ON YAMABE TYPE PROBLEMS ON RIEMANNIAN MANIFOLDS WITH BOUNDARY

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Abstract. Let \((M, g)\) be a \(n\)-dimensional compact Riemannian manifold with boundary. We consider the Yamabe type problem

\[
\begin{cases}
-\Delta_g u + au = 0 & \text{on } M \\
\partial_n u + \frac{\nu}{2} bu = u^{n-2} \pm \varepsilon & \text{on } \partial M
\end{cases}
\]

where \(a \in C^1(M)\), \(b \in C^1(\partial M)\), \(\nu\) is the outward pointing unit normal to \(\partial M\) and \(\varepsilon\) is a small positive parameter. We build solutions which blow-up at a point of the boundary as \(\varepsilon\) goes to zero. The blowing-up behavior is ruled by the function \(b - H_g\), where \(H_g\) is the boundary mean curvature.

1. Introduction

Let \((M, g)\) be a smooth, compact Riemannian manifold of dimension \(n \geq 3\) with a boundary \(\partial M\) which is the union of a finite number of smooth closed compact submanifolds embedded in \(M\).

A well known problem in differential geometry is whether \((M, g)\) is necessarily conformally equivalent to a manifold of constant scalar curvature whose boundary is minimal. When the boundary is empty this is called the Yamabe problem (see Yamabe [27]), which has been completely solved by Aubin [A], Schoen [25] and Trudinger [26]. Cherrier [8] and Escobar [13, 14] studied the problem in the context of manifolds with boundary and gave an affirmative solution to the question in almost every case. The remaining cases where studied by Marques [21, 22], by Almaraz [1] and by Brendle and Chen [6].

Once the problem is solvable, a natural question about compactness of the full set of solutions arises. Concerning the Yamabe problem, it was first raised by Schoen in a topic course at Stanford University in 1988. A necessary condition is that the manifold is not conformally equivalent to the standard sphere \(S^n\), since the group of conformal transformation of the round sphere is not compact itself. The problem of compactness has been widely studied in the last years and it has been completely solved by Brendle [5], Brendle and Marques [17] and Khuri, Marques and Schoen [20].

In the presence of a boundary, a necessary condition is that \(M\) is not conformally equivalent to the standard ball \(B^n\). The problem when the boundary of the manifold is not empty has been studied by V. Felli and M. Ould Ahmedou [17, 18], Han and Li [19] and by Almaraz [2, 3]. In particular, Almaraz studied the compactness property in the case of scalar-flat metrics. Indeed the zero scalar curvature case is

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particularly interesting because it leads to study a linear equation in the interior with a critical Neumann-type nonlinear boundary condition

\[
\begin{cases}
-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = 0 & \text{on } M, \quad u > 0 \text{ in } M \\
\partial_\nu u + \frac{n-2}{2} H_g u = u^\frac{n}{n-2} & \text{on } \partial M
\end{cases}
\]

where \( \nu \) is the outward pointing unit normal to \( \partial M \), \( R_g \) is the scalar curvature of \( M \) with respect to \( g \) and \( H_g \) is the boundary mean curvature with respect to \( g \).

We note that in this case compactness of solutions is equivalent to establish a priori estimates for solutions to equation (1.1). Almaraz in \[3\] proved that compactness holds for a generic metric \( g \). On the other hand, in \[2\] proved that if the dimension of the manifold is \( n \geq 25 \) compactness does not hold because it is possible to build blowing-up solutions to problem (1.1) for a suitable metric \( g \).

We point out that the problem of compactness when the dimension of the manifold \( n \leq 24 \) is still not completely understood.

An interesting issue, closely related to compactness property, is the stability problem. One can ask whether or not the compactness property is preserved under perturbations of the equation, which is equivalent to have or not uniform a-priori estimates for solutions of the perturbed problem. Let us consider the more general problem

\[
\begin{cases}
-\Delta_g u + a(x)u = 0 & \text{in } M, \quad u > 0 \text{ in } M \\
\partial_\nu u + b(x)u = u^\frac{n}{n-2} & \text{on } \partial M
\end{cases}
\]

We say that problem (1.2) is stable if for any sequences of \( C^1 \) functions \( a_\varepsilon : M \to \mathbb{R} \) and \( b_\varepsilon : \partial M \to \mathbb{R} \) converging in \( C^1 \) to functions \( a : M \to \mathbb{R} \) and \( b : \partial M \to \mathbb{R} \), for any sequence of exponents \( p_\varepsilon := \frac{n}{n-2} \pm \varepsilon \) converging to the critical one \( \frac{n}{n-2} \) and for any sequence of associated solutions \( u_\varepsilon \) bounded in \( H^1(M) \) of the perturbed problems

\[
\begin{cases}
-\Delta_g u + a_\varepsilon(x)u = 0 & \text{in } M, \quad u_\varepsilon > 0 \text{ in } M \\
\partial_\nu u + \frac{n-2}{2} b_\varepsilon(x)u = u_\varepsilon^\frac{n}{n-2} & \text{on } \partial M
\end{cases}
\]

there is a subsequence \( u_{\varepsilon_k} \) that converges in \( C^2 \) to a solution to the limit problem (1.2). The stability of the Yamabe problem has been introduced and studied by Druet in \[9, 10\] and by Druet and Hebey in \[11, 12\]. Recently, Esposito Pistoia and Vetois \[15\], Micheletti, Pistoia and Vetois \[23\] and Esposito and Pistoia \[16\] prove that a priori estimates fail for perturbations of the linear potential or of the exponent.

In the present paper, we investigate the question of stability of problem (1.2). It is clear that it is not stable if it possible to build solutions \( u_\varepsilon \) to perturbed problems (1.3) which blow-up at one or more points of the manifold as the parameter \( \varepsilon \) goes to zero. Here, we show that the behavior of the sequence \( u_\varepsilon \) is dictated by the difference

\[
\varphi(q) = b(q) - H_g(q) \quad \text{for } q \in \partial M,
\]

More precisely, we will consider the problem

\[
\begin{cases}
-\Delta_g u + a(x)u = 0 & \text{on } M, \quad u > 0 \text{ in } M \\
\partial_\nu u + \frac{n-2}{2} b(x)u = u^\frac{n}{n-2} & \text{on } \partial M
\end{cases}
\]
We will assume that $a \in C^1(M)$, $b \in C^1(\partial M)$ are such that the linear operator $Lu := -\Delta_g u + au$ with Neumann boundary condition $Bu := \partial_{\nu} u + \frac{n-2}{2} bu$ is coercive, namely there exists a constant $c > 0$ so that

$$\int_M (|\nabla_g u|^2 + a(x)u^2) \mu_g + \frac{n-2}{2} \int_{\partial M} b(x)u^2 d\sigma \geq c\|u\|^2_{H^1(M)}.$$  

Here $\varepsilon$ is a small positive parameter. The problem (1.5) turns out to be either slightly subcritical or slightly supercritical if the exponent in the nonlinearity is either $\frac{n}{n-2} - \varepsilon$ or $\frac{n}{n-2} + \varepsilon$, respectively. Let us state our main result.

**Theorem 1.** Assume (1.6) and $n \geq 7$.

(i) If $q_0 \in \partial M$ is a strict local minimum point of the function $\varphi$ defined in (1.4) with $\varphi(q_0) > 0$, then provided $\varepsilon > 0$ is small enough there exists a solution $u_\varepsilon$ of (1.5) in the slightly subcritical case such that $u_\varepsilon$ blows up at a boundary point when $\varepsilon \to 0^+$.

(ii) If $q_0 \in \partial M$ is a strict local maximum point of the function $\varphi$ defined in (1.4) with $\varphi(q_0) > 0$, then provided $\varepsilon > 0$ is small enough there exists a solution $u_\varepsilon$ of (1.5) in the supercritical case such that $u_\varepsilon$ blows up at a boundary point when $\varepsilon \to 0^+$.

Our result does not concern the stability of the geometric Yamabe problem (1.1). Indeed, the function $\varphi$ in (1.4) turns out to be identically zero. In this case it is interesting to discover the function which rules the behavior of blowin up sequences in this case. We expect that it depends on trace-free 2nd fundamental form as it is suggested by Almaraz in [3], where a compactness result in the subcritical case is established.

It also remains open the case of low dimension, where we expect that the function $\varphi$ in (1.4) should be replaced by a function which depends on the Weyl tensor of the boundary, as it is suggested by Escobar in [13, 14].

The proof of our result relies on a very well known Ljapunov-Schmidt procedure. In Section 2 we set the problem, in Section 3 we reduce the problem to a finite dimensional one, which is studied in Section 4.

2. SETTING OF THE PROBLEM

Let us rewrite problem (1.5) in a more convenient way.

First of all, assumption (1.6) allows to endow the Hilbert space $H := H^1(M)$ with the following scalar product

$$\langle \langle u, v \rangle \rangle_H := \int_M (\nabla_g u \nabla_g v + a(x)uv) \mu_g + \frac{n-2}{2} \int_{\partial M} b(x)uv d\sigma$$

and the induced norm $\|u\|^2_H := \langle \langle u, u \rangle \rangle_H$. We define the exponent

$$s_\varepsilon = \begin{cases} 
\frac{2(n-1)}{n-2} & \text{in the subcritical case} \\
\frac{2(n-1)}{n-2} + n\varepsilon & \text{in the supercritical case}
\end{cases}$$

and the Banach space $\mathcal{H} := H^1(M) \cap L^{s_\varepsilon}(\partial M)$ endowed with norm $\|u\|_H = \|u\|_H + |u|_{L^{s_\varepsilon}(\partial M)}$.

We notice that in the subcritical case $\mathcal{H}$ is nothing but the Hilbert space $H$. 

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By trace theorems, we have the following inclusion $W^{1,\gamma}(M) \subset L^t(\partial M)$ for $t \leq \frac{n-1}{n-\gamma}$.

We consider $i : H^1(M) \to L^\frac{2(n-1)}{n-2}(\partial M)$ and its adjoint with respect to $\langle \cdot, \cdot \rangle_H$

$$i^* : L^\frac{2(n-1)}{n}(\partial M) \to H^1(M)$$

defined by

$$\langle \varphi, i^*(g) \rangle_H = \int_{\partial M} \varphi g d\sigma$$

so that $u = i^*(g)$ is the weak solution of the problem

$$\begin{align*}
&\left\{ \begin{array}{ll}
-\Delta u + a(x)u = 0 & \text{on } M \\
\frac{\partial u}{\partial n} + b(x)u = g & \text{on } \partial M
\end{array} \right.
\end{align*}$$

We recall that by (see [24]) we have that, if $u \in H^1$ is a solution of (2.1), then for $\frac{2n}{n+2} \leq q \leq \frac{n}{2}$ and $r > 0$ it holds

$$\|u\|_{L^\frac{2(n-1)}{n-2r}((\partial M))} = \|i^*(g)\|_{L^\frac{2(n-1)}{n-2q}((\partial M))} \leq \|g\|_{L^\frac{2(n-1)}{n-2q+r}((\partial M))}.$$  

By this result, we can choose $q, r$ such that

$$\frac{(n-1)q}{n-2q} = \frac{2(n-1)}{n-2} + \varepsilon$$

and

$$\frac{(n-1)q}{n-q} \leq \frac{n}{2}$$

so that

$$q = \frac{2n + n^2 \left( \frac{n-2}{n} \right)}{n+2} \varepsilon$$

and

$$r = \frac{2(n-1) + n(n-2)\varepsilon}{n(n-2)-\varepsilon},$$

so we have that, if $u \in L^\frac{2(n-1)}{n-2}((\partial M))$, then $|u|^{\frac{n-2}{n}+\varepsilon}$ is a solution of (2.2), that also $i^*(|u|^{\frac{n-2}{n}+\varepsilon}) \in L^\frac{2(n-1)}{n-2}((\partial M))$.

Finally, we rewrite problem (1.5) -both in the subcritical and in the supercritical case- as

$$u = i^*(f(x)), \ u \in \mathcal{H},$$

where the nonlinearity $f(x)$ is defined as $f(x) := (n-2)(u^+)^{\frac{n-2}{n}+\varepsilon}$ in the supercritical case or $f(x) := (n-2)(u^-)^{\frac{n-2}{n}+\varepsilon}$ in the subcritical case. Here $u^+(x) := \max\{0, u(x)\}$. By assumption (1.9), a solution to problem (2.4) is strictly positive and actually it is a solution to problem (1.5). Therefore, we are led to build solutions to problem (2.4), which blow-up at a boundary point as $\varepsilon$ goes to zero.

The main ingredient to cook up our solutions are the standard bubbles

$$U_{\delta, \xi}(x, t) := \frac{\delta^{\frac{n-2}{2}}}{((\delta + t)^2 + |x - \xi|^2)^\frac{n-2}{2}}, \ (x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}^+, \ \delta > 0, \ \xi \in \mathbb{R}^{n-1},$$

which are all the solutions to the limit problem

$$\begin{align*}
&\left\{ \begin{array}{ll}
-\Delta U = 0 & \text{on } \mathbb{R}^{n-1} \times \mathbb{R}_+ \\
\partial_{t} U = (n-2)U^{\frac{n-2}{n}} & \text{on } \mathbb{R}^{n-1} \times \{t = 0\}
\end{array} \right.
\end{align*}$$

We set $U_{\delta}(x, t) := U_{\delta, 0}(x, t)$.

We also need to introduce the linear problem

$$\begin{align*}
&\left\{ \begin{array}{ll}
-\Delta V = 0 & \text{on } \mathbb{R}^{n-1} \times \mathbb{R}_+ \\
\partial_{t} V = nU_{1}^{\frac{n-2}{n}}V & \text{on } \mathbb{R}^{n-1} \times \{t = 0\}
\end{array} \right.
\end{align*}$$
In [3] it has been proved that the $n$–dimensional space of solutions of (2.6) is generated by the functions

$$V_i = \frac{\partial U_1}{\partial x_i} = (2 - n) \frac{x_i}{((1 + t)^2 + |x|^2)^\frac{n}{2}} \quad \text{for} \quad i = 1, \ldots n - 1$$

$$V_0 = \frac{\partial U_\delta}{\partial \delta}_{|_{\delta=1}} = \frac{n - 2}{2} \left( \frac{1}{(1 + t)^2 + |x|^2} \right)^\frac{n}{2} (t^2 + |x|^2 - 1)$$

Next, given a point $q \in \partial M$, we introduce the Fermi coordinates $\psi^\delta_q : B^{n-1}(0, R) \times [0, R) \to M$, where $B^{n-1}(0, R)$ is the $n - 1$ dimensional unitary ball in $\mathbb{R}^{n-1}$ and we read the bubble on the manifold as the function

$$W_{\delta,q}(\xi) = U_\delta \left( (\psi^\delta_q)^{-1}\xi \right) \chi \left( (\psi^\delta_q)^{-1}\xi \right),$$

and the functions $V_i$’s on the manifold as the functions

$$Z^i_{\delta,q}(\xi) = \frac{1}{\delta^{-\frac{n-2}{2}}} V_i \left( \frac{1}{\delta} (\psi^\delta_q)^{-1}\xi \right) \chi \left( (\psi^\delta_q)^{-1}\xi \right) \quad i = 0, \ldots n - 1,$$

where $\chi(x, t) = \tilde{\chi}(|x|) \tilde{\gamma}(t)$, being $\tilde{\chi}$ a smooth cut off function, $\tilde{\gamma}(s) \equiv 1$ for $0 \leq s < R/2$ and $\tilde{\gamma}(s) \equiv 0$ for $s \geq R$. Then, it is necessary to split the Hilbert space $H$ into the sum of the orthogonal spaces

$$K_{\delta,q} = \text{Span} \left\{ Z^0_{\delta,q}, \ldots, Z^{n-1}_{\delta,q} \right\}$$

and

$$K^+_{\delta,q} = \left\{ \varphi \in H^1(M) \mid \langle \varphi, Z^i_{\delta,q} \rangle_H = 0 \text{ for all } i = 0, \ldots, n - 1 \right\}.$$  

Finally, we can look for a solution to problem (2.4) as

$$u_\epsilon(x) = W_{\delta,q}(x) + \phi(x)$$

where the blow-up point $q \in \partial M$, the blowing-up rate $\delta$ satisfies

(2.7) $$\delta := d \epsilon$$

for some $d > 0$

and the remainder term $\phi$ belongs to the infinite dimensional space $K^+_{\delta,q} \cap H$ of codimension $n$. We are led to solve the system

(2.8) $$\Pi^+_{\delta,q} \left\{ W_{\delta,q}(x) + \phi(x) - i^* (f_\epsilon(W_{\delta,q}(x) + \phi(x))) \right\} = 0$$

(2.9) $$\Pi_{\delta,q} \left\{ W_{\delta,q}(x) + \phi(x) - i^* (f_\epsilon(W_{\delta,q}(x) + \phi(x))) \right\} = 0$$

being $\Pi^+_{\delta,q}$ and $\Pi_{\delta,q}$ the projection respectively on $K^+_{\delta,q}$ and $K_{\delta,q}$.

3. The finite dimensional reduction

In this section we perform the finite dimensional reduction. We rewrite the auxiliary equation (2.8) in the equivalent form

(3.1) $$L(\phi) = N(\phi) + R$$

where $L = L_{\delta,q} : K^+_{\delta,q} \cap H \to K^+_{\delta,q} \cap H$ is the linear operator

$$L(\phi) = \Pi^+_{\delta,q} \left\{ \phi(x) - i^* (f_\epsilon(W_{\delta,q}[\phi])) \right\},$$

$N(\phi)$ is the nonlinear term

(3.2) $$N(\phi) = \Pi^+_{\delta,q} \left\{ i^* (f_\epsilon(W_{\delta,q}(x) + \phi(x))) - i^* (f_\epsilon(W_{\delta,q}(x))) - i^* (f_\epsilon(W_{\delta,q}[\phi])) \right\}$$
and the error term \( R \) is defined by
\[
R = \Pi_{\delta,q}^+ \{ i^* (f_\varepsilon(W_{\delta,q}(x))) - W_{\delta,q}(x) \}.
\]

3.1. The invertibility of the linear operator \( L \).

Lemma 2. For \( a, b \in \mathbb{R} \), \( 0 < a < b \) there exists a positive constant \( C_0 = C_0(a,b) \) such that, for \( \varepsilon \) small, for any \( q \in \partial M \), for any \( d \in [a,b] \) and for any \( \phi \in K^1_{\delta,q} \cap \mathcal{H} \) there holds
\[
\| L_{\delta,q}(\phi) \| \mathcal{H} \geq C_0 \| \phi \| \mathcal{H}.
\]

Proof. We argue by contradiction. We suppose that there exist two sequences of real numbers \( \varepsilon_m \to 0, d_m \in [a,b] \) a sequence of points \( q_m \in \partial M \) and a sequence of functions \( \phi_{\varepsilon_m d_m q_m} \in K^1_{\varepsilon_m d_m q_m} \cap \mathcal{H} \) such that
\[
\| \phi_{\varepsilon_m d_m q_m} \| \mathcal{H} = 1 \quad \text{and} \quad \| L_{\varepsilon_m d_m q_m} (\phi_{\varepsilon_m d_m q_m}) \| \mathcal{H} \to 0 \quad \text{as} \quad m \to +\infty.
\]

For the sake of simplicity, we set \( \delta_m = \varepsilon_m d_m \) and we define
\[
\tilde{\phi}_m := \delta_m^{-1} \phi_{\delta_m q_m} (\psi_{q_m}^\delta (\delta_m \eta)) \chi(\delta_m \eta) \quad \text{for} \quad \eta = (z,t) \in \mathbb{R}^n_+, \quad \text{with} \quad z \in \mathbb{R}^{n-1} \quad \text{and} \quad t \geq 0.
\]

Since \( \| \phi_{\varepsilon_m d_m q_m} \| H \leq 1 \), by change of variables we easily get that \( \{ \tilde{\phi}_m \}_m \) is bounded in \( D^{1,2}(\mathbb{R}^n_+) \) (but not in \( H^1(\mathbb{R}^n_+) \)). Thus there exists \( \tilde{\phi} \in D^{1,2}(\mathbb{R}^n_+) \) such that \( \tilde{\phi}_m \to \tilde{\phi} \) weakly in \( D^{1,2}(\mathbb{R}^n_+) \), in \( L^{\infty,1}_w(\mathbb{R}^n_+) \), strongly in \( L^{\infty,1}_{loc}(\partial \mathbb{R}^n_+) \) and almost everywhere.

Since \( \phi_{\delta_m q_m} \in K^1_{\delta_m q_m} \), and taking in account (2.16) we get, for \( i = 0, \ldots, n-1 \),
\[
0 = \left( \left( \phi_{\delta_m q_m}, Z_{\delta_m q_m} \right) \right)_{\mathcal{H}} = \int_M (\nabla g \phi_{\delta_m q_m} \nabla g Z_{\delta_m q_m} + a(x) \phi_{\delta_m q_m} Z_{\delta_m q_m}^i) \, d\mu_g
\]
\[
\quad + \frac{n-2}{2} \int_{\partial M} b(x) \phi_{\delta_m q_m} Z_{\delta_m q_m}^i \, d\sigma
\]
\[
\quad = \int_{\mathbb{R}^n_+} |g_{q_m}(\delta \eta)|^{\frac{2n-2}{2}} g_{q_m}^{\alpha \beta}(\delta \eta) \frac{\partial}{\partial \eta_\alpha} V_i(\eta) \chi(\delta \eta) \frac{\partial}{\partial \eta_\beta} \phi_{\delta_m q_m} (\psi_{q_m}^\delta (\delta_m \eta)) \, d\eta
\]
\[
\quad + \int_{\mathbb{R}^n_+} |g_{q_m}(\delta \eta)|^{\frac{2n-2}{2}} a(\psi_{q_m}^\delta (\delta \eta)) V_i(\eta) \phi_{\delta_m q_m} (\psi_{q_m}^\delta (\delta_m \eta)) \, d\eta
\]
\[
\quad + \int_{\partial \mathbb{R}^n_+} |g_{q_m}(\delta z,0)|^{\frac{2n-2}{2}} b(\psi_{q_m}^\delta (\delta \eta)) \phi_{\delta_m q_m} (\psi_{q_m}^\delta (\delta_m z,0)) V_i(\delta_m z,0) \, dz
\]
\[
\quad = \int_{\mathbb{R}^n_+} \nabla V_i(\eta) \nabla \tilde{\phi}_m(\eta) + \delta^2 a(q_m) V_i(\eta) \tilde{\phi}_m(\eta) \, d\eta
\]
\[
\quad + \delta \int_{\partial \mathbb{R}^n_+} b(q_m) V_i(z,0) \tilde{\phi}_m(z,0) \, d\eta + O(\delta) = \int_{\mathbb{R}^n_+} \nabla V_i(\eta) \nabla \tilde{\phi}_m(\eta) + O(\delta)
\]
\[
\quad = \int_{\mathbb{R}^n_+} \nabla V_i(\eta) \nabla \tilde{\phi}(\eta) + o(1),
\]
By definition of $L_{\delta_m, q_m}$ we have

$$
(3.5) \quad \phi_{\delta_m, q_m} - i^* \left( f_\varepsilon (W_{\delta_m, q_m}) [\phi_{\delta_m, q_m}] \right) - L_{\delta_m, q_m} (\phi_{\delta_m, q_m}) = \sum_{i=0}^{n-1} c_i^m Z_i^m.
$$

We want to prove that, for all $i = 0, \ldots, n - 1$, $c_i^m \to 0$ while $m \to \infty$. Multiplying equation $(3.5)$ by $Z_i^m$ we obtain, by definition of $i^*$,

$$
\sum_{i=0}^{n-1} c_i^m \left\langle Z_i^m, Z_i^m \right\rangle_H = \left\langle i^* ( f_\varepsilon (W_{\delta_m, q_m}) [\phi_{\delta_m, q_m}]), Z_i^m \right\rangle_H = \int_{\partial M} f_\varepsilon (W_{\delta_m, q_m}) [\phi_{\delta_m, q_m}] Z_i^m d\sigma.
$$

Moreover, by multiplying $(3.5)$ by $\phi_{\delta_m, q_m}$ we obtain that

$$
\| \phi_{\delta_m, q_m} \|_H - \int_{\partial M} f_\varepsilon (W_{\delta_m, q_m}) \phi_{\delta_m, q_m}^2 d\sigma \to 0,
$$

thus $(f_\varepsilon (W_{\delta_m, q_m}))^{1/2} \phi_{\delta_m, q_m}$ is bounded and weakly convergent in $L^2(\partial M)$. With this consideration easily we get

$$
\int_{\partial M} f_\varepsilon (W_{\delta_m, q_m}) [\phi_{\delta_m, q_m}] Z_i^m d\sigma
$$

$$
= \int_{\partial M} \left( f_\varepsilon (W_{\delta_m, q_m}) \right)^{1/2} \phi_{\delta_m, q_m} (f_\varepsilon (W_{\delta_m, q_m}))^{1/2} Z_i^m d\sigma
$$

$$
= n \int_{\mathbb{R}^{n-1}} U_1^\infty (z, 0) \tilde{\phi}(z, 0) V_i(z, 0) dz + o(1) = o(1),
$$

once we take in account $(5.4)$. Now, it is easy to prove that

$$
\left\langle Z_i^m, Z_i^m \right\rangle_H = C \delta_{ij} + o(1),
$$

hence we can conclude that $c_i^m \to 0$ while $m \to \infty$ for each $i = 0, \ldots, n - 1$. This, combined with $(3.6)$ and since $\| L_{\varepsilon, d_m, q_m} (\phi_{\varepsilon, d_m, q_m}) \|_H \to 0$ gives us that

$$
(3.6) \quad \| \phi_{\delta_m, q_m} - i^* ( f_\varepsilon (W_{\delta_m, q_m}) [\phi_{\delta_m, q_m}]) \|_H = \sum_{i=0}^{n-1} c_i^m \| Z_i^m \|_H + o(1) = o(1)
$$

Now, choose a smooth function $\varphi \in C_0^\infty (\mathbb{R}^+_2)$ and define

$$
\varphi_m (x) = \frac{1}{\delta_m^2} \varphi \left( \frac{1}{\delta_m} (\psi_{q_m}^{-1}) (x) \right) \chi \left( (\psi_{q_m}^{-1}) (x) \right) \text{ for } x \in M.
$$
We have that \( \| \varphi_m \|_H \) is bounded and, by \((3.6)\), that
\[
\langle \langle \phi_{\delta_m,q_m}, \varphi_m \rangle \rangle_H = \int_{\partial M} f'_{x_m} (W_{\delta_m,q_m}) [\phi_{\delta_m,q_m}] \varphi_m d\sigma \\
+ \langle \langle \phi_{\delta_m,q_m} - i^* (f'_{x_m} (W_{\delta_m,q_m}) [\phi_{\delta_m,q_m}]), \varphi_m \rangle \rangle_H \\
= \int_{\partial M} f'_{x_m} (W_{\delta_m,q_m}) [\phi_{\delta_m,q_m}] \varphi_m d\sigma + o(1) \\
= (n \pm \varepsilon_m (n - 2)) \int_{\mathbb{R}^{n-1}} \frac{1}{\delta_m^{n-2}} U_{1/\delta_m}^{2 \pm \varepsilon_m} (z,0) \hat{\varphi}_m(z,0) \varphi dz + o(1) \\
= n \int_{\mathbb{R}^{n-1}} U_{1}^{2 \pm \varepsilon_m} (z,0) \hat{\varphi}(z,0) \varphi(z,0) dz + o(1),
\]
by the strong \( L^2_{\text{loc}} (\partial \mathbb{R}^n_+) \) convergence of \( \hat{\varphi}_m \). On the other hand
\[
\langle \langle \phi_{\delta_m,q_m}, \varphi_m \rangle \rangle_H = \int_{\mathbb{R}^n_+} \nabla \hat{\varphi} \nabla \varphi d\eta + o(1),
\]
so \( \hat{\varphi} \) is a weak solution of \((2.5)\) and we conclude that
\[
\hat{\varphi} \in \text{Span} \{ V_0, V_1, \ldots, V_n \}.
\]

This, combined with \((3.4)\) gives that \( \hat{\varphi} = 0 \). Proceeding as before we have
\[
\langle \langle \phi_{\delta_m,q_m}, \phi_{\delta_m,q_m} \rangle \rangle_H = \int_{\partial M} f'_{x_m} (W_{\delta_m,q_m}) [\phi_{\delta_m,q_m}] \phi_{\delta_m,q_m} d\sigma + o(1) \\
= (n \pm \varepsilon_m (n - 2)) \int_{\mathbb{R}^{n-1}} \frac{1}{\delta_m^{n-2}} U_{1/\delta_m}^{2 \pm \varepsilon_m} (z,0) \hat{\varphi}_m^2(z,0) \varphi dz + o(1) = o(1)
\]
In a similar way, by \((3.6)\) we have
\[
| \phi_{\delta_m,q_m} |_{L^\infty} = | i^* (f'_{x}(W_{\delta_m,q_m}) [\phi_{\delta_m,q_m}]) |_{L^\infty} + o(1) = o(1)
\]
which gives \( \| \phi_{\delta_m,q_m} \|_H \to 0 \) that is a contradiction. \( \square \)

3.2. The estimate of the error term \( R \).

**Lemma 3.** For \( a, b \in \mathbb{R} \), \( 0 < a < b \) there exists a positive constant \( C_1 = C_1(a, b) \) such that, for \( \varepsilon \) small, for any \( q \in \partial M \) and for any \( d \in [a, b] \) holds
\[
\| R_{\varepsilon, \delta, q} \|_H \leq C_1 \varepsilon | \ln \varepsilon |
\]

**Proof.** We estimate
\[
\| i^* (f_{x}(W_{\delta,q}(x)) - W_{\delta,q}(x)) \|_H \leq \| i^* (f_{x}(W_{\delta,q}(x)) - i^* (f_0(W_{\delta,q}(x))) \|_H \\
+ \| i^* (f_0(W_{\delta,q}(x)) - W_{\delta,q}(x)) \|_H.
\]
By definition of \( i^* \) there exists \( \Gamma \) which solves the equation
\[
\begin{cases}
-\Delta \Gamma + a(x) \Gamma = 0 & \text{on } M \\
\frac{\partial \Gamma}{\partial n} + \frac{n-2}{2} b(x) \Gamma = f_0(W_{\delta,q}) & \text{on } \partial M.
\end{cases}
\]

\[\text{(3.7)}\]
so, by (3.7), we have
\[
\| i^* (f_0(W_{\delta,q}(x)) - W_{\delta,q}(x)) \|_H = \| \Gamma(x) - W_{\delta,q}(x) \|^2_H
\]
\[
= \int_M \left[ - \Delta_g (\Gamma - W_{\delta,q}) + a (\Gamma - W_{\delta,q}) \right] (\Gamma - W_{\delta,q}) \, d\mu_g
\]
\[
+ \int_{\partial M} \left[ \frac{\partial}{\partial v} (\Gamma - W_{\delta,q}) + b(x)(\Gamma - W_{\delta,q}) \right] (\Gamma - W_{\delta,q}) \, d\mu_g
\]
\[
= \int_M [\Delta_g W_{\delta,q} - a W_{\delta,q}] (\Gamma - W_{\delta,q}) \, d\mu_g
\]
\[
+ \int_{\partial M} \left[ f_0(W_{\delta,q}) - \frac{\partial}{\partial v} W_{\delta,q} \right] (\Gamma - W_{\delta,q}) \, d\mu_g
\]
\[
- \frac{(n - 2)}{2} \int_{\partial M} b(x) W_{\delta,q}(\Gamma - W_{\delta,q}) \, d\mu_g := I_1 + I_2 + I_3
\]

We obtain
\[
I_1 = \| \Gamma - W_{\delta,q} \|_H O(\delta).
\]

Infact
\[
I_1 \leq \| \Delta_g W_{\delta,q} - a W_{\delta,q} \|_{L^{2n/(n-2)}(M)} \| \Gamma - W_{\delta,q} \|_{L^{2n/(n-2)}(M)} \leq \| \Delta_g W_{\delta,q} - a W_{\delta,q} \|_{L^{2n/(n-2)}(\partial M)} \| \Gamma - W_{\delta,q} \|_H.
\]

Easily we have that \( |W_{\delta,q}|_{L^{2n/(n-2)}} = O(\delta^2) \). For the other term we have, in coordinates,

\[
\Delta_g W_{\delta,q} = \Delta [U_{\delta} \chi] + (g^{ab} - \delta_{ab}) \partial_a U_{\delta} \chi - g^{ab} \Gamma^k_{ab} \partial_k U_{\delta} \chi,
\]

\( \Gamma_k^{ab} \) being the Christoffel symbols. Using the expansion of the metric \( g^{ab} \) given by (4.2) and (4.3) we have that

\[
\| (g^{ab} - \delta_{ab}) \partial_a U_{\delta} \chi \|_{L^{2n/(n-2)}(M)} = O(\delta) \quad \text{and} \quad \| g^{ab} \Gamma^k_{ab} \partial_k U_{\delta} \chi \|_{L^{2n/(n-2)}(M)} = O(\delta^2)
\]

Since \( U_{\delta} \) is a harmonic function we deduce

\[
\| \Delta [U_{\delta} \chi] \|_{L^{2n/(n-2)}(M)} = \| U_{\delta} \Delta \chi + 2 \nabla U_{\delta} \nabla \chi \|_{L^{2n/(n-2)}(M)} = O(\delta^2).
\]

For the second integral \( I_2 \) we have

\[
I_2 = \| \Gamma - W_{\delta,q} \|_H O(\delta^2).
\]

since

\[
I_2 \leq \left\| f_0(W_{\delta,q}) - \frac{\partial}{\partial v} W_{\delta,q} \right\|_{L^{2(n-1)/(n-2)}(\partial M)} \| \Gamma - W_{\delta,q} \|_{L^{2n/(n-2)}(\partial M)}
\]
\[
\leq C \left\| f_0(W_{\delta,q}) - \frac{\partial}{\partial v} W_{\delta,q} \right\|_{L^{2(n-1)/(n-2)}(\partial M)} \| \Gamma - W_{\delta,q} \|_H,
\]
and, using the boundary condition for (2.15) we have

\begin{equation}
\left(3.13\right) \left| f_0(W_{\delta,q}) - \frac{\partial}{\partial \nu} W_{\delta,q}\right| \left.\right|_{L^{\frac{2(n-1)}{n}}(\partial M)} = \frac{1}{\delta^2} \left( \int_{\mathbb{R}^{n-1}} |g(\delta z, 0)|^{\frac{1}{2}} \left[(n-2)U_{\frac{n}{n-2}}(z, 0)\chi_{\frac{n}{n-2}}(\delta z, 0) - \chi(\delta z, 0)\frac{\partial U}{\partial \nu}(z, 0)\right] \delta^{n-1} dz \right) \frac{\delta^{n-1}}{\frac{2(n-1)}{n}} \leq C \left( \int_{\mathbb{R}^{n-1}} [(n-2)U_{\frac{n}{n-2}}(z, 0)\chi_{\frac{n}{n-2}}(\delta z, 0) - \chi(\delta z, 0)] \frac{2(n-1)}{n} \delta^{n-1} dz \right) \frac{\delta^{n-1}}{\frac{2(n-1)}{n}} = O(\delta^2),
\end{equation}

Lastly,

\begin{equation}
\left(3.14\right) I_3 \leq \left| W_{\delta,q}\right| \left.\right|_{L^{\frac{2(n-1)}{n}}(\partial M)} \left| \Gamma - W_{\delta,q}\right| \left.\right|_{L^{\frac{2(n-1)}{n}}(\partial M)} = \| \Gamma - W_{\delta,q} \|_H O(\delta).
\end{equation}

By (3.13), (3.12) and (3.14) we conclude that

\[ \| i^* (f_0(W_{\delta,q}(x)) - W_{\delta,q}(x)) \|_H = \| \Gamma(x) - W_{\delta,q}(x) \|_H = O(\delta). \]

To conclude the proof we estimate the term \( \| i^* (f_\varepsilon(W_{\delta,q}(x)) - i^* (f_0(W_{\delta,q}(x))) \|_H \). We have, by the properties of \( i^* \), that

\[ \| i^* (f_\varepsilon(W_{\delta,q}(x)) - i^* (f_0(W_{\delta,q}(x))) \|_H \leq \left| W_{\delta,q}(x) \right| \left.\right|_{L^{\frac{2(n-1)}{n}}(\partial M)} \left| W_{\delta,q}(x) \right| \left.\right|_{L^{\frac{2(n-1)}{n}}(\partial M)} = \int_{\mathbb{R}^{n-1}} \left[ \left( \frac{1}{\delta^{\frac{n}{n-2}}} U^{\frac{n}{n-2}}(z, 0) - 1 \right) U_{\frac{n}{n-2}}(z, 0) \right] \frac{2(n-1)}{n} \delta^{n-1} dz + O(\delta^2) \]

To estimate the last integral, we first recall two Taylor expansions with respect to \( \varepsilon \)

\begin{equation}
\left(3.15\right) U^{\pm \varepsilon} = 1 \pm \varepsilon \ln U + \frac{1}{2} \varepsilon^2 \ln^2 U + o(\varepsilon^2)
\end{equation}

\begin{equation}
\left(3.16\right) \delta^{\frac{n}{n-2}} = 1 \mp \frac{n-2}{n} \ln \delta + \varepsilon \left( \ln^2 \delta - \frac{2}{n} \ln^2 \delta \right) + o(\varepsilon \ln^2 \delta)
\end{equation}

In light of (3.15) and (3.16) we have

\begin{equation}
\left(3.17\right) \| i^* (f_\varepsilon(W_{\delta,q})) - i^* (f_0(W_{\delta,q})) \|_H \leq \int_{\mathbb{R}^{n-1}} \left[ \left( \frac{n}{2} \varepsilon \ln \delta \pm \varepsilon \ln U(z, 0) + O(\varepsilon^2) + O(\varepsilon \ln \delta) \right) U_{\frac{n}{n-2}}(z, 0) \right] \frac{2(n-1)}{n} \delta^{n-1} dz + O(\delta^2)
\end{equation}

Choosing \( \delta = d\varepsilon \) concludes the proof of Lemma 3 for the supercritical case. For the supercritical case, we have to control \( |R_{\varepsilon,\delta,q}|_{L^\infty(\partial M)} \). As in the previous case we consider

\[ |R_{\varepsilon,\delta,q}|_{L^\infty(\partial M)} \leq |i^* (f_\varepsilon(W_{\delta,q}(x)) - i^* (f_0(W_{\delta,q}(x)))|_{L^\infty(\partial M)} + |i^* (f_0(W_{\delta,q}(x)) - W_{\delta,q}(x))|_{L^\infty(\partial M)}. \]
As before, set $\Gamma = i^* (f_0(W_{\delta,q}(x)))$. Since $\Gamma$ solves (3.7), $\Gamma - W_{\delta,q}$ solves
\[
\begin{align*}
&-\Delta_g (\Gamma - W_{\delta,q}) + a(x) (\Gamma - W_{\delta,q}) = -\Delta_g W_{\delta,q} + a(x) W_{\delta,q} & \text{ on } M, \\
&\frac{\partial}{\partial \nu} (\Gamma - W_{\delta,q}) + \frac{n-2}{2} b(x) (\Gamma - W_{\delta,q}) = f_0(\Gamma) + \frac{\partial}{\partial \nu} W_{\delta,q} + \frac{n-2}{2} b(x) W_{\delta,q} & \text{ on } \partial M.
\end{align*}
\]
We choose $q$ as in (2.3), and $r = \varepsilon$, thus, by Theorem 3.14 in [24], we have
\[
|\Gamma - W_{\delta,q}|_{L^{r+\varepsilon}(\partial M)} \leq | -\Delta_g W_{\delta,q} + a(x) W_{\delta,q}|_{L^{r+\varepsilon}(M)}.
\]
We remark that
\[
q = \frac{2n + n^2 \left( \frac{n-2}{n-1} \right) \varepsilon}{n + 2 + n \left( \frac{n-2}{n-1} \right) \varepsilon} = \frac{2n}{n + 2} + O^+ (\varepsilon) \text{ with } 0 < O^+ (\varepsilon) < C\varepsilon
\]
for some positive constant $C$. By direct computation we have
\[
|a(x) W_{\delta,q}|_{L^{r+\varepsilon}(M)} \leq C\delta^{2-O^+ (\varepsilon)},
\]
\[
|b(x) W_{\delta,q}|_{L^\left( \frac{2n}{n-1+\varepsilon}\right)(\partial M)} \leq C\delta^{1-O^+ (\varepsilon)}.
\]
Moreover, proceeding as in (3.9), (3.10), (3.11), and as in (3.13) we get
\[
|\Delta_g W_{\delta,q}|_{L^{r+\varepsilon}(M)} \leq C\delta^{2-O^+ (\varepsilon)};
\]
\[
|f_0(\Gamma) + \frac{\partial}{\partial \nu} W_{\delta,q}|_{L^\left( \frac{2n}{n-1+\varepsilon}\right)(\partial M)} \leq C\delta^{1-O^+ (\varepsilon)}.
\]
Since $i^* (f_\varepsilon(W_{\delta,q}))$ solves (1.5), and $i^* (f_\varepsilon|u^{n-1+\varepsilon}(W_{\delta,q}))$ solves (1.6), we again use Theorem 3.14 in [24]. Taking in account (3.15) and (3.16) finally we get
\[
(3.18)
\]
\[
|i^* (f_\varepsilon(W_{\delta,q})) - i^* (f_0(W_{\delta,q}))|_{L^{r+\varepsilon}(\partial M)} \leq |f_\varepsilon(W_{\delta,q}) - f_0(W_{\delta,q})|_{L^\left( \frac{2n}{n-1+\varepsilon}\right)(\partial M)},
\]
\[
\leq \delta^{-O^+ (\varepsilon)} \left\{ \int_{\mathbb{R}^n} \left[ \frac{1}{\delta^{\frac{n-2}{2}} U^\varepsilon(z,0) - 1} U^\frac{n-2}{2}(z,0) \right]^\left( \frac{2(n-1)}{n} + O^+ (\varepsilon) \right) dz \right\}^{\frac{2(n-1)}{n} + O^+ (\varepsilon)} + O(\delta^2)
\]
\[
= \delta^{-O^+ (\varepsilon)} \{ O(\varepsilon |\ln \delta|) + O(\varepsilon) \} + O(\delta^2).
\]
Now, choosing $\delta = d\varepsilon$, we can conclude the proof, since
\[
\delta^{-O^+ (\varepsilon)} = 1 + O^+ (\varepsilon |\ln (d\varepsilon)|) = 1 + O^+ (\varepsilon |\ln \varepsilon|) = O(1).
\]

3.3. Solving equation (2.8): the remainder term $\phi$. 

**Proposition 4.** For $a, b \in \mathbb{R}$, $0 < a < b$ there exists a positive constant $C = C(a,b)$ such that, for $\varepsilon$ small, for any $q \in \partial M$ and for any $d \in [a,b]$ there exists a unique $\phi_{\delta,q}$ which solves (2.3)
\[
\|\phi_{\delta,q}\|_H \leq C\varepsilon |\ln \varepsilon|.
\]
Moreover the map $q \mapsto \phi_{\delta,q}$ is a $C^1(\partial M, H)$ map.
Proof. First of all, we point out that $N$ is a contraction mapping. We remark that the conjugate exponent of $s_\varepsilon$ is

$$s_\varepsilon' = \begin{cases} \frac{2(n-1)}{n} & \text{in the subcritical case} \\ \frac{2(n-1)+cn(n-2)}{n+cn(n-2)} & \text{in the supercritical case} \end{cases},$$

By the properties of $i^*$ and using the expansion of $f_\varepsilon(W_{\delta,q} + \phi_1)$ centered in $W_{\delta,q} + \phi_2$ we have

$$\|N(\phi_1) - N(\phi_2)\|_H \leq \|f_\varepsilon(W_{\delta,q} + \phi_1) - f_\varepsilon(W_{\delta,q} + \phi_2) - f'_\varepsilon(W_{\delta,q})[\phi_1 - \phi_2]\|_{L^{s_\varepsilon'}(\partial M)}$$

$$\leq \|(f'_\varepsilon(W_{\delta,q} + \theta \phi_1 + (1 - \theta)\phi_2) - f'_\varepsilon(W_{\delta,q})) [\phi_1 - \phi_2]\|_{L^{s_\varepsilon'}(\partial M)}$$

and, since $|\phi_1 - \phi_2|^{s_\varepsilon'} \in L^{s_\varepsilon'/s_\varepsilon'}(\partial M)$ and $|f'_\varepsilon(\cdot)|^{s_\varepsilon'} \in L^\left(\frac{p'}{p}\right)'(\partial M)$ since $f'_\varepsilon(\cdot) \in L^{s_\varepsilon}(\partial M)$, we have

$$\|N(\phi_1) - N(\phi_2)\|_H$$

$$\leq \|(f'_\varepsilon(W_{\delta,q} + \theta \phi_1 + (1 - \theta)\phi_2) - f'_\varepsilon(W_{\delta,q}))\|_{L^{s_\varepsilon'}(\partial M)} \|\phi_1 - \phi_2\|_{L^{s_\varepsilon'}(\partial M)}$$

$$= \gamma \|\phi_1 - \phi_2\|_H$$

where

$$\gamma = \|(f'_\varepsilon(W_{\delta,q} + \theta \phi_1 + (1 - \theta)\phi_2) - f'_\varepsilon(W_{\delta,q}))\|_{L^{s_\varepsilon'}(\partial M)} < 1$$

provided $\|\phi_1\|_H$ and $\|\phi_2\|_H$ sufficiently small.

In the same way we can prove that $\|N(\phi)\|_H \leq \gamma \|\phi\|_H$ with $\gamma < 1$ if $\|\phi\|_H$ is sufficiently small.

Next, by Lemma 2 and by Lemma 3 we have

$$\|L^{-1}(N(\phi) + R_{\varepsilon,\delta,q})\|_H \leq C (\|\phi\|_H + \varepsilon |\ln \varepsilon|)$$

where $C = \max\{C_0, C_0 C_1\} > 0$ being $C_0, C_1$ the constants which appear in Lemma 2 and in Lemma 3. Notice that, given $C > 0$, it is possible (up to choose $\|\phi\|_H$ sufficiently small) to choose $0 < C\gamma < 1/2$.

Now, if $\|\phi\|_H \leq 2C\varepsilon |\ln \varepsilon|$, then the map

$$T(\phi) := L^{-1}(N(\phi) + R_{\varepsilon,\delta,q})$$

is a contraction from the ball $\|\phi\|_H \leq 2C\varepsilon |\ln \varepsilon|$ in itself, so, by the fixed point Theorem, there exists a unique $\phi_{\delta,q}$ with $\|\phi_{\delta,q}\|_H \leq 2C\varepsilon |\ln \varepsilon|$ solving (3.1) and hence (2.8). The regularity of the map $q \mapsto \phi_{b,q}$ can be proven via the implicit function Theorem.  \hfill \square

4. The reduced problem

Problem (1.5) has a variational structure. Weak solutions to (1.5) are critical points of the energy functional $J_\varepsilon : H \to \mathbb{R}$

$$J_\varepsilon(u) = \frac{1}{2} \int_M (|\nabla u|^2 + a(x)u^2) \, d\mu_g + \frac{n-2}{4} \int_{\partial M} b(x)u^2 \, d\sigma$$

$$- \frac{(n-2)^2}{2n-2 \pm \varepsilon(n-2)} \int_{\partial M} u^{2n-2 \pm \varepsilon} \, d\sigma$$

where
Let us introduce the reduced energy $I_\varepsilon : (0, +\infty) \times \partial M \to \mathbb{R}$ by

$$I_\varepsilon(d, q) := J_\varepsilon(W_{ed, q} + \phi_{ed, q}).$$

where the remainder term $\phi_{ed, q}$ has been found in 1.

4.1. The reduced energy. Here we will use the following expansion for the metric tensor on $M$.

$$g^{ij}(y) = \delta_{ij} + 2h_{ij}(0)y + O(|y|^2) \text{ for } i, j = 1, \ldots, n-1,$$

$$g^{in}(y) = \delta_{in} \text{ for } i = 1, \ldots, n-1,$$

$$\sqrt{g}(y) = 1 - (n-1)H(0)y + O(|y|^2)$$

where $(y_1, \ldots, y_n)$ are the Fermi coordinates and, by definition of $h_{ij}$,

$$H = \frac{1}{n-1}\sum_{i} h_{ii}.$$

We also recall that on $\partial M$ the Fermi coordinates coincide with the exponential ones, so we have that

$$\sqrt{g}(y_1, \ldots, y_{n-1}, 0) = 1 + O(|y|^2).$$

To improve the readability of this paper, thereafter we will introduce $z = (z_1, \ldots, z_{n-1})$ to indicate the first $n-1$ Fermi coordinates and $t$ to indicate the last one, so $(y_1, \ldots, y_{n-1}, y_n) = (z, t)$. Moreover, indices $i, j$ conventionally refer to sums from 1 to $n-1$, while $l, m$ usually refer to sums from 1 to $n$.

**Proposition 5.**

(i) If $(d_0, q_0) \in (0, +\infty) \times \partial M$ is a critical point for the reduced energy $I_\varepsilon$ defined in (1.1), then $W_{ed_0, q_0} + \phi_{ed_0, q_0} \in H$ solves problem (1.3).

(ii) It holds true that

$$I_\varepsilon(d, q) = c_n(\varepsilon) + \varepsilon[\alpha d \varphi(q) - \beta_n \ln d] + o(\varepsilon) \text{ in the subcritical case}$$

and

$$I_\varepsilon(d, q) = c_n(\varepsilon) + \varepsilon[\alpha d \varphi(q) + \beta_n \ln d] + o(\varepsilon) \text{ in the supercritical case}$$

$C^0$-uniformly with respect to $d$ in compact sets of $(0, +\infty)$ and $q \in \partial M$. Here $c_n(\varepsilon)$ is a constant which only depends on $\varepsilon$ and $n$, $\alpha$ and $\beta_n$ are positive constants which only depend on $n$ and $\varphi(q) = h(q) - H_g(q)$ is the function defined in (1.4).

**Proof.** Proof of (i).
Set $q := q(y) = \psi^\partial_{q_0}(y)$. Since $(d_0, q_0)$ is a critical point, we have, for any $h \in 1, \ldots, n - 1$,

$$0 = \left. \frac{\partial}{\partial y_h} I_\epsilon(d, \psi^\partial_{q_0}(y)) \right|_{y=0}$$

$$= \left\langle \left( W_{\epsilon, d,q}(y) + \phi_{\epsilon, d,q}(y) - i^* (f_\epsilon(W_{\epsilon, d,q}(y) + \phi_{\epsilon, d,q}(y))) \right), \left( \frac{\partial}{\partial y_h} W_{\epsilon, d,q}(y) + \frac{\partial}{\partial y_h} \phi_{\epsilon, d,q}(y) \right) \right\rangle \bigg|_{y=0}$$

$$= \sum_{i=0}^{n-1} c^i \epsilon \left\langle \left( Z^i_{\epsilon, d,q}(y) \cdot \frac{\partial}{\partial y_h} W_{\epsilon, d,q}(y) \right) \right\rangle \bigg|_{y=0} - \sum_{i=0}^{n-1} c^i \epsilon \left\langle \left( \frac{\partial}{\partial y_h} Z^i_{\epsilon, d,q}(y), \phi_{\epsilon, d,q}(y) \right) \right\rangle \bigg|_{y=0}$$

using that $\phi_{\epsilon, d,q}(y)$ is a solution of $\mathbf{(2.8)}$ and that $\left\langle \left( Z^i_{\epsilon, d,q}(y), \frac{\partial}{\partial y_h} \phi_{\epsilon, d,q}(y) \right) \right\rangle = - \left\langle \left( \frac{\partial}{\partial y_h} Z^i_{\epsilon, d,q}(y), \phi_{\epsilon, d,q}(y) \right) \right\rangle$

since $\phi_{\epsilon, d,q}(y) \in K_{\epsilon, d,q}(y)$ for all $y$. Now it is enough to observe that

$$\left\langle \left( \frac{\partial}{\partial y_h} Z^i_{\epsilon, d,q}(y), \phi_{\epsilon, d,q}(y) \right) \right\rangle \leq \left\| \frac{\partial}{\partial y_h} Z^i_{\epsilon, d,q}(y) \right\|_{H} \left\| \phi_{\epsilon, d,q}(y) \right\|_{H} = o(1)$$

$$\left\langle \left( Z^i_{\epsilon, d,q}(y) \cdot \frac{\partial}{\partial y_h} W_{\epsilon, d,q}(y) \right) \right\rangle \bigg|_{y=0} = \frac{1}{\epsilon d} \left\langle \left( Z^i_{\epsilon, d,q}(y), Z^i_{\epsilon, d,q}(y) \right) \right\rangle \bigg|_{y=0} = \frac{1}{\epsilon d} \delta_i + o(1)$$

to conclude that

$$0 = \frac{1}{\epsilon d} \sum_{i=0}^{n-1} c^i \epsilon (\delta_i + o(1))$$

and so $c^i = 0$ for all $i = 0, \ldots, n - 1$. This conclude the proof.

**Proof of (ii).**

*Step 1:* we prove that for $\epsilon$ small enough and for any $q \in \partial M$,

$$|J_\epsilon(W_{\delta,q} + \phi_{\delta,q}) - J_\epsilon(W_{\delta,q})| \leq \|\phi_{\delta,q}\|_H^2 + C\epsilon |\ln \epsilon| \|\phi_{\delta,q}\|_H = o(\epsilon)$$

We have

$$|J_\epsilon(W_{\delta,q} + \phi_{\delta,q}) - J_\epsilon(W_{\delta,q})| = \int_M \left[ -\Delta_g W_{\delta,q} + a(x)W_{\delta,q} \right] \phi_{\delta,q} d\mu_g + \frac{1}{2} \|\phi_{\delta,q}\|_H^2$$

$$+ \int_{\partial M} \left[ \frac{\partial}{\partial \nu} W_{\delta,q} + \frac{n-2}{2} b(x)W_{\delta,q} - f_0(W_{\delta,q}) \right] \phi_{\delta,q} d\sigma + \int_{\partial M} \left[ f_0(W_{\delta,q}) - f_\epsilon(W_{\delta,q}) \right] \phi_{\delta,q} d\sigma$$

$$+ \int_{\partial M} \frac{(n-2)^2}{2n-2} \left( (W_{\delta,q} + \phi_{\delta,q})^{\frac{2(n-2)}{n-2}} - W_{\delta,q}^{\frac{2(n-2)}{n-2}} - f_\epsilon(W_{\delta,q}) \phi_{\delta,q} \right) d\sigma.$$

With the same estimate of $I_1$ in Lemma 8 we obtain that

$$\int_M \left[ -\Delta_g W_{\delta,q} + a(x)W_{\delta,q} \right] \phi_{\delta,q} d\mu_g = O(\delta) \|\phi_{\delta,q}\|_H,$$

and in light of the estimate of $I_2$ and $I_3$ in Lemma 8 we get

$$\int_{\partial M} \left[ \frac{\partial}{\partial \nu} W_{\delta,q} + \frac{n-2}{2} b(x)W_{\delta,q} - f_0(W_{\delta,q}) \right] \phi_{\delta,q} d\sigma = O(\delta) \|\phi_{\delta,q}\|_H.$$
In the subcritical case, following the computation in (3.17) we obtain
\[
\left| \int_{\partial M} [f_0(W_{\delta,q}) - f_\varepsilon(W_{\delta,q})] \phi_{\delta,q} d\sigma \right| \\
\leq C |f_0(W_{\delta,q}) - f_\varepsilon(W_{\delta,q})|_{L^{2(n-1)}(\partial M)} |\phi_{\delta,q}|_{L^{2(n-1)}(\partial M)} \\
= |O(\varepsilon) + O(\varepsilon \ln \delta)| \|\phi_{\delta,q}\|_H = O(\varepsilon |\ln \varepsilon|) \|\phi_{\delta,q}\|_H
\]
and in a similar way, for the supercritical case we get, in light of (3.18)
\[
\left| \int_{\partial M} [f_0(W_{\delta,q}) - f_\varepsilon(W_{\delta,q})] \phi_{\delta,q} d\sigma \right| \\
\leq \left( \delta^{-O^+(\varepsilon)} (O(\varepsilon \ln \delta) + O(\varepsilon)) + O(\delta^2) \right) \|\phi_{\delta,q}\|_H = O(\varepsilon |\ln \varepsilon|) \|\phi_{\delta,q}\|_H
\]
Finally, using Taylor expansion formula we have immediately, for some \( \theta \in (0,1) \),
\[
\left| \int_{\partial M} \frac{(n-2)^2}{2n-2 \pm \varepsilon(n-2)} \left[ (W_{\delta,q} + \theta \phi_{\delta,q}) \frac{2n-2 \pm \varepsilon}{2n-2} - W_{\delta,q} \frac{2n-2 \pm \varepsilon}{2n-2} \right] - f_\varepsilon(W_{\delta,q}) \phi_{\delta,q} d\sigma \right| \\
= \left| \frac{n \pm \varepsilon(n-2)}{2} \int_{\partial M} (W_{\delta,q} + \theta \phi_{\delta,q}) \frac{2n-2 \pm \varepsilon}{2n-2} \phi_{\delta,q} d\sigma \right| \\
\leq C \left[ \int_{\partial M} |W_{\delta,q} + \theta \phi_{\delta,q}| \left( \frac{2n-2 \pm \varepsilon}{2n-2} \right)^2 d\sigma \right] \left[ \int_{\partial M} |\phi_{\delta,q}|^2 d\sigma \right]^\frac{1}{2} \\
\leq C |W_{\delta,q} + \theta \phi_{\delta,q}|_{L^{2(n-2)}(\partial M)} \|\phi_{\delta,q}\|_H^2 
\]
Choosing \( \delta = d\varepsilon \), and recalling that, by Proposition 4 \( \|\phi_{\delta,q}\|_H = O(\varepsilon |\ln \varepsilon|) \) concludes the proof.

**Step 2:** we prove that
\[ J_\varepsilon(W_{\delta,q}) = C(\varepsilon) + \varepsilon \left\{ (\frac{n-2}{4} - 2) [b(q) - H(q)] \pm \ln d \frac{(n-2)(n-3)}{4(n-2)(2n-2)} \right\} \omega_{n-1} \mu_{n-2} + o(\varepsilon) \]

\( C^0 \) uniformly with respect to \( d \) in compact sets of \((0, +\infty)\) and \( q \in \partial M \), where
\[
C(\varepsilon) = \frac{1}{2} \int_{\mathbb{R}^n} \left| \nabla U(y) \right|^2 dy - \left( \frac{n-2}{2n-2} \right) \int_{\mathbb{R}^{n-1}} U_{\frac{2n-2}{2n-2}}(z,0) dz \\
\pm \varepsilon (\frac{n-2}{2n-2}) \int_{\mathbb{R}^{n-1}} U_{\frac{2n-2}{2n-2}}(z,0) dz + \varepsilon (\frac{n-2}{2n-2}) \int_{\mathbb{R}^{n-1}} U_{\frac{2n-2}{2n-2}}(z,0) ln U(z,0) dz \\
\pm \varepsilon |\ln \varepsilon| (\frac{n-2}{2(2n-2)}) \int_{\mathbb{R}^{n-1}} U_{\frac{2n-2}{2n-2}}(z,0) dz,
\]
and
\[
\mu_{n-2} = \int_{0}^{\infty} \frac{s^{n-2}}{(1 + s^2)^{n-2}} ds
\]
and \( \omega_{n-1} \) is the volume of the \( n-1 \) dimensional unit ball.
We compute each term separately. First, we have, by change of variables and by (4.2), (4.3), (4.4),
\[
\int_M |\nabla W_{\delta,q}|^2 d\mu_g = \sum_{l,m=1}^{n} \int_{\mathbb{R}^+_n} g^{lm}(\delta y) \frac{\partial}{\partial y_l} U(y) \frac{\partial}{\partial y_m} U(y) \sqrt{g(\delta y)} dy + o(\delta)
\]
\[
= \int_{\mathbb{R}^+_n} |\nabla U(y)|^2 dy - \delta(n-1)H(q) \int_{\mathbb{R}^+_n} y_n |\nabla U(y)|^2 dy
\]
\[
+ 2\delta \sum_{i,j=1}^{n-1} \int_{\mathbb{R}^+_n} y_n h_{ij}(q) \frac{\partial}{\partial y_i} U(y) \frac{\partial}{\partial y_j} U(y) dy + o(\delta).
\]
By simmetry argument we can simplify the last integral to obtain, in a more compact form
\[
\frac{1}{2} \int_M |\nabla W_{\delta,q}|^2 d\mu_g = \frac{1}{2} \int_{\mathbb{R}^+_n} |\nabla U|^2 - \delta(n-1)H(q) \int_{\mathbb{R}^+_n} y_n |\nabla U|^2
\]
\[
+ \delta \sum_{i=1}^{n-1} h_{ii}(q) \int_{\mathbb{R}^+_n} y_n \left( \frac{\partial U}{\partial y_i}(y) \right)^2 dy + o(\delta).
\]
Since \( \frac{\partial U}{\partial y_i} = \frac{\partial U}{\partial y_l} \) for all \( i, l = 1, \ldots, n-1 \) and by (4.9) we get
\[
\sum_{i=1}^{n-1} h_{ii}(q) \int_{\mathbb{R}^+_n} y_n \left( \frac{\partial U}{\partial y_i}(y) \right)^2 dy = \frac{1}{n-1} \sum_{i=1}^{n-1} h_{ii}(q) \int_{\mathbb{R}^+_n} y_n \sum_{l=1}^{n-1} \left( \frac{\partial U}{\partial y_l}(y) \right)^2 dy
\]
\[
= \frac{H(q)}{4} \int_{\mathbb{R}^{n-1}} U^2(z,0) dz,
\]
and in light of (4.7), we conclude that
\[
\frac{1}{2} \int_M |\nabla W_{\delta,q}|^2 d\mu_g = \frac{1}{2} \int_{\mathbb{R}^+_n} |\nabla U|^2 - \delta(n-2)H(q) \int_{\mathbb{R}^{n-1}} U^2(z,0) dz + o(\delta).
\]
By change of variables, immediately we obtain
\[
\frac{1}{2} \int_M a(x)|W_{\delta,q}|^2 d\mu_g = \frac{\delta^2}{2} \int_{\mathbb{R}^+_n} a(x)U^2(y)\sqrt{g(\delta y)} dy + o(\delta^2) = O(\delta^2).
\]
Coming to boundary integral, we get, by change of variables, by (4.6), and by expanding \( b, \)
\[
\frac{n-2}{4} \int_{\partial M} b(z)|W_{\delta,q}|^2 d\sigma = \delta \frac{n-2}{4} \int_{\mathbb{R}^{n-1}} b(\delta z)U^2(z,0)\sqrt{g(\delta z)} dz + O(\delta^2)
\]
\[
= \delta b(q) \frac{n-2}{4} \int_{\mathbb{R}^{n-1}} U^2(z,0) dz + O(\delta^2).
\]
By (3.15), (3.16) and (4.6), we have
\[
\int_{\partial M} |W_{\delta,q}|^{2n-2} \delta^\varepsilon d\sigma = \int_{\mathbb{R}^{n-1}} \delta^{\varepsilon} \frac{2n-2}{4} U^{2n-2}(z,0)U^{\pm\varepsilon}(z,0)\sqrt{g(\delta z)} dz + o(\delta)
\]
\[
= \int_{\mathbb{R}^{n-1}} U^{2n-2}(z,0) dz \pm \varepsilon \int_{\mathbb{R}^{n-1}} U^{\frac{2n-2}{n-1}}(z,0) \ln U(z,0) dz
\]
\[
\pm \frac{n-2}{2} \varepsilon \ln \delta \int_{\mathbb{R}^{n-1}} U^{\frac{2n-2}{n-1}}(z,0) dz + o(\delta) + O(\varepsilon^2) + O(\varepsilon^2 \ln \delta)
\]
Proof.

Let us introduce the proof of Theorem 1: completed.

4.2. Some technicalities.

Notice that, with the choice \( \delta = \varepsilon \) it holds \( o(\delta) + O(\varepsilon^2) + O(\varepsilon^2 \ln \delta) = o(\varepsilon) \) and \( \varepsilon \ln \delta = \varepsilon \ln d - \varepsilon [\ln \varepsilon] \). At this point we have

\[
J_c(W_{d,q}) = C(\varepsilon) + \varepsilon d \frac{n-2}{4} [b(q) - H(q)] \int_{\mathbb{R}^n} U^2(z,0) dz
\]

\[
\pm \varepsilon \frac{(n-2)^3}{2(2n-2)} \ln d \int_{\mathbb{R}^n} U^2(n-2) \ln (z,0) dz + O(\varepsilon ^{2} \ln \varepsilon )
\]

To conclude observe that

\[
\int_{\mathbb{R}^n} U^2(z,0) dz = \omega_n - 1 I_{n-2}^n \quad \text{and} \quad \int_{\mathbb{R}^n} U^2(n-2) \ln (z,0) dz = \omega_n - 1 I_{n-1}^n
\]

where \( I_{n-2}^n = \int_0^\infty \frac{\varepsilon}{(1+s)^{n-1}} ds \). The thesis follows after that we observe that \( I_{n-2}^n = \frac{n-3}{2(n-2)} I_{n-2}^n \) (for a proof, see [3], Lemma 9.4 (b)). \( \square \)

4.2. Proof of Theorem 1: completed.

Proof. Let us introduce

\[
\hat{I}(d,q) = \alpha_n d \varphi(q) - \beta_n \ln d.
\]

If \( q_0 \) is a local minimizer of \( \varphi(q) \) with \( \varphi(q_0) > 0 \), set \( d_0 = \frac{\beta_n}{\alpha_n \varphi(q_0)} > 0 \). Thus the pair \((d_0, q_0)\) is a critical point for \( \hat{I} \). Moreover, since there exists a neighborhood \( B \) such that \( \varphi(q) > \varphi(q_0) \) on \( \partial B \), it is possible to find a neighborhood \( \hat{B} \subset [a,b] \times \partial M \), \((d_0, q_0) \in \hat{B} \) such that \( \hat{I}(d,q) > \hat{I}(d_0,q_0) \) for \((d,q) \in \partial \hat{B} \). Since, in the subcritical case, by (i) of Proposition 5 we have

\[
I_c(d,q) = c_n(\varepsilon) + \varepsilon \hat{I}(d,q) + o(\varepsilon)
\]

we get that, for \( \varepsilon \) sufficiently small there exists a \((d^*, q^*) \in \hat{B} \) such that \( W_{d^*, q^*} + \varphi_{d^*, q^*} \) is a critical point for \( I_c \). Then, by (i) of Proposition 5 \( W_{d^*, q^*} + \varphi_{d^*, q^*} \in \mathcal{H} \) is a solution for problem \((1.3)\) in the subcritical case.

The proof for the supercritical case follows in a similar way. \( \square \)

4.3. Some technicalities. If \( U \) is a solution of \((2.3)\), the following equalities hold

\[
(4.7) \quad \int_{\mathbb{R}^n_+} t|\nabla U|^2 dz dt = \frac{1}{2} \int_{\mathbb{R}^n} U^2(z,0) dz
\]

\[
(4.8) \quad \int_{\mathbb{R}^n_+} t|\nabla U|^2 dz dt = \int_{\mathbb{R}^n} t|\partial_t U|^2 dz dt
\]

\[
(4.9) \quad \int_{\mathbb{R}^n_+} t \sum_{i=1}^{n-1} |\partial_{z_i} U|^2 dz dt = \frac{1}{4} \int_{\mathbb{R}^n} U^2(z,0) dz
\]
Proof. To simplify the notation, we set
\[ \eta = (z,t) \in \mathbb{R}^n_+, \text{ where } z \in \mathbb{R}^{n-1} \text{ and } t \geq 0. \]
The first estimate can be obtained by integration by parts, and taking into account that \( \Delta U = 0 \); indeed
\[
\int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 d\eta = -\sum_{i=1}^n \int_{\mathbb{R}^n_+} U \partial_i [\eta_n \partial_i U] d\eta
\]
\[ = -\int_{\mathbb{R}^n_+} U \partial_n U d\eta - \int_{\mathbb{R}^n_+} \eta_n \Delta U d\eta
\]
\[ = -\frac{1}{2} \int_{\mathbb{R}^n_+} \partial_n [U^2] d\eta = \frac{1}{2} \int_{\mathbb{R}^{n-1}} U^2(z,0) dz. \]
To obtain (4.8) we proceed in a similar way: since \( \Delta U = 0 \) we have
\[
0 = -\int_{\mathbb{R}^n_+} \Delta U \eta_n^2 \partial_n U d\eta = \sum_{i=1}^n \int_{\mathbb{R}^n_+} \partial_i U \partial_i [\eta_n^2 \partial_n U] d\eta
\]
\[ = \int_{\mathbb{R}^n_+} 2\eta_n |\partial_n U|^2 d\eta + \sum_{i=1}^n \int_{\mathbb{R}^n_+} \eta_n^2 \partial_i U \partial_i^2 U d\eta
\]
\[ = \int_{\mathbb{R}^n_+} 2\eta_n |\partial_n U|^2 d\eta + \frac{1}{2} \int_{\mathbb{R}^n_+} \eta_n^2 |\nabla U|^2 d\eta
\]
\[ = \int_{\mathbb{R}^n_+} 2\eta_n |\partial_t U|^2 d\eta - \int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 d\eta
\]
so (4.8) is proved. Equation (4.9) is a direct consequence of the first two equalities.
In fact by (4.8) we have
\[
\int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 d\eta = \int_{\mathbb{R}^n_+} \eta_n \sum_{i=1}^{n-1} |\partial_i U|^2 d\eta + \int_{\mathbb{R}^n_+} \eta_n |\partial_n U|^2 d\eta
\]
\[ = \int_{\mathbb{R}^n_+} \eta_n \sum_{i=1}^{n-1} |\partial_i U|^2 d\eta + \frac{1}{2} \int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 d\eta
\]
thus
\[
\int_{\mathbb{R}^n_+} \eta_n \sum_{i=1}^{n-1} |\partial_i U|^2 d\eta = \frac{1}{2} \int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 d\eta
\]
and in light of (4.7) we get the proof. \( \square \)

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