                       | $SW3(4,5) \cdot F\left(\frac{\pi}{2}\right) \cdot NOT(0) \cdot RR3 \cdot NOT(0) \cdot W_+(\pi, 0)$ |